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Veronika Pedić Tomić

Whittakerovi moduli i pravila fuzije za Weylovu verteks-algebru, afine verteks-agebre i njihove invarijantne podalgebre

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SUMMARY

The topic of this thesis are two problems in the vertex operator algebra theory: determination of fusion rules and the orbifold problem. For the fusion rules problem we study the example of Weyl vertex algebra, also known as the $\beta\gamma$ ghost system. This is a non-rational vertex algebra, hence we give a first proof of a Verlinde formula for non-rational VOAs and confirm the Verlinde type conjecture given by D. Ridout and S. Wood in [70]. For the orbifold problem, we extend a theorem given in the Dong-Mason quantum Galois theory paper [41], from the category of ordinary modules to the whole category of weak modules. The proof given by Dong and Mason necessarily involves Zhu's theory, and therefore can not be extended to the category of weak modules. In particular, we study the example of Whittaker modules for the Weyl vertex algebra and Heisenberg VOA.

In the first part of this thesis, we calculate fusion rules in the category of weight modules for the Weyl vertex algebra. Our proof is entirely vertex-algebraic and it uses the theory of intertwining operators for vertex algebras and the fusion rules for the affine vertex superalgebra $L_1(\mathfrak{gl}(1|1))$. Moreover, we explicitly construct the intertwining operators involved. We also prove a general irreducibility result which relates irreducible weight modules for the Weyl vertex algebra M to irreducible weight modules for $L_1(\mathfrak{gl}(1|1))$.

In the second part of this thesis we prove a theorem on irreducible weak V-module W and an automorphism g of finite order. Here either $W \circ g^i \ncong W$ for all i, in which case W is an irreducible $V^{\langle g \rangle}$–module, or $W \cong W \circ g$ in which case W is a direct sum of p irreducible $V^{\langle g \rangle}$–modules. The key idea of our proof was to consider a "big" module for the vertex algebra, constructed as a direct sum of modules $\bigoplus_i W \circ g^i$. Moreover, we present a counterexample for the expansion of our theorem to the case of infinite-dimensional group G for the irreducible Weyl algebra modules of Whittaker type.

Key words: vertex algebra, vertex operator algebra, ordinary modules, weak modules, Whittaker modules, Weyl vertex algebra, Heisenberg vertex operator algebra, quantum Galois theory, orbifold problem, fusion rules, intertwining operators, Verlinde formula, fusion algebra, simple current, vertex algebra automorphism, lattice vertex superalgebra, Verma module, Clifford vertex algebra

SAŽETAK

U ovoj disertaciji proučavamo dvije teme teorije verteks-algebri: računanje pravila fuzije te problem podalgebre fiksnih točaka. Verteks-algebra za koju računamo pravila fuzije je Weylova verteks-algebra ili $\beta\gamma$ sistem. To je iracionalna verteks-algebra i naš dokaz je prvi dokaz Verlindeove formule za slučaj iracionalnih verteks-algebri te potvrđujemo slutnju iznesenu u članku D. Ridouta i S. Wooda [70]. Za problem podalgebre fiksnih točaka proširujemo teorem C. Donga i G. Masona iz njihova članka o kvantnoj Galoisovoj teoriji [41], s kategorije jakih modula na cijelu kategoriju slabih modula. Dokaz iznesen u [41] ne može se proširiti na slabe module jer koristi Zhuovu teoriju. Svoj rezultat primjenjujemo na Weylovu verteks-algebru, ali i Heisenbergovu algebru verteks-operatora te za obje promatramo kategoriju Whittakerovih modula.

U prvom dijelu disertacije računamo pravila fuzije u kategoriji težinskih modula Weylove verteks-algebre. Naš je dokaz potpuno uklopljen u teoriju verteks-algebri te koristi teoriju operatora ispreplitanja verteks-algebri i pravila fuzije za afinu verteks-superalgebru $L_1(\mathfrak{gl}(1|1))$. Štoviše, eksplicitno smo konstruirali operatore ispreplitanja koji se javljaju u iskazu. Također, pokazali smo općeniti rezultat koji povezuje ireducibilne težinske module Weylove verteks-algebre M s ireducibilnim težinskim modulima za $L_1(\mathfrak{gl}(1|1))$.

U drugom dijelu disertacije dokazali smo teorem o ireducibilnim slabim V–modulima W i automorfizmu g konačnog reda. Naime, pokazali smo da je ili $W \circ g^i \ncong W$, za sve i, i u tom slučaju je W ireducibilan $V^{\langle g \rangle}$–modul, ili je $W \cong W \circ g$, i u tom slučaju je W direktna suma p ireducibilnih $V^{\langle g \rangle}$–modula. Glavna ideja našeg dokaza je bila konstruirati "veliki" modul za verteks-algebru, tako da uzmemo direktnu sumu modula $\bigoplus_i W \circ g^i$. Nadalje, dajemo protuprimjer za proširenje našeg teorema na slučaj beskonačno-dimenzionalne grupe automorfizama G za ireducibilne Whittakerove module Weylove verteks-algebre.

Ključne riječi: verteks-algebra, algebra verteks-operatora, jaki moduli, slabi moduli, Whittakerovi moduli, Weylova verteks-algebra, Heisenbergova algebra verteks-operatora, kvantna Galoisova teorija, problem podalgebre fiksnih točaka, pravila fuzije, operatori ispreplitanja, Verlindeova formula, fuzijska algebra, prosta struja, automorfizam verteks-algebri, verteks-superalgebra pridružena rešetci, Vermaov modul, Cliffordova verteks-algebra

Contents

1. Introduction

The theory of vertex algebras is a modern mathematical theory with beginnings dating back to the 1980s. It was then that Richard Borcherds (cf. [24]), inspired by I. Frenkel, J. Lepowsky and A. Muerman's moonshine module $V^\natural$ (cf. [51]), proved the moonshine conjecture on the Monster group proposed by J. H. Conway and S.P. Norton (cf. [25]). For this proof he used new objects which he called vertex algebras. These new objects were in fact a rigorous mathematical description of chiral algebras appearing in conformal field theory. Since then the theory of vertex algebras flourished and many interesting connections have been made to various other mathematical theories.

For example, there is a strong connection between vertex algebraic theory and the number theory via the modular forms. If we take a vertex operator algebra V of central charge c such that $V = \bigoplus_{n\geq 0} V_n$, where $\dim V_n < \infty$, and define

$$\mathrm{ch}[V](q) = q^{-\frac{c}{24}} \sum_{n=0}^{\infty} (\dim V_n) q^n,$$

where $q = e^{2\pi i\tau}$, and Re(τ)>0, then for certain vertex algebras (e.g. for holomorphic VOAs), we will get a modular form.

We should also mention that vertex operators appeared in the mathematical literature before the formal definition of a vertex algebra was given, in particular in the work of J. Lepowsky and R. Wilson (cf. [63]), where they presented an explicit free-field realization of the basic representation of the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$.

There is also a connection of vertex algebra theory to the Langlands program. Namely, vertex algebra theory can be a useful tool in the construction of some conjectural geometric Langlands correspondences (cf. [48]).

In this thesis we study two important problems of the vertex algebra theory: **determination of fusion rules for a vertex algebra** and the **Orbifold subalgebras of vertex**

algebras and their Whittaker modules.

Fusion rules for vertex algebras

Fusion rules are defined in [50] as the dimension of vector spaces of intertwining operators. Fusion rules for rational vertex algebras can be obtained using the Verlinde formula, proposed by physicists and proved by Y. Z. Huang in [57]. For non-rational vertex algebras which have infinitely many irreducible modules, the determination of fusion rules is still an open problem in most cases. A natural method would be to determine the category of modules which are closed under fusion rules, and try to determine fusion rules in that category. It is a recent trend that researchers first propose a Verlinde type formula, although this procedure is still not completely rigourous. This way we can get some conjectures which can then be verified. Since the Verlinde formula is based on modular transformation of irreducible characters, it is necessary to consider weight modules with finite-dimensional weight spaces. It turns out that this category should, as an addition to the usual highest weight modules, also include the relaxed highest weight modules, which appeared in the cases of Weyl vertex algebra and affine vertex algebras on non-integral levels.

Let us mention some examples and recent progress in this direction.

1. The singlet vertex algebra $\mathscr{M}(p)$ associated to $(1,p)$–modules for the Virasoro algebras. The irreducible modules were classified in [8]. The Verlinde type conjecture is presented in [29], and the case $p=2$ is proved in [15]. The fusion rules for $p>2$ are still not determined. A vertex tensor category approach to to the representation theory of singlet algebras was recently proposed in [28].

2. The affine vertex algebra $L_k(\mathfrak{sl}(2))$. The irreducible representations are classified in [14]. The Verlinde type formula is presented in [31]. These fusion rules are still unproved. Some intertwining operators predicted by the Verlinde formula are constructed in [11].

3. Weyl vertex algebra. The Verlinde type formula is presented in [70]. The proof is given in [16], and it is included in the Chapter 3 of this dissertation. A vertex

tensor categorical approach to the representation theory of Weyl vertex algebra has appeared in a recent preprint [21].

Orbifold subalgebras of vertex algebras and their Whittaker modules

Coset and orbifold constructions are two basic ways to construct new vertex algebras from the given ones. In this thesis we are focused more on the orbifold constructions.

Orbifolds have long been an important part of mathematics and physics as generalizations of manifolds. However, recently a new meaning has been given to the word orbifold as a part of the string theory. They were first introduced in [32], where the authors constructed new conformal field theories from the old ones by applying automorphisms.

Mathematically formulated, the idea of orbifold theory of vertex algebras is to take a vertex algebra V and some group of its automorphisms G, and study the representation theory of the fixed point subalgebra V^G. This theory was initiated in 1994 by C. Dong and G. Mason in [41], and a rigorous mathematical foundation to the theory of rational orbifold models in conformal field theory was given in [40]. However, the quantum Galois theory established by Dong and Mason relies on the $\mathbb{Z}$–gradation of modules in the representation theory of the orbifolds, semi-simple action of the Virasoro algebra and uses Zhu's theory. Therefore, it cannot be applied to weak modules, such as the Whittaker modules.

Whittaker modules originate from complex Lie theory, but recently they have become a matter of great interest in the vertex operator algebra theory as well. Let V be a vertex algebra and let $\mathfrak{L}$ be the Lie algebra associated to V or the Lie algebra of modes of generating fields of V. Let $\mathfrak{n}$ be a nilpotent subalgebra of $\mathfrak{L}$. Then Whittaker modules are defined by using a Lie algebra homomorphism $\lambda : \mathfrak{n} \to \mathbb{C}$, and there is no constraint on the action of the Cartan subalgebra $\mathfrak{h}$. Therefore, Whittaker modules are not weight modules, and the Dong-Mason theory can not be applied to them.

Let us mention some examples and recent progress regarding the irreducibility of Whittaker modules for vertex algebras.

1. Whittaker modules for affine vertex algebras. The case of affine Lie algebra $\widehat{\mathfrak{sl}_2}$ of type $A_1^{(1)}$ with noncritical level is proved in [17]. A family of Whittaker modules

for $\widehat{\mathfrak{sl}(2)}$ and $\widehat{\mathfrak{osp}(1|2)}$, and all irreducible degenarate Whittaker modules for $V_k(\mathfrak{sl}_2)$ are constructed in [11].

2. Whittaker modules for Heisenberg vertex operator algebra. In [71] it is proved that any simple weak module for the Heisenberg vertex algebra orbifold $M(1)^+$ with at least one Whittaker vector is isomorphic to some simple weak $M(1)$–module or to some θ–twisted simple weak $M(1)$–module. In [56], previous result is generalized to Heisenberg algebras of higher rank.

3. Whittaker modules for Virasoro vertex operator algebras. In [69] irreducible Whittaker modules for the Virasoro algebra are classified.

Let us now present the contents of this thesis.

Chapter 2: Preliminaries

In this chapter we give basic definitions of the theory of vertex algebras necessary for our work. The literature we follow in this chapter are monographies by J. Lepowsky, H. Li (cf. [62]), V. Kac (cf. [58]) and E. Frenkel, D. Ben-Zvi (cf. [48]). We start with a brief overview of formal calculus underlying the definition of a vertex algebra. Next we recall the notion of vertex algebra and vertex operator algebra, the notion of weak and ordinary vertex algebra modules, and vertex algebra automorphisms. Finally, we define universal and simple affine vertex algebras.

Chapter 3: Fusion rules and intertwining operators for the Weyl vertex algebra

This chapter is joint work with D. Adamović and it is published in Journal of Mathematical Physics [16].

The notion of fusion rules is one of the most important concepts both in the mathematical theory of vertex algebras, but also in its physical counterpart - conformal field theory. In the language of theoretical physics, fusion rules determine which fields appear in the operator product expansion of two primary fields, and in the language of mathematics fusion rules determine the dimension of the space of intertwining operators between three vertex algebra modules.

One of the deepest results in conformal field theory is the so called Verlinde formula. It connects the local and global properties of the conformal field theory by establishing a relation between the modular S–matrix and the fusion algebra for the operator product expansion on the sphere (cf. [72]). In mathematical language, fusion rules can be determined by using the Verlinde formula for rational vertex algebras, as was proved by Y. Z. Huang [57]. However, so far there is no proof that fusion rules for non-rational vertex algebras can be determined by using the Verlinde formula.

In this chapter we study the case of Weyl vertex algebra M, also known as the $\beta\gamma$ system in physics literature. This is a non-rational vertex algebra and it admits a category of relaxed modules $\mathscr{R}$, whose objects are $\mathbb{Z}_{+}$–graded modules. In this chapter, we construct modules $\widetilde{U(\lambda)}$ from that category and discuss their irreducibility. However, contrary to the case of rational vertex algebras, the category of $\mathbb{Z}_{+}$–graded modules is not closed under fusion. Therefore, it is natural to consider a larger category $\mathscr{F}$, obtained by applying the spectral flow automorphism ρ_s. In their article [70], D. Ridout and S. Wood give a conjecture on fusion rules for the category $\mathscr{F}$, and this is a Verlinde type conjecture for fusion rules.

Our first main result of this chapter is proving this conjecture:

Main Theorem 1. Assume that $\lambda, \mu, \lambda+\mu \in \mathbb{C} \setminus \mathbb{Z}$. Then we have:

$$\rho_{\ell_1}(M) \times \rho_{\ell_2}(\widetilde{U(\lambda)}) = \rho_{\ell_1+\ell_2}(\widetilde{U(\lambda)}), \tag{1.1}$$

$$\rho_{\ell_1}(\widetilde{U(\lambda)}) \times \rho_{\ell_2}(\widetilde{U(\mu)}) = \rho_{\ell_1+\ell_2}(\widetilde{U(\lambda+\mu)}) + \rho_{\ell_1+\ell_2-1}(\widetilde{U(\lambda+\mu)}), \tag{1.2}$$

where $\widetilde{U(\lambda)}$ is an irreducible weight module, and ρ_ℓ, $\ell \in \mathbb{Z}$, are the spectral flow automorphisms defined by (3.8).

The fusion rule (1.1) is proved in Proposition 3.3.5, and it is a direct consequence of a construction by H. Li (cf. [65]). The main contribution of our work is a vertex-algebraic proof of (1.2) (cf. Corollary 3.6.4) which uses the theory of intertwining operators for vertex algebras and fusion rules for the affine vertex superalgebra $L_1(\mathfrak{gl}(1|1))$.

We also prove a general irreducibility result which relates irreducible weight modules for the Weyl vertex algebra M to irreducible weight modules for $L_1(\mathfrak{gl}(1|1))$ (see Theorem 3.5.3).

Main Theorem 2. Assume that $\mathscr{N}$ is an irreducible weight M–module. Then $\mathscr{N} \otimes F$ is a completely reducible $L_1(\mathfrak{gl}(1|1))$–module.

It is known that fusion rules can be determined by using fusion rules for the singlet vertex algebra (cf. [15], [29]). However, we believe that our methods, which use $L_1(\mathfrak{gl}(1|1))$, can be generalized to a wider class of vertex algebras.

In Section 3.2. we list some important basic definitions necessary to understand fusion rules, such as the intertwining operators and fusion coefficients. We also give a very important proposition (cf. 3.2.1) which allows us to produce a new intertwining operator from an existing one via an appropriate vertex algebra automorphism.

In Section 3.3. we define the Weyl vertex algebra M. We list a family of Virasoro vectors ω_μ and a Heisenberg vector β for M. This way we can define the notion of a weight module for M to be the one where the operators $\beta(0)$ and $L(0)$ act semisimply. We give two proofs of an important lemma regarding the action of $L(0)$ on the top level of a graded M–module. One of the proofs of this lemma also uses Zhu's theory. We prove the first part of our Main theorem 1 by slightly modifying the proof by H. Li in [64]. We define spectral flow automorphisms on M. We construct a category $\mathscr{K}$ of irrreducible weight modules for M. We describe a family of weight modules with infinite-dimensional weight spaces, which are not in $\mathscr{K}$.

In Section 3.4. we construct intertwining operators appearing in the second part of our Main theorem 1. For this purpose we introduce a subalgebra $\Pi(0)$ of a lattice vertex superalgebra V_L, into which M is embedded. Following [36], we construct one intertwining operator in the category of $\Pi(0)$–modules, and restrict it to M. Then we obtain a second intertwining operator using an automorphism of M.

In Section 3.5. we setup the necessary background for calculating the fusion rules. We take Verma modules for the Lie superalgebra $\mathfrak{gl}(1|1)$ and induce them to Verma modules $\widehat{\mathscr{V}_{r,s}}$ of level 1 for the simple affine vertex algebra of rank 1 associated to $\mathfrak{gl}(1|1)$. We obtain a result on fusion rules for $L_1(\mathfrak{gl}(1|1))$–modules $\widehat{\mathscr{V}_{r,s}}$. We tensor the M with the Clifford vertex algebra F, and the zero level of this product is isomorphic to $L_1(\mathfrak{gl}(1|1))$.

In Section 3.6. we finish the calculation of fusion rules for M. We prove the fusion rules result in the category of modules for the tensor product $M \otimes F$, and then shift this result to the category of modules for M by using a natural isomorphism of the spaces of intertwining operators.

Chapter 4: Irreducibility of modules of Whittaker type for cyclic orbifold vertex algebras

This chapter is joint work with D. Adamović, C.-H. Lam and N. Yu and it is published in Journal of Algebra (cf. [12]).

In this chapter we study the orbifold problem of the theory of vertex algebras, a very important tool in understanding their structure theory. In a nutshell, given a vertex algebra V and some finite group of automorphisms G of V, one wants to draw conclusions on the representation theory of the fixed-point vertex subalgebra V^G. Therefore, orbifold theory provides new examples of vertex algebras by looking at the existing ones. Since it is rather difficult to find nontrivial examples of vertex algebras, orbifold subalgebras are a wonderful source of possibly interesting examples.

In their article [41], C. Dong and G. Mason founded the study of Galois theory for vertex operator algebras, and as a key point in establishing this theory, they presented the following result in orbifold theory:

Theorem 1.0.1 (cf. [41])**.** Let V be a vertex algebra, g be an automorphism of finite primitive order, and $V^{\langle g\rangle}$ be the vertex subalgebra of fixed points under g of V. Let M be an irreducible ordinary module for V such that it is not isomorphic to $M\circ g^i$. Then M is also an irreducible module for the subalgebra $V^{\langle g\rangle}$.

This result has already proven useful in the construction of irreducible modules for vertex algebras $M(1)^+$ and V_L^+ (cf. [1–5, 33–35, 42–44]), and recently also for irreducible modules for subalgebras of the triplet vertex algebra $\mathscr{W}(p)$ (cf. [13]).

It is natural to ask whether the Dong-Mason theorem holds for any weak V–module. The proof presented by Dong and Mason cannot be extended to the case of arbitrary weak modules since it necessarily involves Zhu's algebra theory. However, in this chapter we prove the complete extension of the Dong-Mason theorem to the weak modules, and present an important application to the weak modules of Whittaker type.

Recently, Whittaker modules have been studied within the vertex operator algebra theory in [17, 56, 71], and the latter two give a Lie theoretic proof of irreducibility of some Whittaker modules for the Heisenberg vertex algebra orbifold. This was another motivation for our work as we give a vertex algebraic proof of these statements, which can be applied more generally. Here we present only the result on non-twisted modules and we study only basic examples of Heisenberg and Weyl vertex algebras. However, in the future we expect to prove a more general result which would include twisted modules and $\mathscr{W}$–algebras.

Our first main theorem of this chapter is the following (see Theorem 4.5.3 for part (1) and Theorem 4.6.3 for part (2)):

Main Theorem 3. Let W be an irreducible weak V–module and g an automorphism of finite order p.

(1) Assume that $W \circ g^i \ncong W$ for all i. Then W is an irreducible $V^{\langle g \rangle}$–module.

(2) Assume that $W \cong W \circ g$. Then W is a direct sum of p irreducible $V^{\langle g \rangle}$–modules.

The most convenient category of modules on which we apply our first main theorem are the Whittaker modules for certain infinite-dimensional Lie algebras, due to being uniquely described by their Whittaker function. Therefore, we have our second main result (cf. Theorem 4.7.10):

Main Theorem 4. Let W be an irreducible weak V–module such that all $W_i = W \circ g^i$ are Whittaker modules whose Whittaker functions $\lambda^{(i)} = \mathfrak{n} \to \mathbb{C}$ are mutually distinct. Then W is an irreducible weak $V^{\langle g \rangle}$–module.

In particular, we construct a family of Whittaker modules $M_1(1, \boldsymbol{\lambda})$ for the Heisenberg vertex algebra, where $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_r, 0, \cdots)$ is a sequence of elements of $\mathfrak{h}$ with at least one nonzero entry, and $\lambda_n = 0$ for $n \gg 0$. (cf. Proposition 4.8.1):

Main Theorem 5. (1) Assume that $g \in O(\ell)$ is of finite order such that $\boldsymbol{\lambda} \circ g^i \neq \boldsymbol{\lambda}$ for all i. Then $M(1, \boldsymbol{\lambda})$ is an irreducible $M(1)^{\langle g \rangle}$–module.

(2) Assume that $\boldsymbol{\lambda} \circ \sigma \neq \boldsymbol{\lambda}$ for any 2–cycle $\sigma \in S_\ell$. Then $M(1, \boldsymbol{\lambda})$ is an irreducible $M(1)^{\langle g \rangle}$–module for any $g \in S_\ell$.

Also, we we construct a family of Whittaker modules $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ where $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{C}^n \times \mathbb{C}^n$ for Weyl vertex algebra:

Main Theorem 6 (cf. Theorem 4.9.3)**.** Assume that $\Lambda = (\boldsymbol{\lambda}, \boldsymbol{\mu}) \neq 0$. Then $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible weak module for the orbifold subalgebra $M^{\mathbb{Z}_p}$, for each $p \geq 1$.

In Section 4.2. we recall the Dong-Mason theorem which we generalize.

In Section 4.3. we prove a technical result on cyclic vectors in a direct sum of non-isomorphic weak modules for a vertex operator algebra. To prove this result, we use a similar result for associative algebras and then we use the Lie algebra $\mathfrak{g}(V)$ associated to the vertex operator algebra to lift the associative algebra result to the vertex algebra case. This will be a lemma for the order two case of our Main theorem 3.

In Section 4.4. we prove the case of automorphisms of order two of the first part of our Main Theorem 3. The key idea is to consider a "big" module for the vertex algebra constructed as a direct sum of both the V–module W and the V–module $W \circ \theta$, where θ is an order two automorphism.

In Section 4.5. we prove the first part of our Main Theorem 3. The proof is a generalized proof of the one given in Section 4.4 and here we have an automorphism g of arbitrary order p (not necessarily prime). We use a generalized version of the technical lemma from Section 4.3. and the "big" module is now a direct sum of p factors, $\mathscr{M} = W_0 \oplus W_1 \oplus \cdots \oplus W_{p-1}$, where $W_i = W \circ g^i, i = 0, 1, \cdots, p-1$.

In Section 4.6. we prove the second part of our Main Theorem 3. For the proof we use Schur's Lemma and the fact that every irreducible weak V–module W is countable dimensional.

In Section 4.7. we give some structural results on Whittaker modules. We recall the Lie algebra $\mathfrak{g}(V)$ associated to vertex algebra V and we discuss some cases in which it is possible to replace it with a smaller Lie algebra (cf. 4.7.3). We define the standard (universal) Whittaker $\mathfrak{L}$–module M_λ and modules of Whittaker type λ. We show how the Whittaker module of type λ is uniquely determined by the function λ. Finally, we prove our main result 2.

In Section 4.8. we apply our main theorem 3 to the case of Heisenberg vertex algebra and prove Main Theorem 5. We recall basic definitions regarding the Heisenberg Lie algebra and Heisenberg vertex algebra and we prove irreducibility of certain Whittaker modules $M(1,\lambda)$ as modules for the Heisenberg vertex algebra orbifold.

In Section 4.9. we apply our main theorem to the case of Weyl vertex algebra. We recall basic definitions regarding the Weyl associative algebra with generators and relations and Weyl vertex algebra. We prove our Main Theorem 6 and we demonstrate how this result gives a construction of new irrreducible modules for affine Lie algebra $\widehat{\mathfrak{sl}(2)}$ associated to $\mathfrak{sl}(2)$.

Chapter 5: New results on the structure of Whittaker modules for certain vertex algebras

This chapter is based on a preprint (cf. [20]) and it is joint work with D. Adamović.

In this chapter we continue where we left of in Chapter 4. We study irreducible Whittaker modules $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ with non-trivial Whittaker functions for the Weyl vertex algebra M as modules for its orbifold. However, in this chapter we consider the limit case of infinite-dimensional group of automorphisms, and prove that irreducible Weyl vertex algebra modules of Whittaker type are actually always reducible as M^G–modules. Therefore, this is a counterexample to the generalization of our work in Chapter 4 to infinite-dimensional group of automorphisms.

Our main theorem of this chapter is the following (see Theorem 5.2.3):

Main Theorem 7.

(1) $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a reducible $\widehat{\mathfrak{g}l}$–module.

(2) $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a reducible $\mathscr{W}_{1+\infty}$–module at central charge $c = -1$.

The statement (2) of the main theorem will then imply that $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is reducible also as an M^0–module.

In Section 5.1., we recall two important realizations of the Weyl vertex algebra orbifold M^0:

- M^0 is isomorphic to the vertex algebra $\mathscr{W}_{1+\infty}$-algebra at central charge $c = -1$.
- M^0 is isomorphic to the simple module for the Lie algebra $\widehat{\mathfrak{g}l}$, which is the central extension of the Lie algebra of infinite matrices.

In Section 5.2. we introduce a Casimir element I of $\widehat{\mathfrak{gl}}$, and we show it acts non-trivially on Whittaker vectors $\mathbf{w}_{\boldsymbol{\lambda}, \boldsymbol{\mu}}$. We use I to prove our main result.

2. PRELIMINARIES

In this chapter we list all of the important basic definitions that appear in our work, such as the definition of vertex algebras and vertex operator algebras, weak and ordinary modules for vertex operator algebras and vertex algebra automorphisms. For more details and for an overview of the structural theory of vertex algebras we refer the reader to monographies J. Lepowsky, H. Li (cf. [62]), V. Kac (cf. [58]) and E. Frenkel, D. Ben Zvi (cf. [48]).

2.1. FORMAL CALCULUS

In this section we follow [62] to present the formal calculus necessary for defining a vertex algebra.

Let $x, y, z, z_0, \dots$ denote the commuting formal variables.

We define several useful objects:

- $V\{z\} = \left\{ \sum_{\alpha\in\mathbb{C}} v_\alpha z^\alpha \;\middle|\; v_\alpha \in V \right\}$,
- $V[[z, z^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \;\middle|\; v_n \in V \right\}$ formal Laurent series,
- $V[[z]] = \left\{ \sum_{n=0}^{\infty} v_n z^{-n-1} \;\middle|\; v_n \in V \right\}$ formal Taylor series,
- $V((z)) = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \;\middle|\; v_n \in V,\ v_n = 0, \text{ for sufficiently negative } n \right\}$ truncated formal Laurent series,
- $V[z] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \;\middle|\; v_n \in V,\ \exists n_0 \in \mathbb{Z}_{\geq 0},\ \forall n \geq n_0,\ v_n = 0, \right\}$ Taylor polynomials,
- $V[z, z^{-1}] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \;\middle|\; v_n \in V,\ \exists N \in \mathbb{Z},\ \forall n \leq N,\ v_n = 0 \right\}$ Laurent polynomials.
- $R(z_1, z_2)$ rational function with poles $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$,

- $\iota_{z_1,z2}R(z_1,z_2)$ Laurent expansion for $|z_1|>|z_2|$, where ι_{z_1,z_2} is a linear operator from rational functions to Laurent series,
- $\iota_{z_2,z1}R(z_1,z_2)$ Laurent expansion for $|z_2|>|z_1|$.

We calculate

$$\frac{1}{z_1-z_2}=\frac{1}{z_1}\cdot\frac{1}{1-\frac{z_2}{z_1}}=\frac{1}{z_1}\sum_{m=0}^{\infty}\left(\frac{z_2}{z_1}\right)^m=\sum_{m=0}^{\infty}\frac{z_2^m}{z_1^{m+1}}=\sum_{m=0}^{\infty}z_2^m z_1^{-m-1}.$$

Therefore,

- $\iota_{z_1,z_2}\frac{1}{z_1-z_2}=\sum_{m=0}^{\infty}z_2^m z_1^{-m-1}$,
- $\iota_{z_2,z_1}\frac{1}{z_1-z_2}=-\sum_{m=0}^{\infty}z_1^m z_2^{-m-1}$.

In other words, we have two different power series expansions for a single rational function, depending on the domain of the function.

Also,

$$\iota_{z_2,z_1}\frac{1}{(z_1-z_2)^{j+1}}=-\sum_{m=-1}^{-\infty}\binom{m}{j}z_1^{-m-1}z_2^{m-j}, \tag{2.1}$$

$$\text{and}\quad \iota_{z_1,z_2}(z_1-z_2)^r=\sum_{l=0}^{\infty}\binom{r}{l}(-1)^l z_1^{r-l}z_2^l\in\mathbb{C}z_1,z_2. \tag{2.2}$$

The consensus on the notation is the following

$$(z_1-z_2)^r=\iota_{z_1,z_2}(z_1-z_2)^r, \tag{2.3}$$

and we have that

$$(z_1-z_2)^r=(-z_2+z_1)^r \iff r\in\mathbb{Z}_{\geq 0}.$$

In general (if $r\notin\mathbb{Z}_{\geq 0}$), we have that $(z_1-z_2)^r\neq(-z_2+z_1)^r$.

- $\delta(z)=\sum_{n\in\mathbb{Z}}z^n$ Delta function, formal Laurent series

$$\frac{1}{z_1-z_2}-\frac{1}{-z_2+z_1}=z_2^{-1}\delta(\frac{z_1}{z_2}). \tag{2.4}$$

Similarly,

$$\frac{1}{(z_1-z_2)^{j+1}} - \frac{1}{|-z_2+z_1|^{j+1}} = \frac{1}{j!}\partial_{z_2}^{j}(z_2^{-1}\delta(\frac{z_1}{z_2}). \tag{2.5}$$

Let $f \in \mathbb{C}[z_1, z_2, z_1^{-1}, z_2^{-1}]$.

We have

$$f(z_1,z_2)\delta(\frac{z_1}{z_2}) = f(z_1,z_1)\delta(\frac{z_1}{z_2}) = f(z_2,z_1)\delta(\frac{z_1}{z_2}). \tag{2.6}$$

Lemma 2.1.1. Let $m, n \in \mathbb{Z}_{\geq 0}$, $m > n$. We have

$$(z_1-z_2)^m\delta^{(n)}(\frac{z_1}{z_2}) = 0,$$

where the n-th derivative is partial derivative over z_1.

Definition 2.1.2. Let $f(z) = \sum v_n z^n$. *Residue of f at z* is defined as

$$Res_z f(z) = v_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)dz.$$

Let $f(z) \in \mathbb{C}((z))$, $g(z) \in V((z))$. Then $f(z)g(z)$ is well-defined and we have

$$Res_z(f^{'}(z)g(z)) = -Res_z(f(z)g^{'}(z)). \tag{2.7}$$

Lemma 2.1.3. (cf. [62]) Let $f(z) \in V\{z\}$. We have the so called *Taylor's formula*

$$e^{z_0\partial_z}f(z) = f(z+z_0).$$

Lemma 2.1.4. (cf. [62])

(i) $z_0^{-1}\delta(\frac{z_1-z_2}{z_0}) - z_0^{-1}\delta(\frac{-z_2+z_1}{-z_0}) = z_2^{-1}\delta(\frac{z_1-z_0}{z_2})$.

(ii) $z_2^{-1}\delta(\frac{z_1-z_0}{z_2}) = z_1^{-1}\delta(\frac{z_2+z_0}{z_1})$.

2.2. VERTEX ALGEBRA AND VERTEX OPERATOR ALGEBRA

In this section we follow [62] in defining vertex algebra, vertex operator algebra and related objects.

Let $V = V^{\bar{0}} \oplus V^{\bar{1}}$ be a $\mathbb{Z}/2\mathbb{Z}$–graded vector space. A vector $v \in V^{\bar{0}}$ (resp. $v \in V^{\bar{0}}$) is called even (resp. odd). We write $\bar{v} = 0$ if $v \in V^{\bar{0}}$, and $\bar{v} = 1$ if $v \in V^{\bar{1}}$.

Definition 2.2.1. *Vertex superalgebra* is a triple $(V, Y, \mathbf{1})$, where $V = V^0 \oplus V^1$ is a $\mathbb{Z}/2\mathbb{Z}$–graded vector space, Y is a linear map,

$$Y : V \to (\text{End } V)[[z, z^{-1}]],$$
$$v \mapsto \sum_{n \in \mathbb{Z}} v_n z^{-n-1} = Y(v, z),$$

and $\mathbf{1}$ is a distinguished vector called *vacuum vector*, such that it satisfies the following axioms on $\mathbb{Z}_2$–homogeneous elements $a, b \in V$:

- Y is a parity preserving, i.e. $Y(a,z)b \in V^{\bar{a}+\bar{b}}((z))$,
- $Y(a,z)b = \sum_{n \in \mathbb{Z}} a_n b z^{-n} \in V((z))$, i.e. $a_n b = 0$, for $n \gg 0$.
- (vacuum) $Y(\mathbf{1}, z) = \text{Id}$, i.e. $\mathbf{1}_n = \delta_{n,-1}\text{Id}$.
- (creation) $Y(a,z)\mathbf{1} \in (\text{End } V)[[z]]$ and $\lim_{z \to 0} Y(a,z)\mathbf{1} = a$, i.e. $a_n \mathbf{1} = 0$, $n \geq 0$, $a_{-1} = a$.
- $[D, Y(a,z)] = \frac{d}{dz} Y(a,z)$, where $D \in End(V)$, defined by $Da = a_{-2}$, for $a \in V$.
- (locality) $\exists n \in \mathbb{Z}_{\geq 0}$, depending on a and b,

$$(z_1 - z_2)^n [Y(a, z_1), Y(b, z_2)] = 0.$$

where $[Y(a,z_1), Y(b,z_2)]$ is denotes the commutator if a or b are even, and super commutator if a and b are odd.

If $V = V^{\bar{0}}$, we say that V is a *vertex algebra*.

Remark 2.2.2. We call $Y(v,x)$ the *vertex operator* or a *field*. Therefore, we sometimes call the mapping $v \mapsto Y(v,x)$ the *state-field correspondence*.

Definition 2.2.3. Let V be a vertex algebra, ω be a vector in V and $L(n) = \omega_{n+1}$. We say that ω is the *Virasoro* or *conformal vector* if

$$[L(n), L(m)] = (n-m)L(n+m) + \frac{n^3-n}{12}\delta_{n+m,0}c,$$

that is, the linear operators associated to ω satisfy the relations for the Virasoro algebra.

Definition 2.2.4. A vertex algebra V is a *conformal vertex algebra* if:

(i) there exists a conformal vector $\omega \in V$,

(ii) $D = L(-1) = \omega_0$,

(iii) $V = \bigoplus_{r\in\mathbb{C}} V^{(r)}$, $V^{(r)} = \{v \in V \mid L(0)v = rv\}$.

Definition 2.2.5. A vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_{(n)}$ such that

- $\mathrm{wt}(v) = n$, for $v \in V_{(n)}$,
- $\dim V_{(n)} < \infty$, for $n \in \mathbb{Z}$,
- and $V_{(n)} = 0$ for n sufficiently small,

equipped with a linear map $V \otimes V \to V[[z, z^{-1}]]$, or equivalently,

$$\begin{aligned} V &\to (EndV)[[z, z^{-1}]] \\ v &\mapsto Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \text{ (where } v_n \in EndV), \end{aligned}$$

$Y(v,z)$ denoting the vertex operator associated with v, and equipped also with two distinguished homogenous vectors $\mathbf{1} \in V_{(0)}$ (the vacuum) and $\omega \in V_{(2)}$. The following conditions are assumed for $u, v \in V$:

- $u_n v = 0$ for n sufficiently large (the lower truncation condition),
- $Y(\mathbf{1}, z) = \mathrm{Id}$,

- $Y(v,z)\mathbf{1} \in V[[z]]$ and $\lim_{z\to 0} Y(v,z)\mathbf{1} = v$ (creation property),

and the Jacobi identity holds

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2).$$

Also, the Virasoro algebra relations hold (acting on V):

$$[L(m),L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3-m)\delta_{n+m,0}(\mathrm{rk}\, V)\mathbf{1},$$

for $m,n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \text{ for } n \in \mathbb{Z} \text{ i.e., } Y(\omega,z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$$

and

$$\mathrm{rk}\, V \in \mathbb{C},$$

$$L(0)v = nv = (\mathrm{wt}\ v)v \text{ for } n \in \mathbb{Z} \text{ and } v \in V_{(n)},$$
$$\frac{d}{dz}Y(v,z) = Y(L(-1)v,z).$$

In other words, we say that a confomal vertex algebra is a *vertex operator algebra* if:

(i) $V = \bigoplus_{n\in\mathbb{Z}} V^{(n)}$,

(ii) $\dim V_n < \infty$, for all n

(iii) there exists a large enough integer $n \in \mathbb{Z}$ such that $V^{(n)} = \{0\}$, for all $n \leq N$.

Definition 2.2.6. *Vertex subalgebra $U \subset V$ is a D–invariant subspace, such that $\mathbf{1} \in U$ and*

$$a_n U = \{a_n u \mid u \in U\} \subseteq U, \quad a \in U.$$

In other words, $(U, Y\big|_U, \mathbf{1})$ is a vertex algebra.

For any group G of automorphisms of V, we have the *orbifold vertex algebra* or the *fixed point subalgebra* $V^G = \{v \in V \mid g(v) = v,\ g \in G\}$, which is a vertex subalgebra of V. If $G = \langle g\rangle$ is cyclic, we write $V^{\langle g\rangle}$ for V^G.

2.3. Modules for vertex algebras and vertex operator algebras

In this section we give the definition of weak and ordinary modules for vertex operator algebras.

Definition 2.3.1. A weak V–module is a pair (W, Y_W) where W is a complex vector space, and $Y_W(\cdot, z)$ is a linear map

$$\begin{aligned} Y_W:\ V &\to \mathrm{End}(W)[[z, z^{-1}]], \\ a &\mapsto Y_W(a,z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \end{aligned}$$

which satisfies the following conditions for $a, b \in V$ and $v \in W$:

- $a_n v = 0$ for n sufficiently large.
- $Y_W(\mathbf{1}, z) = I_W$.
- The following Jacobi identity holds:

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_W(a,z_1)Y_W(b,z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_W(b,z_2)Y_W(a,z_1) \\ = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_W(Y(a,z_0)b, z_2). \end{aligned}$$

Let $L(z) = Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$. Note that every weak V–module is a module for the Virasoro algebra generated by $L(n)$, $n \in \mathbb{Z}$.

Definition 2.3.2. A weak V–module (W, Y_W) is called an ordinary V–module if the following conditions hold:

- The $L(-1)$-derivative property: for any $a \in V$,

$$Y_W(L(-1)a, z) = \frac{d}{dz} Y_W(a,z).$$

- The grading property:

$$W = \oplus_{\alpha\in\mathbb{C}} W(\alpha), \quad W(\alpha) = \{v \in W \mid L(0)v = \alpha v\}$$

such that for every α, $\dim W(\alpha) < \infty$ and $W(\alpha+n) = 0$ for sufficiently negative $n \in \mathbb{Z}$.

Next, we define the notions of vertex algebra automorphism and vertex operator algebra automorphism.

Definition 2.3.3. We say that $g \in Aut_{\mathbb{C}}(V)$ is an automorphism of a vertex algebra V if $g(a_n b) = g(a)_n g(b)$ for all $a, b \in V$, $n \in \mathbb{Z}$.

Definition 2.3.4. We say that $g \in Aut_{\mathbb{C}}(V)$ is an automorphism of a vertex operator algebra V if

- $g(a_n b) = g(a)_n g(b)$ for all $a, b \in V$, $n \in \mathbb{Z}$.
- $g(\omega) = \omega$.

In other words, a vertex algebra automorphism with an additional property of preserving the conformal vector becomes a vertex operator algebra automorphism.

The following definition was introduced in [52]:

Definition 2.3.5. Vertex operator algebra is rational if it has finitely many irreducible modules and every finitely generated module is completely reducible.

2.4. AFFINE VERTEX ALGEBRAS

In this section we define universal and simple affine vertex (super)algebras associated to Lie (super)algebras. For more details see [58], [62]. Let $\mathfrak{g}$ be a complex simple Lie superalgebra with non-degenerate super-symmetric invariant bilinear form $(\cdot,\cdot)$. Let $\widehat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c$ be the associated affine Kac-Moody superalgebra. The commutation relations on $\widehat{\mathfrak{g}}$ are given by

$$[x(n),y(m)] = [x,y](n+m)+n(x,y)\delta_{n+m,0}c,$$

where $x(n) = x\otimes t^n$, $[c,\widehat{\mathfrak{g}}] = 0$. Identify $\mathfrak{g}$ with $\mathfrak{g}\otimes t^0$.

Consider the following Lie subalgebra $\mathfrak{p} = \mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}c$ of $\widehat{\mathfrak{g}}$. For any $k\in\mathbb{C}$, any $\mathfrak{g}$–module U can be equipped with the structure of a $\mathfrak{p}$–module such that for $v\in U$:

$$cv = kv, \quad (\mathfrak{g}\otimes t\mathbb{C}[t])v = 0.$$

Therefore, we have the following induced $\widehat{\mathfrak{g}}$–module:

$$N_{\widehat{\mathfrak{g}}}(k,U) = U(\widehat{\mathfrak{g}})\otimes_{U(\mathfrak{p})}U,$$

where $U(\widehat{\mathfrak{g}})$ is the universal enveloping algebra of $\widehat{\mathfrak{g}}$. The module $N_{\widehat{\mathfrak{g}}}(k,U)$ is sometimes called *generalized Verma module*.

In the special case where $U = \mathbb{C}v_0$ is a one-dimensional representation of $\mathfrak{g}$, we denote

$$V^k(\mathfrak{g}) = N_{\widehat{\mathfrak{g}}}(k,0),$$

and we say that $V^k(\mathfrak{g})$ is the universal affine vertex algebra associated with $\mathfrak{g}$ at level k. For any $x\in\mathfrak{g}$, we defined the field

$$x(z) = \sum_{n\in\mathbb{Z}} x(n)z^{-n-1}.$$

The fields $\{x(z) \mid x\in\mathfrak{g}\}$ are local on $V^k(\mathfrak{g})$, and applying the theorem on generating fields (cf. [58], [62]), $V^k(\mathfrak{g})$ becomes a vertex algebra, which we call the universal affine vertex algebra of level k. When the level k is not critical, $V^k(\mathfrak{g})$ has a unique irreducible quotient, which we denote by $L_k(\mathfrak{g})$.

The main example of affine vertex superalgebra appearing in this thesis is affine vertex algebra $L_1(\mathfrak{g})$ associated to the Lie superalgebra $\mathfrak{gl}(1|1)$ in Chapter 3.

3. Fusion rules and intertwining operators for the Weyl vertex algebra

3.1. Introduction

In the theory of vertex algebras and conformal field theory, determination of fusion rules is one of the most important problems. By a result by Y. Z. Huang (cf. [57]) for a rational vertex algebra, fusion rules can be determined by using the Verlinde formula. However, although there are certain versions of Verlinde formula for a broad class of non-rational vertex algebras, so far there is no proof that fusion rules for such algebras can be determined by using the Verlinde formula. One important example is the singlet vertex algebra for $(1,p)$–models whose irreducible representations were classified in [8]. Verlinde formula for fusion rules was also presented by T. Creutzig and A. Milas in [29], but so far the proof was only given for the case $p = 2$ in [15]. We should also mention that the fusion rules and intertwining operators for some affine and superconformal vertex algebras were studied in [7], [11], [27] and [60].

In this paper we study the case of the Weyl vertex algebra, which we denote by M, also called the $\beta\gamma$ system in the physics literature. Its Verlinde type conjecture for fusion rules was presented by S. Wood and D. Ridout in [70]. Here, we present a short proof of Verlinde conjecture in this case. We prove the following fusion rules result:

Theorem 3.1.1. Assume that $\lambda, \mu, \lambda+\mu \in \mathbb{C} \setminus \mathbb{Z}$. Then we have:

$$\rho_{\ell_1}(M) \times \rho_{\ell_2}(\widetilde{U(\lambda)}) = \rho_{\ell_1+\ell_2}(\widetilde{U(\lambda)}), \tag{3.1}$$

$$\rho_{\ell_1}(\widetilde{U(\lambda)}) \times \rho_{\ell_2}(\widetilde{U(\mu)}) = \rho_{\ell_1+\ell_2}(\widetilde{U(\lambda+\mu)}) + \rho_{\ell_1+\ell_2-1}(\widetilde{U(\lambda+\mu)}), \tag{3.2}$$

where $\widetilde{U(\lambda)}$ is an irreducible weight module, and ρ_ℓ, $\ell \in \mathbb{Z}$, are the spectral flow automorphisms defined by (3.8).

The fusion rule (3.1) is proved in Proposition 3.3.5, and it is a direct consequence of the construction of H. Li (cf. [65]). The main contribution of our paper is vertex-algebraic proof of (3.2) which uses the theory of intertwining operators for vertex algebras and the fusion rules for the affine vertex superalgebra $L_1(\mathfrak{gl}(1|1))$.

We also prove a general irreducibility result which relates irreducible weight modules for the Weyl vertex algebra M to irreducible weight modules for $L_1(\mathfrak{gl}(1|1))$ (see Theorem 3.5.3).

Theorem 3.1.2. Assume that $\mathscr{N}$ is an irreducible weight M–module. Then $\mathscr{N} \otimes F$ is a completely reducible $L_1(\mathfrak{gl}(1|1))$–module.

The construction of intertwining operators appearing in the fusion rules is based on two different embeddings of the Weyl vertex algebra M into the lattice vertex algebra $\Pi(0)$. Then one $\Pi(0)$–intertwining operator gives two different M–intertwining operators. Therefore, both intertwining operators are realized as $\Pi(0)$–intertwining operators. Once we tensor the Weyl vertex algebra M with the Clifford vertex algebra F, we can use the fusion rules for $L_1(\mathfrak{gl}(1|1))$ to calculate the fusion rules for M.

It is known that fusion rules can be determined by using fusion rules for the singlet vertex algebra (cf. [15], [29]). However, we believe that our methods, which use $L_1(\mathfrak{gl}(1|1))$, can be generalized to a wider class of vertex algebras. In our future work we plan to study the following related fusion rules problems:

- Connect fusion rules for higher rank Weyl vertex algebra with fusion rules for $L_1(\mathfrak{gl}(n|m))$.
- Extend fusion ring with weight modules having infinite-dimensional weight spaces (cf. Subsection 3.3.4) and possibly with irreducible Whittaker modules (cf. [12]).

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3.2. Fusion rules and intertwining operators

In this section we recall the definition of intertwining operators and fusion rules. More details can be found in [52], [50], [37], [26]. We also prove an important result on the action of certain automorphisms on intertwining operators. This result will enable us to produce new intertwining operators from the existing one.

Let V be a conformal vertex algebra with the conformal vector ω and let $Y(\omega,z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$. We assume that the derivation in the vertex algebra V is $D = L(-1)$. A V-module (cf. [62]) is a vector space M endowed with a linear map Y_M from V to the space of $End(M)$-valued fields

$$a \mapsto Y_M(a,z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}$$

such that:

1. $Y_M(|0\rangle, z) = I_M$,
2. for $a, b \in V$,

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_M(a,z_1)Y_M(b,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_M(b,z_2)Y_M(a,z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_M(Y(a,z_0)b,z_2).$$

Given three V-modules M_1, M_2, M_3, an *intertwining operator of type* $\begin{pmatrix} M_3 \\ M_1\ M_2 \end{pmatrix}$ (cf. [50], [52]) is a map $I : a \mapsto I(a,z) = \sum_{n\in\mathbb{Z}} a^I_{(n)} z^{-n-1}$ from M_1 to the space of $Hom(M_2, M_3)$-valued fields such that:

1. for $a \in V$, $b \in M_1$, $c \in M_2$, the following Jacobi identity holds:

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_{M_3}(a,z_1)I(b,z_2)c - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)I(b,z_2)Y_{M_2}(a,z_1)c$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)I(Y_{M_1}(a,z_0)b,z_2)c,$$

2. for every $a \in M_1$,

$$I(L(-1)a,z) = \frac{d}{dz}I(a,z).$$

We let $I\binom{M_3}{M_1\ M_2}$ denote the space of intertwining operators of type $\binom{M_3}{M_1\ M_2}$, and set

$$N_{M_1,M_2}^{M_3} = \dim I \begin{pmatrix} M_3 \\ M_1\ M_2 \end{pmatrix}.$$

When $N_{M_1,M_2}^{M_3}$ is finite, it is usually called a *fusion coefficient*.

Assume that in the category K of $L(0)$-diagonalizable V-modules, irreducible modules $\{M_i \mid i \in I\}$, where I is an index set, have the following properties

(1) for every $i, j \in I$, $N_{M_i,M_j}^{M_k}$ is finite for any $k \in I$;

(2) $N_{M_i,M_j}^{M_k} = 0$ for all but finitely many $k \in I$.

Then the algebra with basis $\{e_i \in I\}$ and product

$$e_i \cdot e_j = \sum_{k \in I} N_{M_i,M_j}^{M_k} e_k$$

is called the *fusion algebra* of V, K.

Let M_1, M_2 be irreducible V-modules in K. Given an irreducible V-module M_3 in K, we will say that the fusion rule

$$M_1 \times M_2 = M_3 \tag{3.3}$$

holds in K if $N_{M_1,M_2}^{M_3} = 1$ and $N_{M_1,M_2}^{R} = 0$ for any other irreducible V-module R in K which is not isomorphic to M_3.

We say that an irreducible V-module M_1 is a simple current in K if M_1 is in K and, for every irreducible V-module M_2 in K, there is an irreducible V-module M_3 in K, such that the fusion rule (3.3) holds in K (cf. [37]).

Recall that for any automorphism g of V, and any V–module $(M, Y_M(\cdot, z))$, we have a new V–module $M \circ g = M^g$, such that $M^g \cong M$ as a vector space and the vertex operator Y_M^g is given by $Y_M^g(v, z) := Y_M(gv, z)$, for $v \in V$. Namely, the only axiom we have to check is the Jacobi identity, and we have:

$$\begin{aligned}
& z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M^g(a, z_1) Y_M^g(b, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M^g(b, z_2) Y_M^g(a, z_1) \\
&= z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(ga, z_1) Y_M(gb, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(gb, z_2) Y_M(ga, z_1) \\
&= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(ga, z_0) gb, z_2) \\
&= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M^g(Y(a, z_0) b, z_2).
\end{aligned}$$

Therefore, M^g is a V–module. The following proposition shows that automorphism g also produces a new intertwining operator.

Proposition 3.2.1. Let g be an automorphism of the vertex algebra V satisfying the condition

$$\omega - g(\omega), \omega - g^{-1}(\omega) \in \mathrm{Im}(D). \tag{3.4}$$

Let M_1, M_2, M_3 be V–modules and $I(\cdot, z)$ an intertwining operator of type $\binom{M_3}{M_1\ M_2}$. Then we have an intertwining operator I^g of type $\binom{M_3^g}{M_1^g\ M_2^g}$, such that $I^g(b, z_1) = I(b, z_1)$, for all $b \in M_1$. Moreover,

$$N^{M_3}_{M_1,M_2} = N^{M_3^g}_{M_1^g,M_2^g}.$$

Proof. We have:

$$\begin{aligned}
&z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y^g_{M_3}(a, z_1)I^g(b, z_2)c - z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)I^g(b, z_2)Y^g_{M_2}(a, z_1)c\\
&= z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y_{M_3}(ga, z_1)I(b, z_2)c - z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)I(b, z_2)Y_{M_2}(ga, z_1)c\\
&= z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)I(Y_{M_1}(ga, z_0)b, z_2)c\\
&= z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)I(Y^g_{M_1}(a, z_0)b, z_2)c.
\end{aligned}$$

Set

$$Y^g(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)^g z^{-n-2}.$$

Since $g(\omega) = \omega + Dv$ for a certain $v \in V$, we have that

$$g(\omega)_0 = \omega_0 + (Dv)_0 = \omega_0 = L(-1).$$

This implies that $L(-1)^g = L(-1)$. Hence for $a \in M_1$ we have

$$I^g(L(-1)^g a, z) = I^g(L(-1)a, z) = I(L(-1)a, z) = \frac{d}{dz}I(a, z) = \frac{d}{dz}I^g(a, z).$$

Therefore, I^g has the $L(-1)$–derivation property and I^g is an intertwining operator of type $\binom{M_3^g}{M_1^g\ M_2^g}$.

■

Remark 3.2.2. If V is a vertex operator algebra and g an automorphism of V, then $g(\omega) = \omega$ and the condition (3.4) is automatically satisfied. In our applications, g will only be a vertex algebra automorphism such that $g(\omega) \neq \omega$, yet the condition (3.4) will be satisfied.

3.3. THE WEYL VERTEX ALGEBRA

3.3.1. The Weyl vertex algebra

The *Weyl algebra* $\widehat{\mathscr{A}}$ is an associative algebra with generators

$$a(n), a^*(n) \quad (n \in \mathbb{Z})$$

and relations

$$[a(n), a^*(m)] = \delta_{n+m,0}, \ \ [a(n), a(m)] = [a^*(m), a^*(n)] = 0 \ \ (n, m \in \mathbb{Z}). \tag{3.5}$$

Let M denote the simple *Weyl module* generated by the cyclic vector $\mathbf{1}$ such that

$$a(n)\mathbf{1} = a^*(n+1)\mathbf{1} = 0 \quad (n \geq 0).$$

As a vector space,

$$M \cong \mathbb{C}[a(-n), a^*(-m) \mid n > 0, \ m \geq 0].$$

We set $a := a(-1)\mathbf{1}$, $a^* := a^*(0)\mathbf{1}$. There is a unique vertex algebra $(M, Y, \mathbf{1})$ (cf. [48], [59], [47]) where the vertex operator map is

$$Y : M \to \mathrm{End}(M)[[z, z^{-1}]]$$

such that

$$Y(a, z) = a(z), \quad Y(a^*, z) = a^*(z),$$

$$a(z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1}, \ \ a^*(z) = \sum_{n\in\mathbb{Z}} a^*(n) z^{-n}.$$

In particular we have:

$$Y(a(-1)a^*(0)\mathbf{1}, z) = a(z)^+ a^*(z) + a^*(z) a(z)^-,$$

where

$$a(z)^+ = \sum_{n\leq -1} a(n) z^{-n-1}, \quad a(z)^- = \sum_{n\geq 0} a(n) z^{-n-1}.$$

Let $\beta := a(-1)a^*(0)\mathbf{1}$. Set $\beta(z) = Y(\beta, z) = \sum_{n\in\mathbb{Z}} \beta(n) z^{-n-1}$. Then β is a Heisenberg vector in M of level -1. This means that the components of the field $\beta(z)$ satisfy the commutation relations

$$[\beta(n), \beta(m)] = -n\delta_{n+m,0} \quad (n, m \in \mathbb{Z}).$$

Also, we have the following formula

$$[\beta(n), a(m)] = -a(n+m), \quad [\beta(n), a^*(m)] = a^*(n+m).$$

The vertex algebra M admits a family of Virasoro vectors

$$\omega_\mu = (1-\mu)a(-1)a^*(-1)\mathbf{1} - \mu a(-2)a^*(0)\mathbf{1} \quad (\mu \in \mathbb{C})$$

of central charge $c_\mu = 2(6\mu(\mu-1)+1)$. Let

$$L^\mu(z) = Y(\omega_\mu, z) = \sum_{n\in\mathbb{Z}} L^\mu(n) z^{-n-2}.$$

This means that the components of the field $L(z)$ satisfy the relations

$$[L^\mu(n), L^\mu(m)] = (n-m)L^\mu(m+n) + \frac{n^3-n}{12}\delta_{n+m,0}c_\mu.$$

For $\mu = 0$, we write $\omega = \omega_\mu$, $L^\mu(n) = L(n)$, $c = c_\mu$. Then $c = 2$. Clearly

$$\omega_\mu = \omega - \mu\beta(-2)\mathbf{1}.$$

Since $(\beta(-2)\mathbf{1})_0 = (D\beta)_0$, we have that

$$L^\mu(-1) = L(-1), \quad \text{for every } \mu \in \mathbb{C}. \tag{3.6}$$

For $n, m \in \mathbb{Z}$ we have

$$[L(n), a(m)] = -ma(n+m), \quad [L(n), a^*(m)] = -(m+n)a^*(n+m).$$

In particular,

$$[L(0), a(m)] = -ma(m), [L(0), a^*(m)] = -ma^*(m).$$

Therefore, the vertex algebra M is $\mathbb{Z}_{\geq 0}$–graded:

$$M = \bigoplus_{\ell\in\mathbb{Z}_{\geq 0}} M(\ell), \quad L(0)|M(\ell) \equiv \ell \,\mathrm{Id}.$$

We say that M–module $W = \bigoplus_{\ell\in\mathbb{Z}_{\geq 0}} W(\ell)$ is $\mathbb{Z}_{\geq 0}$–graded if for $v \in M(r)$:

$$v_m W(\ell) \subset W(\ell + r - m - 1).$$

(In the terminology of [38], this means that W is admissible M–module.)

Lemma 3.3.1. Assume that $W=\bigoplus_{\ell\in\mathbb{Z}_{\geq 0}}W(\ell)$ is a $\mathbb{Z}_{\geq 0}$–graded M–module. Then

$$L(0)\equiv 0 \quad \text{on } W(0).$$

Proof. Since W is $\mathbb{Z}_{\geq 0}$–graded with top component $W(0)$, the operators $a(n), a^*(n)$ must act trivially on $W(0)$ for all $n\in\mathbb{Z}_{>0}$. Since

$$L(z)=:(\frac{d}{dz}a^*(z))a(z):=\sum_{n\in\mathbb{Z}}L(n)z^{-n-2},$$

we have

$$L(0)=\sum_{n=1}^{\infty}n(a^*(-n)a(n)-a(-n)a^*(n))\equiv 0 \quad (\text{on } W(0)).$$

The Lemma holds. ■

Recall that the first Weyl algebra A_1 is generated by x,∂_x with the commutation relation $[\partial_x,x]=1$. It is isomorphic to a subalgebra of $\widehat{\mathscr{A}}$ generated by $a(0)$ and $a^*(0)$.

Remark 3.3.2. Let us recall that the Zhu's algebra $A(M)$ is defined as a quotient $A(M)=M/O(M)$, where $O(M)$ is the subspace of M spanned by elements of the form

$$u\circ v=Res_z\left(Y(u,z)\frac{(z+1)^{\mathrm{wt}\,u}}{z^2}v\right),$$

for a homogeneous $u\in M$ and any $v\in M$. One can show that Zhu's algebra $A(M)\cong A_1$. Let us prove that $[\omega]=0$ in $A(M)$. This way we will have a second proof of Lemma 3.3.1. We calculate:

$$\begin{aligned}
a^*\circ a &= Res_z\left(\frac{(z+1)^0}{z^2}Y(a^*,z)a\right)\\
&= Res_z\left(\sum_{n\in\mathbb{Z}}a^*(n)az^{-n-2}\right)\\
&= a^*(-1)a(-1)\mathbf{1}\\
&= a(-1)a^*(-1)\mathbf{1}\\
&= \omega.
\end{aligned}$$

Therefore, $\omega\in O(M)$, and on the quotient $A(M)$ we have $[\omega]=0$. By [73, Theorem 2.1.2.] we have that $[\omega]$ acts on $W(0)$ as $o(\omega)$, where $o(a)=a_{wt(a)-1}$, for $a\in M$. Since

$$o(\omega)=\omega_{2-1}=\omega_1=L(0),$$

it follows that $L(0)$ acts trivially on $W(0)$.

Definition 3.3.3. A module W for the Weyl algebra $\widehat{\mathscr{A}}$ is called restricted if the following condition holds:

- For every $w \in W$, there is $N \in \mathbb{Z}_{\geq 0}$ such that

$$a(n)w = a^*(n)w = 0, \quad \text{for } n \geq N.$$

Definition 3.3.4. A module W for the Weyl vertex algebra M is called **weight** if the operators $\beta(0)$ and $L(0)$ act semisimply on W.

3.3.2. Automorphisms of the Weyl algebra

Denote by $\mathrm{Aut}(\widehat{\mathscr{A}})$ the group of automorphisms of the Weyl algebra $\widehat{\mathscr{A}}$. For any $f \in \mathrm{Aut}(\widehat{\mathscr{A}})$, and $\widehat{\mathscr{A}}$–module N, one can construct $\widehat{\mathscr{A}}$–module $f(N)$ as follows:

$$f(N) := N \quad \text{as vector space, and action is } \; x.v = f(x)v \quad (v \in N).$$

For $f, g \in \mathrm{Aut}(\widehat{\mathscr{A}})$, we have

$$(f \circ g)(N) = g(f(N)). \tag{3.7}$$

For every $s \in \mathbb{Z}$ the Weyl algebra $\widehat{\mathscr{A}}$ admits the following automorphism

$$\rho_s(a(n)) = a(n+s), \quad \rho_s(a^*(n)) = a^*(n-s). \tag{3.8}$$

Then ρ_s is an automorphism of $\widehat{\mathscr{A}}$ which can be lifted to an automorphism of the vertex algebra M. Automorphism ρ_s is called spectral flow automorphism.

Assume that U is any restricted module for $\widehat{\mathscr{A}}$. Then $\rho_s(U)$ is also a restricted module for $\widehat{\mathscr{A}}$ and $\rho_s(U)$ is a module for the vertex algebra M.

Let $\mathscr{K}$ be the category of weight M–modules such that the operators $\beta(n)$, $n \geq 1$, act locally nilpotent on each module N in $\mathscr{K}$. Applying the automorphism ρ_s to the vertex algebra M, we get an M–module $\rho_s(M)$, which is a simple current in the category $\mathscr{K}$. The proof is essentially given by H. Li in [65, Theorem 2.15] in a slightly different setting.

Proposition 3.3.5. (cf. [65]) Assume that N is an irreducible weight M–module in the category $\mathscr{K}$. Then the following fusion rules hold:

$$\rho_{s_1}(M) \times \rho_{s_2}(N) = \rho_{s_1+s_2}(N) \quad (s_1, s_2 \in \mathbb{Z}). \tag{3.9}$$

Proof. First we notice that if N is an irreducible M–module in $\mathscr{K}$, we have the following fusion rule

$$M \times N = N. \tag{3.10}$$

Using [65], one can prove that $\rho_s(M)$ is constructed from M as:

$$(\rho_s(M), Y_s(\cdot,z)) := (M, Y(\Delta(-s\beta,z)\cdot,z)),$$

where

$$\Delta(v,z) := z^{v_0}\exp\left(\sum_{n=1}^{\infty}\frac{v_n}{-n}(-z)^{-n}\right). \tag{3.11}$$

Note that in our case $v = -s\beta$, and $v_0 = -s\beta(0)$ acts semisimply on M with integer values.

The $\widehat{\mathscr{A}}$–action on $\rho_s(M)$ is uniquely determined by

$$\begin{aligned}\rho_s(a(z)) &= \sum_{i\in\mathbb{Z}}\rho_s(a(i))z^{-i-1} = z^s a(z),\\ \rho_s(a^*(z)) &= \sum_{i\in\mathbb{Z}}\rho_s(a^*(i))z^{-i} = z^{-s}a^*(z).\end{aligned}$$

Assume that N_i, $i = 1,2,3$ are irreducible modules in $\mathscr{K}$. By [65, Proposition 2.4] from an intertwining operator $I(\cdot,z)$ of type $\binom{N_3}{N_1\ N_2}$, one can construct intertwining operator $I_{s_2}(\cdot,z)$ of type $\binom{\rho_{s_2}(N_3)}{N_1\ \rho_{s_2}(N_2)}$, where

$$I_{s_2}(v,z) := I(\Delta(-s_2\beta,z)v,z) \quad (v \in N_1).$$

Take $N_1 = M$, $N_2 = N$, and using skew-symmetry, we conclude that

$$\dim I\binom{\rho_{s_2}(N)}{\rho_{s_2}(N)\ \ M} = 1,$$

and using (3.10) the fusion rules $\rho_{s_2}(N) \times M = \rho_{s_2}(N)$.

Applying argument as above we get

$$\dim I\binom{\rho_{s_1+s_2}(N_2)}{\rho_{s_2}(N)\ \ \rho_{s_1}(M)} = 1,$$

and the fusion rules $\rho_{s_2}(N) \times \rho_{s_1}(M) = \rho_{s_1+s_2}(N)$.

Applying skew symmetry again, we get fusion rules (3.9). The proof follows. ∎

Consider the following automorphism of the Weyl vertex algebra

$$\begin{aligned} g: \quad & M \to M \\ & a \mapsto -a^*, \;\; a^* \mapsto a \end{aligned}$$

Assume that U is any M–module. Then $U^g = U \circ g$ is generated by the following fields

$$a_g(z) = -a^*(z), \; a_g^*(z) = a(z).$$

As an $\widehat{\mathscr{A}}$–module, U^g is obtained from U by applying the following automorphism g:

$$a(n) \mapsto -a^*(n+1), \;\; a^*(n) \mapsto a(n-1). \tag{3.12}$$

This implies that

$$g = \rho_{-1} \circ \sigma = \sigma \circ \rho_1 \tag{3.13}$$

where σ is the automorphism of $\widehat{\mathscr{A}}$ determined by

$$a(n) \mapsto -a^*(n), \;\; a^*(n) \mapsto a(n). \tag{3.14}$$

The automorphism g is then a vertex algebra automorphism of order 4.

Denote by σ_0 the restriction of σ of the A_1. Using (3.12)-(3.14) we get the following result:

Lemma 3.3.6. Assume that W is an irreducible M–module. Then

$$W^g \cong \rho_1(\sigma(W)).$$

Proof. We have that as an $\widehat{\mathscr{A}}$–module:

$$W^g = (\rho_{-1} \circ \sigma)(W).$$

Since $\rho_{-1} \circ \sigma = \sigma \circ \rho_1$, by applying (3.7) we get:

$$W^g = (\sigma \circ \rho_1)(W) = \rho_1(\sigma(W)).$$

The proof follows. ∎

3.3.3. Weight modules for the Weyl vertex algebra.

Recall the definition of weight M–module from Definition 3.3.4 and the definition of the first Weyl algebra from Subsection 3.3.1. Clearly, the vertex algebra M is a weight M–module. We will now construct a family of weight modules.

- For every $\lambda \in \mathbb{C}$,
$$U(\lambda) := x^{\lambda}\mathbb{C}[x,x^{-1}]$$
has the structure of an A_1–module.

- $U(\lambda)$ is irreducible if and only if $\lambda \in \mathbb{C}\setminus\mathbb{Z}$.

- Note that $a(0), a^*(0)$ generate a subalgebra of the Weyl algebra, which is isomorphic to the first Weyl algebra A_1. Therefore $U(\lambda)$ can be treated as an A_1–module by letting $a(0) = \partial_x$, $a^*(0) = x$.

- By applying the automorphism σ_0 on $U(\lambda)$ we get
$$\sigma_0(U(\lambda)) \cong U(-\lambda).$$
Indeed, let $z_{-\mu-1} = \frac{x^{\mu}}{\Gamma(\mu+1)}$, where $\mu \in \mathbb{C}\setminus\mathbb{Z}$. Then
$$\sigma_0(a(0)).z_{-\mu} = -\frac{x^{\mu}}{\Gamma(\mu)} = -\mu\frac{x^{\mu}}{\Gamma(\mu+1)} = -\mu z_{-\mu-1}.$$
$$\sigma_0(a^*(0)).z_{-\mu} = (\mu-1)\frac{x^{\mu-2}}{\Gamma(\mu)} = \frac{x^{\mu-2}}{\Gamma(\mu-1)} = z_{-\mu+1}.$$

- Define the following subalgebras of $\widehat{\mathscr{A}}$:
$$\widehat{\mathscr{A}_{\geq 0}} = \mathbb{C}[a(n), a^*(m) \mid n,m \in \mathbb{Z}_{\geq 0}],$$
$$\widehat{\mathscr{A}_{<0}} = \mathbb{C}[a(-n), a^*(-n) \mid n \in \mathbb{Z}_{\geq 1}].$$

- The A_1–module structure on $U(\lambda)$ can be extended to a structure of $\widehat{\mathscr{A}_{\geq 0}}$–module by defining
$$a(n)\big|_{U(\lambda)} = a^*(n)\big|_{U(\lambda)} \equiv 0 \qquad (n \geq 1).$$

- Then we have the induced module for the Weyl algebra:

$$\widetilde{U(\lambda)} = \widehat{\mathscr{A}} \otimes_{\widehat{\mathscr{A}}_{\geq 0}} U(\lambda)$$

which is isomorphic to

$$\mathbb{C}[a(-n), a^*(-n) \, | n \geq 1] \otimes U(\lambda)$$

as a vector space.

Proposition 3.3.7. For every $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, $\widetilde{U(\lambda)}$ is an irreducible weight module for the Weyl vertex algebra M.

Proof. The proof follows from Lemma 3.3.1 and the fact that $\widetilde{U(\lambda)}$ is a $\mathbb{Z}_{\geq 0}$–graded M–module whose top component is an irreducible module for A_1. ■

Applying Lemma 3.3.6 we get:

Corollary 3.3.8. For every $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ and $s \in \mathbb{Z}$ we have

$$\widetilde{U(\lambda)}^g \cong \rho_1(\widetilde{U(-\lambda)}), \quad \left(\rho_{-s+1}(\widetilde{U(\lambda)})\right)^g \cong \rho_s(\widetilde{U(-\lambda)}).$$

3.3.4. More general weight modules

A classification of irreducible weight modules for the Weyl algebra $\widehat{\mathscr{A}}$ is given in [53]. Let us describe here a family of weight modules having infinite–dimensional weight spaces.

Take $\lambda, \mu \in \mathbb{C} \setminus \mathbb{Z}$. Let

$$U(\lambda, \mu) = x_1^{\lambda} x_2^{\mu} \mathbb{C}[x_1, x_2, x_1^{-1}, x_2^{-1}].$$

Then $U(\lambda, \mu)$ is an irreducible module for the second Weyl algebra A_2 generated by $\partial_1, \partial_2, x_1, x_2$. Note that A_2 can be realized as a subalgebra of $\widehat{\mathscr{A}}$ generated by $\partial_2 = a(1), \partial_1 = a(0), x_2 = a^*(-1), x_1 = a^*(0)$. Then we have the irreducible $\widehat{\mathscr{A}}$–module $\widetilde{U(\lambda, \mu)}$ as follows. Let $\mathscr{B}$ be the subalgebra of $\widehat{\mathscr{A}}$ generated by $a(i), a^*(j)$, $i \geq 0, j \geq -1$. Consider $U(\lambda, \mu)$ as a $\mathscr{B}$–module such that $a(n)$, $a^*(m)$ act trivially for $n \geq 2$, $m \geq 1$. Then by [53],

$$\widetilde{U(\lambda, \mu)} = \widehat{\mathscr{A}} \otimes_{\mathscr{B}} U(\lambda, \mu)$$

is an irreducible $\widehat{\mathscr{A}}$–module. As a vector space:

$$\begin{aligned}\widetilde{U(\lambda,\mu)} &\cong \mathbb{C}[a(-n-1),a^*(-m-2) \mid n,m\in\mathbb{Z}_{\geq 0}]\otimes U(\lambda,\mu)\\ &\cong a^*(0)^{\lambda}a^*(-1)^{\mu}\mathbb{C}[a(-n-1),a^*(-m)\mid n,m\in\mathbb{Z}_{\geq 0}].\end{aligned}$$

Since $\widetilde{U(\lambda,\mu)}$ is a restricted $\widehat{\mathscr{A}}$–module we get:

Proposition 3.3.9. $\widetilde{U(\lambda,\mu)}$ is an irreducible weight module for the Weyl vertex algebra M.

One can see that the weight spaces of the module $\widetilde{U(\lambda,\mu)}$ are all infinite-dimensional with respect to $(\beta(0),L(0))$. In particular, vectors

$$a(-1)^m a^*(0)^{\lambda+2m}a^*(-1)^{\mu-m},\quad m\in\mathbb{Z}_{\geq 0},$$

are linearly independent and they belong to the same weight space.

Remark 3.3.10. Note that modules $\widetilde{U(\lambda,\mu)}$ are not in the category $\mathscr{K}$, and therefore Proposition 3.3.5 can not be applied in this case.

3.4. The vertex algebra $\Pi(0)$ and the construction of intertwining operators

In this section we present a bosonic realization of the weight modules for the Weyl vertex algebra. We also construct intertwining operators using this bosonic realization.

3.4.1. The vertex algebra $\Pi(0)$ and its modules

Let L be the lattice

$$L = \mathbb{Z}\alpha + \mathbb{Z}\beta,\ \langle\alpha,\alpha\rangle = -\langle\beta,\beta\rangle = 1, \quad \langle\alpha,\beta\rangle = 0,$$

and $V_L = M_{\alpha,\beta}(1)\otimes\mathbb{C}[L]$ the associated lattice vertex superalgebra, where $M_{\alpha,\beta}(1)$ is the Heisenberg vertex algebra generated by fields $\alpha(z)$ and $\beta(z)$ and $\mathbb{C}[L]$ is the group algebra of L.

We have the following vertex subalgebra of V_L:

$$\Pi(0) = M_{\alpha,\beta}(1)\otimes\mathbb{C}[\mathbb{Z}(\alpha+\beta)] \subset V_L.$$

By [6], [47] we see that there is an injective vertex algebra homomorphism $f : M \to \Pi(0)$ such that

$$f(a) = e^{\alpha+\beta},\ f(a^*) = -\alpha(-1)e^{-\alpha-\beta}.$$

Note that since M is a simple vertex algebra, a homomorphism f must be injective. We identify a, a^* with their image in $\Pi(0)$. We have (cf. [47])

$$M \cong \mathrm{Ker}_{\Pi(0)} e_0^{\alpha}.$$

Let us recall some useful formulas in lattice vertex algebras (cf. [58], [62]) which we will apply on vertex algebra $\Pi(0)$ and its modules. Let $\gamma\in\mathbb{Z}(\alpha+\beta)$. Then

$$Y(e^{\gamma},z) = e^{\gamma}z^{\gamma(0)}\exp\left(-\sum_{j<0}\frac{\gamma(j)}{j}z^{-j}\right)\exp\left(-\sum_{j>0}^{\infty}\frac{\gamma(j)}{j}z^{-j}\right),$$

$$Y(h^1(-n_1-1)\cdots h^r(-n_r-1)\otimes e^{\gamma},z) = \frac{:\partial_z^{n_1}h^1(z)\cdots\partial_z^{n_r}h^r(z)Y(e^{\gamma},z):}{n_1!\cdots n_r!}.$$

The Virasoro vector ω is mapped to

$$\omega = a(-1)a^*(-1)\mathbf{1} = \frac{1}{2}(\alpha(-1)^2 - \alpha(-2) - \beta(-1)^2 + \beta(-2))\mathbf{1}.$$

Note also that

$$g(\omega) = -a(-2)a^* = \omega_\mu = \frac{1}{2}(\alpha(-1)^2 - \alpha(-2) - \beta(-1)^2 - \beta(-2))\mathbf{1}. \quad (\mu = 1).$$

Since

$$\Pi(0) = \mathbb{C}[\mathbb{Z}(\alpha+\beta)] \otimes M_{\alpha,\beta}(1) \tag{3.15}$$

is a vertex subalgebra of V_L, for every $\lambda \in \mathbb{C}$ and $r \in \mathbb{Z}$,

$$\Pi_r(\lambda) = \mathbb{C}[r\beta + (\mathbb{Z}+\lambda)(\alpha+\beta)] \otimes M_{\alpha,\beta}(1) = \Pi(0).e^{r\beta+\lambda(\alpha+\beta)}$$

is an irreducible $\Pi(0)$–module.

We have

$$L(0)e^{r\beta+(n+\lambda)(\alpha+\beta)} \quad = \quad \frac{1-r}{2}(r+2(n+\lambda))e^{r\beta+(n+\lambda)(\alpha+\beta)},$$

and for $\mu = 1$

$$L^\mu(0)e^{r\beta+(n+\lambda)(\alpha+\beta)} \quad = \quad -\frac{1}{2}r(1+r+2(n+\lambda))e^{r\beta+(n+\lambda)(\alpha+\beta)}.$$

For $s \in \mathbb{Z}$ we have the following operator [1]

$$[e^{s\beta}] = 1 \otimes e^{s\beta} \in \mathrm{Hom}(\Pi_r(\lambda), \Pi_{r+s}(\lambda)) \quad (r \in \mathbb{Z}).$$

Lemma 3.4.1. We have:

$$[e^{s\beta}]\Phi.v = \rho_s(\Phi).[e^{s\beta}]v,$$

where $\Phi \in \widehat{\mathscr{A}}$, $v \in \Pi_r(\lambda)$.

Proof. Let $v \in \Pi_r(\lambda)$. Using explicit lattice realization we have:

$$\begin{aligned} a(n)[e^{s\beta}]v &= [e^{s\beta}]a(n+\langle\beta, s\beta\rangle)v = [e^{s\beta}]a(n-s)v, \\ a^*(n)[e^{s\beta}]v &= [e^{s\beta}]a^*(n+\langle-\beta, s\beta\rangle)v = [e^{s\beta}]a^*(n+s)v. \end{aligned}$$

The proof follows. ∎

[1] $[e^{s\beta}]$ can also be interpreted as the coefficient of the constant term of the intertwining operator of type $\binom{\Pi_{r+s}(\lambda))}{\Pi_s(0) \;\; \Pi_r(\lambda)}$.

Proposition 3.4.2. Assume that $\ell \in \mathbb{Z}$, $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. Then as M–modules:

(1) $\Pi_\ell(\lambda) \cong \rho_{-\ell+1}(\widetilde{U(-\lambda)})$,

(2) $\Pi_\ell(\lambda)^g \cong \rho_\ell(\widetilde{U(\lambda)})$.

Proof. Assume first that $\ell = 1$. Then $\Pi_1(-\lambda)$ is a $\mathbb{Z}_{\geq 0}$–graded M–module whose lowest component is

$$\Pi_1(-\lambda)(0) \cong \mathbb{C}[\beta + (\mathbb{Z} - \lambda)(\alpha+\beta)] \cong U(\lambda).$$

As a vector space

$$\Pi_1(-\lambda)(0) = \mathrm{span}_{\mathbb{C}}\{E_i \mid i \in \mathbb{Z}\}, \quad E_i = e^{\beta+(-\lambda+i)(\alpha+\beta)}.$$

We have

$$a(0)E_i = e_0^{\alpha+\beta} e^{\beta+(-\lambda+i)(\alpha+\beta)} = e^{\beta+(-\lambda+i+1)(\alpha+\beta)} = E_{i+1}. \tag{3.16}$$

$$\begin{aligned}
a^*(0)E_i &= Res_z z^{-1} a^*(z) E_i \\
&= Res_z z^{-1} Y(-\alpha(-1)e^{-\alpha-\beta}, z) E_i \\
&= -Res_z z^{-1} : \alpha(z) Y(e^{-\alpha-\beta}, z) : E_i \\
&= -Res_z z^{-1} \left(\alpha(z)^+ Y(e^{-\alpha-\beta}, z) + Y(e^{-\alpha-\beta}, z)\alpha(z)^- \right) \\
&= -Res_z z^{-1} \left(\sum_{j<0} \alpha(j) z^{-j-1} Y(e^{-\alpha-\beta}, z) + Y(e^{-\alpha-\beta}, z) \sum_{j\geq 0} \alpha(j) z^{-j-1} \right) E_i \\
&= -\sum_{j>0} \alpha(-j) e^{-\alpha-\beta}_{j-2} E_i - e^{-\alpha-\beta}_{-2} \alpha(0) E_i \\
&= -(-\lambda+i)E_{i-1} = (\lambda - i)E_{i-1}.
\end{aligned} \tag{3.17}$$

Using (3.16) and (3.17) we conclude that as A_1–modules

$$\Pi_1(-\lambda)(0) \cong \sigma_0^3(U(-\lambda)) \cong \sigma_0(U(-\lambda)) \cong U(\lambda).$$

(Here we used the fact that $\sigma_0^2 = -\mathrm{Id}$ and that $\sigma_0^2(U(-\lambda)) \cong U(-\lambda)$.)

Now Lemma 3.3.1 and Corollary 3.3.8 imply that $\Pi_1(-\lambda)$ is an irreducible M–module isomorphic to $\widetilde{U(\lambda)}$.

Next we prove that the module $\Pi_\ell(-\lambda)$ is obtained from $\Pi_1(-\lambda)$ by applying the spectral flow automorphism $\rho_{-\ell+1}$. Set $N = \Pi_1(-\lambda)$. Then N is in the category $\mathscr{K}$. We have the linear isomorphism

$$[e^{(\ell-1)\beta}] : N \to \Pi_\ell(-\lambda).$$

Using Lemma 3.4.1 we get

$$[e^{(\ell-1)\beta}]\Phi.e^{\beta-\lambda(\alpha+\beta)} = \rho_{\ell-1}(\Phi).e^{\ell\beta-\lambda(\alpha+\beta)} \quad (\Phi \in U(\widehat{\mathscr{A}}))$$

which implies that $N \cong \rho_{\ell-1}(\Pi_\ell(-\lambda))$ as M–modules. Since $\rho_{\ell-1}$ is invertible, we conclude that $\Pi_\ell(-\lambda) = \rho_{-\ell+1}(N)$. [2]

Using Corollary 3.3.8 we get

$$\begin{aligned}\Pi_\ell(\lambda)^g &= \left(\rho_{-\ell+1}(\widetilde{U(-\lambda)})\right)^g = \rho_1\sigma\rho_{-\ell+1}(\widetilde{U(-\lambda)}) \\ &= \rho_\ell\sigma(\widetilde{U(-\lambda)}) = \rho_\ell(\widetilde{U(\lambda)}).\end{aligned}$$

The proof follows. ∎

3.4.2. Construction of intertwining operators

Proposition 3.4.3. For every $\ell_1, \ell_2 \in \mathbb{Z}$ and $\lambda, \mu \in \mathbb{C}$ there exist non-zero intertwining operators of types

$$\binom{\rho_{\ell_1+\ell_2-1}(\widetilde{U(\lambda+\mu)})}{\rho_{\ell_1}(\widetilde{U(\lambda)}) \ \ \rho_{\ell_2}(\widetilde{U(\mu)})}, \quad \binom{\rho_{\ell_1+\ell_2}(\widetilde{U(\lambda+\mu)})}{\rho_{\ell_1}(\widetilde{U(\lambda)}) \ \ \rho_{\ell_2}(\widetilde{U(\mu)})} \tag{3.18}$$

in the category of weight M–modules.

Proof. By using explicit bosonic realization, as in [36], one can construct a unique non-zero intertwining operator $I(\cdot, z)$ of type

$$\binom{\Pi_{s_1+s_2}(\lambda_1+\lambda_2)}{\Pi_{s_1}(\lambda_1) \ \ \Pi_{s_2}(\lambda_2)} \tag{3.19}$$

[2] The same claim can be proved using results from [65]. As before we have that

$$(\rho_s(N), Y_s(\cdot, z)) := (N, Y(\Delta(-s\beta, z)\cdot, z))$$

where $\Delta(v, z)$ is given by (3.11). By [65, Proposition 3.4] we have that

$$(\Pi_\ell(-\lambda), Y_{\Pi_\ell(-\lambda)}(\cdot, z)) := (N, Y_N(\Delta((\ell-1)\beta, z)\cdot z)).$$

This proves that $\Pi_\ell(-\lambda) = \rho_{-\ell+1}(N)$.

in the category of $\Pi(0)$–modules such that

$$e_{\nu}^{s_1\beta+\lambda_1(\alpha+\beta)}e^{s_2\beta+\lambda_2(\alpha+\beta)} = e^{(s_1+s_2)\beta+(\lambda_1+\lambda_2)(\alpha+\beta)}$$

where $\nu \in \mathbb{C}$ is given by

$$\begin{aligned} \nu &= -(s_1\beta+\lambda_1(\alpha+\beta), s_2\beta+\lambda_2(\alpha+\beta))-1 \\ &= s_1\lambda_2+s_2\lambda_1+s_1s_2-1. \end{aligned}$$

By restriction, this gives a non-trivial intertwining operator in the category of weight M–modules. Taking the embedding $f : M \to \Pi(0)$ and applying Corollary 3.4.2, we conclude the operator (3.19) gives the intertwining operator of type

$$\begin{pmatrix} \rho_{-s_1-s_2+1}(\widetilde{U(-\lambda_1-\lambda_2)}) \\ \rho_{-s_1+1}(\widetilde{U(-\lambda_1)}) \;\; \rho_{-s_2+1}(\widetilde{U(-\lambda_2)}) \end{pmatrix},$$

which for $\ell_1 = -s_1+1$, $\ell_2 = -s_2+1$, $\lambda = -\lambda_1$, $\mu = -\lambda_2$ gives the first intertwining operator. By using action of the automorphism g and Corollary 3.4.2 we get the following intertwining operator

$$\begin{pmatrix} \rho_{s_1+s_2}(\widetilde{U(\lambda_1+\lambda_2)}) \\ \rho_{s_1}(\widetilde{U(\lambda_1)}) \;\; \rho_{s_2}(\widetilde{U(\lambda_2)}) \end{pmatrix},$$

which for $\ell_1 = s_1$, $\ell_2 = s_2$, $\lambda = \lambda_1$, $\mu = \lambda_2$ gives the second intertwining operator. ■

Remark 3.4.4. Intertwining operators in Proposition 3.4.3 are realized on irreducible $\Pi(0)$–modules. This result can be read as

$$\Pi_{\ell_1}(\lambda) \times \Pi_{\ell_2}(\mu) \supseteq \Pi_{\ell_1+\ell_2-1}(\lambda+\mu) + \Pi_{\ell_1+\ell_2}(\lambda+\mu).$$

In the category of M–modules, we have non-trivial intertwining operators

$$\begin{pmatrix} \Pi_{\ell_1+\ell_2-1}(\lambda+\mu) \\ \Pi_{\ell_1}(\lambda) \;\; \Pi_{\ell_2}(\mu) \end{pmatrix}, \tag{3.20}$$

which are not $\Pi(0)$–intertwining operators.

Remark 3.4.5. Note that $(M, Y, \mathbf{1})$ is a conformal vertex algebra with the conformal vector

$$\omega = a(-1)a^*(-1)\mathbf{1}.$$

Note that the intertwining operators (3.18) satisfy the $L(-1)$-derivative property. Intertwining operators (3.19) satisfy this property by using the lattice realization as before, and intertwining operators (3.20) satisfy it by Proposition 3.2.1, using the facts that $g(\omega) = \omega_1$ and $L^{\mu}(-1) = L(-1)$, for $\mu = 1$. Moreover, using relation (3.6) we see that the $L^{\mu}(-1)$–derivation property holds for every $\mu \in \mathbb{C}$, for all intertwining operators constructed above.

3.5. THE VERTEX ALGEBRA $L_1(\mathfrak{gl}(1|1))$ AND ITS MODULES

3.5.1. On the vertex algebra $L_1(\mathfrak{gl}(1|1))$.

We now recall some results on the representation theory of $\mathfrak{gl}(1|1)$ and $\widehat{\mathfrak{gl}(1|1)}$. The terminology follows [30, Section 5].

Let $\mathfrak{g} = \mathfrak{gl}(1|1)$ be the complex Lie superalgebra generated by two even elements E, N and two odd elements $\Psi^{\pm}$ with the following (super)commutation relations:

$$[\Psi^+,\Psi^-] = E,\ [E,\Psi^{\pm}] = [E,N] = 0,\ [N,\Psi^{\pm}] = \pm\Psi^{\pm}.$$

Other (super)commutators are trivial. Let $(\cdot,\cdot)$ be the invariant super-symmetric bilinear form such that

$$(\Psi^+,\Psi^-) = -(\Psi^-,\Psi^+) = 1,\ (N,E) = (E,N) = 1.$$

All other products are zero.

Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{gl}(1|1)} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}] + \mathbb{C}K$ be the associated affine Lie superalgebra with the commutation relations

$$[x(n),y(m)] = [x,y](n+m) + n(x|y)\delta_{n+m,0}K,$$

where K is central and for $x \in \mathfrak{g}$ we set $x(n) = x\otimes t^n$. Let $L_k(\mathfrak{g})$ be the associated simple affine vertex algebra of level k. One can show that the universal affine vertex algebra associated to $\mathfrak{g}$ is simple if $k \neq 0$. Therefore, every restricted $\widehat{\mathfrak{g}}$–module at non–zero level k is a module for $L_k(\mathfrak{g})$.

Let $\mathscr{V}_{r,s}$ be the Verma module for the Lie superalgebra $\mathfrak{g}$ generated by the vector $v_{r,s}$ such that $Nv_{r,s} = rv_{r,s}$, $Ev_{r,s} = sv_{r,s}$. This module is a 2–dimensional module and it is irreducible iff $s \neq 0$. If $s = 0$, $\mathscr{V}_{r,s}$ has a 1–dimensional irreducible quotient, which we denote by $\mathscr{A}_r$.

We will need the following tensor product decompositions:

$$\mathscr{A}_{r_1} \otimes \mathscr{A}_{r_2} = \mathscr{A}_{r_1+r_2}, \quad \mathscr{A}_{r_1} \otimes \mathscr{V}_{r_2,s_2} = \mathscr{V}_{r_1+r_2,s_2}, \tag{3.21}$$

$$\mathscr{V}_{r_1,s_1} \otimes \mathscr{V}_{r_2,s_2} = \mathscr{V}_{r_1+r_2,s_1+s_2} \oplus \mathscr{V}_{r_1+r_2-1,s_1+s_2} \quad (s_1+s_2 \neq 0), \tag{3.22}$$

$$\mathscr{V}_{r_1,s_1} \otimes \mathscr{V}_{r_2,-s_1} = \mathscr{P}_{r_1+r_2}, \tag{3.23}$$

where $\mathscr{P}_r$ is the 4–dimensional indecomposable module which appears in the following extension

$$0 \to \mathscr{V}_{r,0} \to \mathscr{P}_r \to \mathscr{V}_{r-1,0} \to 0.$$

Let $\widehat{\mathscr{V}_{r,s}}$ denote the Verma module of level 1 induced from the irreducible $\mathfrak{gl}(1|1)$–module $\mathscr{V}_{r,s}$. If $s \notin \mathbb{Z}$, then $\widehat{\mathscr{V}_{r,s}}$ is an irreducible $L_1(\mathfrak{gl}(1|1))$–module. If $s \in \mathbb{Z}$, $\widehat{\mathscr{V}_{r,s}}$ is reducible and its structure is described in [30, Section 5.3].

By using the tensor product decomposition (3.22) we get the following result on fusion rules for $L_1(\mathfrak{g})$–modules:

Proposition 3.5.1. Let $r_1, r_2, s_1, s_2 \in \mathbb{C}$, $s_1, s_2, s_1+s_2 \notin \mathbb{Z}$. Then

$$\dim I \binom{\widehat{\mathscr{V}_{r_3,s_3}}}{\widehat{\mathscr{V}_{r_1,s_1}} \;\; \widehat{\mathscr{V}_{r_2,s_2}}} \leq 1.$$

Assume that there is a non-zero intertwining operator

$$\binom{\widehat{\mathscr{V}_{r_3,s_3}}}{\widehat{\mathscr{V}_{r_1,s_1}} \;\; \widehat{\mathscr{V}_{r_2,s_2}}}$$

in the category of $L_1(\mathfrak{g})$–modules. Then $s_3 = s_1+s_2$ and $r_3 = r_1+r_2$, or $r_3 = r_1+r_2-1$.

Recall that the Clifford algebra is generated by $\psi(r), \psi^*(s)$, where $r,s \in \mathbb{Z}+\frac{1}{2}$, with relations

$$\begin{aligned} &[\psi(r), \psi^*(s)] = \delta_{r+s,0}, \\ &[\psi(r), \psi(s)] = [\psi^*(r), \psi^*(s)] = 0, \text{ for all } r,s. \end{aligned} \tag{3.24}$$

Note that the commutators (3.24) are actually anticommutators because both $\psi(r)$ and $\psi^*(s)$ are odd for every r and s.

The Clifford vertex algebra F is generated by the fields

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi(n+\frac{1}{2})z^{-n-1},$$
$$\psi^*(z) = \sum_{n\in\mathbb{Z}} \psi^*(n+\frac{1}{2})z^{-n-1}.$$

As a vector space,

$$F \cong \bigwedge \left(\left\{ \psi(r), \psi^*(s) \mid r,s<0 \right\} \right)$$

Let $V_{\mathbb{Z}\gamma}$ be the lattice vertex algebra associated to the lattice $\mathbb{Z}\gamma \cong \mathbb{Z}$, $\langle \gamma, \gamma \rangle = 1$. By using the boson–fermion correspondence, we have that $F \cong V_{\mathbb{Z}\gamma}$, and we can identify the generators of the Clifford vertex algebra as follows (cf. [58]):

$$\psi := e^{\gamma}, \psi^* = e^{-\gamma}$$

.

Now we define the following vertex superalgebra:

$$S\Pi(0) = \Pi(0) \otimes F \subset V_L \otimes F = V_{L+\mathbb{Z}\gamma},$$

and its irreducible modules

$$S\Pi_r(\lambda) = \Pi_r(\lambda) \otimes F = S\Pi(0).e^{r\beta+\lambda(\alpha+\beta)}.$$

Let $\mathscr{U} = M \otimes F$. Using [58, Section 5.8] we define the vectors

$$\begin{aligned} &\Psi^+ := e^{\alpha+\beta+\gamma} = a(-1)\psi,\ \Psi^- := -\alpha(-1)e^{-\alpha-\beta-\gamma} = a^*(0)\psi^*, \\ &E := \gamma+\beta,\ N := \frac{1}{2}(\gamma-\beta). \end{aligned}$$

Then the components of the fields

$$X(z) = Y(X,z) = \sum_{n\in\mathbb{Z}} X(n) z^{-n-1},\ \ X \in \{\Psi^+, \Psi^-, E, N\}$$

satisfy the commutation relations for the affine Lie algebra $\widehat{\mathfrak{g}} = \widehat{\mathfrak{gl}(1|1)}$, so that $M \otimes F$ is a $\widehat{\mathfrak{g}}$–module of level 1. (See also [10] for a realization of $\widehat{\mathfrak{gl}(1|1)}$ at the critical level).

The Sugawara conformal vector is

$$\begin{aligned} \omega_{c=0} &= \tfrac{1}{2}(N(-1)E(-1) + E(-1)N(-1) - \Psi^+(-1)\Psi^-(-1) + \\ &\qquad \Psi^-(-1)\Psi^+(-1) + E(-1)^2)\mathbf{1} \qquad (3.25) \\ &= \tfrac{1}{2}(\beta(-1)+\gamma(-1))(\gamma(-1)-\beta(-1)) + \alpha(-1)(\alpha(-1)+\beta(-1)+\gamma(-1)) \\ &\quad -\tfrac{1}{2}((\alpha(-1)+\beta(-1)+\gamma(-1))^2 + (\alpha(-2)+\beta(-2)+\gamma(-2))) \\ &\quad +\tfrac{1}{2}(\beta(-1)+\gamma(-1))^2 + \tfrac{1}{2}(\beta(-2)+\gamma(-2)) \\ &= \tfrac{1}{2}(\alpha(-1)^2 - \alpha(-2) - \beta(-1)^2 + \gamma(-1)^2) \\ &= \omega_{c=-1} + \tfrac{1}{2}\gamma(-1)^2 \quad (\omega_{c=-1} = \omega_{\frac{1}{2}}) \end{aligned}$$

3.5.2. Construction of irreducible $L_1(\mathfrak{g})$–modules from irreducible M–modules.

Let $L_1(\mathfrak{g})$ be the simple affine vertex algebra of level 1 associated to $\mathfrak{g}$.

We have the following gradation:

$$\mathscr{U} = \bigoplus \mathscr{U}^\ell, \quad E(0)|_{\mathscr{U}^\ell} = \ell \,\mathrm{Id}.$$

We will present an alternative proof of the following result:

Proposition 3.5.2. (cf. [58]) We have:

$$L_1(\mathfrak{g}) \cong \mathscr{U}^0 = \mathrm{Ker}_{M\otimes F} E(0).$$

Proof. Let $\widetilde{L}_1(\mathfrak{g})$ be the vertex subalgebra of $\mathscr{U}^0$ generated by $\mathfrak{g}$. Assume that $\widetilde{L}_1(\mathfrak{g}) \neq \mathscr{U}^0$. Then there is a subsingular vector $v_{r,s} \notin \mathbb{C}\mathbf{1}$ for $\widehat{\mathfrak{g}}$ of weight (r,s) such that for $n>0$:

$$\Psi^+(0)v_{r,s} \in \widetilde{L}_1(\mathfrak{g}), \quad X(n)v_{r,s} \in \widetilde{L}_1(\mathfrak{g}), \quad X \in \{E,N,\Psi^\pm\}$$
$$E(0)v_{r,s} = s v_{r,s}, \; N(0)v_{r,s} = r v_{r,s}.$$

In other words, $v_{r,s}$ is a singular vector in the quotient $\widetilde{\mathscr{U}^0} = \mathscr{U}^0/\widetilde{L}_1(\mathfrak{g})$. Since $E(0)$ acts trivially on $\mathscr{U}^0$, we conclude that $s=0$. Recalling the expression for the Virasoro conformal vector (3.25), we get that in $\widetilde{\mathscr{U}^0}$:

$$L^{c=0}(0)v_{r,0} = (\omega_{c=0})_1 v_{r,0} = \tfrac{1}{2}(2N(0)E(0) - E(0) + E(0)^2)v_{r,0} = 0.$$

This implies that $v_{r,0}$ has the conformal weight 0 and hence must be proportional to $\mathbf{1}$. A contradiction. Therefore, $\mathscr{U}^0 = \widetilde{L}_1(\mathfrak{g})$. Since $\mathscr{U}^0$ is a simple vertex algebra, we have that $\widetilde{L}_1(\mathfrak{g}) = L_1(\mathfrak{g})$. ■

We can extend this irreducibility result to a wide class of weight modules. The proof is similar to the one given in [9, Theorem 6.2].

Theorem 3.5.3. Assume that $\mathscr{N}$ is an irreducible weight module for the Weyl vertex algebra M, such that $\beta(0)$ acts semisimply on $\mathscr{N}$:

$$\mathscr{N} = \bigoplus_{s\in\mathbb{Z}+\Delta} \mathscr{N}^s, \quad \beta(0)|N^s \equiv s\mathrm{Id} \quad (\Delta\in\mathbb{C}).$$

Then $\mathscr{N}\otimes F$ is a completely reducible $L_1(\mathfrak{g})$–module:

$$\mathscr{N}\otimes F = \bigoplus_{s\in\mathbb{Z}} \mathscr{L}_s(N) \quad \mathscr{L}_s(N) = \{v\in \mathscr{N}\otimes F \,|E(0)v = (s+\Delta)v\},$$

and each $\mathscr{L}_s(N)$ is irreducible $L_1(\mathfrak{g})$–module.

Proof. Clearly $\mathscr{L}_s(N)$ is a $\mathscr{U}^0(= L_1(\mathfrak{g}))$–module. It suffices to prove that each vector $w \in \mathscr{L}_s(N)$ is cyclic. Since $\mathscr{N} \otimes F$ is a simple $\mathscr{U}$–module, we have that $\mathscr{U}.w = \mathscr{N} \otimes F$. On the other hand, $\mathscr{N} \otimes F$ is $\mathbb{Z}$–graded $\mathscr{U}$–module so that

$$\mathscr{U}^r \cdot \mathscr{L}_s(N) \subset \mathscr{L}_{r+s}(N), \quad (r,s \in \mathbb{Z}).$$

This implies that $\mathscr{U}^r.w \subset \mathscr{L}_{r+s}(N)$ for each $r \in \mathbb{Z}$. Theferore $\mathscr{U}^0.w = \mathscr{L}_r(N)$. The proof follows. ■

As a consequence we get a family of irreducible $L_1(\mathfrak{g})$–modules:

Corollary 3.5.4. Assume that $\lambda, \mu \in \mathbb{C} \setminus \mathbb{Z}$. Then for each $s \in \mathbb{Z}$ we have:

(1) $\mathscr{L}_s(\widetilde{U(\lambda)})$ is an irreducible $L_1(\mathfrak{g})$–module,

(2) $\mathscr{L}_s(\widetilde{U(\lambda,\mu)})$ is an irreducible $L_1(\mathfrak{g})$–module.

We will prove in the next section that $\mathscr{L}_s(\widetilde{U(\lambda)})$ are irreducible highest weight modules. But one can see that $\mathscr{L}_s(\widetilde{U(\lambda,\mu)})$ have infinite-dimensional weight spaces. A detailed analysis of the structure of these modules will appear in our forthcoming papers (cf. [19]).

3.6. THE CALCULATION OF FUSION RULES

In this section we will finish the calculation of fusion rules for the Weyl vertex algebra M.

We will first identify certain irreducible highest weight $\widehat{\mathfrak{g}}$–modules.

Lemma 3.6.1. Assume that $r,n \in \mathbb{Z}$, $\lambda \in \mathbb{C}$, $\lambda + n \notin \mathbb{Z}$. Then $e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)}$ is a singular vector in $S\Pi_r(\lambda)$ and

$$U(\widehat{\mathfrak{g}}).e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)} \cong \widehat{\mathscr{V}}_{r+\frac{1}{2}(\lambda+n),-\lambda-n}, \tag{3.26}$$

$$L(0)e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)} = \frac{1}{2}(1-2r)(n+\lambda)e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)}. \tag{3.27}$$

Proof. By using standard calculation in lattice vertex algebras we get for $m \geq 0$

$$\begin{aligned}
\Psi^+(m)e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)} &= e_m^{\alpha+\beta+\gamma}e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)} = 0,\\
\Psi^-(m+1)e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)} &= \left(-\alpha(-1)e^{-\alpha-\beta-\gamma}\right)_{m+1}e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)} = 0,\\
E(m)e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)} &= -(\lambda+n)\delta_{m,0}e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)},\\
N(m)e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)} &= \frac{1}{2}(2r+\lambda+n)\delta_{m,0}e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)}.
\end{aligned}$$

Therefore $e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)}$ is a highest weight vector for $\widehat{\mathfrak{g}}$ with highest weight $(r+\frac{1}{2}(\lambda+n),-\lambda-n)$ with respect to $(N(0),E(0))$. This implies that $U(\widehat{\mathfrak{g}})$ is isomorphic to a certain quotient of the Verma module $\widehat{\mathscr{V}}_{r+\frac{1}{2}(\lambda+n),-\lambda-n}$. But since, $\lambda+n \notin \mathbb{Z}$, $\widehat{\mathscr{V}}_{r+\frac{1}{2}(\lambda+n),-\lambda-n}$ is irreducible and therefore (3.26) holds. Relation (3.27) follows by applying the expression $\omega = \frac{1}{2}(\alpha(-1)^2 - \alpha(-2) - \beta(-1)^2 + \gamma(-1)^2)$:

$$\begin{aligned}
& L(0)e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)}\\
= \;& \frac{1}{2}\left((\lambda+n)^2-(\lambda+n+r)^2+r^2+(\lambda+n)\right)e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)}\\
= \;& \frac{1}{2}(1-2r)(n+\lambda)e^{r(\beta+\gamma)+(\lambda+n)(\alpha+\beta)}.
\end{aligned}$$

∎

Theorem 3.6.2. Assume that $r \in \mathbb{Z}$, $\lambda \in \mathbb{C}\setminus\mathbb{Z}$. Then we have:

(1) $S\Pi_r(\lambda)$ is an irreducible $M \otimes F$–module,

(2) $S\Pi_r(\lambda)$ is a completely reducible $\widehat{\mathfrak{gl}(1|1)}$–module:

$$\begin{aligned} S\Pi_r(\lambda) &\cong \bigoplus_{s\in\mathbb{Z}} U(\widehat{\mathfrak{g}}).e^{r(\beta+\gamma)+(\lambda+s)(\alpha+\beta)} \\ &\cong \bigoplus_{s\in\mathbb{Z}} \widehat{\mathscr{V}}_{r+\frac{1}{2}(\lambda+s),-\lambda-s}. \end{aligned} \tag{3.28}$$

Proof. The assertion (1) follows from the fact that $\Pi_r(\lambda)$ is an irreducible M–module (cf. Proposition 3.4.2). Note next that the operator $E(0)=\beta(0)+\gamma(0)$ acts semi–simply on $M\otimes F$:

$$M\otimes F=\bigoplus_{s\in\mathbb{Z}}(M\otimes F)^{(s)}, \quad (M\otimes F)^{(s)}=\{v\in M\otimes F \mid E(0)v=-sv\}.$$

In particular, $(M\otimes F)^{(0)}\cong L_1(\mathfrak{g})$ (cf. [58] and Proposition 3.5.2). But $E(0)$ also defines the following $\mathbb{Z}$–gradation on $S\Pi_r(\lambda)$:

$$S\Pi_r(\lambda)=\bigoplus_{s\in\mathbb{Z}} S\Pi_r(\lambda)^{(s)}, \quad S\Pi_r(\lambda)^{(s)}=\{v\in S\Pi_r(\lambda) \mid E(0)v=(-s-\lambda)v\}.$$

Applying Theorem 3.5.3 we see that each $S\Pi_r(\lambda)^{(s)}$ is an irreducible $(M\otimes F)^{(0)}\cong L_1(\mathfrak{g})$–module. Using Lemma 3.6.1 we see that it is an irreducible highest weight $\widehat{\mathfrak{g}}$–module with highest weight vector $e^{r(\beta+\gamma)+(\lambda+s)(\alpha+\beta)}$. The proof follows. ∎

Theorem 3.6.3. Assume that $\lambda_1,\lambda_2,\lambda_1+\lambda_2\in\mathbb{C}\setminus\mathbb{Z}$, $r_1,r_2,r_3\in\mathbb{Z}$. Then

$$\dim I\binom{S\Pi_{r_3}(\lambda_3)}{S\Pi_{r_1}(\lambda_1)\;\; S\Pi_{r_2}(\lambda_2)}\leq 1.$$

Assume that there is a non-zero intertwining operator of type

$$\binom{S\Pi_{r_3}(\lambda_3)}{S\Pi_{r_1}(\lambda_1)\;\; S\Pi_{r_2}(\lambda_2)}$$

in the category of $M\otimes F$–modules. Then $\lambda_3=\lambda_1+\lambda_2$ and $r_3=r_1+r_2$, or $r_3=r_1+r_2-1$.

Proof. Assume that I is an non-zero intertwining operator of type

$$\binom{S\Pi_{r_3}(\lambda_3)}{S\Pi_{r_1}(\lambda_1)\;\; S\Pi_{r_2}(\lambda_2)},$$

Since $S\Pi_r(\lambda)$ are simple $M\otimes F$–modules, we have that for every $s_1,s_2\in\mathbb{Z}$:

$$I(e^{r_1(\beta+\gamma)+(\lambda_1+s_1)(\alpha+\beta)},z)e^{r_2(\beta+\gamma)+(\lambda_2+s_2)(\alpha+\beta)}\neq 0.$$

Here we use the well-known result which states that for every non-trivial intertwining operator I between three irreducible modules we have that $I(v,z)w \neq 0$ (cf. [36, Proposition 11.9]). Note that $e^{r_i(\beta+\gamma)+\lambda_i(\alpha+\beta)}$ is a singular vector for $\widehat{\mathfrak{g}}$ which generates $L_1(\mathfrak{g})$–module $\widehat{\mathscr{V}}_{r_i+\frac{1}{2}\lambda_i,-\lambda_i}$, $i=1,2$. The restriction of $I(\cdot,z)$ on

$$\widehat{\mathscr{V}}_{r_1+\frac{1}{2}\lambda_1,-\lambda_1} \otimes \widehat{\mathscr{V}}_{r_2+\frac{1}{2}\lambda_2,-\lambda_2}$$

gives a non-trivial intertwining operator

$$\binom{S\Pi_{r_3}(\lambda_3)}{\widehat{\mathscr{V}}_{r_1+\frac{1}{2}\lambda_1,-\lambda_1} \quad \widehat{\mathscr{V}}_{r_2+\frac{1}{2}\lambda_2,-\lambda_2}}$$

in the category of $L_1(\mathfrak{g})$–modules. Proposition 3.5.1 implies that then

$$\widehat{\mathscr{V}}_{r_1+r_2+\frac{1}{2}(\lambda_1+\lambda_2),-\lambda_1-\lambda_2} \quad \text{or} \quad \widehat{\mathscr{V}}_{r_1+r_2+\frac{1}{2}(\lambda_1+\lambda_2)-1,-\lambda_1-\lambda_2}$$

has to appear in the decomposition of $S\Pi_{r_3}(\lambda_3)$ as a $L_1(\mathfrak{g})$–module. Using decomposition (3.28) we get that there is $s \in \mathbb{Z}$ such that

$$r_1+r_2+\tfrac{1}{2}(\lambda_1+\lambda_2) = r_3+\tfrac{1}{2}(\lambda_3+s), \quad -\lambda_1-\lambda_2 = -\lambda_3-s \tag{3.29}$$

or

$$r_1+r_2-1+\tfrac{1}{2}(\lambda_1+\lambda_2) = r_3+\tfrac{1}{2}(\lambda_3+s), \quad -\lambda_1-\lambda_2 = -\lambda_3-s. \tag{3.30}$$

Solution of (3.29) is

$$\lambda_3+s=\lambda_1+\lambda_2,\ r_3=r_1+r_2,$$

and of (3.30) is

$$\lambda_3+s=\lambda_1+\lambda_2, \quad r_3=r_1+r_2-1.$$

Since $S\Pi_r(\lambda_3) \cong S\Pi_r(\lambda_3+s)$ for $s\in\mathbb{Z}$, we can take $s=0$. Thus, $\lambda_3=\lambda_1+\lambda_2$ and $r_3=r_1+r_2$ or $r_3=r_1+r_2-1$. This way we prove that there is a linear embedding

$$I\binom{S\Pi_{r_3}(\lambda_3)}{S\Pi_{r_1}(\lambda_1) \quad S\Pi_{r_2}(\lambda_2)} \hookrightarrow I\binom{\widehat{\mathscr{V}}_{r_3,s_3}}{\widehat{\mathscr{V}}_{r_1,s_1} \quad \widehat{\mathscr{V}}_{r_2,s_2}}.$$

Using Proposition 3.5.1 we see that in these cases

$$\dim I\binom{S\Pi_{r_3}(\lambda_3)}{S\Pi_{r_1}(\lambda_1) \quad S\Pi_{r_2}(\lambda_2)} \leq 1.$$

The claim holds. ■

By using the following natural isomorphism of the spaces of intertwining operators (cf. [5, Section 2]):

$$\mathrm{I}_{M\otimes F}\begin{pmatrix} S\Pi_{r_3}(\lambda_3) \\ S\Pi_{r_1}(\lambda_1)\ \ S\Pi_{r_2}(\lambda_2)\end{pmatrix} \cong \mathrm{I}_M\begin{pmatrix} \Pi_{r_3}(\lambda_3) \\ \Pi_{r_1}(\lambda_1)\ \ \Pi_{r_2}(\lambda_2)\end{pmatrix},$$

Theorem 3.6.3 implies the fusion rules result in the category of modules for the Weyl vertex algebra M (see also [70, Corollary 6.7], for a derivation of the same fusion rules using Verlinde formula).

Corollary 3.6.4. Assume that $\lambda_1, \lambda_2, \lambda_1+\lambda_2 \in \mathbb{C}\setminus\mathbb{Z}$, $r_1, r_2, r_3 \in \mathbb{Z}$. There exists a non-zero intertwining operator of type

$$\begin{pmatrix} \Pi_{r_3}(\lambda_3) \\ \Pi_{r_1}(\lambda_1)\ \ \Pi_{r_2}(\lambda_2)\end{pmatrix}$$

in the category of M–modules if and only if $\lambda_3 = \lambda_1+\lambda_2$ and $r_3 = r_1+r_2$ or $r_3 = r_1+r_2-1$.

The fusion rules in the category of weight M–modules are given by

$$\Pi_{r_1}(\lambda_1)\times\Pi_{r_2}(\lambda_2) = \Pi_{r_1+r_2}(\lambda_1+\lambda_2) + \Pi_{r_1+r_2-1}(\lambda_1+\lambda_2).$$

4. IRREDUCIBILITY OF MODULES OF WHITTAKER TYPE FOR CYCLIC ORBIFOLD VERTEX ALGEBRAS

This chapter is joint work with D. Adamović, C.-H. Lam and N. Yu and it is published in Journal of Algebra (cf. [12]). In this chapter we extend the Dong-Mason theorem on irreducibility of modules for orbifold vertex algebras (cf. [41]) to the category of weak modules. Although the theorem holds for any category of weak modules for a vertex algebra, we find Whittaker modules to be of greatest interest due to their unique determination via their Whittaker functions. We present certain applications in the cases of Heisenberg and Weyl vertex algebras.

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4.1. Introduction

4.1.1. Irreducibility of orbifolds.

Let W be an irreducible weak V-module and g an automorphism of V with order p. Let $Y_W(v,z)$ be the vertex operator of $v \in V$ operating on W. Recall that $W \circ g$ is defined in [41] to be the space W with the vertex operator given by

$$Y_{W \circ g}(v,z) = Y_W(gv,z), \forall v \in V.$$

It is clear that $W \circ g$ is also a V-module.

The following is our first main result (see Theorem 4.5.3 for part (1) and Theorem 4.6.3 for part (2)).

Main Theorem 1. Let W be an irreducible weak V–module and g an automorphism of finite order.

(1) Assume that $W \circ g^i \ncong W$ for all i. Then W is an irreducible $V^{\langle g \rangle}$–module.

(2) Assume that $W \cong W \circ g$. Then W is a direct sum of p irreducible $V^{\langle g \rangle}$–modules.

Let us explain the main new ideas of our proof. For (1), we construct a graded module

$$\mathscr{M} = \bigoplus_{i=0}^{p-1} W \circ g^i = \bigoplus_{i=0}^{p-1} \Delta^{p,i}(W),$$

compatible with the action of the automorphism g, such that each component is isomorphic to W as $V^{\langle g \rangle}$–module. Then we take any non-trivial submodule S of W and identify it with a submodule of $\Delta^{p,0}(W)$. It is then sufficient to prove the following claim:

(1.1) For each $w \neq 0$, a vector of the form $(w,\dots,w) \in \mathscr{M}$ is cyclic in $\mathscr{M}$.

The advantages of our approach are the fact that we do not need Zhu's algebra and the fact that this approach can be applied for non-weight modules. In Lemma 4.3.3, we prove relation (1.1) for arbitrary weak module by using the Lie algebra $\mathfrak{g}(V)$ associated to V and its universal enveloping associative algebra. It turns out that (1.1) is just a consequence of a similar statement for associative algebras (cf. Lemma 4.3.1).

For proof of the part (2) (cf. Theorem 4.6.3), we slightly modify the methods of [41] and [45] by applying a general version of Schur's Lemma on the action of the group $G = \mathbb{Z}_p$ on W.

4.1.2. Role of the Whittaker modules in the paper

Although Theorem 1 holds for arbitrary weak V–modules, it is not easy to construct examples of modules satisfying the conditions of the theorem. It turns out that these conditions can be checked for a large class of Whittaker modules for certain infinite-dimensional Lie algebras. We use concepts of Whittaker categories which appear in the paper [22] (see also [67]). Since any weak module for a vertex algebra is automatically a module for an infinite-dimensional Lie algebra, such an approach gives a framework for studying many examples. We just need to assume that each module $W \circ g^i$ belongs to a different Whittaker block. This means that each module $W \circ g^i$ has a different Whittaker function. The following is our second main result (see Theorem 4.7.10) which gives most new applications of our construction.

Main Theorem 2. Let W be an irreducible weak V–module such that all $W_i = W \circ g^i$ are Whittaker modules whose Whittaker functions $\lambda^{(i)} = \mathfrak{n} \to \mathbb{C}$ are mutually distinct. Then W is an irreducible weak $V^{\langle g \rangle}$–module.

4.1.3. Examples

We construct a family of Whittaker modules for Heisenberg and Weyl vertex algebra, and apply our new result to prove irreducibility of orbifold subalgebras. In particular, we show that in these cases, standard (= universal) Whittaker modules are irreducible.

In the case of Heisenberg vertex algebra, we use the new method and present an alternative proof of the $\mathbb{Z}_2$–orbifolds of Heisenberg vertex algebra (cf. [56]).

In the case of Weyl vertex algebra M, we construct a family of Whittaker modules $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ where $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{C}^n \times \mathbb{C}^n$. We prove:

Main Theorem 3 (see Theorem 4.9.3)**.** Assume that $\Lambda = (\boldsymbol{\lambda}, \boldsymbol{\mu}) \neq 0$. Then $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible weak module for the orbifold subalgebra $M^{\mathbb{Z}_p}$, for each $p \geq 1$.

4.2. THE DONG-MASON THEOREM

The following result was proved in [41].

Theorem 4.2.1. (cf. [41, Theorem 6.1]) Assume that (W, Y_W) is an **ordinary** module for the vertex operator algebra V. Assume that g is an automorphism of V of prime order p such that $W \circ g \ncong W$. Then W is an irreducible module for the orbifold subalgebra $V^{\langle g \rangle}$.

The goal of this chapter is to extend this result for **weak** modules for vertex operator algebras.

It is truly a matter of great interest to extend this result to weak modules because it allows us to study any module category for vertex algebras and this opens up many possibilities. In particular, we can now look at the category of Whittaker modules for vertex algebras. These modules have long been an important part of the theory of complex semisimple Lie algebras (cf. [61]), and later they were generalized by Batra and Mazorchuk (cf. [22]), but recently they have sparkled a lot of interest among scientists working in the vertex operator algebra theory. Among others, irreducibility of Whittaker modules for vertex operator algebras hase been studied by Adamović, Lu and Zhao for the affine vertex operator algebra case (cf. [17]), Mazorchuk and Zhao (cf. [67]) and Ondrus and Wiesner (cf. [69]) for the Virasoro vertex operator algebra case and Hartwig and Yu (cf. [56]) and Tanabe (cf. [71]) for the Heisenberg vertex operator algebra cases.

In this chapter we will concentrate on the examples of Heisenberg and Weyl vertex operator algebras and their Whittaker modules.

4.3. On cyclic vectors in a direct sum of irreducible weak modules

In this section, we prove one basic, but important technical result on cyclic vectors in a direct sum of non-isomorphic weak modules for a vertex operator algebra. It turns out that the result can be proved much more easily in the context of associative algebras.

First we include the following result for associative algebras:

Lemma 4.3.1. Let $\mathscr{A}$ be an associative algebra with unity. Assume that L_i, $i = 1, \ldots, t$, are non-isomorphic irreducible $\mathscr{A}$-modules and $\mathscr{L} = \bigoplus_{i=1}^{t} L_i$. Then for each $w_i \neq 0$, $w_i \in L_i$, a vector of the form $(w_1, w_2, \ldots, w_t)$ is cyclic in $\mathscr{L}$.

Proof. Let $\mathscr{U} = \mathscr{A}.(w_1, w_2, \ldots, w_t)$ be the $\mathscr{A}$-module generated by $(w_1, w_2, \ldots, w_t)$. Let $J_i = \mathrm{Ann}(w_i) = \{a \in \mathscr{A} \mid a.w_i = 0\}$ for $1 \leq i \leq t$. Then J_i is a left ideal in $\mathscr{A}$ and $\mathscr{A}/J_i \cong L_i$. Since L_i's are irreducible $\mathscr{A}$–modules which are mutually non-isomorphic, we conclude:

- ideals J_i, $i = 1, \ldots, t$, are maximal left ideals,
- $J_i \neq J_j$ for $i \neq j$,
- $\mathscr{A}/\bigcap_{i=1}^{t} J_i \cong \mathscr{L}$.

Note that in the last conclusion we use the fact that J_i's are maximal left ideals of $\mathscr{A}$ and apply Chinese Remainder Theorem. This implies that there is an element

$$u_i \in \bigcap_{1 \leq j \leq t,\ j \neq i} J_j,\ u_i \notin J_i.$$

Then one can construct the vector

$$u_i(w_1, \ldots, w_i, \ldots, w_t) = (0, \ldots 0, u_i w_i, 0, \ldots, 0),$$

which belongs to L_i, so $L_i \subset \mathscr{U}$ for all i. Therefore $\mathscr{L} \subset \mathscr{U}$, which implies that $\mathscr{L} = \mathscr{U}$. ∎

We want to show the analogous result for weak modules for a vertex operator algebra V. For this purpose, we use the Lie algebra $\mathfrak{g}(V)$ associated to the vertex operator algebra V (cf. [24], [39]).

The Lie algebra $\mathfrak{g}(V)$ is realized on the vector space

$$\mathfrak{g}(V) = \frac{V \otimes \mathbb{C}[t,t^{-1}]}{(L(-1)\otimes 1 + 1 \otimes \frac{d}{dt}).V \otimes \mathbb{C}[t,t^{-1}]},$$

where the commutator is given by

$$[a\otimes t^n, b\otimes t^m] = \sum_{i=0}^{\infty} \binom{n}{i} (a_i b)\otimes t^{n+m-i}.$$

Then by [39, Lemma 5.1] we have:

Lemma 4.3.2. Let V be a vertex operator algebra. We have:

- Every weak V–module W is a $\mathfrak{g}(V)$–module with the action

$$v\otimes t^n \mapsto v_n \quad (v\in V,\ n\in\mathbb{Z}).$$

- If W is an irreducible weak V–module, then W is also an irreducible $\mathfrak{g}(V)$–module.

Lemma 4.3.3. Assume that L_i, $i=1,\dots,t$, are non-isomorphic irreducible weak V–modules and $\mathscr{L} = \bigoplus_{i=1}^{t} L_i$. Then for each $w_i \neq 0$, $w_i \in L_i$, a vector of the form $(w_1, w_2, \dots, w_t)$ is cyclic in $\mathscr{L}$.

Proof. By Lemma 4.3.2, we have that L_i, $i=1,\dots,t$ are irreducible modules for the associative algebra $\mathscr{A} = U(\mathfrak{g}(V))$. Then the assertion follows by applying Lemma 4.3.1. ■

4.4. MAIN RESULT: ORDER 2 CASE

We shall first consider the case of automorphisms of order two.

Let θ be an order two automorphism of V. Let

$$V^+ = \{v \in V \mid \theta(v) = v\}, \quad V^- = \{v \in V \mid \theta(v) = -v\}.$$

Then V^+ is a vertex subalgebra of V and V^- is a V^+-module.

Theorem 4.4.1. Let V be a vertex operator algebra and W be an irreducible weak V–module such that $W_\theta = W \circ \theta \ncong W$. Then W is an irreducible weak V^+–module.

Proof. Consider a V–module $\mathscr{M} = W \oplus W_\theta$. Define now the map

$$\Delta^\pm : W \to \mathscr{M}, \quad w \mapsto (w, \pm w).$$

Let

$$\Delta^\pm(W) = \{(w, \pm w) \mid w \in W\}.$$

Then we have

$$\mathscr{M} = \Delta^+(W) \bigoplus \Delta^-(W).$$

Moreover, $\Delta^\pm$ are V^+–homomorphisms. Next we notice that

$$\begin{aligned}
&V^+.\Delta^+(W) \\
&= \mathrm{Span}_{\mathbb{C}} \left\{ (v_n w, \theta(v)_n w), v \in V^+, w \in W \right\} \\
&= \mathrm{Span}_{\mathbb{C}} \left\{ \left((v_n w, v_n w), v \in V^+, w \in W \right) \right\} \\
&= \Delta^+(W)
\end{aligned}$$

and

$$\begin{aligned}
&V^-.\Delta^+(W) \\
&= \mathrm{Span}_{\mathbb{C}} \left\{ (v_n w, \theta(v)_n w), v \in V^-, w \in W \right\} \\
&= \mathrm{Span}_{\mathbb{C}} \left\{ \left((v_n w, -v_n w), v \in V^-, w \in W \right) \right\} \\
&= \Delta^-(W)
\end{aligned}$$

Assume that W is not irreducible V^+–module. Then there is a V^+–submodule $0 \neq S \subsetneqq W$. In particular, $0 \neq \Delta^+(S) \subsetneqq \Delta^+(W)$. But Lemma 4.3.3 implies that $V.\Delta^+(S) = \mathscr{M}$. Since

$$V^{\pm}.\Delta^+(S) \subset \Delta^{\pm}(W),$$

we must have that $V^+.\Delta^+(S) = \Delta^+(W)$ which is a contradiction. The proof follows. ∎

4.5. GENERAL CASE

Assume that g is an automorphism of arbitrary (not necessarily prime) order p. Then

$$V = V^0 \oplus V^1 \oplus \cdots \oplus V^{p-1} \tag{4.1}$$

where

$$V^i = \{ v \in V \mid gv = \zeta^i v \}$$

and ζ is a primitive p–th root of unity.

Let W be a weak V–module. Let

$$\mathscr{M} = W_0 \oplus W_1 \oplus \cdots \oplus W_{p-1},$$

where $W_i = W \circ g^i, i = 0, 1, \cdots, p-1$. Let $\Delta^{(p,i)}$ be the V^0–homomorphism defined by

$$w \mapsto (w, (\zeta^i)w, \cdots, (\zeta^i)^{p-1} w).$$

Lemma 4.5.1. We have:

(1) $\mathscr{M} = \bigoplus_{i=0}^{p-1} \Delta^{(p,i)}(W)$.

(2) $V^j . \Delta^{(p,i)}(W) \subset \Delta^{(p,i+j)}(W)$.

Proof. The proof of (1) is easy. Let us prove (2).

Take $v \in V^j$. For $w' \in W_r$ we have

$$v_n w' = \zeta^{rj} v_n w',$$

which implies

$$v_n(w, (\zeta^i)w, \cdots, (\zeta^i)^{p-1} w) = (v_n w, \zeta^{i+j} v_n w, \cdots, (\zeta^{i+j})^{p-1} v_n w) \in \Delta^{(p,i+j)}(W).$$

The Lemma holds. ■

Lemma 4.5.2. Let W be an irreducible weak V–module and $\mathscr{M}$ be as above. Assume that for every $w \in W$, $w \neq 0$,

$$(w, \cdots, w) \quad \text{is cyclic in } \mathscr{M}$$

Then W is an irreducible weak V^0–module.

Proof. Assume that W is not a simple V^0–module. Then there is a V^0–submodule $0 \neq S \subsetneqq W$. In particular,

$$0 \neq \Delta^{(p,0)}(S) \subsetneqq \Delta^{(p,0)}(W).$$

Since each $(w, \cdots, w)$, with $w \neq 0$, is a cyclic vector in $\mathscr{M}$, we get

$$\mathscr{M} = V.\Delta^{(p,0)}(S).$$

This implies that $V^0.\Delta^{(p,0)}(S) = \Delta^{(p,0)}(W)$. A contradiction. The proof follows. ■

Lemmas 4.3.3 and 4.5.2 imply our main result.

Theorem 4.5.3. Let W be an irreducible weak V–module, and g an automorphism of finite order such that $W \circ g^i \ncong W$ for all i. Then W is an irreducible weak V^0–module.

4.6. On complete reducibility of certain $V^{\langle g \rangle}$–modules

In this section we shall use the following very general version of Schur's Lemma. Proof can be found in [54, Section 4.1.2].

Lemma 4.6.1. Assume that W_1 and W_2 are irreducible modules for an associative algebra A. Assume that W_1 and W_2 have countable dimensions over $\mathbb{C}$. Then $\dim Hom_A(W_1, W_2) \leq 1$ and $\dim Hom_A(W_1, W_2) = 1$ if and only if $W_1 \cong W_2$.

Lemma 4.6.2. Assume that V is a vertex operator algebra. Then every irreducible weak V-module W is countable dimensional.

Proof. Note that the vertex operator algebra V is countable dimensional. Take any $w \in W$. Then by [64, Proposition 6.1] (see also [41, Proposition 4.1]),

$$W = V.w = \mathrm{Span}_{\mathbb{C}}\{u_n w \mid u \in V, n \in \mathbb{Z}\},$$

which implies that W is also countable dimensional.

∎

Assume that g is an automorphism of arbitrary order p. Then we have the decomposition (4.1).

Theorem 4.6.3. Assume that g is an automorphism of V of finite order p as above. Assume that W is an irreducible weak V–module such that $W \circ g \cong W$. Then W is completely reducible weak V^0–module such that

(1) $W = \bigoplus_{i=0}^{p} W^i$, $V^i.W^j \subset W^{i+j \bmod (p)}$, where W^j, $j = 1, \ldots, p$ are eigenspaces of certain linear isomorphism $\Phi(g) : W \to W$.

(2) Each W^i is an irreducible weak V^0–module.

(3) The modules W^i, $i = 0, \ldots p-1$, are non-isomorphic as weak V^0–modules.

Proof. By Lemma 4.6.2, W is countable dimensional. Let $f : W \to W \circ g$ be a V–isomorphism . Then we have

$$f(u_n w) = (gu)_n f(w) \quad \forall u \in V,\ w \in W.$$

Applying f p-times, we get

$$f^p(u_n w) = (g^p u)_n f^p(w) = u_n f^p(w).$$

Therefore f^p is a V–endomorphism. Applying Schur's Lemma for the associative algebra $A = U(\mathfrak{g}(V))$, we get that $f^p = a\mathrm{Id}_W$, where a is a non-zero constant. By rescaling f one gets an isomorphism $\Phi(g) : W \to W \circ g$ such that $\Phi(g)^p = \mathrm{Id}$. Next we consider $\Phi(g)$ as a linear operator on W with the property $\Phi(g)^p = \mathrm{Id}_W$.

That means W is a $\langle g \rangle$–module and there is $0 \le j \le p-1$ and a vector $0 \ne w_j \in W$ such that $\Phi(g)(w_j) = \zeta^j w_j$. Clearly, for any $x \in V^i.w_j = \mathrm{Span}_{\mathbb{C}}\{v_n w_j | v \in V^j, n \in \mathbb{Z}\}$ and $0 \le i \le p-1$, we have $\Phi(g)(x) = \zeta^{i+j} x$. Without loss of generality, we may assume there is a $0 \ne w \in W$ such that $\Phi(g)(w) = w$.

Now define $W^j = V^j.w = \mathrm{Span}_{\mathbb{C}}\{v_n w | v \in V^j, n \in \mathbb{Z}\}$. Then

- $\Phi(g)(w_j) = \zeta^j w_j$ for each $w_j \in W^j$.
- $\Phi(g)(u_n w_j) = \zeta^j g(u)_n w_j = \zeta^{i+j} u_n w_j$ for $u \in V^i$, $w_j \in W^j$.

This implies that $W = \bigoplus_{i=0}^{p} W^i$, $V^i.W^j \subset W^{i+j \ \mathrm{mod}(p)}$ and (1) holds.

Assertion (2) follows from (1).

Let $0 \ne U \ne W^j$ be a proper V^0–submodule of W^j and consider the V–submodule $X = V.U$. Then

$$X = V^0.U + V^1.U + \ldots + V^{p-1}.U$$

Since U is a proper V^0–submodule of W^j, then $V^i.U \subset W^{i+j}$ implies that X is a proper V–submodule of W. This is impossible since W is a simple V–module. Hence W^j is irreducible V^0–module for each j.

Proof of assertion (3) is completely analogous to that of [41, Theorem 5.1] and it uses Lemma 4.6.1. For completeness, we shall include it.

Suppose we have a V^0–isomorphism $p : W^i \to W^j$, $i \ne j$. Take a nonzero $w \in W^i$ and consider the following V–submodule $\mathscr{U}$ of $W \oplus W$

$$\mathscr{U} = V.(w, p(w)) = \mathrm{Span}_{\mathbb{C}}\{(v_n w, v_n p(w)) \mid v \in V\}.$$

Then $W^i \oplus W^i$ is not in $\mathscr{U}$ and hence $\mathscr{U}$ is a proper submodule of $W \oplus W$. Since $W \oplus W$ is a $U(\mathfrak{g}(V))$–module of finite length, the Jordan-Hölder theorem can be applied. Comparing filtrations

$$(0) \to W \to W \oplus W, \quad (0) \to \mathscr{U} \to W \oplus W,$$

and using simplicity of W, we get that $\mathscr{U} \cong W$ as $U(\mathfrak{g}(V))$–modules. This implies that $\mathscr{U} \cong W$ as V–modules.

Then both projection maps from $\mathscr{U} \to W \oplus (0)$ and $\mathscr{U} \to (0) \oplus W$ are V–isomorphisms. Hence the map

$$\Phi : u_n w \mapsto u_n p(w), \quad (u \in V)$$

is also an isomorphism. Using Schur's lemma we get $\Phi = a\mathrm{Id}$ for $a \in \mathbb{C}$, which implies that $p(w) = aw \in W^i$. This implies $i = j$. A contradiction.

■

4.7. WHITTAKER MODULES: SOME STRUCTURAL RESULTS

First we recall some basic notions from [22].

Definition 4.7.1. For a Lie algebra $\mathfrak{n}$, define ideals $\mathfrak{n}_0 := \mathfrak{n}$ and $\mathfrak{n}_i := [\mathfrak{n}_{i-1}, \mathfrak{n}], i > 0$. Then we have a sequence of ideals

$$\mathfrak{n} = \mathfrak{n}_0 \supset \mathfrak{n}_1 \supset \mathfrak{n}_2 \supset \cdots .$$

We say that $\mathfrak{n}$ is *quasi-nilpotent* if $\cap_{i=0}^{\infty} \mathfrak{n}_i = 0$. Obviously, any nilpotent Lie algebra is quasi-nilpotent.

Definition 4.7.2. Let $\mathfrak{g}$ be a nonzero complex Lie algebra and let $\mathfrak{n}$ be a subalgebra of $\mathfrak{g}$. If M is a $\mathfrak{g}$–module, then we say that the action of $\mathfrak{n}$ on M is *locally finite* provided that $U(\mathfrak{n})v$ is finite dimensional for all $v \in M$. Let $\mathscr{W}h(\mathfrak{g}, \mathfrak{n})$ denote the full subcategory of the category $\mathfrak{g}$–Mod of all $\mathfrak{g}$–modules, which consists of all $\mathfrak{g}$–modules, the action of $\mathfrak{n}$ on which is locally finite.

Let V be a vertex algebra. Assume that the Lie algebra $\mathfrak{L}$ is one of the following two types:

(1) $\mathscr{L} = \mathfrak{g}(V)$, or

(2) $\mathscr{L}$ is the Lie algebra of modes of generating fields of the vertex algebra V.

Remark 4.7.3. In many cases it is possible to replace $\mathfrak{g}(V)$ with much smaller Lie algebra. For example, this happens in the following cases:

- If V is the universal affine vertex algebra $V^k(\mathfrak{g})$, then we can take $\mathfrak{L} = \widehat{\mathfrak{g}}$, where $\widehat{\mathfrak{g}}$ is the affine Lie algebra associated to the simple Lie algebra $\mathfrak{g}$ (cf. [17], [11] for studying Whittaker modules in this case).
- If V is the Heisenberg vertex algebra $M(1)$, we can take $\mathfrak{L} = \hat{\mathfrak{h}}$ (cf. Section 4.8 below).

- If V is the Weyl vertex algebra, we can take Lie algebra $\mathfrak{L}$ such that the Weyl algebra $\widehat{\mathscr{A}} = U(\mathfrak{L})$ (cf. Section 4.9 below).

Note that by our assumptions on the vertex algebra, every weak V–module is a module for the Lie algebra $\mathfrak{L}$. We also assume the following:

- Let $\mathfrak{n}$ be a nilpotent subalgebra of $\mathfrak{L}$.
- Let $\mathscr{W}h(\mathfrak{L},\mathfrak{n})$ denote the full category of $\mathfrak{L}$–modules W such that $\mathfrak{n}$ acts locally finitely on W (cf. [22]).

Definition 4.7.4. Let $W \in \mathscr{W}h(\mathfrak{L},\mathfrak{n})$. A vector $v \in W$ is called a Whittaker vector provided that $\langle v \rangle$ is an $\mathfrak{n}$-submodule of W.

Let $\lambda : \mathfrak{n} \to \mathbb{C}$ be a Lie algebra homomorphism which will be called a *Whittaker function*. Let $U_\lambda = \mathbb{C}u_\lambda$ be the 1–dimensional $\mathfrak{n}$–module such that

$$xu_\lambda = \lambda(x)u_\lambda \quad (x \in \mathfrak{n}).$$

Consider the standard (universal) Whittaker $\mathfrak{L}$–module

$$M_\lambda = U(\mathfrak{L}) \otimes_{U(\mathfrak{n})} U_\lambda.$$

Definition 4.7.5. We say that an irreducible V–module W is of Whittaker type λ if W is an irreducible quotient of the standard Whittaker module M_λ.

Lemma 4.7.6 (cf. [22])**.** Assume that W is an irreducible V–module of Whittaker type λ. Then

$$W = \{w \in W \mid \forall x \in \mathfrak{n}, \quad (x-\lambda(x))^k w = 0 \text{ for } k >> 0\}.$$

Proof. Let us first prove that

$$M_\lambda = \{w \in M_\lambda \mid \forall x \in \mathfrak{n}, \quad (x-\lambda(x))^k w = 0 \text{ for } k >> 0\}.$$

Take an arbitrary element $w_1 \in U(\mathfrak{L})$. Since $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{L}$, for $x \in \mathfrak{n}$, we have

$$\mathrm{ad}_x^k(w_1) = 0 \quad \text{for } k >> 0.$$

Next we notice that

$$(x-\lambda(x))w_1 \otimes u_\lambda = [x, w_1] \otimes u_\lambda$$

which implies that

$$(x-\lambda(x))^k w_1 \otimes u_\lambda = \mathrm{ad}_x^k(w_1) \otimes u_\lambda = 0 \quad \text{for } k >> 0.$$

The claim now follows from the fact that W is a quotient of the universal Whittaker module M_λ. ■

Lemma 4.7.7. Assume that $\lambda, \mu : \mathfrak{n} \to \mathbb{C}$ are Whittaker functions such that $\lambda \neq \mu$. Assume that W_λ and W_μ are irreducible Whittaker modules of types λ and μ respectively. Then

(1) W_λ and W_μ are inequivalent as V–modules.

(2) Let $(w_1, w_2) \in W_\lambda \oplus W_\mu$, $w_1 \neq 0, w_2 \neq 0$. Then

$$V.(w_1, w_2) = W_\lambda \oplus W_\mu.$$

Proof. (1) Assume that $f : W_\lambda \to W_\mu$ is an isomorphism. Then $f(w_\lambda)$ is a non-trivial Whittaker vector in W_μ such that

$$(x-\lambda(x))f(w_\lambda) = 0, \quad \forall x \in \mathfrak{n}.$$

Take $x \in \mathfrak{n}$ such that $\lambda(x) \neq \mu(x)$. The for every $k > 0$ we have

$$(x-\mu(x))^k f(w_\lambda) = (x-\lambda(x)+\lambda(x)-\mu(x))^k f(w_\lambda) = (\lambda(x)-\mu(x))^k f(w_\lambda) \neq 0.$$

This contradicts with Lemma 4.7.6. So (1) holds.

Now let us prove (2).

Claim: There exist $k > 0$ and $x \in \mathfrak{n}$ such that

$$(x-\lambda(x))^k w_1 = 0 \quad \text{and} \quad (x-\lambda(x))^k w_2 = z' \neq 0.$$

Indeed, take $x \in \mathfrak{n}$ such that $\mu(x) - \lambda(x) = A \neq 0$. Let k_1, k_2 be the smallest positive integer such that

$$(x-\lambda(x))^{k_1} w_1 = 0 \text{ and } (x-\mu(x))^{k_2} w_2 = 0.$$

Let $k = \max\{k_1, k_2\}$. We have

$$\begin{aligned}
&(x-\lambda(x))^k w_2 \\
&= (x-\mu(x)+A)^k w_2 \\
&= \sum_{p=0}^{k-1} \binom{k}{p} (x-\mu(x))^p A^{k-p} w_2 \\
&= \sum_{p=0}^{k_2-1} \binom{k}{p} (x-\mu(x))^p A^{k-p} w_2 \\
&= A^k w_2 + kA^{k-1}(x-\mu(x))\, w_2 + \cdots + \binom{k}{k_2-1} A^{k-k_2+1} (x-\mu(x))^{k_2-1} w_2.
\end{aligned}$$

Note that $w_2, (x-\mu(x))\, w_2, \cdots, (x-\mu(x))^{k_2-1} w_2$ are linearly independent. Otherwise, there exist $p_0, p_1, \cdots, p_{k_2-1}$ such that

$$p_0 w_2 + p_1 (x-\mu(x))\, w_2 + \cdots + p_{k_2-1} (x-\mu(x))^{k_2-1} w_2 = 0.$$

Let $I = \{i = 0, 1, \cdots, k_2-1 | p_i \neq 0\}$ and $s = \max I$. Then

$$(x-\mu(x))^s w_2 = \sum_{i=0}^{s-1} \frac{p_i}{p_s} (x-\mu(x))^i w_2. \tag{4.2}$$

If $(x-\mu(x))^{k_2-s} w_2$, $(x-\mu(x))^{k_2-s+1} w_2, \cdots, (x-\mu(x))^{k_2-1} w_2$ are linearly independent, then $p_i = 0$, $i = 0, 1, \cdots, s-1$. By (4.2), we obtain $(x-\mu(x))^s w_2 = 0$ where $s < k_2$, which is a contradiction. So$(x-\mu(x))^{k_2-s} w_2$, $(x-\mu(x))^{k_2-s+1} w_2$, $\cdots$, $(x-\mu(x))^{k_2-1} w_2$ (at most k_2-1 terms) are linearly dependent. Repeating similar argument, we can prove that there exists $q < k_2$ such that $(x-\mu(x))^q w_2 = 0$, which is a contradiction. Thus we proved w_2, $(x-\mu(x))\, w_2, \cdots, (x-\mu(x))^{k_2-1} w_2$ are linearly independent and hence

$$A^k w_2 + kA^{k-1}(x-\mu(x))\, w_2 + \cdots + \binom{k}{k_2-1} A^{k-k_2+1} (x-\mu(x))^{k_2-1} w_2 \neq 0.$$

Now we have

$$(x-\lambda(x))^k (w_1, w_2) = (0, z'), \quad z' \in W_\mu, z' \neq 0.$$

Irreducibility of W_μ now gives that $W_\mu \subset V.(w_1, w_2)$. Similarly we prove that $W_\lambda \subset V.(w_1, w_2)$. So (2) holds. ■

Remark 4.7.8. The assertion (2) is a consequence of (1) and Lemmas 4.3.1 and 4.3.3. So we could omit its proof. But since Whittaker modules are the main new examples on which we can apply Theorem 4.5.3, we think that it is important to keep an independent proof which contains explicit construction of elements in maximal left ideals $J_i = Ann(w_i)$, $i = 1,2$. In particular, we have elements $(x - \lambda(x))^k \in J_1 \setminus J_2$ which correspond to element u_i (for $i = 2$) constructed in Lemma 4.3.1 by using abstract arguments.

We can easily generalize the previous lemma:

Lemma 4.7.9. Assume that $\lambda_1, \cdots, \lambda_n : \mathfrak{n} \to \mathbb{C}$ are Whittaker functions such that $\lambda_i \neq \lambda_j$ for $i \neq j$. Assume that W_{λ_i}, $i = 1, \ldots, n$ are irreducible Whittaker modules of types λ_i. Then

(1) All W_{λ_i} are inequivalent as V–modules.

(2) Let $\mathbf{w} = (w_1, w_2, \cdots w_n) \in W_{\lambda_1} \oplus \cdots \oplus W_{\lambda_n}$, where $0 \neq w_i \in W_{\lambda_i}$. Then

$$V.\mathbf{w} = W_{\lambda_1} \oplus \cdots \oplus W_{\lambda_n}.$$

Theorem 4.7.10. Assume that W is a V–module in the Whittaker category $\mathscr{W}h(L, \mathfrak{n})$ as before. Assume also that $W_i = W \circ g^i$ has the Whittaker function $\lambda^{(i)} = \mathfrak{n} \to \mathbb{C}$ and that all $\lambda^{(i)}$ are distinct. Then W is an irreducible V^0–module.

Proof. The proof follows immediately from Lemma 4.7.9 and Theorem 4.5.2. ■

4.8. EXAMPLE: HEISENBERG VERTEX ALGEBRA

First we recall the definition of the Heisenberg Lie algebra $\hat{\mathfrak{h}}$. Let $\mathfrak{h}$ be complex ℓ-dimensional vector spaces with respect to the non-degenerate bilinear form $(\cdot,\cdot)$. Fix an orthonormal basis $\{h_1, h_2, \ldots, h_\ell\}$ with respect to form $(\cdot,\cdot)$. The Heisenberg Lie algebra $\hat{\mathfrak{h}}$ is defined as

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}\left[t, t^{-1}\right] \oplus \mathbb{C}K$$

with the commutator relations

$$[K, \hat{\mathfrak{h}}] = 0, \quad \text{and} \quad [a(m), b(n)] = m\delta_{m+n,0}(a,b)K$$

for $a, b \in \mathfrak{h}$, $m, n \in \mathbb{Z}$ and $a(n) = a \otimes t^n$. We identify $\mathfrak{h}$ with its dual space $\mathfrak{h}^*$ by the form $(\cdot,\cdot)$. Let $\mathbb{C}e^0$ be the one-dimensional module over the Lie algebra $\hat{\mathfrak{h}}$ with action given by

$$h(n)e^0 = 0, \quad \forall h \in \mathfrak{h}, n \geq 0; \qquad Ke^0 = e^0.$$

Define the vector space $M(1)$ by

$$M(1) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h}\otimes\mathbb{C}[t]\oplus\mathbb{C}K)} \mathbb{C}e^0. \tag{4.3}$$

On $M(1)$ define the state-field correspondence by

$$Y(a^{(1)}(n_1)\cdots a^{(r)}(n_r)e^0, z) = a^{(1)}(z)_{n_1}\cdots a^{(r)}(z)_{n_r}\mathrm{Id}_{M(1)} \tag{4.4}$$

for $a^{(i)} \in \mathfrak{h}$ and $n_i \in \mathbb{Z}$. The vacuum vector is $\mathbf{1} = e^0$ and the conformal vector is given by

$$\omega = \frac{1}{2}\sum_{i=1}^{\ell} h_i(-1)^2\mathbf{1}.$$

In particular,

$$Y(\omega, z) = L(z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}, \qquad L(n) = \frac{1}{2}\sum_{m\in\mathbb{Z}} {}^{\circ}_{\circ} h_i(-m)h_i(m+n) {}^{\circ}_{\circ}.$$

Then $(M(1), Y, \mathbf{1}, \omega)$ is a vertex operator algebra (cf. [62] for details). Now consider the order two automorphism θ of the vector space $M(1)$ given by

$$\theta\left(h_{i_1}(-n_1)h_{i_2}(-n_2)\cdots h_{i_k}(-n_k)\mathbf{1}\right) = (-1)^k h_{i_1}(-n_1)h_{i_2}(-n_2)\cdots h_{i_k}(-n_k)\mathbf{1}$$

for $i_j \in \{1,2,\ldots,\ell\}$ for all j and $n_1 \geq n_2 \geq \cdots \geq n_k > 0$. Let $M(1)^+$ be the corresponding subspace of fixed-points with respect to θ:

$$M(1)^+ = \left\{ v \in M(1) \mid \theta(v) = v \right\}. \tag{4.5}$$

Then $M(1)^+$ is a vertex operator algebra and its structure and representation are well studied (cf. [1, 5, 42, 44]).

Let $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_r, 0, \cdots)$ be a sequence of elements of $\mathfrak{h}$ with at least one nonzero entry, and $\lambda_n = 0$ for $n \gg 0$. Let $\mathbb{C}e^{\boldsymbol{\lambda}}$ be the one-dimensional module over the Lie algebra $\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}K$ with action given by

$$h(n)e^{\boldsymbol{\lambda}} = (h, \lambda_n)e^{\boldsymbol{\lambda}},\ h \in \mathfrak{h}, n \geq 0; \quad Ke^{\boldsymbol{\lambda}} = e^{\boldsymbol{\lambda}}.$$

Consider the corresponding induced $U(\widehat{\mathfrak{h}})$-module

$$M(1, \boldsymbol{\lambda}) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}K)} \mathbb{C}e^{\boldsymbol{\lambda}}.$$

Let $\mathfrak{L} = \hat{\mathfrak{h}}$ and $\mathfrak{n} = \mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}K$, then it is clear that $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{L}$ and hence $M(1, \boldsymbol{\lambda})$ is the standard (universal) Whittaker $\mathfrak{L}$-module $U(\mathfrak{L}) \otimes_{U(\mathfrak{n})} \mathbb{C}e^{\boldsymbol{\lambda}}$. Define the Whittaker function $\Lambda : \mathfrak{n} \to \mathbb{C}$ by

$$\Lambda(h(n)) = (h, \lambda_n), n = 0, 1, \cdots, r;\ \Lambda(h(k)) = 0, \forall k > r.$$

Then we see that $\mathbb{C}e^{\boldsymbol{\lambda}}$ is a 1-dimensional $\mathfrak{n}$-module such that $xe^{\lambda} = \Lambda(x) e^{\boldsymbol{\lambda}}$ for any $x \in \mathfrak{n}$. By Definition 4.7.5, $M(1, \boldsymbol{\lambda}) \in \mathscr{W}h(\mathfrak{L}, \mathfrak{n})$ is an irreducible *Whittaker module* for $M(1)$ of Whittaker type $\boldsymbol{\Lambda}$.

Now we see that $M(1, \boldsymbol{\lambda}) \circ \theta$ is a Whittaker module for $M(1)$ with type $-\Lambda$. By Theorem 4.7.10, $M(1, \boldsymbol{\lambda})$ is irreducible as Whittaker module for $M(1)^+$. This gives another proof of Theorem 6.1 in [56].

4.8.1. On cyclic orbifolds of $M(1)$

The orbifolds of $M(1)$ were studied by A. Linshaw in [66] using invariant theory. We can now prove irreducibility of certain Whittaker modules for $M(1)$ for cyclic orbifolds.

Let $O(\ell)$ be orthogonal group. It acts naturally on the vector space $\mathfrak{h}$ by preserving form $(\cdot, \cdot)$:

$$(gh, gh') = (h, h') \quad \forall h, h' \in \mathfrak{h},\ g \in O(\ell).$$

The action of $O(\ell)$ on $\mathfrak{h}$ can be uniquely extended to the action on the vertex operator algebra $M(1)$. Moreover, $O(\ell)$ is the full automorphism group of $M(1)$. Let $\boldsymbol{\lambda}:\mathfrak{n}\to\mathbb{C}$ be a Whittaker function. The action of $O(\ell)$ on $M(1,\boldsymbol{\lambda})$ is given by

$$M(1,\boldsymbol{\lambda})\circ g = M(1,\boldsymbol{\lambda}\circ g),$$

$$(\boldsymbol{\lambda}\circ g)(h(n)) = \boldsymbol{\lambda}(gh)(n), \quad g\in O(\ell),\ h\in\mathfrak{h},\ n\geq 0.$$

It is important to consider orbifolds of certain finite subgroups of $O(\ell)$. A particularly interesting subgroup of $O(\ell)$ is the symmetric group S_ℓ of ℓ letters. A detailed example of orbifold $M(1)^{S_3}$, in the case of rank three Heisenberg algebra, were presented in a recent paper by A. Milas, M. Penn and H. Shao (cf. [68]).

Let $h_1,\dots,h_\ell$ be the basis of $\mathfrak{h}$ as above. Let $\boldsymbol{\lambda}:\mathfrak{n}\to\mathbb{C}$ be a Whittaker function. Then

$$\boldsymbol{\lambda} = (\lambda^1,\dots,\lambda^\ell), \quad \lambda^i = (\lambda(h_i(0)),\lambda(h_i(1)),\cdots).$$

The action of S_ℓ on $M(1,\boldsymbol{\lambda})$ is given by

$$M(1,\boldsymbol{\lambda})\circ g = M(1,\boldsymbol{\lambda}\circ g),$$

$$\boldsymbol{\lambda}\circ g = (\lambda^{g(1)},\dots,\lambda^{g(\ell)}) \quad \forall g\in S_\ell.$$

Proposition 4.8.1. (1) Assume that $g\in O(\ell)$ is of finite order such that $\boldsymbol{\lambda}\circ g^i\neq\boldsymbol{\lambda}$ for all i. Then $M(1,\boldsymbol{\lambda})$ is an irreducible $M(1)^{\langle g\rangle}$–module.

(2) Assume that $\boldsymbol{\lambda}\circ\sigma\neq\boldsymbol{\lambda}$ for any 2–cycle $\sigma\in S_\ell$. Then $M(1,\boldsymbol{\lambda})$ is an irreducible $M(1)^{\langle g\rangle}$–module for any $g\in S_\ell$.

Proof. The proof of assertion (1) follows easily by using Theorem 4.5.2. Since for a 2-cycle σ, we have

$$\boldsymbol{\lambda}\circ\sigma\neq\boldsymbol{\lambda},\ \forall\sigma \iff \forall i,j, \quad 1\leq i<j\leq\ell, \quad \lambda_i\neq\lambda_j$$

$$\iff (\lambda^{h(1)},\dots,\lambda^{h(\ell)})\neq(\lambda^1,\dots,\lambda^\ell), \quad \forall h\in S_\ell.$$

Therefore for any $g\in S_\ell$ we have $\boldsymbol{\lambda}\circ g^i\neq\boldsymbol{\lambda}$. Now assertion (2) follows directly from (1). ■

We also have the following conjecture.

Conjecture 4.8.2. Assume that $\boldsymbol{\lambda} \circ \sigma \neq \boldsymbol{\lambda}$ for any 2–cycle $\sigma \in S_\ell$. Then $M(1, \boldsymbol{\lambda})$ is an irreducible $M(1)^{S_\ell}$–module.

The proof of the conjecture requires certain extension of methods used in the paper. We plan to study the proof of this conjecture in our future work.

4.9. EXAMPLE: WEYL VERTEX ALGEBRA

The Weyl algebra $\widehat{\mathscr{A}}$ is the associative algebra with generators $a(n), a^*(n), n \in \mathbb{Z}$ and relations

$$[a(n), a^*(m)] = \delta_{n+m,0}, \quad [a(n), a(m)] = [a^*(m), a^*(n)] = 0, \quad n, m \in \mathbb{Z}.$$

Let M be the simple *Weyl module* generated by the cyclic vector $\mathbf{1}$ such that

$$a(n)\mathbf{1} = a^*(n+1)\mathbf{1} = 0 \quad (n \geq 0).$$

As a vector space,

$$M \cong \mathbb{C}[a(-n), a^*(-m) \mid n > 0,\ m \geq 0].$$

There is a unique vertex algebra $(M, Y, \mathbf{1})$ (cf. [47, 48, 59]) where the vertex operator is given by

$$Y : M \to \mathrm{End}(M)[[z, z^{-1}]]$$

such that

$$Y(a(-1)\mathbf{1}, z) = a(z), \quad Y(a^*(0)\mathbf{1}, z) = a^*(z),$$
$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1},\ \ a^*(z) = \sum_{n \in \mathbb{Z}} a^*(n) z^{-n}.$$

We choose the following conformal vector of central charge $c = -1$ (cf. [59]):

$$\omega = \frac{1}{2}(a(-1)a^*(-1) - a(-2)a^*(0))\mathbf{1}.$$

Then $(M, Y, \mathbf{1}, \omega)$ has the structure of a $\frac{1}{2}\mathbb{Z}_{\geq 0}$–graded vertex operator algebra. We can define weak and ordinary modules for $(M, Y, \mathbf{1}, \omega)$ as in the case of $\mathbb{Z}$–graded vertex operator algebras.

We define the Whittaker module for $\widehat{\mathscr{A}}$ to be the quotient

$$M_1(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \widehat{\mathscr{A}}/I,$$

where $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_n)$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and I is the left ideal

$$I = \langle a(0) - \lambda_0, \ldots, a(n) - \lambda_n, a^*(1) - \mu_1, \ldots, a^*(n) - \mu_n, a(n+1), \ldots, a^*(n+1), \ldots \rangle.$$

Proposition 4.9.1. We have:

(1) $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ is an irreducible $\widehat{\mathscr{A}}$–module.

(2) $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ is an irreducible weak module for the Weyl vertex operator algebra M.

Proof. It is straightforward to check that the ideal I defined above is a maximal left ideal in $\mathscr{A}$ (cf. [23], [53]) and therefore the quotient $M_1(\boldsymbol{\lambda},\boldsymbol{\mu}) = \mathscr{A}/I$ is an simple $\widehat{\mathscr{A}}$–module. Since by construction, $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ is a restricted $\widehat{\mathscr{A}}$–module, it is an irreducible M–module. ■

Let $w = 1+I \in M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$. Then w is a cyclic vector and

$$a(0)w = \lambda_0 w, \dots, a(n)w = \lambda_n w, a^*(1)w = \mu_1 w, \dots, a^*(n)w = \mu_n w$$

and $a^*(k)w = a(k)w = 0$ for $k > n$.

Now we want to identify $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ as a Whittaker module for certain Whittaker pair. Let $\mathfrak{L}$ be the Lie algebra with generators $a(n), a^*(n), K$, $n \in \mathbb{Z}$ such that K is central and

$$[a(n),a^*(m)] = \delta_{n+m,0}K, \quad [a(n),a(m)] = [a^*(m),a^*(n)] = 0, \quad n,m \in \mathbb{Z}.$$

Then $M_1(\lambda,\mu)$ is an irreducible $\mathfrak{L}$–module of level 1 (i.e., K acts as the multiplication with 1).

Let $\mathfrak{n}$ be the subalgebra of $\mathfrak{L}$ generated by $a(n), a^*(n+1)$ for $n \geq 0$. Then $\mathfrak{n}$ is commutative, and therefore a nilpotent subalgebra of $\mathfrak{L}$.

Define the Whittaker function $\Lambda = (\boldsymbol{\lambda},\boldsymbol{\mu}) : \mathfrak{n} \to \mathbb{C}$ by

$$\Lambda(a(0)) = \lambda_0, \dots, \Lambda(a(n)) = \lambda_n, \Lambda(a(k)) = 0 \quad (k > n),$$

$$\Lambda(a^*(1)) = \mu_1, \dots, \Lambda(a^*(n)) = \mu_n, \Lambda(a^*(k)) = 0 \quad (k > n).$$

Proposition 4.9.2. $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ is a standard Whittaker module of level 1 for the Whittaker pair $(\mathfrak{L},\mathfrak{n})$ with Whittaker function $\Lambda = (\lambda,\mu)$.

Let $\zeta_p = e^{2\pi i/p}$ be p-th root of unity. Let g_p be the automorphism of the vertex operator algebra M which is uniquely determined by the following automorphism of the Weyl algebra $\widehat{\mathscr{A}}$:

$$a(n) \mapsto \zeta_p a(n), \quad a^*(n) \mapsto \zeta_p^{-1} a^*(n) \quad (n \in \mathbb{Z}).$$

Then g_p is the automorphism of M of order p.

Theorem 4.9.3. Assume that $\Lambda = (\boldsymbol{\lambda}, \boldsymbol{\mu}) \neq 0$. Then $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible weak module for the orbifold subalgebra $M^{\mathbb{Z}_p} = M^{\langle g_p \rangle}$ for each $p \geq 1$.

Proof. First we notice that $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu}) \circ g^i = M_1(\zeta_p^i \boldsymbol{\lambda}, \zeta_p^{-i} \boldsymbol{\mu})$ and therefore modules $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu}) \circ g^i$ have different Whittaker functions for $i = 0, \ldots, p-1$. Now assertion follows from Theorem 4.7.10. ■

4.9.1. An application to affine VOA

Let $\mathfrak{g}$ be a simple Lie algebra and let $V^k(\mathfrak{g})$ be its universal affine vertex algebra of level k. Let $L_k(\mathfrak{g})$ be its simple quotient. The following result is well-known:

Lemma 4.9.4. If W is an irreducible weak $L_k(\mathfrak{g})$-module, then M is an irreducible module for the affine Lie algebra $\hat{\mathfrak{g}}$-module of level k.

Next we show how the Theorem 4.9.3 gives a construction of new irreducible modules for affine Lie algebra $\widehat{\mathfrak{sl}(2)}$ associated to $\mathfrak{sl}(2)$. In the case $p = 2$, $\mathbb{Z}_2$–orbifold $M^{\mathbb{Z}_2}$ is isomorphic to a simple affine VOA $L_{-\frac{1}{2}}(\mathfrak{sl}(2))$ (cf. [46] and also [18, Section 6]) associated to affine Lie algebra $\widehat{\mathfrak{sl}(2)}$ at level $-\frac{1}{2}$. The previous theorem gives a realization of large family irreducible modules for VOA $L_{-\frac{1}{2}}(\mathfrak{sl}(2))$.

Corollary 4.9.5. Assume that $\Lambda = (\boldsymbol{\lambda}, \boldsymbol{\mu}) \neq 0$. Then $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible module for the affine Lie algebra $\widehat{\mathfrak{sl}(2)}$ at the level $k = -\frac{1}{2}$.

Proof. For $p = 2$, $M^{\mathbb{Z}_2}$ is isomorphic to the affine VOA $L_{-\frac{1}{2}}(\mathfrak{sl}(2))$. Therefore, module $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is irreducible for $L_{-\frac{1}{2}}(\mathfrak{sl}(2))$. Now Lemma 4.9.4 implies that $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible module for affine Lie algebra $\widehat{\mathfrak{sl}(2)}$. ■

Remark 4.9.6. The irreducible weight modules for the Weyl vertex algebra were analysed in [16]. One can easily show that weight modules, denoted by $\widetilde{U(\lambda)}$, have the property $\widetilde{U(\lambda)} \circ g_p \cong \widetilde{U(\lambda)}$. Then Theorem 4.6.3 implies that they are direct sum of two irreducible relaxed weight modules for the affine vertex algebra $L_{-\frac{1}{2}}(\mathfrak{sl}(2))$ (see also [11], [60]).

5. NEW RESULTS ON THE STRUCTURE OF WHITTAKER MODULES FOR CERTAIN VERTEX ALGEBRAS

In this chapter we present some new results on Whittaker modules for the Weyl vertex algebra. These and other important results on Whittaker modules will be included in a forthcoming paper (cf. [20]). We have proved in chapter 4 that irreducible Whittaker modules $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ (with non-trivial Whittaker functions) for the Weyl vertex algebra M are also irreducible for the orbifolds $M^{\mathbb{Z}_p} = M^{\langle g_p \rangle}$, for each $p \geq 1$. Here we will consider the limit case of infinite-dimensional group $G = \langle g \rangle$ together with its associated orbifold M^G, and prove that irreducible Weyl vertex algebra modules of Whittaker type are always reducible as M^G–modules.

Let us first recall that in chapter 4 we proved the following proposition:

Proposition 5.0.1 (Theorem 4.9.1)**.** We have:

(1) $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible $\widehat{\mathscr{A}}$–module.

(2) $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible weak module for the Weyl vertex operator algebra M.

Let $\zeta_p = e^{2\pi i/p}$ again be p-th root of unity and let g_p again be the automorphism of the vertex operator algebra M which is uniquely determined by the following automorphism of the Weyl algebra $\widehat{\mathscr{A}}$:

$$a(n) \mapsto \zeta_p a(n), \quad a^*(n) \mapsto \zeta_p^{-1} a^*(n) \quad (n \in \mathbb{Z}).$$

As before, g_p is the automorphism of M of order p.

In chapter 4 we also proved the following theorem:

Theorem 5.0.2 (Theorem 4.9.3). Assume that $\Lambda = (\boldsymbol{\lambda}, \boldsymbol{\mu}) \neq 0$. Then $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible weak module for the orbifold subalgebra $M^{\mathbb{Z}_p} = M^{\langle g_p \rangle}$, for each $p \geq 1$.

Let us now define an operator $J^0 := a(-1)a^*$. Then the components of the field

$$Y(J^0, z) = \sum_{n \in \mathbb{Z}} J^0(n) z^{-n-1}$$

satisfisfy the commutation relations for the Heisenberg algebra at level -1. We have the following subalgebra of M:

$$M^0 = \mathrm{Ker}_M J^0(0).$$

Let $\zeta \in \mathbb{C}$, $|\zeta| = 1$, which is **not a root of unity**. Let g be the automorphism of the vertex operator algebra M uniquely determined by the following automorphism of the Weyl algebra $\widehat{\mathscr{A}}$:

$$a(n) \mapsto \zeta a(n), \quad a^*(n) \mapsto \zeta^{-1} a^*(n) \quad (n \in \mathbb{Z}).$$

Then g is the automorphism of M of infinite order, and the group $G = \langle g \rangle$ is isomorphic to $\mathbb{Z}$. Clearly, we have

$$M^0 \cong M^G.$$

Note also that

$$M^0 = \bigcap_{p=1}^{\infty} M^{\mathbb{Z}_p}, \quad M \supset M^{\mathbb{Z}_2} \supset \cdots M^{\mathbb{Z}_p} \supset \cdots \supset M^0.$$

Since $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible $M^{\mathbb{Z}_p}$–module for each $p \geq 1$, it is natural to ask if $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is irreducible also as an M^0–module. However, we prove that $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a reducible and indecomposable M^0–module.

5.1. $\mathscr{W}_{1+\infty}$-ALGEBRA AT CENTRAL CHARGE $c = -1$ AND ITS WHITTAKER MODULES

A peculiarity of the orbifold M^0 is that it has two additional important realizations which we plan to explore in detail in our forthcoming paper [19]:

- M^0 is isomorphic to the vertex algebra $\mathscr{W}_{1+\infty}$ at central charge $c = -1$.

- M^0 is isomorphic to the simple module for the Lie algebra $\widehat{\mathfrak{g}l}$, which is the central extension of the Lie algebra of infinite matrices.

5.1.1. The $\mathscr{W}_{1+\infty}$-algebra approach

The universal vertex algebra $\mathscr{W}^c_{1+\infty}$ is generated by the fields

$$J^k(z) = \sum_{n\in\mathbb{Z}} J^k(n) z^{-n-k-1} \quad (k \in \mathbb{Z}_{\geq 0}),$$

whose components satisfy the commutation relations for the Lie algebra $\widehat{\mathscr{D}}$ at central charge c, which is a central extension of the Lie algebra of complex regular differential operators on $\mathbb{C}^*$. (cf. [49], [59]). It has a simple quotient, which we denote by $\mathscr{W}_{1+\infty,c}$.

It was proved by V. Kac and A. Radul in [59] that $M^0 \cong \mathscr{W}_{1+\infty,c}$ for $c = -1$. As a consequence, we have that M^0 is generated by the fields

$$J^k(z) = Y(a^*(-k)a, z) =: \left(\partial_z^k a^*(z)\right) a(z) := \sum_{n\in\mathbb{Z}} J^k(n) z^{-n-k-1} \quad (k \in \mathbb{Z}_{\geq 0}).$$

So our Whittaker modules for the Weyl vertex algebra are automatically Whittaker modules for $\mathscr{W}_{1+\infty,c=-1}$.

5.1.2. Approach using the Lie algebra $\widehat{\mathfrak{g}l}$

Define the generating function

$$E(z,w) =: a(z)a^*(w) := \sum_{i,j\in\mathbb{Z}} E_{i,j} z^{i-1} w^{-j}.$$

In other words, the operators $E_{i,j}$ are defined as

$$E_{i,j} =: a(-i)a^*(j) : \tag{5.1}$$

These operators endow M with the structure of a $\widehat{\mathfrak{g}l}$–module at central charge $K = -1$ (see formula (2.7) in [59] for commutation relations for $\widehat{\mathfrak{g}l}$).

We have (cf. [59]):

$$[E_{i,j}, a(-m)] = \delta_{j,m} a(-i), \quad [E_{i,j}, a^*(m)] = -\delta_{i,m} a^*(j).$$

5.2. THE STRUCTURE OF THE WHITTAKER MODULE $M_1(\lambda,\mu)$ AS A $\widehat{\mathfrak{gl}}$–MODULE

In this section we finally prove that $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ is not an irreducible $\widehat{\mathfrak{gl}}$–module. We shall use the following Casimir element.

Define

$$I := \sum_{j\in\mathbb{Z}} E_{j,j}.$$

As far we can see, I is introduced in [55] for a slightly different category of modules. But it is well defined on a family of Whittaker $\widehat{\mathfrak{g}l}$–modules which we considered.

Lemma 5.2.1. We have:

(1) $I \in \mathrm{End}(M_1(\boldsymbol{\lambda},\boldsymbol{\mu}))$ is well-defined.

(2) The action of I commutes with the action of $\widehat{\mathfrak{g}l}$ on $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$.

(3) The action of I commutes with the action of $\mathscr{W}_{1+\infty}$ on $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$.

Proof. Let $v \in M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$. Since $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ is a restricted module for the Weyl alegbra, there is $N \in \mathbb{Z}_{\geq 0}$ such that

$$a(n)v = a^*(n)v = 0 \quad \text{for } n > N.$$

This implies that $E_{n,n}v = 0$ if $|n| > N$. Therefore

$$Iv = \sum_{j\in\mathbb{Z}} E_{j,j}v = \sum_{j=-N}^{N} E_{j,j}v,$$

and we conclude that I is well defined on $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$. Thus (1) holds.

For a proof of assertion (2) we use commutation relations for $\widehat{\mathfrak{g}l}$:

$$[E_{i,j},E_{s,t}] = \delta_{j,s}E_{i,t} - \delta_{i,t}E_{s,j} - C\Phi(E_{i,j},E_{s,t}).$$

Then

$$\begin{aligned}
[E_{i,j},I]v &= \left(\sum_{k\in\mathbb{Z}}[E_{i,j},E_{k,k}]\right)v \\
&= \left(E_{i,j} - E_{i,j}\right)v \\
&= 0.
\end{aligned}$$

The assertion (2) holds.

(3) Since $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ is an M^0–module, it is also a $\mathscr{W}_{1+\infty}$–module at central charge $c=-1$. The action of $\mathscr{W}_{1+\infty}$ is generated by the fields

$$J^k(z)=\sum_{n\in\mathbb{Z}} J^k(n)z^{-n-k-1} =: \left(\partial_z^k a^*(z)\right) a(z):$$

which can be expressed as

$$J^k(n)=\sum_{j\in\mathbb{Z}} b_j E_{j,j-n} \quad (b_j\in\mathbb{C}).$$

Now (2) implies that

$$\begin{aligned} IJ^k(n)v &= I\sum_{j\in\mathbb{Z}} b_j E_{j,j-n}v \\ &= \sum_{j\in\mathbb{Z}} b_j I E_{j,j-n}v \\ &= \sum_{j\in\mathbb{Z}} b_j E_{j,j-n} Iv \\ &= J^k(n)Iv \end{aligned}$$

■

Here we will denote the appropriate Whittaker vector by $\mathbf{w}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$.

Lemma 5.2.2. (1) For every $n\in\mathbb{Z}_{\geq 1}$, $I^n\mathbf{w}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a non-trivial Whittaker vector in $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$.

(2) For any $S\subset\mathbb{C}[I]\mathbf{w}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$, let $\langle S\rangle$ be the submodule generated by all of the Whittaker vectors in S. Then the module

$$\left\langle (I-d)\ \mathbb{C}[I]\ \mathbf{w}_{\boldsymbol{\lambda},\boldsymbol{\mu}} \right\rangle$$

is a proper submodule of $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$, for each $d\in\mathbb{C}$.

Finally, we have proved the main result of this chapter.

Theorem 5.2.3.

(1) $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ is a reducible $\widehat{\mathfrak{g}l}$–module.

(2) $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ is a reducible $\mathscr{W}_{1+\infty}$–module at central charge $c=-1$.

Proof. Let us assume to the contrary, i.e. $M_1(\lambda,\mu)$ is irreducible. Schur's lemma implies that the Casimir element I has to act as a constant, $I \equiv \alpha$, $\alpha \in \mathbb{C}$. However, we have seen that $I\mathbf{w}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a non-trivial vector in $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$, so we have a contradiction. ■

Remark 5.2.4. Note that the statement (2) of the previous theorem shows that $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ is a reducible M^0–module. Thus, the main theorem of [12] can not be extended for infinite cyclic groups.

Let $L_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ be an irreducible quotient of $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$. Here we list some of the very important problems which we will address in our future work:

(A) Prove $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ is a cyclic $\widehat{\mathfrak{gl}}$–module.

(B) Realize $L_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ by free-fields.

(C) Determine the complete set of Whittaker vectors in $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ which generate the maximal submodule of $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$.

Conclusion

In this thesis we have dealt with two important problems in the vertex algebra theory: the problem of determining fusion rules in some category of modules for a vertex algebra and the orbifold problem.

For the fusion rules problem, we have shown that it is possible to prove Verlinde formula with vertex-algebraic methods. We anticipate that our methods can be applied to the affine vertex operator algebras and the singlet vertex operator algebras. We also believe there is a vertex tensor category for the relaxed modules category.

For the orbifold problem, we have proven a result generalizing Dong-Mason theorem from quantum Galois theory. Furthermore, we have presented a counterexample demonstrating this theorem can not be extended to the case of infinte-dimensional group of automorphisms. However, we believe we can extend our theorem to the case of arbitrary commutative finite-dimensional group of automorphisms and to the case of twisted modules.

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Curriculum Vitae

Veronika Pedić Tomić was born on October 6, 1993 in Split, Croatia. She received her primary and secondary education in Zagreb, finishing both mathematical "V. gymnasium" and a music high school in Zagreb. During this time she competed in the Croatian state finals in five different disciplines, including in mathematics (winning 2^{nd} prize) and physics (winning 2^{nd} prize).

In 2017, she graduated *summa cum laude* at the Department of Mathematics, Faculty of Science, University of Zagreb under the supervision of prof. Vedran Krčadinac. During her studies, she received Rector's award for joint work *Rational functions on curves and application to the field of complex numbers* under the supervision of prof. Goran Muić. She also received Department's awards for best students of Bachelor's and of Master's Degree.

In the year 2017/18, she enrolled in the doctoral program at the same Department under the supervision of prof. Dražen Adamović. There she was employed as a research and teaching assistant, within the project of Scientific Center of Excellence for Quantum and Complex Systems, and Representations of Lie Algebras (QuantiXLie), as a member of prof. Adamović work group.

During her doctoral studies, she attended a number of conferences and workshops where she gave numerous talks, including *Vertex Operator Algebras and Related Topics*, NYC College of Technology, USA, *Representation Theory and Integrable Systems*, ETH, Switzerland, *Representation Theory XVI*, IUC Dubrovnik, Croatia, *The Mathematical Foundations of CFT and Related Topics*, Nankai University, China and *Geometric and automorphic aspects of W-algebras*, Universite de Lille, France.

Together with prof. Adamović, she published two impactful papers, one of which was a result of their international collaboration with prof. Ching Hung Lam and prof. Nina

Yu, established during their stay in Kyoto in 2018:

- D. Adamović, C.H. Lam, V. Pedić, N. Yu, *On irreducibility of modules of Whittaker type for cyclic orbifold vertex algebra*, Journal of Algebra 539 (2019) 1-23.
- D. Adamović, V. Pedić, *On fusion rules and intertwining operators for the Weyl vertex algebra*, Journal of Mathematical Physics 60 (2019), no. 8, 081701, 18 pp.