

Generating functions for the Racah decomposition

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Generating functions are given for the angular momentum states of prescribed seniority of an assembly of identical particles, both for bosons and for fermions. The method is extended for multilevel configurations. Possible applications are discussed.

The problem of determining the multiplicities of angular momentum in a system of identical particles is an old one and standard solutions appear in textbooks.^{2,8} In mathematical terms it corresponds to finding the multiplicities of SU(2) representations appearing in the decomposition

$$SU(2j+1) \supset SU(2) \supset SO(2) \tag{1}$$

of the fully symmetric $\{n\}$ or fully antisymmetric representation $\{1^n\}$ of SU(2j+1). This problem has recently been solved with the use of generating functions.¹

The purpose of this article is to report on a further development of the generating function method. It is now possible to treat the full Racah decomposition²

$$SU(2j+1) \supset R(2j+1) \supset SU(2) \supset SO(2) \tag{2}$$

with R the seniority group, orthogonal for bosons and symplectic for fermions.^{2,3}

It has been shown¹ that the generating function corresponding to (1) is

$$G(r, n; q) = q^{-n(r-n)/2} \begin{bmatrix} r \\ n \end{bmatrix}_q, \tag{3}$$

with

$$r = 2j + 1 \text{ for fermions,}$$

$$r = 2l + n \text{ for bosons.}$$

The square brackets in (3) denote a Gaussian polynomial⁴ in q of order $2J_{\max} = n(r-n)$:

$$\begin{bmatrix} r \\ n \end{bmatrix}_q = \frac{(1-q^r)(1-q^{r-1}) \cdots (1-q^{r-n+1})}{(1-q)(1-q^2) \cdots (1-q^n)}. \tag{4}$$

Then the coefficient of q^M in (3) is equal to the multiplicity of angular momentum projection M appearing in (1).

The extension to (2) is based on a simple observation, often mentioned in the literature: If an angular momentum state of an n -particle configuration contains at least one zero-coupled pair, then that state also appears in the $(n-2)$ -particle configuration, since the zero-coupled pair does not contribute to the angular momentum. Therefore the generating functions are

$$V_b(2l+n, n; q) = G(2l+n, n; q) - G(2l+n-2, n-2; q), \tag{5}$$

$$V_f(2j+1, n; q) = G(2j+1, n; q) - G(2j+1, n-2; q) \tag{6}$$

for bosons and fermions, respectively, with $n \leq \frac{1}{2}(2j+1)$ for fermions, because of Pauli blocking. Incidentally, we note that the binomial coefficient symmetry $n \rightarrow r-n$ is valid for (3), giving the particle-hole symmetry for fermions. [The bosonic analog is a symmetry between n and $2l$. The author is unaware of any physical interpretation of this. Also, note that (3) allows one to construct a system of bosons with the same angular momentum distribution as a given system of fermions, and vice versa. The spin-statistics relation is not always satisfied in this construction.]

The coefficient of q^M in (5) or (6) gives the multiplicity of SO(2) states with weight equal to M and seniority $s=n$. In this way the multiplicity problem of Eq. (2) is solved.

The polynomials (5) and (6) are well defined under the usual convention that (4) vanishes when r or n become negative. By the interpretation given above, they must be unimodal, the coefficient of q^0 being largest. Interestingly enough, they are unimodal only for the correct spin-statistics relation. The reason is that a zero-coupled pair vanishes identically for particles violating spin statistics, and the interpretation leading to (5) and (6) fails.

Assuming $s \geq 2$, the seniority generating functions are explicitly

$$V_b(r, s; q) = q^{-s(r-s)/2} \begin{bmatrix} r-2 \\ s-2 \end{bmatrix}_q \times \frac{(1-q^{r+s-1})(1-q^{r-s})}{(1-q^{s-1})(1-q^s)}, \tag{5'}$$

$$V_f(r, s; q) = q^{-s(r-s)/2} \begin{bmatrix} r \\ s-2 \end{bmatrix}_q \times \frac{(1-q^{r-2s+2})(1-q^{r+1})}{(1-q^{s-1})(1-q^s)}. \tag{6'}$$

Note the formal similarity with (4). Only the first two (last two) terms in the numerator of the Gaussian polynomial (4) have been changed in V_b (V_f).

An algorithm was previously given¹ to calculate the coefficients in (3). If the coefficient of $q^{m-J_{\max}}$ in (3), (5),

or (6) is called c_m , the general recursion is

$$mc_m = p_1 c_{m-1} + p_2 c_{m-2} + \dots + p_{m-1} c_1 + p_m, \quad (7)$$

with $c_0 = 1$. To obtain the coefficients in (5) or (6), the p_k are specialized:

$$p_k = \sum_{a=1}^s a - \sum_{a=r-s}^{r-2} a - \{r+s-1; (r+s-1) | k\}, \quad (5'')$$

$$p_k = \sum_{a=1}^s a - \sum_{a=r-s+3}^{r+1} a - \{r-2s+2; (r-2s+2) | k\}, \quad (6'')$$

where $a | k$ means that a should divide k in order to contribute to p_k , and similarly for the term in curly brackets. This should be compared with the p_k that give¹ (3):

$$p_k = \sum_{a=1}^s a - \sum_{a=r-s+1}^r a, \quad (3')$$

where we have set $n = s$ in (3) to emphasize the similarity.

The extension to the multilevel case works exactly as in Ref. 5.⁶ If there are z levels, then

$$p_k = \sum_{i=1}^z p_k^{(i)} \quad (8)$$

will give the multiplicities for the total system via (7), where now the levels are specified by seniority and not by number of particles. This means that a given configuration will give rise to as many distributions as there are combinations of seniority among the levels.

To calculate the distribution, the p_k must be obtained up to $k = J_{\max}$. Then, since the recursion (7) must be executed *da capo* for each new c_m , the algorithm is quadratic⁶ in J_{\max} . If only the J multiplicities are required (as is usual), it may be noted that the necessary subtraction of successive c_m 's corresponds to a multiplication of the generating function by $(1-q)$, as far as the first J_{\max} coefficients are concerned. This, in turn, corresponds to replacing

$$p_k \rightarrow p_k - 1 \quad (9)$$

in (7). Then the c_m will give the J multiplicities directly.

As an exercise, the reader may wish to extend the example of Ref. 5, which gave the total multiplicities for the configuration $(\frac{7}{2})^3(\frac{3}{2})^2(\frac{5}{2})^2(\frac{1}{2})^1(\frac{9}{2})^2$. There are 16 distinct seniority configurations.

The second example follows from the observation that the method just described constructs the basis which diagonalizes the Hamiltonian⁷

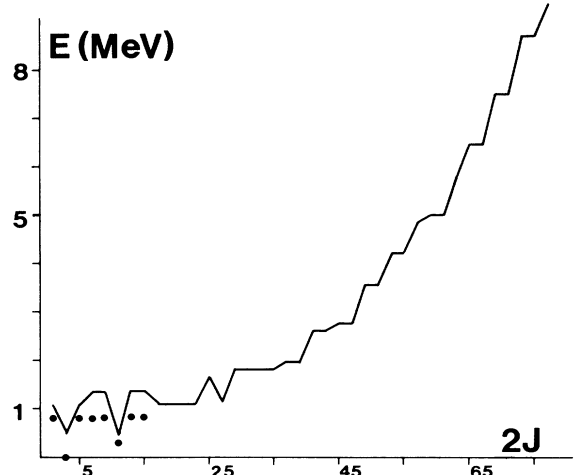


FIG. 1. Shell-model ground-state band for ^{137}Ce (solid line). Dots show yrast states that do not belong to it.

$$H = \sum_i n_i \epsilon_i + G \sum_i S_+^{(i)} S_-^{(i)}, \quad (10)$$

with the usual single-particle part and a pairing force within levels but not among levels. This means that the total seniority is conserved, which is not very realistic,⁸ but is the price paid for a considerable computational advantage, since the energy of (10) is given by a simple expression.⁷ A valence-shell demonstration was made for ^{137}Ce . It has eight protons and three neutron holes distributed over the five levels in the 50–82 shell. The complete spectrum of (10), containing 2.68×10^9 states distributed over 72 451 levels, was generated in 2.9 min of CPU time on a UNIVAC 1100. With Kisslinger-Sorensen energies⁹ and $G = \frac{23}{137}$, part of the results is shown in Fig. 1. The yrast band is indeed “wiggly and kinky,”¹⁰ but our point here is primarily in the states below the shell model ground-state band. They show that the correlations in (10) may be sufficient to destabilize the shell model ground state, and thus model the essential feature of superfluidity.

In conclusion, the method presented is computationally efficient enough to hold promise as an alternative approach to the shell-model calculation of level densities. Since it is exact, it could also be used to investigate the particle-number dependence of the normal-to-superfluid phase transition. Work along these lines is in progress.

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¹D. K. Sunko and D. Svrtnan, Phys. Rev. C 31, 1929 (1985).

²A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic, New York, 1963).

³J. C. Parikh, *Group Symmetries in Nuclear Structure* (Plenum, New York, 1978).

⁴G. Andrews, *Theory of Partitions* (Addison-Wesley, New York, 1976).

⁵D. K. Sunko, Phys. Rev. C 33, 1811 (1986).

⁶Reference 5 contains an error, the algorithm given being quadratic in J_{\max} and not linear as claimed.

⁷P. Ring and P. Schuck, *The Nuclear Many Body Problem* (Springer-Verlag, New York, 1980), pp. 223 and 449.

⁸R. D. Lawson, *Theory of the Nuclear Shell Model* (Clarendon, Oxford, 1980), p. 83.

⁹L. A. Kisslinger and R. A. Sorensen, *Rev. Mod. Phys.* **35**, 853 (1963).

¹⁰J. R. Grover, *Phys. Rev.* **157**, 832 (1966), especially p. 839.