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FACULTY OF SCIENCE

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Subordinate random walks

DOCTORAL DISSERTATION

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PRIRODOSLOVNO–MATEMATIČKI FAKULTET

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Stjepan Šebek

Subordinirane slučajne šetnje

DOKTORSKI RAD

Mentor:

prof. dr. sc. Zoran Vondraček

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SUMMARY

In this thesis, we consider a large class of subordinate random walks on the integer lattice $\mathbb{Z}^d$ via subordinators with Laplace exponents which are complete Bernstein functions satisfying some mild scaling conditions at zero. Subordination is a procedure for obtaining new process from the original one. The new process may differ very much from the original process, but the properties of this new process can be understood in terms of the original process.

Main results of the thesis are the elliptic Harnack inequality and n-step transition probability estimates for subordinate random walks. In order to obtain the elliptic Harnack inequality, we first establish estimates for one-step transition probabilities, the Green function and the Green function of a ball.

The main tools we apply to get n-step transition probability estimates for subordinate random walks are the parabolic Harnack inequality and appropriate bounds for the transition kernel of the corresponding continuous time random walk.

SAŽETAK

U ovoj disertaciji promatramo veliku klasu subordiniranih slučajnih šetnji na cjelobrojnoj mreži $\mathbb{Z}^d$ dobivenih pomoću subordinatora s Laplaceovim eksponentima koji su potpune Bernsteinove funkcije koje zadovoljavaju neke blage uvjete skaliranja u nuli. Subordinacija je procedura dobivanja novog procesa na temelju originalnog procesa. Iako se novi proces može dosta razlikovati od originalnog, svojstva dobivenog procesa mogu se shvatiti u terminima originalnog procesa.

Glavni rezultati do kojih dolazimo su eliptička Harnackova nejednakost te ocjene na prijelazne vjerojatnosti za subordinirane slučajne šetnje. Kako bismo dobili eliptičku Harnackovu nejednakost, prvo dokazujemo ocjene za jednokoračne prijelazne vjerojatnosti, Greenovu funkciju te Greenovu funkciju kugle.

Glavne tehnike koje koristimo kako bismo dobili ocjene za n-koračne prijelazne vjerojatnosti za subordinirane slučajne šetnje su parabolička Harnackova nejednakost i odgovarajuće ocjene za prijelaznu jezgru pripadajuće neprekidno vremenske slučajne šetnje.

Contents

1. Introduction

1.1. Motivation

In the case of continuous time Markov processes, subordination is a well-known and useful procedure of obtaining new process from the original process. The new process may differ very much from the original process, but the properties of this new process can be understood in the terms of the original process. The best known application of this concept is obtaining the symmetric stable process from the Brownian motion. A lot of work has been done concerning subordination of continuous time Markov processes. On the other hand, discrete subordination was introduced only in 2011 by A. Bendikov and L. Saloff-Coste in their paper *Random walks on groups and discrete subordination*, Mathematische Nachrichten no. 285, 580 – 605. Since the discrete subordination is a relatively new technique, not much is known about subordinate random walks, even though it is a very natural technique of obtaining new random walks from the existing ones.

1.2. Subordinate random walks

In this section we introduce subordinate random walks starting from the simple symmetric random walk and using a Bernstein function. For the definition and some details about simple symmetric random walks, see Section 2.1 and for short overview of Bernstein functions, see Section 2.2.

Let $S_n = X_1 + X_2 + \cdots + X_n$ be the simple symmetric random walk in $\mathbb{Z}^d$ which starts from the origin and let ϕ be a Bernstein function such that $\phi(0) = 0$ and $\phi(1) = 1$. Such a function admits the following integral representation

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} \left(1 - e^{-\lambda t}\right)\mu(dt), \tag{1.1}$$

for $b \geqslant 0$ and a measure μ on $(0,\infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\mu(dt) < \infty$, see [25, Sec. 3].

We consider a sequence of positive numbers a_m^ϕ which is related to the function ϕ and is defined as

$$a_m^\phi = b\delta_1(m) + \frac{1}{m!}\int_{(0,\infty)} t^m e^{-t}\mu(dt), \quad m \geq 1, \tag{1.2}$$

where δ_x is the Dirac measure at x. One easily verifies that

$$\sum_{m=1}^{\infty} a_m^\phi = b + \int_{(0,\infty)} (e^t - 1)e^{-t}\mu(dt) = b + \int_{(0,\infty)} (1 - e^{-t})\mu(dt) = \phi(1) = 1.$$

Let $\tau_n = R_1 + R_2 + \cdots + R_n$ be a random walk on $\mathbb{Z}_+$ with increments R_i that are independent of the random walk S_n and have the distribution given by $\mathbb{P}(R_1 = m) = a_m^\phi$. A subordinate random walk is defined as $S_n^\phi := S_{\tau_n}$, for all $n \geqslant 0$. It is straightforward to see that the subordinate random walk is indeed a random walk. Since $\tau_0 = 0$ and $S_0 = 0$, for any $n \in \mathbb{N}$ we can write

$$S_n^\phi = S_{\tau_n} = \sum_{k=1}^{n} (S_{\tau_k} - S_{\tau_{k-1}}) \overset{d}{=} \sum_{k=1}^{n} S_{\tau_k - \tau_{k-1}} = \sum_{k=1}^{n} S_{R_k} = \sum_{k=1}^{n} \xi_k, \tag{1.3}$$

where $(\xi_k)_{k \geqslant 1}$ is a sequence of independent, identically distributed random variables with the same distribution as S_1^ϕ. Notice that the one-step transition probability $p^\phi(1,x,y)$ of the random walk S_n^ϕ is of the form

$$p^\phi(1,x,y) = \mathbb{P}^x(S_1^\phi = y) = \sum_{m=1}^{\infty} \mathbb{P}^x(S_{R_1} = y \mid R_1 = m)a_m^\phi = \sum_{m=1}^{\infty} p(m,x,y)a_m^\phi, \tag{1.4}$$

where $p(n,x,y) = \mathbb{P}^x(S_n = y)$ stands for the n-step transition probability of the simple random walk S_n. We use the notation $p^\phi(n,x,y) = \mathbb{P}^x(S_n^\phi = y)$, $p^\phi(n,x-y) = p^\phi(n,x,y)$ and $p^\phi(1,x,y) = p^\phi(x,y) = p^\phi(x-y)$.

1.3. Overview

As we have already mentioned, subordinate random walks were introduced in [9]. As authors state in the paper, one of the very important characteristics of a random walk is the probability of return to the starting point at time n. The main motivation for introducing the discrete subordination was to find a new class of random walks for which one can estimate the behavior of those probabilities. After that, subordinate random walks were studied in [6] and [7] where authors were interested in massive (recurrent) sets for subordinate random walks. In [21], author proved that the appropriately scaled subordinate random walk converges in the Skorohod space to the stable process if and only if the Bernstein function that is used to define that particular subordinate random walk is regularly varying at zero with index $\alpha \in (0,1]$. Authors in [8] were also dealing with the convergence of subordinate random walks in the Skorohod space and they found estimates for the transition probabilities of the subordinate random walks, but only in some special regions, not global estimates. In all this papers, authors assumed that the Bernstein function ϕ is regularly varying.

In this thesis we are concerned with the transition probabilities of the random walk S_n^ϕ which are defined as $p^\phi(n,x,y) = \mathbb{P}^x(S_n^\phi = y)$. In the course of study we assume that ϕ is a complete Bernstein function. Our second assumption is the *scaling condition*. We require that for some constants $c_*, c^* > 0$ and $0 < \alpha_* \leqslant \alpha^* < 1$ the function ϕ satisfies

$$c_* \left(\frac{R}{r}\right)^{\alpha_*} \leqslant \frac{\phi(R)}{\phi(r)} \leqslant c^* \left(\frac{R}{r}\right)^{\alpha^*}, \quad 0 < r \leqslant R \leqslant 1.$$

Under these two assumptions we establish global estimates for the function $p^\phi(n,x,y)$, that is we prove that for all $x,y \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ it holds

$$p^\phi(n,x,y) \asymp \min\left\{\left(\phi^{-1}(n^{-1})\right)^{d/2}, \frac{n\phi(|x-y|^{-2})}{|x-y|^d}\right\},$$

see Theorem 4.1, Theorem 6.1 and Theorem 6.18. In the above relation, the symbol $\asymp$ means that the ratio of the two expressions is bounded from below and from above by some positive constants.

Similar questions have already been addressed in the literature. In [5] the authors found global estimates for transition probabilities of stable-like random walks. Recently, in [22] the similar problem was solved on uniformly discrete metric measure spaces. We mention here related papers and monographs [2], [3], [4], [14], [18], [19], [26], [28], [30].

We notice that the scaling condition means that the function ϕ is an O-regularly varying function at 0 with Matuszewska indices contained in $(0,1)$, see [10, Sec. 2]. Complete Bernstein functions with such behaviour at zero can be found in the closing table of [25] and include functions: $\phi(\lambda) = \lambda^\alpha + \lambda^\beta$, $\alpha, \beta \in (0,1)$; $\phi(\lambda) = \lambda^\alpha(\log(1+\lambda))^\beta$, $\alpha \in (0,1)$, $\beta \in (0, 1-\alpha)$; $\phi(\lambda) = (\log(\cosh(\sqrt{\lambda})))^\alpha$, for $\alpha \in (0,1)$ etc. It is possible, however, to construct examples of complete Bernstein functions that satisfy scaling conditions and that are not comparable to any regularly varying function, see e.g. [15].

1.4. NOTATION

Throughout the paper $c, c_1, c_2, \ldots$ will denote generic constants. Their labeling starts anew in each statement and their dependence on the function ϕ and on the dimension d will not be mentioned explicitly. The cardinality of a set $A \subset \mathbb{Z}^d$ is denoted by $|A|$. The Euclidean distance between x and y is denoted by $|x-y|$. For $x \in \mathbb{R}^d$ and $r > 0$, we write $B(x,r) = \{y \in \mathbb{Z}^d : |y-x| < r\}$ and $B_r = B(0,r)$. We use notation $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$. For any two positive functions f and g, we write $f \asymp g$ if there exist constants $c_1, c_2 > 0$ such that $c_1 \leqslant g/f \leqslant c_2$.

2. Preparatory material

2.1. Simple random walk

Let $(X_n)_{n\geqslant 1}$ be independent, identically distributed random variables defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ taking values in the integer lattice $\mathbb{Z}^d$ with

$$\mathbb{P}(X_k = e_i) = \mathbb{P}(X_k = -e_i) = \frac{1}{2d}, \quad i \in \{1, 2, \ldots, d\},$$

where e_i is the i^{th} unit vector in $\mathbb{Z}^d$. A *simple random walk* starting at $x \in \mathbb{Z}^d$ is a stochastic process $S = (S_n)_{n\geqslant 0}$ with $S_0 = x$ and

$$S_n = x + X_1 + X_2 + \cdots + X_n.$$

The probability distribution of S_n is denoted by

$$p_n(x,y) = \mathbb{P}^x(S_n = y).$$

Here we have written $\mathbb{P}^x$ to indicate that the random walk starts at the point x. We will similarly write $\mathbb{E}^x$ to denote expectations assuming $S_0 = x$. If x is missing, it will be assumed that $S_0 = 0$. We write $p_n(x)$ for $p_n(0,x)$.

The most important result about simple symmetric random walks on $\mathbb{Z}^d$ that we use in this thesis are Gaussian bounds for the n-step transition probabilities of S. Using the result from [14, Theorem 5.1.] and adjusting it to the case of the simple symmetric random walk which has the period 2, we get

$$\begin{aligned} p_n(x) &\leqslant Cn^{-\frac{d}{2}} e^{-\frac{|x|^2}{Cn}}, \quad x \in \mathbb{Z}^d,\, n \in \mathbb{N}, \\ p_n(x) + p_{n+1}(x) &\geqslant cn^{-\frac{d}{2}} e^{-\frac{|x|^2}{cn}}, \quad |x| \leqslant n,\, n \in \mathbb{N}. \end{aligned} \tag{2.1}$$

2.2. BERNSTEIN FUNCTIONS

Definition 2.1. A function $\phi : (0,\infty) \to (0,\infty)$ is called a *Bernstein function* if ϕ is of class $C^\infty((0,\infty))$ and

$$(-1)^n \phi^{(n)} \leqslant 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Here $\phi^{(n)}$ denotes the n-th derivative of ϕ. It is known (see [25, Theorem 3.2]) that ϕ is a Bernstein function if and only if it is of the form

$$\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t}) \mu(dt),$$

where $a, b \geqslant 0$ and μ is a measure on $(0,\infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge t) \mu(dt) < \infty,$$

called the Lévy measure.

Definition 2.2. A function $m : (0,\infty) \to (0,\infty)$ is a *completely monotone function* if m is of class $C^\infty((0,\infty))$ and

$$(-1)^n m^{(n)} \geqslant 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Remark 2.3. Equivalently, a function m is a completely monotone function if it is a Laplace transform of a measure, see [25, Theorem 1.4].

Definition 2.4. A Bernstein function ϕ is said to be a *complete Bernstein function* if its Lévy measure μ has a completely monotone density $m(t)$ with respect to the Lebesgue measure,

$$\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t}) m(t) dt.$$

One important property of complete Bernstein functions is formulated in the following proposition.

Proposition 2.5. Function $\phi \not\equiv 0$ is a complete Bernstein function if and only if the function $\phi^\star(\lambda) := \lambda/\phi(\lambda)$ is a complete Bernstein function.

A proof can be found in [25, Proposition 7.1]. Generalizing the property from Proposition 2.5 leads to the larger class of special Bernstein functions.

Definition 2.6. A Bernstein function ϕ is said to be a *special Bernstein function* if the function $\phi^\star(\lambda) = \lambda/\phi(\lambda)$ is again a Bernstein function.

It is clear from Proposition 2.5 that complete Bernstein functions are a subset of special Bernstein functions. It can be shown that the family of all special Bernstein functions is strictly larger than the family of all complete Bernstein functions (see [25, Example 11.18]).

It is well known that, if ϕ is a Bernstein function, then $\phi(\lambda t) \leqslant \lambda \phi(t)$ for all $\lambda \geqslant 1$, $t > 0$, which implies

$$\frac{\phi(v)}{\phi(u)} \leqslant \frac{v}{u}, \quad 0 < u \leqslant v. \tag{2.2}$$

2.3. Scaling condition

We need some additional assumptions on the behavior of the Bernstein function ϕ that we use to define the subordinate random walk. As we already mentioned in the Overview, the assumption in some of the pioneer papers was that $\phi(\lambda) = \lambda^{\alpha}$, $\alpha \in (0,1)$. A generalization of that approach was the assumption that ϕ is a Bernstein function which is regularly varying at zero with index $\alpha \in (0,1)$. Even more general assumption is that ϕ is a Bernstein function which satisfies a scaling condition at zero. This means that for some constants $c_*, c^* > 0$ and $0 < \alpha_* \leqslant \alpha^* < 1$ the function ϕ satisfies

$$c_* \left(\frac{R}{r}\right)^{\alpha_*} \leqslant \frac{\phi(R)}{\phi(r)} \leqslant c^* \left(\frac{R}{r}\right)^{\alpha^*}, \quad 0 < r \leqslant R \leqslant 1. \tag{2.3}$$

That this is really more general assumption than regular variation, one can see in the example at the end of [15].

Using (2.3), we can easily obtain the bounds for the inverse function ϕ^{-1} which take the form

$$(1/c^*)^{1/\alpha^*} \left(\frac{R}{r}\right)^{1/\alpha^*} \leqslant \frac{\phi^{-1}(R)}{\phi^{-1}(r)} \leqslant (1/c_*)^{1/\alpha_*} \left(\frac{R}{r}\right)^{1/\alpha_*}, \quad 0 < r \leqslant R \leqslant 1. \tag{2.4}$$

We only show how to get the first inequality since the second one is obtained in a completely analogous way. Take $0 < r \leqslant R \leqslant 1$. Since ϕ is an increasing function which satisfies $\phi(0) = 0$ and $\phi(1) = 1$, we have that ϕ^{-1} is also increasing, $\phi^{-1}(0) = 0$ and $\phi^{-1}(1) = 1$. From the upper bound in (2.3) we get

$$\frac{\phi(\phi^{-1}(R))}{\phi(\phi^{-1}(r))} \leqslant c^* \left(\frac{\phi^{-1}(R)}{\phi^{-1}(r)}\right)^{\alpha^*}.$$

From this we clearly have

$$\frac{\phi^{-1}(R)}{\phi^{-1}(r)} \geqslant (1/c^*)^{1/\alpha^*} \left(\frac{R}{r}\right)^{1/\alpha^*}.$$

2.4. Transience of subordinate random walks

We are only interested in transient random walks. Since we explore transition probability estimates which are closely related to the Green function of our walk, transience is necessary for us to have finiteness of the Green function. We use Chung-Fuchs theorem to show under which condition a subordinate random walk is transient. Denote with Ψ^ϕ the characteristic function of the one step of a subordinate random walk. We want to prove that there exists $\delta > 0$ such that

$$\int_{(-\delta,\delta)^d} \mathrm{Re}\left(\frac{1}{1-\Psi^\phi(\theta)}\right) d\theta < \infty.$$

The law of the variable S_1^ϕ is given with (1.4). We denote the one step of the simple symmetric random walk $(S_n)_{n\geqslant 0}$ with X_1 and the characteristic function of that random variable with Ψ. Assuming $|\theta| < 1$ we have

$$\begin{aligned}
\Psi^\phi(\theta) &= \mathbb{E}[e^{i\theta\cdot S_1^\phi}] = \sum_{x\in\mathbb{Z}^d} e^{i\theta\cdot x} \sum_{m=1}^{\infty} \int_{(0,+\infty)} \frac{t^m}{m!} e^{-t} \mu(dt) \mathbb{P}(S_m = x) \\
&= \sum_{m=1}^{\infty} \int_{(0,+\infty)} \frac{t^m}{m!} e^{-t} \mu(dt) \sum_{x\in\mathbb{Z}^d} e^{i\theta\cdot x} \mathbb{P}(S_m = x) = \sum_{m=1}^{\infty} \int_{(0,+\infty)} \frac{t^m}{m!} e^{-t} \mu(dt) (\Psi(\theta))^m \\
&= \int_{(0,+\infty)} (e^{t\Psi(\theta)} - 1) e^{-t} \mu(dt) = \phi(1) - \phi(1-\Psi(\theta)) = 1 - \phi(1-\Psi(\theta)). \qquad (2.5)
\end{aligned}$$

From [18, Section 1.2, page 13] we have

$$\Psi(\theta) = \frac{1}{d} \sum_{m=1}^{d} \cos(\theta_m), \quad \theta = (\theta_1, \theta_2, \ldots, \theta_d).$$

That is function with real values so

$$\int_{(-\delta,\delta)^d} \mathrm{Re}\left(\frac{1}{1-\Psi^\phi(\theta)}\right) d\theta = \int_{(-\delta,\delta)^d} \frac{1}{\phi(1-\Psi(\theta))} d\theta.$$

From Taylor's theorem it follows that there exists $a \leqslant 1$ such that

$$|\Psi(\theta)| = \Psi(\theta) \leqslant 1 - \frac{1}{4d}|\theta|^2, \quad \theta \in B(0,a). \qquad (2.6)$$

Now we take δ such that $(-\delta,\delta)^d \subset B(0,a)$. From (2.6), using the fact that ϕ is increasing, we get

$$\frac{1}{\phi\left(1-\Psi(\theta)\right)} \leqslant \frac{1}{\phi\left(|\theta|^2/4d\right)}, \quad \theta \in B(0,a).$$

Hence,

$$\int_{(-\delta,\delta)^d} \frac{1}{\phi(1-\Psi(\theta))} d\theta \leqslant \int_{(-\delta,\delta)^d} \frac{1}{\phi\left(|\theta|^2/4d\right)} d\theta \leqslant \int_{B(0,a)} \frac{\phi(|\theta|^2)}{\phi\left(|\theta|^2/4d\right)} \frac{1}{\phi(|\theta|^2)} d\theta$$

$$\leqslant c^*(4d)^{\alpha^*} \int_{B(0,a)} \frac{1}{\phi(|\theta|^2)} d\theta = c_1(4d)^{\alpha^*} \int_0^a \frac{r^{d-1}}{\phi(r^2)} dr$$

$$= \frac{c_1(4d)^{\alpha^*}}{\phi(a)} \int_0^a r^{d-1} \frac{\phi(a)}{\phi(r^2)} dr \leqslant \frac{c_1 c^*(4ad)^{\alpha^*}}{\phi(a)} \int_0^a r^{d-2\alpha^*-1} dr$$

and the last integral converges for $d-2\alpha^*-1>-1$. Hence, the subordinate random walk is transient for $d>2\alpha^*$. In the rest of the thesis, we always assume that we have transient subordinate random walk.

2.5. Functions g and j

Throughout the thesis, we often use the following two functions

$$g:(0,\infty)\to(0,\infty), \quad g(r)=r^{-d}\phi(r^{-2})^{-1}, \tag{2.7}$$

$$j:(0,\infty)\to(0,\infty), \quad j(r)=r^{-d}\phi(r^{-2}). \tag{2.8}$$

In this section, we present their properties that we need later.

It is clear that j is a decreasing function. For function g we prove the following lemma

Lemma 2.7. Let $1 \leqslant r \leqslant q$. Then $g(r) \geqslant (c^*)^{-1} g(q)$.

Proof. Using (2.3) and $d>2\alpha^*$ we get

$$g(r)=r^{-d}\phi(r^{-2})^{-1}=q^{-d}\phi(q^{-2})^{-1}\left(\frac{q}{r}\right)^d \frac{\phi(q^{-2})}{\phi(r^{-2})} \geqslant (c^*)^{-1} g(q) \left(\frac{q}{r}\right)^{d-2\alpha^*} \geqslant (c^*)^{-1} g(q).$$

■

Lemma 2.8. Let $r>0$. If $0<a\leqslant 1$ then

$$j(ar) \leqslant a^{-d-2} j(r), \tag{2.9}$$

$$g(ar) \geqslant a^{-d+2} g(r). \tag{2.10}$$

If $a \geqslant 1$ then

$$j(ar) \geqslant a^{-d-2} j(r). \tag{2.11}$$

Proof. In the proof of this lemma, we only use (2.2):

$$j(ar) = (ar)^{-d}\phi((ar)^{-2}) = (ar)^{-d}\phi(r^{-2})\frac{\phi((ar)^{-2})}{\phi(r^{-2})} \leqslant a^{-d-2}j(r).$$

Relations (2.10) and (2.11) are proved in a completely analogous way. ■

Lemma 2.9. Let $r \geqslant 1$. If $0 < a \leqslant 1$ such that $ar \geqslant 1$ then

$$g(ar) \leqslant \frac{g(r)}{c_* a^{d-2\alpha_*}}, \tag{2.12}$$

$$g(ar) \geqslant \frac{g(r)}{c^* a^{d-2\alpha^*}}. \tag{2.13}$$

If $a \geqslant 1$ then

$$g(ar) \leqslant \frac{c^*}{a^{d-2\alpha^*}} g(r). \tag{2.14}$$

Proof.

$$g(ar) = (ar)^{-d}\phi((ar)^{-2})^{-1} = (ar)^{-d}\frac{1}{\phi(r^{-2})}\frac{\phi(r^{-2})}{\phi((ar)^{-2})} \leqslant \frac{g(r)}{c_* a^{d-2\alpha_*}}.$$

Relations (2.13) and (2.14) are proved in a completely analogous way. ■

2.6. Harmonic functions

In this section we do not restrict ourselves only to subordinate random walks. To stress that, we use notation $X = (X_n)_{n\geqslant 0}$ for a general random walk. We also use notation $p(x,y) = \mathbb{P}^x(X_1 = y)$ for one-step transition probabilities and

$$Pf(x) = \sum_{y\in\mathbb{Z}^d} p(x,y)f(y)$$

for the transition operator.

Definition 2.10. We say that a function $f : \mathbb{Z}^d \to [0,\infty)$ is *harmonic* in $B \subseteq \mathbb{Z}^d$, with respect to X, if

$$f(x) = Pf(x) = \sum_{y\in\mathbb{Z}^d} p(x,y)f(y), \quad \forall x \in B. \tag{2.15}$$

It is sometimes convenient to work with the operator $A := P - I$. Notice that relation (2.15) is equivalent to $Af(x) = 0$ for every $x \in B$. There is a strong connection between martingales and harmonic functions. Denote $\mathscr{F}_n := \sigma\{X_0, X_1, \ldots, X_n\}$, $n \geqslant 0$.

Lemma 2.11. Let $f : \mathbb{Z}^d \to [0,\infty)$ be a harmonic function in B and $\tau_B = \inf\{n \geqslant 0 : X_n \notin B\}$. Then $M_n := f(X_{n\wedge\tau_B})$ is a martingale with respect to $\mathscr{F}_n$.

A proof can be found in [18, Proposition 1.4.1]. We are now ready to prove that Definition 2.10 is equivalent to the mean-value property in terms of the exit from a finite subset of $\mathbb{Z}^d$.

Lemma 2.12. Let B be a finite subset of $\mathbb{Z}^d$. Then $f : \mathbb{Z}^d \to [0,\infty)$ is harmonic in B, with respect to X, if and only if $f(x) = \mathbb{E}^x[f(X_{\tau_B})]$ for every $x \in B$.

Proof. Notice that X_{τ_B} is well defined since $\mathbb{P}^x(\tau_B < \infty) = 1$, which is true because B is a finite set. Let us first assume that $f : \mathbb{Z}^d \to [0,\infty)$ is harmonic in B, with respect to X. We take arbitrary $x \in B$. By the martingale property $f(x) = \mathbb{E}^x[f(X_{n\wedge\tau_B})]$, for all $n \geqslant 1$. First, by Fatou's lemma we have $\mathbb{E}^x[f(X_{\tau_B})] \leqslant f(x)$ so $f(X_{\tau_B})$ is a $\mathbb{P}^x$-integrable random variable. Since B is a finite set, we have $f \leqslant M$ on B, for some constant $M > 0$. Using these two facts, we get

$$f(X_{n\wedge\tau_B}) = f(X_n)\mathbb{1}_{\{n<\tau_B\}} + f(X_{\tau_B})\mathbb{1}_{\{\tau_B\leqslant n\}} \leqslant M + f(X_{\tau_B}).$$

Since the right hand side is $\mathbb{P}^x$-integrable, we can use the dominated convergence theorem and we get

$$f(x) = \lim_{n\to\infty} \mathbb{E}^x[f(X_{n\wedge\tau_B})] = \mathbb{E}^x[\lim_{n\to\infty} f(X_{n\wedge\tau_B})] = \mathbb{E}^x[f(X_{\tau_B})].$$

On the other hand, if $f(x) = \mathbb{E}^x[f(X_{\tau_B})]$, for every $x \in B$, then for $x \in B$ we have

$$f(x) = \sum_{y\in\mathbb{Z}^d} \mathbb{E}^x\left[f(X_{\tau_B}) \mid X_1 = y\right]\mathbb{P}^x(X_1 = y) = \sum_{y\in\mathbb{Z}^d} p(x,y)\mathbb{E}^y[f(X_{\tau_B})] = \sum_{y\in\mathbb{Z}^d} p(x,y)f(y).$$

■

The last thing we prove in this section is the maximum principle for the operator A.

Proposition 2.13. Assume that there exists $x \in \mathbb{Z}^d$ such that $f(x) \leqslant f(y)$ for all $y \in \mathbb{Z}^d$. Then

$$(Af)(x) \geqslant 0. \tag{2.16}$$

Proof. Since $f(x) \leqslant f(y)$ for all $y \in \mathbb{Z}^d$, we have

$$(Pf)(x) = \sum_{y\in\mathbb{Z}^d} \mathbb{P}^x(X_1 = y)f(y) \geqslant f(x) \sum_{y\in\mathbb{Z}^d} \mathbb{P}^x(X_1 = y) = f(x).$$

This implies $(Af)(x) = (Pf)(x) - f(x) \geqslant 0$. ■

2.7. AUXILIARY RESULTS

We repeatedly use the fact that

$$c'r^d \leqslant |B(x,r)| \leqslant c''r^d, \quad x \in \mathbb{Z}^d, \tag{2.17}$$

for constants $c', c'' > 0$ which depend only on the dimension d.

Lemma 2.14. Let $\Gamma(x,a)=\int_a^\infty t^{x-1}e^{-t}dt$ and $\Gamma(x)=\Gamma(x,0)$. Then

$$\lim_{x\to\infty}\frac{\Gamma(x+1,x)}{\Gamma(x+1)}=\frac{1}{2}.$$

Proof. Using a well-known Stirling's formula

$$\Gamma(x+1)\sim\sqrt{2\pi x}\,x^xe^{-x},\quad x\to\infty \tag{2.18}$$

and [1, Formula 6.5.35] that states

$$\Gamma(x+1,x)\sim\sqrt{2^{-1}\pi x}\,x^xe^{-x},\quad x\to\infty,$$

we get

$$\lim_{x\to\infty}\frac{\Gamma(x+1,x)}{\Gamma(x+1)}=\lim_{x\to\infty}\frac{\sqrt{2^{-1}\pi x}\,x^xe^{-x}}{\sqrt{2\pi x}\,x^xe^{-x}}=\frac{1}{2}.$$

■

Lemma 2.15. Let $(U_i)_{i\in\mathbb{N}}$ be a sequence of independent, identically distributed exponential random variables with parameter 1 and let $T_n=\sum_{i=1}^n U_i$. Then for all $n\in\mathbb{N}$ and for all $t>0$ we have

$$\mathbb{P}(T_n\leqslant t)\leqslant t.$$

Proof. Notice that T_n is the sum of n independent exponential random variables with parameter 1. Hence $T_n\sim\Gamma(n,1)$. Denote with $F_{T_n}(t)=\mathbb{P}(T_n\leqslant t)$ the distribution function of the random variable T_n and with f_{T_n} the density function of T_n. We want to prove $F_{T_n}(t)\leqslant t$ for all $t>0$.

Let $g(t):=t-F_{T_n}(t)$. We will now prove that $g(t)\geqslant 0$ for all $t>0$. Since $g(0)=0$ it is enough to prove that g is increasing on $(0,\infty)$. Hence, we want to prove that $g'(t)\geqslant 0$ for all $t>0$. Since $g'(t)=1-f_{T_n}(t)$, it is enough to prove that $f_{T_n}(t)\leqslant 1$ for all $t>0$.

In the case $n=1$ the result is trivial since $T_1\sim Exp(1)$ so $F_{T_1}(t)=1-e^{-t}\leqslant t$. For $n\geqslant 2$ it is easy to check that the function f_{T_n} obtains maximum for $t=n-1$ and that maximum is

$$\frac{(n-1)^{n-1}e^{-(n-1)}}{(n-1)!}.$$

Notice that the only thing left to prove is that $n^ne^{-n}/n!\leqslant 1$ for all $n\in\mathbb{N}$. Using Stirling's approximation, we obtain

$$\frac{n!}{\sqrt{2\pi n}n^ne^{-n}}\geqslant 1\Rightarrow\frac{n!}{n^ne^{-n}}\geqslant\sqrt{2\pi n}\Rightarrow\frac{n^ne^{-n}}{n!}\leqslant\frac{1}{\sqrt{2\pi n}}\leqslant 1.$$

■

Lemma 2.16. Let $L \geqslant 1$. Then for all $0 < r \leqslant 1 \wedge R \leqslant R \leqslant L$ we have

$$\frac{c_*}{L^{\alpha_*}}\left(\frac{R}{r}\right)^{\alpha_*} \leqslant \frac{\phi(R)}{\phi(r)} \leqslant \phi(L)c^*\left(\frac{R}{r}\right)^{\alpha^*}. \tag{2.19}$$

Proof. Since $L \geqslant 1$, relation (2.19) follows directly from (2.3) in the case $R \leqslant 1$. For $0 < r \leqslant 1 < R \leqslant L$ (using (2.3) and the fact that ϕ is increasing) we have

$$\frac{\phi(R)}{\phi(r)} \leqslant \frac{\phi(L)}{\phi(r)} \leqslant \phi(L)c^*\left(\frac{1}{r}\right)^{\alpha^*} \leqslant \phi(L)c^*\left(\frac{R}{r}\right)^{\alpha^*},$$

and similarly

$$\frac{\phi(R)}{\phi(r)} \geqslant \frac{\phi(1)}{\phi(r)} \geqslant c_*\left(\frac{1}{r}\right)^{\alpha_*} \geqslant \frac{c_*}{L^{\alpha_*}}\left(\frac{R}{r}\right)^{\alpha_*},$$

as desired. ■

Lemma 2.17. There exists a constant $c > 0$ such that

$$\sum_{y \in B(x,r)^c} j(|x-y|) \leqslant c\phi(r^{-2})$$

for every $x \in \mathbb{Z}^d$ and $r > 0$.

Proof. Assume that $r \geqslant 1$. By (2.3) and (2.17), we have

$$\begin{aligned}\sum_{y \in B(x,r)^c} j(|x-y|) &\leqslant \sum_{i=0}^{\infty} \sum_{2^i r \leqslant |x-y| < 2^{i+1} r} j(2^i r) \\ &\leqslant c'' 2^d \phi(r^{-2}) \sum_{i=0}^{\infty} \frac{\phi((2^i r)^{-2})}{\phi(r^{-2})} \leqslant c\phi(r^{-2}).\end{aligned}$$

If $r \in (0,1)$ then $B(x,r)^c = B(x,1)^c$. Therefore

$$\sum_{y \in B(x,r)^c} j(|x-y|) = \sum_{y \in B(x,1)^c} j(|x-y|) \leqslant c\phi(1^{-2}) \leqslant c\phi(r^{-2}),$$

what finishes the proof. ■

2.8. Concrete examples of subordinate random walks

Example 1. As we have already commented in Section 1.2, to define the subordinate random walk we need a Bernstein function satisfying some conditions. The canonical example of a Bernstein function satisfying all of our assumptions is $\phi(\lambda) = \lambda^\alpha$, $\alpha \in (0,1)$. Since this is a complete Bernstein function, its Levy measure has a completely monotone density $m(t)$ and from [25, tables on pages 304 and 305] we know the explicit formula for m. Using this formula, we can calculate coefficients a_m^ϕ defined in (1.2):

$$a_m^\phi = \frac{1}{m!}\int_0^\infty t^m e^{-t}\frac{\alpha}{\Gamma(1-\alpha)}t^{-1-\alpha}dt = \frac{\alpha\Gamma(m-\alpha)}{\Gamma(m+1)\Gamma(1-\alpha)}.$$

Using the standard result about asymptotics of the ratio of gamma functions that can be found in [27], we obtain

$$a_m^\phi \sim \frac{\alpha}{\Gamma(1-\alpha)}m^{-\alpha-1} = \frac{\alpha}{\Gamma(1-\alpha)}\frac{\phi(m^{-1})}{m}.$$

We write the last equality because this is precisely the shape of estimates that we will obtain for coefficients a_m^ϕ with our assumptions on the Bernstein function ϕ.

Example 2. We show one more interesting example of a subordinate random walk. In this case, the Bernstein function ϕ will not satisfy the scaling condition, but coefficients a_m^ϕ will have very nice distribution. Again using [25, tables on pages 304 and 305] we know that for $a > 0$

$$\phi(\lambda) = \frac{(1+a)\lambda}{\lambda + a}$$

is a complete Bernstein function satisfying $\phi(0) = 0$ and $\phi(1) = 1$ and that its Lévy measure has a completely monotone density given with

$$m(t) = (1+a)ae^{-at}.$$

We can now calculate coefficients a_m^ϕ:

$$\begin{aligned}
a_m^\phi &= \frac{1}{m!}\int_0^\infty t^m e^{-t}(1+a)ae^{-at}dt = \frac{(1+a)a}{m!}\int_0^\infty t^m e^{-(1+a)t}dt \\
&= \frac{(1+a)a}{\Gamma(m+1)}\int_0^\infty \frac{u^m}{(1+a)^m}e^{-u}\frac{du}{1+a} = \frac{a}{\Gamma(m+1)(1+a)^m}\int_0^\infty u^m e^{-u}du = \frac{a}{(1+a)^m} \\
&= \frac{1}{(1+a)^{m-1}}\left(1-\frac{1}{1+a}\right).
\end{aligned}$$

Hence, in this case, variables $(R_n)_{n\geqslant 1}$ have geometric distribution with the parameter $a/(a+1)$.

3. Elliptic Harnack inequality

The main result of this chapter is the scale-invariant elliptic Harnack inequality for subordinate random walks.

Theorem 3.1 (Elliptic Harnack inequality)**.** Let $S^\phi = (S_n^\phi)_{n\geqslant 0}$ be a subordinate random walk in $\mathbb{Z}^d$. For each $a < 1$, there exists a constant $c_a < \infty$ such that if $f : \mathbb{Z}^d \to [0,\infty)$ is harmonic on $B(x,n)$, with respect to S^ϕ, for $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, then

$$f(x_1) \leqslant c_a f(x_2), \quad x_1, x_2 \in B(x,an).$$

Remark 3.2. Notice that the constant c_a is uniform for all $n \in \mathbb{N}$. That is why we call this result the scale-invariant elliptic Harnack inequality.

The proof of Theorem 3.1 will be given in the last section of this chapter.

3.1. One-step transition probability estimates

In this section, we establish estimates for one-step transition probabilities of the subordinate random walk S^ϕ.

Proposition 3.3. Let S^ϕ be a subordinate random walk in $\mathbb{Z}^d$. Then

$$p^\phi(x,y) \asymp j(|x-y|), \quad x \neq y,$$

where $j(r) = r^{-d}\phi(r^{-2})$ was defined in (2.8).

Before the proof of Proposition 3.3, we need to examine the behavior of the sequence $(a_m^\phi)_{m\geqslant 1}$.

Lemma 3.4. Let a_m^ϕ be as in (1.2). Then

$$a_m^\phi \asymp m^{-1}\phi(m^{-1}), \quad m \in \mathbb{N}. \tag{3.1}$$

Proof. Since ϕ is a complete Bernstein function, there exists a completely monotone density $\mu(t)$ such that

$$a_m^\phi = \frac{1}{m!}\int_{(0,\infty)} t^m e^{-t}\mu(t)dt, \quad m \geqslant 2.$$

From [17, Proposition 2.5] we have

$$\mu(t) \leqslant (1-2e^{-1})^{-1}t^{-1}\phi(t^{-1}) = c_1 t^{-1}\phi(t^{-1}), \quad t > 0 \tag{3.2}$$

and

$$\mu(t) \geqslant c_2 t^{-1}\phi(t^{-1}), \quad t \geqslant 1. \tag{3.3}$$

Inequality (3.3) holds if (2.3) is satisfied and for inequality (3.2) we do not need the scaling condition. Using monotonicity of μ, Lemma 2.14 and (3.3) we get

$$\begin{aligned} a_m^\phi &\geqslant \frac{1}{m!}\int_0^m t^m e^{-t}\mu(t)dt \geqslant \frac{\mu(m)}{m!}\int_0^m t^m e^{-t}dt = \frac{\mu(m)}{m!}(\Gamma(m+1)-\Gamma(m+1,m)) \\ &= \mu(m)\left(1-\frac{\Gamma(m+1,m)}{\Gamma(m+1)}\right) \geqslant \frac{1}{4}\mu(m) \geqslant \frac{c_2}{4}\frac{\phi(m^{-1})}{m}, \end{aligned}$$

for m large enough. On the other hand, using inequality (3.2), monotonicity of μ and (2.2), we get for $m \geqslant 2$

$$\begin{aligned} a_m^\phi &= \frac{1}{m!}\int_0^m t^m e^{-t}\mu(t)dt + \frac{1}{m!}\int_m^\infty t^m e^{-t}\mu(t)dt \leqslant \frac{c_1}{m!}\int_0^m t^m e^{-t}\frac{\phi(t^{-1})}{t}dt + \frac{\mu(m)}{m!}\int_m^\infty t^m e^{-t}dt \\ &\leqslant \frac{c_1\phi(m^{-1})}{m!}\int_0^m t^{m-1}e^{-t}\frac{\phi(t^{-1})}{\phi(m^{-1})}dt + \frac{\mu(m)}{m!}\int_0^\infty t^m e^{-t}dt \leqslant \frac{c_1\phi(m^{-1})}{(m-1)!}\int_0^m t^{m-2}e^{-t}dt + \mu(m) \\ &= \frac{c_1\phi(m^{-1})}{\Gamma(m)}\int_0^\infty t^{m-2}e^{-t}dt + \mu(m) \leqslant \frac{c_1\phi(m^{-1})}{m-1} + c_1\frac{\phi(m^{-1})}{m} \leqslant \frac{3c_1\phi(m^{-1})}{m}. \end{aligned}$$

Hence, we have

$$\frac{c_2}{4}\frac{\phi(m^{-1})}{m} \leqslant a_m^\phi \leqslant 3c_1\frac{\phi(m^{-1})}{m}$$

for m large enough. By modifying constants we obtain (3.1) for all $m \in \mathbb{N}$. ■

Proof of Proposition 3.3. Using (1.4) and the fact that $\mathbb{P}(S_m = z) = 0$ for $|z| > m$, we have

$$\mathbb{P}(S_1^\phi = z) = \sum_{m \geqslant |z|} a_m^\phi \mathbb{P}(S_m = z).$$

Combining Lemma 3.4 and (2.1) we get

$$\begin{aligned}
\mathbb{P}(S_1^\phi = z) &= \sum_{m\geqslant |z|} a_m^\phi \mathbb{P}(S_m = z) \leqslant c_1 \sum_{m\geqslant |z|} \frac{\phi(m^{-1})}{m} m^{-\frac{d}{2}} e^{-\frac{|z|^2}{c_2 m}} \leqslant c_3 \int_{|z|}^{\infty} \phi(t^{-1}) t^{-\frac{d}{2}-1} e^{-\frac{|z|^2}{c_2 t}} dt \\
&= c_3 \int_0^{\frac{|z|}{c_2}} \phi(c_2 s|z|^{-2}) \left(\frac{|z|^2}{c_2 s}\right)^{-\frac{d}{2}-1} e^{-s} \frac{|z|^2}{c_2 s^2} ds = c_4 |z|^{-d} \int_0^{\frac{|z|}{c_2}} \phi(c_2 s|z|^{-2}) s^{\frac{d}{2}-1} e^{-s} ds \\
&= c_4 |z|^{-d} \left(\int_0^{\frac{1}{c_2}} \phi(c_2 s|z|^{-2}) s^{\frac{d}{2}-1} e^{-s} ds + \int_{\frac{1}{c_2}}^{\frac{|z|}{c_2}} \phi(c_2 s|z|^{-2}) s^{\frac{d}{2}-1} e^{-s} ds \right) \\
&=: c_4 |z|^{-d} (I_1(z) + I_2(z)).
\end{aligned}$$

We now show that $I_1(z)$ and $I_2(z)$ have upper bounds of the shape $\phi(|z|^{-2})$. For $I_1(z)$ we use lower scaling to get

$$I_1(z) = \phi(|z|^{-2}) \int_0^{\frac{1}{c_2}} \frac{\phi(c_2 s|z|^{-2})}{\phi(|z|^{-2})} s^{\frac{d}{2}-1} e^{-s} ds \leqslant \phi(|z|^{-2}) \int_0^{\frac{1}{c_2}} \frac{(c_2 s)^{\alpha_*}}{c_*} s^{\frac{d}{2}-1} e^{-s} ds = c_5 \phi(|z|^{-2}).$$

For $I_2(z)$ we use (2.2) to get

$$I_2(z) = \phi(|z|^{-2}) \int_{\frac{1}{c_2}}^{\frac{|z|}{c_2}} \frac{\phi(c_2 s|z|^{-2})}{\phi(|z|^{-2})} s^{\frac{d}{2}-1} e^{-s} ds \leqslant \phi(|z|^{-2}) \int_{\frac{1}{c_2}}^{\infty} c_2 s s^{\frac{d}{2}-1} e^{-s} ds = c_6 \phi(|z|^{-2}).$$

Hence, $\mathbb{P}(S_1^\phi = z) \leqslant c_7 |z|^{-d} \phi(|z|^{-2})$. Similarly, using Lemma 3.4, (2.1), monotonicity of ϕ and (2.2), we get

$$\begin{aligned}
\mathbb{P}(S_1^\phi = z) &\geqslant \sum_{m\geqslant |z|^2} a_m^\phi \mathbb{P}(S_m = z) = \sum_{m\geqslant |z|^2/2} (a_{2m}^\phi \mathbb{P}(S_{2m} = z) + a_{2m+1}^\phi \mathbb{P}(S_{2m+1} = z)) \\
&\geqslant c_8 \sum_{m\geqslant |z|^2/2} \left(\frac{\phi((2m)^{-1})}{2m} \mathbb{P}(S_{2m} = z) + \frac{\phi((2m+1)^{-1})}{2m+1} \mathbb{P}(S_{2m+1} = z) \right) \\
&\geqslant c_8 \sum_{m\geqslant |z|^2/2} \frac{\phi((2m+1)^{-1})}{2m+1} (\mathbb{P}(S_{2m} = z) + \mathbb{P}(S_{2m+1} = z)) \\
&\geqslant \frac{c_8}{4} \sum_{m\geqslant |z|^2/2} \frac{\phi((2m)^{-1})}{2m} c_9 (2m)^{-\frac{d}{2}} e^{-\frac{|z|^2}{c_9 2m}} \geqslant c_{10} \int_{|z|^2}^{\infty} \phi(t^{-1}) t^{-\frac{d}{2}-1} e^{-\frac{|z|^2}{c_9 t}} dt \\
&= c_{10} \int_0^{1/c_9} \phi(c_9 s|z|^{-2}) \left(\frac{|z|^2}{c_9 s}\right)^{-\frac{d}{2}-1} e^{-s} \frac{|z|^2}{c_2 s^2} ds \\
&= c_{11} |z|^{-d} \phi(|z|^{-2}) \int_0^{1/c_9} \frac{\phi(c_9 s|z|^{-2})}{\phi(|z|^{-2})} s^{\frac{d}{2}-1} e^{-s} ds \geqslant c_{12} |z|^{-d} \phi(|z|^{-2}).
\end{aligned}$$

■

Remark 3.5. It follows immediately from Proposition 3.3 that the second moment of the step S_1^ϕ is infinite.

Proposition 3.3 gives us estimates of probability that the random walk S^ϕ jumps in one step from x to y for any $x,y \in \mathbb{Z}^d$, $x \neq y$. We will also need lower bound for the probability that the subordinate random walk stays at the same place.

Lemma 3.6. There exists a constant $c > 0$ such that

$$p^\phi(x,x) \geqslant c, \quad x \in \mathbb{Z}^d.$$

Proof. By [18, Thm. 1.2.1],

$$\mathbb{P}(S_{2m} = 0) \asymp m^{-d/2}, \quad m \in \mathbb{N}.$$

This and the fact that $\mathbb{P}(S_{2m-1} = 0) = 0$ combined with (1.4), Lemma 3.4 and (2.3) yield for all $x \in \mathbb{Z}^d$

$$p^\phi(x,x) \geqslant c_1 \sum_{m=1}^{\infty} \frac{\phi((2m)^{-1})}{2m} m^{-d/2} \geqslant \frac{c_1}{c^* 2^{\alpha^*+1}} \sum_{m=1}^{\infty} m^{-\alpha^*-d/2-1} =: c > 0,$$

as desired. ■

3.2. Green function estimates

The Green function of S^ϕ is defined by $G(x,y) = G(y-x)$, where

$$G(y) = \mathbb{E}[\sum_{n=0}^{\infty} \mathbb{1}_{\{S_n^\phi = y\}}]. \tag{3.4}$$

We first state the main theorem of this section.

Theorem 3.7. Let G be as in (3.4). Then

$$G(x) \asymp g(|x|), \quad x \neq 0, \tag{3.5}$$

where $g(r) = r^{-d}\phi(r^{-2})^{-1}$ was defined in (2.7).

A proof will be given at the end of the section. We can rewrite (3.4) in the following way

$$\begin{aligned} G(y) &= \sum_{n=0}^{\infty} \mathbb{P}(S_n^\phi = y) = \sum_{n=0}^{\infty} \mathbb{P}(S_{\tau_n} = y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathbb{P}(S_m = y)\mathbb{P}(\tau_n = m) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathbb{P}(\tau_n = m)\mathbb{P}(S_m = y) = \sum_{m=0}^{\infty} c(m)\mathbb{P}(S_m = y) \end{aligned} \tag{3.6}$$

where

$$c(m) = \sum_{n=0}^{\infty} \mathbb{P}(\tau_n = m), \tag{3.7}$$

and τ_n is as before. We now investigate the behavior of the sequence $(c(m))_{m\geqslant 0}$. Here we only need the assumption that ϕ is a special Bernstein function which is weaker assumption than ϕ being a complete Bernstein function by Proposition 2.5 and [25, Example 11.18]. Using the assumption that ϕ is a special Bernstein function, we have

$$\frac{1}{\phi(\lambda)} = c + \int_{(0,\infty)} e^{-\lambda t} u(t) dt \tag{3.8}$$

for some $c \geqslant 0$ and some non-increasing function $u : (0,\infty) \to (0,\infty)$ satisfying $\int_0^1 u(t)dt < \infty$, see [25, Theorem 11.3.].

Lemma 3.8. Let $c(m)$ be as in (3.7). Then

$$c(0) = 1, \quad c(m) = \frac{1}{m!}\int_{(0,\infty)} t^m e^{-t} u(t) dt, \quad m \in \mathbb{N}. \tag{3.9}$$

Proof. Since $\tau_0 = 0$ and $\tau_n > 0$ for all $n \in \mathbb{N}$ it is clear from (3.7) that $c(0) = 1$. We now follow the proof of [6, Theorem 2.3]. Define $M(x) = \sum_{m\leqslant x} c(m)$, $x \in \mathbb{R}$. The Laplace transformation $\mathscr{L}(M)$ of the measure generated by M is equal to

$$\begin{aligned}\mathscr{L}(M)(\lambda) &= \int_{[0,\infty)} e^{-\lambda x} dM(x) = \sum_{m=0}^{\infty} c(m)e^{-\lambda m} = \sum_{m=0}^{\infty} e^{-\lambda m}\sum_{n=0}^{\infty}\mathbb{P}(\tau_n = m)\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{-\lambda m}\mathbb{P}(\tau_n = m) = \sum_{n=0}^{\infty}\mathbb{E}[e^{-\lambda\tau_n}] = \sum_{n=0}^{\infty}\left(\mathbb{E}[e^{-\lambda R_1}]\right)^n = \frac{1}{1-\mathbb{E}[e^{-\lambda R_1}]}.\end{aligned} \tag{3.10}$$

Now we calculate $\mathbb{E}[e^{-\lambda R_1}]$:

$$\begin{aligned}\mathbb{E}[e^{-\lambda R_1}] &= \sum_{m=1}^{\infty} e^{-\lambda m} a_m^{\phi} = be^{-\lambda} + \sum_{m=1}^{\infty} e^{-\lambda m}\int_{(0,\infty)}\frac{t^m}{m!}e^{-t}\mu(dt)\\ &= be^{-\lambda} + \int_{(0,\infty)}\sum_{m=1}^{\infty}\frac{(te^{-\lambda})^m}{m!}e^{-t}\mu(dt)\\ &= be^{-\lambda} + \int_{(0,\infty)}(e^{te^{-\lambda}}-1)e^{-t}\mu(dt)\\ &= be^{-\lambda} + \int_{(0,\infty)}(e^{-t(1-e^{-\lambda})}-e^{-t})\mu(dt)\\ &= b + \int_{(0,\infty)}(1-e^{-t})\mu(dt) - [b(1-e^{-\lambda}) + \int_{(0,\infty)}(1-e^{-t(1-e^{-\lambda})})\mu(dt)]\\ &= 1-\phi(1-e^{-\lambda}),\end{aligned}$$

where in the last equality we used $\phi(1) = 1$. Hence, $\mathscr{L}(M)(\lambda) = 1/\phi(1-e^{-\lambda})$. On the other hand, using (3.8), we get

$$
\begin{aligned}
1+\sum_{m=1}^{\infty}\frac{1}{m!}\int_{(0,\infty)} t^m e^{-t}u(t)dt\, e^{-\lambda m} &= 1+\int_{(0,\infty)} e^{-t}\sum_{m=1}^{\infty}\frac{(te^{-\lambda})^m}{m!}u(t)dt \\
&= 1+\int_{(0,\infty)} e^{-t}(e^{te^{-\lambda}}-1)u(t)dt = 1+\int_{(0,\infty)}(e^{-t(1-e^{-\lambda})}-e^{-t})u(t)dt \\
&= 1+\int_{(0,\infty)} e^{-t(1-e^{-\lambda})}u(t)dt-\int_{(0,\infty)} e^{-t}u(t)dt \\
&= 1+\frac{1}{\phi(1-e^{-\lambda})}-c-\frac{1}{\phi(1)}+c=\frac{1}{\phi(1-e^{-\lambda})}. \qquad (3.11)
\end{aligned}
$$

Since $\mathscr{L}(M)(\lambda)=1/\phi(1-e^{-\lambda})$, from calculations (3.10) and (3.11) we have

$$
\sum_{m=0}^{\infty} c(m)e^{-\lambda m}=1+\sum_{m=1}^{\infty}\frac{1}{m!}\int_{(0,\infty)} t^m e^{-t}u(t)dt\, e^{-\lambda m}.
$$

The statement of this lemma follows by the uniqueness of the Laplace transformation. ∎

Lemma 3.9. Let $c(m)$ be as in (3.7). Then

$$
c(m)\asymp\frac{1}{m\phi(m^{-1})},\quad m\in\mathbb{N}.
$$

Proof. Let u be the function from (3.8). From [17, Corollary 2.4.] we have

$$
u(t)\leqslant(1-e^{-1})^{-1}t^{-1}\phi(t^{-1})^{-1}=c_1t^{-1}\phi(t^{-1})^{-1},\quad t>0. \qquad (3.12)
$$

and

$$
u(t)\geqslant c_2t^{-1}\phi(t^{-1})^{-1},\quad t\geqslant 1. \qquad (3.13)
$$

Inequality (3.13) holds if (2.3) is satisfied and for inequality (3.12) we do not need any scaling conditions. Using monotonicity of u, Lemma 2.14 and (3.13), we get

$$
c(m)\geqslant\frac{u(m)}{m!}\int_0^m t^m e^{-t}dt=u(m)\left(1-\frac{\Gamma(m+1,m)}{\Gamma(m+1)}\right)\geqslant\frac{1}{4}u(m)\geqslant\frac{c_3}{m\phi(m^{-1})},
$$

for m large enough. For the upper bound of $c(m)$ we use (3.12), monotonicity of u and monotonicity of ϕ.

$$
\begin{aligned}
c(m) &\leqslant\frac{c_1}{m!}\int_0^m t^m e^{-t}\frac{1}{t\phi(t^{-1})}dt+\frac{u(m)}{m!}\int_m^{\infty} t^m e^{-t}dt \\
&\leqslant\frac{c_1}{m!\phi(m^{-1})}\int_0^m t^{m-1}e^{-t}dt+u(m)\leqslant\frac{c_4}{m\phi(m^{-1})}
\end{aligned}
$$

Hence,

$$
\frac{c_3}{m\phi(m^{-1})}\leqslant c(m)\leqslant\frac{c_4}{m\phi(m^{-1})}
$$

for m large enough. We can now change constants in such a way that the statement of this lemma is true for every $m\in\mathbb{N}$. ∎

Proof of Theorem 3.7. Using (3.6), for $x \neq 0$ we get $G(x) = \sum_{m=1}^{\infty} c(m)p(m,x)$, $p(m,x) = \mathbb{P}(S_m = x)$. Let $q(m,x) = 2(d/(2\pi m))^{\frac{d}{2}} e^{-d|x|^2/2m}$ and $E(m,x) = p(m,x) - q(m,x)$. By [18, Theorem 1.2.1]

$$|E(m,x)| \leqslant c_1 m^{-d/2}/|x|^2. \tag{3.14}$$

Since $p(m,x) = 0$ for $m < |x|$, we have

$$G(x) = \sum_{m>|x|^2} c(m)p(m,x) + \sum_{|x|\leqslant m\leqslant|x|^2} c(m)p(m,x) =: J_1(x) + J_2(x).$$

We first estimate

$$J_1(x) = \sum_{m>|x|^2} c(m)q(m,x) + \sum_{m>|x|^2} c(m)E(m,x) =: J_{11}(x) + J_{12}(x).$$

Combining Lemma 3.9, (3.14) and (2.3) we get

$$\begin{aligned}|J_{12}(x)| &\leqslant c_2 \sum_{m>|x|^2} \frac{1}{m\phi(m^{-1})} \frac{m^{-\frac{d}{2}}}{|x|^2} = \frac{c_2}{|x|^2\phi(|x|^{-2})} \sum_{m>|x|^2} \frac{\phi(|x|^{-2})}{\phi(m^{-1})} m^{-\frac{d}{2}-1} \\ &\leqslant \frac{c_3|x|^{-2\alpha^*}}{|x|^2\phi(|x|^{-2})} \int_{|x|^2}^{\infty} t^{\alpha^*-\frac{d}{2}-1} dt = \frac{c_4}{|x|^2} \frac{1}{|x|^d\phi(|x|^{-2})}.\end{aligned}$$

Now we have

$$\lim_{|x|\to\infty} |x|^d \phi(|x|^{-2})|J_{12}(x)| = 0.$$

By Lemma 3.9 and (2.3)

$$\begin{aligned}J_{11}(x) &\leqslant c_5 \int_{|x|^2}^{\infty} \frac{1}{t\phi(t^{-1})} t^{-\frac{d}{2}} e^{-\frac{d|x|^2}{2t}} dt = \frac{c_5}{\phi(|x|^{-2})} \int_{|x|^2}^{\infty} \frac{\phi(|x|^{-2})}{\phi(t^{-1})} t^{-\frac{d}{2}-1} e^{-\frac{d|x|^2}{2t}} dt \\ &\leqslant \frac{c_6|x|^{-2\alpha^*}}{\phi(|x|^{-2})} \int_{|x|^2}^{\infty} t^{\alpha^*-\frac{d}{2}-1} e^{-\frac{d|x|^2}{2t}} dt = \frac{c_7}{|x|^d\phi(|x|^{-2})} \int_0^{\frac{d}{2}} s^{\frac{d}{2}-\alpha^*-1} e^{-s} ds = \frac{c_8}{|x|^d\phi(|x|^{-2})},\end{aligned}$$

where the last integral converges because of the condition $d > 2\alpha^*$. In a completely analogous way, using lower scaling instead of upper scaling and using $d > 2\alpha_*$, we obtain

$$J_{11}(x) \geqslant \frac{c_9}{|x|^d\phi(|x|^{-2})}.$$

We estimate $J_2(x)$ using (2.1) and (2.3).

$$\begin{aligned}J_2(x) &\leqslant c_{10} \int_{|x|}^{|x|^2} \frac{1}{t\phi(t^{-1})} t^{-\frac{d}{2}} e^{-\frac{|x|^2}{c_{11}t}} dt = \frac{c_{10}}{\phi(|x|^{-2})} \int_{|x|}^{|x|^2} \frac{\phi(|x|^{-2})}{\phi(t^{-1})} t^{-\frac{d}{2}-1} e^{-\frac{|x|^2}{c_{11}t}} dt \\ &\leqslant \frac{c_{10}|x|^{-2\alpha_*}}{c_*\phi(|x|^{-2})} \int_{|x|}^{|x|^2} t^{\alpha_*-\frac{d}{2}-1} e^{-\frac{|x|^2}{c_{11}t}} dt = \frac{c_{10}|x|^{-2\alpha_*}}{c_*\phi(|x|^{-2})} \int_{1/c_{11}}^{|x|/c_{11}} \left(\frac{|x|^2}{c_{11}s}\right)^{\alpha_*-\frac{d}{2}-1} e^{-s} \frac{|x|^2}{c_{11}s^2} ds \\ &\leqslant \frac{c_{12}}{|x|^d\phi(|x|^{-2})} \int_0^{\infty} s^{\frac{d}{2}-\alpha_*-1} e^{-s} ds = \frac{c_{13}}{|x|^d\phi(|x|^{-2})}\end{aligned}$$

Using the above results we get for x large enough

$$G(x) \geqslant J_{11}(x) + J_{12}(x) \geqslant \frac{c_9}{|x|^d \phi(|x|^{-2})} - \frac{c_9/2}{|x|^d \phi(|x|^{-2})} = \frac{c_9/2}{|x|^d \phi(|x|^{-2})},$$

$$G(x) = J_{11}(x) + J_{12}(x) + J_2(x) \leqslant \frac{c_8}{|x|^d \phi(|x|^{-2})} + \frac{c_8}{|x|^d \phi(|x|^{-2})} + \frac{c_{13}}{|x|^d \phi(|x|^{-2})} = \frac{c_{14}}{|x|^d \phi(|x|^{-2})}.$$

We can now change constants to obtain

$$G(x) \asymp |x|^{-d} \phi(|x|^{-2})^{-1}, \quad x \neq 0.$$

■

3.3. Estimates of the Green function of a ball

For $B \subseteq \mathbb{Z}^d$ we define

$$G_B(x,y) = \mathbb{E}^x[\sum_{n=0}^{\tau_B - 1} \mathbb{1}_{\{S_n^\phi = y\}}],$$

where $\tau_B := \min\{n \geqslant 0 : S_n^\phi \notin B\}$. We call G_B the Green function of the set B. In this section we find estimates of the function G_{B_n}. The main result that we prove at the end of this section is the following theorem.

Theorem 3.10. There exist constants $b_1, b_2 \in (0, 1/2)$, $2b_1 \leqslant b_2$, such that for all $n \in \mathbb{N}$

$$G_{B_n}(x,y) \asymp n^{-d} \mathbb{E}^y[\tau_{B_n}], \quad x \in B_{b_1 n},\ y \in A(b_2 n, n). \tag{3.15}$$

A well-known result about the Green function of a set is formulated in the following lemma.

Lemma 3.11. Let B be a finite subset of $\mathbb{Z}^d$. Then

$$G_B(x,y) = G(x,y) - \mathbb{E}^x[G(S_{\tau_B}^\phi, y)], \quad x, y \in B,$$
$$G_B(x,x) = \mathbb{P}^x(\tau_B < \sigma_x)^{-1}, \quad x \in B,$$

where $\sigma_x := \inf\{n \geqslant 1 : S_n^\phi = x\}$.

Proof. Using definitions of functions G and G_B, we get

$$G_B(x,y) = \mathbb{E}^x[\sum_{n=0}^{\infty} \mathbb{1}_{\{S_n^\phi = y\}} - \sum_{n=\tau_B}^{\infty} \mathbb{1}_{\{S_n^\phi = y\}}] = G(x,y) - \sum_{n=0}^{\infty} \mathbb{P}^x(S_{n+\tau_B}^\phi = y).$$

We now examine the expression $\mathbb{P}^x(S^\phi_{n+\tau_B}=y)$:

$$\mathbb{P}^x(S^\phi_{n+\tau_B}=y)=\sum_{z\in B^c}\mathbb{P}^x(S^\phi_{n+\tau_B}=y\mid S^\phi_{\tau_B}=z)\mathbb{P}^x(S^\phi_{\tau_B}=z)=\sum_{z\in B^c}\mathbb{P}_z(S^\phi_n=y)\mathbb{P}^x(S^\phi_{\tau_B}=z).$$

Hence,

$$\begin{aligned}G_B(x,y)&=G(x,y)-\sum_{n=0}^\infty\sum_{z\in B^c}\mathbb{P}_z(S^\phi_n=y)\mathbb{P}^x(S^\phi_{\tau_B}=z)=G(x,y)-\sum_{z\in B^c}G(z,y)\mathbb{P}^x(S^\phi_{\tau_B}=z)\\&=G(x,y)-\mathbb{E}^x[G(S^\phi_{\tau_B},y)].\end{aligned}$$

On the other hand,

$$\begin{aligned}G_B(x,x)&=\sum_{n=0}^\infty\mathbb{P}^x(S^\phi_n=x,n<\tau_B)\\&=1+\sum_{n=1}^\infty\sum_{m=1}^n\mathbb{P}^x(S^\phi_n=x,\sigma_x=m,n<\tau_B)\\&=1+\sum_{m=1}^\infty\sum_{n=m}^\infty\mathbb{P}^x(S^\phi_1,\dots,S^\phi_{m-1}\in B\setminus\{x\},S^\phi_m=x,S^\phi_{m+1},\dots,S^\phi_{n-1}\in B,S^\phi_n=x)\\&=1+\sum_{m=1}^\infty\sum_{n=m}^\infty\mathbb{P}^x(S^\phi_{m+1},\dots,S^\phi_{n-1}\in B,S^\phi_n=x\mid S^\phi_m=x,S^\phi_1,\dots,S^\phi_{m-1}\in B\setminus\{x\})\\&\qquad\mathbb{P}^x(S^\phi_1,\dots,S^\phi_{m-1}\in B\setminus\{x\},S^\phi_m=x)\\&=1+\sum_{m=1}^\infty\sum_{n=m}^\infty\mathbb{P}^x(S^\phi_1,\dots,S^\phi_{n-m-1}\in B,S^\phi_{n-m}=x)\mathbb{P}^x(\sigma_x=m,\sigma_x<\tau_B)\\&=1+\sum_{m=1}^\infty\sum_{n=0}^\infty\mathbb{P}^x(S^\phi_n=x,n<\tau_B)\mathbb{P}^x(\sigma_x=m,\sigma_x<\tau_B)\\&=1+G_B(x,x)\mathbb{P}^x(\sigma_x<\tau_B),\end{aligned}$$

which gives us precisely

$$G_B(x,x)=\mathbb{P}^x(\tau_B<\sigma_x)^{-1}.$$

■

Throughout the rest of this section, we follow [16, Section 4].

Lemma 3.12. There exist $a\in(0,1/3)$ and $C_1>0$ such that for every $n\in\mathbb{N}$

$$G_{B_n}(x,y)\geqslant C_1G(x,y),\quad x,y\in B_{an}. \tag{3.16}$$

Proof. From Lemma 3.11 we have

$$G_{B_n}(x,y)=G(x,y)-\mathbb{E}^x[G(S^\phi_{\tau_{B_n}},y)].$$

First we prove this lemma in the case $x \neq y$. Notice that if we show $\mathbb{E}^x[G(S^{\phi}_{\tau_{B_n}}, y)] \leqslant c_1 G(x,y)$ for some $c_1 \in (0,1)$ we will have (3.16) with the constant $c_2 = 1 - c_1$. Let $a \in (0, 1/3)$. This implies $(1-a)/(2a) > 1$. Take $x, y \in B_{an}$. In this case, we have $|x-y| \leqslant 2an$. Combining $S^{\phi}_{\tau_{B_n}} \notin B_n$, $x \neq y$ and $(1-a)/(2a) > 1$, we have

$$|y - S^{\phi}_{\tau_{B_n}}| \geqslant (1-a)n = \frac{1-a}{2a} 2an \geqslant \frac{1-a}{2a}|x-y| \geqslant 1. \tag{3.17}$$

Using Theorem 3.7, (3.17), Lemma 2.7 and (2.14), we get

$$\begin{aligned} G(S^{\phi}_{\tau_{B_n}}, y) &\asymp g(|y - S^{\phi}_{\tau_{B_n}}|) \leqslant c^* g\left(\frac{1-a}{2a}|x-y|\right) \\ &\leqslant (c^*)^2 \left(\frac{2a}{1-a}\right)^{d-2\alpha^*} g(|x-y|) \asymp (c^*)^2 \left(\frac{2a}{1-a}\right)^{d-2\alpha^*} G(x,y). \end{aligned}$$

Since $2a/(1-a) \longrightarrow 0$ when $a \to 0$ and $d > 2\alpha^*$, if we take a small enough and then fix it, we have $\mathbb{E}^x[G(S^{\phi}_{\tau_{B_n}}, y)] \leqslant c_1 G(x,y)$ for $c_1 \in (0,1)$ and that is exactly what we wanted to prove. Now we deal with the case $x = y$. From Lemma 3.11 we have $G_{B_n}(x,x) = (\mathbb{P}(\tau_{B_n} < \sigma_x))^{-1}$ and from the definition of the function G and transience of the random walk S^{ϕ}, we get $G(x,x) = G(0) \in [1, \infty)$. Now, we can conclude that

$$G_{B_n}(x,x) \geqslant 1 = (G(0))^{-1} G(0) = (G(0))^{-1} G(x,x).$$

Setting $C_1 := \min\{c_2, (G(0))^{-1}\}$ gives us (3.16). ■

Proposition 3.13. There exists a constant $C_2 > 0$ such that for all $n \in \mathbb{N}$

$$\mathbb{E}^x[\tau_{B_n}] \geqslant \frac{C_2}{\phi(n^{-2})}, \quad x \in B_{\frac{an}{2}}, \tag{3.18}$$

where $a \in (0, 1/3)$ is as in Lemma 3.12.

Proof. Let $x \in B_{\frac{an}{2}}$. For such x, we have $B(x, an/2) \subseteq B_{an}$. We set $b := a/2$ for easier notation. Notice that $\mathbb{E}^x[\tau_{B_n}] = \sum_{y \in B_n} G_{B_n}(x,y)$. Combining this equality, Lemma 3.12, Theorem 3.7 and (2.2), we get

$$\begin{aligned} \mathbb{E}^x[\tau_{B_n}] &\geqslant \sum_{y \in B(x,bn)} G_{B_n}(x,y) \geqslant \sum_{y \in B(x,bn) \setminus \{x\}} C_1 G(x,y) \asymp \sum_{y \in B(x,bn) \setminus \{x\}} g(|x-y|) \\ &\asymp \int_1^{bn} g(r) r^{d-1} dr = \int_1^{bn} \frac{1}{r\phi(r^{-2})} dr = \frac{1}{\phi(n^{-2})} \int_1^{bn} \frac{1}{r} \frac{\phi(n^{-2})}{\phi(r^{-2})} dr \\ &\geqslant \frac{1}{c^* \phi(n^{-2}) n^{2\alpha^*}} \int_1^{bn} r^{2\alpha^* - 1} dr \\ &= \frac{1}{2c^* \alpha^* \phi(n^{-2})} \left[b^{2\alpha^*} - \frac{1}{n^{2\alpha^*}} \right] \geqslant \frac{b^{2\alpha^*}}{4c^* \alpha^* \phi(n^{-2})}, \end{aligned}$$

for n large enough. Hence, we proved that $\mathbb{E}^x[\tau_{B_n}] \geqslant C_2/\phi(n^{-2})$, for all $x \in B_{\frac{an}{2}}$, for n large enough and for some $C_2 > 0$. As usual, we can adjust the constant to get the statement of this proposition for every $n \in \mathbb{N}$. ■

Lemma 3.14. There exists a constant $C_3 > 0$ such that for all $n \in \mathbb{N}$

$$\mathbb{E}^x[\tau_{B_n}] \leqslant \frac{C_3}{\phi(n^{-2})}, \quad x \in B_n. \tag{3.19}$$

Proof. We define the process $M^f = (M_n^f)_{n\geqslant 0}$ as

$$M_n^f := f(S_n^\phi) - f(S_0^\phi) - \sum_{k=0}^{n-1} (A^\phi f)(S_k^\phi)$$

where f is a function defined on $\mathbb{Z}^d$ with values in $\mathbb{R}$ and A^ϕ is defined as in Section 2.6. By [23, Theorem 4.1.2], the process M^f is a martingale for every bounded function f. Applying the optional stopping theorem, we get

$$\mathbb{E}_x[M_{\tau_{B_n}}^f] = \mathbb{E}^x[f(S_{\tau_{B_n}}^\phi) - f(S_0^\phi) - \sum_{k=0}^{\tau_{B_n}-1} (A^\phi f)(S_k^\phi)] = \mathbb{E}^x[M_0^f] = 0.$$

Therefore

$$\mathbb{E}^x[f(S_{\tau_{B_n}}^\phi) - f(S_0^\phi)] = \mathbb{E}^x\left[\sum_{k=0}^{\tau_{B_n}-1} (A^\phi f)(S_k^\phi)\right]. \tag{3.20}$$

Let $f := \mathbb{1}_{B_{2n}}$ and $x \in B_n$. We now investigate both sides of relation (3.20). Using Proposition 3.3, for every $y \in B_n$ we have

$$\begin{aligned}
(A^\phi f)(y) &= \sum_{u\in\mathbb{Z}^d} \mathbb{P}^y(S_1^\phi = u)(f(u) - f(y)) \asymp -\sum_{u\in B_{2n}^c} |u-y|^{-d}\phi(|u-y|^{-2}) \\
&\asymp -\int_n^\infty r^{-d}\phi(r^{-2})r^{d-1}dr = -\phi(n^{-2})\int_n^\infty r^{-1}\frac{\phi(r^{-2})}{\phi(n^{-2})}dr \\
&\leqslant -\frac{\phi(n^{-2})n^{2\alpha_*}}{c_*}\int_n^\infty r^{-2\alpha_*-1}dr = -\frac{\phi(n^{-2})}{2c_*\alpha_*},
\end{aligned}$$

where in the last line we used lower scaling condition. Repeating the calculation with upper scaling condition, we get lower bound. Hence $(A^\phi f)(y) \asymp -\phi(n^{-2})$ for $y \in B_n$. Notice that for every $k < \tau_{B_n}$, $S_k^\phi \in B_n$. This gives us

$$\mathbb{E}^x[\sum_{k=0}^{\tau_{B_n}-1} (A^\phi f)(S_k^\phi)] \asymp \mathbb{E}^x[-\sum_{k=0}^{\tau_{B_n}-1} \phi(n^{-2})] = -\phi(n^{-2})\mathbb{E}^x[\tau_{B_n}]. \tag{3.21}$$

Using (3.20), (3.21) and $\mathbb{E}^x[f(S_{\tau_{B_n}}^\phi) - f(S_0^\phi)] = \mathbb{P}^x(S_{\tau_{B_n}}^\phi \in B_{2n}) - 1 = -\mathbb{P}^x(S_{\tau_{B_n}}^\phi \in B_{2n}^c)$, we get

$$\mathbb{P}^x(S_{\tau_{B_n}}^\phi \in B_{2n}^c) \asymp \phi(n^{-2})\mathbb{E}^x[\tau_{B_n}]$$

and this implies

$$\mathbb{E}^x[\tau_{B_n}] \leqslant \frac{C_3 \mathbb{P}^x(X_{\tau_{B_n}} \in B^c_{2n})}{\phi(n^{-2})} \leqslant \frac{C_3}{\phi(n^{-2})}.$$

■

We now make one small observation that we use in the results that follow. Denote with $\eta(x) = \mathbb{E}^x[\tau_{B_n}]$. Let $x \in B_n$. Then

$$\begin{aligned}\eta(x) &= \sum_{y\in\mathbb{Z}^d} \mathbb{E}^x[\tau_{B_n} \mid S^\phi_1 = y]\mathbb{P}^x(S^\phi_1 = y)\\ &= \sum_{y\in\mathbb{Z}^d} (1+\mathbb{E}^y[\tau_{B_n}])\mathbb{P}^x(S^\phi_1 = y) = 1 + (P^\phi \eta)(x).\end{aligned}$$

Using notation $A^\phi = P^\phi - I$ as before, this means that $(A^\phi \eta)(x) = -1$ for every $x \in B_n$. We also introduce notation $A(r,s) = \{x \in \mathbb{Z}^d : r \leqslant |x| < s\}$, for $0 < r < s$.

Proposition 3.15. There exists a constant $C_4 > 0$ such that for all $n \in \mathbb{N}$

$$G_{B_n}(x,y) \leqslant C_4 n^{-d} \eta(y), \quad x \in B_{\frac{an}{4}}, y \in A(an/2, n), \tag{3.22}$$

where $\eta(y) = \mathbb{E}^y[\tau_{B_n}]$ and $a \in (0,1/3)$ is as in Lemma 3.12.

Proof. Let $x \in B_{\frac{an}{4}}$ and $y \in A(an/2,n)$. We define the function $h(z) := G_{B_n}(x,z)$. Notice that for $z \in B_n \setminus \{x\}$ we have

$$h(z) = G_{B_n}(x,z) = G_{B_n}(z,x) = \sum_{y\in\mathbb{Z}^d} \mathbb{P}_z(S^\phi_1 = y) G_{B_n}(y,x) = \sum_{y\in\mathbb{Z}^d} \mathbb{P}_z(S^\phi_1 = y) h(y).$$

Hence, h is a harmonic function on $B_n \setminus \{x\}$. We now take $z \in B(x, an/16)^c$. For n large enough, we have $|z-x| \geqslant an/16 \geqslant 1$. Combining Lemma 2.7 and Theorem 3.7, we get

$$g(an/16) \geqslant (c^*)^{-1} g(|z-x|) \asymp G(x,z) \geqslant G_{B_n}(x,z) = h(z).$$

Thus, $h(z) \leqslant kg(an/16)$ for $z \in B(x,an/16)^c$ and for some constant $k > 0$. It is clear that $A(an/2,n) \subseteq B(x,an/16)^c$. Hence, $y \in B(x,an/16)^c$. Using these facts together with Proposition 3.3, we have

$$
\begin{aligned}
A^{\phi}(h\wedge kg(an/16))(y) &= A^{\phi}(h\wedge kg(an/16)-h)(y)\\
&= \sum_{v\in\mathbb{Z}^d}\mathbb{P}^y(S_1^{\phi}=v)(h(v)\wedge kg(an/16)-h(v)-h(y)\wedge kg(an/16)+h(y))\\
&\asymp \sum_{v\in B(x,an/16)} j(|v-y|)(h(v)\wedge kg(an/16)-h(v))\\
&\geqslant -\sum_{v\in B(x,an/16)} j(|v-y|)h(v)\geqslant -\sum_{v\in B(x,an/16)} j(an/16)h(v)\\
&= -j(an/16)\sum_{v\in B(x,an/16)} G_{B_n}(x,v)\geqslant -j(an/16)\eta(x),
\end{aligned}
$$

where we used monotonicity of j together with $|v-y|\geqslant an/16\geqslant 1$ for $v\in B(x,an/16)$ and for n large enough. Using (2.9) we get $j(an/16)\leqslant (a/16)^{-d-2}j(n)$. Hence, using Lemma 3.14, we have

$$
\begin{aligned}
A^{\phi}(h\wedge kg(an/16))(y) &\geqslant -c_1 n^{-d}\phi(n^{-2})\eta(x)\\
&\geqslant -c_1 n^{-d}\phi(n^{-2})C_3(\phi(n^{-2}))^{-1} = -c_2 n^{-d}
\end{aligned}
$$

for some $c_1,c_2>0$. On the other hand, using (2.12) and Proposition 3.13, we get

$$
\begin{aligned}
g(an/16) &\leqslant (c_*)^{-1}(a/16)^{-d+2\alpha_*}g(n) = (c_*)^{-1}(a/16)^{-d+2\alpha_*}(\phi(n^{-2}))^{-1}n^{-d}\\
&\leqslant (c_*C_2)^{-1}(a/16)^{-d+2\alpha_*}n^{-d}\eta(z) = c_3 n^{-d}\eta(z), \quad \forall z\in B_{an/2}.
\end{aligned}
$$

Now we define $C_4 := (c_2\vee kc_3)+1$ and using

$$
h(z)\wedge kg(an/16)\leqslant kg(an/16)\leqslant kc_3 n^{-d}\eta(z)
$$

we get

$$
C_4 n^{-d}\eta(z)-h(z)\wedge kg(an/16)\geqslant (C_4-kc_3)n^{-d}\eta(z)\geqslant 0, \quad \forall z\in B_{an/2}
$$

Thus, for function u defined as $u(\cdot):=C_4 n^{-d}\eta(\cdot)-h(\cdot)\wedge kg(an/16)$, we showed that u is non-negative on $B_{an/2}$. It obviously vanishes on B_n^c and for $y\in A(an/2,n)$ we have

$$
(A^{\phi}u)(y) = C_4 n^{-d}(A^{\phi}\eta)(y)-A^{\phi}(h\wedge kg(an/16))(y)\leqslant -C_4 n^{-d}+c_2 n^{-d}<0.
$$

Since $u\geqslant 0$ on $B_{\frac{an}{2}}$ and u vanishes on B_n^c, if $\inf_{y\in\mathbb{Z}^d}u(y)<0$ then there would exist $y_0\in A(an/2,n)$ such that $u(y_0)=\inf_{y\in\mathbb{Z}^d}u(y)$. But then, by Proposition 2.13, $(A^{\phi}u)(y_0)\geqslant 0$ which is a contradiction with $(A^{\phi}u)(y)<0$ for $y\in A(an/2,n)$. Hence,

$$
u(y) = C_4 n^{-d}\eta(y)-h(y)\wedge kg(an/16)\geqslant 0, \quad \forall y\in\mathbb{Z}^d
$$

and then, because $h(y) \leqslant kg(an/16)$ for $y \in A(an/2,n)$ we get

$$G_{B_n}(x,y) = h(y) \leqslant C_4 n^{-d} \eta(y), \quad \forall x \in B_{\frac{an}{4}}, y \in A(an/2,n).$$

■

Proposition 3.16. There exist constants $C_5 > 0$ and $b \leqslant a/4$ such that for all $n \in \mathbb{N}$

$$G_{B_n}(x,y) \geqslant C_5 n^{-d} \eta(y), \quad x \in B_{bn}, y \in A(an/2,n), \tag{3.23}$$

where a is as in Lemma 3.12 and $\eta(y) = \mathbb{E}^y[\tau_{B_n}]$.

Proof. Let $a \in (0,1/3)$ be as in Lemma 3.12. Then

$$G_{B_n}(x,v) \geqslant C_1 G(x,v), \quad x,v \in B_{an}, \tag{3.24}$$

where $C_1 > 0$ is the constant from Lemma 3.12. From Proposition 3.15 it follows that

$$G_{B_n}(x,v) \leqslant C_4 n^{-d} \eta(v), \quad x \in B_{an/4}, v \in A(an/2,n), \tag{3.25}$$

for some constant $C_4 > 0$. From Lemma 3.14 we have

$$\eta(v) \leqslant \frac{C_3}{\phi(n^{-2})}, \quad v \in B_n, \tag{3.26}$$

for some constant $C_3 > 0$. By Theorem 3.7 there exists a constant $c_1 > 0$ such that $G(x) \geqslant c_1 g(|x|)$, $x \neq 0$. Now we take

$$b := \min\left\{\frac{a}{4}, \left(\frac{C_1 c_1}{2(c^*)^2 C_3 C_4}\right)^{\frac{1}{d-2\alpha^*}}\right\}.$$

Let $x \in B_{bn}$, $v \in B(x,bn)$. Since $b \leqslant a/4$, we have $x,v \in B_{an}$. We want to prove that $G_{B_n}(x,v) \geqslant 2C_4 n^{-d}\eta(v)$. We first prove that assertion for $x \neq v$. In that case we have $1 \leqslant |x-v|$. Since $v \in B(x,bn)$, we have $|x-v| \leqslant bn$ so we can use (3.24), Lemma 2.7, (2.13) and (3.26) to get

$$G_{B_n}(x,v) \geqslant C_1 G(x,v) \geqslant \frac{C_1 c_1}{c^*} g(bn) \geqslant \frac{C_1 c_1}{(c^*)^2 b^{d-2\alpha^*}} g(n) \geqslant \frac{2C_3 C_4}{n^d \phi(n^{-2})} \geqslant 2C_4 n^{-d} \eta(v). \tag{3.27}$$

Hence, we obtained $G_{B_n}(x,v) \geqslant 2C_4 n^{-d}\eta(v)$ for $x \neq v$. Now we prove $G_{B_n}(x,x) \geqslant 2C_4 n^{-d}\eta(x)$, for $x \in B_{bn}$ and for n large enough. First note that

$$\lim_{n\to\infty} n^d \phi(n^{-2}) = \lim_{n\to\infty} n^d \, \frac{\phi(n^{-2})}{\phi(1)} \geqslant \lim_{n\to\infty} n^d \frac{1}{c^* n^{2\alpha^*}} = \lim_{n\to\infty} \frac{1}{c^*} n^{d-2\alpha^*} = \infty,$$

since $d - 2\alpha^* > 0$. Therefore

$$2C_4 n^{-d} \eta(x) \leqslant \frac{2C_4 C_3}{n^d \phi(n^{-2})} \leqslant 1 \leqslant G_{B_n}(x,x)$$

for n large enough. Hence,

$$C_4 n^{-d}\eta(v) \leqslant \frac{1}{2} G_{B_n}(x,v), \quad x \in B_{bn}, v \in B(x,bn). \tag{3.28}$$

Now we fix $x \in B_{bn}$ and define the function

$$h(v) := G_{B_n}(x,v) \wedge (C_4 n^{-d}\eta(v)).$$

From (3.28) we have $h(v) \leqslant \frac{1}{2} G_{B_n}(x,v)$ for $v \in B(x,bn)$. Recall that $G_{B_n}(x,\cdot)$ is harmonic in $B_n \setminus \{x\} \supseteq A(an/2,n)$. Using (3.25) we get $h(y) = G_{B_n}(x,y)$ for $y \in A(an/2,n)$. Hence, for $y \in A(an/2,n)$

$$\begin{aligned}
(A^\phi h)(y) &= A^\phi (h(\cdot) - G_{B_n}(x,\cdot))(y) \\
&= \sum_{v\in\mathbb{Z}^d} \mathbb{P}^y(S_1^\phi = v)(h(v) - G_{B_n}(x,v) - h(y) + G_{B_n}(x,y)) \\
&\leqslant c_2 \sum_{v\in B(x,bn)} j(|v-y|)(h(v) - G_{B_n}(x,v)) \\
&\leqslant -\frac{c_2}{2} \sum_{v\in B(x,bn)} j(|v-y|) G_{B_n}(x,v) \\
&\leqslant -\frac{c_2 j(2n)}{2} \sum_{v\in B(x,bn)} G_{B_n}(x,v),
\end{aligned} \tag{3.29}$$

where we used Proposition 3.3 and monotonicity of j together with $1 \leqslant |v-y| \leqslant 2n$. Combining (3.27) and (2.17), we get

$$\sum_{v\in B(x,bn)} G_{B_n}(x,v) \geqslant \frac{2C_3C_4}{n^d\phi(n^{-2})}|B_{bn}| \geqslant \frac{2c' C_3 C_4}{n^d\phi(n^{-2})}(bn)^d = \frac{c_3}{\phi(n^{-2})}. \tag{3.30}$$

Using (2.11) we get $j(2n) \geqslant 2^{-d-2} j(n)$. When we put this together with (3.29) and (3.30), we get

$$(A^\phi h)(y) \leqslant -c_4 n^{-d}.$$

Define $u(\cdot) := h(\cdot) - \kappa\eta(\cdot)$, where

$$\kappa := \min\left\{\frac{c_4}{2}, \frac{c_5}{2}, \frac{C_4}{2}\right\} n^{-d},$$

where $c_5 > 0$ will be specified later. For $y \in A(an/2,n)$

$$(A^\phi u)(y) = (A^\phi h)(y) - \kappa(A^\phi\eta)(y) \leqslant -c_4 n^{-d} + \kappa \leqslant -c_4 n^{-d} + \frac{c_4}{2} n^{-d} = -\frac{c_4}{2} n^{-d} < 0.$$

Now we want to prove that there exists a constant $c_5 > 0$ such that $G_{B_n}(x,v) \geqslant c_5 n^{-d}\eta(v)$ for all $x \in B_{bn}$, $v \in B_{an/2}$ and for n large enough. For $x \in B_{bn} \subseteq B_{an/2}$ and $v \in B_{an/2}$ we have

$|x-v| \leqslant an \leqslant n$. We first assume that $x \neq v$. Combining Theorem 3.7, Lemma 2.7, (2.13) and (3.26), we get

$$G_{B_n}(x,v) \geqslant C_1 G(x,v) \asymp g(|x-v|) \geqslant \frac{1}{c^*} g(an) \geqslant \frac{1}{(c^*)^2 a^{d-2\alpha^*}} g(n) \geqslant \frac{1}{(c^*)^2 C_3 a^{d-2\alpha^*}} n^{-d} \eta(v).$$

Thus, $G_{B_n}(x,v) \geqslant c_5 n^{-d}\eta(v)$ for some constant $c_5 > 0$ and for $x \neq v$. For the case $x = v$ we can use the same arguments that we used when we were proving that $G_{B_n}(x,x) \geqslant 2C_4 n^{-d}\eta(x)$ for n large enough. Hence, $G_{B_n}(x,v) \geqslant c_5 n^{-d}\eta(v)$ for all $x \in B_{bn}$, $v \in B_{an/2}$ and for n large enough. Now we have

$$h(v) = G_{B_n}(x,v) \wedge \left(C_4 n^{-d}\eta(v)\right) \geqslant \left(c_5 n^{-d}\eta(v)\right) \wedge \left(C_4 n^{-d}\eta(v)\right) = (C_4 \wedge c_5) n^{-d}\eta(v).$$

Hence,

$$u(v) = h(v) - \kappa\eta(v) \geqslant (C_4 \wedge c_5) n^{-d}\eta(v) - \left(\frac{C_4}{2} \wedge \frac{c_5}{2}\right) n^{-d}\eta(v) \geqslant 0.$$

Since $u(v) \geqslant 0$ for $v \in B_{an/2}$, $u(v) = 0$ for $v \in B_n^c$ and $(Au)(v) < 0$ for $v \in A(an/2,n)$ we can use the same argument as in Proposition 3.15 to conclude by Proposition 2.13 that $u(y) \geqslant 0$ for all $y \in \mathbb{Z}^d$. Since $G_{B_n}(x,y) \leqslant C_4 n^{-d}\eta(y)$ for $x \in B_{an/4}, y \in A(an/2,n)$ we have $h(y) = G_{B_n}(x,y)$ for $x \in B_{bn}$ and $y \in A(an/2,n)$. Using that, we have

$$G_{B_n}(x,y) \geqslant \kappa\eta(y) = C_5 n^{-d}\eta(y), \quad x \in B_{bn}, y \in A(an/2,n),$$

for n large enough. As before, we can change the constant and get (3.23) for all $n \in \mathbb{N}$. ∎

Proof of Theorem 3.10. The result follows directly from Proposition 3.15 and Proposition 3.16. We set $b_2 = a/2$ where $a \in (0,1/3)$ is as in Lemma 3.12 and $b_1 = b$ where $b \leqslant a/4$ is as in Proposition 3.16. ∎

3.4. Proof of the elliptic Harnack inequality

At the end of this section we finally prove Theorem 3.1.

Proposition 3.17. Let $f : \mathbb{Z}^d \times \mathbb{Z}^d \to [0,\infty)$ be a function and $B \subset \mathbb{Z}^d$ a finite set. For every $x \in B$ we have

$$\mathbb{E}^x[f(S^\phi_{\tau_B - 1}, S^\phi_{\tau_B})] = \sum_{y \in B} G_B(x,y)\mathbb{E}[f(y, y+S^\phi_1)\mathbb{1}_{\{y+S^\phi_1 \notin B\}}]. \tag{3.31}$$

Proof.

$$\mathbb{E}^x\left[f(S^{\phi}_{\tau_B-1},S^{\phi}_{\tau_B})\right]=\sum_{y\in B,z\in B^c}\mathbb{P}^x(S^{\phi}_{\tau_B-1}=y,S^{\phi}_{\tau_B}=z)f(y,z).$$

Using (1.3), we get

$$\begin{aligned}\mathbb{P}^x(S^{\phi}_{\tau_B-1}=y,S^{\phi}_{\tau_B}=z)&=\sum_{m=1}^{\infty}\mathbb{P}^x(S^{\phi}_{\tau_B-1}=y,S^{\phi}_{\tau_B}=z,\tau_B=m)\\&=\sum_{m=1}^{\infty}\mathbb{P}^x(S^{\phi}_{m-1}+\xi_m=z,S^{\phi}_{m-1}=y,S^{\phi}_1,\dots,S^{\phi}_{m-2}\in B)\\&=\sum_{m=1}^{\infty}\mathbb{P}(\xi_m=z-y)\mathbb{P}^x(S^{\phi}_{m-1}=y,S^{\phi}_1,\dots,S^{\phi}_{m-2}\in B)\\&=\mathbb{P}(\xi_1=z-y)\sum_{m=1}^{\infty}\mathbb{P}^x(S^{\phi}_{m-1}=y,S^{\phi}_1,\dots,S^{\phi}_{m-2}\in B)\\&=\mathbb{P}(S^{\phi}_1=z-y)\sum_{m=1}^{\infty}\mathbb{P}^x(S^{\phi}_{m-1}=y,\tau_B>m-1)=\mathbb{P}(S^{\phi}_1=z-y)G_B(x,y).\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}^x[f(S^{\phi}_{\tau_B-1},S^{\phi}_{\tau_B})]&=\sum_{y\in B,z\in B^c}f(y,z)G_B(x,y)\mathbb{P}(y+S^{\phi}_1=z)\\&=\sum_{y\in B}G_B(x,y)\mathbb{E}[f(y,y+S^{\phi}_1)\mathbb{1}_{\{y+S^{\phi}_1\notin B\}}].\end{aligned}$$

■

Remark 3.18. Formula (3.31) can be considered as a discrete counterpart of the continuous-time Ikeda-Watanabe formula. We will refer to it as discrete Ikeda-Watanabe formula.

We now introduce the Poisson kernel of a finite set $B\subseteq\mathbb{Z}^d$.

$$K_B(x,z):=\mathbb{P}^x(S^{\phi}_{\tau_B}=z),\quad x\in B,z\in B^c. \tag{3.32}$$

Using the discrete Ikeda-Watanabe formula for function $f=\mathbb{1}_z$, $z\in B^c$ we get

$$\begin{aligned}\mathbb{P}^x(S^{\phi}_{\tau_B}=z)=\mathbb{E}^x[\mathbb{1}_z(S^{\phi}_{\tau_B})]&=\sum_{y\in B}G_B(x,y)\mathbb{E}[\mathbb{1}_z(y+S^{\phi}_1)\mathbb{1}_{\{y+S^{\phi}_1\notin B\}}]\\&=\sum_{y\in B}G_B(x,y)\mathbb{P}(S^{\phi}_1=z-y). \end{aligned}\tag{3.33}$$

If the function f is non-negative and harmonic in B_n, with respect to S^{ϕ}, combining Lemma

2.12 and (3.33), we obtain

$$\begin{aligned} f(x) = \mathbb{E}^x[f(X_{\tau_{B_n}})] &= \sum_{y\in B_n} G_{B_n}(x,y)\mathbb{E}[f(y+S_1^\phi)\mathbb{1}_{\{y+S_1^\phi \notin B_n\}}] \\ &= \sum_{y\in B_n} G_{B_n}(x,y) \sum_{z\in B_n^c} \mathbb{E}[f(y+S_1^\phi)\mathbb{1}_{\{y+S_1^\phi\notin B_n\}} \mid S_1^\phi = z-y]\mathbb{P}(S_1^\phi = z-y) \\ &= \sum_{z\in B_n^c}\sum_{y\in B_n} G_{B_n}(x,y)\mathbb{E}[f(y+z-y)\mathbb{1}_{\{y+z-y\notin B_n\}}]\mathbb{P}(S_1^\phi = z-y) \\ &= \sum_{z\in B_n^c} f(z)(\sum_{y\in B_n} G_{B_n}(x,y)\mathbb{P}(S_1^\phi = z-y)) = \sum_{z\in B_n^c} f(z)K_{B_n}(x,z). \end{aligned} \tag{3.34}$$

The idea now is to find sharp estimates for the Poisson kernel $K_{B_n}(x,z)$ that are independent of x and then use those estimates together with formula (3.34) to get the elliptic Harnack inequality.

Lemma 3.19. Let $b_1, b_2 \in (0, \frac{1}{2})$ be as in Theorem 3.10. Then $K_{B_n}(x,z) \asymp l(z)$ for all $x \in B_{b_1 n}$, where

$$l(z) = \frac{j(|z|)}{\phi(n^{-2})} + n^{-d} \sum_{y\in A(b_2 n,n)} \mathbb{E}^y[\tau_{B_n}]j(|z-y|).$$

Proof. Splitting the expression (3.33) for the Poisson kernel in two parts and using Proposition 3.3, we get

$$K_{B_n}(x,z) \asymp \sum_{y\in B_{b_2 n}} G_{B_n}(x,y)j(|z-y|) + \sum_{y\in A(b_2 n,n)} G_{B_n}(x,y)j(|z-y|).$$

Since $G_{B_n}(x,y) \asymp n^{-d}\mathbb{E}^y[\tau_{B_n}]$ for $x \in B_{b_1 n}$, $y \in A(b_2 n, n)$, for the second sum in the upper expression we have

$$\sum_{y\in A(b_2 n,n)} G_{B_n}(x,y)j(|z-y|) \asymp n^{-d} \sum_{y\in A(b_2 n,n)} \mathbb{E}^y[\tau_{B_n}]j(|z-y|). \tag{3.35}$$

Now we look closely at the expression $\sum_{y\in B_{b_2 n}} G_{B_n}(x,y)j(|z-y|)$. Using the fact that $y \in B_{b_2 n}$, $b_2 \in (0, \frac{1}{2})$ and $z \in B_n^c$, we have

$$|z-y| \leqslant |z|+|y| \leqslant |z|+b_2 n \leqslant |z|+b_2|z| \leqslant (1+b_2)|z| \leqslant 2|z|. \tag{3.36}$$

On the other hand

$$|z| \leqslant |z-y|+|y| \leqslant |z-y|+b_2 n \leqslant |z-y|+b_2|z|.$$

Hence,

$$\frac{1}{2}|z| \leqslant (1-b_2)|z| \leqslant |z-y|. \tag{3.37}$$

Combining (3.36), (3.37) and using monotonicity of j, we have

$$j(|z|/2) \geqslant j(|z-y|) \geqslant j(2|z|).$$

Using (2.9), we get $j(|z|/2) \leqslant 2^{d+2} j(|z|)$. Similarly, from (2.11), we get $j(2|z|) \geqslant 2^{-d-2} j(|z|)$. Hence,

$$2^{-d-2} j(|z|) \leqslant j(2|z|) \leqslant j(|z-y|) \leqslant j(|z|/2) \leqslant 2^{d+2} j(|z|). \tag{3.38}$$

This gives us

$$\sum_{y \in B_{b_2 n}} G_{B_n}(x,y) j(|z-y|) \asymp \sum_{y \in B_{b_2 n}} G_{B_n}(x,y) j(|z|) = j(|z|) \sum_{y \in B_{b_2 n}} G_{B_n}(x,y).$$

Now we want to show that $\sum_{y \in B_{b_2 n}} G_{B_n}(x,y) \asymp 1/\phi(n^{-2})$. Using the fact that G_{B_n} is a non-negative function and $\mathbb{E}^x[\tau_{B_n}] \leqslant C_3/\phi\left(n^{-2}\right)$ for $x \in B_n$ we have

$$\sum_{y \in B_{b_2 n}} G_{B_n}(x,y) \leqslant \sum_{y \in B_n} G_{B_n}(x,y) = \mathbb{E}^x[\tau_{B_n}] \leqslant \frac{C_3}{\phi\left(n^{-2}\right)}. \tag{3.39}$$

To prove the other inequality we use Lemma 3.12, Theorem 3.7, Lemma 2.7, (2.17) and (2.2).

$$\begin{aligned}
\sum_{y \in B_{b_2 n}} G_{B_n}(x,y) &\geqslant C_1 \sum_{y \in B_{b_2 n} \setminus \{x\}} G(x,y) \geqslant C_1 c_1 \sum_{y \in B_{b_2 n} \setminus \{x\}} g(|x-y|) \\
&\geqslant C_1 c_1 (c^*)^{-1} \sum_{y \in B_{b_2 n} \setminus \{x\}} g(2 b_2 n) = C_1 c_1 (c^*)^{-1} g(2 b_2 n)(|B_{b_2 n}| - 1) \\
&\geqslant \frac{C_1 c_1 c'}{2c^*} \frac{1}{(2 b_2 n)^d \phi((2 b_2 n)^{-2})} (b_2 n)^d = \frac{C_1 c_1 c'}{2^{d+1} c^*} \frac{1}{\phi(n^{-2})} \frac{\phi(n^{-2})}{\phi((2 b_2 n)^{-2})} \\
&\geqslant \frac{C_1 c_1 c' (2 b_2)^2}{2^{d+1} c^*} \frac{1}{\phi(n^{-2})} = \frac{c_2}{\phi(n^{-2})}.
\end{aligned}$$

Together with (3.39) this gives us

$$\sum_{y \in B_{b_2 n}} G_{B_n}(x,y) \asymp \frac{1}{\phi(n^{-2})}. \tag{3.40}$$

Finally, using (3.38) and (3.40) we have

$$\sum_{y \in B_{b_2 n}} G_{B_n}(x,y) j(|z-y|) \asymp \frac{j(|z|)}{\phi(n^{-2})}. \tag{3.41}$$

And now, from (3.41) and (3.35) we have the statement of the lemma. ■

Proof of Theorem 3.1. Notice that, because of the spatial homogeneity, it is enough to prove this result for balls centered at the origin. We first prove the theorem for $a = b_1$, where b_1 is as in Theorem 3.10. General case follows using the standard Harnack chain argument. Let $x_1, x_2 \in B_{b_1 n}$. Using Lemma 3.19 we get

$$K_{B_n}(x_1, z) \leqslant c_1 l(z) \leqslant c_2 K_{B_n}(x_2, z).$$

Now we multiply both sides with $f(z) \geqslant 0$ and sum over all $z \in B_n^c$ and then use (3.34) to get

$$f(x_1) = \sum_{z \in B_n} f(z) K_{B_n}(x_1, z) \leqslant c_2 \sum_{z \in B_n} f(z) K_{B_n}(x_2, z) = c_2 f(x_2). \tag{3.42}$$

The result is obviously true for all $a \leqslant b_1$. If we take any $a < 1$ and $x_1, x_2 \in B_{an}$, we can find chain of $k = k(a)$ balls of radius $b_1 n$ with nonempty intersections and apply (3.42) k times to obtain

$$f(x_1) \leqslant c_2^k f(x_2), \quad x_1, x_2 \in B_{an}.$$

■

4. On-diagonal bounds

In this section we establish the on-diagonal bounds for the n-step transition probabilities of the subordinate random walk S^{ϕ}. We apply a Fourier analytic method which is extracted from [8].

Theorem 4.1. For all $n \in \mathbb{N}$ it holds

$$p^{\phi}(n,0) \asymp \left(\phi^{-1}(n^{-1})\right)^{d/2}. \tag{4.1}$$

Proof. Let Ψ be the characteristic function of the simple random walk S. We already proved in (2.5) that the characteristic function of S^{ϕ} is $\Psi^{\phi}(\theta) = 1 - \phi(1 - \Psi(\theta))$. Thus, by the Fourier inversion formula,

$$p^{\phi}(n,0) = \frac{1}{(2\pi)^d} \int_{\mathscr{D}_d} (1 - \phi(1 - \Psi(\theta)))^n d\theta, \tag{4.2}$$

where $\mathscr{D}_d = [-\pi, \pi)^d$. We fix $\varepsilon > 0$ and first we estimate the integral in (4.2) over the set $\mathscr{D}_d^{\varepsilon} := \{\theta \in \mathscr{D}_d : |\theta| \geqslant \varepsilon\}$. Since $|1 - \phi(1 - \Psi(\theta))| = 1$ if and only if $\theta \in 2\pi\mathbb{Z}^d$, see [8, Claim 2], it holds that $|1 - \phi(1 - \Psi(\theta))| < 1 - \eta$ for all $\theta \in \mathscr{D}_d^{\varepsilon}$ and for some $\eta \in (0,1)$. Hence

$$\frac{1}{(2\pi)^d} \int_{\mathscr{D}_d^{\varepsilon}} |1 - \phi(1 - \Psi(\theta))|^n d\theta \leqslant (1 - \eta)^n.$$

Next, we consider the remaining part of the integral in (4.2), which is the integral over the ball B_{ε}. We set $a_n = \left(\phi^{-1}(n^{-1})\right)^{1/2}$ and by the change of variable we get

$$a_n^{-d} \int_{|\theta| < \varepsilon} (1 - \phi(1 - \Psi(\theta)))^n \, d\theta = \int_{|\xi| < \varepsilon / a_n} (1 - \phi(1 - \Psi(a_n \xi)))^n \, d\xi.$$

To finish the proof we need to show that for some $c_1, c_2 > 0$

$$c_1 \leqslant \int_{|\xi| < \varepsilon / a_n} (1 - \phi(1 - \Psi(a_n \xi)))^n \, d\xi \leqslant c_2. \tag{4.3}$$

Notice that it suffices to prove (4.3) only for n large enough, as the integrand in (4.3) is strictly positive if ε is small enough, and thus in the end of the proof we can change constants appropriately to estimate the expression in (4.2) for all n.

Claim 1.

$$\lim_{n\to\infty} \frac{1-\Psi(a_n\xi)}{|a_n\xi|^2/d} = \frac{1}{2}.$$

Proof of Claim 1. From [18, Section 1.2, page 13] we have

$$\Psi(\theta) = \frac{1}{d}\sum_{m=1}^{d} \cos(\theta_m), \quad \theta = (\theta_1, \theta_2, \dots, \theta_d).$$

Using the Taylor expansion of the cosine function, we get that there exists a constant $c_3 > 0$ such that for every $x \in \mathbb{R}$ we have

$$|1-\cos(x) - x^2/2| \leqslant c_3 x^4.$$

For any $\theta \in \mathscr{D}_d$ we have

$$\begin{aligned}
\left|1-\Psi(\theta) - \frac{1}{2d}|\theta|^2\right| &= \left|1 - \frac{1}{d}\sum_{m=1}^{d}\cos(\theta_m) - \frac{1}{2d}\sum_{m=1}^{d}\theta_m^2\right| \\
&= \left|\frac{1}{d}\sum_{m=1}^{d}\left(1-\cos(\theta_m) - \frac{\theta_m^2}{2}\right)\right| \\
&\leqslant \frac{1}{d}\sum_{m=1}^{d}\left|1-\cos\theta_m - \frac{\theta_m^2}{2}\right| \leqslant \frac{c_3}{d}\sum_{m=1}^{d}\theta_m^4 \\
&\leqslant \frac{c_3}{d}|\theta|^2\sum_{m=1}^{d}\theta_m^2 = \frac{c_3}{d}|\theta|^4 \leqslant \frac{c_4}{d}|\theta|^3, \qquad (4.4)
\end{aligned}$$

where in the last inequality we used that $|\theta|$ is less than some constant for all $\theta \in \mathscr{D}_d$. Using this we get

$$\begin{aligned}
\left|(1-\Psi(a_n\xi)) - \frac{1}{2d}|a_n\xi|^2\right| \leqslant \frac{c_4}{d}|a_n\xi|^3 &\Rightarrow \left|\frac{1-\Psi(a_n\xi)}{|a_n\xi|^2/d} - \frac{1}{2}\right| \leqslant c_4|a_n\xi| \\
&\Rightarrow \lim_{n\to\infty}\left|\frac{1-\Psi(a_n\xi)}{|a_n\xi|^2/d} - \frac{1}{2}\right| \leqslant \lim_{n\to\infty} c_4|a_n\xi| = 0 \\
&\Rightarrow \lim_{n\to\infty}\frac{1-\Psi(a_n\xi)}{|a_n\xi|^2/d} = \frac{1}{2}.
\end{aligned}$$

We next prove that for some $c_5, c_6 > 0$ and for all $n \in \mathbb{N}$

$$c_5\left(|\xi|^{2\alpha_*} \wedge |\xi|^{2\alpha^*}\right) \leqslant n\phi(1-\Psi(a_n\xi)) \leqslant c_6\left(|\xi|^{2\alpha_*} \vee |\xi|^{2\alpha^*}\right). \qquad (4.5)$$

For that we establish the following simple result.

Claim 2. Let (a_n) and (b_n) be two sequences of positive numbers both tending to zero and such that $\lim_{n\to\infty}(a_n/b_n) = 1$. Then there exists a constant $c_7 > 0$ such that

$$c_7^{-1} \leqslant \frac{\phi(a_n)}{\phi(b_n)} \leqslant c_7, \quad n \in \mathbb{N}. \qquad (4.6)$$

Proof of Claim 2. Scaling condition (2.3) implies that, for some $c_8 > 0$,

$$c_8^{-1}\left((x/y)^{\alpha_*} \wedge (x/y)^{\alpha^*}\right) \leqslant \frac{\phi(x)}{\phi(y)} \leqslant c_8\left((x/y)^{\alpha_*} \vee (x/y)^{\alpha^*}\right), \quad x,y \in (0,1).$$

With this inequality it is straightforward to obtain (4.6).

By Claim 1 and Claim 2,

$$c_9^{-1} \leqslant \frac{\phi\left(1-\Psi(a_n\xi)\right)}{\phi\left(|a_n\xi|^2/2d\right)} \leqslant c_9$$

and whence

$$n\phi(1-\Psi(a_n\xi)) = \frac{\phi\left(1-\Psi(a_n\xi)\right)}{\phi\left(|a_n\xi|^2/2d\right)} \frac{\phi\left(|a_n\xi|^2/2d\right)}{n^{-1}} \asymp \frac{\phi\left(a_n^2|\xi|^2/2d\right)}{\phi(a_n^2)}. \tag{4.7}$$

We have $|a_n\xi| < \varepsilon < 1$ so $|a_n\xi|^2/2d \leqslant 1$. This is why we are able to use (2.3) to bound the expression on the right hand side of (4.7) from below and above. First we consider the case $|\xi|^2/2d \geqslant 1$. In this case we have

$$0 < a_n^2 \leqslant \frac{a_n^2|\xi|^2}{2d} \leqslant 1 \Rightarrow c_*\left(\frac{|\xi|^2}{2d}\right)^{\alpha_*} \leqslant \frac{\phi\left(a_n^2|\xi|^2/2d\right)}{\phi(a_n^2)} \leqslant c^*\left(\frac{|\xi|^2}{2d}\right)^{\alpha^*}.$$

Now we consider the case $|\xi|^2/2d \leqslant 1$. Here we have

$$0 < \frac{a_n^2|\xi|^2}{2d} \leqslant a_n^2 \leqslant 1 \Rightarrow \frac{1}{c^*}\left(\frac{|\xi|^2}{2d}\right)^{\alpha^*} \leqslant \frac{\phi\left(a_n^2|\xi|^2/2d\right)}{\phi(a_n^2)} \leqslant \frac{1}{c_*}\left(\frac{|\xi|^2}{2d}\right)^{\alpha_*}.$$

Hence,

$$c_*\left(\frac{|\xi|^2}{2d}\right)^{\alpha_*} \wedge \frac{1}{c^*}\left(\frac{|\xi|^2}{2d}\right)^{\alpha^*} \leqslant \frac{\phi\left(a_n^2|\xi|^2/2d\right)}{\phi(a_n^2)} \leqslant c^*\left(\frac{|\xi|^2}{2d}\right)^{\alpha^*} \vee \frac{1}{c_*}\left(\frac{|\xi|^2}{2d}\right)^{\alpha_*}. \tag{4.8}$$

Using (4.7) and (4.8) we get (4.5).

Next, we notice that

$$\lim_{n\to\infty} \frac{n\log\left(1-\phi(1-\Psi(a_n\xi))\right)}{-n\phi\left(1-\Psi(a_n\xi)\right)} = 1.$$

Thus, by (4.5), for n large enough,

$$\int_{|\xi|<\varepsilon/a_n} e^{-c_{10}\left(|\xi|^{2\alpha_*}\vee|\xi|^{2\alpha^*}\right)} d\xi \leqslant \int_{|\xi|<\varepsilon/a_n} (1-\phi(1-\Psi(a_n\xi)))^n \, d\xi \leqslant \int_{|\xi|<\varepsilon/a_n} e^{-c_{11}\left(|\xi|^{2\alpha_*}\wedge|\xi|^{2\alpha^*}\right)} d\xi.$$

Since both of the side integrals converge to positive constants as n goes to infinity, we conclude that (4.3) is valid for n large enough and the proof is finished. ■

Corollary 4.2. There exists a constant $c > 0$ such that

$$p^{\phi}(n,x,y) \leqslant c\left(\phi^{-1}(n^{-1})\right)^{d/2}, \quad \text{for } n \in \mathbb{N} \text{ and } x,y \in \mathbb{Z}^d.$$

Proof. Let n be even. Combining Cauchy-Schwarz inequality and Theorem 4.1 we get

$$\begin{aligned} p^\phi(n,x,y) &= \sum_{z\in\mathbb{Z}^d} p^\phi(n/2,x,z)p^\phi(n/2,z,y) \leqslant \sqrt{\sum_{z\in\mathbb{Z}^d} p^\phi(n/2,x,z)^2}\sqrt{\sum_{z\in\mathbb{Z}^d} p^\phi(n/2,z,y)^2} \\ &= \sqrt{\sum_{z\in\mathbb{Z}^d} p^\phi(n/2,x,z)p^\phi(n/2,z,x)}\sqrt{\sum_{z\in\mathbb{Z}^d} p^\phi(n/2,y,z)p^\phi(n/2,z,y)} \\ &= \sqrt{p^\phi(n,x,x)}\sqrt{p^\phi(n,y,y)} \leqslant c_1\left(\phi^{-1}(n^{-1})\right)^{d/2}. \end{aligned}$$

For n odd we first use Lemma 3.6 to obtain

$$p^\phi(n+1,x,y) = \sum_{z\in\mathbb{Z}^d} p^\phi(n,x,z)p^\phi(z,y) \geqslant p^\phi(n,x,y)p^\phi(y,y) \geqslant c_2 p^\phi(n,x,y)$$

and now combining this with what we have already proved for n even, we have for n odd

$$p^\phi(n,x,y) \leqslant c_2^{-1}p^\phi(n+1,x,y) \leqslant c_1c_2^{-1}\left(\phi^{-1}((n+1)^{-1})\right)^{d/2} \leqslant c_1c_2^{-1}\left(\phi^{-1}(n^{-1})\right)^{d/2}$$

where in the last inequality we used that ϕ^{-1} is increasing. ■

5. Parabolic Harnack inequality

The main result of this chapter is the parabolic Harnack inequality. In the first section, we find the estimate for the probability of leaving a ball which is then used in the proof of the parabolic Harnack inequality that can be found in the second section.

5.1. Estimate for probability of leaving a ball

In this section we establish the following result:

Theorem 5.1. There exists a constant $\gamma \in (0,1)$ such that for all $r > 0$

$$\mathbb{P}^x\Big(\max_{k \leqslant \lfloor \gamma/\phi(r^{-2}) \rfloor} |S_k^\phi - x| \geqslant r/2\Big) \leqslant 1/4. \tag{5.1}$$

Our approach is based on the application of the concentration inequality from [24], see (5.3), which provides a bound for the maximum of the random walk in terms of the function h which in our case is of the form

$$h(x) = \mathbb{P}(|S_1^\phi| > x) + x^{-2} \int_{|y| \leqslant x} |y|^2 dF(y), \tag{5.2}$$

where F is the distribution of the random variable S_1^ϕ. Before we prove Theorem 5.1, we show that under the scaling condition (2.3) the function h is dominated by the function ϕ.

Lemma 5.2. In the above notation, there exists a constant $c \geqslant 1$ such that

$$h(x) \leqslant c\phi(x^{-2}), \quad x > 0.$$

Proof. First observe that if $x \in (0,1)$ then

$$h(x) = \mathbb{P}(S_1^\phi \neq 0) \leqslant 1 \leqslant \phi(x^{-2}).$$

Assume next that $x \geq 1$. Using Proposition 3.3 and (2.3), we get

$$\begin{aligned}\mathbb{P}(|S_1^\phi| > x) &\leqslant c_1 \sum_{|y|>x} |y|^{-d} \phi(|y|^{-2}) \leqslant \frac{c_1}{c_*} \phi(x^{-2}) \sum_{|y|>x} |y|^{-d} (x/|y|)^{2\alpha_*} \\ &\leqslant c_2 x^{2\alpha_*} \phi(x^{-2}) \int_x^\infty r^{-d-2\alpha_*} r^{d-1}\, dr = c_3 \phi(x^{-2}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}x^{-2} \int_{|y|\leqslant x} |y|^2 dF(y) = x^{-2} \sum_{1\leqslant|y|\leqslant x} |y|^2 \mathbb{P}(S_1^\phi = y) &\leqslant c_4 x^{-2} \sum_{1\leqslant|y|\leqslant x} |y|^{2-d} \phi(|y|^{-2}) \\ &\leqslant c_5 x^{-2} \phi(x^{-2}) \sum_{1\leqslant|y|\leqslant x} |y|^{2-d} (|y|/x)^{-2\alpha^*} \\ &\leqslant c_6 x^{2\alpha^*-2} \phi(x^{-2}) \int_1^x r^{2-d-2\alpha^*} r^{d-1} dr \\ &\leqslant c_6 x^{2\alpha^*-2} \phi(x^{-2}) \int_0^x r^{1-2\alpha^*} dr = c_7 \phi(x^{-2}),\end{aligned}$$

for some constant $c_7 > 0$. Plugging these bounds into (5.2) finishes the proof. ■

Proof of Theorem 5.1. We first consider the case $r < 1$. Since ϕ is increasing and $\phi(1) = 1$, we have $\gamma/\phi(r^{-2}) < 1$, for any $\gamma \in (0,1)$. Therefore

$$\max_{k\leqslant\lfloor\gamma/\phi(r^{-2})\rfloor} |S_k^\phi - x| = |S_0^\phi - x|$$

and thus for any $r < 1$ it holds

$$\mathbb{P}^x \left(\max_{k\leqslant\lfloor\gamma/\phi(r^{-2})\rfloor} |S_k^\phi - x| \geqslant r/2 \right) = 0.$$

Assume that $r \geqslant 1$. Applying the result from [24, Lemma on page 949] we get

$$\mathbb{P}^x \left(\max_{k\leqslant\lfloor\gamma/\phi(r^{-2})\rfloor} |S_k^\phi - x| \geqslant r/2 \right) \leqslant c_1 \lfloor\gamma/\phi(r^{-2})\rfloor h(r/2), \tag{5.3}$$

where c_1 depends only on the dimension d. By Lemma 5.2 and (2.2),

$$\mathbb{P}^x \left(\max_{k\leqslant\lfloor\gamma/\phi(r^{-2})\rfloor} |S_k^\phi - x| \geqslant r/2 \right) \leqslant 4c_1 C \lfloor\gamma/\phi(r^{-2})\rfloor \phi(r^{-2}) \leqslant 4c_1 C\gamma.$$

Choosing $\gamma = \frac{1}{2} \wedge \frac{1}{16c_1C}$ we obtain (5.1) for all $r > 0$. ■

5.2. Parabolic Harnack inequality

In this section we prove the parabolic Harnack inequality which is the main tool that we will use in Chapter 6 to obtain off-diagonal bounds for n-step transition probabilities of subordinate random walk S^ϕ. We follow closely the elegant approach of [5] but we emphasize that for the case that we undertake, it requires numerous adjustments and alterations.

Let $\mathscr{P} = \mathbb{N}_0 \times \mathbb{Z}^d$ and consider the $\mathscr{P}$-valued Markov chain $(V_k, S_k^\phi)_{k\geqslant 0}$, where $V_k = V_0 + k$. We write $\mathbb{P}^{(j,x)}$ for the law of (V_k, S_k^ϕ) when it starts from (j,x) and we set $\mathscr{F}_j = \sigma\{(V_k, S_k^\phi) : k \leqslant j\}$. A bounded function q defined on $\mathscr{P}$ is called *parabolic* on a subset $D \subseteq \mathscr{P}$ if $q(V_{k\wedge\tau_D}, S^\phi_{k\wedge\tau_D})$ is a martingale, where τ_D denotes the exit time of the Markov chain (V_k, S_k^ϕ) from the set D. We now prove the following important observation.

Lemma 5.3. For each $n_0 \in \mathbb{N}$ and $x_0 \in \mathbb{Z}^d$ the function $q(k,x) = p^\phi(n_0 - k, x, x_0)$ is parabolic on the set $\{0,1,2,\dots,n_0\} \times \mathbb{Z}^d$.

Proof. By the Markov property,

$$\begin{aligned}\mathbb{E}[q(V_{k+1}, S^\phi_{k+1}) \mid \mathscr{F}_k] &= \mathbb{E}^{(V_k, S_k^\phi)}[p^\phi(n_0 - V_1, S_1^\phi, x_0)] \\ &= \sum_{x\in\mathbb{Z}^d} p^\phi(1, S_k^\phi, x) p^\phi(n_0 - V_k - 1, x, x_0) = q(V_k, S_k^\phi),\end{aligned}$$

where the last equality follows by the semigroup relation. ■

We introduce the notation

$$Q(k,x,r) = \{k, k+1, \dots, k + \lfloor \gamma/\phi(r^{-2}) \rfloor\} \times B(x,r),$$

where γ is the constant from Theorem 5.1. We fix the following two constants

$$B = 3 \vee (2/c_*)^{1/2\alpha_*}, \qquad b = 3 \vee \big(\lfloor (3/c_*)^{1/\alpha_*} \rfloor + 1\big). \tag{5.4}$$

The main result of this section is the following theorem.

Theorem 5.4. There exists a constant $C_{PH} > 0$ such that for every non-negative, bounded function q on $\mathscr{P}$ which is parabolic on the set $\{0,1,2,\dots,\lfloor \gamma/\phi((\sqrt{b}R)^{-2}) \rfloor\} \times \mathbb{Z}^d$, the following inequality holds

$$\max_{(k,y)\in Q(\lfloor \gamma/\phi(R^{-2})\rfloor, z, R/B)} q(k,y) \leqslant C_{PH} \min_{w\in B(z,R/B)} q(0,w) \tag{5.5}$$

for all $z \in \mathbb{Z}^d$ and for R large enough.

Before we prove this theorem, we need to establish a series of lemmas. Let

$$\tau(k,x,r) := \min\{l \geqslant 0 : (V_l, S_l^\phi) \notin Q(k,x,r)\}$$

and put $\tau(x,r) = \tau(0,x,r)$. We observe that $\tau(k,x,r) \leqslant \lfloor \gamma/\phi(r^{-2}) \rfloor + 1$. For a non-empty set $A \subseteq Q(0,x,r)$, we define

$$A(k) = \{y \in \mathbb{Z}^d : (k,y) \in A\} \subset \mathbb{Z}^d.$$

We now fix a non-empty $A \subseteq Q(0,x,r)$ such that $A(0) = \emptyset$ and we set

$$N(k,x) = \mathbb{P}^{(k,x)}(S_1^\phi \in A(k+1)) \mathbb{1}_{A^c}(k,x).$$

For any $A \subset \mathscr{P}$ we also define

$$T_A = \min\{n \geqslant 0 : (V_n, S_n^\phi) \in A\}, \text{ and } T_\emptyset = \infty.$$

Lemma 5.5. In the above notation, let

$$J_n = \mathbb{1}_A(V_n, S_n^\phi) - \mathbb{1}_A(V_0, S_0^\phi) - \sum_{k=0}^{n-1} N(V_k, S_k^\phi).$$

The process $J_{n \wedge T_A}$ is an $\mathscr{F}$-martingale.

Proof. If $T_A \leqslant k-1$, we have

$$J_{(k+1)\wedge T_A} - J_{k \wedge T_A} = 0.$$

For $T_A = k$ we get

$$J_{(k+1)\wedge T_A} - J_{k \wedge T_A} = N(V_{T_A}, S_{T_A}^\phi) = 0,$$

by the definition of $N(k,x)$. If $T_A > k$ then

$$\begin{aligned}\mathbb{E}[J_{(k+1)\wedge T_A} - J_{k \wedge T_A} \mid \mathscr{F}_k] &= \mathbb{E}[\mathbb{1}_A(V_{k+1}, S_{k+1}^\phi) \mid \mathscr{F}_k] - N(V_k, S_k^\phi) \\ &= \mathbb{P}^{(V_k, S_k^\phi)}(S_1^\phi \in A(V_k + 1)) - N(V_k, S_k^\phi) = 0,\end{aligned}$$

as desired. ∎

Proposition 5.6. There exists a constant $\theta_1 \in (0,1)$ such that

$$\mathbb{P}^{(0,x)}(T_A < \tau(x,r)) \geqslant \theta_1 |A| j(r). \tag{5.6}$$

Proof. We claim that $\lfloor \gamma/\phi(r^{-2}) \rfloor + 1 \leqslant 2\gamma/\phi(r^{-2})$. Indeed, we have $A(0) = \emptyset$ and $A \neq \emptyset$ so it follows that $A(k) \neq \emptyset$, for some $k \geqslant 1$. Thus $\gamma/\phi(r^{-2}) \geqslant 1$, which clearly yields the claim.

We first assume that $\mathbb{P}^{(0,x)}(T_A \leqslant \tau(x,r)) \geqslant 1/4$. Since $A \subseteq Q(0,x,r)$, using (2.17) we get

$$|A|j(r) \leqslant |Q(0,x,r)|j(r) \leqslant c''(\lfloor \gamma/\phi(r^{-2})\rfloor + 1)\phi(r^{-2}) \leqslant 2c''\gamma.$$

Hence

$$\mathbb{P}^{(0,x)}(T_A \leqslant \tau(x,r)) \geqslant \frac{1}{4} = \frac{1}{8c''\gamma}2c''\gamma \geqslant \frac{1}{8c''\gamma}|A|j(r).$$

Assume that $\mathbb{P}^{(0,x)}(T_A \leqslant \tau(x,r)) < 1/4$. Let $M := T_A \wedge \tau(x,r)$. By Lemma 5.5 and the Optional Stopping Theorem, $\mathbb{E}[J_M] = \mathbb{E}[J_0] = 0$. This and the fact that $(0,X_0) \notin A$ imply

$$\mathbb{E}^{(0,x)}[\mathbb{1}_A(M,S^{\phi}_M)] = \mathbb{E}^{(0,x)}\Big[\sum_{k=0}^{M-1} N(k,S^{\phi}_k)\Big].$$

By Proposition 3.3, Lemma 3.6, monotonicity of the function j and (2.11), we get for $(k,w) \in Q(0,x,r) \cap A^c$

$$\begin{aligned}N(k,w) &= \sum_{y \in A(k+1)\setminus\{w\}} p^{\phi}(w,y) + p^{\phi}(w,w)\mathbb{1}_{A(k+1)}(w)\\ &\geq c_1 j(2r)|A(k+1)\setminus\{w\}| + c_2\,\mathbb{1}_{A(k+1)}(w) \geq c_3 j(r)|A(k+1)|.\end{aligned}$$

Observe that if $k < M$ then $(k,S^{\phi}_k) \in Q(0,x,r) \cap A^c$ and if $M \geqslant \lfloor \gamma/\phi(r^{-2})\rfloor$ then $\sum_{k=0}^{M-1}|A(k+1)| = |A|$. Hence, on the set $\{M \geqslant \lfloor \gamma/\phi(r^{-2})\rfloor\}$ we have

$$\sum_{k=0}^{M-1} N(k,S^{\phi}_k) \geqslant \sum_{k=0}^{M-1} c_3|A(k+1)|j(r) = c_3|A|j(r).$$

Since $\mathbb{P}^{(0,x)}(T_A \leqslant \tau(x,r)) = \mathbb{E}^{(0,x)}[\mathbb{1}_A(M,S^{\phi}_M)]$, we get

$$\begin{aligned}\mathbb{P}^{(0,x)}(T_A \leqslant \tau(x,r)) &\geqslant \mathbb{E}^{(0,x)}\Big[\sum_{k=0}^{M-1} N(k,S^{\phi}_k)\mathbb{1}_{\{M \geqslant \lfloor \gamma/\phi(r^{-2})\rfloor\}}\Big]\\ &\geq c_3|A|j(r)\mathbb{P}^{(0,x)}(M \geqslant \lfloor \gamma/\phi(r^{-2})\rfloor)\\ &= c_3|A|j(r)\Big(1 - \mathbb{P}^{(0,x)}(T_A < \tau(x,r), T_A < \lfloor \gamma/\phi(r^{-2})\rfloor)\\ &\qquad - \mathbb{P}^{(0,x)}(\tau(x,r) < T_A, \tau(x,r) < \lfloor \gamma/\phi(r^{-2})\rfloor)\Big)\\ &\geqslant c_3|A|j(r)\Big(1 - \mathbb{P}^{(0,x)}(T_A \leqslant \tau(x,r))\\ &\qquad - \mathbb{P}^{(0,x)}\big(\tau(x,r) \leqslant \lfloor \gamma/\phi(r^{-2})\rfloor\big)\Big).\end{aligned}$$

We notice that if $\tau(x,r) \leqslant \lfloor \gamma/\phi(r^{-2})\rfloor$ then $\max_{k \leqslant \lfloor \gamma/\phi(r^{-2})\rfloor}|S^{\phi}_k - x| \geqslant r/2$. Thus (5.1) implies

$$\mathbb{P}^{(0,x)}\big(\tau(x,r) \leqslant \lfloor \gamma/\phi(r^{-2})\rfloor\big) \leqslant \mathbb{P}^{(0,x)}\big(\max_{k \leqslant \lfloor \gamma/\phi(r^{-2})\rfloor}|S^{\phi}_k - x| \geqslant r/2\big) \leqslant 1/4.$$

We conclude the desired result with $\theta_1 = \frac{1}{2} \wedge \frac{1}{8c''\gamma} \wedge \frac{c_3}{2}$. ■

Lemma 5.7. There exists a constant $\theta_2 > 0$ such that for $(k,x) \in Q(0,z,R/2)$ and for $r > 0$ such that $k \geqslant \lfloor \gamma/\phi(r^{-2}) \rfloor + 1$ we have

$$\mathbb{P}^{(0,x)}\left(T_{U(k,x,r)} < \tau(z,R)\right) \geqslant \theta_2 \frac{j(R)}{j(r)},$$

where $U(k,x,r) = \{k\} \times B(x,r)$.

Proof. Let $Q' = \{k, k-1, \ldots, k - \lfloor \gamma/\phi(r^{-2}) \rfloor\} \times B(x,r/2)$. We want to apply Proposition 5.6 to sets Q' and $Q(0,z,R)$. To be able to do that, we have to show $Q'(0) = \emptyset$ and $Q' \subseteq Q(0,z,R)$. Since $r > 0$ satisfies $k \geqslant \lfloor \gamma/\phi(r^{-2}) \rfloor + 1$ it is clear that $k - \lfloor \gamma/\phi(r^{-2}) \rfloor \geqslant 1$. Hence, $Q'(0) = \emptyset$. Now we just need to check whether $B(x,r/2) \subseteq B(z,R)$. From $(k,x) \in Q(0,z,R/2)$ it follows that $k \leqslant \lfloor \gamma/\phi((R/2)^{-2}) \rfloor$. Therefore

$$\lfloor \gamma/\phi(r^{-2}) \rfloor + 1 \leqslant \lfloor \gamma/\phi((R/2)^{-2}) \rfloor.$$

Since $x \mapsto \gamma/\phi(x^{-2})$ is an increasing function, we have $r \leqslant R/2$. It is now clear that $B(x,r/2) \subseteq B(z,R)$ because $x \in B(z,R/2)$. By Proposition 5.6, we get

$$\begin{aligned}\mathbb{P}^{(0,x)}\left(T_{Q'} < \tau(z,R)\right) &\geqslant \theta_1 |Q'| j(R) \geqslant \theta_1 c' (\lfloor \gamma/\phi(r^{-2}) \rfloor + 1)(r/2)^d j(R) \\ &\geqslant \frac{\theta_1 c'}{2^d} \frac{\gamma}{\phi(r^{-2})} r^d j(R) = c_1 \frac{j(R)}{j(r)}.\end{aligned}$$

The strong Markov property yields

$$\begin{aligned}\mathbb{P}^{(0,x)}\left(T_{U(k,x,r)} < \tau(z,R)\right) &\geqslant \mathbb{P}^{(0,x)}\left(T_{U(k,x,r)} < \tau(z,R), T_{Q'} < \tau(z,R)\right) \\ &= \mathbb{P}^{(T_{Q'}, S^{\phi}_{T_{Q'}})}\left(T_{U(k,x,r)} < \tau(z,R)\right) \mathbb{P}^{(0,x)}\left(T_{Q'} < \tau(z,R)\right). \qquad (5.7)\end{aligned}$$

We are left to bound from below the first term in (5.7). Observe that if the process (V_k, S^{ϕ}_k) starts from the point $(T_{Q'}, S^{\phi}_{T_{Q'}})$ and the S^{ϕ}-coordinate stays in $B(x,r)$ for at least $\lfloor \gamma/\phi(r^{-2}) \rfloor$ steps, then (V_k, S^{ϕ}_k) hits $U(k,x,r)$ before exiting $Q(0,z,R)$. We also notice that the S^{ϕ}-coordinate stays in $B(x,r)$ for at least $\lfloor \gamma/\phi(r^{-2}) \rfloor$ steps if for all $T_{Q'} \leqslant k \leqslant T_{Q'} + \lfloor \gamma/\phi(r^{-2}) \rfloor$ it holds $|S^{\phi}_k - S^{\phi}_{T_{Q'}}| < \frac{r}{2}$. Thus, using Theorem 5.1, we get

$$\mathbb{P}^{(T_{Q'}, S^{\phi}_{T_{Q'}})}\left(T_{U(k,x,r)} < \tau(z,R)\right) \geqslant 3/4$$

and we conclude that

$$\mathbb{P}^{(0,x)}\left(T_{U(k,x,r)} < \tau(z,R)\right) \geqslant \theta_2 \frac{j(R)}{j(r)},$$

where $\theta_2 = \frac{3c_1}{4}$. ∎

Lemma 5.8. Let $H(k,w) \geq 0$ be a function on $\mathscr{P}$ such that $H(k,w)\mathbb{1}_{B(x,2r)}(w) = 0$. There exists a constant $\theta_3 > 0$ which does not depend on x, r and H and such that

$$\mathbb{E}^{(0,x)}[H(V_{\tau(x,r)}, S^{\phi}_{\tau(x,r)})] \leqslant \theta_3 \mathbb{E}^{(0,y)}[H(V_{\tau(x,r)}, S^{\phi}_{\tau(x,r)})], \tag{5.8}$$

for all $y \in B(x,r/2)$.

Proof. It suffices to check the validity of (5.8) for $H = \mathbb{1}_{(k,w)}$ if $y \in B(x,r/2)$, $w \notin B(x,2r)$ and $1 \leqslant k \leqslant \lfloor \gamma/\phi(r^{-2}) \rfloor + 1$. With such a choice we have

$$\begin{aligned}\mathbb{E}^{(0,y)}[\mathbb{1}_{(k,w)}(V_{\tau(x,r)}, S^{\phi}_{\tau(x,r)})] &= \mathbb{E}^{(0,y)}[\mathbb{E}^{(0,y)}[\mathbb{1}_{(k,w)}(V_{\tau(x,r)}, S^{\phi}_{\tau(x,r)}) \mid \mathscr{F}_{k-1}]] \\ &= \mathbb{E}^{(0,y)}[\mathbb{1}_{\{\tau(x,r)>k-1\}} p^{\phi}(S^{\phi}_{k-1}, w)],\end{aligned} \tag{5.9}$$

Since $S^{\phi}_{k-1} \in B(x,r)$, we have $p^{\phi}(S^{\phi}_{k-1}, w) \geqslant \inf_{z \in B(x,r)} p^{\phi}(z,w)$. For $z \in B(x,r)$ and $w \notin B(x,2r)$, $z \neq w$ and whence Proposition 3.3 implies

$$\mathbb{E}^{(0,y)}[\mathbb{1}_{(k,w)}(V_{\tau(x,r)}, S^{\phi}_{\tau(x,r)})] \geqslant c_1 \mathbb{P}^{(0,y)}\left(\tau(x,r) = \lfloor \gamma/\phi(r^{-2}) \rfloor + 1\right) \inf_{z \in B(x,r)} j(|z-w|).$$

If (V_k, S^{ϕ}_k) starts from $(0,y)$ and the S^{ϕ}-coordinate stays in $B(y,r/2)$ for $\lfloor \gamma/\phi(r^{-2}) \rfloor$ steps then at the same time it also stays in $B(x,r)$. Hence

$$\frac{3}{4} \leqslant \mathbb{P}^{(0,y)}\left(\max_{k \leqslant \lfloor \gamma/\phi(r^{-2}) \rfloor} |S^{\phi}_k - y| < \frac{r}{2}\right) \leqslant \mathbb{P}^{(0,y)}(\tau(x,r) = \lfloor \gamma/\phi(r^{-2}) \rfloor + 1).$$

For every $z \in B(x,r)$ we have $|z-w| \leqslant 2|x-w|$. By monotonicity of j and (2.11), we get

$$\inf_{z \in B(x,r)} j(|z-w|) \geqslant j(2|x-w|) \geqslant 2^{-d-2} j(|x-w|).$$

Combining upper relations, we obtain

$$\mathbb{E}^{(0,y)}[\mathbb{1}_{(k,w)}(V_{\tau(x,r)}, S^{\phi}_{\tau(x,r)})] \geqslant c_2 j(|x-w|). \tag{5.10}$$

Notice that (5.9) remains valid if the process starts from $(0,x)$ instead of $(0,y)$. Using similar arguments as in proving (5.10) we get

$$\begin{aligned}\mathbb{E}^{(0,x)}[\mathbb{1}_{(k,w)}(V_{\tau(x,r)}, S^{\phi}_{\tau(x,r)})] &= \mathbb{E}^{(0,x)}[\mathbb{1}_{\{\tau(x,r)>k-1\}} p^{\phi}(S^{\phi}_{k-1}, w)] \\ &\leqslant \mathbb{E}^{(0,x)}[\mathbb{1}_{\{\tau(x,r)>k-1\}} \sup_{z \in B(x,r)} p^{\phi}(z,w)] \\ &\leqslant c_3 \sup_{z \in B(x,r)} j(|z-w|) \leqslant c_3 j(|x-w|/2) \\ &\leqslant c_3 (1/2)^{-d-2} j(|x-w|) = c_4 j(|x-w|).\end{aligned} \tag{5.11}$$

From (5.10) and (5.11) follows the statement of this lemma with $\theta_3 = c_4/c_2$. ■

Lemma 5.9. There exists a constant $R_0 \geqslant B$ such that

$$\lfloor \gamma/\phi(R^{-2}) \rfloor \geqslant \lfloor \gamma/\phi((R/B)^{-2}) \rfloor + 1, \quad R \geqslant R_0,$$

where B is defined at (5.4).

Proof. For every $x \in \mathbb{R}$ we write $\lfloor x \rfloor = x - m(x)$, $m(x) \in [0,1)$. Thus, we look for R_0 such that

$$\frac{\gamma}{\phi(R^{-2})} - \frac{\gamma}{\phi(B^2R^{-2})} \geqslant 1 + m(\gamma/\phi(R^{-2})) - m\big(\gamma/\phi((R/B)^{-2})\big), \quad R \geqslant R_0.$$

Observe that $1 + m(\gamma/\phi(R^{-2})) - m\big(\gamma/\phi((R/B)^{-2})\big) \leqslant 2$. Hence, it is enough to find R_0 large enough and such that

$$\frac{\gamma}{\phi(R^{-2})} - \frac{\gamma}{\phi(B^2R^{-2})} \geqslant 2, \quad R \geqslant R_0.$$

By (2.3), we get

$$\frac{\gamma}{\phi(R^{-2})} - \frac{\gamma}{\phi(B^2R^{-2})} \geqslant \frac{\gamma}{\phi(B^2R^{-2})}\left(c_*B^{2\alpha_*} - 1\right) \geqslant \frac{\gamma}{\phi(B^2R^{-2})} \overset{R\to\infty}{\longrightarrow} \infty. \tag{5.12}$$

Therefore, there exists $R_0 \geqslant B$ such that

$$\frac{\gamma}{\phi(B^2R^{-2})} \geqslant 2, \quad R \geqslant R_0 \tag{5.13}$$

and the proof is finished. ■

We can now prove the parabolic Harnack inequality.

Proof of Theorem 5.4. By multiplying the function q by a constant, we can assume that

$$\min_{w \in B(z,R/B)} q(0,w) = q(0,v) = 1. \tag{5.14}$$

Notice that if $q(0,x) = 0$ for some $x \in B(z,R/B)$ then (5.5) is trivially satisfied, as the parabolicity of q implies that

$$\max_{(k,y) \in Q(\lfloor \gamma/\phi(R^{-2}) \rfloor, z, R/B)} q(k,y) = 0.$$

Let B be the constant defined at (5.4). By Lemma 5.9, there exists a constant $R_0 \geqslant B$ such that

$$\lfloor \gamma/\phi(r^{-2}) \rfloor \geqslant \lfloor \gamma/\phi((r/B)^{-2}) \rfloor + 1, \quad r \geqslant R_0. \tag{5.15}$$

Let us fix $r \geqslant R_0$, $(k,x) \in \mathscr{P}$ and a set $G \subseteq Q(k+1,x,r/B)$ for which it holds

$$\frac{|G|}{|Q(k+1,x,r/B)|} \geqslant \frac{1}{3}.$$

We claim that for such a set G there is a constant $c_1 \in (0,1)$ such that

$$\mathbb{P}^{(k,x)}(T_G < \tau(k,x,r)) \geqslant c_1. \tag{5.16}$$

Indeed, by our choice $G \subseteq Q(k,x,r)$ and $G(k) = \emptyset$. Therefore, Proposition 5.6 and relation (2.2) yield

$$\begin{aligned}
\mathbb{P}^{(k,x)}(T_G < \tau(k,x,r)) &\geqslant \theta_1 |G| j(r) = \theta_1 \frac{|G|}{|Q(k+1,x,r/B)|} |Q(k+1,x,r/B)| j(r) \\
&\geqslant \frac{\theta_1}{3} |\{k+1, k+1+1, \ldots, k+1+\lfloor \gamma/\phi((r/B)^{-2}) \rfloor\}| |B(x,r/B)| j(r) \\
&\geqslant \frac{\theta_1}{3} \frac{\gamma}{\phi((r/B)^{-2})} c' \left(\frac{r}{B}\right)^d r^{-d} \phi(r^{-2}) = \frac{\theta_1 \gamma c'}{3B^d} \frac{\phi(r^{-2})}{\phi(B^2 r^{-2})} \geqslant \frac{\theta_1 \gamma c'}{3B^d} \frac{1}{B^2} = \frac{\theta_1 \gamma c'}{3B^{d+2}} =: c_1,
\end{aligned}$$

where we can achieve that $c_1 < 1$ by decreasing c' in (2.17) if necessary.

Let θ_1, θ_2 and θ_3 be the constants from Proposition 5.6, Lemma 5.7 and Lemma 5.8 respectively. We set

$$\eta = \frac{c_1}{3}, \quad \zeta = \frac{c_1}{3} \wedge \frac{\eta}{\theta_3}, \quad a = 2 \vee \left(\frac{2}{c_*}\right)^{1/\alpha_*}, \tag{5.17}$$

where c_1 is the constant from relation (5.16) and $c_*, \alpha_* \in (0,1)$ are the constants from the scaling condition (2.3).

Claim 3. There exists a constant $c_2 > 0$ such that for all $r, R, K > 0$ which satisfy

$$\frac{r}{R} < 1 \quad \text{and} \quad \frac{r}{R} K^{1/(d+2)} \geqslant c_2, \tag{5.18}$$

the following two inequalities hold

$$\frac{j(2\sqrt{a}R)}{j(r/R_0)} > \frac{1}{\theta_2 \zeta K}, \tag{5.19}$$

$$|Q(0,x,r/B)| j(\sqrt{b}R) > \frac{3}{\theta_1 \zeta K}. \tag{5.20}$$

We prove this claim in the end of the proof of the theorem and the value of the constant c_2 is specified there, see (5.35).

Let us choose $(k_1, x_1) \in Q(\lfloor \gamma/\phi(R^{-2}) \rfloor, z, R)$ such that it holds

$$K_1 = q(k_1, x_1) = \max_{(k,y) \in Q(\lfloor \gamma/\phi(R^{-2}) \rfloor, z, R/B)} q(k,y).$$

We construct a sequence of points (k_i, x_i) such that $K_1 = q(k_1, x_1)$ is bigger than some constant and under this condition the sequence $K_i = q(k_i, x_i)$ is increasing and tends to infinity, *cf.* (5.25). *This will finally contradict the fact that q is bounded. Therefore, we will be able to conclude that K_1 is bounded by some constant and that is precisely what we need to prove because of the assumption* (5.14).

If $c_2 K_1^{-1/(d+2)} \geqslant 1/B$ then relation (5.5) holds with the constant $C_{PH} = (Bc_2)^{d+2}$. That is why it suffices to study the case $c_2 K_1^{-1/(d+2)} < 1/B$. Suppose that we have already defined the points $(k_1,x_1),(k_2,x_2),\dots,(k_i,x_i) \in Q(\lfloor\gamma/\phi(R^{-2})\rfloor,z,R)$. We describe the procedure how to obtain $(k_{i+1},x_{i+1}) \in Q(\lfloor\gamma/\phi(R^{-2})\rfloor,z,R)$. We first define r_i by

$$\frac{r_i}{R} = c_2 K_i^{-1/(d+2)}. \tag{5.21}$$

In what follows, we want to use Lemma 5.7 so we need to show

$$(k_i,x_i) \in Q(0,v,\sqrt{a}R) \quad \text{and} \quad k_i \geqslant 1 + \lfloor\gamma/\phi((r_i/R_0)^{-2})\rfloor \tag{5.22}$$

for v defined in (5.14). To show $(k_i,x_i) \in Q(0,v,\sqrt{a}R)$ we need to prove $k_i \leqslant \lfloor\gamma/\phi((\sqrt{a}R)^{-2})\rfloor$ and $x_i \in B(v,\sqrt{a}R)$. Since $(k_i,x_i) \in Q(\lfloor\gamma/\phi(R^{-2})\rfloor,z,R)$ we have

$$\begin{aligned} k_i &\leqslant \left\lfloor\frac{\gamma}{\phi(R^{-2})}\right\rfloor + \left\lfloor\frac{\gamma}{\phi(R^{-2})}\right\rfloor \leqslant \frac{2\gamma}{\phi(R^{-2})} \leqslant \frac{\gamma}{\phi(R^{-2})} c_* a^{\alpha_*} \\ &\leqslant \frac{\gamma}{\phi(R^{-2})}\frac{\phi(R^{-2})}{\phi(R^{-2}/a)} = \frac{\gamma}{\phi((\sqrt{a}R)^{-2})}. \end{aligned}$$

From this, we clearly have $k_i \leqslant \lfloor\gamma/\phi((\sqrt{a}R)^{-2})\rfloor$. Again using $(k_i,x_i) \in Q(\lfloor\gamma/\phi(R^{-2})\rfloor,z,R)$ we have

$$|x_i - v| \leqslant |x_i - z| + |z - v| \leqslant R + \frac{R}{B} \leqslant R + \frac{R}{3} = \frac{4R}{3} \leqslant \sqrt{a}R,$$

where we used $v \in B(z,R/B)$, $B \geqslant 3$ and $a \geqslant 2$. The inequality $k_i \geqslant 1 + \lfloor\gamma/\phi((r_i/R_0)^{-2})\rfloor$ holds because $(k_i,x_i) \in Q(\lfloor\gamma/\phi(R^{-2})\rfloor,z,R)$ so

$$k_i \geqslant \lfloor\gamma/\phi(R^{-2})\rfloor \geqslant 1 + \lfloor\gamma/\phi((R/B)^{-2})\rfloor \geqslant 1 + \lfloor\gamma/\phi((r_i/R_0)^{-2})\rfloor,$$

where in the second inequality we used Lemma 5.9 and in the third one we used that $r_i/R_0 \leqslant R/B$ and that $x \mapsto \lfloor\gamma/\phi(x^{-2})\rfloor$ is an increasing function. Now, suppose that $q \geqslant \zeta K_i$ on the set $U_i := \{k_i\} \times B(x_i, r_i/R_0)$. Since q is parabolic on $D = \{0,1,2,\dots,\lfloor\gamma/\phi((\sqrt{b}R)^{-2})\rfloor\} \times \mathbb{Z}^d$, we know that $(q(V_{k\wedge\tau_D}, S^{\phi}_{k\wedge\tau_D}))_{k\geqslant 0}$ is a martingale. Thus (5.19) and Lemma 5.7 imply

$$\begin{aligned} 1 = q(0,v) &= \mathbb{E}^{(0,v)}[q(V_{T_{U_i}\wedge\tau(v,2\sqrt{a}R)}, S^{\phi}_{T_{U_i}\wedge\tau(v,2\sqrt{a}R)})] \\ &\geqslant \mathbb{E}^{(0,v)}[q(V_{T_{U_i}\wedge\tau(v,2\sqrt{a}R)}, S^{\phi}_{T_{U_i}\wedge\tau(v,2\sqrt{a}R)})\mathbb{1}_{\{T_{U_i}<\tau(v,2\sqrt{a}R)\}}] \\ &= \mathbb{E}^{(0,v)}[q(V_{T_{U_i}}, S^{\phi}_{T_{U_i}})\mathbb{1}_{\{T_{U_i}<\tau(v,2\sqrt{a}R)\}}] \geqslant \zeta K_i \mathbb{P}^{(0,v)}(T_{U_i} < \tau(v,2\sqrt{a}R)) \\ &\geqslant \zeta K_i \theta_2 \frac{j(2\sqrt{a}R)}{j(r_i/R_0)} > \zeta K_i \theta_2 \frac{1}{\zeta K_i \theta_2} = 1, \end{aligned}$$

and we mention that we could apply Lemma 5.7 because of (5.22). Thus we get a contradiction, so there must exist $y_i \in B(x_i, r_i/R_0)$ such that $q(k_i, y_i) < \zeta K_i$. Observe that

$$q(k_i,y_i) < \zeta K_i \leqslant (c_1/3)K_i < K_i/3$$

and whence $x_i \neq y_i$. This in turn implies

$$r_i \geqslant R_0. \tag{5.23}$$

Suppose next that

$$\mathbb{E}^{(k_i,x_i)}[q(V_{\tau(k_i,x_i,r_i)}, S^{\phi}_{\tau(k_i,x_i,r_i)})\mathbb{1}_{\{S^{\phi}_{\tau(k_i,x_i,r_i)} \notin B(x_i,2r_i)\}}] \geqslant \eta K_i.$$

By Lemma 5.8 we have

$$\begin{aligned}
\zeta K_i > q(k_i,y_i) &= \mathbb{E}^{(k_i,y_i)}[q(V_{\tau(k_i,x_i,r_i)}, S^{\phi}_{\tau(k_i,x_i,r_i)})] \\
&\geqslant \mathbb{E}^{(k_i,y_i)}[q(V_{\tau(k_i,x_i,r_i)}, S^{\phi}_{\tau(k_i,x_i,r_i)})\mathbb{1}_{\{S^{\phi}_{\tau(k_i,x_i,r_i)} \notin B(x_i,2r_i)\}}] \\
&\geqslant \theta_3^{-1}\mathbb{E}^{(k_i,x_i)}[q(V_{\tau(k_i,x_i,r_i)}, S^{\phi}_{\tau(k_i,x_i,r_i)})\mathbb{1}_{\{S^{\phi}_{\tau(k_i,x_i,r_i)} \notin B(x_i,2r_i)\}}] \\
&\geqslant \frac{\eta}{\theta_3}K_i \geqslant \zeta K_i,
\end{aligned}$$

which again gives a contradiction. Therefore

$$\mathbb{E}^{(k_i,x_i)}[q(V_{\tau(k_i,x_i,r_i)}, S^{\phi}_{\tau(k_i,x_i,r_i)})\mathbb{1}_{\{S^{\phi}_{\tau(k_i,x_i,r_i)} \notin B(x_i,2r_i)\}}] < \eta K_i. \tag{5.24}$$

Define the set

$$A_i = \{(j,y) \in Q(k_i+1, x_i, r_i/B) : q(j,y) \geqslant \zeta K_i\}.$$

We want to apply Proposition 5.6 for A_i and $Q(0,v,\sqrt{b}R)$. Clearly, from the definition of the set A_i, we have $A_i \subseteq Q(k_i+1,x_i,r_i/B)$ and $A_i(0) = \emptyset$. We next show $Q(k_i+1,x_i,r_i/B) \subseteq Q(0,v,\sqrt{b}R)$. We prove that $k_i + 1 + \lfloor \gamma/\phi((r_i/B)^{-2})\rfloor \leqslant \lfloor \gamma/\phi((\sqrt{b}R)^{-2})\rfloor$ using $(k_i,x_i) \in Q(\lfloor \gamma/\phi(R^{-2})\rfloor, z, R)$, $r_i \leqslant R$, Lemma 5.9, $R \geqslant R_0$ and lower scaling:

$$\begin{aligned}
k_i + 1 + \left\lfloor \frac{\gamma}{\phi((r_i/B)^{-2})} \right\rfloor &\leqslant \left\lfloor \frac{\gamma}{\phi(R^{-2})} \right\rfloor + \left\lfloor \frac{\gamma}{\phi(R^{-2})} \right\rfloor + 1 + \left\lfloor \frac{\gamma}{\phi((R/B)^{-2})} \right\rfloor \\
&\leqslant \frac{\gamma}{\phi(R^{-2})} + \frac{\gamma}{\phi(R^{-2})} + \frac{\gamma}{\phi(R^{-2})} = \frac{3\gamma}{\phi(R^{-2})} \leqslant \frac{\gamma}{\phi(R^{-2})} c_* b^{\alpha_*} \\
&\leqslant \frac{\gamma}{\phi(R^{-2})} \frac{\phi(R^{-2})}{\phi((\sqrt{b}R)^{-2})} = \frac{\gamma}{\phi((\sqrt{b}R)^{-2})}.
\end{aligned}$$

The second thing we have to show is that $B(x_i, r_i/B) \subseteq B(v, \sqrt{b}R)$. For that, we use $r_i \leqslant R$, $B \geqslant 3$ and $b \geqslant 3$. Let $w \in B(x_i, r_i/B)$.

$$|w-v| \leqslant |w-x_i| + |x_i - z| + |z-v| \leqslant R/B + R + R/B \leqslant R/3 + R + R/3 = 5R/3 \leqslant \sqrt{b}R.$$

Therefore

$$\begin{aligned}
1 = q(0,v) &= \mathbb{E}^{(0,v)}[q(V_{T_{A_i}\wedge\tau(v,\sqrt{b}R)}, X_{T_{A_i}\wedge\tau(v,\sqrt{b}R)})] \\
&\geqslant \mathbb{E}^{(0,v)}[q(V_{T_{A_i}\wedge\tau(v,\sqrt{b}R)}, X_{T_{A_i}\wedge\tau(v,\sqrt{b}R)})\mathbb{1}_{\{T_{A_i}<\tau(v,\sqrt{b}R)\}}] \\
&= \mathbb{E}^{(0,v)}[q(V_{T_{A_i}}, X_{T_{A_i}})\mathbb{1}_{\{T_{A_i}<\tau(v,\sqrt{b}R)\}}] \geqslant \zeta K_i \mathbb{P}^{(0,v)}(T_{A_i} < \tau(v,\sqrt{b}R)) \\
&\geqslant \zeta K_i \theta_1 |A_i| j(\sqrt{b}R) \geqslant \zeta K_i \theta_1 \frac{|A_i|}{|Q(k_i+1, x_i, r_i/B)|} \frac{3}{\zeta K_i \theta_1},
\end{aligned}$$

where we used (5.20) in the last line. We conclude that

$$\frac{|A_i|}{|Q(k_i+1, x_i, r_i/B)|} \leqslant \frac{1}{3}.$$

Define next

$$D_i = Q(k_i+1, x_i, r_i/B) \setminus A_i \quad \text{and} \quad M_i = \max_{Q(k_i+1, x_i, 2r_i)} q.$$

By (5.24) combined with (5.16), we obtain

$$\begin{aligned}
K_i &= \mathbb{E}^{(k_i,x_i)}[q(V_{T_{D_i}}, X_{T_{D_i}})\mathbb{1}_{\{T_{D_i}<\tau(k_i,x_i,r_i)\}}] \\
&\quad + \mathbb{E}^{(k_i,x_i)}[q(V_{\tau(k_i,x_i,r_i)}, X_{\tau(k_i,x_i,r_i)})\mathbb{1}_{\{\tau(k_i,x_i,r_i)<T_{D_i}\}}\mathbb{1}_{\{X_{\tau(k_i,x_i,r_i)}\notin B(x_i,2r_i)\}}] \\
&\quad + \mathbb{E}^{(k_i,x_i)}[q(V_{\tau(k_i,x_i,r_i)}, X_{\tau(k_i,x_i,r_i)})\mathbb{1}_{\{\tau(k_i,x_i,r_i)<T_{D_i}\}}\mathbb{1}_{\{X_{\tau(k_i,x_i,r_i)}\in B(x_i,2r_i)\}}] \\
&\leqslant \zeta K_i + \eta K_i + M_i(1 - \mathbb{P}^{(k_i,x_i)}(T_{D_i} < \tau(k_i,x_i,r_i))) \\
&\leqslant \frac{c_1}{3}K_i + \frac{c_1}{3}K_i + M_i(1-c_1) = \frac{2c_1}{3}K_i + M_i(1-c_1).
\end{aligned}$$

Hence $M_i/K_i \geqslant 1+\rho$, where $\rho = c_1/(3(1-c_1)) > 0$. Finally, the point $(k_{i+1}, x_{i+1}) \in Q(k_i + 1, x_i, 2r_i)$ is chosen such that

$$K_{i+1} = q(k_{i+1}, x_{i+1}) = M_i.$$

This implies

$$K_{i+1} \geqslant (1+\rho)K_i. \tag{5.25}$$

which together with (5.21) gives

$$r_{i+1} \leqslant r_i(1+\rho)^{-1/(d+2)}. \tag{5.26}$$

We finally want to show that if K_1 is chosen to be sufficiently large then the new point (k_{i+1}, x_{i+1}) will lie in $Q(\lfloor \gamma/\phi(R^{-2}) \rfloor, z, R)$. Indeed, $(k_{i+1}, x_{i+1}) \in Q(k_i+1, x_i, 2r_i)$ and $r_i \geqslant R_0 \geqslant B \geqslant 3$. Therefore

$$\begin{aligned} k_{i+1} &\leqslant k_i + 1 + \left\lfloor \frac{\gamma}{\phi((2r_i)^{-2})} \right\rfloor \leqslant k_i + 1 + \frac{\gamma}{\phi(r_i^{-2})} \frac{\phi(r_i^{-2})}{\phi((2r_i)^{-2})} \\ &\leqslant k_i + \frac{1}{\phi(r_i^{-2})} + \frac{4\gamma}{\phi(r_i^{-2})} \leqslant k_i + \frac{5}{\phi(r_i^{-2})} \end{aligned} \tag{5.27}$$

where we used (2.2) and $\gamma < 1$. We also have $|x_{i+1} - x_i| \leqslant 2r_i$ since $(k_{i+1}, x_{i+1}) \in Q(k_i + 1, x_i, 2r_i)$. Iterating (5.26) we get

$$r_{i+1} \leqslant r_i (1+\rho)^{-1/(d+2)} \leqslant r_{i-1}(1+\rho)^{-2/(d+2)} \leqslant \ldots \leqslant r_1 (1+\rho)^{-i/(d+2)}. \tag{5.28}$$

Hence, for every $j \in \{1, 2, \ldots, i+1\}$ we have

$$\begin{aligned} r_j \leqslant r_1 (1+\rho)^{-(j-1)/(d+2)} &\Rightarrow r_j (1+\rho)^{(j-1)/(d+2)} \leqslant r_1 \\ &\Rightarrow r_j^{-2} (1+\rho)^{-2(j-1)/(d+2)} \geqslant r_1^{-2} \\ &\Rightarrow \phi(r_1^{-2}) \leqslant \phi\left(r_j^{-2}(1+\rho)^{-2(j-1)/(d+2)}\right). \end{aligned} \tag{5.29}$$

Notice also that from $r_j \leqslant r_{j-1}(1+\rho)^{-1/(d+2)}$ we have that $r_j \leqslant r_{j-1}$. Therefore, by (5.23), for all $j \in \{1, 2, \ldots, i\}$ we have

$$\begin{aligned} r_j \geqslant r_i \geqslant R_0 \geqslant 3 &\Rightarrow r_j (1+\rho)^{(j-1)/(d+2)} \geqslant r_j \geqslant r_i \geqslant 1 \\ &\Rightarrow r_j^{-2}(1+\rho)^{-2(j-1)/(d+2)} \leqslant r_j^{-2} \leqslant 1 \\ &\Rightarrow \frac{\phi(r_j^{-2})}{\phi\left(r_j^{-2}(1+\rho)^{-2(j-1)/(d+2)}\right)} \geqslant c_* \left((1+\rho)^{\frac{2\alpha_*}{d+2}}\right)^{j-1} \\ &\Rightarrow \frac{\phi\left(r_j^{-2}(1+\rho)^{-2(j-1)/(d+2)}\right)}{\phi(r_j^{-2})} \leqslant c_*^{-1} \left((1+\rho)^{\frac{-2\alpha_*}{d+2}}\right)^{j-1} \end{aligned} \tag{5.30}$$

Using (5.27), (5.29), (5.30) and $(k_1, x_1) \in Q(\lfloor \gamma/\phi(R^{-2}) \rfloor, z, R/B)$ we get

$$\begin{aligned} k_{i+1} &\leqslant k_i + \frac{5}{\phi(r_i^{-2})} \leqslant k_{i-1} + \frac{5}{\phi(r_{i-1}^{-2})} + \frac{5}{\phi(r_i^{-2})} \leqslant \ldots \leqslant k_1 + 5 \sum_{j=1}^{i} \frac{1}{\phi(r_j^{-2})} \\ &= k_1 + \frac{5}{\phi(r_1^{-2})} \sum_{j=1}^{i} \frac{\phi(r_1^{-2})}{\phi(r_j^{-2})} \leqslant k_1 + \frac{5}{\phi(r_1^{-2})} \sum_{j=1}^{i} \frac{\phi\left(r_j^{-2}(1+\rho)^{-2(j-1)/(d+2)}\right)}{\phi(r_j^{-2})} \\ &\leqslant k_1 + \frac{5}{\phi(r_1^{-2})} \sum_{j=1}^{i} c_*^{-1} \left((1+\rho)^{\frac{-2\alpha_*}{d+2}}\right)^{j-1} \leqslant k_1 + \frac{5c_*^{-1}}{\phi(r_1^{-2})} \sum_{j=0}^{\infty} \left((1+\rho)^{\frac{-2\alpha_*}{d+2}}\right)^{j} \\ &\leqslant \left\lfloor \frac{\gamma}{\phi(R^{-2})} \right\rfloor + \left\lfloor \frac{\gamma}{\phi((R/B)^{-2})} \right\rfloor + \frac{5c_*^{-1}}{1-\kappa^{2\alpha_*}} \frac{1}{\phi(r_1^{-2})}, \end{aligned} \tag{5.31}$$

with $\kappa = (1+\rho)^{-1/(d+2)}$. We have similar calculation for $|x_{i+1}-z|$,

$$
\begin{aligned}
|x_{i+1}-z| &\leqslant |x_{i+1}-x_i| + |x_i - x_{i-1}| + \cdots + |x_2 - x_1| + |x_1 - z| \\
&\leqslant 2r_i + 2r_{i-1} + \cdots + 2r_1 + \frac{R}{B} = \frac{R}{B} + 2\sum_{j=1}^{i} r_j \\
&\leqslant \frac{R}{B} + 2\sum_{j=1}^{i} r_1 (1+\rho)^{-(j-1)/(d+2)} \\
&\leqslant \frac{R}{B} + 2r_1 \sum_{j=0}^{\infty} ((1+\rho)^{-1/(d+2)})^j = \frac{R}{B} + \frac{2r_1}{1-\kappa}.
\end{aligned} \tag{5.32}
$$

We next need the following technical result which we prove later.

Claim 4. There is a constant $c_3 > 0$ such that the following two relation hold for all R sufficiently large

$$
\lfloor \gamma/\phi((R/B)^{-2}) \rfloor + \frac{5c_*^{-1}}{1-\kappa^{2\alpha_*}} \frac{1}{\phi((c_3 R)^{-2})} \leqslant \lfloor \gamma/\phi(R^{-2}) \rfloor \tag{5.33}
$$

and

$$
\frac{R}{B} + \frac{2c_3 R}{1-\kappa} < R. \tag{5.34}
$$

At last, let c_3 be the constant as in Claim 4 and suppose that $K_1 \geqslant (c_2/c_3)^{d+2}$. This would mean that $r_1 \leqslant c_3 R$. By (5.31), (5.32) and Claim 4, $(k_{i+1}, x_{i+1}) \in Q(\lfloor \gamma/\phi(R^{-2}) \rfloor, z, R)$. However, by (5.23) $r_i \geqslant 3$ for all i. On the other hand, if we let i tend to infinity in (5.28), we would obtain that r_i approaches zero. This is a contradiction and whence $K_1 \leqslant (c_2/c_3)^{d+2}$, which means that (5.5) holds with $C_{PH} = (c_2/c_3)^{d+2}$ and for all R large enough. To finish the prove we are left to establish Claims 3 and 4.

Proof of Claim 3. We set

$$
c_2 = 2R_0\sqrt{a} \left(\frac{1}{\theta_2 \zeta} \right)^{1/(d+2)} \vee B\sqrt{b} \left(\frac{3}{\theta_1 \zeta \gamma c'} \right)^{1/(d+2)}, \tag{5.35}
$$

where γ is the constant from Theorem 5.1, c' is the constant from (2.17) and b is defined in (5.4). We show that the claim is true with such a constant. We start by showing (5.19). Combining (2.2) and (5.18) we get

$$
\begin{aligned}
\frac{j(2\sqrt{a}R)}{j(r/R_0)} &= (2R_0\sqrt{a})^{-d} \left(\frac{R}{r} \right)^{-d} \frac{\phi((2\sqrt{a}R)^{-2})}{\phi((r/R_0)^{-2})} \geqslant \frac{1}{(2R_0\sqrt{a})^{d+2}} \left(\frac{r}{R} \right)^{d+2} \\
&> \frac{1}{(2R_0\sqrt{a})^{d+2}} \frac{(2R_0\sqrt{a})^{d+2}}{\theta_2 \zeta} K^{-1} = \frac{1}{\theta_2 \zeta K}.
\end{aligned}
$$

Similarly, to prove (5.20) we apply (2.17) and (2.2) and obtain

$$|Q(0,x,r/B)|j(\sqrt{b}R) \geq \frac{\gamma c' b^{-d/2}}{B^d}\left(\frac{r}{R}\right)^d \frac{\phi((\sqrt{b}R)^{-2})}{\phi((r/B)^{-2})}$$
$$\geqslant \frac{\gamma c'}{(B\sqrt{b})^{d+2}} c_2^{d+2} K^{-1} > \frac{\gamma c'}{(B\sqrt{b})^{d+2}} \frac{3(B\sqrt{b})^{d+2}}{\theta_1 \zeta \gamma c'} K^{-1} = \frac{3}{\theta_1 \zeta K}.$$

Proof of Claim 4. Notice that (5.33) is equivalent to

$$\frac{5c_*^{-1}}{1-\kappa^{2\alpha_*}} \frac{1}{\phi((c_3R)^{-2})} \leqslant \lfloor \gamma/\phi(R^{-2}) \rfloor - \lfloor \gamma/\phi((R/B)^{-2}) \rfloor.$$

Using (5.12) and (5.13) we get

$$\lfloor \gamma/\phi(R^{-2}) \rfloor - \lfloor \gamma/\phi((R/B)^{-2}) \rfloor \geqslant \frac{\gamma}{2\phi(B^2R^{-2})}.$$

Hence, it is enough to define c_3 for which

$$\frac{\phi(B^2R^{-2})}{\phi(c_3^{-2}R^{-2})} \leqslant \frac{\gamma c_*(1-\kappa^{2\alpha_*})}{10}. \tag{5.36}$$

This can be achieved by setting

$$c_3 := B^{-1}\left(1 \wedge \left(\gamma c_*^2(1-\kappa^{2\alpha_*})/10\right)^{1/2\alpha_*} \wedge (B-1)(1-\kappa)/3\right).$$

Indeed, with such a choice, for R sufficiently large we apply the scaling condition and get

$$\frac{\phi(B^2R^{-2})}{\phi(c_3^{-2}R^{-2})} \leqslant \frac{1}{c_*}(c_3B)^{2\alpha_*}.$$

Clearly (5.36) follows. With such c_3 the validity of (5.34) is obvious. ■

6. Off-diagonal bounds

In this chapter we establish global estimates for the function $p^{\phi}(n,x,y)$, that is, we prove that for all $x,y \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ it holds

$$p^{\phi}(n,x,y) \asymp \min\left\{\left(\phi^{-1}(n^{-1})\right)^{d/2}, \frac{n\,\phi(|x-y|^{-2})}{|x-y|^d}\right\}.$$

We split the proof in two sections. In Section 6.1 we find the lower bound for the heat kernel of the subordinate random walk S^{ϕ} and in Section 6.2 we find the upper bound for $p^{\phi}(n,x,y)$.

6.1. Lower bound

The aim of this section is to prove the global lower estimate. We use a probabilistic method based on the parabolic Harnack inequality.

Theorem 6.1. Under our assumptions, for some constant $C>0$

$$p^{\phi}(n,x,y) \geqslant C\Big(\left(\phi^{-1}(n^{-1})\right)^{d/2} \wedge \frac{n}{|x-y|^d}\phi(|x-y|^{-2})\Big), \tag{6.1}$$

for all $x,y \in \mathbb{Z}^d$ and for all $n \in \mathbb{N}$.

Proof. Let us set

$$r_n = \frac{1}{\sqrt{\phi^{-1}(n^{-1})}}, \quad n \geq 1.$$

Near-diagonal bound: We start by proving that there exists a constant $C>0$ such that

$$p^{\phi}(n,x,y) \geqslant C\left(\phi^{-1}(n^{-1})\right)^{d/2}, \tag{6.2}$$

for $n \in \mathbb{N}$ and $|x-y| \leqslant d_1 r_n$, where $d_1>0$ is a constant to be specified. We take $n \in \mathbb{N}$ and choose R to satisfy $n = \gamma/\phi(R^{-2})$, where γ is the constant from Theorem 5.1. Let $q(k,w) = p^{\phi}(bn-$

$k,x,w)$, where b is the constant from (5.4). By Lemma 5.3, q is parabolic on $\{0,1,2,\dots,bn\}\times\mathbb{Z}^d$. Since $b\geqslant 1$, using (2.2) we have

$$\frac{\gamma}{\phi((\sqrt{b}R)^{-2})}=\frac{\gamma}{\phi(R^{-2})}\frac{\phi(R^{-2})}{\phi(b^{-1}R^{-2})}\leqslant\frac{\gamma}{\phi(R^{-2})}\frac{1}{b^{-1}}=\frac{b\gamma}{\phi(R^{-2})}\Rightarrow bn\geqslant\left\lfloor\frac{\gamma}{\phi((\sqrt{b}R)^{-2})}\right\rfloor. \quad (6.3)$$

Hence, specially, q is parabolic on $\{0,1,2,\dots,\lfloor\gamma/\phi((\sqrt{b}R)^{-2})\rfloor\}\times\mathbb{Z}^d$. Now we want to find a constant $d_1>0$ such that

$$B(y,d_1r_n)\subseteq B(y,R/B).$$

Using $n=\gamma/\phi(R^{-2})$ we get

$$\phi(R^{-2})=\gamma n^{-1}\Rightarrow R^{-2}=\phi^{-1}(\gamma n^{-1})\leqslant\phi^{-1}(n^{-1})\Rightarrow R\geqslant r_n,$$

where we used monotonicity of the function ϕ and $\gamma<1$. We now choose $d_1=1/B$ which implies that $B(y,d_1r_n)\subseteq B(y,R/B)$ and whence $(n,x)\in Q(\lfloor\gamma/\phi(R^{-2})\rfloor,y,R/B)$. By choosing n big enough we can make R large enough and this allows us to apply Theorem 5.4. Thus, there is $n_0\geq 1$ such that for all $n\geq n_0$,

$$\begin{aligned}\min_{z\in B(y,d_1r_n)}p^{\phi}(bn,x,z)\geqslant\min_{z\in B(y,R/B)}p^{\phi}(bn,x,z)&=\min_{z\in B(y,R/B)}q(0,z)\\&\geqslant C_{PH}^{-1}\max_{(k,z)\in Q(\lfloor\gamma/\phi(R^{-2})\rfloor,y,R/B)}q(k,z)\\&\geqslant C_{PH}^{-1}q(n,x).\end{aligned}$$

Hence, by Theorem 4.1,

$$\begin{aligned}\min_{z\in B(y,d_1r_n)}p^{\phi}(bn,x,z)\geqslant C_{PH}^{-1}q(n,x)&=C_{PH}^{-1}p^{\phi}((b-1)n,x,x)\\&\geqslant C_{PH}^{-1}c_1\left(\phi^{-1}(((b-1)n)^{-1})\right)^{d/2}\\&\geqslant C_{PH}^{-1}c_1\left(\phi^{-1}((bn)^{-1})\right)^{d/2},\end{aligned}$$

for all $x\in\mathbb{Z}^d$ and $n\geq n_0$. Hence, we have proved (6.2) for all integers of the form bn with $n\geq n_0$. For the remaining values of n between bn_0 and $b(n_0+1)$ (and so forth) we use Lemma 3.6 to get

$$\begin{aligned}p^{\phi}(bn+1,x,y)=\sum_{z\in\mathbb{Z}^d}p^{\phi}(bn,x,z)p^{\phi}(z,y)&\geqslant p^{\phi}(bn,x,y)p^{\phi}(y,y)\\&\geqslant c_2\left(\phi^{-1}((bn)^{-1})\right)^{d/2}\geqslant c_2\left(\phi^{-1}((bn+1)^{-1})\right)^{d/2}.\end{aligned}$$

For $n<bn_0$ we apply the above procedure together with Proposition 3.3. For $|x-y|\leqslant d_1r_n$ we have

$$p^{\phi}(x,y)\geqslant c_3j(|x-y|)\geqslant c_3j(d_1r_n)\geqslant c_3j(d_1r_{bn_0})=c_4=c_4(\phi^{-1}(1^{-1}))^{d/2}.$$

Now using Lemma 3.6 together with the Chapman-Kolmogorov equality and monotonicity of the function $n \mapsto (\phi^{-1}(n^{-1}))^{d/2}$ we get

$$p^{\phi}(2,x,y) = \sum_{z\in\mathbb{Z}^d} p^{\phi}(x,z)p^{\phi}(z,y) \geqslant p^{\phi}(x,y)p^{\phi}(y,y) \geqslant c_5(\phi^{-1}(1^{-1}))^{d/2} \geqslant c_5(\phi^{-1}(2^{-1}))^{d/2}.$$

Since we only have to make finite number of steps, this finishes the proof.

Estimate away from the diagonal: Let $j(r)$ be the function defined at (2.8). We now show that there is $C>0$ such that

$$p^{\phi}(n,x,y) \geqslant Cnj(|x-y|), \tag{6.4}$$

for all $n \in \mathbb{N}$ and $|x-y| \geqslant d_2 r_n$, where a constant $d_2 > 0$ will be specified later. We first claim that there is a constant $c_3 > 0$ such that for all $x \in \mathbb{Z}^d$ and for all $k,n \in \mathbb{N}$

$$\mathbb{P}^x\Big(\max_{j\leqslant k}|S_j^{\phi}-x| \geqslant c_3 r_n\Big) \leqslant \frac{1}{2}\frac{k}{n}. \tag{6.5}$$

By Lemma 5.2 we get

$$\mathbb{P}^x(\max_{j\leqslant k}|S_j^{\phi}-x| \geqslant c_3 r_n) \leqslant c_4 k\phi(c_3^{-2}r_n^{-2}).$$

This is true for all constants $c_3 > 0$. We define a specific constant c_3 as

$$c_3 = 1 \vee (2c_4/c_*)^{1/2\alpha_*}.$$

Since $c_3 \geqslant 1$ we can use lower scaling to obtain

$$c_4 k\phi(c_3^{-2}r_n^{-2}) = c_4 k\phi(r_n^{-2})\frac{\phi(c_3^{-2}r_n^{-2})}{\phi(r_n^{-2})} \leqslant \frac{c_4}{c_* c_3^{2\alpha_*}}\frac{k}{n} \leqslant \frac{1}{2}\frac{k}{n}.$$

Last two relations give us (6.5). We now set $d_2 = 3c_3$ and we notice that $d_1 < d_2$, as $d_1 = 1/B \leqslant 1/3$. Let

$$\tau(x,r) = \inf\{k : S_k^{\phi} \notin B(x,r)\}$$

and consider a family of sets

$$A_k = \{\tau(x,c_3 r_n) = k, S_k^{\phi}, S_{k+1}^{\phi}, \dots, S_{n-1}^{\phi} \in B(y,c_3 r_n), S_n^{\phi} = y\}, \tag{6.6}$$

for $k = 1,2,\dots,n$. Observe that

$$p^{\phi}(n,x,y) = \mathbb{P}^x(S_n^{\phi} = y) \geqslant \sum_{k=1}^{n} \mathbb{P}^x(A_k)$$

and our task is to estimate the last sum from below. By the time reversal of the random walk we get

$$\mathbb{P}^x(A_k) = \sum_{\substack{x_{k-1}\in B(x,c_3r_n)\\ x_k\in B(y,c_3r_n)}} \Big(\mathbb{P}^x(\tau(x,c_3r_n)>k-1, S^\phi_{k-1}=x_{k-1})p^\phi(x_{k-1},x_k)$$
$$\times\mathbb{P}^y(\tau(y,c_3r_n)>n-k, S^\phi_{n-k}=x_k)\Big)\,. \tag{6.7}$$

For $x_{k-1}\in B(x,c_3r_n)$, $x_k\in B(y,c_3r_n)$ and $|x-y|\geqslant d_2r_n=3c_3r_n$, we have

$$|x_{k-1}-x_k|\leqslant 2c_3r_n+|x-y|\leqslant 2|x-y|,$$

and whence, for $|x-y|\geqslant d_2r_n$, by using Proposition 3.3, monotonicity of j and (2.11) we get

$$p^\phi(x_{k-1},x_k)\geqslant c_5 j(|x-y|). \tag{6.8}$$

Thus

$$\begin{aligned}\mathbb{P}^x(A_k) &\geqslant c_5 j(|x-y|)\mathbb{P}^x(\tau(x,c_3r_n)>k-1)\mathbb{P}^y(\tau(y,c_3r_n)>n-k)\\ &= c_5 j(|x-y|)\mathbb{P}^x(\max_{j\leqslant k-1}|S^\phi_j-x|<c_3r_n)\mathbb{P}^x(\max_{j\leqslant n-k}|S^\phi_j-x|<c_3r_n).\end{aligned} \tag{6.9}$$

Using (6.5) we get

$$\mathbb{P}^x(A_k)\geqslant c_5\left(1-\frac{1}{2}\frac{k-1}{n}\right)\left(1-\frac{1}{2}\frac{n-k}{n}\right)j(|x-y|)\geqslant\frac{c_5}{4}j(|x-y|)$$

and (6.4) follows for all $n\in\mathbb{N}$ and $|x-y|\geqslant d_2r_n$.

Intermediate estimate: We finally show that

$$p^\phi(n,x,y)\geqslant C\left(\phi^{-1}(n^{-1})\right)^{d/2}, \tag{6.10}$$

for all $n\in\mathbb{N}$ and for $d_1r_n<|x-y|<d_2r_n$. For any $1\leqslant K\leqslant n$ we can write

$$p^\phi(n,x,y)\geqslant\sum_{z\in B(y,d_1r_n/2)}p^\phi(\lfloor n/K\rfloor,x,z)p^\phi(n-\lfloor n/K\rfloor,z,y).$$

We now state the claim which we prove later.

Claim 5. Let us set

$$K=2\vee c^*\left(\frac{2d_2}{d_1}\right)^{2\alpha^*}\vee\left(1-\frac{4^{-\alpha_*}}{c_*}\right)^{-1}. \tag{6.11}$$

Then for all $n\geq K$ the following inequalities hold

$$\frac{d_1r_n}{2}\geqslant d_2r_{\lfloor n/K\rfloor},\quad r_{n-\lfloor n/K\rfloor}\geqslant\frac{r_n}{2}.$$

Thus, if $|x-y| > d_1 r_n$ and $z \in B(y, d_1 r_n/2)$ then

$$|x-z| \geqslant d_2 r_{\lfloor n/K \rfloor} \quad \text{and} \quad |y-z| \leqslant d_1 r_{n-\lfloor n/K \rfloor}.$$

Combining this with (6.2) and (6.4) we get

$$p^{\phi}(n,x,y) \geqslant c_6 \sum_{z \in B(y, d_1 r_n/2)} \lfloor n/K \rfloor j(|x-z|) \left(\phi^{-1}((n-\lfloor n/K \rfloor)^{-1})\right)^{d/2}.$$

Since $|x-y| < d_2 r_n$, for every $z \in B(y, d_1 r_n/2)$ we get $|x-z| \leqslant c_7 r_n$, where $c_7 = d_1/2 + d_2 \geqslant 1$. By (2.11) and (2.2) we get

$$j(|x-z|) \geq c_7^{-d-2} \left(\phi^{-1}(n^{-1})\right)^{d/2} n^{-1}$$

and whence

$$\begin{aligned} p^{\phi}(n,x,y) &\geqslant c_8 \lfloor n/K \rfloor n^{-1} \left(\phi^{-1}(n^{-1})\right)^{d/2} \left(\phi^{-1}\left((n-\lfloor n/K \rfloor)^{-1}\right)\right)^{d/2} |B(y, d_1 r_n/2)| \\ &\geq c_9 \lfloor n/K \rfloor n^{-1} \left(\frac{\phi^{-1}\left((n-\lfloor n/K \rfloor)^{-1}\right)}{\phi^{-1}(n^{-1})}\right)^{d/2} \left(\phi^{-1}(n^{-1})\right)^{d/2}. \end{aligned} \tag{6.12}$$

Since $n/K \geqslant 1$ we have $\lfloor n/K \rfloor \geqslant n/(2K)$. Hence $\lfloor n/K \rfloor n^{-1} \geqslant \frac{1}{2K}$ and, by (2.4),

$$\begin{aligned} \frac{\phi^{-1}\left((n-\lfloor n/K \rfloor)^{-1}\right)}{\phi^{-1}(n^{-1})} &\geqslant \left(\frac{1}{c^*}\right)^{1/\alpha^*} \left(\frac{n-\lfloor n/K \rfloor}{n}\right)^{-1/\alpha^*} \\ &\geqslant \left(\frac{1}{c^* - c^*/(2K)}\right)^{1/\alpha^*}. \end{aligned}$$

Combining these two bounds with (6.12) we obtain (6.10) for all $n \geqslant K$ and for $d_1 r_n < |x-y| < d_2 r_n$. For $n < K$ we proceed as in the end of the proof of the near-diagonal bound.

Proof of Claim 5. Since $r_{n/K} \geqslant r_{\lfloor n/K \rfloor}$, it is enough to find K such that

$$\frac{d_1}{2} r_n \geqslant d_2 r_{n/K} \iff \frac{\phi^{-1}((n/K)^{-1})}{\phi^{-1}(n^{-1})} \geqslant \left(\frac{2d_2}{d_1}\right)^2.$$

By (2.4), for $n \geqslant K$,

$$\frac{\phi^{-1}((n/K)^{-1})}{\phi^{-1}(n^{-1})} \geqslant \left(\frac{1}{c^*}\right)^{1/\alpha^*} \left(\frac{(n/K)^{-1}}{n^{-1}}\right)^{1/\alpha^*} = \left(\frac{K}{c^*}\right)^{1/\alpha^*},$$

and whence K has to satisfy $K \geqslant c^* \left(\frac{2d_2}{d_1}\right)^{2\alpha^*}$. Similarly, as $r_{n-\lfloor n/K \rfloor} \geqslant r_{n-n/K}$, it is enough to have K such that

$$r_{n-n/K} \geqslant \frac{1}{2} r_n \iff \frac{\phi^{-1}((n-n/K)^{-1})}{\phi^{-1}(n^{-1})} \leqslant 4.$$

We assume that $K \geqslant 2$ and thus (2.4) implies

$$\frac{\phi^{-1}((n-n/K)^{-1})}{\phi^{-1}(n^{-1})} \leqslant \left(\frac{1}{c_*}\right)^{1/\alpha_*}\left(\frac{(n-n/K)^{-1}}{n^{-1}}\right)^{1/\alpha_*} = c_*^{-1/\alpha_*}(1-1/K)^{-1/\alpha_*}.$$

We conclude that K has to be such that $K \geqslant \left(1-\frac{4^{-\alpha_*}}{c_*}\right)^{-1}$.

Finally, combining inequalities (6.2), (6.4) and (6.10) we obtain (6.1) and the proof is finished.

■

6.2. UPPER BOUND

In this final section we aim at proving the global upper estimates for the transition probabilities of the random walk S_n^ϕ. Our strategy is to study the continuous time random walk and to estimate its transition kernel and hitting time of a ball, and then to use these results to get similar identities in the discrete time.

6.2.1. Estimates for the continuous time random walk

We study the continuous time version of the random walk S_n^ϕ which is constructed in the standard way. We take $(U_i)_{i\in\mathbb{N}}$ to be a sequence of independent, identically distributed exponential random variables with parameter 1 which are independent of S^ϕ. Let $T_0 = 0$ and $T_k = \sum_{i=1}^k U_i$. Then we define $Y_t = S_n^\phi$ if $T_n \leqslant t < T_{n+1}$. Equivalently, we can take $(N_t)_{t\geqslant 0}$ to be a homogeneous Poisson process with intensity 1 independent of the random walk S^ϕ and then $Y_t = S_{N_t}^\phi$. The transition probability of the process Y is denoted by $q(t,x,y) = \mathbb{P}^x(Y_t = y)$. We want to find the upper bound for $q(t,x,y)$. The main result of this subsection is formulated in the following proposition:

Proposition 6.2. There is a constant $C_1 > 0$ such that

$$q(t,x,y) \leqslant C_1\Big(\left(\phi^{-1}(t^{-1})\right)^{d/2} \wedge \frac{t}{|x-y|^d}\phi(|x-y|^{-2})\Big), \tag{6.13}$$

for all $x,y \in \mathbb{Z}^d$ and for all $t \geqslant 1$.

The proof will be given at the end of this subsection. We first handle the on-diagonal part.

Lemma 6.3. There exists a constant $C_2 > 0$ such that for all $t > 0$ and all $x,y \in \mathbb{Z}^d$

$$q(t,x,y) \leqslant C_2\left(\phi^{-1}(t^{-1})\right)^{d/2}. \tag{6.14}$$

Proof. By the independence and Theorem 4.1 we get

$$
\begin{aligned}
q(t,x,x) &= \mathbb{P}^x(S^{\phi}_{N_t}=x) = \sum_{k=0}^{\infty} \mathbb{P}^x(S^{\phi}_k = x)\mathbb{P}^x(N_t=k) = \sum_{k=0}^{\infty} \frac{t^k e^{-t}}{k!}\mathbb{P}^x(S^{\phi}_k=x) \\
&\leqslant e^{-t} + c_1 e^{-t} \sum_{k=1}^{\infty} \frac{t^k}{k!} \left(\phi^{-1}(k^{-1})\right)^{d/2} \\
&= e^{-t} + c_1 e^{-t}\left(\phi^{-1}(t^{-1})\right)^{d/2} \Big(\sum_{k>t} + \sum_{1\leqslant k\leqslant t}\Big) \frac{t^k}{k!} \frac{\left(\phi^{-1}(k^{-1})\right)^{d/2}}{\left(\phi^{-1}(t^{-1})\right)^{d/2}} \\
&= e^{-t} + c_1 e^{-t}\left(\phi^{-1}(t^{-1})\right)^{d/2}(\Sigma_1+\Sigma_2). \qquad (6.15)
\end{aligned}
$$

Since ϕ^{-1} is increasing, we obtain

$$
\Sigma_1 = \sum_{k>t} \frac{t^k}{k!} \frac{\left(\phi^{-1}(k^{-1})\right)^{d/2}}{\left(\phi^{-1}(t^{-1})\right)^{d/2}} \leqslant \sum_{k>t} \frac{t^k}{k!} \leqslant \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t.
$$

We next find a bound for Σ_2 and after that, we will show that $e^{-t} \leqslant c_4\left(\phi^{-1}(t^{-1})\right)^{d/2}$ for all $t>0$ and for some constant $c_4>0$. Observe that $\Sigma_2 = 0$ for $t<1$. By (2.4) we get

$$
\Sigma_2 \leqslant c_2 t^{d/2\alpha_*} \sum_{1\leqslant k\leqslant t} \frac{t^k}{k!} \frac{1}{k^{d/2\alpha_*}} \leqslant c_3 e^t,
$$

where in the last inequality we applied [29, Cor. 3]. The only thing left to prove is that $e^{-t} \leqslant c_4\left(\phi^{-1}(t^{-1})\right)^{d/2}$ for all $t>0$. For $t\geqslant 1$, using (2.4) we get

$$
\begin{aligned}
\frac{\phi^{-1}(1)}{\phi^{-1}(t^{-1})} \leqslant \left(\frac{1}{c_*}\right)^{1/\alpha_*}\left(\frac{1}{t^{-1}}\right)^{1/\alpha_*} &\Rightarrow \phi^{-1}(t^{-1}) \geqslant c_*^{1/\alpha_*} t^{-1/\alpha_*} \\
&\Rightarrow \left(\phi^{-1}(t^{-1})\right)^{d/2} \geqslant c_*^{d/2\alpha_*} t^{-d/2\alpha_*} \\
&\Rightarrow \frac{e^{-t}}{\left(\phi^{-1}(t^{-1})\right)^{d/2}} \leqslant \frac{e^{-t}}{c_*^{d/2\alpha_*} t^{-d/2\alpha_*}} \\
&\Rightarrow \lim_{t\to\infty} \frac{e^{-t}}{\left(\phi^{-1}(t^{-1})\right)^{d/2}} = 0.
\end{aligned}
$$

Hence, there is a constant $c_4 \geqslant 1$ such that $e^{-t} \leqslant c_4\left(\phi^{-1}(t^{-1})\right)^{d/2}$, for all $t\geqslant 1$. If $t\in(0,1)$ we have $\left(\phi^{-1}(t^{-1})\right)^{-d/2} \leqslant 1$ and $e^{-t}\leqslant 1$. Therefore,

$$
\left(\phi^{-1}(t^{-1})\right)^{-d/2} e^{-t} \leqslant 1 \Rightarrow e^{-t} \leqslant \left(\phi^{-1}(t^{-1})\right)^{d/2}.
$$

Plugging the bounds for Σ_1, Σ_2 and e^{-t} into (6.15), we get

$$
q(t,x,x) \leqslant C_2(\phi^{-1}(t^{-1}))^{d/2}.
$$

Finally, by the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} q(t,x,y) &= \sum_{z\in\mathbb{Z}^d} q(t/2,x,z)q(t/2,y,z) \\ &\leqslant \Big(\sum_{z\in\mathbb{Z}^d} q(t/2,x,z)^2\Big)^{1/2}\Big(\sum_{z\in\mathbb{Z}^d} q(t/2,y,z)^2\Big)^{1/2} \leqslant C_2\left(\phi^{-1}(t^{-1})\right)^{d/2} \end{aligned}$$

and the proof of (6.14) is finished. ■

Before we prove the off-diagonal estimate in (6.13), we establish a series of auxiliary results. We follow here the elaborate approach of [11]. We use the notation

$$\tau^Y(x,r) = \inf\{t \geqslant 0 : Y_t \notin B(x,r)\}.$$

Lemma 6.4. For all $r \geqslant 1$ it holds

$$\mathbb{E}^x[\tau^Y(x,r)] \asymp \frac{1}{\phi(r^{-2})}.$$

Proof. Let

$$\tau^{S^\phi}(x,r) = \inf\{k \geqslant 0 : S^\phi_k \notin B(x,r)\}.$$

By Proposition 3.13 and Lemma 3.14,

$$\mathbb{E}^x[\tau^{S^\phi}(x,n)] \asymp \frac{1}{\phi(n^{-2})}, \quad n \in \mathbb{N}.$$

Then, by Wald's identity,

$$\mathbb{E}^x[\tau^Y(x,n)] = \mathbb{E}^x\left(U_1 + \ldots + U_{\tau^{S^\phi}(x,n)}\right) = \mathbb{E}^x[\tau^{S^\phi}(x,n)].$$

Hence, for every $n \in \mathbb{N}$ we have

$$\frac{c_1}{\phi(n^{-2})} \leqslant \mathbb{E}^x[\tau^Y(x,n)] \leqslant \frac{c_2}{\phi(n^{-2})}.$$

Take $r \geqslant 1$. There exists $n \in \mathbb{N}$ such that $r \in [n, n+1)$. Since ϕ is increasing, we have

$$n \leqslant r < n+1 \Rightarrow \frac{1}{\phi(n^{-2})} \leqslant \frac{1}{\phi(r^{-2})} \leqslant \frac{1}{\phi((n+1)^{-2})}.$$

Using (2.2) and $n/(n+1) \geqslant 1/2$, which is true for all $n \in \mathbb{N}$, we get

$$\frac{\phi(n^{-2})}{\phi((n+1)^{-2})} \leqslant \left(\frac{n}{n+1}\right)^{-2} \leqslant 4.$$

Now we have

$$\mathbb{E}^x[\tau^Y(x,r)] \geqslant \mathbb{E}^x[\tau^Y(x,n)] \geqslant \frac{c_1}{\phi(n^{-2})} = \frac{c_1}{\phi((n+1)^{-2})}\frac{\phi((n+1)^{-2})}{\phi(n^{-2})} \geqslant \frac{c_1}{4}\frac{1}{\phi(r^{-2})}.$$

Similarly, for the upper bound we have

$$\mathbb{E}^x[\tau^Y(x,r)] \leqslant \mathbb{E}^x[\tau^Y(x,n+1)] \leqslant \frac{c_2}{\phi((n+1)^{-2})} = \frac{c_2}{\phi(n^{-2})}\frac{\phi(n^{-2})}{\phi((n+1)^{-2})} \leqslant 4c_2\frac{1}{\phi(r^{-2})}.$$

■

Lemma 6.5. There exist constants $C_3, C_4 > 0$ such that

$$\mathbb{P}^x(\tau^Y(x,r) \leqslant t) \leqslant 1 - \frac{C_3\phi((2r)^{-2})}{\phi(r^{-2})} + C_4 t\phi((2r)^{-2}), \tag{6.16}$$

for all $x \in \mathbb{Z}^d$ and for all $r,t > 0$

Proof. We first consider the case $r \in (0,1)$. Then the process Y exits from the ball $B(x,r)$ as soon as it jumps to some point other than x. Observe that

$$\{\tau^Y(x,r) \leqslant t\} = \bigcup_{n=1}^{\infty}\{T_n \leqslant t, S_1^\phi = S_2^\phi = \cdots = S_{n-1}^\phi = x, S_n^\phi \neq x\}.$$

Hence

$$\mathbb{P}^x(\tau^Y(x,r) \leqslant t) = \sum_{n=1}^{\infty}\mathbb{P}(T_n \leqslant t)\left(\mathbb{P}(S_1^\phi = 0)\right)^{n-1}\mathbb{P}(S_1^\phi \neq 0) \leqslant t,$$

where we used Lemma 2.15. Choosing $C_3' = 1/2$ we have

$$1 - \frac{C_3'\phi((2r)^{-2})}{\phi(r^{-2})} \geqslant \frac{1}{2}.$$

If we set $C_4' = 1/\phi(1/4)$ we have $t \leqslant C_4' t\phi((2r)^{-2})$. Hence, for $r < 1$ we have

$$\mathbb{P}^x(\tau^Y(x,r) \leqslant t) \leqslant 1 - \frac{C_3'\phi((2r)^{-2})}{\phi(r^{-2})} + C_4' t\phi((2r)^{-2}),$$

and this is precisely (6.16) with C_3' and C_4'.

Next, assume that $r \geqslant 1$. Since for any $t > 0$

$$\tau^Y(x,r) \leqslant t + (\tau^Y(x,r) - t)\mathbb{1}_{\{\tau^Y(x,r)>t\}},$$

by Markov property and Lemma 6.4 we get

$$\begin{aligned}
\mathbb{E}^x[\tau^Y(x,r)] &\leqslant t + \mathbb{E}^x\left[\mathbb{1}_{\{\tau^Y(x,r)>t\}}\mathbb{E}^{Y_t}[\tau^Y(x,r) - t]\right] \\
&\leqslant t + \sup_{z\in B(x,r)}\mathbb{E}^z[\tau^Y(x,r)]\mathbb{P}^x(\tau^Y(x,r) > t) \\
&\leqslant t + \sup_{z\in B(x,r)}\mathbb{E}^z[\tau^Y(z,2r)]\mathbb{P}^x(\tau^Y(x,r) > t) \\
&\leqslant t + \frac{c_2}{\phi((2r)^{-2})}\mathbb{P}^x(\tau^Y(x,r) > t).
\end{aligned}$$

Using again Lemma 6.4 we have

$$\frac{c_1}{\phi(r^{-2})} \leqslant \mathbb{E}^x[\tau^Y(x,r)] \leqslant t + \frac{c_2}{\phi((2r)^{-2})}\mathbb{P}^x(\tau^Y(x,r) > t)$$

and whence

$$1 - \mathbb{P}^x(\tau^Y(x,r) \leqslant t) \geqslant \frac{c_1\phi((2r)^{-2})}{c_2\phi(r^{-2})} - \frac{t\phi((2r)^{-2})}{c_2}.$$

If we set $C_3 = \min\{C_3', c_1/c_2\} \leqslant 1/2$ and $C_4 = \max\{C_4', 1/c_2\}$ we obtain (6.16) and the proof is finished. ■

We now study the truncated process which is built upon the process Y. For any $\rho > 0$ we denote by $Y^{(\rho)}$ the process obtained by removing from Y the jumps of the size larger than ρ. More precisely, the process $Y^{(\rho)}$ is associated with the following Dirichlet form

$$\mathscr{E}^{(\rho)}(u,v) = \sum_{|x-y|\leqslant\rho} (u(x)-u(y))(v(x)-v(y))p^{\phi}(x,y),$$

which is defined for functions u, v from the domain of the Dirichlet form of the random walk S^{ϕ}, cf. [3, Sec. 5]. We write $q^{(\rho)}(t,x,y)$ for the transition probability of $Y^{(\rho)}$ and $Q_t^{(\rho)}$ for its semigroup. We will also work with killed processes. For any non-empty $D \subseteq \mathbb{Z}^d$, we denote by $(Q_t^D)_{t\geqslant 0}$ the semigroup of the killed process Y^D. Similarly we write $(Q_t^{(\rho),D})_{t\geqslant 0}$ for the semigroups of $Y^{(\rho),D}$. Let

$$\tau^{(\rho)}(x,r) = \inf\{t \geqslant 0 : Y_t^{(\rho)} \notin B(x,r)\}.$$

Lemma 6.6. There exist constants $C_5 \in (0,1)$ and $C_6 > 0$ such that for any $r,t,\rho > 0$

$$\mathbb{P}^x(\tau^{(\rho)}(x,r) \leqslant t) \leqslant 1 - C_5 + C_6 t\left(\phi((2r)^{-2}) \vee \phi(\rho^{-2})\right).$$

Proof. By Lemma 6.5 and (2.2) we get that for all $x \in \mathbb{Z}^d$ and $r,t > 0$

$$\mathbb{P}^x(\tau^Y(x,r) \leqslant t) \leqslant 1 - C_3/4 + C_4 t\phi((2r)^{-2}).$$

According to [11, Lemma 7.8], for all $t > 0$

$$Q_t^{B(x,r)}\mathbb{1}_{B(x,r)}(x) \leqslant Q_t^{(\rho),B(x,r)}\mathbb{1}_{B(x,r)}(x) + c_1 t\phi(\rho^{-2}). \tag{6.17}$$

Remark. In [11, Lemma 7.8] the authors assume more restrictive assumption on the function ϕ then our condition (2.3), namely they require the global scaling. The key tool to prove (6.17) is, however, [11, Lemma 2.1] which in our case is covered by Lemma 2.17.

We notice that

$$Q_t^{B(x,r)}\mathbb{1}_{B(x,r)}(x) = \mathbb{E}^x\left[\mathbb{1}_{B(x,r)}(Y_t)\mathbb{1}_{\{\tau^Y(x,r)>t\}}\right] = \mathbb{P}^x(\tau^Y(x,r)>t),$$
$$Q_t^{(\rho),B(x,r)}\mathbb{1}_{B(x,r)}(x) = \mathbb{E}^x\left[\mathbb{1}_{B(x,r)}(Y_t^{(\rho)})\mathbb{1}_{\{\tau^{(\rho)}(x,r)>t\}}\right] = \mathbb{P}^x(\tau^{(\rho)}(x,r)>t)$$

and whence

$$\mathbb{P}^x(\tau^Y(x,r)>t) \leqslant \mathbb{P}^x(\tau^{(\rho)}(x,r)>t) + c_1 t\phi(\rho^{-2}).$$

This and Lemma 6.5 imply

$$\mathbb{P}^x(\tau^{(\rho)}(x,r)\leqslant t) \leqslant 1 - \frac{C_3}{4} + C_4 t\phi((2r)^{-2}) + c_1 t\phi(\rho^{-2})$$

and the result follows if we choose $C_5 = C_3/4 < 1$ and $C_6 = C_4 + c_1$. ■

Lemma 6.7. There exist constants $\varepsilon \in (0,1)$ and $C_7 > 0$ such that for $x \in \mathbb{Z}^d$ and all $r, \lambda, \rho > 0$ with $\lambda \geqslant C_7\phi((r\wedge\rho)^{-2})$ it holds

$$\mathbb{E}^x\left[e^{-\lambda\tau^{(\rho)}(x,r)}\right] \leqslant 1-\varepsilon. \tag{6.18}$$

Proof. By Lemma 6.6, for any $t>0$ and $x\in\mathbb{Z}^d$,

$$\begin{aligned}\mathbb{E}^x\left[e^{-\lambda\tau^{(\rho)}(x,r)}\right] &= \mathbb{E}^x\left[e^{-\lambda\tau^{(\rho)}(x,r)}\mathbb{1}_{\{\tau^{(\rho)}(x.r)\leqslant t\}}\right] + \mathbb{E}^x\left[e^{-\lambda\tau^{(\rho)}(x,r)}\mathbb{1}_{\{\tau^{(\rho)}(x,r)>t\}}\right]\\ &\leqslant \mathbb{P}^x(\tau^{(\rho)}(x,r)\leqslant t) + e^{-\lambda t}\\ &\leqslant 1 - C_5 + C_6 t\left(\phi((2r)^{-2})\vee\phi(\rho^{-2})\right) + e^{-\lambda t}.\end{aligned}$$

We now choose $\varepsilon = C_5/4 \in (0,1)$. We next take $t = c_1/\phi((r\wedge\rho)^{-2})$, for some $c_1>0$, in such a way that $C_6 t\phi((2r)^{-2}) + C_6 t\phi(\rho^{-2}) \leqslant 2\varepsilon$. Hence, we need to choose $c_1>0$ such that

$$\frac{C_6c_1\phi((2r)^{-2})}{\phi((r\wedge\rho)^{-2})} + \frac{C_6c_1\phi(\rho^{-2})}{\phi((r\wedge\rho)^{-2})} \leqslant 2\varepsilon.$$

Since ϕ is increasing,

$$\frac{\phi((2r)^{-2})}{\phi((r\wedge\rho)^{-2})} \leqslant 1 \quad\text{and}\quad \frac{\phi(\rho^{-2})}{\phi((r\wedge\rho)^{-2})} \leqslant 1$$

and thus it suffices to choose $c_1 \leqslant \varepsilon/C_6$. At last, we claim that there is $C_7>0$ such that for $\lambda \geqslant C_7\phi((r\wedge\rho)^{-2})$ we will have $e^{-\lambda t}\leqslant\varepsilon$. Indeed, with such a choice we get that $\lambda t \geq C_7c_1$ and thus we can choose C_7 so big that $e^{-\lambda t} \leqslant C_5/4 = \varepsilon$. We finally obtain

$$\mathbb{E}^x\left[e^{-\lambda\tau^{(\rho)}(x,r)}\right] \leqslant 1 - C_5 + C_6t(\phi((2r)^{-2}) + \phi(\rho^{-2})) + e^{-\lambda t} \leqslant 1-\varepsilon,$$

as desired. ■

Lemma 6.8. There exist constants $C_8, C_9 > 0$ such that for $x \in \mathbb{Z}^d$ and $R, \rho > 0$

$$\mathbb{E}^x\left[e^{-C_7\phi(\rho^{-2})\tau^{(\rho)}(x,R)}\right] \leqslant C_8 e^{-C_9 R/\rho},$$

where $C_7 > 0$ is the constant from Lemma 6.7.

Proof. We first observe that if $\rho \geqslant R/2$ then we can choose constants C_8 and C_9 such that $C_8\exp(-2C_9) \geqslant 1$ and the result follows. Thus we study the case $\rho \in (0, R/2)$. Let $z \in \mathbb{Z}^d$, $R > 0$ be fixed. We write for simplicity $\tau = \tau^{(\rho)}(z,R)$. For any fixed $0 < r < R/2$ we set $n = \lfloor R/2r \rfloor$. Let

$$u(x) = \mathbb{E}^x[e^{-\lambda\tau}] \quad \text{and} \quad m_k = \|u\|_{L^\infty(B(z,kr))}, \ \ k \in \{1, 2, \ldots, n\}.$$

We fix ε from Lemma 6.7 and for any $0 < \varepsilon' < \varepsilon$ we choose $x_k \in B(z,kr)$ such that

$$(1-\varepsilon')m_k < u(x_k) = m_k.$$

Since $x_k \in B(z,kr)$ and $n = \lfloor R/2r \rfloor$ it is easy to see that for any $k \leqslant n-1$

$$B(x_k, r) \subseteq B(z, (k+1)r) \subseteq B(z,R).$$

Next we consider the following function

$$v_k(x) = \mathbb{E}^x[e^{-\lambda\tau_k}], \quad x \in B(x_k, r),$$

where we write $\tau_k = \tau^{(\rho)}(x_k, r)$. By the strong Markov property, for any $x \in B(x_k, r)$,

$$u(x) = \mathbb{E}^x[e^{-\lambda\tau_k}e^{-\lambda(\tau-\tau_k)}] = \mathbb{E}^x\left[e^{-\lambda\tau_k}\mathbb{E}^{Y^{(\rho)}_{\tau_k}}(e^{-\lambda\tau})\right] = \mathbb{E}^x\left[e^{-\lambda\tau_k}u(Y^{(\rho)}_{\tau_k})\right].$$

Since $Y^{(\rho)}_{\tau_k} \in B(x_k, r+\rho)$, we get that for every $x \in B(x_k, r)$

$$u(x) \leqslant v_k(x)\,\|u\|_{L^\infty(B(x_k, r+\rho))}.$$

It follows that for any $0 < \rho \leqslant r$

$$u(x_k) \leqslant v_k(x_k)\,\|u\|_{L^\infty(B(x_k, r+\rho))} \leqslant v_k(x_k)m_{k+2}.$$

Since $u(x_k) > (1-\varepsilon')m_k$, we have

$$(1-\varepsilon')m_k \leqslant v_k(x_k)m_{k+2}.$$

In view of Lemma 6.7, if $\lambda \geqslant C_7\phi(\rho^{-2})$ and $0<\rho\leqslant r$ then $v_k(x_k)\leqslant 1-\varepsilon$. Hence

$$m_k \leqslant \left(\frac{1-\varepsilon}{1-\varepsilon'}\right) m_{k+2}$$

and iterating yields

$$u(z)\leqslant m_1 \leqslant \left(\frac{1-\varepsilon}{1-\varepsilon'}\right) m_3 \leqslant \left(\frac{1-\varepsilon}{1-\varepsilon'}\right)^2 m_5 \leqslant \ldots \leqslant \left(\frac{1-\varepsilon}{1-\varepsilon'}\right)^{n-1} m_{2n-1}.$$

Since $u(x)\leqslant 1$, we have $m_{2n-1}\leqslant 1$. Together with $n=\lfloor R/2r\rfloor$ this gives us

$$u(z)\leqslant \left(\frac{1-\varepsilon}{1-\varepsilon'}\right)^{n-1} \leqslant \left(\frac{1-\varepsilon}{1-\varepsilon'}\right)^{R/2r-2}.$$

Setting $2C_9=\log((1-\varepsilon')/(1-\varepsilon))$ we get

$$u(z)\leqslant C_8 \exp\left(-C_9\frac{R}{r}\right),$$

with $C_8=e^{4C_9}$. If we set $\lambda=C_7\phi(\rho^{-2})$ and $\rho=r$ we conclude the result. ■

Corollary 6.9. For any $R,\rho,t>0$ and all $x\in\mathbb{Z}^d$

$$\mathbb{P}^x(\tau^{(\rho)}(x,R)\leqslant t)\leqslant C_8 e^{-C_9\frac{R}{\rho}+C_7 t\phi(\rho^{-2})},$$

where $C_7>0$ is the constant from Lemma 6.7 and $C_8,C_9>0$ from Lemma 6.8.

Proof. By Lemma 6.8,

$$\begin{aligned}\mathbb{P}^x(\tau^{(\rho)}(x,R)\leqslant t) &= \mathbb{P}^x\left(e^{-C_7\phi(\rho^{-2})\tau^{(\rho)}(x,R)} \geqslant e^{-C_7\phi(\rho^{-2})t}\right)\\ &\leqslant e^{C_7\phi(\rho^{-2})t}\mathbb{E}^x\left[e^{-C_7\phi(\rho^{-2})\tau^{(\rho)}(x,R)}\right]\leqslant C_8 e^{-C_9\frac{R}{\rho}+C_7 t\phi(\rho^{-2})},\end{aligned}$$

as desired. ■

For any $\rho>0$ and $x,y\in\mathbb{Z}^d$, we define

$$J_\rho(x,y)=p^\phi(x,y)\mathbb{1}_{\{|x-y|>\rho\}}.$$

By Meyer's decomposition and [11, Lemma 7.2(1)], the following estimate holds

$$q(t,x,y)\leqslant q^{(\rho)}(t,x,y)+\mathbb{E}^x\Big[\int_0^t \sum_{z\in\mathbb{Z}^d} J_\rho(Y_s^{(\rho)},z)q(t-s,z,y)ds\Big],\quad x,y\in\mathbb{Z}^d. \tag{6.19}$$

Proposition 6.10. There exists $C_{10}>0$ such that for all $t,\rho>0$ and $x\in\mathbb{Z}^d$

$$\mathbb{E}^x\Big[\int_0^t \sum_{z\in\mathbb{Z}^d} J_\rho(Y_s^{(\rho)},z)q(t-s,z,y)ds\Big]\leqslant C_{10}t\rho^{-d}\phi(\rho^{-2}).$$

Proof. By monotonicity and Proposition 3.3 we get $J_\rho(x,y) \leqslant C_{10}\rho^{-d}\phi(\rho^{-2})$, for some $C_{10} > 0$. This and symmetry imply the result. ■

In the next Lemma we prove the upper bound for the transition kernel of the truncated process.

Lemma 6.11. For all $t \geqslant 1$ and $x,y \in \mathbb{Z}^d$

$$q^{(\rho)}(t,x,y) \leqslant C_{11}\left(\phi^{-1}(t^{-1})\right)^{d/2}\exp\left(C_{12}t\phi(\rho^{-2}) - C_{13}\frac{|x-y|}{\rho}\right), \tag{6.20}$$

where $C_{11}, C_{12}, C_{13} > 0$ are constants independent of ρ.

Proof. A direct application of [11, Lemma 7.2(2)] combined with Lemma 2.17 and Lemma 6.3, imply that for all $t > 0$ and $x,y \in \mathbb{Z}^d$ we have

$$q^{(\rho)}(t,x,y) \leqslant q(t,x,y)e^{tc_1\phi(\rho^{-2})} \leqslant C_2\left(\phi^{-1}(t^{-1})\right)^{d/2}\exp(c_1t\phi(\rho^{-2})). \tag{6.21}$$

We first observe that for $|x-y| < 2\rho$ relation (6.20) is trivial. Indeed, since

$$\exp\left(\frac{-C_{13}|x-y|}{\rho}\right) > \exp(-2C_{13}),$$

for any $C_{13} > 0$, we get

$$\begin{aligned} q^{(\rho)}(t,x,y) &\leqslant C_2\left(\phi^{-1}(t^{-1})\right)^{d/2}\exp(c_1t\phi(\rho^{-2}))\frac{\exp(-2C_{13})}{\exp(-2C_{13})} \\ &\leqslant C_{11}\left(\phi^{-1}(t^{-1})\right)^{d/2}\exp\left(C_{12}t\phi(\rho^{-2}) - C_{13}\frac{|x-y|}{\rho}\right), \end{aligned} \tag{6.22}$$

for any $C_{11} \geqslant C_2/\exp(-2C_{13})$, $C_{12} \geqslant c_1$.

Assume that $|x-y| \geqslant 2\rho$. By Corollary 6.9,

$$Q_t^{(\rho)}\mathbb{1}_{B(x,r)^c}(x) \leqslant \mathbb{P}^x(\tau^{(\rho)}(x,r) \leqslant t) \leqslant C_8\exp\Big(-C_9\frac{r}{\rho} + C_7t\phi(\rho^{-2})\Big). \tag{6.23}$$

We set $r = |x-y|/2$ and write

$$\begin{aligned} q^{(\rho)}(2t,x,y) &= \sum_{z\in\mathbb{Z}^d} q^{(\rho)}(t,x,z)q^{(\rho)}(t,z,y) \\ &\leqslant \sum_{z\in B(x,r)^c} q^{(\rho)}(t,x,z)q^{(\rho)}(t,z,y) + \sum_{z\in B(y,r)^c} q^{(\rho)}(t,x,z)q^{(\rho)}(t,z,y). \end{aligned}$$

By (6.21) and (6.23) we get

$$\begin{aligned} \sum_{z\in B(x,r)^c} q^{(\rho)}(t,x,z)q^{(\rho)}(t,z,y) &\leqslant C_2\left(\phi^{-1}(t^{-1})\right)^{d/2}e^{c_1t\phi(\rho^{-2})}\sum_{z\in B(x,r)^c} q^{(\rho)}(t,x,z) \\ &\leqslant C_2C_8\left(\phi^{-1}(t^{-1})\right)^{d/2}e^{c_1t\phi(\rho^{-2})}e^{-C_9\frac{r}{\rho}+C_7t\phi(\rho^{-2})} \\ &= C_2C_8\left(\phi^{-1}(t^{-1})\right)^{d/2}e^{(c_1+C_7)t\phi(\rho^{-2})-\frac{C_9}{2}\frac{|x-y|}{\rho}}. \end{aligned}$$

We can show the same bound for $z \in B(y,r)^c$ and thus, for every $t > 0$ and $|x-y| \geqslant 2\rho$ we have

$$q^{(\rho)}(2t,x,y) \leqslant 2C_2C_8\left(\phi^{-1}(t^{-1})\right)^{d/2} e^{(c_1+C_7)t\phi(\rho^{-2})-\frac{C_9}{2}\frac{|x-y|}{\rho}}.$$

Replacing t with $t/2$ yields (6.20). It only remains to show that

$$\frac{\phi^{-1}((t/2)^{-1})}{\phi^{-1}(t^{-1})} \leqslant c_2, \tag{6.24}$$

for some constant $c_2 > 0$. *To prove* (6.24) *we have to apply scaling condition* (2.4) *and this is the reason why estimate* (6.20) *works only for* $t \geq 1$. Indeed, for $t \geqslant 2$, by (2.4) we get

$$\frac{\phi^{-1}((t/2)^{-1})}{\phi^{-1}(t^{-1})} \leqslant \left(\frac{2}{c_*}\right)^{1/\alpha_*}.$$

For $1 \leqslant t \leqslant 2$ we simply use monotonicity and (6.24) follows. ■

In the rest of this section we use the notation

$$r_t = \frac{1}{\sqrt{\phi^{-1}(t^{-1})}}, \quad t \geq 1.$$

Lemma 6.12. There are $N \in \mathbb{N}$ with $N > (2\alpha_* + d)/(2\alpha_*)$ and $c_1 \geqslant 1$ such that for all $r > 0$, $t \geqslant 1$ and $x \in \mathbb{Z}^d$

$$\sum_{y\in B(x,r)^c} q(t,x,y) \leqslant c_1 r^{-\theta}\left(\phi^{-1}(t^{-1})\right)^{-\theta/2}, \tag{6.25}$$

where $0 < \theta = 2\alpha_* - (2\alpha_* + d)/N$ and α_* is the constant from (2.3).

Proof. We first observe that for $r \leqslant r_t$ relation (6.25) is trivially satisfied, as in this case $r_t/r \geqslant 1$.

We assume that $r > r_t$. We set

$$N = \lfloor 2 + d/(2\alpha_*)\rfloor \tag{6.26}$$

and with this N we define a sequence

$$\rho_n = 2^{n\alpha} r^{1-1/N} r_t^{1/N}, \quad n \in \mathbb{N},$$

where

$$\left(\frac{d}{d+2\alpha_*} \vee \frac{1}{2}\right) < \alpha < 1. \tag{6.27}$$

We now show that under this choice we have

$$\frac{2^n r}{\rho_n} \leqslant \frac{\rho_n}{r_t} \tag{6.28}$$

and

$$t\phi(\rho_n^{-2}) \leqslant 1. \tag{6.29}$$

Indeed, (6.28) follows from (6.26) and from the fact that $\alpha \geq 1/2$, and

$$\frac{2^n r}{\rho_n} = 2^{n(1-\alpha)}\left(\frac{r}{r_t}\right)^{1/N}, \text{ and } \quad \frac{\rho_n}{r_t} = 2^{n\alpha}\left(\frac{r}{r_t}\right)^{1-1/N}.$$

Similarly, (6.29) follows, since under our choice we see that $\rho_n \geq r_t$.

Recall that by (6.19) and Proposition 6.10 we have

$$q(t,x,y) \leqslant q^{(\rho)}(t,x,y) + C_{10} t j(\rho), \tag{6.30}$$

for all $\rho, t > 0$ and $x, y \in \mathbb{Z}^d$. Next, by Lemma 6.11, for all $t \geqslant 1$, $x, y \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, we have

$$q^{(\rho_n)}(t,x,y) \leqslant C_{11}\left(\phi^{-1}(t^{-1})\right)^{d/2} \exp\left(C_{12} t\phi(\rho_n^{-2}) - C_{13}\frac{|x-y|}{\rho_n}\right),$$

where $C_{11}, C_{12}, C_{13} > 0$ are constants independent of ρ_n. Hence, for all $2^n r \leqslant |x-y| < 2^{n+1} r$ and all $t \geqslant 1$ we have

$$q^{(\rho_n)}(t,x,y) \leqslant C_{11}\left(\phi^{-1}(t^{-1})\right)^{d/2} \exp\left(C_{12} t\phi(\rho_n^{-2}) - C_{13}\frac{2^n r}{\rho_n}\right).$$

By (6.29) we get

$$q^{(\rho_n)}(t,x,y) \leqslant c_2\left(\phi^{-1}(t^{-1})\right)^{d/2} \exp\left(-C_{13}\frac{2^n r}{\rho_n}\right). \tag{6.31}$$

Thus, by (6.30) and (6.31) we get, for $t \geqslant 1$ and $x \in \mathbb{Z}^d$

$$\begin{aligned}
\sum_{y \in B(x,r)^c} q(t,x,y) &\leqslant \sum_{n=0}^{\infty} \sum_{2^n r \leqslant |x-y| < 2^{n+1} r} \left(q^{(\rho_n)}(t,x,y) + C_{10} t j(\rho_n)\right) \\
&\leqslant c_3 \sum_{n=0}^{\infty} (2^n r)^d \left(\phi^{-1}(t^{-1})\right)^{d/2} e^{-C_{13}\frac{2^n r}{\rho_n}} \\
&\qquad + c_4 \sum_{n=0}^{\infty} (2^n r)^d t j(\rho_n) = I_1 + I_2.
\end{aligned}$$

We first estimate I_2. Since $\rho_n^{-2} \leqslant \phi^{-1}(t^{-1}) \leqslant 1$, we can use (2.3) to get

$$t\phi(\rho_n^{-2}) \leqslant \frac{1}{c_*}\left(\frac{r_t}{\rho_n}\right)^{2\alpha_*}.$$

This implies

$$
\begin{aligned}
I_2 &= \sum_{n=0}^{\infty} c_4 (2^n r)^d t\phi(\rho_n^{-2})\rho_n^{-d} \leqslant \sum_{n=0}^{\infty} c_4 \left(\frac{2^n r}{\rho_n}\right)^d \frac{1}{c_*}\left(\frac{r_t}{\rho_n}\right)^{2\alpha_*} \\
&= \frac{c_4}{c_*}\left(\frac{r_t}{r}\right)^{2\alpha_*-(2\alpha_*+d)/N} \sum_{n=0}^{\infty} \left(\frac{r_t}{\rho_n}\right)^{2\alpha_*} \left(\frac{r_t}{r}\right)^{(2\alpha_*+d)/N-2\alpha_*} 2^{n(d-\alpha(d+2\alpha_*))} 2^{n\alpha(d+2\alpha_*)} \left(\frac{r}{\rho_n}\right)^d \\
&= \frac{c_4}{c_*}\left(\frac{r_t}{r}\right)^{2\alpha_*-(2\alpha_*+d)/N} \sum_{n=0}^{\infty} \left(2^{n\alpha} r^{1-1/N} r_t^{1/N}\right)^{d+2\alpha_*} \frac{1}{\rho_n^{d+2\alpha_*}} 2^{n(d-\alpha(d+2\alpha_*))} \\
&= \frac{c_4}{c_*}\left(\frac{r_t}{r}\right)^{2\alpha_*-(2\alpha_*+d)/N} \sum_{n=0}^{\infty} 2^{n(d-\alpha(d+2\alpha_*))}.
\end{aligned}
$$

By (6.27), $d-\alpha(d+2\alpha_*)<0$ and whence

$$
I_2 \leqslant c_5 \left(\frac{r_t}{r}\right)^{2\alpha_*-(2\alpha_*+d)/N}. \tag{6.32}
$$

We proceed to estimate I_1. There exists a constant $c_K>0$ such that for $x \geqslant C_{13}$ we have $e^{-x} \leqslant c_K x^{-K}$. Applying this, we get

$$
\exp\left(-C_{13}\frac{2^n r}{\rho_n}\right) \leqslant c_K \left(\frac{C_{13} 2^n r}{\rho_n}\right)^{-K}, \quad K>0.
$$

We set

$$
K = 1 + N(d+2\alpha_*) \vee \frac{d}{1-\alpha}.
$$

For such K we have $K/N > d+2\alpha_*$ and $(1-\alpha)K > d$ and this yields

$$
\begin{aligned}
I_1 &= \sum_{n=0}^{\infty} c_6 (2^n r)^d \left(\phi^{-1}(t^{-1})\right)^{d/2} \exp\left(-C_{13}\frac{2^n r}{\rho_n}\right) \\
&\leqslant \sum_{n=0}^{\infty} c_6 (2^n r)^d r_t^{-d} c_K \left(\frac{C_{13} 2^n r}{\rho_n}\right)^{-K} \\
&= \sum_{n=0}^{\infty} c_6 c_K C_{13}^{-K} \left(\frac{2^n r}{r_t}\right)^d \left(\frac{2^{n\alpha} r^{1-1/N} r_t^{1/N}}{2^n r}\right)^K \\
&= c_7 \left(\frac{r}{r_t}\right)^d \left(\frac{r_t}{r}\right)^{K/N} \sum_{n=0}^{\infty} 2^{n(d-(1-\alpha)K)} \\
&= c_7 \left(\frac{r_t}{r}\right)^{K/N-d} \sum_{n=0}^{\infty} \left(2^{d-(1-\alpha)K}\right)^n = c_8 \left(\frac{r_t}{r}\right)^{K/N-d} \\
&\leqslant c_8 \left(\frac{r_t}{r}\right)^{2\alpha_*} \leqslant c_8 \left(\frac{r_t}{r}\right)^{2\alpha_*} \left(\frac{r_t}{r}\right)^{-(2\alpha_*+d)/N} \\
&= c_8 \left(\frac{r_t}{r}\right)^{2\alpha_*-(2\alpha_*+d)/N}.
\end{aligned} \tag{6.33}
$$

Using the definition of θ, (6.32), (6.33) and setting $c_1 = c_5 + c_8$, we conclude (6.25). ∎

Lemma 6.13. Assume that condition (6.25) holds with some $\theta > 0$. Then there exists a constant $c_2 > 0$ such that for any ball $B(x_0, r)$ and for any $t \geqslant 1$

$$\mathbb{P}^x(\tau^Y(x_0,r) \leqslant t) \leqslant c_2 r^{-\theta} \left(\phi^{-1}(t^{-1})\right)^{-\theta/2}, \quad x \in B(x_0, r/4).$$

Proof. For $x \in B(x_0, r/4)$, we have $B(x, 3r/4) \subseteq B(x_0, r)$. Using (6.25) we get

$$\begin{aligned}
\mathbb{P}^x(\tau^Y(x_0,r) \leqslant t) &\leqslant \mathbb{P}^x(\tau^Y(x,3r/4) \leqslant t) \\
&= \mathbb{P}^x(\tau^Y(x,3r/4) \leqslant t, Y_{2t} \in B(x,r/2)^c) + \mathbb{P}^x(\tau^Y(x,3r/4) \leqslant t, Y_{2t} \in B(x,r/2)) \\
&\leqslant \mathbb{P}^x(Y_{2t} \in B(x,r/2)^c) + \sup_{\substack{z \in B(x,3r/4)^c \\ s \leqslant t}} \mathbb{P}^z(Y_{2t-s} \in B(x,r/2)) \\
&\leqslant \sum_{y \in B(x,r/2)^c} q(2t,x,y) + \sup_{\substack{z \in B(x,3r/4)^c \\ s \leqslant t}} \sum_{y \in B(z,r/4)^c} q(2t-s,z,y) \\
&\leqslant c_1 \left(\frac{r_{2t}}{r/2}\right)^{\theta} + c_1 \sup_{s \leqslant t} \left(\frac{r_{2t-s}}{r/4}\right)^{\theta}. \qquad (6.34)
\end{aligned}$$

Since $t \geqslant 1$, we can use (2.4) to obtain

$$r_{2t} \leqslant \left(\frac{2}{c_*}\right)^{1/2\alpha_*} r_t.$$

Since $s \leqslant t$, we have

$$\sup_{s \leqslant t} r_{2t-s} \leqslant \left(\frac{2}{c_*}\right)^{1/2\alpha_*} r_t.$$

With these estimates used in (6.34) we get

$$\mathbb{P}^x(\tau^Y(x_0,r) \leqslant t) \leqslant c_1 2^{\theta} \left(\frac{2}{c_*}\right)^{\theta/2\alpha_*} \left(\frac{r_t}{r}\right)^{\theta} + c_1 4^{\theta} \left(\frac{2}{c_*}\right)^{\theta/2\alpha_*} \left(\frac{r_t}{r}\right)^{\theta} = c_2 \left(\frac{r_t}{r}\right)^{\theta},$$

for all $x \in B(x_0, r/4)$. ■

Lemma 6.14. Assume that condition (6.25) holds with $0 < \theta = 2\alpha_* - (2\alpha_* + d)/N$. Then for all $t \geqslant 1$, $k \geqslant 1$ and $|x_0 - y_0| > 4k\rho$ it holds

$$q^{(\rho)}(t, x_0, y_0) \leqslant c(k) \left(\phi^{-1}(t^{-1})\right)^{d/2} \exp\left(c_0 t \phi(\rho^{-2})\right) \left(1 + \frac{\rho}{r_t}\right)^{-(k-1)\theta}. \qquad (6.35)$$

Proof. As observed in the proof of Lemma 6.6, for all $t > 0$,

$$Q_t^B \mathbb{1}_B(x) \leqslant Q_t^{(\rho),B} \mathbb{1}_B(x) + c_1 t \phi(\rho^{-2})$$

and

$$\mathbb{P}^x(\tau^Y(x_0,r) \leqslant t) = 1 - Q_t^B \mathbb{1}_B(x).$$

This and Lemma 6.13 imply

$$1-Q_t^{(\rho),B}\mathbb{1}_B(x)-c_1t\phi(\rho^{-2})\leqslant 1-Q_t^B\mathbb{1}_B(x)\leqslant c_2\left(\frac{r}{r_t}\right)^{-\theta}.$$

Hence

$$1-Q_t^{(\rho),B}\mathbb{1}_B(x)\leqslant c_3\Big[\left(\frac{r}{r_t}\right)^{-\theta}+t\phi(\rho^{-2})\Big],\quad x\in B(x_0,r/4). \tag{6.36}$$

We now proceed to prove (6.35). If $\rho<r_t$ then clearly

$$\Big(1+\frac{\rho}{r_t}\Big)^{(k-1)\theta}<2^{(k-1)\theta}.$$

and, by (6.21),

$$q^{(\rho)}(t,x_0,y_0)\leqslant C_2 2^{(k-1)\theta}\left(\phi^{-1}(t^{-1})\right)^{d/2}\exp(c_0t\phi(\rho^{-2}))\Big(1+\frac{\rho}{r_t}\Big)^{-(k-1)\theta},$$

as claimed.

Let us now consider the case $\rho\geqslant r_t$. Fix $k\geqslant 1, t\geqslant 1$ and $x_0,y_0\in\mathbb{Z}^d$ such that $|x_0-y_0|>4k\rho$. Set $r=|x_0-y_0|/2>2k\rho$ and

$$\psi(r,t)=c_3\Big[\left(\frac{r}{r_t}\right)^{-\theta}+t\phi(\rho^{-2})\Big]. \tag{6.37}$$

Notice that $\psi(r,t)$ is non-decreasing in t. We take $R=r/k>2\rho$ and apply [11, Lemma 7.11] to get

$$Q_t^{(\rho)}\mathbb{1}_{B(x_0,r)^c}(x)\leqslant\Big\{c_4\Big[\left(\frac{r/k-\rho}{r_t}\right)^{-\theta}+t\phi(\rho^{-2})\Big]\Big\}^{k-1},\quad x\in B(x_0,R).$$

Remark. In our case the assumption of [11, Lemma 7.11] is valid only for $t\geqslant 1$. Since the lemma is proven by induction, we could repeat the argument and get the same result.

Notice that

$$\Big(\frac{r}{k}-\rho\Big)^{-\theta}<\rho^{-\theta}.$$

Using this and the fact that $R>\rho$, we obtain

$$Q_t^{(\rho)}\mathbb{1}_{B(x_0,r)^c}(x)\leqslant c_1(k)\Big\{\big(\frac{\rho}{r_t}\big)^{-\theta}+t\phi(\rho^{-2})\Big\}^{k-1},\quad x\in B(x_0,\rho). \tag{6.38}$$

We notice that

$$t\phi(\rho^{-2})\leqslant\frac{1}{c_*}\left(\frac{\rho}{r_t}\right)^{-\theta},\quad \rho\geqslant r_t.$$

This follows easily by (2.3). Combining this with (6.38), we get

$$Q_t^{(\rho)}\mathbb{1}_{B(x_0,r)^c}(x)\leqslant c_2(k)\Big(\frac{\rho}{r_t}\Big)^{-(k-1)\theta},\quad x\in B(x_0,\rho). \tag{6.39}$$

Moreover, since $\rho \geqslant r_t$, we have

$$\left(\frac{\rho}{r_t}\right)^{-(k-1)\theta} \leqslant 2^{(k-1)\theta}\left(1+\frac{\rho}{r_t}\right)^{-(k-1)\theta}.$$

Hence, by (6.39),

$$Q_t^{(\rho)}\mathbb{1}_{B(x_0,r)^c}(x_0) \leqslant c_3(k)\left(1+\frac{\rho}{r_t}\right)^{-(k-1)\theta}. \tag{6.40}$$

Further, observe that

$$Q_t^{(\rho)}\mathbb{1}_{B(x_0,r)^c}(x_0) = \mathbb{P}^{x_0}(Y_t^{(\rho)} \in B(x_0,r)^c) = \sum_{z\in B(x_0,r)^c} q^{(\rho)}(t,x_0,z)$$

and, by the semigroup property,

$$\begin{aligned} q^{(\rho)}(2t,x_0,y_0) &= \sum_{z\in\mathbb{Z}^d} q^{(\rho)}(t,x_0,z)q^{(\rho)}(t,z,y_0) \\ &\leqslant \sum_{z\in B(x_0,r)^c} q^{(\rho)}(t,x_0,z)q^{(\rho)}(t,z,y_0) + \sum_{z\in B(y_0,r)^c} q^{(\rho)}(t,x_0,z)q^{(\rho)}(t,z,y_0). \end{aligned}$$

Using (6.21) and (6.40) we obtain

$$\begin{aligned} \sum_{z\in B(x_0,r)^c} q^{(\rho)}(t,x_0,z)q^{(\rho)}(t,z,y_0) &\leqslant C_2\left(\phi^{-1}(t^{-1})\right)^{d/2}\exp(c_0t\phi(\rho^{-2}))Q_t^{(\rho)}\mathbb{1}_{B(x_0,r)^c}(x_0) \\ &\leqslant c_4(k)\left(\phi^{-1}(t^{-1})\right)^{d/2}\exp(c_0t\phi(\rho^{-2}))\left(1+\frac{\rho}{r_t}\right)^{-(k-1)\theta}. \end{aligned}$$

Similarly, we show that

$$\sum_{z\in B(y_0,r)^c} q^{(\rho)}(t,x_0,z)q^{(\rho)}(t,z,y_0) \leqslant c_4(k)\left(\phi^{-1}(t^{-1})\right)^{d/2}\exp(c_0t\phi(\rho^{-2}))\left(1+\frac{\rho}{r_t}\right)^{-(k-1)\theta}.$$

This yields

$$q^{(\rho)}(2t,x_0,y_0) \leqslant c_5(k)\left(\phi^{-1}(t^{-1})\right)^{d/2}\exp(c_0t\phi(\rho^{-2}))\left(1+\frac{\rho}{r_t}\right)^{-(k-1)\theta}.$$

As in the proof of Lemma 6.11, we can replace $2t$ with t and the proof is finished. ■

We now finally prove the upper bound for the heat kernel of the process Y_t.

Proof of Proposition 6.2. Our aim is to prove that for all $t \geqslant 1$

$$q(t,x,y) \leqslant c_1t|x-y|^{-d}\phi(|x-y|^{-2}), \quad x \neq y. \tag{6.41}$$

We take arbitrary $x_0, y_0 \in \mathbb{Z}^d$ such that $x_0 \neq y_0$ and we set $r := |x_0-y_0|/2$. Assume that $r < r_t$. We show that in this case the on-diagonal bound from Lemma 6.3 is smaller than the bound in (6.41), that is

$$\left(\phi^{-1}(t^{-1})\right)^{d/2} \leqslant c_2tr^{-d}\phi(r^{-2}). \tag{6.42}$$

Indeed, since $1/2 \leqslant r < r_t$, we can use Lemma 2.16 (with $L = 4$) to obtain

$$\frac{\left(\phi^{-1}(t^{-1})\right)^{d/2}}{tr^{-d}\phi(r^{-2})} \leqslant \frac{4^{\alpha_*}}{c_*}\left(\frac{r_t}{r}\right)^{-2\alpha_*}\left(\frac{r_t}{r}\right)^{-d} \leqslant \frac{4^{\alpha_*}}{c_*}.$$

Combining (6.42) with Lemma 6.3 and using (2.2) we get

$$q(t,x_0,y_0) \leqslant C_2 c_2 2^d t|x_0-y_0|^{-d}\phi(4|x_0-y_0|^{-2}) \leqslant c_3 t|x_0-y_0|^{-d}\phi(|x_0-y_0|^{-2}). \tag{6.43}$$

We next consider the case $r \geqslant r_t$. We set $k = 1+(d+2\alpha^*)/\theta$ and $\rho = r/(8k)$. By (6.19), Proposition 6.10 and (6.35),

$$q(t,x_0,y_0) \leqslant c(k)\left(\phi^{-1}(t^{-1})\right)^{d/2}\exp\left(c_0 t\phi(\rho^{-2})\right)\left(1+\frac{\rho}{r_t}\right)^{-(k-1)\theta}+C_{10}t\rho^{-d}\phi(\rho^{-2}).$$

We observe that $t\phi(\rho^{-2})$ is bounded. This follows as $r \geqslant r_t$ implies $t\phi(r^{-2}) \leqslant 1$, and we use $\rho = r/(8k)$ with (2.2) to get

$$t\phi(\rho^{-2}) = t\phi(64k^2 r^{-2}) \leqslant 64k^2 t\phi(r^{-2}) \leqslant 64k^2.$$

Hence

$$\begin{aligned} q(t,x_0,y_0) &\leqslant c(k)\left(\phi^{-1}(t^{-1})\right)^{d/2}\exp(c_0 64k^2)\left(1+\frac{\rho}{r_t}\right)^{-(k-1)\theta}+C_{10}t\rho^{-d}\phi(\rho^{-2}) \\ &\leqslant c_6(k)\left(\phi^{-1}(t^{-1})\right)^{d/2}\left(1+\frac{\rho}{r_t}\right)^{-(k-1)\theta}+C_{10}t\rho^{-d}\phi(\rho^{-2}). \end{aligned} \tag{6.44}$$

Since $\rho = r/(8k)$ and $r_t/r > 0$, we get

$$\left(1+\frac{\rho}{r_t}\right)^{-(k-1)\theta} \leqslant c_7(k)\left(\frac{r}{r_t}\right)^{-(k-1)\theta},$$

and, by (2.2),

$$\rho^{-d}\phi(\rho^{-2}) = (r/(8k))^{-d}\phi\left((r/(8k))^{-2}\right) \leqslant (8k)^{d+2} r^{-d}\phi(r^{-2}).$$

These inequalities together with (6.44) yield

$$\begin{aligned} q(t,x_0,y_0) &\leqslant c_8(k)\left(\phi^{-1}(t^{-1})\right)^{d/2}\left(\frac{r}{r_t}\right)^{-(k-1)\theta}+ c_8(k)tr^{-d}\phi(r^{-2}) \\ &= c_8(k)tr^{-d}\phi(r^{-2})\Big[\frac{t^{-1}}{\phi(r^{-2})}\left(\frac{r}{r_t}\right)^{-2\alpha^*}+1\Big]. \end{aligned} \tag{6.45}$$

By $r^{-2} \leqslant r_t^{-2} \leqslant 1$ and (2.3), we get

$$\frac{t^{-1}}{\phi(r^{-2})}\left(\frac{r}{r_t}\right)^{-2\alpha^*} \leqslant c^*.$$

Thus, (6.45) implies

$$q(t,x_0,y_0) \leqslant c_9(k)2^{d+2}t|x_0-y_0|^{-d}\phi(|x_0-y_0|^{-2}). \tag{6.46}$$

Finally, (6.43) and (6.46) yield relation (6.41) for all $t \geqslant 1$ and $x \neq y$. Keeping in mind Lemma 6.3 we conclude the result. ■

6.2.2. Full upper estimate

In this paragraph we establish the upper bound for the transition probability of the random walk S_n^ϕ. We follow approach of [5], cf. also [22], which is based on the application of the hitting time estimates. We start with results for the process Y and then we exploit them to obtain bounds for S_n^ϕ. Recall that $\tau^Y(x,r) = \inf\{t \geqslant 0 : Y_t \notin B(x,r)\}$.

Proposition 6.15. There exists a constant $C_{14} > 0$ such that

$$\mathbb{P}^x(\tau^Y(x,r) \leqslant t) \leqslant C_{14} t \phi(r^{-2}),$$

for all $x \in \mathbb{Z}^d$, $r > 0$ and $t \geqslant 1$.

Proof. By Proposition 6.2 and Lemma 2.17, we get

$$\mathbb{P}^x(|Y_t - x| \geqslant r) \leqslant c_1 t \sum_{y \in B(x,r)^c} |x-y|^{-d} \phi(|x-y|^{-2}) \leqslant c_2 t \phi(r^{-2}),$$

for all $x \in \mathbb{Z}^d$, $r > 0$ and $t \geqslant 1$. For simplicity we write $\tau = \tau^Y(x,r)$. Thus, by (2.2),

$$\begin{aligned}
\mathbb{P}^x(\tau \leqslant t) &= \mathbb{P}^x(\tau \leqslant t, |Y_{2t} - x| \leqslant r/2) + \mathbb{P}^x(\tau \leqslant t, |Y_{2t} - x| > r/2) \\
&\leqslant \mathbb{P}^x(\tau \leqslant t, |Y_{2t} - Y_\tau| \geqslant r/2) + \mathbb{P}^x(|Y_{2t} - x| > r/2) \\
&\leqslant \mathbb{E}^x\left[\mathbb{1}_{\{\tau \leqslant t\}} \mathbb{P}^{Y_\tau}(|Y_{2t-\tau} - Y_0| \geqslant r/2)\right] + c_2 2t \phi((r/2)^{-2}) \\
&\leqslant \mathbb{E}^x\Big[\mathbb{1}_{\{\tau \leqslant t\}} \sup_{y \in B(x,r)^c} \sup_{s \leqslant t} \mathbb{P}^y(|Y_{2t-s} - y| \geqslant r/2)\Big] + 2c_2 t \phi(4r^{-2}) \\
&\leqslant 2c_2 t \phi(4r^{-2}) \mathbb{E}^x\left[\mathbb{1}_{\{\tau \leqslant t\}}\right] + 2c_2 t \phi(4r^{-2}) \leqslant C_{14} t \phi(r^{-2}),
\end{aligned}$$

as desired. ■

We use the notation

$$\mathscr{T}^Y(x,r) = \inf\{t \geqslant 0 : Y_t \in B(x,r)\} \quad \text{and} \quad \mathscr{T}^{S^\phi}(x,r) = \inf\{k \in \mathbb{N}_0 : S_k^\phi \in B(x,r)\}$$

and we recall that $r_t = \left(\phi^{-1}(t^{-1})\right)^{-1/2}$, for $t \geq 1$.

Lemma 6.16. There exists a constant $C_{15} > 0$ such that

$$\mathbb{P}^x(\mathscr{T}^Y(y, r_t) \leqslant t) \leqslant C_{15} t r_t^d j(|x-y|), \tag{6.47}$$

for all $x, y \in \mathbb{Z}^d$ and $t \geqslant 1$.

Proof. We first show that there is $c_1 > 0$ such that

$$\mathbb{P}^z(\tau^Y(z, c_1 r_t) > t) \geqslant 1/2. \tag{6.48}$$

Indeed, we set

$$c_1 = 1 \vee \left(\frac{2C_{14}}{c_*}\right)^{1/2\alpha_*},$$

where C_{14} comes from Proposition 6.15. Using Proposition 6.15 and (2.3) we get

$$\mathbb{P}^z(\tau^Y(z, c_1 r_t) \leqslant t) \leqslant C_{14} t \phi((c_1 r_t)^{-2}) \leqslant \frac{C_{14}}{c_* c_1^{2\alpha_*}} \leqslant \frac{1}{2}.$$

We now consider the case $|x-y| \leqslant 2(1+c_1)r_t$. By monotonicity of $j(r)$ and relation (2.11), we get

$$\begin{aligned} t r_t^d j(|x-y|) &\geqslant t r_t^d j(2(1+c_1)r_t) \geqslant (2(1+c_1))^{-(d+2)} \\ &\geq (2(1+c_1))^{-(d+2)} \mathbb{P}^x(\mathscr{T}^Y(y, r_t) \leqslant t). \end{aligned}$$

Therefore

$$\mathbb{P}^x(\mathscr{T}^Y(y, r_t) \leqslant t) \leqslant C'_{15} t r_t^d j(|x-y|), \tag{6.49}$$

with $C'_{15} = (2(1+c_1))^{d+2}$.

Next, we consider the case $|x-y| > 2(1+c_1)r_t$. We write $\mathscr{T} = \mathscr{T}^Y(y, r_t)$. Using the strong Markov property and (6.48) we get

$$\begin{aligned} \mathbb{P}^x\Big(\mathscr{T} \leqslant t, \sup_{\mathscr{T} \leqslant s \leqslant \mathscr{T}+t} |Y_s - Y_{\mathscr{T}}| \leqslant c_1 r_t\Big) &= \mathbb{P}^{Y_{\mathscr{T}}}\Big(\sup_{s \leqslant t} |Y_s - Y_0| \leqslant c_1 r_t\Big) \mathbb{P}^x(\mathscr{T} \leqslant t) \\ &\geqslant \frac{1}{2} \mathbb{P}^x(\mathscr{T} \leqslant t). \end{aligned} \tag{6.50}$$

If $\mathscr{T} \leqslant t$ and $\sup_{\mathscr{T} \leqslant s \leqslant \mathscr{T}+t} |Y_s - Y_{\mathscr{T}}| \leqslant c_1 r_t$ then $|Y_t - Y_{\mathscr{T}}| \leqslant c_1 r_t$. As $\mathscr{T}$ is the first moment when the process Y_t hits the ball $B(y, r_t)$, it follows that

$$|Y_t - y| \leqslant |Y_t - Y_{\mathscr{T}}| + |Y_{\mathscr{T}} - y| \leqslant c_1 r_t + r_t = (1+c_1) r_t.$$

Combining these two inequalities with (6.50), we get

$$\mathbb{P}^x(\mathscr{T} \leqslant t) \leqslant 2\mathbb{P}^x(|Y_t - y| \leqslant (1+c_1)r_t) \leqslant 2 \sum_{z \in B(y,(1+c_1)r_t)} q(t,x,z). \tag{6.51}$$

Since $x \notin B(y, 2(1+c_1)r_t)$ and $z \in B(y, (1+c_1)r_t)$, we have $x \neq z$ and thus we can use (6.41). Notice also that $|x-z| \geqslant |x-y|/2$. This, monotonicity of j, (2.9) and (6.51) imply

$$\mathbb{P}^x(\mathscr{T} \leqslant t) \leqslant c_2 t \sum_{z \in B(y,(1+c_1)r_t)} j(|x-z|) \leqslant C''_{15} t r_t^d j(|x-y|). \tag{6.52}$$

Relations (6.49) and (6.52) yield the result. ■

Proposition 6.17. There exists a constant $C_{16} > 0$ such that

$$\mathbb{P}^x(\mathscr{T}^{S^\phi}(y,r_n) \leqslant n) \leqslant C_{16} n r_n^d j(|x-y|),$$

for all $x, y \in \mathbb{Z}^d$ and $n \in \mathbb{N}$.

Proof. As before $(T_k)_{k\in\mathbb{N}_0}$ stand for the arrival times of the Poisson process $(N_t)_{t\geqslant 0}$ that was used to define the process Y. More precisely, $N_t = k$ for all $T_k \leqslant t < T_{k+1}$. Using the Markov inequality, we easily get that $\mathbb{P}(T_n \leqslant 2n) \geqslant \frac{1}{2}$. By independence, Lemma 6.16 and (2.4), we obtain

$$\begin{aligned}\frac{1}{2}\mathbb{P}^x\big(\mathscr{T}^{S^\phi}(y,r_n) \leqslant n\big) &\leqslant \mathbb{P}^x\big(\mathscr{T}^{S^\phi}(y,r_n) \leqslant n, T_n \leqslant 2n\big) \leqslant \mathbb{P}^x\big(\mathscr{T}^{Y}(y,r_n) \leqslant 2n\big) \\ &\leqslant \mathbb{P}^x\big(\mathscr{T}^{Y}(y,r_{2n}) \leqslant 2n\big) \leqslant 2C_{15} n r_{2n}^d j(|x-y|) = C_{16} n r_n^d j(|x-y|),\end{aligned}$$

as claimed. ■

In the following theorem we finally prove the upper bound for the transition probability of the random walk S^ϕ. In the proof we again apply the parabolic Harnack inequality.

Theorem 6.18. There exists a constant $C > 0$ such that

$$p^\phi(n,x,y) \leqslant C\Big(\big(\phi^{-1}(n^{-1})\big)^{d/2} \wedge \frac{n}{|x-y|^d}\phi(|x-y|^{-2})\Big),$$

for all $x, y \in \mathbb{Z}^d$ and $n \in \mathbb{N}$.

Proof. By Proposition 6.17 we have for all $k \in \mathbb{N}$

$$\sum_{z\in B(y,r_k)} p^\phi(k,x,z) \leqslant \mathbb{P}^x(\mathscr{T}^{S^\phi}(y,r_k) \leqslant k) \leqslant C_{16} k r_k^d j(|x-y|).$$

On the other hand

$$\sum_{z\in B(y,r_k)} p^\phi(k,x,z) \geqslant c' r_k^d \min_{z\in B(y,r_k)} p^\phi(k,x,z).$$

Hence

$$\min_{z\in B(y,r_k)} p^\phi(k,x,z) \leqslant c_1 k j(|x-y|). \tag{6.53}$$

Next we apply the parabolic Harnack inequality. We choose $R > 0$ to satisfy $\gamma/\phi(R^{-2}) = n$, where γ is the constant from Theorem 5.1. Remember that we can choose γ to be even smaller than specified in the theorem. Thus we take $\gamma \leqslant B^{-2}$ where B is the constant defined in (5.4). By (2.2) we easily get that $r_n \leqslant R/B$. By Lemma 5.3, the function $q(k,w) = p^\phi(bn -$

$k,x,w)$ is parabolic on $\{0,1,2,\dots,bn\}\times\mathbb{Z}^d$, where b is defined at (5.4). With our choice $bn \geqslant \lfloor \gamma/\phi((\sqrt{b}R)^{-2})\rfloor$ and thus the function q is parabolic on $\{0,1,2,\dots,\lfloor\gamma/\phi((\sqrt{b}R)^{-2})\rfloor\}\times\mathbb{Z}^d$. By (6.53), we get

$$\min_{z\in B(y,R/B)} q(0,z) = \min_{z\in B(y,R/B)} p^{\phi}(bn,x,z) \leqslant \min_{z\in B(y,r_n)} p^{\phi}(bn,x,z) \leqslant c_1 bnj(|x-y|). \tag{6.54}$$

Choosing n big enough we can enlarge R so that we can apply Theorem 5.4. Hence

$$\max_{(k,z)\in Q(\lfloor\gamma/\phi(R^{-2})\rfloor,y,R/B)} q(k,z) \leqslant C_{PH} \min_{z\in B(y,R/B)} q(0,z).$$

Since $n=\gamma/\phi(R^{-2})$, it is clear that $(n,y)\in Q(\lfloor\gamma/\phi(R^{-2})\rfloor,y,R/B)$. Combining this with (6.54), we obtain

$$\begin{aligned} p^{\phi}((b-1)n,x,y) = q(n,y) &\leqslant \max_{(k,z)\in Q(\lfloor\gamma/\phi(R^{-2})\rfloor,y,R/B)} q(k,z) \leqslant C_{PH} \min_{z\in B(y,R/B)} q(0,z) \\ &\leqslant C_{PH}c_1 bnj(|x-y|) = c_2(b-1)nj(|x-y|). \end{aligned} \tag{6.55}$$

Similarly as in the proof of Theorem 6.1, we can show that this is enough to get the desired upper bound for all $n\in\mathbb{N}$. Finally, we have

$$p^{\phi}(n,x,y) \leqslant c_3 nj(|x-y|),$$

for all $x,y\in\mathbb{Z}^d$, $x\neq y$ and $n\in\mathbb{N}$. This combined with Corollary 4.2 yields the result. ■

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CURRICULUM VITAE

Stjepan Šebek was born on August 14th 1990 in Sisak, Croatia. After primary school, he moves to Zagreb where he graduates from 15th Gymnasium in 2009. In 2012 he receives a bachelor degree in Mathematics and in 2014 a masters degree in Mathematical Statistics from the Department of Mathematics, Faculty of Science, University of Zagreb, under the supervision of prof. dr. sc. Zoran Vondraček. During his studies, he is helping assistant at the courses of Statistics, Probability, Linear algebra 1 and Linear algebra 2 and he receives awards for exceptional success during the bachelor and master program.

In 2014 he enrolls in the PhD program in Mathematics at the University of Zagreb and becomes a member of the Zagreb Probability theory seminar. He also starts working as a research and teaching assistant at the Faculty of Electrical Engineering and Computing, University of Zagreb.

He participated in the project *Stochastic methods in analytical and applied problems*, funded by the Croatian Science Foundation (project leader: prof. dr. sc. Zoran Vondraček) from 2014 to 2018. He is currently a member of the project *Discontinuous stochastic processes and nonlocal operators*, funded by the Croatian Science Foundation (project leader: prof. dr. sc. Zoran Vondraček) and of the MSES-DAAD bilateral project with Germany *Random time-change and jump processes* (project leaders: prof. dr. sc. René Schilling and dr. sc. Nikola Sandrić).

Talks and posters

04/2019	*CLT for the capacity of the range of stable random walks*, Graz University of Technology, Graz, Austria
10/2018	*Mathematics in the service of successful business*, Power of Data 2018, Zagreb, Croatia
09/2018	*Harnack inequality for subordinate random walks*, 9th International Conference on Stochastic Analysis and Its Applications, Bielefeld University, Bielefeld, Germany
07/2018	*Harnack inequality for subordinate random walks*, 48th Probability Summer School, Saint-Flour, France
05/2018	*Subordinate random walks and Harnack inequality*, Probabilistic Aspects of Harmonic Analysis 2018, Bedlewo, Poland
02/2018	*Harnack inequality for subordinate random walks*, Symposium of students of doctoral studies of Faculty of Science, University of Zagreb, Zagreb, Croatia
12/2017	*Subordinate random walks and Harnack inequality*, Technische Universität Dresden, Dresden, Germany
07/2017	*Subordinate random walks*, Graz University of Technology, Graz, Austria
06/2017	*Harnack inequality for subordinate random walks*, PIMS-CRM Summer School in Probability 2017, University of British Columbia, Vancouver, Canada
05/2017	*Harnack inequality for subordinate random walks*, Probability and Analysis 2017, Bedlewo, Poland
10/2016	*Harnack inequality for subordinate random walks*, DK Seminar, Technische Universität Wien, Vienna, Austria

Preprints

1. W. Cygan, N. Sandrić and S. Šebek: *CLT for the capacity of the range of stable random walks*, 2019, https://arxiv.org/abs/1904.05695

2. W. Cygan and S. Šebek: *Transition probability estimates for subordinate random walks*, 2018, https://arxiv.org/abs/1812.03471

Papers

1. A. Mimica and S. Šebek: *Harnack inequality for subordinate random walks*, Journal of Theoretical Probability, Volume 32(2), 737 – 764, 2019, https://doi.org/10.1007/s10959-018-0821-5

ŽIVOTOPIS

Stjepan Šebek rođen je 14.8.1990. u Sisku. Nakon osnovne škole, seli u Zagreb gdje 2009. godine završava 15. gimnaziju. 2012. godine završava preddiplomski studij Matematike, a 2014. diplomski studij Matematičke statistike na Matematičkom odsjeku Prirodoslovno-matematičkog fakulteta Sveučilišta u Zagrebu, pod mentorstvom prof. dr. sc. Zorana Vondračeka. Tijekom studija drži demonstrature iz kolegija Statistika, Vjerojatnost, Linearna algebra 1 i Linearna algebra 2 te dobiva nagrade za izvrstan uspjeh tijekom preddiplomskog i diplomskog programa.

2014. godine upisuje Poslijediplomski doktorski studij matematike na Sveučilištu u Zagrebu i postaje član seminara za Teoriju vjerojatnosti. Iste godine počinje raditi i kao asistent na Fakultetu elektrotehnike i računarstva u Zagrebu, Sveučilište u Zagrebu.

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