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Closed $SU(2)_q$ invariant spin chain and its operator content

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We derive the operator content of the closed $SU(2)_q$ invariant quantum chain for generic values of the deformation parameter q. [S0556-2821(98)05724-5]

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I. INTRODUCTION

Integrable models and, in particular, spin chains are very interesting in studying various theoretical ideas. In particular in the continuum limit they describe various relativistic field theories; e.g., XYZ spin chain is connected to massive Thirring model [1]. The possible relevance of spin chains for high energy QCD was suggested in [2]. Recently their connection to matrix models was emphasized [3].

Among spin chains a particularly interesting class is the quantum group invariant spin chains. The closed $SU(2)_q$ invariant Hamiltonian was constructed in Ref. [4]. Further investigation showed that this Hamiltonian implied boundary conditions which depend on the coupling constant and quantum numbers of the sector [5,6]. This second property together with the property that the conformal anomaly was found to be smaller than one made this Hamiltonian different from the XXZ chain with the toroidal boundary conditions [7,9]. The model showed also the interesting properties of level crossing including the change in properties of the ground state [5]. The properties of the ground state have been investigated by Bethe ansatz methods. Bethe constraint equations have also been written for other groups [6,10,11]. The excited spectrum for $SU(2)_q$ was previously studied for rational values of the deformation parameter q [12] and a particular class of statistical models belonging to a unitary minimal series was projected. Here we want to address the question of the operator content for generic values of the parameter q. Here, as opposed to the rational case, there are no null states corresponding to ''bad'' representations of $SU(2)_q$ and we shall be able to exploit that representation theory as isomorphic to the ordinary $SU(2)$.

II. STATISTICAL SYSTEMS AND THE QUANTUM CHAIN

We start with the Hamiltonian for the closed $SU(2)_q$ invariant chain [4,5]

$$H=Lq-\sum_{i=1}^{L-1} R_i-R_0 \tag{1}$$

$$R_0=GR_{L-1}G^{-1}$$

$$G=R_1\cdots R_{L-1},$$

where R_i are 4×4 matrices

$$R_i=\sigma_i^+\sigma_{i+1}^-+\sigma_i^-\sigma_{i+1}^+ + \frac{q+q^{-1}}{4}(\sigma_i^3\sigma_{i+1}^3+1)-\frac{q-q^{-1}}{4}(\sigma_i^3-\sigma_{i+1}^3-2).$$

We choose the quantum group parameter q to be on the unit circle:

$$q=e^{i\varphi}$$

$$\frac{q+q^{-1}}{2}=\cos\varphi=-\cos\gamma.$$

The operator G plays the role of the translation operator

$$GR_iG^{-1}=R_{i+1},\quad R_L=R_0,\quad i=1,\ldots,L-1$$

and also commutes with the Hamiltonian. Contrary to [12], here we shall be interested in the generic case:

$$q^n\neq\pm1,\quad \forall n\in\mathbf{Z}.$$

In this case, one can decompose the space of states into the direct sum of irreducible representations of the quantum group which are in one-to-one correspondence with the usual $SU(2)$ representations. It is therefore sufficient to treat the highest weight states. All other states can be obtained with the action of the S^- operator. We derived the Bethe ansatz (BA) equation in [5]. The energy eigenvalues are given by

$$E=2\sum_{i=1}^{M}(\cos\varphi-\cos k_i),\quad M=\frac{L}{2}-Q.$$

Here Q is the eigenvalue of S^3 and k_i satisfy the BA constraints

$$Lk_i=2\pi I_i+2\varphi(Q+1)-\sum_{\substack{j=1\\ j\neq i}}^{M}\Theta(k_i,k_j),\quad k_i\neq\varphi, \tag{2}$$

where I_i are integers (half-integers) if M is odd (even), and $\Theta(k_i,k_j)$ is the usual two-particle phase defined in [5].

Owing to the antisymmetry of phase shifts, from Eq. (2) it follows that

$$\sum_{i=1}^{M}k_i=\frac{2\pi}{L}\sum_{i=1}^{M}I_i+\frac{2M}{L}\varphi(Q+1).$$

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This allows us to determine the eigenvalues of the translation operator G or equivalently of the operator P

$$P = i \ln G .$$

In fact,

$$P = \sum_{i=1}^{M} k_i - \varphi\left(Q - 1 + \frac{L}{2}\right)$$

$$= \frac{2\pi}{L}\sum_{i=1}^{M} I_i + \varphi\left[-\frac{L}{2} - Q + 1 + \frac{2M}{L}(Q+1)\right].$$

It was also shown in [5] that the finite-size correction to the thermodynamic limit of the ground-state energy was given by (L even)

$$E_0(L) = E_0(\infty) - \frac{\pi c\, \zeta}{6L} + O\left(\frac{1}{L}\right),$$

where

$$\zeta = \frac{\pi \sin\gamma}{\gamma}.$$

The conformal anomaly c was found to be

$$c = 1 - \frac{6(\pi - \varphi)^2}{\pi\varphi}$$

for $\varphi \in [\pi/2, \pi]$. If we parametrize φ as

$$\varphi = \frac{\pi m}{m+1}, \quad m \geqslant 1 , \tag{3}$$

then we have (m is irrational)

$$c = 1 - \frac{6}{m(m+1)}. \tag{4}$$

Now we define scaled gaps

$$\overline{E}_n = \frac{L}{2\pi\zeta}(E_n - E_0)$$

$$\overline{P}_n = \frac{L}{2\pi}(P_n - P_0 + \varphi Q) .$$

We introduce the partition function in some sector $Q \geqslant 0$:

$$\mathcal{F}_Q(z,\overline{z},L) = \sum_{\text{all states}} z^{(1/2)(\overline{E}_n + \overline{P}_n)}\, \overline{z}^{(1/2)(\overline{E}_n - \overline{P}_n)}.$$

As mentioned above, we are interested in the partition function for the highest weight states:

$$\mathcal{D}_Q(z,\overline{z},L) = \sum_{\substack{\text{highest} \\ \text{weight states}}} z^{(1/2)(\overline{E}_n + \overline{P}_n)}\, \overline{z}^{(1/2)(\overline{E}_n - \overline{P}_n)}$$

$$= \mathcal{F}_Q(z,\overline{z},L) - \mathcal{F}_{Q+1}(z,\overline{z},L) .$$

III. QUANTUM CHAIN AND THE XXZ CHAIN WITH A TOROIDAL BOUNDARY CONDITION

To determine $\mathcal{D}_Q$ in the limit $L \to \infty$, we can use results for the XXZ chain with a toroidal boundary condition [7,9]. The Hamiltonian is defined by

$$H(q,\phi,L) = -\sum_{i=1}^{L}\left\{\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{q+q^{-1}}{4}\sigma_i^3\sigma_{i+1}^3\right\} \tag{5}$$

$$\frac{q+q^{-1}}{2} = \cos\varphi = -\cos\gamma$$

and

$$\sigma_{L+1}^{\pm} = e^{\mp i\phi}\sigma_1^{\pm}, \quad \phi \in (-\pi, \pi].$$

This Hamiltonian commutes with

$$S^z = \sum_{i=1}^{L} \sigma_i^3$$

and with the translation operator

$$T = e^{-i\phi\sigma_1^3/2} P_1 P_2 \cdots P_{L-1},$$

where P_i, $i = 1, \ldots, L-1$ are permutation operators

$$P_i = \sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{1}{2}(\sigma_i^3\sigma_{i+1}^3 + 1) .$$

The momentum operator is then

$$P = i \ln T .$$

The BA constraints for this system are [8]

$$Lk_i = 2\pi I_i + \phi - \sum_{\substack{j=1 \\ j\neq i}}^{M} \Theta(k_i, k_j), \quad i = 1, \ldots, M \tag{6}$$

and give

$$E = -\frac{L}{2}\cos\varphi + 2\sum_{i=1}^{M}(\cos\varphi - \cos k_i)$$

$$P = \sum_{i=1}^{M} k_i = \frac{2\pi}{L}\sum_{i=1}^{M} I_i + \frac{M}{L}\phi .$$

We define

$$\phi = 2\pi l, \quad -\frac{1}{2} < l \leqslant \frac{1}{2} \tag{7}$$

and

$$h = \frac{\pi}{4\varphi}.$$

Let us denote by $E^{l}_{Q;j}(L)$ and $P^{l}_{Q;j}(L)$ the eigenvalues of H and P in the sector $S^z=Q$ with a boundary condition l. It was shown [9] that it was possible to project theories with $c<1$ by choosing a new ground state with energy $E^{l_0}_{0;j_0}(L)$. The number $j_0 \geqslant 1$ was chosen in such a way that the new ground state gave the contribution $(z\overline{z})^{h(l_0+\nu_0)^2} = (z\overline{z})^{(1-c)/24}$ in the partition function. So the quantity $(l_0+\nu_0)$ is related to h by the condition

$$c = 1 - 24h(l_0+\nu_0)^2,$$

where $-\frac{1}{2} < l_0 \leqslant \frac{1}{2}$ and $\nu_0 \in \mathbf{Z}$. From Eq. (4) it follows that

$$l_0+\nu_0 = [4hm(m+1)]^{-1/2}.$$

Now new scaled gaps can be defined as

$$\overline{F}^{k}_{Q;j}(L) = \frac{L}{2\pi}\left(E^{k(l_0+\nu_0)}_{Q;j}(L) - E^{l_0}_{0;j_0}(L)\right) \tag{8}$$

$$\overline{P}^{k}_{Q;j}(L) = \frac{L}{2\pi} P^{k(l_0+\nu_0)}_{Q;j}(L).$$

The corresponding finite-size scaling partition function is[1]

$$\mathcal{F}^{k}_{Q}(z,\overline{z}) = \lim_{L\to\infty} \mathcal{F}^{k}_{Q}(z,\overline{z},L) = \lim_{L\to\infty} \sum_{j=1}^{\binom{L}{Q+L/2}} z^{(1/2)(\overline{F}^{k}_{Q;j}+\overline{P}^{k}_{Q;j})}\,\overline{z}^{(1/2)(\overline{F}^{k}_{Q;j}-\overline{P}^{k}_{Q;j})}.$$

According to [9,12], one subset of $c<1$ models compatible with Eq. (3) can be projected by imposing

$$l_0+\nu_0 = 1 - \frac{1}{4h} = \frac{1}{m+1}. \tag{9}$$

If we define

$$\mathcal{D}^{k}_{Q}(z,\overline{z},L) \equiv \mathcal{F}^{k}_{Q}(z,\overline{z},L) - \mathcal{F}^{Q}_{k}(z,\overline{z},L) \tag{10}$$

we obtain [9]

$$\mathcal{D}^{k}_{Q}(z,\overline{z}) \equiv \lim_{L\to\infty} \mathcal{D}^{k}_{Q}(z,\overline{z},L) = \sum_{r=1}^{\infty} \chi_{r,k-Q}(z)\,\chi_{r,k+Q}(\overline{z}). \tag{11}$$

[1]Partition functions connected to the toroidal XXZ chain (5) have superscript in addition to subscript, in contrast to those connected to our $SU(2)_q$ invariant chain (1).

Here $\chi_{r,s}$ are character functions of irreducible representations of the Virasoro algebra with highest weights $\Delta_{r,s}$ given by

$$\Delta_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}. \tag{12}$$

We should emphasize here that k gives the boundary condition, according to Eqs. (7), (8) and (9), to be

$$\phi = 2\pi l = 2\pi k(l_0+\nu_0) = 2\pi\frac{k}{m+1}. \tag{13}$$

IV. QUANTUM CHAIN AND CFT

Now we are going to make a connection with the quantum chain. To do that, first we must have the same Bethe equations. Comparing Eqs. (2) and (6), and using Eq. (3), we see that

$$\phi = 2\varphi(Q+1) \pmod{2\pi} = 2\pi\frac{m(Q+1)}{m+1} \pmod{2\pi} = -2\pi\frac{Q+1}{m+1} \pmod{2\pi}. \tag{14}$$

Comparing now Eqs. (14) and (13) it follows that we should take

$$k = -(Q+1). \tag{15}$$

If we define

$$H^{k}_{Q}(L) = H(q,\phi,L)\mathcal{P}_{Q}(L),$$

where $H(q,\phi,L)$ is the toroidal XXZ Hamiltonian (5) [with ϕ given by Eq. (13)] and $\mathcal{P}_Q$ is the projection operator on sector $S^z=Q$, from [13] and the Appendix of [9], it follows that

$$S^{+}H^{-(Q+1)}_{Q} = H^{-Q}_{Q+1}S^{+}, \qquad S^{+}T^{-(Q+1)}_{Q} = \pm T^{-Q}_{Q+1}S^{+}. \tag{16}$$

Therefore, due to the fact that when we make a difference in

$$\mathcal{D}^{-(Q+1)}_{Q} = \mathcal{F}^{-(Q+1)}_{Q} - \mathcal{F}^{Q}_{-(Q+1)} = \mathcal{F}^{-(Q+1)}_{Q} - \mathcal{F}^{-Q}_{Q+1}$$

[we have used charge symmetry generated by $C=\Pi\sigma^{x}_{j}$ which changes signs of S_z and ϕ; see Eq. (2.12) in [9]] we make a difference between spectra of $H^{-(Q+1)}_{Q}$ and H^{-Q}_{Q+1}, from Eq. (16) it follows that we obtain only the highest weight states of $SU(2)_q$. And these are exactly solutions of

Bethe equations which are the same for both chains. Thus the quantum chain partition function is

$$\begin{aligned}\mathcal{D}_Q(z,\bar{z}) &= \lim_{L\to\infty} \mathcal{D}_Q(z,\bar{z},L) \\ &= \mathcal{D}_Q^{-(Q+1)}(z,\bar{z}) \\ &= \sum_{r=1}^{\infty} \chi_{r,-(2Q+1)}(z)\, \chi_{r,-1}(\bar{z})\,.\end{aligned}$$

Using the formal relation $\chi_{r,-s} = -\chi_{r,s}$ we can write the final result

$$\mathcal{D}_Q(z,\bar{z}) = \sum_{r=1}^{\infty} \chi_{r,2Q+1}(z)\, \chi_{r,1}(\bar{z})\,. \tag{17}$$

In our case the number m appearing in Virasoro characters [see Eq. (12)] is connected to the $q=\exp(i\varphi)$ parameter with relation (3).

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