Higgs masses in the minimal supersymmetric SO(10) grand unified theory

Fukuyama, Takeshi; Ilakovac, Amon; Kikuchi, Tatsuru; Meljanac, Stjepan; Okada, Nobuchika

PHYSICAL REVIEW D **72,** 051701(R) (2005)

Higgs masses in the minimal supersymmetric $SO(10)$ grand unified theory

Takeshi Fukuyama,[1] Amon Ilakovac,[2] Tatsuru Kikuchi,[1] Stjepan Meljanac,[3] and Nobuchika Okada[4]
[1]*Department of Physics, Ritsumeikan University, Kusatsu, Shiga, 525-8577, Japan*
[2]*University of Zagreb, Department of Physics, P.O. Box 331, Bijenička cesta 32, HR-10002 Zagreb, Croatia*
[3]*Institut Rudjer Bošković, Bijenička cesta 54, P.O. Box 180, HR-10002 Zagreb, Croatia*
[4]*Theory Group, KEK, Oho 1-1, Tsukuba, Ibaraki 305-0801, Japan*
(Received 22 December 2004; published 14 September 2005)

We explicitly show that minimal supersymmetric $SO(10)$ Higgs-Higgsino mass matrices evaluated by various groups are mutually consistent and correct. We comment on the corresponding results of other authors. We construct one-to-one mappings of our approach to the approaches of other authors.

DOI: 10.1103/PhysRevD.72.051701 PACS numbers: 12.10.Kt, 02.20.Qs, 12.60.Jv

I. INTRODUCTION

There is a large interest in the minimal supersymmetric $SO(10)$ grand unified theory (GUT) [1–4] concerning neutrino masses [2–4], lepton-flavor violation processes [5], and proton decay [6,7]. To study the proton decay lifetime it is important to know Higgs-Higgsino masses which were analyzed in Refs. [6–11]. However, there are apparently different results for corresponding mass matrices [6–11] for $SO(10) \to G_{321}$ symmetry breaking with unbroken supersymmetry [$G_{321} \equiv SU(3)_c \times SU(2)_L \times U(1)_Y$ is the standard model gauge group].

In order to prove that the mass matrices in Ref. [6] are correct, we present a set of universal consistency checks that the mass matrices must satisfy in our approach. These are the trace of the total Higgs mass matrix, the hermiticity condition of the matrix of Clebsch-Gordan coefficients in each mass matrix and the higher symmetry group checks containing the standard model gauge group G_{321}. Further, finding explicit one-to-one correspondences between the results of Refs. [6–12], we prove the consistencies between the approaches considered.

II. SHORT SUMMARY OF THE MINIMAL SUSY $SO(10)$ GUT

The Yukawa sector of the minimal SUSY $SO(10)$ GUT has couplings of each generation of the matter multiplet with only the **10** and $\overline{\mathbf{126}}$ Higgs multiplets. The Higgs sector contains $\mathbf{10} \equiv H$, $\overline{\mathbf{126}} \equiv \overline{\Delta}$, $\mathbf{126} \equiv \Delta$, and $\mathbf{210} \equiv \Phi$ multiplets. The last two multiplets are necessary to achieve the correct $SO(10) \to G_{321}$ breaking. The Higgs superpotential reads [6]

$$W = m_1\Phi^2 + m_2\Delta\overline{\Delta} + m_3H^2 + \lambda_1\Phi^3 + \lambda_2\Phi\Delta\overline{\Delta} + \lambda_3\Phi\Delta H + \lambda_4\Phi\overline{\Delta}H. \tag{1}$$

We are interested in symmetry breaking $SO(10) \to G_{321}$. The G_{321} invariant vacuum expectation values (VEVs) are

$$\langle\Phi\rangle = \sum_{i=1}^{3}\phi_i\hat{\phi}_i; \qquad \langle\Delta\rangle = v_R\widehat{v_R}; \qquad \langle\overline{\Delta}\rangle = \overline{v_R}\,\widehat{\overline{v_R}}; \beta \tag{2}$$

where ϕ_i, $i = 1, 2, 3$, v_R, and $\overline{v_R}$ are complex VEV variables and $\hat{\phi}_i$, $i = 1, 2, 3$, $\widehat{v_R}$, and $\widehat{\overline{v_R}}$ are unit G_{321} invariant vectors, satisfying $\hat{\phi}_i\hat{\phi}_j = \delta_{ij}$, $\widehat{v_R}\,\widehat{\overline{v_R}} = 1$, $\widehat{v_R}^2 = \widehat{\overline{v_R}}^2 = 0$, described in Y diagonal basis [6,8] as

$$\begin{aligned}
\widehat{v_R} &= \frac{1}{\sqrt{120}}(24\,680), \qquad \widehat{\overline{v_R}} = \frac{1}{\sqrt{120}}(13\,579),\\
\hat{\phi}_1 &= -\frac{1}{\sqrt{24}}(1234),\\
\hat{\phi}_2 &= -\frac{1}{\sqrt{72}}(5678 + 5690 + 7890),\\
\hat{\phi}_3 &= -\frac{1}{12}([12 + 34][56 + 78 + 90]).
\end{aligned} \tag{3}$$

Inserting the VEVs (2) into the superpotential (1), one obtains

$$\begin{aligned}
\langle W\rangle &= m_1[\phi_1^2 + \phi_2^2 + \phi_3^2] + m_2 v_R\overline{v_R}\\
&\quad + \lambda_1\left[\phi_2^3\frac{1}{9\sqrt{2}} + 3\phi_1\phi_3^2\frac{1}{6\sqrt{6}} + 3\phi_2\phi_3^2\frac{1}{9\sqrt{2}}\right]\\
&\quad + \lambda_2\left[\phi_1\frac{1}{10\sqrt{6}} + \phi_2\frac{1}{10\sqrt{2}} + \phi_3\frac{1}{10}\right]v_R\overline{v_R},
\end{aligned} \tag{4}$$

which determines the VEV equations,

$$\begin{aligned}
0 &= 2m_1\phi_1 + \frac{\lambda_1\phi_3^2}{2\sqrt{6}} + \frac{\lambda_2 v_R\overline{v_R}}{10\sqrt{6}},\\
0 &= 2m_1\phi_2 + \frac{\lambda_1\phi_2^2}{3\sqrt{2}} + \frac{\lambda_1\phi_3^2}{3\sqrt{2}} + \frac{\lambda_2 v_R\overline{v_R}}{10\sqrt{2}},\\
0 &= 2m_1\phi_3 + \frac{\lambda_1\phi_1\phi_3}{\sqrt{6}} + \frac{\sqrt{2}\lambda_1\phi_2\phi_3}{3} + \frac{\lambda_2 v_R\overline{v_R}}{10},\\
0 &= v_R\overline{v_R}\left[m_2 + \frac{\lambda_2\phi_1}{10\sqrt{6}} + \frac{\lambda_2\phi_2}{10\sqrt{2}} + \frac{\lambda_2\phi_3}{10}\right].
\end{aligned} \tag{5}$$

In the following we will assume that $|v_R| = |\overline{v_R}|$ [6,9].

For $v_R = 0$ the solutions of the VEV Eqs. (5) are vacuum minima with $SU(5) \times U(1)$, $SU(5) \times U(1)^{\text{flipped}}$, G_{3221} and G_{3211} symmetry.

For $v_R \neq 0$ the VEV Eqs. (5) lead to the fourth-order equation in ϕ_1 (or ϕ_2 or ϕ_3). One of the solutions of that equation corresponds to the $SU(5)$ symmetry, while the remaining three have G_{321} symmetry [6].

The $SU(5)$ solution is given by

$$\phi_1 = -\frac{\sqrt{6}m_2}{\lambda_2}, \qquad \phi_2 = -\frac{3\sqrt{2}m_2}{\lambda_2},$$
$$\phi_3 = -\frac{6m_2}{\lambda_2}, \qquad v_R\overline{v_R} = \frac{60m_2^2}{\lambda_2^2}\left[2\left(\frac{m_1}{m_2}\right) - 3\left(\frac{\lambda_1}{\lambda_2}\right)\right]. \quad (6)$$

III. HIGGS MASS MATRICES

The mass matrices are defined as

$$\mathcal{M}_{ij} = \frac{\partial^2 W}{\partial\varphi_i \partial\overline{\varphi}_j}\bigg|_{\rm VEV}, \quad (7)$$

where φ_i represents any G_{321} multiplet. We point out that the physical masses squared are eigenvalues of $\mathcal{M}^\dagger\mathcal{M}$ and $\mathcal{M}\mathcal{M}^\dagger$ matrices.

The G_{321} mass matrices [6,8] that we use here are given in Ref. [6] [see formulas (4.1)–(4.5), (5.3), (6.4) and Tables I and II]. Phenomenologically the most interesting doublet and triplet mass matrices are given by equations (5.3) and (6.4) in Ref. [6], respectively.

IV. CONSISTENCY CHECKS

In Ref. [8] a detailed explanation of a method for calculation of the above matrices is given, and all possible consistency checks are briefly explained.

There are three main consistency checks.

The first is that the trace of the total Higgs mass matrix does not depend on the coupling constants λ_i, $i = 1, 2, 3, 4$. It depends only on mass parameters m_i, $i = 1, 2, 3$ and the dimensions of the corresponding $SO(10)$ representations. The sum rule for the Higgs-Higgsino mass matrices is

$$\mathrm{Tr}\,\mathcal{M} = 2m_1 \times 210 + m_2 \times 252 + 2m_3 \times 10. \quad (8)$$

The second is that the Clebsch-Gordan coefficients in all mass matrices satisfy hermiticity property.

The mass sum rule and the hermiticity property for the mass matrices are easily verified for the results for the mass matrices given in Ref. [6].

The third and main check is the $SU(5)$ check briefly described in the paper [8]. Here we explicitly prove that mass matrices in Ref. [6] satisfy this highly nontrivial test.

Let us insert the SU(5) solution (6) into mass matrices in Ref. [6] for general mass parameters m_i, $i = 1, 2, 3$ and coupling constants λ_i, $i = 1, 2, 3, 4$. Note that $\mathbf{10}$, $\overline{\mathbf{126}}$, $\mathbf{126}$, and $\mathbf{210}$ decompose under the $SU(5)$ symmetry as

$$\mathbf{10} = \mathbf{5} + \overline{\mathbf{5}}, \qquad \mathbf{126} = \mathbf{1} + \overline{\mathbf{5}} + \mathbf{10} + \overline{\mathbf{15}} + \mathbf{45} + \overline{\mathbf{50}},$$
$$\overline{\mathbf{126}} = \mathbf{1} + \mathbf{5} + \overline{\mathbf{10}} + \mathbf{15} + \overline{\mathbf{45}} + \mathbf{50},$$
$$\mathbf{210} = \mathbf{1} + \mathbf{5} + \overline{\mathbf{5}} + \mathbf{10} + \overline{\mathbf{10}} + \mathbf{24} + \mathbf{40} + \overline{\mathbf{40}} + \mathbf{75}. \quad (9)$$

In total, there are three singlets, three ($\mathbf{5} + \overline{\mathbf{5}}$), two ($\mathbf{10} + \overline{\mathbf{10}}$), one ($\mathbf{15} + \overline{\mathbf{15}}$), one $\mathbf{24}$, one ($\mathbf{40} + \overline{\mathbf{40}}$), one ($\mathbf{45} + \overline{\mathbf{45}}$), one ($\mathbf{50} + \overline{\mathbf{50}}$), and one $\mathbf{75}$. All together there are 14 $SU(5)$ representations which can form mass terms.

The corresponding mass-matrix eigenvalues are $m_1^G(\mathbf{1}) = m_1^G(\mathbf{10}) = 0$, corresponding to 21 would-be Goldstone modes, $m_{2,3}(\mathbf{1})$, $m_{1,2,3}(\mathbf{5})$, $m_2(\mathbf{10})$, $m_\Delta(\mathbf{15})$, $m_\phi(\mathbf{24})$, $m_\phi(\mathbf{40})$, $m_\Delta(\mathbf{45})$, $m_\Delta(\mathbf{50})$, and $m_\phi(\mathbf{75})$. Therefore, the $SU(5)$ decomposition of $\mathbf{10}$, $\overline{\mathbf{126}}$, $\mathbf{126}$, and $\mathbf{210}$ implies there are at most 13 different mass-matrix eigenvalues.

We found the $SU(5)$ mass-matrix eigenvalues analytically. These eigenvalues are obtained diagonalizing the 26 mass matrices corresponding to the 26 G_{321} multiplets contained in $\mathbf{10}$, $\overline{\mathbf{126}}$, $\mathbf{126}$, and $\mathbf{210}$. The results are given in Table I. In Table I the mass-matrix eigenvalues read

TABLE I. $SU(5)$ mass-matrix eigenvalues obtained from G_{321} mass matrices.

G_{321}	$SU(5)$ mass-matrix eigenvalues	G_{321}	$SU(5)$ mass-matrix eigenvalues
$(\mathbf{3}, \mathbf{2}, -\frac{5}{6})$	$m_\phi(\mathbf{24})$,$m_\phi(\mathbf{75})$	$(\mathbf{3}, \mathbf{2}, \frac{7}{6})$	$m_\Delta(\mathbf{45})$,$m_\Delta(\mathbf{50})$
$(\mathbf{3}, \mathbf{2}, \frac{1}{6})$	$m_1^G(\mathbf{10})$,$m_2(\mathbf{10})$,$m_\phi(\mathbf{40})$,$m_\Delta(\mathbf{15})$	$(\overline{\mathbf{3}}, \mathbf{1}, \frac{4}{3})$	$m_\Delta(\mathbf{45})$
$(\mathbf{3}, \mathbf{1}, \frac{2}{3})$	$m_1^G(\mathbf{10})$,$m_2(\mathbf{10})$,$m_\phi(\mathbf{40})$	$(\mathbf{1}, \mathbf{3}, 1)$	$m_\Delta(\mathbf{15})$
$(\mathbf{1}, \mathbf{1}, 1)$	$m_1^G(\mathbf{10})$,$m_2(\mathbf{10})$	$(\mathbf{1}, \mathbf{1}, 2)$	$m_\Delta(\mathbf{50})$
$(\mathbf{1}, \mathbf{1}, 0)$	$m_1^G(\mathbf{1})$,$m_2(\mathbf{1})$,$m_3(\mathbf{1})$,$m_\phi(\mathbf{24})$,$m_\phi(\mathbf{75})$	$(\mathbf{8}, \mathbf{3}, 0)$	$m_\phi(\mathbf{75})$
$(\mathbf{1}, \mathbf{2}, \frac{1}{2})_D$	$m_\Delta(\mathbf{45})$,$m_1(\mathbf{5})$,$m_2(\mathbf{5})$,$m_3(\mathbf{5})$	$(\mathbf{8}, \mathbf{1}, 1)$	$m_\phi(\mathbf{40})$
$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_T$	$m_\Delta(\mathbf{50})$,$m_\Delta(\mathbf{45})$,$m_1(\mathbf{5})$,$m_2(\mathbf{5})$,$m_3(\mathbf{5})$	$(\mathbf{8}, \mathbf{1}, 0)$	$m_\phi(\mathbf{24})$,$m_\phi(\mathbf{75})$
$(\mathbf{8}, \mathbf{2}, \frac{1}{2})$	$m_\Delta(\mathbf{45})$,$m_\Delta(\mathbf{50})$	$(\mathbf{6}, \mathbf{2}, \frac{5}{6})$	$m_\phi(\mathbf{75})$
$(\mathbf{6}, \mathbf{3}, \frac{1}{3})$	$m_\Delta(\mathbf{50})$	$(\mathbf{6}, \mathbf{2}, \frac{1}{6})$	$m_\phi(\mathbf{40})$
$(\mathbf{6}, \mathbf{1}, \frac{4}{3})$	$m_\Delta(\mathbf{50})$	$(\overline{\mathbf{3}}, \mathbf{3}, \frac{2}{3})$	$m_\phi(\mathbf{40})$
$(\overline{\mathbf{6}}, \mathbf{1}, \frac{2}{3})$	$m_\Delta(\mathbf{15})$	$(\mathbf{3}, \mathbf{1}, \frac{5}{3})$	$m_\phi(\mathbf{75})$
$(\mathbf{6}, \mathbf{1}, \frac{1}{3})$	$m_\Delta(\mathbf{45})$	$(\mathbf{1}, \mathbf{3}, 0)$	$m_\phi(\mathbf{24})$
$(\overline{\mathbf{3}}, \mathbf{3}, \frac{1}{3})$	$m_\Delta(\mathbf{45})$	$(\mathbf{1}, \mathbf{2}, \frac{3}{2})$	$m_\phi(\mathbf{40})$

$$m_\Delta(\mathbf{50}) = \frac{6}{5}m_2, \qquad m_\Delta(\mathbf{45}) = m_2, \qquad m_\Delta(\mathbf{15}) = \frac{4}{5}m_2, \qquad m_2(\mathbf{10}) = 2m_1 - 3m_2\frac{\lambda_1}{\lambda_2} + \frac{3}{5}m_2,$$
$$m_\phi(\mathbf{75}) = 2m_1 + 2m_2\frac{\lambda_1}{\lambda_2}, \qquad m_\phi(\mathbf{40}) = 2m_1, \qquad m_\phi(\mathbf{24}) = 2m_1 - m_2\frac{\lambda_1}{\lambda_2}, \tag{10}$$
$$m_{2,3}(\mathbf{1}) = m_1 - 3m_2\frac{\lambda_1}{\lambda_2} \pm \left[\left(m_1 - 3m_2\frac{\lambda_1}{\lambda_2}\right)^2 + 4m_1m_2 - 6m_2^2\frac{\lambda_1}{\lambda_2}\right]^{1/2}.$$

The remaining three mass-matrix eigenvalues $m_{1,2,3}(\mathbf{5})$ are solutions of the following cubic equation

$$0 = x^3 - x^2\left[2m_1 + 2m_3 + \frac{3m_2}{5} - \frac{6\lambda_1 m_2}{\lambda_2}\right] - x\left[\frac{36\lambda_3\lambda_4 m_2^2}{5\lambda_2^2} - \frac{36\lambda_1\lambda_3\lambda_4 m_2^2}{\lambda_2^3} + \frac{9\lambda_1 m_2^2}{5\lambda_2} + \frac{24\lambda_3\lambda_4 m_1 m_2}{\lambda_2^2} + \frac{12\lambda_1 m_3 m_2}{\lambda_2} - \frac{6m_3m_2}{5} - 4m_1m_3\right] - \left[\frac{108\lambda_1\lambda_3\lambda_4 m_2^3}{\lambda_2^3} - \frac{288\lambda_3\lambda_4 m_1 m_2^2}{5\lambda_2^2} - \frac{18\lambda_1 m_3 m_2^2}{5\lambda_2}\right]. \tag{11}$$

We point out that when the $SU(5)$ solution for VEVs (6) is inserted in 26 matrices in Ref. [6] G_{321} mass matrices we obtain 13 different mass-matrix eigenvalues, as predicted counting the $SU(5)$ multiplets in $\mathbf{10}$, $\overline{\mathbf{126}}$, $\mathbf{126}$, and $\mathbf{210}$, with different mass-matrix eigenvalues. That is a nontrivial test of the G_{321} mass matrices.

Moreover, the sum rule for the $SU(5)$ mass-matrix eigenvalues also holds

$$\begin{aligned}\mathrm{Tr}\mathcal{M} &= [m_2(\mathbf{1}) + m_3(\mathbf{1})] + [m_1(\mathbf{5}) + m_2(\mathbf{5}) + m_3(\mathbf{5})] \\ &\quad\times 10 + m_2(\mathbf{10}) \times 20 + m_\Delta(\mathbf{15}) \times 30 + m_\Delta(\mathbf{45}) \\ &\quad\times 90 + m_\Delta(\mathbf{50}) \times 100 + m_\phi(\mathbf{24}) \times 24 \\ &\quad+ m_\phi(\mathbf{40}) \times 80 + m_\phi(\mathbf{75}) \times 75 \\ &= 2m_1 \times 210 + m_2 \times 252 + 2m_3 \times 10.\end{aligned} \tag{12}$$

Substitution of the $SU(5)$ solution into G_{321} Higgs mass matrices leads, for example, to the following mass matrices for Higgs doublets $(\mathbf{1}, \mathbf{2}, \frac{1}{2})$

$$M_{\text{doublet}} \equiv \begin{pmatrix} 2m_3 & 0 & \frac{6\lambda_3 m_2}{\sqrt{5}\lambda_2} & \frac{2\sqrt{3}\lambda_4 m_2}{\lambda_2}A \\ 0 & m_2 & 0 & 0 \\ \frac{6\lambda_3 m_2}{\sqrt{5}\lambda_2} & 0 & \frac{3m_2}{5} & -\frac{\sqrt{3}m_2}{\sqrt{5}}A \\ \frac{2\sqrt{3}\lambda_3 m_2}{\lambda_2}A & 0 & -\frac{\sqrt{3}m_2}{\sqrt{5}}A & 2m_1 - \frac{6\lambda_1 m_2}{\lambda_2}\end{pmatrix}, \tag{13}$$

where $A = (\frac{2m_1}{m_2} - \frac{3\lambda_1}{\lambda_2})^{1/2}$, and color triplets $(\mathbf{3}, \mathbf{1}, -\frac{1}{3})$

$$M_{\text{triplet}} \equiv \begin{pmatrix} 2m_3 & M_{12} & 0 & M_{14} & M_{51} \\ M_{21} & m_2 & 0 & -\frac{m_2}{\sqrt{5}}A & -\frac{\sqrt{2}m_2}{5} \\ 0 & 0 & m_2 & 0 & 0 \\ M_{41} & -\frac{m_2}{\sqrt{5}}A & 0 & M_{44} & -\frac{\sqrt{2}m_2 A}{\sqrt{5}} \\ M_{51} & -\frac{\sqrt{2}m_2}{5} & 0 & -\frac{\sqrt{2}m_2 A}{\sqrt{5}} & \frac{4m_2}{5}\end{pmatrix}, \tag{14}$$

where

$$\begin{aligned} &M_{12} = \frac{2\sqrt{3}\lambda_4 m_2}{\sqrt{5}\lambda_2}, \qquad M_{21} = \frac{2\sqrt{3}\lambda_3 m_2}{\sqrt{5}\lambda_2}, \\ &M_{14} = 2\sqrt{3}\lambda_4 m_2 A, \qquad M_{41} = 2\sqrt{3}\lambda_3 m_2 A, \\ &M_{44} = 2m_1 - \frac{6\lambda_1 m_2}{\lambda_2}, \qquad M_{15} = \frac{2\sqrt{6}\lambda_4 m_2}{\sqrt{5}\lambda_2}, \\ &M_{51} = \frac{2\sqrt{6}\lambda_3 m_2}{\sqrt{5}\lambda_2}.\end{aligned} \tag{15}$$

Note that

$$\begin{aligned}\mathrm{Tr}\,\mathcal{M}_{\text{triplet}} &= m_\Delta(\mathbf{50}) + \mathrm{Tr}\mathcal{M}_{\text{doublet}}, \\ \det\mathcal{M}_{\text{triplet}} &= m_\Delta(\mathbf{50})\det\mathcal{M}_{\text{doublet}}.\end{aligned} \tag{16}$$

V. EQUIVALENCE TO OTHER APPROACHES

We show that the results of the Refs. [6–12] are consistent with each other, by giving the unique correspondences between the results of different authors. In Ref. [12] it was suggested that the issue may be connected to the different definitions of the fields. We show explicitly that with the correct field identifications the apparently different results are in accord with each other.

In order to make a contact with the results for the G_{321} mass matrices found in Ref. [9], we compare the VEV equations (6)–(9) of Ref. [9] and mass matrices given in Table XI and in Eqs. (B12)–(B19) of Ref. [9] with our VEVs Eqs. (5) (see also (3.10)–(3.13) in Ref. [6]) and mass matrices given in Ref. [6] by Eqs. (4.1)–(4.5), (5.3), (6.4) and in Tables I and II. From a comparison of these results in the two papers one finds the unique correspondence between parameters of the two papers,

$$\begin{aligned} &m_1 = m_\phi, \qquad m_2 = m_\Sigma, \qquad m_3 = m_H, \\ &\lambda_1 = \sqrt{24}\lambda, \qquad \lambda_2 = 10\sqrt{6}\eta, \qquad \lambda_3 = \sqrt{5}\alpha, \\ &\lambda_4 = \sqrt{5}\overline{\alpha}, \qquad \phi_1 = p, \qquad \phi_2 = \sqrt{3}a, \\ &\phi_3 = -\sqrt{6}\omega, \qquad v_R = \sigma_B \qquad \overline{v_R} = \overline{\sigma}_B.\end{aligned} \tag{17}$$

(Label B is introduced to distinguish the quantities of Ref. [9] from the equally named quantities in Ref. [11] which will be denoted by label A). Namely, if one performs the above substitution in our VEV equations and mass matrices, one gets the VEV equations as in Ref. [9]. Also, up to the phase redefinitions and simultaneous permutations of rows and columns, the same mass matrices as in Ref. [9] are obtained, except for the doublet $(\mathbf{1}, \mathbf{2}, \frac{1}{2})$ mass matrix. There is the reverse sign in all matrix elements in the fourth row of $(\mathbf{1}, \mathbf{2}, \frac{1}{2})$ mass matrix. This difference comes from an arbitrary choice of phases for states conjugated to each other. In our approach the phases of conjugate states are chosen to be related by complex conjugation.

Namely, if we multiply our results for the total mass matrix by a diagonal matrix of arbitrary phases $\mathcal{D}$ preserving G_{321} symmetry, we obtain matrix $\mathcal{M}'$, ($\mathcal{M}' = \mathcal{D}\mathcal{M}$ or $\mathcal{M}' = \mathcal{M}\mathcal{D}$) which then spoils all our consistency checks for $\mathcal{M}'$ but preserves validity of all our higher symmetry checks, except the trace check (16), for $(\mathcal{M}')^\dagger\mathcal{M}'$ and $\mathcal{M}'(\mathcal{M}')^\dagger$ matrices. The maximal number of arbitrary phases is equal to the number of G_{321} multiplets. The matrices $\mathcal{M}'$ and $\mathcal{M}$ are physically equivalent. Hence we agree with [12] that there should be an equivalence i.e. one-to-one correspondence between all results of all groups.

From the substitution (17) we see that there is one-to-one correspondence between VEV equations and mass matrices (up to phases), but the superpotential can be identified only after the following rescaling of the $\mathbf{210} - \Phi$ and $\mathbf{126} + \overline{\mathbf{126}} = \Delta + \overline{\Delta}$ fields

$$\Phi = \frac{\Phi_B}{\sqrt{24}}, \qquad \Delta = \frac{\Sigma_B}{\sqrt{120}}, \qquad \overline{\Delta} = \frac{\overline{\Sigma}_B}{\sqrt{120}}. \tag{18}$$

Only after substitutions (17) and the above rescalings of the fields (18) there is one-to-one correspondence of all our results and results of Ref. [9].

The similar equivalence holds between our results and the results of Ref. [11]. But this is not the correspondence given in Ref. [11] which does not map our G_{321} mass matrices to those of Ref. [11]. The correspondence between our results and those of Refs. [11,12] is

$$\begin{aligned} &m_1 = m, \qquad m_2 = 2M, \qquad m_3 = \frac{1}{2}M_H, \\ &\lambda_1 = \sqrt{24}\lambda, \qquad \lambda_2 = 20\sqrt{6}\eta, \qquad \lambda_3 = \sqrt{10}\gamma, \\ &\lambda_4 = \sqrt{10}\overline{\gamma}, \qquad \phi_1 = p, \qquad \phi_2 = \sqrt{3}a, \\ &\phi_3 = -\sqrt{6}\omega, \qquad v_R = \frac{\sigma_A}{\sqrt{2}}, \qquad \overline{v_R} = \frac{\overline{\sigma}_A}{\sqrt{2}}, \\ &\Phi = \frac{\Phi_A}{\sqrt{24}}, \qquad \Delta = \frac{\Sigma_A}{\sqrt{240}}, \qquad \overline{\Delta} = \frac{\overline{\Sigma}_A}{\sqrt{240}}. \end{aligned} \tag{19}$$

Finally, we have shown that our results are internally consistent and correct. By constructing unique mappings from our results to the results of the Refs. [9] and [11,12] we have also shown that the results of Refs. [6–12] are mutually consistent. The advantages of our method are that it is very simple and necessarily incorporates a set of strong consistency checks, proposed in Ref. [8] that apply to the total mass matrix $\mathcal{M}$. As a consequence, $\mathcal{M}^\dagger\mathcal{M}$ and $\mathcal{M}\mathcal{M}^\dagger$ automatically satisfy all the higher symmetry tests, except the trace test. Furthermore, our method can easily be programmed for tensor representations and can easily be extended to spinor representations. Therefore, it is suitable for a broad use in the model building.

[1] C. S. Aulakh and R. N. Mohapatra, Phys. Rev. D **28**, 217 (1983); T. E. Clark, T. K. Kuo, and N. Nakagawa, Phys. Lett. B **115**, 26 (1982); C. S. Aulakh, B. Bajc, A. Melfo, G. Senjanović, and F. Vissani, Phys. Lett. B **588**, 196 (2004).

[2] K. S. Babu and R. N. Mohapatra, Phys. Rev. Lett. **70**, 2845 (1993).

[3] T. Fukuyama and N. Okada, J. High Energy Phys. 11 (2002) 011; K. Matsuda, Y. Koide, T. Fukuyama, and H. Nishiura, Phys. Rev. D **65**, 033008 (2002); **65**, 079904(E) (2002); K. Matsuda, Y. Koide, and T. Fukuyama, Phys. Rev. D **64**, 053015 (2001).

[4] B. Bajc, G. Senjanović, and F. Vissani, Phys. Rev. Lett. **90**, 051 802 (2003); H. S. Goh, R. N. Mohapatra, and S. P. Ng, Phys. Lett. B **570**, 215 (2003); B. Dutta, Y. Mimura, and R. N. Mohapatra, Phys. Rev. D **69**, 115014 (2004).

[5] T. Fukuyama, T. Kikuchi, and N. Okada, Phys. Rev. D **68**, 033012 (2003).

[6] T. Fukuyama, A. Ilakovac, T. Kikuchi, S. Meljanac, and N. Okada, Eur. Phys. J. C **42**, 191 (2005).

[7] T. Fukuyama, A. Ilakovac, T. Kikuchi, S. Meljanac, and N. Okada, J. High Energy Phys. 09 (2004) 052.

[8] T. Fukuyama, A. Ilakovac, T. Kikuchi, S. Meljanac, and N. Okada, J. Math. Phys. (N.Y.) **46**, 033505 (2005).

[9] B. Bajc, A. Melfo, G. Senjanović, and F. Vissani, Phys. Rev. D **70**, 035007 (2004).

[10] C. S. Aulakh and A. Girdhar, Int. J. Mod. Phys. A **20**, 865 (2005).

[11] C. S. Aulakh and A. Girdhar, Nucl. Phys. **B711**, 275 (2005).

[12] C. S. Aulakh, following Article, Phys. Rev. D **72**, 051702 (2005).