Normal forms for transseries and Dulac germs

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Normalne forme za transredove i Dulacove klice

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To my family, all my mentors and friends.

SUMMARY

Transseries are formal (possibly infinite) sums of monomials that are formal products of iterated exponentials, powers and iterated logarithms, with real coefficients (see e.g. [6], [21]). We consider here a subclass of *logarithmic transseries* which contain only powers and iterated logarithms. Transseries appear in many problems in mathematics ([3], [11]) and physics ([1]) as asymptotic expansions of certain important maps. In dynamics, for example, transseries are related to the famous *Dulac's problem* ([7]) of non-accumulation of limit cycles on a hyperbolic or semi-hyperbolic polycycle of a planar analytic vector field. The problem was solved independently by *Ilyashenko* ([10], [11], [12]) and *Écalle* ([3]), but the proofs are so far not well-understood, at least in the semi-hyperbolic case. The study of the accumulation of limit cycles on a polycycle is naturally related to the study of fixed points of the *first return map* of a polycycle (see e.g. [32]). The first return map of a hyperbolic polycycle is an analytic map on interval $(0,d)$, $d>0$, which has a *transserial* asymptotic expansion at zero. In particular, its asymptotic expansion at zero is a logarithmic series involving only polynomials in logarithms attached to each power, which is called a *Dulac series* (see e.g. [12], [32]). The proof of the Dulac problem strongly relies on the existence of a logarithmic asymptotic expansion of the first return map. Although Dulac gave the proof ([7]) of the mentioned problem, there was an imprecision in his proof. In particular, at some point in the proof, the statement that every first return map of a hyperbolic polycycle is uniquely determined by its asymptotic expansion is used. This is not correct in general for non-analytic maps on the real line, due to the possibility of adding exponentially *small* terms, as opposed to the case of analytic maps and their Taylor expansions. Ilyashenko corrected this imprecision in [11] by proving that every such map can be analytically extended to a sufficiently large complex domain called a *standard quadratic domain* and by applying the *Phragmen-Lindelöf Theorem* (a

maximum modulus principle on an unbounded complex domain, see e.g. [11], [12]). The existence of such analytic extension makes possible to conclude that the first return map is uniquely determined by its logarithmic asymptotic expansion.

In this dissertation, we consider the so-called *Dulac germs* (called *almost regular germs* in [12]), i.e., analytic germs on $(0,d)$, $d>0$, that have a Dulac series as their asymptotic expansion at zero, and that can be analytically extended to a standard quadratic domain. On the one hand, we consider normal forms of logarithmic transseries (the *formal part*), and, on the other hand, analytic normalizations of hyperbolic and strongly hyperbolic Dulac germs (the *analytic part*). We also generalize to their complex counterparts, called hyperbolic and strongly hyperbolic *complex Dulac germs*. We prove as well that, for hyperbolic and strongly hyperbolic Dulac germs, the formal normalizations are asymptotic expansions of their analytic normalizations.

In proving that the formal transserial normalization is an asymptotic expansion of the analytic normalization, in general, there is a problem of a choice of the *summation rule* at the limit ordinal steps. In particular, given some map f on open interval $(0,d)$, $d>0$, we want to assign to the map f its asymptotic expansion at zero in power-iterated logarithm scale. Up to the first limit ordinal it can be done following the usual *Poincaré algorithm*, contrary to the limit ordinal steps where we have multiple choices of *intermediate sums*. Therefore, we have to determine a summation rule at limit ordinal steps, which vary from problem to problem (see e.g. *integral summation rule* in [20], [22]). Luckily, for hyperbolic and strongly hyperbolic Dulac germs the formal normalizations are Dulac series, so standard Poincaré algorithm suffices. On the other hand, in case of parabolic Dulac germs, it is proved in [22] that, in general, the formal normalization is of order type strictly bigger than ω.

Normal forms and normalizations of standard power series are already known (see e.g. [4], [12], [16]). Furthermore, in previous papers ([21], [22]), normal forms for logarithmic transseries were obtained only for power-logarithm transseries, i.e., logarithmic transseries that contain only powers and the first iterate of logarithm. The techniques used in [21] are based on a transfinite algorithm of successive changes of variables. Here, we

generalize these results to a larger class of logarithmic transseries containing also iterated logarithms. Additionally, as a normalization process we use fixed point theorems on various complete metric spaces of logarithmic transseries. In this way, normalizations are given as limits (in appropriate topologies) of *Picard sequences*. This is important for the future work because we think that this approach is better for revealing the summation rule at limit ordinal steps.

In proving the existence of the analytic normalization of a hyperbolic Dulac germ, we generalize the classical *Koenigs Theorem* (see e.g. [4], [14], [24]) for linearization of analytic hyperbolic diffeomorphisms at zero. Recently, there have been some improvements of this result for various classes of maps not necessarily analytic at 0. One such generalization is a result of *Dewsnap* and *Fischer* [5] for C^1 real maps on an open interval around zero with *power-logarithmic asymptotic bounds*. In this dissertation, we prove a linearization theorem for analytic maps with power-logarithmic asymptotic bounds on invariant complex domains, that can be seen as a generalization of both Koenigs Theorem and the result of Dewsnap and Fischer from [5, Theorem 2.2].

In particular, we apply the mentioned linearization theorem to obtain the analytic linearization of hyperbolic (complex) Dulac germs.

Finally, we also generalize the *Böttcher Theorem* (see e.g. [4], [24]) for germs of strongly hyperbolic analytic diffeomorphisms at zero to strongly hyperbolic complex Dulac germs on standard quadratic domains.

Key words: logarithmic transseries, order of transseries, normal forms, normalization, linearization, formal and analytic classification, (complex) Dulac germs, Dulac series, standard quadratic domains, local fixed point theory, fixed point theorems, iteration theory, Koenigs sequence

SAŽETAK

Transredovi su formalne (beskonačne) sume formalnih umnožaka iteriranih eksponencijalnih funkcija, općih potencija i iteriranih logaritama (koje nazivamo monomi) s realnim koeficijentima (vidjeti npr. [6], [21]). U ovoj disertaciji bavimo se podklasom takozvanih *logaritamskih transredova* čiji monomi sadrže samo opće potencije i iterirane logaritme. Transredovi se pojavljuju u mnogim problemima u matematici ([3], [11]) i fizici ([1]) kao asimptotski razvoji nekih značajnih preslikavanja. U dinamičkim sustavima transredovi su primjerice povezani s poznatim *Dulacovim problemom* ([7]) o neakumulaciji graničnih ciklusa na hiperbolički ili semi-hiperbolički policiklus ravninskog analitičkog vektorskog polja. Iako su navedeni problem nezavisno riješili *Ilyashenko* ([10], [11], [12]) i *Écalle* ([3]), rješenja semi-hiperboličkog slučaja i dalje nisu u potpunosti shvaćena. Akumulacija graničnih ciklusa na policiklus se prirodno povezuje s teorijom fiksnih točaka *preslikavanja povrata* (ili *Poincaréovog preslikavanja*) danog policiklusa (vidjeti npr. [32]). Preslikavanje povrata hiperboličkog policiklusa je analitičko preslikavanje na intervalu $(0,d)$, $d>0$, s transredom kao asimptotskim razvojem u nuli. Preciznije, njegov asimptotski razvoj u nuli je logaritamski red u kojem, uz svaku opću potenciju, stoji polinom u logaritmima. Takav red nazivamo *Dulacov red* (vidjeti npr. [12], [32]). Rješenje Dulacovog problema uvelike se oslanja na postojanje logaritamskog asimptotskog razvoja preslikavanja povrata u nuli. Iako je Dulac dao rješenje navedenog problema, u njegovom rješenju ([7]) je postojala nepreciznost. Naime, bez dokaza je korištena tvrdnja da je svako preslikavanje povrata hiperboličkog policiklusa jedinstveno određeno svojim asimptotskim razvojem. Navedena tvrdnja općenito nije istinita za realna preslikavanja koja nisu analitička u nuli zbog mogućnosti dodavanja eksponencijalno *malih* članova u razvoju, za razliku od analitičkih preslikavanja u nuli i njihovih Taylorovih razvoja. Ilyashenko je u [11] otklonio navedenu nepreciznost dokazavši da se svako preslikavanje

povrata može analitički proširiti na dovoljno veliku kompleksnu domenu koju nazivamo *standardna kvadratna domena*. *Phragmen-Lindelöfov teorem* (varijanta principa maksimuma na neomeđenoj kompleksnoj domeni, vidjeti npr. [11], [12]) tada daje injektivnost asimptotskog razvoja za preslikavanja povrata. Na taj način možemo zaključiti da je preslikavanje povrata jedinstveno određeno svojim logaritamskim asimptotskim razvojem.

U ovoj disertaciji promatramo takozvane *Dulacove klice* (*skoro regularne klice* u [12]), tj. analitičke klice na $(0,d)$, $d>0$, kojima je asimptotski razvoj u nuli Dulacov red te koje se mogu analitički proširiti na neku standardnu kvadratnu domenu. S jedne strane, promatramo normalne forme logaritamskih transredova (*formalni dio*). S druge strane, promatramo analitičke normalizacije (jako) hiperboličkih Dulacovih klica (*analitički dio*) te njihovih generalizacija koje nazivamo (jako) hiperboličkim *kompleksnim Dulacovim klicama*. Također dokazujemo da je formalna normalizacija asimptotski razvoj analitičke normalizacije (jako) hiperboličkih Dulacovih klica.

U dokazu da je formalna normalizacija asimptotski razvoj analitičke normalizacije općenito se javlja problem izbora *sumacijskog pravila* u koracima indeksiranim graničnim ordinalima. Naime, pretpostavimo da želimo odrediti asimptotski razvoj u nuli u skali općih potencija i iteriranih logaritama za dano preslikavanje f na otvorenom intervalu $(0,d)$, $d>0$. Koristeći standardni *Poincaréov algoritam* to se može napraviti do prvog graničnog ordinala. U koracima određenim graničnim ordinalima postoje višestruki izbori takozvanih *međusuma*. Upravo zbog toga je potrebno odrediti pravila sumacije, koja ovise o problemu kojeg promatramo (vidjeti *integralno pravilo sumacije* u [20], [22]). Kod hiperboličkih i jako hiperboličkih Dulacovih klica, formalne normalizacije su, srećom, Dulacovi redovi, pa nam je dovoljan Poincaréov algoritam za asimptotski razvoj. S druge strane, u [22] je dokazano da je formalna normalizacija paraboličkih Dulacovih klica transred indeksiran ordinalom strogo većim od ω.

Normalne forme i normalizacije standardnih redova potencija su otprije poznate (vidjeti npr. [4], [12], [16]). Nadalje, u prijašnjim radovima ([21], [22]) normalne forme su određene samo za transredove tipa potencija-logaritam, tj. za logaritamske transredove koji sadržavaju samo opće potencije i prvu iteraciju logaritma. Tehnike korištene u [21] temelje se na transfinitoj kompoziciji elementarnih zamjena varijabli. U ovoj disertaciji generaliziramo navedene rezultate na širu klasu svih logaritamskih transredova koji mogu sadržavati i iterirane logaritme. Nadalje, u postupku normalizacije koristimo teoreme fiksne točke na potpunim metričkim prostorima logaritamskih transredova. Time su normalizacije dane kao limesi (u odgovarajućim topologijama) *Picardovih iteracija*. Smatramo da je ovaj pristup problemu normalizacije bolji pri određivanju sumacijskog pravila na mjestima graničnih ordinala, što ga čini bitnim za naš budući rad.

U dokazu postojanja analitičke normalizacije hiperboličkih Dulacovih klica, generaliziramo klasični *Koenigsov teorem* (vidjeti npr. [4], [14], [24]) koji daje linearizaciju analitičkih hiperboličkih difeomorfizama u nuli. Nedavno je ovaj rezultat poboljšan za razne klase preslikavanja koja nisu nužno analitička u nuli. Jedno takvo poboljšanje je *Dewsnap-Fischerov* rezultat [5] za realna preslikavanja klase C^1 na otvorenom intervalu oko nule, s asimptotskim ocjenama tipa potencija-logaritam. U ovoj disertaciji dokazujemo linearizacijski teorem za analitička preslikavanja s asimptotskim ocjenama tipa potencija-logaritam na invariantnim kompleksnim domenama, koji možemo smatrati generalizacijom i Koenigsovog teorema i Dewsnap-Fischerovog rezultata iz [5, Theorem 2.2].

Nadalje, navedeni linearizacijski teorem primjenjujemo prilikom analitičke linearizacije hiperboličkih (kompleksnih) Dulacovih klica.

Naposljetku, generaliziramo *Böttcherov teorem* (vidjeti npr. [4], [24]) za jako hiperboličke difeomorfizme u nuli, za klasu jako hiperboličkih kompleksnih Dulacovih klica na standardnim kvadratnim domenama.

Ključne riječi: logaritamski transredovi, red logaritamskih transredova, normalne forme, normalizacija, linearizacija, formalna i analitička klasifikacija, (kompleksne) Dulacove klice, Dulacovi redovi, standardne kvadratne domene, lokalna teorija fiksne točke, teoremi fiksne točke, teorija iteracija, Koenigsov niz

Sažetak

CONTENTS

LIST OF SYMBOLS

$\mathbb{N}$	the set of all natural numbers $\{0, 1, 2, 3, \ldots\}$
$\mathbb{N}_{\geq a}$	the set of all natural numbers n such that $n \geq a$
$\mathbb{Z}$	the set of all integers
$\mathbb{R}$	the set of all real numbers
$\mathbb{R}_{>a}$	the set of all real numbers x such that $x > a$
$\mathbb{R}_{\geq a}$	the set of all real numbers x such that $x \geq a$
$\mathbb{C}$	the set of all complex numbers
$\mathbb{C}^+$	the set of all complex numbers z such that $\Re(z) > 0$
$\widetilde{\mathbb{C}}$	the Riemann surface of the logarithm, see Definition 3.1.1
$\mathbb{R}[x]$	the space of all polynomials with real coefficients
$\mathbb{R}[[x]]$	the space of all power series with real coefficients
$x^n\mathbb{R}[[x]]$	the space of all power series f with real coefficients of order bigger than or equal to n
$\mathbb{C}[x]$	the space of all polynomials with complex coefficients
$\mathbb{C}[[x]]$	the space of all power series with complex coefficients
$\mathscr{L}_k$	the differential algebra of all logarithmic transseries of depth $k \in \mathbb{N}$, see Subsection 1.1.2
$\mathscr{L}_k^\infty$	the differential algebra of all generalized logarithmic transseries of depth $k \in \mathbb{N}$, see Subsection 1.1.2
$\mathfrak{L}$	the differential algebra of all logarithmic transseries, see Subsection 1.1.2
$\mathfrak{L}^\infty$	the differential algebra of all generalized logarithmic transseries, see Subsection 1.1.2
$\mathscr{L}_k^H$	the group of all $f \in \mathscr{L}_k$ without logarithms in their leading terms, see Subsection 1.1.5

$\mathfrak{L}^H$	the group of all $f \in \mathfrak{L}$ without logarithms in their leading terms, see Subsection 1.1.5
$\mathscr{L}_k^0$	the group of all parabolic $f \in \mathscr{L}_k^H$, see Subsection 1.1.5
$\mathfrak{L}^0$	the group of all parabolic $f \in \mathfrak{L}^H$, see Subsection 1.1.5
$\mathscr{B}_m$	the space of all blocks of level m, for $1 \leq m \leq k$ and $k \in \mathbb{N}_{\geq 1}$, see Remark 1.1.21, 2.
$\mathscr{B}_m[[x]]$	the space of all power series with coefficients in $\mathscr{B}_m$
$x^n\mathscr{B}_m[[x]]$	the space of all power series with coefficients in $\mathscr{B}_m$ multiplied by x^n, for $n \in \mathbb{N}$
$\mathrm{Supp}(f)$	the support of a logarithmic transseries f, see Definition 1.1.3
$\mathrm{Supp}_z(f)$	the support in the variable z of a logarithmic transseries f, see Subsection 1.1.2
$\mathrm{Supp}_{z,\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_m}(f)$	the support in variables $z,\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_m$ of a logarithmic transseries f, see Subsection 1.1.2
$\mathrm{ord}(f)$	the order of a logarithmic transseries f, see Definition 1.1.5
$\mathrm{ord}_z(f)$	the order in the variable z of a logarithmic transseries f, see Subsection 1.1.2
$\mathrm{ord}_{z,\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_m}(f)$	the order in variables $z,\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_m$ of a logarithmic transseries f, see Subsection 1.1.2
$\mathrm{Lt}(f)$	the leading term of a logarithmic transseries f, see Definition 1.1.8
$\mathrm{Lm}(f)$	the leading monomial of a logarithmic transseries f, see Definition 1.1.8
$\mathrm{Lb}_z(f)$	the leading block in the variable z of a logarithmic transseries f, see Subsection 1.1.2
$[f]_{\alpha,\mathbf{n}}$	the coefficient in front of the monomial of order $(\alpha,\mathbf{n})$ in a logarithmic transseries f, see Definition 1.1.7
$\mathrm{Res}(f)$	the residual monomial of a parabolic logarithmic transseries f, see Definition 2.3.3

$\mathrm{Res}_t\,(f)$	the residual term of a parabolic logarithmic transseries f, see Definition 2.3.3
$\mathrm{ord}_{\boldsymbol{\ell}_m}(R)$	the order in the variable $\boldsymbol{\ell}_m$ of a block $R \in \mathscr{B}_m$, see Definition 1.1.23
$\mathrm{Lb}_{\boldsymbol{\ell}_m}(R)$	the leading block in the variable $\boldsymbol{\ell}_m$ of a block $R \in \mathscr{B}_m$, see Definition 1.1.24
$\mathscr{L}_k^W$	the space of all $f \in \mathscr{L}_k$ such that $\mathrm{Supp}\,(f) \subseteq W$, see Definition 1.1.12
$\mathscr{L}_k^\beta$	the space of all $f \in \mathscr{L}_k$ such that $\mathrm{ord}_z(f) \geq \beta$, for $\beta \in \mathbb{R}_{>0}$, see Subsection 1.1.2
$\mathscr{L}_k^{>\beta}$	the space of all $f \in \mathscr{L}_k$ such that $\mathrm{ord}_z(f) > \beta$, for $\beta \in \mathbb{R}_{>0}$, see Subsection 1.1.2
$\mathfrak{L}_{>\mathrm{id}}$	the space of all $f \in \mathfrak{L}$ such that $\mathrm{ord}\,(f) > \mathbf{0}$, see Subsection 2.1.2
$\mathscr{B}_m^W$	the space of all $R \in \mathscr{B}_m$ such that $\mathrm{Supp}\,(R) \subseteq W$, see Subsection 1.1.4
$\mathscr{B}_m^+$	the space of all $R \in \mathscr{B}_m$ such that $\mathrm{ord}_{\boldsymbol{\ell}_m}(R) \geq 1$, see Subsection 1.1.4
$\mathscr{B}_{\geq m}^+$	the space of all $R \in \mathscr{B}_m$ such that $\mathrm{ord}\,(R) > \mathbf{0}_{k+1}$, for $1 \leq m \leq k$, $k \in \mathbb{N}_{\geq 1}$, see Subsection 1.1.4
h.o.t.	higher order terms, see Subsection 1.1.2
$\mathrm{h.o.b.}(z)$	higher order blocks in the variable z, see Subsection 1.1.2
$\mathrm{h.o.b.}\,(\boldsymbol{\ell}_m)$	higher order blocks in the variable $\boldsymbol{\ell}_m$, see Subsection 1.1.4
d_z	the power-metric, see Subsection 1.1.3
d_m	the m-metric, see Definition 1.1.26
D_m	the derivation $\boldsymbol{\ell}_m^2 \cdot \frac{d}{d\boldsymbol{\ell}_m}$ on the space $\mathscr{B}_m$, see Definition 1.1.22
$\mathscr{P}_f$	the generalized Koenigs operator (see Definition 2.1.8) of a hyperbolic logarithmic transseries f in Section 2.1, or the Böttcher operator (see Definition 2.2.3) of a strongly hyperbolic logarithmic transseries f in Section 2.2

$\mathscr{R}_f$	the Böttcher operator on the space $\mathrm{id}+z\mathscr{B}^+_{\geq 1}$ of a strongly hyperbolic logarithmic transseries f, see Definition 2.2.7
$\mathrm{Supp}\,(f(\zeta))$	the support of an exponential transseries $f(\zeta)$, see Definition 3.1.11
$\mathrm{Supp}_{\mathrm{e}^{-1}}\,(f(\zeta))$	the support in the variable e^{-1} of an exponential transseries $f(\zeta)$, see Subsection 3.1.2
$\mathrm{ord}_{\mathrm{e}^{-1}}\,(f(\zeta))$	the order in the variable e^{-1} of an exponential transseries $f(\zeta)$, see Subsection 3.1.2
$\widehat{f}(z)$	the asymptotic expansion of a complex Dulac germ f in the z-chart, see Remark 3.1.14
$\widehat{f}(\zeta)$	the asymptotic expansion of a complex Dulac germ f in the ζ-chart, see Definition 3.1.13
$f \sim \widehat{f}$	the series $\widehat{f}$ is the asymptotic expansion of a germ f, see Definition 3.1.13
$\mathscr{R}_C$	the standard quadratic domain with coefficient $C \in \mathbb{R}_{>0}$, see Definition 3.1.8
D_R	the complex domain $D \cap ([R,+\infty) \times \mathbb{R})$, for $R \in \mathbb{R}_{>0}$ and $D \subseteq \mathbb{C}$, see Subsection 3.2.1
D_{h_l,h_u}	the complex domain *bounded* by the graphs of continuous maps $h_l, h_u : [t,+\infty) \to \mathbb{R}$, $t > 0$, from *below* and *above* respectively, see Subsection 3.2.1
$\overline{D}$	the maximal subdomain of type (β, ε, k) of an admissible domain D, see Subsection 3.2.1
D^f	the maximal f-invariant subdomain of a complex domain $D \subseteq \mathbb{C}$, see Subsection 3.2.1

INTRODUCTION

Transseries are formal sums of products of powers, iterated logarithms and iterated exponentials with real coefficients and well-ordered *supports* (i.e., well-ordered index sets). Through the years they have become a very useful tool for tackling a great variety of problems in mathematics (see e.g. [3], [11]) and physics (see e.g. [1]). Transseries in the broader sense are studied in [6]. In this dissertation, the main object of our interest are *logarithmic transseries*, i.e., formal sums, with real (or complex) coefficients, of products of powers and iterated logarithms with well-ordered supports.

The reason for that restriction comes from a particular problem in dynamics. Logarithmic transeries appear in the solution of the *Dulac problem* of non-accumulation of limit cycles on hyperbolic or semi-hyperbolic polycycles of analytic planar vector fields ([7]). In particular, the *first return map* (or *Poincaré map*) (see e.g. [31]) of a hyperbolic polycycle has a logarithmic transseries as its asymptotic expansion at zero, see e.g. [32]. In the case of hyperbolic polycycles these asymptotic expansions are logarithmic series of a particular type, and we call them *Dulac series*. One of the important properties of Dulac series is that they do not involve iterated logarithms and that every power of the variable is multiplied by a *polynomial* in the logarithm. However, the original Dulac's proof of non-accumulation theorem (see [7]) was incomplete, because he assumed (without proving it) the nontrivial fact that every first return map of a hyperbolic polycycle of an analytic planar vector field is uniquely determined by its asymptotic expansion. *Ilyashenko* ([10], [11]) and *Écalle* ([3]) independently solved the Dulac problem. In particular, Ilyashenko, in the proof of the hyperbolic case, completed Dulac's proof by showing a nontrivial fact that every first return map can be analytically extended to a sufficiently large complex domain. Such domains are called the *standard quadratic domains*, since their boundaries are asymptotic to the graphs of quadratic real maps. Using the

Phragmen-Lindelöf Theorem (a maximum modulus principle on a particular unbounded complex domain), Ilyashenko proved that the first return map of a hyperbolic polycycle is uniquely determined by its asymptotic expansion, see [12]). This property is called the *quasi-analyticity*. Analytic germs on standard quadratic domains with Dulac series as their asymptotic expansions are called the *almost regular germs* in [10], [12]), but, in this dissertation, we call them simply *Dulac germs*. By the above argument, they are uniquely determined by their Dulac asymptotic expansion.

In this dissertation, the main objects of our interest are the logarithmic transseries and the Dulac germs.

In the first part of the dissertation, we are interested in logarithmic transseries that do not involve logarithms in their leading terms, i.e., that are of the form $f = \lambda z^{\alpha} +$ "higher order terms", for $\lambda, \alpha > 0$. By definition, the Dulac series (i.e., the asymptotic expansions of the Dulac germs) are a subclass of the logarithmic transseries. Following [21, Definition 1.1], we distinguish three types of logarithmic transseries with real coefficients: *parabolic* ($\alpha = \lambda = 1$), *hyperbolic* ($\lambda \neq 1$, $\alpha = 1$) and *strongly hyperbolic* ($\alpha \neq 1$).

We consider the problem of finding the *normal forms* with the smallest number of terms (or "short" normal forms). More precisely, for a logarithmic transseries $f = \lambda z^{\alpha} +$ h.o.t., $\lambda, \alpha > 0$, we find a logarithmic transseries g with the smallest number of terms, such that the *conjugacy equation* $\varphi \circ f \circ \varphi^{-1} = g$ has a solution φ in the space of parabolic logarithmic transseries. We call such g the *normal form* of f, and we call φ the *normalization* of f.

For standard power series the normal forms and normalizations are well-understood (see e.g. [4], [12], [16]). Furthermore, normal forms for power-logarithm transseries, i.e., transseries which do not involve iterated logarithms, were already found in [21, Theorem A]. Here, we generalize these results to logarithmic transseries involving iterated logarithms, but using a different method. The method used in [21] is based on transfinite compositions of parabolic elementary changes of variables, which are chosen *step-by-step* in order to eliminate *term-by-term* in the original transseries. A generalization of this method to logarithmic transseries involving iterated logarithms seems too complicated.

In the case of logarithmic transseries involving iterated logarithms, there is also an interesting phenomenon, which is new with respect to the non-iterated logarithmic case. In [21], the ”short” normal forms of power-logarithm transseries are finite. However, this is not the case even if we allow only two iterations of the logarithm. Normal forms for hyperbolic and parabolic logarithmic transseries have, in general, infinitely many terms.

All this motivated us to develop a ”less transfinite” method based on fixed point theorems. We prove in Proposition 1.2.12 an easy consequence of the Banach Fixed Point Theorem motivated by the *Krasnoselskii Fixed Point Theorem* (see e.g. [36]), which plays a crucial role in our proofs of the normalization theorems. More precisely, we transform the conjugacy equation $\varphi \circ f \circ \varphi^{-1} = g$ into a fixed point equation and prove the existence and the uniqueness of the solution using a fixed point theorem. These fixed point methods allow us to have better control of the support of the normalization, which might also be useful to define the notion of transserial asymptotic expansions in future work. More precisely, for logarithmic transseries with supports of order type ω we apply the standard Poincaré term-by-term algorithm to get the unique logarithmic asymptotic expansion. However, if the order type of logarithmic transseries is strictly bigger than ω, for a well-defined logarithmic asymptotic expansion, we have to specify a *summation rule*, i.e., a *canonical choice* of germs at limit ordinal steps of the expansion. For example, the *integral summation rule* from [20], [22] is an example of such summation rule.

In the case of hyperbolic and strongly hyperbolic logarithmic transseries, we generalize two classical theorems from local complex dynamics: the *Koenigs Theorem* (see e.g. [4], [14], [24]) and the *Böttcher Theorem* (see e.g. [4], [24]). These are normalization theorems for hyperbolic and strongly hyperbolic germs of analytic diffeomorphisms at zero. Here, for a hyperbolic logarithmic transseries f and its normal form f_0, we prove that the so-called *generalized Koenigs sequence* $\left(f_0^{\circ(-n)} \circ h \circ f^{\circ n}\right)_n$ converges to the normalization φ in an appropriate topology on the space of logarithmic transseries. We prove a similar statement for strongly hyperbolic logarithmic transseries, motivated by the Böttcher Theorem. Our results for hyperbolic logarithmic transseries are given in preprint [29].

In the second part of the dissertation we consider the so-called complex Dulac germs, i.e., Dulac germs on standard quadratic domains with complex coefficients in their Dulac asymptotic expansions. The partial results are published in [30]. We consider hyperbolic and strongly hyperbolic complex Dulac germs and solve the analytic *linearization* (or normalization) problem in the class of holomorphic Dulac germs. The linearization problem for a real or complex map f in one variable, with an isolated fixed point zero, consists in finding a number λ and a change of coordinates φ that satisfy the *Schröder's equation* ([33])

$$\varphi(f(z)) = \lambda\varphi(z).$$

In the proof of the classical Koenigs Theorem, the change of coordinates $\varphi(z) = z + o(z)$ is obtained as the uniform limit of the so-called *Koenigs sequence* $\left(\frac{1}{\lambda^n}\cdot f^{\circ n}\right)_n$. Throughout the years many proofs have been given for the convergence of the Koenigs sequence for different classes of germs, not necessarily analytic at the fixed point. For example, *Knaser* ([13]) proved the convergence of the Koenigs sequence for a hyperbolic attracting real germ of the form $f(x) = \lambda x + O(|x|^{1+\varepsilon})$, as $x \to 0$, where $\varepsilon > 0$. In [35], *Szekeres* proved the convergence of the Koenigs sequence for a continuous germ f which has a strictly increasing differentiable representative on an open interval $(0,d)$, $d > 0$, such that $0 < f(x) < x$, $x \in (0,d)$, and $f'(x) = \lambda + O(x^{\varepsilon})$, as $x \to 0$, for $0 < \lambda < 1$ and $\varepsilon > 0$. Also, in [26], [34], *Sternberg* gave a proof of the convergence of the Koenigs sequence for real germs of class C^n, $n \in \mathbb{N}_{\geq 2}$. For our application, the most interesting was the result of *Dewsnap* and *Fischer* (see [5, Theorem 2.2]), which proves the convergence of the Koenigs sequence for C^1 real germs admitting logarithmic asymptotic behavior at zero of the form:

$$f(x) = f'(0)\cdot x + O\left(\frac{x}{y\log(y)\cdots\log^{\circ p-1}(y)\,(\log^{\circ p}(y))^{1+\varepsilon}}\right),$$

as $x \to 0$, for $\varepsilon > 0$ and $p \in \mathbb{N}$. Here, we denote $y := -\log(|x|)$. The addition of the small shift $\varepsilon > 0$ in the exponent above seems to be important for the convergence (i.e., for the linearization) for two reasons. The first reason is more an indication than a proof. It is proved in the first part of the dissertation that a hyperbolic logarithmic transseries $f = \lambda z +$ "higher order terms" can be *linearized* if and only if $f = \lambda z + a\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_p^{1+n} +$ h.o.t., where $\boldsymbol{\ell}_1 := -\frac{1}{\log z}$, and inductively $\boldsymbol{\ell}_i := -\frac{1}{\log(\boldsymbol{\ell}_{i-1})}$, $i = 2,\ldots,p$, for $a \in \mathbb{R}$ and

$n \in \mathbb{N}_{\geq 1}$. The second reason is the counter-example given by Sternberg in [34]: Let $f(x) = x\left(\lambda - \frac{1}{\log(x)}\right)$, for $x \in (0,d]$, $d > 0$, and $f(0) = 0$. Its Koenigs sequence diverges on $(0,d]$ (see also [5], [26]).

Additionally, as an intermediate step of the proof of the linearization theorem for hyperbolic complex Dulac germs, in the second part of the thesis, we prove the convergence of the Koenigs sequence for analytic maps with hyperbolic logarithmic asymptotic behavior on their invariant complex domains. In this sense, our linearization result can be viewed as a generalization to complex domains of both the Koenigs Theorem and the result of Dewsnap and Fischer from [5].

Similarly, motivated by the Böttcher Theorem, we prove that, for a strongly hyperbolic complex Dulac germ $f(z) = z^\alpha + o(z^\alpha)$, $\alpha > 1$, there exists a unique parabolic complex Dulac germ φ holomorphic on a standard quadratic domain, which *normalizes* f to its first term, i.e., such that

$$\varphi(f(z)) = (\varphi(z))^\alpha.$$

Finally, note that, in this thesis, we do not consider the analytic normalization of parabolic Dulac germs using the fixed point theory. Other than difference in dynamics that is sectorially attractive/repulsive, and the fact that the parabolic Dulac germs are far from being globally linearizable on standard quadratic domains (by [20], their analytic classes are given by a variant of *Écalle-Voronin moduli*), the additional difference between (strongly) hyperbolic and parabolic Dulac germs is that the formal normalization of a (strongly) hyperbolic Dulac germ is a Dulac series, while for a parabolic Dulac germ, in general, it is a much more complicated logarithmic transseries of order typer strictly bigger than ω. For more details, see [20] and [22].

Overview of the main results in the thesis

The main results of the thesis are in Chapter 2 and Chapter 3. The main theorems from Chapter 2 are formal normalization theorems for hyperbolic, strongly hyperbolic and parabolic logarithmic transseries stated in Theorems A, B and C, respectively. We state below their ”short” forms.

Theorem A (short form). Every hyperbolic logarithmic transseries $f = \lambda z + o(z)$, $0 < \lambda < 1$, can be formally linearized by a parabolic logarithmic change of variables φ if and only if $f = \lambda z + o(z\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_k^{1+n})$, $n \geq 1$.

Moreover, the parabolic linearization φ is unique and is given as the limit of the so-called *Koenigs sequence*

$$\left(\frac{1}{\lambda^n} \cdot f^{\circ n}\right)_n$$

in the appropriate formal topology.

In the general version of Theorem A in Section 2.1 we give the explicit ”short” normal form and convergence of the *generalized Koenigs sequence* in non-linearizable case.

Theorem B (short form). Every strongly hyperbolic logarithmic transseries $f = z^\alpha + o(z^\alpha)$, $\alpha > 0$, $\alpha \neq 1$, can be formally normalized to its first term $f_0 := z^\alpha$ by the unique parabolic logarithmic change of variables φ.

Furthermore, if $\alpha > 1$, φ is the limit of the so-called *Böttcher sequence*

$$\left(z^{\frac{1}{\alpha^n}} \circ h \circ f^{\circ n}\right)_n$$

in the appropriate formal topology, for every initial parabolic condition h.

Theorem C (short form). Every parabolic logarithmic transseries

$$f = z + az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + \text{higher order terms}, \quad a \neq 1, \beta > 1,$$

can be formally reduced to its ”short” normal form

$$f_c := z + az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + cz^{2\beta-1} \boldsymbol{\ell}_1^{2n_1+1} \cdots \boldsymbol{\ell}_k^{2n_k+1}, \quad c \in \mathbb{R},$$

by parabolic logarithmic change of variables φ that is of the same depth in logarithm as f. Moreover, the so-called *residual coefficient* c is unique.

In Section 2.3 Theorem C is stated also for parabolic logarithmic transseries where $\beta = 1$. The initial part of the ”short” normal form f_c is more complicated in that case.

The main results from Chapter 3 are analytic normalization theorems for (complex) Dulac germs stated in Theorem D and Theorem E. We state here their short forms.

Theorem D (short form). Every hyperbolic (complex) Dulac germ $f(z) = \lambda z + o(z)$, $0 < |\lambda| < 1$, on a standard quadratic domain, can be analytically linearized by the unique parabolic (complex) Dulac germ.

Theorem E (short form). Every strongly hyperbolic (complex) Dulac germ $f(z) = z^\alpha + o(z^\alpha)$, $\alpha > 1$, on a standard quadratic domain, can be analytically normalized to the map $z \mapsto z^\alpha$ by the unique parabolic (complex) Dulac germ.

The structure of the thesis

The thesis is divided in three chapters.

Chapter 1 serves as a prerequisite for the remaining chapters. More precisely, in Section 1.1 of Chapter 1 we define the differential algebra of logarithmic transseries, some basic notions such as composition and *blockwise notation*, and prove the *Taylor Theorem* in this formal setting. Section 1.2 is dedicated to proving a fixed point theorem (stated in Proposition 1.2.12). The tools from Section 1.2 and Appendix B are the main tools for solving the normalization equations in the next chapter.

In Chapter 2 we find normal forms of hyperbolic, strongly hyperbolic and parabolic logarithmic transseries by solving the appropriate normalization equations. We use the fixed point method from Chapter 1. The main normalization theorems for the three types of the logarithmic transseries (Theorems A, B and C) are stated in Sections 2.1, 2.2 and 2.3 respectively.

The first two chapters represent the *formal* part of the thesis: the results are obtained in the *formal setting* (in the differential algebra of logarithmic transseries). On the other hand, Chapter 3 is the *analytic* counterpart to Chapter 2. In particular, in Chapter 3 we apply the formal normalization results from Chapter 2 in order to obtain the analytic normal forms for (strongly) hyperbolic complex Dulac germs on standard quadratic domains. Chapter 3 is divided into three sections. Section 3.1 serves as a prerequisite for the remaining two sections. In particular, in Section 3.1 we define basic notions such as analytic germs on spiraling subdomains of the Riemann surface of the logarithm and complex Dulac germs (series) on standard quadratic domains. In Sections 3.2 and 3.3 we solve the normalization equations for hyperbolic (Theorem D) and strongly hyperbolic complex Dulac germs (Theorem E) respectively. In both sections we first solve the normalization equations on appropriate invariant domains and then relate the solutions with formal solutions obtained in Chapter 2 via asymptotic expansions.

Finally, in Appendix A and Appendix B we prove some useful technical results that are used throughout the thesis. In Appendix A we give a list of useful formulas in differential algebras of logarithmic transseries, while in Appendix B we solve various differential equations in differential algebras of transseries. Technical results from Appendix A are used to transform normalization equations in Chapter 2 into appropriate fixed point equations, and to solve differential equations in Appendix B. The solutions to various differential equations from Appendix B are used to apply fixed point theorem to appropriate fixed point equations.

1. Preliminaries

The main object of our study in this chapter are logarithmic transseries which are, roughly speaking, formal sums of formal product of powers and iterated logarithms. They are studied in [6] in more general form. On the other hand, in [21] and [22] they are studied under additional restrictions and are used as tools for solving the particular dynamical problems. In particular, we study here the logarithmic transseries without logarithms in their leading terms. Among them we distinguish: parabolic, hyperbolic and strongly hyperbolic logarithmic transseries. This chapter serves as a prerequisite for Chapter 2 and Chapter 3. In Section 1.1 we introduce differential algebras of logarithmic transseries and some basic notions. It is not necessary for a familiar reader to read this section in detail. On the other hand, Section 1.2 is dedicated to developing the fixed point techniques that are crucial for proving the normalization theorems in Chapter 2.

1.1. Differential algebras of logarithmic transseries

We first recall in Subsection 1.1.1 the notions of well-ordered sets and of basic ordered algebraic structures which are used to define the differential algebras of logarithmic transseries $\mathfrak{L}$, $\mathfrak{L}^{\infty}$, and their subalgebras $\mathscr{L}_k$ and $\mathscr{L}_k^{\infty}$ respectively, in Subsection 1.1.2. Furthermore, in Subsection 1.1.3 and in preprint [29] we introduce the *power-metric topology* (the *valuation topology* from [6]), the *product topology* and the *weak topology* on these algebras. The same topologies were already introduced in [21] for power-logarithmic transseries without iterated logarithms.

We introduce the *blockwise notation*, where we consider a logarithmic transseries as a power series with *blocks* of logarithms as its *coefficients*, see also [29]. In Subsection 1.1.4, we define differential algebras of blocks. We consider blocks as elements of $\mathfrak{L}^\infty$, but in the variable $-\frac{1}{\log z}$ instead of z. Therefore, all notions in Subsection 1.1.4 are similar to those defined in Subsection 1.1.2. Finally, in Subsection 1.1.5 and Appendix A we define the composition of logarithmic transseries and prove the Taylor Theorem in the formal setting.

1.1.1. Well-ordered sets and basic ordered algebraic structures

Well-ordered sets

We recall shortly the following standard definitions that can be found e.g. in [15].

Let $(W,\leq)$ be an ordered set. We say that $(W,\leq)$ is *well-ordered* if every nonempty subset of W has a minimum.

The *lexicographic order* $\leq$ on the product $W_1\times\cdots\times W_n$ of ordered sets $(W_i,\leq_i)$, $i=1,\ldots,n$, is the relation on $W_1\times\cdots\times W_n$ defined by: $(w_1,\ldots w_n)\leq(v_1,\ldots,v_n)$ if $(w_1,\ldots w_n)=(v_1,\ldots,v_n)$ or if there exists $i\in\{1,\ldots,n-1\}$ such that $w_1=v_1,\ldots,w_i=v_i$ and $w_{i+1}<_{i+1}v_{i+1}$.

In particular, in this thesis we consider the well-ordered subsets of $\mathbb{R}_{\geq 0}\times\mathbb{Z}^k$, $k\in\mathbb{N}$, with respect to the lexicographic order.

Example 1.1.1. Suppose that (α_n) is a strictly increasing sequence of real numbers tending to $+\infty$. Then:

1. $\{(\alpha_n,-n):n\in\mathbb{N}\}\subseteq\mathbb{R}_{\geq 0}\times\mathbb{Z}$ is a well-ordered subset of $\mathbb{R}_{\geq 0}\times\mathbb{Z}$, with respect to the lexicographic order.

2. $\{(2^{-\alpha_n},n):n\in\mathbb{N}\}\subseteq\mathbb{R}_{\geq 0}\times\mathbb{Z}$ is not a well-ordered subset of $\mathbb{R}_{\geq 0}\times\mathbb{Z}$, with respect to the lexicographic order.

Ordered semigroups, monoids and groups

Recall that nonempty set S equipped with associative operation $+:S\times S\to S$ is called a *semigroup*.

Let $\leq$ be a total order on a semigroup S, that satisfies the property that $a \leq b$ implies that, for each $c \in S$, $a+c \leq b+c$. Then $(S,+,\leq)$ is called *ordered semigroup*.

If S is additionally a monoid (a semigroup with neutral element), then we call S an *ordered monoid*. If S is a group, we call S an *ordered group*.

For example, the set $\mathbb{R} \times \mathbb{Z}^k$, $k \in \mathbb{N}$, with component-wise addition, with $(0,0,\ldots,0) \in \mathbb{R} \times \mathbb{Z}^k$ as the neutral element, and with the lexicographic order, is an ordered Abelian group. The set $\mathbb{R}_{\geq 0} \times \mathbb{Z}^k$, $k \in \mathbb{N}$, with the same structure as in 1, is an ordered Abelian monoid.

Let S be an ordered semigroup and $a \in S$. We set:

$$
\begin{aligned}
S_{\geq a} &:= \{s \in S : s \geq a\}, \\
S_{>a} &:= \{s \in S : s > a\}, \\
S_{\leq a} &:= \{s \in S : s \leq a\}, \\
S_{<a} &:= \{s \in S : s < a\}.
\end{aligned}
$$

Note that, if S is an ordered monoid and $a > 0$, then $S_{\geq a}$ and $S_{>a}$ are ordered semigroups.

Let S be a semigroup and $A \subseteq S$. The intersection of all sub-semigroups of S that contain A is called the *sub-semigroup of S generated by* A, and is denoted by $\langle A \rangle$.

The set A is called a *set of generators for* $\langle A \rangle$.

Let S be an Abelian semigroup and $A \subseteq S$. It is easy to see that:

$$\langle A \rangle = \{n_1 a_1 + \cdots + n_m a_m : m \in \mathbb{N}_{\geq 1},\, n_i \in \mathbb{N}_{\geq 1},\, a_i \in A,\, i = 1,\ldots,m\}.$$

Let G be an ordered group and $\langle A \rangle$ its sub-semigroup generated by $A \subseteq G$. It is natural to ask the question: If A is a well-ordered subset of G, is $\langle A \rangle$ also well-ordered? The answer to this question in the class of Abelian groups is given by the *Neumann Lemma* (see e.g. [27]), which we state here without proof and use several times throughout the thesis.

Theorem 1.1.2 (The Neumann Lemma, [27])**.** Let G be an ordered Abelian group and $A,B \subseteq G$ well-ordered subsets of G. Then:

1. $A+B$[1] is a well-ordered subset of G,

2. for each $g \in A+B$ there exist only finitely many pairs $(a,b) \in A \times B$ such that $g = a+b$,

3. if $A \subseteq G_{>0}$ is a well-ordered subset of $G_{>0}$, then $\langle A \rangle$ is also well-ordered. Moreover, for each $g \in \langle A \rangle$ there are only finitely many $n \in \mathbb{N}_{\geq 1}$ and finitely many n-tuples $(a_1, \ldots, a_n) \in A^n$ such that $g = a_1 + \cdots + a_n$.

1.1.2. Differential algebra of logarithmic transseries

From now on, we work in the ordered Abelian group $\mathbb{R} \times \mathbb{Z}^k$, for $k \in \mathbb{N}$, with respect to the standard component-wise addition, and equipped with the lexicographic order.

In the sequel, we use the following notation for multi-indices:

1. $\mathbf{n} := (n_1, \ldots n_k)$, for $n_i \in \mathbb{Z}$, and $k \in \mathbb{N}_{\geq 1}$ (it will be always clear from the context which $k \in \mathbb{N}_{\geq 1}$ we consider),

2. $\mathbf{a}_k := (a, \ldots, a)_k \in \mathbb{R}^k$, for $k \in \mathbb{N}_{\geq 1}$, where the subscript k means that $(a, \ldots, a)_k$ is a k-tuple,

3. for $\mathbf{n} = (n_1, \ldots, n_k)$ we put $(\mathbf{n}, \mathbf{a}_m) := (n_1, \ldots, n_k, \underbrace{a, \ldots, a}_{m \text{ times}})$, for $m \in \mathbb{N}_{\geq 1}$.

Furthermore, we often consider $\mathbf{n} := (n_1, \ldots, n_k)$ as an element of $\mathbb{R}^{k+m}$ by the usual identification $\mathbf{n} := (\mathbf{n}, \mathbf{0}_m)$, for $k, m \in \mathbb{N}_{\geq 1}$. With that identification we have the inclusion $\mathbb{R} \times \mathbb{Z}^k \subseteq \mathbb{R} \times \mathbb{Z}^{k+m}$, for $k, m \in \mathbb{N}$. Similarly, for $\mathbf{n} := (n_1, \ldots, n_k)$, $k \in \mathbb{N}_{\geq 1}$, using the identification $\mathbf{n} := (\mathbf{n}, 0, 0, \ldots) \in \mathbb{R}^{\mathbb{N}_{\geq 1}}$, we consider $\mathbf{n}$ as an element of $\mathbb{R}^{\mathbb{N}_{\geq 1}}$.

This section represents a generalization of the notions introduced in [21] for the differential algebra $\mathscr{L}_1$ of power-logarithm transseries. Almost all notions from this section are introduced in the preprint [29, Sections 2, 3]. In this chapter, z is a *formal variable* and $\log z$ is a *formal logarithm*. Put $\boldsymbol{\ell}_0 := z$, and inductively put

$$\boldsymbol{\ell}_{m+1} := -\frac{1}{\log(\boldsymbol{\ell}_m)}, \quad m \in \mathbb{N}.$$

[1] Let S be a semigroup, $n \in \mathbb{N}_{\geq 2}$, and $S_1, \ldots, S_n \subseteq S$. Then we define the set $S_1 + \cdots + S_n \subseteq S$ by $S_1 + \cdots + S_n := \{s_1 + \cdots + s_n : s_i \in S_i,\, i = 1, \ldots, n\}$.

Definition 1.1.3 (Logarithmic transseries of depth k, [29])**.** For every $k \in \mathbb{N}$ we define the *logarithmic transseries of depth k* as a formal sum:

$$f = \sum_{(\alpha,\mathbf{n}) \in \mathbb{R}\times\mathbb{Z}^k} a_{\alpha,\mathbf{n}} z^{\alpha} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}, \tag{1.1}$$

where $a_{\alpha,\mathbf{n}} \in \mathbb{R}$ and

$$\mathrm{Supp}\,(f) := \left\{(\alpha,\mathbf{n}) \in \mathbb{R}\times\mathbb{Z}^k : a_{\alpha,\mathbf{n}} \neq 0\right\}$$

is a well-ordered subset of $\mathbb{R}\times\mathbb{Z}^k$ that contains only elements strictly bigger than $\mathbf{0}_{k+1}$. We call $\mathrm{Supp}\,(f)$ the *support* of f. If $\mathrm{Supp}\,(f) = \emptyset$, we call f the *zero transseries* and denote it by 0.

Remark 1.1.4.

1. Note that a transseries of depth zero is just a formal sum of powers with a well-ordered subset of exponents with respect to the standard order on $\mathbb{R}_{>0}$.

2. Since $\mathrm{Supp}\,(f)$, for a logarithmic transseries f of depth $k \in \mathbb{N}$, is a well-ordered subset of $\mathbb{R}\times\mathbb{Z}^k$, we can write:

$$f = \sum_{\alpha \in A_f} \sum_{n_1 = N_\alpha}^{+\infty} \sum_{n_2 = N_{\alpha,n_1}}^{+\infty} \cdots \sum_{n_k = N_{\alpha,n_1,\ldots,n_{k-1}}}^{+\infty} a_{\alpha,\mathbf{n}} z^{\alpha} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k},$$

   where A_f is the projection of $\mathrm{Supp}\,(f)$ on the first coordinate. Hence, A_f is a well-ordered subset of $\mathrm{Supp}\,(f)$. Note that some coefficients in above formal sum may be equal to zero.

In the sequel we define important notions such as order, leading monomial, leading term of a logarithmic transseries, etc., which are useful while working with logarithmic transseries.

Definition 1.1.5 (Order of a logarithmic transseries, [29])**.** Let f be a logarithmic transseries of depth $k \in \mathbb{N}$. If $f = 0$, we say that the *order* of f is infinity, and denote it by $\mathrm{ord}\,(f) = \infty$. If $f \neq 0$, then the minimum of $\mathrm{Supp}\,(f)$ is called the *order* of f and denoted by $\mathrm{ord}\,(f)$.

Remark 1.1.6. To be precise, in Definition 1.1.5 above, we consider the extension of the lexicographic order to the set $\left(\mathbb{R}\times\mathbb{Z}^k\right) \cup \{\infty\}$, by posing $(\alpha,\mathbf{n}) < \infty$, for every $(\alpha,\mathbf{n}) \in \mathbb{R}\times\mathbb{Z}^k$, $k \in \mathbb{N}$. Note that the zero transseries has the maximal order in the set of all logarithmic transseries.

Definition 1.1.7 (The coefficient of order $(\alpha,\mathbf{n})$ in a logarithmic transseries, [29]). Let

$$f=\sum_{(\alpha,\mathbf{n})\in\mathbb{R}\times\mathbb{Z}^k} a_{\alpha,\mathbf{n}} z^{\alpha}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}$$

be a logarithmic transseries of depth $k\in\mathbb{N}$. We call $a_{\alpha,\mathbf{n}}$ the *coefficient of order* $(\alpha,\mathbf{n})$ in the logarithmic transseries f, and denote it by $[f]_{\alpha,\mathbf{n}}$.

We call $z^{\alpha}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}$ the *monomial of order* $(\alpha,\mathbf{n})$ and $a_{\alpha,\mathbf{n}}z^{\alpha}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}$, $a_{\alpha,\mathbf{n}}\neq 0$, the *term of order* $(\alpha,\mathbf{n})$. Furthermore, we call a term of order $\mathbf{0}_{k+1}$ a *constant*. Note that the zero transseries is not a constant, since it is of order ∞.

For transseries f and g of depth $k\in\mathbb{N}$, we use the notation

$$f=g+\text{h.o.t.}$$

(which means: *higher order terms*) if every term in $f-g$ is of order that is strictly bigger than the order of every term in g.

Definition 1.1.8 (Leading monomial and leading term of a logarithmic transseries, [29]). Let f be a nonzero logarithmic transseries of depth $k\in\mathbb{N}$. We call the monomial (term) of order $\operatorname{ord}(f)$ in f the *leading monomial* (*leading term*) of f, and denote it by $\operatorname{Lm}(f)$ ($\operatorname{Lt}(f)$).

We denote by $\mathscr{L}_k$ the set of all logarithmic transseries of depth $k\in\mathbb{N}$. Adopting the notation from the beginning of the section for the multiindices, note that $\mathscr{L}_k\subseteq\mathscr{L}_{k+1}$, for $k\in\mathbb{N}$. Put

$$\mathfrak{L}:=\bigcup_{k\in\mathbb{N}}\mathscr{L}_k.$$

We call $\mathfrak{L}$ the *set of logarithmic transseries*.

By $\mathscr{L}_k^{\infty}$, $k\in\mathbb{N}$, we denote the set of all logarithmic transseries f as in (1.1), where we allow that the support of f contains elements in $\mathbb{R}\times\mathbb{Z}^k$ that are not necessarily strictly bigger than $\mathbf{0}_{k+1}$. Note that $\mathscr{L}_k^{\infty}\subseteq\mathscr{L}_{k+1}^{\infty}$, for each $k\in\mathbb{N}$. Furthermore, let

$$\mathfrak{L}^{\infty}:=\bigcup_{k\in\mathbb{N}}\mathscr{L}_k^{\infty}.$$

Note that $\mathscr{L}_k \subseteq \mathscr{L}_k^\infty$, for $k \in \mathbb{N}$, and $\mathfrak{L} \subseteq \mathfrak{L}^\infty$. As for logarithmic transseries, we analogously define the order, the leading term and the leading monomial of $f \in \mathfrak{L}^\infty$.

Note that $\mathscr{L}_k$, $\mathscr{L}_k^\infty$, $k \in \mathbb{N}$, and $\mathfrak{L}$, $\mathfrak{L}^\infty$ are real linear spaces with respect to the usual *termwise* addition and scalar multiplication. Furthermore, $\mathscr{L}_k$ ($\mathscr{L}_k^\infty$) is a subspace of $\mathscr{L}_{k+m}$ ($\mathscr{L}_{k+m}^\infty$), $m \in \mathbb{N}_{\geq 1}$, and of $\mathfrak{L}$ ($\mathfrak{L}^\infty$).

Multiplication in $\mathfrak{L}^\infty$

Let $f, g \in \mathfrak{L}^\infty$ be arbitrary. Suppose that $f \in \mathscr{L}_{k_1}^\infty$ and $g \in \mathscr{L}_{k_2}^\infty$, for $k_1, k_2 \in \mathbb{N}$. Now, set $k := \max\{k_1, k_2\}$. Since $\mathrm{Supp}\,(f) \subseteq \mathbb{R} \times \mathbb{Z}^{k_1}$ and $\mathrm{Supp}\,(g) \subseteq \mathbb{R} \times \mathbb{Z}^{k_2}$ are well-ordered subsets of $\mathbb{R} \times \mathbb{Z}^k$ (by the usual identification), by the Neumann Lemma (Theorem 1.1.2) it follows that the multiplication on $\mathfrak{L}^\infty$ can be defined *termwise*. It is classically called the *convolution product*.

If $f = 0$ or $g = 0$, we define $f \cdot g := 0$. Now, suppose that $f, g \neq 0$. Since $\mathscr{L}_{k_1}^\infty, \mathscr{L}_{k_2}^\infty \subseteq \mathscr{L}_k^\infty$, we write f and g in the *form*:

$$f = \sum_{(\alpha,\mathbf{n}) \in \mathrm{Supp}\,(f)} a_{\alpha,\mathbf{n}} z^\alpha \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}, \quad g = \sum_{(\beta,\mathbf{m}) \in \mathrm{Supp}\,(g)} b_{\beta,\mathbf{m}} z^\beta \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_k^{m_k}.$$

The *product* of f and g is defined by:

$$f \cdot g := \sum_{(\gamma,\mathbf{u}) \in \mathbb{R} \times \mathbb{Z}^k} \Big(\sum_{(\alpha,\mathbf{n})+(\beta,\mathbf{m})=(\gamma,\mathbf{u})} a_{\alpha,\mathbf{n}} b_{\beta,\mathbf{m}} \Big) z^\gamma \boldsymbol{\ell}_1^{u_1} \cdots \boldsymbol{\ell}_k^{u_k}$$

Now, it is easy to see that $\mathfrak{L}^\infty$ is an associative commutative $\mathbb{R}$-algebra with unity, and a field. Note that $\mathscr{L}_k^\infty$, for $k \in \mathbb{N}$, are subalgebras and subfields of $\mathfrak{L}^\infty$. Furthermore, $\mathfrak{L}$ is a subalgebra (without unity) of $\mathfrak{L}^\infty$, and $\mathscr{L}_k$ is a subalgebra (without unity) of $\mathscr{L}_k^\infty$, for each $k \in \mathbb{N}$.

Remark 1.1.9 ($\mathscr{L}_k^\infty$, $k \in \mathbb{N}$, as Hahn fields, see [6])**.** Let G be an ordered Abelian group with neutral element 0. Let $\mathbb{F}$ be a field and let $\mathbb{F}((G))$ be the set of all formal sums:

$$f = \sum_{g \in G} f_g \cdot g,$$

where $f_g \in \mathbb{F}$, for each $g \in G$, and $\mathrm{Supp}\,(f) := \left\{ g \in G : f_g \neq 0_{\mathbb{F}} \right\}$ is a well-ordered subset of G called the *support* of f. Then $\mathbb{F}((G))$ is a linear space with respect to the usual addition and scalar multiplication.

We define the multiplication in $\mathbb{F}((G))$ as:

$$f \cdot h := \sum_{g \in G} \Big(\sum_{g_1 + g_2 = g} (f_{g_1} \cdot_{\mathbb{F}} h_{g_2}) \cdot g \Big),$$

for each $f, h \in \mathbb{F}((G))$. By the Neumann Lemma, it follows that the multiplication defined above is a well-defined operation. It can be shown that every nonzero element in $\mathbb{F}((G))$ has a multiplicative inverse. Consequently, $\mathbb{F}((G))$ is an associative commutative algebra, and a field. For $\mathbb{F} := \mathbb{R}$, we call $\mathbb{R}((G))$ the *Hahn field* (see e.g. [21]).

Note that $\mathscr{L}_k^\infty$, $k \in \mathbb{N}$, is a particular Hahn field, where we set G to be the set of all monomials in $\mathscr{L}_k^\infty$ with the multiplication as a commutative operation and with the order induced by the lexicographic order on $\mathbb{R} \times \mathbb{Z}^k$, i.e.

$$z^\alpha \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} \preceq z^\beta \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_v^{m_k} \quad \text{if and only if} \quad (\alpha, n_1, \cdots, n_k) \leq (\beta, m_1, \ldots, m_k).$$

Remark 1.1.10 (Remark 2.1, [29])**.** The collection $\mathfrak{L}$ introduced above is a subset of the field of logarithmic-exponential series $\mathbb{R}((t))^{\mathrm{LE}}$ defined in [6]. Notice that, while the variable t in $\mathbb{R}((t))^{\mathrm{LE}}$ is infinite, we prefer here to work with the infinitesimal variable $z = t^{-1}$, which is more convenient in the framework of iteration theory. Actually, $\mathfrak{L}$ is even contained in the subfield $\mathbb{T}_{\log}$ of “purely logarithmic transseries” introduced in [2] and studied from a model-theoretic point of view in [8]. More precisely, in $\mathfrak{L}$ the iterated logarithms are raised to integer powers, whereas they are raised to arbitrary real powers in $\mathbb{T}_{\log}$.

Differential algebra $\mathfrak{L}^\infty$

Note that $\mathfrak{L}^\infty$ is a differential algebra with respect to the usual derivation $\frac{d}{dz}$ (*termwise*), and $\mathfrak{L}$, $\mathscr{L}_k^\infty$, $\mathscr{L}_k$, $k \in \mathbb{N}$, are its subalgebras.

Suppose that $f \in \mathfrak{L}^\infty$ has an antiderivative F. There exists $c \in \mathbb{R}$, such that the antiderivative $F - c$ does not contain a constant term. For simpler computations in the sequel, we use the following convention: by $\int f dz$ we denote the antiderivative of f without a constant term. For more about the antiderivative in $\mathfrak{L}^\infty$ and $\mathscr{L}_k^\infty$, $k \in \mathbb{N}$, see Proposition B.2.1 in Section B.2.

The following definition is a generalization of the definition given in [21, Subsection 3.3] in the differential algebra $\mathscr{L}_1$.

Definition 1.1.11 (The Lie bracket operator)**.** Let $[\cdot,\cdot] : \mathfrak{L}^\infty \times \mathfrak{L}^\infty \to \mathfrak{L}^\infty$ be defined by

$$[f,g] := g \cdot \frac{df}{dz} - f \cdot \frac{dg}{dz}, \quad f,g \in \mathfrak{L}^\infty.$$

We call $[\cdot,\cdot]$ the *Lie bracket operator*.

For some properties of the Lie bracket operator, see Lemma A.2.3 in Appendix A.

The blockwise notation for logarithmic transseries and its generalization

We call (1.1) the *termwise* notation for logarithmic transseries. It is often useful to re-write logarithmic transseries in the so-called *blockwise notation* (see [29, Subsection 2.1]):

$$\begin{aligned} f &= \sum_{(\alpha,\mathbf{n})\in\mathbb{R}\times\mathbb{Z}^k} a_{\alpha,\mathbf{n}} z^\alpha \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} \\ &= \sum_{\alpha\in\mathbb{R}} z^\alpha \Big(\sum_{\mathbf{n}\in\mathbb{Z}^k} a_{\alpha,\mathbf{n}} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} \Big) \\ &= \sum_{\alpha\in\mathbb{R}} z^\alpha R_\alpha, \end{aligned}$$

where

$$R_\alpha := \sum_{\mathbf{n}\in\mathbb{Z}^k} a_{\alpha,\mathbf{n}} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}.$$

Note that R_α can be considered as an element of $\mathscr{L}_{k-1}^\infty$ in the formal variable $\boldsymbol{\ell}_1$ instead of z.

We call $z^\alpha R_\alpha$ the α*-block* of f (or the *block of order* α in z), for $\alpha \in \mathbb{R}$. Moreover, we call

$$\operatorname{Supp}_z(f) := \{\alpha \in \mathbb{R} : R_\alpha \neq 0\}$$

the *support of* f *in* z.

Since $\operatorname{Supp}_z(f)$ is the projection of $\operatorname{Supp}(f)$ onto the first coordinate, $\operatorname{Supp}_z(f)$ is a well-ordered subset of $\mathbb{R}$.

Similarly as before, for a logarithmic transseries f we define the *order of* f *in* z by putting $\operatorname{ord}_z(f) := \infty$, if $f = 0$, and $\operatorname{ord}_z(f) := \min \operatorname{Supp}_z(f)$ otherwise. In this notation, the zero transseries has the maximal order in z in the differential algebra $\mathfrak{L}$.

If $f \neq 0$, we call the block of order $\mathrm{ord}_z(f)$ in f the *leading block of f in z*, and denote it by $\mathrm{Lb}_z(f)$. All these notions have an analogue in the larger differential algebra $\mathfrak{L}^\infty$.

For $f, g \in \mathfrak{L}^\infty$ we put

$$f = g + \text{h.o.b.}(z)$$

(which means: *higher order blocks*) if every block in $f - g$ is of order in z that is strictly bigger than the order of every block in g.

Finally, for $f \in \mathscr{L}_k^\infty$, we generalize the *blockwise* notation:

$$\begin{aligned} f &= \sum_{(\alpha,\mathbf{n})\in\mathbb{R}\times\mathbb{Z}^k} a_{\alpha,\mathbf{n}} z^\alpha \boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k} \\ &= \sum_{(\alpha,n_1,\ldots,n_m)\in\mathbb{R}\times\mathbb{Z}^m} z^\alpha \boldsymbol{\ell}_1^{n_1}\ldots\boldsymbol{\ell}_m^{n_m} \Big(\sum_{(m_{m+1},\ldots,n_k)\in\mathbb{Z}^{k-m}} a_{\alpha,\mathbf{n}} \boldsymbol{\ell}_{m+1}^{n_{m+1}}\cdots\boldsymbol{\ell}_k^{n_k}\Big) \\ &= \sum_{(\alpha,n_1,\ldots,n_m)\in\mathbb{R}\times\mathbb{Z}^m} z^\alpha \boldsymbol{\ell}_1^{n_1}\ldots\boldsymbol{\ell}_m^{n_m} R_{\alpha,n_1,\ldots,n_m}, \end{aligned}$$

where

$$R_{\alpha,n_1,\ldots,n_m} := \sum_{(n_{m+1},\ldots,n_k)\in\mathbb{Z}^{k-m}} a_{\alpha,\mathbf{n}} \boldsymbol{\ell}_{m+1}^{n_{m+1}}\cdots\boldsymbol{\ell}_k^{n_k}.$$

We call $z^\alpha \boldsymbol{\ell}_1^{n_1}\ldots\boldsymbol{\ell}_m^{n_m} R_{\alpha,n_1,\ldots,n_m}$ the $(\alpha,n_1,\ldots,n_m)$*-block* of f (or the *block of order* $(\alpha,n_1,\ldots,n_m)$) in variables $z,\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_m$, $(\alpha,n_1,\ldots,n_m)\in\mathbb{R}\times\mathbb{Z}^m$, for $0 \leq m \leq k$.
Since $\mathrm{Supp}(f)$ is a well-ordered subset of $\mathbb{R}\times\mathbb{Z}^k$, it can be proven that the set

$$\mathrm{Supp}_{z,\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_m}(f) := \{(\alpha,n_1,\ldots,n_m)\in\mathbb{R}\times\mathbb{Z}^m : R_{\alpha,n_1,\ldots,n_m}\neq 0\}$$

is a well-ordered subset of $\mathbb{R}\times\mathbb{Z}^m$ (with respect to the lexicographic order) which we call the *support of f in the variables $z,\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_m$*. Note that $R_{\alpha,n_1,\ldots,n_m}$ can be considered as an element of $\mathscr{L}_{k-m}^\infty$ in the formal variable $\boldsymbol{\ell}_{m+1}$ instead of z. Similarly as in case of the variable z, we define the order and the leading block of $f \in \mathfrak{L}^\infty$ in the variables $z,\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_m$.

The subspaces $\mathscr{L}_k^W \subseteq \mathscr{L}_k^\infty$

Definition 1.1.12 (see [29])**.** Let $W \subseteq \mathbb{R}\times\mathbb{Z}^k$, for $k\in\mathbb{N}$. We define

$$\mathscr{L}_k^W := \{f \in \mathscr{L}_k^\infty : \mathrm{Supp}(f) \subseteq W\}.$$

Since the zero transseries has the empty support, it follows that the zero transseries is an element of $\mathscr{L}_k^W$, for $W \subseteq \mathbb{R} \times \mathbb{Z}^k$. Note that $\mathscr{L}_k^W$ is a linear subspace of $\mathscr{L}_k^\infty$.

The proof of the next proposition is elementary, and is, therefore, left to the reader.

Proposition 1.1.13 (Properties of spaces $\mathscr{L}_k^W$, $k \in \mathbb{N}$)**.**

1. If W is a sub-semigroup of $\mathbb{R} \times \mathbb{Z}^k$, then $\mathscr{L}_k^W$ is a subalgebra of $\mathscr{L}_k^\infty$, for $k \in \mathbb{N}$.

2. If $W_1 \subseteq W_2 \subseteq \mathbb{R} \times \mathbb{Z}^k$, for $k \in \mathbb{N}$, then $\mathscr{L}_k^{W_1}$ is a subspace of $\mathscr{L}_k^{W_2}$.

3. Let $(W_i, i \in I)$ be a family of subsets of $\mathbb{R} \times \mathbb{Z}^k$, for $k \in \mathbb{N}$. If $(W_i, i \in I)$ are pairwise disjoint, then $\mathscr{L}_k^{\bigcup_{i\in I} W_i}$ is a direct sum of the family $(\mathscr{L}_k^{W_i}, i \in I)$, i.e.
$$\mathscr{L}_k^{\bigcup_{i\in I} W_i} = \bigoplus_{i\in I} \mathscr{L}_k^{W_i}.$$

For every real number $\alpha \geq 1$ and $k \in \mathbb{N}$, we define $W := \left\{(\beta, \mathbf{n}) \in \mathbb{R}_{\geq\alpha} \times \mathbb{Z}^k : (\beta, \mathbf{n}) > (1, \mathbf{0}_k)\right\}$ and:

$$\begin{aligned} \mathscr{L}_k^\alpha &:= \mathscr{L}_k^W, \\ \mathscr{L}_k^{>\alpha} &:= \mathscr{L}_k^{\mathbb{R}_{>\alpha} \times \mathbb{Z}^k}. \end{aligned}$$

Note that $\mathscr{L}_k^\alpha$ and $\mathscr{L}_k^{>\alpha}$ are subalgebras of $\mathscr{L}_k$, for $\alpha > 0$, $k \in \mathbb{N}$.

Moreover, for every $az^\alpha \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}$, $a \in \mathbb{R} \setminus \{0\}$, $(\alpha, n_1, \ldots, n_k) \in \mathbb{R} \times \mathbb{Z}^k$, and $W \subseteq \mathbb{R} \times \mathbb{Z}^k$ such that $(\alpha, n_1, \ldots, n_k) < w$, for each $w \in W$, we use the following notation:

$$az^\alpha \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + \mathscr{L}_k^W := \left\{az^\alpha \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + \varepsilon : \varepsilon \in \mathscr{L}_k^W\right\}.$$

Superlinear operators on $\mathfrak{L}^\infty$

Definition 1.1.14 (Superlinear operators on $\mathfrak{L}^\infty$)**.** Let $\mathscr{A}$ be some subspace of $\mathfrak{L}^\infty$ and let $\mathscr{S} : \mathscr{A} \to \mathfrak{L}^\infty$ be an operator such that

$$\mathscr{S}(f) = \sum_{(\alpha,\mathbf{n}) \in \mathrm{Supp}(f)} a_{\alpha,\mathbf{n}} \mathscr{S}(z^\alpha \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}),$$

for each $f := \sum_{(\alpha,\mathbf{n}) \in \mathrm{Supp}(f)} a_{\alpha,\mathbf{n}} z^\alpha \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} \in \mathscr{A}$. We call $\mathscr{S}$ a *superlinear operator on $\mathscr{A}$*.

Note that a superlinear operator is a linear operator, but the converse is not true in general.

1.1.3. The three topologies on the differential algebra $\mathfrak{L}^\infty$

In the previous section we defined the differential algebra $\mathfrak{L}^\infty$ and its differential subalgebras $\mathfrak{L}$, $\mathscr{L}_k^\infty$, and $\mathscr{L}_k$, for $k \in \mathbb{N}$. In this section (see also [29]) we introduce three topologies on the differential algebra $\mathfrak{L}^\infty$: *power-metric (or valuation) topology* (see [6]), *product topology* and *weak topology*. They represent generalizations of the corresponding topologies introduced in [21, Subsection 4.2] on the differential algebra $\mathscr{L}_1$.

Power-metric topology

Let $d_z : \mathfrak{L}^\infty \times \mathfrak{L}^\infty \to \mathbb{R}$ be a map defined by:

$$d_z(f,g) := \begin{cases} 2^{-\mathrm{ord}_z(f-g)}, & f \neq g, \\ 0, & f = g. \end{cases}$$

It is obvious that d_z is a metric on $\mathfrak{L}^\infty$ and $(\mathfrak{L}^\infty, d_z)$ is a metric space. We call d_z the *power-metric*, and the induced topology $\mathscr{T}_{d_z}$ the *power-metric topology* on $\mathfrak{L}^\infty$ ([29, Subsection 2.3]). It is easy to see that the power-metric topology is the same as the *valuation topology* defined in [6] and the *formal topology* defined in [21, Subsection 4.2] on $\mathscr{L}_1$. Now, the subalgebras $\mathfrak{L}$, $\mathscr{L}_k^\infty$, $\mathscr{L}_k$, and the linear subspaces $\mathscr{L}_k^W$, for $W \subseteq \mathbb{R} \times \mathbb{Z}^k$, $k \in \mathbb{N}$, are metric spaces with respect to the appropriate restrictions of the power-metric d_z.

In Example 1.1.15 and Proposition 1.1.16 we discuss the completeness of the metric spaces $(\mathfrak{L}^\infty, d_z)$, $(\mathfrak{L}, d_z)$ and also of subspaces $\mathscr{L}_k^\infty$, $\mathscr{L}_k$, for $k \in \mathbb{N}$.

Example 1.1.15. Consider the sequence (φ_n) in $\mathfrak{L}$ defined by $\varphi_n := \sum_{i=0}^n z^i \boldsymbol{\ell}_i$, for $n \in \mathbb{N}$. It is easy to check that (φ_n) is a Cauchy sequence on $(\mathfrak{L}, d_z)$. Indeed, $d_z(\varphi_n, \varphi_{n+m}) = \frac{1}{2^{\mathrm{ord}_z(z^{n+1}\boldsymbol{\ell}_{n+1})}} = \frac{1}{2^{n+1}}$, for $n \in \mathbb{N}$ and $m \in \mathbb{N}_{\geq 1}$. Now, suppose that (φ_n) converges in the space $(\mathfrak{L}^\infty, d_z)$ to some $\varphi \in \mathfrak{L}^\infty$. Consequently, there exists $k \in \mathbb{N}$ such that $\varphi \in \mathscr{L}_k^\infty$. Therefore, $\mathrm{ord}_z(\varphi - \varphi_n) \leq k+1$, i.e., $d_z(\varphi, \varphi_n) \geq \frac{1}{2^{k+1}}$, for every $n \geq k+1$, which is a contradiction with the assumption that (φ_n) converges to φ in $(\mathfrak{L}^\infty, d_z)$. By the definition of $\mathfrak{L}^\infty$, it follows that $(\mathfrak{L}^\infty, d_z)$ is not a complete metric space. Since (φ_n) is also a sequence in $\mathfrak{L}$ and $\mathfrak{L} \subseteq \mathfrak{L}^\infty$, it follows that $(\mathfrak{L}, d_z)$ is not a complete metric space.

In the previous example we showed that $(\mathfrak{L}, d_z)$ is not a complete metric space, by constructing the Cauchy sequence (φ_n) which does not converge. A crucial fact in the previous example is that there is no $k \in \mathbb{N}$, such that $\varphi_n \in \mathscr{L}_k$, for all $n \in \mathbb{N}$. This leads us to the next proposition.

Proposition 1.1.16 (Completeness of spaces $\mathscr{L}_k^\infty$, $\mathscr{L}_k$ and $\mathscr{L}_k^W$, Proposition 3.6, [29])**.** The spaces $(\mathscr{L}_k^\infty, d_z)$, $(\mathscr{L}_k, d_z)$ and $(\mathscr{L}_k^W, d_z)$, for $W \subseteq \mathbb{R} \times \mathbb{Z}^k$, $k \in \mathbb{N}$, are complete.

Proof. We first prove that $\left(\mathscr{L}_k^\infty, d_z\right)$ is a complete space. Suppose that (g_n) is a Cauchy sequence in the space $\left(\mathscr{L}_k^\infty, d_z\right)$. Hence, for every $\alpha \in \mathbb{R}$ there exists $n_\alpha \in \mathbb{N}$, such that $p, q \geq n_\alpha$ implies $\operatorname{ord}_z\left(g_p - g_q\right) > \alpha$. So, for every $n \geq n_\alpha$, every $\beta \leq \alpha$ and every $\mathbf{m} \in \mathbb{Z}^k$:

$$[g_n]_{\beta,\mathbf{m}} = [g_{n_\alpha}]_{\beta,\mathbf{m}}\,. \tag{1.2}$$

We define an element $g \in \mathbb{R}^{\mathbb{R}\times\mathbb{Z}^k}$ by setting, for every $(\alpha, \mathbf{m}) \in \mathbb{R} \times \mathbb{Z}^k$,

$$[g]_{\alpha,\mathbf{m}} = [g_{n_\alpha}]_{\alpha,\mathbf{m}}\,. \tag{1.3}$$

It remains to prove that g is indeed an element of $\mathscr{L}_k^\infty$, and that $(g_n) \to g$ in $\left(\mathscr{L}_k^\infty, d_z\right)$. In order to prove that $g \in \mathscr{L}_k^\infty$, it is enough to prove that $\operatorname{Supp}(g) \subseteq \mathbb{R} \times \mathbb{Z}^k$ is well-ordered. Let A be a nonempty subset of $\operatorname{Supp}(g)$ and let $(\alpha, \mathbf{m}) \in A$. It follows from the definition of g that there exists $n_\alpha \in \mathbb{N}$ such that $[g]_{\alpha,\mathbf{m}} = [g_{n_\alpha}]_{\alpha,\mathbf{m}}$. By (1.2) we have

$$[g]_{\beta,\mathbf{k}} = \left[g_{n_\beta}\right]_{\beta,\mathbf{k}} = [g_{n_\alpha}]_{\beta,\mathbf{k}}\,, \tag{1.4}$$

for every $\beta \leq \alpha$ and $\mathbf{k} \in \mathbb{Z}^k$. From (1.4) we deduce that

$$\{(\beta, \mathbf{n}) \in A : (\beta, \mathbf{n}) \leq (\alpha, \mathbf{m})\} \subseteq \operatorname{Supp}(g_{n_\alpha})\,.$$

Since $\operatorname{Supp}(g_{n_\alpha})$ is well-ordered, the set A admits a minimum element $\min A$. This implies that $\operatorname{Supp}(g)$ is a well-ordered subset of $\mathbb{R} \times \mathbb{Z}^k$. Finally, it follows easily from (1.2) and (1.3) that $(g_n) \to g$ in $\left(\mathscr{L}_k^\infty, d_z\right)$.

Now we prove that $(\mathscr{L}_k^W, d_z)$ is a complete space. Since $\mathscr{L}_k^W \subseteq \mathscr{L}_k^\infty$ and $(\mathscr{L}_k^\infty, d_z)$ is a complete space, it is sufficient to prove that $\mathscr{L}_k^W$ is a closed subset of $\mathscr{L}_k^\infty$ with respect to the power-metric topology. If $\mathscr{L}_k^W = \mathscr{L}_k^\infty$, then $\mathscr{L}_k^W$ is closed. Therefore, suppose that $\mathscr{L}_k^W \neq \mathscr{L}_k^\infty$. Let $f \in \mathscr{L}_k^\infty \setminus \mathscr{L}_k^W$ be arbitrary. Consequently, $\operatorname{Supp}(f) \nsubseteq W$. Since, $\operatorname{Supp}(f)$ is a well-ordered set, let $(\alpha, \mathbf{n}) := \min(\operatorname{Supp}(f) \setminus W)$. It is easy to check that the open ball $B(f, \frac{1}{2^\alpha})$ is a subset of $\mathscr{L}_k^\infty \setminus \mathscr{L}_k^W$. Indeed, for each $g \in B(f, \frac{1}{2^\alpha})$ it follows that

$\mathrm{ord}_z(f-g) > \alpha$. Therefore, $[g]_{\alpha,\mathbf{n}} = [f]_{\alpha,\mathbf{n}} \neq 0$, which implies that $(\alpha,\mathbf{n}) \in \mathrm{Supp}(g) \setminus W$. Consequently, it follows that $g \in \mathscr{L}_k^\infty \setminus \mathscr{L}_k^W$. Therefore, $\mathscr{L}_k^\infty \setminus \mathscr{L}_k^W$ is open, i.e., $\mathscr{L}_k^W$ is closed in $(\mathscr{L}_k^\infty, d_z)$.

Since $\mathscr{L}_k = \mathscr{L}_k^W$, for the set W of all elements of $\mathbb{R}\times\mathbb{Z}^k$ that are strictly bigger than $\mathbf{0}_{k+1}$, it follows that $(\mathscr{L}_k, d_z)$ is a complete metric space, for $k \in \mathbb{N}$. ■

Remark 1.1.17 (A sufficient condition for convergence of a sum of logarithmic transseries in $(\mathscr{L}_k, d_z)$, [29])**.** Let (φ_n) be a sequence in $\mathscr{L}_k$, $k \in \mathbb{N}$, such that $(\mathrm{ord}_z(\varphi_n))$ is a strictly increasing sequence of real numbers tending to $+\infty$. Then the series $\sum_{n=0}^{\infty} \varphi_n$ converges in $(\mathscr{L}_k, d_z)$. Indeed, since

$$\mathrm{ord}_z\left(\sum_{i=0}^{n+m} \varphi_i - \sum_{i=0}^{n} \varphi_i\right) = \mathrm{ord}_z(\varphi_{n+1}),$$

it follows that $d_z(\sum_{i=0}^{n+m} \varphi_i, \sum_{i=0}^{n} \varphi_i) = \frac{1}{2^{\mathrm{ord}(\varphi_{n+1})}}$, which implies that the sequence of the partial sums is Cauchy in $(\mathscr{L}_k, d_z)$. This implies that the series $\sum_{n=0}^{\infty} \varphi_n$ converges in $(\mathscr{L}_k, d_z)$, since $(\mathscr{L}_k, d_z)$ is complete, by Proposition 1.1.16.

The product topology

Let us consider $\mathfrak{L}^\infty$ as a subspace of $\mathbb{R}^{\mathbb{R}\times\mathbb{Z}^{\mathbb{N}_{\geq 1}}}$, equipped by the product topology, where the discrete topology is taken on each coordinate space $\mathbb{R}$. We call the induced relative topology on $\mathfrak{L}^\infty$ the *product topology* on $\mathfrak{L}^\infty$, and denote it by $\mathscr{T}_p$. In $\mathscr{L}_1$, it was already introduced in [21, Subsection 4.2].

By the usual identification of $\mathbb{R}\times\mathbb{Z}^k$ with a subset of $\mathbb{R}\times\mathbb{Z}^{\mathbb{N}_{\geq 1}}$, we consider $\mathscr{L}_k^\infty$, $k \in \mathbb{N}$, as a subspace of the product $\mathbb{R}^{\mathbb{R}\times\mathbb{Z}^{\mathbb{N}_{\geq 1}}}$. We call the induced relative topology on $\mathscr{L}_k^\infty$ the *product topology* on $\mathscr{L}_k^\infty$, $k \in \mathbb{N}$. Since $(\mathfrak{L}^\infty, \mathscr{T}_p)$ is not first countable, it follows that $(\mathfrak{L}^\infty, \mathscr{T}_p)$ is not a metrizable space. The same holds for its topological subspaces $\mathfrak{L}$ and $\mathscr{L}_k^\infty$, $k \in \mathbb{N}$.

In the sequel we often consider the product topology on subspaces $\mathfrak{L}$ and $\mathscr{L}_k$, $k \in \mathbb{N}$, of space $\mathfrak{L}^\infty$. It can be proven that $\mathfrak{L}$ and $\mathscr{L}_k$, $k \in \mathbb{N}$, are not metrizable, since they are not first countable (see e.g. [25]).

By the definition of product topology, it is clear that a sequence (φ_n) in $\mathfrak{L}^\infty$ converges to $\varphi \in \mathfrak{L}^\infty$ if and only if, for each $(\alpha,\mathbf{n}) \in \mathbb{R}\times\mathbb{Z}^{\mathbb{N}_{\geq 1}}$, there exists $n_{\alpha,\mathbf{n}} \in \mathbb{N}$ such that, for every $n \geq n_{\alpha,\mathbf{n}}$, $[\varphi_n]_{\alpha,\mathbf{n}} = [\varphi]_{\alpha,\mathbf{n}}$. The same holds for $\mathscr{L}_k^\infty$ and $\mathscr{L}_k$, $k \in \mathbb{N}$. Therefore, to check the convergence of a sequence in $\mathfrak{L}^\infty$ in the product topology, one can equivalently check that the sequences of coefficients eventually become stationary and that $\bigcup_{n\in\mathbb{N}} \operatorname{Supp}(\varphi_n)$ is a well-ordered subset of $\mathbb{R} \times \mathbb{Z}^k$, for some $k \in \mathbb{N}$. Their limits then represent the appropriate coefficients of the limit transseries.

The following remark will be used for the definition of a composition in Subsection 1.1.5. It relates the notion of *summable families* to the convergence of series in $\mathfrak{L}$, with respect to the product topology.

Remark 1.1.18 (Summable families and product topology, see [6])**.** Motivated by [29] where the weak topology and *summable families* were related, here we relate the product topology to the notion of summable families introduced in [6].

Let G be an ordered group of logarithmic monomials and let $\mathbb{R}((G))$ be the Hahn field (see Remark 1.1.9 or [6]). A family $(f_i, i \in I)$ of elements of $\mathbb{R}((G))$ is called *summable* if:

1. the union $\bigcup_{i\in I} \operatorname{Supp}(f_i)$ is a well-ordered subset of G,

2. for every $g \in G$, there exist only finitely many elements $i \in I$ such that $g \in \operatorname{Supp}(f_i)$.

If $(f_n, n \in \mathbb{N})$ is a summable family in $\mathfrak{L} \subseteq \mathbb{R}((G))$, it is easy to prove that the series $\sum f_n$ converges in the product topology on $\mathfrak{L}$ to the sum of the family.

The weak topology

Let us consider $\mathfrak{L}^\infty$ as a subspace of $\mathbb{R}^{\mathbb{R}\times\mathbb{Z}^{\mathbb{N}_{\geq 1}}}$ equipped by the product topology, where the Euclidean topology is taken on every coordinate space $\mathbb{R}$. The induced topology on $\mathfrak{L}^\infty$ is called the *weak topology* on $\mathfrak{L}^\infty$, and denoted by $\mathscr{T}_w$. In $\mathscr{L}_1$ it was introduced in [29, Subsection 2.3]. Analogously as in the definition of the product topology, we consider the induced topology on $\mathscr{L}_k^\infty$, $\mathfrak{L}$ and $\mathscr{L}_k$, $k \in \mathbb{N}$, of $\mathbb{R}^{\mathbb{R}\times\mathbb{Z}^{\mathbb{N}_{\geq 1}}}$. The space $\mathfrak{L}^\infty$ is not first countable (see e.g. [25]), and therefore, $\mathfrak{L}^\infty$ is not metrizable. The same holds for $\mathscr{L}_k^\infty$, $\mathscr{L}_k$, $k \in \mathbb{N}$, and $\mathfrak{L}$.

By the definition of the product topology, it is clear that a sequence (φ_n) in $(\mathfrak{L}^\infty, \mathscr{T}_w)$ converges to $\varphi \in \mathfrak{L}^\infty$ if and only if the sequence of coefficients $([\varphi_n]_{\alpha,\mathbf{n}})_n$ converges to

$[\varphi]_{\alpha,\mathbf{n}}$ in the Euclidean topology on $\mathbb{R}$, for every $(\alpha,\mathbf{n}) \in \mathbb{R}\times\mathbb{Z}^{\mathbb{N}_{\geq 1}}$. Now, suppose that the limit φ is an element of $\mathbb{R}^{\mathbb{R}\times\mathbb{Z}^{\mathbb{N}_{\geq 1}}}$. If $\bigcup_{n\in\mathbb{N}}\operatorname{Supp}(\varphi_n)$ is a well-ordered subset of $\mathbb{R}\times\mathbb{Z}^k$, for some $k \in \mathbb{N}$, then $\varphi \in \mathscr{L}_k^\infty \subseteq \mathfrak{L}^\infty$.

Properties and relations between the three topologies

Note that the power-metric topology is finer than the product topology which is finer than the weak topology. In the sequel we give examples that show that $\mathscr{T}_w \subsetneqq \mathscr{T}_p \subsetneqq \mathscr{T}_{d_z}$ on the differental algebra $\mathfrak{L}^\infty$ and its subalgebras $\mathfrak{L}$ and $\mathscr{L}_k^\infty$, $\mathscr{L}_k$, for $k \in \mathbb{N}$. Moreover, we show that $\mathscr{T}_{d_z} = \mathscr{T}_p$ on $\mathscr{L}_0^W$, in the case that $W \subseteq \mathbb{R}_{\geq 0}$ has no accumulation points with respect to Euclidean topology on $\mathbb{R}_{\geq 0}$.

Example 1.1.19. (1) Let (φ_n) be a sequence of logarithmic transseries in $\mathscr{L}_k$, $k \in \mathbb{N}_{\geq 1}$, such that $\varphi_n := \boldsymbol{\ell}_k^n + z$, for $n \in \mathbb{N}$. Now, put $\varphi := \mathrm{id} \in \mathscr{L}_k$. For an arbitrary element $(\alpha,\mathbf{m}) \in \mathbb{R}_{\geq 0}\times\mathbb{Z}^k$ such that $(\alpha,\mathbf{m}) \neq (1,\mathbf{0}_k)$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $[\varphi_n]_{\alpha,\mathbf{m}} = 0$ and $[\varphi_n]_{1,\mathbf{0}_k} = 1$. Therefore, it follows that (φ_n) converges to φ in the product topology in $\mathscr{L}_k$ (or in $\mathfrak{L}^\infty$).

On the other hand, since $d_z(\varphi_n,\varphi_{n+m}) = \frac{1}{2^0} = 1$, for every $n,m \in \mathbb{N}$, $m \geq 1$, it follows that (φ_n) does not converge in the power-metric topology in $\mathfrak{L}^\infty$ (or in $\mathscr{L}_k^\infty$, $\mathscr{L}_k$).

(2) Example (1) shows that the power-metric and the product topology are not equal on $\mathscr{L}_k$, if $k \in \mathbb{N}_{\geq 1}$. Consequently, the power-metric and the product topology are not equal on $\mathfrak{L}$. On the other hand, suppose that $k = 0$. Let us define the sequence (φ_n) in $\mathscr{L}_0$ such that $\varphi_n := z^{3-\frac{1}{n+1}} + z^4$, for $n \in \mathbb{N}$. As in Example (1), it is easy to see that (φ_n) converges in the product topology in $\mathscr{L}_0$ to the transseries $\varphi = z^4 \in \mathscr{L}_0$. Since $d_z(\varphi_n,\varphi_{n+m}) = \frac{1}{2^{3-\frac{1}{n+1}}} > \frac{1}{2^3}$, for $n,m \in \mathbb{N}$, $m \geq 1$, it follows that (φ_n) does not converge in the power-metric topology on $\mathscr{L}_0$.

(3) In Example (2) we constructed a sequence (φ_n) in $\mathscr{L}_0$ which converges in the product topology and does not converge in the power-metric topology. Note that 3 is an accumulation point of $\bigcup_{n\in\mathbb{N}}\operatorname{Supp}(\varphi_n)$ in the Euclidean space $\mathbb{R}$. Now, suppose that (φ_n) is a sequence in $\mathscr{L}_0$ which converges in the product topology to some $\varphi \in \mathscr{L}_0$, and such that

$\bigcup_{n\in\mathbb{N}} \mathrm{Supp}\,(\varphi_n)$ has no accumulation points in $\mathbb{R}$. We prove that (φ_n) converges to φ in the power-metric topology on $\mathscr{L}_0$.

Let $\alpha \in \mathbb{R}_{>0}$ be arbitrary. Since there are no accumulation points of $\bigcup_{n\in\mathbb{N}} \mathrm{Supp}\,(\varphi_n)$ in $\mathbb{R}$, there are only finitely many elements $\{\alpha_1,\ldots,\alpha_m\}$ of $\bigcup_{n\in\mathbb{N}} \mathrm{Supp}\,(\varphi_n)$ which are smaller than or equal to α. Since (φ_n) converges in the product topology, it follows that there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $[\varphi_n]_{\alpha_i} = [\varphi]_{\alpha_i}$, for $i = 1,\ldots,m$. It implies that $\mathrm{ord}_z\,(\varphi-\varphi_n) > \alpha$, i.e., $d_z(\varphi,\varphi_n) < \frac{1}{2^\alpha}$, for all $n \geq n_0$. Since $\alpha \in \mathbb{R}_{>0}$ is arbitrary, we get that (φ_n) converges to φ in the power-metric topology on $\mathscr{L}_0$.

(4) The product and the power-metric topology are equal on $\mathscr{L}_0^W$ if $W \subseteq \mathbb{R}_{\geq 0}$ has no accumulation points with respect to the Euclidean topology on $\mathbb{R}_{\geq 0}$.

Indeed, suppose that $W \subseteq \mathbb{R}_{\geq 0}$ has no accumulation points with respect to the Euclidean topology on $\mathbb{R}_{\geq 0}$. We distinguish two cases. If W is finite, then it is clear that $\mathscr{L}_0^W$ is homeomorphic to the product $\mathbb{R}^{\mathrm{card}\,(W)}$ of discrete spaces. Therefore, suppose that W is infinite. Then there exists the strictly increasing sequence (α_n) of nonnegative real numbers tending to $+\infty$ such that $W = \{\alpha_n : n \in \mathbb{N}\}$. Let $F : \mathscr{L}_0^W \to z\mathbb{R}\,[[z]]$ be defined by:

$$F\Big(\sum_{n=1}^{+\infty} a_n\, z^{\alpha_n}\Big) := \sum_{n=1}^{+\infty} a_n\, z^n,$$

where $z\mathbb{R}\,[[z]] := \mathscr{L}_0^{\mathbb{N}}$ is the set of all power series f in the formal variable z, such that $f(0)=0$. It is easy to see that F is a homeomorphism in both cases: if we consider the power-metric topology on $\mathscr{L}_0^W$ and $\mathbb{R}\,[[z]]$, or the product topology. Therefore, *identify* $\mathscr{L}_0^W$ with the space of all power series $\mathbb{R}\,[[z]]$. By Example (3), the product topology and the power-metric topology are equal on $\mathbb{R}\,[[z]]$.

(5) Let (φ_n) be the sequence of logarithmic transseries in $\mathscr{L}_k$, $k \in \mathbb{N}$, such that $\varphi_n := \frac{1}{n+1}\cdot z + z^2$, for $n \in \mathbb{N}$. Since $\left(\frac{1}{n+1}\right) \to 0$ in the Euclidean topology on $\mathbb{R}$, it is easy to see that (φ_n) converges to $\varphi := z^2$ in the weak topology on $\mathscr{L}_k$. On the other hand, (φ_n) does not converge in the product topology on $\mathscr{L}_k$ because the sequence $([\varphi_n]_{1,\mathbf{0}_k}) = \left(\frac{1}{n+1}\right)$ does not eventually become stationary.

(6) We show here that, although $\mathfrak{L}$ and $\mathscr{L}_k$, $k \in \mathbb{N}$, are not metrizable with respect to the weak topology, the space $\mathscr{L}_0^W$ is metrizable for $W \subseteq \mathbb{R}_{\geq 0}$ with no accumulation points in $\mathbb{R}_{\geq 0}$.

Suppose that $W \subseteq \mathbb{R}_{\geq 0}$ has no accumulation points with respect to the Euclidean topology on $\mathbb{R}_{\geq 0}$. It was shown in Example (4) that $\mathscr{L}_0^W$ and $\mathbb{R}[[z]]$ are homeomorphic, with respect to the weak topology. The weak topology on $\mathbb{R}[[z]]$ is metrizable by e.g. the metric

$$d(f,g) := \sum_{i=0}^{+\infty} \left(\frac{1}{2^i} \cdot \frac{|a_i - b_i|}{1 + |a_i - b_i|} \right),$$

for $f := \sum_{i=0}^{+\infty} a_i z^i$ and $g := \sum_{i=0}^{+\infty} b_i z^i$.

1.1.4. Differential algebras of blocks

In this subsection, for a fixed $k \in \mathbb{N}_{\geq 1}$, we introduce and prove some basic properties of what we call the *differential algebras of blocks* $\mathscr{B}_m$, for $1 \leq m \leq k$. This subsection is partially taken from [29, Subsection 3.4], and used mostly in Subsections 2.1.2, 2.2.2 and 2.3.4 as a technical prerequisite for proving the steps of the *normalizing algorithm*, and is not to be read independently. The definitions and statements in this subsection are similar to those in Subsection 1.1.2.

Definition 1.1.20 (Block of level m)**.** Let $k \in \mathbb{N}_{\geq 1}$, $1 \leq m \leq k$. A logarithmic transseries $K \in \mathscr{L}_k^\infty$ of the form

$$K := \sum_{(\mathbf{0}_m, n_m, \ldots, n_k) \in \mathbb{R} \times \mathbb{Z}^k} a_{n_m,\ldots,n_k} \boldsymbol{\ell}_m^{n_m} \cdots \boldsymbol{\ell}_k^{n_k},$$

where $a_{n_m,\ldots,n_k} \in \mathbb{R}$, is called the *block of level* m in $\mathscr{L}_k^\infty$.

For $k \in \mathbb{N}_{\geq 1}$, we denote by $\mathscr{B}_m \subseteq \mathscr{L}_k^\infty$, $1 \leq m \leq k$, the set of all blocks of level m.

Remark 1.1.21.

1. Note that the zero transseries and every constant are blocks of level m, for $1 \leq m \leq k$, and $k \in \mathbb{N}_{\geq 1}$.

2. For $k \in \mathbb{N}_{\geq 1}$, note that $\mathscr{B}_m$ is a subalgebra and a subfield of $\mathscr{L}_k^\infty$, with respect to the standard addition and multiplication, for every $1 \leq m \leq k$. Furthermore, $\mathscr{B}_{m+1}$ is a subfield of the field $\mathscr{B}_m$, for $1 \leq m \leq k-1$.

3. Note that $\mathscr{B}_m \subseteq \mathscr{L}_k^\infty$, and thus depends on $k \in \mathbb{N}_{\geq 1}$, but in the sequel it is always clear which $k \in \mathbb{N}_{\geq 1}$ we consider.

4. Every block in $\mathscr{B}_m \subseteq \mathscr{L}_k^\infty$, $1 \leq m \leq k-1$, $k \in \mathbb{N}_{\geq 1}$, can be viewed as a Laurent series in the variable $\boldsymbol{\ell}_m$, with coefficients in the field $\mathscr{B}_{m+1}$, and every block in $\mathscr{B}_k$ can be viewed as a Laurent series in the variable $\boldsymbol{\ell}_k$, with real coefficients.

Although $\mathscr{B}_m$ is a differential algebra with respect to the derivation $\frac{d}{d\boldsymbol{\ell}_m}$, we do not consider the derivation $\frac{d}{d\boldsymbol{\ell}_m}$ because it is not a contraction on the space $\mathscr{B}_m$. The contraction property will be important for applications of fixed point theorems in Sections 2.1, 2.2 and 2.3. Therefore, in the next definition we introduce a slightly modified derivation operator D_m.

Definition 1.1.22 (Derivation operator D_m, [29]). Let $k \in \mathbb{N}_{\geq 1}$, and let $m \in \mathbb{N}$, $1 \leq m \leq k$. We define the operator $D_m : \mathscr{B}_m \to \mathscr{B}_m$ by:

$$D_m := \boldsymbol{\ell}_m^2 \cdot \frac{d}{d\boldsymbol{\ell}_m}. \tag{1.5}$$

It is easy to check that D_m is a derivation operator on $\mathscr{B}_m \subseteq \mathscr{L}_k^\infty$, for $1 \leq m \leq k$.

The associative commutative algebra $\mathscr{B}_m$ equipped with the derivation D_m will be called the *differential algebra of blocks of level m* in $\mathscr{L}_k^\infty$.

Using convention $\boldsymbol{\ell}_0 := z$ and putting $\mathscr{B}_0 := \mathscr{L}_k^\infty$ and $D_0 := z^2 \cdot \frac{d}{dz}$, we get:

$$D_m(\boldsymbol{\ell}_{m+1}^n) = \boldsymbol{\ell}_m D_{m+1}(\boldsymbol{\ell}_{m+1}^n),$$

for every $0 \leq m \leq k-1$ and $n \in \mathbb{Z}$. For additional properties of derivations D_m, see Section A.

In the sequel we generalize the results obtained in Subsection 1.1.2 for $\mathscr{B}_0 = \mathscr{L}_k^\infty$ to $\mathscr{B}_m \subseteq \mathscr{L}_k^\infty$, $1 \leq m \leq k$. As in Definition 1.1.11 in Section 1.1, we define the *Lie bracket operator* (see [21, Subsection 3.3]) on differential algebra $\mathscr{B}_m$, for $1 \leq m \leq k$, and $k \in \mathbb{N}_{\geq 1}$, by

$$[K,G] := G \cdot D_m(K) - K \cdot D_m(G), \quad K,G \in \mathscr{B}_m.$$

For some properties of the Lie bracket operator on $\mathscr{B}_m$ see Lemma A.2.5 in Appendix A.

As before, by $\int K \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}$ we denote the antiderivative (with respect to D_m) of $K \in \mathscr{B}_m \subseteq \mathscr{L}_k^\infty$ without a constant term.

Similarly as ord_z in $\mathscr{L}_k^\infty$, we define the order of an element of $\mathscr{B}_m \subseteq \mathscr{L}_k^\infty$ in the variable $\boldsymbol{\ell}_m$ (see [29]).

Definition 1.1.23 (Order of a block of level m in $\boldsymbol{\ell}_m$, [29])**.** Let $k \in \mathbb{N}_{\geq 1}$ and let $K \in \mathscr{B}_m \subseteq \mathscr{L}_k^\infty$, $1 \leq m \leq k$. If $K = 0$, then we define the *order of K in $\boldsymbol{\ell}_m$* as infinity, and write $\mathrm{ord}_{\boldsymbol{\ell}_m}(K) = \infty$. If $K \neq 0$, we define the *order of K in $\boldsymbol{\ell}_m$* as the minimal exponent of $\boldsymbol{\ell}_m$ and denote it by $\mathrm{ord}_{\boldsymbol{\ell}_m}(K)$.

Note that $\mathrm{ord}_{\boldsymbol{\ell}_m}(K)$, $K \in \mathscr{B}_m \subseteq \mathscr{L}_k^\infty$, belongs to the extended set $\mathbb{Z} \cup \{\infty\}$, where ∞ is an element such that $a < \infty$, for every $a \in \mathbb{Z}$. The zero transseries has thus the maximal order in $\boldsymbol{\ell}_m$, in space $\mathscr{B}_m$.

Definition 1.1.24 (The leading block in $\boldsymbol{\ell}_m$, [29])**.** Let $k \in \mathbb{N}_{\geq 1}$ and $K \in \mathscr{B}_m \setminus \{0\} \subseteq \mathscr{L}_k^\infty$, $1 \leq m \leq k$. Put $K := \sum_{i=n_m}^{+\infty} \boldsymbol{\ell}_m^i K_i$, for $K_i \in \mathscr{B}_{m+1}$ and $n_m \in \mathbb{Z}$. We call $\boldsymbol{\ell}_m^i K_i$ the *i-block* of *K in $\boldsymbol{\ell}_m$*, for each $i \geq n_m$. Furthermore, we call $\boldsymbol{\ell}_m^{n_m} K_{n_m}$ the *leading block of K in $\boldsymbol{\ell}_m$*, and denote it by $\mathrm{Lb}_{\boldsymbol{\ell}_m}(K)$.

For $K, G \in \mathscr{B}_m \subseteq \mathscr{L}_k^\infty$, $k \in \mathbb{N}_{\geq 1}$, we write

$$K = G + \mathrm{h.o.b.}(\boldsymbol{\ell}_m)$$

(which means: *higher order blocks in $\boldsymbol{\ell}_m$*) if every block (in $\boldsymbol{\ell}_m$) in $K - G$ is of order in $\boldsymbol{\ell}_m$ that is strictly bigger than the order in $\boldsymbol{\ell}_m$ of every block (in $\boldsymbol{\ell}_m$) in G.

Let $k \in \mathbb{N}_{\geq 1}$, $1 \leq m \leq k$, and let $W \subseteq \{0\}^m \times \mathbb{Z}^{k-m}$. Similarly as in Subsection 1.1.2 we denote by $\mathscr{B}_m^W \subseteq \mathscr{L}_k^\infty$ the set:

$$\mathscr{B}_m^W := \{K \in \mathscr{B}_m \subseteq \mathscr{L}_k^\infty : \mathrm{Supp}(K) \subseteq W\}.$$

Since the zero transseries has empty support, it follows that the zero transseries is an element of $\mathscr{B}_m^W$.

Let $k \in \mathbb{N}_{\geq 1}$ and $1 \leq m \leq k$. The following spaces will often be used in the sequel:

$$\mathscr{B}_{\geq m}^{+} := \{K \in \mathscr{B}_m \subseteq \mathscr{L}_k^{\infty} : \operatorname{ord}(K) > \mathbf{0}_{k+1}\}, \tag{1.6}$$

$$\mathscr{B}_{m}^{+} := \left\{K \in \mathscr{B}_m \subseteq \mathscr{L}_k^{\infty} : \operatorname{ord}_{\boldsymbol{\ell}_m}(K) > 0\right\}. \tag{1.7}$$

For $\alpha \in \mathbb{R}_{\geq 0}$ and $W \subseteq \{0\}^{m+1} \times \mathbb{Z}^{k-m}$, we denote:

$$z^{\alpha}\mathscr{B}_m^W := \left\{z^{\alpha} \cdot K : K \in \mathscr{B}_m^W \subseteq \mathscr{L}_k^{\infty}\right\}.$$

Remark 1.1.25 (Properties of spaces of blocks)**.**

1. For $k \in \mathbb{N}_{\geq 1}$ and $1 \leq m \leq k$, note that $\mathscr{B}_m^+ = \mathscr{B}_m^W$, for $W = \{0\}^m \times \mathbb{N}_{\geq 1} \times \mathbb{Z}^{k-m}$, and $\mathscr{B}_{\geq m}^+ = \mathscr{B}_m^V$, for $V = \left\{(\alpha, \mathbf{n}) \in \mathbb{R} \times \mathbb{Z}^k : (\alpha, \mathbf{n}) > \mathbf{0}_{k+1}\right\}$.

2. For $k \in \mathbb{N}_{\geq 1}$ and $1 \leq m \leq k$, note that $\mathscr{B}_{\geq m}^+ = \mathscr{B}_m \cap \mathscr{L}_k$ and $\mathscr{B}_m^+ \subseteq \mathscr{B}_{\geq m}^+$. Furthermore, $\mathscr{B}_m^+ = \mathscr{B}_{\geq m}^+$ if and only if $m = k$.

3. For $k \in \mathbb{N}_{\geq 1}$ and $1 \leq m \leq k-1$, note that $\mathscr{B}_{\geq m+1}^+ \subseteq \mathscr{B}_{\geq m}^+$, but
$$\mathscr{B}_m^+ \cap \mathscr{B}_{\geq m+1}^+ = \{0\}.$$

4. For $k \in \mathbb{N}_{\geq 1}$ and $1 \leq m \leq k$, if W is a sub-semigroup of $\{0\}^{m+1} \times \mathbb{Z}^{k-m}$, then $\mathscr{B}_m^W$ is a subalgebra of $\mathscr{B}_m$. In particular, $\mathscr{B}_{\geq m}^+$ and $\mathscr{B}_m^+$ are subalgebras of $\mathscr{B}_m$.

5. For $k \in \mathbb{N}_{\geq 1}$, and for $1 \leq m \leq k$, if $W_1 \subseteq W_2 \subseteq \mathbb{R} \times \mathbb{Z}^k$, then $\mathscr{B}_m^{W_1}$ is a subspace of $\mathscr{B}_m^{W_2}$.

6. Let $(W_i, i \in I)$ be a family of pairwise disjoint subsets of $\mathbb{R} \times \mathbb{Z}^k$, for $k \in \mathbb{N}_{\geq 1}$. Then $\mathscr{B}_m^{\bigcup_{i \in I} W_i}$ is a direct sum of the family $(\mathscr{B}_m^{W_i}, i \in I)$, i.e.
$$\mathscr{B}_m^{\bigcup_{i \in I} W_i} = \bigoplus_{i \in I} \mathscr{B}_m^{W_i},$$
for each $1 \leq m \leq k$.

A metric on spaces of blocks

In Subsection 1.1.3 we defined the power-metric d_z on the differential algebra $\mathfrak{L}^{\infty}$. Observe the sequence (K_n) in $\mathscr{B}_1 \subseteq \mathscr{L}_k^{\infty}$, $k \in \mathbb{N}_{\geq 1}$, given by $K_n := \boldsymbol{\ell}_1^n$, for $n \in \mathbb{N}$. Note that:

$$d_z(K_n, K_{n+m}) = \frac{1}{2^{\operatorname{ord}_z(\boldsymbol{\ell}_1^n - \boldsymbol{\ell}_1^{n+m})}} = \frac{1}{2^0} = 1,$$

for each $n, m \in \mathbb{N}_{\geq 1}$. From this example we deduce that the metric d_z is useless on differential algebras $\mathscr{B}_m \subseteq \mathscr{L}_k$, $1 \leq m \leq k$, $k \in \mathbb{N}_{\geq 1}$.

Therefore, in the next definition we define the *m-metric* on the differential algebra $\mathscr{B}_m$, for $1 \leq m \leq k$, and $k \in \mathbb{N}_{\geq 1}$.

Definition 1.1.26 (m-metric, [29])**.** Let $k, m \in \mathbb{N}_{\geq 1}$ be such that $1 \leq m \leq k$. The *m-metric* $d_m : \mathscr{B}_m \times \mathscr{B}_m \to \mathbb{R}$ on the differential algebra $\mathscr{B}_m$ is defined as:

$$d_m(K_1, K_2) := \begin{cases} 2^{-\operatorname{ord}_{\boldsymbol{\ell}_m}(K_1 - K_2)}, & K_1 \neq K_2, \\ 0, & K_1 = K_2. \end{cases}$$

The space $(\mathscr{B}_m, d_m)$ is called the *(metric) space of blocks of level m.*

Similarly as in Proposition 1.1.16, it can be proven that metric spaces $(\mathscr{B}_m, d_m)$ and $(\mathscr{B}_m^W, d_m)$, for $W \subseteq \mathbb{R} \times \mathbb{Z}^k$, are complete, for $1 \leq m \leq k$, $k \in \mathbb{N}_{\geq 1}$. In particular, $(\mathscr{B}^+_{\geq m}, d_m)$ and $(\mathscr{B}^+_m, d_m)$ are complete metric spaces.

1.1.5. Composition of logarithmic transseries

In [6, Section 6] the general result on composition of transseries is proven. In our setting of logarithmic transseries, we do not need the definition of composition in full generality. Therefore, we define a composition only for the logarithmic transseries in the spirit of the definition given in [21, Section 2] only for the logarithmic transseries of depth 1. For our purpose in Sections 2.1, 2.2 and 2.3, it is important that the composition of two logarithmic transseries in $\mathscr{L}_k$ is again an element of $\mathscr{L}_k$. This is not always the case. Therefore, we restrict ourselves to the set $\mathscr{L}_k^H$ of all logarithmic transseries in $\mathscr{L}_k$ without logarithms in their leading term, as was done in [21] for power-logarithm transseries and in [29, Subsection 3.2] for logarithmic transseries. More precisely, we denote by $\mathscr{L}_k^H$, $k \in \mathbb{N}$, the set of all logarithmic transseries $f \in \mathscr{L}_k$, such that

$$f = \lambda z^\alpha + \text{h.o.t.},$$

for $\lambda, \alpha > 0$. Furthermore, we put $\mathfrak{L}^H := \bigcup_{k \in \mathbb{N}} \mathscr{L}_k^H$. We distinguish three types of logarithmic transseries in $\mathfrak{L}^H$ (see [21, Definition 1.1], [29, Subsection 2.1]).

Definition 1.1.27 (Parabolic, hyperbolic and strongly hyperbolic transseries in $\mathfrak{L}^H$)**.** Let $f \in \mathfrak{L}^H$ such that $f = \lambda z^\alpha + \text{h.o.t.}$, $\alpha, \lambda > 0$. We say that f is:

1. *parabolic*, if $\alpha = \lambda = 1$,

2. *hyperbolic*, if $\alpha = 1$ and $\lambda > 0$, $\lambda \neq 1$,

3. *strongly hyperbolic*, if $\alpha > 0$, $\alpha \neq 1$, and $\lambda > 0$.

We denote by $\mathscr{L}_k^0$ the set of all parabolic logarithmic transseries in $\mathscr{L}_k^H$. Furthermore, we put $\mathfrak{L}^0 := \bigcup_{k \in \mathbb{N}} \mathscr{L}_k^0$.

Finally, we define a composition of logarithmic transseries.

Definition 1.1.28 (Composition of logarithmic transseries)**.** Let $f \in \mathfrak{L}$ and $g \in \mathscr{L}_k^H$, for $k \in \mathbb{N}$. Let $g = \lambda z^\alpha + g_1$, for $\alpha, \lambda > 0$, and $\operatorname{ord}(g_1) > (\alpha, \mathbf{0}_k)$. Then, the *composition of f and g*, denoted by $f \circ g$, is defined as:

$$f \circ g := f(\lambda z^\alpha) + \sum_{i \geq 1} \frac{f^{(i)}(\lambda z^\alpha)}{i!} (g_1)^i, \tag{1.8}$$

where the series on the right-hand side of (1.8) converges in the product topology.

In Proposition A.1.1 in Appendix A we prove that the series on the right-hand side of (1.8) converges in the product topology, which implies that the composition of logarithmic transseries from Definition 1.1.28 is well-defined. In Appendix A we also prove the formal Taylor Theorem (Proposition A.1.6) and the fact that we can compose logarithmic transseries *term-wise* using formulas (A.1).

1.2. Fixed point theorems

In this section we state and prove a version of a fixed point theorem (Proposition 1.2.12) (see [29, Proposition 3.2]), that we use in Sections 2.1, 2.2 and 2.3 to solve normalization equations. In particular, we use the mentioned fixed point theorem on complete metric spaces introduced in Subsections 1.1.3 and 1.1.4. The section is divided into three subsections: Lipschitz map and homothety, (μ_1,μ_2)-Lipschitz maps and Fixed point theorems.

1.2.1. Lipschitz map and homothety

In this subsection we recall some basic notions from [29, Subsection 3.1].

Definition 1.2.1 (Homothety, Definition 3.1, [29])**.** Let (X,d) and (Y,ρ) be metric spaces. The map $\mathscr{T}: X \to Y$ such there exists $\lambda \in \mathbb{R}_{>0}$ with property that

$$\rho(\mathscr{T}(x_1),\mathscr{T}(x_2)) = \lambda d(x_1,x_2),$$

for each $x_1,x_2 \in X$, is called the λ-*homothety*. The coefficient λ is called the *coefficient of homothety* $\mathscr{T}$.
If $\lambda = 1$, then $\mathscr{T}$ is called the *isometry*.

Remark 1.2.2. Note that, for the given λ-homothety $\mathscr{T}$, the coefficient λ is unique and $\mathscr{T}$ is injective. Therefore, there exists the compositional inverse $\mathscr{T}^{-1}: \mathscr{T}(X) \to X$ which is a $\frac{1}{\lambda}$-homothety.

Definition 1.2.3 (Lipschitz map, Definition 3.1, [29])**.** Let (X,d) and (Y,ρ) be metric spaces. The map $\mathscr{S}: X \to Y$ such there exists $\mu \in \mathbb{R}_{>0}$ with the property that

$$\rho(\mathscr{S}(x_1),\mathscr{S}(x_2)) \le \mu d(x_1,x_2), \tag{1.9}$$

for all $x_1,x_2 \in X$, is called the μ-*Lipschitz map*. The coefficient μ is called the *Lipschitz coefficient* (or the *Lipschitz constant*) of $\mathscr{S}$.

We call the smallest $\mu \in \mathbb{R}_{>0}$ (if such exists) such that (1.9) holds, the *minimal* Lipschitz coefficient of $\mathscr{S}$.

In particular, if $\mu < 1$, then $\mathscr{S}$ is called the μ-*contraction* and the coefficient μ is called the *coefficient of contraction* of $\mathscr{S}$.

Example 1.2.4. Let $\mathscr{A}$ be a subspace of $\mathfrak{L}^{\infty}$ and $\mathscr{T} : \mathscr{A} \to \mathfrak{L}^{\infty}$ a linear operator. If there exists $C \in \mathbb{R}_{>0}$ such that:

1. $\mathrm{ord}_z(\mathscr{T}(g)) = \mathrm{ord}_z(g) + C$, for every $g \in \mathscr{A}$, then $\mathscr{T}$ is a $\frac{1}{2^C}$-homothety, with respect to the metric d_z,

2. $\mathrm{ord}_z(\mathscr{T}(g)) \geq \mathrm{ord}_z(g) + C$, for every $g \in \mathscr{A}$, then $\mathscr{T}$ is a $\frac{1}{2^C}$-Lipschitz map, with respect to the metric d_z.

Similarly, let $\widetilde{\mathscr{B}}$ be a subspace of $\mathscr{B}_m$, for $1 \leq m \leq k$, $k \in \mathbb{N}_{\geq 1}$, and $\mathscr{T} : \widetilde{\mathscr{B}} \to \mathscr{B}_m$ a linear operator which satisfies above (in)equality for $\mathrm{ord}_{\boldsymbol{\ell}_m}$ instead of ord_z, then $\mathscr{T}$ is a $\frac{1}{2^C}$-homothety (resp. Lipschitz), with respect to the metric d_m. These statements will be used throughout the thesis.

Example 1.2.5. Let $k \in \mathbb{N}$ and $W \subseteq \mathbb{R} \times \mathbb{Z}^k$. Denote by $\mathscr{P}_W : \mathscr{L}_k^{\infty} \to \mathscr{L}_k^W$ the projection operator to the subspace $\mathscr{L}_k^W \subseteq \mathscr{L}_k^{\infty}$. Then the projection operator $\mathscr{P}_W$ is a superlinear 1-Lipschitz operator, with respect to the metric d_z.

Indeed, for every term $M \in \mathscr{L}_k^{\infty}$ it follows that $\mathscr{P}_W(M) = 0$, if $M \notin \mathscr{L}_k^W$, and $\mathscr{P}_W(M) = M$, if $M \in \mathscr{L}_k^W$. Now, superlinearity follows immediately.

For every $g \in \mathscr{L}_k^{\infty}$ it follows that

$$\mathrm{ord}\,(\mathscr{P}_W(g) = \mathrm{ord}\,(\mathscr{P}_W(g)) \geq \mathrm{ord}\,(g).$$

By Example 1.2.4, it follows that $\mathscr{P}_W$ is an 1-Lipschitz map.

Example 1.2.6. Let $k \in \mathbb{N}_{\geq 1}$ and $1 \leq m \leq k$. By Example 1.2.4, note that the derivation $D_m : \mathscr{B}_m \to \mathscr{B}_m$ defined in (1.5) is a $\frac{1}{2}$-contraction on the space $(\mathscr{B}_m, d_m)$, since $\mathrm{ord}_{\boldsymbol{\ell}_m}(D_m(K)) = 1 + \mathrm{ord}_{\boldsymbol{\ell}_m}(K)$, for every $K \in \mathscr{B}_m$ that is not a constant.

Furthermore, the restriction of D_m on the space $\widetilde{\mathscr{B}}$ of all $K \in \mathscr{B}_m$ which do not contain constants is a $\frac{1}{2}$-homothety. Therefore, there exists the inverse D_m^{-1} of the restriction $D_m|_{\widetilde{\mathscr{B}}}$, which is a 2-homothety, with respect to the metric d_m.

Now we state two examples of contractions that will be used throughout the Sections 2.1, 2.2 and 2.3 for solving normalization equations.

Example 1.2.7. Let $\varphi \in \mathscr{L}_k^{\gamma}$ (see Subsection 1.1.2), for $\gamma \geq 1$ and $k \in \mathbb{N}$. Let $\delta \geq 1$ and let the operator $\mathscr{S} : \mathscr{L}_k^{\delta} \to \mathscr{L}_k^{\delta}$ be defined as:

$$\mathscr{S}(\varepsilon) := \sum_{i \geq 2} \varphi^{(i)} \cdot \varepsilon^i, \tag{1.10}$$

for $\varepsilon \in \mathscr{L}_k^{\delta}$. Then the operator $\mathscr{S}$ is $\frac{1}{2^{\delta+\gamma-2}}$-Lipschitz on the space $\left(\mathscr{L}_k^{\delta}, d_z\right)$.

Moreover, $\frac{1}{2^{\delta+\alpha-2}}$ is the minimal Lipschitz coefficient of $\mathscr{S}$.

Proof. By the definition of the set $\mathscr{L}_k^{\delta}$ and since $\delta, \gamma \geq 1$, it can be shown that the series in (1.10) converges in the product topology. Therefore, the operator $\mathscr{S}$ is well-defined. Let $\varepsilon_1, \varepsilon_2 \in \mathscr{L}_k^{\delta}$, such that $\varepsilon_1 \neq \varepsilon_2$. Then:

$$\begin{aligned}
\mathscr{S}(\varepsilon_1) - \mathscr{S}(\varepsilon_2) &= \sum_{i \geq 2} \left(\varphi^{(i)} \cdot (\varepsilon_1^i - \varepsilon_2^i)\right) \\
&= \sum_{i \geq 2} \left(\varphi^{(i)} \cdot (\varepsilon_1 - \varepsilon_2) \cdot \left(\sum_{j=0}^{i-1} \varepsilon_1^j \varepsilon_2^{i-1-j}\right)\right).
\end{aligned}$$

Now, $\mathrm{ord}_z(\mathscr{S}(\varepsilon_1) - \mathscr{S}(\varepsilon_2)) \geq \mathrm{ord}_z(\varepsilon_1 - \varepsilon_2) + \delta + \gamma - 2$, which implies that $\mathscr{S}$ is a $\frac{1}{2^{\delta+\gamma-2}}$-Lipschitz operator.

If $\varepsilon \in \mathscr{L}_k^{\delta}$ such that $\mathrm{ord}_z(\varepsilon) = \delta$, then $\mathrm{ord}(\mathscr{S}(\varepsilon) - \mathscr{S}(0)) = \mathrm{ord}(\varepsilon - 0) + \delta + \alpha - 2$. Thus, $\frac{1}{2^{\delta+\alpha-2}}$ is the minimal Lipschitz coefficient of $\mathscr{S}$. ■

Since $\gamma, \delta \geq 1$, note that $\mathscr{S}$ is a $\frac{1}{2^{\delta+\gamma-2}}$-contraction if and only if $\gamma > 1$ or $\delta > 1$.

Example 1.2.8. Let $k \in \mathbb{N}_{\geq 1}$, $1 \leq m \leq k$, and $G_i \in \mathscr{B}_m \setminus \{0\} \subseteq \mathscr{L}_k^{\infty}$, such that $\mathrm{ord}(G_i) \geq \mathbf{0}_{k+1}$, for every $i \geq 2$. Let $\mathscr{S} : \mathscr{B}_m^+ \to \mathscr{B}_m^+$ be defined as:

$$\mathscr{S}(Q) := \sum_{i \geq 2} G_i \cdot Q^i,$$

for $Q \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$. The map $\mathscr{S}$ is a $\frac{1}{2}$-contraction on the space $(\mathscr{B}_m^+, d_m)$.

Proof. Note that the series above converge in $\mathscr{B}_m$, with respect to the metric d_m. Let $Q_1, Q_2 \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$ such that $Q_1 \neq Q_2$. We have:

$$\begin{aligned}
\mathscr{S}(Q_1) - \mathscr{S}(Q_2) &= \sum_{i \geq 2} G_i \cdot (Q_1^i - Q_2^i) \\
&= \sum_{i \geq 2} \left(G_i \cdot (Q_1 - Q_2) \cdot \left(\sum_{j=0}^{i-1} Q_1^j Q_2^{i-1-j}\right)\right).
\end{aligned}$$

Using the following facts

$$\mathrm{ord}(G_i) \geq \mathbf{0}_{k+1}, \quad \mathrm{ord}_{\boldsymbol{\ell}_m}(Q_1), \mathrm{ord}_{\boldsymbol{\ell}_m}(Q_2) \geq 1,$$

and $i \geq 2$, we conclude that

$$\mathrm{ord}_{\ell_m}(\mathscr{S}(Q_1) - \mathscr{S}(Q_2)) \geq \mathrm{ord}_{\ell_m}(Q_1 - Q_2) + 1.$$

This implies that

$$d_m\big(\mathscr{S}(Q_1), \mathscr{S}(Q_2)\big) \leq \frac{1}{2} \cdot d_m(Q_1, Q_2),$$

that is, $\mathscr{S}$ is a $\frac{1}{2}$-contraction on the space $(\mathscr{B}_m^+, d_m)$. ■

1.2.2. (μ_1, μ_2)-Lipschitz map

In this subsection we define a (μ_1, μ_2)-Lipschitz map, which is, in some sense, a generalization of the standard definition of a μ-Lipschitz map on a metric space to a Cartesian product of metric spaces. This notion is not needed in the proof of the fixed point theorem stated in Proposition 1.2.12, but it will be needed in the sequel as a natural generalization of Lipschitz maps.

Definition 1.2.9 ((μ_1, μ_2)-Lipschitz map)**.** Let X_1, X_2 and Y be metric spaces, and let $\mathscr{C} : X_1 \times X_2 \to Y$ be a map. Let $\mu_i : X_i \to \mathbb{R}_{>0}$ be maps, for $i = 1, 2$, such that $\mathscr{C}(x_1, \cdot) : X_2 \to Y$ is a $\mu_1(x_1)$-Lipschitz map and $\mathscr{C}(\cdot, x_2) : X_1 \to Y$ is a $\mu_2(x_2)$-Lipschitz map, for every $(x_1, x_2) \in X_1 \times X_2$. Then we call $\mathscr{C}$ a (μ_1, μ_2)-*Lipschitz map*.

If, additionally, in the above definition it holds that $\mu_1(x_1), \mu_2(x_2) < 1$, for every $(x_1, x_2) \in X_1 \times X_2$, then we call $\mathscr{C}$ a (μ_1, μ_2)-*contraction*.

Example 1.2.10. Let $k \in \mathbb{N}_{\geq 1}$, $\mathscr{B}_1 \subseteq \mathscr{L}_k^\infty$, and $\mathscr{C} : \mathscr{B}_1 \times \mathscr{B}_1 \to \mathscr{B}_1$ be a map defined by $\mathscr{C}(K, G) := K \cdot D_1(G)$. By linearity of derivation D_1 and of multiplication, it follows that $\mathscr{C}$ is a $\left(\frac{1}{2^{1+\mathrm{ord}_{\ell_1}(K)}}, \frac{1}{2^{1+\mathrm{ord}_{\ell_1}(G)}}\right)$-Lipschitz map with respect to the metric d_1 on $\mathscr{B}_1$.

In particular, if we restrict to $\mathscr{B}_{\geq 1}^+ \times \mathscr{B}_{\geq 1}^+$, then $\mathscr{C}$ is a $\left(\frac{1}{2^{1+\mathrm{ord}_{\ell_1}(K)}}, \frac{1}{2^{1+\mathrm{ord}_{\ell_1}(G)}}\right)$-contraction.

1.2.3. Fixed point theorems

Recall the classical Banach Fixed Point Theorem.

Theorem 1.2.11 (Banach Fixed Point Theorem, see e.g. [19])**.** Let X be a complete metric space and $\mathscr{S} : X \to X$ a contraction. There exists a unique fixed point $x \in X$ of $\mathscr{S}$, i.e., $\mathscr{S}(x) = x$.

Furthermore, x is given as the limit of the *Picard sequence*:

$$x = \lim_n (\mathscr{S}^{\circ n}(x_0)),$$

for any initial point $x_0 \in X$.

Now we state a fixed point theorem which is an easy consequence of the Banach Fixed Point Theorem. The idea of the proof is motivated by the Krasnoselskii's Fixed Point Theorem (see e.g. [36]). Proposition 1.2.12 is frequently used in proofs in Chapter 2.

Proposition 1.2.12 (Fixed point theorem, Proposition 3.2, [29])**.** Let X, Y be two metric spaces and let X be complete. Let $\mathscr{S}, \mathscr{T} : X \to Y$, such that:

1. $\mathscr{S}$ is a μ-Lipschitz map,
2. $\mathscr{T}$ is a λ-homothety,
3. $\mu < \lambda$,
4. $\mathscr{S}(X) \subseteq \mathscr{T}(X)$.

There exists a unique point $x \in X$ such that $\mathscr{T}(x) = \mathscr{S}(x)$.

Furthermore, x is the limit of the Picard sequence:

$$x = \lim_n ((\mathscr{T}^{-1} \circ \mathscr{S})^{\circ n}(x_0)),$$

for any initial point $x_0 \in X$.

Proof. Since $\mathscr{S}(X) \subseteq \mathscr{T}(X)$, $\mathscr{T}^{-1} \circ \mathscr{S} : X \to X$ is well defined. The map $\mathscr{T}$ is a λ-homothety, so its inverse $\mathscr{T}^{-1}$ is a $\frac{1}{\lambda}$-homothety on $\mathscr{S}(X)$. Therefore, since $\frac{\mu}{\lambda} < 1$, $\mathscr{T}^{-1} \circ \mathscr{S} : X \to X$ is a $\frac{\mu}{\lambda}$-contraction on X. We conclude by the Banach Fixed Point Theorem (Theorem 1.2.11). ∎

2. Normal forms of logarithmic transseries

In this chapter the main object of our study is the conjugacy equation:

$$\varphi \circ f \circ \varphi^{-1} = g, \tag{2.1}$$

in the *variable* $\varphi \in \mathfrak{L}^0$, where $f, g \in \mathfrak{L}^H$ are given. Equation (2.1) is solved in [21, Theorem A] using a transfinite algorithm of elementary parabolic changes of variables, but only for the logarithmic transseries of depth 1 (i.e., only one iteration of the logarithm). We generalize these results for hyperbolic, strongly hyperbolic and parabolic logarithmic transseries of an arbitrary depth. Our algorithm is *less transfinite*, and based on a fixed point theorem stated in Proposition 1.2.12 in Section 1.2.

Note that equation (2.1) is equivalent to the equation:

$$\varphi \circ f = g \circ \varphi. \tag{2.2}$$

In the next proposition we give a necessary condition on f and g for solvability of the conjugacy equation (2.1).

Proposition 2.0.1 (Necessary condition for solvability of the conjugacy equation)**.** Let $f, g \in \mathfrak{L}^H$ and let $\varphi \in \mathfrak{L}^0$ be a solution to the conjugacy equation $\varphi \circ f \circ \varphi^{-1} = g$. Then, $\mathrm{Lt}(f) = \mathrm{Lt}(g)$.

Proof. Let $\varphi = \mathrm{id} + \varphi_1$, $f = \mathrm{Lt}(f) + f_1$, and let $g = \mathrm{Lt}(g) + g_1$, for $\varphi_1, f_1, g_1 \in \mathfrak{L}$. By the definition of the composition, it follows that:

$$\varphi \circ f = f + \varphi_1 \circ f = \mathrm{Lt}(f) + f_1 + \varphi_1(\mathrm{Lt}(f)) + \sum_{i\geq 1} \frac{\varphi_1^{(i)}(\mathrm{Lt}(f))}{i!}(f_1)^i,$$

$$g \circ \varphi = \mathrm{Lt}(g) + g_1 + \sum_{i\geq 1} \frac{g^{(i)}}{i!}(\varphi_1)^i,$$

which implies that $\mathrm{Lt}(\varphi \circ f) = \mathrm{Lt}(f)$ and $\mathrm{Lt}(g \circ \varphi) = \mathrm{Lt}(g)$. Since $\varphi \circ f = g \circ \varphi$, it follows that $\mathrm{Lt}(f) = \mathrm{Lt}(g)$. ■

By Proposition 2.0.1, it follows that g in the conjugacy equation (2.1) is hyperbolic (strongly hyperbolic, parabolic) if and only if f is hyperbolic (strongly hyperbolic, parabolic). Moreover, we ask of g in equation (2.1) to be *minimal* in $\mathfrak{L}^H$, i.e., to have as little number of terms as possible. We call such g the *normal form* of f, and denote it by f_0. In that case, the conjugacy equation $\varphi \circ f \circ \varphi^{-1} = f_0$ is called the *normalization equation*, and its solutions are called the *normalizations* of f.

In Sections 2.1, 2.2 and 2.3 we prove normalization theorems in all three cases: hyperbolic (Theorem A), strongly hyperbolic (Theorem B) and parabolic (Theorem C). These theorems are proved by transforming a normalization equation to appropriate fixed point equations on complete metric spaces and applying fixed point theorem stated in Proposition 1.2.12 on these spaces.

In order to use the fixed point theorem from Proposition 1.2.12, for $f \in \mathfrak{L}^H$, we define the operators $\mathscr{T}_f$ and $\mathscr{S}_f$ on the appropriate spaces. We use the results from Appendix B for solving linear and various nonlinear equations to prove that $\mathscr{T}_f$ and $\mathscr{S}_f$ satisfy all assumptions of the fixed point theorem.

2.1. Normal forms of hyperbolic logarithmic transseries

In this section we present our results from [29]. They represent a generalization of the results obtained in [21] for hyperbolic logarithmic transseries of depth 1, to hyperbolic logarithmic transseries of an arbitrary depth, using fixed point theorems.

In the sequel we assume that $f \in \mathfrak{L}^H$, $f = \lambda z +$ h.o.t., $\lambda \in \mathbb{R}_{>0}$, $\lambda \neq 1$, is a *hyperbolic* logarithmic transseries.

In this section, we solve the conjugacy equation:

$$\varphi \circ f \circ \varphi^{-1} = g, \tag{2.3}$$

in variable $\varphi \in \mathfrak{L}^0$, for given $f, g \in \mathfrak{L}^H$. By Proposition 2.0.1, it follows that, if the conjugacy equation above has a solution in $\mathfrak{L}^0$, then $g = \lambda z +$ h.o.t. In Proposition 2.1.1 (see [29, Subsection 4.3]) below, we generalize Proposition 2.0.1 for hyperbolic logarithmic transseries.

In Subsection 2.1.1 we state the complete normalization theorem (Theorem A) for hyperbolic logarithmic transseries, which is the main theorem of this section. Furthermore, in Subsection 2.1.2 we prove statements 1 and 2, and in Subsection 2.1.4 we prove statement 3 of the normalization theorem. The normalization theorem is constructive: in Subsection 2.1.5, we give two algorithms for obtaining normal forms and normalizations. Finally, in Subsection 2.1.6 we give the description of the support of the normalization and prove that the support of the normalization depends only on the support of the initial hyperbolic logarithmic transseries.

Proposition 2.1.1 (Necessary condition for solvability of normalization equation, [29])**.** Let $f \in \mathfrak{L}_k^H$, $k \in \mathbb{N}$, such that $f = f_0 +$ h.o.t., for

$$f_0 := \lambda z + \sum_{\mathbf{0}_k < \mathbf{m} \leq \mathbf{1}_k} a_{\mathbf{m}} z \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_k^{m_k} \tag{2.4}$$

and $\mathbf{m} = (m_1, \ldots, m_k)$. Let $f_1 := f - f_0$ and $g \in \mathfrak{L}^H$. If the conjugacy equation $\varphi \circ f \circ \varphi^{-1} = g$ is solvable in $\mathfrak{L}^0$, then $g = f_0 +$ h.o.t.

Proof. Let $k \in \mathbb{N}$ be minimal such that $f, g \in \mathscr{L}_k^H$. Note that the conjugacy equation $\varphi \circ f \circ \varphi^{-1} = g$, $\varphi \in \mathfrak{L}^0$, is equivalent to the equation $\varphi \circ f = g \circ \varphi$. Let $\varphi = \mathrm{id} + \varphi_1$, $g = g_0 + g_1$, where the order of every term in g_0 is smaller than or equal to $\mathbf{1}_{k+1}$, and $\operatorname{ord}(g_1) > \mathbf{1}_{k+1}$. Now, by the Taylor Theorem (see Proposition A.1.6), we get the equivalent equation:

$$f_0 + f_1 + \varphi_1 \circ f_0 + \sum_{i \geq 1} \frac{\varphi_1^{(i)}(f_0)}{i!} (f_1)^i = g_0 + g_1 + \sum_{i \geq 1} \frac{(g_0 + g_1)^{(i)}}{i!} (\varphi_1)^i.$$

Since $\mathrm{ord}\left(\frac{g_1^{(i)}}{i!}(\varphi_1)^i\right) > \mathbf{1}_{k+1}$ and $\frac{\varphi_1^{(i)}(f_0)}{i!}(f_1)^i > \mathbf{1}_{k+1}$, for $i \in \mathbb{N}_{\geq 1}$, it is sufficient to consider the equation:

$$f_0 + \varphi_1 \circ f_0 = g_0 + \sum_{i\geq 1} \frac{g_0^{(i)}}{i!}(\varphi_1)^i.$$

Now, put $f_0 := \lambda z + zK$, $K \in \mathscr{B}_{\geq 1}^{+}$, $g_0 := \lambda z + zQ$, $Q \in \mathscr{B}_{\geq 1}^{+}$, and $\varphi_1 := zG + \text{h.o.b.}$, $G \in \mathscr{B}_{\geq 1}^{+}$. We get the equation:

$$\lambda z + zK + (zG)(\lambda z) + \sum_{i\geq 1} \frac{(zG)^{(i)}(\lambda z)}{i!}(zK)^i = \lambda z + zQ + \sum_{i\geq 1} \frac{(\lambda z + zQ)^{(i)}}{i!}(zG)^i$$

Now, by Lemmas A.2.8, A.3.1, A.3.3, and by the fact that $\mathrm{ord}(D_1(G)), \mathrm{ord}(D_1(G)) > \mathbf{1}_{k+1}$, after dividing by z, it follows that:

$$K + \lambda G + G \cdot K = Q + \lambda G + Q \cdot G.$$

Therefore, $K \cdot (1+G) = Q \cdot (1+G)$ and $K = Q$, since $1 + G \neq 0$. This implies that $f_0 = g_0$. ■

Proposition 2.1.1 suggests the normal form of a hyperbolic transseries. Indeed, we know that the initial part of f, f_0 as in (2.4), remains intact in the normal form. On the other hand, it is proved in [21, Theorem A] that the normal form of hyperbolic transseries $f \in \mathscr{L}_1^H$, $f = \lambda z + az\boldsymbol{\ell}_1 + \text{h.o.t.}$, $\lambda \in \mathbb{R}_{>0}$, $\lambda \neq 1$, $a \neq 0$, in $\mathscr{L}_1$ is $f_0 = \lambda z + az\boldsymbol{\ell}_1$.

This suggests putting $g := f_0$ and seeking in the following subsection for a solution of the conjugacy equation (2.3) with $g = f_0$ in $\mathfrak{L}^0$. If such a solution exists, then, by Proposition 2.1.1, f_0 is the normal form of the hyperbolic logarithmic transseries f.

Note that logarithmic transseries f_0 defined in (2.4) can be infinite. That depends on the initial part of the original transseries f. This can be seen in the next example.

Example 2.1.2. Let $\lambda \in \mathbb{R}_{>0}$ such that $\lambda \neq 1$.

1. $f(z) = \lambda z + 3z\boldsymbol{\ell}_1 + \text{h.o.t.} \in \mathscr{L}_1$,
   $f_0(z) = \lambda z + 3z\boldsymbol{\ell}_1$, $\varphi \in \mathscr{L}_1^0$,

2. $f(z) = \lambda z + 2z\boldsymbol{\ell}_1^2 + \text{h.o.t.} \in \mathscr{L}_1$,
   $f_0(z) = \lambda z$, $\varphi \in \mathscr{L}_1^0$,

3. $f(z) = \lambda z + z(\boldsymbol{\ell}_1\boldsymbol{\ell}_2 + \boldsymbol{\ell}_1\boldsymbol{\ell}_2^2 + \boldsymbol{\ell}_1^2\boldsymbol{\ell}_2^{-5}) + z^2\boldsymbol{\ell}_1^{-3} + z^3 \in \mathscr{L}_2$,
   $f_0(z) = \lambda z + z\boldsymbol{\ell}_1\boldsymbol{\ell}_2$, $\varphi \in \mathscr{L}_2^0$,

4. $f(z) = \lambda z + z(\boldsymbol{\ell}_3^{10} + \boldsymbol{\ell}_2^2\boldsymbol{\ell}_3^{-2} + \boldsymbol{\ell}_1\boldsymbol{\ell}_2 + \boldsymbol{\ell}_1\boldsymbol{\ell}_2\boldsymbol{\ell}_3 + \boldsymbol{\ell}_1\boldsymbol{\ell}_2\boldsymbol{\ell}_3^2 + \boldsymbol{\ell}_1\boldsymbol{\ell}_2^2\boldsymbol{\ell}_3^{-3}) + z^3\boldsymbol{\ell}_2^{-6} + z^4\boldsymbol{\ell}_3^{-2} \in \mathscr{L}_3$,
   $f_0(z) = \lambda z + z(\boldsymbol{\ell}_3^{10} + \boldsymbol{\ell}_2^2\boldsymbol{\ell}_3^{-2} + \boldsymbol{\ell}_1\boldsymbol{\ell}_2 + \boldsymbol{\ell}_1\boldsymbol{\ell}_2\boldsymbol{\ell}_3)$, $\varphi \in \mathscr{L}_3^0$,

5. $f(z) = \lambda z + z\Big(\sum_{i=2}^{+\infty} \boldsymbol{\ell}_2^i\boldsymbol{\ell}_3^{-i} + \boldsymbol{\ell}_1\boldsymbol{\ell}_4^{-3} + \boldsymbol{\ell}_1^2\boldsymbol{\ell}_4\Big) + \sum_{k\geq 2} z^k\boldsymbol{\ell}_4^{-k} \in \mathscr{L}_4$,
   $f_0(z) = \lambda z + z\Big(\sum_{i=2}^{+\infty} \boldsymbol{\ell}_2^i\boldsymbol{\ell}_3^{-i} + \boldsymbol{\ell}_1\boldsymbol{\ell}_4^{-3}\Big)$, $\varphi \in \mathscr{L}_4^0$.

For more examples see [29, Example 2.3].

2.1.1. Normalization theorem for hyperbolic logarithmic transseries

We consider the *normalization equation* $\varphi \circ f \circ \varphi^{-1} = f_0$, in the *variable* $\varphi \in \mathfrak{L}^0$, for a hyperbolic logarithmic transseries f, and f_0 its initial part as defined in (2.4).

Proposition 2.1.3. Suppose that the conjugacy equation $\varphi \circ f \circ \varphi^{-1} = f_0$ is solvable in $\mathfrak{L}^0$ for every hyperbolic logarithmic transseries $f = \lambda z + \text{h.o.t.}$, $0 < \lambda < 1$, and f_0 as defined in (2.4). Then the same holds for all hyperbolic logarithmic transseries with the leading term μz, for $\mu > 1$.

Proof. Let $f = \mu z + \text{h.o.t.}$, for $\mu > 1$. Then $f^{-1} = \frac{1}{\mu}z + \text{h.o.t.}$ and $f_0^{-1} = \frac{1}{\mu}z + \text{h.o.t.}$ Now, put $\lambda := \frac{1}{\mu}$ and $(f^{-1})_0$ be as defined in (2.4) with f^{-1} instead of f. It can be shown that $(f_0)^{-1} = (f^{-1})_0 + \text{h.o.t.}$. Since $0 < \lambda < 1$, by assumption, there exist $\varphi_1, \varphi_2 \in \mathfrak{L}^0$ such that $\varphi_1 \circ f^{-1} \circ \varphi_1^{-1} = (f^{-1})_0$ and $\varphi_2 \circ (f_0)^{-1} \circ \varphi_2^{-1} = (f^{-1})_0$. Now, for $\psi := \varphi_2^{-1} \circ \varphi_1$ we get $\psi \circ f^{-1} \circ \psi^{-1} = (f_0)^{-1}$. Taking compositional inverses on both sides of the above equation, it follows that $\psi \circ f \circ \psi^{-1} = f_0$. ∎

By Proposition 2.1.3, it follows that it is sufficient to consider hyperbolic logarithmic transseries $f = \lambda z + \text{h.o.t.}$, for $0 < \lambda < 1$.

Theorem A (Normalization theorem for hyperbolic logarithmic transseries, Main Theorem, [29])**.** Let $f = \lambda z + \text{h.o.t.} \in \mathscr{L}_k$, $k \in \mathbb{N}$, with $\lambda \in \mathbb{R}_{>0}$, $\lambda \neq 1$, be a hyperbolic logarithmic transseries. Write

$$f = \lambda z + \sum_{\mathbf{0}_k < \mathbf{m} \leq \mathbf{1}_k} a_{\mathbf{m}} z \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_k^{m_k} + \text{h.o.t.}, \quad \mathbf{m} = (m_1, \ldots, m_k),$$

and

$$f_0 := \lambda z + \sum_{\mathbf{0}_k < \mathbf{m} \leq \mathbf{1}_k} a_{\mathbf{m}} z \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_k^{m_k}. \tag{2.5}$$

Then:

1. There exists a unique parabolic logarithmic transseries $\varphi = z + \text{h.o.t.} \in \mathfrak{L}^0$, called *the normalizing transformation,* such that:

$$\varphi \circ f \circ \varphi^{-1} = f_0. \tag{2.6}$$

   Moreover, $\varphi \in \mathscr{L}_k^0$.

2. The logarithmic transseries f_0 is *minimal* with respect to the inclusion of the supports, and the coefficients of f_0 are invariant, within the conjugacy class of f by parabolic transformations in $\mathfrak{L}^0$. Therefore, f_0 is a *normal form* of f.

   In particular, f can be *linearized* in $\mathfrak{L}^0$ if and only if $\operatorname{ord}(f - \lambda \cdot \mathrm{id}) > \mathbf{1}_{k+1}$.

3. Let $0 < \lambda < 1$. For a parabolic initial condition $h \in \mathfrak{L}^0$, the *generalized Koenigs sequence*

$$\left(f_0^{\circ(-n)} \circ h \circ f^{\circ n} \right)_n \tag{2.7}$$

   converges to φ in the weak topology if and only if $\mathrm{Lb}_z(h) = \mathrm{Lb}_z(\varphi)$.

In statement 3 of Theorem A, the generalized Koenigs sequence converges in the weak topology which is the weakest of all topologies defined in Subsection 1.1.3. Indeed, in the following example we give an example of a hyperbolic logarithmic transseries f whose Koenigs sequence does not converge neither in the product nor in the power-metric topology, which are finer than the weak topology.

Example 2.1.4 (Remark 2.2, [29]). Consider the logarithmic transseries $f := \lambda z + z^2$ with $0 < \lambda < 1$. Note that $f \in \mathscr{L}_0^H$ is a formal power series in the *variable* z with real coefficients, and $f_0 = \lambda \cdot \mathrm{id}$. Consequently, if we choose the initial condition $h := \mathrm{id}$, the generalized Koenigs sequence is, in fact, the standard Koenigs sequence

$$\frac{1}{\lambda^n} \cdot (h \circ f^{\circ n}) = z + \left(\frac{1}{\lambda} + 1 + \lambda + \lambda^2 + \cdots + \lambda^{n-2} \right) z^2 + \text{h.o.t.}$$

Since the coefficients of z^2 do not eventually become stationary, the generalized Koenigs sequence does not converge in the product topology. Obviously, the sequence does not converge in the power-metric topology. On the other hand, note that the sequence $(\frac{1}{\lambda} + 1 + \lambda + \lambda^2 + \cdots + \lambda^{n-2})_n$ converges to $\frac{1}{\lambda} + \frac{1}{1-\lambda}$ in the Euclidean topology.

Remark 2.1.5 (Remark 2.2, [29])**.**

1. The normalization φ for hyperbolic logarithmic transseries $f \in \mathfrak{L}$ will be obtained in Subsection 2.1.2 using the fixed point theorem stated in Proposition 1.2.12 on suitable subspaces of $\mathscr{L}_k$, for the minimal $k \in \mathbb{N}$ such that $f \in \mathscr{L}_k$. We show that the normalization φ belongs to such $\mathscr{L}_k$ and, additionally, satisfies $\mathrm{ord}_z(\varphi - \mathrm{id}) \geq \mathrm{ord}_z(f - \lambda \cdot \mathrm{id})$.

2. In Subsection 2.1.6 we prove that the support of the normalization depends only on the support of the original hyperbolic logarithmic transseries $f \in \mathfrak{L}$, which means that the support of the normalization does not depend on the chosen initial condition $h \in \mathfrak{L}^0$.

3. The proof of the existence of the normalization relies on a fixed point theorem stated in Proposition 1.2.12. The normalization is given explicitely as the limit (in the power-metric topology) of the Picard sequence related to the certain contraction operator. However, this Picard sequence is not the generalized Koenigs sequence given in (2.7). Nevertheless, in Subsection 2.1.4 we prove the convergence of the generalized Koenigs sequence towards the normalization, for the appropriate initial conditions. In Subsection 2.1.5 we explain in detail these two different algorithms for obtaining the normalization.

Remark 2.1.6 (Remark 2.2, [29])**.** Let $f = \lambda z + \text{h.o.t.} \in \mathscr{L}_k$, $k \in \mathbb{N}$, for $0 < \lambda < 1$, be such that $\mathrm{ord}(f - \lambda \cdot \mathrm{id}) > \mathbf{1}_{k+1}$, i.e., $f_0 = \lambda \cdot \mathrm{id}$. By Theorem A, there exists the unique normalization $\varphi \in \mathfrak{L}^0$, such that

$$\varphi \circ f \circ \varphi^{-1} = \lambda \cdot \mathrm{id}.$$

By Theorem A, 1, it follows that $\varphi \in \mathscr{L}_k^0$. In this case, we call φ the *linearization* of the hyperbolic logarithmic transseries f and we say that f is *linearizable*.

In the linearizable case, the generalized Koenigs sequence (2.7) becomes the standard Koenigs sequence

$$\left(\frac{1}{\lambda^n}\cdot(h\circ f^{\circ n})\right)_n,$$

where $h \in \mathfrak{L}^0$ is the initial condition. By Theorem A, 3, the Koenigs sequence $\left(\frac{1}{\lambda^n}\cdot(h\circ f^{\circ n})\right)_n$ converges in the weak topology to the linearization φ if and only if $\mathrm{Lb}_z(h) = \mathrm{Lb}_z(\varphi)$.

In particular, if f satisfies that $\mathrm{ord}_z(f-\lambda\cdot\mathrm{id}) > 1$, then by Theorem A, 3, and Remark 2.1.5, 1, the Koenigs sequence $\left(\frac{1}{\lambda^n}\cdot(h\circ f^{\circ n})\right)_n$ converges in the weak topology to the linearization φ for any initial condition $h\in\mathfrak{L}^0$ such that $\mathrm{ord}_z(h-\mathrm{id}) > 1$. In particular, the sequence $\left(\frac{1}{\lambda^n}f^{\circ n}\right)_n$ converges to the normalization φ in the weak topology.

2.1.2. Existence and uniqueness of the normalization

Transforming the normalization equation to fixed point equations

The idea of transforming the normalization equation to a fixed point equation came from the classical Koenigs Theorem for complex hyperbolic germs of diffeomorphisms at zero. Therefore, we first state the Koenigs Theorem (without the proof) and then proceed to the transformation of our normalization equation.

Theorem 2.1.7 (Koenigs Theorem, see e.g. [4], [14], [24])**.** Let $f \in \mathrm{Diff}(\mathbb{C},0)$ be a hyperbolic analytic germ of diffeomorphism at zero such that $f(z) = \lambda z + o(z)$, for $\lambda\in\mathbb{C}$, $0<|\lambda|<1$. Then there exists an open neighbourhood U of 0 and a parabolic change of variables $\varphi\in\mathrm{Diff}(\mathbb{C},0)$, $\varphi(z) = z+o(z)$ such that $(\varphi\circ f)(z) = \lambda\varphi(z)$ on U. Moreover, the *Koenigs sequence* $\left(\frac{1}{\lambda^n}f^{\circ n}\right)_n$ converges uniformly to φ on U.

Since the parabolic change of variables φ in the Koenigs Theorem satisfies the conjugacy equation $\frac{1}{\lambda}\varphi\circ f = \varphi$, it is natural to consider the operator $\mathscr{P}_f(h) := \frac{1}{\lambda}\circ h\circ f$, for $h\in\mathrm{Diff}(\mathbb{C},0)$ tangent to the identity, which we call the *Koenigs operator*, and to transform the equation to the fixed point equation $\mathscr{P}_f(\varphi) = \varphi$. By the Koenigs Theorem, it follows that the sequence of iterations $(\mathscr{P}_f^{\circ n}(\mathrm{id}))$ converges uniformly to φ.

Our goal here is to generalize the Koenigs Theorem to hyperbolic logarithmic transseries. From Proposition 2.1.1, it follows that the normal form for a hyperbolic logarithmic transseries f is of the form $g = f_0 + \text{h.o.t.}$, where f_0 is given in (2.5). Since we want this normal form to be *minimal*, we try to solve the conjugacy equation $\varphi \circ f \circ \varphi^{-1} = g$, for $g := f_0$. Therefore, we have to adapt the Koenigs operator for f_0. Note that in the Koenigs Theorem, the normal form is $f_0 = \lambda \cdot \mathrm{id}$. Since $f_0^{-1} = \frac{1}{\lambda} \cdot \mathrm{id}$, the Koenigs operator is, in fact, $\mathscr{P}_f(h) = f_0^{-1} \circ h \circ f$.

Definition 2.1.8 (Generalized Koenigs operator and sequence, [29])**.** Let $f \in \mathfrak{L}^H$, $f = \lambda z + \text{h.o.t.}$, $0 < \lambda < 1$, be a hyperbolic logarithmic transseries and f_0 as in (2.5). Let $\mathscr{P}_f : \mathfrak{L}^0 \to \mathfrak{L}^0$ be the operator defined by:

$$\mathscr{P}_f(h) := f_0^{-1} \circ h \circ f, \quad h \in \mathfrak{L}^0. \tag{2.8}$$

We call $\mathscr{P}_f$ the *generalized Koenigs operator*. In particular, if $f_0 = \lambda \cdot \mathrm{id}$, then we call $\mathscr{P}_f$ the *Koenigs operator*.

Moreover, we call $\left(\mathscr{P}_f^{\circ n}(h)\right)_n$ the *(generalized) Koenigs sequence* with the *initial condition* $h \in \mathfrak{L}^0$.

In the next example we show that $\mathscr{P}_f$ is not a contraction even on $\mathscr{L}_0^0$, with respect to some standard metrics.

Example 2.1.9 (Noncontractibility of operator$\mathscr{P}_f$ in standard metrics)**.** For simplicity, we consider the space of formal power series $z\mathbb{R}[[z]] \subseteq \mathscr{L}_0^H$ in the formal variable z. By Example 1.1.19, (4), on $z\mathbb{R}[[z]]$ the product topology is the same as the power-metric topology, so they are both metrizable by the power-metric d_z defined in Subsection 1.1.3. By Example 1.1.19, (6), the weak topology on $z\mathbb{R}[[z]]$ is also metrizable by e.g. the *weak metric*:

$$d_w(h_1, h_2) := \sum_{i=1}^{\infty} \frac{|a_i - b_i|}{2^i \cdot (1 + |a_i - b_i|)},$$

where $h_1, h_2 \in z\mathbb{R}[[z]]$, $h_1 := \sum_{i=1}^{+\infty} a_i z^i, h_2 := \sum_{i=1}^{+\infty} b_i z^i$, for $a_i, b_i \in \mathbb{R}$ and $i \in \mathbb{N}$.

We give below some examples that show that the operator $\mathscr{P}_f$, $f \in z\mathbb{R}[[z]]$, $f = \lambda z + \text{h.o.t}$, $0 < \lambda < 1$, is not necessarily a contraction with respect to the power-metric d_z, nor with respect to the weak metric d_w on $z + z^2\mathbb{R}[[z]] \subseteq \mathscr{L}_0^0$.

(i) Take e.g. $f := \lambda z + z^2$ and $g := \mathrm{id}$. We get:

$$d_z(0,g) = d_z\big(0, \mathscr{P}_f(g)\big) = \frac{1}{2}.$$

(ii) For f and g as above,

$$d_w(0,g) = \frac{1}{4} < \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{1+\lambda} = d_w\big(0, \mathscr{P}_f(g)\big).$$

Since $\mathscr{P}_f$ is not a contraction in any of previously introduced metrics, not even on $\mathscr{L}_0^0$, we adapt the idea from [26, Chapter 3]. For a hyperbolic $f \in \mathfrak{L}^H$, we define the operator $\mathscr{H}_f : \mathfrak{L}_{>\mathrm{id}} \to \mathfrak{L}_{>\mathrm{id}}$ such that

$$\mathscr{H}_f(h) := \mathscr{P}_f(\mathrm{id} + h) - \mathrm{id},$$

where $\mathfrak{L}_{>\mathrm{id}} := \bigcup_{k\in\mathbb{N}} \mathscr{L}_k^{W_k}$, for $W_k := \big\{(\alpha, \mathbf{n}) \in \mathbb{R} \times \mathbb{Z}^k : (\alpha, \mathbf{n}) > (1, \mathbf{0}_k)\big\}$, $k \in \mathbb{N}$.

Example 2.1.10 (Noncontractibility of the operator $\mathscr{H}_f$ in standard metrics)**.** Let d_z be the power-metric and d_w the weak metric defined in Example 2.1.9 on $z\mathbb{R}[[z]]$. We show below that the operator $\mathscr{H}_f$, $f \in z\mathbb{R}[[z]]$, for $f = \lambda z + \text{h.o.t}$, $0 < \lambda < 1$, is not a contraction with respect to the power-metric d_z, nor with respect to the weak metric d_w on $z^2\mathbb{R}[[z]]$.

(i) Take e.g. $f := \lambda z + z^2 + \text{h.o.t}$, $0 < \lambda < 1$, $g := z^2$. Since $\mathrm{ord}_z\big(\mathscr{H}_f(0) - \mathscr{H}_f(g)\big) = \mathrm{ord}_z(g) = 2$, we get:

$$d_z(0,g) = d_z\big(\mathscr{H}_f(0), \mathscr{H}_f(g)\big) = \frac{1}{4}.$$

(ii) In the weak metric d_w, for f and g as in (i), we get:

$$\begin{aligned} &d_w(0,g) = \frac{1}{8}, \\ &d_w\big(\mathscr{H}_f(0), \mathscr{H}_f(g)\big) = \frac{1}{4} \cdot \frac{\lambda}{1+\lambda} + \frac{1}{8} \cdot \frac{2}{1+2} + \frac{1}{16} \cdot \frac{1}{\lambda+1}. \end{aligned} \tag{2.9}$$

If we put $\lambda = 1$, we get:

$$d_w\big(\mathscr{H}_f(0), \mathscr{H}_f(g)\big) = \frac{23}{96} > \frac{1}{8} = d_w(0,g).$$

Therefore, by continuity of (2.9) in the variable λ, it follows that there exists $0 < \lambda < 1$ sufficiently close to 1, such that $d_w\big(\mathscr{H}_f(0), \mathscr{H}_f(g)\big) > \frac{1}{8}$, which implies that $\mathscr{H}_f$ is not a contraction in general.

Since the operators $\mathscr{P}_f$ and $\mathscr{H}_f$ are not contractions in any of our usual metrics, in order to apply a fixed point theorem stated in Proposition 1.2.12, in the next proposition we transform the equation $\mathscr{H}_f(\varphi) = \varphi$ to another fixed point equation $\mathscr{T}_f(\varphi) = \mathscr{S}_f(\varphi)$ for suitable operators $\mathscr{T}_f$ and $\mathscr{S}_f$. Such transformation of the equation is motivated by the Krasnoselskii Fixed Point Theorem (see e.g. [36]).

Proposition 2.1.11 (Proposition 3.4, [29])**.** Let $k \in \mathbb{N}$ and $f \in \mathscr{L}_k$, $f = \lambda z + \text{h.o.t.}$, with $0 < \lambda < 1$. Let $f_1 := f - f_0$, for f_0 as in (2.5). For $\varphi \in \mathfrak{L}^0$ and $h := \varphi - \mathrm{id} \in \mathfrak{L}_{>\mathrm{id}}$, the following equations are equivalent:

1. $\varphi \circ f \circ \varphi^{-1} = f_0$,
2. $\mathscr{T}_f(h) = \mathscr{S}_f(h)$,

where the operators $\mathscr{S}_f, \mathscr{T}_f \colon \mathfrak{L}_{>\mathrm{id}} \to \mathfrak{L}_{>\mathrm{id}}$ are given by:

$$\begin{aligned} \mathscr{S}_f(h) &:= \frac{1}{\lambda}\Big(f_1 + (h \circ f - h \circ f_0) - \big(g_0\,(\mathrm{id} + h) - g_0 - g_0' \cdot h \big) \Big), \\ \mathscr{T}_f(h) &:= \frac{1}{\lambda}\Big((\lambda \cdot h - h(\lambda \cdot \mathrm{id})) - (h \circ f_0 - h(\lambda \cdot \mathrm{id})) + g_0' \cdot h \Big). \end{aligned} \tag{2.10}$$

Here, $g_0 := f_0 - \lambda \cdot \mathrm{id}$.

Proof. Note that the fixed point equation $\mathscr{T}_f(h) = \mathscr{S}_f(h)$ is equivalent to the equation $f_1 + h \circ f - g_0(\mathrm{id} + h) + g_0 = \lambda h$. From the last equation, since $f = \mathrm{id} + g_0 + f_1$ and $\varphi = \mathrm{id} + h$, we get the equivalent equation $\varphi \circ f - g_0 \circ \varphi = \lambda \varphi$, i.e., $\varphi \circ f \circ \varphi^{-1} = f_0$. ■

By superlinearity of derivation, left multiplication and right composition (Proposition A.1.4), note that $\mathscr{T}_f$ is a superlinear operator.

Remark 2.1.12 (Expansions of the operators $\mathscr{S}_f$ and $\mathscr{T}_f$)**.** Let f, f_0, f_1 and g_0 be as in Proposition 2.1.11. By the Taylor Theorem (Proposition A.1.6) we have the following expansions:

$$\begin{aligned} \mathscr{S}_f(h) &= \frac{1}{\lambda}\Big(f_1 + \sum_{i \geq 1} \frac{h^{(i)}(f_0)}{i!}(f_1)^i - \sum_{i \geq 2} \frac{g_0^{(i)}}{i!} h^i \Big), \\ \mathscr{T}_f(h) &= h - \frac{1}{\lambda} h(\lambda \cdot \mathrm{id}) - \frac{1}{\lambda} \sum_{i \geq 1} \frac{h^{(i)}(\lambda \cdot \mathrm{id})}{i!}(g_0)^i + \frac{1}{\lambda} g_0' \cdot h, \end{aligned} \tag{2.11}$$

for $h \in \mathfrak{L}_{>\mathrm{id}}$.

By (2.11) and Example 1.2.7, if $\mathrm{ord}_z(f_1) > 1$, then $\mathscr{S}_f$ increases the order (in z) by at least $\mathrm{ord}_z(f_1) - 1$ on the space $(\mathscr{L}_k^{\mathrm{ord}_z(f_1)}, d_z)$. On the other hand, $\mathscr{T}_f$ is a superlinear isometry on space $(\mathscr{L}_k^{\mathrm{ord}_z(f_1)}, d_z)$. In that case, we can apply fixed point theorem stated in Proposition 1.2.12 to the existence and the uniqueness of the solution of the fixed point equation $\mathscr{T}_f(h) = \mathscr{S}_f(h)$ on the space $(\mathscr{L}_k^{\mathrm{ord}_z(f_1)}, d_z)$.

However, if $\mathrm{ord}_z(f_1) = 1$, then $\mathrm{ord}_z(\mathscr{T}_f(h)) = \mathrm{ord}_z(\mathscr{S}_f(h)) = \mathrm{ord}_z(h)$. Since the coefficient of the homothety $\mathscr{T}_f$ and the minimal coefficient of the Lipschitz map $\mathscr{S}_f$ are equal, we cannot apply the fixed point theorem from Proposition 1.2.12 directly. However, $\mathscr{S}_f$ increases the order by at least $(0, \mathbf{1}_{k-1}, 2)$ and, since $\mathrm{ord}(g_0) < (0, \mathbf{1}_k)$, $\mathscr{T}_f$ increases the order of h by at most $(0, \mathbf{1}_k)$. Therefore, the idea is to use a metric which captures the increase in order, no matter how small (in which variable), even though there is no increase in ord_z. This motivates the definition of the so-called *r-preserving metric*.

Definition 2.1.13 (r-preserving metric). Let $r : \mathscr{L}_k \to \mathbb{R}^p$, for $1 \le p \le k+1$. We say that a metric d on the space $\mathscr{L}_k$, $k \in \mathbb{N}$, is *r-preserving* if, for every $M > \mathbf{0}_p$ lexicographically, there exists a constant $0 < \mu_M < 1$, such that, for every g_1, g_2 such that $r(g_1) + M < r(g_2)$, it holds that $d(0, g_2) \le \mu_M \cdot d(0, g_1)$.

Proposition 2.1.14.

1. The power-metric d_z on $\mathscr{L}_k$, for $k \in \mathbb{N}$, is ord_z-preserving.

2. There are no ord-preserving metrics on $\mathscr{L}_k$, $k \in \mathbb{N}_{\ge 1}$.

Proof. 1. Let $M > \mathbf{0}_p$ and let $g_1, g_2 \in \mathscr{L}_k$ such that $\mathrm{ord}_z(g_1) + M < \mathrm{ord}_z(g_2)$. Then

$$d_z(0, g_2) = 2^{-\mathrm{ord}_z(g_2)} \le 2^{-M} 2^{-\mathrm{ord}_z(g_1)} = 2^{-M} d_z(0, g_1).$$

Put $\mu_M := 2^{-M} < 1$. Thus, we proved statement 1.

2. Suppose that d is an r-preserving metric on $\mathscr{L}_k$, $k \ge 1$, with respect to the $r := \mathrm{ord} : \mathscr{L}_k \to \mathbb{R}^{k+1}$. Take the sequence (g_n) in $\mathscr{L}_k$ defined by $g_n := z\boldsymbol{\ell}_1^n$, for $n \in \mathbb{N}$. Evidently,

$$\mathrm{ord}(g_n) = \mathrm{ord}(g_0) + (0, n, \mathbf{0}_{k-1}), \tag{2.12}$$

for $n \in \mathbb{N}$. Since d is ord-preserving, for $M = (0, 1, \mathbf{0}_{k-1})$ there exists $0 < \mu_M < 1$ such that $d(0, g_{n+1}) \le \mu_M \cdot d(0, g_n)$, for $n \in \mathbb{N}$. Then, $d(0, g_n) \le \mu_M^n d(0, g_0)$, for $n \in \mathbb{N}$. Taking the limit as $n \to +\infty$, we get that

$$\lim_{n\to+\infty} d(0, g_n) = 0. \tag{2.13}$$

However, by the definition of g_n,

$$\operatorname{ord}(z^2) \geq \operatorname{ord}(g_n) + \left(\frac{1}{2}, \mathbf{0}_k\right), \tag{2.14}$$

for $n \in \mathbb{N}$. Since d is ord-preserving, there exists $\mu_{\frac{1}{2}} > 0$ such that $d(z^2, 0) \leq \mu_{\frac{1}{2}}\, d(g_n, 0)$, for $n \in \mathbb{N}$. Passing to the limit as $n \to +\infty$ and using (2.13), we get $d(z^2, 0) = 0$. This is a contradiction with the definition of a metric. Therefore, an ord-preserving metric does not exist. ■

By Proposition 2.1.14, it seems useless to try to come up with a metric in which $\mathscr{S}_f$ is a contraction and $\mathscr{T}_f$ is an isometry. Therefore, we split the proof of Theorem A in two cases:

(a) $\operatorname{ord}_z(f - f_0) > 1$,

(b) $\operatorname{ord}_z(f - f_0) = 1$.

In case (a) we proceed directly by the fixed point theorem from Proposition 1.2.12 to prove the existence and the uniqueness of the solution h of the fixed point equation $\mathscr{T}_f(h) = \mathscr{S}_f(h)$, for $h \in \mathfrak{L}_{>\mathrm{id}}$. Then, by Proposition 2.1.11 $\varphi = \mathrm{id} + h$ is the unique normalization satisfying equation $\varphi \circ f \circ \varphi^{-1} = f_0$. In case (b), we first *prenormalize* the hyperbolic logarithmic transseries f and then apply case (a) to the prenormalized f.

Proof of case (a)**:** $\operatorname{ord}_z(f - f_0) > 1$

In the following lemma we verify that the operators $\mathscr{T}_f$ and $\mathscr{S}_f$ satisfy the assumptions of the fixed point theorem from Proposition 1.2.12.

Lemma 2.1.15 (Properties of the operators $\mathscr{T}_f$ and $\mathscr{S}_f$, Lemma 4.1, [29])**.** Let $k \in \mathbb{N}$ and $f = \lambda z + \text{h.o.t.} \in \mathscr{L}_k^H$, with $0 < \lambda < 1$. Let f_0 be as defined in (2.5) and $f_1 := f - f_0$. Let $\beta := \operatorname{ord}_z(f_1) > 1$ and let $\mathscr{T}_f$ and $\mathscr{S}_f$ be the operators defined in (2.10). Then:

1. $\mathscr{L}_k^\beta$ is invariant under $\mathscr{T}_f$ and $\mathscr{S}_f$,

2. $\mathscr{S}_f$ is a $\frac{1}{2\beta-1}$-contraction on the space $\left(\mathscr{L}_k^\beta, d_z\right)$,

3. $\mathscr{T}_f$ is an isometry and a surjection on the space $\left(\mathscr{L}_k^\beta, d_z\right)$.

The same holds for the spaces $\mathscr{L}_m^\beta$, in place of $\mathscr{L}_k^\beta$, for all $m \geq k$.

Proof. 1. Note that $f_1 \in \mathscr{L}_m^\beta$, for every $m \geq k$. The invariance of $\mathscr{L}_m$, $m \geq k$, and of the subspaces $\mathscr{L}_m^\beta$ under $\mathscr{T}_f$ and $\mathscr{S}_f$ follows easily from Remark 2.1.12.

2. In order to prove statement 2, we consider the *expansion* of the operator $\mathscr{S}_f$ from Remark 2.1.12. Let $h_1, h_2 \in \mathscr{L}_m^\beta$, $m \geq k$. Then $\operatorname{ord}_z(h_1), \operatorname{ord}_z(h_1) \geq \beta$. Since $\beta = \operatorname{ord}_z(f_1)$, we obtain

$$\begin{aligned}\operatorname{ord}_z\left(\sum_{i\geq 1}\frac{h_1^{(i)}\circ f_0}{i!}(f_1)^i - \sum_{i\geq 1}\frac{h_2^{(i)}\circ f_0}{i!}(f_1)^i\right) &= \operatorname{ord}_z\left(\sum_{i\geq 1}\frac{(h_1-h_2)^{(i)}\circ f_0}{i!}(f_1)^i\right) \\ &= \operatorname{ord}_z(h_1-h_2)+\beta-1,\end{aligned} \tag{2.15}$$

for the *linear part* of the operator $\mathscr{S}_f$. For the *non-linear part* of $\mathscr{S}_f$, as in Example 1.2.7, we get:

$$\begin{aligned}\operatorname{ord}_z\left(\sum_{i\geq 2}\frac{g_0^{(i)}}{i!}h_1^i - \sum_{i\geq 2}\frac{g_0^{(i)}}{i!}h_2^i\right) &= \operatorname{ord}_z\left(\sum_{i\geq 2}\frac{g_0^{(i)}}{i!}(h_1-h_2)\left(\sum_{j=0}^{i-1}h_1^j h_2^{i-j-1}\right)\right) \\ &\geq \operatorname{ord}_z(h_1-h_2)+\beta-1.\end{aligned} \tag{2.16}$$

The equations (2.15) and (2.16) imply that $\mathscr{S}_f$ is a $\frac{1}{2\beta-1}$-contraction on the space $(\mathscr{L}_k, d_z)$, as well as on the spaces $(\mathscr{L}_m^\beta, d_z)$, $m \geq k$.

3. We first prove that $\mathscr{T}_f$ is an isometry on $\left(\mathscr{L}_m^\beta, d_z\right)$, $m \geq k$. We use the expansion of the operator $\mathscr{T}_f$ from Remark 2.1.12. Let $h = z^\alpha H_\alpha + \text{h.o.b.}(z) \in \mathscr{L}_m^\beta$, $m \geq k$, where $H_\alpha \in \mathscr{B}_1$, $\alpha \geq \beta$. Analyzing the orders of the terms of $\mathscr{T}_f(h)$ in the expansion from Remark 2.1.12, expanding $h(\lambda z)$ by Lemma A.3.1, and using the fact that $\operatorname{ord}(g_0) > (1,0,\ldots,0)_{m+1}$ and $\lambda \neq 1$, $\alpha > 1$, we conclude that

$$\operatorname{ord}(\mathscr{T}_f(h)) = \operatorname{ord}(z^\alpha H_\alpha).$$

Hence,

$$\operatorname{ord}_z(\mathscr{T}_f(h)) = \operatorname{ord}_z(z^\alpha H_\alpha) = \operatorname{ord}_z(h). \tag{2.17}$$

Therefore, $\mathscr{T}_f$ is a superlinear isometry on the spaces $\left(\mathscr{L}_m^\beta, d_z\right)$, $m \geq k$.

It remains to prove that $\mathscr{T}_f : \mathscr{L}_k^\beta \to \mathscr{L}_k^\beta$ is a surjection, and that this also holds if we replace $\mathscr{L}_k^\beta$ by $\mathscr{L}_m^\beta$, $m \geq k$. Due to the superlinearity of $\mathscr{T}_f$, it is sufficient to prove that, for every block $z^\gamma M_\gamma \in \mathscr{L}_k^\beta$, $M_\gamma \in \mathscr{B}_1$, there exists a block $z^\alpha H_\alpha \in \mathscr{L}_k^\beta$, $H_\alpha \in \mathscr{B}_1$, such that

$$\mathscr{T}_f\big(z^\alpha H_\alpha\big) = z^\gamma M_\gamma. \tag{2.18}$$

The idea there is to prove the existence of a solution to (2.18) by reformulating this equation as a fixed point equation for a suitable contraction on the complete space $(\mathscr{B}_1, d_1)$.

First, as $\mathscr{T}_f$ is an isometry, $\alpha = \gamma$. Write $g_0 = zQ$, with $Q \in \mathscr{B}_{\geq 1}^+$. Using Lemma A.3.1, Lemma A.2.7 and Lemma A.3.3, we regroup the elements of the left-hand side $\mathscr{T}_f\big(z^\gamma H_\gamma\big)$ of (2.18) as

$$\begin{aligned}
\lambda z^\gamma H_\gamma - \big(z^\gamma H_\gamma\big)(\lambda z) &= \big(\lambda - \lambda^\gamma\big) z^\gamma H_\gamma - \lambda^\gamma z^\gamma \big(\log\lambda \cdot D_1(H_\gamma) + \mathscr{C}_\lambda(H_\gamma)\big),\\
\big(zQ\big)' \cdot z^\gamma H_\gamma &= z^\gamma \big(Q + D_1(Q)\big) H_\gamma,\\
\sum_{i\geq 1} \frac{(z^\gamma H_\gamma)^{(i)}(\lambda z)}{i!} (zQ)^i &= z^\gamma \lambda^\gamma H_\gamma \sum_{i\geq 1} \binom{\gamma}{i} \left(\frac{Q}{\lambda}\right)^i + z^\gamma \mathscr{K}_Q(H_\gamma),
\end{aligned}$$

where $\mathscr{C}_\lambda$ from Lemma A.3.1 is a superlinear $\frac{1}{4}$-contraction on $(\mathscr{B}_1, d_1)$ and $\mathscr{K}_Q := \mathscr{K}(\cdot, Q)$ from Lemma A.3.3 is a superlinear $\frac{1}{2^{1+\mathrm{ord}_{\boldsymbol{\ell}_1}(Q)}}$-contraction on $(\mathscr{B}_1, d_1)$:

$$\mathrm{ord}_{\boldsymbol{\ell}_1}\left(\mathscr{C}_Q(H_\gamma)\right) \geq \mathrm{ord}_{\boldsymbol{\ell}_1}(H_\gamma) + 1.$$

Finally, the operators D_1, $\mathscr{C}_\lambda$ and $\mathscr{K}_Q$ do not decrease the powers of the variables $\boldsymbol{\ell}_m$, for $1 \leq m \leq k$. Hence, after dividing by z^γ, these identities allow to rewrite (2.18) as the following fixed point equation:

$$H_\gamma = \mathscr{S}_1(H_\gamma),$$

where $\mathscr{S}_1 : \mathscr{B}_1 \to \mathscr{B}_1$ is the operator defined by

$$\mathscr{S}_1(H) := \frac{\lambda^\gamma\big(\log\lambda \cdot D_1(H) + \mathscr{C}_\lambda(H)\big) - H \cdot D_1(Q) + \mathscr{K}_Q(H) + M_\gamma}{\lambda - \lambda^\gamma + Q - \lambda^\gamma \sum_{i\geq 1}\binom{\gamma}{i}\left(\frac{Q}{\lambda}\right)^i},\ H \in \mathscr{B}_1. \tag{2.19}$$

Note that $\gamma \geq \beta > 1$, so $\lambda - \lambda^\gamma \neq 0$. Hence, thanks to Example 1.2.6, $\mathscr{S}_1$ is an affine $\frac{1}{2}$-contraction on the space $\big(\mathscr{B}_1, d_1\big)$, which is complete by Proposition 1.1.16. It follows from the Banach Fixed Point Theorem (Theorem 1.2.11) that $\mathscr{S}_1$ has a unique fixed point in $\mathscr{B}_1$, so that the block $z^\gamma M_\gamma$ has a unique preimage $z^\gamma H_\gamma \in \mathscr{L}_m^\beta$ by $\mathscr{T}_f$.

Finally, thanks to the superlinearity of $\mathscr{T}_f$, we conclude that $\mathscr{T}_f$ is surjective on $\mathscr{L}_m^\beta$.

■

Finally, we prove the case (a) of Theorem A.

Proof of case (a) of Theorem A. Let $\beta := \mathrm{ord}_z(f - f_0)$. By Proposition 1.1.16, the space $\mathscr{L}_k^\beta$ is complete. By Lemma 2.1.15 and the fixed point theorem from Proposition 1.2.12, the equation $\mathscr{T}_f(h) = \mathscr{S}_f(h)$ has a unique solution $h \in \mathscr{L}_k^\beta$. Since $\beta > 1$, it follows that $h \in \mathfrak{L}_{>\mathrm{id}}$.

We now prove the uniqueness of the solution of

$$\mathscr{T}_f(h) = \mathscr{S}_f(h) \tag{2.20}$$

in the larger space $\mathfrak{L}_{>\mathrm{id}}$. Suppose that there exists another solution $h_1 \in \mathfrak{L}_{>\mathrm{id}}$ of (2.20), such that $h_1 \neq h$. There exists the minimal $m \geq k$ such that $h_1 \in \mathscr{L}_m$.

We prove that $\mathrm{ord}_z(h_1) \geq \beta$. To this end we introduce the operators

$$\widetilde{\mathscr{S}_f}(h) := \frac{1}{\lambda}\left(f_1 + (h \circ f - h \circ f_0)\right)$$

and

$$\widetilde{\mathscr{T}_f}(h) := \frac{1}{\lambda}\Big(\left(\lambda h - h(\lambda z)\right) - \left(h \circ f_0 - h(\lambda z)\right) + g_0' \cdot h + \big(g_0(\mathrm{id} + h) - g_0 - g_0' \cdot h\big)\Big)$$

obtained by moving the last term of $\mathscr{S}_f(h)$ to $\mathscr{T}_f(h)$ (in (2.10)). So we have

$$\widetilde{\mathscr{S}_f}(h_1) = \widetilde{\mathscr{T}_f}(h_1). \tag{2.21}$$

Since $\beta = \mathrm{ord}_z(f_1)$ and $\mathrm{ord}_z(h_1) \geq 1$ for $h_1 \in \mathfrak{L}_{>\mathrm{id}}$, by the Taylor Theorem (Proposition A.1.6) it follows that

$$\mathrm{ord}_z\left(\widetilde{\mathscr{S}_f}(h_1)\right) \geq \min\left\{\beta, \mathrm{ord}_z(h_1) + \beta - 1\right\} = \beta.$$

On the other hand, it can be seen as in (2.17) in the proof of Lemma 2.1.15 that we have the identity

$$\mathrm{ord}_z(\widetilde{\mathscr{T}_f}(h_1)) = \mathrm{ord}_z(h_1).$$

Comparing the orders of the left and the right-hand sides of (2.21), we obtain

$$\mathrm{ord}_z(h_1) \geq \beta.$$

That is, $h_1 \in \mathscr{L}_m^\beta$. Recall that the space $\mathscr{L}_m^\beta$, $m \in \mathbb{N}$, is complete by Proposition 1.1.16. Hence Lemma 2.1.15 and the fixed point theorem from Proposition 1.2.12 give the uniqueness of the solution of (2.20) in $\mathscr{L}_m^\beta$. Now, since $\mathscr{L}_k^\beta \subseteq \mathscr{L}_m^\beta$, both h and h_1 belong to $\mathscr{L}_m^\beta$, which contradicts the uniqueness of the solution in $\mathscr{L}_m^\beta$.

Finally, by Proposition 1.2.12, the uniqueness of the solution h of the equation $\mathscr{S}_f(h) = \mathscr{T}_f(h)$ in $\mathfrak{L}_{>\mathrm{id}}$ implies the uniqueness of the normalizing change of variables $\varphi = \mathrm{id} + h$ in the space $\mathfrak{L}^0$. This proves the case (a). ■

Proof of case (b)**:** $\mathrm{ord}_z(f - f_0) = 1$

Let $f \in \mathfrak{L}^H$ be a hyperbolic logarithmic transseries and f_0 be the initial part of f as defined in (2.5). Suppose that $\mathrm{ord}_z(f - f_0) = 1$. Since in this case $\mathscr{T}_f$ is an isometry and $\mathscr{S}_f$ is a 1-Lipschitz map (where 1 is its minimal Lipschitz coefficient), we cannot apply the fixed point theorem stated in Proposition 1.2.12 directly. Therefore, we proceed in the following steps:

Step 1. We *prenormalize* the hyperbolic logarithmic transseries f, i.e., we solve a conjugacy equation:

$$\varphi_0 \circ f \circ \varphi_0^{-1} = f_0 + \mathrm{h.o.b.}(z), \tag{2.22}$$

in the *variable* $\varphi_0 \in \mathfrak{L}^0$. The solution is not unique in the space $\mathfrak{L}^0$, since it obviously depends on higher order blocks of the right-hand side of (2.22). On the other hand, in the proof of case (b) below, we show that, if we impose the *canonical form* of φ_0, i.e., $\varphi_0 = \mathrm{id} + zH$, for $H \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, then φ_0 is unique.

We call such φ_0 the *prenormalization* (or *prenormalizing transformation*) of f and equation (2.22) the *prenormalization equation*. The prenormalization φ_0 is obtained by transforming the prenormalization equation to the *fixed point equation* and using the fixed point theorem stated in Proposition 1.2.12.

Step 2. We apply the procedure from case (a) to solve the conjugacy equation:

$$\varphi_1 \circ \left(\varphi_0 \circ f \circ \varphi_0^{-1}\right) \circ \varphi_1^{-1} = f_0,$$

in the *variable* $\varphi_1 \in \mathfrak{L}^0$. By case (a), it follows that such $\varphi_1 \in \mathfrak{L}^0$ is unique.

In the next lemma we first define the operators $\mathscr{T}_0$ and $\mathscr{S}_0$ on the complete space $(\mathscr{B}_{\geq 1}^+, d_1)$ that transform a prenormalization equation (2.22) to the *fixed point equation* (2.23).

Lemma 2.1.16 (Transforming a prenormalization equation to the fixed point equation, Lemma 4.2, [29]). Let $k \in \mathbb{N}_{\geq 1}$ and $f(z) = \lambda z + \text{h.o.t.} \in \mathscr{L}_k^H$, with $0 < \lambda < 1$. Write $f = f_0 + f_1$, where f_0 is defined as in (2.5), and $\mathrm{ord}_z(f_1) = 1$.

A transseries $\varphi_0 = z + zH + \text{h.o.b.}(z)$, with $H \in \mathscr{B}_{\geq 1}^+$, satisfies a prenormalization equation (2.22) if and only if H satisfies the equation

$$\mathscr{T}_0(H) = \mathscr{S}_0\big(H\big), \tag{2.23}$$

where the operators $\mathscr{T}_0, \mathscr{S}_0 : \mathscr{B}_{\geq 1}^+ \to \mathscr{B}_1^+ \subset \mathscr{B}_{\geq 1}^+$, are defined by:

$$\begin{aligned} \mathscr{T}_0(H) := \Big(-\lambda \log \lambda - (1+\log \lambda) R_0 - \lambda \sum_{i \geq 2} \frac{(-1)^i}{i(i-1)} \Big(\frac{R_0}{\lambda}\Big)^i \Big) \cdot D_1\big(H\big) + \\ + D_1(R_0) \cdot \Big(H + \sum_{i \geq 2} \frac{(-1)^i}{i(i-1)} H^i \Big) - H \cdot R, \\ \mathscr{S}_0(H) := \lambda \mathscr{C}_\lambda(H) + \mathscr{C}_{R_0}(H) - \mathscr{K}_{R_0}(H) + \mathscr{K}_R(H) + R. \end{aligned} \tag{2.24}$$

Here, $R \in \mathscr{B}_1^+ \subseteq \mathscr{L}_k$ is defined by $f_1 = zR + \text{h.o.b.}(z)$, and $R_0 \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$ is defined by $f_0 = \lambda z + zR_0$. The operators $\mathscr{C}_\lambda, \mathscr{C}_{R_0}, \mathscr{K}_{R_0}, \mathscr{K}_R : \mathscr{B}_{\geq 1}^+ \to \mathscr{B}_1^+ \subset \mathscr{B}_{\geq 1}^+$ are suitable $\frac{1}{4}$-contractions with respect to the metric d_1.

The above $\frac{1}{4}$-contractions are obtained from the appropriate contractions from Lemma A.2.8, Lemma A.3.1 and Lemma A.3.3. The precise definition of these contraction operators is visible in the proof.

Proof. Setting $\varphi_0 = z + zH + \text{h.o.b.}(z)$, where $H \in \mathscr{B}_{\geq 1}^+$, $f_0 = \lambda z + zR_0$ and $f = f_0 + zR + \text{h.o.b.}(z)$, we rewrite prenormalization equation (2.22) as an equation satisfied by H, R and R_0. To this end, we use the Taylor Theorem (Proposition A.1.6) to expand the compositions in (2.22) and compare the leading blocks (namely the blocks with ord_z equal to 1) of both sides of the equation. We obtain:

$$\begin{aligned} \lambda zH - \lambda zH(\lambda z) - \sum_{i \geq 1} \frac{\big(zH\big)^{(i)}(\lambda z)}{i!} \big(zR_0\big)^i \\ = \sum_{i \geq 1} \frac{\big(zH\big)^{(i)}(\lambda z + zR_0)}{i!} \big(zR\big)^i - \sum_{i \geq 1} \frac{\big(zR_0\big)^{(i)}}{i!} \big(zH\big)^i + zR. \end{aligned} \tag{2.25}$$

By Lemma A.3.1 and Lemma A.3.3 we have:

$$
\begin{aligned}
&\lambda zH-\lambda zH(\lambda z)=-\lambda\log\lambda\cdot zD_1(H)-\lambda z\mathscr{C}_\lambda(H),\\
&\sum_{i\geq1}\frac{\big(zH\big)^{(i)}(\lambda z)}{i!}\big(zR_0\big)^i\\
&\qquad=zH\cdot R_0+zD_1(H)\cdot\Big((1+\log\lambda)R_0+\lambda\sum_{i\geq2}\frac{(-1)^i}{i(i-1)}\Big(\frac{R_0}{\lambda}\Big)^i\Big)+z\mathscr{C}_{R_0}(H),
\end{aligned}
\tag{2.26}
$$

where $\mathscr{C}_\lambda$ and $\mathscr{C}_{R_0}:=\mathscr{C}(\cdot,R_0)$ are superlinear $\frac{1}{4}$-contractions on the space $\big(\mathscr{B}^+_{\geq1},d_1\big)$ from Lemma A.3.1 and Lemma A.3.3. Moreover, since $\operatorname{ord}(R)>(0,\mathbf{1}_k)$, by Lemma A.2.8, Lemma A.3.1 and Lemma A.3.3, it follows that:

$$
\sum_{i\geq1}\frac{\big(zR_0\big)^{(i)}}{i!}\big(zH\big)^i=zR_0\cdot H+zD_1(R_0)\cdot\Big(H+\sum_{i\geq2}\frac{(-1)^i}{i(i-1)}H^i\Big)+z\mathscr{K}_{R_0}(H),
\tag{2.27}
$$

$$
\begin{aligned}
\sum_{i\geq1}\frac{\big(zH\big)^{(i)}\big(\lambda z+zR_0\big)}{i!}\big(zR\big)^i&=\big(zH\big)'\big(\lambda z+zR_0\big)\cdot zR+\sum_{i\geq2}\frac{\big(zH\big)^{(i)}\big(\lambda z+zR_0\big)}{i!}\big(zR\big)^i=\\
&=zH\cdot R+z\mathscr{K}_R(H),
\end{aligned}
$$

where the operator $\mathscr{K}_{R_0}:=\mathscr{C}(R_0,\cdot)$ is a $\frac{1}{4}$-contraction on the space $\big(\mathscr{B}^+_{\geq1},d_1\big)$ from Lemma A.2.8, and $\mathscr{K}_R$ is a superlinear $\frac{1}{4}$-contraction on the space $\big(\mathscr{B}^+_{\geq1},d_1\big)$.

Now, eliminating z and using (2.26) and (2.27) in (2.25), we obtain:

$$
\begin{aligned}
\Big(-\lambda\log\lambda-(1+\log\lambda)R_0-\lambda\sum_{i\geq2}\frac{(-1)^i}{i(i-1)}\Big(\frac{R_0}{\lambda}\Big)^i\Big)\cdot D_1(H)+&\\
+D_1(R_0)\cdot\Big(H+\sum_{i\geq2}\frac{(-1)^i}{i(i-1)}H^i\Big)-H\cdot R=&\\
=\lambda\mathscr{C}_\lambda(H)+\mathscr{C}_{R_0}(H)-\mathscr{K}_{R_0}(H)+\mathscr{K}_R(H)+R.&
\end{aligned}
\tag{2.28}
$$

It follows that (2.28) is equivalent to $\mathscr{S}_0(H)=\mathscr{T}_0(H)$, where $\mathscr{S}_0$ and $\mathscr{T}_0$ are defined in (2.24). ■

In the next lemma we prove that operators $\mathscr{T}_0$ and $\mathscr{S}_0$ defined in (2.24) satisfy all the assumptions of the fixed point theorem stated in Proposition 1.2.12.

Lemma 2.1.17 (Properties of the operators $\mathscr{T}_0$ and $\mathscr{S}_0$, Lemma 4.3, [29])**.** Let $k\in\mathbb{N}_{\geq1}$ and $f(z)=\lambda z+\text{h.o.t.}\in\mathscr{L}_k^H$, with $0<\lambda<1$. Let $f=f_0+g$, where f_0 is defined as in

(2.5), and $\operatorname{ord}(g) > \mathbf{1}_{k+1}$, $\operatorname{ord}_z(g) = 1$. Let the operators $\mathscr{T}_0, \mathscr{S}_0 : \mathscr{B}^+_{\geq 1} \to \mathscr{B}^+_1$, $\mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_k$, be defined as in (2.24). Then:

1. $\mathscr{T}_0$ is a $\frac{1}{2}$-homothety on $\mathscr{B}^+_{\geq 1}$ with respect to the metric d_1,
2. $\mathscr{S}_0$ is a $\frac{1}{4}$-contraction on $\mathscr{B}^+_{\geq 1}$ with respect to the metric d_1,
3. $\mathscr{S}_0(\mathscr{B}^+_{\geq 1}) \subseteq \mathscr{T}_0(\mathscr{B}^+_{\geq 1})$.

Proof. 1. From the definition of $\mathscr{T}_0$ in (2.24), by Example 1.2.8 and Lemma A.2.11, we deduce that

$$\operatorname{ord}\big(\mathscr{T}_0(H_1) - \mathscr{T}_0(H_2)\big) = \operatorname{ord}\big(D_1(H_1 - H_2)\big), \quad H_1, H_2 \in \mathscr{B}^+_{\geq 1}, H_1 \neq H_2.$$

In particular,

$$\operatorname{ord}_{\boldsymbol{\ell}_1}\big(\mathscr{T}_0(H_1 - \mathscr{T}_0(H_2)\big) = \operatorname{ord}_{\boldsymbol{\ell}_1}\big(D_1(H_1 - H_2)\big) = \operatorname{ord}_{\boldsymbol{\ell}_1}(H_1 - H_2) + 1,$$

for $H_1, H_2 \in \mathscr{B}^+_{\geq 1}$ such that $H_1 \neq H_2$. Therefore, $\mathscr{T}_0$ is a $\frac{1}{2}$-homothety.

2. Part 2 follows directly from the definition of the operator $\mathscr{S}_0$ in (2.24).

3. Let us prove that $\mathscr{S}_0(\mathscr{B}^+_{\geq 1}) \subseteq \mathscr{T}_0(\mathscr{B}^+_{\geq 1})$. Let $M \in \mathscr{S}_0(\mathscr{B}^+_{\geq 1})$. By (2.24), since $\operatorname{ord}(zR) > \mathbf{1}_{k+1}$, it follows that $\operatorname{ord}(zM) > \mathbf{1}_{k+1}$. We prove that $M \in \mathscr{T}_0(\mathscr{B}^+_{\geq 1})$. Indeed, dividing both sides of

$$\mathscr{T}_0(H) = M \tag{2.29}$$

by $-\lambda \log\lambda - (1+\log\lambda)R_0 - \lambda\sum_{i\geq 2}\frac{(-1)^i}{i(i-1)}\left(\frac{R_0}{\lambda}\right)^i$ and applying Proposition B.5.1, we obtain that there exists a $H \in \mathscr{B}^+_{\geq 1}$ such that (2.29) holds. ■

Finally, we use Lemma 2.1.16 and Lemma 2.1.17 to prove the case (b) of Theorem A (see [29, Subsection 4.2.3]).

Proof of case (b) of Theorem A. Note that the logarithmic transseries $\varphi_0 \in \mathfrak{L}^0$ satisfying (2.22) is necessarily of the form

$$\varphi_0(z) = z + zH + \text{h.o.b.}(z), \quad H \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_m,$$

for some $m \in \mathbb{N}_{\geq 1}$.

By Lemma 2.1.16, to prove the existence of a solution φ_0 of equation (2.22), it is enough to prove the existence of a solution $H \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_m$ of fixed point equation (2.23).

By Lemma 2.1.17, equation (2.23) satisfies all the assumptions of the fixed point theorem from Proposition 1.2.12 on spaces $\mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_m$, for every $m \geq k$. Therefore, there exists a unique solution $H \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_m$, $m \geq k$, of equation (2.23). By Lemma 2.1.16, $\varphi_0(z) = \mathrm{id} + zH + \text{h.o.b.}(z)$ is a solution of equation (2.22), which is unique in $\mathfrak{L}^0$ up to h.o.b.(z). Therefore, we set $\varphi_0 := \mathrm{id} + zH$ as the *canonical form*. With this convention, the prenormalization φ_0 in the canonical form is unique. Moreover, φ_0 belongs to $\mathscr{L}_k$ for the smallest $k \in \mathbb{N}_{\geq 1}$ such that $f \in \mathscr{L}_k$.

Now we have that

$$\varphi_0 \circ f \circ \varphi_0^{-1} = f_0 + f_2, \text{ for some } f_2 \in \mathscr{L}_k \text{ such that } \mathrm{ord}_z(f_2) > 1.$$

Hence, we can apply case (a) to reduce $f_0 + f_2$ to the normal form f_0. By the proof of case (a), we know that there exists a unique $\varphi_1 \in \mathfrak{L}^0$, such that

$$\varphi_1 \circ \left(\varphi_0 \circ f \circ \varphi_0^{-1}\right) \circ \varphi_1^{-1} = f_0.$$

Moreover, $\varphi_1 \in \mathscr{L}_k$ such that $\mathrm{ord}_z(\varphi_1) \geq \mathrm{ord}_z(f_2)$.

Now, $\varphi := \varphi_1 \circ \varphi_0 \in \mathscr{L}_k^0$ is the normalization in case (b), *i.e.*

$$\varphi \circ f \circ \varphi^{-1} = f_0. \tag{2.30}$$

It remains to prove the uniqueness of the whole normalization $\varphi \in \mathfrak{L}^0$. Suppose that there exists another parabolic logarithmic transseries $\psi \neq \varphi$, $\psi \in \mathfrak{L}^0$, satisfying (2.30). Then $\psi \in \mathscr{L}_m^0$, for some $m \geq k$. Let us decompose ψ as $\psi = \psi_1 \circ \psi_0$, where ψ_0 is of the form $\psi_0 = \mathrm{id} + zV + \text{h.o.b.}(z)$, $V \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_m$, and $\mathrm{ord}_z(\psi_1 - \mathrm{id}) > 1$. It is easy to see that such V is unique. Now,

$$\psi_0 \circ f \circ \psi_0^{-1} = \psi_1^{-1} \circ f_0 \circ \psi_1 = f_0 + g_1,$$
$$\varphi_0 \circ f \circ \varphi_0^{-1} = \varphi_1^{-1} \circ f_0 \circ \varphi_1 = f_0 + g_2,$$

where $\mathrm{ord}_z(g_1)$, $\mathrm{ord}_z(g_2) > 1$, are both prenormalization equations for f. By the proof of case (b), it follows that $V = H$.

Now put $\widetilde{\psi}_1 := \psi \circ (\mathrm{id} + zV)^{-1} \in \mathfrak{L}^0$ and $\widetilde{\varphi}_1 := \varphi \circ (\mathrm{id} + zV)^{-1} \in \mathfrak{L}^0$, where $(\mathrm{id} + zV)^{-1}$ is the compositional inverse of the logarithmic transseries $\mathrm{id} + zV$. Set $f_1 := (\mathrm{id} + zV) \circ$

$f \circ (\mathrm{id} + zV)^{-1} = f_0 + h$, $\mathrm{ord}_z(h) > 1$. Then:

$$\widetilde{\psi}_1 \circ f_1 \circ \widetilde{\psi}_1^{-1} = f_0,$$
$$\widetilde{\varphi}_1 \circ f_1 \circ \widetilde{\varphi}_1^{-1} = f_0.$$

By the uniqueness in the proof of case (a), $\widetilde{\psi}_1 = \widetilde{\varphi}_1$. Therefore, $\psi = \varphi$. ■

2.1.3. Proof of the minimality of the normal form f_0

Let $f \in \mathfrak{L}^H$ be a hyperbolic logarithmic transseries and let f_0 be its initial part as defined in (2.5). By Proposition 2.1.1 it follows that, for every $g \in \mathfrak{L}^H$ such that there exists a solution of conjugacy equation $\varphi \circ f \circ \varphi^{-1} = g$, $\varphi \in \mathfrak{L}^0$, we have that $g = f_0 + \text{h.o.t.}$ On the other hand, in Subsection 2.1.2 we have proved the existence of a parabolic $\varphi \in \mathfrak{L}^0$ such that $\varphi \circ f \circ \varphi^{-1} = f_0$. This implies that f_0 is indeed the minimal logarithmic transseries to which f can be conjugated via parabolic change of variables $\varphi \in \mathfrak{L}^0$.

2.1.4. Proof of the convergence of the generalized Koenigs sequence

In the previous subsections, for every hyperbolic logarithmic transseries $f \in \mathscr{L}_k^H$, we have constructed its normalization $\varphi \in \mathscr{L}_k^0$, which reduces f to its normal form f_0 given in (2.5). Moreover, φ was obtained by a two-step algorithm: 1. the prenormalization, and 2. the normalization of the prenormalized hyperbolic logarithmic transseries.

Each of these steps can be realized as the limit of a Picard sequence of an appropriate contraction on the appropriate complete metric space. In the prenormalization step, the sequence consists of the forward iterations of the contraction $\mathscr{T}_0^{-1} \circ \mathscr{S}_0$, and, in the normalization step, the sequence consists of the forward iterations of the contraction $\mathscr{T}_f^{-1} \circ \mathscr{S}_f$, see [29]. Here we pose the following question: Although $\mathscr{P}_f$ is not a contraction in any previously introduced metric on $\mathscr{L}_k$, do the forward iterations of $\mathscr{P}_f$ converge in some *natural* topology on $\mathscr{L}_k$? If the answer is positive, is their limit the unique normalization φ?

In what follows, we give sufficient and necessary conditions for the convergence of the forward iterations of $\mathscr{P}_f$ in the weak topology. The proof relies on a transfinite induction and is given in Proposition 2.1.25 at the end of the subsection. First we give the definition of the *sequential continuity* which is a special case of the *transfinite sequential continuity* defined in [21, Subsection 4.2]).

Definition 2.1.18 (Sequential continuity, see e.g. [21])**.** Let $\mathscr{G}:\mathscr{A}\to\mathscr{L}_k$, $k\in\mathbb{N}$, be an operator and let $\mathscr{A}$ be a differential subalgebra of $\mathscr{L}_k$. We say that $\mathscr{G}$ is the *sequentially continuous operator* on $\mathscr{L}_k$ if for every sequence (g_n) in $\mathscr{A}$, with property that $\operatorname{Supp}(g_n)\subseteq W$, for each $n\in\mathbb{N}$, where W is a common well-ordered subset of $\mathbb{R}\times\mathbb{Z}^k$ such that $\min W>\mathbf{0}_{k+1}$, and such that (g_n) converges in the weak topology to $g\in\mathscr{A}$, it follows that $(\mathscr{G}(g_n))$ converges to $\mathscr{G}(g)$ (in the weak topology).

Remark 2.1.19. Suppose that $\mathscr{G}:\mathscr{A}\to\mathscr{L}_k$, for a differential subalgebra $\mathscr{A}$ of $\mathscr{L}_k$, is a sequentially continuous bijection, such that its inverse $\mathscr{G}^{-1}$ is a sequentially continuous. Then every sequence (g_n) in $\mathscr{A}$, with property that $\operatorname{Supp}(g_n)\subseteq W$, for each $n\in\mathbb{N}$, where W is a common well-ordered subset of $\mathbb{R}\times\mathbb{Z}^k$ such that $\min W>\mathbf{0}_{k+1}$, converges in the weak topology to $g\in\mathscr{A}$ if and only if the sequence $(\mathscr{G}(g_n))$ converges to $\mathscr{G}(g)$ (in the weak topology).

Definition 2.1.20 (Right composition operator)**.** Let $\varphi\in\mathfrak{L}^H$ and let $\mathscr{R}_\varphi:\mathfrak{L}\to\mathfrak{L}$ be the operator defined by $\mathscr{R}_\varphi(g)=g\circ\varphi$, for each $g\in\mathfrak{L}$. We call $\mathscr{R}_\varphi$ the *right composition operator* with respect to φ.

Remark 2.1.21. Let $\varphi\in\mathfrak{L}^H$. The right composition operator $\mathscr{R}_\varphi$ is a bijection and its compositional inverse is equal to $\mathscr{R}_{\varphi^{-1}}$, i.e., $(\mathscr{R}_\varphi)^{-1}=\mathscr{R}_{\varphi^{-1}}$.

The following proposition is needed for the proof of Proposition 2.1.25.

Proposition 2.1.22 (Sequential continuity of the right composition operator, Lemma 4.10, [29])**.** Let $\varphi\in\mathfrak{L}^H$ and suppose that $k\in\mathbb{N}$ is minimal such that $\varphi\in\mathscr{L}_k^H$. The right composition operator $\mathscr{R}_\varphi$ is sequentially continuous on $\mathscr{L}_r$, for each $r\geq k$.

Proof. Let $r\geq k$ be arbitrary. Since $\mathscr{L}_k\subseteq\mathscr{L}_r$, we have $\varphi\in\mathscr{L}_r$. Therefore, our *ambient space* in this proof is $\mathscr{L}_r$. Let (g_n) be a sequence in $\mathscr{L}_r$ and W a well-ordered subset of $\mathbb{R}\times\mathbb{Z}^r$ such that $\min W>\mathbf{0}_{r+1}$ and $\operatorname{Supp}(g_n)\subseteq W$, for every $n\in\mathbb{N}$. Suppose that $g_n\to g_0$, as $n\to\infty$, with respect to the weak topology. Put $\varphi:=\mathrm{id}+\varphi_1$, where $\varphi_1\in\mathscr{L}_r$, $\operatorname{ord}(\varphi_1)>(1,\mathbf{0}_r)$. By the definition of a composition, we have:

$$g_n\circ\varphi=g_n\big(\mathrm{id}+\varphi_1\big)=g_n+\sum_{i\geq1}\frac{g_n^{(i)}}{i!}\varphi_1^i,\ \text{for } n\in\mathbb{N}.$$

By Proposition A.3.5, it follows that, for $n \in \mathbb{N}$, the supports of $g_n \circ \varphi$ are contained in the semigroup G generated by W and

$$\{(\alpha-1,\mathbf{m}),(0,1,0,\dots,0)_{r+1},\dots,(0,0,\dots,0,1)_{r+1}\},$$

for $(\alpha,\mathbf{m}) \in \mathrm{Supp}(\varphi_1)$. Let $\mathbf{w} \in G$. By the Neumann Lemma (Theorem 1.1.2), there exists $k_{\mathbf{w}} \in \mathbb{N}$ and a linear real polynomial $P_{\mathbf{w}}$ in $k_{\mathbf{w}}$ variables (the coefficients of which depend on φ_1 but not on $n \in \mathbb{N}$), such that

$$[g_n \circ \varphi]_{\mathbf{w}} = P_{\mathbf{w}}\left([g_n]_{\mathbf{w}_1},\dots,[g_n]_{\mathbf{w}_{k_{\mathbf{w}}}}\right), \text{ for } n \in \mathbb{N}. \tag{2.31}$$

Here, $\mathbf{w}_1,\dots,\mathbf{w}_{k_{\mathbf{w}}}$ are finitely many elements of G, independent of $n \in \mathbb{N}$. By continuity of polynomial functions, we have:

$$P_{\mathbf{w}}\left([g_n]_{\mathbf{w}_1},\dots,[g_n]_{\mathbf{w}_{k_{\mathbf{w}}}}\right) \underset{n\to\infty}{\longrightarrow} P_{\mathbf{w}}\left([g_0]_{\mathbf{w}_1},\dots,[g_0]_{\mathbf{w}_{k_{\mathbf{w}}}}\right). \tag{2.32}$$

Thus, using (2.31) and (2.32), we obtain, for every $\mathbf{w} \in G$:

$$[g_n \circ \varphi]_{\mathbf{w}} \underset{n\to\infty}{\longrightarrow} [g_0 \circ \varphi]_{\mathbf{w}}.$$

■

Remark 2.1.23. Let $\varphi \in \mathscr{L}_k^H$, for $k \in \mathbb{N}$. By Proposition 2.1.22, Remark 2.1.19 and Remark 2.1.21, it follows that:
Every sequence (g_n) in $\mathscr{L}_r$, $r \in \mathbb{N}$, with property that $\mathrm{Supp}(g_n) \subseteq W$, for each $n \in \mathbb{N}$, where W is a common well-ordered subset of $\mathbb{R}\times\mathbb{Z}^r$ such that $\min W > \mathbf{0}_{r+1}$, converges in the weak topology to $g \in \mathscr{L}_r$ *if and only if* the sequence $(g_n \circ \varphi)$ converges to $g \circ \varphi$ (in the weak topology).

The next lemma is an auxiliary technical lemma for the proof of Proposition 2.1.25. We suggest the reader to skip it and read its proof only when it is required in the proof of Proposition 2.1.25.

Lemma 2.1.24 (Lemma 4.11, [29])**.** Let $m \in \mathbb{N}$ and $f \in \mathscr{L}_m^H$ be hyperbolic logarithmic transseries. Let f_0 be its normal form given in (2.5) and let $\varphi \in \mathscr{L}_m^0$ be its normalization. Let $h \in \mathscr{L}_m^0$ be such that $\mathrm{Lb}_z(h) = \mathrm{Lb}_z(\varphi)$ and let $\widetilde{\mathscr{H}}$ be as in (2.40). Let $P_{\beta,\mathbf{m}}^n$, for $(\beta,\mathbf{m}) \in \widetilde{\mathscr{H}}$ and $n \in \mathbb{N}$, be as in (2.42). Then

$$P_{\beta,\mathbf{m}}^{n+1} = A_{\beta,\mathbf{m}}\left(\{P_{\gamma,\mathbf{n}}^n :\ (\gamma,\mathbf{n}) \in \widetilde{\mathscr{H}},\ (\gamma,\mathbf{n}) < (\beta,\mathbf{m})\}\right) + \lambda^{\beta-1} P_{\beta,\mathbf{m}}^n, \text{ for } n \in \mathbb{N}.$$

Here, for $(\beta,\mathbf{m}) \in \widetilde{\mathscr{H}}$, the $A_{\beta,\mathbf{m}}$ are linear polynomials in the variables $\{P^n_{\gamma,\mathbf{n}} :\ (\gamma,\mathbf{n}) \in \widetilde{\mathscr{H}},\ (\gamma,\mathbf{n}) < (\beta,\mathbf{m})\}$, whose coefficients are independent of n. The coefficients depend only on f, h and φ.

Proof. Let $P^n_{\gamma,\mathbf{r}}$, $(\gamma,\mathbf{r}) \in \widetilde{\mathscr{H}}$, $n \in \mathbb{N}$, be as in (2.42). We define, for $(\gamma,\mathbf{r}) \in \widetilde{\mathscr{H}}$, $n \in \mathbb{N}$,

$$R^n_{\gamma,\mathbf{r}} := \left[f_0^{\circ(-n+1)} \circ (\mathrm{id} + h_1) \circ f_0^{\circ n}\right]_{\gamma,\mathbf{r}}. \tag{2.33}$$

First, we prove that

$$R^{n+1}_{\beta,\mathbf{m}} = B_{\beta,\mathbf{m}}\big(\{P^n_{\beta,\mathbf{n}} :\ (\beta,\mathbf{n}) \in \widetilde{\mathscr{H}},\ \mathbf{n} < \mathbf{m}\}\big) + \lambda^{\beta} \cdot P^n_{\beta,\mathbf{m}},\ (\beta,\mathbf{m}) \in \widetilde{\mathscr{H}},\ n \in \mathbb{N}, \tag{2.34}$$

where $B_{\beta,\mathbf{m}}$, $(\beta,\mathbf{m}) \in \widetilde{\mathscr{H}}$, are linear real polynomials in the variables $\{P^n_{\beta,\mathbf{n}} \in \widetilde{\mathscr{H}} : \mathbf{n} < \mathbf{m}\}$, whose coefficients are independent of n. Indeed, for any $(\beta,\mathbf{m}) \in \widetilde{\mathscr{H}}$, $\mathbf{m} = (m_1,\ldots,m_m) \in \mathbb{Z}^m$, and $n \in \mathbb{N}$,

$$\left(P^n_{\beta,\mathbf{m}} z^{\beta} \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_m^{m_m}\right) \circ f_0 = P^n_{\beta,\mathbf{m}} \lambda^{\beta} z^{\beta} \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_m^{m_m}\big(1 + \text{h.o.t.}(\mathscr{B}_1)\big).$$

Here, the notation h.o.t.$(\mathscr{B}_1)$ means *higher order terms lying in* $\mathscr{B}_1$. The statement (2.34) then follows simply by $f_0^{-n} \circ (\mathrm{id} + h_1) \circ f_0^{n+1} = \left(f_0^{-n} \circ (\mathrm{id} + h_1) \circ f_0^{n}\right) \circ f_0$ and the Neumann Lemma (Theorem 1.1.2). Indeed, it can be seen that the coefficient $R^{n+1}_{\beta,\mathbf{m}}$ in step $n+1$ can be expressed as a linear real polynomial, with coefficients independent of n, of finitely many coefficients $P^n_{\beta,\mathbf{n}}$, $\mathbf{n} < \mathbf{m}$, from previous step and of $P^n_{\beta,\mathbf{m}}$. Here, h_1 is as in Proposition 2.1.25, defined by $\mathrm{id} + h_1 = h \circ \varphi^{-1}$.

Let us prove that:

$$P^{n+1}_{\beta,\mathbf{m}} = C_{\beta,\mathbf{m}}\big(\{R^{n+1}_{\gamma,\mathbf{n}} :\ (\gamma,\mathbf{n}) \in \widetilde{\mathscr{H}},\ (\gamma,\mathbf{n}) < (\beta,\mathbf{m})\}\big) + \frac{1}{\lambda} R^{n+1}_{\beta,\mathbf{m}}. \tag{2.35}$$

Then, by (2.34), it follows that, for $(\beta,\mathbf{m}) \in \widetilde{\mathscr{H}}$ and $n \in \mathbb{N}$,

$$\begin{aligned} P^{n+1}_{\beta,\mathbf{m}} &= C_{\beta,\mathbf{m}}\big(\{R^{n+1}_{\gamma,\mathbf{n}} :\ (\gamma,\mathbf{n}) \in \widetilde{\mathscr{H}},\ (\gamma,\mathbf{n}) < (\beta,\mathbf{m})\}\big) \\ &\qquad + \frac{1}{\lambda} \cdot B_{\beta,\mathbf{m}}\big(\{P^n_{\beta,\mathbf{n}} :\ (\beta,\mathbf{n}) \in \widetilde{\mathscr{H}},\ \mathbf{n} < \mathbf{m}\}\big) + \lambda^{\beta-1} P^n_{\beta,\mathbf{m}} \\ &= A_{\beta,\mathbf{m}}(\{P^n_{\gamma,\mathbf{n}} :\ (\gamma,\mathbf{n}) \in \widetilde{\mathscr{H}},\ (\gamma,\mathbf{n}) < (\beta,\mathbf{m})\}) + \lambda^{\beta-1} P^n_{\beta,\mathbf{m}}, \end{aligned}$$

where $C_{\beta,\mathbf{m}}$ and $A_{\beta,\mathbf{m}}$ are linear real polynomials in the variables $\{P^n_{\gamma,\mathbf{n}} : (\gamma,\mathbf{n}) \in \widetilde{\mathscr{H}},\ (\gamma,\mathbf{n}) < (\beta,\mathbf{m})\}$, whose coefficients are independent of n. This proves the lemma.

In order to prove (2.35), let $k_0 := f_0^{-1} - \frac{1}{\lambda}\mathrm{id}$. It is easy to see that k_0 contains only monomials which are of order 1 (in z). Let h_1, as before, be defined as $\mathrm{id} + h_1 = h \circ \varphi^{-1}$, and r by:

$$\left(f_0^{\circ(-n)} \circ (\mathrm{id}+h_1) \circ f_0^{\circ(n+1)}\right) = f_0 + r.$$

Then $\mathrm{ord}_z(r) > 1$ since $\mathrm{ord}_z(h_1) > 1$. By the Taylor Theorem (Proposition A.1.6) we obtain:

$$\begin{aligned} f_0^{-1} \circ \left(f_0^{\circ(-n)} \circ (\mathrm{id}+h_1) \circ f_0^{\circ(n+1)}\right) &= \mathrm{id} + \sum_{i\geq 1} \frac{\left(f_0^{-1}\right)^{(i)}(f_0)}{i!} r^i \\ &= \mathrm{id} + \left(\frac{1}{\lambda} + k_0'(f_0)\right) r + \sum_{i\geq 2} \frac{k_0^{(i)}(f_0)}{i!} r^i \\ &= \mathrm{id} + \frac{1}{\lambda} r + \sum_{i\geq 1} \frac{k_0^{(i)}(f_0)}{i!} r^i. \end{aligned}$$

Clearly, by definition (2.33) of $R^{n+1}_{\beta,\mathbf{m}}$,

$$\left[\frac{1}{\lambda} r\right]_{\beta,\mathbf{m}} = \frac{1}{\lambda} R^{n+1}_{\beta,\mathbf{m}}, \tag{2.36}$$

and $(\beta,\mathbf{m})$ in $\sum_{i\geq 1} \frac{k_0^{(i)}(f_0)}{i!} r^i$ can be realized as follows:

$$\begin{aligned} (\beta,\mathbf{m}) &= (\gamma_1,\mathbf{n}_1) + \cdots + (\gamma_i,\mathbf{n}_i) + (1-i,\mathbf{v}), \\ &= (\gamma_1,\mathbf{n}_1) + (\gamma_2 - 1,\mathbf{n}_2) + \cdots + (\gamma_i - 1,\mathbf{n}_i) + (0,\mathbf{v}), \end{aligned} \tag{2.37}$$

where $(1-i,\mathbf{v}) \in \mathrm{Supp}\left(k_0^{(i)}(f_0)\right)$ and $(\gamma_1,\mathbf{n}_1),\ldots,(\gamma_i,\mathbf{n}_i) \in \mathrm{Supp}(r)$. Note that $\gamma_j > 1$, $j = 1,\ldots,i$, and $\mathbf{v} > \mathbf{0}_m$. Note that, in (2.37), we can subtract -1 from any $(i-1)$ elements γ_k, $k = 1,\ldots,i$. Therefore, it follows from (2.37) that

$$(\gamma_1,\mathbf{n}_1),\ldots,(\gamma_i,\mathbf{n}_i) < (\beta,\mathbf{m}).$$

Now (2.35) follows from the Neumann Lemma (Theorem 1.1.2), (2.36) and (2.37). ■

In the next proposition we prove the statement 3 of Theorem A.

Proposition 2.1.25 (Convergence of the generalized Koenigs sequence, Lemma 4.9, [29])**.** Let $k \in \mathbb{N}$ and $f \in \mathscr{L}_k^H$ be hyperbolic, and let f_0 be its formal normal form from (2.5). For a parabolic initial condition $h \in \mathfrak{L}^0$, the generalized Koenigs sequence

$$\mathscr{P}_f^{\circ n}(h) := \left(f_0^{\circ(-n)} \circ h \circ f^{\circ n}\right)_n \tag{2.38}$$

converges to the normalization $\varphi \in \mathfrak{L}^0$ in the weak topology, as $n \to \infty$, if and only if $\mathrm{Lb}_z(h) = \mathrm{Lb}_z(\varphi)$.

Proof. Let $m \in \mathbb{N}$ be minimal such that $f, h \in \mathscr{L}_m$. We first prove the following: for $h, \varphi \in \mathfrak{L}^0$, $\mathrm{Lb}_z(h) = \mathrm{Lb}_z(\varphi)$ if and only if $\mathrm{ord}_z\big(h \circ \varphi^{-1} - \mathrm{id}\big) > 1$.

($\Leftarrow$) Let $h \in \mathfrak{L}^0$ and let $\varphi \in \mathscr{L}_k^0$ be the normalization of f. Let $m \in \mathbb{N}$ be the smallest integer such that $h \in \mathscr{L}_m^0$ and $m \geq k$. Let $\mathrm{ord}_z\big(h \circ \varphi^{-1} - \mathrm{id}\big) > 1$.

Since $\varphi \circ f^{\circ n} \circ \varphi^{-1} = f_0^{\circ n}$ for every $n \in \mathbb{N}$, it is easy to show that

$$\mathscr{P}_f^{\circ n}(h) \circ \varphi^{-1} = f_0^{\circ(-n)} \circ (h \circ \varphi^{-1}) \circ f_0^{\circ n}, \quad \text{for every } n \in \mathbb{N}.$$

Note that $\mathscr{P}_f^{\circ n}(h) \circ \varphi^{-1} \in \mathscr{L}_m^0$. Therefore, by Remark 2.1.23 applied to $g_n := \mathscr{P}_f^{\circ n}(h) \circ \varphi^{-1}$ (which have support in the common well-ordered set $\widetilde{\mathscr{H}} \cup \{(1, \mathbf{0}_m)\}$, for $\widetilde{\mathscr{H}}$ given below by (2.40)), in order to prove the convergence of (2.38) to the normalization φ, it is sufficient to prove the equivalent statement

$$f_0^{\circ(-n)} \circ (h \circ \varphi^{-1}) \circ f_0^{\circ n} \underset{n\to\infty}{\longrightarrow} \mathrm{id}$$

in the weak topology in $\mathscr{L}_m^0$.

Let h_1 be defined by $h \circ \varphi^{-1} = \mathrm{id} + h_1$. Then, by assumption, $\mathrm{ord}_z(h_1) > 1$. By the Taylor Theorem (Proposition A.1.6), it follows that

$$f_0^{\circ(-n)} \circ (h \circ \varphi^{-1}) \circ f_0^{\circ n} = f_0^{\circ(-n)} \circ (\mathrm{id} + h_1) \circ f_0^{\circ n} = \mathrm{id} + \sum_{i \geq 1} \frac{\big(f_0^{\circ(-n)}\big)^{(i)}(f_0^{\circ n})}{i!} h_1^i(f_0^{\circ n}). \tag{2.39}$$

By (2.39), for $n \in \mathbb{N}$, the leading block of $f_0^{\circ(-n)} \circ (h \circ \varphi^{-1}) \circ f_0^{\circ n}$ is equal to z. Let us define the set $\widetilde{\mathscr{H}} \subseteq \mathbb{R}_{>1} \times \mathbb{Z}^m$ by

$$\widetilde{\mathscr{H}} := \bigcup_{n \in \mathbb{N}} \mathrm{Supp}\big(f_0^{\circ(-n)} \circ (h \circ \varphi^{-1}) \circ f_0^{\circ n} - \mathrm{id}\big). \tag{2.40}$$

It can be shown, using (2.39) and the fact that f_0 contains only block of order 1 in z, that $\widetilde{\mathscr{H}}$ is well-ordered. It remains to prove that

$$\big[f_0^{\circ(-n)} \circ (h \circ \varphi^{-1}) \circ f_0^{\circ n}\big]_{\alpha, \mathbf{k}} \underset{n\to\infty}{\longrightarrow} 0$$

for every $(\alpha, \mathbf{k}) \in \widetilde{\mathscr{H}}$. We prove this by transfinite induction on $\widetilde{\mathscr{H}}$.

The induction basis. Let $(\alpha_0,\mathbf{k}_0):=\min\widetilde{\mathscr{H}}$, $\mathbf{k}_0\in\mathbb{Z}^m$. By (2.39), since $f_0=\lambda\cdot\mathrm{id}+$ h.o.t.,

$$\mathrm{Lt}\big(f_0^{\circ(-n)}\circ(\mathrm{id}+h_1)\circ f_0^{\circ n}-\mathrm{id}\big)=\lambda^{n(\alpha-1)}\mathrm{Lt}(h_1),\ \text{for } n\in\mathbb{N}, \tag{2.41}$$

where $\alpha:=\mathrm{ord}_z(h_1)$. Therefore, by definition of $\widetilde{\mathscr{H}}$, $\mathrm{ord}(h_1)=(\alpha_0,\mathbf{k}_0)$. Now, by (2.41),

$$\big[f_0^{\circ(-n)}\circ(h\circ\varphi^{-1})\circ f_0^{\circ n}\big]_{\alpha_0,\mathbf{k}_0}=\lambda^{n(\alpha_0-1)}\big[h_1\big]_{\alpha_0,\mathbf{k}_0}\to 0,\ \text{as } n\to\infty,$$

since $\alpha_0-1>0$ and $0<\lambda<1$.

The induction step. For simplicity, let us denote, for $(\gamma,\mathbf{r})\in\widetilde{\mathscr{H}}$ and $n\in\mathbb{N}$,

$$P^n_{\gamma,\mathbf{r}}:=\big[f_0^{\circ(-n)}\circ(\mathrm{id}+h_1)\circ f_0^{\circ n}\big]_{\gamma,\mathbf{r}}. \tag{2.42}$$

Suppose that $(\beta,\mathbf{m})\in\widetilde{\mathscr{H}}$ and $(\beta,\mathbf{m})>(\alpha_0,\mathbf{k}_0)$ and that

$$P^n_{\gamma,\mathbf{r}}\to 0,\ \text{as } n\to\infty,\quad \text{for every } (\gamma,\mathbf{r})\in\widetilde{\mathscr{H}},\ \text{such that } (\gamma,\mathbf{r})<(\beta,\mathbf{m}).$$

We prove that $P^n_{\beta,\mathbf{m}}\to 0,\ \text{as } n\to\infty.$

Using inductively Lemma 2.1.24, we obtain:

$$\begin{aligned}P^{n+1}_{\beta,\mathbf{m}}&=A_{\beta,\mathbf{m}}\big(\{P^n_{\gamma,\mathbf{n}}:\ (\gamma,\mathbf{n})\in\widetilde{\mathscr{H}},\ (\gamma,\mathbf{n})<(\beta,\mathbf{m})\}\big)+\\ &\quad+\lambda^{\beta-1}A_{\beta,\mathbf{m}}\big(\{P^{n-1}_{\gamma,\mathbf{n}}:\ (\gamma,\mathbf{n})\in\widetilde{\mathscr{H}},\ (\gamma,\mathbf{n})<(\beta,\mathbf{m})\}\big)+\ldots+\\ &\quad+\lambda^{n(\beta-1)}A_{\beta,\mathbf{m}}\big(\{P^0_{\gamma,\mathbf{n}}:\ (\gamma,\mathbf{n})\in\widetilde{\mathscr{H}},\ (\gamma,\mathbf{n})<(\beta,\mathbf{m})\}\big)+\lambda^{(n+1)(\beta-1)}P^0_{\beta,\mathbf{m}}\\ &=\sum_{i=0}^{n}\lambda^{i(\beta-1)}A_{\beta,\mathbf{m}}\big(\{P^{n-i}_{\gamma,\mathbf{n}}:\ (\gamma,\mathbf{n})\in\widetilde{\mathscr{H}},\ (\gamma,\mathbf{n})<(\beta,\mathbf{m})\}\big)+\lambda^{(n+1)(\beta-1)}[h_1]_{\beta,\mathbf{m}}.\end{aligned} \tag{2.43}$$

Note that, for $(\beta,\mathbf{m})>(1,\mathbf{0}_m)$, $P^0_{\beta,\mathbf{m}}=[h_1]_{\beta,\mathbf{m}}$. Let $a_{\gamma,\mathbf{n}}\in\mathbb{R}$ be the nonzero coefficient of $P^n_{\gamma,\mathbf{n}}$ (it does not depend on $n\in\mathbb{N}$) in the polynomial $A_{\beta,\mathbf{m}}$, $(1,\mathbf{0}_m)<(\gamma,\mathbf{n})<(\beta,\mathbf{m})$. We prove that the sum

$$a_{\gamma,\mathbf{n}}\sum_{i=0}^{n}\lambda^{i(\beta-1)}P^{n-i}_{\gamma,\mathbf{n}} \tag{2.44}$$

converges to 0, as $n\to\infty$. Then, since the first sum in the last row of (2.43) is a sum of finitely many sums of type (2.44), it converges to 0, as $n\to\infty$. Moreover, since $0<|\lambda|<1$, $\lambda^{(\beta-1)(n+1)}\to 0$, as $n\to\infty$. Therefore, by (2.43), $P^{n+1}_{\beta,\mathbf{m}}\to 0$, as $n\to\infty$. This proves the induction step.

It remains to prove the convergence of (2.44) to 0 as $n \to \infty$. We observe that (2.44) is, up to the factor $a_{\gamma,\mathbf{n}}$, the general term of the discrete convolution product of the sequence $\left(\lambda^{i(\beta-1)}\right)_{i\in\mathbb{N}}$ and the sequence $\left(P^i_{\gamma,\mathbf{n}}\right)_{i\in\mathbb{N}}$. The series $\sum_{i\in\mathbb{N}} \lambda^{i(\beta-1)}$ is absolutely convergent and $\sum_{i\in\mathbb{N}} \left|\lambda^{i(\beta-1)}\right| = \dfrac{1}{1-|\lambda|^{\beta-1}}$. Hence, up to the multiplication by $1-|\lambda|^{\beta-1}$, this convolution can be expressed as the product of the infinite vector $\left(P^i_{\gamma,\mathbf{n}}\right)_{i\in\mathbb{N}}$ by an infinite *Toeplitz matrix*. It is well known (see for example [9, Section 2.16, Theorem 1]) that such a product is a *regular method of summability*, which respects the limits of convergent sequences. Since, by hypothesis, $P^i_{\gamma,\mathbf{n}} \underset{i\to\infty}{\to} 0$, it follows that (2.44) tends to 0 as $n \to \infty$.

($\Rightarrow$) Conversely, let $h \in \mathfrak{L}^0$ and let $m \in \mathbb{N}$ be the minimal integer such that $h,\ f \in \mathscr{L}_m$. Suppose that

$$f_0^{\circ(-n)} \circ (h \circ \varphi^{-1}) \circ f_0^{\circ n} \underset{n\to\infty}{\longrightarrow} \mathrm{id},$$

in the weak topology in $\mathscr{L}^0_m$, and that $\mathrm{ord}_z \left(h \circ \varphi^{-1} - \mathrm{id}\right) = 1$. Setting $h \circ \varphi^{-1} = z + zR + \mathrm{h.o.b.}(z)$, where $R \in \mathscr{B}^+_{\geq 1}$, $R \neq 0$, by the Taylor Theorem (Proposition A.1.6) we get:

$$f_0^{\circ(-n)} \circ (z + zR + \mathrm{h.o.b.}(z)) \circ f_0^{\circ n} = \mathrm{id} + \frac{f_0^{\circ n} \cdot R(f_0^{\circ n})}{\frac{d}{dz} f_0^{\circ n}} + \cdots \tag{2.45}$$

Now, since $f_0^{\circ n} = \lambda^n z + \mathrm{h.o.t.}$ and since $\mathrm{Lt}\left(R\left(f_0^{\circ n}\right)\right) = \mathrm{Lt}(R)$, by (2.45) we get that

$$f_0{}^{\circ(-n)} \circ (z + zR + \mathrm{h.o.b.}(z)) \circ f_0^{\circ n} - \mathrm{id} = z\mathrm{Lt}(R) + \mathrm{h.o.t.} \tag{2.46}$$

The first term does not change with n, and therefore the right-hand side of (2.46) does not converge to 0 in the weak topology. ■

2.1.5. Two normalizing sequences

In this subsection we explain two different algorithms for obtaining the normalization of a hyperbolic logarithmic transseries $f \in \mathfrak{L}^H$ to its normal form f_0 given in (2.5). In each algorithm, the normalization is obtained as the limit of an appropriate sequence in appropriate spaces. Both algorithms begin with the same first step which is a *prenormalization*.

Step 1. - Prenormalization. By the proof of case (b) of Theorem A, the unique prenormalization φ_0 is given in the *canonical form* by $\varphi_0 = \mathrm{id} + zH$, where $H \in \mathscr{B}^+_{\geq 1}$ is a unique solution of the fixed point equation $\mathscr{T}_0(H) = \mathscr{S}_0(H)$. Here, $\mathscr{T}_0, \mathscr{S}_0 : \mathscr{B}^+_{\geq 1} \to \mathscr{B}^+_{\geq 1}$ are the operators given in (2.24). By the fixed point theorem from Proposition 1.2.12, H is the limit of the Picard sequence

$$\big((\mathscr{T}_0^{-1} \circ \mathscr{S}_0)^{\circ n}(Q)\big)_n, \tag{2.47}$$

with respect to the metric d_1 on the space $\mathscr{B}^+_{\geq 1}$, and for any *initial condition* $Q \in \mathscr{B}^+_{\geq 1}$.

Step 2. - Normalization

- *Algorithm 1*

  Let $f_1 := \varphi_0 \circ f \circ \varphi_0^{-1}$ be the prenormalized logarithmic transseries from *Step 1.* By the proof of case (a) of Theorem A, there exists the unique solution $\varphi_1 := \mathrm{id} + \varepsilon$ of the conjugacy equation $\varphi_1 \circ f_1 \circ \varphi_1^{-1} = f_0$ in the space $\mathfrak{L}^0$, where $\varepsilon \in \mathfrak{L}_{>\mathrm{id}}$ is the unique solution of the fixed point equation $\mathscr{T}_{f_1}(\varepsilon) = \mathscr{S}_{f_1}(\varepsilon)$ in the space $\mathfrak{L}_{>\mathrm{id}}$, for operators $\mathscr{T}_{f_1}, \mathscr{S}_{f_1} : \mathfrak{L}_{>\mathrm{id}} \to \mathfrak{L}_{>\mathrm{id}}$ given in (2.10). By the fixed point theorem from Proposition 1.2.12, it follows that ε is the limit of the Picard sequence

  $$\big((\mathscr{T}_{f_1}^{-1} \circ \mathscr{S}_{f_1})^{\circ n}(h)\big)_n,$$

  with respect to the metric d_z, for any initial condition $h \in \mathfrak{L}_{>\mathrm{id}}$ such that $\mathrm{ord}_z(h) \geq \mathrm{ord}_z(f_1 - f_0)$.

  Finally, we put $\varphi := \varphi_1 \circ \varphi_0$, to get the normalization φ which reduces f to the normal form f_0.

- *Algorithm 2*

  Let $f_1 := \varphi_0 \circ f \circ \varphi_0^{-1}$ be as above. Since $\varphi_0 = \mathrm{id} + zH$, for $H \in \mathscr{B}^+_{\geq 1}$, is the prenormalization, it follows that φ_0 is the leading block (in z) of the whole normalization φ. By statement 3 of Theorem A, it follows that the generalized Koenigs sequence $\big(\mathscr{P}_{f_1}^{\circ n}(h)\big)_n$, where $\mathscr{P}_{f_1}$ is defined as in (2.8), converges in the weak topology to the normalization φ, for any *initial condition* $h \in \mathfrak{L}^0$ such that $\mathrm{Lb}_z(h) = \varphi_0$.

Remark 2.1.26 (The difference between the two algorithms)**.** Note that the Picard sequence from the *Step 2.* of the first algorithm, in general, differs from the generalized

Koenigs sequence from the *Step 2.* of the second algorithm. In particular, the first sequence converges in the power-metric topology and the second sequence converges in the weak topology.

As opposed to the Picard sequence from the first algorithm which is deduced by the Banach Fixed Point Theorem, the generalized Koenigs operator $\mathscr{P}_{f_1}$, as defined in (2.8), is not a contraction in any of the introduced metrics (see Example 2.1.9). Furthermore, in Subsection 2.1.4 we proved the convergence of the generalized Koenigs sequence in the weak topology. It is proved using a transfinite induction and not as a direct consequence of a fixed point theorem.

2.1.6. Control of the support of the normalization

In Theorem 2.1.28 at the end of the subsection, for a hyperbolic logarithmic transseries $f \in \mathfrak{L}^H$, we determine the support of its normalizing change of variables $\varphi \in \mathfrak{L}^0$. Moreover, we prove that the support of φ depends only on f.

Let $f \in \mathscr{L}_k$, for minimal $k \in \mathbb{N}$. The idea of the proof is to find a restricted space $\mathscr{L}_k^W$, for a well-ordered $W \subseteq \mathbb{R} \times \mathbb{Z}^k$, $\min W > \mathbf{0}_{k+1}$, which depends only on f, and such that the proofs of case (a) and case (b) of Theorem A remain valid if we replace the space $\mathscr{L}_k$ by its subspace $\mathscr{L}_k^W \subseteq \mathscr{L}_k$.

In order to estimate the support of normalization φ, we introduce a well-ordered set $W_\beta \subseteq \mathbb{R}_{\geq 0} \times \mathbb{Z}^k$. Let W_1 be the semigroup generated by

$$\operatorname{Supp}(f) \cup \big\{ (\alpha - 1, \mathbf{m}) : \ (\alpha, \mathbf{m}) \in \operatorname{Supp}(f - \lambda \cdot \mathrm{id}) \big\} \cup \{(0,1,\dots,0)_{k+1}, \dots, (0,\dots,0,1)_{k+1}\}. \tag{2.48}$$

Inductively, we define the sequence W_n, $n \in \mathbb{N}_{\geq 1}$ of semigroups, such that W_{n+1} is generated by

$$W_n \cup \{(\beta_1, \mathbf{m}_1) + \cdots + (\beta_{n+1}, \mathbf{m}_{n+1}) - (n, \mathbf{0}_k) : (\beta_i, \mathbf{m}_i) \in W_n, \ \beta_i > 1\}. \tag{2.49}$$

By the Neumann Lemma (Theorem 1.1.2), it follows that W_n is well-ordered, for each $n \in \mathbb{N}_{\geq 1}$. Put:

$$W := \bigcup_{n=1}^{\infty} W_n$$

and

$$W_\beta := W \cap \left(\mathbb{R}_{\geq\beta} \times \mathbb{Z}^k\right). \tag{2.50}$$

In the next proposition we prove that the set W_β is well-ordered.

Proposition 2.1.27 (Proposition 5.2, [29])**.** Let $\beta \in \mathbb{R}_{>1}$ and suppose that W_β is as in (2.50). Then W_β is a well-ordered set with the property that

$$(\beta_1, \mathbf{m}_1) + \cdots + (\beta_m, \mathbf{m}_m) - (m-1, \mathbf{0}_k) \in W_\beta, \tag{2.51}$$

for all $(\beta_1, \mathbf{m}_1), \ldots, (\beta_m, \mathbf{m}_m) \in W_\beta$, $m \geq 2$.

Proof. Property (2.51) follows directly from (2.49) and the fact that W is the increasing union of the sets W_n, $n \in \mathbb{N}_{\geq 1}$.

Hence, we only need to prove that W_β is well-ordered. Since $W_\beta = (\mathbb{R}_{\geq\beta} \times \mathbb{Z}^k) \cap W$, it is sufficient to prove that W is well-ordered. In general, an increasing union of well-ordered sets may not be well-ordered. So, we give a proof based on the specific properties of the sets W_n. Let A be a nonempty subset of W, and let us prove that A admits a minimal element.

Set $W_0 := \emptyset$ and let I be the set of all $n \in \mathbb{N}$ such that $A \cap (W_n \setminus W_{n-1}) \neq \emptyset$. Let $\mathbf{w}_n := \min\left(A \cap (W_n \setminus W_{n-1})\right)$, $n \in I$. Such a minimum exists because the sets W_n, for $n \in I$, are well-ordered. We have now constructed a sequence $(\mathbf{w}_n)_n$ of minimal elements of the sets $A \cap (W_n \setminus W_{n-1})$, $n \in I$. Clearly, $\min A = \min\{\mathbf{w}_n : n \in I\}$. Therefore, it is enough to prove that the family $\{\mathbf{w}_n : n \in I\}$ has the smallest element. Note that, by (2.49), $\min(W_n \cap (\mathbb{R}_{>1} \times \mathbb{Z}^k)) = \min(W_{n+1} \cap (\mathbb{R}_{>1} \times \mathbb{Z}^k)), n \in \mathbb{N}$, and, therefore, $\min(W \cap (\mathbb{R}_{>1} \times \mathbb{Z}^k)) = \min(W_1 \cap (\mathbb{R}_{>1} \times \mathbb{Z}^k))$. Now let

$$\mathbf{w} := \min\left(W_1 \cap (\mathbb{R}_{>1} \times \mathbb{Z}^k)\right). \tag{2.52}$$

Take $m_0 \in I$. By Archimedes' Axiom and since the first coordinate of $\mathbf{w}$ is strictly greater than 1, there exists $n_0 \geq m_0$, such that $n \cdot \mathbf{w} - (n-1, \mathbf{0}_k) > \mathbf{w}_{m_0}$, for all $n > n_0$. By (2.52), $\mathbf{w}_n \geq n \cdot \mathbf{w} - (n-1, \mathbf{0}_k) > \mathbf{w}_{m_0}$, for every $n \in I$, $n > n_0$. This implies that

$$\min A = \min\{\mathbf{w}_n : n \in I\} = \min\left(\{\mathbf{w}_i : i \in I, i \leq n_0\} \cup \{\mathbf{w}_{m_0}\}\right).$$

The latter set is finite, therefore, the minimum exists. ∎

Theorem 2.1.28 (Control of the support of the normalization, Proposition 5.4, [29])**.** Let $f(z) = \lambda z + \text{h.o.t.} \in \mathscr{L}_k$, $k \in \mathbb{N}$, $0 < \lambda < 1$, and let $\varphi \in \mathscr{L}_k^0$ be the normalization of f to its normal form f_0 from Theorem A.[1] Then

$$\varphi - \mathrm{id} \in \mathscr{L}_k^{\widetilde{W_0} \cup \widetilde{W_\beta}},$$

where $\beta := \mathrm{ord}_z(f_1 - f_0)$, $\beta > 1$, $W_0 := \mathrm{Supp}\left(z^{-1}(\varphi_0 - \mathrm{id})\right)$,

$$\widetilde{W_\beta} := \left\langle W_0 \cup W_\beta \right\rangle \cap (\mathbb{R}_{\geq \beta} \times \mathbb{Z}^k), \tag{2.53}$$

and

$$\widetilde{W_0} := (1, \mathbf{0}_k) + W_0. \tag{2.54}$$

Here, in case (a) of Theorem A, we simply put $W_0 := \emptyset$.

In particular, the support of the normalizing change of variables φ depends only on the support of the initial logarithmic transseries f.

Proof. Case (a). Here, we put $W_0 := \emptyset$, and therefore, $\widetilde{W_0} = \emptyset$ and $\widetilde{W_\beta} = W_\beta$. It is enough to check that the proof of Lemma 2.1.15 works the same if, in Lemma 2.1.15, we replace $\mathscr{L}_k^\beta$ by $\mathscr{L}_k^{W_\beta}$, where $\beta := \mathrm{ord}_z(f - f_0)$. First, we easily check, by Proposition 2.1.27 and by the Taylor Theorem (Proposition A.1.6) of the operators $\mathscr{S}_f$ and $\mathscr{T}_f$ given in (2.10), that they leave the spaces $\mathscr{L}_k^{W_\beta}$ invariant. Then we have to prove that $\mathscr{T}_f$ is a surjection on $\mathscr{L}_k^{W_\beta}$. That is, for a given block $z^\gamma M_\gamma \in \mathscr{L}_k^{W_\beta}$, we need to prove that its preimage by $\mathscr{T}_f$ belongs to $\mathscr{L}_k^{W_\beta}$ as well.

To this end, define, for a well-ordered subset V of $\{0\} \times \mathbb{Z}^k$, the set

$$H(V) := \left\langle V \cup \mathrm{Supp}\left(z^{-1} g_0\right) \cup \left\{(0,1,0,\ldots,0)_{k+1}, \ldots, (0,0,\ldots,1)_{k+1}\right\} \right\rangle.$$

$H(V)$ is also well-ordered, by the Neumann Lemma (Theorem 1.1.2). It is easy to see that

$$\mathscr{B}_1^{H\left(\mathrm{Supp}\left(M_\gamma\right)\right)} \subseteq \mathscr{B}_1$$

is invariant under the action of $\mathscr{S}_1$, where $\mathscr{S}_1$ is as defined in (2.19). Note that $z^{-1} g_0$ is nothing but Q in the proof of Lemma 2.1.15. As the space $\left(\mathscr{B}_1^{H\left(\mathrm{Supp}\left(M_\gamma\right)\right)}, d_1\right)$ is

[1] In case (a) of Theorem A, let $f_1 := f$. In case (b), let $f_1 \in \mathscr{L}_k$ be the prenormalized transseries.

complete, it follows from the proof of Lemma 2.1.15 that $\mathscr{S}_1$ has a unique fixed point in $\mathscr{B}_1^{H(\mathrm{Supp}(M_\gamma))}$. Hence, the preimage $z^\gamma H_\gamma$ of $z^\gamma M_\gamma$ by $\mathscr{T}_f$ belongs to $\mathscr{L}_k^{W_\beta}$.

Therefore, we can apply Lemma 2.1.15 to restricted spaces $\mathscr{L}_k^{W_\beta}$ instead of $\mathscr{L}_k^{\geq\beta}$ to conclude that $\varphi - \mathrm{id} \in \mathscr{L}_k^{W_\beta}$.

Case (b). Let φ_0 be the *prenormalizing* transformation of transseries $f = f_0 + \mathrm{h.o.t.}$ (where f and f_0 are as defined in the Theorem A), which contains only the leading block. It is easy to see that $W_0 = \mathrm{Supp}\left(z^{-1}(\varphi_0 - \mathrm{id})\right)$ depends only on the leading block of the initial transseries f. Indeed, it is obtained as the limit of a Picard sequence (2.47) with contraction operator depending only on the leading block of f, where the initial condition $H_0 \in \mathscr{B}_{\geq 1}^{+}$ can be chosen arbitrarily.

Let now $\beta := \mathrm{ord}_z(\varphi_0 \circ f \circ \varphi_0^{-1} - f_0)$. Obviously, $\beta > 1$. Let W_β be as defined in (2.50), where the initial f (before prenormalization) is used in definition (2.48) of W_1. It can be checked by the Taylor Theorem (Proposition A.1.6) that

$$\mathrm{Supp}\left(z^{-1}(\varphi_0^{-1} - \mathrm{id})\right) \in \left\langle W_0 \cup \{(0,1,\ldots,0)_{k+1},\ldots,(0,\ldots,0,1)_{k+1}\}\right\rangle.$$

Then, for $f_1 := \varphi_0 \circ f \circ \varphi_0^{-1}$,

$$f_1 - f_0 \in \mathscr{L}_k^{\widetilde{W}_\beta}.$$

It can be checked that the set $\widetilde{W}_\beta$ satisfies property (2.51) from Proposition 2.1.27 and, by the same reasoning as in case (a), that $\mathscr{L}_k^{\widetilde{W}_\beta}$ is invariant under $\mathscr{S}_{f_1}$ and $\mathscr{T}_{f_1}$. Therefore, the normalization φ_1 reducing $f_1 = \varphi_0 \circ f \circ \varphi_0^{-1}$ to the normal form f_0 belongs to $\varphi_1 - \mathrm{id} \in \mathscr{L}_k^{\widetilde{W}_\beta}$.

Finally, by (2.54), it holds that $\varphi - \mathrm{id} \in \mathscr{L}_k^{\widetilde{W}_0 \cup \widetilde{W}_\beta}$. ■

2.2. Normal forms of strongly hyperbolic logarithmic transseries

This section represents a generalization of the results obtained in [21, Theorem A] for strongly hyperbolic logarithmic transseries of depth 1 to strongly hyperbolic logarithmic transseries of an arbitrary depth. Instead of transfinite compositions of elementary changes of variables used in [21], we use the fixed point techniques to obtain normal forms.

We consider a strongly hyperbolic logarithmic transseries $f \in \mathfrak{L}^H$, i.e., $f = \lambda z^\alpha +$ h.o.t., $\lambda, \alpha \in \mathbb{R}_{>0} \setminus \{1\}$. Note that $\psi \circ f \circ \psi^{-1} = z^\alpha +$ h.o.t., where $\psi := \lambda^{\frac{1}{\alpha-1}} \cdot \mathrm{id}$ is a homothety. Therefore, without the loss of generality, we assume that $f = z^\alpha +$ h.o.t. We consider the conjugacy equation

$$\varphi \circ f \circ \varphi^{-1} = g, \quad \varphi \in \mathfrak{L}^0, \tag{2.55}$$

for g minimal in $\mathfrak{L}$ in the sense that g has as little number of terms as possible. By Proposition 2.0.1, if conjugacy equation (2.55) has a solution, it follows that $g = z^\alpha +$ h.o.t. We set $g := z^\alpha$ and, in the sequel, we consider the conjugacy equation:

$$\varphi \circ f \circ \varphi^{-1} = z^\alpha, \quad \varphi \in \mathfrak{L}^0.$$

Every solution φ of the above equation will, as before, be called a *normalization* of the strongly hyperbolic logarithmic transseries f. Furthermore, logarithmic transseries z^α will be called the *normal form* of f.

In Subsection 2.2.1 we state the normalization theorem for strongly hyperbolic logarithmic transseries. Subsections 2.2.2, 2.2.3 and 2.2.4 are dedicated to proving all three statements of the normalization theorem, respectively.

2.2.1. Normalization theorem for strongly hyperbolic logarithmic transseries

In this subsection we state the normalization theorem for strongly hyperbolic logarithmic transseries.

Theorem B (Normalization theorem for strongly hyperbolic logarithmic transseries)**.** Let $f \in \mathfrak{L}^H$ be such that $f = z^\alpha + \text{h.o.t.}$, for $\alpha \in \mathbb{R}_{>0}$, $\alpha \neq 1$. Then:

1. There exists a unique solution $\varphi \in \mathfrak{L}^0$ of the *normalization equation*:

$$\varphi \circ f \circ \varphi^{-1} = z^\alpha. \tag{2.56}$$

   Moreover, $\text{ord}_z(\varphi - \text{id}) \geq \text{ord}_z(f - z^\alpha) - \alpha + 1$.

   Additionally, if $f \in \mathscr{L}_k^H$, then $\varphi \in \mathscr{L}_k^0$.

2. If $\alpha > 1$, then, for every initial condition $h \in \mathfrak{L}^0$, the *Böttcher sequence*

$$(z^{\frac{1}{\alpha^n}} \circ h \circ f^{\circ n})_n \tag{2.57}$$

   converges to the normalization φ in the weak topology on $\mathfrak{L}^0$ as n tends to $+\infty$. Moreover, the sequence (2.57) converges in the power-metric topology on $\mathfrak{L}^0$ if and only if the initial condition h is such that $\text{Lb}_z(h) = \text{Lb}_z(\varphi)$.

3. Let $f \in \mathscr{L}_k$, $k \in \mathbb{N}$. The support $\text{Supp}(\varphi)$ is contained in the semigroup generated by $(\alpha^p, \mathbf{0}_k)$, $p \in \mathbb{N}$, $(0,1,0,\ldots,0)_{k+1}$, $\ldots$, $(0,0,\ldots,0,1)_{k+1}$, and $(\alpha^m(\gamma - \alpha), \mathbf{n})$, for $(\gamma, \mathbf{n}) \in \text{Supp}(f - z^\alpha)$, $m \in \mathbb{N}$.

Remark 2.2.1.

1. Let $f \in \mathfrak{L}^H$, $f = z^\alpha + \text{h.o.t.}$, $\alpha \in \mathbb{R}_{>0}$, $\alpha \neq 1$, be a strongly hyperbolic logarithmic transseries such that $\text{ord}_z(f - z^\alpha) > \alpha$, by statement 1 of Theorem B, it follows that there exists a unique normalization $\varphi \in \mathfrak{L}^0$. It satisfies $\text{ord}_z(\varphi - \text{id}) \geq \text{ord}_z(f - z^\alpha) - \alpha + 1 > 1$. Therefore, $\text{ord}_z(\varphi - \text{id}) > 1$ and, consequently, $\text{Lb}_z(\varphi) = \text{id}$. By statement 2 of Theorem B, it follows that the Böttcher sequence $(z^{\frac{1}{\alpha^n}} \circ h \circ f^{\circ n})_n$ converges to the normalization φ in the power-metric topology on $\mathfrak{L}^0$, for each $h \in \mathfrak{L}^0$ such that $\text{ord}_z(h - \text{id}) > 1$. In particular, putting $h := \text{id}$, $(z^{\frac{1}{\alpha^n}} \circ f^{\circ n})_n$ converges to φ in the power-metric topology on $\mathfrak{L}^0$.

2. Notice that, by statement 3 of Theorem B, the support $\text{Supp}(\varphi)$ depends only on $\text{Supp}(f)$. In particular, $\text{Supp}(\varphi)$ is independent of $\text{Supp}(h)$, for the initial condition $h \in \mathfrak{L}^0$.

2.2.2. Proof of statement 1 of Theorem B

We first explain that, without the loss of generality, we can consider only the case when $\mathrm{ord}_z(f) > 1$, for $f \in \mathfrak{L}^H$.

Suppose that $f = z^\alpha + \text{h.o.t.}$, for $0 < \alpha < 1$. Note that $f^{-1} = z^{\frac{1}{\alpha}} + \text{h.o.t.}$ If $\varphi \in \mathfrak{L}^0$ is a solution of the conjugacy equation $\varphi \circ f \circ \varphi^{-1} = z^\alpha$ then

$$\varphi \circ f^{-1} \circ \varphi^{-1} = (\varphi \circ f \circ \varphi^{-1})^{-1} = (z^\alpha)^{-1} = z^{\frac{1}{\alpha}}.$$

Furthermore, it can be proven that $\mathrm{ord}_z(f - z^\alpha) - \alpha = \mathrm{ord}_z\left(f^{-1} - z^{\frac{1}{\alpha}}\right) - \frac{1}{\alpha}$, which is important for statement 1 of Theorem B.

The structure of the proof is similar to the proof of the normalization theorem for hyperbolic logarithmic transseries stated in Theorem A in Section 2.1. We first transform normalization equation (2.56) to the equivalent fixed point equation using the so-called *Böttcher operator* in Lemma 2.2.4. Then, in Lemma 2.2.6, we give a sufficient and necessary conditions for the contractibility of the Böttcher operator, with respect to the metric d_z. This forces us, as in Section 2.1, to distinguish two cases: $\mathrm{ord}_z(f - z^\alpha) > 1$ and $\mathrm{ord}_z(f - z^\alpha) = 1$. In the first case, the Böttcher operator is a contraction and therefore, the corresponding Picard sequence converges towards the normalization in the power-metric topology. However, in the second case, as in Section 2.1, we first *prenormalize* the initial strongly hyperbolic logarithmic transseries f, i.e., eliminate every term of order 1 (in z) except z^α. After the prenormalization, we normalize the prenormalized transseries f, as in the first case, using a variant of the Böttcher operator on the space of blocks.

Transforming the conjugacy equation to the fixed point equation

In this subsection, for a strongly hyperbolic logarithimc transseries $f = z^\alpha + \text{h.o.t.}$, $\alpha \in \mathbb{R}_{>1}$, we transform the conjugacy equation (2.56) to the fixed point equation, using the so-called *Böttcher operator* which is motivated by the *Böttcher Theorem*.

Theorem 2.2.2 (Böttcher Theorem, see e.g. Theorem 4.1, [4], Theorem 9.1, [24])**.** Let $f \in \mathrm{Diff}(\mathbb{C}, 0)$ be a strongly hyperbolic complex analytic germ of diffeomorphism at zero, i.e., $f(z) = z^n + o(z^n)$, for $n \geq 2$. There exists a parabolic analytic change of variables $\varphi \in \mathrm{Diff}(\mathbb{C}, 0)$, $\varphi(z) = z + o(z)$, such that $\varphi(f(z)) = (\varphi(z))^n$.

Motivated by the Böttcher Theorem, and the Koenigs Theorem stated in Theorem 2.1.7, we define an analogue of the generalized Koenigs operator from (2.8) for strongly hyperbolic logarithmic transseries, which we call the *Böttcher operator*.

Definition 2.2.3 (Böttcher operator and Böttcher sequence)**.** Let $f = z^\alpha + \text{h.o.t.}$, $\alpha \in \mathbb{R}_{>1}$, and let $\mathscr{P}_f : \mathfrak{L}^0 \to \mathfrak{L}^0$, be defined by:

$$\mathscr{P}_f(h) := z^{\frac{1}{\alpha}} \circ h \circ f, \quad h \in \mathfrak{L}^0. \tag{2.58}$$

The operator $\mathscr{P}_f$ is called the *Böttcher operator*.

We call $\left(\mathscr{P}_f^{\circ n}(h)\right)_n$ the *Böttcher sequence* of the strongly hyperbolic logarithmic transseries f with the *initial condition* $h \in \mathfrak{L}^0$.

Lemma 2.2.4 (Transformation of the conjugacy equation to a fixed point equation)**.** Let $f \in \mathfrak{L}^H$, $f = z^\alpha + \text{h.o.t.}$, $\alpha \in \mathbb{R}_{>1}$, and let $\mathscr{P}_f$ be the Böttcher operator defined in (2.58). Then, $\varphi \in \mathfrak{L}^0$ is a solution of the normalization equation (2.56) if and only if φ is a fixed point of the operator $\mathscr{P}_f$.

Proof. Directly from (2.58) and normalization equation (2.56). ∎

Lemma 2.2.5. Let $f \in \mathfrak{L}^H$ be such that $f = z^\alpha + \text{h.o.t.}$, $\alpha \in \mathbb{R}_{>1}$, and let $\beta := \text{ord}_z\left(f - z^\alpha\right) - \alpha + 1$. Let $\varphi \in \mathfrak{L}^0$ be a solution of the normalization equation (2.56). Then $\text{ord}_z\left(\varphi - \text{id}\right) \geq \beta$.

Proof. If $\beta = 1$, then obviously $\text{ord}_z\left(\varphi - \text{id}\right) \geq \beta$. Suppose that $\beta > 1$. Let φ be a solution of the normalization equation (2.56). Put $\varphi_1 := \varphi - \text{id}$ and $f_1 := f - z^\alpha$. Note that $\text{ord}_z\left(f_1\right) = \alpha + \beta - 1$. From the normalization equation (2.56) we get:

$$\begin{aligned} z^{\frac{1}{\alpha}} \circ \left(z^\alpha + f_1 + \varphi_1 \circ f\right) &= \text{id} + \varphi_1, \\ z\Big(1 + \frac{f_1}{z^\alpha} + \frac{\varphi_1(z^\alpha)}{z^\alpha} + \sum_{i \geq 1} \frac{\varphi_1^{(i)}(z^\alpha)}{z^\alpha i!} f_1^i\Big)^{\frac{1}{\alpha}} &= \text{id} + \varphi_1. \end{aligned} \tag{2.59}$$

Suppose that $\text{ord}_z\left(\varphi - \text{id}\right) < \beta$, i.e., $\text{ord}_z\left(\varphi_1\right) < \beta$. Note that $\text{ord}\left(\frac{f_1}{z^{\alpha-1}}\right) = \beta$. Since $\text{ord}_z\left(\varphi - \text{id}\right) < \beta$, by (2.59) and the first formula in (A.1), it follows that

$$\text{id} + \text{Lt}\left(\frac{\varphi_1(z^\alpha)}{\alpha z^{\alpha-1}}\right) + \text{h.o.t.} = \text{id} + \varphi_1.$$

Therefore, the order (in z) of $\frac{\varphi_1(z^\alpha)}{\alpha z^{\alpha-1}}$ is equal to the order (in z) of φ_1, which implies that $\alpha \cdot \mathrm{ord}_z(\varphi_1) - \alpha + 1 = \mathrm{ord}_z(\varphi_1)$, i.e., $\mathrm{ord}_z(\varphi_1) = 1$. Since $\mathrm{ord}(\varphi_1) > (1, \mathbf{0}_k)$, by Lemma A.3.2 and by multiplication by $\frac{1}{\alpha}$, we get $\mathrm{Lt}\left(\frac{\varphi_1(z^\alpha)}{\alpha z^{\alpha-1}}\right) = \frac{1}{\alpha^{n+1}}\mathrm{Lt}(\varphi_1)$, for some $n \in \mathbb{N}$, which is a contradiction. Therefore, $\mathrm{ord}_z(\varphi - \mathrm{id}) \geq \beta$. ■

In the next proposition we give a sufficient and necessary condition for contractibility of the Böttcher operator defined in (2.58).

Lemma 2.2.6 (Contractibility of the Böttcher operator $\mathscr{P}_f$)**.** Let $f \in \mathscr{L}_m^H$, $m \in \mathbb{N}$, be such that $f = z^\alpha + \text{h.o.t.}$, $\alpha \in \mathbb{R}_{>1}$, and let $\mathscr{P}_f$ be the Böttcher operator defined by (2.58). Then:

1. The space $\mathrm{id} + \mathscr{L}_k^\beta$ is $\mathscr{P}_f$-invariant, for every $\beta \geq 1$ and $k \geq m$.
2. For every $k \geq m$, the operator $\mathscr{P}_f$ is a contraction on the space $(\mathrm{id} + \mathscr{L}_k^\beta, d_z)$ if and only if $\beta > 1$. In that case, the operator $\mathscr{P}_f$ is a $\frac{1}{2^{(\alpha-1)(\beta-1)}}$-contraction on the space $(\mathrm{id} + \mathscr{L}_k^\beta, d_z)$, $k \geq m$.

Proof. 1. Let $f_1 := f - z^\alpha$ and let $h \in \mathrm{id} + \mathscr{L}_k^\beta$. Put $h_1 := h - \mathrm{id}$. Now we get:

$$\begin{aligned}
\mathscr{P}_f(h) &= z^{\frac{1}{\alpha}} \circ h \circ f \\
&= z^{\frac{1}{\alpha}} \circ (z^\alpha + f_1 + h_1 \circ f) \\
&= z\left(1 + \frac{f_1}{z^\alpha} + \frac{h_1(z^\alpha)}{z^\alpha} + \sum_{i \geq 1} \frac{h_1^{(i)}(z^\alpha)}{z^\alpha i!} f_1^i\right)^{\frac{1}{\alpha}}.
\end{aligned} \tag{2.60}$$

Let

$$\mathscr{N}(h) := \frac{f_1}{z^\alpha} + \frac{h_1(z^\alpha)}{z^\alpha} + \sum_{i \geq 1} \frac{h_1^{(i)}(z^\alpha)}{z^\alpha i!} f_1^i.$$

By the first formula in (A.1), we get:

$$\mathscr{P}_f(h) = \mathrm{id} + z \sum_{j \geq 1} \binom{\frac{1}{\alpha}}{j} (\mathscr{N}(h))^j. \tag{2.61}$$

It is obvious that $\mathscr{P}_f(h) \in \mathscr{L}_k^0$ and since $\mathrm{ord}_z(h_1) \geq \beta$ and $\mathrm{ord}_z(f_1) = \alpha + \beta - 1$, it follows that:

$$\mathrm{ord}_z\left(\frac{f_1}{z^{\alpha-1}}\right) = \beta, \tag{2.62}$$

$$\begin{aligned}\operatorname{ord}_z\left(\frac{h_1(z^\alpha)}{z^{\alpha-1}}\right) &= \alpha\cdot\operatorname{ord}_z(h_1)-\alpha+1\\ &= \operatorname{ord}_z(h_1)+(\alpha-1)\cdot\operatorname{ord}_z(h_1)-(\alpha-1)\\ &\geq \operatorname{ord}_z(h_1)+(\alpha-1)(\beta-1) \end{aligned}\tag{2.63}$$

and

$$\begin{aligned}\operatorname{ord}_z\left(\sum_{i\geq 1}\frac{h_1^{(i)}(z^\alpha)}{z^{\alpha-1}i!}f_1^i\right) &\geq \alpha\cdot(\operatorname{ord}_z(h_1)-1)-(\alpha-1)+\operatorname{ord}_z(f_1)\\ &> \alpha\cdot\operatorname{ord}_z(h_1)\\ &= \operatorname{ord}_z(h_1)+(\alpha-1)\cdot\operatorname{ord}_z(h_1)\\ &\geq \operatorname{ord}_z(h_1)+(\alpha-1)(\beta-1). \end{aligned}\tag{2.64}$$

Note that, if $\operatorname{ord}_z(h_1)=\beta$, equality holds in (2.63). From (2.62), (2.63) and (2.64), we get that $\operatorname{ord}_z(z\mathscr{N}(h))\geq\beta$, and, therefore, by (2.61),

$$\operatorname{ord}_z\left(\mathscr{P}_f(h)-\mathrm{id}\right)\geq\beta.$$

This implies that $\mathscr{P}_f(\mathrm{id}+\mathscr{L}_k^\beta)\subseteq\mathrm{id}+\mathscr{L}_k^\beta$.

2. Let $\mathrm{id}+h_1,\mathrm{id}+h_2\in\mathrm{id}+\mathscr{L}_k^\beta$. Using (2.61), (2.63), (2.64), we get:

$$\begin{aligned}\operatorname{ord}_z\left(\mathscr{P}_f(\mathrm{id}+h_1)-\mathscr{P}_f(\mathrm{id}+h_2)\right) &= \operatorname{ord}_z\left(\frac{1}{\alpha}z\mathscr{N}(h_1)-\frac{1}{\alpha}z\mathscr{N}(h_2)\right)\\ &\geq \operatorname{ord}_z(h_1-h_2)+(\alpha-1)(\beta-1). \end{aligned}\tag{2.65}$$

From (2.65), we conclude that $\mathscr{P}_f$ is a $\frac{1}{2^{(\alpha-1)(\beta-1)}}$-Lipschitz map. Suppose that $\operatorname{ord}_z(h_1-h_2)=\beta$. Then, the equality holds in (2.63). Hence, the equality holds also in (2.65). Therefore, $\frac{1}{2^{(\alpha-1)(\beta-1)}}$ is the minimal Lipschitz coefficient of $\mathscr{P}_f$. Consequently, it follows that $\mathscr{P}_f$ is a contraction on the space $\mathrm{id}+\mathscr{L}_k^\beta$ if and only if $\beta>1$. In that case, $\mathscr{P}_f$ is a $\frac{1}{2^{(\alpha-1)(\beta-1)}}$-contraction on the space $\mathrm{id}+\mathscr{L}_k^\beta$. ■

Let $f\in\mathscr{L}_k^H$, $k\in\mathbb{N}$, be such that $f=z^\alpha+\text{h.o.t.}$, $\alpha\in\mathbb{R}_{>1}$, and let $f_1:=f-z^\alpha$. If $\operatorname{ord}_z(f_1)>\alpha$, then $\beta>1$. By Lemma 2.2.6 and the Banach Fixed Point Theorem (Theorem 1.2.11), it follows that there exists a unique fixed point φ of the Böttcher operator in every space $\mathrm{id}+\mathscr{L}_m^\beta$, $m\geq k$. Since, by Lemma 2.2.5, every solution of the normalization equation (2.56) belongs to some $\mathrm{id}+\mathscr{L}_m^\beta$, $m\geq k$, by contractivity of the Böttcher operator, it follows that φ is unique in $\mathfrak{L}^0$. Moreover, φ is obtained as the limit of the corresponding Picard sequence, with respect to the power-metric topology, for any initial condition.

However, if $\mathrm{ord}_z(f_1) = \alpha$, then $\beta = 1$. By Lemma 2.2.6, the operator $\mathscr{P}_f$ is not a contraction on the space $\mathscr{L}_m^1$, $m \geq k$, and we cannot apply the Banach Fixed Point Theorem. Thus, we proceed in two steps:

Step 1. We *prenormalize* f, i.e., we solve a *prenormalization equation*:

$$\varphi_0 \circ f \circ \varphi_0^{-1} = z^\alpha + \text{h.o.b.}(z), \quad \varphi_0 \in \mathfrak{L}^0, \tag{2.66}$$

We call a solution φ_0 a *prenormalization* of f. We prove that φ_0 is unique up to blocks of higher order. Thus, if we impose the *canonical form* $\varphi_0 = \mathrm{id} + zH$, $H \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, then φ_0 is unique in $\mathfrak{L}^0$.

Step 2. We solve the *normalization equation* in the *variable* φ_1:

$$\varphi_1 \circ (\varphi_0 \circ f \circ \varphi_0^{-1}) \circ \varphi_1^{-1} = z^\alpha$$

in the space $\mathfrak{L}^0$ using the discussion above.

Finally, $\varphi := \varphi_1 \circ \varphi_0$ is the desired normalization in $\mathfrak{L}^0$.

Proof of step 1 (prenormalization)

In order to apply Lemma 2.2.6, we first prenormalize logarithmic transseries $f \in \mathfrak{L}^H$, $f = z^\alpha + \text{h.o.t.}$, $\alpha \in \mathbb{R}_{>1}$. That is, we solve prenormalization equation (2.66).

Definition 2.2.7 (Böttcher operator on the space $\mathrm{id} + z\mathscr{B}_{\geq 1}$)**.** Let $f \in \mathscr{L}_k^H$, $k \in \mathbb{N}_{\geq 1}$, be such that $f = z^\alpha + z^\alpha R_\alpha + \text{h.o.b.}(z)$, $\alpha \in \mathbb{R}_{>1}$, and $R_\alpha \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$. Let $\mathscr{R}_f : \mathrm{id} + z\mathscr{B}_{\geq 1}^+ \to \mathrm{id} + z\mathscr{B}_{\geq 1}^+$, be the operator defined by:

$$\mathscr{R}_f(\mathrm{id} + H) := z^{\frac{1}{\alpha}} \circ (\mathrm{id} + zH) \circ (z^\alpha + z^\alpha R_\alpha), \quad H \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k. \tag{2.67}$$

The operator $\mathscr{R}_f$ is called the *Böttcher operator on the space* $\mathrm{id} + z\mathscr{B}_{\geq 1}^+$.

In the next lemma we transform prenormalization equation (2.66) (where we consider only canonical solutions) to the fixed point equation on the space $\mathrm{id} + z\mathscr{B}_{\geq 1}^+$.

Lemma 2.2.8. Let $f \in \mathscr{L}_k^H$, $k \in \mathbb{N}_{\geq 1}$, be such that $f = z^\alpha + z^\alpha R_\alpha + \text{h.o.b.}(z)$, $\alpha \in \mathbb{R}_{>1}$, $R_\alpha \in \mathscr{B}_{\geq 1}^+ \setminus \{0\} \subseteq \mathscr{L}_k$, and let $\mathscr{R}_f$ be the Böttcher operator on the space $\mathrm{id} + z\mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k^0$ defined in (2.67). Parabolic logarithmic transseries $\varphi_0 = \mathrm{id} + zH$, $H \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, is a solution of prenormalization equation (2.66), if and only if φ_0 is a fixed point of the operator $\mathscr{R}_f$.

Proof. Let $\varphi_0 = \mathrm{id} + zH$, $H \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$ be a solution of prenormalization equation (2.66). Put $g := \varphi_0 \circ f \circ \varphi_0^{-1}$ and $f_1 := f - (z^\alpha + z^\alpha R_\alpha)$. By the Taylor Theorem (Proposition A.1.6), it follows that:

$$\begin{aligned}(z^\alpha + z^\alpha R_\alpha)^{-1} \circ f &= (z^\alpha + z^\alpha R_\alpha)^{-1} \circ (z^\alpha + z^\alpha R_\alpha + f_1)\\ &= \mathrm{id} + \sum_{i \geq 1} \frac{((z^\alpha + z^\alpha R_\alpha)^{-1})^{(i)}(z^\alpha + z^\alpha R_\alpha)}{i!} f_1^i,\end{aligned}$$

so $\mathrm{ord}_z\big((z^\alpha + z^\alpha R_\alpha)^{-1} \circ f - \mathrm{id}\big) > 1$. Put $f_2 := (z^\alpha + z^\alpha R_\alpha)^{-1} \circ f$. Note that $\mathrm{ord}_z(f_2 - \mathrm{id}) > 1$ and $f = (z^\alpha + z^\alpha R_\alpha) \circ f_2$. Since $g = \varphi_0 \circ f \circ \varphi_0^{-1}$, we get $\varphi_0 \circ f = g \circ \varphi_0$, i.e.

$$z^{\frac{1}{\alpha}} \circ \varphi_0 \circ (z^\alpha + z^\alpha R_\alpha) = z^{\frac{1}{\alpha}} \circ g \circ \varphi_0 \circ f_2^{-1}. \tag{2.68}$$

Note that the left-hand side of equation (2.68) consists only of a 1-block, so $z^{\frac{1}{\alpha}} \circ g \circ \varphi_0 \circ f_2^{-1}$ also consists only of a 1-block. Furthermore, since $g = z^\alpha + \text{h.o.b.}(z)$, it follows that $z^{\frac{1}{\alpha}} \circ g = \mathrm{id} + \text{h.o.b.}(z)$. Since $\mathrm{ord}_z(f_2 - \mathrm{id}) > 1$, we conclude that $\mathrm{ord}_z(f_2^{-1} - \mathrm{id}) > 1$, and therefore, $\varphi_0 \circ f_2^{-1} = \varphi_0 + \text{h.o.b.}(z)$. Now, from (2.68) we conclude that $z^{\frac{1}{\alpha}} \circ \varphi_0 \circ (z^\alpha + z^\alpha R_\alpha) = \varphi_0$, i.e., $\mathscr{R}_f(\varphi_0) = \varphi_0$.

To prove the converse, suppose that $\varphi_0 = \mathrm{id} + zH$, $H \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, is such that $\mathscr{R}_f(\varphi_0) = \varphi_0$, i.e., $\varphi_0 \circ (z^\alpha + z^\alpha R_\alpha) = z^\alpha \circ \varphi_0$. Since $\varphi_0 \circ f = \varphi_0 \circ (z^\alpha + z^\alpha R_\alpha) + \text{h.o.b.}(z)$, it follows that $\varphi_0 \circ f = z^\alpha \circ \varphi_0 + \text{h.o.b.}(z)$, i.e., $\varphi \circ f \circ \varphi_0^{-1} = z^\alpha + \text{h.o.b.}(z)$. ■

Unfortunately, the following examples show that the Böttcher operator on the space $\mathrm{id} + z\mathscr{B}_{\geq 1}^+$, defined in (2.67), is not a contraction, in general, in any of the metrics introduced in Section 2.1.

Example 2.2.9 (Non-contractibility of the Böttcher operator $\mathscr{R}_f$ on the space $\mathrm{id} + z\mathscr{B}_{\geq 1}^+$)**.**
1. Let $k \in \mathbb{N}_{\geq 1}$ be arbitrary. Take metric d on the space $\mathrm{id} + z\mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k^0$, defined by:

$$d(\mathrm{id} + zR, \mathrm{id} + zS) := \begin{cases} 2^{-\mathrm{ord}_{\ell_1}(R-S)}, & R \neq S,\\ 0, & R = S.\end{cases}$$

Now, consider id and $\mathrm{id}+z\boldsymbol{\ell}_1$ in $\mathrm{id}+z\mathscr{B}_{\geq 1}^{+}$. It is easy to see that $d(\mathrm{id}+z\boldsymbol{\ell}_1,\mathrm{id})=\frac{1}{2}$. Take $f=z^\alpha+z^\alpha R_\alpha+\text{h.o.b}(z)$, $\alpha\in\mathbb{R}_{>1}$, and $R_\alpha\in\mathscr{B}_{\geq 1}^{+}\subseteq\mathscr{L}_k$, $\mathrm{ord}_{\boldsymbol{\ell}_1}(R_\alpha)\geq 2$. Notice that:

$$\mathscr{R}_f(\mathrm{id})=\mathrm{id}+z\sum_{i\geq 1}\binom{\frac{1}{\alpha}}{i}R_\alpha^i,$$

and

$$\begin{aligned}\mathscr{R}_f(\mathrm{id}+z\boldsymbol{\ell}_1)&=z^{\frac{1}{\alpha}}\circ(\mathrm{id}+z\boldsymbol{\ell}_1)\circ(z^\alpha+z^\alpha R_\alpha)\\&=\mathrm{id}+\frac{1}{\alpha}z\boldsymbol{\ell}_1+\text{h.o.t.}\end{aligned}$$

Therefore, $\mathscr{R}_f(\mathrm{id}+z\boldsymbol{\ell}_1)-\mathscr{R}_f(\mathrm{id})=\frac{1}{\alpha}z\boldsymbol{\ell}_1+\text{h.o.t.}$, which implies that

$$d\big(\mathscr{R}_f(\mathrm{id}+z\boldsymbol{\ell}_1),\mathscr{R}_f(\mathrm{id})\big)=\frac{1}{2}=d(\mathrm{id}+z\boldsymbol{\ell}_1,\mathrm{id}).$$

Thus, $\mathscr{R}_f$ is not a contraction on the space $\mathrm{id}+z\mathscr{B}_{\geq 1}^{+}\subseteq\mathscr{L}_k^0$, with respect to the metric d.

2. Let d be the metric on the space $\mathrm{id}+z\mathscr{B}_{\geq 1}^{+}$, defined by:

$$d(\mathrm{id}+zT_1,\mathrm{id}+zT_2):=\sum_{i=1}^{+\infty}\frac{1}{2^i}\cdot\frac{|a_{1,i}-a_{2,i}|}{1+|a_{1,i}-a_{2,i}|},$$

where $T_1:=\sum_{i=1}^{+\infty}a_{1,i}\boldsymbol{\ell}_1^i$, $T_2:=\sum_{i=1}^{+\infty}a_{2,i}\boldsymbol{\ell}_1^i$. Let $f:=z^\alpha+az^\alpha\boldsymbol{\ell}_1$, $\alpha\in\mathbb{R}_{>1}$, $a\in\mathbb{R}_{>0}$. It is easy to see that $d(\mathrm{id},\mathrm{id}+z\boldsymbol{\ell}_1)=\frac{1}{4}$. Notice that:

$$\mathscr{R}_f(\mathrm{id})=\mathrm{id}+z\sum_{i\geq 1}\binom{\frac{1}{\alpha}}{i}(a\boldsymbol{\ell}_1)^i=\mathrm{id}+\frac{a}{\alpha}z\boldsymbol{\ell}_1+\frac{1}{2\alpha}\Big(\frac{1}{\alpha}-1\Big)a^2z\boldsymbol{\ell}_1^2+\text{h.o.t.}\tag{2.69}$$

and

$$\begin{aligned}\mathscr{R}_f(\mathrm{id}+z\boldsymbol{\ell}_1)&=z^{\frac{1}{\alpha}}\circ(\mathrm{id}+z\boldsymbol{\ell}_1)\circ(z^\alpha+az^\alpha\boldsymbol{\ell}_1)\\&=\mathrm{id}+\frac{1}{\alpha}\Big(a+\frac{1}{\alpha}\Big)z\boldsymbol{\ell}_1+\left(\frac{1}{2\alpha}\Big(\frac{1}{\alpha}-1\Big)\Big(a+\frac{1}{\alpha}\Big)^2+\frac{a}{\alpha^2}\right)z\boldsymbol{\ell}_1^2+\text{h.o.t.}\end{aligned}\tag{2.70}$$

By (2.69) and (2.70), it follows that:

$$d\big(\mathscr{R}_f(\mathrm{id}),\mathscr{R}_f(\mathrm{id}+z\boldsymbol{\ell}_1)\big)\geq\frac{1}{2}\cdot\frac{\frac{1}{\alpha^2}}{1+\frac{1}{\alpha^2}}+\frac{1}{4}\cdot\frac{\Big(\frac{1}{\alpha}-1\Big)\Big(\frac{2a}{\alpha^2}+\frac{1}{\alpha^3}\Big)+\frac{a}{\alpha^2}}{1+\Big(\frac{1}{\alpha}-1\Big)\Big(\frac{2a}{\alpha^2}+\frac{1}{\alpha^3}\Big)+\frac{a}{\alpha^2}}.$$

As α tends to 1 and a tends to $+\infty$, $d(\mathscr{R}_f(\mathrm{id}),\mathscr{R}_f(\mathrm{id}+z\boldsymbol{\ell}_1))$ tends to $\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{4}\cdot 1=\frac{1}{2}$. Since $d(\mathrm{id},\mathrm{id}+z\boldsymbol{\ell}_1)=\frac{1}{4}$, there exists sufficiently small $\alpha\in\mathbb{R}_{>1}$ and sufficiently big $a\in\mathbb{R}_{>0}$, such that $\mathscr{R}_f$ is not a contraction on the space $(\mathrm{id}+z\mathscr{B}_{\geq 1}^{+},d)$.

Now, similarly as in Section 2.1 for hyperbolic logarithmic transseries, we transform prenormalization equation (2.66) to a fixed point equation, in order to use the fixed point theorem stated in Proposition 1.2.12 to prove the existence and the uniqueness of the solution.

Lemma 2.2.10 (A fixed point equation for the *canonical* prenormalization)**.** Let $f \in \mathscr{L}_k^H$, $k \in \mathbb{N}_{\geq 1}$, be such that $f = z^\alpha + z^\alpha R_\alpha + \text{h.o.b}(z)$, $\alpha \in \mathbb{R}_{>1}$, and $R_\alpha \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$. A logarithmic transseries $\varphi_0 = \mathrm{id} + zH$, $H \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, is a solution of prenormalization equation (2.66) if and only if H is a solution of the *fixed point equation*

$$\mathscr{T}_f(H) = \mathscr{S}_f(H), \tag{2.71}$$

where $\mathscr{T}_f, \mathscr{S}_f : \mathscr{B}_{\geq 1}^+ \to \mathscr{B}_{\geq 1}^+$ are the operators defined by:

$$\mathscr{T}_f(H) := H \circ z^\alpha + (H \circ z^\alpha) \cdot R_\alpha - \sum_{i \geq 1} \binom{\alpha}{i} H^i, \quad H \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k, \tag{2.72}$$

and

$$\mathscr{S}_f(H) := -R_\alpha - \big((D_1(H)) \circ z^\alpha\big) \cdot \Big(R_\alpha + \sum_{i \geq 2} \frac{(-1)^{i-2}}{i(i-1)} R_\alpha^i\Big) - \mathscr{C}_{R_\alpha}(H), \quad H \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k, \tag{2.73}$$

where $\mathscr{C}_{R_\alpha} := \mathscr{C}(\cdot, R_\alpha)$ is a superlinear $\frac{1}{4}$-contraction on the space $(\mathscr{B}_{\geq 1}^+, d_1)$ from Lemma A.3.4.

Proof. Let $f = z^\alpha + z^\alpha R_\alpha + \text{h.o.b.}(z)$, $\alpha \in \mathbb{R}_{>1}$, $R_\alpha \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$. By Lemma 2.2.8, it follows that $\varphi_0 = \mathrm{id} + zH$, $H \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, is a solution of prenormalization equation (2.66) if and only if φ_0 is a fixed point of the operator $\mathscr{R}_f$ defined by (2.67), i.e.

$$\begin{aligned} \mathscr{R}_f(\varphi_0) &= \varphi_0, \\ (\mathrm{id} + zH) \circ (z^\alpha + z^\alpha R_\alpha) &= (\mathrm{id} + zH)^\alpha, \\ z^\alpha + z^\alpha R_\alpha + z^\alpha \cdot H(z^\alpha) + \sum_{i \geq 1} \frac{(zH)^{(i)}(z^\alpha)}{i!} (z^\alpha R_\alpha)^i &= z^\alpha + z^\alpha \sum_{i \geq 1} \binom{\alpha}{i} H^i. \end{aligned} \tag{2.74}$$

From the last line of (2.74) and Lemma A.3.4, we conclude that:

$$R_\alpha + H(z^\alpha) + H(z^\alpha) \cdot R_\alpha + \big((D_1(H)) \circ z^\alpha\big) \cdot \Big(R_\alpha + \sum_{i \geq 2} \frac{(-1)^{i-2}}{i(i-1)} R_\alpha^i\Big) + \mathscr{C}_{R_\alpha}(H) = \sum_{i \geq 1} \binom{\alpha}{i} H^i, \tag{2.75}$$

for a superlinear $\frac{1}{4}$-contraction $\mathscr{C}_{R_\alpha}:\mathscr{B}^+_{\geq 1}\to\mathscr{B}_1$ such that $\mathscr{C}_{R_\alpha}:=\mathscr{C}(\cdot,R_\alpha)$, where $\mathscr{C}$ is the operator from Lemma A.3.4. By (2.75), it follows that:

$$H\circ z^\alpha+(H\circ z^\alpha)\cdot R_\alpha-\sum_{i\geq 1}\binom{\alpha}{i}H^i=-R_\alpha-((D_1(H))\circ z^\alpha)\cdot\Big(R_\alpha+\sum_{i\geq 2}\frac{(-1)^{i-2}}{i(i-1)}R_\alpha^i\Big)-\mathscr{C}_{R_\alpha}(H),$$

that is,

$$\mathscr{T}_f(H)=\mathscr{S}_f(H),\quad \text{for } H\in\mathscr{B}^+_{\geq 1}\subseteq\mathscr{L}_k.$$

$\blacksquare$

Proposition 2.2.11 (Properties of the operators $\mathscr{T}_f$ and $\mathscr{S}_f$)**.** Let $f\in\mathscr{L}_k^H$, $k\in\mathbb{N}_{\geq 1}$, be such that $f=z^\alpha+z^\alpha R_\alpha+\text{h.o.b.}(z)$, $\alpha\in\mathbb{R}_{>1}$, and $R_\alpha\in\mathscr{B}^+_{\geq 1}\subseteq\mathscr{L}_k$. Let $\mathscr{T}_f,\mathscr{S}_f:\mathscr{B}^+_{\geq 1}\to\mathscr{B}^+_{\geq 1}$ be the operators defined by (2.72) and (2.73) respectively. Then:

1. $\mathscr{T}_f$ is an isometry and a surjection on the space $(\mathscr{B}^+_{\geq 1},d_1)$,
2. $\mathscr{S}_f$ is a $\frac{1}{2}$-contraction on the space $(\mathscr{B}^+_{\geq 1},d_1)$.

Here we state and prove a technical lemma that is needed in the proof of Proposition 2.2.11.

Lemma 2.2.12. Let $R,M\in\mathscr{B}^+_{\geq 2}\subseteq\mathscr{L}_k$, $k\in\mathbb{N}_{\geq 2}$, and let $h:=\sum_{i\geq 2}\binom{\alpha}{i}x^i$ be formal power series in the *variable* x, for $\alpha\in\mathbb{R}_{>1}$. Then, there exists $H\in\mathscr{B}^+_{\geq 2}$, such that:

$$(H\circ z^\alpha)\cdot(1+R)-\alpha H-h(H)=M. \tag{2.76}$$

Proof. Let $M_2,R_2\in\mathscr{B}^+_2$ and $M_3,R_3\in\mathscr{B}^+_{\geq 3}$ be such that:

$$M=M_2+M_3,$$
$$R=R_2+R_3.$$

Put $H:=H_2+H_3$, $H_2\in\mathscr{B}^+_2$, $H_3\in\mathscr{B}^+_{\geq 3}$. By the Taylor Theorem, note that $h(H)=h(H_3)+\sum_{i\geq 1}\frac{h^{(i)}(H_3)}{i!}H_2^i$. Now, equation (2.76) is equivalent to the equation:

$$((H_2+H_3)\circ z^\alpha)\cdot(1+R_2+R_3)-\alpha(H_2+H_3)-h(H_3)-\sum_{i\geq 1}\frac{h^{(i)}(H_3)}{i!}H_2^i=M_2+M_3. \tag{2.77}$$

By Remark B.5.6 there exists $H_3 \in \mathscr{B}^+_{\geq 3}$ such that:

$$(1-\alpha+R_3)\cdot H_3 - h(H_3) = M_3. \tag{2.78}$$

By Lemma A.3.2, it follows that $T\circ z^\alpha = T+\mathscr{K}(T)$, $T\in\mathscr{B}^+_{\geq 2}$, where $\mathscr{K}:\mathscr{B}^+_{\geq 2}\to\mathscr{B}^+_{\geq 2}$ is a superlinear operator and a $\frac{1}{2}$-contraction on the space $(\mathscr{B}^+_{\geq 2},d_2)$. Now, from (2.77) and (2.78) we get the equation:

$$H_2\cdot(1-\alpha+R-h'(H_3))+\big(\mathscr{K}(H_2)+\mathscr{K}(H_3)\big)\cdot(1+R)-\sum_{i\geq 2}\frac{h^{(i)}(H_3)}{i!}H_2^i = M_2-R_2\cdot H_3, \tag{2.79}$$

in the *variable* $H_2\in\mathscr{B}^+_2$. Let $\mathscr{T}_2,\mathscr{S}_2:\mathscr{B}^+_2\to\mathscr{B}^+_2$ be the operators defined by:

$$\begin{aligned}\mathscr{S}_2(T) &:= -(1+R)\cdot\big(\mathscr{K}(T)+\mathscr{K}(H_3)\big)+\sum_{i\geq 2}\frac{h^{(i)}(H_3)}{i!}T^i+M_2-R_2\cdot H_3,\\ \mathscr{T}_2(T) &:= (1-\alpha+R-h'(H_3))\cdot T,\quad T\in\mathscr{B}^+_2.\end{aligned}$$

Now equation (2.79) is equivalent to the *fixed point equation* $\mathscr{T}_2(T)=\mathscr{S}_2(T)$, $T\in\mathscr{B}^+_2$. By Example 1.2.8, it is easy to see that $\mathscr{S}_2$ is a $\frac{1}{2}$-contraction and, since $\alpha>1$, $\mathscr{T}_2$ is an isometry and a surjection on the space $(\mathscr{B}^+_2,d_2)$. By the fixed point theorem stated in Proposition 1.2.12, there exists $H_2\in\mathscr{B}^+_{\geq 2}$, such that $\mathscr{T}_2(H_2)=\mathscr{S}_2(H_2)$. Now, $H:=H_2+H_3$ is a solution of equation (2.76). ∎

Proof of Proposition 2.2.11. Statement 2 follows directly from (2.73).

1. Let us prove that $\mathscr{T}_f$ is an isometry on the space $(\mathscr{B}^+_{\geq 1},d_1)$. Let $H_1,H_2\in\mathscr{B}^+_{\geq 1}$, $H_1\neq H_2$. By Lemma A.3.2, it follows that $(H_1-H_2)\circ z^\alpha=\frac{1}{\alpha^n}\mathrm{Lt}(H_1-H_2)+\text{h.o.t.}$, for $n:=\mathrm{ord}_{\boldsymbol{\ell}_1}(H_1-H_2)\geq 0$. Thus,

$$\mathscr{T}_f(H_1)-\mathscr{T}_f(H_2)=\Big(\frac{1}{\alpha^n}-\alpha\Big)\mathrm{Lt}(H_1-H_2)+\text{h.o.t.}$$

Since $n\geq 0$, it follows that $\frac{1}{\alpha^n}-\alpha\neq 0$, and, therefore, $\mathrm{ord}_{\boldsymbol{\ell}_1}(\mathscr{T}_f(H_1)-\mathscr{T}_f(H_2))=\mathrm{ord}_{\boldsymbol{\ell}_1}(H_1-H_2)$, which implies that $\mathscr{T}_f$ is an isometry on the space $(\mathscr{B}^+_{\geq 1},d_1)$.

Let us prove that $\mathscr{T}_f$ is a surjection. Let $M\in\mathscr{B}^+_{\geq 1}$, and let $M_1,R_1\in\mathscr{B}^+_1$ and $M_2,R_2\in\mathscr{B}^+_{\geq 2}$ be such that $M=M_1+M_2$ and $R_\alpha=R_1+R_2$. Put $h:=\sum_{i\geq 2}\binom{\alpha}{i}x^i$, where x is a formal variable. We prove that the equation $\mathscr{T}_f(H)=M$, i.e.

$$H\circ z^\alpha+(H\circ z^\alpha)\cdot R_\alpha-\alpha H-h(H)=M, \tag{2.80}$$

has a solution $H \in \mathscr{B}^+_{\geq 1}$. Let us decompose H as $H = H_1 + H_2$, where $H_1 \in \mathscr{B}^+_1$ and $H_2 \in \mathscr{B}^+_{\geq 2}$. By the Taylor Theorem (Proposition A.1.6), it follows that (2.80) is equivalent to:

$$
\begin{aligned}
&H_1 \circ z^\alpha \cdot (1+R_\alpha) - \alpha H_1 + (H_2 \circ z^\alpha) \cdot (1+R_2) - \alpha H_2 - h(H_2) - \sum_{i\geq 1} \frac{h^{(i)}(H_2)}{i!} H_1^i \\
&= M_1 + M_2 - (H_2 \circ z^\alpha) \cdot R_1. \qquad (2.81)
\end{aligned}
$$

By Lemma 2.2.12, there exists $H_2 \in \mathscr{B}^+_{\geq 2}$, such that:

$$
(H_2 \circ z^\alpha) \cdot (1+R_2) - \alpha H_2 - h(H_2) = M_2. \qquad (2.82)
$$

Now, by putting (2.82) in (2.81), we get the equation:

$$
H_1 \circ z^\alpha \cdot (1+R_\alpha) - \alpha H_1 - \sum_{i\geq 1} \frac{h^{(i)}(H_2)}{i!} H_1^i = M_1 - (H_2 \circ z^\alpha) \cdot R_1. \qquad (2.83)
$$

in the *variable* $H_1 \in \mathscr{B}^+_1$. Let $\mathscr{S}_1, \mathscr{T}_1 : \mathscr{B}^+_1 \to \mathscr{B}^+_1$ be the operators defined by:

$$
\begin{aligned}
\mathscr{S}_1(T) &:= -(T \circ z^\alpha) \cdot R_1 + \sum_{i\geq 2} \frac{h^{(i)}(H_2)}{i!} T^i + M_1 - (H_2 \circ z^\alpha) \cdot R_1, \\
\mathscr{T}_1(T) &:= (T \circ z^\alpha) \cdot (1+R_2) - \alpha T - h'(H_2) \cdot T, \quad T \in \mathscr{B}^+_1. \qquad (2.84)
\end{aligned}
$$

By Example 1.2.8 and since $\mathrm{ord}_{\ell_1}(R_1) \geq 1$, it is easy to see that $\mathscr{S}_1$ is a $\frac{1}{2}$-contraction on the space $(\mathscr{B}^+_1, d_1)$. Furthermore, it is easy to see that $\mathscr{T}_1$ is a superlinear isometry on the space $(\mathscr{B}^+_1, d_1)$. By similar analysis as in Lemma 2.2.12, applied on each term of $\mathscr{T}_1(T)$, we deduce that $\mathscr{T}_1$ is a surjection. By the fixed point theorem stated in Proposition 1.2.12, there exists $H_1 \in \mathscr{B}^+_1$, such that $\mathscr{S}_1(H_1) = \mathscr{T}_1(H_1)$. To conclude, $H \in \mathscr{B}^+_{\geq 1}$, given by $H := H_1 + H_2$ is a solution of equation (2.80). This proves that $\mathscr{T}_f$ is a surjection. ∎

Proposition 2.2.13 (The unique *canonical* solution of the prenormalization equation)**.** Let $f \in \mathscr{L}^H_k$, $k \in \mathbb{N}_{\geq 1}$, be such that $f = z^\alpha + z^\alpha R_\alpha + \text{h.o.b.}(z)$, $\alpha > 1$ and $R_\alpha \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_k$. There exists the unique logarithmic transseries $\varphi_0 \in \mathscr{L}^0_k$ of the form $\varphi_0 = \mathrm{id} + zH$, $H \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_k$, such that $\varphi_0 \circ f \circ \varphi_0^{-1} = z^\alpha + \text{h.o.b.}(z)$.

Proof. Directly from Lemma 2.2.10, Proposition 2.2.11 and the fixed point theorem from Proposition 1.2.12. ∎

Proof of statement 1 of Theorem B (normalization)

Proof. Existence. Let $f \in \mathscr{L}_k^H$, $k \in \mathbb{N}$, be such that $f = z^\alpha + z^\alpha R_\alpha + \text{h.o.b.}(z)$, $\alpha \in \mathbb{R}_{>1}$ and $R_\alpha \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$.

By Proposition 2.2.13, there exists the unique $\varphi_0 = \mathrm{id} + zH$, $H \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, such that $\varphi_0 \circ f \circ \varphi_0^{-1} = z^\alpha + \text{h.o.b.}(z)$. Notice that $\varphi_0 = \mathrm{id}$ and $\varphi_0 \circ f \circ \varphi^{-1} = f$, if $z^\alpha R_\alpha = 0$. By Lemma 2.2.6, the operator $\mathscr{P}_{\varphi_0 \circ f \circ \varphi_0^{-1}}$ is a contraction on the space $(\mathrm{id} + \mathscr{L}_k^\beta, d_z)$, where $\beta := \mathrm{ord}_z(\varphi_0 \circ f \circ \varphi_0^{-1} - z^\alpha) - \alpha + 1$. By the Banach Fixed Point Theorem and Lemma 2.2.4, there exists the unique $\varphi_1 \in \mathrm{id} + \mathscr{L}_k^\beta$, such that:

$$\varphi_1 \circ (\varphi_0 \circ f \circ \varphi_0^{-1}) \circ \varphi_1^{-1} = z^\alpha. \tag{2.85}$$

Now, by putting $\varphi := \varphi_1 \circ \varphi_0$, we get $\varphi \in \mathscr{L}_k^0$ and $\varphi \circ f \circ \varphi^{-1} = z^\alpha$.

Uniqueness. Suppose that $\varphi, \psi \in \mathfrak{L}^0$ such that $\varphi \neq \psi$, $\psi \circ f \circ \psi^{-1} = z^\alpha$, and $\varphi \circ f \circ \varphi^{-1} = z^\alpha$. Then there exists $k_0 \in \mathbb{N}$, such that $f, \varphi, \psi \in \mathscr{L}_{k_0}^0$. Let $T \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_{k_0}$ be such that $\psi = \mathrm{id} + zT + \text{h.o.b.}(z)$. Put $\psi_0 := \mathrm{id} + zT$ and $\psi_1 := \psi \circ \psi_0^{-1}$. Note that $\mathrm{ord}_z(\psi_1 - \mathrm{id}) > 1$ and $\psi_1 \circ \psi_0 = \psi$. Let us decompose $\varphi := \mathrm{id} + zH + \text{h.o.b.}(z)$, for $H \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$. Similarly, let $\varphi_0 := \mathrm{id} + zH$ and $\varphi_1 := \varphi \circ \varphi_0^{-1}$. Since

$$z^\alpha = \psi \circ f \circ \psi^{-1} = \psi_1 \circ \psi_0 \circ f \circ \psi_0^{-1} \circ \psi_1^{-1}$$

and $\mathrm{ord}_z(\psi_1 - \mathrm{id}) > 1$, it follows that:

$$\psi_0 \circ f \circ \psi_0^{-1} = \psi_1^{-1} \circ z^\alpha \circ \psi_1 = z^\alpha + \text{h.o.b.}(z).$$

Similarly, for $\varphi = \varphi_1 \circ \varphi_0$ we get:

$$\varphi_0 \circ f \circ \varphi_0^{-1} = \varphi_1^{-1} \circ z^\alpha \circ \varphi_1 = z^\alpha + \text{h.o.b.}(z).$$

By Proposition 2.2.13 (the uniqueness of the prenormalization in the *canonical form*) we get that $\psi_0 = \varphi_0$. Put $\beta := \mathrm{ord}_z(\varphi_0 \circ f \circ \varphi_0^{-1} - \mathrm{id})$. By Lemma 2.2.5, it follows that $\mathrm{ord}_z(\psi_1 - \mathrm{id}) \geq \beta$, so $\psi_1 \in \mathrm{id} + \mathscr{L}_{k_0}^\beta$ and $\mathscr{P}_{\varphi_0 \circ f \circ \varphi_0^{-1}}(\psi_1) = \psi_1$. Similarly, we get $\varphi_1 \in \mathrm{id} + \mathscr{L}_{k_0}^\beta$ and $\mathscr{P}_{\varphi_0 \circ f \circ \varphi_0^{-1}}(\varphi_1) = \varphi_1$. Since $\mathscr{P}_{\varphi_0 \circ f \circ \varphi_0^{-1}}$ is a contraction on the space $\mathrm{id} + \mathscr{L}_{k_0}^\beta$ we conclude that $\psi_1 = \varphi_1$. Hence, $\psi = \varphi$. ■

2.2.3. Proof of convergence of the Böttcher sequence

In this subsection we prove statement 2 of Theorem B. That is, we prove that, for a strongly hyperbolic transseries $f \in \mathfrak{L}^H$, $f = z^\alpha + \text{h.o.t.}$, for $\alpha > 1$, the Böttcher sequence $(\mathscr{P}_f^{\circ n}(h))_n$ defined in (2.58) converges in the weak topology on $\mathfrak{L}^0$ to the normalization φ obtained in the previous subsection, for any initial condition $h \in \mathfrak{L}^0$. Moreover, the convergence is in the finer power-metric topology on $\mathfrak{L}^0$ if and only if $\mathrm{Lb}_z(h) = \mathrm{Lb}_z(\varphi)$. This is proved at the end of the subsection using Proposition 2.2.15 and Lemma 2.2.17 below.

The next proposition is a generalization of Proposition 2.1.22 and is needed in the proof of Proposition 2.2.15.

Proposition 2.2.14. Let (ψ_n), (φ_n) be sequences in $\mathscr{L}_k^0$, $k \in \mathbb{N}$, which converge in the weak topology to ψ, $\varphi \in \mathscr{L}_k^0$ respectively, as $n \to +\infty$. Suppose that there exists a well-ordered subset $W \subseteq \mathbb{R}_{>0} \times \mathbb{Z}^k$, such that $\mathrm{Supp}(\psi_n), \mathrm{Supp}(\varphi_n) \subseteq W$, $n \in \mathbb{N}$. Then the sequence $(\psi_n \circ \varphi_n)$ converges to $\psi \circ \varphi$ in the weak topology on $\mathscr{L}_k^0$. In particular, the sequences, $(\psi_n \circ \varphi)$ and $(\psi \circ \varphi_n)$ converge to $\psi \circ \varphi$.

Proof. Before the proof, we give a general estimate of the support of a composition of two logarithmic transseries whose supports belong to a well-ordered set W. Let $S \subseteq \mathbb{R}_{>0} \times \mathbb{Z}^k$ be the semigroup generated with $(0,1,0,\dots,0)_{k+1} \dots, (0,0,\dots,0,1)_{k+1}$ and let $W \subseteq \mathbb{R}_{>0} \times \mathbb{Z}^k$ be a well-ordered subset. Let $g, h \in \mathscr{L}_k^0$ be such that $\mathrm{Supp}(g), \mathrm{Supp}(h) \subseteq W$ and let $g_1 := g - \mathrm{id}$, $h_1 := h - \mathrm{id}$. By the definition of the composition, it follows that:

$$g \circ h = h + g_1 + \sum_{i \geq 1} \frac{g_1^{(i)}}{i!} h_1^i.$$

Hence, every element of $\mathrm{Supp}(g \circ h)$ is an element of $\mathrm{Supp}(h) \cup \mathrm{Supp}(g_1)$ or can be written in the form:

$$(\beta, \mathbf{n}) + (0, \mathbf{m}) + (\beta_1 - 1, \mathbf{n}_1) + \cdots + (\beta_i - 1, \mathbf{n}_i),$$

where $(\beta, \mathbf{n}) \in \mathrm{Supp}(g_1)$, $(0, \mathbf{m}) \in S$, $(\beta_1, \mathbf{n}_1), \dots, (\beta_i, \mathbf{n}_i) \in \mathrm{Supp}(h_1)$ and $i \in \mathbb{N}_{\geq 1}$.

Let W_1 be the semigroup generated by $(\beta - 1, \mathbf{n})$, for $(\beta, \mathbf{n}) \in W$ such that $(\beta, \mathbf{n}) > (1, \mathbf{0}_k)$, and let $\overline{W}$ be the semigroup generated by $W \cup W_1 \cup S$. By the Neumann Lemma

(Theorem 1.1.2), it follows that W_1 and $\overline{W}$ are well-ordered sets. By the above analysis, it follows that $\operatorname{Supp}(g \circ h) \subseteq \overline{W}$.

Now, for every $f \in \mathscr{L}_k$ such that $\operatorname{Supp}(f) \subseteq \overline{W}$ and every $w \in \overline{W}$ we define $[f]_w$ to be: the coefficient of f if $w \in \operatorname{Supp}(f)$, and 0 otherwise. By the Neumann Lemma (Theorem 1.1.2), for every $w \in \overline{W}$ there exist finitely many tuples with elements in $W \cup W_1 \cup S$ whose sum eguals to w. Hence, there exist $m, r \in \mathbb{N}$ and $w_1, \ldots, w_m, t_1, \ldots, t_r \in \overline{W}$, such that $[g \circ h]_w$ depends only on $[g]_{w_1}, \ldots, [g]_{w_m}, [h]_{t_1}, \ldots, [h]_{t_r}$. That is, there exists a polynomial P_w in $m + r$ variables, with real coefficients, which does not depend on g or h, such that

$$[g \circ h]_w = P_w\big([g]_{w_1}, \ldots, [g]_{w_m}, [h]_{t_1}, \ldots, [h]_{t_r}\big).$$

Now we prove the proposition. Recall that $\operatorname{Supp}(\psi_n), \operatorname{Supp}(\varphi_n) \subseteq W$, for every $n \in \mathbb{N}$. Since $(\psi_n) \to \psi$ and $(\varphi_n) \to \varphi$ in the weak topology, it follows that $\operatorname{Supp}(\psi), \operatorname{Supp}(\varphi) \subseteq W$. From the consideration above, we conclude that $\operatorname{Supp}(\psi \circ \varphi) \subseteq \overline{W}$ and $\operatorname{Supp}(\psi_n \circ \varphi_n) \subseteq \overline{W}$, for every $n \in \mathbb{N}$. Since the polynomial P_w, $w \in \overline{W}$, defined above, depends only on $w_1, \ldots, w_m, t_1, \ldots, t_r \in \overline{W}$ above, we get:

$$[\psi_n \circ \varphi_n]_w = P_w\big([\psi_n]_{w_1}, \ldots, [\psi_n]_{w_m}, [\varphi_n]_{t_1}, \ldots, [\varphi_n]_{t_r}\big)$$

and

$$[\psi \circ \varphi]_w = P_w\big([\psi]_{w_1}, \ldots, [\psi]_{w_m}, [\varphi]_{t_1}, \ldots, [\varphi]_{t_r}\big).$$

Hence, by the continuity of the polynomial map P_w, it follows that:

$$\begin{aligned}
\lim_{n \to +\infty} [\psi_n \circ \varphi_n]_w &= \lim_{n \to +\infty} P_w([\psi_n]_{w_1}, \ldots, [\psi_n]_{w_m}, [\varphi_n]_{t_1}, \ldots, [\varphi_n]_{t_r}) \\
&= P_w([\psi]_{w_1}, \ldots, [\psi]_{w_m}, [\varphi]_{t_1}, \ldots, [\varphi]_{t_r}) \\
&= [\psi \circ \varphi]_w, \quad w \in \overline{W}.
\end{aligned}$$

This proves the convergence of the sequence $(\psi_n \circ \varphi_n)_n$ to $\psi \circ \varphi$ in the weak topology on the space $\mathscr{L}_k^0$. ■

Let $f \in \mathscr{L}_k$, $k \in \mathbb{N}$, be a strongly hyperbolic logarithmic transseries. Although the Böttcher operator $\mathscr{R}_f$ on $\mathrm{id} + z\mathscr{B}^+_{\geq 1}$, defined in (2.67), is not a contraction in any of introduced metrics (Example 2.2.9), we prove in the next proposition that the Böttcher sequence $(\mathscr{R}_f^{\circ n}(\mathrm{id}+h))_n$ on $\mathrm{id}+z\mathscr{B}_{\geq 1}^{+}$ converges to the *canonical* prenormalization φ_0 of strongly hyperbolic logarithmic transseries f in the weak topology, for any initial condition $\mathrm{id}+h \in \mathrm{id}+z\mathscr{B}_{\geq 1}^{+}$. Proposition 2.2.15 is needed in the proof of statement 2 of Theorem B.

Proposition 2.2.15 (Convergence of the Böttcher sequence on $\mathrm{id}+z\mathscr{B}_{\geq 1}^{+}$)**.** Let $f \in \mathscr{L}_k^H$, $k \in \mathbb{N}$, $f = z^\alpha + z^\alpha R_\alpha + \text{h.o.b.}(z)$, $\alpha \in \mathbb{R}_{>1}$, $R_\alpha \in \mathscr{B}_{\geq 1}^{+} \subseteq \mathscr{L}_k$, and let $h = \mathrm{id}+zH$, $H \in \mathscr{B}_{\geq 1}^{+} \subseteq \mathscr{L}_k$. Then the sequence $(\mathscr{R}_f^{\circ n}(\mathrm{id}+zH))_n$ converges to the *canonical* prenormalization $\varphi_0 = \mathrm{id}+zS$, $S \in \mathscr{B}_{\geq 1}^{+} \subseteq \mathscr{L}_k$, in the weak topology on the space $\mathrm{id}+z\mathscr{B}_{\geq 1}^{+} \subseteq \mathscr{L}_k^0$, where $\mathscr{R}_f$ is the Böttcher operator on $\mathrm{id}+z\mathscr{B}_{\geq 1}^{+}$, defined in (2.67).

Proof. Note that:

$$\begin{aligned}\mathscr{R}_f^{\circ n}(h)\circ\varphi_0^{-1} &= z^{\frac{1}{\alpha^n}}\circ(h\circ\varphi_0^{-1})\circ\left(\varphi_0\circ(z^\alpha+z^\alpha R_\alpha)\circ\varphi_0^{-1}\right)^{\circ n}\\ &= z^{\frac{1}{\alpha^n}}\circ(h\circ\varphi_0^{-1})\circ z^{\alpha^n},\end{aligned}$$

for $n \in \mathbb{N}$. By Proposition 2.2.14, it follows that $(\mathscr{R}_f^{\circ n}(\mathrm{id}+zH))_n$ converges to φ_0 if $(z^{\frac{1}{\alpha^n}}\circ(h\circ\varphi_0^{-1})\circ z^{\alpha^n})_n$ converges to id in the weak topology on the space $\mathrm{id}+z\mathscr{B}_{\geq 1}^{+} \subseteq \mathscr{L}_k^0$. Now suppose that $h\circ\varphi_0^{-1} = \mathrm{id}+zK$, $K \in \mathscr{B}_{\geq 1}^{+} \subseteq \mathscr{L}_k$. We prove that $(z^{\frac{1}{\alpha^n}}\circ(\mathrm{id}+zK)\circ z^{\alpha^n})_n$ converges to id in the weak topology on $\mathrm{id}+z\mathscr{B}_{\geq 1}^{+}$. Note that:

$$\begin{aligned}z^{\frac{1}{\alpha^n}}\circ(\mathrm{id}+zK)\circ z^{\alpha^n} &= \left(z^{\alpha^n}+z^{\alpha^n}K(z^{\alpha^n})\right)^{\frac{1}{\alpha^n}}\\ &= \mathrm{id}+z\sum_{i\geq 1}\binom{\frac{1}{\alpha^n}}{i}\left(K(z^{\alpha^n})\right)^i.\end{aligned}\tag{2.86}$$

Note that:

$$\begin{aligned}\binom{\frac{1}{\alpha^n}}{i} &= \frac{\frac{1}{\alpha^n}\cdot(\frac{1}{\alpha^n}-1)\cdots(\frac{1}{\alpha^n}-(i-1))}{i!}\\ &= \frac{1}{i\alpha^n}\cdot\frac{\frac{1}{\alpha^n}-1}{1}\cdots\frac{\frac{1}{\alpha^n}-(i-1)}{i-1},\end{aligned}\tag{2.87}$$

for $i \in \mathbb{N}_{\geq 1}$ and $n \in \mathbb{N}$. Since, $0 < \frac{j-\frac{1}{\alpha^n}}{j} < 1$, $j \in \{1,\ldots,i-1\}$, it follows that:

$$\left|\binom{\frac{1}{\alpha^n}}{i}\right| \leq \frac{1}{\alpha^n}.\tag{2.88}$$

for $i \in \mathbb{N}_{\geq 1}$ and $n \in \mathbb{N}$. By Lemma A.3.2, it follows that every coefficient of $\boldsymbol{\ell}_m(z^{\alpha^n})$, $m \in \{1, \ldots, k\}$, is a polynomial in the variables $\frac{1}{\alpha^n}$ and $n \log \alpha$, for $n \in \mathbb{N}$. Let $\mathbf{n} \in \mathbb{Z}^k$ be such that $\mathbf{n} > \mathbf{0}_k$. Thus, for $n \in \mathbb{N}$, it follows that $\left[(K(z^{\alpha^n}))^i\right]_{(1,\mathbf{n})}$, $i \in \mathbb{N}_{\geq 1}$, is a polynomial $P_i(\frac{1}{\alpha^n}, n \log \alpha)$ in the variables $\frac{1}{\alpha^n}$ and $n \log \alpha$ that does not depend on n. By the Neumann Lemma (Theorem 1.1.2) and (2.86), there exist $m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \mathbb{N}_{\geq 1}$, such that:

$$\left[((\mathrm{id} + zK)(z^{\alpha^n}))^{\frac{1}{\alpha^n}}\right]_{1,\mathbf{n}} = \binom{\frac{1}{\alpha^n}}{i_1} P_{i_1}\left(\frac{1}{\alpha^n}, n \log \alpha\right) + \cdots + \binom{\frac{1}{\alpha^n}}{i_m} P_{i_m}\left(\frac{1}{\alpha^n}, n \log \alpha\right), \tag{2.89}$$

for every $n \in \mathbb{N}$. From (2.88) and (2.89), it follows that:

$$\begin{aligned} \left| \left[((\mathrm{id} + zK)(z^{\alpha^n}))^{\frac{1}{\alpha^n}}\right]_{1,\mathbf{n}} \right| &\leq \left| \binom{\frac{1}{\alpha^n}}{i_1} \right| \left| P_{i_1}\left(\frac{1}{\alpha^n}, n \log \alpha\right) \right| + \cdots + \left| \binom{\frac{1}{\alpha^n}}{i_m} \right| \left| P_{i_m}\left(\frac{1}{\alpha^n}, n \log \alpha\right) \right| \\ &\leq \frac{1}{\alpha^n} \left(\sum_{j=1}^{m} \left| P_{i_j}\left(\frac{1}{\alpha^n}, n \log \alpha\right) \right| \right), \end{aligned} \tag{2.90}$$

for $n \in \mathbb{N}$. Now, passing to the limit as n goes to infinity, we get:

$$\left| \lim_{n \to +\infty} \left[((\mathrm{id} + zK)(z^{\alpha^n}))^{\frac{1}{\alpha^n}}\right]_{1,\mathbf{n}} \right| \leq \lim_{n \to +\infty} \frac{\sum_{j=1}^{m} \left| P_{i_j}(\frac{1}{\alpha^n}, n \log \alpha) \right|}{\alpha^n} = 0. \tag{2.91}$$

From (2.91) and $\left[((\mathrm{id} + zK)(z^{\alpha^n}))^{\frac{1}{\alpha^n}}\right]_{1,\mathbf{0}_k} = 1$, for each $n \in \mathbb{N}$, it follows that the sequence $(z^{\frac{1}{\alpha^n}} \circ (\mathrm{id} + zK) \circ z^{\alpha^n})_n$ converges to id in the weak topology on the space $\mathrm{id} + z\mathscr{B}_{\geq 1}^{+} \subseteq \mathscr{L}_k^0$. ■

Remark 2.2.16. By the proof of Proposition 2.2.15, it follows that the sequence $(z^{\frac{1}{\alpha^n}} \circ (\mathrm{id} + zH) \circ z^{\alpha^n})_n$ converges to id in the weak topology on the space $\mathrm{id} + z\mathscr{B}_{\geq 1}^{+}$, for each $H \in \mathscr{B}_{\geq 1}^{+}$.

The following lemma is an auxiliary lemma in the proof of statement 2 of Theorem B.

Lemma 2.2.17.

1. For every $f \in \mathfrak{L}^0$ there exist $f_1 \in \mathrm{id} + z\mathscr{B}_{\geq 1}^{+}$ and $f_2 \in \mathfrak{L}^0$ such that $\mathrm{ord}_z(f_2) > 1$ and $f = f_1 \circ f_2$.

2. Let $g, h \in \mathfrak{L}^0$. Then $\mathrm{Lb}_z(g \circ h^{-1}) = \mathrm{id}$, if and only if $\mathrm{Lb}_z(g) = \mathrm{Lb}_z(h)$.

3. Let $f \in \mathfrak{L}^0$ be a parabolic logarithmic transseries. If a sequence (φ_n) converges to φ in the space $(\mathscr{L}_k, d_z)$, $k \in \mathbb{N}$, then the sequence $(\varphi_n \circ f)$ converges to $\varphi \circ f$ in the space $(\mathscr{L}_m, d_z)$, where $m \in \mathbb{N}$ is minimal such that $m \geq k$ and $f \in \mathscr{L}_m^0$.

Proof. 1. Let $k \in \mathbb{N}$ be minimal such $f \in \mathscr{L}_k$ and let $f = \mathrm{id} + zK + g$, where $K \in \mathscr{B}_{\geq 1}^{+} \subseteq \mathscr{L}_k$ and $g \in \mathscr{L}_k$ such that $\mathrm{ord}_z(g) > 1$. Put $f_1 := \mathrm{id} + zK$ and $f_2 := f_1^{-1} \circ f$. Then $f_1 \in \mathrm{id} + z\mathscr{B}_{\geq 1}^{+}$ and, by the Taylor Theorem (Proposition A.1.6), it follows that:

$$f_2 = f_1^{-1} \circ f = f_1^{-1} \circ (f_1 + g) = \mathrm{id} + \sum_{i \geq 1} \frac{\left(f_1^{-1}\right)^{(i)}(f_1)}{i!} g^i.$$

Consequently, it follows that $f_2 \in \mathscr{L}_k^0$ such that $\mathrm{ord}_z(f_2 - \mathrm{id}) > 1$.

2. Let $k \in \mathbb{N}$ be minimal such that $g, h \in \mathscr{L}_k^0$. Suppose that $\mathrm{Lb}_z(g \circ h^{-1}) = \mathrm{id}$. As in statement 1, let $g = g_1 \circ g_2$ and $h = h_1 \circ h_2$, for $g_1, h_1 \in \mathrm{id} + z\mathscr{B}_{\geq 1}^{+} \subseteq \mathscr{L}_k^0$ and $g_1, g_2 \in \mathscr{L}_k^0$ such that $\mathrm{ord}_z(g_2 - \mathrm{id}), \mathrm{ord}_z(h_2 - \mathrm{id}) > 1$. Now, $\mathrm{Lb}_z(h) = h_1$, $\mathrm{Lb}_z(g) = g_1$ and $g \circ h^{-1} = g_1 \circ g_2 \circ h_2^{-1} \circ h_1^{-1}$. Since $\mathrm{ord}_z(g_2), \mathrm{ord}_z(h_2) > 1$, it follows that $\mathrm{ord}_z(g_2 \circ h_2^{-1} - \mathrm{id}) > 1$. This implies that $g_2 \circ h_2^{-1} \circ h_1^{-1} = h_1^{-1} + \mathrm{h.o.b.}(z)$ and, consequently, $g_1 \circ (g_2 \circ h_2^{-1} \circ h_1^{-1}) = g_1 \circ h_1^{-1} + \mathrm{h.o.b.}(z)$. Since $\mathrm{Lb}_z(g \circ h^{-1}) = \mathrm{id}$, it follows that $g_1 = h_1$, i.e., $\mathrm{Lb}_z(g) = \mathrm{Lb}_z(h)$.

Suppose the contrary, that is, $\mathrm{Lb}_z(g) = \mathrm{Lb}_z(h)$ and let g_1, g_2, h_1, h_2 be as above. Then $g_1 = h_1$, and therefore,

$$g \circ h^{-1} = g_1 \circ g_2 \circ h_2^{-1} \circ h_1^{-1} = g_1(h_1^{-1} + \mathrm{h.o.b.}(z)) = \mathrm{id} + \mathrm{h.o.b.}(z).$$

Hence, $\mathrm{Lb}_z(g \circ h^{-1}) = \mathrm{id}$.

3. Since the sequence (φ_n) converges to φ in $(\mathscr{L}_k, d_z)$, it follows that the same holds in the larger space $(\mathscr{L}_m, d_z)$. Now, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, $n \geq n_0$, we have

$$[\varphi_n]_{\alpha,\mathbf{n}} = [\varphi]_{\alpha,\mathbf{n}}, \tag{2.92}$$

for $(\alpha, \mathbf{n}) \in \mathbb{R}_{>0} \times \mathbb{Z}^m$ such that $\frac{1}{2^\alpha} < \varepsilon$. Let $f := \mathrm{id} + f_1$, where $f_1 \in \mathscr{L}_m$ such that $\mathrm{ord}(f_1) > (1, \mathbf{0}_m)$. By the definition of the composition, it follows that:

$$\varphi \circ f = \varphi + \sum_{i \geq 1} \frac{\varphi^{(i)}}{i!}(f_1)^i,$$

$$\varphi_n \circ f = \varphi_n + \sum_{i \geq 1} \frac{\varphi_n^{(i)}}{i!}(f_1)^i.$$

Let $\varphi_n := \sum_{\beta \in \mathrm{Supp}_z(\varphi_n)} z^\beta P_{n,\beta}$, for $P_{n,\beta} \in \mathscr{B}_1 \subseteq \mathscr{L}_m^\infty$ and $n \in \mathbb{N}$. Since

$$\mathrm{ord}_z\left(z^\alpha \cdot \frac{P_{n,\alpha}^{(i)}}{i!}(f_1)^i\right) = \mathrm{ord}_z(P_{n,\alpha}) + i \cdot (\mathrm{ord}_z(f_1) - 1) \geq \mathrm{ord}_z(P_{n,\alpha}),$$

for $\alpha \in \mathrm{Supp}_z(\varphi_n)$, it follows that the α-block of $\varphi_n \circ f$ depends only on the blocks of φ_n of the order lower than or equal to α, for $\alpha \in \mathrm{Supp}_z(\varphi_n)$, $n \in \mathbb{N}$. Similarly, we get that α-block of $\varphi \circ f$ depends only on the blocks of φ of the order lower than or equal to α, for $\alpha \in \mathrm{Supp}_z(\varphi)$. Now, the statement follows by (2.92). ■

Proof of statement 2 of Theorem B. Let $\varphi = \varphi_1 \circ \varphi_0$ be the unique solution of the normalization equation (2.56), where φ_0 is the solution of the prenormalization equation (2.66), and φ_1 is the fixed point of the Böttcher operator $\mathscr{P}_{\varphi_0 \circ f \circ \varphi_0^{-1}}$. Let $h \in \mathfrak{L}^0$ be arbitrary. Suppose that $m \in \mathbb{N}$ is minimal such that $f, h \in \mathscr{L}_m$. For $n \in \mathbb{N}$, it follows that:

$$z^{\frac{1}{\alpha^n}} \circ h \circ f^{\circ n} = z^{\frac{1}{\alpha^n}} \circ (h \circ \varphi_0^{-1}) \circ (\varphi_0 \circ f \circ \varphi_0^{-1})^{\circ n} \circ \varphi_0. \tag{2.93}$$

Put $h \circ \varphi_0^{-1} = \mathrm{id} + zK + g$, where $g \in \mathscr{L}_m$ such that $\mathrm{ord}_z(g) > 1$, and $K \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_m$. Put $h_1 := \mathrm{id} + zK$ and $h_2 := h_1^{-1} \circ h \circ \varphi_0^{-1}$. Note that $h \circ \varphi_0^{-1} = h_1 \circ h_2$, $\mathrm{ord}_z(h_2 - \mathrm{id}) > 1$, and $\mathrm{ord}_z(h_1 - \mathrm{id}) = 1$. By (2.93), it follows that:

$$\begin{aligned} z^{\frac{1}{\alpha^n}} \circ h \circ f^{\circ n} &= \left(z^{\frac{1}{\alpha^n}} \circ h_1 \circ z^{\alpha^n}\right) \circ \left(z^{\frac{1}{\alpha^n}} \circ h_2 \circ (\varphi_0 \circ f \circ \varphi_0^{-1})^{\circ n}\right) \circ \varphi_0 \\ &= \left(z^{\frac{1}{\alpha^n}} \circ h_1 \circ z^{\alpha^n}\right) \circ \mathscr{P}^{\circ n}_{\varphi_0 \circ f \circ \varphi_0^{-1}}(h_2) \circ \varphi_0. \end{aligned} \tag{2.94}$$

It is easy to see that the supports of $z^{\frac{1}{\alpha^n}} \circ h_1 \circ z^{\alpha^n}$ and $\mathscr{P}^{\circ n}_{\varphi_0 \circ f \circ \varphi_0^{-1}}(h_2)$, for $n \in \mathbb{N}$, are subsets of a common well-ordered set (see the proof in the next subsection). By Remark 2.2.16, Lemma 2.2.6 and Proposition 2.2.14, it follows that the Böttcher sequence $(z^{\frac{1}{\alpha^n}} \circ h \circ f^{\circ n})_n$ converges to $\varphi = \varphi_1 \circ \varphi_0$ in the weak topology on $\mathfrak{L}^0$.

Now, suppose that $\mathrm{Lb}_z(h) = \mathrm{Lb}_z(\varphi)$. Let $\varphi = \varphi_1 \circ \varphi_0$, where φ_0 is the *canonical* prenormalization, if $\mathrm{ord}_z(f - z^\alpha) = \alpha$, or $\varphi_0 := \mathrm{id}$, if $\mathrm{ord}_z(f - z^\alpha) > \alpha$, and φ_1 is the normalization of $\varphi_0 \circ f \circ \varphi_0^{-1}$. Now, since $\mathrm{ord}_z(\varphi_1) > 1$, it follows that $\mathrm{Lb}_z(\varphi) = \varphi_0$. Thus, $\mathrm{Lb}_z(h) = \varphi_0$. Therefore, $\mathrm{ord}_z(h \circ \varphi_0^{-1} - \mathrm{id}) > 1$. By (2.93), it follows that

$$z^{\frac{1}{\alpha^n}} \circ h \circ f^{\circ n} = \mathscr{P}^{\circ n}_{\varphi_0 \circ f \circ \varphi_0^{-1}}(h \circ \varphi_0^{-1}) \circ \varphi_0.$$

The Böttcher operator $\mathscr{P}_{\varphi_0\circ f\circ\varphi_0^{-1}}$ is a contraction on the space $(\mathscr{L}_m^\beta, d_z)$, for each $\beta > 1$. Since $\mathrm{ord}_z(h\circ\varphi_0^{-1}-\mathrm{id}) > 1$, by the Banach Fixed Point Theorem, the sequence of the Picard iterations $(\mathscr{P}^{\circ n}_{\varphi_0\circ f\circ\varphi_0^{-1}}(h\circ\varphi_0^{-1}))_n$ converges to φ_1 in the power-metric topology on the space $\mathrm{id}+\mathscr{L}_m^\beta$. By Lemma 2.2.17, 3, it follows that $(z^{\frac{1}{\alpha^n}}\circ h\circ f^{\circ n})_n$ converges to $\varphi=\varphi_1\circ\varphi_0$ in the power-metric topology.

Conversely, suppose that $(z^{\frac{1}{\alpha^n}}\circ h\circ f^{\circ n})_n$ converges to the normalization $\varphi := \varphi_1\circ\varphi_0$ in the power-metric topology and that $\mathrm{Lb}_z(\varphi)\neq\mathrm{Lb}_z(h)$, where φ_0 is the *canonical* prenormalization and φ_1 is the normalization of the prenormalized transseries $\varphi_0\circ f\circ\varphi_0^{-1}$. By Lemma 2.2.17, 2, it follows that $\mathrm{Lb}_z(h\circ\varphi_0^{-1})\neq\mathrm{id}$. Let $\mathrm{Lb}_z(h\circ\varphi_0^{-1})=\mathrm{id}+zK$, for $K\in\mathscr{B}^+_{\geq 1}\subseteq\mathscr{L}_m$, $K\neq 0$. Put $h_1 := \mathrm{id}+zK$ and $h_2 := h_1^{-1}\circ h\circ\varphi_0^{-1}$. Consequently, it follows that $\mathrm{ord}_z(h_2-\mathrm{id})>1$ and $h\circ\varphi_0^{-1}=h_1\circ h_2$. Now, by (2.94) and Lemma 2.2.17, 3, it follows that $\left((z^{\frac{1}{\alpha^n}}\circ h_1\circ z^{\alpha^n})\circ\mathscr{P}^{\circ n}_{\varphi_0\circ f\circ\varphi_0^{-1}}(h_2)\right)_n$ converges to $\varphi_1=\varphi\circ\varphi_0^{-1}$ in the power-metric topology. Note that

$$(z^{\frac{1}{\alpha^n}}\circ h_1\circ z^{\alpha^n})\circ\mathscr{P}^{\circ n}_{\varphi_0\circ f\circ\varphi_0^{-1}}(h_2)=z^{\frac{1}{\alpha^n}}\circ h_1\circ z^{\alpha^n}+\text{h.o.b.}(z),$$

for every $n\in\mathbb{N}$. Since $\mathrm{ord}_z(\varphi_1)>1$, by the Taylor Theorem (Proposition A.1.6) and the convergence in the power-metric topology, it follows that there exists $n_0\in\mathbb{N}$ such that $z^{\frac{1}{\alpha^n}}\circ h_1\circ z^{\alpha^n}=\mathrm{id}$, for every $n\geq n_0$. By (2.86) and Lemma A.3.2, it follows that

$$\mathrm{Lt}\left(z^{\frac{1}{\alpha^n}}\circ h_1\circ z^{\alpha^n}-\mathrm{id}\right)=\left(\frac{1}{\alpha^n}\right)^{1+\mathrm{ord}_{\ell_1}(K)}\mathrm{Lt}(zK),$$

for each $n\in\mathbb{N}$. Therefore, $\mathrm{ord}_z(z^{\frac{1}{\alpha^n}}\circ h_1\circ z^{\alpha^n}-\mathrm{id})=1$, for each $n\in\mathbb{N}$. Since $z^{\frac{1}{\alpha^n}}\circ h_1\circ z^{\alpha^n}=\mathrm{id}$, for each $n\geq n_0$, it follows that $K=0$, i.e., $\mathrm{Lb}_z(\varphi)=\mathrm{Lb}_z(h)$. ■

2.2.4. Control of the support of the normalization

Proof of statement 3 of Theorem B. Let $f\in\mathscr{L}_k^H$, $k\in\mathbb{N}$, be such that $f=z^\alpha+\text{h.o.t.}$, for $\alpha\in\mathbb{R}_{>1}$. Let W be the set of elements $(\alpha^p,\mathbf{0}_k)$, $p\in\mathbb{N}$, $(0,1,0,\ldots,0)_{k+1}$,...,$(0,0,\ldots,0,1)_{k+1}$, and

$$(\alpha^m(\gamma-\alpha),\mathbf{n}),\tag{2.95}$$

for $(\gamma,\mathbf{n})\in\mathrm{Supp}(f-z^\alpha)$, $m\in\mathbb{N}$. Since $\mathrm{Supp}(f-z^\alpha)$ is a well-ordered set and $\alpha>1$, it is easy to see that W is a well-ordered set. Now, the semigroup $\langle W\rangle$ generated by W is well-ordered by the Neumann Lemma (Theorem 1.1.2).

Notice that $\operatorname{Supp}(f) \subseteq \langle W \rangle$ and

$$(\alpha\delta, \mathbf{n}) \in \langle W \rangle, \tag{2.96}$$

for each $(\delta, \mathbf{n}) \in \langle W \rangle$.

Since, by statement 2 of Theorem B (taking $h := \mathrm{id}$), the sequence $(z^{\frac{1}{\alpha^n}} \circ f^{\circ n})_n$ converges to φ in the weak topology, it is sufficient to prove that $\operatorname{Supp}(z^{\frac{1}{\alpha^n}} \circ f^{\circ n}) \subseteq \langle W \rangle$, for each $n \in \mathbb{N}_{\geq 1}$.

We prove that $\operatorname{Supp}(f^{\circ n}) \subseteq \langle W \rangle$, for $n \in \mathbb{N}_{\geq 1}$. First, let $g \in \mathscr{L}_k$ be such that $\operatorname{Supp}(g) \subseteq \langle W \rangle$. By Lemma A.3.5, the support $\operatorname{Supp}(g \circ f)$ is contained in the sub-semigroup of $\mathbb{R}_{\geq 0} \times \mathbb{Z}^k$ generated by the elements:

$$\begin{aligned}
&(0,1,0,\dots,0)_{k+1}, \dots, (0,0,\dots,0,1)_{k+1},\\
&(\alpha\delta, \mathbf{m}),\ \text{for each } (\delta, \mathbf{m}) \in \operatorname{Supp}(g),\\
&(\beta - \alpha, \mathbf{n}),\ \text{for each } (\beta, \mathbf{n}) \in \operatorname{Supp}(f - z^{\alpha}).
\end{aligned} \tag{2.97}$$

Since $\operatorname{Supp}(g), \operatorname{Supp}(f) \subseteq \langle W \rangle$, by (2.95), (2.96) and (2.97), it follows that $\operatorname{Supp}(g \circ f) \subseteq \langle W \rangle$. Since $\operatorname{Supp}(f) \subseteq \langle W \rangle$, it follows inductively that $\operatorname{Supp}(f^{\circ n}) \subseteq \langle W \rangle$, for $n \in \mathbb{N}_{\geq 1}$.

Now we prove that $\operatorname{Supp}(z^{\frac{1}{\alpha^n}} \circ f^{\circ n}) \subseteq \langle W \rangle$, for $n \in \mathbb{N}_{\geq 1}$. By (A.1) it follows that:

$$z^{\frac{1}{\alpha^n}} \circ f^{\circ n} = \mathrm{id} + z \sum_{i \geq 1} \binom{\frac{1}{\alpha^n}}{i} \left(\frac{f^{\circ n} - z^{\alpha^n}}{z^{\alpha^n}} \right)^i, \quad n \in \mathbb{N}_{\geq 1}. \tag{2.98}$$

Therefore, in order to prove that

$$\operatorname{Supp}(z^{\frac{1}{\alpha^n}} \circ f^{\circ n}) \subseteq \langle W \rangle, \tag{2.99}$$

for $n \in \mathbb{N}_{\geq 1}$, it is sufficient to prove that $(\gamma - \alpha^n, \mathbf{n}) \in \langle W \rangle$, for every $(\gamma, \mathbf{n}) \in \operatorname{Supp}(f^{\circ n} - z^{\alpha^n})$, $n \in \mathbb{N}_{\geq 1}$. By the proof of Lemma A.3.5 every element of the support of $f^{\circ n} = f^{\circ(n-1)} \circ f$, $n \in \mathbb{N}_{\geq 1}$, is of the form:

$$(\delta\alpha, \mathbf{v}) + (0, \mathbf{u}),$$

or

$$(\delta\alpha, \mathbf{v}) + (0, \mathbf{u}) + (\beta_1 - \alpha, \mathbf{n}_1) + \cdots + (\beta_i - \alpha, \mathbf{n}_i),$$

for $i \in \mathbb{N}_{\geq 1}$, $(\delta, \mathbf{v}) \in \mathrm{Supp}\,(f^{\circ(n-1)})$, $(0, \mathbf{u})$ is a linear combination (with coefficients in $\mathbb{N}_{\geq 1}$) of $(0,1,0,\ldots,0)_{k+1},\ldots,(0,0,\ldots,0,1)_{k+1}$, and $(\beta_j, \mathbf{n}_j) \in \mathrm{Supp}\,(f - z^{\alpha})$, $1 \leq j \leq i$. From this fact, (2.96) and $\mathrm{Supp}\,(f^{\circ(n-1)}) \subseteq \langle W \rangle$ statement (2.99) follows immediately.

■

2.3. Normal forms of parabolic logarithmic transseries

In this section we generalize the results obtained in [21, Theorem A] for parabolic logarithmic transseries of depth 1 to parabolic logarithmic transseries of an arbitrary depth, using fixed point theorems.

We consider the conjugation equation:

$$\varphi \circ f \circ \varphi^{-1} = g, \quad \varphi \in \mathfrak{L}^0, \tag{2.100}$$

for $f \in \mathfrak{L}^0$ and $g \in \mathfrak{L}$. If the conjugation equation is solvable in $\mathfrak{L}^0$, by Proposition 2.0.1, since $f \in \mathfrak{L}^0$, it follows that $g \in \mathfrak{L}^0$.

Suppose that $f \in \mathscr{L}_k^0$, $k \in \mathbb{N}$. As opposed to the hyperbolic or strongly hyperbolic case, where it is equivalent to solve the equation in $\mathscr{L}_k^0$ for $g \in \mathscr{L}_k^0$, or in the larger group $\mathfrak{L}^0$ for $g \in \mathfrak{L}^0$, in the parabolic case a solution depends on chosen $k \in \mathbb{N}$ such that $f \in \mathscr{L}_k^0$. In Theorem C we solve the equation (2.100) in the group $\mathscr{L}_k^0$ for arbitrary $k \in \mathbb{N}$ such that $f \in \mathscr{L}_k^0$. Furthermore, in Corollary 2.3.7 we solve the equation (2.100) in the larger group $\mathfrak{L}^0$.

The results in this section represent the results from the preprint in preparation [28].

2.3.1. Normalization theorem for parabolic logarithmic transseries

Before stating the normalization theorem for parabolic logarithmic transseries, we first define the *shift by* D_1 of the tuple of exponents $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k$, $k \in \mathbb{N}_{\geq 1}$, and a *residual term* (*monomial*, *coefficient*).

Definition 2.3.1 (Shift by D_1). Let $\mathbf{n} := (n_1, \ldots, n_k) \in \mathbb{Z}^k$, $k \in \mathbb{N}_{\geq 1}$. We define the *shift by* D_1 *of* $\mathbf{n}$ as $\mathbf{n}' \in \mathbb{Z}^k$ such that:

$$(0, \mathbf{n}') := \operatorname{ord}\left(D_1(\boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k})\right). \tag{2.101}$$

Remark 2.3.2. Let $\mathbf{n} := (n_1, \ldots, n_k) \in \mathbb{Z}^k$, $k \in \mathbb{N}_{\geq 1}$. By Lemma A.2.11 and the previous definition, it follows that $\mathbf{n}' = (\mathbf{1}_{m-1}, n_m + 1, n_{m+1}, \ldots, n_k)$ if $n_1 = \ldots = n_{m-1} = 0$ and $n_m \neq 0$, for $1 \leq m \leq k$.

Definition 2.3.3 (Residual term (resp. monomial, coefficient) of a parabolic logarithmic transseries)**.** Let $f \in \mathfrak{L}^0$ be a parabolic logarithmic transseries and $k \in \mathbb{N}$ minimal such that $f \in \mathscr{L}_k^0$. Let $(\alpha, n_1, \ldots, n_k) := \operatorname{ord}(f - \mathrm{id})$. The term (resp. monomial) in f of order $(2\alpha - 1, 2n_1 + 1, \ldots, 2n_k + 1)$ will be called the *residual term* (resp. *monomial*) *in* f and denoted by $\mathrm{Res}_t(f)$ ($\mathrm{Res}(f)$).

We call $[f]_{2\alpha-1, 2n_1+1, \ldots, 2n_k+1}$ the *residual coefficient of* f.

Example 2.3.4. Note that the residual term is defined with respect to the *ambient* space $\mathscr{L}_k$.

1. Let $\mathbf{n} := (1,2,3) \in \mathbb{Z}^3$. Then $\mathbf{n}' = (2,2,3)$.

   Let $\mathbf{n} := (0,0,-3,3,6) \in \mathbb{Z}^5$. Then $\mathbf{n}' = (1,1,-2,3,6)$.

2. Let $f_1 := \mathrm{id} + 2z^2\boldsymbol{\ell}_2^4 + z^2\boldsymbol{\ell}_1 + 5z^3\boldsymbol{\ell}_1\boldsymbol{\ell}_2^9 + z^5\boldsymbol{\ell}_1^{-5}$ and $f_2 := \mathrm{id} + 2z^2\boldsymbol{\ell}_2^4 + z^2\boldsymbol{\ell}_1 + 5z^3\boldsymbol{\ell}_1\boldsymbol{\ell}_2^9 + z^5\boldsymbol{\ell}_3^{-5}$ be parabolic transseries in $\mathfrak{L}^0$. Note that $f_1 \in \mathscr{L}_2^0$. Therefore, $\mathrm{Res}_t(f_1) = 5z^3\boldsymbol{\ell}_1\boldsymbol{\ell}_2^9$, where $\mathrm{Res}(f_1) = z^3\boldsymbol{\ell}_1\boldsymbol{\ell}_2^9$ is the residual monomial and 5 is the residual coefficient of f_1 in the *ambient space* $\mathscr{L}_2$.

   If we choose $\mathscr{L}_3$ for the ambient space, we get that $\mathrm{Res}(f) = z^3\boldsymbol{\ell}_1\boldsymbol{\ell}_2^9\boldsymbol{\ell}_3$ and residual coefficient equals to 0.

   Note that $f_2 \in \mathscr{L}_3^0$. It follows that $\mathrm{Res}_t(f_2) = 0$, where $\mathrm{Res}(f_2) = z^3\boldsymbol{\ell}_1\boldsymbol{\ell}_2^9\boldsymbol{\ell}_3$ is the residual monomial and 0 is the residual coefficient of f_2 in the ambient space $\mathscr{L}_3$.

Theorem C (Normalization theorem for parabolic logarithmic transseries)**.** Let $f \in \mathscr{L}_k^0$, $k \in \mathbb{N}$, be a parabolic logarithmic transseries. Let $(\beta, \mathbf{n}) := \operatorname{ord}(f - \mathrm{id}) > (1, \mathbf{0}_k)$ and let $\mathbf{n}'$ be the shift by D_1 of $\mathbf{n}$ as in (2.101). For

$$f = \mathrm{id} + \sum_{\mathbf{n} \le \mathbf{m} \le \mathbf{n}'} a_{\mathbf{m}} z^{\beta} \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_k^{m_k} + \text{h.o.t.},$$

where $\mathbf{m} = (m_1, \ldots, m_k)$, put:

$$L := \begin{cases} a_{\mathbf{n}} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}, & \text{if } \beta > 1, \\ \\ \displaystyle\sum_{\mathbf{n} \le \mathbf{m} \le \mathbf{n}'} a_{\mathbf{m}} \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_k^{m_k}, & \text{if } \beta = 1, \end{cases}$$

and let

$$f_c := \mathrm{id} + z^{\beta} L + c\mathrm{Res}(f), \tag{2.102}$$

for each $c \in \mathbb{R}$.

1. There exists $c \in \mathbb{R}$, such that the *conjugacy equation*:

$$\varphi \circ f \circ \varphi^{-1} = f_c \tag{2.103}$$

   has a solution $\varphi \in \mathscr{L}_k^0$.

2. The real number c is unique such that (2.103) has a solution in $\mathscr{L}_k^0$. Moreover, the *residual coefficient* c is explicitely given by:

$$c = \left[\frac{a_{\mathbf{n}}^2}{f - \mathrm{id}}\right]_{-1,\mathbf{1}_k} - \left[\frac{a_{\mathbf{n}}^2}{zL}\right]_{-1,\mathbf{1}_k} = \left[a_{\mathbf{n}}^2 \int \frac{dz}{f - \mathrm{id}}\right]_{\mathbf{0}_{k+1},-1} - \left[\frac{a_{\mathbf{n}}^2}{zL}\right]_{-1,\mathbf{1}_k}, \tag{2.104}$$

   if $\beta = 1$, and

$$c = \left[\frac{a_{\mathbf{n}}^2}{f - \mathrm{id}}\right]_{-1,\mathbf{1}_k} = \left[a_{\mathbf{n}}^2 \int \frac{dz}{f - \mathrm{id}}\right]_{\mathbf{0}_{k+1},-1}, \tag{2.105}$$

   if $\beta > 1$.

3. The *normal form* f_c is minimal[2] in $\mathscr{L}_k^0$.

Remark 2.3.5. Let f and c be as in Theorem C. Put:

$$X_c := \begin{cases} \dfrac{a_{\mathbf{n}} z^{\beta} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}}{1 + \frac{\beta a_{\mathbf{n}}}{2}\left(z^{\beta} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}\right)' - \frac{c}{a_{\mathbf{n}}} z^{\beta-1} \boldsymbol{\ell}_1^{n_1+1} \cdots \boldsymbol{\ell}_k^{n_k+1}} \dfrac{d}{dz}, & \text{if } \beta > 1, \\[2ex] \dfrac{z \log(1+L)}{1 - \frac{c\mathrm{Res}(f)}{z \log(1+L)} + \frac{\boldsymbol{\ell}_1^2 \frac{d}{d\boldsymbol{\ell}_1}(L)}{1+L}} \dfrac{d}{dz}, & \text{if } \beta = 1. \end{cases}$$

Then there exists a parabolic logarithmic transseries $\psi \in \mathscr{L}_k^0$ that reduces f to a normal form given as the time-one map of the vector field X_c. That is, such that $\psi \circ f \circ \psi^{-1} = \exp(X_c).\mathrm{id}$. This represents a generalization of [21, Theorem A] for parabolic logarithmic transseries of depth 1.

[2] In the sense it cannot be further reduced by changes of variables from $\mathscr{L}_k^0$. Moreover, the coefficients of f_c cannot be changed by changes of variables from $\mathscr{L}_k^0$.

The proof of Remark 2.3.5 is in Subsection 2.3.8.

Remark 2.3.6.

1. Note that in Theorem C we do not claim the uniqueness of a normalization. The non-uniqueness of normalization follows from the fact that ψ from Remark 2.3.5 is unique only up to the precomposition with a time-t map of the vector field X_c (for more details see [21]).

2. The quadruple $(\beta, \mathbf{n}, a_{\mathbf{n}}, c)$ determines the conjugacy class of the logarithmic transseries f in the case $\beta > 1$.

   On the other hand, in the case $\beta = 1$, the conjugacy class of f is determined by the (possibly infinite) sequence $(1, \mathbf{n}, (a_{\mathbf{m}})_{\mathbf{n} \leq \mathbf{m} \leq \mathbf{n}'}, c)$.

3. Note that (2.104) is a generalization of the residual formula obtained in [23, Proposition 9.3] for the class $\mathscr{L}_1$.

4. Note that we find normal form in Theorem C for parabolic logarithmic transseries f in every space $\mathscr{L}_k$ such that $f \in \mathscr{L}_k$. In Corollary 2.3.7 we prove that there are only two different normal forms. One is obtained for minimal $k \in \mathbb{N}$ such that $f \in \mathscr{L}_k$, and the other for every $m \geq k+1$. The last one coincides with the normal form in the larger space $\mathfrak{L}$.

Corollary 2.3.7 (The normal form in the larger differential algebra $\mathfrak{L}$)**.** Let $f \in \mathfrak{L}^0$ and let $k \in \mathbb{N}$ be minimal such that $f \in \mathscr{L}_k^0$. By a parabolic change of variables φ in the larger differential algebra $\mathfrak{L}^0$, a parabolic logarithmic transseries $f \in \mathscr{L}_k^0$, $k \in \mathbb{N}$, can be further reduced to

$$f_0 := \mathrm{id} + zL,$$

for L as defined in Theorem C.

Moreover, normalization φ belongs to $\mathscr{L}_{k+1}^0$ and f_0 is minimal[3] in $\mathfrak{L}^0$.

The proof of Corollary 2.3.7 is in Subsection 2.3.8.

Example 2.3.8.

[3] In the sense that it cannot be further reduced in $\mathfrak{L}^0$, nor can the coefficients be changed.

1. Let $f = z + a_1 z\boldsymbol{\ell}_1 + a_2 z\boldsymbol{\ell}_1^2 + a_3 z\boldsymbol{\ell}_1^3 + \text{h.o.t.} \in \mathscr{L}_1$, for $a_1, a_2, a_3 \in \mathbb{R} \setminus \{0\}$. Note that $L = a_1\boldsymbol{\ell}_1 + a_2\boldsymbol{\ell}_1^2$ and $\operatorname{Res}(f) = z\boldsymbol{\ell}_1^3$. By (2.104), it follows that
$$c := \left[\frac{a_1^2}{f - \text{id}}\right]_{-1,1} - \left[\frac{a_1^2}{zL}\right]_{-1,1} = -a_3 + \frac{a_2^2}{a_1} - \frac{a_2^2}{a_1} = -a_3.$$
Therefore, by Theorem C, it follows that
$$f_c = \text{id} + a_1 z\boldsymbol{\ell}_1 + a_2 z\boldsymbol{\ell}_1^2 + \left(-a_3 + \frac{a_2^2}{a_1} - \left(\frac{a_2}{a_1}\right)^2\right) z\boldsymbol{\ell}_1^3$$
is the normal form in $\mathscr{L}_1$ of the logarithmic transseries f and the normalization φ belongs to $\mathscr{L}_1^0$.

   Furthermore, by Corollary 2.3.7, $f_0 = \text{id} + a_1 z\boldsymbol{\ell}_1 + a_2 z\boldsymbol{\ell}_1^2$ is the normal form of f in the larger space $\mathfrak{L}$.

2. Let $f = z + a_1 z^2\boldsymbol{\ell}_2 + a_2 z^2\boldsymbol{\ell}_1 + a_3 z^3\boldsymbol{\ell}_1\boldsymbol{\ell}_2^2 + z^4 + \text{h.o.t.} \in \mathscr{L}_2$, for $a_1, a_2, a_3 \in \mathbb{R} \setminus \{0\}$. Note that $L = a_1\boldsymbol{\ell}_2$ and $\operatorname{Res}(f) = z^3\boldsymbol{\ell}_1\boldsymbol{\ell}_2^3$. By (2.105), it follows that $c = \left[\frac{a_1^2}{f - \text{id}}\right]_{-1,\mathbf{1}_2} = 0$. Therefore, by Theorem C, it follows that $f_c = \text{id} + a_1 z^2\boldsymbol{\ell}_2$ is the normal form in $\mathscr{L}_2$ of the logarithmic transseries f and the normalization φ belongs to $\mathscr{L}_2^0$.

   Furthermore, by Corollary 2.3.7, $f_0 = \text{id} + a_1 z^2\boldsymbol{\ell}_2$ is the normal form of f in the larger space $\mathfrak{L}$. Note that $f_c = f_0$, since $c = 0$. In this case normal forms of f in $\mathscr{L}_2$ and $\mathfrak{L}$ coincide.

2.3.2. Transforming the conjugacy equation to a fixed point equation

The main strategy of the proof of Theorem C is to transform the conjugacy equation (2.100) to a fixed point equation using operators $\mathscr{S}_f, \mathscr{T}_f : \mathscr{L}_k \to \mathscr{L}_k$, $k \in \mathbb{N}$, in order to apply the fixed point theorem stated in Proposition 1.2.12.

Lemma 2.3.9 (Transformation of the conjugacy equation to the fixed point equation)**.** Let $f, g \in \mathscr{L}_k^0$, $k \in \mathbb{N}$, $f = \text{id} + z^\beta R_\beta + \text{h.o.b.}(z)$ and $g = \text{id} + z^\alpha S_\alpha + \text{h.o.b.}(z)$, for $\alpha, \beta \geq 1$, and $S_\alpha, R_\beta \in \mathscr{B}_1 \subseteq \mathscr{L}_k^\infty$. A logarithmic transseries $\varphi \in \mathscr{L}_k^0$, such that $\varphi = \text{id} + \varepsilon$ and $\operatorname{ord}(\varepsilon) > (1, \mathbf{0}_k)$, satisfies the *conjugacy equation*
$$\varphi \circ f \circ \varphi^{-1} = g \tag{2.106}$$

if and only if the logarithmic transseries $\varepsilon \in \mathscr{L}_k$ satisfies the *fixed point equation*:

$$\mathscr{S}_f(\varepsilon) = \mathscr{T}_f(\varepsilon), \tag{2.107}$$

where the operators $\mathscr{S}_f, \mathscr{T}_f : \mathscr{L}_k \to \mathscr{L}_k$ are defined as:

$$\mathscr{S}_f(\varepsilon) := \psi - \mu - \varepsilon' \cdot \mu_1 + \psi_1' \cdot \varepsilon - \sum_{i \geq 2} \frac{\varepsilon^{(i)}}{i!} \mu^i + \sum_{i \geq 2} \frac{\psi^{(i)}}{i!} \varepsilon^i, \tag{2.108}$$

and

$$\mathscr{T}_f(\varepsilon) := \varepsilon' \cdot z^\beta R_\beta - (z^\alpha S_\alpha)' \cdot \varepsilon, \tag{2.109}$$

for $\varepsilon \in \mathscr{L}_k$. Here,

$$\begin{aligned} &\mu := f - \mathrm{id}, \quad \mu_1 := f - \mathrm{id} - z^\beta R_\beta, \\ &\psi := g - \mathrm{id}, \quad \psi_1 := g - \mathrm{id} - z^\alpha S_\alpha. \end{aligned} \tag{2.110}$$

Proof. By composing the conjugacy equation (2.106) from the right with φ, we get the equivalent equation:

$$\varphi \circ f - g \circ \varphi = 0. \tag{2.111}$$

Recall the notation from (2.110). From equation (2.111), using the Taylor Theorem (Proposition A.1.6), we get that:

$$\begin{aligned} 0 &= \varphi \circ f - g \circ \varphi \\ &= (z + \varepsilon) \circ (z + \mu) - (z + \psi) \circ (z + \varepsilon) \\ &= z + \mu + \varepsilon + \sum_{i \geq 1} \frac{\varepsilon^{(i)}}{i!} \mu^i - (z + \varepsilon) - \psi - \sum_{i \geq 1} \frac{\psi^{(i)}}{i!} \varepsilon^i \\ &= \sum_{i \geq 1} \frac{\varepsilon^{(i)}}{i!} \mu^i + \mu - \psi - \sum_{i \geq 1} \frac{\psi^{(i)}}{i!} \varepsilon^i \\ &= \varepsilon' \cdot \mu - \psi' \cdot \varepsilon + \sum_{i \geq 2} \frac{\varepsilon^{(i)}}{i!} \mu^i + \mu - \psi - \sum_{i \geq 2} \frac{\psi^{(i)}}{i!} \varepsilon^i \\ &= \varepsilon' \cdot z^\beta R_\beta - (z^\alpha S_\alpha)' \cdot \varepsilon + \mu - \psi + \varepsilon' \cdot \mu_1 - \psi_1' \cdot \varepsilon + \sum_{i \geq 2} \frac{\varepsilon^{(i)}}{i!} \mu^i - \sum_{i \geq 2} \frac{\psi^{(i)}}{i!} \varepsilon^i. \end{aligned} \tag{2.112}$$

Equation (2.112) is equivalent to the following equation:

$$\varepsilon' \cdot z^\beta R_\beta - (z^\alpha S_\alpha)' \cdot \varepsilon = \psi - \mu - \varepsilon' \cdot \mu_1 + \psi_1' \cdot \varepsilon - \sum_{i \geq 2} \frac{\varepsilon^{(i)}}{i!} \mu^i + \sum_{i \geq 2} \frac{\psi^{(i)}}{i!} \varepsilon^i, \tag{2.113}$$

i.e., $\mathscr{T}_f(\varepsilon) = \mathscr{S}_f(\varepsilon)$. ∎

Remark 2.3.10. In Lemma 2.3.9 we transformed the conjugacy equation (2.100) to the fixed point equation $\mathscr{T}_f(\varepsilon) = \mathscr{S}_f(\varepsilon)$. Note that $\mathscr{T}_f(\varepsilon)$ is a superlinear operator which is an analogue to the Lie bracket operator from Definition 1.1.11.

In the next proposition, among other properties of the operator $\mathscr{S}_f$, for $f \in \mathscr{L}_k^0$, $k \in \mathbb{N}$, we prove that $\mathscr{S}_f$ is Lipschitz on the space $\mathscr{L}_k^\delta$, where the Lipschitz coefficient of $\mathscr{S}_f$ depends on $\delta \geq 1$.

Proposition 2.3.11 (Properties of the operator $\mathscr{S}_f$)**.** Suppose that $f, g \in \mathscr{L}_k^0$, $k \in \mathbb{N}$, $f = \mathrm{id} + z^\beta R_\beta + \text{h.o.b.}(z)$ and $g = \mathrm{id} + z^\alpha S_\alpha + \text{h.o.b.}(z)$, for $\alpha, \beta \geq 1$, $S_\alpha, R_\beta \in \mathscr{B}_1 \subseteq \mathscr{L}_k^\infty$, and (2.110). Let $\delta \geq 1$ and put:

$$\begin{aligned} \gamma &:= \min\left\{\mathrm{ord}_z\left(f - \mathrm{id} - z^\beta R_\beta\right), \mathrm{ord}_z\left(g - \mathrm{id} - z^\alpha S_\alpha\right)\right\}, \\ \rho(\delta) &:= \min\left\{\gamma - 1, 2(\beta - 1), \delta + \alpha - 2\right\}. \end{aligned} \tag{2.114}$$

Let $\mathscr{S}_f : \mathscr{L}_k^1 \to \mathscr{L}_k^1$ be the operator defined as in (2.108).

1. The operator $\mathscr{S}_f$ is $\frac{1}{2^{\rho(\delta)}}$-Lipschitz[4] on the space $\mathscr{L}_k^\delta$, for every $\delta \geq 1$.

2. If $z^\alpha S_\alpha = z^\beta R_\beta$ and $\delta = \gamma - \beta + 1$, then $\mathscr{S}_f(\mathscr{L}_k^\delta) \subseteq \mathscr{L}_k^\gamma$ and $\rho(\delta) = \min\{\gamma - 1, 2(\beta - 1)\}$.

Proof. 1. Let $\delta \geq 1$ and $\varepsilon \in \mathscr{L}_k^\delta$. By (2.110) and (2.114), it follows that $\mathrm{ord}_z(\mu_1), \mathrm{ord}_z(\psi_1) \geq \gamma$. For the linear parts of $\mathscr{S}$, from (2.110) and (2.114), we have the following bounds:

$$\begin{aligned} \mathrm{ord}_z\left(-\varepsilon' \mu_1 + \psi_1' \varepsilon\right) &= \mathrm{ord}_z(\varepsilon) + \gamma - 1, \\ \mathrm{ord}_z\left(-\sum_{i \geq 2} \frac{\varepsilon^{(i)}}{i!} \mu^i\right) &= \mathrm{ord}_z(\varepsilon) + 2(\beta - 1). \end{aligned} \tag{2.115}$$

For the nonlinear part of $\mathscr{S}_f$, we use Example 1.2.8.

Now, by (2.108), Example 1.2.8, (2.114), and (2.115) we conclude that $\mathscr{S}_f$ is $\frac{1}{2^{\rho(\delta)}}$-Lipschitz and $\frac{1}{2^{\rho(\delta)}}$ is the minimal Lipschitz coefficient.

2. From $z^\alpha S_\alpha = z^\beta R_\beta$ and $\delta = \gamma - (\beta - 1)$, we conclude that $\delta + \alpha - 2 = \gamma - 1$ and that, consequently,

$$\rho(\delta) = \min\{\gamma - 1, 2(\beta - 1)\}.$$

Note that $\rho(\delta) + \delta \geq \gamma$. This and $\mathrm{ord}_z(\psi - \mu) \geq \gamma$ implies that $\mathscr{S}_f(\mathscr{L}_k^\delta) \subseteq \mathscr{L}_k^\gamma$. ■

[4] In the sense that $\mathscr{S}_f$ is $\frac{1}{2^{\rho(\delta)}}$-Lipschitz and there is no strictly smaller Lipschitz coefficient of $\mathscr{S}_f$.

2.3.3. A necessary condition for solvability of the conjugacy equation

In the next proposition we give a necessary condition on $g \in \mathfrak{L}^0$ such that conjugacy equation (2.100) is solvable in $\mathfrak{L}^0$. It can be seen as a generalization of Proposition 2.0.1 for parabolic logarithmic transseries.

Proposition 2.3.12 (Necessary condition for solvability of the conjugacy equation)**.** Let $f, g \in \mathscr{L}_k^0$, $k \in \mathbb{N}$, $f = \mathrm{id} + z^\beta R_\beta + \mathrm{h.o.b.}(z)$, $g = \mathrm{id} + z^\alpha S_\alpha + \mathrm{h.o.b.}(z)$, where $\alpha, \beta \geq 1$ and $S_\alpha, R_\beta \in \mathscr{B}_1 \subseteq \mathscr{L}_k^\infty$. Let $(\beta, \mathbf{n}) := \mathrm{ord}\,(f - \mathrm{id})$ and let $\mathbf{n}'$ be as in (2.101). Let $\varphi \in \mathfrak{L}^0$, such that $\varphi \circ f \circ \varphi^{-1} = g$. Then:

1. $\mathrm{Lt}\,(f - \mathrm{id}) = \mathrm{Lt}\,(g - \mathrm{id})$.

2. Additionally, if $\beta = 1$, then $[f]_{1,\mathbf{m}} = [g]_{1,\mathbf{m}}$, for every $\mathbf{n} \leq \mathbf{m} \leq \mathbf{n}'$.

Proof. 1. Suppose that $\mathrm{Lt}(f - \mathrm{id}) \neq \mathrm{Lt}(g - \mathrm{id})$. The conjugacy equation (2.106) is equivalent to the fixed point equation (2.107). From (2.108) and (2.109) we conclude that

$$\mathrm{ord}\,(\mathscr{S}_f(\varepsilon)) = \min\{\mathrm{ord}\,(f - \mathrm{id}), \mathrm{ord}\,(g - \mathrm{id})\} \tag{2.116}$$

and

$$\mathrm{ord}\,(\mathscr{T}_f(\varepsilon)) > \min\{\mathrm{ord}\,(f - \mathrm{id}), \mathrm{ord}\,(g - \mathrm{id})\}\,. \tag{2.117}$$

Now, (2.116) and (2.117) are in contradiction with $\mathrm{ord}\,(\mathscr{T}_f(\varepsilon)) = \mathrm{ord}\,(\mathscr{S}_f(\varepsilon))$. This implies that $\mathrm{Lt}(f - \mathrm{id}) = \mathrm{Lt}(g - \mathrm{id})$.

2. Let $\beta = 1$ and $(1, \mathbf{n}) := \mathrm{ord}\,(f - \mathrm{id})$. By 1., it follows that $\mathrm{Lt}(f - \mathrm{id}) = \mathrm{Lt}(g - \mathrm{id})$, which implies that $\alpha = \beta = 1$ and $[f]_{1,\mathbf{n}} = [g]_{1,\mathbf{n}}$. Suppose that there exists $\mathbf{m} \in \mathbb{Z}^k$ such that $\mathbf{n} < \mathbf{m} \leq \mathbf{n}'$ and

$$[f]_{1,\mathbf{m}} \neq [g]_{1,\mathbf{m}}. \tag{2.118}$$

Since $\mathrm{Supp}(f)$ and $\mathrm{Supp}(g)$ are well-ordered subsets of $\mathbb{R}_{>0} \times \mathbb{Z}^k$, we can suppose that such an $\mathbf{m}$ is minimal such that (2.118) holds. Now, suppose that there exists a solution $\varphi \in \mathscr{L}_k^0$ of the conjugacy equation (2.100). By Lemma 2.3.9, the conjugacy equation is equivalent to the fixed point equation $\mathscr{T}_f(\varepsilon) = \mathscr{S}_f(\varepsilon)$, for $\varepsilon := \varphi - \mathrm{id}$, where $\mathscr{S}_f$ and $\mathscr{T}_f$ are given in (2.108) and (2.109). From (2.108), it follows that $\mathrm{ord}\,(\mathscr{S}_f(\varepsilon)) = (1, \mathbf{m})$. So,

$$\mathrm{ord}\,(\mathscr{T}_f(\varepsilon)) = (1, \mathbf{m}), \tag{2.119}$$

which implies that $\mathrm{ord}_z(\varepsilon) = 1$. Let $\mathrm{Lb}_z(\varepsilon) := zT$. Since $\alpha = \beta = 1$, let us use the notation $R := R_\beta$ and $S := S_\alpha$. From (2.109), it follows that:

$$\begin{aligned}\mathscr{T}_f(\varepsilon) &= \varepsilon' \cdot zR - (zS)' \cdot \varepsilon \\ &= (zT)' \cdot zR - (zS)' \cdot zT + \text{h.o.b.}(z) \\ &= (T + D_1(T)) \cdot zR - (S + D_1(S)) \cdot zT + \text{h.o.b.}(z) \\ &= zT \cdot (R - S) + (zR \cdot D_1(T) - zT \cdot D_1(S)) + \text{h.o.b.}(z). \end{aligned} \tag{2.120}$$

Since $\mathrm{Lt}(f - \mathrm{id}) = \mathrm{Lt}(g - \mathrm{id})$, it follows that $\mathrm{ord}\,(zR) = \mathrm{ord}\,(zS) = (1, \mathbf{n})$. Hence,

$$\mathrm{ord}(zR \cdot D_1(T) - zT \cdot D_1(S)) > (1, \mathbf{n}'). \tag{2.121}$$

Now, from the minimality of $\mathbf{m}$ and the fact that $\mathrm{ord}(zT) > (1, \mathbf{0}_k)$, it follows that

$$\mathrm{ord}(zT \cdot (R - S)) > (1, \mathbf{m}). \tag{2.122}$$

Now, by (2.120), (2.121) and (2.122), it follows that $\mathrm{ord}(\mathscr{T}_f(\varepsilon)) > (1, \mathbf{m})$, which is a contradiction with (2.119). ■

2.3.4. Sketch of the proof of statement 1 of Theorem C

Let $f \in \mathscr{L}_k^0$, $k \in \mathbb{N}$, such that $f = \mathrm{id} + z^\beta R_\beta + \text{h.o.b.}(z)$, for $\beta \geq 1$ and $R_\beta \in \mathscr{B}_1 \subseteq \mathscr{L}_k^\infty$. We proved in Lemma 2.3.9 that the conjugacy equation (2.106) is equivalent to the fixed point equation (2.107), by introducing the operators $\mathscr{S}_f$ and $\mathscr{T}_f$, given in (2.108) and (2.109) respectively. In order to apply Proposition 1.2.12 on appropriate spaces, the operators $\mathscr{S}_f$ and $\mathscr{T}_f$ have to satisfy the assumptions of Proposition 1.2.12. Based on that, we distinguish two cases: $\beta > 1$ and $\beta = 1$.

Case $\beta > 1$. Since $\varphi \circ f \circ \varphi^{-1} = g$, by Proposition 2.3.12, it follows that $\mathrm{ord}_z(f - \mathrm{id}) = \mathrm{ord}_z(g - \mathrm{id})$. Note that the operator $\mathscr{T}_f$ is a $\frac{1}{2^{\beta-1}}$-homothety on the set of all logarithmic transseries in $\mathscr{L}_k^1$ which do not contain the term of order $\mathrm{ord}\,(f - \mathrm{id})$. On the other hand, the operator $\mathscr{S}_f$ is a $\frac{1}{2^{\rho(\delta)}}$-Lipschitz on space $\mathscr{L}_k^\delta$, for $\delta \geq 1$, by Proposition 2.3.11. In order to apply the fixed point theorem stated in Proposition 1.2.12, the minimal Lipschitz coefficient of $\mathscr{S}_f$ and the coefficient of the homothety of $\mathscr{T}_f$ have to satisfy the inequality $\frac{1}{2^{\rho(\delta)}} < \frac{1}{2^{\beta-1}}$, which is satisfied if and only if $\delta > 1$. Therefore, we are obliged to work on the space $\mathscr{L}_k^{\delta}$, for $\delta > 1$, which is impossible if we have other terms in the leading block (in z) except the leading term of $f - \mathrm{id}$. Therefore, we first *prenormalize* the logarithmic transseries f, i.e., eliminate each term in the leading block (in z) of $f - \mathrm{id}$ except the leading term of $f - \mathrm{id}$, and then apply Proposition 1.2.12.

Case $\beta = 1$. Note that the operator $\mathscr{S}_f$ is a 1-Lipschitz map on the space $(\mathscr{L}_k^{\delta}, d_z)$ and $\mathscr{T}_f$ is a 1-homothety on a subset of $\mathscr{L}_k^{\delta}$, for every $\delta \geq 1$. Since 1 is the minimal Lipschitz coefficient of $\mathscr{S}_f$, which is not strictly smaller than the coefficient of homothety $\mathscr{T}_f$, we conclude that we cannot apply the fixed point theorem stated in Proposition 1.2.12 directly. Therefore, in this case, we change the operators $\mathscr{T}_f$ and $\mathscr{S}_f$.

Since cases $\beta > 1$ and $\beta = 1$ are significantly different, we split the proof of Theorem C in two different cases (see Figure 2.1):

(**a**) $\beta = \mathrm{ord}_z(f - \mathrm{id}) > 1$,

(**b**) $\beta = \mathrm{ord}_z(f - \mathrm{id}) = 1$.

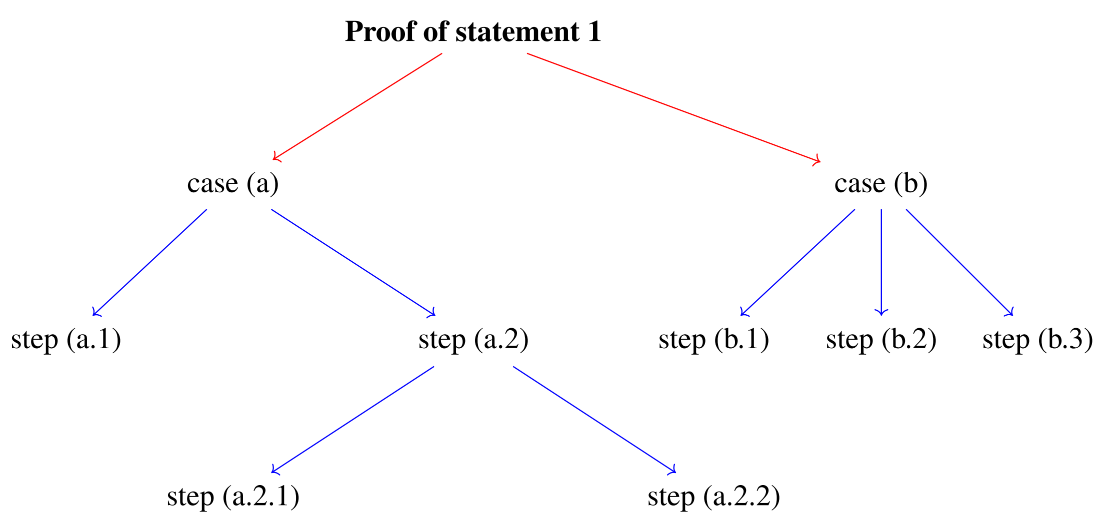

Figure 2.1: Diagram of the proof of statement 1 of Theorem C, [28]

2.3.5. Case $\mathrm{ord}_z\,(f-\mathrm{id})>1$ of the proof of statement 1

Sketch of the proof

Let $f\in\mathscr{L}_k^0$, $k\in\mathbb{N}$, such that $f=\mathrm{id}+z^\beta R_\beta+\text{h.o.b.}(z)$, for $R_\beta\in\mathscr{B}_1\subseteq\mathscr{L}_k^\infty$, and $\beta>1$. Proposition 2.3.12 gives a necessary condition for solvability of the conjugacy equation

$$\varphi\circ f\circ\varphi^{-1}=g,\quad \varphi\in\mathscr{L}_k^0, \tag{2.123}$$

where $g\in\mathscr{L}_k$:

$$g=\mathrm{id}+\mathrm{Lt}(f-\mathrm{id})+\text{h.o.t.}$$

Consequently, it follows that $g=\mathrm{id}+z^\beta M_\beta+\text{h.o.b.}(z)$, for $M_\beta\in\mathscr{B}_1\subseteq\mathscr{L}_k^\infty$, such that $\mathrm{Lt}\,(z^\beta M_\beta)=\mathrm{Lt}\,(z^\beta R_\beta)$. By Proposition 2.3.9, we transformed the conjugacy equation (2.123) to the fixed point equation

$$\mathscr{S}_f(\varepsilon)=\mathscr{T}_f(\varepsilon), \tag{2.124}$$

where $\mathscr{S}_f,\mathscr{T}_f:\mathscr{L}_k\to\mathscr{L}_k$ are operators given in (2.108) and (2.109) respectively, and $\varepsilon:=\varphi-\mathrm{id}$.

If $z^\beta M_\beta\neq z^\beta R_\beta$, by (2.108) and (2.109), it follows that:

$$\begin{aligned}&\mathrm{ord}_z\,(\mathscr{T}_f(\varepsilon))\geq\mathrm{ord}_z\,(\varepsilon)+\beta-1,\\&\mathrm{ord}_z\,(\mathscr{S}_f(\varepsilon))=\beta.\end{aligned} \tag{2.125}$$

Since $\mathscr{T}_f(\varepsilon)=\mathscr{S}_f(\varepsilon)$, by (2.125) it follows that $\mathrm{ord}_z\,(\varepsilon)+\beta-1=\beta$, i.e., $\mathrm{ord}_z\,(\varepsilon)=1$. Note that $\mathscr{S}_f$ is $\frac{1}{2^{\beta-1}}$-Lipschitz ($\frac{1}{2^{\beta-1}}$ is the minimal Lipschitz coefficient of $\mathscr{S}_f$) on the space $\mathscr{L}_k^1$, by Proposition 2.3.11. Since the minimal Lipschitz coefficient of $\mathscr{S}_f$ is equal to the coefficient of $\mathscr{T}_f$, we cannot apply Proposition 1.2.12 directly. Therefore, if the leading block of $f-\mathrm{id}$ in z contains more terms than just the leading term, we first *prenormalize* the logarithmic transseries f, i.e., eliminate each term in the leading block (in z) of $f-\mathrm{id}$ except the leading term. That is, we proceed in two steps:

(a.1) We prove the existence of a parabolic conjugacy $\varphi_1=\mathrm{id}+zT\in\mathscr{L}_k^0$, $T\in\mathscr{B}_{\geq1}^+\subseteq\mathscr{L}_k$, that solves the *prenormalization equation*:

$$\varphi_1\circ f\circ\varphi_1^{-1}=\mathrm{id}+\mathrm{Lt}(f-\mathrm{id})+\text{h.o.b.}(z). \tag{2.126}$$

That is, we eliminate every term in the leading block (in z) of $f-\mathrm{id}$ except its leading term. We prove that such a $T\in\mathscr{B}_{\geq 1}^{+}\subseteq\mathscr{L}_k$ is unique in $\mathscr{B}_{\geq 1}^{+}$.

(a.2) We prove the existence of a parabolic conjugacy $\varphi_2\in\mathscr{L}_k^0$ satisfying $\mathrm{ord}_z(\varphi_1-\mathrm{id})>1$, which solves the *normalization equation*:

$$\varphi_2\circ(\varphi_1\circ f\circ\varphi_1^{-1})\circ\varphi_2^{-1}=\mathrm{id}+\mathrm{Lt}(f-\mathrm{id})+c\mathrm{Res}(f), \tag{2.127}$$

for the unique choice of the residual coefficient $c\in\mathbb{R}$. That is, we eliminate every term in $\varphi_1\circ f\circ\varphi_1^{-1}-\mathrm{id}-\mathrm{Lt}(f-\mathrm{id})$ except for the residual term.

After the prenormalization of f, we apply Proposition 1.2.12 twice: firstly to eliminate all terms between the leading and the residual term, and secondly to eliminate all terms after the residual term. Therefore, we split the proof of the step $(a.2)$ in two steps.

Finally, $\varphi:=\varphi_2\circ\varphi_1$, where φ_1 is from step (a.1), and φ_2 from step (a.2), is a solution of the normalization equation (2.123), with $g:=f_c=\mathrm{id}+\mathrm{Lt}(f-\mathrm{id})+c\mathrm{Res}(f)$.

Proof of (a.1) (prenormalization)

In the next lemma we transform a prenormalization equation (2.126) into a fixed point equation, and then apply Propositions 2.3.14 and 1.2.12 to solve the fixed point equation.

Lemma 2.3.13 (Fixed point equation for the prenormalization)**.** Let $f\in\mathscr{L}_k$, $k\in\mathbb{N}$, such that $f=\mathrm{id}+z^{\beta}R_{\beta}+\mathrm{h.o.b.}(z)$, for $\beta>1$ and $R_{\beta}\in\mathscr{B}_1\subseteq\mathscr{L}_k^{\infty}$. Let $az^{\beta}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}$ be the leading term of $f-\mathrm{id}$ and let $\mathscr{T}_1,\mathscr{S}_1:\mathscr{B}_{\geq 1}^{+}\to\mathscr{B}_{\geq 1}^{+}$ be the operators defined by:

$$\mathscr{T}_1(T):=\Big(1-\frac{\beta a\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}}{R_{\beta}}\Big)\cdot T-\frac{a\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}}{R_{\beta}}\cdot\sum_{i\geq 2}\binom{\beta}{i}T^i, \tag{2.128}$$

$$\mathscr{S}_1(T):=\frac{\mathscr{K}_L(T)}{R_{\beta}}-D_1(T)+\frac{a\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}}{R_{\beta}}-1\,, \tag{2.129}$$

for $T\in\mathscr{B}_{\geq 1}^{+}\subseteq\mathscr{L}_k$, where $\mathscr{K}_L:\mathscr{B}_{\geq 1}^{+}\to\mathscr{B}_1$ is a suitable $\frac{1}{2^{1+n_1}}$-Lipschitz map on the space $\big(\mathscr{B}_{\geq 1}^{+},d_1\big)$. Then $\varphi_1=\mathrm{id}+zT$, $T\in\mathscr{B}_{\geq 1}^{+}\subseteq\mathscr{L}_k$, is a solution of a prenormalization equation (2.126) if and only if

$$\mathscr{T}_1(T)=\mathscr{S}_1(T).$$

Proof. The conjugacy equation (2.126) with $g := z + az^{\beta}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k} + \text{h.o.b.}(z)$ can be equivalently written as:

$$\varphi \circ f - g \circ \varphi = 0. \tag{2.130}$$

This implies that

$$\mathrm{Lb}\big(\varphi \circ f - g \circ \varphi\big) = 0. \tag{2.131}$$

Let $L := a\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}$. From (2.113) we get that (2.131) is equivalent to the equation:

$$(zT)' \cdot z^{\beta} R_{\beta} - \sum_{i \geq 1} \frac{(z^{\beta}L)^{(i)}}{i!}(zT)^i + z^{\beta} \cdot (R_{\beta} - L) = 0. \tag{2.132}$$

Dividing by z^{β} in equation (2.132), and by Lemma A.2.7 and Lemma A.2.8, we get that:

$$(T + D_1(T)) \cdot R_{\beta} - L \sum_{i \geq 1} \binom{\beta}{i} T^i - \mathscr{K}_L(T) + R_{\beta} - L = 0, \tag{2.133}$$

where $\mathscr{K}_L := \mathscr{K}(L, \cdot) : \mathscr{B}_{\geq 1}^{+} \to \mathscr{B}_1$ is a $\frac{1}{2^{1+n_1}}$-Lipschitz map from Lemma A.2.8, 2. We divide the equation by R_{β} and we get the following equation:

$$\Big(1 - \frac{\beta \cdot L}{R_{\beta}}\Big) \cdot T - \frac{L}{R_{\beta}} \cdot \sum_{i \geq 2} \binom{\beta}{i} T^i = \frac{\mathscr{K}_L(T)}{R_{\beta}} - D_1(T) + \frac{L}{R_{\beta}} - 1\,, \tag{2.134}$$

i.e., $\mathscr{T}_1(T) = \mathscr{S}_1(T)$. ■

In the next proposition we prove that the operators $\mathscr{T}_1$ and $\mathscr{S}_1$ defined in (2.128) and (2.129) satisfy assumptions of Proposition 1.2.12.

Proposition 2.3.14 (Properties of the operators $\mathscr{S}_1$ and $\mathscr{T}_1$)**.** Let $f \in \mathscr{L}_k$, $k \in \mathbb{N}$, such that $f = \mathrm{id} + z^{\beta}R_{\beta} + \text{h.o.b.}(z)$, for $\beta > 1$ and $R_{\beta} \in \mathscr{B}_1 \setminus \{0\} \subseteq \mathscr{L}_k^{\infty}$. Let $az^{\beta}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}$ be the leading term of $f - \mathrm{id}$ and let $\mathscr{T}_1, \mathscr{S}_1 : \mathscr{B}_{\geq 1}^{+} \to \mathscr{B}_{\geq 1}^{+}$ be the operators defined in (2.128) and (2.129). Then:

1. $\mathscr{S}_1$ is a $\frac{1}{2}$-contraction on the space $\big(\mathscr{B}_{\geq 1}^{+}, d_1\big)$.

2. $\mathscr{T}_1$ is an isometry on the space $\big(\mathscr{B}_{\geq 1}^{+}, d_1\big)$.

3. $\mathscr{T}_1$ is a surjection.

Proof. 1. Recall that $\mathscr{K}_L$ in (2.129) is a $\frac{1}{2^{1+n_1}}$-Lipschitz map. Since $\mathrm{ord}\,(R_\beta)=(0,n_1,\dots,n_k)$, it follows that

$$\mathrm{ord}_{\boldsymbol{\ell}_1}\Big(\frac{\mathscr{K}_L(T)}{R_\beta}\Big)\geq \mathrm{ord}_{\boldsymbol{\ell}_1}(T)+1\,. \tag{2.135}$$

Since D_1 is a $\frac{1}{2}$-contraction, from (2.129) it follows that $\mathscr{S}_1$ is a $\frac{1}{2}$-contraction on the space $(\mathscr{B}^+_{\geq 1},d_1)$.

2. From (2.128), since $\beta>1$, we see that

$$\mathrm{ord}_{\boldsymbol{\ell}_1}(\mathscr{T}_1(T))=\mathrm{ord}_{\boldsymbol{\ell}_1}(T),$$

which implies that $\mathscr{T}_1$ is an isometry.

3. Recall that $\beta=\mathrm{ord}_z\,(f-\mathrm{id})$. Take $h\in x^2\,\mathbb{R}\,[[x]]$ such that

$$h:=\sum_{i\geq 2}\binom{\beta}{i}x^i. \tag{2.136}$$

Rewrite (2.128) as:

$$\mathscr{T}_1(T)=\Big(1-\frac{\beta a\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}}{R_\beta}\Big)\cdot T-\frac{a\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}}{R_\beta}\cdot h(T),\quad T\in\mathscr{B}^+_{\geq 1}\subseteq\mathscr{L}_k. \tag{2.137}$$

Since $az^\beta\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}$ is the leading term of $z^\beta R_\beta$, and $\beta>1$, it follows that

$$\mathrm{ord}\Big(1-\frac{\beta a\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}}{R_\beta}\Big),\,\mathrm{ord}\Big(\frac{a\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}}{R_\beta}\Big)=\mathbf{0}_{k+1},$$

which is the order of a constant term. Surjectivity of $\mathscr{T}_1$ now follows from Lemma B.5.4. ■

Proof of step $(a.1)$ of statement 1 of Theorem C. By Lemma 2.3.13, we transform the prenormalization equation

$$\varphi_1^{-1}\circ f\circ\varphi_1=\mathrm{id}+\mathrm{Lt}\,(f-\mathrm{id})+\mathrm{h.o.b.}(z)\,, \tag{2.138}$$

for $\varphi_1=\mathrm{id}+zT$, into the equivalent fixed point equation $\mathscr{T}_1(T)=\mathscr{S}_1(T)$. By Proposition 2.3.14 and Proposition 1.2.12, there exists a unique $T\in\mathscr{B}^+_{\geq 1}\subseteq\mathscr{L}_k$, such that $\mathscr{T}_1(T)=\mathscr{S}_1(T)$. So, $\varphi_1:=\mathrm{id}+zT$ is the unique solution of the prenormalization equation (2.138), after we impose that the solution consists only of the block of order 1 in z. ■

Remark 2.3.15 (Minimality of the prenormalization)**.** Proposition 2.3.12 implies that $\operatorname{Lt}(f-\mathrm{id})=\operatorname{Lt}(\varphi_1\circ f\circ\varphi_1^{-1}-\mathrm{id})$. Consequently, it follows that the prenormalization $\varphi_1\circ f\circ\varphi_1^{-1}=\mathrm{id}+az^{\beta}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}+\text{h.o.b.}(z)$ is minimal in the sense that the leading term of $\varphi_1\circ f\circ\varphi_1^{-1}-\mathrm{id}$ cannot be eliminated, nor its coefficients changed, while all other terms of the first block can be eliminated.

Proof of (a.2) (normalization)

In the previous subsection we proved that every parabolic transseries $f\in\mathfrak{L}^0$ such that $\operatorname{ord}_z(f-\mathrm{id})>1$, can be prenormalized. In this subsection we suppose that $f\in\mathscr{L}_k$, $k\in\mathbb{N}$ such that

$$f=\mathrm{id}+az^{\beta}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}+\text{h.o.b.}(z),\quad \beta>1,\, a\neq 0,$$

is a prenormalized parabolic logarithmic transseries, which means that the leading block (in z) of $f-\mathrm{id}$ contains only the leading term $az^{\beta}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}$. In Proposition 2.3.11 we proved several properties of the operator $\mathscr{S}_f$. In Remark 2.3.16 and Proposition 2.3.17 we prove some properties of the superlinear operator $\mathscr{T}_f$. We use these results to apply Proposition 1.2.12 to appropriate complete metric spaces.

Remark 2.3.16 (Injectivity of the operator $\mathscr{T}_f$)**.** Let $f\in\mathscr{L}_k$, $k\in\mathbb{N}$, such that $f=\mathrm{id}+az^{\beta}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}+\text{h.o.b.}(z)$, for $\beta>1$ and $a\neq0$. Let $g:=\mathrm{id}+az^{\beta}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}$. Put $z^{\alpha}S_{\alpha}=z^{\beta}R_{\beta}=az^{\beta}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}$ in (2.109). It follows that $\mathscr{T}_f$ is the Lie bracket operator defined in (1.1.11). By Remark B.3.2 it follows that the kernel of $\mathscr{T}_f$ equals to

$$\ker(\mathscr{T}_f)=\left\{Cz^{\beta}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}:C\in\mathbb{R}\right\}.$$

The operator $\mathscr{T}_f$ is not injective, and therefore, also not a homothety on any subspace of $\mathscr{L}_k^1$ that contains any of the terms belonging to $\ker(\mathscr{T}_f)\setminus\{0\}$.

Let $f\in\mathscr{L}_k$, $k\in\mathbb{N}$, such that $f=\mathrm{id}+az^{\beta}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}+\text{h.o.t.}$, for $\beta>1$. We define the spaces $\mathscr{L}_k^{\delta,1}:=\mathscr{L}_k^{W_1}$, for $W_1:=([\delta,+\infty)\times\mathbb{Z}^k)\setminus\{(\beta,\mathbf{n})\}$, and $\mathscr{L}_k^{\delta,2}:=\mathscr{L}_k^{W_2}$, for $W_2:=([\delta,+\infty)\times\mathbb{Z}^k)\setminus\{(2\beta-1,2\mathbf{n}+\mathbf{1}_k)\}$, for $\delta\geq1$. By Proposition 1.1.16, it follows that $\mathscr{L}_k^{\delta,1}$ and $\mathscr{L}_k^{\delta,2}$ are complete subspaces of $\mathscr{L}_k$, for $\delta\geq1$.

Since $\mathscr{L}_k^{\delta,1}$, $\delta\geq1$, does not contain any term in $\ker(\mathscr{T}_f)\setminus\{0\}$, by Remark 2.3.16, the restriction $\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}:\mathscr{L}_k^{\delta,1}\to\mathscr{L}_k^{\delta,1}$ is injective.

Proposition 2.3.17 (Properties of the operator $\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}$)**.** Let $f \in \mathscr{L}_k$, $k \in \mathbb{N}$, such that $f = \mathrm{id} + az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + \mathrm{h.o.b.}(z)$, for $\beta > 1$ and $a \neq 0$. Let $g \in \mathscr{L}_k^0$ be such that $\mathrm{Lb}_z(g - \mathrm{id}) = az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}$. Let $\delta \geq 1$ and let the operator $\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}} : \mathscr{L}_k^{\delta,1} \to \mathscr{L}_k^{\delta,1}$ be as in (2.109). Then:

1. $\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}$ is a superlinear $\frac{1}{2^{\beta-1}}$-homothety in the space $\mathscr{L}_k^{\delta,1}$, for $\delta \geq 1$,
2. the image of $\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}$ equals to $\mathscr{L}_k^{\delta+\beta-1,2}$, for $\delta \geq 1$.

Proof. 1. Let $\delta \geq 1$, $\varepsilon \in \mathscr{L}_k^{\delta,1}$, $\varepsilon \neq 0$. Put $\varepsilon := z^\alpha T_\alpha + \mathrm{h.o.b.}(z)$, for $\alpha \geq \delta$. From (2.109) we get:

$$\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}(\varepsilon) = \mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}(z^\alpha T_\alpha) + \mathrm{h.o.b.}(z). \tag{2.139}$$

From (2.139) and the fact that $z^\alpha T_\alpha$ does not contain a term of order $(\beta, \mathbf{n})$, it follows that:

$$\mathrm{ord}_z\left(\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}(\varepsilon)\right) = \mathrm{ord}_z(z^\alpha T_\alpha) + \beta - 1 = \mathrm{ord}_z(\varepsilon) + \beta - 1. \tag{2.140}$$

This implies that:

$$d_z\left(0, \mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}(\varepsilon)\right) = \frac{1}{2^{\beta-1}} d_z(0, \varepsilon). \tag{2.141}$$

By linearity of $\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}$ and (2.141), we conclude that $\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}$ is a $\frac{1}{2^{\beta-1}}$-homothety.

2. Let $\delta \geq 1$. First, we prove that $\mathscr{L}_k^{\delta+\beta-1,2}$ is a subset of the image of $\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}$. Let $g \in \mathscr{L}_k^{\delta+\beta-1,2}$ be arbitrary. Consider the equation $\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}(\varepsilon) = g$, i.e.,

$$\varepsilon' - \frac{(az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k})'}{az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}} \cdot \varepsilon = \frac{g}{az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}}. \tag{2.142}$$

This is a linear ordinary differential equation of order one. By Proposition B.3.1, its solutions are given by:

$$\begin{aligned} \varepsilon_C &= \exp\left(\int \frac{(az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k})'}{az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}} dz\right) \cdot \left(C + \int \frac{g}{az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}} \exp\left(-\int \frac{(az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k})'}{az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}} dz\right) dz\right) \\ &= az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} \cdot \left(C + \int \frac{g}{(az^\beta \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k})^2} dz\right), \end{aligned} \tag{2.143}$$

for $C \in \mathbb{R}$. Now we choose $C \in \mathbb{R}$, such that ε_C does not contain a term of order $(\beta, \mathbf{n})$ and put $\varepsilon := \varepsilon_C$. Since $g \in \mathscr{L}_k^{\delta+\beta-1,2}$, then g does not contain a term of order $(2\beta - 1, 2\mathbf{n} +$

$\mathbf{1}_k)$. Therefore, the transseries $\frac{g}{(z^\beta \boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k})^2}$ does not contain a term of order $(-1,\mathbf{1}_k)$. By Lemma B.2.3, we conclude that

$$\int \frac{g}{(az^\beta \boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k})^2}dz$$

belongs to $\mathscr{L}_k^\infty$. By formal integration we get that:

$$\mathrm{ord}_z\Big(\int \frac{g}{(az^\beta \boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k})^2}dz\Big) = \mathrm{ord}_z(g) - 2\beta + 1,$$

which implies, by (2.143), that:

$$\mathrm{ord}_z(\varepsilon) = \mathrm{ord}_z(g) - \beta + 1\,. \tag{2.144}$$

Since $g \in \mathscr{L}_k^{\delta+\beta-1,2}$, it follows that $\varepsilon \in \mathscr{L}_k^\delta$. Recall that ε does not contain a term of order $(\beta,\mathbf{n})$, which implies that $\varepsilon \in \mathscr{L}_k^{\delta,1}$.

It is left to prove that the image of $\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}$ is a subset of $\mathscr{L}_k^{\delta+\beta-1,2}$. For an arbitrary term $bz^\alpha \boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k} \in \mathscr{L}_k^{\delta,1}$, $b\neq 0$, by Lemma A.2.3, we get that $\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}(bz^\alpha \boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k})$ does not contain the residual term. In particular,

$$\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}(bz^\alpha \boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k}) \in \mathscr{L}^{\delta+\beta-1,2},$$

for any term $bz^\alpha \boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k} \in \mathscr{L}_k^{\delta,1}$, $b\neq 0$. Since $\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}$ is superlinear, the image of $\mathscr{T}_f|_{\mathscr{L}_k^{\delta,1}}$ is a subset of $\mathscr{L}_k^{\delta+\beta-1,2}$. ■

Let $f\in\mathscr{L}_k^0$, $k\in\mathbb{N}$, such that $f = \mathrm{id} + az^\beta \boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k} + \text{h.o.b.}(z)$, for $\beta>1$ and $a\neq 0$. Since the logarithmic transseries in $\mathscr{S}_f(\mathscr{L}_k^{\delta,1})$, $\delta\geq 1$, in general contain residual terms (i.e., terms of order $(2\beta-1, 2\mathbf{n}+\mathbf{1}_k)$), by Proposition 2.3.17, 2, and Remark 2.3.16, we cannot directly apply Proposition 1.2.12. Therefore, we split the proof of step $(a.2)$ in two substeps:

(a.2.1). We obtain $\varphi_{2,1}\in\mathscr{L}_k^0$ and $c\in\mathbb{R}$ such that

$$\varphi_{2,1}\circ f\circ\varphi_{2,1}^{-1} = \mathrm{id} + az^\beta \boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k} + c\mathrm{Res}(f) + \text{h.o.b.}(z), \tag{2.145}$$

which means that we eliminate all blocks of f between the first and the residual block (i.e., the $2\beta-1$-block) and all terms in the residual block except the residual term. In that process, the residual coefficient in general changes.

(a.2.2). We obtain $\varphi_{2,2} \in \mathcal{L}_k^0$ such that

$$\varphi_{2,2} \circ (\varphi_{2,1} \circ f \circ \varphi_{2,1}^{-1}) \circ \varphi_{2,2}^{-1} = \mathrm{id} + az^{\beta} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + c\mathrm{Res}\,(f), \tag{2.146}$$

which means that we eliminate all terms in $\varphi_{2,1} \circ f \circ \varphi_{2,1}^{-1}$ after the residual term.

Finally, $\varphi_2 := \varphi_{2,2} \circ \varphi_{2,1}$ is a solution of the normalization equation $\varphi \circ f \circ \varphi^{-1} = f_c$, where $f_c := \mathrm{id} + az^{\beta} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + c\mathrm{Res}\,(f)$.

The general strategy of the proofs of steps $(a.2.1)$ and $(a.2.2)$ is also to transform equations (2.145) and (2.146) to appropriate fixed point equations and then use the fixed point theorem stated in Proposition 1.2.12 to solve them.

Proof of step $(a.2.1)$. Let $f \in \mathcal{L}_k$, $k \in \mathbb{N}$, $f = \mathrm{id} + az^{\beta} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + \mathrm{h.o.b.}(z)$, $\beta > 1$, $a \neq 0$, be prenormalized. We prove that there exists $c \in \mathbb{R}$ and $\varphi_{2,1} \in \mathcal{L}_k^0$, such that:

$$\varphi_{2,1} \circ f \circ \varphi_{2,1}^{-1} = \mathrm{id} + az^{\beta} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + c\mathrm{Res}\,(f) + \mathrm{h.o.b.}(z). \tag{2.147}$$

Put

$$g := \mathrm{id} + az^{\beta} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}.$$

Suppose for a moment that we want to solve the conjugacy equation $\varphi \circ f \circ \varphi^{-1} = g$, which is, by Proposition 2.3.11, equivalent to the fixed point equation $\mathcal{T}_f(\varepsilon) = \mathcal{S}_f(\varepsilon)$, where $\varepsilon := \varphi - \mathrm{id}$, $\varepsilon \in \mathcal{L}_k^1$ and $\mathrm{ord}\,(\varepsilon) > (1, \mathbf{0}_k)$. The operators $\mathcal{S}_f$ and $\mathcal{T}_f$ here are as in (2.108) and (2.109). Put

$$\gamma := \mathrm{ord}_z\,(f - \mathrm{id} - az^{\beta} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}).$$

Notice that $\gamma > \beta > 1$.

Note that $\mathrm{ord}_z\,(\mathcal{S}_f(\varepsilon)) = \gamma$. If the fixed point equation $\mathcal{T}_f(\varepsilon) = \mathcal{S}_f(\varepsilon)$ is solvable, it necessarily follows that $\mathrm{ord}_z\,(\mathcal{T}_f(\varepsilon)) = \gamma$. By Proposition 2.3.17, the operator $\mathcal{T}_f$ is a superlinear $\frac{1}{2^{\beta-1}}$-homothety on the space $\mathcal{L}_k^{1,1}$. To conclude, if ε is a solution of the fixed point equation, it necessarily follows that $\mathrm{ord}_z\,(\varepsilon) = \gamma - (\beta - 1) > 1$. Therefore, set:

$$\delta := \gamma - (\beta - 1).$$

By Proposition 2.3.17 and Proposition 2.3.11, the operator $\mathscr{T}_f$ is a superlinear $\frac{1}{2^{\beta-1}}$-homothety on the space $\mathscr{L}_k^{\delta,1}$ and the operator $\mathscr{S}_f$ is a $\frac{1}{2^{\rho}}$-contraction on the space $\mathscr{L}_k^{\delta,1}$, where

$$\rho := \min\{\gamma-1, 2(\beta-1)\}.$$

By Proposition 2.3.11, it follows that $\mathscr{S}_f(\mathscr{L}_k^{\delta,1}) \subseteq \mathscr{L}_k^{\gamma}$, and, by Proposition 2.3.17, $\mathscr{T}_f(\mathscr{L}_k^{\delta,1}) = \mathscr{L}_k^{\gamma,2}$. Notice that, in general, $\mathscr{S}_f(\mathscr{L}_k^{\delta,1}) \nsubseteq \mathscr{L}_k^{\gamma,2}$. So, in order to apply Proposition 1.2.12, we compose the operator $\mathscr{S}_f$ with the projection operator $[\,\cdot\,] : \mathscr{L}_k^{\gamma} \to \mathscr{L}_k^{\gamma,2}$, defined as follows: $[h]$ is the logarithmic transseries obtained from h by removing its term of order $(2\beta-1, 2\mathbf{n}+\mathbf{1}_k)$. So, instead of the fixed point equation $\mathscr{T}_f(\varepsilon) = \mathscr{S}_f(\varepsilon)$, we solve the modified fixed point equation $\mathscr{T}_f(\varepsilon) = \left[\mathscr{S}_f(\varepsilon)\right]$, for $\varepsilon \in \mathscr{L}_k^{\delta,1}$. Notice that $[\,\cdot\,]$ is a 1-Lipschitz linear operator on $\mathscr{L}_k^{\gamma}$, i.e.

$$d_z([g_1],[g_2]) \leq d_z(g_1,g_2), \quad g_1, g_2 \in \mathscr{L}_k^{\gamma}. \tag{2.148}$$

We define the operator $\left[\mathscr{S}_f\right] : \mathscr{L}_k^{\delta,1} \to \mathscr{L}_k^{\gamma,2}$, as a composition $\left[\mathscr{S}_f\right] = [\,\cdot\,] \circ \mathscr{S}_f$. Using (2.148) and Proposition 2.3.11, we conclude that the operator $\left[\mathscr{S}_f\right]$ is a $\frac{1}{2^{\rho}}$-contraction on $\mathscr{L}_k^{\delta,1}$. Furthermore, $\left[\mathscr{S}_f\right](\mathscr{L}_k^{\delta,1}) \subseteq \mathscr{L}_k^{\gamma,2}$, which is, by Proposition 2.3.17, equal to $\mathscr{T}_f(\mathscr{L}_k^{\delta,1})$. By definition, $\rho > \beta-1$, which implies that $\frac{1}{2^{\rho}} < \frac{1}{2^{\beta-1}}$. By Proposition 1.2.12, there exists a unique $\varepsilon \in \mathscr{L}_k^{\delta,1}$, such that $\mathscr{T}_f(\varepsilon) = \left[\mathscr{S}_f\right](\varepsilon)$ (see Figure 2.2).

$$\begin{tikzcd}
\mathscr{L}_k^{\delta,1} \arrow[r, "\mathscr{T}"] & \mathscr{L}_k^{\gamma,2} & \mathscr{L}_k^{\gamma} \arrow[l, "{[\,\cdot\,]}"'] & \mathscr{L}_k^{\delta,1} \arrow[l, "\mathscr{S}"'] \arrow[ll, bend left, "{[\mathscr{S}_f]}"]
\end{tikzcd}$$

Figure 2.2: Relation between the operators $\left[\mathscr{S}_f\right]$ and $\mathscr{T}_f$, [28]

Note that $\left[\mathscr{T}_f(\varepsilon)\right] = \mathscr{T}_f(\varepsilon)$, because $\mathscr{T}_f(\mathscr{L}_k^{\delta,1}) = \mathscr{L}_k^{\gamma,2}$. Therefore, solving the equation $\mathscr{T}_f(\varepsilon) = \left[\mathscr{S}_f\right](\varepsilon)$ in $\mathscr{L}_k^{\delta,1}$ is equivalent to solving the equation

$$\left[\mathscr{T}_f(\varepsilon)\right] = \left[\mathscr{S}_f(\varepsilon)\right].$$

The operator $[\,\cdot\,]$ is linear, so the above equation is equivalent to

$$\left[\mathscr{T}_f(\varepsilon) - \mathscr{S}_f(\varepsilon)\right] = 0,$$

which is equivalent to the existence of $c \in \mathbb{R}$, such that

$$\mathscr{T}_f(\varepsilon) - \mathscr{S}_f(\varepsilon) = c\mathrm{Res}(f). \tag{2.149}$$

Finally, put $\varphi_{2,1} := \mathrm{id} + \varepsilon$. As in (2.112), we get

$$\mathscr{T}_f(\varepsilon) - \mathscr{S}_f(\varepsilon) = \varphi_{2,1} \circ f - g \circ \varphi_{2,1}. \tag{2.150}$$

From (2.149) and (2.150), it follows that there exists $c \in \mathbb{R}$ such that

$$\varphi_{2,1} \circ f - g \circ \varphi_{2,1} = c\mathrm{Res}(f). \tag{2.151}$$

Now, by transforming (2.151), since $\mathrm{ord}_z(\varepsilon) > 1$, by Taylor Theorem (Proposition A.1.6), we get that

$$\begin{aligned}\varphi_{2,1} \circ f \circ \varphi_{2,1}^{-1} &= g + c\mathrm{Res}(f) \circ \varphi_{2,1}^{-1} \\ &= \mathrm{id} + az^{\beta}\boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + c\mathrm{Res}(f) + \mathrm{h.o.b.}(z).\end{aligned}$$

Thus, we have eliminated all blocks in f of order (in z) between β and $2\beta - 1$, and all terms in the residual block, except the residual term. ∎

Proof of step $(a.2.2)$. Let f, $\varphi_{2,1}$ and $a, c \in \mathbb{R}$ be as in step $(a.2.1)$ above. We prove that there exists $\varphi_{2,2} \in \mathscr{L}_k^0$, such that:

$$\varphi_{2,2} \circ (\varphi_{2,1} \circ f \circ \varphi_{2,1}^{-1}) \circ \varphi_{2,2}^{-1} = \mathrm{id} + az^{\beta}\boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + c\mathrm{Res}(f).$$

Put:

$$\begin{aligned}h &:= \varphi_{2,1} \circ f \circ \varphi_{2,1}^{-1}, \\ g &:= \mathrm{id} + az^{\beta}\boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + c\mathrm{Res}(f), \\ \gamma &:= \mathrm{ord}_z(h - g), \\ \delta &:= \gamma - (\beta - 1).\end{aligned}$$

The conjugacy equation

$$\varphi_{2,2} \circ h \circ \varphi_{2,2}^{-1} = g, \tag{2.152}$$

is, by Lemma 2.3.9, equivalent to the fixed point equation $\mathscr{S}_f(\varepsilon) = \mathscr{T}_f(\varepsilon)$, where $\varepsilon := \varphi_{2,2} - \mathrm{id}$. As in the proof of step $(a.2.1)$, if ε is a solution of the equation $\mathscr{S}_f(\varepsilon) = \mathscr{T}_f(\varepsilon)$, we conclude that $\mathrm{ord}_z(\mathscr{T}_f(\varepsilon)) = \gamma$. Therefore, if ε is a solution, necessarily $\mathrm{ord}_z(\varepsilon) = \delta$. Since $\gamma > 2\beta - 1$ and $\delta > \beta$, it follows that $\mathscr{L}_k^{\delta} = \mathscr{L}_k^{\delta,1}$ and $\mathscr{L}_k^{\gamma} = \mathscr{L}_k^{\gamma,2}$. By Proposition 2.3.17, the operator $\mathscr{T}_f : \mathscr{L}_k^{\delta} \to \mathscr{L}_k^{\gamma}$ is a $\frac{1}{2^{\beta-1}}$-homothety and $\mathscr{T}_f(\mathscr{L}_k^{\delta}) = \mathscr{L}_k^{\gamma}$. By Proposition 2.3.11, the operator $\mathscr{S}_f : \mathscr{L}_k^{\delta} \to \mathscr{L}_k^{\gamma}$ is a $\frac{1}{2^{\rho}}$-contraction, with $\rho := \min\{\gamma - 1, 2(\beta - 1)\}$. Since $\mathscr{S}_f(\mathscr{L}_k^{\delta}) \subseteq \mathscr{L}_k^{\gamma} = \mathscr{T}_f(\mathscr{L}_k^{\delta})$ and $\frac{1}{2^{\rho}} < \frac{1}{2^{\beta-1}}$, by Proposition 1.2.12, it follows that there exists a unique $\varepsilon \in \mathscr{L}_k^{\delta}$, such that $\mathscr{T}_f(\varepsilon) = \mathscr{S}_f(\varepsilon)$. Now, $\varphi_{2,2} := \mathrm{id} + \varepsilon$ is a solution of the conjugacy equation (2.152). ■

2.3.6. Case $\mathrm{ord}_z(f - \mathrm{id}) = 1$ of the proof of statement 1

Let $f \in \mathscr{L}_k^0$, $k \in \mathbb{N}_{\geq 1}$, such that $f = \mathrm{id} + zL + \text{h.o.t.}$, where L is as defined in Theorem C. Put $\mathbf{n} := \mathrm{ord}\left(\frac{f-\mathrm{id}}{z}\right)$. Suppose that $\mathbf{n} = (\mathbf{0}_{m-1}, n_m, \ldots, n_k)$, where $n_m \geq 1$, for some $1 \leq m \leq k$. As opposed to the case $\mathrm{ord}_z(f - \mathrm{id}) > 1$, here we have three steps. The reason for that is that the residual term of f is in the leading block (in z) of f.

For that purpose we define the following:

1. Let $\mathscr{L}_k^{<(\alpha,n_1,\ldots,n_m)}$ (resp. $\mathscr{L}_k^{>(\alpha,n_1,\ldots,n_m)}$) be the space of all transseries in $\mathscr{L}_k$, $k \in \mathbb{N}_{\geq 1}$, of order strictly smaller (resp. bigger) than $(\alpha, n_1, \ldots, n_m) \in \mathbb{R} \times \mathbb{Z}^m$ in variables $z, \boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_m$, for $1 \leq m \leq k$.

2. Let $\mathscr{L}_k^{\leq(\alpha,n_1,\ldots,n_m)}$ (resp. $\mathscr{L}_k^{\geq(\alpha,n_1,\ldots,n_m)}$) be the space of all transseries in $\mathscr{L}_k$, $k \in \mathbb{N}_{\geq 1}$, of order smaller (resp. bigger) than or equal to $(\alpha, n_1, \ldots, n_m)$ in variables $z, \boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_m$, for $1 \leq m \leq k$.

3. Let $\mathscr{P}_{<(\alpha,n_1,\ldots,n_m)} : \mathscr{L}_k \to \mathscr{L}_k^{<(\alpha,n_1,\ldots,n_m)}$ be the projection operator on the space $\mathscr{L}_k^{<(\alpha,n_1,\ldots,n_m)}$. Similarly we define the projection operators:

$$\mathscr{P}_{\leq(\alpha,n_1,\ldots,n_m)} : \mathscr{L}_k \to \mathscr{L}_k^{\leq(\alpha,n_1,\ldots,n_m)},$$
$$\mathscr{P}_{>(\alpha,n_1,\ldots,n_m)} : \mathscr{L}_k \to \mathscr{L}_k^{>(\alpha,n_1,\ldots,n_m)},$$
$$\mathscr{P}_{\geq(\alpha,n_1,\ldots,n_m)} : \mathscr{L}_k \to \mathscr{L}_k^{\geq(\alpha,n_1,\ldots,n_m)}.$$

Let $f \in \mathscr{L}_k$, $k \in \mathbb{N}_{\geq 1}$, such that $f = \mathrm{id} + az\boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + \text{h.o.t.}$ Suppose that $n_1 = \cdots = n_{m-1} = 0$ and $n_m \in \mathbb{N}_{\geq 1}$, for $1 \leq m \leq k$. Let L be defined as in Theorem C. In the sequel, we use the notation $r := \mathrm{ord}\,(\mathrm{Res}\,(f))$ for the order of the residual monomial in the space $\mathscr{L}_k$.

We proceed in three steps:

(b.1) We eliminate all the terms in the leading block (in z) of $f-\mathrm{id}$, which are not in zL and are of order less than or equal to $(1,\mathbf{1}_{m-1},n_m+1)$ in the first $m+1$ variables $z,\boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_m$. That is, we find a solution $\varphi_1\in\mathscr{L}_k^0$, of the equation

$$\mathscr{P}_{\leq(1,\mathbf{1}_{m-1},n_m+1)}(\varphi_1\circ f\circ\varphi_1^{-1})=\mathrm{id}+zL. \tag{2.153}$$

A solution φ_1 is not unique in $\mathscr{L}_k^0$, but, if we impose the so-called *canonical form* of φ_1, i.e., $\varphi_1=\mathrm{id}+zS_{m+1}$, $S_{m+1}\in\mathscr{B}^+_{\geq m+1}\subseteq\mathscr{L}_k$, then φ_1 is unique. This uniqueness is proven in Remark 2.3.21. Note that we can skip this step if $m=k$, since then $\mathscr{P}_{\leq(1,\mathbf{1}_{m-1},n_m+1)}(f)=\mathrm{id}+zL$.

(b.2) We eliminate all the terms in the leading block (in z) of $f-\mathrm{id}$, which are not in zL, up to the residual term. That is, we find a solution $\varphi_2\in\mathscr{L}_k^0$ of the equation

$$\mathscr{P}_{<r}(\varphi_2\circ(\varphi_1\circ f\circ\varphi_1^{-1})\circ\varphi_2^{-1})=\mathrm{id}+zL. \tag{2.154}$$

This implies that there exists $c\in\mathbb{R}$ such that:

$$\varphi_2\circ(\varphi_1\circ f\circ\varphi_1^{-1})\circ\varphi_2^{-1}=\mathrm{id}+zL+c\mathrm{Res}\,(f)+\text{h.o.t.}$$

The solution φ_2 is not unique in $\mathscr{L}_k^0$, but, if we impose the *canonical form* of φ_2, i.e., $\varphi_2=\mathrm{id}+zS$, for $S\in\widetilde{\mathscr{B}}$, where $\widetilde{\mathscr{B}}$ is the set of all logarithmic transseries in $\mathscr{B}_m^+\subseteq\mathscr{L}_k$ that contain only terms of order strictly smaller than $(\mathbf{0}_m,n_m,\dots,n_k)$, then φ_2 is unique. This uniqueness is proven in Remark 2.3.25.

(b.3) Let $c\in\mathbb{R}$ be the residual coefficient in $\varphi_2\circ\varphi_1\circ f\circ\varphi_1^{-1}\circ\varphi_2^{-1}$. We eliminate all the terms after the residual term. In that process c remains unchanged. That is, we find a solution $\varphi_3\in\mathscr{L}_k^0$ of the equation

$$\varphi_3\circ(\varphi_2\circ(\varphi_1\circ f\circ\varphi_1^{-1})\circ\varphi_2^{-1})\circ\varphi_3^{-1}=\mathrm{id}+zL+c\mathrm{Res}\,(f). \tag{2.155}$$

Put $f_1:=\varphi_2\circ(\varphi_1\circ f\circ\varphi_1^{-1})\circ\varphi_2^{-1}$. The solution φ_3 is not unique in $\mathscr{L}_k^0$. However, if we impose the *cononical form* of φ_3, that is $\varphi_3=\mathrm{id}+zS+\varepsilon$, where $S\in\mathscr{B}^+_{\geq 1}\subseteq\mathscr{L}_k$ such that $\mathrm{ord}\,(zS)>r$, and $\varepsilon\in\mathscr{L}_k$ such that $\mathrm{ord}_z\,(\varepsilon)\geq\mathrm{ord}_z\,\big(f_1-\mathrm{Lb}_z\,(f_1)\big)$, then φ_3 is unique.

The general strategy in all three steps is to transform the appropriate conjugacy equation into the fixed point equation and apply the fixed point theorem stated in Proposition 1.2.12. In order to apply Proposition 1.2.12, we define suitable operators $\mathscr{T}$ and $\mathscr{S}$, and use the results from Subsection B.5 to prove requested properties of the mentioned operators.

We illustrate steps $(b.1)-(b.3)$ on the following example.

Example 2.3.18. Let $f \in \mathscr{L}_3$ be given by:

$$\begin{aligned} f = z + &\overbrace{z\boldsymbol{\ell}_2\Big(\sum_{i=-2}^{+\infty}\boldsymbol{\ell}_3^i\Big) + z\boldsymbol{\ell}_1 + z\boldsymbol{\ell}_1\boldsymbol{\ell}_2^2\boldsymbol{\ell}_3^{-2}}^{zL} \\ &+ \underbrace{z\boldsymbol{\ell}_1\boldsymbol{\ell}_2^2(\boldsymbol{\ell}_3^{-1}+1+\boldsymbol{\ell}_3^6+\boldsymbol{\ell}_3^{14})}_{\text{elimination of this part in step } (b.1)} + \underbrace{z\boldsymbol{\ell}_1\boldsymbol{\ell}_2^3\boldsymbol{\ell}_3^{-3}}_{\text{residual term}} + z^3\boldsymbol{\ell}_2^{-5} + \text{h.o.t.} \end{aligned}$$

Note that $\mathbf{n} = (0,1,-2)$ and $m = 2$. Therefore, by (2.101), it follows that $\mathbf{n}' = (1,2,-2)$ and

$$L = \boldsymbol{\ell}_2\sum_{i=-2}^{+\infty}\boldsymbol{\ell}_3^i + \boldsymbol{\ell}_1 + \boldsymbol{\ell}_1\boldsymbol{\ell}_2^2\boldsymbol{\ell}_3^{-2}.$$

Furthermore, $\mathrm{Res}(f) = z\boldsymbol{\ell}_1\boldsymbol{\ell}_2^3\boldsymbol{\ell}_3^{-3}$. In step $(b.1)$ we find a solution $\varphi_1 \in \mathscr{L}_3^0$ of the equation (2.153).

Now,

$$\varphi_1 \circ f \circ \varphi_1^{-1} = z + zL + \underbrace{\sum_{u_3=v_3}^{-4} a_{1,1,3,u_3} z\boldsymbol{\ell}_1\boldsymbol{\ell}_2^3\boldsymbol{\ell}_3^{u_3}}_{\text{elimination of this part in step } (b.2)} + b\mathrm{Res}(f) + \text{h.o.t.},$$

for some $v_3 \leq -4$. Some of these coefficients $a_{1,1,3,u_3}$ and b can be equal to zero.

Note that $r = \mathrm{ord}(\mathrm{Res}(f)) = (1,1,3,-3)$. Now, we apply step $(b.2)$ and obtain $\varphi_2 \in \mathscr{L}_3^0$ and $c \in \mathbb{R}$ such that (2.154) holds, i.e.

$$\varphi_2 \circ (\varphi_1 \circ f \circ \varphi_1^{-1}) \circ \varphi_2^{-1} = z + zL + c\mathrm{Res}(f) + \underbrace{\text{h.o.t.}}_{\text{elimination } (b.3)}.$$

Finally, we apply step $(b.3)$ to eliminate h.o.t by $\varphi_3 \in \mathscr{L}_3^0$, not changing any terms in $\mathrm{id} + zL + c\mathrm{Res}(f)$.

Proof of step (b.1)

In Lemma 2.3.19 below we transform the conjugation equation (2.153) into a fixed point equation. Then we use Proposition 2.3.20 and Proposition 1.2.12 to prove step $(b.1)$. As explained above, if $m=k$ we can skip this step. Therefore, we assume that $1\leq m\leq k-1$ and $k\geq 2$.

Lemma 2.3.19 (Transforming equation (2.153) to a differential equation)**.** Let $f\in\mathscr{L}_k^0$, $k\in\mathbb{N}_{\geq 1}$, such that $f=\mathrm{id}+zR+\text{h.o.b.}(z)$, $R\in\mathscr{B}_{\geq 1}^{+}\subseteq\mathscr{L}_k$. Let $R\neq 0$ and $(1,\mathbf{n}):=\mathrm{ord}(f-\mathrm{id})$, where $\mathbf{n}=(\mathbf{0}_{m-1},n_m,\ldots,n_k)$, $n_m\in\mathbb{N}_{\geq 1}$, $1\leq m\leq k-1$. Let L be defined as in Theorem C and suppose that $L_{m+1},T_{m+1}\in\mathscr{B}_{m+1}\setminus\{0\}$[5] are such that:

$$\begin{aligned} L&=\boldsymbol{\ell}_m^{n_m}L_{m+1}+\text{h.o.t.},\\ R-L&=\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{m-1}\boldsymbol{\ell}_m^{n_m+1}T_{m+1}+\text{h.o.t.}\end{aligned}$$

The logarithmic transseries $\varphi\in\mathscr{L}_k$ is a solution of the equation

$$\mathscr{P}_{\leq(\mathbf{1}_m,n_m+1)}(\varphi\circ f\circ\varphi^{-1})=\mathrm{id}+zL \tag{2.156}$$

if and only if S_{m+1} satisfies the differential equation

$$\begin{aligned} &L_{m+1}\cdot D_{m+1}(S_{m+1})-\log(1+S_{m+1})\cdot(n_m\cdot L_{m+1}+D_{m+1}(L_{m+1}))\cdot(1+S_{m+1})\\ &+T_{m+1}\cdot S_{m+1}=-T_{m+1}.\end{aligned} \tag{2.157}$$

Here, we write $\varphi=\mathrm{id}+zS+\varepsilon$, $S\in\mathscr{B}_{\geq 1}^{+}$, $\varepsilon\in\mathscr{L}_k$ such that $\mathrm{ord}_z(\varepsilon)>1$, and we decompose $S=S_{m+1}+S_m+\cdots+S_1$, for $S_i\in\mathscr{B}_i^{+}$, $1\leq i\leq m$, and $S_{m+1}\in\mathscr{B}_{\geq m+1}^{+}$.

Proof. The conjugacy equation (2.156) with $\varphi\in\mathscr{L}_k^0$ is equivalent to the equation

$$\mathscr{P}_{\leq(\mathbf{1}_m,n_m+1)}(\varphi\circ f)=\mathscr{P}_{\leq(\mathbf{1}_m,n_m+1)}\big((\mathrm{id}+zL)\circ\varphi\big),\quad \varphi\in\mathscr{L}_k^0. \tag{2.158}$$

Put $\mu:=f-(\mathrm{id}+zR)$. Then $\mathrm{ord}_z(\mu)>1$. Put $\varphi=\mathrm{id}+zS+\varepsilon$, for $S\in\mathscr{B}_{\geq 1}^{+}\subseteq\mathscr{L}_k$, and $\varepsilon\in\mathscr{L}_k$ such that $\mathrm{ord}_z(\varepsilon)>1$. By the Taylor Theorem we get

$$\begin{aligned} \varphi\circ f&=zS+\varepsilon+\sum_{i\geq 1}\frac{(zS+\varepsilon)^{(i)}(\mathrm{id}+zR)}{i!}\mu^i+\mathrm{id}+zR+\mu+\sum_{i\geq 1}\frac{(zS)^{(i)}}{i!}(zR)^i+\sum_{i\geq 1}\frac{\varepsilon^{(i)}}{i!}(zR)^i\\ (\mathrm{id}+zL)\circ\varphi&=\mathrm{id}+zS+\varepsilon+zL+\sum_{i\geq 1}\frac{(zL)^{(i)}(\mathrm{id}+zS)}{i!}\varepsilon^i+\sum_{i\geq 1}\frac{(zL)^{(i)}}{i!}(zS)^i,\end{aligned} \tag{2.159}$$

[5]Note that $L_{m+1}\neq 0$ since $n_m\neq 0$. Without the loss of generality, we may assume that $T_{m+1}\neq 0$. Otherwise step $(b.1)$ is omitted.

for $S \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_k$, $\varepsilon \in \mathscr{L}_k$, $\mathrm{ord}_z(\varepsilon) > 1$. Analyzing orders and applying the operator $\mathscr{P}_{\leq(\mathbf{1}_m,n_m+1)}$ on (2.159), we get that equation (2.158) is equivalent to the equation

$$\mathscr{P}_{\leq(\mathbf{1}_m,n_m+1)}\Big(\sum_{i\geq 1}\frac{(zS)^{(i)}}{i!}(zR)^i - \sum_{i\geq 1}\frac{(zL)^{(i)}}{i!}(zS)^i + z(R-L)\Big) = 0. \tag{2.160}$$

By Lemma A.2.8, it follows that

$$\begin{aligned}\sum_{i\geq 1}\frac{(zL)^{(i)}}{i!}(zS)^i &= z\left(L\cdot S + D_1(L)\cdot(1+S)\cdot\log(1+S) + \mathscr{C}_1(S)\right),\\ \sum_{i\geq 1}\frac{(zS)^{(i)}}{i!}(zR)^i &= z\left(S\cdot R + D_1(S)\cdot(1+R)\cdot\log(1+R) + \mathscr{K}_1(S)\right),\end{aligned} \tag{2.161}$$

for suitable $\frac{1}{2^{2+\mathrm{ord}_{\boldsymbol{\ell}_1}(R)}}$-contractions $\mathscr{C}_1, \mathscr{K}_1 : (\mathscr{B}^+_{\geq 1}, d_1) \to (\mathscr{B}^+_{\geq 1}, d_1)$. Note that

$$\mathscr{P}_{\leq(\mathbf{1}_m,n_m+1)}(\mathscr{K}_1(S)) = \mathscr{P}_{\leq(\mathbf{1}_m,n_m+1)}(\mathscr{C}_1(S)) = 0, \tag{2.162}$$

for each $S \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_k$. Applying $\mathscr{P}_{\leq(\mathbf{1}_m,n_m+1)}$ to equation (2.160) and using (2.161) and (2.162) , after dividing by z, we get the following equivalent equation

$$\mathscr{P}_{\leq(0,\mathbf{1}_{m-1},n_m+1)}\Big(S\cdot(R-L) + (R-L) + (1+R)\cdot\log(1+R)\cdot D_1(S) - (1+S)\cdot\log(1+S)\cdot D_1(L)\Big) = 0. \tag{2.163}$$

Since we eliminated z from the equation (2.160), we use the projection operator $\mathscr{P}_{\leq(0,\mathbf{1}_{m-1},n_m+1)}$ instead of $\mathscr{P}_{\leq(\mathbf{1}_m,n_m+1)}$ in the equation (2.163).

Let us decompose S as $S = S_{m+1} + S_m + \cdots + S_1$. Note that:

$$\begin{aligned} D_1(S_{m+1}) &= \boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{m-1}D_m(S_{m+1}),\\ D_m(S_{m+1}) &= \boldsymbol{\ell}_m D_{m+1}(S_{m+1}).\end{aligned}$$

Furthermore, $\mathrm{Lb}_{\boldsymbol{\ell}_m}(R) = \mathrm{Lb}_{\boldsymbol{\ell}_m}(L) = \boldsymbol{\ell}_m^{n_m}L_{m+1}$ and $\mathrm{Lb}_{\boldsymbol{\ell}_m}\left(\frac{R-L}{\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{m-1}}\right) = \boldsymbol{\ell}_m^{n_m+1}T_{m+1}$. Using this and analyzing the orders of the terms in (2.163), we get

$$\begin{aligned}&\mathscr{P}_{\leq(0,\mathbf{1}_{m-1},n_m+1)}\Big((1+R)\cdot\log(1+R)\cdot D_1(S) - (1+S)\cdot\log(1+S)\cdot D_1(L) + S\cdot(R-L) + (R-L)\Big)\\ &= \boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{m-1}\Big(\boldsymbol{\ell}_m^{n_m}L_{m+1}\cdot D_m(S_{m+1}) - (1+S_{m+1})\cdot\log(1+S_{m+1})\cdot D_m(\boldsymbol{\ell}_m^{n_m}L_{m+1})\Big)\\ &+ \boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{m-1}\Big(S_{m+1}\boldsymbol{\ell}_m^{n_m+1}\cdot T_{m+1} + \boldsymbol{\ell}_m^{n_m+1}T_{m+1}\Big).\end{aligned} \tag{2.164}$$

Dividing by $\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{m-1}$ and using (2.164), equation (2.163) is equivalent to the following equation:

$$\begin{aligned}&\boldsymbol{\ell}_m^{n_m}L_{m+1}\cdot D_m(S_{m+1}) - (1+S_{m+1})\cdot\log(1+S_{m+1})\cdot D_m(\boldsymbol{\ell}_m^{n_m}L_{m+1}) + S_{m+1}\cdot\boldsymbol{\ell}_m^{n_m+1}T_{m+1}\\ &= -\boldsymbol{\ell}_m^{n_m+1}T_{m+1}.\end{aligned} \tag{2.165}$$

Using Lemma A.2.10, we get that equation (2.165) is equivalent to the equation

$$\begin{aligned}&L_{m+1}\cdot D_{m+1}(S_{m+1})-(1+S_{m+1})\cdot\log(1+S_{m+1})\cdot(n_m L_{m+1}+D_{m+1}(L_{m+1}))\\&+S_{m+1}\cdot T_{m+1}=-T_{m+1}.\end{aligned}$$

■

Proposition 2.3.20 (A solution of a differential equation in $\mathscr{B}^+_{\geq m+1}$). Let $N,T\in\mathscr{B}_{m+1}\subseteq\mathscr{L}_k^\infty$, $k\in\mathbb{N}_{\geq1}$, such that $N\neq0$ and $\operatorname{ord}(T)>\operatorname{ord}(N)$. Let $n\in\mathbb{N}_{\geq1}$ and let $h\in x^2\mathbb{R}[[x]]$ be a power series in the variable x, with real coefficients, such that $h(0)=h'(0)=0$. Then there exists a unique solution $S\in\mathscr{B}^+_{\geq m+1}\subseteq\mathscr{L}_k$ of the equation

$$N\cdot D_{m+1}(S)-(n\cdot N+D_{m+1}(N))\cdot(S+h(S))+T\cdot S=T. \tag{2.166}$$

Proof. Dividing by $N\neq0$ both sides of equation (2.166), we get the equivalent equation:

$$\left(\frac{T}{N}-n\right)\cdot S-n\cdot h(S)=(S+h(S))\cdot\frac{D_{m+1}(N)}{N}-D_{m+1}(S)+\frac{T}{N}.$$

Let $\mathscr{T}_{m+1},\mathscr{S}_{m+1}:\mathscr{B}^+_{\geq m+1}\to\mathscr{B}^+_{\geq m+1}$ be the operators defined by

$$\begin{aligned}\mathscr{S}_{m+1}(S)&:=(S+h(S))\cdot\frac{D_{m+1}(N)}{N}-D_{m+1}(S)+\frac{T}{N},\\\mathscr{T}_{m+1}(S)&:=\left(\frac{T}{N}-n\right)\cdot S-n\cdot h(S),\quad S\in\mathscr{B}^+_{\geq m+1}\subseteq\mathscr{L}_k.\end{aligned}$$

Let $S_1,S_2\in\mathscr{B}^+_{\geq m+1}\subseteq\mathscr{L}_k$ be arbitrary such that $S_1\neq S_2$. Since $\operatorname{ord}(T)>\operatorname{ord}(N)$, it follows that $\operatorname{ord}\left(\frac{T}{N}\right)>\mathbf{0}_{k+1}$. Note that

$$\begin{aligned}\mathscr{T}_{m+1}(S_1)-\mathscr{T}_{m+1}(S_2)&=\left(\frac{T}{N}-n\right)(S_1-S_2)-n\cdot(S_1-S_2)\cdot\sum_{i\geq2}\Big(\sum_{j=0}^{i-1}S_1^{i-j}\cdot S_2^j\Big)\\&=-n\mathrm{Lt}(S_1-S_2)+\text{h.o.t.}\end{aligned}$$

Therefore, $\operatorname{ord}_{\boldsymbol{\ell}_{m+1}}(\mathscr{T}_{m+1}(S_1-S_2))=\operatorname{ord}_{\boldsymbol{\ell}_{m+1}}(S_1-S_2)$. This implies that $\mathscr{T}_{m+1}$ is an isometry on the space $(\mathscr{B}^+_{\geq m+1},d_{m+1})$. By Remark B.5.6, it follows that $\mathscr{T}_{m+1}$ is a surjection. Since

$$\begin{aligned}&\operatorname{ord}_{\boldsymbol{\ell}_{m+1}}\Big(\frac{D_{m+1}(N)}{N}\Big)\geq1,\\&\operatorname{ord}_{\boldsymbol{\ell}_{m+1}}(D_{m+1}(S))\geq\operatorname{ord}_{\boldsymbol{\ell}_{m+1}}(S)+1,\end{aligned}$$

it follows that $\mathscr{S}_{m+1}$ is a $\frac{1}{2}$-contraction on the space $(\mathscr{B}^+_{\geq m+1},d_{m+1})$. By Proposition 1.2.12, it follows that there exists a unique solution $S\in\mathscr{B}^+_{\geq m+1}\subseteq\mathscr{L}_k$ of the equation $\mathscr{T}_{m+1}(S)=\mathscr{S}_{m+1}(S)$, i.e., equation (2.166). ■

Proof of step $(b.1)$. Suppose that $f \in \mathscr{L}_k^0$, $k \in \mathbb{N}_{\geq 1}$, such that $f = \mathrm{id} + zR + \text{h.o.b.}(z)$, for $R \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$. We assume here that $R \neq 0$, since otherwise, it follows that $\mathrm{ord}_z(f - \mathrm{id}) > 1$ and we apply case (a) instead. Let $\mathrm{ord}\,(f - \mathrm{id}) = (1, \mathbf{0}_{m-1}, n_m, \ldots, n_k)$, $n_m \geq 1$, $1 \leq m \leq k-1$. Let L be as defined in Theorem C. In this step we eliminate all terms in $f - \mathrm{id}$ except the terms in zL, up to order $(\mathbf{1}_m, n_m + 1)$ in the first $m+1$ variables $z, \boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_m$. Note that, if $m = k$, all such terms are in zL. In this case we have nothing to eliminate and we skip case $(b.1)$.

Suppose that $m < k$. By Lemma 2.3.19 we transform equation (2.153) into the differential equation (2.157). Let us decompose L and $R - L$ as in Lemma 2.3.19. If $T_{m+1} = 0$, then we have nothing to eliminate and we skip case $(b.1)$. Suppose that $T_{m+1} \neq 0$. Now, put $T := T_{m+1}$, $N := L_{m+1}$ and $h := (1+x)\log(1+x) - x$. Since $N \neq 0$ and $\mathrm{ord}(T) > \mathrm{ord}(N)$, by Proposition 2.3.20, there exists a unique solution $S_{m+1} \in \mathscr{B}_{\geq m+1}^+ \subseteq \mathscr{L}_k$ of equation (2.157). Now, we put $\varphi_1 := \mathrm{id} + zS_{m+1}$ and, by Lemma 2.3.19, $\varphi_1 \in \mathscr{L}_k^0$ is a solution of the conjugacy equation (2.153). ■

Remark 2.3.21 (Non-uniqueness of the conjugacy φ_1)**.** Let $\varphi_1 := \mathrm{id} + zS + \varepsilon$, $\varepsilon \in \mathscr{L}_k$, $\mathrm{ord}_z(\varepsilon) > 1$, and $S \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, be a solution of the conjugacy equation (2.153) for a logarithmic transseries $f \in \mathscr{L}_k^0$, $k \in \mathbb{N}_{\geq 1}$. Now, put $\mathrm{ord}\,(f - \mathrm{id}) := (1, \mathbf{0}_{m-1}, n_m, \ldots, n_k)$, for $n_m \geq 1$ and $1 \leq m \leq k-1$. We decompose S as $S = S_1 + \cdots + S_m + S_{m+1}$, for $S_i \in \mathscr{B}_i^+ \subseteq \mathscr{L}_k$, $1 \leq i \leq m$, and $S_{m+1} \in \mathscr{B}_{\geq m+1}^+ \subseteq \mathscr{L}_k$. By Lemma 2.3.19, φ_1 is a solution of the conjugacy equation (2.153) if and only if S_{m+1} is a solution of equation (2.157). Therefore, we can choose arbitrary $S_i \in \mathscr{B}_i^+ \subseteq \mathscr{L}_k$, $1 \leq i \leq m$, and $\varepsilon \in \mathscr{L}_k$, $\mathrm{ord}_z(\varepsilon) > 1$, such that φ_1 is still a solution of the conjugacy equation (2.153). Although φ_1 is obviously not the unique solution of the conjugacy equation (2.153), if we request the *canonical form* of φ_1, i.e., $\varphi_1 := \mathrm{id} + zS_{m+1}$, where $S_{m+1} \in \mathscr{B}_{\geq m+1}^+ \subseteq \mathscr{L}_k$, then, by Lemma 2.3.19 and Proposition 2.3.20, it follows that $S_{m+1} \in \mathscr{B}_{\geq m+1}^+ \subseteq \mathscr{L}_k$ is the unique solution of equation (2.157). Consequently, $\varphi_1 = \mathrm{id} + zS_{m+1}$ is the unique *canonical* solution of the conjugacy equation (2.153).

Proof of step (b.2)

Here we suppose that we have already applied step $(b.1)$ on $f \in \mathscr{L}_k^0$, $k \in \mathbb{N}_{\geq 1}$, such that $\mathrm{ord}\,(f - \mathrm{id}) = (1, \mathbf{0}_{m-1}, n_m, \ldots, n_k)$, $n_m \geq 1$. That is, $\mathscr{P}_{\leq (\mathbf{1}_m, n_m+1)}(f) = \mathrm{id} + zL$, for L as defined in Theorem C.

The general strategy is to transform the conjugacy equation

$$\mathscr{P}_{<r}(\varphi \circ f \circ \varphi^{-1}) = \mathrm{id} + zL, \quad \varphi \in \mathscr{L}_k^0, \tag{2.167}$$

into the differential equation stated in the following Lemma 2.3.22. Then we use Proposition 2.3.23 below and Proposition 1.2.12 to solve it.

Lemma 2.3.22 (Transforming equation (2.167) into a differential equation)**.** Let $f \in \mathscr{L}_k^0$, $k \in \mathbb{N}_{\geq 1}$, such that $f = \mathrm{id} + zR + \mathrm{h.o.b.}(z)$, for $R \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$. Put $(1, \mathbf{n}) := \mathrm{ord}(f - \mathrm{id})$ and $\mathbf{n} := (\mathbf{0}_{m-1}, n_m, \ldots, n_k)$, $n_m \geq 1$, for $1 \leq m \leq k$. Let L be as defined in Theorem C. Suppose that $z(R-L)$ contains at least one term of order strictly smaller than $\mathrm{ord}\,(\mathrm{Res}\,(f))$[6]. Put[7]

$$\begin{aligned} L &= L_m + \cdots + L_1, \\ R - L &= \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_m T_m + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1} T_{m-1} + \cdots + \boldsymbol{\ell}_1 T_1, \end{aligned} \tag{2.168}$$

where $L_i, T_i \in \mathscr{B}_i^+ \subseteq \mathscr{L}_k$, for $i = 1, \ldots, m$. Put $r := \mathrm{ord}(\mathrm{Res}(f)) = (1, 2\mathbf{n} + \mathbf{1}_k)$ and $r_0 := (\mathbf{0}_m, 2n_m + 1, \ldots, 2n_k + 1)$.
Here, we put $\varphi := \mathrm{id} + zS + \varepsilon$, for $\varepsilon \in \mathscr{L}_k$ such that $\mathrm{ord}_z(\varepsilon) > 1$, $S \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$ and decompose S as $S = S_m + \cdots + S_1$, for $S_i \in \mathscr{B}_i^+ \subseteq \mathscr{L}_k$, $1 \leq i \leq m-1$, and $S_m \in \mathscr{B}_{\geq m}^+ \subseteq \mathscr{L}_k$. The logarithmic transseries $\varphi \in \mathscr{L}_k^0$ is a solution of the equation

$$\mathscr{P}_{<r}(\varphi \circ f \circ \varphi^{-1}) = \mathrm{id} + zL$$

if and only if S_m belongs to $\mathscr{B}_m^+ \setminus \{0\} \subseteq \mathscr{L}_k$ and S_m satisfies the equation

$$\begin{aligned} &\mathscr{P}_{<r_0}\Big((1+L_m) \cdot \log(1+L_m) \cdot D_m(S_m) - D_m(L_m) \cdot (1+S_m) \cdot \log(1+S_m) + \boldsymbol{\ell}_m T_m S_m - \boldsymbol{\ell}_m T_m\Big) \\ &= \mathscr{P}_{<r_0}(\mathscr{C}(S_m)), \end{aligned} \tag{2.169}$$

where $\mathscr{C} : \mathscr{B}_{\geq 1}^+ \to \mathscr{B}_{\geq 1}^+$ is a $\frac{1}{2^{2+\mathrm{ord}_{\boldsymbol{\ell}_1}(R)}}$-contraction on the space $(\mathscr{B}_{\geq 1}^+, d_1)$.

Proof. We first apply a procedure similar to the one in the proof of Lemma 2.3.19. By comparing orders and applying the projection operator $\mathscr{P}_{<r}$, it is easy to see that the conjugacy equation $\mathscr{P}_{<r}(\varphi \circ f \circ \varphi^{-1}) = \mathrm{id} + zL$ is equivalent to the equation

$$\mathscr{P}_{<r}(\varphi \circ f) = \mathscr{P}_{<r}\big((\mathrm{id} + zL) \circ \varphi\big),$$

[6] Otherwise we skip case $(b.2)$.

[7] Note that $\mathrm{ord}(R-L) > (\mathbf{1}_m, n_m + 1, n_{m+1}, \ldots, n_k)$.

where $\varphi := \mathrm{id}+zS+\varepsilon$, for $S\in\mathscr{B}^+_{\geq 1}\subseteq\mathscr{L}_k$ and $\varepsilon\in\mathscr{L}_k$, $\mathrm{ord}_z(\varepsilon)>1$. Put $\mu := f-(\mathrm{id}+zR)$. By the Taylor Theorem we get the equivalent equation

$$\begin{aligned}&\mathscr{P}_{<r}\Big(zS+\varepsilon+\sum_{i\geq 1}\frac{(zS+\varepsilon)^{(i)}(\mathrm{id}+zR)}{i!}\mu^i+\mathrm{id}+zR+\mu+\sum_{i\geq 1}\frac{(zS)^{(i)}}{i!}(zR)^i+\sum_{i\geq 1}\frac{\varepsilon^{(i)}}{i!}(zR)^i\Big)\\&=\mathscr{P}_{<r}\Big(\mathrm{id}+zS+\varepsilon+zL+\sum_{i\geq 1}\frac{(zL)^{(i)}(\mathrm{id}+zS)}{i!}\varepsilon^i+\sum_{i\geq 1}\frac{(zL)^{(i)}}{i!}(zS)^i\Big).\end{aligned}\tag{2.170}$$

Subtracting and applying the operator $\mathscr{P}_{<r}$ to (2.170) and analyzing the orders of terms, we get that the equation $\mathscr{P}_{<r}(\varphi\circ f\circ\varphi^{-1})=\mathrm{id}+zL$ is equivalent to the equation

$$\mathscr{P}_{<r}\Big(\sum_{i\geq 1}\frac{(zS)^{(i)}}{i!}(zR)^i-\sum_{i\geq 1}\frac{(zL)^{(i)}}{i!}(zS)^i+z(R-L)\Big)=0.\tag{2.171}$$

By Lemma A.2.8, it follows that

$$\begin{aligned}&\sum_{i\geq 1}\frac{(zL)^{(i)}}{i!}(zS)^i=z\left(L\cdot S+D_1(L)\cdot(1+S)\cdot\log(1+S)+\mathscr{C}_L(S)\right),\\&\sum_{i\geq 1}\frac{(zS)^{(i)}}{i!}(zR)^i=z\left(S\cdot R+D_1(S)\cdot(1+R)\cdot\log(1+R)+\mathscr{K}_R(S)\right),\end{aligned}\tag{2.172}$$

for suitable $\frac{1}{2^{2+\mathrm{ord}_{\boldsymbol{\ell}_1}(R)}}$-contraction $\mathscr{C}_L:(\mathscr{B}^+_{\geq 1},d_1)\to(\mathscr{B}^+_{\geq 1},d_1)$ and $\frac{1}{2^{2+2\cdot\mathrm{ord}_{\boldsymbol{\ell}_1}(R)}}$-contraction $\mathscr{K}_R:(\mathscr{B}^+_{\geq 1},d_1)\to(\mathscr{B}^+_{\geq 1},d_1)$.

Since

$$\mathrm{ord}_{\boldsymbol{\ell}_1}(\mathscr{K}_R(S))\geq 2+2\cdot\mathrm{ord}_{\boldsymbol{\ell}_1}(R),$$

it follows that $\mathscr{P}_{<r}(\mathscr{K}_R(S))=0$. Consequently, by (2.171) and (2.172), we get that the equation $\mathscr{P}_{<r}(\varphi\circ f\circ\varphi^{-1})=\mathrm{id}+zL$ is equivalent to the equation

$$\begin{aligned}&\mathscr{P}_{<r}\Big(z\Big(S\cdot(R-L)+(R-L)+(1+R)\cdot\log(1+R)\cdot D_1(S)-(1+S)\cdot\log(1+S)\cdot D_1(L)\Big)\Big)\\&=\mathscr{P}_{<r}(z\mathscr{C}_L(S)).\end{aligned}\tag{2.173}$$

Now decompose $S=S_m+\cdots+S_1$, for $S_i\in\mathscr{B}^+_i\subseteq\mathscr{L}_k$, $1\leq i\leq m-1$, $S_m\in\mathscr{B}^+_{\geq m}\subseteq\mathscr{L}_k$, and $R-L$ as in (2.168). It follows from (2.173) that $S_m\in\mathscr{B}^+_m\setminus\{0\}\subseteq\mathscr{L}_k$. Suppose for a contradiction that $S_m=0$ or $\mathrm{ord}_{\boldsymbol{\ell}_m}(S_m)=0$, i.e., $S_m\in\mathscr{B}^+_{\geq m+1}\subseteq\mathscr{L}_k$.

Suppose first that $m=1$. Then $S=S_1$. Note that $\mathrm{ord}_{\boldsymbol{\ell}_1}(S\cdot(R-L))\geq 2+\mathrm{ord}_{\boldsymbol{\ell}_1}(R)$ and $\mathrm{ord}_{\boldsymbol{\ell}_1}(\mathscr{C}_L(S))\geq 2+\mathrm{ord}_{\boldsymbol{\ell}_1}(R)$. Since $S\in\mathscr{B}^+_{\geq 2}\subseteq\mathscr{L}_k$ and $R\in\mathscr{B}^+_1\subseteq\mathscr{L}_k$, it follows that

$$\mathrm{ord}_{\boldsymbol{\ell}_1}\big((1+R)\cdot\log(1+R)\cdot D_1(S)-(1+S)\cdot\log(1+S)\cdot D_1(L)\big)=\mathrm{ord}_{\boldsymbol{\ell}_1}(R)+1.$$

Now we get that the order of the left-hand side of equation (2.173) is strictly smaller than the order of the right-hand side of the same equation, which is a contradiction.

Now suppose that $m > 1$. It can be seen that $\mathrm{ord}(z\mathscr{C}_L(S)) \geq r$, and therefore, the right-hand side of equation (2.173) is equal to zero. Note that $\mathrm{ord}\,(S \cdot (R-L)) > \mathrm{ord}\,(R) + \mathrm{ord}\,(S) + (0, \mathbf{1}_m, \mathbf{0}_{k-m})$. Since $S_m \in \mathscr{B}^+_{\geq m+1} \subseteq \mathscr{L}_k$ and $L_m \in \mathscr{B}^+_m \subseteq \mathscr{L}_k$, it follows that

$$\begin{aligned}
&\mathrm{ord}\,\big((1+R)\cdot\log(1+R)\cdot D_1(S) - (1+S)\cdot\log(1+S)\cdot D_1(L)\big)\\
&= \mathrm{ord}\,(R) + \mathrm{ord}\,(S) + (0, \mathbf{1}_m, \mathbf{0}_{k-m}).
\end{aligned}$$

Now, the order of the left-hand side of equation (2.173) is strictly smaller than the order of the right-hand side of the same equation, which is a contradiction. Thus, we proved that $S_m \in \mathscr{B}^+_m \setminus \{0\} \subseteq \mathscr{L}_k$.

We finish the proof considering separately the cases $m = 1$ and $m > 1$. Suppose that $m = 1$. Then we have $S = S_1$, $R - L = \boldsymbol{\ell}_1 T_1$ and $L = L_m$. Therefore, equation (2.169) follows after dividing equation (2.173) by z.

Now, suppose that $m > 1$. Note that $D_1(S_i) = \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{i-1} D_i(S_i)$, for $2 \leq i \leq m$, $(1+R)\cdot\log(1+R) = R + \sum_{i\geq 2} \frac{(-1)^i}{i(i-1)} R^i$ and $R = L_m + L_{m-1} + \cdots + L_1 + \text{h.o.t.}$ Note that $S_m \neq 0$ by the above discussion, and $L_m \neq 0$. Analyzing the orders of the terms, it follows that

$$\begin{aligned}
&\mathscr{P}_{<r}(z(1+R)\cdot\log(1+R)\cdot D_1(S))\\
&= \mathscr{P}_{<r}(z(1+L_m)\cdot\log(1+L_m)\cdot D_1(S_m))\\
&= \mathscr{P}_{<r}(z\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{m-1}(1+L_m)\cdot\log(1+L_m)\cdot D_m(S_m)). \qquad (2.174)
\end{aligned}$$

From (2.168), it follows that

$$\mathscr{P}_{<r}(z(R-L)) = \mathscr{P}_{<r}(z\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_m T_m). \qquad (2.175)$$

Since $\mathrm{ord}_{\boldsymbol{\ell}_1}(\mathscr{C}_L(S)) \geq 2 + \mathrm{ord}_{\boldsymbol{\ell}_1}(R) > 1$, it follows that $\mathscr{P}_{<r}(z\mathscr{C}_L(S)) = 0$, $S \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_k$. Therefore,

$$\begin{aligned}
&\mathscr{P}_{<r}(z(1+S)\cdot\log(1+S)\cdot D_1(R) + \mathscr{C}_L(S))\\
&= \mathscr{P}_{<r}\big(z(1+S_m)\cdot\log(1+S_m)\cdot D_1(L_m)\big)\\
&= \mathscr{P}_{<r}(z\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{m-1}(1+S_m)\cdot\log(1+S_m)\cdot D_m(L_m)). \qquad (2.176)
\end{aligned}$$

After dividing by $z\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1}$ and using (2.174), (2.175) and (2.176), we get that equation (2.173) is equivalent to equation (2.169). Therefore, the equation (2.169) is equivalent to the equation $\mathscr{P}_{<r}(\varphi \circ f \circ \varphi^{-1}) = \mathrm{id} + zL$. ■

Proposition 2.3.23 (Solution of a differential equation)**.** Let $L \in \mathscr{B}_m^+ \setminus \{0\} \subseteq \mathscr{L}_k$, $\mathrm{ord}(L) = (\mathbf{0}_m, n_m, \ldots, n_k)$, $n_m \geq 1$, and $r_0 := (\mathbf{0}_m, 2n_m+1, \ldots, 2n_k+1)$, $1 \leq m \leq k$, $k \in \mathbb{N}_{\geq 1}$. Let $V \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, $\mathrm{ord}_{\boldsymbol{\ell}_m}(V) \geq n_m + 2$, $\mathrm{ord}(V) < r_0$. Furthermore, let $h \in x^2\mathbb{R}[[x]]$ be a power series in the variable x, with real coefficients, such that $h(0) = h'(0) = 0$, and let $\mathscr{C}_m : \mathscr{B}_m^+ \to \mathscr{B}_m^+$ be a $\frac{1}{2^{2+n_m}}$-contraction, with respect to the metric d_m. Then there exists a solution $S \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, $\mathrm{ord}(S) < \mathrm{ord}(L)$, of the equation

$$\mathscr{P}_{<r_0}\Big((L+h(L)) \cdot D_m(S) - D_m(L) \cdot (S+h(S)) + V \cdot S\Big) = \mathscr{P}_{<r_0}(V + \mathscr{C}_m(S)). \quad (2.177)$$

In the proof of Proposition 2.3.23, we need the following technical lemma.

Lemma 2.3.24. Let $K, P \in \mathscr{B}_i \subseteq \mathscr{L}_k^\infty$, $P \neq 0$, $1 \leq i \leq k$, $k \in \mathbb{N}_{\geq 1}$, and let $n_i := \mathrm{ord}_{\boldsymbol{\ell}_i}(P)$. If the order in $\boldsymbol{\ell}_i$ of each term in K is strictly smaller than $2n_i + 1$, then there exists $S \in \mathscr{B}_i \subseteq \mathscr{L}_k^\infty$ such that

$$\mathscr{P}_{<(\mathbf{0}_i, 2n_i+1)}\big(P \cdot D_i(S) - S \cdot D_i(P)\big) = K. \quad (2.178)$$

Moreover, if we impose the condition that the order in $\boldsymbol{\ell}_i$ of each term in S is strictly smaller than n_i, then such an S is unique.

Proof. Let

$$P = \boldsymbol{\ell}_i^{n_i} P_{i+1} + P_i,$$

where $P_{i+1} \in \mathscr{B}_{i+1} \subseteq \mathscr{L}_k^\infty$, and $P_i \in \mathscr{B}_i \subseteq \mathscr{L}_k^\infty$ such that $\mathrm{ord}_{\boldsymbol{\ell}_i}(P_i) \geq n_i + 1$. Let $\mathscr{S}, \mathscr{T} : \mathscr{B}_i \to \mathscr{B}_i$ be the operators defined by:

$$\mathscr{S}(S) := \mathscr{P}_{<(\mathbf{0}_i, 2n_i+1)}\big(K - P_i \cdot D_i(S) + S \cdot D_i(P_i)\big),$$
$$\mathscr{T}(S) := \mathscr{P}_{<(\mathbf{0}_i, 2n_i+1)}\big(\boldsymbol{\ell}_i^{n_i} P_{i+1} \cdot D_i(S) - S \cdot D_i(\boldsymbol{\ell}_i^{n_i} P_{i+1})\big).$$

Note that $\mathscr{S}$ is an affine $\frac{1}{2^{n_i+2}}$-contraction on the space $(\mathscr{B}_i, d_i)$ and $\mathscr{T}$ is linear. Furthermore,

$$\mathrm{ord}_{\boldsymbol{\ell}_i}\big(\boldsymbol{\ell}_i^{n_i} P_{i+1} \cdot D_i(S) - S \cdot D_i(\boldsymbol{\ell}_i^{n_i} P_{i+1})\big) = \mathrm{ord}_{\boldsymbol{\ell}_i}(S) + n_i + 1, \quad (2.179)$$

if and only if $\boldsymbol{\ell}_i^{n_i}P_{i+1}\cdot D_i(S)-S\cdot D_i(\boldsymbol{\ell}_i^{n_i}P_{i+1})\neq 0$. Now, by solving the linear differential equation, we get that $\boldsymbol{\ell}_i^{n_i}P_{i+1}\cdot D_i(S)-S\cdot D_i(\boldsymbol{\ell}_i^{n_i}P_{i+1})=0$ if and only if $S=C\cdot\boldsymbol{\ell}_i^{n_i}P_{i+1}$, $C\in\mathbb{R}$. Therefore, (2.179) holds if and only if $S\neq C\cdot\boldsymbol{\ell}_i^{n_i}P_{i+1}$, for each $C\in\mathbb{R}$.

Let $\widetilde{\mathscr{B}}$ be the space of all $S\in\mathscr{B}_i\subseteq\mathscr{L}_k^\infty$ such that either $S=0$, or every term in S has order in $\boldsymbol{\ell}_i$ strictly smaller than n_i. By (2.179), the order in $\boldsymbol{\ell}_i$ of each term in $\boldsymbol{\ell}_i^{n_i}P_{i+1}D_i(S)-SD_i(\boldsymbol{\ell}_i^{n_i}P_{i+1})$, $S\in\widetilde{\mathscr{B}}$, is strictly smaller than $2n_i+1$. Consequently, it follows that

$$\mathscr{T}|_{\widetilde{\mathscr{B}}}(S)=\boldsymbol{\ell}_i^{n_i}P_{i+1}\cdot D_i(S)-S\cdot D_i(\boldsymbol{\ell}_i^{n_i}P_{i+1}),\quad S\in\widetilde{\mathscr{B}}. \tag{2.180}$$

Let $S_1,S_2\in\widetilde{\mathscr{B}}$ such that $S_1\neq S_2$. Suppose that $\mathscr{T}|_{\widetilde{\mathscr{B}}}(S_1-S_2)=0$. Solving the linear differential equation as above, we get that $S_1-S_2=C\cdot\boldsymbol{\ell}_i^{n_i}P_{i+1}$, for some $C\in\mathbb{R}$. Since, $S_1,S_2\in\widetilde{\mathscr{B}}$, we get $S_1=S_2$, which is a contradiction. Therefore, $\mathscr{T}|_{\widetilde{\mathscr{B}}}(S_1-S_2)\neq 0$. By (2.179) and (2.180), it follows that

$$\mathrm{ord}_{\boldsymbol{\ell}_i}(\mathscr{T}|_{\widetilde{\mathscr{B}}}(S_1-S_2))=\mathrm{ord}_{\boldsymbol{\ell}_i}(S_1-S_2)+n_i+1.$$

Now we get that $\mathscr{T}|_{\widetilde{\mathscr{B}}}:\widetilde{\mathscr{B}}\to\mathscr{B}_i$ is a linear $\frac{1}{2^{n_i+1}}$-homothety, with respect to the metric d_i. Now, equation (2.178) is equivalent to the fixed point equation

$$\mathscr{T}|_{\widetilde{\mathscr{B}}}(S)=\mathscr{S}|_{\widetilde{\mathscr{B}}}(S),\quad S\in\widetilde{\mathscr{B}}.$$

We now prove that $\mathscr{S}(\widetilde{\mathscr{B}})\subseteq\mathscr{T}(\widetilde{\mathscr{B}})$. Then we conclude by Proposition 1.2.12.

Suppose that $M\in\mathscr{S}(\widetilde{\mathscr{B}})$. Then, by definition of $\mathscr{S}$, each term in M is of order in $\boldsymbol{\ell}_i$ strictly smaller than $2n_i+1$. We solve the equation $\mathscr{T}|_{\widetilde{\mathscr{B}}}(S)=M$, i.e.

$$\boldsymbol{\ell}_i^{n_i}P_{i+1}D_i(S)-SD_i(\boldsymbol{\ell}_i^{n_i}P_{i+1})=M,\quad S\in\widetilde{\mathscr{B}}.$$

This is a linear ordinary differential equation whose solutions are given by:

$$\begin{aligned}S&=\exp\Big(\int\frac{D_i(\boldsymbol{\ell}_i^{n_i}P_{i+1})}{\boldsymbol{\ell}_i^{n_i}P_{i+1}}\frac{d\boldsymbol{\ell}_i}{\boldsymbol{\ell}_i^2}\Big)\Big(C+\int\Big(\frac{M}{\boldsymbol{\ell}_i^{n_i}P_{i+1}}\cdot\exp\Big(-\int\frac{D_i(\boldsymbol{\ell}_i^{n_i}P_{i+1})}{\boldsymbol{\ell}_i^{n_i}P_{i+1}}\frac{d\boldsymbol{\ell}_i}{\boldsymbol{\ell}_i^2}\Big)\frac{d\boldsymbol{\ell}_i}{\boldsymbol{\ell}_i^2}\Big)\Big)\\&=\exp\big(\log(\boldsymbol{\ell}_i^{n_i}P_{i+1})\big)\Big(C+\int\frac{M}{\boldsymbol{\ell}_i^{n_i}P_{i+1}}\cdot\big(\exp(\log(\boldsymbol{\ell}_i^{n_i}P_{i+1}))\big)^{-1}\frac{d\boldsymbol{\ell}_i}{\boldsymbol{\ell}_i^2}\Big)\\&=\boldsymbol{\ell}_i^{n_i}P_{i+1}\Big(C+\int\frac{M}{(\boldsymbol{\ell}_i^{n_i}P_{i+1})^2}\frac{d\boldsymbol{\ell}_i}{\boldsymbol{\ell}_i^2}\Big),\quad C\in\mathbb{R}.\end{aligned} \tag{2.181}$$

By Lemma B.2.6, it follows that $S\in\mathscr{B}_i\subseteq\mathscr{L}_k^\infty$, for each $C\in\mathbb{R}$. Moreover, taking $C=0$, we get $S\in\widetilde{\mathscr{B}}\subseteq\mathscr{B}_i\subseteq\mathscr{L}_k^\infty$. ■

Proof of Proposition 2.3.23. Note that:

$$L = \boldsymbol{\ell}_m^{n_m} P_{m+1} + P_m, \quad n_m \geq 1,$$

where $P_{m+1} \in \mathscr{B}_{m+1} \subseteq \mathscr{L}_k^\infty$ and $P_m \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, $\mathrm{ord}_{\boldsymbol{\ell}_m}(P_m) \geq n_m + 1$. Put:

$$\begin{aligned}
\mathscr{S}_m(S) &:= \mathscr{P}_{<r_0}\Big(V + \mathscr{C}_m(S) - h(L)\cdot D_m(S) + h(S)\cdot D_m(L) - SV\Big),\\
\mathscr{T}_m(S) &:= \mathscr{P}_{<r_0}\Big(L\cdot D_m(S) - S\cdot D_m(L)\Big).
\end{aligned} \tag{2.182}$$

Note that (2.177), for $S \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, is equivalent to

$$\mathscr{T}_m(S) = \mathscr{S}_m(S). \tag{2.183}$$

Let $S_1, S_2 \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, $S_1 \neq S_2$. Note that:

$$\begin{aligned}
&\mathrm{ord}_{\boldsymbol{\ell}_m}(\mathscr{C}_m(S_1) - \mathscr{C}_m(S_2)) \geq \mathrm{ord}_{\boldsymbol{\ell}_m}(S_1 - S_2) + n_m + 2,\\
&\mathrm{ord}_{\boldsymbol{\ell}_m}((S_1 - S_2)\cdot V) \geq \mathrm{ord}_{\boldsymbol{\ell}_m}(S_1 - S_2) + n_m + 2,\\
&\mathrm{ord}_{\boldsymbol{\ell}_m}(h(L)\cdot D_m(S_1 - S_2)) \geq \mathrm{ord}_{\boldsymbol{\ell}_m}(S_1 - S_2) + 2n_m + 1,\\
&\mathrm{ord}_{\boldsymbol{\ell}_m}((h(S_1) - h(S_2))\cdot D_m(L)) \geq \mathrm{ord}_{\boldsymbol{\ell}_m}(S_1 - S_2) + \min\big\{\mathrm{ord}_{\boldsymbol{\ell}_m}(S_1), \mathrm{ord}_{\boldsymbol{\ell}_m}(S_2)\big\} + n_m + 1,
\end{aligned} \tag{2.184}$$

The last inequality in (2.184) is obtained using the identity

$$h(S_1) - h(S_2) = (S_1 - S_2)\cdot \sum_{i\geq 2} h_i\Big(\sum_{j=0}^{i-1} S_1^{i-j}\cdot S_2^j\Big),$$

where $h(x) := \sum_{i\geq 2} h_i x^i$, $h_i \in \mathbb{R}$, $i \geq 2$. By (2.184) and since $n_m \geq 1$ and $\mathrm{ord}_{\boldsymbol{\ell}_m}(S_1), \mathrm{ord}_{\boldsymbol{\ell}_m}(S_1) \geq 1$, it follows that $\mathscr{S}_m$ is a $\frac{1}{2^{n_m+2}}$-contraction on the space $(\mathscr{B}_m^+, d_m)$.

Let $\widetilde{\mathscr{B}} \subseteq \mathscr{B}_m^+ \subseteq \mathscr{L}_k$ be the space that contains 0 and all logarithmic transseries in $\mathscr{B}_m^+ \subseteq \mathscr{L}_k$ that contain only terms of order strictly smaller than $(\mathbf{0}_m, n_m, \ldots, n_k) = \mathrm{ord}(L)$. Note that the restriction $\mathscr{T}_m|_{\widetilde{\mathscr{B}}} : \widetilde{\mathscr{B}} \to \mathscr{B}_m^+$ is a linear operator. For $S \in \widetilde{\mathscr{B}} \setminus \{0\}$ we obtain that

$$\mathrm{ord}_{\boldsymbol{\ell}_m}(\mathscr{T}_m(S)) = \mathrm{ord}_{\boldsymbol{\ell}_m}(S) + n_m + 1, \tag{2.185}$$

if $\mathscr{T}_m(S) \neq 0$. Suppose that $\mathscr{T}_m(S) = 0$. Then there exists $N \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, $\mathrm{ord}(N) \geq r_0$, such that $L\cdot D_m(S) - S\cdot D_m(L) = N$. Dividing by L and solving the linear differential equation, we get

$$S = L\cdot\Big(C + \int \frac{N}{L^2}\cdot\frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\Big),$$

for $C \in \mathbb{R}$. Since $S \in \widetilde{\mathscr{B}}$, we get that $N = 0$ and $C = 0$, which implies that $S = 0$. Consequently, if $S \in \widetilde{\mathscr{B}} \setminus \{0\}$, then $\mathscr{T}_m(S) \neq 0$, and therefore, (2.185) holds. Since $\mathscr{T}_m|_{\widetilde{\mathscr{B}}}$ is linear, it follows that $\mathscr{T}_m|_{\widetilde{\mathscr{B}}}$ is a $\frac{1}{2^{n_m+1}}$-homothety, with respect to the metric d_m.

Let us prove that $\mathscr{S}_m(\widetilde{\mathscr{B}}) \subseteq \mathscr{T}_m(\widetilde{\mathscr{B}})$. Let $K \in \mathscr{S}_m(\widetilde{\mathscr{B}}) \subseteq \mathscr{B}_m^+ \subseteq \mathscr{L}_k$ be arbitrary. By (2.182) and (2.184), we get $\mathrm{ord}_{\boldsymbol{\ell}_m}(K) \geq n_m + 2$ and $\mathrm{ord}(K) < r_0$. We find $S \in \widetilde{\mathscr{B}}$, such that $\mathscr{T}_m(S) = K$, i.e.

$$\mathscr{P}_{<r_0}(L \cdot D_m(S) - S \cdot D_m(L)) = K. \tag{2.186}$$

We have the following decomposition:

$$K = K_m + \boldsymbol{\ell}_m^{2n_m+1} K_{m+1} + \boldsymbol{\ell}_m^{2n_m+1} \boldsymbol{\ell}_{m+1}^{2n_{m+1}+1} K_{m+2} + \cdots + \boldsymbol{\ell}_m^{2n_m+1} \cdots \boldsymbol{\ell}_{k-1}^{2n_{k-1}+1} K_k,$$

where $K_m \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, $n_m + 2 \leq \mathrm{ord}_{\boldsymbol{\ell}_m}(K_m) < 2n_m + 1$, and $K_{m+i} \in \mathscr{B}_{m+i} \subseteq \mathscr{L}_k^\infty$, $\mathrm{ord}_{\boldsymbol{\ell}_{m+i}}(K_{m+i}) < 2n_{m+i} + 1$, $1 \leq i \leq k - m$.

Now we proceed to solving (2.186) inductively in $k - m + 1$ steps.

Step 1. We first solve the equation $\mathscr{P}_{<(\mathbf{0}_m, 2n_m+1)}(\mathscr{T}_m(S)) = K_m$, i.e.

$$\mathscr{P}_{<(\mathbf{0}_m, 2n_m+1)}(L \cdot D_m(S) - S \cdot D_m(L)) = K_m.$$

By Lemma 2.3.24, there exists a solution $S_m \in \mathscr{B}_m \subseteq \mathscr{L}_k^\infty$ of the previous equation, such that the order (in $\boldsymbol{\ell}_m$) of every term in S_m is strictly smaller than n_m. Since $\mathrm{ord}_{\boldsymbol{\ell}_m}(K_m) \geq n_m + 2$, it follows, by (2.181), that $S_m \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$. Now, put

$$\boldsymbol{\ell}_m^{2n_m+1} G_{m+1} := \mathscr{P}_{<r_0}(\mathscr{T}_m(S_m) - K_m), \tag{2.187}$$

where $G_{m+1} \in \mathscr{B}_{m+1} \subseteq \mathscr{L}_k^\infty$ is such that $\mathrm{ord}(\boldsymbol{\ell}_m^{2n_m+1} G_{m+1}) < r_0$.

Step 2. Now, with $S_m \in \mathscr{B}_m \subseteq \mathscr{L}_k^\infty$ from the previous step, we solve in the variable $S \in \mathscr{B}_{m+1} \subseteq \mathscr{L}_k^\infty$ the equation

$$\begin{aligned} &\mathscr{P}_{<(0, \mathbf{0}_{m-1}, 2n_m+1, 2n_{m+1}+1)}\Big(L \cdot D_m(S_m + \boldsymbol{\ell}_m^{n_m} S) - (S_m + \boldsymbol{\ell}_m^{n_m} S) \cdot D_m(L)\Big) \\ &= K_m + \boldsymbol{\ell}_m^{2n_m+1} K_{m+1}. \end{aligned} \tag{2.188}$$

Let us decompose L as

$$L = L_m + \boldsymbol{\ell}_m^{n_m} L_{m+1},$$

where $L_m \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, $n_m + 1 \leq \mathrm{ord}_{\boldsymbol{\ell}_m}(L_m)$, and $L_{m+1} \in \mathscr{B}_{m+1} \subseteq \mathscr{L}_k^\infty$. By (2.187) from the previous step, we get:

$$\begin{aligned} &\mathscr{P}_{<(\mathbf{0}_m, 2n_m+1, 2n_{m+1}+1)}(L \cdot D_m(S_m) - S_m \cdot D_m(L)) \\ &= K_m + \mathscr{P}_{<(\mathbf{0}_m, 2n_m+1, 2n_{m+1}+1)}(\boldsymbol{\ell}_m^{2n_m+1} G_{m+1}) \\ &= K_m + \boldsymbol{\ell}_m^{2n_m+1} \mathscr{P}_{<(\mathbf{0}_{m+1}, 2n_{m+1}+1)}(G_{m+1}). \end{aligned} \tag{2.189}$$

By (2.189), it follows that equation (2.188) is equivalent to the equation:

$$\begin{aligned} &\mathscr{P}_{<(\mathbf{0}_{m+1}, 2n_{m+1}+1)}\big(L_{m+1} \cdot D_{m+1}(S) - S \cdot D_{m+1}(L_{m+1})\big) \\ &= K_{m+1} - \mathscr{P}_{<(\mathbf{0}_{m+1}, 2n_{m+1}+1)}(G_{m+1}). \end{aligned}$$

By Lemma 2.3.24, there exists a solution $S := S_{m+1} \in \mathscr{B}_{m+1} \subseteq \mathscr{L}_k^\infty$ of the previous equation, such that each term in S_{m+1} is of order in $\boldsymbol{\ell}_{m+1}$ strictly smaller than n_{m+1}. Now, put:

$$\boldsymbol{\ell}_m^{2n_m+1} \boldsymbol{\ell}_{m+1}^{2n_{m+1}+1} G_{m+2} := \mathscr{P}_{<r_0}\big(\mathscr{T}(S_m + S_{m+1}) - K_m - \boldsymbol{\ell}_m^{2n_m+1} K_{m+1}\big),$$

where $G_{m+2} \in \mathscr{B}_{m+2} \subseteq \mathscr{L}_k^\infty$ such that $\mathrm{ord}(\boldsymbol{\ell}_m^{2n_m+1} \boldsymbol{\ell}_{m+1}^{2n_{m+1}+1} G_{m+2}) < r_0$.

Inductively, in $k - m + 1$ steps, we find $S_m, \ldots, S_k$. Now, put:

$$S := S_m + \boldsymbol{\ell}_m^{n_m} S_{m+1} \cdots + \boldsymbol{\ell}_m^{n_m} \cdots \boldsymbol{\ell}_{k-1}^{n_{k-1}} S_k.$$

Note that $S \in \widetilde{\mathscr{B}}$, and, by the induction, S is a solution of the equation $\mathscr{T}_m|_{\widetilde{\mathscr{B}}}(S) = K$. Therefore, $\mathscr{S}_m(\widetilde{\mathscr{B}}) \subseteq \mathscr{T}_m(\widetilde{\mathscr{B}})$.

Now we conclude by Proposition 1.2.12. ■

Proof of step $(b.2)$. For simplicity of notation we denote again by f the logarithmic transseries $\varphi_1 \circ f \circ \varphi_1^{-1}$, with $\varphi_1 \in \mathscr{L}_k^0$ obtained in step $(b.1)$. Now, put

$$f = \mathrm{id} + zR + \mathrm{h.o.b.}(z), \quad R \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k.$$

Let $\mathrm{ord}(f - \mathrm{id}) := (1, \mathbf{0}_{m-1}, n_m, \ldots, n_k)$, for $n_m \geq 1$, $1 \leq m \leq k$, and let L be as defined in Theorem C. Suppose additionally that $R \neq L$ and that $\mathrm{ord}(z(R - L)) < \mathrm{ord}(\mathrm{Res}(f))$. Otherwise, we skip step $(b.2)$ and proceed directly to step $(b.3)$.

Now, f satisfies the assumptions of Lemma 2.3.22. Therefore, we transform the conjugacy equation $\mathscr{P}_{<r}(\varphi_2 \circ f \circ \varphi_2^{-1}) = \mathrm{id} + zL$ to equation (2.169).

Now, put $V := \boldsymbol{\ell}_m T_m \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, where T_m is from the decomposition (2.168) in Lemma 2.3.22. Since step $(b.1)$ has been applied on f, note that $\mathscr{P}_{\leq(\mathbf{1}_m, n_m+1)}(f) = \mathrm{id} + zL$, and therefore, $\mathrm{ord}_{\boldsymbol{\ell}_m}(V) \geq \mathrm{ord}_{\boldsymbol{\ell}_m}(L) + 2$. Let $h := (1+x)\log(1+x) - x \in x^2\mathbb{R}[[x]]$. If $m = 1$, we take $\mathscr{C}_1 := \mathscr{C} : \mathscr{B}_1^+ \to \mathscr{B}_1^+$, for $\frac{1}{2^{2+n_1}}$-contraction $\mathscr{C}$, with respect to the metric d_1, from equation (2.169). If $m > 1$, take $\mathscr{C}_m : \mathscr{B}_m^+ \to \mathscr{B}_m^+$, such that $\mathscr{C}_m(S) := 0$, for each $S \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$. Evidently, for $m \geq 1$, $\mathscr{C}_m : \mathscr{B}_m^+ \to \mathscr{B}_m^+$ is a $\frac{1}{2^{2+n_m}}$-contraction, with respect to the metric d_m. Therefore, we can apply Proposition 2.3.23, so there exists a unique solution $S \in \widetilde{\mathscr{B}} \subseteq \mathscr{B}_m^+ \subseteq \mathscr{L}_k$ (for $\widetilde{\mathscr{B}}$ as defined in the proof of Proposition 2.3.23) of equation (2.169). Now, by Lemma 2.3.22, it follows that $\varphi_2 := \mathrm{id} + zS$ (here, we take the solution with the simplest choice $\varepsilon := 0$ in Lemma 2.3.22) is a solution of the equation

$$\mathscr{P}_{<r}(\varphi_2 \circ f \circ \varphi_2^{-1}) = \mathrm{id} + zL.$$

Therefore, there exists $c \in \mathbb{R}$ such that:

$$\varphi_2 \circ f \circ \varphi_2^{-1} = \mathrm{id} + zL + c\mathrm{Res}(f) + \text{h.o.t.}$$

This completes step $(b.2)$. ■

Remark 2.3.25 (Non-uniqueness of a conjugacy φ_2 in step $(b.2)$)**.** Let $\varphi_2 := \mathrm{id} + zS + \varepsilon$, for $S \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$ and $\varepsilon \in \mathscr{L}_k$, $\mathrm{ord}_z(\varepsilon) > 1$, be a parabolic solution of the conjugacy equation (2.154). We do not claim the uniqueness of the solution φ_2. Let us decompose S as $S = S_m + S_{m-1} + \cdots + S_1$, for $S_i \in \mathscr{B}_i^+ \subseteq \mathscr{L}_k$, $1 \leq i \leq m-1$, and $S_m \in \mathscr{B}_{\geq m}^+ \subseteq \mathscr{L}_k$. By Lemma 2.3.22 it follows that φ_2 is a solution of equation (2.154) if and only if $S_m \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$ and if S_m is a solution of the differential equation given in (2.169). Therefore, we can choose arbitrarily $S_i \in \mathscr{B}_i^+ \subseteq \mathscr{L}_k$, $1 \leq i \leq m-1$, and $\varepsilon \in \mathscr{L}_k$, $\mathrm{ord}_z(\varepsilon) > 1$, such that φ_2 is still a solution of (2.154). Although φ_2 is not unique, if we request the *canonical form* of φ_2, that is, $\varphi_2 := \mathrm{id} + zS_m$, for $S_m \in \widetilde{\mathscr{B}} \subseteq \mathscr{B}_m^+ \subseteq \mathscr{L}_k$ (for $\widetilde{\mathscr{B}}$ as defined in the proof of Proposition 2.3.23), then, by Lemma 2.3.22, S_m satisfies the differential equation (2.169) and, by Proposition 2.3.23, such an S_m is unique in $\widetilde{\mathscr{B}}$. Therefore, if φ_2 is a *canonical* solution of the conjugacy equation (2.154), then it is unique.

Proof of step (b.3)

We first transform the conjugacy equation (2.155) into the fixed point equation given in Lemma 2.3.26, and then use Proposition 2.3.27 and Proposition 1.2.12 to solve it.

Lemma 2.3.26 (Transforming the conjugacy equation (2.155) to a fixed point equation)**.** Let $f \in \mathscr{L}_k^0$, $k \in \mathbb{N}_{\geq 1}$, such that $f := \mathrm{id} + zR + \mu$, $R \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, $\mu \in \mathscr{L}_k$, $\mathrm{ord}_z(\mu) > 1$, and $g := \mathrm{id} + zT$, $T \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$. The conjugacy equation

$$\varphi \circ f \circ \varphi^{-1} = g,$$

where we write $\varphi := \mathrm{id} + zS + \varepsilon$, for $S \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, $\varepsilon \in \mathscr{L}_k$, $\mathrm{ord}_z(\varepsilon) > 1$, is equivalent to the fixed point equation

$$\mathscr{T}_f(zS + \varepsilon) = \mathscr{S}_f(zS + \varepsilon)$$

on the space $\mathscr{L}_k^1$, where:

$$\begin{aligned} \mathscr{S}_f(zS+\varepsilon) &:= \sum_{i\geq 1} \frac{(zS+\varepsilon)^{(i)}(\mathrm{id}+zR)}{i!}\mu^i - \sum_{i\geq 2} \frac{(zT)^{(i)}(\mathrm{id}+zS)}{i!}\varepsilon^i + z(R-T) + \mu, \\ \mathscr{T}_f(zS+\varepsilon) &:= (zT)' \circ (\mathrm{id}+zS)\cdot\varepsilon - \sum_{i\geq 1} \frac{\varepsilon^{(i)}}{i!}(zR)^i + \sum_{i\geq 1} \frac{(zT)^{(i)}}{i!}(zS)^i - \sum_{i\geq 1} \frac{(zS)^{(i)}}{i!}(zR)^i. \end{aligned} \tag{2.190}$$

Proof. The conjugacy equation $\varphi \circ f \circ \varphi^{-1} = g$ is equivalent to the equation:

$$\varphi \circ f = g \circ \varphi,$$

for $\varphi := \mathrm{id} + zS + \varepsilon$, $S \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, $\varepsilon \in \mathscr{L}_k$, $\mathrm{ord}_z(\varepsilon) > 1$. By the Taylor Theorem (Proposition A.1.6) we get that:

$$\begin{aligned} &zS + \varepsilon + \sum_{i\geq 1} \frac{(zS+\varepsilon)^{(i)}(\mathrm{id}+zR)}{i!}\mu^i + \mathrm{id} + zR + \mu + \sum_{i\geq 1}\frac{(zS)^{(i)}}{i!}(zR)^i + \sum_{i\geq 1}\frac{\varepsilon^{(i)}}{i!}(zR)^i \\ &\quad = \mathrm{id} + zS + \varepsilon + zT + \sum_{i\geq 1} \frac{(zT)^{(i)}(\mathrm{id}+zS)}{i!}\varepsilon^i + \sum_{i\geq 1}\frac{(zT)^{(i)}}{i!}(zS)^i. \end{aligned} \tag{2.191}$$

Let us define the operators $\mathscr{S}_f, \mathscr{T}_f : \mathscr{L}_k^1 \to \mathscr{L}_k^1$ as in (2.190). By (2.191), it follows that the conjugacy equation is equivalent to the equation $\mathscr{S}_f(zS+\varepsilon) = \mathscr{T}_f(zS+\varepsilon)$, for $S \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, $\varepsilon \in \mathscr{L}_k$, $\mathrm{ord}_z(\varepsilon) > 1$. ■

Proposition 2.3.27 (Properties of $\mathscr{S}_f$ and $\mathscr{T}_f$)**.** Let $f \in \mathscr{L}_k^0$, $k \in \mathbb{N}_{\geq 1}$, such that $f := \mathrm{id} + zR + \mu$, for $R \in \mathscr{B}_{\geq 1}^+ \setminus \{0\} \subseteq \mathscr{L}_k$, and let $\alpha := \mathrm{ord}_z(\mu) > 1$. Let $g := \mathrm{id} + zT$, $T \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, and let $\mathscr{S}_f, \mathscr{T}_f : z\mathscr{B}_{\geq 1}^+ \oplus \mathscr{L}_k^\alpha \to z\mathscr{B}_{\geq 1}^+ \oplus \mathscr{L}_k^\alpha$ be operators defined in (2.190). If $\mathrm{ord}(z(R-T)) > \mathrm{ord}(\mathrm{Res}(f))$, then:

1. the operator $\mathscr{S}_f$ is a $\frac{1}{2^{\alpha-1}}$-contraction on the space $(z\mathscr{B}^+_{\geq 1} \oplus \mathscr{L}^\alpha_k, d_z)$,

2. the restriction of the operator $\mathscr{T}_f$ to the space

$$\widetilde{\mathscr{L}} := \left\{ h \in z\mathscr{B}^+_{\geq 1} \oplus \mathscr{L}^\alpha_k : \mathrm{ord}\,(h) > \mathrm{ord}\,(f - \mathrm{id}) \right\}$$

is an isometry, with respect to the metric d_z,

3. $\mathscr{S}_f(\widetilde{\mathscr{L}}) \subseteq \mathscr{T}_f(\widetilde{\mathscr{L}})$.

The following Proposition 2.3.28 is needed in the proof of statement 3 in Proposition 2.3.27.

Proposition 2.3.28. Let $M, R, V \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_k$, $k \in \mathbb{N}_{\geq 1}$, such that $R \neq 0$ and $\mathrm{ord}\,(V), \mathrm{ord}\,(M) > 2 \cdot \mathrm{ord}\,(R) + (0, 1_k)$. Let

$$\widetilde{\mathscr{B}} := \left\{ K \in \mathscr{B}^+_{\geq 1} : \mathrm{ord}\,(K) > \mathrm{ord}\,(R) \right\},$$

and let $\mathscr{T}, \mathscr{S} : \widetilde{\mathscr{B}} \to \mathscr{B}^+_{\geq 1}$ be the operators defined by

$$\begin{aligned} \mathscr{T}(S) &:= -(1+R) \cdot \log(1+R) \cdot D_1(S) + (1+S) \cdot \log(1+S) \cdot D_1(R) + S \cdot V, \\ \mathscr{S}(S) &:= M + \mathscr{K}_1(S) - \mathscr{C}_1(S) - (1+S) \cdot \log(1+S) \cdot D_1(V), \quad S \in \widetilde{\mathscr{B}}, \end{aligned} \tag{2.192}$$

where $\mathscr{C}_1, \mathscr{K}_1 : \mathscr{B}^+_{\geq 1} \to \mathscr{B}^+_{\geq 1}$ are $\frac{1}{2^{2+\mathrm{ord}_{\boldsymbol{\ell}_1}(R)}}$-contractions on the space $(\mathscr{B}^+_{\geq 1}, d_1)$. Then:

1. the operator $\mathscr{S}$ is a $\frac{1}{2^{2+\mathrm{ord}_{\boldsymbol{\ell}_1}(R)}}$-contraction, with respect to the metric d_1,

2. the operator $\mathscr{T}$ is a $\frac{1}{2^{1+\mathrm{ord}_{\boldsymbol{\ell}_1}(R)}}$-homothety, with respect to the metric d_1,

3. $\mathscr{S}(\widetilde{\mathscr{B}}) \subseteq \mathscr{T}(\widetilde{\mathscr{B}})$.

Proof. 1. Let $S_1, S_2 \in \widetilde{\mathscr{B}}$, $S_1 \neq S_2$, be arbitrary. Since $\mathrm{ord}\,(V) > 2 \cdot \mathrm{ord}\,(R) + (0, \mathbf{1}_k)$ and $(1+x)\log(1+x) = x + \sum_{i \geq 2} \frac{(-1)^i}{i(i-1)} x^i$, it follows that

$$\begin{aligned} &\mathrm{ord}_{\boldsymbol{\ell}_1}\Big((1+S_1) \cdot \log(1+S_1) \cdot D_1(V) - (1+S_2) \cdot \log(1+S_2) \cdot D_1(V) \Big) \\ &\geq \mathrm{ord}_{\boldsymbol{\ell}_1}(S_1 - S_2) + 2 \cdot \mathrm{ord}_{\boldsymbol{\ell}_1}(R) + 2. \end{aligned}$$

Since $\mathscr{C}_1$ and $\mathscr{K}_1$ are $\frac{1}{2^{2+\mathrm{ord}_{\boldsymbol{\ell}_1}(R)}}$-contractions, by (2.192), it follows that $\mathscr{S}$ is a $\frac{1}{2^{2+\mathrm{ord}_{\boldsymbol{\ell}_1}(R)}}$-contraction on the space $(\widetilde{\mathscr{B}}, d_1)$.

2. Let $S_1, S_2 \in \widetilde{\mathscr{B}}$, $S_1 \neq S_2$, be arbitrary. Since $\mathrm{ord}(V) > 2\cdot\mathrm{ord}(R) + (0, \mathbf{1}_k)$ and $(1+x)\log(1+x) = x + \sum_{i\geq 2} \frac{(-1)^i}{i(i-1)} x^i$ by (2.192), it follows that:

$$\begin{aligned}\mathscr{T}(S_1) - \mathscr{T}(S_2) &= -(1+R)\cdot\log(1+R)\cdot D_1(S_1-S_2) + (S_1-S_2)\cdot V \\ &\quad + \big((1+S_1)\cdot\log(1+S_1) - (1+S_2)\cdot\log(1+S_2)\big)\cdot D_1(R) \\ &= \mathrm{Lt}\big((S_1-S_2)\cdot D_1(R) - R\cdot D_1(S_1-S_2)\big) + \text{h.o.t.}, \end{aligned} \tag{2.193}$$

if $(S_1-S_2)D_1(R) - RD_1(S_1-S_2) \neq 0$. Suppose that $(S_1-S_2)D_1(R) - RD_1(S_1-S_2) = 0$. By solving the linear differential equation, we get

$$S_1 - S_2 = C\cdot R, \quad C \in \mathbb{R}.$$

Since $S_1 - S_2 \in \widetilde{\mathscr{B}}$, we get that $C = 0$, i.e., $S_1 = S_2$, which is a contradiction. Therefore, $(S_1-S_2)\cdot D_1(R) - R\cdot D_1(S_1-S_2) \neq 0$, and (2.193) holds. By (2.193), we get that:

$$\mathrm{ord}_{\boldsymbol{\ell}_1}\big(\mathscr{T}(S_1) - \mathscr{T}(S_2)\big) = \mathrm{ord}_{\boldsymbol{\ell}_1}(S_1-S_2) + \mathrm{ord}_{\boldsymbol{\ell}_1}(R) + 1.$$

Therefore, $\mathscr{T}$ is a $\frac{1}{2^{1+\mathrm{ord}_{\boldsymbol{\ell}_1}(R)}}$-homothety on the space $(\widetilde{\mathscr{B}}, d_1)$.

3. Let $W \in \mathscr{S}(\widetilde{\mathscr{B}})$. Since $\mathrm{ord}(M), \mathrm{ord}(V) > 2\cdot\mathrm{ord}(R) + (0, \mathbf{1}_k)$ and $\mathscr{C}_1, \mathscr{K}_1 : \mathscr{B}^+_{\geq 1} \to \mathscr{B}^+_{\geq 1}$ are $\frac{1}{2^{2+\mathrm{ord}_{\boldsymbol{\ell}_1}(R)}}$-contractions on $(\mathscr{B}^+_{\geq 1}, d_1)$, it follows that W belongs to $\mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_k$ and satisfies $\mathrm{ord}(W) > 2\cdot\mathrm{ord}(R) + (0, \mathbf{1}_k)$. We solve the equation $\mathscr{T}(S) = W$ in the space $\widetilde{\mathscr{B}}$. Using (2.192), this equation becomes

$$-(1+R)\cdot\log(1+R)\cdot D_1(S) + (1+S)\cdot\log(1+S)\cdot D_1(R) + S\cdot V = W.$$

Since $R \neq 0$, after dividing by $-(1+R)\log(1+R)$, we get the equivalent equation:

$$D_1(S) - S\cdot\frac{(D_1(R)+V)}{(1+R)\cdot\log(1+R)} - \frac{((1+S)\cdot\log(1+S) - S)\cdot D_1(R)}{(1+R)\cdot\log(1+R)} = -\frac{W}{(1+R)\cdot\log(1+R)}. \tag{2.194}$$

For the purpose of applying Proposition B.5.7, put:

$$\begin{aligned} h &:= (1+x)\log(1+x) - x, \\ N &:= \log(1+R), \\ K &:= \frac{V}{(1+R)\cdot\log(1+R)}, \\ T &:= -\frac{D_1(R)}{(1+R)\cdot\log(1+R)} = -\frac{D_1(N)}{N}, \\ M &:= -\frac{W}{(1+R)\cdot\log(1+R)}. \end{aligned}$$

Since $\mathrm{ord}(V) > 2\cdot\mathrm{ord}(R)+(0,\mathbf{1}_k)$, it follows that $\mathrm{ord}(K) > (0,\mathbf{1}_k)$. Since $\mathrm{ord}(W) > 2\cdot\mathrm{ord}(R)+(0,\mathbf{1}_k)$ and $\mathrm{ord}(N)=\mathrm{ord}(R)$, it follows that $\mathrm{ord}(M)-\mathrm{ord}(N) > (0,\mathbf{1}_k)$. By Proposition B.5.7, there exists the unique solution $S\in\widetilde{\mathscr{B}}$, of the differential equation (2.194). Therefore, $S\in\widetilde{\mathscr{B}}$ is the unique solution of $\mathscr{T}(S)=W$ on the space $\widetilde{\mathscr{B}}$. This implies that $\mathscr{S}(\widetilde{\mathscr{B}})\subseteq\mathscr{T}(\widetilde{\mathscr{B}})$. ■

Proof of Proposition 2.3.28. 1. First, note that $\mathscr{S}_f(z\mathscr{B}_{\geq 1}^{+}\oplus\mathscr{L}_k^{\alpha}),\mathscr{T}_f(z\mathscr{B}_{\geq 1}^{+}\oplus\mathscr{L}_k^{\alpha})\subseteq z\mathscr{B}_{\geq 1}^{+}\oplus\mathscr{L}_k^{\alpha}$. Indeed, let $S\in\mathscr{B}_{\geq 1}^{+}$ and $\varepsilon\in\mathscr{L}_k^{\alpha}$. Since $(zT)^{(i)}(\mathrm{id}+zS)=\mathrm{Lt}\big((zT)^{(i)}\big)+$ h.o.t., $(zS+\varepsilon)^{(i)}(\mathrm{id}+zR)=\mathrm{Lt}\big((zS)^{(i)}\big)+$h.o.t., it follows that:

$$\begin{aligned}
&\mathrm{ord}_z\Big(\sum_{i\geq 1}\frac{(zS+\varepsilon)^{(i)}(\mathrm{id}+zR)}{i!}\mu^i\Big)\geq\mathrm{ord}_z(\mu)=\alpha,\\
&\mathrm{ord}_z\Big(\sum_{i\geq 2}\frac{(zT)^{(i)}(\mathrm{id}+zS)}{i!}\varepsilon^i\Big)\geq 2\cdot\mathrm{ord}_z(\varepsilon)-1\geq\alpha.
\end{aligned}\tag{2.195}$$

Therefore, $\mathscr{S}_f(zS+\varepsilon)\in z\mathscr{B}_{\geq 1}^{+}\oplus\mathscr{L}_k^{\alpha}$, and, consequently, $\mathscr{S}_f(z\mathscr{B}_{\geq 1}^{+}\oplus\mathscr{L}_k^{\alpha})\subseteq z\mathscr{B}_{\geq 1}^{+}\oplus\mathscr{L}_k^{\alpha}$.

Similarly, since $(zT)'\circ(\mathrm{id}+zS)\cdot\varepsilon=\mathrm{Lt}(T\cdot\varepsilon)+$h.o.t. and $\sum_{i\geq 1}\frac{\varepsilon^{(i)}}{i!}(zR)^i=\mathrm{Lt}(\varepsilon'\cdot zR)+$ h.o.t., it follows that:

$$\begin{aligned}
&\mathrm{ord}_z\Big((zT)'\circ(\mathrm{id}+zS)\cdot\varepsilon\Big)=\mathrm{ord}_z(\varepsilon)\geq\alpha,\\
&\mathrm{ord}_z\Big(\sum_{i\geq 1}\frac{\varepsilon^{(i)}}{i!}(zR)^i\Big)=\mathrm{ord}_z(\varepsilon)\geq\alpha.
\end{aligned}$$

Therefore, $\mathscr{T}_f(zS+\varepsilon)\in z\mathscr{B}_{\geq 1}^{+}\oplus\mathscr{L}_k^{\alpha}$, and, consequently, $\mathscr{T}_f(z\mathscr{B}_{\geq 1}^{+}\oplus\mathscr{L}_k^{\alpha})\subseteq z\mathscr{B}_{\geq 1}^{+}\oplus\mathscr{L}_k^{\alpha}$.

Now we prove that $\mathscr{S}_f$ is a contraction. Let $zS_1+\varepsilon_1,zS_2+\varepsilon_2\in z\mathscr{B}_{\geq 1}^{+}\oplus\mathscr{L}_k^{\alpha}$, $S_1,S_2\in\mathscr{B}_{\geq 1}^{+}\subseteq\mathscr{L}_k$, $\varepsilon_1,\varepsilon_2\in\mathscr{L}_k^{\alpha}$, such that $zS_1+\varepsilon_1\neq zS_2+\varepsilon_2$. We distinguish two cases.

(a) *Case* $\varepsilon_1\neq\varepsilon_2$. Since $(zT)^{(i)}(\mathrm{id}+zS_1)=\mathrm{Lt}\big((zT)^{(i)}\big)+$h.o.t. and $\varepsilon_1,\varepsilon_2\in\mathscr{L}_k^{\alpha}$, we get:

$$\begin{aligned}
&\mathrm{ord}_z\Big(\sum_{i\geq 2}\frac{(zT)^{(i)}(\mathrm{id}+zS_1)}{i!}\varepsilon_1^i-\sum_{i\geq 2}\frac{(zT)^{(i)}(\mathrm{id}+zS_2)}{i!}\varepsilon_2^i\Big)\\
&\geq\mathrm{ord}_z(\varepsilon_1-\varepsilon_2)+(\alpha-1)\\
&\geq\mathrm{ord}_z\big((zS_1+\varepsilon_1)-(zS_2+\varepsilon_2)\big)+(\alpha-1).
\end{aligned}$$

(b) *Case* $\varepsilon_1 = \varepsilon_2$. Then $S_1 \neq S_2$ and:

$$\begin{aligned}&\sum_{i\geq 2}\frac{(zT)^{(i)}(\mathrm{id}+zS_1)}{i!}\varepsilon_1^i - \sum_{i\geq 2}\frac{(zT)^{(i)}(\mathrm{id}+zS_2)}{i!}\varepsilon_1^i\\ &= -\mathrm{Lt}\Big(\frac{1}{2}z^{-1}D_1(T)\cdot(S_1-S_2)\cdot\varepsilon_1^2\Big)+\mathrm{h.o.t.}\end{aligned}$$

That is:

$$\begin{aligned}&\mathrm{ord}_z\Big(\sum_{i\geq 2}\frac{(zT)^{(i)}(\mathrm{id}+zS_1)}{i!}\varepsilon_1^i - \sum_{i\geq 2}\frac{(zT)^{(i)}(\mathrm{id}+zS_2)}{i!}\varepsilon_1^i\Big)\\ &\geq 2\alpha-1\\ &\geq \mathrm{ord}_z\big((zS_1+\varepsilon_1)-(zS_2+\varepsilon_1)\big)+(\alpha-1).\end{aligned}$$

Therefore, in both cases, it follows that:

$$\begin{aligned}&\mathrm{ord}_z\Big(\sum_{i\geq 2}\frac{(zT)^{(i)}(\mathrm{id}+zS_1)}{i!}\varepsilon_1^i - \sum_{i\geq 2}\frac{(zT)^{(i)}(\mathrm{id}+zS_2)}{i!}\varepsilon_2^i\Big)\\ &\geq \mathrm{ord}_z\big((zS_1+\varepsilon_1)-(zS_2+\varepsilon_2)\big)+(\alpha-1).\end{aligned}\tag{2.196}$$

Note that:

$$\begin{aligned}&\mathrm{ord}_z\Big(\sum_{i\geq 1}\frac{(zS_1+\varepsilon_1)^{(i)}(\mathrm{id}+zR)}{i!}\mu^i - \sum_{i\geq 1}\frac{(zS_2+\varepsilon_2)^{(i)}(\mathrm{id}+zR)}{i!}\mu^i\Big)\\ &= \mathrm{ord}_z\Big(\sum_{i\geq 1}\frac{(z(S_1-S_2)+(\varepsilon_1-\varepsilon_2))^{(i)}(\mathrm{id}+zR)}{i!}\mu^i\Big)\\ &\geq \mathrm{ord}_z\big(z(S_1-S_2)+(\varepsilon_1-\varepsilon_2)\big)+\mathrm{ord}_z(\mu)-1\\ &\geq \mathrm{ord}_z\big((zS_1+\varepsilon_1)-(zS_2+\varepsilon_2)\big)+\alpha-1.\end{aligned}\tag{2.197}$$

Finally, statement 1 follows from (2.196) and (2.197).

3. Let $h\in\mathscr{S}_f(\widetilde{\mathscr{L}})$, where $\widetilde{\mathscr{L}}\subseteq z\mathscr{B}_{\geq 1}^{+}\oplus\mathscr{L}_k^{\alpha}$ is as defined in Proposition 2.3.27. Since $\mathscr{S}_f(\widetilde{\mathscr{L}})\subseteq z\mathscr{B}_{\geq 1}^{+}\oplus\mathscr{L}_k^{\alpha}$ and $\mathrm{ord}(z(R-T))>\mathrm{ord}(\mathrm{Res}(f))$, by formula (2.190) for $\mathscr{S}_f$, h can be written in the *form*:

$$h = zM+\sum_{\beta\geq\alpha}z^{\beta}M_{\beta},$$

where $M\in\mathscr{B}_{\geq 1}^{+}\subseteq\mathscr{L}_k$, $\mathrm{ord}(zM)>\mathrm{ord}(\mathrm{Res}(f))$, $M_\beta\in\mathscr{B}_1$, $\beta\geq\alpha$. We prove that there exists $zS+\varepsilon\in\widetilde{\mathscr{L}}$ such that:

$$\mathscr{T}_f(zS+\varepsilon)=h=zM+\sum_{\beta\geq\alpha}z^{\beta}M_{\beta}.\tag{2.198}$$

Therefore, we search for solution of (2.198) in the *form*:

$$zS+\sum_{\beta\geq\alpha} z^{\beta}S_{\beta}, \quad S\in\mathscr{B}_{\geq 1}^{+}\subseteq\mathscr{L}_k, \operatorname{ord}(S)>\operatorname{ord}(R), S_{\beta}\in\mathscr{B}_1\subseteq\mathscr{L}_k^{\infty}. \tag{2.199}$$

From (2.190) and by comparing the blocks with the same order in z, it follows that equation (2.198) is equivalent to the following system of equations:

$$(zT)'\circ(\mathrm{id}+zS)\cdot z^{\beta}S_{\beta}-\sum_{i\geq 1}\frac{(z^{\beta}S_{\beta})^{(i)}}{i!}(zR)^{i}=z^{\beta}M_{\beta}, \quad \beta\geq\alpha, \tag{2.200}$$

$$\sum_{i\geq 1}\frac{(zT)^{(i)}}{i!}(zS)^{i}-\sum_{i\geq 1}\frac{(zS)^{(i)}}{i!}(zR)^{i}=zM. \tag{2.201}$$

By Lemma A.2.8 it follows that:

$$\begin{aligned}
&\sum_{i\geq 1}\frac{(z^{\beta}S_{\beta})^{(i)}}{i!}(zR)^{i}=z^{\beta}\Big(S_{\beta}\cdot\sum_{i\geq 1}\binom{\beta}{i}R^{i}+\mathscr{C}_{\beta}(S_{\beta})\Big),\\
&\sum_{i\geq 1}\frac{(zT)^{(i)}}{i!}(zS)^{i}=z\left(T\cdot S+D_1(T)\cdot(1+S)\cdot\log(1+S)+\mathscr{C}_1(S)\right),\\
&\sum_{i\geq 1}\frac{(zS)^{(i)}}{i!}(zR)^{i}=z\left(S\cdot R+D_1(S)\cdot(1+R)\cdot\log(1+R)+\mathscr{K}_1(S)\right),
\end{aligned} \tag{2.202}$$

for linear $\frac{1}{2^{1+\operatorname{ord}_{\ell_1}(R)}}$-contractions $\mathscr{C}_{\beta}:(\mathscr{B}_1,d_1)\to(\mathscr{B}_1,d_1)$, $\beta\geq\alpha$, and $\frac{1}{2^{2+\operatorname{ord}_{\ell_1}(R)}}$-contractions $\mathscr{C}_1,\mathscr{K}_1:(\mathscr{B}_{\geq 1}^{+},d_1)\to(\mathscr{B}_{\geq 1}^{+},d_1)$. By (2.202), after eliminating the variable z, we get that solving (2.200) and (2.201) is equivalent to solving:

$$\mathscr{S}_{\beta}(S_{\beta})=S_{\beta}, \quad S_{\beta}\in\mathscr{B}_1\subseteq\mathscr{L}_k^{\infty}, \beta\geq\alpha, \tag{2.203}$$

and

$$\mathscr{T}_1(S)=\mathscr{S}_1(S), \quad S\in\mathscr{B}_{\geq 1}^{+}\subseteq\mathscr{L}_k, \operatorname{ord}(S)>\operatorname{ord}(R), \tag{2.204}$$

where:

$$\mathscr{S}_{\beta}(S_{\beta}):=\frac{M_{\beta}-S_{\beta}\cdot D_1(T)\circ(\mathrm{id}+zS)+\mathscr{C}_{\beta}(S_{\beta})}{T\circ(\mathrm{id}+zS)-\sum_{i\geq 1}\binom{\beta}{i}R^{i}}, \tag{2.205}$$

and

$$\begin{aligned}
&\mathscr{S}_1(S):=M+\mathscr{K}_1(S)-\mathscr{C}_1(S)-(1+S)\cdot\log(1+S)\cdot D_1(T-R),\\
&\mathscr{T}_1(S):=-(1+R)\cdot\log(1+R)\cdot D_1(S)+(1+S)\cdot\log(1+S)\cdot D_1(R)+S\cdot(T-R).
\end{aligned} \tag{2.206}$$

Since fixed point equations $\mathscr{S}_\beta(S_\beta)=S_\beta$, $\beta\geq\alpha$, depend on the solutions $S\in\mathscr{B}^+_{\geq 1}\subseteq\mathscr{L}_k$, $\operatorname{ord}(S)>\operatorname{ord}(R)$, of the fixed point equation $\mathscr{T}_1(S)=\mathscr{S}_1(S)$, we first determine the unique solution S of equation (2.204).

Solving equation (2.204). Let

$$\widetilde{\mathscr{B}}:=\left\{K\in\mathscr{B}^+_{\geq 1}\subseteq\mathscr{L}_k:\operatorname{ord}(K)>\operatorname{ord}(R)\right\}.$$

Now, for the purpose of applying Proposition 2.3.28 to the operators $\mathscr{T}_1,\mathscr{S}_1:\widetilde{\mathscr{B}}\to\mathscr{B}^+_{\geq 1}$ defined by (2.206), put $V:=T-R$. Since $\operatorname{ord}\big(z(R-T)\big),\operatorname{ord}(zM)>\operatorname{ord}(\operatorname{Res}(f))$, we get $\operatorname{ord}(V),\operatorname{ord}(M)>2\cdot\operatorname{ord}(R)+(0,\mathbf{1}_k)$.

By Proposition 2.3.28, the operators $\mathscr{T}_1$ and $\mathscr{S}_1$ satisfy the assumptions of Proposition 1.2.12 on the complete space $(\widetilde{\mathscr{B}},d_1)$. Therefore, by Proposition 1.2.12, there exists a unique $S\in\widetilde{\mathscr{B}}$ such that $\mathscr{T}_1(S)=\mathscr{S}_1(S)$.

Solving equation (2.203). For the unique solution $S\in\widetilde{\mathscr{B}}$ of the fixed point equation $\mathscr{T}_1(S)=\mathscr{S}_1(S)$ we prove that operators $\mathscr{S}_\beta:\mathscr{B}_1\to\mathscr{B}_1$, $\beta\geq\alpha$, defined by (2.205), are $\frac{1}{2}$-contractions on the space $(\mathscr{B}_1,d_1)$.

Since $\operatorname{ord}\big(z(R-T)\big)>\operatorname{ord}(\operatorname{Res}(f))$, it follows that $\operatorname{Lt}(T)=\operatorname{Lt}(R)$. Using $\binom{\beta}{1}=\beta\geq\alpha>1$ and $T\circ(\mathrm{id}+zS)=T+\text{h.o.t.}$, we get:

$$\frac{S_\beta\cdot D_1(T)\circ(\mathrm{id}+zS)+\mathscr{C}_\beta(S_\beta)}{T\circ(\mathrm{id}+zS)-\sum_{i\geq 1}\binom{\beta}{i}R^i}=\frac{S_\beta\cdot D_1(T)\circ(\mathrm{id}+zS)+\mathscr{C}_\beta(S_\beta)}{(1-\beta)\operatorname{Lt}(R)+\text{h.o.t.}},$$

for $S_\beta\in\mathscr{B}_1\subseteq\mathscr{L}_k^\infty$ and each $\beta\geq\alpha$. From this, using the facts that $\mathscr{C}_\beta$ is a linear $\frac{1}{2^{1+\operatorname{ord}_{\ell_1}(R)}}$-contraction on $(\mathscr{B}_1,d_1)$ and that $\operatorname{ord}_{\ell_1}\big(D_1(T)\circ(\mathrm{id}+zS)\big)=\operatorname{ord}_{\ell_1}(R)+1$, we get:

$$\operatorname{ord}_{\ell_1}\left(\frac{S_\beta\cdot D_1(T)\circ(\mathrm{id}+zS)+\mathscr{C}_\beta(S_\beta)}{T\circ(\mathrm{id}+zS)-\sum_{i\geq 1}\binom{\beta}{i}R^i}\right)\geq\operatorname{ord}_{\ell_1}(S_\beta)+1,$$

for $S_\beta\in\mathscr{B}_1\subseteq\mathscr{L}_k^\infty$ and each $\beta\geq\alpha$. Since $\mathscr{S}_\beta$, $\beta\geq\alpha$, are affine operators, it follows that $\mathscr{S}_\beta$, $\beta\geq\alpha$, are $\frac{1}{2}$-contractions on the space $(\mathscr{B}_1,d_1)$.
By the Banach Fixed Point Theorem, there exists the unique solution $S_\beta\in\mathscr{B}_1\subseteq\mathscr{L}_k^\infty$ of equation (2.200), for each $\beta\geq\alpha$.

Finally, by putting $\varepsilon := \sum_{\beta \geq \alpha} z^\beta S_\beta$, where $S_\beta \in \mathscr{B}_1 \subseteq \mathscr{L}_k^\infty$ are the unique solutions of (2.200), for $\beta \geq \alpha$, and taking the unique solution $S \in \widetilde{\mathscr{B}}$ of (2.204), the system of equations (2.200), (2.201) is satisfied, and therefore, we get $\mathscr{T}_f(zS+\varepsilon) = h$, for $h \in \mathscr{S}_f(\widetilde{\mathscr{L}})$ chosen in (2.198). Since $\operatorname{ord}(S) > \operatorname{ord}(R)$, it follows that $zS+\varepsilon \in \widetilde{\mathscr{L}}$. Consequently, it follows that $\mathscr{S}_f(\widetilde{\mathscr{L}}) \subseteq \mathscr{T}_f(\widetilde{\mathscr{L}})$.

2. Let $zS_i + \varepsilon_i \in \widetilde{\mathscr{L}}$, for $i = 1,2$, be distinct, and written in *form* (2.199), i.e.

$$zS_i + \varepsilon_i = zS_i + \sum_{\beta \geq \alpha} z^\beta S_{\beta,i},$$

where $S_i \in \widetilde{\mathscr{B}}$, and $S_{\beta,i} \in \mathscr{B}_1 \subseteq \mathscr{L}_k^\infty$, $\beta \geq \alpha$, for $i = 1,2$. By putting such decompositions in (2.190) and using (2.202), we get:

$$\begin{aligned}
&\mathscr{T}_f(zS_i+\varepsilon_i)\\
&= z\big((T-R)\cdot S_i + D_1(T)\cdot(1+S_i)\cdot\log(1+S_i) - D_1(S_i)\cdot(1+R)\cdot\log(1+R) + \mathscr{C}_1(S_i) - \mathscr{K}_1(S_i)\big)\\
&+ \sum_{\beta \geq \alpha} z^\beta \left(S_{\beta,i}\cdot\Big(T\circ(\mathrm{id}+zS_i) - \sum_{i\geq 1}\binom{\beta}{i}R^i + D_1(T)\circ(\mathrm{id}+zS_i)\Big) - \mathscr{C}_\beta(S_{\beta,i})\right),
\end{aligned} \tag{2.207}$$

for $i = 1,2$. Now, we consider two cases: $S_1 \neq S_2$ and $S_1 = S_2$, and prove that the restriction $\mathscr{T}_f|_{\widetilde{\mathscr{L}}}$ is an isometry.

If $S_1 \neq S_2$, since $\operatorname{Lt}(T) = \operatorname{Lt}(R)$, we get that:

$$\mathscr{T}_f(zS_1+\varepsilon_1) - \mathscr{T}_f(zS_2+\varepsilon_2) = z\operatorname{Lt}\big((S_1-S_2)\cdot D_1(R) - R\cdot D_1(S_1-S_2)\big) + \text{h.o.t.}, \tag{2.208}$$

assuming that $(S_1-S_2)\cdot D_1(R) - R\cdot D_1(S_1-S_2) \neq 0$. Now, suppose that $(S_1-S_2)\cdot D_1(R) - R\cdot D_1(S_1-S_2) = 0$. Solving the linear differential equation, we get $S_1 - S_2 = C\cdot R$, for $C \in \mathbb{R}$. Since $\operatorname{ord}(S_i) > \operatorname{ord}(R)$, $i = 1,2$, we conclude that $C = 0$, i.e., $S_1 = S_2$, which is a contradiction. This implies that:

$$\operatorname{ord}_z\big(\mathscr{T}_f(zS_1+\varepsilon_1) - \mathscr{T}_f(zS_2+\varepsilon_2)\big) = 1 = \operatorname{ord}_z\big((zS_1+\varepsilon_1) - (zS_2+\varepsilon_2)\big).$$

If $S_1 = S_2$, then $\varepsilon_1 \neq \varepsilon_2$. Put $\beta_0 := \operatorname{ord}_z(\varepsilon_1 - \varepsilon_2)$. By (2.207), and since $\mathscr{C}_\beta$ is a linear $\frac{1}{2^{1+\operatorname{ord}_{\ell_1}(R)}}$-contraction (with respect to the metric d_1), we get that:

$$\mathscr{T}_f(zS_1+\varepsilon_1) - \mathscr{T}_f(zS_1+\varepsilon_2) = z^{\beta_0}(1-\beta_0)\operatorname{Lt}(R)\operatorname{Lt}(S_{\beta_0,1} - S_{\beta_0,2}) + \text{h.o.t.}$$

Since $S_{\beta_0,1} \neq S_{\beta_0,2}$, $R \neq 0$, and $\beta_0 \geq \alpha > 1$, we have:

$$\mathrm{ord}_z\big(\mathscr{T}_f(zS_1+\varepsilon_1) - \mathscr{T}_f(zS_1+\varepsilon_2)\big) = \beta_0 = \mathrm{ord}_z\big((zS_1+\varepsilon_1)-(zS_1+\varepsilon_2)\big).$$

Therefore, the restriction $\mathscr{T}_f|_{\widetilde{\mathscr{L}}}$ is an isometry, with respect to the metric d_z. ■

Proof of step $(b.3)$. In steps $(b.1)$ and $(b.2)$ we obtained logarithmic transseries $\varphi_1, \varphi_2 \in \mathscr{L}_k^0$, such that:

$$\varphi_2 \circ \varphi_1 \circ f \circ \varphi_1^{-1} \circ \varphi_2^{-1} = \mathrm{id} + zL + c\mathrm{Res}\,(f) + \text{h.o.t.},$$

where L is as defined in Theorem C and the unique c is given by (2.104). For a simpler notation denote by f the whole composition $\varphi_2 \circ \varphi_1 \circ f \circ \varphi_1^{-1} \circ \varphi_2^{-1}$. Note that now $f = \mathrm{id} + zL + c\mathrm{Res}\,(f) + \text{h.o.t.}$ Put $zR := zL + c\mathrm{Res}\,(f) + \text{h.o.t.}$ and $zT := zL + c\mathrm{Res}\,(f)$. Let $g := \mathrm{id} + zL + c\mathrm{Res}\,(f)$. By Lemma 2.3.26, we define the operators $\mathscr{T}_f, \mathscr{S}_f : \mathscr{L}_k^1 \to \mathscr{L}_k^1$ and transform the conjugacy equation

$$\varphi_3 \circ f \circ \varphi_3^{-1} = g \tag{2.209}$$

to the equivalent fixed point equation

$$\mathscr{T}_f(zS+\varepsilon) = \mathscr{S}_f(zS+\varepsilon), \tag{2.210}$$

where $\varphi_3 := \mathrm{id} + zS + \varepsilon \in \mathscr{L}_k^0$. Put $\alpha := \mathrm{ord}_z\,(f - \mathrm{id} - zR)$ and consider the restrictions of the operators $\mathscr{T}_f$ and $\mathscr{S}_f$ on the subspace $z\mathscr{B}_{\geq 1}^+ \oplus \mathscr{L}_k^\alpha \subseteq \mathscr{L}_k^1$. Since $\mathrm{ord}\,\big(z(R-T)\big) > \mathrm{ord}\,(\mathrm{Res}\,(f))$, by Proposition 2.3.27 and Proposition 1.2.12, there exists the unique solution $\varphi_3 \in z\mathscr{B}_{\geq 1}^+ \oplus \mathscr{L}_k^\alpha$, $\varphi_3 := \mathrm{id} + zS + \varepsilon$, of (2.210), satisfying $\mathrm{ord}\,(S) > \mathrm{ord}\,(R)$. As a consequence, by Lemma 2.3.26, there exists a solution $\varphi_3 \in \mathscr{L}_k^0$ of the conjugacy equation (2.209). This completes step $(b.3)$.

Note that φ_3 is the unique solution of the conjugacy equation (2.209), if we additionally impose the condition that $\mathrm{ord}\,(\varphi_3 - \mathrm{id}) > \mathrm{ord}\,(f - \mathrm{id})$ and $\mathrm{ord}_z\,(\varepsilon) \geq \mathrm{ord}_z\,(f - \mathrm{id} - zR)$. ■

2.3.7. Proof of statements 2 and 3 of Theorem C

Let $f \in \mathscr{L}_k$, $k \in \mathbb{N}$, such that $f = \mathrm{id} + z^\beta L + \text{h.o.t.}$, for $\beta \geq 1$. Let $L = a_{\mathbf{n}} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} + \text{h.o.t.}$, $a \neq 0$, $\mathbf{n} := (n_1, \ldots, n_k)$, be as in Theorem C. Let $f_c = \mathrm{id} + z^\beta L + c\mathrm{Res}(f)$ be the formal normal form of f from statement 1 of Theorem C.

Proof of statement 2: The formula for the residual coefficient. The proof is adapted from [23, Proposition 9.3].

Note that:

$$\begin{aligned}\frac{1}{f_c-\mathrm{id}}&=\frac{1}{z^\beta L+c\mathrm{Res}(f)}\\&=\frac{1}{z^\beta L}\cdot\frac{1}{1+\frac{c\mathrm{Res}(f)}{z^\beta L}}\\&=\frac{1}{z^\beta L}\cdot\Big(1+\sum_{i\geq 1}\Big(\frac{c\mathrm{Res}(f)}{z^\beta L}\Big)^i\Big)\\&=\frac{1}{z^\beta L}+\big(\frac{c}{a_{\mathbf{n}}^2}z^{-1}\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_k+\text{h.o.t.}\big).\end{aligned}\tag{2.211}$$

From (2.211) we get:

$$c=\left[\frac{a_{\mathbf{n}}^2}{f_c-\mathrm{id}}\right]_{-1,\mathbf{1}_k}-\left[\frac{a_{\mathbf{n}}^2}{z^\beta L}\right]_{-1,\mathbf{1}_k}.\tag{2.212}$$

For every $f\in\mathscr{L}_k^0$, by Proposition B.2.3, it follows that:

$$\left[\frac{1}{f-\mathrm{id}}\right]_{-1,\mathbf{1}_k}=\left[\int\frac{dz}{f-\mathrm{id}}\right]_{\mathbf{0}_{k+1},-1}\tag{2.213}$$

and, in particular, for its normal form f_c,

$$\left[\frac{1}{f_c-\mathrm{id}}\right]_{-1,\mathbf{1}_k}=\left[\int\frac{dz}{f_c-\mathrm{id}}\right]_{\mathbf{0}_{k+1},-1}.\tag{2.214}$$

We prove that, for every $f\in\mathscr{L}_k^0$ and its normal form f_c given by (2.102),

$$\left[\int\frac{dz}{f_c-\mathrm{id}}\right]_{\mathbf{0}_{k+1},-1}=\left[\int\frac{dz}{f-\mathrm{id}}\right]_{\mathbf{0}_{k+1},-1}.\tag{2.215}$$

Put $g:=f-\mathrm{id}$. Let us use the following notation

$$\int^{\varphi^{-1}}h(s)ds:=\int h(s)ds\Big|_{s=\varphi^{-1}},\quad h\in\mathscr{L}_k,\ \varphi\in\mathscr{L}_k^0.$$

From $\varphi\circ f\circ\varphi^{-1}=f_0$, it follows that $\varphi\circ f=f_c\circ\varphi$. Then, by the change of variable of the integration $z=\varphi(s)$ and the Taylor Theorem (Proposition A.1.6), it follows that:

$$\begin{aligned}\int \frac{dz}{f_c-\mathrm{id}} &= \int^{\varphi^{-1}} \frac{\varphi'(s)ds}{(f_c\circ\varphi)(s)-\varphi(s)} \\ &= \int^{\varphi^{-1}} \frac{\varphi'(s)ds}{\varphi(f(s))-\varphi(s)} \\ &= \int^{\varphi^{-1}} \frac{\varphi'(s)ds}{\sum_{i\geq 1}\frac{\varphi^{(i)}(s)}{i!}(g(s))^i} \\ &= \int^{\varphi^{-1}} \frac{\varphi'(s)ds}{\varphi'(s)g(s)\cdot\Big(1+\sum_{i\geq 2}\frac{\varphi^{(i)}(s)}{i!\varphi'(s)}(g(s))^{i-1}\Big)} \\ &= \int^{\varphi^{-1}} \frac{ds}{g(s)}\cdot\Big(1-\mathrm{Lt}\Big(\frac{\varphi''(s)}{2\varphi'(s)}g(s)\Big)+\text{h.o.t.}\Big) \\ &= \int^{\varphi^{-1}} \frac{ds}{g(s)}+\int^{\varphi^{-1}}\Big(-\frac{1}{2}\mathrm{Lt}\Big(\frac{\varphi''(s)}{\varphi'(s)}\Big)+\text{h.o.t.}\Big)ds.\end{aligned} \tag{2.216}$$

Put $\varepsilon:=\varphi-\mathrm{id}$. Note that:

$$\frac{\varphi''(s)}{\varphi'(s)}=\frac{d}{ds}(\log(1+\varepsilon'(s))). \tag{2.217}$$

Since $\mathrm{ord}(\varepsilon)>(1,\mathbf{0}_k)$, it follows that $\mathrm{ord}(\log(1+\varepsilon'(s)))>\mathbf{0}_{k+1}$. Therefore,

$$\mathrm{ord}\Big(\frac{d}{ds}(\log(1+\varepsilon'(s)))\Big)>(-1,\mathbf{1}_k).$$

By Proposition B.2.3, it follows that

$$\int^{\varphi^{-1}}\Big(-\frac{1}{2}\mathrm{Lt}\Big(\frac{\varphi''(s)}{\varphi'(s)}\Big)+\text{h.o.t.}\Big)ds$$

is an element of $\mathscr{L}_k$. Thus, we proved:

$$\Big[\int\frac{dz}{f_c-\mathrm{id}}\Big]_{\mathbf{0}_{k+1},-1}=\Big[\int^{\varphi^{-1}}\frac{ds}{g(s)}\Big]_{\mathbf{0}_{k+1},-1}=\Big[\int^{\varphi^{-1}}\frac{ds}{(f-\mathrm{id})(s)}\Big]_{\mathbf{0}_{k+1},-1}. \tag{2.218}$$

Put $h(s):=\int\frac{ds}{(f-\mathrm{id})(s)}$. Note that $h=h(z)=\int^z\frac{ds}{(f-\mathrm{id})(s)}=\int\frac{dz}{f-\mathrm{id}}$. Now, we get:

$$\int^{\varphi^{-1}}\frac{ds}{(f-\mathrm{id})(s)}-\int^{z}\frac{ds}{(f-\mathrm{id})(s)}=h\left(\varphi^{-1}\right)-h=\sum_{i\geq 1}\frac{h^{(i)}}{i!}(\varphi^{-1}-\mathrm{id})^i. \tag{2.219}$$

Note that $\left[h^{(i)}\right]_{\mathbf{0}_{k+1},-1}=0$ and $(\varphi^{-1}-\mathrm{id})^i\in\mathscr{L}_k$, for $i\geq 1$. Therefore, by (2.219), it follows that:

$$\Big[\int^{\varphi^{-1}}\frac{ds}{(f-\mathrm{id})(s)}\Big]_{\mathbf{0}_{k+1},-1}-\Big[\int^{z}\frac{ds}{(f-\mathrm{id})(s)}\Big]_{\mathbf{0}_{k+1},-1}=0,$$

which implies that:

$$\left[\int^{\varphi^{-1}} \frac{ds}{(f-\mathrm{id})(s)}\right]_{\mathbf{0}_{k+1},-1} = \left[\int^{z} \frac{ds}{(f-\mathrm{id})(s)}\right]_{\mathbf{0}_{k+1},-1} = \left[\int \frac{dz}{f-\mathrm{id}}\right]_{\mathbf{0}_{k+1},-1}. \tag{2.220}$$

Now, (2.215) follows from (2.218) and (2.220). By (2.213), (2.214) and (2.215), it follows that

$$\left[\frac{1}{f-\mathrm{id}}\right]_{-1,\mathbf{1}_k} = \left[\frac{1}{f_c-\mathrm{id}}\right]_{-1,\mathbf{1}_k}.$$

Therefore, (2.104) follows from (2.212). If $\beta > 1$, note that the second term in (2.212) vanishes, which implies (2.105).

Since c is explicitely given by formula (2.104), it is unique. ■

Proof of statement 3: The minimality of f_c. The minimality of the normal form f_c in $\mathscr{L}_k^0$ follows from the uniqueness of $c \in \mathbb{R}$ and by Proposition 2.3.12. ■

2.3.8. Proofs of Remark 2.3.5 and Corollary 2.3.7

Proof of Remark 2.3.5. By a simple calculation it can be shown that the time-one map of the vector field X_c is an element of $\mathscr{L}_k^0$ that has the initial part equal to f_c. Therefore, by Theorem C, the time-one map of the vector field X_c can be reduced to f_c by a change of variables form $\mathscr{L}_k^0$. Since $\mathscr{L}_k^0$ is a group, it follows that f can be reduced to the time-one map of the vector field X_c by a change of variables from $\mathscr{L}_k^0$. ■

Proof of Corollary 2.3.7. Let $k \in \mathbb{N}$ be minimal such that $f \in \mathscr{L}_k^0$. Let $f := \mathrm{id} + z^{\beta} L +$ h.o.t., for $\beta \geq 1$, where L is as defined in Theorem C. By Theorem C, for $m \geq k+1$, there exists the unique $c \in \mathbb{R}$ such that f can be reduced to $f_c = \mathrm{id} + zL + c\mathrm{Res}(f)$, where $\mathrm{Res}(f)$ is the residual monomial of f in the differential algebra $\mathscr{L}_m$. By statement 2 of Theorem C, it follows that:

$$c = \left[\frac{a_{\mathbf{n}}^2}{f-\mathrm{id}}\right]_{-1,\mathbf{1}_m} - \left[\frac{a_{\mathbf{n}}^2}{zL}\right]_{-1,\mathbf{1}_m}, \tag{2.221}$$

if $\beta = 1$, and

$$c = \left[\frac{a_{\mathbf{n}}^2}{f-\mathrm{id}}\right]_{-1,\mathbf{1}_m}, \tag{2.222}$$

if $\beta > 1$. Since $f \in \mathscr{L}_k^0 \subseteq \mathscr{L}_k^\infty$ and $\mathscr{L}_k^\infty$ is an algebra and a field, it follows that $\frac{a_{\mathbf{n}}^2}{f-\mathrm{id}} - \frac{a_{\mathbf{n}}^2}{zL}$ if $\beta = 1$, and $\frac{a_{\mathbf{n}}^2}{f-\mathrm{id}}$ if $\beta > 1$, are elements of $\mathscr{L}_k^\infty$. Since $m \geq k+1$, by (2.221) and (2.222), it follows that $c = 0$ in both cases. Therefore, $f_c = \mathrm{id} + zL$, for every $m \geq k+1$. Now, we put $f_0 := \mathrm{id} + zL$. It therefore suffices to take a normalization in $\mathscr{L}_{k+1}^0$ to eliminate the residual term.

The minimality of f_0 in $\mathfrak{L}$ now follows directly by Proposition 2.3.12. ■

3. Normal forms for Dulac germs

In [10] the *almost regular germs* (or *Dulac germs*) are introduced (see also [11] and [12, Definition 24.27]), related to Dulac's problem of non-accumulation of limit cycles to elementary polycycles in the plane (see e.g. [3], [11]). The Dulac germs are analytic germs on an open interval $(0,d) \subseteq \mathbb{R}$, $d > 0$, with a particular logarithmic asymptotic expansion at zero. Furthermore, switching from the standard chart to the logarithmic chart (which is a global chart for the Riemann surface of the logarithm), Dulac germs can be analytically extended to particular *spiraling domains* around the origin of the *Riemann surface of the logarithm* called the *standard quadratic domains* (see e.g. [12, Definition 24.25]). In addition, their asymptotic expansions are uniform on standard quadratic domains. This extension property allows us to consider Dulac germs as analytic germs on spiraling subdomains of the Riemann surface of the logarithm with transserial asymptotic expansions around the origin.

Moreover, we allow complex coefficients in the asymptotic expansions of Dulac germs. We call such germs the *complex Dulac germs* (see [30, Subsection 2.1]).

The main goal of this chapter is to solve analytic normalization equations for hyperbolic and strongly hyperbolic complex Dulac germs on standard quadratic domains, and to prove that the normalization lies again inside the class of parabolic complex Dulac germs. The main strategy is to prove *analytic normalization theorems* (Theorem 3.2.11 and Theorem 3.3.5) for analytic maps with hyperbolic and strongly hyperbolic asymptotic bounds on their invariant complex domains called *admissible domains*. Furthermore, we use the formal normalization theorems (Theorem A and Theorem B) proved in the previous chapter, and prove in Theorems D and E that formal normalizations for hyperbolic and strongly hyperbolic Dulac germs are asymptotic expansions of their analytic normalizations, thereby ensuring that their analytic normalizations remain of the *Dulac type*. The connection between the formal and the analytic normalization is established through a certain homological equation, which we solve adapting the procedure from [17].

The analytic normalization theorem (Theorem 3.2.11) for analytic germs with hyperbolic asymptotic bounds is motivated by the linearization result of *Dewsnap* and *Fischer* (see [5, Theorem 2.2]), and the techiques used in the proof are motivated by the proof of the classical *Koenigs Theorem* (see e.g. [4, Theorem 2.1], [14], [24, Theorem 8.2]). Similarly, Theorem 3.3.5 for analytic maps with strongly hyperbolic asymptotic bounds is motivated by the classical *Böttcher Theorem* (see e.g. [4, Theorem 4.1], [24, Theorem 9.1]).

This chapter is divided into three sections. First, in Section 3.1 we define spiraling domains, analytic germs on the Riemann surface of the logarithm, and complex Dulac germs on standard quadratic domains. Then we solve the analytic normalization equation in Theorem D, in Section 3.2, for hyperbolic complex Dulac germs, and in Theorem E, in Section 3.3, for strongly hyperbolic complex Dulac germs.

3.1. Dulac germs and Dulac series

The main goal of this section is to introduce complex Dulac germs on standard quadratic subdomains (see e.g. [12, Definition 24.25]) of the Riemann surface of the logarithm. In Subsection 3.1.1 we introduce the general notion of analytic germs on the spiraling domains around the origin of the Riemann surface of the logarithm. In Subsection 3.1.2 we introduce *complex Dulac germs* as particular analytic germs on spiraling domains called the *standard quadratic domains* and with particular logarithmic asymptotic expansions in the form of complex Dulac series (see [30, Subsection 2.1]).

3.1.1. Analytic germs on the Riemann surface of the logarithm

In this subsection the domains of interest are spiraling neighbourhoods around the origin of the Riemann surface of the logarithm $\widetilde{\mathbb{C}}$. This allows us to introduce the notion of analytic *germ* on $\widetilde{\mathbb{C}}$. Moreover, particular spiraling neighborhoods are natural domains of the holomorphic extension from the real line of the first return maps of hyperbolic polycycles. Such extensions belong to the class of the so-called Dulac germs, and contain logarithms in their asymptotic expansions. We recall some basic notions introduced in [30, Subsection 2.1].

Definition 3.1.1 (Riemann surface of the logarithm, [30])**.** We denote by $\widetilde{\mathbb{C}}$ the set

$$\widetilde{\mathbb{C}} := \{(r,\theta) : r \in \mathbb{R}_{>0}, \theta \in \mathbb{R}\},$$

endowed with a structure of one-dimensional analytic Riemann manifold whose atlas consists of a single chart, called the *logarithmic chart*, given by:

$$\begin{aligned} &-\log : \widetilde{\mathbb{C}} \to \mathbb{C}, \\ &z := (r,\theta) \mapsto \zeta := -\log z = -\log r - \mathrm{i}\theta. \end{aligned}$$

We call $\widetilde{\mathbb{C}}$ the *Riemann surface of the logarithm*.

Remark 3.1.2 ([30])**.** In the sequel, by an abuse of notation we often identify (r,θ) with the formal product $z := r\cdot \mathrm{e}^{\theta\cdot i}$, where we do not identify $\mathrm{e}^{\theta\cdot i}$ with $\mathrm{e}^{(\theta+2k\pi)\cdot i}$, $k\in\mathbb{Z}$. In this notation we have $z = \exp(-\zeta)$, $\zeta\in\mathbb{C}$, for $\exp : \mathbb{C}\to\widetilde{\mathbb{C}}$, where $\exp(-\zeta)$ depends on the determination of the logarithm of ζ.

Definition 3.1.3 (Spiraling neighborhood around the origin of $\widetilde{\mathbb{C}}$, [30])**.** We say that $\mathscr{R}\subseteq\widetilde{\mathbb{C}}$ is a *spiraling neighborhood (or domain) around the origin of* $\widetilde{\mathbb{C}}$ if there exists a continuous map $\eta : \mathbb{R}\to(0,+\infty)$ such that:

$$\left\{r\cdot \mathrm{e}^{\theta\cdot i} : 0 < r < \eta(\theta),\, \theta\in\mathbb{R}\right\} \subseteq \mathscr{R}.$$

Remark 3.1.4. Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be two spiraling neighborhoods around the origin of $\widetilde{\mathbb{C}}$. Then $\mathscr{R}_1\cap\mathscr{R}_2$ is a spiraling neighborhood around the origin of $\widetilde{\mathbb{C}}$. Indeed, let η_j be a continuous map such that

$$\left\{r\cdot \mathrm{e}^{\theta\cdot i} : 0 < r < \eta_j(\theta),\, \theta\in\mathbb{R}\right\} \subseteq \mathscr{R}_j,$$

for each $j = 1,2$. Then

$$\left\{r\cdot \mathrm{e}^{\theta\cdot i} : 0 < r < \min\{\eta_1(\theta),\eta_2(\theta)\},\, \theta\in\mathbb{R}\right\} \subseteq \mathscr{R}_1\cap\mathscr{R}_2.$$

Definition 3.1.5 (Germs on the Riemann surface of the logarithm)**.** Let $D_1, D_2 \subseteq \widetilde{\mathbb{C}}$ be two spiraling neighborhoods around the origin of $\widetilde{\mathbb{C}}$ and $f_i : D_i \to \widetilde{\mathbb{C}}$, $i = 1,2$. We say that f_1 and f_2 are equivalent if there exists a spiraling neighborhood D_3 around the origin of $\widetilde{\mathbb{C}}$ such that $D_3 \subseteq D_1 \cap D_2$ and $f_1|_{D_3} = f_2|_{D_3}$. The equivalence classes by this relation are called *germs on the Riemann surface of the logarithm*. By a standard abuse of notation and terminology we denote the class of f again by f and call it the germ f.

A germ on the Riemann surface of the logarithm is said to be *analytic* if it admits a representative which is analytic in the logaritmic ζ-chart, given in Definition 3.1.1.

Let f, g, h be some germs on $\widetilde{\mathbb{C}}$ in the ζ-chart. We write:

1. $f(\zeta) = g(\zeta) + o(h(\zeta))$ as $|\zeta| \to +\infty$ if
$$\lim_{|\zeta|\to+\infty} \frac{|f(\zeta) - g(\zeta)|}{|h(\zeta)|} = 0.$$
2. $f(\zeta) = g(\zeta) + O(h(\zeta))$ if there exist $M, R \in \mathbb{R}_{>0}$ such that
$$|f(\zeta) - g(\zeta)| \leq M\,|h(\zeta)|,$$
for every $\zeta \in \mathbb{C}$, $|\zeta| \geq R$.

Remark 3.1.6. Let f, g, h be some germs on $\widetilde{\mathbb{C}}$ in the ζ-chart. If $f(\zeta) = g(\zeta) + o(h(\zeta))$, then
$$\lim_{\Re(\zeta)\to+\infty} \frac{|f(\zeta) - g(\zeta)|}{|h(\zeta)|} = 0.$$
The converse is not true in general. However, if we consider the subdomain $D \subseteq \mathbb{C}^+$ such that $h_l(\Re(\zeta)) \leq \Im(\zeta) \leq h_u(\Re(\zeta))$, $\zeta \in D$, where $h_l, h_u : [R, +\infty)$ are continuous maps, for some $R \in \mathbb{R}_{>0}$, then the converse holds. The same holds for $f(\zeta) = g(\zeta) + O(h(\zeta))$. We use this frequently in the sequel.

Let $\alpha \in \mathbb{R}_{>0}$, $\beta \in \mathbb{C}$, and let f be an analytic germ $f(\zeta) = \alpha\zeta + \beta + o(1)$, as $|\zeta| \to +\infty$, on $\widetilde{\mathbb{C}}$, given in the ζ-chart. Note that $f(z) = \mathrm{e}^{-\beta} \cdot z^{\alpha} + o(z^{\alpha})$ as $|z| \to 0$, in the z-chart. This motivates us to give the following definitions which are generalizations of the standard parabolic, hyperbolic and strongly hyperbolic analytic complex diffeomorphisms at zero (see e.g. [4], [24]).

Definition 3.1.7 (Parabolic, hyperbolic and strongly hyperbolic analytic germs, [30])**.** Consider an analytic germ f on $\widetilde{\mathbb{C}}$ in the ζ-chart. We say that f is:

1. *parabolic* if $f(\zeta) = \zeta + o(1)$ as $|\zeta| \to +\infty$, and if $f^{\circ q} \neq \mathrm{id}$, for all $q \in \mathbb{N}_{\geq 1}$,

2. *hyperbolic* if $f(\zeta) = \zeta + \beta + o(1)$, as $|\zeta| \to +\infty$, for some $\beta \in \mathbb{C}$ (we can always suppose that $\beta \in \mathbb{C}^+$ up to replacing f by f^{-1}),

3. *strongly hyperbolic* if $f(\zeta) = \alpha\zeta + \beta + o(1)$, as $|\zeta| \to +\infty$, for some $\alpha \in \mathbb{R}_{>0} \setminus \{1\}$, $\beta \in \mathbb{C}$ (we can always suppose that $\alpha > 1$ up to replacing f by f^{-1}).

3.1.2. Dulac germs and Dulac series

In this subsection we consider complex and real Dulac germs which are analytic germs on the Riemann surface of the logarithm with representatives defined on the so-called *standard quadratic domains*, and which have complex (resp. real) *Dulac series* as their asymptotic expansions. For convenience, we work here usually in the logarithmic chart ζ. As in [30, Subsection 2.1], we first define standard quadratic domains in the logarithmic chart, exponential transseries, complex Dulac series as particular exponential transseries, and, finally, complex Dulac germs.

Standard quadratic domains

Definition 3.1.8 (Standard quadratic domains, see Definition 24.25, [12], Subsection 2.1, [30])**.** A *standard quadratic domain* $\mathscr{R}_C \subset \widetilde{\mathbb{C}}$, $C \in \mathbb{R}_{>0}$, is the set defined in the logarithmic ζ-chart as

$$\kappa\big(\mathbb{C}^+\big), \text{ where } \kappa(\zeta) = \zeta + C(\zeta+1)^{\frac{1}{2}} \tag{3.1}$$

(see Figure 3.1).

A standard quadratic domain $\mathscr{R}_C$, $C \in \mathbb{R}_{>0}$, is a spiraling neighborhood of the origin of $\widetilde{\mathbb{C}}$, which can be seen e.g. from the parametrization of its border in Example 3.2.5.

For our needs, we *germify* a standard quadratic domain at infinity, by intersecting it with half-planes $[R, +\infty) \times \mathbb{R}$, as R tends to $+\infty$.

Definition 3.1.9 (Representative of a standard quadratic domain $\mathscr{R}_C$, Definition 2.1, [30])**.** For $C \in \mathbb{R}_{>0}$, let $\mathscr{R}_C \subseteq \widetilde{\mathbb{C}}$ be the standard quadratic domain given by (3.1) in the ζ-chart. We call the elements of the collection of sets

$$\{(\mathscr{R}_C)_R := \mathscr{R}_C \cap ([R, +\infty) \times \mathbb{R}) : R > 0\},$$

the *representatives* of the standard quadratic domain $\mathscr{R}_C$.

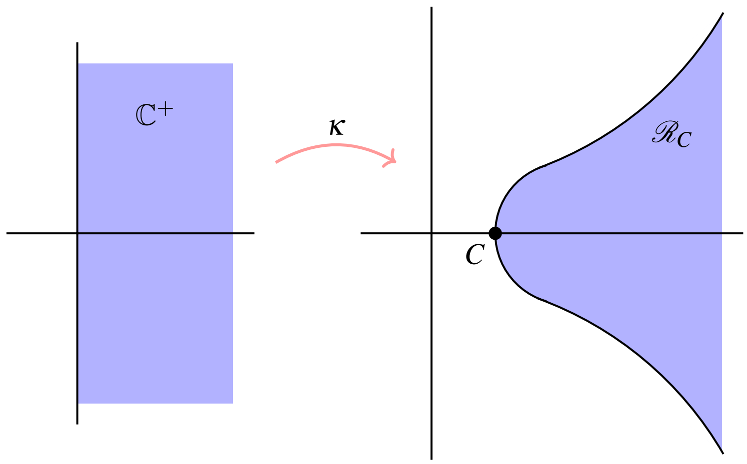

Figure 3.1: A standard quadratic domain $\mathscr{R}_C$, for $C > 0$, in the ζ-chart (see [30, Figure 1]).

In the following remark we emphasize some property of standard quadratic domains which will be important later for proving analytic normalization theorems.

Remark 3.1.10 (Remark 2.2, [30])**.** Let $C > 0$. For each representative $(\mathscr{R}_C)_R$ of the standard quadratic domain $\mathscr{R}_C$, there exists sufficiently big $C_0 > R, C$ such that for every $C' > C_0$, the standard quadratic domain $\mathscr{R}_{C'}$ is entirely contained in $(\mathscr{R}_C)_R$. Indeed, for $C' > C_0$, where C_0 is sufficiently big, it can be seen that $\mathscr{R}_{C'} \subseteq \mathscr{R}_C$. Also, for every $\zeta \in \mathscr{R}_{C'}$, it holds that $\Re(\zeta) > C' > C_0 > R$.

Exponential transseries

In order to define a complex Dulac series we first introduce the notion of exponential transseries. For a notion of a more general exponential-logarithmic series, see [6].

Definition 3.1.11 (Exponential transseries)**.** Let ζ be a formal variable at infinity. The *exponential transseries* is defined as a formal sum of the *form*

$$\widehat{f}(\zeta) := \sum_{(\alpha,n)\in\mathbb{R}_{\geq 0}\times\mathbb{Z}} a_{\alpha,n}\cdot\zeta^{-n}\mathrm{e}^{-\alpha\zeta}, \tag{3.2}$$

where $a_{\alpha,n}\in\mathbb{C}$, for each $(\alpha,n)\in\mathbb{R}_{\geq 0}\times\mathbb{Z}$, and $\mathrm{Supp}(\widehat{f}(\zeta)) := \{(\alpha,n)\in\mathbb{R}_{\geq 0}\times\mathbb{Z} : a_{\alpha,n}\neq 0\}$ is a well-ordered subset of $\mathbb{R}_{\geq 0}\times\mathbb{Z}$ such that $\min\mathrm{Supp}(\widehat{f}(\zeta)) > (0,0)$ (with respect to the lexicographic order). We standardly call $\mathrm{Supp}(\widehat{f}(\zeta))$ the *support* of $\widehat{f}(\zeta)$.

Note that exponential transseries are defined in analogy with the logarithmic transseries (see Definition 1.1.3), when seen in the ζ-chart (see [30, Remark 2.3]).

Analogously as before for logarithmic transseries $\widehat{f}(\zeta)$ can be written in the *block-wise form*

$$\widehat{f}(\zeta) = \sum_{\alpha\in\mathbb{R}_{\geq 0}} R_\alpha(\zeta^{-1})\cdot\mathrm{e}^{-\alpha\zeta},$$

where $R_\alpha(\zeta^{-1})$ is a Laurent series in the variable ζ^{-1} with complex coefficients, for each $\alpha\in\mathbb{R}_{\geq 0}$, and $\left\{\alpha\in\mathbb{R}_{\geq 0} : R(\zeta^{-1})\neq 0\right\}$ is a well-ordered subset of $\mathrm{R}_{\geq 0}$. We call it the *support* of $\widehat{f}(\zeta)$ *in* $\mathrm{e}^{-\zeta}$ and denote it by $\mathrm{Supp}_{\mathrm{e}^{-\zeta}}(\widehat{f}(\zeta))$.

The minimum of $\mathrm{Supp}_{\mathrm{e}^{-\zeta}}(\widehat{f}(\zeta))$ is called the *order* of the exponential transseries $\widehat{f}(\zeta)$ *in* $\mathrm{e}^{-\zeta}$, and denoted by $\mathrm{ord}_{\mathrm{e}^{-1}}(\widehat{f}(\zeta))$.

Moreover, if $R_\alpha(\zeta^{-1})$ is a polynomial in ζ, for each $\alpha\in\mathbb{R}_{\geq 0}$, then we say that $\widehat{f}(\zeta)$ satisfies the *polynomial property* (see [30, Subsection 4.1]).

Using the identification $z=\mathrm{e}^{-\zeta}$ one can see that $\widehat{f}(z)$ is a logarithmic transseries with complex coefficients, such that $\mathrm{Supp}_z(\widehat{f}(z))$ and $\mathrm{Supp}_{\mathrm{e}^{-1}}(\widehat{f}(\zeta))$ coincide. Moreover, the orders $\mathrm{ord}_z(\widehat{f}(z))$ and $\mathrm{ord}_{\mathrm{e}^{-1}}(\widehat{f}(\zeta))$ coincide.

Complex Dulac series and complex Dulac germs

In analogy with [11] and [12, Definition 24.26], we define complex and real Dulac series (see [30, Subsection 2.1]). Note that the real Dulac series is already defined in [11], [12], see e.g. [12, Definition 24.26].

Definition 3.1.12 (Complex and real Dulac series)**.** A *complex Dulac series* is a complex exponential transseries of the *form*

$$\widehat{f}(\zeta) := \alpha\zeta + \beta + \sum_{i=1}^{\infty} P_i(\zeta) \cdot \mathrm{e}^{-\alpha_i \zeta}, \quad \alpha \in \mathbb{R}_{>0}, \beta \in \mathbb{C}, P_i \in \mathbb{C}[\zeta], \tag{3.3}$$

where $(\alpha_i)_{i\geq 1}$ is a strictly increasing sequence of positive real numbers belonging to a finitely generated sub-semigroup of $(\mathbb{R}_{>0},+)$, such that $(\alpha_i)_i \to +\infty$.

If $\beta \in \mathbb{R}$ and $P_i \in \mathbb{R}[\zeta]$, for each $i \geq 1$, as in the case in [11], [12, Definition 24.26], then we call $\widehat{f}$ a *real Dulac series*.

Following the notation from the beginning of the section, the complex (real) Dulac series $\widehat{f} = \alpha\zeta + \beta + o(1)$, $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{C}$, is called:

1. *parabolic* if $\alpha = 1$ and $\beta = 0$,
2. *hyperbolic* if $\alpha = 1$ and $\Re(\beta) \neq 0$,
3. *strongly hyperbolic* if $\alpha \in \mathbb{R}_{>0}$, $\alpha \neq 1$.

In analogy with [11] and [12, Definition 24.27], we define complex and real Dulac germs (see [30, Subsection 2.1]). In [11], [12, Definition 24.27] the real Dulac germs are called the *almost regular germs*.

Definition 3.1.13 (Complex and real Dulac germ)**.** A *complex Dulac germ* is a holomorphic germ f on a standard quadratic domain $\mathscr{R}_C$, $C > 0$, which admits on $\mathscr{R}_C$ an asymptotic expansion given by a complex Dulac series (3.3), uniformly on $\mathscr{R}_C$ in the following sense: for every $\nu > 0$, there exists $N_\nu \in \mathbb{N}$, such that

$$\left| f(\zeta) - \alpha\zeta - \beta - \sum_{i=1}^{N_\nu} P_i(\zeta) \cdot \mathrm{e}^{-\alpha_i \zeta} \right| = o\big(\mathrm{e}^{-\nu\zeta}\big), \tag{3.4}$$

uniformly on $\mathscr{R}_C$ as $\Re(\zeta) \to +\infty$ in $\mathscr{R}_C$. In this case, we write $f \sim \widehat{f}$ and say that $\widehat{f}$ is the (unique) *Dulac asymptotic expansion of* f.

If additionally the set $\{\zeta \in \mathscr{R}_C : \Im(\zeta) = 0\}$ is f-invariant, we say that f is a *real Dulac germ*. It implies that its Dulac asymptotic expansion is a real Dulac series.

Note that a complex (real) Dulac germ f is parabolic (hyperbolic, strongly hyperbolic) if and only if its asymptotic expansion $\widehat{f}$ is a parabolic (hyperbolic, strongly hyperbolic) complex (real) Dulac series.

Remark 3.1.14. In the z-chart, a complex Dulac germ f is represented by the germ $\widetilde{f}$ which admits as $|z| \to 0$ an asymptotic expansion which is a *logarithmic complex Dulac series*, that is, a series of the form:

$$\widehat{f}(z) := \exp(-\widehat{f}(-\log z)) = \lambda z^{\alpha} + \sum_{i=1}^{+\infty} z^{\beta_i} Q_i(-\log z), \quad \alpha > 0, \lambda \in \widetilde{\mathbb{C}}, Q_i \in \mathbb{C}[X], \tag{3.5}$$

where $(\beta_i)_{i\geq 1}$ is a strictly increasing sequence of real numbers strictly bigger than α, belonging to a finitely generated sub-semigroup of $(\mathbb{R}_{>0}, +)$, and which tends to $+\infty$ (see [30, Remark 2.3]). Therefore, in the z-chart, a complex Dulac series is a logarithmic transseries of depth 1 with complex coefficients, i.e., an element of the space $\mathscr{L}_1^H(\mathbb{C})$, which additionally, satisfies the *polynomial property* (see the previous subsection). Here, $\mathscr{L}_1(\mathbb{C})$ denotes the space of all logarithmic transseries of depth 1 as defined in Subsection 1.1.2, but with complex coefficients.

In the sequel we conform to the following convention: for a complex (real) Dulac series $\widehat{f}$ we denote by $\widehat{f}(z)$ its form in the standard z-chart, and by $\widehat{f}(\zeta)$ its form in the ζ-chart. Note that $\widehat{f}(z)$ is a complex (real) logarithmic transseries of depth 1, and $\widehat{f}(\zeta)$ is a complex (real) exponential transseries.

Without proof we state the next theorem which is an easy analogue of the same real-case theorem for real Dulac germs from [11] or [12, Theorem 24.29]. It is a consequence of the *Phragmen-Lindelöf Theorem* (a version of a *maximum modulus principle* on an unbounded complex domain), see e.g. [12, Theorem 24.36] (see also [11]).

Theorem 3.1.15 (*Quasi-analyticity* of complex Dulac germs)**.** Let f be a complex Dulac germ such that $\widehat{f} = 0$ is its asymptotic expansion. Then $f \equiv 0$.

The above theorem easily implies that every complex Dulac germ is uniquely determined by its asymptotic expansion. This property is called the *quasi-analyticity* property of complex Dulac germs.

3.2. NORMAL FORMS FOR HYPERBOLIC DULAC GERMS

In this section we present our results from [30]. For a given hyperbolic complex Dulac germ $f(\zeta) = \zeta + \beta + o(1)$, $\beta \in \mathbb{C}^+$, we solve the following *linearization equation*:

$$(\varphi \circ f)(\zeta) = \varphi(\zeta) + \beta, \quad \varphi(\zeta) = \zeta + o(1), \tag{3.6}$$

on some standard quadratic domain (Definition 3.1.8), given in the ζ-chart. We prove the existence and the uniqueness of the parabolic complex Dulac germ φ that satisfies (3.6). Equation (3.6) in the standard z-chart becomes the equation

$$\left(\widetilde{\varphi} \circ \widetilde{f}\right)(z) = \mathrm{e}^{-\beta} \cdot \widetilde{\varphi}(z), \tag{3.7}$$

where $\widetilde{\varphi}(z) := \exp(-\varphi(-\log z))$ and $\widetilde{f}(z) := \exp(-f(-\log z))$. Therefore, we call both (3.6) and (3.7) the *linearization equations*. In particular, we call (3.6) a *Schröder-type linearization equation* (see e.g. [33]), and (3.7) an *Abel-type linearization equation* (see e.g. [18], [24]).

The existence and the uniqueness of the solution φ of linearization equation (3.6) is proven in Theorem D in Subsection 3.2.3. We split the mentioned proof into two separate problems:

1. Finding a solution of linearization equation (3.6) in the class of analytic maps on an f-invariant complex domain:

   $$(\varphi \circ f)(\zeta) = \varphi(\zeta) + \beta,$$

   where f is an analytic map with a certain hyperbolic logarithmic asymptotic bound, but not necessarily having the full logarithmic asymptotic expansion (as is case with the complex Dulac germs). This is proven in Theorem 3.2.11, on the so-called *admissible domains* introduced in Subsection 3.2.1.

2. Finding a solution of the linearization equation (3.6) for hyperbolic complex (real) Dulac germs f, in the class of parabolic complex (real) Dulac germs. Here, we

use the analytic solution of the general linearization equation obtained in 1 and the formal linearization result for hyperbolic Dulac series $\widehat{f}$ from Theorem A which we prove to be an asymptotic expansion of f.

Remark 3.2.1. Note that, in the z-chart, a complex Dulac series is an element of the space $\mathscr{L}_1(\mathbb{C})$ of all logarithmic transseries of depth 1 with complex coefficients, which additionally, satisfies the *polynomial property*. Since the proof of the formal normalization theorem (Theorem A) only uses algebraic properties of the field of real numbers $\mathbb{R}$ which hold also in the field of complex numbers $\mathbb{C}$, it can be proven that Theorem A holds in the space $\mathscr{L}_1(\mathbb{C})$. Therefore, in the z-chart, a hyperbolic complex Dulac series $f(z)$ has a unique parabolic linearization $\varphi(z) \in \mathscr{L}_1(\mathbb{C})$. We ask the following question: Is $\varphi(z)$ a complex Dulac series in the z-chart? This is answered in Subsection 3.2.2.

3.2.1. Linearization theorem on complex domains

This subsection is dedicated to solving the *linearization equation* (3.6) in the class of analytic maps on particular domains around the origin of the Riemann surface of the logarithm $\widetilde{\mathbb{C}}$ (not necessary spiraling domains), for a map f satisfying certain asymptotics in the *exponential-logarithmic scale*.

Note that equation (3.6) can be satisfied only on some f-invariant subdomain of $\widetilde{\mathbb{C}}$. In order to find f-invariant domains, we introduce the so-called *admissible subdomains* of $\widetilde{\mathbb{C}}$ and prove that the admissible domains contain f-invariant subdomains (Proposition 3.2.10). Finally we prove the linearization theorem (Theorem 3.2.11).

In the sequel we work in the ζ-chart, since it is a global chart for $\widetilde{\mathbb{C}}$, which makes our calculations easier.

Admissible complex domains

In this subsection we recall the notion of admissible domains from [30, Subsection 3.1]. Let $\beta \in \mathbb{C}^+$, $\varepsilon > 0$ and $k \in \mathbb{N}$. Let

$$\begin{aligned} M_{\varepsilon,k}(x) &:= \frac{1}{x \log x \cdots (\log^{\circ k} x)^{1+\varepsilon}}, \qquad (3.8) \\ \rho^{\pm}_{\beta,\varepsilon,k}(x) &:= \Re(\beta) \pm M_{\varepsilon,k}(x), \quad \text{for } x \in \left(\exp^{\circ k}(0), +\infty\right). \end{aligned}$$

The map $M_{\varepsilon,k}$ is a positive, strictly decreasing, tending to 0, as $x \to +\infty$. Consequently, it follows that $\rho^{-}_{\beta,\varepsilon,k}$ is a strictly increasing map and $\rho^{+}_{\beta,\varepsilon,k}$ is a strictly decreasing map, both tending to $\Re(\beta)$ at infinity. Note that the series $\sum_{n\in\mathbb{N}} M_{\varepsilon,k}(x+ny)$ converges for every $x, y > 0$ (see e.g. [5, Section 2]).

As in [30, Subsection 3.1], we first define two functions h_l and h_u, whose graphs are used to define admissible domain of type (β, ε, k). In particular, these graphs are, roughly speaking, used to *control* the domain from *below* and from *above*. We distinguish three cases: $\Im(\beta) > 0$, $\Im(\beta) = 0$ and $\Im(\beta) < 0$:

(i) **Case** $\Im(\beta) > 0$. Let $t > \exp^{\circ k}(0)$ such that $\rho^{-}_{\beta,\varepsilon,k}(x) > 0$ and $\Im(\beta) - M_{\varepsilon,k}(x) > 0$, $x \in [t, +\infty)$. Let $h_l,\ h_u : [t, +\infty) \to \mathbb{R}$ be any two functions satisfying:

(a) $h_l(x) < h_u(x)$, $x \in [t, +\infty)$;

(b) h_l is a decreasing map on $[t, +\infty)$, or h_l is an increasing map with property:

$$h_l(x + \rho^{+}_{\beta,\varepsilon,k}(x)) - h_l(x) \leq \Im(\beta) - M_{\varepsilon,k}(x),\ x \in [t, +\infty);$$

(c) h_u is an increasing map with property:

$$h_u(x + \rho^{-}_{\beta,\varepsilon,k}(x)) - h_u(x) \geq \Im(\beta) + M_{\varepsilon,k}(x),\ x \in [t, +\infty).$$

(ii) **Case** $\Im(\beta) = 0$. Let $t > \exp^{\circ k}(0)$ such that $\rho^{-}_{\beta,\varepsilon,k}(x) > 0$, $x \in [t, +\infty)$. Let $h_l, h_u : [t, +\infty) \to \mathbb{R}$ be any two functions satisfying:

(a) $h_l(x) < h_u(x)$, $x \in [t, +\infty)$;

(b) h_l is a decreasing map with property:

$$h_l(x + \rho^{-}_{\beta,\varepsilon,k}(x)) - h_l(x) \leq -M_{\varepsilon,k}(x),\ x \in [t, +\infty);$$

(c) h_u is an increasing map with property:

$$h_u(x + \rho^{-}_{\beta,\varepsilon,k}(x)) - h_u(x) \geq M_{\varepsilon,k}(x),\ x \in [t, +\infty).$$

(iii) **Case** $\Im(\beta) < 0$. Let $t > \exp^{\circ k}(0)$ such that $\rho^{-}_{\beta,\varepsilon,k}(x) > 0$ and $-\Im(\beta) - M_{\varepsilon,k}(x) > 0$, $x \in [t, +\infty)$. Let $h_l,\ h_u : [t, +\infty) \to \mathbb{R}$ be any two functions satisfying:

(a) $h_l(x) < h_u(x),\, x \in [t, +\infty)$;

(b) h_l is a decreasing map on $[t, +\infty)$:

$$h_l(x + \rho^-_{\beta,\varepsilon,k}(x)) - h_l(x) \leq \Im(\beta) - M_{\varepsilon,k}(x),\ x \in [t, +\infty);$$

(c) h_u is an increasing map, or a decreasing map with property:

$$h_u(x + \rho^+_{\beta,\varepsilon,k}(x)) - h_u(x) \geq \Im(\beta) + M_{\varepsilon,k}(x),\ x \in [t, +\infty).$$

A map $h_l : [t, +\infty) \to \mathbb{R}$ with property (2) is called a *lower map of type* (β, ε, k). A map $h_u : [t, +\infty) \to \mathbb{R}$ with property (3) is called an *upper map of type* (β, ε, k). A pair (h_l, h_u) of maps $h_l, h_u : [t, +\infty) \to \mathbb{R}$, satisfying conditions $(1)-(3)$ above, is called a *lower-upper pair of type* (β, ε, k). Notice that the opposite of an upper map of type (β, ε, k) is a lower map of type (β, ε, k).

Finally, let

$$D_{h_l,h_u} := \left\{\zeta \in \mathbb{C}^+ : \Re(\zeta) \geq t,\ h_l(\Re(\zeta)) < \Im(\zeta) < h_u(\Re(\zeta))\right\}. \tag{3.9}$$

The importance of the conditions above for lower-upper pairs will be apparent in Proposition 3.2.10. Following [30], we define *admissible domains*.

Definition 3.2.2 (Admissible domain, Definition 3.1, [30])**.** Let $\beta \in \mathbb{C}^+$, $\varepsilon > 0$ and $k \in \mathbb{N}$. A *domain of type* (β, ε, k) (or (β, ε, k)*-domain*) is defined as a union of an arbitrary nonempty collection of subsets of the form $D_{h_l,h_u} \subseteq \mathbb{C}$ defined above. Similarly, a subset $D \subseteq \mathbb{C}$ which contains a (β, ε, k)-domain is called an *admissible domain of type* (β, ε, k) (or (β, ε, k)*-admissible domain*).

Remark 3.2.3.

1. It follows from Definition 3.2.2 that an arbitrary union of domains of type (β, ε, k) is again a domain of type (β, ε, k) (see [30, Remark 3.2]).

2. Note that the domain D_{h_l,h_u} for a lower-upper pair (h_l, h_u) is a spiraling subdomain of the Riemann surface of the logarithm if h_l and h_u are continuous injective maps with continuous inverses, the union of the images of which equals $\mathbb{R}$.

Examples of admissible complex domains

Following [30, Subsection 3.2], in this subsection we give several examples of upper and lower maps of type (β,ε,k) and prove in Example 3.2.5 that a standard quadratic domain, given in Definition 3.1.8, is a (β,ε,k)-domain (see [30, Example (3)]), for every $\beta\in\mathbb{C}^+$, $\varepsilon>0$ and $k\in\mathbb{N}$. We first provide some general sufficient conditions on C^n-maps to be upper (lower) maps.

Proposition 3.2.4 (Sufficient condition for C^1-maps, Example (1), [30])**.** Let $h:[t,+\infty)\to\mathbb{R}$, $t\in\mathbb{R}_{>0}$, be an increasing map of class C^1.

1. Let $\beta\in\mathbb{C}^+$ such that $\Im(\beta)\geq 0$. If $h':[t,+\infty)\to\mathbb{R}$ tends to $\lambda'>\frac{\Im(\beta)}{\Re(\beta)}$, as $x\to+\infty$, then for each $\varepsilon\in\mathbb{R}_{>0}$ and $k\in\mathbb{N}$, there exists $t'\geq t$ such that the restriction $h|_{[t',+\infty)}$ is an upper map of type (β,ε,k).
2. If $h':[t,+\infty)\to\mathbb{R}$ tends to $+\infty$, as $x\to+\infty$, then for each $\beta\in\mathbb{C}^+$, $\Im(\beta)\geq 0$, $\varepsilon\in\mathbb{R}_{>0}$ and $k\in\mathbb{N}$, there exists $t'\geq t$ such that the restriction $h|_{[t',+\infty)}$ is an upper map of type (β,ε,k).

Proof. 1. Since $M_{\varepsilon,k}(x)\to 0$ and $\rho^-_{\beta,\varepsilon,k}\to\Re(\beta)$ as $x\to+\infty$, there exists $t'\geq t$ sufficiently large such that

$$h'(x)\geq\frac{\Im(\beta)+M_{\varepsilon,k}(t')}{\rho^-_{\beta,\varepsilon,k}(t')},\tag{3.10}$$

for every $x\geq t'$. Since $M_{\varepsilon,k}$ is decreasing and $\rho^-_{\beta,\varepsilon,k}$ is increasing, for every $v\in(0,1)$ and $x>t'$,

$$h'(x+v\rho^-_{\beta,\varepsilon,k}(x))\cdot\rho^-_{\beta,\varepsilon,k}(x)\geq\Im(\beta)+M_{\varepsilon,k}(x).$$

Hence, by the Mean Value Theorem, the restriction $h|_{[t',+\infty\rangle}$ is an upper map of type (β,ε,k).

2. We choose $t'\geq t$ sufficiently large such that (3.10) holds. The restriction $h|_{[t',+\infty\rangle}$ is, therefore, an upper map of type (β,ε,k). ■

Analogously, a similar sufficient condition as in the previous proposition can be deduced for lower maps of type (β,ε,k).

Example 3.2.5 (Standard quadratic domains, Example (3), [30])**.** Let $\beta\in\mathbb{C}^+$, $\varepsilon>0$ and $k\in\mathbb{N}$. For $C>0$, let $\mathscr{R}_C\subseteq\mathbb{C}$ be the standard quadratic domain defined in Definition 3.1.8.

The upper half of the boundary of $\mathscr{R}_C$ is described by a smooth function which satisfies assumptions of statement 2 of Proposition 3.2.4 (and the lower half satisfies the symmetric statement). Hence, such a domain is admissible.

Indeed, a direct computation shows that the boundary

$$\partial\left(\mathscr{R}_C \cap \{\zeta \in \mathbb{C} : \Im(\zeta) \geq 0\}\right)$$

can be parameterized by:

$$\begin{aligned} r \mapsto x(r) + \mathrm{i}\cdot y(r) = {} & C\sqrt[4]{r^2+1}\cos\left(\frac{1}{2}\mathrm{arctg}\, r\right) + \\ & + \mathrm{i}\cdot\left(r + C\sqrt[4]{r^2+1}\sin\left(\frac{1}{2}\mathrm{arctg}\, r\right)\right),\ r \in [0,+\infty). \end{aligned}$$

Note that $y : [0,+\infty) \to \mathbb{R}$ is strictly increasing. Let $t > 0$ be such that $x(t) > \exp^{\circ k}(0)$ and x is strictly increasing on $[t,+\infty)$. Therefore, $h_u := y \circ x^{-1}$ is strictly increasing on $[x(t),+\infty)$. By direct computation, if can be shown that:

$$h_u'(r) = \frac{dy(r)}{dx(r)} = \frac{2(r^2+1)^{\frac{3}{4}}}{C(rs_2 - s_1)} + \frac{rs_1 + s_2}{rs_2 - s_1}, \tag{3.11}$$

where $s_1 := \sin(\frac{1}{2}\mathrm{arctg}\, r)$ and $s_2 := \cos(\frac{1}{2}\mathrm{arctg}\, r)$, for each $r \in [0,+\infty)$. Consequently, the derivative of h_u on $[x(t),+\infty)$ tends to $+\infty$, as $x \to +\infty$. Therefore, by statement 2 of Proposition 3.2.4, there exists $x' > 0$ such that the restriction $h_u|_{[x',+\infty\rangle}$ is an upper map of type (β,ε,k). A similar argument can be repeated to show that an appropriate restriction of the lower boundary of $\mathscr{R}_C$ is the graph of a lower map of type (β,ε,k).

Therefore, for $(\beta,\varepsilon,k) \in \mathbb{C}^+ \times \mathbb{R}_{>0} \times \mathbb{N}$, there exists $R > 0$ such that the representative $(\mathscr{R}_C)_R$ of a standard quadratic domain $\mathscr{R}_C$, $C > 0$, is a domain of type (β,ε,k).

In the next example we state some examples of lower-upper maps with power asymptotic bounds.

Example 3.2.6 (Maps of type $h(x) \sim x^r,\ r > 0$, Example (2), [30])**.** Let $\beta \in \mathbb{C}^+$, $\varepsilon > 0$ and $k \in \mathbb{N}$.

1. *Case* $r > 1$. Let $h : [t,+\infty) \to \mathbb{R}$, $t > 0$, be an increasing map of class C^1 such that[1]

$$h(x) \sim ax^r,\ h'(x) \sim arx^{r-1},\ a > 0,\ r > 1,\ x \to +\infty. \tag{3.12}$$

[1] We write $f \sim g$, $x \to \infty$, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

By Proposition 3.2.4, there exists $t > 0$ big enough such that h is an upper map of type (β, ε, k).

Let $h : [t, +\infty) \to \mathbb{R}$, $t > 0$, be a decreasing C^1-map , such that

$$h(x) \sim -ax^r,\ h'(x) \sim -arx^{r-1},\ a > 0,\ r > 1,\ x \to +\infty. \tag{3.13}$$

By a similar argument, there exists $t > 0$ big enough such that h is a lower map of type (β, ε, k).

2. *Case* $r = 1$. Let $h : [t, +\infty) \to \mathbb{R}$, $t > 0$, be an increasing C^1-map such that

$$h(x) \sim ax,\ h'(x) \sim a,\ a > 0,\ x \to +\infty.$$

Then, by statement 1 of Proposition 3.2.4, if $\Im(\beta) < 0$, or if $\Im(\beta) \geq 0$ and $\frac{\Im(\beta)}{\Re(\beta)} < a$, there exists $t > 0$ big enough such that h is an upper map of type (β, ε, k). Let $h : [t, +\infty) \to \mathbb{R}$, $t > 0$, be a decreasing C^1-map satisfying $h(x) \sim -ax$, $h'(x) \sim -a$, $a > 0$, $x \to +\infty$. Similarly, if $\Im(\beta) > 0$, or if $\Im(\beta) \leq 0$ and $\frac{\Im(\beta)}{\Re(\beta)} > -a$, there exists big enough $t > 0$ such that h is a lower map of type (β, ε, k).

3. *Case* $0 < r < 1$. The assumptions of Proposition 3.2.4 are not satisfied for $0 < r < 1$. Therefore, we give the second necessary condition in the next proposition (see [30, Subsection 3.2]). Using that new condition we finish this part in Example 3.2.8.

Proposition 3.2.7 (Sufficient condition for C^n-maps, [30])**.** Let $\beta \in \mathbb{C}^+$, $\varepsilon \in \mathbb{R}_{>0}$, and $k \in \mathbb{N}$.

1. **Upper map condition in case $\Im(\beta) \geq 0$.** Let $\Im(\beta) \geq 0$, $t > \exp^{\circ k}(0)$ such that $\rho^-_{\beta,\varepsilon,k}(t) > 0$, and let $h \in C^n([t, +\infty))$, for some $n \in \mathbb{N}_{\geq 1}$, be an increasing map. Suppose that there exists a positive number $0 < \rho < \rho^-_{\beta,\varepsilon,k}(t)$ such that

$$\sum_{i=1}^{n} \frac{h^{(i)}(x)}{i!}\rho^i \geq \Im(\beta) + M_{\varepsilon,k}(x),\ \text{for all } x \geq t, \tag{3.14}$$

and that $h^{(n)} : [t, +\infty) \to \mathbb{R}$ is increasing. Then h is an upper map of type (β, ε, k).

2. **Upper map condition in case $\Im(\beta) < 0$.** Let $\Im(\beta) < 0$ and $t > \exp^{\circ k}(0)$. Suppose that $h : [t, +\infty) \to \mathbb{R}$ is either an increasing map, or a decreasing map belonging to $C^n([t, +\infty))$, for some $n \in \mathbb{N}$, which satisfies, for some positive number $\rho >$

$\rho^+_{\beta,\varepsilon,k}(t)$:

$$\sum_{i=1}^{n} \frac{h^{(i)}(x)}{i!}\rho^i \geq \Im(\beta) + M_{\varepsilon,k}(x), \text{ for all } x \geq t, \tag{3.15}$$

and that $h^{(n)} : [t,+\infty) \to \mathbb{R}$ is increasing. Then h is an upper map of type (β,ε,k).

3. **Lower map condition in case** $\Im(\beta) > 0$. Let $\Im(\beta > 0)$ and $t > \exp^{\circ k}(0)$ such that $\Im(\beta) - M_{\varepsilon,k}(t) > 0$. Suppose that h is either decreasing on $[t,+\infty)$, or increasing belonging to $C^n([t,+\infty))$, for some $n \in \mathbb{N}_{\geq 1}$, and satisfying the property

$$\sum_{i=1}^{n} \frac{h^{(i)}(x)}{i!}\rho^i \leq \Im(\beta) - M_{\varepsilon,k}(x), \text{ for all } x \geq t,$$

for some $\rho > \rho^+_{\beta,\varepsilon,k}(t)$, and with $h^{(n)}$ decreasing on $[t,+\infty)$. Then h is a lower map of type (β,ε,k).

4. **Lower map condition in case** $\Im(\beta) \leq 0$. Let $\Im(\beta) \leq 0$ and $t > \exp^{\circ k}(0)$ such that $\rho^-_{\beta,\varepsilon,k}(t) > 0$. Suppose that h is decreasing map belonging to $C^n([t,+\infty))$, for some $n \in \mathbb{N}_{\geq 1}$, and satisfies the property

$$\sum_{i=1}^{n} \frac{h^{(i)}(x)}{i!}\rho^i \leq \Im(\beta) - M_{\varepsilon,k}(x), \text{ for all } x \geq t, \tag{3.16}$$

for some $0 < \rho < \rho^-_{\beta,\varepsilon,k}(t)$, and $h^{(n)}$ is decreasing. Then h is a lower map of type (β,ε,k).

Proof. 1. Note that $\rho^-_{\beta,\varepsilon,k}(t) \leq \rho^-_{\beta,\varepsilon,k}(x)$, for $x \in [t,+\infty)$. Since h and $h^{(n)}$ are increasing, it follows from the Taylor Theorem and then (3.14) that

$$\begin{aligned} h(x+\rho^-_{\beta,\varepsilon,k}(x)) - h(x) \geq h(x+\rho) - h(x) &\geq \sum_{i=1}^{n} \frac{h^{(i)}(x)}{i!}\rho^i \\ &\geq \Im(\beta) + M_{\varepsilon,k}(x),\ x \in [t,+\infty). \end{aligned}$$

Statements 2-4 can be proven similarly. ■

Example 3.2.8 (Maps of type $h(x) \sim x^r$, $0 < r < 1$, Example (2), [30]). Let $\beta \in \mathbb{C}^+$, $\varepsilon > 0$ and $k \in \mathbb{N}$ and let $h : [t,+\infty) \to \mathbb{R}$, $t > 0$, be an increasing map of class C^1 such that (3.12) (resp. (3.13)) holds. The assumptions of Proposition 3.2.4 are not satisfied for $0 < r < 1$. We use Proposition 3.2.7 to finish Example 3.2.6.

Suppose that h is increasing (*resp.* decreasing) of class C^2, satisfying (3.12) (*resp.* (3.13)) with $r<1$, with the additional property that $h''(x)\sim ar(r-1)x^{r-2}$ (*resp.* $h''(x)\sim -ar(r-1)x^{r-2}$) and h'' is increasing (*resp.* decreasing). It follows from conditions (3.14) and (3.16), in a similar way as in Example 3.2.9 below, that if $\Im(\beta)=0$, then h is an upper (*resp.* lower) map of type (β,ε,k), $\varepsilon>0$, $k\in\mathbb{N}$.

Example 3.2.9 (Logarithmic upper/lower maps, Example (4), [30])**.** Let $\beta\in\mathbb{C}^+$, $\varepsilon>0$ and $k\in\mathbb{N}$.

Logarithmic upper maps. Let $\Im(\beta)=0$ and let $h:[t,+\infty)\to\mathbb{R}$, $h(x):=(\log x)^\delta$, $\delta\in\mathbb{R}_{>0}$, $t\in\mathbb{R}_{>1}$. Note that:

$$\begin{aligned}h'(x)&=\frac{\delta\,(\log x)^\delta}{x\log x},\\ h''(x)&=\frac{\delta\,(\log x)^\delta}{x\log x}\cdot\Big(\frac{\delta-1}{x\log x}-\frac{1}{x}\Big),\quad x\geq t.\end{aligned}$$

For every $0<\rho<\rho^-_{\beta,\varepsilon,k}$, there exists $t>\exp^{\circ k}(0)$ big enough such that

$$\begin{aligned}&h'(x)\rho+\frac{1}{2}h''(x)\rho^2=\frac{1}{x\log x}\cdot\rho\,\delta(\log x)^\delta\cdot\Big(1+\frac{\rho\,(\delta-1)}{2x\log x}-\frac{\rho}{2x}\Big)\\ &\geq\frac{1}{x\log x}\cdot\frac{1}{\log^{\circ 2}x\cdots(\log^{\circ k}x)^{1+\varepsilon}}=M_{\varepsilon,k}(x),\end{aligned}$$

for $x\geq t$. It can be proven that we can take $t>\exp^{\circ k}(0)$ large enough such that $h'''(x)>0$, for each $x\geq t$. This implies that restriction of h'' on $[t,+\infty)$ is an increasing map. Therefore, since h is increasing, it follows from sufficient upper map condition (3.14) that there exists $t>\exp^{\circ k}(0)$ such that the restriction $h|_{[t,+\infty)}$ is an upper map of type (β,ε,k).

Logarithmic lower maps. Let $\Im(\beta)=0$ and let h be as defined above. It follows that $g:[t,+\infty)\to\mathbb{R}$, defined by $g(x):=-h(x)$, $x\in[t,+\infty)$, is a lower map of type (β,ε,k).

Linearization of hyperbolic maps on admissible domains

Let $f(\zeta)=\zeta+\beta+o(1)$, $\beta\in\mathbb{C}^+$, be a hyperbolic analytic map on domain $D\subseteq\widetilde{\mathbb{C}}$, given in the ζ-chart. As in [30, Subsection 3.3], we consider the *linearization equation*:

$$\varphi\circ f=\varphi+\beta, \tag{3.17}$$

and ask about the existence and the uniqueness of the analytic parabolic change of variables φ defined on some f-invariant subdomain of D that satisfies (3.17). The main result of this subsection is given in Theorem 3.2.11 below. The idea of the proof is the following. Recall the Koenigs sequence used in the proof of the classical Koenigs Theorem for hyperbolic analytic diffeomorphisms, stated in Theorem 2.1.7 (see e.g. [4, Theorem 2.1], [14], [24, Theorem 8.2]). In the ζ-chart, the Koenigs sequence becomes the sequence:

$$(f^{\circ n} - n\beta)_n$$

We obtain a solution φ of linearization equation (3.17) as the uniform limit of the Koenigs sequence.

First, in Proposition 3.2.10, we find a maximal f-invariant subdomain of the domain D. Let us denote by D^f the union of all f-invariant subdomains of the domain D. We call D^f the *maximal f-invariant subdomain* od the domain D.

If we assume that D is an admissible domain, Proposition 3.2.10 below guarantees that $D^f \neq \emptyset$. This is the main reason why we introduce the notion of an admissible domain.

Let $D \subseteq \mathbb{C}$ and $R \in \mathbb{R}_{>0}$. We denote by D_R the subdomain

$$D_R := D \cap \{\zeta \in \mathbb{C} : \Re(\zeta) \geq R\},$$

in the ζ-chart. Let $\beta \in \mathbb{C}^+$, $\varepsilon \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}$.

Let D be an admissible domain of type (β, ε, k). We denote by $\overline{D}$ the union of all subdomains of D of type (β, ε, k). Note that $D_1 \subseteq \overline{D}$ for every subdomain $D_1 \subseteq D$ of type (β, ε, k). Therefore, we call $\overline{D}$ the *maximal subdomain of type* (β, ε, k) of the admissible domain D. For every $R \in \mathbb{R}_{>0}$ we put:

$$\overline{D}_R := (\overline{D})_R,$$
$$D^f := (D^f)_R.$$

Proposition 3.2.10 (Proposition 3.4, [30])**.** Let $\beta \in \mathbb{C}^+$, $\varepsilon > 0$ and $k \in \mathbb{N}$. Let $D \subseteq \mathbb{C}^+$ be an admissible domain of type (β, ε, k). Let $f : D_C \to \mathbb{C}$, $C > \exp^{\circ k}(0)$, be an analytic map, such that

$$f(\zeta) = \zeta + \beta + o(\zeta^{-1} \boldsymbol{L}_1^{-1} \cdots \boldsymbol{L}_k^{-(1+\varepsilon)}), \text{ as } \Re(\zeta) \to +\infty \text{ uniformly on } D_C. \tag{3.18}$$

Here,

$$\boldsymbol{L}_1 := \log(\zeta), \ldots, \boldsymbol{L}_k := \log(\boldsymbol{L}_{k-1}),$$

where log represents the principal branch of the logarithm[2]. Then, for every $R \geq C$ sufficiently large, the domain $\overline{D}_R$ is f-invariant. In particular, $\overline{D}_R \subseteq D_R^f$ and $D_R^f \neq \emptyset$.

Proof. By asymptotics (3.18),

$$\lim_{\Re(\zeta)\to+\infty} \frac{f(\zeta)-(\zeta+\beta)}{\zeta^{-1}\boldsymbol{L}_1^{-1}\cdots\boldsymbol{L}_k^{-(1+\varepsilon)}} = 0, \tag{3.19}$$

uniformly on D.

Let $\rho^{\pm}_{\beta,\varepsilon,k}$ and $M_{\varepsilon,k}$ be as defined in (3.8). By (3.19), there exists $R \geq C$ such that $\rho^{-}_{\beta,\varepsilon,k}(R) > 0$, $\rho^{-}_{\beta,\varepsilon,k}$ is increasing on $[R,+\infty)$ and, for all $\zeta \in D_R$,

$$|f(\zeta)-(\zeta+\beta)| \leq \frac{1}{\left|\zeta\boldsymbol{L}_1\cdots\boldsymbol{L}_k^{1+\varepsilon}\right|}. \tag{3.20}$$

Since $R > \exp^{\circ k}(0)$ and $|\log\zeta| \geq \log|\zeta| \geq \log(|\Re(\zeta)|) = \log(\Re(\zeta))$, we inductively get:

$$|\boldsymbol{L}_m| \geq \log^{\circ m}(\Re(\zeta)), \text{ for } 1 \leq m \leq k,\ \zeta \in D_R. \tag{3.21}$$

Now, by (3.20) and (3.21), we get, for $\zeta \in D_R$:

$$|f(\zeta)-(\zeta+\beta)| \leq \frac{1}{\Re(\zeta)\cdot\log(\Re(\zeta))\cdots(\log^{\circ k}(\Re(\zeta)))^{1+\varepsilon}}. \tag{3.22}$$

Therefore, for $\zeta \in D_R$:

$$\begin{aligned}\Re(f(\zeta))-\Re(\zeta) &\geq \Re(\beta) - \frac{1}{\Re(\zeta)\cdot\log(\Re(\zeta))\cdots(\log^{\circ k}(\Re(\zeta)))^{1+\varepsilon}}\\ &= \rho^{-}_{\beta,\varepsilon,k}(\Re(\zeta)),\end{aligned} \tag{3.23}$$

$$\begin{aligned}\Re(f(\zeta))-\Re(\zeta) &\leq \Re(\beta) + \frac{1}{\Re(\zeta)\cdot\log(\Re(\zeta))\cdots(\log^{\circ k}(\Re(\zeta)))^{1+\varepsilon}}\\ &= \rho^{+}_{\beta,\varepsilon,k}(\Re(\zeta)),\end{aligned} \tag{3.24}$$

and

[2]Note here that, for $C > \exp^{\circ k}(0)$, the iterated logarithms $\boldsymbol{L}_1,\ldots,\mathbf{L}_k$ are well-defined on D_C (using only the principal branch of the logarithm), since $\Re(\zeta) > \exp^{\circ k}(0)$.

$$\begin{aligned}\Im(f(\zeta))-\Im(\zeta) &\geq \Im(\beta)-\frac{1}{\Re(\zeta)\cdot\log(\Re(\zeta))\cdots(\log^{\circ k}(\Re(\zeta)))^{1+\varepsilon}} \\ &= \Im(\beta)-M_{\varepsilon,k}(\Re(\zeta)),\end{aligned} \tag{3.25}$$

$$\begin{aligned}\Im(f(\zeta))-\Im(\zeta) &\leq \Im(\beta)+\frac{1}{\Re(\zeta)\cdot\log(\Re(\zeta))\cdots(\log^{\circ k}(\Re(\zeta)))^{1+\varepsilon}} \\ &= \Im(\beta)+M_{\varepsilon,k}(\Re(\zeta)).\end{aligned} \tag{3.26}$$

Since $\rho^-_{\beta,\varepsilon,k}$ is an increasing function, it follows that $\rho^-_{\beta,\varepsilon,k}(\Re(\zeta)) \geq \rho^-_{\beta,\varepsilon,k}(R) > 0$, for every $\zeta \in D_R$. Let, for $\zeta \in D_R$,

$$\begin{aligned}\mathscr{S}_{\beta,\varepsilon,k}(\zeta) := &\left[\Re(\zeta)+\rho^-_{\beta,\varepsilon,k}(\Re(\zeta)),\, \Re(\zeta)+\rho^+_{\beta,\varepsilon,k}(\Re(\zeta))\right] \\ &\times\left[\Im(\zeta)+\Im(\beta)-M_{\varepsilon,k}(\Re(\zeta)), \Im(\zeta)+\Im(\beta)+M_{\varepsilon,k}(\Re(\zeta))\right].\end{aligned}$$

By (3.23)-(3.26), we get that, for $\zeta \in D_R$,

$$f(\zeta) \in \mathscr{S}_{\beta,\varepsilon,k}(\zeta).$$

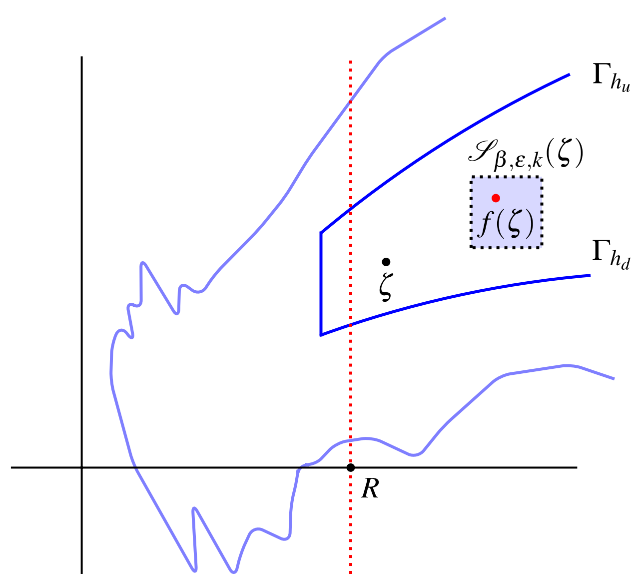

Figure 3.2: For R sufficiently large, $\mathscr{S}_{\beta,\varepsilon,k}(\zeta) \subseteq \overline{D}_R$, $\zeta \in \overline{D}_R$ ([30, Figure 2]).

Now take $\zeta \in \overline{D}_R$. It is left to prove that then $f(\zeta) \in \overline{D}_R$, that is, that $\overline{D}_R$ is f-invariant. By the definition of $\overline{D}_R$, there exists a (β,ε,k)-domain $(D_{h_l,h_u})_R \subseteq \overline{D}_R$, such that $\zeta \in$

$(D_{h_l,h_u})_R$. Now, by properties (2) and (3) in the definition of lower-upper pair of type (β,ε,k), it follows that $\mathscr{S}_{\beta,\varepsilon,k}(\zeta)\subseteq D_{h_l,h_u}$, see Figure 3.2. Therefore, $f(\zeta)\in D_{h_l,h_u}\subseteq\overline{D}$. Since $\zeta\in D_R$ and $\rho^{-}_{\beta,\varepsilon,k}(\Re(\zeta))>0$ for $\zeta\in D_R$, by (3.23) it follows that $\Re(f(\zeta))>\Re(\zeta)\geq R$. Therefore, $f(\zeta)\in\overline{D}_R$. ■

Finally, we state and prove the linearization theorem for analytic maps with hyperbolic (logarithmic) asymptotic bounds on admissible domains.

Theorem 3.2.11 (Linearization theorem for analytic maps on admissible domains, Theorem A, [30])**.** Let $\beta\in\mathbb{C}^+$, $\varepsilon>0$, $k\in\mathbb{N}$. Let $D\subseteq\mathbb{C}^+$ be an admissible domain of type (β,ε,k). For $C>\exp^{\circ k}(0)$, let $f:D_C\to\mathbb{C}$ be an analytic map such that

$$f(\zeta)=\zeta+\beta+o(\zeta^{-1}\boldsymbol{L}_1^{-1}\cdots\boldsymbol{L}_k^{-(1+\varepsilon)}),\ \text{ as } \Re(\zeta)\to+\infty \text{ uniformly on } D_C. \tag{3.27}$$

Here, the iterated logarithms $\boldsymbol{L}_1,\ldots\boldsymbol{L}_k$ are defined as in Proposition 3.2.10. Then:

1. *(Existence)* For a sufficiently large $R\geq C$ there exists an analytic linearizing map φ on the f-invariant subdomain $D_R^f\subseteq D$. That is, φ satisfies

$$(\varphi\circ f)(\zeta)=\varphi(\zeta)+\beta,\ \text{for all } \zeta\in D_R^f. \tag{3.28}$$

   Moreover, φ is the uniform limit on D_R^f of the Koenigs sequence

$$(f^{\circ n}-n\beta)_n.$$

2. If $D_R^f\cap\{\zeta\in\mathbb{C}^+:\Im(\zeta)=0\}$ is f-invariant, so is φ-invariant.

3. *(Asymptotics)* The linearization φ is *tangent to identity*, i.e., $\varphi(\zeta)=\zeta+o(1)$, uniformly on $D_R^f\subseteq\mathbb{C}^+$, as $\Re(\zeta)\to+\infty$.

   In particular, $\varphi(\zeta)=\zeta+o(\boldsymbol{L}_k^{-\nu})$, for every $\nu\in(0,\varepsilon)$, uniformly as $\Re(\zeta)\to+\infty$, on every subdomain $D_{h_l,h_u}\subseteq D_R^f$ such that $h_l(x)=O(x)$ and $h_u(x)=O(x)$.

4. *(Uniqueness)* Let $\psi:D_1\to\mathbb{C}$, be a linearization of f on an f-invariant subset $D_1\subseteq D$, such that $\psi(\zeta)=\zeta+o(1)$ uniformly on D_1, as $\Re(\zeta)\to+\infty$. Then $\psi\equiv\varphi$ on $(D_1)_R$, for R from statement 1.

Proof. 1. By Proposition 3.2.10, there exists $R \geq C$ sufficiently large, such that $\overline{D}_R \subseteq D_R^f$, where D^f is the maximal f-invariant subdomain of $D \subseteq \mathbb{C}^+$. Therefore, $D_{R'}^f \neq \emptyset$, for all $R' \geq R$. Let $\zeta \in D_R^f$ and let $\rho_{\beta,\varepsilon,k}^{\pm}$ be the increasing (decreasing) maps defined in (3.8). From (3.23), for $\zeta \in D_R^f$, with R sufficiently large, since $\rho_{\beta,\varepsilon,k}^{-}$ is increasing on $[R, +\infty)$, it follows that:

$$\Re(f^{\circ n}(\zeta)) \geq \Re(\zeta) + n\rho_{\beta,\varepsilon,k}^{-}(\Re(\zeta)) \geq R + n\rho_{\beta,\varepsilon,k}^{-}(R),\ n \in \mathbb{N}_{\geq 1}. \tag{3.29}$$

By (3.22), for $\zeta \in D_R^f$ it holds that:

$$|f(\zeta) - (\zeta + \beta)| \leq \frac{1}{\Re(\zeta) \cdot \log(\Re(\zeta)) \cdots (\log^{\circ k}(\Re(\zeta)))^{1+\varepsilon}} = M_{\varepsilon,k}(\Re(\zeta)). \tag{3.30}$$

From (3.29) and (3.30) and since $M_{\varepsilon,k}$ is decreasing on $[R, +\infty)$, we inductively obtain, for $n \in \mathbb{N}_{\geq 1}$ and $\zeta \in D_R^f$:

$$\begin{aligned}\left| f^{\circ(n+1)}(\zeta) - (n+1)\beta - (f^{\circ n}(\zeta) - n\beta) \right| &= |f(f^{\circ n}(\zeta)) - (f^{\circ n}(\zeta) + \beta)| \\ &\leq M_{\varepsilon,k}\big(\Re(f^{\circ n}(\zeta))\big) \\ &\leq M_{\varepsilon,k}\Big(\Re(\zeta) + n\rho_{\beta,\varepsilon,k}^{-}(\Re(\zeta))\Big) \\ &\leq M_{\varepsilon,k}\Big(R + n\rho_{\beta,\varepsilon,k}^{-}(R)\Big). \end{aligned} \tag{3.31}$$

As stated in Subsection 3.2.1, the series $\sum_{n\geq 0} M_{\varepsilon,k}\big(R + n\rho_{\beta,\varepsilon,k}^{-}(R)\big)$ converges. Therefore, the Koenigs sequence $(f^{\circ n} - n\beta)_n$ is uniformly Cauchy, hence, converges uniformly on D_R^f. Denote by φ its uniform limit on the domain D_R^f. By the Weierstrass Theorem, it follows that φ is analytic on D_R^f.

Finally, we compute:

$$\begin{aligned}(\varphi \circ f)(\zeta) &= \lim_n (f^{\circ n}(f(\zeta)) - n\beta) \\ &= \lim_n \Big(f^{\circ(n+1)}(\zeta) - (n+1)\beta\Big) + \beta \\ &= \varphi(\zeta) + \beta, \quad \zeta \in D_R^f.\end{aligned}$$

Therefore, φ is an analytic linearization of f on D_R^f, obtained as a uniform limit of the Koenigs sequence.

2. Suppose that $D_R^f \cap \{\zeta \in \mathbb{C} : \Im(\zeta) = 0\}$ is invariant under $f(\zeta) = \zeta + \beta + o(1)$. Then obviously $\Im(\beta) = 0$. Now consider the pointwise limit

$$\varphi(\zeta) := \lim_{n\to\infty} (f^{\circ n}(\zeta) - n\beta),\ \zeta \in D_R^f \cap \{\zeta \in \mathbb{C} : \Im(\zeta) = 0\}. \tag{3.32}$$

Since $D_R^f \cap \{\zeta \in \mathbb{C}^+ : \Im(\zeta) = 0\}$ is invariant for (all iterates of) f, $\{\zeta \in \mathbb{C} : \Im(\zeta) = 0\}$ closed in $\mathbb{C}^+$, and since $\Im(\beta) = 0$, (3.32) implies that $\{D_R^f \cap \{\zeta \in \mathbb{C} : \Im(\zeta) = 0\}$ is invariant for φ.

3. For $0 < \nu < \varepsilon$, $\zeta \in D_R^f$ and $m \in \mathbb{N}_{\geq 1}$, taking the sum of the terms

$$f^{\circ(n+1)}(\zeta) - (n+1)\beta - (f^{\circ n}(\zeta) - n\beta)$$

in (3.31) for n ranging from 0 to $m-1$ it follows that

$$\begin{aligned}
|f^{\circ m}(\zeta) - m\beta - \zeta| &\leq \sum_{n=0}^{m-1} M_{\varepsilon,k}\left(\Re(\zeta) + n\rho^-_{\beta,\varepsilon,k}(\Re(\zeta))\right) \\
&= \sum_{n=0}^{m-1} \frac{M_{\nu,k}\left(\Re(\zeta) + n\rho^-_{\beta,\varepsilon,k}(\Re(\zeta))\right)}{\left(\log^{\circ k}\left(\Re(\zeta) + n\rho^-_{\beta,\varepsilon,k}(\Re(\zeta))\right)\right)^{\varepsilon-\nu}} \\
&\leq \frac{1}{\left(\log^{\circ k}(\Re(\zeta))\right)^{\varepsilon-\nu}} \cdot \sum_{n=0}^{+\infty} M_{\nu,k}(R + n\rho^-_{\beta,\varepsilon,k}(R)) \\
&\leq \frac{K}{\left(\log^{\circ k}(\Re(\zeta))\right)^{\varepsilon-\nu}},\ K > 0,
\end{aligned} \tag{3.33}$$

where the last sum converges to $K > 0$. Taking the pointwise limit for $m \to +\infty$ in (3.33), it follows that

$$|\varphi(\zeta) - \zeta| \leq \frac{K}{\left(\log^{\circ k}(\Re(\zeta))\right)^{\varepsilon-\nu}},\ K > 0,\ \zeta \in D_R^f. \tag{3.34}$$

Therefore, $\varphi(\zeta) = \zeta + o(1)$, uniformly on D_R^f as $\Re(\zeta) \to +\infty$.

To get a more rigorous estimate, using the elementary properties of the logarithm, we easily see that, for every domain $D_{h_l,h_u} \subseteq \mathbb{C}_+$ such that $h_l(x) = O(x)$ and $h_u(x) = O(x)$, as $x \to +\infty$, there exists $N > 0$, such that

$$|\boldsymbol{L}_k| \leq N \cdot \log^{\circ k}(\Re(\zeta)),\ \text{for}\ \zeta \in (D_{h_l,h_u})_R, \tag{3.35}$$

for sufficiently large $R > \exp^{\circ k}(0)$, $k \in \mathbb{N}_{\geq 1}$. Indeed, the conditions $h_l(x) = O(x)$ and $h_u(x) = O(x)$ imply that there exists $d > 0$ such that $|\Im(\zeta)| \leq d \cdot \Re(\zeta)$, $\zeta \in (D_{h_l,h_u})_R$, for a sufficiently large $R > 0$.

Using (3.35) in (3.34), it now follows that, for any $0 < \nu < \nu' < \varepsilon$, there exists $E > 0$ such that:

$$|\varphi(\zeta) - \zeta| \leq \frac{E}{\left(\log^{\circ k}(\Re(\zeta))\right)^{\nu'-\nu}} |\boldsymbol{L}_k|^{\nu'-\varepsilon}, \ \zeta \in (D_{h_l,h_u})_R. \tag{3.36}$$

This implies that, for every $0 < \nu' < \varepsilon$, on $(D_{h_l,h_u})_R$ it holds that:

$$\lim_{\Re(\zeta)\to+\infty} \frac{\varphi(\zeta) - \zeta}{\frac{1}{(\boldsymbol{L}_k)^{\varepsilon-\nu'}}} = 0.$$

Therefore, for any $0 < \nu < \varepsilon$, $\varphi(\zeta) = \zeta + o\left(\boldsymbol{L}_k^{-\nu}\right)$, as $\Re(\zeta) \to +\infty$ in $(D_{h_l,h_u})_R$.

4. Suppose that ψ is an analytic linearizing germ, i.e., $\psi \circ f = \psi + \beta$, on f-invariant subset $D_1 \subseteq D$, such that $\psi(\zeta) = \zeta + o(1)$ uniformly on D_1 as $\Re(\zeta) \to +\infty$. Since D_1 is f-invariant and D^f is a maximal f-invariant subdomain of D, obviously $(D_1)_R \subseteq D_R^f$. Clearly, $(D_1)_R$ is also f-invariant, and by (3.29), non-empty. Recall from statement 1 that φ is the analytic linearization constructed on whole D_R^f as the limit of the Koenigs sequence, for sufficiently large $R > \exp^{\circ k}(0)$. It satisfies $\varphi(\zeta) = \zeta + o(1)$, uniformly as $\Re(\zeta) \to +\infty$ on D_R^f. We now show that $\psi \equiv \varphi$ on $(D_1)_R$.

Put

$$E(\zeta) := \varphi(\zeta) - \psi(\zeta), \ \zeta \in (D_1)_R.$$

Then E is analytic on $(D_1)_R$ and $E(\zeta) = o(1)$, as $\Re(\zeta) \to +\infty$ uniformly on $(D_1)_R$. Moreover, $(E \circ f)(\zeta) = E(\zeta)$, $\zeta \in (D_1)_R$. Inductively, since $(D_1)_R$ is f-invariant, we obtain

$$E(f^{\circ n}(\zeta)) = E(\zeta), \ \zeta \in (D_1)_R, \ n \in \mathbb{N}. \tag{3.37}$$

By (3.29), $\Re(f^{\circ n}(\zeta)) \geq R + n\rho^-_{\beta,\varepsilon,k}(R)$, for $n \in \mathbb{N}$ and $\zeta \in (D_1)_R \subseteq D_R^f$. It follows that

$$\lim_n \Re(f^{\circ n}(\zeta)) = +\infty, \ \zeta \in (D_1)_R. \tag{3.38}$$

Passing to limit, as $n \to \infty$, in (3.37), and using (3.38) and the fact that $E(\zeta) = o(1)$, as $\Re(\zeta) \to +\infty$, we get that $E(\zeta) = 0$, for every $\zeta \in (D_1)_R$. That is, $\varphi \equiv \psi$ on $(D_1)_R$. ■

3.2.2. Formal linearization of hyperbolic Dulac germs

In this subsection we prove that the formal linearization of a hyperbolic complex Dulac series is again a Dulac series (see [30, Subsection 4.1]).

Let $\widehat{f}(z) = \lambda z + \text{h.o.t.}$, $\lambda \in \mathbb{R}_{>0}$, be a hyperbolic complex (real) Dulac series given in the standard z-chart. Since $\operatorname{ord}(f - \mathrm{id}) > (1,1)$, by Theorem A and Remark 3.2.1, it follows that there exists a unique solution $\widehat{\varphi} \in \mathscr{L}_1^0(\mathbb{C})$ to the conjugacy equation:

$$(\widehat{\varphi} \circ \widehat{f} \circ \widehat{\varphi}^{-1})(z) = \lambda z. \tag{3.39}$$

Moreover, $\widehat{\varphi} - \mathrm{id}$ is the limit of the Picard sequence $\left((\mathscr{T}_{\widehat{f}}^{-1} \circ \mathscr{S}_{\widehat{f}})^{\circ n}(\widehat{h})\right)_n$, for any initial condition $\widehat{h} \in \mathscr{L}_1(\mathbb{C})$, $\operatorname{ord}(\widehat{h}) > (1,0)$, with respect to the power-metric topology given in Section 1.1.3 of Chapter 1.

Note that conjugacy equation (3.39) in the ζ-chart becomes the equation:

$$(\widehat{\varphi}^{-1} \circ \widehat{f} \circ \widehat{\varphi})(\zeta) = \zeta - \log \lambda.$$

In the next proposition we prove that the linearization $\widehat{\varphi}$ is a parabolic complex (real) Dulac series.

Proposition 3.2.12 (Formal linearization, Lemma 4.2, [30])**.** Let $\widehat{f}(\zeta) = \zeta + \beta + \text{h.o.t.}$, $\beta \in \mathbb{C}^+$, be a hyperbolic complex Dulac series given in the ζ-chart. Then there exists a unique parabolic complex exponential transseries $\widehat{\varphi}$ such that $\widehat{\varphi} \circ \widehat{f} = \widehat{\varphi} + \beta$. Moreover, $\widehat{\varphi}$ is a complex Dulac series. Finally, if the coefficients of $\widehat{f}$ are real, then so are those of $\widehat{\varphi}$.

Proof. Existence. Let $\widehat{f}(\zeta) := \zeta + \beta + \sum_{i=1}^{\infty} \exp(-\nu_i \zeta) R_i(\zeta)$, for $\beta \in \mathbb{C}^+$, $R_i \in \mathbb{C}[X]$, and (ν_i) a finitely generated sequence of strictly positive real numbers tending to $+\infty$. Let $\lambda := \exp(-\beta) \in \mathbb{C}$, where exp is the complex exponential function and not the compositional inverse of the logarithmic chart. Now, we get:

$$\begin{aligned}
\widehat{f}_1(z) &:= \exp\left(-\widehat{f}(-\log z)\right) \\
&= \exp\left(-\left(-\log z+\beta+\sum_{i=1}^{\infty}\exp(\nu_i \log z) R_i(-\log z)\right)\right), R_i \in \mathbb{C}[X], \nu_i > 0, \\
&= \lambda z \exp\left(\sum_{i=1}^{\infty} z^{\nu_i} R_i(-\log z)\right) \\
&= \lambda z + \sum_{i=1}^{\infty} z^{\alpha_i} P_i(-\log z), \quad \alpha_i > 1, P_i \in \mathbb{C}[X],
\end{aligned}$$

so that $\widehat{f}_1 \in \mathscr{L}_1(\mathbb{C})$ is a complex logarithmic Dulac series in the formal variable z. Notice that here λ is a complex number, and is not seen as the element of $\widetilde{\mathbb{C}}$ parameterized by β in the logarithmic chart. It is indeed important that all the coefficients of $\widehat{f}_1$ are complex numbers and not elements of $\widetilde{\mathbb{C}}$, in order to apply to them all the algebraic computations involved in the proof of the Theorem 2.1.1.

The latter implies that $\widehat{f}_1$ admits a unique parabolic linearization $\widehat{\varphi}_1 \in \mathscr{L}_1(\mathbb{C})$. Let $\widehat{g}_1 := \widehat{f}_1 - \lambda \cdot \mathrm{id}$ and $\gamma := \mathrm{ord}_z(\widehat{g}_1)$. As $\widehat{f}_1$ is a Dulac series, we have $\gamma > 1$. Moreover, recall that the exponents of z in $\widehat{f}_1$ form a finitely generated strictly positive sequence which tends to $+\infty$. Hence, we deduce from the description of the support of $\widehat{\varphi}_1$ given in Theorem 2.1.28, in Subsection 2.1.6, that the exponents of z in $\widehat{\varphi}_1$ also form a finitely generated strictly positive sequence which tends to $+\infty$.

We now prove the *polynomial property* (see Subsection 3.1.2) for blocks of the linearization $\widehat{\varphi}_1$: that each monomial z^{α} in $\widehat{\varphi}_1$ is multiplied by a complex *polynomial in* $\boldsymbol{\ell}_1^{-1} = -\log z$. By the proof of Theorem A, the linearization $\widehat{\varphi}_1$ is given by

$$\widehat{\varphi}_1 := \mathrm{id} + \widehat{h}_1,$$

where $\widehat{h}_1 \in \mathscr{L}_1(\mathbb{C})$, $\mathrm{ord}_z(\widehat{h}_1) > 1$, is the limit of the Picard sequence $(\widehat{\psi}_n)_{n\in\mathbb{N}}$ defined by

$$\widehat{\psi}_n := (\mathscr{T}_{\widehat{f}_1}^{-1} \circ \mathscr{S}_{\widehat{f}_1})^{\circ n}(0). \tag{3.40}$$

Here, the limit is taken in the sense of the power-metric topology: $\mathrm{ord}_z(\widehat{\psi}_n - \widehat{\varphi}_1)$ tends to $+\infty$, as $n \to +\infty$. The operators $\mathscr{T}_{\widehat{f}_1}$ and $\mathscr{S}_{\widehat{f}_1}$ (case $\beta > 1$) are expanded in Remark 2.1.12 as

$$\mathscr{S}_{\widehat{f}_1}(\widehat{h}) = \frac{1}{\lambda}\widehat{g}_1 + \frac{1}{\lambda}\sum_{i\geq 1}\frac{\widehat{h}^{(i)}(\lambda z)}{i!}\widehat{g}_1^i, \text{ and } \mathscr{T}_{\widehat{f}_1}(\widehat{h}) = \widehat{h} - \frac{1}{\lambda}\widehat{h}(\lambda z), \tag{3.41}$$

since $\widehat{f}_0 = \lambda \cdot \mathrm{id}$, i.e., $\widehat{g}_0 = 0$. In (3.41), logarithmic transseries are applied to λz. This means, using the following compositional rules, that

$$\log(\lambda z) = \log(\lambda) + \log(z) = -\beta + \log(z)$$

and, for $\alpha > 0$,

$$(\lambda z)^\alpha = \lambda^\alpha z^\alpha = \exp(\alpha \log(\lambda)) z^\alpha = \exp(-\alpha\beta) z^\alpha.$$

In particular, we see that, in this proof, λ is the only complex number for which we have to impose a determination of the logarithm. We chose $\log(\lambda) = -\beta$ in view of the final step of the proof, in which we deduce the linearization of $\widehat{f}$ from the linearization of $\widehat{f}_1$.

Now, due to the convergence of (3.40) to the linearization $\widehat{\varphi}_1$ in the power-metric topology, it suffices to prove the following: if $\widehat{h} \in \mathscr{L}_1(\mathbb{C})$ with $\mathrm{ord}_z(\widehat{h}) > 1$ satisfies the polynomial property, the same holds for $\mathscr{T}_{\widehat{f}_1}^{-1}(\widehat{h})$ and for $\mathscr{S}_{\widehat{f}_1}(\widehat{h})$.

Notice that the polynomial property is preserved under differentiation, under multiplication by a complex Dulac series, as well as under precomposition with λz. Therefore, if $\widehat{h}$ has the polynomial property, then so does $\mathscr{S}_{\widehat{f}_1}(\widehat{h})$, thanks to (3.41), the previous remark and the fact that, as $\gamma > 1$, $\mathscr{S}_{\widehat{f}_1}$ is an infinite sum of operators which strictly increase ord_z.

Let us now check the polynomial property for $\mathscr{T}_{\widehat{f}_1}^{-1}(\widehat{h})$. Suppose the contrary, that is, that there exists $\widehat{h} \in \mathscr{L}_1(\mathbb{C})$, $\mathrm{ord}_z(\widehat{h}) > 1$, which satisfies the polynomial property, while $\mathscr{T}_{\widehat{f}_1}^{-1}(\widehat{h})$ does not. Then $\mathscr{T}_{\widehat{f}_1}^{-1}(\widehat{h})$ admits a block $\widehat{R}_\nu(\boldsymbol{\ell}_1) \in \mathbb{C}((\boldsymbol{\ell}_1))$, for some $\nu > 1$, which is not a polynomial in $\boldsymbol{\ell}_1^{-1}$. Hence, we can write

$$\widehat{R}_\nu(\boldsymbol{\ell}_1) = Q\left(\boldsymbol{\ell}_1^{-1}\right) + a(\boldsymbol{\ell}_1), \text{ with } a(\boldsymbol{\ell}_1) = \sum_{n \geq n_0} a_n \boldsymbol{\ell}_1^n,\ a_n \in \mathbb{C},$$

where $Q \in \mathbb{C}\left[\boldsymbol{\ell}_1^{-1}\right]$ is a polynomial and $a \in \mathbb{C}[[\boldsymbol{\ell}_1]]$ is a nonzero power series such that $a_{n_0} \neq 0$, $n_0 \in \mathbb{N}_{\geq 1}$. Now apply $\mathscr{T}_{\widehat{f}_1}$ from (3.41) to such $\mathscr{T}_{\widehat{f}_1}^{-1}(\widehat{h})$, to obtain $\widehat{h}$. However, using (3.41) and the relations proved in Lemma A.3.1

$$\begin{aligned}
&\boldsymbol{\ell}_1^{-1}(\lambda z) = \boldsymbol{\ell}_1^{-1} - \log\lambda = \boldsymbol{\ell}_1^{-1} + \beta,\\
&\boldsymbol{\ell}_1(\lambda z) = \boldsymbol{\ell}_1 \cdot (1 + \varepsilon(\boldsymbol{\ell}_1)), \text{ for some } \varepsilon \in \mathbb{C}[[\ell_1]],\ \varepsilon(0) = 0,\\
&\boldsymbol{\ell}_1^n(\lambda z) = \boldsymbol{\ell}_1^n \cdot (1 + \varepsilon_n(\boldsymbol{\ell}_1)), \text{ for some } \varepsilon_n \in \mathbb{C}[[\ell_1]],\ \varepsilon_n(0) = 0\ (n \in \mathbb{N}_{\geq 1}),
\end{aligned}$$

it is easy to see that the block of index ν in $\widehat{h} = \mathscr{T}_{\widehat{f}_1}\left(\mathscr{T}_{\widehat{f}_1}^{-1}(\widehat{h})\right)$ is

$$Q\left(\boldsymbol{\ell}_1^{-1}\right) + a(\boldsymbol{\ell}_1) - \lambda^{\nu-1}\left(Q\left(\boldsymbol{\ell}_1^{-1} + \beta\right) + \widetilde{a}(\boldsymbol{\ell}_1)\right), \tag{3.42}$$

where $\widetilde{a} \in \mathbb{C}[[\ell_1]]$ is a power series such that $\widetilde{a}(0) = 0$ and with $a_{n_0}\boldsymbol{\ell}_1^{n_0}$ as leading term. The power series $a - \lambda^{\nu-1}\widetilde{a} \in \mathbb{C}[[\boldsymbol{\ell}_1]]$ does not have a constant term, but is nonzero because its smallest coefficient is equal to $\left(1-\lambda^{\nu-1}\right)a_{n_0} \neq 0$. This contradicts the fact that the block of index ν of $\widehat{h}$ is a polynomial in $\boldsymbol{\ell}_1^{-1}$. Therefore, $\widehat{\varphi}_1$ is a logarithmic Dulac series.

Finally, let $\widehat{\varphi}(\zeta) := -\log(\widehat{\varphi}_1(\exp(-\zeta)))$. Since we chose $\log(\lambda) = -\beta$, we have that $\widehat{f}(\zeta) = -\log\left(\widehat{f}_1(\exp(-\zeta))\right)$ and $-\log(\lambda\widehat{\varphi}_1(\exp(-\zeta))) = \widehat{\varphi}(\zeta) + \beta$. Hence, we deduce from $\widehat{\varphi}_1 \circ \widehat{f}_1 = \lambda\widehat{\varphi}_1$ that $\widehat{\varphi} \circ \widehat{f} = \widehat{\varphi} + \beta$.

Notice that in this proof, if the coefficients of $\widehat{f}$ are real, so are the coefficients of $\widehat{f}_1$, $\widehat{\varphi}_1$, and $\widehat{\varphi}$.

The uniqueness follows directly by Theorem 2.1.1. ■

3.2.3. Analytic linearization of hyperbolic Dulac germs

In this subsection we prove the analytic linearization theorem for hyperbolic complex (real) Dulac germs on standard quadratic domains, see [30, Subsections 4.2 and 4.3].

Let f be a hyperbolic complex (real) Dulac germ and let $\widehat{f}(\zeta) = \zeta + \beta + \text{h.o.t.}$, $\beta \in \mathbb{C}^+$, be its asymptotic expansion in the ζ-chart. We prove its linearization by a parabolic complex (real) Dulac germ using the results from Subsection 3.2.1 and Subsection 3.2.2. In Subsection 3.2.1 we proved that the formal linearization of hyperbolic complex (real) Dulac series $\widehat{f}$ is a parabolic complex (real) Dulac series. On the other hand, since the standard quadratic domain $\mathscr{R}_C$, $C \in \mathbb{R}_{>0}$, is an admissible domain of type (β, ε, k) (see Example 3.2.5), for each $\beta \in \mathbb{C}^+$, $\varepsilon \in \mathbb{R}_{>0}$, $k \in \mathbb{N}$, by Theorem 3.2.11 in Subsection 3.2.1, it follows that there exists $R \in \mathbb{R}_{>0}$ and a unique parabolic analytic germ $\varphi(\zeta) = \zeta + o(1)$ on the domain $(\mathscr{R}_C^f)_R$, such that:

$$(\varphi \circ f)(\zeta) = \varphi(\zeta) + \beta.$$

Since $(\mathscr{R}_C)_R$ is a domain of type (β, ε, k) (Example 3.2.5), for sufficiently large $R > 0$, then, by Proposition 3.2.10, $(\mathscr{R}_C)_R$ is f-invariant which implies that $(\mathscr{R}_C)_R = (\mathscr{R}_C^f)_R$.

Therefore, on the one hand, we obtain the formal linearization $\widehat{\varphi}$, and, on the other hand, the analytic linearization φ on $(\mathscr{R}_C)_R$. In the next theorem we prove that φ is a parabolic complex (real) Dulac germ, by proving that $\widehat{\varphi}$ is the asymptotic expansion of φ uniformly on the domain $(\mathscr{R}_C)_R$, for some $R \in \mathbb{R}_{>0}$.

Theorem D (Linearization of hyperbolic complex (real) Dulac germs, Theorem B, [30])**.** Let $f(\zeta) = \zeta + \beta + o(1)$, $\beta \in \mathbb{C}^+$, be a hyperbolic complex Dulac germ on a standard quadratic domain $\mathscr{R}_C$. Then there exists a unique parabolic germ φ satisfying

$$\varphi \circ f = \varphi + \beta, \tag{3.43}$$

on f-invariant germs of $\mathscr{R}_C$. Moreover, φ is a complex parabolic Dulac germ (possibly on a smaller standard quadratic subdomain $\mathscr{R}_{C'} \subset \mathscr{R}_C$). Furthermore, if f is a real Dulac germ, then φ is also a real Dulac germ.

The proof is given at the end of the section. For the sake of the proof, we first introduce some definitions and lemmas.

Definition 3.2.13. (Formal and analytic partial linearizations, see [30, Subsection 4.2]) Let $\widehat{f}(\zeta)$ be a hyperbolic complex (real) Dulac series and let

$$\widehat{\varphi}(\zeta) := \zeta + \sum_{i=1}^{\infty} \mathrm{e}^{-\beta_i \zeta} Q_i(\zeta)$$

be its formal linearization given in Proposition 3.2.12, in the ζ-chart. The sequence $(\widehat{\varphi}_n)$ given by:

$$\begin{aligned} &\widehat{\varphi}_0 := \zeta, \\ &\widehat{\varphi}_n := \zeta + \sum_{i=1}^{n} \mathrm{e}^{-\beta_i \zeta} Q_i(\zeta),\ n \in \mathbb{N}_{\geq 1}, \end{aligned} \tag{3.44}$$

is called the *sequence of formal partial linearizations* of $\widehat{f}$.

Since each $\widehat{\varphi}_n$ is a finite formal sum, we denote by φ_n the canonically associated map defined on whole $\mathbb{C}^+$ in the ζ-chart, for each $n \in \mathbb{N}$. Note that every φ_n, $n \in \mathbb{N}$, is analytic on $\mathbb{C}^+$ in the ζ-chart. We call (φ_n) the *sequence of analytic partial linearizations* of f.

Let f be a hyperbolic complex (real) Dulac germ. The conjugacy equation $\varphi \circ f = \varphi + \beta$, for analytic germ φ in the ζ-chart, is equivalent to the equation $\varphi \circ f - \varphi - \beta = 0$. In Lemma 3.2.14 below, we determine the asymptotics of $\varphi_n \circ f - \varphi_n - \beta$, for each $n \in \mathbb{N}$, where (φ_n) is the sequence of analytic partial linearizations of f.

Lemma 3.2.14 (Lemma 4.3, [30]). Let $f(\zeta)=\zeta+\beta+o(1)$, $\beta\in\mathbb{C}^{+}$, be a nontrivial hyperbolic complex Dulac germ defined on a standard quadratic domain $\mathscr{R}_C$, $C>0$, and let $\widehat{\varphi}$ be its formal linearization from Proposition 3.2.12, given in the ζ-chart as:

$$\widehat{\varphi}(\zeta)=\zeta+\sum_{i=1}^{\infty}\mathrm{e}^{-\beta_i\zeta}Q_i(\zeta),\tag{3.45}$$

where $Q_i\in\mathbb{C}[\zeta]$ and $(\beta_i)_i$ is a strictly increasing sequence of positive real numbers tending to $+\infty$. Here, if $\widehat{\varphi}$ is a finite sum, that is, if there exists $i_0\in\mathbb{N}$ such that $Q_i=0$ for $i>i_0$, we take any strictly increasing sequence $(\beta_i)_{i>i_0}$ such that $\beta_i>\beta_{i_0}$ and $\beta_i\to+\infty$. Let $(\widehat{\varphi}_n)$ be the sequence of partial linearizations of $\widehat{f}$ defined in (3.44), and (φ_n) be the related sequence of analytic partial linearizations. Then, for every $n\in\mathbb{N}$, there exists $\nu_n>0$, such that

$$(\varphi_n\circ f)(\zeta)-\varphi_n(\zeta)=\beta+o\left(\mathrm{e}^{-(\beta_n+\nu_n)\zeta}\right),\tag{3.46}$$

uniformly on $\mathscr{R}_C$ as $\Re(\zeta)\to+\infty$. Here, $\beta_0=0$.

Note that, if $\widehat{\varphi}$ is a finite sum, the sequence (φ_n) eventually stabilizes.

Proof. Let

$$\widehat{f}(\zeta)=\zeta+\beta+\sum_{i=1}^{\infty}\mathrm{e}^{-\alpha_i\zeta}P_i(\zeta),\, P_i\in\mathbb{C}[\zeta],\ i\in\mathbb{N}_{\geq 1},$$

where $(\alpha_i)_i$ is a strictly increasing sequence of strictly positive real numbers tending to $+\infty$, be the complex Dulac asymptotic expansion of f. Recall that this expansion of f is uniform on the standard quadratic domain $\mathscr{R}_C$, as $\Re(\zeta)\to+\infty$. For $n\in\mathbb{N}$, let

$$\begin{aligned}\widehat{f}_0&:=\zeta+\beta,\\ \widehat{f}_n&:=\zeta+\beta+\sum_{i\in\mathbb{N}_{\geq 1}:\,\alpha_i\leq\beta_n}\mathrm{e}^{-\alpha_i\zeta}P_i(\zeta),\quad n\in\mathbb{N}_{\geq 1},\end{aligned}$$

be the *partial sums* of $\widehat{f}$. Furthermore, let $\widehat{g}_n:=\widehat{f}-\widehat{f}_n$, for $n\in\mathbb{N}$. The composition $\widehat{\varphi}\circ\widehat{f}$ can be computed as

$$\widehat{\varphi}\circ\widehat{f}=\widehat{\varphi}\left(\widehat{f}_n+\widehat{g}_n\right)=\widehat{\varphi}\circ\widehat{f}_n+\sum_{i\geq 1}\frac{\widehat{\varphi}^{(i)}\circ\widehat{f}_n}{i!}\widehat{g}_n^{i},\tag{3.47}$$

as the series in (3.47) converges for the power-metric topology. Obviously,

$$\widehat{\varphi}\circ\widehat{f}_n=\widehat{\varphi}_n\circ\widehat{f}_n+(\widehat{\varphi}-\widehat{\varphi}_n)\circ\widehat{f}_n,\quad n\in\mathbb{N}.\tag{3.48}$$

Now, using (3.47) and (3.48) and the fact that $\widehat{\varphi}$ is the formal linearization of $\widehat{f}$, we get:

$$\begin{aligned}0 &= \widehat{\varphi}\circ\widehat{f}-\widehat{\varphi}-\beta\\ &= \widehat{\varphi}_n\circ\widehat{f}_n-\widehat{\varphi}_n+(\widehat{\varphi}-\widehat{\varphi}_n)\circ\widehat{f}_n-(\widehat{\varphi}-\widehat{\varphi}_n)+\sum_{i\geq 1}\frac{\widehat{\varphi}^{(i)}\circ\widehat{f}_n}{i!}\widehat{g}_n^i-\beta,\end{aligned}\tag{3.49}$$

for $n\in\mathbb{N}$. For every $n\in\mathbb{N}$, it can easily be seen that there exists $\mu_n>0$, such that:

$$\operatorname{ord}_{e^{-\zeta}}\left((\widehat{\varphi}-\widehat{\varphi}_n)\circ\widehat{f}_n-(\widehat{\varphi}-\widehat{\varphi}_n)+\sum_{i\geq 1}\frac{\widehat{\varphi}^{(i)}\circ\widehat{f}_n}{i!}\widehat{g}_n^i\right)>\beta_n+\mu_n. \tag{3.50}$$

From (3.49) and (3.50), we obtain that

$$\operatorname{ord}_{e^{-\zeta}}\left(\widehat{\varphi}_n\circ\widehat{f}_n-\widehat{\varphi}_n-\beta\right)>\beta_n+\mu_n,\quad n\in\mathbb{N}. \tag{3.51}$$

As the sums in $\widehat{\varphi}_n$ and $\widehat{f}_n$ are finite, they define analytic germs φ_n and f_n on $\mathbb{C}^+$, in the ζ-chart. This implies that

$$\varphi_n\circ f_n-\varphi_n-\beta=o\left(e^{-(\beta_n+\mu_n)\zeta}\right),\ \Re(\zeta)\to+\infty \text{ on } \mathbb{C}^+, \tag{3.52}$$

for $n\in\mathbb{N}$. Moreover, due to the fact that f_n and φ_n, $n\in\mathbb{N}$, are *finite* sums of power-exponential monomials, the convergence is *uniform* if we restrict to the standard quadratic domain $\mathscr{R}_C$ (since the imaginary part is bounded by a power of the real part along this domain).

Now put $g_n(\zeta):=f(\zeta)-f_n(\zeta)$, for $\zeta\in\mathscr{R}_C$ and $n\in\mathbb{N}$. It is obvious that $g_n\sim\widehat{g}_n$ uniformly on $\mathscr{R}_C$ as $\Re(\zeta)\to+\infty$. By Taylor's Theorem, (3.52), and since φ_n is a finite sum of monomials with uniform asymptotics on $\mathscr{R}_C$, it follows that for every $n\in\mathbb{N}$ there exists some $\nu_n>0$ such that

$$\begin{aligned}\varphi_n\circ f-\varphi_n-\beta &= \varphi_n(f_n+g_n)-\varphi_n-\beta\\ &= \varphi_n\circ f_n-\varphi_n-\beta+\sum_{i=1}^{+\infty}\frac{\varphi_n^{(i)}(f_n)}{i!}g_n^i\\ &= o(e^{-(\beta_n+\nu_n)\zeta}),\end{aligned}$$

uniformly on the standard quadratic domain $\mathscr{R}_C$, as $\Re(\zeta)\to+\infty$. ■

Let h be some analytic germ on a standard quadratic domain $\mathscr{R}_C$, $C\in\mathbb{R}_{>0}$, and f a hyperbolic complex Dulac germ on $\mathscr{R}_C$. The equation

$$(\psi\circ f)(\zeta)-\psi(\zeta)=h(\zeta), \tag{3.53}$$

is called the *Abel-type homological equation* for f. This is a generalization of the standard notion of the Abel equation where $h \equiv 1$ (see e.g. [18], [24]). Our goal in the next lemma, whose proof is motivated by a solution of certain equation in [18], is to find an analytic solution ψ on some f-invariant subdomain of $\mathscr{R}_C$.

Lemma 3.2.15 (Explicit analytic solutions to Abel-type homological equations, Lemma 4.4, [30])**.** Let f be a hyperbolic complex Dulac germ defined on a standard quadratic domain $\mathscr{R}_C \subseteq \mathbb{C}^+$. Let h be an analytic map on $\mathscr{R}_C$, such that $h(\zeta) = o\left(\mathrm{e}^{-\alpha\zeta}\right)$ for some $\alpha > 0$, uniformly on $\mathscr{R}_C$ as $\Re(\zeta) \to +\infty$. Then:

1. *(Existence of an analytic solution to a homological equation)* There exist $R > 0$ such that $D := (\mathscr{R}_C)_R$ is an f-invariant subdomain $D \subseteq \mathscr{R}_C$, and an analytic solution ψ of the Abel-type homological equation (3.53) on the subdomain D.

2. *(Estimate of the solution)* The following estimate holds:

$$\psi(\zeta) = O(\mathrm{e}^{-\alpha\zeta}), \tag{3.54}$$

uniformly on D as $\Re(\zeta) \to +\infty$.

3. *(Uniqueness of the solution)* If ψ_1 is an analytic solution of Abel-type homological equation (3.53) on an f-invariant subdomain $D_1 \subseteq \mathscr{R}_C$, such that $\psi_1(\zeta) = o(1)$ uniformly on D_1 as $\Re(\zeta) \to +\infty$, then

$$\psi_1 \equiv \psi \text{ on } (D_1)_R = D \cap D_1.$$

Proof. 1. *Existence of a solution*. We prove that the following series:

$$\psi(\zeta) := -\sum_{n=0}^{+\infty} h\left(f^{\circ n}(\zeta)\right) \tag{3.55}$$

converges uniformly on D (in the ζ-chart) to an analytic map ψ which satisfies equation (3.53).

Since $h(\zeta) = o\left(\mathrm{e}^{-\alpha\zeta}\right)$ uniformly on $\mathscr{R}_C$ as $\Re(\zeta) \to +\infty$, there exists $R > 0$ such that

$$|h(\zeta)| \leq |\mathrm{e}^{-\alpha\zeta}| = \frac{1}{\mathrm{e}^{\alpha\Re(\zeta)}} \leq \frac{1}{\mathrm{e}^{\alpha R}}, \tag{3.56}$$

for $\zeta \in \mathscr{R}_C$, $\Re(\zeta) \geq R$.

Let $\varepsilon > 0$ and $k \in \mathbb{N}$ be arbitrary. By the discussion at the beginning of Subsection 3.2.3, we take $R > 0$ sufficiently large such that $(\mathscr{R}_C)_R = (\mathscr{R}_C)_R^f$, that is, such that the whole of $(\mathscr{R}_C)_R$ is f-invariant. Now, put $D := (\mathscr{R}_C)_R$. From (3.29) it follows that

$$\Re(f^{\circ n}(\zeta)) \geq \Re(\zeta) + n\rho^-_{\beta,\varepsilon,k}(R) \geq R + n\rho^-_{\beta,\varepsilon,k}(R), \text{ for } \zeta \in D, \tag{3.57}$$

for $n \in \mathbb{N}$. Now, from (3.56) and (3.57), it follows that, for $n \in \mathbb{N}$ and $\zeta \in D$

$$|h(f^{\circ n}(\zeta))| \leq \frac{1}{\mathrm{e}^{\alpha\Re(f^{\circ n}(\zeta))}} \leq \frac{1}{\mathrm{e}^{\alpha R}} \cdot \left(\frac{1}{\mathrm{e}^{\alpha\cdot\rho^-_{\beta,\varepsilon,k}(R)}}\right)^n.$$

This implies that sum (3.55) converges uniformly on D. By the Weierstrass Theorem, it follows that ψ defined by (3.55) is analytic on D. Now (3.53) follows easily:

$$\begin{aligned}\psi(f(\zeta)) &= -\sum_{n=0}^{+\infty} h\big(f^{\circ(n+1)}(\zeta)\big)\\ &= -\sum_{n=1}^{+\infty} h\left(f^{\circ n}(\zeta)\right)\\ &= -\big(-\psi(\zeta) - h(\zeta)\big)\\ &= \psi(\zeta) + h(\zeta), \text{ for } \zeta \in D.\end{aligned}$$

2. *Estimate of the solution* ψ. From (3.55), (3.56) and (3.57) it follows that

$$|\psi(\zeta)| \leq \sum_{n=0}^{\infty} |h\left(f^{\circ n}(\zeta)\right)| \leq \mathrm{e}^{-\alpha\Re(\zeta)} \cdot \frac{1}{1 - \frac{1}{\mathrm{e}^{\alpha\cdot\rho^-_{\beta,\varepsilon,k}(R)}}},$$

for $\zeta \in D$. This implies that $\psi(\zeta) = O\left(\mathrm{e}^{-\alpha\zeta}\right)$ uniformly on D as $\Re(\zeta) \to +\infty$.

3. *Uniqueness of the solution.* Suppose that there exists an analytic solution ψ_1 to homological equation (3.53), defined on an f-invariant subdomain $D_1 \subseteq \mathscr{R}_C$, such that $\psi_1(\zeta) = o(1)$ uniformly on D_1 as $\Re(\zeta) \to +\infty$. By (3.57), note that $(D_1)_R = D_1 \cap D$ is a nonempty f-invariant subdomain of D. Let

$$\psi_2(\zeta) := \psi(\zeta) - \psi_1(\zeta),\ \zeta \in (D_1)_R.$$

Since both ψ and ψ_1 satisfy equation (3.53) on $(D_1)_R$ and $\psi(\zeta) = o(1)$, $\psi_1(\zeta) = o(1)$, we have that $\psi_2(f(\zeta)) = \psi_2(\zeta)$, for $\zeta \in (D_1)_R$, and $\psi_2(\zeta) = o(1)$ uniformly on $(D_1)_R$ as $\Re(\zeta) \to +\infty$. Therefore,

$$\psi_2(f^{\circ n}(\zeta)) = \psi_2(\zeta),\ \zeta \in (D_1)_R,\ n \in \mathbb{N}. \tag{3.58}$$

Note that $\psi_2(\zeta) = o(1)$, as $\Re(\zeta) \to +\infty$ uniformly on $(D_1)_R$. From (3.57) it follows that $\Re(f^{\circ n}(\zeta)) \to +\infty$ as $n \to +\infty$, for every $\zeta \in (D_1)_R$. Therefore, passing to the limit as $n \to +\infty$ in (3.58), we obtain that $\psi_2 \equiv 0$ on $(D_1)_R$. Therefore, $\psi \equiv \psi_1$ on $(D_1)_R$. ■

Finally, we use Theorem 3.2.11, Proposition 3.2.12, Lemma 3.2.14 and Lemma 3.2.15 to prove Theorem D.

Proof of Theorem D. Let $f(\zeta) = \zeta + \beta + o(1)$, for $\beta \in \mathbb{C}^+$, be a hyperbolic complex Dulac germ on a standard quadratic domain $\mathscr{R}_C$, $C > 0$, given in the ζ-chart, and let $\widehat{f}$ be its complex Dulac expansion.

If $\widehat{f}(\zeta) = \zeta + \beta$, then, by the *quasi-analyticity* stated in Theorem 3.1.15, it follows that $f(\zeta) = \zeta + \beta$ for $\zeta \in \mathscr{R}_C$, so that f is already linearized.

Now, suppose that both f and $\widehat{f}$ are nontrivial. By Proposition 3.2.12, there exists a unique formal linearization $\widehat{\varphi}$ of $\widehat{f}$ which is a parabolic complex Dulac series. By Theorem 3.2.11, since $\widehat{f}$ is nontrivial, there exist sufficiently large $R > 0$ and a parabolic analytic linearization φ of f on the f-invariant subdomain $(\mathscr{R}_C)_R^f$, given as the uniform limit on $(\mathscr{R}_C)_R^f$ of the Koenigs sequence for f. By the discussion at the beginning of Subsection 3.2.3, it follows that $(\mathscr{R}_C)_R^f = (\mathscr{R}_C)_R$. Now, we put $D := (\mathscr{R}_C)_R$.

To prove that φ is a complex Dulac germ, we prove that it admits $\widehat{\varphi}$ as its asymptotic expansion, uniformly on some standard quadratic subdomain $\mathscr{R}_{C'}$ of D, as $\Re(\zeta) \to +\infty$.

Let (φ_n) be the sequence of analytic partial linearizations of f defined by (3.44), and let

$$\psi_n(\zeta) := \varphi(\zeta) - \varphi_n(\zeta),\ \zeta \in D, n \in \mathbb{N}. \tag{3.59}$$

Note that, by Theorem 3.2.11, 3, and (3.44), for every $\delta > 0$ such that $\beta_1 - \delta > 0$ it holds that

$$\psi_n(\zeta) = \zeta + o(1) - \zeta - o(\mathrm{e}^{-(\beta_1-\delta)\zeta}) = o(1), \tag{3.60}$$

uniformly on D as $\Re(\zeta) \to +\infty$. Since φ is an analytic linearization of f on D, by (3.59) the following holds:

$$\psi_n(f(\zeta)) - \psi_n(\zeta) = -\varphi_n(f(\zeta)) + \varphi_n(\zeta) + \beta,\ \text{for } \zeta \in D \text{ and } n \in \mathbb{N}. \tag{3.61}$$

By Lemma 3.2.14, for every $n \in \mathbb{N}$ there exists $\nu_n > 0$ such that $(\varphi_n \circ f)(\zeta) - \varphi_n(\zeta) = \beta + o\left(\mathrm{e}^{-(\beta_n+\nu_n)\zeta}\right)$, uniformly on $\mathscr{R}_C$ as $\Re(\zeta) \to +\infty$. Here, $\beta_n > 0$ ($n \in \mathbb{N}$) are the exponents in the complex Dulac series $\widehat{\varphi}$, as in (3.45). Now applying Lemma 3.2.15 to (3.61), for every $n \in \mathbb{N}$, the Abel-type homological equation (3.61) admits a unique solution η_n analytic on D, such that $\eta_n(\zeta) = O(\mathrm{e}^{-(\beta_n+\nu_n)\zeta})$ uniformly on D, as $\Re(\zeta) \to +\infty$.

Since $\psi_n = o(1)$ by (3.60), it follows from Lemma 3.2.15, 3, that $\psi_n \equiv \eta_n$ on D, $n \in \mathbb{N}$. Therefore,

$$\psi_n(\zeta) = O(\mathrm{e}^{-(\beta_n+\nu_n)\zeta}),\ n \in \mathbb{N}.$$

This implies, by (3.59), that $\widehat{\varphi}$ is the asymptotic expansion of the linearization φ on D. Recall that D is a representative of the standard quadratic domain $\mathscr{R}_C$, by Definition 3.1.9.

By Remark 3.1.10, there exists a standard quadratic subdomain $\mathscr{R}_{C'}$, where $C' > R, C$, that is contained in D. Therefore, $\widehat{\varphi}$ is the Dulac asymptotic expansion of φ also on the standard quadratic subdomain $\mathscr{R}_{C'}$. Thus, φ is a parabolic complex Dulac germ (its domain of definition contains a standard quadratic domain).

Finally, the uniqueness of the linearization φ follows from Theorem 3.2.11, 4. The statement about real Dulac linearizations of real Dulac germs follows from Theorem 3.2.11, 2. ■

3.3. Normal forms for strongly hyperbolic Dulac germs

This section is dedicated to the analytic normalization of strongly hyperbolic complex Dulac germs. The idea and the strategy of the proofs in this section are similar to those in Section 3.2. The main results of this section are Theorem 3.3.5, which can be viewed as a generalization of the Böttcher Theorem (see e.g. [4, Theorem 4.1], [24, Theorem 9.1]) on complex subdomains of the Riemann surface of the logarithm, and its particular case, Theorem E, which is a normalization theorem for strongly hyperbolic complex Dulac germs. Note that Theorem E completes the analytic normalization results for Dulac germs, since parabolic real Dulac germs on the real line are treated in [20] and [22] and hyperbolic Dulac germs are treated in Theorem D.

In Section 3.3.1 we define admissible domains and prove a general result for analytic normalization of analytic maps with strongly hyperbolic (logarithmic) asymptotic bounds on their invariant domains (Theorem 3.3.5). In Subsection 3.3.3 we apply this result to the class of complex Dulac germs, and prove in Theorem E that the analytic normalization is again a complex Dulac germ.

3.3.1. Analytic normalizations of strongly hyperbolic analytic maps on admissible domains

This section is dedicated to the normalization of analytic maps

$$f(\zeta) = \alpha\zeta + o(1),$$

as $\Re(\zeta) \to +\infty$, for $\alpha > 1$, with certain strongly hyperbolic logarithmic asymptotics on the so-called *admissible domains*. We call such a map f a *strongly hyperbolic* analytic map on an admissible domain. Being adapted for strongly hyperbolic analytic maps, admissible domains in this section differ from the similar notion of admissible domains defined in Definition 3.2.2, corresponding to analytic maps with hyperbolic asymptotic bounds. For simplicity of notation, we use the same name, since it is clear from the context which type of admissible domain we refer to. In Proposition 3.3.4, we prove their invariance under strongly hyperbolic maps. At the end of the subsection we prove the normalization theorem for analytic maps with strongly hyperbolic (logarithmic) asymptotic bounds (Theorem 3.3.5).

Admissible complex domains

Similarly as in Subsection 3.2.1, we define admissible domains which are adapted to analytic maps with strongly hyperbolic asymptotic bounds. Let $\alpha \in \mathbb{R}_{>1}$, $\varepsilon \in \mathbb{R}_{>0}$, $k \in \mathbb{N}$, and let

$$M_{\varepsilon,k}(x) := \frac{1}{(\log^{\circ k} x)^{\varepsilon}}, \tag{3.62}$$
$$\rho_{\alpha,\varepsilon,k}(x) := (\alpha-1)x - M_{\varepsilon,k}(x),$$

for $x \in \left(\exp^{\circ k}(0), +\infty\right)$. The map $M_{\varepsilon,k}$ is positive, strictly decreasing, tending to 0, as $x \to +\infty$, and $\rho_{\alpha,\varepsilon,k}$ is a strictly increasing map for sufficiently large $x \in \left(\exp^{\circ k}(0), +\infty\right)$. Furthermore, $\rho_{\alpha,\varepsilon,k}(x)$ tends to $+\infty$, as $x \to +\infty$.

As before in Subsection 3.2.1, we define the lower and the upper map h_l resp. h_u, whose graphs are used to define admissible domains of type (α, ε, k), for $\alpha \in \mathbb{R}_{>1}$, $\varepsilon \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}$. In particular, we use them to *control* an admissible domain from *below* and from *above*, so that it remains invariant under the corresponding strongly hyperbolic analytic maps.

Let $t > \exp^{\circ k}(0)$ be such that $\rho_{\alpha,\varepsilon,k}(x) > 0$, for $x \in [t, +\infty)$. Let $h_l,\ h_u : [t, +\infty) \to \mathbb{R}$ be two maps satisfying:

1. $h_l(x) < h_u(x)$, $x \in [t, +\infty)$;

2. h_l is a decreasing map on $[t, +\infty)$ with the property:

$$h_l(x + \rho_{\alpha,\varepsilon,k}(x)) - h_l(x) \leq (\alpha-1) \cdot h_l(x) - M_{\varepsilon,k}(x),\ x \in [t, +\infty);$$

3. h_u is an increasing map with the property:

$$h_u(x + \rho_{\alpha,\varepsilon,k}(x)) - h_u(x) \geq (\alpha-1) \cdot h_u(x) + M_{\varepsilon,k}(x),\ x \in [t, +\infty).$$

A map $h_l : [t,+\infty) \to \mathbb{R}$ satisfying property 2. is called a *lower map of type* (α,ε,k), and a map $h_u : [t,+\infty) \to \mathbb{R}$ satisfying property 3. is called an *upper map of type* (α,ε,k). A pair (h_l,h_u) of maps h_l, $h_u : [t,+\infty) \to \mathbb{R}$, satisfying properties 1.-3. is called a *lower-upper pair of type* (α,ε,k).

The definitions of D_{h_l,h_u}, (α,ε,k)-domain and (α,ε,k)-admissible domain are the direct analogues of Definition 3.2.2, so we omit them.

Remark 3.3.1. Note that an arbitrary union of domains of type (α,ε,k) is again a domain of type (α,ε,k).

In the next proposition we give a sufficient condition for upper maps, which we use later in Example 3.3.3 to prove that the standard quadratic domains are admissible. It is an analogue of Proposition 3.2.4 in Subsection 3.2.1, and is proven similary. Therefore, we omit the proof.

Proposition 3.3.2 (Sufficient condition for the upper maps)**.** Let $h : [t,+\infty) \to \mathbb{R}$, $t > 0$, be a C^1-map such that $x \mapsto \frac{h(x)}{x}$ is an increasing map on $[t,+\infty)$. Let $d > 0$ be such that

$$h'(x) \geq d + \frac{h(x)}{x}, \tag{3.63}$$

for $x \geq t$. Then, for every $(\alpha,\varepsilon,k) \in \mathbb{R}_{>1} \times \mathbb{R}_{>0} \times \mathbb{N}$, there exists $t' \geq t$ large enough such that, the restriction $h|_{[t',+\infty)}$ is an upper map of type (α,ε,k).

Similarly, let $h : [t,+\infty) \to \mathbb{R}$, $t > 0$, be a C^1-map such that $x \mapsto \frac{h(x)}{x}$ is a decreasing map on $[t,+\infty)$. Let $d > 0$ be such that

$$h'(x) \leq \frac{h(x)}{x} - d,$$

for $x \geq t$. Then, for $(\alpha,\varepsilon,k) \in \mathbb{R}_{>1} \times \mathbb{R}_{>0} \times \mathbb{N}$, there exists $t' \geq t$ large enough such that the restriction $h|_{[t',+\infty)}$ is a lower map of type (α,ε,k).

In the next example, using Proposition 3.3.2, we prove that standard quadratic domains, defined in Definition 3.1.8, are (α,ε,k)-admissible, for every $(\alpha,\varepsilon,k) \in \mathbb{R}_{>1} \times \mathbb{R}_{>0} \times \mathbb{N}$.

Example 3.3.3 (Standard quadratic domains). Let $\mathscr{R}_C \subseteq \mathbb{C}$, $C > 0$, be a standard quadratic domain defined in Definition 3.1.8 and let $\alpha \in \mathbb{R}_{>1}$, $\varepsilon \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}$. As in Example 3.2.5 it can be shown that the upper half of the boundary of $\mathscr{R}_C$ is parametrized by:

$$\begin{aligned} r \mapsto x(r) + \mathrm{i} \cdot y(r) = \ & C\sqrt[4]{r^2+1}\cos\left(\frac{1}{2}\operatorname{arctg} r\right) + \\ & + \mathrm{i} \cdot \left(r + C\sqrt[4]{r^2+1}\sin\left(\frac{1}{2}\operatorname{arctg} r\right)\right), \ r \in [0, +\infty). \end{aligned}$$

Furthermore, as in Example 3.2.5, it can be shown that $r \mapsto y(r)$ and $r \mapsto x(r)$ are strictly increasing maps. Now, there exists $t \in \mathbb{R}_{>0}$ such that $x(t) > \exp^{\circ k}(0)$, $h_u := y \circ x^{-1}$ is strictly increasing C^1-map on $[x(t), +\infty)$, and

$$h_u'(r) = \frac{dy(r)}{dx(r)} = \frac{2(r^2+1)^{\frac{3}{4}}}{C(rs_2 - s_1)} + \frac{rs_1 + s_2}{rs_2 - s_1} \geq \frac{2}{Cs_2}\sqrt{r} + \frac{s_1}{s_2}, \tag{3.64}$$

where $s_1 := \sin(\frac{1}{2}\operatorname{arctg} r)$ and $s_2 := \cos(\frac{1}{2}\operatorname{arctg} r)$, for each $r \in [0, +\infty)$.

Note that:

$$\frac{h_u(x(r))}{x(r)} = \frac{y(r)}{x(r)} = \frac{r + Cs_1\sqrt[4]{r^2+1}}{Cs_2\sqrt[4]{r^2+1}} = \frac{r}{Cs_2\sqrt[4]{r^2+1}} + \frac{s_1}{s_2} \leq \frac{1}{Cs_2}\sqrt{r} + \frac{s_1}{s_2}, \tag{3.65}$$

for each $r \in [0, +\infty)$ such that $x(r) \geq t$. Since $r \mapsto x(r)$ is strictly increasing, from (3.65), we see that $x \mapsto \frac{h_u(x)}{x}$ is an increasing map on $[x(t), +\infty)$. Furthermore, from (3.64) and (3.65), it follows that (3.63) holds for $d := \frac{1}{Cs_2(t)}\sqrt{t} > 0$ and the restriction $h_u|_{[x(t),+\infty)}$. By Proposition 3.3.2, there exists $t' \geq t$ large enough such that the restriction $h_u|_{[x(t),+\infty)}$ is an upper map of type (α, ε, k).

Since the lower half of the boundary of $\mathscr{R}_C$ is symmetric to the upper half, using the argumentation below Proposition 3.3.2, similarly we can show that an appropriate restriction of the lower half of the boundary of $\mathscr{R}_C$ represents the graph of a lower map of type (α, ε, k).

Therefore, $\mathscr{R}_C$ is an admissible domain of type (α, ε, k). Furthermore, there exists $R > 0$ such that $(\mathscr{R}_C)_R := \mathscr{R}_C \cap ([R, +\infty) \times \mathbb{R})$ is a domain of type (α, ε, k).

Analytic normalization of strongly hyperbolic analytic germs

In analogy with Subsection 3.2.1, in this subsection we find a unique analytic solution of the normalization equation

$$\varphi \circ f = \alpha \cdot \varphi,$$

where f is an analytic map satisfying certain logarithmic asymptotics of strongly hyperbolic type, on some f-invariant complex domain $D \subseteq \mathbb{C}^+$ in the ζ-chart.

The main idea is to prove the convergence of the Böttcher sequence[3] $\left(\frac{1}{\alpha^n} \cdot f^{\circ n}\right)_n$, as $n \to +\infty$. Therefore, we first find the maximal f-invariant subdomain D^f of the domain D. In general D^f and D_R^f, $R > 0$ (see Subsection 3.2.1 for their definitions), can be empty, but this is not the case if D is an admissible domain, as stated in the following proposition which is the strongly hyperbolic variant of Proposition 3.2.10.

Proposition 3.3.4. Let $\alpha \in \mathbb{R}_{>1}$, $\varepsilon \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}$. Let $D \subseteq \mathbb{C}^+$ be an admissible domain of type (α, ε, k) and let $f : D_C \to \mathbb{C}$, $C > \exp^{\circ k}(0)$, be an analytic map with the following asymptotic behaviour:

$$f(\zeta) = \alpha\zeta + o(\boldsymbol{L}_k^{-\varepsilon}), \ \text{ as } \Re(\zeta) \to +\infty \text{ uniformly on } D_C. \tag{3.66}$$

Here, $\boldsymbol{L}_k$ is a logarithmic term defined as in Proposition 3.2.10. Then, for every $R \geq C$ sufficiently large, the domain $\overline{D}_R$ is f-invariant. In particular, $\overline{D}_R \subseteq D_R^f$ and $D_R^f \neq \emptyset$.

Note that (3.66) gives the so-called *strongly hyperbolic* asymptotic bound since, in the z-chart, condition (3.66) becomes $f(z) = z^\alpha\Big(1 + o\big((\boldsymbol{\ell}_{k+1})^\varepsilon\big)\Big)$, where the leading monomial is of strongly hyperbolic type.

Proof. By (3.66), it follows that

$$\lim_{\Re(\zeta)\to+\infty} \frac{f(\zeta) - \alpha\zeta}{\boldsymbol{L}_k^{-\varepsilon}} = 0, \tag{3.67}$$

uniformly on D_C.

Now, there exists $R \geq C$ large enough such that $\rho_{\alpha,\varepsilon,k}(R) > 0$, $\rho_{\alpha,\varepsilon,k}$ is increasing on $[R, +\infty)$ and

$$|f(\zeta) - \alpha\zeta| \leq \frac{1}{|\boldsymbol{L}_k^\varepsilon|}, \tag{3.68}$$

for $\zeta \in D_R$. As in the proof of Proposition 3.2.10 we inductively get:

$$|\boldsymbol{L}_m| \geq \log^{\circ m}(\Re(\zeta)), \tag{3.69}$$

[3] We call it here the Böttcher sequence because in the z-chart it coincides with the standard form of the Böttcher sequence as defined in Definition 2.2.3.

for $1 \leq m \leq k$ and $\zeta \in D_R$. From (3.68) and (3.69), it follows that:

$$|f(\zeta) - \alpha\zeta| \leq \frac{1}{(\log^{\circ k}(\Re(\zeta)))^{\varepsilon}}, \tag{3.70}$$

for $\zeta \in D_R$. Now, we get that:

$$\begin{aligned} \Re(f(\zeta)) - \Re(\zeta) &\geq (\alpha-1)\Re(\zeta) - \frac{1}{(\log^{\circ k}(\Re(\zeta)))^{\varepsilon}} \\ &= \rho_{\alpha,\varepsilon,k}(\Re(\zeta)), \end{aligned} \tag{3.71}$$

$$\begin{aligned} \Re(f(\zeta)) - \Re(\zeta) &\leq (\alpha-1)\Re(\zeta) + \frac{1}{(\log^{\circ k}(\Re(\zeta)))^{\varepsilon}} \\ &= (\alpha-1)\Re(\zeta) + M_{\varepsilon,k}(\Re(\zeta)), \end{aligned} \tag{3.72}$$

and

$$\begin{aligned} \Im(f(\zeta)) - \Im(\zeta) &\geq (\alpha-1)\Im(\zeta) - \frac{1}{(\log^{\circ k}(\Re(\zeta)))^{\varepsilon}} \\ &= (\alpha-1)\Im(\zeta) - M_{\varepsilon,k}(\Re(\zeta)), \end{aligned} \tag{3.73}$$

$$\begin{aligned} \Im(f(\zeta)) - \Im(\zeta) &\leq (\alpha-1)\Im(\zeta) + \frac{1}{(\log^{\circ k}(\Re(\zeta)))^{\varepsilon}} \\ &= (\alpha-1)\Im(\zeta) + M_{\varepsilon,k}(\Re(\zeta)), \end{aligned} \tag{3.74}$$

for $\zeta \in D_R$. Note that $\rho_{\alpha,\varepsilon,k}(\Re(\zeta)) \geq \rho_{\alpha,\varepsilon,k}(R) > 0$, for every $\zeta \in D_R$, since $\rho_{\alpha,\varepsilon,k}$ is an increasing map on $[R, +\infty)$. Now, put

$$\begin{aligned} \mathscr{S}_{\alpha,\varepsilon,k}(\zeta) := &\left[\Re(\zeta) + \rho_{\alpha,\varepsilon,k}(\Re(\zeta)),\, \alpha \cdot \Re(\zeta) + M_{\varepsilon,k}(\Re(\zeta))\right] \\ &\times \left[\alpha \cdot \Im(\zeta) - M_{\varepsilon,k}(\Re(\zeta)), \alpha \cdot \Im(\zeta) + M_{\varepsilon,k}(\Re(\zeta))\right], \end{aligned}$$

for each $\zeta \in D_R$. By (3.71)-(3.74), we get that

$$f(\zeta) \in \mathscr{S}_{\alpha,\varepsilon,k}(\zeta), \text{for each } \zeta \in D_R.$$

Now we prove that $\overline{D}_R$ is f-invariant. Let $\zeta \in \overline{D}_R$ be arbitrary. Now, there exists an (α, ε, k)-domain $(D_{h_l,h_u})_R \subseteq \overline{D}_R$, such that $\zeta \in (D_{h_l,h_u})_R$. By properties 2. and 3. in the definition of lower-upper pair of type (α, ε, k) it is easy to see that $\mathscr{S}_{\alpha,\varepsilon,k}(\zeta) \subseteq D_{h_l,h_u}$. Consequently, it follows that $f(\zeta) \in D_{h_l,h_u} \subseteq \overline{D}$. By (3.71), since $\zeta \in D_R$ and $\rho_{\alpha,\varepsilon,k}(\Re(\zeta)) > 0$, for $\zeta \in D_R$, it follows that $\Re(f(\zeta)) > \Re(\zeta) \geq R$. Since $f(\zeta) \in D_{h_l,h_u} \subseteq \overline{D}$, we get that $f \in \overline{D}_R$. ■

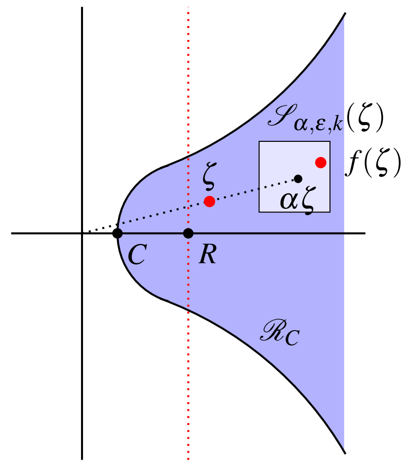

Figure 3.3: Application of Proposition 3.3.4 for $D := \mathscr{R}_C$, $C > 0$.

For visualisation of the application of Proposition 3.3.4 to a standard quadratic domain, see Figure 3.3.

In the next theorem we give the normalization result for analytic maps with strongly hyperbolic (logarithmic) asymptotic bounds on admissible domains. It is the strongly hyperbolic analogue of Theorem 3.2.11.

Theorem 3.3.5 (Normalization theorem for strongly hyperbolic analytic maps on admissible domains)**.** Let $\alpha \in \mathbb{R}_{>1}$, $\varepsilon \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}$. Let $D \subseteq \mathbb{C}^+$ be an admissible domain of type (α, ε, k). For $C > \exp^{\circ k}(0)$, let $f : D_C \to \mathbb{C}$ be an analytic map such that

$$f(\zeta) = \alpha\zeta + o(\boldsymbol{L}_k^{-\varepsilon}), \text{ as } \Re(\zeta) \to +\infty \text{ uniformly on } D_C. \tag{3.75}$$

Here, the iterated logarithm $\boldsymbol{L}_k$ is defined as in Proposition 3.2.10. Then:

1. *(Existence)* For a sufficiently large $R \geq C$ there exists an analytic normalizing map φ on the f-invariant subdomain $D_R^f \subseteq D$. That is, φ satisfies

$$(\varphi \circ f)(\zeta) = \alpha \cdot \varphi(\zeta), \text{ for all } \zeta \in D_R^f. \tag{3.76}$$

Moreover, φ is the uniform limit on D_R^f of the Böttcher sequence

$$\left(\frac{1}{\alpha^n} f^{\circ n}\right)_n$$

in the ζ-chart.

2. If $D_R^f \cap \{\zeta \in \mathbb{C}^+ : \Im(\zeta) = 0\}$ is f-invariant, then it is also φ-invariant.

3. *(Asymptotics)* The linearization φ is *tangent to identity*, i.e., $\varphi(\zeta) = \zeta + o(1)$, uniformly on $D_R^f \subseteq \mathbb{C}^+$, as $\Re(\zeta) \to +\infty$.

   In particular, for every $0 < \nu < \varepsilon$, it follows that $\varphi(\zeta) = \zeta + o(\boldsymbol{L}_k^{-\nu})$, uniformly as $\Re(\zeta) \to +\infty$, on every subdomain $D_{h_l,h_u} \subseteq D_R^f$ such that $h_l(x) = O(x)$ and $h_u(x) = O(x)$.

4. *(Uniqueness)* Let $\psi : D_1 \to \mathbb{C}$, be a normalization of f on an f-invariant subset $D_1 \subseteq D$, such that $\psi(\zeta) = \zeta + o(1)$ uniformly on D_1, as $\Re(\zeta) \to +\infty$. Then $\psi \equiv \varphi$ on $(D_1)_R$, where R is from statement 1.

Proof. 1. By Proposition 3.3.4, there exists $R \geq C$ such that $\overline{D}_R \subseteq D_R^f$, $\rho_{\alpha,\varepsilon,k}(R) > 0$ and $\rho_{\alpha,\varepsilon,k}$ is strictly increasing. Consequently, it follows that $D_{R'}^f \neq \emptyset$, for all $R' \geq R$. Let $\zeta \in D_R^f$. By (3.71), since $\rho_{\alpha,\varepsilon,k}$ is increasing on $[R, +\infty)$, it follows that:

$$\Re(f^{\circ n}(\zeta)) \geq \Re(\zeta) + n\rho_{\alpha,\varepsilon,k}(\Re(\zeta)) \geq R + n\rho_{\alpha,\varepsilon,k}(R),\ n \in \mathbb{N}_{\geq 1}. \tag{3.77}$$

By (3.70), it follows that

$$|f(\zeta) - \alpha\zeta| \leq \frac{1}{(\log^{\circ k}(\Re(\zeta)))^{\varepsilon}} = M_{\varepsilon,k}(\Re(\zeta)), \tag{3.78}$$

for every $\zeta \in D_R^f$. From (3.77) and (3.78), since $M_{\varepsilon,k}$ is decreasing on $[R, +\infty)$, for every $n \in \mathbb{N}_{\geq 1}$ and $\zeta \in D_R^f$ we get that:

$$\begin{aligned} \left| \frac{1}{\alpha^{n+1}} f^{\circ(n+1)}(\zeta) - \frac{1}{\alpha^n} f^{\circ n}(\zeta) \right| &= \frac{1}{\alpha^{n+1}} \left| f(f^{\circ n}(\zeta)) - \alpha f^{\circ n}(\zeta) \right| \\ &\leq \frac{1}{\alpha^{n+1}} M_{\varepsilon,k}\big(\Re(f^{\circ n}(\zeta))\big) \\ &\leq \frac{1}{\alpha^{n+1}} M_{\varepsilon,k}\left(R + n\rho_{\alpha,\varepsilon,k}(R)\right). \end{aligned} \tag{3.79}$$

Consequently, the Böttcher sequence $\left(\frac{1}{\alpha^n} f^{\circ n}\right)_n$ is uniformly Cauchy, and therefore, converges uniformly on D_R^f. Let φ be its uniform limit on D_R^f. By the Weierstrass Theorem, φ is analytic on D_R^f.

In the end, we check that φ is a solution of the normalization equation:

$$\begin{aligned}(\varphi\circ f)(\zeta) &= \lim_n\left(\frac{1}{\alpha^n}f^{\circ n}(f(\zeta))\right)\\ &= \alpha\cdot\lim_n\left(\frac{1}{\alpha^{n+1}}f^{\circ(n+1)}(\zeta)\right)\\ &= \alpha\cdot\varphi(\zeta),\quad \zeta\in D_R^f.\end{aligned}$$

2. Let $D_R^f\cap\{\zeta\in\mathbb{C}:\Im(\zeta)=0\}$ be f-invariant. Recall that

$$\varphi(\zeta)=\lim_{n\to\infty}\left(\frac{1}{\alpha^n}f^{\circ n}(\zeta)\right), \tag{3.80}$$

for every $\zeta\in D_R^f\cap\{\zeta\in\mathbb{C}:\Im(\zeta)=0\}$. Since $\{\zeta\in\mathbb{C}:\Im(\zeta)=0\}$ is closed in $\mathbb{C}$, by (3.80), it follows that $D_R^f\cap\{\zeta\in\mathbb{C}:\Im(\zeta)=0\}$ is φ-invariant.

3. For every $m\in\mathbb{N}_{\geq 1}$, $\zeta\in D_R^f$, since $M_{\varepsilon,k}$ is decreasing, it follows that:

$$\begin{aligned}\left|\frac{1}{\alpha^m}f^{\circ m}(\zeta)-\zeta\right| &\leq \sum_{n=0}^{m-1}\left|\frac{1}{\alpha^{n+1}}f^{\circ(n+1)}(\zeta)-\frac{1}{\alpha^n}f^{\circ n}(\zeta)\right|\\ &\leq \sum_{n=0}^{m-1}\frac{1}{\alpha^{n+1}}M_{\varepsilon,k}\left(\Re(\zeta)+n\rho_{\alpha,\varepsilon,k}(\Re(\zeta))\right)\\ &\leq M_{\varepsilon,k}(\Re(\zeta))\cdot\sum_{n=0}^{m-1}\frac{1}{\alpha^{n+1}}\\ &\leq M_{\varepsilon,k}(\Re(\zeta))\cdot\frac{1}{1-\frac{1}{\alpha}}.\end{aligned} \tag{3.81}$$

Now, as $m\to+\infty$, by (3.81), it follows that

$$|\varphi(\zeta)-\zeta|\leq\frac{1}{1-\frac{1}{\alpha}}\cdot\frac{1}{\left(\log^{\circ k}(\Re(\zeta))\right)^{\varepsilon}}, \tag{3.82}$$

for each $\zeta\in D_R^f$. From (3.82) we conclude that $\varphi(\zeta)=\zeta+o(1)$, uniformly on D_R^f as $\Re(\zeta)\to+\infty$.

Now, the remainder of the proof of statement 3 follows as in the proof of statement 3 of Theorem 3.2.11.

4. Let ψ be an analytic germ such that $\psi\circ f=\alpha\cdot\psi$, on f-invariant subset $D_1\subseteq D$, and $\psi(\zeta)=\zeta+o(1)$ uniformly on D_1, as $\Re(\zeta)\to+\infty$. Since D^f is a maximal f-invariant subdomain of D, it follows that $D_1\subseteq D^f$, and, consequently, $(D_1)_R\subseteq D_R^f$. Since D_1 is

f-invariant, by (3.71), it follows that $(D_1)_R$ is f-invariant, and by (3.77), nonempty. By statement 1, φ is the analytic linearization on D_R^f obtained as the limit of the Böttcher sequence $\left(\frac{1}{\alpha^n} f^{\circ n}\right)_n$, for sufficiently large $R \geq C$, which satisfies $\varphi(\zeta) = \zeta + o(1)$, uniformly as $\Re(\zeta) \to +\infty$ on D_R^f. Now, put

$$E(\zeta) := \varphi(\zeta) - \psi(\zeta),$$

for every $\zeta \in (D_1)_R$. Note that E is analytic on $(D_1)_R$, and $E(\zeta) = o(1)$, uniformly on $(D_1)_R$, as $\Re(\zeta) \to +\infty$. It satisfies

$$\frac{1}{\alpha} \cdot (E \circ f)(\zeta) = E(\zeta),$$

for every $\zeta \in (D_1)_R$. Inductively, we get:

$$\frac{1}{\alpha^n} \cdot E(f^{\circ n}(\zeta)) = E(\zeta), \tag{3.83}$$

for $\zeta \in (D_1)_R$, $n \in \mathbb{N}$. By (3.77), $\Re\left(f^{\circ n}(\zeta)\right) \geq R + n\rho_{\alpha,\varepsilon,k}(R)$, for $n \in \mathbb{N}$, and $\zeta \in (D_1)_R \subseteq D_R^f$, it follows that

$$\lim_n \Re\left(f^{\circ n}(\zeta)\right) = +\infty,\ \zeta \in (D_1)_R. \tag{3.84}$$

Passing to the limit, as $n \to \infty$, in (3.83), using (3.84) and the fact that $E(\zeta) = o(1)$, as $\Re(\zeta) \to +\infty$, we get that $E(\zeta) = 0$, for each $\zeta \in (D_1)_R$. That is, $\varphi \equiv \psi$ on $(D_1)_R$. ■

3.3.2. Formal normalization of strongly hyperbolic Dulac germs

In the next proposition which can be seen as the strongly hyperbolic variant of Proposition 3.2.12, we prove that the formal normalization of a strongly hyperbolic real or complex Dulac series is a parabolic real or complex Dulac series.

The strongly hyperbolic complex Dulac series are a particular type of strongly hyperbolic logarithmic transseries of depth 1, with complex coefficients, satisfying additional properties from Definition 3.1.12. Therefore, in Proposition 3.3.6 below, we apply Theorem B and prove additional properties from Definition 3.1.12. Note that Theorem B is originally stated for the spaces of *real* transseries, $\mathscr{L}_k$, $k \in \mathbb{N}$. Since the proof of Theorem B uses only algebraic properties of $\mathbb{R}$ that also hold in $\mathbb{C}$, the analogue of Theorem B holds for the spaces of *complex* logarithmic transseries $\mathscr{L}_k(\mathbb{C})$, $k \in \mathbb{N}$.

Proposition 3.3.6 (Formal normalization of strongly hyperbolic complex Dulac series)**.** Let $\widehat{f} = z^\alpha + \text{h.o.t.}$, $\alpha > 1$, be a strongly hyperbolic complex (real) Dulac series and let $\widehat{\varphi}$ be its parabolic normalization obtained as in Theorem B. Then $\widehat{\varphi}$ is a parabolic complex (real) Dulac series.

Proof. Let $\mathscr{P}_{\widehat{f}}$ be the Böttcher operator defined in (2.58), in Section 2.2. Recall from Theorem B that $\mathscr{P}_{\widehat{f}}^{\circ n}(\mathrm{id}) = z^{\frac{1}{\alpha^n}} \circ \widehat{f}^{\circ n}$, $n \in \mathbb{N}$. Since $\widehat{f}$ is a complex Dulac series and the set of all complex Dulac series is a subgroup of $\mathscr{L}_1^H(\mathbb{C})$, we deduce that $\mathscr{P}_{\widehat{f}}^{\circ n}(\mathrm{id})$ is a complex Dulac series, for every $n \in \mathbb{N}$. Since f is a complex Dulac series, note that $\mathrm{ord}_z(\widehat{f} - z^\alpha) > \alpha$. By statement 2 of Theorem B, it follows that $(\mathscr{P}_{\widehat{f}}(\mathrm{id}))_n$ converges to the parabolic normalization $\widehat{\varphi}$ in the power-metric topology on the space $\mathscr{L}_1^H(\mathbb{C})$. Let

$$\widehat{\varphi} := \mathrm{id} + \sum_{\beta \in \mathrm{Supp}_z(\widehat{\varphi})} z^\beta R_\beta, \tag{3.85}$$

and let $\gamma > 1$. Then there exists $n \in \mathbb{N}$, such that $\mathrm{ord}_z(\mathscr{P}_{\widehat{f}}^{\circ n}(\mathrm{id}) - \widehat{\varphi}) > \gamma$. Since $\mathscr{P}_{\widehat{f}}^{\circ n}(\mathrm{id})$ is a complex Dulac series, it follows that R_β is a polynomial in variable $\boldsymbol{\ell}_1^{-1} = -\log z$, for $1 < \beta \leq \gamma$, and there are only finitely many $1 < \beta \leq \gamma$, such that $R_\beta \neq 0$. Furthermore, from the description of the support of the normalization $\widehat{\varphi}$ in Theorem B, we deduce that the support of $\widehat{\varphi}$ is contained in a finitely generated sub-semigroup of $\mathbb{R}_{>0} \times \mathbb{Z}$. Thus, $\widehat{\varphi}$ is a parabolic complex Dulac series.

Moreover, if f is a real Dulac series, then φ is also a real Dulac series. ■

3.3.3. Analytic normalization of strongly hyperbolic Dulac germs

Here, we state and prove the main theorem of this section.

Theorem E (Analytic normalization of strongly hyperbolic complex Dulac germs)**.** Let f be a strongly hyperbolic complex Dulac germ and let $\widehat{f}(\zeta) = \alpha\zeta + o(1)$, $\alpha \in \mathbb{R}_{>1}$, be its asymptotic expansion in the ζ-chart. Then:

1. There exists a unique parabolic complex Dulac germ φ (given in the ζ-chart) which is a solution of the analytic normalization equation:

   $$\varphi \circ f = \alpha \cdot \varphi. \tag{3.86}$$

   Furthermore, if f is a real Dulac germ, so is φ.

2. $\varphi \sim \widehat{\varphi}$, uniformly on a standard quadratic domain, as $\Re(\zeta) \to +\infty$, where $\widehat{\varphi}$ is the unique solution of the formal normalization equation (2.56) in Proposition 3.3.6 given in the ζ-chart.

Before proving Theorem E, we first state Lemma 3.3.7 and Lemma 3.3.8 that will be used in the proof.

Let f be a hyperbolic complex (real) Dulac germ. The analytic normalization equation $\varphi \circ f = \alpha \cdot \varphi$, for an analytic germ φ, is equivalent to the equation $\varphi \circ f - \alpha \cdot \varphi = 0$. In Lemma 3.3.7 below, we determine the asymptotics of $\varphi_n \circ f - \alpha \cdot \varphi_n$, where, for $n \in \mathbb{N}$, φ_n is the partial analytic normalization as defined in Definition 3.2.13.

Lemma 3.3.7. Let f be a strongly hyperbolic complex Dulac germ defined on a standard quadratic domain $\mathscr{R}_C$, $C > 0$, and let $\widehat{f}(\zeta) = \alpha\zeta + o(1)$, $\alpha \in \mathbb{R}_{>1}$, be its complex Dulac asymptotic expansion in the ζ-chart. Let $\widehat{\varphi}(\zeta) = \zeta + \sum_{n\in\mathbb{N}_{\geq 1}} Q_n(\zeta)\mathrm{e}^{-\beta_n\zeta}$ be the formal normalization of $\widehat{f}$ from Proposition 3.3.6 given in the ζ-chart. Let (φ_n) be the related sequence of analytic partial normalizations of f, as defined in Definition 3.2.13. Then, for every $n \in \mathbb{N}$, there exists $\varepsilon_n > 0$ such that

$$(\varphi_n \circ f - \alpha\varphi_n)(\zeta) = o\big(\mathrm{e}^{-(\beta_n+\varepsilon_n)\zeta}\big), \tag{3.87}$$

uniformly on $\mathscr{R}_C$, as $\Re(\zeta) \to +\infty$. Here, $\beta_0 := 0$.

Proof. Let $\widehat{f} = \alpha\zeta + \sum_{n\geq 1} R_n(\zeta)\mathrm{e}^{-\alpha_n\zeta}$, where $\alpha > 1$, (R_n) is a sequence of polynomials in the variable ζ and (α_n) a strictly increasing sequence of positive real numbers tending to $+\infty$. Put:

$$\begin{aligned} \widehat{f}_0 &:= \alpha\zeta, \\ \widehat{f}_n &:= \alpha\zeta + \sum_{i\in\mathbb{N},\, \alpha_i \leq \beta_n} R_i(\zeta)\mathrm{e}^{-\alpha_i\zeta}, \quad n \in \mathbb{N}_{\geq 1}. \end{aligned}$$

Since $\widehat{f}_n$, $n \in \mathbb{N}$, are finite sums of terms, we denote the related sequence of maps as (f_n). Put $\widehat{\varphi}_{>n} := \widehat{\varphi} - \widehat{\varphi}_n$ and $\widehat{f}_{>n} := \widehat{f} - \widehat{f}_n$, $n \in \mathbb{N}$. Since $\widehat{\varphi} \circ \widehat{f} = \alpha \cdot \widehat{\varphi}$, it follows that:

$$\begin{aligned} (\widehat{\varphi}_n + \widehat{\varphi}_{>n}) \circ (\widehat{f}_n + \widehat{f}_{>n}) &= \alpha \cdot (\widehat{\varphi}_n + \widehat{\varphi}_{>n}), \\ \widehat{\varphi}_n \circ (\widehat{f}_n + \widehat{f}_{>n}) + \widehat{\varphi}_{>n} \circ (\widehat{f}_n + \widehat{f}_{>n}) &= \alpha \cdot \widehat{\varphi}_n + \alpha \cdot \widehat{\varphi}_{>n}, \quad n \in \mathbb{N}. \end{aligned} \tag{3.88}$$

This implies that, for every $n \in \mathbb{N}$, there exists $\mu_n > 0$ such that:

$$\operatorname{ord}_{e^{-\zeta}}(\widehat{\varphi}_n \circ \widehat{f}_n - \alpha\widehat{\varphi}_n) = \beta_n + \mu_n.$$

Therefore, for every $n \in \mathbb{N}$, it follows that:

$$\varphi_n(f_n(\zeta)) - \alpha\varphi_n(\zeta) = o\big(\mathrm{e}^{-(\beta_n+\mu_n)\zeta}\big). \tag{3.89}$$

Put

$$f_{>n}(\zeta) := (f - f_n)(\zeta), \quad \zeta \in \mathscr{R}_C, n \in \mathbb{N}.$$

It is easy to see that $f_{>n} \sim \widehat{f}_{>n}$, and therefore, for every $n \in \mathbb{N}$, there exists $\eta_n > 0$ such that:

$$f_{>n}(\zeta) = o\big(\mathrm{e}^{-(\beta_n+\eta_n)\zeta}\big),$$

uniformly on $\mathscr{R}_C$, as $\Re(\zeta) \to +\infty$. Since φ_n is analytic on $\mathscr{R}_C$, by the Taylor Theorem, we get:

$$\begin{aligned}
\varphi_n(f(\zeta)) &= \varphi_n((f_n + f_{>n})(\zeta)) \\
&= \varphi_n(f_n(\zeta)) + \sum_{i\geq 1} \frac{\varphi_n^{(i)}(f_n(\zeta))}{i!}(f_{>n}(\zeta))^i \\
&= \varphi_n(f_n(\zeta)) + o(\mathrm{e}^{-(\beta_n+\eta_n)\zeta}),
\end{aligned}$$

uniformly on $\mathscr{R}_C$, as $\Re(\zeta) \to +\infty$, for each $n \in \mathbb{N}$. This, together with (3.89), implies that, for every $n \in \mathbb{N}$, and $\varepsilon_n := \min\{\mu_n, \eta_n\}$, (3.87) holds. ■

Lemma 3.3.8 (A solution of a Schröder's type homological equation)**.** Let f be a strongly hyperbolic complex Dulac germ defined on a standard quadratic domain $\mathscr{R}_C$, $C > 0$, and let $\widehat{f}(\zeta) = \alpha\zeta + o(1)$, $\alpha \in \mathbb{R}_{>1}$, be its asymptotic expansion in the ζ-chart. Let $g(\zeta) = o(\mathrm{e}^{-\nu\zeta})$, uniformly on $\mathscr{R}_C$ as $\Re(\zeta) \to +\infty$, for $\nu \in \mathbb{R}_{>0}$, be an analytic germ defined on $\mathscr{R}_C$, in the ζ-chart. Then:

1. (*Existence*) There exist $R > 0$ such that $D := (\mathscr{R}_C)_R$ is an f-invariant, and an analytic solution

$$\varphi_g(\zeta) := -\sum_{n=0}^{+\infty} \frac{1}{\alpha^{n+1}}(g \circ f^{\circ n})(\zeta) \tag{3.90}$$

of the *Schröder's type homological equation*:

$$\varphi_g(f(\zeta)) - \alpha f(\zeta) = g(\zeta), \quad \zeta \in D. \tag{3.91}$$

2. (*Asymptotics*) $\varphi_g(\zeta) = O(\mathrm{e}^{-\nu\zeta})$, uniformly on D, as $\Re(\zeta) \to +\infty$.

3. (*Uniqueness*) If $\psi_g = o(1)$ is a solution of homological equation (3.91) on an f-invariant subdomain $D_1 \subseteq \mathscr{R}_C$ then $\psi_g \equiv \varphi_g$ on $(D_1)_R = D \cap D_1$, for R from statement 1.

The proof of Lemma 3.3.8 is motivated by a solution to the similar equation in [17].

Proof. Let $\varepsilon > 0$ and $k \in \mathbb{N}$ be arbitrary. Since $g(\zeta) = o(\mathrm{e}^{-\nu\zeta})$, $\nu > 0$, uniformly as $\Re(\zeta) \to +\infty$, it follows that there exists $R > 0$ large enough, such that $|g(\zeta)| \leq \frac{1}{\mathrm{e}^{\nu\cdot\Re(\zeta)}}$, $\zeta \in D$. By a similar discussion as the one at the beginning of Subsection 3.2.3, we take $R > 0$ sufficiently large such that $(\mathscr{R}_C)_R = (\mathscr{R}_C)_R^f$, that is, such that $(\mathscr{R}_C)_R$ is f-invariant. Now, put $D := (\mathscr{R}_C)_R$. By (3.71), it follows that:

$$\begin{aligned}\left|\sum_{i=0}^{n} \frac{1}{\alpha^{i+1}}(g \circ f^{\circ i})(\zeta) - \sum_{i=0}^{n-1} \frac{1}{\alpha^{i+1}}(g \circ f^{\circ i})(\zeta)\right| &\leq \frac{1}{\alpha^{n+1}} \cdot \frac{1}{\mathrm{e}^{\nu\cdot\Re(f^{\circ n}(\zeta))}} \\ &\leq \frac{1}{\alpha^{n+1}} \cdot \frac{1}{\mathrm{e}^{\nu R}}, \quad \zeta \in D,\, n \in \mathbb{N}. \qquad (3.92)\end{aligned}$$

This implies that $\sum \frac{1}{\alpha^{n+1}}(g \circ f^{\circ n})$ converges uniformly on D. Put

$$\varphi_g(\zeta) := -\sum_{n=0}^{+\infty} \frac{1}{\alpha^{n+1}}(g \circ f^{\circ n})(\zeta), \quad \zeta \in D. \qquad (3.93)$$

By the Weierstrass Theorem, φ_g is analytic on D. Now, it is easy to see that φ_g is a solution of homological equation (3.91).

Statement 2. follows similarly as the proof of statement 2 of Lemma 3.2.15.

3. Suppose that $\psi_g(\zeta) = o(1)$, uniformly as $\Re(\zeta) \to +\infty$, is a solution of homological equation (3.91) on D_1. Put $E(\zeta) := \varphi_g(\zeta) - \psi_g(\zeta)$, $\zeta \in D_1 \cap D$. Now, $E \circ f = \alpha E$ and $E(\zeta) = o(1)$, uniformly on $D_1 \cap D$ as $\Re(\zeta) \to +\infty$. Similarly as the proof of statement 4 of Theorem 3.3.5, it follows that $E \equiv 0$, i.e., $\varphi_g \equiv \psi_g$. ∎

Proof of Theorem E. Let $f(\zeta) = \alpha\zeta + o(1)$, $\alpha \in \mathbb{R}_{>1}$, be a strongly hyperbolic complex Dulac germ on a standard quadratic domain $\mathscr{R}_C$, $C > 0$, given in the ζ-chart, and let $\widehat{f}$ be its complex Dulac asymptotic expansion. We distinguish two cases.

If $\widehat{f} = \alpha\zeta$, by quasi-analyticity (Theorem 3.1.15), it follows that $f(\zeta) = \alpha\zeta$. In this case f is already normalized, so we put $\varphi := \mathrm{id}$. Since the formal normalization $\widehat{\varphi}$ obviously equals to id, it follows $\varphi \sim \widehat{\varphi}$.

Now suppose that $\widehat{f}$ is nontrivial. This implies that f satisfies the assumptions of Theorem 3.3.5, since the standard quadratic domain $\mathscr{R}_C$ is an admissible domain of any type (Example 3.3.3). Therefore, by Theorem 3.3.5, for f we obtain an analytic normalization φ on the maximal f-invariant subdomain $(\mathscr{R}_C)_R^f$ of the standard quadratic domain $\mathscr{R}_C$. By a similar discussion as the one at the beginning of Subsection 3.2.3, it follows that we can take $R > 0$ sufficiently large such that $(\mathscr{R}_C)_R = (\mathscr{R}_C)_R^f$, that is, such that $(\mathscr{R}_C)_R$ is f-invariant.

On the other hand, by Proposition 3.3.6, in Subsection 3.3.2, the normalization of strongly hyperbolic complex Dulac series $\widehat{f}$ is a parabolic complex Dulac series $\widehat{\varphi}$.

By Remark 3.1.10, there exists $C_1 > 0$ large enough such that $\mathscr{R}_{C_1} \subseteq (\mathscr{R}_C)_R$. Therefore, it is left to prove that $\widehat{\varphi}$ is the asymptotic expansion of φ on the standard quadratic domain $\mathscr{R}_{C_1}$. Hence, φ is a parabolic complex Dulac germ. Furthermore, since $\varphi \sim \widehat{\varphi}$ and $\widehat{\varphi}$ is a complex Dulac series, by Theorem 3.3.5, 4., it follows that φ is the unique analytic solution of the analytic normalization equation (3.86) which is a parabolic complex Dulac germ.

Let us prove that $\widehat{\varphi}$ is the asymptotic expansion of φ (in the ζ-chart). Let

$$\widehat{\varphi} := \zeta + \sum_{n \in \mathbb{N}} Q_n(\zeta) \mathrm{e}^{-\beta_n \zeta}$$

and let (φ_n) be the sequence of partial analytic normalizations defined in Definition 3.2.13. Furthermore, let

$$g_n(\zeta) := -(\varphi_n(f(\zeta)) - \alpha \varphi_n(\zeta)), \quad \zeta \in \mathscr{R}_{C_1}, n \in \mathbb{N}. \tag{3.94}$$

Since φ is a normalization of f, we have:

$$\varphi(f(\zeta)) - \alpha \varphi(\zeta) = 0, \quad \zeta \in \mathscr{R}_{C_1}. \tag{3.95}$$

Note that, by (3.94) and (3.95),

$$(\varphi - \varphi_n)(f(\zeta)) - \alpha(\varphi - \varphi_n)(\zeta) = g_n(\zeta), \quad n \in \mathbb{N}. \tag{3.96}$$

By Lemma 3.3.7, it follows that $g_n(\zeta) = o(\mathrm{e}^{-(\beta_n + \varepsilon_n)\zeta})$, for some $\varepsilon_n > 0$, uniformly on $\mathscr{R}_{C_1}$, as $\Re(\zeta) \to +\infty$, for every $n \in \mathbb{N}$. From statement 3 of Theorem 3.3.5, it follows that $(\varphi - \varphi_n)(\zeta) = o(1)$, for each $n \in \mathbb{N}$. Since $\varphi - \varphi_n$ is a solution of the homological equation

(3.96), by Lemma 3.3.8, 3., it follows that $(\varphi - \varphi_n)(\zeta) = O(\mathrm{e}^{-(\beta_n+\varepsilon_n)\zeta})$, uniformly on $\mathscr{R}_{C_1}$, as $\Re(\zeta) \to +\infty$, for $\varepsilon_n > 0$, $n \in \mathbb{N}$. Therefore, $(\varphi - \varphi_n)(\zeta) = o(\mathrm{e}^{-\beta_n\zeta})$, uniformly on $\mathscr{R}_{C_1}$, as $\Re(\zeta) \to +\infty$, for every $n \in \mathbb{N}$. This proves that $\widehat{\varphi}$ is the asymptotic expansion of φ on the standard quadratic domain $\mathscr{R}_{C_1}$. Therefore, φ is a parabolic complex Dulac germ.

Furthermore, by statement 2 of Theorem 3.3.5, it follows that φ is a real Dulac germ if f is a real Dulac germ. ■

A. Composition and the list of important identities

In Section 1.1 we have defined differential algebra $\mathfrak{L}^\infty$ and its subalgebras $\mathfrak{L}$, and $\mathscr{L}_k^\infty$, $\mathscr{L}_k$, for $k \in \mathbb{N}$. In Subsection 1.1.4 we have defined differential algebras of blocks $\mathscr{B}_m$, $1 \leq m \leq k$, for $k \in \mathbb{N}_{\geq 1}$. Here, in Appendix A, we precisely define the composition of logarithmic transseries and state and prove the Taylor Theorem in this formal setting (see [29, Proposition 3.3]). Furthermore, we state and prove some important computational identities in the mentioned differential algebras, which are also introduced in [29, Subsection 3.6], and which we use throughout the thesis. In particular, they are mainly used in Chapter 2 to transform normalization equations into fixed point equations and in Appendix B to solve various differential equations in differential algebras $\mathfrak{L}^\infty$ and $\mathscr{B}_m$. Appendix A is divided into three sections: Composition of logarithmic transseries, Derivation identities and Composition identities.

A.1. Composition of logarithmic transseries and Taylor Theorem

In this section we prove that the composition of logarithmic transseries defined in Definition 1.1.28 in Section 1.1.5 is again a logarithmic transseries. First we define $f(\lambda z^\alpha)$, for $\alpha, \lambda > 0$, and $f \in \mathfrak{L}$. Using the standard Taylor formula for powers and logarithms (see [21, Section 2]), we put:

$$\begin{aligned} z^{\beta}\circ g = z^{\beta}\circ(\lambda z^{\alpha}\cdot(1+\frac{g_1}{\lambda z^{\alpha}})) &:= \lambda^{\beta} z^{\alpha\beta}\cdot\sum_{i\geq 1}\binom{\beta}{i}\left(\frac{g_1}{\lambda z^{\alpha}}\right)^i, \\ (\log z)\circ g = (\log z)\circ(\lambda z^{\alpha}\cdot(1+\frac{g_1}{\lambda z^{\alpha}})) &:= \log\lambda+\alpha\log z+\sum_{i\geq 1}\frac{(-1)^{i-1}}{i}\left(\frac{g_1}{\lambda z^{\alpha}}\right)^i, \end{aligned} \tag{A.1}$$

where $g\in\mathscr{L}_k^H$, $k\in\mathbb{N}$, such that $g=\lambda z^{\alpha}+g_1$, $g_1\in\mathscr{L}_k$, $\operatorname{ord}(g_1)>(\alpha,\mathbf{0}_k)$. Using the Neumann Lemma (Theorem 1.1.2) it can be seen that both series on the right-hand side of (A.1) form summable families. By Remark 1.1.18, it follows that both series converge in the product topology on $\mathfrak{L}^{\infty}$.

Now, $f(\lambda z^{\alpha})$ is defined *termwise* using (A.1). For exact calculations of $(z^{\beta}\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_k)\circ(\lambda z^{\alpha})$ see Lemma A.3.1 and Lemma A.3.2.

In the next proposition, which is a special case of the more general statement [29, Proposition 3.3], we prove that the series in the *formal Taylor formula* for logarithmic transseries converges in appropriate topologies. This allows us to define the composition of logarithmic transseries in Definition 1.1.28.

Proposition A.1.1 (see [29]). Let $f\in\mathscr{L}_m$ and $g\in\mathscr{L}_k^H$, for $m,k\in\mathbb{N}$. Let $g=\lambda z^{\alpha}+g_1$, for $\alpha,\lambda>0$ and $\operatorname{ord}(g_1)>(\alpha,\mathbf{0}_k)$. Then the series

$$\sum_{i\geq 1}\frac{f^{(i)}(\lambda z^{\alpha})}{i!}(g_1)^i \tag{A.2}$$

converges in the product topology on $\mathscr{L}_r$, for $r\geq\max\{m,k\}$.

If moreover $\operatorname{ord}_z(g_1)>\alpha$, then the convergence holds also in the finer power-metric topology on $\mathscr{L}_r$, for $r\geq\max\{m,k\}$.

Proof. It follows from the properties of the composition of transseries proved in [6], as a consequence of the Neumann Lemma (Theorem 1.1.2), that the family

$$\left(\frac{f^{(i)}(\lambda z^{\alpha})}{i!}(g_1)^i\right)_{i\in\mathbb{N}}$$

is summable in the sense recalled in Remark 1.1.18. Hence, as it was already noticed in the same remark, the series in (A.2) converges in the product topology.

For the second part of the statement, set $\mu := \mathrm{ord}_z\,(f)$ and $r := \mathrm{ord}_z\,(g_1) - \alpha$, $r > 0$. It is easy to see that

$$\mathrm{ord}_z\left(\frac{f^{(i)}(\lambda z^{\alpha})}{i!}(g_1)^i\right) = (\mu - i)\,\alpha + (\alpha + r)\,i = \mu\alpha + ri$$

tends to $+\infty$ when $i \to +\infty$. We conclude by Remark 1.1.17. ■

Remark A.1.2. Note that

$$f(\lambda z^{\alpha}) + \sum_{i\geq 1} \frac{f^{(i)}(\lambda z^{\alpha})}{i!}(g_1)^i, \tag{A.3}$$

is just the *formal Taylor expansion* of $f(\lambda z^{\alpha} + g_1)$ at λz^{α}.

Note that, in Definition 1.1.28, only g is requested to be in $\mathscr{L}_k^H$, $k \in \mathbb{N}$. The reason behind this is a closedness of $\mathscr{L}_k^H$, $k \in \mathbb{N}$, under the composition. Without proof we state the following proposition (for more details see e.g. [6], [21]).

Proposition A.1.3 (Groups $\mathfrak{L}^H$ and $\mathfrak{L}^0$)**.**

1. $\mathfrak{L}^H$ is a group and $\mathscr{L}_k^H$ are its subgroups under the composition.
2. $\mathfrak{L}^0$ is a normal subgroup of $\mathfrak{L}$, $\mathscr{L}_k^0$ is a subgroup of $\mathfrak{L}^0$ and a normal subgroup of $\mathscr{L}_k^H$, for $k \in \mathbb{N}$, under the composition.

Proposition A.1.4 (Superlinearity of the composition)**.** Let $g \in \mathfrak{L}^H$ and $\mathscr{R}_g : \mathfrak{L} \to \mathfrak{L}$ be the operator defined by $\mathscr{R}_g(f) := f \circ g$, for each $f \in \mathfrak{L}$. Then $\mathscr{R}_g$ is a superlinear operator.

Proof. Since $g \in \mathfrak{L}^H$, there exists $k \in \mathbb{N}$ such that $g \in \mathscr{L}_k^H$. By the definition of the composition, it follows that:

$$\mathscr{R}_g(f) = f(\lambda z^{\alpha}) + \sum_{i\geq 1} \frac{f^{(i)}(\lambda z^{\alpha})}{i!}(g_1)^i,$$

where $g := \lambda z^{\alpha} + g_1$, $\alpha, \lambda > 0$, $g_1 \in \mathscr{L}_k$, $\mathrm{ord}\,(g_1) > (\alpha, \mathbf{0}_k)$. Since derivation, multiplication by $(g_1)^i$ and composition by λz^{α} are superlinear operators, we conclude that $\mathscr{R}_g$ is a superlinear operator. ■

Remark A.1.5. Note that $(f_1 \cdot f_2) \circ g = (f_1 \circ g) \cdot (f_2 \circ g)$, for each $f_1, f_2 \in \mathfrak{L}$ and $g \in \mathfrak{L}^H$.

In particular, the operator of the composition $\mathscr{R}_g$, for $g \in \mathscr{L}_k^H$, $k \in \mathbb{N}$, defined in Proposition A.1.4 is an automorphism of algebra $\mathfrak{L}$ and its restriction on $\mathscr{L}_k$ is an automorphism of subalgebra $\mathscr{L}_k \subseteq \mathfrak{L}$.

By Proposition A.1.4 and Remark A.1.5, it follows that formula (1.8) in Definition 1.1.28 can be evaluated composing *term-by-term* using the standard Taylor formula for powers and logarithms stated in (A.1).

In the next proposition we prove the formal Taylor Theorem (see [29, Proposition 3.3]).

Proposition A.1.6 (The Taylor Theorem)**.** Let $f \in \mathfrak{L}$ and $g \in \mathscr{L}_k^H$, $k \in \mathbb{N}$, such that $g := g_0 + \text{h.o.t.}$, for $g_0 \in \mathscr{L}_k^H$. Let $g_1 := g - g_0$. Then:

$$f \circ g = f(g_0) + \sum_{i \geq 1} \frac{f^{(i)}(g_0)}{i!}(g_1)^i, \tag{A.4}$$

where the right-hand side converges in the product topology in $\mathfrak{L}$. Furthermore, if $\text{ord}_z(g_1) > 1$, then the right-hand side of (A.4) converges in the power-metric topology.

We call identity (A.4) the *Taylor formula* or the *Taylor expansion* of f at g_0.

Proof. By the same arguments as in Proposition A.1.1, we prove that sum on the right-hand side of (A.4) converges in the product topology (resp. the power-metric topology).

Since $g_0 \in \mathscr{L}_k^H$, there exists $\alpha, \lambda > 0$ such that $g_0 = \lambda z^\alpha + h$, $h \in \mathscr{L}_k$, $\text{ord}(h) > (\alpha, \mathbf{0}_k)$. Now, by (1.8), we have:

$$f(g_0) + \sum_{i \geq 1} \frac{f^{(i)}(g_0)}{i!}(g_1)^i = f(\lambda z^\alpha) + \sum_{j_0 \geq 1} \frac{f^{(j_0)}(\lambda z^\alpha)}{j_0!} h^{j_0} + \sum_{i \geq 1} \frac{f^{(i)}(g_0)}{i!}(g_1)^i.$$

Note that for each $i \in \mathbb{N}_{\geq 1}$, by (1.8), we have:

$$\begin{aligned} f^{(i)}(g_0) &= f^{(i)}(\lambda z^\alpha) + \sum_{j \geq 1} \frac{(f^{(i)})^{(j)}(\lambda z^\alpha)}{j!} h^j \\ &= f^{(i)}(\lambda z^\alpha) + \sum_{j \geq 1} \frac{f^{(i+j)}(\lambda z^\alpha)}{j!} h^j. \end{aligned}$$

Now, from the last two identities, we have:

$$
\begin{aligned}
f(g_0)+\sum_{i\geq 1}\frac{f^{(i)}(g_0)}{i!}(g_1)^i &= \sum_{i\geq 0}\sum_{j\geq 0}\frac{f^{(i+j)}(\lambda z^{\alpha})}{j!i!}h^j g_1^i \\
&= f(\lambda z^{\alpha})+\sum_{p\geq 1}\sum_{j=0}^{p}\frac{f^{(p)}(\lambda z^{\alpha})}{j!(p-j)!}h^j g_1^{p-j} \\
&= f(\lambda z^{\alpha})+\sum_{p\geq 1}\frac{f^{(p)}(\lambda z^{\alpha})}{p!}\Big(\sum_{j=0}^{p}\binom{p}{j}h^j g_1^{p-j}\Big) \\
&= f(\lambda z^{\alpha})+\sum_{p\geq 1}\frac{f^{(p)}(\lambda z^{\alpha})}{p!}(h+g_1)^p \\
&= f\circ g.
\end{aligned}
$$

■

A.2. DERIVATION IDENTITIES

In the lemma below we prove the usual formula for the derivation of a composition.

Lemma A.2.1 (Derivation of a composition)**.** Let $f \in \mathfrak{L}$ and $g \in \mathscr{L}_k^H$ such that $g = \lambda z^\alpha + g_1$, for $\alpha, \lambda > 0$, $g_1 \in \mathscr{L}_k$, $\operatorname{ord}(g_1) > (\alpha, \mathbf{0}_k)$. Then:

$$\frac{d}{dz}(f \circ g) = \Big(\frac{df}{dz}\Big) \circ g \cdot \frac{dg}{dz}.$$

Proof. It can be proven using the Neumann Lemma (Theorem 1.1.2) that the derivation $\frac{d}{dz} : \mathfrak{L}^\infty \to \mathfrak{L}^\infty$ is continuous with respect to the product topology. Since the series (A.2) in the definition of the composition $f \circ g$ converges in the product topology on $\mathfrak{L}^\infty$ and the derivation is continuous, we have the following:

$$\begin{aligned}
\frac{d}{dz}(f \circ g) &= \frac{d}{dz}\left(f(\lambda z^\alpha) + \sum_{i \geq 1} \frac{f^{(i)}(\lambda z^\alpha)}{i!}(g_1)^i\right)\\
&= \lambda\alpha z^{\alpha-1} f'(\lambda z^\alpha) + \sum_{i\geq 1}\Big(\frac{\lambda\alpha z^{\alpha-1} f^{(i+1)}(\lambda z^\alpha)}{i!}(g_1)^i + \frac{f^{(i)}(\lambda z^\alpha)}{(i-1)!}(g_1)^{i-1}(g_1)'\Big)\\
&= \lambda\alpha z^{\alpha-1} f'(\lambda z^\alpha) + f'(\lambda z^\alpha)(g_1)' + \sum_{i\geq 1}\frac{\lambda\alpha z^{\alpha-1} f^{(i+1)}(\lambda z^\alpha)}{i!}(g_1)^i + \sum_{i\geq 1}\frac{f^{(i+1)}(\lambda z^\alpha)}{i!}(g_1)^i(g_1)'\\
&= \lambda\alpha z^{\alpha-1}\Big(f'(\lambda z^\alpha) + \sum_{i\geq 1}\frac{f^{(i+1)}(\lambda z^\alpha)}{i!}(g_1)^i\Big) + (g_1)'\Big(f'(\lambda z^\alpha) + \sum_{i\geq 1}\frac{f^{(i+1)}(\lambda z^\alpha)}{i!}(g_1)^i\Big)\\
&= g' \cdot \left(f'(\lambda z^\alpha) + \sum_{i\geq 1}\frac{(f')^{(i)}(\lambda z^\alpha)}{i!}(g_1)^i\right)\\
&= \Big(\frac{df}{dz}\Big) \circ g \cdot \frac{dg}{dz}.
\end{aligned}$$

■

Lemma A.2.2 (Properties of the derivation $\frac{d}{dz}$)**.** Let $k \in \mathbb{N}$ and $(\alpha, n_1, \ldots n_k) \in \big(\mathbb{R} \times \mathbb{Z}^k\big) \setminus \{\mathbf{0}_{k+1}\}$. Then

$$\begin{aligned}
\frac{d}{dz}(z^\alpha \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}) &= z^{\alpha-1}\Big(\alpha \boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k} + n_1\boldsymbol{\ell}_1^{n_1+1}\boldsymbol{\ell}_2^{n_2}\cdots\boldsymbol{\ell}_k^{n_k} + \cdots + n_k\boldsymbol{\ell}_1^{n_1+1}\boldsymbol{\ell}_2^{n_2+1}\cdots\boldsymbol{\ell}_k^{n_k+1}\Big)\\
&= z^{\alpha-1}\Big(\alpha \boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k} + \sum_{i=1}^k n_i \boldsymbol{\ell}_1^{n_1+1}\ldots\boldsymbol{\ell}_i^{n_i+1}\boldsymbol{\ell}_{i+1}^{n_{i+1}}\cdots\boldsymbol{\ell}_k^{n_k}\Big). \qquad \text{(A.5)}
\end{aligned}$$

Proof. Note that $\frac{d}{dz}(z^\alpha) = \alpha z^{\alpha-1}$ and $\frac{d}{dz}(\boldsymbol{\ell}_1^{n_1}) = \frac{d}{dz}\Big(\big(-\frac{1}{\log z}\big)^{n_1}\Big) = n_1 z^{-1}\boldsymbol{\ell}_1^{n_1+1}$. Inductively, we get:

$$\begin{aligned}\frac{d}{dz}(\boldsymbol{\ell}_m^{n_m}) &= n_m \boldsymbol{\ell}_m^{n_m-1}\frac{d}{dz}(\boldsymbol{\ell}_1\circ\boldsymbol{\ell}_{m-1})\\ &= n_m \boldsymbol{\ell}_m^{n_m-1}\cdot(z^{-1}\boldsymbol{\ell}_1^2)\circ\boldsymbol{\ell}_{m-1}\cdot\frac{d}{dz}(\boldsymbol{\ell}_{m-1})\\ &= n_m \boldsymbol{\ell}_m^{n_m-1}\cdot\frac{1}{\boldsymbol{\ell}_{m-1}}\cdot\boldsymbol{\ell}_m^2\cdot z^{-1}\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{m-2}\boldsymbol{\ell}_{m-1}^2\\ &= n_m z^{-1}\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{m-1}\boldsymbol{\ell}_m^{n_m+1}, \end{aligned} \tag{A.6}$$

for every $2\leq m\leq k$. Consequently, identity (A.5) follows from the Newton-Leibnitz rule. ■

Lemma A.2.3 (Properties of the Lie bracket operator)**.** Let $k\in\mathbb{N}$ and let $(\alpha,n_1,\ldots n_k),(\beta,m_1,\ldots m_k)\in\left(\mathbb{R}\times\mathbb{Z}^k\right)\setminus\{\mathbf{0}_{k+1}\}$. Then:

1.

$$\begin{aligned}&\left[z^\alpha\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k},z^\beta\boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k}\right]\\ &= z^\alpha\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}\cdot\frac{d}{dz}(z^\beta\boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k})-z^\beta\boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k}\cdot\frac{d}{dz}(z^\alpha\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k})\\ &= z^{\alpha+\beta-1}\Big((\alpha-\beta)\boldsymbol{\ell}_1^{n_1+m_1}\cdots\boldsymbol{\ell}_k^{n_k+m_k}+\sum_{i=1}^k(n_i-m_i)\boldsymbol{\ell}_1^{n_1+m_1+1}\ldots\boldsymbol{\ell}_i^{n_i+m_i+1}\boldsymbol{\ell}_{i+1}^{n_{i+1}+m_{i+1}}\cdots\boldsymbol{\ell}_k^{n_k+m_k}\Big).\end{aligned} \tag{A.7}$$

2. Let $z^\alpha\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}\in\mathscr{L}_k^\infty$ such that $\alpha\neq0$ if $k=0$, and $n_k\neq0$ if $k\geq1$. Then $\left[z^\alpha\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k},z^\beta\boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k}\right]$ does not contain a monomial $z^{2\alpha-1}\boldsymbol{\ell}_1^{2n_1+1}\cdots\boldsymbol{\ell}_k^{2n_k+1}$, for any $z^\beta\boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k}\in\mathscr{L}_k$.
3. Let $z^\alpha\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}\in\mathscr{L}_k^\infty$. Then, for every $c\in\mathbb{R}\setminus\{0\}$:

$$\left[z^\alpha\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k},cz^\alpha\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}\boldsymbol{\ell}_{k+1}^{-1}\right]=cz^{2\alpha-1}\boldsymbol{\ell}_1^{2n_1+1}\cdots\boldsymbol{\ell}_k^{2n_k+1}.$$

4. Let $z^\alpha\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}\in\mathscr{L}_k^\infty$. For every $cz^\gamma\boldsymbol{\ell}_1^{\nu_1}\cdots\boldsymbol{\ell}_k^{\nu_k}\in\mathscr{L}_k^\infty$, $c\in\mathbb{R}\setminus\{0\}$, such that $(\gamma,\nu_1,\ldots,\nu_k)\neq(2\alpha-1,2n_1+1,\ldots,2n_k+1)$, there exists $bz^\beta\boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k}\in\mathscr{L}_k^\infty$, $b\in\mathbb{R}\setminus\{0\}$, such that $\left[z^\alpha\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k},bz^\beta\boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k}\right]$ contains the term $cz^\gamma\boldsymbol{\ell}_1^{\nu_1}\cdots\boldsymbol{\ell}_k^{\nu_k}$.

The proof of Lemma A.2.3 is motivated by similar properties of the Lie bracket operator on the differential algebra $\mathscr{L}_1$, proved in [21, Section 3].

Proof. Statement 1 follows directly from Lemma A.2.2.

2. On the contrary, suppose that such $z^{\beta}\boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k}\in\mathscr{L}_k^{\infty}$ exists. By (A.7), it follows that $\alpha+\beta-1=2\alpha-1$, i.e., $\alpha=\beta$. Now, $n_1+m_1+1=2n_1+1$, i.e., $n_1=m_1$. Inductively, we get $n_i=m_i$, for each $1\leq i\leq k$, but then $\left[z^{\alpha}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}, bz^{\beta}\boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k}\right]=0$, which is a contradiction. Therefore, such $z^{\beta}\boldsymbol{\ell}_1^{m_1}\cdots\boldsymbol{\ell}_k^{m_k}\in\mathscr{L}_k^{\infty}$ does not exist.

Statement 3 follows directly by (A.7).

4. If $\gamma\neq 2\alpha-1$, then put $\beta:=\gamma-\alpha+1$, $b:=\frac{c}{2\alpha-1-\gamma}$, and $m_i:=v_i-n_i$, for each $1\leq i\leq k$. By (A.7), it easy to check that statement 4 holds. Now, suppose that $\gamma=2\alpha-1$, and there exists $r\in\mathbb{N}_{\geq 1}$ such that $1\leq r\leq k$, $v_i=2n_i+1$, for each $1\leq i\leq r-1$, and $v_r\neq 2n_r+1$. Put $\beta:=\alpha$, $m_i:=n_i$, for each $1\leq i\leq r-1$, and $m_r:=v_r-n_r-1$. Finally, put $m_j:=v_j-n_j$, for each $r+1\leq j\leq k$, and $b:=\frac{c}{2n_r+1-v_r}$. By (A.7), it easy to check that statement 4 holds. ■

The following Lemmas A.2.4 and A.2.5 are analogues of Lemmas A.2.2 and A.2.3 in differential algebras $\mathscr{B}_m\subseteq\mathscr{L}_k^{\infty}$, for $1\leq m\leq k$, $k\in\mathbb{N}_{\geq 1}$.

Lemma A.2.4 (Properties of derivations D_m)**.** Let $k,m\in\mathbb{N}_{\geq 1}$ such that $1\leq m\leq k$, and $(n_m,\ldots n_k)\in\mathbb{Z}^{k-m+1}$. Then

$$\begin{aligned}&D_m(\boldsymbol{\ell}_m^{n_m}\cdots\boldsymbol{\ell}_k^{n_k})\\&=n_m\boldsymbol{\ell}_m^{n_m+1}\boldsymbol{\ell}_{m+1}^{n_{m+1}}\cdots\boldsymbol{\ell}_k^{n_k}+n_{m+1}\boldsymbol{\ell}_m^{n_m+1}\boldsymbol{\ell}_{m+1}^{n_{m+1}+1}\boldsymbol{\ell}_{m+2}^{n_{m+2}}\cdots\boldsymbol{\ell}_k^{n_k}+\cdots+n_k\boldsymbol{\ell}_m^{n_m+1}\boldsymbol{\ell}_{m+1}^{n_{m+1}+1}\cdots\boldsymbol{\ell}_k^{n_k+1}\\&=\sum_{i=m}^{k}n_i\boldsymbol{\ell}_m^{n_m+1}\ldots\boldsymbol{\ell}_i^{n_i+1}\boldsymbol{\ell}_{i+1}^{n_{i+1}}\cdots\boldsymbol{\ell}_k^{n_k}.\end{aligned}\tag{A.8}$$

Proof. Directly by the definition of the derivation D_m (see (1.5)) and the Newton-Leibnitz rule. ■

Lemma A.2.5 (Properties of the Lie bracket operator on $\mathscr{B}_m$)**.** Let $k,m\in\mathbb{N}_{\geq 1}$ such that $1\leq m\leq k$, and let $(n_m,\ldots n_k),(v_m,\ldots v_k)\in\mathbb{Z}^{k-m+1}$. Then:

1.

$$\begin{aligned}&\left[\boldsymbol{\ell}_m^{n_m}\cdots\boldsymbol{\ell}_k^{n_k},\boldsymbol{\ell}_m^{v_m}\cdots\boldsymbol{\ell}_k^{v_k}\right]\\&=\boldsymbol{\ell}_m^{n_m}\cdots\boldsymbol{\ell}_k^{n_k}\cdot D_m(\boldsymbol{\ell}_m^{v_1}\cdots\boldsymbol{\ell}_k^{v_k})-\boldsymbol{\ell}_m^{v_m}\cdots\boldsymbol{\ell}_k^{v_k}\cdot D_m(\boldsymbol{\ell}_m^{n_m}\cdots\boldsymbol{\ell}_k^{n_k})\\&=\sum_{i=m}^{k}(n_i-v_i)\,\boldsymbol{\ell}_m^{n_m+v_m+1}\ldots\boldsymbol{\ell}_i^{n_i+v_i+1}\boldsymbol{\ell}_{i+1}^{n_{i+1}+v_{i+1}}\cdots\boldsymbol{\ell}_k^{n_k+v_k}.\end{aligned}\tag{A.9}$$

2. Let $\boldsymbol{\ell}_m^{n_m}\cdots\boldsymbol{\ell}_k^{n_k}\in\mathscr{B}_m$ such that $n_k\neq 0$. Then $\left[\boldsymbol{\ell}_m^{n_m}\cdots\boldsymbol{\ell}_k^{n_k},\boldsymbol{\ell}_m^{v_m}\cdots\boldsymbol{\ell}_k^{v_k}\right]$ does not contain a monomial $\boldsymbol{\ell}_m^{2n_m+1}\cdots\boldsymbol{\ell}_k^{2n_k+1}$, for any $\boldsymbol{\ell}_m^{v_m}\cdots\boldsymbol{\ell}_k^{v_k}\in\mathscr{B}_m$.

3. Let $\boldsymbol{\ell}_m^{n_m}\cdots\boldsymbol{\ell}_k^{n_k}\in\mathscr{B}_m$. Then, for every $c\in\mathbb{R}\setminus\{0\}$:

$$\left[\boldsymbol{\ell}_m^{n_m}\cdots\boldsymbol{\ell}_k^{n_k},-c\boldsymbol{\ell}_m^{n_m}\cdots\boldsymbol{\ell}_k^{n_k}\boldsymbol{\ell}_{k+1}^{-1}\right]=c\boldsymbol{\ell}_m^{2n_m+1}\cdots\boldsymbol{\ell}_k^{2n_k+1}.$$

4. Let $\boldsymbol{\ell}_m^{n_m}\cdots\boldsymbol{\ell}_k^{n_k}\in\mathscr{B}_m$. For every $c\boldsymbol{\ell}_m^{v_m}\cdots\boldsymbol{\ell}_k^{v_k}\in\mathscr{B}_m$, $c\in\mathbb{R}\setminus\{0\}$, such that $(v_m,\dots,v_k)\neq(2n_m+1,\dots,2n_k+1)$, there exists $b\boldsymbol{\ell}_m^{u_m}\cdots\boldsymbol{\ell}_k^{u_k}\in\mathscr{B}_m$, $b\in\mathbb{R}\setminus\{0\}$, such that $\left[\boldsymbol{\ell}_m^{n_m}\cdots\boldsymbol{\ell}_k^{n_k},b\boldsymbol{\ell}_m^{u_m}\cdots\boldsymbol{\ell}_k^{u_k}\right]$ contains the term $c\boldsymbol{\ell}_m^{v_m}\cdots\boldsymbol{\ell}_k^{v_k}$.

Proof. Similarly as the proof of Lemma A.2.3. ■

In the following lemmas we prove the identities relating derivations $\frac{d}{dz}$ and D_m, for $1\leq m\leq k$, and $k\in\mathbb{N}_{\geq 1}$.

Lemma A.2.6 (Derivation of α-block, for $\alpha\in\mathbb{R}$)**.** Let $k\in\mathbb{N}_{\geq 1}$, $\alpha\in\mathbb{R}$, and let $K\in\mathscr{B}_1\subseteq\mathscr{L}_k^\infty$. Let $I:\mathscr{B}_1\to\mathscr{B}_1$ be the identity operator. Then:

$$\begin{aligned}(z^\alpha K)'&=z^{\alpha-1}\cdot(\alpha I+D_1)(K),\\ (z^\alpha K)^{(i)}&=z^{\alpha-i}\cdot\Big(\big((\alpha-i+1)I+D_1\big)\circ\cdots\circ\big((\alpha-1)I+D_1\big)\circ(\alpha I+D_1)\Big)(K)\\ &=z^{\alpha-i}\cdot\Big((\alpha-i+1)\cdots(\alpha-1)\alpha K+\mathscr{C}_i(K)\Big),\end{aligned}\tag{A.10}$$

for every $i\in\mathbb{N}_{\geq 2}$, where $\mathscr{C}_i:\mathscr{B}_1\to\mathscr{B}_1$ is a superlinear $\frac{1}{2}$-contraction on the space $(\mathscr{B}_1,d_1)$.

Proof. By $\boldsymbol{\ell}_m=\boldsymbol{\ell}_{m-1}\circ\boldsymbol{\ell}_1$ and (A.6), it follows that:

$$\begin{aligned}D_1(\boldsymbol{\ell}_m^{n_m})&=\boldsymbol{\ell}_1^2\cdot\frac{d}{d\boldsymbol{\ell}_1}(\boldsymbol{\ell}_m^{n_m})\\ &=\boldsymbol{\ell}_1^2\cdot\big(\frac{d}{dz}(\boldsymbol{\ell}_{m-1}^{n_m})\big)\circ\boldsymbol{\ell}_1\\ &=\boldsymbol{\ell}_1^2\cdot\big(n_m z^{-1}\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{m-2}\boldsymbol{\ell}_{m-1}^{n_m+1}\big)\circ\boldsymbol{\ell}_1\\ &=n_m\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{m-1}\boldsymbol{\ell}_m^{n_m+1},\end{aligned}\tag{A.11}$$

for every $1\leq m\leq k$ and $n_m\in\mathbb{Z}$. Now, by the Newton-Leibniz rule and Lemma A.2.2, it follows that $\frac{d}{dz}(\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k})=z^{-1}D_1(\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k})$. Since D_1 is superlinear, we get that

$\frac{d}{dz}(K) = z^{-1}D_1(K)$, for every $K \in \mathscr{B}_1$. Consequently, it follows that:

$$\frac{d}{dz}(z^\alpha K) = \alpha z^{\alpha-1}K + z^\alpha \frac{d}{dz}(K) \tag{A.12}$$

$$= z^{\alpha-1} \cdot (\alpha I + D_1)(K), \tag{A.13}$$

for every $K \in \mathscr{B}_1$ and $\alpha \in \mathbb{R}$. Note that:

$$\begin{aligned}(z^\alpha K)^{(2)} &= \frac{d}{dz}(z^{\alpha-1} \cdot (\alpha I + D_1(K))) \\ &= z^{\alpha-2} \cdot \big(((\alpha-1)I + D_1) \circ (\alpha I + D_1)\big)(K) \\ &= z^{\alpha-2} \cdot \big(\alpha(\alpha-1)K + \mathscr{C}_2(K)\big),\end{aligned}$$

where $\mathscr{C}_2(K) := (2\alpha - 1)D_1(K) + D_1^2(K)$, for every $K \in \mathscr{B}_1$ and $\alpha \in \mathbb{R}$. Since D_1 is a superlinear $\frac{1}{2}$-contraction on the space $(\mathscr{B}_1, d_1)$, it follows that $\mathscr{C}_2 : \mathscr{B}_1 \to \mathscr{B}_1$ is a superlinear $\frac{1}{2}$-contraction on the space $(\mathscr{B}_1, d_1)$. Now, the general identity follows inductively for $i \geq 3$:

$$\begin{aligned}(z^\alpha K)^{(i)} &= ((z^\alpha K)^{(i-1)})' \\ &= \Big(z^{\alpha-(i-1)}\big(((\alpha-i+2)I + D_1) \circ \cdots \circ (\alpha I + D_1)\big)(K)\Big)' \\ &= z^{\alpha-i} \cdot \Big(((\alpha-i+1)I + D_1) \circ \cdots \circ (\alpha I + D_1)\Big)(K) \\ &= z^{\alpha-i} \cdot \Big((\alpha-i+1)\cdots(\alpha-1)\alpha K + \mathscr{C}_i(K)\Big),\end{aligned}$$

where $\mathscr{C}_i : \mathscr{B}_1 \to \mathscr{B}_1$ is a superlinear $\frac{1}{2}$-contraction on the space $(\mathscr{B}_1, d_1)$. ■

By Lemma A.2.6 it follows that $(zK)^{(i)} = z \cdot \mathscr{C}_i(K)$, for every $K \in \mathscr{B}_1 \subseteq \mathscr{L}_k^\infty$, $k \in \mathbb{N}_{\geq 1}$, and $i \geq 2$, where $\mathscr{C}_i : \mathscr{B}_1 \to \mathscr{B}_1$ is a $\frac{1}{2}$-contraction. In the next lemma we investigate more precisely the *structure* of the $\frac{1}{2}$-contraction $\mathscr{C}_i$, $i \geq 2$.

Lemma A.2.7 (Derivation of a 1-block)**.** Let $k \in \mathbb{N}_{\geq 1}$ and $K \in \mathscr{B}_1 \subseteq \mathscr{L}_k^\infty$, $k \in \mathbb{N}_{\geq 1}$. Let $I : \mathscr{B}_1 \to \mathscr{B}_1$ be the identity operator. Then:

$$\begin{aligned}(zK)' &= (I + D_1)(K), \\ (zK)'' &= z^{-1} \cdot (D_1 + D_1^2)(K) = z^{-1} \cdot (D_1(K) + \mathscr{C}_2(K)), \\ (zK)^{(i)} &= z^{-(i-1)} \cdot \Big((-(i-2)I + D_1) \circ \cdots \circ (-I + D_1) \circ (D_1 + D_1^2)\Big)(K), \\ &= z^{-(i-1)} \cdot \Big((-1)^{i-2}(i-2)!D_1(K) + \mathscr{C}_i(K)\Big),\end{aligned} \tag{A.14}$$

for every $i \in \mathbb{N}_{\geq 3}$, where $\mathscr{C}_i : \mathscr{B}_1 \to \mathscr{B}_1$, $i \in \mathbb{N}_{\geq 2}$, is a superlinear $\frac{1}{4}$-contraction on the space $(\mathscr{B}_1, d_1)$.

Proof. The first identity follows directly from the first identity in (A.10). Now, we get that:

$$\begin{aligned}(zK)'' &= \frac{d}{dz}(K + D_1(K)) \\ &= z^{-1}(D_1(K) + D_1^2(K)),\end{aligned}$$

and, by (A.10):

$$\begin{aligned}(zK)^{(3)} &= \frac{d}{dz}\big(z^{-1}(D_1(K) + D_1^2(K))\big) \\ &= z^{-2}(-D_1(K) + D_1^3(K)),\end{aligned}$$

for every $K \in \mathscr{B}_1$. Put $\mathscr{C}_3(K) := D_1^3(K)$, for every $K \in \mathscr{B}_1$. Since D_1 is a superlinear $\frac{1}{2}$-contraction, it follows that $\mathscr{C}_3 : \mathscr{B}_1 \to \mathscr{B}_1$ is a superlinear $\frac{1}{4}$-contraction (even $\frac{1}{6}$-contraction) on the space $(\mathscr{B}_1, d_1)$. Now, the rest of identity (A.14) follows inductively. ■

Lemma A.2.8. Let $k \in \mathbb{N}_{\geq 1}$, $\alpha \in \mathbb{R}_{>0}$, $\alpha \neq 1$, and let $K \in \mathscr{B}_1 \subseteq \mathscr{L}_k^\infty$ and $G \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$. Then:

1.
$$\begin{aligned}\sum_{i\geq 1} \frac{(zK)^{(i)}}{i!}(zG)^i &= zKG + zD_1(K)\cdot\Big(G + \sum_{i\geq 2}\frac{(-1)^{i-2}}{(i-1)i}G^i\Big) + z\mathscr{C}(K,G) \\ &= zKG + z(1+G)\log(1+G)D_1(K) + z\mathscr{C}(K,G), \qquad \text{(A.15)}\end{aligned}$$

where $\mathscr{C} : \mathscr{B}_1 \times \mathscr{B}_{\geq 1}^+ \to \mathscr{B}_1$ is a $\left(\frac{1}{2^{2+2\mathrm{ord}_{\ell_1}(G)}} \frac{1}{2^{2+\mathrm{ord}_{\ell_1}(K)}}\right)$-Lipschitz map (in the sense of Definition 1.2.9), with respect to the metric d_1,

2.
$$\begin{aligned}\sum_{i\geq 1} \frac{(z^\alpha K)^{(i)}}{i!}(zG)^i &= z^\alpha K \cdot \sum_{i\geq 1}\binom{\alpha}{i} G^i + z^\alpha \mathscr{K}(K,G) \\ &= z^\alpha((1+G)^\alpha - G) + z^\alpha \mathscr{K}(K,G), \qquad \text{(A.16)}\end{aligned}$$

where $\mathscr{K} : \mathscr{B}_1 \times \mathscr{B}_{\geq 1}^+ \to \mathscr{B}_1$ is a $\left(\frac{1}{2^{1+\mathrm{ord}_{\ell_1}(G)}}, \frac{1}{2^{1+\mathrm{ord}_{\ell_1}(K)}}\right)$-Lipschitz map, with respect to the metric d_1.

Proof. 1. By (A.14) it follows that $\frac{1}{1!}(zK)' \cdot zG = zKG + zGD_1(K)$, $\frac{1}{2!}(zK)'' \cdot zG^2 = \frac{1}{2}zG^2(D_1(K) + \mathscr{C}_2(K))$, and:

$$\begin{aligned}\frac{(zK)^{(i)}}{i!}(zG)^i &= \frac{(-1)^{i-2}(i-2)!}{i!} \cdot z^{-(i-1)+i}G^iD_1(K) + \frac{1}{i!} \cdot z^{-(i-1)+i}G^i\mathscr{C}_i(K)\\ &= \frac{(-1)^{i-2}}{i(i-1)} \cdot zG^iD_1(K) + \frac{1}{i!}zG^i\mathscr{C}_i(K),\end{aligned}$$

for every $i \in \mathbb{N}_{\geq 3}$. Put $\mathscr{C}(K,G) := \sum_{i\geq 2} \frac{\mathscr{C}_i(K)}{i!}G^i$, for every $(K,G) \in \mathscr{B}_1 \times \mathscr{B}^+_{\geq 1}$. The first line of (A.15) follows directly. By analyzing the order in $\boldsymbol{\ell}_1$ and by Example 1.2.8, we get that $\mathscr{C} : \mathscr{B}_1 \times \mathscr{B}^+_{\geq 1} \to \mathscr{B}_1$ is a $\left(\frac{1}{2^{2+2\mathrm{ord}_{\boldsymbol{\ell}_1}(G)}}, \frac{1}{2^{2+\mathrm{ord}_{\boldsymbol{\ell}_1}(K)}}\right)$-Lipschitz map, with respect to the metric d_1.

Note that:

$$\begin{aligned}(1+G)\log(1+G) &= (1+G) \cdot \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i}G^i\\ &= G + \sum_{i=2}^{\infty}\left(\frac{(-1)^{i-1}}{i} + \frac{(-1)^{i-2}}{i-1}\right)G^i\\ &= G + \sum_{i=2}^{\infty} \frac{(-1)^{i-2}}{i(i-1)}G^i.\end{aligned}$$

Now, the last line of (A.15) follows directly.

Statement 2 follows similarly by (A.10) and the fact that:

$$(1+G)^\alpha - G = \sum_{i\geq 1}\binom{\alpha}{i}G^i.$$

■

Remark A.2.9. Note that $\mathscr{C}$ and $\mathscr{K}$ in Lemma A.2.8 are superlinear in the first variable and nonlinear in the second variable.

Lemma A.2.10 (Relations between derivations D_m)**.** Let $k,n,m \in \mathbb{N}_{\geq 1}$ such that $1 \leq n \leq k-1$ and $1 \leq m \leq k-n$.

1. If $K \in \mathscr{B}_{m+n} \subseteq \mathscr{L}_k^\infty$, then:

$$D_m(K) = \boldsymbol{\ell}_m \cdots \boldsymbol{\ell}_{m+n-1}D_{m+n}(K). \tag{A.17}$$

2. If $G \in \mathscr{B}_m \subseteq \mathscr{L}_k^\infty$, then:

$$(D_m(G)) \circ \boldsymbol{\ell}_n = D_{m+n}(G \circ \boldsymbol{\ell}_n). \tag{A.18}$$

Proof. 1. By (A.6), it follows that:

$$\begin{aligned}
D_m(\boldsymbol{\ell}_{m+n}^u) &= \boldsymbol{\ell}_m^2 \cdot \frac{d}{d\boldsymbol{\ell}_m}(\boldsymbol{\ell}_{m+n}^u)\\
&= \boldsymbol{\ell}_m^2 \cdot \Big(\frac{d}{dz}(\boldsymbol{\ell}_n^u)\Big) \circ \boldsymbol{\ell}_m\\
&= \boldsymbol{\ell}_m^2 \cdot \big(uz^{-1}\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{n-1}\boldsymbol{\ell}_n^{u+1}\big) \circ \boldsymbol{\ell}_m\\
&= u\boldsymbol{\ell}_m \cdots \boldsymbol{\ell}_{m+n-1}\boldsymbol{\ell}_{m+n}^{u+1}\\
&= \boldsymbol{\ell}_m \cdots \boldsymbol{\ell}_{m+n-1} D_{m+n}(\boldsymbol{\ell}_{m+n}^u),
\end{aligned}$$

for every $u \in \mathbb{Z}$. Now, statement 1 follows from superlinearity and the Newton-Leibnitz rule.

Statement 2 follows from the fact that $\frac{d}{d\boldsymbol{\ell}_{m+n}}(G \circ \boldsymbol{\ell}_n) = \Big(\frac{d}{d\boldsymbol{\ell}_m}(G)\Big) \circ \boldsymbol{\ell}_n$ and from (1.5). ■

Lemma A.2.11 (Properties of the derivation D_1)**.** Let $k,m \in \mathbb{N}_{\geq 1}$ such that $1 \leq m \leq k$, and let $K \in \mathscr{B}_1 \subseteq \mathscr{L}_k^\infty$. If $\operatorname{ord}(K) = (\mathbf{0}_m, n_m, n_{m+1}, \ldots, n_k)$, where $n_m \neq 0$, then

$$\operatorname{ord}(D_1(K)) = (0, \mathbf{1}_{m-1}, n_m + 1, n_{m+1}, \ldots n_k) = \operatorname{ord}(K) + (0, \mathbf{1}_m, \mathbf{0}_{k-m}). \tag{A.19}$$

Proof. Since $\operatorname{ord}(K) = (0, \mathbf{0}_{m-1}, n_m, n_{m+1}, \ldots, n_k)$, where $n_m \neq 0$, it follows that there exists $a \in \mathbb{R} \setminus \{0\}$ such that $a\boldsymbol{\ell}_m^{n_m} \cdots \boldsymbol{\ell}_k^{n_k}$ is the leading term of K. By (A.11), it follows that:

$$D_1(a\boldsymbol{\ell}_m^{n_m} \cdots \boldsymbol{\ell}_k^{n_k}) = n_m a \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1}\boldsymbol{\ell}_m^{n_m+1}\boldsymbol{\ell}_{m+1}^{n_{m+1}} \cdots \boldsymbol{\ell}_k^{n_k} + \text{h.o.t.}$$

Now, the claim follows from:

$$\operatorname{ord}(D_1(K)) = \operatorname{ord}(D_1(a\boldsymbol{\ell}_m^{n_m} \cdots \boldsymbol{\ell}_k^{n_k})) = \operatorname{ord}(n_m a \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1}\boldsymbol{\ell}_m^{n_m+1}\boldsymbol{\ell}_{m+1}^{n_{m+1}} \cdots \boldsymbol{\ell}_k^{n_k}).$$

■

A.3. COMPOSITION IDENTITIES

Lemma A.3.1 (Composition with $\lambda\cdot\mathrm{id}$, [29])**.** Let $k\in\mathbb{N}_{\geq 1}$, $\alpha\in\mathbb{R}_{\geq 0}$, $\lambda\in\mathbb{R}_{>0}$, $\lambda\neq 1$, and let $K\in\mathscr{B}_1\subseteq\mathscr{L}_k^\infty$ such that $\operatorname{ord}(z^\alpha K)>\mathbf{0}_{k+1}$. Then:

1. $$(z^\alpha K)(\lambda z)=\lambda^\alpha z^\alpha\big(I+\log\lambda\cdot D_1(K)+\mathscr{C}_\lambda(K)\big), \tag{A.20}$$

   where $\mathscr{C}_\lambda:\mathscr{B}_1\to\mathscr{B}_1$ is a superlinear $\frac{1}{4}$-contraction and $I:\mathscr{B}_1\to\mathscr{B}_1$ an identity operator on the space $(\mathscr{B}_1,d_1)$.

2. The support $\operatorname{Supp}\big((z^\alpha K)(\lambda z)\big)$ is contained in the sub-semigroup of $\mathbb{R}\times\mathbb{Z}^k$ generated by elements of $\operatorname{Supp}(z^\alpha K)$ and $(0,1,0,\ldots,0)_{k+1},\ldots,(0,0,\ldots,0,1)_{k+1}$.

3. Every coefficient of $z^\alpha K(\lambda z)$, $K\in\mathscr{B}_1$, is equal to the value of some polynomial with real coefficients in the variable $\log\lambda$.

Proof. 1. We follow the proof from [29, Subsection 3.6]. Using (A.1), it follows that:

$$\begin{aligned}\boldsymbol{\ell}_1(\lambda z)&=-\frac{1}{\log\lambda+\log z}\\&=-\frac{1}{\log z\big(1+\frac{\log\lambda}{\log z}\big)}\\&=\boldsymbol{\ell}_1\cdot\frac{1}{1-\log\lambda\cdot\boldsymbol{\ell}_1}\\&=\boldsymbol{\ell}_1\cdot\Big(\sum_{i=0}^{\infty}(\log\lambda\cdot\boldsymbol{\ell}_1)^i\Big)\\&=\sum_{i=0}^{\infty}(\log\lambda)^i\boldsymbol{\ell}_1^{i+1}\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\ell}_1^{n_1}(\lambda z)&=\Big(\boldsymbol{\ell}_1+\sum_{i=1}^{\infty}(\log\lambda)^i\boldsymbol{\ell}_1^{i+1}\Big)^{n_1}\\&=\boldsymbol{\ell}_1^{n_1}\cdot\Big(1+\sum_{i=1}^{\infty}(\log\lambda)^i\boldsymbol{\ell}_1^{i}\Big)^{n_1}\\&=\boldsymbol{\ell}_1^{n_1}\cdot\sum_{j=0}^{n_1}\binom{n_1}{j}\Big(\sum_{i=1}^{\infty}(\log\lambda)^i\boldsymbol{\ell}_1^{i}\Big)^{j}\\&=\boldsymbol{\ell}_1^{n_1}+n_1\log\lambda\cdot\boldsymbol{\ell}_1^{n_1+1}+\text{h.o.b.}(\boldsymbol{\ell}_1)\\&=\boldsymbol{\ell}_1^{n_1}+\log\lambda\cdot D_1(\boldsymbol{\ell}_1^{n_1})+\text{h.o.b.}(\boldsymbol{\ell}_1),\quad n_1\in\mathbb{Z}.\end{aligned}\tag{A.21}$$

Similarly, we get that:

$$\begin{aligned}
\boldsymbol{\ell}_2(\lambda z) &= -\frac{1}{\log\left(\boldsymbol{\ell}_1\cdot(1+\sum_{i=1}^{\infty}(\log\lambda\cdot\boldsymbol{\ell}_1)^i)\right)}\\
&= -\frac{1}{\log(\boldsymbol{\ell}_1)+\log\left(1+\sum_{i=1}^{\infty}(\log\lambda\cdot\boldsymbol{\ell}_1)^i\right)}\\
&= \boldsymbol{\ell}_2\cdot\frac{1}{1+\frac{\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}(\sum_{i=1}^{\infty}(\log\lambda\cdot\boldsymbol{\ell}_1)^i)^j}{\log(\boldsymbol{\ell}_1)}}\\
&= \boldsymbol{\ell}_2\cdot\frac{1}{1-((\log\lambda)\boldsymbol{\ell}_1+\text{h.o.b.}(\boldsymbol{\ell}_1))\boldsymbol{\ell}_2}\\
&= \sum_{i=0}^{\infty}\left((\log\lambda)\boldsymbol{\ell}_1+\text{h.o.b.}(\boldsymbol{\ell}_1)\right)^i\boldsymbol{\ell}_2^{i+1}\\
&= \boldsymbol{\ell}_2+\log\lambda\cdot\boldsymbol{\ell}_1\boldsymbol{\ell}_2^2+\text{h.o.b.}(\boldsymbol{\ell}_1)
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\ell}_2^{n_2}(\lambda z) &= \boldsymbol{\ell}_2^{n_2}+n_2\log\lambda\cdot\boldsymbol{\ell}_1\boldsymbol{\ell}_2^{n_2+1}+\text{h.o.b.}(\boldsymbol{\ell}_1)\\
&= \boldsymbol{\ell}_2^{n_2}+\log\lambda\cdot D_1\boldsymbol{\ell}_2^{n_2}+\text{h.o.b.}(\boldsymbol{\ell}_1),\quad n_2\in\mathbb{Z}.
\end{aligned}$$

Inductively, it follows that:

$$\begin{aligned}
\boldsymbol{\ell}_m^{n_m}(\lambda z) &= \boldsymbol{\ell}_m^{n_m}+n_m\log\lambda\cdot\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{m-1}\boldsymbol{\ell}_m^{n_m+1}+\text{h.o.b.}(\boldsymbol{\ell}_1)\\
&= \boldsymbol{\ell}_m^{n_m}+\log\lambda\cdot D_1(\boldsymbol{\ell}_m^{n_m})+\text{h.o.b.}(\boldsymbol{\ell}_1),\quad n_m\in\mathbb{Z},\, 1\le m\le k.
\end{aligned}$$

Therefore, by the Newton-Leibnitz rule, it follows that:

$$\begin{aligned}
(az^{\alpha}\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k})(\lambda z) &= a\lambda^{\alpha}z^{\alpha}(\boldsymbol{\ell}_1^{n_1}(\lambda z)\cdots\boldsymbol{\ell}_k^{n_k}(\lambda z))\\
&= a\lambda^{\alpha}z^{\alpha}\left(\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}+\log\lambda\cdot D_1(\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k})+\text{h.o.b}(\boldsymbol{\ell}_1)\right),\qquad \text{(A.22)}
\end{aligned}$$

for $(n_1,\dots,n_k)\in\mathbb{Z}^k$. By the superlinearity of the composition by $\lambda\cdot z$, we get that $z^{\alpha}K(\lambda z)=\lambda^{\alpha}z^{\alpha}\cdot(I+\log\lambda\cdot D_1)(K)+\mathscr{C}_{\lambda}(K)$, where $\mathscr{C}_{\lambda}(K):=K(\lambda z)-\lambda^{\alpha}\cdot(I+\log\lambda\cdot D_1)(K)$, for every $K\in\mathscr{B}_1$. Since the composition by $\lambda\cdot\mathrm{id}$ and the operator $I+\log\lambda\cdot D_1$ are superlinear, it follows that $\mathscr{C}_{\lambda}$ is a superlinear operator on $\mathscr{B}_1$. From (A.22) and linearity, it follows that $\mathscr{C}_{\lambda}:\mathscr{B}_1\to\mathscr{B}_1$ is a superlinear $\frac{1}{4}$-contraction on the space $(\mathscr{B}_1,d_1)$.

Statements 2 and 3 follow directly by induction using (A.21) and the Neumann Lemma (Theorem 1.1.2). ■

Lemma A.3.2 (Composition with z^β)**.** Let $k \in \mathbb{N}_{\geq 1}$, $\alpha \in \mathbb{R}_{\geq 0}$, $\beta \in \mathbb{R}_{>0}$, $\beta \neq 1$, $n_1 \in \mathbb{Z}$, and let $K \in \mathscr{B}_2 \subseteq \mathscr{L}_k^\infty$ such that $\operatorname{ord}(z^\alpha \boldsymbol{\ell}_1^{n_1} K) > \mathbf{0}_{k+1}$. Then:

1.
$$\big(z^\alpha \boldsymbol{\ell}_1^{n_1} K\big)(z^\beta) = z^{\alpha\beta} \frac{1}{\beta^{n_1}} \boldsymbol{\ell}_1^{n_1} \Big(\big(I - \log(\beta) \cdot D_2 + \mathscr{K}_\beta\big)(K)\Big), \tag{A.23}$$
where $\mathscr{K}_\beta : \mathscr{B}_2 \to \mathscr{B}_2$ is a superlinear $\frac{1}{4}$-contraction and $I : \mathscr{B}_2 \to \mathscr{B}_2$ an identity operator on the space $(\mathscr{B}_2, d_2)$.

2. The support $\operatorname{Supp}\big((z^\alpha \boldsymbol{\ell}_1^{n_1} K) \circ z^\beta\big)$ is contained in the sub-semigroup of $\mathbb{R} \times \mathbb{Z}^k$ generated by elements of $\operatorname{Supp}(z^\alpha \boldsymbol{\ell}_1^{n_1} K)$ and $(0,1,0,\dots,0)_{k+1}, \dots, (0,0,\dots,0,1)_{k+1}$.

3. Every coefficient of $(z^\alpha \boldsymbol{\ell}_1^{n_1} K) \circ z^\beta$ is equal to the value of some polynomial with real coefficients in variables $\frac{1}{\beta}$ and $\log\beta$.

Proof. 1. Note that $\boldsymbol{\ell}_1^{n_1}(z^\beta) = (-\frac{1}{\beta \log z})^{n_1} = \frac{1}{\beta^{n_1}} \boldsymbol{\ell}_1^{n_1}$, for $\beta \in \mathbb{R}_{>0} \setminus \{1\}$ and $n_1 \in \mathbb{Z}$. Let $(n_2, \dots, n_k) \in \mathbb{Z}^{k-1}$. Recall from Lemma A.3.1 that

$$\big(\boldsymbol{\ell}_2^{n_2} \cdots \boldsymbol{\ell}_k^{n_k}\big)(\lambda z) = \boldsymbol{\ell}_2^{n_2} \cdots \boldsymbol{\ell}_k^{n_k} + \log\lambda \cdot D_1(\boldsymbol{\ell}_2^{n_2} \cdots \boldsymbol{\ell}_k^{n_k}) + \mathscr{C}_\lambda(\boldsymbol{\ell}_2^{n_2} \cdots \boldsymbol{\ell}_k^{n_k}),$$

for $\lambda > 0$, $\lambda \neq 1$, and a superlinear $\frac{1}{4}$-contraction $\mathscr{C}_\lambda : \mathscr{B}_1 \to \mathscr{B}_1$ on the space $(\mathscr{B}_1, d_1)$. Note that $\frac{1}{\beta} > 0$ and $\frac{1}{\beta} \neq 1$. Furthermore, note that:

$$\begin{aligned} &\boldsymbol{\ell}_2^{n_2} \cdots \boldsymbol{\ell}_k^{n_k} = \big(\boldsymbol{\ell}_1^{n_2} \cdots \boldsymbol{\ell}_{k-1}^{n_k}\big) \circ \boldsymbol{\ell}_1, \\ &\big(D_1(\boldsymbol{\ell}_1^{n_2} \cdots \boldsymbol{\ell}_{k-1}^{n_k})\big) \circ \boldsymbol{\ell}_1 = D_2(\boldsymbol{\ell}_2^{n_2} \cdots \boldsymbol{\ell}_k^{n_k}). \end{aligned}$$

The last identity follows directly by Lemma A.2.10. Now we have:

$$\begin{aligned} \big(\boldsymbol{\ell}_2^{n_2} \cdots \boldsymbol{\ell}_k^{n_k}\big)(z^\beta) &= \boldsymbol{\ell}_1^{n_2} \cdots \boldsymbol{\ell}_{k-1}^{n_k} \circ \boldsymbol{\ell}_1(z^\beta) \\ &= \boldsymbol{\ell}_1^{n_2} \cdots \boldsymbol{\ell}_{k-1}^{n_k} \circ \big(\frac{1}{\beta} \boldsymbol{\ell}_1\big) \\ &= \boldsymbol{\ell}_1^{n_2} \cdots \boldsymbol{\ell}_{k-1}^{n_k} \circ \big(\frac{1}{\beta} z\big) \circ \boldsymbol{\ell}_1 \\ &= \Big(\boldsymbol{\ell}_1^{n_2} \cdots \boldsymbol{\ell}_{k-1}^{n_k} - \log(\beta) \cdot D_1(\boldsymbol{\ell}_1^{n_2} \cdots \boldsymbol{\ell}_{k-1}^{n_k}) + \mathscr{C}_{\frac{1}{\beta}}(\boldsymbol{\ell}_1^{n_2} \cdots \boldsymbol{\ell}_{k-1}^{n_k})\Big) \circ \boldsymbol{\ell}_1 \\ &= \boldsymbol{\ell}_2^{n_2} \cdots \boldsymbol{\ell}_k^{n_k} - \log(\beta) \cdot D_2(\boldsymbol{\ell}_2^{n_2} \cdots \boldsymbol{\ell}_k^{n_k}) + \big(\mathscr{C}_{\frac{1}{\beta}}(\boldsymbol{\ell}_1^{n_2} \cdots \boldsymbol{\ell}_{k-1}^{n_k})\big) \circ \boldsymbol{\ell}_1. \end{aligned}$$

Put $\mathscr{K}_\beta(G) := \big(\mathscr{C}_{\frac{1}{\beta}}(G\circ \exp\big(-\frac{1}{z}\big)\big)\circ \boldsymbol{\ell}_1,\ G\in\mathscr{B}_2$. Note that $\exp\big(-\frac{1}{z}\big)$ is the formal inverse of $\boldsymbol{\ell}_1$ and $G\circ\exp\big(-\frac{1}{z}\big)\in\mathscr{B}_1\subseteq\mathscr{L}_k^\infty$, for $G\in\mathscr{B}_2$. Note that $\mathscr{K}_\beta$ is a superlinear $\frac{1}{4}$-contraction on the space $(\mathscr{B}_2,d_2)$. Now, we get the identity:

$$\big(\boldsymbol{\ell}_1^{n_1}\boldsymbol{\ell}_2^{n_2}\cdots\boldsymbol{\ell}_k^{n_k}\big)(z^\beta)=\frac{1}{\beta^{n_1}}\boldsymbol{\ell}_1^{n_1}\Big(\boldsymbol{\ell}_2^{n_2}\cdots\boldsymbol{\ell}_k^{n_k}-\log(\beta)\cdot D_2(\boldsymbol{\ell}_2^{n_2}\cdots\boldsymbol{\ell}_k^{n_k})+\mathscr{K}_\beta(\boldsymbol{\ell}_2^{n_2}\cdots\boldsymbol{\ell}_k^{n_k})\Big).$$

The statement follows directly from the last identity and superlinearity of I, D_2 and $\mathscr{K}_\beta$.

Statements 2 and 3 follow directly from Lemma A.3.2. ■

Lemma A.3.3. Let $k\in\mathbb{N}_{\geq1}$, $\alpha\in\mathbb{R}_{>0}$, $\alpha\neq1$, $\lambda\in\mathbb{R}_{>0}$, and let $K\in\mathscr{B}_1\subseteq\mathscr{L}_k^\infty$ and $G\in\mathscr{B}_{\geq1}^+\subseteq\mathscr{L}_k$. Then:

1.
$$\begin{aligned}\sum_{i\geq1}\frac{(zK)^{(i)}(\lambda z)}{i!}(zG)^i &= zKG+\log\lambda\cdot zD_1(K)\cdot G\\ &\quad+\lambda zD_1(K)\cdot\left(\frac{G}{\lambda}+\sum_{i\geq2}\frac{(-1)^{i-2}}{(i-1)i}\Big(\frac{G}{\lambda}\Big)^i\right)+z\mathscr{C}(K,G)\\ &= zKG+\log\lambda\cdot zD_1(K)\cdot G\\ &\quad+\lambda z\Big(1+\frac{G}{\lambda}\Big)\log\Big(1+\frac{G}{\lambda}\Big)D_1(K)+z\mathscr{C}(K,G), \qquad \text{(A.24)}\end{aligned}$$

where $\mathscr{C}:\mathscr{B}_1\times\mathscr{B}_{\geq1}^+\to\mathscr{B}_1$ is a $\Big(\frac{1}{2^{2+2\mathrm{ord}_{\ell_1}(G)}}\frac{1}{2^{2+\mathrm{ord}_{\ell_1}(K)}}\Big)$-Lipschitz map, with respect to the metric d_1,

2.
$$\begin{aligned}\sum_{i\geq1}\frac{(z^\alpha K)^{(i)}(\lambda z)}{i!}(zG)^i &= \lambda^\alpha z^\alpha K\cdot\sum_{i\geq1}\binom{\alpha}{i}\Big(\frac{G}{\lambda}\Big)^i+z^\alpha\mathscr{K}(K,G)\\ &= \lambda^\alpha z^\alpha\Big(\Big(1+\frac{G}{\lambda}\Big)^\alpha-\frac{G}{\lambda}\Big)+z^\alpha\mathscr{K}(K,G), \qquad \text{(A.25)}\end{aligned}$$

where $\mathscr{K}:\mathscr{B}_1\times\mathscr{B}_{\geq1}^+\to\mathscr{B}_1$ is a $\Big(\frac{1}{2^{1+\mathrm{ord}_{\ell_1}(G)}},\frac{1}{2^{1+\mathrm{ord}_{\ell_1}(K)}}\Big)$-Lipschitz map, with respect to the metric d_1.

Proof. Directly by Lemma A.2.8 and Lemma A.3.1. ■

Lemma A.3.4. Let $k\in\mathbb{N}_{\geq1}$, $\alpha\in\mathbb{R}_{>0}$, $\alpha\neq1$, and let $K,G\in\mathscr{B}_{\geq1}^+\subseteq\mathscr{L}_k$. Then:

$$\begin{aligned}\sum_{i\geq1}\frac{(zK)^{(i)}(z^\alpha)}{i!}(z^\alpha G)^i &= z^\alpha\cdot(K\circ z^\alpha)\cdot G+z^\alpha\cdot(D_1(K)\circ z^\alpha)\cdot\Big(G+\sum_{i\geq2}\frac{(-1)^{i-2}}{(i-1)i}G^i\Big)+z^\alpha\mathscr{C}(K,G)\\ &= z^\alpha\cdot(K\circ z^\alpha)\cdot G+z^\alpha\cdot\Big(1+G\Big)\log\Big(1+G\Big)\cdot(D_1(K)\circ z^\alpha)+z^\alpha\mathscr{C}(K,G),\end{aligned}$$
(A.26)

where $\mathscr{C}:\mathscr{B}_{\geq 1}^{+}\times\mathscr{B}_{\geq 1}^{+}\to\mathscr{B}_1$ is a $\left(\frac{1}{2^{2+2\mathrm{ord}_{\ell_1}(G)}},\frac{1}{2^{2+\mathrm{ord}_{\ell_1}(K)}}\right)$-Lipschitz map, with respect to the metric d_1.

Proof. Directly by Lemma A.2.8 and Lemma A.3.2. ■

In the next lemma we use Lemmas A.3.1 and A.3.2 to control the support of a composition of logarithmic transseries.

Lemma A.3.5 (Control of the support of a composition)**.** Let $f\in\mathfrak{L}$ and $g\in\mathfrak{L}^H$. Let $r\in\mathbb{N}$ be minimal such that $f,g\in\mathscr{L}_r$. Let $g:=\lambda z^{\alpha}+g_1$, for $\alpha,\lambda>0$, and $g_1\in\mathscr{L}_r$ such that $\mathrm{ord}\,(g_1)>(\alpha,\mathbf{0}_r)$. Then the support $\mathrm{Supp}\,(f\circ g)$ is contained in the sub-semigroup of $\mathbb{R}_{\geq 0}\times\mathbb{Z}^r$ generated by the elements:

$$\begin{aligned}&(0,1,0,\dots,0)_{r+1},\dots,(0,0,\dots,0,1)_{r+1},\\&(\alpha\delta,\mathbf{m}),\ \text{for each } (\delta,\mathbf{m})\in\mathrm{Supp}\,(f),\\&(\beta-\alpha,\mathbf{n}),\ \text{for each } (\beta,\mathbf{n})\in\mathrm{Supp}\,(g_1).\end{aligned}$$

Proof. By the definition of a composition we have:

$$f\circ g=f(\lambda z^{\alpha})+\sum_{i\geq 1}\frac{f^{(i)}(\lambda z^{\alpha})}{i!}g_1^i. \tag{A.27}$$

By Lemma A.3.1, 2., and Lemma A.3.2, 2., it follows that every element $(\rho,\mathbf{r})$ of the support of $f^{(i)}(\lambda z^{\alpha})$, $i\in\mathbb{N}_{\geq 1}$, can be obtained as:

$$(\rho,\mathbf{r})=((\delta-i)\alpha,\mathbf{v})+(0,\mathbf{u}), \tag{A.28}$$

where $(\delta,\mathbf{v})\in\mathrm{Supp}\,(f)$ and $(0,\mathbf{u})$ is linear combination (with coefficients in $\mathbb{N}_{\geq 1}$) of elements $(0,1,0,\dots,0)_{r+1},\dots,(0,0,\dots,0,1)_{r+1}$.

From (A.28) we conclude that every element $(\gamma,\mathbf{m})$ of the support of the sum on the right-hand side of (A.27) can be obtained in the following way:

$$\begin{aligned}(\gamma,\mathbf{m})&=((\delta-i)\alpha,\mathbf{v})+(0,\mathbf{u})+(\beta_1,\mathbf{n}_1)+\cdots+(\beta_i,\mathbf{n}_i)\\&=(\delta\alpha,\mathbf{v})+(0,\mathbf{u})+(\beta_1-\alpha,\mathbf{n}_1)+\cdots+(\beta_i-\alpha,\mathbf{n}_i),\end{aligned}$$

where $i\in\mathbb{N}$, $(\delta,\mathbf{v})\in\mathrm{Supp}\,(f)$, $(0,\mathbf{u})$ is a linear combination (with coefficients in $\mathbb{N}_{\geq 1}$) of $(0,1,0,\dots,0)_{r+1},\dots,(0,0,\dots,0,1)_{r+1}$, and $(\beta_j,\mathbf{n}_j)\in\mathrm{Supp}\,(g_1)$, $1\leq j\leq i$. ■

B. Various differential equations in $\mathfrak{L}^\infty$ and $\mathscr{B}_m$

The main objective in this chapter is to solve linear differential equations on differential algebras $\mathfrak{L}^\infty$ and $\mathscr{B}_m \subseteq \mathscr{L}_k^\infty$, for $1 \leq m \leq k$, $k \in \mathbb{N}_{\geq 1}$, and various types of nonlinear differential equations on $\mathscr{B}_m$. These results are important for solving normalization equations in Sections 2.1, 2.2 and 2.3, using the fixed point theorem stated in Proposition 1.2.12.

In Section B.3 and Section B.4 we give explicit formulas for solutions of linear differential equations on $\mathfrak{L}^\infty$ and $\mathscr{B}_m$ respectively. They are then used to solve nonlinear differential equations in Section B.5. As a prerequisite for solving linear differential equations, in Section B.1 we define exponential and logarithmic operators, and in Section B.2 we consider the stability of spaces $\mathscr{L}_k^\infty$ and $\mathscr{L}_k$, $k \in \mathbb{N}$, under integration.

B.1. Exponential and logarithmic operators

Motivated by [29, Lemma 4.5] we define exponential and logarithmic operators on spaces of logarithmic transseries.

Proposition B.1.1 (Composition of a power series and a logarithmic transseries)**.** Let $h := \sum_{i=0}^\infty h_i x^i$ be a formal power series in the variable x with coefficients $h_i \in \mathscr{L}_m^\infty$, $i \in \mathbb{N}$, $m \in \mathbb{N}$, such that $\operatorname{ord}(h_i) \geq \mathbf{0}_{m+1}$, for each $i \in \mathbb{N}$. Let $f \in \mathfrak{L}$. Then the series $h(f) := \sum_{i=0}^\infty h_i \cdot f^i$ converges in the product topology on $\mathfrak{L}^\infty$. Moreover, if $\operatorname{ord}_z(f) \geq 1$, then series $h(f)$ converges in the power-metric topology on $\mathfrak{L}^\infty$.

Proof. Since $f \in \mathfrak{L}$, there exists $k \in \mathbb{N}$, such that $k \geq m$ and $f \in \mathscr{L}_k$. The statement follows directly by the Neumann Lemma (Theorem 1.1.2) since $\operatorname{ord}(f) > \mathbf{0}_{k+1}$ and $\operatorname{ord}(h_i) \geq \mathbf{0}_{k+1}$. Indeed, by the Neumann Lemma, every coefficient in $h(f)$ can be obtained as a finite sum of products of lower order coefficients. Therefore, $(h_i \cdot f^i)_{i\in\mathbb{N}}$ is a summable family. By Remark 1.1.18, the series converges in the product topology in $\mathscr{L}_k^\infty$, and, therefore, in $\mathfrak{L}^\infty$.

Now, suppose that $\operatorname{ord}_z(f) \geq 1$. Then $\operatorname{ord}_z(h_i \cdot f^i) \geq i$, for every $i \in \mathbb{N}$. By Remark 1.1.17, it follows that $\sum_{i=0}^\infty h_i \cdot f^i$ converges in $\mathscr{L}_k^\infty$ and therefore, in $\mathfrak{L}^\infty$, with respect to the power-metric topology. ■

Definition B.1.2 (Exponential operator, [29])**.** Let $\exp : \mathfrak{L} \to \mathfrak{L}$ be an operator defined by

$$\exp(f) := \sum_{i=0}^{+\infty} \frac{f^i}{i!},$$

for $f \in \mathfrak{L}$. We call exp the *exponential operator*.

By Proposition B.1.1, it follows that the exponential operator is well defined. We often use the following notation $\mathrm{e}^f := \exp(f)$, for $f \in \mathfrak{L}$. By convention, we define $\exp(\boldsymbol{\ell}_k^{-1}) := \boldsymbol{\ell}_{k-1}^{-1}$, for $k \in \mathbb{N}_{\geq 1}$. The definition of the exponential operator can be extended from $\mathfrak{L}$ to a particular subset of $\mathfrak{L}^\infty$ by putting:

$$\exp(\alpha\boldsymbol{\ell}_1^{-1} + n_1\boldsymbol{\ell}_2^{-1} \ldots + n_{k-1}\boldsymbol{\ell}_k^{-1} + c + f) := \exp(c) \cdot z^{-\alpha}\boldsymbol{\ell}_1^{-n_1} \cdots \boldsymbol{\ell}_{k-1}^{-n_{k-1}} \cdot \exp(f),$$

for $f \in \mathfrak{L}$, $\alpha \in \mathbb{R}$, $n_1, \ldots, n_{k-1} \in \mathbb{Z}$, $c \in \mathbb{R}$, and $k \in \mathbb{N}_{\geq 1}$.

Let us now define the *logarithmic operator*. Let

$$\log(1+f) := \sum_{i=1}^{+\infty} \frac{(-1)^{i-1}}{i} f^i, \tag{B.1}$$

for every $f \in \mathfrak{L}$. By Proposition B.1.1, it follows that $\log(1+f)$ is a well-defined element of $\mathfrak{L}$.

Now we define $\log(f)$ for every $f \in \mathfrak{L}^\infty$. First, by convention, for every $az^\alpha\boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} \in \mathscr{L}_k^\infty$, $a \in \mathbb{R} \setminus \{0\}$, and $k \in \mathbb{N}$, we define:

$$\log(az^\alpha\boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k}) := \log a + (-\alpha)\boldsymbol{\ell}_1^{-1} + (-n_1)\boldsymbol{\ell}_2^{-1} + \cdots + (-n_k)\boldsymbol{\ell}_{k+1}^{-1}. \tag{B.2}$$

Note that $\log a$ is in $\mathbb{C}\setminus\mathbb{R}$ if $a<0$. Now, we extend the definition of the logarithmic operator to whole $\mathfrak{L}^\infty$, by writing $f=\mathrm{Lt}(f)\cdot\left(1+\frac{f-\mathrm{Lt}(f)}{\mathrm{Lt}(f)}\right)$, where $f\in\mathfrak{L}^\infty$ and $\frac{f-\mathrm{Lt}(f)}{\mathrm{Lt}(f)}$ obviously does not contain a constant term. Using (B.1) and (B.2), we define:

$$\log(f):=\log(\mathrm{Lt}(f))+\sum_{i=1}^{+\infty}\frac{(-1)^{i-1}}{i}\left(\frac{f-\mathrm{Lt}(f)}{\mathrm{Lt}(f)}\right)^i.$$

Proposition B.1.3. For every $f\in\mathfrak{L}^\infty$ it follows that $\exp(\log(f))=f$.

Proof. By straightforward calculation and regrouping of terms we get that $\exp(\log(1+f))=1+f$, for every $f\in\mathfrak{L}$. Let $f\in\mathfrak{L}^\infty$ and let $\mathrm{Lt}(f):=az^\alpha\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}$, for $a\in\mathbb{R}\setminus\{0\}$ and $(\alpha,n_1,\dots,n_k)\in\mathbb{R}\times\mathbb{Z}^k$, $k\in\mathbb{N}$. Note that:

$$\begin{aligned}\exp(\log(\mathrm{Lt}(f)))&=\exp(\log(a))\exp(\log(z^\alpha))\cdots\exp(\log(\boldsymbol{\ell}_k^{n_k}))\\&=az^\alpha\boldsymbol{\ell}_1^{n_1}\cdots\boldsymbol{\ell}_k^{n_k}\\&=\mathrm{Lt}(f).\end{aligned}$$

Consequently, it follows that:

$$\begin{aligned}\exp(\log(f))&=\exp\left(\log\left(\mathrm{Lt}(f)+\log\left(1+\frac{f-\mathrm{Lt}(f)}{\mathrm{Lt}(f)}\right)\right)\right)\\&=\exp\big(\log(\mathrm{Lt}(f))\big)\cdot\exp\left(\log\left(1+\frac{f-\mathrm{Lt}(f)}{\mathrm{Lt}(f)}\right)\right)\\&=\mathrm{Lt}(f)\cdot\left(1+\frac{f-\mathrm{Lt}(f)}{\mathrm{Lt}(f)}\right)=f.\end{aligned}$$

■

Proposition B.1.4 (Properties of derivations of exponential and logarithmic operators)**.**

1. For $f\in\mathfrak{L}$, $c\in\mathbb{R}$, $\alpha\in\mathbb{R}$, $n_1,\dots,n_{k-1}\in\mathbb{Z}$, and $k\in\mathbb{N}_{\geq1}$, it follows that

$$\begin{aligned}&\frac{d}{dz}\Big(\exp(\alpha\boldsymbol{\ell}_1^{-1}+n_1\boldsymbol{\ell}_2^{-1}\ldots+n_{k-1}\boldsymbol{\ell}_k^{-1}+c+f)\Big)\\&=\exp(\alpha\boldsymbol{\ell}_1^{-1}+n_1\boldsymbol{\ell}_2^{-1}\ldots+n_{k-1}\boldsymbol{\ell}_k^{-1}+c+f)\cdot\frac{d}{dz}(\alpha\boldsymbol{\ell}_1^{-1}+n_1\boldsymbol{\ell}_2^{-1}\ldots+n_{k-1}\boldsymbol{\ell}_k^{-1}+c+f).\end{aligned}$$

2. For $f\in\mathfrak{L}^\infty$ it follows that $\frac{d}{dz}(\log(f))=\frac{\frac{df}{dz}}{f}$.

Proof. 1. Directly from the definition of $\exp(f)$, it follows that $\exp(f) = \exp(f) \cdot \frac{df}{dz}$. Note that:

$$
\begin{aligned}
&\frac{d}{dz}\Big(\exp(\alpha\boldsymbol{\ell}_1^{-1} + n_1\boldsymbol{\ell}_2^{-1} \ldots + n_{k-1}\boldsymbol{\ell}_k^{-1})\Big)\\
&= \frac{d}{dz}(z^{-\alpha}\boldsymbol{\ell}_1^{-n_1} \cdots \boldsymbol{\ell}_{k-1}^{-n_{k-1}})\\
&= z^{-\alpha-1}\Big(-\alpha\boldsymbol{\ell}_1^{-n_1} \cdots \boldsymbol{\ell}_k^{-n_{k-1}} - \sum_{i=1}^{k-1} n_i \boldsymbol{\ell}_1^{-n_1+1} \ldots \boldsymbol{\ell}_i^{-n_i+1}\boldsymbol{\ell}_{i+1}^{-n_{i+1}} \cdots \boldsymbol{\ell}_{k-1}^{-n_{k-1}}\Big)\\
&= -z^{-\alpha}\boldsymbol{\ell}_1^{-n_1} \cdots \boldsymbol{\ell}_{k-1}^{-n_{k-1}} \cdot \Big(\alpha z^{-1} + \sum_{i=1}^{k-1} n_i \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_i\Big)\\
&= z^{-\alpha}\boldsymbol{\ell}_1^{-n_1} \cdots \boldsymbol{\ell}_{k-1}^{-n_{k-1}} \cdot \frac{d}{dz}\Big(\alpha\boldsymbol{\ell}_1^{-1} + \sum_{i=1}^{k-1} n_i \boldsymbol{\ell}_{i+1}^{-1}\Big)\\
&= \exp(\alpha\boldsymbol{\ell}_1^{-1} + n_1\boldsymbol{\ell}_2^{-1} \ldots + n_{k-1}\boldsymbol{\ell}_k^{-1}) \cdot \frac{d}{dz}(\alpha\boldsymbol{\ell}_1^{-1} + n_1\boldsymbol{\ell}_2^{-1} \ldots + n_{k-1}\boldsymbol{\ell}_k^{-1}).
\end{aligned} \tag{B.3}
$$

By the extended definition of the exponential operator and by (B.3), we have:

$$
\begin{aligned}
&\frac{d}{dz}\Big(\exp(\alpha\boldsymbol{\ell}_1^{-1} + n_1\boldsymbol{\ell}_2^{-1} \ldots + n_{k-1}\boldsymbol{\ell}_k^{-1} + c + f)\Big)\\
&= \frac{d}{dz}\Big(\exp(c) \cdot \exp(\alpha\boldsymbol{\ell}_1^{-1} + n_1\boldsymbol{\ell}_2^{-1} \ldots + n_{k-1}\boldsymbol{\ell}_k^{-1}) \cdot \exp(f)\Big)\\
&= \exp(f) \cdot \exp(c) \cdot \frac{d}{dz}\Big(\exp(\alpha\boldsymbol{\ell}_1^{-1} + n_1\boldsymbol{\ell}_2^{-1} \ldots + n_{k-1}\boldsymbol{\ell}_k^{-1})\Big) +\\
&+ \exp(c) \cdot \exp(\alpha\boldsymbol{\ell}_1^{-1} + n_1\boldsymbol{\ell}_2^{-1} \ldots + n_{k-1}\boldsymbol{\ell}_k^{-1}) \cdot \exp(f) \cdot \frac{df}{dz}\\
&= \exp(\alpha\boldsymbol{\ell}_1^{-1} + n_1\boldsymbol{\ell}_2^{-1} \ldots + n_{k-1}\boldsymbol{\ell}_k^{-1} + c + f) \cdot \frac{d}{dz}(\alpha\boldsymbol{\ell}_1^{-1} + n_1\boldsymbol{\ell}_2^{-1} \ldots + n_{k-1}\boldsymbol{\ell}_k^{-1} + c + f).
\end{aligned}
$$

2. Suppose that $f \in \mathfrak{L}^\infty$. By Proposition B.1.3 and statement 1, it follows that:

$$
\begin{aligned}
\frac{d}{dz}\Big(\exp(\log(f))\Big) &= \frac{df}{dz}\\
\exp(\log(f)) \cdot \frac{d}{dz}(\log(f)) &= \frac{df}{dz}\\
f \cdot \frac{d}{dz}(\log(f)) &= \frac{df}{dz}\\
\frac{d}{dz}(\log(f)) &= \frac{\frac{df}{dz}}{f}.
\end{aligned}
$$

■

B.2. STABILITY OF CERTAIN DIFFERENTIAL ALGEBRAS UNDER INTEGRATION

B.2.1. Differential algebras $\mathscr{L}_k^\infty$ and $\mathscr{L}_k$

In Section 1.1, $\int f dz$, $f \in \mathfrak{L}^\infty$, is defined as the particular antiderivative of f without the constant term. The next proposition deals with the *stability* i.e., *closedness* of differential algebras $\mathscr{L}_k^\infty$, $k \in \mathbb{N}$, under *integration*.

Proposition B.2.1 (Stability of the differential algebra $\mathscr{L}_k^\infty$ under integration, Lemma 4.4, [29])**.**

1. $\displaystyle\int z^{-1}\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_k \, dz = -\boldsymbol{\ell}_{k+1}^{-1}$, for $k \in \mathbb{N}$.
2. For $k \in \mathbb{N}$ and $(\delta, m_1, \ldots, m_k) \in \mathbb{R} \times \mathbb{Z}^k$ such that $(\delta, m_1, \ldots, m_k) \neq (-1, \mathbf{1}_k)$, we have:
$$\int z^\delta \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_k^{m_k} \, dz \in \mathscr{L}_k^\infty.$$

Proof. We prove 1 by direct computation of the derivative of $-\boldsymbol{\ell}_{k+1}^{-1}$.

Let us now prove 2. When $\delta \neq -1$, firstly we notice that the series

$$G := \sum_{i=0}^{\infty} \Big(\frac{-1}{\delta+1}\Big)^i D_1^i(\boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_k^{m_k})$$

converges in the complete space $(\mathscr{B}_1, d_1)$, where D_1 is a derivation defined in (1.5).

Indeed, D_1 is a $\frac{1}{2}$-contraction on $\mathscr{B}_1$ (see Example 1.2.6). Secondly, using Lemma A.2.6 we prove that $\frac{z^{\delta+1}}{\delta+1} G$ is an antiderivative of $z^\delta \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_k^{m_k}$, by computing its derivative term by term.

Similarly, if $\delta = -1$, suppose that $m_1 = m_2 = \cdots = m_{r-1} = 1$ and $m_r \neq 1$, for some $r \in \{1, \ldots, k\}$, and let $K := \boldsymbol{\ell}_{r+1}^{m_{r+1}} \cdots \boldsymbol{\ell}_k^{m_k}$ if $r < k$ (and $K = 1$ if $r = k$). Since D_{r+1} is a $\frac{1}{2}$-contraction on the complete space $(\mathscr{B}_{r+1}, d_{r+1})$ (by Example 1.2.6), the series $\sum_{i=0}^{\infty} \frac{(-1)^i D_{r+1}^i(K)}{(m_r-1)^i}$ converges in this space. A direct computation, using Lemma A.2.4, shows that the derivative of

$$\frac{\boldsymbol{\ell}_r^{m_r-1}}{m_r-1} \sum_{i=0}^{\infty} \frac{(-1)^i D_{r+1}^i(K)}{(m_r-1)^i} \tag{B.4}$$

is $z^{-1}\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{r-1} \ell_r^{m_r} K$. Note that (B.4) remains in $\mathscr{L}_k^\infty$. ∎

Remark B.2.2. Let $k \in \mathbb{N}$. From the proof of Proposition B.2.1 we get the following formulas which will be used in the proof of Proposition B.2.3:

1. $$\int \sum_{\delta \in A} z^\delta R_\delta \, dz = \sum_{\delta \in A} \Big(\frac{z^{\delta+1}}{\delta+1} \sum_{i=0}^{\infty} (\frac{-1}{\delta+1})^i D_1^i(R_\delta) \Big), \tag{B.5}$$

   where $A \subseteq \mathbb{R} \setminus \{-1\}$ is a well-ordered subset of $\mathbb{R} \setminus \{-1\}$, and $R_\delta \in \mathscr{B}_1 \subseteq \mathscr{L}_k^\infty$, for $\delta \in A$.

2. $$\int \Big(\sum_{n \in A} z^{-1} \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{r-1} \boldsymbol{\ell}_r^n K_n \Big) dz = \sum_{n \in A} \Big(\frac{\boldsymbol{\ell}_r^{n-1}}{n-1} \sum_{i=0}^{\infty} \frac{(-1)^i D_{r+1}^i (K_n)}{(n-1)^i} \Big), \tag{B.6}$$

   where $A \subseteq \mathbb{Z} \setminus \{1\}$ is a well-ordered subset of $\mathbb{Z} \setminus \{1\}$, and $K_n \in \mathscr{B}_{r+1} \subseteq \mathscr{L}_k^\infty$, for $n \in A$.

3. $$\int z^{-1} \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_k \, dz = -\boldsymbol{\ell}_{k+1}^{-1}.$$

4. To conclude, $\int f dz \in \mathscr{L}_{k+1}^\infty$, for $f \in \mathscr{L}_k^\infty$, $k \in \mathbb{N}$. Moreover, $f \in \mathscr{L}_k^\infty$ does not contain a term of order $(-1, \mathbf{1}_k)$ if and only if $\int f dz \in \mathscr{L}_k^\infty$, $k \in \mathbb{N}$. In particular, $\mathfrak{L}^\infty$ is closed (i.e., stable) under integration.

Although $\mathscr{L}_k$, $k \in \mathbb{N}$, are not closed under integration, in the next proposition we find a subspace of $\mathscr{L}_k$ such that the integrals remain in $\mathscr{L}_k$.

Proposition B.2.3 (Stability of the differential algebra $\mathscr{L}_k$ under integration)**.** Let $f \in \mathscr{L}_k$, $k \in \mathbb{N}$, such that $\operatorname{ord}(f) > (-1, \mathbf{1}_k)$. Then $\int f dz$ belongs to $\mathscr{L}_k$.

Proof. Let $az^\delta \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_k^{m_k}$ be any term in f, for $a \in \mathbb{R} \setminus \{0\}$ and $(\delta, m_1, \ldots, m_k) \in \mathbb{R} \times \mathbb{Z}^k$. Since $\operatorname{ord}(f) > (-1, \mathbf{1}_k)$, it follows that $(\delta, m_1, \ldots, m_k) > (-1, \mathbf{1}_k)$. If $\delta > -1$, then by (B.5) and $\delta + 1 > 0$, it follows that $\int az^\delta \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_k^{m_k} dz \in \mathscr{L}_k$. Now, suppose that $\delta = -1$. Since $(\delta, m_1, \ldots, m_k) > (-1, \mathbf{1}_k)$, there exists $r \in \mathbb{N}$ such that $1 \le r \le k$, $m_1 = \cdots = m_{r-1} = 1$, and $m_r > 1$. By (B.6) and $m_r - 1 > 0$, we conclude that $\int az^\delta \boldsymbol{\ell}_1^{m_1} \cdots \boldsymbol{\ell}_k^{m_k} dz \in \mathscr{L}_k$. ■

Remark B.2.4. Let $f \in \mathscr{L}_k$, for $k \in \mathbb{N}$. If $\operatorname{ord}(f) > (-1, \mathbf{1}_k)$, then:

$$\exp\Big(\int f dz\Big) - 1 \in \mathscr{L}_k. \tag{B.7}$$

The statement follows directly by Proposition B.2.3 and the definition of the exponential operator.

B.2.2. Differential algebras $\mathscr{B}_m$ and $\mathscr{B}^+_{\geq m}$

Recall that, in Subsection 1.1.4, we have defined $\int K \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}$, $K \in \mathscr{B}_m$, $1 \leq m \leq k$, as the particular antiderivative of K without the constant term.

Proposition B.2.5 (*Stability of $\mathscr{B}_m \subseteq \mathscr{L}_k^\infty$ under integration*)**.**

1. $\int \boldsymbol{\ell}_m \boldsymbol{\ell}_{m+1} \cdots \boldsymbol{\ell}_k \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2} = -\boldsymbol{\ell}_{k+1}^{-1}$, for $1 \leq m \leq k$ and $k \in \mathbb{N}_{\geq 1}$.
2. For $k \in \mathbb{N}_{\geq 1}$, $1 \leq m \leq k$, and $(n_m, \ldots, n_k) \in \mathbb{Z}^{k-m+1}$ such that $(n_m, \ldots, n_k) \neq \mathbf{1}_{k-m+1}$, it follows that:
$$\int \boldsymbol{\ell}_m^{n_m} \cdots \boldsymbol{\ell}_k^{n_k} \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2} \in \mathscr{B}_m \subseteq \mathscr{L}_k^\infty.$$

Proof. Note that $D_m(-\boldsymbol{\ell}_{k+1}^{-1}) = \boldsymbol{\ell}_m \cdots \boldsymbol{\ell}_k$, which implies statement 1.

Let us prove statement 2. Note that $\boldsymbol{\ell}_m^{n_m} \cdots \boldsymbol{\ell}_k^{n_k} = (z^{n_m} \boldsymbol{\ell}_1^{n_{m+1}} \cdots \boldsymbol{\ell}_{k-m}^{n_k}) \circ \boldsymbol{\ell}_m$. Now we use the substitution $t := \boldsymbol{\ell}_m$, i.e.

$$\int \boldsymbol{\ell}_m^{n_m} \cdots \boldsymbol{\ell}_k^{n_k} \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2} = \Big(\int t^{n_m-2} \boldsymbol{\ell}_1^{n_{m+1}}(t) \cdots \boldsymbol{\ell}_{k-m}^{n_k}(t) dt\Big)\Big|_{\boldsymbol{\ell}_m}.$$

Since $(n_m, \ldots, n_k) \neq \mathbf{1}_{k-m+1}$ it follows that $(n_m - 2, n_{m+1}, \ldots, n_k) \neq (-1, \mathbf{1}_{k-m})$. Now, by Proposition B.2.1, it follows that $\Big(\int t^{n_m-2} \boldsymbol{\ell}_1^{n_{m+1}}(t) \cdots \boldsymbol{\ell}_{k-m}^{n_k}(t) dt\Big)\Big|_z \in \mathscr{L}_{k-m}^\infty$. Therefore, $\Big(\int t^{n_m-2} \boldsymbol{\ell}_1^{n_{m+1}}(t) \cdots \boldsymbol{\ell}_{k-m}^{n_k}(t) dt\Big) \circ \boldsymbol{\ell}_m \in \mathscr{B}_m \subseteq \mathscr{L}_k^\infty$. ■

Proposition B.2.6 (Stability of $\mathscr{B}^+_{\geq m} \subseteq \mathscr{L}_k$ under integration)**.** Let $K \in \mathscr{B}^+_{\geq m} \subseteq \mathscr{L}_k$, $k \in \mathbb{N}_{\geq 1}$, $1 \leq m \leq k$, such that $\operatorname{ord}(K) > (\mathbf{0}_m, \mathbf{1}_{k-m+1})$. Then $\int K \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}$ is an element of $\mathscr{B}^+_{\geq m} \subseteq \mathscr{L}_k$.

Proof. Similarly as the proof of Proposition B.2.5 using Proposition B.2.3. ■

Remark B.2.7. Let $K \in \mathscr{B}_m \subseteq \mathscr{L}_k$, for $1 \leq m \leq k$ and $k \in \mathbb{N}_{\geq 1}$. If $\operatorname{ord}(K) > (\mathbf{0}_m, \mathbf{1}_{k-m+1})$, then:

$$\exp\left(\int K \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\right) - 1 \in \mathscr{B}_{\geq m}^{+} \subseteq \mathscr{L}_k. \tag{B.8}$$

It follows from Proposition B.2.6 and the definition of the exponential operator.

B.3. Linear differential equations in $\mathfrak{L}^\infty$

In this section we consider the existence and the uniqueness of solutions of the *linear differential equations* in $\mathfrak{L}^\infty$:

$$g_1 \cdot \frac{df}{dz} + g_2 \cdot f = h, \quad f \in \mathfrak{L}^\infty, \tag{B.9}$$

for given $g_1, g_2, h \in \mathfrak{L}^\infty$. Note that there exists the minimal $k \in \mathbb{N}$ such that $g_1, g_2, h \in \mathscr{L}_k^\infty$. Therefore, we assume in the sequel that $g_1, g_2, h \in \mathscr{L}_k^\infty$, for such $k \in \mathbb{N}$.

If $g_1 = 0$, then we have the equation $g_2 \cdot f = h$, and $f := \frac{h}{g_2}$ is its unique solution (if $g_2 \neq 0$). Therefore, we assume that $g_1 \neq 0$. Then equation (B.9) is equivalent to the equation:

$$\frac{df}{dz} + \frac{g_2}{g_1} \cdot f = \frac{h}{g_1}, \quad f \in \mathfrak{L}^\infty. \tag{B.10}$$

We state the theorem about the existence of a solution of linear differential equation (B.10) in $\mathfrak{L}^\infty$, which will be used throughout Sections 2.1, 2.2 and 2.3 for applying the fixed point theorem from Proposition 1.2.12.

Theorem B.3.1 (Solutions of linear differential equation (B.10))**.** Let $k \in \mathbb{N}$ and $g_1, g_2, h \in \mathscr{L}_k^\infty$, $g_1 \neq 0$. Let $\operatorname{ord}\left(\frac{g_2}{g_1}\right) > (-1, \mathbf{1}_k)$ or $g_2 = -\frac{dg_1}{dz}$. Then:

1. the space of all solutions $f \in \mathfrak{L}^\infty$ of linear differential equation (B.10) is given by $\{f_c : c \in \mathbb{R}\}$, where:

$$f_c := \exp\Big(-\int \frac{g_2}{g_1} dz\Big) \cdot \Big(c + \int \Big(\frac{h}{g_1} \cdot \exp(\int \frac{g_2}{g_1} dz)\Big) dz\Big), \quad c \in \mathbb{R}, \tag{B.11}$$

2. f_c belongs to $\mathscr{L}_{k+1}^\infty$, for each $c \in \mathbb{R}$,

3. f_c belongs to $\mathscr{L}_k^\infty$, for each $c \in \mathbb{R}$, if and only if $\frac{h}{g_1} \cdot \exp(\int \frac{g_2}{g_1} dz)$ does not contain a term of order $(-1, \mathbf{1}_k)$.

Proof. We first prove that f_c given by (B.11) belongs to $\mathscr{L}_{k+1}$, i.e., $\mathscr{L}_k$ if the condition in 3. is satisfied. Suppose that $\operatorname{ord}\left(\frac{g_2}{g_1}\right) > (-1, \mathbf{1}_k)$. By Remark B.2.4, it follows that $\exp(\int \frac{g_2}{g_1} dz) - 1 \in \mathscr{L}_k$ and $\exp(-\int \frac{g_2}{g_1} dz) - 1 \in \mathscr{L}_k$. Therefore, $\frac{h}{g_1} \cdot \exp(\int \frac{g_2}{g_1} dz) \in \mathscr{L}_k^\infty$. By Proposition B.2.1, it follows that $f_c \in \mathscr{L}_{k+1}^\infty$, $c \in \mathbb{R}$. Moreover, if $\frac{h}{g_1} \cdot \exp(\int \frac{g_2}{g_1} dz)$ does not contain a term of order $(-1,\mathbf{1}_k)$, then, by Remark B.2.2, 4., we deduce that $f_c \in \mathscr{L}_k^\infty$, $c\in\mathbb{R}$. On the other hand, if $f_c \in \mathscr{L}_k^\infty$, $c\in\mathbb{R}$, then $\int\Big(\frac{h}{g_1}\cdot\exp(\int\frac{g_2}{g_1}dz)\Big)dz$ belongs to $\mathscr{L}_k^\infty$, and, consequently, by Remark B.2.2, 4., it follows that $\frac{h}{g_1}\cdot\exp(\int\frac{g_2}{g_1}dz)$ does not contain a term of order $(-1,\mathbf{1}_k)$.

Now, suppose that $g_2=-\frac{dg_1}{dz}$. Then $\exp\Big(-\int\frac{g_2}{g_1}dz\Big)=\exp(\log(g_1))=g_1$, by Proposition B.1.3 and Proposition B.1.4, 2. Consequently, it follows that:

$$f_c = g_1\cdot\Big(c+\int\frac{h}{g_1^2}dz\Big),\quad c\in\mathbb{R}.$$

Since $\frac{h}{g_1^2}\in\mathscr{L}_k^\infty$, by Proposition B.2.1, it follows that $f_c\in\mathscr{L}_{k+1}^\infty$, $c\in\mathbb{R}$. Moreover, by Remark B.2.2, 4., $\frac{h}{g_1^2}$ does not contain a term of order $(-1,\mathbf{1}_k)$ if and only if $f_c\in\mathscr{L}_k^\infty$, $c\in\mathbb{R}$. Thus, we proved statements 2 and 3.

Now, for every $c\in\mathbb{R}$, by Proposition B.1.4 and by a straightforward computation, it follows that $\{f_c : c\in\mathbb{R}\}$ is a set of solutions of equation (B.10). On the other hand, by linearity, it follows that every solution of equation (B.10) is of the *form* $f_c = c\cdot\exp\Big(-\int\frac{g_2}{g_1}dz\Big)+f_0$, for $c\in\mathbb{R}$. This proves (B.11). ■

Remark B.3.2 (Application: the image and the kernel of the Lie bracket operator)**.** For $g,h\in\mathscr{L}_k^\infty$, $k\in\mathbb{N}$, note that $[f,g]=h$ is a linear differential equation of the form (B.9), where $g_1:=g$ and $g_2:=-\frac{dg}{dz}$. By Theorem B.3.1, it follows that

$$f_c := g\cdot\Big(c+\int\frac{h}{g^2}dz\Big),\quad c\in\mathbb{R},$$

are all solutions of the equation $[f,g]=h$ in the differential algebra $\mathfrak{L}^\infty$. Moreover, $f_c\in\mathscr{L}_{k+1}^\infty$, $c\in\mathbb{R}$, and $f_c\in\mathscr{L}_k^\infty$ if and only if $\frac{h}{g^2}$ does not contain a term of order $(-1,\mathbf{1}_k)$. Furthermore, if $h:=0$, then $f_c:=cg$, $c\in\mathbb{R}$, are all solutions of the equation $[f,g]=0$. Therefore, $\{cg : c\in\mathbb{R}\}$ is the kernel of the Lie bracket operator, for a given $g\in\mathscr{L}_k^\infty$.

B.4. LINEAR DIFFERENTIAL EQUATIONS IN $\mathscr{B}_m$

Let $k \in \mathbb{N}_{\geq 1}$ and $1 \leq m \leq k$. In this section we consider the existence and the uniqueness of solutions of the *linear differential equations* in $\mathscr{B}_m \subseteq \mathscr{L}_{k+1}^\infty$:

$$G_1 \cdot D_m(Q) + G_2 \cdot Q = H, \quad Q \in \mathscr{B}_m \subseteq \mathscr{L}_{k+1}^\infty, \tag{B.12}$$

for given $G_1, G_2, H \in \mathscr{B}_m \cap \mathscr{L}_k^\infty$. If $G_1 = 0$, then we have the equation $G_2 \cdot Q = H$, whose unique solution is $Q := \frac{H}{G_2}$ (if $G_2 \neq 0$). Therefore, we assume in the sequel that $G_1 \neq 0$. Then, equation (B.12) is equivalent to the equation:

$$D_m(Q) + \frac{G_2}{G_1} \cdot Q = \frac{H}{G_1}, \quad Q \in \mathscr{B}_m \subseteq \mathscr{L}_{k+1}^\infty. \tag{B.13}$$

The following theorem is an analogue of Theorem B.3.1.

Theorem B.4.1 (Solution of the linear differential equation (B.12))**.** Let $k \in \mathbb{N}_{\geq 1}$, $1 \leq m \leq k$, and $G_1, G_2, H \in \mathscr{B}_m \cap \mathscr{L}_k^\infty$, $G_1 \neq 0$. If $\operatorname{ord}\left(\frac{G_2}{G_1}\right) > (\mathbf{0}_m, \mathbf{1}_{k-m+1})$ or $G_2 = -D_m(G_1)$, then:

1. the space of all solutions in $\mathscr{B}_m \subseteq \mathscr{L}_{k+1}^\infty$ of the linear differential equation

   $$G_1 \cdot D_m(Q) + G_2 \cdot Q = H$$

   is given by $\{Q_c : c \in \mathbb{R}\}$, where:

   $$Q_c := \exp\left(-\int \frac{G_2}{G_1} \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\right) \cdot \left(c + \int \frac{H}{G_1} \cdot \exp\left(\int \frac{G_2}{G_1} \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\right) \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\right), \quad c \in \mathbb{R}, \tag{B.14}$$

2. $Q_c \in \mathscr{B}_m \cap \mathscr{L}_k^\infty$, for $c \in \mathbb{R}$, if and only if $\frac{H}{G_1} \cdot \exp\left(-\int \frac{G_2}{G_1} \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\right)$ does not contain a term of order $(\mathbf{0}_m, \mathbf{1}_{k-m+1})$.

Proof. Similarly as the proof of Theorem B.3.1. ■

B.5. VARIOUS NONLINEAR DIFFERENTIAL EQUATIONS IN $\mathscr{B}_m$

Let $k \in \mathbb{N}_{\geq 1}$ and $1 \leq m \leq k$. As opposed to the previous section, where we considered linear differential equations, in this section we consider particular *nonlinear differential equations* in the variable $Q \in \mathscr{B}_{\geq m}^{+} \subseteq \mathscr{L}_k$ of the *form*:

$$G_1 \cdot D_m(Q) + G_2 \cdot Q + G_3 \cdot h(Q) = H, \tag{B.15}$$

where $G_1, G_2, G_3, H \in \mathscr{B}_m \subseteq \mathscr{L}_k^\infty$, and $h \in x^2\mathbb{R}[[x]]$ is a power series with real coefficients such that $h(0) = 0$ and $h'(0) = 0$. We consider solutions $Q \in \mathscr{B}_{\geq m}^{+} \subseteq \mathscr{B}_m$ since, by Proposition B.1.1, $h(Q)$ is convergent for $Q \in \mathscr{B}_{\geq m}^{+}$, but not in general for $Q \in \mathscr{B}_m$.

We solve here the three types of equation (B.15) which depend on the form of G_i, $i = 1,2,3$, and H. These types are explicitly given and proven in Proposition B.5.1, Proposition B.5.4 and Proposition B.5.7. The main strategy of the proof of each proposition is to solve the equation inductively. The inductive step is proven in auxiliary lemmas, using the fixed point theorem from Proposition 1.2.12.

B.5.1. Type I

Let us first consider the equation (B.15) for $G_1 \neq 0$. Then we have the equivalent equation:

$$D_m(Q) + \frac{G_2}{G_1} \cdot Q + \frac{G_3}{G_1} \cdot h(Q) = \frac{H}{G_1}, \quad Q \in \mathscr{B}_{\geq m}^{+} \subseteq \mathscr{L}_k. \tag{B.16}$$

In Proposition B.5.1 (see [29, Proposition 4.7]) and Remark B.5.3, using the fixed point theorem stated in Proposition 1.2.12, we prove the existence and the uniqueness (under some additional assumptions, see Proposition B.5.1 below) of solutions of equations of the form (B.16), for $m = 1$. We generalize it in Remark B.5.6 for arbitrary m such that $1 \leq m \leq k$.

This type of a nonlinear equation appears in Lemma 2.1.17 in Section 2.1, and is used to apply the fixed point theorem stated in Proposition 1.2.12 for solving the normalization equation. The following results represent a part of the results obtained in [29].

Proposition B.5.1 (Lemma 4.7, [29])**.** Let $G_1, G_2,\ V \in \mathscr{B}_1^+ \subseteq \mathscr{L}_k$, $k \in \mathbb{N}_{\geq 1}$, such that $\operatorname{ord}(zG_1)$, $\operatorname{ord}(zG_2) > \mathbf{1}_{k+1}$ and $\operatorname{ord}(zV) > \mathbf{1}_{k+1}$. Let $h \in x^2\mathbb{R}[[x]]$ be a formal power series in the variable x with real coefficients, such that $h(0) = h'(0) = 0$. Then the equation

$$D_1(Q) + G_1 \cdot Q + G_2 \cdot h(Q) = V \tag{B.17}$$

admits a unique solution $Q \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$.

The following lemma is an auxiliary lemma in the proof of Proposition B.5.1.

Lemma B.5.2 (Lemma 4.6, [29])**.** Let G_1, G_2, $H \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, $1 \leq m \leq k$, $k \in \mathbb{N}_{\geq 1}$, such that

$$\operatorname{ord}\big(z\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1} G_1\big) > \mathbf{1}_{k+1},\ \operatorname{ord}\big(z\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1} G_2\big) > \mathbf{1}_{k+1},$$

and $\operatorname{ord}_{\boldsymbol{\ell}_m}(H) \geq 2$. Let $h \in x^2\,\mathscr{B}_{\geq m+1}^+[[x]]$ such that $h(0) = h'(0) = 0$[1]. Then the differential equation

$$D_m(Q) + G_1 \cdot Q + G_2 \cdot h(Q) = H \tag{B.18}$$

admits a unique solution $Q \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$.

Proof. Let $h := \sum_{n\geq 2} H_n x^n$, with $H_n \in \mathscr{B}_{\geq m+1}^+$, $n \geq 2$. Define the operators $\mathscr{S} : \mathscr{B}_m^+ \to \mathscr{B}_m^+$ and $\mathscr{T} : \mathscr{B}_m^+ \to \mathscr{B}_m^+$ by

$$\begin{aligned} \mathscr{S}(Q) &:= H - G_2 \cdot \sum_{n\geq 2} H_n \cdot Q^n, \\ \mathscr{T}(Q) &:= D_m(Q) + G_1 \cdot Q, \quad Q \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k. \end{aligned} \tag{B.19}$$

Note that $\mathscr{T}$ is a linear operator, while $\mathscr{S}$ is not. Now (B.18) is equivalent to the fixed-point equation

$$\mathscr{S}(Q) = \mathscr{T}(Q), \quad Q \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k. \tag{B.20}$$

By (B.19), Example 1.2.6 and Example 1.2.8 it is easy to see that $\mathscr{S}$ is a $\frac{1}{4}$-contraction and $\mathscr{T}$ is a $\frac{1}{2}$-homothety on the space $\big(\mathscr{B}_m^+, d_m\big)$.

Let us now prove that $\mathscr{S}(\mathscr{B}_m^+) \subseteq \mathscr{T}(\mathscr{B}_m^+)$. Let $M \in \mathscr{S}(\mathscr{B}_m^+)$. Note that, since $\operatorname{ord}_{\boldsymbol{\ell}_m}(H) \geq 2$, $M \in \mathscr{S}(\mathscr{B}_m^+)$ implies that $\operatorname{ord}_{\boldsymbol{\ell}_m}(M) \geq 2$. We prove now that $M \in \mathscr{T}(\mathscr{B}_m^+)$. That is, by (B.19), that the equation

$$D_m(Q) + G_1 \cdot Q = M$$

[1] h is a formal power series in the formal variable x with coefficients in $\mathscr{B}_{\geq m+1}^+$.

has a solution $Q \in \mathscr{B}_m^+$. By Theorem B.4.1, the linear differential equation above admits a unique solution in $\mathscr{B}_m \subseteq \mathscr{L}_{k+1}^\infty$ given by:

$$Q := \exp\left(-\int \frac{G_1}{\boldsymbol{\ell}_m^2} d\boldsymbol{\ell}_m\right) \cdot \int \frac{M}{\boldsymbol{\ell}_m^2} \cdot \exp\left(\int \frac{G_1}{\boldsymbol{\ell}_m^2} d\boldsymbol{\ell}_m\right) d\boldsymbol{\ell}_m. \tag{B.21}$$

Since $\operatorname{ord}\big(z\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1} G_1\big) > \mathbf{1}_{k+1}$, it follows $\frac{M}{\boldsymbol{\ell}_m^2} \cdot \exp\left(\int \frac{G_1}{\boldsymbol{\ell}_m^2} d\boldsymbol{\ell}_m\right)$ does not contain a term of order $(\mathbf{0}_m, \mathbf{1}_{k-m+1})$. Now, by Theorem B.4.1, it follows that $Q \in \mathscr{B}_m \cap \mathscr{L}_k^\infty$. For the same reasons, analyzing (B.21), and since $\operatorname{ord}_{\boldsymbol{\ell}_m}\big(\frac{M}{\boldsymbol{\ell}_m^2}\big) \geq 0$, we obtain that $Q \in \mathscr{B}_m^+$.

Finally, Proposition 1.2.12 guarantees a unique solution in $\mathscr{B}_m^+$ to (B.20). It is then the unique solution of (B.18). ■

Proof of Proposition B.5.1. Since $\operatorname{ord}_{\boldsymbol{\ell}_1}(V)$ in (B.17) is not necessarily strictly bigger than 1 (unlike in the hypotheses of Lemma B.5.2) and since h has real coefficients, we cannot directly apply Lemma B.5.2 to prove the existence of a solution. Therefore, we decompose adequately both sides of (B.17), and then we successively apply Lemma B.5.2 in the subspaces $\mathscr{B}_m^+$, $1 \leq m \leq k$.

Since $\operatorname{ord}\big(zG_1\big) > \mathbf{1}_{k+1}$, $\operatorname{ord}\big(zG_2\big) > \mathbf{1}_{k+1}$ and $\operatorname{ord}\big(zV\big) > \mathbf{1}_{k+1}$, we have the unique decompositions

$$\begin{aligned} V &= \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_k V_k + \cdots + \boldsymbol{\ell}_1 V_1, \\ G_1 &= \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_k G_{1,k} + \cdots + \boldsymbol{\ell}_1 G_{1,1}, \\ G_2 &= \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_k G_{2,k} + \cdots + \boldsymbol{\ell}_1 G_{2,1}, \end{aligned}$$

where $V_i, G_{1,i}, G_{2,i} \in \mathscr{B}_i^+ \subseteq \mathscr{L}_k$, $i = 1, \ldots, k$. We proceed inductively.

Step 1. We assume, without loss of generality, that $V_k \neq 0$ (otherwise, simply consider the lowest m, $1 \leq m < k$, such that $V_m \neq 0$). By Lemma B.5.2 and Lemma A.2.10, the nonlinear differential equation (in the variable Q_k)

$$\begin{aligned} D_1(Q_k) + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{k-1}\boldsymbol{\ell}_k \left(G_{1,k} \cdot Q_k + G_{2,k} \cdot h(Q_k)\right) &= \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{k-1}\boldsymbol{\ell}_k V_k, \qquad \text{(B.22)} \\ \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{k-1} D_k(Q_k) + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{k-1}\boldsymbol{\ell}_k \left(G_{1,k} \cdot Q_k + G_{2,k} \cdot h(Q_k)\right) &= \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{k-1}\boldsymbol{\ell}_k V_k, \\ D_k(Q_k) + \boldsymbol{\ell}_k G_{1,k} \cdot Q_k + \boldsymbol{\ell}_k G_{2,k} \cdot h(Q_k) &= \boldsymbol{\ell}_k V_k. \end{aligned}$$

admits a unique solution $Q_k \in \mathscr{B}_k^+$.

Step 2. Let $Q_k \in \mathscr{B}_k^+$ be the solution of (B.22) as above. In the next step, consider the nonlinear differential equation (in the variable Q_{k-1})

$$\begin{aligned} D_1\big(Q_k+Q_{k-1}\big)+\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{k-2}\left(\boldsymbol{\ell}_{k-1}\boldsymbol{\ell}_k G_{1,k}+\boldsymbol{\ell}_{k-1}G_{1,k-1}\right)\cdot\left(Q_k+Q_{k-1}\right)+ \\ +\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{k-2}\left(\boldsymbol{\ell}_{k-1}\boldsymbol{\ell}_k G_{2,k}+\boldsymbol{\ell}_{k-1}G_{2,k-1}\right)\cdot h\left(Q_k+Q_{k-1}\right)= \\ =\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{k-2}\left(\boldsymbol{\ell}_{k-1}\boldsymbol{\ell}_k V_k+\boldsymbol{\ell}_{k-1}V_{k-1}\right), \end{aligned} \tag{B.23}$$

$$\begin{aligned} D_1(Q_{k-1})+\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{k-2}\Big(\boldsymbol{\ell}_{k-1}\boldsymbol{\ell}_k G_{1,k}+\boldsymbol{\ell}_{k-1}G_{1,k-1}+\big(\boldsymbol{\ell}_{k-1}\boldsymbol{\ell}_k G_{2,k}+\boldsymbol{\ell}_{k-1}G_{2,k-1}\big)\cdot h'(Q_k)\Big)\cdot Q_{k-1}+ \\ +\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{k-2}\left(\boldsymbol{\ell}_{k-1}\boldsymbol{\ell}_k G_{2,k}+\boldsymbol{\ell}_{k-1}G_{2,k-1}\right)\cdot\sum_{i\geq 2}\frac{h^{(i)}(Q_k)}{i!}Q_{k-1}^i= \\ =\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{k-2}\left(\boldsymbol{\ell}_{k-1}V_{k-1}-\boldsymbol{\ell}_{k-1}G_{1,k-1}Q_k-h(Q_k)\boldsymbol{\ell}_{k-1}G_{2,k-1}\right). \end{aligned} \tag{B.24}$$

Equation (B.24) is obtained from (B.23) by using (B.22) to eliminate $D_1\left(Q_k\right)$, and by the Taylor expansion of h. Using Lemma A.2.10 and dividing (B.24) by $\boldsymbol{\ell}_1\cdots\boldsymbol{\ell}_{k-2}$, we get the equivalent equation:

$$\begin{aligned} D_{k-1}(Q_{k-1})+\Big(\boldsymbol{\ell}_{k-1}\boldsymbol{\ell}_k G_{1,k}+\boldsymbol{\ell}_{k-1}G_{1,k-1}+\big(\boldsymbol{\ell}_{k-1}\boldsymbol{\ell}_k G_{2,k}+\boldsymbol{\ell}_{k-1}G_{2,k-1}\big)\cdot h'(Q_k)\Big)\cdot Q_{k-1}+ \\ +\left(\boldsymbol{\ell}_{k-1}\boldsymbol{\ell}_k G_{2,k}+\boldsymbol{\ell}_{k-1}G_{2,k-1}\right)\cdot\sum_{i\geq 2}\frac{h^{(i)}(Q_k)}{i!}Q_{k-1}^i= \\ =\boldsymbol{\ell}_{k-1}V_{k-1}-\boldsymbol{\ell}_{k-1}G_{1,k-1}Q_k-h(Q_k)\boldsymbol{\ell}_{k-1}G_{2,k-1}. \end{aligned} \tag{B.25}$$

Notice that the order $\operatorname{ord}_{\boldsymbol{\ell}_{k-1}}$ of the right-hand side of (B.25) is bigger than or equal to 2. Therefore, by Lemma B.5.2, equation (B.25) admits a unique solution $Q_{k-1}\in\mathscr{B}_{k-1}^+$.

Step 3. Proceeding inductively in k steps, we prove that (B.17) admits a solution $Q := Q_1+\cdots+Q_k\in\mathscr{B}_{\geq 1}^+$. It is the unique solution since we can decompose every solution as we did for Q and every Q_i is unique in that decomposition. ■

Remark B.5.3. Note that the initial equation (B.15) with $G_1,G_2,G_3,H\in\mathscr{B}_1\subseteq\mathscr{L}_k^\infty$, $k\in\mathbb{N}_{\geq 1}$, and $h\in x^2\,\mathbb{R}\,[[x]]$, can, dividing by $G_1\neq 0$, be brought to the form:

$$D_1(Q)+\frac{G_2}{G_1}\cdot Q+\frac{G_3}{G_1}\cdot h(Q)=\frac{H}{G_1}.$$

If $\operatorname{ord}\left(\frac{G_2}{G_1}\right),\operatorname{ord}\left(\frac{G_3}{G_1}\right),\operatorname{ord}\left(\frac{H}{G_1}\right)>(0,\mathbf{1}_k)$, then, by Proposition B.5.1, there exists a unique solution $Q\in\mathscr{B}_{\geq 1}^+$ of differential equation (B.15).

B.5.2. Type II

In Remark B.5.3 we suppose that $G_1 \neq 0$ in differential equation (B.15). In Proposition B.5.4 below we solve equation (B.15), but with the assumption $G_1 = 0$. As in the proof of Proposition B.5.1, we first prove an auxiliary lemma (for the inductive step of the algorithm) and inductively solve equation (B.15), with the assumption $G_1 = 0$ and $m = 1$. We generalize it in Remark B.5.6 for arbitrary m such that $1 \leq m \leq k$, $k \in \mathbb{N}_{\geq 1}$.

This type of a nonlinear equation appears in Proposition 2.3.14 and Proposition 2.3.20 in Section 2.3, and is used to apply the fixed point theorem stated in Proposition 1.2.12 for solving the normalization equation. The following results represent a part of the results obtained in the preprint in preparation [28].

Proposition B.5.4. Let $V \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_k$, $k \in \mathbb{N}_{\geq 1}$, and let $G_1, G_2 \in \mathscr{B}_1 \setminus \{0\} \subseteq \mathscr{L}^\infty_k$ such that $\operatorname{ord}(G_1), \operatorname{ord}(G_2) = \mathbf{0}_{k+1}$, i.e., G_1 and G_2 have non-zero constants as leading terms. Let $h \in x^2\,\mathbb{R}[[x]]$ be a formal power series in the variable x, with real coefficients, such that $h(0) = h'(0) = 0$. Then there exists a unique $Q \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_k$, such that

$$G_1 \cdot Q + G_2 \cdot h(Q) = V. \tag{B.26}$$

The following lemma is an auxiliary lemma in the proof of Proposition B.5.4.

Lemma B.5.5. Let $H \in \mathscr{B}^+_m \subseteq \mathscr{L}_k$, $k \in \mathbb{N}_{\geq 1}$, and let $G_1, G_2 \in \mathscr{B}_m \setminus \{0\} \subseteq \mathscr{L}^\infty_k$ such that $\operatorname{ord}(G_1), \operatorname{ord}(G_2) = \mathbf{0}_{k+1}$, i.e., G_1 and G_2 have non-zero constants as the leading terms. Let $h \in x^2\,\mathscr{B}^+_{\geq m}[[x]]$ be a formal power series in the variable x, with coefficients in $\mathscr{B}^+_{\geq m}$, such that $h(0) = h'(0) = 0$. Then there exists a unique solution $Q \in \mathscr{B}^+_m \subseteq \mathscr{L}_k$ of the equation

$$G_1 \cdot Q + G_2 \cdot h(Q) = H. \tag{B.27}$$

Proof. Let $\mathscr{T}, \mathscr{S} : \mathscr{B}^+_m \to \mathscr{B}^+_m$, such that $\mathscr{T}(Q) = G_1 \cdot Q$ and $\mathscr{S}(Q) = H - G_2 \cdot h(Q)$, for every $Q \in \mathscr{B}^+_m$. Equation (B.27) is equivalent to the fixed point equation $\mathscr{T}(Q) = \mathscr{S}(Q)$. Let $h := \sum_{i \geq 2} H_i \cdot x^i$, where $H_i \in \mathscr{B}^+_{\geq m}$, $i \geq 2$. Notice that:

$$\operatorname{ord}_{\boldsymbol{\ell}_m}(\mathscr{T}(Q)) = \operatorname{ord}_{\boldsymbol{\ell}_m}(Q) + \operatorname{ord}_{\boldsymbol{\ell}_m}(G_1) = \operatorname{ord}_{\boldsymbol{\ell}_m}(Q).$$

So, $\mathscr{T}$ is an isometry on the space $(\mathscr{B}_m^+, d_m)$. By Example 1.2.8, it follows that $\mathscr{S}$ is a $\frac{1}{2}$-contraction on the space $(\mathscr{B}_m^+, d_m)$. It is obvious that $\mathscr{T}$ is a surjection, since $\frac{1}{G_1} \in \mathscr{B}_{\geq m}^+$ (due to $\operatorname{ord}(G_1) = \mathbf{0}_{k+1}$). By the fixed point theorem stated in Proposition 1.2.12, there exists a unique $Q \in \mathscr{B}_m^+$, such that $\mathscr{T}(Q) = \mathscr{S}(Q)$. ■

Proof of Proposition B.5.4. We have the following decomposition:

$$V = V_1 + \cdots + V_k,$$

where $V_i \in \mathscr{B}_i^+ \subseteq \mathscr{L}_k$, for $1 \leq i \leq k$. Similarly, we have the following decompositions:

$$G_1 = G_{1,1} + \cdots + G_{1,k}$$

and

$$G_2 = G_{2,1} + \cdots + G_{2,k}\,,$$

where $G_{1,i}, G_{2,i} \in \mathscr{B}_i^+ \subseteq \mathscr{L}_k$, for $1 \leq i \leq k-1$, and $G_{1,k}, G_{2,k} \in \mathscr{B}_k \subseteq \mathscr{L}_k^\infty$ such that $\operatorname{ord}(G_{1,k}) = \operatorname{ord}(G_{2,k}) = \mathbf{0}_{k+1}$. Without loss of generality assume that $V_k \neq 0$ (if $V_k = 0$, then replace k with the biggest m such that $1 \leq m < k$ and $V_m \neq 0$). By Lemma B.5.5, there exists a unique solution $Q_k \in \mathscr{B}_k^+ \subseteq \mathscr{L}_k$ of the equation:

$$G_{1,k} \cdot Q_k + G_{2,k} \cdot h(Q_k) = V_k\,. \tag{B.28}$$

Let us now consider the equation:

$$(G_{1,k} + G_{1,k-1}) \cdot (Q_k + Q_{k-1}) + (G_{2,k} + G_{2,k-1}) \cdot h(Q_k + Q_{k-1}) = V_k + V_{k-1}$$

in the variable $Q_{k-1} \in \mathscr{B}_{k-1}^+$. By the Taylor Theorem, it follows that:

$$h(Q_k + Q_{k-1}) = h(Q_k) + \sum_{i \geq 1} \frac{h^{(i)}(Q_k)}{i!} Q_{k-1}^i. \tag{B.29}$$

Using (B.28) and (B.29), we get the equation:

$$\begin{aligned} &\big(G_{1,k} + G_{1,k-1} + (G_{2,k} + G_{2,k-1}) \cdot h'(Q_k)\big) \cdot Q_{k-1} + (G_{2,k} + G_{2,k-1}) \cdot \Big(\sum_{i \geq 2} \frac{h^{(i)}(Q_k)}{i!} Q_{k-1}^i\Big) \\ &= V_{k-1} - G_{1,k-1} \cdot Q_k - G_{2,k-1} \cdot h(Q_k). \end{aligned} \tag{B.30}$$

Since $G_{1,k}$ has a non-zero constant as the leading term, $G_{1,k-1}, G_{2,k-1} \in \mathscr{B}_{k-1}^+ \subseteq \mathscr{L}_k$, and $G_{2,k} \cdot h'(Q_k) \in \mathscr{B}_k^+ \subseteq \mathscr{L}_k$, it follows that $G_{1,k} + G_{1,k-1} + (G_{2,k} + G_{2,k-1}) \cdot h'(Q_k) \neq 0$.

By Lemma B.5.5, there exists a unique solution $Q_{k-1} \in \mathscr{B}_{k-1}^+ \subseteq \mathscr{L}_k$ of equation (B.30). Proceeding inductively, there exists a solution $Q := Q_1 + \cdots + Q_k \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$ of equation (B.26). The uniqueness follows similarly as in the proof of Proposition B.5.1. ■

Remark B.5.6. Let $k \in \mathbb{N}_{\geq 1}$. Note that the analogues of Proposition B.5.1 and Proposition B.5.4 hold in $(\mathscr{B}_{\geq m}^+, d_m)$, for $1 \leq m \leq k$. Those analogues can be proven similarly as the mentioned propositions, by replacing $\boldsymbol{\ell}_1$ by $\boldsymbol{\ell}_m$, and using D_m on the space $(\mathscr{B}_{\geq m}^+, d_m)$ instead of D_1 on the space $(\mathscr{B}_{\geq 1}^+, d_1)$.

B.5.3. Type III

In Proposition B.5.1 we solve the nonlinear differential equation (in the variable Q)

$$D_1(Q) + G_1 \cdot Q + G_2 \cdot h(Q) = V, \quad Q \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k,$$

for $G_1, G_2, V \in \mathscr{B}_1^+ \subseteq \mathscr{L}_k$, $k \in \mathbb{N}_{\geq 1}$, such that $\operatorname{ord}(zG_1), \operatorname{ord}(zG_2), \operatorname{ord}(zV) > \mathbf{1}_{k+1}$ and $h \in x^2\mathbb{R}[[x]]$. Now we generalize this result for $G_1 := \frac{D_1(G)}{G} + P$, where $G, P \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, $\operatorname{ord}(zP) > \mathbf{1}_{k+1}$, $\operatorname{ord}(G_2) \geq \operatorname{ord}(G_1)$ and for $V \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$ such that $\operatorname{ord}(V) - \operatorname{ord}(G) > (0, \mathbf{1}_k)$.

This type of a nonlinear equation appears in Proposition 2.3.28 in Section 2.3 and is used to apply the fixed point theorem stated in Proposition 1.2.12 for solving the normalization equation. The following results represent a part of the results obtained in the preprint in preparation [28].

Proposition B.5.7 (Solution of a differential equation in $\mathscr{B}_{\geq 1}^+$)**.** Let $K, M, N, T \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$, $k \in \mathbb{N}_{\geq 1}$, such that $N \neq 0$, $\operatorname{ord}(T) \geq \operatorname{ord}(\frac{D_1(N)}{N})$, $\operatorname{ord}(K) > (0, \mathbf{1}_k)$, and $\operatorname{ord}(M) - \operatorname{ord}(N) > (0, \mathbf{1}_k)$. Let $h \in x^2\mathbb{R}[[x]]$ be a power series in the variable x, with real coefficients, such that $h(0) = h'(0) = 0$. Then there exists a unique solution $Q \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$ satisfying $\operatorname{ord}(Q) > \operatorname{ord}(N)$ of the differential equation:

$$D_1(Q) - \Big(\frac{D_1(N)}{N} + K\Big) \cdot Q + T \cdot h(Q) = M. \tag{B.31}$$

Using the fixed point theorem stated in Proposition 1.2.12 we first prove Lemma B.5.8 and Lemma B.5.9 which are auxiliary lemmas in the proof of Proposition B.5.7. We prove Proposition B.5.7 inductively, and Lemmas B.5.8 and B.5.9 are used for proving the inductive step.

Lemma B.5.8 (Solution of a differential equation in $\mathscr{B}_m^+$)**.** Let $M,N,P,T \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, $N \neq 0$, such that $\operatorname{ord}(P) > (\mathbf{0}_m, \mathbf{1}_{k-m+1})$, for $1 \leq m \leq k$, $k \in \mathbb{N}_{\geq 1}$, and let $h \in x^2\, \mathscr{B}_{\geq m}^+ [[x]]$ be a power series in the variable x, with coefficients in $\mathscr{B}_{\geq m}^+ \subseteq \mathscr{L}_k$, such that $h(0) = h'(0) = 0$. Suppose that

$$\operatorname{ord}(M) - \operatorname{ord}(N) > (\mathbf{0}_m, \mathbf{1}_{k-m+1}).$$

Then there exists a unique solution $Q \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$ satisfying $\operatorname{ord}(Q) > \operatorname{ord}(N)$ of the differential equation:

$$D_m(Q) - \Big(\frac{D_m(N)}{N} + P\Big) \cdot Q + T \cdot h(Q) = M. \tag{B.32}$$

Proof. Let $\mathscr{T}_m, \mathscr{S}_m : \mathscr{B}_m^+ \to \mathscr{B}_m^+$ be operators defined by:

$$\begin{aligned} \mathscr{T}_m(Q) &:= D_m(Q) - \Big(\frac{D_m(N)}{N} + P\Big) \cdot Q, \\ \mathscr{S}_m(Q) &:= M - T \cdot h(Q), \quad Q \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k. \end{aligned} \tag{B.33}$$

Now, the equation $\mathscr{T}_m(Q) = \mathscr{S}_m(Q)$ becomes equation (B.32), for $Q \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$.

We prove that the operators $\mathscr{T}_m$ and $\mathscr{S}_m$ satisfy the assumptions of Proposition 1.2.12.

For arbitrary $Q_1, Q_2 \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, note that:

$$\begin{aligned} &\operatorname{ord}_{\boldsymbol{\ell}_m}(\mathscr{S}_m(Q_1) - \mathscr{S}_m(Q_2)) \\ &\geq \operatorname{ord}_{\boldsymbol{\ell}_m}(Q_1 - Q_2) + \operatorname{ord}_{\boldsymbol{\ell}_m}(T) + \min\big\{\operatorname{ord}_{\boldsymbol{\ell}_m}(Q_1), \operatorname{ord}_{\boldsymbol{\ell}_m}(Q_2)\big\} \\ &\geq \operatorname{ord}_{\boldsymbol{\ell}_m}(Q_1 - Q_2) + 2. \end{aligned}$$

Therefore, $\mathscr{S}_m$ is a $\frac{1}{4}$-contraction on the space $(\mathscr{B}_m^+, d_m)$.

Suppose that $\mathscr{T}_m(Q) = 0$, i.e., $D_m(Q) - \Big(\frac{D_m(N)}{N} + P\Big) \cdot Q = 0$, for some $Q \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$. Solving the linear differential equation, we get:

$$\begin{aligned} Q &= C \cdot \exp\Big(\log(N) + \int P \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\Big) \\ &= C \cdot N \cdot \exp\Big(\int P \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\Big), \quad C \in \mathbb{R}. \end{aligned}$$

Since $\mathrm{ord}(P) > (\mathbf{0}_m, \mathbf{1}_{k-m+1})$, using the substitution $z := \boldsymbol{\ell}_m$, by Lemma B.2.6, it follows that

$$\int P \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2} \in \mathscr{B}_{\geq m}^{+}.$$

This implies that

$$\exp\Big(\int P \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\Big) = 1 + \text{h.o.t.} \tag{B.34}$$

We conclude that $Q \in \ker(\mathscr{T}_m)$ implies that $\mathrm{ord}(Q) = \mathrm{ord}(N)$. To avoid terms in the kernel of $\mathscr{T}_m$, we restrict ourselves to the space

$$\widetilde{\mathscr{B}} := \left\{ V \in \mathscr{B}_m^{+} \subseteq \mathscr{L}_k : \mathrm{ord}(V) > \mathrm{ord}(N) \right\}.$$

Then, for every $Q_1, Q_2 \in \widetilde{\mathscr{B}}$, $Q_1 \neq Q_2$, it follows that $Q_1 - Q_2 \notin \ker(\mathscr{T}_m)$. Now, by (B.33), it follows that

$$\mathrm{ord}_{\boldsymbol{\ell}_m}(\mathscr{T}_m(Q_1 - Q_2)) = \mathrm{ord}_{\boldsymbol{\ell}_m}(Q_1 - Q_2) + 1.$$

By the linearity of the operator $\mathscr{T}_m$, it follows that the restriction $\mathscr{T}_m|_{\widetilde{\mathscr{B}}}$ is a $\frac{1}{2}$-homothety on the space $\widetilde{\mathscr{B}}$, with respect to the metric d_m.

It is left to prove that $\mathscr{S}_m(\widetilde{\mathscr{B}}) \subseteq \mathscr{T}_m(\widetilde{\mathscr{B}})$. By definition (B.33) of $\mathscr{S}_m$, note that $\mathscr{S}_m(\widetilde{\mathscr{B}})$ is contained in the space

$$\overline{\mathscr{B}} := \left\{ K \in \mathscr{B}_m^{+} \subseteq \mathscr{L}_k : \mathrm{ord}(K) > \mathrm{ord}(N) + (\mathbf{0}_m, \mathbf{1}_{k-m+1}) \right\}.$$

We prove that $\overline{\mathscr{B}} \subseteq \mathscr{T}_m(\widetilde{\mathscr{B}})$. Let $K \in \overline{\mathscr{B}}$ be arbitrary. We solve in $\widetilde{\mathscr{B}}$ the equation $\mathscr{T}_m(Q) = K$, i.e.

$$D_m(Q) - \Big(\frac{D_m(N)}{N} + P\Big) \cdot Q = K.$$

Solving the linear differential equation above, we get:

$$\begin{aligned} Q &= \exp\Big(\log(N) + \int P \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\Big) \cdot \Big(C + \int \Big(\frac{K}{\exp\big(\log(N) + \int P \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\big)}\Big) \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\Big) \\ &= \Big(N \cdot \exp \int P \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\Big) \cdot \Big(C + \int \Big(\frac{K}{N \cdot \exp \int P \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}}\Big) \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}\Big), \quad C \in \mathbb{R}. \end{aligned} \tag{B.35}$$

Since $Q \in \widetilde{\mathscr{B}}$ we choose the solution with $C = 0$. Now, from (B.34) and (B.35) we get:

$$Q = (\mathrm{Lt}(N) + \text{h.o.t.}) \int \Big(\frac{K}{\mathrm{Lt}(N) + \text{h.o.t.}}\Big) \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2}. \tag{B.36}$$

Since $K \in \mathscr{B}_m^+$, $\mathrm{ord}(K) > \mathrm{ord}(N) + (\mathbf{0}_m, \mathbf{1}_{k-m+1})$, it follows that

$$\mathrm{ord}\Big(\frac{K}{\mathrm{Lt}(N) + \text{h.o.t.}}\Big) > (\mathbf{0}_m, \mathbf{1}_{k-m+1}),$$

and, by Lemma B.2.6,

$$\int \Big(\frac{K}{\mathrm{Lt}(N) + \text{h.o.t.}}\Big) \frac{d\boldsymbol{\ell}_m}{\boldsymbol{\ell}_m^2} \in \mathscr{B}_{\geq m}^+ \subseteq \mathscr{L}_k.$$

By (B.36), $\mathrm{ord}(Q) > \mathrm{ord}(N)$, i.e., the solution Q given by (B.36) belongs to $\widetilde{\mathscr{B}}$.

Thus, we proved that $\mathscr{S}_m(\widetilde{\mathscr{B}}) \subseteq \overline{\mathscr{B}} \subseteq \mathscr{T}_m(\widetilde{\mathscr{B}})$.

Finally, by Proposition 1.2.12 applied to the operators $\mathscr{T}_m|_{\widetilde{\mathscr{B}}} : \widetilde{\mathscr{B}} \to \mathscr{B}_{\geq m}^+$ and $\mathscr{S}_m|_{\widetilde{\mathscr{B}}} : \widetilde{\mathscr{B}} \to \mathscr{B}_{\geq m}^+$, there exists a unique $Q \in \widetilde{\mathscr{B}}$ such that $\mathscr{T}_m(Q) = \mathscr{S}_m(Q)$. ■

Lemma B.5.9 (Solution of a differential equation in $\mathscr{B}_m^+$)**.** Let $N \in \mathscr{B}_{\geq m+1}^+ \setminus \{0\} \subseteq \mathscr{L}_k$, and $M, P, T \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$ such that $\mathrm{ord}\,(P) > (\mathbf{0}_m, \mathbf{1}_{k-m+1})$ and $\mathrm{ord}_{\boldsymbol{\ell}_m}(M) \geq 2$, for $1 \leq m \leq k$, $k \in \mathbb{N}_{\geq 1}$. Let $h \in x^2 \mathscr{B}_{\geq m}^+[[x]]$ be a power series in the variable x, with coefficients in $\mathscr{B}_{\geq m}^+ \subseteq \mathscr{L}_k$, such that $h(0) = h'(0) = 0$. There exists a unique solution $Q \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$ of the differential equation:

$$D_m(Q) - \Big(\frac{D_m(N)}{N} + P\Big) \cdot Q + T \cdot h(Q) = M.$$

Proof. Similarly as the proof of Lemma B.5.8. ■

Proof of Proposition B.5.7. We first transform equation (B.31) to an equivalent *simpler* differential equation. Let

$$N = N_k + \cdots + N_1, \tag{B.37}$$

where $N_i \in \mathscr{B}_i^+ \subseteq \mathscr{L}_k$, $1 \leq i \leq k$. Let $1 \leq m \leq k$ be the biggest m such that $N_m \neq 0$. Note that $N = N_m + \cdots + N_1$ and

$$\begin{aligned}
\frac{D_1(N)}{N} &= \frac{D_1(N)}{N_m \cdot \Big(1 + \frac{N_{m-1}}{N_m} + \cdots + \frac{N_1}{N_m}\Big)} \\
&= \Big(\sum_{i=1}^{m} \frac{D_1(N_i)}{N_m}\Big) \cdot \Big(\sum_{i \geq 0} (-1)^i \Big(\frac{N_{m-1}}{N_m} + \cdots + \frac{N_1}{N_m}\Big)^i\Big) \\
&= \Big(\frac{D_1(N_m)}{N_m} + \sum_{i=1}^{m-1} \frac{D_1(N_i)}{N_m}\Big) \cdot \Big(1 + \sum_{i \geq 1} (-1)^i \Big(\frac{N_{m-1}}{N_m} + \cdots + \frac{N_1}{N_m}\Big)^i\Big).
\end{aligned} \tag{B.38}$$

Note that:

$$\begin{aligned} &\mathrm{ord}\Big(\frac{D_1(N_i)}{N_m}\Big) > (0, \mathbf{1}_k), \\ &\mathrm{ord}\left(\frac{N_i}{N_m}\right) \geq (\mathbf{0}_i, 1, v_{i+1}, \dots, v_k), \end{aligned} \tag{B.39}$$

where $(v_{i+1}, \dots, v_k) \in \mathbb{Z}^{k-i}$, for $1 \leq i \leq m-1$. Since, in addition, $\mathrm{ord}\Big(\frac{D_1(N_m)}{N_m}\Big) = (0, \mathbf{1}_m, \mathbf{0}_{k-m})$, we get that:

$$\mathrm{ord}\left(\frac{D_1(N_m)}{N_m} \cdot \sum_{i \geq 1} (-1)^i \Big(\frac{N_{m-1}}{N_m} + \cdots + \frac{N_1}{N_m}\Big)^i\right) > (0, \mathbf{1}_k). \tag{B.40}$$

By (B.38), (B.39) and (B.40), we get that:

$$\mathrm{ord}\Big(\frac{D_1(N)}{N} - \frac{D_1(N_m)}{N_m}\Big) > (0, \mathbf{1}_k). \tag{B.41}$$

Put $P := K + \frac{D_1(N)}{N} - \frac{D_1(N_m)}{N_m}$. By (B.41) and since $\mathrm{ord}\,(K) > (0, \mathbf{1}_k)$ (by assumption), it follows that $P \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_k$ and $\mathrm{ord}\,(P) > (0, \mathbf{1}_k)$. Note that it is also possible that $P = 0$. In this notation, equation (B.31) is equivalent to the equation:

$$D_1(Q) - \Big(\frac{D_1(N_m)}{N_m} + P\Big) \cdot Q + T \cdot h(Q) = M. \tag{B.42}$$

Proof of the existence of a solution: Now we prove the existence of a solution $Q \in \mathscr{B}^+_{\geq 1} \subseteq \mathscr{L}_k$ of equation (B.42). Moreover, we prove that Q satisfies $\mathrm{ord}\,(Q) > \mathrm{ord}\,(N)$.

Note that $Q := 0$ is a solution of (B.42) if $M = 0$. Moreover, Q is the unique solution that satisfies $\mathrm{ord}\,(Q) > \mathrm{ord}\,(N)$.

Suppose that $M \neq 0$. Since $\mathrm{ord}(M) - \mathrm{ord}(N) > (0, \mathbf{1}_k)$ and $N \in \mathscr{B}^+_{\geq 1}$ (therefore, $\mathrm{ord}\,(N) > \mathbf{0}_{k+1}$), we have the following decomposition of M:

$$M = \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_m M_m + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1} M_{m-1} + \cdots + \boldsymbol{\ell}_1 M_1, \tag{B.43}$$

where $M_i \in \mathscr{B}^+_i \subseteq \mathscr{L}_k$, $1 \leq i \leq m-1$, and $M_m \in \mathscr{B}^+_{\geq m} \subseteq \mathscr{L}_k$, $\mathrm{ord}(\boldsymbol{\ell}_m M_m) > \mathrm{ord}(N_m) + (\mathbf{0}_m, \mathbf{1}_{k-m+1})$.

By (B.41), and since $\mathrm{ord}(K) > (0, \mathbf{1}_k)$ by assumption, it follows that $P \in \mathscr{B}^+_{\geq 1}$ and $\mathrm{ord}(P) > (0, \mathbf{1}_k)$. Now, decompose:

$$P = \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_m P_m + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1} P_{m-1} + \cdots + \boldsymbol{\ell}_1 P_1, \tag{B.44}$$

where $P_i \in \mathscr{B}_i^+ \subseteq \mathscr{L}_k$, $1 \leq i \leq m-1$, and $P_m \in \mathscr{B}_{\geq m}^+ \subseteq \mathscr{L}_k$, $\mathrm{ord}(P_m) > (\mathbf{0}_{m+1}, \mathbf{1}_{k-m})$. If $P = 0$, we simply put $P_i := 0$, for each $1 \leq i \leq m$.

By (B.38), it follows that $\mathrm{ord}(\frac{D_1(N)}{N}) = \mathrm{ord}(\frac{D_1(N_m)}{N_m})$. Since

$$\mathrm{ord}(T) \geq \mathrm{ord}\Big(\frac{D_1(N)}{N}\Big) = \mathrm{ord}\Big(\frac{D_1(N_m)}{N_m}\Big) = (0, \mathbf{1}_m, \mathbf{0}_{k-m}),$$

we have the following decomposition of T:

$$T = \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_m T_m + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1} T_{m-1} + \cdots + \boldsymbol{\ell}_1 T_1, \tag{B.45}$$

where $T_i \in \mathscr{B}_i^+ \subseteq \mathscr{L}_k$, $1 \leq i \leq m-1$, and $T_m \in \mathscr{B}_m \subseteq \mathscr{L}_k^\infty$, $\mathrm{ord}(T_m) \geq \mathbf{0}_{k+1}$. If $T = 0$, we simply put $T_i := 0$, for each $1 \leq i \leq m$.

The proof of the existence of a solution of equation (B.42) is inductive. In the i-th step $(1 \leq i \leq m)$, instead of equation (B.42), we consider the equation:

$$\begin{aligned}
&D_1(Q_m + \cdots + Q_{m-i+1}) \\
&-\Big(\frac{D_1(N_m)}{N_m} + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_m P_m + \cdots + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-i+1} P_{m-i+1}\Big) \cdot (Q_m + \cdots + Q_{m-i+1}) \\
&+ (\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_m T_m + \cdots \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-i+1} T_{m-i+1}) \cdot h(Q_m + \cdots + Q_{m-i+1}) \\
&= \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_m M_m + \cdots + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-i+1} M_{m-i+1},
\end{aligned} \tag{B.46}$$

with appropriate partial sums from decompositions (B.43), (B.44), and (B.45) instead of the whole M, P, T, where Q_i is an unknown *variable* and $Q_m, \ldots, Q_{i-1}$ are obtained in the previous steps. For the proof of the existence a solution $Q_m \in \mathscr{B}_{\geq m}^+ \subseteq \mathscr{L}_k$ in the first step, we use Lemma B.5.8, and for the proof of the existence of a solution $Q_{m-i+1} \in \mathscr{B}_{m-i+1}^+ \subseteq \mathscr{L}_k$ in steps $2 \leq i \leq m$, we use Lemma B.5.9. Finally, we put $Q := Q_m + \cdots + Q_1$ ($Q \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$).

Step 1. Consider the equation:

$$D_1(Q) - \Big(\frac{D_1(N_m)}{N_m} + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_m P_m\Big) \cdot Q + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_m T_m \cdot h(Q) = \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_m M_m. \tag{B.47}$$

where $Q \in \mathscr{B}_{\geq m}^+ \subseteq \mathscr{L}_k$. By Lemma A.2.10, it follows that $\frac{D_1(N_m)}{N_m} = \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1} \frac{D_m(N_m)}{N_m}$ and $D_1(Q) = \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1} D_m(Q)$. Dividing by $\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1}$, we get the equivalent equation:

$$D_m(Q) - \Big(\frac{D_m(N_m)}{N_m} + \boldsymbol{\ell}_m P_m\Big) \cdot Q + \boldsymbol{\ell}_m T_m \cdot h(Q) = \boldsymbol{\ell}_m M_m. \tag{B.48}$$

Now, just for the purpose of applying Lemma B.5.8, put $N := N_m$, $P := \boldsymbol{\ell}_m P_m$, $T := \boldsymbol{\ell}_m T_m$, and $M := \boldsymbol{\ell}_m M_m$. Since $N \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, $\operatorname{ord}(P) > (\mathbf{0}_m, \mathbf{1}_{k-m+1})$, $T \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, and $\operatorname{ord}(M) > \operatorname{ord}(N) + (\mathbf{0}_m, \mathbf{1}_{k-m+1})$, by Lemma B.5.8 it follows that there exists a unique solution $Q_m \in \mathscr{B}_{\geq m}^+ \subseteq \mathscr{L}_k$ of equation (B.48), i.e., equation (B.47), satisfying $\operatorname{ord}(Q_m) > \operatorname{ord}(N_m)$.

Step 2. Let $Q_m \in \mathscr{B}_{\geq m}^+ \subseteq \mathscr{L}_k$ be the solution of equation (B.47) obtained in *Step* 1. Now, consider the equation (in the variable Q):

$$D_1(Q_m + Q) - \Big(\frac{D_1(N_m)}{N_m} + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_m P_m + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1} P_{m-1}\Big) \cdot (Q_m + Q)$$
$$+ (\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_m T_m + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1} T_{m-1}) \cdot h(Q_m + Q) = \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_m M_m + \boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-1} M_{m-1}, \quad \text{(B.49)}$$

where $Q \in \mathscr{B}_{m-1}^+ \subseteq \mathscr{L}_k$. By the Taylor Theorem, it follows that:

$$h(Q_m + Q) = h(Q_m) + h'(Q_m) \cdot Q + \sum_{i \geq 2} \frac{h^{(i)}(Q_m)}{i!} Q^i. \quad \text{(B.50)}$$

Since Q_m is the solution of equation (B.47), and by Lemma A.2.10, after dividing by $\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{m-2}$ equation (B.49) becomes:

$$D_{m-1}(Q) - \Big(\frac{D_{m-1}(N_m)}{N_m} + \boldsymbol{\ell}_{m-1}\boldsymbol{\ell}_m P_m + \boldsymbol{\ell}_{m-1} P_{m-1} - (\boldsymbol{\ell}_{m-1}\boldsymbol{\ell}_m T_m + \boldsymbol{\ell}_{m-1} T_{m-1}) \cdot h'(Q_m)\Big) \cdot Q$$
$$+ (\boldsymbol{\ell}_{m-1}\boldsymbol{\ell}_m T_m + \boldsymbol{\ell}_{m-1} T_{m-1}) \cdot \Big(\sum_{i \geq 2} \frac{h^{(i)}(Q_m)}{i!} Q^i\Big) = \boldsymbol{\ell}_{m-1}\big(M_{m-1} + P_{m-1} \cdot Q_m - T_{m-1} \cdot h(Q_m)\big).$$
$$\text{(B.51)}$$

Now, for the purpose of applying Lemma B.5.9, put:

$$N := N_m,$$
$$P := \boldsymbol{\ell}_{m-1}\boldsymbol{\ell}_m P_m + \boldsymbol{\ell}_{m-1} P_{m-1} - (\boldsymbol{\ell}_{m-1}\boldsymbol{\ell}_m T_m + \boldsymbol{\ell}_{m-1} T_{m-1}) \cdot h'(Q_m),$$
$$T := \boldsymbol{\ell}_{m-1}\boldsymbol{\ell}_m T_m + \boldsymbol{\ell}_{m-1} T_{m-1},$$
$$M := \boldsymbol{\ell}_{m-1}\big(M_{m-1} + P_{m-1} \cdot Q_m - T_{m-1} \cdot h(Q_m)\big),$$
$$h := \sum_{i \geq 2} \frac{h^{(i)}(Q_m)}{i!} x^i. \quad \text{(B.52)}$$

Note that $h \in x^2 \mathscr{B}_{\geq m}^+[[x]]$. Since $N \in \mathscr{B}_m^+ \subseteq \mathscr{B}_{\geq m}^+ \subseteq \mathscr{L}_k$ and $\operatorname{ord}(h'(Q_m)) \geq \operatorname{ord}(Q_m) > \operatorname{ord}(N_m)$, it follows that $P \in \mathscr{B}_{m-1}^+ \subseteq \mathscr{L}_k$ with

$$\operatorname{ord}(P) > (\mathbf{0}_{m-1}, \mathbf{1}_{k-m+2}).$$

Note, from (B.52), that $T \in \mathscr{B}_{m-1}^+ \subseteq \mathscr{L}_k$ and $M \in \mathscr{B}_{m-1}^+ \subseteq \mathscr{L}_k$, $\mathrm{ord}_{\boldsymbol{\ell}_{m-1}}(M) \geq 2$. By Lemma B.5.9 (for $m-1$), it follows that there exists a unique solution $Q := Q_{m-1} \in \mathscr{B}_{m-1}^+ \subseteq \mathscr{L}_k$ of equation (B.51).

Inductively, we find $Q_{m-2} \in \mathscr{B}_{m-2}^+, \ldots, Q_1 \in \mathscr{B}_1^+$. Put $Q := Q_m + \cdots + Q_1$. By construction, it follows that $Q \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$ is a solution of equation (B.31). It satisfies the property $\mathrm{ord}(Q) = \mathrm{ord}(Q_m) > \mathrm{ord}(N_m) = \mathrm{ord}(N)$.

Proof of the uniqueness of the solution: Suppose that $Q_1, Q_2 \in \mathscr{B}_{\geq 1}^+ \subseteq \mathscr{L}_k$ are distinct solutions of equation (B.31) satisfying $\mathrm{ord}(Q_1), \mathrm{ord}(Q_2) > \mathrm{ord}(N)$. Therefore, Q_1 and Q_2 are solutions of equation (B.42) satisfying $\mathrm{ord}(Q_i) > \mathrm{ord}(N_m)$, for $i = 1, 2$, where N_m is given in the decomposition (B.37). By putting Q_1 and Q_2 respectively in (B.42), then subtracting the two equations and multiplying by N_m, we get:

$$N_m \cdot D_1(Q_1 - Q_2) - (Q_1 - Q_2) \cdot D_1(N_m) = N_m \cdot P \cdot (Q_1 - Q_2) - N_m \cdot T \cdot (h(Q_1) - h(Q_2)). \tag{B.53}$$

Since $\mathrm{ord}(Q_1 - Q_2) \geq \min\{\mathrm{ord}(Q_1), \mathrm{ord}(Q_2)\} > \mathrm{ord}(N_m)$, by solving the differential equation, it is easy to see that $\mathrm{Lt}(N_m) \cdot D_1(\mathrm{Lt}(Q_1 - Q_2)) - \mathrm{Lt}(Q_1 - Q_2) \cdot D_1(\mathrm{Lt}(N_m)) \neq 0$. Using this fact and analyzing orders of terms in $N_m \cdot D_1(Q_1 - Q_2) - (Q_1 - Q_2) \cdot D_1(N_m)$, it can be proven that:

$$\begin{aligned} &\mathrm{Lt}\big(N_m \cdot D_1(Q_1 - Q_2) - (Q_1 - Q_2) \cdot D_1(N_m)\big) \\ &= \mathrm{Lt}\Big(\mathrm{Lt}(N_m) \cdot D_1(\mathrm{Lt}(Q_1 - Q_2)) - \mathrm{Lt}(Q_1 - Q_2) \cdot D_1(\mathrm{Lt}(N_m))\Big). \end{aligned} \tag{B.54}$$

By (B.54), it follows that

$$\mathrm{ord}\big(N_m \cdot D_1(Q_1 - Q_2) - (Q_1 - Q_2) \cdot D_1(N_m)\big) \leq \mathrm{ord}(Q_1 - Q_2) + \mathrm{ord}(N_m) + (0, \mathbf{1}_k). \tag{B.55}$$

Since $\mathrm{ord}(P) > (0, \mathbf{1}_k)$, it follows that

$$\mathrm{ord}\big(N_m \cdot P \cdot (Q_1 - Q_2)\big) > \mathrm{ord}(Q_1 - Q_2) + \mathrm{ord}(N_m) + (0, \mathbf{1}_k). \tag{B.56}$$

By (B.55) and (B.56), it follows that

$$\mathrm{ord}\big(N_m \cdot D_1(Q_1 - Q_2) - (Q_1 - Q_2) \cdot D_1(N_m)\big) < \mathrm{ord}\big(N_m \cdot P \cdot (Q_1 - Q_2)\big). \tag{B.57}$$

On the other hand, since $N_m \in \mathscr{B}_m^+ \subseteq \mathscr{L}_k$, $\mathrm{ord}\,(T) \geq \mathrm{ord}\,(\frac{D_1(N)}{N}) = \mathrm{ord}\,(\frac{D_1(N_m)}{N_m})$, it follows that $\mathrm{ord}\,(N_m \cdot T) \geq \mathrm{ord}\,(D_1(N_m))$. Since $\mathrm{ord}\,(Q_i) > \mathrm{ord}\,(N_m)$, $i = 1, 2$, it follows that:

$$\mathrm{ord}\,(h(Q_1) - h(Q_2)) \geq \mathrm{ord}\,(Q_1 - Q_2) + \min\{\mathrm{ord}\,(Q_1), \mathrm{ord}\,(Q_2)\} > \mathrm{ord}\,(Q_1 - Q_2) + \mathrm{ord}\,(N_m).$$

Now, we get:

$$\begin{aligned} \mathrm{ord}\,\big(N_m \cdot T \cdot (h(Q_1) - h(Q_2))\big) &> \mathrm{ord}\,(Q_1 - Q_2) + \mathrm{ord}\,(N_m) + \mathrm{ord}\,(D_1(N_m)) \\ &> \mathrm{ord}\,(Q_1 - Q_2) + \mathrm{ord}\,(N_m) + (0, \mathbf{1}_k). \end{aligned} \tag{B.58}$$

Now, by (B.55) and (B.58), we get:

$$\mathrm{ord}\,\big(N_m \cdot D_1(Q_1 - Q_2) - (Q_1 - Q_2) \cdot D_1(N_m)\big) < \mathrm{ord}\,\big(N_m \cdot T \cdot (h(Q_1) - h(Q_2))\big). \tag{B.59}$$

By (B.57) and (B.59), we proved that the order of the left-hand side of equation (B.53) is strictly less than the order of the right-hand side of equation (B.53), which is a contradiction. Therefore, the solution Q of equation (B.31), satisfying $\mathrm{ord}\,(Q) > \mathrm{ord}\,(N)$, obtained in the first part of the proof, is unique. ∎

Conclusion

The purpose of this thesis is twofold. Firstly, in Chapter 1 we obtain the formal normal forms and normalizations for logarithmic transseries of parabolic, hyperbolic and strongly hyperbolic type. These results represent a generalization of the results obtained in [21] only for logarithmic transseries that do not contain iterated logarithms.

In [21] the normal forms are obtained using a transfinite algorithm: normalizations are transfinite compositions of elementary parabolic changes of variables used for *term-by-term* eliminations. On the other hand, our method is based on the fixed point theorems. Therefore, the normalizations are obtained as limits of the corresponding Picard sequences in appropriate formal topologies. This method allows us to have better control of the support of the normalization at the limit ordinal steps. It can be described as *block-wise* instead of *term-wise*.

In the second part of the thesis, we use the formal results from Chapter 2 to obtain the analytic normal forms and normalizations for hyperbolic and strongly hyperbolic complex Dulac germs defined on standard quadratic domains ([12]). In particular, we prove that the normalization of a (strongly) hyperbolic complex Dulac germ is again a complex Dulac germ of parabolic type. These germs are, by *quasi-analyticity* ([12]), uniquely determined by their asymptotic expansions which are logarithmic series. Therefore, the formal normalization uniquely determines the analytic normalization. In the intermediate step of the proof of analytic normalization of (strongly) hyperbolic complex Dulac germs we obtain the normalization results for a more general class of (strongly) hyperbolic analytic germs with a logarithmic asymptotic behavior (not necessarily having full logarithmic asymptotic expansion) on their invariant complex domains. These analytic normalization results can be seen as generalizations of *Koenigs Theorem* (see e.g. [4], [14], [24]) and *Böttcher Theorem* (see e.g. [4], [24]), and the result of *Dewsnap* and *Fischer* ([5, Theorem

2.2]).

For our future work, we plan to use our formal results for normalizations of the first return maps of semi-hyperbolic polycycles of analytic planar vector fields (see e.g. [11]). Although treated in [11], the problem of non-accumulation of limit cycles in semi-hyperbolic case is not fully understood. The first return maps are more complicated than Dulac germs, since, in general, they have more complicated transasymptotic expansions (see e.g. [3], [11]). Moreover, the complex domains of their analytic extensions are not clear.

BIBLIOGRAPHY

[1] Aniceto, I., G. Başar, and R. Schiappa: *A primer on resurgent transseries and their asymptotics*. Phys. Rep., 809:1–135, 2019. ↑ iii, vii, 1.

[2] Aschenbrenner, M., L. van den Dries, and J. van der Hoeven: *Towards a model theory for transseries*. Notre Dame J. Form. Log., 54(3-4):279–310, 2013, ISSN 0029-4527. ↑ 16.

[3] Écalle, J.: *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*. Actualités Mathématiques. Hermann, Paris, 1992, ISBN 2-7056-6199-9. ↑ iii, vii, 1, 143, 240.

[4] Carleson, L. and T. W. Gamelin: *Complex dynamics*. Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993. ↑ iv, v, viii, ix, 2, 3, 44, 73, 144, 146, 161, 179, 239.

[5] Dewsnap, D. J. and P. Fischer: *Convergence of Koenigs' sequences*. Real Anal. Exchange, 28(1):111–120, 2002/03. ↑ v, ix, 4, 5, 144, 154, 239.

[6] Dries, L. van den, A. Macintyre, and D. Marker: *Logarithmic-exponential series*. In *Proceedings of the Internaltional Conference "Analyse & Logique" (Mons, 1997)*, volume 111, pages 61–113, 2001. ↑ iii, vii, 1, 9, 15, 16, 20, 23, 30, 148, 196, 197.

[7] Dulac, H.: *Sur les cycles limites*. Bull. Soc. Math. France, 51:45–188, 1923. ↑ iii, vii, 1.

[8] Gehret, A.: *NIP for the asymptotic couple of the field of logarithmic transseries*. J. Symb. Log., 82(1):35–61, 2017. ↑ 16.

[9] Goffman, C. and G. Pedrick: *First course in functional analysis*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965. ↑ 65.

[10] Il'yashenko, Y.: *Limit cycles of polynomial vector fields with nondegenerate singular points on the real plane*. Functional Anal. Appl., 18(3):199–209, 1984. ↑ iii, vii, 1, 2, 143.

[11] Il'yashenko, Y.: *Finitness theorems for limit cycles*, volume 94 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1991. ↑ iii, iv, vii, viii, 1, 143, 149, 150, 151, 240.

[12] Ilyashenko, Y. and S. Yakovenko: *Lectures on analytic differential equations*, volume 86 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008. ↑ iii, iv, vii, viii, 2, 143, 144, 147, 149, 150, 151, 239.

[13] Kneser, H.: *Reelle analytische Lösungen der Gleichung* $\varphi(\varphi(x)) = e^x$ *und verwandter Funktional-gleichungen*. J. Reine Angew. Math., 187:56–67, 1949. ↑ 4.

[14] Koenigs, G.: *Recherches sur les intégrales de certaines équations fonctionnelles*. Ann. Sci. École Norm. Sup. (3), 1:3–41, 1884. ↑ v, ix, 3, 44, 144, 161, 239.

[15] Lang, S.: *Algebra*. Graduate Texts in Mathematics. Springer New York, 2005, ISBN 9780387953854. ↑ 10.

[16] Loray, F.: *Analyse des séries divergentes*, volume Quelques aspects des mathématiques actuelles of *Mathématiques pour le 2e cycle*. Ellipses, 1998, ISBN 9782729868352. ↑ iv, viii, 2.

[17] Loray, F.: *Pseudo-groupe d'une singularité de feuilletage holomorphe en dimension deux*. HAL, 00016434, 2005. ↑ 144, 192.

[18] Loray, F.: *Pseudo-groupe d'une singularité de feuilletage holomorphe en dimension deux*. Prépublication IRMAR, 2005. ↑ 152, 175.

[19] Mardešić, S.: *Matematička analiza u n-dimenzionalnom realnom prostoru I*. Školska knjiga, Zagreb, 1974. ↑ 35.

[20] Mardešić, P. and M. Resman: *Analytic moduli for parabolic Dulac germs*. Russian mathematical surveys, 76(3):13–92, 2021, ISSN 0036-0279. ↑ iv, viii, 3, 5, 179.

[21] Mardešić, P., M. Resman, J. P. Rolin, and V. Županović: *Normal forms and embeddings for power-log transseries*. Adv. Math., 303:888–953, 2016. ↑ iii, iv, vii, viii, ix, 2, 3, 9, 12, 16, 20, 22, 27, 30, 37, 38, 40, 59, 71, 94, 96, 97, 196, 197, 201, 239.

[22] Mardešić, P., M. Resman, J. P. Rolin, and V. Županović: *The Fatou coordinate for parabolic Dulac germs*. J. Differential Equations, 266(6):3479–3513, 2019. ↑ iv, viii, 3, 5, 9, 179.

[23] Mardešić, P., M. Resman, J. P. Rolin, and V. Županović: *Tubular neighborhoods of orbits of power-logarithmic germs*. Journal of Dynamics and Differential Equations, 2019, ISSN 1572-9222. `https://doi.org/10.1007/s10884-019-09812-8`. ↑ 97, 139.

[24] Milnor, J.: *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006. ↑ v, ix, 3, 44, 73, 144, 146, 152, 161, 175, 179, 239.

[25] Munkres, J.R.: *Topology*. Featured Titles for Topology. Prentice Hall, Incorporated, 2000, ISBN 9780131816299. ↑ 22, 23.

[26] Navas, A.: *Groups of circle diffeomorphisms*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2011. ↑ 4, 5, 46.

[27] Neumann, B. H.: *On ordered division rings*. Trans. Amer. Math. Soc., 66:202–252, 1949. ↑ 11.

[28] Peran, D.: *Normal forms of parabolic logarithmic transseries*. preprint in preparation, 2021. ↑ 94, 103, 112, 228, 230.

[29] Peran, D., M. Resman, J. P. Rolin, and T. Servi: *Normal forms of hyperbolic logarithmic transseries*. `https://arxiv.org/pdf/2105.10660.pdf`, submitted, 2021. ↑ 3, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 23, 26, 27, 28, 30, 32, 36, 38, 39, 41, 42, 43, 45, 47, 49, 54, 55, 56, 58, 59, 60, 62, 68, 69, 195, 196, 198, 208, 213, 214, 217, 224, 225.

[30] Peran, D., M. Resman, J.P. Rolin, and T. Servi: *Linearization of complex hyperbolic Dulac germs*. Journal of Mathematical Analysis and Applications, 508(1):1–27, 2022. `https://doi.org/10.1016/j.jmaa.2021.125833`. ↑ 4, 143, 144, 145, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 163, 164, 168, 171, 172, 173, 175.

[31] Perko, L.: *Differential Equations and Dynamical Systems*. Texts in Applied Mathematics. Springer, New York, NY, 2001. ↑ 1.

[32] Roussarie, R.: *Bifurcation of planar vector fields and Hilbert's sixteenth problem*, volume 164 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1998, ISBN 3-7643-5900-5. ↑ iii, vii, 1.

[33] Schröder, E.: *Ueber iterirte Functionen*. Math. Ann., 3(2):296–322, 1870. ↑ 4, 152.

[34] Sternberg, S.: *Local c^n transformations of the real line*. Duke Math. J., 24:97–102, 1957. ↑ 4, 5.

[35] Szekeres, G.: *Regular iteration of real and complex functions*. Acta Math., 100:203–258, 1958. ↑ 4.

[36] Xiang, T. and S. G. Georgiev: *Noncompact-type Krasnoselskii fixed-point theorems and their applications*. Math. Methods Appl. Sci., 39(4):833–863, 2016. ↑ 3, 36, 47.

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- D. Peran, M. Resman, J.-P. Rolin, T. Servi, *Normal forms of hyperbolic logarithmic transseries*, poslano, 2021., https://arxiv.org/pdf/2105.10660.pdf

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