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A $U_A(1)$ symmetry restoration scenario supported by the generalized Witten–Veneziano relation and its analytic solution

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ABSTRACT

The Witten–Veneziano relation, or, alternatively, its generalization proposed by Shore, facilitates understanding and describing the complex of η and η' mesons. We present an analytic, closed-form solution to Shore's equations which gives results on the η–η' complex in full agreement with results previously obtained numerically. Although the Witten–Veneziano relation and Shore's equations are related, the ways they were previously used in the context of dynamical models to calculate η and η' properties, were rather different. However, with the analytic solution, the calculation can be formulated similarly to the approach through the Witten–Veneziano relation, and with some conceptual improvements. In the process, one strengthens the arguments in favor of a possible relation between the $U_A(1)$ and $SU_A(3)$ chiral symmetry breaking and restoration. To test this scenario, the experiments such as those at RHIC, NICA and FAIR, which extend the RHIC (and LHC) high-temperature scans also to the finite-density parts of the QCD phase diagram, should pay particular attention to the signatures from the η'–η complex indicating the symmetry restoration.

1. Introduction

Among the most intriguing results from RHIC are those on the increased η' multiplicities found in the $\sqrt{s_{NN}} = 200$ GeV central Au + Au reactions [1,2], since Csörgő et al. [3–5] established that this implied that the vacuum value of the η' meson mass, $M_{\eta'} = 957.8$ MeV, was reduced by at least 200 MeV inside the fireball. This was interpreted as the "return of the prodigal Goldstone boson" predicted as a signal of the $U_A(1)$ symmetry restoration [6]. Namely, $M_{\eta'}$ is so very high due to the nonabelian, "gluon" axial anomaly breaking the $U_A(1)$ symmetry and so precluding η' from being the 9th (almost-)Goldstone boson of QCD.

Nevertheless, it may seem somewhat surprising that the $U_A(1)$ symmetry restoration is observed before deconfinement or the restoration of the [$SU_A(N_f)$ flavor] chiral symmetry of QCD (whose dynamical breaking results in light, (almost-)Goldstone pseudoscalar meson octet $P = \pi^0, \pi^\pm, K^0, \bar{K}^0, K^\pm, \eta$ for $N_f = 3$). Namely, the $U_A(1)$ symmetry restoration was expected to occur last, at the temperature (T) scale characterizing the pure-gauge, Yang–Mills (YM) theory, which is significantly higher (by some 100 MeV) than the T scale characterizing the full QCD. For example, consider the Witten–Veneziano relation (WVR) [7,8], seemingly peculiarly relating the four quantities of the full QCD, namely the pion decay constant f_π and η', η and K-meson masses $M_{\eta',\eta,K}$, to YM topological susceptibility $\chi_{\rm YM}$:

$$M_{\eta'}^2 + M_\eta^2 - 2M_K^2 = 2N_f \frac{\chi_{\rm YM}}{f_\pi^2}. \tag{1}$$

It shows that the anomalously large mass of η', nonzero even in the chiral limit due to the breaking of the $U_A(1)$ symmetry, is determined by the ratio of $\chi_{\rm YM}$ and f_π.

WVR is well satisfied at $T = 0$ for $\chi_{\rm YM}$ obtained by lattice calculations (e.g., [9–12]). Nevertheless, the T-dependence of $\chi_{\rm YM}$ is such that the straightforward extension of Eq. (1) to $T > 0$ [13], *i.e.*, replacement of all quantities[1] therein by their respective T-dependent versions $M_{\eta'}(T)$, $M_\eta(T)$, $M_K(T)$, $f_\pi(T)$ *and* $\chi_{\rm YM}(T)$, then follows the (naive) expectation that chiral symmetry restoration occurs significantly before the $\chi_{\rm YM}(T)$ "melting" and the partial $U_A(1)$ symmetry restoration, which leads to a conflict with experiment [3,4].

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[1] Throughout this paper, all quantities are for definiteness assumed at $T = 0$ unless their T-dependence is specifically indicated in formulas or in the text.

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Such a conflict is expected at high T, since WVR relates the four full-QCD quantities ($M_{\eta',\eta,K}$, f_π) with $\chi_{\rm YM}$, a quantity from the pure-gauge, YM theory, where one finds a much larger resilience to increasing T than in QCD, which contains also quark degrees of freedom. We thus conjectured [14] that the experimentally observed η' multiplicities can be explained by invoking the Leutwyler–Smilga (LS) relation [15]. It expresses the full-QCD topological susceptibility χ through the YM topological susceptibility $\chi_{\rm YM}$ (equal to χ in the limit of *quenched* QCD), the *chiral-limit* quark condensate $\langle\bar{q}q\rangle_0$, and the current masses m_q of the $N_f = 3$ light quark flavors. Inverted, and in our notation, the LS relation is

$$\frac{\chi}{1+\frac{\chi}{\langle\bar{q}q\rangle_0}\sum_{q=u,d,s}\frac{1}{m_q}} \equiv \widetilde{\chi} = \chi_{\rm YM} \quad (\text{at } T=0). \tag{2}$$

Ref. [14] proposed that the presence of $\chi_{\rm YM}$ in WVR should be understood in the light of the LS relation (2); *i.e.*, the successful zero-temperature WVR is retained, since $\chi_{\rm YM} = \widetilde{\chi}$ at $T = 0$, but at $T > 0$, $\widetilde{\chi}(T)$ should be used instead of $\chi_{\rm YM}(T)$, avoiding the mismatch of the T-dependences of QCD and YM theory. Since $\widetilde{\chi}$ is a combination of the quantities of the full QCD, it should be much less T-resistant than $\chi_{\rm YM}$. Indeed, employing the light-quark-sector result [15,16] appropriate for the topological susceptibility[2] of the full QCD,

$$\chi = -\frac{\langle\bar{q}q\rangle_0}{\sum_{q=u,d,s}\frac{1}{m_q}} + \mathcal{C}_m \tag{3}$$

(where $\mathcal{C}_m$ denotes corrections of higher orders[3] in small m_q), yields $\widetilde{\chi}(T)$ which falls with T proportionally to the chiral quark condensate $\langle\bar{q}q\rangle_0$ [14]. This way, the (partial) restoration of $U_A(1)$ symmetry is naturally tied to the restoration of the $SU_A(3)$ flavor chiral symmetry and to its characteristic temperature $T_{\rm Ch}$. This scenario enabled Ref. [14] to provide the first explanation of the findings of Csörgő et al. [3,4], since the anomalous part of the η' mass falls together with $\langle\bar{q}q\rangle_0(T)$ as $T \to T_{\rm Ch}$. Some other approaches [18–20] have also provided indirect support to this scenario, but it remained just a conjecture on the T-dependence of WVR, until the present paper. The paper is organized as follows. Section 2 recalls Shore's [21,22] generalization of WVR, and how it was adapted to $q\bar{q}$ bound-state calculations [23]. This entails replacing $\chi_{\rm YM}$ by a quantity to which the LS relation (2), *i.e.*, $\widetilde{\chi}$, is just a large-N_c approximation. This confirms the conjecture of Ref. [14], especially after we present an analytic solution of Shore's equations, and discuss its implications in Section 3. We summarize in Section 4.

2. Analytic solution to generalized Witten–Veneziano relations

Like the LS relation (2), WVR (1) was derived in the lowest-order approximation in the large N_c expansion. Its generalization by Shore is however valid to all orders in $1/N_c$ [21,22]. It consists of several relations, and the ones pertinent for the present paper are those containing the masses of the pseudoscalar nonet mesons:

$$\left(f^0_{\eta'}\right)^2 M^2_{\eta'} + \left(f^0_\eta\right)^2 M^2_\eta = \frac{1}{3}\left(f^2_\pi M^2_\pi + 2 f^2_K M^2_K\right) + 6A, \tag{4}$$

$$f^0_{\eta'} f^8_{\eta'} M^2_{\eta'} + f^0_\eta f^8_\eta M^2_\eta = \frac{2\sqrt{2}}{3}\left(f^2_\pi M^2_\pi - f^2_K M^2_K\right), \tag{5}$$

$$\left(f^8_{\eta'}\right)^2 M^2_{\eta'} + \left(f^8_\eta\right)^2 M^2_\eta = -\frac{1}{3}\left(f^2_\pi M^2_\pi - 4 f^2_K M^2_K\right). \tag{6}$$

Here, A is the full QCD topological charge parameter, namely the quantity which takes the role of $\chi_{\rm YM}$ in WVR,

$$A = \frac{\chi}{1+\chi\left(\frac{1}{\langle\bar{u}u\rangle m_u} + \frac{1}{\langle\bar{d}d\rangle m_d} + \frac{1}{\langle\bar{s}s\rangle m_s}\right)}, \tag{7}$$

given by the full QCD topological susceptibility χ, the current quark masses m_q, and the three condensates $\langle q\bar{q}\rangle$ which differ from each other for different flavors $q = u, d, s$. Since they are not known well enough, Shore himself [21,22], as well as Ref. [23] which adapted his generalization of WVR to the Dyson–Schwinger (DS) bound-state approach, had to approximate A by values of $\chi_{\rm YM}$ [21–23] found on lattice. This is a good approximation at least in the large N_c limit, as $A = \chi_{\rm YM} + \mathcal{O}(1/N_c)$ [consistent with, *e.g.*, the large-N_c relation (2)]. On the other hand, in the chiral limit ($m_q \to 0, \forall q$), all $\langle q\bar{q}\rangle$ condensates tend to the chiral one, $\langle q\bar{q}\rangle_0$. Since this limit is not far from the real world even when the strange flavor is included, the possibility of confirming the conjecture of Ref. [14], and thus of better understanding the experimental results [3,4] signaling the restoration of $U_A(1)$ symmetry, becomes apparent. Namely, when $\langle\bar{u}u\rangle, \langle\bar{d}d\rangle, \langle\bar{s}s\rangle \to \langle q\bar{q}\rangle_0$, the topological charge parameter A (7) reduces to the quantity $\widetilde{\chi}$ defined by Eq. (2). This obviously supports our conjecture [14] that $\widetilde{\chi}(T)$ determines the T-dependence for the anomalous mass.

Eqs. (4)–(6) take into account that η and η' possess *two* decay constants each [24–26], *i.e.*, f^0_η, f^8_η and $f^0_{\eta'}, f^8_{\eta'}$, since η and η' are mixtures of the $SU(3)$ singlet and octet basis states η_0 and η_8. These four decay constants can be parametrized in terms of two auxiliary decay constants and two angles; *e.g.*, the purely octet and singlet decay constants f_8 and f_0, and the mixing angles θ_8 and θ_0:

$$\begin{bmatrix} f^8_\eta & f^0_\eta \\ f^8_{\eta'} & f^0_{\eta'} \end{bmatrix} = \begin{bmatrix} \cos\theta_8 & -\sin\theta_0 \\ \sin\theta_8 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} f_8 & 0 \\ 0 & f_0 \end{bmatrix}. \tag{8}$$

For realistic quark masses, θ_8 and θ_0 are rather different both from each other and from θ, the mixing angle of the states η_8 and η_0 into η and η' [24–30]. Only in the limit of the exact $SU(3)$ flavor symmetry, $\theta_8 = \theta_0 = \theta = 0$.

If, instead of the $SU(3)$ basis states η_0 and η_8, one uses the nonstrange–strange (NS–S) basis, $\eta_{\rm NS} = (u\bar{u} + d\bar{d})/\sqrt{2}$ and $\eta_S = s\bar{s}$, one obtains

$$\begin{bmatrix} f^{\rm NS}_\eta & f^{\rm S}_\eta \\ f^{\rm NS}_{\eta'} & f^{\rm S}_{\eta'} \end{bmatrix} = \begin{bmatrix} \cos\phi_{\rm NS} & -\sin\phi_{\rm S} \\ \sin\phi_{\rm NS} & \cos\phi_{\rm S} \end{bmatrix} \begin{bmatrix} f_{\rm NS} & 0 \\ 0 & f_{\rm S} \end{bmatrix}. \tag{9}$$

where $f_{\rm NS}$ and f_S are given by the matrix elements of $A_{\rm NS}$ and $A_{\rm S}$, the NS and S axial currents of quarks:

$$\langle 0|A^\mu_{\rm NS(S)}(x)\big|\eta_{\rm NS(S)}(p)\big\rangle = i f_{\rm NS(S)}\, p^\mu e^{-ip\cdot x}, \tag{10}$$

whereas $\langle 0|A^\mu_{\rm NS}(x)|\eta_{\rm S}(p)\rangle = 0$, $\langle 0|A^\mu_{\rm S}(x)|\eta_{\rm NS}(p)\rangle = 0$.

Differing just by the choices of bases, these two sets of decay constants are simply related (*e.g.*, see [30,31]):

$$\begin{bmatrix} f^{\rm NS}_\eta & f^{\rm S}_\eta \\ f^{\rm NS}_{\eta'} & f^{\rm S}_{\eta'} \end{bmatrix} = \begin{bmatrix} f^8_\eta & f^0_\eta \\ f^8_{\eta'} & f^0_{\eta'} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \tag{11}$$

and completely equivalent *in principle*. Still, there is a big *practical difference*: in the NS–S basis, FKS [28–30] managed to recover a scheme with a single angle ϕ, which also plays the familiar role of the state mixing angle describing the rotation of the NS–S basis states into the mass (squared) eigenstates – the physical η and η' mesons:

$$\eta = \cos\phi\,\eta_{\rm NS} - \sin\phi\,\eta_{\rm S}, \qquad \eta' = \sin\phi\,\eta_{\rm NS} + \cos\phi\,\eta_{\rm S}. \tag{12}$$

[2] Although lattice calculations of the topological susceptibility with light quarks are much harder than those of $\chi_{\rm YM} = \chi_{\rm quenched}$, note that recent lattice calculations also yield the full QCD topological susceptibility which vanishes in the limit of a vanishing quark mass – for example, see [17].

[3] Nevertheless, having $\mathcal{C}_m \neq 0$ is essential so that the LS relation (2) with Eq. (3) can yield relatively large but finite $\chi_{\rm YM}$.

Of course, this is done at the expense of the full generality, but also without losing essential physics, making reasonable approximations by applying the Okubo–Zweig–Iizuka (OZI) rule [28–30]; *e.g.*, $f_{\rm NS} f_{\rm S} \sin(\phi_{\rm NS} - \phi_{\rm S})$ differs from zero just by an OZI-suppressed term [30]. Neglecting it therefore implies $\phi_{\rm NS} = \phi_{\rm S}$. That is, applications of the OZI rule lead to the FKS approximation scheme [28–30], which exploits the practical difference between the parameterizations (8) and (9): θ_8 and θ_0 much differ from each other and from the η_8–η_0 *state* mixing angle $\theta \approx (\theta_8 + \theta_0)/2$, but the NS–S decay-constant mixing angles are very close to each other and both can be approximated by the state mixing angle: $\phi_{\rm NS} \approx \phi_{\rm S} \approx \phi$. It is thus a reasonable approximation to use only this one angle, ϕ, and express (see, *e.g.*, [23,30,31]) the physical η–η' decay constants as

$$\begin{bmatrix} f_\eta^8 & f_\eta^0 \\ f_{\eta'}^8 & f_{\eta'}^0 \end{bmatrix} = \begin{bmatrix} f_{\rm NS}\cos\phi & -f_{\rm S}\sin\phi \\ f_{\rm NS}\sin\phi & f_{\rm S}\cos\phi \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}. \tag{13}$$

This result of the FKS scheme was inserted in Shore's equations (4)–(6) already in Ref. [23], but only numerical solutions were found there (after some additional assumptions, see below). In contrast, now we have found analytic, closed-form solutions of combined Eqs. (4)–(6) and (13) for ϕ and the masses of η and η'. The relevant set of solutions, where $M_{\eta'} > M_\eta$, is:

$$\tan\phi = \frac{-f_{\rm NS}^2 + 2f_{\rm S}^2}{2\sqrt{2} f_{\rm NS} f_{\rm S}} - \frac{2f_{\rm NS}^2 f_K^2 M_K^2 - f_{\rm NS}^2 f_\pi^2 M_\pi^2 - f_{\rm S}^2 f_\pi^2 M_\pi^2 - \Delta}{4\sqrt{2} A f_{\rm NS} f_{\rm S}}, \tag{14}$$

$$M^2_{\eta'(\eta)} = \frac{A}{f_{\rm S}^2} + \frac{2A}{f_{\rm NS}^2} + \frac{2f_{\rm NS}^2 f_K^2 M_K^2 - f_{\rm NS}^2 f_\pi^2 M_\pi^2 + f_{\rm S}^2 f_\pi^2 M_\pi^2 + (-)\Delta}{2 f_{\rm NS}^2 f_{\rm S}^2}, \tag{15}$$

$$\Delta^2 = 32 A^2 f_{\rm NS}^2 f_{\rm S}^2 + \left[2A\left(f_{\rm NS}^2 - 2f_{\rm S}^2\right) + 2f_K^2 f_{\rm NS}^2 M_K^2 - f_\pi^2\left(f_{\rm NS}^2 + f_{\rm S}^2\right)M_\pi^2\right]^2. \tag{16}$$

The major obstacle to evaluating these results may seem to be the lack of information[4] on $f_{\rm NS}$ and $f_{\rm S}$, the decay constants of the unphysical pseudoscalars $\eta_{\rm NS}$ and $\eta_{\rm S}$. However, the guidance is provided by the nature of the FKS scheme, which neglects OZI-violating contributions, *i.e.*, possible gluonium admixtures in $\eta_{\rm NS}$ and $\eta_{\rm S}$. It is then reasonable to treat them as pure $q\bar{q}$ states, whereby $f_{\rm NS} = f_{u\bar{u}} = f_{d\bar{d}} = f_\pi$ (in the isospin symmetry limit), and $f_{\rm S} = f_{s\bar{s}}$, the decay constant of the fictitious $s\bar{s}$ pseudoscalar meson. Then the analytic, closed-form solutions (14), (15) and (16) become

$$\tan\phi = \frac{-f_\pi^2 + 2f_{s\bar{s}}^2}{2\sqrt{2} f_\pi f_{s\bar{s}}} - \frac{2f_K^2 f_\pi^2 M_K^2 - f_\pi^4 M_\pi^2 - f_\pi^2 f_{s\bar{s}}^2 M_\pi^2 - \Delta}{4\sqrt{2} A f_\pi f_{s\bar{s}}}, \tag{17}$$

$$M^2_{\eta'(\eta)} = \frac{A}{f_{s\bar{s}}^2} + \frac{2A}{f_\pi^2} + \frac{2f_K^2 f_\pi^2 M_K^2 - f_\pi^4 M_\pi^2 + f_\pi^2 f_{s\bar{s}}^2 M_\pi^2 + (-)\Delta}{2 f_\pi^2 f_{s\bar{s}}^2}, \tag{18}$$

$$\Delta^2 = 32 A^2 f_\pi^2 f_{s\bar{s}}^2 + \left\{-2A\left(f_\pi^2 - 2f_{s\bar{s}}^2\right) + f_\pi^2\left[-2f_K^2 M_K^2 + \left(f_\pi^2 + f_{s\bar{s}}^2\right)M_\pi^2\right]\right\}^2. \tag{19}$$

The situation that $\eta_{\rm NS}$ and $\eta_{\rm S}$ are pure $q\bar{q}$ states, so that $f_{\rm NS} = f_\pi$ and $f_{\rm S} = f_{s\bar{s}}$, is realized, for example, in the DS approach in the rainbow-ladder approximation (RLA). There, mesons are pure $q\bar{q}$ solutions (of Bethe–Salpeter equations), without any gluonium admixtures, which would be prominent possible sources of OZI violations. The FKS scheme is well-suited for the usage in such a context which is in agreement with the OZI rule. In a bound-state approach, notably the DS approach used in Ref. [23], the decay constants are quantities calculated from the $q\bar{q}$ substructure of mesons. Ref. [23] used three different dynamical models for the nonperturbative gluon interactions (in the DS gap and Bethe–Salpeter equations) yielding $q\bar{q}$ meson solutions reproducing well the empirical light meson masses and decay constants including the presently important M_π, M_K, f_π and f_K. Along with the auxiliary but unphysical $f_{s\bar{s}}$, these results enabled the numerical solutions describing well the η–η' complex in Ref. [23]. Now, the same model results used in the analytic solutions (17)–(18) reproduce accurately these numerical solutions of Ref. [23] for all DS models used, and for the same lattice values for $\chi_{\rm YM}$ used in Ref. [23] in the approximation $A \approx \chi_{\rm YM}$. (For comparisons with Ref. [23], we use the same $\chi_{\rm YM}$ to evaluate Table I, *i.e.*, *the weighted average* $\chi_{\rm YM} = (0.1757\ {\rm GeV})^4$ as in Refs. [13,33] and $\chi_{\rm YM} = (0.191\ {\rm GeV})^4$ [10] (used by Shore [21,22]).)

Moreover, the need for a specific dynamical model to evaluate the auxiliary, unphysical quantity $f_{s\bar{s}}$ can be circumvented, since the chiral expansion indicates that it is a good approximation to express $f_{s\bar{s}} = 2f_K - f_\pi$. Then the analytic solutions (17)–(18) can be evaluated by inserting exclusively the empirical values of the masses M_π, M_K and decay constants f_π, f_K, yielding a model-independent description of the η–η' complex and suffering theoretical uncertainties only due to choosing $A \approx \chi_{\rm YM}$ and the FKS scheme, including the OZI rule.

The results obtained in this way are summarized in Table 1, showing they are quite similar to those of Ref. [23] for all rather different bound-state models used there.

Table 1
The values adopted for A are the two lattice values of $\chi_{\rm YM}$ used in Ref. [23]. The other inputs (in the small table on the left, taken from Ref. [34]) are the experimental values for $M_{\pi^0,K}$, $f_{\pi^-,K}$. Everything else, starting with M_η and $M_{\eta'}$, are the calculated quantities. All quantities are given in MeV, except the angles ϕ, θ, θ_0 and θ_8, which are in degrees.

M_{π^0}	134.977
M_K	493.677
f_{π^-}	92.2138
f_K	110.379

$A^{1/4}$	175.7	191
M_η	488.4	503.2
$M_{\eta'}$	832.7	949.8
ϕ	45.07°	50.92°
θ	−9.664°	−3.816°
θ_0	−0.341°	5.507°
θ_8	−18.03°	−12.18°
f_0	105.7	105.7
f_8	117.7	117.7
f_η^0	0.630	10.15
$f_{\eta'}^0$	105.7	105.2
f_η^8	111.9	115.0
$f_{\eta'}^8$	−36.43	−24.84

[4] Recently, Ref. [32] presented a lattice analysis of $f_{\rm NS}$ and $f_{\rm S}$ (denoted by f_l and f_s there), which is nevertheless still affected by residual lattice artefacts, quark mass dependence and chiral perturbation theory approximation used there.

3. Discussion of results

For the choice of larger χ_{YM}, the results in Table 1 are also reasonably close to the η–η' studies, such as [31,33,35], using the same dynamical DS models as Ref. [23] to get the pion and kaon masses and decay constants, but the standard η–η' mass matrix in conjunction with WVR to describe η–η' complex.

Thanks to the existence of the analytic solutions (14)–(18), we can now understand both the similarities and differences between the two η–η' descriptions: the one using (*e.g.*, in [14,33]) the standard WVR, and the other, using Shore's generalization [21,22] thereof in conjunction with the FKS scheme, as in Ref. [23] and here.

In the latter approach, the η–η' mass matrix was not needed [23], but it can readily be constructed; its matrix elements in the NS–S basis are:

$$M_{\mathrm{NS}}^2 = M_\eta^2 \cos^2\phi + M_{\eta'}^2 \sin^2\phi = M_\pi^2 + \frac{4A}{f_\pi^2}, \tag{20}$$

$$M_{\mathrm{S}}^2 = M_\eta^2 \sin^2\phi + M_{\eta'}^2 \cos^2\phi = \frac{1}{f_{s\bar{s}}^2}\left[2f_K^2 M_K^2 - f_\pi^2 M_\pi^2\right] + \frac{2A}{f_{s\bar{s}}^2}, \tag{21}$$

$$M_{\mathrm{NSS}}^2 = \sin\phi\cos\phi\left(M_\eta^2 - M_{\eta'}^2\right) = \frac{2\sqrt{2}A}{f_\pi f_{s\bar{s}}}, \tag{22}$$

where the second equalities in the Eqs. (20), (21) and (22) are obtained through inserting the analytic solutions (14), (15) and (16) with $f_{\mathrm{NS}} = f_\pi$ and $f_{\mathrm{S}} = f_{s\bar{s}}$. Using the DGMOR relations like Shore for $f_\pi^2 M_\pi^2$ and $f_K^2 M_K^2$, enables one to express the decay constant and mass of the unphysical $s\bar{s}$ almost-Goldstone pseudoscalar as

$$2f_K^2 M_K^2 - f_\pi^2 M_\pi^2 = f_{s\bar{s}}^2 M_{s\bar{s}}^2, \tag{23}$$

whereby Eq. (21) becomes $M_{\mathrm{S}}^2 = M_{s\bar{s}}^2 + 2A/f_{s\bar{s}}^2$.

The η–η' mass matrix is [31,33]

$$\hat{M}^2 = \begin{bmatrix} M_{\mathrm{NS}}^2 & M_{\mathrm{NSS}}^2 \\ M_{\mathrm{NSS}}^2 & M_{\mathrm{S}}^2 \end{bmatrix} = \begin{bmatrix} M_\pi^2 + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M_{s\bar{s}}^2 + \beta X^2 \end{bmatrix}, \tag{24}$$

where X is the flavor $SU(3)$-breaking parameter, and $\beta \equiv \Delta M_{\eta_0}/3$ denotes $\frac{1}{3}$ of the $U_A(1)$-anomalous, chiral-limit-nonvanishing part of the mass of the flavor singlet pseudoscalar η_0 in the $SU(3)$-symmetric limit ($X = 1$).

The matrix (24) implies the η and η' masses

$$M_{\eta'(\eta)}^2 = \frac{M_{\mathrm{NS}}^2 + M_{\mathrm{S}}^2 + (-)\sqrt{(M_{\mathrm{NS}}^2 - M_{\mathrm{S}}^2)^2 + 8\beta^2 X^2}}{2},$$

which can also be obtained from the closed-form solutions (14), (15) and (16) using Eqs. (20)–(23), providing a good consistency check.

One of the advantages of the present approach is that the comparison of the matrix elements (20), (21) [inserting (23)] and (22) with the matrix (24) shows that

$$X = \frac{f_\pi}{f_{s\bar{s}}}, \qquad \beta = \frac{2A}{f_\pi^2} \equiv \beta_{\mathrm{S}} \tag{25}$$

follows necessarily. In contrast, Eq. (25) for X is usually just an educated estimate [36], see, e.g. Refs. [28,30,31,33]. Eq. (25) for β also explains why one needs higher values of χ_{YM} (if one approximates $A \approx \chi_{\mathrm{YM}}$) than in the approach employing WVR [14,33]. Namely, there the mass matrix yields β which is larger for the same χ_{YM}. That is,

$$\beta_{\mathrm{WV}} = \frac{6\chi_{\mathrm{YM}}}{(2+X^2)f_\pi^2} > \frac{2\chi_{\mathrm{YM}}}{f_\pi^2}, \tag{26}$$

since $X < 1$ for any realistic flavor symmetry breaking.

4. Summary and outlook

Shore's generalization [21,22] of WVR provides a description of the η–η' complex which, in its original form, is valid to all orders in the large-N_c expansion. We have presented the analytic, closed-form solutions of Shore's equations (4)–(6) combined with the FKS scheme. This was previously solved only numerically [23] (for several $q\bar{q}$ bound-state models), while now we have closed-form, analytic expressions (14)–(20) for the masses and the mixing angle in the η–η' complex, leading to the mass matrix elements (20)–(22). They show explicitly, *e.g.*, why the flavor breaking is necessarily given by $X = f_\pi/f_{s\bar{s}}$ and how the full QCD topological charge parameter A (7) replaces the YM topological susceptibility χ_{YM} appearing in the standard WVR. In general, both the present η–η' description and the corresponding numerical ones in Ref. [23], are much better understood now, as the analytic solutions have in the previous section exposed clearly both similarities and differences with respect to the descriptions of the η–η' complex through the mass matrix and standard WVR (in, *e.g.*, [14,33]). Obviously, some of the generality of the original Shore's approach has been reduced due to the approximations present in the FKS scheme. This scheme is however well founded and, through numerous phenomenological applications, has also been proven to preserve the essential physics of the η–η' complex, which in this context is also an argument for reliability of the η–η' description exploiting WVR [14,33].

It is important to note that in the present paper, the YM topological susceptibility χ_{YM} is used (at $T = 0 = \mu$) only as an approximation of the full QCD topological charge parameter A (7). The latter is, however, not a pure-gauge quantity, but a full QCD quantity, and the LS (2) quantity $\widetilde{\chi}$ is its approximation recovered from A (7) by replacing $\langle u\bar{u}\rangle, \langle d\bar{d}\rangle, \langle s\bar{s}\rangle \to \langle q\bar{q}\rangle_0$.

This relationship between A, Eq. (7), appearing as the fundamental quantity of the WVR generalization [21,22], and $\widetilde{\chi}$, Eq. (2), supports our explanation [14] of the data on the enhanced η'-multiplicity [3,4] in RHIC experiments at $T > 0$, where we replace the T-dependence of χ_{YM} by that of $\widetilde{\chi}(T)$ (2), which is, in essence, the T-dependence of the chiral quark condensate $\langle q\bar{q}\rangle_0(T)$. Such relationship of $U_A(1)$ symmetry breaking to the order parameter of dynamical chiral symmetry breaking indicates more strongly the possibility that similar experimental signals of $U_A(1)$ symmetry restoration be observed in experiments at finite matter density ($\mu > 0$) [37]. Namely, the quark condensate $\langle q\bar{q}(T,\mu)\rangle$ should drop significantly not only with T, but also after the chemical potential μ exceeds some critical value. This motivates the theoretical work [38] on extending the approach of Ref. [14] to $\mu > 0$.

Previous experimental studies at RHIC have already been extended from the high-temperature regime also to the finite density (*e.g.*, see [39]), and more studies, including detailed scans of the μ–T QCD phase diagram, are planned at RHIC, GSI/FAIR, and NICA [37]. The present paper stresses it is important that such experiments look for signatures (primarily related to the η'–η complex) that would test the scenario of Ref. [14] (and related ideas [19,20,40–42]) on the relationship between the $U_A(1)$ and $SU_A(3)$ chiral symmetry breaking and restoration.

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