Renormalization and Observables in the Standard Model and Beyond in Higher Orders of Perturbation Theory

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Renormalizacija i opservable u standardnom modelu i izvan njega u višim redovima računa smetnje

DOKTORSKI RAD

Mentor:
prof. dr. sc. Amon Ilakovac

Zagreb, 2022.

Best Witchcraft is Geometry
To the magician's mind —
His ordinary acts are feats
To thinking of mankind.

Emily Dickinson

I would like to express my special appreciation and thanks to my advisor Amon Ilakovac and my collaborators.

ABSTRACT

We study the application of the Breitenlohner–Maison–'t Hooft–Veltman (BMHV) scheme of Dimensional Regularization to the renormalization of massless chiral gauge theories; chiral Yang-Mills theory, and chiral Quantum Electrodynamics, being the main ingredients of the Standard Model, up to one and two-loop level, respectively. We focus on the counterterm structure specific to the BHMV scheme induced by the non-anticommuting Dirac γ_5 matrix and the breaking of the Becchi-Rouet-Stora-Tyutin (BRST) invariance. We find the singular counterterms and BRST symmetry restoring counterterms needed for the complete renormalization of the models. This study is based on the symmetry requirements coming from the Algebraic Renormalization and regularized Quantum Action Principle. We find the Renormalization Group Equations for the Yang-Mills model at the first order within this framework and compare the procedure with standard symmetry-invariant regularizations. For chiral Quantum Electrodynamics, we determine the full structure of symmetry-restoring counterterms up to the 2-loop level and find no discrepancy from the 1-loop level structures. The counterterms have interpretation in terms of restoration of Ward identities that play the role of the benchmark for this 2-loop study. Hence we prove the correctness of results within this framework, what is the property of a self-consistent scheme. We impose future proposals for the application of the BMHV scheme to phenomenological studies.

Keywords: Anomalies in Field Theories, BRST Symmetry, Gauge Symmetry, Renormalization, Perturbation Theory, Algebraic Renormalization, BMHV scheme, Chiral Theory, Yang-Mills Model, Quantum Electrodynamics, Slavnov-Taylor identities, Ward identities, Standard Model

PROŠIRENI SAŽETAK

Standardni model fizike elementarnih čestica toliko uspješno[1] objašnjava strukturu materije da je već gotovo desetljećima duboko u tzv. eri preciznosti. Mjerljive opservable precizno se procjenjuju na temelju obrade sada već otprilike dvjesto petabajta dostupnih podataka iz velikih svjetskih sudarivača čestica. Preciznost koja se stalno povećava zahtijeva teorijski dobro postavljene proračune mjerljivih opservabli u visokim redovima računa smetnje. Teorijski izračuni ovih preciznih opservabli u velikoj su mjeri provedeni u dimenzionalnoj regularizaciji (DReg), regularizacijskoj shemi koja je dovela do revolucije u proračunima na visokim redovima. DReg čuva simetrije vektorskih teorija, uključujući Lorentzovu kovarijantnost, baždarnu i BRST invarijantnost i translacijske simetrije. Kada se primjeni na vektorski model, ova shema divergencije proizašle iz računa dijagrama s petljama izolira kao polove koje je moguće otkloniti uvođenjem lokalnih kontračlanova. Zbog svoje teorijske konzistentnosti, praktičnosti i korisnih svojstava dosegla je široku upotrebu u računima i implementirana u računalne kodove koji djelomično automatiziraju račune u višim redovima.

Nažalost, kada dimenzionalnu regularizaciju primjenimo na klasu tzv. kiralnih teorija, koje posjeduju asimetriju interakcije lijevih-kiralnih i desnih-kiralnih fermiona s bozonima, javljaju se problemi. Naime, kiralne teorije sadrže matematičke objekte koji postoje strogo u 4 dimenzije i njihova definicija ne može se proširiti u d-dimenzija što je jedan od zahtijeva dimenzionalne regularizacije. Primjeri takvih objekata su matrica γ_5 i Levi-Civita tenzor $\epsilon_{\mu\nu\rho\sigma}$. Klasu kiralnih teorija na žalost ne možemo gurnuti pod tepih, budući da je sam standardni model kiralna teorija u elektroslabom sektoru, budući da nabijeni bozoni međudjeluju samo s lijevim fermionima. Drugim riječima, lijevi lepton je SU(2) dublet, a desni lepton je singlet. Ovaj problem može se riješiti na nekoliko načina. Sljedeća tri četverodimenzionalna svojstva

$$\{\gamma^5, \gamma^\mu\} = 0, \quad \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4i\epsilon^{\mu\nu\rho\sigma}, \quad \mathrm{Tr}(ab) = \mathrm{Tr}(ba),$$

[1]Ipak, ne i savršeno. Nedostaje npr. tamna materija, bariogeneza i neutrinske mase s fenomenološkog aspekta, i gravitacija s teorijskog.

ne mogu biti istovremeno ispunjena u d dimenzija bez da se ne uvedu algebarske nekonzistentnosti ili uvede nova algebra. Napuštanje bilo kojeg od ovih svojstava vodi na različite renormalizacijske sheme. Jedina potpuno matematički konzistentna među njima je Breitenlohner-Maison-'t Hooft-Veltmanova shema koja je dobro uspostavljena za sve redove računa smetnje. Računi u ovoj shemi dovode do rezultata koji su samokonzistentni, tj. njihova je provjera moguća unutar same sheme, što kod drugih izbora za dimenzijske regularizacijske sheme nije slučaj. Zašto onda svi ne računaju u BMHV shemi? Razlog leži u percipiranoj nepraktičnosti ove sheme, koja na žalost u međukoracima ruši baždarnu i BRST simetriju koja se mora vratiti u sustav izračunom i uvođenjem tzv. konačnih kontračlanova. Ova shema nema ni privilegiju multiplikativne renormalizabilnosti, što znači da skup operatora polja koji postoji na drvastom nivou nije naslijeđen u višim redovima računa smetnje, nego se u sustavu javljaju novi operatori. Oni uzrokuju različite vrste anomalija koje je potrebno otkloniti u fizikalnom limesu razmatranih modela. Ipak, potpuno konzistentni proračuni u standardnom modelu trebali bi se temeljiti na konzistentnim shemama koje daju rezultate provjerljive unutar same sheme i koji leže na fundamentalnim simetrijama koje teorija posjeduje, što u konačnici povećava preciznost opservabli. Ova doktorska teza opisuje BMHV renormalizaciju dva opća baždarna fundamentalna modela, do drugog reda računa smetnje, što je prva takva sveobuhvatna studija kiralnog modela u području. Zbog inicijalne ideje da se renormalizacija provodi nad cijelim modelom, u ovoj tezi nastoje se iznijeti svi važni detalji potrebni za izračune. Pokazali smo da se modeli daju renormalizirati konačnim lokalnim kontračlanovima i da su teorije sigurne od anomalija.
U BMHV shemi ekstenzija u d dimenzija oblika $d = 4 - 2\epsilon$ provodi se tako da se d-dimenzionalni prostor razdvaja na direktan zbroj 4-dimenzionalnog i ϵ-dimenzionalnog prostora sa pripadajućim metrikama

$$d\text{-dim.}: g_{\mu\nu}\,, \qquad 4\text{-dim.}: \bar{g}_{\mu\nu}\,, \qquad (-2\epsilon)\text{-dim.}: \ \hat{g}_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}\,,$$

i pri čemu γ_5 više ne antikomutira sa γ^μ matricama nego slijedi pravilo

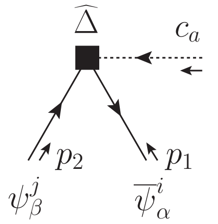
$$\{\gamma_5, \gamma^\mu\} = \{\gamma_5, \hat{\gamma}^\mu\} = 2\gamma_5\hat{\gamma}^\mu.$$

Za naše izračune neophodan je regularizirani princip kvantne akcije. Narušenje BRST simetrije odgovara narušenju Slavnov-Taylorove jednadžbe, a regularizirani princip kvantne akcije garantira nam da se ta narušenja mogu prikazati kao lokalno umetanje operatora u tzv. efektivnu akciju. Efektivna akcija u sebi sadrži sve relevantne jednočestične ireducibilne dijagrame potrebne za renormalizaciju. To znači da konačne kontračlanove koji vraćaju BRST simetriju u sustav ne moramo tražiti tako da prvo izračunamo sve konačne dijelove amplituda i onda tražimo njihove BRST transformacije (što je, subjektivno gledano, noćna mora za onu osobu koja te račune mora provesti), nego je dovoljno naći

dijagrame sa spomenutim umetanjima i iz njih rekonstruirati kontračlanove koji će vratiti simetriju u sustav. Napomenimo još da uporaba BMHV sheme proizvodi lažne ili irelevantne anomalije, koji se tako zovu jer ih je moguće otkloniti kontračlanovima. Ako se jave anomalije kod kojih to nije moguće, što ćemo kasnije vidjeti, riječ je o esencijalnim ili stvarnim anomalijama, koje se mogu otkloniti samo posebnim uvjetima koji obično daju restrikcije nad grupnim strukturama ili nameću fermionski sadržaj modelu.
BMHV schemu prvo primjenjujemo na opći kiralni Yang-Millsov model sa bezmasenim skalarima. Naravno, u 4 dimenzije model ima svojstvo BRST invarijantnosti. Kada ga poopćavamo na d dimenzija, prvo primjećujemo da se 4-dimenzionalni identični objekti

$$\overline{\psi}_i \gamma^\mu \mathbb{P}_\mathrm{R} \psi_j \,, \qquad \overline{\psi}_i \mathbb{P}_\mathrm{L} \gamma^\mu \psi_j \,, \qquad \overline{\psi}_i \mathbb{P}_\mathrm{L} \gamma^\mu \mathbb{P}_\mathrm{R} \psi_j \,.$$

u poopćenju međusobno razlikuju (pri čemu biramo onaj koji daje najjednostavnije kontračlanove). Drugo, mnogo važnije je da uvođenje lijevo-kiralnih fermiona u kinetički član lagranžijana potreban da bi imali regularizirane propagatore proizvodi narušenje BRST simetrije na drvastom nivou sa Feynmanovim pravilom



$$= \frac{g}{2} T_{R\,ij}^{\,a} \left((\widehat{\not{p}_1} + \widehat{\not{p}_2}) + (\widehat{\not{p}_1} - \widehat{\not{p}_2})\gamma_5 \right)_{\alpha\beta}$$
$$= g T_{R\,ij}^{\,a} \left(\widehat{\not{p}_1} \mathbb{P}_\mathrm{R} + \widehat{\not{p}_2} \mathbb{P}_\mathrm{L} \right)_{\alpha\beta} \,.$$

Standardna multiplikativna renormalizacija npr. vektorskih teorija rezultira u singularnim (divergentnim) i konačnim kontračlanovima, koji se u višim redovima računa smetnje javljaju kao redom divergentni i konačni predfaktori operatora akcije koji formiraju bazu već na drvastom nivou. U BMHV shemi, osim ovih simetrijski invarijantnih kontračlanova, javljaju se neinvarijantni kontračlanovi, koji isto mogu biti i divergentni i konačni, ali i evanescentni, što znači da su njihovi pripadni renormalizacijski faktori pomnoženi novim operatorima akcije koji iščezavaju u četverodimenzionalnom limesu. Svi divergentni kontračlanovi pronađeni su za Yang-Millsov model na nivou jedne petlje. Osim standardnih, pojavili su se i evanescentni kontračlanovi. U idućem koraku potrebno je pronaći konačne kontračlanove koji slomljenu BRST simetriju vraćaju u model. Zahvaljujući regulariziranom principu kvantne akcije, to smo napravili preko izračuna svih dijagrama na jednoj petlji koji imaju umetnut vrh koji se na drvastom nivou javlja za BRST narušenje. Ti dijagrami svojim singularnim dijelom poništavaju singularne evanescentne dijelove akcije koji ruše BRST simetriju, a njihov konačni dio služi za rekonstrukciju simetrijskih kontračlanova. Jednom kada su svi kontračlanovi definirani, teorija je renormalizirana. Skup kontračlanova koji vraćaju BRST simetriju u Yang-Millsov model na nivou jedne petlje u četiri dimenzije dan je sa

$$S^{(1)}_{\text{fct,restore}} = \frac{\hbar}{16\pi^2}\Bigg\{ g^2\frac{S_2(R)}{6}\left(5S_{GG} + S_{GGG} - \int \mathrm{d}^4x\, G^{a\,\mu}\partial^2 G^a_\mu\right) + \frac{Y_2(S)}{3}S_{\Phi\Phi}$$
$$+g^2\frac{(T_R)^{abcd}}{3}\int \mathrm{d}^4x\,\frac{g^2}{4}G^a_\mu G^{b\,\mu}G^c_\nu G^{d\,\nu} - \frac{(\mathcal{C}_R)^{ab}_{mn}}{3}\int \mathrm{d}^4x\,\frac{g^2}{2}G^a_\mu G^{b\,\mu}\Phi^m\Phi^n$$
$$+g^2\left(1+\frac{\xi-1}{6}\right)C_2(R)S_{\overline{\psi}\psi} - \frac{((Y^m_R)^* T_{\overline{R}}{}^a Y^m_R)_{ij}}{2}\int \mathrm{d}^4x\, g\overline{\psi}_i \not{G}^a \mathbb{P}_{\mathrm{R}}\psi_j$$
$$-g^2\frac{\xi C_2(G)}{4}(S_{\bar{R}c\psi_R} + S_{Rc\overline{\psi_R}})\Bigg\}\,,$$

gdje je $(\mathcal{C}_R)^{ab}_{mn} \equiv \mathrm{Tr}\left[2\{T_R{}^a, T_R{}^b\}(Y^m_R)^* Y^n_R - T_R{}^a (Y^m_R)^* T_{\overline{R}}{}^b Y^n_R\right]$. Izbor ovih kontračlanova nije jedinstven, budući da dodavanje BRST invarijantnih konačnih kontračlanova ne narušava ovu ponovno uspostavljenu simetriju. Esencijalne anomalije koje su također pronađene u dijagramima s umetanjem otklonjene su odgovarajućim uvjetima. Za isti model smo dodatno izračunali renormalizacijske grupne jednadžbe na nivou jedne petlje preko BMHV procedure i pokazali da se rezultati poklapaju s onima koji bi proizišli iz naivne sheme. Ipak, postoje naznake da je ovo svojstvo posljedica drastičnih pojednostavljenja na nivou jedne petlje. Konačno, razmotrili smo ekvivalentan model s lijevo-kiralnim fermionima, što je dio koji nam je potreban za daljnja fenomenološka istraživanja, i to prvenstveno u standardnom modelu.

U fazi istraživanja kada je Yang-Millsov model potpuno renormaliziran na nivou jedne petlje, vlastito teoretsko i praktično iskustvo s BMHV shemom dovelo nas je u fazu da smo bili spremni prijeći na drugi red računa smetnje. Tu je u velikoj mjeri ulogu odigrala automatizacija - računalne kodove i algoritme uspjeli smo u velikoj mjeri prilagoditi BMHV shemi. Idući model, koji se nameće kao neophodan za renormalizaciju standardnog modela, jest kiralna elektrodinamika χQED, čiji je lagranžijan zadan sa

$$\mathcal{L} = \imath\overline{\psi_R}_i \not{D}_{ij}\psi_{Rj} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \bar{c}\partial^2 c + \rho^\mu s A_\mu + \bar{R}^i s\psi_{Ri} + R^i s\overline{\psi_{R}}_i,$$

a nužan i dovoljan uvjet za poništenje esencijalnih anomalija dan je sa

$$\mathrm{Tr}(\mathcal{Y}^3_R) = 0\,.$$

Uporabom BMHV sheme BRST simetrija slomljena je već na drvastom nivou, analogno slučaju u Yang-Millsovom modelu. Pronalazimo da uvjet renormalizabilnosti proizašao iz Slavnov-Taylorovog identiteta za posljedicu daje Wardove identitete koji diktiraju odnose među Greenovim funkcijama i da sljedeće mora biti zadovoljeno:

1. Transverzalnost vlastite energije fotona,

$$ip_\nu \frac{\delta^2 \widetilde{\Gamma}_{\text{ren}}}{\delta A_\mu(p)\delta A_\nu(-p)} = 0\,;$$

2. Transverzalnost višefotonskih vrhova, u ovom slučaju vrha sa četiri fotona,

$$i(p_1+p_2+p_3)_\sigma \frac{\delta^4 \widetilde{\Gamma}_{\text{ren}}}{\delta A_\rho(p_3)\delta A_\nu(p_2)\delta A_\mu(p_1)\delta A_\sigma(-p_1-p_2-p_3)} = 0\,;$$

3. Veza fermionske vlastite energije i fermion-foton interakcije za iščezavajući fotonski impuls $q=0$,

$$-ie\mathcal{Y}_R \frac{\partial}{\partial p_\mu}\frac{\delta^2 \widetilde{\Gamma}_{\text{ren}}}{\delta\overline{\psi}(-p)\delta\psi(p)} + i\,\frac{\delta^3 \widetilde{\Gamma}_{\text{ren}}}{\delta A_\mu(0)\delta\overline{\psi}(-p)\delta\psi(p)} = 0\,.$$

Za kvantnu elektrodinamiku, Wardovi identitieti u praktičnom su smislu zlatni standard provjere točnosti kontračlanova koje ćemo izračunati na nivou dvije petlje. Analizirajući renormalizaciju u višim redovima smetnje preko algebarskih metoda, dolazimo do renormalizacijskog zahtjeva koji mora vrijediti za sve redove računa smetnje,

$$\text{LIM}_{d\to 4}\left(\widehat{\Delta}\cdot\Gamma^i_{\text{DReg}} + \sum_{k=1}^{i-1}\Delta^k_{\text{ct}}\cdot\Gamma^{i-k}_{\text{DReg}} + \Delta^i_{\text{ct}}\right) = 0\,.$$

Izračunata je singularna kontračlanska akcija na nivou jedne petlje,

$$S^1_{\text{sct}} = \frac{-\hbar\, e^2}{16\pi^2\epsilon}\left(\frac{2\text{Tr}(\mathcal{Y}_R^2)}{3}\overline{S_{AA}} + \xi\sum_j(\mathcal{Y}_R^j)^2\left(\overline{S^j_{\overline{\psi}\psi_R}} + \overline{S^j_{\overline{\psi_R}A\psi_R}}\right)\right.$$
$$\left. + \frac{\text{Tr}(\mathcal{Y}_R^2)}{3}\int \mathrm{d}^d\, x\ \frac{1}{2}\bar{A}_\mu\widehat{\partial}^2\bar{A}^\mu\right)\,,$$

i konačna akcija koja na istom nivou vraća BRST simetriju u sustav,

$$S^1_{\text{fct}} = \frac{\hbar}{16\pi^2}\int \mathrm{d}^4\, x\left\{\frac{-e^2\text{Tr}(\mathcal{Y}_R^2)}{6}\bar{A}_\mu\bar{\partial}^2\bar{A}^\mu + \frac{e^4\text{Tr}(\mathcal{Y}_R^4)}{12}\bar{A}_\mu\bar{A}^\mu\bar{A}_\nu\bar{A}^\nu\right.$$
$$\left. + \frac{5+\xi}{6}e^2\sum_j(\mathcal{Y}_R^j)^2 i\overline{\psi}_j\overline{\gamma}^\mu\overline{\partial}_\mu\mathbb{P}_{\text{R}}\psi_j\right\}\,,$$

koja je konstruirana iz dijagrama sa umetanjima, koji proizlaze iz regulariziranog principa kvantne akcije. Analogno, ovaj put uz mnogo bogatiju strukturu i veći broj dijagrama, kao i implementaciju novih tehnika u račun, izračunata je singularna akcija na nivou dvije petlje,

$$
\begin{aligned}
S^2_{\text{sct}} = &-\left(\frac{\hbar\, e^2}{16\pi^2}\right)^2 \frac{\mathrm{Tr}(\mathcal{Y}_R^4)}{3}\left[\frac{2}{\epsilon}\overline{S_{AA}} + \left(\frac{1}{4\epsilon^2} - \frac{17}{48\epsilon}\right)\int \mathrm{d}^d\, x\, \overline{A}_\mu \widehat{\partial}^2 \overline{A}^\mu\right] \\
&+\left(\frac{\hbar\, e^2}{16\pi^2}\right)^2 \sum_j (\mathcal{Y}_R^j)^2 \left[\left(\frac{1}{2\epsilon^2} + \frac{17}{12\epsilon}\right)(\mathcal{Y}_R^j)^2 - \frac{1}{9\epsilon}\mathrm{Tr}(\mathcal{Y}_R^2)\right]\left(\overline{S^j_{\overline{\psi}\psi_R}} + \overline{S^j_{\overline{\psi_R}A\psi_R}}\right) \\
&-\left(\frac{\hbar\, e^2}{16\pi^2}\right)^2 \sum_j \frac{(\mathcal{Y}_R^j)^2}{3\epsilon}\left(\frac{5}{2}(\mathcal{Y}_R^j)^2 - \frac{2}{3}\mathrm{Tr}(\mathcal{Y}_R^2)\right)\overline{S^j_{\overline{\psi}\psi_R}}\,.
\end{aligned}
$$

Njezin evaescentni dio prirodno će se poništiti kada iz dijagrama s umetanjima rekonstruiramo divergentni evanescentni dio, što daje dodatnu provjeru na naše račune. Konačno pronalazimo i konačnu kontračlansku akciju na nivou dvije petlje koja vraća BRST simetriju i time zatvara renormalizaciju teorije u drugom redu. Unatoč velikom broju dijagrama koji su komplicirani na individualnom nivou, konačne sume i rekonstrukcija daju relativno jednostavan izraz,

$$
\begin{aligned}
S^2_{\text{fct}} = &\left(\frac{\hbar}{16\pi^2}\right)^2 \int \mathrm{d}^4\, x\; e^4 \Bigg\{ \mathrm{Tr}(\mathcal{Y}_R^4)\frac{11}{48}\bar{A}_\mu \bar{\partial}^2 \bar{A}^\mu + 3\, e^2 \frac{\mathrm{Tr}(\mathcal{Y}_R^6)}{8}\,\bar{A}_\mu \bar{A}^\mu \bar{A}_\nu \bar{A}^\nu \\
&- \sum_j (\mathcal{Y}_R^j)^2 \left(\frac{127}{36}(\mathcal{Y}_R^j)^2 - \frac{1}{27}\mathrm{Tr}(\mathcal{Y}_R^2)\right)\left(\overline{\psi}_j i \bar{\partial\!\!\!/}\, \mathbb{P}_{\mathrm{R}}\, \psi_j\right)\Bigg\} \\
&+\ \text{BRST-simetrični članovi}\,,
\end{aligned}
$$

čime vidimo da se struktura kontračlanova na razini dvije petlje nije promijenila u odnosu na prvi red računa smetnje. Rezultati su potvrđeni provjerom Wardovih identiteta.
U idućim fazama istraživanja planirano je BMHV shemu primijeniti na Yang-Millsov model na razini dvije petlje, na opći U(1) model i SU(2) model s primjenom na standardni model do drugog reda računa smetnje, uključujući i renormalizacijske grupne jednadžbe. To nas dovodi do rezultata koji su konzistentni i nema prostora za narušenja simetrije, anomalije ili nekonzistentnosti proizišle iz odabira renormalizacijske sheme, s konačnim ciljem primjene i upotrebe sheme na široki spektar fenomenoloških istraživanja.

Ključne riječi: anomalije u teoriji polja, BRST simetrija, baždarna simetrija, renormalizacija, perturbacijska teorija, algebarska renormalizacija, BMHV shema, kiralna teorija, Yang-Millsov model, kvantna elektrodinamika, Slavnov-Taylorovi identiteti, Wardovi identitieti, standardni model

CONTENTS

CHAPTER 1

INTRODUCTION

Modern particle physics, as we know it today, was established in the 20th century, where the first half was devoted to the foundation of quantum mechanics that led, after unification with special relativity, to modern quantum field theories. The second half of the last century singled out one particular quantum field theory, known as the electroweak Standard Model (SM), as the best available theoretical description of particles and forces found in nature. The Standard Model has been extremely successful in the prediction and interpretation of experimental data but lacks to encompass some possible beyond SM indications and gravity. From this starting point, modern particle physics theoreticians can proceed in two directions: follow increasing precision of the experiment to achieve high-precision higher order results for SM observables, or impose and investigate beyond SM theories to provide their own predictions for those observables. At the end of the day, experiments will decide, and so far, they are almost always on the SM side.

Precise predictions in the SM wouldn't be possible without perturbation theory and renormalizability. Fortunately, relevant coupling constants in nature are small enough in their value that perturbative expansion is possible. Perturbative expansion implies that observables are calculated as expectation values of time-ordered products of fields expanded in terms of numbers of loops or in powers of $\hbar$. The physical information of scattering process is extracted via S-matrix, the operator valued formal power series in the coupling constants g, that is required to satisfy the following properties [1]:

1. Poincaré invariance:
$$U(a,\Lambda)\, S(g)\, U^{-1}(a,\Lambda) = S({}^{a,\Lambda}g),$$
where $U(a,\Lambda)$ is the representation of the Poincaré group acting in Fock space and transformation ${}^{a,\Lambda}g(x)$ of the coupling constant is $g(\Lambda^{-1}(x-a))$.

2. S-matrix is invertible i.e. $S^{-1}(g)$ exists, and is continuous with the respect of the

coupling constant g.

3. S-matrix in the absence of interaction is

$$S(0) = 1.$$

4. S-matrix is unitary operator,

$$S^{-1}(g) = S^{*}(g).$$

5. For the closure of the future light cone $\bar{V}^{+}$ and the support of the causal shadow of G, causality holds:

$$S(G+g) = S(G)\,S(g) \quad \text{if} \quad \operatorname{supp} g \cap \operatorname{supp}(G+\bar{V}^{+}) = \emptyset.$$

Higher loop order calculations result in increasing precision of the observables, unfortunately with increasing complexity. Thanks to the Dimensional Regularization scheme (DReg), precise calculation of observables is accessible in, relatively speaking, a simple manner. DReg preserves the BRST symmetry of vector-like theories and leads to multiplicative renormalization of the tree-level action.

The formal proofs of SM renormalizability rely on the assumption that symmetry preserving and gauge-invariant regularization exists, but as stated, DReg preserves symmetries and gauge invariance of vector-like theories. SM is not a vector-like theory due to the chiral structure of the fermions involved, so its treatment in dimensional regularization is not simple nor straightforward. Nevertheless, we still want to keep our calculation procedure in the DReg scheme due to its simplicity, accessibility, and theoretical rigor. The price we pay, as we will explain in detail in this thesis, is the breaking of symmetries that have to be restored order by order in perturbation expansion.

As we have already stated and emphasize again, the existence of chiral fermions is a empirical fact of nature and hence implemented in SM fermion content. This leads to the phenomenon of chiral anomalies [2,3] manifested e.g. in pion decays or baryon number non-conservation in the Standard model (SM). Gauge theories involving chiral fermions are only well-defined in absence of chiral gauge anomalies, which is assured thanks to the Adler-Bardeen theorem [4]. In a practical sense, chiral anomalies are related to the impossibility to find a chiral symmetry preserving regularization scheme.

In practical calculations, Dimensional Regularization (DReg) [5–9] is by far the most used scheme. In chiral theories, like the SM, the algebraic properties that the matrix γ_5 has in 4 dimensions cannot be maintained without introducing algebraic inconsistencies, or alternatively, without breaking of chiral gauge symmetries [10]. If one uses so called "Naive Dimensional Regularization" i.e. extends anticommuting property

$$\{\gamma_\mu, \gamma^5\} = 0, \tag{1.0.1}$$

to d-dimensions, soon finds that property is in the inconsistence with

$$\lim_{d\to 4} \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma^5) = 4i\epsilon_{\mu\nu\rho\sigma}. \tag{1.0.2}$$

Despite algebraic inconsistencies, a large set of treatments of γ_5 in DReg has been proposed which retain the anticommutativity of γ_5 in $d \neq 4$ dimensions; these treatments are typically either defined only for subclasses of diagrams [10, 11] or for specific objects like fermion traces [12] or give up other properties such as cyclicity of the trace [13–15]. The anticommutative definition of γ_5 is advantageous in practical calculations; however, these anticommuting schemes have not reached the same level of mathematical rigor as the original scheme by 't Hooft and Veltman [8] (see also Refs. [16–18]), for which perturbative all-order consistency with fundamental field theoretical properties has been prooved by Breitenlohner and Maison [19–22].

In the work this thesis is based on, we decided to keep uncompromised mathematical rigor using the *"Breitenlohner–Maison–'t Hooft–Veltman"* (BMHV) scheme. In this scheme, γ_5 loses non-anticommuting property in d dimensions, but the scheme is rigorously established to all orders of perturbation theory, ensuring renormalizability. Gauge invariance is broken in intermediate steps but can be restored order by order by adding suitable counterterms, i.e. gain in mathematical rigor is charged by an increase in complexity of already involved calculations. For this reason, the usual procedure of generating counterterms by a renormalization transformation is not sufficient and multiplicative renormalization no longer holds. Use of BMHV scheme results in three additional types of counterterms along with the ones in standard multiplicative renormalization: *(i)* UV divergent counterterms cancelling "evanescent" divergences that emerge from this scheme, *(ii)* the finite symmetry-restoring counterterms which restore BRST (and underlying gauge) invariance, and *(iii)* finite BRST invariant evanescent counterterms, which can optionally be added without spoiling the restored symmetry. The existence of local symmetry-restoring counterterms follows in complete generality from the renormalizability of the theory, which can be established in scheme independent way e.g. using purely algebraic methods [1, 23–26]. To highlight the previous work in the field, symmetry-restoring counterterms for the BMHV scheme have been considered already for gauge theories without scalar fields [27] and for abelian gauge theories [28] up to 1-loop level, in the evaluation of flavor-changing neutral processes at 1-loop [29], for supersymmetric QED [30], and different practical strategies for their determination have been developed e.g. in Refs. [27, 31–33].

Due to the facts stated above, it is worthwhile or even inevitable for the calculations of observables in SM and beyond, to apply BMHV scheme to general chiral gauge theories without compromises and work out its properties in detail. In the first part of this thesis, we focus on the 1-loop level of a general gauge theory with purely right-handed chiral fermions and evaluate the full counterterm structure that will be used as the foundation for the generalization to the full electroweak Standard Model. We expose the technical details of the BMHV scheme and the determination of the counterterms in a way that is familiar to phenomenology practitioners, with the hope to help bridge the gap between purely algebraic approaches and phenomenological applications. Our study is, among many other reasons, motivated by the increasing need for high-precision (multi-loop) electroweak calculations, discussed e.g. in Ref. [34]. In the first part of this thesis main goal is, therefore, to present detailed discussions and 1-loop results which will be vital ingredients in analyses of the BMHV scheme for 2-loop calculations in chiral gauge theories, presented in the second part of this thesis. We will present also this 2-loop calculation in detail.

One can argue that the origin of our problems lies in the forcing of treatment of strictly 4-dimensional objects like γ_5 in d dimensions, i.e. one could think that if we abandon DReg with all its practicality and multi-loop convenience, and choose some strictly 4-dimensional scheme, we would not get the symmetry breaking that must be restored. Unfortunately, this is not the case. Let's explore several alternative options. If one discards continuous space-time dimensions, and instead uses quantum field theories on the lattice, it is shown that chiral invariance is preserved for both abelian and nonabelian theories on lattice [35]. Back to continuum limit, the other option is the use of practically tedious BPHZ framework that is mathematically well established but breaks the symmetry in intermediate steps where [36] is the example of all-orders SM study. The work of [37] has considered a few strictly 4-dimensional schemes as alternatives to dimensional regularization, in the hope that these schemes might give practical advantages with respect to the treatment of γ_5. However, this study find that these fixed-dimension methods face the same difficulties as the different versions of dimensional regularization. Ref. [38] considers γ_5 in various versions of dimensional schemes, including the so-called four-dimensional formulation (FDF) of DReg [39]; this reference showed that FDF may be understood as a particularly efficient implementation of the BMHV scheme at the 1-loop level, at least for the four-dimensional helicity version of DReg [38].

Despite the special status the BMHV scheme enjoys among regularization schemes, at the beginning of research presented in this thesis, the scheme was thoroughly used only in a general chiral gauge theory without scalars [27], and in an Abelian gauge theory [28], both at the 1-loop level. Particular uses include an application in a range of calculations and practical procedures, see e.g. [40–42]; still it was often considered as rather impractical and less preferable than its alternatives, see e.g. Refs. [43, 44]. But given the result of Ref. [38], the general computer-algebraic progress, and the ambiguities present in other schemes, we decided a new thorough and general study of the BMHV scheme was timely and crucial, among other things, for breakthrough beyond 1-loop level.

The structure of this thesis is as follows. In Chapter 2 we introduce the reader to the generalities in dimensional regularization and we set up BMHV algebra, and in Chapter 3 we give brief introduction to algebraic renormalization needed for renormalizability study of chiral theories. In Chapter 4 we define in detail the chiral gauge theory we consider; we provide formulations using Weyl spinors and using Dirac spinors; the latter is the one we promote to d dimensions. We exhibit in detail the symmetry properties with respect to gauge invariance, BRST invariance, and the functional form of the Slavnov-Taylor identity and its breaking in d dimensions. Chapter 5 begins the study of renormalization in the BMHV scheme. It first collects known results from the standard case where gauge invariance is preserved by the regularization; then it describes the differences appearing in the BMHV scheme.

The central results for renormalization of chiral Yang-Mills model are presented in Chapter 6 and Chapter 7. The UV divergent, singular counterterms (regular and evanescent) are computed and discussed in Chapter 6. The symmetry-restoring counterterms are determined in Chapter 7 using the procedure based on the Bonneau identities [45, 46].

In Chapter 8 and Chapter 9 we evaluate the 1-loop RGEs for chiral Yang-Mills model and show that the obtained results are the standard, known ones. We focus on explaining how these results are obtained in spite of the necessity of non-standard divergent and finite counterterms. These two sections thus provide a check of the procedure and prepare multi-loop applications. Both sections use different methods to derive the β functions, and each case leads to valuable insights on expected issues in 2-loop BMHV calculations.

In Chapter 10 we expose the changes in our main results that would appear if one uses a left-handed model instead of a right-handed one. This gives another missing part for phenomenological Standard Model study. The study of chiral Yang-Mills model was published in [47].

The second part of this thesis goes beyond the 1-loop level. Starting from the the case of an abelian gauge theory with chiral fermions, a chiral QED (“χQED”) model, we apply BMHV scheme once again to obtain proper renormalization up to 2-loop level.

We determined the full 2-loop structure of the special counterterms in the BMHV scheme, i.e. the determine evanescent UV divergences, the deviations from parameter and field renormalization, and ultimately the symmetry-restoring counterterms.

In Chapter 11 we define in detail abelian gauge theory with chiral fermions and its extension to d dimensions, and we also set up the Slavnov-Taylor identity corresponding to

BRST invariance and show that it is already broken at tree-level in the BMHV scheme as it was for the Yang-Mills case. Chapter 12 summarizes the general strategy of renormalization beyond the 1-loop level and lays out the general procedure for finding UV divergent and finite symmetry-restoring counterterms. All-order symmetry requirements are then easily applied to the specific loop order we need. Chapters 13 and 14 contain the 1-loop counterterm results for this model. Both the singular, including the evanescent ones, as well as the BRST-restoring finite counterterms can also be derived by particularizing to this model our previously obtained generic results we presented in Chapters 6 and 7.

Chapter 15 presents detailed results for the UV divergences of subrenormalized 2-loop Green functions, and determines the required singular 2-loop counterterms and their relationship to field and parameter renormalization, for the first time at the 2-loop level. Chapter 16 presents first the evaluation of the 2-loop breaking of the Slavnov-Taylor identity by the regularization, using the method described in Chapter 12 and Ref. [47]. Also, 2-loop symmetry-restoring counterterms are presented. We also provide a consistency check by explicitly evaluating the analog of the usual QED Ward identities for two-, three- and four-point functions and checking that they are correctly restored as well, demonstrating the correctness of our results, which is of great importance since the 2-loop chiral-model result in BMHV scheme is unknown besides our work. The results for this chiral QED study were published in [48]. In Chapter 17 we discuss the chirality problem in the context of physical observables and make a future proposal for this research. We summarize and conclude in the last section.

CHAPTER 2

DIMENSIONAL REGULARIZATION IN BMHV SCHEME

No one can deny that dimensional regularization (DReg) played one of the most important roles in leading particle physics to the precision era we witness today. When it was first realized that perturbation theory above the lowest order introduces the divergences and calculations blow up, numerous ways to isolate and cancel those divergences came to existence. As we have already said, the discovery of equivalence among those schemes helped to establish algebraic renormalization.

2.1 | General properties of dimensional regularization

For dimensional regularization, the scheme that treats divergent integrals as integrals over d-dimensional momenta, it turns out that those singularities of graphs are simple poles in $d-4$ that can be isolated by performing Laurent expansion. The scheme is both mathematically rigorous, practical, and widely used today, especially in computer packages applied for the automatization of calculations. The dimensional continuation of integrals is defined by this conditions [49]

$$\int d^d p\, F(p+q) = \int d^d p\, F(p)\,, \tag{2.1.1a}$$

$$\int d^d p\, F(\lambda p) = |\lambda|^{-d} \int d^d p\, F(p)\,, \tag{2.1.1b}$$

$$\int d^d p\, d^{d'} q\, f(p) g(q) = \int d^d p\, f(p) \int d^{d'} q\, g(q)\,, \tag{2.1.1c}$$

i.e. properties of translation, dilatation, and factorization, which is already enough to formalize scalar theory. Today the basis of these scalar integrals is incorporated in computational packages e.g. Tarcer [50] to the 2-loop level. However, we must also implement tensor structures, but fortunately any diagram that is not scalar can be expanded on a set of fixed tensors with scalar coefficients, like

$$\int d^d p\, p_\mu p_\nu\, f(p^2, q^2, p\cdot q) = A(q^2) q_\mu q_\nu + B(q^2)\delta_{\mu\nu}, \tag{2.1.2}$$

where A and B are part of the basis of scalar integrals, that is implemented in the calculation packages. The practicality of the scheme comes from the fact that it explicitly preserves symmetries e.g. Lorentz covariance or gauge and BRST invariance, but this preservation holds for vector-like theories only. The Lorentz-covariant objects like scalar, vector and spinor fields, momenta, derivatives, and spinor matrices are extended in a formal "d"-dimensional space. Unfortunately, in chiral theories intrinsically 4-dimensional objects exist, like the γ_5 Dirac matrix and the Levi-Civita $\epsilon_{\mu\nu\rho\sigma}$, that can not be extended to d-dimensions. Chiral theories usually exhibit gauge anomalies (the Adler-Bell-Jackiw anomaly) that are generated by the presence of these objects, as well as by their fermion content. The existence of strictly 4-dimensional objects can be solved in many ways with some examples mentioned in the introduction. However, we proceed with the BMHV scheme willing to pay the price for mathematical rigor we preserve by this choice.

2.2 | BMHV d-dimensional covariants

To prepare the playground for 4-dimensional objects in d-dimensional scheme, the formal d-dimensional space is separated into 4-dimensional and $d-4 \equiv -2\epsilon$-dimensional subspaces as direct sum. Lorentz covariants extended into this d-dimensional space possess 4-dimensional (denoted by bars: $\bar{\cdot}$) and (-2ϵ)-dimensional (also called "evanescent", denoted by hats: $\hat{\cdot}$) components. Metric tensors on these subspaces are defined as

$$d\text{-dim.}: g_{\mu\nu}\,, \qquad 4\text{-dim.}: \bar{g}_{\mu\nu}\,, \qquad (-2\epsilon)\text{-dim.}: \ \hat{g}_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}\,. \tag{2.2.1}$$

The existence of these objects and their inverse (with upper indices) has been shown by explicit construction in Ref. [51]; they are defined such that

$$g_{\mu\nu} g^{\nu\mu} = d\,, \qquad \bar{g}_{\mu\nu}\bar{g}^{\nu\mu} = 4\,, \qquad \hat{g}_{\mu\nu}\hat{g}^{\nu\mu} = d-4 \equiv -2\epsilon\,, \tag{2.2.2}$$

and also

$$g_{\mu\nu} g^{\nu\rho} = g_\mu^{\ \rho} \equiv \delta_\mu^{\ \rho}\,, \qquad \bar{g}_{\mu\nu}\bar{g}^{\nu\rho} = \bar{g}_\mu^{\ \rho} = \bar{g}_{\mu\nu} g^{\nu\rho} = g_{\mu\nu}\bar{g}^{\nu\rho}\,, \tag{2.2.3}$$

$$\hat{g}_{\mu\nu}\hat{g}^{\nu\rho} = \hat{g}_\mu^{\ \rho} = \hat{g}_{\mu\nu} g^{\nu\rho} = g_{\mu\nu}\hat{g}^{\nu\rho}\,, \qquad \bar{g}_{\mu\nu}\hat{g}^{\nu\rho} = 0 = \hat{g}_{\mu\nu}\bar{g}^{\nu\rho}\,, \tag{2.2.4}$$

since the quasi-d-dimensional space is a direct sum of the actual 4-dimensional space and a quasi-(-2ϵ)-dimensional space. Our convention for the 4-dimensional metric signature is mostly minus, i.e. $(+1,-1,-1,-1)$. When being extended to the d-dimensional formalism, Lorentz indices become formal symbols that cannot take any particular value, still they obey Einstein summation convention for repeated indices, while lowering and raising indices is done using the metric tensors. Notice that the metric tensors act similarly as projectors onto these different subspaces. As an illustration for 4-vectors, the following behaviour is exhibited:

$$k^\mu = g^{\mu\nu} k_\nu \ , \quad k_\mu = g_{\mu\nu} k^\nu \ , \quad \bar{k}_\mu = \bar{g}_{\mu\nu} k^\nu \ , \quad \hat{k}_\mu = \hat{g}_{\mu\nu} k^\nu \ , \quad k^2 = \bar{k}^2 + \hat{k}^2 \ , \tag{2.2.5}$$

$$k^2 = k^\mu k_\mu = g^{\mu\nu} k_\nu k_\mu = g_{\mu\nu} k^\nu k^\mu \ , \qquad \bar{k}^2 = \bar{k}^\mu \bar{k}_\mu = \bar{g}^{\mu\nu} k_\nu k_\mu = \bar{g}_{\mu\nu} k^\nu k^\mu \ , \tag{2.2.6}$$

$$\hat{k}^2 = \hat{k}^\mu \hat{k}_\mu = \hat{g}^{\mu\nu} k_\nu k_\mu = \hat{g}_{\mu\nu} k^\nu k^\mu \ , \qquad \bar{g}_{\mu\nu} \hat{k}^\mu = 0 \ , \qquad \hat{g}_{\mu\nu} \bar{k}^\mu = 0 \ , \tag{2.2.7}$$

with similar extensions due to the fact that the different metrics, and as extension, the different contracted indices, project onto their associated subspaces.

For the usual γ^μ matrices extended to d-dimensional space, one can similarly define their 4-dimensional and (-2ϵ)-dimensional versions $\bar{\gamma}^\mu$ and $\hat{\gamma}^\mu$ respectively, including the anticommutation relations between matrices of same space-time dimensionality, the anticommutation relations between matrices of different space-time dimensionalities, their contractions and their traces:

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{1} \, , \qquad \{\gamma^\mu, \bar{\gamma}^\nu\} = \{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = 2 \bar{g}^{\mu\nu} \mathbb{1} \, , \qquad \gamma_\mu \gamma^\mu = d \, \mathbb{1} \, , \tag{2.2.8a}$$

$$\{\bar{\gamma}^\mu, \hat{\gamma}^\nu\} = 0 \, , \qquad \{\gamma^\mu, \hat{\gamma}^\nu\} = \{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2 \hat{g}^{\mu\nu} \mathbb{1} \, , \qquad \gamma_\mu \bar{\gamma}^\mu = \bar{\gamma}_\mu \bar{\gamma}^\mu = 4 \, \mathbb{1} \, , \tag{2.2.8b}$$

$$\gamma_\mu \hat{\gamma}^\mu = \hat{\gamma}_\mu \hat{\gamma}^\mu = (d-4) \mathbb{1} \, , \qquad \bar{\gamma}_\mu \hat{\gamma}^\mu = 0 \, , \tag{2.2.8c}$$

$$\mathrm{Tr} \gamma^\mu = 0 \, , \qquad \mathrm{Tr} \bar{\gamma}^\mu = 0 \, , \qquad \mathrm{Tr} \hat{\gamma}^\mu = 0 \, . \tag{2.2.8d}$$

The real question is now how to define in DReg the Levi-Civita symbol ϵ and the γ_5 matrix, which are *intrinsically 4-dimensional quantities*. In this work we adopt the "*Breitenlohner–Maison–'t Hooft–Veltman*" (BMHV) scheme for treating γ_5 and $\epsilon_{\mu\nu\rho\sigma}$, whose consistency in perturbative renormalization has been proved by Breitenlohner and Maison [19–22], and that is able to reproduce the ABJ anomaly [16–18, 52, 53]. The ϵ symbol is defined by its product with the metric tensor, and the product of two ϵ symbols together,

$$g_\mu^{\mu_1} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} = \epsilon_{\mu \mu_2 \mu_3 \mu_4} \ , \tag{2.2.9}$$

$$\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\nu_1 \nu_2 \nu_3 \nu_4} = - \sum_{\pi \in S_4} \mathrm{sgn}\,(\pi) \prod_{i=1}^{4} \bar{g}_{\mu_i \nu_{\pi(i)}} \ , \tag{2.2.10}$$

from which its other properties can be obtained,

$$\begin{aligned} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} &= \mathrm{sgn}\,(\pi) \, \epsilon_{\mu_{\pi(1)} \mu_{\pi(2)} \mu_{\pi(3)} \mu_{\pi(4)}} \ , \\ \sum_{\pi \in S_5} \mathrm{sgn}\,(\pi) \, \epsilon_{\mu_{\pi(1)} \mu_{\pi(2)} \mu_{\pi(3)} \mu_{\pi(4)}} \bar{g}^{\mu_{\pi(5)} \nu} &= 0 \, . \end{aligned} \tag{2.2.11}$$

Here, π is a permutation belonging to the permutation group of n elements S_n indicated in the corresponding expression. Here we use the $\epsilon^{0123} = +1$ convention. The γ_5 matrix is defined to be anticommuting with Dirac matrices in the 4-dimensional subspace, and commuting in the (-2ϵ)-dimensional subspace:

$$\{\gamma_5, \bar{\gamma}^\mu\} = 0\,, \quad [\gamma_5, \hat{\gamma}^\mu] = 0\,, \quad \boxed{\{\gamma_5, \gamma^\mu\} = \{\gamma_5, \hat{\gamma}^\mu\} = 2\gamma_5\hat{\gamma}^\mu}\,, \quad [\gamma_5, \gamma^\mu] = [\gamma_5, \bar{\gamma}^\mu] = 2\gamma_5\bar{\gamma}^\mu \tag{2.2.12}$$

and γ_5 otherwise keeps its usual 4-dimensional behaviour. The boxed equation shows the direct difference from the naive scheme where this anticommutator vanishes. The last of the equations (2.2.12) follows from the explicit definition of γ_5, and its square,

$$\gamma_5 = \frac{-i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\,, \qquad \gamma_5^2 = \mathbb{1}\,, \tag{2.2.13}$$

leading to the trace important to realize the Adler-Bell-Jackiw (ABJ) anomaly

$$\mathrm{Tr}(\{\gamma^\alpha, \gamma_5\}\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 8i(d-4)\epsilon_{\mu\nu\rho\sigma}\,. \tag{2.2.14}$$

2.2.1 | Amplitudes in d dimensions and the 4-dimensional limit

Once an amplitude has been defined, its evaluation in d dimensions is performed using standard techniques for loop calculations. Its actual Laurent expansion in $4 - d = 2\epsilon$ is determined only after having completely reduced and simplified its Lorentz structures: fully evaluating Dirac γ traces (cyclicity of the trace is of course valid in this scheme), fully contracting any vector, tensor and Levi-Civita symbol using the properties defined above. Any γ_5 matrix and pair of ϵ symbols can be further removed by using Eqs. (2.2.9) and (2.2.13). This defines a unique "normal form" [19] for the amplitude.

This allows one to define the regularized version of the amplitude via its Laurent expansion in $4 - d = 2\epsilon$. From there one can define its divergent part and the associated counterterms, as well as its finite part and its evanescent part that may be neglected in the $d \to 4$ limit. The renormalized value of an amplitude is obtained after performing all the necessary subtractions of the divergences of its sub-diagrams, and the resulting finite expression is interpreted in the physical 4-dimensional space by setting all quantities to their 4-dimensional values, i.e. first taking the $d \to 4$ limit and then, setting all remaining evanescent objects to zero. This operation will be denoted by $\mathrm{LIM}_{d\to4}$ or as the *big limit* in the rest of this thesis.

2.2.2 | Charge conjugation in d dimensions

Phenomenological models may contain, for example in their Yukawa sector, fermions as well as their corresponding charge-conjugated partners. This is precisely the case in the chiral models we study. Thus the question concerning the definition of the charge-conjugation operation in the framework of dimensional regularization arises.

In usual integer dimensions the charge-conjugation operation $\widehat{\mathcal{C}}$ can always be defined, and a corresponding matrix representation C explicitly constructed. For example, in 4 dimensions such a matrix, with antihermitean property, can be constructed as to be numerically equal to $C = \imath\gamma^0\gamma^2$, and satisfies the usual relations:

$$C^{-1}\gamma^\mu C = -\gamma^{\mu T} \, , \quad C^{-1} = C^\dagger = C^T \, , \quad C^T = -C \, , \quad \text{and: } C^{-1}\gamma_5 C = \gamma_5^T \, . \tag{2.2.15}$$

In even dimensions one can construct another matrix representation that provides $C^{-1}\gamma^\mu C = +\gamma^{\mu T}$ instead, while in odd dimensions either one or the other representation can exist at the same time: for example in $d = 5$ we can only construct instead $C^{-1}\gamma^\mu C = +\gamma^{\mu T}$ since now γ_5 is part of the corresponding Clifford algebra. Note also that the sign in $C^T = -C$ does change depending on the dimensionality of the space-time considered.

One can wonder whether in the continuous dimensionality of the dimensional regularization such a construction is still possible. As it turns out, an explicit construction via a matrix representation has been provided in Appendix A of [54], based on the construction of Dirac γ matrices in d dimensions given by Collins in [51]. Alternatively, one can define the charge-conjugation operation based only on its properties on the set of Dirac matrices and on its action on the d-dimensional spinors. For this purpose, since we work in dimension $d = 4 - 2\epsilon$ around 4, we postulate that the relations given in Eq. (2.2.15) also hold in $d \approx 4$ (see Appendix A of [55] for a motivation[1]). Obviously, this would not be true anymore if d was to be pushed to a different integer dimension.

Our final choice for the charge-conjugation matrix in $d \approx 4$ dimension employs the same definitions as in 4 dimensions Eq. (2.2.15), together with the following properties:

$$C^{-1}\Gamma C = \eta_\Gamma \Gamma^T \Rightarrow C\Gamma^T C^{-1} = \eta_\Gamma \Gamma \, , \text{ with } \eta_\Gamma = \begin{cases} +1 & \text{for } \Gamma = \mathbb{1} \, , \, \gamma_5 \, , \\ -1 & \text{for } \Gamma = \gamma^\mu \, , \, \sigma^{\mu\nu} \, , \end{cases} \tag{2.2.16}$$

and in the presence of anticommuting fermions (see also Appendix G.1 of [57]):

$$\widehat{\mathcal{C}}\Psi\widehat{\mathcal{C}}^{-1} \equiv \Psi^C = C\overline{\Psi}^T \, , \qquad (\Psi^C)^C = \Psi \, , \qquad \widehat{\mathcal{C}}\overline{\Psi}\widehat{\mathcal{C}}^{-1} \equiv \overline{\Psi}^C = -\Psi^T C^{-1} = \overline{\Psi^C} \, , \tag{2.2.17}$$

$$\overline{\Psi}_i^C \Gamma \Psi_j^C = -\Psi_i^T C^{-1}\Gamma C \overline{\Psi}_j^T = \overline{\Psi}_j C\Gamma^T C^{-1}\Psi_i = \eta_\Gamma \overline{\Psi}_j \Gamma \Psi_i \, . \tag{2.2.18}$$

Note that employing Eq. (2.2.16) in d dimensions has an extra subtlety: while it is true

[1] As an alternative definition, Appendix A of [56] instead postulates a different action of the charge-conjugation operation, on a product of Dirac matrices, as being equal to minus the product of the same Dirac matrices taken in the opposite order, and not transposed. This latter definition is still satisfactory since ultimately, in most of the resulting amplitudes, the internal gamma matrices attached to loops appear inside traces.

that when using these definitions in 4 dimensions, we have: $C^{-1}(\gamma^\mu\gamma_5)C = +(\gamma^\mu\gamma_5)^T$, it is not so in d dimensions in the BMHV scheme due to the γ_5 matrix:

$$C^{-1}(\gamma^\mu\gamma_5)C = (C^{-1}\gamma^\mu C)(C^{-1}\gamma_5 C) = -(\gamma^\mu)^T\gamma_5^T = -(\gamma_5\gamma^\mu)^T = (\overline{\gamma}^\mu\gamma_5)^T - (\widehat{\gamma}^\mu\gamma_5)^T\,, \tag{2.2.19}$$

while, of course, we have:

$$C^{-1}(-\gamma_5\gamma^\mu)C = \gamma_5^T(\gamma^\mu)^T = (\gamma^\mu\gamma_5)^T\,. \tag{2.2.20}$$

With the covariants, γ algebra, and charge conjugation operation defined, we can now proceed to the renormalization in the BMHV scheme. Due to the aim of doing the complete study completely applicable to models in phenomenology, the natural choice is the general gauge model with the scalar fields, since it is the key ingredient of the Standard Model. Following this logic, the reader will later see that the next model of our interest self-imposes.

CHAPTER 3

ALGEBRAIC RENORMALIZATION

Today the quantum field theory is the main framework for describing the structure of matter. Most successful examples like quantum electrodynamics, quantum chromodynamics, and standard model that incorporates both of them, brought particle physics to today's precision era. However, since the beginnings of quantum field theory, it was clear that it suffers from a dangerous disease, the problem of ultraviolet (UV) divergences. As soon as computation goes beyond the lowest order of perturbation theory, the result becomes divergent, since the momenta that appear in the loops of diagrams must be integrated over all possible options, including the ones at high energies that blow up the calculation. The subtraction of these divergences led to 30 years long research starting from the 1940s that set up a rigorous mathematical and theoretical framework known today as renormalization theory. While this solved problem of divergences, it also resulted in spectacular agreements of theory and experiment, and general proof of renormalizability of non-abelian gauge theories [8]. All present theories used for the description of nature must pass the renormalizability criteria. UV subtraction was performed in many different regularization schemes, that are in principle equivalent. It was realized that their common properties come from the so-called quantum action principle. Becchi, Rouet, and Stora [23] then figured out that the quantum action principle gives purely algebraic proof for renormalizability of theory with the set of local or rigid invariances. This means that symmetry properties of classical action can be implemented to all orders of perturbation expansion if the theory itself is renormalizable. This we recognize as a revolutionary result since now we no longer need a regularization scheme preserving the symmetries, they can be broken (in a setting controlled by quantum action principle) and restored by proper counterterms. Algebraic renormalization includes all the frameworks for establishing the renormalizability of theory and restoring the symmetries, that depend only on the theory itself and are independent of schemes used for subtraction. Now, when we recall that theories where no invariant regularization is known not only exist but the standard model itself is chiral, the algebraic renormalization becomes crucial. We briefly introduce the reader to the key aspects needed for our study.

3.1 | The generating functionals and effective action

All relevant physical quantities, like the field operators or S-matrix elements can be computed from Green functions, i.e. vacuum expectation values of time-ordered products of field operators, given by the Gell-Mann Low formula [1],

$$\langle\Omega\, T\phi(x_1)...\phi(x_n)\,\Omega\rangle = \frac{\sum_{m=0}^{\infty} i^m/m! \int d^4x_1...d^4x_m \langle 0|T\mathcal{M}_1(x_1)...\mathcal{M}_n(x_n)\mathcal{L}^{\text{bare}}(x_1,g)...\mathcal{L}^{\text{bare}}(x_m,g)|0\rangle}{\sum_{m=0}^{\infty} i^m/m! \int d^4x_1...d^4x_m \langle 0|T\mathcal{L}^{\text{bare}}(x_1,g)...\mathcal{L}^{\text{bare}}(x_m,g)|0\rangle}, \tag{3.1.1}$$

where $\mathcal{L}^{\text{bare}}(x_m,g)$ is bare Lagrangian of the theory, and $\mathcal{M}_i(x_i)$ are some local polynomials in the fields and derivatives. Green functions can be collected in the generating functional $\mathcal{Z}(J)$, a formal series in so-called classical sources $J(x_i)$ as

$$\mathcal{Z}(J) = \sum_{n=1}^{\infty} \frac{(i/\hbar)^n}{n!} \int d^4x_1...d^4x_n J(x_1)...J(x_n)\langle\Omega\, T\phi(x_1)...\phi(x_n)\,\Omega\rangle, \tag{3.1.2}$$

where we consider perturbation expansion in the powers of $\hbar$, and emphasize that Green function contains all Feynman graphs[1]. If we instead want the functional that contains only the connected Feynman graphs, it is given by connected functional $\mathcal{W}(J)$ defined via generating functional as[2]

$$\mathcal{Z}(J) = e^{\frac{i}{\hbar}\mathcal{W}(J)}, \tag{3.1.3}$$

and given by the relation [25]

$$\mathcal{W}(J) = \sum_{n=1}^{\infty} \frac{(i/\hbar)^{(n-1)}}{n!} \int d^4x_1...d^4x_n J(x_1)...J(x_n)\langle\Omega\, T\phi(x_1)...\phi(x_n)\,\Omega\rangle^c, \tag{3.1.4}$$

where Green functions are now connected and defined order by order by the sum of all possible connected Feynman graphs. Central functional needed for our study is now Legendre transform of connected generating functional,

$$\Gamma(\phi) = \mathcal{W}(J) - \int d^4x J(x)\phi(x)\Big|_{J(x)=-\frac{\delta\Gamma}{\delta\phi(x)}} \quad \text{with} \quad \frac{\delta\Gamma}{\delta\phi(x)}\bigg|_{\phi=0} = 0, \tag{3.1.5}$$

or in the inverted version

$$\mathcal{W}(J) = \Gamma(\phi) + \int d^4x J(x)\phi(x)\Big|_{\phi(x)=\frac{\delta\mathcal{W}}{\delta J(x)}} \quad \text{with} \quad \frac{\delta\mathcal{W}}{\delta J(x)}\bigg|_{J=0} = 0. \tag{3.1.6}$$

[1] All means disconnected, too.

[2] Notice the analogy with the free energy in statistical physics.

The generating functional of the vertex functions $\Gamma(\phi)$ [25] better known as *the effective action* is now given by

$$\Gamma(\phi) = \sum_{n=2}^{\infty} \frac{i}{n!} \int d^4x_1 ... d^4x_n \phi(x_1)...\phi(x_n) \langle \Omega \, T\phi(x_1)...\phi(x_n) \, \Omega \rangle^{1PI}, \tag{3.1.7}$$

where 1PI in the Green function index denotes 1-particle irreducible Feynman graphs, elementary objects that must be calculated for each order of perturbation theory and that incorporate all the quantum corrections. It is a functional of "classical fields" defined as the vacuum expectation values of their corresponding field operators in presence of suitable external currents. Notice that the sum starts from 2 since tadpoles can be eliminated [58]. Another important property proved in [25] is that effective action can be written as a power series in $\hbar$ as

$$\Gamma(\phi) = \sum_{n=0}^{\infty} \hbar^n \Gamma^n = S_0(\phi) + \Gamma^1(\phi) + \, ... \tag{3.1.8}$$

where S_0 is the classical action of the theory.
All the above relations are general and may be used both for the theory with or without counterterms. Now we introduce specific notation for regularized and (partially or fully) renormalized quantities, since we will need this in our study. In the context of DReg, the effective action is first defined for $d \neq 4$ and obtained from genuine loop diagrams and diagrams involving counterterm insertions. At the e.g. 1-loop level we use the notation $\Gamma^{(1)}$ for the effective action including tree-level and genuine 1-loop contributions Γ^1, but no counterterms; the object $\Gamma^{(1)}_{\text{DReg}}$ contains also 1-loop counterterms. Hence, we can write

$$\Gamma^{(1)} = S_0 + \Gamma^1 \,, \tag{3.1.9a}$$

$$\Gamma^{(1)}_{\text{DReg}} = \Gamma^{(1)} + S_{\text{ct}} \,, \tag{3.1.9b}$$

where S_0 and S_{ct} denote the tree-level and the 1-loop counterterm action, respectively, and where the argument (ϕ) is dropped. All these quantities are ϵ-dependent and contain evanescent objects. The quantity $\Gamma^{(1)}_{\text{DReg}}$ contains counterterms, which by construction must cancel the UV $1/\epsilon$ divergences; hence this quantity allows the limit $\epsilon \to 0$.

The final, fully renormalized effective action at the 1-loop level is then defined by taking the operation $\text{LIM}_{d\to 4}$ described in Chapter 2, i.e. by setting $\epsilon = 0$ and neglecting all the evanescent objects:

$$\Gamma^{(1)}_{\text{Ren}}[\phi, g_i, \xi, \mu] = \underset{d\to 4}{\text{LIM}} \, \Gamma^{(1)}_{\text{DReg}}[\phi, g_i, \xi, \mu] \,, \tag{3.1.10}$$

where in this equation we emphasised the fact that the effective action, both in the dimensionally-regularized and the renormalized cases, depends on the fields, the coupling constants of the theory, the gauge fixing parameter ξ and the renormalization scale μ. At the 2-loop level analogous notation holds. Now we introduce the coefficient of correlation function as

$$\Gamma_{\phi_n\cdots\phi_1}(x_1,\ldots,x_n) = \left.\frac{\delta^n\Gamma(\phi)}{\delta\phi_n(x_n)\cdots\delta\phi_1(x_1)}\right|_{\phi_i=0} = -i\langle\Omega\, T\,\phi_n(x_n)\cdots\phi_1(x_1)\Omega\rangle^{\,1\mathrm{PI}}\,. \tag{3.1.11}$$

Note that the order of the fields in the functional derivative matters in the case of anti-commuting fields, so that

$$\Gamma_{\phi_n\cdots\phi_{i+1}\phi_i\cdots\phi_1}(x_1,\ldots,x_n) = -\Gamma_{\phi_n\cdots\phi_i\phi_{i+1}\cdots\phi_1}(x_1,\ldots,x_n)$$

if ϕ_i anticommutes with ϕ_{i+1}.

Since the calculations of 1PI Feynman diagrams are performed in momentum space, we have to define the Fourier transform of this coefficient,

$$\Gamma(\phi) = \sum_{n=2}^{\infty}\frac{1}{n!}\int\left(\prod_{i=1}^{n}\frac{\mathrm{d}^d\,p_i}{(2\pi)^d}\widetilde{\phi_i}(p_i)\right)\widetilde{\Gamma}_{\phi_n\cdots\phi_1}(p_1,\ldots,p_n)(2\pi)^d\delta^d(\textstyle\sum_{j=1}^n p_j)\,, \tag{3.1.12}$$

where the tilde over the fields indicate that they have been Fourier-transformed and we made the formal transition to d dimensions. The coefficients $\widetilde{\Gamma}_{\phi_n\cdots\phi_1}(p_1,\ldots,p_n)$ are the Green's functions in momentum space, with all the momenta taken to be incoming:

$$\begin{gathered}\widetilde{\Gamma}_{\phi_n\cdots\phi_1}(p_1,\ldots,p_n)(2\pi)^d\delta^d(\textstyle\sum_{j=1}^n p_j) = (2\pi)^{d\times n}\left.\dfrac{\delta^n\Gamma(\phi)}{\delta\widetilde{\phi_n}(p_n)\cdots\delta\widetilde{\phi_1}(p_1)}\right|_{\widetilde{\phi_i}=0}\,,\\ \widetilde{\Gamma}_{\phi_n\cdots\phi_1}(p_1,\ldots,p_n) \equiv -i\langle\widetilde{\phi_n}(p_n)\cdots\widetilde{\phi_1}(p_1)\rangle^{\,1\mathrm{PI}}\,,\end{gathered} \tag{3.1.13}$$

and the delta-distribution ensures momentum conservation for these Green's functions (originating from their invariance under spatial translations, in coordinate space). When there is no ambiguity, we adopt the shortened notation $\widetilde{\Gamma}_{\phi_n\cdots\phi_1}$ in place of $\widetilde{\Gamma}_{\phi_n\cdots\phi_1}(p_1,\ldots,p_n)$. Under these definitions, the evaluation of $\langle\phi_n\cdots\phi_1\rangle^{\,1\mathrm{PI}}$ is done using the standard diagrammatic method, and the Feynman rules for the vertex with ordered fields $\phi_1\cdots\phi_n$ are given by the value of $i\widetilde{\Gamma}_{\phi_n\cdots\phi_1} = \langle\phi_n\cdots\phi_1\rangle^{\,1\mathrm{PI}}$.

An *insertion* of a local field-operator $\mathcal{O}(x)$ in Γ, denoted by $\mathcal{O}(x)\cdot\Gamma$, is defined by the set of all Feynman diagrams where $\mathcal{O}(x)$ is inserted as an "interaction vertex", or equivalently by the generating functional (see Ref. [1])

$$\mathcal{O}(x)\cdot\Gamma(\phi) = \sum_{n=2}^{\infty}\frac{-i}{n!}\int\left(\prod_{i=1}^{n}\mathrm{d}^d\,x_i\;\phi_i(x_i)\right)\langle\Omega\,T\,\mathcal{O}(x)\phi_n(x_n)\cdots\phi_1(x_1)\Omega\,\rangle^{\,1\mathrm{PI}}\,. \tag{3.1.14}$$

The *integrated insertion* $\mathcal{O} \cdot \Gamma$ is defined by

$$\mathcal{O} \cdot \Gamma(\phi) = \int \mathrm{d}^d\, x\; \mathcal{O}(x) \cdot \Gamma(\phi)\,, \tag{3.1.15}$$

and thus invariance under spatial translations will ensure momentum conservation at the "vertex" $\mathcal{O}$ in momentum space.

3.2 | Slavnov-Taylor identity

In the previous chapter, we introduced the effective action that incorporates quantum corrections. However, the classical action possess symmetries that via Noether theorems lead to conserved currents. The question is, does the effective action inherit those classical symmetries?

In Eq. (3.1.2) we defined the generating functional as power expansion of classical sources $J(x_i)$. Instead, we can define $\mathcal{Z}(J)$ for interacting theory in the functional integral representation,

$$\mathcal{Z}(J) = \int \mathcal{D}\phi\, e^{i\left(S(\phi)+\int d^4x\, J(x)\phi(x)\right)}. \tag{3.2.1}$$

Now suppose there exists the infinitesimal field transformation

$$\phi(x) \to \phi' = \phi(x) + \epsilon X(x,\phi), \tag{3.2.2}$$

that leaves both the action and integral measure invariant:

$$S(\phi') = S(\phi), \quad \text{and} \quad \mathcal{D}\phi' = \mathcal{D}\phi. \tag{3.2.3}$$

Then the generating functional is transformed as:

$$\int \mathcal{D}\phi'\, e^{i\left(S(\phi')+\int d^4x\, J(x)\phi'(x)\right)} = \int \mathcal{D}\phi\, e^{i\left(S(\phi)+\int d^4x\, J(x)\phi(x)\right)}\Big(1 + i\epsilon \int d^4x X(x,\phi)J(x)\Big), \tag{3.2.4}$$

where we used the expansion in infinitesimal parametar ϵ and kept first two terms. Requirement of generating functional invariance hence reduces to:

$$\int \mathcal{D}\phi \int d^4x X(x,\phi)J(x) e^{i\left(S(\phi)+\int d^4x\, J(x)\phi(x)\right)} = \mathcal{Z}(J) \int d^4x \langle X(x)\rangle_J J(x) \tag{3.2.5}$$

for every source J, where $\langle ...\rangle_J$ denotes the quantum expectation value of field operator in presence of source J. If we recall Eq. (3.1.5) we have

$$\boxed{\int d^4x \langle X(x)\rangle_{J_\phi} \frac{\delta\Gamma}{\delta\phi(x)} = 0} \tag{3.2.6}$$

the so-called Slavnov-Taylor identity [59], ensuring the quantum effective action invariance under

$$\phi(x) \to \phi' = \phi(x) + \epsilon \langle X(x) \rangle_{J_\phi}. \tag{3.2.7}$$

3.3 | Quantum action principle

Quantum action principle (QAP) was first stated by Schwinger for quantum mechanics [60]. Suppose we have two states defined by the values of a complete set of commuting operators at two different times. Let $|a\rangle$ and $|b\rangle$ be early and late state, respectively. If there is the parameter in Lagrangian that can be varied, then

$$\delta\langle b|a\rangle = i\langle b|\delta S|a\rangle, \tag{3.3.1}$$

with the derivative with respect to changes in the Lagrangian parameter. It means that QAP describes the behavior of Green functions under infinitesimal variations of fields and Lagrangian parameters. The renormalized versions of QAP are given in [61] and in our notation from previous section are given by

$$\boxed{\frac{\delta\Gamma(\phi)}{\delta\lambda} = \Delta_\lambda \cdot \Gamma(\phi)} \tag{3.3.2}$$

for the variations of some parameter λ where Δ_λ is the normal product operator with the UV degree 4, and by

$$\boxed{\frac{\delta\Gamma(\phi)}{\delta E(x)} = \Delta_E(x) \cdot \Gamma(\phi)} \tag{3.3.3}$$

for the variations of external field $E(x)$ where $\Delta_E(x)$ is the normal product operator with the UV degree $4 - dim(E(x))$. Furthermore, if the $\delta\phi$ is an infinitesimal variation of the field coupled to external source $\eta(x)$ then

$$\boxed{\int d^4x \, \frac{\delta\Gamma(\phi)}{\delta\phi(x)} \frac{\delta\Gamma(\phi)}{\delta\eta(x)} = \Delta \cdot \Gamma(\phi)} \tag{3.3.4}$$

holds, where Δ is a normal product of degree 4-dim(ϕ)+dim($\delta\phi$). Those quantum action principles in the dimensionally renormalized theory lead to these functional equations [20, 27, 45, 53]

$$\frac{\delta\Gamma_{\text{Ren}}}{\delta\lambda} = N\Big[\frac{\partial(S_0 + S_{\text{fct}})}{\partial\lambda}\Big] \cdot \Gamma_{\text{Ren}}, \tag{3.3.5}$$

for the parameter λ and

$$\frac{\delta\Gamma_{\text{Ren}}}{\delta E(x)} = N\Big[\frac{\partial(S_0 + S_{\text{fct}})}{\partial E(x)}\Big] \cdot \Gamma_{\text{Ren}}, \tag{3.3.6}$$

for the external field variation, and finally to

$$\frac{\delta\Gamma_{\text{Ren}}}{\delta\phi(x)}\frac{\delta\Gamma_{\text{Ren}}}{\delta\eta(x)} = N\Big[\mathcal{O}(x)\Big]\cdot\Gamma_{\text{Ren}}, \tag{3.3.7}$$

for non-linear field transformation, where $N[\mathcal{O}]$ denotes the Zimmermann-like definition [1, 62–64] of a renormalized local operator (also called *"normal product"*[3]), defined as an insertion of a local operator $\mathcal{O}$ and followed, in the context[4] of dimensional regularization and renormalization, by a minimal subtraction prescription [65]. Since chiral theories suffer from the non-existence of the regularization scheme that preserves chiral gauge symmetries, we will need the QAP to remedy the breaking of symmetries that we will introduce in the next section.

3.4 | BRST invariance and the interpretation of breaking

The BRST symmetry was first introduced in [23] to deal with the quantization of gauge theories. Concerning algebraic renormalization, BRST symmetry preservation at all orders of perturbation theory is the crucial ingredient in the proof of the unitarity of the S-matrix and construction of gauge-invariant operators.

Consider the simplest gauge theory as in [25] in 4-dimensional space-time based on a simple compact Lie group, with gauge fields and left-handed fermion content $\psi \equiv \mathbb{P}_{\text{L}}\psi$. The gauge fields of this group of course belong to adjoint representation so that

$$G_\mu = G_\mu^a \tau_a, \tag{3.4.1}$$

i.e. they are Lie-algebra valued objects whose generators obey

$$[\tau_a, \tau_b] = i f_{abc}\tau^c, \quad \operatorname{Tr}\tau_a\tau_b = \delta_{ab}. \tag{3.4.2}$$

Now consider finite gauge transformations in fundamental and adjoint representation $\mathcal{U} = e^{i\omega(x)}$,

$$G_\mu(x) \quad \to \mathcal{U}\, G_\mu(x)\mathcal{U}^{-1} + \frac{i}{g}\mathcal{U}\,\partial_\mu\mathcal{U}^{-1} \tag{3.4.3}$$

$$\psi(x) \quad \to \mathcal{U}\,\psi(x), \tag{3.4.4}$$

[3]Do not confuse with the Wick normal product with the leftmost creation operators.

[4]The actual definition for a "normal product" depends on the chosen renormalization procedure: for example in BPHZ renormalization, where the renormalization is performed by subtracting the first terms of a Taylor expansion of loop integrands up to a given order (called "degree" of subtraction), different normal products are associated to the choice of the "degree" of subtraction [1].

in the infinitesimal limit

$$\delta G_\mu(x) \quad = \partial_\mu \omega(x) + ig[\omega(x), A_\mu(x)] = D_\mu \omega(x), \tag{3.4.5}$$

$$\delta\psi(x) \quad = i\omega_a(x) T^a \psi(x). \tag{3.4.6}$$

Gauge invariant Lagrangian for this field content is then given by

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + i\overline{\psi} \not{D} \psi. \tag{3.4.7}$$

Now let's introduce the gauge fixing in the BRST fashion, where we now define three new Lie-algebra valued fields in the adjoint representation, ghost $c(x)$, antighost $\bar{c}(x)$ and Nakanishi-Lautrup field [66] B, so that

$$\omega_a(x) \to c_a(x)$$

so the gauge parameter generalizes to anticommuting number called the ghost field, with the ghost number one (and of course Lagrangian being ghostless). Gauge transformations are then replaced by the BRST transformations,

$$sG^a_\mu = D^{ab}_\mu c^b = \partial_\mu c^a + g f^{abc} G^b_\mu c^c \,, \tag{3.4.8a}$$

$$s\psi = ic^a g T_a \psi \,, \tag{3.4.8b}$$

$$s\overline{\psi} = i\overline{\psi} c^a g T_a \,, \tag{3.4.8c}$$

$$sc^a = -\frac{1}{2} f^{abc} g c^b c^c \equiv igc^2 \,, \tag{3.4.8d}$$

$$s\bar{c}^a = B^a \,, \tag{3.4.8e}$$

$$sB^a = 0 \,, \tag{3.4.8f}$$

with the important property of nilpotence,

$$\boxed{s^2 = 0} \tag{3.4.9}$$

Starting from the fact that action defined by this Lagrangian is BRST invariant, and from the property of nilpotence, it is easy to see that adding to Lagrangian terms of $K_\phi s\phi$ will not spoil the invariance if those external fields K_ϕ are itself BRST invariant. BRST invariance of this action can thus be expressed in the form of functional identity called the *Slavnov-Taylor identity*,

$$sS = \mathcal{S}(S) = 0, \tag{3.4.10}$$

where Slavnov-Taylor operator for functional $\mathcal{F}$ is given by

$$\mathcal{S}(\mathcal{F}) = \int \mathrm{d}^4\, x\, \left(\sum_\phi \frac{\delta \mathcal{F}}{\delta K_\phi} \frac{\delta \mathcal{F}}{\delta \phi} + B^a \frac{\delta \mathcal{F}}{\delta \bar{c}_a} \right) . \tag{3.4.11}$$

Finally, going back to the renormalized effective action, its Slavnov-Taylor operator is given by

$$\mathcal{S}(\Gamma_{\mathrm{Ren}}) = \int \mathrm{d}^4\, x\, \left(\sum_\phi \frac{\delta \Gamma_{\mathrm{Ren}}}{\delta K_\phi} \frac{\delta \Gamma_{\mathrm{Ren}}}{\delta \phi} + B^a \frac{\delta \Gamma_{\mathrm{Ren}}}{\delta \bar{c}_a} \right) . \tag{3.4.12}$$

If the breaking of BRST symmetry Δ then happened in the theory e.g. because of the non-invariant renormalization scheme, the quantum action principle implies

$$\mathcal{S}(\Gamma_{\mathrm{Ren}}) = \Delta = \mathcal{O} \cdot \Gamma_{\mathrm{Ren}}, \tag{3.4.13}$$

so quantum action principle guarantee that the breaking of such symmetry is given by the insertion into the renormalized effective action of the local operator of dimension $4 - dim(\phi) + dim(s\phi)$ and ghost number 1.
Now recall that the renormalized effective action is indefinite pool of extremely complicated mathematical objects hidden in the finite counterterms of the theory, while you are interested only in ones among them that break BRST symmetry, since you need to restore it to calculate physical quantities. To translate it into analogy, quantum action principle applied in this context is like you need to find the needle in the haystack, where QAP is like the giant ferromagnet in your hands. Later we will use the regularized quantum action principle, the version applied to dimensional regularization, first proved in [20] and then applied in the dimensional reduction in [54]. Those particular versions of quantum action principle need this kind of tailor-made proofs since the general derivation starting from the functional integral $\mathcal{Z}(J)$ assumes that integral measure from the functional is invariant under the field or variable transformation, what needs to be proved for particular regularization scheme.

It is important to emphasize that the renormalization of theories where no invariant regularization is known would not be possible without algebraic tools like action principles and BRST transformations.

CHAPTER 4

THE RIGHT-HANDED YANG-MILLS MODEL AND ITS EXTENSION TO D DIMENSIONS

Due its rigorously consistent framework Breitenlohner-Maison-t'Hooft-Veltman (BMHV) scheme has a distinct status among dimensional regularization schemes. Despite this fact, it is still not widely used, since its renormalization procedure is more complicated than in many other schemes. Since it is not widely used in the community of renormalization practitioners, we will apply the BMHV scheme in a complete and systematic way to the general gauge theory with wide practical use.

We will start the investigation of the Dirac γ_5 matrix in the BMHV scheme in a general massless chiral gauge theory with scalar fields. In the present chapter we define the model of our interest in 4 dimensions, then extend it to d dimensions, show that this extension is not unique, and for the most symmetric of these extensions, we provide the respective Lagrangian, BRST transformations, and Slavnov-Taylor identities of the model. The d-dimensional extension requires the usage of Dirac fermions instead of Weyl fermions, requires purely d-dimensional kinetic terms, while in a choice of interaction terms there is some freedom regarding their evanescent terms. We then examine the breaking of BRST invariance of the tree-level action, which is caused only by the evanescent part of the fermionic kinetic term. The tree-level gauge and BRST invariance are broken only by the evanescent part of the fermionic kinetic term. The BRST breaking of the tree-level action is converted into the Feynman rule used later for the proper renormalization of the theory.

4.1 | The right-handed Yang-Mills model in 4 dimensions

When we consider the non-abelian gauge theories, the best balance between the generality and practical use we can think of is given in the chiral modification of models presented in the famous papers of Machacek and Vaughn, Refs. [67–69]. The model of our choice is a Yang-Mills gauge theory with matter fields, based on a simple gauge Lie group $\mathcal{G}$, with gauge fields G^a_μ in the adjoint representation of $\mathcal{G}$ with the structure constants f^{abc}, that also define the generators $T_{G}{}^a_{bc} \equiv i f^{acb}$ of the adjoint representation. This model was presented and renormalized in [47].

The model incorporates real massless scalars Φ^m and massless right-handed fermion fields described, in the 4-dimensional formulation, using Weyl spinors[1] $\xi^i_\alpha \equiv \psi^C_R$. Scalar and fermion fields are charged under the gauge group $\mathcal{G}$ and we take their group representations to be irreducible. We denote their representations by 'S' for the scalars and 'R' for the fermions, and their associated generator matrices by θ^a_{mn} and $(T^a_R)_{ij}$, respectively. The scalar representation is imaginary and antisymmetric, $\theta^a_{mn} = -\theta^a_{nm}$.[2]

The 4-dimensional classical Lagrangian of the model can be separated into four terms:

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{scalars}} + \mathcal{L}_{\text{Yukawa}}\,, \tag{4.1.1}$$

where the pieces of the Lagrangian are given by

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}\,, \tag{4.1.2a}$$

$$\mathcal{L}_{\text{fermions}} = i\xi\sigma^\mu D_\mu \bar{\xi}\,, \tag{4.1.2b}$$

$$\mathcal{L}_{\text{scalars}} = \frac{1}{2}(D_\mu \Phi^m)^2 - \frac{\lambda^{mnop}}{4!}\Phi_m\Phi_n\Phi_o\Phi_p\,, \tag{4.1.2c}$$

$$\mathcal{L}_{\text{Yukawa}} = -\frac{(Y_R)^m_{ij}}{2}\Phi_m \bar{\xi}_i \bar{\xi}_j + \text{h.c.}\,, \tag{4.1.2d}$$

where the last equation[3] uses an index-free notation for the Lorentz invariant contraction of two Weyl spinors.

The covariant derivatives acting on the fermion fields and scalar fields are defined as:

$$D_{ij\,\mu} = \partial_\mu \delta_{ij} - igG^a_\mu T_R{}^a_{ij}\,, \tag{4.1.3a}$$

$$D_{mn\,\mu} = \partial_\mu \delta_{mn} - igG^a_\mu \theta^a_{mn}\,. \tag{4.1.3b}$$

We choose to introduce the coupling constant g in the minimal coupling term of the covariant derivative. The minus sign in front of the coupling term is part of our conventions, and the reason for this choice (as in e.g. [58] for covariant derivative in non-Abelian models) is mainly practical since it is part of the FeynRules [73] conventions, one of the programs

[1] Transition of Weyl spinors to Dirac 4-component notation is given by $\psi^i = (0, \bar{\xi}^{i\,\dot{\alpha}})^T$

[2] The model may be generalized to products of (semi-)simple gauge groups and to reducible representations. In this case one needs to consider all the possible mixings for each set of irreducible representations that have equal quantum numbers (see e.g. [70, 71]).

[3] Note that contrary to Refs. [67–69] the Yukawa term has a normalisation factor 1/2 since the two 2-component fields are identical – the corresponding Feynman rule would generate the compensating factor 2. This is the notation also used in e.g. [57, 72].

we will later use in our calculations. From the commutator of the covariant derivatives acting on a given type of field, the field strength tensor for G is defined as:

$$F^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^{abc} G^b_\mu G^c_\nu \,. \tag{4.1.4}$$

In $\mathcal{L}_{\text{scalars}}$ the scalar potential does not contain any quadratic term $\mu^2|\Phi|^2$, since we are working in the framework of a massless theory; the scalar fields do not acquire a vacuum expectation value and the fields remain perturbatively massless. Also, in a purely right or left-handed model, Dirac-type Yukawa terms $\bar{\psi}\psi$ are forbidden by chiral symmetry; only Majorana-type Yukawa terms $\bar{\psi}^C\psi$ do survive. The form of the Yukawa interaction implies that the Yukawa matrix $(Y_R)^m_{ij}$ is symmetric in fermion-group indices i, j. Since the Weyl spinor formalism is intrinsically tied to 4-dimensional space, as a natural preparation for the d-dimensional regularization we replace the Weyl spinors by projections of Dirac spinors, which can be generalized to d dimensions. Specifically we promote the right-handed Weyl fermion $\bar{\xi}$ to

$$\bar{\xi} \to \mathbb{P}_\mathrm{R}\psi \equiv \psi_R \,, \tag{4.1.5}$$

where ψ is a Dirac spinor whose left-handed part is understood to be *sterile*, decoupled from the theory. Right and left chirality operators (projectors) are defined as

$$\mathbb{P}_\mathrm{R} = \frac{\mathbb{1} + \gamma_5}{2}, \quad \mathbb{P}_\mathrm{L} = \frac{\mathbb{1} - \gamma_5}{2}. \tag{4.1.6}$$

At this point we rewrite matter content of this model, $\mathcal{L}_{\text{fermions}}$ and $\mathcal{L}_{\text{Yukawa}}$ (we recall that $\overline{\psi_R} = \overline{\psi}_L \equiv \overline{\psi}\mathbb{P}_\mathrm{L}$) as:

$$\mathcal{L}_{\text{fermions}} = i\overline{\psi_R}_i \not{D}^{ij} \psi_{Rj} = i\overline{\psi_R}_i \not{\partial} \psi_{Ri} + g T^a_{Rij} \overline{\psi_R}_i \not{G}^a \psi_{Rj} \,, \tag{4.1.7a}$$

$$\mathcal{L}_{\text{Yukawa}} = -\frac{(Y_R)^m_{ij}}{2} \Phi_m \overline{\psi_R}_i^C \psi_{Rj} - \frac{(Y_R)^{m\,*}_{ij}}{2} \Phi_m \overline{\psi_R}_i \psi_{Rj}^C \,. \tag{4.1.7b}$$

Notice that in this setting the left-handed part $P_L\psi = \psi_L$ entirely decouples and does not appear at all in this Lagrangian. It is important to emphasize that the Lagrangian defined so far is gauge invariant.

Gauge-fixing

For quantization and later renormalization of right-handed Yang-Mills model, we promote gauge invariance to BRST invariance in the usual way and write the Slavnov-Taylor identity [23, 24]. The BRST transformations of ordinary fields[4] are defined as standard infinitesimal gauge transformations, where the transformation parameter is replaced by a

[4]Ordinary fields are quantum fields in the sense that they, unlike external fields, have propagators.

Faddeev-Popov anticommuting ghost field c^a (in the adjoint representation):

$$
\begin{aligned}
sG^a_\mu &= D^{ab}_\mu c^b = \partial_\mu c^a + g f^{abc} G^b_\mu c^c \,, && \text{(4.1.8a)}\\
s\psi_i &= s\psi_{Ri} = i c^a g T^{\,a}_{Rij} \psi_{Rj} \,, && \text{(4.1.8b)}\\
s\overline{\psi}_i &= s\overline{\psi_{Ri}} = i\overline{\psi_{Rj}} c^a g T^{\,a}_{Rji} \,, && \text{(4.1.8c)}\\
s\psi_{Li} &= 0 \,, && \text{(4.1.8d)}\\
s\overline{\psi_{Li}} &= 0 \,, && \text{(4.1.8e)}\\
s\Phi_m &= i c^a g \theta^a_{mn} \Phi_n \,. && \text{(4.1.8f)}
\end{aligned}
$$

Fermionic differential operator s represents the generator of the BRST transformation. The BRST transformations of ghost and antighost fields c^a and $\bar{c}^a$ and the auxiliary Nakanishi-Lautrup [66, 74] field B^a are given by:

$$
\begin{aligned}
sc^a &= -\frac{1}{2} g f^{abc} c^b c^c \equiv i g c^2 \,, && \text{(4.1.9a)}\\
s\bar{c}^a &= B^a \,, && \text{(4.1.9b)}\\
sB^a &= 0 \,. && \text{(4.1.9c)}
\end{aligned}
$$

The BRST operator s is nilpotent: $s^2\phi = 0$ for any field or linear combination of fields.

The Lagrangian of the theory is then extended with the ghost and the gauge-fixing terms, obtained as the BRST transformation of the expression $\bar{c}^a(\xi B^a/2 + \partial^\mu G^a_\mu)$, resulting in (up to total derivatives)

$$
\begin{aligned}
\mathcal{L}_{\text{ghost}} &= \partial^\mu \bar{c}_a \cdot D^{ab}_\mu c_b \equiv -\bar{c}_a \partial^\mu D^{ab}_\mu c_b \,, && \text{(4.1.10a)}\\
\mathcal{L}_{\text{g-fix}} &= \frac{\xi}{2} B^a B_a + B^a \partial^\mu G^a_\mu \,. && \text{(4.1.10b)}
\end{aligned}
$$

The gauge-fixing Lagrangian $\mathcal{L}_{\text{g-fix}}$ written here is equivalent to the common form:

$$
\mathcal{L}_{\text{g-fix}} = -\frac{1}{2\xi} (\partial^\mu G^a_\mu)^2, \tag{4.1.11}
$$

obtained after integrating out the Nakanishi-Lautrup B^a field. Finally, it is useful to couple the non-linear BRST transformations to external sources (or Batalin-Vilkovsky “anti-fields”, [75–77]) and add corresponding terms to the Lagrangian (see e.g. [25] and references therein),

$$
\mathcal{L}_{\text{ext}} = \rho^\mu_a s G^a_\mu + \zeta_a s c^a + \bar{R}^i s\psi_{Ri} + R^i s\overline{\psi_{Ri}} + \mathcal{Y}^m s\Phi_m \,, \tag{4.1.12}
$$

where the external sources do not transform under BRST transformations: $s\mathcal{J} = 0$ for $\mathcal{J} = \rho^\mu_a, \zeta_a, R, \bar{R}, \mathcal{Y}^m$, ensuring BRST invariance of the corresponding Lagrangian terms.

The tree-level action in 4 dimensions, a starting point for the extension to d dimensions, quantization and renormalization procedure is then finally given by

$$S_0^{(4d)} = \int \mathrm{d}^4 x \, (\mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{scalars}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{g-fix}} + \mathcal{L}_{\text{ext}}) \,. \quad (4.1.13)$$

This tree-level action satisfies the Slavnov-Taylor identity

$$\mathcal{S}(S_0^{(4d)}) = 0 \,, \quad (4.1.14)$$

where the Slavnov-Taylor operation is defined for a general functional (with this field content) $\mathcal{F}$ as

$$\mathcal{S}(\mathcal{F}) = \int \mathrm{d}^4 x \, \left(\frac{\delta \mathcal{F}}{\delta \rho_a^\mu} \frac{\delta \mathcal{F}}{\delta G_\mu^a} + \frac{\delta \mathcal{F}}{\delta \zeta_a} \frac{\delta \mathcal{F}}{\delta c^a} + \frac{\delta \mathcal{F}}{\delta \mathcal{Y}^m} \frac{\delta \mathcal{F}}{\delta \Phi_m} + \frac{\delta \mathcal{F}}{\delta \bar{R}^i} \frac{\delta \mathcal{F}}{\delta \psi_i} + \frac{\delta \mathcal{F}}{\delta R^i} \frac{\delta \mathcal{F}}{\delta \overline{\psi}_i} + B^a \frac{\delta \mathcal{F}}{\delta \bar{c}_a} \right) . \quad (4.1.15)$$

The Slavnov-Taylor identity is the defining symmetry property of the theory. At the very end, we must require that the Slavnov-Taylor identity $\mathcal{S}(\Gamma_{\text{Ren}}) = 0$ is satisfied for the fully renormalized, finite effective action Γ_{Ren} (which incorporates the tree-level action, loop corrections and counterterm contributions). On the level of the 4-dimensional tree-level action, the Slavnov-Taylor identity summarizes the gauge invariance of the physical part of the Lagrangian, the BRST invariance of the gauge-fixing and ghost Lagrangian, and the nilpotency of the BRST transformations.

Quantum numbers and constraints from gauge-invariance

In Table 4.1 we list the quantum numbers (mass dimension, ghost number and (anti)commutativity) of the fields and the external sources (BV "anti-fields") of the Yang-Mills model, that are necessary for building the whole set of all possible renormalizable mass-dimension ≤ 4 field-monomial operators with a given ghost number.

Table 4.1
List of fields, external sources and operators, and their quantum numbers.

	G_μ^a	$\bar{\psi}_i$, ψ_i	Φ_m	c^a	$\bar{c}^a$	B^a	ρ_a^μ	ζ_a	R^i, $\bar{R}^i$	$\mathcal{Y}^m$	∂_μ	s
mass dim.	1	3/2	1	0	2	2	3	4	5/2	3	1	0
ghost #	0	0	0	1	-1	0	-1	-2	-1	-1	0	1
comm.	+1	-1	+1	-1	-1	+1	-1	+1	+1	-1	+1	-1

Concerning the gauge transformations under the group $\mathcal{G}$, the mentioned gauge invariance of the terms in Eq. (4.1.1) implies two consequences (that can be proved alternatively by imposing their BRST invariance) for the fermionic and scalar sectors, that generate two particular restrictions among the generators and matrices.

- Gauge-invariance of the Yukawa interaction implies that the Yukawa matrices satisfy this constraint:

$$(Y_R)^n_{ij}\theta^a_{nm} + (Y_R)^m_{ik} T_{R\,kj}^{\,a} - T_{\overline{R}\,ik}^{\,a} (Y_R)^m_{kj} = 0\,. \tag{4.1.16a}$$

  The generators ${T_R}^a$ verify ${T_R}^{a\,\dagger} = {T_R}^a$, and from them the conjugate representation $\overline{R}$ is defined with generators $T_{\overline{R}}^{\,a} \equiv -{T_R}^{a\,T} = -{T_R}^{a\,*}$. The complex-conjugate counterpart of this equation is

$$(Y_R)^{n\,*}_{ij}\theta^a_{nm} + (Y_R)^{m\,*}_{ik} T_{\overline{R}\,kj}^{\,a} - T_{R\,ik}^{\,a} (Y_R)^{m\,*}_{kj} = 0\,. \tag{4.1.16b}$$

- Gauge-invariance of the scalar self-coupling interaction implies that the scalar quartic coupling matrix λ satisfies the constraint:

$$\theta^a_{mq}\lambda^{qnop} + \theta^a_{nq}\lambda^{mqop} + \theta^a_{oq}\lambda^{mnqp} + \theta^a_{pq}\lambda^{mnoq} = 0\,. \tag{4.1.17}$$

If the gauge group representations of the quantum fields are reducible and contain two different, but group theoretically identical irreducible representations, the mixings between group theoretically identical irreducible representations might appear through Yukawa couplings, see [70,71]. For that reason, in the following, we consider only irreducible gauge boson, fermion and scalar group representations, if not stated otherwise.

Group invariants

Here we summarize the group invariants that are used in all of calculations for right-handed Yang-Mills model. We remind the reader that the right-handed fermions are in an irreducible representation R of the gauge group $\mathcal{G}$ with corresponding hermitian group generators ${T_R}^a$, and the real scalar fields are in an irreducible representation S of $\mathcal{G}$ with imaginary and antisymmetric generators θ^a. The adjoint representation of the gauge group $\mathcal{G}$ is denoted by G, with corresponding Casimir index $C_2(G)$.

We list the Casimir and Dynkin indices for these representations, as well as invariants built out of the Yukawa matrices, that will be used in loop calculations:

$$C_2(R)\mathbb{1} = T^a_R T^a_R\,, \tag{4.1.18}$$
$$S_2(R)\delta^{ab} = \mathrm{Tr}(T^a_R T^b_R)\,, \tag{4.1.19}$$
$$Y_2(R)_{ij} = (Y^m_R Y^{m\,\dagger}_R)_{ij} \equiv Y_2(R)\delta_{ij}\,, \tag{4.1.20}$$
$$C_2(S)\mathbb{1} = \theta^a\theta^a\,, \tag{4.1.21}$$
$$S_2(S)\delta^{ab} = \mathrm{Tr}(\theta^a\theta^b)\,, \tag{4.1.22}$$
$$Y_2(S)^{mn} = \frac{1}{2}\mathrm{Tr}(Y^m_R Y^{n\,\dagger}_R + Y^{m\,\dagger}_R Y^n_R) \equiv Y_2(S)\delta^{mn}\,. \tag{4.1.23}$$

Due to the presence of charge-conjugated fermions[5], we also introduce the corresponding complex-conjugate fermion representation $\overline{R}$ associated with group generators $T_{\overline{R}}{}^{a} \equiv -T_R{}^{a\,*} = -T_R{}^{a\,T}$, since the generators themselves are hermitian: $T_R{}^{a\,\dagger} = T_R{}^{a}$. We define the Yukawa matrices for the conjugate representation as: $Y^m_{\overline{R}} \equiv (Y^m_R)^\dagger = (Y^m_R)^*$ since the Yukawa matrix $(Y_R)^m_{ij}$ is symmetric in its fermion-group indices i, j. We then obtain the group invariants for complex-conjugate $\overline{R}$ representation:

$$C_2(\overline{R})\mathbb{1} = T_{\overline{R}}{}^{a} T_{\overline{R}}{}^{a} = (-T_R{}^{a\,T})(-T_R{}^{a\,T}) = T_R{}^{a} T_R{}^{a} = C_2(R)\mathbb{1}\,, \tag{4.1.24}$$

$$S_2(\overline{R})\delta^{ab} = \mathrm{Tr}(T_{\overline{R}}{}^{b} T_{\overline{R}}{}^{a}) = \mathrm{Tr}((-T_R{}^{b\,T})(-T_R{}^{a\,T})) = \mathrm{Tr}(T_R{}^{a} T_R{}^{b}) = S_2(R)\delta^{ab}\,, \tag{4.1.25}$$

$$Y_2(\overline{R})_{ij} = ((Y_{\overline{R}})^m (Y_{\overline{R}})^{m\,\dagger})_{ij} = (Y^{m\,\dagger}_R Y^m_R)_{ij} = (Y^m_R Y^{m\,\dagger}_R)_{ji} = Y_2(R)_{ji} \equiv Y_2(R)_{ij}\,. \tag{4.1.26}$$

It can be shown, using restriction in Eq. (4.1.16a), that:

$$\mathrm{Tr}(Y^m_R T_R{}^{a} Y^{n\,\dagger}_R) = \mathrm{Tr}(Y^{m\,\dagger}_R T_{\overline{R}}{}^{a} Y^n_R) = \frac{Y_2(S)}{2}\theta^a_{mn}\,, \tag{4.1.27}$$

what closes the list of group invariants that appear in the right-handed Yang-Mills model up to the 1-loop level.

4.2 | Promoting the right-handed model to d dimensions

Since we will work within the framework of dimensional regularization, the next step is the extension of the right-handed Yang-Mills model from 4 to d dimensions. While it is straightforward to do so for the bosonic fields, the extension for fermionic fields is not trivial and requires elaboration, even if we start from the version Eq. (4.1.7) of the Lagrangian in terms of Dirac spinors.

Fermion-gauge interaction term given in Eq. (4.1.7a) involves the right-handed chiral current $\overline{\psi_R}_i \gamma^\mu \psi_{Rj}$ in 4 dimensions. The extension to d dimensions of this term has three *inequivalent choices*, each of them *equally correct*:

$$\overline{\psi}_i \gamma^\mu \mathbb{P}_\mathrm{R} \psi_j\,, \qquad \overline{\psi}_i \mathbb{P}_\mathrm{L} \gamma^\mu \psi_j\,, \qquad \overline{\psi}_i \mathbb{P}_\mathrm{L} \gamma^\mu \mathbb{P}_\mathrm{R} \psi_j\,. \tag{4.2.1}$$

They are different because $\mathbb{P}_\mathrm{L}\gamma^\mu \neq \gamma^\mu \mathbb{P}_\mathrm{R}$ in d dimensions, what can be proven starting from Eq. (2.2.12). Each of these choices does lead to a valid d-dimensional extension of the model that is renormalizable using dimensional regularization and the BMHV scheme and produces the same final results in physical 4 dimensions after the renormalization procedure is performed. However, the intermediate calculations and the d-dimensional results will differ, depending on the choice for this interaction term.

[5] Or equivalently, when mapping a left-handed model to its corresponding right-handed model by interpreting left-handed fermions as charge-conjugated right-handed fermions.

Our choice for the extension is to use the third option, which is equal to

$$\overline{\psi}\mathbb{P}_{\mathrm{L}}\gamma^{\mu}\mathbb{P}_{\mathrm{R}}\psi = \overline{\psi}\mathbb{P}_{\mathrm{L}}\overline{\gamma}^{\mu}\mathbb{P}_{\mathrm{R}}\psi = \overline{\psi_R}\overline{\gamma}^{\mu}\psi_R\,, \tag{4.2.2}$$

since from our calculation experience, this is the most symmetric one and leads to the simplest intermediate expressions (see also the discussions in Refs. [10, 27]). Notice that this choice is actually the most straightforward one since it preserves the information that right-handed fermions were present on the left and on the right sides of the interaction term before the extension.

The second, more serious problem, is that as it stands the pure fermionic kinetic term $i\overline{\psi_{Ri}}\not{\partial}\psi_{Ri} = i\overline{\psi}_i\mathbb{P}_{\mathrm{L}}\not{\partial}\mathbb{P}_{\mathrm{R}}\psi_i$ projects only the purely 4-dimensional derivative, leading to a purely 4-dimensional propagator [6]

$$\Delta_F(p) = \frac{i\,\mathbb{P}_{\mathrm{R}}\,\overline{\not{p}}\,\mathbb{P}_{\mathrm{L}}}{\overline{p}^2}, \tag{4.2.3}$$

and to unregularized loop diagrams. The only valid choice for propagator in d dimensional theory in the contex of dimensional regularization is

$$\Delta_F(p) = \frac{i\,\not{p}}{p^2}, \tag{4.2.4}$$

so we are thus led to consider the full Dirac fermion ψ with both left and right-handed component, and use instead the fully d dimensional covariant kinetic term $i\overline{\psi}_i\not{\partial}\psi_i$. It can be re-expressed in terms of projectors as follows:

$$\begin{aligned} i\overline{\psi}_i\not{\partial}\psi_i &= i\overline{\psi}_i\overline{\not{\partial}}\psi_i + i\overline{\psi}_i\widehat{\not{\partial}}\psi_i \\ &= i(\overline{\psi}_i\mathbb{P}_{\mathrm{L}}\not{\partial}\mathbb{P}_{\mathrm{R}}\psi_i + \overline{\psi}_i\mathbb{P}_{\mathrm{R}}\not{\partial}\mathbb{P}_{\mathrm{L}}\psi_i) + i(\overline{\psi}_i\mathbb{P}_{\mathrm{L}}\not{\partial}\mathbb{P}_{\mathrm{L}}\psi_i + \overline{\psi}_i\mathbb{P}_{\mathrm{R}}\not{\partial}\mathbb{P}_{\mathrm{R}}\psi_i)\,. \end{aligned} \tag{4.2.5}$$

Notice that the fictitious, *sterile* left-chiral field ψ_L is introduced, which appears only within the kinetic term and nowhere else, it does not interact so it does not couple in particular to the gauge bosons of the theory, and we enforce it to be invariant under gauge transformations.

For all the reasons just explained, our final choice for the d-dimensionally regularized fermionic kinetic and gauge interaction terms is given by:

$$\mathcal{L}_{\text{fermions}} = i\overline{\psi}_i\not{\partial}\psi_i + gT^a_{Rij}\overline{\psi_{Ri}}\not{G}^a\psi_{Rj}\,. \tag{4.2.6}$$

Unfortunately, the choice of d dimensional propagator, crucial for loop regularization, that

[6] Expressing the Fourier-transformed kinetic term as $\widetilde{\overline{\psi}}_i\mathcal{K}(p)\widetilde{\psi}_i = \widetilde{\overline{\psi}}_i\mathbb{P}_{\mathrm{L}}\not{p}\mathbb{P}_{\mathrm{R}}\widetilde{\psi}_i$, the expression for the propagator $\Delta(p)$ is the only possibility such that: $\Delta(p)\mathcal{K}(p) = \mathbb{P}_{\mathrm{R}}$ and $\mathcal{K}(p)\Delta(p) = \mathbb{P}_{\mathrm{L}}$. The problematic term is then the $\overline{p}^2$, i.e. the 4-dimensional scalar product in the denominator, which cancels a similar term coming from the Dirac matrices contractions between the projectors, according to Eq. (2.2.12).

led to the introduction to the left-handed component in the kinetic term, broke the gauge invariance of the fermionic part, which is evident if we separate it in this way:

$$\mathcal{L}_{\text{fermions}} = \mathcal{L}_{\text{fermions,inv}} + \mathcal{L}_{\text{fermions,evan}} \,, \tag{4.2.7}$$

$$\mathcal{L}_{\text{fermions,inv}} = i\overline{\psi}_i \overline{\partial\!\!\!/} \psi_i + g T^a_{R\,ij} \overline{\psi_{R}}_i \overline{G\!\!\!/}^a \psi_{Rj} \,, \tag{4.2.8}$$

$$\mathcal{L}_{\text{fermions,evan}} = i\overline{\psi}_i \widehat{\partial\!\!\!/} \psi_i \,, \tag{4.2.9}$$

where the first term contains purely 4-dimensional derivatives and gauge fields and preserves the gauge and BRST invariance, since the fictitious left-chiral field ψ_L is a gauge singlet. The invariant term can also be written as a sum of purely left-chiral and purely right-chiral terms involving the 4-dimensional covariant derivative as

$$\begin{aligned} \mathcal{L}_{\text{fermions,inv}} &= i\overline{\psi_{L}}_i \overline{\partial\!\!\!/} \psi_{Li} + i\overline{\psi_{R}}_i \overline{\partial\!\!\!/} \psi_{Ri} + g T^a_{R\,ij} \overline{\psi_{R}}_i \overline{G\!\!\!/}^a \psi_{Rj} & (4.2.10) \\ &= i\overline{\psi_{L}}_i \overline{\partial\!\!\!/} \psi_{Li} + i\overline{\psi_{R}}_i \overline{D\!\!\!\!/} \psi_{Ri} \,, & (4.2.11) \end{aligned}$$

where the gauge invariance is obvious. The second term in Eq. (4.2.7) is purely evanescent, i.e. it vanishes in 4-dimensional limit. If we rewrite the evanescent term as

$$\mathcal{L}_{\text{fermions,evan}} = i\overline{\psi_{L}}_i \widehat{\partial\!\!\!/} \psi_{Ri} + i\overline{\psi_{R}}_i \widehat{\partial\!\!\!/} \psi_{Li} \,, \tag{4.2.12}$$

it can be easily seen that it mixes left- and right-chiral fields with different gauge transformation properties. This causes *the breaking of gauge and BRST invariance* – the central difficulty and the negative consequence of the BMHV scheme usage.

Now when the fermionic part is established, the rest of the model is straightforwardly extended to d dimensions. We define the d-dimensional BRST transformations on the fields formally in the same way as in 4 dimensions:

$$\begin{aligned} s_d G^a_\mu &= D^{ab}_\mu c^b = \partial_\mu c^a + g f^{abc} G^b_\mu c^c \,, & (4.2.13a) \\ s_d \psi_i &= s_d \psi_{Ri} = i c^a g T^a_{R\,ij} \psi_{Rj} \,, & (4.2.13b) \\ s_d \overline{\psi}_i &= s_d \overline{\psi_{R}}_i = +i\overline{\psi_{R}}_j c^a g T^a_{R\,ji} \,, & (4.2.13c) \\ s_d \psi_{Li} &= 0 \,, & (4.2.13d) \\ s_d \overline{\psi_{L}}_i &= 0 \,, & (4.2.13e) \\ s_d \Phi_m &= i c^a g \theta^a_{mn} \Phi_n \,, & (4.2.13f) \\ s_d c^a &= -\frac{1}{2} g f^{abc} c^b c^c \equiv i g c^2 \,, & (4.2.13g) \\ s_d \overline{c}^a &= B^a \,, & (4.2.13h) \\ s_d B^a &= 0 \,, & (4.2.13i) \end{aligned}$$

and again the external sources are invariant under BRST transformations. This version of the BRST operator s_d preserves nilpotence. Notice that the right-hand sides of these equations contain no d-dependent prefactors or evanescent objects.

The full d-dimensional tree-level action S_0 of the model thus reads:

$$S_0 = \int \mathrm{d}^d x \left(\mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{scalars}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{g-fix}} + \mathcal{L}_{\text{ext}}\right), \tag{4.2.14}$$

where all terms except $\mathcal{L}_{\text{fermions}}$ remain formally exactly as before, with all Lorentz indices interpreted in d dimensions.

Properties and classification of the d-dimensional tree-level action

We now provide two ways to rewrite the d-dimensional classical action, which will be needed in the discussion of the calculation at the higher orders and renormalization. First, we note that we can naturally decompose S_0 according to the split of the fermion Lagrangian (4.2.7) into

$$S_0 = S_{0,\text{inv}} + S_{0,\text{evan}}, \tag{4.2.15a}$$

i.e. into a BRST-invariant and a purely evanescent part, with

$$\begin{aligned} S_{0,\text{inv}} = \int \mathrm{d}^d x \Big(&\mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{fermions,inv}} + \mathcal{L}_{\text{scalars}} + \mathcal{L}_{\text{Yukawa}} \\ &+ \mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{g-fix}} + \mathcal{L}_{\text{ext}}\Big), \end{aligned} \tag{4.2.15b}$$

$$S_{0,\text{evan}} = \int \mathrm{d}^d x \, \mathcal{L}_{\text{fermions,evan}} , \tag{4.2.15c}$$

where the first part of the action contains everything except the evanescent part of the d-dimensional fermion kinetic term. The first part is by construction BRST-invariant since the 4-dimensional part of the fermion covariant derivative term is gauge and BRST-invariant and all other sectors of the theory are unaffected to the transition from 4 to d dimensions.

Since this will be needed later, we will also write the d-dimensional action of the model as a sum of integrated field monomials and we will define each of them:

$$\begin{aligned} S_0 = {} & (S_{GG} + S_{GGG} + S_{GGGG}) + (S_{\overline{\psi}\psi} + \overline{S_{\overline{\psi}G\psi_R}}) + (S_{\Phi\Phi} + S_{\Phi G\Phi} + S_{\Phi GG\Phi}) \\ & + ((Y_R)^m_{ij} S_{\overline{\psi_{Ri}}^C \Phi^m \psi_{Rj}} + \text{h.c.}) + \lambda_{mnop} S_{\Phi^4_{mnop}} \\ & + S_{\text{g-fix}} + (S_{\bar{c}c} + S_{\bar{c}Gc}) + (S_{\rho c} + S_{\rho Gc}) + S_{\zeta cc} + S_{\bar{R}c\psi_R} + S_{Rc\overline{\psi_R}} + S_{\mathcal{Y}c\Phi} , \end{aligned} \tag{4.2.16a}$$

with the gauge kinetic and self-interaction terms

$$\begin{aligned}\int \mathrm{d}^d x\, \frac{-1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} &= S_{GG} + S_{GGG} + S_{GGGG}\,, \text{ with:}\\ S_{GG} &= \int \mathrm{d}^d x\, \frac{1}{2} G^a_\mu (g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu) G^a_\nu\,,\\ S_{GGG} &= \int \mathrm{d}^d x\, (-g) f^{abc} (\partial_\mu G^a_\nu) G^{b\,\mu} G^{c\,\nu}\,,\\ S_{GGGG} &= \int \mathrm{d}^d x\, \frac{-g^2}{4} f^{eac} f^{ebd} G^a_\mu G^{b\,\mu} G^c_\nu G^{d\,\nu}\,,\end{aligned} \tag{4.2.16b}$$

the fermion kinetic and interaction terms (using property $A\overset{\leftrightarrow}{\partial}B \equiv A(\partial B) - (\partial A)B$)

$$\begin{aligned} S_{\overline{\psi}\psi} &= \int \mathrm{d}^d x\, i\overline{\psi}_i \partial\!\!\!/ \psi_i \equiv \int \mathrm{d}^d x\, \frac{i}{2}\overline{\psi}_i \overset{\leftrightarrow}{\partial\!\!\!/} \psi_i\,,\\ \overline{S_{\overline{\psi}G\psi_R}} &= \int \mathrm{d}^d x\, g T^a_{Rij} \overline{\psi}_i \mathbb{P}_{\mathrm{L}} G\!\!\!/^a \mathbb{P}_{\mathrm{R}} \psi_j = \int \mathrm{d}^d x\, g T^a_{Rij} \overline{\psi}_i \overline{G\!\!\!/^a} \mathbb{P}_{\mathrm{R}} \psi_j\,,\end{aligned} \tag{4.2.16c}$$

the scalar kinetic and interaction terms

$$\begin{aligned}\int \mathrm{d}^d x\, \frac{1}{2} (D_\mu \Phi^m)^2 &= S_{\Phi\Phi} + S_{\Phi G\Phi} + S_{\Phi GG\Phi}\,, \text{ with:}\\ S_{\Phi\Phi} &= \int \mathrm{d}^d x\, \frac{1}{2}(\partial_\mu \Phi^m)^2 \equiv \int \mathrm{d}^d x\, \frac{-1}{2} \Phi^m \partial^2 \Phi^m\,,\\ S_{\Phi G\Phi} &= \int \mathrm{d}^d x\, -ig\theta^a_{mn} (\partial^\mu \Phi^m) G^a_\mu \Phi^n\,,\\ S_{\Phi GG\Phi} &= \int \mathrm{d}^d x\, \frac{g^2}{2} (\theta^a\theta^b)_{mn} \Phi^m G^a_\mu G^{b\,\mu} \Phi^n\,,\end{aligned} \tag{4.2.16d}$$

the Yukawa and the scalar quartic self-coupling terms

$$\begin{aligned}(Y_R)^m_{ij} S_{\overline{\psi_{Ri}}^C \Phi^m \psi_{Rj}} + \text{h.c.} &= \int \mathrm{d}^d x\, \left(-\frac{(Y_R)^m_{ij}}{2} \Phi_m \overline{\psi_{Ri}}^C \psi_{Rj} - \frac{(Y_R)^{m\,*}_{ij}}{2} \Phi_m \overline{\psi_{Ri}} \psi_{Rj}^C \right)\,,\\ \lambda_{mnop} S_{\Phi^4_{mnop}} &= \int \mathrm{d}^d x\, \frac{-\lambda_{mnop}}{4!} \Phi^m \Phi^n \Phi^o \Phi^p\,,\end{aligned} \tag{4.2.16e}$$

the gauge-fixing terms

$$S_{\text{g-fix}} = \int \mathrm{d}^d x\, \frac{\xi}{2} B^a B_a + B_a \partial^\mu G^a_\mu\,, \tag{4.2.16f}$$

the ghost kinetic and interaction terms

$$\begin{aligned}\int \mathrm{d}^d x\, (\partial^\mu \bar{c}^a)(D_\mu c_a) &= S_{\bar{c}c} + S_{\bar{c}Gc}\,, \text{ with:}\\ S_{\bar{c}c} &= \int \mathrm{d}^d x\, (\partial^\mu \bar{c}^a)(\partial_\mu c_a) \equiv \int \mathrm{d}^d x\, (-\bar{c}^a \partial^2 c_a)\,,\\ S_{\bar{c}Gc} &= \int \mathrm{d}^d x\, g f^{abc} (\partial_\mu \bar{c}_a) G^\mu_b c_c\,,\end{aligned} \tag{4.2.16g}$$

and finally the external BRST source terms

$$\begin{aligned}\int \mathrm{d}^d\, x\ \rho_a^\mu s_d G_\mu^a = \int \mathrm{d}^d\, x\ \rho_a^\mu D_\mu^{ab} c_b = S_{\rho c} + S_{\rho G c}\,,\ \text{with:}\\ S_{\rho c} = \int \mathrm{d}^d\, x\ \rho_a^\mu (\partial_\mu c_a)\,,\\ S_{\rho G c} = \int \mathrm{d}^d\, x\ g f^{abc} \rho_a^\mu G_\mu^b c_c\,,\end{aligned} \tag{4.2.16h}$$

and

$$\begin{aligned} S_{\zeta cc} &= \int \mathrm{d}^d\, x\ \zeta_a s_d c^a = \int \mathrm{d}^d\, x\ \frac{-1}{2} g f^{abc} \zeta_a c^b c^c\,,\\ S_{\bar{R} c \psi_R} &= \int \mathrm{d}^d\, x\ \bar{R}^i s_d \psi_i = \int \mathrm{d}^d\, x\ ig \bar{R}^i c^a T_{R ij}^a \psi_{Rj} \equiv \int \mathrm{d}^d\, x\ ig \bar{R}^i c^a T_{R ij}^a \mathbb{P}_\mathrm{R} \psi_j\,,\\ S_{R c \overline{\psi_R}} &= \int \mathrm{d}^d\, x\ R^i s_d \overline{\psi}_i \equiv \int \mathrm{d}^d\, x\ s_d \overline{\psi}_i R^i = \int \mathrm{d}^d\, x\ ig \overline{\psi_R}_j c^a T_{R ji}^a R^i \equiv \int \mathrm{d}^d\, x\ ig \overline{\psi}_j \mathbb{P}_\mathrm{L} c^a T_{R ji}^a R^i\,,\\ S_{\mathcal{Y} c \Phi} &= \int \mathrm{d}^d\, x\ \mathcal{Y}^m s_d \Phi_m = \int \mathrm{d}^d\, x\ ig \mathcal{Y}^m c^a \theta_{mn}^a \Phi_n\,. \end{aligned} \tag{4.2.16i}$$

Notice that these field monomials are convenient for the BRST symmetry study. For illustration, notice that gauge-fixing monomial together with ghost kinetic and interaction terms form a BRST invariant structure,

$$S_{\text{g-fix}} + S_{\bar{c}c} + S_{\bar{c}Gc} = s_d \left(\bar{c}_a \left(\frac{1}{2\xi} B^a + \partial^\mu G_\mu^a \right) \right),$$

while other invariant monomials satisfy BRST invariance individually. The breaking is coming from one term, evanescent fermion kinetic term, and we examine this breaking presently.

4.3 | BRST breaking of the right-handed model in d dimensions

In this section we will determine to what extent our choice of the d-dimensional action S_0 given in Eqs. (4.2.14), (4.2.15a) and (4.2.16a) breaks the defining BRST invariance and the Slavnov-Taylor identity. Using the Eq. (4.2.13), the BRST transformation operator in d dimensions acting on the classical action S_0 can be written as

$$s_d = \int \mathrm{d}^d\, x\ \left(\frac{\delta S_0}{\delta \rho_a^\mu} \frac{\delta}{\delta G_\mu^a} + \frac{\delta S_0}{\delta \zeta_a} \frac{\delta}{\delta c^a} + \frac{\delta S_0}{\delta \mathcal{Y}^m} \frac{\delta}{\delta \Phi_m} + \frac{\delta S_0}{\delta \bar{R}^i} \frac{\delta}{\delta \psi_i} + \frac{\delta S_0}{\delta R^i} \frac{\delta}{\delta \overline{\psi}_i} + B^a \frac{\delta}{\delta \bar{c}_a} \right). \tag{4.3.1}$$

At this point we will introduce the so-called linearized Slavnov-Taylor operator b_d, since it will be useful later for higher-order calculations. In our later applications we will require the Slavnov-Taylor identity at higher orders in the form $\mathcal{S}(S_0 + \mathcal{F})$, where the functional $\mathcal{F}$ might be the 1-loop regularized or renormalized effective action or the 1-loop counterterm action. We can then write to first order in $\mathcal{F}$,

$$\mathcal{S}_d(S_0 + \mathcal{F}) = \mathcal{S}_d(S_0) + b_d\mathcal{F} + \mathcal{O}(\mathcal{F}^2) \tag{4.3.2}$$

where b_d can be written in functional form as

$$\begin{aligned} b_d = \int \mathrm{d}^d x \, \Bigg(& \frac{\delta S_0}{\delta \rho_a^\mu}\frac{\delta}{\delta G_\mu^a} + \frac{\delta S_0}{\delta G_\mu^a}\frac{\delta}{\delta \rho_a^\mu} + \frac{\delta S_0}{\delta \zeta_a}\frac{\delta}{\delta c^a} + \frac{\delta S_0}{\delta c^a}\frac{\delta}{\delta \zeta_a} + \frac{\delta S_0}{\delta \mathcal{Y}^m}\frac{\delta}{\delta \Phi_m} + \frac{\delta S_0}{\delta \Phi_m}\frac{\delta}{\delta \mathcal{Y}^m} \\ & + \frac{\delta S_0}{\delta \bar{R}^i}\frac{\delta}{\delta \psi_i} + \frac{\delta S_0}{\delta \psi_i}\frac{\delta}{\delta \bar{R}^i} + \frac{\delta S_0}{\delta R^i}\frac{\delta}{\delta \overline{\psi}_i} + \frac{\delta S_0}{\delta \overline{\psi}_i}\frac{\delta}{\delta R^i} + B^a\frac{\delta}{\delta \bar{c}_a}\Bigg) \, . \end{aligned} \tag{4.3.3}$$

The linearized Slavnov-Taylor operator is an extension of the BRST transformations in the sense that

$$b_d = s_d + \int \mathrm{d}^d x \, \left(\frac{\delta S_0}{\delta G_\mu^a}\frac{\delta}{\delta \rho_a^\mu} + \frac{\delta S_0}{\delta c^a}\frac{\delta}{\delta \zeta_a} + \frac{\delta S_0}{\delta \Phi_m}\frac{\delta}{\delta \mathcal{Y}^m} + \frac{\delta S_0}{\delta \psi_i}\frac{\delta}{\delta \bar{R}^i} + \frac{\delta S_0}{\delta \overline{\psi}_i}\frac{\delta}{\delta R^i} \right) , \tag{4.3.4}$$

so b_d and s_d act in the same way on fields but only b_d acts in a non-trivial way on the sources. A subtlety, compared to the standard situation with symmetry-preserving regularization, is that b_d is not nilpotent, ${b_d}^2 \neq 0$. The reason is that the d-dimensional action S_0 is not BRST-invariant, but both $s_d S_0 \neq 0$ and $s_d^2 S_0 \neq 0$. For later usage it is advantageous to also define the 4-dimensional linearized Slavnov-Taylor operator, b, as the restriction to 4 dimensions of d-dimensional operator b_d, based on the Slavnov-Taylor operation Eq. (11.2.8) and on the 4-dimensional action $S_0^{(4d)}$. Its functional form is then:

$$b = s + \int \mathrm{d}^4 x \, \left(\frac{\delta S_0^{(4d)}}{\delta G_\mu^a}\frac{\delta}{\delta \rho_a^\mu} + \frac{\delta S_0^{(4d)}}{\delta c^a}\frac{\delta}{\delta \zeta_a} + \frac{\delta S_0^{(4d)}}{\delta \Phi_m}\frac{\delta}{\delta \mathcal{Y}^m} + \frac{\delta S_0^{(4d)}}{\delta \psi_i}\frac{\delta}{\delta \bar{R}^i} + \frac{\delta S_0^{(4d)}}{\delta \overline{\psi}_i}\frac{\delta}{\delta R^i} \right) . \tag{4.3.5}$$

Contrary to its d-dimensional counterpart b_d, the operator b is nilpotent because the 4-dimensional action $S_0^{(4d)}$ is BRST-invariant [1].

Now we will examine the BRST invariance of d-dimensional tree-level action. Starting from the split of the d- dimensional action in (4.2.15a) it is easy to see that the part $S_{0,\text{inv}}$ on its own was constructed in the way it satisfies

$$s_d S_{0,\text{inv}} = \mathcal{S}_d(S_{0,\text{inv}}) = 0, \tag{4.3.6}$$

what follows from the definition of Slavnov-Taylor operator, where the Slavnov-Taylor operator $\mathcal{S}_d$ is of the same form as its 4-dimensional version in Eq. (4.1.15) with all 4-dimensional objects replaced by d-dimensional ones. The relation Eq. (4.3.6) may easily be checked applying explicitly the operator Eq. (4.3.2) and the operator S_d defined in the previous sentence to $S_{0,\text{inv}}$.

The evanescent part of the action $S_{0,\text{evan}}$ is not BRST-invariant since it mixes left- and right-chiral fermions with different gauge transformation properties. This breaking of BRST invariance

$$s_d S_0 = s_d S_{0,\text{evan}} \equiv \widehat{\Delta}\,, \tag{4.3.7}$$

due to the Quantum Action Principle leads to a breaking of the Slavnov-Taylor identity in the form

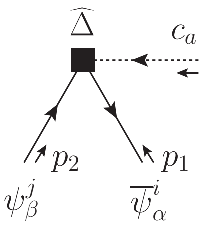
$$\mathcal{S}_d(S_0) = \widehat{\Delta}\,. \tag{4.3.8}$$

The breaking for fermions and charge-conjugated fermions is given by

$$\begin{aligned}
\widehat{\Delta} &= \int \mathrm{d}^d x\ (g T^a_{Rij}) c^a \left\{ \overline{\psi}_i \left(\overleftarrow{\widehat{\partial\!\!\!/}} \mathbb{P}_\mathrm{R} + \overrightarrow{\widehat{\partial\!\!\!/}} \mathbb{P}_\mathrm{L} \right) \psi_j \right\} \equiv \int \mathrm{d}^d x\ \widehat{\Delta}(x) \\
&= \int \mathrm{d}^d x\ \left(\frac{g}{2} T^a_{Rij} \right) c^a \left\{ \partial_\mu (\overline{\psi}_i \widehat{\gamma}^\mu \psi_j) - \overline{\psi}_i \overleftrightarrow{\widehat{\partial\!\!\!/}} \gamma_5 \psi_j \right\},
\end{aligned} \tag{4.3.9a}$$

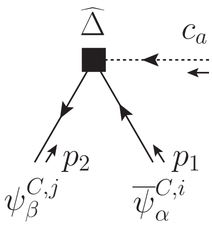
$$\begin{aligned}
\widehat{\Delta} &= \int \mathrm{d}^d x\ (g T^a_{\overline{R}ij}) c^a \left\{ \overline{\psi^C}_i \left(\overleftarrow{\widehat{\partial\!\!\!/}} \mathbb{P}_\mathrm{L} + \overrightarrow{\widehat{\partial\!\!\!/}} \mathbb{P}_\mathrm{R} \right) \psi^C_j \right\} \equiv \int \mathrm{d}^d x\ \widehat{\Delta}(x) \\
&= \int \mathrm{d}^d x\ \left(\frac{g}{2} T^a_{\overline{R}ij} \right) c^a \left\{ \partial_\mu (\overline{\psi^C}_i \widehat{\gamma}^\mu \psi^C_j) + \overline{\psi^C}_i \overleftrightarrow{\widehat{\partial\!\!\!/}} \gamma_5 \psi^C_j \right\}.
\end{aligned} \tag{4.3.9b}$$

For the renormalization procedure that must result in restoring of the BRST symmetry, as we will see in Chapter 7. The evaluation of Feynman diagrams with insertion of this breaking will be required, so we derive its Feynman rule as:



$$\begin{aligned}
&= \frac{g}{2} T^a_{Rij} \left((\widehat{p\!\!\!/_1} + \widehat{p\!\!\!/_2}) + (\widehat{p\!\!\!/_1} - \widehat{p\!\!\!/_2}) \gamma_5 \right)_{\alpha\beta} \\
&= g T^a_{Rij} \left(\widehat{p\!\!\!/_1} \mathbb{P}_\mathrm{R} + \widehat{p\!\!\!/_2} \mathbb{P}_\mathrm{L} \right)_{\alpha\beta},
\end{aligned} \tag{4.3.10}$$

and as well the Feynman rule corresponding to the charge-conjugated fermions, since they appear in the Yukawa couplings (applying flipping rules as in [78, 79]),



$$\begin{aligned}
&= \frac{g}{2} T^a_{\overline{R}ij} \left((\widehat{p\!\!\!/_1} + \widehat{p\!\!\!/_2}) - (\widehat{p\!\!\!/_1} - \widehat{p\!\!\!/_2}) \gamma_5 \right)_{\alpha\beta} \\
&= g T^a_{\overline{R}ij} \left(\widehat{p\!\!\!/_1} \mathbb{P}_\mathrm{L} + \widehat{p\!\!\!/_2} \mathbb{P}_\mathrm{R} \right)_{\alpha\beta}.
\end{aligned} \tag{4.3.11}$$

We emphasize to the reader a very important fact: BRST invariance is *broken already at the tree-level* action in d-dimensions and this is inevitable consequence of the BHMV scheme. The only option for the regularized loop integrals is d-dimensional propagator and this results in the tree-level breaking of gauge and BRST invariance.

CHAPTER 5

STANDARD RENORMALIZATION TRANSFORMATIONS VERSUS GENERAL COUNTERTERM STRUCTURE

Elaborated multiloop calculations of the parameters in gauge theories, e.g. Standard Model, are usually performed in regularization schemes that preserve gauge and BRST invariance of the theory. In those cases, and primarily for vector-like theories, the basis of the operators of the tree-level action remains constant: loop results are equal to exactly the same operators multiplied by (divergent or finite) factor - what leads us to so-called multiplicative renormalization. Counterterm structure then can simply be obtained from the classical action by applying a renormalization transformation. Despite we do not have the privilege of multiplicative renormalization in the BMHV scheme, we recall the structure of the required renormalization transformation for the right-handed Yang-Mills model; this will provide a useful benchmark against which the counterterm structure in the BMHV scheme can be compared, which will, as we hope, make the BMHV scheme more user-friendly to renormalization practitioners.

5.1 | Renormalization transformation of fields and parameters

The renormalization transformation consists of the renormalization of physical parameters of right-handed Yang-Mills model, where we introduce additive renormalization since we will later show that multiplicative renormalization is not sufficient (because of non-trivial group structures, see e.g. in 1-loop Yukawa vertex),

$$g \to g + \delta g\,, \tag{5.1.1a}$$

$$(Y_R)^m_{ij} \to (Y_R)^m_{ij} + \delta(Y_R)^m_{ij}\,, \tag{5.1.1b}$$

$$\lambda^{mnop} \to \lambda^{mnop} + \delta\lambda^{mnop}\,. \tag{5.1.1c}$$

On the contrary, for ordinary fields, multiplicative renormalization is sufficient,

$$G_\mu^a \to \sqrt{Z_G} G_\mu^a \,, \tag{5.1.1d}$$
$$(\psi_{Ri}, \overline{\psi_{Ri}}) \to \sqrt{Z_{\psi_R}} (\psi_{Ri}, \overline{\psi_{Ri}}) \,, \tag{5.1.1e}$$
$$(\psi_{Li}, \overline{\psi_{Li}}) \to \quad (\psi_{Li}, \overline{\psi_{Li}}) \,, \tag{5.1.1f}$$
$$\Phi_m \to \sqrt{Z_\Phi} \Phi_m \,, \tag{5.1.1g}$$
$$c^a \to \sqrt{Z_c} c^a \,. \tag{5.1.1h}$$

Notice that the sterile left-handed fermion field does not renormalize, right-handed fermion and anti-fermion fields renormalize in the same way and we have used a ghost field renormalization that is different from the antighost field one. The remaining fields, the gauge parameter and the external sources renormalize in a way dependent of the renormalization of ordinary fields, as

$$\{B^a, \bar{c}^a, \xi\} \to \left\{ \sqrt{Z_G}^{-1} B^a, \sqrt{Z_G}^{-1} \bar{c}^a, Z_G \xi \right\} \,, \tag{5.1.1i}$$
$$\rho_a^\mu \to \sqrt{Z_G}^{-1} \rho_a^\mu \,, \tag{5.1.1j}$$
$$\zeta_a \to \sqrt{Z_c}^{-1} \zeta_a \,, \tag{5.1.1k}$$
$$(R^i, \bar{R}^i) \to \sqrt{Z_{\psi_R}}^{-1} (R^i, \bar{R}^i) \,, \tag{5.1.1l}$$
$$\mathcal{Y}^m \to \sqrt{Z_\Phi}^{-1} \mathcal{Y}^m \,, \tag{5.1.1m}$$

where is evident that terms involving external fields and gauge-fixing terms in the Lagrangian do not renormalize, as they shouldn't by construction.

5.2 | Counterterm structure in the BMHV scheme

If the renormalization transformation from previous section is applied on the BRST invariant part of the tree-level action we obtain an invariant counterterm action $S_{\text{ct,inv}}$,

$$S_{0,\text{inv}} \longrightarrow S_{0,\text{inv}} + S_{\text{ct,inv}} \,, \tag{5.2.1}$$

so the Slavnov-Taylor identity

$$\mathcal{S}_d(S_{0,\text{inv}} + S_{\text{ct,inv}}) = 0 \tag{5.2.2}$$

holds.

This structure will later be compared to the actual counterterm structure obtained in the BMHV scheme. Unlike in this naive approach, the BMHV scheme usage results in five different types of counterterms. The further analysis is model independent and these types of the counterterms are the BHMV scheme feature. We will briefly introduce the reader to their structure. In general, in the BMHV scheme we have:

$$S_{\text{sct,inv}} + S_{\text{sct,evan}} + S_{\text{fct,inv}} + S_{\text{fct,restore}} + S_{\text{fct,evan}}\,, \tag{5.2.3}$$

where

$$S_{\text{sct,inv}} \quad \text{and} \quad S_{\text{fct,inv}}$$

correspond to the invariant counterterms generated by a renormalization transformation as in Eq. (5.2.1). The subscripts "sct" and "fct" denote singular counterterms (i.e. involving $1/\epsilon$ poles) and finite counterterms, respectively. Those counterterms are of the form found in usual counterterms of multiplicative renormalization for vector-like theories or for chiral theories with the naive prescription for γ_5. Singular invariant counterterms contain tree-level action operators, while finite counterterms in principle don't have to, due to the loop-induced processes. The next type of counterterms,

$$S_{\text{sct,evan}}$$

corresponds to additional singular counterterms needed to cancel additional $1/\epsilon$ poles of loop diagrams that have additional BHMV-induced contributions, and they contain so-called hat objects, that are part of the (-2ϵ)-dimensional space. We will see that these counterterms are purely evanescent. Similarly, evanescent divergent counterterms are also familiar from computations using regularization by dimensional reduction (see [9] for a recent review). We will later see that these types of counterterms are canceled by specific objects coming from the renormalization conditions.

$$S_{\text{fct,restore}}$$

corresponds to finite counterterms needed to restore the BRST symmetry and underlying gauge invariance. Determining these counterterms is one of the central goals of this research, and is presented in Chapter 7 for right-handed Yang-Mills model at 1-loop level, and in Chapter 14 and Chapter 16 for chiral quantum electrodynamics at 1- and 2-loop level, respectively. Once those counterterms are found, the theory is considered to be renormalized. In other words, symmetries broken by the regularization scheme must be restored. These counterterms are by themselves BRST and gauge non-invariant since their role is to retrieve missing symmetry to the action. Finally,

$$S_{\text{fct,evan}}$$

corresponds to additional counterterms which are both finite and evanescent. Adding or changing such counterterms can swap e.g. between different options as in Eq. (4.2.1); these counterterms vanish in the 4-dimensional limit, but they can affect calculations at higher orders. In our research we discard these type of counterterms, i.e. our finite symmetry-restoring counterterms are 4-dimensional. From the practical point of view, this reduces the number of objects introduced in renormalization, which makes things more manageable at higher orders.

Now we will further examine the structure of invariant counterterms by re-expressing operators of the action that form a basis as functionals L_φ, that can be constructed with field-numbering operators acting on the tree-level action or derivatives with the respect of the coupling constants acting on the tree-level action. Counterterms arising in the first order of the renormalization constants δg, δY, $\delta\lambda$ and δZ_φ[1] are in fact numerical factors (divergent or not) multiplying basis functionals L_φ,

$$\begin{aligned} S_{\mathrm{ct,inv}} = \frac{\delta Z_G}{2} L_G + \frac{\delta Z_{\psi_R}}{2} \overline{L_{\psi_R}} + \frac{\delta Z_\Phi}{2} L_\Phi + \frac{\delta Z_c}{2} L_c \\ + \frac{\delta g}{g} L_g + \Big(\delta(Y_R)^m_{ij} L_{Y_R}{}^m_{ij} + \text{h.c.} \Big) + \delta\lambda^{mnop} L_{\lambda^{mnop}} \,, \end{aligned} \tag{5.2.4}$$

where functionals are defined via the field-numbering operators

$$N_\varphi = \int \mathrm{d}^d x \; \varphi_i(x) \frac{\delta}{\delta\varphi_i(x)} \,, \text{ for } \varphi_i \in \{G^a_\mu, \Phi^m, c_a, \bar{c}_a, B^a, \rho^\mu_a, \zeta_a, R^i, \bar{R}^i, \mathcal{Y}^m\} \;, \tag{5.2.5a}$$

$$N_\psi^{R/L} = \int \mathrm{d}^d x \; (\mathbb{P}_{\mathrm{R/L}} \psi_i(x))_s \frac{\delta}{\delta\psi_i(x)_s} \,, \tag{5.2.5b}$$

$$N_{\overline{\psi}}^{L/R} = \int \mathrm{d}^d x \; (\overline{\psi}_i(x) \mathbb{P}_{\mathrm{L/R}})^s \frac{\delta}{\delta\overline{\psi}_i(x)^s} \,, \tag{5.2.5c}$$

(and sum over repeated generic group index i and spinor index s is taken into account) as derivatives of the tree-level action:

$$\begin{aligned} L_G &= (N_G - N_{\bar{c}} - N_B - N_\rho + 2\xi \frac{\partial}{\partial\xi}) S_0 \equiv \mathcal{N}_G S_0 \,, \\ L_c &= (N_c - N_\zeta)\, S_0 \equiv \mathcal{N}_c S_0 \,, \\ L_\Phi &= (N_\Phi - N_{\mathcal{Y}})\, S_0 \equiv \mathcal{N}_\Phi S_0 \,, \\ \overline{L_{\psi_R}} &= -(N_\psi^R + N_{\overline{\psi}}^L - N_{\bar{R}} - N_R) S_{0,\mathrm{inv}} \equiv \mathcal{N}_\psi^R S_{0,\mathrm{inv}} \,, \\ L_{\psi_R} &= -(N_\psi^R + N_{\overline{\psi}}^L - N_{\bar{R}} - N_R) S_0 \equiv \mathcal{N}_\psi^R S_0 \\ &= \overline{L_{\psi_R}} + S_{0,\mathrm{evan}} \,, \end{aligned} \tag{5.2.6}$$

[1]Renormalization constant δZ_φ is $Z_\varphi - 1$.

and we introduce funcionals connected with the coupling constant

$$L_g \equiv g\frac{\partial S_0}{\partial g}\,, \qquad L_{Y_R}{}^{m}_{ij} \equiv \frac{\partial S_0}{\partial (Y_R)^m_{ij}}\,, \qquad L_{\lambda_{mnop}} \equiv \frac{\partial S_0}{\partial \lambda_{mnop}}\,. \tag{5.2.7}$$

Notice that the L_φ functionals corresponding to field renormalization can be written as a total b_d-variation and in terms of the monomials of Section 4.2 as

$$\begin{aligned} L_G &= b_d \int \mathrm{d}^d\, x\ \widetilde{\rho}^\mu_a G^a_\mu \\ &= 2S_{GG} + 3S_{GGG} + 4S_{GGGG} + \overline{S_{\overline{\psi} G \psi_R}} + S_{\Phi G \Phi} + 2S_{\Phi G G \Phi} - S_{\bar{c}c} - S_{\rho c}\,, \end{aligned} \tag{5.2.8}$$

where $\widetilde{\rho}^\mu_a = \rho^\mu_a + \partial^\mu \bar{c}_a$ is the natural combination arising from the *ghost equation*

$$\begin{aligned} L_c &= -b_d \int \mathrm{d}^d\, x\ \zeta_a c^a \\ &= S_{\bar{c}c} + S_{\bar{c}Gc} + S_{\rho c} + S_{\rho G c} + S_{\zeta cc} + S_{\bar{R}c\psi_R} + S_{Rc\overline{\psi_R}} + S_{\mathcal{Y}c\Phi}\,, \end{aligned} \tag{5.2.9}$$

$$\begin{aligned} L_\Phi &= b_d \int \mathrm{d}^d\, x\ \mathcal{Y}^m \Phi_m \\ &= 2\left(S_{\Phi\Phi} + S_{\Phi G \Phi} + S_{\Phi G G \Phi}\right) + 4\lambda_{mnop} S_{\Phi^4_{mnop}} + \left((Y_R)^m_{ij} S_{\overline{\psi_{R}}_i^C \Phi^m \psi_{Rj}} + \text{h.c.}\right), \end{aligned} \tag{5.2.10}$$

$$\begin{aligned} L_{\psi_R} &= -b_d \int \mathrm{d}^d\, x\ (\bar{R}^i \mathbb{P}_\mathrm{R} \psi_i + \overline{\psi}_i \mathbb{P}_\mathrm{L} R^i) \\ {}^{[2]} &= \left(2\int \mathrm{d}^d\, x\ \frac{i}{2}\overline{\psi}_i (\overleftarrow{\partial\!\!\!/} \mathbb{P}_\mathrm{R} + \mathbb{P}_\mathrm{L} \partial\!\!\!/)\psi_i\right) + 2\overline{S_{\overline{\psi} G \psi_R}} + 2((Y_R)^m_{ij} S_{\overline{\psi_{R}}_i^C \Phi^m \psi_{Rj}} + \text{h.c.})\,, \end{aligned} \tag{5.2.11}$$

while the L_φ functionals corresponding to renormalization of physical couplings can be expressed in terms of the monomials of Section 4.2 as

$$\begin{aligned} L_g = S_{GGG} + 2S_{GGGG} + S_{\Phi G \Phi} + 2S_{\Phi G G \Phi} + \overline{S_{\overline{\psi} G \psi_R}} \\ + S_{\bar{c}Gc} + S_{\rho G c} + S_{\zeta cc} + S_{\bar{R}c\psi_R} + S_{Rc\overline{\psi_R}} + S_{\mathcal{Y}c\Phi}\,, \end{aligned} \tag{5.2.12}$$

$$L_{Y_R}{}^{m}_{ij} = S_{\overline{\psi_{R}}_i^C \Phi^m \psi_{Rj}}\,, \tag{5.2.13}$$

$$L_{\lambda_{mnop}} = S_{\Phi^4_{mnop}}\,. \tag{5.2.14}$$

Despite the non-nilpotency of b_d, several of the L_φ are actually b_d-invariant in the following sense:

$$b_d L_\varphi = 0 \quad \text{for} \quad \varphi = G, \Phi\,, \tag{5.2.15}$$

$$b_d \overline{L_{\psi_R}} = 0\,, \tag{5.2.16}$$

$$b_d \left[\delta (Y_R)^m_{ij} L_{Y_R}{}^{m}_{ij}\right] = 0\,, \tag{5.2.17}$$

$$b_d \left[\delta \lambda^{mnop} L_{\lambda^{mnop}}\right] = 0\,, \tag{5.2.18}$$

[2] Observing that $i\overline{\psi}_i(\overleftarrow{\partial\!\!\!/}\mathbb{P}_\mathrm{R} + \mathbb{P}_\mathrm{L}\partial\!\!\!/)\psi_i = 2i\overline{\psi}_i \overline{\partial\!\!\!/} \mathbb{P}_\mathrm{R}\psi_i + i\overline{\psi}_i \widehat{\partial\!\!\!/}\psi_i$, we note that there exists a difference between this calculation and the result given in [27], amounting to: $L^{\mathrm{CPM}}_{\psi_R} - L^{\mathrm{ours}}_{\psi_R} = i\int \mathrm{d}^d\, x\ \overline{\psi}_i \widehat{\partial\!\!\!/}\gamma_5 \psi_i$.

where the last two equations nontrivially hold with requirement that the renormalization constants $\delta(Y_R)$ and $\delta\lambda$ satisfy the analogous gauge invariance constraints as Eqs. (4.1.17) and (4.1.16a). In contrast, the functional L_c is not b_d-invariant[3] in this sense. Instead, it is easy to see that

$$b_d L_c = \widehat{\Delta} \tag{5.2.19}$$

with the same breaking as in Eq. (4.3.9). As a result, also L_g, corresponding to gauge coupling renormalization, is not b_d-invariant. However, one may define the quantity L_{F^2} corresponding to the field strength tensor, that will be b_d-invariant. This quantity has the useful properties

$$L_{F^2} = \frac{-1}{4}\int \mathrm{d}^d x\, F^a_{\mu\nu} F^{a\,\mu\nu} = S_{GG} + S_{GGG} + S_{GGGG}\,, \tag{5.2.20}$$

$$b_d L_{F^2} = 0\,, \tag{5.2.21}$$

$$L_g = L_c + L_G - 2L_{F^2}\,. \tag{5.2.22}$$

Note, however, that in the limit $d \to 4$ and evanescent (hat) terms vanishing, all the L_φ functionals presented here become invariant under the linear b transformation in 4 dimensions. When our task comes to the point where we must restore BRST symmetry in the theory, the reader will understand the convenience of introducing these L_φ and L_g invariants. For now, keep in mind that they are b_d variations of some action monomials.

As a takeaway for the reader, the BMHV scheme usage results in new types of counterterms due to the emergence of evanescent objects starting from the tree-level action. While in multiplicative renormalization of vector-like theories we are concerned with the counterterms

$$S_{\text{sct,inv}} + S_{\text{fct,inv}},$$

meanwhile in the BHMV scheme treatment we would have

$$S_{\text{sct,inv}} + S_{\text{sct,evan}} + S_{\text{fct,inv}} + S_{\text{fct,restore}} + S_{\text{fct,evan}}\,.$$

[3]This fact appears to be in contradiction with a claim made in [27].

CHAPTER 6

THE ONE-LOOP SINGULAR COUNTERTERM ACTION IN THE YANG-MILLS RIGHT-HANDED MODEL

In this chapter, we will present the complete list of the 1-loop (order $\hbar^1$) contributions that define the singular counterterm action $S_{\text{sct}}^{(1)}$ for right-handed Yang-Mills model, results first presented in [47]. The calculations are performed in $d = 4 - 2\epsilon$ dimensions. Since the tree-level action S_0 also contains vertex terms $K_\phi s_d \phi$ with BRST sources K_ϕ, their loop corrections are computed as well. Together with the tree-level action S_0, the singular counterterm action is a part of the dimensionally-regularized effective action Γ_{DReg}. This action, once counterterms are fixed, is no longer divergent, but is not yet BRST-invariant, and thus additional *finite counterterms* will be necessary to restore the BRST symmetry thus completing the definition of Γ_{DReg}. This finite counterterms will remedy *spurious* anomalies that break BRST symmetry and the next chapter is devoted to this task. It is important to emphasize that so-called *non-spurious* or *essential* anomalies can appear, and it is not possible to cancel them by choice of proper counterterms, but they are canceled by anomaly cancelation conditions emerging from the matter content of the theory. Supposing anomalous terms have been properly cancelled so as BRST symmetry is restored, the renormalized effective action Γ_{Ren} is then defined from Γ_{DReg} at the loop-order of interest by taking the renormalized limit, i.e. the limit $d \to 4$ and all remaining evanescent terms vanishing.

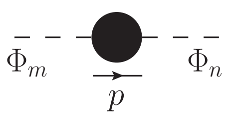
6.1 | Technical notes and remarks about the calculation

The amplitudes of the necessary Feynman diagrams have been computed using `Mathematica` packages `FeynRules` [73], `FeynArts` [80] and `FeynCalc` [81–83]; the ϵ-expansion of the amplitudes has been cross-checked using the `FeynCalc`'s interface `FeynHelpers` [84] to `Package-X` [85]. Now let's briefly explain how all those results are obtained. Our computation algorithm has several stages. Model of interest is defined via `FeynRules` where gauge groups, parameters, and particle content are defined. Feynman rules are then generated after the Lagrangian properties check, and counterterm expansion is performed in the fashion of multiplicative renormalization (where additive parts are added later manually). `FeynRules` then generates model files that we adjust for BMHV scheme calculation and d-dimensional extension is defined. Starting with the defined model, `FeynArts` is used to construct the topologies, generate Feynman diagrams and write down the amplitudes for them. The amplitudes are then exported as lists and used as calculation input. Finally, `FeynCalc` is used to calculate loop integrals and to simplify tensor structures, and finaly sums of diagrams contributing to self-energies and vertices are summed and their group structures are simplified. The process is automated as much as possible, but e.g. insertion of evanescent operators and simplification of group structures requires manual interventions and/or calculation by hand. The BHMV scheme is (yet) not widely used, so the calculation procedure requires adjustments in codes that are implemented in the most elegant way we could think of. Yet we believe there is a lot of room for improvement and this is also part of our ongoing research.

6.2 | Calculation of the one-loop divergent terms

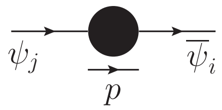
We present now the full list of the results of the divergent parts of the self-energies and vertices of the theory, evaluated at 1-loop order. In the following calculations, all momenta in vertices are taken incoming. The blobs shown in the diagrams represent the collection of the 1-loop corrections not explicitly shown, that can be easily obtained diagrammatically via the standard methods using the possible interactions in the theory.

6.2.1 | Self-energies

Scalar field: 

$$i\widetilde{\Gamma}^{nm}_{\Phi\Phi}(p,-p)|^{1}_{\text{div}} = \frac{i\hbar}{16\pi^2\epsilon}\left(\Big(g^2(\xi-3)C_2(S)\Big)\delta^{mn}p^2 + Y_2(S)\delta^{mn}\overline{p}^2 + \frac{2Y_2(S)}{3}\delta^{mn}\widehat{p}^2\right). \tag{6.2.1}$$

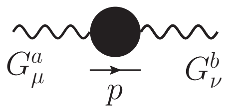
Here we introduce the reader with the first evanescent object emerging from 1-loop diagram, the last term in scalar-field self energy, coming from the chiral-fermion loop.

Fermion field: 

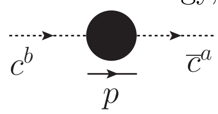
$$i\widetilde{\Gamma}^{ji}_{\psi\bar{\psi}}(-p,p)|^1_{\text{div}} = \frac{i\hbar}{16\pi^2\epsilon}\left(g^2\xi C_2(R) + \frac{Y_2(R)}{2}\right)\delta^{ij}\,\overline{\not p}\,\mathbb{P}_{\text{R}}\,, \tag{6.2.2}$$

and for the charge-conjugated fermion field:

$$i\widetilde{\Gamma}^{ji}_{\psi^C\bar{\psi}^C}(-p,p)|^1_{\text{div}} = \frac{i\hbar}{16\pi^2\epsilon}\left(g^2\xi C_2(R) + \frac{Y_2(R)}{2}\right)\delta^{ij}\,\overline{\not p}\,\mathbb{P}_{\text{L}}\,. \tag{6.2.3}$$

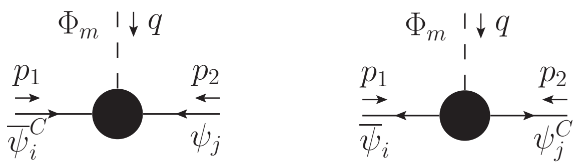
Since fermion interaction vertices have projectors, self-energy result is 4-dimensional.
Gauge boson: 

$$\begin{aligned} i\widetilde{\Gamma}^{ba,\nu\mu}_{GG}(p,-p)|^1_{\text{div}} = &-\frac{i\hbar g^2}{16\pi^2\epsilon}\frac{(13-3\xi)C_2(G)-S_2(S)}{6}\delta^{ab}(p^\mu p^\nu - p^2 g^{\mu\nu}) \\ &+\frac{i\hbar g^2}{16\pi^2\epsilon}\frac{2S_2(R)}{3}\delta^{ab}(\overline{p}^\mu\overline{p}^\nu - \overline{p}^2\overline{g}^{\mu\nu}) - \frac{i\hbar g^2}{16\pi^2\epsilon}\frac{S_2(R)}{3}\delta^{ab}\widehat{p}^2\overline{g}^{\mu\nu}\,. \end{aligned} \tag{6.2.4}$$

For the gauge boson self-energy, notice that the transversality of the photon is lost.
Ghost field: 

$$i\widetilde{\Gamma}^{ba}_{c\bar{c}}(-p,p)|^1_{\text{div}} = \frac{i\hbar g^2}{16\pi^2\epsilon}\frac{\xi-3}{4}C_2(G)\delta^{ab}p^2\,. \tag{6.2.5}$$

6.2.2 | Standard vertices

Yukawa vertex: 

$$\begin{aligned} i\widetilde{\Gamma}^{ji,m}_{\psi\overline{\psi^C}\Phi}|^1_{\text{div}} &= \frac{i\hbar}{16\pi^2\epsilon}\Big(Y^n_R(Y^m_R)^*Y^n_R - g^2\xi C_2(S)Y^m_R - g^2(3+\xi)T_{\overline{R}}{}^a Y^m_R T_R{}^a\Big)_{ij}\mathbb{P}_{\text{R}} \\ &= \frac{i\hbar}{16\pi^2\epsilon}\left(Y^n_R(Y^m_R)^*Y^n_R - g^2\frac{2C_2(R)(3+\xi)-C_2(S)(3-\xi)}{2}Y^m_R\right)_{ij}\mathbb{P}_{\text{R}}\,, \end{aligned} \tag{6.2.6}$$

where the last line is obtained by evaluating $(T_{\overline{R}}{}^a Y^m_R T_R{}^a)_{ij}$, using Eq. (4.1.16a):

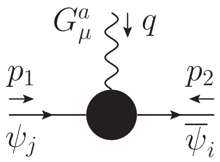
$$(T_{\overline{R}}{}^a Y^m_R T_R{}^a)_{ij} = (C_2(R) - C_2(S)/2)(Y_R)^m_{ij},$$

and the divergent result for charge-conjugated Yukawa vertex is given by:

$$i\widetilde{\Gamma}^{ji,m}_{\psi^C\bar{\psi}\Phi}|^1_{\text{div}} = \frac{i\hbar}{16\pi^2\epsilon}\left((Y^n_R)^* Y^m_R (Y^n_R)^* - g^2\frac{2C_2(R)(3+\xi) - C_2(S)(3-\xi)}{2}(Y^m_R)^*\right)_{ij}\mathbb{P}_{\text{L}}\,. \tag{6.2.7}$$

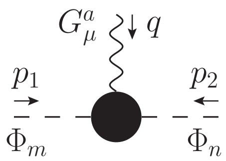
Notice that the 1-loop coupling coefficient is not proportional to tree-level one i.e. Yukawa matrix.

Fermion-gauge boson interaction:



$$i\widetilde{\Gamma}^{ji,a,\mu}_{\psi\bar{\psi}G}|^1_{\text{div}} = \frac{i\hbar g}{16\pi^2\epsilon}\left(g^2\frac{(3+\xi)C_2(G) + 4\xi C_2(R)}{4} + \frac{Y_2(R)}{2}\right)T_{R\,ij}^{\,a}\overline{\gamma}^\mu\mathbb{P}_{\text{R}}\,. \tag{6.2.8}$$

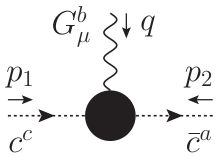
Scalar-gauge boson interaction:



$+\,(p_1, m) \leftrightarrow (p_2, n)$ permutation.

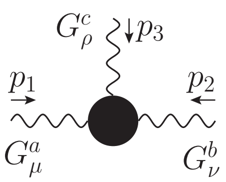
$$i\widetilde{\Gamma}^{nm,a,\mu}_{\Phi\Phi G}(q = -p_1 - p_2, p_1, p_2)|^1_{\text{div}} =$$
$$\frac{i\hbar g^3}{16\pi^2\epsilon}\left(\frac{3+\xi}{4}C_2(G) - (3-\xi)C_2(S)\right)\theta^a_{nm}(p_1 - p_2)^\mu + \frac{i\hbar g}{16\pi^2\epsilon}Y_2(S)\theta^a_{nm}\overline{(p_1 - p_2)}^\mu\,. \tag{6.2.9}$$

Ghost-gauge boson interaction:



$$i\widetilde{\Gamma}^{cba}_{cG\bar{c}}(p_2, q = -p_1 - p_2, p_1)|^1_{\text{div}} = \frac{\hbar g^3}{16\pi^2\epsilon}\frac{\xi C_2(G)}{2}f^{abc}p_2^\mu\,. \tag{6.2.10}$$

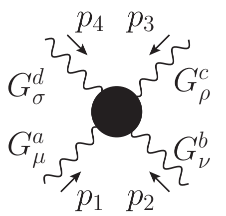
Triple gauge boson vertex:



$+\{(p_1, \mu, a)\,,\,(p_2, \nu, b)\,,\,(p_3, \rho, c)\}$ permutations.

$$
\begin{aligned}
&i\widetilde{\Gamma}^{cba,\rho\nu\mu}_{GGG}(p_1,p_2,p_3=-p_1-p_2)|^1_{\text{div}} = \\
&\frac{-\hbar g^3}{16\pi^2\epsilon} f^{abc}\frac{(17-9\xi)C_2(G)-2S_2(S)}{12}\left((p_2-p_3)^\mu g^{\nu\rho}+(p_3-p_1)^\nu g^{\mu\rho}+(p_1-p_2)^\rho g^{\mu\nu}\right) \\
&\qquad + \frac{\hbar g^3}{16\pi^2\epsilon} f^{abc}\frac{2S_2(R)}{3}\left(\overline{(p_2-p_3)}^\mu \overline{g}^{\nu\rho}+\overline{(p_3-p_1)}^\nu \overline{g}^{\mu\rho}+\overline{(p_1-p_2)}^\rho \overline{g}^{\mu\nu}\right). \qquad (6.2.11)
\end{aligned}
$$

Quartic gauge boson vertex:



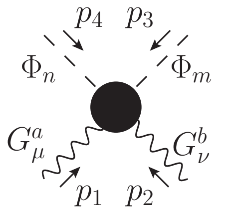
$+\{(p_1,\mu,a)\,,\,(p_2,\nu,b)\,,\,(p_3,\rho,c)\,,\,(p_4,\sigma,d)\}$ permutations.

$$
\begin{aligned}
&i\widetilde{\Gamma}^{abcd,\mu\nu\rho\sigma}_{GGGG}|^1_{\text{div}} = \\
&\frac{i\hbar g^4}{16\pi^2\epsilon}\frac{2(2-3\xi)C_2(G)-S_2(S)}{6}\left(g_{\mu\nu}g_{\rho\sigma},\quad g_{\mu\rho},g_{\nu\sigma},\quad g_{\mu\sigma}g_{\nu\rho}\right)\cdot\begin{pmatrix} f^{eac}f^{ebd}+f^{ead}f^{ebc} \\ f^{eab}f^{ecd}+f^{ead}f^{ecb} \\ f^{eab}f^{edc}+f^{eac}f^{edb}\end{pmatrix} \\
&\qquad -\frac{i\hbar g^4}{16\pi^2\epsilon}\frac{2S_2(R)}{3}\left(\overline{g}_{\mu\nu}\overline{g}_{\rho\sigma},\quad \overline{g}_{\mu\rho}\overline{g}_{\nu\sigma},\quad \overline{g}_{\mu\sigma}\overline{g}_{\nu\rho}\right)\cdot\begin{pmatrix} f^{eac}f^{ebd}+f^{ead}f^{ebc} \\ f^{eab}f^{ecd}+f^{ead}f^{ecb} \\ f^{eab}f^{edc}+f^{eac}f^{edb}\end{pmatrix}. \qquad (6.2.12)
\end{aligned}
$$

We employed here a matrix-like "scalar product" to express in a compact form the result and to indicate how the Lorentz tensors are associated with the corresponding group structures.

Tadpoles, and interactions with an odd number of scalar fields: For triple scalar vertex, scalar-gauge boson vertices with one or three scalar fields, at 1-loop the only possibility is that all the scalar fields are connected to a single internal fermion loop; since we are studying a massless theory these contributions vanish. The same reason also apply for tadpoles in DReg.

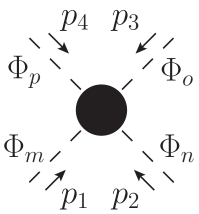
Scalar-gauge boson interaction:



$+\{(p_1,\mu,a)\,,\,(p_2,\nu,b)\}$ and $\{(p_3,m)\,,\,(p_4,n)\}$ permutations.

$$
\begin{aligned}
i\widetilde{\Gamma}^{mnab,\mu\nu}_{\Phi\Phi GG}|^1_{\text{div}} = {}& \frac{i\hbar g^4}{16\pi^2\epsilon}\left(\frac{3+\xi}{2}C_2(G)-(3-\xi)C_2(S)\right)\{\theta^a,\theta^b\}_{mn}g_{\mu\nu} \\
&+\frac{i\hbar}{16\pi^2\epsilon}Y_2(S)g^2\{\theta^a,\theta^b\}_{mn}\overline{g}_{\mu\nu}\,.
\end{aligned} \qquad (6.2.13)
$$

Quartic scalar vertex:



$+\{(p_1, m)\,,\,(p_2, n)\,,\,(p_3, o)\,,\,(p_4, p)\}$ permutations.

$$i\widetilde{\Gamma}^{mnop}_{\Phi\Phi\Phi\Phi}|^{1}_{\rm div} = \frac{i\hbar}{16\pi^2\epsilon}\frac{1}{2}(3g^4 A - g^2\xi\Lambda^S - 4H + \Lambda^2)_{mnop}\,, \tag{6.2.14}$$

using the following group invariants, as defined by Eqs. (2.16), (2.17), (2.18) and (2.19) in [69] and employing the same conventions:

$$\begin{aligned} A_{mnop} &= \frac{1}{8}\sum_{\rm perms}\{\theta^a,\theta^b\}_{mn}\{\theta^a,\theta^b\}_{op}\,, & H_{mnop} &= \frac{1}{4}\sum_{\rm perms}{\rm Tr} Y^m_R Y^{\dagger\, n}_R Y^o_R Y^{\dagger\, p}_R\,, \\ \Lambda^2_{mnop} &= \frac{1}{8}\sum_{\rm perms}\lambda_{mnqr}\lambda_{qrop}\,, & \Lambda^S_{mnop} &= \lambda_{mnop}\sum_{k=m,n,o,p} C_2(k)\,, \end{aligned} \tag{6.2.15}$$

where in the definition of Λ^S_{mnop} the sum is performed on each scalar line represented by the index k, and $C_2(k)$ is the eigenvalue of the Casimir operator $(\theta^a\theta^a)_{mn}$ for the scalar representation of line k. In our case the scalar fields are in the same scalar (and irreducible) representation, therefore we have $\Lambda^S_{mnop} = 4C_2(S)\lambda_{mnop}$. Also, note that multiplicative renormalization transformation for λ_{mnop} is not sufficient.

6.2.3 | Vertices with external BRST sources

Since the diagrams with the BRST-source vertex insertions necessary for this formalism are not conventional ones, instead of representing them as blobs we explicitly list all the contributions. To further illustrate the technical problems that can appear when using this formalism let's discuss the implementation of these BRST-source vertex insertions in calculation codes. Starting from the fact that those types of fields are not part of standard particle content in e.g. `FeynArts` [80], we have to translate them to language codes will understand. Starting from the Table 4.1 and keeping in mind that we can impose objects like auxiliary U(1) gauge bosons and ghosts, we overcame this difficulty by the construction:

$$\zeta_a \to \bar{c}\bar{c}_a,$$

$$\mathcal{Y}_m \to \Phi_m\bar{c},$$

$$\rho^a_\mu \to A_\mu\bar{c}^a,$$

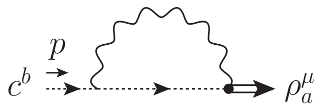
$$R^i \to \psi^i\bar{c},$$

$$\bar{R}^i \to \bar{\psi}^i\bar{c},$$

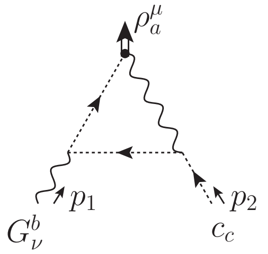
where one can see that it is not possible to reproduce mass-dimension of the fermionic BRST sources so we have to keep that in mind while doing calculations.

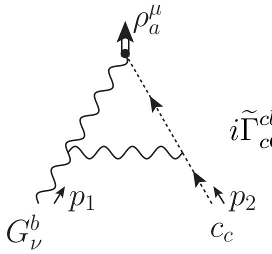
From $\rho^a_\mu s_d G^a_\mu$:

there exist two different Green's functions involving this insertion, whose divergent parts are:



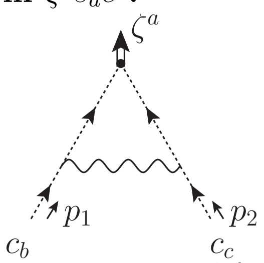
$$i\widetilde{\Gamma}^{ba,\mu}_{c\rho}(-p,p)|^1_{\text{div}} = -\frac{\hbar g^2}{16\pi^2\epsilon}\frac{3-\xi}{4}C_2(G)\delta^{ab}p^\mu\,, \tag{6.2.16}$$





$$i\widetilde{\Gamma}^{cba,\nu\mu}_{cG\rho}|^1_{\text{div}} = \frac{i\hbar g^3}{16\pi^2\epsilon}\frac{\xi C_2(G)}{2}f^{abc}g^{\mu\nu}\,. \tag{6.2.17}$$

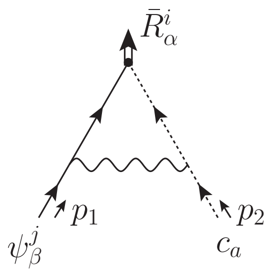
From $\zeta^a s_d c^a$:



$$i\widetilde{\Gamma}^{cba}_{cc\zeta}|^{(1)}_{\text{div}} = -\frac{i\hbar g^3}{16\pi^2\epsilon}\frac{\xi C_2(G)}{2}f^{abc}\,, \tag{6.2.18}$$

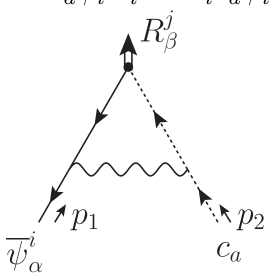
where we accounted for the diagram's symmetry factor $= 2$ due to the fact there are two interchangeable vertices – the $(\bar{c}Gc)$ vertices – leaving the diagram invariant.

From $\bar{R}_i s_d \psi_i$:



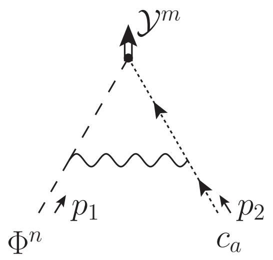
$$i\widetilde{\Gamma}^{jai,\beta\alpha}_{\psi c\bar{R}}|^{(1)}_{\text{div}} = -\frac{\hbar g^3}{16\pi^2\epsilon}\frac{\xi C_2(G)}{2}T^a_{Rij}\mathbb{P}_{\text{R}\alpha\beta}\,. \tag{6.2.19}$$

From $s_d\bar{\psi}_i R_i \equiv R_i s_d \bar{\psi}_i$:



$$i\widetilde{\Gamma}^{jai,\beta\alpha}_{Rc\bar{\psi}}|^{(1)}_{\text{div}} = -\frac{\hbar g^3}{16\pi^2\epsilon}\frac{\xi C_2(G)}{2}T^a_{Rij}\mathbb{P}_{\text{L}\alpha\beta}\,. \tag{6.2.20}$$

From $\mathcal{Y}_m s_d \Phi_m$:



$$i\widetilde{\Gamma}^{nam}_{\Phi c \mathcal{Y}}\big|^{(1)}_{\text{div}} = -\frac{\hbar g^3}{16\pi^2\epsilon}\frac{\xi C_2(G)}{2}\theta^a_{mn}\,. \tag{6.2.21}$$

6.3 | The one-loop singular counterterm action

After computing all UV divergent 1-loop Feynman diagrams, we can determine the singular 1-loop counterterm action needed for regularization. It is defined such that the divergent parts of the 1-loop vertices cancel:

$$S^{(1)}_{\text{sct}} = -\Gamma\big|^{(1)}_{\text{div}}\,. \tag{6.3.1}$$

Since here we present the first result with the chiral Yang-Mills model with scalar fields, we will separate it into the non-scalar and scalar sector for clarity. First, we provide the contributions with and without scalar fields separately[1],

$$S^{(1)}_{\text{sct}} = S^{(1)\,\text{NS}}_{\text{sct}} + S^{(1)\,\text{S}}_{\text{sct}}\,, \tag{6.3.2}$$

where $S^{(1)\,\text{NS}}_{\text{sct}}$ represents the terms without any contribution from the scalar fields, and agrees with Eq. (37) of [27], and reads:

$$\begin{aligned}
S^{(1)\,\text{NS}}_{\text{sct}} = \;& \frac{\hbar g^2}{16\pi^2\epsilon}\bigg\{\frac{13-3\xi}{6}C_2(G)S_{GG} + \frac{17-9\xi}{12}C_2(G)S_{GGG} + \frac{2-3\xi}{3}C_2(G)S_{GGGG} \\
& -\frac{2S_2(R)}{3}\left(\overline{S_{GG}} + \overline{S_{GGG}} + \overline{S_{GGGG}}\right) - \xi C_2(R)\left(\overline{S_{\bar{\psi}\psi_R}} + \overline{S_{\overline{\psi}G\psi_R}}\right) - \frac{3+\xi}{4}C_2(G)\overline{S_{\overline{\psi}G\psi_R}} \\
& +\frac{3-\xi}{4}C_2(G)\left(S_{\bar{c}c} + S_{\rho c}\right) - \frac{\xi C_2(G)}{2}\Big(S_{\bar{c}Gc} + S_{\rho Gc} + S_{\zeta cc} + S_{\bar{R}c\psi_R} + S_{Rc\overline{\psi_R}}\Big)\bigg\} \\
& -\frac{\hbar g^2}{16\pi^2\epsilon}\frac{S_2(R)}{3}\int \mathrm{d}^d\,x\;\frac{1}{2}\bar{G}^{a\,\mu}\widehat{\partial}^2\bar{G}^a_\mu\,.
\end{aligned} \tag{6.3.3}$$

[1] Since tree-level action is finite, notice that superscripts 1 meaning the explicit 1-loop, and (1) meaning up to 1-loop do not differ in this case.

The counterterm action $S_{\text{sct}}^{(1)\,\text{S}}$ represents the terms generated from the scalar contributions, and reads:

$$\begin{aligned}S_{\text{sct}}^{(1)\,\text{S}} = \; & \frac{\hbar}{16\pi^2\epsilon}\Bigg\{-g^2\frac{S_2(S)}{6}(S_{GG}+S_{GGG}+S_{GGGG}) - \frac{Y_2(R)}{2}\left(\overline{S_{\bar{\psi}\psi_R}} + \overline{S_{\overline{\psi}G\psi_R}}\right) \\ & +g^2(3-\xi)C_2(S)\left(S_{\Phi\Phi}+S_{\Phi G\Phi}+S_{\Phi GG\Phi}\right) - g^2\frac{3+\xi}{4}C_2(G)\left(S_{\Phi G\Phi}+2S_{\Phi GG\Phi}\right) \\ & -Y_2(S)\left(\overline{S_{\Phi\Phi}}+\overline{S_{\Phi G\Phi}}+\overline{S_{\Phi GG\Phi}}\right) + \frac{1}{2}(3g^4A - g^2\xi\Lambda^S - 4H + \Lambda^2)_{mnop}S_{\Phi^4_{mnop}} \\ & + \left(Y_R^n(Y_R^m)^* Y_R^n - g^2\frac{2C_2(R)(3+\xi)-C_2(S)(3-\xi)}{2}Y_R^m\right)_{ij} S_{\overline{\psi_{Ri}}^C\Phi^m\psi_{Rj}} + \text{h.c.} \\ & -g^2\frac{\xi C_2(G)}{2}S_{\mathcal{Y}_c\Phi}\Bigg\} - \frac{\hbar}{16\pi^2\epsilon}\frac{2Y_2(S)}{3}\widehat{S_{\Phi\Phi}}\,. \end{aligned} \tag{6.3.4}$$

Notice it contains both additional contributions to the operators without scalar fields and contributions to additional operators involving scalar fields. In both equations the monomials introduced in Eq. (4.2.16a) have been used; a bar such as in $\overline{S_{GG}}$ corresponds to taking all Lorentz indices in the respective monomial only in purely 4 dimensions; a hat such as in $\widehat{S_{\Phi\Phi}}$ corresponds to taking all Lorentz indices purely in $d-4$ dimensions. The new object

$$\overline{S_{\bar{\psi}\psi_R}} = \int d^d x\, i\overline{\psi}_i\bar{\partial\!\!\!/}\mathbb{P}_\text{R}\psi_i \equiv \int d^d x\, \frac{i}{2}\overline{\psi}_i\overset{\leftrightarrow}{\bar{\partial\!\!\!/}}\mathbb{P}_\text{R}\psi_i$$

corresponds to the 4-dimensional kinetic term of the purely right-handed fermion. It differs from its d-dimensional equivalent $S_{\overline{\psi}\psi}$. Its appearance reflects the fact that only the right-handed fermion component renormalizes, while the fictitious left-handed component required to properly extend the 4-dimensional chiral fermion kinetic term to d dimensions, see Section 4.2, does not renormalize. This is expected since all fermion interaction vertices in the model are explicitly chiral (contain the right-handed projector $\mathbb{P}_\text{R}$), thus any fermion propagator connecting such vertices get their extra left-handed component projected out. Any loop correction to a fermion propagator contains at least one such vertex connected to the fermion line, therefore such correction will only contribute to the renormalization of the right-handed part of the fermion kinetic term. The 1-loop result produced two evanescent operators with their corresponding Feynman rules,

$$\int \mathrm{d}^d\, x\; \frac{1}{2}\bar{G}^{a\,\mu}\widehat{\partial}^2\bar{G}^b_\mu \;\Rightarrow\; -i\widehat{p}^2\overline{g}_{\mu\nu}\delta^{ab},$$

$$\widehat{S_{\Phi\Phi}} = -\int \mathrm{d}^d\, x\; \frac{1}{2}\Phi^m\widehat{\partial}^2\Phi^m \;\Rightarrow\; i\widehat{p}^2\delta^{mn}\,.$$

We observe that, should we have used instead another d-dimensional choice for the fermion-gauge interaction term with a $\gamma^\mu\mathbb{P}_\text{R}$, we would have obtained *many more* evanescent operators, so even those choices are equally correct, we prefer the most elegant one in terms of simplicity. However, it would be interesting to see how the results behave for the general choice of this interaction. That question is part of the current research for the abelian case.

We will now re-express the result for the singular counterterms in the structure announced in Chapter 5 and make contact to the usual renormalization transformation. The sum of the singular counterterms can be written as

$$S^{(1)}_{\text{sct}} = S^{(1)}_{\text{sct,inv}} + S^{(1)}_{\text{sct,evan}}\,, \tag{6.3.5}$$

where the first term arises from renormalization transformation as in Eq. (5.2.1) and is given by Eq. (5.2.4):

$$\begin{aligned}S_{\text{ct,inv}} = {} & \frac{\delta Z_G}{2} L_G + \frac{\delta Z_{\psi_R}}{2}\overline{L_{\psi_R}} + \frac{\delta Z_\Phi}{2} L_\Phi + \frac{\delta Z_c}{2} L_c \\ & + \frac{\delta g}{g} L_g + \left(\delta(Y_R)^m_{ij} L_{Y_R}{}^m_{ij} + \text{h.c.}\right) + \delta\lambda^{mnop} L_{\lambda^{mnop}}\,,\end{aligned}$$

while the second term contains purely evanescent quantities. The renormalization constants needed in Eq. (5.1.1) agree with the usual ones obtained for non-chiral theory (where we keep in mind we have just right-handed fermions in loops) (see e.g. [67–69]) and read

$$\delta Z^{(1)}_G = \frac{\hbar}{16\pi^2\epsilon} g^2 \frac{(13-3\xi)C_2(G) - 4S_2(R) - S_2(S)}{6}\,, \tag{6.3.6}$$

$$\delta Z^{(1)}_{\psi_R} = \frac{-\hbar}{16\pi^2\epsilon}\left(g^2\xi C_2(R) + \frac{Y_2(R)}{2}\right)\,, \tag{6.3.7}$$

$$\delta Z^{(1)}_\Phi = \frac{\hbar}{16\pi^2\epsilon}\left(g^2(3-\xi)C_2(S) - Y_2(S)\right)\,, \tag{6.3.8}$$

$$\delta Z^{(1)}_c = 2\delta Z^{(1)}_{\rho c} + \delta Z^{(1)}_G = \frac{\hbar}{16\pi^2\epsilon} g^2 \frac{(22-6\xi)C_2(G) - 4S_2(R) - S_2(S)}{6}\,, \tag{6.3.9}$$

where $\delta Z^{(1)}_{\rho c}$ is the coefficient of $S_{\rho c}$ in $S^{(1)}_{\text{sct}}$:

$$\delta Z^{(1)}_{\rho c} \equiv \frac{\hbar}{16\pi^2\epsilon} g^2 \frac{3-\xi}{4} C_2(G)\,;$$

$$\delta g^{(1)}/g = \frac{-\hbar}{16\pi^2\epsilon} g^2 \frac{22C_2(G) - 4S_2(R) - S_2(S)}{12}\,, \tag{6.3.10}$$

$$\delta(Y_R)^{m,(1)}_{ij} = \delta Z^{m,(1)}_{Y,ij} - (\delta Z^{(1)}_{\psi_R} + \delta Z^{(1)}_\Phi/2)(Y_R)^m_{ij}\,, \tag{6.3.11}$$

where $\delta Z^{m,(1)}_{Y,ij}$ is the coefficient of $S_{\overline{\psi_{Ri}}{}^C \Phi^m \psi_{Rj}}$ in $S^{(1)}_{\text{sct}}$:

$$\delta Z^{m,(1)}_{Y,ij} \equiv \frac{\hbar}{16\pi^2\epsilon}\left((Y^n_R (Y^m_R)^* Y^n_R) - g^2 \frac{2C_2(R)(3+\xi) - C_2(S)(3-\xi)}{2} Y^m_R\right)_{ij}\,;$$

$$\delta\lambda^{(1)}_{mnop} = \delta Z^{(1)}_{4\Phi,mnop} - 2\delta Z^{(1)}_\Phi \lambda_{mnop}\,, \tag{6.3.12}$$

where $\delta Z^{(1)}_{4\Phi,mnop}$ is the coefficient of $S_{\Phi^4_{mnop}}$ in $S^{(1)}_{\text{sct}}$:

$$\delta Z^{(1)}_{4\Phi,mnop} \equiv \frac{\hbar}{16\pi^2\epsilon}\frac{1}{2}(3g^4 A - g^2\xi\Lambda^S - 4H + \Lambda^2)_{mnop}\,.$$

The evanescent counterterms appearing in Eq. (6.3.5) can be written as

$$\begin{aligned} S^{(1)}_{\text{sct,evan}} = \frac{-\hbar}{16\pi^2\epsilon}\Bigg\{ & g^2\frac{S_2(R)}{3}\left(2(\widetilde{S}_{GG}+\widetilde{S}_{GGG}+\widetilde{S}_{GGGG})+\int \mathrm{d}^d\,x\;\frac{1}{2}\bar{G}^{a\,\mu}\hat{\partial}^2\bar{G}^a_\mu\right) \\ & + Y_2(S)\left((\widetilde{S}_{\Phi\Phi}+\widetilde{S}_{\Phi G\Phi}+\widetilde{S}_{\Phi GG\Phi})+\frac{2}{3}\widehat{S_{\Phi\Phi}}\right)\Bigg\}\,, \end{aligned} \tag{6.3.13}$$

where we introduce new notation for evanescent operators as

$$\widetilde{S}_{\mathcal{O}} = \overline{S}_{\mathcal{O}} - S_{\mathcal{O}} \quad \text{for } \mathcal{O} = GG, GGG, GGGG, \Phi\Phi, \Phi G\Phi, \Phi GG\Phi\,. \tag{6.3.14}$$

We close this chapter with the following imporant remarks:

1. The renormalization transformation even in the BMHV treatment provides most of the counterterms at the 1-loop level. It must be applied to the invariant part of the tree-level action, not to the evanescent part which contains the d-dimensional extension of the fermion kinetic term. As a result the counterterms $S^{(1)}_{\text{sct,inv}}$ contain only purely 4-dimensional fermion terms.

2. The remaining evanescent counterterms are specific to the BMHV scheme and do not have an equivalent in naive schemes. They involve all vertices of scalars and vectors with up to 4 legs. The evanescent terms of the form $\widetilde{S}^{(1)}_{\mathcal{O}}$ are gauge invariant and evanescent; however, the two additional evanescent terms present in Eq. (6.3.13), contributions to the gauge boson and scalar two-point function counterterms, are not gauge invariant.

3. The corresponding result for a gauge theory without scalars has already been obtained in Ref. [27]; since our non-scalar contributions agree with what is already known, it is an additional check to the consistency of our calculations. The scalars contribute in two ways: they provide additional contributions to the invariant counterterms $S^{(1)}_{\text{sct,inv}}$ and thus to the renormalization constants in Eqs. (6.3.6) to (6.3.12). These contributions are equal to the case without the BMHV scheme e.g in vector-like theory or naive treatment. Second, there is an explicit evanescent scalar operator present in Eq. (6.3.13) that originates from fermion loop contributions to the scalar self-energy.

4. The result presented here is specific to our choice of the regularized, d-dimensional theory Eq. (4.2.14), based on Eq. (4.2.6). This choice does not generate an extra evanescent counterterm to the fermion two-point function. Had we used another choice out of the options indicated in Eq. (4.2.1), the result would have been different. As an illustration that shows why this choice was made we provide here the results for the

self-energies corresponding to replacing the object $\mathbb{P}_{\rm L}\gamma_\mu\mathbb{P}_{\rm R}$ by $\gamma_\mu\mathbb{P}_{\rm R}$ (choice designated by "Alt") in the fermion-gauge boson interaction. The scalar self-energy does not change, but the fermion and gauge boson self-energies change as

$$i\widetilde{\Gamma}^{ji}_{\psi\bar{\psi}}(p)|^{\rm Alt,(1)}_{\rm div} = i\widetilde{\Gamma}^{ji}_{\psi\bar{\psi}}(p)|^{(1)}_{\rm div} - \frac{i\hbar g^2}{16\pi^2\epsilon}C_2(R)\delta^{ij}\,\widehat{\not p}\,\mathbb{P}_{\rm R}\,, \tag{6.3.15}$$

$$i\widetilde{\Gamma}^{ba,\nu\mu}_{GG}(p)|^{\rm Alt,(1)}_{\rm div} = i\widetilde{\Gamma}^{ba,\nu\mu}_{GG}(p)|^{(1)}_{\rm div} + \frac{i\hbar g^2}{16\pi^2\epsilon}\frac{S_2(R)}{3}\delta^{ab}(\overline{p}^\mu\widehat{p}^\nu + 2\widehat{p}^\mu\widehat{p}^\nu + \widehat{p}^\mu\overline{p}^\nu + \overline{p}^2\widehat{g}^{\mu\nu})\,. \tag{6.3.16}$$

We see that both self-energies receive additional evanescent contributions and the structure of the resulting $S^{(1)}_{\rm sct,evan}$ will become considerably more complicated. In particular, a new evanescent counterterm to the fermion two-point function would have appeared,

$$S^{\rm Alt,(1)}_{\rm sct,evan} \supset \frac{\hbar}{16\pi^2\epsilon}g^2C_2(R)\int d^4x\, i\overline{\psi}_i\widehat{\not\partial}\mathbb{P}_{\rm R}\psi_i. \tag{6.3.17}$$

At the very end, when theory is renormalized, final results in physical limit must all agree, so the choice is the question of the simplicity of the middle steps of the procedure. Let's also mention that this choice is not always obvious, but is evident after the practitioner tries several options and checks the outcome. For the reader interested in technical details, this alternative choice corresponds to vertex generated by default by `FeynRules`. No matter how one defines fermion-gauge boson interaction vertex, the program will interpret it in 4-dimensions and automatically use anticommuting γ_5. We outsmarted this by altering interactions manually in the output model files.

CHAPTER 7

BRST SYMMETRY BREAKING AND ITS RESTORATION

In the previous chapter, we evaluated and listed all singular counterterms needed to regularize the right-handed Yang-Mills model. However, at the end of the day, the theory is considered renormalized when all symmetries of the tree-level action are again valid after, in this particular case, 1-loop evaluation. Since the BMHV scheme broke both gauge and BRST invariance, symmetries must be restored by proper counterterms. This chapter is devoted to that task.

7.1 | Renormalization condition

The ultimate requirement is that after renormalization at the some particular loop-order, the finite effective action Γ_{Ren} satisfies the Slavnov-Taylor identity,

$$\mathcal{S}(\Gamma_{\text{Ren}}) = 0\,. \tag{7.1.1}$$

Here we come to the central problem of the BMHV scheme – restoration of the BRST symmetry and underlying gauge invariance. To restore original symmetries, we have to determine finite counterterms, where the information we need is contained in the effective action at the 1-loop level. Following the expansion in Eq. (3.1.9), we see that effective action up to 1-loop level in total contains

$$\Gamma^{(1)}_{\text{DReg}} = \Gamma^{(1)} + S^{(1)}_{\text{sct}} + S^{(1)}_{\text{fct}}\,, \tag{7.1.2}$$

where $\Gamma^{(1)}$ denotes the effective action from tree-level and genuine 1-loop diagrams, without counterterms. Regularized action in principle still contains evanescent objects, so when the limit $d \to 4$ is performed, the renormalized 1-loop effective action is obtained by

$$\Gamma^{(1)}_{\text{Ren}} = \underset{d\to 4}{\text{LIM}}\, \Gamma^{(1)}_{\text{DReg}}, \tag{7.1.3}$$

as defined in Section 2.2.1 and Eq. (3.1.10). If we perform Slavnov-Taylor operator to the regularized action, we get the Slavnov-Taylor identity in d dimensions at the 1-loop level as

$$\mathcal{S}_d(\Gamma^{(1)}_{\mathrm{DReg}}) = \mathcal{S}_d(\Gamma^{(1)}) + b_d S^{(1)}_{\mathrm{sct}} + b_d S^{(1)}_{\mathrm{fct}}\,; \tag{7.1.4}$$

here the linearized operator b_d of Eq. (4.3.3) has been used and terms of higher loop order have been dismissed.

The first term on the right-hand side of equation (7.1.4) is expected to be nonzero in the BMHV scheme and corresponds to the breaking of the Slavnov-Taylor identity by 1-loop regularized Green's functions. The second term by construction cancels any UV divergences present in the first term, as we will explicitely show. The last term contains the finite counterterms that will be evaluated and discussed in the present section. These finite counterterms must be chosen in a way that the finite parts of the first term are cancelled in the $\mathrm{LIM}_{d\to 4}$.

The determination of the symmetry-restoring finite counterterms thus requires three technical steps:

1. Evaluate the symmetry breaking caused by the genuine 1-loop diagrams and the required singular counterterms coming from the insertion diagrams, i.e. evaluate $\mathcal{S}_d(\Gamma^{(1)})$ and $b_d S^{(1)}_{\mathrm{sct}}$. Check the consistency by the cancellation of these two terms.

2. Find the symmetry-restoring counterterms $S^{(1)}_{\mathrm{fct}}$, so that their b_d-variation cancels the symmetry breaking.

3. Check the Slavnov-Taylor identity in the renormalized limit after BRST restoring counterterms are taken into account. If an anomaly is still present, it is a *non-spurious* or *essential* anomaly, which means it can not be canceled by choice of proper counterterm, but by meeting the anomaly cancellation condition (usually in the form of restrictions on group structures).

Before presenting the procedure for obtaining the finite counterterms in detail we provide several remarks we find important to the reader. First, we emphasize that the set of symmetry-restoring finite counterterms is *not unique*. In general, the finite counterterms can always be written as (see also Chapter 5)

$$S^{(1)}_{\mathrm{fct}} = S^{(1)}_{\mathrm{fct,inv}} + S^{(1)}_{\mathrm{fct,restore}} + S^{(1)}_{\mathrm{fct,evan}}\,. \tag{7.1.5}$$

Here $S^{(1)}_{\mathrm{fct,inv}}$ originates from the renormalization transformation (5.2.1) and is symmetry invariant in the sense of (5.2.2); the evanescent counterterms $S^{(1)}_{\mathrm{fct,evan}}$ vanish in the $\mathrm{LIM}_{d\to 4}$ by definition and are therefore can not spoil the symmetry restoration at the 1-loop level[1]. Therefore, the symmetry-restoring 1-loop counterterms are given by $S^{(1)}_{\text{fct,restore}}$. So total set of finite counterterms is only unambiguous up to shifting around terms obtained by different renormalization transformations and/or evanescent terms. What we will provide in the present section is one particular representative the choice for these symmetry-restoring counterterms, most suitable, up to our knowledge, to higher-order calculations.

Regarding the evaluation of the symmetry breaking caused by the first and second[2] terms on the r.h.s. of (7.1.4), there are several methods to determine the breaking of the symmetry. The most straightforward way is to directly compute all the required Green's functions in divergent and finite part and insert them into the Slavnov-Taylor identity. Such a direct approach was used e.g. in Ref. [29] for comparing the BMHV *vs.* the naive γ_5 schemes in flavor-changing neutral processes, in Refs. [31, 32] in the study of chiral gauge theories and e.g. in Refs. [30, 33, 86] in similar applications on supersymmetric gauge theories. An advantage of this method is the direct connection to Green's functions appearing in physical processes and the explicit control over the symmetry breaking. From the practical point of view, these calculations can easily become cumbersome and of an unnecessary extent, especially in finite parts, and especially in the extensive studies where renormalization of all theory is of interest. In that way practitioner unnecessarily complicates his or her task since only symmetry-breaking parts of Green functions are of interest, not all of them.

Fortunately, this can be avoided thanks to second, more indirect method based on the *regularized quantum action principle*, established for dimensional regularization in Ref. [20] and stated in Chapter 3. This regularized quantum action principle implies d-dimensional breaking of the form

$$\mathcal{S}_d(\Gamma^{(1)}) = \widehat{\Delta} \cdot \Gamma^{(1)} = [\widehat{\Delta} \cdot \Gamma_{\text{DReg}}]^{(1)} = [\widehat{\Delta} \cdot \Gamma_{\text{DReg}}]^{(1)}_{\text{div}} + [\widehat{\Delta} \cdot \Gamma_{\text{DReg}}]^{(1)}_{\text{fin}}, \tag{7.1.6}$$

where $\widehat{\Delta} = s_d S_0$ is the original tree-level BRST symmetry breaking Eq. (4.3.8), while the full r.h.s. denotes the generating functional of 1-loop regularized Green's functions with one insertion corresponding to $\widehat{\Delta}$. The r.h.s. of Eq. (7.1.6) also contains the tree-level result Eq. (4.3.8), but this tree-level result will be irrelevant in the following when we take only the UV divergent part and/or the $\text{LIM}_{d\to 4}$ of Eq. (7.1.6). Using this relation, the computation is dramatically simplified since the r.h.s. involves far fewer, and simpler Feynman diagrams than the left-hand side. The reason is because l.h.s. formally holds all the information contained in effective action, and then the Slavnov-Taylor operator is performed on it. Furthermore, it does not involve the evaluation of products of 1PI Green's

[1]The choice of 1-loop evanescent counterterms will have an impact on two- and higher-loop calculations.
[2]Symmetry-restoring counterterms are by themselves, obviously, BRST non-invariant, since when added to BRST non-invariant structures they form BRST invariance.

functions, as would be the case in the direct approach, when the Slavnov-Taylor operator is applied. This indirect method has been applied in the literature, e.g. in Ref. [20] to scale invariance, in [27, 28] to chiral non-abelian and abelian gauge theories at the 1-loop level, and in Refs. [54, 87, 88] in a similar way to supersymmetric theories at the 2- and 3-loop level.

In this work, we will apply the second method, which we find more suitable for our overall type of studies. In Section 7.3 we will also present additional reasons why this holds.

The condition that the Slavnov-Taylor identity is satisfied at the 1-loop level in the 4-dimensional limit is now

$$\boxed{0 = \underset{d\to 4}{\mathrm{LIM}}\left([\widehat{\Delta}\cdot\Gamma^{(1)}]_{\mathrm{div}} + b_d S^{(1)}_{\mathrm{sct}} + [\widehat{\Delta}\cdot\Gamma^{(1)}]_{\mathrm{fin}} + b_d S^{(1)}_{\mathrm{fct,restore}}\right)} \tag{7.1.7}$$

where the subscripts "div"/"fin" denote the divergent and finite parts, respectively. Notice that we regrouped singular and finite parts since they can only cancel first divergent parts (with possible finite residue) and than all finite parts. This is the defining condition for the 1-loop symmetry-restoring counterterms and once it is satisfied (and all anomaly cancellation conditions are met) the model is renormalized at 1-loop order. The following Section 7.2 will present the evaluation of the first two, divergent quantities, and Section 7.3 will present the evaluation of the finite parts of $[\widehat{\Delta}\cdot\Gamma^{(1)}]_{\mathrm{fin}}$. In Section 7.4 we will determine and present the required finite, symmetry-restoring counterterms.

7.2 | Evaluation of divergent insertion and comparison with singular breaking

In this section we present the evaluation of the divergent quantities appearing in Eq. (7.1.7), i.e. first and the second term on the l.h.s. By construction, it is clear that these two quantities must add up at least to something finite since their divergent parts must cancel; however, we will see in the following that they actually add up to zero. The basic reason is that both quantities are pure divergences, and no terms of the form ϵ/ϵ can not be generated from combining evanescent terms with UV singularities, up to the 1-loop level.

First we will calculate the BRST breaking of the singular counterterm action, $b_d S^{(1)}_{\mathrm{sct}}$. The form of the action constructed just for this purpose, evaluation of the BRST breaking, is the one introduced in Chapter 5 in the fashion of L_ϕ invariants. As we evaluated, the L_ϕ terms present in the invariant part of the singular counterterms in Eqs. (5.2.4) and (6.3.5) are b_d-invariant, except for L_c and L_g where $b_d L_{c,g} = \widehat{\Delta}$. Several of the evanescent terms specified in Eq. (6.3.13) are b_d-invariant as well.

For the final aim of this section, evaluation of the BRST breaking of the singular action, the missing ingredients are b_d transformations of the fermion–gauge-boson interaction and scalar quartic interaction at the 1-loop level. For the first transformation we get

$$b_d((Y_R^n(Y_R^m)^*Y_R^n)_{ij}S_{\overline{\psi_{Ri}}^C\Phi^m\psi_{Rj}}+\text{h.c.})\propto\theta^a_{no}(Y_R^n(Y_R^m)^*Y_R^o+Y_R^o(Y_R^m)^*Y_R^n)_{ij}=0\,,\quad(7.2.1)$$

where vanishing is due to a group structure that, when simplified using the gauge-invariance property Eq. (4.1.16a) reduces to the structure that cancels due to the antisymmetry of θ^a. Notice that the invariance is now satisfied via group structure, not by operators themselves. Similar will happen in the next term. For the BRST transformation of the part including four-scalar interaction, we get

$$b_d((3g^4A-4H+\Lambda^2)_{mnop}S_{\Phi^4_{mnop}})=4(3g^4A-4H+\Lambda^2)_{qnop}\theta^a_{qm}\frac{ig}{2}\int\mathrm{d}^dx\;c_aS_{\Phi^4_{mnop}}=0\,.\quad(7.2.2)$$

The group factor $(3g^4A-4H+\Lambda^2)_{mnop}$ is completely symmetric in its indices, same as the tree-level scalar self-coupling λ_{mnop}, and its contraction with θ^a_{qm} can be rewritten similarly to Eq. (4.1.17). Notice in this example the fact that multiplicative renormalization transformation for scalar self-coupling λ_{mnop} would not be sufficient, as we assumed. So, we do not explicitly obtain λ_{mnop} proportionality at the 1-loop level, but the object is still symmetric in its indices. For the reader interested in details, for each term involved: $A_{qnop}\theta^a_{qm}S_{\Phi^4_{mnop}}$, $\Lambda^2_{qnop}\theta^a_{qm}S_{\Phi^4_{mnop}}$ and $H_{qnop}\theta^a_{qm}S_{\Phi^4_{mnop}}$, we throughly exploit the allowed symmetrizations in group indices so as to exhibit contractions between symmetric and antisymmetric symbols or internal cancellations, leading to the complete cancellation of these three terms. Finally, the last term in H_{qnop} furthermore requires the usage of Eq. (4.1.16a).

Non-vanishing transformations form the symmetry breaking in the singular part,

$$b_dS^{(1)}_{\text{sct}}=\frac{-\hbar}{16\pi^2\epsilon}\left\{g^2\frac{\xi C_2(G)}{2}\widehat{\Delta}+g^2\frac{S_2(R)}{3}b_d\int\mathrm{d}^dx\;\frac{1}{2}\bar{G}^{a\,\mu}\widehat{\partial}^2\bar{G}^a_\mu+\frac{2Y_2(S)}{3}b_d\widehat{S_{\Phi\Phi}}\right\}\,,\quad(7.2.3)$$

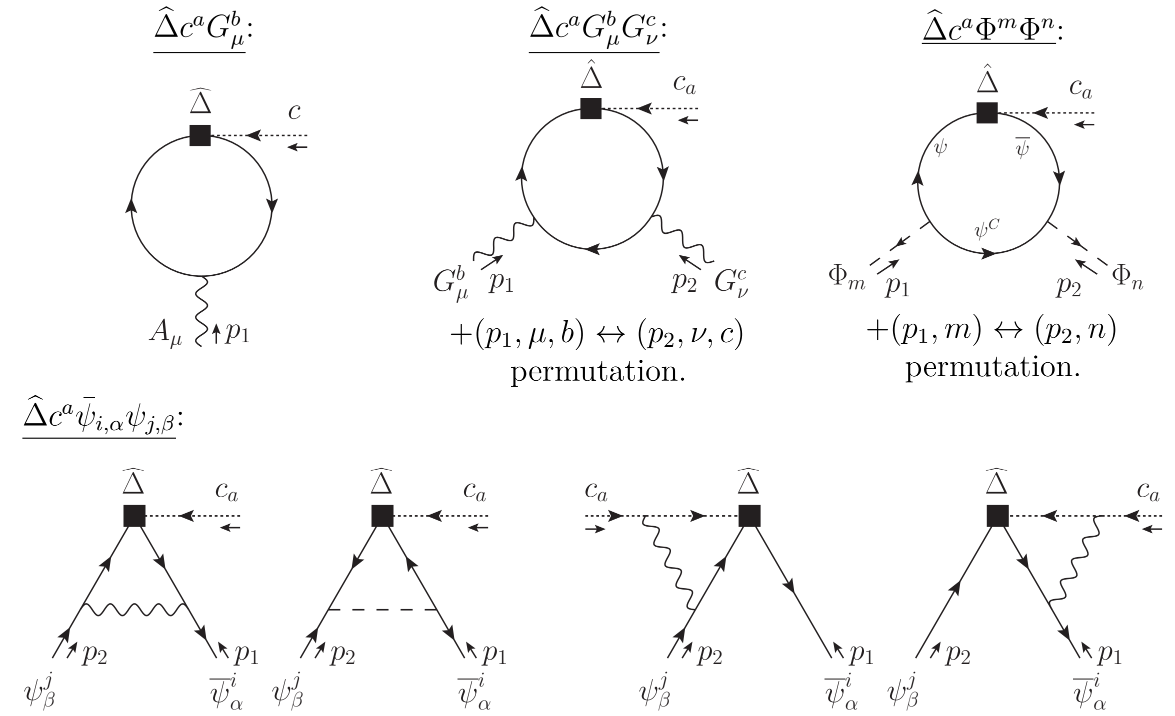
where, in the last two terms, b_d actually acts like the BRST transformation, leading to:

$$b_d\int\mathrm{d}^dx\;\frac{1}{2}\bar{G}^{a\,\mu}\widehat{\partial}^2\bar{G}^a_\mu=\int\mathrm{d}^dx\;(s_d\bar{G}^{a\,\mu})\widehat{\partial}^2\bar{G}^a_\mu=\int\mathrm{d}^dx\;(\bar{\partial}^\mu c_a+gf^{abc}\bar{G}^{b\,\mu}c_c)\widehat{\partial}^2\bar{G}^a_\mu\,,\quad(7.2.4a)$$

$$b_d\widehat{S_{\Phi\Phi}}=b_d\int\mathrm{d}^dx\;\frac{-1}{2}\Phi_m\widehat{\partial}^2\Phi_m=-\int\mathrm{d}^dx\;(s_d\Phi_m)\widehat{\partial}^2\Phi_m=\int\mathrm{d}^dx\;ig\theta^a_{mn}c^a\Phi_m\widehat{\partial}^2\Phi_n\,.\quad(7.2.4b)$$

In total we have three breaking terms: one for the fermions (proportional to the tree-level breaking $\widehat{\Delta}$), one for the gauge bosons and one for the scalars. Notice that, as announced, Eq. (7.2.3) is a pure $1/\epsilon$ singular term; no finite terms are generated by applying the d-dimensional operator b_d onto the singular counterterm action, at the 1-loop level, i.e. it is impossible to generate finite breakings from the b_d transformation of the singular part of the action.

Now we proceed with the evaluation of first term of Eq. (7.1.7). For evaluating $[\widehat{\Delta} \cdot \Gamma^{(1)}]_{\text{div}}$ we calculate all possible 1-loop vertex corrections with insertion of the $\widehat{\Delta}$ evanescent operator. We usually refer to these types of diagrams as *insertion diagrams*. In our prescription all momenta are incoming and all the results use $d = 4 - 2\epsilon$. Below is the list of all diagrams with a $\widehat{\Delta}$ insertion that have a non-vanishing divergent part:



(a) Vanishing diagrams.

(b) Diagrams giving the $\mathbb{P}_{\text{R}}$ and $\mathbb{P}_{\text{L}}$ contributions respectively.

and their result is the following,

$$i[\widehat{\Delta} \cdot \widetilde{\Gamma}^{ba,\mu}_{Gc}]^{(1)}_{\text{div}} = \frac{\hbar g^2}{16\pi^2 \epsilon} \frac{S_2(R)}{3} \delta^{ab} \widehat{p_1}^2 \overline{p_1}^\mu \,, \tag{7.2.5a}$$

$$i[\widehat{\Delta} \cdot \widetilde{\Gamma}^{cba,\nu\mu}_{GGc}]^{(1)}_{\text{div}} = \frac{-i\hbar g^3}{16\pi^2 \epsilon} \frac{S_2(R)}{3} f^{abc} (\widehat{p_1}^2 - \widehat{p_2}^2) \overline{g}^{\mu\nu} \,, \tag{7.2.5b}$$

$$i[\widehat{\Delta} \cdot \widetilde{\Gamma}^{nm,a}_{\Phi\Phi c}]^{(1)}_{\text{div}} = \frac{-\hbar g}{16\pi^2 \epsilon} \frac{2Y_2(S)}{3} \theta^a_{mn} (\widehat{p_1}^2 - \widehat{p_2}^2) \,, \tag{7.2.5c}$$

$$i[\widehat{\Delta} \cdot \widetilde{\Gamma}^{ji,a}_{\psi\bar{\psi}c}]^{(1)}_{\text{div}} = \frac{\hbar g^3}{16\pi^2 \epsilon} \frac{\xi C_2(G)}{2} T_{R}{}^a_{ij} (\widehat{\not{p}_1} \mathbb{P}_{\text{R}} + \widehat{\not{p}_2} \mathbb{P}_{\text{L}}) \,. \tag{7.2.5d}$$

The sum of these 1PI contributions evaluated in this section, when Fourirer transformed to coordinate space, constitutes the non-vanishing contribution to $[\widehat{\Delta}\cdot\Gamma]^{(1)}_{\text{div}}$:

$$[\widehat{\Delta}\cdot\Gamma]^{(1)}_{\text{div}} = \frac{\hbar}{16\pi^2\epsilon}\Bigg\{g^2\frac{\xi C_2(G)}{2}\widehat{\Delta} + g^2\frac{S_2(R)}{3}\int \mathrm{d}^d x\, (\bar{\partial}^\mu c_a + g f^{abc}\bar{G}^{b\,\mu} c_c)\widehat{\partial}^2\bar{G}^a_\mu \\ + \frac{2Y_2(S)}{3}\int \mathrm{d}^d x\; ig\theta^a_{mn} c^a\Phi_m\widehat{\partial}^2\Phi_n\Bigg\}\,, \quad (7.2.6)$$

and by comparing with Eq. (7.2.3) that provides the expression of $b_d S^{(1)}_{\text{sct}}$, we conclude that there exists a *complete cancellation*,

$$b_d S^{(1)}_{\text{sct}} + [\widehat{\Delta}\cdot\Gamma]^{(1)}_{\text{div}} = 0\,, \quad (7.2.7)$$

even before performing a 4-dimensional limit.
While this result is expected, in the practical sense it is welcome, since it confirms that practitioners procedure of calculating insertion diagrams (what usually includes manual implementation of $\widehat{\Delta}$) is correct, and we can safely proceed to the calculation we present in the next section.

7.3 | Evaluation of finite insertion

This section presents the evaluation of the finite quantity appearing in Eq. (7.1.7), i.e. $\text{LIM}_{d\to 4}[\widehat{\Delta}\cdot\Gamma^{(1)}]_{\text{fin}}$, and is closely following procedure from [47]. This is the central quantity which describes the 1-loop symmetry breaking caused by the BMHV scheme for γ_5, but also detects the essential anomalies present in the theory that are not a consequence of the scheme usage. This calculation will provide a particularly efficient way to evaluate the symmetry breaking and will provide the information how to fix the last term in Eq. (7.1.7). Indeed, this finite quantity accounts for the finite part of the Slavnov-Taylor identity breaking which, if we were using the direct method instead, would be evaluated using products of 1PI Green's functions, including their finite parts, which is in general a difficult matter, as we stated before. Here instead, only UV-divergent parts of specific Green's functions will be required, as we will see, thanks to so-called Bonneau identities.

At first order in $\hbar$, our quantity of interest may be expressed as

$$\underset{d\to 4}{\text{LIM}}[\widehat{\Delta}\cdot\Gamma^{(1)}]_{\text{fin}} = [N[\widehat{\Delta}]\cdot\Gamma_{\text{Ren}}]^{(1)}\,, \quad (7.3.1)$$

where the subscript "Ren" implies minimal subtraction and taking the $\text{LIM}_{d\to 4}$. Here $N[\mathcal{O}]$ denotes the Zimmermann-like definition introduced in Chapter 3. Let us begin with further comments on how to evaluate $[N[\widehat{\Delta}]\cdot\Gamma_{\text{Ren}}]^{(1)}$. At the 1-loop level, it is reasonably straightforward to carry out a direct computation, extending the computation of the divergent parts in the previous section. However, it is useful to first discuss the structure of the computation in more detail.

The BRST breaking vertex operator $\widehat{\Delta}$ in its local form is proportional to the evanescent metric:

$$\widehat{\Delta} = \hat{g}_{\mu\nu}\Delta^{\mu\nu} = (g_{\mu\nu} - \bar{g}_{\mu\nu})\Delta^{\mu\nu}, \tag{7.3.2}$$

where we used the fact that $\Delta^{\mu\nu}$ contains $\partial^\mu\gamma^\nu$ covariants, what is evident from Eq. (4.3.9). Finite contributions are generated once $\widehat{\Delta}$ is inserted into loop diagrams. This insertion will provide the evanescent numerator, and then the evanescent numerator combines with a $1/\epsilon$ singularity to form a finite term that behaves schematically as ϵ/ϵ. Using this simple fact, we can expect that the finite symmetry breaking can also be obtained, with the proper normalization, from extracting only the singular parts of suitable diagrams. Such a relationship is provided by an identity due to Bonneau [45, 46]. The general form of this identity for the insertion of our interest has the form

$$N[\widehat{\Delta}_{\mu\nu}\mathcal{O}^{\mu\nu}](x)\cdot\Gamma_{\text{Ren}} = \sum_{n=2}^{n_{\max}}\sum_{r}^{4-n}\sum_{\substack{\{i_1,\dots,i_r\}\\ 1\le i_j\le n}}\Bigg\{\frac{i^r}{r!}\frac{\partial^r}{\partial p_{i_1}^{\mu_1}\dots\partial p_{i_r}^{\mu_r}}$$
$$\times\,\text{r.s.p.}\overline{\langle N[\check{\Delta}_{\mu\nu}\mathcal{O}^{\mu\nu}]\tilde{\phi}_{j_1}(p_1)\dots\tilde{\phi}_{j_n}(p_n)\rangle}^{1PI}\Bigg|_{p_i=\check{g}=0}\Bigg\} N\left[\frac{-i}{n!}\prod_{k=n}^{1}\prod_{\alpha/i_\alpha=k}\partial_{\mu_\alpha}\phi_{j_k}(x)\right]\cdot\Gamma_{\text{Ren}}\,,$$

where the bar implies that minimal subtraction of the subdivergences has been done. On the right-hand side "r.s.p." means the residue of the simple pole in $\nu = 4-d = 2\epsilon$. Bonneau identity derivation can be also found in [27] and in [89]. What is of importance for us in this identity is that the n is the number of fields in the monomial and it is bounded by 2 and $n_{\max}$. Here we discuss the essence of this identity and its form applied to our 1-loop case. This will provide valuable additional understanding of the symmetry breaking.

The essential property contained in the Bonneau identity can be explained with the help of the equation

$$N[\widehat{\Delta}(x)] = N[g_{\mu\nu}\Delta^{\mu\nu}(x)] - N[\bar{g}_{\mu\nu}\Delta^{\mu\nu}(x)] = N[g_{\mu\nu}\Delta^{\mu\nu}(x)] - \bar{g}_{\mu\nu}N[\Delta^{\mu\nu}(x)]\,. \tag{7.3.3}$$

The first equation in (7.3.3) makes explicit the appearance of the evanescent metric, which is decomposed as $g_{\mu\nu} - \bar{g}_{\mu\nu}$. The second equation highlights that pulling the metric out of the minimal subtraction procedure is possible only for the purely 4-dimensional metric, but not for the d-dimensional metric where doing this operation would not commute with the minimal subtraction procedure, and therefore Eq. (7.3.3) does not vanish. Note that $N[\Delta^{\mu\nu}(x)]$ is a 4-dimensional object since it has been submitted to the renormalization procedure, therefore its contraction with $\bar{g}_{\mu\nu}$ is the same as its contraction with $g_{\mu\nu}$ from outside, i.e. in total we have

$$N[\widehat{\Delta}(x)] = N[g_{\mu\nu}\Delta^{\mu\nu}(x)] - g_{\mu\nu}N[\Delta^{\mu\nu}(x)]\,. \tag{7.3.4}$$

The 1-loop version of the Bonneau identity is given by

$$[N[\widehat{\mathcal{O}}]\cdot\Gamma_{\text{Ren}}]^{(1)}=\underset{d\to 4}{\text{LIM}}\left(-\text{r.s.p.}\left[\check{\mathcal{O}}\cdot\Gamma\right]^{(1)}_{\check{g}=0}\right)\,. \tag{7.3.5}$$

Let's explain this equation in detail. Here again on the right-hand side "r.s.p." means the residue of the simple pole in $\nu=4-d=2\epsilon$ of the 1PI Green's function under consideration[3]. The Feynman rules corresponding to the operator $\check{\mathcal{O}}$ are obtained from the ones for $\widehat{\mathcal{O}}$ by formally replacing all the evanescent Lorentz structures by their corresponding d-dimensional versions contracted, e.g.

$$\widehat{p}^2=p_\mu p_\nu \hat{g}^{\mu\nu}\to p_\mu p_\nu \check{g}^{\mu\nu}\equiv \check{p}^2,$$

with the symmetric "metric"-tensor $\check{g}_{\mu\nu}$, possessing the following properties:

$$\check{g}_{\mu\nu}g^{\nu\rho}=\check{g}_{\mu\nu}\hat{g}^{\nu\rho}=\check{g}_\mu^{\;\rho}\ ,\qquad \check{g}_{\mu\nu}\bar{g}^{\nu\rho}=0\ ,\qquad \check{g}_\mu^{\;\mu}=1\ . \tag{7.3.6}$$

This symbol can be understood as corresponding to the evanescent metric $\hat{g}_{\mu\nu}$ such that its trace has been normalized to one. This explains also the appearance of the minus sign on the right-hand-side of Eq. (7.3.5): its left-hand-side is proportional to $\hat{g}_{\mu\nu}$ which satisfies $\hat{g}_{\mu\nu}\hat{g}^{\nu\mu}=-2\epsilon$. The equality Eq. (7.3.5) implements the intuition developed above: the finite part of the breaking can be obtained by evaluating the UV singularity of suitable diagrams, involving the object $\check{g}_{\mu\nu}$, i.e. fishing out UV divergences proportional to 2ϵ in numerator. The significant advantage of using the Bonneau identity is that it further simplifies the evaluation of the required finite part of the insertion to an evaluation of residue of simple pole,

$$\underset{d\to 4}{\text{LIM}}[\widehat{\Delta}\cdot\Gamma^{(1)}]_{\text{fin}}=[N[\widehat{\Delta}]\cdot\Gamma_{\text{Ren}}]^{(1)}=\underset{d\to 4}{\text{LIM}}\left(-\text{r.s.p.}[\check{\Delta}\cdot\Gamma]^{(1)}_{\check{g}=0}\right)\,, \tag{7.3.7}$$

i.e. we need to determine all UV-divergent 1PI 1-loop diagrams with an insertion of $\check{\Delta}$. From now on we will refer to this object as *inverse breaking*. Clearly, at fixed loop order by power counting there is only a limited finite number of UV-singular diagrams to be evaluated. This constitutes the main advantage of this method. The number of external legs starts from $n=2$ for the graphs to be 1PI, and ends at n_{max} the maximal number of external lines for a 1PI graph with insertion of $\widehat{\mathcal{O}}$ to be superficially divergent. More precisely,

$$n_{\text{max}}=4-\sum_i\delta_{\phi_i}+(\delta_{\mathcal{O}}-4)=\delta_{\mathcal{O}}-\sum_i\delta_{\phi_i}\,, \tag{7.3.8}$$

where δ_{ϕ_i} is the canonical dimension of the field ϕ_i and $\delta_{\mathcal{O}}$ the canonical dimension of the inserted operator $\mathcal{O}$. This equation defines important restriction on the mass-dimension

[3] I.e. since we evaluate the divergent parts of the 1PI Green's functions in $d=4-2\epsilon$, we will have to take a factor 2 into account.

of the objects involved in the calculation.
In the following, we will present an exhaustive list of all diagrams contributing to the breaking and determine their values.

7.3.1 | One-loop vertices with insertion of inverse breaking

As presented above, we need to evaluate all the non-vanishing contributions to the finite breaking of the Slavnov-Taylor identity at the 1-loop level, i.e. all the non-vanishing contributions to Eq. (7.3.7). This requires evaluating the contributions to the breaking functional $[N[\widehat{\Delta}] \cdot \Gamma_{\text{Ren}}]^{(1)}$, see Eqs. (7.3.5) and (7.3.7).

We now discuss in details how this quantity is evaluated in practice, at 1-loop level. Starting with the Eq. (7.3.7) we first evaluate $[\check{\Delta} \cdot \Gamma]^{(1)}_{\check{g}=0}$, i.e. all the 1PI 1-loop diagrams with an insertion of $\check{\Delta}$, that are UV-divergent since we will extract the r.s.p. This procedure includes severals steps:

1. The complete set of non-vanishing diagrams is restricted in this case by ghost number 1 (since those objects are breaking insertions or if you like the breaking of Slavnov-Taylor identity) and bounded by Eq. (7.3.8) i.e. mass-dimension 4. The reader can easily construct this list with the help of Table 4.1.

2. At the level of Feynman rules $\check{\Delta}$ is obtained from $\widehat{\Delta}$ by converting all occurrences of evanescent Lorentz symbols inside it into contractions of their corresponding d-dimensional versions with the $\check{g}_{\mu\nu}$ symbol.

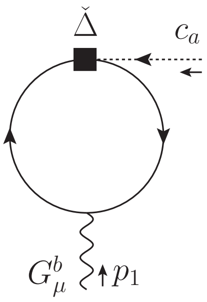
3. Evaluation of the obtained diagrams is then performed using standard loop techniques as in Chapter 6, and is followed by a *complete* tensor contraction and simplification (including Dirac structures) so as to eliminate as many $\check{g}_{\mu\nu}$ symbols as possible, using the properties Eq. (7.3.6). The property $\check{g}_\mu^{\ \mu} = 1$ of the $\check{g}_{\mu\nu}$ symbol has the effect of selecting the contributions of interest originally coming from the evanescent operator $\widehat{\Delta}$, that would have otherwise been absorbed into the finite part, if the $\check{g}_{\mu\nu}$ symbol was not used and the original evanescent metric $\hat{g}_{\mu\nu}$ was used instead.

4. When the result is simplified as much as possible, an ϵ-expansion is performed to keep only the simple-pole terms.

5. The remaining $\check{g}_{\mu\nu}$ symbols that have not been already eliminated (signalling the contribution of higher-order evanescent quantities) have to be discarded: indeed, according to the Bonneau identity, these remaining contributions would be one $\hbar$-order higher.

6. The different Lorentz structures arising from the calculation of the Green's function can be obtained and their corresponding coefficients can be extracted out. Here we have to keep in mind that the r.s.p. extracts -2ϵ factor[4].

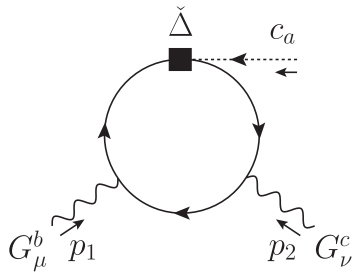
In the following, we provide the list of all these non-vanishing contributions. For each contribution, we provide the associated Feynman diagram, its result, and the corresponding contribution to the breaking functional $[N[\widehat{\Delta}]\cdot\Gamma_{\text{Ren}}]^{(1)}$. Since the operators contained in this functional are fully expressed in 4 space-time dimensions, for the sake of simplicity we will omit all the "overlines" that would otherwise be present over all the Lorentz covariants (vectors, tensors, fields, to symbolize their 4-dimensionality). We are as well employing the same notations for the integrated field monomials as in Eq. (4.2.16a), but now all defined purely in 4 dimensions.
List is the following:



$$i[\check{\Delta}\cdot\widetilde{\Gamma}^{ba,\mu}_{Gc}]^{(1)}_{\text{div}} = \frac{-\hbar g^2}{16\pi^2\epsilon}\frac{S_2(R)}{6}\delta^{ab}\overline{p_1}^2\overline{p_1}^{\mu}\,, \tag{7.3.9a}$$

results in the the contribution

$$[N[\widehat{\Delta}]\cdot\Gamma_{\text{Ren}}]^{(1)} \supset \frac{\hbar g^2}{16\pi^2}\frac{S_2(R)}{3}\int \mathrm{d}^4\,x\;(\partial^\mu c_a)(\partial^2 G^a_\mu)\,. \tag{7.3.9b}$$



$+(p_1,\mu,b)\leftrightarrow(p_2,\nu,c)$ permutation.

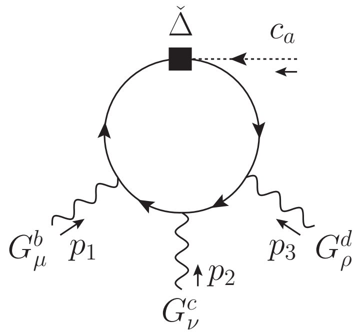
$$\begin{aligned} i[\check{\Delta}\cdot\widetilde{\Gamma}^{cba,\nu\mu}_{GGc}]^{(1)}_{\text{div}} = \frac{-i\hbar}{16\pi^2\epsilon}\frac{g^3}{6}\Big[&S_2(R)f^{abc}((\overline{p_1}^2-\overline{p_2}^2)\overline{g}^{\mu\nu} \\ &-2\overline{p_1}^{\mu}\overline{p_1}^{\nu}+2\overline{p_2}^{\mu}\overline{p_2}^{\nu})+2d^{abc}_R\epsilon^{\mu\nu\rho\sigma}\overline{p_1}_{\rho}\overline{p_2}_{\sigma}\Big]\,, \end{aligned} \tag{7.3.10a}$$

where $d^{abc}_R = \mathrm{Tr}[T_R{}^a\{T_R{}^b, T_R{}^c\}]$ is the fully symmetric symbol for the R-representation. The Green function corresponds to the following contribution in the Bonneau identity and exhibits an anomalous contribution (second line):

$$\begin{aligned} [N[\widehat{\Delta}]\cdot\Gamma_{\text{Ren}}]^{(1)} \supset\; &\frac{\hbar g^2}{16\pi^2}\frac{S_2(R)}{3}\int \mathrm{d}^4\,x\; g f^{abc} c_a G^b_\mu(\partial^2 g^{\mu\nu}-2\partial^\mu\partial^\nu)G^c_\nu \\ &-\frac{\hbar g^2}{16\pi^2}\frac{d^{abc}_R}{3}\int \mathrm{d}^4\,x\; g\epsilon^{\mu\nu\rho\sigma}c_a(\partial_\rho G^b_\mu)(\partial_\sigma G^c_\nu)\,. \end{aligned} \tag{7.3.10b}$$

[4]Finally, when transitioning from $\widetilde{\Gamma}$ momentum representation to coordinate representation Γ, keep in mind that $p\to -i\partial$ and the possible sign flips from partial integration.

This anomalous contribution, as we will see later, is not the consequence of the BMHV scheme usage so it can not be canceled by finite counterterm. For the next contribution we have:



$$+\left\{(p_1,\mu,b)\,,\,(p_2,\nu,c)\,,\,(p_3,\rho,d)\right\}\text{ permutations}^{5}.$$

$$i[\check{\Delta}\cdot\widetilde{\Gamma}^{dcba,\rho\nu\mu}_{GGGc}]^{(1)}_{\text{div}}=\frac{-\hbar}{16\pi^2\epsilon}\frac{g^4}{6}\overline{(p_1+p_2+p_3)}_\sigma\Big[\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma}(\mathcal{A}^{abcd}_R+\mathcal{A}^{acbd}_R)/2+\overline{g}^{\mu\rho}\overline{g}^{\nu\sigma}(\mathcal{A}^{abdc}_R+\mathcal{A}^{adbc}_R)/2$$
$$+\overline{g}^{\mu\sigma}\overline{g}^{\nu\rho}(\mathcal{A}^{acdb}_R+\mathcal{A}^{adcb}_R)/2-\mathcal{D}^{abcd}_R\epsilon^{\mu\nu\rho\sigma}\Big]\,. \quad (7.3.11)$$

Introducing the notation $(T_R)^{a_1\cdots a_n}=\text{Tr}[T_R{}^{a_1}\cdots T_R{}^{a_n}]$ for the trace of a product of same group generators $T_R{}^a$, we have employed in the previous equation the group factor

$$\begin{aligned}\mathcal{A}^{abcd}_R&=(T_R)^{abcd}-(T_R)^{abdc}+(T_R)^{acbd}-(T_R)^{acdb}+(T_R)^{adbc}+(T_R)^{adcb}\\&=(T_R)^{abcd}+(T_R)^{adcb}-S_2(R)f^{ace}f^{bde}=(T_R)^{acbd}+(T_R)^{adbc}-S_2(R)f^{abe}f^{cde}\\&=(T_R)^{abdc}+(T_R)^{acdb}-S_2(R)(f^{abe}f^{cde}+f^{ace}f^{bde})\\&=\frac{1}{2}((T_R)^{abcd}+(T_R)^{adcb}+(T_R)^{acbd}+(T_R)^{adbc})-\frac{S_2(R)}{2}(f^{abe}f^{cde}+f^{ace}f^{bde})\,,\end{aligned} \quad (7.3.12)$$

and we have defined the fully antisymmetric symbol[6]

$$\mathcal{D}^{abcd}_R=(-i)3!\text{Tr}[T_R{}^aT_R{}^{[b}T_R{}^cT_R{}^{d]}]=\frac{1}{2}(d^{abe}_Rf^{ecd}+d^{ace}_Rf^{edb}+d^{ade}_Rf^{ebc}),$$

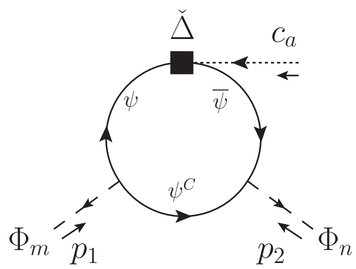
for the R-representation, in the fashion of Ref. [27]. The 1PI Green's function Eq. (7.3.11) corresponds to the contribution

$$[N[\widehat{\Delta}]\cdot\Gamma_{\text{Ren}}]^{(1)}\supset\frac{\hbar g^4}{16\pi^2}\frac{\mathcal{A}^{abcd}_R}{6}\int \mathrm{d}^4\,x\;c_a\partial^\nu\left(G^b_\mu G^{c\,\mu}G^d_\nu\right)$$
$$-\frac{\hbar g^4}{16\pi^2}\frac{\mathcal{D}^{abcd}_R}{3\times 3!}\int \mathrm{d}^4\,x\;c_a\epsilon^{\mu\nu\rho\sigma}\partial_\sigma\left(G^b_\mu G^c_\nu G^d_\rho\right)\,, \quad (7.3.13)$$

and also exhibits an non-spurious anomaly given by the last term.

[5]The third term of our calculation ($\propto\overline{g}^{\mu\sigma}\overline{g}^{\nu\rho}$) agrees with equation (53) of [27]; however, an apparent discrepancy arises when comparing the first two terms ($\propto\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma}$ and $\propto\overline{g}^{\mu\rho}\overline{g}^{\nu\sigma}$ with different group factors) with equation (54) that tells that both $\overline{p_1}_\nu\overline{g}^{\mu\rho}$ and $\overline{p_1}_\rho\overline{g}^{\mu\nu}$ acquire the very same coefficient.

[6]Here and in what follows, we employ the standard indicial notation for the (anti-)symmetrization of tensor indices (or subset thereof): $T^{[a_1\cdots a_n]}=\frac{1}{n!}\sum_\pi\sigma(\pi)T^{a_{\pi(1)}}\cdots T^{a_{\pi(n)}}$, and $T^{\{a_1\cdots a_n\}}=\frac{1}{n!}\sum_\pi T^{a_{\pi(1)}}\cdots T^{a_{\pi(n)}}$.

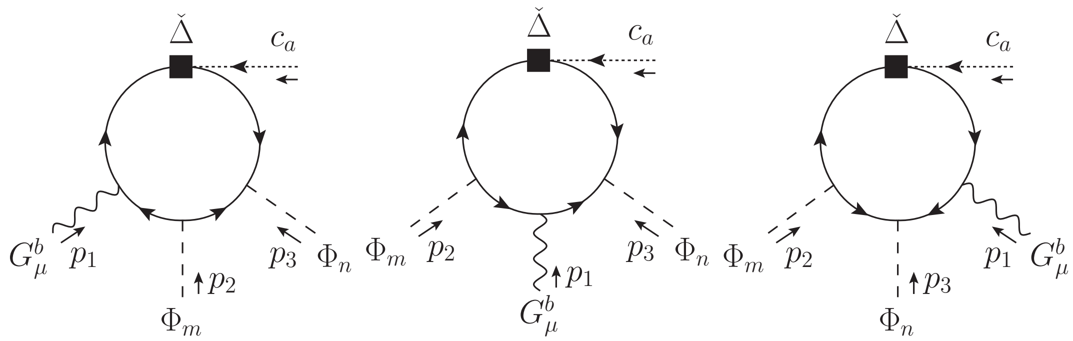


$+(p_1, m) \leftrightarrow (p_2, n)$ permutation.

$$i[\check{\Delta} \cdot \widetilde{\Gamma}^{nm,a}_{\Phi\Phi c}]^{(1)}_{\text{div}} = \frac{-\hbar g}{16\pi^2 \epsilon} \frac{Y_2(S)}{6} \theta^a_{mn} (\overline{p_1}^2 - \overline{p_2}^2) \, , \tag{7.3.14a}$$

correspons to the contribution

$$[N[\widehat{\Delta}] \cdot \Gamma_{\text{Ren}}]^{(1)} \supset -\frac{\hbar}{16\pi^2} \frac{Y_2(S)}{3} \int \mathrm{d}^4 \, x \; ig\theta^a_{mn} c^a \Phi_m \partial^2 \Phi_n \, . \tag{7.3.14b}$$



$+(p_2, m) \leftrightarrow (p_3, n)$ permutation.

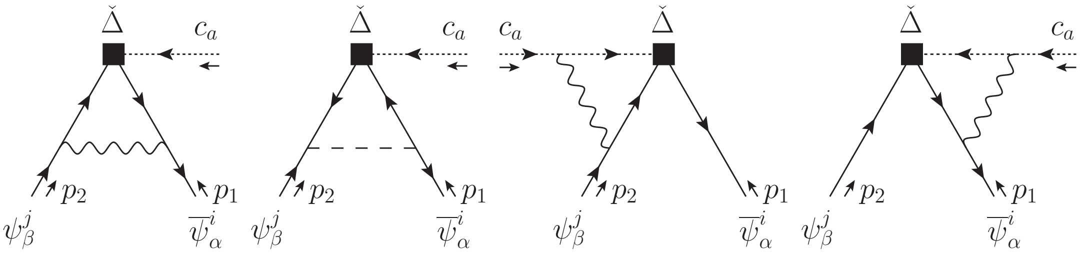
$$\begin{aligned} i[\check{\Delta} \cdot \widetilde{\Gamma}^{nm,ba,\mu}_{\Phi\Phi G c}]^{(1)}_{\text{div}} = \frac{\hbar g^2}{16\pi^2 \epsilon} \frac{1}{6} \overline{(p_1 + p_2 + p_3)}^{\mu} \text{Tr} \Big[& 2\{T_R{}^a, T_R{}^b\}((Y^m_R)^* Y^n_R + (Y^n_R)^* Y^m_R) \\ & - T_R{}^a (Y^m_R)^* T_{\overline{R}}{}^b Y^n_R - T_R{}^a (Y^n_R)^* T_{\overline{R}}{}^b Y^m_R \Big] \, , \end{aligned} \tag{7.3.15a}$$

where the different ways of inserting the fields in the fermion loop, as well as the permutations of field legs of the same type, are considered.

The trace is equal to $(\mathcal{S}_R)^{ab}_{mn} \equiv ((\mathcal{C}_R)^{ab}_{mn} + (\mathcal{C}_R)^{ba}_{mn} + m \leftrightarrow n)/2$, completely symmetric by exchanges $a \leftrightarrow b$ and $m \leftrightarrow n$, and $(\mathcal{C}_R)^{ab}_{mn} \equiv \text{Tr}\left[2\{T_R{}^a, T_R{}^b\}(Y^m_R)^* Y^n_R - T_R{}^a (Y^m_R)^* T_{\overline{R}}{}^b Y^n_R\right]$. Thus, the 1PI Green's function Eq. (7.3.15a) corresponds to the contribution

$$[N[\widehat{\Delta}] \cdot \Gamma_{\text{Ren}}]^{(1)} \supset -\frac{\hbar}{16\pi^2} \frac{(\mathcal{S}_R)^{ab}_{mn}}{3} \int \mathrm{d}^4 \, x \; \frac{g^2}{2} c_a \partial^\mu \left(G^b_\mu \Phi^m \Phi^n \right) \, . \tag{7.3.15b}$$

Besides, it is interesting to note that $\text{Tr}\left[T_R{}^a (Y^m_R)^* T_{\overline{R}}{}^b Y^n_R\right] = \text{Tr}\left[T_{\overline{R}}{}^a Y^n_R T_R{}^b (Y^m_R)^*\right]$, due to the symmetry properties of the Yukawa matrices and the definition of the generators in the conjugate representation. For the next contribution we have:



$$i[\check{\Delta}\cdot\widetilde{\Gamma}^{ji,a}_{\psi\bar{\psi}c}]^{(1)}_{\mathrm{div}} = \frac{\hbar g^3}{16\pi^2\epsilon}\left[\frac{C_2(R)-C_2(G)/4}{2} + (\xi-1)\frac{C_2(R)/6-C_2(G)/4}{2}\right] T_R{}^a_{ij}\overline{(\not{p}_1+\not{p}_2)}\mathbb{P}_{\mathrm{R}}$$
$$+\frac{\hbar g}{16\pi^2\epsilon}\frac{1}{4}((Y_R^m)^* T_{\overline{R}}{}^a Y_R^m)_{ij}\overline{(\not{p}_1+\not{p}_2)}\mathbb{P}_{\mathrm{R}}\,. \quad (7.3.16a)$$

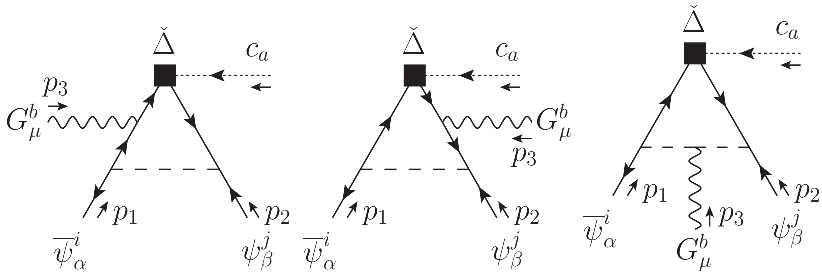
Note that here, contrary to the previous case when we inserted the evanescent $\widehat{\Delta}$ operator Eq. (7.2.5d), the first two diagrams do not vanish, and the one with the scalar propagator provides the last scalar contribution in Eq. (7.3.16a). Using charge-conjugated fermionic legs, the scalar part becomes:

$$\frac{\hbar g}{16\pi^2\epsilon}(Y_R^m T_R{}^a (Y_R^m)^*)_{ji}\overline{(\not{p}_1+\not{p}_2)}\mathbb{P}_{\mathrm{L}} = -\frac{\hbar g}{16\pi^2\epsilon}((Y_R^m)^* T_{\overline{R}}{}^a Y_R^m)_{ij}\overline{(\not{p}_1+\not{p}_2)}\mathbb{P}_{\mathrm{L}}.$$

This 1PI Green's function corresponds to the contribution

$$[N[\widehat{\Delta}]\cdot\Gamma_{\mathrm{Ren}}]^{(1)} \supset -\frac{\hbar g}{16\pi^2}\left\{g^2\left[C_2(R)-\frac{C_2(G)}{4}+(\xi-1)\left(\frac{C_2(R)}{6}-\frac{C_2(G)}{4}\right)\right]T_R{}^a_{ij}\right.$$
$$\left.+\frac{1}{2}((Y_R^m)^* T_{\overline{R}}{}^a Y_R^m)_{ij}\right\}\int \mathrm{d}^4x\; c_a\partial_\mu(\overline{\psi}_i\gamma^\mu\mathbb{P}_{\mathrm{R}}\psi_j)\,. \quad (7.3.16b)$$

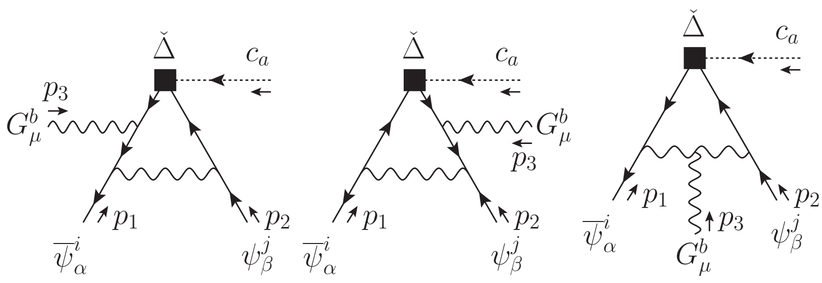
For the insertion in the fermion-gauge interactions we have:



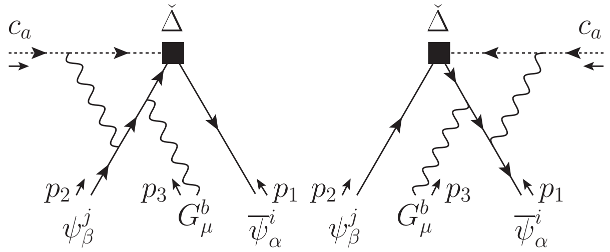
(a) Vanishing diagrams with fermion-scalar interactions.

$$i[\check{\Delta}\cdot\widetilde{\Gamma}^{ji,ba,\mu}_{\psi\bar{\psi}Gc}]^{(1)}_{\mathrm{div}} = \frac{-\hbar g^4}{16\pi^2\epsilon}\frac{\xi C_2(G)}{8}if^{abc}T_R{}^c_{ij}\overline{\gamma}^\mu\mathbb{P}_{\mathrm{R}} = \frac{-\hbar g^4}{16\pi^2\epsilon}\frac{\xi C_2(G)}{8}[T_R{}^a,T_R{}^b]_{ij}\overline{\gamma}^\mu\mathbb{P}_{\mathrm{R}}\,. \quad (7.3.17a)$$

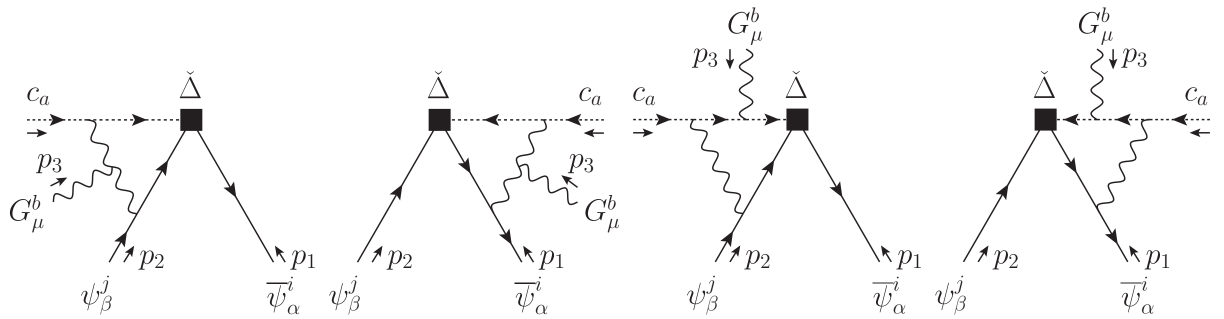
Note that both the diagrams with the scalar propagators, and the diagrams with a gluon



(b) Vanishing diagrams with fermion-gauge boson interactions.



(c) Diagrams cancelling with each other.



(d) The four contributing diagrams; their group structures simplify considerably when summing the first two (and last two) diagrams together.

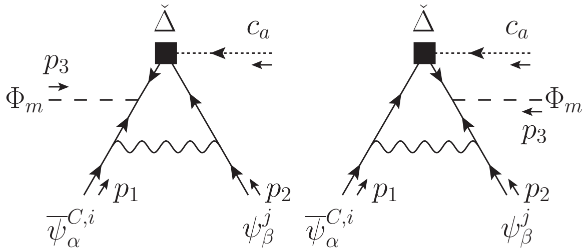
propagator connecting the fermions, are finite and thus do not contribute. Also, in our model there is no $GG\Phi$ vertex. The two diagrams with a gluon propagator connecting a fermion and the ghost leg cancel each other. The four remaining diagrams sum in pairs and their group structure simplify to get the simple result quoted above.
This 1PI Green's function corresponds to the contribution

$$[N[\widehat{\Delta}] \cdot \Gamma_{\mathrm{Ren}}]^{(1)} \supset \frac{-i\hbar g^2}{16\pi^2}\frac{\xi C_2(G)}{4}\int \mathrm{d}^4\, x\; i g^2 f^{abc} T_{R}{}^{c}_{ij} c_a \overline{\psi}_i \not{G}^b \mathbb{P}_{\mathrm{R}} \psi_j\,. \tag{7.3.17b}$$

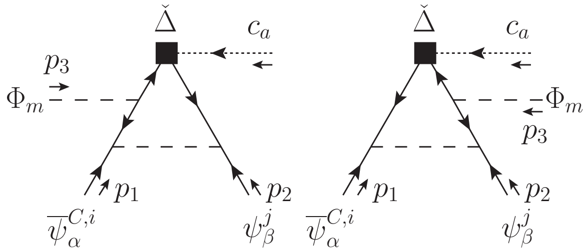
For the insertion in fermion-scalar interaction we have

$$i[\check{\Delta} \cdot \widetilde{\Gamma}^{ji,m,a}_{\psi\overline{\psi^C}\Phi c}]^{(1)}_{\mathrm{div}} = \frac{\hbar g^3}{16\pi^2\epsilon}\frac{\xi C_2(G)}{8}(Y_R)^n_{ij}\theta^a_{nm}\mathbb{P}_{\mathrm{R}} = \frac{\hbar g^3}{16\pi^2\epsilon}\frac{\xi C_2(G)}{8}(T_{\overline{R}}{}^{a} Y^m_R - Y^m_R T_R{}^{a})_{ij}\mathbb{P}_{\mathrm{R}}\,. \tag{7.3.18a}$$

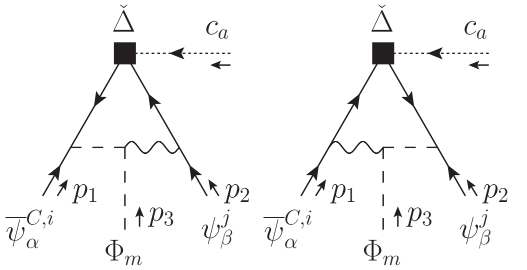
Similarly to the previous case $\check{\Delta} c^a G^b_\mu \bar{\psi}_{i,\alpha}\psi_{j,\beta}$, the diagrams with scalar or gluonic propagators between the fermions, and also those with the vertex $G\Phi\Phi$ and scalar/gluon



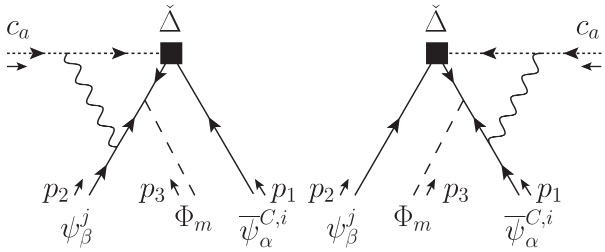
(a) Vanishing diagrams with fermion-scalar interactions.



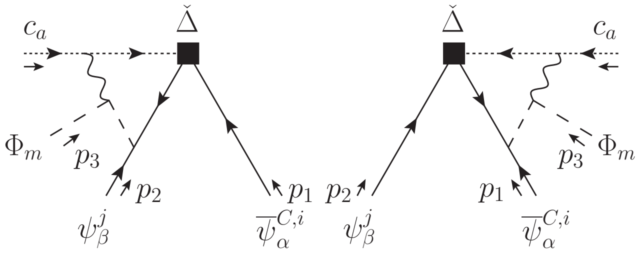
(b) Vanishing diagrams with fermion-gauge boson interactions.



(c) Vanishing diagrams with fermion-gauge boson + fermion-scalar interactions.



(d) Diagrams cancelling with each other.



(e) The two contributing diagrams.

propagator between the fermions, are finite and thus do not contribute. Also, the two diagrams with a gluon propagator between a fermion and the ghost leg cancel each other. The two remaining diagrams form a pair whose total amplitude acquires a simpler group structure, after using the relation coming from the gauge-invariance of the Yukawa Lagrangian Eq. (4.1.16a). Thus, the 1PI Green's function Eq. (7.3.18a) corresponds to the contribution

$$[N[\widehat{\Delta}]\cdot\Gamma_{\mathrm{Ren}}]^{(1)} \supset \frac{i\hbar g^2}{16\pi^2}\frac{\xi C_2(G)}{4}\int \mathrm{d}^4 x\, \frac{g}{2}(Y_R)^n_{ij}\theta^a_{nm}c_a\Phi^m\overline{\psi^C}_i\mathbb{P}_{\mathrm{R}}\psi_j\,. \tag{7.3.18b}$$

Associated with this term is the complex conjugate process that generates a similar contribution to the Bonneau identity:

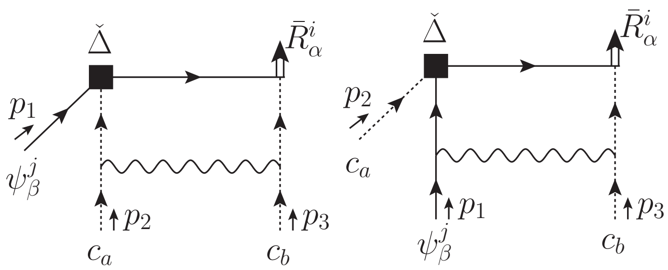
$$[N[\widehat{\Delta}]\cdot\Gamma_{\mathrm{Ren}}]^{(1)} \supset \frac{i\hbar g^2}{16\pi^2}\frac{\xi C_2(G)}{4}\int \mathrm{d}^4 x\, \frac{g}{2}(Y_R)^{n\,*}_{ij}\theta^a_{nm}c_a\Phi^m\overline{\psi}_i\mathbb{P}_{\mathrm{L}}\psi^C_j\,. \tag{7.3.18c}$$

We have exhausted all the possibilities for insertion diagrams up to mass-dimension 4 coming from the generic Lagrangian terms. In the next subsection, we evaluate the terms with external-field insertions.

7.3.2 | One-loop vertices with insertion of one BRST-source-vertex and inverse breaking

At 1-loop, and up to mass-dimension 4, the only 1PI diagrams containing a single insertion of $\check{\Delta}$ and one BRST-source-vertex are those that only have one insertion of $\bar{R}s_d\psi$ or $Rs_d\overline{\psi}$ BRST-source-vertex; these diagrams have mass-dimension four. The reasons are as follows.

These diagrams have ghost number one since they have a breaking insertion and are constituents of the Slavnov-Taylor identity. Also, Eq. (7.3.8) imposes that the sum of the mass-dimensions of their incoming and outgoing fields and derivatives, has to be smaller than or equal to four. The BRST sources are external fields without (propagators and) loop contributions, and their mass-dimensions are relatively large (see Table 4.1). Furthermore, both the operator $\check{\Delta}$ and any of the BRST-source-vertices contain only ghost fields, therefore all ghost lines from $\check{\Delta}$ and any of the BRST-source-vertices give rise to an external ghost line. Thus the mass-dimension and the ghost number constraints allow only the following operators: ρGcc, $\rho\partial cc$, ζccc, $\bar{R}\psi cc$, $R\overline{\psi}cc$ and $\mathcal{Y}\Phi cc$. The operators ρGcc, $\rho\partial cc$, ζccc and $\mathcal{Y}\Phi cc$ imply that the fermions from $\check{\Delta}$ are enclosed into a loop, in which case one cannot form at 1-loop level a 1PI diagram with the BRST-source-vertex. The remaining operators $\bar{R}\psi cc$ and $R\overline{\psi}cc$ may arise from 1-loop contributions if one of the fermions of $\check{\Delta}$ is contracted with a fermion from one of the operators $\bar{R}s_d\psi$ or $Rs_d\overline{\psi}$. Only the following diagrams are therefore generated:



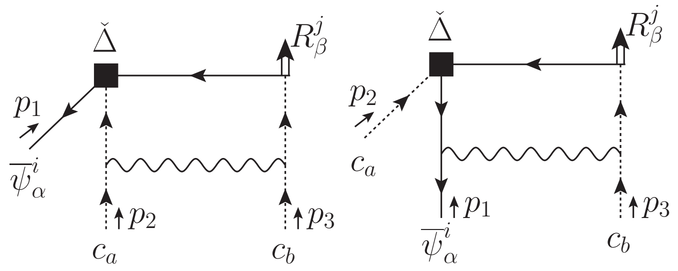
$+(p_2, a) \leftrightarrow (p_3, b)$ permutation.

$$i[\check{\Delta} \cdot \widetilde{\Gamma}^{ji,ba}_{\psi\bar{R}cc}]^{(1)}_{\text{div}} = \frac{-i\hbar g^4}{16\pi^2\epsilon}\frac{\xi C_2(G)}{8} i f^{abc} {T_R}^c_{ij}\mathbb{P}_\text{R} = \frac{-i\hbar g^4}{16\pi^2\epsilon}\frac{\xi C_2(G)}{8}[{T_R}^a, {T_R}^b]_{ij}\mathbb{P}_\text{R}\,, \tag{7.3.19a}$$

corresponding to the contribution

$$[N[\widehat{\Delta}] \cdot \Gamma_\text{Ren}]^{(1)} \supset \frac{\hbar g^2}{16\pi^2}\frac{\xi C_2(G)}{4}\int \mathrm{d}^4 x\, i\frac{g^2}{2} f^{abc} {T_R}^c_{ij} c^a c^b \bar{R}_i \mathbb{P}_\text{R}\psi_j\,. \tag{7.3.19b}$$

And finally, the charge-conjugated interaction is given by



$+(p_2, a) \leftrightarrow (p_3, b)$ permutation.

$$i[\check{\Delta} \cdot \widetilde{\Gamma}^{ji,ba}_{R\bar{\psi}cc}]^{(1)}_{\text{div}} = \frac{i\hbar g^4}{16\pi^2\epsilon} \frac{\xi C_2(G)}{8} i f^{abc} {T_R}^c_{ij} \mathbb{P}_\text{L} = \frac{i\hbar g^4}{16\pi^2\epsilon} \frac{\xi C_2(G)}{8} [{T_R}^a, {T_R}^b]_{ij} \mathbb{P}_\text{L} \,, \tag{7.3.20a}$$

corresponding to the contribution

$$[N[\widehat{\Delta}] \cdot \Gamma_\text{Ren}]^{(1)} \supset -\frac{\hbar g^2}{16\pi^2} \frac{\xi C_2(G)}{4} \int \mathrm{d}^4 x \, i \frac{g^2}{2} f^{abc} {T_R}^c_{ij} c^a c^b \bar{\psi}_i \mathbb{P}_\text{L} R_j \,. \tag{7.3.20b}$$

Note that only the diagrams with a gluon propagator connecting the two ghost lines do contribute, while those where the gluon propagator connects one ghost line with a fermion line do not.

7.4 | Construction of BRST-restoring finite one-loop counterterms

In the present section we evaluate the BRST-restoring finite 1-loop counterterms $S^{(1)}_{\text{fct,restore}}$. At this point, we have collected all the insertion diagrams possible (by mass-dimension restriction) at the 1-loop level.
First, recall that divergent terms in Eq. (7.1.7) cancelled completely. Since they did not produce any finite contribution to the breaking of Slavnov-Taylor identity, we see from Eq. (7.1.7) that finite counterterms are defined such that their 4-dimensional linear BRST transformation completely cancels insertion diagrams, which have been evaluated in the previous Section 7.3, i.e.

$$\underset{d\to 4}{\text{LIM}} \left([\widehat{\Delta} \cdot \Gamma^{(1)}]_\text{fin} + b_d S^{(1)}_{\text{fct,restore}} \right) = 0 \tag{7.4.1}$$

must hold. We calculate these counterterms without imposing constraints on the fermion group representations and we also obtain the expression for the gauge anomalies as a by-product. These finite counterterms will be sufficient to restore the BRST invariance broken by the BMHV scheme if the anomaly cancellation condition is met [25].
So now, our final task is to construct symmetry-restoring counterterms in a way that their b_d transformation cancels insertion diagrams from the previous chapter. In order to prepare our calculations, we make an assumption that $S^{(1)}_{\text{fct,restore}}$ will be a linear combination of all

possible mass-dimension ≤ 4 field monomials that emerged from the 1-loop calculations. We therefore first evaluate all the linear BRST transformations of these monomials in Section 7.4.1, then we combine these results and directly compare them in Section 7.4.2 with the terms from $[N[\widehat{\Delta}] \cdot \Gamma_{\text{Ren}}]^{(1)}$ so as to find the finite counterterms $S^{(1)}_{\text{fct,restore}}$.

7.4.1 | Evaluation of linear BRST transformation for some field monomials

The following calculations are also performed in 4 dimensions, so we will again omit all the "overlines" over all the Lorentz covariants so as to simplify the notation. The notations for the integrated field monomials are the same as in Eq. (4.2.16a) (Section 4.2), but now all defined purely in 4 dimensions. We obtain:

$$b \int \mathrm{d}^4\, x\; \frac{1}{2} G^{a\,\mu} \partial^2 G^a_\mu = \int \mathrm{d}^4\, x\; (\partial^\mu c_a + g f^{abc} G^{b\,\mu} c_c) \partial^2 G^a_\mu \,, \tag{7.4.2}$$

$$bS_{GG} = b \int \mathrm{d}^4\, x\; \frac{1}{2} G^a_\mu (g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu) G^a_\nu = -g f^{abc} \int \mathrm{d}^4\, x\; c^a G^b_\mu (g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu) G^c_\nu \,, \tag{7.4.3}$$

where we used the fact that $(\partial_\mu c^a)(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu) G^a_\nu = 0$ when using integrations by parts.

$$bS_{GGGG} = -\frac{g^2}{2} (f^{abe} f^{cde} + f^{ace} f^{bde}) \int \mathrm{d}^4\, x\; c^a \partial_\nu \left(G^b_\mu G^{c\,\mu} G^{d\,\nu} \right) . \tag{7.4.4}$$

In this calculation, a term proportional to $\int \mathrm{d}^4\, x\; c^f G^e_\mu G^{b\,\mu} G^c_\nu G^{d\,\nu}$ actually cancels. Indeed, its prefactor is given by: $(f^{acg} f^{bdg} + f^{adg} f^{bcg}) f^{aef}$, which vanishes after symmetrizing with respect to the group indices $e \leftrightarrow b$, $c \leftrightarrow d$, and the set $(e,b) \leftrightarrow (c,d)$. Also, because $\frac{-1}{4} \int \mathrm{d}^4\, x\; F^a_{\mu\nu} F^{a\,\mu\nu} = S_{GG} + S_{GGG} + S_{GGGG}$ is gauge-invariant, $b \int \mathrm{d}^4\, x\; F^a_{\mu\nu} F^{a\,\mu\nu} = 0$ and we have:

$$bS_{GGG} = -bS_{GG} - bS_{GGGG} \,. \tag{7.4.5}$$

$$\begin{aligned} & b (T_R)^{abcd} \int \mathrm{d}^4\, x\; G^a_\mu G^{b\,\mu} G^c_\nu G^{d\,\nu} = \\ & \quad - \left((T_R)^{abcd} + (T_R)^{acbd} + (T_R)^{adbc} + (T_R)^{adcb} \right) \int \mathrm{d}^4\, x\; c^a \partial_\nu \left(G^b_\mu G^{c\,\mu} G^{d\,\nu} \right) . \end{aligned} \tag{7.4.6}$$

As before, a term proportional to $\int \mathrm{d}^4\, x\; c^f G^e_\mu G^{b\,\mu} G^c_\nu G^{d\,\nu}$ cancels. Its prefactor is given by: $((T_R)^{abcd} + (T_R)^{acbd} + (T_R)^{adbc} + (T_R)^{adcb}) f^{aef}$ (using the shorthand notation $(T_R)^{abcd} \equiv \mathrm{Tr}[{T_R}^a \cdots {T_R}^d]$), and vanishes after symmetrization with respect to the group indices $e \leftrightarrow b$,

$c \leftrightarrow d$, and the set $(e,b) \leftrightarrow (c,d)$.

$$bS_{\Phi\Phi} = b \int \mathrm{d}^4\, x\, \frac{-1}{2} \Phi_m \partial^2 \Phi_m = \int \mathrm{d}^4\, x\; ig\theta^a_{mn} c^a \Phi_m \partial^2 \Phi_n\,, \tag{7.4.7}$$

$$bS_{\Phi GG\Phi} = -\frac{g^2}{2} \{\theta^a, \theta^b\}_{mn} \int \mathrm{d}^4\, x\; (\partial^\mu c^a) G^b_\mu \Phi^m \Phi^n\,, \tag{7.4.8}$$

and because $\frac{1}{2}(D_\mu \Phi^m)^2 = S_{\Phi\Phi} + S_{\Phi G\Phi} + S_{\Phi GG\Phi}$ is gauge-invariant, $b(D_\mu \Phi^m)^2 = 0$ and we have:

$$bS_{\Phi G\Phi} = -bS_{\Phi\Phi} - bS_{\Phi GG\Phi}\,. \tag{7.4.9}$$

For an arbitrary group symbol $\mathcal{C}^a_{mn}$,

$$\begin{aligned} b\mathcal{C}^a_{mn} \int \mathrm{d}^4\, x\; (\partial^\mu \Phi^m) G^a_\mu \Phi^n = & -\mathcal{C}^a_{mn} \int \mathrm{d}^4\, x\; c^a (\partial^2 \Phi^m) \Phi^n \\ & - \frac{1}{2}(\mathcal{C}^a_{mn} + \mathcal{C}^a_{nm}) \int \mathrm{d}^4\, x\; c^a (\partial^\mu \Phi^m)(\partial_\mu \Phi^n) \\ & + ig \left[i f^{abc} \mathcal{C}^c_{nm} + \theta^a_{mo} (\mathcal{C}^b_{on} - \mathcal{C}^b_{no}) \right] \int \mathrm{d}^4\, x\; c^a G^b_\mu \Phi^m (\partial^\mu \Phi^n) \\ & + \frac{ig}{2} (\theta^a_{mo} \mathcal{C}^b_{on} + \theta^a_{no} \mathcal{C}^b_{om}) \int \mathrm{d}^4\, x\; c^a (\partial^\mu G^b_\mu) \Phi^m \Phi^n\,, \end{aligned} \tag{7.4.10}$$

and, for an arbitrary group symbol $\mathcal{C}^{ab}_{mn}$,

$$b\mathcal{C}^{ab}_{mn} \int \mathrm{d}^4\, x\; G^a_\mu G^{b\,\mu} \Phi^m \Phi^n = -\mathcal{S}^{ab}_{mn} \int \mathrm{d}^4\, x\; c^a \partial_\mu \left(G^{b\,\mu} \Phi^m \Phi^n \right)\,, \tag{7.4.11}$$

where $\mathcal{S}^{ab}_{mn} = (\mathcal{C}^{ab}_{mn} + \mathcal{C}^{ba}_{mn} + m \leftrightarrow n)/2$, completely symmetric by exchanges $a \leftrightarrow b$ and $m \leftrightarrow n$. In this calculation, a term proportional to the field monomial $\int \mathrm{d}^4\, x\; c^a G^b_\mu G^{d\,\mu} \Phi^m \Phi^n$ actually cancels. Indeed, its prefactor is given by: $f^{acd} \mathcal{S}^{bc}_{mn} - i\theta^a_{mo} \mathcal{S}^{bd}_{on}$, and one can show that its contraction with the field monomial vanishes after symmetrizing with respect to the group indices (b,d) and (m,n).

We explicitly evaluate in addition the following 4-dimensional linear BRST transformations of the following fermionic operators, as these are the ones being involved in the definition of the finite counterterm action, which is naturally defined in 4 dimensions. (Note that if we were interested in their d-dimensional version, these would contain extra evanescent contributions.)

$$bS_{\overline{\psi}\psi} = b \int \mathrm{d}^4\, x\; i\overline{\psi}_i \not{\partial} \psi_i = g T_R{}^a_{ij} \int \mathrm{d}^4\, x\; c^a \partial_\mu \left(\overline{\psi}_i \gamma^\mu \mathbb{P}_\mathrm{R} \psi_j \right)\,, \tag{7.4.12}$$

$$
\begin{aligned}
b(S_{\bar{R}c\psi_R}+S_{Rc\overline{\psi_R}}) = \;& + i\frac{g}{2}\theta^a_{nm}\int \mathrm{d}^4\,x\; c^a\Phi^m\left((Y_R)^n_{ij}\overline{\psi}^C_i\mathbb{P}_\mathrm{R}\psi_j+(Y_R)^{n\,*}_{ij}\overline{\psi}_i\mathbb{P}_\mathrm{L}\psi^C_j\right)\\
&+i\frac{g^2}{2}f^{abc}T_{R}{}^{c}_{ij}\int \mathrm{d}^4\,x\; c^ac^b\left(\bar{R}^i\mathbb{P}_\mathrm{R}\psi_j-\overline{\psi}_i\mathbb{P}_\mathrm{L}R^j\right)\\
&+g^2f^{abc}T_{R}{}^{c}_{ij}\int \mathrm{d}^4\,x\; c^a\overline{\psi}_i\not{G}^a\mathbb{P}_\mathrm{R}\psi_j\\
&+gT_{R}{}^{a}_{ij}\int \mathrm{d}^4\,x\; c^a\partial_\mu\left(\overline{\psi}_i\gamma^\mu\mathbb{P}_\mathrm{R}\psi_j\right)\,.
\end{aligned}
\tag{7.4.13}
$$

7.4.2 | The finite one-loop counterterm action

Now when we explicitly know the b transformation of the action operators, we can group the results coming from different insertion diagrams and write them as a b-variation of the action operators. The total contribution from Eqs. (7.3.13), (7.3.9b) and (7.3.10b) is equal to:

$$
-\frac{\hbar g^2}{16\pi^2}\left(\frac{S_2(R)}{6}b\left(5S_{GG}+S_{GGG}-\int \mathrm{d}^4\,x\; G^{a\,\mu}\partial^2G^a_\mu\right)+\frac{g^2}{12}(T_R)^{abcd}b\int \mathrm{d}^4\,x\; G^a_\mu G^{b\,\mu}G^c_\nu G^{d\,\nu}\right)\,,
\tag{7.4.14}
$$

together with essential (non-spurious) anomalies

$$
-\frac{\hbar g^2}{16\pi^2}\left(\frac{S_2(R)}{3}d^{abc}_R\int \mathrm{d}^4\,x\; g\epsilon^{\mu\nu\rho\sigma}c_a(\partial_\rho G^b_\mu)(\partial_\sigma G^c_\nu)+\frac{\mathcal{D}^{abcd}_R}{3\times 3!}\int \mathrm{d}^4\,x\; g^2c_a\epsilon^{\mu\nu\rho\sigma}\partial_\sigma\left(G^b_\mu G^c_\nu G^d_\rho\right)\right)\,.
\tag{7.4.15}
$$

The contribution of Eq. (7.3.14b) is equal to:

$$
-\frac{\hbar}{16\pi^2}\frac{Y_2(S)}{3}b\overline{S_{\Phi\Phi}}\,.
\tag{7.4.16}
$$

The contribution of Eq. (7.3.15b) is equal to:

$$
\frac{\hbar}{16\pi^2}\frac{(\mathcal{C}_R)^{ab}_{mn}}{3}b\int \mathrm{d}^4\,x\;\frac{g^2}{2}G^a_\mu G^{b\,\mu}\Phi^m\Phi^n\,,
\tag{7.4.17}
$$

with $(\mathcal{C}_R)^{ab}_{mn}\equiv \mathrm{Tr}\left[2\{T_R{}^a,T_R{}^b\}(Y^m_R)^*Y^n_R-T_R{}^a(Y^m_R)^*T_{\overline{R}}{}^bY^n_R\right]$.
The total contribution of coming from Eqs. (7.3.16b), (7.3.17b), (7.3.18b), (7.3.18c), (7.3.19b) and (7.3.20b) is equal to:

$$
\begin{aligned}
\frac{-\hbar g^2}{16\pi^2}\left(1+\frac{\xi-1}{6}\right)C_2(R)bS_{\overline{\psi}\psi}+\frac{\hbar}{16\pi^2}\frac{((Y^m_R)^*T_{\overline{R}}{}^aY^m_R)_{ij}}{2}b\int \mathrm{d}^4\,x\; g\overline{\psi}_i\not{G}^a\mathbb{P}_\mathrm{R}\psi_j&\\
+\frac{\hbar g^2}{16\pi^2}\frac{\xi C_2(G)}{4}b(S_{\bar{R}c\psi_R}+S_{Rc\overline{\psi_R}})\,.&
\end{aligned}
\tag{7.4.18}
$$

Finally, the BRST-restoring finite counterterms defined in 4 dimensions such as

$bS^{(1)}_{\text{fct,restore}}$ cancels the contributions from $[N[\widehat{\Delta}]\cdot\Gamma_{\text{Ren}}]^{(1)}$ are:

$$
\begin{aligned}
S^{(1)}_{\text{fct,restore}} = \frac{\hbar}{16\pi^2}\Bigg\{ & g^2\frac{S_2(R)}{6}\left(5S_{GG}+S_{GGG}-\int \mathrm{d}^4x\; G^{a\,\mu}\partial^2 G^a_\mu\right)+\frac{Y_2(S)}{3}S_{\Phi\Phi} \\
& +g^2\frac{(T_R)^{abcd}}{3}\int \mathrm{d}^4x\;\frac{g^2}{4}G^a_\mu G^{b\,\mu}G^c_\nu G^{d\,\nu}-\frac{(\mathcal{C}_R)^{ab}_{mn}}{3}\int \mathrm{d}^4x\;\frac{g^2}{2}G^a_\mu G^{b\,\mu}\Phi^m\Phi^n \\
& +g^2\left(1+\frac{\xi-1}{6}\right)C_2(R)S_{\overline{\psi}\psi}-\frac{((Y^m_R)^* T_{\overline{R}}{}^a Y^m_R)_{ij}}{2}\int \mathrm{d}^4x\; g\overline{\psi}_i \not{G}^a \mathbb{P}_{\mathrm{R}}\psi_j \\
& -g^2\frac{\xi C_2(G)}{4}(S_{\bar{R}c\psi_R}+S_{Rc\overline{\psi_R}})\Bigg\}\,,
\end{aligned}
\tag{7.4.19}
$$

with $(\mathcal{C}_R)^{ab}_{mn} \equiv \mathrm{Tr}\left[2\{T_R{}^a,T_R{}^b\}(Y^m_R)^*Y^n_R - T_R{}^a(Y^m_R)^*T_{\overline{R}}{}^bY^n_R\right]$, and the essential (non-spurious) anomalies that can not be written as the b-variation of the action operators, are:

$$
-\frac{\hbar g^2}{16\pi^2}\left(\frac{S_2(R)}{3}d^{abc}_R\int \mathrm{d}^4x\; g\epsilon^{\mu\nu\rho\sigma}c_a(\partial_\rho G^b_\mu)(\partial_\sigma G^c_\nu)+\frac{\mathcal{D}^{abcd}_R}{3\times 3!}\int \mathrm{d}^4x\; g^2 c_a\epsilon^{\mu\nu\rho\sigma}\partial_\sigma\left(G^b_\mu G^c_\nu G^d_\rho\right)\right)\,,
\tag{7.4.20}
$$

with the fully symmetric symbol $d^{abc}_R = \mathrm{Tr}[T_R{}^a\{T_R{}^b,T_R{}^c\}]$, and the fully antisymmetric symbol $\mathcal{D}^{abcd}_R = (-i)3!\mathrm{Tr}[T_R{}^aT_R{}^{[b}T_R{}^cT_R{}^{d]}]$ for the R-representation. In realistic renormalizable models, the fermionic content and the associated group representations are chosen so as to cancel these anomalies, i.e. by cancelling separately both

$$
\boxed{\sum_R S_2(R)d^{abc}_R = 0}
\tag{7.4.21a}
$$

$$
\boxed{\sum_R \mathcal{D}^{abcd}_R = 0}
\tag{7.4.21b}
$$

the theory is finally free of anomalies. Eq. (7.4.21) are called *anomaly cancellation conditions*. Notice that the first structure is proportional to the usual triangle anomaly.

The equation Eq. (7.4.19) thus represents the crucial result for the renormalization in BMHV scheme. If the anomalies Eq. (7.4.20) are canceled, these finite counterterms are *necessary and sufficient* to restore the BRST symmetry at 1-loop level in the BMHV scheme. They are necessary building blocks for a consistent 1-loop applications of the scheme, and they are vital ingredients in 2-loop and higher-loop order calculations. It should be noted that these finite counterterms, purely 4-dimensional and non-evanescent, are *not* gauge- and BRST invariant, which is expected since they complete gauge and BRST invariance when they are added to the action. They modify all the self-energies, as well as some specific interactions: the gauge-boson self-interactions, and the interactions between gauge-boson and scalars or fermions.

As previously mentioned in the remarks around Eq. (7.1.5), one can also add, to these BRST-restoring finite counterterms, any other finite counterterms that are BRST-invariant, or even that are evanescent (because they will nonetheless vanish after taking the $\mathrm{LIM}_{d\to4}$), when being defined in d dimensions. However, both of these will not contribute to BRST restoration; they will instead only correspond to a change of renormalization prescription for higher-order calculations, see discussion below Eq. (5.2.3) in Chapter 5. For example, the BRST-invariant finite counterterms could contain a linear combination of the L_φ functionals defined in Eqs. (5.2.6) and (5.2.7). Once again, the question of this choice is answered in a way most suitable for a practitioner.

To summarize for the reader, we found that anomalies must be canceled by appropriate BRST restoring counterterms constructed from breaking insertions. Those anomalies are a product of the BMHV scheme and are called spurious (or non-essential or unphysical). Anomalies that can not be canceled by this procedure are not the result of BMHV scheme use, but the fermion content of the theory. They are called non-spurious (or essential or physical) and must be canceled with proper anomaly cancellation conditions or specific fermion content.

CHAPTER 8

THE RENORMALIZATION GROUP EQUATION IN THE RENORMALIZED MODEL

In this chapter, we derive the renormalization group equation in the BMHV scheme. Since in this scheme the multiplicative renormalization no longer holds, we again use the quantum action principle and formalism of algebraic renormalization. In this formulation, we avoid the introduction of bare fields and bare couplings. We will focus on the new type of counterterms emerging from the BMHV scheme and later compare them with the treatment in the fashion of standard renormalization group equations derivation we study in the next chapter. In both cases, we will see that evanescent contributions play no role at the 1-loop level but will have an influence on higher orders. We hence list them for future 2-loop renormalization group equations studies.

8.1 | Components of the renormalization group system

As we have shown in the previous sections, the set of operators in the tree-level action is not the same set that exists at the 1-loop level when using the BMHV dimensional renormalization scheme. Due to the presence of evanescent operators and finite non-evanescent counterterms needed to restore the BRST symmetry, the formalism of multiplicative renormalization (with bare fields, bare coupling constants and Z-factors) will not straightforwardly lead to the true renormalization group equation, that involves only fields and parameters of the original 4-dimensional tree-level action (see also discussion in Ref. [27]). This will be briefly overviewed in Chapter 9.

Instead if we start with the dimensionally renormalized 1PI functional Γ_{Ren}, see Eq. (3.1.10), and we use the Quantum Action Principle and the Bonneau identities, the formalism of bare objects and Z-factors can be avoided. From now on we take this effective action to be anomaly free, i.e. the anomalies described by Eq. (7.4.20) are cancelled.

The Bonneau identities [45, 46] form a linear system whose unique solution provides an expansion of any anomalous (e.g. evanescent) operator in terms of a quantum basis of standard (hence 4-dimensional) operators. More precisely, any anomalous normal product can be re-expressed as a linear combination of standard and evanescent monomial normal products [27]:

$$N[\hat{g}_{\mu\nu}\mathcal{O}^{\mu\nu}](x)\cdot\Gamma_{\text{Ren}} = \sum_i \alpha_i N[\bar{\mathcal{M}}^i](x)\cdot\Gamma_{\text{Ren}} + \sum_j \hat{\alpha}_j N[\hat{\mathcal{M}}^j](x)\cdot\Gamma_{\text{Ren}}\,, \tag{8.1.1}$$

where the latter term can be further reduced to pure standard operators thanks to the Bonneau identities, so as the anomalous normal product ultimately reduces to

$$N[\hat{g}_{\mu\nu}\mathcal{O}^{\mu\nu}](x)\cdot\Gamma_{\text{Ren}} = \sum_i q_i N[\bar{\mathcal{M}}^i](x)\cdot\Gamma_{\text{Ren}}\,, \tag{8.1.2}$$

where the coefficient functions q_i are formal series in $\hbar$. Fortunately, at lowest order in $\hbar$ the linear system is trivial and decoupled, i.e. loops with anomalous insertions can be transformed in sum of tree-level diagrams with insertions of standard operators.
In general, for the calculation of the coefficient q_i at order $\hbar^n$, we need the coefficients α_i up to order $\hbar^n$ and $\hat{\alpha}_j$ up to order $\hbar^{n-1}$, since the evanescent operators count for an order $\hbar$ higher, according to the Bonneau identities, what is of crucial importance in particular for the calculation at 1-loop level.
It can be shown [25] that the Renormalization Group Equation corresponds to the expansion of the insertion

$$\mu\frac{\partial}{\partial\mu}\Gamma_{\text{Ren}} = \Delta\cdot\Gamma_{\text{Ren}} \tag{8.1.3}$$

in a suitable basis of operators of ultraviolet dimension 4, ghost number 0, with contracted Lorentz indices but free gauge indices (later contracted with group factors from the associated coefficients). On the other side, (8.1.3) is expanded in a basis of operators that respect the same symmetries as the functional $\mu\partial/\partial\mu\,\Gamma_{\text{Ren}}$, that are, generally speaking, operators comprising derivatives with respect to the parameters of the theory, and field-counting operators,

$$\mu\frac{\partial}{\partial\mu}\Gamma_{\text{Ren}} = \Big(-\sum_g \beta_g\frac{\partial}{\partial g} + \sum_\phi N_\phi\gamma_\phi\Big)\cdot\Gamma_{\text{Ren}}\,. \tag{8.1.4}$$

Calculating independently RHS of (8.1.4), and LHS of same equation that corresponds to the (8.2.4) as we will see soon, will result in a system of equations, overdetermined and solvable by direct comparison of their coefficients.

8.2 | Calculation of the one-loop coefficients

The first non-trivial contribution to the functional $\mu\partial/\partial\mu\,\Gamma_{\mathrm{Ren}}$ is always of order $\hbar$, since the tree-level action does not depend on the renormalization scale μ. The problem of expressing $\mu\partial/\partial\mu\,\Gamma_{\mathrm{Ren}}$ as an insertion of normal product operators into Γ_{Ren} (keep in mind that μ is not a parameter of the action) was solved by Bonneau [45] and generalized by Martin [27] due to the presence of different types of fields and external sources, evanescent contributions and finite counterterms, what resulted in rather involved expression

$$
\begin{aligned}
&\mu\frac{\partial}{\partial\mu}\Gamma_{\mathrm{Ren}} = \sum_{n=0}^{4}\sum_{\{j_1\dots j_n\}} N_\ell\Bigg[\\
&\sum_{r=0}^{\omega(J)}\sum_{\substack{i_1\dots i_r\\ 1\le i_j\le n-1}}\left\{\text{r.s.p.}\frac{\imath^r}{r!}\frac{\partial^r}{\partial p_{i_1}^{\mu_1}\dots\partial p_{i_r}^{\mu_r}}(-i\hbar)\overline{\langle\tilde{\phi}_{j_1}(p_1)\dots\tilde{\phi}_{j_n}(p_n)(p_n=-\textstyle\sum p_i)\rangle}^{1\mathrm{PI},(N_\ell)}_{K=0}\Big|_{p_i=0}\right\}\\
&\times\frac{1}{n!}N\left[\phi_{j_n}\prod_{k=n-1}^{1}\left\{\left(\prod_{\{\alpha/i_\alpha=k\}}\partial_{\mu_\alpha}\right)\phi_{j_k}\right\}\right]\cdot\Gamma_{\mathrm{Ren}}+\sum_{\Phi}\sum_{r=0}^{\omega(J;\Phi)}\\
&\sum_{\substack{i_1\dots i_r\\ 1\le i_j\le n}}\left\{\text{r.s.p.}\frac{\imath^r}{r!}\frac{\partial^r}{\partial p_{i_1}^{\mu_1}\dots\partial p_{i_r}^{\mu_r}}(-i\hbar)\overline{\langle\tilde{\phi}_{j_1}(p_1)\dots\tilde{\phi}_{j_n}(p_n)N[s\Phi](p_{n+1}=-\textstyle\sum p_i)\rangle}^{1\mathrm{PI},(N_\ell)}_{K=0}\Big|_{p_i=0}\right\}\\
&\times\frac{1}{n!}N\left[K_\Phi(x)\prod_{k=n}^{1}\left\{\left(\prod_{\{\alpha/i_\alpha=k\}}\partial_{\mu_\alpha}\right)\phi_{j_k}\right\}\right]\cdot\Gamma_{\mathrm{Ren}}\Bigg],
\end{aligned}
\tag{8.2.1}
$$

where $J=\{j_1\dots j_n\}$, $\omega(J)$ and $\omega(J;\Phi)$ are overall ultraviolet degrees of divergence of 1PI Green's functions. The bar over 1PI Green's functions stands for subtractions of sub-divergences from the contributing Feynman diagrams. Fortunately, *restriction to 1-loop ($\hbar$) order*, particularly for the Yang-Mills model we use, contains only the first of these sums and reads:

$$\mu\frac{\partial}{\partial\mu}\Gamma_{\mathrm{Ren}} = N\Big[\text{r.s.p.}\,\Gamma^{(1)}\Big]\cdot\Gamma_{\mathrm{Ren}}. \tag{8.2.2}$$

Notice that, since the singular parts of Feynman diagrams contributing to 1PI Green's functions are local polynomials in external momenta expressed in d, 4 and/or ϵ (i.e. evanescent) dimensions, the results generally contain evanescent contributions. Hence, the expansion

$$\mu\frac{\partial}{\partial\mu}\Gamma_{\mathrm{Ren}} = \sum_i \bar{r}_i N[\bar{\mathcal{W}}_i]\cdot\Gamma_{\mathrm{Ren}} + \sum_j \hat{r}_j N[\hat{\mathcal{W}}_j]\cdot\Gamma_{\mathrm{Ren}} \tag{8.2.3}$$

holds, and can be re-expressed, thanks to Bonneau identities, in the form

$$\mu\frac{\partial}{\partial\mu}\Gamma_{\mathrm{Ren}} = \sum_i r_i N[\bar{\mathcal{W}}_i]\cdot\Gamma_{\mathrm{Ren}}\,, \tag{8.2.4}$$

where plain, barred and hatted objects correspond to d, 4 and ϵ-dimensional objects, respectively, and every evanescent insertion is re-expressed as [27]

$$N[\hat{\mathcal{W}}_j] \cdot \Gamma_{\text{Ren}} = \sum_i c_{ji} N[\bar{\mathcal{W}}_i] \cdot \Gamma_{\text{Ren}}\,, \tag{8.2.5}$$

where c_{ji} are formal expansions in $\hbar$, having no order $\hbar^0$ contribution due to the r.s.p. extractions. So, at $\hbar$ order we have:

$$\mu\frac{\partial}{\partial\mu}\Gamma_{\text{Ren}} = \sum_i r_i N[\bar{\mathcal{W}}_i] \cdot \Gamma_{\text{Ren}} = \sum_i (\bar{r}_i + \textstyle\sum_j \hat{r}_j c_{ji}) N[\bar{\mathcal{W}}_i] \cdot \Gamma_{\text{Ren}} \overset{\mathcal{O}(\hbar)}{=} \displaystyle\sum_i \bar{r}_i \bar{\mathcal{W}}_i\,, \tag{8.2.6}$$

where in the last step, the non-zero contributions at lowest $\hbar$ order come from the coefficients $\bar{r}_i$; therefore the corresponding field product insertions $N[\bar{\mathcal{W}}_i] \cdot \Gamma_{\text{Ren}}$ are tree-level $\hbar^0$ insertions simply equal to $\bar{\mathcal{W}}_i$. The general algorithm for calculating of the $\bar{r}_i$ and $\hat{r}_j$ coefficients at any order is explained in [27].

Starting from the previous equation, keep in mind that group structures are contained in the coefficients $\bar{r}_i$. Our coefficients are, in a practical sense, extracted from the divergent part of the 1-loop 1PI Green's functions as r.s.p.-s. The set of monomials differs from the set of operators contained in S_0 in a fact that S_{GG} contains combination of two field monomials. If we define in the same manner as [27] the field monomials that contain two gauge fields

$$\bar{\mathcal{W}}_1^{ab} \equiv \int \mathrm{d}^4\,x\;(\partial_{\bar{\mu}}\partial_{\bar{\nu}} G^{a\bar{\mu}})G^{b\bar{\nu}}\ \ ,\ \bar{\mathcal{W}}_2^{ab} \equiv \int \mathrm{d}^4\,x\;(\bar{\Box} G^a_{\bar{\mu}})G^{b\bar{\mu}}\,,$$

then according to (8.2.1) the first two coefficients are (with incoming momenta)

$$\begin{aligned}
\bar{r}_1^{ab} &= \frac{i^2}{2}\,\text{coef. of r.s.p}(-i\hbar)\overline{\langle G^a_\mu(p_1)G^b_\nu(p_2=-p_1)\rangle}^{\text{1PI},(1)} \text{ of } \bar{p}_1^\mu \bar{p}_1^\nu\,,\\
\bar{r}_2^{ab} &= \frac{i^2}{2}\,\text{coef. of r.s.p}(-i\hbar)\overline{\langle G^a_\mu(p_1)G^b_\nu(p_2=-p_1)\rangle}^{\text{1PI},(1)} \text{ of } \bar{p}_1^2 \bar{g}^{\mu\nu}\,,
\end{aligned}$$

where r.s.p. is residue of simple pole in $\nu = 4 - d = 2\epsilon$. According to 1-loop results we obtained in Chapter 6 the results for coefficients are

$$\begin{aligned}
\bar{r}_1^{ab} &= \frac{\hbar}{16\pi^2} g^2 \frac{(13-3\xi)C_2(G) - S_2(S) - 4S_2(R)}{3}\,,\\
\bar{r}_2^{ab} &= \frac{-\hbar}{16\pi^2} g^2 \frac{(13-3\xi)C_2(G) - S_2(S) - 4S_2(R)}{3}\,.
\end{aligned}$$

Therefore, at lowest order the gauge-boson propagator is transverse, $\overline{S_{GG}} \propto \bar{p}_1^\mu \bar{p}_1^\nu - \bar{p}_1^2 \bar{g}^{\mu\nu}$, and therefore,

$$\mu\frac{\partial}{\partial\mu}\Gamma_{\text{Ren}} = \sum_{i\in f.b.}\sum_{a_i} \bar{r}_i^{(1),a_i}\bar{\mathcal{W}}_i^{a_i} = \bar{r}_1^{ab}\bar{\mathcal{W}}_1^{ab} + \bar{r}_2^{ab}\bar{\mathcal{W}}_2^{ab} + \cdots \supset c_{GG}\overline{S_{GG}} + \dots\,, \tag{8.2.7}$$

where $f.b.$ denotes a full basis of monomials. In other words, the basis of monomials is equivalent to the corresponding classical basis, which agrees with the statement that any such basis of renormalized insertions is completely characterized by the corresponding classical basis [25]. If

$$\left\{\Delta^p \cdot \Gamma = \Delta^p_{class} + \mathcal{O}(\hbar) \mid p = 1, 2, \ldots; \dim(\Delta^p) \leq d\right\} \tag{8.2.8}$$

is the set of insertions whose classical approximations form a basis for classical insertions up to dimension d, then the same set is a basis for the quantum insertions bounded by d. This means that a convenient choice for our set of monomials are the field operators that are contained in the tree-level action S_0. Our insertion than can be chosen as a linear combination of operators of 4-dimensional action

$$\mu\frac{\partial}{\partial\mu}\Gamma_{\text{Ren}} = \sum_{i\in f.b.}\sum_{a_i}\bar{c}^{(1),a_i}_{\phi_1\phi_2\ldots}\,\overline{S^{0,a_i}_{\phi_1\phi_2\ldots}}, \tag{8.2.9}$$

where $f.b.$ denotes the full basis of operators $\phi_1, \phi_2, \ldots$ in the tree-level action S_0. In our notation for Green's functions the Bonneau insertions have the form

$$\begin{aligned}&\text{r.s.p.}(-i\hbar)\overline{\left\langle\widetilde{\phi_n}(p_n)\cdots\widetilde{\phi_1}(p_1)\right\rangle}^{\text{1PI}} \times N[\textstyle\prod\partial\phi_i]\cdot\Gamma\\ &\quad= \text{r.s.p.}\Gamma_{\phi_n\cdots\phi_1}(\bar{p}_1,\ldots,\bar{p}_n) \times N[\textstyle\prod\partial\phi_i]\cdot\Gamma \overset{\mathcal{O}(\hbar)}{=} \text{r.s.p.}(-S^{(1),4d}_{\text{sct}}) \overset{\mathcal{O}(\hbar)}{=} -2\epsilon S^{(1),4d}_{\text{sct}}.\end{aligned} \tag{8.2.10}$$

where $(1), 4d$ denotes $\hbar$ order and 4-dimensional space, respectively. Therefore, at $\hbar$ order the Renormalization Group equation acquires the simple form

$$\mu\frac{\partial\Gamma_{\text{Ren}}}{\partial\mu} \equiv -2\epsilon S^{(1),4d}_{\text{sct}}, \tag{8.2.11}$$

where $S^{(1),4d}_{\text{sct}}$ is just equal to Eqs. (6.3.2) to (6.3.4) but projected onto 4 dimensions only (thus there are no appearance of evanescent operators).
We again emphasize that no evanescent contribution is a 1-loop effect.

8.3 | Basis of insertions

The anomaly-free renormalization group equation (8.1.4) is, as we have noted, an expansion of the renormalization group functional in terms of a basis of quantum insertions with the operators being of ultraviolet dimension 4, ghost number 0, contracted Lorentz indices and free gauge indices. Such basis satisfies the same equations as Γ_{Ren},

$$\mathcal{S}_{\Gamma_{\text{Ren}}}\,\mu\frac{\partial\Gamma_{\text{Ren}}}{\partial\mu} = 0\,,\ \frac{\delta}{\delta B}\,\mu\frac{\partial\Gamma_{\text{Ren}}}{\partial\mu} = 0\,,\ \mathcal{G}\,\mu\frac{\partial\Gamma_{\text{Ren}}}{\partial\mu} = 0\,,$$

i.e. the BRST equation, the gauge-fixing condition and the ghost equation [27], where

$$\mathcal{S}_{\Gamma_{\mathrm{Ren}}} = \int \mathrm{d}^4 x \, \Bigg(\frac{\delta\Gamma_{\mathrm{Ren}}}{\delta\rho^\mu_a}\frac{\delta}{\delta G^a_\mu} + \frac{\delta\Gamma_{\mathrm{Ren}}}{\delta G^a_\mu}\frac{\delta}{\delta\rho^\mu_a} + \frac{\delta\Gamma_{\mathrm{Ren}}}{\delta\zeta_a}\frac{\delta}{\delta c^a} + \frac{\delta\Gamma_{\mathrm{Ren}}}{\delta c^a}\frac{\delta}{\delta\zeta_a} + B^a\frac{\delta\Gamma_{\mathrm{Ren}}}{\delta\bar{c}_a}$$
$$+\frac{\delta\Gamma_{\mathrm{Ren}}}{\delta\mathcal{Y}^m}\frac{\delta}{\delta\Phi_m} + \frac{\delta\Gamma_{\mathrm{Ren}}}{\delta\Phi_m}\frac{\delta}{\delta\mathcal{Y}^m} + \left(\frac{\delta\Gamma_{\mathrm{Ren}}}{\delta\bar{R}^i}\frac{\delta}{\delta\psi_i} + \frac{\delta\Gamma_{\mathrm{Ren}}}{\delta\psi_i}\frac{\delta}{\delta\bar{R}^i}\right) + \left(\frac{\delta\Gamma_{\mathrm{Ren}}}{\delta R^i}\frac{\delta}{\delta\overline{\psi}_i} + \frac{\delta\Gamma_{\mathrm{Ren}}}{\delta\overline{\psi}_i}\frac{\delta}{\delta R^i}\right)\Bigg) \, ,$$

is the linearized BRST operator of our model. The basis that respects those equations is constructed from its classical approximation in the sense of (8.2.8), by employing the operators L_G, L_c, L_Φ, L_{ψ_R} that are b-invariant in 4 dimensions, whose definitions have been introduced in Chapter 5 Eq. (5.2.6) and can be expressed as linear combinations of field-counting operators for $d = 4$ acting on the tree-level action: $L_\varphi \equiv \mathcal{N}_\varphi S_0$ for $\varphi = G, c, \Phi, \psi_R$, as well as the operators L_g, $L_{Y^m_{ij}}$ and $L_{\lambda_{mnop}}$ defined by differentiating the action with respect to the coupling parameters of the theory, Eq. (5.2.7).

A quantum extension of this classical basis is constructed [25] by the action on Γ_{Ren} of the symmetric differential operators we have just introduced (see ref. [27] for the details), and up to order $\hbar^n$ the following equation holds:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\, g\frac{\partial}{\partial g} + (\beta_Y)^m_{ij}\frac{\partial}{\partial Y^m_{ij}} + \beta_\lambda\frac{\partial}{\partial\lambda} - \gamma_G\mathcal{N}_G - \gamma_c\mathcal{N}_c - \gamma_\Phi\mathcal{N}_\Phi - \gamma_\psi\mathcal{N}^R_\psi\right]\Gamma_{\mathrm{Ren}} = 0. \quad (8.3.1)$$

This is the renormalization group equation of our theory. Now, thanks to the consequence of the Quantum Action Principle that any differential operator contained in our quantum basis can be expressed as insertions of normal products in Γ_{Ren}, and the fact that the first non-vanishing contribution to these expansions is of order $\hbar$ [69], at 1-loop level we have:

$$\mu\frac{\partial\Gamma_{\mathrm{Ren}}}{\partial\mu} \overset{\mathcal{O}(\hbar)}{=} -\beta^{(1)}g\frac{\partial S_0^{(4d)}}{\partial g} - (\beta^{(1)}_Y)^m_{ij}\frac{\partial S_0^{(4d)}}{\partial Y^m_{ij}} - \beta^{(1)}_{\lambda_{mnop}}\frac{\partial S_0^{(4d)}}{\partial\lambda_{mnop}} + \sum_\phi \gamma^{(1)}_\phi\mathcal{N}_\phi S_0^{(4d)} \, , \quad (8.3.2)$$

where $S_0^{(4d)}$ symbolizes the 4-dimensional restriction of the tree-level action of our model, Eq. (4.1.13). The RHS of equation (8.3.2) is the constituent needed in the construction of our system of the renormalization group equations.

8.4 | Solution of the system

By direct comparison of (8.3.2) with (8.2.11) we obtain the following system of equations:

$$S_{GG} \to 2\gamma^{(1)}_G = \frac{-2\hbar}{16\pi^2}g^2\frac{(13-3\xi)C_2(G) - 4S_2(R) - S_2(S)}{6} \, , \quad (8.4.1)$$

$$S_{GGG} \to -\beta^{(1)} + 3\gamma^{(1)}_G = \frac{-2\hbar}{16\pi^2}g^2\frac{(17-9\xi)C_2(G) - 8S_2(R) - 2S_2(S)}{12} \, , \quad (8.4.2)$$

$$S_{GGGG} \to -2\beta^{(1)} + 4\gamma_G^{(1)} = \frac{-2\hbar}{16\pi^2} g^2 \frac{2(2-3\xi)C_2(G) - 4S_2(R) - S_2(S)}{6}\,, \tag{8.4.3}$$

$$S_{\bar\psi\psi_R} \to 2\gamma_\psi^{(1)} = \frac{2\hbar}{16\pi^2}\left(g^2\xi C_2(R) + \frac{Y_2(R)}{2}\right)\,, \tag{8.4.4}$$

$$S_{\bar\psi G\psi_R} \to -\beta^{(1)} + \gamma_G^{(1)} + 2\gamma_\psi^{(1)} = \frac{2\hbar}{16\pi^2}\left(g^2\frac{(3+\xi)C_2(G) + 4\xi C_2(R)}{4} + \frac{Y_2(R)}{2}\right)\,, \tag{8.4.5}$$

$$S_{\Phi\Phi} \to 2\gamma_\Phi^{(1)} = \frac{-2\hbar}{16\pi^2}\left(g^2(3-\xi)C_2(S) - Y_2(S)\right)\,, \tag{8.4.6}$$

$$S_{\Phi G\Phi} \to -\beta^{(1)} + \gamma_G^{(1)} + 2\gamma_\Phi^{(1)} = \frac{-2\hbar}{16\pi^2}\left(g^2\left((3-\xi)C_2(S) - \frac{3+\xi}{4}C_2(G)\right) - Y_2(S)\right)\,, \tag{8.4.7}$$

$$S_{\Phi GG\Phi} \to -2\beta^{(1)} + 2\gamma_G^{(1)} + 2\gamma_\Phi^{(1)} = \frac{-2\hbar}{16\pi^2}\left(g^2\left((3-\xi)C_2(S) - \frac{3+\xi}{2}C_2(G)\right) - Y_2(S)\right)\,, \tag{8.4.8}$$

$$S_{\Phi^4_{mnop}} \to -\beta_{\lambda_{mnop}}^{(1)} + 4\gamma_\Phi^{(1)}\lambda_{mnop} = \frac{-2\hbar}{16\pi^2}\frac{1}{2}(3g^4A - g^2\xi\Lambda^S - 4H + \Lambda^2)_{mnop}\,, \tag{8.4.9}$$

$$\begin{aligned} S_{\overline{\psi_{Ri}^C}\Phi^m\psi_{Rj}} &\to -(\beta_Y^{(1)})^m_{ij} + (Y_R)^m_{ij}(\gamma_\Phi^{(1)} + 2\gamma_\psi^{(1)}) \\ &= \frac{-2\hbar}{16\pi^2}\left((Y_R^n(Y_R^m)^*Y_R^n) - g^2\frac{2C_2(R)(3+\xi) - C_2(S)(3-\xi)}{2}Y_R^m\right)_{ij}\,, \end{aligned} \tag{8.4.10}$$

$$S_{\bar cc}, S_{\rho c} \to -\gamma_G^{(1)} + \gamma_c^{(1)} = \frac{-2\hbar}{16\pi^2}g^2\frac{3-\xi}{4}C_2(G)\,, \tag{8.4.11}$$

$$S_{\bar cGc}, S_{\rho Gc}, S_{\zeta cc}, S_{\bar Rc\psi_R}, S_{Rc\overline{\psi_R}}, S_{\mathcal{Y}c\Phi} \to -\beta^{(1)} + \gamma_c^{(1)} = \frac{2\hbar}{16\pi^2}g^2\frac{\xi C_2(G)}{2}\,. \tag{8.4.12}$$

This is an overdetermined system of equations that provides the following solutions for the β-functions and anomalous dimensions at 1-loop level:

$$\beta = \frac{\hbar}{16\pi^2}g^2\left(\frac{-22C_2(G) + 4S_2(R) + S_2(S)}{6}\right)\,, \tag{8.4.13}$$

$$\beta_{\lambda_{mnop}} = \frac{\hbar}{16\pi^2}(3g^4A_{mnop} - 4H_{mnop} + \Lambda^2_{mnop} + \Lambda^Y_{mnop} - 3g^2\Lambda^S_{mnop})\,, \tag{8.4.14}$$

$$\begin{aligned} \beta_{Y^m_{ij}} = \frac{\hbar}{16\pi^2}\Big(&2(Y_R^n(Y_R^m)^*Y_R^n)_{ij} - 3g^2\{C_2(R), Y_R^m\}_{ij} + (Y_R)^m_{ij}Y_2(S) \\ &+\frac{1}{2}((Y_R)^m_{ij}Y_2(R) + Y_2(\bar R)(Y_R)^m_{ij})\Big)\,, \end{aligned} \tag{8.4.15}$$

$$\gamma_G = \frac{\hbar}{16\pi^2}g^2\frac{(3\xi-13)C_2(G) + 4S_2(R) + S_2(S)}{6}\,, \tag{8.4.16}$$

$$\gamma_\psi = \frac{\hbar}{16\pi^2}\frac{2g^2\xi C_2(R) + Y_2(R)}{2}\,, \tag{8.4.17}$$

$$\gamma_\Phi = \frac{\hbar}{16\pi^2}\left(g^2(\xi-3)C_2(S) + Y_2(S)\right)\,, \tag{8.4.18}$$

$$\gamma_c = \frac{\hbar}{16\pi^2}g^2\frac{(6\xi-22)C_2(G) + 4S_2(R) + S_2(S)}{6}\,. \tag{8.4.19}$$

In the next chapter, we will solve the same system in the standard approach via renormalization constants and compare the results.

CHAPTER 9

COMPARISON WITH THE STANDARD MULTIPLICATIVE RENORMALIZATION APPROACH IN THE BMHV SCHEME

In this chapter, we explain the derivation of the RGE using the standard approach based on divergences of renormalization constants, that are used in the textbook's approach for vector-like theories, e.g. the one in the fashion of [68]. In the BMHV scheme, there are extra divergences for evanescent operators, so we first analyze their influence at the 1-loop level. All these new objects present must be properly treated so that *true* renormalization group equation is derived, i.e. the one that has only anomalous dimensions and beta functions present in the classical limit. We will see that at the 1-loop level this rather involved task reduces to the discarding of evanescent objects.

9.1 | Renormalization group equations in BMHV scheme

The standard textbook approach to deriving RGEs in the context of DReg was developed in Ref. [90] and applied e.g. in Refs. [67–69] for general gauge model up to 2-loop level. DReg, besides keeping symmetries intact for vector-like theories, gives a transparent answer to how the theory behaves under spacetime scale transformations. This information is stored in the renormalization group equation that relates beta functions for couplings and anomalous dimensions in the theory.

Derivation starts from the fact that the bare action (i.e. the sum of tree-level and counterterm action) can be written in terms of bare fields and parameters which depend on the $\overline{\text{MS}}$-renormalization scale μ. For a generic bare parameter $g_{i,\text{bare}}$ in a massless theory defined in $d = 4 - 2\epsilon$, and in the $\overline{\text{MS}}$-renormalization scheme, this may be written as

$$g_{i,\text{bare}}\, \mu^{-\rho_i \epsilon} = g_i + \delta g_i \,, \qquad \delta g_i = \sum_{n=1}^{\infty} \frac{a_i^{(n)}}{\epsilon^n} \,, \tag{9.1.1}$$

where ρ_i is a constant dependent of the fields coupled in the vertex (equal to 1 for gauge and Yukawa couplings and 2 for quartic coupling), g_i the renormalized parameter and δg_i the renormalization constant, which contains divergent poles in ϵ. The coefficients $a_i^{(n)}$ depend explicitly on the parameters of the theory and on μ implicitly via the μ-dependence of these parameters. On which parameters these coefficients depend depends on the types of interactions present in the theory. The β function for the parameter g_i is given by

$$\beta_i(\epsilon) \equiv \frac{\partial g_i}{\partial \ln \mu} = -\rho_i \epsilon g_i - \rho_i a_i^{(1)} + \sum_k \rho_k g_k \frac{\partial a_i^{(1)}}{\partial g_k} \tag{9.1.2}$$

where the sum runs over all parameters g_k of the theory. Notice that the β function is completely determined by the coefficients of the *single pole* term in Eq. (9.1.1). The anomalous dimension is obtained from the renormalization constant associated with a self-energy Green's function for fields ϕ, which has the expansion in ϵ-poles as

$$Z_\phi = 1 + \sum_{n=1}^{\infty} \frac{a_\phi^{(n)}}{\epsilon^n}\,, \tag{9.1.3}$$

and, assuming equal renormalization of the fields contained in self-energy Green's function, is equal to

$$\gamma_\phi(\epsilon) = \frac{1}{2}\mu \frac{d}{d\mu} \ln Z_\phi\,. \tag{9.1.4}$$

If we take into account all coupling constants and full field content of the theory, β and γ functions represent coefficients of operators acting on the effective action so that

$$\mu \frac{\partial}{\partial \mu} \Gamma_{\text{DReg}} = \Big(- \sum_k \beta_k(\epsilon) \frac{\partial}{\partial g_k} + \sum_\phi \gamma_\phi(\epsilon) N_\phi \Big) \Gamma_{\text{DReg}}\,, \tag{9.1.5}$$

holds, where N_ϕ is number-counting operator for a generic field ϕ. This is the RGE for regularized action and it holds even for $\epsilon \neq 0$. Notice that at this level of regularized action the β and γ functions are ϵ-dependent and have the structure

$$\beta_i(\epsilon), \gamma_i(\epsilon) = \mathcal{O}(\epsilon) \times (\text{tree-level}) + \mathcal{O}(\epsilon^0) \times (\geq \text{1-loop level})\,. \tag{9.1.6}$$

For the RGE derivation in vector-like theories or naive treatment, basis of operators need for insertion is the one present at the tree-level. Multiplicative renormalization is sufficient and one derives γ and β functions as linear combination of respective renormalization constants Z-s coming from the calculation and cancellation of divergent parts of loop diagrams at the order of interest that are part of the $S_{\text{sct, inv}}$. In the BMHV scheme, however, the action contains additional types of counterterms as we discussed in Chapter 5, one of them being evanescent divergent counterterms, see Eq. (6.3.13). These have no tree-level counterpart, so our tree-level action is not sufficient to construct the basis of operators and we have to extend it with evanescent operators that appear at higher orders. New operators contain new parameters $\hat{g}_i$ that generally also have some renormalization transformation

$$\hat{g}_i \to \hat{g}_i + \delta\hat{g}_i\,. \tag{9.1.7}$$

and hence their own β functions. Also, the additional renormalization constants $\hat{Z}$-s multiply the emerging evanescent operators. Using standard renormalization transformations in the fashion of Chapter 5, we would get all the singular countertems (now including evanescent counterterms too) and our RGE now have additional term,

$$\mu\frac{\partial}{\partial\mu}\Gamma_{\text{DReg}} = \Big(- \sum_k \beta_k(\epsilon)\frac{\partial}{\partial g_k} - \sum_k \hat{\beta}_k(\epsilon)\frac{\partial}{\partial \hat{g}_k} + \sum_\phi \gamma_\phi(\epsilon) N_\phi \Big)\Gamma_{\text{DReg}}\,, \tag{9.1.8}$$

where the second sum on the right-hand side is over all parameters $\hat{g}_k$ of the evanescent additional action.

In the following, we will discuss the influence of these additional "evanescent" parameters $\hat{g}_k$. We emphasize to the reader that such or similar parameters have been discussed in various contexts before. Ref. [91] considered the same problem as the present section, but in the context of a non-gauge theory, hence the BHMV scheme treatment shouldn't influence the final result. It is found that if these parameters would be omitted, that would lead to incorrect RGE in general. The β functions for evanescent parameters influence the ones for regular parameters if the limit where evanescent parameters vanish is taken at the very end of the calculation.

Similarly, in the context of regularization by dimensional reduction (DRed), evanescent quantities do not correspond to γ_5 but to the extra $(4-d)$ degrees of freedom of the gauge fields, the so-called "ϵ-scalars". Accordingly, the impact of the ϵ-scalar mass term on the 2-loop RGE of softly broken supersymmetric gauge theories has been discussed in Ref. [92]. Finally, in applications of DRed to non-supersymmetric QCD, the evanescent coupling α_e between the ϵ-scalar and quarks appears. The need for treating this coupling and its β function as independent has been explained first in Ref. [93], for a further overview and references see [9].

Our original formulation of the theory in Chapters 4 to 6 corresponds to setting the evanescent parameters $\hat{g}_k = 0$ at tree-level. This is compatible with the RGE in $\epsilon \neq 0$ only at one particular renormalization scale μ. At other scales μ', the RGE generates non-vanishing tree-level values, i.e.

$$\hat{g}_k(\mu) = 0; \quad \hat{g}_k(\mu') \neq 0. \tag{9.1.9}$$

Non-vanishing $\hat{g}_i$ will contribute up to the tree level in the purely evanescent part, at the 1-loop level in singular contributions to evanescent Green's functions and in finite contributions to $\hat{\beta}$ functions of evanescent parameters, and finally at the 1-loop level in finite contributions to standard (non-evanescent) Green's functions, what is of great importance at higher orders. Now, applying the $\mathrm{LIM}_{d\to 4}$ operation to the generic RGE we get at the 1-loop level

$$\underset{d\to 4}{\mathrm{LIM}}\, \mu\frac{\partial}{\partial\mu}\Gamma_{\mathrm{DReg}} = \Big(-\sum_k \beta_k(0)\frac{\partial}{\partial g_k} - \sum_k \hat{\beta}_k(0)\frac{\partial}{\partial \hat{g}_k} + \sum_\phi \gamma_\phi(0) N_\phi \Big)\Gamma_{\mathrm{Ren}}\,, \tag{9.1.10}$$

with important remarks:

1. the derivative $\frac{\partial}{\partial \hat{g}_k}\Gamma_{\mathrm{DReg}}$ reduces to a finite, pure 1-loop quantity, since the tree-level action is free from evanescent parameters;
2. all coefficients $\beta_k(\epsilon)$, $\gamma_\phi(\epsilon)$, and $\hat{\beta}_k(\epsilon)$ vanish at tree-level and become quantities of 1-loop order; in the $\epsilon \to 0$ limit we denote $\beta_k(0) \equiv \beta_k$, $\gamma_\phi(0) \equiv \gamma_\phi$;
3. the coefficients β_k and γ_ϕ corresponding to non-evanescent operators are independent of the evanescent $\hat{g}_k$.

In renormalized limit at the 1-loop order we then have

$$\mu\frac{\partial\Gamma_{\mathrm{Ren}}}{\partial\mu} = \Big(-\sum_k \beta_k\frac{\partial}{\partial g_k} + \sum_\phi \gamma_\phi N_\phi \Big)\Gamma_{\mathrm{Ren}}\,, \tag{9.1.11}$$

where both sides are evaluated up to the 1-loop level. We get independence on evanescent parameters $\hat{g}_i$, and the non-evanescent coefficients β_k, γ_ϕ may be evaluated without any impact on calculation by setting the $\hat{g}_i = 0$. This shows that the correct 1-loop and 1-loop only RGE in the BMHV context may be obtained by the the usual procedure of Refs. [68, 90] from the divergences of renormalization constants, discarding the additional evanescent objects contained in the amended tree-level action, and instead taking only the theory as defined in Chapters 4 to 6. We strongly emphasize again that this statement is valid only at the 1-loop level, where at the higher orders $\hat{g}_i$ parameters start to influence non-evanescent parts of the action. E.g. at the 2-loop level and further we expect:

1. The term
$$\hat{\beta}_k\frac{\partial}{\partial \hat{g}_k}\Gamma_{\mathrm{DReg}}$$
can be expected to provide finite, non-vanishing 2-loop contributions.
2. The β_i, γ_ϕ coefficients might too depend on the evanescent parameters $\hat{g}_i$.

Both effects have appeared in the contexts of Refs. [44, 91–93] mentioned above, and additional calculations are required to replace the dependence on the $\hat{g}_i$ by modifications of the β_i, γ_ϕ. Ignoring the $\hat{g}_i$ parameters is justified exclusively at the 1-loop order and in general leads to incorrect RGE. The 2-loop study of RGE-s is hence of great importance and work is in progress.

9.2 | Self-energies and anomalous dimensions

Since at the 1-loop level, the ignoring of evanescent parameters is allowed by the particular choice of renormalization scheme, and the procedure follows to one usual for vector-like theories, for the sake of comparison we now evaluate γ and β functions for chiral Yang-Mills in this way.
Starting from the Green's function for scalar self energy at the 1-loop level, the corresponding anomalous dimension can be derived from the $1/\epsilon$ pole, using the standard approach [90] (see also Machacek & Vaughn [68]). From the general definition of the anomalous dimension Eq. (9.1.4) we derive the 1-loop anomalous dimension as

$$\gamma_\phi = \frac{1}{2}\mu\frac{d}{d\mu}\ln Z_\phi = -a_\phi^{(1)} \equiv -\epsilon\delta Z_\phi^{(1)}\,, \tag{9.2.1}$$

where we assumed equal renormalization of the fields in self-energy Green's function and took into account that scale dependence is implicitly coming from the renormalization constant coefficients. We obtain the 1-loop level anomalous dimensions respectively for the scalar γ_Φ, the fermion γ_ψ, and the gauge γ_G fields:

$$\gamma_\Phi^{mn\,(1)} = -\frac{1}{16\pi^2}(g^2(3-\xi)C_2(S) - Y_2(S))\delta^{mn}\,, \tag{9.2.2}$$

$$\gamma_\psi^{ij\,(1)} = \frac{1}{16\pi^2}\left(g^2\xi C_2(R) + \frac{Y_2(R)}{2}\right)\delta^{ij}\,, \tag{9.2.3}$$

$$\gamma_G^{ab\,(1)} = -\frac{g^2}{16\pi^2}\frac{(13-3\xi)C_2(G) - S_2(S) - 4S_2(R)}{6}\delta^{ab}\,. \tag{9.2.4}$$

In a choice of ghost renormalization where both the ghost and the antighost are chosen to renormalize equally, their anomalous dimension γ_c is then:

$$\gamma_c^{ab\,(1)} = -\frac{g^2}{16\pi^2}\frac{3-\xi}{4}C_2(G)\delta^{ab}\,. \tag{9.2.5}$$

9.3 | Vertex counterterms and beta functions

The renormalization of the coupling constant g for our Yang-Mills model is defined by:

$$g = Z_{\text{vertex}}^{-1}\prod Z_{\text{field}}^{1/2}\, g_{\text{bare}}\,\mu^{-\epsilon}\,, \tag{9.3.1}$$

in terms of its "bare" value g_{bare} and of the associated renormalization constants Z_{field} of the fields attached to the considered vertex. The coupling constant is expanded as an infinite Laurent series in ϵ,

$$g_{\text{bare}}\mu^{-\epsilon} = g + a^{(1)}\frac{g^3}{(4\pi)^2}\frac{1}{\epsilon} + \dots\,, \tag{9.3.2}$$

where μ is the renormalization scale parameter with mass dimension, so as the coupling constant remains dimensionless, and $a^{(1)}$ is the 1-loop contribution. Using the definition Eq. (9.1.2), the derivation of the β-function for g is then straightforward:

$$\begin{aligned}\mu\frac{d}{d\mu}(g_{\text{bare}}\mu^{-\epsilon}) = -\epsilon(g_{\text{bare}}\mu^{-\epsilon}) &= -\epsilon(g + a^{(1)}\frac{g^3}{(4\pi)^2}\frac{1}{\epsilon} + \dots) \\ &= \beta_g - \epsilon g + a^{(1)}\frac{3g^2}{(4\pi)^2}\frac{1}{\epsilon}(\beta_g - \epsilon g) + \dots\,.\end{aligned} \tag{9.3.3}$$

Taking the terms of order ϵ^0 we obtain the β-function for the coupling constant g at 1-loop level:

$$\beta_g = 2a^{(1)}\frac{g^3}{(4\pi)^2} \equiv g\,\beta\,. \tag{9.3.4}$$

Beta function for the gauge coupling constant

The gauge coupling constant renormalization can be obtained via the scalar sector according to:

$$g_{\text{bare}}\,\mu^{-\epsilon} = g\,Z_G^{-1/2}Z_\Phi^{-1}Z_{\Phi\Phi G}\,. \tag{9.3.5}$$

The renormalization constant for the scalar-scalar-gauge boson vertex $\imath\widetilde{\Gamma}^{mn,a,\mu}_{\Phi\Phi G}$ is defined by:

$$\delta Z^{(1)}_{\Phi\Phi G} = \frac{g^2}{16\pi^2\epsilon}\left(-\frac{3+\xi}{4}C_2(G) + (3-\xi)C_2(S) - Y_2(S)\right)\,. \tag{9.3.6}$$

The coupling constant renormalization up to 1-loop level is then:

$$\begin{aligned}g_{\text{bare}}\,\mu^{-\epsilon} &= g\,(1 - \frac{1}{2}\delta Z_G - \delta Z_\Phi + \delta Z_{\Phi\Phi G}) \\ &= \frac{g^3}{16\pi^2\epsilon}\frac{-22C_2(G) + S_2(S) + 4S_2(R)}{12}\end{aligned} \tag{9.3.7}$$

From the fermion sector we have:

$$g_{\text{bare}}\,\mu^{-\epsilon} = g\,Z_G^{-1/2}Z_\psi^{-1}Z_{\psi\bar{\psi}G}\,. \tag{9.3.8}$$

Thus, the renormalization constant for fermion-fermion-gauge boson vertex $\imath\widetilde{\Gamma}^{ij,a,\mu}_{\psi\bar{\psi}G}$ is defined by:

$$\delta Z^{(1)}_{\psi\bar{\psi}G} = \frac{1}{16\pi^2\epsilon}\left(-g^2\frac{(3+\xi)C_2(G) + 4\xi C_2(R)}{4} - \frac{Y_2(R)}{2}\right)\,. \tag{9.3.9}$$

The coupling constant renormalization up to 1-loop level is then:

$$
\begin{aligned}
g_{\text{bare}}\,\mu^{-\epsilon} &= g\,(1-\frac{1}{2}\delta Z_G - \delta Z_\psi + \delta Z_{\psi\bar{\psi}G}) \\
&= \frac{g^3}{16\pi^2\epsilon}\frac{-22C_2(G)+S_2(S)+4S_2(R)}{12}\,.
\end{aligned}
\tag{9.3.10}
$$

We also obtain the same result from the gauge sector, where the vertex counterterm is defined via:

$$
\delta Z^{(1)}_{GGG} = \frac{g^2}{16\pi^2\epsilon}\left(\frac{17-9\xi}{12}C_2(G) - \frac{S_2(S)}{6} - \frac{2S_2(R)}{3}\right)\,. \tag{9.3.11}
$$

The β-function for the coupling constant at the 1-loop level is then

$$
\beta^{(1)}_g = \frac{g^3}{16\pi^2}\frac{-22C_2(G)+S_2(S)+4S_2(R)}{6} \equiv g\,\beta^{(1)}\,. \tag{9.3.12}
$$

Notice that the gauge dependence present in particular renormalization constants gets canceled in their linear combination present in the derivation of the beta function. Similarly, beta functions neither depend on Yukawa coupling, while renormalization constants do. All these results are consistent with each other, and they agree as expected with the result obtained from the Background-field method [94], where there is a nice[1] property that the β function for the coupling is proportional to the anomalous dimension of the background gauge field, or more precisely, equal to $g\gamma_{Gb}$.

Beta function for Yukawa coupling

From the Green's function for the Yukawa vertex

$$
i\widetilde{\Gamma}^{ji,m}_{\overline{\psi\psi^C}\Phi}(p_1,p_2)|^{(1)}_{\text{div}} = \frac{\imath}{16\pi^2\epsilon}\left(Y^n_R(Y^m_R)^*Y^n_R - g^2\frac{2C_2(R)(3+\xi)-C_2(S)(3-\xi)}{2}Y^m_R\right)_{ij}\mathbb{P}_{\text{R}}\,, \tag{9.3.13}
$$

we obtain the vertex renormalization constant:

$$
\delta Z^{ji,m\,(1)}_{\overline{\psi\psi^C}\Phi} = \frac{1}{16\pi^2\epsilon}\left(Y^n_R(Y^m_R)^*Y^n_R - g^2\frac{2C_2(R)(3+\xi)-C_2(S)(3-\xi)}{2}Y^m_R\right)_{ij}\,. \tag{9.3.14}
$$

The renormalization of the Yukawa coupling is obtained via:

$$
(Y^m_{\text{R, bare}})_{ij}\mu^{-\epsilon} = \left(Z_\Phi^{-1/2}Z_\psi^{\dagger-1/2}Z_{\psi\bar{\psi}\Phi}Y^m_R Z_\psi^{-1/2}\right)_{ij} = (Y_R)^m_{ij} + (a^m_Y)_{ij}\frac{1}{\epsilon}\,, \tag{9.3.15}
$$

[1]For very high order β function calculations, e.g. 4-loop level, this property is not just nice but crucial.

what at the 1-loop level results in

$$(Y^m_{\mathrm{R,\ bare}})_{ij}\mu^{-\epsilon} = \Big(Y^m_R(1-\frac{1}{2}\delta Z_\Phi - \frac{1}{2}\delta Z^\dagger_\psi - \frac{1}{2}\delta Z_\psi) + \delta Z^m_{\psi\bar{\psi}\Phi}\Big)_{ij} = (Y_R)^m_{ij} + (a^m_Y)^{(1)}_{ij}\frac{1}{\epsilon}\,. \quad (9.3.16)$$

The β-function is then obtained straightforwardly

$$\begin{aligned}\beta_{Y^m_{ij}} =& 2(a^m_Y)^{(1)}_{ij}\\ =& \frac{1}{16\pi^2}\left(2(Y^n_R(Y^m_R)^*Y^n_R)_{ij} - 3g^2\{C_2(R), Y^m_R\}_{ij} + Y^m_{ij}Y_2(S) + \frac{1}{2}(Y^m_{ij}Y_2(R) + Y_2(\bar{R})Y^m_{ij})\right),\end{aligned} \quad (9.3.17)$$

and it depends both on Yukawa couplings and gauge coupling.

Beta function for scalar quartic coupling

The scalar quartic coupling λ_{abcd} is renormalized as follows:

$$\begin{aligned}\lambda_{abcd,\mathrm{bare}}\,\mu^{-2\epsilon} &= Z_\Phi^{-1/2}(a)Z_\Phi^{-1/2}(b)Z_\Phi^{-1/2}(c)Z_\Phi^{-1/2}(d)Z_{4\Phi}\,\lambda_{abcd}\\ &= \lambda_{abcd} + a^{(1)}_{\lambda,abcd}\frac{1}{\epsilon} + \dots\,,\end{aligned} \quad (9.3.18)$$

with $\lambda_{abcd,\mathrm{bare}}$ its corresponding "bare" coupling. We read off the vertex renormalization factor from $\imath\widetilde{\Gamma}^{abcd}_{\Phi\Phi\Phi\Phi}$ as:

$$\delta Z_{4\Phi,abcd} = \frac{1}{16\pi^2\epsilon}\frac{1}{2}(3g^4A_{abcd} - g^2\xi\Lambda^S_{abcd} - 4H_{abcd} + \Lambda^2_{abcd})\,. \quad (9.3.19)$$

The β-function for the scalar quartic coupling at the 1-loop level is:

$$\beta_\lambda = 2a^{(1)}_{\lambda,abcd} = \frac{1}{16\pi^2}(3g^4A_{abcd} - 4H_{abcd} + \Lambda^2_{abcd} + \Lambda^Y_{abcd} - 3g^2\Lambda^S_{abcd})\,, \quad (9.3.20)$$

being dependent on scalar, gauge, and Yukawa couplings. Notice the 1/2 difference with the respect to the non-chiral model, in the contribution for H_{abcd} coming from the fermion loops, expected since we only have right-handed fermions in our model, i.e. half of fermionic content "is missing".

All results for anomalous dimensions and all beta functions are in agreement with the BMHV approach and [67–69].

9.4 | Full system of renormalization group equations

In Chapters 6 and 7 we explicitly renormalized the theory at the 1-loop level by calculating and introducing the singular and finite counterterms in the action:

$$S_{\text{Ren}}^{(1)} = \underset{d\to 4}{\text{LIM}}(S_0 + S_{\text{sct}}^{(1)} + S_{\text{fct}}^{(1)})\,, \tag{9.4.1}$$

what rendered the theory finite and restored BRST invariance. If we focus on singular part $S_{\text{sct}}^{(1)}$, we can interpret divergent factors as Z renormalization constants:

$$\begin{aligned} S_{\text{sct}}^{(1)} &= \delta Z_G^{(1)}\, S_{GG} + \delta Z_{3G}^{(1)} S_{GGG} + \delta Z_{4G}^{(1)}\, S_{GGGG} + \delta \overline{Z}_G^{(1)}\, \overline{S_{GG}} + \delta \overline{Z}_{3G}^{(1)}\, \overline{S_{GGG}} + \delta \overline{Z}_{4G}^{(1)}\, \overline{S_{GGGG}} \\ &+ \delta \overline{Z}_\psi^{(1)}\, \overline{S_{\bar\psi\psi_R}} + \delta \overline{Z}_{\bar\psi G\psi_R}^{(1)}\, \overline{S_{\bar\psi G\psi_R}} + \delta Z_{\bar c c}^{(1)}\, S_{\bar c c} + \delta Z_{\rho c}^{(1)} S_{\rho c} + \delta Z_{\bar c G c}^{(1)}\, S_{\bar c G c} \\ &+ \delta Z_{\rho G c}^{(1)}\, S_{\rho G c} + \delta Z_{\zeta cc}^{(1)}\, S_{\zeta cc} + \delta Z_{\bar R\psi_R}^{(1)}\, S_{\bar R\psi_R} + \delta Z_{R\overline{\psi_R}}^{(1)}\, S_{R\overline{\psi_R}} + \delta Z_{\mathcal{Y}c\Phi}^{(1)}\, S_{\mathcal{Y}c\Phi} \\ &+ \delta \hat{Z}_G^{(1)} \int \mathrm{d}^d\, x\, \frac{1}{2}\bar{G}^{a\,\mu}\hat{\partial}^2\bar{G}^a_\mu + \delta Z_\Phi^{(1)}\, S_{\Phi\Phi} + \delta Z_{\Phi G\Phi}^{(1)} S_{\Phi G\Phi} + \delta Z_{\Phi GG\Phi}^{(1)}\, S_{\Phi GG\Phi} \\ &+ \delta \overline{Z}_\Phi^{(1)}\, \overline{S_{\Phi\Phi}} + \delta \overline{Z}_{\Phi G\Phi}^{(1)}\, \overline{S_{\Phi G\Phi}} + \delta \overline{Z}_{\Phi GG\Phi}^{(1)}\, \overline{S_{\Phi GG\Phi}} + \delta Z_{4\Phi,mnop}^{(1)}\, S_{\Phi^4_{mnop}} \\ &+ \delta Z_{Y,ij}^{m,(1)} S_{\overline{\psi_{Ri}}{}^C\Phi^m\psi_{Rj}} + \text{h.c.} + \delta \hat{Z}_\phi^{(1)}\, \widehat{S_{\Phi\Phi}}\,. \end{aligned} \tag{9.4.2}$$

The naive dimensional regularization result would correspond to d-dimensional part of (9.4.2) (meaning we interpret $\delta\bar{Z}$ as $\delta Z + \delta\hat{Z}$ and add d-dimensional contribution to the corresponding δZ). To obtain the β and γ functions from renormalization constants, we have to solve the following system of equations:

$$\begin{aligned} \delta Z_G^{(1)} &\to -\gamma_G^{(1)} = \frac{\hbar}{16\pi^2} g^2 \frac{(13-3\xi)C_2(G) - S_2(S) - 4S_2(R)}{6}\,, \\ \delta Z_{3G}^{(1)} &\to \frac{1}{2}\beta^{(1)} - \frac{3}{2}\gamma_G^{(1)} = \frac{\hbar}{16\pi^2} g^2 \frac{(17-9\xi)C_2(G) - 2S_2(S) - 8S_2(R)}{12}\,, \\ \delta Z_{4G}^{(1)} &\to \beta^{(1)} - 2\gamma_G^{(1)} = \frac{\hbar}{16\pi^2} g^2 \Big(\frac{(2-3\xi)C_2(G) - 2S_2(R)}{3} - \frac{S_2(S)}{6}\Big)\,, \\ \delta Z_\psi^{(1)} &\to -\gamma_\psi^{(1)} = \frac{-\hbar}{16\pi^2}\left(g^2\xi C_2(R) + \frac{Y_2(R)}{2}\right), \\ \delta Z_{\bar\psi G\psi_R}^{(1)} &\to \frac{1}{2}\beta^{(1)} - \frac{1}{2}\gamma_G^{(1)} - \gamma_\psi^{(1)} = \frac{-\hbar}{16\pi^2}\left(g^2 \frac{(3+\xi)C_2(G) + 4\xi C_2(R)}{4} + \frac{Y_2(R)}{2}\right), \\ \delta Z_\Phi^{(1)} &\to -\gamma_\Phi^{(1)} = \frac{\hbar}{16\pi^2}\left(g^2(3-\xi)C_2(S) - Y_2(S)\right), \\ \delta Z_{\Phi G\Phi}^{(1)} &\to \frac{1}{2}\beta^{(1)} - \frac{1}{2}\gamma_G^{(1)} - \gamma_\Phi^{(1)} = \frac{\hbar}{16\pi^2}\left(g^2\left((3-\xi)C_2(S) - \frac{3+\xi}{4}C_2(G)\right) - Y_2(S)\right), \\ \delta Z_{\Phi GG\Phi}^{(1)} &\to \beta^{(1)} - \gamma_G^{(1)} - \gamma_\Phi^{(1)} = \frac{\hbar}{16\pi^2}\left(g^2\left((3-\xi)C_2(S) - \frac{3+\xi}{2}C_2(G)\right) - Y_2(S)\right), \\ \delta Z_{4\Phi,mnop}^{(1)} &\to \frac{1}{2}\beta_{\lambda_{mnop}}^{(1)} - 2\gamma_\Phi^{(1)}\lambda_{mnop} = \frac{\hbar}{16\pi^2}\frac{1}{2}(3g^4 A - g^2\xi\Lambda^S - 4H + \Lambda^2)_{mnop}\,, \end{aligned}$$

$$
\begin{aligned}
\delta Z^{m,(1)}_{Y,ij} &\to \frac{1}{2}(\beta^{(1)}_Y)^m_{ij} - Y^m_{ij}(\frac{1}{2}\gamma^{(1)}_\Phi + \gamma^{(1)}_\psi) \\
&= \frac{\hbar}{16\pi^2}\left((Y^n_R(Y^m_R)^* Y^n_R) - g^2\frac{2C_2(R)(3+\xi) - C_2(S)(3-\xi)}{2}Y^m_R\right)_{ij}, \\
\delta Z^{(1)}_{\bar{c}c} &\to -\frac{1}{2}\gamma^{(1)}_c + \frac{1}{2}\gamma^{(1)}_G = \frac{\hbar}{16\pi^2}g^2\frac{3-\xi}{4}C_2(G), \\
\delta Z^{(1)}_{\bar{c}Gc} &\to \frac{1}{2}\beta^{(1)} - \frac{1}{2}\gamma^{(1)}_c = \frac{-\hbar}{16\pi^2}g^2\frac{\xi}{2}C_2(G), \\
\delta Z^{(1)}_{\rho c} &\to -\frac{1}{2}\gamma^{(1)}_\rho - \frac{1}{2}\gamma^{(1)}_c = \frac{\hbar}{16\pi^2}g^2\frac{3-\xi}{4}C_2(G), \\
\delta Z^{(1)}_{\rho Gc} &\to \frac{1}{2}\beta^{(1)} - \frac{1}{2}\gamma^{(1)}_\rho - \frac{1}{2}\gamma^{(1)}_G - \frac{1}{2}\gamma^{(1)}_c = \frac{-\hbar}{16\pi^2}g^2\frac{\xi}{2}C_2(G), \\
\delta Z^{(1)}_{\zeta cc} &\to \frac{1}{2}\beta^{(1)} - \frac{1}{2}\gamma^{(1)}_\zeta - \gamma^{(1)}_c = \frac{-\hbar}{16\pi^2}g^2\frac{\xi}{2}C_2(G), \\
\delta Z^{(1)}_{\bar{R}\psi_R} &\to \frac{1}{2}\beta^{(1)} - \frac{1}{2}\gamma^{(1)}_R - \frac{1}{2}\gamma^{(1)}_c - \frac{1}{2}\gamma^{(1)}_\psi = \frac{-\hbar}{16\pi^2}g^2\frac{\xi}{2}C_2(G), \\
\delta Z^{(1)}_{R\overline{\psi_R}} &\to \frac{1}{2}\beta^{(1)} - \frac{1}{2}\gamma^{(1)}_R - \frac{1}{2}\gamma^{(1)}_c - \frac{1}{2}\gamma^{(1)}_\psi = \frac{-\hbar}{16\pi^2}g^2\frac{\xi}{2}C_2(G), \\
\delta Z^{(1)}_{\mathcal{Y}c\Phi} &\to \frac{1}{2}\beta^{(1)} - \frac{1}{2}\gamma^{(1)}_{\mathcal{Y}} - \frac{1}{2}\gamma^{(1)}_c - \frac{1}{2}\gamma^{(1)}_\Phi = \frac{-\hbar}{16\pi^2}g^2\frac{\xi}{2}C_2(G),
\end{aligned}
$$

again overdetermined and with the solution at 1-loop level:

$$
\begin{aligned}
\beta &= \frac{\hbar}{16\pi^2}g^2\left(\frac{-22C_2(G) + 4S_2(R) + S_2(S)}{6}\right), \\
\beta_\lambda &= \frac{\hbar}{16\pi^2}(3g^4 A_{abcd} - 4H_{abcd} + \Lambda^2_{abcd} + \Lambda^Y_{abcd} - 3g^2\Lambda^S_{abcd}), \\
\beta_{Y^m_{ij}} &= \frac{\hbar}{16\pi^2}\left(2(Y^n_R(Y^m_R)^*Y^n_R)_{ij} - 3g^2\{C_2(R), Y^m_R\}_{ij} + Y^m_{ij}Y_2(S) + \frac{1}{2}(Y^m_{ij}Y_2(R) + Y_2(\bar{R})Y^m_{ij})\right), \\
\gamma_G &= \frac{\hbar}{16\pi^2}g^2\frac{(3\xi - 13)C_2(G) + 4S_2(R) + S_2(S)}{6} = -\gamma_{\bar{c}} = -\gamma_\rho, \\
\gamma_\psi &= \frac{\hbar}{16\pi^2}\frac{2g^2\xi C_2(R) + Y_2(R)}{2} = -\gamma_R, \\
\gamma_\Phi &= \frac{\hbar}{16\pi^2}\left(g^2(\xi-3)C_2(S) + Y_2(S)\right) = -\gamma_{\mathcal{Y}}, \\
\gamma_c &= \frac{\hbar}{16\pi^2}g^2\frac{(6\xi - 22)C_2(G) + 4S_2(R) + S_2(S)}{6} = -\gamma_\zeta.
\end{aligned}
$$

If we compare solution of the system with the one obtained in Section 8.4, the β-functions and anomalous dimensions do not differ.

The result is the same as the one obtained in Chapter 8, demonstrating that both methods may be applied to obtain the correct 1-loop RGE in the BMHV scheme. We again strongly emphasize that this is valid exclusively at the 1-loop level. For the higher orders, explicit calculation is needed and is recommended as future proposal of this research.

CHAPTER 10

THE LEFT-HANDED (L) MODEL

In this chapter, we indicate how our previous results adapt for a model including only left-handed fermions. Since the result is a straightforward list of the results analogous to the right-handed chiral theory, we strictly follow the [47]. We define the Left-handed (L) model to be the same as the Right-handed (R) model studied so far, except now with the usage of purely left-handed fermions $\psi_L \equiv \mathbb{P}_{\rm L}\psi$: the gauge, scalar, and gauge-scalars sectors remain unchanged, while only the fermion kinetic and Yukawa terms get modified. We aim to know how our results derived so far change when considering these left-handed fermions. This is needed as an ingredient for future phenomenological studies in SM and beyond.

10.1 | Transition to Left-handed model

It is possible to construct a mapping between the L-model and the R-model indeed, using the charge-conjugation construction from Section 2.2.2, the charge-conjugate of a left-handed fermion is a right-handed fermion:

$$
{\psi_L}^C = C\overline{\psi_L}^T = C(\overline{\psi}\mathbb{P}_{\rm R})^T = C{\mathbb{P}_{\rm R}}^T\overline{\psi}^T = C{\mathbb{P}_{\rm R}}^T C^{-1}C\overline{\psi}^T = \mathbb{P}_{\rm R}C\overline{\psi}^T = \mathbb{P}_{\rm R}\psi^C \equiv \mathbb{P}_{\rm R}\widehat{\psi} \equiv \widehat{\psi}_R\,, \tag{10.1.1}
$$

with the definition $\widehat{\psi} \equiv \psi^C$.

The same discussion as in Section 4.2 holds and we can promote this L-model to d dimensions. The left-handed fermion-kinetic term is:

$$
\mathcal{L}_{\text{fermions}} = i\overline{\psi}_i\partial\!\!\!/\psi_i + gT^a_{L\,ij}\overline{\psi_{L}}_i G\!\!\!/^a\psi_{Lj}\,, \tag{10.1.2}
$$

where T_L is the generator for their corresponding representation. Since the kinetic term is a scalar function, it is also equal to its transpose in spinor space, and thus we obtain:

$$
\begin{aligned}
\mathcal{L}_{\text{fermions}} &= i(\overline{\psi}_i \not{\partial} \psi_i)^T + g T_{L}{}^a_{ij} (\overline{\psi_L}_i \not{G}^a \psi_{Lj})^T = -i \psi_i^T \overleftarrow{\not{\partial}}^T \overline{\psi}_i^T - g T_{L}{}^a_{ij} G^a_\mu \psi_{L}{}_j^T (\gamma^\mu)^T \overline{\psi_L}_i^T \\
&= -i \psi_i^T C^{-1} C \overleftarrow{\not{\partial}}^T C^{-1} C \overline{\psi}_i^T - g T_{L}{}^a_{ij} G^a_\mu \psi_{L}{}_j^T C^{-1} C (\gamma^\mu)^T C^{-1} C \overline{\psi_L}_i^T \\
&= i \overline{\psi}_i^C C \overleftarrow{\not{\partial}}^T C^{-1} \psi_i^C + g T_{L}{}^a_{ij} G^a_\mu \overline{\psi_L}_j^C C (\gamma^\mu)^T C^{-1} \psi_{L}{}_i^C \\
&= -i \overline{\widehat{\psi}}_i \overleftarrow{\not{\partial}} \widehat{\psi}_i + g(-T_{L}{}^a_{ij}) G^a_\mu \overline{\widehat{\psi}_R}_j \gamma^\mu \widehat{\psi}_{Ri} = i \overline{\widehat{\psi}}_i \not{\partial} \widehat{\psi}_i + g T_{\overline{L}}{}^a_{ij} \overline{\widehat{\psi}_R}_i \not{G}^a \widehat{\psi}_{Rj} \, ,
\end{aligned}
\tag{10.1.3}
$$

where in the second equality we used the anticommutativity of the fermion fields, in the second line we inserted $\mathbb{1} = C^{-1}C$ and used the properties of the charge-conjugation as defined in Section 2.2.2, and in the last line we used an integration by parts (supposing the absence of surface terms) to rewrite the pure kinetic (first) term, and defined in the interaction term $T_{\overline{L}}{}^a_{ij} = -T_{L}{}^a_{ji}$ corresponding to the complex-conjugated representation of the left-handed fermions. Posing $T_{\widehat{R}}{}^a_{ij} \equiv T_{\overline{L}}{}^a_{ij}$, we see that this conjugated L-representation corresponds to the representation for the associated right-handed fermions.

Let us now turn to the Yukawa term, which is a real number and therefore equals to its hermitian conjugate:

$$
2 \times \mathcal{L}_{\text{Yukawa}} = -(Y_L)^m_{ij} \Phi_m \overline{\psi_L}_i^C \psi_{Lj} - (Y_L)^{m\,*}_{ij} \Phi^\dagger_m \overline{\psi_L}_i \psi_{L}{}_j^C = -(Y_L)^{m\,*}_{ij} \Phi^\dagger_m \overline{\widehat{\psi}_R}_i^C \widehat{\psi}_{Rj} + \text{h.c.} \, ,
\tag{10.1.4}
$$

and we can define $(Y_{\widehat{R}})^m_{ij} \equiv (Y_L)^{m\,*}_{ij}$ the corresponding Yukawa matrix for the associated right-handed fermions, which is just the complex conjugate of the one for the left-handed fermions.

External sources for the fermion fields need to be introduced in the L-model due to the BRST quantization procedure:

$$
\begin{aligned}
S_{\bar{L}c\psi_L} &= \bar{L}^i s_d \psi_i = i g \bar{L}^i c^a T_{L}{}^a_{ij} \psi_{Lj} \equiv i g \bar{L}^i c^a T_{L}{}^a_{ij} \mathbb{P}_{\mathrm{L}} \psi_j \, , \\
S_{Lc\overline{\psi_L}} &= s_d \overline{\psi}_i L^i = i g \overline{\psi_L}_j c^a T_{L}{}^a_{ji} L^i \equiv i g \overline{\psi}_j \mathbb{P}_{\mathrm{L}} c^a T_{L}{}^a_{ji} L^i \, .
\end{aligned}
\tag{10.1.5}
$$

Since these are scalar functions, we can take their transpose, and use the fact that L and $\bar{L}$ are *commuting* fermions (their ghost number is equal to 1) to obtain:

$$
\begin{aligned}
S_{\bar{L}c\psi_L} &= i g c^a T_{L}{}^a_{ij} (\bar{L}^i \psi_{Lj})^T = i g c^a T_{L}{}^a_{ij} \psi_{L}{}_j^T \bar{L}_i^T = i g c^a T_{L}{}^a_{ij} \psi_{L}{}_j^T C^{-1} C \bar{L}_i^T \\
&= i g \overline{\psi_L}_j^C c^a (-T_{L}{}^a_{ij})(-C \bar{L}_i^T) = i g \overline{\widehat{\psi}_R}_j c^a T_{R}{}^a_{ji} \widehat{R}_i \equiv S_{\widehat{R}c\overline{\widehat{\psi}_R}} \, ,
\end{aligned}
\tag{10.1.6}
$$

where we have employed the notations introduced above and have defined the external source $\widehat{R}_i$ for the corresponding right-handed fermions: $\widehat{R}_i \equiv -C\bar{L}_i^T = -L^C{}_i$. Similarly we obtain for the other source term:

$$\begin{aligned} S_{Lc\overline{\psi_L}} &= ig(\overline{\psi_L}_j L^i)^T c^a T_L{}^a_{ji} = igL^{Ti}\overline{\psi_L}_j^T c^a T_L{}^a_{ji} = igL^{Ti}C^{-1}C\overline{\psi_L}_j^T c^a T_L{}^a_{ji} \\ &= ig(-T_L{}^a_{ji})L^{Ti}C^{-1}c^a \psi_L{}_j^C = igT_R{}^a_{ij}\overline{\widehat{R}}^i c^a \widehat{\psi}_{Rj} \equiv S_{\overline{\widehat{R}}c\widehat{\psi}_R}\,, \end{aligned} \tag{10.1.7}$$

where we used that $\overline{\widehat{R}}_i = -\overline{L^C}_i = L_i^T C^{-1}$, stemming from the properties of C.

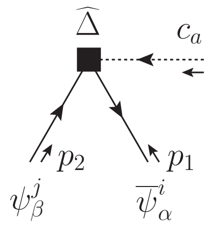
These calculations demonstrate that we can establish a one-to-one mapping between a left-handed model with fermions ψ ($\psi_L \equiv \mathbb{P}_\mathrm{L}\psi$) defined in a left-representation of the considered gauge group with generators T_L that couple to scalar fields with the Yukawa interaction $(Y_L)^m_{ij}$, and a right-handed model with fermions $\widehat{\psi}$ related via charge-conjugation: $\widehat{\psi} \equiv \psi^C$ ($\widehat{\psi}_R \equiv \mathbb{P}_\mathrm{R}\widehat{\psi}$), in a right-representation $T_{\widehat{R}}{}^a_{ij} \equiv T_{\overline{L}}{}^a_{ij}$ that couple to the scalar fields with the Yukawa interaction $(Y_{\widehat{R}})^m_{ij} \equiv (Y_L)^{m\,*}_{ij}$. Therefore, all of our calculations derived so far in this work apply to the left-handed model as well.

10.2 | Evaluation of the tree-level breaking in left-handed model

We will evaluate the tree-level breaking of the BRST symmetry by the action of this Left-handed model, similarly to what has been done in Section 4.3. We find that the breaking $\widehat{\Delta} = s_d S_0$ is given by:

$$\widehat{\Delta} = \int \mathrm{d}^d\, x\; (gT_L{}^a_{ij})c^a \left\{\overline{\psi}_i \left(\overleftarrow{\widehat{\partial\!\!\!/}}\mathbb{P}_\mathrm{L} + \overrightarrow{\widehat{\partial\!\!\!/}}\mathbb{P}_\mathrm{R}\right)\psi_j\right\} \equiv \int \mathrm{d}^d\, x\; \widehat{\Delta}(x)\,, \tag{10.2.1}$$

generating a corresponding Feynman rule:

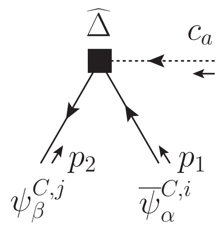


$$\begin{aligned} &= \frac{g}{2}T_L{}^a_{ij}\left((\widehat{p\!\!\!/_1} + \widehat{p\!\!\!/_2}) - (\widehat{p\!\!\!/_1} - \widehat{p\!\!\!/_2})\gamma_5\right)_{\alpha\beta} \\ &= gT_L{}^a_{ij}\left(\widehat{p\!\!\!/_1}\mathbb{P}_\mathrm{L} + \widehat{p\!\!\!/_2}\mathbb{P}_\mathrm{R}\right)_{\alpha\beta}\,, \end{aligned} \tag{10.2.2}$$

and the one corresponding to the charge-conjugated fermions:

$$\widehat{\Delta} = \int \mathrm{d}^d\, x\; (gT_{\overline{L}}{}^a_{ij})c^a \left\{\overline{\psi^C}_i \left(\overleftarrow{\widehat{\partial\!\!\!/}}\mathbb{P}_\mathrm{R} + \overrightarrow{\widehat{\partial\!\!\!/}}\mathbb{P}_\mathrm{L}\right)\psi_j^C\right\}\,, \tag{10.2.3}$$

generating the Feynman rule:



$$\begin{aligned} &= \frac{g}{2} T^{\ a}_{\overline{L}ij} \left((\widehat{\not{p}_1} + \widehat{\not{p}_2}) + (\widehat{\not{p}_1} - \widehat{\not{p}_2})\gamma_5 \right)_{\alpha\beta} \\ &= g T^{\ a}_{\overline{L}ij} \left(\widehat{\not{p}_1}\mathbb{P}_{\mathrm{R}} + \widehat{\not{p}_2}\mathbb{P}_{\mathrm{L}} \right)_{\alpha\beta} . \end{aligned} \tag{10.2.4}$$

where the difference with the previous result is in the appearance of the generator $T_{\overline{L}}{}^a$ for the fermionic conjugate representation L.

10.3 | The group invariants

The group invariants related to the scalar-fields representation $C_2(S), S_2(S), Y_2(S)$ and those defined in Eq. (6.2.15): $A_{mnop}, H_{mnop}, \Lambda^2_{mnop}, \Lambda^S_{mnop}$ all remain the same. The group invariants $C_2(L), S_2(L), Y_2(L)$ of the left-representation are actually equal to those of the corresponding right-representation:

$$C_2(L)\mathbb{1} = T_L{}^a T_L{}^a = (-T_L{}^{a\,T})(-T_L{}^{a\,T}) = T_{\overline{L}}{}^a T_{\overline{L}}{}^a = C_2(\overline{L})\mathbb{1} \equiv C_2(\widehat{R})\mathbb{1};$$

$$\begin{aligned} S_2(L)\delta^{ab} &= \mathrm{Tr}(T_L{}^a T_L{}^b) = \mathrm{Tr}((-T_L{}^{b\,T})(-T_L{}^{a\,T})) \\ &= \mathrm{Tr}(T_{\overline{L}}{}^b T_{\overline{L}}{}^a) = S_2(\overline{L})\delta^{ab} = \mathrm{Tr}(T_{\widehat{R}}{}^a T_{\widehat{R}}{}^b) = S_2(\widehat{R})\delta^{ab}; \end{aligned}$$

and

$$Y_2(L)\mathbb{1} = (Y^m_L)^* Y^m_L = Y^m_{\widehat{R}} (Y^m_{\widehat{R}})^* = (Y^m_{\widehat{R}})^* Y^m_{\widehat{R}} \equiv Y_2(\widehat{R})\mathbb{1}$$

by using the symmetry of the Yukawa matrices.

10.4 | Counterterms in left-handed model

The singular counterterms are now obtained, and are the same as in Eqs. (6.3.3), (6.3.4) and (13.2.2), except for the replacements of group invariants:

$$\begin{aligned} &S_2(R) \to S_2(L)\,, \\ &C_2(R) \to C_2(L)\,, \\ &Y_2(R) \to Y_2(L)\,, \\ &Y^m_R \to Y^m_L\,, \end{aligned}$$

and of course action operators involving fermion fields,

$$\overline{S_{\bar\psi\psi_R}} \to \overline{S_{\bar\psi\psi_L}}\,,$$

$$
\begin{aligned}
&\overline{S_{\overline{\psi}G\psi_R}} \to \overline{S_{\overline{\psi}G\psi_L}}\,, \\
&S_{\overline{\psi_R}_i^C \Phi^m \psi_{Rj}} \to S_{\overline{\psi_L}_i^C \Phi^m \psi_{Lj}}\,, \\
&S_{\bar{R}c\psi_R} \to S_{\bar{L}c\psi_L}\,, \\
&S_{Rc\overline{\psi_R}} \to S_{Lc\overline{\psi_L}}\,.
\end{aligned}
\tag{10.4.1a}
$$

Again, we can make contact to the usual renormalization transformation, and express the singular counterterms as follows:

$$
S_{\text{sct}}^{(1)} = S_{\text{sct,inv}}^{(1)} + S_{\text{sct,evan}}^{(1)}\,.
\tag{10.4.2}
$$

The invariant counterterms $S_{\text{sct,inv}}^{(1)}$ acquire the same form as those from Eq. (5.2.4), in terms of the functionals L_φ, and with the changes:

$$
\delta Z_{\psi_R}\overline{L_{\psi_R}} \to \delta Z_{\psi_L}\overline{L_{\psi_L}}\,, \qquad \delta(Y_R)_{ij}^m L_{Y_R}{}_{ij}^{m} \to \delta(Y_L)_{ij}^m L_{Y_L}{}_{ij}^{m}\,,
\tag{10.4.3}
$$

and the corresponding δZ_φ, δg_i renormalization constants are again the same as their counterparts Eqs. (6.3.6) to (6.3.12), but with the coefficients changed according to Eq. (10.4.1). The purely evanescent counterterms $S_{\text{sct,evan}}^{(1)}$ Eqs. (6.3.13) and (6.3.14) are also expressed in the same way, with the substitution $S_2(R) \to S_2(L)$.

Therefore, following the explanations given in Chapters 8 and 9, the resulting renormalization group equations for the Left-handed model are the very same ones as those for the Right-handed model, with the obvious changes $R \leftrightarrow L$.

The BRST-restoring finite counterterms Eq. (7.4.19) now read:

$$
\begin{aligned}
S_{\text{fct,restore}}^{(1)} = \frac{\hbar}{16\pi^2}\Bigg\{ & g^2\frac{S_2(L)}{6}\left(5S_{GG} + S_{GGG} - \int \mathrm{d}^4x\, G^{a\,\mu}\partial^2 G_\mu^a\right) + \frac{Y_2(S)}{3}S_{\Phi\Phi} \\
&+ g^2\frac{(T_L)^{abcd}}{3}\int \mathrm{d}^4x\, \frac{g^2}{4}G_\mu^a G^{b\,\mu}G_\nu^c G^{d\,\nu} - \frac{(\mathcal{C}_L)_{mn}^{ab}}{3}\int \mathrm{d}^4x\, \frac{g^2}{2}G_\mu^a G^{b\,\mu}\Phi^m\Phi^n \\
&+ g^2\left(1+\frac{\xi-1}{6}\right)C_2(L)S_{\overline{\psi}\psi} - \frac{((Y_L^m)^* T_{\overline{L}}{}^{a} Y_L^m)_{ij}}{2}\int \mathrm{d}^4x\, g\overline{\psi}_i \not{G}^a \mathbb{P}_{\mathrm{L}}\psi_j \\
&- g^2\frac{\xi C_2(G)}{4}\left(S_{\bar{L}c\psi_L} + S_{Lc\overline{\psi_L}}\right)\Bigg\}\,,
\end{aligned}
\tag{10.4.4}
$$

where we have used the following group factors:

$$
\begin{aligned}
(T_{\widehat{R}})^{a_1\cdots a_n} &= \mathrm{Tr}[T_{\overline{L}}{}^{a_1}\cdots T_{\overline{L}}{}^{a_n}] = \mathrm{Tr}\left[(-T_L{}^{a_1\,T})\cdots(-T_L{}^{a_n\,T})\right] \\
&= (-1)^n \mathrm{Tr}[T_L{}^{a_n}\cdots T_L{}^{a_1}] = (-1)^n (T_L)^{a_n\cdots a_1}\,,
\end{aligned}
\tag{10.4.5a}
$$

$$
(\mathcal{C}_L)_{mn}^{ab} \equiv \mathrm{Tr}\left[2\{T_L{}^a, T_L{}^b\}(Y_L^n)^* Y_L^m - T_L{}^a (Y_L^n)^* T_{\overline{L}}{}^b Y_L^m\right]\,,
\tag{10.4.5b}
$$

Again, this expression is formally completely unchanged with respect to Eq. (7.4.19), with the only change $R \leftrightarrow L$. However, the relevant (non-spurious) anomalies Eq. (7.4.20) now become:

$$\frac{\hbar g^2}{16\pi^2}\left(\frac{S_2(L)}{3} d_L^{abc} \int \mathrm{d}^4\, x\; g\epsilon^{\mu\nu\rho\sigma} c_a (\partial_\rho G_\mu^b)(\partial_\sigma G_\nu^c) + \frac{\mathcal{D}_L^{abcd}}{3 \times 3!} \int \mathrm{d}^4\, x\; g^2 c_a \epsilon^{\mu\nu\rho\sigma} \partial_\sigma \left(G_\mu^b G_\nu^c G_\rho^d\right)\right), \tag{10.4.6}$$

where the group factors are the fully symmetric symbol

$$d_L^{abc} = \mathrm{Tr}[T_L{}^a \{T_L{}^b, T_L{}^c\}]$$

and

$$\mathcal{D}_L^{abcd} = \frac{1}{2}(d_L^{abe} f^{ecd} + d_L^{ace} f^{edb} + d_L^{ade} f^{ebc})$$

for the L-representation. The opposite of sign in front of the equation, with respect to the one in Eq. (7.4.20), comes from the fact that these group factors for the L-representation are related to the corresponding ones in the corresponding right-handed model by two important relations:

$$d_L^{abc} = -d_{\widehat{R}}^{abc},$$

$$\mathcal{D}_L^{abcd} = -\mathcal{D}_{\widehat{R}}^{abcd}.$$

We note to the reader this has important phenomenological consequences for model-building: relevant anomalies can be cancelled in a given model if ones includes both right-handed and left-handed fermions whose representations are the complex-conjugate of the other. This gives one possible realization of anomaly cancellation condition Eq. (7.4.21).

CHAPTER 11

RIGHT-HANDED CHIRAL QED (χQED) AND ITS EXTENSION TO D DIMENSIONS

In the first part of this thesis, we have shown the BMHV scheme application on the Yang-Mills model with a general gauge group. The use of the BMHV scheme resulted in several types of counterterms, including one need for restoration of gauge and BRST symmetry. We have restored the symmetry at the 1-loop level, showed that anomalies cancel, and evaluated renormalization group equations for this model. During that research first published in [47] we found out that situations at the 1-loop level do not differ from one we would get if we treat the same model in a naive scheme, but we would expect severe change once we treat higher orders. The next step in our research was then determined by these important guidelines:

1. Since 2-loop treatment requires a significant increase of complications in calculations, it would be useful to decrease the complexity of the group structure in a model.

2. Since we want to do a complete study at the 2-loop level, it is important to have a benchmark model to compare the results. At the 1-loop level, a great source of information would be the results from our previous work in [47], so the next model should be contained in the general model we have already investigated.

3. Since we do not have any direct comparison available in literature, we should look for an additional symmetry check or property arising from the Slavnov-Taylor identities of the considered model.

4. Due to the comprehensiveness present in this kind of calculation, it would be useful that it is possible to write and run the computer codes to automatize the calculation as much as possible. From the previous experience, we can estimate the possibility and duration of these calculations, so it is important to make a reasonable choice from this point of view, too.

5. We didn't forget the phenomenology. It would be useful if the model is embedded in SM.

11.1 | The new model proposal

The best possible model enabling to follow the mentioned guidelines is the chiral quantum electrodynamics, with right-handed fermion fields but without scalar fields. It may be understood as a chiral version of QED with right-handed couplings only, or as a variant of the U(1) part of the electroweak standard model, or if you want, as a U(1) limit of previously defined and studied general Yang-Mills model with replacements

$$T^a_{Rij} \to \mathcal{Y}_{Rij}, \tag{11.1.1}$$

$$T^a_{\bar{R}ij} \to \mathcal{Y}_{\bar{R}ij} \equiv -\mathcal{Y}_{Rij}, \tag{11.1.2}$$

$$f^{abc} = 0, \tag{11.1.3}$$

and all scalars and their interactions are absent. We note this model as χQED. Since the adjoint representation is trivial, trilinear and quartic gauge boson interactions as well as ghost-gauge interaction are absent. In this chapter, we again define the model as we did for the Yang-Mills case. The theory is defined firstly in 4 dimensions: its action, BRST transformations rules, and basic Slavnov-Taylor identity. Then the theory is defined in d-dimensions: its action with d dimensional kinetic terms and the most symmetric right-handed fermion–gauge boson vertex, BRST transformations and Slavnov-Taylor identities in analogy with the 4-dimensional case. The breaking of the BRST symmetry of the action and Slavnov-Taylor identity for the action were identified with the evanescent part of the fermion kinetic term, and this is followed by detailed analysis and restoration of these breakings.

11.2 | χQED in 4 dimensions

In χQED, the only generator is the hypercharge, which we assume to be diagonal and real,

$$\mathcal{Y}_{Rij} \equiv (\mathrm{diag}\{\mathcal{Y}^1_R, \dots, \mathcal{Y}^{N_f}_R\})_{ij},$$

where N_f is the number of fermion flavours. The 4-dimensional classical Lagrangian of the model reads:

$$\mathcal{L} = i\overline{\psi_{Ri}}\not{D}_{ij}\psi_{Rj} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \bar{c}\partial^2 c + \rho^\mu s A_\mu + \bar{R}^i s\psi_{Ri} + R^i s\overline{\psi_{Ri}}, \tag{11.2.1}$$

where only purely right-handed fermions ψ_R appear. In general, we use the standard chirality projectors Eq. (4.1.6) and abbreviate $\psi_{R/L} = \mathbb{P}_{\mathrm{R/L}}\psi$. The covariant derivative acting on the fermion field is defined in the diagonal basis for couplings by

$$D^{\mu}_{ij} = \partial^{\mu}\delta_{ij} - ieA^{\mu}\mathcal{Y}_{Rij}\,, \tag{11.2.2}$$

and the field strength tensor is defined as

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}\,. \tag{11.2.3}$$

In order not to have anomalies in χQED, the following *anomaly cancellation condition* is imposed on the hypercharge couplings,

$$\boxed{\mathrm{Tr}(\mathcal{Y}_R^3) = 0} \tag{11.2.4}$$

We emhasize to the reader that this condition cancels 3-photon interaction at 1-loop level, also known as the *triangle diagram.* Next, the Lagrangian contains an R_ξ gauge fixing term with gauge parameter ξ and a corresponding Faddeev-Popov ghost kinetic term. The last three terms of Eq. (11.2.1) are the BRST transformations of the physical fields, coupled to external sources (or Batalin-Vilkovisky "anti-fields", [75–77]), where the external sources do not transform under BRST transformations by design. The non-vanishing BRST transformations are

$$\begin{aligned}
sA_{\mu} &= \partial_{\mu}c\,, && (11.2.5a)\\
s\psi_i &= s\psi_{Ri} = i\,e\,c\,\mathcal{Y}_{Rij}\psi_{Rj}\,, && (11.2.5b)\\
s\overline{\psi}_i &= s\overline{\psi_{Ri}} = i\,e\,\overline{\psi_{Rj}}c\mathcal{Y}_{Rji} && (11.2.5c)\\
s\bar{c} &= B \equiv -\frac{1}{\xi}\partial A, && (11.2.5d)
\end{aligned}$$

where "s" is the nilpotent generator of the BRST transformation, which acts as a fermionic differential operator. The last of these equations also introduces the auxiliary Nakanishi-Lautrup field B, which is integrated out from the action in Eq. (11.2.1) and in the rest of the χQED study. The 4-dimensional tree-level action

$$S_0^{(4d)} = \int \mathrm{d}^4\,x\,\mathcal{L} \tag{11.2.6}$$

satisfies the following Slavnov-Taylor identity

$$\mathcal{S}(S_0^{(4d)}) = 0\,, \tag{11.2.7}$$

where the Slavnov-Taylor operation is given for a general functional $\mathcal{F}$ as

$$\mathcal{S}(\mathcal{F}) = \int \mathrm{d}^4\, x \, \left(\frac{\delta \mathcal{F}}{\delta \rho^\mu} \frac{\delta \mathcal{F}}{\delta A_\mu} + \frac{\delta \mathcal{F}}{\delta \bar{R}^i} \frac{\delta \mathcal{F}}{\delta \psi_i} + \frac{\delta \mathcal{F}}{\delta R^i} \frac{\delta \mathcal{F}}{\delta \overline{\psi}_i} + B \frac{\delta \mathcal{F}}{\delta \bar{c}} \right) , \tag{11.2.8}$$

where again B is treated as an abbreviation to its value given in Eq. (11.2.5d). As usual in the context of algebraic renormalization, several additional functional identities hold. In particular all functional derivatives of $S_0^{(4d)}$ with respect to the fields c, $\bar{c}$ or ρ^μ are linear in the propagating fields, and one may write down identities of the form $\delta S_0^{(4d)}/\delta c(x) =$ (linear expression). Such identities may be required to hold at all orders as part of the definition of the theory.[1] We highlight first the so-called ghost equation

$$\left(\frac{\delta}{\delta \bar{c}} + \partial_\mu \frac{\delta}{\delta \rho_\mu} \right) S_0^{(4d)} = 0 \, , \tag{11.2.9}$$

which is a linear combination which has analogues also in the non-abelian case.[2] Second, starting from the Slavnov-Taylor Eq. (11.2.8) operator performed on the χQED 4-dimensional action (for simplicity we discard $4d$ action superscript till the end of this section)

$$\mathcal{S}(S_0) = \int d^4x \left(\frac{\delta S_0}{\delta \rho^\mu} \frac{\delta S_0}{\delta A_\mu} + \frac{\delta S_0}{\delta \bar{R}^i} \frac{\delta S_0}{\delta \psi_{R_i}} + \frac{\delta S_0}{\delta R^i} \frac{\delta S_0}{\delta \overline{\psi_{R_i}}} + B \frac{\delta S_0}{\delta \bar{c}} \right), \tag{11.2.10}$$

and then applying variation with the respect to the ghost field, where non-vanishing functional derivatives appearing in the Slavnov-Taylor operation are the following,

$$\begin{aligned}
\frac{\delta S_0}{\delta \rho^\mu} &= s A_\mu = \partial_\mu c \, , & \frac{\delta S_0}{\delta \bar{R}^i} &= s \psi_{R_i} = i e \mathcal{Y}_{Ri} c \psi_{R_i} \, , \\
\frac{\delta S_0}{\delta R^i} &= s \overline{\psi_{R_i}} = i e \mathcal{Y}_{Ri} \overline{\psi_{R_i}} c \, , & \frac{\delta S_0}{\delta \bar{c}} &= -\partial^2 c \, , \\
\frac{\delta S_0}{\delta \psi_{R_i}} &= -e \mathcal{Y}_{Ri} \overline{\psi_{R_i}} \not{A} - i e \bar{R}^i c \mathcal{Y}_{Ri} \, , & \frac{\delta S_0}{\delta \overline{\psi_{R_i}}} &= e \mathcal{Y}_{Ri} \not{A} \psi_{R_i} + i e c \mathcal{Y}_{Ri} R^i \, .
\end{aligned}$$

we get for the functional derivative of $\mathcal{S}(S_0)$ with respect to c this expression:

$$\begin{aligned}
\frac{\delta \mathcal{S}(S_0)}{\delta c_x} = \int d^4y \Bigg[& \left(\frac{\delta}{\delta c_x} \frac{\delta S_0}{\delta \rho^\mu_y} \right) \frac{\delta S_0}{\delta A^y_\mu} + \left(\frac{\delta}{\delta c_x} \frac{\delta S_0}{\delta \bar{R}^i_y} \right) \frac{\delta S_0}{\delta \psi^y_{R_i}} + \frac{\delta S_0}{\delta \bar{R}^i_y} \left(\frac{\delta}{\delta c_x} \frac{\delta S_0}{\delta \psi^y_{R_i}} \right) \\
& + \left(\frac{\delta}{\delta c_x} \frac{\delta S_0}{\delta R^i_y} \right) \frac{\delta S_0}{\delta \overline{\psi^y_{R_i}}} + \frac{\delta S_0}{\delta R^i_y} \left(\frac{\delta}{\delta c_x} \frac{\delta S_0}{\delta \overline{\psi^y_{R_i}}} \right) + B_y \left(\frac{\delta}{\delta c_x} \frac{\delta S_0}{\delta \bar{c}_y} \right) \Bigg],
\end{aligned} \tag{11.2.11}$$

[1] If B is not integrated out, the same is true for the functional derivative $\delta S_0^{(4d)}/\delta B(x)$.

[2] It can be obtained in general from evaluating $\delta \mathcal{S}(S_0^{(4d)})/\delta B$ if the field B is not eliminated.

where the coordinate dependence is now explicitly written in (as an index x or y). After applying the results for functional derivatives,

$$\begin{aligned}
\frac{\delta \mathcal{S}(S_0)}{\delta c_x} &= -\partial^x_\mu \frac{\delta S_0}{\delta A^x_\mu} + ie\mathcal{Y}_{Ri}\psi^x_{R_i}\frac{\delta S_0}{\delta \psi^x_{R_i}} + \frac{\delta S_0}{\delta \bar{R}^i_x}(-ie\bar{R}^i_x\mathcal{Y}_{Ri}) \\
&- ie\mathcal{Y}_{Ri}\overline{\psi^x_{R_i}}\frac{\delta S_0}{\delta \overline{\psi^x_{R_i}}} + \frac{\delta S_0}{\delta R^i_x}(ie\mathcal{Y}_{Ri}R^i_x) - \partial^2_x B_x \\
&= -\partial^x_\mu \frac{\delta S_0}{\delta A^x_\mu} - \partial^2_x B_x + ie\mathcal{Y}_{Ri}\left(\psi^x_{R_i}\frac{\delta}{\delta \psi^x_{R_i}} - \overline{\psi^x_{R_i}}\frac{\delta}{\delta \overline{\psi^x_{R_i}}} + R^i_x \frac{\delta}{\delta R^i_x} - \bar{R}^i_x \frac{\delta}{\delta \bar{R}^i_x}\right) S_0, \qquad (11.2.12)
\end{aligned}$$

and taking Eq. (11.2.7) into account, we obtain the functional form of the abelian Ward identity

$$\left(\partial^\mu \frac{\delta}{\delta A^\mu(x)} - ie\mathcal{Y}^j_R \sum_\Psi (\pm)\Psi(x)\frac{\delta}{\delta \Psi(x)}\right) S_0^{(4d)} = -\partial^2 B(x)\,. \qquad (11.2.13)$$

The summation extends over the charged fermions and their sources, $\Psi \in \{\psi_{Rj}, \overline{\psi_{Rj}}, R^j, \bar{R}^j\}$, and the signs are $+, -, +, -$, respectively. For extensive discussions of Ward identity of the more general case and the importance to the Standard Model and extensions see e.g. [36, 95, 96].
Finally, we summarize in Table 11.1 a list of the quantum numbers (mass dimension, ghost number and (anti)commutativity) of the fields and the external sources of the theory, that are necessary for building the whole set of all possible renormalizable mass-dimension ≤ 4 field-monomial operators with a given ghost number.

Table 11.1
List of fields, external sources and operators, and their quantum numbers for χQED model.

	A_μ	$\overline{\psi}_i$, ψ_i	c	$\bar{c}$	B	ρ^μ	R^i, $\bar{R}^i$	∂_μ	s
mass dimension	1	3/2	0	2	2	3	5/2	1	0
ghost number	0	0	1	-1	0	-1	-1	0	1
(anti)commutativity	+1	-1	-1	-1	+1	-1	+1	+1	-1

11.3 | The χQED in d dimensions and its BRST breaking

The extension of the χQED model Eq. (11.2.1) to d dimensions is not unique, due to the fermionic interaction terms. Again, we can use the experience from Yang-Mills model renormalization, since now it is of even greater importance to use the most symmetric interaction vertex due to the complexity of 2-loop calculations. We follow the procedure used in Ref. [47]. Again, the extension to d dimensions requires fully d-dimensional fermion propagators, so as to ensure that Feynman diagrams involving fermions can be regularized. This is achieved by introducing a left-chiral $U(1)$-singlet fermion into the kinetic part of the Lagrangian, thus promoting the intrisically 4-dimensional χQED fermionic kinetic term to a full d-dimensional one. On the other side the fermion-gauge boson interaction is chosen to be fully chiral-projected, with right-handed fermions only. This procedure, together with the straightforward extension of the other terms in Eq. (11.2.1) to d dimensions, leads to the tree-level action S_0,

$$\begin{aligned} S_0 = & \int \mathrm{d}^d\, x \left(i\overline{\psi}_i \partial\!\!\!/ \psi_i + e\mathcal{Y}_{Rij}\overline{\psi_{R}}_i A\!\!\!/ \psi_{Rj} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \right. \\ & \left. - \bar{c}\partial^2 c + \rho^\mu(\partial_\mu c) + i\, e\, \bar{R}^i c\, \mathcal{Y}_{Rij}\psi_{Rj} + i\, e\, \overline{\psi_{R}}_i c \mathcal{Y}_{Rij} R^j \right) \\ \equiv & \sum_i S^i_{\overline{\psi}\psi} + \sum_i \overline{S^i_{\overline{\psi}_R A\psi_R}} + S_{AA} + S_{\text{g-fix}} + S_{\bar{c}c} + S_{\rho c} + S_{\bar{R}c\psi_R} + S_{Rc\overline{\psi_R}} \,. \end{aligned} \tag{11.3.1}$$

In Eq. (11.3.1) a set of abbreviations for the S_0 terms is introduced. To define some of the expressions for these terms we use either the notation $A\overleftrightarrow{\partial}B \equiv A(\partial B) - (\partial A)B$ or the gauge-fixing Eq. (4.1.11) is used or the fact the BRST transformations Eq. (11.2.5) retain the same form in d dimensions up to the replacement of 4-dimensional objects by corresponding d-dimensional ones, where we introduced the following abbreviations for the action operators and we list them explicitly for the later use:

$$S_{AA} = \int \mathrm{d}^d\, x\ \frac{-1}{4}F_{\mu\nu}F^{\mu\nu} = \int \mathrm{d}^d\, x\ \frac{1}{2}A_\mu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu \,, \tag{11.3.2}$$

$$S_{\overline{\psi}\psi} = \int \mathrm{d}^d\, x\ i\overline{\psi}_i\partial\!\!\!/\psi_i \equiv \int \mathrm{d}^d\, x\ \frac{i}{2}\overline{\psi}_i \overleftrightarrow{\partial\!\!\!/}\psi_i \,, \tag{11.3.3}$$

$$\overline{S_{\overline{\psi_R}A\psi_R}} = \int \mathrm{d}^d\, x\ e\,\mathcal{Y}_{Rij}\overline{\psi}_i \mathbb{P}_{\mathrm{L}} A\!\!\!/\, \mathbb{P}_{\mathrm{R}}\psi_j = \int \mathrm{d}^d\, x\ e\,\mathcal{Y}_{Rij}\overline{\psi}_i \overline{A\!\!\!/}\, \mathbb{P}_{\mathrm{R}}\psi_j \,, \tag{11.3.4}$$

$$S_{\text{g-fix}} = \int \mathrm{d}^d\, x \left(\frac{\xi}{2}B^2 + B\partial^\mu A_\mu\right) = \int \mathrm{d}^d\, x\ \frac{-1}{2\xi}(\partial_\mu A^\mu)^2 \,, \tag{11.3.5}$$

$$S_{\bar{c}c} = \int \mathrm{d}^d\, x\ (\partial^\mu \bar{c})(\partial_\mu c) \equiv \int \mathrm{d}^d\, x\ (-\bar{c}\partial^2 c) \,, \tag{11.3.6}$$

$$S_{\rho c} = \int \mathrm{d}^d\, x\ \rho^\mu s_d A_\mu = \int \mathrm{d}^d\, x\ \rho^\mu(\partial_\mu c) \,, \tag{11.3.7}$$

$$S_{\bar{R}c\psi_R} = \int \mathrm{d}^d\, x\ \bar{R}^i s_d\psi_i = \int \mathrm{d}^d\, x\ i\, e\, \bar{R}^i c\mathcal{Y}_{Rij}\psi_{Rj} \equiv \int \mathrm{d}^d\, x\ i\, e\, \bar{R}^i c\mathcal{Y}_{Rij}\mathbb{P}_{\mathrm{R}}\psi_j \,, \tag{11.3.8}$$

$$\begin{aligned} S_{Rc\overline{\psi_R}} = & \int \mathrm{d}^d\, x\ R^i s_d\overline{\psi}_i \equiv \int \mathrm{d}^d\, x\ s_d\overline{\psi}_i R^i = \int \mathrm{d}^d\, x\ i\, e\, \overline{\psi_{R}}_j c\mathcal{Y}_{Rji}R^i \\ \equiv & \int \mathrm{d}^d\, x\ i\, e\, \overline{\psi}_j \mathbb{P}_{\mathrm{L}} c\mathcal{Y}_{Rji}R^i \,, \end{aligned} \tag{11.3.9}$$

Following the discussion of the [47] in the rest of this section we investigate the BRST symmetry (breaking) of the tree-level action S_0. We define the d-dimensional Slavnov-Taylor operation $\mathcal{S}_d$ by straightforward extension of its 4-dimensional version to d-dimensions, as an extension of the four-dimensional Slavnov-Taylor operator $\mathcal{S}$ from

Eq. (11.2.8) to d dimensions,

$$\mathcal{S}_d(S_0) = \int \mathrm{d}^d\, x\, \left(\frac{\delta S_0}{\delta \rho^\mu} \frac{\delta S_0}{\delta A_\mu} + \frac{\delta S_0}{\delta \bar{R}^i} \frac{\delta S_0}{\delta \psi_i} + \frac{\delta S_0}{\delta R^i} \frac{\delta S_0}{\delta \overline{\psi}_i} + B \frac{\delta S_0}{\delta \bar{c}} \right) , \tag{11.3.10}$$

where there is an analogous definition for any functional $\mathcal{F}$. The d-dimensional action may be written as the sum of two parts, an "invariant" and an "evanescent" part,

$$S_0 = S_{0,\mathrm{inv}} + S_{0,\mathrm{evan}}\, , \tag{11.3.11a}$$

$$S_{0,\mathrm{evan}} = \int \mathrm{d}^d\, x\; i \overline{\psi}_i \widehat{\partial\!\!\!/} \psi_i\, . \tag{11.3.11b}$$

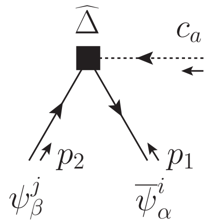
The second part $S_{0,\mathrm{evan}}$ consists solely of one single, evanescent fermion kinetic term, the reminiscent of d-dimensional propagator. Acting with the d-dimensional BRST operator on the tree-level action Eq. (11.3.1) gives:

$$s_d S_0 = s_d S_{0,\mathrm{inv}} + s_d S_{0,\mathrm{evan}} = 0 + s_d \int \mathrm{d}^d\, x\; i \overline{\psi}_i \widehat{\partial\!\!\!/} \psi_i \equiv \widehat{\Delta}, \tag{11.3.12}$$

where non-vanishing integrated breaking term $\widehat{\Delta}$ is given by

$$\widehat{\Delta} = \int \mathrm{d}^d\, x\; e\, \mathcal{Y}_{Rij}\, c \left\{ \overline{\psi}_i \left(\overleftarrow{\widehat{\partial\!\!\!/}} \mathbb{P}_\mathrm{R} + \overrightarrow{\widehat{\partial\!\!\!/}} \mathbb{P}_\mathrm{L} \right) \psi_j \right\} \equiv \int \mathrm{d}^d\, x\; \widehat{\Delta}(x). \tag{11.3.13}$$

This breaking generates an interaction vertex whose Feynman rule (with all momenta incoming) is:

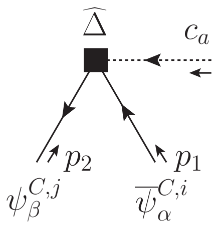


$$\begin{aligned} &= \frac{e}{2}\, \mathcal{Y}_{Rij} \left((\widehat{p\!\!\!/_1} + \widehat{p\!\!\!/_2}) + (\widehat{p\!\!\!/_1} - \widehat{p\!\!\!/_2}) \gamma_5 \right)_{\alpha\beta} \\ &= e\, \mathcal{Y}_{Rij} \left(\widehat{p\!\!\!/_1} \mathbb{P}_\mathrm{R} + \widehat{p\!\!\!/_2} \mathbb{P}_\mathrm{L} \right)_{\alpha\beta} . \end{aligned} \tag{11.3.14}$$

The Feynman rule corresponding to the charge-conjugated fermions (applying flipping rules as in [78, 79] and charge conjugation as in [47]) follows from the breaking term rewritten as:

$$\widehat{\Delta} = \int \mathrm{d}^d\, x\; e\, \mathcal{Y}_{Rij}\, c \left\{ \overline{\psi^C}_i \left(\overleftarrow{\widehat{\partial\!\!\!/}} \mathbb{P}_\mathrm{L} + \overrightarrow{\widehat{\partial\!\!\!/}} \mathbb{P}_\mathrm{R} \right) \psi^C_j \right\} , \tag{11.3.15}$$

and leads to the Feynman rule:



$$\begin{aligned} &= -\frac{e}{2}\, \mathcal{Y}_{Rij} \left((\widehat{p\!\!\!/_1} + \widehat{p\!\!\!/_2}) - (\widehat{p\!\!\!/_1} - \widehat{p\!\!\!/_2}) \gamma_5 \right)_{\alpha\beta} \\ &= -e\, \mathcal{Y}_{Rij} \left(\widehat{p\!\!\!/_1} \mathbb{P}_\mathrm{L} + \widehat{p\!\!\!/_2} \mathbb{P}_\mathrm{R} \right)_{\alpha\beta} . \end{aligned} \tag{11.3.16}$$

Acting with the d-dimensional Slavnov-Taylor operator $\mathcal{S}_d$ on the tree-level action, we obtain the BRST breaking

$$\mathcal{S}_d(S_0) = \mathcal{S}_d(S_{0,\text{inv}}) + \mathcal{S}_d(S_{0,\text{evan}}) = 0 + \widehat{\Delta}\,. \tag{11.3.17}$$

At this point it is again convenient to introduce the linearized Slavnov-Taylor operator b_d. In later applications we use the Slavnov-Taylor identity at higher orders in the form $\mathcal{S}(S_0 + \mathcal{F})$, where the functional $\mathcal{F}$ is e.g. the 1-loop regularized or renormalized effective action or the 1-loop counterterm action. We can then expand to first order in $\mathcal{F}$ according to (4.3.2),

$$\mathcal{S}_d(S_0 + \mathcal{F}) = \mathcal{S}_d(S_0) + b_d\mathcal{F} + \mathcal{O}(\mathcal{F}^2), \tag{11.3.18}$$

where b_d can be written in functional form as

$$\begin{aligned} b_d &= \int \mathrm{d}^d x \, \left(\frac{\delta S_0}{\delta \rho^\mu}\frac{\delta}{\delta A_\mu} + \frac{\delta S_0}{\delta A_\mu}\frac{\delta}{\delta \rho^\mu} + \left(\frac{\delta S_0}{\delta \bar{R}^i}\frac{\delta}{\delta \psi_i} + \frac{\delta S_0}{\delta \psi_i}\frac{\delta}{\delta \bar{R}^i} \right) + \left(\frac{\delta S_0}{\delta R^i}\frac{\delta}{\delta \overline{\psi}_i} + \frac{\delta S_0}{\delta \overline{\psi}_i}\frac{\delta}{\delta R^i} \right) + B\frac{\delta}{\delta \bar{c}} \right) \\ &= s_d + \int \mathrm{d}^d x \, \left(\frac{\delta S_0}{\delta A_\mu}\frac{\delta}{\delta \rho^\mu} + \frac{\delta S_0}{\delta \psi_i}\frac{\delta}{\delta \bar{R}^i} + \frac{\delta S_0}{\delta \overline{\psi}_i}\frac{\delta}{\delta R^i} \right)\,, \end{aligned} \tag{11.3.19}$$

i.e. it is extended BRST transformation operator that acts also on the source fields.

11.4 | Defining symmetry requirements for the renormalized theory

Symmetry identities valid at the tree level must be fulfilled at all orders of perturbation theory. If the symmetries are broken in the regularization and renormalization procedure, as is the case when we use the BMHV scheme, they must be restored order by order in perturbation theory. Possibility of this restoration at higher orders comes from the structure and the properties of chiral QED. Arguments are coming from the general analysis of algebraic renormalization of gauge theories[3] and the anomaly condition Eq. (11.2.4).

We will briefly collect the relevant symmetry identities for chiral QED. These identities may be viewed as part of the definition of the model; they constrain the regularization/renormalization procedure and particularly determine the symmetry-restoring counterterms.

The symmetry requirements are defined for renormalized and finite 4-dimensional effective action of form

$$\Gamma_{\text{Ren}} = S_0^{(4d)} + \mathcal{O}(\hbar). \tag{11.4.1}$$

[3] See Refs. [1, 97–99] for important treatments of abelian theories in such contexts and Refs. [1, 25] for general overviews.

The first and most important symmetry requirement is BRST invariance, which is expressed as the Slavnov-Taylor identity

$$\boxed{\mathcal{S}(\Gamma_{\mathrm{Ren}}) = 0} \tag{11.4.2}$$

for the renormalized theory. Notice that the fields c, $\bar{c}$ and ρ^μ do not have higher order corrections, so relations

$$\frac{\delta\Gamma_{\mathrm{Ren}}}{\delta c(x)} = \frac{\delta S_0^{(4d)}}{\delta c(x)}\,, \qquad \frac{\delta\Gamma_{\mathrm{Ren}}}{\delta \bar{c}(x)} = \frac{\delta S_0^{(4d)}}{\delta \bar{c}(x)}\,, \qquad \frac{\delta\Gamma_{\mathrm{Ren}}}{\delta \rho^\mu(x)} = \frac{\delta S_0^{(4d)}}{\delta \rho^\mu(x)}\,, \tag{11.4.3}$$

hold trivially, since the respective derivatives of the tree-level action are linear in the dynamical fields as described between Eqs. (11.2.8) and (11.2.13). Fact that the ghost doesn't have higher loop corrections will of course also play the part in reducing the number of diagrams appearing in higher orders.

The Ward identity

$$\left(\partial^\mu \frac{\delta}{\delta A^\mu(x)} - ie\mathcal{Y}_R^j \sum_\Psi (\pm)\Psi(x)\frac{\delta}{\delta\Psi(x)}\right)\Gamma_{\mathrm{Ren}} = -\partial^2 B(x)\,, \tag{11.4.4}$$

is an automatic consequence of the Slavnov-Taylor identity Eq. (11.4.2) combined with the antighost equation in Eq. (11.4.3). It is not manifestly valid at higher orders but it will be automatically valid once the Slavnov-Taylor identity holds, i.e. when symmetry is restored by proper counterterms. In fact we will see that the breaking and restoration of the Slavnov-Taylor identity can be well interpreted in terms of the Ward identity.

Eq. (11.4.4) supplies us with three well-known QED Ward identities for renormalized Green functions (in momentum space):

1. The transversality of the photon self energy,

$$ip_\nu \frac{\delta^2\widetilde{\Gamma}_{\mathrm{Ren}}}{\delta A_\mu(p)\delta A_\nu(-p)} = 0\,; \tag{11.4.5}$$

2. The transversality of multi-photon vertices, and in particular the photon 4-point amplitude,

$$i(p_{1+2+3})_\sigma \frac{\delta^4\widetilde{\Gamma}_{\mathrm{Ren}}}{\delta A_\rho(p_3)\delta A_\nu(p_2)\delta A_\mu(p_1)\delta A_\sigma(-p_{1+2+3})} = 0 \tag{11.4.6}$$

(denoting $p_{1+2+3} \equiv p_1 + p_2 + p_3$);

3. The relation between fermion self energy and fermion-photon interaction for vanishing photon momentum $q = 0$,

$$-ie\mathcal{Y}_R \frac{\partial}{\partial p_\mu}\frac{\delta^2\widetilde{\Gamma}_{\mathrm{Ren}}}{\delta\overline{\psi}(-p)\delta\psi(p)} + i\,\frac{\delta^3\widetilde{\Gamma}_{\mathrm{Ren}}}{\delta A_\mu(0)\delta\overline{\psi}(-p)\delta\psi(p)} = 0\,. \tag{11.4.7}$$

The $\widetilde{\Gamma}_{\mathrm{Ren}}$ as usual denotes the renormalized effective action in momentum representation. Those equations will provide important check for finite counterterms. In what follows we will only refer to BRST invariance and the Slavnov-Taylor identity, which are the most important symmetry requirements. The requirements Eq. (11.4.3) are manifestly valid at all steps and individually for the regularized Green functions and for the counterterms.

CHAPTER 12

MULTILOOP REGULARIZATION AND RENORMALIZATION FORMULAE

In the first part of this thesis, we were dealing with 1-loop breaking and restoration of BRST symmetry for the chiral Yang-Mills model. For that purpose, we studied the renormalization in the BMHV scheme up to 1-loop order, but now we will expand our study to a general, multiloop case and then restrict it to a 2-loop level.

12.1 | General action and effective action structure

Recall that the BMHV scheme introduces several new types of counterterms, listed in Eq. (5.2.3)

$$S_{\text{ct}} = S_{\text{sct,inv}} + S_{\text{sct,noninv}} + S_{\text{fct,inv}} + S_{\text{fct,restore}} + S_{\text{fct,evan}} \equiv S_{\text{sct}} + S_{\text{fct}} \,, \tag{12.1.1}$$

that can be splitted into singular (divergent) and finite counterterms. We recall that in our notation symbols without indices denote all-order quantities. For the following perturbative expressions, we will also use an upper index i for quantities of precisely order i, and upper index (i) for quantities up to and including order i. For example, the bare action and the counterterm action split as

$$S_{\text{bare}} = S_0 + S_{\text{ct}}, \qquad S_{\text{ct}} = \sum_{i=1}^{\infty} S_{\text{ct}}^{i}, \qquad S_{\text{ct}}^{(i)} = \sum_{j=1}^{i} S_{\text{ct}}^{j} \,. \tag{12.1.2}$$

The effective action in dimensional regularization and renormalization is constructed iteratively at each order of $\hbar$, starting from the tree-level action S_0 of order $\hbar^0$. At each higher loop order $i \geq 1$ a counterterm action S_{ct}^{i} has to be constructed from the calculation of counterterms. The counterterms are subject to the two conditions that the renormalized theory is UV finite and in agreement with all required symmetries listed in Section 11.4.

In general, at each order i one may distinguish Green functions at various levels of regularization, partial or full renormalization. Of particular importance are "subloop-renormalized" Green functions and the corresponding effective action. For simplicity we use the symbol Γ^i for this subloop-renormalized effective action of order i. By definition this is obtained at order i by using Feynman rules from the tree-level action up to and including order $i-1$ counterterms. As usual, by constructing and including singular counterterms of the order i we obtain the finite quantity

$$\Gamma^i + S^i_{\text{sct}} = \text{finite for } \epsilon \to 0\,. \tag{12.1.3}$$

This equation fixes the singular counterterms unambiguously, including their evanescent parts, introduced in Eq. (12.1.1). By also including additional, finite counterterms of the order i we obtain

$$\Gamma^i_{\text{DReg}} := \Gamma^i + S^i_{\text{sct}} + S^i_{\text{fct}}\,. \tag{12.1.4}$$

This resulting effective action is finite at this order and essentially renormalized but still contains the variable ϵ and evanescent quantities, so for final procedure includes taking the limit $d \to 4$ and by setting all evanescent quantities to zero. This operation is denoted as

$$\Gamma^i_{\text{Ren}} := \text{LIM}_{d\to 4}\Gamma^i_{\text{DReg}}\,. \tag{12.1.5}$$

It is important to note that this limit is performed at the respective loop order at the very end when all quantities are calculated and all counterterms are known. E.g. if we are dealing with 2-loop calculation, quantities from 1-loop order must be inserted in the calculation in their original form.

12.2 | All-orders symmetry requirements

Once we broke gauge and BRST invariance using the BMHV scheme, our main task is to determine the finite counterterms needed to restore the symmetries. The ultimate symmetry requirement is the Slavnov-Taylor identity expressing BRST invariance for the fully renormalized theory, which can be written as

$$\mathcal{S}(\Gamma_{\text{Ren}}) = \text{LIM}_{d\to 4}\Big(\mathcal{S}_d(\Gamma_{\text{DReg}})\Big) = 0\,. \tag{12.2.1}$$

As we discussed in detail in Ref. [47] there are several possibilities to extract the symmetry-restoring counterterms from this equation. The first option is to calculate the complete finite part of the action at the loop order of interest and then find which part breaks BRST invariance and then impose counterterms to remedy this breaking. This straightforward or brute-force way complicates significantly already complicated calculations and is not pragmatic from the practitioner's point of view. Fortunately, the regularized quantum action principle [20] gives the possibility to extract only the symmetry breaking parts of finite action and we can understand the breaking parts as local insertions into Γ_{Dreg}[1]

$$
\begin{aligned}
\mathcal{S}_d(\Gamma_{\text{DReg}}) &= (\widehat{\Delta} + \Delta_{\text{ct}}) \cdot \Gamma_{\text{DReg}} \\
&= \widehat{\Delta} + \sum_{i=1}^{\infty} \hbar^i \left(\widehat{\Delta} \cdot \Gamma^i_{\text{DReg}} + \sum_{k=1}^{i-1} \Delta^k_{\text{ct}} \cdot \Gamma^{(i-k)}_{\text{DReg}} + \Delta^i_{\text{ct}} \right),
\end{aligned}
\tag{12.2.2}
$$

where the insertions or breakings $\widehat{\Delta}$ and Δ_{ct} in the present abelian theory are given as

$$\widehat{\Delta} = \mathcal{S}_d(S_0) = s_d S_0 \,, \tag{12.2.3a}$$

$$\widehat{\Delta} + \Delta_{\text{ct}} = \mathcal{S}_d(S_0 + S_{\text{ct}}) \,, \tag{12.2.3b}$$

$$\Delta_{\text{ct}} \equiv s_d S_{\text{ct}} \,. \tag{12.2.3c}$$

The first two equations are completely general, the third one is valid in the present context because our external fields do not have loop corrections, so there will be no counterterms involving external fields, hence b_d operator reduces to s_d. If the regularized quantum action principle is inserted into Eq. (12.2.1) and perturbatively expanded at the order i, we get crucial equation for symmetry restoration at the i-th loop order

$$\boxed{\text{LIM}_{d\to 4} \left(\widehat{\Delta} \cdot \Gamma^i_{\text{DReg}} + \sum_{k=1}^{i-1} \Delta^k_{\text{ct}} \cdot \Gamma^{i-k}_{\text{DReg}} + \Delta^i_{\text{ct}} \right) = 0} \tag{12.2.4}$$

valid for $i \geq 1$, which explicitly exhibits the genuine i-loop counterterm via Δ^i_{ct} (see Eq. (12.2.3)). The fact that the limit $d \to 4$ exists provides a consistency check on the divergent part of Δ^i_{ct}, which contains the singular counterterms S^i_{sct}. The finite part of the equation determines the finite part of Δ^i_{ct}. This equation extracts the finite counterterms of our interest – the ones that break BRST symmetry. It is possible to add any finite BRST invariant counterterms to this set without spoiling the symmetry to e.g. adjust renormalization conditions, but they are not of our interest (as long as they do not induce non-spurious anomalies in the theory). Now when we know all-order Eq. (12.2.4), we can apply it to the 2-loop order to restore the symmetries broken in BMHV treated χQED.

[1]The same equation has been presented specifically for the 1-loop case in Ref. [47] and for the general case in Ref. [54]. Ref. [27] presents a slightly different version where external source fields are taken into account. All versions of the equation become equal in the present context of an abelian gauge theory where there are no counterterms involving external fields.

CHAPTER 13

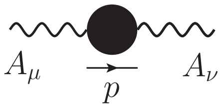
EVALUATION OF THE ONE-LOOP SINGULAR COUNTERTERM ACTION IN χQED VERSUS QED

Our regularization procedure of χQED starts with the evaluation of the 1-loop (order $\hbar^1$) singular counterterm action S^1_{sct} defined from the divergent parts of the 1-loop diagrams. Counterterms, once determined, will become part of 1-loop singular conterterm action S_{ct} which, canceling the divergent terms in the 1-loop effective action $\Gamma^{(1)}$, lead to the 1-loop χQED regularized action $\Gamma^{(1)}_{\text{DReg}}$. We will also evaluate the same diagrams in standard quantum electrodynamics (QED) where it will be interesting to see where are the differences coming from and why. This, we hope, can serve a pedagogical purpose since the QED is the common textbook example used in loop calculations. The calculations are again performed in $d = 4 - 2\epsilon$ dimensions, and calculation procedure is similar to one explained in Chapter 6.

13.1 | List of the one-loop divergent terms

We present in this section the results of the divergent parts of the self-energies and vertex of the theory, evaluated at 1-loop order, both for the χ-QED and QED. In the following calculations all momenta are taken as incoming. The blobs shown in the diagrams represent the sum of the all possible 1-loop corrections within our theory.
First we list all self-energy results for 2-point Green functions.

Gauge boson:



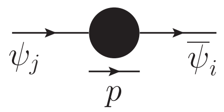
$$
\begin{aligned}
i\widetilde{\Gamma}^{\nu\mu}_{AA}(p)|^{1}_{\text{div, }\chi\text{QED}} &= \frac{ie^2}{16\pi^2\epsilon}\frac{2\,\mathrm{Tr}(\mathcal{Y}_R^2)}{3}(\overline{p}^{\mu}\overline{p}^{\nu} - \overline{p}^2\overline{g}^{\mu\nu}) - \frac{ie^2}{16\pi^2\epsilon}\frac{\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\widehat{p}^2\overline{g}^{\mu\nu}\,, \\
i\widetilde{\Gamma}^{\nu\mu}_{AA}(p)|^{1}_{\text{div, QED}} &= \frac{ie^2}{16\pi^2\epsilon}\frac{4\,\mathrm{Tr}(\mathcal{Y}^2)}{3}(p^{\mu}p^{\nu} - p^2 g^{\mu\nu})\,.
\end{aligned}
\tag{13.1.1}
$$

Notice that, since the interaction vertex in the χQED differs from the one given in the standard QED by

$$
V_{\text{QED}} = ie\gamma^{\mu}\mathcal{Y}_{ij}, \quad V_{\chi\text{QED}} = ie\bar{\gamma}^{\mu}\mathbb{P}_{\text{R}}\mathcal{Y}_{R,ij},
$$

it will project the fermion loop content, so the transversal part becomes 4-dimensional. Due to the half of the fermion degrees of freedom in the loop for the chiral case, there is a relative factor of 2 with respect to the QED. In addition, gauge boson self-energy breaks gauge invariance with the second, evanescent term.

Fermion field:



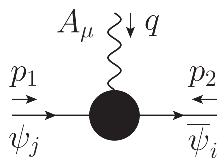
$$
\begin{aligned}
i\widetilde{\Gamma}^{ji}_{\psi\bar{\psi}}(p)|^{1}_{\text{div, }\chi\text{QED}} &= \frac{ie^2\xi}{16\pi^2\epsilon}(\mathcal{Y}_R^2)^{ij}\ \overline{\not{p}}\ \mathbb{P}_{\text{R}}\,, \\
i\widetilde{\Gamma}^{ji}_{\psi\bar{\psi}}(p)|^{1}_{\text{div, QED}} &= \frac{ie^2\xi}{16\pi^2\epsilon}(\mathcal{Y}^2)^{ij}\ \ \not{p}\,,
\end{aligned}
\tag{13.1.2}
$$

and for the charge-conjugated fermion field:

$$
\begin{aligned}
i\widetilde{\Gamma}^{ji}_{\psi^C\bar{\psi}^C}(p)|^{1}_{\text{div, }\chi\text{QED}} &= \frac{ie^2\xi}{16\pi^2\epsilon}(\mathcal{Y}_R^2)^{ij}\ \overline{\not{p}}\ \mathbb{P}_{\text{L}}\,, \\
i\widetilde{\Gamma}^{ji}_{\psi^C\bar{\psi}^C}(p)|^{1}_{\text{div, QED}} &= \frac{ie^2\xi}{16\pi^2\epsilon}(\mathcal{Y}^2)^{ij}\ \ \not{p}\ \ .
\end{aligned}
\tag{13.1.3}
$$

Notice that again projectors in vertices make the momentum 4-dimensional.
Since the theory does not comprise a ghost-gauge boson interaction, ghost field can't obtain loop corrections, and therefore there are no self-energy contributions at any loop order. That confirms that symmetry requirements in Eq. (11.4.3) hold.
The only vertex in the χQED that has singular part at the 1-loop level is fermion-gauge boson interaction.

Fermion-gauge boson interaction:



$$
\begin{aligned}
i\widetilde{\Gamma}^{ji,\mu}_{\psi\bar{\psi}A}(p_1,p_2)\big|^1_{\text{div, }\chi\text{QED}} &= \frac{ie^3\xi}{16\pi^2\epsilon}\,(\mathcal{Y}_R^3)^{ij}\;\overline{\gamma}^\mu\,\mathbb{P}_\mathrm{R}\,,\\
i\widetilde{\Gamma}^{ji,\mu}_{\psi\bar{\psi}A}(p_1,p_2)\big|^1_{\text{div, QED}} &= \frac{ie^3\xi}{16\pi^2\epsilon}\,(\mathcal{Y}^3)^{ij}\;\gamma^\mu\;.
\end{aligned}
\tag{13.1.4}
$$

Vertices with external BRST source fields ρ, R and $\bar{R}$ can't obtain loop corrections at any loop order, since there is no ghost-gauge boson interaction in the theory. Interactions of three and four gauge bosons do not have singular part at the 1-loop level, as it is expected for loop-induced interactions. In all above expressions one has to take into account that $\mathcal{Y}_R$ and $\mathcal{Y}$ are diagonal matrices.

13.2 | The one-loop singular counterterm action

The singular 1-loop counterterm action is defined as usual such as to cancel the divergent parts of the 1-loop vertices:

$$
S^1_{\text{sct}} = -\Gamma\big|^1_{\text{div}}\,. \tag{13.2.1}
$$

After restoring the powers of $\hbar$, it reads at the 1-loop level

$$
\begin{aligned}
S^1_{\text{sct},\chi\text{QED}} = \frac{-\hbar\,e^2}{16\pi^2\epsilon}\Bigg(&\frac{2\text{Tr}(\mathcal{Y}_R^2)}{3}\overline{S_{AA}} + \xi\sum_j(\mathcal{Y}_R^j)^2\left(\overline{S^j_{\overline{\psi}\psi_R}} + \overline{S^j_{\overline{\psi_R}A\psi_R}}\right)\\
&+\frac{\text{Tr}(\mathcal{Y}_R^2)}{3}\int \mathrm{d}^d\,x\;\frac{1}{2}\bar{A}_\mu\widehat{\partial}^2\bar{A}^\mu\Bigg)\,,
\end{aligned}
\tag{13.2.2}
$$

and it may be compared to the corresponding result of ordinary QED with Dirac fermions of (hyper)charges $\mathcal{Y}$,

$$
S^1_{\text{sct,QED}} = \frac{-\hbar\,e^2}{16\pi^2\epsilon}\left(\frac{4\text{Tr}(\mathcal{Y}^2)}{3}S_{AA} + \xi\sum_j(\mathcal{Y}^j)^2\left(S^j_{\overline{\psi}\psi} + S^j_{\overline{\psi}A\psi}\right)\right)\,. \tag{13.2.3}
$$

Notice the difference in 4-dimensional and d-dimensional operators respectively and evanescent part in the chiral version of QED. The monomials not yet introduced are terms $\overline{S^i_{\overline{\psi}\psi_R}}$, $\overline{S^i_{\overline{\psi_R}A\psi_R}}$, the fully right-chiral-projected equivalents to their usual d-dimensional versions,

$$
\overline{S^i_{\overline{\psi}\psi_R}} = \int \mathrm{d}^d\,x\;i\overline{\psi}_i\overline{\partial\!\!\!/}\mathbb{P}_\mathrm{R}\psi_i \equiv \int \mathrm{d}^d\,x\;\frac{i}{2}\overline{\psi}_i\overset{\leftrightarrow}{\overline{\partial\!\!\!/}}\mathbb{P}_\mathrm{R}\psi_i\,, \tag{13.2.4a}
$$

$$
\overline{S^i_{\overline{\psi_R}A\psi_R}} = \int \mathrm{d}^d\,x\;e\mathcal{Y}_R^i\overline{\psi}_i\overline{A\!\!\!/}\mathbb{P}_\mathrm{R}\psi_i\,, \tag{13.2.4b}
$$

and the bar in $\overline{S_{AA}}$ designates the fully 4-dimensional version of S_{AA}.

The results Eqs. (13.2.2) and (13.2.3) differ in three characteristic ways respectively:

1. χQED has half as many fermionic degrees of freedom, hence the fermion loop contributions to the photon self energy generate the prefactor 2/3 instead of 4/3.

2. The fermion self-energy and the fermion-photon interaction receive only purely 4-dimensional right-handed corrections in χQED, while in (non-chiral) QED these contributions remain d-dimensional.

3. The purely right-handed nature of the boson-fermion interaction leads to a purely evanescent divergent non-transverse contribution to the photon self energy in χQED.

Also, since Eq. (11.4.3) holds, this implies in particular that the linearized Slavnov-Taylor operator b_d reduces to the BRST operator s_d when acting on the loop contributions of the effective action, justifying Eq. (12.2.3c).

13.3 | The BRST breaking of one-loop singular counterterm action

As in Ref. [47] we can re-express the result for the singular 1-loop counterterms S^1_{sct} in a fashion of the usual renormalization transformations, where fields renormalize multiplicatively as

$$\phi \to \sqrt{Z_\phi}\,\phi, \quad Z_\phi \equiv 1 + \delta Z_\phi,$$

and the coupling constant renormalizes additively as

$$e \to e + \delta e.$$

We will split the sum of singular counterterms into invariant and evanescent (BRST breaking) part,

$$S^1_{\text{sct}} = S^1_{\text{sct,inv}} + S^1_{\text{sct,evan}}\,, \tag{13.3.1}$$

where the first term arises in the usual way from a renormalization transformation used in non-chiral theories, while the second term has a different structure. The first term can be obtained by applying the renormalization transformation

$$S_{0,\text{inv}} \longrightarrow S_{0,\text{inv}} + S_{\text{ct,inv}},$$

and it is given by

$$S^1_{\text{ct,inv}} = \frac{\delta Z^1_A}{2} L_A + \frac{\delta Z^1_c}{2} L_c + \frac{\delta Z^1_{\psi_{Rj}}}{2} \overline{L_{\psi_{Rj}}} + \frac{\delta e^1}{e} L_e\,, \tag{13.3.2}$$

with the L_φ functionals corresponding to field renormalizations that can be written as a field-numbering operators acting on the tree-level action as we did for Yang-Mills model in Chapter 5. The 1-loop renormalization constants δZ_φ, δe agree with the usual ones for QED up to the different fermion content (see e.g. [67–69]) and read

$$\delta Z_A^1 = \delta Z_c^1 = -2\frac{\delta e^1}{e}\,, \tag{13.3.3a}$$

$$\delta Z_A^1 = \frac{-\hbar\, e^2}{16\pi^2\epsilon}\frac{2\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\,, \tag{13.3.3b}$$

$$\delta Z_{\psi_{R\,j}}^1 = \frac{-\hbar\, e^2}{16\pi^2\epsilon}\xi(\mathcal{Y}_R^j)^2\,. \tag{13.3.3c}$$

The first of these relations again reflects Eq. (11.4.3) as in ordinary QED, what is transparent if we write the invariant counterterms as

$$\begin{aligned} S_{\text{sct,inv}}^1 = \delta Z_A^1 S_{AA} &+ \left(\frac{\delta Z_c^1}{2} - \frac{\delta Z_A^1}{2}\right)(S_{\bar{c}c} + S_{\rho c}) + \left(\frac{\delta Z_c^1}{2} + \frac{\delta e^1}{e}\right)(S_{\overline{R}c\psi_R} + S_{Rc\overline{\psi_R}}) \\ &+ \delta Z_{\psi_R}^1\left(\overline{S_{\overline{\psi}\psi_R}} + \overline{S_{\overline{\psi}A\psi_R}}\right) + \left(\frac{\delta Z_A^1}{2} + \frac{\delta e^1}{e}\right)\overline{S_{\overline{\psi}A\psi_R}}\,, \end{aligned} \tag{13.3.4}$$

where it is interesting to see that fermion-self energy counterterm and fermion-gauge boson vertex renormalize with the same renormalization constant (since the last term in Eq. (13.3.4) vanishes), what agrees with the preservation of the Ward identity at 1-loop level. The L_φ functionals as field-numbering operators acting on the tree-level action in terms of the monomials of Eq. (11.3.1) (or total b_d-variations if possible) are given by

$$L_A = b_d \int \mathrm{d}^d\, x\; \widetilde{\rho}^\mu A_\mu = 2S_{AA} + \overline{S_{\overline{\psi}A\psi_R}} - S_{\bar{c}c} - S_{\rho c}\,, \tag{13.3.5a}$$

where $\widetilde{\rho}^\mu = \rho^\mu + \partial^\mu \bar{c}$ is the natural combination arising from the ghost equation (11.2.9);

$$L_c = \int \mathrm{d}^d\, x\; c(x)\frac{\delta S_0}{\delta c(x)} = S_{\bar{c}c} + S_{\rho c} + S_{\bar{R}c\psi_R} + S_{Rc\overline{\psi_R}}\,, \tag{13.3.5b}$$

$$\begin{aligned} L_{\psi_R} &= -b_d \int \mathrm{d}^d\, x\; (\bar{R}^i \mathbb{P}_\mathrm{R}\psi_i + \overline{\psi}_i \mathbb{P}_\mathrm{L} R^i) \\ &= 2\left(\int \mathrm{d}^d\, x\; i\overline{\psi}_i \overline{\partial\!\!\!/} \mathbb{P}_\mathrm{R}\psi_i + \overline{S_{\overline{\psi}A\psi_R}}\right) + \int \mathrm{d}^d\, x\; i\overline{\psi}_i \widehat{\partial\!\!\!/}\psi_i \equiv \overline{L_{\psi_R}} + S_{0,\text{evan}} = \sum_i L_{\psi_{Ri}}\,. \end{aligned} \tag{13.3.5c}$$

The L_e functional corresponding to renormalization of the physical coupling can be expressed in terms of the monomials of Eq. (11.3.1) or related to the field renormalization functionals as

$$L_e = e\frac{\partial S_0}{\partial e} = \overline{S_{\overline{\psi}A\psi_R}} + S_{\bar{R}c\psi_R} + S_{Rc\overline{\psi_R}} = L_c + L_A - 2S_{AA}\,. \tag{13.3.6}$$

It is interesting to note that despite the non-nilpotency of b_d, several of the L_φ are actually b_d-invariant:

$$b_d L_A = 0\,, \qquad\qquad b_d \overline{L_{\psi_R}} = 0\,. \tag{13.3.7}$$

while L_c is not b_d-invariant in this sense, since

$$b_d L_c = \widehat{\Delta}\,, \tag{13.3.8}$$

with the same breaking as in Eq. (11.3.13). Since L_c is contained in the L_e, also L_e corresponding to gauge coupling renormalization, is not b_d-invariant. Again we have that in the limit $d \to 4$ and evanescent terms vanishing, all the L_φ functionals presented here become invariant under the linear b transformation in 4 dimensions. Finally, the evanescent counterterms appearing in Eq. (13.3.1) can be written in the form

$$S^1_{\text{sct,evan}} = \frac{-\hbar\, e^2}{16\pi^2\epsilon}\frac{\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\left(2(\overline{S_{AA}} - S_{AA}) + \int \mathrm{d}^d\, x\ \frac{1}{2}\bar{A}^\mu \widehat{\partial}^2 \bar{A}_\mu\right)\,. \tag{13.3.9}$$

The BRST breaking of the singular 1-loop counterterms originates solely from the evanescent non-invariant second term of $S^1_{\text{sct,evan}}$ and is given by

$$\Delta^1_{\text{ct}}\Big|_{\text{div}} = s_d S^1_{\text{sct}} = -\frac{\hbar}{16\pi^2\epsilon}\frac{e^2\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\int \mathrm{d}^d\, x\ (\bar{\partial}_\mu c)\,(\widehat{\partial}^2 \bar{A}^\mu)\,. \tag{13.3.10}$$

With the help of the regularized quantum action principle, we have to prove that this singular breaking corresponds to some local insertion. This is the case as we will see soon.

CHAPTER 14

BRST SYMMETRY BREAKING AND ITS RESTORATION AT THE ONE-LOOP LEVEL FOR χQED

In the previous chapter, we determined the singular counterterms action S^1_{sct}, Eq. (13.2.2), where we found out there is the BRST symmetry breaking term already at the singular level. To restore the BRST symmetry, we must first prove that the singular part of Eq. (12.2.4) vanishes, and we need to determine a set of finite symmetry-restoring counterterms S^1_{fct} at the 1-loop level. Here we will apply the general procedure outlined in Chapter 12 to the 1-loop case.

14.1 | Renormalization conditions

At the 1-loop level the regularized action and breaking are given by

$$\Gamma^{(1)}_{\text{DReg}} = \Gamma^{(1)} + S^1_{\text{sct}} + S^1_{\text{fct}}\,, \tag{14.1.1a}$$
$$\Delta^1_{\text{ct}} = \mathcal{S}_d(S_0 + S_{\text{ct}})^1\,, \tag{14.1.1b}$$

where we apply discussion given in Chapter 12 what at the 1-loop level leads us to the following renormalization conditions:

$$S^1_{\text{sct}} + \Gamma^1_{\text{div}} = 0\,, \tag{14.1.2a}$$
$$\left(\widehat{\Delta}\cdot\Gamma^1 + \Delta^1_{\text{ct}}\right)_{\text{div}} = 0\,, \tag{14.1.2b}$$
$$\text{LIM}_{d\to4}\left(\widehat{\Delta}\cdot\Gamma^1 + \Delta^1_{\text{ct}}\right)_{\text{fin}} = 0\,. \tag{14.1.2c}$$

Here the subscripts 'div,fin' refer to the pure $1/\epsilon$ pole part and the ϵ-independent finite part, respectively[1]. Equation (14.1.2a) is regularizing condition at the 1-loop level and

[1]Compared to Eq. (12.2.4) we dropped the index 'DReg' because the 1-loop insertions arise from genuine 1-loop diagrams and not from 1-loop counterterms.

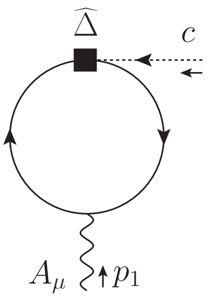
has already been satisfied in the previous section. Eq. (14.1.2b) should automatically hold by construction, providing a consistency check and is a demonstration of the regularized quantum action principle. The last equation determines the finite symmetry-restoring counterterms, with a remaining ambiguity of adding finite symmetric or evanescent counterterms without spoiling restored symmetries. If we write the last equation as

$$N\left[\widehat{\Delta}\cdot\Gamma^1\right] + \Delta^1_{\text{fct}} = 0\,, \tag{14.1.3}$$

this form implicitly fixes the choice of the finite, evanescent counterterms. This version of the equation uses the result (13.3.10) that the BRST variation of the 1-loop singular counterterms contains no finite term (which could in principle arise from the evaluation of s_d), hence $\Delta_{\text{ct}}|_{\text{fin}} = \Delta_{\text{fct}}$. The symbol $N[\mathcal{O}]$ denotes the Zimmermann-like definition of a renormalized local operator introduced in the Chapter 3.

14.2 | The divergent one-loop breaking

For evaluating $[\widehat{\Delta}\cdot\Gamma^{(1)}]_{\text{div}}$ we calculate the 1-loop vertex correction with insertion of the $\widehat{\Delta}$ evanescent operator. All momenta are incoming and $d = 4-2\epsilon$. For χQED we have only one non-vanishing diagram, $\widehat{\Delta}\, c\, A^\mu$:



$$i[\widehat{\Delta}\cdot\widetilde{\Gamma}^{\mu}_{Ac}]^{(1)}_{\text{div}} = \frac{\hbar\, e^2}{16\pi^2\epsilon}\,\frac{\text{Tr}(\mathcal{Y}_R^2)}{3}\,\widehat{p_1}^{\,2}\,\overline{p_1}^{\mu}\,, \tag{14.2.1}$$

producing the only non-vanishing contribution to $[\widehat{\Delta}\cdot\Gamma]^{(1)}_{\text{div}}$:

$$[\widehat{\Delta}\cdot\Gamma]^{(1)}_{\text{div}} = \frac{\hbar\, e^2}{16\pi^2\epsilon}\,\frac{\text{Tr}(\mathcal{Y}_R^2)}{3}\int \mathrm{d}^d x\; \bar{\partial}_\mu c\,\widehat{\partial}^2\bar{A}^\mu, \tag{14.2.2}$$

so we have, as expected,

$$[\widehat{\Delta}\cdot\Gamma^{(1)}]_{\text{div}} + s_d S^{(1)}_{\text{sct}} = 0. \tag{14.2.3}$$

So, evanescent breaking found in Eq. (13.3.10) is completely cancelled by the insertion diagram $\widehat{\Delta}\, c\, A^\mu$, what satisfied second renormalization condition Eq. (14.1.2).

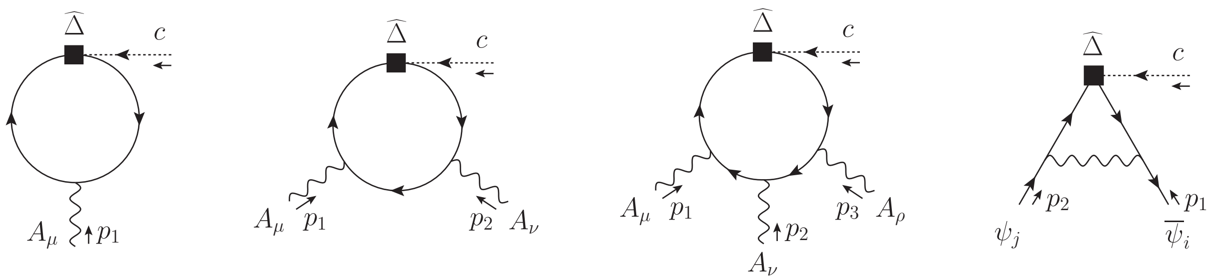


Figure 14.1: The four diagrams contributing to $\widehat{\Delta} \cdot \Gamma^1$ and determining 1-loop symmetry restoring counterterms.

14.3 | Finite symmetry restoring counterterms at one-loop

In order to determine the finite symmetry restoring counterterms we need to compute the quantity $\widehat{\Delta} \cdot \Gamma^1$, corresponding to the breaking of the Slavnov-Taylor identity or BRST symmetry by 1-loop regularized Green functions. This is given by 1-loop Feynman diagrams with one insertion of the vertex $\widehat{\Delta}$, the BRST breaking of the d-dimensional action given in Eq. (12.2.3). Theoretically speaking, infinitely many Feynman diagrams can give a nonzero result but fortunately, in most cases the result is purely evanescent or of order ϵ. Only power-counting divergent diagram can lead to a result which contributes to the above equations, i.e. which contains either a $1/\epsilon$ pole or which is finite and survives in the big limit $\mathrm{LIM}_{d\to 4}$. For the χQED there are only four possible diagrams at the 1-loop level and they are given in Fig. 14.1. The finite results for contributing diagrams are given by:

$$i\,[\widehat{\Delta} \cdot \widetilde{\Gamma}]^1_{A^\mu c} = \frac{e^2}{16\pi^2} \frac{\mathrm{Tr}(\mathcal{Y}_R^2)}{3} \overline{p}_1^2 \overline{p}_1^\mu \,, \tag{14.3.1}$$

$$i\,[\widehat{\Delta} \cdot \widetilde{\Gamma}]^1_{A^\mu A^\nu c} = \frac{e^3}{16\pi^2} \frac{2i\mathrm{Tr}(\mathcal{Y}_R^3)}{3} \epsilon^{\mu\nu\alpha\beta} \overline{p}_{1\alpha} \overline{p}_{2\beta} \,, \tag{14.3.2}$$

$$i\,[\widehat{\Delta} \cdot \widetilde{\Gamma}]^1_{A^\mu A^\nu A^\rho c} = \frac{e^4}{16\pi^2} \frac{2\mathrm{Tr}(\mathcal{Y}_R^4)}{3} (\overline{g}^{\mu\sigma}\overline{g}^{\nu\rho} + \overline{g}^{\mu\rho}\overline{g}^{\nu\sigma} + \overline{g}^{\mu\nu}\overline{g}^{\rho\sigma})(\overline{p}_{1\sigma} + \overline{p}_{2\sigma} + \overline{p}_{3\sigma}) \,, \tag{14.3.3}$$

$$i\,[\widehat{\Delta} \cdot \widetilde{\Gamma}]^1_{\overline{\psi}\psi c} = \frac{e^3}{16\pi^2} \frac{-\mathcal{Y}_R^3(\xi + 5)}{6} (\overline{\not{p}}_1 + \overline{\not{p}}_2)\, \mathbb{P}_{\mathrm{R}} \,. \tag{14.3.4}$$

The result of these diagrams can be written in the field-operator representation in the coordinate space, as an insertion in the effective action in terms of field monomials of ghost number one, as[2]

$$\begin{aligned}\widehat{\Delta}\cdot\Gamma^1 = \frac{\hbar}{16\pi^2}\int \mathrm{d}^d x \Bigg[&\frac{e^2\mathrm{Tr}(\mathcal{Y}_R^2)}{3}(\overline{\partial}_\mu c)(\overline{\partial}^2 \bar{A}^\mu) \\ &+ \frac{e^4\mathrm{Tr}(\mathcal{Y}_R^4)}{3}\, c\,\overline{\partial}_\mu(\bar{A}^\mu \bar{A}^2) - \frac{5+\xi}{6}e^3 \sum_j (\mathcal{Y}_R^j)^3\, c\,\overline{\partial}^\mu(\overline{\psi}_j\overline{\gamma}_\mu \mathbb{P}_\mathrm{R}\psi_j) \\ &- \frac{2\,e^3\mathrm{Tr}(\mathcal{Y}_R^3)}{3}\,\epsilon^{\mu\nu\rho\sigma} c(\overline{\partial}_\rho \overline{A}_\mu)(\overline{\partial}_\sigma \overline{A}_\nu)\Bigg],\end{aligned} \tag{14.3.5}$$

where further terms of order ϵ and evanescent terms of order ϵ^0 have been omitted. The first term correspond to the first diagram of Fig. 14.1 and represents the the violation of the Slavnov-Taylor identity for the photon self-energy (describing essentially its transversality). The divergent part of same diagram was discussed in the previous section. The next two terms are UV finite and non-evanescent. They correspond in an obvious way to the third and fourth diagrams of Fig. 14.1, and they represent the violation of the Slavnov-Taylor identities involving the photon 4-point function and the fermion–photon interaction, respectively.

Notice that the last term in Eq. (14.3.5), arising from the second diagram in Fig. 14.1, cannot be written as the BRST transformation of any local field operator in the action, hence it cannot be removed by any counterterm we can possibly construct. Counterterm that cancels this contribution would have to be proportional to a structure like

$$\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu A_\rho A_\sigma$$

which however vanishes because of its BRST invariance. The last term represents the non-spurious or *essential anomaly*, but since it is proportional to the $\mathrm{Tr}(\mathcal{Y}_R^3)$, it will vanish due to the anomaly cancelation condition Eq. (11.2.4), ensuring that the theory is anomaly free[3]. In general, Eq. (14.3.5) reflects important statements established in the context of algebraic renormalization [1, 25, 95–98]. The breaking of the Slavnov-Taylor identity at any order is a *local, power-counting renormalizable expression* with ghost number one. The term in the last line is the unique and only kind of term that can possibly represent a true anomaly that cannot be canceled by symmetry-restoring counterterms, i.e. this object belongs to the non-trivial cohomology class of the Slavnov-Taylor operator. It is known that if the term vanishes at 1-loop order, like here, it vanishes at all orders and the theory is free of anomalies.

Determination of the finite counterterms $S_{\mathrm{fct}}^{(1)}$, which serve to restore the Slavnov-Taylor identity in the form of Eqs. (14.1.2c) or (14.1.3) is now proceeded in a few steps. Since we

[2] We restore $\hbar$ explicitly at the end result.

[3] Such a condition can be realized in realistic models by either, having multiple families of fermions of same chirality with property that the cube of their hypercharges sum up to zero, or by having the same number of both right and left handed fermions whose hypercharges are equal but of opposite signs.

have the result for breaking, we recall that it must correspond to the BRST transformation of the wanted counterterm action,

$$s_d S^1_{\text{fct}} = -N[\widehat{\Delta}] \cdot \Gamma^1 \,, \tag{14.3.6}$$

where the right-hand side corresponds to the part of Eq. (14.3.5) without the essential anomaly (cancelled by corresponding condition). Further, the counterterms do not depend on external source fields, which implies

$$s_d S^1_{\text{fct}} = b_d S^1_{\text{fct}} = \mathcal{S}_d(S_0 + S_{\text{fct}})^1 = \Delta^1_{\text{fct}} \,. \tag{14.3.7}$$

Now, proceeding as we did for the Yang-Mills model, we assume that the finite counterterm action contains operators i.e. local polynomials that emerge from loop calculations. By s_d variation of these polynomials and direct comparison with the the breaking (14.3.5) we obtain the finite 1-loop counterterm action,

$$\begin{aligned} S^1_{\text{fct}} = \frac{\hbar}{16\pi^2} \int \mathrm{d}^4 x \, \Bigg\{ & \frac{-e^2 \text{Tr}(\mathcal{Y}_R^2)}{6} \bar{A}_\mu \bar{\partial}^2 \bar{A}^\mu + \frac{e^4 \text{Tr}(\mathcal{Y}_R^4)}{12} \bar{A}_\mu \bar{A}^\mu \bar{A}_\nu \bar{A}^\nu \\ & + \frac{5+\xi}{6} e^2 \sum_j (\mathcal{Y}_R^j)^2 \imath \overline{\psi}_j \bar{\gamma}^\mu \bar{\partial}_\mu \mathbb{P}_{\mathrm{R}} \psi_j \Bigg\} \,. \end{aligned} \tag{14.3.8}$$

The first finite counterterm restores transversality of the photon self energy, and the second restores the Ward identity relation for the quartic photon interaction. The third term restores the Ward identity between the fermion self energy and its photon interaction. The counterterms S^1_{fct} restore the symmetry, and all equations (14.1.2c, 14.1.3) and ultimately (12.2.1) are valid at the 1-loop level and all renormalization conditions at the 1-loop level are satisfied at this point.

We emphasize again that finite counterterms are not uniquely fixed. One can add any BRST-symmetric term to these finite counterterms without spoiling the restoration of the BRST symmetry. Further, the finite counterterms are defined as purely 4-dimensional quantities. This corresponds to our requirement (14.1.3). As discussed there, one may change the finite counterterms by evanescent contributions which vanish in the $\text{LIM}_{d\to 4}$.

This means that it would be allowed e.g. to change the counterterms by extending them to d dimensions, i.e. to replace some or all of the $\bar{A}_\mu$ and $\bar{\partial}_\mu$ by full A_μ and ∂_μ and so on. Such changes are irrelevant for pure 1-loop discussions, however at the higher orders the changes matter and might change the form of 2-loop results since for higher-loop order calculations, it is required to insert these counterterms in loop diagrams. Such a choice will differ by the addition of an evanescent part added to these counterterms, and as a result, will affect either the evanescent singular counterterms (case $\sim \hat{g}_{\mu\nu}/\epsilon$) or the finite parts (case $\sim \epsilon/\epsilon$). However, our choice is finite counterterms in 4-dimensional form and that is how we will implement them in 2-loop calculations.

CHAPTER 15

EVALUATION OF THE TWO-LOOP SINGULAR COUNTERTERM ACTION IN χQED VERSUS QED AND ITS BRST BREAKING

In this chapter, we list all the UV divergences in χQED and we compare the results with the standard QED. Calculation includes photon and fermion self-energy and the interaction vertex. Since now we are heading at the 2-loop order, a lot of technical difficulties must be overcome. We used a wide range of computation tools available and adjusted the codes, implemented different techniques, and constructed the algorithms needed for 2-loop renormalization and the complete counterterm action S^2_{sct}.

15.1 | Calculation procedure at two-loop order

Now for the first time in this research, we have to deal with the regularization of the 2-loop divergent integrals, which is not a trivial task, technically speaking. Packages available for the 2-loop calculations, e.g. `TARCER` [50], contain the basis of 2-loop self-energy type scalar integrals (that is integrals depending on two propagators only). Therefore, to calculate our quantities of interest, we must reduce our objects to those 2-loop self-energy scalar integrals. This is performed in two steps. First, one has to perform tensor decomposition of tensor integrals to scalar integrals in the fashion of Eq. (2.1.2). Then the scalar integrals have to be reduced to the integrals od self-energy (type) diagrams. This is even more difficult when we are dealing with vertex diagrams proportional to momenta. We will briefly explain how we managed to overcome those problems.

The general calculation procedure uses the same tools already mentioned in Chapter 6, which is sufficient to define the model, calculate Feynman rules, create topologies, calculate Feynman diagrams and prepare them for calculation. In addition, we need to compute 2-loop self-energy integrals what was done using `TARCER`, that implements the basis of

2-loop scalar integrals where we manually added solutions to integrals that were missing in our calculations.

Divergences of three-point functions can not be calculated directly so we have to reduce them to two-point functions. The first solution is to set one external momentum to zero where the diagram effectively reduces to self-energy. This is possible for diagrams with the UV divergences which are local and independent of external momenta. However, this approach fails in case zero external momenta induce infrared divergences or in case the diagram is itself momentum dependent. In this case, we use a UV/IR-decomposition [100–102] where effectively all external momenta vanish and propagators become massive. For example, consider the expansion of propagator with loop momentum q and external momentum p,

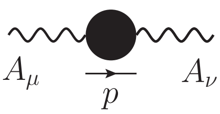
$$\frac{1}{(q+p)^2-m^2} = \frac{1}{q^2-m^2} + \frac{-2qp-p^2}{(q^2-m^2)((q+p)^2-m^2)}, \tag{15.1.1}$$

where the splitting results in a last part that contributes to integrands with a lower degree of divergence and is polynomial in external momenta. Performing this expansion iteratively many times we end with massive self energies without external momenta, i.e. massive vacuum integrals. Reduction to self-energy again makes calculation possible. Wherever it was possible, the calculation was done in both ways and the results agree.

15.2 | List and interpretation of divergent two-loop Green functions

The calculations are performed in $d = 4 - 2\epsilon$ dimensions, and in the Feynman gauge $\xi = 1$, and the results are compared with the corresponding results for QED. Same as at the 1-loop level we have three types of UV divergent Green functions, corresponding to the photon self energy, the fermion self energy and the fermion–photon interaction. Here we first present the explicit results for each subrenormalized 2-loop Green function separately and both for χQED and ordinary QED. The blobs shown in the diagrams represent the sum of the all possible subrenormalized 2-loop corrections, i.e. 2-loop diagrams with tree-level vertices and 1-loop diagrams with singular and finite BRST-restoring counterterm insertions. The latter represent diagrams that are specific to the BMHV scheme, since 1-loop finite counterterms generate Feynman rules for new finite insertions.

Gauge boson self energy:



$$i\widetilde{\Gamma}^{\nu\mu}_{AA}(p)\big|^2_{\text{div},\,\chi\text{QED}} = \frac{ie^4}{256\pi^4}\frac{\text{Tr}(\mathcal{Y}_R^4)}{3}\left[\frac{2}{\epsilon}(\overline{p}^\mu\overline{p}^\nu - \overline{p}^2\overline{g}^{\mu\nu}) + \left(\frac{17}{24\epsilon}-\frac{1}{2\epsilon^2}\right)\widehat{p}^2\widehat{g}^{\mu\nu}\right], \tag{15.2.1a}$$

$$i\widetilde{\Gamma}^{\nu\mu}_{AA}(p)|^2_{\text{div, QED}} = \frac{ie^4}{256\pi^4\epsilon} 2\,\mathrm{Tr}(\mathcal{Y}^4)(p^\mu p^\nu - p^2 g^{\mu\nu})\,. \tag{15.2.1b}$$

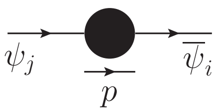
Notice again that d-dimensional transversal part for QED is projected to 4 dimensions in the chiral case where also the new evanescent term is present, again spoiling gauge and BRST invariance. Also, unlike at the 1-loop case, the global factor in front of the chiral transversal part is not half of the content of the QED case, since now in the loop we have an additional diagram with finite 1-loop counterterm insertion.
When performing renormalization in the BMHV scheme, one question that usually comes to mind is: Would the direct sum of left and right-handed model reproduce the coefficients for the regularization of corresponding vector-like theory? From the example of gauge boson self energy we see that 1-loop comparison

$$(\widetilde{\Gamma}^{\nu\mu}_{AA}(p)|^1_{\text{div, }\chi\text{QED}})_{\text{L+R}} = \widetilde{\Gamma}^{\nu\mu}_{AA}(p)|^1_{\text{div, 4dQED}}\,, \tag{15.2.2}$$

fails at 2-loop level:

$$(\widetilde{\Gamma}^{\nu\mu}_{AA}(p)|^2_{\text{div, }\chi\text{QED}})_{\text{L+R}} \neq \widetilde{\Gamma}^{\nu\mu}_{AA}(p)|^2_{\text{div, 4dQED}}\,. \tag{15.2.3}$$

Fermion self energy:

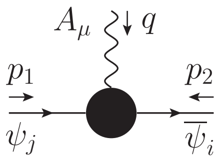


$$i\widetilde{\Gamma}^{ji}_{\psi\overline{\psi}}(p)|^2_{\text{div, }\chi\text{QED}} = \frac{-ie^4}{256\pi^4}\left[\frac{(\mathcal{Y}^2_R)^{ij}\,\mathrm{Tr}(\mathcal{Y}^2_R)}{9\epsilon} + (\mathcal{Y}^4_R)^{ij}\left(\frac{7}{12\epsilon}+\frac{1}{2\epsilon^2}\right)\right]\overline{\not p}\,\mathbb{P}_\text{R}\,, \tag{15.2.4a}$$

$$i\widetilde{\Gamma}^{ji}_{\psi\overline{\psi}}(p)|^2_{\text{div, QED}} = \frac{-ie^4}{256\pi^4}\left[\frac{(\mathcal{Y}^2)^{ij}\,\mathrm{Tr}(\mathcal{Y}^2)}{\epsilon} + (\mathcal{Y}^4)^{ij}\left(\frac{3}{4\epsilon}+\frac{1}{2\epsilon^2}\right)\right]\not p\,. \tag{15.2.4b}$$

Notice again that the chiral result is projected onto 4-dimensions.

Fermion-gauge boson interaction:



$$i\widetilde{\Gamma}^{ji,\mu}_{\psi\overline{\psi}A}|^2_{\text{div, }\chi\text{QED}} = \frac{-ie^5}{256\pi^4}\left[\frac{(\mathcal{Y}^2_R)^{ij}\,\mathrm{Tr}(\mathcal{Y}^3_R)}{\epsilon} - \frac{(\mathcal{Y}^3_R)^{ij}\,\mathrm{Tr}(\mathcal{Y}^2_R)}{9\epsilon} + (\mathcal{Y}^5_R)^{ij}\left(\frac{17}{12\epsilon}+\frac{1}{2\epsilon^2}\right)\right]\overline{\gamma}^\mu\,\mathbb{P}_\text{R}\,, \tag{15.2.5a}$$

$$i\widetilde{\Gamma}^{ji,\mu}_{\psi\overline{\psi}A}|^2_{\text{div, QED}} = \frac{-ie^5}{256\pi^4}\left[\frac{(\mathcal{Y}^3)^{ij}\,\mathrm{Tr}(\mathcal{Y}^2)}{\epsilon} + (\mathcal{Y}^5)^{ij}\left(\frac{3}{4\epsilon}+\frac{1}{2\epsilon^2}\right)\right]\gamma^\mu\,. \tag{15.2.5b}$$

The first term with $\mathrm{Tr}(\mathcal{Y}_R^3) = 0$ does not contribute due to the previously imposed anomaly cancellation condition.

Fermion self-energy and fermion gauge-boson interaction are interesting due to the Ward identity. We will illustrate it in the following discussion. If we redefine Green functions as

$$\widetilde{\Gamma}^{ji}_{\psi\overline{\psi}} \equiv \widetilde{\Gamma}_{\psi\overline{\psi}}\,\delta^{ji} \quad \text{and} \quad \widetilde{\Gamma}^{ji,\mu}_{\psi\overline{\psi}A} \equiv e\,\delta^{ji}\,\mathcal{Y}\,\widetilde{\Gamma}^{\mu}_{\psi\overline{\psi}A},$$

Ward identity can be written for momentum representation in the form

$$\frac{\partial}{\partial p^\mu}\widetilde{\Gamma}_{\psi\overline{\psi}} - \widetilde{\Gamma}^{\mu}_{\psi\overline{\psi}A} = 0, \tag{15.2.6}$$

where we take p, $-p$ and $q = 0$ for fermion, antifermion, and photon momenta respectively. The reader can easily see that Ward identity is trivially satisfied for respective Green functions at the 1-loop level using the results from Chapter 13 both for chiral and generic QED. If we apply same relation at the 2-loop level, we get

$$\begin{aligned}
&\frac{\partial}{\partial p^\mu}\widetilde{\Gamma}_{\psi\overline{\psi}}\big|^{(2)}_{\text{div, }\chi\text{QED}} - \widetilde{\Gamma}^{\mu}_{\psi\overline{\psi}A}\big|^{(2)}_{\text{div, }\chi\text{QED}} \\
&\quad = -\frac{e^4}{256\pi^4\epsilon}\overline{\gamma}^\mu\,\mathbb{P}_{\mathrm{R}}\left(\frac{\mathcal{Y}_R^2\,\mathrm{Tr}(\mathcal{Y}_R^2)}{9} + \frac{7\,\mathcal{Y}_R^4}{12} + \frac{1}{\epsilon}\frac{\mathcal{Y}_R^4}{2} + \frac{\mathcal{Y}_R^2\,\mathrm{Tr}(\mathcal{Y}_R^2)}{9} - \frac{17\,\mathcal{Y}_R^4}{12} - \frac{1}{\epsilon}\frac{\mathcal{Y}_R^4}{2}\right) \\
&\quad = -\frac{e^4}{256\pi^4\epsilon}\overline{\gamma}^\mu\,\mathbb{P}_{\mathrm{R}}\left(\frac{2\mathcal{Y}_R^2\,\mathrm{Tr}(\mathcal{Y}_R^2)}{9} - \frac{5\mathcal{Y}_R^4}{6}\right),
\end{aligned} \tag{15.2.7}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial p^\mu}\widetilde{\Gamma}_{\psi\overline{\psi}}\big|^{(2)}_{\text{div, QED}} - \widetilde{\Gamma}^{\mu}_{\psi\overline{\psi}A}\big|^{(2)}_{\text{div, QED}} \\
&\quad = -\frac{e^4}{256\pi^4\epsilon}\overline{\gamma}^\mu\left(\mathcal{Y}^2\,\mathrm{Tr}(\mathcal{Y}^2) + \frac{3\,\mathcal{Y}^4}{4} + \frac{\mathcal{Y}^4}{2\epsilon} - \mathcal{Y}^2\,\mathrm{Tr}(\mathcal{Y}^2) - \frac{3\,\mathcal{Y}^4}{4} - \frac{\mathcal{Y}^4}{2\epsilon}\right) = 0,
\end{aligned} \tag{15.2.8}$$

as expected for non-chiral model. We will soon find out what is correct interpretation of this breaking of Ward identity in χQED[1].

Due to the completeness of results, we note that triple-photon interaction amplitude is equal to zero for QED models, while it is finite and purely evanescent for χQED. The four-photon interaction amplitude is finite.

[1] When this result was calculated for the first time, at the first sight we suspected that the calculations were incorrect. However, we will soon see that this is not the case, since at this point we are dealing with the Green functions at the point where symmetries are not restored yet.

15.3 | Singular two-loop counterterms

From the singular parts of the 2-loop diagrams listed above we reconstruct Green functions in field-operator representation in the coordinate space and get

$$\Gamma_{\text{div}}^{2,AA} = \frac{e^4}{256\pi^4}\frac{\text{Tr}(\mathcal{Y}_R^4)}{3}\Big[\frac{1}{\epsilon}\overline{A}_\mu(\overline{\partial}^2\overline{g}^{\mu\nu} - \overline{\partial}^\mu\overline{\partial}^\nu)\overline{A}_\nu + \overline{A}_\mu\widehat{\partial}^2\overline{A}^\mu\Big(\frac{1}{4\epsilon^2} - \frac{17}{48\epsilon}\Big)\Big], \tag{15.3.1a}$$

$$\Gamma_{\text{div}}^{2,\overline{\psi}\psi} = \frac{-e^4(\mathcal{Y}_R^j)^2}{256\pi^4}\overline{\psi}_j i\overline{\partial\!\!\!/}\,\mathbb{P}_{\text{R}}\psi_j\Big[\Big(\frac{1}{2\epsilon^2} + \frac{7}{12\epsilon}\Big)(\mathcal{Y}_R^j)^2 + \frac{1}{9\epsilon}\text{Tr}(\mathcal{Y}_R^2)\Big], \tag{15.3.1b}$$

$$\Gamma_{\text{div}}^{2,A\overline{\psi}\psi} = \frac{-e^5(\mathcal{Y}_R^j)^3}{256\pi^4}\overline{\psi}_j\overline{A\!\!\!/}\,\mathbb{P}_{\text{R}}\psi_j\Big[\Big(\frac{1}{2\epsilon^2} + \frac{17}{12\epsilon}\Big)(\mathcal{Y}_R^j)^2 - \frac{1}{9\epsilon}\text{Tr}(\mathcal{Y}_R^2)\Big]. \tag{15.3.1c}$$

The singular counterterm action at the 2-loop level is then defined to cancel this divergences

$$\begin{aligned}
S_{\text{sct}}^2 = & -\Gamma_{\text{div}}^2 \\
= & -\left(\frac{\hbar\, e^2}{16\pi^2}\right)^2\frac{\text{Tr}(\mathcal{Y}_R^4)}{3}\left[\frac{2}{\epsilon}\overline{S_{AA}} + \left(\frac{1}{4\epsilon^2} - \frac{17}{48\epsilon}\right)\int \mathrm{d}^d x\, \overline{A}_\mu\widehat{\partial}^2\overline{A}^\mu\right] \\
& + \left(\frac{\hbar\, e^2}{16\pi^2}\right)^2\sum_j(\mathcal{Y}_R^j)^2\left[\left(\frac{1}{2\epsilon^2} + \frac{17}{12\epsilon}\right)(\mathcal{Y}_R^j)^2 - \frac{1}{9\epsilon}\text{Tr}(\mathcal{Y}_R^2)\right]\left(\overline{S_{\overline{\psi}\psi_R}^j} + \overline{S_{\overline{\psi}_R A\psi_R}^j}\right) \\
& - \left(\frac{\hbar\, e^2}{16\pi^2}\right)^2\sum_j\frac{(\mathcal{Y}_R^j)^2}{3\epsilon}\left(\frac{5}{2}(\mathcal{Y}_R^j)^2 - \frac{2}{3}\text{Tr}(\mathcal{Y}_R^2)\right)\overline{S_{\overline{\psi}\psi_R}^j}\,.
\end{aligned} \tag{15.3.2}$$

First, notice that the operator content is the same as at the 1-loop level, see Eq. (13.2.2). A conceptually new feature compared to the 1-loop case is the term in the last line of S_{sct}^2, which breaks BRST invariance by a non-evanescent amount, which was already detected as the Ward identity breaking. If we re-express the result using renormalization transformations, now we have a structure comparable to those of the 1-loop singular counterterms, Eq. (13.3.1), except this time, the BRST-breaking, non-invariant term also contains non-evanescent part:

$$S_{\text{sct}}^2 = S_{\text{sct,inv}}^2 + S_{\text{sct,break}}^2\,. \tag{15.3.3}$$

Again, the first term is BRST invariant and arises from the renormalization transformation

$$S_{0,\text{inv}} \longrightarrow S_{0,\text{inv}} + S_{\text{ct,inv}},$$

and is given by

$$
\begin{aligned}
S^2_{\text{sct,inv}} &= \frac{\delta Z^2_A}{2} L_A + \frac{\delta Z^2_c}{2} L_c + \frac{\delta Z^2_{\psi_{Rj}}}{2} \overline{L_{\psi_{Rj}}} + \frac{\delta(e)^2}{e} L_e \\
&= \delta Z^2_A S_{AA} + \left(\frac{\delta Z^2_c}{2} - \frac{\delta Z^2_A}{2}\right)(S_{\bar{c}c} + S_{\rho c}) + \left(\frac{\delta Z^2_c}{2} + \frac{\delta(e)^2}{e}\right)(S_{\overline{R}c\psi_R} + S_{Rc\overline{\psi_R}}) \\
&\quad + \delta Z^2_{\psi_R}\left(\overline{S_{\overline{\psi}\psi_R}} + \overline{S_{\overline{\psi}A\psi_R}}\right) + \left(\frac{\delta Z^2_A}{2} + \frac{\delta(e)^2}{e}\right)\overline{S_{\overline{\psi}A\psi_R}}\,,
\end{aligned}
\tag{15.3.4}
$$

with 2-loop renormalization constants δZ^2_φ and $\delta(e)^2$. The renormalization constants are given by

$$
\delta Z^2_A = \delta Z^2_c = -2\frac{\delta(e)^2}{e}\,, \tag{15.3.5a}
$$

$$
\delta Z^2_A = -\frac{e^4}{256\pi^4\epsilon}\frac{2\mathrm{Tr}(\mathcal{Y}^4_R)}{3}\,, \tag{15.3.5b}
$$

$$
\delta Z^2_{\psi_{Rj}} = \frac{e^4}{256\pi^4}(\mathcal{Y}^j_R)^2\left[\left(\frac{1}{2\epsilon^2} + \frac{17}{12\epsilon}\right)(\mathcal{Y}^j_R)^2 - \frac{1}{9\epsilon}\mathrm{Tr}(\mathcal{Y}^2_R)\right]. \tag{15.3.5c}
$$

If we recall 1-loop interpretation of renormalization constants, the first equation (15.3.5a) here shows the validity of the trivial identities equation (11.4.3) and the analog of the usual QED Ward identity on the level of $S^2_{\text{sct,inv}}$, because the fermion self-energy and fermion-boson vertex now renormalize with the common factor $\delta Z^2_{\psi_{Rj}}$. Nevertheless, the results for the renormalization constants differ from the ones in the literature obtained without the BMHV scheme, see e.g. Refs. [67–69]. This difference implies the modified relationship between the renormalization-group β functions and singular counterterms in the BMHV scheme, see the discussions in Refs. [47, 103]. A detailed investigation of this issue is of great importance and is recommend as future proposal.

The BRST-breaking singular counterterms appearing in Eq. (15.3.3) is given by

$$
\begin{aligned}
S^2_{\text{sct,break}} = &-\left(\frac{\hbar e^2}{16\pi^2}\right)^2 \frac{1}{\epsilon}\frac{\mathrm{Tr}(\mathcal{Y}^4_R)}{3}\left(2(\overline{S_{AA}} - S_{AA}) + \left(\frac{1}{2\epsilon} - \frac{17}{24}\right)\int \mathrm{d}^d x\, \frac{1}{2}\overline{A}^\mu \widehat{\partial}^2 \overline{A}_\mu\right) \\
&-\left(\frac{\hbar e^2}{16\pi^2}\right)^2 \frac{1}{3\epsilon}\sum_j (\mathcal{Y}^j_R)^2\left(\frac{5}{2}(\mathcal{Y}^j_R)^2 - \frac{2}{3}\mathrm{Tr}(\mathcal{Y}^2_R)\right)\overline{S^j_{\overline{\psi}\psi_R}}\,,
\end{aligned}
\tag{15.3.6}
$$

what generates a BRST breaking,

$$\Delta^2_{\text{sct}} = s_d S^2_{\text{sct}} = \frac{-\hbar^2 e^4}{256\pi^4}\frac{\mathrm{Tr}(\mathcal{Y}_R^4)}{6}\left(\frac{1}{\epsilon^2}-\frac{17}{12\epsilon}\right)\int \mathrm{d}^d\, x\, (\overline{\partial}_\mu c)(\widehat{\partial}^2\overline{A}^\mu)$$
$$-\frac{\hbar^2 e^5}{256\pi^4}\frac{1}{3\epsilon}\sum_j (\mathcal{Y}_R^j)^3\left(\frac{5}{2}(\mathcal{Y}_R^j)^2-\frac{2}{3}\mathrm{Tr}(\mathcal{Y}_R^2)\right)\int \mathrm{d}^d\, x\, c\,\overline{\partial}_\mu(\overline{\psi}\overline{\gamma}^\mu \mathbb{P}_\mathrm{R}\psi)\,, \tag{15.3.7}$$

where we have used the BRST invariance of $\overline{S_{AA}}$ and S_{AA}. We again emphasize that at the 2-loop level we obtain non-evanescent BRST breaking for the first time in the regularization procedure.

Table 15.1
Types of singular BRST breaking counterterms in χQED.

	evanescent	non-evanescent
1-loop	✓	✗
2-loop	✓	✓

CHAPTER 16

TWO-LOOP FINITE COUNTERTERM ACTION AND BRST SYMMETRY RESTORATION

Once we know the 2-loop singular counterterm action, the final missing piece of the puzzle in the renormalization procedure is a set of finite counterterms needed to restore broken gauge and BRST symmetry. At this point, we have a set of strong theoretical tools (like regularized quantum action principle and Ward identities) and practical algorithms for 2-loop calculations we have established at the singular level. Thanks to the general set of identities in Chapter 12 we can now restrict the renormalization conditions to this loop order.

16.1 | Renormalization conditions

The two loop renormalization is defined by regularization of the 2-loop effective action and the BRST transform of the counterterm,

$$\Gamma^{(2)}_{\text{DReg}} = \Gamma^{(2)} + S^2_{\text{sct}} + S^2_{\text{fct}}\,, \tag{16.1.1a}$$

$$\Delta^2_{\text{ct}} = (\mathcal{S}_d(S_0 + S_{\text{ct}}))^2\,. \tag{16.1.1b}$$

Using the conditions on conterterms given by Eq. (12.1.3) and Eq. (12.2.4) at the 2-loop order one obtains the following equations for the counterterms that must hold:

$$S^2_{\text{sct}} + \Gamma^2_{\text{div}} = 0\,, \tag{16.1.2a}$$

$$\left(\widehat{\Delta}\cdot\Gamma^2 + \Delta^1_{\text{ct}}\cdot\Gamma^1 + \Delta^2_{\text{ct}}\right)_{\text{div}} = 0\,, \tag{16.1.2b}$$

$$\text{LIM}_{d\to 4}\left(\widehat{\Delta}\cdot\Gamma^2 + \Delta^1_{\text{ct}}\cdot\Gamma^1 + \Delta^2_{\text{ct}}\right)_{\text{fin}} = 0\,. \tag{16.1.2c}$$

The first equation represents a regularization condition that renders effective action finite by imposing a set of singular counterterms. Procedure corresponds to standard regularization in vector-like theories. The second equation must hold by construction and provides a useful consistency check (that is so welcome at this loop level since we do not have a comparison from literature). The third equation determines the finite symmetry-restoring counterterms: once it is satisfied, χQED renormalization task is done up to 2-loop order. Using the operation $\mathrm{LIM}_{d\to 4}$ the first two terms in Eq. (16.1.2c) become normal products of corresponding operators, while last term is replaced by finite part of Δ^2_{ct}, leading to equation

$$N\Big[\widehat{\Delta}\cdot\Gamma^2+\Delta^1_{\mathrm{ct}}\cdot\Gamma^1\Big]+\Delta^2_{\mathrm{fct}}=0\,, \tag{16.1.3}$$

what implicitly discards finite, evanescent counterterms. Once again, the breaking of the Slavnov-Taylor identity is given via the quantum action principle by Green functions with breaking insertions and the finite symmetry-restoring counterterms are defined such that they cancel the finite, purely 4-dimensional part of the breaking. Same as at the 1-loop level, we have used that the BRST variation of the singular counterterms Δ^2_{sct} contains no finite terms and we could drop the index 'DReg'.

In the following, we first describe the required Feynman diagrammatic computation that has some insertions that appear for the first time at this loop level. The second task is to check Eq. (16.1.2b) in the singular sector. Finally, we determine the finite, symmetry-restoring counterterms. At the very end of this chapter, we provide the ultimate test of correctness: validity of corresponding Ward identities.

16.2 | Computation of the full two-loop breaking of BRST symmetry

Feynman diagrams describing the 2-loop symmetry breakings are, by quantum action principle, the diagrams with insertions of the symmetry breaking of the tree-level and (for the first time at this loop order) counterterm action. Eqs. (16.1.3) and (16.1.2b) imply the ingredients of diagrams, e.g.

$$\widehat{\Delta}\cdot\Gamma^2$$

$$\widehat{\Delta}\cdot(2\mathrm{L}) \qquad \widehat{\Delta}\cdot(1\mathrm{L}+1\mathrm{LCT})$$

where 1L/2L denotes the 1/2-loop diagrams and 1LCT denote generic 1-loop counterterm, and also

$$
\begin{array}{ccc}
& \Delta^1_{\mathrm{ct}} \cdot \Gamma^1 & \\
\swarrow & & \searrow \\
\Delta^1_{\mathrm{sct}} \cdot \Gamma^1 & & \Delta^1_{\mathrm{fct}} \cdot \Gamma^1
\end{array}
$$

Note that in the case of this model for finite insertions we have

$$(\Delta^1_{\mathrm{sct}} \cdot \Gamma^1)^2 = 0 \quad \Longrightarrow \quad (\Delta^1_{\mathrm{ct}} \cdot \Gamma^1)^2 = (\Delta^1_{\mathrm{fct}} \cdot \Gamma^1)^2,$$

since in χQED there are no ghost loop corrections.

The only relevant results are the ones that are either divergent or finite but not evanescent (since the latter will not survive the 4-dimensional limit). Since the breaking insertions $\widehat{\Delta}$ are by themselves evanescent, non-vanishing results can only arise from power-counting divergent Feynman diagrams. To convince the reader this is the case we recall the 1-loop Bonneau identity Eq. (7.3.5) for $\widehat{\Delta}$ insertion,

$$[N[\widehat{\Delta}] \cdot \Gamma_{\mathrm{Ren}}]^{(1)} = \underset{d\to 4}{\mathrm{LIM}} \left(-\text{r.s.p.} \left[\check{\Delta} \cdot \Gamma_{\mathrm{Dreg}}\right]^{(1)}_{\check{g}=0}\right) \equiv N\Big[-\text{r.s.p.}\Big[N[\check{\Delta}] \cdot \Gamma_{\mathrm{Dreg}}\Big]^{(1)}_{\check{g}=0}\Big], \quad (16.2.1)$$

that resulted in a convenient procedure for poles extraction and discarded evanescent symmetry-restoring counterterms. However, the hidden beauty of Bonneau identity manifests at the 2-loop level where all possible insertions are given by

$$[N[\widehat{\Delta}] \cdot \Gamma_{\mathrm{Ren}}]^{(2)} = \underset{d\to 4}{\mathrm{LIM}} \left(-\text{r.s.p.} \left[\check{\Delta} \cdot \Gamma\right]^{(2)}_{\check{g}=0}\right) + N\Big[-\text{r.s.p.}\Big[N[\check{\Delta}] \cdot \Gamma_{\mathrm{Dreg}}\Big]^{(1)}_{\check{g}=0}\Big] \cdot \Gamma^{(1)}_{\mathrm{Ren}}, \, , \quad (16.2.2)$$

where in the last term we see that 1-loop Bonneau insertion is now inserted in effective action and is equal

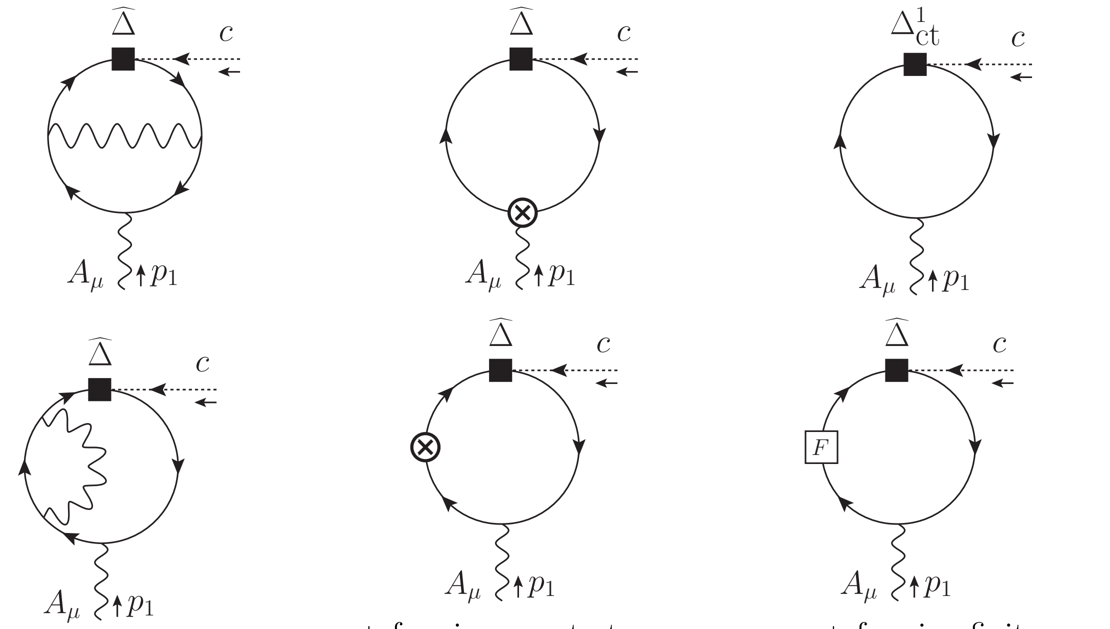
$$N\Big[-\text{r.s.p.}\Big[N[\check{\Delta}] \cdot \Gamma_{\mathrm{Dreg}}\Big]^{(1)}_{\check{g}=0}\Big] \cdot \Gamma^{(1)}_{\mathrm{Ren}} = [N[\widehat{\Delta}] \cdot \Gamma_{\mathrm{Ren}}]^{(1)} \cdot \Gamma^{(1)}_{\mathrm{Ren}} = -N[\Delta^{(1)}_{\mathrm{fct}}] \cdot \Gamma^{(1)}_{\mathrm{Ren}}. \quad (16.2.3)$$

Now, the breaking of the Slavnov-Taylor identity at the 2-loop level contains

$$[N[\widehat{\Delta} + \Delta^{(1)}_{\mathrm{fct}}] \cdot \Gamma_{\mathrm{Ren}}]^{(2)} = \underset{d\to 4}{\mathrm{LIM}} \left(-\text{r.s.p.} \left[\check{\Delta} \cdot \Gamma\right]^{(2)}_{\check{g}=0}\right), \quad (16.2.4)$$

again the diagrams with the UV poles and possible evanescent finite symmetry-restoring counterterms are discarded. For this reason, fortunately only a finite number of Feynman diagrams with a specific set of external fields need to be computed.
The relevant diagrams with non-vanishing contributions are shown in Figs. 16.1 to 16.3. The first contributing diagram is the ghost-photon breaking interaction.

The total ghost-gauge boson contribution from the diagrams with external fields cA



+ loop on the other fermion propagator.

+ fermion counterterm on the other fermion propagator.

+ fermion finite counterterm on the other fermion propagator.

Figure 16.1: List of Feynman diagrams for the ghost–photon breaking contribution given in Eq. (16.2.5).

shown in Fig. 16.1 is

$$i\left([\widehat{\Delta}+\Delta^1_{\mathrm{ct}}]\cdot\widetilde{\Gamma}\right)^2_{A_\mu c} = \frac{1}{256\pi^4}\frac{e^4\mathrm{Tr}(\mathcal{Y}_R^4)}{6}\left[\left(\frac{1}{\epsilon^2}-\frac{17}{12\epsilon}\right)\hat{p}_1^2\overline{p}_1^\mu - \frac{11}{4}\overline{p}_1^2\overline{p}_1^\mu + \mathcal{O}(\hat{\cdot})\right], \tag{16.2.5}$$

where in figure the diagrams in the first column are 2-loop diagrams with one insertion of the tree-level breaking $\widehat{\Delta}$. The diagrams in the second column are 1-loop diagrams with one insertion of a 1-loop singular counterterm, denoted as a circled cross. The third column contains a 1-loop diagram with an insertion of a 1-loop symmetry-restoring counterterm obtained from the fermion self-energy operator, denoted by a boxed F, and a 1-loop diagram with an insertion of the 1-loop breaking Δ^1_{ct}. The result contains $1/\epsilon^2$ poles and $1/\epsilon$ poles with local, evanescent coefficients and finite, non-evanescent term. Finite but evanescent terms $\mathcal{O}(\hat{\cdot})$ may be and are suppressed in Eq. (16.2.5) as well as in following equations due to procedure described by Eq. (16.1.3), hence we discard them in the following results.

The ghost-fermion-fermion contribution from the diagrams with external fields $c\overline{\psi}\psi$ shown in Fig. 16.2 is

$$\begin{aligned} i\left([\widehat{\Delta}+\Delta^1_{\mathrm{ct}}]\cdot\widetilde{\Gamma}\right)^2_{\psi\overline{\psi}c} = \frac{1}{256\pi^4}\frac{e^5(\mathcal{Y}_R^j)^3}{3}(\bar{\not{p}}_1+\bar{\not{p}}_2)\mathbb{P}_\mathrm{R}\times \\ \left[\frac{1}{\epsilon}\left(\frac{5}{2}(\mathcal{Y}_R^j)^2-\frac{2}{3}\mathrm{Tr}(\mathcal{Y}_R^2)\right)+\frac{127}{12}(\mathcal{Y}_R^j)^2-\frac{1}{9}\mathrm{Tr}(\mathcal{Y}_R^2)\right]. \end{aligned} \tag{16.2.6}$$

The result contains $1/\epsilon$ poles with local, evanescent coefficients and finite, non-evanescent terms.

The ghost-two gauge bosons contribution from diagrams with external fields cAA turns out to vanish. Hence

$$i\left([\widehat{\Delta}+\Delta^1_{\rm ct}]\cdot\widetilde{\Gamma}\right)^2_{AAc}=0\,, \tag{16.2.7}$$

and the 1-loop essential anomaly does not appear at the higher orders, as it shouldn't, keeping the theory anomaly free.

The ghost-three gauge bosons contribution from the diagrams with external fields $cAAA$ shown in Fig. 16.3 is

$$\begin{aligned} i\left([\widehat{\Delta}+\Delta^1_{\rm ct}]\cdot\widetilde{\Gamma}\right)^2_{A_\rho A_\nu A_\mu c}=\frac{1}{256\pi^4}&3e^6\mathrm{Tr}(\mathcal{Y}_R^6)(\overline{p}_1+\overline{p}_2+\overline{p}_3)_\sigma\\ &(\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma}+\overline{g}^{\mu\rho}\overline{g}^{\nu\sigma}+\overline{g}^{\mu\sigma}\overline{g}^{\nu\rho})\,. \end{aligned} \tag{16.2.8}$$

We emphasize that this result contains no UV divergence but only a finite term. At this point, we encourage the reader to try to construct any other possible insertion diagram with ghost number 1 and mass-dimension 4 with the help of Table 11.1[1].

Collecting the results of Eqs. (16.2.5) to (16.2.8), we obtain the result for the 2-loop breaking of the Slavnov-Taylor identity of 2-loop subrenormalized Green functions:

$$\begin{aligned} \left([\widehat{\Delta}+\Delta^1_{\rm ct}]\cdot\Gamma\right)^2=\frac{\hbar^2e^4}{256\pi^4}&\int \mathrm{d}^d\,x\\ &\Bigg\{-\frac{\mathrm{Tr}(\mathcal{Y}_R^4)}{6}\left[\left(\frac{1}{\epsilon^2}-\frac{17}{12\epsilon}\right)c\,\overline{\partial}_\mu\widehat{\partial}^2\bar{A}^\mu-\frac{11}{4}c\,\overline{\partial}_\mu\overline{\partial}^2\bar{A}^\mu\right]\\ &+e\sum_j\frac{(\mathcal{Y}_R^j)^3}{3}\Bigg[\frac{1}{\epsilon}\left(\frac{5}{2}(\mathcal{Y}_R^j)^2-\frac{2}{3}\mathrm{Tr}(\mathcal{Y}_R^2)\right)\\ &+\frac{127}{12}(\mathcal{Y}_R^j)^2-\frac{1}{9}\mathrm{Tr}(\mathcal{Y}_R^2)\Bigg]c\,\overline{\partial}_\mu(\overline{\psi}_j\bar{\gamma}^\mu\,\mathbb{P}_{\rm R}\,\psi_j)\\ &+\frac{3\,e^2\mathrm{Tr}(\mathcal{Y}_R^6)}{2}c\,\overline{\partial}_\mu(\bar{A}^\mu\bar{A}_\nu\bar{A}^\nu)\Bigg\}+\mathcal{O}(\hat{\cdot})\,. \end{aligned} \tag{16.2.9}$$

Despite significant complications in calculations at higher-order, the term structure remained the same as it was at the 1-loop level.

16.3 | Two-loop singular breaking

Like at the 1-loop level, we first use the result to check the cancellation of the UV divergences as prescribed by Eq. (16.1.2b). A expected, this cancellation with $s_dS^{(2)}_{\rm sct}$ given

[1] Better luck next time.

in Eq. (15.3.7) occurs as

$$\Delta^2_{\mathrm{sct}} = s_d S^{(2)}_{\mathrm{sct}} = -\left([\widehat{\Delta} + \Delta^1_{\mathrm{ct}}] \cdot \Gamma\right)^2_{\mathrm{div}} \, , \tag{16.3.1}$$

confirming the computation. This check ensures both correctness of counterterms we evaluated at the 1-loop and 2-loop order and of 1-loop BRST breaking term Δ^1_{ct}, as well as of the calculation procedure we implemented in codes for insertion evaluation. Furthermore, this also implies the correctness of the finite part of our breaking diagrams, since each diagram is evaluated without approximations, and the expansion in powers of ϵ is performed at very end of calculation.

16.4 | Two-loop Finite Symmetry-Restoring Counterterms

The 2-loop symmetry-restoring counterterms are calculated using Eq. (16.1.3), with the result

$$\begin{aligned}
\Delta^2_{\mathrm{fct}} =& - N\Big[\widehat{\Delta} \cdot \Gamma^2_{\mathrm{DReg}} + \Delta^1_{\mathrm{ct}} \cdot \Gamma^1_{\mathrm{DReg}}\Big] \\
=& - \underset{d\to 4}{\mathrm{LIM}} \left\{ \left([\widehat{\Delta} + \Delta^{(1)}_{\mathrm{ct}}] \cdot \Gamma\right)^{(2)} + s_d S^{(2)}_{\mathrm{sct}} \right\} \\
=& + \frac{\hbar^2 e^4}{256\pi^4} \mathrm{Tr}(\mathcal{Y}_R^4) \; s \left(\frac{11}{48} \int \mathrm{d}^4 x \; \bar{A}_\mu \overline{\partial}^2 \bar{A}^\mu \right) \\
& - \frac{\hbar^2 e^4}{256\pi^4} \sum_j (\mathcal{Y}_R^j)^2 \; \left(\frac{127}{36} (\mathcal{Y}_R^j)^2 - \frac{1}{27} \mathrm{Tr}(\mathcal{Y}_R^2) \right) s \int \mathrm{d}^4 x \; \overline{\psi}_j i \bar{\not{\partial}} \, \mathbb{P}_{\mathrm{R}} \, \psi_j \\
& + \frac{\hbar^2 e^6}{256\pi^4} \frac{3\, \mathrm{Tr}(\mathcal{Y}_R^6)}{8} \; s \int \mathrm{d}^4 x \; \bar{A}_\mu \bar{A}^\mu \bar{A}_\nu \bar{A}^\nu \, .
\end{aligned} \tag{16.4.1}$$

From the BRST restoration requirement that $\mathcal{S}(\Gamma_{\mathrm{Ren}}) = 0$ in the last of Eq. (16.1.2), this enables one to define the 2-loop finite counterterm action which cancels the finite BRST breaking, up to any additional BRST-symmetric terms that can be added at will,

$$\begin{aligned}
S^2_{\mathrm{fct}} =& \left(\frac{\hbar}{16\pi^2}\right)^2 \int \mathrm{d}^4 x \; e^4 \Bigg\{ \mathrm{Tr}(\mathcal{Y}_R^4) \frac{11}{48} \bar{A}_\mu \overline{\partial}^2 \bar{A}^\mu + 3\, e^2 \frac{\mathrm{Tr}(\mathcal{Y}_R^6)}{8} \; \bar{A}_\mu \bar{A}^\mu \bar{A}_\nu \bar{A}^\nu \\
& - \sum_j (\mathcal{Y}_R^j)^2 \; \left(\frac{127}{36} (\mathcal{Y}_R^j)^2 - \frac{1}{27} \mathrm{Tr}(\mathcal{Y}_R^2) \right) \left(\overline{\psi}_j i \bar{\not{\partial}} \, \mathbb{P}_{\mathrm{R}} \, \psi_j \right) \Bigg\} \, .
\end{aligned} \tag{16.4.2}$$

The terms we obtain correspond to the restoration of the Ward identity relations for the photon self-energy, the photon 4-point function, and the fermion self-energy/photon interaction, as we will explicitly show in the next section. Also, the set of operators remains the same at the 2-loop level.

16.5 | Tests of Ward identities

Since here we present a complete 2-loop study of renormalization in the BMHV scheme, it would be useful to provide the additional check for counterterm for which we claim that restore gauge and BRST invariance. Fortunately, we can make use of Ward identities which express relations of Green's functions and their properties due to gauge invariance of the theory. In Section 11.4 we have seen that in our $U(1)$ model the Slavnov-Taylor identity straightforwardly leads to Ward identities since certain functional relations trivially survive renormalization. Once the Slavnov-Taylor identity is satisfied, the Ward identities will likewise be valid, but they provide a check that is independent of breaking diagrams. Ward identities given in Eqs. (11.4.5) to (11.4.7) are used to extract their breaking of the finite parts of Green functions we successfully avoided using the quantum action principle. However, we will calculate the relevant part once to provide this ultimate check. We begin with the example of the 2-loop divergent part of the photon self-energy. If we contract it with one momentum, what we obtain is

$$i\, p_\nu\, \widetilde{\Gamma}^{\mu\nu}_{A(-p)A(p)}\Big|^2_{\text{div}} = \frac{ie^4}{256\pi^4}\frac{\text{Tr}(\mathcal{Y}_R^4)}{6}\left(\frac{17}{12\epsilon}-\frac{1}{\epsilon^2}\right)\widehat{p}^2\overline{p}^\mu = -\left([\widehat{\Delta}+\Delta^1_{\text{ct}}]\cdot\widetilde{\Gamma}\right)^2_{\text{div},\,A_\mu(-p)c(p)} . \tag{16.5.1}$$

The first of these equations is obtained by direct computation of the appropriate 2-loop diagrams. The second equation is then an observation using Eq. (15.3.7) and Eq. (16.3.1). These equations confirm that the part of the divergent photon self-energy that would violate transversality is cancelled by the divergent counterterm[2] calculated from the breaking insertion, restoring gauge invariance.

The finite part of photon self-energy at the two loop level is given by

$$i\widetilde{\Gamma}^{\mu\nu}_{AA}(p)\Big|^2_{\text{fin}} = \frac{ie^4}{256\pi^4}\frac{\text{Tr}(\mathcal{Y}_R^4)}{3}\left[\left(\frac{673}{23}-6\log(-\overline{p}^2)-24\zeta(3)\right)(\overline{p}^\mu\overline{p}^\nu-\overline{p}^2\overline{g}^{\mu\nu})+\frac{11}{8}\overline{p}^\mu\overline{p}^\nu\right] , \tag{16.5.2}$$

and after the momentum contraction we obtain

$$i\, p_\nu\, \widetilde{\Gamma}^{\mu\nu}_{A(-p)A(p)}\Big|^2_{\text{fin}} = \frac{ie^4}{256\pi^4}\frac{\text{Tr}(\mathcal{Y}_R^4)}{6}\frac{11}{4}\overline{p}^2\overline{p}^\mu = -\left([\widehat{\Delta}+\Delta^1_{\text{ct}}]\cdot\widetilde{\Gamma}\right)^2_{\text{fin},\,A_\mu(-p)c(p)} . \tag{16.5.3}$$

The first of these equations is again obtained by direct computation of the finite parts of diagrams. It illustrates that the non-local $\log(-\overline{p}^2)$ and transcendental $\zeta(3)$ parts are by themselves transversal and so can not break the gauge invariance. The second equation is then observed by comparison with Eq. (16.4.1). Hence we confirm that the violation of the symmetry is restored by our finite counterterm evaluated from breaking diagrams.

The 4-photon amplitude is completely finite. A direct, explicit manipulation of the corresponding Feynman diagrams shows that we can relate the breaking of the Ward

[2] Of course, to restore gauge invariance, counterterm must by itself be gauge non-invariant.

identity to the breaking of the Slavnov-Taylor identity as

$$
\begin{aligned}
&- i\, p_\nu\, \widetilde{\Gamma}^{\mu_1\mu_2\mu_3\nu}_{A(-p_1)A(-p_2)A(-p_3)A(p)}\Big|^2_{\text{fin}} = \Big([\widehat{\Delta} + \Delta^1_{\text{ct}}] \cdot \widetilde{\Gamma}\Big)_{\text{fin},\, A_{\mu_1}(-p_1)A_{\mu_2}(-p_2)A_{\mu_3}(-p_3)c(p)} \\
&= \frac{ie^6}{256\pi^4}\, 3\text{Tr}(\mathcal{Y}_R^6)\, (\overline{p}_1 + \overline{p}_2 + \overline{p}_3)_\nu (\overline{g}^{\nu\mu_1}\overline{g}^{\mu_2\mu_3} + \overline{g}^{\nu\mu_2}\overline{g}^{\mu_1\mu_3} + \overline{g}^{\nu\mu_3}\overline{g}^{\mu_1\mu_2})\,.
\end{aligned}
\tag{16.5.4}
$$

Via Eq. (16.4.1) this shows again that the counterterms of Eq. (16.4.2) appropriately restore this Ward identity.

We can investigate the Ward identity between the fermion self energy and fermion-photon interaction Eq. (11.4.7) in a similar way. The divergent 2-loop violation is given by

$$
\begin{aligned}
&- i\, e\mathcal{Y}_R \frac{\partial}{\partial p_\mu} \widetilde{\Gamma}_{\psi(p)\overline{\psi}(-p)}\Big|^2_{\text{div}} + i\, \widetilde{\Gamma}^\mu_{\psi(p)\overline{\psi}(-p)A(0)}\Big|^2_{\text{div}} \\
&= -\frac{e^4}{256\pi^4\epsilon}\overline{\gamma}^\mu\, \mathbb{P}_{\text{R}} \left(\frac{\mathcal{Y}_R^2\,\text{Tr}(\mathcal{Y}_R^2)}{9} + \frac{7\,\mathcal{Y}_R^4}{12} + \frac{1}{\epsilon}\frac{\mathcal{Y}_R^4}{2} + \frac{\mathcal{Y}_R^2\,\text{Tr}(\mathcal{Y}_R^2)}{9} - \frac{17\,\mathcal{Y}_R^4}{12} - \frac{1}{\epsilon}\frac{\mathcal{Y}_R^4}{2}\right) \\
&= i\mathcal{Y}_R \frac{e^5}{256\pi^4\epsilon}\overline{\gamma}^\mu\, \mathbb{P}_{\text{R}} \left(\frac{2\mathcal{Y}_R^2\,\text{Tr}(\mathcal{Y}_R^2)}{9} - \frac{5\mathcal{Y}_R^4}{6}\right) \\
&= -\frac{\partial}{\partial q_\mu}\Big([\widehat{\Delta} + \Delta^1_{\text{ct}}] \cdot \widetilde{\Gamma}\Big)^2_{\text{div},\, \psi(p-q)\overline{\psi}(-p)c(q)} (q=0)\,,
\end{aligned}
\tag{16.5.5}
$$

so it is directly cancelled by 2-loop singular counterterm that emerges from insertion procedure. The finite 2-loop violation of this Ward identity is

$$
\begin{aligned}
&- i\, e\mathcal{Y}_R \frac{\partial}{\partial p_\mu} \widetilde{\Gamma}_{\psi(p)\overline{\psi}(-p)}\Big|^2_{\text{fin}} + i\, \widetilde{\Gamma}^\mu_{\psi(p)\overline{\psi}(-p)A(0)}\Big|^2_{\text{fin}} \\
&= i\mathcal{Y}_R \frac{e^5}{256\pi^4}\overline{\gamma}^\mu\, \mathbb{P}_{\text{R}} \Bigg(\log(-\overline{p}^2)\left(\frac{3}{2}\mathcal{Y}_R^4 - \mathcal{Y}_R^2\text{Tr}(\mathcal{Y}_R^2)\right) + \frac{62}{27}\mathcal{Y}_R^2\text{Tr}(\mathcal{Y}_R^2) - \frac{109}{72}\mathcal{Y}_R^4 \\
&\qquad\qquad - \log(-\overline{p}^2)\left(\frac{3}{2}\mathcal{Y}_R^4 - \mathcal{Y}_R^2\text{Tr}(\mathcal{Y}_R^2)\right) - \frac{61}{27}\mathcal{Y}_R^2\text{Tr}(\mathcal{Y}_R^2) - \frac{145}{72}\mathcal{Y}_R^4\Bigg) \\
&= i\mathcal{Y}_R \frac{e^5}{256\pi^4}\overline{\gamma}^\mu\, \mathbb{P}_{\text{R}} \left(\frac{\mathcal{Y}_R^2\,\text{Tr}(\mathcal{Y}_R^2)}{27} - \frac{127\mathcal{Y}_R^4}{36}\right) \\
&= -\frac{\partial}{\partial q_\mu}\Big([\widehat{\Delta} + \Delta^1_{\text{ct}}] \cdot \widetilde{\Gamma}\Big)^2_{\text{fin},\, \psi(p-q)\overline{\psi}(-p)c(q)} (q=0)\,.
\end{aligned}
\tag{16.5.6}
$$

In each case again the first equations are obtained from explicit computation of the Feynman diagrams, and the last equations are obtained by comparing with Eq. (15.3.7), Eq. (16.3.1) and Eq. (16.4.1).[3] This is direct confirmation that the counterterms in Eq. (16.4.2) restore all Ward identities at the 2-loop level.

[3]The divergent $1/\epsilon^2$ poles in (16.5.5) are omitted since they cancel completely. The second and third rows in (16.5.6) represent the full results for finite (momentum-differentiated) photon self energy and vertex interaction, respectively.

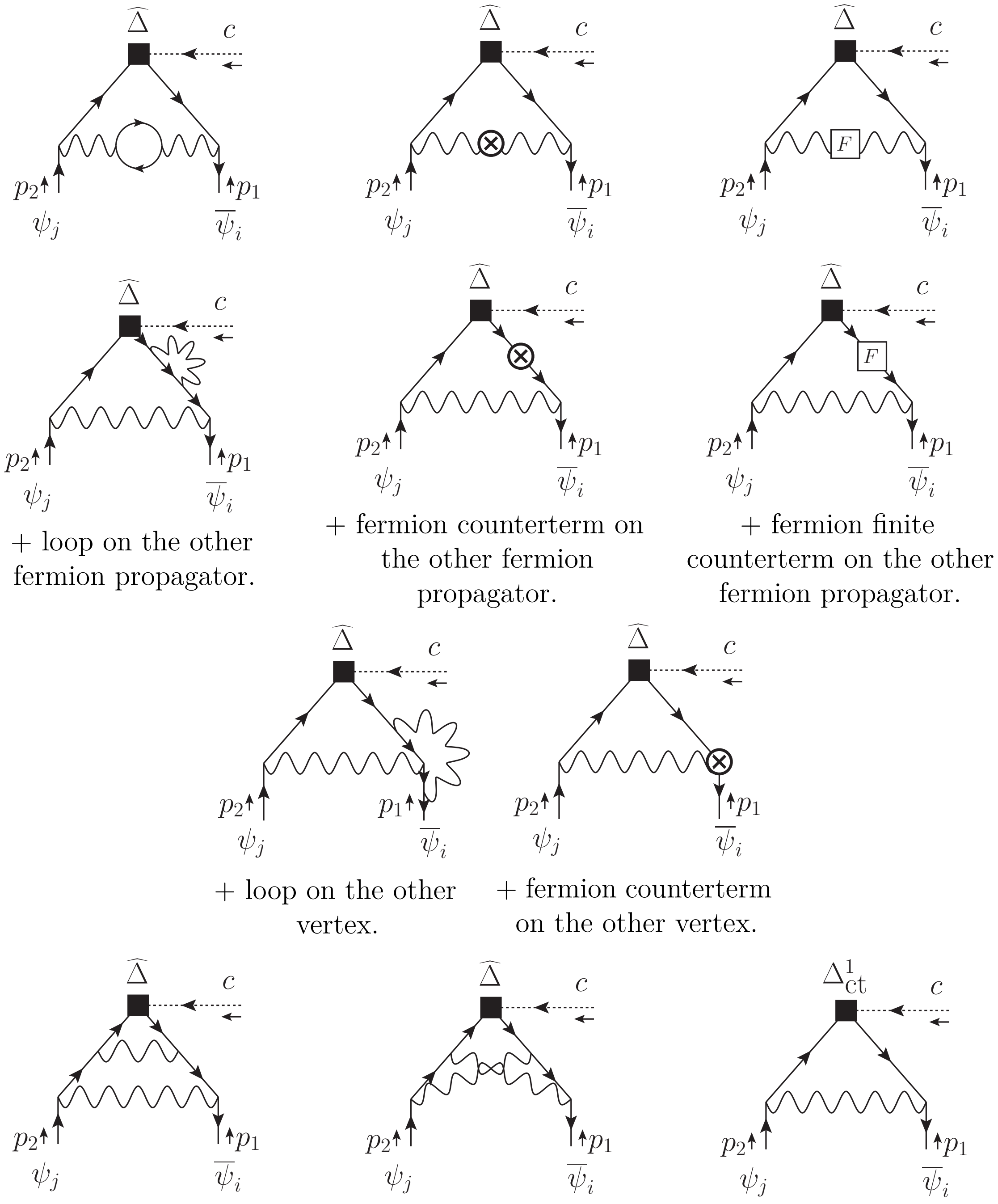


Figure 16.2: List of Feynman diagrams for the ghost–fermion–fermion breaking contribution.The results are given in Eq. (16.2.6).

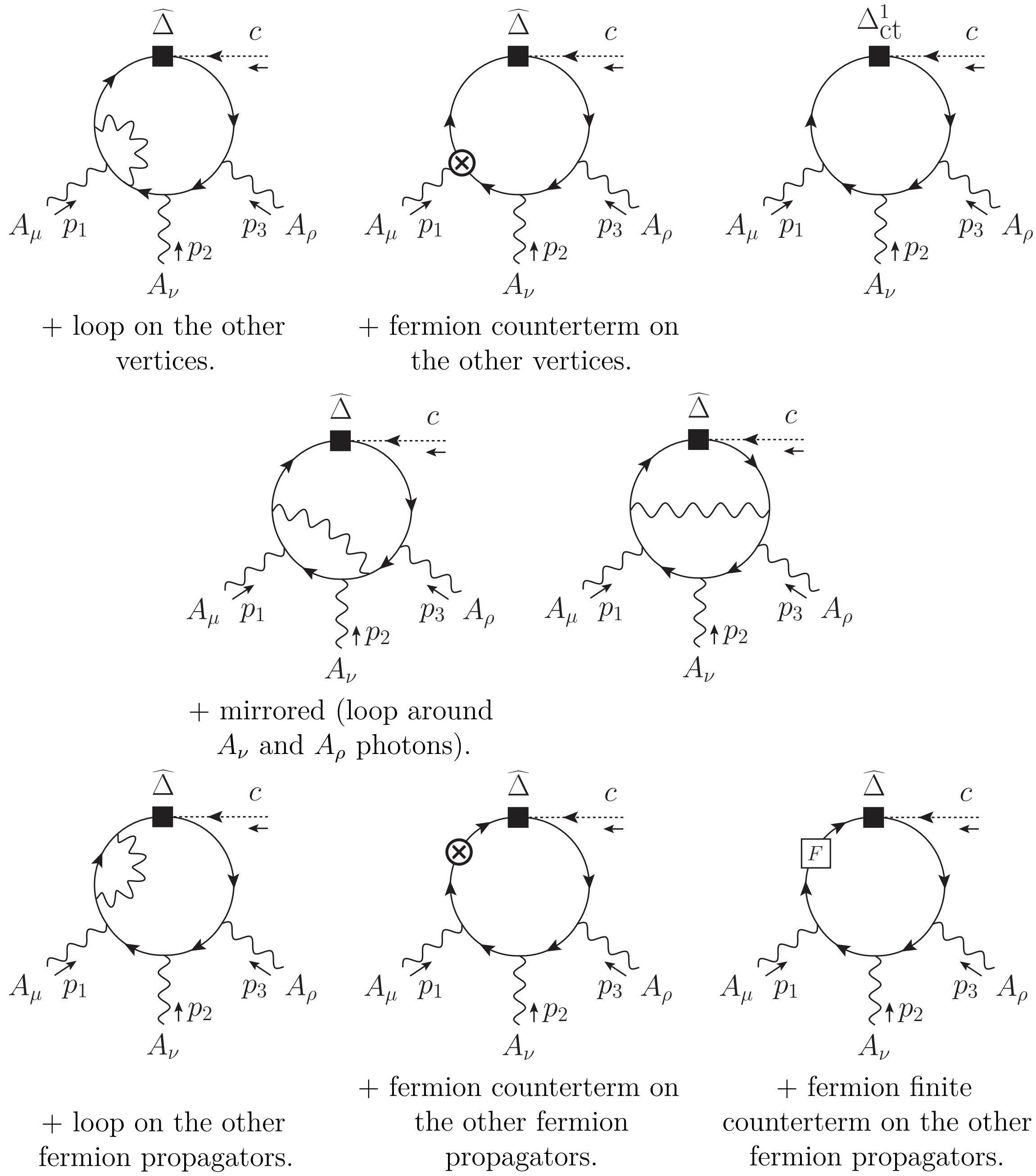


Figure 16.3: List of Feynman diagrams for the Ghost–three gauge bosons breaking contribution (additional diagrams corresponding to $\{(p_1, \mu), (p_2, \nu), (p_3, \rho)\}$ permutations are not shown). The symbols are as in Fig. 16.1 and the results are given in Eq. (16.2.8).

CHAPTER 17

CHIRALITY PROBLEM AND PHYSICAL OBSERVABLES

In this chapter, we reflect on our research up to this point and put it in the context of modern particle physics theory and phenomenology. In the first section, we explain the importance of this work and the results as well as the observables of interest which have been computed and which are of interest. In the second section, we propose possible next steps in the research.

17.1 | Renormalization and observables, so far and so forth

At the time the research described in this thesis started, in 2017, there was no complete study of a general chiral gauge theory with scalar fields in the BMHV scheme even at the 1-loop level. The problem first occurred when we wanted to evaluate in Standard Model and Minimal Supersymmetric Standard Model diagrams involving fermion loops. The question was, do we implement naive prescription, or do we continue with axiomatically correct and mathematically rigorous BMHV scheme and proceeded with the complete study. We decided to use BMHV and apply it first to the general gauge model, where the work of [27] presented the very nice and detailed benchmark for the scalarless limit of our model. The renormalization and symmetry restoration of this SU(N) model proceed with the complete 2-loop study of U(1), which gave the important foundation for the complete study of Standard Model, being a direct product of $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. In this research, we calculated a full list of the counterterms for the models, with the aim, among others, to make BMHV more useful to renormalization practitioners. In addition, at the 1-loop level, we found renormalization group equations and prooved the agreement with the naive prescription, justifying it.

Although Standard Model has shown enormous success in predicting and explaining experimental data (where no significant deviations from SM have been found), it fails to incorporate e.g. dark matter, baryogenesis, and neutrino masses from phenomenological, and e.g. theory of gravitation, hierarchies, and naturalness from a theoretical point of view. It is then of great importance to treat the calculation of observables properly in the consistent scheme, since results from the other approaches can not be checked without comparison with the other schemes – the privilege possessed by the BMHV treatment. From the other point, a promising solution to SM problems is the supersymmetric class of theories (SUSY). Although there is still no experimental signature of SUSY [104], a belief that it may play a fundamental role in particle physics is based on the Haag-Lopuszanski-Sohnius theorem which proves that SUSY algebra is the only graded Lie algebra of symmetries of the S-matrix consistent with relativistic quantum field theory [105]. Physical QFTs have the property of renormalizability – all infinities that appear in higher orders of perturbation theory can be canceled in the procedure of regularization and renormalization (where usually different counterterms are introduced) leaving theory finite i.e. divergences never show up in observable quantities. If a theory is renormalizable it is possible to investigate its behavior at energy scales inaccessible to experiments using a theoretical tool called renormalization group (RG). The evolution of parameters of the theory (e.g. coupling constants and masses) is described by a set of linear differential equations called Renormalization Group Equations (RGEs), where low-energy (experimental) data is used as their initial condition. To extrapolate theory to the unexplored scales, investigate its asymptotic behavior and test the possibility of unification of parameters, it is inevitable to calculate RGEs. For the SU(N) gauge model (referred to as Yang-Mills model in this thesis) we confirmed the naive result at the 1-loop level in the chiral theory.

As we have stated many times, no consistent regularization scheme is known that preserves BRST symmetry and chiral gauge symmetries simultaneously. There is a similar problem in SUSY theories: there is no consistent regularization scheme that preserves both gauge invariance and supersymmetry. For example, the dimensional regularization (DREG) scheme respects gauge symmetry to all orders in perturbation theory but breaks SUSY already at one loop. The algebraic renormalization method is introduced to remedy this problem [54]. In the approach, the Slavnov-Taylor identity and its derivatives are used to determine the non-supersymmetric counterterms introduced to remedy the SUSY noninvariance amplitudes. At the 1-loop level, the algebraic procedure singles out the dimensional reduction (DRED) scheme as a SUSY preserving and, i.e. no extra SUSY restoring term is needed. Nevertheless, its full compatibility at the 2-loop level is not completely tested, although it has been shown DRED can be defined in a mathematically consistent way.

Once the theoretical problems related to regularization and renormalization are solved, RGEs can be calculated and the evolution of parameters for (BSM) model is known. Model is in practice tested by calculation of physical observables which means that the amplitudes for relevant processes must be calculated. Physical low-energy observables of our interest which can give insight to physics beyond SM can be grouped into the three classes: forbidden, precision, and suppressed observables. For precision and suppressed observables, it is mandatory to calculate them in higher orders of perturbation theory. Forbidden observables include e.g. lepton flavor violation processes (LFV) and lepton number violation processes (LNV). It is interesting that with the expected sensitivities of next-generation experiments, CLFV will become the most powerful probe of new physics signals [106]. For example, the sensitivity of the process muon to three electrons is expected to improve for four orders of magnitude [107], while the sensitivity of muon to electron conversion on nuclei is expected to be improved four orders of magnitude in the next generation experiments [108].

In the recent past, electroweak precision observables have played a key role in constraining new physics, which resulted in e.g. predictions of the top quark and Higgs masses. Currently, one of the most interesting precision observables is Higgs mass. Measurements of the Higgs boson mass had reached a precision of 0.2 % in 2015 [109], and are expected to reach 0.1 % in the second 13 TeV run of LHC. Current status for Higgs boson mass is 125.25 ± 0.17 GeV [110]. From the other side, the theoretical precision of Higgs boson mass evaluation is 3 %, so the improvement of theoretical evaluation of Higgs boson mass in the SM and BSM became mandatory.

Magnetic moments of leptons are precision observables in the SM, and they are, together with electric dipole moments of leptons (which are suppressed observables in the SM) described by the same Lorentz operator as the LFV processes lepton to lepton photon. Experimental value (116 591 821 (45) $\times 10^{-10}$ [111]) and theoretical value in the SM (116 0591 821 (45) $\times 10^{-10}$ [112]) of the anomalous magnetic moment of muon amu differ by 3.4 σ CL. A new measurement of the amu is being performed at the FERMILAB where data taking started in 2017. Experiment will improve precision on amu by factor four [113].

Examples of suppressed observables in the SM are electric dipole moments of leptons and decays of B mesons into two muons. Upper limit of electric dipole moment is known to be $< 0.11 \times 10^{-28}$ e cm with 90% confidence [110]. This measurement of atto-electronvolt energy shifts in a molecule probes new physics at the tera-electronvolt energy scale.

Although current experimental results are more or less consistent with the SM predictions, it cannot be a complete theory of elementary particles due to neutrino oscillation experiments and the existence of dark matter. Therefore the right question is not whether new physics will be found but what is its scale. The precision observables and processes and observables suppressed/forbidden in the Standard Model are tools for detection of new physics scale and they are very intensively investigated. From the theoretical and phenomenological point of view, the results are the most reliable if those observables are calculated in a completely consistent way. We hope that here we have helped and will help to overcome this gap between the theory and phenomenology.

17.2 | Future proposal

So far we have applied a consistent BMHV scheme to general gauge the Yang-Mills model up to first order in the perturbation theory and found the renormalization group equations of the model. Further, we renormalized chiral quantum electrodynamics up to second order in perturbation theory. A full list of renormalization coefficients is provided for both models. The natural next steps in this research include:

1. This research is at the stage where the collaboration gained some theoretical knowledge, calculation experience and has set up the semi-automated algorithms and codes for application of the BMHV scheme. It is interesting to renormalize the general U(1) model where we keep all possible interaction vertices and compare the results with the ones we already have.

2. Application of the BMHV scheme in chiral Yang-Mills model up to 2-loop order and the restoration of symmetries is planned.

3. The renormalization group equations of the above model at the 2-loop order are of interest because of the comparison with the naive approach.

4. The renormalization group equations of the abelian model at the 2-loop order are needed for the same comparison.

5. Those calculations are key ingredients for Standard Model application at the both 1 and 2-loop level.

6. By systematic studies of the BMHV scheme, we hope to bridge the gap between the theory and phenomenology. By providing full lists or renormalization coefficients, we hope that the particle physics community will embrace the scheme.

7. The calculation of the wide range of observables in this treatment is of great interest. At this stage, we hope this study will be pursued not only by this group but also by other theoretical and phenomenology collaborations.

CHAPTER 18

CONCLUSIONS

Today, in the era of high precision particle physics, the increasing accuracy of many experiments leads to the necessity of theoretically well-established calculations in higher and higher orders of perturbation theory in the Standard Model and beyond. A great amount of these calculations were performed in dimensional regularization, which preserves symmetries of vector-like theories and makes higher-order calculations feasible. Today dimensional regularization is mathematically well established, widely used, and incorporated in computer codes. However, this scheme faces problems when applied to chiral theories, that possess strictly 4-dimensional objects. Unfortunately, this class of theories can not be swept under the carpet, since the electroweak Standard Model is chiral itself. This problem can be attacked in different ways, wherein this work we choose the Breitenlohner-Maison-t'Hooft-Veltman scheme, which possesses unmatched mathematical rigor and consistency when compared to other schemes. That is the reason why the results of calculations coming from this scheme can be checked for correctness without using the information from other schemes. Its understanding is thus not only important for practical BMHV calculations but also as a point of reference and benchmark for the study of alternative γ_5 schemes. However, the high cost of this choice comes because this scheme in its intermediate steps breaks gauge and BRST invariance, introducing the new set of so-called evanescent operators. That is the reason why we, besides the usual counterterms that must be computed and implemented in renormalization, must find the set of symmetry-restoring counterterms for each order of perturbation theory. Thanks to the regularized quantum action principle, those counterterms can be constructed from so-called breaking diagrams, constructed in a way that breaking of BRST invariance from the previous orders is as a Feynman rule inserted in the effective action diagrams. This procedure was applied to the general gauge Yang-Mills model up to 1-loop level and chiral quantum electrodynamics up to 2-loop level. It is shown that the BRST symmetry breaking can be canceled order by order by proper local finite symmetry restoring counterterms. To remind the reader,

we briefly recall our main steps and the results discussed in this thesis.
The present thesis starts a systematic study of the BMHV scheme for γ_5 and its application to chiral gauge theories such as the electroweak Standard Model. We first apply the BMHV scheme to chiral gauge theory with massless chiral fermions and scalars, so-called right-handed Yang-Mills theory, and for simplicity restrict the model to irreducible representations and a simple gauge group. We find that the breaking of BRST invariance at the tree-level is localized in one single term, the evanescent part of the fermion kinetic term; the breaking has been expressed in a set of Feynman rules. We proceed with the 1-loop study: we provide a detailed overview of the different renormalization and counterterm structure in the BMHV scheme compared to the usual case where counterterms can be generated by a renormalization transformation. Even in the BMHV scheme, a large part of the counterterms can be generated by the usual renormalization transformation, but there are several additional, BMHV-specific new counterterm structures. We evaluate all 1-loop singular counterterms in this model and find that most of the evanescent counterterms are still BRST invariant (despite being evanescent), but there are two non-BRST invariant evanescent counterterms, related to the scalar and vector self-energies, respectively. Evanescent counterterms, despite their vanishing in the 4-dimensional limit, must be implemented in higher orderer calculations where this limit is performed at the very end. We proceed with the evaluation of BRST symmetry restoring counterterms at this level. We have explained in detail the role and the structure of these counterterms and described various possible ways of how these counterterms may be determined. Our calculation is based on the regularized quantum action principle. The list of symmetry restoring counterterms is not unique, since it is always possible to add evanescent or BRST invariant finite counterterms without spoiling the restored symmetry. Our choice is particularly simple and is constructed to the largest possible extent from objects which appear already in the tree-level action. In other words, we shrink from the extending of our basis of operators from the form that is originally present at the tree level. We evaluate renormalization group equations for this model. We demonstrate in two different ways that despite the extra, BMHV-specific counterterms the 1-loop RGE is unchanged compared to the familiar case of using a symmetry-invariant regularization. Keep in mind that this result is the 1-loop privilege.
Starting from the 1-loop experience, we decide that it is now theoretically and practically possible to proceed with this investigation to the 2-loop level. We apply the BMHV scheme to the chiral gauge theory i.e. chiral QED or χQED at the 2-loop level. We again follow the same procedure but now for the first time, we do a systematic study to the second order. Again, the application of the BMHV scheme leads to several specific kinds of counterterms: the ultraviolet (UV) divergences cannot be canceled by counterterms generated by field and parameter renormalization; additional, UV divergent evanescent counterterms are needed; and the breaking of BRST symmetry needs to be repaired by

adding finite, symmetry-restoring counterterms. We have evaluated all these counterterms explicitly at the 1-loop and 2-loop level. We confirm an important result: the structure at the 1-loop and 2-loop levels of those restoring counterterms are essentially the same. The UV divergences arise in the fermion and the photon self-energy and in the fermion–photon interaction. The triple and quartic photon self interactions are UV finite. However, there are purely evanescent divergences in the photon self-energy, and at the 2-loop level there is a non-evanescent divergence in the fermion self-energy, both of which require an extra counterterm which cannot be obtained from field or parameter renormalization. Symmetry- restoring counterterms have simple physical interpretations: counterterm to the photon self-energy restores transversality of the renormalized photon self-energy, a counterterm to the photon 4-point function restores the Ward identity for this Green function and a counterterm to the fermion self-energy restores its Ward identity-like relation to the fermion–photon interaction. An important outcome is that the precise form of these counterterms is now known, and it is established that this is the complete set of symmetry-restoring counterterms for arbitrary 2-loop calculations in the model. The 2-loop level renormalization involves Feynman diagrams of four different kinds: genuine 2-loop diagrams with insertion of the tree-level breaking, 1-loop diagrams with insertions of the 1-loop breaking or of the 1-loop divergent or finite counterterms. The χQED study is restricted to an abelian gauge theory with right-handed fermions and establishes the methodology. However, the same method will be applicable to general non-abelian gauge theories with scalar and fermionic matter. Several future extensions are of interest, especially for phenomenological studies. First, the results can be slightly extended and specialized to the case of the electroweak SM, which has a non-semisimple gauge group, reducible representations and both right-handed and left-handed chiral fermions. This is work in progress. Since the method is now established and not restricted to abelian theories, it will be possible to apply it to general non-abelian chiral gauge theories and to the Standard Model at the 2-loop level. In this way, 2-loop Standard Model calculations will become feasible in the BMHV scheme without worrying about symmetry violations or scheme inconsistencies. As a further outlook, it will be of interest to explore in detail the relationship between the modified counterterm structure (with additional UV divergent and non-symmetric finite terms) and the renormalization group.
Since the BMHV renormalization scheme is usually (unnecessarily) avoided, in this thesis and in the main publications it is based on, we tried to present our results in a detailed and systematic way, using among the other things the language of renormalization factors familiar to the renormalization practitioners. We sincerely hope we gave some foundations to ourselves and to anybody who uses our result as the starting point of consistent multiloop renormalization of the Standard Model and chiral theories in general.

CURICULUM VITAE

Marija Mađor-Božinović was born on the 28th of July 1991 in Split, Croatia. After completing her secondary education at "Gimnazija Dinka Šimunovića, Sinj" in 2010 she enrolled in the Physics program at the Department of Physics, Faculty of Science, University of Zagreb. During her faculty studies, she was active in the popularization of science as an organizer and lecturer in Science Festivals in Sinj and Zagreb. She was a student teaching assistant in Quantum physics and Relativistic quantum physics, student representative on the faculty board, and president of the Student section of Croatian physical society. In 2016 she graduated on the subject *Renormalization group* under the supervision of prof. Amon Ilakovac and obtained the title of Master of Science in Physics. That same year she enrolled in the doctoral program at the Department of Physics, Faculty of Science, University of Zagreb and started to work as a research and teaching assistant. She is a full-time participant in the scientific project Precise Computation of Physical Quantities in Supersymmetric Particle Physics Models, funded by the Croatian Science Foundation. Marija's teaching duties include auditory and practical exercises as well as administering written exams to undergraduate courses Quantum Field Theory I and II and Statistics. Her native language is Croatian while being fluent in English and having elementary proficiency in Italian and Swedish.

List of publications and conference proceedings

- Bélusca-Maïto, Hermès; Ilakovac, Amon; Kühler, Paul; Mađor-Božinović, Marija; Stöckinger, Dominik *Two-loop application of the Breitenlohner- Maison/'t Hooft-Veltman scheme with non- anticommuting γ_5: full renormalization and symmetry-restoring counterterms in an abelian chiral gauge theory* **Journal of High Energy Physics**, 2021 (2021), 11; 159 (2021), 32 doi:10.1007/jhep11(2021)159

- Bélusca-Maïto, Hermès; Ilakovac, Amon; Mađor- Božinović, Marija; Stöckinger, Dominik *Dimensional regularization and Breitenlohner- Maison/'t Hooft-Veltman scheme for γ_5 applied to chiral YM theories: full one-loop counterterm and RGE structure* **Journal of High Energy Physics**, 2020 (2020), 8; 24, 71 doi:10.1007/jhep08(2020)024

- Bélusca-Maïto, Hermès; Ilakovac, Amon; Mađor- Božinović, Marija; Stöckinger, Dominik *Treatment of γ_5 in Dimensionally-Regularized Chiral Yang-Mills Theory with Scalar Fields* **Proceedings, 19th Hellenic School and Workshops on Elementary Particle Physics and Gravity** (CORFU2019), CORFU2019, 090 (2020), 090, 18 doi:10.22323/1.376.0090

- Bélusca-Maïto, Hermès; Ilakovac, Amon; Mađor- Božinović, Marija; Stöckinger, Dominik *Bonneau Identities* **Proceedings, 19th Hellenic School and Workshops on Elementary Particle Physics and Gravity** (CORFU2019), CORFU2019 (2020), 089, 22 doi:10.22323/1.376.0089

Invited talks and seminars

- Charged LFV, RGEs and algebraic renormalization, TU Dresden (with A. Ilakovac and J. You) (2017)
- Day of Science and Technology, Faculty of Electrical Engineering and Computing, University of Zagreb (2019)
- Dimensional Regularization and Chiral Theories, Fermilab virtual seminar (2022)

Conferences and presentations

- LHC days in Split (2016)
- First Zagreb School of Theoretical Physics (2017)
- Corfu Summer School: Workshop on Standard Model and Beyond (2017)
- 14th Central European Seminar on Particle Physics and Quantum Field Theory (VCES) Vienna (2018)
- Portorož 2019: Precision Era in High Energy Physics, poster presentation
- THOR Winter School 2020 Jahorina
- ACHT 2021: Perspectives in Particle, Cosmo- and Astroparticle Theory, talk presentation

SUPERVISOR'S CURICULUM VITAE

Amon Ilakovac is a full professor with tenure at the Faculty of Science of the University of Zagreb since 2016. According to the Web of Science database, he has 39 published works with 1361 citations (1301 citations without self-citations) with an h-index 17 and 34.9 citations per paper. According to the hep-inspire database he has 34 published works, 47 cited works with 1795 citations, 34 published works with 1772 citations, in total he has 50 papers, and an overall h-index 18. He was head of the international project KRO-005-95 "Leptonen" (6B0A1A "Leptonen") in the period 1995-1998 and head of the HRZZ project 2016-06-IP-7460 "Precise Computation of Physical Quantities in Supersymmetric Particle Physics Models" (PRECIOUS). In addition, he was a collaborator on several Croatian projects. He has published articles in the following areas of elementary particle physics: Nonrelativistic models of heavy mesons, Relativistic formulation of quark models. Lepton flavour/number violation (LFV/LNV) and CP violation in the framework of the Standard model extended by heavy neutrinos, Exact solutions of QCD2, Mathematical analysis of the Higgs sector in MSSM SO(10) model and its applications to proton decay and to LFV/LNV. LFV in extensions of MSSM, Analysis of noncommutative field theories with Moyal product to any order with respect to noncommutative parameters, Renormalization and renormalization group equations in axiomatically correct BMHV renormalization scheme in the general-chiral model, and $U(1)$-chiral model. Currently, he is working on the axiomatically correct renormalization of the Standard model. He was a supervisor of one Ph.D. student. At the moment he is a supervisor to Ph.D. student Marija Mađor-Božinović who wrote this thesis and who will receive her doctoral degree within the HRZZ project 2016-06-IP-7460 PRECIOUS.

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