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Currently he is the Principal Investigator of the Croatian Science Foundation project “New Geometries for Gravity and Spacetime” (IP-2018-01-7615), within which the present thesis was completed.

For more information, visit his personal website.

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ABSTRACT

This thesis is devoted to the study of physical problems in Gauge Theory and Gravity, within the mathematical framework of Graded Geometry. The main advantage of this framework is that it can be used to describe mixed-symmetry tensor fields of any type in a very general and universal way. As such, it is ideal for exploiting similarities and formal analogies between Gauge Theories and Gravity, as well as for putting the latter on equal footing and for studying them in parallel.

After reviewing some basic notions in $\mathbb{Z}_2$-graded geometry, we show how mixed-symmetry tensor fields can be alternatively described as functions on a graded manifold. Subsequently, we define a differential calculus and an integration theory on this graded manifold. Finally, we use this machinery to construct gauge invariant Lagrangians for physical theories in any number of spacetime dimensions, involving gauge fields in arbitrary mixed-symmetry tensor representations of the general linear group. These include kinetic, mass and Galileon interaction terms.

Subsequently, we present an overview of the Hodge dualities present in two of the most fundamental theories of Nature; Electromagnetism and General Relativity (in its linearized limit). In both cases, we review the concept of duality from the on-shell level of the field equation to its off-shell implementation through a first-order parent Lagrangian. On top of that, we extensively discuss the effects induced by a topological ϑ-term in these Hodge dualities.

As a key result, we present a general graded geometric parent Lagrangian encoding all types of Hodge dualities involving any bipartite mixed-symmetry tensor gauge field. This Lagrangian gets further generalized to account for dualities between theories involving multiple gauge fields, in self-dual dimensions. We find that there exist two distinct possibilities for the underlying duality group, depending solely on the type of the dualized gauge field.

Finally, we introduce more formally the gravitational ϑ-term that we used extensively in the dualization discussions and trace its nonlinear origin. We conclude the thesis by mentioning two physical effects induced by this topological term, illustrated through two simple examples.

Keywords: graded geometry, duality, galileons, ϑ-terms, gauge theory, gravity.

PROŠIRENI SAŽETAK NA HRVATSKOM JEZIKU

Ovaj rad posvećen je proučavanju fizičkih problema u baždarnoj teoriji i gravitaciji, koristeći matematičke metode gradirane geometrije. Glavna prednost tog pristupa je u tome što se može koristiti za opisivanje tenzorskih polja mješovite simetrije i bilo kojeg tipa na vrlo općenit i univerzalan način. Kao takav, idealan je za iskorištavanje sličnosti i formalnih analogija između baždarnih teorija i gravitacije, kao i za stavljanje potonjih u ravnopravni položaj, te za njihovo usporedno proučavanje.

U poglavlju 2 uvodimo neke osnovne pojmove u gradiranoj geometriji s fokusom na $\mathbb{Z}_2$-gradiranu ili supergeometriju. Pokazujemo kako se tenzorska polja mješovite simetrije i proizvoljnog tipa na glatkoj mnogostrukosti M mogu opisati na alternativan način, pomoću posebnih izomorfizama između mnogostrukosti M i gradirane (super) mnogostrukosti $\mathcal{M}$. Zatim definiramo diferencijalni račun i teoriju integracije na toj gradiranoj mnogostrukosti. Konačno, koristeći upravo definirane osnovne alate gradirane geometrije, konstruiramo baždarno invarijantne lagranžijane za fizičke teorije u bilo kojem broju prostornovremenskih dimenzija, uključujući baždarna polja u proizvoljnoj reprezentaciji opće linearne grupe tenzorom mješovite simetrije. Tu spadaju kinetički, maseni i galilejevski interakcijski članovi. Naši rezultati pokazuju da svi ti lagranžijani imaju isti, jednostavni, geometrijski oblik, bez obzira na vrstu baždarnog polja. Kao dodatni rezultat naše analize, također dobivamo snažne dokaze koji potkrepljuju tvrdnju da se baždarno invarijantne i lokalne galilejevske interakcije ne mogu konstruirati za baždarna polja višeg spina.

U poglavljima 3 & 4 donosimo pregled Hodgeovih dualnosti prisutnih u dvjema temeljnima teorijama prirode, elektromagnetizmu i općoj relativnosti u njezinom lineariziranom limesu. U oba slučaja dajemo prikaz koncepta dualnosti od razine jednadžbe polja do implementacije dualnosti kroz roditeljski (engl. *parent*) lagranžijan prvog reda. Povrh toga, opširno raspravljamo o učincima induciranim topološkim članom u tim Hodgeovim dualnostima. U slučaju elektromagnetizma, za ovaj član uzima se standardni ϑ-član koji je formiran kao skalarni produkt električnog i magnetskog polja. Za lineariziranu gravitaciju, uzimamo u obzir gravitacijski analog tog topološkog člana, formiran kao skalarni produkt gravitoelektričnog (njutnovskog) i gravitomagnetskog polja. Oba ova topološka člana postoje samo u četiri prostornovremenske dimenzije, a to su dimenzije s kojima radimo u ovim poglavljima.

Što se tiče samih Hodgeovih dualnosti, klasificiramo ih koristeći standardnu terminologiju koja se pojavljuje u literaturi. Posebni slučajevi uključuju električno-magnetsku, egzotičnu i dvostruku dualnost. I električno-magnetska i egzotična dualnost Maxwellovog polja opisane su u poglavlju 3, dok se poglavlje 4 bavi električno-magnetskim i dvostrukim dualnostima lineariziranog gravitona. U tim poglavljima također se usredotočujemo na identificiranje analogija i sličnosti postupaka dualizacije i rezultirajućih formula između elektromagnetizma i linearizirane gravitacije. A posteriori, naši rezultati vode na zaključak da bi se ove dvije naizgled različite analize mogle provesti na općenitiji način.

Ovaj zaključak vodi do razmatranja u poglavlju 5, gdje predstavljamo općeniti gradirani geometrijski roditeljski lagranžijan koji uključuje sve tipove Hodgeovih dualnosti koje uključuju bilo koje tenzorsko baždarno polje mješane simetrije tipa $(p, 1)$. To se postiže uvođenjem dva parametra u lagranžijan, koji mogu poprimiti vrijednosti u četiri određene domene. Svaka od tih domena odgovara drugoj vrsti Hodgeovih dualnosti i, kao takav, roditeljski lagranžijan ima unificirajuću prirodu. Tehnički, ove su izjave formulirane kroz dva teorema, a oba dokazujemo rigorozno. Naši se konačni rezultati podudaraju s onima iz prethodnih poglavlja za određene vrijednosti dvaju parametara, obuhvaćaju sve poznate rezultate u literaturi i, nadalje, protežu se na prije toga nepoznate slučajeve.

U poglavlju 6 dalje poopćujemo našu analizu kako bismo uključili dualnost između teorija koje uključuju više baždarnih polja, u samodualnim dimenzijama. Polazeći od primjera više skalarnih polja u dvije dimenzije, izvodimo postupak dualizacije i dobivamo poznata Buscherova pravila. Ona odgovaraju transformacijama pozadinskih polja generiranih s T-dualnošću u okviru teorije struna. Kao nastavak, provodimo isti postupak za slučaj više Abelovih 1-formi (Maxwellova polja) u četiri dimenzije i dobivamo odgovarajuća "viša" Buscherova pravila za pozadinska polja. Motivirani vezom T-dualnosti i sigma modela za strune, također komentiramo nelinearni sigma model za potonji slučaj Maxwellovih polja.

Koristeći poopćenje univerzalnog roditeljskog lagranžijana predstavljenu u poglavlju 5, također razrađujemo slučaj više lineariziranih gravitona u četiri dimenzije. Naši rezultati ukazuju da postoje samo dvije vrste pravila transformacija. Za baždarna polja koja su diferencijalne p-forme i za bipartitne tenzore tipa $(p, 1)$ nalazimo da je osnovna grupa dualnosti ortogonalna grupa $O(d, d; \mathbb{R})$ za parne p i simplektička grupa $Sp(2d; \mathbb{R})$ za neparne p.

Konačno, poglavlje 7 posvećeno je gravitacijskom ϑ-članu, koji je analiziran u prethodnim poglavljima. Uvodimo ovaj član u okviru gravitoelektromagnetizma, alternativnog (ekvivalentnog) opisa linearizirane gravitacije. Koristeći pseudoskalarno pozadinsko polje $\vartheta(x)$ kao njegovu konstantu vezanja, ϑ-član je tada topološki član, kvadratičan u parcijalnim derivacijama lineariziranog gravitona. Koristeći to jednostavno zapažanje, pokazujemo da opća relativnost ne može opisati nelinearnu verziju tog člana; umjesto toga, povezan je s topološkom invarijantom, kvadratičnom u torzijskoj 2-formi. Kao takav, može se opisati alternativnim teorijama nelinearne gravitacije koje se temelje na koneksijama s torzijom.

Ovo poglavlje zaključujemo predstavljanjem dvaju jednostavnih primjera koji ilustriraju učinak ϑ-člana u fizičkim situacijama. U prvom primjeru opisujemo kako Newtonov zakon poprima relativističku korekciju zbog raspodjela neiščezavajućih ϑ u prostoru. U drugom primjeru predstavljamo gravitacijski analog Wittenovog efekta, koji kaže da čisto gravitomagnetski naboj (gravitipole) poprima masu i postaje gravitacijski dion, kada je postavljen u područje prostora s neiščezavajućom $\vartheta(x)$.

Disertaciju završavamo zaključkom.

Ključne riječi: gradirana geometrija, dualnost, galileoni, ϑ-članovi, baždarna teorija, gravitacija.

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Chapter 1

INTRODUCTION

Two of the biggest scientific breakthroughs of the last century were, undoubtedly, the discoveries of General Relativity (GR) and Yang-Mills (YM) theory. The former is the commonly accepted classical theory of Gravitation, while the latter describes the strong and weak nuclear interactions and, in its Abelian version, the electromagnetic interaction.

General Relativity appears to have remarkable accuracy in describing and predicting numerous astrophysical and astronomical effects (including the recently detected gravitational waves from the LIGO collaboration [1]). On the other hand, it fails to describe the effects induced by Dark Matter (DM) and, in addition, the physics close to spacetime singularities, e.g. inside the horizons of black holes. In the latter situations of singularities, GR seems to be insufficient and a quantum description of Gravity becomes necessary.

Despite the successful efforts on quantizing YM theory, a quantum theory of Gravity continues to be elusive and its discovery has been rightfully established as one of the most important problems in modern Physics. With this general goal in mind, finding new ways to apply technology acquired from Gauge Theory to Gravity is very well-motivated.

One way of action in this direction is to seek for formal analogies between Gauge Theory and Gravity. There are two main types of such analogies:

- Similarities between the mathematical description and treatment of these theories, and
- Physical properties of Gauge Theory that can, in principle, extend to Gravity.

The former heavily depend on the choice of mathematical formalism one employs. Therefore, the most suitable such formalism should be the one that exploits most of these similarities. The latter analogies are related to the most fundamental aspects of Gauge Theory, which are its symmetries at the quantum level. One of the most efficient ways to discover these is through dualities.

The choice of formalism

To choose the optimal mathematical formalism, one has to first inspect the field content of the theories under consideration. In this respect, Gauge Theory and Gravity differ significantly. Gauge theories typically involve a Lie algebra valued 1-form gauge field[1] and are invariant under some gauge transformations involving local parameters. On the other hand, Gravity describes the dynamics of a symmetric 2-tensor (in its metric formulation) and is invariant under arbitrary differentiable coordinate transformations (diffeomorphisms).

The metric tensor can be alternatively seen as the simplest example of a so-called *mixed-symmetry tensor*. These are generalizations of differential forms, in that their local components contain more than one set of antisymmetrized indices. Then, the metric 2-tensor is an irreducible[2] tensor with two such sets, where each set contains a single spacetime index. In addition, any differential form can be viewed as a degenerate mixed-symmetry tensor, with one nonempty set of antisymmetrized indices and (potentially infinite) empty sets without any index. The same holds for scalars, in which case all the sets are empty. These considerations lead us to an interesting observation; the fundamental fields in both Gauge Theory and Gravity can all be seen as special cases of mixed-symmetry tensors.

The mathematical framework used in this thesis is based on Graded Geometry. This is a generalization of standard differential geometry, whose main advantage for our purposes is that it can be used to describe mixed-symmetry tensor fields of any type in a very general and universal way. As such, it is ideal for exploiting similarities and formal analogies between Gauge Theories and Gravity, as well as for putting them on equal footing and for studying them in parallel.

One example of such a similarity can be found within the context of *Galileons* [3]. These are effective field theories, whose construction ensures that they are free of the Ostrogradsky instability [4]. Using Graded Geometry, we will observe that the Galileon theory for the metric tensor, also known as Lovelock's theory [5], can be written in a form that strongly resembles the scalar [3] and differential form Galileons [6].

[1] Or a collection of differential forms with different degrees, in the more general setting of Higher Gauge Theory. For a review on the latter, see [2].

[2] Here, irreducibility refers to Young tableau representations of the general linear group $GL(D,\mathbb{R})$. We will give the precise definitions in the main text.

Dualities in Gauge Theory

The prototype example of a duality invariant gauge theory is Maxwell's Electromagnetism in four dimensions. This was first noted by Heaviside in [7], who observed that there exists a symmetry of the Maxwell equations under an $SO(2)$ rotation on the space of electric and magnetic fields

$$\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} . \tag{1.1}$$

From a different point of view, this rotation mixes the 2-form field strength (Faraday tensor) $F = \mathrm{d}A$ with its Hodge dual 2-form $*F$. A special feature of this Hodge symmetry transformation is that it exchanges the Bianchi identity with the equation of motion. In fact, this is precisely the standard definition of an *electric-magnetic duality* transformation. As we will show in detail, the fact that Maxwell's theory is duality invariant indicates that if A is a Maxwell 1-form (i.e. it satisfies the Maxwell equation of motion $\mathrm{d} * \mathrm{d}A = 0$), then any 1-form $\widehat{A}$ related to A through

$$\mathrm{d}\widehat{A} = * \, \mathrm{d}A \tag{1.2}$$

is also a Maxwell 1-form. This equation is often called the *duality relation* and obviously holds up to physically irrelevant gauge transformations. In addition, it is a nonlocal relation and, as such, does not relate the two dual 1-forms through an algebraic field redefinition.

At first glance, this story looks quite trivial. In fact, it implies that there exist two independent (dual) ways to describe classical Electromagnetism in four dimensions; either by using the "electric" 1-form A, or the "magnetic" one $\widehat{A}$. Due to the fact that both dual fields are 1-forms, one often talks about *self-duality* and triviality lies on the fact that there does not seem to exist any physical difference between the two descriptions. There is, however, a catch here. Although self-dualities are not physically interesting when they appear in free theories, they become extremely important in interacting ones. Since the underlying principles and features of self-duality are common to both free and interacting theories, the former serve as invaluable toy models and have been rightfully the subject of extensive study.

At the off-shell level, electric-magnetic duality relates two Maxwell Lagrangians with inverse couplings $\frac{1}{e^2}$ and e^2. Despite the fact that these couplings can be absorbed into the definitions of the gauge fields, this observation underlies the notion of a *strong-weak duality*. Indeed, in interacting theories whose couplings cannot be absorbed by field redefinitions, self-duality relates the weak coupling regime of a theory with the strong coupling regime of its dual. As such, it provides us with an indirect way to study the nonperturbative limit of a given theory, by using standard perturbation techniques in its dual theory.

There exist two main types of duality invariant interacting generalizations of Maxwell's theory in four dimensions. The first type concerns nonlinear theories involving an Abelian 1-form gauge field, whose Lagrangians are specific[3] algebraic functions of the two Lorentz invariants $x = F_{\mu\nu}F^{\mu\nu}$ and $y = \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$. Two such examples are the famous Born-Infeld theory [9] and the most recently constructed ModMax theory of [10]. Both these theories admit $SO(2)$ duality rotations, see e.g. [11] for Born-Infeld theory. The second type corresponds to the nonabelian generalization of Maxwell theory, which is the famous Yang-Mills (YM) theory [12].

Four-dimensional quantum YM theory is the arena for the first nonabelian generalization of Maxwell theory's electric-magnetic duality, which was first formulated as a conjecture by Montonen and Olive [13]. Although Montonen and Olive originally conjectured about a $\mathbb{Z}_2$ duality symmetry of YM theory at the quantum level, it was later realized by Witten and Olive [14] that supersymmetry should also be involved. Subsequently, Osborn [15] showed that the amount of supersymmetry should be $N = 4$ and, finally, it was observed by Cardy, Cardy-Rabinovici and Shapere-Wilzcek [16–18] that the originally conjectured $\mathbb{Z}_2$ duality symmetry should get extended to $PSL(2, \mathbb{Z})$. Nowadays, the latter goes by the code name *S-duality*.

A remarkable feature of the (refined) Montonen-Olive conjecture is that the aforementioned S-duality is expected to be present only at the quantum level. Indeed, the classical $N = 4$ super-YM field equations do not exhibit such a duality invariance. As we will see, it is only its Abelian classical version (i.e. Maxwell's theory equipped with the electromagnetic ϑ-term) that does. At that level, supersymmetry is not a requirement.

Motivation for dualities can also be found within the context of String Theory, which is described by a two-dimensional interacting theory involving $d = 26$ (or $d = 10$ for the superstring) compact scalar fields. The common viewpoint is that of the string sigma model (see e.g. [19–21]) for a map from the two-dimensional world-sheet (spacetime) to the d-dimensional target space. It is

[3] See [8] for a discussion on which Lagrangians admit such duality rotations.

well-known that this is the arena for T-duality, which inverts the radii of the target space manifold and simply corresponds to the electric-magnetic self-duality of the scalars from the viewpoint of the string world-sheet. For a review, see e.g. [22].

The example of T-duality reveals that electric-magnetic duality can also exist for arbitrary differential p-form gauge fields (for T-duality we have $p = 0$) and in arbitrary spacetime dimensions. Due to the fact that the Hodge star operator $*$ maps a $(p+1)$-form field strength to a $(D-p-1)$-form one (in D spacetime dimensions), it becomes clear that the dual of a p-form gauge field will be a $(D-p-2)$-form. That is, self-duality only appears in the special number of dimensions $D = 2p + 2$. Away from these self-dual dimensions, it is obvious that electric-magnetic dualities cannot be viewed as a strong-weak ones. However, they are still important because they provide links between seemingly inequivalent theories. These may have different field content, but still propagate exactly the same number of physical degrees of freedom.

As a final note, we should mention that there also exists a different type of Hodge duality, which differs from the standard electric-magnetic one in that it does not exchange Bianchi identities with equations of motion. These are the so-called *exotic* dualities and were first discussed by Deser, Townsend and Siegel in [23]. Their signature feature is that they link free theories involving different types of gauge fields; for example, Maxwell's theory in four dimensions with a theory involving a Curtright gauge field [24] of Young type $(2, 1)$ and a different 1-form gauge field (see e.g. [25]). Recently, they have been the subject of extensive study mainly due to their relevance in the E_{11} approach of [26] and to the natural importance of the exotic dual fields in the framework of higher-spin theories (see e.g. [25, 27]). In addition, these dual fields are conjectured to couple electrically to special low codimensional branes in String Theory (see [28] for a review).

Despite the fact that exotic dualities are not really independent from the standard electric-magnetic ones [29, 30] (at least at the on-shell level of the field equations and when the dualized theory involves a single field), we will show that this is not the case when one includes a topological ϑ-term in the dualization procedure. As such, these dualities fully deserve the attention they have been enjoying and should be understood more deeply.

Duality in Gravity

The examples of T-duality in String Theory and S-duality in four-dimensional $N = 4$ super-YM theory teach us a valuable lesson; dualities are extremely important when it comes to quantum theories. Classical dualities can, in principle, survive at the quantum level, where the initial continuous duality group is expected to break down to one of its discrete subgroups. In different words, knowing that a classical theory is invariant under a specific duality group may provide us with information about the duality group (if it exists) and, consequently, about the symmetries of the quantum theory. If the quantum theory is unknown, knowledge of its symmetries should greatly help in discovering it.

These considerations can be used in the direction of discovering the quantum theory of Gravity. That is, finding dual descriptions of GR could help elucidate the symmetries that Quantum Gravity should have and, in that way, help in its discovery.

The first steps towards this direction were carried out by Hull [31–33], who observed that linearized Gravity is symmetric under $SO(2)$ electric-magnetic duality rotations mixing the Riemann tensor with its Hodge duals. Subsequently, this duality was realized at the off-shell level by West [26, 34], who constructed a first-order duality invariant parent action. Finally, generalizations of this parent action were found in [35]. These extended the spin-2 case to higher-spin fields in arbitrary spacetime dimensions.

In addition to the standard $SO(2)$ duality invariance, it was already shown in [31] that linearized Gravity is invariant under two distinct simultaneous Hodge dualizations of the Riemann tensor. This *double duality* was the key object under examination in the recent work of [29], where it was shown that the double dual linearized graviton is algebraically related on-shell to the original dualized graviton. As such, the double dual picture does not provide with an independent description of the gravitational degrees of freedom.

To be more precise, the latter statement holds in the simplest setting where no additional fields or topological terms are present. This fact was already known to Hull [31], who worked out the forms of the on-shell duality relations in the presence of a topological term analogous to the electromagnetic ϑ-term in Maxwell's theory. The precise Lagrangian form of this topological term was later found in [36] in a different setting and, subsequently, tested within the duality framework in [37]. We will devote a significant part of this thesis in reviewing the results of these last works. In particular, we will also find the nonlinear origin of this topological term and examine some of its consequences in physical situations.

Structure of the thesis & Summary of results

In Chapter 2, we review some basic notions in Graded Geometry; in particular, we restrict ourselves to one of its simplest forms, that is the $\mathbb{Z}_2$-graded or supergeometry. We show how mixed-symmetry tensor fields of arbitrary type on a smooth manifold M can be alternatively described within this framework, by means of particular isomorphisms between the manifold M and a graded (super)manifold $\mathcal{M}$. Subsequently, we define a differential calculus and an integration theory on this graded manifold. Finally, we use this machinery to construct gauge invariant Lagrangians for physical theories in any number of spacetime dimensions, involving gauge fields in arbitrary mixed-symmetry tensor representations of the general linear group. These include kinetic, mass and Galileon interaction terms. Our results show that all these Lagrangians have the same simple geometric form, irrespective of the type of the gauge field. As a byproduct of our analysis, we also obtain strong evidence supporting that gauge invariant and local Galileon interactions cannot be constructed for higher-spin gauge fields.

In Chapters 3 & 4, we present an overview of the Hodge dualities present in two of the most fundamental theories of Nature; Electromagnetism and General Relativity (in its linearized limit). In both cases, we review the concept of duality from the on-shell level of the field equation to its off-shell implementation through a first-order parent Lagrangian. On top of that, we extensively discuss the effects induced by a topological term in these Hodge dualities. In the case of Electromagnetism, this term is taken to be the standard ϑ-term formed as the dot product of the electric and magnetic fields. For linearized Gravity, we consider the gravitational analogue of this topological term formed as the dot product of the gravitoelectric (Newtonian) and gravitomagnetic fields. Both these topological terms exist only in four spacetime dimensions, which are our working dimensions in these chapters.

Regarding the Hodge dualities themselves, we classify them using the standard terminology that appears in the literature. Special cases include the electric-magnetic, exotic and double dualities. Both the electric-magnetic and exotic dualities of the Maxwell field are described in Chapter 3, while Chapter 4 deals with the electric-magnetic and double dualities of the linearized graviton. Throughout these chapters, we also focus on identifying the analogies and similarities of both the dualization procedures and the resulting formulae between Electromagnetism and linearized Gravity. A posteriori, our results lead to the conclusion that these two seemingly different discussions could be carried out in a more general manner.

This conclusion leads to the considerations of Chapter 5, where we present a general graded geometric parent Lagrangian encoding all types of Hodge dualities involving any bipartite mixed-symmetry tensor gauge field. This is effectuated by introducing two parameters in this Lagrangian, which can take values in four specific domains. Each of these domains corresponds to a different kind of Hodge duality and, as such, the parent Lagrangian has a unifying nature. More technically, these statements are formulated in terms of two theorems, both of which we subsequently prove in a rigorous way. Our final results coincide with the ones found in the previous chapters for specific values of the two parameters, encompass all known results in the literature and, furthermore, extend to previously unknown cases.

In Chapter 6, we further generalize our discussion to account for the duality between theories involving multiple gauge fields, in self-dual dimensions. Starting from the example of multiple scalar fields in two dimensions, we perform the dualization procedure and obtain the well-known Buscher rules. These correspond to the transformations of the background fields under T-duality, in the setting of String Theory. As a follow-up, we carry out the same procedure for the case of multiple Abelian 1-forms (Maxwell fields) in four dimensions and obtain the corresponding "higher" Buscher rules for the background fields. Motivated by the relation between T-duality and the string sigma model, we also comment on the nonlinear sigma model picture for the latter case of Maxwell fields.

Using a generalization of the universal parent Lagrangian presented in Chapter 5, we also work out the case of multiple linearized gravitons in four dimensions. Our results indicate that there only exist two different types of transformation rules. For differential p-form gauge fields and bipartite tensors of type $(p, 1)$, we find that the underlying duality group is either the orthogonal group $O(d, d; \mathbb{R})$ for even p or the symplectic group $Sp(2d; \mathbb{R})$ for odd p.

Finally, Chapter 7 is devoted to the gravitational ϑ-term, which was used excessively in the previous chapters. We introduce this term within the framework of Gravitoelectromagnetism, which is an alternative (equivalent) description of linearized Gravity. Using a pseudoscalar background field $\vartheta(x)$ as its coupling, the ϑ-term is then a topological term quadratic in the partial derivatives of the linearized graviton. Using this simple observation, we show that the nonlinear version of this term cannot be described by General Relativity; instead, it is related to a topological invariant quadratic in the torsion 2-form. As such, it can be described by alternative theories of nonlinear Gravity that are based on torsionful connections.

We conclude this chapter by presenting two simple examples that illustrate the effect of the ϑ-term in physical situations. In the first example, we describe how Newton's law receives a relativistic correction due to distributions of nonvanishing ϑ in space. In the second example we present the gravitational analogue of Witten's effect, which states that a purely gravitomagnetic charge (gravitipole) acquires mass and becomes a gravitational dyon, when placed inside a region of space where $\vartheta(x)$ is nonzero.

We finish the thesis with our Conclusions, where we also mention some interesting research directions for future.

Chapter 2

GRADED GEOMETRY AND MIXED-SYMMETRY TENSORS

In this chapter, we will review some basics of graded geometry in its simplest forms and explain why it is a suitable framework for the study of mixed-symmetry tensor fields. For a general review on graded geometry, we refer the interested reader to [38–42].

2.1 Basic notions in $\mathbb{Z}_2$-graded geometry

Let us begin by defining the notion of a *real $\mathbb{Z}_2$-graded vector space*. This is any real vector space V that can be decomposed as a direct sum

$$V = \bigoplus_{i \in \mathbb{Z}_2} V_i = V_0 \oplus V_1 \,, \qquad \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \tag{2.1}$$

of two real vector spaces V_0 and V_1. These are often called the even and odd components of V, respectively. A nonzero element $v \in V$ that belongs to V_i is said to be a homogeneous element of V of degree $|v| = i$. For the purposes of this thesis we will restrict to finite-dimensional $\mathbb{Z}_2$-graded vector spaces. The requirement for this is that both its subspaces V_0 and V_1 should be finite-dimensional[1].

Now that we have introduced the notion of a $\mathbb{Z}_2$-graded vector space it is useful to define an important operation, namely degree-shifting. Let us assume that V is a real vector space (not necessarily graded) and assign the degree 0 to all its elements. Then, we can degree-shift these elements by a positive integer $k \in \mathbb{Z}$ and denote the resulting vector space by $V[k]$. If V is $\mathbb{Z}_2$-graded, then degree-shifting acts on its components as

$$V[k] = \bigoplus_{i \in \mathbb{Z}_2} V_{i+k} = V_k \oplus V_{k+1} \tag{2.2}$$

and $V[k]$ is clearly a graded vector space itself. It should be clear that it also has a $\mathbb{Z}_2$ grading, in that the degrees of its homogeneous elements are simply the shifted $\mathbb{Z}_2$ integers $\mathbb{Z}_2 + k$.

[1] As a simple example of a real, finite-dimensional $\mathbb{Z}_2$-graded vector space one can consider the set of complex numbers $\mathbb{C}$ equipped with the usual addition and scalar multiplication. This vector space can be decomposed into the direct sum $\mathbb{C} = \mathbb{C}_0 \oplus \mathbb{C}_1$, where $\mathbb{C}_0$ is the vector space containing the real and $\mathbb{C}_1$ is the vector space containing the imaginary numbers. Then, $\mathbb{C}$ becomes $\mathbb{Z}_2$-graded if we assign the degree 0 to the real numbers and the degree 1 to the imaginary ones, treating them respectively as degree-0 and degree-1 homogeneous elements of $\mathbb{C}$. In this picture, a complex number containing both a real and an imaginary part is an inhomogeneous element of $\mathbb{C}$.

Given two $\mathbb{Z}_2$-graded vector spaces V and W, their direct sum and tensor product are given by

$$\begin{aligned} V \oplus W &= \bigoplus_{i \in \mathbb{Z}_2} V_i \oplus W_i = V_0 \oplus W_0 \oplus V_1 \oplus W_1 \,, \\ V \otimes W &= \bigoplus_{i,j \in \mathbb{Z}_2} V_j \otimes W_{i-j} = (V_0 \otimes W_0) \oplus (V_0 \otimes W_1) \oplus (V_1 \otimes W_0) \,. \end{aligned} \tag{2.3}$$

One can easily check that both these constructions are also $\mathbb{Z}_2$-graded vector spaces.

Let us now define the $\mathbb{Z}_2$-graded algebra. This is a $\mathbb{Z}_2$-graded vector space A together with an associative bilinear multiplication $\circ : A \times A \to A$ that preserves the degree. We can also refer to this property by saying that the multiplication $\circ$ has degree 0. This means that for homogeneous elements $a_i, a_j \in A$ of degrees $|a_i| = i \in \mathbb{Z}_2$ and $|a_j| = j \in \mathbb{Z}_2$, their multiplication $a_i \circ a_j$ should be a homogeneous element of A with degree $|a_i \circ a_j| = (i + j)\text{mod}2$. Moreover, the $\mathbb{Z}_2$-graded algebra A is also graded-commutative if the multiplication satisfies

$$a_i \circ a_j = (-1)^{|a_i|\,|a_j|} a_j \circ a_i \tag{2.4}$$

for any two homogeneous elements[2].

The tensor algebra of a real $\mathbb{Z}_2$-graded vector space V is defined as usual. In particular, it is the vector space given by the direct sum of all the tensor powers $V^n = V \otimes \cdots \otimes V$ as

$$T(V) = \bigoplus_{n=0}^{\infty} V^n \,, \qquad V^0 \equiv \mathbb{R} \tag{2.5}$$

with the tensor product $\otimes : V^n \times V^m \to V^{n+m}$ being its associative bilinear multiplication. Since the vector space $T(V)$ is linear in the tensor powers of V, this multiplication can be extended to $\otimes : T(V) \times T(V) \to T(V)$. As we already saw, the direct sum and tensor product of two $\mathbb{Z}_2$-graded vector spaces is also a $\mathbb{Z}_2$-graded vector space. Thus, both V^n and $T(V)$ are also $\mathbb{Z}_2$-graded.

In general, the tensor algebra of a vector space is of great importance, mainly because other important objects can be defined directly from it. Two examples are the symmetric and exterior algebras, which are based on specific quotients of the vector space $T(V)$.

[2]Let us think again of the set of complex numbers $\mathbb{C}$ as a $\mathbb{Z}_2$-graded vector space and equip it with the standard vector multiplication between real and imaginary numbers. This is a real, finite-dimensional $\mathbb{Z}_2$-graded algebra since it is easy to see that the multiplication preserves the degree. In other words, the product of two real numbers is again a real number $(0 + 0 = 0\text{mod}2 = 0)$, the product of a real with an imaginary number is imaginary $(0 + 1 = 1\text{mod}2 = 1)$ and the product of two imaginary numbers is real $(1 + 1 = 2\text{mod}2 = 0)$. Finally, despite being $\mathbb{Z}_2$-graded, it is clear that this algebra is not graded-commutative.

To describe these, one has to introduce the graded symmetric and graded wedge product as the associative bilinear maps $\vee, \wedge : T(V) \times T(V) \to T(V)$ defined by

$$v \vee u = v \otimes u + (-1)^{|v|\,|u|} u \otimes v\,, \qquad v \wedge u = v \otimes u - (-1)^{|v|\,|u|} u \otimes v\,, \tag{2.6}$$

for any two $v, u \in T(V)$ that are homogeneous elements of V. In addition, we will denote by $I_\vee$ (resp. $I_\wedge$) the subspace of $T(V)$ containing all elements that can be expressed as the graded symmetric (resp. wedge) product of any two elements of $T(V)$ (that are also homogeneous elements of V).

These being said, the symmetric algebra on V is the vector space $S(V) = T(V)/I_\wedge$ equipped with the multiplication $\vee$, while the exterior algebra is the vector space $\bigwedge(V) = T(V)/I_\vee$ equipped with the multiplication $\wedge$. The underlying vector spaces are naturally decomposed into the direct sums

$$S(V) = \bigoplus_{n=0}^{\infty} S^n V\,, \qquad \bigwedge(V) = \bigoplus_{n=0}^{\infty} \bigwedge{}^n V\,, \tag{2.7}$$

in terms of all the symmetric and exterior tensor powers of V. Each of these summands corresponds to a $\mathbb{Z}_2$-graded vector space, as a tensor power of the $\mathbb{Z}_2$-graded vector space V, and thus both $S(V)$ and $\bigwedge(V)$ are also $\mathbb{Z}_2$-graded vector spaces as their direct sums.

To conclude this Section, let us introduce the notion of a $\mathbb{Z}_2$-graded manifold with dimension (D, d). This is a smooth D-dimensional manifold $\mathcal{M}$, whose algebra of functions $C^\infty(\mathcal{M})$ is locally isomorphic to $C^\infty(U_0) \otimes \bigwedge(\mathbb{R}^d)^*$, where U_0 is an open subset of $\mathbb{R}^D$. For the purposes of this thesis, it will be sufficient to work with graded manifolds of dimension (D, D).

2.2 Mixed-symmetry tensors in graded geometry

Mixed-symmetry tensors are generalizations of differential forms, in that their local components possess more than one sets of antisymmetrized indices. Differential forms live on the cotangent bundle of some smooth D-dimensional manifold M and, thus, they can be described by means of standard differential geometry. However, in this Section we will explain why this standard formalism is not enough for the study of mixed-symmetry tensors and present a suitable extension that suffices. In physical settings, this extended formalism was considered before for the study of gauge theories involving mixed-symmetry tensors, e.g. in [43–46]. Following the results of [47, 48] (see also [49]), we will show that this extension has a straightforward graded geometric analogue by means of particular isomorphisms between mixed-symmetry tensors on M and functions on a graded manifold.

Let us begin our discussion by considering differential forms on M. These are sections of the exterior algebra of its cotangent bundle, i.e. if $\overline{\omega}$ is a differential p-form then

$$\overline{\omega} \in \Gamma(\textstyle\bigwedge^p T^*M) \equiv \Omega^p(M)\,. \tag{2.8}$$

The local coordinates on T^*M are the ordinary differentials $\mathrm{d}x^\mu$, on which we can expand $\overline{\omega}$ as

$$\overline{\omega}(x) = \frac{1}{p!}\,\omega_{\mu_1\dots\mu_p}(x)\,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}\,. \tag{2.9}$$

These basic facts can be translated into graded geometric language by observing that the cotangent bundle of M is isomorphic to the degree-shifted by 1 tangent bundle of M, namely $T^*M \simeq T[1]M$ (see, e.g. [41]). Using this, degree-p functions on $T[1]M$ are naturally identified with differential p-forms on M, that is[3]

$$C^\infty(T[1]M)|_p \simeq \Omega^p(M)\,. \tag{2.10}$$

Note that this is the case only for $p \leq D$. The graded manifold $T[1]M$ is equipped with degree-0 and degree-1 local coordinates x^μ and θ^μ, respectively, where $\mu = 1, \dots, D$. The former are simply the coordinates on M, while the latter are the degree-shifted fiber coordinates that correspond to the graded geometric version of the 1-forms $\mathrm{d}x^\mu$. Since the wedge product is antisymmetric, i.e. $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$, the degree-1 coordinates θ^μ are also mutually anticommuting

$$\theta^\mu\theta^\nu = -\theta^\nu\theta^\mu\,. \tag{2.11}$$

Finally, if $\omega \in C^\infty(T[1]M)|_p$ is a degree-p function on $T[1]M$ then it can be expanded as

$$\omega(x,\theta) = \frac{1}{p!}\,\omega_{\mu_1\dots\mu_p}(x)\,\theta^{\mu_1}\dots\theta^{\mu_p}\,. \tag{2.12}$$

In this picture, the ordinary composition of two differential forms $\overline{\omega} \in \Omega^p(M)$ and $\overline{\zeta} \in \Omega^q(M)$ through the wedge product translates into the pointwise multiplication of the functions $\omega \in C^\infty(T[1]M)|_p$ and $\zeta \in C^\infty(T[1]M)|_q$, i.e.

$$\overline{\omega} \wedge \overline{\zeta} \simeq \omega\,\zeta\,. \tag{2.13}$$

Let us now proceed by considering mixed-symmetry tensors. As we already mentioned, these are generalized objects whose local components contain more than one sets of antisymmetrized indices.

[3] We can make contact with our previous definition of a graded manifold by observing that locally $\Omega^p(U_0) = C^\infty(U_0) \otimes \bigwedge^p(\mathbb{R}^D)^*$, see e.g. [41]. Then, the isomorphism (2.10) implies that $T[1]M$ is a graded manifold with dimension (D, D).

As such, they cannot be described by the standard differential geometry based on T^*M. In this thesis, we will follow the terminology of [48] and refer to a tensor with N such sets of indices as *N-partite*[4].

For $N = 1$, it should be clear that the mixed-symmetry tensor is identified with a differential form. Let us stick to the simplest nontrivial case of $N = 2$, which corresponds to a 2-partite or *bipartite* tensor $\overline{\omega}$. A bipartite tensor $\overline{\omega}$ of type (p_1, p_2) lives in the product space $\Omega^{p_1}(M) \otimes \Omega^{p_2}(M)$ of differential p_1-forms and p_2-forms. That is, it can be expanded in the local basis of this space as

$$\overline{\omega}(x) = \frac{1}{p_1!p_2!}\omega_{\mu_1\dots\mu_{p_1}\nu_1\dots\nu_{p_2}}(x)\,\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_{p_1}}\otimes\mathrm{d}x^{\nu_1}\wedge\dots\wedge\mathrm{d}x^{\nu_{p_2}}\,, \tag{2.14}$$

which is a direct generalization of the expansion (2.9). Due to the wedge products, one can observe that the components $\omega_{\mu_1\dots\mu_{p_1}\nu_1\dots\nu_{p_2}}$ are antisymmetric both in the $\{\mu\}$ and in the $\{\nu\}$ indices separately, but there is no index symmetry between these two sets. In fact, this indicates that $\omega_{\mu_1\dots\mu_{p_1}\nu_1\dots\nu_{p_2}}$ corresponds to a *reducible* representation of the general linear group $GL(D)$.

Let us now translate these data into graded geometric language. To do this, one has to consider the graded manifold $\mathcal{M}_2 := T[1]M \oplus T[1]M$. This is equipped with the standard coordinates x^μ of M that are assigned a degree $(0, 0)$, as well as two different types of anticommuting coordinates θ^μ and χ^μ of degrees $(1, 0)$ and $(0, 1)$. These correspond to the degree-shifted fiber coordinates of the two distinct copies of $T[1]M$ and satisfy the relations

$$\theta^\mu\theta^\nu = -\theta^\nu\theta^\mu\,, \quad \chi^\mu\chi^\nu = -\chi^\nu\chi^\mu\,, \quad \theta^\mu\chi^\nu = \chi^\nu\theta^\mu\,. \tag{2.15}$$

The last condition is purely conventional and does not affect the results of the present work.

Then, the relation between bipartite tensors and graded geometry can be established due to the isomorphism (which holds only when $p_1 + p_2 \leq D$)

$$C^\infty(\mathcal{M}_2)|_{(p_1,p_2)} \simeq \Omega^{p_1}(M) \otimes \Omega^{p_2}(M) \tag{2.16}$$

between functions of degree (p_1, p_2) on the graded manifold $\mathcal{M}_2$ and bipartite tensors of type (p_1, p_2) on M. In analogy to their tensor counterparts, we will call these functions 2-partite or

[4] Note, however, there do exist different terminologies in the literature. For example, the term *multiform* was used in the works [43–46].

bipartite[5]. According to (2.16), a bipartite function ω of degree (p_1, p_2) on $\mathcal{M}_2$ can be expanded locally as

$$\omega(x, \theta, \chi) = \frac{1}{p_1! p_2!}\, \omega_{\mu_1 \dots \mu_{p_1} \nu_1 \dots \nu_{p_2}}(x)\, \theta^{\mu_1} \dots \theta^{\mu_{p_1}} \chi^{\nu_1} \dots \chi^{\nu_{p_2}} . \tag{2.17}$$

Then, the ordinary composition of two bipartite tensors $\overline{\omega} \in \Omega^p(M) \otimes \Omega^q(M)$ and $\overline{\zeta} \in \Omega^r(M) \otimes \Omega^s(M)$ through the tensor product translates into the pointwise multiplication between the bipartite functions $\omega \in C^\infty(T[1]M)|_{(p,q)}$ and $\zeta \in C^\infty(T[1]M)|_{(r,s)}$, i.e.

$$\overline{\omega} \otimes \overline{\zeta} \simeq \omega\, \zeta\, . \tag{2.18}$$

Finally, let us mention that generalization to the graded geometric description of N-partite tensors as N-partite functions is really straightforward. Indeed, the latter are functions of the extended graded manifold

$$\mathcal{M}_N = T[1]M \oplus \dots \oplus T[1]M \tag{2.19}$$

obtained by the direct sum of N copies of $T[1]M$. This is equipped with the commuting coordinates x^μ of M, as well as a collection of N degree-shifted fiber coordinates θ^μ_A for $A = 1, \dots, N$. These have degrees $(0, \dots, 0, 1_A, 0, \dots, 0)$, where 1_A indicates that 1 is located at the A-th slot, and satisfy the (anti)commutation relations

$$\theta^\mu_A \theta^\nu_B = (-1)^{\delta_{AB}} \theta^\nu_B \theta^\mu_A\, . \tag{2.20}$$

In our previous example for $N = 2$, the role of θ^μ_1 (resp. θ^μ_2) was played by θ^μ (resp. χ^μ).

2.3 Defining the differential calculus

To make contact with Physics, one has to define a suitable differential calculus on the graded manifold $\mathcal{M}_N$. In this thesis, we will mostly consider physical applications of graded geometry at the $N = 2$ level of bipartite functions, but we will present the main ingredients of this calculus for the more general case of arbitrary N. Finally, we will assume that the smooth manifold M is the D-dimensional Minkowski spacetime $\mathbb{R}^{1,D-1}$.

The calculus we will construct is based on a collection of differential and algebraic maps and was originally developed in [43–46], in the context of the extended differential geometry on $\Omega^{p_1}(M) \otimes \dots \otimes \Omega^{p_N}(M)$ described in the previous Section. Here, we will follow the alternative picture

[5]Under this terminology, the counterpart of a differential p-form is a 1-partite function of degree p.

of [47, 48] and show how this calculus gets translated into the graded geometric language. The main difference will be that the differential and algebraic maps acting on N-partite tensors in $\Omega^{p_1}(M)\otimes\cdots\otimes\Omega^{p_N}(M)$ will be viewed as operators acting on the N-partite functions of the graded manifold $\mathcal{M}_N$.

We begin by considering derivative operators with respect to the anticommuting coordinates

$$\bar{\theta}^A_\mu := \frac{\partial}{\partial\theta^\mu_A} \, . \tag{2.21}$$

These operators obey exactly the same relations (2.20) as the coordinates θ^μ_A, while there also exist canonical (anti)commutation relations between them of the form

$$\{\bar{\theta}^A_\mu, \theta^\nu_A\} = \delta^\nu_\mu \qquad \& \qquad [\bar{\theta}^A_\mu, \theta^\nu_B] = 0 \quad \text{for} \quad B \neq A \, . \tag{2.22}$$

The differential and algebraic maps required for our purposes can all be constructed as specific combinations that are bilinear in $(\theta^\mu_A, \bar{\theta}^A_\mu)$.

For example, one can define N distinct algebraic operators $\widehat{p}_A : C^\infty(\mathcal{M}_N) \mapsto C^\infty(\mathcal{M}_N)$ by

$$\widehat{p}_A := \theta^\mu_A \bar{\theta}^A_\mu \, , \tag{2.23}$$

where no summation in A is implied. These operators act on any function $\omega \in C^\infty(\mathcal{M}_N)$ with degree $(p_1, \ldots, p_N)$ by the usual pointwise multiplication of functions. By virtue of the definition (2.23) and the (anti)commutation relations between θ's and $\bar{\theta}$'s, this action will result in the eigenvalue equation $\widehat{p}_A\omega = p_A\omega$. Because of this, the operators $\widehat{p}_A$ are naturally identified with *number operators*.

In addition, the Minkowski metric and the corresponding trace map can be cast into this formalism by means of the algebraic operators

$$\eta_{AB} := \eta_{\mu\nu}\theta^\mu_A\theta^\nu_B \qquad \& \qquad \mathrm{tr}_{AB} := \eta^{\mu\nu}\bar{\theta}^A_\mu\bar{\theta}^B_\nu \, . \tag{2.24}$$

In fact, the local expansion (2.14) implies that each of the operators η_{AB} can also be seen as a bipartite function of degree $(1,1)$. The local components $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ correspond to the Minkowski metric and its inverse. Both of these are symmetric 2-tensors and, thus, irreducible representations of $GL(D)$. In the ensuing, we will call functions with irreducible local components as *irreducible functions*, in analogy to the standard terminology that exists for mixed-symmetry tensors.

As we will see shortly, reducible and irreducible functions can be distinguished using the algebraic operators defined by

$$\sigma_{AB} := -\theta^{\mu}_{A}\bar{\theta}^{B}_{\mu} \,. \tag{2.25}$$

After this last input, one can easily see that the $\binom{2N}{2}$ algebraic maps defined in (2.23)-(2.24)-(2.25) satisfy certain (anti)commutation relations, which follow naturally from the defining properties (2.20)-(2.22). Operators with all of their indices distinct commute, while for the rest of them we easily find the relations

$$[\sigma_{AB}, \sigma_{BA}] = \hat{p}_A - \hat{p}_B \,, \quad \{\sigma_{AB}, \sigma_{BC}\} = -\sigma_{AC} \,, \quad \{\sigma_{AB}, \sigma_{CB}\} = 0 \,, \tag{2.26}$$

$$[tr_{AB}, \eta_{AB}] = d - \hat{p}_A - \hat{p}_B \,, \quad \{tr_{AB}, \eta_{BC}\} = -\sigma_{CA} \,, \tag{2.27}$$

$$\{\eta_{AB}, \eta_{BC}\} = 0 \,, \quad \{tr_{AB}, tr_{BC}\} = 0 \,, \tag{2.28}$$

$$[\hat{p}_A, tr_{AB}] = -tr_{AB} \,, \quad [\hat{p}_A, \eta_{AB}] = \eta_{AB} \,, \quad [\hat{p}_A, \sigma_{AB}] = -[\hat{p}_B, \sigma_{AB}] = \sigma_{AB} \,, \tag{2.29}$$

$$[\sigma_{AB}, \eta_{AB}] = 0 \,, \quad \{\sigma_{AB}, \eta_{AC}\} = 0 \,, \quad \{\sigma_{AB}, \eta_{BC}\} = -\eta_{AC} \,, \tag{2.30}$$

$$[\sigma_{AB}, tr_{AB}] = 0 \,, \quad \{\sigma_{AB}, tr_{AC}\} = -tr_{BC} \,, \quad \{\sigma_{AB}, tr_{BC}\} = 0 \,. \tag{2.31}$$

It is interesting to note that these relations can be seen as the commutation relations for the $\binom{2N}{2}$ generators of the Lie algebra $\mathfrak{so}(2N)$. For example, we can restrict to the $N = 2$ case and define the six combinations (see [50, 51])

$$\begin{aligned}
&J_{13} = -\widehat{p}_1 \,, \quad J_{24} = -\widehat{p}_2 \,, \\
&J_{12} = -\frac{i}{2}(\sigma_{12} - \sigma_{21} + \eta_{12} - \mathrm{tr}_{12}) \,, \quad J_{14} = -\frac{1}{2}(\sigma_{12} + \sigma_{21} - \eta_{12} - \mathrm{tr}_{12}) \,, \\
&J_{23} = -\frac{1}{2}(\eta_{12} + \mathrm{tr}_{12} + \sigma_{12} + \sigma_{21}) \,, \quad J_{34} = -\frac{i}{2}(\eta_{12} - \mathrm{tr}_{12} - \sigma_{12} + \sigma_{21}) \,,
\end{aligned} \tag{2.32}$$

In terms of these objects, the (anti)commutation relations above simply become

$$[J_{ab}, J_{cd}] = 2i\delta_{c[a}J_{b]d} + 2i\delta_{a[c}J_{d]b} \,, \qquad a, b, c, d = 1, \ldots, 4 \,. \tag{2.33}$$

Finally, the algebraic maps defined in (2.25) are related to the ones of (2.24) by means of the standard Hodge star operators $*_A$. We mention these relations in Appendix A.1, for the simplest case of $N = 2$.

We will now move on and discuss the criteria under which a function is irreducible. Let $\omega \in C^{\infty}(\mathcal{M}_N)$ be a function of degree $(p_1, \ldots, p_N)$ and, furthermore, assume that $p_1 \geq p_2 \geq \cdots \geq p_N$

without loss of generality. Then, ω is an irreducible function if and only if it satisfies the algebraic constraints

$$\sigma_{AB}\,\omega = 0\,, \quad \forall\, A \leq B \qquad \& \qquad \omega^{\top_{AB}} = \omega\,, \quad \forall\, A, B \quad \text{with} \quad p_A = p_B\,. \tag{2.34}$$

The second condition above involves the transposition operator $\top_{AB}$, which acts on the local expansion of a function by swapping the coordinates θ_A and θ_B. The constraints above constitute the so-called Young symmetry and, therefore, the $GL(D)$-irreducible subspace of $C^\infty(\mathcal{M}_N)$ is the space of functions whose local components possess the index symmetries of a Young tableau with N columns of respective lengths $p_1, p_2, \ldots, p_N$.

Given an arbitrary reducible function $\omega \in C^\infty(\mathcal{M}_N)$, there is a unique way to obtain a an irreducible one of the same degree. This can be done by acting on ω with the Young projection denoted by $\mathbb{P}_{(p_1,\ldots,p_N)}$. In general, this is an algebraic operator constructed as a polynomial in the operators σ_{AB}. For the special case of $N = 2$, it takes the form (see, e.g. [44])

$$\mathbb{P}_{(p_1,p_2)} = \begin{cases} \mathbb{I} + \sum\limits_{n=1}^{p_2} c_n(p_1, p_2)\sigma_{21}^n\sigma_{12}^n\,, & \text{for} \quad p_1 \geq p_2 \\ \\ \mathbb{I} + \sum\limits_{n=1}^{p_1} c_n(p_2, p_1)\sigma_{12}^n\sigma_{21}^n\,, & \text{for} \quad p_1 \leq p_2 \end{cases} \tag{2.35}$$

with coefficients given by

$$c_n(p_1, p_2) = \frac{(-1)^n}{\prod\limits_{r=1}^{n} r(p_1 - p_2 + r + 1)}\,. \tag{2.36}$$

Note that the $p_1 = p_2$ case is covered by both entries due to the first identity in (2.26), which implies that $[\sigma_{12}, \sigma_{21}]$ vanishes when it acts on a function of degree (p, p). Evidently, the projection is trivial for $\min(p_1, p_2) = 0$, since a differential form is already an irreducible function. More generally and for arbitrary N, the Young projection $\mathbb{P}_{(p_1,\ldots,p_N)}$ is idempotent when acting on any function of degree $(p_1, \ldots, p_N)$ by construction, i.e. $\mathbb{P}^2_{(p_1,\ldots,p_N)}\omega = \mathbb{P}_{(p_1,\ldots,p_N)}\omega$. This relation holds trivially, as can be seen by use of the identities (2.26) and the fact that $\mathbb{P}_{(p_1,\ldots,p_N)}\omega$ is irreducible and, thus, satisfies the algebraic constraints (2.34).

In addition to the algebraic operators, one can also define differential ones. For example, the exterior derivatives, their codifferentials and the Laplacian are naturally defined as

$$\mathrm{d}_A := \theta^\mu_A\partial_\mu\,, \qquad \mathrm{d}^\dagger_A := \bar{\theta}^A_\mu\partial^\mu\,, \qquad \Box := \mathrm{d}_A\mathrm{d}^\dagger_A + \mathrm{d}^\dagger_A\mathrm{d}_A\,. \tag{2.37}$$

For distinct labels the above maps commute, i.e. $[\mathrm{d}_A, \mathrm{d}_B^\dagger] = 0$ for $B \neq A$, while both d_A and $\mathrm{d}_A^\dagger$ square to zero for every A. In fact, the codifferentials are related to their corresponding exterior derivatives by means of the Hodge star operators $*_A$. We mention their relation in Appendix A.1.

The relations between the algebraic operators defined in (2.23)-(2.25) and differential ones defined above also follow from the (anti)commutation relations (2.20) and (2.22). In particular, one finds

$$\{tr_{AB}, \mathrm{d}_A\} = \mathrm{d}_B^\dagger\,, \tag{2.38}$$

$$\{tr_{AB}, \mathrm{d}_A^\dagger\} = 0\,, \tag{2.39}$$

$$\{\eta_{AB}, \mathrm{d}_A\} = 0\,, \tag{2.40}$$

$$\{\eta_{AB}, \mathrm{d}_A^\dagger\} = \mathrm{d}_B\,, \tag{2.41}$$

$$\{\sigma_{AB}, \mathrm{d}_A\} = 0\,, \tag{2.42}$$

$$\{\sigma_{AB}, \mathrm{d}_B\} = -\mathrm{d}_A\,, \tag{2.43}$$

$$\{\sigma_{AB}, \mathrm{d}_A^\dagger\} = \mathrm{d}_B^\dagger\,, \tag{2.44}$$

$$\{\sigma_{AB}, \mathrm{d}_B^\dagger\} = 0\,. \tag{2.45}$$

Just like the algebraic operators and their relation to the generators of $\mathfrak{so}(2N)$, the $2N$ differential operators (2.37) are related to $2N$ supercharges Q_A defined by [51]

$$Q_A := \begin{cases} \frac{-i}{\sqrt{2}}(\mathrm{d}_A + \mathrm{d}_A^\dagger) & \text{for} \quad A = 1, \ldots, N \\ \frac{1}{\sqrt{2}}(\mathrm{d}_A - \mathrm{d}_A^\dagger) & \text{for} \quad A = N+1, \ldots, 2N\,. \end{cases} \tag{2.46}$$

These supercharges satisfy $Q_A^2 = H$ and $[Q_A, H] = 0$, where H is the Hamiltonian $-\frac{\Box}{2}$. Moreover, the supercharges are related to one another by

$$\{Q_A, Q_{N+A}\} = 0 \qquad \& \qquad [Q_A, Q_B] = 0 \quad \text{for} \quad B \neq N + A\,.$$

Note that typically supercharges anticommute with each other. The apparent difference here is due to the choice of sign convention, i.e. the second relation in (2.22), which however is more convenient for the considerations of this thesis. Had we chosen the alternative convention, namely that $\{\overline{\theta}^A_\mu, \theta^\nu_B\} = \delta^\nu_\mu$, we would indeed find that all supercharges defined by (2.46) mutually anticommute. These being said, we can see that they generate a $\mathcal{N} = 2N$ supersymmetry algebra with $H = -\frac{\Box}{2}$ being the bosonic operator. In addition, the aforementioned $\mathfrak{so}(2N)$ algebra generated by the algebraic operators corresponds to its R-symmetry, as can be seen from the anticommutation relations (2.38)-(2.45).

2.4 Tensor gauge theories

As we already mentioned, throughout this thesis we are going to study applications of graded geometry in physical theories defined on the Minkowski spacetime of arbitrary dimensions. The respective metric will be in the mostly plus signature convention $(-, +, \ldots, +)$. However, this choice will only be important in the last Chapter 7, since we will generally refrain from writing down local expressions elsewhere. In general, we will be working in the units system where $\varepsilon_0 = c = \hbar = G = 1$, except for our discussion in Chapter 7.

One of the main ingredients of a physical theory is an action functional or, equivalently, a Lagrangian (assuming, of course, that the theory at hand admits a Lagrangian formulation). For the special case of gauge theories, the aforementioned Lagrangian should also be invariant under the gauge transformations of the fields it contains. For differential form gauge fields, the construction of a gauge invariant Lagrangian is very simple and it involves the standard Hodge operator. This simplicity is mainly due to the fact that differential forms (as well as their gauge invariant field strengths) do not posses traces, since their local components correspond to antisymmetric tensors.

In fact, this last point reveals the main difficulty in constructing theories for mixed-symmetry tensor gauge fields. As we have already seen, these objects have nonvanishing traces and, consequently, the corresponding Lagrangians cannot be obtained using the standard Hodge operator. To account for this difficulty, one needs to define an inner product on the graded manifold $\mathcal{M}_N$ that will take traces into account in a natural way.

The two basic ingredients for any inner product over a graded manifold are a suitable Hodge star operator and an integral over the anticommuting coordinates of the manifold. For the latter, we will use the standard Berezin integral. Over one set of anticommuting coordinates θ^μ, this is defined by

$$\int_\theta \theta^{\mu_1} \ldots \theta^{\mu_D} \equiv \int d^D\theta\, \theta^{\mu_1} \ldots \theta^{\mu_D} = \varepsilon^{\mu_1 \cdots \mu_D} \tag{2.47}$$

in terms of the Levi-Civita tensor of the Minkowski space $\varepsilon^{\mu_1 \cdots \mu_D}$.

Let us now discuss about the Hodge star operator we will use. This was constructed in [47] and further studied in [48, 49]. Following these works, we define the collection of star operators $\star_{AB}$

which act as

$$\star_{AB}\,\omega := \frac{1}{(D-p_A-p_B)!}\,\eta_{AB}^{D-p_A-p_B}\,\omega^{\top_{AB}} \tag{2.48}$$

on an arbitrary function ω of degree $(p_1,\ldots,p_N)$ with $p_A+p_B \leq D$. By simple degree counting, one can check that the degree of the function $\star_{AB}\,\omega$ is (assuming $A>B$ without loss of generality)

$$(p_1,\ldots,D-p_A,\ldots,D-p_B,\ldots,p_N) \tag{2.49}$$

and, as such, $\star_{AB}$ resembles the combined action $*_A *_B$ of two usual Hodge operators[6]. However, this is not the case. In fact, it was shown in [48] that the two operations are related via

$$\star_{AB}\,\omega = *_A *_B\,\overline{\overline{\omega}}\,, \qquad \text{where} \qquad \overline{\overline{\omega}} := (-1)^{\epsilon}\sum_{n=0}^{\min(p_A,p_B)}\frac{(-1)^n}{(n!)^2}\,\eta_{AB}^n\,\mathrm{tr}_{AB}^n\,\omega\,. \tag{2.50}$$

The parity above reads as $\epsilon=(D-1)(p_A+p_B)+p_Ap_B+1$ and we have also used the notation $\eta_{AB}^0\mathrm{tr}_{AB}^0 \equiv \mathbb{I}$. The expression above reveals that action of the Hodge star operator (2.48) on a function ω results in an expression containing all possible traces of ω.

Have both an integral and a star operator, we can now form an inner product. To this end, we first use the floor function and consider an even number $k := 2\lfloor\frac{N+1}{2}\rfloor$ of anticommuting coordinates. Using these, we define the bilinear symmetric[7] map $(\cdot,\cdot): C^\infty(\mathcal{M}_k)\times C^\infty(\mathcal{M}_k)\mapsto\mathbb{R}$ by

$$(\omega,\omega') := \int_{\theta_1,\ldots,\theta_k}\omega\ \star_{12}\star_{34}\cdots\star_{k-1\ k}\ \omega'\,. \tag{2.51}$$

Obviously, this is well-defined only if $p_A+p_{A+1}\leq D$ for any $A=1,3,5,\ldots,k-1$. Note that although we considered ω and ω' to be k-partite functions with k even, they can still be degenerate such functions having degrees of the form $(p_1,\ldots,p_{k-1},0)$.

Now that an inner product is defined, we can begin the construction of gauge invariant Lagrangians. In fact, one could even start at the on-shell level and assume that there exists an *irreducible* function ω (the gauge field) of degree $(p_1,\ldots,p_N)$ with a local (gauge) transformation of the form[8]

$$\delta\omega = \mathbb{P}_{(p_1,\ldots,p_N)}\left(\sum_{A=1}^N \mathrm{d}_A\lambda^{(A)}\right)\,. \tag{2.52}$$

[6]These are the standard Hodge operators used in differential geometry. For more details on their definition, the reader may consult Appendix A.1.

[7]Note that η_{AB} is invariant under the transposition operator $\top_{AB}$ for any A, B. In addition, any scalar function in $\mathbb{R}$ is also invariant and, therefore, $(\omega,\omega')=(\omega,\omega')^{\top_{AB}}$. Using the definition (2.48), one can then see that the bilinear map defined above is symmetric, i.e. $(\omega,\omega')=(\omega',\omega)$. For a more rigorous proof in the $N=2$ case, see the Appendix A.2.

[8]Note that we restrict here to linear transformations. Nonlinear gauge theories, such as Yang-Mills, will not be discussed in this thesis.

Each function $\lambda^{(A)}$ is of degree $(p_1, \ldots, p_A - 1, \ldots, p_N)$ and corresponds to a reducible gauge parameter. Furthermore, the Young projection is required to ensure that the local components of the r.h.s. possess the same index symmetries as the gauge field ω. An additional important element in any gauge theory involving ω is the gauge invariant *field strength* defined by

$$F := \mathrm{d}_1 \ldots \mathrm{d}_N\, \omega\,. \tag{2.53}$$

This is an N-partite function of degree $(p_1 + 1, \ldots, p_N + 1)$. Its invariance under (2.52) is obvious, since all exterior derivatives d_A are nilpotent. Moreover, F is irreducible due to the irreducibility of ω, as can be seen by the fact that it satisfies the Young constraints (2.34).

The general goal in constructing a gauge theory for ω at the off-shell level is to find a Lagrangian that is invariant under the gauge transformation (2.52). Let us start by considering a Lagrangian kinetic term for ω, which should be a scalar function bilinear in ω *and* of second-order in spacetime derivatives. In addition, the kinetic term should be a local function or, at least, lead to local field equations after fixing some part of the gauge symmetry. This assumption was originally made in [52], where locality of the field equations was required after application of the partial gauge fixing conditions[9]

$$\mathrm{d}_A^\dagger \omega \overset{!}{=} 0\,, \quad \forall\, A \in [1, N]\,. \tag{2.54}$$

In light of these requirements, one can use the inner product (2.51) to define the term [49]

$$\begin{aligned} \mathcal{L}_{(0)}(\omega) &= \left(\mathrm{d}_1 \mathrm{d}_3 \ldots \mathrm{d}_{k-1} \omega\ ,\ \frac{1}{\Box^{\frac{k}{2}-1}}\, \mathrm{d}_1 \mathrm{d}_3 \ldots \mathrm{d}_{k-1} \omega \right) \\ &= \int_{\theta_1, \ldots, \theta_k} \mathrm{d}_1 \mathrm{d}_3 \ldots \mathrm{d}_{k-1} \omega\ \star_{12} \cdots \star_{k-1\, k}\ \frac{1}{\Box^{\frac{k}{2}-1}}\, \mathrm{d}_1 \mathrm{d}_3 \ldots \mathrm{d}_{k-1} \omega\,. \end{aligned} \tag{2.55}$$

One can easily see that this term is indeed invariant under (2.52), up to total derivative terms. Moreover, it can be shown that it reproduces the standard kinetic terms for all types of gauge fields up to total derivative terms[10].

For $N = 0$ ($k = 0$), there is no integration over anticommuting variables and (2.55) reduces to the kinetic term for the scalar field ω

$$\mathcal{L}_{(0)}(\omega) = \omega \frac{1}{\Box^{-1}} \omega = \omega \Box \omega\,. \tag{2.56}$$

[9] Combined with the tracelessness conditions $\mathrm{tr}_{AB}\,\omega \overset{!}{=} 0$ for all $A \neq B$, the conditions (2.54) lead to the transverse traceless or to the so-called physical gauge in [46]. The latter fixes the gauge symmetry completely.

[10] A small subtlety here is that (2.55) does not contain any physical constants or overall signs related to unitarity. These are ignored here for brevity, but we will include them in the ensuing Chapters.

For $N = 1$ ($k = 2$), there only exist two anticommuting variables θ_1^μ and θ_2^μ. In addition, integration over the latter is trivial and simply leads to an additional minus sign. That is, (2.55) coincides with the standard kinetic term for the differential p_1-form gauge field ω (more precisely, the degree-p_1 function) with field strength $F := \mathrm{d}_1\omega$. That is

$$\mathcal{L}_{(0)}(\omega) = \int_{\theta_1,\theta_2} F \star_{12} F \overset{(2.50)}{=} (-1)^\epsilon \int_{\theta_1,\theta_2} F *_1 *_2 F = (-1)^{\epsilon+1} \int_{\theta_1} F * F\,, \tag{2.57}$$

in terms of a parity ϵ that reads as in (2.50).

For $N = 2$ ($k = 2$), the result is the standard kinetic term for the bipartite tensor gauge field ω of type (p_1, p_2) (more precisely, the bipartite function of degree (p_1, p_2)) with field strength $F := \mathrm{d}_1\mathrm{d}_2\omega$. These terms were first constructed in [47] and reduce, for example, to the linearized Einstein-Hilbert term for $p_1 = p_2 = 1$,

$$\mathcal{L}_{(0)}(\omega) = -\frac{1}{4}\omega^{\mu|}{}_\mu \Box \omega^{\nu|}{}_\nu + \frac{1}{2}\omega^{\mu|}{}_\mu \partial_\nu\partial_\rho\omega^{\nu|\rho} - \frac{1}{2}\omega_{\mu|\nu}\partial^\nu\partial_\rho\omega^{\mu|\rho} + \frac{1}{4}\omega_{\mu|\nu}\Box\omega^{\mu|\nu}\,, \tag{2.58}$$

as well as to the Curtright kinetic term of [24] for $p_1 = 2$ and $p_2 = 1$,

$$\begin{aligned}\mathcal{L}_{(0)}(\omega) = {}& \frac{1}{2}\partial_\mu\omega_{\nu\rho|\sigma}\partial^\mu\omega^{\nu\rho|\sigma} - \partial_\mu\omega^{\mu\nu|\rho}\partial^\sigma\omega_{\sigma\nu|\rho} - \frac{1}{2}\partial_\mu\omega^{\nu\rho|\mu}\partial^\sigma\omega_{\nu\rho|\sigma} \\ & - 2\omega_\mu{}^{\nu|\mu}\partial^\rho\partial^\sigma\omega_{\rho\nu|\sigma} - \partial_\mu\omega_\nu{}^{\rho|\nu}\partial^\mu\omega^\sigma{}_{\rho|\sigma} + \partial_\mu\omega_\nu{}^{\mu|\nu}\partial^\rho\omega^\sigma{}_{\rho|\sigma}\,.\end{aligned} \tag{2.59}$$

All factors and signs of the several terms in these Lagrangians are encoded in the Berezin integration of the simple geometric term (2.55).

For $N \geq 3$, the kinetic term (2.55) describes the dynamics of free massless gauge fields of spin higher than 2. In fact, it reproduces precisely the coordinate expressions constructed in [52] upon performing the Berezin integrations. On the other hand, this term is obviously nonlocal due to the inverse box operator that remains in the expression. Despite this nonlocality, it is well-defined assuming suitable boundary conditions. Finally, it leads to local field equations after the gauge condition (2.54) is taken into account, as desired. For more details, the reader may consult the original paper of Francia and Sagnotti [52].

Before moving on to construct interactions, let us mention that mass terms for the gauge fields ω can be also easily defined. In fact, they are all contained in the simple graded geometric expression of the form [48]

$$\mathcal{L}_{m^2}(\omega) = m^2(\omega,\omega) = m^2\int_{\theta_1,\dots,\theta_k} \omega \star_{12}\cdots\star_{k-1\,k}\ \omega\,. \tag{2.60}$$

Let us now switch gears and see how one can define meaningful interactions for an arbitrary gauge field ω. To this end, we must pose some general physical requirements that will have to be satisfied. First of all, any interaction term should retain the invariance under the gauge transformations (2.52). In addition, we will allow these terms to be nonlocal, as long as they lead to local contributions to the field equations after imposing the gauge fixing condition (2.54). Finally, we will require that the interaction term does not spoil the unitarity of the free theory.

The last requirement is equivalent to the assumption that the theory does not propagate Ostrogradsky ghosts [4] and, thus, that the respective Hamiltonian is bounded from below. In general, it is easy to check whether this is the case or not, by examining the corresponding field equations. If the field equations do not contain terms having more than two *time* derivatives acting on the gauge field, then the Ostrogradsky instability is avoided. In this thesis we are concerned about relativistic theories, for which there should not exist terms having more than two *partial* derivatives acting on the respective field.

In the special case of field equations containing *exactly* two partial derivatives acting on the gauge field, a new symmetry arises. Denoting the total degree of the gauge field by $P := p_1 + \cdots + p_N$, the corresponding transformation has the form[11]

$$\omega \quad \mapsto \quad \omega + B_{\mu_0\mu_1\ldots\mu_P} x^{\mu_0} \theta_1^{\mu_1} \ldots \theta_1^{\mu_{p_1}} \ldots \theta_N^{\mu_{p_1+\cdots+p_{N-1}+1}} \ldots \theta_N^{\mu_P}, \tag{2.61}$$

in terms of a fully antisymmetric constant $(P+1)$-tensor $B_{\mu_0\mu_1\ldots\mu_P}$. This is a global transformation but contains the spacetime coordinates explicitly. This indicates that it does not commute with the Poincaré group, which is the group of isometries of the Minkowski spacetime. Theories that are invariant under the global transformation (2.61) are called *Galileons*, since the transformation rule (2.61) directly generalizes the ordinary Galilean transformation relating two reference frames that move at a constant velocity with respect to one another.

Galileon interactions, i.e. interaction terms that are invariant under (2.61), were first constructed and studied for scalar fields in [3] (see also [53]). Since these original works, there have been studied extensively and generalized in various directions. Regarding the type of the gauge field, Galileon theories were constructed for differential form gauge fields of even degree in [6] and for bipartite tensor gauge fields of even total degree in [47].

[11] For $N = 2$, this transformation was proven to be a symmetry of the bipartite tensor Galileon constructed in [47].

In the case of differential p-forms, these interactions generally contain an even number of field appearances (the only counter-example is the 4-form cubic vertex discussed in [54]) and it was already noticed in [6] that odd degree forms (like the Maxwell field) do not admit gauge invariant Galileon interactions that are not total derivatives[12]. This observation was extended to the bipartite mixed-symmetry tensor case of [47], where it was shown that tensors with odd total degree $p_1 + p_2$ also fall into this category. Furthermore, a special case of Galileon interactions for bipartite tensors of type (p, p) exists and these can also contain an odd number of fields. For example, these interactions exist for the scalar field and for the graviton, corresponding to the odd vertices in the Galileon theory of [3] (like the cubic scalar vertex of the DGP model [56]) and in Lovelock's theory of Gravity [5] (like the Gauss-Bonnet term) respectively.

Finally, Galileon interactions involving a single species of N-partite tensor gauge field with $N \geq 3$ were constructed in [49]. As we will see shortly, the main difference in this case is that locality and gauge invariance of these interactions are in direct conflict with each other.

Let us now move on by defining the *generalized* kinetic term constructed in [49]. To this end, we will introduce the shorthand notation for the total exterior derivative and its odd part, namely

$$D_{\text{tot}} := \mathrm{d}_1 \ldots \mathrm{d}_k \qquad \& \qquad D_{\text{odd}} := \mathrm{d}_1 \mathrm{d}_3 \ldots \mathrm{d}_{k-1} \,, \tag{2.62}$$

and write down the following term:

$$\begin{aligned} \mathcal{L}_{(n)}(\omega) &= \left(D_{\text{odd}} \left[\omega \, \frac{1}{\Box^{\frac{k}{2}-1}} (D_{\text{tot}} \omega)^n \right] \, , \, \frac{1}{\Box^{\frac{k}{2}-1}} D_{\text{odd}} \left[\omega \, \frac{1}{\Box^{\frac{k}{2}-1}} (D_{\text{tot}} \omega)^n \right] \right) \\ &= \int_{\theta_1, \ldots, \theta_k} D_{\text{odd}} \left[\omega \, \frac{1}{\Box^{\frac{k}{2}-1}} (D_{\text{tot}} \omega)^n \right] \star_{12} \cdots \star_{k-1\, k} \, \frac{1}{\Box^{\frac{k}{2}-1}} D_{\text{odd}} \left[\omega \, \frac{1}{\Box^{\frac{k}{2}-1}} (D_{\text{tot}} \omega)^n \right] \end{aligned} \tag{2.63}$$

It is easy to check that the term $\mathcal{L}_{(n)}(\omega)$ is invariant under the gauge transformation (2.52) and contains an even number, namely $2n+2$, of appearances of the field ω. Furthermore, it is well-defined only for N-partite functions ω with degree satisfying

$$D \geq (n+1)(p_A + p_{A+1} + 1) + n \,, \quad \forall\, A = 1, 3, 5, \ldots, k-1 \tag{2.64}$$

due to the definition of the generalized Hodge operator (2.48). For a gauge field of given type and in given number of spacetime dimensions, the above conditions put an upper bound, n_{max}, on n

[12] To be more precise, this observation concerns only the gauge invariance of the Lagrangian and does not refer to the equations of motion. Indeed, there has been an example of a 3-form theory with a Lagrangian that is not gauge invariant, but with gauge invariant field equations that exhibit the Galilean symmetry [55].

corresponding to the maximum integer obeying all the above $k/2$ inequalities. Finally, these terms reduce to the standard kinetic terms (2.55) upon setting $n = 0$.

Moreover, there exists an enhancement of Galileon interactions for the special case of gauge fields with type $(p, p, \dots, p)$. These interactions allow for odd appearances of the gauge field and their general form reads as

$$\overline{\mathcal{L}_{(\overline{n})}}(\omega) = \int_{\theta_1,\dots,\theta_k} H^{[D-(\overline{n}-1)(1-p),\,\overline{n}p]}\,\omega\,\left(\frac{1}{\Box^{\frac{k}{2}-1}} D_{\text{tot}}\omega\right)^{\overline{n}}, \tag{2.65}$$

in terms of the newly defined quantity $H^{[D,p]} := \prod_{A=1,3,5,\dots}^{k-1} \eta_{A\,A+1}^{D-p_A-p_{A+1}-1}$. For $\overline{n} = 1$ they reduce (up to a total derivative term) to the respective kinetic term (2.55), while for higher odd values of $\overline{n}$ they can be brought to the form (2.63) after some partial integrations. For even $\overline{n}$, the above interaction terms contain an odd number of appearances of ω and, as such, they cannot be identified with any term in (2.63).

At this stage, one has to also make sure that the interactions defined above do not correspond to total derivative terms. To this end, let us assume that the total degree $P = p_1 + \cdots + p_N$ of the gauge field ω is an odd integer. This implies that its total exterior derivative $D_{\text{tot}}\omega$ also has odd total degree. Then, one can perform some partial integrations in (2.63) and observe that the quantity $(D_{\text{tot}}\omega)^{n+1}$ appears. This quantity obviously vanishes for $n \geq 1$ and, thus, the above interactions are total derivatives if the total degree of ω is odd[13]. Finally, the same argument can be used for the special case of Galileon interactions (2.65). In the case of odd P, the quantity $\left(\frac{1}{\Box^{\frac{k}{2}-1}} D_{\text{tot}}\omega\right)^{\overline{n}}$ vanishes identically for $\overline{n} \geq 2$, which implies that (2.65) are identically zero (not even total derivatives).

The property described above was already observed in [6] for the $N = 1$ case of differential form and, also, in [47] for the $N = 2$ case of bipartite tensors where it was termed *evenophilia*. It is only for gauge fields with even total degree that the Galileon interactions in (2.63) and (2.65) are nontrivial.

Just like the kinetic and mass terms presented before, the standard forms of the Galileon Lagrangians can all be obtained by expanding the term (2.63) (or (2.65)) in local coordinates and, subsequently, by carrying out the Berezin integrations. The simplest example is the scalar Galileon of [3], which

[13] Of course, note that this does not hold for $n = 0$. In that case, the interactions (2.63) reduce to the corresponding kinetic terms (2.55) which do not correspond to total derivatives even for gauge fields with odd P.

contains the following terms in $D = 4$:

$$
\begin{aligned}
\overline{\mathcal{L}_{(0)}}(\omega) &\propto \omega\,, \\
\overline{\mathcal{L}_{(1)}}(\omega) &\propto \omega\Box\omega\,, \\
\overline{\mathcal{L}_{(2)}}(\omega) &\propto \partial^{\mu}\omega\partial^{\nu}\omega\partial_{\mu}\partial_{\nu}\omega - \partial^{\mu}\omega\partial_{\mu}\omega\Box\omega\,, \\
\overline{\mathcal{L}_{(3)}}(\omega) &\propto -(\Box\omega)^2\partial_{\mu}\omega\partial^{\mu}\omega + 2\,\Box\omega\partial_{\mu}\omega\partial_{\nu}\omega\partial^{\mu}\partial^{\nu}\omega \\
&\quad + \partial_{\mu}\partial_{\nu}\omega\partial^{\mu}\partial^{\nu}\omega\partial_{\rho}\omega\,\partial^{\rho}\omega - 2\,\partial_{\mu}\omega\partial^{\mu}\partial^{\nu}\omega\partial_{\nu}\partial_{\rho}\omega\partial^{\rho}\omega\,, \\
\overline{\mathcal{L}_{(4)}}(\omega) &\propto -(\Box\omega)^3\partial_{\mu}\omega\partial^{\mu}\omega + 3\,(\Box\omega)^2\partial_{\mu}\omega\partial_{\nu}\omega\partial^{\mu}\partial^{\nu}\omega \\
&\quad + 3\,\Box\omega\partial_{\mu}\partial_{\nu}\omega\partial^{\mu}\partial^{\nu}\omega\partial_{\rho}\omega\partial^{\rho}\omega - 6\,\Box\omega\partial_{\mu}\omega\partial^{\mu}\partial^{\nu}\omega\partial_{\nu}\partial_{\rho}\omega\partial^{\rho}\omega \\
&\quad - 2\,\partial_{\mu}\partial^{\nu}\omega\partial_{\nu}\partial^{\rho}\omega\partial_{\rho}\partial^{\mu}\omega\partial_{\sigma}\omega\partial^{\sigma}\omega - 3\,\partial_{\mu}\partial_{\nu}\omega\partial^{\mu}\partial^{\nu}\omega\partial_{\rho}\omega\partial_{\sigma}\omega\partial^{\rho}\partial^{\sigma}\omega \\
&\quad + 6\,\partial_{\mu}\omega\partial^{\mu}\partial^{\nu}\omega\partial_{\nu}\partial_{\rho}\omega\partial^{\rho}\partial^{\sigma}\omega\partial_{\sigma}\omega\,.
\end{aligned}
\tag{2.66}
$$

These result from the term (2.65) for $k = 2$ and $p = 0$, up to an irrelevant overall factor and total derivative terms. Since there can be no 1-form Galileon due to evenophilia, the second simplest nontrivial example can be obtained from (2.65) for $k = 2$ and $p = 1$. In arbitrary number of dimensions D, the corresponding local expressions have the form (once again, up to an overall factor and total derivative terms)

$$
\overline{\mathcal{L}_{(n)}}(\omega) \;\propto\; \delta^{\mu_1\ldots\mu_{2n+1}}_{\nu_1\ldots\nu_{2n+1}}\,\omega^{\nu_1}_{\mu_1}\prod_{i=1}^{n} R_{\mu_{2i}\mu_{2i+1}}{}^{\nu_{2i}\nu_{2i+1}}\,, \qquad n_{max} = \lfloor\frac{D-1}{2}\rfloor
\tag{2.67}
$$

in terms of the generalized Kronecker delta $\delta^{\dots}_{\dots}$ and the linearized Riemann tensor $R^{\mu\nu|}{}_{\rho\sigma} = 2!2!\partial^{[\mu}\partial_{[\rho}\omega^{\nu]}_{\sigma]}$. These are precisely the Lovelock invariants [5] for the linearized graviton $\omega_{\mu\nu}$.

Let us now recall our initial physical requirement of locality. It is clear that the Galileon interactions we considered here are nonlocal for $N \geq 3$, just like the corresponding kinetic terms in (2.55). However, the difference with the interactions is that they do not lead to local field equations, after imposing the gauge condition (2.54). In fact, the field equations remain nonlocal even after imposing a condition that fixes the gauge symmetry fully. This last point was illustrated through a simple example in [49]. We conclude that the unique terms for $N \geq 3$ satisfying our locality criterion are the kinetic terms (2.55). A further possibility for obtaining local field equations was explained in [49], where it was shown that one can trade a part of the gauge symmetry (2.52) in order to force locality. In any case, this discussion provides strong evidence in favor of the following statement:

"*Galileon interactions for a single N-partite field leading to local, second-order field equations and respecting the gauge symmetries* (2.52) *cannot be constructed for* $N \geq 3$".

Important Note on Terminology & Notation

In this Chapter, we developed a formalism that enables one to think of differential forms and mixed-symmetry tensors on a smooth manifold M as functions of specific degree on the graded manifold $\mathcal{M}_N$. This was made possible due to an isomorphism, which indicates that there is a one-to-one correspondence between the elements of the former and the latter. In the following Chapters we will comply with the standard terminology in Physics literature and use the terms "differential forms" and "N-partite mixed-symmetry tensors", rather than "1-partite" and "N-partite" functions. However, it should be clear that we will always refer to the latter.

In Chapters 3, 4, 5 and 6, we will be dealing exclusively with differential forms and bipartite mixed-symmetry tensors. To this end, we will simplify the notation regarding the relevant algebraic and differential maps defined earlier. More precisely, we will make the following identifications:

$$\begin{aligned}
&\eta_{12} \equiv \eta\,, \quad \mathrm{tr}_{12} \equiv \mathrm{tr}\,, \quad \sigma_{12} \equiv \sigma\,, \quad \sigma_{21} \equiv \widetilde{\sigma}\,,\\
&\mathrm{d}_1 \equiv \mathrm{d}\,, \quad \mathrm{d}_2 \equiv \widetilde{\mathrm{d}}\,, \quad \mathrm{d}_1^\dagger \equiv \mathrm{d}^\dagger\,, \quad \mathrm{d}_2^\dagger \equiv \widetilde{\mathrm{d}}^\dagger\,,\\
&\theta_1^\mu \equiv \theta^\mu\,, \quad \theta_2^\mu \equiv \chi^\mu\,, \quad \bar{\theta}^1_\mu \equiv \bar{\theta}_\mu\,, \quad \bar{\theta}^2_\mu \equiv \bar{\chi}_\mu\,,\\
&*_1 \equiv *\,, \quad *_2 \equiv \widetilde{*}\,, \quad \star_{12} \equiv \star\,, \quad \top_{12} \equiv \top\,.
\end{aligned}$$

Using this notation, the relations (2.26)-(2.31) for the $N = 2$ maps are written as

$$[\sigma, \widetilde{\sigma}] = \hat{p}_1 - \hat{p}_2\,, \tag{2.68}$$

$$[\mathrm{tr}, \eta] = d - \hat{p}_1 - \hat{p}_2\,, \tag{2.69}$$

$$[\hat{p}_1, \mathrm{tr}] = -\mathrm{tr}\,, \quad [\hat{p}_1, \eta] = \eta\,, \tag{2.70}$$

$$[\hat{p}_1, \sigma] = -[\hat{p}_2, \sigma] = \sigma\,, \tag{2.71}$$

$$[\sigma, \eta] = 0\,, \tag{2.72}$$

$$[\sigma, \mathrm{tr}] = 0\,, \tag{2.73}$$

while the relations (2.38)-(2.45) as

$$\{\mathrm{tr}, \mathrm{d}\} = \widetilde{\mathrm{d}}^{\dagger}\,, \tag{2.74}$$

$$\{\mathrm{tr}, \mathrm{d}^{\dagger}\} = 0\,, \tag{2.75}$$

$$\{\eta, \mathrm{d}\} = 0\,, \tag{2.76}$$

$$\{\eta, \mathrm{d}^{\dagger}\} = \widetilde{\mathrm{d}}\,, \tag{2.77}$$

$$\{\sigma, \mathrm{d}\} = 0\,, \tag{2.78}$$

$$\{\sigma, \widetilde{\mathrm{d}}\} = -\mathrm{d}\,, \tag{2.79}$$

$$\{\sigma, \mathrm{d}^{\dagger}\} = \widetilde{\mathrm{d}}^{\dagger}\,, \tag{2.80}$$

$$\{\sigma, \widetilde{\mathrm{d}}^{\dagger}\} = 0\,. \tag{2.81}$$

It should be clear that the transposed relations also hold.

Chapter 3

DUALITIES IN MAXWELL'S THEORY

Maxwell's theory of Electromagnetism is one of the oldest and most celebrated physical theories. It is also the simplest Gauge Theory, as it is based on an Abelian 1-form gauge field propagating in the 4-dimensional Minkowski spacetime $\mathbb{R}^{1,3}$. In this Chapter, we will review the Hodge dualities in Maxwell's theory and set the stage for a generalization to more general Gauge Theories involving a p-form gauge field in arbitrary spacetime dimensions. The latter will appear in Chapter 5.

3.1 Electric-magnetic duality

Let us begin our discussion by considering a 1-form field A, with 2-form field $F = \mathrm{d}A$, satisfying the Maxwell equations in vacuum

$$\mathrm{d}F = 0 \qquad \& \qquad \mathrm{d} * F = 0\,. \tag{3.1}$$

The first is the Bianchi identity on F, while the second corresponds to the equation of motion obeyed by the Maxwell field A. In four dimensions the Hodge star operator $*$ squares to -1 when acting on a 2-form, which implies that the above equations are invariant under the general linear transformation

$$\begin{pmatrix} F \\ *F \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} F \\ *F \end{pmatrix}, \tag{3.2}$$

for two arbitrary numbers $a, b \in \mathbb{R}$. Indeed, it is easy to see that the transformed system of equations becomes

$$\mathrm{d} * F = 0 \qquad \& \qquad \mathrm{d}F = 0 \tag{3.3}$$

and we further observe that the two initial equations (3.1) have been interchanged. Such a transformation that exchanges the Bianchi identity with the equation of motion is called an *electric-magnetic duality transformation*. More generally, it is a special case of a Hodge transformation as it involves the corresponding star operation.

The transformation (3.2) is a symmetry of the Maxwell equations, but not of the Maxwell Hamiltonian defined by the purely timelike component of the respective energy-momentum tensor $H = T^{00}$.

In fact, it can be easily shown that (3.2) leaves the Hamiltonian invariant only for parameters satisfying $a^2 + b^2 = 1$. To this end, we say that Maxwell's theory in four dimensions is invariant under the electric-magnetic duality transformations

$$\begin{pmatrix} F \\ *F \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} F \\ *F \end{pmatrix}, \tag{3.4}$$

where we used a different parametrization for the transformation to also enforce the invariance of the Hamiltonian. The above transformation corresponds to an arbitrary $SO(2)$ rotation and, thus, we identify the so-called duality group with $SO(2)$.

As a final comment, note that the Maxwell Lagrangian is invariant under the above transformation only for angles satisfying $\cos^2\alpha = 1 + \sin^2\alpha$. This equation is satisfied only when $\alpha = 0$, in which case the duality transformation (3.4) becomes trivial. However, there exists a specific angle α for which the transformed Maxwell Lagrangian simply changes sign[1] and is, thus, equivalent to the original one. This value is $\alpha = \frac{\pi}{2}$, upon which the transformation (3.4) becomes

$$\begin{pmatrix} F \\ *F \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F \\ *F \end{pmatrix}. \tag{3.5}$$

The above discussion shows that the Maxwell Lagrangian is not invariant under the full $SO(2)$ duality rotations (3.4). This naturally leads to the question of whether one can realize such a duality purely at the off-shell Lagrangian level. As shown in [57], the answer to this question is affirmative and it involves a specific nonlocal (in space) transformation of the Maxwell field A itself (see also [58]-[59] for more discussions).

Let us now treat the 2-form $\widehat{F} := *F$ as independent of F. Upon doing so, the first equations in (3.1) and (3.3) can be seen as two distinct Bianchi identities $\mathrm{d}F = 0 = \mathrm{d}\widehat{F}$. One can then use the Poincaré lemma to locally solve these on a contractible patch by

$$F := \mathrm{d}A \qquad \& \qquad \widehat{F} := \mathrm{d}\widehat{A}\,, \tag{3.6}$$

in terms of two distinct 1-form fields A and $\widehat{A}$. With these identifications, the remaining second equations in (3.1) and (3.3) can be seen as two independent copies of the Maxwell equations of

[1]The sign flip of the Lagrangian is a consequence of Hodge duality in Minkowski spacetime.

motion in vacuum, $\mathrm{d} * F = 0 = \mathrm{d} * \widehat{F}$. Finally, taking into account the defining relation $\widehat{F} := *F$ reveals that the two 1-forms are related through

$$\mathrm{d}\widehat{A} = * \, \mathrm{d}A \, . \tag{3.7}$$

This relation is very important since it implies that if A is a Maxwell 1-form, i.e. if it satisfies the equations of motion $\mathrm{d} * \mathrm{d}A = 0$, then the 1-form $\widehat{A}$ defined by (3.7) is also a Maxwell 1-form, and vice versa. An important observation here is that equation (3.7) is nonlocal. Indeed, an algebraic relation between the dual fields would imply that $\widehat{A}$ and A are related by a field redefinition and, as such, they would fail to provide distinct independent descriptions of Maxwell's theory.

At first glance, electric-magnetic duality in Maxwell's theory can seem quite trivial, in the sense that the dual gauge fields A and $\widehat{A}$ are both 1-forms. The reason for this is that in four dimensions the Hodge dual of the 2-form F is also a 2-form, leading to two distinct dual 1-form gauge fields. Had we started with Maxwell's equations in D dimensions, we would have found the dual gauge field to be a $(D-3)$-form. This fact leads to the notion of *self-duality* of Maxwell's theory, which is only present in four dimensions. To be more precise, one would say that this is an example of *twisted* self-duality [60], where the adjective "twisted" indicates that the dual fields A and $\widehat{A}$ are *not* Hodge duals to one another (see also [61]). We will refrain from using this adjective in the ensuing for brevity.

In general, the importance of self-duality becomes clear when one deals with interacting theories, in which case the duality transformation can be seen as a transformation of the theory's couplings. Thus, the existence of self-duality implies a symmetry between two specific coupling regimes of the interacting theory. In the case when the duality transformation inverts the original coupling, one talks about a self-duality of *strong-weak* type. That is because knowledge of the weak coupling regime (e.g. through perturbation theory) grants access to the strong coupling one (where perturbative methods break down). This justifies why strong-weak dualities are a very natural and important tool in studying non-perturbative Physics.

Self-duality of Maxwell's theory in four dimensions is clearly not of the type discussed above, since it involves a free theory. However, it provides us with a simple toy model that already possesses the key features underlying a strong-weak duality and, as such, it is very useful to study it. To explore this picture, the on-shell discussion we presented so far does not suffice and one has to realize the duality at the off-shell level of the Lagrangians.

3.2 Off-shell realization and the parent action approach

The most common procedure for exploiting an off-shell duality is the parent action approach. In this paragraph, we will demonstrate its key features in the simple example of Maxwell's theory in four dimensions. This procedure can be done entirely using the standard differential geometry language, since there are no mixed-symmetry tensors involved at any stage. However, we will make use the graded formalism described in the previous Chapter right away. The reason is that we want to familiarize the reader with the more practical and computational aspects of this formalism, since we are going to actually need it soon in the following Sections.

The starting point for this procedure will be a parent action, or rather a Lagrangian, defined as

$$\mathcal{L}_{\mathrm{p}}(F, \widehat{A}) = \frac{1}{2e^2} \int_\theta F * F - \frac{1}{4\pi} \int_\theta F \, \mathrm{d}\widehat{A} \,, \tag{3.8}$$

depending on a 2-form F (which can be expanded as in (2.12)) and a 1-form $\widehat{A}$. In natural units $\epsilon_0 = c = \hbar = 1$, the unit of electric charge e is the dimensionless coupling related to the fine structure constant α by $e^2 = 4\pi\alpha$. Clearly, this Lagrangian is of first order in derivatives and is not related to any physical theory at this level. Regarding the gauge symmetries present, it is easy to check that both

$$\delta_\epsilon F = \mathrm{d}^2\epsilon \,, \qquad \delta_{\widehat{\epsilon}} \widehat{A} = \mathrm{d}\widehat{\epsilon} \,, \tag{3.9}$$

for some arbitrary scalar gauge parameters ϵ and $\widehat{\epsilon}$, leave $\mathcal{L}_{\mathrm{p}}$ invariant. The gauge transformation of F is, of course, trivial due to the nilpotency of the exterior derivative $\mathrm{d}^2 = 0$, but it is important to mention it for a reason that will become transparent later.

The idea now is that $\widehat{A}$ can be thought of as a Lagrange multiplier field and, thus, one can integrate it out using its equation of motion. To this end, we can vary $\mathcal{L}_{\mathrm{p}}$ with respect to $\widehat{A}$ and F, and obtain the respective Euler-Lagrange equations

$$\mathrm{d}F = 0 \qquad \& \qquad * F = \frac{e^2}{4\pi} \mathrm{d}\widehat{A} \,. \tag{3.10}$$

The first is the Bianchi identity on F and, as we already mentioned, can be solved locally by means of the Poincaré lemma to give $F := \mathrm{d}A$ for a new 1-form A. Plugging this solution back into the parent Lagrangian, one gets

$$\mathcal{L}(A) = \frac{1}{2e^2}\int_\theta \mathrm{d}A * \mathrm{d}A \tag{3.11}$$

up to a total derivative term that contains the auxiliary field $\widehat{A}$, which we can then integrate out by assuming appropriate boundary conditions. The final Lagrangian (3.11) is now of second order in derivatives and only involves the 1-form A. In addition, we can easily check that this Lagrangian is invariant under the the Abelian gauge transformation $\delta_\epsilon A = \mathrm{d}\epsilon$. This invariance was inherited to $\mathcal{L}$ by the parent Lagrangian, through the (trivial) gauge invariance of the latter under $\delta_\epsilon F = \mathrm{d}^2\epsilon$.

The second field equation in (3.10) is the off-shell version of the duality relation presented in (3.7), in the sense that F is not yet related to the original gauge field A at this level. Indeed, going fully on-shell (taking into account also the Bianchi identity on F, which implies that $F := \mathrm{d}A$) one recovers (3.7) modulo the overall factor $\frac{e^2}{4\pi}$ that we omitted in our previous discussion for simplicity. These being said, the second equation in (3.10) can be substituted back into $\mathcal{L}_\mathrm{p}$ to give

$$\widehat{\mathcal{L}}(\widehat{A}) = \frac{1}{2\widehat{e}^2}\int_\theta \mathrm{d}\widehat{A} * \mathrm{d}\widehat{A}\,, \qquad \widehat{e} = \frac{4\pi}{e} \tag{3.12}$$

This is the dual Lagrangian that is of second order and only depends on the dual gauge field $\widehat{A}$. Once again, the gauge invariance of $\widehat{\mathcal{L}}$ under $\delta_{\widehat{\epsilon}}\widehat{A} = \mathrm{d}\widehat{\epsilon}$ is directly inherited from its parent.

We have found that the parent Lagrangian (3.8) is equivalent on-shell to two second-order "child" Lagrangians (3.11) and (3.12) that solely involve the two dual 1-form gauge fields A and $\widehat{A}$. In addition, both of them inherit their Abelian gauge invariance directly through the initial invariance of their parent. The only difference between the two, which was hidden in the previous on-shell discussion, is that they have inverse couplings. In other words, at the off-shell level, self-duality of Maxwell's theory becomes manifest as a symmetry under the transformation of the coupling

$$e \longrightarrow \widehat{e} = \frac{4\pi}{e}\,. \tag{3.13}$$

As we already mentioned, this symmetry does not indicate a strong-weak duality, simply because Maxwell's theory is non-interacting. In other words, both couplings e and $\widehat{e}$ can be absorbed into the gauge fields A and $\widehat{A}$ through a field redefinition.

3.3 The effect of the electromagnetic ϑ-term

Let us know generalize our discussion and include the well-known electromagnetic ϑ-term in the parent Lagrangian (3.8). This term is the second gauge and Lorentz invariant scalar in four dimensions, reading as $\vartheta \int_\theta F^2 \sim \vartheta \vec{E} \cdot \vec{B}$ in terms of the electric and magnetic fields. If the coupling ϑ is a spacetime constant, then this corresponds to a topological term, being simply the derivative of the Chern-Simons 3-form.

In addition, it is easy to see that the ϑ-term violates both parity $\mathcal{P}$ and time reversal $\mathcal{T}$ symmetries, for arbitrary values of ϑ. To see this, remember that the electric (magnetic) field is odd (even) under $\mathcal{P}$ and even (odd) under $\mathcal{T}$ transformations. This implies that the ϑ-term will violate $\mathcal{PT}$ unless ϑ is odd under both $\mathcal{P}$ and $\mathcal{T}$, i.e. it must be a pseudoscalar. In light of these, we will demand that ϑ is periodic and takes values in $[0, 2\pi)$. This is the only way of having the possibility to obtain a nonzero value of ϑ that is odd under both $\mathcal{P}$ and $\mathcal{T}$. This nonzero value is $\vartheta = \pi$.

Now that we have laid the ground, let us discuss the duality of such a model. Our starting point will be the parent Lagrangian

$$\mathcal{L}_{\mathrm{p}}(F, \widehat{A}) = \frac{1}{2e^2} \int_\theta F * F + \frac{\vartheta}{16\pi^2} \int_\theta F^2 - \frac{1}{4\pi} \int_\theta F \, \mathrm{d}\widehat{A} \,, \tag{3.14}$$

where the only difference with (3.8) is the inclusion of the ϑ-term (in all that follows, we will treat the parameter ϑ as a spacetime constant). Indeed, the two first-order Lagrangians even share the same gauge symmetries (3.9). Just like before, variation with respect to $\widehat{A}$ and F will give the Bianchi identity and the duality relation

$$\mathrm{d}F = 0 \qquad \& \qquad \frac{4\pi}{e^2} * F + \frac{\vartheta}{2\pi} F = \mathrm{d}\widehat{A} \,. \tag{3.15}$$

Again, the first equation can be solved locally on a contractible patch using Poincaré's lemma and one obtains $F := \mathrm{d}A$, for a new 1-form field A. Plugging this solution back into $\mathcal{L}_{\mathrm{p}}$, we find the first child Lagrangian

$$\mathcal{L}(A) = \frac{1}{2e^2} \int_\theta \mathrm{d}A * \mathrm{d}A + \frac{\vartheta}{16\pi^2} \int_\theta (\mathrm{d}A)^2 \tag{3.16}$$

up to a total derivative containing the Lagrange multiplier field $\widehat{A}$. Note that the ϑ-term above is also a total derivative since we have assumed that ϑ is spacetime independent. However, we will not drop this term from the Lagrangian since we wish to find how ϑ transforms under duality.

As is always the case, the Bianchi identity leads to a second order theory and the duality relation leads to its dual. To this end, we wish to solve the second equation in (3.15) in terms of F. By acting on both sides of it with $*$, one easily obtains

$$*F = \frac{2\pi}{\vartheta} * \mathrm{d}\widehat{A} + \frac{8\pi^2}{\vartheta e^2} F\,. \tag{3.17}$$

This intermediate relation can be used to integrate out $*F$. Indeed, substituting it back into the duality relation leads to

$$F = \frac{2\pi\vartheta e^4}{64\pi^4 + \vartheta^2 e^4}\,\mathrm{d}\widehat{A} - \frac{16\pi^3 e^2}{64\pi^4 + \vartheta^2 e^4} * \mathrm{d}\widehat{A} \tag{3.18}$$

and we have obtained the solution in terms of F. After substitution in the parent Lagrangian and a straightforward calculation, we find the dual Lagrangian

$$\widehat{\mathcal{L}}(\widehat{A}) = \frac{1}{2\widehat{e}^2}\int_\theta \mathrm{d}\widehat{A} * \mathrm{d}\widehat{A} + \frac{\widehat{\vartheta}}{16\pi^2}\int_\theta (\mathrm{d}\widehat{A})^2 \tag{3.19}$$

where the new couplings read as

$$\widehat{e}^2 = \frac{64\pi^4 + \vartheta^2 e^4}{4\pi^2 e^2}\,, \qquad \widehat{\vartheta} = -\frac{4\pi^2\vartheta e^4}{64\pi^4 + \vartheta^2 e^4}\,. \tag{3.20}$$

We have found that the parent Lagrangian (3.14) is equivalent on-shell to the two children second order Lagrangians (3.16) and (3.19). In addition, we have found that the transformation of couplings connecting the two Lagrangians is $e \to \widehat{e}$ and $\vartheta \to \widehat{\vartheta}$, with the dual couplings given by (3.20). At face value, these transformations are more involved than the one we found earlier in (3.13) and do not invert the couplings. However, there is still a notion of coupling inversion in this case. To convince ourselves, let us define the complex coupling

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi}{e^2} i\,. \tag{3.21}$$

As it turns out, the dual complex coupling is simply the inverse of the original modulo a minus sign

$$\widehat{\tau} = \frac{\widehat{\vartheta}}{2\pi} + \frac{4\pi}{\widehat{e}^2} i = -\frac{1}{\tau}\,, \tag{3.22}$$

which indicates that the transformation $S : \tau \to -\frac{1}{\tau}$ is a symmetry. Furthermore, one can see that

$$\tau + 1 = \frac{\vartheta + 2\pi}{2\pi} + \frac{4\pi}{e^2} i = \frac{\vartheta}{2\pi} + \frac{4\pi}{e^2} i = \tau \tag{3.23}$$

due to the periodicity of ϑ and, thus, we also have a symmetry under the transformation $T : \tau \longrightarrow \tau+1$. It is well-known that the transformations S and T generate the modular group Γ of linear fractional transformations of the upper half of the complex plane, which have the form

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \qquad ad - bc = 1\,, \qquad a, b, c, d \in \mathbb{Z}\,. \tag{3.24}$$

That is, every element in Γ can be obtained (in a non-unique way) by acting with specific powers of S and T on another element. The modular group Γ is isomorphic to the group $PSL(2, \mathbb{Z})$ of all 2×2 matrices with integer entries and unit determinant, with the additional identification that $\mathbb{M} = -\mathbb{M}$ for any matrix $\mathbb{M} \in PSL(2, \mathbb{Z})$. In fact, this is the group underlying the coupling space in the oldest example of an actual strong-weak duality, which is the S-duality of quantum $N = 4$ super Yang-Mills theory first conjectured by Montonen and Olive in [13].

According to our discussion so far, we have seen that the duality group at the space of couplings is the S-duality group $PSL(2, \mathbb{Z})$. Let us now take a step back and examine the duality group underlying the above system from the perspective of spacetime. At the on-shell level, the effect of the ϑ-term is to modify the Maxwell equations (3.1) into the system

$$\mathrm{d}F = 0 \qquad \& \qquad \mathrm{d}\widehat{F} = 0\,, \tag{3.25}$$

where we have made the identification $\widehat{F} := \frac{4\pi}{e^2} * F + \frac{\vartheta}{2\pi} F$. To proceed, we assume that the field strength transforms according to $F \mapsto aF - b\widehat{F}$ with $b \neq 0$, which implies that

$$\widehat{F} \quad \mapsto \quad \frac{64\pi^4 + \vartheta^2 e^4}{4\pi^2 e^4}\, bF + \left(a - \frac{b\vartheta}{\pi}\right) \widehat{F}\,. \tag{3.26}$$

One can then observe that $\frac{64\pi^4+\vartheta^2 e^4}{4\pi^2 e^4} = |\tau|^2$ in terms of the complex coupling we defined in (3.21), which implies that we have a transformation of the form

$$\begin{pmatrix} F \\ \widehat{F} \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} a & -b \\ b|\tau|^2 & a - \frac{b\vartheta}{\pi} \end{pmatrix} \begin{pmatrix} F \\ \widehat{F} \end{pmatrix}. \tag{3.27}$$

It is easy to check that this is an electric-magnetic duality transformation, in that it maps the first equation in (3.25) to the second, and vice versa.

Let us now see how the first-order form of the Lagrangian (3.16) (i.e. the parent Lagrangian (3.14) without the Lagrange multiplier term) transform under (3.27). After a short computation we find

$$\mathcal{L} \quad \mapsto \quad \left(a^2 - b^2|\tau|^2 + \frac{b\vartheta}{\pi^2}(b\vartheta - 2\pi a)\right) \mathcal{L} + \frac{b}{8\pi^2}(2\pi a - b\vartheta)|\tau|^2 \int_\theta F^2\,, \tag{3.28}$$

and it becomes clear that (3.27) is a symmetry only if

$$a^2 - b^2|\tau|^2 + \frac{b\vartheta}{\pi^2}(b\vartheta - 2\pi a) = -1 \qquad \& \qquad 2\pi a - b\vartheta = 0\,. \tag{3.29}$$

Note that, by setting the r.h.s. of the first condition to -1, we have again assumed invariance of $\mathcal{L}$ modulo a sign flip, in the same way that the standard Maxwell Lagrangian is invariant under the duality transformation (3.5). In fact, had we set the above r.h.s. to be $+1$ we would have found that the system of the two equations does not have a solution for real a and b. The solution of (3.29) reads as $a = \frac{\vartheta e^2}{8\pi^2}$ and $b = \frac{e^2}{4\pi}$, while applying these values into (3.27) gives

$$\begin{pmatrix} F \\ \widehat{F} \end{pmatrix} \quad \mapsto \quad \frac{e^2}{4\pi}\begin{pmatrix} \frac{\vartheta}{2\pi} & -1 \\ |\tau|^2 & -\frac{\vartheta}{2\pi} \end{pmatrix}\begin{pmatrix} F \\ \widehat{F} \end{pmatrix}. \tag{3.30}$$

It is easy to check that the determinant of the above matrix is equal to 1 and, thus, the effect of adding the ϑ-term in the Maxwell Lagrangian is that the initial $SO(2)$ electric-magnetic duality group gets extended to $SL(2,\mathbb{R})$ [62]. Indeed, one can rewrite the above matrix in the form

$$\frac{e^2}{4\pi}\begin{pmatrix} \frac{\vartheta}{2\pi} & -1 \\ |\tau|^2 & -\frac{\vartheta}{2\pi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\vartheta}{2\pi} & \frac{4\pi}{e^2} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{\vartheta}{2\pi} & \frac{4\pi}{e^2} \end{pmatrix}^{-1} \tag{3.31}$$

and observe that the original $SO(2)$ rotation $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ gets conjugated by the matrix $\begin{pmatrix} 1 & 0 \\ \frac{\vartheta}{2\pi} & \frac{4\pi}{e^2} \end{pmatrix}$. As we already saw in previous discussions, the Maxwell Hamiltonian is invariant under the full $SO(2)$ duality transformation given by (3.4) and, thus, inclusion of the ϑ-term leads to the extended $SL(2,\mathbb{R})$ invariance

$$\begin{pmatrix} 1 & 0 \\ \frac{\vartheta}{2\pi} & \frac{4\pi}{e^2} \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{\vartheta}{2\pi} & \frac{4\pi}{e^2} \end{pmatrix}^{-1} = \frac{e^2}{4\pi}\begin{pmatrix} \frac{4\pi}{e^2}\cos\alpha + \frac{\vartheta}{2\pi}\sin\alpha & -\sin\alpha \\ |\tau|^2\sin\alpha & \frac{4\pi}{e^2}\cos\alpha - \frac{\vartheta}{2\pi}\sin\alpha \end{pmatrix} \tag{3.32}$$

of the corresponding Hamiltonian (see, e.g. [8]). For a review on the general theory of electric-magnetic duality rotations, the reader may consult [63] and references therein.

3.4 Exotic duality

Besides the standard electric-magnetic duality that we reviewed so far, there also exists a different type of Hodge duality. It was first suggested by Deser, Townsend and Siegel [23], and later studied by Hull [33], that there exists a duality between a free theory involving a differential p-form gauge field in flat D-dimensional spacetime and a free theory involving an irreducible bipartite tensor gauge field of type $(D-2, p)$. Although the dual field is seemingly of spin 2, it obeys a particular field equation that is "weaker" than the expected one and, thus, propagates the same physical degrees of freedom (d.o.f.) as the p-form. In this setting, it is clear that any notion of self-duality is lost, simply because the dual fields are, by default, of different type.

This is often termed as *exotic* duality and it shares lots of common features with electric-magnetic one, including the fact that it can be realized off-shell by means of a parent action [48, 64, 65]. However, obtaining the aforementioned weaker field equation for the exotic dual field requires the presence of additional fields at the off-shell level. These fields also appear naturally in the dual action, but do not propagate any physical d.o.f. (see also [27] for a discussion on the role of these additional fields). Indeed, we will show that, upon gauge fixing, the on-shell duality relation implies that these fields are pure gauge.

Exotic duality can be generalized to free gauge theories involving an n-partite tensor gauge field of arbitrary type $(p_1, \dots, p_n)$, in any number of spacetime dimensions. The exotic dual field is then an $(n+1)$-partite tensor of type $(D-2, p_1, \dots, p_n)$. Such tensor gauge fields of higher spin arise naturally in the context of maximal supergravity and String Theory, the latter containing infinitely many such massive excitations. For a review on the interplay between higher-spin fields and String Theory, the reader may consult [66]. In addition, exotic duals of differential forms and higher-spin fields are abundant in the E_{11} approach of [26], where they correspond to charges of branes [67, 68]. Because of this, it has also been conjectured that they couple electrically to low codimensional branes [69–74], including domain walls, solitonic and spacetime-filling branes, as well as the so-called "exotic branes" (see [28] for a review on the latter).

From the above discussion it becomes clear that exotic dual fields are *never* differential forms. This fact creates an obstacle for a geometric formulation of exotic duality, since there is no way to describe mixed-symmetry tensor gauge fields geometrically using standard differential geometry. This is the reason why one needs to utilize graded geometry, which becomes the natural mathematical framework for the description of exotic duality.

Let us begin by briefly reviewing the on-shell exotic duality of the Maxwell field in four dimensions. The exotic dual is an irreducible bipartite tensor gauge field A^{ex} of type $(2, 1)$ and, thus, its local components have the index symmetry of the Young tableau $\yng(2,1)$. The gauge transformations for such a field were first studied by Curtright in [24] and their graded geometric form is

$$\delta A^{\mathrm{ex}} = \mathrm{d}\alpha - \mathrm{d}\widetilde{\sigma}\beta + 2\widetilde{\mathrm{d}}\beta\,, \tag{3.33}$$

in terms of an irreducible $(1,1)$ gauge parameter α, carrying the index symmetries of $\yng(2)$, and a 2-form parameter β.

We now turn to the question of the equation of motion that such a field should obey. To answer that, it is useful to make a remark regarding the equation of motion of the Maxwell field A itself. Using the identities (2.74) and (A.6), it is easy to check that the Maxwell equation of motion can be written in the equivalent form

$$\mathrm{d} * \mathrm{d}A = 0 \quad \Leftrightarrow \quad \mathrm{tr}\,\mathrm{d}\widetilde{\mathrm{d}}A = 0\,. \tag{3.34}$$

In fact, the field equations of any differential form gauge field in any number of dimensions can be written in this equivalent way. To this end, it is natural to expect that the same equation $\mathrm{tr}\,\mathrm{d}\widetilde{\mathrm{d}}A^{\mathrm{ex}} = 0$ holds for the exotic dual field too. However, it was pointed out by Hull [33] (see also [43]-[44] for a more general discussion) that a traceless bipartite tensor of type (p,q) is identically zero in spacetime dimensions $D < p + q$. According to this statement, one can easily check that the field equation $\mathrm{tr}\,\mathrm{d}\widetilde{\mathrm{d}}A^{\mathrm{ex}} = 0$ is trivial in four dimensions. This is because it implies that $\mathrm{d}\widetilde{\mathrm{d}}A^{\mathrm{ex}} = 0$, meaning that the exotic dual field A^{ex} is pure gauge.

This last fact was already known to Curtright [24], who showed that if such a field equation is obeyed in four dimensions, then there are no propagating degrees of freedom in the theory[2]. To overcome this, one has to assume that A^{ex} satisfies the weaker equation

$$\mathrm{tr}^2\mathrm{d}\widetilde{\mathrm{d}}A^{\mathrm{ex}} = 0\,. \tag{3.35}$$

This relation implies the vanishing of a 1-form quantity in four dimensions and, as such, corresponds to 4 independent equations. Upon gauge fixing, these equations reduce to 2 and count the propagating

[2]Note however that the field equation $\mathrm{tr}\,\mathrm{d}\widetilde{\mathrm{d}}A^{\mathrm{ex}} = 0$ is *not* trivial in five dimensions or higher, since it does not imply that $\mathrm{d}\widetilde{\mathrm{d}}A^{\mathrm{ex}} = 0$. In the case of five dimensions, Curtright showed that such a field equation leads to propagation of 5 degrees of freedom (d.o.f.), which coincides with the d.o.f. propagated by the linearized graviton. This fact indicates that the standard (non-exotic) dual of the latter in five dimensions is a $(2,1)$ Curtright field satisfying precisely this field equation.

d.o.f. of the exotic dual field. These clearly coincide with the d.o.f. propagated by the Maxwell field A, as they should. Finally, one should note that, in contrast to $\mathrm{tr}\, \mathrm{d}\widetilde{\mathrm{d}}A^{\mathrm{ex}} = 0$ which follows from the Curtright action, equation (3.35) *cannot* be obtained as an Euler-Lagrange equation from any action that solely involves A^{ex}. This is the reason why the off-shell formulation of exotic duality must involve additional fields.

Another crucial difference between the standard electric-magnetic duality and exotic duality is the form of the relation between the dual fields. In particular, we will show that the exotic dual field A^{ex} is related to A through the nonlocal relation

$$\mathrm{d}\widetilde{\mathrm{d}}A^{\mathrm{ex}} = *\, \mathrm{d}\widetilde{\mathrm{d}}A^{\top} . \tag{3.36}$$

When compared to (3.7), we observe that this relation contains two derivatives instead of one. In addition, it indicates that on-shell exotic duality can be understood as a Hodge duality involving the transposition of A, rather than A itself. In the ensuing, we will show that a relation like (3.36) forces the exotic dual field A^{ex} to be algebraically related to the standard electric-magnetic dual $\widehat{A}$ of A at the on-shell level. The implication is that the dual fields $\widehat{A}$ and A^{ex} do not provide independent descriptions of Maxwell's theory. This result was obtained in [30] for arbitrary p-form gauge fields in any spacetime dimension and was based on a very similar result concerning the double dual linearized graviton [29]. We will discuss the latter in more detail in the next Chapter.

Let us perform the exotic dualization procedure at the off-shell level, through the parent action approach. We will use the following first-order Lagrangian

$$\mathcal{L}_{\mathrm{p}}(F, \Lambda) = -\frac{1}{2e^2} \int_{\theta,\chi} F \star F + \frac{1}{4\pi} \int_{\theta,\chi} F \,\widetilde{*}\, \mathrm{d}\Lambda \,, \tag{3.37}$$

which depends on two *reducible* bipartite tensor fields F and Λ of types $(1, 1)$ and $(2, 1)$, respectively. As we already explained, the parent action should contain the exotic dual field and, in general, additional fields that will enforce the correct equation of motion (3.35) on-shell. This is the reason why Λ has to be chosen reducible; its irreducible $(2, 1)$ component will be naturally identified with the exotic dual field A^{ex}, while its 3-form component will play the role of these additional fields. Regarding the choice of reducibility for F, we will provide the explanation momentarily. The parent Lagrangian above is a simple generalization of the one presented in [48], which in turn corresponds to the graded geometric form of the one presented in [25].

We will begin by identifying the gauge symmetries of $\mathcal{L}_{\text{p}}$, which read as

$$\delta_\xi F = \mathrm{d}\widetilde{\mathrm{d}}\xi\,, \qquad \delta_\xi \Lambda = \frac{2\pi}{e^2}\,\widetilde{*}\,\eta^2\widetilde{\mathrm{d}}\xi\,, \qquad \delta_\epsilon \Lambda = \mathrm{d}\epsilon\,, \tag{3.38}$$

in terms of a scalar ξ and a reducible bipartite tensor gauge parameter ϵ of type $(1,1)$. The symmetry parametrized by ϵ can be seen immediately, since the Lagrange multiplier term is invariant due to the nilpotency of d. Under the ξ-symmetry, the transformation of (3.37) contains three terms and one of them is the total derivative $\int_{\theta,\chi} \delta_\xi F\,\widetilde{*}\,\mathrm{d}\Lambda$. The remaining terms read as $\int_{\theta,\chi} F\delta_\xi I$, where

$$\delta_\xi I = -\frac{1}{e^2}\star\delta_\xi F + \frac{1}{4\pi}\,\widetilde{*}\,\mathrm{d}\delta_\xi\Lambda = -\frac{1}{e^2}\star\mathrm{d}\widetilde{\mathrm{d}}\xi + \frac{1}{2e^2}\eta^2\mathrm{d}\widetilde{\mathrm{d}}\xi = -\frac{1}{e^2}\star\mathrm{d}\widetilde{\mathrm{d}}\xi + \frac{1}{e^2}\star\mathrm{d}\widetilde{\mathrm{d}}\xi = 0\,. \tag{3.39}$$

This shows that (3.37) is indeed invariant under (3.38). In agreement to what happened in the previous cases, it is natural to expect that the parent Lagrangian will inherit the above gauge symmetries to its children theories.

Let us now vary $\mathcal{L}_{\text{p}}$ with respect to Λ and obtain the Bianchi identity $\mathrm{d}F = 0$. As usual, this can be locally solved by $F := \mathrm{d}A^\top$ for a new field of type $(0,1)$, which we have conveniently identified with the transposed Maxwell 1-form. Substituting this solution back into $\mathcal{L}_{\text{p}}$, one obtains the second order Lagrangian

$$\mathcal{L}(A) = -\frac{1}{2e^2}\int_{\theta,\chi}\mathrm{d}A^\top\star\mathrm{d}A^\top, \tag{3.40}$$

up to a total derivative containing the Lagrange multiplier Λ. Despite appearances, one can show that this Lagrangian is exactly the same as (3.11) up to a total derivative term. This can be easily seen by noting that $\star\,\mathrm{d}A^\top = \frac{1}{2}\eta^2\widetilde{\mathrm{d}}A$. The gauge invariance of (3.40) reads as $\delta_\xi A = \mathrm{d}\xi$ and is inherited from the gauge invariance $\delta_\xi F = \mathrm{d}\widetilde{\mathrm{d}}\xi$ of its parent through the identification $F := \mathrm{d}A^\top$. The above discussion also provides the reason why F had to be a reducible tensor in the first place. Indeed, had F been irreducible then σF (and $\widetilde{\sigma}F$) would vanish. However, as we can see $\sigma F = \sigma\mathrm{d}A^\top = -\mathrm{d}\sigma A^\top = \mathrm{d}A$ meaning that the child Lagrangian (3.40) would correspond to a total derivative (equivalently, A would be pure gauge).

We move on by finding the duality relation. As usual, this results from variation of (3.37) with respect to F and has the very simple form

$$\frac{1}{e^2}\star F = \frac{1}{4\pi}\,\widetilde{*}\,\mathrm{d}\Lambda\,. \tag{3.41}$$

In order to find the dual theory one has to first solve the above equation in terms of F. Using the relation $\star F = *\widetilde{*}(F - \eta\,\mathrm{tr}F)$ and subsequently acting with $*\widetilde{*}$, we can rewrite the above equation as

$$\frac{1}{e^2}F - \frac{1}{e^2}\,\eta\,\mathrm{tr}F = \frac{1}{4\pi} * \mathrm{d}\Lambda\,. \tag{3.42}$$

One can now act on both sides of this with the trace map and, making use of the identities (A.4) and (2.69), find that $\mathrm{tr}F = -\frac{e^2}{12\pi} * \sigma\mathrm{d}\Lambda$. This relation can be used to integrate out $\mathrm{tr}F$ in (3.42), which leads to the solution of (3.41) in terms of F,

$$F = \frac{e^2}{4\pi} * \mathrm{d}\Lambda - \frac{e^2}{12\pi}\,\eta * \sigma\mathrm{d}\Lambda\,. \tag{3.43}$$

Finally, we can use the identities (A.4), (2.78) and (2.79) to rewrite the above solution in the more suggestive form

$$F = \frac{e^2}{4\pi} * \mathrm{d}\,\mathbb{P}_{(2,1)}\Lambda - \frac{e^2}{12\pi} * \widetilde{\mathrm{d}}\sigma\Lambda\,, \tag{3.44}$$

where $\mathbb{P}_{(2,1)} = \mathbb{I} - \frac{1}{3}\,\widetilde{\sigma}\sigma$ is the Young projection onto the symmetries of the $(2,1)$ Young tableau $\yng(2,1)$. In our picture, the exotic dual A^{ex} can be naturally identified with the irreducible component of Λ possessing the index symmetries of $\yng(2,1)$, that is $A^{\mathrm{ex}} := \mathbb{P}_{(2,1)}\Lambda$. Furthermore, we will denote the 3-form component of Λ by $H := \frac{1}{3}\sigma\Lambda$, so that we have the irreducible decomposition

$$\Lambda := A^{\mathrm{ex}} + \widetilde{\sigma}H\,. \tag{3.45}$$

In addition, we can decompose the reducible gauge parameter ϵ into irreducible components $\alpha := \mathbb{P}_{(1,1)}\epsilon$ and $\beta := -\frac{1}{6}\sigma\epsilon$, so that $\epsilon = \alpha - 3\widetilde{\sigma}\beta$. Using these decompositions, the gauge symmetries $\delta_\xi\Lambda$ and $\delta_\epsilon\Lambda$ can also be decomposed and seen as transformations of A^{ex} and H,

$$\begin{aligned} &\delta_\xi A^{\mathrm{ex}} = \mathbb{P}_{(2,1)}\delta_\xi\Lambda = 0\,, && \delta_\xi H = \frac{1}{3}\sigma\delta_\xi\Lambda = \frac{4\pi}{e^2} * \mathrm{d}\xi\,,\\ &\delta_\epsilon A^{\mathrm{ex}} = \mathbb{P}_{(2,1)}\delta_\epsilon\Lambda = \mathrm{d}\alpha - \mathrm{d}\widetilde{\sigma}\beta + 2\widetilde{\mathrm{d}}\beta\,, && \delta_\epsilon H = \frac{1}{3}\sigma\delta_\epsilon\Lambda = 2\mathrm{d}\beta\,. \end{aligned} \tag{3.46}$$

These decompositions follow directly using the map identities (A.4), (2.68), (2.69), (2.78), (2.79) and (2.50). It is important to note here that A^{ex} has, indeed, the correct gauge symmetry given by (3.33).

Let us now open a parenthesis and analyze the duality relation (3.44). In terms of the irreducible fields defined by (3.45), this takes the form

$$F = \frac{e^2}{4\pi} * \mathrm{d}A^{\mathrm{ex}} - \frac{e^2}{4\pi} * \widetilde{\mathrm{d}}H\,. \tag{3.47}$$

Let us for the moment restrict to the fully on-shell level, i.e. assume that $F = \mathrm{d}A^\top$. There are some very interesting observations to be made, by manipulating the above equation in appropriate ways. For instance, acting on both sides of it with $\widetilde{\mathrm{d}}$ gives

$$\mathrm{d}\widetilde{\mathrm{d}}A^\top = \frac{e^2}{4\pi} * \mathrm{d}\widetilde{\mathrm{d}}A^{\mathrm{ex}}, \tag{3.48}$$

which is precisely the exotic duality relation (3.36) up to an unimportant factor. As we already mentioned, equation (3.36) implies that the exotic dual field is algebraically related to the standard dual on-shell. To see that this is the case, one can act with $\widetilde{\mathrm{d}}$ on the duality relation (3.7) and transpose both sides of the resulting equation. Then, this can be combined with (3.36) and leads to

$$\mathrm{d}\widetilde{\mathrm{d}}A^{\mathrm{ex}} = - * \widetilde{*}\, \mathrm{d}\widetilde{\mathrm{d}}\widehat{A}^\top = - \star\, \mathrm{d}\widetilde{\mathrm{d}}\widehat{A}^\top - * \widetilde{*}\, \eta \mathrm{tr}\, \mathrm{d}\widetilde{\mathrm{d}}\widehat{A}^\top, \tag{3.49}$$

where we also used the identity (2.50). However, if the standard dual field $\widehat{A}$ satisfies its field equation $\mathrm{tr}\, \mathrm{d}\widetilde{\mathrm{d}}\widehat{A} = 0$, then the second term on the r.h.s. of (3.49) vanishes identically as it involves the transposed quantity $\mathrm{tr}\, \mathrm{d}\widetilde{\mathrm{d}}\widehat{A}^\top = (\mathrm{tr}\, \mathrm{d}\widetilde{\mathrm{d}}\widehat{A})^\top$. Finally, one can note that $\star\, \mathrm{d}\widetilde{\mathrm{d}}\widehat{A}^\top = \eta\, \mathrm{d}\widetilde{\mathrm{d}}\widehat{A}$ by definition, which reveals that

$$\mathrm{d}\widetilde{\mathrm{d}}(A^{\mathrm{ex}} + \eta \widehat{A}) = 0\,. \tag{3.50}$$

According to a generalization of the Poincaré lemma that was presented in [47] and proven in [48], one can locally solve the above equation on a contractible patch and obtain[3]

$$A^{\mathrm{ex}} + \eta \widehat{A} = \mathrm{d}X + 2\widetilde{\mathrm{d}}Y\,, \tag{3.51}$$

for some arbitrary reducible tensor X of type $(1,1)$ and a 2-form Y. However, the l.h.s. is clearly irreducible and so must be the combination on the r.h.s. This argument sets the restriction $X = -\widetilde{\sigma}Y$, which is easily obtained by acting with σ on both sides of (3.51) and making use of the identities (2.68), (2.72), (2.78) and (2.79). Thus, we end up with

$$A^{\mathrm{ex}} + \eta \widehat{A} = 2\widetilde{\mathrm{d}}Y - \mathrm{d}\widetilde{\sigma}Y \tag{3.52}$$

and we can see that the l.h.s. of this equation can be set to zero if we perform a gauge transformation $\delta_\epsilon A^{\mathrm{ex}}$, given by (3.46), with parameter $\beta \equiv Y$. Therefore, we have shown that the exotic and standard

[3] In fact, this is true up to a constant antisymmetric 3-tensor that we ignore in the present discussion.

duals of the Maxwell 1-form are algebraically related on-shell, up to a physically irrelevant gauge transformation.

Let us now return to the fully on-shell version of the duality relation (3.47) and make a second observation. To this end, we can act on both sides of it with the combination of maps $\mathrm{tr} * \mathrm{d}$ and use the map identities (2.74), (2.75) and (A.6). Upon doing so, one will find precisely the field equation (3.35) that the exotic dual field A^{ex} should obey. This is an important result since it guarantees that (3.35) will be implied by the Euler-Lagrange equations of the dual Lagrangian itself, by construction.

Finally, one can trace equation (3.47) and obtain

$$\widetilde{\mathrm{d}}^{\dagger} A^{\top} = \frac{e^2}{4\pi} * \mathrm{d}H \,. \tag{3.53}$$

The l.h.s. of this equation is put to zero on-shell, as a result of the Lorentz gauge fixing condition $\widetilde{\mathrm{d}}^{\dagger} A^{\top} = \mathrm{d}^{\dagger} A \stackrel{!}{=} 0$. Thus, equation (3.53) implies that the Lorentz gauge condition on A forces $\mathrm{d}H = 0$. This proves that, on-shell, H has to be pure gauge for consistency and explains why this field does not propagate any physical degrees of freedom.

Closing the parenthesis, we can now substitute (3.47) (without the assumption $F = \mathrm{d}A^{\top}$) back into the parent Lagrangian (3.37) to obtain the exotic dual theory. Making use of (2.80), as well as the integral identities (A.8) and (A.10), one easily finds that

$$\begin{aligned} \mathcal{L}^{\mathrm{ex}}(A^{\mathrm{ex}}, H) = &- \frac{e^2}{32\pi^2} \int_{\theta,\chi} \mathrm{d}A^{\mathrm{ex}} * \widetilde{*}\, \mathrm{d}A^{\mathrm{ex}} + \frac{e^2}{32\pi^2} \int_{\theta,\chi} \mathrm{d}^{\dagger} H * \widetilde{*}\, \mathrm{d}^{\dagger} H \\ &- \frac{e^2}{32\pi^2} \int_{\theta,\chi} \mathrm{d}A^{\mathrm{ex}} * \widetilde{*}\, \mathrm{d}\widetilde{\sigma} H + \frac{e^2}{32\pi^2} \int_{\theta,\chi} \mathrm{d}A^{\mathrm{ex}} * \widetilde{*}\, \widetilde{\mathrm{d}} H \,. \end{aligned} \tag{3.54}$$

In contrast to what happens in the standard electric-magnetic duality, we observe that the dual Lagrangian $\mathcal{L}^{\mathrm{ex}}$ contains both the dual field A^{ex} and the 3-form H. Furthermore, it is invariant under the gauge transformations (3.46) and implies the field equation (3.35) by construction.

3.5 Exotic dualization in presence of ϑ

In this Section we will enjoy the possibility of including the topological ϑ-term in the off-shell exotic dualization procedure. Our goal is to find the respective on-shell duality relation that replaces (3.36) in that case. The results here are new and have not appeared in any published work.

Our starting point will be the parent Lagrangian

$$\mathcal{L}_{\mathrm{p}}(F,\Lambda) = -\frac{1}{2e^2}\int_{\theta,\chi} F\star F - \frac{\vartheta}{16\pi^2}\int_{\theta,\chi}\sigma F\,\widetilde{*}\,\sigma F + \frac{1}{4\pi}\int_{\theta,\chi} F\,\widetilde{*}\,\mathrm{d}\Lambda\,, \tag{3.55}$$

where the only difference with respect to (3.37) is the inclusion of the second term. As usual, we will assume that the parameter ϑ is a (periodic) spacetime constant that takes values in $[0,2\pi)$. The gauge symmetries of this Lagrangian are precisely (3.38), since it is straightforward to use the identities (2.78)-(2.79) and show that $\sigma\delta_\xi F = 0$ identically.

Moving on, variation of (3.55) with respect to Λ will give the Bianchi $\mathrm{d}F = 0$ and its solution will read as $F := \mathrm{d}A^\top$. Using this, one obtains the second order Lagrangian

$$\mathcal{L}(A) = -\frac{1}{2e^2}\int_{\theta,\chi}\mathrm{d}A^\top\star\mathrm{d}A^\top - \frac{\vartheta}{2\pi}\int_{\theta,\chi}\sigma\mathrm{d}A^\top\widetilde{*}\,\sigma\mathrm{d}A^\top \tag{3.56}$$

instead of (3.40). As explained earlier, the first term is the same as (3.11) up to a total derivative. The second term is a total derivative itself and, using (2.78) and the fact that $\sigma A^\top = -A$, one can show that it precisely matches the electromagnetic ϑ-term[4]. Thus, the Lagrangian above matches (3.16).

The Euler-Lagrange variation for F will result in the central object of this Section, i.e. the duality relation. To obtain this, we first note that the novel term is quadratic in F and, thus, its variation will read as

$$\begin{aligned}
-\frac{\vartheta}{8\pi^2}\int_{\theta,\chi}\sigma\delta F\,\widetilde{*}\,\sigma F &\overset{(A.4)}{=} \frac{\vartheta}{8\pi^2}\int_{\theta,\chi} *\,\mathrm{tr}\,*\,\delta F\,\widetilde{*}\,\sigma F\\
&\overset{(A.8)}{=} \frac{\vartheta}{8\pi^2}\int_{\theta,\chi}\mathrm{tr}\,*\,\delta F\ *\widetilde{*}\,\sigma F\\
&\overset{(A.9)}{=} \frac{\vartheta}{8\pi^2}\int_{\theta,\chi} *\,\delta F\ *\widetilde{*}\,\eta\sigma F \overset{(A.8)}{=} -\frac{\vartheta}{8\pi^2}\int_{\theta,\chi}\delta F\,\widetilde{*}\,\eta\sigma F\,.
\end{aligned} \tag{3.57}$$

Using this, one finds that the duality relation has the form

$$\frac{1}{e^2}\star F + \frac{\vartheta}{8\pi^2}\,\widetilde{*}\,\eta\sigma F = \frac{1}{4\pi}\,\widetilde{*}\,\mathrm{d}\Lambda \tag{3.58}$$

and we can immediately observe that the only difference with respect to (3.41) is the second term on the l.h.s. The solution of (3.58) in terms of F can be obtained by following a procedure very

[4]Note that there seems to exist an extra minus sign. However, this corresponds to the trivial integration over the variable χ, i.e. $\int_\chi \widetilde{*}\,\mathbb{I} = -1$, for the top form $\widetilde{*}\,\mathbb{I} = \frac{1}{4!}\varepsilon_{\mu_1\ldots\mu_4}\chi^{\mu_1}\ldots\chi^{\mu_4}$.

similar to the one we carried out in the previous Section. Namely, we can use the relation (2.50) and subsequently act on (3.58) with $*\widetilde{*}$, to obtain the equivalent form

$$\frac{1}{e^2}F - \frac{1}{e^2}\,\eta\mathrm{tr}F + \frac{\vartheta}{8\pi^2} * \eta\sigma F = \frac{1}{4\pi} * \mathrm{d}\Lambda\,. \tag{3.59}$$

Tracing both sides of this relation, while making use (2.50) and the map identities (2.69), (2.72) and (2.73), will lead to $\mathrm{tr}F = -\frac{e^2}{12\pi} * \sigma\mathrm{d}\Lambda$. This is, in fact, the same trace condition on F as the one we found in Section 3.4. We can now substitute this condition back into (3.59) to integrate out $\mathrm{tr}F$. The resulting equation will take the form

$$F + \frac{e^2\vartheta}{8\pi^2} * \eta\sigma F = \frac{e^2}{4\pi} * \mathrm{d}\,\mathbb{P}_{(2,1)}\Lambda - \frac{e^2}{12\pi} * \widetilde{\mathrm{d}}\sigma\Lambda\,, \tag{3.60}$$

where the identities (A.4), (2.78) and (2.79) were also used. Notice that the only difference between this and the solution (3.44) to the previous duality relation lies on the additional term on the l.h.s.

We are now ready to perform the decomposition (3.45) of Λ into its irreducible components A^{ex} and H, and also go fully on-shell by assuming that $F := \mathrm{d}A^{\top}$. Remembering that $\sigma\mathrm{d}A^{\top} = -\mathrm{d}\sigma A^{\top} = \mathrm{d}A$, we can immediately see that the resulting relation will read as

$$\mathrm{d}A^{\top} + \frac{e^2\vartheta}{8\pi^2} * \eta\,\mathrm{d}A = \frac{e^2}{4\pi} * \mathrm{d}A^{\mathrm{ex}} - \frac{e^2}{4\pi} * \widetilde{\mathrm{d}}H\,. \tag{3.61}$$

As a consistency check, we can act on this relation with the operator $\mathrm{tr} * \mathrm{d}$ and, assuming that A satisfies its field equation $\mathrm{tr}\mathrm{d}\widetilde{\mathrm{d}}A = 0$ (which remains the free Maxwell equation since the ϑ-term is a total derivative for constant ϑ), obtain the expected field equation (3.35) for the exotic dual field A^{ex}. In addition, tracing (3.61) will still lead to the relation (3.53) that we found earlier. This implies that the 3-form H (which will still appear in the exotic dual Lagrangian) will not propagate any physical degrees of freedom. For the short computations above, we used the map identities (A.6), (2.74) and (2.75).

Finally, we can act with $\widetilde{\mathrm{d}}$ on both sides of equation (3.61). Using (2.50), the defining relation $\eta\,\mathrm{d}\widetilde{\mathrm{d}}A^{\top} = \star\,\mathrm{d}\widetilde{\mathrm{d}}A$ and the field equation of A, we obtain

$$\mathrm{d}\widetilde{\mathrm{d}}A^{\top} - \frac{e^2\vartheta}{8\pi^2}\widetilde{*}\,\mathrm{d}\widetilde{\mathrm{d}}A^{\top} = \frac{e^2}{4\pi} * \mathrm{d}\widetilde{\mathrm{d}}A^{\mathrm{ex}}\,, \tag{3.62}$$

which is the main result of this Section. Alternatively, this relation can be inverted and written in the equivalent form

$$\mathrm{d}\widetilde{\mathrm{d}}A^{\top} = \frac{16\pi^3 e^2}{64\pi^4 + \vartheta^2 e^4} * \mathrm{d}\widetilde{\mathrm{d}}A^{\mathrm{ex}} + \frac{2\pi\vartheta e^4}{64\pi^4 + \vartheta^2 e^4} * \widetilde{*}\,\mathrm{d}\widetilde{\mathrm{d}}A^{\mathrm{ex}}\,, \tag{3.63}$$

which clearly coincides with (3.48) for $\vartheta = 0$. However and in contrast to (3.48), the duality relation above *does not* imply that A^{ex} is algebraically related on-shell to the electric-magnetic dual $\widehat{A}$ of A. On the contrary, the duality relation (3.63) is *nonlocal* which means that the effect of the ϑ-term is to promote the exotic dual theory to an independent description of Maxwell's theory in terms of the gauge field A^{ex}. Therefore, exotic duality of the Maxwell field in four dimensions is special, in that the electromagnetic ϑ-term is admissible.

As we will see in Chapter 5, exotic dualities exist for any p-form gauge field in any number of spacetime dimensions but they all suffer from the algebraic on-shell duality relation described in the previous Section [30]. The novel result of this Section is that this no longer holds for the Maxwell 1-form in four dimensions, when the topological electromagnetic ϑ-term is present. One would expect that this observation can be extended to the exotic duality of any odd $(2p+1)$-form in its self-dual dimensions $D = 4p + 4$ (since the ϑ-terms of even differential forms in their self-dual dimensions vanish identically and cannot possibly contribute).

Chapter 4

DUALITY IN LINEARIZED GRAVITY

In this Chapter, we will review the electric-magnetic duality present in the linearized limit of Gravity. The idea that linearized Gravity exhibits the same electric-magnetic duality as the one in Maxwell's theory originates from the work of Hull [31–33], where it was shown that an $SO(2)$ rotation mixing the Riemann tensor with its Hodge dual exchanges the Bianchi identities with the linearized Einstein equations. Later on, this duality was realized at the off-shell level by West [26, 34] in terms of a parent action that links the two dual linearized Einstein-Hilbert actions. This parent action was then studied in more detail in [35] and extended to higher-spin fields in arbitrary spacetime dimensions. Finally, all these parent Lagrangians and the corresponding off-shell dualization procedures were put in a simple geometric form in [48] using graded geometry. In this Chapter, we will follow this last geometric approach.

4.1 Electric-magnetic duality of the Einstein equations

At the weak or linearized limit, one can think of Gravity as a 4-dimensional theory that involves an irreducible mixed-symmetry tensor gauge field h of type $(1,1)$, with gauge transformation given by

$$\delta_\xi h = \frac{1}{2}\mathrm{d}\xi^\top + \frac{1}{2}\widetilde{\mathrm{d}}\xi\,, \tag{4.1}$$

for a 1-form parameter ξ. The local components of the gauge field h correspond to a symmetric 2-tensor that is naturally identified with the linearized graviton, while the gauge transformation above is the graded geometric version of the linearized diffeomorphisms. Finally, one should note that irreducibility of h implies the conditions $\sigma h = 0 = \widetilde{\sigma} h$ and $h = h^\top$, in agreement with the Young constraints (2.34).

The gauge invariant field strength of h is the irreducible $(2,2)$ tensor of Young symmetry $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}$ defined by $R := \mathrm{d}\widetilde{\mathrm{d}}h$, whose local components are identified with the linearized Riemann tensor. Due to the nilpotency of the exterior derivatives and its irreducibility, the Riemann tensor satisfies the condition $R = R^\top$, as well as the differential and algebraic Bianchi identities

$$\mathrm{d}R = 0 = \widetilde{\mathrm{d}}R \qquad \& \qquad \sigma R = 0 = \widetilde{\sigma} R \quad \overset{(A.4)}{\Leftrightarrow} \quad \mathrm{tr} * R = 0 = \mathrm{tr}\,\widetilde{*}\, R\,. \tag{4.2}$$

Moreover, the linearized Einstein equation obeyed by h on-shell can be written as the vanishing of the Ricci tensor

$$\mathrm{tr}R := \mathrm{tr}\,\mathrm{d}\widetilde{\mathrm{d}}h = 0\,. \tag{4.3}$$

Using the map identity (2.74), the differential Bianchi identities and the above trace-free condition, one can easily show that the Riemann tensor is divergenceless on-shell, i.e.

$$\mathrm{d}^\dagger R = 0 = \widetilde{\mathrm{d}}^\dagger R \quad \overset{(A.6)}{\Leftrightarrow} \quad \mathrm{d} * R = 0 = \widetilde{\mathrm{d}}\,\widetilde{*}\, R\,. \tag{4.4}$$

In full analogy to the invariance of Maxwell's equations, it was first noticed by Hull [33] (see also [32]) that the system of equations (4.2)-(4.3)-(4.4) is invariant under the transformation

$$\begin{pmatrix} R & \widetilde{*}R \\ *R & *\widetilde{*}R \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} R & \widetilde{*}R \\ *R & *\widetilde{*}R \end{pmatrix}, \tag{4.5}$$

for arbitrary constants a, b. In fact, it can be easily seen that this transformation exchanges the differential Bianchi identities with the divergenceless conditions, as well as the algebraic Bianchi identities with the Einstein equation. It is an electric-magnetic self-duality transformation.

In full analogy to the previous case of Maxwell's theory, the Hamiltonian is invariant under the transformation above only if $a^2 + b^2 = 1$, i.e. if $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$. This was shown by Henneaux and Teitelboim in [75][1], where it was also proven that this $SO(2)$ transformation is a symmetry of the linearized Einstein-Hilbert action, realized by means of a nonlocal transformation of h (following exactly the same procedure as the one carried out in [57] for the Maxwell Lagrangian).

Let us now treat the Hodge duals of R as independent fields and write

$$\mathcal{R} = \begin{pmatrix} R & \widehat{R}^\top \\ \widehat{R} & R^{\mathrm{doub}} \end{pmatrix}. \tag{4.6}$$

Then, the aforementioned invariance under (4.5) implies that every matrix component of $\mathcal{R}$ is an irreducible $(2,2)$ tensor satisfying the differential Bianchi identities and the linearized Einstein

[1] We thank Arash Ranjbar for bringing this reference to our attention.

equation, namely

$$\sigma\mathcal{R}_{ij} = 0 = \widetilde{\sigma}\mathcal{R}_{ij}\,, \qquad \mathcal{R}_{ij} = \mathcal{R}_{ij}^{\top}\,, \qquad \mathrm{d}\mathcal{R}_{ij} = 0 = \widetilde{\mathrm{d}}\mathcal{R}_{ij} \qquad \& \qquad \mathrm{tr}\mathcal{R}_{ij} = 0\,. \tag{4.7}$$

As such, every component $\mathcal{R}_{ij}$ can be seen as a linearized Riemann tensor for an independent linearized graviton. Indeed, one can solve the differential Bianchi identities above using the Poincaré lemma in a contractible patch to find

$$\mathcal{R}_{00} = R := \mathrm{d}\widetilde{\mathrm{d}}h\,, \qquad \mathcal{R}_{10} = \widehat{R} := \mathrm{d}\widetilde{\mathrm{d}}\widehat{h}\,, \qquad \mathcal{R}_{11} = R^{\mathrm{doub}} := \mathrm{d}\widetilde{\mathrm{d}}h^{\mathrm{doub}}\,, \tag{4.8}$$

in terms of the irreducible $(1,1)$ tensor fields h, $\widehat{h}$ and h^{doub}. Finally, recalling the defining relations $\widehat{R} := *R$ and $R^{\mathrm{doub}} := *\widetilde{*}R$ between the Riemann tensors, we immediately obtain the on-shell duality relations between the dual gravitons

$$\mathrm{d}\widetilde{\mathrm{d}}\,\widehat{h} = *\,\mathrm{d}\widetilde{\mathrm{d}}h \qquad \& \qquad \mathrm{d}\widetilde{\mathrm{d}}h^{\mathrm{doub}} = *\widetilde{*}\,\mathrm{d}\widetilde{\mathrm{d}}h\,. \tag{4.9}$$

The field $\widehat{h}$ will be referred to as the dual linearized graviton, while h^{doub} will be called the double dual. In the following Section, we will review the off-shell dualization procedure linking h with $\widehat{h}$. Double duality deserves separate treatment and will be discussed in more detail in Section 4.4.

4.2 Off-shell dualization procedure

Let us now move on and show how one can realize the electric-magnetic duality between h and $\widehat{h}$ at the off-shell level. To this end, we will need a parent Lagrangian of the form

$$\mathcal{L}_{\mathrm{p}}(f,\Lambda) = -\frac{1}{2g^2}\int_{\theta,\chi} f \star \mathcal{O} f + \frac{1}{4\pi}\int_{\theta,\chi} f\,\widetilde{*}\,\mathrm{d}\Lambda\,, \tag{4.10}$$

where f and Λ are two independent reducible bipartite tensors of types $(2,1)$ and $(1,1)$ respectively. Moreover, we work in the natural units system with unit Newton's constant and Planck mass $G = M_{\mathrm{P}} = 1$, while g is a dimensionless reference constant. Note that we have also included the algebraic operator $\mathcal{O} := \mathbb{I} - \frac{1}{2}\widetilde{\sigma}\sigma$, for reasons that will become clear soon. An immediate observation is that $\mathcal{O}$ is formally the same as the Young projection $\mathbb{P}_{(1,1)}$, but acts on a field of type $(2,1)$ instead of $(1,1)$. As such, it cannot be identified with a projection onto the index symmetries of the Young tableaux $\square\square$.

The parent Lagrangian above is the same as the one constructed in [48] (and later used in [37]), which in turn is the graded geometric form of the original parent Lagrangian used by West in [26].

Its gauge symmetries were initially spelled out in [35] and their graded geometric form read as

$$\begin{aligned} \delta_\xi f &= \mathrm{d}\widetilde{\mathrm{d}}\xi\,, & \delta_\xi\Lambda &= \frac{4\pi}{g^2} * \eta\,\mathrm{d}\xi\,, \\ \delta_\omega f &= \mathrm{d}\widetilde{\sigma}\omega\,, & \delta_\omega\Lambda &= -\frac{4\pi}{g^2} * \eta\omega\,, \\ \delta_{\widehat{\xi}} f &= 0\,, & \delta_{\widehat{\xi}}\Lambda &= \mathrm{d}\widehat{\xi}^{\top}\,, \end{aligned} \tag{4.11}$$

in terms of the 1-form parameters $\xi, \widehat{\xi}$ and a 2-form ω. Let us now open a small parenthesis and show that these transformations are indeed symmetries of $\mathcal{L}_{\mathrm{p}}$. To this end, one can use the identity[2] $\int_{\theta,\chi} \delta(f \star O f) = 2\int_{\theta,\chi} f \star O\delta f$ and show that (up to the total derivative term $\int_{\theta,\chi}\delta_\xi f\,\widetilde{*}\,\mathrm{d}\Lambda$) the δ_ξ-variation reads as $\delta_\xi\mathcal{L}_{\mathrm{p}} = \int_{\theta,\chi} f * \widetilde{*}\,\delta_\xi I$, where

$$\delta_\xi I = -\frac{1}{g^2}\left(\mathbb{I} - \eta\mathrm{tr}\right)\mathrm{d}\widetilde{\mathrm{d}}\xi + \frac{1}{g^2}\mathrm{d}^\dagger\eta\mathrm{d}\xi \overset{(2.77)}{=} -\frac{1}{g^2}\left(\mathbb{I} - \eta\mathrm{tr}\right)\mathrm{d}\widetilde{\mathrm{d}}\xi + \frac{1}{g^2}\left(\widetilde{\mathrm{d}} - \eta\mathrm{d}^\dagger\right)\mathrm{d}\xi \overset{(2.74)}{=} 0\,. \tag{4.12}$$

In the computation above we also used the relations (2.50), (A.6) and the fact that $\widetilde{\sigma}\sigma\mathrm{d}\widetilde{\mathrm{d}}\xi = 0$, which follows directly from the identities (2.78) and (2.79). For the transformation parametrized by ω, we can similarly show that $\delta_\omega\mathcal{L}_{\mathrm{p}} = \int_{\theta,\chi} f * \widetilde{*}\,\delta_\omega I$ where

$$\delta_\omega I = \frac{1}{g^2}\left(\mathbb{I} - \eta\mathrm{tr}\right)\widetilde{\mathrm{d}}\omega - \frac{1}{g^2}\mathrm{d}^\dagger\eta\omega \overset{(2.77)}{=} \frac{1}{g^2}\left(\mathbb{I} - \eta\mathrm{tr}\right)\widetilde{\mathrm{d}}\omega - \frac{1}{g^2}\left(\widetilde{\mathrm{d}} - \eta\mathrm{d}^\dagger\right)\omega \overset{(2.74)}{=} 0\,. \tag{4.13}$$

For this computation, we also used the identity (2.68). Finally, the transformation parametrized by $\widehat{\xi}$ is a symmetry in a trivial way due to the nilpotency of the exterior derivatives. In analogy to our previous discussion on the off-shell dualization of the Maxwell field, the parent Lagrangian (4.10) is expected to inherit the symmetries (4.11) to its children theories.

As usual, the second order children theories will be obtained by solving the Euler-Lagrange equations of their parent. To this end, we can use $\int_{\theta,\chi}\delta(f \star O f) = 2\int_{\theta,\chi}\delta f \star O f$ to vary (4.10) with respect to to Λ and f. This will result, respectively, in the Bianchi identity and the duality relation, which read as

$$\mathrm{d}f = 0 \qquad \& \qquad \star O f = \frac{g^2}{4\pi}\,\widetilde{*}\,\mathrm{d}\Lambda\,. \tag{4.14}$$

The Bianchi identity can be solved in the standard way using the Poincaré lemma. This will result in the solution $f := \mathrm{d}e$, where e is a reducible tensor of type $(1,1)$. Substituting this solution back

[2] For a justification as to why this identity holds, the reader may consult the discussion around equation (5.16) in Chapter 5.

into the parent Lagrangian gives the second order theory

$$\mathcal{L}(e) = -\frac{1}{2g^2}\int_{\theta,\chi} \mathrm{d}e \star O \mathrm{d}e\,, \tag{4.15}$$

up to a total derivative term containing Λ. One can now decompose e into its irreducible components $h := \mathbb{P}_{(1,1)}e$ and $b := \frac{1}{2}\sigma e$, such that $e := h + \widetilde{\sigma} b$. These components correspond to the irreducible linearized graviton h of type $(1,1)$ and a 2-form field b. After a short calculation, one can then see that this 2-form component decouples from $\mathcal{L}$ as it hides inside total derivative terms (which vanish under appropriate boundary conditions). Finally, the resulting Lagrangian coincides with the linearized Einstein-Hilbert one

$$\mathcal{L}(e) = \mathcal{L}(h) = -\frac{1}{2g^2}\int_{\theta,\chi} \mathrm{d}h \star \mathrm{d}h\,, \tag{4.16}$$

as it should. The above effect is due to the algebraic operator O that we included in the parent Lagrangian. More precisely, O has the role of ensuring that only the irreducible component of e with the same type (i.e. the linearized graviton h) survives in the second-order Lagrangian. We will show how this happens in full technical detail in the next Chapter.

The underlying reason why b decouples from $\mathcal{L}$ can also be seen by inspection of the gauge symmetries (4.11). At the on-shell level where $f := \mathrm{d}e$, the gauge symmetries $\delta_\xi f = \mathrm{d}\widetilde{\mathrm{d}}\xi$ and $\delta_\omega f = \mathrm{d}\widetilde{\sigma}\omega$ become $\mathrm{d}_\xi e = \widetilde{\mathrm{d}}\xi$ and $\delta_\omega e = \widetilde{\sigma}\omega$. In terms of the irreducible fields h and b, these symmetries are decomposed as

$$\begin{aligned} &\delta_\xi h = \mathbb{P}_{(1,1)}\delta_\xi e = \frac{1}{2}\mathrm{d}\xi^\top + \frac{1}{2}\widetilde{\mathrm{d}}\xi\,, \qquad \delta_\xi b = \frac{1}{2}\sigma\delta_\xi e = -\frac{1}{2}\mathrm{d}\xi\,,\\ &\delta_\omega h = \mathbb{P}_{(1,1)}\delta_\omega e = 0\,, \qquad \delta_\omega b = \frac{1}{2}\sigma\delta_\omega e = \omega\,. \end{aligned} \tag{4.17}$$

For the decompositions above, we used the map identities (2.68), (2.78), (2.79) and the fact that $\widetilde{\sigma}\xi = -\xi^\top$. From the last one, we can see that the Lagrangian (4.15) is invariant under an arbitrary translation $b \mapsto b + \omega$ and, thus, the 2-form b is expected to hide inside total derivative terms. Finally, one should note that the first transformation matches with the linearized diffeomorphisms (4.1), which implies that (4.16) has the correct gauge symmetries.

Before moving on to the dual theory, let us prove for completeness that the Euler-Lagrange equation obtained by variation of $\mathcal{L}(h)$ with respect to h coincides with the tracelessness condition (4.3) on the Riemann tensor. The Euler-Lagrange equation reads as

$$\mathbb{P}_{(1,1)}\mathrm{d}\star\mathrm{d}h = 0 \qquad \Leftrightarrow \qquad \mathrm{d}\star\mathrm{d}h = 0\,, \tag{4.18}$$

where the equivalence relies on the irreducibility conditions $\sigma h = 0 = \widetilde{\sigma} h$. Then, one can use the definition of $\star$ and note that $\mathrm{d}\star\mathrm{d}h = -\eta R$. Finally, acting on both sides of $\eta R = 0$ with tr^2 and using the identity (2.69) will result in equation

$$\mathrm{tr}R + \frac{1}{2}\eta\mathrm{tr}^2 R = 0\,. \tag{4.19}$$

This is the standard form of the linearized four-dimensional Einstein equation in vacuum. Tracing this equation once more (and using (2.69)) leads to the well-known condition $\mathrm{tr}^2 R = 0$, namely to the vanishing of the Ricci scalar. Then, equation (4.3) follows directly upon substitution of this condition back into (4.19).

Let us now focus on the second equation in (4.14), i.e. the duality relation. To obtain the dual Lagrangian, this equation has to be solved in terms of f. Using the relation (2.50), one can rewrite it in the equivalent form

$$(\mathbb{I} - \eta\mathrm{tr})\,\mathcal{O}f = -\frac{g^2}{4\pi} * \mathrm{d}\Lambda\,. \tag{4.20}$$

Furthermore, taking the trace of this relation and using the identity (2.69) leads to the condition $\mathrm{tr}\,\mathcal{O}f = \frac{g^2}{8\pi}\mathrm{tr} * \mathrm{d}\Lambda$. Substituting this condition back into (4.20) gives

$$\mathcal{O}f = -\frac{g^2}{4\pi} * \mathrm{d}\Lambda + \frac{g^2}{8\pi}\eta\mathrm{tr} * \mathrm{d}\Lambda \tag{4.21}$$

and, using the identity (A.5), we find that

$$\mathcal{O}f = -\frac{g^2}{4\pi} * \mathcal{O}\mathrm{d}\Lambda\,. \tag{4.22}$$

Moreover, one can use the identities (2.78) and (2.79) to show that $\mathcal{O}\mathrm{d}\Lambda = \mathrm{d}\mathbb{P}_{(1,1)}\Lambda - \frac{1}{2}\widetilde{\mathrm{d}}\sigma\Lambda$. Finally, there exists the unique operator $\mathcal{O}^{-1} = \mathbb{I} - \widetilde{\sigma}\sigma$ defined by the property $\mathcal{O}^{-1}\mathcal{O} = \mathcal{O}\mathcal{O}^{-1} = \mathbb{I}$. Acting with $\mathcal{O}^{-1}$ on both sides and putting everything together, the equation above becomes

$$f = -\frac{g^2}{4\pi}\mathcal{O}^{-1} * \left(\mathrm{d}\mathbb{P}_{(1,1)}\Lambda - \frac{1}{2}\widetilde{\mathrm{d}}\sigma\Lambda\right)\,. \tag{4.23}$$

Using the duality relation and its solution, one obtains the dual Lagrangian

$$\widehat{\mathcal{L}}(\Lambda) = -\frac{g^2}{32\pi^2}\int_{\theta,\chi}\mathcal{O}^{-1} * \left(\mathrm{d}\mathbb{P}_{(1,1)}\Lambda - \frac{1}{2}\widetilde{\mathrm{d}}\sigma\Lambda\right)\widetilde{*}\,\mathrm{d}\Lambda\,. \tag{4.24}$$

This can be put into a simpler form by use of the integral identity (A.13). Up to a total derivative term $\int \widetilde{\mathrm{d}}\sigma\Lambda \star \mathrm{d}\Lambda$, the above Lagrangian will then read as

$$\begin{aligned}\widehat{\mathcal{L}}(\Lambda) &= -\frac{g^2}{32\pi^2}\int_{\theta,\chi} \mathrm{d}\mathbb{P}_{(1,1)}\Lambda \star \mathrm{d}\Lambda \\ &= -\frac{g^2}{32\pi^2}\int_{\theta,\chi} \mathrm{d}\mathbb{P}_{(1,1)}\Lambda \star \mathrm{d}\mathbb{P}_{(1,1)}\Lambda + \frac{g^2}{64\pi^2}\int_{\theta,\chi} \mathrm{d}\mathbb{P}_{(1,1)}\Lambda \star \mathrm{d}\widetilde{\sigma}\sigma\Lambda\,.\end{aligned} \tag{4.25}$$

The first part corresponds to the linearized Einstein-Hilbert Lagrangian for the dual graviton defined as the irreducible part of Λ with the same type, i.e. as $\widehat{h} := \mathbb{P}_{(1,1)}\Lambda$. The second summand above vanishes identically. This can be seen easily; let us call it $\mathcal{L}_{\text{rest}}$ and compute (ignoring the unimportant overall factor)

$$\begin{aligned}\mathcal{L}_{\text{rest}} &= \int_{\theta,\chi} \mathrm{d}\mathbb{P}_{(1,1)}\Lambda \star \mathrm{d}\widetilde{\sigma}\sigma\Lambda \\ &\overset{(2.79)}{=} -\int_{\theta,\chi} \mathrm{d}\mathbb{P}_{(1,1)}\Lambda \star \widetilde{\sigma}\mathrm{d}\sigma\Lambda - \int_{\theta,\chi} \mathrm{d}\mathbb{P}_{(1,1)}\Lambda \star \widetilde{\mathrm{d}}\sigma\Lambda \\ &\overset{(2.78)}{=} \int_{\theta,\chi} \mathrm{d}\mathbb{P}_{(1,1)}\Lambda \star \widetilde{\sigma}\sigma\mathrm{d}\Lambda \overset{(2.35)}{=} 3\int_{\theta,\chi} \mathrm{d}\mathbb{P}_{(1,1)}\Lambda \star \left(\mathrm{d}\Lambda - \mathbb{P}_{(2,1)}\mathrm{d}\Lambda\right) \overset{(5.15)}{=} 0\,,\end{aligned} \tag{4.26}$$

where in the second line we dropped the total derivative term $\int \mathrm{d}\mathbb{P}_{(1,1)}\Lambda \star \widetilde{\mathrm{d}}\sigma\Lambda$. We have, thus, found that the dual Lagrangian is

$$\widehat{\mathcal{L}}(\widehat{h}) = -\frac{1}{2\widehat{g}^2}\int_{\theta,\chi} \mathrm{d}\widehat{h} \star \mathrm{d}\widehat{h}\,, \qquad \widehat{g} = \frac{4\pi}{g} \tag{4.27}$$

and solely involves the dual linearized graviton $\widehat{h}$. Moreover, we can see that there is an inversion of the coupling as expected[3]. To understand why the 2-form component of Λ decouples from $\widehat{\mathcal{L}}$, one has to examine the gauge symmetries (4.11) of its parent. First of all, we can decompose Λ into its irreducible components as $\Lambda := \widehat{h} + \widetilde{\sigma}\widehat{b}$, where we have introduced the dual fields $\widehat{h} := \mathbb{P}_{(1,1)}\Lambda$ and $\widehat{b} := \frac{1}{2}\sigma\Lambda$. The gauge transformations of these fields correspond to

$$\begin{aligned}&\delta_{\widehat{\xi}}\widehat{h} = \mathbb{P}_{(1,1)}\delta_{\widehat{\xi}}\Lambda = \frac{1}{2}\mathrm{d}\widehat{\xi}^{\top} + \frac{1}{2}\widetilde{\mathrm{d}}\widehat{\xi}\,, \qquad \delta_{\widehat{\xi}}\widehat{b} = \frac{1}{2}\sigma\delta_{\widehat{\xi}}\Lambda = \frac{1}{2}\mathrm{d}\widehat{\xi}\,, \\ &\delta_{\xi}\widehat{h} = \mathbb{P}_{(1,1)}\delta_{\xi}\Lambda = 0\,, \qquad \delta_{\xi}\widehat{b} = \frac{1}{2}\sigma\delta_{\xi}\Lambda = \frac{4\pi}{g^2} * \mathrm{d}\xi\,, \\ &\delta_{\omega}\widehat{h} = \mathbb{P}_{(1,1)}\delta_{\omega}\Lambda = 0\,, \qquad \delta_{\omega}\widehat{b} = \frac{1}{2}\sigma\delta_{\omega}\Lambda = \frac{4\pi}{g^2} * \omega\,.\end{aligned} \tag{4.28}$$

[3] We note here that it is not Newton's constant G that gets inverted, but the dimensionless reference coupling g. As we have already mentioned, we are working in the unit system where $G = M_{\text{P}} = 1$.

For the above decompositions, we made use of the map identities (2.68), (2.78), (2.79), (A.4) and the fact that $\sigma\widehat{\xi}^{\top} = -\widehat{\xi}$. From the last transformation, we observe that the dual Lagrangian is invariant under an arbitrary translation $\widehat{b} \mapsto \widehat{b} + \frac{4\pi}{g^2} * \omega$, which implies that $\widehat{b}$ is expected to decouple. This is very similar to what happened in the original Lagrangian (4.15), in which the 2-form component of e does not contribute due to its gauge transformation $\delta_\omega b = \omega$. Finally, the first transformation above indicates that the dual Lagrangian is indeed invariant under linearized diffeomorphisms as it should.

As a final remark on the dualization procedure, let us consider the duality relation (4.23) at the full on-shell level. In terms of the gauge fields, it will then read as

$$\mathrm{d}\,(h + \widetilde{\sigma} b) = -\frac{g^2}{4\pi} O^{-1} * \left(\mathrm{d}\widehat{h} - \widetilde{\mathrm{d}}\widehat{b}\right) . \tag{4.29}$$

It is then straightforward to act on both sides of this relation with the operator $\widetilde{\mathrm{d}}\,O$ and, making use of the identities (2.68) and (2.79), obtain the on-shell duality relation (4.9) (modulo the overall factor $\frac{4\pi}{g^2}$ that we omitted in our previous discussion).

Finally, it is important to note that the first-order form of the linearized Einstein-Hilbert Lagrangian $\mathcal{L} = -\frac{1}{2g^2}\int_{\theta,\chi} f \star O f$ is invariant under some specific "restricted" $SO(2)$ rotation. To see this, let us assume the transformation

$$\begin{pmatrix} O f \\ * O f \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} O f \\ * O f \end{pmatrix} . \tag{4.30}$$

Using the fact that $\star * O f = -\widetilde{*} f$ (which, in fact, holds for a bipartite tensor f of arbitrary type) and the identity (A.13), one can show that $\mathcal{L} = -\frac{1}{2g^2}\int_{\theta,\chi} f \star O f = -\frac{1}{2g^2}\int_{\theta,\chi} O^{-1} O f \star O f$ is invariant (or flips sign) under this rotation provided

$$\cos^2\alpha - \sin^2\alpha = \pm 1 \quad \& \quad \cos\alpha \sin\alpha = 0\,. \tag{4.31}$$

Similarly to our discussion on the Maxwell Lagrangian, if the r.h.s. of the first equation is $+1$ then $\alpha = 0$ and the rotation becomes the identity transformation. However, the choice of -1 leads to the (equivalent) Lagrangian with flipped sign. For this choice, the rotation angle is $\alpha = \frac{\pi}{2}$. We have, thus, found that $\mathcal{L}$ is invariant under the restricted $SO(2)$ duality rotation

$$\begin{pmatrix} O f \\ * O f \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} O f \\ * O f \end{pmatrix} , \tag{4.32}$$

just like in the Maxwell theory. The above transformation is a generalization of (3.5), in that the Maxwellian field strength F is a 2-form and $\mathcal{O}F = F$ identically. Finally, it is straigthforward to act with $\widetilde{\mathrm{d}}$ on (4.32) and show that it coincides (on-shell) with the transformation $\begin{pmatrix} R \\ *R \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} R \\ *R \end{pmatrix}$ suggested by Hull in [33].

4.3 The gravitational ϑ-term

In analogy to our discussion in Section 3.3 regarding the electromagnetic ϑ-term, we will now study the effect of its gravitational analogue in the electric-magnetic duality of linearized Gravity. This is a topological term of the form $\vartheta \int_{\theta,\chi} \mathrm{d}h \,\widetilde{*}\, \mathrm{d}h$, which is invariant under $\mathcal{P}$ and $\mathcal{T}$ transformations only if the gravitational theta angle ϑ is a pseudoscalar. Following closely our analysis of Section 3.3, we will further assume that this parameter is a periodic spacetime constant taking values in $[0, 2\pi)$. We will not expand on the nature of this topological gravitational term here; Chapter 7 has been reserved for this purpose.

To study the effect of ϑ in the electric-magnetic duality discussed in the previous Section, one has to find a way to include it in the parent Lagrangian (4.10). To this end, we will employ a new first-order theory of the form

$$\mathcal{L}_{\mathrm{p}}(f, \Lambda) = -\frac{1}{2g^2} \int_{\theta,\chi} f \star \mathcal{O} f - \frac{\vartheta}{16\pi^2} \int_{\theta,\chi} f \,\widetilde{*}\, f + \frac{1}{4\pi} \int_{\theta,\chi} f \,\widetilde{*}\, \mathrm{d}\Lambda \,. \tag{4.33}$$

The types of the reducible tensors f and Λ are the same as in the previous Section, namely $(2, 1)$ and $(1, 1)$ respectively. This is precisely the parent Lagrangian constructed in [37]. Inclusion of the additional second term in the parent Lagrangian affects its gauge symmetries, which now differ from (4.11) and read as

$$\begin{aligned} \delta_\xi f &= \mathrm{d}\widetilde{\mathrm{d}}\xi \,, & \delta_\xi \Lambda &= \frac{4\pi}{g^2} * \eta \,\mathrm{d}\xi + \frac{\vartheta}{2\pi} \widetilde{\mathrm{d}}\xi \,, \\ \delta_\omega f &= \mathrm{d}\widetilde{\sigma}\omega \,, & \delta_\omega \Lambda &= -\frac{4\pi}{g^2} * \eta\omega + \frac{\vartheta}{2\pi} \widetilde{\sigma}\omega \,, \\ \delta_{\widehat{\xi}} f &= 0 \,, & \delta_{\widehat{\xi}} \Lambda &= \mathrm{d}\widehat{\xi}^{\top} \,, \end{aligned} \tag{4.34}$$

for arbitrary gauge parameters of the same type as the ones in (4.11). Finally, the Euler-Lagrange equations obtained by variation with respect to Λ and f read as

$$\mathrm{d}f = 0 \qquad \& \qquad \star \left(\frac{4\pi}{g^2} \mathbb{I} - \frac{\vartheta}{2\pi} * \right) \mathcal{O} f = \widetilde{*}\, \mathrm{d}\Lambda \,, \tag{4.35}$$

where we made use of the relation $\widetilde{*}f = -\star * \mathcal{O}f$. One can easily see that this formula holds by using the relations (2.68), (2.50) and (A.4).

The Bianchi identity $\mathrm{d}f = 0$ can be solved, as usual, through the Poincaré lemma and implies that $f := \mathrm{d}e$ for the reducible tensor e of type $(1,1)$. Like before, this tensor admits an irreducible decomposition $e := h + \widetilde{\sigma}b$ in terms of the fields $h := \mathbb{P}_{(1,1)}e$ and $b := \frac{1}{2}\sigma e$. Then, substitution of this solution back into the parent Lagrangian will result in the second order theory

$$\mathcal{L}(e) = -\frac{1}{2g^2}\int_{\theta,\chi} \mathrm{d}e \star \mathcal{O}\mathrm{d}e - \frac{\vartheta}{16\pi^2}\int_{\theta,\chi} \mathrm{d}e\,\widetilde{*}\,\mathrm{d}e\,, \tag{4.36}$$

up to a total derivative term involving Λ. In our previous discussion around equation (4.15), we already explained how the first term in the above Lagrangian corresponds to the linearized Einstein-Hilbert term for h, up to a total derivative containing the field b. In addition, the second term above is clearly a total derivative itself since ϑ is a spacetime constant. Since we are interested in examining how ϑ transforms under electric-magnetic duality transformations, we are going to keep this second term and drop the term that contains b. We will thus write

$$\mathcal{L}(h) = -\frac{1}{2g^2}\int_{\theta,\chi} \mathrm{d}h \star \mathrm{d}h - \frac{\vartheta}{16\pi^2}\int_{\theta,\chi} \mathrm{d}e\,\widetilde{*}\,\mathrm{d}e\,, \tag{4.37}$$

with the understanding that the above second-order Lagrangian is equivalent to (4.16). Finally, it is clear that this Lagrangian inherits its gauge symmetry under linearized diffeomorphisms parametrized by ξ through the ξ-invariance (4.34) of its parent (in the same way that (4.16) does).

The second equation in (4.35) is the duality relation. In order to find the dual theory, this has to be solved in terms of f. In fact, one can follow the exactly same procedure as the one we performed in the previous Section and obtain the relation

$$\left(\frac{4\pi}{g^2}\mathbb{I} - \frac{\vartheta}{2\pi}*\right)\mathcal{O}f = -*\mathcal{O}\mathrm{d}\Lambda\,, \tag{4.38}$$

which is the equivalent to (4.22) in absence of the ϑ-term. To obtain the final result, we can act with $*$ on both sides of this equation and use the result to integrate $*\mathcal{O}f$ out from (4.38). Upon doing so (and consequently acting with $\mathcal{O}^{-1}$), one obtains the solution

$$f = \frac{2\pi\vartheta g^4}{64\pi^4 + \vartheta^2 g^4}\,\mathrm{d}\Lambda - \frac{16\pi^3 g^2}{64\pi^4 + \vartheta^2 g^4}\,\mathcal{O}^{-1} * \mathcal{O}\mathrm{d}\Lambda\,. \tag{4.39}$$

Finally, substituting this solution back into the parent Lagrangian (4.33) will result in the dual second order theory. From the technical point of view, this is obtained in the same way that (4.27) was obtained in the previous Section. In terms of the dual graviton defined by $\widehat{h} := \mathbb{P}_{(1,1)}\Lambda$, the dual Lagrangian is found to be

$$\widehat{\mathcal{L}}(\widehat{h}) = -\frac{1}{2\widehat{g}^2}\int_{\theta,\chi} \mathrm{d}\widehat{h} \star \mathrm{d}\widehat{h} - \frac{\widehat{\vartheta}}{16\pi^2}\int_{\theta,\chi} \mathrm{d}\Lambda \,\widetilde{*}\, \mathrm{d}\Lambda\,, \tag{4.40}$$

where the new couplings are given by

$$\widehat{g}^2 = \frac{64\pi^4 + \vartheta^2 g^4}{4\pi^2 g^2}\,, \qquad \widehat{\vartheta} = -\frac{4\pi^2\vartheta g^4}{64\pi^4 + \vartheta^2 g^4}\,. \tag{4.41}$$

This is the linearized Einstein-Hilbert Lagrangian for $\widehat{h}$, with the addition of a total derivative term. In terms of the gauge symmetries, it is easy to use the identities (2.78)-(2.79) and check that $\widehat{h}$ inherits the invariance under linearized diffeomorphisms of the form

$$\left(\delta_\xi + \delta_{\widehat{\xi}}\right)\widehat{h} = \frac{1}{2}\,\mathrm{d}\left(\widehat{\xi} + \frac{\vartheta}{2\pi}\xi\right)^{\top} + \frac{1}{2}\,\widetilde{\mathrm{d}}\left(\widehat{\xi} + \frac{\vartheta}{2\pi}\xi\right)\,. \tag{4.42}$$

Moving on, it is interesting to note that the new couplings (4.41) are formally the same as the ones obtained in Maxwell's theory, i.e. (3.20). This indicates that electric-magnetic duality in the presence of the ϑ-term acts as $S : \tau \mapsto -\frac{1}{\tau}$ on the "gravitational" complex coupling

$$\tau := \frac{\vartheta}{2\pi} + i\,\frac{4\pi}{g^2}\,, \tag{4.43}$$

which defined in full analogy to the electromagnetic complex coupling (3.21). Together with the symmetry transformation $T : \tau \mapsto \tau + 1$ induced by the periodicity of ϑ, this implies that the duality group in coupling space is, once again, $PSL(2,\mathbb{Z})$.

Let us now examine the duality group from the perspective of spacetime. To this end, we will introduce a specific linear combination of Of and $*Of$, namely

$$\widehat{Of} := \frac{4\pi}{g^2} * Of + \frac{\vartheta}{2\pi} Of\,, \tag{4.44}$$

and assume a transformation of the form $Of \mapsto a\,Of - b\,\widehat{Of}$ for some arbitrary constants a, b with $b \neq 0$. The definition (4.44) comes from identifying $O\mathrm{d}\Lambda \equiv \widehat{Of}$ in (4.38). Under this transformation, it is simple to show that $\widehat{Of}$ transforms as

$$\widehat{Of} \quad \mapsto \quad b|\tau|^2 Of + \left(a - \frac{b\vartheta}{\pi}\right)\widehat{Of} \tag{4.45}$$

and, thus, one ends up with

$$\begin{pmatrix} \mathcal{O}f \\ \widehat{\mathcal{O}f} \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} a & -b \\ b|\tau|^2 & a - \frac{b\vartheta}{\pi} \end{pmatrix} \begin{pmatrix} \mathcal{O}f \\ \widehat{\mathcal{O}f} \end{pmatrix} . \tag{4.46}$$

This is the gravitational analogue of the transformation (3.27) between the Maxwellian field strength F and its dual $\widehat{F}$, in the presence of the electromagnetic ϑ-term.

The final step is to find the conditions, under which the first-order Lagrangian of linearized Gravity (that is, the parent Lagrangian (4.33) without the Lagrange multiplier term) is invariant under the transformation (4.46). To easily perform this transformation, one can use the relations $\widetilde{*}f = -\star * \mathcal{O}f$ and $\mathcal{O}^{-1}\mathcal{O} = \mathbb{I}$ to rewrite the aforementioned Lagrangian in the more suggestive form

$$\mathcal{L} = -\frac{1}{2g^2} \int_{\theta,\chi} \mathcal{O}^{-1}\mathcal{O}f \star \mathcal{O}f + \frac{\vartheta}{16\pi^2} \int_{\theta,\chi} \mathcal{O}^{-1}\mathcal{O}f \star * \mathcal{O}f = \frac{1}{8\pi} \int_{\theta,\chi} \mathcal{O}^{-1}\mathcal{O}f \star * \widehat{\mathcal{O}f} . \tag{4.47}$$

Then, a straightforward calculation[4] reveals that $\mathcal{L}$ transforms as

$$\mathcal{L} \quad \mapsto \quad \left(a^2 - b^2|\tau|^2 + \frac{b\vartheta}{\pi^2}(b\vartheta - 2\pi a) \right) \mathcal{L} - \frac{b}{8\pi^2}(2\pi a - b\vartheta)|\tau|^2 \int_{\theta,\chi} f \,\widetilde{*} f . \tag{4.48}$$

Therefore, the Lagrangian transforms in exactly the same way under (4.46) as the first-order Maxwell Lagrangian does under the respective transformation (3.27). This implies that $\mathcal{L}$ is invariant (more precisely, it changes sign) under (4.46) only if the set of conditions (3.29) holds. That is, we have to demand that the parameters a, b are given by $a = \frac{\vartheta g^2}{8\pi^2}$ and $b = \frac{g^2}{4\pi}$.

Assuming these values, the 2×2 matrix parametrizing the transformation takes the form

$$\begin{pmatrix} a & -b \\ b|\tau|^2 & a - \frac{b\vartheta}{\pi} \end{pmatrix} = \frac{g^2}{4\pi} \begin{pmatrix} \frac{\vartheta}{2\pi} & -1 \\ |\tau|^2 & -\frac{\vartheta}{2\pi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\vartheta}{2\pi} & \frac{4\pi}{g^2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\vartheta}{2\pi} & \frac{4\pi}{g^2} \end{pmatrix}^{-1} , \tag{4.49}$$

which contains the restricted $SO(2)$ rotation $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ conjugated by the matrix $\begin{pmatrix} 1 & 0 \\ \frac{\vartheta}{2\pi} & \frac{4\pi}{g^2} \end{pmatrix}$. The discussion above shows that, in addition to the $PSL(2,\mathbb{Z})$ duality on the coupling space, the first-order form of linearized Gravity equipped with the ϑ-term enjoys an $SL(2,\mathbb{R})$ duality on the spacetime. This becomes manifest through the transformation parametrized by (4.49) at the space of the "field

[4] For this calculation, one should make use of the integral identity (A.13) as well as the additional relation $\int_{\theta,\chi} \mathcal{O}f \star \mathcal{O}^{-1} * \mathcal{O}f = -\int_{\theta,\chi} f \,\widetilde{*} f$. One can see that the latter holds, e.g., by expanding the integrands in local coordinates and performing the Berezin integration.

strength" $\mathcal{O}f$ and its dual $\widehat{\mathcal{O}f}$, while its on-shell version corresponds to the same transformation at the space of R and $\widehat{R}$. The latter was suggested by Hull in [33].

As we already mentioned, it was proven in [75] that (in absence of the ϑ-term) the Hamiltonian of linearized Gravity is invariant under the full $SO(2)$ rotations. It is then natural to conjecture that, when ϑ is present, the Hamiltonian will enjoy an extended invariance under the $SL(2,\mathbb{R})$ transformation of the form

$$\begin{pmatrix} \mathcal{O}f \\ \widehat{\mathcal{O}f} \end{pmatrix} \quad \mapsto \quad \frac{g^2}{4\pi} \begin{pmatrix} \frac{4\pi}{g^2}\cos\alpha + \frac{\vartheta}{2\pi}\sin\alpha & -\sin\alpha \\ |\tau|^2 \sin\alpha & \frac{4\pi}{g^2}\cos\alpha - \frac{\vartheta}{2\pi}\sin\alpha \end{pmatrix} \begin{pmatrix} \mathcal{O}f \\ \widehat{\mathcal{O}f} \end{pmatrix}. \tag{4.50}$$

4.4 Double duality

Let us now discuss in more detail the double duality in four-dimensional linearized Gravity. This corresponds to the electric-magnetic duality realized by Hodge dualizing the Riemann tensor twice, i.e. with respect to both star operators $*$ and $\widetilde{*}$ [33]. To justify our focus in four dimensions, we will begin by explaining the two main reasons why double duality in four dimensions is so special.

The first obvious reason is that it is only in four dimensions that it becomes a self-duality. That is because the double dual Riemann tensor $R^{\text{doub}} = *\widetilde{*}R$ is an irreducible tensor of type $(2,2)$, just like R. As we already saw, the double dual graviton h^{doub} satisfies on-shell the standard Einstein equation $\text{tr}\,\text{d}\widetilde{\text{d}}h^{\text{doub}} = 0$ and, in addition, it is related to the original graviton h through the second duality relation in (4.9).

The second reason why the four-dimensional case is particularly interesting is related to a recent observation made by Henneaux, Lekeu and Leonard in [29]. In this work, the authors showed that (in arbitrary dimensions) a duality relation of the form $\text{d}\widetilde{\text{d}}h^{\text{doub}} = *\widetilde{*}\text{d}\widetilde{\text{d}}h$ would force the double dual graviton h^{doub} to be algebraically related on-shell to the original graviton h [5]. In analogy to our previous discussion on the exotic duality in Maxwell's theory, this observation implies that the double dual picture of linearized Gravity fails to provide an independent description of the theory. An earlier hint about this fact was already presented by Hull in [33], where it was shown that there can only be at most two (and not three) independent types of sources in this setting.

[5]This statement can be proven easily by following a procedure very similar to the one we carried out for the exotic dual of the Maxwell field, around equation (3.49).

However, we saw earlier in Section 3.5 that inclusion of the electromagnetic ϑ-term lifted this obstacle and led to an independent exotic dual description of Maxwell's electrodynamics. This is precisely what happens in the setting of double duality of linearized Gravity, but only in four dimensions where the gravitational ϑ-term is admissible. This observation was already implied by the on-shell analysis of [33], where a "θ-parameter" of unknown off-shell origin was introduced, with the effect of deforming the $SO(2)$ duality rotation into the $SL(2,\mathbb{R})$ transformation we obtained in the previous Section. In [37], the observation was made explicit and the aforementioned parameter was identified off-shell with the coupling of the gravitational ϑ-term.

Historically, the first proposal for a parent Lagrangian that realizes the double duality in linearized Gravity at the off-shell level was presented by Boulanger, Cook and Ponomarev in [64] (see also [65] for a more general construction). In fact, this was carried out in arbitrary spacetime dimensions and it was shown that, in dimensions higher than four, the dualization procedure required the presence of additional fields at the off-shell level. This was of course to be expected since, as we explained, double duality is not a self-duality in dimensions higher than four. More precisely, the double dual graviton h^{doub} is no longer of type $(1,1)$ and has to obey equations of motion that are "weaker" than usual, in order to propagate the correct degrees of freedom. These weaker equations cannot be obtained from any action principle that solely contains h^{doub} and, thus, additional fields have to appear in the off-shell formulation. This is in full analogy to what happens in the off-shell exotic dualization of differential forms. The main difference is that exotic duality is not a self-duality in any number of dimensions and, thus, these additional fields are always present.

The graded geometric form of the parent Lagrangian proposed in [64] was constructed in [48]. The four-dimensional double dualization procedure in the presence of the gravitational ϑ-term was later explained in [37], where it was noted that the parent Lagrangian one should use is exactly the same as the one we used earlier for the single dualization, namely (4.33). The only difference lies in the identifications of the dual fields, which follow the irreducible decompositions of f and Λ. Combining the single and double dualities, one will then find the duality symmetry

$$\begin{pmatrix} \mathcal{O}f & \widehat{\mathcal{O}f} \\ \widehat{\mathcal{O}f} & (\mathcal{O}f)^{\text{doub}} \end{pmatrix} \quad \mapsto \quad \frac{g^2}{4\pi}\begin{pmatrix} \frac{\vartheta}{2\pi} & -1 \\ |\tau|^2 & -\frac{\vartheta}{2\pi} \end{pmatrix}\begin{pmatrix} \mathcal{O}f & \widehat{\mathcal{O}f} \\ \widehat{\mathcal{O}f} & (\mathcal{O}f)^{\text{doub}} \end{pmatrix}, \tag{4.51}$$

where

$$
\begin{aligned}
\widehat{\mathcal{O}f} &:= \frac{4\pi}{g^2} * \mathcal{O}f + \frac{\vartheta}{2\pi}\mathcal{O}f\,, \\
(\mathcal{O}f)^{\mathrm{doub}} &:= \frac{4\pi}{g^2} * \widehat{\mathcal{O}f} + \frac{\vartheta}{2\pi}\widehat{\mathcal{O}f} = \frac{4\vartheta}{g^2} * \mathcal{O}f + \left(\frac{\vartheta^2}{4\pi^2} - \frac{16\pi^2}{g^4}\right)\mathcal{O}f\,.
\end{aligned}
\tag{4.52}
$$

These relations coincide on-shell (and upon differentiation using $\widetilde{\mathrm{d}}$) with the ones conjectured by Hull [33] in terms of the respective Riemann tensors.

Chapter 5

UNIFICATION OF HODGE DUALITIES THROUGH GRADED GEOMETRY

In this Chapter we will present the main result of the work carried out in [48], which is a graded geometric parent Lagrangian that unifies all types of Hodge dualities discussed in the previous Chapters 3 & 4. In addition to the cases we already reviewed, i.e. the electric-magnetic and exotic duality in Maxwell's theory and the electric-magnetic and double duality in linearized Gravity in four dimensions, this Lagrangian encodes the off-shell dualization procedures for theories involving a single p-form or mixed-symmetry tensor gauge field of type $(p, 1)$ in arbitrary spacetime dimensions.

Let us first present this universal parent theory and, subsequently, discuss some of its features. To this end, we work in Minkowski spacetime of dimension D and define the parametric Lagrangian

$$\mathcal{L}_{\mathrm{p}}^{(p+1,q)}(F_{p+1,q}\,,\Lambda_{D-p-2,q}) = \frac{(-1)^q G}{2}\int_{\theta,\chi} F\star O^{(p+1,q)}F + \frac{1}{4\pi}\int_{\theta,\chi} F\,\widetilde{*}\,\mathrm{d}\Lambda\,. \tag{5.1}$$

The parameters $\{p, q\}$ are positive reals and they are required to satisfy the condition $p+q \leq D-2$. The fields F and Λ are taken to be reducible mixed-symmetry tensors of respective types $(p+1, q)$ and $(D-p-2, q)$, as indicated. In addition, we have included an arbitrary positive-definite coupling constant $G > 0$ in the first term, so that we can make contact with our previous discussions in Chapters 3 & 4. Finally, the sign $(-1)^q$ in front of the first term has been added so that the corresponding second order children Lagrangians respect unitarity.

The operator $O^{(p+1,q)}$ is a simple generalization of the algebraic operator O used throughout the previous Chapter for the case of the linearized graviton. In particular, it has the same role; it ensures that one of the children Lagrangians will eventually contain only the irreducible piece of F with the same type (i.e. the original gauge field that we dualize). In more technical terms, the sufficient condition for this is that

$$O^{(p+1,q)}\mathrm{d}\omega \stackrel{!}{=} \mathrm{d}\,\mathbb{P}_{(p,q)}\omega + \widetilde{\mathrm{d}}\,Y \tag{5.2}$$

for *any* mixed-symmetry tensor ω of type (p, q), where Y is an arbitrary tensor of type $(p+1, q-1)$. Equation (5.2) will also serve as the defining relation for $O^{(p+1,q)}$ which, as we will show shortly, is then forced to have the form

$$\mathcal{O}^{(p+1,q)} = \begin{cases} \mathbb{I} + \sum_{n=1}^{q} c_n(p,q)\,\widetilde{\sigma}^n\,\sigma^n\,, & p \geq q \\ \mathbb{I} + \sum_{n=1}^{p} c_n(q,p)\left(\sigma^n\,\widetilde{\sigma}^n + \sum_{k=1}^{n}(-1)^k\prod_{m=0}^{k-1}(n-m)^2\sigma^{n-k}\,\widetilde{\sigma}^{n-k}\right), & p < q \end{cases} \tag{5.3}$$

in terms of the coefficients c_n that appear in (2.36). Note that, for $p \geq q$, the operator $\mathcal{O}^{(p+1,q)}$ is formally the same as the Young projection $\mathbb{P}_{(p,q)}$ defined in (2.35). However, it acts on a field of type $(p+1,q)$ and, as such, it cannot be identified with a Young projection.

Let us now prove that the algebraic operator $\mathcal{O}^{(p+1,q)}$ defined by the relation (5.2) must be given by (5.3). For $p \geq q$ and an arbitrary mixed-symmetry tensor ω of type (p,q), we directly compute

$$\begin{aligned} \mathcal{O}^{(p+1,q)}\mathrm{d}\omega - \mathrm{d}\mathbb{P}_{(p,q)}\omega &= \mathcal{O}^{(p+1,q)}\mathrm{d}\omega - \mathrm{d}\left(\mathbb{I} + \sum_{n=1}^{q} c_n(p,q)\,\widetilde{\sigma}^n\sigma^n\right)\omega \\ &= \left(\mathcal{O}^{(p+1,q)} - \mathbb{I}\right)\mathrm{d}\omega - \sum_{n=1}^{q} c_n(p,q)\,\mathrm{d}\,\widetilde{\sigma}^n\sigma^n\,\omega \\ &= \left(\mathcal{O}^{(p+1,q)} - \mathbb{I} - \sum_{n=1}^{q} c_n(p,q)\,\widetilde{\sigma}^n\sigma^n\right)\mathrm{d}\omega + \widetilde{\mathrm{d}}Y\,, \end{aligned} \tag{5.4}$$

for some calculable Y. In the last step, we used the identities (2.78) and (2.79). Finally, by inspection of (5.2) we conclude that

$$\mathcal{O}^{(p+1,q)} = \mathbb{I} + \sum_{n=1}^{q} c_n(p,q)\,\widetilde{\sigma}^n\sigma^n\,, \quad \text{for} \quad p \geq q\,. \tag{5.5}$$

The computation proceeds differently in the $p < q$ case and one should first collect some intermediate results. Using the map identities (2.78) and (2.79), we compute

$$\begin{aligned} \mathrm{d}\,\sigma^n\,\widetilde{\sigma}^n &= (-1)^n\,\sigma^n\,\mathrm{d}\,\widetilde{\sigma}^n \\ &= (-1)^n\,\sigma^n\left(-(-1)^{n+1}\,\widetilde{\sigma}^n\,\mathrm{d} - n\,\widetilde{\mathrm{d}}\,\widetilde{\sigma}^{n-1}\right) \\ &= \sigma^n\,\widetilde{\sigma}^n\,\mathrm{d} + (-1)^{n+1}n\,\sigma^n\,\widetilde{\mathrm{d}}\,\widetilde{\sigma}^{n-1} \\ &= \sigma^n\,\widetilde{\sigma}^n\,\mathrm{d} + (-1)^{n+1}n\left((-1)^n\widetilde{\mathrm{d}}\,\sigma^n + (-1)^n n\,\mathrm{d}\,\sigma^{n-1}\right)\widetilde{\sigma}^{n-1} \\ &= \sigma^n\,\widetilde{\sigma}^n\,\mathrm{d} + \widetilde{\mathrm{d}}Y - n^2\mathrm{d}\,\sigma^{n-1}\,\widetilde{\sigma}^{n-1}, \end{aligned} \tag{5.6}$$

which directly leads to the formula

$$\mathrm{d}\left(\sigma^n\,\widetilde{\sigma}^n + n^2\,\sigma^{n-1}\,\widetilde{\sigma}^{n-1}\right)\omega = \sigma^n\,\widetilde{\sigma}^n\mathrm{d}\omega + \widetilde{\mathrm{d}}Y_1\,. \tag{5.7}$$

Here $Y_1 = -n\,\sigma^n\,\widetilde{\sigma}^{n-1}\omega$, although it is unnecessary to present explicitly such quantities for our purposes, since they never influence the results. The formula (5.7) is essentially a recursion relation, which can be used to obtain a second intermediate relation. First we define for brevity

$$P(n) := \mathrm{d}\,\sigma^n\,\widetilde{\sigma}^n\,, \qquad Q(n) = \sigma^n\,\widetilde{\sigma}^n\,\mathrm{d} \tag{5.8}$$

and then we write (5.7) for every n as follows

$$\begin{aligned}
P(n)\,\omega + n^2 P(n-1)\,\omega &= Q(n)\,\omega + \widetilde{\mathrm{d}}Y_1 \\
-\,n^2\left(P(n-1) + (n-1)^2 P(n-2)\right)\omega &= -\,n^2 Q(n-1)\,\omega + \widetilde{\mathrm{d}}Y_2 \\
+\,n^2(n-1)^2\left(P(n-2) + (n-2)^2 P(n-3)\right)\omega &= +\,n^2(n-1)^2 Q(n-2)\,\omega + \widetilde{\mathrm{d}}Y_3 \\
\vdots \qquad &= \qquad \vdots \\
+\,(-1)^{n-1} n^2 (n-1)^2 \ldots 2^2\,(P(1) + P(0))\,\omega &= +\,(-1)^{n-1} n^2(n-1)^2 \ldots 2^2 Q(1)\,\omega + \widetilde{\mathrm{d}}Y_n \\
+\,(-1)^n (n!)^2\,P(0)\,\omega &= +\,(-1)^n (n!)^2\,Q(0)\,\omega + \widetilde{\mathrm{d}}Y_{n+1}\;.
\end{aligned}$$

Summing up these $n+1$ equations, one easily obtains

$$P(n)\,\omega = Q(n)\,\omega + \sum_{k=1}^{n} (-1)^k \prod_{m=0}^{k-1} (n-m)^2\,Q(n-k)\,\omega + \widetilde{\mathrm{d}}Y' \tag{5.9}$$

for some calculable Y'. This corresponds precisely to the second intermediate relation that we need

$$\mathrm{d}\sigma^n\,\widetilde{\sigma}^n\omega = \sigma^n\,\widetilde{\sigma}^n\,\mathrm{d}\omega + \sum_{k=1}^{n} (-1)^k \prod_{m=0}^{k-1} (n-m)^2 \sigma^{n-k}\,\widetilde{\sigma}^{n-k}\,\mathrm{d}\omega + \widetilde{\mathrm{d}}Y'\,, \tag{5.10}$$

which can be used to compute $O^{(p+1,q)}$ via the defining relation (5.2). Collecting the previous result (5.5) too, we find the form (5.3) as expected.

5.1 The original theory

The dualization procedure is the same as the ones discussed in the previous Chapters. That is, integrating out the Lagrange multiplier field Λ yields the the original second-order theory involving the gauge field we wish to dualize, while integrating out F through the duality relation gives the dual theory. The first statement can be implemented by means of the following Theorem:

Theorem 5.11 *(Ref. [48]) The parent Lagrangian (5.1) equipped with the algebraic operator (5.3) is equivalent on-shell to the (second-order) gauge invariant kinetic term for an irreducible gauge field of type (p, q), for parameters p and q taking values in the domain $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$ where*

$$\mathcal{D}_1 = \{p \in [0, D-2], q = 0\}\,, \quad \mathcal{D}_2 = \{p \in [1, D-3], q = 1\}\,,$$
$$\mathcal{D}_3 = \{p = 0, q \in [1, D-2]\}\,, \quad \mathcal{D}_4 = \{p = 1, q \in [2, D-3]\}\,.$$

Let us now prove this Theorem. Suppose that one varies the parent Lagrangian (5.1) with respect to the Lagrange multiplier field Λ and obtains, as usual, the Bianchi identity $\mathrm{d}F = 0$. Then, using the Poincaré lemma yields the local solution $F := \mathrm{d}\omega$, in terms of a *reducible* tensor field ω of type (p, q). Substituting this solution back into $\mathcal{L}_{\mathrm{p}}^{(p+1,q)}$, and taking into account the defining relation (5.2) of $\mathcal{O}^{(p+1,q)}$, will yield the second-order child Lagrangian

$$\mathcal{L}^{(p+1,q)}(\omega) = \frac{(-1)^q G}{2} \int_{\theta,\chi} \mathrm{d}\omega \star \mathrm{d}\mathbb{P}_{(p,q)}\omega\,. \tag{5.12}$$

We notice that the second occurrence of the field contains only its desired irreducible component, but the first does not. In close analogy to our discussion around equations (4.25)-(4.26), let us now assume that $p \geq q$ (which is the case for $\{p, q\}$ taking values in the domains $\mathcal{D}_1$ or $\mathcal{D}_2$) and rewrite the Lagrangian above as

$$\begin{aligned}\mathcal{L}^{(p+1,q)}(\omega) &\overset{(2.35)}{=} \frac{(-1)^q G}{2} \int_{\theta,\chi} \mathrm{d}\left(\mathbb{P}_{(p,q)}\omega - \sum_{n=1}^{q} c_n(p,q)\,\widetilde{\sigma}^n\sigma^n\omega\right) \star \mathrm{d}\mathbb{P}_{(p,q)}\omega \\ &= \frac{(-1)^q G}{2} \int_{\theta,\chi} \mathrm{d}\mathbb{P}_{(p,q)}\omega \star \mathrm{d}\mathbb{P}_{(p,q)}\omega - \mathcal{L}_{\mathrm{rest}}\,,\end{aligned} \tag{5.13}$$

where we defined

$$\mathcal{L}_{\mathrm{rest}} = \frac{(-1)^q G}{2} \int_{\theta,\chi} \sum_{n=1}^{q} c_n(p,q)\,\mathrm{d}\,\widetilde{\sigma}^n\sigma^n\omega \star \mathrm{d}\mathbb{P}_{(p,q)}\omega\,. \tag{5.14}$$

We immediately observe that the first term on the r.h.s. of (5.13) is precisely the kinetic term for an irreducible gauge field of type (p, q), namely $\mathbb{P}_{(p,q)}\omega$. That is, in order to fully prove Theorem 5.11 we need to show that the second term, namely $\mathcal{L}_{\mathrm{rest}}$, does not contribute. Instead, it has to vanish identically and this does not happen in general. In fact, one can show that this is the case *only* for parameters $\{p, q\}$ in $\mathcal{D}_1$ or $\mathcal{D}_2$.

For $q = 0$, it is easy to see that $\mathcal{L}_{\mathrm{rest}}$ vanishes since $\sigma\omega = 0$ for a p-form field ω. This proves the Theorem for the first domain of values $\mathcal{D}_1$. For the case of $q = 1$, we can use the identities (2.78) and (2.79) to compute

$$\begin{aligned}
\mathcal{L}_{\text{rest}} &= \frac{(-1)^q G}{2} \int_{\theta,\chi} c_1(p,1)\, \widetilde{\sigma}\sigma \mathrm{d}\omega \star \mathrm{d}\mathbb{P}_{(p,q)}\omega \\
&= \frac{(-1)^q G}{2} \frac{c_1(p,1)}{c_1(p+1,1)} \int_{\theta,\chi} c_1(p+1,1)\, \widetilde{\sigma}\sigma \mathrm{d}\omega \star \mathrm{d}\mathbb{P}_{(p,q)}\omega \\
&\overset{(2.35)}{=} \frac{(-1)^q G}{2} \frac{c_1(p,1)}{c_1(p+1,1)} \int_{\theta,\chi} \left(\mathbb{P}_{(p+1,1)} - \mathbb{I}\right) \mathrm{d}\omega \star \mathrm{d}\mathbb{P}_{(p,q)}\omega = 0\,,
\end{aligned}$$

where in the last step we used the following two general relations that hold for any tensor fields ω and ξ of arbitrary type (p,q)

$$\int_{\theta,\chi} \mathbb{P}_{(p,q)}\omega \star \xi = \int_{\theta,\chi} \omega \star \mathbb{P}_{(p,q)}\xi \qquad \& \qquad \mathbb{P}_{(p+1,q)}(\mathrm{d}\mathbb{P}_{(p,q)}\omega) = \mathrm{d}\mathbb{P}_{(p,q)}\omega\,. \tag{5.15}$$

The first relation is the graded geometric version of an identity that was proven in [76], while the second one is obvious since the exterior derivative of an irreducible field yields an irreducible field too. In fact, the first relation can be generalized to the following integral identity

$$\int_{\theta,\chi} \Sigma\omega \star \xi = \int_{\theta,\chi} \omega \star \Sigma\xi\,, \tag{5.16}$$

where Σ is any algebraic operator that is written as a polynomial in powers of $\widetilde{\sigma}\sigma$ (or $\sigma\widetilde{\sigma}$). We will not present the proof of this statement as it is straightforward to show that it holds by using (A.5) and the fact that the σ-maps commute with η and tr.

Thus, we have proven Theorem 5.11 also for the second domain of values $\mathcal{D}_2$. One can convince themselves that $\mathcal{L}_{\text{rest}}$ is non vanishing in any other domain with $p \geq q$ and $q \geq 2$ (assuming, of course, that there are no further restrictions on the field ω).

Up to now, we have considered the case $p \geq q$ and found that the Theorem holds only for the domains $\mathcal{D}_1$ and $\mathcal{D}_2$. The proof for the case where $p < q$ is completely analogous and one finds that the corresponding $\mathcal{L}_{\text{rest}}$ (which differs from the one in (5.14)) vanishes only for $p = 0$ and $p = 1$. These values naturally correspond to the third and fourth domains $\mathcal{D}_3$ and $\mathcal{D}_4$.

5.2 The dual theory

For $\{p,q\} \in \mathcal{D}_1$ it is clear that the gauge field mentioned in Theorem 5.11 will be a differential p-form, while for $\{p,q\} \in \mathcal{D}_3$ it will be a q-form (more precisely, a transposed q-form). Similarly, for $\{p,q\} \in \mathcal{D}_2$ or $\mathcal{D}_4$ the gauge field will correspond to an irreducible $(p,1)$ tensor or to a transposed $(q, 1)$ tensor. As we have seen, exotic duality can be seen effectively as a Hodge duality of the transposed field (see e.g. the duality relation (3.36)). This is also true for double duality of $(p, 1)$ tensors, although it was not transparent in our previous discussion about the graviton. The reason is that the linearized graviton (and the Riemann tensor) is an irreducible tensor with index symmetries of an equal-length Young tableaux and, as such, it is invariant under transposition.

The above discussion gives us a hint regarding the domains of applicability of Theorem 5.11. As we will show, the parent Lagrangian $\mathcal{L}_{\mathrm{p}}^{(p+1,q)}$ implements the standard electric-magnetic and exotic duality for differential forms if $\{p, q\}$ take values in $\mathcal{D}_1$ and $\mathcal{D}_3$, respectively, while it describes the single and double electric-magnetic duality for $(p, 1)$ mixed-symmetry tensors if $\{p, q\}$ take values in $\mathcal{D}_2$ and $\mathcal{D}_4$. In more rigorous terms, these observations are equivalent to the following Theorem:

Theorem 5.17 *(Ref. [48]) For parameters $\{p, q\}$ taking values in $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$, the parent Lagrangian (5.1) equipped with (5.3) is equivalent on-shell to a second-order Lagrangian involving an irreducible gauge field of type $(D - p - 2, q)$ (and possibly other fields), which corresponds to the irreducible piece of Λ with the same type (and possibly its remaining irreducible components). After fixing all the gauge symmetry, it is only the aforementioned $(D - p - 2, q)$ field that propagates and the number of physical d.o.f. equals the d.o.f. propagated by an irreducible tensor gauge field of type (p, q).*

Let us now provide the proof of this Theorem in all its domains of applicability $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$. To this end, we have to vary the parent Lagrangian (5.1) with respect to the field F and obtain the corresponding duality relation in each domain.

Using the integral identity (5.16), one can see that the variation of the first term in (5.1) reads as

$$\delta\left(\int_{\theta,\chi} F \star O^{(p+1,q)} F\right) = 2\int_{\theta,\chi} \delta F \star O^{(p+1,q)} F \tag{5.18}$$

irrespective of the domain. Therefore, the general duality relation resulting as the Euler-Lagrange equation for F will have the form

$$\star\, O^{(p+1,q)} F = \frac{(-1)^{q+1}}{4\pi G} \,\widetilde{*}\, \mathrm{d}\Lambda \,. \tag{5.19}$$

Using the relation (2.50), this equation can be written in the equivalent form

$$O^{(p+1,q)} F - \eta \mathrm{tr}\, O^{(p+1,q)} F + \frac{1}{4}\eta^2 \mathrm{tr}^2 O^{(p+1,q)} F = \frac{(-1)^{\alpha(p,q)}}{4\pi G} * \mathrm{d}\Lambda \,, \tag{5.20}$$

where we introduced the parity $\alpha(p,q) := q(D-p-1)+1$ for brevity. Note that we have only kept up to second traces of $\mathcal{O}^{(p+1,q)}F$, since a $(p+1,q)$ tensor with $\{p,q\} \in \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$ can have at most two traces. Upon tracing this relation twice, we can use the first relation in (A.4) and the map identity (2.69) consecutively to obtain the double trace condition

$$\mathrm{tr}^2 \mathcal{O}^{(p+1,q)}F = \frac{(-1)^{\alpha(p,q)}}{2\pi G(D-p-q)(D-p-q+1)} * \sigma^2 \mathrm{d}\Lambda \,. \tag{5.21}$$

This relation can be substituted back into (5.20) and, upon tracing the resulting expression and using (A.4), we obtain the single trace condition

$$\mathrm{tr}\, \mathcal{O}^{(p+1,q)}F = \frac{(-1)^{\alpha(p,q)+D+1}}{4\pi G(D-p-q)} \left(* \,\sigma \mathrm{d}\Lambda - \frac{1}{D-p-q+1} * \widetilde{\sigma}\sigma^2 \mathrm{d}\Lambda \right). \tag{5.22}$$

Finally, we can substitute both trace conditions back into (5.20) and, after a short computation that once again involves the map identity (A.4), obtain the elegant formula

$$\mathcal{O}^{(p+1,q)}F = \frac{(-1)^{\alpha(p,q)}}{4\pi G} * \mathcal{O}^{(D-p-1,q)} \mathrm{d}\Lambda \,, \tag{5.23}$$

in terms of the operator

$$\mathcal{O}^{(D-p-1,q)} = \mathbb{I} - \frac{1}{D-p-q} \widetilde{\sigma}\sigma + \frac{1}{2(D-p-q)(D-p-q+1)} \widetilde{\sigma}^2 \sigma^2 \,. \tag{5.24}$$

Note that the r.h.s. of the above equation is identified with the l.h.s. simply because we have required that $p+q \leq D-2 < D-1$, which implies that $\mathcal{O}^{(D-p-1,q)}$ will always take the first form in (5.3). Then, the final solution of (5.19) in terms of F can be obtained from (5.23) by acting with an appropriate inverse operator $(\mathcal{O}^{-1})^{(p+1,q)}$, which can be defined always through the relation $(\mathcal{O}^{-1})^{(p+1,q)}\mathcal{O}^{(p+1,q)} = \mathcal{O}^{(p+1,q)}(\mathcal{O}^{-1})^{(p+1,q)} = \mathbb{I}$, and we obtain

$$F = \frac{(-1)^{\alpha(p,q)}}{4\pi G} (\mathcal{O}^{-1})^{(p+1,q)} * \mathcal{O}^{(D-p-1,q)} \mathrm{d}\Lambda \,. \tag{5.25}$$

Finally, one can use this solution to obtain the dual Lagrangian. By direct substitution back into the parent Lagrangian (5.1), we obtain the result

$$\widehat{\mathcal{L}}^{(p+1,q)}(\Lambda) = \frac{(-1)^{\alpha(p,q)}}{32\pi^2 G} \int_{\theta,\chi} (\mathcal{O}^{-1})^{(p+1,q)} * \mathcal{O}^{(D-p-1,q)} \mathrm{d}\Lambda \,\widetilde{*}\, \mathrm{d}\Lambda \tag{5.26}$$

in terms of the reducible Lagrange multiplier field Λ. The formula above essentially proves the first part of Theorem 5.17; for parameters $\{p,q\}$ taking values in the domain $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$, we have found that the parent theory (5.1) is equivalent on-shell to a second order Lagrangian that involves an irreducible tensor field of type $(D-p-2,q)$ and possibly other fields. This irreducible field is naturally identified with the component $\mathbb{P}_{(D-p-2,q)}\Lambda$, while the additional fields correspond to the irreducible components of Λ with different type.

Before moving on to prove the second part of the Theorem, namely the fact that the additional fields (when present) do not propagate and that the only physical d.o.f. in the dual theory are equal to the ones propagated by an irreducible gauge field of type (p,q), let us find the explicit forms of the dual Lagrangian (5.26) in each domain. To this end, we are going to need the respective forms of $(\mathcal{O}^{-1})^{(p+1,q)}$. Knowledge of the operator $\mathcal{O}^{(p+1,q)}$ through (5.3) allows one to obtain these forms using the defining relations $(\mathcal{O}^{-1})^{(p+1,q)}\mathcal{O}^{(p+1,q)} = \mathcal{O}^{(p+1,q)}(\mathcal{O}^{-1})^{(p+1,q)} = \mathbb{I}$ in each case. The results read as [48]

$$\begin{aligned}
\mathcal{D}_1 : &\quad (\mathcal{O}^{-1})^{(p+1,q)} = (\mathcal{O}^{-1})^{(p+1,0)} = \mathbb{I}\,,\\
\mathcal{D}_2 : &\quad (\mathcal{O}^{-1})^{(p+1,q)} = (\mathcal{O}^{-1})^{(p+1,1)} = \mathbb{I} - \widetilde{\sigma}\sigma\,,\\
\mathcal{D}_3 : &\quad (\mathcal{O}^{-1})^{(p+1,q)} = (\mathcal{O}^{-1})^{(1,q)} = \mathbb{I}\,,\\
\mathcal{D}_4 : &\quad (\mathcal{O}^{-1})^{(p+1,q)} = (\mathcal{O}^{-1})^{(2,q)} = \frac{q+1}{q+2}\,\mathbb{I} + \frac{q+1}{2(q+2)}\,\sigma\widetilde{\sigma} - \frac{q+1}{2q(q+2)}\,\sigma^2\widetilde{\sigma}^2\,.
\end{aligned} \tag{5.27}$$

In addition to these, let us decompose the Lagrange multiplier field Λ into its irreducible parts as

$$\Lambda := \mathbb{P}_{(D-p-2,q)}\Lambda + \widetilde{\sigma}X\,, \qquad X := -\sum_{n=1}^{q} c_n(D-p-2,q)\,\widetilde{\sigma}^{n-1}\sigma^n\Lambda \tag{5.28}$$

where we have gathered all the remaining irreducible components of Λ into the reducible object X of type $(D-p-1,q-1)$. Finally, the defining relation (5.2) implies that

$$\mathcal{O}^{(D-p-1,q)}\mathrm{d}\Lambda = \mathrm{d}\mathbb{P}_{(D-p-2,q)}\Lambda - \widetilde{\mathrm{d}}\,Y\,, \qquad Y := -\sum_{n=1}^{q} n\,c_n(D-p-2,q)\,\widetilde{\sigma}^{n-1}\sigma^n\Lambda \tag{5.29}$$

which was found easily by using the map identities (2.78) and (2.79) consecutively.

Let us now put all these results together and find the explicit form of the dual Lagrangian (5.26) in each domain. For the first domain $\mathcal{D}_1$, we have $(\mathcal{O}^{-1})^{(p+1,0)} = \mathbb{I}$ and $X = 0$, the latter being true due to degree reasons. This implies that $\Lambda = \mathbb{P}_{(D-p-2,0)}\Lambda$ and the dual Lagrangian reads as

$$\begin{aligned}\widehat{\mathcal{L}}^{(p+1,0)}(\Lambda) &= \frac{-1}{32\pi^2 G}\int_{\theta,\chi} * \,\mathrm{d}\mathbb{P}_{(D-p-2,0)}\Lambda \,\widetilde{*}\, \mathrm{d}\mathbb{P}_{(D-p-2,0)}\Lambda \\ &\overset{(\mathrm{A.8})}{=} \frac{(-1)^{D(p+1)+p}}{32\pi^2 G}\int_{\theta,\chi} \mathrm{d}\mathbb{P}_{(D-p-2,0)}\Lambda \, * \widetilde{*}\, \mathrm{d}\mathbb{P}_{(D-p-2,0)}\Lambda \\ &= \frac{(-1)^{(D+1)(p+1)}}{32\pi^2 G}\int_{\theta} \mathrm{d}\mathbb{P}_{(D-p-2,0)}\Lambda \, * \,\mathrm{d}\mathbb{P}_{(D-p-2,0)}\Lambda = \widehat{\mathcal{L}}^{(p+1,0)}(\mathbb{P}_{(D-p-2,0)}\Lambda)\,,\end{aligned} \tag{5.30}$$

where in the last step we performed the trivial integration over the χ-variable, resulting in an additional minus sign. This is the generalization of the dual Lagrangian (3.12) that we found in four-dimensional Maxwell's theory. In fact, it is easy to check that the latter is obtained by setting $D = 4$, $p = 1$, $G = \frac{1}{e^2}$ and $\mathbb{P}_{(1,0)}\Lambda \equiv \widehat{A}$ in $\widehat{\mathcal{L}}^{(p+1,0)}$. Using these identifications, it is also trivial to show that the universal parent Lagrangian (5.1) coincides with the one in (3.8).

For the second domain $\mathcal{D}_2$, the algebraic operator (5.27) is nontrivial and reads as $(O^{-1})^{(p+1,1)} = \mathbb{I} - \widetilde{\sigma}\sigma$. We directly compute

$$\begin{aligned}\widehat{\mathcal{L}}^{(p+1,1)}(\Lambda) &= \frac{(-1)^{D+p}}{32\pi^2 G}\int_{\theta,\chi} (\mathbb{I} - \widetilde{\sigma}\sigma) * \left(\mathrm{d}\mathbb{P}_{(D-p-2,1)}\Lambda - \widetilde{\mathrm{d}}Y\right) \widetilde{*}\, \mathrm{d}\left(\mathbb{P}_{(D-p-2,1)}\Lambda + \widetilde{\sigma}X\right) \\ &\overset{(\mathrm{A.13})}{=} \frac{(-1)^{D+1}}{32\pi^2 G}\int_{\theta,\chi} \left(\mathrm{d}\mathbb{P}_{(D-p-2,1)}\Lambda - \widetilde{\mathrm{d}}Y\right) \star \mathrm{d}\left(\mathbb{P}_{(D-p-2,1)}\Lambda + \widetilde{\sigma}X\right) \\ &= \frac{(-1)^{D+1}}{32\pi^2 G}\int_{\theta,\chi} \mathrm{d}\mathbb{P}_{(D-p-2,1)}\Lambda \star \mathrm{d}\left(\mathbb{P}_{(D-p-2,1)}\Lambda + \widetilde{\sigma}X\right) \\ &= \frac{(-1)^{D+1}}{32\pi^2 G}\int_{\theta,\chi} \mathrm{d}\mathbb{P}_{(D-p-2,1)}\Lambda \star \mathrm{d}\mathbb{P}_{(D-p-2,1)}\Lambda = \widehat{\mathcal{L}}^{(p+1,1)}(\mathbb{P}_{(D-p-2,1)}\Lambda)\,,\end{aligned} \tag{5.31}$$

where we discarded the total derivative term associated to Y. To obtain the result in the last line, we followed a procedure completely analogous to the one carried out around equation (4.26). As we also found in Section 4.2 for the case of single duality in linearized Gravity, the additional fields (contained into X and Y) disappear from the resulting second-order dual Lagrangian. In fact, one can easily see that the universal parent Lagrangian (5.1) reduces to the one in (4.10) for $D = 4$, $p = 1$ and $G = \frac{1}{g^2}$ (and, obviously, under the identification $F \equiv f$). The same holds between the dual Lagrangian obtained above and the one found in (4.27).

For the third domain $\mathcal{D}_3$, the technicalities are much simpler since $(O^{-1})^{(1,q)} = \mathbb{I}$. Moreover, remembering our discussion on exotic duality in Sections 3.4 & 3.5, it is expected that the additional fields X, Y will *not* decouple from the dual Lagrangian. In fact, it is crucial that they remain in the dual theory since they are needed to enforce the correct "weaker" field equations on the exotic dual field. Having these in mind and using the decomposition (5.28), we directly compute

$$
\begin{aligned}
\widehat{\mathcal{L}}^{(1,q)}(\Lambda) &= \frac{(-1)^{q(D-1)+1}}{32\pi^2 G} \int_{\theta,\chi} * \left(\mathrm{d}\mathbb{P}_{(D-2,q)}\Lambda - \widetilde{\mathrm{d}}Y\right) \widetilde{*}\, \mathrm{d}\left(\mathbb{P}_{(D-2,q)}\Lambda + \widetilde{\sigma}X\right) \\
&\overset{(A.8)}{=} \frac{(-1)^{q(D-1)+D}}{32\pi^2 G} \int_{\theta,\chi} \left(\mathrm{d}\mathbb{P}_{(D-2,q)}\Lambda - \widetilde{\mathrm{d}}Y\right) * \widetilde{*}\, \mathrm{d}\left(\mathbb{P}_{(D-2,q)}\Lambda + \widetilde{\sigma}X\right) \,.
\end{aligned}
\tag{5.32}
$$

This Lagrangian reduces to the one in (3.54) for $D = 4$, $q = 1$ and $G = \frac{1}{e^2}$. In this example, one should also note that the field X corresponds to the additional 3-form field H that we encountered in Section 3.4. Finally, these identifications can be used to reduce the universal parent Lagrangian (5.1) to the one in (3.37).

Finally, let us discuss in more detail the dual Lagrangian in the fourth domain $\mathcal{D}_4$. For parameters $\{p, q\}$ taking values in this domain, the form of the algebraic operator $(O^{-1})^{(p+1,q)}$ is more complicated and reads as in (5.27). Moreover, the additional fields X and Y neither cancel algebraically nor hide inside total derivative terms.

As we explained earlier, double duality of mixed-symmetry tensor gauge fields generally requires the existence of these additional off-shell fields, in close analogy to the exotic duality of differential forms. However, we already saw from our previous example of double duality in four-dimensional linearized Gravity that these additional fields decouple. We discussed that the underlying reason for this is that double duality can be seen as a self-duality in this case (that is, there is no need for additional off-shell fields simply because the double dual tensor will obey the same field equations as the original one). This also implied that the parent Lagrangians for single and double dualizations would coincide.

This is a special feature of double duality in four-dimensional linearized Gravity and does not appear in the double dualization of any other field of type $(p, 1)$ (in any number of spacetime dimensions). To see that this is indeed the case, one can observe that the domains $\mathcal{D}_2$ and $\mathcal{D}_4$ posses a nonempty intersection for a very specific value of spacetime dimensions D, namely for $D = 4$. In this case, both domains $\mathcal{D}_2$ and $\mathcal{D}_4$ reduce to their common element $\{p, q\} = \{1, 1\}$ and, thus, the four-dimensional universal parent Lagrangian $\mathcal{L}_{\mathrm{p}}^{(1,1)}$ describes both the single and the double dualization of an irreducible gauge field of type $(1, 1)$ (i.e., linearized graviton). Finally, it is easy to check that this matching of parent Lagrangians is unique in $D = 4$ linearized Gravity by checking that $\mathcal{D}_2 \cap \mathcal{D}_4 = \varnothing$ for any $D \neq 4$.

To conclude the proof of Theorem 5.17, we have to show that the dual theories found above propagate the same physical d.o.f. as their dual counterparts from Theorem 5.11. We will show this collectively by manipulating the fully on-shell version of the duality relation (5.23), which holds true for any domain. To this end, we use the defining relation (5.2) of the algebraic operator $O^{(p+1,q)}$ and the relation (5.29) to transform (5.23) into[1]

$$\mathrm{d}\mathbb{P}_{(p,q)}\omega = \frac{(-1)^{\alpha(p,q)}}{4\pi G} * \left(\mathrm{d}\mathbb{P}_{(D-p-2,q)}\Lambda - \widetilde{\mathrm{d}}Y\right). \tag{5.33}$$

As usual, tensors with types containing at least one negative input are assumed to vanish identically. This implies that (5.33) gets simplified even more in the first domain $\mathcal{D}_1$, since $Y = 0$ for $q = 0$. In this case, one can either act with d or with d$*$ on both sides of (5.33) and obtain the standard field equation for the original ($\mathbb{P}_{(p,0)}\omega$) or the dual ($\mathbb{P}_{(D-p-2,0)}\Lambda$) differential form, respectively. Note that, at this level, the field equations for the p-form can be written in local components and are $\binom{D}{p}$ in number. This is simply because an antisymmetric p-tensor in D dimensions has $\binom{D}{p}$ independent elements. Equivalently, the dual equations for the $(D-p-2)$-form are $\binom{D}{D-p-2}$. Although $\binom{D}{p} \neq \binom{D}{D-p-2}$, after full gauge fixing the gauge fields reduce to the respective representations of the little group $SO(D-2) \subset SO(1, D-1) \subset GL(D, \mathbb{R})$. That is, the physical d.o.f. propagated will be $\binom{D-2}{p}$ for the p-form theory and $\binom{D-2}{D-p-2}$ for its dual. Therefore, the two dual theories propagate the same number of physical d.o.f., as it is trivial to check that $\binom{D-2}{p} = \binom{D-2}{D-p-2}$.

For the domain $\mathcal{D}_2$, we already saw in (5.31) that the extra fields decouple from the dual Lagrangian. The underlying reason is a specific gauge symmetry of the parent Lagrangian (5.1) that forces these fields to hide inside total derivative terms in both children actions. This is exactly the effect we discussed about in Section 4.2, where the parent theory was invariant under an arbitrary shift of the 2-form component of e. To obtain the original (or dual) field equations, one has to act on both sides of (5.33) with the operator $\mathrm{tr}\widetilde{\mathrm{d}}$ (or $\mathrm{tr}\widetilde{\mathrm{d}}*$). Then, using the map identities (2.78) and (2.79) will

[1]To be more precise, the l.h.s. of equation (5.33) should also contain a term $\widetilde{\mathrm{d}}Z$, for a reducible tensor Z of type $(p+1, q-1)$ containing the possible additional irreducible parts of ω with different types. However, we already proved in Theorem 5.11 that all such components of ω decouple from the original second-order Lagrangian and, as such, we can ignore them without loss of generality.

finally yield

$$\mathrm{tr}\,\mathrm{d}\widetilde{\mathrm{d}}\mathbb{P}_{(p,1)}\omega = 0 \quad \left(\text{or} \quad \mathrm{tr}\,\mathrm{d}\widetilde{\mathrm{d}}\mathbb{P}_{(D-p-2,1)}\Lambda = 0 \right). \tag{5.34}$$

Before gauge fixing[2], these field equations correspond to the vanishing of two irreducible tensor representation of $SO(1, D-1)$ with types $(p, 1)$ and $(D-p-2, 1)$, respectively, which have $\frac{p(D+1)}{p+1}\binom{D}{p}$ and $\frac{(D+1)(D-p-2)}{D-p-1}\binom{D}{D-p-2}$ independent components [77]. Clearly, these two numbers do not match. However, upon gauge fixing the tensors reduce to the same representations of the little group $SO(D-2)$ with independent components $\frac{p(D-1)}{p+1}\binom{D-2}{p} - \binom{D-2}{p-1}$ and $\frac{(D-1)(D-p-2)}{D-p-1}\binom{D-2}{D-p-2} - \binom{D-2}{D-p-3}$. These two last numbers are actually the same and, therefore, we can see that the dual theory propagates exactly the same number of physical d.o.f. as the original.

Finally, the dual equations of motion for the domains $\mathcal{D}_3$ (or $\mathcal{D}_4$) can be found by acting on both sides of equation (5.33) with the algebraic operator $\sigma^{q+1}\widetilde{\mathrm{d}}$ (or $\sigma^{q}\widetilde{\mathrm{d}}$). In both cases, one can use the map identities (2.78) and (2.79) to see that the l.h.s. will vanish identically, leaving us with the dual equations of motion

$$\mathrm{tr}^{q+1}\mathrm{d}\widetilde{\mathrm{d}}\mathbb{P}_{(D-2,q)}\Lambda = 0 \quad \left(\text{or} \quad \mathrm{tr}^{q}\mathrm{d}\widetilde{\mathrm{d}}\mathbb{P}_{(D-3,q)}\Lambda = 0 \right). \tag{5.35}$$

The first equation, which corresponds to $\mathcal{D}_3$, implies the vanishing of a q-form and, as such, its independent local components are $\binom{D}{q}$ in number. As expected, this number is the same as the independent components of the original field equation for the (transposed) q-form ω, even before gauge fixing. This proves Theorem 5.17 for the third domain.

On the other hand, the second equation above implies the vanishing of an irreducible tensor of type $(D-q-2, 1)$ and has $\frac{(D+1)(D-q-2)}{D-q-1}\binom{D}{D-q-2}$ independent local components. In analogy to what happens in the second domain, this number does not match the number of independent components of the original field equation for the irreducible field ω of type $(1, q)$, which is $\frac{q(D+1)}{q+1}\binom{D}{q}$. However, fixing the gauge symmetry requires us to think of these tensors as representations of the little group $SO(D-2)$, under which they have precisely the same number of independent components (as shown above for the case of $\mathcal{D}_2$). This proves Theorem 5.17 for the fourth domain $\mathcal{D}_4$ and concludes our general proof.

[2] See also [44, 46] for useful discussions regarding gauge fixing in general tensor gauge theories.

5.3 Some examples in local coordinates

As a final note, let us complement our previous discussions by presenting the local expressions of the parent Lagrangian (5.1) in some simple cases. Following [49] (where a slightly different parent Lagrangian was used with a Lagrange multiplier of different bidegree), the simplest examples are:

Standard duality of a scalar, domain $\mathcal{D}_1$

$$\mathcal{L}_{\mathrm{P}}^{(1,0)} = -\frac{G}{2}\, F_\mu F^\mu - \frac{1}{4\pi(D-2)!}\varepsilon^{\mu_1\dots\mu_D} F_{\mu_1}\partial_{\mu_2}\Lambda_{\mu_3\dots\mu_D} \tag{5.36}$$

Standard duality of a 1-form, domain $\mathcal{D}_1$

$$\mathcal{L}_{\mathrm{P}}^{(2,0)} = -\frac{G}{2\cdot 2!}\, F_{\mu\nu} F^{\mu\nu} - \frac{1}{4\pi\cdot 2!(D-3)!}\varepsilon^{\mu_1\dots\mu_D} F_{\mu_1\mu_2}\partial_{\mu_3}\Lambda_{\mu_4\dots\mu_D} \tag{5.37}$$

Exotic duality of a 1-form, domain $\mathcal{D}_3$

$$\mathcal{L}_{\mathrm{P}}^{(1,1)} = -\frac{G}{2}\left(F_{\mu\nu} F^{\mu\nu} - F_\mu{}^\mu F_\nu{}^\nu\right) - \frac{1}{4\pi(D-2)!}\varepsilon^{\mu_1\dots\mu_D} F_{\mu_1\alpha}\partial_{\mu_2}\Lambda_{\mu_3\dots\mu_D}{}^\alpha \tag{5.38}$$

Standard duality of a $(1,1)$ tensor, domain $\mathcal{D}_2$

$$\begin{aligned}\mathcal{L}_{\mathrm{P}}^{(2,1)} = -\frac{G}{2\cdot 2!}&\left(2F_{\mu\nu}{}^\nu F^{\mu\rho}{}_\rho - F_{\mu\nu\rho}F^{\mu\rho\nu} - \frac{1}{2}\, F_{\mu\nu\rho}F^{\mu\nu\rho}\right)\\ &- \frac{1}{4\pi\cdot 2!(D-3)!}\varepsilon^{\mu_1\dots\mu_D} F_{\mu_1\mu_2\alpha}\partial_{\mu_3}\Lambda_{\mu_4\dots\mu_D}{}^\alpha\end{aligned} \tag{5.39}$$

Standard duality of a $(2,1)$ tensor, domain $\mathcal{D}_2$

$$\begin{aligned}\mathcal{L}_{\mathrm{P}}^{(3,1)} = -\frac{G}{2\cdot 3!}&\left(\frac{2}{3}\, F_{\mu\nu\rho\kappa}F^{\mu\nu\rho\kappa} + F_{\mu\nu\rho\kappa}F^{\mu\nu\kappa\rho} - 3\, F_{\mu\nu\rho}{}^\rho F^{\mu\nu\kappa}{}_\kappa\right)\\ &- \frac{1}{4\pi\cdot 3!(D-4)!}\varepsilon^{\mu_1\dots\mu_D} F_{\mu_1\mu_2\mu_3\alpha}\partial_{\mu_4}\Lambda_{\mu_5\dots\mu_D}{}^\alpha\end{aligned} \tag{5.40}$$

By simple observation, it becomes clear that these parent Lagrangians become increasingly more complicated as the values of the parameters increase, mainly due to the overall and relative coefficients in the term multiplied by the coupling constant G (recall that the coefficient of the Lagrange multiplier term is completely irrelevant). Therefore, it is very interesting that all these expressions originate from the simple geometric Lagrangian (5.1) through the generalized $\star$-operator.

Chapter 6

THEORIES WITH MULTIPLE FIELDS IN SELF-DUAL DIMENSIONS

In the previous Chapter, we described the dualization procedures for differential p-form and $(p, 1)$ mixed-symmetry tensor gauge fields, and showed how all the seemingly different starting points for each case can be unified by means of the graded geometric parent Lagrangian (5.1). In this Chapter, we will generalize and discuss the (standard) electric-magnetic dualization for theories involving multiple gauge fields (of the aforementioned types) defined on Minkowski spacetime of self-dual dimensions.

The reason for restricting to self-dual dimensions is that we are mostly interested in examining the underlying self-duality groups, in a way similar to our discussions about the Maxwell field (in Sections 3.1, 3.2 & 3.3) and the linearized graviton (in Sections 4.1, 4.2 & 4.3) in four dimensions. In the ensuing, we will be mainly following the discussion presented in [37].

6.1 Two-dimensional scalar fields

Let us now consider the electric-magnetic self-duality of a free scalar field ϕ, propagating in two-dimensional Minkowski spacetime. This duality can be explored using the universal parent Lagrangian (5.1), which was the key object of the previous Chapter. More precisely, the scalar field can be thought of as a 0-form gauge field and, as such, the relevant starting point for its standard dualization is

$$\mathcal{L}_{\mathrm{p}}^{(1,0)}(F_1, \Lambda) = -\frac{G}{4\pi\alpha'}\int_\theta F * F + \int_\theta F \mathrm{d}\Lambda \overset{!}{=} -\frac{G}{2}\int_\theta F * F + \int_\theta F \mathrm{d}\Lambda\,, \tag{6.1}$$

in terms of the 1-form field F and the scalar Lagrange multiplier[1] Λ. It is worth mentioning here that F is assumed to have dimensions of inverse length and G is a coupling constant with dimensions of length squared. The α'-parameter also has dimensions of length squared and has been chosen to satisfy $2\pi\alpha' \overset{!}{=} 1$. As usual, the Euler-Lagrange equations for F and Λ correspond to the Bianchi

[1] Note that there is a slight difference between this parent Lagrangian and the one defined in (5.1), in that here we are working with a rescaled Lagrange multiplier field $\Lambda \mapsto -4\pi\Lambda$. Rescalings like these are always possible since the overall scaling factors can be always absorbed into the definition of the dual field.

identity and duality relation

$$\mathrm{d}F = 0 \qquad \& \qquad * F = \frac{1}{G}\,\mathrm{d}\Lambda \tag{6.2}$$

which lead, respectively, to the original (using the solution $F := \mathrm{d}\phi$) and dual children theories

$$\mathcal{L}(\phi) = -\frac{G}{2}\int_\theta \mathrm{d}\phi * \mathrm{d}\phi \qquad \& \qquad \widehat{\mathcal{L}}(\Lambda) = -\frac{1}{2G}\int_\theta \mathrm{d}\Lambda * \mathrm{d}\Lambda\,. \tag{6.3}$$

Similarly to our discussion on the self-duality in four-dimensional Maxwell theory in Sections 3.1 & 3.2, we observe that the two scalar theories above have inverse couplings. In other words, electric-magnetic duality acts (in coupling space) as the transformation

$$G \mapsto \frac{1}{G}\,. \tag{6.4}$$

The main difference between the two-dimensional scalar and the four-dimensional Maxwell cases is that there is no notion of a ϑ-term in the first. Indeed, it is easy to see that a term of the form $\int_\theta (\mathrm{d}\phi)^2$ vanishes identically since $(\mathrm{d}\phi)^2 = -(\mathrm{d}\phi)^2$.

Things change, however, when one considers a theory containing multiple (say d) scalars ϕ^M, with $M = 1,\ldots,d$. In such cases, one can consider the general second-order Lagrangian

$$\mathcal{L}(\phi^M) = -\frac{G_{MN}(\phi)}{2}\int_\theta \mathrm{d}\phi^M * \mathrm{d}\phi^N - \frac{B_{MN}(\phi)}{2}\int_\theta \mathrm{d}\phi^M \mathrm{d}\phi^N, \tag{6.5}$$

in terms of a symmetric (G_{MN}) and an antisymmetric (B_{MN}) $d \times d$ matrices that can be arbitrary functions of the scalars. The first term generalizes the kinetic term in the single scalar case, while the latter is the topological term that, as we explained, is trivially absent for a single scalar. This is encoded into the condition $B_{MM} = 0$, which holds for any $M = 1,\ldots,d$.

The theory defined by (6.5) can be dualized with respect to any of the scalar fields involved, assuming that neither G_{MN} nor B_{MN} depends on this particular field. Therefore, let us choose the field that we wish to dualize, say $\phi^d \equiv \phi$, and perform the splitting $\phi^M = (\phi^m, \phi)$ where $m = 1,\ldots,d-1$. To this end, we will demand that the couplings can depend on any scalar other than ϕ, i.e. $G_{MN} \overset{!}{=} G_{MN}(\phi^m)$ and $B_{MN} \overset{!}{=} B_{MN}(\phi^m)$. For the sake of brevity, we are not going to mention these dependencies explicitly in the following discussion.

The starting point for the dualization procedure will be a parent Lagrangian of the form

$$\begin{aligned}\mathcal{L}_\mathrm{p}(\phi^m, F, \Lambda) = &-\frac{1}{2}\int_\theta (G_{mn}\,\mathrm{d}\phi^m * \mathrm{d}\phi^n + 2\,G_{md}\,F * \mathrm{d}\phi^m + G_{dd}\,F * F)\\ &-\frac{1}{2}\int_\theta (B_{mn}\,\mathrm{d}\phi^m \mathrm{d}\phi^n + 2\,B_{md}\,F\mathrm{d}\phi^m) + \int_\theta F\,\mathrm{d}\Lambda\,,\end{aligned} \tag{6.6}$$

containing the "spectator" scalars ϕ^m as well as a 1-form field F and a scalar Lagrange multiplier Λ. Variation with respect to the latter gives the Bianchi identity $\mathrm{d}F = 0$ which can be solved locally by $F := \mathrm{d}\phi$. Substituting this solution back into the parent Lagrangian (6.6) will give, as usual, a second-order child theory. It is trivial to check that this will be precisely (6.5), as expected.

Alternatively, the duality relation resulting as the Euler-Lagrange equation for F will read as

$$F = \frac{1}{G_{dd}} * \mathrm{d}\Lambda - \frac{G_{md}}{G_{dd}}\,\mathrm{d}\phi^m - \frac{B_{md}}{G_{dd}} * \mathrm{d}\phi^m . \tag{6.7}$$

Inserting this equation back into the parent Lagrangian (6.6) and introducing the notation $\widehat{\phi}^M = (\phi^m, \Lambda)$ leads to a dual second-order Lagrangian of the desired form

$$\widehat{\mathcal{L}}(\widehat{\phi}^M) = -\frac{\widehat{G}_{MN}(\phi^m)}{2}\int_\theta \mathrm{d}\widehat{\phi}^M * \mathrm{d}\widehat{\phi}^N - \frac{\widehat{B}_{MN}(\phi^m)}{2}\int_\theta \mathrm{d}\widehat{\phi}^M \mathrm{d}\widehat{\phi}^N , \tag{6.8}$$

where the new couplings are related to the original ones through

$$\begin{aligned} &\widehat{G}_{dd} = \frac{1}{G_{dd}} , \qquad \widehat{G}_{md} = \frac{B_{md}}{G_{dd}} , \qquad \widehat{B}_{md} = \frac{G_{md}}{G_{dd}} , \\ &\widehat{G}_{mn} = G_{mn} - \frac{G_{md}G_{nd} - B_{md}B_{nd}}{G_{dd}} , \qquad \widehat{B}_{mn} = B_{mn} - \frac{B_{md}G_{nd} - G_{md}B_{nd}}{G_{dd}} . \end{aligned} \tag{6.9}$$

Furthermore, these rather complicated expressions for the transformation of couplings under electric-magnetic duality acquire a more transparent form in terms of the new coupling

$$E_{MN} = G_{MN} + B_{MN} , \tag{6.10}$$

which is often called the generalized metric. In terms of this quantity, the transformation rules above read as

$$\widehat{E}_{mn} = E_{mn} - E_{md}\,\frac{1}{E_{dd}}\,E_{nd} , \quad \widehat{E}_{md} = E_{md}\,\frac{1}{E_{dd}} , \quad \widehat{E}_{dd} = \frac{1}{E_{dd}} . \tag{6.11}$$

In fact, the above expressions are very suggestive and have a straightforward generalization to the case of multiple dualities. Had we split $\phi^M = (\phi^m, \phi^\alpha)$ and dualized all fields ϕ^α, we would have found the transformation rules [22]

$$\widehat{E}_{mn} = E_{mn} - E_{m\alpha}E^{-1}_{\alpha\beta}E_{\beta n} , \quad \widehat{E}_{m\alpha} = E_{m\beta}E^{-1}_{\alpha\beta} , \quad \widehat{E}_{\alpha\beta} = E^{-1}_{\alpha\beta} , \tag{6.12}$$

where E^{-1} is the matrix inverse of E. These rules and, in particular, the ones in (6.9) are the famous Buscher rules [78] that describe the transformation of the background fields of String Theory under

T-duality. In addition, it is well-known (see [22, 79]) that they are associated to the group of orthogonal transformations $O(d, d, \mathbb{R})$. We will expand on this point in Section 6.3.

Before moving on to actually discuss T-duality in String Theory, let us make a final comment; the Lagrangian (6.5) (as well as the dual one in (6.8)) defines a nonlinear sigma model.

More precisely, one can think of the multiple scalars ϕ^M as the components of a sigma model map $\phi : \Sigma_2 \mapsto \mathcal{N}_d$ from the two-dimensional Riemann surface of Lorentzian signature Σ_2 (also called the source manifold or spacetime) to a d-dimensional manifold $\mathcal{N}_d$ (the target space). Then, the scalar fields ϕ^M can be seen as the pull-back of the coordinates y^M of the target space under the sigma model map, i.e. $\phi^M = \phi^*(y^M)$. In this picture, the couplings G_{MN} and B_{MN} correspond to background fields on the spacetime Σ_2 but are the dynamical metric tensor and B-field on the target space $\mathcal{N}_d$. In contrast to the source manifold, the target space does not (in general) have a physical interpretation as a spacetime. In fact, one of the most exciting and nontrivial features of String Theory is precisely the fact that the underlying target space is meaningful itself as a physical spacetime[2].

The string sigma model [19] (see also [20, 21]) is given by a Lagrangian very similar to (6.5) (in addition to higher-order terms like the dilaton coupling), with the additional assumptions that the scalars are $d = 26$ (or $d = 10$ for the superstring)[3] in number and some of them can be periodic with period 2π. In that setting, our analysis here would be very similar to the T-dualization procedure carried out in [80]. The two-dimensional spacetime Σ_2 would correspond to the string worldsheet, while our assumption that the background fields G_{MN} and B_{MN} are independent of the dualized scalars ϕ^α, $\alpha = 1, \ldots, s$, would be equivalent to the assumption that the target space $\mathcal{N}_d$ has s isometries. That is, there exist s Killing vectors ρ^α such that

$$\mathcal{L}_{\rho^\alpha} G_{MN} = 0 \qquad \& \qquad \mathcal{L}_{\rho^\alpha} B_{MN} = \mathrm{d}\beta^\alpha \,, \qquad \forall \quad \alpha = 1, \ldots, s \tag{6.13}$$

where β^α are arbitrary 1-forms and $\mathcal{L}_{\rho^\alpha}$ is the Lie derivative along the Killing vectors.

For the single scalar case ($d = 1$) considered before, the target space is obviously a one-dimensional manifold. In the context of String Theory, periodicity of the scalar field would imply that this

[2] This is because the vanishing of the beta functions of the string sigma model corresponds to the Einstein equations for the low energy field theory on the target space. For a review, check e.g. [20] and references therein.

[3] Of course, the restriction on the number of scalars is naturally put upon quantization. The classical string sigma model can contain an arbitrary number of scalars.

manifold is a circle $S^1(R)$ with radius R. In fact, this radius is naturally matched with the square root of the positive-definite coupling G which, as we saw earlier in (6.4), gets inverted under electric-magnetic duality. From the perspective of the two-dimensional worldsheet, T-duality is simply this electric-magnetic self-duality of the scalar. From the alternative viewpoint of the target space, T-duality relates the two sigma models defined by (6.3) with maps $\phi : \Sigma_2 \mapsto S^1(R)$ and $\Lambda : \Sigma_2 \mapsto S^1(1/R)$. That is, it acts on the original target space by inverting its radius.

In the case of multiple scalars, the original target space is a higher-dimensional manifold. Assuming that it contains a sufficient number of Killing directions, one can perform the electric-magnetic dualization procedure discussed above and obtain the $O(d, d, \mathbb{R})$ transformation rules (6.12) for the generalized metric E_{MN}. These describe how T-duality acts on the dynamical metric and the B-field of the original target space.

6.2 Maxwell fields in four dimensions

Let us now move on and describe how the discussion of the previous Section can be relevant for the case of multiple Maxwell fields in four dimensions. The main difference between this setting and the multiple scalars in two dimensions is that now there exists a nonvanishing topological term, namely the electromagnetic ϑ-term we encountered in Section 3.3. In that Section, we presented the off-shell dualization procedure for a single Maxwell field and showed that the transformation of the complex coupling τ under electric-magnetic duality is an element of the group $PSL(2, \mathbb{Z})$. Our goal here is to generalize this procedure in the case where multiple Maxwell fields are present and, upon dualizing a specific number of those, find the duality group at the space of couplings.

To this end, let us start with a $U(1)^d$ gauge theory described by the Lagrangian

$$\mathcal{L}(A^M) = \frac{G_{MN}}{2} \int_\theta \mathrm{d}A^M * \mathrm{d}A^N + \frac{B_{MN}}{2} \int_\theta \mathrm{d}A^M \mathrm{d}A^N \,, \tag{6.14}$$

in terms of d distinct Abelian 1-form gauge fields A^M, for $M = 1, \dots, d$. Note that both couplings G_{MN} and B_{MN} have to be symmetric $d \times d$ matrices for the above Lagrangian to be nonzero. This is another difference between this Lagrangian and the one in (6.5) for multiple scalars; the background field B_{MN} had to be antisymmetric in the latter (in agreement with the absence of a topological term for a single scalar). In the single field case of $d = 1$, we can easily see that the above Lagrangian coincides with (3.16) if we make the identifications $G_{11} = \frac{1}{e^2}$ and $B_{11} = \frac{\vartheta}{8\pi^2}$. In addition, we will allow both couplings to be gauge invariant functions of any of the 1-forms A^M except for the ones that we will dualize.

Multiple remarks are in order regarding this Lagrangian and its comparison to the scalar sigma model of the previous Section. Indeed, the Lagrangian (6.14) can be considered as a sigma model for a map[4] $A : T\mathbb{R}^{1,3} \mapsto \mathcal{N}_d$, with components A^M, from the tangent bundle of the four-dimensional Minkowski space to a d-dimensional generalized target space $\mathcal{N}_d$. To provide a geometric justification of this statement, it would be useful to think of the source manifold and the target space as the degree-shifted graded bundles $T^*[1]\mathbb{R}^{1,3}$ and $\overline{\mathcal{N}_d}[1]$, where the latter is some manifold isomorphic to $\mathcal{N}_d$. Recall that $T^*[1]\mathbb{R}^{1,3}$ is isomorphic to $T\mathbb{R}^{1,3}$ and thus degree-1 functions on it are ordinary 1-forms. Choosing degree-1 local coordinates ξ^M on the target space $\overline{\mathcal{N}_d}[1]$, the fields A^M are obtained as a pull-back under the sigma model map A, i.e. $A^M = A^*(\xi^M)$. For a more general discussion in this direction, see e.g. [81].

Let us now move on with the dualization procedure. For simplicity, we will focus on dualizing one of the 1-form fields, say $A^d \equiv A$, and split $A^M = (A^m, A)$ where $m = 1, \ldots, d-1$. Then, we consider the parent Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{p}}(A^m, F, \Lambda) = \frac{1}{2}\int_\theta (G_{mn}\,\mathrm{d}A^m * \mathrm{d}A^n + 2\,G_{md}\,\mathrm{d}A^m * F + G_{dd}\,F * F) \\ + \frac{1}{2}\int_\theta \Big(B_{mn}\,\mathrm{d}A^m \mathrm{d}A^n + 2\,B_{md}\,\mathrm{d}A^m F + B_{dd}\,F^2\Big) - \int_\theta F\,\mathrm{d}\Lambda\,,\end{aligned} \tag{6.15}$$

which contains a 2-form F and a Lagrange multiplier 1-form Λ. As usual, variation with respect to the latter will give the Bianchi identity $\mathrm{d}F = 0$, which can be locally solved by $F := \mathrm{d}A$. Then, substitution back into $\mathcal{L}_{\mathrm{p}}$ will give the second order theory (6.14) up to a total derivative term containing the Lagrange multiplier Λ.

Let us now find the dual theory in full analogy to the previous cases we described. First, we have to vary the parent Lagrangian with respect to F and obtain the duality relation

$$(G_{dd} * + B_{dd})\,F = \mathrm{d}\Lambda - (G_{md} * + B_{md})\,\mathrm{d}A^m\,. \tag{6.16}$$

[4] Note that there is a slight abuse of notation here, in that we use the same symbol for the Lie algebra valued degree-1 function in $C^\infty(T[1]\mathbb{R}^{1,3}; \mathfrak{u}(1)^d)$, or equivalently a Lie algebra valued 1-form in $\Omega^1\left(\mathbb{R}^{1,3}; \mathfrak{u}(1)^d\right)$, and a map $A : T\mathbb{R}^{1,3} \mapsto \mathcal{N}_d$.

To solve this equation in terms of F, one must act on both sides with $*$ and obtain the intermediate relation

$$*F = \frac{1}{B_{dd}} * \mathrm{d}\Lambda + \frac{G_{md} - B_{md}*}{B_{dd}} \mathrm{d}A^m + \frac{G_{dd}}{B_{dd}} F \,. \tag{6.17}$$

This expression can be used to integrate out $*F$ in (6.16), leading to its solution

$$F = \frac{B_{dd} - G_{dd}*}{G_{dd}^2 + B_{dd}^2} \mathrm{d}\Lambda - \frac{(G_{dd}G_{md} + B_{dd}B_{md}) + (B_{dd}G_{md} - G_{dd}B_{md})*}{G_{dd}^2 + B_{dd}^2} \mathrm{d}A^m. \tag{6.18}$$

The dual Lagrangian can be easily found after a long but straightforward calculation, after substitution of this solution back into (6.15). By denoting $\widehat{A}^M = (A^m, \Lambda)$ we find

$$\widehat{\mathcal{L}}(A^M) = \frac{1}{2}\int_\theta \widehat{G}_{MN}\, \mathrm{d}\widehat{A}^M * \mathrm{d}\widehat{A}^N + \frac{1}{2}\int_\theta \widehat{B}_{MN}\, \mathrm{d}\widehat{A}^M \mathrm{d}\widehat{A}^N \,, \tag{6.19}$$

where the new couplings are related to the initial ones by

$$\begin{aligned}
\widehat{G}_{dd} &= \frac{G_{dd}}{G_{dd}^2 + B_{dd}^2}\,, \qquad \widehat{B}_{dd} = -\frac{B_{dd}}{G_{dd}^2 + B_{dd}^2}\,,\\
\widehat{G}_{md} &= \frac{B_{md}G_{dd} - G_{md}B_{dd}}{G_{dd}^2 + B_{dd}^2}\,, \qquad \widehat{B}_{md} = -\frac{G_{md}G_{dd} + B_{md}B_{dd}}{G_{dd}^2 + B_{dd}^2}\,,\\
\widehat{G}_{mn} &= G_{mn} - \frac{G_{dd}\,(G_{md}G_{nd} - B_{md}B_{nd}) + B_{dd}\,(B_{md}G_{nd} + G_{md}B_{nd})}{G_{dd}^2 + B_{dd}^2}\,,\\
\widehat{B}_{mn} &= B_{mn} - \frac{G_{dd}\,(G_{md}B_{nd} + B_{md}G_{nd}) - B_{dd}\,(G_{md}G_{nd} - B_{md}B_{nd})}{G_{dd}^2 + B_{dd}^2}\,.
\end{aligned} \tag{6.20}$$

These are the "generalized Buscher rules" for the Maxwell fields [37]. In analogy to our previous discussion, one can now define the complex coupling

$$\tau_{MN} = B_{MN} + G_{MN}\, i \tag{6.21}$$

that generalizes the one defined in (3.21) for the single Maxwell field. It is then easy to see that, under the transformations (6.20), the components of this complex coupling transform as

$$\widehat{\tau}_{mn} = \tau_{mn} - \tau_{md}\,\frac{1}{\tau_{dd}}\,\tau_{nd}\,, \quad \widehat{\tau}_{md} = -\tau_{md}\,\frac{1}{\tau_{dd}}\,, \quad \widehat{\tau}_{dd} = -\frac{1}{\tau_{dd}}\,. \tag{6.22}$$

This already suggests the result for the transformation of the couplings when we dualize multiple fields in (6.14). If we do the splitting $A^M = (A^m, A^\alpha)$ and dualize all fields A^α following the same procedure, we will find that the above transformation generalizes to

$$\widehat{\tau}_{mn} = \tau_{mn} - \tau_{m\alpha}\,\tau^{-1}_{\alpha\beta}\,\tau_{\beta n}\,, \quad \widehat{\tau}_{m\alpha} = -\tau_{m\beta}\,\tau^{-1}_{\beta\alpha}\,, \quad \widehat{\tau}_{\alpha\beta} = -\tau^{-1}_{\alpha\beta}\,, \tag{6.23}$$

where τ^{-1} is the matrix inverse of τ. Note that these transformations are very similar to the ones we obtained for the generalized metric (6.12) in the scalar case, but there exist some sign differences between them. These indicate that the underlying duality groups are different; in fact, we will show in the upcoming Section that the rules (6.23) are associated to the group of symplectic transformations $Sp(2d, \mathbb{R})$. This will be in full agreement with the well-known results of Gaillard and Zumino [62].

6.3 Arbitrary differential forms and bipartite tensors

In our approach, the starting point for the dualization of either multiple p-forms or bipartite tensor fields of type $(p, 1)$ in the self-dual dimensions $D = 2p + 2$ is universal and therefore the corresponding generalization of the analysis of the previous Sections is fairly straightforward. The parent Lagrangian for the dualization of a single field (in the presence of $d-1$ additional fields of the same type) is a generalization of (5.1) for $\{p, q\}$ taking values in either $\mathcal{D}_1$ or $\mathcal{D}_2$, namely

$$\mathcal{L}_{\mathrm{p}}(F^M_{p+1,q}, \Lambda_{p,q}) = \frac{(-1)^q}{2} \int_{\theta,\chi} F^M \star \mathcal{U}_{MN} \mathcal{O}^{(p+1,q)} F^N + \int_{\theta,\chi} F \,\widetilde{*}\, \mathrm{d}\Lambda \,, \tag{6.24}$$

where

$$\mathcal{U}_{MN} = G_{MN} + (-1)^{p+q} B_{MN} * \qquad \text{and} \qquad G_{mm} > 0 \quad \forall\, m = 1, \ldots, d \,. \tag{6.25}$$

This is a functional of two fields F^M, $M = 1, \ldots d$ and Λ, which are in principle reducible under the general linear group, with types as indicated in (6.24). The field F^M is decomposed as $F^M = (F^m = \mathrm{d}\omega^m, F)$, where $m = 1 \ldots, d-1$ and $F \equiv F^d$, with ω^m being $d-1$ irreducible gauge fields of type (p, q).

Following the procedure described in all previous Sections, one easily finds out that the Lagrangian (6.24) gives rise to two second-order Lagrangians for two dual sets of fields $\omega^M = (\omega^m, \omega)$ and $\widehat{\omega}^M = (\omega^m, \widehat{\omega})$, both of which being irreducible with type (p, q). The dualization procedure can be depicted as

$$\mathcal{L}(\omega^m, \omega) \xleftarrow{\Lambda \text{ on-shell}} \mathcal{L}_{\mathrm{p}}(F^M, \Lambda) \xrightarrow{F \text{ on-shell}} \widehat{\mathcal{L}}(\omega^m, \widehat{\omega}) \,, \tag{6.26}$$

where ω is the original field introduced by the solution of the Bianchi identity $F := \mathrm{d}\omega$ and $\widehat{\omega}$ is the dual field defined, as usual, to be the irreducible piece of Λ with the same type, i.e. $\widehat{\omega} := \mathbb{P}_{(p,q)}\Lambda$.

When compared to the $D = 2p+2$ version of (5.1), there exist some obvious differences. First of all, note that we have performed an unimportant rescalling of the Lagrange multiplier $\Lambda \mapsto 4\pi\Lambda$, just like we did in the previous Sections. Secondly, we have included the additional "theta" terms whose coupling is the $d \times d$ matrix B_{MN}. For even (odd) p this matrix is antisymmetric (symmetric), in agreement with the fact that $2k$-forms and $(2k, 1)$ tensors have vanishing ϑ-terms. It should be clear that such a term would not make sense in (5.1), where the number of spacetime dimensions was arbitrary. As it turns out, the fact that B_{MN} is either symmetric or antisymmetric leads to two distinct possibilities regarding the duality group underlying the coupling space. We will expand on this point shortly.

Let us now make contact with our previous examples. For parameters taking values in the first domain $\mathcal{D}_1$, we have $q = 0$ and, thus, $O^{(p+1,0)} = \mathbb{I}$. Applying these in (6.24) and performing the trivial integration over the variable χ (which results in an additional minus sign), will give

$$\mathcal{L}_{\mathrm{p}}(F^M, \Lambda) = \frac{(-1)^{p+1}}{2} \int_\theta F^M * (G_{MN} + (-1)^p B_{MN} *)\ F^N - \int_\theta F\, \mathrm{d}\Lambda\,, \tag{6.27}$$

in terms of the $(p+1)$-forms F^M and the p-form Λ. It is easy to check that this parent Lagrangian coincides both with the one in (6.6) for $p = 0$ and with the one in (6.15) for $p = 1$.

The story is very similar for bipartite tensors, for which the parameters $\{p, q\}$ take values in the second domain $\mathcal{D}_2$. Although we have not presented any example for the dualization of multiple such gauge fields, the procedure is straightforward and follows precisely the recipe described above. In fact, it is easy to check that the parent Lagrangian (6.24) coincides with the one in (4.33) for the single graviton case in four dimensions, upon setting $d = p = q = 1$, rescaling $\Lambda \mapsto 4\pi\Lambda$ and identifying $G_{11} = \frac{1}{g^2}$, $B_{11} = -\frac{\vartheta}{8\pi^2}$. Upon performing the dualization procedure, one will find that the background fields G_{MN} and B_{MN} transform exactly as in (6.20), i.e. one can define the complex coupling $\tau_{MN} = B_{MN} + G_{MN} i$ and see that the rules (6.23) hold.

Generally speaking, the background fields of the two children theories related through (6.24) will always be connected via either of the two sets of Buscher rules (6.12) or (6.23) depending on the value of p. For even p one obtains the rules (6.12), whereas for odd p one obtains the alternative set (6.23). Let us now see how the first set is associated to orthogonal transformations in $O(d, d; \mathbb{R})$ and the second set to symplectic transformations in $Sp(2d; \mathbb{R})$.

In the present context, we can discuss both cases in a parallel fashion. In the following, we assume there are s Killing directions in the total d-dimensional target space. We first consider a $d \times d$ block matrix

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}_1 & \mathcal{E}_2 \\ \mathcal{E}_3 & \mathcal{E}_4 \end{pmatrix}, \tag{6.28}$$

with block sizes $s \times s$ for $\mathcal{E}_1$, $s \times (d-s)$ for $\mathcal{E}_2$, $(d-s) \times s$ for $\mathcal{E}_3$ and $(d-s) \times (d-s)$ for $\mathcal{E}_4$. In the two separate cases we have discussed, the components of this matrix can be identified as

$$\mathcal{E}_{MN} = \begin{cases} E_{MN}\,, & p = 2k \\ \tau_{MN}\,, & p = 2k+1 \end{cases}, \tag{6.29}$$

meaning that $\mathcal{E}_{ij} = (\mathcal{E}_1)_{ij}$ is E_{ij} or τ_{ij}, $\mathcal{E}_{im} = (\mathcal{E}_2)_{im}$ is E_{im} or τ_{im} etc. Following closely the analyses in [22, 79], we will next consider block matrices of the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G}\,, \tag{6.30}$$

where $\mathcal{G}$ is either the orthogonal group $O(s,s;\mathbb{R})$ for $p = 2k$ or the symplectic group $Sp(2s;\mathbb{R})$ for $p = 2k+1$. This means that the matrices g should satisfy the condition

$$g^t \mathcal{J} g = \mathcal{J}\,, \tag{6.31}$$

for a constant $2s \times 2s$ matrix

$$\mathcal{J} = \begin{cases} J = \begin{pmatrix} 0 & \mathbb{1}_s \\ \mathbb{1}_s & 0 \end{pmatrix}, & p = 2k \\ \Omega = \begin{pmatrix} 0 & \mathbb{1}_s \\ -\mathbb{1}_s & 0 \end{pmatrix}, & p = 2k+1 \end{cases} \tag{6.32}$$

given in terms of the $s \times s$ identity matrix $\mathbb{1}_s$. These can be embedded in the larger groups $O(d,d;\mathbb{R})$ and $Sp(2d;\mathbb{R})$, collectively denoted by $\widehat{\mathcal{G}}$, by means of elements

$$\widehat{g} = \begin{pmatrix} \widehat{a} & \widehat{b} \\ \widehat{c} & \widehat{d} \end{pmatrix} \in \widehat{\mathcal{G}}\,, \tag{6.33}$$

with blocks given as

$$\widehat{a} = \begin{pmatrix} a & 0 \\ 0 & \mathbb{1}_{d-s} \end{pmatrix}, \quad \widehat{b} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, \quad \widehat{c} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}, \quad \widehat{d} = \begin{pmatrix} d & 0 \\ 0 & \mathbb{1}_{d-s} \end{pmatrix}. \tag{6.34}$$

Clearly $\widehat{g}$ satisfies (6.31) too, this time for the higher-dimensional $\widehat{\mathcal{J}}$ given by (6.32) with $d \times d$ blocks instead. The two sets of equations (6.12) and (6.23) are now associated to fractional linear transformations of $\mathcal{E}$ as

$$\widehat{\mathcal{E}} = \widehat{g}(\mathcal{E}) = \left(\widehat{a}\,\mathcal{E} + \widehat{b}\right)\left(\widehat{c}\,\mathcal{E} + \widehat{d}\right)^{-1} . \tag{6.35}$$

This may be seen from the fact that when $\widetilde{\mathcal{E}}$ is decomposed according to (6.28), one finds

$$\begin{aligned}
\widehat{\mathcal{E}}_1 &= (a\mathcal{E}_1 + b)(c\mathcal{E}_1 + d)^{-1} , && (6.36a)\\
\widehat{\mathcal{E}}_2 &= (a - (a\mathcal{E}_1 + b)(c\mathcal{E}_1 + d)^{-1} c)\,\mathcal{E}_2 , && (6.36b)\\
\widehat{\mathcal{E}}_3 &= \mathcal{E}_3(c\mathcal{E}_1 + d)^{-1} , && (6.36c)\\
\widehat{\mathcal{E}}_4 &= \mathcal{E}_4 - \mathcal{E}_3(c\mathcal{E}_1 + d)^{-1} c\mathcal{E}_2 . && (6.36d)
\end{aligned}$$

It is then straightforward to see that the transformations (6.12) and (6.23) are identified with elements of $\mathcal{G}$ corresponding to factorized dualities, namely

$$g = \begin{pmatrix} \mathbb{1} - e_i & e_i \\ (-1)^p e_i & \mathbb{1} - e_1 \end{pmatrix} , \tag{6.37}$$

where $e_i = \mathrm{diag}(0, \ldots, 0, 1_i, 0, \ldots, 0)$ and 1_i denotes that the entry is in the i-th position. For scalars where $p = 0$ this is the usual story of factorized T-dualities [22, 79], where the discrete T-duality group is $O(s, s; \mathbb{Z})$. Here, we have followed [37] and discussed how this generalizes to all p, both for multiple p-forms and for multiple $(p, 1)$ tensor gauge fields.

As a final remark, let us mention a subtlety that we have kept under the rug in our analysis. Throughout this Chapter, we considered the couplings G_{MN} and B_{MN} to be, in general, functions of scalar fields (and, whenever possible, of gauge invariant combinations of the spectator fields). However, we did not discuss the dynamics of those scalars since we treated them as background fields and, as such, one can eliminate them by means of their equations of motion. Upon doing so, the Lagrangians can be rewritten in terms of new couplings that are scalar independent, i.e. constant. Such cases have been discussed in [62] where it was shown that, in absence of scalars, the duality group is a compact group. In fact, for the cases we have considered in this Chapter, the duality group in absence of scalars is the maximal compact subgroup of the relevant non-compact duality group in the presence of scalars [59], namely $U(1)^d$ (maximal compact subgroup of $Sp(2d, \mathbb{R})$) for odd p and $O(d; \mathbb{R})$ (maximal compact subgroup of $O(d, d; \mathbb{R})$) for even p. In the presence of scalars, one has to require that they transform in an appropriate way in order to keep the full Lagrangian (including the scalar sector) invariant. This lifts the compact duality groups to the non-compact ones that we have found. In the above discussion, we are actually implying that the scalar fields are dynamical even though we are not explicitly writing down the scalar Lagrangian.

Chapter 7

THE GRAVITATIONAL ϑ-TERM

In this last Chapter, we will discuss more thoroughly the gravitational ϑ-term introduced in Chapter 4. This term was first studied very recently in [36] in the context of Gravitoelectromagnetism (GEM), where some of its physical implications were also discussed. Subsequently, cosmological applications of the gravitational ϑ-term were studied in [82] and [83]. Finally, the role of this term in electric-magnetic duality of linearized Gravity was highlighted in [37], as we already saw in Chapters 4 & 6.

In the ensuing, we will follow the original discussion of [36]. After a short review of the basic concepts in the theory of Gravitoelectromagnetism (GEM), we will elaborate on the modifications induced by the inclusion of the ϑ-term. Afterwards, we will present two physical applications and conclude this Chapter by tracing the origin of this term in nonlinear Gravity.

For the purposes of this Chapter, we will reintroduce the physical constants (c, G) and frequently use the standard notation $\vec{A}$ for a 3-vector A. Moreover, we will mostly work with local expressions instead of geometric quantities. We will assume the mostly plus $(-, +, +, +)$ signature convention for the Minkowski metric. We will use Greek indices for the Minkowski spacetime, while Latin indices will parametrize the spatial Euclidean subspace.

7.1 Gravitoelectromagnetism

Gravitoelectromagnetism (GEM) refers to an alternative (but equivalent) form of linearized General Relativity, by exploiting the fact that there exists a formal analogy between Maxwell's equations and the linearized Einstein ones. For a standard review on GEM, we refer the reader to the textbook review of Mashhoon [84].

Let us begin by considering the linearized graviton $h_{\mu\nu}$, introduced by perturbing the metric tensor around the flat Minkowski spacetime $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$. In addition, we will define the trace-reversed linearized graviton $\overline{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. The terminology stems from the fact that its trace is the opposite of the trace of $h_{\mu\nu}$, i.e. $\overline{h} = -h$. Upon assuming the de Donder gauge condition $\partial^\mu \overline{h}_{\mu\nu} \overset{!}{=} 0$, one can easily find that the linearized Einstein equations take the form

$$\Box\,\overline{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}\,. \tag{7.1}$$

The symmetric 2-tensor $T_{\mu\nu}$ on the r.h.s. corresponds to the stress-energy tensor for the matter which, in general, can be chosen to be of any kind. For our purposes, we will assume that the matter generating the gravitational field is a finite distribution of slowly-moving dust. The stress-energy tensor describing a distribution of dust has the form

$$T_{\mu\nu} = \rho\, v_\mu v_\nu\,, \tag{7.2}$$

where ρ is the mass density and v_μ the 4-velocity, whose integral curves correspond to the world-lines of the dust particles. Under the additional assumption that the dust is slowly-moving, i.e. $|\vec{v}| \ll c$, the components of the stress-energy tensor will read as

$$T_{00} \simeq \rho c^2\,, \quad T_{i0} \simeq -\rho v_i c\,, \quad T_{ij} \simeq \rho v_i v_j\,. \tag{7.3}$$

Consequently, the components of any solution of the linearized Einstein equations (7.1) will scale with the speed of light as

$$\overline{h}_{00} = O(c^{-2})\,, \quad \overline{h}_{i0} = O(c^{-3})\,, \quad \overline{h}_{ij} = O(c^{-4})\,. \tag{7.4}$$

In the ensuing, we will ignore all terms and effects that are of order $O(c^{-4})$, including the purely spacelike components $\overline{h}_{ij}$.

Let us now make contact with GEM. To this end, we can use the components $\overline{h}_{00}$ and $\overline{h}_{i0}$ to introduce the new quantities ϕ and A_i as

$$\phi := -\frac{c^2}{4}\overline{h}_{00} \qquad \& \qquad A_i := \frac{c^2}{2}\overline{h}_{i0}\,. \tag{7.5}$$

These are the GEM potentials; the scalar ϕ will be called the *gravitoelectric*, while the vector $\vec{A} = (A_1, A_2, A_3)$ will be the *gravitomagnetic* potential. In the standard terminology, the scalar ϕ is simply the Newtonian potential induced by the mass of an object, while $\vec{A}$ corresponds to the vector potential induced by a rotating object's angular momentum. Both GEM potentials have dimensions of energy over mass, while they scale with the speed of light as $\phi = O(c^0)$ and $\vec{A} = O(c^{-1})$.

Now that we have a pair of potentials, we can mimic the standard procedure in Electromagnetism and construct a pair of fields. In fact, we will use precisely the same definitions and introduce the *gravitoelectric* and *gravitomagnetic* fields as[1]

$$\vec{E} := -\vec{\nabla}\phi - \frac{1}{2c}\frac{\partial \vec{A}}{\partial t} \qquad \& \qquad \vec{B} := \vec{\nabla} \times \vec{A}\,. \tag{7.6}$$

It is straightforward to check that both these fields have dimensions of acceleration and scale with the speed of light as $\vec{E} = O(c^0)$ and $\vec{B} = O(c^{-1})$. Note that, in standard terminology, the gravitoelectric field $\vec{E}$ coincides with the gravitational field (sometimes denoted by $\vec{g}$).

Using the newly defined potentials (7.5), the de Donder gauge $\partial^\mu \overline{h}_{\mu\nu} \overset{!}{=} 0$ can be rewritten into two independent conditions. These read as

$$\frac{2}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} \overset{!}{=} 0 \qquad \& \qquad \frac{1}{c^3}\frac{\partial \vec{A}}{\partial t} \overset{!}{=} \vec{0} \tag{7.7}$$

and correspond to its temporal ($\partial^\mu \overline{h}_{\mu 0} \overset{!}{=} 0$) and spatial ($\partial^\mu \overline{h}_{\mu i} \overset{!}{=} 0$) components, respectively. The first condition is strikingly similar the the Lorentz gauge in Electromagnetism, while the second is of order $O(c^{-4})$ and will thus be ignored. However, note that this condition implies that the second term in the definition of $\vec{E}$ does not contribute at lowest order.

To complete the picture and fully justify the formal analogy between Electromagnetism and linearized General Relativity, we use the definitions of the GEM potentials and fields (7.5)-(7.6) and the gauge conditions (7.7) to rewrite the linearized Einstein equations (7.1) as:[2]

$$\begin{aligned}
&\vec{\nabla} \cdot \vec{E} = -4\pi G \rho \\
&\vec{\nabla} \cdot \vec{B} = 0 \\
&\vec{\nabla} \times \vec{E} + \frac{1}{2c}\frac{\partial \vec{B}}{\partial t} = \vec{0} \\
&\vec{\nabla} \times \vec{B} - \frac{2}{c}\frac{\partial \vec{E}}{\partial t} = -\frac{8\pi G}{c}\vec{j}
\end{aligned} \tag{7.8}$$

These are the *GEM equations* and their resemblance to the Maxwell's equations is uncanny. The first corresponds to the GEM Gauss' law, indicating that mass densities induce gravitoelectric (Newtonian) fields. The fourth equation is the GEM analogue of Ampère's law, stating that mass

[1] We are using the identification $x^0 \equiv ct = -x_0$, which implies that $\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t} = -\partial^0$.

[2] We have introduced the notation $\vec{j} := \rho\vec{v}$ for the mass current (or mass flux).

currents generate gravitomagnetic fields. Finally, the second and third equations above result trivially from the definitions (7.6) and correspond to the Bianchi identities on the GEM fields (equivalently, the Bianchi identities on the linearized Riemann tensor).

To conclude this Section, it is interesting to note that the GEM Gauss' and Ampère's laws in (7.8) can be obtained as the Euler-Lagrange equations, upon varying the (gauge-fixed by (7.7)) action

$$\mathcal{S}_{\text{GEM}}(\phi, \vec{A}) = \frac{1}{8\pi G} \int d^4x \left(E^2 - B^2\right) + \mathcal{S}_M \tag{7.9}$$

with respect to the GEM potentials ϕ and $\vec{A}$. Equivalently, had we started from the standard form of the (gauge-fixed by the de Donder condition) linearized Einstein-Hilbert action

$$\mathcal{S}_{\text{lin.EH}}(h_{\mu\nu}) = \frac{c^4}{64\pi G} \int d^4x\, h^{\mu\nu} \Box\, \overline{h}_{\mu\nu} + \mathcal{S}_M \tag{7.10}$$

and redefined the components of $h_{\mu\nu}$ according to (7.5), we would have found precisely the GEM action (7.9). In both expressions above the term $\mathcal{S}_M$ corresponds to the matter sector of the theory, variation of which would lead to the stress-energy tensor at hand.

7.2 GEM and the gravitational ϑ-term

Our goal now is to introduce the gravitational ϑ-term at the level of linearized Gravity and find its implications in the GEM picture. To this end, we will follow [36] and modify the linearized Einstein-Hilbert action (7.10) into

$$\mathcal{S}_{\text{lin.}\vartheta\text{EH}}(h_{\mu\nu}) = \frac{c^4}{64\pi G} \int d^4x \left(h^{\mu\nu} \Box\, \overline{h}_{\mu\nu} + \vartheta \varepsilon^{\sigma\rho\lambda\mu} \partial_\rho h_{\mu\nu} \partial_\lambda \overline{h}^{\nu}_{\sigma}\right) + \mathcal{S}_M\,. \tag{7.11}$$

The difference lies in the second term, which is a total derivative if its coupling ϑ is a spacetime constant. Recall that we had previously made this assumption when discussing this term in the context of electric-magnetic duality. However, in this Chapter we are interested in studying physical effects of this term, which requires a modification to the field equations. This forces us to assume that ϑ is a dimensionless background scalar that depends on the spacetime coordinates. In fact, one could also consider adding a kinetic term for ϑ and promote it to a scalar field (axion), but these considerations are outside the scope of this thesis.

At this point, the term that we added by hand in (7.11) looks completely random. Indeed, we still lack the justification as to why this particular term is the analogue of the electromagnetic ϑ-term.

To demystify our choice, we can write this new action in terms of the GEM fields as

$$\mathcal{S}_{\vartheta\mathrm{GEM}}(\phi,\vec{A}) = \frac{1}{8\pi G}\int d^4x\left(E^2 - B^2 - 2\vartheta\vec{E}\cdot\vec{B}\right) + \mathcal{S}_M\,. \tag{7.12}$$

That is, the term we added takes the form of the dot product between the two GEM fields, in full analogy to the electromagnetic case where the ϑ-term reads as the dot product of the electric and magnetic fields.

Let us now see what are the effects of the gravitational ϑ-term in the dynamics of the theory. This can be done in two equivalent ways; we can either find the Euler-Lagrange equations for $h_{\mu\nu}$ from (7.11) and, subsequently, rewrite them in terms of the GEM fields, or vary (7.12) directly in terms of ϕ and $\vec{A}$. Both procedures will lead to the same equations. Since the second and third equations in (7.8) are independent of the action, one expects that the presence of the ϑ-term will only affect the GEM Gauss' and Ampère's laws. Indeed, we find[3]

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= -4\pi G\rho - \frac{1}{2}\vec{\nabla}\vartheta\cdot\vec{B}\\
\vec{\nabla}\cdot\vec{B} &= 0\\
\vec{\nabla}\times\vec{E} + \frac{1}{2c}\frac{\partial\vec{B}}{\partial t} &= \vec{0}\\
\vec{\nabla}\times\vec{B} - \frac{2}{c}\frac{\partial\vec{E}}{\partial t} &= -\frac{8\pi G}{c}\vec{j} + \vec{\nabla}\vartheta\times\vec{E} + \frac{1}{2c}\frac{\partial\vartheta}{\partial t}\vec{B}
\end{aligned} \tag{7.14}$$

which are the *modified GEM equations* [36]. Therefore, by examining these equations we observe that inclusion of the ϑ-term has the following effects:

1. In regions of space where ϑ varies, namely where $\vec{\nabla}\vartheta \neq \vec{0}$, a gravitomagnetic field $\vec{B}$ induces an effective mass density given by $\rho_{\mathrm{eff}} = \frac{1}{8\pi G}\vec{\nabla}\vartheta\cdot\vec{B}$. This effect is due to the modification to Gauss' law.

2. In the same regions of space, a gravitoelectric (Newtonian) field $\vec{E}$ induces an effective mass current given by $\vec{j}_{\mathrm{eff}} = -\frac{c}{8\pi G}\vec{\nabla}\vartheta\times\vec{E}$. This effect is due to the modification to Ampère's law.

[3] For completeness, let us mention that variation of (7.11) with respect to $h_{\mu\nu}$ will result in the modified linearized Einstein equations

$$\Box\,\overline{h}_{\mu\nu} - \varepsilon_{\alpha\beta\gamma(\mu}\partial^{\beta}\overline{h}^{\gamma}_{\nu)}\partial^{\alpha}\vartheta = -\frac{16\pi G}{c^4}T_{\mu\nu}\,. \tag{7.13}$$

The (00) and $(i0)$ components of these can be rewritten, respectively, as the first and fourth equations of (7.14), using the identifications (7.5) and (7.6).

3. If ϑ varies in time, then a gravitomagnetic field $\vec{B}$ induces an effective mass current given by $\vec{j}'_{\text{eff}} = -\frac{1}{16\pi G}\frac{\partial\vartheta}{\partial t}\vec{B}$. Once again, this effect is due to the modified Ampère's law.

Note that both ρ_{eff} and $\vec{j}'_{\text{eff}}$ scale with the speed of light as $O(c^{-1})$, due to the scaling behavior of $\vec{B}$. Thus, one expects that physical effects related to these induced quantities will be very small and become manifest as relativistic corrections[4]. Indeed, the two applications presented in Section 7.4 will be related to relativistic corrections induced by the modified Gauss' law and the effective mass density ρ_{eff}. On the other hand, the effective current $\vec{j}_{\text{eff}}$ induced by the gravitoelectric field scales as $O(c)$ and, thus, is expected to contribute much more than a relativistic correction. It would be interesting to explore physical applications related to this modification of the Ampère's law.

7.3 Nonlinear origin of the gravitational ϑ-term

Our goal in this Section is to find an action for a nonlinear theory of Gravity, which reduces to the action (7.11) upon linearization. In fact, we are only interested in the second term in (7.11) since the first one obviously results from linearization of the Einstein-Hilbert action. Note that we used the phrasing "a nonlinear theory of Gravity", instead of "General Relativity". This is because, as we will see shortly, the aforementioned nonlinear term is trivial in GR.

Before we proceed, we can already make an important observation regarding the linearized ϑ-term. It contains two partial derivatives, exactly like the linearized Einstein-Hilbert term. The derivatives of the latter originate from the spacetime curvature, since the nonlinear Einstein-Hilbert term is proportional to $\sqrt{-g}R$. That is, it is proportional to the Ricci scalar R, which corresponds to the second trace of the Riemann tensor. The linearized curvature 2-form, whose local components are identified with the linearized Riemann tensor, contains precisely two partial derivatives.

On the other hand, the linearized ϑ-term also contains two partial derivatives. Given the fact that it is clearly independent from the linearized Einstein-Hilbert term, this observation indicates that its nonlinear version should not depend on the curvature. Rather, it should be a term quadratic

[4]Let us mention here that the effects of $\vec{j}'_{\text{eff}}$ were studied in [85], but in a slightly different context. Instead of the gravitational ϑ-term discussed in this thesis, the work of [85] revolved around the more common gravitational Chern-Simons (CS), which is a topological term quadratic in the curvature 2-form (see, e.g. [86]). When compared to the gravitational ϑ-term, the CS term contains two additional partial derivatives and, thus, Ampère's law gets modified into a second-order differential equation for $\vec{B}$. Finally, the coupling θ of this term was taken to be a quintessence-like scalar field (equipped with its own kinetic term) and, furthermore, it was assumed to only depend on time.

in the torsion 2-form. It is a well-known fact that General Relativity is based on the Levi-Cività connection, which has vanishing torsion. Because of this, one expects that the nonlinear origin of the ϑ-term cannot be traced within the standard GR framework.

This expectation is not unreasonable. Indeed, there exist several well-studied nonlinear theories of Gravity based on torsionful connections that are either alternatives to or modifications of General Relativity. Two important examples of such theories are the Teleparallel Equivalent of General Relativity (TEGR) and the Einstein-Cartan (EC) theory[5]. The main difference between these theories is that TEGR is based on the Weitzenböck connection, which has torsion but is curvature-free, while the Cartan connection of the EC theory contains both. Therefore, it is expected that the nonlinear gravitational ϑ-term can be found within these kind of frameworks.

Let us now present a more detailed discussion. To this end, we will follow the excellent book of Ortin [89] and consider the simpler case of TEGR. We begin by choosing an orthonormal basis of vectors $\{e_i\}$ and their dual basis of 1-forms $\{\hat{e}^i\}$. These are expanded in the bases of coordinates of the tangent and cotangent space of the Minkowski spacetime as

$$e_i = e_i{}^{\mu}\partial_\mu \qquad \& \qquad \hat{e}^i = \hat{e}^i{}_\mu \mathrm{d}x^\mu \,. \tag{7.15}$$

Their local components $\hat{e}^i{}_\mu(x)$ and $e_i{}^\mu(x)$ are the vielbein and its inverse, satisfying the relations $\hat{e}^i{}_\mu e_j{}^\mu = \delta^i_j$ and $\hat{e}^i{}_\mu e_i{}^\nu = \delta^\nu_\mu$. The Ricci rotation (or anholonomy) coefficients $\Omega_{ij}{}^k$ related to this basis are defined through the commutator of the vectors e_i

$$[e_i, e_j] \equiv -2\Omega_{ij}{}^k e_k \qquad \Rightarrow \qquad \Omega_{ij}{}^k := e_i{}^\mu e_j{}^\nu \partial_{[\mu}\hat{e}^k{}_{\nu]} \,. \tag{7.16}$$

As always, square brackets containing indices correspond to the antisymmetrization rule $[\mu\nu] = \frac{1}{2}(\mu\nu - \nu\mu)$. The local expressions for the Weitzenböck connection and the Ricci rotation coefficients on the holonomic bases $\{\partial_\mu\}$ and $\{\mathrm{d}x^\mu\}$ read as

$$W_{\mu\nu}{}^\rho := e_k{}^\rho \partial_\mu \hat{e}^k{}_\nu \qquad \& \qquad \Omega_{\mu\nu}{}^\rho = e_k{}^\rho\, \hat{e}^i{}_\mu\, \hat{e}^j{}_\nu\, \Omega_{ij}{}^k = e_k{}^\rho\, \partial_{[\mu}\hat{e}^k{}_{\nu]} \,, \tag{7.17}$$

which implies that $\Omega_{\mu\nu}{}^\rho = W_{[\mu\nu]}{}^\rho$. The torsion tensor is always defined to be the antisymmetric part of the connection, i.e. $T_{\mu\nu}{}^\rho := -2W_{[\mu\nu]}{}^\rho$. This reveals that, in TEGR, the torsion of the Weitzenböck connection is related to the Ricci rotation coefficients through $T_{\mu\nu}{}^\rho = -2\Omega_{\mu\nu}{}^\rho$.

[5] For a review on EC theory, see [87, 88]

In arbitrary number of dimensions, there exist three objects that transform as densities under general coordinate transformations, as scalars under Lorentz transformations and they are quadratic in first partial derivatives of the vielbeins. These are the so-called *Weitzenböck invariants*, given in terms of the Ricci rotation coefficients and vielbein determinant[6] as

$$I_1 = \hat{e}\,\Omega_{\mu\nu\rho}\Omega^{\mu\nu\rho}\,, \qquad I_2 = \hat{e}\,\Omega_{\mu\nu\rho}\Omega^{\rho\nu\mu}\,, \qquad I_3 = \hat{e}\,\Omega_{\mu\rho}{}^{\rho}\Omega^{\mu\sigma}{}_{\sigma}\,. \tag{7.18}$$

Note also that there exists a fourth invariant, which is quadratic only in four dimensions and takes the form $I_4 = \hat{e}\,\epsilon^{\mu\nu\rho\sigma}\Omega_{\mu\lambda}{}^{\lambda}\Omega_{\nu\rho\sigma}$. The object $\epsilon^{\mu\nu\rho\sigma}$ corresponds to the Levi-Cività symbol[7], related to the respective tensor by $\varepsilon^{\mu\nu\rho\sigma} = \hat{e}\,\epsilon^{\mu\nu\rho\sigma}$. It is interesting to point out that, in contrast to the three Weitzenböck invariants (7.18), the invariant I_4 violates parity.

The action of TEGR is nothing but a particular linear combination of the three Weitzenböck invariants. In particular, it reads as

$$\mathcal{S}_{\mathrm{TEGR}}(e_i{}^{\mu}, \hat{e}^i{}_{\mu}) = \frac{c^4}{16\pi G}\int d^4x\ (I_1 + 2I_2 - 4I_3) \tag{7.19}$$

and the relative coefficients are fixed by the requirement that $\mathcal{S}_{\mathrm{TEGR}}$ being identical to the Einstein-Hilbert action $\mathcal{S}_{\mathrm{EH}}(g_{\mu\nu}) = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}R$, up to total derivative terms. The latter can be written in terms of the vielbeins through the relation $g_{\mu\nu} = \hat{e}^i{}_{\mu}\hat{e}^j{}_{\nu}\eta_{ij}$. To see that this is indeed the case, one can use the contorsion tensor

$$K_{\mu\nu\rho} := \Omega_{\mu\nu\rho} - \Omega_{\nu\rho\mu} + \Omega_{\rho\mu\nu} \tag{7.20}$$

and rewrite (7.19) in the alternative form

$$\mathcal{S}_{\mathrm{TEGR}}(e_i{}^{\mu}, \hat{e}^i{}_{\mu}) = \frac{c^4}{16\pi G}\int d^4x\,\hat{e}\left(K_{\nu\lambda}{}^{\nu}K_{\mu}{}^{\mu\lambda} - K_{\mu\lambda\nu}K^{\nu\mu\lambda}\right)\,. \tag{7.21}$$

Then, one can use the identity $\frac{1}{2}R^{\mathrm{LC}}_{\mu\rho\sigma\nu} = K_{[\mu|\nu\lambda}K_{|\rho]\sigma}{}^{\lambda} - \nabla_{[\mu}K_{\rho]\sigma\nu}$, involving the Riemann tensor of the Levi-Cività connection and the contorsion tensor, to show that $\mathcal{S}_{\mathrm{TEGR}}$ is equal to $\mathcal{S}_{\mathrm{EH}}$ plus the total derivative term $2\nabla_{\mu}K_{\nu}{}^{\nu\mu} = 4\nabla_{\mu}\Omega^{\mu\nu}{}_{\nu}$. Therefore, the TEGR action is classically equivalent to the standard Einstein-Hilbert one.

Having said these, we are now ready to construct the nonlinear gravitational ϑ-term

$$\mathcal{S}_{\vartheta}(e_i{}^{\mu}, \hat{e}^i{}_{\mu}) = \frac{c^4}{16\pi G}\int d^4x\,\hat{e}\,\vartheta\,\epsilon^{\mu\nu\rho\sigma}K_{\mu\nu\lambda}K_{\rho\sigma}{}^{\lambda} = \frac{c^4}{16\pi G}\int d^4x\,\hat{e}\,\vartheta\,\epsilon^{\mu\nu\rho\sigma}\Omega_{\mu\nu\lambda}\Omega_{\rho\sigma}{}^{\lambda}\,. \tag{7.22}$$

[6] The determinant of the vielbein $\hat{e} := \det(\hat{e}^i{}_{\mu})$ is related to the metric determinant by $\widehat{e} = \sqrt{-g}$.

[7] Note that the reversed notation is used in [36].

Let us make some observations. First of all, the term defined above violates parity but it is not related to the fourth Weitzenböck invariant I_4. Secondly, it corresponds to a total spacetime derivative for constant ϑ. This can be seen by contracting the identity $\frac{1}{2}R^{\mathrm{LC}}_{\mu\rho\sigma\nu} = K_{[\mu|\nu\lambda}K_{|\rho]\sigma}{}^{\lambda} - \nabla_{[\mu}K_{\rho]\sigma\nu}$ with the Levi-Cività symbol $\epsilon^{\mu\rho\sigma\nu}$. The l.h.s. vanishes identically and we obtain the result

$$\epsilon^{\mu\rho\sigma\nu}K_{\mu\nu\lambda}K_{\rho\sigma}{}^{\lambda} = \epsilon^{\mu\rho\sigma\nu}\nabla_{\mu}K_{\rho\sigma\nu} = \nabla_{\mu}\left(\epsilon^{\mu\rho\sigma\nu}K_{\rho\sigma\nu}\right) . \tag{7.23}$$

For completeness, we also note that the term $\mathcal{S}_{\vartheta}$ corresponds to the Nieh-Yan topological invariant [90] for a connection that has vanishing curvature (e.g. for the Weitzenböck connection that we are using). The fact that this nonlinear term is topological provides us with a direct justification as to why its linearized version considered before is also a topological term.

Finally, it is straightforward to check that at the linearized limit[8] $\hat{e}^{i}{}_{\mu}\delta_{i\nu} \simeq \delta_{\mu\nu} + \frac{1}{2}(h_{\mu\nu} + b_{\mu\nu})$ one has $2K_{[\mu\nu]\rho} = 2\Omega_{\mu\nu\rho} \simeq \partial_{[\mu}h_{\nu]\rho} \simeq -K_{\rho\mu\nu}$. Using these identifications and the de Donder gauge condition, it becomes clear that (7.11) coincides with the linearized version of the action

$$\mathcal{S} = \mathcal{S}_G + \mathcal{S}_{\vartheta} + \mathcal{S}_M \, , \tag{7.24}$$

where $\mathcal{S}_G$ is any gravity action that can be (re-)written in terms of vielbeins (e.g. $\mathcal{S}_{\mathrm{EH}}$ or $\mathcal{S}_{\mathrm{TEGR}}$).

7.4 Two physical applications

We conclude this last Chapter by presenting two physical applications of the ϑ-term, which were originally presented in [36]. These are given in the form of simple examples, but capture the essence of the underlying effects induced by the ϑ.

Relativistic correction to Newton's law

Let us consider the example of a uniform spherical distribution of radius R and $\vartheta \neq 0$, and place a rotating massive body of mass M and angular momentum $\vec{S}$ at its center. Assuming that $\vec{S}$ points to the north and denoting the polar angle by θ, one gets the relation $\vec{S}\cdot\hat{r} = S\cos\theta$. It is also obvious that the gradient of ϑ reads as $\vec{\nabla}\vartheta(r) = -\vartheta\,\delta(R-r)\,\hat{r}$.

[8] We have denoted by $b_{\mu\nu}$ the local components of the Kalb-Ramond 2-form. This decouples from the action (7.21) due to the latter's invariance under local Lorentz transformations. In fact, the nonlinear gravitational ϑ-term (7.22) is also invariant under such transformations. However, this does not hold for its linearized version appearing in the action (7.11). Despite this fact, we will ignore the contribution of the linearized Kalb-Ramond field in the applications we are about to present.

Furthermore, we will assume that the radius R is much larger than the characteristic scale of the rotating body, ensuring that the GEM fields it generates will have their standard far-field form (see e.g. [91])

$$\vec{E}(\vec{r}) \simeq -GM\frac{\hat{r}}{r^2}, \quad \vec{B}(\vec{r}) \simeq \frac{G}{cr^3}\left[\vec{S} - 3(\vec{S}\cdot\hat{r})\,\hat{r}\right] \tag{7.25}$$

in the internal vicinity of the spherical boundary. Then the modified Gauss' law in (7.14) implies that there is an effective accumulation of mass density on the spherical shell induced by the gravitomagnetic field of the rotating body,

$$\rho_{\text{eff}}(r,\theta) = \frac{1}{8\pi G}\,\vec{\nabla}\vartheta(r)\cdot\vec{B}(r,\theta) = \frac{S\vartheta}{4\pi c}\,\frac{\delta(r-R)}{r^3}\,\cos\theta\,, \tag{7.26}$$

which in turn induces an effective gravitoelectric (Newtonian) field outside the distribution. Since the induced mass density above scales as $O(c^{-1})$, we expect that the effective gravitoelectric field it generates will correspond to a relativistic correction to Newton's law.

To calculate this correction, one needs to consider a spherical shell that is homocentric to the ϑ-distribution but has larger radius $r > R$. This will serve as the Gaussian surface. Integrating the Gauss' law (7.14) over the region enclosed by this surface, we obtain the (full) gravitoelectric field

$$\vec{E}(r,\theta) \simeq -G\left(M - \frac{S\vartheta}{cR}\cos\theta\right)\frac{\hat{r}}{r^2}\,, \qquad r > R\,. \tag{7.27}$$

The second term corresponds to the relativistic correction due to the nonvanishing ϑ parameter.

Gravitational analogue of the Witten effect

In analogy to the discussions on magnetic monopoles in Electromagnetism, one could speculate the existence of a purely gravitomagnetic source in GEM. Such kind of pointlike sources were originally considered by Zee in [92], where they were coined as *gravitipoles*. The starting point for considering these objects is to modify the second equation in (7.14) into

$$\vec{\nabla}\cdot\vec{B} = 4\pi G\,\widetilde{M}\,\delta^{(3)}(\vec{r})\,, \quad \widetilde{M} > 0 \tag{7.28}$$

where the gravitomagnetic charge $\widetilde{M}$ also has dimensions of mass. As it was argued by Zee, the existence of such purely gravitomagnetic charges would lead to mass quantization, in the same way that pointlike magnetic monopoles would lead to quantization of the electric charge [93]. Finally, it was also shown that if such a gravitipole lied in the center of the Sun then the orbits of the planets would experience a small lift.

Let us now also indulge in speculation regarding the existence of gravitipoles and examine what would be the effect of the including the ϑ-term. To this end, we can assume that $\vartheta \neq 0$ everywhere apart from a hollow sphere of radius R, where $\vartheta = 0$, and thus $\vec{\nabla}\vartheta(r) = \vartheta\,\delta(r-R)\,\hat{r}$. We will now place a gravitipole at the center of the hollow sphere, which will induce a radial gravitomagnetic field of the form

$$\vec{B}(r) = G\widetilde{M}\,\frac{\hat{r}}{r^2}\,. \tag{7.29}$$

This gravitomagnetic field falls perpendicularly to the boundary of the hollow sphere and, according to the modified Gauss' law in (7.14), it induces an effective mass density

$$\rho_{\text{eff}}(r) = \frac{1}{8\pi G}\,\vec{\nabla}\vartheta(r)\cdot\vec{B}(r) = \frac{\vartheta\widetilde{M}}{8\pi}\,\frac{\delta(r-R)}{r^2} \tag{7.30}$$

on this boundary. This will, in turn, generate an effective gravitoelectric (Newtonian) field outside the sphere. We can choose a spherical shell homocentric to the hollow sphere with radius $r > R$ as our Gaussian surface and obtain the integrate the modified Gauss' law over the region enclosed by this surface. Upon doing so, we easily find the induced Newtonian field $\vec{E}_{\text{eff}}(r) = -\frac{\vartheta}{2}\vec{B}(r)$.

This is an interesting result on its own, but we can push forward by observing that neither $\vec{E}_{\text{eff}}$ nor $\vec{B}$ depend on the radius R of the hollow sphere. This is a crucial observation, since it allows us to shrink this sphere until it becomes a point by taking the limit $R \to 0$. Then, all of space has $\vartheta \neq 0$ and we observe that the gravitipole acquires mass. This is the gravitational analogue of the Witten effect for magnetic monopoles [94], which indicates that a gravitipole becomes a gravitational dyon when placed inside a region of space with $\vartheta \neq 0$. The mass of the dyon is related to its gravitomagnetic charge by $M = \frac{\vartheta}{2}\widetilde{M}$.

Chapter 8

CONCLUSIONS

One of our main results of this thesis was the development of a mathematical formalism [47–49], based on which one can view mixed-symmetry tensors on a smooth manifold M as functions of arbitrary degree on the graded supermanifold $\mathcal{M}_N = T[1]M \oplus \cdots \oplus T[1]M$. Within this framework and taking M to be the D-dimensional Minkowski spacetime, we were able to unify all kinetic, mass and Galileon interactions for arbitrary such gauge fields into simple, geometric expressions.

An obvious question is whether such a graded geometric formalism can be constructed to describe gauge theories in curved spacetimes. A first step towards this direction was already made in [47], with the aim of covariantizing the mixed-symmetry tensor Galileon actions.

A second observation stems from our choice of graded manifold. Although sufficient for our purposes, one could in principle choose a different manifold to begin with. For example, for theories involving mixed multivector and multiform fields one could alternatively use the graded manifold $T[1]M \oplus T^*[1]M$, or generalizations of it. A different manifold would potentially enhance the allowed field content of the theory under consideration, due to the inclusion of additional coordinates of different degrees.

Besides these mathematical contributions, we reviewed the Hodge duality in a wide variety of gauge theories, as well as in linearized Gravity. We saw that the graded formalism becomes very useful, as it allows one to construct a universal 2-parameter parent Lagrangian [48]. This encodes all different types of Hodge duality between bipartite gauge fields of any type, in general number of dimensions. One could possibly extend this parent Lagrangian into an N-parameter one, with the goal of accommodating also gauge fields of higher spin. This would either require a graded geometric translation of the general frame-like construction of [65], or a "metric-like" action (as the one we already presented) constructed independently by use of the graded formalism.

In the special case of Maxwell's theory, we also carried out the exotic dualization procedure [25]. Besides the mathematical formulation, one of the novelties was that we included the electromagnetic

ϑ-term. More precisely, we showed that this is sufficient in order to promote exotic dual theory, from an equivalent to a fully-fledged independent description of Maxwell's electrodynamics. Similar results were obtained for the double dual linearized graviton, where the original discussion was presented in [37]. Motivated by these, a rigorous and more systematic study on the effects of topological terms in exotic and double dualities is an imperative need.

In addition, we considered multiple dualizations in theories containing multiple gauge fields of the same type. Our analyses, lead to identifying the background field transformation rules under these dualities, which we coined "higher Buscher rules" [37]. In the case of scalar fields (relevant in String Theory), these are the T-duality rules inverting the radii of the target space manifold. An interesting observation is that, for the more general cases of Abelian p-forms and $(p, 1)$ mixed-symmetry tensor gauge fields, this target space must be a graded manifold. These graded target spaces, as well as the related nonlinear sigma models, deserve to be studied in detail.

Finally, we reviewed in detail the gravitational theta term discussed in [36] and we studied some of its physical applications within the framework of Gravitoelectromagnetism (GEM). In particular, both applications we presented are related to a modification of the gravitational Gauss' law; this predicts an effective mass density induced by the gravitomagnetic field, in the presence of nonvanishing ϑ-parameter. We mentioned, however, that there exist two additional effects in the context of ϑ-GEM. These are related to the effective mass currents induced by gravitoelectric (resp. gravitomagnetic) fields in regions of space where the derivatives of ϑ are nonvanishing (resp. when ϑ is time-dependent or promoted to an axion field). Effects related to these currents should also be studied more thoroughly.

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Appendix A

SOME DEFINITIONS, IDENTITIES AND PROOFS

A.1 Hodge relations between the operators

The Hodge star operators $*_A$ are defined in the standard way. For $N = 2$, there exist two such operators and their action on the local expansion of a bipartite function ω with degree (p_1, p_2) reads as

$$\begin{aligned} *_1 \omega &= \frac{1}{p_1!(D-p_1)!p_2!}\, \varepsilon^{\mu_1\dots\mu_{p_1}}{}_{\mu_{p_1+1}\dots\mu_D} \omega_{\mu_1\dots\mu_{p_1}\nu_1\dots\nu_{p_2}} \theta_1^{\mu_{p_1+1}} \dots \theta_1^{\mu_D} \theta_2^{\nu_1} \dots \theta_2^{\nu_{p_2}} \\ *_2 \omega &= \frac{1}{p_1!(D-p_2)!p_2!}\, \varepsilon^{\nu_1\dots\nu_{p_2}}{}_{\nu_{p_2+1}\dots\nu_D} \omega_{\mu_1\dots\mu_{p_1}\nu_1\dots\nu_{p_2}} \theta_1^{\mu_1} \dots \theta_1^{\mu_{p_1}} \theta_2^{\nu_{p_2+1}} \dots \theta_2^{\nu_D} \,. \end{aligned} \tag{A.1}$$

The quantity $\varepsilon_{\mu_1\dots\mu_D}$ is the Levi-Civita tensor of the Minkowski spacetime satisfying

$$\varepsilon^{\mu_1\dots\mu_p\kappa_1\dots\kappa_{D-p}} \varepsilon_{\nu_1\dots\nu_p\kappa_1\dots\kappa_{D-p}} = -p!(D-p)!\, \delta^{[\mu_1}_{\nu_1} \dots \delta^{\mu_p]}_{\nu_p} \,, \tag{A.2}$$

where square brackets denote antisymmetrization of weight one. Due to the above definitions, it is easy to check that

$$*_1 *_2 \omega = *_2 *_1 \omega \quad \& \quad (*_1)^2 \omega = (-1)^{1+p_1(D-p_1)} \omega \quad \& \quad (*_2)^2 \omega = (-1)^{1+p_2(D-p_2)} \omega \tag{A.3}$$

for any function ω of degree (p_1, p_2). In addition, the definitions (2.25) and (2.24) imply that the σ operators are related to η and tr through [37, 44]

$$\begin{aligned} \sigma_{12}\omega &= (-1)^{1+D(p_1+1)} *_1 \mathrm{tr}_{12} *_1 \omega = (-1)^{p_2(D-p_2)} *_2 \eta_{12} *_2 \omega \,, \\ \sigma_{21}\omega &= (-1)^{1+D(p_2+1)} *_2 \mathrm{tr}_{12} *_2 \omega = (-1)^{p_1(D-p_1)} *_1 \eta_{12} *_1 \omega \,. \end{aligned} \tag{A.4}$$

Finally, one can use these and the (anti)commutation relations (2.68) and (2.69) to prove the additional identities [37]

$$\begin{aligned} *_1 \sigma_{21}\sigma_{12} *_1 \omega &= (-1)^{1+p_1(D-p_1)} \eta_{12} \mathrm{tr}_{12} \omega \,, \\ *_2 \sigma_{12}\sigma_{21} *_2 \omega &= (-1)^{1+p_2(D-p_2)} \eta_{12} \mathrm{tr}_{12} \omega \,. \end{aligned} \tag{A.5}$$

In addition, the codifferential operators defined in (2.37) are related to the exterior derivatives as

$$\begin{aligned} \mathrm{d}_1^\dagger \omega &= (-1)^{1+D(p_1+1)} *_1 \mathrm{d}_1 *_1 \omega \,, \\ \mathrm{d}_2^\dagger \omega &= (-1)^{1+D(p_2+1)} *_2 \mathrm{d}_2 *_2 \omega \,. \end{aligned} \tag{A.6}$$

As a final note, let us mention that the Hodge operators (A.1) can be defined in a purely graded geometric way, i.e. without reference to their action on the local components of a function. This can be done using Berezin integration over an auxiliary coordinate, namely [48]

$$\begin{aligned} *_1\,\omega &:= \frac{1}{(D-p_1)!}\int_{\theta_3}\omega^{\top_{13}}\eta_{13}^{D-p_1}\,, \\ *_2\,\omega &:= \frac{1}{(D-p_2)!}\int_{\theta_3}\omega^{\top_{23}}\eta_{23}^{D-p_2}\,, \end{aligned} \tag{A.7}$$

for the case of $N = 2$.

A.2 Identities involving the Berezin integral

In this Appendix, we have collected a number of integral identities that will be useful for the analyses of this thesis. We will restrict to the $N = 2$ case of bipartite functions.

Assuming that ω and ξ are bipartite functions of respective degrees (p_1, p_2) and $(D - p_1, D - p_2)$, one such identity reads as

$$\int_{\theta_1,\theta_2}\omega\,\xi = -\int_{\theta_1,\theta_2} *_1\omega *_1 \xi = -\int_{\theta_1,\theta_2} *_2\,\omega *_2\,\xi\,, \tag{A.8}$$

which can be easily proven using the definitions (A.7). A second identity we will use appears in [76] and reads as

$$\int_{\theta_1,\theta_2}\omega *_1 *_2\,\eta_{12}\,\zeta = \int_{\theta_1,\theta_2}\mathrm{tr}_{12}\,\omega *_1 *_2\,\zeta\,, \tag{A.9}$$

in terms of an arbitrary bipartite function ζ of degree $(p_1 - 1, p_2 - 1)$. In addition, we will make use of the following identity [76]

$$\int_{\theta_1,\theta_2}\mathrm{d}_1\omega *_1 *_2\,\kappa = -\int_{\theta_1,\theta_2}\omega *_1 *_2\,\mathrm{d}_1^{\dagger}\kappa \tag{A.10}$$

involving the differential operators and a bipartite function κ of degree $(p_1 + 1, p_2)$. An equivalent identity involving the operators d_2 and $\mathrm{d}_2^{\dagger}$ also holds. In fact, these follow directly from (A.6) and the fact that the exterior derivatives satisfy the graded Leibniz rule of the form $\mathrm{d}_1(\omega *_1 *_2\,\kappa) = \mathrm{d}_1\omega *_1 *_2\,\kappa + (-1)^{p_1}\omega\,\mathrm{d}_1 *_1 *_2\,\kappa$.

More involved integral identities can also be proven. For example, one can show that [48]

$$\begin{aligned} \int_{\theta_1,\theta_2}\sigma_{21}^{n}\sigma_{12}^{n} *_1\,\omega *_2\,\omega' &= (-1)^{p_1(D-p_1)}\int_{\theta_1,\theta_2}\eta_{12}^{n}\,\mathrm{tr}_{12}^{n}\,\omega *_1 *_2\,\omega' \\ \int_{\theta_1,\theta_2}\sigma_{12}^{n}\sigma_{21}^{n} *_2\,\omega *_1\,\omega' &= (-1)^{p_2(D-p_2)}\int_{\theta_1,\theta_2}\eta_{12}^{n}\,\mathrm{tr}_{12}^{n}\,\omega *_1 *_2\,\omega' \end{aligned} \tag{A.11}$$

also hold for any n, in terms of an arbitrary bipartite function ω' of degree (p_1, p_2). We will now prove the first of these identities for $n = 1$; the other cases, as well as the second identity, can be proven in exactly the same manner. Using the definitions of the maps appearing in the left hand side, we find

$$\begin{aligned}
\int_{\theta_1,\theta_2} \sigma_{21}\sigma_{12} *_1 \omega *_2 \omega' &= (-1)^{p_2 D + p_1 + 1} \int_{\theta_1,\theta_2} *_2(\mathrm{tr}_{12} *_1 *_2 \mathrm{tr}_{12}\, \omega) \; *_2 \omega' \\
&\overset{(A.8)}{=} (-1)^{p_2 D + p_1} \int_{\theta_1,\theta_2} \mathrm{tr}_{12}(*_1 *_2 \mathrm{tr}_{12}\, \omega)\, \omega' \\
&= (-1)^{p_1 D + p_2} \int_{\theta_1,\theta_2} \mathrm{tr}_{12}(*_1 *_2 \mathrm{tr}_{12}\, \omega) *_1 *_2 (*_1 *_2 \omega') \\
&\overset{(A.9)}{=} (-1)^{p_1 D + p_2} \int_{\theta_1,\theta_2} *_1 *_2 \mathrm{tr}_{12}\, \omega *_1 *_2 (\eta_{12} *_1 *_2 \omega') \\
&\overset{(A.8)}{=} (-1)^{p_1 D + p_2} \int_{\theta_1,\theta_2} \mathrm{tr}_{12}\, \omega\, \eta_{12} *_1 *_2 \omega' \\
&= (-1)^{p_1(D - p_1)} \int_{\theta_1,\theta_2} \eta_{12}\, \mathrm{tr}_{12}\, \omega *_1 *_2 \omega' \,,
\end{aligned} \tag{A.12}$$

which is indeed the first identity in (A.9) for $n = 1$. In fact, one can use this identity and show that the following important relation [37]

$$\int_{\theta_1,\theta_2} (\mathbb{I} - \sigma_{21}\sigma_{12}) *_1 \omega *_2 \omega' = (-1)^l \int_{\theta_1,\theta_2} \omega \star_{12} \omega', \qquad l = p_2(D - p_2) + p_1 p_2 + 1 \tag{A.13}$$

holds when $\min(p_1, p_2) = 1$. Note that the operator $\mathbb{I} - \sigma_{21}\sigma_{12}$ on the l.h.s. can be readily identified with the inverse of the algebraic operator $O^{(p+1,1)}$ (e.g., see equation (5.27)) used in the dualization procedures of Chapters 4 and 5.

Let us now proceed by showing that the bilinear map defined in (2.51) is symmetric. We will do this for $k = N = 2$ since generalization to arbitrary k is straightforward. The map between two bipartite functions ω and ω' of degree (p_1, p_2) is a scalar function and, thus, invariant under the transposition operator $\top_{12}$. We directly compute

$$\begin{aligned}
\int_{\theta_1,\theta_2} \omega \star_{12} \omega' \quad &= \quad \left(\int_{\theta_1,\theta_2} \omega \star_{12} \omega'\right)^{\top_{12}} = \int_{\theta_1,\theta_2} \omega^{\top_{12}} (\star_{12}\omega')^{\top_{12}} \\
&\overset{\eta^\top=\eta}{=} \quad \int_{\theta_1,\theta_2} \omega^{\top_{12}} \frac{1}{(D-p_1-p_2)!}\, \eta_{12}^{D-p_1-p_2} (\omega'^{\top_{12}})^{\top_{12}} \\
&\overset{(\omega'^\top)^\top=\omega'}{=} \quad \int_{\theta_1,\theta_2} \omega' \frac{1}{(D-p_1-p_2)!}\, \eta_{12}^{D-p_1-p_2} \omega^{\top_{12}} \\
&= \quad \int_{\theta_1,\theta_2} \omega' \star_{12} \omega\,,
\end{aligned} \tag{A.14}$$

which proves that $(\omega, \omega') = (\omega', \omega)$.

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