Application of H-measures to evolution equations

Grbac, Matko

University of Zagreb

FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS

Matko Grbac

Application of H-measures to evolution equations

DOCTORAL DISSERTATION

Zagreb, 2024.

University of Zagreb

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS

Matko Grbac

Application of H-measures to evolution equations

DOCTORAL DISSERTATION

Supervisor:

Nenad Antonić

Zagreb, 2024.

Sveučilište u Zagrebu

PRIRODOSLOVNO–MATEMATIČKI FAKULTET

MATEMATIČKI ODSJEK

Matko Grbac

Primjena H-mjera na evolucijske jednadžbe

DOKTORSKI RAD

Mentor:

Nenad Antonić

Zagreb, 2024.

ACKNOWLEDGEMENTS

First and foremost, I am thankful to my advisor, Nenad Antonić, for giving me opportunity to engage in this research and his guidance and encouragement, which were instrumental in shaping this thesis.

I am also thankful to the committee members — Marko Erceg, Ivan Ivec and Marko Vrdoljak — for reviewing the thesis and providing valuable advice throughout its completion.

Furthermore, I extend my gratitude to my colleagues and friends who have enriched my academic journey, with special thanks to my closest friend, Aleksandar Bulj.

Finally, my deepest gratitude goes to my parents, Dragica and Željko, for their unconditional support and belief in me throughout my whole life.

SUMMARY

In the thesis evolution equations of second order in t are studied. Some standard results regarding the existence, uniqueness, and well-posedness of the semilinear wave equation are revisited in the first part and generalised to the case where the coefficients have only bounded variation with respect to time.

The second part begins with an introduction to the theory of pseudodifferential operators and microlocal analysis tools, with an emphasis on the notion of H-measures and their main properties. The theory is then applied to both linear and semilinear wave equations. The results in this part generalize the work of Francfort and Murat, allowing the coefficients of the equation to depend on the time variable as well.

Keywords: H-measures, pseudodifferential operators, semilinear wave equation

Sažetak

Disertacija se bavi proučavanjem evolucijskih jednadžbi drugog reda u t. U prvom dijelu disertacije revidiraju se i generaliziraju standardni rezultati o postojanju i jedinstvenosti rješenja, te dobroj postavljenosti polulinearne valne jednadžbe na slučaj kada koeficijenti imaju samo ograničenu varijaciju s obzirom na vremensku varijablu.

Drugi dio disertacije započinje uvodom u teoriju pseudodiferencijalnih operatora i alata mikrolokalne analize, s naglaskom na pojam H-mjera i njihovih glavnih svojstava. Zatim se teorija primjenjuje na slučajeve linearne i polulinearne valne jednadžbe. Rezultati u ovom dijelu generaliziraju rad Francforta i Murata, omogućujući da koeficijenti jednadžbe također ovise o vremenskoj varijabli.

Ključne riječi: H-mjere, pseudodiferencijalni operatori, polulinearna valna jednadžba

Contents

Introduction

The thesis is divided into two parts, connected by the common theme of studying evolution equations. First part is contained in Chapter 1, and it is devoted to the study of semilinear wave equation, which in its simplest form is given by

$$u'' - \Delta u + F(u) = 0, \tag{0.1}$$

where F represents the nonlinear part of the equation. Most common cases include nonlinearities of the form $F(u) = u|u|^p$ for some (integer) exponent p. Existence, smoothness and even uniqueness of solutions has in early work been known only under severe growth restrictions on F and its derivatives, for instance, if F is Lipschitz continuous (v. [67] and references therein). Starting as early as in 1970., Strauss showed the existence of the weak solution of (0.1) coupled with initial conditions $u(0) = u^0 \in \mathrm{H}^1$ and $u'(0) = u^1 \in \mathrm{L}^2$. The nonlinear part F was assumed to be a continuous function satisfying the *sign condition*, meaning that $F(u)$ has the same sign as u, alongside the integrability of $G(u^0)$, where G is the primitive function of F satisfying $G(0) = 0$. These assumptions will be the basis for our further work.

On the other hand, Casado-Díaz et al. dealt in their work on homogenisation for the wave equation [25] with *linear* wave equation of the more general form

$$(\rho u')' - \operatorname{div}(\mathbf{A}\nabla u) = f, \tag{0.2}$$

but instead of classical assumptions that required coefficients to be Lipschitz-continuous in time, they considered a more general case involving coefficents which are functions of *bounded variation* in time.

For the wave equation one assumes that $\mathbf{A}$ takes values in the space of positively definite symmetric matrices. However, there are physically relevant situations (e.g. related to the Hall effect [24]), as well as some man-made materials that have recently been constructed [55, 56], where the corresponding coefficients are not symmetric any more. This prompted our study of more general wave-like evolution equations.

The main result of the first chapter consists in combining these earlier approaches in order to obtain the existence and uniqueness results for the initial value problem for the equation

$$(\rho u')' - \operatorname{div}(\mathbf{A}\nabla u) + F(u) = 0. \tag{0.3}$$

The second part of the thesis begins with Chapter 2, an introduction to the theory of H-measures, which was independently introduced by Tartar in [71] and Gérard in [36]. The difference in their approach lies mainly in the regularity assumptions of the framework. For this matter, appropriate classes of symbols and associated pseudodifferential operators are being introduced beforehand.

Since the main application of H-measures was in connection to sequences of solutions of certain partial differential equations motivated by continuum physics, Tartar wanted to minimise the regularity hypotheses when introducing the pseudodifferential calculus, in order to better capture the real-life needs for coefficients which do not need to be smooth. However, even though this appears as an obvious advantage to the approach, the main setback lies in the fact that the mentioned pseudodifferential calculus had to be reinvented for his cause, since the majority of the known results at the time were done for operators associated with smooth symbols. This is precisely the framework Gérard uses in his introduction of H-measures, which allows for definition of H-measures on more abstract spaces. However, as it has already been mentioned, this restricts the PDE framework only to those having smooth coefficients.

After the overview of these two approaches, we give some final remarks and considerations about pseudodifferential operators with symbols that can be considered as spatial pseudodifferential operators with time as parameter. This will serve as an addition to already known pseudodifferential calculus in order to carry out the proof of the main result of Chapter 3.

The third, and final chapter, is devoted to the study of transport properties of H-measures associated to the wave equation. More precisely, we are considering the linear wave equation, with variable time-dependent coefficients, coupled with an oscillating sequence of initial data $(u_n(0), u_n'(0)) = (g_n, h_n)$, weakly converging in $\mathrm{H}^1(\mathbb{R}^d) \times \mathrm{L}^2(\mathbb{R}^d)$. Associated with the sequence of gradients of solutions is an H-measure μ, which, due to the fact that it arises from a gradient, has a specific form that allows its study to be reduced to a certain scalar Radon measure ν. The main part then consists in answering the following three questions:

(a) can we obtain the transport equation for the measure ν?

(b) can the trace of the measure at time $t = 0$ be defined in some sense?

(c) can the trace of ν at $t = 0$ be computed only through knowledge of initial data g_n and h_n?

The (positive) answer to all of these question was already given by Francfort and Murat in [34] for the linear wave equation with smooth coefficients depending *only on* x variable. Their approach has its basis in the work done by Tartar in regard to the first order equation in [71]. However, a step-up from first to second order equation would, as it seems at this point, require some additional reinvention of the pseudodifferential calculus for non-smooth symbols. For that reason, we will only take smooth coefficients into consideration, and mostly follow Gérard's approach.

We then generalise the aforementioned result of Francfort and Murat to the case of the linear wave equation with coefficients that depend on both t and x. Finally, we draw some comparision with the semilinear wave equation studied in Chapter 1. We are mainly motivated by the work of Gérard in [37], which studied the three-dimensional semilinear wave equation (with constant coefficients and) with smooth nonlinearity satisfying the growth conditions $|F^{(j)}(u)| \leq C(1+|u|)^{p-j}$ for $1 < p \leq 5$ and $j \in \mathbb{N}_0$ (i.e. up to the critical exponent). Since the study of strong solutions and their behaviour is not of our focus here, we consider the corresponding case in general dimension d where $p \leq \frac{d}{d-2}$, and observe whether these types of nonlinearities contribute to the energy densities in the case with variable coefficients.

1. SEMILINEAR WAVE-LIKE EQUATION

1.1. PRELIMINARIES

1.1.1. Function spaces

First and foremost, let us state that the reader is expected to be familiar with all the standard spaces that will appear not only in this chapter but throughout the entire thesis, as well as with the basics of theory of distributions (see [65], [23], [84]). Below, we briefly introduce the notation used for the spaces of interest:

- Space of all k-times (strongly) differentiable functions on $\Omega \subseteq \mathbb{R}^d$ denoted by $C^k(\Omega)$, $k \in \mathbb{N}_0 \cup \{\infty\}$.

- Space of *test functions* denoted by $C_c^\infty(\Omega)$ (or sometimes $\mathscr{D}(\Omega)$ as well), consisting of all compactly supported functions in $C^\infty(\Omega)$.

- *Lebesgue spaces* $\mathrm{L}^p(\Omega)$ with $\Omega \subseteq \mathbb{R}^d$ (underlying measure is understood to be standard Lebesgue measure) equipped with respective norms

$$\|u\|_{\mathrm{L}^p(\Omega)} = \begin{cases} \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}, & 1 \le p < \infty \\ \operatorname*{ess\,sup}_{x\in\Omega} u(x), & p = \infty. \end{cases}$$

- *Sobolev spaces* for *nonnegative integer k* defined as spaces of L^p functions whose weak derivatives of order less than or equal k belong to L^p and denoted by $\mathrm{W}^{k,p}(\Omega)$, equipped with respective norms

$$\|u\|_{\mathrm{W}^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\le k} \|\partial^\alpha u\|_{\mathrm{L}^p(\Omega)}\right)^{\frac{1}{p}}, & 1 \le p < \infty \\ \max_{|\alpha|\le k} \operatorname*{ess\,sup}_{x\in\Omega} |\partial^\alpha(x)|, & p = \infty. \end{cases}$$

In particular, we write $\mathrm{H}^k(\Omega)$ instead of $\mathrm{W}^{k,2}(\Omega)$.

- Spaces $\mathrm{W}_0^{k,p}(\Omega)$ defined as the closure of the space $C_c^\infty(\Omega)$ in $\mathrm{W}^{k,p}(\Omega)$.

- Spaces $\mathrm{W}^{-k,p'}(\Omega)$ defined as the duals of the corresponding $\mathrm{W}_0^{k,p}(\Omega)$ space, where p' denotes the conjugate exponent of p.

- *Bochner spaces* $\mathrm{L}^p(I;X)$, for an interval $I \subseteq \mathbb{R}$ and a Banach space X, defined as the set of all Bochner measurable functions $u : I \to X$ such that the corresponding norms are finite:

$$\|u\|_{\mathrm{L}^p(I;X)} = \begin{cases} \left(\int_I \|u(t)\|_X^p dt\right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \|u\|_{\mathrm{L}^\infty(I;X)} = \operatorname*{ess\,sup}_{t \in I} \|u(t)\|_X, & p = \infty. \end{cases}$$

- *Bochner-Sobolev spaces* $\mathrm{W}^{k,p}(I;X)$ defined analogously as the set of all Bochner measurable functions whose derivatives of order less than or equal to k belong to $\mathrm{L}^p(I;X)$, and have finite norms:

$$\|u\|_{\mathrm{W}^{k,p}(I;X)} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{\mathrm{L}^p(I;X)}.$$

  The most common case will be when $I = (0,T)$ for some $T > 0$, in which case we write $\mathrm{W}^{k,p}(0,T;X)$ instead of $\mathrm{W}^{k,p}((0,T);X)$.

One can find detailed discussion on properties of the Sobolev spaces in [32, Chapter 5] or [23]. It is also worth noting that the Bochner (and Bochner–Sobolev) spaces, which can be viewed as the vector-valued versions of the corresponding Lebesgue and Sobolev spaces, share many similarities with them, but there are still some important differences one must take into account. Although there are plenty of sources one may look upon in order to go into further details ([22], [23], [32] among others), we will refer the reader to [50] for a very detailed and self-sufficient overview of this topic.

1.1.2. Preparatory remarks

We now proceed to introduce the necessary ingredients in order to state the main problems and results of this chapter.

Let Ω be a bounded open subset of $\mathbb{R}^d$ with a smooth boundary. Denote $H = \mathrm{L}^2(\Omega)$ and $V = \mathrm{H}^1_0(\Omega)$, with $V \hookrightarrow H$ being continous dense injection which is compact due to the Rellich-Kondrachov theorem [2, Theorem 6.3]. The dual of V is given by $V' = \mathrm{H}^{-1}(\Omega)$. By further identifying H with its dual H' we get

$$V \hookrightarrow H = H' \hookrightarrow V',$$

where the second inclusion is also a continuous dense injection. The triple (V, H, V') is often called the *Gelfand triple* [86, Chapter 17.1].

The scalar product in H will be denoted by $(\cdot,\cdot)_H = (\cdot,\cdot)$ and the dual pairing between V and V' by ${}_{V'}\langle\cdot,\cdot\rangle_V = \langle\cdot,\cdot\rangle$. All the spaces in this chapter are considered to be real.

Let $T > 0$ be fixed. We now introduce families of operators $\mathbf{R}(t)$, $\mathbf{A}_0(t)$ and $\mathbf{A}_1(t)$ for $t \in [0,T]$, satisfying properties that follow. For $\mathbf{R}$ we assume

$$\mathbf{R} \in \mathrm{W}^{2,1}(0,T;\mathscr{L}(H)) \tag{1.1}$$

$$(\mathbf{R}(t)u, v) = (\mathbf{R}(t)v, u), \qquad u,v \in H, t \in [0,T] \tag{1.2}$$

$$(\mathbf{R}(t)u, u) \geq \alpha \|u\|^2_{\mathrm{H}}, \qquad u \in H, t \in [0,T], \tag{1.3}$$

while for $\mathbf{A}_0$ we assume

$$\mathbf{A}_0 \in \mathrm{W}^{2,1}(0,T;\mathscr{L}(V,V')) \tag{1.4}$$

$$\langle \mathbf{A}_0(t)u, v\rangle = \langle \mathbf{A}_0(t)v, u\rangle, \qquad u,v \in V, t \in [0,T] \tag{1.5}$$

$$\langle \mathbf{A}_0(t)u, u\rangle \geq \alpha \|u\|^2_V, \qquad u \in V, t \in [0,T]. \tag{1.6}$$

Finally, for $\mathbf{A}_1$ we assume

$$\mathbf{A}_1 \in \mathrm{W}^{1,1}(0,T;\mathscr{L}(V,H)). \tag{1.7}$$

Remark 1.1.1. (a) Given the continuous inclusion $\mathrm{W}^{1,1}(0,T;X) \hookrightarrow C([0,T];X)$ is valid for each Banach space X, (1.1) and (1.4) allow us to deduce that $\mathbf{R}$ belongs to the space $C^1([0,T];\mathscr{L}(H))$, $\mathbf{A}_0$ belongs to the space $C^1([0,T];\mathscr{L}(V,V'))$ and we deduce the existence of constant $\beta > 0$ such that

$$\|\mathbf{R}\|_{\mathrm{W}^{1,\infty}(0,T;\mathscr{L}(H))} \leq \beta \tag{1.8}$$

$$\|\mathbf{A}_0\|_{\mathrm{W}^{1,\infty}(0,T;\mathscr{L}(V,V'))} \leq \beta\ . \tag{1.9}$$

(b) When there is enough regularity in functions $u \in H$ and $v \in V$, standard formulae for the (weak) derivative of products $\mathbf{R}u$ and $\mathbf{A}_0 v$ hold, where we denote $(\mathbf{R}u)(t) = \mathbf{R}(t)u(t)$ (analogously for $\mathbf{A}_0$).

Namely, if u belongs additionally to $\mathrm{W}^{1,1}(0,T;H)$, (a) implies $\mathbf{R}'u$ and $\mathbf{R}u' \in \mathrm{L}^1(0,T;H)$ and consequently we have $\mathbf{R}u \in \mathrm{W}^{1,1}(0,T;H)$ with

$$(\mathbf{R}u)' = \mathbf{R}'u + \mathbf{R}u'.$$

while for $v \in \mathrm{W}^{1,1}(0,T;V)$ we have $\mathbf{A}_0 v \in \mathrm{W}^{1,1}(0,T;V')$ with

$$(\mathbf{A}_0 v)' = \mathbf{A}_0' v + \mathbf{A}_0 v'.$$

Of course, higher order derivatives are obtained inductively when there is more regularity in both of the factors.

(c) Symmetric operators $\mathbf{R}(t)$, $t \in [0,T]$ satisfy

$$\alpha \mathbf{I} \le \mathbf{R}(t) \le \beta \mathbf{I},$$

and as such are invertible symmetric operators on H, with symmetric inverse $\mathbf{R}(t)^{-1}$. For each $t \in [0,T]$ one has for the inverses

$$\beta^{-1}\mathbf{I} \le \mathbf{R}(t)^{-1} \le \alpha^{-1}\mathbf{I}, \tag{1.10}$$

hence they are bounded. Observing the operator identity valid in $\mathscr{L}(H)$

$$\mathbf{R}(t+h)^{-1} - \mathbf{R}(t)^{-1} = \mathbf{R}(t+h)^{-1}\left[\mathbf{R}(t) - \mathbf{R}(t+h)\right]\mathbf{R}(t)^{-1},$$

we first note that the first and the last term are bounded in norm by α^{-1}, whereas the middle term tends to 0 as $h \to 0$. Therefore, after denoting $\mathbf{R}^{-1}$ the map $t \mapsto \mathbf{R}(t)^{-1}$, we see that $\mathbf{R}^{-1}$ is continuous. Additionally, from the very same operator identity we obtain after dividing by h and letting $h \to 0$

$$\frac{d}{dt}\left(\mathbf{R}(t)^{-1}\right) = -\mathbf{R}^{-1}(t)\mathbf{R}'(t)\mathbf{R}^{-1}(t).$$

Hence, we deduce that $\mathbf{R}^{-1} \in C^1([0,T];\mathscr{L}(H))$, where we now denote by $\mathbf{R}^{-1}$ the map $t \mapsto \mathbf{R}(t)^{-1}$.

(d) If we denote $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1 \in \mathrm{W}^{1,1}(0,T;\mathscr{L}(V,V'))$, we can think of $\mathbf{A}_0$ as a symmetric and coercive part of an operator $\mathbf{A}$, while $\mathbf{A}_1$ represents a possible non-symmetric part.

(e) Finally, let us mention a slight change of notation which will be used throught the first chapter. For brevity, we will occasionally ommit $0,T$ when writing corresponding Bochner spaces and respective components, and write $\mathrm{W}^{k,p}(X)$ instead of $\mathrm{W}^{k,p}(0,T;X)$.

1.2. Lipschitz nonlinearity with sign condition

We begin by studying the initial-boundary value problem in $(0,T)\times\Omega$

$$\begin{cases} (\mathbf{R}u')' + (\mathbf{A}_0 + \mathbf{A}_1)u + \mathbf{F}(u) = f + g \\ u(0) = u^0 \\ u'(0) = u^1. \end{cases} \tag{1.11}$$

Here, the we are given $u^0 \in V$, $u^1 \in H$, $f \in \mathrm{W}^{1,1}(0,T;H)$ and $g \in \mathrm{W}^{2,1}(0,T;V')$, while the non-linear part of the equation arises from the function $F : \mathbb{R} \to \mathbb{R}$, and is given by $\mathbf{F}(u)(t,x) = F(u(t,x))$. We will also sometimes use $\mathbf{F}(u(t))$ or $\mathbf{F}(u)(t)$ to denote the function $x \mapsto F(u(t,x))$ for fixed t.

We will now procede with stating our assumptions on the functions involved in a problem. Higher regularity in the coefficients is assumed here in order to obtain the solution of higher regularity, one which will be used in order to find approximate solutions of the problems considered in the following sections. However, we do first start with usual, more restrictive assumption on F. Assume that F is Lipschitz continuous with constant denoted by $\mathrm{Lip}(F)$, that is

$$|F(z) - F(w)| \leq \mathrm{Lip}(F)|z-w|, \quad z,w \in \mathbb{R}. \tag{1.12}$$

Consider its primitive function

$$G(z) = \int_0^z F(w)dw,$$

so that $G' = F$ and $G(0) = 0$. Assume further that F satisfies the *sign condition*

$$zF(z) \geq 0, \qquad z \in \mathbb{R}. \tag{1.13}$$

Note that the sign condition (1.13) also gives

$$G(z) \geq 0, \qquad z \in \mathbb{R}, \tag{1.14}$$

and together with the continuity of F we have as well that $F(0) = 0$. Hence, from (1.12) we have the bound

$$|F(z)| \leq \mathrm{Lip}(F)|z|, \qquad z \in \mathbb{R},$$

which in turn implies

$$\|\mathbf{F}(u)\|_H \le \mathrm{Lip}(F)\|u\|_H, \qquad u \in H, \tag{1.15}$$

as well as

$$G(z) \le \int_0^{|z|} |F(w)|dw \le \frac{\mathrm{Lip}(F)}{2}|z|^2, \qquad z \in \mathbb{R}. \tag{1.16}$$

Now we are ready to prove the existence result given by the next theorem.

Theorem 1.2.1. Consider $\mathbf{R}$ satisfying (1.1)–(1.3), $\mathbf{A}_0$ satisfying (1.4)–(1.6), $\mathbf{A}_1$ satisfying (1.7), F satisfying (1.12) and (1.13). Let $u^0, u^1 \in V$, $f \in \mathrm{W}^{1,1}(0,T;H)$ and $g \in \mathrm{W}^{2,1}(0,T;V')$, with $\mathbf{A}_0(0)u^0 - g(0) \in H$. Then there exists a solution $u \in \mathrm{W}^{1,\infty}(0,T;V) \cap \mathrm{W}^{2,\infty}(0,T;H)$ with $\mathbf{G}(u) \in \mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega))$ satisfying

$$(\mathbf{R}u')' + (\mathbf{A}_0 + \mathbf{A}_1)u + \mathbf{F}(u) = f + g \qquad \text{in } \mathrm{L}^2(0,T;V') \tag{1.17}$$

with initial conditions

$$u(0) = u^0, \qquad u'(0) = u^1.$$

Remark 1.2.2. (a) Note that the initial condition $u(0) = u^0$ makes sense for such a solution u; the Aubin-Lions compactness lemma [79, Lecture 24] gives us $u \in C([0,T];V)$.

(b) Since $u' \in \mathrm{W}^{1,\infty}(0,T;H)$, it follows in the same manner that $u' \in C([0,T];H)$. Hence the initial condition $u'(0) = u^1$ makes sense at least in the sense of H.

1.2.1. Galërkin approximations

In order to prove the existence of such a solution, we employ the standard Galërkin method, with slightly changed orthogonal basis. We form orthonormal basis (w_k) for V in the following way: in case $u^0, u^1 \neq 0$ first let

$$\tilde{w}_1 = \frac{u^0}{\|u^0\|_V}, \qquad \tilde{w}_2 = \frac{u^1}{\|u^1\|_V}, \qquad \tilde{w}_{2+k} = \frac{b_k}{\|b_k\|_V},$$

where (b_k) is an orthonormal basis for H (and therefore orthogonal for V) consisting of, for example, eigenvectors of $-\Delta$ (which are additionally of class $C^\infty(\overline{\Omega})$ due to the elliptic regularity), and then apply Gram-Schmidt's orthonormalization process to obtain the desired basis. If u^1 is already proportional to u^0, skip $\tilde{w}_2$, and in case $u^0 = 0$ skip $\tilde{w}_1$ (obviously skip both in case both $u^0 = u^1 = 0$).

For any $m \in \mathbb{N}$, $m \geq 2$, denote $V_m = \operatorname{span}\{w_1, \ldots, w_m\}$. Note that $u^0, u^1 \in V_m$, $m \geq 2$. We first show that there exists $u_m \in \mathrm{W}^{3,1}(0,T;V_m)$ which solves the projected problem

$$\begin{cases} ((\mathbf{R}(t)u_m'(t))', w_j) + \langle \mathbf{A}_0(t)u_m(t), w_j\rangle + \\ \qquad (\mathbf{A}_1(t)u_m(t), w_j) + (\mathbf{F}(u_m(t)), w_j) = (f(t), w_j) + \langle g(t), w_j\rangle, \quad 1 \leq j \leq m \\ \qquad u_m(0) = u^0 \\ \qquad u_m'(0) = u^1. \end{cases} \tag{1.18}$$

We seek a solution u_m of the projected problem in the form

$$u_m(t) = \sum_{j=1}^{m} d_j(t) w_j. \tag{1.19}$$

This leads to the following system of ODEs

$$\frac{d}{dt}[\mathbf{C}(t)\mathbf{d}'(t)] + \mathbf{M}(t)\mathbf{d}(t) + \mathbf{H}(\mathbf{d}(t)) = \mathbf{v}(t), \tag{1.20}$$

where

$$\begin{aligned} \mathbf{d}(t)_j &= d_j(t) \\ \mathbf{C}(t)_{ij} &= (\mathbf{R}(t)w_j, w_i) \\ \mathbf{M}(t)_{ij} &= \langle \mathbf{A}_0(t)w_j, w_i\rangle + (\mathbf{A}_1(t)w_j, w_i) \\ \mathbf{H}_j(a_1, \ldots, a_m) &= \left(\mathbf{F}\left(\sum_{i=1}^{m} a_i w_i\right)), w_j\right) \\ \mathbf{v}(t)_j &= (f(t), w_j) + \langle g(t), w_j\rangle. \end{aligned}$$

Note that $\mathbf{C}(t)$ is the the matrix representation of $\mathbf{R}(t)$ in the basis $(w_k)_{k=1}^m$. Hence, it is invertible for each $t \in [0,T]$. Moreover, because of (1.1) we have that $t \mapsto \mathbf{C}(t)$ is in $\mathrm{W}^{2,1}(0,T;\mathbb{R}^{m\times m})$, and since $\mathbf{C}(t)^{-1}$ is just a matrix representation of $\mathbf{R}(t)^{-1}$, we deduce the same for $\mathbf{C}(t)^{-1}$. Next, we easily see that $\mathbf{M} \in \mathrm{W}^{1,1}(0,T;\mathbb{R}^{m\times m})$, and similarly that $\mathbf{v} \in \mathrm{W}^{1,1}(0,T;\mathbb{R}^m)$. Finally, since F is Lipschitz, we have after denoting $\mathbf{a} = (a_1, \ldots, a_m)$,

$\mathbf{b} = (b_1,\dots,b_m) \in \mathbb{R}^m$, $\mathbf{w} = (w_1,\dots,w_m) \in \mathrm{H}^1_0(\Omega;\mathbb{R}^m)$ and using the fact that the components of $\mathbf{w}$ are unit vectors in V, and therefore $\|w_j\|_H \le 1$, we obtain

$$|\mathbf{H}(\mathbf{a}-\mathbf{b})| = |(\mathbf{F}((\mathbf{a}-\mathbf{b})\cdot\mathbf{w}),\mathbf{w})| \le \|\mathbf{F}((\mathbf{a}-\mathbf{b})\cdot\mathbf{w})\|_H \|\mathbf{w}\|_H \le \mathrm{Lip}(F)|\mathbf{a}-\mathbf{b}|_\infty.$$

Therefore, **H** is also a Lipschitz function.

By introducing a new variable $\mathbf{e} = \mathbf{d}'$, (1.20) can be rewritten as an equivalent first order system of ODEs

$$\begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{C} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{d} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M} & -\mathbf{C}' \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{v} \end{bmatrix} - \begin{bmatrix} 0 \\ \mathbf{H}(\mathbf{d}) \end{bmatrix},$$

which, due to the previous remark about invertibility of **C** can further be written as

$$\frac{d}{dt} \begin{bmatrix} \mathbf{d} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{C}^{-1} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M} & -\mathbf{C}' \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{v} \end{bmatrix} - \begin{bmatrix} 0 \\ \mathbf{H}(\mathbf{d}) \end{bmatrix}, \tag{1.21}$$

subject to initial conditions

$$\begin{aligned} \mathbf{d}(0)_j &= (u^0, w_j)_V \\ \mathbf{e}(0)_j &= (u^1, w_j)_V. \end{aligned}$$

Since the right-hand side of the system (1.21) is in $\mathrm{L}^2(0,T;\mathbb{R}^m)$, and Lipschitz continuous in $(\mathbf{d},\mathbf{e})$ (it consists of a linear part and a Lipschitz nonlinear part), Carathéodory's existence theorem hence yields a unique global absolutely continuous solution $\mathbf{d},\mathbf{e} \in \mathrm{W}^{1,1}(0,T;\mathbb{R}^m)$. Since we also have $\mathbf{d}' = \mathbf{e}$, it follows that $\mathbf{d} \in \mathrm{W}^{2,1}(0,T;\mathbb{R}^m)$.

From the second equation of the system (1.21)

$$\mathbf{e}' = -\mathbf{C}^{-1}\mathbf{M}\mathbf{d} - \mathbf{C}^{-1}\mathbf{C}'\mathbf{e} + \mathbf{v} - \mathbf{H}(\mathbf{d})$$

we also deduce that $e' \in \mathrm{W}^{1,1}(0,T;\mathbb{R}^m)$. Since the product of two $\mathrm{W}^{1,1}(0,T)$ elements is once again $\mathrm{W}^{1,1}(0,T)$, we have that the first two factors, consisting of product of three matrices in $\mathrm{W}^{1,1}$, belong to $\mathrm{W}^{1,1}(0,T;\mathbb{R}^m)$ due to remarks that precede the introduction of the system (1.21). For the last term, we have used the fact that **H** is Lipschitz continuous (and hence differentiable almost everywhere) so that the composition $\mathbf{H}(\mathbf{d})$ satisfies $(\mathbf{H}(\mathbf{d}))' = \mathbf{H}'(\mathbf{d})\mathbf{d}' \in \mathrm{L}^1(0,T;\mathbb{R}^m)$. Hence, $\mathbf{e} \in \mathrm{W}^{2,1}(0,T;\mathbb{R}^m)$.

We may now return to the first equation in (1.21) given by

$$\mathbf{d}' = \mathbf{e},$$

that yields $\mathbf{d}' \in \mathrm{W}^{2,1}(0,T;\mathbb{R}^m)$, and subsequently, $\mathbf{d} \in \mathrm{W}^{3,1}(0,T;\mathbb{R}^m)$.

Finally, this implies that u_m, being defined in (1.19), is in $\mathrm{W}^{3,1}(0,T;V_m)$ and indeed a solution of (1.18).

As a consequence, we also obtain

$$\mathbf{R}u_m' \in \mathrm{W}^{2,1}(0,T;H), \quad \mathbf{A}_0 u_m \in \mathrm{W}^{2,1}(0,T;V') \quad \text{and} \quad \mathbf{A}_1 u_m \in \mathrm{W}^{1,1}(0,T;H)\,. \tag{1.22}$$

1.2.2. A priori estimates

We now proceed with obtaining some a priori estimates of the sequence of solutions (u_m). Multiplying $(1.18)_1$ by $d_j'(t)$ and summing in j we obtain

$$((\mathbf{R}u_m')', u_m') + \langle \mathbf{A}_0 u_m, u_m' \rangle + (\mathbf{A}_1 u_m, u_m') + (\mathbf{F}(u_m), u_m') = (f, u_m') + \langle g, u_m' \rangle. \tag{1.23}$$

We make use of the following identities, which are valid because of the smoothness assumptions on $\mathbf{R}, \mathbf{A}_0$, as well as (1.22):

$$\begin{aligned}
((\mathbf{R}u_m')', u_m') &= \frac{1}{2}\Big(\frac{d}{dt}(\mathbf{R}u_m', u_m') + (\mathbf{R}'u_m', u_m')\Big) \\
\langle \mathbf{A}_0 u_m, u_m' \rangle &= \frac{1}{2}\Big(\frac{d}{dt}\langle \mathbf{A}_0 u_m, u_m \rangle - \langle \mathbf{A}_0' u_m, u_m \rangle\Big) \\
(\mathbf{F}(u_m), u_m') &= \frac{d}{dt}\|\mathbf{G}(u_m)\|_{\mathrm{L}^1(\Omega)}\,.
\end{aligned}$$

Note that we have used $\mathbf{G} \geq 0$ in the last equality, so the L^1 norm of $\mathbf{G}$ is in fact equal to the integral of $\mathbf{G}$ without the absolute sign. After denoting the *energy* at time t by

$$E_m(t) := \frac{1}{2}(\mathbf{R}(t)u_m'(t), u_m'(t)) + \frac{1}{2}\langle \mathbf{A}_0(t)u_m(t), u_m(t) \rangle + \|\mathbf{G}(u_m(t))\|_{\mathrm{L}^1(\Omega)}, \tag{1.24}$$

we rewrite equation (1.23) in the form

$$E_m'(t) = -\frac{1}{2}(\mathbf{R}'u_m', u_m') + \frac{1}{2}\langle \mathbf{A}_0' u_m, u_m \rangle - (\mathbf{A}_1 u_m, u_m') + (f, u_m') + \langle g, u_m' \rangle. \tag{1.25}$$

We now proceed with obtaining the estimates of the terms on the right hand side of (1.25). Beforehand, let us emphasise the following relation between the energy $E_m(t)$ and the norms $\|u_m(t)\|_V$ and $\|u_m'(t)\|_H$: due to coercivity properties (1.3) and (1.6) of $\mathbf{R}$ and $\mathbf{A}_0$ respectively, we have

$$E_m(t) \geq \frac{\alpha}{2}\left(\|u_m'(t)\|_{\mathrm{H}}^2 + \|u_m(t)\|_V^2\right). \tag{1.26}$$

Using this inequality, we obtain

$$\begin{aligned}
(\mathbf{R}'(t)u_m'(t), u_m'(t)) &\leq \|\mathbf{R}'(t)\|_{\mathscr{L}(H)}\|u_m'(t)\|_{\mathrm{H}}^2 \leq \frac{\|\mathbf{R}'(t)\|_{\mathscr{L}(H)}}{\alpha/2}E_m(t)\\
\langle \mathbf{A}_0' u_m(t), u_m(t)\rangle &\leq \|\mathbf{A}_0'(t)\|_{\mathscr{L}(V,V')}\|u_m(t)\|_V^2 \leq \frac{\|\mathbf{A}_0'(t)\|_{\mathscr{L}(V,V')}}{\alpha/2}E_m(t)\\
(\mathbf{A}_1(t)u_m, u_m') &\leq \|\mathbf{A}_1(t)\|_{\mathscr{L}(V,H)}(\|u_m'(t)\|_{\mathrm{H}}^2 + \|u_m(t)\|_V^2)\\
&\leq \frac{\|\mathbf{A}_1(t)\|_{\mathscr{L}(V,H)}}{\alpha/2}E_m(t)\\
(f(t), u_m'(t)) &\leq \|f(t)\|_H\|u_m'(t)\|_H \leq \|f(t)\|_H(1 + \|u_m'(t)\|_{\mathrm{H}}^2)\\
&\leq \|f(t)\|_H + \frac{\|f(t)\|_H}{\alpha/2}E_m(t)\ .
\end{aligned} \tag{1.27}$$

From (1.27) it follows that for each $0 \leq t \leq T$

$$E_m'(t) \leq \phi(t)E_m(t) + \|f(t)\|_H + \langle g(t), u_m'(t)\rangle,$$

where

$$\phi(t) = \frac{2}{\alpha}\left(\|\mathbf{R}'(t)\|_{\mathscr{L}(H)} + \|\mathbf{A}_0'(t)\|_{\mathscr{L}(V,V')} + \|\mathbf{A}_1(t)\|_{\mathscr{L}(V,H)} + \|f(t)\|_H\right) \in \mathrm{L}^1(0,T).$$

Integrating from 0 to t we obtain

$$E_m(t) \leq E_m(0) + \int_0^t \phi(s)E_m(s)ds + \int_0^t \|f(s)\|_{\mathrm{H}}ds + \int_0^t \langle g(s), u_m'(s)\rangle ds$$

To estimate the term $\int_0^t \langle g, u_m'\rangle$, we first perform integration by parts to obtain

$$\int_0^t \langle g(s), u_m'(s)\rangle ds = \langle g(t), u_m(t)\rangle - \langle g(0), u_m(0)\rangle - \int_0^t \langle g'(s), u_m(s)\rangle ds.$$

Recall the continuous inclusion $\mathrm{W}^{1,1}(0,T;V') \hookrightarrow C([0,T];V')$, and deduce that for each $s \in [0,T]$ we have the estimate

$$\langle g(s), u_m(s)\rangle \leq \|g(s)\|_{V'}\|u_m(s)\|_V \leq \|g\|_{\mathrm{W}^{1,1}(V')}\|u_m(s)\|_V \leq \kappa\|g\|_{\mathrm{W}^{1,1}(V')}^2 + \frac{1}{4\kappa}\|u_m(s)\|_V^2,$$

where $\kappa > 0$ is an arbitrary constant. We can now apply this inequality for the first two terms, where κ is chosen large enough so that $\frac{1}{2\kappa\alpha}M \leq \frac{1}{2}$

$$
\begin{aligned}
\int_0^t \langle g(s), u_m'(s)\rangle ds &= \langle g(t), u_m(t)\rangle - \langle g(0), u_m(0)\rangle - \int_0^t \langle g'(s), u_m(s)\rangle ds \\
&\leq 2\kappa \|g\|^2_{\mathrm{W}^{1,1}(V')} + \frac{1}{4\kappa}\left(\|u_m(t)\|_V^2 + \|u_m(0)\|_V^2\right) \\
&\quad + \int_0^t \|g'(s)\|_{V'} \|u_m(s)\|_V ds \\
&\leq 2\kappa \|g\|^2_{\mathrm{W}^{1,1}(V')} + \frac{1}{2\kappa\alpha} E_m(t) + \frac{1}{2\kappa\alpha} E_m(0) \\
&\quad + \int_0^t \|g'(s)\|_{V'} (1 + \|u_m(s)\|_V^2) ds \\
&\leq 2\kappa \|g\|^2_{\mathrm{W}^{1,1}(V')} + \|g'\|_{\mathrm{L}^1(V')} + \frac{1}{2\kappa\alpha} E_m(t) \\
&\quad + \frac{1}{2\kappa\alpha} E_m(0) + \int_0^t \frac{\|g'(s)\|_{V'}}{\alpha/2} E_m(s) ds.
\end{aligned} \tag{1.28}
$$

We thus obtain the inequality

$$
\begin{aligned}
\left(1 - \frac{1}{2\kappa\alpha}\right) E_m(t) \leq \left(1 + \frac{1}{2\kappa\alpha}\right) E_m(0)& \\
+ 2\kappa \|g\|^2_{\mathrm{W}^{1,1}(V')} + \|g'\|_{\mathrm{L}^1(V')} + \|f\|_{\mathrm{L}^1(H)}& \\
+ \int_0^t \tilde{\phi}(s) E_m(s) ds,&
\end{aligned} \tag{1.29}
$$

where $\tilde{\phi}(s) = \phi(s) + \frac{2}{\alpha}\|g'(s)\|_{V'} \in \mathrm{L}^1(0,T)$. At this point we can choose κ such that $1 \geq \frac{1}{2\kappa\alpha} \geq \frac{1}{2}$. For the initial term

$$
E_m(0) = \frac{1}{2}(\mathbf{R}u_m'(0), u_m'(0)) + \frac{1}{2}\langle \mathbf{A}_0 u_m(0), u_m(0)\rangle + \|\mathbf{G}(u_m(0))\|_{\mathrm{L}^1(\Omega)},
$$

we first recall (1.8), (1.9) and initial conditions (1.18) to obtain

$$
(\mathbf{R}u_m'(0), u_m'(0)) \leq \beta \|u_m'(0)\|^2_{\mathrm{H}} \leq \beta \|u^1\|^2_{\mathrm{H}}, \tag{1.30}
$$

$$
\langle \mathbf{A}_0 u_m(0), u_m(0)\rangle \leq \beta \|u_m(0)\|_V^2 \leq \beta \|u^0\|_V^2. \tag{1.31}
$$

Next, we recall (1.16) in order to estimate

$$
\|\mathbf{G}(u_m(0))\|_{\mathrm{L}^1(\Omega)} \leq \mathrm{Lip}(F)\|u^0\|^2_{\mathrm{H}} \leq \mathrm{Lip}(F)\|u^0\|_V^2.
$$

Putting the pieces together yields the inequality of the form

$$
E_m(t) \leq C\left(1 + \int_0^t \tilde{\phi}(s) E_m(s) ds\right),
$$

where the constant C depends on respective norms $\|u^0\|_V, \|u^1\|_H, \|\mathbf{R}\|_{\mathrm{W}^{1,1}(\mathscr{L}(H))}$, $\|\mathbf{A}_0\|_{\mathrm{W}^{1,1}(\mathscr{L}(V,V'))}, \|\mathbf{A}_1\|_{\mathrm{W}^{1,1}(\mathscr{L}(V,H))}$, $\|f\|_{\mathrm{L}^1(H)}, \|g\|_{\mathrm{W}^{1,1}(V')}$, α and $\mathrm{Lip}(F)$. Finally, we apply Gronwall's inequality once more to obtain the estimate

$$E_m(t) \leq C\exp\left(\int_0^t \tilde{\phi}(s)ds\right) \leq C\exp\|\tilde{\phi}\|_{\mathrm{L}^1(0,T)},$$

which gives the uniform bound on $E_m(t)$. Consequently, we have obtained

$$\begin{aligned} &u_m \text{ is bounded in } \mathrm{L}^\infty(0,T;V) \\ &u_m' \text{ is bounded in } \mathrm{L}^\infty(0,T;H). \end{aligned} \tag{1.32}$$

To obtain further a priori estimates on the sequence (u_m), we return to the approximate equation

$$((\mathbf{R}u_m')',v) + \langle \mathbf{A}_0 u_m, v\rangle + (\mathbf{A}_1 u_m, v) + (\mathbf{F}(u_m),v) = (f,v) + \langle g, v\rangle\ , \tag{1.33}$$

valid for every $v \in V_m$, differentiate it with respect to t (there is enough smoothness for this to be valid due to (1.22) as well as our initial assumptions) and insert $v = u_m''(t)$. Thus we obtain for each $0 \leq t \leq T$

$$((\mathbf{R}u_m')'',u_m'') + \langle (\mathbf{A}_0 u_m)', u_m''\rangle + ((\mathbf{A}_1 u_m)', u_m'') + ((\mathbf{F}(u_m))',u_m'') = (f',u_m'') + \langle g', u_m''\rangle\ . \tag{1.34}$$

Some of the terms that appear can be transformed in the following way

$$\begin{aligned} ((\mathbf{R}u_m')'',u_m'') &= (\mathbf{R}''u_m',u_m'') + 2(\mathbf{R}'u_m'',u_m'') + (\mathbf{R}u_m''',u_m'') \\ &= \frac{1}{2}\frac{d}{dt}(\mathbf{R}u_m'',u_m'') + \frac{3}{2}(\mathbf{R}'u_m'',u_m'') + (\mathbf{R}''u_m',u_m'') \\ \langle (\mathbf{A}_0 u_m)', u_m''\rangle &= \langle \mathbf{A}_0' u_m, u_m''\rangle + \langle \mathbf{A}_0 u_m', u_m''\rangle \\ &= \frac{1}{2}\frac{d}{dt}\langle \mathbf{A}_0 u_m', u_m'\rangle - \frac{1}{2}\langle \mathbf{A}_0' u_m', u_m'\rangle + \langle \mathbf{A}_0' u_m, u_m''\rangle \\ ((\mathbf{A}_1 u_m)', u_m'') &= (\mathbf{A}_1 u_m', u_m'') + (\mathbf{A}_1' u_m, u_m'') \\ ((\mathbf{F}(u_m))',u_m'') &= (\mathbf{F}'(u_m)u_m',u_m'') \end{aligned} \tag{1.35}$$

After denoting

$$\tilde{E}_m(t) := \frac{1}{2}(\mathbf{R}(t)u_m''(t),u_m''(t)) + \frac{1}{2}\langle \mathbf{A}_0(t)u_m'(t),u_m'(t)\rangle,$$

we are able to rewrite (1.34) as

$$\begin{aligned}
\tilde{E}_m'(t) &= \frac{d}{dt}\frac{1}{2}(\mathbf{R}u_m'', u_m'') + \frac{d}{dt}\frac{1}{2}\langle \mathbf{A}_0 u_m', u_m'\rangle \\
&= ((\mathbf{R}u_m')'', u_m'') - \frac{3}{2}(\mathbf{R}'u_m'', u_m'') - (\mathbf{R}''u_m', u_m'') \\
&\quad + \langle (\mathbf{A}_0 u_m)', u_m''\rangle + \frac{1}{2}\langle \mathbf{A}_0' u_m', u_m'\rangle - \langle \mathbf{A}_0' u_m, u_m''\rangle \\
&= (f', u_m'') + \langle g', u_m''\rangle - ((\mathbf{A}_1 u_m)', u_m'') - ((\mathbf{F}(u_m))', u_m'') \\
&\quad - \frac{3}{2}(\mathbf{R}'u_m'', u_m'') - (\mathbf{R}''u_m', u_m'') + \frac{1}{2}\langle \mathbf{A}_0' u_m', u_m'\rangle - \langle \mathbf{A}_0' u_m, u_m''\rangle \\
&= (f', u_m'') + \langle g', u_m''\rangle \\
&\quad - \frac{3}{2}(\mathbf{R}'u_m'', u_m'') - (\mathbf{R}''u_m', u_m'') + \frac{1}{2}\langle \mathbf{A}_0' u_m', u_m'\rangle - \langle \mathbf{A}_0' u_m, u_m''\rangle \\
&\quad - (\mathbf{A}_1 u_m', u_m'') - (\mathbf{A}_1' u_m, u_m'') - (\mathbf{F}'(u_m)u_m', u_m'').
\end{aligned} \tag{1.36}$$

It remains to estimate the terms appearing on the right hand side of (1.36). Let us begin by noting that the energy $\tilde{E}_m$ satisfies the inequality similar to (1.26)

$$\tilde{E}_m(t) \geq \frac{\alpha}{2}\left(\|u_m''(t)\|_{\mathrm{H}}^2 + \|u_m'(t)\|_V^2\right). \tag{1.37}$$

We now proceed to estimate the aforementioned terms in (1.34) as follows

$$\begin{aligned}
(f'(t), u_m''(t)) &\leq \|f'(t)\|_H \left(1 + \frac{1}{\alpha/2}\tilde{E}_m(t)\right) \\
(\mathbf{R}'(t)u_m''(t), u_m''(t)) &\leq \frac{\|\mathbf{R}'(t)\|_{\mathscr{L}(H)}}{\alpha/2}\tilde{E}_m(t) \\
(\mathbf{R}''(t)u_m'(t), u_m''(t)) &\leq \|\mathbf{R}''(t)\|_{\mathscr{L}(H)}\left(\|u_m'(t)\|_{\mathrm{H}}^2 + \|u_m''(t)\|_{\mathrm{H}}^2\right) \\
&\leq \frac{\|\mathbf{R}''(t)\|_{\mathscr{L}(H)}}{\alpha/2}\tilde{E}_m(t) \\
\langle \mathbf{A}_0(t)u_m'(t), u_m'(t)\rangle &\leq \frac{\|\mathbf{A}_0(t)\|_{\mathscr{L}(V,V')}}{\alpha/2}\tilde{E}_m(t) \\
(\mathbf{A}_1(t)u_m'(t), u_m''(t)) &\leq \frac{\|\mathbf{A}_1(t)\|_{\mathscr{L}(V,H)}}{\alpha/2}\tilde{E}_m(t) \\
(\mathbf{A}_1'(t)u_m(t), u_m''(t)) &\leq \|\mathbf{A}_1'(t)\|_{\mathscr{L}(V,H)}(\|u_m(t)\|_V^2 + \|u_m''(t)\|_{\mathrm{H}}^2) \\
&\leq \frac{\|\mathbf{A}_1'(t)\|_{\mathscr{L}(V,H)}}{\alpha/2}\left(E_m(t) + \tilde{E}_m(t)\right) \\
(\mathbf{F}'(u_m(t))u_m'(t), u_m''(t)) &\leq \frac{\mathrm{Lip}(F)}{\alpha/2}\tilde{E}_m(t)\ .
\end{aligned} \tag{1.38}$$

Combining estimates in (1.38) and recalling the uniform bound $E_m(t) \le C_E$ obtained in the previous step, we reduce (1.36) to

$$\tilde{E}_m'(t) \le \frac{1}{\alpha/2}\tilde{\phi}(t)\tilde{E}_m(t) + \frac{C_E}{\alpha/2}\|\mathbf{A}_1'(t)\|_{\mathscr{L}(V,H)} + \|f'(t)\|_H + \langle g', u_m''\rangle - \langle \mathbf{A}_0' u_m, u_m''\rangle, \tag{1.39}$$

where $\tilde{\phi}$ is given by

$$\begin{aligned}\tilde{\phi}(t) :=& \|\mathbf{R}'(t)\|_{(\mathscr{L}(H))} + \|\mathbf{R}''(t)\|_{\mathscr{L}(H)} + \|\mathbf{A}_0(t)\|_{\mathscr{L}(V,V')} \\ &+ \|\mathbf{A}_1(t)\|_{\mathscr{L}(V,H)} + \|\mathbf{A}_1'(t)\|_{\mathscr{L}(V,H)} + \mathrm{Lip}(F) \in \mathrm{L}^1(0,T).\end{aligned}$$

Integrating from 0 to $t \le T$ we obtain

$$\begin{aligned}\tilde{E}_m(t) \le& \tilde{E}_m(0) + \frac{1}{\alpha/2}\int_0^t \tilde{\phi}(s)\tilde{E}_m(s)ds + \frac{C}{\alpha/2}\int_0^t \|\mathbf{A}_1'(s)\|_{\mathscr{L}(V,H)}ds \\ &+ \int_0^t \|f'(s)\|_H ds + \int_0^t \langle g'(s), u_m''(s)\rangle ds - \int_0^t \langle \mathbf{A}_0'(s)u_m(s), u_m''(s)\rangle ds.\end{aligned} \tag{1.40}$$

The last two terms are transformed via integration by parts:

$$\begin{aligned}\int_0^t \langle g'(s), u_m''(s)\rangle ds =& -\int_0^t \langle g''(s), u_m'(s)\rangle ds \\ &+ \langle g'(t), u_m'(t)\rangle - \langle g'(0), u_m'(0)\rangle \\ \le& \int_0^t \|g''(s)\|_{V'}\left(1 + \frac{1}{\alpha/2}\tilde{E}_m(s)\right) ds \\ &+ \frac{1+\alpha}{2\alpha}\|g'\|^2_{\mathrm{W}^{1,1}(V')} + \frac{1}{4}\tilde{E}_m(t) + \frac{1}{2}\tilde{E}_m(0),\end{aligned} \tag{1.41}$$

and

$$\begin{aligned}\int_0^t \langle \mathbf{A}_0'(s)u_m(s), u_m''(s)\rangle ds =& -\int_0^t \langle \mathbf{A}_0''(s)u_m(s), u_m'(s)\rangle ds \\ &- \int_0^t \langle \mathbf{A}_0(s)u_m'(s), u_m'(s)\rangle ds \\ &+ \langle \mathbf{A}_0'(t)u_m(t), u_m'(t)\rangle \\ &- \langle \mathbf{A}_0'(0)u_m(0), u_m'(0)\rangle \\ \le& \frac{1}{\alpha/2}\int_0^t \|\mathbf{A}_0''(s)\|_{\mathscr{L}(V,V')}\left(E_m(s) + \tilde{E}_m(s)\right) ds \\ &+ \int_0^t \|\mathbf{A}_0\|_{\mathscr{L}(V,V')}\left(1 + \frac{1}{\alpha/2}\tilde{E}_m(s)\right) ds \\ &+ \frac{1+\alpha}{2\alpha}\|\mathbf{A}_0'\|^2_{\mathrm{W}^{1,1}(\mathscr{L}(V,V'))} + \frac{1}{4}\tilde{E}_m(t) + \frac{1}{2}\tilde{E}_m(0)\ .\end{aligned} \tag{1.42}$$

Next we need an estimate of the term

$$\begin{aligned}\tilde{E}_m(0) &= \frac{1}{2}(\mathbf{R}(0)u_m''(0), u_m''(0)) + \frac{1}{2}\langle \mathbf{A}_0(0)u_m'(0), u_m'(0)\rangle \\ &\le \frac{1}{2}\|\mathbf{R}\|_{\mathrm{W}^{1,1}(\mathscr{L}(H))}\|u_m''(0)\|_{\mathrm{H}}^2 + \frac{1}{2}\|\mathbf{A}_0\|_{\mathrm{W}^{1,1}(\mathscr{L}(V,V'))}\|u_m'(0)\|_V^2,\end{aligned}$$

which obviously boils down to estimating the term $\|u_m''(0)\|_H$, since $u_m'(0) = u^1$ is trivially bounded in V. In order to do so, we return back to (1.33), take $t = 0$ and insert $v = u_m''(0)$, thus obtaining

$$\begin{aligned}(\mathbf{R}(0)u_m''(0), u_m''(0)) = (f(0), u_m''(0)) &+ \langle g(0) - \mathbf{A}_0(0)u_m(0), u_m''(0)\rangle \\ &- (\mathbf{R}'(0)u_m'(0), u_m''(0)) - (\mathbf{A}_1(0)u_m(0), u_m''(0)) \\ &- (\mathbf{F}(u_m(0)), u_m''(0)).\end{aligned} \tag{1.43}$$

We have the following estimates

$$\begin{aligned}(f(0), u_m''(0)) &\le C\|f\|_{\mathrm{W}^{1,1}(H)}^2 + \frac{\alpha}{8}\|u_m''(0)\|_{\mathrm{H}}^2 \\ (\mathbf{R}'(0)u_m'(0), u_m''(0)) &\le C\|\mathbf{R}\|_{\mathrm{W}^{2,1}(\mathscr{L}(H))}^2\|u_m'(0)\|_V^2 + \frac{\alpha}{8}\|u_m''(0)\|_{\mathrm{H}}^2 \\ (\mathbf{A}_1(0)u_m(0), u_m''(0)) &\le C\|\mathbf{A}_1\|_{\mathrm{W}^{1,1}(\mathscr{L}(V,H))}^2\|u_m(0)\|_V^2 + \frac{\alpha}{8}\|u_m''(0)\|_{\mathrm{H}}^2 \\ (\mathbf{F}(u_m(0)), u_m''(0)) &\le C\mathrm{Lip}(F)^2\|u_m(0)\|_H^2 + \frac{\alpha}{8}\|u_m''(0)\|_{\mathrm{H}}^2\end{aligned} \tag{1.44}$$

Recalling the assumption $\mathbf{A}_0(0)u_m(0) - g(0) = \mathbf{A}_0(0)u^0 - g(0) \in H$, we also have

$$\langle g(0) - \mathbf{A}_0(0)u_m(0), u_m''(0)\rangle \le C\|\mathbf{A}_0(0)u^0 - g(0)\|_{\mathrm{H}}^2 + \frac{\alpha}{8}\|u_m''(0)\|_{\mathrm{H}}^2. \tag{1.45}$$

Collecting (1.44)–(1.45) we obtain the estimate

$$\begin{aligned}\|u_m''(0)\|_{\mathrm{H}}^2 &\le \frac{1}{\alpha}(\mathbf{R}(0)u_m''(0), u_m''(0)) \\ &\le C\Big(\|f\|_{\mathrm{W}^{1,1}(H)}^2 + \|\mathbf{R}\|_{\mathrm{W}^{2,1}(\mathscr{L}(H))}^2\|u^1\|_V^2 + \|\mathbf{A}_0(0)u^0 - g(0)\|_{\mathrm{H}}^2 \\ &\quad + \|\mathbf{A}_1\|_{\mathrm{W}^{1,1}(\mathscr{L}(V,H))}^2\|u^0\|_V^2 + \mathrm{Lip}(F)^2\|u^0\|_H^2\Big) + \frac{5}{8}\|u_m''(0)\|_{\mathrm{H}}^2,\end{aligned} \tag{1.46}$$

from where we deduce that

$$\|u_m''(0)\|_H \text{ is uniformly bounded.} \tag{1.47}$$

This allows us to finally obtain the uniform bound on $\tilde{E}_m$, via an application of the Gronwall's inequality to (1.40) after taking into account estimates given by (1.41)–(1.47).

1.2.3. Solution

We are now in a position to prove the existence of a solution to (1.17). Let u_m be a solution of the projected problem (1.18). According to the a priori estimates, we have that the sequence of solutions (u_m) satisfies the following:

$$(u_m) \text{ is bounded in } \mathrm{L}^\infty(0,T;V) \tag{1.48}$$

$$(u_m') \text{ is bounded in } \mathrm{L}^\infty(0,T;V) \tag{1.49}$$

$$(u_m'') \text{ is bounded in } \mathrm{L}^\infty(0,T;H). \tag{1.50}$$

We can then extract a subsequence (which we still denote by (u_m)) which satisfies (as the derivative is a continuous operator on $\mathscr{D}'$)

$$\begin{aligned} u_m &\overset{*}{\rightharpoonup} u && \text{in } \mathrm{L}^\infty(0,T;V) \\ u_m' &\overset{*}{\rightharpoonup} u' && \text{in } \mathrm{L}^\infty(0,T;V) \\ u_m'' &\overset{*}{\rightharpoonup} u'' && \text{in } \mathrm{L}^\infty(0,T;H). \end{aligned} \tag{1.51}$$

Additionally, we have

$$\mathbf{R}u_m' \overset{*}{\rightharpoonup} \mathbf{R}u' \quad \text{in } \mathrm{L}^\infty(0,T;H) \tag{1.52}$$

$$(\mathbf{R}u_m')' \overset{*}{\rightharpoonup} (\mathbf{R}u')' \quad \text{in } \mathrm{L}^\infty(0,T;H) \tag{1.53}$$

$$\mathbf{A}_0 u_m \overset{*}{\rightharpoonup} \mathbf{A}_0 u \quad \text{in } \mathrm{L}^\infty(0,T;V') \tag{1.54}$$

$$\mathbf{A}_1 u_m \overset{*}{\rightharpoonup} \mathbf{A}_1 u \quad \text{in } \mathrm{L}^\infty(0,T;H). \tag{1.55}$$

Lastly, we examine the convergence of the nonlinear part, $\mathbf{F}(u_m)$. First we deduce for all $t \in [0,T]$

$$\|\mathbf{F}(u_m(t)) - \mathbf{F}(u(t))\|_{\mathrm{H}} \leq \mathrm{Lip}(\mathbf{F})\|u_m(t) - u(t)\|_H \longrightarrow 0,$$

which gives

$$\mathbf{F}(u_m) \longrightarrow \mathbf{F}(u) \qquad \text{in } \mathrm{L}^\infty(0,T;H). \tag{1.56}$$

Take $n \in \mathbb{N}$ and $v_n \in V_n$, and let $m \geq n$. Since u_m is a solution of projected problem (1.18) on V_m, which contains V_n, we have for each $t \in [0,T]$

$$\begin{aligned} ((\mathbf{R}(t)u_m'(t))', v_n) + \langle \mathbf{A}_0(t)u_m(t), v_n\rangle + (\mathbf{A}_1(t)u_m(t), v_n) \\ + (\mathbf{F}(u_m(t)), v_n) = (f(t), v_n) + \langle g(t), v_n\rangle. \end{aligned} \tag{1.57}$$

Multiplying (1.57) by $\vartheta \in \mathscr{D}(0,T)$ and integrating over $[0,T]$ we obtain for $\psi_n := \vartheta \boxtimes v_n$

$$\int_0^T \langle (\mathbf{R}u_m')', \psi_n\rangle + \langle \mathbf{A}_0 u_m, \psi_n\rangle + (\mathbf{A}_1 u_m, \psi_n) + (\mathbf{F}(u_m), \psi_n)\, dt = \int_0^T (f, \psi_n) + \langle g, \psi_n\rangle dt. \tag{1.58}$$

We then pass to the limit $m \to \infty$, while using (1.51), (1.52), (1.53), (1.54), (1.55) and (1.56) to obtain

$$\int_0^T ((\mathbf{R}u')', \psi_n) + \langle \mathbf{A}_0 u, \psi_n\rangle + (\mathbf{A}_1 u, \psi_n) + (\mathbf{F}(u), \psi_n)\, dt = \int_0^T (f, \psi_n) + \langle g, \psi_n\rangle dt. \tag{1.59}$$

Since $\operatorname{span}\{\mathscr{D}(0,T) \boxtimes \bigcup_n V_n\}$ is dense in $\mathrm{L}^2(0,T;V)$, we can now deduce that

$$\int_0^T -(\mathbf{R}u', \psi') + \langle \mathbf{A}_0 u, \psi\rangle + (\mathbf{A}_1 u, \psi) + (\mathbf{F}(u), \psi)\, dt = \int_0^T (f, \psi) + \langle g, \psi\rangle dt$$

holds for each $\psi \in \mathrm{L}^2(0,T;V)$, that is, the equality

$$(\mathbf{R}u')' + (\mathbf{A}_0 + \mathbf{A}_1)u + \mathbf{F}(u) = f + g$$

holds in the sense of $\mathscr{D}'(0,T;V')$. From here, due to the density of functions of the form $\mathscr{D}(0,T;V)$ in $\mathrm{L}^2(0,T;V)$ which is a predual of $\mathrm{L}^2(0,T;V')$, we deduce (1.17).

The fact that $\mathbf{G}(u) \in \mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega))$ follows from (1.16) and (1.51), giving

$$\|\mathbf{G}(u(t))\|_H \le \mathrm{Lip}(F)\|u(t)\|_{\mathrm{H}}^2 \le C, \qquad t \in [0,T].$$

It remains to be shown that u satisfies initial conditions. First we check that $u'(0) = u^1$. Take $\varphi \in C^\infty([0,T])$ such that $\varphi(0) = 1$ and $\varphi(T) = 0$ and let $v \in V_m$ for some m. Choose $\psi = \varphi \boxtimes v \in C^\infty([0,T];V)$ and insert it into (1.59), thus obtaining

$$\int_0^T ((\mathbf{R}u')', \psi) + \langle \mathbf{A}_0 u, \psi\rangle + (\mathbf{A}_1 u, \psi) + (\mathbf{F}(u), \psi)\, dt = \int_0^T (f, \psi) dt.$$

After integrating by parts in the first term it follows

$$\int_0^T -(\mathbf{R}u', \psi') + \langle \mathbf{A}_0 u, \psi\rangle + (\mathbf{A}_1 u, \psi) + (\mathbf{F}(u), \psi)\, dt + (\mathbf{R}u'(0), \psi(0)) = \int_0^T (f, \psi) dt.$$

On the other hand, by inserting ψ into (1.58) and once again integrating by parts we get

$$\int_0^T -((\mathbf{R}u_m'), \psi') + \langle \mathbf{A}_0 u_m, \psi\rangle + (\mathbf{A}_1 u_m, \psi) + (\mathbf{F}(u_m), \psi)\, dt + (\mathbf{R}u_m'(0), \psi(0)) = \int_0^T (f, \psi) dt. \tag{1.60}$$

Now recall that $u_m'(0) = u^1$, and take into account (1.52), (1.54), (1.55) and (1.56), in order to obtain

$$\int_0^T (\mathbf{R}u', \psi') + \langle \mathbf{A}_0 u, \psi \rangle + (\mathbf{A}_1 u, \psi) + (\mathbf{F}(u), \psi)\, dt + (\mathbf{R}(0)u^1, \psi(0)) = \int_0^T (f, \psi) dt. \tag{1.61}$$

Comparing (1.60) and (1.61) we deduce

$$(\mathbf{R}u'(0), v) = (\mathbf{R}(0)u^1, v).$$

Since m and $v \in V_m$ were arbitrary, we conclude that

$$\mathbf{R}(0)(u'(0) - u^1) = 0.,$$

and since $\mathbf{R}(0)$ is an isomorphism, we conclude $u'(0) = u^1$. In order to prove $u(0) = u^0$, we additionally assume $\varphi'(0) = 1$ and $\varphi'(T) = 0$ and integrate by parts once more in first terms of both (1.60) and (1.61), while using symmetricity of $\mathbf{R}$ to obtain

$$\int_0^T (\mathbf{R}u', \psi') dt = -\int_0^T (u, (\mathbf{R}\psi')') dt - (u(0), \mathbf{R}\psi(0)),$$

$$\int_0^T (\mathbf{R}u_m', \psi') dt = -\int_0^T (u_m, (\mathbf{R}\psi')') dt - (u_m(0), \mathbf{R}\psi(0)).$$

Recalling that $u_m(0) = u^0$ and (1.52) we get by comparison

$$(u(0), \mathbf{R}(0)v) = (u^0, \mathbf{R}(0)v).$$

Once again, using the fact $\mathbf{R}(0)$ is an isomorphism and m, v are arbitrary, we finally conclude

$$u(0) = u^0.$$

Proposition 1.2.3. The solution of (1.11) is unique.

Proof. Assume u_1 and u_2 are two solutions of the problem, and denote $u := u_1 - u_2$. We wish to prove that $u \equiv 0$ is the only solution to the problem

$$\begin{cases} (\mathbf{R}u')' + (\mathbf{A}_0 + \mathbf{A}_1)u + \mathbf{F}(u_1) - \mathbf{F}(u_2) = 0, \\ \qquad\qquad\qquad\qquad u(0) = u'(0) = 0, \end{cases}$$

Since $u' \in \mathrm{L}^2(0, T; V)$, we can use it as a test function for the equation and obtain

$$((\mathbf{R}u')', u') + \langle \mathbf{A}_0 u, u' \rangle + (\mathbf{A}_1 u, u') + (\mathbf{F}(u_1) - \mathbf{F}(u_2), u') = 0, \tag{1.62}$$

which can then be rewritten as

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left((\mathbf{R}(t)u'(t),u'(t))+\langle \mathbf{A}_0(t)u(t),u(t)\rangle\right) &= (\mathbf{R}'(t)u',u'(t))+\langle \mathbf{A}_0'(t)u(t),u(t)\rangle \\
&\quad +2(\mathbf{A}_1(t)u(t),u'(t))+(\mathbf{F}(u_2(t))-\mathbf{F}(u_1(t)),u'(t)) \\
&\leq \|\mathbf{R}'(t)\|_{\mathscr{L}(H)}\|u'(t)\|_{\mathrm{H}}^2+\|\mathbf{A}_0'(t)\|_{\mathscr{L}(V,V')}\|u(t)\|V^2 \\
&\quad +2\|\mathbf{A}_1(t)\|_{\mathscr{L}(V,H)}\|u(t)\|_V\|u'(t)\|_H \\
&\quad +(\mathbf{F}(u_2)-\mathbf{F}(u_1),u') \\
&\leq C(\|u'(t)\|_{\mathrm{H}}^2+\|u(t)\|_V^2)+(\mathbf{F}(u_2)-\mathbf{F}(u_1),u'),
\end{aligned}
$$

where the constant C can be expressed via corresponding norms of $\|\mathbf{R}\|_{\mathrm{W}^{2,1}(\mathscr{L}(H))}$, $\|\mathbf{A}_0\|_{\mathrm{W}^{2,1}(\mathscr{L}(V,V'))}$ and $\|\mathbf{A}_1\|_{\mathrm{W}^{1,1}(\mathscr{L}(V,H))}$, similarly as it was done for a priori estimates. It remains to estimate the term involving nonlinearity, for which we have

$$
\begin{aligned}
(\mathbf{F}(u_1)-\mathbf{F}(u_2),u') &\leq \mathrm{Lip}(\mathbf{F})\|u_1(t)-u_2(t)\|_H\|u'(t)\|_H \\
&\leq \mathrm{Lip}(\mathbf{F})\left(\|u(t)\|_{\mathrm{V}}^2+\|u'(t)\|_{\mathrm{H}}^2\right).
\end{aligned}
$$

Hence, using (1.26), we obtain the inequality of the form

$$
\frac{1}{2}\frac{d}{dt}\left((\mathbf{R}u',u')+\langle \mathbf{A}_0u,u\rangle\right) \leq C\left((\mathbf{R}u',u')+\langle \mathbf{A}_0u,u\rangle\right)
$$

for constant C now depending on the norms of $\mathbf{R},\mathbf{A}_0,\mathbf{A}_1$ and $\mathrm{Lip}(F)$. Applying the Gronwall lemma, combined with the fact that $u(0)=u'(0)=0$, gives $u\equiv 0$. ∎

1.3. Continuous nonlinearity with sign condition

In this section, we wish to obtain similar existence results on the solution of the problem of the same form as in the previous section, but with less regularity on the right hand side, the initial conditions and the nonlinearity. Namely, we keep the same regularity assumptions on $\mathbf{R}, \mathbf{A}_0$ and $\mathbf{A}_1$, with addition of the following assumption on $\mathbf{A}_0(0)$: assume that it is an isomorphism of V and V'. Let $\mathbf{F} : \mathbb{R} \to \mathbb{R}$ only be a continuous function satisfying the sign condition (1.13), with additional assumption

$$\mathbf{G}(u^0) \in \mathrm{L}^1(\Omega). \tag{1.63}$$

Finally, assume $f \in \mathrm{L}^1(0,T;H)$, $g \in \mathrm{W}^{1,1}(0,T;V')$, with initial conditions satisfying $u^0 \in V$ and $u^1 \in H$.

We first state the following approximation lemma for the nonlinear part, the proof of which can be found in [67, Lemma 2.2.].

Lemma 1.3.1. Continuous function $F : \mathbb{R} \to \mathbb{R}$ which satisfies the sign condition can be locally uniformly approximated by a sequence of Lipschitz continuous functions that satisfy the sign condition.

Denote the members of such a sequence by F_k, and analogously as before let us define G_k by $G_k(z) := \int_0^z F_k(w)dw$. Then sequences (F_k) and (G_k) satisfy properties analogous to (1.12)–(1.16) with constants $\mathrm{Lip}(F_k)$.

Next, since u^0 is not necessarily bounded, we approximate it by a sequence of functions u_j^0 given by

$$u_j^0(x) = \xi_j(u^0(x)), \qquad \text{a.e. } x \in \Omega,$$

where ξ_j is defined by

$$\xi_j(x) := \begin{cases} -j, & x < -j \\ x, & |x| \le j \\ j, & x > j. \end{cases}$$

Due to Corollary A.5. of [48], this sequence satisfies

$$u_j^0 \longrightarrow u^0 \quad \text{strongly in } V,$$

as well as

$$|u_j^0(x)| \leq |u^0(x)| \quad \text{and} \quad u_j^0(x)u^0(x) \geq 0 \quad \text{for a.e } x \in \Omega. \tag{1.64}$$

Additionally, we can extract a subsequence (still dentoted by u_j^0) such that

$$u_j^0 \longrightarrow u^0 \quad \text{strongly in } V \text{ and a.e.} \tag{1.65}$$

Furthermore, let the sequences of functions $(u_p^1)_p \in V$, $(f_p)_p \in \mathrm{W}^{1,1}(0,T;H)$ and $(g_p)_p \in \mathrm{W}^{2,1}(0,T;V')$ be such that

$$\begin{aligned} u_p^1 &\to u^1 && \text{strongly in } H \\ f_p &\to f && \text{strongly in } \mathrm{L}^1(0,T;H) \\ g_p &\to g && \text{strongly in } \mathrm{W}^{1,1}(0,T;V'). \end{aligned} \tag{1.66}$$

Finally, choose a sequence $(u_{jp}^0)_p \in V$ such that

$$u_{jp}^0 \to u_j^0 \qquad \text{strongly in } V \tag{1.67}$$

and that $\mathbf{A}_0(0)u_{jp}^0 - g_p(0) \in H$ holds. Such a sequence can be obtained as follows. As $\mathbf{A}_0(0)$ is an isomorphism of V and V', we have that $\mathbf{A}_0(0)^{-1}H$ is dense in V. First, choose a sequence (φ_k) $\mathbf{A}_0(0)^{-1}H$ such that $\varphi_k \to u_j^0$ strongly in V. Then, for fixed $p \in \mathbb{N}$, take sequence (g_{pk}) in H such that $g_{pk} \to g_p(0)$ strongly in V'. Define $u_{jpk}^0 := \varphi_k + \mathbf{A}_0(0)^{-1}(g_p(0) - g_{pk})$, and finally take the diagonal sequence $u_{jp}^0 := u_{jpp}^0$.

For each $u_{jp}^0, u_p^1, f_p, g_p$ and F_k, Theorem 1.2.1 yields a solution u_{jpk} satisfying

$$\begin{cases} u_{jpk} \in \mathrm{L}^\infty(0,T;V) \\ u'_{jpk} \in \mathrm{L}^\infty(0,T;V) \\ u''_{jpk} \in \mathrm{L}^2(0,T;H) \\ \mathbf{G}_k(u_{jpk}) \in \mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega)), \end{cases}$$

of each of the approximate problems

$$\begin{cases} (\mathbf{R}u'_{jpk})' + (\mathbf{A}_0 + \mathbf{A}_1)u_{jpk} + \mathbf{F}_k(u_{jpk}) = f_p + g_p \\ u_{jpk}(0) = u_{jp}^0 \\ u'_{jpk}(0) = u_p^1, \end{cases} \tag{1.68}$$

Using the same procedure as in obtaining a priori estimates in the previous section, we first obtain, after denoting

$$E_{jpk}(t) := \frac{1}{2}(\mathbf{R}u'_{jpk}, u'_{jpk}) + \frac{1}{2}\langle \mathbf{A}_0 u_{jpk}, u_{jpk}\rangle + \|\mathbf{G}_k(u_{jpk})\|_{\mathrm{L}^1(\Omega)},$$

the following inequality

$$E'_{jpk}(t) \leq \phi(t)E_{jpk}(t) + \|f_p(t)\|_H + \langle g_p(t), u'_{jpk}(t)\rangle, \tag{1.69}$$

where

$$\phi(t) = \frac{1}{\alpha/2}\left(\|\mathbf{R}'(t)\|_{\mathscr{L}(H)} + \|\mathbf{A}'_0(t)\|_{\mathscr{L}(V,V')} + \|\mathbf{A}_1(t)\|_{\mathscr{L}(V,H)} + \|f_p(t)\|_H\right).$$

Integrating from 0 to $t \leq T$, we obtain

$$E_{jpk}(t) \leq E_{jpk}(0) + \|f_p\|_{\mathrm{L}^1(H)} + \int_0^t \langle g_p(s), u'_{jpk}(s)\rangle + \int_0^t \phi(s)E_{jpk}(s). \tag{1.70}$$

Since $\|f_p\|$ converges strongly in $\mathrm{L}^1(H)$, term $\|f_p\|_{\mathrm{L}^1(H)}$ is bounded by a constant independent of j, p and k. The term involving g_p can be integrated by parts, yielding in the same manner as in (1.28)

$$\begin{aligned}\int_0^t \langle g_p(s), u'_{jpk}(s)\rangle ds &= -\int_0^t \langle g'_p(s), u_{jpk}(s)\rangle ds + \langle g_p(t), u_{jpk}(t)\rangle - \langle g_p(0), u_{jpk}(0)\rangle \\ &\leq 2\kappa\|g_p\|^2_{\mathrm{W}^{1,1}(V')} + \|g'_p\|_{\mathrm{L}^1(V')} + \frac{1}{2\kappa\alpha}E_{jpk}(t) \\ &\quad + \frac{1}{2\kappa\alpha}E_{jpk}(0) + \int_0^t \frac{\|g'_p(s)\|_{V'}}{\alpha/2}E_{jpk}(s)ds.\end{aligned}$$

First two terms are now bounded due to the convergence of g_p in $\mathrm{W}^{1,1}(0,T;V')$. The constant κ is again chose so that $\geq \frac{1}{2\kappa\alpha} \geq \frac{1}{2}$ holds.

It remains to provide estimates on $E_{jpk}(0)$, that is, to analyse the terms $\|u^0_{jp}\|_V, \|u^1_p\|_H$ and $\|\mathbf{G}_k(u^0_{jp})\|_{\mathrm{L}^1(\Omega)}$. Since u^1_p converges strongly in H, it is obviously bounded. Furthermore, from the construction of u^0_j it follows $\|u^0_j\|_V \leq \|u^0\|_V$ for each $j \in \mathbb{N}$, and the convergence of u^0_{jp} to u^0_j then implies boundedness of $\|u^0_{jp}\|$ independent of j, p. Let us now deal with the term $\|\mathbf{G}_k(u^0_{jp})\|_{\mathrm{L}^1(\Omega)}$. Since for each $j \in \mathbb{N}$, u^0_j is a bounded function, $\mathbf{F}_k(u^0_j)$ converges uniformly in Ω to $\mathbf{F}(u^0_j)$ and hence we have

$$\begin{aligned}|G_k(u^0_j(x)) - G(u^0_j(x))| &\leq \left|\int_0^{u^0_j(x)} F_k(u^0_j(w)) - F(u^0_j(w))dw\right| \\ &\leq j \sup_{w\in[-j,j]} |F_k(w) - F(w)| \xrightarrow{k\to\infty} 0, \quad \text{a.e. } x \in \Omega,\end{aligned}$$

deducing

$$\mathbf{G}_k(u_j^0) \xrightarrow{k\to\infty} \mathbf{G}(u_j^0) \quad \text{in } \mathrm{L}^\infty(\Omega). \tag{1.71}$$

As a consequence, since Ω is bounded, for each $j \in \mathbb{N}$ we have

$$\|\mathbf{G}_k(u_j^0) - \mathbf{G}(u_j^0)\|_{\mathrm{L}^1(\Omega)} \le |\Omega| \|\mathbf{G}_k(u_j^0) - \mathbf{G}(u_j^0)\|_{\mathrm{L}^\infty(\Omega)} \xrightarrow{k\to\infty} 0.$$

Hence, we can extract a diagonal subsequence $\mathbf{G}_j(u_j^0) := \mathbf{G}_{k(j)}(u_j^0)$ of $\mathbf{G}_k(u_j^0)$ (now denoted simply as $\mathbf{G}_j$) such that

$$\int_\Omega |\mathbf{G}_j(u_j^0) - \mathbf{G}(u_j^0)| \to 0.$$

Furthermore, since $\mathbf{G}$ is continuous, (1.65) gives

$$\mathbf{G}(u_j^0) \longrightarrow \mathbf{G}(u^0) \text{ a.e.}$$

Now, the sign condition on $\mathbf{F}$ implies that $\mathbf{G}$ is nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0, \infty)$, which combined with (1.64) yields $\mathbf{G}(u_j^0) \le \mathbf{G}(u^0)$ a.e. Assumption (1.63) therefore allows us to apply the Lebesgue dominated convergence theorem in order to obtain

$$\mathbf{G}(u_j^0) \longrightarrow \mathbf{G}(u^0) \quad \text{in } \mathrm{L}^1(\Omega).$$

Thus, we conclude

$$\mathbf{G}_j(u_j^0) \to \mathbf{G}(u^0) \quad \text{strongly in } \mathrm{L}^1(\Omega) \text{ as well as a.e.}$$

Finally, from the elementary inequality valid for each $j \in \mathbb{N}$

$$|G_j(z_2) - G_j(z_1)| \le \mathrm{Lip}(F_j)(|z_1| + |z_2|)|z_2 - z_1|,$$

we have for each $p \in \mathbb{N}$

$$\|\mathbf{G}_j(u_{jp}^0) - \mathbf{G}_j(u_j^0)\|_{\mathrm{L}^1(\Omega)} \le \mathrm{Lip}(F_j) \left(\|u_{jp}^0\|_H + \|u_j^0\|_H \right) \|u_{jp}^0 - u_j^0\|_H.$$

Therefore, for each $j \in \mathbb{N}$

$$\mathbf{G}_j(u_{jp}^0) \xrightarrow{p\to\infty} \mathbf{G}_j(u_j^0) \quad \text{in } \mathrm{L}^1(\Omega).$$

We can then once again choose a subsequence of u_{jp}^0 denoted by u_{jj}^0 such that

$$\mathbf{G}_j(u_{jj}^0) \longrightarrow \mathbf{G}(u^0) \quad \text{in } \mathrm{L}^1(\Omega). \tag{1.72}$$

As a consequence, we also get uniform boundedness of term $\|\mathbf{G}_j(u_j^0)\|_{L^1(\Omega)}$. Let us now denote the (diagonal) subsequence of u_{jp}^0 obtained above with $u_k^0 := u_{kk}^0$ and, accordingly, denote by u_k the sequence corresponding to u_{kk}. Taking previous remarks into consideration, upon returning to (1.69), we obtain the inequality of the form

$$E_k(t) \le C(1 + \int_0^t (\phi(s) + \|g'(s)\|_{V'})E_k(s)ds),$$

and a simple application of the Gronwall's lemma yields the uniform bound of E_k. Thus, we have

$$\begin{aligned} (u_k) &\text{ is bounded in } L^\infty(0,T;V) \\ (u_k') &\text{ is bounded in } L^\infty(0,T;H) \\ \mathbf{G}_k(u_k) &\text{ is bounded in } L^\infty(0,T;L^1(\Omega)) . \end{aligned} \tag{1.73}$$

Once again, we can extract a subsequence such that

$$\begin{aligned} u_k &\overset{*}{\rightharpoonup} u \qquad \text{in } L^\infty(0,T;V) \\ u_k' &\overset{*}{\rightharpoonup} u' \qquad \text{in } L^\infty(0,T;H). \end{aligned} \tag{1.74}$$

Recall that u_k satisfies

$$(\mathbf{R}u_k')' + (\mathbf{A}_0 + \mathbf{A}_1)u_k + \mathbf{F}_k(u_k) = f_k + g_k \qquad \text{in } L^2(0,T;V'). \tag{1.75}$$

We now wish to test this equation against arbitrary $\psi \in \mathscr{D}((0,T)\times\Omega)$ and pass to the limit $k \to \infty$. In order to do that, it remains to check the convergence of the term $F_k(u_k)$ in some sense. For this, we refer to [67, Theorem 1.1], now stated.

Theorem 1.3.2. Let $\tilde{\Omega}$ be a finite measure space and X and Y Banach spaces. Let (u_k) be a sequence of strongly measurable functions from $\tilde{\Omega}$ to X. Let $(\mathbf{F}_k)$ be a sequence of functions from X to Y such that

(a) (F_k) is uniformly bounded in Y on B for any bounded subset B of X,

(b) $F_k(u_k(x))$ is strongly measurable and

$$\sup_k \int_{\tilde{\Omega}} \|u_k(x)\|_X \|F_k(u_k(x))\|_Y dx < \infty,$$

(c) $\|F_k(u_k(x)) - v(x)\|_Y \to 0$ a.e.

Then $v \in \mathrm{L}^1(\tilde{\Omega};Y)$, and

$$\|F_k(u_k) - v\|_{\mathrm{L}^1(\tilde{\Omega},Y)} \to 0.$$

We tend to apply this theorem to sequences (u_k) and (F_k) with $\tilde{\Omega} = [0,T] \times \Omega$, $X = Y = \mathbb{R}$. In order to do so, we need to find a uniform bound (without the absolute value, owing to the sign condition) of the term

$$\int_0^T (\mathbf{F}_k(u_k), u_k)dt,$$

and the corresponding almost-everywhere convergence.

Recalling the equation in (1.68), multiplying it by u_k and integrating over $[0,T]$ we obtain

$$\int_0^T (\mathbf{F}_k(u_k), u_k)dt = \int_0^T (f, u_k) - \langle (\mathbf{R}u_k')', u_k \rangle - \langle \mathbf{A}_0 u_k, u_k \rangle - \langle \mathbf{A}_1 u_k, u_k \rangle \, dt.$$

After integrating by parts in the term involving $\mathbf{R}$ we get

$$\begin{aligned}\int_0^T (\mathbf{F}_k(u_k), u_k)dt = &\int_0^T (f, u_k) + \langle g, u_k \rangle dt \\ &- \int_0^T (\mathbf{R}u_k', u_k') + \langle \mathbf{A}_0 u_k, u_k \rangle + \langle \mathbf{A}_1 u_k, u_k \rangle \, dt \\ &+ (\mathbf{R}u_k'(0), u_k(0)) - (\mathbf{R}u_k'(T), u_k(T)),\end{aligned} \tag{1.76}$$

it follows

$$\begin{aligned}\int_0^T (\mathbf{F}_k(u_k), u_k)dt \leq &\|f\|_{\mathrm{L}^1(H)}(1 + \|u_k\|^2_{\mathrm{L}^\infty(H)}) + \|g\|_{\mathrm{L}^2}\|u_k\|_{\mathrm{L}^\infty(H)} \\ &+ \|\mathbf{R}\|_{\mathrm{L}^\infty(\mathscr{L}(H))}\|u_k'\|^2_{\mathrm{L}^2(H)} + \|\mathbf{A}_0\|_{\mathrm{L}^\infty(\mathscr{L}(V,V'))}\|u_k\|^2_{\mathrm{L}^2(V)} \\ &+ \|\mathbf{A}_1\|_{\mathrm{L}^\infty(\mathscr{L}(V,H))}\|u_k\|_{\mathrm{L}^2(V)}\|u_k\|_{\mathrm{L}^2(H)} \\ &+ 2\|\mathbf{R}u_k'\|_{\mathrm{L}^\infty(H)}\|u_k\|_{\mathrm{L}^\infty(H)}.\end{aligned}$$

The uniform bound now follows directly from (1.73). Since $\mathbf{F}_k$ converges to $\mathbf{F}$ pointwise, we have

$$\mathbf{F}_k(u_k) \to \mathbf{F}(u_k) \qquad \text{a.e. in } (0,T) \times \Omega.$$

Because of compact injection $V \hookrightarrow H$, an application of the Aubin–Lions lemma yields the strong convergence

$$u_k \to u, \quad \text{in } \mathrm{L}^2((0,T) \times \Omega),$$

which, after possibly extracting another subsequence, yields

$$u_k \to u \quad \text{a.e. in } (0,T)\times\Omega.$$

Since $\mathbf{F}$ is continuous, this also implies

$$\mathbf{F}(u_k) \to \mathbf{F}(u) \quad \text{a.e. in } (0,T)\times\Omega.$$

Therefore, we conclude that

$$\mathbf{F}_k(u_k) \to \mathbf{F}(u) \quad \text{a.e. in } (0,T)\times\Omega. \tag{1.77}$$

We are now in a position to apply the aforementioned theorem to conclude

$$\mathbf{F}_k(u_k) \longrightarrow \mathbf{F}(u) \text{ strongly in } \mathrm{L}^1((0,T)\times\Omega). \tag{1.78}$$

Finally, we can now proceed by testing the equation (1.75) against an arbitrary $\psi \in \mathscr{D}((0,T)\times\Omega$. Using convergences (1.74), (1.77) and (1.66) we obtain

$$(\mathbf{R}u')' + (\mathbf{A}_0+\mathbf{A}_1)u + \mathbf{F}(u) = f + g \qquad \text{in } \mathscr{D}'((0,T)\times\Omega),$$

and therefore

$$(\mathbf{R}u')' + (\mathbf{A}_0+\mathbf{A}_1)u + \mathbf{F}(u) = f + g \qquad \text{in } \mathrm{L}^1(0,T;\mathrm{L}^1(\Omega)+V'). \tag{1.79}$$

We summarise the discussion in the following result.

Theorem 1.3.3. Consider $\mathbf{R}$ satisfying (1.1)–(1.3), $\mathbf{A}_0$ satisfying (1.4)–(1.6), $\mathbf{A}_1$ satisfying (1.7), $\mathbf{F}$ a continuous function satisfying (1.13) and (1.63), $f \in \mathrm{L}^1(0,T;H)$ and $g \in \mathrm{W}^{1,1}(V')$. Let $u^0 \in V$ and $u^1 \in H$. Then there exists a solution $u \in \mathrm{L}^\infty(0,T;V)$, with $u' \in \mathrm{L}^\infty(0,T;H)$ such that $\mathbf{G}(u) \in \mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega))$, satisfying the equation

$$(\mathbf{R}u')' + (\mathbf{A}_0+\mathbf{A}_1)u + \mathbf{F}(u) = f + g \qquad \text{in } \mathrm{L}^1(0,T;V'+\mathrm{L}^1(\Omega)), \tag{1.80}$$

with initial conditions

$$u(0) = u^0, \qquad u'(0) = u^1.$$

As was the case in the first section, we can obtain uniqueness of the solution with some additional assumptions.

Proposition 1.3.4. In the case $d \geq 3$, assume additionally that $\mathbf{A}_1$ satisfies

$$\mathbf{A}_1 \in \mathrm{L}^1(0,T;\mathscr{L}(H,V')), \tag{1.81}$$

and $\mathbf{F}$ is a function of class C^1 satisfying growth condition

$$|\mathbf{F}'(z)| \leq C|z|^p, \tag{1.82}$$

for some $0 \leq p \leq \frac{2}{d-2}$. Then the solution of (1.80) is unique.

Proof. Once again assume u_1, u_2 are solutions and denote $u = u_1 - u_2$. Since there is not enough regularity in u' to use it as a test function, we circumvent this problem by using the standard argument. Fix $0 < s < T$ and define

$$v(t) := \begin{cases} -\int_s^t u(\tau)d\tau, & 0 \leq t \leq s, \\ 0, & s < t \leq T. \end{cases}$$

Then $v \in \mathrm{W}^{1,\infty}(0,T;V)$, so we get

$$\int_0^s \langle (\mathbf{R}u')', v\rangle + \langle \mathbf{A}_0 u, v\rangle + (\mathbf{A}_1 u, v) + (\mathbf{F}(u_1) - \mathbf{F}(u_2), v)\, dt = 0.$$

Since $u'(0) = v(s) = 0$, the integration by parts in the first term gives us

$$\int_0^s -(\mathbf{R}u', v') + \langle \mathbf{A}_0 u, v\rangle + (\mathbf{A}_1 u, v) + (\mathbf{F}(u_1) - \mathbf{F}(u_2), v)\, dt = 0.$$

As $v' = -u$ for $0 \leq t \leq s$, we can write

$$\int_0^s (\mathbf{R}u', u) - \langle \mathbf{A}_0 v', v\rangle + (\mathbf{A}_1 u, v) + (\mathbf{F}(u_1) - \mathbf{F}(u_2), v)\, dt = 0.$$

It follows that

$$\begin{aligned}(\mathbf{R}(s)u(s), u(s)) + \langle \mathbf{A}_0(0)v(0), v(0)\rangle &= \int_0^s \frac{d}{dt}\frac{1}{2}\Big((\mathbf{R}u, u) - \langle \mathbf{A}_0 v, v\rangle\Big)\, dt \\ &= \int_0^s -(\mathbf{R}'u, u) - \langle \mathbf{A}_0' v, v\rangle + (\mathbf{A}_1 u, v) \\ &\qquad - (\mathbf{F}(u_1) - \mathbf{F}(u_2), v)\, dt.\end{aligned}$$

The left hand side can then be estimated from below:

$$\|u(s)\|_{\mathrm{H}}^2 + \|v(0)\|_V^2 \leq \frac{1}{\alpha}\left((\mathbf{R}(s)u(s), u(s)) + \langle \mathbf{A}_0(0)v(0), v(0)\rangle\right). \tag{1.83}$$

For the right hand side, we first deal with the nonlinear part. Condition (1.82) and the mean-value theorem yield the inequality

$$|\mathbf{F}(z_1) - \mathbf{F}(z_2)| \leq C \max\{|z_1|^p, |z_2|^p\} |z_1 - z_2|.$$

Therefore, the extended Hőlder inequality applied for $\frac{1}{q} + \frac{1}{d} + \frac{1}{2} = 1$ gives

$$\begin{aligned}
\int_0^s (\mathbf{F}(u_1) - \mathbf{F}(u_2), v) dt &\leq C \int_0^s (\max\{|u_1|^p, |u_2|^p\} |u_1 - u_2|, |v|) dt \\
&\leq C \int_0^s \big((|u_1|^p + |u_2|^p) |u_1 - u_2|, |v| \big) dt \\
&\leq C \int_0^s \| |u_1(t)|^p + |u_2(t)|^p \|_{\mathrm{L}^d(\Omega)} \|u(t)\|_{\mathrm{L}^q(\Omega)} \|v(t)\|_H \, dt.
\end{aligned}$$

Since $q > 2$, we have $\|u(t)\|_{\mathrm{L}^q(\Omega)} \leq \|u(t)\|_H$. Next, since $dp \leq q = \frac{2d}{d-2}$, the Sobolev embedding theorem gives

$$\begin{aligned}
\| |u_1(t)|^p + |u_2(t)|^p \|_{\mathrm{L}^d(\Omega)} &\leq \| |u_1(t)|^p \|_{\mathrm{L}^d(\Omega)} + \| |u_2(t)|^p \|_{\mathrm{L}^d(\Omega)} \\
&\leq C \big(\|u_1(t)\|_V^p + \|u_2(t)\|_V^p \big) \\
&\leq C \Big(\|u_1\|_{\mathrm{L}^\infty(0,T;V)}^p + \|u_2\|_{\mathrm{L}^\infty(0,T;V)}^p \Big).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\int_0^s \| |u_1(t)|^p + |u_2(t)|^p \|_{\mathrm{L}^d(\Omega)} \|u(t)\|_{\mathrm{L}^q(\Omega)} \|v(t)\|_H \, dt \\
&\quad \leq C \Big(\|u_1\|_{\mathrm{L}^\infty(0,T;V)}^p + \|u_2\|_{\mathrm{L}^\infty(0,T;V)}^p \Big) \int_0^s \|u(t)\|_H \|v(t)\|_H \, dt \\
&\quad \leq C \Big(\|u_1\|_{\mathrm{L}^\infty(0,T;V)}^p + \|u_2\|_{\mathrm{L}^\infty(0,T;V)}^p \Big) \int_0^s \|u(t)\|_{\mathrm{H}}^2 + \|v(t)\|_V^2 \, dt.
\end{aligned}$$

Recalling (1.81), we further estimate

$$\begin{aligned}
&\int_0^s - (\mathbf{R}'u, u) - \langle \mathbf{A}_0' v, v \rangle + (\mathbf{A}_1 u, v) \\
&\quad \leq \int_0^s \|\mathbf{R}'(t)\|_{\mathscr{L}(H)} \|u\|_{\mathrm{H}}^2 + \|\mathbf{A}_0'(t)\|_{\mathscr{L}(V,V')} \|v(t)\|_V^2 + \|\mathbf{A}_1(t)\|_{\mathscr{L}(V,H)} \|u\|_H \|v\|_V \\
&\quad \leq \int_0^s \Big(\|\mathbf{R}'(t)\|_{\mathscr{L}(H)} + \|\mathbf{A}_0(t)\|_{\mathscr{L}(V,V')} + \|\mathbf{A}_1(t)\|_{\mathscr{L}(V,H)} \Big) \Big(\|u(t)\|_{\mathrm{H}}^2 + \|v(t)\|_V^2 \Big)
\end{aligned}$$

Combining it all together yields

$$\|u(s)\|_{\mathrm{H}}^2 + \|v(0)\|_V^2 \leq C \int_0^s \beta(t) \left(\|u(t)\|_{\mathrm{H}}^2 + \|v(t)\|_V^2 \right) dt, \tag{1.84}$$

where β denotes the (positive) function

$$\beta(t) = \|\mathbf{R}'(t)\|_{\mathscr{L}(H)} + \|\mathbf{A}_0(t)\|_{\mathscr{L}(V,V')} + \|\mathbf{A}_1(t)\|_{\mathscr{L}(V,H)} + \mathrm{Lip}(F).$$

Note that we have $\beta \in \mathrm{L}^1(0,T)$.

For $0 \le t \le T$ define

$$w(t) := \int_0^t u(\tau)d\tau\ .$$

The inequality in (1.84) then becomes

$$\|u(s)\|_{\mathrm{H}}^2 + \|w(s)\|_V^2 \le C\int_0^s \beta(t)\left(\|u(t)\|_{\mathrm{H}}^2 + \|w(s)-w(t)\|_V^2\right)dt. \tag{1.85}$$

However

$$\|w(t)-w(s)\|_V^2 \le 2(\|w(t)\|_V^2 + \|w(s)\|_V^2),$$

so (1.85) becomes

$$\|u(s)\|_{\mathrm{H}}^2 + (1-2C\|\beta\|_{\mathrm{L}^1(0,s)})\|w(s)\|_V^2 \le C\Big(\int_0^s \beta(t)\left(\|u(t)\|_{\mathrm{H}}^2 + \|w(t)\|_V^2\right)dt\Big). \tag{1.86}$$

Choose T_1 such that $(1-2C\|\beta\|_{\mathrm{L}^1(0,T_1)}) \ge \frac{1}{2}$. Then for each $0 \le s \le T_1$ we have

$$\|u(s)\|_{\mathrm{H}}^2 + \|w(s)\|_V^2 \le C\Big(\int_0^s \beta(t)\left(\|u(t)\|_{\mathrm{H}}^2 + \|w(t)\|_V^2\right)dt\Big). \tag{1.87}$$

Applying Gronwall's inequality we get $u \equiv 0$ on $[0,T_1]$. By using the same argument on intervals $[T_1,2T_1],[2T_1,3T_1]$, etc. we eventually get $u \equiv 0$ on $[0,T]$. ■

1.4. DISCONTINUOUS COEFFICIENTS WITH LIPSCHITZ NONLINEARITY

1.4.1. Functions of bounded variation

We are interested in such functions defined on a segment $I \subseteq \mathbf{R}$ (i.e. a closed convex subset of $\mathbf{R}$), taking values in a Banach space X. As for our purposes this segment will be the domain of time variable t in the wave equation, we shall, more often than not, take $I = [0,T]$, for a $T > 0$. At this point, we will introduce the notion of functions of bounded variation and state some of the main properties which will be of use in this section. For more details, one can refer to [22] or [41].

Following [52], we say that a (pointwisely defined) function $f : [0,T] \to X$ is of *bounded pointwise variation* (BPV) if ($0 = t_0 < t_1 < \cdots < t_n = T$ being a partition of $[0,T]$)

$$\sup_{0=t_0<t_1<\cdots<t_n=T} \sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|_X < \infty. \tag{1.88}$$

This quantity will be denoted by $\mathrm{Var}(f,[0,T])$, while the vector space of all such functions we denote by $\mathrm{BPV}([0,T];X)$.

Of course, it is possible to extend the definition to a semi-closed or an open interval I, possibly unbounded. For example, for $I = (a,b)$, one can consider the quantity

$$\sup_{a<t_0<t_1<\cdots<t_n<b} \sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|_X$$

instead. In that case, we would denote it by $\mathrm{Var}(f,(a,b))$, and the corresponding space would be denoted by $\mathrm{BPV}((0,T);X)$.

The above definition can immediately be extended to a metric space (X,d), by replacing the norm by a metric. This might be useful when considering different topologies on the Banach space X.

Let us recall some immediate results on functions of bounded pointwise variation (valued in metric spaces).

Proposition 1.4.1. [52, 2.17] Any $f \in \mathrm{BPV}((0,T);X)$ with values in a complete metric space X is continuous up to a countable subset of $(0,T)$, and f has both left and right sided limits

$$f(t^-) := \lim_{s \to t^-} f(s), \qquad f(t^+) := \lim_{s \to t^+} f(s)$$

at any $t \in (0,T)$, and one-sided limits at the end points.

As a consequence of the previous proposition, we may always assume $f \in \mathrm{BPV}((0,T);X)$ is defined on the whole of $[0,T]$ by extending it to the end points via $f(0) := f(0^+)$ and $f(T) := f(T^-)$. Of course, it holds

$$\mathrm{Var}(f;(0,T)) = \mathrm{Var}(f;[0,T]).$$

Proposition 1.4.2. [52, 2.12] Any function of bounded pointwise variation is bounded; in fact $d(f(t), f(t_0)) \leq \mathrm{Var}(f,[0,T])$. For a normed space X this gives the estimate

$$\|f(t)\|_X \leq \|f(t_0)\|_X + \mathrm{Var}(f,[0,T]).$$

Proposition 1.4.3. For any $f \in \mathrm{BPV}([0,T];X)$ we can define its *indefinite pointwise variation* $\varphi(t) := \mathrm{Var}(f,[0,t])$. Then φ is non-decreasing, continuous in the same points of $(0,T)$ as f and defines a unique Stieltjes measure on $[0,T]$ such that

$$\mu([a,b)) = \varphi(b) - \varphi(a) = \mathrm{Var}(f,[a,b)), \quad a,b \in [0,T].$$

The vector space $\mathrm{BPV}(0,T;X)$ (here we have to assume that X is a Banach space) is usually equipped with the norm

$$\|f\|_{\mathrm{BPV}(0,T;X)} := \|f(0)\|_X + \mathrm{Var}(f,[0,T]),$$

and then becomes a Banach space [52, 2.42].

However, this space is not separable (even for real functions, cf. [1]), but with a different norm the same vector space can be made separable.

The pointwise definition above also has an important drawback: one cannot define weak derivatives of such functions. In order to overcome that, we define the space of *functions of bounded variation* as a subspace of $\mathrm{L}^1(0,T;X)$ (cf. [41]). The definition of the latter space is clear for a Banach space X; however, if (X,d) is only a metric space, it consists of equivalence classes of measurable functions in $\mathscr{L}^0(0,T;X)$.

First, we define the semimetric (possibly taking value $+\infty$) on $\mathscr{L}^0(0,T;X)$ by

$$\rho(f,g) := \int_0^T d(f(t),g(t))dt\ .$$

In order to make it a metric, we identify the functions having zero distance one from another, and denote this equivalence relation by $\sim$. We also choose a particular (constant) $x_0 \in X$, and identify it with a constant function on $[0,T]$ taking that value. Now we can define the metric space

$$\mathrm{L}^1(0,T;X) := \left\{ f \in \mathscr{L}^0(0,T;X) : \rho(x_0, f) < \infty \right\} / \sim ,$$

which is complete if X is such.

On that space we can define the *essential pointwise variation* by

$$\mathrm{epvar} f := \inf_{g \sim f} \mathrm{Var} g ,$$

and by $\mathrm{BV}(0,T;X)$ we denote the space of all (classes of) L^1 functions having essential pointwise variation finite. The infimum above is actually a minimum: one can take representative of f defined by either $g(t) = f(t^-)$ or $g(t) = f(t^+)$ ([41] Propositions 2.2 and 2.3). It is worth noting here that the left and right limits are independent of the choice of the representative.

As a consequence, we will further assume that for $f \in \mathrm{BV}(0,T;X)$ we are taking its representative $f(\cdot^-)$ which is left continuous in $(0,T]$ and right continuous in 0 (due to our previous correction of replacing $f(0)$ with $f(0^+)$).

If X is a Banach space, $\mathrm{BV}(0,T;X)$ is a vector space, and equipped with the norm

$$\|f\|_{\mathrm{BV}(0,T;X)} := \|f\|_{\mathrm{L}^1(0,T;X)} + \mathrm{epvar} f$$

it is a Banach space.

Equivalently, the elements of the space $\mathrm{BV}(0,T;X)$ can also be defined as the functions $f \in \mathrm{L}^1_{\mathrm{loc}}(0,T;X)$ such that

(1) $\sup_{h>0} \int_0^{T-h} \left\| \frac{f(t+h)-f(t)}{h} \right\|_X dt < \infty$,

(2) $\sup \left\{ \int_a^b \langle \psi', f \rangle dt \,:\, \psi \in \mathscr{D}((0,T);X'), \quad \|\psi\|_{\mathrm{L}^\infty(0,T;X')} \le 1 \right\} < \infty$.

In case of (1) and (2) the supremums coincide with $\mathrm{Var}(f;[0,T])$. Note that the space $\mathrm{W}^{1,1}(0,T;X)$ is included in $\mathrm{BV}(0,T;X)$ and it holds for each $f \in \mathrm{W}^{1,1}(0,T;X)$

$$\mathrm{Var}(f;[0,T]) = \int_a^b \|f'(t)\|_X dt.$$

1.4.2. Coefficients of bounded variation

We now reduce our assumptions on the coefficients of the equation to the following. Assume that $\mathbf{R} \in \mathrm{BV}(0,T;\mathscr{L}(H))$ satisfies

$$(\mathbf{R}(t)u,v) = (\mathbf{R}(t)v,u) \qquad u,v \in H, \text{ a.e. } t \in (0,T) \tag{1.89}$$

$$(\mathbf{R}(t)u,u) \geq \alpha \|u\|_{\mathrm{H}}^2 \qquad u \in H, \text{ a.e. } t \in (0,T)\ . \tag{1.90}$$

and $\mathbf{A} \in \mathrm{BV}(0,T;\mathscr{L}(V,V'))$ satisfies

$$\langle \mathbf{A}(t)u,v\rangle = \langle \mathbf{A}(t)v,u\rangle \qquad u,v \in V, \text{ a.e. } t \in (0,T) \tag{1.91}$$

$$\langle \mathbf{A}(t)u,u\rangle \geq \alpha \|u\|_V^2 \qquad u \in V, \text{ a.e. } t \in (0,T). \tag{1.92}$$

For f and g we now assume

$$f \in \mathrm{L}^1(0,T;H), \qquad g \in \mathrm{BV}(0,T;V'). \tag{1.93}$$

As remarked earlier, we can consider the corresponding representatives of $\mathbf{R},\mathbf{A}$ and g which are left-continuous and extended to endpoints via corresponding one-sided limits. Taking this consideration into account, we add the additional assumption of $\mathbf{A}(0) = \mathbf{A}(0^+)$ being an isomorphism of V and V'. Let us state our main existence result.

Theorem 1.4.4. Let $\mathbf{R},\mathbf{A},f$ and g be as stated before the theorem. Let $\mathbf{F} \in \mathrm{W}^{1,\infty}(\mathbb{R})$ satisfy sign condition and $u^0 \in V$, $u^1 \in H$. Then there exists unique $u \in \mathrm{L}^\infty(0,T;V)$, with $u' \in \mathrm{L}^\infty(0,T;H)$ satisfying, in the sense of $\mathrm{L}^1(0,T;V')$, the equation:

$$(\mathbf{R}u')' + \mathbf{A}u + \mathbf{F}(u) = f + g, \tag{1.94}$$

with initial conditions

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1.$$

The first step of the proof consists of approximating $\mathbf{R},\mathbf{A},f$ and g with respective sequences belonging to spaces of higher regularity in time.

In order to do that, let us extend $\mathbf{R},\mathbf{A}$ and f to the whole of $\mathbb{R}$ by defining

$$\tilde{\mathbf{R}}(t) = \begin{cases} \mathbf{R}(0^+), & t < 0 \\ \mathbf{R}(t), & 0 \leq t \leq T\ , \\ \mathbf{R}(T^-), & t > T \end{cases} \qquad \tilde{\mathbf{A}}(t) = \begin{cases} \mathbf{A}(0^+), & t < 0 \\ \mathbf{A}(t), & 0 \leq t \leq T\ , \\ \mathbf{A}(T^-), & t > T \end{cases}$$

$$\tilde{f}(t) = \begin{cases} f(t), & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}, \qquad \tilde{g}(t) = \begin{cases} g(0^+), & t < 0 \\ g(t), & 0 \le t \le T \\ g(T^-), & t > T. \end{cases}$$

Obviously, $\tilde{\mathbf{R}} \in \mathrm{BV}(\mathbb{R};\mathscr{L}(H))$, $\tilde{\mathbf{A}} \in \mathrm{BV}(\mathbb{R};\mathscr{L}(V,V'))$, $\tilde{f} \in \mathrm{L}^1(\mathbb{R};H)$ and $\tilde{g} \in \mathrm{BV}(\mathbb{R};V')$.

Next we define the standard molifier sequence: let $\psi \in \mathscr{D}(\mathbb{R})$ be such that

$$\psi \ge 0, \qquad \operatorname{supp}\psi \subseteq [0,1], \qquad \int_{\mathbb{R}} \psi = 1,$$

and put $\psi_n(t) = n\psi(nt)$. Define sequences $\mathbf{R}_n \in C^\infty(\mathbb{R};\mathscr{L}(H)), \mathbf{A}_n \in C^\infty(\mathbb{R};\mathscr{L}(V,V'))$, $f_n \in C^\infty(\mathbb{R};H)$ and $g_n \in C^\infty(\mathbb{R};V')$ by

$$\mathbf{R}_n := \tilde{\mathbf{R}} * \psi_n, \qquad \mathbf{A}_n := \tilde{\mathbf{A}} * \psi_n, \qquad f_n = \tilde{f} * \psi_n, \qquad g_n = \tilde{g} * \psi_n.$$

The approximating sequence $\mathbf{R}_n$ now satisfies the following properties:

$$\begin{cases} (\mathbf{R}_n(t)u,v) = (\mathbf{R}_n(t)v,u) & u,v \in H, t \ge 0 \\ (\mathbf{R}_n(t)u,u) \ge \alpha\|u\|_H & u \in H, t \ge 0 \\ \mathbf{R}_n(0) = \mathbf{R}(0^+), \quad \mathbf{R}_n'(0) = 0 \\ \|\mathbf{R}_n(t) - \mathbf{R}(t)\|_{\mathscr{L}(H)} \to 0 & t > 0 \\ \mathbf{R}_n \to \mathbf{R} \text{ in } \mathrm{L}^1(0,T;\mathscr{L}(H)) \\ \|\mathbf{R}_n'\|_{\mathrm{L}^1(0,t;\mathscr{L}(H))} \le \mathrm{Var}(\mathbf{R};(0,t)) & t > 0. \end{cases} \tag{1.95}$$

The first property follows from the fact that $\tilde{\mathbf{R}}(t)$ is symmetric for each $t \in \mathbb{R}$ by definition and the identity

$$(\mathbf{R}_n(t)u,v) = \left(\left(\int_{\mathbb{R}} \tilde{\mathbf{R}}(t-s)\rho_n(s)ds\right)u,v\right) = \int_{\mathbb{R}} (\tilde{\mathbf{R}}(t-s)u,v)\rho_n(s)ds.$$

The second property follows from the identity above with $v = u$ together with the coercivity property of $\tilde{\mathbf{R}}$:

$$(\mathbf{R}_n(t)u,u) = \int_{\mathbb{R}} (\tilde{\mathbf{R}}(t-s)u,u)\rho_n(s)ds \ge \alpha\|u\|_{\mathrm{H}}^2 \int_{\mathbb{R}} \rho_n(s)ds = \alpha\|u\|_{\mathrm{H}}^2.$$

It is important to note that the coercivity coefficient remained the same for each $n \in \mathbb{N}$. The following two properties are obtained easily from the fact that mollification is performed from the left. Finally, in order to prove the last property we have the following inequalities

$$\begin{aligned}\|\mathbf{R}_n'\|_{\mathrm{L}^1(0,t;\mathscr{L}(H)} &= \mathrm{Var}(\mathbf{R}_n;(0,t)) \leq \mathrm{Var}(\mathbf{R}_n;(-\infty,t)) \\ &\leq \mathrm{Var}(\tilde{\mathbf{R}};(-\infty,t)) = \mathrm{Var}(\mathbf{R};(0,t)).\end{aligned}$$

The only nontrivial inequality is

$$\mathrm{Var}(\mathbf{R}_n;(-\infty,t)) \leq \mathrm{Var}(\tilde{\mathbf{R}};(-\infty,t)).$$

In order to prove it, take $-\infty < t_0 < t_1 < \cdots < t_k < t$. Then

$$\begin{aligned}\sum_{j=1}^{k} \|\mathbf{R}_n(t_j) - \mathbf{R}_n(t_{j-1})\|_{\mathscr{L}(H)} &= \sum_{j=1}^{k} \left\| \int_{\mathbb{R}} (\tilde{\mathbf{R}}(t_j - s) - \tilde{\mathbf{R}}(t_{j-1} - s))\rho_n(s)ds \right\|_{\mathscr{L}(H)} \\ &\leq \int_{\mathbb{R}} \sum_{j=1}^{k} \|\tilde{\mathbf{R}}(t_j - s) - \tilde{\mathbf{R}}(t_{j-1} - s)\|_{\mathscr{L}(H)}\rho_n(s)ds \\ &\leq \mathrm{Var}(\tilde{\mathbf{R}};(-\infty,t)).\end{aligned}$$

In a similar fashion we can deduce that the following holds for the approximating sequence $\mathbf{A}_n$:

$$\begin{cases} \langle \mathbf{A}_n(t)u, v\rangle = \langle u, \mathbf{A}_n(t)v\rangle \qquad u,v \in V, t \geq 0 \\ \langle \mathbf{A}_n(t)u, u\rangle \geq \alpha\|u\|_V, \qquad u \in V, t \geq 0 \\ \mathbf{A}_n(0) = \mathbf{A}(0^+), \quad \mathbf{A}_n'(0) = 0 \\ \|\mathbf{A}_n(t) - \mathbf{A}(t)\|_{\mathscr{L}(V,V')} \to 0 \qquad t > 0 \\ \mathbf{A}_n \to \mathbf{A} \text{ in } \mathrm{L}^1(0,T;\mathscr{L}(V,V')) \\ \|\mathbf{A}_n'\|_{\mathrm{L}^1(0,t;\mathscr{L}(V,V'))} \leq \mathrm{Var}(\mathbf{A};(0,t)), \quad t > 0, \end{cases} \tag{1.96}$$

and for g_n:

$$\begin{cases} g_n(0) = g(0^+), \quad g_n'(0) = 0 \\ \|g_n(t) - g(t)\|_{V'} \to 0 \qquad t > 0 \\ g_n \to g \text{ in } \mathrm{L}^1(0,T;V') \\ \|g_n'\|_{\mathrm{L}^1(0,T;V')} \leq \mathrm{Var}(g;(0,t)), \quad t > 0. \end{cases} \tag{1.97}$$

Let us now return to the construction of the solution of (1.94) with given initial conditions. According to Theorem 1.2.1, there exists the unique solution $u_n \in \mathrm{W}^{1,\infty}(0,T;V)$, with $u_n'' \in \mathrm{L}^2(0,T;H)$ such that

$$\begin{cases} (\mathbf{R}_n u_n')' + \mathbf{A}_n u_n + \mathbf{F}(u_n) = f_n + g_n, \\ u_n(0) = u_n^0, \\ u_n'(0) = u_n^1, \end{cases} \tag{1.98}$$

where the initial data $u_n^0, u_n^1 \in V$ is chosen in such manner a that it holds

$$u_n^0 \xrightarrow{V} u^0, \qquad u_n^1 \xrightarrow{H} u^1, \qquad \mathbf{A}(0^+)u_n^0 - g(0^+) \in H. \tag{1.99}$$

If we denote

$$E_n(t) := \frac{1}{2}(\mathbf{R}_n u_n', u_n') + \frac{1}{2}\langle \mathbf{A}_n u_n, u_n \rangle,$$

we can multiply the equation in 1.98 by u_n' (in the sense of duality of $\mathrm{L}^2(0,T;V)$ and $\mathrm{L}^2(0,T;V')$), and in the same manner as in the Subsection 1.2.2 obtain the inequality

$$\begin{aligned} E_n(t) \leq \frac{1}{\alpha/2} \int_0^t \phi_n(s) E_n(s) + E_n(0) + \|f_n\|_{\mathrm{L}^1(H)} + \|g_n'\|_{\mathrm{L}^1(V')} \\ + \|g_n(t)\|_{V'} \|u_n(t)\|_V + \|g_n(0)\|_{V'} \|u_n(0)\|_V \end{aligned} \tag{1.100}$$

where

$$\phi_n(t) = \|\mathbf{R}_n(t)\|_{\mathscr{L}(H)} + \|\mathbf{A}_n(t)\|_{\mathscr{L}(V,V')} + \|f_n(t)\|_H + \mathrm{Lip}(F)$$

is in $\mathrm{L}^1(0,T)$ satisfying

$$\int_0^t \phi_n(t) \leq \|\mathbf{R}\|_{\mathrm{L}^1(H))} + \|\mathbf{A}\|_{\mathrm{L}^1(\mathscr{L}(V,V'))} + \|f\|_{\mathrm{L}^1(H)} + \mathrm{Lip}(F).$$

Note that for each $0 \leq t \leq T$ we have

$$g_n(t) = g_n(0) + \int_0^t g_n'(s)ds = g(0^+) + \int_0^t g_n'(s)ds, \tag{1.101}$$

from where it follows

$$\|g_n(t)\|_{V'} \leq \|g(0^+)\|_{V'} + \mathrm{Var}(g;(0,T)), \tag{1.102}$$

and subsequently

$$\|g_n(t)\|_{V'} \|u_n(t)\|_V \leq 2C\left(\|g(0^+)\|_{V'}^2 + \mathrm{Var}(g;(0,T))^2\right) + \frac{1}{2}E_n(t).$$

Hence, (1.100) becomes

$$E_n(t) \leq \frac{2}{\alpha}\int_0^t \phi_n(s)E_n(s) + E_n(0) + \|f\|_{\mathrm{L}^1(H)} + \mathrm{Var}(g;(0,T)) \\ + C\left(\|g(0+)\|_{V'}^2 + \mathrm{Var}(g;(0,T))^2\right) + 2E_n(0). \tag{1.103}$$

For the term $E_n(0)$ we estimate

$$(\mathbf{R}(0)u_n'(0), u_n'(0)) = (\mathbf{R}(0^+)u_n^1, u_n^1) \leq \|\mathbf{R}(0^+)\|_{\mathscr{L}(H)}\|u_n^1\|_{\mathrm{H}}^2,$$

and similarly

$$\langle \mathbf{A}_n(0)u_n(0), u_n(0)\rangle \leq \|\mathbf{A}(0^+)\|_{\mathscr{L}(V,V')}\|u_n^0\|_V^2,$$

which are both bounded sequences due to (1.99). Hence, an application of the Gronwall's lemma allows us to deduce

$$u_n \text{ lies in a bounded set of } \mathrm{L}^\infty(0,T;V)$$
$$u_n' \text{ lies in a bounded set of } \mathrm{L}^\infty(0,T;H).$$

Hence, there exists $u \in \mathrm{L}^\infty(0,T;V)$ with $u' \in \mathrm{L}^\infty(0,T;H)$ such that

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } \mathrm{L}^\infty(0,T;V)$$
$$u_n' \overset{*}{\rightharpoonup} u' \quad \text{in } \mathrm{L}^\infty(0,T;H).$$

Recalling (1.95) we deduce

$$\mathbf{R}_n u_n' \rightharpoonup \mathbf{R}u' \quad \text{in } \mathrm{L}^1(0,T;H). \tag{1.104}$$

Indeed, for $v \in \mathrm{L}^\infty(0,T;H)$ we have using the dual product of $\mathrm{L}^1(0,T;H)$ and $\mathrm{L}^\infty(0,T;H)$

$$\left|\int_0^T \left(\mathbf{R}_n u_n' - \mathbf{R}u', v\right)\right| \leq \int_0^T \left|\left((\mathbf{R}_n - \mathbf{R})u_n', v\right)\right| + \int_0^T \left|\left(\mathbf{R}(u_n' - u'), v\right)\right| \\ \leq \|\mathbf{R}_n - \mathbf{R}\|_{\mathrm{L}^1(\mathscr{L}(H))}\|u_n\|_{\mathrm{L}^\infty(H)}\|v\|_{\mathrm{L}^\infty(H)} \\ + \|\mathbf{R}v\|_{\mathrm{L}^1(H)}\|u_n' - u'\|_{\mathrm{L}^\infty(H)}.$$

Since u_n' is a weakly convergent sequence in $\mathrm{L}^\infty(0,T;H)$, it is bounded, so the first term goes to 0. The second term goes to zero because of $\mathbf{R}v \in \mathrm{L}^1(0,T;H)$ and the weak-$*$ convergence of $u_n' - u'$ to 0 in $\mathrm{L}^\infty(0,T;H)$. Similar arguments show that

$$\mathbf{A}_n u_n \rightharpoonup \mathbf{A}u \quad \text{in } \mathrm{L}^1(0,T;V'). \tag{1.105}$$

Finally, we have

$$\|\mathbf{F}(u_n)-\mathbf{F}(u)\|_{\mathrm{L}^1(H)} \leq \mathrm{Lip}(F)\|u_n-u\|_{\mathrm{L}^1(H)} \to 0,$$

so we deduce that

$$\mathbf{F}(u_n) \to \mathbf{F}(u) \quad \text{ in } \mathrm{L}^1(0,T;H). \tag{1.106}$$

Using convergences (1.104), (1.105), (1.106), as well as $f_n \xrightarrow{\mathrm{L}^1(H)} f$ and $g_n \xrightarrow{\mathrm{L}^1(V')} g$, we deduce

$$(\mathbf{R}u')' + \mathbf{A}u + \mathbf{F}(u) = f+g \qquad \text{in } \mathscr{D}'(0,T;V'),$$

and therefore

$$(\mathbf{R}u')' + \mathbf{A}u + \mathbf{F}(u) = f+g \qquad \text{in } \mathrm{L}^1(0,T;V'). \tag{1.107}$$

Initial conditions can then be checked in the same manner as in the proof of Theorem 1.2.1.

Finally, let us prove the uniqueness of the solution. Assume $u_1, u_2 \in \mathrm{L}^\infty(0,T;V)$ are two solutions of the problem 1.94, and denote $v := u_1 - u_2$. To begin with, we fix $0 < s < T$ and define

$$v(t) := \begin{cases} -\int_s^t u(\tau)d\tau, & 0 \leq t \leq s, \\ 0, & s < t \leq T. \end{cases}$$

Note that $v \in \mathrm{W}^{1,\infty}(0,T;V)$, and it satisfies $v(s) = u'(0) = 0$. Hence, using the fact that u_1 and u_2 are both solutions of (1.94), we obtain

$$\int_0^s -(\mathbf{R}u',v') + \langle \mathbf{A}u, v\rangle + (\mathbf{F}(u_1)-\mathbf{F}(u_2),v)\,dt = 0.$$

As $v' = -u$ for $0 \leq t \leq s$, we can write

$$\int_0^s (\mathbf{R}u',u) - \langle \mathbf{A}v', v\rangle + (\mathbf{F}(u_2)-\mathbf{F}(u_1),v)\,dt = 0.$$

Rewriting the previous equation as

$$\int_0^s (\mathbf{R}u',u) = \int_0^s ((\mathbf{R}-\mathbf{R}_n)u',u) + \int_0^s (\mathbf{R}_n u',u), \tag{1.108}$$

we can now use the identity

$$\frac{d}{dt}(\mathbf{R}_n u,u) = 2(\mathbf{R}_n u',u) + (\mathbf{R}_n' u,u)$$

in order to obtain

$$\int_0^s (\mathbf{R}u',u) = \int_0^s ((\mathbf{R}-\mathbf{R}_n)u',u) - \frac{1}{2}(\mathbf{R}_n(s)u(s),u(s)) + \frac{1}{2}\int_0^s (\mathbf{R}_n'u,u) \tag{1.109}$$

Analogously we can deduce that

$$\int_0^s \langle \mathbf{A}v',v\rangle = \int_0^s \langle (\mathbf{A}-\mathbf{A}_n)v',v\rangle + \frac{1}{2}\langle \mathbf{A}_n(0)u(0),u(0)\rangle + \frac{1}{2}\int_0^s \langle \mathbf{A}_n'v,v\rangle. \tag{1.110}$$

From the above it follows that

$$\begin{aligned}(\mathbf{R}_n(s)u(s),u(s)) + \langle \mathbf{A}_n(0)v(0),v(0)\rangle = &\int_0^s ((\mathbf{R}-\mathbf{R}_n)u',u) - \langle (\mathbf{A}-\mathbf{A}_n)v',v\rangle \\ &+ \int_0^s (\mathbf{R}_n'u,u) - \langle \mathbf{A}_n'v,v\rangle \\ &+ \int_0^s (\mathbf{F}(u_2)-\mathbf{F}(u_1),v).\end{aligned} \tag{1.111}$$

The left hand side can then be estimated from below:

$$\|u(s)\|_{\mathrm{H}}^2 + \|v(0)\|_V^2 \le \frac{1}{\alpha}\left((\mathbf{R}_n(s)u(s),u(s)) + \langle \mathbf{A}_n(0)v(0),v(0)\rangle\right). \tag{1.112}$$

To estimate the right hand side, we first obtain by using the Lipschitz property (1.12)

$$(\mathbf{F}(u_1)-\mathbf{F}(u_2),v) \le \mathrm{Lip}(\mathbf{F})\|u_1(t)-u_2(t)\|_H\|v(t)\|_H \le \mathrm{Lip}(\mathbf{F})\left(\|u(t)\|_{\mathrm{H}}^2 + \|v(t)\|_V^2\right)$$

Next we have obvious estimates

$$\begin{aligned}\int_0^s (\mathbf{R}_n'u,u)dt &\le \int_0^s \|\mathbf{R}_n'(t)\|_{\mathscr{L}(H)}\|u(t)\|_{\mathrm{H}}^2 dt \\ \int_0^s \langle \mathbf{A}_n'v,v\rangle dt &\le \int_0^s \|\mathbf{A}_n'(t)\|_{\mathscr{L}(V,V')}\|v(t)\|_V^2 dt \\ \int_0^s ((\mathbf{R}-\mathbf{R}_n)u',u)dt &\le \|u\|_{\mathrm{L}^\infty(H)}\|u'\|_{\mathrm{L}^\infty(H)}\int_0^s \|(\mathbf{R}-\mathbf{R}_n)(t)\|_{\mathscr{L}(H)}dt \\ \int_0^s \langle (\mathbf{A}-\mathbf{A}_n)v',v\rangle dt &\le \|v\|_{\mathrm{L}^\infty(V)}\|v'\|_{\mathrm{L}^\infty(V)}\int_0^s \|(\mathbf{A}-\mathbf{A}_n)(t)\|_{\mathscr{L}(V,V')}dt\end{aligned} \tag{1.113}$$

Hence we obtain

$$\begin{aligned}\|u(s)\|_{\mathrm{H}}^2 + \|v(0)\|_V^2 \le C&\int_0^s \beta_n(t)\left(\|u(t)\|_{\mathrm{H}}^2 + \|v(t)\|_V^2\right)dt \\ &+ \|u\|_{\mathrm{L}^\infty(H)}\|u'\|_{\mathrm{L}^\infty(H)}\int_0^s \|(\mathbf{R}-\mathbf{R}_n)(t)\|_{\mathscr{L}(H)}dt \\ &+ \|v\|_{\mathrm{L}^\infty(V)}\|v'\|_{\mathrm{L}^\infty(V)}\int_0^s \|(\mathbf{A}-\mathbf{A}_n)(t)\|_{\mathscr{L}(V,V')}dt,\end{aligned} \tag{1.114}$$

where β_n denotes the (positive) function

$$\beta_n(t) = \|\mathbf{R}_n'(t)\|_{\mathscr{L}(H)} + \|\mathbf{A}_n'(t)\|_{\mathscr{L}(V,V')} + \mathrm{Lip}(F).$$

Recalling properties (1.95), (1.96) we note

$$\int_0^s \beta_n(t)dt \leq \mathrm{Var}(\mathbf{R};(0,s)) + \mathrm{Var}(\mathbf{A};(0,s)) + s\mathrm{Lip}(F) =: \beta(s).$$

Now let $n \to \infty$ in (1.114). The terms involving $\mathbf{R} - \mathbf{R}_n$ and $\mathbf{A} - \mathbf{A}_n$ disappear, and we deduce

$$\|u(s)\|_{\mathrm{H}}^2 + \|v(0)\|_V^2 \leq C\beta(s) \int_0^s \|u(t)\|_{\mathrm{H}}^2 + \|v(t)\|_V^2 \, dt.$$

For $0 \leq t \leq T$ define

$$w(t) := \int_0^t u(\tau)d\tau$$

The inequality in (1.84) then becomes

$$\|u(s)\|_{\mathrm{H}}^2 + \|w(s)\|_V^2 \leq C\beta(s) \int_0^s \|u(t)\|_{\mathrm{H}}^2 + \|w(s) - w(t)\|_V^2 \, dt. \tag{1.115}$$

However

$$\|w(t) - w(s)\|_V^2 \leq 2(\|w(t)\|_V^2 + \|w(s)\|_V^2),$$

so (1.115) becomes

$$\|u(s)\|_{\mathrm{H}}^2 + (1 - 2C\beta(s))\|w(s)\|_V^2 \leq C\Big(\int_0^s \|u(t)\|_{\mathrm{H}}^2 + \|w(t)\|_V^2 \, dt\Big). \tag{1.116}$$

Finally, choose T_1 small enough such that $(1 - 2C\beta(T_1)) \geq \frac{1}{2}$. Then for each $0 \leq s \leq T_1$ we have

$$\|u(s)\|_{\mathrm{H}}^2 + \|w(s)\|_V^2 \leq C\Big(\int_0^s \|u(t)\|_{\mathrm{H}}^2 + \|w(t)\|_V^2 \, dt\Big). \tag{1.117}$$

Applying Gronwall's inequality we get $u \equiv 0$ on $[0,T_1]$. By using the same argument on intervals $[T_1, 2T_1], [2T_1, 3T_1]$, etc. we eventually get $u \equiv 0$ on $[0,T]$.

2. H-MEASURES

H-measures, also called microlocal defect measures, were introduced in the late eighties independently by Tartar [71] and Gérard [36]. Since they first appeared in relation to some homogenisation problems, Tartar called them H-measures, while Gérard's scientific background prompted the name microlocal defect measures, since they are a microlocal refinement of (reduced) defect measures of DiPerna and Majda [27, 32] which encode information about the extent to which strong convergence fails. In either case, they were Radon measures designed to describe quadratic limits of weakly convergent L^2 sequences. If one has a weakly convergent L^2 sequence $u_n \rightharpoonup u$, it is natural to observe the bounded L^1 sequence $|u_n - u|^2$. After possibly passing onto a subsequence, one obtains Radon measure ν, a *defect measure* of u_n. However, this measure depends on the variable x only, and it is not well suited to describe any effect which depends on a particular direction in space, for example, for the study of propagation of oscillation effects. To demonstrate this deficiency, we can observe the following example.

Example 2.0.1. Take $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\|\varphi\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$, $\xi_0 \in \mathbb{R}^d \setminus \{0\}$ and a real sequence $\varepsilon_n \to 0^+$. Define sequence $u_n(x) = \varphi(x)e^{\frac{2\pi i x\cdot\xi_0}{\varepsilon_n}}$. Then $u_n \overset{\mathrm{L}^2}{\rightharpoonup} 0$, and it can be easily checked that the defect measure it defines is $\nu = \varphi^2\lambda$, where λ denotes the Lebesgue measure on $\mathbb{R}^d$. As we can see, the information about the direction of oscillations is lost, since the defect measure obtained in this way is the same for any choice of ξ_0.

The remainder of this chapter is split into presentation of these two variant descriptions of H-measures as well as making some comparison on the subject. At this point we can note that the main difference in corresponding approaches is based on the regularity assumptions in their framework. Accordingly, we will briefly introduce the preliminary requirements.

2.1. PRELIMINARIES

Before stating the existence result for H-measures, which will also serve as the defining theorem, we will briefly recall some of the basic facts regarding the *Fourier transform*. We adopt the following normalization: for $u \in \mathscr{S}(\mathbb{R}^d)$ (the Schwartz space) let

$$(\mathscr{F}u)(\xi) = \widehat{u}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} u(x)dx.$$

The Fourier transform is an isomorphism on $\mathscr{S}(\mathbb{R}^d)$ equipped with standard Fréchet topology [39, 42], and its inverse is given by the *inverse Fourier transform* given by

$$(\overline{\mathscr{F}}u)(x) = \check{u}(x) := \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} u(\xi)d\xi,$$

that is, for each $u \in \mathscr{S}(\mathbb{R}^d)$ the Fourier inversion formula holds

$$u(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \widehat{u}(\xi)d\xi.$$

Both operators $\mathscr{F}$ and $\overline{\mathscr{F}}$ can be extended to unitary operators on $\mathrm{L}^2(\mathbb{R}^d)$ by density. Additionally, they can also be extended by duality to operators acting on the space of tempered distributions $\mathscr{S}'(\mathbb{R}^d)$. More precisely, for $u \in \mathscr{S}'(\mathbb{R}^d)$ and $\varphi \in \mathscr{S}(\mathbb{R}^d)$ we define $\langle \hat{u}, \varphi\rangle := \langle u, \hat{\varphi}\rangle$, as well as $\langle \check{u}, \varphi\rangle := \langle u, \check{\varphi}\rangle$.

We will not go into details about the elementary properties of Fourier transform, proofs of which can be found in, for example [39, 42]. However, we shall emphasise the following property which is important in connection with the theory of partial differential equations, and that is the fact that the Fourier transform turns differentiation in the physical space into the multiplication by a polynomial in the phase space. Namely, we have

$$(\widehat{\partial^\alpha u})(\xi) = (2\pi i\xi)^\alpha \widehat{u}(\xi), \qquad u \in \mathscr{S}(\mathbb{R}^d), \alpha \in \mathbb{N}_0^d. \tag{2.1}$$

Together with the Fourier transform, we introduce two basic operators acting on functions in $\mathrm{L}^2(\mathbb{R}^d)$. If a is a function in ξ, and b is a function in x, we consider linear operators on functions defined in x:

$$
\begin{aligned}
A_a u(x) &:= \overline{\mathscr{F}}\left(a\left(\frac{\xi}{|\xi|}\right)\hat{u}(\xi)\right)\\
B_b u(x) &:= b(x)u(x).
\end{aligned}
$$

Operator A is called the (L^2) *Fourier multiplier* [39, 2.5] associated with symbol a, while the operator B is a simple *multiplication* by b (sometimes called the *Sobolev multiplier* [54]). If we assume additionally that both functions a and b are in $\mathrm{L}^\infty(\mathbb{R}^d)$, it immediately follows that operators A and B are bounded operators on $\mathrm{L}^2(\mathbb{R}^d)$. We will later deal with operators of a more general form when dealing with Gérard's approach to H-measures.

2.2. TARTAR'S APPROACH

An H-measure is a Radon measure on the cospherical bundle $\mathbb{S}^*\Omega$ over a domain Ω in consideration. If $\Omega \subseteq \mathbb{R}^d$ is an open set, then it is just a measure on the product $\Omega \times \mathbb{S}^{d-1}$. Since its definition requires the Fourier transform, which in turn requires concerned functions to be defined on the whole of $\mathbb{R}^d$, one can circumvent this issue by extending functions by zero outside of the domain. We will therefore state the following results in a manner that considers $\Omega = \mathbb{R}^d$. First in line is the existence theorem for H-measures introduced by Tartar [71, Theorem 1.1] (see also [80, Theorem 28.5]), which also serves as a definition.

Before stating the existence theorem, let us introduce some notation which will also be of use in the latter sections of this chapter.

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For two vectors $u = (u_i)_{i=1}^d, v = (v_j)_{j=1}^d \in \mathbb{F}^d$, we denote with $u \otimes v$ the *tensor product* of u and v given by

$$u \otimes v \in M_d(\mathbb{F}), \qquad (u \otimes v)_{ij} = u_i v_j, \quad 1 \leq i, j \leq d.$$

For two functions $f : \mathbb{F}^k \to \mathbb{F}$, $g : \mathbb{F}^l \to \mathbb{F}$, we denote with $f \boxtimes g$ the *tensor product* of f and g given by

$$f \boxtimes g : \mathbb{F}^{k+l} \to \mathbb{F}, \qquad (f \boxtimes g)(x, y) = f(x)g(y), \quad x \in \mathbb{F}^k, y \in \mathbb{F}^l.$$

With $C_0(\mathbb{R}^d)$ we denote the closure of $C_c^\infty(\mathbb{R}^d)$ in $C(\mathbb{R}^d)$ equipped with the standard sup-norm.

The unit sphere in the d-dimensional space will be denoted by

$$\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}.$$

The existence theorem for H-measures is now introduced.

Theorem 2.2.1. (**Existence of H-measures**) If (u_n) is a sequence in $\mathrm{L}^2(\mathbb{R}^d; \mathbb{R}^r)$, such that $u_n \xrightarrow{\mathrm{L}^2} 0$ (weakly), then there exists a subsequence $(u_{n'})$ and a complex matrix Radon measure μ on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ such that for all $\varphi_1, \varphi_2 \in C_0(\mathbb{R}^d)$ and $\psi \in C(\mathbb{S}^{d-1})$:

$$\lim_{n'} \int_{\mathbb{R}^d} \widehat{(\varphi_1 u_{n'})} \otimes \widehat{(\varphi_2 u_{n'})} \psi\left(\frac{\xi}{|\xi|}\right) d\xi = \langle \mu, (\varphi_1 \overline{\varphi_2}) \boxtimes \psi \rangle \tag{2.2}$$

$$= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \varphi_1(x) \overline{\varphi_2}(x) \psi(\xi) d\mu(x, \xi). \tag{2.3}$$

Measure μ is called the H-measure associated to the (sub)sequence $(u_{n'})$.

For simplicity, we will often denote the subsequence also by (u_n) and refer to μ as the H-measure associated to the sequence (u_n). Sequences for which the relation (2.2) is valid without passing to a subsequence are called *pure*.

Remark 2.2.2. The definition of H-measures can also be extended to sequences converging weakly to zero in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$. In that case, one has to change requirements for test functions to $\varphi_1, \varphi_2 \in C_c(\mathbb{R}^d)$. The resulting measure, however, is in general infinite (Example 2.2.10 below). More precisely, we distinguish between bounded Radon measures denoted by $\mathscr{M}_b(\mathbb{R}^d \times \mathbb{S}^{d-1}) = C_0(\mathbb{R}^d \times \mathbb{S}^{d-1})'$ and unbounded ones $\mathscr{M}(\mathbb{R}^d \times \mathbb{S}^{d-1}) = C_c(\mathbb{R}^d \times \mathbb{S}^{d-1})'$.

Remark 2.2.3. The above definition of H-measures can be extended in various directions. By changing the projection to the sphere one can get parabolic H-measures [11, 14, 78], adopted to equations with different order of derivatives in different variables (including ultraparabolic H-measures [61], and fractional H-measures [28,29]). Another extension is to one-scale H-measures [6,80], which at the same time extend the notion of semiclassical measures [36], also known as Wigner measures.

If L^2 is replaced by L^p and $\mathrm{L}^{p'}$, one does not get Radon measures, but distributions of higher, though still finite order, called H-distributions [7, 15], which are related [16] to microlocal compactness forms [63], a generalisation o Young measures. Further generalisations involve $\mathrm{L}^p/\mathrm{L}^{p'}$ setting combined with different scales, leading to one-scale H-distributions [5].

2.2.1. A class of symbols and associated operators

A simple application of the Plancherel theorem allows us to rewrite (2.2) in terms of the aforementioned elementary operators as

$$\lim_n \int_{\mathbb{R}^d} (A_\psi B_{\varphi_1} u_n) \otimes (B_{\varphi_2} u_n) dx = \langle \mu, \varphi_1 \overline{\varphi_2} \boxtimes \psi \rangle. \tag{2.4}$$

Note that the H-measure, as defined by (2.2), is a trilinear form in φ_1, φ_2 and ψ. However, the following commutation lemma [71, Lemma 1.7] (see also [17, 80]) allowed rewriting μ as a bilinear form which could, due to the Schwartz kernel theorem [7, 84], be represented as a distribution of order 0, for which it is easy to check that it is positive, thus making it a Radon measure. Tartar wanted to avoid using a sofisticated result as the Kernel theorem, hence he used a construction relying on Hilbert-Schmidt operators.

Lemma 2.2.4. (**First commutation lemma**) If $a \in C(\mathbb{S}^{d-1})$ and $b \in C_0(\mathbb{R}^d)$ then the above defined operators belong to $\mathscr{L}(\mathrm{L}^2(\mathbb{R}^d))$, and their norms coincide with supremum norms of a and b respectively. Moreover, the commutator $C := [A,B] = AB - BA$ is a compact operator on $\mathrm{L}^2(\mathbb{R}^d)$.

As a consequence, a class of *admissible symbols* is now ready to be defined. A function $p \in C(\mathbb{R}^d \times \mathbb{S}^{d-1})$ is an admissible symbol if it can be written in the form of a series

$$p(x,\xi) = \sum_k b_k(x) a_k(\xi), \tag{2.5}$$

with $a_k \in C(\mathbb{S}^{d-1})$ and $b_k \in C_0(\mathbb{R}^d)$, and such that the following boundedness condition is satisfied: $\sum_k \|a_k\|_\infty \|b_k\|_\infty < \infty$.

We then introduce *standard operator* $S_p \in \mathscr{L}(\mathrm{L}^2(\mathbb{R}^d))$ associated with symbol p, which is defined by

$$S_p = \sum_k A_k B_k, \tag{2.6}$$

where operators A_k and B_k are defined as above (the order of operations, first multiplication by b_k, and then the Fourier multiplier with a_k, corresponds to the so called standard, or Kohn-Nirenberg [49] quantisation). Of course, one needs to check that this object is well-defined, namely, that S_p does not depend upon the decomposition (2.5). Indeed, for each decomposition (2.5) S_p satisfies for each $u \in \mathscr{S}(\mathbb{R}^d)$:

$$\widehat{S_p u}(\xi) = \sum_k a_k\left(\frac{\xi}{|\xi|}\right) \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} b_k(x) u(x) dx = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} p\left(x, \frac{\xi}{|\xi|}\right) u(x) dx.$$

Thus, S_p does not depend on the choice of the representation for p.

Finally, an *operator with symbol p* is defined to be any operator $L_p \in \mathscr{L}(\mathrm{L}^2(\mathbb{R}^d))$ such that $L_p = S_p + K$, where $K \in \mathscr{K}(\mathrm{L}^2(\mathbb{R}^d))$ is a compact operator. One such example can be given by the operator (here the adjoint quantisation is applied)

$$L_p = \sum_k B_k A_k, \tag{2.7}$$

that differs from the standard operator S_p by an operator

$$L_p - S_p = \sum_k B_k A_k - A_k B_k = \sum_k [B_k, A_k],$$

which is compact due to the first commutation lemma and the fact that it is the uniform limit of compact operators.

Remark 2.2.5. Let us once again revisit the defining expression (2.2) for the H-measure and its variant given by (2.4). Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $\varphi_1 \in C_c^\infty(\mathbb{R}^d)$ be such that $\varphi_1 \equiv 1$ on $\operatorname{supp}\varphi$, while $\psi \in C^\infty(\mathbb{S}^{d-1})$, and finally take $p(x,\xi) = \varphi(x)\psi(\xi)$. Then for $u \in \mathscr{S}(\mathbb{R}^d)$:

$$\lim_n \int_{\mathbb{R}^d} L_P u_n \cdot u_n = \lim_n \int_{\mathbb{R}^d} L_P u_n \cdot (\varphi_1 u_n) = \langle \mu, p \rangle. \tag{2.8}$$

We will later use identity (2.8) in order to compare the defining results with Gérard's approach.

2.2.2. Immediate consequences and examples

Once the existence of H-measures has been proved, a few simple consequences can immediately be obtained.

Corollary 2.2.6. H-measure μ is hermitian and nonnegative:

$$\mu = \mu^* \qquad \text{and} \qquad \langle \mu, \varphi \boxtimes \varphi \rangle \geq 0, \quad \varphi \in C_0(\mathbb{R}^d; \mathbb{R}^r).$$

Corollary 2.2.7. Let sequence (u_n) define an H-measure μ. If all the components $u_n \cdot e_j$ have their supports in closed sets $K_j \subseteq \mathbb{R}^d$ respectively, then the support of the component μ_{ij} is contained in $(K_i \cap K_j) \times \mathbb{S}^{d-1}$. Specifically, if $\operatorname{supp} u_n \subseteq K$, for some closed $K \subseteq \mathbb{R}^d$, then $\operatorname{supp}\mu \subseteq K \times \mathbb{S}^{d-1}$.

Note that a strongly convergent sequence (u_n) defines an H-measure $\mu = 0$. This is an immediate consequence of the following corollary of Theorem 2.2.1.

Corollary 2.2.8. If $u_n \otimes u_n$ converges weakly $*$ (i.e. vaguely) to a measure ν, then for every $\varphi \in C_0(\mathbb{R}^d)$ one has:

$$\langle \nu, \varphi \rangle = \langle \mu, \varphi \boxtimes 1 \rangle.$$

However, it is important to note that if a sequence defines an H-measure 0, we can only deduce that it converges strongly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$. An example which exhibits this effect is given in [51, Example I.3].

We can now showcase the difference in the information obtained by computing an H-measure of a sequence as opposed to a defect measure of the same sequence. For that matter, we recall Example 2.0.1.

Example 2.2.9. (**Oscillation**) Under the same assumptions as in Example 2.0.1, we have that $u_n \xrightarrow{\mathrm{L}^2(\mathbb{R}^d)} 0$. Hence, it defines an H-measure μ. In this case it turns out that the sequence is pure, and the (unique) H-measure is given by

$$\mu = \lambda \boxtimes \delta_{\frac{\xi_0}{|\xi_0|}},$$

where $\delta_{\frac{\xi_0}{|\xi_0|}}$ denotes the Dirac measure concentrated in $\frac{\xi_0}{|\xi_0|}$ on the unit sphere. As we can now see, the direction in which the oscillations occur is captured by the resulting H-measure.

Example 2.2.10. (**Concentration**) Given $u \in \mathrm{L}^2(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ and a sequence $\varepsilon \to 0^+$, observe the sequence (u_n) defined by $u_n(x) = \varepsilon_n^{-\frac{d}{2}} u\left(\frac{x-x_0}{\varepsilon_n}\right)$. It can easily be seen that u_n is bounded in $\mathrm{L}^2(\mathbb{R}^d)$, and that it converges weakly to zero. This sequence is pure as well and the H-measure it defines is given by

$$\mu = \delta_{x_0} \boxtimes \nu,$$

where ν is the measure which is absolutely continuous with respect to the surface measure on the sphere with density N:

$$N(\eta) = \int_0^\infty |\hat{u}(t\eta)|^2 t^{d-1} dt.$$

While Corollary 2.2.7 already gives some constraints on the support of the H-measure, an even stronger result can be obtained by using the information contained in linear balance equations, thus providing a compactness by compensation result for variable coefficients.

Theorem 2.2.11. (**Localisation principle**) Let the sequence (u_n) define an H-measure μ. If it additionally satisfies, for some $m \in \mathbb{N}$:

$$\sum_{|\alpha| \leq m} \partial^\alpha (\mathbf{A}_\alpha u_n) \xrightarrow{\mathrm{H}^{-m}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{C}^q)} 0, \tag{2.9}$$

for given $\mathbf{A}_\alpha \in C(\mathbb{R}^d; M_{q\times r}(\mathbb{C}))$, then

$$\mathbf{p}\mu = 0, \tag{2.10}$$

where

$$\mathbf{p}(x,\xi) = (2\pi i)^m \sum_{|\alpha|=m} \left(\frac{\xi}{|\xi|}\right)^\alpha \mathbf{A}_\alpha(x),$$

is the principal symbol of the above differential operator.

2.3. Pseudodifferential operators

While Tartar's approach has its obvious advantages in terms of lower regularity assumptions on symbols and functions, it also requires reinventing the *pseudodifferential calculus* for the operators at stake, a theory that has been widely developed for more appropriate classes of symbols. We will now introduce one of the most standard ones, the class of Hörmander's symbols, and recall some of the basic calculus which will be of use. For more details, the reader can refer to [3, 43, 62, 64].

At this point we also remind the reader of the *Sobolev spaces* H^s defined now *only* on $\mathbb{R}^d$, but for arbitrary real number s. Namely, we denote by

$$\mathrm{H}^s(\mathbb{R}^d) = \{u \in \mathscr{S}'(\mathbb{R}^d) : \langle\xi\rangle^s \hat{u}(\xi) \in \mathrm{L}^2(\mathbb{R}^d)\}.$$

Here $\langle\cdot\rangle$ denotes the *japanese bracket* given by $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$, which serves as a smooth version of the function $\xi \mapsto |\xi|$.

For further details and properties of this spaces, as well as their relation with the Sobolev spaces introduced in the beginning of the first chapter, the reader can refer to [64] or [32]. However, it is worth noting that for $\Omega = \mathbb{R}^d$ and $k \in \mathbb{N}_0$ two spaces coincide (topologically).

2.3.1. Class of Hörmander's symbols

As a consequence of the Fourier inversion formula and (2.1), we have the following important identity,

$$D^\alpha u(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \xi^\alpha \widehat{u}(\xi) d\xi, \qquad u \in \mathscr{S}(\mathbb{R}^d) \tag{2.11}$$

where D denotes the reduced differential operator $D = \frac{1}{2\pi i}\partial$. If $L = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ is a linear partial differential operator of order m, the linearity of the Fourier transform and (2.11) allow us to rewrite its action as

$$Lu(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} a(x,\xi)\hat{u}(\xi) d\xi, \tag{2.12}$$

where $a(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$ denotes the *symbol* of operator L. In order to further illustrate the connection between the operator and its symbol, it is customary to write

$a(x,D)$ for the operator defined by (2.12) with symbol $a = a(x,\xi)$. The theory of pseudodifferential operators investigates tools allowing for the definition of operators of the form (2.12) for more general symbols a, in a class which will, at least aproximately, yield the possiblity of inverting these operators.

Definition 2.3.1. For $m \in \mathbb{Z}$, the vector space $S^m(\mathbb{R}^d \times \mathbb{R}^d)$ of symbols of order m consists of those functions $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying

$$\sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|}, \quad \alpha,\beta \in \mathbb{N}_0^d. \tag{2.13}$$

When there is no risk of confusion, we will write S^m instead of $S^m(\mathbb{R}^d \times \mathbb{R}^d)$ for brevity. Since one obviously has $S^m \le S^l$ for $m \le l$, we can additionally introduce $S^{-\infty} := \cap_{m\in\mathbb{Z}} S^m$ and $S^\infty := \cup_{m\in\mathbb{Z}} S^m$.

Remark 2.3.2. The above definition covers a particular case of Hörmander's symbol $S^m_{\rho,\delta}$, where $\rho = 1$ and $\delta = 0$.

Denote with $\|a\|_{\alpha,\beta}$ the seminorm given by the smallest constant $C_{\alpha,\beta}$ for which (2.13) holds, i.e.

$$\|a\|_{\alpha,\beta} = \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \langle\xi\rangle^{-m+|\beta|}. \tag{2.14}$$

The vector space S^m endowed with this (countable) family of seminorms (for $\alpha,\beta \in \mathbb{N}^d$) becomes a Fréchet space.

Example 2.3.3. (a) An obvious starting example of a symbol $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$ is given as above by

$$a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha,$$

where $a_\alpha \in \mathrm{H}^\infty(\mathbb{R}^d) = \cap_{s>0}\mathrm{H}^s(\mathbb{R}^d)$.

(b) Another example of a symbol in S^m is the japanese bracket $\langle\xi\rangle^m = (1+|\xi|^2)^{\frac{m}{2}}$.

(c) The third example is related to functions $a(x,\xi)$ which are compactly supported in x, positively homogeneous of order m in ξ and of class C^∞ outside some (small) neighbourhood of $\xi = 0$. One can then choose a symbol $b \in S^m$ such that $b(x,\xi) = a(x,\xi)$ for all $|\xi| \ge 1$. Such b is uniquely determined modulo $S^{-\infty}$. Indeed, one can choose a smooth cut-off function $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi \equiv 1$ on $B_{\frac{1}{2}}$ and $\operatorname{supp}\chi \subseteq$

B_1 (here B_r denotes the ball of radius r around the origin) and then define $b(x,\xi) = (1-\chi(\xi))a(x,\xi)$. Note in particular that this modification does not expand the support in the x variable.

Remark 2.3.4. (a) It is not inherently important that there are as many x variables as there are ξ variables; Definition 2.3.1 can be stated in the same manner for symbols $a \in S^\infty(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$, with obvious modifications. We will discuss this in more detail later for the specific case of time dependent symbols.

(b) While the number of variables may differ, it is important to note that symbols cannot in general be independent of some of the ξ variables. That is the case if and only if a is a polynomial in ξ. Indeed, assume that $a(x,\xi) = a(x,\xi')$, where $\xi = (\xi', \xi_d)$. From (2.13) it follows, for $\beta = (\beta', 0) \in \mathbb{N}_0^d$ such that $|\beta'| > m$, that

$$\sup_{(x,\xi')\in\mathbb{R}^d\times\mathbb{R}^{d-1}} \left|\partial^\beta a(x,\xi')\right| \le C_N \langle(\xi',\xi_d)\rangle^{m-|\beta'|}.$$

By letting $\xi_d \longrightarrow \infty$ we get $\partial^{\beta'} a = 0$, hence a is a polynomial in ξ'.

As it has already been noted at the beginning of this section, we will associate these symbols to a certain class of operators. The main idea of developing the calculus for such operators consists in transferring operations on operators to operations on their symbols. In order to develop such calculus in a way that will also allow for certain approximations, it is necessary to follow the notions such as the summation of a series. The following lemma provides such a result (see [43, Proposition 18.1.4] or [64, Lemma 2.2]).

Proposition 2.3.5. Let $a_j \in S^{m_j}$, $j \in \mathbb{N}_0$ and assume that $m_j \to -\infty$ as $j \to \infty$. Set $m'_k = \max_{j\ge k} m_j$. Then there exists $a \in S^{m'_0}$ such that $\operatorname{supp} a \subseteq \cup_j \operatorname{supp} a_j$ and for every k

$$a - \sum_{j<k} a_j \in S^{m'_k}. \tag{2.15}$$

The function a is uniquely determined modulo $S^{-\infty}$ and has the same property relative to any rearrangement of the series $\sum_j a_j$; we write

$$a \sim \sum_j a_j.$$

As an example of such construction, we also mention a special case of symbols, called *polyhomogeneous symbols* (sometimes also referred to as *classical symbols*).

Definition 2.3.6. The set of all $a \in S^m$ such that

$$a \sim \sum_{j=0}^{\infty} a_{m-j},$$

where a_{m-j} is homogeneous of degree $m-j$, meaning that they satisfy

$$a_{m-j}(x,\lambda\xi) = \lambda^{m-j} a_{m-j}(x,\xi), \quad |\xi| \geq 1, \lambda \geq 1, \tag{2.16}$$

will be called polyhomogeneous of degree m. The set of all polyhomogeneous symbols of order m is denoted with S^m_{phg}.

If a is a polyhomogeneous symbol of order m and $(a_{m-j})_{j\geq 0}$ as above, we will refer to a_m as the *principal symbol*. It will often also be denoted by $\sigma(a)$.

Remark 2.3.7. (a) Note that the functions a_j cannot at the same time be C^∞ and homogeneous for every $\xi \neq 0$, unless they are polynomials in ξ. Nevertheless, this does not pose any obstruction, since the construction of the asymptotic expansion is based on the process described in Example 2.3.3 (c).

(b) The homogeneity condition immediately implies $a_{m-j} \in S^{m-j}(\mathbb{R}^d)$.

2.3.2. Operator calculus

We are now in a position to define pseudodifferential operators associated to symbols in Hörmander's class. The reader can find more details on results stated in this section in [43, Chapter 18] or [64, Chapters 2–3].

Theorem 2.3.8. If $a \in S^m(\mathbb{R}^d)$ and $u \in \mathscr{S}(\mathbb{R}^d)$, then

$$a(x,D)u(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} a(x,\xi)\widehat{u}(\xi)d\xi \tag{2.17}$$

defines a function $a(x,D)u \in \mathscr{S}$, and the bilinear map $(a,u) \mapsto a(x,D)u$ is continuous. One calls $a(x,D)$ a *pseudodifferential operator* of order m.

The space of all pseudodifferential operators of order m will be denoted by $\Psi^m(\mathbb{R}^d)$, or just Ψ^m when there is no fear of confusion. Since $\Psi^m \leq \Psi^l$ for $m \leq l$, we will also write $\Psi^{-\infty} := \cap_m \Psi^m$ and $\Psi^\infty = \cup_m \Psi^m$. Morover, we will denoted by Ψ^m_c the subspace of Ψ^m consisting of operators whose symbols are compactly supported functions in x.

Remark 2.3.9. Note that for $a(x,D) \in \Psi_c^m$, one has that $a(x,D)$ maps $\mathscr{S}$ to C_c^∞. This follows immediately from (2.17).

The basic operator calculus involves, of course, taking adjoints and composing operators. As usual, the usage of the adjoint operator also allows us to extend pseudodifferential operators to continuous maps on $\mathscr{S}'$. Let us recall some basic results.

Theorem 2.3.10. (a) If $a \in S^m$, then the function a^* defined as an oscillatory integral

$$a^*(x,\xi) := e^{2\pi i D_x \cdot D_\xi}\overline{a}(x,\xi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{2\pi i y\cdot\eta}\overline{a}(x-y,\xi-\eta)\,dy\,d\eta \tag{2.18}$$

belongs to $S^m(\mathbb{R}^d)$ and has asymptotic expansion

$$a^*(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^d} \frac{1}{(2\pi i)^{|\alpha|}\alpha!}\partial_x^\alpha\partial_\xi^\alpha\overline{a}(x,\xi). \tag{2.19}$$

Moreover, it holds

$$(a(x,D)u,v) = (u,a^*(x,D)v), \qquad u,v\in\mathscr{S},$$

and $a(x,D)$ can be extended to a continuous map from $\mathscr{S}'$ to $\mathscr{S}'$, as the adjoint of $a^*(x,D)$.

(b) If $a \in S^m$ and $b \in S^l$, then as operators in $\mathscr{S}$ or $\mathscr{S}'$

$$a(x,D)b(x,D) = (a\#b)(x,D),$$

where $a\#b \in S^{m+l}$ is defined as an oscillatory integral

$$\begin{aligned}(a\#b)(x,\xi) :&= e^{2\pi i D_y\cdot D_\eta}a(x,\eta)b(y,\xi)\Big|_{y=x,\eta=\xi}\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{2\pi i y\cdot\eta}a(x-y,\xi)b(y,\xi-\eta)\,dy\,d\eta\end{aligned}$$

and has the asymptotic expansion

$$(a\#b)(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^d}\frac{1}{(2\pi i)^{|\alpha|}\alpha!}\partial_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi). \tag{2.20}$$

When working with pseudodifferential operators, the importance of the commutator has already been seen in Tartar's approach, in view of being compact on L^2. The same notation is adopted here: we define

$$[a(x,D),b(x,D)] := a(x,D)b(x,D) - b(x,D)a(x,D), \qquad \text{for } a(x,D),b(x,D)\in\Psi^\infty.$$

It follows immediately from Theorem 2.3.10 (b) that for $a \in S^m$, $b \in S^l$ and their associated operators $a(x,D)$ and $b(x,D)$, one has $[a(x,D),b(x,D)] \in \psi^{m+l}$. However, a consequence of asymptotic expansion of its symbol also given by Theorem 2.3.10 (b) is that the commutator is actually an operator of order $m+l-1$, and its principal symbol is given by $\frac{1}{2\pi i}\{\sigma(a),\sigma(b)\}$, where $\{\cdot,\cdot\}$ denotes the *Poisson bracket* of two functions f,g of (x,ξ) given by

$$\{f,g\} := \nabla_\xi f \cdot \nabla_x g - \nabla_x f \cdot \nabla_\xi g.$$

2.3.3. Action on Lebesgue and Sobolev spaces

In this section we shall state some of the most important results regarding the action of pseudodifferential operators on Lebesgue spaces, as well as Sobolev spaces.

Theorem 2.3.11. If $a \in S^m$, then $a(x,D)$ is a continuous operator from H^s to H^{s-m} for every $s \in \mathbb{R}$.

Remark 2.3.12. As a consequence of Theorem 2.3.11, we have that symbols $a \in S^{-\infty}$ define operators which map $\mathrm{H}^{-\infty}$ into H^{∞}. Therefore, the operators in $\Psi^{-\infty}$ are also referred to as *smoothing operators*.

More on this topic was discovered recently in [9], where some results regarding continuity of pseudodifferential operators on *mixed-norm* Lebesgue and Sobolev spaces have been obtained. For any $\mathbf{p} = (p_1,\dots,p_d) \in [1,\infty)^d$, $\mathrm{L}^{\mathbf{p}}(\mathbb{R}^d)$ denotes (with identification of almost everywhere equal functions) the space of all measurable functions f on $\mathbb{R}^d$ for which we have

$$\|f\|_{\mathbf{p}} = \left(\int_{\mathbb{R}} \dots \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(x_1,\dots,x_d)|^{p_1} dx_1\right)^{p_2/p_1} dx_2\right)^{p_3/p_2} \dots dx_d\right)^{1/p_d} < \infty. \tag{2.21}$$

In other words, for $j = 1,\dots,d$ the norms $\|\cdot\|_{\mathrm{L}^{p_j}}$ in variable x_j are computed in that exact order. Simple adjustments extend this definition to the case when some $p_j = \infty$. For $\mathbf{p} \in [1,\infty]^d$ thus defined $\|\cdot\|_{\mathbf{p}}$ is a norm on $\mathrm{L}^{\mathbf{p}}(\mathbb{R}^d)$, which becomes a Banach space.

Similarly, we can define mixed-norm Sobolev spaces: for $k \in \mathbb{N}_0$ and $\mathbf{p} \in (1,\infty)^d$ let

$$\mathrm{W}^{k,\mathbf{p}}(\mathbb{R}^d) = \left\{u \in \mathscr{S}' : (\forall \alpha \in \mathbb{N}_0^d)\, |\alpha| \le k \implies \partial^\alpha u \in \mathrm{L}^{\mathbf{p}}(\mathbb{R}^d)\right\},$$

with the norm

$$\|u\|_{\mathrm{W}^{k,\mathbf{p}}} = \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{\mathbf{p}}.$$

This definition can easily be extended to an open set $\Omega \subseteq \mathbb{R}^d$, with derivatives being understood in the sense of distributions.

Since our focus is not on general mixed-norm spaces, we will single out the one specific case, which appears naturally in the study of evolution PDEs, and that is the case of $\mathbf{p} = (p, 2, \ldots, 2) \in \mathbb{R}^{d+1}$, where $p \in [1, \infty]$. In this case, the space $\mathrm{L}^{\mathbf{p}}(\mathbb{R}^{d+1})$ obviously coincides with the Bôchner space $\mathrm{L}^p(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d))$ introduced above in Chapter 1.

We now state the main result on continuity of pseudodifferential operators on mixed-norm Sobolev spaces [9, Corollary 1].

Theorem 2.3.13. Let $a(x,D)$ be a pseudodifferential operator from Ψ^m. Then for any $\mathbf{p} \in (1,\infty)^d$ and any integer $k \geq m \in \mathbb{N}_0$ the operator $a(x,D) : \mathrm{W}^{k,\mathbf{p}}(\mathbb{R}^d) \to \mathrm{W}^{k-m,\mathbf{p}}(\mathbb{R}^d)$ is bounded.

As a special case for $k = m = 0$ we get the following result.

Corollary 2.3.14. Pseudodifferential operators of class Ψ^0 are bounded on $\mathrm{L}^{\mathbf{p}}(\mathbb{R}^d)$, $\mathbf{p} \in (1,\infty)^d$.

Note also that for $\mathbf{p} = (p, \ldots, p) \in \mathbb{R}^d$ the previous theorem can be restated as follows.

Theorem 2.3.15. If $a \in S^0$, then $a(x,D)$ is a continuous operator on L^p, for $1 < p < \infty$.

These results will be used in the final chapter, as they will enable us to obtain better estimates with respect to time, a feat that was beforehand a difficult task because of the need to lower the regularity of the framework to L^2 in all of the variables.

2.4. GÉRARD'S APPROACH

In this section, we will briefly introduce Gérard's definition of microlocal defect measures. To fully emphasise the difference in approaches, the existence theorem, which also serves as a definition, is here stated in its full generality. For that purpose, we need to introduce some additional components.

Let H be a separable Hilbert space. For a positive bounded operator $A \in \mathscr{L}(H)$, the *trace of A* is defined as

$$\mathrm{tr}(A) := \sum_{k=1}^{\infty} \langle Ae_k, e_k \rangle_H,$$

where (e_k) is an arbitrary orthonormal basis for H, and this quantity does not depend on the choice of the orthonormal basis $(e_k)_k$ for H.

For $A \in \mathscr{L}(H)$ not necessarily positive, we say it is of *trace class* if $\mathrm{tr}(\sqrt{A^*A}) < \infty$. The space of all trace class operators is denoted with $\mathscr{L}^1(H)$. With $\mathscr{K}(H)$ we denote the space of all compact operators on H.

Next in line, with $\mathscr{M}(\mathbb{R}^d \times \mathbb{S}^{d-1}; \mathscr{L}^1(H))$ is denoted the space of *trace operator-valued Radon measures* on $\mathbb{R}^d \times \mathbb{S}^{d-1}$, i.e. the space of linear operators $\mu : C_0(\mathbb{R}^d \times \mathbb{S}^{d-1}) \to \mathscr{L}^1(H)$ satisfying the following boundedness condition: for any compact $K \subseteq \mathbb{R}^d \times \mathbb{S}^{d-1}$ there exists constant $C_K > 0$ such that

$$(\forall \varphi \in C_0(K)) \qquad |\langle \mu, \varphi \rangle|_{\mathscr{L}^1(H)} \leq C_K \|\varphi\|_\infty.$$

The subset of all positive elements of $\mathscr{M}(\mathbb{R}^d \times \mathbb{S}^{d-1}; \mathscr{L}^1(H))$ is denoted by $\mathscr{M}_+(\mathbb{R}^d \times \mathbb{S}^{d-1}; \mathscr{L}^1(H))$. Finally, if $\mu \in \mathscr{M}_+(\mathbb{R}^d \times \mathbb{S}^{d-1}; \mathscr{L}^1(H))$, we denote by $\mathrm{tr}(\mu)$ the positive scalar Radon measure defined by $\langle \mathrm{tr}(\mu), \varphi \rangle := \mathrm{tr}\big(\langle \mu, \varphi \rangle\big)$.

The existence theorem given by Gérard [36, Theorem 1] can now be stated.

Theorem 2.4.1. Let (u_n) be a bounded sequence in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d; H)$ (in the sense of locally convex topological vector spaces) and assume $u_n \rightharpoonup 0$. Then there exists a subsequence $(u_{n'})$ and a measure $\mu \in \mathscr{M}_+(\mathbb{R}^d \times \mathbb{S}^{d-1}; \mathscr{L}^1(H))$ such that, for any $A \in \Psi^0_c(\mathbb{R}^d, \mathscr{K}(H))$

$$\lim_{n'} \int_{\mathbb{R}^d} Au_{n'} \overline{u_{n'}} = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{tr}\Big(\sigma(a) d\mu\Big). \tag{2.22}$$

The case of our primary interest is $H = \mathbb{R}^r$. Since it is of finite dimension, measure μ can be identified with an $r \times r$ matrix valued Radon measure μ on $\mathbb{R}^d \times \mathbb{S}^{d-1}$. Additionally, the notion of the trace defined prior to the theorem obviously coincides with the notion of the trace of a matrix. Taking these remarks into account, we see that (2.22) can be rewritten in the same manner as (2.8).

Of course, the main difference now lies in the fact that the requirements on the class of operators appear to be more strict here; one may think that the class of symbols introduced in 2.2.1 generalizes the class Ψ_c^0, given its lesser restraints on regularity. However, it is worth noting that it is not the case. The example we give now can also be found in [51].

Indeed, take $a \in C_c^\infty(\mathbb{R})$ and $b \in C_c^\infty(\mathbb{R}^2)$ such that $\operatorname{supp} a \subseteq [0,1]$ and $\operatorname{supp} b \subseteq [0,1]^2$, with $\|a\|_\infty = \|b\|_\infty = 1$. Define sequences

$$\begin{aligned} a_n(\xi) &:= \frac{1}{n^2} a\left(n^2\left(\pi(\xi) - \sum_{j=1}^{n-1} \frac{1}{j^2}\right)\right), \\ b_n(x) &:= nb\left(\frac{1}{n}\left(x_1 - \sum_{j=1}^{n-1} j, x_2 - \sum_{j=1}^{n-1} j\right)\right), \end{aligned}$$

where π denotes the stereographic projection of the circle $\mathbb{S}^1$ to the real line. The function p defined by $p(x,\xi) = \sum_n a_n(\xi) b_n(x)$ belongs to the space S^0, but on the other hand we have $\sum_n \|a_n\|_\infty \|b_n\|_\infty = \sum_n \frac{1}{n} = \infty$.

Hence, we see that the class of Hörmander's symbols S^0 and class of symbols used by Tartar are in general position.

Remark 2.4.2. Although Tartar's approach may be more practical and realistic when dealing with the non-smooth nature of partial differential equations, we do not posses enough information on pseudodifferential calculus (mostly related to higher order operators) to apply that approach successfully in our further work. Therefore, in the following chapter we will primarily rely on Gérard's approach, and the theory of pseudodifferential operators presented throughout this chapter.

2.5. FURTHER REMARKS ON PSEUDODIFFERENTIAL OPERATORS

As the final part of this chapter, we will present some additional remarks regarding the theory of pseudodifferential operators and the symbolic calculus presented in the last section.

Firstly, let us observe that similarly to the way the Fourier transform was defined, one can define partial Fourier transform with respect to subset of variables; for simplicity, we can observe the case where x is split into $x = (x', x'')$ with $x' \in \mathbb{R}^k, x'' \in \mathbb{R}^{d-k}$ and define for $u \in \mathscr{S}(\mathbb{R}^d)$:

$$(\mathscr{F}_{x'}u)(\xi) = (\mathscr{F}_{x'}u)(\xi', \xi'') := \int_{\mathbb{R}^k} e^{-2\pi i x'\cdot\xi'} u(x', \xi'')dx'.$$

It is straightforward to check that $\mathscr{F}_{x'}\mathscr{F}_{x''} = \mathscr{F}$, and that $\mathscr{F}_{x'}$ maps $\mathscr{S}(\mathbb{R}^d)$ into itself. Since evolution equations are central to this disertation, the case we wish to emphasise is the one that involves separating the time variable t from the space variables x, and performing partial Fourier transform only with respect to x variable; which will be denoted by $\mathscr{F}_x$.

Next we turn back to the topic opened in 2.3.4 (a). As was mentioned there, it is irrelevant for the definition of symbols for functions to have the same number of variables in x and ξ. The definition of symbols remain the same with obvious adjustments.

Functions that belong to $C^\infty(\mathbb{R}^{d+1} \times \mathbb{R}^d)$, where one additional variable in the first argument will correspond to time-dependence, will be of interest in the remainder of the thesis. Here, (t, x) will play the role of spatial variable in $\mathbb{R}^{d+1}$, while the covariable will be $\xi \in \mathbb{R}^d$. We would like to associate partial pseudodifferential operators with such symbols, the ones that will act in a "pseudodifferential way" on the spatial variable x, with some additional dependence on t in terms of multiplication. Remaining remarks are mostly dedicated to comparing some basic results regarding boundedness of such operators to those introduced in 2.3.2.

In what follows, we assume additionally that $a = a(t, x, \xi) \in C^\infty(\mathbb{R}^{d+1} \times \mathbb{R}^d)$ satisfies for all $k \in \mathbb{N}_0, \alpha, \beta \in \mathbb{N}_0^d$ and some constants $C_{k,\alpha,\beta} > 0$

$$\sup_{(t,x,\xi)\in\mathbb{R}^{d+1}\times\mathbb{R}^d} \langle\xi\rangle^{|\beta|-m}|\partial_t^k\partial_x^\alpha\partial_\xi^\beta a(t,x,\xi)| \le C_{k,\alpha,\beta}. \tag{2.23}$$

Then a is a symbol of order m; here denoted by $S^m(\mathbb{R}^{d+1}\times\mathbb{R}^d)$ for clarity.

We first remark that the asymptotic expansion result given by Proposition 2.3.5 for such symbols is still valid. For the sake of completeness, we reiterate the Proposition.

Proposition 2.5.1. Let $a_j \in S^{m_j}(\mathbb{R}^{d+1}\times\mathbb{R}^d)$, $j\in\mathbb{N}_0$ and assume that $m_j \to -\infty$ as $j\to\infty$. Set $m'_k = \max_{j\ge k} m_j$. Then there exists $a\in S^{m'_0}(\mathbb{R}^{d+1}\times\mathbb{R}^d)$ such that $\operatorname{supp} a \subseteq \cup_j \operatorname{supp} a_j$ and for every k

$$a(t,x,\xi) - \sum_{j<k} a_j(t,x,\xi) \in S^{m'_k}(\mathbb{R}^{d+1}\times\mathbb{R}^d). \tag{2.24}$$

The function a is uniquely determined modulo $S^{-\infty}(\mathbb{R}^{d+1}\times\mathbb{R}^d)$ and has the same property relative to any rearrangement of the series $\sum_j a_j$; we write

$$a \sim \sum_j a_j.$$

We now associate with $a\in S^\infty(\mathbb{R}^{d+1}\times\mathbb{R}^d)$ a linear operator $a(t;x,D)$ defined by

$$a(t;x,D)u(t,x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} a(t,x,\xi)(\mathscr{F}_x u)(t,\xi)d\xi, \qquad u\in\mathscr{S}(\mathbb{R}^{d+1}). \tag{2.25}$$

As $\mathscr{F}_x u$ is again an element of $\mathscr{S}(\mathbb{R}^{d+1})$, same arguments as in the proof of Theorem 2.3.8 (see [43]) show that this operator maps $\mathscr{S}(\mathbb{R}^{d+1})$ into $\mathscr{S}(\mathbb{R}^{d+1})$. As a matter of fact, the same proof yields the bound of the functions $(t,x)^\alpha\partial_t^k\partial_x^\beta a(t;x,D)u$ in terms of a semi-norm of u in $\mathscr{S}(\mathbb{R}^{d+1})$ and a seminorm of $a\in S^m(\mathbb{R}^{d+1}\times\mathbb{R}^d)$.

We would now like to deal with the question of boundedness of such operators, at least of some mixed type in time and space, on Sobolev spaces. For that purpose, let us consider for $a\in S^\infty(\mathbb{R}^{d+1}\times\mathbb{R}^d)$ operator $a(t;x,D)$ as an operator acting on $\mathbb{R}^d$ with t as a parameter, i.e., we "freeze" the time variable. According to Theorem 2.3.11, there is a constant $C = C(t) > 0$ such that for any $s\in\mathbb{R}$ and $u\in\mathscr{S}(\mathbb{R}^{d+1})$ it holds:

$$\|a(t;\cdot,D)u(t)\|_{\mathrm{H}^{s-m}(\mathbb{R}^d)} \le C(t)\|u(t)\|_{\mathrm{H}^s(\mathbb{R}^d)}. \tag{2.26}$$

Here we have implicitly used the fact that $u\in\mathscr{S}(\mathbb{R}^{d+1})$ implies $u(t)\in\mathscr{S}(\mathbb{R}^d)$ for each $t\in\mathbb{R}$. The constant $C(t)$ can now be bounded via the proof of that very theorem (see [43], Theorem 18.1.11 and Theorem 18.1.13 and remarks following the proof of Theorem 18.1.11) by seminorms of a in $S(\mathbb{R}^{d+1}\times\mathbb{R}^d)$. Consequently, constants $C(t)$ can be uniformly bounded in t due to (2.23), which yields the following bound:

$$\|a(\cdot;\cdot,D)u\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{s-m}(\mathbb{R}^d))}\leq C\|u\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^s(\mathbb{R}^d))},\qquad u\in\mathscr{S}(\mathbb{R}^{d+1}). \tag{2.27}$$

If given two symbols $a\in S^m(\mathbb{R}^{d+1}\times\mathbb{R}^d)$ and $b\in S^l(\mathbb{R}^{d+1}\times\mathbb{R}^d)$, composition $a(t;\cdot,D)b(t;\cdot,D)$ of corresponding operators is again an operator of the same type, and asymptotic expansion of the form as in (2.20) holds for each $t\in\mathbb{R}$. Finally, commutator $[a(t;\cdot,D),b(t;\cdot,D)]$ is then an operator which satisfies

$$\|[a(\cdot;\cdot,D),b(\cdot;\cdot,D)]u\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{s-(m+l-1)}(\mathbb{R}^d)}\leq C\|u\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^s(\mathbb{R}^d)},\qquad u\in\mathscr{S}(\mathbb{R}^{d+1}),\ s\in\mathbb{R}.$$

As a final remark, we will touch upon the composition of aforementioned operators with standard operators in $\Psi^\infty(\mathbb{R}^{d+1})$. One has to be careful here, as Remark 2.3.4 (b) tells us that we cannot just consider partial operators with symbols in $S^\infty(\mathbb{R}^{d+1}\times\mathbb{R}^d)$ as operators from $\Psi^\infty(\mathbb{R}^{d+1})$ that are independent of the covariable corresponding to t. However, there is a way to circumvent this issue in order to have the composition of one operator associated with symbol in $S^\infty(\mathbb{R}^{d+1}\times\mathbb{R}^d)$ with another associated to a symbol in $S^\infty(\mathbb{R}^{d+1}\times\mathbb{R}^{d+1})$ well defined. This is stated in the following theorem (see [43, Theorem 18.1.35]).

Theorem 2.5.2. Let $a\in S^m(\mathbb{R}^{d+1}\times\mathbb{R}^{d+1})$, $b\in S^{m'}(\mathbb{R}^{d+1}\times\mathbb{R}^d)$, and assume that for some $\varepsilon>0$ we have

$$a(x,\xi)=0 \text{ if } |\xi_{d+1}|<\frac{1}{\varepsilon} \text{ and } |\xi'|\leq\varepsilon|\xi_{d+1}|. \tag{2.28}$$

Then $a(x,D)b(x,D')$ and $b(x,D')a(x,D)$ are in $\Psi^{m+m'}(\mathbb{R}^{d+1})$, and the asymptotic expansion of the symbols can be carried out in standard way.

3. Transport properties of H-measures

3.1. Wave equation setting

Consider the Cauchy problem for the linear wave equation, with oscillating initial data and smooth non-oscillating coefficients

$$\begin{cases} (\rho u_n')' - \operatorname{div}(\mathbf{A}\nabla u_n) = 0, \\ u_n(0) = g_n, \\ u_n'(0) = h_n. \end{cases} \tag{3.1}$$

We assume $\rho \in C^\infty(\mathbb{R}^{d+1};\mathbb{R})$ and $\mathbf{A} \in C^\infty(\mathbb{R}^{d+1};\mathrm{M}_d^{\mathrm{psym}})$, where $\mathrm{M}_d^{\mathrm{psym}}$ denotes the space of all positive definite symmetric matrices of order d. Moreover, we assume all corresponding derivatives to be bounded, that is,

$$\|\rho^{(k)}\|_{\mathrm{L}^\infty(\mathbb{R}^d)}, \|\partial^\alpha \mathbf{A}\|_{\mathrm{L}^\infty(\mathbb{R}^d;\mathrm{M}_d^{\mathrm{psym}})} < \infty, \qquad k \in \mathbb{N}_0 \text{ and } \alpha \in \mathbb{N}_0^d. \tag{3.2}$$

Additionally, let ρ and $\mathbf{A}$ satisfy the following coercivity properties: for some $\alpha > 0$ it holds

$$\rho(t,x) \geq \alpha > 0, \qquad \mathbf{A}(t,x) \geq \alpha \mathbf{I}, \quad (t,x) \in \mathbb{R}^{1+d}. \tag{3.3}$$

The initial conditions g_n and h_n are both assumed to be $C_c^\infty(\mathbb{R}^d)$ with a common compact support $K \subseteq \mathbb{R}^d$. They further satisfy

$$\begin{aligned} g_n &\rightharpoonup 0, \quad \text{in } \mathrm{H}^1(\mathbb{R}^d), \\ h_n &\rightharpoonup 0, \quad \text{in } \mathrm{L}^2(\mathbb{R}^d). \end{aligned} \tag{3.4}$$

Remark 3.1.1. (a) For $t \geq 0$ denote by $\mathbf{R}(t) \in \mathscr{L}(H)$ and $\mathbf{A}_0(t) \in \mathscr{L}(V,V')$ the operators given by

$$\mathbf{R}(t)u(t,x) = \rho(t,x)u(t,x)$$
$$\mathbf{A}_0(t)u(t,x) = -\mathrm{div}\,(\mathbf{A}(t,x)\nabla u(t,x)).$$

Due to (3.2) and (3.3), these are just a special case of operators denoted equally in Chapter 1. Note that the operator norms here coincide with the sup-norms of ρ and $\mathbf{A}$; namely, we have

$$\|\mathbf{R}\|_{\mathrm{L}^\infty(0,T;\mathscr{L}(H))} = \|\rho\|_{\mathrm{L}^\infty((0,T)\times\Omega)}, \qquad \|\mathbf{A}_0\|_{\mathrm{L}^\infty(0,T;\mathscr{L}(V,V'))} = \|\mathbf{A}\|_{\mathrm{L}^\infty((0,T)\times\Omega;\mathrm{M}_d^{\mathrm{psym}})}.$$

Similar estimates follow for the time derivatives of $\mathbf{R}$ and $\mathbf{A}_0$, which are obviously then given by corresponding time derivatives of ρ and $\mathbf{A}$.

(b) Owing to the regularity of coefficients, as well as the initial data, there exists a unique smooth solution u_n to the corresponding Cauchy problem. Moreover, because of the finite speed of propagation for the wave equation (see [60, Theorem 6.10]), for each $T > 0$ we have that the projection of u_n to the spatial domain is compactly supported in some K' independent of n. Therefore, we may also interpret (3.1) as a Cauchy problem on $(0,T)\times\Omega$ with the Dirichlet boundary condition, where Ω is chosen such that $K' \subseteq \Omega$. This also allows us to interpret given functions as elements of $\mathrm{H}^1_0(\Omega)$ and $\mathrm{H}^1(\mathbb{R}^d)$ interchangeably, enabling us to apply certain results regarding Sobolev spaces that hold true only for bounded sets.

(c) Using energy estimates obtained in the first chapter, we deduce the bound of

$$u_n \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\mathbb{R}^d)) \cap \mathrm{W}^{1,\infty}(0,T;\mathrm{L}^2(\mathbb{R}^d))$$

in terms of respective norms of $\rho, \mathbf{A}$ (and their time derivatives), as well as the norms of $\|g_n\|_{\mathrm{H}^1(\mathbb{R}^d)}$ and $\|h_n\|_{\mathrm{L}^2(\mathbb{R}^d)}$.

Lemma 3.1.2. For each $T > 0$ the sequence of solutions u_n to (3.1) satisfy

$$\begin{aligned} u_n &\overset{*}{\rightharpoonup} 0 \quad \text{in } \mathrm{L}^\infty(0,T;\mathrm{H}^1(\mathbb{R}^d)) \\ u_n' &\overset{*}{\rightharpoonup} 0 \quad \text{in } \mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^d)). \end{aligned} \tag{3.5}$$

Proof. Remarks preceding this lemma, together with the boundedness of initial data following from (3.4) yield boundedness of the sequence of solutions u_n in given spaces. Hence, there is an $u \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\mathbb{R}^d))$ with $u' \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^d))$ such that

$$
\begin{aligned}
u_n &\overset{*}{\rightharpoonup} u \quad \text{in } \mathrm{L}^\infty(0,T;\mathrm{H}^1(\mathbb{R}^d)) \\
u_n' &\overset{*}{\rightharpoonup} u' \quad \text{in } \mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^d)).
\end{aligned}
$$

In the same vein as in the first chapter, it can then be easily checked that u solves the limit problem

$$
\begin{cases}
(\rho u')' - \operatorname{div}(\mathbf{A}\nabla u) = 0 \\
u(0) = 0 \\
u'(0) = 0.
\end{cases}
$$

Uniqueness of the solution then implies $u \equiv 0$, which concludes the proof of the lemma.

■

3.2. H-MEASURE ASSOCIATED TO THE SEQUENCE OF SOLUTIONS

In order to apply the theory of H-measures, we need to ensure that we are in the L^2 framework. In order to achieve that, time truncation is necessary. Let $\vartheta \in C_c^\infty(\mathbb{R})$ be a time-cutoff function such that $0 \le \vartheta \le 1$ and $\vartheta \equiv 1$ on $[0,T]$ for some fixed $T > 0$. Set $v_n = \vartheta u_n$ and

$$V_n = (V_n^0, V_n^x) := (v_n', \nabla v_n).$$

The convergence in (3.5) now implies

$$v_n \overset{*}{\rightharpoonup} 0 \quad \text{in } \mathrm{L}^\infty(\mathbb{R}; \mathrm{H}^1(\mathbb{R}^d)) \tag{3.6}$$

$$V_n \overset{*}{\rightharpoonup} 0 \quad \text{in } \mathrm{L}^\infty(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d)), \tag{3.7}$$

but the additional compactness of the support in time now also yields

$$V_n \overset{*}{\rightharpoonup} 0 \quad \text{in } \mathrm{L}^2(\mathbb{R}^{d+1}). \tag{3.8}$$

Therefore, V_n defines (up to a subsequence) an H-measure μ. Note that v_n satisfies the equation

$$(\rho v_n')' - \operatorname{div}(\mathbf{A}\nabla v_n) = f_n, \tag{3.9}$$

where

$$f_n = 2\vartheta'\rho' u_n + \vartheta'\rho u_n' + \vartheta''\rho u_n \overset{*}{\rightharpoonup} 0 \quad \text{in } \mathrm{L}^\infty(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d)).$$

First, we can show that H-measure μ associated to the (sub)sequence V_n has a specific form given by the following theorem.

Theorem 3.2.1. The H-measure μ has the form

$$\mu = ((\tau, \xi) \otimes (\tau, \xi))\, \nu \tag{3.10}$$

where $(\tau, \xi) \in \mathbb{S}^d$ and ν is a non-negative scalar-valued measure satisfying

$$Q\nu = 0, \tag{3.11}$$

with

$$Q(t,x,\tau,\xi) = \frac{1}{2}\rho(t,x)\tau^2 - \frac{1}{2}\mathbf{A}(t,x)\xi \cdot \xi \tag{3.12}$$

being the principal symbol of the wave operator, up to a constant factor.

The proof is essentially the same as in [4]; since the sequence V_n is in fact a (time-space) gradient of a smooth function, it is first shown through symmetries of second order partial derivatives that μ is of the form (3.10). Then we use (3.9) to obtain

$$\mathrm{div}_{t,x}(\rho V_n^0, -\mathbf{A}V_n^x) = f_n \to 0 \quad \text{strongly in } \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbb{R}^d),$$

where the strong convergence of f_n follows from the compact support (independent of n) of f_n, weak convergence of f_n in $\mathrm{L}^2(\mathbb{R}^{d+1})$ and the fact that $\mathrm{L}^2(\mathbb{R}^{d+1})$ is compactly embedded into $\mathrm{H}^{-1}_{\mathrm{loc}}(\mathbb{R}^{d+1})$. The localization principle now yields (3.11).

Remark 3.2.2. (a) As a consequence of Theorem 3.2.1 we have that the support of the H-measure μ is contained in the null set of Q.

(b) It is important to observe that $(\tau,\xi) \notin \operatorname{supp}\nu$ in the neighbourhood of points where $\tau = \pm 1$ or 0. This follows immediately from (3.11) and coercivity properties of ρ and $\mathbf{A}$.

Our main goal is to obtain the transport equation for scalar measure ν, which will in turn yield a transport equation for H-measure μ, while also incorporating initial conditions for μ, by allowing test functions to be supported in $t = 0$. This will pave the way to determination of ν only from the H-measure associated to initial data without tedious computations of the solution to (3.1).

Since the transport equation is first to be obtained for a suitable dense class of functions, namely, those which are in the form of tensor products $\psi(t) \boxtimes \varphi(x) \boxtimes p(\tau,\xi)$, our first step will be to associate p with a symbol of a pseudodifferential operator of order 0 (hence, P is in fact a Fourier multiplier). For that purpose, let $p = p(\tau,\xi) \in C^\infty(\mathbb{R}^{d+1})$ be a polyhomogenous symbol of order 0 independent of (t,x). Denote with $P \in \Psi^0(\mathbb{R}^{d+1})$ the pseudodifferential operator associated with p.

It is important to emphasise the fact that sequences v_n and V_n are both contained in $C_c^\infty(\mathbb{R}^{d+1})$. Therefore, an application of standard pseudodifferential operators to either one of them will ensure that there is still enough regularity (as a matter of fact, the result will be a Schwartz function) so that the calculations that follow are indeed licit.

We now begin by applying P to equation (3.9). Since P is a Fourier multiplier, it commutes with all the derivatives, so we obtain

$$(\rho PV_n^0)' - \mathrm{div}\,(\mathbf{A}PV_n^x) - Pf_n + K\cdot V_n = 0, \tag{3.13}$$

where $K := (K_0, -K_x)$ is a pseudodifferential operator of order 0 defined by

$$K_0 := \partial_t \circ [P,\rho], \qquad K_x = \mathrm{div}\circ[P,A].$$

Here we have also used the shortened notation $PV_n^x = \nabla(Pv_n)$. Next we multiply (3.13) by $\overline{PV_n^0}$, and take real parts. We make use of following elementary identities

$$\mathrm{Re}\,(\rho PV_n^0)'\overline{PV_n^0} = \frac{d}{dt}\frac{1}{2}\left(\rho|PV_n^0|^2\right) + \frac{1}{2}\rho'|PV_n^0|^2,$$

$$\mathrm{Re}\left(\mathrm{div}\,(\mathbf{A}PV_n^x)\overline{PV_n^0}\right) = -\frac{d}{dt}\frac{1}{2}\mathbf{A}PV_n^x\cdot PV_n^x + \frac{1}{2}\mathbf{A}'PV_n^x\cdot PV_n^x + \mathrm{Re}\left(\mathrm{div}\,(\mathbf{A}PV_n^x\overline{PV_n^0})\right),$$

in order to obtain

$$\frac{d}{dt}\left(\frac{1}{2}\rho|PV_n^0|^2 + \frac{1}{2}\mathbf{A}PV_n^x\cdot PV_n^x\right) - \mathrm{Re}\left(\mathrm{div}\,(\mathbf{A}PV_n^x\overline{PV_n^0})\right) + \mathrm{Re}\left((K\cdot V_n)\overline{PV_n^0}\right)$$
$$- \mathrm{Re}\,Pf_n\overline{PV_n^0} + \left(\frac{1}{2}\rho'|PV_n^0|^2 - \frac{1}{2}\mathbf{A}'PV_n^x\cdot PV_n^x\right) = 0.$$

Now let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be an arbitrary real-valued test function. We multiply the equality above by φ, integrate over $\mathbb{R}^d$ and perform integration by parts of the divergence term. Let us set

$$R_n(t) := \int_{\mathbb{R}^d} (q(PV_n))(t)\varphi,$$

where q denotes the positive quadratic form

$$q(\mathbf{v};t,x) = \frac{1}{2}\rho|v_0|^2 + \frac{1}{2}\mathbf{A}v'\cdot v' \qquad \mathbf{v} = (v_0, v') \in \mathbb{C}^{d+1}.$$

The resulting equation can then be written in the following form

$$R_n' + \mathrm{Re}\int_{\mathbb{R}^d} \overline{PV_n^0}\mathbf{A}PV_n^x\cdot\nabla\varphi$$
$$+ \mathrm{Re}\int_{\mathbb{R}^d}\left((K\cdot V_n)\overline{PV_n^0}\right)\varphi + \int_{\mathbb{R}^d} Q'(PV_n)\varphi - \mathrm{Re}\int_{\mathbb{R}^d} Pf_n\overline{PV_n^0}\varphi = 0. \tag{3.14}$$

Lemma 3.2.3. Up to an extraction of a subsequence, R_n converges uniformly to some R on compact intervals of $\mathbb{R}$.

Proof. Let $I \subseteq \mathbb{R}$ be a compact interval. We aim to prove that R_n is bounded in $\mathrm{H}^1(I)$. This will in turn imply the statement of the Lemma, since the inclusion $\mathrm{H}^1(I) \hookrightarrow C(I)$ is compact. Recalling (3.7), we have that

$$V_n \text{ is bounded in } \mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d)). \tag{3.15}$$

In particular, because of the uniform in n compactness of the support of V_n, we have that

$$V_n \text{ is bounded in } \mathrm{L}^4(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d)). \tag{3.16}$$

We now refer to the Corollary 2.3.14 to deduce

$$PV_n \text{ is bounded in } \mathrm{L}^4(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d)). \tag{3.17}$$

From here, using (3.2) and the fact that q is a positive quadratic form, it follows that

$$q(PV_n) \text{ is bounded in } \mathrm{L}^2(\mathbb{R};\mathrm{L}^1(\mathbb{R}^d)). \tag{3.18}$$

Therefore

$$\|R_n\|_{\mathrm{L}^2(I)} \leq \|q(PV_n)\|_{\mathrm{L}^2(\mathbb{R};\mathrm{L}^1(\mathbb{R}^d))}\|\varphi\|_{\mathrm{L}^\infty(\mathbb{R}^d)},$$

the last expression being bounded uniformly in n. Hence,

$$R_n \text{ is bounded in } \mathrm{L}^2(I).$$

Returning back to (3.14) we once again encounter expressions which are quadratic in PV_n, for which the similar reasoning can be applied in order to deduce that R_n' is also bounded in $\mathrm{L}^2(I)$, and subsequently, R_n is bounded in $\mathrm{H}^1(I)$. ■

Remark 3.2.4. Note that the preceeding discussion does not depend on the choice of φ. This allows us to deduce that, for any compact $K \subseteq \mathbb{R}^d$ it holds

$$PV_n \text{ is bounded in } \mathrm{L}^\infty(0,T;\mathrm{L}^2(K)). \tag{3.19}$$

Indeed, by taking $\varphi \geq 0$ with $\varphi \equiv 1$ on K in the definition of R_n, we obtain for $t \in [0,T]$

$$\int_K q(PV_n)(t) \leq \int_{\mathbb{R}^d} q(PV_n)(t)\varphi = R_n(t).$$

Hence, $t \mapsto q(PV_n)(t)$ is bounded in $\mathrm{L}^1(K)$, uniformly in n, and the positive quadratic nature of q once again allows us to deduce (3.19).

We now multiply (3.14) by an arbitrary real-valued $\psi \in C^\infty([0,T))$ and perform integration by parts on $[0,T]$ in order to obtain

$$\begin{aligned} &-\psi(0)R_n(0) - \int_0^T R_n(t)\psi'(t)dt + \mathrm{Re}\int_0^T \left(\mathbf{A}PV_n^x, PV_n^0 \nabla\varphi\right)_{\mathrm{L}^2(\mathbb{R}^d)} \psi dt \\ &+ \mathrm{Re}\int_0^T \left((K\cdot V_n), PV_n^0\varphi\right)_{\mathrm{L}^2(\mathbb{R}^d)} \psi dt + \int_0^T \left(Q'(PV_n),\varphi\right)_{\mathrm{L}^2(\mathbb{R}^d)} \psi dt \\ &+ \int_0^T \left(Pf_n, PV_n^0\varphi\right)_{\mathrm{L}^2(\mathbb{R}^d)} \psi dt = 0. \end{aligned} \tag{3.20}$$

We now wish to pass to the limit in each of the terms in (3.20). As a direct consequence of Lemma 3.2.3, we get for the limit of the first term

$$-\psi(0)R(0). \tag{3.21}$$

The term involving Pf_n vanishes. Indeed, each of the terms in f_n involves derivatives of ϑ, and can be treated in the same way, as we are going to show for the term $\vartheta'\rho u_n'$. We can rewrite it as

$$P(\rho\vartheta' u_n') = \vartheta' P(\rho u_n') + [P,\vartheta'](\rho u_n').$$

Since $\vartheta' \equiv 0$ on $[0,T]$, the first term does not contribute to the computations, while the second term tends strongly to 0 in $\mathrm{L}^2(0,T;\mathbb{R}^d)$ since $[P,\vartheta']$ is of order -1 and $\rho u_n'$ is bounded in $\mathrm{L}^2(0,T;\mathbb{R}^d)$. Hence, the sequence $P(\rho\vartheta' u_n')$ converges strongly to 0 in $\mathrm{L}^2(\mathbb{R}^{d+1})$, and as it has already been mentioned, the other terms can be treated in a similar fashion in order to obtain

$$Pf_n \to 0 \quad \text{strongly in } \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d)).$$

As a product of one strongly convergent sequence converging to zero and one weakly convergent sequence, the last term then vanishes, that is

$$\lim_n \int_0^T \left(Pf_n, PV_n^0\varphi\right)_{\mathrm{L}^2(\mathbb{R}^d)} \psi dt = 0. \tag{3.22}$$

For all of the remaining terms we wish to pass to the limit using their quadratic nature and the definition of H-measure. In order to do so, we introduce a sequence of smooth functions $\phi_m = \phi(m\cdot)$ where ϕ is given by

$$\begin{cases} 0 \le \phi \le 1, \\ \phi(t) = 0, & t \le \frac{1}{2}, \\ \phi(t) = 1, & t \ge 1. \end{cases}$$

We showcase the process for the term $\int_0^T R_n\psi'$; the other terms can be treated in a similar manner. We begin by rewriting

$$\int_0^T R_n\psi' = \int_0^T R_n\psi'\phi_m + \int_0^T R_n\psi'(1-\phi_m).$$

Since $\psi'\phi_m\varphi \in C_c^\infty((0,T)\times\mathbb{R}^d)$, we may pass to the limit (in n, for a fixed m) in the first term, which yields

$$\left\langle \nu, \frac{1}{2}(\rho\tau^2 + \mathbf{A}\xi\cdot\xi)|p|^2\varphi\psi'\phi_m \right\rangle.$$

Letting $m\to\infty$ now yields

$$\lim_m \left\langle \nu, \frac{1}{2}(\rho\tau^2 + \mathbf{A}\xi\cdot\xi)|p|^2\psi'\phi_m \right\rangle = \left\langle \nu, \rho\tau^2|p|^2\varphi\psi'\mathbb{1}_{(0,\infty)} \right\rangle.$$

In order to treat the first term above, we first note that (3.19) implies that $t\mapsto R_n\psi'(1-\phi_m)$ is uniformly bounded in n, for

$$\begin{aligned}\int_0^T R_n(t)\psi'(t)(1-\phi_m(t)) &= \int_0^T\int_{\mathbb{R}^d} q(PV_n)(t)\psi'(t)(1-\phi_m(t))\\ &\leq \|\varphi\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\|\psi'\|_{\mathrm{L}^\infty(\mathbb{R})}\int_0^{\frac{1}{m}}\int_{\operatorname{supp}\varphi} q(PV_n)(t) \leq C\frac{1}{m},\end{aligned}$$

with C independent of n and m. Letting $m\to\infty$ we deduce that the second term tends to zero.

With the notation $\langle\langle \nu,\Phi\rangle\rangle := \left\langle \nu, \Phi\mathbb{1}_{(0,\infty)}\right\rangle$, the limits of the remaining terms of (3.20) can be written as

$$\begin{aligned}\lim_n \int_0^T \left(Q'(PV_n),\varphi\right)_{\mathrm{L}^2(\mathbb{R}^d)}\psi dt &= \left\langle\left\langle \nu, (\xi\cdot k)\tau p^0\varphi\psi\right\rangle\right\rangle\\ \lim_n \int_0^T R_n(t)\psi'(t)dt &= \left\langle\left\langle \nu, \frac{1}{2}(\rho\tau^2+\mathbf{A}\xi\cdot\xi)|p^0|^2\varphi\psi'\right\rangle\right\rangle\\ \lim_n \int_0^T \left(\mathbf{A}PV_n^x, PV_n^0\nabla\varphi\right)_{\mathrm{L}^2(\mathbb{R}^d)}\psi dt &= \left\langle\left\langle \nu, \tau(\mathbf{A}\xi\cdot\nabla\varphi)\psi\tau|p^0|^2\right\rangle\right\rangle\\ \lim_n \int_0^T \left((K\cdot V_n), PV_n^0\varphi\right)_{\mathrm{L}^2(\mathbb{R}^d)}\psi dt &= \left\langle\left\langle \nu, (\xi\cdot k)\tau p^0\varphi\psi\right\rangle\right\rangle.\end{aligned} \tag{3.23}$$

Collecting the results (3.21) to (3.23) together yields the equation

$$\langle\langle \nu, \{\tau\Phi, Q\}\rangle\rangle = \psi(0)R(0), \tag{3.24}$$

where $\Phi(t,x,\tau,\xi)=\psi(t)\varphi(x)|p(\tau,\xi)|^2$. Specifically, if we were to take $\Phi\in C_c^1((0,T)\times\mathbb{R}^d\times\mathbb{S}^d)$, a test function that is compactly supported away from $t=0$, the resulting equation would be

$$\langle\nu,\{\tau\Phi,Q\}\rangle=0. \tag{3.25}$$

Furthermore, we can deduce that (3.25) holds if we replace $\tau\Phi$ by Φ. Indeed, in view of Remark 3.2.2, we can choose a smooth cut-off function θ on the sphere that satisfies $\theta\equiv 0$ in the neighbourhood of $\tau=0$ and $\theta\equiv 1$ on the projection of $\operatorname{supp}\nu$ to $\mathbb{S}^d$. By setting $\tilde\Phi=\Phi\theta/\tau$ we obtain

$$0=\langle\nu,\{\tau\tilde\Phi,Q\}\rangle=\langle\nu,\{\Phi,Q\}\rangle \tag{3.26}$$

for all Φ of the form $\Phi(t,x,\tau,\xi)=\psi(t)\varphi(x)|p(\tau,\xi)|^2$. Since functions of this form are dense in the space $C_c^1((0,T)\times\mathbb{R}^d\times\mathbb{S}^d)$, the following result is valid.

Theorem 3.2.5. The measure ν defined by Theorem 3.2.1 satisfies the equation

$$\langle\nu,\{\Phi,Q\}\rangle=0,\qquad \Phi\in C_c^1((0,T)\times\mathbb{R}^d\times\mathbb{S}^d). \tag{3.27}$$

However, we will now turn our attention back to the case where test functions are allowed to have support in $t=0$ as well, which will allow us to capture the initial value of ν.

3.3. TRACE OF MEASURE

For the sake of simplicity, we introduce some additional notation. Let $x_0 = t$, $\mathbf{x} = (x_0, x)$, $\xi_0 = \tau$ and $\boldsymbol{\xi} = (\xi_0, \xi)$. Equation (3.25) is written in a weak formulation. To obtain the first order equation for ν, one must perform various integrations by parts to transform the expression

$$\begin{aligned}\langle \nu, \{\xi_0\Phi, Q\}\rangle &= \int_0^T \int_{\mathbb{R}^d\times\mathbb{S}^d} \nu\{\xi_0\Phi, Q\}\\ &= \int_0^T \int_{\mathbb{R}^d\times\mathbb{S}^d} \xi_0\nu\left(\nabla_{\boldsymbol{\xi}}\Phi\cdot\nabla_{\mathbf{x}}Q - \nabla_{\mathbf{x}}\Phi\cdot\nabla_{\boldsymbol{\xi}}Q\right) + \int_0^T \int_{\mathbb{R}^d\times\mathbb{S}^d} (\nu Q')\Phi.\end{aligned}$$

Of course, this formulation makes sense when measure ν is absolutely continuous with respect to the product of the Lebesgue measure on $\mathbb{R}^{d+1}$ and the surface measure on $\mathbb{S}^d$, with density $\nu = \nu(\mathbf{x}, \boldsymbol{\xi})$. The general case follows in the same manner, with operations performed being made in the distributional sense. We state the result, a proof of which can be found in [4].

Theorem 3.3.1. The measure ν satisfies a first-order partial differential equation on $(0,T)\times\mathbb{R}^d\times\mathbb{S}^d$:

$$\{Q, \xi_0\nu\} + (\nabla_{\mathbf{x}}Q\cdot\xi)\left(\nabla_{\boldsymbol{\xi}}(\xi_0\nu)\cdot\xi + (d+2)(\xi_0\nu)\right) + Q'\nu = 0. \tag{3.28}$$

If we were to allow test functions $\Phi\in C_c^1([0,T)\times\mathbb{R}^d\times\mathbb{S}^d)$, as it was the case when we inferred equation (3.24), the calculations which led to the transport equation for measure ν would yield

$$\begin{aligned}\langle\langle \nu, \{\xi_0\Phi, Q\}\rangle\rangle &= \int_0^T \int_{\mathbb{R}^d\times\mathbb{S}^d} \{\text{LHS of (3.28)}\}\Phi + \int_{\mathbb{R}^d\times\mathbb{S}^d} \left(\rho(0)\xi_0^2\nu(0)\right)\Phi(0)\\ &= \int_{\mathbb{R}^d\times\mathbb{S}^d} \left(\rho(0)\xi_0^2\nu(0)\right)\Phi(0).\end{aligned}$$

Since $\rho(0)\xi_0^2$ is strictly positive and bounded from below by a positive constant on $\operatorname{supp}\nu$, the last expression defines the trace of measure ν at time $t = 0$. Therefore, in view of equation (3.24), in order to recover the trace of measure ν at $t = 0$, it remains to compute the quantity $R(0)$. In addition, our goal is also to connect that quantity with sequences of initial conditions g_n and h_n, and to prove that it is possible to recover that quantity solely through them.

3.4. CONNECTION WITH THE SEQUENCE OF INITIAL CONDITIONS

From the very definition of H-measure ν, it follows that the sequence R_n satisfies

$$\lim_n \int_{\mathbb{R}} R_n \psi dt = \left\langle \nu, \frac{1}{2}(\rho\tau^2 + \mathbf{A}\xi\cdot\xi)|p|^2\varphi\psi \right\rangle. \tag{3.29}$$

Our aim now is to find another sequence, $\tilde{R}_n$, such that (3.29) holds for $\tilde{R}_n$ in place of R_n with $\tilde{R}_n$ also converging uniformly (on some compact interval around 0). Of course, the quantity $\tilde{R}_n(0)$ is intended to be computable solely based on the information given by the initial conditions g_n and h_n. The main ingredient is the fact that we can factor the expression $\frac{1}{2}\left(\rho\tau^2 + \mathbf{A}\xi\cdot\xi\right)|p|^2$ into two first-order parts which will then in turn be connected to $V_n(0) = (h_n, \nabla g_n)$.

Recall Remark 3.2.2 and the fact that points $(\tau,\xi) = (\pm 1, 0)$ are not in the projection of $\operatorname{supp}\nu$ on $\mathbb{S}^d$. In the sequel, we will therefore assume that p additionally satisfies $p(\tau,\xi) = 0$ on a neighbourhood of $(\pm 1, 0)$, since the expression on the left hand side of (3.24) does not depend on its behaviour there. More precisely, one could decompose tensor product

$$\psi\boxtimes\varphi\boxtimes p = \psi\boxtimes\varphi\boxtimes p\omega + \psi\boxtimes\varphi\boxtimes p(1-\omega),$$

where ω (introduced shortly) is a conic cutoff in (τ,ξ) such that $\omega \equiv 1$ on the projection of $\operatorname{supp}\nu$ on $\mathbb{S}^d$ and $\omega \equiv 0$ on aforementioned neighbourhoods of poles $(\pm 1, 0)$. Then the left hand side of (3.24) reduces to the term $\langle\langle \nu, \{\tau\psi\boxtimes\varphi\boxtimes p, Q\}\rangle\rangle$.

In order to construct such a symbol ω, we proceed as follows. Consider a cone C_α of angle α and direction $\tau = \pm 1$, $\xi = 0$ in $\mathbb{R}^{d+1}$. Then, if $(\tau,\xi) \in \mathbb{R}^{d+1}$ lies outside of C_α, we have

$$\frac{|\xi|}{|\tau|} \geq \tan\alpha,$$

in which case

$$\frac{|\xi|}{\sqrt{\tau^2 + |\xi|^2}} \geq \sin\alpha.$$

For any small α we are at liberty to choose χ to satisfy

$$\chi(s) = \begin{cases} 1, & |s| \leq \sin\alpha, \\ 0, & |s| \geq \sin 2\alpha, \end{cases}$$

in which case

$$\chi\left(\frac{|\xi|}{\sqrt{\tau^2+|\xi|^2}}\right) = \begin{cases} 1, & (\tau,\xi) \in C_\alpha, \\ 0, & (\tau,\xi) \notin C_{2\alpha}, \end{cases}$$

so that

$$\omega(\tau,\xi) = \begin{cases} 0, & (\tau,\xi) \in C_\alpha, \\ 1, & (\tau,\xi) \notin C_{2\alpha}. \end{cases} \tag{3.30}$$

Of course, ω is then to be multiplied by a smooth cut-off around 0.

We now begin our process of computing the expression $R(0)$ by introducing cut-off function $\zeta \in C_c^\infty(\mathbb{R}^{d+1})$ such that $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ on $[0,T] \times K'$, where $K' \subseteq \mathbb{R}^d$ is a compact containing the joint support of spatial projections of v_n, so that $\zeta \equiv 1$ on $\operatorname{supp} V_n$, and subsequently on $\operatorname{supp} \nu$. Define

$$\lambda(t,x,\xi) = -2\pi i \zeta(t,x)\left(\mathbf{A}(t,x)\xi\cdot\xi\right)^{1/2}\chi(\xi),$$

where χ is a smooth cutoff around $\xi = 0$, in order to have $\lambda \in C^\infty(\mathbb{R}^{d+1}\times\mathbb{R}^d)$. Note that $\lambda \in S_c^1(\mathbb{R}^{d+1}\times\mathbb{R}^d)$. Associated with it, we define time-dependent pseudodifferential operator $\Lambda = \lambda(t,x,D_x)$ given by

$$\lambda(t,x,D_x)u(t,x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\lambda(t,x,\xi)\mathscr{F}_x u(t,\xi)d\xi. \tag{3.31}$$

We shall sometimes also write $\Lambda(t) = \lambda(t,x,D_x)$.

Remark 3.4.1. (a) As it has already been noted in remarks following (2.25), Λ sends $\mathscr{S}(\mathbb{R}^{d+1})$ into $C_c^\infty(\mathbb{R}^{d+1})$, due to the compact support of ζ in (t,x).

(b) Since the time derivatives of symbol λ are also bounded due to (3.2) and the coercivity property of $\mathbf{A}$, we can perform differentiation under the integral sign in (3.31), thus obtaining

$$\frac{d}{dt}\Lambda(t)u = \Lambda'(t)u + \Lambda(t)u', \tag{3.32}$$

where with $\Lambda'(t)$ we denote the operator whose symbol is given with λ'.

(c) Furthermore, due to (2.27), Λ satisfies

$$\sup_{t\in\mathbb{R}} \|\Lambda(t)u(t,\cdot)\|_{\mathrm{H}^{m-1}(\mathbb{R}^d)} \leq C \sup_{t\in\mathbb{R}} \|u(t,\cdot)\|_{\mathrm{H}^m(\mathbb{R}^d)}. \tag{3.33}$$

Next we set

$$v_n^\pm = \zeta\sqrt{\rho}v_n' \pm \Lambda v_n. \tag{3.34}$$

Previous remarks allow us to conclude that $v_n^\pm \in C_c^\infty(\mathbb{R}^{d+1})$ and

$$v_n^\pm \overset{*}{\rightharpoonup} 0 \quad \text{in } \mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d)). \tag{3.35}$$

Moreover, we can recover the initial data of $v_n^\pm$ as well, and they are given by formulae

$$v_n^\pm(0) = \zeta(0)\sqrt{\rho(0)}h_n \pm \Lambda(0)g_n. \tag{3.36}$$

After denoting $v_n^\pm(0) := v_{n0}^\pm$, the compactness of the support of $v_n^\pm$ also yields

$$v_{n0}^\pm \rightharpoonup 0 \quad \text{in } \mathrm{L}^2(\mathbb{R}^d). \tag{3.37}$$

In turn, these sequences allow for extraction of associated H-measures, to which the expression $R(0)$ will be connected. Note that the computation of these two H-measures relies completely upon the initial conditions g_n and h_n, which is our goal.

Since v_n is compactly supported in $\mathbb{R}^{d+1}$, we can rewrite it as

$$v_n = \Delta^{-1}\left(\operatorname{div} V_n^x\right), \tag{3.38}$$

with Δ^{-1} being an inverse of the Laplacian with Dirichlet boundary conditions on a domain large enough that it contains the spatial projection of the common support of v_n, K'. Hence, we can rewrite v_n^+ as

$$v_n^+ = \zeta\sqrt{\rho}V_n^0 + \tilde{\Lambda}V_n^x, \tag{3.39}$$

where $\tilde{\Lambda}$ is given with

$$\tilde{\Lambda} = \Lambda \circ \Delta^{-1} \circ \operatorname{div}, \tag{3.40}$$

and its symbol is then in turn given by

$$\tilde{\lambda}(t,x,\xi) = -\zeta(t,x)\sqrt{\mathbf{A}(t,x)\xi\cdot\xi}\,\frac{\xi}{|\xi|}\chi(\xi). \tag{3.41}$$

Now, according to Theorem 2.5.2, $P\tilde{\Lambda}$ is a well defined pseudodifferential operator in $\Psi_c^0(\mathbb{R}^{d+1})$. Hence, we can compute the limit of the expressions

$$\int_{\mathbb{R}^{d+1}} |Pv_n^{\pm}|^2 \varphi\psi$$

through H-measure of the sequence V_n: we obtain

$$\lim_n \int_{\mathbb{R}^{d+1}} |Pv_n^{\pm}|^2 \varphi\psi = \left\langle \nu, \zeta^2 \left(\sqrt{\rho}\tau \mp \sqrt{\mathbf{A}\xi\cdot\xi}\chi\right)^2 |p|^2 \varphi\psi \right\rangle. \tag{3.42}$$

Recalling the fact that $\zeta \equiv 1$ on $\operatorname{supp}\nu$ and $\chi \equiv 1$ on $\mathbb{S}^d$, we obtain the following relation

$$\lim_n R_n = \lim_n \frac{1}{4} \int_{\mathbb{R}^d} \left(|Pv_n^+|^2 + |Pv_n^-|^2\right) \varphi \quad \text{ in } \mathscr{D}'(\mathbb{R}). \tag{3.43}$$

For the next step, we notice that (3.35) allows us to determine, for an extracted subsequence, H-measures $\nu^{\pm}$. By introducing pseudodifferential operator $\Omega \in \Psi^0(\mathbb{R}^{d+1})$ whose symbol is given by ω defined in (3.30), we can rewrite $v_n^{\pm}$ as

$$v_n^{\pm} = \zeta\sqrt{\rho}\Omega V_n^0 \pm \tilde{\Lambda}\Omega V_n^x + \zeta\sqrt{\rho}(1-\Omega)V_n^0 \pm \tilde{\Lambda}(1-\Omega)V_n^x. \tag{3.44}$$

Note that because of the choice of ω we have for every $\phi \in C_c^\infty(\mathbb{R}^{d+1})$

$$\lim_n \|\phi(1-\Omega)V_n\|^2_{\mathrm{L}^2(\mathbb{R}^{d+1})} = \langle \nu, \phi^2(1-\omega)^2 \rangle = 0 \tag{3.45}$$

Since $\tilde{\Lambda}$ is a continuous operator on $\mathrm{L}^2(\mathbb{R}^{d+1})$ which includes cutoff $\zeta \in C_c^\infty(\mathbb{R}^{d+1})$ in its definition, it follows that

$$\zeta\sqrt{\rho}(1-\Omega)V_n^0 \pm \tilde{\Lambda}(1-\Omega)V_n^x \longrightarrow 0 \quad \text{ strongly in } \mathrm{L}^2(\mathbb{R}^{d+1}). \tag{3.46}$$

Therefore, H-measures $\nu^{\pm}$ associated to sequences $v_n^{\pm}$ coincide with those of sequences given by

$$\zeta\sqrt{\rho}\Omega V_n^0 \pm \tilde{\Lambda}\Omega V_n^x, \tag{3.47}$$

which allows us to express them in the following way

$$\nu^{\pm} = \zeta^2 \left(\sqrt{\rho}\tau \mp \sqrt{\mathbf{A}\xi\cdot\xi}\chi\right)^2 \nu = \left(\sqrt{\rho}\tau \mp \sqrt{\mathbf{A}\xi\cdot\xi}\right)^2 \nu, \tag{3.48}$$

where the last equation follows from the fact that $\zeta, \chi \equiv 1$ on $\operatorname{supp}\nu$.

Recalling the fact that $\operatorname{supp}\nu \subseteq \{(t,x,\tau,\xi) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} : \rho\tau^2 - \mathbf{A}\xi\cdot\xi = 0\}$, it follows from (3.48) that

$$\operatorname{supp}\nu^{\pm} \subseteq \left\{(t,x,\tau,\xi) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} : \tau = \mp\sqrt{\frac{\mathbf{A}(t,x)\xi\cdot\xi}{\rho(t,x)}}\right\}. \tag{3.49}$$

As a consequence, we would now like to replace τ in $p(\tau,\xi)$ using (3.49). We therefore introduce $p^{\pm}$ given with

$$p^{\pm}(t,x,\xi) = p\left(\mp\sqrt{\frac{\mathbf{A}(t,x)\xi\cdot\xi}{\rho(t,x)}},\xi\right). \tag{3.50}$$

Accordingly, we define operators $P^{\pm}$ whose symbols will be precisely $p^{\pm}$; the operators $P^{\pm}$ are once again to be considered as spatial pseudodifferential operators dependent on time parameter. Define

$$\tilde{R}_n(t) = \frac{1}{4}\int_{\mathbb{R}^d}\left(|P^+v_n^+|^2 + |P^-v_n^-|^2\right)\varphi. \tag{3.51}$$

Our goal now is to show that $\tilde{R}_n$ is indeed the desired function mentioned in the introduction. First we show that it satisfies the limit equality similar to (3.43). It is sufficient to show that

$$Pv_n^{\pm} - P^{\pm}v_n^{\pm} \longrightarrow 0 \quad \text{strongly in } \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{d+1}). \tag{3.52}$$

We once again make use of the operator Ω in order to make $P^{\pm}\Omega$ a viable pseudodifferential operator on $\mathbb{R}^{d+1}$ and to decompose (3.52) into

$$\left(P - P^{\pm}\Omega v_n^{\pm}\right)v_n^{\pm} - P^{\pm}(1-\Omega)v_n^{\pm}. \tag{3.53}$$

Multiplying by $\phi \in C_c^{\infty}(\mathbb{R}^{d+1})$ and taking L^2 norm gives

$$\left\|\phi\left(P^{\pm}v_n^{\pm} - Pv_n^{\pm}\right)\right\|_{\mathrm{L}^2(\mathbb{R}^{d+1})} \leq \left\|\phi\left(P - P^{\pm}\Omega\right)v_n^{\pm}\right\|_{\mathrm{L}^2(\mathbb{R}^{d+1})} + \left\|\phi P^{\pm}(1-\Omega)v_n^{\pm}\right\|_{\mathrm{L}^2(\mathbb{R}^{d+1})}. \tag{3.54}$$

Since $\operatorname{supp}\nu^{\pm} \subseteq \operatorname{supp}\nu$, the same argument as in (3.45) yields

$$(1-\Omega)v_n^{\pm} \longrightarrow 0 \quad \text{strongly in } \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{d+1}). \tag{3.55}$$

As $P^{\pm}$ are continuous on $\mathrm{L}^2(\mathbb{R}^{d+1})$, we deduce

$$\left\|\phi P^{\pm}(1-\Omega)v_n^{\pm}\right\|_{\mathrm{L}^2(\mathbb{R}^{d+1})} \longrightarrow 0. \tag{3.56}$$

For the first term in (3.54) we can compute the limit using H-measures of sequences $v_n^\pm$; we obtain

$$\lim_n \left\| \left(P^\pm v_n^\pm - P v_n^\pm\right) \phi \right\|^2_{\mathrm{L}^2(\mathbb{R}^{d+1})} = \langle \nu^\pm, \phi^2 (p - p^\pm \omega)^2 \rangle. \tag{3.57}$$

Since $\omega \equiv 1$ on $\operatorname{supp} \nu^\pm$, and $p = p^\pm$ on $\operatorname{supp} \nu^\pm$ by construction, we conclude

$$\lim_n \left\| \left(P^\pm v_n^\pm - P v_n^\pm\right) \phi \right\|^2_{\mathrm{L}^2(\mathbb{R}^{d+1})} = 0.$$

Thus, we have proven (3.52). From here it easily follows that

$$\lim_n R_n = \lim_n \tilde{R}_n \text{ in } \mathscr{D}'(\mathbb{R}). \tag{3.58}$$

The final step is now devoted to showing that $\tilde{R}_n$ converges uniformly on compact time intervals. Once we have shown that, we can compute quantity $R(0)$ as the limit

$$\lim_n \tilde{R}_n(0) = \lim_n \frac{1}{4} \int_{\mathbb{R}^d} \left(|P_0^+ v_n^+(0)|^2 + |P_0^- v_n^-(0)|^2 \right) \varphi. \tag{3.59}$$

As (3.37) holds, we can extract H-measures $\nu_0^\pm$ associated with subsequences of $v_n^\pm(0)$; the previous expression can thus be computed as

$$\lim_n \frac{1}{4} \int_{\mathbb{R}^d} \left(|P_0^+ v_n^+(0)|^2 + |P_0^- v_n^-(0)|^2 \right) \varphi = \frac{1}{4} \left(\left\langle \nu_0^+, |p^+(0)|^2 \varphi \right\rangle + \left\langle \nu_0^-, |p^-(0)|^2 \varphi \right\rangle \right) \tag{3.60}$$

Our goal now is to mimic the arguments which lead to concluding uniform convergence of R_n, that is, we prove that $\tilde{R}_n$ converges uniformly on compact intervals $I \subseteq \mathbb{R}$ by proving it is a bounded sequence in $\mathrm{H}^1(I)$. The fact that $\tilde{R}_n$ is bounded in $\mathrm{L}^2(I)$ follows immediately. In order to show the bound for the derivatives, first note that we can write $v_n^\pm$ as

$$v_n^\pm = (\zeta \sqrt{\rho} \partial_t \pm \Lambda) v_n, \tag{3.61}$$

and consequently

$$\zeta \sqrt{\rho} (v_n^\pm)' \mp \Lambda v_n^\pm = (\zeta \sqrt{\rho} \partial_t \mp \Lambda)(\zeta \sqrt{\rho} \partial_t \pm \Lambda) v_n = \zeta^2 \rho v_n'' - \Lambda^2 v_n \pm [\zeta \sqrt{\rho} \partial_t, \Lambda] v_n. \tag{3.62}$$

To calculate the term $\Lambda^2 v_n$, we note that the principal symbol of Λ^2 is given by $\lambda^2(t,x,\xi) = -4\pi^2 \zeta^2 \mathbf{A} \xi \cdot \xi \chi^2$, hence we have

$$\widehat{\Lambda^2 v_n} = \lambda^2 \widehat{v_n} + T_{-1} v_n = (-4\pi^2 \zeta^2 \mathbf{A} \xi \cdot \xi) \widehat{v_n} - (4\pi^2 \zeta^2 \mathbf{A} \xi \cdot \xi (1 - \chi^2)) \widehat{v_n} + T_{-1} v_n,$$

where T_{-1} is a time dependent spatial pseudodifferential operator that sends boundedly $\mathrm{H}^1(\mathbb{R}^d)$ into $\mathrm{L}^2(\mathbb{R}^d)$, uniformly in time. Since the second term is associated with an operator with smooth compactly supported kernel, the same conclusion applies. Therefore, we can write

$$\Lambda^2 v_n = \zeta^2 \mathbf{A} : \nabla^2 v_n + r_n, \tag{3.63}$$

with r_n bounded in $\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$. For the last term, note first that we do not consider commmutator $[\zeta\sqrt{\rho}\partial_t,\Lambda]$ as an pseudodifferential operator on $\mathbb{R}^{d+1}$; it is merely a shortened notation. Writing out explicitly that term gives

$$[\zeta\sqrt{\rho}\partial_t,\Lambda]v_n = [\zeta\sqrt{\rho},\Lambda]v_n' + \zeta\sqrt{\rho}\Lambda' v_n. \tag{3.64}$$

First commutator is then viewed as a time dependent spatial pseudodifferential operator, which is now of order 0, and therefore the first term on the right is bounded in $\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$. The same holds for the second term, so we deduce that $[\zeta\sqrt{\rho}\partial_t,\Lambda]$ is bounded in $\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$. Collecting previous conclusions allows us to rewrite (3.62) as

$$\zeta\sqrt{\rho}(v_n^\pm)' \pm \Lambda v_n^\pm = \zeta^2(\rho v_n'' - \mathbf{A} : \nabla^2 v_n) + r_n^\pm, \tag{3.65}$$

with $r_n^\pm$ bounded in $\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$. Finally, since the first term can be rewritten as

$$\zeta^2((\rho v_n')' - \operatorname{div}(\mathbf{A}\nabla v_n)) - \zeta^2(\rho' v_n' - \operatorname{div}\mathbf{A}\cdot\nabla v_n),$$

we conclude that $\zeta\sqrt{\rho}(v_n^\pm)' \pm \Lambda v_n^\pm$ is bounded in $\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$. We will, for simplicity, denote the entire right-hand side of (3.65) with $r_n^\pm$, that is,

$$\zeta\sqrt{\rho}(v_n^\pm)' \pm \Lambda v_n^\pm = r_n^\pm, \tag{3.66}$$

with $r_n^\pm$ bounded in $\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$.

Applying $P^\pm$ to (3.66) yields

$$\zeta\sqrt{\rho}(P^\pm v_n^\pm)' \mp \Lambda P^\pm v_n^\pm = P^\pm r_n^\pm - [P,\zeta\sqrt{\rho}\partial_t \mp \Lambda]v_n^\pm. \tag{3.67}$$

Arguments similar to those used in the last step show that the right-hand side is bounded in $\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$, and we denote it with $\tilde{r}_n^\pm$. Multiplying (3.67) by $\varphi\overline{P^\pm v_n^\pm}$, taking real parts and integrating over $\mathbb{R}^d$ yields

$$\frac{1}{2}\int_{\mathbb{R}^d}\zeta\sqrt{\rho}\left(\frac{d}{dt}|P^\pm v_n^\pm|^2\right)\varphi = \int_{\mathbb{R}^d}\Lambda P^\pm v_n^\pm\overline{P^\pm v_n^\pm}\varphi - \int_{\mathbb{R}^d}\tilde{r}_n^\pm\overline{P^\pm v_n^\pm}\varphi. \tag{3.68}$$

The last term in (3.68) can be bounded as

$$\int_{\mathbb{R}^d} \tilde{r}_n^{\pm} \overline{P^{\pm} v_n^{\pm}} \varphi \leq \|\varphi\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \|\tilde{r}_n^{\pm}(t)\|_{\mathrm{L}^2(\mathbb{R}^d)} \|P^{\pm}(t) v_n^{\pm}(t)\|_{\mathrm{L}^2(\mathbb{R}^d)} \leq C.$$

The remaining term on the right-hand side of (3.68) can first be rewritten as

$$\int_{\mathbb{R}^d} \Lambda P^{\pm} v_n^{\pm} \overline{P^{\pm} v_n^{\pm}} \varphi = \frac{1}{2} \left(\int_{\mathbb{R}^d} \Lambda P^{\pm} v_n^{\pm} \overline{\varphi P^{\pm} v_n^{\pm}} + \int_{\mathbb{R}^d} \Lambda(\varphi P^{\pm} v_n^{\pm}) \overline{P^{\pm} v_n^{\pm}} \right) + \frac{1}{2} \int_{\mathbb{R}^d} [\varphi, \Lambda] P^{\pm} v_n^{\pm} \overline{P^{\pm} v_n^{\pm}}.$$

To treat the first term on the right, we note that $\Lambda(t)$ is an antihermitian operator for each t, up to an operator of order 0; this follows from the asymptotic expansion for the symbol of adjoint, yielding $\sigma(\Lambda(t)^*) = \overline{\lambda(t)} = -\lambda(t)$. Therefore $\Lambda(t) + \Lambda(t)^* \in \Psi_c^0(\mathbb{R}^d)$. The second term is then bounded by $\|P^{\pm} v_n^{\pm}\|^2_{\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))}$ since $[\varphi, \Lambda]$ is of order 0. Therefore, we conclude that

$$\int_{\mathbb{R}^d} \zeta \sqrt{\rho} \left(\frac{d}{dt} \left|P^{\pm} v_n^{\pm}\right|^2 \right) \varphi \text{ is bounded in } \mathbb{R}. \tag{3.69}$$

Taking $I = [0, T]$ we can replace ζ with 1 and together with the coercivity property of $\sqrt{\rho}$ we conclude that $\tilde{R}'_n$ is bounded on $[0, T]$, which proves the claim.

The same density argument used in conclusion of the proof of Theorem 3.2.5 now yields the main result of this chapter given by the following theorem.

Theorem 3.4.2. The measure ν defind in Theorem 3.2.1 satisfies for any $\Phi \in C_c^1([0, T) \times \mathbb{R}^d \times \mathbb{S}^d)$:

$$\langle\langle \nu, \{\tau\Phi, Q\} \rangle\rangle = \frac{1}{4} \Big(\big\langle \nu_0^+, \Phi_0^+ \big\rangle + \big\langle \nu_0^-, \Phi_0^- \big\rangle \Big). \tag{3.70}$$

Here, $\Phi_0^{\pm}$ denotes functions defined on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ given by

$$\Phi_0^{\pm}(x, \xi) = \Phi \left(0, x, \pm \left(\frac{\mathbf{A}(0,x)\xi \cdot \xi}{\rho(0,x) + \mathbf{A}(0,x)\xi \cdot \xi} \right)^{1/2}, \left(\frac{\rho(0,x)}{\rho(0,x) + \mathbf{A}(0,x)\xi \cdot \xi} \right)^{1/2} \right),$$

and $\nu_0^{\pm}$ are the H-measures defined by sequences

$$v_{n0}^{\pm} = \zeta \sqrt{\rho(0)} h_n \pm \Lambda(0) g_n,$$

where ζ is a cut-off function that is equal to 1 on $\operatorname{supp} \nu$, and $\Lambda(0)$ denotes the element of $\Psi_c^1(\mathbb{R}^d)$ associated with symbol $-2\pi i \zeta \left(\mathbf{A}(0,x)\xi \cdot \xi\right)^{1/2}$.

3.5. SEMILINEAR WAVE EQUATION SETTING

Consider now the three-dimensional semilinear wave equation

$$\begin{cases} (\rho u_n')' - \operatorname{div}(\mathbf{A}\nabla u_n) + \mathbf{F}(u_n) = 0, \\ u_n(0) = g_n, \\ u_n'(0) = h_n. \end{cases} \tag{3.71}$$

Here, we assume that $\rho = \rho(t,x)$ and $\mathbf{A} = \mathbf{A}(t,x)$ satisfy same smoothness, coercivity and boundedness properties as in the linear case in Section 3.1. Initial data g_n and h_n are once again assumed to be compactly supported smooth functions with common support K, converging weakly to zero in $\mathrm{H}^1(\mathbb{R}^d)$ and $\mathrm{L}^2(\mathbb{R}^d)$ respectively. Nonlinear function $F : \mathbb{R} \to \mathbb{R}$ is assumed to be a smooth function satisfying sign condition (1.13), with primitive function G such that $G(0) = 0$. Assume additionally that F satisfies the following growth conditions:

$$\left|F^{(j)}(z)\right| \leq C|z|^{\frac{d}{d-2}-j}, \qquad j \in \mathbb{N}_0. \tag{3.72}$$

As a consequnce, we have

$$G(z) = \int_0^z F(w)dw \leq C\int_0^z |w|^{\frac{d}{d-2}}dw \leq C|z|^{\frac{2d-2}{d-2}}. \tag{3.73}$$

Under these conditions, there exists a unique global smooth solution to the initial value problem (3.71) ([37] and references therein). Owing to the finite speed of propagation, we have that for any fixed time t, solutions u_n are supported in the common compact subset of $\mathbb{R}^{d+1}$, and we can once again look at (3.71) as a Cauchy problem with Dirichlet boundary condition for Ω large enough so that the spatial projection of the common compact support of u_n, up until time T, is contained in Ω.

Multiplying the equation (3.71) by u_n', integrating over $\mathbb{R}^d$ and performing integration by parts in the divergence term we obtain for each $t > 0$

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} \rho(u_n')^2 + \mathbf{A}\nabla u_n \cdot \nabla u_n + \mathbf{G}(u_n) &= \int_{\mathbb{R}^d} \rho'(u_n')^2 + \mathbf{A}'\nabla u_n \cdot \nabla u_n \\ &\leq \frac{\|\rho'\|_{\mathrm{L}^\infty(\mathbb{R}^{d+1})} + \|\mathbf{A}'\|_{\mathrm{L}^\infty(\mathbb{R}^{d+1})}}{\alpha} \int_{\mathbb{R}^d} \rho(u_n')^2 + \mathbf{A}\nabla u_n \cdot \nabla u_n + \mathbf{G}(u_n). \end{aligned} \tag{3.74}$$

An application of the Gronwall's lemma yields the bound of the quantities $\|u_n\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^1(\mathbb{R}^d))}$ and $\|u_n\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^d))}$ for each $T > 0$ in terms of norms $\|g_n\|_{\mathrm{H}^1(\mathbb{R}^d)}$ and $\|h_n\|_{\mathrm{L}^2(\mathbb{R}^d)}$, which are uniformly bounded in n due to their respective weak convergences. Note that we have also used the estimate as in (3.73) to uniformly bound $\|\mathbf{G}(u_n(0))\|_{\mathrm{L}^1(\mathbb{R}^d)} = \|\mathbf{G}(g_n)\|_{\mathrm{L}^1(\Omega)} \leq C\|g_n\|_{\mathrm{L}^{(2d-2)/(d-2)}} \leq C\|g_n\|_{\mathrm{H}^1(\mathbb{R}^d)}$ due to continuous embedding of H^1 into $\mathrm{L}^{(2d-2)/(d-2)}$.

Therefore, we obtain the following

$$u_n \text{ is bounded in } \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d)) \cap \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{R};\mathrm{H}^1(\mathbb{R}^d)). \tag{3.75}$$

We first prove the result analogous to that of 3.1.2.

Lemma 3.5.1. For each $T > 0$ the sequence of solutions u_n to (3.71) satisfies

$$u_n \overset{*}{\rightharpoonup} 0 \quad \text{in } \mathrm{L}^{\infty}(0,T;\mathrm{H}^1(\mathbb{R}^d)) \cap \mathrm{W}^{1,\infty}(0,T;\mathrm{L}^2(\mathbb{R}^d)). \tag{3.76}$$

Proof. Given (3.75), there exists $u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d)) \cap \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{R};\mathrm{H}^1(\mathbb{R}^d))$ such that

$$\begin{aligned} u_n &\overset{*}{\rightharpoonup} u \quad \text{in } \mathrm{L}^{\infty}(0,T;\mathrm{H}^1(\mathbb{R}^d)) \\ u_n' &\overset{*}{\rightharpoonup} u' \quad \text{in } \mathrm{L}^{\infty}(0,T;\mathrm{L}^2(\mathbb{R}^d)). \end{aligned} \tag{3.77}$$

Since the first convergence implies strong convergence in $\mathrm{L}^2((0,T)\times\mathbb{R}^d)$, we may as well assume that u_n converges to u almost everywhere. Therefore, $\mathbf{F}(u_n) \to \mathbf{F}(u)$ almost everywhere. As

$$\|\mathbf{F}(u_n)\|_{\mathrm{L}^{\infty}(0,T;\mathrm{L}^2(\mathbb{R}^d))} \leq C\|u_n\|_{\mathrm{L}^{\infty}(0,T;\mathrm{L}^{\frac{2d}{d-2}}(\mathbb{R}^d))} \leq C\|u_n\|_{\mathrm{L}^{\infty}(0,T;\mathrm{H}^1(\mathbb{R}^d))},$$

we deduce $F(u_n)$ converges weakly (after possibly extracting a subsequence) in $\mathrm{L}^2((0,T)\times\mathbb{R}^d)$, and the limit therefore must be $\mathbf{F}(u)$, that is

$$\mathbf{F}(u_n) \rightharpoonup \mathbf{F}(u) \qquad \text{in } \mathrm{L}^2((0,T)\times\mathbb{R}^d). \tag{3.78}$$

Hence, as u_n is a solution of (3.71), through (3.77) and (3.78) we obtain in the sense of distributions

$$(\rho u')' - \operatorname{div}(\mathbf{A}\nabla u) + \mathbf{F}(u) = 0.$$

One then checks that u satisfies intital conditions $u(0) = u'(0) = 0$ in a standard way. Since the solution is unique due to Proposition 1.3.4, the proof is completed. ■

For the sake of simplifying further discussion, we shall instead of applying a smooth cut-off function in time, assume that u_n is already a sequence of compactly supported functions in $\mathbb{R}^{d+1}$ with a common compact support. Consequently, we can replace local spaces with ordinary ones in the energy estimates stated above. Additionally, we now have

$$u_n \overset{*}{\rightharpoonup} 0 \qquad \text{in } \mathrm{L}^\infty(\mathbb{R};\mathrm{H}^1(\mathbb{R}^d)),$$
$$(u_n', \nabla u_n) =: U_n \overset{*}{\rightharpoonup} 0 \qquad \text{in } \mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d)).$$

Sequence U_n now defines an H-measure μ. The localisation principle is once again applied in order to reduce the form of μ: since the vector field U_n has time-space curl equal to zero, it is once again obtained that μ is of the form $\mu = ((\tau,\xi)\otimes(\tau,\xi))\,\nu$ for a positive scalar Radon measure ν. Furthermore, we have

$$\operatorname{div}(\rho U_n^0, -\mathbf{A}U_n^x) = -\mathbf{F}(u_n).$$

As we have already seen, $\mathbf{F}(u_n)$ converges weakly to 0 in $\mathrm{L}^2(\mathbb{R}^{d+1})$. Since we are working on a fixed bounded set in $\mathbb{R}^{d+1}$, this implies $\mathbf{F}(u_n)$ is relatively compact in $\mathrm{H}^{-1}_{\mathrm{loc}}$. Therefore, we have $\mathbf{F}(u_n) \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}(\mathbb{R}^{d+1})} 0$. Another application of localization principle now yields that the scalar measure ν satisfies $Q\nu = 0$, where Q is defined as in previous section.

We now proceed to see the effect this added nonlinearity has on the transport equation. In order to capture the effect, we observe instead the sequence $(U_n, \mathbf{F}(u_n))_n$, which is now a sequence converging weakly to 0 in $\mathrm{L}^2(\mathbb{R}^{d+1};\mathbb{R}^{d+2})$. Denote the H-measure associated to the subsequence by

$$\tilde{\mu} = \begin{bmatrix} \mu & \mu_1 \\ (\mu_1)^* & \mu_2 \end{bmatrix},$$

where μ is the aforementioned H-measure associated to U_n, and $\mu_1 = \begin{bmatrix} \tilde{\mu}_{0,d+1} & \dots & \tilde{\mu}_{d,d+1} \end{bmatrix}^T$ and $\mu_2 = \tilde{\mu}_{d+1,d+1}$ are corresponding Radon measures. We now return to the equation in (3.71), and mimic the procedure of the previous section; notation used will hereby follow the pattern. Apply $P \in \Psi^0(\mathbb{R}^{d+1})$ with symbol $p = p(\tau,\xi)$ and multiply the resulting equation by $\overline{PU_n^0}$ and obtain

$$\frac{1}{2}\frac{d}{dt}\left(\rho|PU_n^0|^2 + \mathbf{A}PU_n^x\cdot PU_n^x\right) - \mathrm{Re}\left(\operatorname{div}(\mathbf{A}PU_n^x\overline{PU_n^0})\right) + \mathrm{Re}\left((K\cdot U_n)\overline{PU_n^0}\right)$$
$$+P(\mathbf{F}(u_n))\overline{PU_n^0} + \frac{1}{2}\left(\rho'|PU_n^0|^2 - \mathbf{A}'PU_n^x\cdot PU_n^x\right) = 0. \tag{3.79}$$

We now multiply (3.79) by $\psi \boxtimes \varphi \in C_c^\infty(0,T) \boxtimes C_c^\infty(\mathbb{R}^d)$ and integrate over $\mathbb{R}^{d+1}$. Performing integration by parts in the first term (in the time variable) and the second term (spatial variables) we obtain

$$\int_0^T \int_{\mathbb{R}^d} q(PU_n)\psi'\varphi + \mathrm{Re}\int_0^T \int_{\mathbb{R}^d} (\mathbf{A}PU_n^x \cdot \nabla\varphi)\overline{PU_n^0}\psi + \mathrm{Re}\int_0^T \int_{\mathbb{R}^d} (K \cdot U_n)\overline{PU_n^0}\psi\varphi$$
$$\int_0^T \int_{\mathbb{R}^d} P(\mathbf{F}(u_n))\overline{PU_n^0}\psi\varphi + \int_0^T \int_{\mathbb{R}^d} Q'(PU_n)\psi\varphi = 0. \tag{3.80}$$

We now pass to the limit $n \to \infty$, and note that every term is treated in the same manner as in the linear case, with the exception of the new term that appears including nonlinearity. Using the definition of H-measure $\tilde{\mu}$, we have

$$\int_0^T \int_{\mathbb{R}^d} P(\mathbf{F}(u_n))\overline{PU_n^0}\psi\varphi \to \langle \tilde{\mu}_{1,d+1}, \psi\varphi|p|^2\rangle.$$

Using density argument therefore yields the following result.

Corollary 3.5.2. The measure $\tilde{\mu}$ satisfies the equation

$$\langle \nu, \{\Phi, Q\}\rangle = \langle \mathrm{Re}\,\tilde{\mu}_{1,d+1}, \Phi\rangle, \qquad \Phi \in C_c^1((0,T)\times\mathbb{R}^d\times\mathbb{S}^d). \tag{3.81}$$

Remark 3.5.3. If we were to assume an even stronger bound on the growth of F , namely $F(z) \leq C|z|^{\frac{d}{d-2}-\varepsilon}$ for some $\varepsilon > 0$, then the sequence $\mathbf{F}(u_n)$ turns out to be strongly convergent in $\mathrm{L}^2(\mathbb{R}^{d+1})$, due to the compact inclusion $\mathrm{H}^1 \hookrightarrow \mathrm{L}^q$ on bounded sets for $q < \frac{2d}{d-2}$. In that case, the nonlinear term in (3.80) converges strongly to 0, and we obtain the same transport equation as for the linear case.

CONCLUSION

This thesis studied evolution equations of second order in t, with emphasis on two examples: the linear wave equation, and the semilinear wave equation.

In the first part of the thesis, we first revisited some standard results regarding the existence, uniqueness and well-posedness of semilinear wave equation with absolutely continuous coefficients in time in the linear part. Afterwards, the standard approach to obtaining weak solutions of such equations (for example, [67]) with the approach used in [25], which allowed to lower the regularity assumptions on the coefficients in the linear part to now only be of bounded variation in time.

In the second part, we introduce first the notion of H-measures in the penultimate chapter, as it was done independently by Tartar [71] and Gerard [36], and draw some comparisons between the approaches. A special part is dedicated to introducing the theory of pseudodifferential operators with symbols in Hörmander classes, as well as some preparatory remarks at the end of Chapter 2, targeted towards specific applications in the last chapter.

The third, and final chapter, is then devoted to studying transport properties of H-measures associated to the sequence of solutions of wave equations with oscillating initial data. Here we generalise the result of Francfort and Murat [34], obtaining the initial-value problem for the H-measure in the case of linear wave equation with varying, time-dependent coefficients. Finally, we conclude the thesis with a discussion on a specific of semilinear wave equation, with additional growth conditions on the nonlinear term. In this case, we show that the added nonlinearity does not contribute to the microlocal properties of the sequence of solutions in a certain case, thus extending the Gérard's [37] result of linearisability to the variable coefficient case.

BIBLIOGRAPHY

[1] C. RAYMOND ADAMS, *The space of functions of bounded variation and certain general spaces*, Trans. Amer. Math. Soc., 40 (1936), pp. 421–438. ↑ 35.

[2] ROBERT A. ADAMS AND JOHN J. F. FOURNIER, *Sobolev spaces*, vol. 140 of Pure and Applied Mathematics (Amsterdam), Elsevier/Academic Press, Amsterdam, second ed., 2003. ↑ 6.

[3] SERGE ALINHAC AND PATRICK GÉRARD, *Pseudo-differential operators and the Nash-Moser theorem*, vol. 82 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2007. Translated from the 1991 French original by Stephen S. Wilson. ↑ 54.

[4] NENAD ANTONIĆ, *H-measures applied to symmetric systems*, Proc. Roy. Soc. Edinburgh Sect. A, 126 (1996), pp. 1133–1155. ↑ 70, 76.

[5] NENAD ANTONIĆ AND MARKO ERCEG, *One-scale H-distributions and variants*, Results Math., 78 (2023), pp. Paper No. 165, 53. ↑ 49.

[6] NENAD ANTONIĆ, MARKO ERCEG, AND MARTIN LAZAR, *Localisation principle for one-scale H-measures*, J. Funct. Anal., 272 (2017), pp. 3410–3454. ↑ 49.

[7] NENAD ANTONIĆ, MARKO ERCEG, AND MARIN MIŠUR, *Distributions of anisotropic order and applications to H-distributions*, Anal. Appl. (Singap.), 19 (2021), pp. 801–843. ↑ 49.

[8] NENAD ANTONIĆ AND IVAN IVEC, *On the Hörmander-Mihlin theorem for mixed-norm Lebesgue spaces*, J. Math. Anal. Appl., 433 (2016), pp. 176–199.

[9] NENAD ANTONIĆ, IVAN IVEC, AND IVANA VOJNOVIĆ, *Continuity of pseudodifferential operators on mixed-norm Lebesgue spaces*, Monatsh. Math., 190 (2019), pp. 657–674. ↑ 59, 60.

[10] NENAD ANTONIĆ AND MARTIN LAZAR, *Microlocal energy density for hyperbolic systems*, in Applied mathematics and scientific computing (Dubrovnik, 2001), Kluwer/Plenum, New York, 2003, pp. 179–190.

[11] NENAD ANTONIĆ AND MARTIN LAZAR, *H-measures and variants applied to parabolic equations*, J. Math. Anal. Appl., 343 (2008), pp. 207–225. ↑ 49.

[12] NENAD ANTONIĆ AND MARTIN LAZAR, *A parabolic variant of H-measures*, Ann. Univ. Ferrara Sez. VII Sci. Mat., 54 (2008), pp. 183–201.

[13] NENAD ANTONIĆ AND MARTIN LAZAR, *Parabolic variant of H-measures in homogenisation of a model problem based on Navier-Stokes equation*, Nonlinear Anal. Real World Appl., 11 (2010), pp. 4500–4512.

[14] NENAD ANTONIĆ AND MARTIN LAZAR, *Parabolic H-measures*, J. Funct. Anal., 265 (2013), pp. 1190–1239. ↑ 49.

[15] NENAD ANTONIĆ AND DARKO MITROVIĆ, *H-distributions: an extension of H-measures to an L^p-L^q setting*, Abstr. Appl. Anal., (2011), pp. Art. ID 901084, 12. ↑ 49.

[16] NENAD ANTONIĆ, DARKO MITROVIĆ, AND LJUDEVIT PALLE, *On relationship between H-distributions and microlocal compactness forms*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 31 (2020), pp. 297–318. ↑ 49.

[17] NENAD ANTONIĆ, MARIN MIŠUR, AND DARKO MITROVIĆ, *On compactness of commutators of multiplications and Fourier multipliers*, Mediterr. J. Math., 15 (2018), pp. Paper No. 170, 13. ↑ 49.

[18] NENAD ANTONIĆ AND MARKO VRDOLJAK, *Parabolic H-convergence and small-amplitude homogenization*, Appl. Anal., 88 (2009), pp. 1493–1508.

[19] Fágner D. Araruna and Aldo B. Maciel, *Existence and boundary stabilization of the semilinear wave equation*, Nonlinear Analysis: Theory, Methods and Applications, 67 (2007), pp. 1288–1305.

[20] Wolfgang Arendt and Marcel Kreuter, *Mapping theorems for sobolev spaces of vector-valued functions*, Studia Mathematica, 240 (2018), pp. 275–299.

[21] Safia Brahim-Otsmane, Gilles A. Francfort, and François Murat, *Correctors for the homogenization of the wave and heat equations*, J. Math. Pures Appl. (9), 71 (1992), pp. 197–231.

[22] Haïm Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, vol. No. 5 of North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. Notas de Matemática, No. 50. [Mathematical Notes]. ↑ 5, 34.

[23] Haïm Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011. ↑ 4, 5.

[24] Marc Briane and Graeme W. Milton, *An antisymmetric effective Hall matrix*, SIAM J. Appl. Math., 70 (2010), pp. 1810–1820. ↑ 2.

[25] Juan Casado-Díaz, Julio Couce-Calvo, Faustino Maestre, and José D. Martín-Gómez, *Homogenization and corrector for the wave equation with discontinuous coefficients in time*, J. Math. Anal. Appl., 379 (2011), pp. 664–681. ↑ 1, 89.

[26] Robert Dautray and Jacques-Louis Lions, *Mathematical analysis and numerical methods for science and technology. Vol. 5*, Springer-Verlag, Berlin, 1992. Evolution problems. I, With the collaboration of Michel Artola, Michel Cessenat and Hélène Lanchon, Translated from the French by Alan Craig.

[27] Ronald J. DiPerna and Andrew Majda, *Reduced Hausdorff dimension and concentration-cancellation for two-dimensional incompressible flow*, J. Amer. Math. Soc., 1 (1988), pp. 59–95. ↑ 45.

[28] MARKO ERCEG AND IVAN IVEC, *On generalisation of H-measures*, Filomat, 31 (2017), pp. 5027–5044. ↑ 49.

[29] MARKO ERCEG AND IVAN IVEC, *Second commutation lemma for fractional H-measures*, J. Pseudo-Differ. Oper. Appl., 9 (2018), pp. 589–613. ↑ 49.

[30] MARKO ERCEG AND MARTIN LAZAR, *Characteristic scales of bounded* L^2 *sequences*, Asymptot. Anal., 109 (2018), pp. 171–192.

[31] MARKO ERCEG, MARIN MIŠUR, AND DARKO MITROVIĆ, *Velocity averaging for diffusive transport equations with discontinuous flux*, J. Lond. Math. Soc. (2), 107 (2023), pp. 658–703.

[32] LAWRENCE C. EVANS, *Weak convergence methods for nonlinear partial differential equations*, vol. 74 of CBMS Regional Conference Series in Mathematics, Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1990. ↑ 5, 45, 54.

[33] GILLES A. FRANCFORT, *An introduction to H-measures and their applications*, in Variational problems in materials science, vol. 68 of Progr. Nonlinear Differential Equations Appl., Birkhäuser, Basel, 2006, pp. 85–110.

[34] GILLES A. FRANCFORT AND FRANÇOIS MURAT, *Oscillations and energy densities in the wave equation*, Comm. Partial Differential Equations, 17 (1992), pp. 1785–1865. ↑ 3, 89.

[35] PATRICK GÉRARD, *Mesures semi-classiques et ondes de Bloch*, in Séminaire sur les Équations aux Dérivées Partielles, 1990–1991, École Polytech., Palaiseau, 1991, pp. Exp. No. XVI, 19.

[36] PATRICK GÉRARD, *Microlocal defect measures*, Comm. Partial Differential Equations, 16 (1991), pp. 1761–1794. ↑ 2, 45, 49, 61, 89.

[37] PATRICK GÉRARD, *Oscillations and concentration effects in semilinear dispersive wave equations*, J. Funct. Anal., 141 (1996), pp. 60–98. ↑ 3, 85, 89.

[38] DAVID GILBARG AND NEIL S. TRUDINGER, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[39] LOUKAS GRAFAKOS, *Classical Fourier analysis*, vol. 249 of Graduate Texts in Mathematics, Springer, New York, third ed., 2014. ↑ 46, 47.

[40] LOUKAS GRAFAKOS, *Modern Fourier analysis*, vol. 250 of Graduate Texts in Mathematics, Springer, New York, third ed., 2014.

[41] MARTIN HEIDA, ROBERT I. A. PATTERSON, AND D. R. MICHIEL RENGER, *Topologies and measures on the space of functions of bounded variation taking values in a Banach or metric space*, J. Evol. Equ., 19 (2019), pp. 111–152. ↑ 34, 35, 36.

[42] LARS HÖRMANDER, *The analysis of linear partial differential operators. I*, Classics in Mathematics, Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)]. ↑ 46.

[43] LARS HÖRMANDER, *The analysis of linear partial differential operators. III*, Classics in Mathematics, Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition. ↑ 54, 56, 57, 64, 65.

[44] IVAN IVEC, *Continuity of pseudodifferential operators with nonsmooth symbols on mixed-norm Lebesgue spaces*, Filomat, 37 (2023), pp. 8205–8212.

[45] IVAN IVEC AND MARTIN LAZAR, *Propagation principle for parabolic H-measures*, J. Pseudo-Differ. Oper. Appl., 11 (2020), pp. 467–489.

[46] IVAN IVEC AND IVANA VOJNOVIĆ, *H-distributions on Hörmander spaces*, J. Math. Anal. Appl., 495 (2021), pp. Paper No. 124717, 17.

[47] IVAN IVEC AND IVANA VOJNOVIĆ, *Basic calculus of pseudodifferential operators with nonsmooth symbols*, J. Pseudo-Differ. Oper. Appl., 13 (2022), pp. Paper No. 13, 34.

[48] DAVID KINDERLEHER AND GUIDO STAMPACCHIA, *An Introduction to Variational Inequalities and Their Applications*, SIAM, 2000. ↑ 25.

[49] JOSEPH J. KOHN AND LOUIS NIRENBERG, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math., 18 (1965), pp. 269–305. ↑ 50.

[50] MARCEL KREUTER, *Sobolev spaces of vector-valued functions*, Master thesis, Ulm University, (2015). ↑ 5.

[51] MARTIN LAZAR, *Poopćenja h-mjera i primjene*, doktorska disertacija, Sveučilište u Zagrebu, (2007). ↑ 52, 62.

[52] GIOVANNI LEONI, *A first course in Sobolev spaces*, vol. 181 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2017. ↑ 34, 35.

[53] J.-L. LIONS AND E. MAGENES, *Non-homogeneous boundary value problems and applications. Vol. I*, vol. Band 181 of Die Grundlehren der mathematischen Wissenschaften, Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth.

[54] VLADIMIR G. MAZ'YA AND TATYANA O. SHAPOSHNIKOVA, *Theory of Sobolev multipliers*, vol. 337 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2009. With applications to differential and integral operators. ↑ 47.

[55] GRAEME W. MILTON, *Some open problems in the theory of composites*, Philos. Trans. Roy. Soc. A, 379 (2021), pp. Paper No. 20200115, 20. ↑ 2.

[56] GRAEME W. MILTON, *The theory of composites*, vol. 88 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, [2023] ©2023. Reprint of the 2002 edition [1899805]. ↑ 2.

[57] DARKO MITROVIĆ AND IVAN IVEC, *A generalization of H-measures and application on purely fractional scalar conservation laws*, Commun. Pure Appl. Anal., 10 (2011), pp. 1617–1627.

[58] MARIN MIŠUR AND DARKO MITROVIĆ, *On a generalization of compensated compactness in the L^p-L^q setting*, J. Funct. Anal., 268 (2015), pp. 1904–1927.

[59] MARIN MIŠUR AND DARKO MITROVIĆ, *On compactness of commutator of multiplication and pseudodifferential operator*, J. Pseudo-Differ. Oper. Appl., 10 (2019), pp. 121–131.

[60] SIGERU MIZOHATA, *The theory of partial differential equations*, Cambridge University Press, New York, 1973. Translated from the Japanese by Katsumi Miyahara. ↑ 67.

[61] EVGENI YU. PANOV, *Ultra-parabolic H-measures and compensated compactness*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 28 (2011), pp. 47–62. ↑ 49.

[62] BENT E. PETERSEN, *Introduction to the Fourier transform & pseudodifferential operators*, vol. 19 of Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, MA, 1983. ↑ 54.

[63] FILIP RINDLER, *Directional oscillations, concentrations, and compensated compactness via microlocal compactness forms*, Arch. Ration. Mech. Anal., 215 (2015), pp. 1–63. ↑ 49.

[64] XAVIER SAINT RAYMOND, *Elementary introduction to the theory of pseudodifferential operators*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1991. ↑ 54, 56, 57.

[65] LAURENT SCHWARTZ, *Théorie des distributions*, vol. IX-X of Publications de l'Institut de Mathématique de l'Université de Strasbourg, Hermann, Paris, 1966. Nouvelle édition, entiérement corrigée, refondue et augmentée. ↑ 4.

[66] ELIAS M. STEIN, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, vol. 43 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[67] WALTER A. STRAUSS, *On weak solutions of semi-linear hyperbolic equations*, An. Acad. Brasil. Ci., 42 (1970), pp. 645–651. ↑ 1, 24, 28, 89.

[68] WALTER A. STRAUSS, *Nonlinear wave equations*, vol. 73 of CBMS Regional Conference Series in Mathematics, Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1989.

[69] LUC TARTAR, *Compensated compactness and applications to partial differential equations*, in Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV, vol. 39 of Res. Notes in Math., Pitman, Boston, Mass.-London, 1979, pp. 136–212.

[70] LUC TARTAR, *H-measures and small amplitude homogenization*, in Random media and composites (Leesburg, VA, 1988), SIAM, Philadelphia, PA, 1989, pp. 89–99.

[71] LUC TARTAR, *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A, 115 (1990), pp. 193–230. ↑ 2, 3, 45, 48, 49, 89.

[72] LUC TARTAR, *H-measures and applications*, in Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 1215–1223.

[73] LUC TARTAR, *Propagation of oscillations and concentration effects in partial differential equations*, in Ordinary and partial differential equations, Vol. III (Dundee, 1990), vol. 254 of Pitman Res. Notes Math. Ser., Longman Sci. Tech., Harlow, 1991, pp. 208–219.

[74] LUC TARTAR, *On mathematical tools for studying partial differential equations of continuum physics: H-measures and Young measures*, in Developments in partial differential equations and applications to mathematical physics (Ferrara, 1991), Plenum, New York, 1992, pp. 201–217.

[75] LUC TARTAR, *Beyond Young measures*, Meccanica, 30 (1995), pp. 505–526. Microstructure and phase transitions in solids (Udine, 1994).

[76] LUC TARTAR, *Homogenization and hyperbolicity*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 25 (1997), pp. 785–805. Dedicated to Ennio De Giorgi.

[77] LUC TARTAR, *Oscillations and concentration effects in partial differential equations: why waves may behave like particles*, in XVII CEDYA: Congress on Differential Equations and Applications/VII CMA: Congress on Applied Mathematics (Spanish) (Salamanca, 2001), Dep. Mat. Apl., Univ. Salamanca, Salamanca, 2001, pp. 179–219.

[78] LUC TARTAR, *Mathematical tools for studying oscillations and concentrations: from Young measures to H-measures and their variants*, in Multiscale problems in science and technology (Dubrovnik, 2000), Springer, Berlin, 2002, pp. 1–84. ↑ 49.

[79] LUC TARTAR, *An introduction to Navier-Stokes equation and oceanography*, vol. 1 of Lecture Notes of the Unione Matematica Italiana, Springer-Verlag, Berlin; UMI, Bologna, 2006. ↑ 10.

[80] LUC TARTAR, *The general theory of homogenization*, vol. 7 of Lecture Notes of the Unione Matematica Italiana, Springer-Verlag, Berlin; UMI, Bologna, 2009. A personalized introduction. ↑ 48, 49.

[81] LUC TARTAR, *Multi-scales H-measures*, Discrete Contin. Dyn. Syst. Ser. S, 8 (2015), pp. 77–90.

[82] LUC TARTAR, *H-measures and propagation effects*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 28 (2017), pp. 701–728.

[83] LUC TARTAR, *Approximation of H-measures and beyond*, Ann. Math. Sci. Appl., 8 (2023), pp. 461–497.

[84] FRANÇOIS TRÈVES, *Topological vector spaces, distributions and kernels*, Dover Publications, Inc., Mineola, NY, 2006. Unabridged republication of the 1967 original. ↑ 4, 49.

[85] NIK WEAVER, *Lipschitz algebras*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second ed., 2018.

[86] JOSEPH WLOKA, *Partial Differential Equations*, Cambridge: Cambridge University Press, 1987. ↑ 6.

Curriculum Vitae

Matko Grbac was born on the 9th of June 1994 in Rijeka, where he completed his elementary and high school. In 2013, he started his studies in mathematics at the Department of Mathematics, University of Zagreb.

In 2018, he graduated with a thesis titled ”Parseval farmes in Hilbert spaces” (”Parsevalovi bazni okviri Hilbertovih prostora”) under the supervision of Damir Bakić. In the same year, he enrolled in the PhD program in mathematics at the University of Zagreb, now under the supervision of Nenad Antonić. In 2019, he started working as a teaching assistant at the Department of Mathematics.