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Because financial crises are characterized by dangerous rare events that occur more frequently than those predicted by models with finite variances, we investigate the underlying stochastic process generating these events. In the 1960s Mandelbrot [Mandelbrot B (1963) *J Bus* 36:394–419] and Fama [Fama EF (1965) *J Bus* 38:34–105] proposed a symmetric Lévy probability distribution function (PDF) to describe the stochastic properties of commodity changes and price changes. We find that an asymmetric Lévy PDF, $\mathcal{L}$, characterized by infinite variance, models several multiple credit ratios used in financial accounting to quantify a firm's financial health, such as the Altman [Altman EI (1968) *J Financ* 23:589–609] *Z* score and the Zmijewski [Zmijewski ME (1984) *J Accounting Res* 22:59–82] score, and models changes of individual financial ratios, ΔX_i. We thus find that Lévy PDFs describe both the static and dynamics of credit ratings. We find that for the majority of ratios, ΔX_i scales with the Lévy parameter $\alpha \approx 1$, even though only a few of the individual ratios are characterized by a PDF with power-law tails $X_i^{-1-\alpha}$ with infinite variance. We also find that α exhibits a striking stability over time. A key element in estimating credit losses is the distribution of credit rating changes, the functional form of which is unknown for alphabetical ratings. For continuous credit ratings, the Altman *Z* score, we find that $P(\Delta Z)$ follows a Lévy PDF with power-law exponent $\alpha \approx 1$, consistent with changes of individual financial ratios. Estimating the conditional $P(\Delta Z|Z)$ versus *Z*, we demonstrate how this continuous credit rating approach and its dynamics can be used to evaluate credit risk.

complex systems | econophysics | rating migrations

Most tests and tools used in statistics assume that any errors in a financial model are Gaussian distributed, and it is a common practice in economics to use a Gaussian distribution to approximate empirical data. Mandelbrot (1) and Fama (2) were among the first to notice that the logarithm of cotton price fluctuations and common stock price fluctuations have fatter tails than those produced by a Gaussian distribution, and they proposed a stable Lévy distribution to model the stochastic properties of the fluctuations. Analyzing high-frequency data, Mantegna and Stanley (3) reported that the stable Lévy distribution accurately models only a broad central region of the probability distribution function (PDF) of stock price changes, whereas Gopikrishnan et al. reported that a power law with an exponent value beyond the Lévy regime is needed to describe the tails (4, 5).

The central limit theorem (CLT) implies that the mean of a sufficiently large number of independent random variables, each with finite variance, will approximately follow a normal distribution (6). A generalization of the CLT shows that the mean of a sufficiently large number of independent random variables, each with infinite variance, approximately follows a stable Lévy distribution $L_{\alpha,\gamma}(x) = (1/\pi)\int_0^\infty dq \exp(-\gamma q^\alpha)\cos(qx)$, where $\gamma > 0$ and $0 < \alpha < 2$ (6, 7). Infinite variances are related to power-law distributions, and the general rule when combining two or more power-law variables, $x^{1+\alpha}$, is that the one with the smallest power-law exponent (the fattest power law) dominates when $x \to \infty$, which holds even if some variables are Gaussian distributed (8, 9). Because in finance one commonly deals with credit ratios defined as multiple financial ratios, if only one ratio is found to be power-law distributed, the credit ratio itself is also power-law distributed.

In contrast to the previous literature on financial ratios, we focus not on ratios, X_i, but on dynamics of credit ratios, represented by their changes, ΔX_i. For each of eight individual ratios X_i comprising the Altman *Z* score (10), the Zmijewski Z_m score (11), and also the Shumway Hazard model (12), we find asymmetric Lévy $\mathscr{L}$ PDFs in changes of financial ratios, ΔX_i, which are related to credit rating changes and thus to credit risk. We find that $\mathscr{L}$ models several multiple credit ratios such as the Altman *Z* score and the Zmijewski score. $P(Z)$ follows an $\mathscr{L}$ PDF with scale parameter $\alpha = 1.06 \pm 0.02$ and skewness parameter $\beta = 0.70 \pm 0.02$. We depart from the usual discrete alphabet credit ratings, such as Moody's (13), and choose the *Z* score as a proxy for the continuous credit rating (14), where the ΔZ quantifies credit rating migrations. We find that $P(\Delta Z)$ follows a Lévy PDF with a power-law exponent $\alpha \approx 1$. We demonstrate how our previous findings can be used to model credit risk.

Methods and Data

In modeling changes of financial ratios and multiple credit scores, we choose the asymmetric Lévy $\mathscr{L}$ because, e.g., multiple credit scores *Z* are characterized by heavy tails in $P(Z)$, and we fit them with a scale parameter α. We model asymmetry in the PDF tails using skewness parameter β, the location (mean) of multiple credit scores using shift parameter μ, and the spread using parameter σ. For both the symmetric Lévy $L_{\alpha,\gamma}$ and its generalization, $\mathscr{L}$, the PDF generally cannot be written analytically. $\mathscr{L}$ is determined by its characteristic function $\varphi(t)$: $\mathscr{L} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(t)e^{-ixt}dt$, where

$$\varphi(t;\mu,c,\alpha,\beta) = \exp\{it\mu - |\sigma t|^\alpha[1 - i\beta \mathrm{sgn}(t)\Phi]\}. \quad [1]$$

In Eq. **1**, $\mathrm{sgn}(t)$ is the sign of *t*, $\Phi = \tan(\pi\alpha/2)$, and $\beta \in [-1,1]$ (15). When $\beta = 0$ and $\alpha = 1$, the $\mathscr{L}$ becomes the Cauchy distribution, the analytic form of which is well-known.

For power-law distributed variables with a cumulative distribution function (CDF), $P(s > x) \sim x^{-\zeta'}$, a Zipf plot of size *s* versus rank *R* asymptotically ($R \gg 1$) follows a power law with exponent ζ (16),

$$\zeta = 1/\zeta'. \quad [2]$$

If CDF is a Lévy distribution, then $\zeta' = \alpha$.

We analyze financial data for each quarter during the period 2000–2009 of 488 publicly traded manufacturing firms. The data are available at http://www.wikinvest.com. Our body of data includes (*i*) working capital to total assets (X_1), (*ii*) retained earnings divided by total assets (X_2), (*iii*) earnings before taxes and interest divided by total assets (X_3), (*iv*) market value of equity divided by book value of total liabilities (X_4), (*v*) sales divided by total assets (X_5), (*vi*) net income divided by total assets (X_6), (*vii*) total liabilities divided by total assets (X_7), and (*viii*) current

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assets divided by current liabilities (X_8). To improve the quality of the statistics, and to assess the common scaling properties of the ratings and rating changes, for each ratio we aggregate all the data into one common dataset.

Analysis

Just as stock market risk is determined by equity price changes, so also is credit risk determined by the changes in credit ratings (10–12, 17–19). To calculate both the distribution of expected credit losses and the credit risk, we first need to know both the distribution of rating changes and how credit (or bond) is calculated for each rating. We can then estimate both the mean of the credit loss and the deviation from the mean—the credit risk (20).

Our interest in studying rare events in rating changes is threefold: (*i*) downgrade events are associated with large financial losses, (*ii*) the distribution of rating changes helps us understand the underlying dynamics of economic systems producing the transitions, and (*iii*) the rarer the event, the larger the error in any estimation of its probability.

The discrete credit ratings of Fitch, Moody's, and Standard & Poor's (S&P) are widely used in assessing a firm's financial health (13, 20). For example, an S&P credit rating is expressed alphabetically and uses discrete values such as *AAA*, *AA*, and *A*. This approach is severely limited because it does not allow an analytical evaluation of the distribution of rating changes, and it is a discrete rating system, it is slow in responding to changes in corporate credit quality, according to recent surveys (14).

In contrast to discrete credit rating systems, the Altman Z score (10) is a continuous numerical score, the calculation of which combines five financial ratios for each manufacturing firm (see *Methods and Data*),

$$Z \equiv 1.2X_1 + 1.4X_2 + 3.3X_3 + 0.6X_4 + 1.0X_5. \qquad [3]$$

The smaller the Z score, the larger the probability that bankruptcy will occur within 2 y. During the 2007–2009 recession, two large financial institutions, J. P. Morgan and Nomura Securities advocated buying stocks of companies with a high Altman Z score and selling stocks of companies with a low Z score (20). Additionally, we note that Goldman Sachs has adopted the Z score model for long-short baskets in 2008–2010 (top 10% and bottom 10% Z scores). The prefactors multiplying the ratios X_i in ref. 3 are determined by employing discriminant analysis, which assumes normality in empirical data. Recently, the impact of outlier observations on the parameter estimation and significance testing of the entire model was addressed in refs. 21–23, where the authors used logarithmic transformations of the financial ratios and truncated values for many ratios at two or three standard deviations from the mean to reduce the outlier impact. Besides, Altman and Saunders proposed using the firms' multiple ratio score as a superior means of addressing the correlation between credit assets in asset management to arrive at an optimal portfolio, both in terms of which securities to include and their weightings (19).

In addition to the Altman Z score, the Zmijewski Z_m score is also a widely employed multiple credit ratio (11), and is a combination of three financial ratios (see *Methods and Data*),

$$Z_m \equiv -4.336 - 4.513X_6 + 5.679X_7 + 0.004X_8. \qquad [4]$$

Another widely employed model, a Hazard model (12), incorporates X_6 and X_7 ratios for forecasting bankruptcy.

Utilizing the map between the Z score and the S&P bond rating developed by Altman and Hotchkiss (14), we take the Z score (14) as a proxy for a continuous credit rating, where the Z changes quantify the rating changes.

For the eleven ratios computed for manufacturing companies over the period 1954–1973, Deakin tested the assumption of normality (24). With the exception of the total debt over total assets ratio, none of the ratios was normally distributed at 5% significance. Clearly, the largest contribution to deviation from normality arises from the data in the tails.

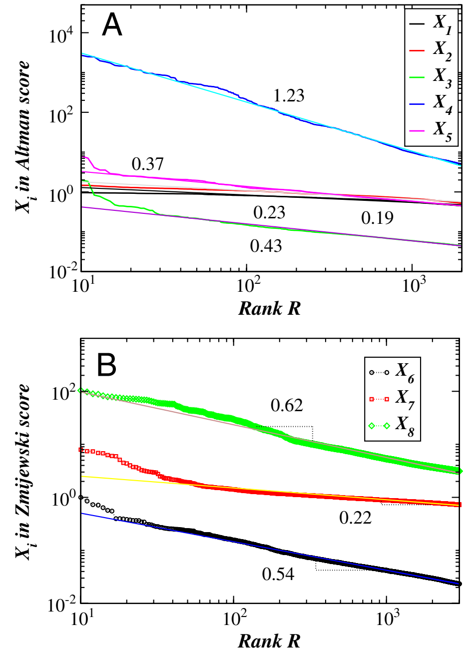

Fig. 1. Power laws in financial accounting ratios. For each of eight financial ratios, X_i, comprising (*A*) the Altman Z and (*B*) the Zmijewski Z_m score, we find a power law in the Zipf plot of ratio. For each X_i, the Zipf plot in the right tail follows a power law. The 2,000 data points represents 13% of all data points.

In Fig. 1, for each individual ratio X_i comprising (*A*) the Altman score and (*B*) the Zmijewski score, we find that the Zipf plot of a ratio approximately follows a power law in the right-hand tail. Note that the power-law Zipf exponents come from a broad range, in agreement with ref. 25. McLeay reported that the t_ν-distribution (with the tails following $x^{-\nu-1}$) with ν degrees of freedom provides a good descriptive model for three financial ratios—ν ranges from 1.1 to 3.7, the last obtained for earnings-to-total assets ratio (25). In addition to the Zipf ranking approach, we also employ the maximum likelihood (ML) approach. We find that three X_i ratios, one in the Altman score and two in the Zmijewski score, follow asymmetric Lévy distributions with the ML values for α and β reported in Table 1. For these ratios we find the Zipf exponents $\zeta > 0.5$, which is a precondition for a Lévy distribution ($\zeta' = \alpha < 2$) (see Eq. **2**). We can understand the importance of this result if we recall the general rule for power laws—when combining two or more power-law variables $x^{1+\alpha_i}$, in the limit $x \to \infty$

$$a_1 x^{1+\alpha_1} + a_2 x^{1+\alpha_1} + \ldots a_n x^{1+\alpha_n} \propto a_a x^{1+\alpha_a}, \qquad [5]$$

the variable with the fattest power law—the smallest power-law exponent or largest Zipf exponent in Eq. **2**—[$\alpha_a = \text{minimum}(\alpha_1, \alpha_2, \ldots, \alpha_n)$] dominates, and this behavior holds even if some variables in [**5**] are Gaussian distributed (those where $\alpha \to \infty$) (8, 9). This behavior implies that even when only one power law $x^{1+\alpha}$ is combined with many Gaussian-distributed vari-

Table 1. Ratio

Ratio	X_4	X_6	X_8
α	0.86	1.14	1.24
	(0.07)	(0.08)	(0.07)
β	0.99	−0.22	0.97
	(0.05)	(0.03)	(0.03)

ables the combination is still a power law in the tails, i.e., power laws dominate any other functional dependence in the tails where rare events occur.

To test [**5**] we next collect 14,779 values for the Altman Z score, and in Fig. 2*A* show the PDF $P(Z)$. Note that Eq. **2** and Fig. 1*A* give for the smallest exponent $\zeta' = \alpha = 0.81$ (corresponding to $\zeta = 1.23$). Applying the ML method we find that the PDF $P(Z)$ is nicely fit by an asymmetric Lévy PDF, $\mathscr{L}$, with a scale parameter $\alpha = 1.06 \pm 0.02$, a skewness parameter $\beta = 0.70 \pm 0.02$, where $\mu = 1.55 \pm 0.02$ and $\sigma = 0.81 \pm 0.02$ (see Eq. **1**). Because $\beta = 0.70 \pm 0.02$ is close to a limit of unity for an $\mathscr{L}$ PDF, the PDF exhibits (significant) asymmetry. We stress that for the Zmijewski score ML gives $\alpha = 1.87 \pm 0.02$ and $\beta = 0.89 \pm 0.08$ in agreement with Fig. 1*B* and [**5**] where the largest $\zeta = 0.62$.

The parameters obtained for the Altman Z score represent the average behavior of the Z score for the data collected over one decade. To test the stability of the parameters, in particular α, which is supposed to be stable in the Lévy formalism, we next fit the Z score on the $\mathscr{L}$ PDF of Eq. **1** for each quarter over the last decade. In Fig. 2*B* we find that both α and β exhibit a striking stability over time. Note that the Lévy distribution was first proposed to describe commodities and the broad central region of the PDF of price changes. Now we find that the generalized asymmetric Lévy distribution describes even multiple credit scores.

In our approach where we take the Z score as a proxy for a continuous credit rating, ΔZ quantifies the credit rating migrations. In Fig. 3*A*, for each of five ratios comprising the Altman score of Eq. **3** and, in Fig. 3*B*, for each of three ratios comprising the Zmijewski score of Eq. **4**, we find that the Zipf plot of changes in ratio, ΔX_i, also follows a power law. For each ratio, first, the positive and negative tails of ΔX_i virtually overlap and both follow a power law. Second, the largest Zipf exponent—the smallest power-law exponent of the PDF (see Eq. **2**)—we find for ΔX_4 corresponding to changes in the market value of equity divided by the book value of total liabilities. In addition to X_4, for the rest of ratios the Zipf $\zeta \approx 1$. Motivated by this finding and the relation between ζ and ζ' in Eq. **2**, for each of eight financial ratios, we find applying the ML method that $P(\Delta X_i)$ is well fit by an $\mathscr{L}$ PDF

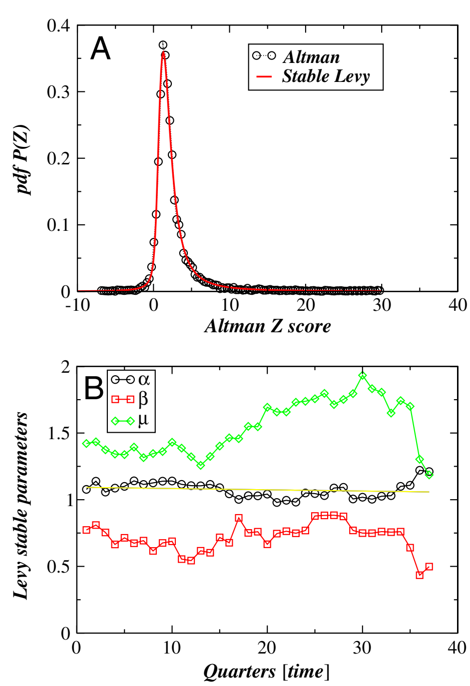

Fig. 2. (*A*) Asymmetric Lévy PDF of the Altman Z score, quarterly recorded. The stable parameter $\alpha = 1.06$ implies infinite variance in Z. (*B*) Stability of asymmetric Lévy parameters calculated for each quarter between 2000 and 2009 supporting the usage of Lévy PDFs.

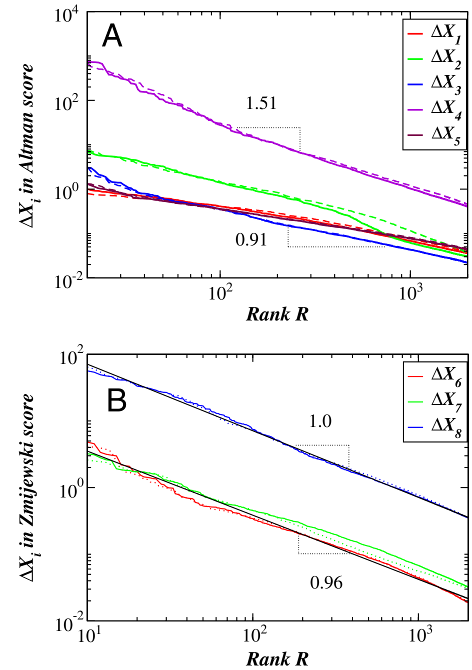

Fig. 3. Power laws in changes of eight financial accounting ratios. For each of the five ratios X_i comprising (*A*) the Altman Z score and each of the three ratios X_i comprising (*B*) the Zmijewski Z_m score, the Zipf plot of ΔX_i in the right and left tails follows a power law.

with parameters α and β reported in Table 2. For each ratio, asymmetry in ΔX_i measured by β is much smaller (significant for X_1, X_2, and X_7) than for X_i, and that the parameter α is relatively close to 1. This result is surprising and reveals the complexity of the stochastic process responsible for ratio changes, because in Fig. 1 we find that the power-law Zipf exponent ζ for eight ratios X_i is highly diversified, where ζ ranges between 0.19 and 1.72. Finally, for each quarter over the last decade, for each of eight ratios, we fit ΔX_i on the $\mathscr{L}$ PDF and find a temporal stability in α and β (in Table 3, we report average values for α and β).

Because ΔX_i for each ratio follows a power law, from [**5**] we expect that changes in the Altman Z score and changes in the Zmijewski score will also follow a power law. First, we analyze $P(\Delta Z)$, disregarding the initial Z score (unconditional analysis), and then we analyze $P(\Delta Z|Z)$ taking into account the initial Z score. Fig. 4*A* shows $P(\Delta Z)$ for varying time horizons ranging from 3 mo to 2 y. For one quarter the ML approach gives $\alpha = 0.92$, $\beta = -0.11$, $\mu = 0.02$, and $\sigma = 0.17$. For $\Delta Z = 1,2,\ldots,8$ we obtain $\langle \alpha \rangle = 0.98 \pm 0.02$. A positive ΔZ is associated with rating upgrades, and a negative ΔZ with rating downgrades, including bankruptcy. For each time horizon Δt, the PDF in the central region exhibits an approximately symmetric form, implying that the probability that a given company will increase its Altman Z score approximately equals the probability that it will decrease its Altman Z score. As expected, if we increase the time horizon, the peak of the PDF will decrease, because the probability that the company will retain its current rating score decreases with Δt.

We next focus on the tails of the distribution of rating changes, $P(\Delta Z)$, for two choices of time horizon: 3 mo (Fig. 4*B*) and 1 y (Fig. 4*C*). We find that negative and positive tails are nearly iden-

Table 2. Difference in ratio

Δ Ratio	ΔX_1	ΔX_2	ΔX_3	ΔX_4	ΔX_5	ΔX_6	ΔX_7	ΔX_8
α	1.19	0.91	1.05	0.80	1.25	0.92	1.17	0.96
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
β	−0.12	−0.12	−0.02	−0.03	−0.05	−0.03	0.14	0.01
	(0.03)	(0.03)	(0.03)	(0.03)	(0.04)	(0.03)	(0.03)	(0.03)

Table 3. Stability in difference of ratio

Δ Ratio	ΔX_1	ΔX_2	ΔX_3	ΔX_4	ΔX_5	ΔX_6	ΔX_7	ΔX_8
α	1.19	0.92	1.06	0.81	1.24	0.93	1.18	0.96
	(0.08)	(0.07)	(0.08)	(0.06)	(0.07)	(0.07)	(0.10)	(0.06)
β	−0.11	−0.14	−0.00	0.01	−0.02	−0.02	0.13	0.01
	(0.11)	(0.13)	(0.16)	(0.33)	(0.22)	(0.15)	(0.16)	(0.09)

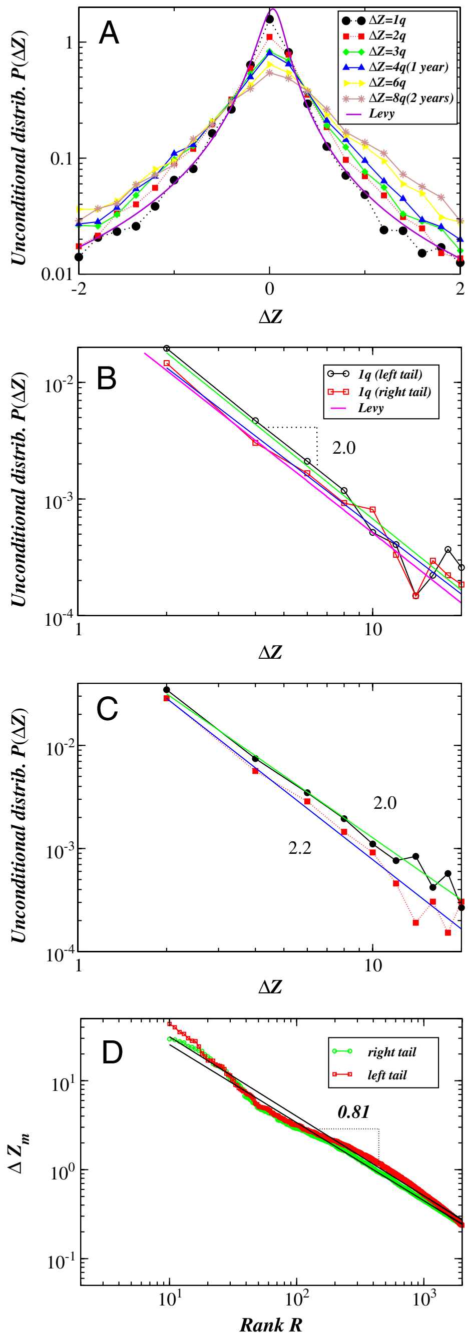

Fig. 4. Power-law tails in $P(\Delta Z)$, for (*A*–*C*) the Altman *Z* score of Eq. **3** and (*D*) the Zmijewski Z_m score of Eq. **4**. (*A*) The central part of $P(\Delta Z)$ for different choices of time horizon Δt in agreement with a Lévy PDF. Right and left tails of $P(\Delta Z)$ for (*B*) 3 mo (a quarter) and (*C*) 1 y time horizon. The tails of the PDF follow a power law with exponent $\alpha \approx 2$. $P(\Delta Z)$ is nicely fit by the single Lévy PDF with $\alpha = 0.93$ and $\gamma = 0.17$ in both (*A*) the central and (*B*) the tail part of PDF. (*D*) For the Zmijewski Z_m score, we find an approximate Zipf law ($\zeta = 0.81$) in the right and left tails of ΔZ_m over the 3 mo time horizon. The Zipf exponent is close to the power-law exponent $\alpha + 1 \approx 2$ found in *B*–*C* for the Altman *Z* score (see Eq. **2**).

tical, and that the tails of $P(\Delta Z)$ follow a power law with exponent $1 + \alpha \approx 2$. For a 1 y time horizon we find that the negative tail still follows a power law with $1 + \alpha \approx 2$ and that the positive tail follows a power law with $1 + \alpha \approx 2.2$. The form of the PDF $P(\Delta Z)$ calculated for different time horizons indicates the relative stability in the tails, and that property is attributed to stable probability distributions. We find that the PDF $P(\Delta Z)$ in both the central region (Fig. 4*B*) and the tails (Fig. 4*B*) follows the Lévy distribution with a power-law exponent $\alpha \approx 1$. We also use the ML approach and again find that $P(\Delta Z)$ is less asymmetric than $P(Z)$, where $\alpha = 0.92 \pm 0.02$, $\beta = -0.11 \pm 0.03$ [$\beta = 0.70 \pm 0.02$ for $P(Z)$], $\mu = 0.02 \pm 0.01$, and $\sigma = 0.17 \pm 0.01$. The power-law exponent α is in agreement with the Zipf exponent $\zeta = 1.21$ obtained for ΔZ (see Eq. **2**).

To find out whether the scaling results reported in Fig. 4*A*–*C* are invariant in the choice of rating, we next study the Zmijewski score Z_m of Eq. **4**. For a 3 mo time horizon, using the Zipf plot of ΔZ_m in Fig. 4*D*, we find that the negative and positive tails follow each other with the Zipf exponent $\zeta = 0.81 \pm 0.003$. We estimate the standard error on the power-law exponent using the method presented in ref. 26. For ΔZ_m, ML approach with $\mathscr{L}$ PDF gives $\alpha = 1.09 \pm 0.02$, $\beta = 0.10 \pm 0.03$, $\mu = -0.03 \pm 0.003$, and $\sigma = 0.10 \pm 0.002$. We demonstrate, using the ML and Zipf ranking approaches, that the empirical power-law regularities appear to be invariant with respect to the choice of rating.

We next test whether the scaling results found in α parameter in Figs. 1–4 are affected by economic crises. We perform the same analysis as in Fig. 1, but now for data covering the crisis period between 2007 and 2009. In Fig. 5*A* we show both the right $1 + \alpha \approx 1.99 \pm 0.15$ and left $1 + \alpha \approx 2.16 \pm 0.29$ tails of the quarterly changes in the Altman *Z* score, ΔZ. The results confirm that even during a period of stock market crashes, the distribution of rating changes in the tails follows a power law with exponent $1 + \alpha \approx 2$. In $P(\Delta Z)$ the temporal stability we find in α is in agreement with the stability we previously found in $P(Z)$ (see Fig. 2*B*).

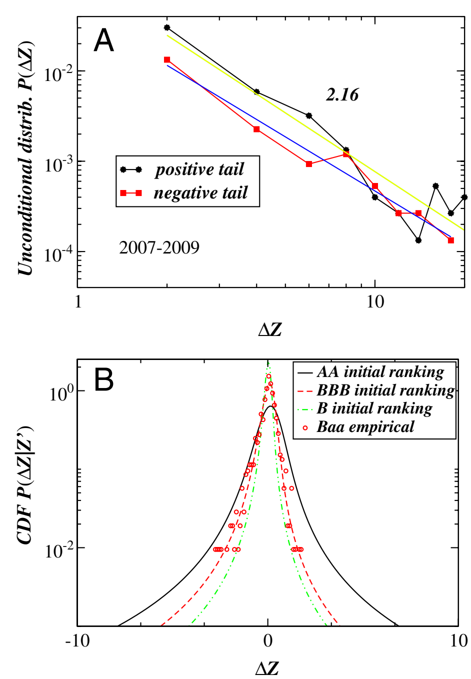

Fig. 5. (*A*) Power-law stability in distribution of rating changes for data covering the period 2007–2009. For the quarterly changes in the Altman *Z* score, shown are both the right and the left tails. During economic crises, the distribution of rating changes in the tails follows a power law with exponent $\alpha \approx 2$, as found for 10-y period in Fig. 1. (*B*) $P(\Delta Z|i)$ for different choice of initial ranking *i* where each is fit on the $\mathscr{L}$ PDF. We also show the empirical data for the initial rating *Baa*.

In finance one commonly calculates not the unconditional but the CDF $P(f|i)$ that the initial credit rating i will change to f over the next period. We set the initial rating to the alphabetic S&P 500 $i = AA$ rating that approximately corresponds to the Z range of 4.25 to 5.5 found in ref. 14. We find that the tails of $P(\Delta Z|AA)$ follows a power law with exponent $1 + \alpha \approx 2.15 \pm 0.21$. The ML approach gives $\alpha = 1.37$, $\beta = -0.18$, $\mu = 0.12$, and $\sigma = 0.46$. In Fig. 5*B*, we show three CDF $P(\Delta Z|i)$ for a different choice of initial ranking $i = Z'$ where each CDF is obtained by fitting empirical data on the $\mathscr{L}$ PDF.

Finally, the previous analysis is accomplished by aggregating the data of 488 manufacturing firms for each ratio implying that the outcomes based on the fitted Lévy distributions reflect the average behavior of the entire sector, not the single firms. Next, for each of the 237 firms for which the data include 37 quarter records, we fit the Z and ΔZ with $\mathscr{L}$ and obtain for the average α values 1.70 ± 0.40 and 1.48 ± 0.34, respectively. For some of individual financial ratios, we also fit the X and ΔX with $\mathscr{L}$. For the average α of X_i and ΔX_i we obtain, X_3 (1.53 ± 0.34, 1.37 ± 0.40), X_6 (1.40 ± 0.35, 1.22 ± 0.38), and X_8 (1.70 ± 0.37, 1.48 ± 0.33). The values are different than those obtained for the aggregated data, but still in the Lévy range. This result opens an interesting and intriguing question about the firm homogeneity across the entire market. Can we expect that each firm, even only manufacturing, is governed by the same dynamics? Clearly, to answer this question we need more data and thus a much longer time series.

Application to Credit Risk Modeling

Credit risk has become perhaps the key risk management challenge of the late 1990s. A firm's credit risk reflects all possible credit migrations, i.e., it is quantified not only in terms of the possibility of a company's filing for bankruptcy but also in terms of any upgrades and downgrades in its credit rating. Suppose a company initially has a Moody's credit rating of $i = Baa$, which corresponds to a range of Z values in our continuous credit rating (14, 20). When a two-year \$100 million (5% loan) is revalued at the end of the first year, after a credit event has occurred during that year, the new loan value (in millions of dollars) (19, 20) will be, as follows:

$$B_{i,f} = 5 + \frac{105}{1 + r + s_f}, \qquad [6]$$

where r is the risk-free rates and s_f is the annual credit spread on zero coupon loans of a particular rating class f. The credit risk is due to all possible credit rating migrations, from i to f, but also because of variations in $B_{i,f}$ values, which depend on the value of the final credit rating f.

In the Moody's discrete alphabet credit ratings, the number of $B_{i,f}$ values is finite because the number of different ratings is finite. In contrast, in our numerical approach the Z score is a proxy for a continuous credit rating, and therefore the ΔZ quantifies the rating transition (migration). To calculate all $B_{i,f}$ values, based on Monte Carlo simulations, we first need to know the CDF of credit rating migration given initial rating Baa, $P(\Delta Z|Baa)$, which we find empirically (Fig. 5*B*). We then need the $r + s$ values for each ΔZ, which quantifies the transition from i to f. In Fig. 6*A*, based on mapping between discrete alphabetic credit ratings and Z values (20), we show the $r + s$ values versus the ΔZ. In calculating $r + s$ versus ΔZ beyond the mapping, we reasonably assume that for companies with the largest possible ΔZ (those that are least risky when $\Delta Z \gg 1$) $r + s$ tends toward a risk-free rate r. In contrast, for companies where the ΔZ goes to extremely negative values ($\Delta Z \ll -1$) we may expect bankruptcy. Using a continuous approach with a smaller number of parameters instead of a discrete rating approach enables us to more clearly distinguish between companies that are close to bankruptcy from those with only a moderately bad ΔZ score. Because we assume two asymp-

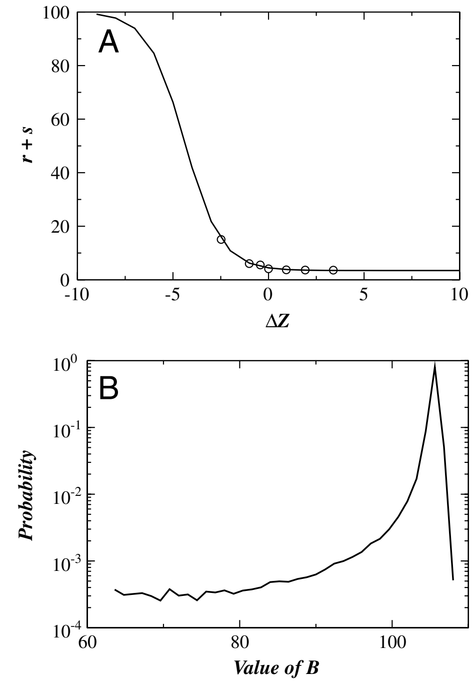

Fig. 6. (*A*) Risk-free rates plus credit spread versus ΔZ. We extrapolate the functional dependence in agreement with hyperbolic tangent. (*B*) The PDF of loan values has a short upside and a long downside tail.

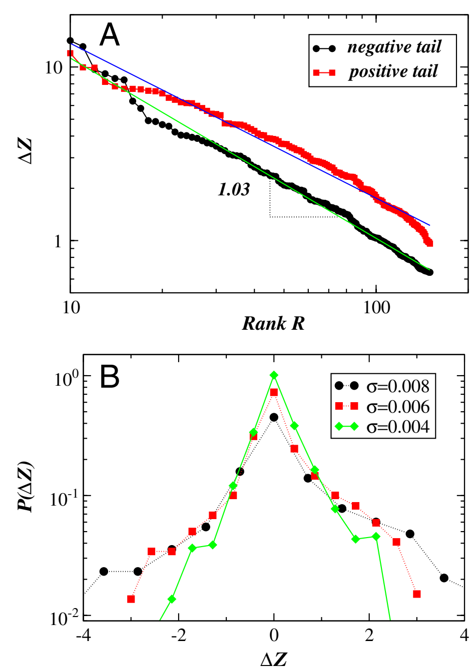

Fig. 7. Model simulations generating changes in Altman ΔZ score in agreement with power-law findings in empirical data. We generate each of eight financial variables needed to generate the Altman Z score: total assets, $A(t_i) = 1$; total liabilities, $D(t_i) = d_2$; retained earnings, $RE(t_i) = d_3$; earnings before interest and taxes, $EBIT(t_i) = d_4$; market value of equity, $MCAP(t_i) = d_5$; sales, $S(t_i) = d_6$; current assets, $CA(t_i) = d_7$; and current liabilities, $CL(t_i) = d_8$. Here d_i are set to the averages across manufacturing companies over the period 2000–2009. Shown are (*A*) the Zipf plots of ΔZ and (*B*) the central part of ΔZ. For geometric Brownian motion we use a Gaussian PDF with mean $\mu = 0$ and standard deviation $\sigma = 0.006$. Because the smallest Δt in simulations is one hour, this hourly σ corresponds to annual $\sigma \approx 0.27$. In *B* we show how the spread of $P(\Delta Z)$ increases with increasing σ.

totic limits for $r+s$ versus ΔZ, we fit this dependence to a hyperbolic tangent,

$$a \tanh(bx+c)+d. \quad [7]$$

We set the lower asymptotic limit ($r+s=1$ when $\Delta Z \ll -1$) to calculate a recovery rate (20, 27) of approximately 50% after bankruptcy is declared ($r+s$ is calculated when $B_{if} \approx 0.5 \cdot 105$ in Eq. **6**).

We next apply the previous approach to assess 1% risk as a specified percentile level for the portfolio value distribution (28). The lowest value that the portfolio will achieve 1% of the time is the first percentile. We then perform Monte Carlo simulations. For each simulation we generate ΔZ from $P(\Delta Z)$, and based on ΔZ we calculate B_{if} in Eq. **6** by using [**7**]. In Fig. 6*B* we show the PDF of loan values due to the increase and decrease of Z values. The PDF has a rapidly decreasing upside tail and a long downside tail, as found in empirical data on loan values with a *Baa* initial rating (20). Having this PDF one may estimate the 1% risk by calculating the B_1 value below, which there are 1% of all B values.

In our approach, stochasticity exists in credit rating migrations, and interest rate and credit rating are deterministically related (**7**). Our approach contradicts, e.g., the Black–Derman–Toy model (29), where the interest rate is stochastically evolved and follows a lognormal process. Now we demonstrate how we calculate the price of a bond maturing at time T, when applying the same approach to bond options (29). We subdivide a period between 0 and T on, e.g., n steps, each representing one quarter. If the option expiration date T is 3 y, then $n=12$. In the first step, having information about the initial ranking i we apply a CDF of migration $P(\Delta Z|i)$ of Fig. 5*B* to determine a new ranking f', where $\Delta Z = f' - i$. The new ranking, f', in the previous step is also the initial ranking i' for the next step. After 12 steps we are able to calculate the final ranking f. By performing Monte Carlo simulations on the exercise date we obtain the final credit ranking, and also the final bond value, using formulas similar to Eq. **6** and [**7**].

Summary and Conclusion

Recently we have witnessed rapid growth in the study of power-law tail phenomena in economics and finance (1, 2, 4, 5, 9, 30–35). We model the power-law scaling properties of credit rating changes using a multivariant Simon model, which is an extension of the Simon model used in the theory of firm growth (36). We perform 100,000 Monte Carlo time steps, and for each existing company, calculate the Z score. We set the time step to be 1 h, define a working day to be eight working hours, and a working year to be ≈250 working days. Hence 100,000 steps represent ≈50 y. We calculate the Z score after 90,000 time steps and after 92,000 steps, a timespan of ≈1 y. Then we calculate $P(\Delta Z)$ over the year. For $\sigma = 0.006$ in Fig. 7*A*, the tail is well fit by a power law with exponent ≈2, as is found in the data. In Fig. 7*B*, using numerical simulations, we calculate that the choice for σ in the Gaussian distribution determines the spread of $P(\Delta Z)$. It is the rich get richer formalism that naturally leads to fat power-law tails in the distribution of rating changes. Geometric Brownian motion is needed to assure the spread in $P(\Delta Z)$.

Lévy PDFs were first proposed in finance to describe the commodity and price changes. We find asymmetric Lévy PDFs, $\mathscr{L}$, in multiple credit ratios and changes of individual financial ratios, ΔX_i, related to credit rating changes and hence credit risk. Although power-law exponents in ratios X_i are highly diversified, surprisingly, ΔX_i are all fit by the power laws of a Lévy stable regime. Existence of the Lévy PDFs in financial ratios has an important implication: It calls for the development of a statistical approach based on infinite variances (37).

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