Cross-correlations between volume change and price change

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In finance, one usually deals not with prices but with growth rates R, defined as the difference in logarithm between two consecutive prices. Here we consider not the trading volume, but rather the volume growth rate $\tilde{R}$, the difference in logarithm between two consecutive values of trading volume. To this end, we use several methods to analyze the properties of volume changes $|\tilde{R}|$, and their relationship to price changes $|R|$. We analyze 14,981 daily recordings of the Standard and Poor's (S & P) 500 Index over the 59-year period 1950–2009, and find power-law cross-correlations between $|R|$ and $|\tilde{R}|$ by using detrended cross-correlation analysis (DCCA). We introduce a joint stochastic process that models these cross-correlations. Motivated by the relationship between $|R|$ and $|\tilde{R}|$, we estimate the tail exponent $\tilde{\alpha}$ of the probability density function $P(|\tilde{R}|) \sim |\tilde{R}|^{-1-\tilde{\alpha}}$ for both the S & P 500 Index as well as the collection of 1819 constituents of the New York Stock Exchange Composite Index on 17 July 2009. As a new method to estimate $\tilde{\alpha}$, we calculate the time intervals τ_q between events where $\tilde{R} > q$. We demonstrate that $\bar{\tau}_q$, the average of τ_q, obeys $\bar{\tau}_q \sim q^{\tilde{\alpha}}$. We find $\tilde{\alpha} \approx 3$. Furthermore, by aggregating all τ_q values of 28 global financial indices, we also observe an approximate inverse cubic law.

econophysics | finance | volatility

There is a saying on Wall Street that "it takes volume to move stock prices." A number of studies have analyzed the relationship between price changes and the trading volume in financial markets (1–14). Some of these studies (1, 3–6) have found a positive relationship between price change and the trading volume. In order to explain this relationship, Clarke assumed that the daily price change is the sum of a random number of uncorrelated intraday price changes (3), so predicted that the variance of the daily price change is proportional to the average number of daily transactions. If the number of transactions is proportional to the trading volume, then the trading volume is proportional to the variance of the daily price change.

The cumulative distribution function (cdf) of the absolute logarithmic price change $|R|$ obeys a power law

$$P(|R| > x) \sim x^{-\alpha}. \quad [1]$$

It is believed (15–18) that $\alpha \approx 3$ ("inverse cubic law"), outside the range $\alpha < 2$ characterizing a Lévy distribution (18, 19). A parallel analysis of Q, the volume traded, yields a power law (20–28)

$$P(Q > x) \sim x^{-\alpha_Q}. \quad [2]$$

To our knowledge, the logarithmic volume change—$\tilde{R}$ and its relation to the logarithmic price change R—has not been analyzed, and this analysis is our focus here.

Data Analyzed

A. We analyze the Standard and Poor's (S & P) 500 Index recorded daily over the 59-year period January 1950–July 2009 (14,981 total data points).

B. We also analyze 1,819 New York Stock Exchange (NYSE) Composite members comprising this index on 17 July 2009, recorded at one-day intervals (6,794,830 total data points). Both data sets are taken from http://finance.yahoo.com. Different companies comprising the NYSE Composite Index have time series of different lengths. The average time series length is 3,735 data points, the shortest time series is 10 data points, and the longest is 11,966 data points. If the data display scale-independence, then the same scaling law should hold for different time periods.

C. We also analyze 28 worldwide financial indices from http://finance.yahoo.com, recorded daily.
   (*i*) Eleven European indices (ATX, BEL20, CAC 40, DAX, AEX General, OSE All Share, MIBTel, Madrid General, Stockholm General, Swiss Market, FTSE 100),
   (*ii*) Twelve Asian indices (All Ordinaries, Shanghai Composite, Hang Seng, BSE 30, Jakarta Composite, KLSE Composite, Nikkei 225, NZSE 50, Straits Times, Seoul Composite, Taiwan Weighted, TA-100), and
   (*iii*) Five American and Latin American indices (MerVal, Bovespa, S & P TSX Composite, IPC, S & P 500 Index).

For each of the 1,819 companies and 28 indices, we calculate over the time interval of one day the logarithmic change in price $S(t)$,

$$R_t \equiv \ln\left(\frac{S(t+1)}{S(t)}\right), \quad [3]$$

and also the logarithmic change in trading volume $Q(t)$ (29),

$$\tilde{R}_t \equiv \ln\left(\frac{Q(t+1)}{Q(t)}\right). \quad [4]$$

For each of the 3,694 time series, we also calculate the absolute values $|R_t|$ and $|\tilde{R}_t|$ and define the "price volatility" (30) and "volume volatility," respectively,

$$V_R \equiv \frac{|R_t|}{\sigma_R} \quad [5]$$

and

$$V_{\tilde{R}} \equiv \frac{|\tilde{R}_t|}{\sigma_{\tilde{R}}}, \quad [6]$$

where $\sigma_R \equiv (\langle |R_t|^2\rangle - \langle |R_t|\rangle^2)^{1/2}$ and $\sigma_{\tilde{R}} \equiv (\langle |\tilde{R}_t|^2\rangle - \langle |\tilde{R}_t|\rangle^2)^{1/2}$ are the respective standard deviations.

Methods

Recently, several papers have studied the return intervals τ between consecutive price fluctuations above a volatility

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threshold q. The probability density function (pdf) of return intervals $P_q(\tau)$ scales with the mean return interval $\bar{\tau}$ as (31–33)

$$P_q(\tau) = \bar{\tau}^{-1} f\left(\frac{\tau}{\bar{\tau}}\right), \quad [7]$$

where $f(x)$ is a stretched exponential. Similar scaling was found on the intratrade time scale for $q = 0$ (34). In this paper, we analyze either (*i*) separate indices or (*ii*) aggregated data mimicking the market as a whole. In case *i*, e.g., the S & P 500 Index for any q, we calculate all the τ values between consecutive index fluctuations and calculate the average return interval $\bar{\tau}$. In case *ii*, we estimate average market behavior, e.g., by analyzing all the 500 members of the S & P 500 Index. For each q and each company, we calculate all τ_q values and their average.

For any given value of Q in order to improve statistics, we aggregate all the τ values in one dataset and calculate $\bar{\tau}$. If the pdf of large volatilities is asymptotically power-law distributed, $P(|x|) \sim |x|^{-1-\alpha}$, and $P(|\tilde{x}|) \sim |\tilde{x}|^{-1-\tilde{\alpha}}$, we propose an estimator that relates the mean return intervals $\bar{\tau}_q$ with α, where $\bar{\tau}_q$ is calculated for both case *i* and case *ii*. Because on average there is one volatility above threshold q for every $\bar{\tau}_q$ volatilities, then

$$1/\bar{\tau}_q \approx \int_q^{\infty} P(|x|) d|x| = P(|x| > q) \sim q^{-\alpha}. \quad [8]$$

For both case *i* and case *ii*, we calculate $\bar{\tau}_q$ for varying q, and obtain an estimate for α through the relationship

$$\bar{\tau}_q \propto q^{\alpha}. \quad [9]$$

We compare our estimate for α in the above procedure with the α value obtained from $P(|R| > Q)$, by using an alternative method of Hill (35). If the pdf follows a power law $P(x) \sim Ax^{-(1+\alpha)}$, we estimate the power-law exponent α by sorting the normalized returns by their size, $x_1 > x_2 > \ldots > x_N$, with the result (35)

$$\alpha = (N-1)\left[\sum_{i=1}^{N-1} \ln \frac{x_i}{x_N}\right]^{-1}, \quad [10]$$

where $N - 1$ is the number of tail data points. We employ the criterion that N does not exceed 10% of the sample size which, to a good extent, ensures that the sample is restricted to the tail part of the pdf (36).

A new method based on detrended covariance, detrended cross-correlations analysis (DCCA), has recently been proposed (37). To quantify power-law cross-correlations in nonstationary time series, consider two long-range cross-correlated time series $\{y_i\}$ and $\{y'_i\}$ of equal length N, and compute two integrated signals $Y_k \equiv \sum_{i=1}^{k} y_i$ and $Y'_k \equiv \sum_{i=1}^{k} y'_i$, where $k = 1, \ldots, N$. We divide the entire time series into $N - n$ overlapping boxes, each containing $n + 1$ values. For both time series, in each box that starts at i and ends at $i + n$, define the "local trend" to be the ordinate of a linear least-squares fit. We define the "detrended walk" as the difference between the original walk and the local trend.

Next, calculate the covariance of the residuals in each box $f^2_{\mathrm{DCCA}}(n,i) \equiv \frac{1}{n-1}\sum_{k=i}^{i+n}(Y_k - Y'_{k,i})(Y_k - Y'_{k,i})$. Calculate the detrended covariance by summing over all overlapping $N - n$ boxes of size n,

$$F^2_{\mathrm{DCCA}}(n) \equiv \sum_{i=1}^{N-n} f^2_{\mathrm{DCCA}}(n,i). \quad [11]$$

If cross-correlations decay as a power law, the corresponding detrended covariances are either always positive or always negative, and the square root of the detrended covariance grows with time window n as

$$F_{\mathrm{DCCA}}(n) \propto n^{\lambda_{\mathrm{DCCA}}}, \quad [12]$$

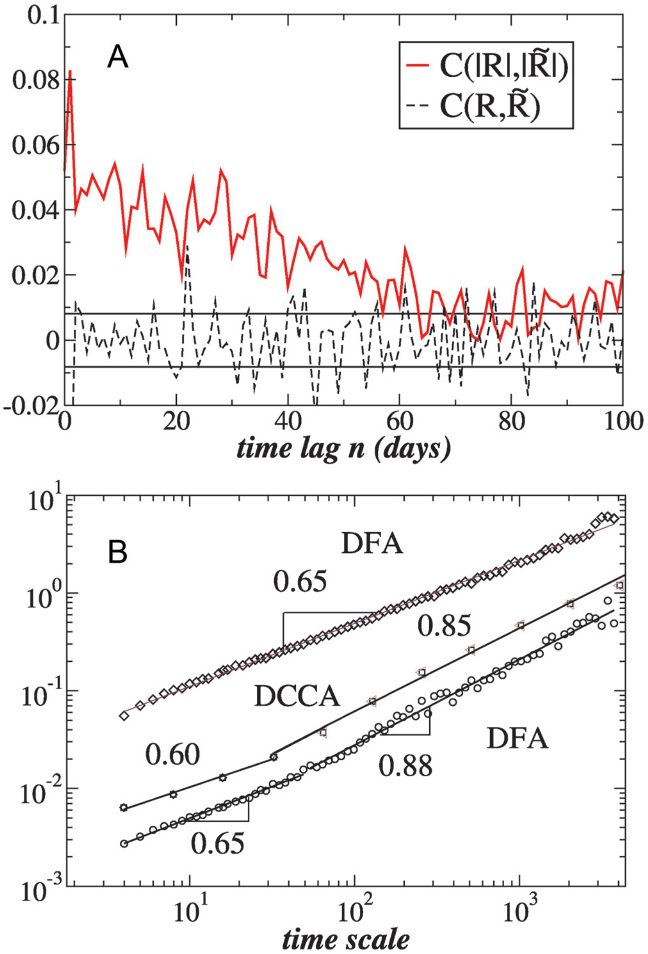

Fig. 1. Autocorrelations and cross-correlations in absolute values of price changes $|R_t|$ of Eq. **3** and trading-volume changes $|\tilde{R}_t|$ of Eq. **4** for daily returns of the S & P 500 Index. (*A*) The cross-correlation function $C(R, \tilde{R})$ between R and $\tilde{R}$, and the cross-correlation function $C(|R|, |\tilde{R}|)$ between $|R|$ and $|\tilde{R}|$. (*B*) For $R(t)$, and $\tilde{R}(t)$, we show the rms of the detrended variance $F_{\mathrm{DFA}}(n)$ for $|R|$ and $|\tilde{R}|$ and also the rms of the detrended covariance (37), $F_{\mathrm{DCCA}}(n)$. The two DFA exponents $\lambda_{|R|}$ and $\lambda_{|\tilde{R}|}$ imply that power-law autocorrelations exist in both $|R|$ and $|\tilde{R}|$. The DCCA exponent implies the presence of power-law cross-correlations. Power-law cross-correlations between $|R|$ and $|\tilde{R}|$ imply that current price changes depend upon previous changes but also upon previous volume changes and vice versa.

where λ_{DCCA} is the cross-correlation exponent. If, however, the detrended covariance oscillates around zero as a function of the time scale n, there are no long-range cross-correlations.

When only one random walk is analyzed ($Y_k = Y'_k$), the detrended covariance $F^2_{\mathrm{DCCA}}(n)$ reduces to the detrended variance

$$F_{\mathrm{DFA}}(n) \propto n^{\lambda_{\mathrm{DFA}}} \quad [13]$$

used in the detrended fluctuation analysis (DFA) method (38).

Results of Analysis

We first investigate the daily closing values of the S & P 500 Index adjusted for stock splits together with their trading volumes. In Fig. 1*A*, we show the cross-correlation function between $|R_t|$ and $|\tilde{R}_t|$ and the cross-correlation function between R_t and $\tilde{R}_t$. The solid lines are 95% confidence interval for the autocorrelations of an independent and identically distributed random variables (i.i.d.) process. The cross-correlation function between R_t and $\tilde{R}_t$ is practically negligible and stays within the 95% confidence interval. On the contrary, the cross-correlation function between $|R_t|$

and $|\tilde{R}_t|$ is significantly different than zero at the 5% level for more than 50 time lags.

In Fig. 1*B* we find, by using the DFA method (38, 39), that not only $|R_t|$ (30, 40), but also $|\tilde{R}_t|$ exhibit power-law autocorrelations. As an indicator that there is an association between $|R_t|$ and $|\tilde{R}_t|$, we note that during market crashes large changes in price are associated with large changes in market volume. To confirm comovement between $|R_t|$ and $|\tilde{R}_t|$, in Fig. 1*B* we demonstrate that $|R_t|$ and $|\tilde{R}_t|$ are power-law cross-correlated with the DCCA cross-correlation exponent (see *Methods* section) close to the DFA exponent (38, 39) corresponding to $|R_t|$. Thus, we find the cross-correlations between $|R_{t+n}|$ and $|\tilde{R}_t|$ not only at zero time scale ($n = 0$) but for a large range of time scales.

Having analyzed cross-correlations between corresponding (absolute) changes in prices and volumes, we now investigate the pdf of the absolute value of $\tilde{R}_t$ of Eq. **4**. In order to test whether exponential or power-law functional form better fits the data, in Figs. 2 *A* and *B* we show the pdf $P(\tilde{R})$ in both linear–log and log–log plot. In Fig. 2*A* we see that the tail substantially deviates from the central part of pdf which we fit by exponential function. In Fig. 2*B*, we find that the tails of the pdf can be well described by a power law $\tilde{R}^{1+\tilde{\alpha}}$ with exponent $\tilde{\alpha} = 3 \pm 0.16$, which supports an inverse cubic law—virtually the same as found for average stock price returns (15–17), and individual companies (18).

In order to justify the previous finding, we employ two additional methods. First, we introduce a method (described in *Methods* by Eqs. **8** and **9**) for a single financial index. We analyze the probability that a trading volume change $\tilde{R}$ has an absolute value larger than a given threshold, q. We analyze the time series of the S & P 500 Index for 14,922 data points. First, we define different thresholds, ranging from 2 σ to 8 σ. For each q, we calculate the mean return interval, $\bar{\tau}$. In Fig. 2*C*, we find that q and $\bar{\tau}$ follow the power law of Eq. **9**, where $\tilde{\alpha} = 2.97 \pm 0.02$. We note that the better the power law relation between $\bar{\tau}_q$ and q in Fig. 2*C*, the better the power-law approximation $P(|\tilde{R}| > x) \approx x^{-\tilde{\alpha}}$ for the tail of the pdf $P(|\tilde{R}|)$. In order to confirm our finding that $P(|\tilde{R}|)$ follows a power law $P(|\tilde{R}|) \approx \tilde{R}^{-\tilde{\alpha}-1}$ where $\tilde{\alpha} \approx 3$ obtained in Figs. 2 *A* and *B*, we also apply a third method, the Hill estimator (35), to a single time series of the S & P 500 Index. We obtain $\tilde{\alpha} = 2.80 \pm 0.07$, consistent with the results in Figs. 2 *A* and *B*.

Next, by using the procedure described in case *ii* of *Methods*, we analyze 1,819 different time series of Eq. **4**, each representing one of the 1,819 members of the NYSE Composite Index. For each company, we calculate the normalized $|\tilde{R}_t|$ volatility of trading volume changes of each company (see Eq. **6**). In Figs. 3 *A* and *B*, we show the pdf in both linear–log and log–log plot. In Fig. 3*A*, we see that the broad central region of the pdf, from 2 σ up to 15 σ, is fit by an exponential function. However, the far tail deviates from the exponential fit. In Fig. 3*B*, we find that the tails of the pdf from 15 σ up to 25 σ, are described by a power law $\tilde{R}^{1+\tilde{\alpha}}$ with exponent $\tilde{\alpha} = 3.65 \pm 1.00$.

Then, by employing the method described by Eqs. **8** and **9** we define different thresholds, q, ranging from 2 σ to 8 σ (different range than in Fig. 3*A*). We choose the lowest q equal to 2 because we employ the criterion that N does not exceed 10% of the sample size (36). For each q, and each company, we calculate the time series of return intervals, τ_q. For a given q, we then collect all the τ values obtained from all companies in one unique dataset— mimicking the market as a whole — and calculate the average return interval, $\bar{\tau}_q$. In Fig. 3*C* we find that q and $\bar{\tau}_q$ follow an approximate inverse cubic law of Eq. **9**, where $\tilde{\alpha} = 3.11 \pm 0.11$. Our method is sensitive to data insufficiency, so we show the results only up to 8 σ. Clearly, this method gives the $\tilde{\alpha}$ value for the market as a whole, not the $\tilde{\alpha}$ values for particular companies. By joining all the normalized volatilities $|V_{\tilde{R}}|$ obtained from 1,819 time series in one unique dataset, we estimate Hill's exponent of Eq. **10**, $\tilde{\alpha} = 2.82 \pm 0.003$,

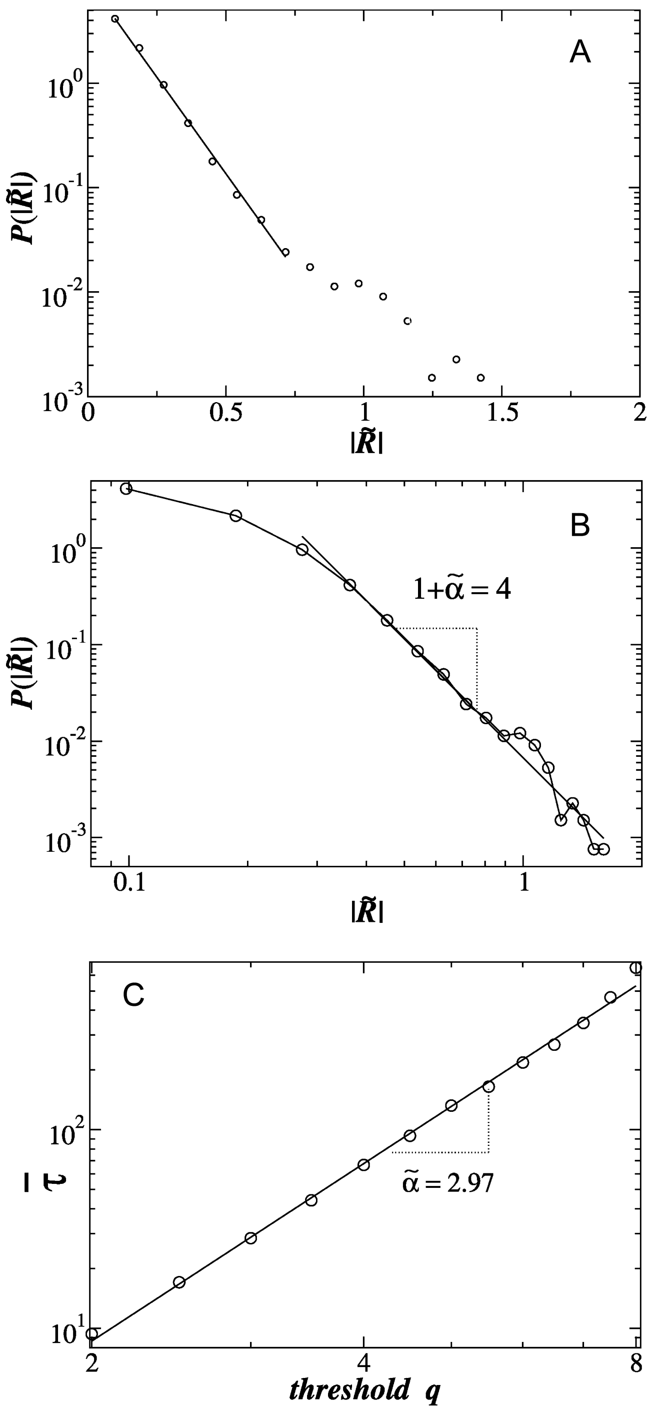

Fig. 2. Pdf $P(|\tilde{R}|)$ of absolute value of differences in logarithm of trading volume, $\tilde{R}$, of Eq. **4** for the S & P 500 Index. (*A*) A log–linear plot $P(|\tilde{R}|)$. The solid line is an exponential fit. The tail part of pdf deviates from the fit in the central part. (*B*) Log–log plot of the pdf. The broad tail part can be explained by a power law $\tilde{R}^{1+\tilde{\alpha}}$ with $\tilde{\alpha} = 3 \pm 0.16$. (*C*) For the absolute values of changes in trading volume (see Eq. **4**), the average return interval τ vs. threshold q (in units of standard deviation σ) follows a power law, with exponent $\tilde{\alpha} = 2.97 \pm 0.02$. The power law is consistent with inverse cubic law of the pdf.

consistent with the value of the exponent obtained by using the method of Eqs. **8** and **9**.

In the previous analysis, we consider time series of the companies comprising the NYSE Composite Index of different lengths (from 10 to 11,966 data points). In order to prove that the Hill exponent of Eq. **10** is not affected by the shortest time series,

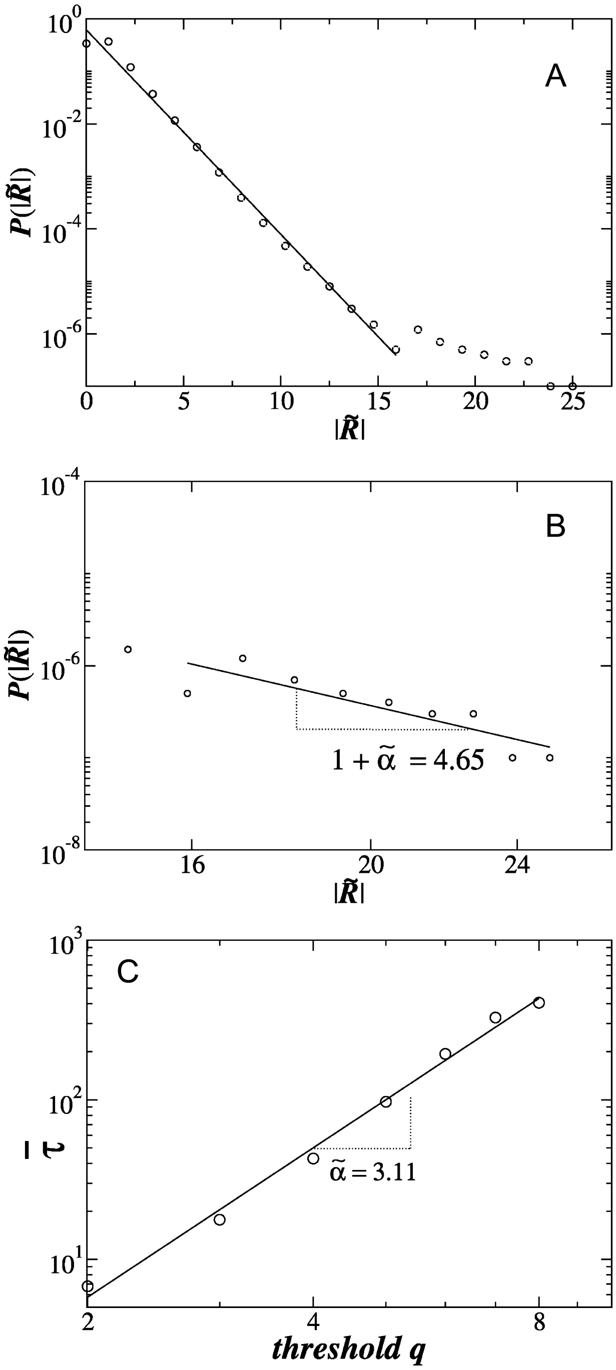

Fig. 3. Pdf of absolute value of differences in logarithm of trading volume, $\tilde{R}$, of Eq. **4** for the members of the NYSE Composite Index. We use the method described in the *Methods* section—case *ii*—for normalized volatilities of Eq. **6**. (*A*) From 1 σ to 15 σ we show the linear–log plot of the pdf $P(\tilde{R})$. The straight line is exponential fit. The far tail of pdf deviates from the fit in the central region of pdf. (*B*) Log–log plot of pdf from 15 σ to 25 σ. The tail part of the pdf can be explained by a power law $\tilde{R}^{1+\tilde{\alpha}}$ with $\tilde{\alpha} = 4.2 \pm 0.26$. (*C*) For the absolute values of changes in trading volume (see Eq. **4**), we show the average return interval $\bar{\tau}_q$ versus threshold q (in units of a standard deviation). Up to 8 σ, we show a power law with exponent $\tilde{\alpha} = 3.11 \pm 0.12$ which leads to the inverse cubic law.

we next analyze only the time series longer than 3,000 data points (1,128 firms in total). For the Hill exponent, we obtain $\tilde{\alpha} = 2.81 \pm 0.003$, which is practically the same value as the one ($\tilde{\alpha} = 2.82 \pm 0.003$) we obtained when short time series were considered as well.

We also perform the method of Hill (35), and the method of Eqs. **8** and **9**, for the 500 members of the S & P 500 Index comprising the index in July 2009. There are in total 2, 601, 247 data points for $\tilde{R}$ of Eq. **6**. For the thresholds, q, ranging from 2 σ to 10 σ, we find that q and $\bar{\tau}$ follow for this range an approximate inverse cubic law of Eq. **9**, where $\tilde{\alpha} = 3.1 \pm 0.12$. We estimate the Hill exponent of Eq. **10** to be $\tilde{\alpha} = 2.86 \pm 0.005$, with the lowest $Q = 2$.

In order to find the functional form for trading-volume changes at the world level, we analyze 28 worldwide financial indices by using the procedure described in *Methods* (case *ii*). For each q, and for each of the 28 indices, we calculate the values for the return interval τ. Then for a given q, we collect all the τ values obtained for all indices and calculate the average return interval $\bar{\tau}_q$. In Fig. 4*A*, we find a functional dependence between q and $\bar{\tau}$, which can be approximated by a power law with exponent $\tilde{\alpha} = 2.41 \pm 0.06$. We also calculate $\bar{\tau}$ vs. q for different levels of financial aggregation.

Finally, in addition to trading-volume changes, we employ for stock price changes our procedure for identifying power-law behavior in the pdf tails described in *Methods* (case *ii*). The pdf of

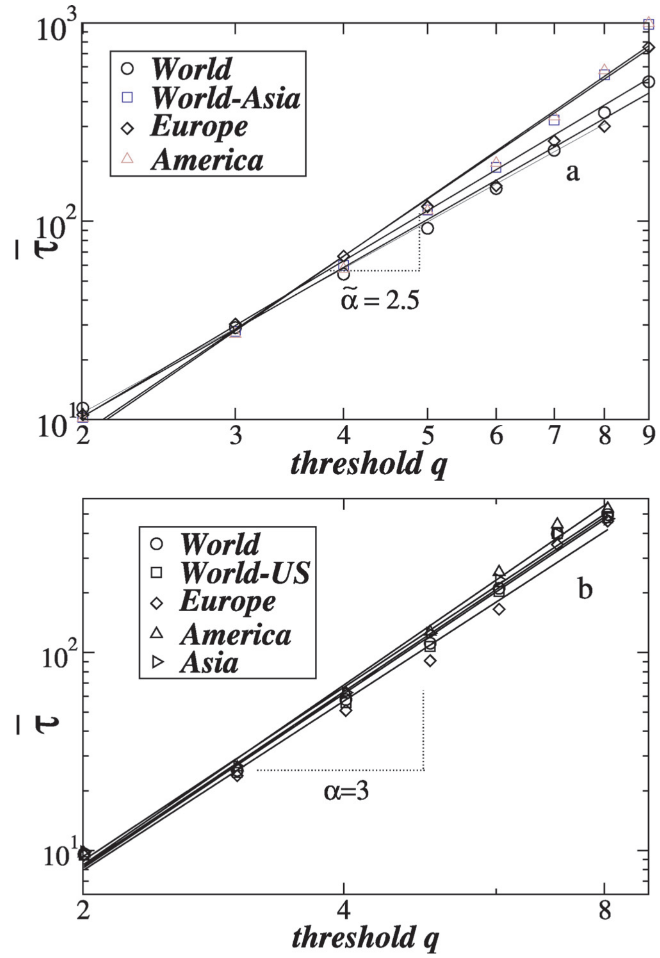

Fig. 4. Power-law correlations for worldwide financial indices in (*Upper*) absolute values of price changes ($|\tilde{R}|$) and (*Lower*) absolute values of trading-volume changes ($|R|$). We use the method described by Eqs. **7** and **8**. (*Upper*) The average return interval $\bar{\tau}$ vs. threshold q (in units of standard deviation) for absolute values of trading-volume changes. For each of 28 worldwide financial indices, we calculate the corresponding $\bar{\tau}_q$ values. Then we collect all the $\bar{\tau}$ values obtained from different indices, and show $\bar{\tau}_q$ versus q. Up to eight standard deviations, we find a power law with exponent $\tilde{\alpha} = 2.41 \pm 0.06$. (*Lower*) The average return interval $\bar{\tau}_q$ vs. threshold q for absolute values of price changes (see Eq. **3**) for different levels of aggregation. For each of five different types of aggregation reported, we find that $\bar{\tau}$ versus q exhibits a power law with an exponent very close to $\alpha = 3$.

stock price changes, calculated for an "average" stock, is believed to follow $P(R) \approx R^{-(1+\alpha)}$ where $\alpha \approx 3$, as empirically found for a wide range of different stock markets (15, 17).

Next, we test whether this law holds more generally. To this end, we analyze the absolute values of price changes, $|R_t|$ (see Eq. **3**), for five different levels of financial aggregation: (*i*) Europe, (*ii*) Asia, (*iii*) North and South America, (*iv*) the world without the U.S.A, and (*v*) the entire world. For each level of aggregation, we find that the average return interval $\bar{\tau}_q \sim q^{-3}$.

Model

In order to model long-range cross-correlations between $|R_t|$ and $|\tilde{R}_t|$, we introduce a new joint process for price changes

$$\epsilon_t = \sigma_t \eta_t \qquad \textbf{[14]}$$

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2 + \tilde{\gamma}\tilde{\epsilon}_{t-1}^2 \qquad \textbf{[15]}$$

and for trading-volume changes

$$\tilde{\epsilon}_t = \tilde{\sigma}_t \tilde{\eta}_t \qquad \textbf{[16]}$$

$$\tilde{\sigma}_t^2 = \tilde{\omega} + \tilde{\alpha}\tilde{\epsilon}_{t-1}^2 + \tilde{\beta}\tilde{\sigma}_{t-1}^2 + \gamma\epsilon_{t-1}^2. \qquad \textbf{[17]}$$

If $\gamma = \tilde{\gamma} = 0$, Eqs. **14–17** reduce to two separate processes of ref. 41. Here, η_t and $\tilde{\eta}_t$ are two i.i.d. stochastic processes each chosen as Gaussian distribution with zero mean and unit variance. In order to fit two time series, we define free parameters ω, α, β, γ, $\tilde{\omega}$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, which we assume to be positive (41). The process of Eqs. **14–17** is based on the generalized autoregressive conditional heteroscedasticity (GARCH) process (obtained from Eqs. **14** and **15** when $\tilde{\gamma} = 0$) introduced to simulate long-range autocorrelations through $\beta \neq 0$. The GARCH process also generates the power-law tails as often found in empirical data (see refs. 15–18), and also Fig. 2*B*. In the process of Eqs. **14–17**, we obtain cross-correlations because the time-dependent standard deviation σ_t for price changes depends not only on its past values (through α and β), but also on past values of trading-volume errors ($\tilde{\gamma}$). Similarly, $\tilde{\sigma}_t$ for trading-volume changes depends not only on its past values (through $\tilde{\alpha}$ and $\tilde{\beta}$) but also on past values of price errors (γ).

For the joint stochastic process of Eqs. **14–17** with $\beta = \tilde{\beta} = 0.65$, $\alpha = \tilde{\alpha} = 0.14$, $\gamma = \tilde{\gamma} = 0.2$, we show in Fig. 5*A* the cross-correlated time series of Eqs. **15** and **17**. In Fig. 5*B*, we show the autocorrelation function for $|\epsilon_t|$ and the cross-correlation function, which practically overlap because of the choice of parameters.

If stationarity is assumed, we calculate the expectation of Eqs. **15** and **17** and because, e.g., $E(\sigma_t^2) = E(\sigma_{t-1}^2) = E(\epsilon_{t-1}^2) = \sigma_0^2$, we obtain $\sigma_0^2(1-\alpha-\beta) = \omega + \tilde{\gamma}\tilde{\sigma}_0^2$ and similarly $\tilde{\sigma}_0^2(1-\tilde{\alpha}-\tilde{\beta}) = \tilde{\omega} + \gamma\sigma_0^2$. So stationarity generally assumes that $\alpha + \beta < 1$ as found for the GARCH process (41). However, for the choice of parameters in the previous paragraph for which $\sigma_0 = \tilde{\sigma}_0$ stationarity assumes that $\sigma_0^2(1 - \alpha - \beta - \tilde{\gamma}) = \omega$. This result explains why the persistence of variance measured by $\alpha + \beta$ should become negligible in the presence of volume in the GARCH process (10). In order to have finite σ_0^2, we must assume $\alpha + \beta + \tilde{\gamma} < 1$.

It is also possible to consider integrated generalized autoregressive conditional heteroskedasticity (IGARCH) and fractionally integrated generalized autoregressive conditional heteroskedasticity (FIGARCH) processes with joint processes for price and volume change, a potential avenue for future research (46).

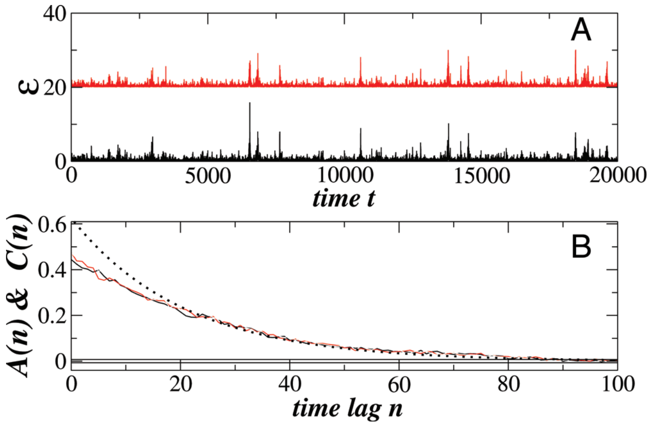

Fig. 5. Cross-correlations between two time series generated from the stochastic process of Eqs. **14–17**, with $\beta = \tilde{\beta} = 0.65$, $\alpha = \tilde{\alpha} = 0.14$, $\gamma = \tilde{\gamma} = 0.2$, and $\omega = \tilde{\omega} = 0.01$. In *A*, we show the time series ϵ and $\tilde{\epsilon}$ of Eqs. **14–17**, where the latter time series is shifted for clarity. These two time series follow each other due to the terms $\gamma \neq 0$ and $\tilde{\gamma} \neq 0$. In *B*, we show the autocorrelation function $A(n)$ for $|\epsilon_t|$ and the cross-correlation function $C(|\tilde{\epsilon}|, |\epsilon|)$. The 95% confidence intervals for no cross-correlations are shown (solid lines) along with the best exponential fit of $A(n)$ (dotted curve).

Summary

In order to investigate possible relations between price changes and volume changes, we analyze the properties of $|\tilde{R}|$, the logarithmic volume change. We hypothesize that the underlying processes for logarithmic price change $|R|$ and logarithmic volume change $|\tilde{R}|$ are similar. Consequently, we use the traditional methods that are used to analyze changes in trading price to analyze changes in trading volume. Two major empirical findings are:

(*i*) we analyze a well-known U.S. financial index, the S & P 500 Index over the 59-year period 1950–2009, and find power-law cross-correlations between $|\tilde{R}|$ and $|R|$. We find no cross-correlations between $\tilde{R}$ and R; and

(*ii*) we demonstrate that, at different levels of aggregation, ranging from the S & P 500 Index to an aggregation of different worldwide financial indices, $|\tilde{R}|$ approximately follows the same cubic law as $|R|$. Also, we find that the central region of the pdf, $P(|\tilde{R}|)$, follows an exponential function as reported for annually recorded variables, such as gross domestic product (42, 43), company sales (44), and stock prices (45).

In addition to empirical findings, we offer two theoretical results:

(*i*) to estimate the tail exponent $\tilde{\alpha}$ for the pdf of $|\tilde{R}|$, we develop an estimator which relates $\tilde{\alpha}$ of the cdf $P(|\tilde{R}| > x) \approx x^{-\tilde{\alpha}}$ to the average return interval $\bar{\tau}_q$ between two consecutive volatilities above a threshold q (31); and

(*ii*) we introduce a joint stochastic process for modeling simultaneously $|R|$ and $|\tilde{R}|$, which generates the cross-correlations between $|R|$ and $|\tilde{R}|$. We also provide conditions for stationarity.

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