Estimates for homogenous Fourier multipliers and multiparameter maximal Fourier restriction

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DOCTORAL DISSERTATION

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Supervisor:

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Ocjene za homogene Fourierove multiplikatore i višeparametarsku maksimalnu Fourierovu restrikciju

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Summary

The thesis studies two problems in harmonic analysis – sharp estimates for the norms of the powers of multipliers associated with unimodular homogeneous symbols and the multi-parameter maximal Fourier restriction.

In the first part, it proves asymptotically sharp estimates for the norms of multipliers associated with unimodular homogeneous symbols of degree 0 and shows that the powers of a generic multiplier in that class exhibit asymptotically maximal order of growth. As a consequence, it disproves Maz'ya's conjecture regarding the asymptotically sharp estimates of such multipliers in all dimensions and solves the problem posed by Dragičević, Petermichl, and Volberg concerning the sharp lower estimate of a certain multiplier falling within the mentioned class.

In the second part, the thesis generalizes the Christ–Kiselev lemma for maximal operators to its multi-parameter version. As a consequence, it solves the multi-parameter maximal Fourier restriction problem in all dimensions, a result that was known only in two dimensions from the work of Müller, Ricci and Wright and proves the multi-parameter version of the Menshov–Paley–Zygmund theorem for the multi-dimensional Fourier transform.

Keywords: Fourier multipliers, singular integrals, Fourier restriction, maximal estimates

SAŽETAK

Disertacija se bavi proučavanjem dvaju problema u harmonijskoj analizi – problemom strogih ocjena normi potencija multiplikatora pridruženih unimodularnim homogenim simbolima i problemom višeparametarske Fourierove restrikcije.

U prvom se dijelu dokazuju asimptotski stroge ocjene za norme potencija multiplikatora pridruženih unimodularnim homogenim simbolima reda 0 i dokazuje se da potencije generičkog multiplikatora u navedenoj klasi imaju asimptotski maksimalan rast. Posljedično, to opovrgava Maz'yinu slutnju o asimptotski strogim ocjenama normi takvih multiplikatora i odgovara na pitanje Dragičevića, Petermichl i Volberga vezano uz stroge donje ocjene za konkretni multiplikator u navedenoj klasi.

U drugom se dijelu generalizira lema Christa i Kiseleva za maksimalne operatore na višeparametarsku varijantu, posljedično rješava problem višeparametarske maksimalne Fourierove restrikcije u svim dimenzijama, koji je prije ove disertacije bio poznat samo u dvije dimenzije iz rada Müllera, Riccija i Wrighta, te dokazuje višeparametarsku verziju Menshov–Paley–Zygmundovog teorema za višedimenzionalnu Fourierovu transformaciju.

Disertacija je organizirana na sljedeći način.

U uvodnom poglavlju ("Introduction") čitatelja se uvodi u kontekst problema kojima se disertacija bavi i navode se poznati rezultati.

U poglavlju 1 ("Preliminaries") uvodi se notacija i prezentira kratak uvod u poznate rezultate koji će biti korišteni u dokazima u nastavku.

U poglavlju 2 ("Powers of homogeneous unimodular multipliers"), koje je bazirano na radu [8], dokazuju se gornje ocjene za norme Fourierovih multiplikatora pridruženih unimodularnim homogenim simbolima reda 0 i pokazuje se da su ocjene stroge u parnim dimenzijama, čime se posljedično opovrgava Maz'yina slutnja u parnim dimenzijama. U nastavku se dokazuju stroge donje ocjene za multiplikator iz rada Dragičevića, Petermichl i Volberga i odgovara na njihovo pitanje.

U poglavlju 3 ("Norm growth of powers of unimodular multipliers is a generic property"), koje je bazirano na radu [6], dokazuje se da je gornja ocjena dokazana u prethodnom poglavlju optimalna za generički multiplikator u svim dimenzijama većim ili jednakim dva i posljedično opovrgava Maz'yina slutnja u svim dimenzijama.

U poglavlju 4 ("Multi-parameter maximal Fourier restriction"), koje je bazirano na radu [7], generalizira se lema Christa i Kiseleva za maksimalne operatore na višeparametarsku varijantu, posljedično se rješava problem višeparametarske Fourierove restrikcije u svim dimenzijama i dokazuje se višeparametarska verzija Menshov–Paley–Zygmundovog teorema za Fourierovu transformaciju u više dimenzija.

Ključne riječi: Fourierovi multiplikatori, singularni integrali, Fourierova restrikcija, maksimalne ocjene

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Introduction

The thesis is thematically divided into two parts. The first part is based on papers [8] and [6] and studies the Fourier multiplier operators with homogeneous symbols. The second part is based on paper [7] and is concerned with the study of the maximal Fourier restriction. Since both Fourier multipliers and Fourier restriction are considered central topics of harmonic analysis, we include a very brief history of problems closely related to each.

The study of Fourier multipliers, at least in the context of the Fourier series, can be traced back to at least as early as the work of J. Marcinkiewicz [47]. That study and the study of more general singular integral operators gained momentum in the 1960s, from which period we mention the works of Calderón and Zygmund [10], Mikhlin [51], Hörmander [38] and Stein [62]. The so-called Calderon–Zygmund theory of singular integrals began to take shape and one can find the basic results from that time in Stein's book [64].

In the 1970s, Stein's observation (recorded by Fefferman in [21]) that for some $p > 1$ one can meaningfully restrict Fourier transform of any $L^p(\mathbb{R}^d)$ function to the sphere $\mathbb{S}^{d-1}$, even though it is the set of measure 0, initiated the study of Fourier restriction problem. The aforementioned paper by Fefferman already made connections between the problem of Fourier restriction and the so-called Bochner–Riesz multipliers, while Strichartz [69] used the progress on the restriction problem to prove estimates for the solution of the Schrödinger partial differential equation and from that point on the studies of these three problems have been closely related. Notably, Tao [71] showed that the Bochner–Riesz multiplier problem implies the Fourier restriction problem for the sphere.

Soon after the initial observation by Stein, Fefferman [22] also discovered that the Besicovitch set from geometric measure theory can be combined with wave packets to disprove the ball multiplier conjecture (which is the extreme case of the Bochner–Riesz multiplier). This connection was used by Bourgain [5] to improve the range of exponents in which the restriction problems are known to hold, but the full range of estimates in the Fourier restriction problem and Bochner–Riesz multiplier is still the topic of an active area of research. The progress on the Fourier restriction problem until 2004 was summarized by Tao in [73]. The current best range of exponents for the restriction problem was proved by Guth [33, 34], who introduced the polynomial partitioning method in the problem of Fourier restriction, and by Hickman and Rogers [35] who improved the techniques in higher dimensions.

We can now turn back to the problems studied in the dissertation. The most studied Fourier multiplier operator is the Hilbert transform, the operator defined on $L^2(\mathbb{R})$ with $Hf := (-i\,\mathrm{sgn}(\cdot)\widehat{f})^\vee$. Natural generalizations of the Hilbert transform to higher dimensions are the multipliers associated with homogeneous functions of degree 0 and such operators were extensively studied by Calderon and Zygmund in [10], L. Hörmander in [38], S. Mikhlin in [51] and by Maz'ya and Haĭkin in [49].

One concrete example of the multiplier that falls in the aforementioned class is the class of Riesz transforms, the sequence of operators defined on $L^2(\mathbb{R}^d)$ with $R_i f := (\xi_i/|\xi|\widehat{f}(\xi))^\vee$, $i = 1, 2, \ldots, d$. The $L^p \to L^p$ boundedness for $p \in (1, \infty)$ of the Riesz multipliers falls under the scope of the aforementioned Calderon–Zygmund theory, but sharp $L^p \to L^p$ norm estimates for the Riesz transform were determined much later by Iwaniec and Martin [40]. Iwaniec and Martin also reduced the estimates for a wider class of operators to estimates for powers of the complex Riesz transform, the operator defined on $L^2(\mathbb{R}^2)$ using Riesz transforms as $R_{\mathbb{C}} f := (R_2 + iR_1)f$. However, the question of sharp estimates of $k \mapsto \|R_{\mathbb{C}}^k\|_{L^p \to L^p}$ asymptotically in $k \to \infty$ and $p \to 1$ remained an open question. This question was addressed by Dragičević, Petermichel, and Volberg [20], and Dragičević [18], who proved the asymptotically sharp estimates for even powers of the complex Riesz transform. A complete asymptotically sharp solution for all powers was given by Carbonaro, Dragičević, and Kovač [11].

In the first part of the thesis, we study the vast generalization of the problem of asymptotically sharp estimates of complex Riesz transform. We will study sharp $L^p \to L^p$ estimates of powers of Fourier multipliers associated with arbitrary unimodular homogeneous symbols of degree 0. This question was initially studied by Maz'ya and Haĭkin in [49] and the question of the asymptotically sharp estimates of such symbols is posed as Problem 15 on Maz'ya's list of 75 open problems in analysis [48]. We will also study the problem of Dragičević, Petermichl and Volberg from [20] regarding the asymptotically sharp estimate in both the exponent and $p \in (1,\infty)$ of one concrete multiplier falling in the aforementioned class.

The second part of the thesis deals with the so-called maximal Fourier restriction. The aforementioned historical work on the Fourier restriction means that for certain surfaces S and exponents p,q the estimate $\|\widehat{f}\|_{L^q(S)} \le C\|f\|_{L^p(\mathbb{R}^d)}$ holds for all Schwartz functions f. Since the Schwartz space is dense in every $L^p(\mathbb{R}^d)$, by a standard result in operator theory, there exists the unique extension of the operator $\mathscr{R}f := \widehat{f}|_S$ to the whole $L^p(\mathbb{R}^d)$. Furthermore, if we take any mollifier χ (a function in $C_c^\infty(\mathbb{R}^d)$ such that $\int \chi = 1$) and denote $\chi_\varepsilon := \varepsilon^{-d}\chi(\cdot/\varepsilon)$, then the observation

$$\widehat{f} * \chi_\varepsilon = \widehat{f\chi(\varepsilon\cdot)},$$

together with the restriction estimate applied to the function $f\chi(\varepsilon\cdot)$, implies that $\widehat{f} * \chi_\varepsilon \xrightarrow{L^p} \mathscr{R}f$. A natural question is whether the almost everywhere convergence holds. From the classical proof of Lebesgue's differentiation theorem (see [64, §1, Theorem 1]) using Hardy–Littlewood maximal function we know that for the proof of the almost convergence it is useful (and by Stein's maximal principle [61] sometimes necessary), to prove the boundedness of the related maximal function, so the question of the almost everywhere convergence of the aforementioned convolutions is called maximal Fourier restriction.

The study of the maximal Fourier restriction problem was initiated by Müller, Ricci, and Wright [52], who proved the almost everywhere convergence in two dimensions by proving the boundedness of a certain two-parameter maximal operator using techniques from the proofs of two-dimensional restriction theorems of Carleson and Sjölin [13] and Sjölin [58]. Later on, Ramos [54, 55], Jesurum [41], and Fraccaroli [26] built upon the suggested method, but the proofs were restricted to either two-dimensional problems or very special curves. A different approach was suggested by Vitturi [76], who proved the maximal Fourier restriction in the special case of the sphere $\mathbb{S}^{d-1}$ in dimension $d \ge 3$ and

Kovač and Oliveira e Silva [44] followed the approach to obtain the variational restriction theorem. Finally, Kovač [43] showed that, under very mild restrictions, in all dimensions, the Fourier restriction theorem implies the maximal Fourier restriction, basically giving a definite answer to the maximal Fourier restriction. However, after the work of Kovač [43], an interesting question regarding the analogue of the multi-parameter maximal function defined by Müller, Ricci, and Wright in [52] remained open. The second part of the thesis studies the multi-parameter maximal estimates and their application to maximal Fourier restriction.

In Chapter 1 we introduce notation and give quick revision of the main tools used in the thesis that are usually not covered in a standard graduate analysis course or a book like [25].

In Chapter 2, which is based on paper [8], we prove the upper bound for the L^p norms of the powers of 0-homogeneous unimodular multipliers and prove that the bound is sharp in even dimensions, disproving the conjecture of Maz'ya from [48] in even dimensions as a corollary. We also answer the question from the work of Dragičević, Petermichl and Volberg [20] regarding one concrete multiplier falling in the aforementioned class.

In Chapter 3, which is based on paper [6], we prove that the powers of a generic 0-homogeneous unimodular symbol in all dimensions greater than one attain the asymptotically maximal L^p norm growth given by the upper bound from the previous chapter, implying that the upper bound from the previous chapter is sharp in all dimensions, thus disproving the conjecture of Maz'ya in all dimensions by a generic symbol in the class.

In Chapter 4, which is based on paper [7], we prove the multiparameter version of the Christ–Kiselev lemma from [15] and as a consequence, we prove that the Fourier restriction estimate implies the multi-parameter Fourier restriction, thus extending the results from [43] and [52].

1. PRELIMINARIES

1.1. NOTATION AND TERMINOLOGY

The content of this section is usually taught in a classical graduate analysis course so we omit full definitions and just fix the notation. The reader can find all definitions in a classical book like [25].

1.1.1. General notation

The *imaginary unit* will be denoted by $\mathbb{i}$. We say that a function $f : X \to \mathbb{C}$ is *unimodular* if $|f(x)| = 1$ for all $x \in X$. Any *logarithm* is having e as its base. We use the notation $\mathbb{1}_A$ for the *indicator function* (i.e., the *characteristic function*) of a set A.

The *Euclidean norm* (i.e., the ℓ^2 *norm*) on $\mathbb{R}^n$ will be written simply as $x \mapsto |x|$, while the *dot product* (i.e., the standard inner product) of $x, y \in \mathbb{R}^n$ is denoted $x \cdot y$ or $\langle x, y \rangle$. When working with matrices, we identify vectors in $\mathbb{R}^n$ with matrices of size $n \times 1$. For a matrix $A \in M_n(\mathbb{R})$, we write $A > 0$ to denote that it is positive definite. The *standard unit sphere* in $\mathbb{R}^n$ is

$$\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}.$$

The *surface measure* on $\mathbb{S}^{n-1}$, i.e., the $(n-1)$-dimensional *spherical measure*, is the restriction of the $(n-1)$-dimensional Hausdorff measure to Borel subsets of $\mathbb{S}^{n-1}$; it is written as σ_{n-1}.

For $x \in \mathbb{R}$ we respectively write $\lfloor x \rfloor$ and $\lceil x \rceil$ for the largest integer k such that $k \leq x$ and the smallest integer l such that $l \geq x$. If $T : X \to Y$ is a linear operator between normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, then we write $\|T\|_{X\to Y}$ for its *operator norm*, defined as

$$\|T\|_{X\to Y} := \sup_{x\in X,\, \|x\|_X=1} \|Tx\|_Y.$$

1.1.2. Asymptotic notation

We use the following variants of the Hardy–Vinogradov and the Bachmann–Landau notations. Let A and B be two complex functions on a set X. We write

$$A(x) \lesssim_P B(x) \quad \text{and} \quad B(x) \gtrsim_P A(x)$$

if the inequality $|A(x)| \leq C_P|B(x)|$ holds for every $x \in X$, with some finite constant C_P depending on a set of parameters P. Moreover,

$$A(x) \sim_P B(x)$$

if both $A(x) \lesssim_P B(x)$ and $B(x) \lesssim_P A(x)$ hold. Next, assume that A and B are, more specifically, complex functions of a single real (or complex) variable x and that $a \in \mathbb{R}\cup\{-\infty,\infty\}$ (or $a \in \mathbb{C}\cup\{\infty\}$) is a fixed point. We write

$$A(x) = O_P^{x\to a}\big(B(x)\big)$$

if $\limsup_{x\to a} |A(x)/B(x)| < \infty$ and

$$A(x) = o_P^{x\to a}\big(B(x)\big)$$

if $\lim_{x\to a} A(x)/B(x) = 0$. Here P in the subscript emphasizes that A and B are also allowed to depend on the parameters from P, but the limits need not be uniform in those parameters.

1.1.3. Functions and function spaces

The set of k-times differentiable functions on $\mathbb{R}^n$ with codomain Y will be denoted as $C^k(\mathbb{R}^n, Y)$. When the functions are complex-valued, we will omit writing the codomain. The set of $C^\infty(\mathbb{R}^n)$ functions with compact support is denoted as $C_c^\infty(\mathbb{R}^n)$ and the *Schwartz class* (see [25, 31]) is denoted as $\mathscr{S}(\mathbb{R}^n)$.

The *Lebesgue norms* $\|\cdot\|_{\mathrm{L}^p}$ and the *Lebesgue spaces* L^p are defined in a standard way; see [25,31]. We often denote the measure space in the parentheses, such as $\mathrm{L}^p(\mathbb{S}^{n-1})$, the underlying measure being understood. The notation for the measure is suppressed in the integral whenever the integrals are evaluated with respect to the Lebesgue measure on $\mathbb{R}^n$. Sometimes, we write the variable in which the L^p norm is taken in the subscript, such as $\|f(x,y)\|_{\mathrm{L}^p_x}$. We say that $p, p' \in [1,\infty]$ are *conjugated exponents* if $1/p + 1/p' = 1$ holds. We use notation p^* for the larger number between $p \in [1,\infty]$ and its conjugate exponent. The *weak Lebesgue quasinorm* $\|\cdot\|_{L^{p,\infty}}$ and *weak Lebesgue space* $L^{p,\infty}$ are defined in standard way; see [25,31]. If an operator T satisfies $\|Tf\|_{L^{q,\infty}} \leq \|f\|_{L^p}$ for all $f \in L^p$, we say that it is of *weak type* (p,q); see [25,31].

A function $f\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is *homogeneous of degree* j if $f(tx) = t^j f(x)$ holds for every $t > 0$ and every $x \in \mathbb{R}^n$. It is simply said to be *homogeneous* if it is homogeneous of degree 0. Thus, a polynomial P of n real variables $x_1, \ldots, x_n$ is homogeneous of degree j precisely when it is a linear combination of the monomials $x_1^{k_1} \cdots x_n^{k_n}$ for nonnegative integers $k_1, \ldots, k_n$ adding up to j.

For a function $f : \mathbb{R}^n \to \mathbb{R}$ we use notation ∇f to denote the $1 \times n$ matrix of a differential Df of a function f in standard coordinates $\nabla_x f := \left[\frac{\partial f}{\partial x_j}\right]_j$ and Hf to denote the $n \times n$ Hessian matrix $H_x f := \left[\frac{\partial^2 f}{\partial x_j \partial x_k}\right]_{j,k}$. We will suppress x in subscript if the variables upon which we differentiate are clear from the context.

1.1.4. Fourier transform

Through the thesis, except for the chapter 3 we use the following widespread normalization for the *Fourier transform* (see [25,31]).

For $f \in \mathrm{L}^1(\mathbb{R}^n)$, we define $\widehat{f}\colon \mathbb{R}^n \to \mathbb{C}$ by the formula

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi \mathrm{i} x \cdot \xi} \, \mathrm{d}x; \quad \xi \in \mathbb{R}^n.$$

It extends to a unitary operator on $\mathrm{L}^2(\mathbb{R}^n)$ and also to bounded linear maps from $\mathrm{L}^p(\mathbb{R}^n)$ to $\mathrm{L}^q(\mathbb{R}^n)$ for any pair of conjugated exponents $p \in [1,2]$ and $p' \in [2,\infty]$. Moreover, it extends to the space of tempered distributions F via the duality formula: $\widehat{F}(\varphi) := F(\widehat{\varphi})$ for every Schwartz function φ. We will also use the notation $\mathscr{F}(f)$ to denote the Fourier transform, while the inverse Fourier transform of a function f will be denoted as $\mathscr{F}^{-1}(f)$ or $f^{\vee}$.

In chapter 3 (and exclusively only there) we will use the following normalization for the Fourier transform that is often used in the field of partial differential equations (see [74]):

$$\widehat{f}(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} \mathrm{d}x$$

to suppress writing 2π when analyzing the phase. The properties of the Fourier transform defined this way are analogous to the ones previously commented, up to dilations and multiplications by a constant.

1.2. PROPERTIES OF SPHERICAL HARMONICS

Since spherical harmonics are usually not a part of standard graduate analysis courses, in this section we review several results on spherical harmonics that will be needed later. Basic properties are taken from the book by Stein and Weiss [67, Sections IV.2–IV.4] and the book by Stein [64, Section III.3]. For more sophisticated L^p estimates concerning spherical harmonics we will recall the work of Sogge [59, 60].

Throughout the chapter we are working in $\mathbb{R}^n$ for a fixed dimension $n \geq 2$. Let us also take a nonnegative integer j. Homogeneous polynomials in n variables of degree j that are also harmonic functions on $\mathbb{R}^n$ (i.e., satisfy the Laplace equation $\Delta u = 0$) are called *solid spherical harmonics of degree* j. Their restrictions to the sphere $\mathbb{S}^{n-1}$ are called *(surface) spherical harmonics of degree* j. Spherical harmonics of distinct degrees are mutually orthogonal and the whole space $\mathrm{L}^2(\mathbb{S}^{n-1})$ can be written as an orthogonal sum of (finite dimensional) spaces of spherical harmonics of degrees $j = 0, 1, 2, \ldots$; see [67, Chapter IV, Corollaries 2.3 and 2.4].

Spherical harmonics play important roles in describing how the Fourier transform acts on many particular types of functions and distributions. If P is a solid spherical harmonic of degree j, then

$$f(x) = P(x)e^{-\pi|x|^2} \implies \widehat{f}(\xi) = \mathbb{i}^{-j}P(\xi)e^{-\pi|\xi|^2}, \tag{1.1}$$

by [67, Chapter IV, Theorem 3.4]. The relevance of constants (2.16) comes from a formula by Bochner [4] (also see [67, Chapter IV, Theorem 4.1]): if Y is a spherical harmonic of degree j and if $0 < \alpha < n$, then

$$K(x) = Y\Big(\frac{x}{|x|}\Big)|x|^{-n+\alpha} \implies \widehat{K}(\xi) = \mathbb{i}^{-j}\gamma_{n,j,\alpha}Y\Big(\frac{\xi}{|\xi|}\Big)|\xi|^{-\alpha}. \tag{1.2}$$

However, the last function K is only locally integrable, so the Fourier transform needs to be understood as acting on the space of tempered distributions. Bochner's formula (1.2) also holds in the limiting case $\alpha = 0$ if $j \geq 1$ and the function K is interpreted as a principal value distribution

$$f \mapsto \mathrm{p.v.}\int_{\mathbb{R}^n} K(x)f(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0+}\int_{\{|x|\geq\varepsilon\}} K(x)f(x)\,\mathrm{d}x.$$

Then it reads

$$K(x) = \mathrm{p.v.}\, Y\Big(\frac{x}{|x|}\Big)|x|^{-n} \implies \widehat{K}(\xi) = \mathbb{i}^{-j}\gamma_{n,j,0} Y\Big(\frac{\xi}{|\xi|}\Big); \tag{1.3}$$

see [67, Chapter IV, Theorem 4.5]. Stein and Weiss also formulated the ultimate consequence of (1.3) as [67, Chapter IV, Theorem 4.7]: if $\Omega, \Omega_0 \in \mathrm{L}^2(\mathbb{S}^{n-1})$ have related expansions into spherical harmonics of the form

$$\Omega = \sum_{j=1}^{\infty} Y_j, \quad \Omega_0 = \sum_{j=1}^{\infty} \mathbb{i}^{-j}\gamma_{n,j,0} Y_j,$$

then

$$K(x) = \mathrm{p.v.}\, \Omega\Big(\frac{x}{|x|}\Big)|x|^{-n} \implies \widehat{K}(\xi) = \Omega_0\Big(\frac{\xi}{|\xi|}\Big). \tag{1.4}$$

Observe that

$$\gamma_{n,j,n-\alpha} = \frac{1}{\gamma_{n,j,\alpha}} \tag{1.5}$$

holds whenever the constants $\gamma_{n,j,\alpha}$ are defined. By writing

$$\log \frac{\Gamma(j/2+\alpha/2)}{(j/2)^{\alpha/2}\Gamma(j/2)} = \int_0^{\alpha/2} \Big(\psi\Big(\frac{j}{2}+t\Big) - \log\frac{j}{2}\Big)\,\mathrm{d}t$$

in terms of the digamma function $\psi = \Gamma'/\Gamma$ and using the asymptotic expansion of ψ, see [17, Eq. 5.11.2] or [1, Eq. 6.3.18], we easily conclude

$$\gamma_{n,j,\alpha} \sim_n j^{\alpha-n/2} \tag{1.6}$$

for $\alpha \in [0,n]$ and a positive integer j. Also, writing

$$\Gamma\Big(\frac{\alpha}{2}\Big) = \frac{\Gamma(\alpha/2+1)}{\alpha/2}, \quad \Gamma\Big(\frac{n-\alpha}{2}\Big) = \frac{\Gamma((n-\alpha)/2+1)}{(n-\alpha)/2},$$

we easily get

$$\gamma_{n,0,\alpha} \sim_n \frac{n}{\alpha} - 1 \tag{1.7}$$

for every $\alpha \in (0,n)$.

Let us also recall the *spherical Laplacean*, which is a particular case of the *Laplace–Beltrami operator*. In the case of the sphere $\mathbb{S}^{n-1}$ we can define $\Delta_{\mathbb{S}^{n-1}} f$ for a C^2 function $f\colon \mathbb{S}^{n-1} \to \mathbb{C}$ by applying the ordinary n-dimensional Laplace operator $\Delta = \Delta_{\mathbb{R}^n}$ to the homogeneous function $\mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{C}$, $x \mapsto f(x/|x|)$ and then restricting back to the sphere

$\mathbb{S}^{n-1}$. Spherical harmonics are eigenfunctions of $\Delta_{\mathbb{S}^{n-1}}$. More precisely, if Y_j is a spherical harmonic of degree j, then

$$\Delta_{\mathbb{S}^{n-1}} Y_j = -j(j+n-2)Y_j; \tag{1.8}$$

see [64, §III.3.1.4].

For an integer $j \geq 0$ let H_j denote the orthogonal projection onto the linear subspace of $\mathrm{L}^2(\mathbb{S}^{n-1})$ consisting of spherical harmonics of degree j (including the zero-function). Sogge [60] established the sharp estimate

$$\|H_j f\|_{\mathrm{L}^2(\mathbb{S}^{n-1})} \lesssim_{n,p} j^{\tau(n,p)} \|f\|_{\mathrm{L}^p(\mathbb{S}^{n-1})}$$

for $j \geq 1$ and $1 \leq p \leq 2$, where the exponent $\tau(n,p)$ is defined as

$$\tau(n,p) := \begin{cases} (n-1)\big(\frac{1}{p} - \frac{1}{2}\big) - \frac{1}{2} & \text{for } 1 \leq p \leq \frac{2n}{n+2}, \\ \frac{1}{2}(n-2)\big(\frac{1}{p} - \frac{1}{2}\big) & \text{for } \frac{2n}{n+2} \leq p \leq 2. \end{cases}$$

Since H_j is self-adjoint, the last estimate has its dual formulation:

$$\|H_j f\|_{\mathrm{L}^q(\mathbb{S}^{n-1})} \lesssim_{n,p} j^{\tau(n,p)} \|f\|_{\mathrm{L}^2(\mathbb{S}^{n-1})},$$

where $2 \leq q \leq \infty$ is the conjugate exponent of p. In particular,

$$\|Y\|_{\mathrm{L}^q(\mathbb{S}^{n-1})} \lesssim_{n,p} j^{\tau(n,p)} \|Y\|_{\mathrm{L}^2(\mathbb{S}^{n-1})} \tag{1.9}$$

for every spherical harmonic Y of degree $j \geq 1$.

1.3. Fourier multipliers and singular integrals

A thorough introduction to Fourier multipliers can be found in [31, 64], but we recall the basic definitions from [31] here for reader's convenience.

Given $1 \leq p < \infty$ and a complex function $m \in L^\infty(\mathbb{R}^n)$, the operator defined on the Schwartz space as

$$T_m f := (m\widehat{f})^\vee, \quad f \in \mathscr{S}(\mathbb{R}^n) \tag{1.10}$$

is called (L^p-) *Fourier multiplier* associated with symbol m if T_m can be extended to a bounded operator on $L^p(\mathbb{R}^n)$, i.e. if there exists $C > 0$ such that for all $f \in \mathscr{S}(\mathbb{R}^n)$ it satisfies

$$\|T_m f\|_{L^p} \leq C\|f\|_{L^p}.$$

On the other hand, if a linear operator T, defined on Schwartz space $\mathscr{S}(\mathbb{R}^n)$ commutes with translations and if for some $p \in (1,\infty)$ it satisfies the bound

$$\|Tf\|_{L^p} \leq C\|f\|_{L^p}, \quad f \in \mathscr{S}(\mathbb{R}^n),$$

then by theorems [31, Theorems 2.5.7, 2.5.10] we know that it is of the form (1.10). This equivalence justifies the assumptions in the definition.

If $m \in L^\infty(\mathbb{R}^n)$ is a homogeneous function of degree 0 such that $m|_{\mathbb{S}^{n-1}} \in C^\infty(\mathbb{S}^{n-1})$, by [64, §III.3.5, Theorem 6] we know that the associated operator T_m has a representation

$$T_m f = a \cdot f + Sf, \tag{1.11}$$

where a is a constant given by

$$a := \frac{1}{\sigma_{n-1}(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} m(\xi)\, \mathrm{d}\sigma_{n-1}(\xi),$$

while S is a singular integral operator defined for $f \in \mathscr{S}(\mathbb{R}^n)$ as

$$(Sf)(x) := \lim_{\varepsilon\to 0+} \int_{|y|\geq\varepsilon} f(x-y)\, \frac{\Omega(y/|y|)}{|y|^n}\, \mathrm{d}y; \quad x \in \mathbb{R}^n, \tag{1.12}$$

where $\Omega \in \mathrm{C}^\infty(\mathbb{S}^{n-1})$ is a function with mean zero, i.e., $\int_{\mathbb{S}^{n-1}} \Omega(y)\, \mathrm{d}\sigma_{n-1}(y) = 0$. Functions $m|_{\mathbb{S}^{n-1}}$ and Ω are related through expansions into spherical harmonics given in (1.4).

By classical results of the Calderón–Zygmund theory (see [64, §II.4.2, Theorem 3] or [10, Theorem 1]) we know that S extends to a bounded linear operator on $\mathrm{L}^p(\mathbb{R}^n)$ for every $p \in (1,\infty)$, and the kernel representation (1.11)–(1.12) remains valid for every $f \in \mathrm{L}^p(\mathbb{R}^n)$.

The weak $(1,1)$ estimates for singular integrals of the form (1.12) that do not depend on the smoothness of Ω were first proved in $n=2$ dimensions independently by Christ and Rubio de Francia [14] and Hofmann [37]. A higher-dimensional analogue was later shown by Seeger [57] and subsequently generalized further by Tao [72]. We will use the following theorem in the chapter 2 to establish sharp upper bounds of the powers of homogeneous unimodular multipliers.

Theorem 1.3.1 (from [57])**.** *Let $\Omega \in \mathrm{L}^1(\mathbb{S}^{n-1})$ be such that $\int_{\mathbb{S}^{n-1}} \Omega(x)\,\mathrm{d}\sigma_{n-1}(x) = 0$. If we denote $K(x) = \Omega(x/|x|)|x|^{-n}$, then the operator S_Ω defined for $f \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)$ as*

$$(S_\Omega f)(x) := \mathrm{p.\,v.} \int_{\mathbb{R}^n} f(x-y)K(y)\,\mathrm{d}y$$

satisfies the bound

$$\|S_\Omega\|_{\mathrm{L}^1(\mathbb{R}^n)\to\mathrm{L}^{1,\infty}(\mathbb{R}^n)} \lesssim_n 1 + \big\|\widehat{K}\big\|_{\mathrm{L}^\infty(\mathbb{R}^n)} + \int_{\mathbb{S}^{n-1}} |\Omega(x)| \Big(1 + \log_+\Big(\frac{|\Omega(x)|}{\|\Omega\|_{\mathrm{L}^1(\mathbb{S}^{n-1})}}\Big)\Big)\,\mathrm{d}\sigma_{n-1}(x).$$

1.4. Differential geometry and Morse theory

An introduction to differential topology and the Morse theory can be found in [36]. We repeat the basic definitions from that book here.

The only manifold, apart from the $\mathbb{R}^n$ itself, that will be needed in the thesis is the standard sphere $\mathbb{S}^{n-1}$, so when talking about manifolds, one can think of $\mathbb{S}^{n-1}$. However, since one cannot essentially reduce the complexity of the definitions in this special case, we state the definitions for the general manifold that is already embedded into $\mathbb{R}^n$.

We say that $M \subset \mathbb{R}^n$ is a k-dimensional *manifold* if there exists an open cover $\mathscr{U} = \{U_i\}_{i\in I}$ of M (in the induced topology from $\mathbb{R}^n$) such that for each $i \in I$ there is a map $\psi_i : U_i \to \mathbb{R}^k$ which maps U_i homeomorphically onto an open subset of $\mathbb{R}^k$. We call (ψ_i, U_i) a *chart*; the set of charts $\{\psi_i, U_i\}$ is an *atlas*. For $1 \leq r \leq \infty$, we say that two charts $(\psi_i, U_i), (\psi_j, U_j)$ have C^r *overlap* if the function

$$\psi_j \psi_i^{-1} : \psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$$

is of differentiability class C^r and $\psi_i \psi_j^{-1}$ is also C^r. The definition makes sense because both $\psi_i(U_i \cap U_j)$ and $\psi_j(U_i \cap U_j)$ are open sets in $\mathbb{R}^k$. If every two charts have C^r overlap, we say that the atlas Ψ is of class C^r. In that case, there exists a maximal C^r atlas α on M (which is the set of all charts that have C^r overlap with every chart in Ψ) and we say that the pair (M, α) is a *manifold of class* C^r.

In the remainder of the section, the manifold M is always a C^∞ manifold to avoid problems with the regularity of charts when talking about functions.

For $1 \leq r \leq \infty$ and a function $f : M \to \mathbb{R}$ we say that it *is of class* $C^r(M, \mathbb{R})$ if for every $x \in M$ there exists a chart ψ at x such that the function $f \circ \psi^{-1} : \psi(U) \to \mathbb{R}$ is of class C^r.

We say that a function $f \in C^1(M, \mathbb{R})$ has a *critical point* at $x \in M$ if there exists a local chart ψ at x such that

$$D(f \circ \psi^{-1})(\psi(x)) = 0.$$

For a function $f \in C^2(M, \mathbb{R})$, we say that the critical point x is *nondegenerate* if also

$$\det(H(f \circ \psi^{-1}))(\psi(x)) \neq 0.$$

Direct calculation shows that the definition is independent of a chosen chart around a critical point. In the case of $M = \mathbb{R}^n$, the definition of nondegenerate critical point $x \in \mathbb{R}^n$ reduces to $\nabla f(x) = 0$ and $\det Hf(x) \neq 0$.

When M is a general manifold, one can endow the set $C^r(M,\mathbb{R})$ with either weak or strong (also called Whitney) topology, but the two topologies coincide when M is compact. As we will need the genericity of Morse functions just for $M = \mathbb{S}^{n-1}$, we give the definitions of the Whitney topology by weak topology to avoid technical details needed for strong Whitney topology on noncompact manifolds.

Let M be a compact C^∞ manifold and $r \in \mathbb{N}$. For $f \in C^r(M,\mathbb{R})$, a chart (ψ,U), compact set $K \subset U$ and $\varepsilon > 0$, we define neighborhood base at f:

$$\mathscr{N}^r(f;(\psi,U),K,\varepsilon) := \left\{ g \in C^r(M,\mathbb{R}) : \forall k = 0,\dots,r \quad \sup_{x\in\phi(K)} |D^k(f\circ\psi^{-1}) - D^k(g\circ\psi^{-1})| < \varepsilon \right\}$$

For $r \in \mathbb{N}$, the *Whitney topology* on $C^r(M,\mathbb{R})$ is topology generated by the given neighborhood bases. *Whitney topology* on $C^\infty(M,\mathbb{R})$ is union of the topologies induced by the inclusion maps $C^\infty(M,\mathbb{R}) \to C^r(M,\mathbb{R})$ for $r \in \mathbb{N}$.

A function $f \in C^2(M,\mathbb{R})$ is called a *Morse function* if all critical points of f are nondegenerate. The main theorem about Morse functions that we will use in the thesis is the following.

Theorem 1.4.1 (§6, Theorem 1.2 form [36])**.** *The set of Morse functions in $C^\infty(\mathbb{S}^{n-1},\mathbb{R})$ is a dense open set in the Whitney topology on $C^\infty(\mathbb{S}^{n-1},\mathbb{R})$.*

1.5. OSCILLATORY INTEGRALS

One can find a thorough introduction to oscillatory integrals in [65, §8], but we state here the theorems that we will use in the thesis.

The theory of oscillatory integrals studies the question of asymptotics of

$$\int_{\mathbb{R}^n} e^{i\lambda\Phi(\xi)}\psi(\xi)d\xi, \quad \lambda\to\infty, \tag{1.13}$$

where $\psi\in C_c^\infty(\mathbb{R}^n)$ is a bump function and $\Phi:\mathbb{R}^n\to\mathbb{R}$ is a smooth real-valued function. The method of nonstationary phase [65, §8, Proposition 4] says that if a function Φ does not have a critical point on the support of ψ, then the expression 1.13 is $O^{\lambda\to\infty}_{\Phi,\psi,N}(\lambda^{-N})$ for any $N\in\mathbb{N}$. On the other hand, if Ψ has a nondegenerate critical point ξ_0, the method of stationary phase [65, §8, Proposition 6] gives the precise asymptotics of the integral when ψ has support in a sufficiently small neighbourhood around ξ_0. Decomposing the integral 1.13 into the part in the sufficiently small neighbourhood of ξ_0 and the complement using a smooth mollifier, one can apply the method of stationary phase in the neighbourhood and the method of nonstationary phase on the complement to obtain the following theorem.

Theorem 1.5.1 (§8 Propositions 4, 6 from [65])**.** *Let $\psi\in C_c^\infty(\mathbb{R}^n,\mathbb{R})$ and let $\Phi\in C^2(\mathbb{R}^n,\mathbb{R})$ be a function that has a unique critical point on the support of the function ψ, call it ξ_0. If ξ_0 is a nondegenerate critical point of Φ, the following holds*

$$\int_{\mathbb{R}^n} e^{i\lambda\Phi(\xi)}\psi(\xi)d\xi = Ce^{i\lambda\Phi(\xi_0)}\lambda^{-\frac{n}{2}} + O^{\lambda\to\infty}_{\Phi,\psi,n}(\lambda^{-\frac{n}{2}-1})$$

where $C=\psi(\xi_0)(2\pi)^{\frac{n}{2}}e^{\frac{i\pi}{4}\operatorname{sgn}(H\Phi(\xi_0))}|\det H\Phi(\xi_0)|^{-\frac{1}{2}}$ and $\operatorname{sgn}(H\Phi(\xi_0))$ denotes the number of positive eigenvalues minus the number of negative eigenvalues of the matrix $H\Phi(\xi_0)$.

Part I

Homogeneous Fourier multipliers

2. POWERS OF HOMOGENEOUS UNIMODULAR MULTIPLIERS

The content of this chapter is based on the paper [8].

We study Fourier multiplier operators associated with symbols $\xi \mapsto \exp(\mathbb{i}\lambda\Phi(\xi/|\xi|))$, where λ is a real number and $\Phi \in C^\infty(S^{n-1})$. For $1 < p < \infty$ we investigate asymptotic behavior of norms of these operators on $\mathrm{L}^p(\mathbb{R}^n)$ as $|\lambda| \to \infty$. In the section 2.2 we show that these norms are always $O((p^*-1)|\lambda|^{n|1/p-1/2|})$. In the section 2.3 we prove the preparation theorem for lower bounds and in the section 2.4 we show that this bound is sharp in for even integers n. In particular, this gives a negative answer to a question posed by Maz'ya. In the section 2.5 we study the two-dimensional Riesz group, given by the symbols $r\exp(\mathbb{i}\varphi) \mapsto \exp(\mathbb{i}\lambda\cos\varphi)$ and show that their L^p norms are comparable to $(p^*-1)|\lambda|^{2|1/p-1/2|}$ for large $|\lambda|$, solving affirmatively a problem suggested in the work of Dragičević, Petermichl, and Volberg.

2.1. INTRODUCTION AND MAIN RESULTS

Consider Fourier multiplier operators T_Φ^λ associated with symbols of the form

$$m_\Phi^\lambda(\xi) := e^{\mathbb{i}\lambda\Phi(\xi/|\xi|)}; \quad \xi \in \mathbb{R}^n \setminus \{\mathbf{0}\},$$

i.e., T_Φ^λ acts on Schwartz functions f on the Fourier side as

$$\widehat{(T_\Phi^\lambda f)}(\xi) = m_\Phi^\lambda(\xi)\widehat{f}(\xi).$$

Here, $\Phi \in \mathrm{C}^\infty(\mathbb{S}^{n-1})$ is a real-valued *phase function* on the unit sphere, while λ is a real parameter. Form section 1.3 we know that T_Φ^λ has a representation

$$T_\Phi^\lambda f = a_\Phi^\lambda f + S_\Phi^\lambda f, \tag{2.1}$$

where a_Φ^λ is a constant given by

$$a_\Phi^\lambda := \frac{1}{\sigma_{n-1}(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} m_\Phi^\lambda(\xi)\,\mathrm{d}\sigma_{n-1}(\xi),$$

while S_Φ^λ is a singular integral operator defined as

$$(S_\Phi^\lambda f)(x) := \lim_{\varepsilon\to 0+} \int_{|y|\geq\varepsilon} f(x-y)\,\frac{\Omega_\Phi^\lambda(y/|y|)}{|y|^n}\,\mathrm{d}y; \quad x\in\mathbb{R}^n \tag{2.2}$$

for some $\Omega_\Phi^\lambda \in \mathrm{C}^\infty(\mathbb{S}^{n-1})$ with mean zero, i.e., $\int_{\mathbb{S}^{n-1}} \Omega_\Phi^\lambda(y)\,\mathrm{d}\sigma_{n-1}(y) = 0$. Here σ_{n-1} denotes the $(n-1)$-dimensional spherical measure. In our case we clearly have

$$|a_\Phi^\lambda| \leq 1. \tag{2.3}$$

Again from section 1.3 we know that T_Φ^λ extends to a bounded linear operator on $\mathrm{L}^p(\mathbb{R}^n)$ for every $p\in(1,\infty)$.

For each $p\in(1,\infty)$ we thus arrived at a one-parameter group of bounded linear operators $(T_\Phi^\lambda)_{\lambda\in\mathbb{R}}$ on the Banach space $\mathrm{L}^p(\mathbb{R}^n)$. Plancherel's theorem and unimodularity of m_Φ^λ give

$$\|T_\Phi^\lambda\|_{\mathrm{L}^2(\mathbb{R}^n)\to\mathrm{L}^2(\mathbb{R}^n)} = 1, \tag{2.4}$$

while for $p\neq 2$ it makes sense to investigate asymptotic behavior of the L^p norms of T_Φ^λ as $|\lambda|\to\infty$. The present chapter is motivated in part by the following question by Vladimir Maz'ya, formulated as Problem 15 on his list of 75 open problems [48].

Problem 2.1.1 (from [48, Subsection 4.2]). Prove or disprove the estimate

$$\|T_\Phi^\lambda\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \leq C_{n,p,\Phi}|\lambda|^{(n-1)|1/p-1/2|}, \tag{2.5}$$

where $|\lambda|\geq 1$ and $1<p<\infty$, while the constant $C_{n,p,\Phi}$ depends on n, p, and Φ.

In Theorem 2.1.2 below we find the largest possible growth in $|\lambda|$ of the L^p norms of multiplier operators T_Φ^λ in every even number of dimensions n. It will turn out that the answer to Problem 2.1.1 is negative in all even-dimensional Euclidean spaces $\mathbb{R}^n$.

Moreover, we will also be concerned with sharp dependence of the constant $C_{n,p,\Phi}$ on the exponent p.

The origins of Problem 2.1.1 trace back to the papers by Maz'ya and Haĭkin [49, 50] on rather general multiplier theorems. Specific operators T_{Φ}^{λ} with the particular phase

$$\Phi(\xi) = \xi_1; \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{S}^{n-1} \tag{2.6}$$

appear in the analysis of the Navier–Stokes equations; see [27, §2], [28, §4], [29, Eq. (1.3)], or [24, Eq. (23)]. This phase leads to a one-parameter uniformly continuous operator group $(T_{\Phi}^{\lambda})_{\lambda\in\mathbb{R}}$ on every $\mathrm{L}^p(\mathbb{R}^n)$, $1 < p < \infty$, called the *Riesz group*. Its infinitesimal generator is simply the *Riesz transform*,

$$(\widehat{R_1 f})(\xi) = -\mathbb{i}\frac{\xi_1}{|\xi|}\widehat{f}(\xi); \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

If $n = 2$, then (2.6) becomes simply the phase $\Phi(e^{\mathbb{i}\varphi}) = \cos\varphi$, which yields the two-dimensional symbol

$$m_{\cos}^{\lambda}(re^{\mathbb{i}\varphi}) := e^{\mathbb{i}\lambda\cos\varphi}; \quad r \in (0,\infty),\ \varphi \in \mathbb{R}. \tag{2.7}$$

One-dimensional case of estimate (2.5) is easily seen to hold, as T_{Φ}^{λ} is always a bounded linear combination of the identity and the Hilbert transform, so it satisfies L^p estimates that are independent of λ. In higher dimensions, the Hörmander-Mihlin theorem (see [31, Theorem 6.2.7]) gives a weak L^1 bound

$$\|T_{\Phi}^{\lambda}\|_{\mathrm{L}^1(\mathbb{R}^n)\to\mathrm{L}^{1,\infty}(\mathbb{R}^n)} \leq C_{n,\Phi}|\lambda|^{\lfloor n/2\rfloor+1},$$

which can then be interpolated with the L^2 identity (2.4) and dualized to deduce

$$\|T_{\Phi}^{\lambda}\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \leq C_{n,\Phi,p}|\lambda|^{(2\lfloor n/2\rfloor+2)|1/p-1/2|}. \tag{2.8}$$

This makes one suspect that the sharp exponent of $|\lambda|$ on the average grows by $|1/p-1/2|$ as we increase the number of dimensions n by 1. Thus, inequality (2.5) is actually a reasonable guess.

Let us now formulate the main result of this chapter. For every $p \in (1,\infty)$ we denote $p^* := \max\{p, p/(p-1)\}$.

Theorem 2.1.2.

(a) Fix an integer $n \geq 2$ and a real-valued phase function $\Phi \in \mathrm{C}^\infty(\mathbb{S}^{n-1})$. There is a finite constant $C_{n,\Phi}$ such that for every exponent $p \in (1,\infty)$ and every $\lambda \in \mathbb{R}$ satisfying $|\lambda| \geq 1$ we have

$$\|T_\Phi^\lambda\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \leq C_{n,\Phi}\,(p^*-1)\,|\lambda|^{n|1/p-1/2|} \tag{2.9}$$

and

$$\|T_\Phi^\lambda\|_{\mathrm{L}^1(\mathbb{R}^n)\to\mathrm{L}^{1,\infty}(\mathbb{R}^n)} \leq C_{n,\Phi}\,|\lambda|^{n/2}. \tag{2.10}$$

(b) Fix an <u>*even*</u> *integer $n \geq 2$. There exist a real-valued phase function $\Phi \in \mathrm{C}^\infty(\mathbb{S}^{n-1})$ and a constant $c_{n,\Phi} > 0$ such that for every exponent $p \in (1,\infty)$ and every nonzero integer k we have*

$$\|T_\Phi^k\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \geq c_{n,\Phi}\,(p^*-1)\,|k|^{n|1/p-1/2|} \tag{2.11}$$

and

$$\|T_\Phi^k\|_{\mathrm{L}^1(\mathbb{R}^n)\to\mathrm{L}^{1,\infty}(\mathbb{R}^n)} \geq c_{n,\Phi}\,|k|^{n/2}. \tag{2.12}$$

In particular, notice that (2.9) improves the "cheap" bound (2.8), while (2.11) disproves the conjectured estimate (2.5) in all even dimensions $n \geq 2$. Let us remark that (2.11) easily extends to non-integer values of k using the group property of the operators T_Φ^λ, but at the cost of possibly losing sharp dependence on p; cf. the comments in [11]. Techniques that we use also allow us to obtain weak L^1 estimates (2.10) and (2.12).

While (2.10) and (2.4) will immediately imply (2.9), there are also other ways to establish upper L^p bounds of that form. The number $2\lfloor n/2\rfloor + 2$ in the exponent on the right hand side of (2.8) can be easily lowered to anything strictly larger than n by considering more sophisticated versions of the Hörmander–Mihlin theorem, such as those in [9, 32, 56], but the optimal exponent is trickier. Shortly after the first preprint of the present chapter was made public, Stolyarov [68] showed us an interpolation argument that gives the same sharp exponent in (2.9). However this argument does not seem to give the sharp order of the constant in terms of p and it misses the weak endpoint (2.10).

Moreover, Stolyarov [68] independently showed lower estimates

$$\|T_\Phi^\lambda\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \geq c_{n,\Phi,p}\,|\lambda|^{n|1/p-1/2|}$$

for a particular choice of the phase Φ in both even and odd dimensions $n \geq 2$ using different techniques from ours. Just as for the upper bound, we do not see how to modify his approach to give sharp dependence on p of the constant $c_{n,\Phi,p}$ in the above lower bound.

Two-dimensional multiplier operators $T^{\lambda}_{\cos}$ with very concrete symbols (2.7) were already studied by Dragičević, Petermichl, and Volberg in [20]. Their paper, which might have been overlooked in [48], claims bounds of the form

$$c_{\delta}\,(p^{*}-1)\,|k|^{2|1/p-1/2|-\delta} \leq \|T^{k}_{\cos}\|_{\mathrm{L}^{p}(\mathbb{R}^{2})\to\mathrm{L}^{p}(\mathbb{R}^{2})} \leq C\,(p^{*}-1)\,|k|^{2|1/p-1/2|} \tag{2.13}$$

for every $\delta > 0$, every $p \in (1,\infty)$, and every nonzero integer k. The lower bound in (2.13) is sketched in the proof of [20, Theorem 6] and it already disproves the estimate (2.5) in $n = 2$ dimensions. Since the authors of [20] remark that they "do not know how to get rid of δ" in (2.13), an optimal growth of the L^{p} norms of $T^{k}_{\cos}$ is an interesting separate problem, which we fully address in the following theorem.

Theorem 2.1.3. *Let $T^{\lambda}_{\cos}$ be the Fourier multiplier operator associated with the symbol (2.7). There exist constants $0 < c \leq C < \infty$ such that for every exponent $p \in (1,\infty)$ and every $\lambda \in \mathbb{R}$ satisfying $|\lambda| \geq 1$ we have*

$$c\,(p^{*}-1)\,|\lambda|^{2|1/p-1/2|} \leq \|T^{\lambda}_{\cos}\|_{\mathrm{L}^{p}(\mathbb{R}^{2})\to\mathrm{L}^{p}(\mathbb{R}^{2})} \leq C\,(p^{*}-1)\,|\lambda|^{2|1/p-1/2|} \tag{2.14}$$

and

$$c\,|\lambda| \leq \|T^{\lambda}_{\cos}\|_{\mathrm{L}^{1}(\mathbb{R}^{2})\to\mathrm{L}^{1,\infty}(\mathbb{R}^{2})} \leq C\,|\lambda|. \tag{2.15}$$

The upper estimates in Theorem 2.1.2, and thus also those in Theorem 2.1.3, will be established in Section 2.2. We use weak L^1 estimates for singular integrals in terms of the size of the kernel alone and no smoothness assumptions imposed; see the series of papers by Christ and Rubio de Francia [14], Hofmann [37], Seeger [57], and Tao [72]. That way we only need to bound $\|\Omega^{\lambda}_{\Phi}\|_{\mathrm{L}^{2}(\mathbb{S}^{n-1})}$ for the singular kernel appearing in the singular integral part (2.2). That is achieved by generalizing the two-dimensional approach of Dragičević, Petermichl, and Volberg [20, Theorem 5] to higher dimensions: replacing the Fourier series expansion by the expansion into spherical harmonics, and replacing one-dimensional derivatives with powers of the spherical Laplacean.

The lower estimates for the L^p norms of operators T_Φ^λ are more substantial results of this chapter. To some extent we generalize an approach by Carbonaro, Dragičević, and Kovač [11]. That paper was only concerned with asymptotics for powers of a particular two-dimensional Fourier multiplier with the complex symbol $\xi \mapsto \xi/|\xi|$. Here we develop a convenient way of bounding L^p norms of more general Fourier multipliers from below by merely choosing two particular spherical functions, u and v, with mutually related expansions into spherical harmonics. Let us already state the result, referring the reader to Subsection 1.2 for a review of spherical harmonics. For an integer $j \geq 0$ and a real parameter $\alpha \in [0,n]$ denote the constants

$$\gamma_{n,j,\alpha} := \pi^{n/2-\alpha} \frac{\Gamma((j+\alpha)/2)}{\Gamma((j+n-\alpha)/2)}. \tag{2.16}$$

Theorem 2.1.4. *Let $p \in [1,2]$ and $q \in [2,\infty]$ be mutually conjugate exponents and let m be a bounded homogeneous Borel-measurable symbol on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Take a sequence $(Y_j)_{j=0}^{\infty}$ such that:*

(a) each Y_j is from the linear space of spherical harmonics on $\mathbb{S}^{n-1}$ of degree j;

(b) the series $\sum_{j=0}^{\infty} Y_j$ converges in $\mathrm{L}^q(\mathbb{S}^{n-1})$ to some function u;

(c) the orthogonal series $\sum_{j=0}^{\infty} \mathbb{i}^{-j} \gamma_{n,j,n/p} Y_j$ converges in $\mathrm{L}^2(\mathbb{S}^{n-1})$ to some function v.

If $p > 1$, $q < \infty$, then the Fourier multiplier operator T_m associated with m satisfies the bound

$$\|T_m\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \geq \frac{\gamma_{n,0,n/q}}{\sigma(\mathbb{S}^{n-1})^{1/p}} \frac{|\langle m,v\rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})}|}{\|u\|_{\mathrm{L}^q(\mathbb{S}^{n-1})}} \tag{2.17}$$

$$\geq c_n\,(q-1)\,\frac{|\langle m,v\rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})}|}{\|u\|_{\mathrm{L}^q(\mathbb{S}^{n-1})}}, \tag{2.18}$$

while in the endpoint case $p = 1$, $q = \infty$ we have

$$\|T_m\|_{\mathrm{L}^1(\mathbb{R}^n)\to\mathrm{L}^{1,\infty}(\mathbb{R}^n)} \geq \frac{c}{n} \frac{|\langle m,v\rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})}|}{\|u\|_{\mathrm{L}^\infty(\mathbb{S}^{n-1})}}. \tag{2.19}$$

Here, $c_n > 0$ is a constant depending on n, while $c > 0$ is an absolute constant.

Theorem 2.1.4 combined with some guessing of appropriate spherical functions u and v is a useful tool for proving lower bounds for multipliers with homogeneous unimodular symbols. In particular, it will be a crucial ingredient in the proof of the part (b) of Theorem 2.1.2 and in the proof of Theorem 2.1.3; see Sections 2.4 and 2.5, respectively. The proof of Theorem 2.1.4 in Section 2.3 will, in turn, build on the approach from [11, Section 6], but with additional complications arising from arbitrary symbols and higher dimensions.

Estimate (2.17) is tailored to exact constants and we believe that it is, in fact, absolutely sharp for many multipliers. For instance, if one considers the two-dimensional complex symbol $m(\xi)=\overline{\xi}/\xi$, $\xi\in\mathbb{C}$, then the underlying operator T_m is the Ahlfors–Beurling operator. By choosing $u=m$ on $\mathbb{S}^1$, the inequality (2.17) simplifies to $\|T_m\|_{\mathrm{L}^p(\mathbb{C})\to\mathrm{L}^p(\mathbb{C})}\geq q-1$, which reproves the result of Lehto [46] and matches the well-known conjecture by Iwaniec [39] on the exact L^p norm of T_m. Estimate (2.17) is also believed to be sharp in the case of $m(\xi)=(\xi/|\xi|)^k$, $\xi\in\mathbb{C}$, for an integer k; see the paper [11] as this estimate generalizes [11, Theorem 6.1]. This potential sharpness of Theorem 2.1.4 can be viewed both as a virtue and as a lack of flexibility, by focusing on global and not local properties of the multiplier. In particular we do not see how to use that theorem to prove lower L^p bounds for T_Φ^λ that are simultaneously sharp in λ and p in odd dimensions $n\geq 3$.

2.2. Proof of the upper bounds in Theorem 2.1.2

Before we start discussing any proofs, let us give a brief remark on symmetries of T_Φ^λ, which needs to be kept in mind throughout the chapter.

Remark 2.2.1. In the proofs of any upper or lower L^p bounds for T_Φ^λ we can focus on the case $\lambda \geq 1$ and $p \leq 2$ only. This fact is an immediate consequence of the duality of L^p spaces and

$$\langle T_\Phi^\lambda f, g\rangle_{\mathrm{L}^2(\mathbb{R}^n)} = \langle f, T_\Phi^{-\lambda} g\rangle_{\mathrm{L}^2(\mathbb{R}^n)} = \langle T_\Phi^\lambda \widetilde{g}, \widetilde{f}\rangle_{\mathrm{L}^2(\mathbb{R}^n)},$$

where $\widetilde{f}(x) = \overline{f(-x)}$, $\widetilde{g}(x) = \overline{g(-x)}$.

The upper bound (2.10) in Theorem 2.1.2 is reduced to the weak L^1 bound for the singular integral given in (2.2), by using representation (2.1) and an obvious bound (2.3). Estimate (2.9) then follows from the Marcinkiewicz interpolation theorem [31, Theorem 1.3.2], which interpolates between the endpoint L^1 case and the trivial L^2 case (2.4), followed by duality observations in Remark 2.2.1.

For the $L^1 \to L^{1,\infty}$ bound, we use the theorem 1.3.1. Since the last integral in the theorem 1.3.1 is difficult to compute for a kernel that is defined implicitly via the corresponding multiplier symbol, we find it convenient that the whole expression on the right hand side is bounded by $\|\Omega\|_{\mathrm{L}^2(\mathbb{S}^{n-1})}$. Indeed, if we define

$$A_0 := \Big\{x \in \mathbb{S}^{n-1} : \frac{|\Omega(x)|}{\|\Omega\|_{\mathrm{L}^1}} \leq 1\Big\} \quad \text{and} \quad A_k := \Big\{x \in \mathbb{S}^{n-1} : 2^{k-1} < \frac{|\Omega(x)|}{\|\Omega\|_{\mathrm{L}^1}} \leq 2^k\Big\}; \quad k \geq 1,$$

then Chebyshev's inequality implies $\sigma_{n-1}(A_k) \lesssim 2^{-k}$. Thus, bounding the logarithm with the upper bound of the function Ω on the set A_k and using the Cauchy–Schwarz inequality, it follows that

$$\int_{\mathbb{S}^{n-1}} |\Omega(x)|\Big(1 + \log_+\Big(\frac{|\Omega(x)|}{\|\Omega\|_{\mathrm{L}^1(\mathbb{S}^{n-1})}}\Big)\Big)\,\mathrm{d}\sigma_{n-1}(x) \lesssim \sum_{k=0}^{\infty}\int_{A_k} |\Omega(x)|(k+1)\,\mathrm{d}\sigma_{n-1}(x)$$
$$\leq \sum_{k=0}^{\infty}(k+1)\|\Omega\|_{\mathrm{L}^2(\mathbb{S}^{n-1})}\sigma_{n-1}(A_k)^{1/2} \lesssim \|\Omega\|_{\mathrm{L}^2(\mathbb{S}^{n-1})}\sum_{k=0}^{\infty}(k+1)2^{-k/2} \lesssim \|\Omega\|_{\mathrm{L}^2(\mathbb{S}^{n-1})}.$$

Therefore, in order to apply Theorem 1.3.1 to the operator kernel Ω_Φ^λ from (2.2), we also observe that for $K_\Phi^\lambda(x) = \mathrm{p.\,v.}\,\Omega_\Phi^\lambda(x/|x|)|x|^{-n}$ we have $\big|\widehat{K_\Phi^\lambda}(\xi)\big| = |m_\Phi^\lambda(\xi) - a_\Phi^\lambda| \leq 2,$

where we recall (2.3). Thus, it remains to prove

$$\|\Omega_\Phi^\lambda\|_{\mathrm{L}^2(\mathbb{S}^{n-1})} \lesssim_{n,\Phi} \lambda^{n/2} \tag{2.20}$$

for $\lambda \geq 1$.

From equation (1.4) and the symmetry property (1.5) we can see that expansions of Ω_Φ^λ and m_Φ^λ into spherical harmonics are related as:

$$m_\Phi^\lambda = \sum_{j=0}^{\infty} Y_j \quad \Longrightarrow \quad \Omega_\Phi^\lambda = \sum_{j=1}^{\infty} \mathbb{i}^j \gamma_{n,j,n} Y_j,$$

where the two series converge in the L^2 sense. Now, the aforementioned asymptotics (1.6) implies $\gamma_{n,j,n} \sim_n j^{n/2}$, so

$$\|\Omega_\Phi^\lambda\|^2_{\mathrm{L}^2(\mathbb{S}^{n-1})} = \sum_{j=1}^{\infty} \gamma_{n,j,n}^2 \|Y_j\|^2_{\mathrm{L}^2(\mathbb{S}^{n-1})} \lesssim_n \sum_{j=0}^{\infty} j^n \|Y_j\|^2_{\mathrm{L}^2(\mathbb{S}^{n-1})}. \tag{2.21}$$

Recalling that the spherical harmonics are eigenfunctions of the spherical Laplacean, namely that (1.8) holds, we arrive at

$$\Delta^r_{\mathbb{S}^{n-1}} m_\Phi^\lambda = \sum_{j=1}^{\infty} \big(-j(j+n-2)\big)^r Y_j,$$

for any positive integer r. Uniform convergence of the above series is needed for justification of the performed term-by-term differentiation, but it is, in turn, guaranteed by the smoothness of m_Φ^λ, Sogge's estimate (1.9), and the standard results on rapid convergence of spherical harmonic expansions of smooth functions in [64, § 3.1.5]. Using (2.21) and Hölder's inequality, and by choosing $r = \lceil n/4 \rceil$ in the previous display, we can write

$$\begin{aligned}
\|\Omega_\Phi^\lambda\|^2_{\mathrm{L}^2(\mathbb{S}^{n-1})} &\lesssim_n \Big(\sum_{j=1}^{\infty} j^{4r} \|Y_j\|^2_{\mathrm{L}^2(\mathbb{S}^{n-1})}\Big)^{n/4r} \Big(\sum_{j=0}^{\infty} \|Y_j\|^2_{\mathrm{L}^2(\mathbb{S}^{n-1})}\Big)^{1-n/4r} \\
&\leq \Big(\sum_{j=1}^{\infty} \big(j(j+n-2)\big)^{2r} \|Y_j\|^2_{\mathrm{L}^2(\mathbb{S}^{n-1})}\Big)^{n/4r} \Big(\sum_{j=0}^{\infty} \|Y_j\|^2_{\mathrm{L}^2(\mathbb{S}^{n-1})}\Big)^{1-n/4r} \\
&= \|\Delta^r_{\mathbb{S}^{n-1}} m_\Phi^\lambda\|^{n/2r}_{\mathrm{L}^2(\mathbb{S}^{n-1})} \|m_\Phi^\lambda\|^{2-n/2r}_{\mathrm{L}^2(\mathbb{S}^{n-1})} = \|\Delta^r_{\mathbb{S}^{n-1}} m_\Phi^\lambda\|^{n/2r}_{\mathrm{L}^2(\mathbb{S}^{n-1})} \\
&\lesssim_n \|\Delta^r_{\mathbb{S}^{n-1}} m_\Phi^\lambda\|^{n/2r}_{\mathrm{L}^\infty(\mathbb{S}^{n-1})}.
\end{aligned} \tag{2.22}$$

Therefore, it remains to bound

$$\|\Delta^r_{\mathbb{S}^{n-1}} e^{\mathbb{i}\lambda\Phi}\|_{\mathrm{L}^\infty(\mathbb{S}^{n-1})}.$$

First observe that, if a function $f \in \mathrm{C}^\infty(\mathbb{S}^{n-1})$ is of the form

$$f = e^{\mathbb{i}\lambda\Phi} \sum_{l=0}^{k} \lambda^l \Phi_l,$$

for some nonnegative integer k and some functions $\Phi, \Phi_0, \ldots, \Phi_k \in \mathrm{C}^\infty(\mathbb{S}^{n-1})$, then there exist functions $\widetilde{\Phi}_0, \ldots, \widetilde{\Phi}_{k+2} \in \mathrm{C}^\infty(\mathbb{S}^{n-1})$ such that

$$\Delta_{\mathbb{S}^{n-1}} f = e^{\mathbb{i}\lambda\Phi} \sum_{l=0}^{k+2} \lambda^l \widetilde{\Phi}_l.$$

The claim is easily seen by calculating the Laplacean $\Delta_{\mathbb{R}^n}$ of the homogenized expression

$$\mathbb{R}^n \setminus \{0\} \ni x \mapsto e^{\mathbb{i}\lambda\Phi(x/|x|)} \sum_{l=0}^{k} \lambda^l \Phi_l\Big(\frac{x}{|x|}\Big)$$

and evaluating it at $x \in \mathbb{S}^{n-1}$. Applying the previous observation inductively, we conclude that there exist functions $\Phi_0, \ldots, \Phi_{2r} \in \mathrm{C}^\infty(\mathbb{S}^{n-1})$ depending only on Φ such that

$$\Delta_{\mathbb{S}^{n-1}}^r e^{\mathbb{i}\lambda\Phi} = e^{\mathbb{i}\lambda\Phi} \sum_{l=0}^{2r} \lambda^l \Phi_l.$$

Since $\mathbb{S}^{n-1}$ is compact and the functions Φ_l are continuous, they are also bounded and, therefore,

$$\|\Delta_{\mathbb{S}^{n-1}}^r e^{\mathbb{i}\lambda\Phi}\|_{\mathrm{L}^\infty(\mathbb{S}^{n-1})} \leq \sum_{l=0}^{2r} \lambda^l \|\Phi_l\|_{\mathrm{L}^\infty(\mathbb{S}^{n-1})} \lesssim_{n,\Phi} \lambda^{2r}$$

for $\lambda \geq 1$. The desired estimate (2.20) now follows from (2.22).

2.3. PROOF OF THEOREM 2.1.4

In this section we develop a somewhat general scheme of bounding norms of Fourier multiplier operators from below by constructing functions on $\mathbb{S}^{n-1}$ from infinite sums of spherical harmonics. The following auxiliary lemma can be thought of as a quantitative refinement of the classical formula (1.2). Recall the constants (2.16).

Lemma 2.3.1. *For $p \in (1,\infty)$, $q = p/(p-1)$, $\varepsilon \in (0,1/2]$, and a spherical harmonic Y of degree $j \geq 0$ one can find a Schwartz function $g_{n,p,\varepsilon,Y}$ such that*

$$\Big\| g_{n,p,\varepsilon,Y}(x) - Y\Big(\frac{x}{|x|}\Big)|x|^{-n/p}\mathbb{1}_{\{\varepsilon\leq|x|\leq1/\varepsilon\}}(x)\Big\|_{\mathrm{L}^p_x(\mathbb{R}^n)} \lesssim_{n,p,Y} 1 \tag{2.23}$$

and

$$\Big\| \widehat{g}_{n,p,\varepsilon,Y}(\xi) - \mathbb{i}^{-j}\gamma_{n,j,n/q}Y\Big(\frac{\xi}{|\xi|}\Big)|\xi|^{-n/q}\mathbb{1}_{\{\varepsilon\leq|\xi|\leq1/\varepsilon\}}(\xi)\Big\|_{\mathrm{L}^q_\xi(\mathbb{R}^n)} \lesssim_{n,p,Y} 1. \tag{2.24}$$

Consequently, also

$$\|g_{n,p,\varepsilon,Y}\|_{\mathrm{L}^p(\mathbb{R}^n)} = \|Y\|_{\mathrm{L}^p(\mathbb{S}^{n-1})}\Big(2\log\frac{1}{\varepsilon}\Big)^{1/p} + O^{\varepsilon\to0+}_{n,p,Y}(1) \tag{2.25}$$

and

$$\big\|\widehat{g}_{n,p,\varepsilon,Y}\big\|_{\mathrm{L}^q(\mathbb{R}^n)} = \gamma_{n,j,n/q}\|Y\|_{\mathrm{L}^q(\mathbb{S}^{n-1})}\Big(2\log\frac{1}{\varepsilon}\Big)^{1/q} + O^{\varepsilon\to0+}_{n,p,Y}(1). \tag{2.26}$$

We emphasize that the implicit constants in (2.23)–(2.26) do not depend on ε.

Proof. Note that Y extends from $\mathbb{S}^{n-1}$ to the unique solid spherical harmonic P of degree j on $\mathbb{R}^n$ via $P(x) = |x|^j Y(x/|x|)$. We will construct $g = g_{n,p,\varepsilon,Y}$ as a superposition of dilated Gaussian functions, very similarly as these were employed in [67, Sections IV.3–IV.4]. Define

$$g(x) := \frac{2\pi^{j/2+n/2p}}{\Gamma(j/2+n/2p)}P(x)\int_\varepsilon^{1/\varepsilon} e^{-\pi t^{-2}|x|^2}t^{-n/p-j-1}\,\mathrm{d}t; \quad x\in\mathbb{R}^n.$$

This is clearly a Schwartz function. Dilating formula (1.1) we get

$$f(x) = t^{-n-j}P(x)e^{-\pi t^{-2}|x|^2} \implies \widehat{f}(\xi) = \mathbb{i}^{-j}t^jP(\xi)e^{-\pi t^2|\xi|^2},$$

which enables us to take the Fourier transform of g:

$$\widehat{g}(\xi) = \frac{2\mathbb{i}^{-j}\pi^{j/2+n/2p}}{\Gamma(j/2+n/2p)}P(\xi)\int_\varepsilon^{1/\varepsilon} e^{-\pi t^2|\xi|^2}t^{n/q+j-1}\,\mathrm{d}t; \quad \xi\in\mathbb{R}^n.$$

An easy computation changing the variables of integration leads to

$$\begin{aligned}
g(x) &= Y\Big(\frac{x}{|x|}\Big)|x|^{-n/p}\frac{1}{\Gamma(j/2+n/2p)}\int_{\pi\varepsilon^2|x|^2}^{\pi\varepsilon^{-2}|x|^2} u^{j/2+n/2p-1}e^{-u}\,\mathrm{d}u,\\
\widehat{g}(\xi) &= \frac{\mathbb{i}^{-j}\pi^{n/2p-n/2q}}{\Gamma(j/2+n/2p)}Y\Big(\frac{\xi}{|\xi|}\Big)|\xi|^{-n/q}\int_{\pi\varepsilon^2|\xi|^2}^{\pi\varepsilon^{-2}|\xi|^2} u^{j/2+n/2q-1}e^{-u}\,\mathrm{d}u\\
&= \mathbb{i}^{-j}\gamma_{n,j,n/q}Y\Big(\frac{\xi}{|\xi|}\Big)|\xi|^{-n/q}\frac{1}{\Gamma(j/2+n/2q)}\int_{\pi\varepsilon^2|\xi|^2}^{\pi\varepsilon^{-2}|\xi|^2} u^{j/2+n/2q-1}e^{-u}\,\mathrm{d}u.
\end{aligned}$$

Since Y is clearly bounded on the unit sphere $\mathbb{S}^{n-1}$, estimate (2.23) is now reduced to

$$\Big\|\Big(\int_{\pi\varepsilon^2|x|^2}^{\pi\varepsilon^{-2}|x|^2} u^{\beta-1}e^{-u}\,\mathrm{d}u-\Gamma(\beta)\mathbb{1}_{\{\varepsilon\le|x|\le1/\varepsilon\}}(x)\Big)|x|^{-n/p}\Big\|_{\mathrm{L}^p_x(\mathbb{R}^n)}\lesssim_{n,p,\beta} 1$$

for some $\beta\in(0,\infty)$, while (2.24) then also follows by interchanging p and q. Moreover, by passing to n-dimensional spherical coordinates and using the definition of the gamma function, we see that it remains to establish the following four elementary estimates:

$$\int_0^{\varepsilon}\Big(\int_{\pi\varepsilon^2r^2}^{\pi\varepsilon^{-2}r^2} u^{\beta-1}e^{-u}\,\mathrm{d}u\Big)^p\frac{\mathrm{d}r}{r}\lesssim_{p,\beta} 1, \tag{2.27a}$$

$$\int_{1/\varepsilon}^{\infty}\Big(\int_{\pi\varepsilon^2r^2}^{\pi\varepsilon^{-2}r^2} u^{\beta-1}e^{-u}\,\mathrm{d}u\Big)^p\frac{\mathrm{d}r}{r}\lesssim_{p,\beta} 1, \tag{2.27b}$$

$$\int_{\varepsilon}^{1/\varepsilon}\Big(\int_0^{\pi\varepsilon^2r^2} u^{\beta-1}e^{-u}\,\mathrm{d}u\Big)^p\frac{\mathrm{d}r}{r}\lesssim_{p,\beta} 1. \tag{2.27c}$$

$$\int_{\varepsilon}^{1/\varepsilon}\Big(\int_{\pi\varepsilon^{-2}r^2}^{\infty} u^{\beta-1}e^{-u}\,\mathrm{d}u\Big)^p\frac{\mathrm{d}r}{r}\lesssim_{p,\beta} 1. \tag{2.27d}$$

We now provide detailed proofs of (2.27a)–(2.27d), even though similar computations appeared in [11, Section 6].

Proof of (2.27a). For a fixed $r\in(0,\varepsilon]$ we estimate the inner integral as

$$\int_{\pi\varepsilon^2r^2}^{\pi\varepsilon^{-2}r^2} u^{\beta-1}e^{-u}\,\mathrm{d}u\lesssim\int_0^{\pi\varepsilon^{-2}r^2} u^{\beta-1}\,\mathrm{d}u\lesssim_\beta(\varepsilon^{-1}r)^{2\beta},$$

then we raise it to the p-th power and integrate in r, getting

$$\varepsilon^{-2\beta p}\int_0^{\varepsilon} r^{2\beta p-1}\,\mathrm{d}r=\frac{1}{2\beta p}<\infty.$$

Proof of (2.27b). Using integration by parts as many times as needed (depending on β), we easily obtain the following estimate for the incomplete gamma function:

$$\int_x^{\infty} u^{\beta-1}e^{-u}\,\mathrm{d}u\lesssim_\beta x^{\beta-1}e^{-x};\quad x\in[1,\infty); \tag{2.28}$$

also see [17, Eq. 8.11.1–8.11.3] or [1, Eq. 6.5.32]. By taking $x = \pi\varepsilon^2 r^2$ for any $r \in [1/\varepsilon, \infty)$, we get

$$\int_{\pi\varepsilon^2 r^2}^{\pi\varepsilon^{-2} r^2} u^{\beta-1} e^{-u}\,\mathrm{d}u \lesssim_\beta (\varepsilon r)^{2\beta-2} e^{-\pi\varepsilon^2 r^2}.$$

This is raised to the p-th power and integrated in r, substituting $s = \pi p\varepsilon^2 r^2$:

$$\int_{1/\varepsilon}^{\infty} (\varepsilon r)^{(2\beta-2)p} e^{-\pi p\varepsilon^2 r^2}\,\frac{\mathrm{d}r}{r} = \frac{1}{2}(\pi p)^{(1-\beta)p} \int_{\pi p}^{\infty} s^{(\beta-1)p-1} e^{-s}\,\mathrm{d}s < \infty.$$

Proof of (2.27c). From

$$\int_0^{\pi\varepsilon^2 r^2} u^{\beta-1} e^{-u}\,\mathrm{d}u \lesssim \int_0^{\pi\varepsilon^2 r^2} u^{\beta-1}\,\mathrm{d}u \lesssim_\beta (\varepsilon r)^{2\beta}$$

we see that the left hand side of (2.27c) is at most a multiple of

$$\varepsilon^{2\beta p} \int_0^{1/\varepsilon} r^{2\beta p-1}\,\mathrm{d}r = \frac{1}{2\beta p} < \infty.$$

Proof of (2.27d). Using (2.28) again we can write

$$\int_{\pi\varepsilon^{-2} r^2}^{\infty} u^{\beta-1} e^{-u}\,\mathrm{d}u \lesssim_\beta (\varepsilon^{-1} r)^{2\beta-2} e^{-\pi\varepsilon^{-2} r^2}.$$

Thus, the left hand side of (2.27d) is at most a constant times

$$\int_{\varepsilon}^{\infty} (\varepsilon^{-1} r)^{(2\beta-2)p} e^{-\pi p\varepsilon^{-2} r^2}\,\frac{\mathrm{d}r}{r} = \frac{1}{2}(\pi p)^{(1-\beta)p} \int_{\pi p}^{\infty} s^{(\beta-1)p-1} e^{-s}\,\mathrm{d}s < \infty.$$

This also completes the proofs of (2.23) and (2.24).

In order to establish (2.25) we only need to combine (2.23) with

$$\Big\| Y\Big(\frac{x}{|x|}\Big) |x|^{-n/p} \mathbb{1}_{\{\varepsilon \le |x| \le 1/\varepsilon\}}(x) \Big\|_{\mathrm{L}^p_x(\mathbb{R}^n)} = \|Y\|_{\mathrm{L}^p(\mathbb{S}^{n-1})} \Big(2\log\frac{1}{\varepsilon}\Big)^{1/p}.$$

In the same way we verify (2.26). □

Observe that the error terms in (2.23) and (2.24) are of smaller order in ε than both of the main terms

$$Y\Big(\frac{x}{|x|}\Big) |x|^{-n/p} \mathbb{1}_{\{\varepsilon \le |x| \le 1/\varepsilon\}}(x) \tag{2.29}$$

and

$$\mathbb{i}^{-j} \gamma_{n,j,n/q} Y\Big(\frac{\xi}{|\xi|}\Big) |\xi|^{-n/q} \mathbb{1}_{\{\varepsilon \le |\xi| \le 1/\varepsilon\}}(\xi), \tag{2.30}$$

norms of which were calculated in (2.25) and (2.26), respectively. Thus, Lemma 2.3.1 enables us to think of (2.30) as an approximation of the Fourier transform of (2.29) up to a small relative error. By letting $\varepsilon \to 0+$ we would formally recover Bochner's distributional identity (1.2), but it is crucial for us to stay within the realm of function spaces $\mathrm{L}^p(\mathbb{R}^n)$ and $\mathrm{L}^q(\mathbb{R}^n)$.

We are finally in a position to prove Theorem 2.1.4.

Proof of Theorem 2.1.4. Let us first assume that $p>1$, $q<\infty$. Take a number $\varepsilon \in (0,1/2]$ and a positive integer J. By several applications of Lemma 2.3.1 we can find a Schwartz function g (depending on n,p,ε,m,J) such that g differs from

$$x \mapsto \Big(\sum_{j=0}^{J} Y_j\Big(\frac{x}{|x|}\Big)\Big)|x|^{-n/q}\mathbb{1}_{\{\varepsilon\le|x|\le 1/\varepsilon\}}(x)$$

in the L^q norm by $O_{n,p,m,J}(1)$ as $\varepsilon \to 0+$, while $\widehat{g}$ differs from

$$\xi \mapsto \Big(\sum_{j=0}^{J} \mathbb{i}^{-j}\gamma_{n,j,n/p}Y_j\Big(\frac{\xi}{|\xi|}\Big)\Big)|\xi|^{-n/p}\mathbb{1}_{\{\varepsilon\le|\xi|\le 1/\varepsilon\}}(\xi)$$

in the L^p norm by $O_{n,p,m,J}(1)$ as $\varepsilon \to 0+$. (Note that the roles of p and q were interchanged here.) From the same lemma we also obtain a Schwartz function f (depending on n,p,ε) such that f differs from

$$x \mapsto |x|^{-n/p}\mathbb{1}_{\{\varepsilon\le|x|\le 1/\varepsilon\}}(x)$$

in the L^p norm by $O_{n,p}(1)$ as $\varepsilon \to 0+$, while $\widehat{f}$ differs from

$$\xi \mapsto \gamma_{n,0,n/q}|\xi|^{-n/q}\mathbb{1}_{\{\varepsilon\le|\xi|\le 1/\varepsilon\}}(\xi)$$

in the L^q norm by $O_{n,p}(1)$ as $\varepsilon \to 0+$. Using Plancherel's theorem we bound

$$\begin{aligned}
\|T_m\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} &\ge \frac{|\langle T_m f, g\rangle_{\mathrm{L}^2(\mathbb{R}^n)}|}{\|f\|_{\mathrm{L}^p(\mathbb{R}^n)}\|g\|_{\mathrm{L}^q(\mathbb{R}^n)}} = \frac{|\langle m\widehat{f}, \widehat{g}\rangle_{\mathrm{L}^2(\mathbb{R}^n)}|}{\|f\|_{\mathrm{L}^p(\mathbb{R}^n)}\|g\|_{\mathrm{L}^q(\mathbb{R}^n)}} \\
&= \frac{|\langle \gamma_{n,0,n/q}m, \sum_{j=0}^{J}\mathbb{i}^{-j}\gamma_{n,j,n/p}Y_j\rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})}|\,2\log(1/\varepsilon) + o^{\varepsilon\to0+}_{n,p,m,J}(\log(1/\varepsilon))}{\sigma(\mathbb{S}^{n-1})^{1/p}\|\sum_{j=0}^{J}Y_j\|_{\mathrm{L}^q(\mathbb{S}^{n-1})}\,2\log(1/\varepsilon) + o^{\varepsilon\to0+}_{n,p,m,J}(\log(1/\varepsilon))} \\
&= \frac{\gamma_{n,0,n/q}|\langle m, \sum_{j=0}^{J}\mathbb{i}^{-j}\gamma_{n,j,n/p}Y_j\rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})}| + o^{\varepsilon\to0+}_{n,p,m,J}(1)}{\sigma(\mathbb{S}^{n-1})^{1/p}\|\sum_{j=0}^{J}Y_j\|_{\mathrm{L}^q(\mathbb{S}^{n-1})} + o^{\varepsilon\to0+}_{n,p,m,J}(1)}.
\end{aligned}$$

We first take the limit as $\varepsilon \to 0+$ to obtain

$$\|T_m\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \ge \frac{\gamma_{n,0,n/q}}{\sigma(\mathbb{S}^{n-1})^{1/p}}\,\frac{|\langle m, \sum_{j=0}^{J}\mathbb{i}^{-j}\gamma_{n,j,n/p}Y_j\rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})}|}{\|\sum_{j=0}^{J}Y_j\|_{\mathrm{L}^q(\mathbb{S}^{n-1})}} \tag{2.31}$$

and then let $J \to \infty$ using conditions (b) and (c). This proves (2.17) and, in combination with (1.7), also (2.18).

In order to prove (2.19), assume hypotheses (a)–(c) of the theorem with $p = 1$, $q = \infty$. For every pair of conjugated exponents $p \in (1,2]$, $q \in [2,\infty)$ we repeat the prevoius part of the proof leading to (2.31). Then we borrow a trick from [19, Subsection 4.2] (also see [20, pp. 496–497]) and use a sharp form of the Marcinkiewicz interpolation theorem (see [31, Theorem 1.3.2]) together with a trivial estimate on $\mathrm{L}^2(\mathbb{R}^n)$ to get

$$\|T_m\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \lesssim \frac{1}{(p-1)^{1/p}} \|T_m\|^{2/p-1}_{\mathrm{L}^1(\mathbb{R}^n)\to\mathrm{L}^{1,\infty}(\mathbb{R}^n)}$$

for $1 < p < 3/2$. Combining this with (2.31), we conclude

$$\begin{aligned} &\|T\|_{\mathrm{L}^1(\mathbb{R}^n)\to\mathrm{L}^{1,\infty}(\mathbb{R}^n)} \\ &\gtrsim \lim_{p\to 1+} \Big((p-1)^{1/p} \frac{\gamma_{n,0,n/q}}{\sigma(\mathbb{S}^{n-1})^{1/p}} \frac{|\langle m, \sum_{j=0}^J \mathbb{i}^{-j}\gamma_{n,j,n/p} Y_j\rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})}|}{\|\sum_{j=0}^J Y_j\|_{\mathrm{L}^q(\mathbb{S}^{n-1})}} \Big)^{p/(2-p)} \\ &= \frac{1}{n} \frac{|\langle m, \sum_{j=0}^J \mathbb{i}^{-j}\gamma_{n,j,n} Y_j\rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})}|}{\|\sum_{j=0}^J Y_j\|_{\mathrm{L}^\infty(\mathbb{S}^{n-1})}}, \end{aligned} \tag{2.32}$$

where we used

$$\begin{aligned} &\lim_{p\to 1+} (p-1)^{1/p}\gamma_{n,0,n/q} = \lim_{p\to 1+} (p-1)^{1/p} \pi^{n/2-n/q} \frac{\Gamma(n/2q)}{\Gamma(n/2p)} \\ &= \Big(\lim_{p\to 1+} (p-1)^{1/p} q \Big) \Big(\lim_{p\to 1+} \frac{2\pi^{n/2-n/q}\Gamma(n/2q+1)}{n\Gamma(n/2p)} \Big) = \frac{2\pi^{n/2}}{n\Gamma(n/2)} \end{aligned}$$

and $\sigma(\mathbb{S}^{n-1}) = 2\pi^{n/2}/\Gamma(n/2)$. Finally, we take limits in (2.32) as $J \to \infty$ using conditions (b) and (c) as before. □

The condition from part (b) of Theorem 2.1.4 is not always easy to verify, since convergence in L^q for $q > 2$ is typically trickier than L^2 convergence. The following remark can often be of some help in that matter.

Remark 2.3.2. If the sequence $(Y_j)_{j=0}^\infty$ from Theorem 2.1.4 satisfies assumption (a) and a stronger condition

$$\sum_{j=1}^\infty j^n \|Y_j\|^2_{\mathrm{L}^2(\mathbb{S}^{n-1})} < \infty, \tag{2.33}$$

then convergence claims from (b) and (c) are automatically satisfied for every pair of conjugated exponents $p \in [1,2]$ and $q \in [2,\infty]$. Indeed, recalling (2.16) and (1.6) we observe

$$\gamma_{n,j,n/p}^2 \|Y_j\|_{\mathrm{L}^2(\mathbb{S}^{n-1})}^2 \lesssim_n j^n \|Y_j\|_{\mathrm{L}^2(\mathbb{S}^{n-1})}^2.$$

Moreover, we apply the endpoint case of Sogge's inequality (1.9) and the Cauchy–Schwarz inequality to get

$$\sum_{j=1}^{\infty} \|Y_j\|_{\mathrm{L}^\infty(\mathbb{S}^{n-1})} \lesssim_n \sum_{j=1}^{\infty} j^{n/2-1} \|Y_j\|_{\mathrm{L}^2(\mathbb{S}^{n-1})} \lesssim_n \Big(\sum_{j=1}^{\infty} j^n \|Y_j\|_{\mathrm{L}^2(\mathbb{S}^{n-1})}^2 \Big)^{1/2}.$$

Then we use (2.33) and remember that convergence in $\mathrm{L}^\infty(\mathbb{S}^{n-1})$ implies convergence in every $\mathrm{L}^q(\mathbb{S}^{n-1})$.

Another interesting observation, which we do not need in the later text, is that, for $1 \leq p < 2(n+2)/(n+4)$, assumptions (a) and (c) imply assumption (b). This is easily seen just as before, only applying a larger range of Sogge's estimates (1.9).

Even though Remark 2.3.2 enables easy verification of the conditions of Theorem 2.1.4, the above reasoning does not necessarily give sharp upper bounds on the quantity $\|u\|_{\mathrm{L}^q(\mathbb{S}^{n-1})}$. Controlling this number sometimes requires significant extra work; see Lemma 2.4.1 below.

2.4. Proof of the lower bounds in Theorem 2.1.2

In this section we apply Theorem 2.1.4 to establish the part (b) of Theorem 2.1.2. Our symbol m_Φ^k will be a "smoothed" version of

$$(\xi_1,\dots,\xi_n) \mapsto \prod_{i=1}^{n/2} \Big(\frac{\xi_{2i-1}+\mathrm{i}\xi_{2i}}{|\xi_{2i-1}+\mathrm{i}\xi_{2i}|}\Big)^k. \tag{2.34}$$

On the one hand, this smoothing is required for two reasons. First, C^∞ smoothness was imposed in the formulation of Problem 2.1.1, which we we address. Second, for $n \geq 4$ the non-smooth symbol appearing in (2.34) is singular on the union of two-dimensional coordinate planes, so the part (a) of Theorem 2.1.2 does not apply and we do not even have clear upper bounds for the associated multiplier operators.

On the other hand, smoothing of the symbol significantly complicates analysis of lower bounds by destroying the tensor product structure. These complications are detailed in Remark 2.6.1, which rules out the possibility of testing T_Φ^k on examples of functions that are elementary tensor products with respect to $\mathbb{R}^2 \times \cdots \times \mathbb{R}^2$.

In accordance with Remark 2.2.1, in the remaining text we always assume

$$\lambda \geq 1, \quad p \in [1,2], \quad q \in [2,\infty],$$

and that p and q are conjugate exponents.

2.4.1. Two dimensions

This short subsection is logically redundant, both because the next subsection covers all even-dimensional spaces $\mathbb{R}^n$, and because Theorem 2.1.3 provides yet another example of a phase that leads to the "worst possible" asymptotics. We include it for reader's convenience: to illustrate the main idea of proof with absence of many subtle technical complications arising in dimensions $n \geq 4$.

Take $\delta \in (0,\pi)$; we will choose it a bit later. Define Φ in polar coordinates as

$$\Phi(e^{\mathrm{i}\varphi}) := \varphi; \quad \varphi \in (-\pi+\delta, \pi-\delta).$$

This still leaves $\Phi\colon \mathbb{S}^1 \to \mathbb{R}$ undefined on a circular arc of length 2δ, but we can define it arbitrarily there, only taking care that Φ is C^∞ on the whole circle $\mathbb{S}^1$. For a positive integer k we also define

$$v^{(k)}(e^{\mathbb{i}\varphi}) := e^{\mathbb{i}k\varphi}; \quad \varphi \in \mathbb{R}.$$

This is clearly a spherical harmonic on $\mathbb{S}^1$ of degree k, as it is obtained by restricting the harmonic polynomial $(\xi_1, \xi_2) \mapsto (\xi_1 + \mathbb{i}\xi_2)^k$ to the unit circle. With the intention of applying Theorem 2.1.4, we also set

$$u^{(k)} := \mathbb{i}^k \gamma_{2,k,2/p}^{-1} v^{(k)},$$

so that functions $u = u^{(k)}$ and $v = v^{(k)}$ trivially satisfy assumptions (a)–(c). Note that for every $\varphi \in (-\pi+\delta, \pi-\delta)$ we have

$$m_\Phi^k(e^{\mathbb{i}\varphi}) = e^{\mathbb{i}k\Phi(e^{\mathbb{i}\varphi})} = e^{\mathbb{i}k\varphi} = v^{(k)}(e^{\mathbb{i}\varphi}),$$

so, in particular,

$$\|m_\Phi^k - v^{(k)}\|_{\mathrm{L}^2(\mathbb{S}^1)} \leq 2\sqrt{\delta}.$$

For any $p \in [1,2]$ we now have

$$\begin{aligned}\frac{|\langle m_\Phi^k, v^{(k)}\rangle_{\mathrm{L}^2(\mathbb{S}^1)}|}{\|u^{(k)}\|_{\mathrm{L}^q(\mathbb{S}^1)}} &\geq \frac{\|v^{(k)}\|_{\mathrm{L}^2(\mathbb{S}^1)}^2 - \|m_\Phi^k - v^{(k)}\|_{\mathrm{L}^2(\mathbb{S}^1)}\|v^{(k)}\|_{\mathrm{L}^2(\mathbb{S}^1)}}{\gamma_{2,k,2/p}^{-1}\|v^{(k)}\|_{\mathrm{L}^q(\mathbb{S}^1)}} \\ &\geq (\sqrt{2\pi} - 2\sqrt{\delta})(2\pi)^{1/2-1/q}\gamma_{2,k,2/p}.\end{aligned}$$

Finally, we take $\delta = \pi/8$ and recall $\gamma_{2,k,2/p} \sim k^{2/p-1}$, by (1.6). Theorem 2.1.4 applies, so (2.18) and (2.19) respectively give

$$\|T_\Phi^k\|_{\mathrm{L}^p(\mathbb{R}^2)\to\mathrm{L}^p(\mathbb{R}^2)} \gtrsim (q-1)k^{2/p-1} = (q-1)\,k^{2(1/p-1/2)}$$

for $p \in (1,2]$ and

$$\|T_\Phi^k\|_{\mathrm{L}^1(\mathbb{R}^2)\to\mathrm{L}^{1,\infty}(\mathbb{R}^2)} \gtrsim k.$$

These are precisely the desired two-dimensional cases of (2.11) and (2.12).

2.4.2. Higher dimensions

Suppose that $n = 2r$ for a positive integer r. We will consider a slightly non-standard coordinatization of $\mathbb{S}^{2r-1} \subset \mathbb{R}^{2r}$. Denote

$$\mathbb{S}_+^{r-1} := \{(\omega_1, \ldots, \omega_r) \in (0,\infty)^r : \omega_1^2 + \cdots + \omega_r^2 = 1\}, \quad D_+ := (-\pi,\pi)^r \times \mathbb{S}_+^{r-1}.$$

Transformation $\Psi\colon D_+ \to \mathbb{S}^{2r-1}$ is defined as

$$\Psi(\varphi_1,\dots,\varphi_r,\omega) = (\omega_1\cos\varphi_1,\, \omega_1\sin\varphi_1,\, \dots,\, \omega_r\cos\varphi_r,\, \omega_r\sin\varphi_r),$$

where $\varphi_1,\dots,\varphi_r \in (-\pi,\pi)$ and $\omega = (\omega_1,\dots,\omega_r) \in \mathbb{S}_+^{r-1}$. This is a C^∞ diffeomorphism onto its image, and the complement of its image is a negligible subset of $\mathbb{S}^{2r-1}$ with respect to the surface measure σ_{2r-1}. Moreover, this parametrization Ψ enables us to write the infinitesimal element of the surface measure on $\mathbb{S}^{2r-1}$ as

$$\mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_r\,\omega_1\cdots\omega_r\,\mathrm{d}\sigma_{r-1}(\omega). \tag{2.35}$$

In the case $r=1$ one simply needs to disregard any occurrence of $\mathbb{S}_+^{r-1}$.

For any $\delta>0$ we also denote

$$\mathbb{S}_\delta^{r-1} := \mathbb{S}_+^{r-1}\cap(\delta,\infty)^r, \quad D_\delta := (-\pi+\delta,\pi-\delta)^r\times\mathbb{S}_\delta^{r-1}.$$

Now fix a parameter $0<\delta<1/r$ (to be chosen later) and define $\Theta\colon D_+\to\mathbb{R}$ by setting

$$\Theta(\varphi_1,\dots,\varphi_r,\omega) := \begin{cases} \varphi_1+\cdots+\varphi_r & \text{on } D_\delta, \\ 0 & \text{on } D_+\setminus D_{\delta/2}, \end{cases}$$

and choosing its values on $D_{\delta/2}\setminus D_\delta$ quite arbitrarily, only taking care that Θ remains C^∞. We can finally define the desired phase function $\Phi\colon\mathbb{S}^{2r-1}\to\mathbb{R}$ as

$$\Phi := \begin{cases} \Theta\circ\Psi^{-1} & \text{on } \Psi(D_+), \\ 0 & \text{on } \mathbb{S}^{2r-1}\setminus\Psi(D_+). \end{cases}$$

Since the composition $\Theta\circ\Psi^{-1}$ vanishes outside the closure of $\Psi(D_{\delta/2})\subset\mathbb{S}^{2r-1}$, we clearly see that Φ is C^∞ on the whole sphere. We can call

$$M_\delta := \Big\{\xi\in\mathbb{R}^n\setminus\{\mathbf{0}\} \,:\, \frac{\xi}{|\xi|}\in\Psi(D_\delta)\Big\}$$

the *major cone*, while its complement $\mathbb{R}^n\setminus M_\delta$ is a certain *exceptional set*. For a positive integer k and any $\xi\in M_\delta$, denoting $\xi/|\xi| = \Psi(\varphi_1,\dots,\varphi_r,\omega)$ we can write

$$m_\Phi^k(\xi) = e^{\mathbb{i}k\Phi(\xi/|\xi|)} = e^{\mathbb{i}k\Theta(\varphi_1,\dots,\varphi_r,\omega)} = \prod_{j=1}^r e^{\mathbb{i}k\varphi_j},$$

so for every $\xi \in M_\delta$ the symbol simplifies as

$$m_\Phi^k(\xi) = \prod_{i=1}^{r} \Big(\frac{\xi_{2i-1} + \mathbb{i}\xi_{2i}}{|\xi_{2i-1} + \mathbb{i}\xi_{2i}|} \Big)^k. \tag{2.36}$$

Once again we remark that it was necessary to smooth out the expression (2.36) on the exceptional set. For a generic point $\xi = (\xi_1, \xi_2, \dots, \xi_{2r-1}, \xi_{2r}) \in \mathbb{R}^{2r}$ we will also write

$$\zeta_i := \xi_{2i-1} + \mathbb{i}\xi_{2i}; \quad j = 1, 2, \dots, r$$

and identify pairs (ξ_{2i-1}, ξ_{2i}) with complex numbers ζ_i.

We choose a particular L^2 function on the sphere, $\widetilde{m}^{(k)}$, acting on r complex variables and defined as

$$\widetilde{m}^{(k)}(\zeta_1, \dots, \zeta_r) := \prod_{i=1}^{r} \Big(\frac{\zeta_i}{|\zeta_i|} \Big)^k.$$

The reader can recognize it simply as the right hand side of (2.36), i.e., the non-smooth version of the symbol. Every homogeneous polynomial on $\mathbb{R}^{2r}$ can be written as a homogeneous polynomial in terms of ζ_1, $\bar{\zeta}_1$, $\dots$, ζ_r, $\bar{\zeta}_r$. By integrating over the sphere we see that $\widetilde{m}^{(k)}$ is orthogonal in $\mathrm{L}^2(\mathbb{S}^{2r-1})$ to any such monomial that is <u>not</u> of the form

$$\begin{aligned} P_{k_1,\dots,k_r}(\zeta_1, \dots, \zeta_r) &:= \zeta_1^{k+k_1} \bar{\zeta}_1^{\,k_1} \cdots \zeta_r^{k+k_r} \bar{\zeta}_r^{\,k_r} \\ &= \Big(\prod_{j=1}^{r} \Big(\frac{\zeta_j}{|\zeta_j|} \Big)^k \Big) |\zeta_1|^{k+2k_1} \cdots |\zeta_r|^{k+2k_r} \end{aligned} \tag{2.37}$$

for some nonnegative integers $k_1, \dots, k_r$. In other words, spherical harmonics from the orthogonal expansion of $\widetilde{m}^{(k)}$ are necessarily linear combinations of (2.37). Also note that $P_{k_1,\dots,k_r}$ has degree $rk + 2k_1 + \cdots + 2k_r$.

By the previous discussion, the expansion of $\widetilde{m}^{(k)}$ into spherical harmonics is of the form

$$\widetilde{m}^{(k)} = \sum_{j=rk}^{\infty} \widetilde{Y}_j^{(k)},$$

where each $\widetilde{Y}_j^{(k)}$ is a spherical harmonic of degree j. Let us set

$$Y_j^{(k)} := \mathbb{i}^j \gamma_{n,j,0} \widetilde{Y}_j^{(k)}$$

for every integer $j \geq rk$ and define another spherical function by

$$u^{(k)} := \sum_{j=rk}^{\infty} Y_j^{(k)}. \tag{2.38}$$

Lemma 2.4.1. *Functions $u^{(k)}$ satisfy*

$$\|u^{(k)}\|_{\mathrm{L}^1(\mathbb{S}^{2r-1})} \sim_r \|u^{(k)}\|_{\mathrm{L}^2(\mathbb{S}^{2r-1})} \sim_r \|u^{(k)}\|_{\mathrm{L}^\infty(\mathbb{S}^{2r-1})} \sim_r k^{-r} \tag{2.39}$$

for every positive integer k.

Proof. By property (1.4), the Fourier transform of

$$K(\xi) = \mathrm{p.\,v.}\,\overline{\widetilde{m}^{(k)}\Big(\frac{\xi}{|\xi|}\Big)}\,|\xi|^{-n}$$

is given by

$$\widehat{K}(\xi) = \overline{u^{(k)}\Big(\frac{\xi}{|\xi|}\Big)}. \tag{2.40}$$

Thus, once we compute $\widehat{K}$, we will be able to read off the function $u^{(k)}$ from (2.40). The following calculations are much in the spirit of the proofs of Theorems 3.3 and 3.10 from [67, Chapter IV].

Recalling

$$K(\zeta_1,\ldots,\zeta_r) = \mathrm{p.\,v.}\Big(\prod_{i=1}^{r}\Big(\frac{\zeta_i}{|\zeta_i|}\Big)^{-k}\Big)\Big(\sum_{i=1}^{r}|\zeta_i|^2\Big)^{-r}$$

and using

$$\big(|\zeta_1|^2+\cdots+|\zeta_r|^2\big)^{-r} = \frac{2\pi^r}{(r-1)!}\int_0^\infty e^{-\pi t^{-2}(|\zeta_1|^2+\cdots+|\zeta_r|^2)}\,\frac{\mathrm{d}t}{t^{2r+1}},$$

we can write

$$K(\zeta_1,\ldots,\zeta_r) = \frac{2\pi^r}{(r-1)!}\int_0^\infty\Big(\prod_{i=1}^{r}\Big(\frac{\zeta_i}{|\zeta_i|}\Big)^{-k}e^{-\pi t^{-2}|\zeta_i|^2}\Big)\,\frac{\mathrm{d}t}{t^{2r+1}}.$$

Let us first compute the Fourier transform of an auxiliary function

$$f\colon \mathbb{C}\to\mathbb{C},\quad f(\zeta) = \Big(\frac{\zeta}{|\zeta|}\Big)^{-k}e^{-\pi t^{-2}|\zeta|^2}$$

by changing variables

$$\zeta = \rho e^{i\varphi},\ \zeta' = \rho' e^{i\varphi'};\quad \rho,\rho'\in(0,\infty),\ \varphi,\varphi'\in[0,2\pi)$$

and writing

$$\begin{aligned}\widehat{f}(\zeta') &= \int_{\mathbb{C}} f(\zeta)e^{-2\pi\mathbb{i}\zeta\cdot\zeta'}\,\mathrm{d}\zeta\\ &= \int_0^\infty\int_0^{2\pi} e^{-\mathbb{i}k\varphi}e^{-\pi t^{-2}\rho^2}e^{-2\pi\mathbb{i}\rho\rho'\cos(\varphi-\varphi')}\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\end{aligned}$$

$$
\begin{aligned}
&\qquad\big[\text{change variable } \varphi \to \varphi + \varphi' + \pi/2\big] \\
&= \mathbb{i}^{-k} e^{-\mathbb{i}k\varphi'} \int_0^\infty \Big(\int_0^{2\pi} e^{-\mathbb{i}k\varphi + 2\pi\mathbb{i}\rho\rho'\sin\varphi}\,\mathrm{d}\varphi \Big) e^{-\pi t^{-2}\rho^2} \rho\,\mathrm{d}\rho \\
&= \mathbb{i}^{-k} e^{-\mathbb{i}k\varphi'} 2\pi \int_0^\infty J_k(2\pi\rho\rho') e^{-\pi t^{-2}\rho^2} \rho\,\mathrm{d}\rho \\
&= \mathbb{i}^{-k} \Big(\frac{\zeta'}{|\zeta'|} \Big)^{-k} 2\pi t^2 \int_0^\infty J_k(2\pi\rho t|\zeta'|) e^{-\pi\rho^2} \rho\,\mathrm{d}\rho.
\end{aligned}
$$

Here J_k denotes the Bessel function of the first kind of an integer order k and we used Bessel's integral formula,

$$
J_k(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\mathbb{i}(x\sin\tau - k\tau)}\,\mathrm{d}\tau = \frac{\mathbb{i}^{-k}}{2\pi} \int_{-\pi}^{\pi} e^{\mathbb{i}(x\cos\tau - k\tau)}\,\mathrm{d}\tau; \quad x \in (0,\infty); \tag{2.41}
$$

see [17, Eq. 10.9.2] or [1, Eq. 9.1.21]. Justifying the interchange of the integral in t and the action the Fourier transform in ζ_i as in the proof of [67, Chapter IV, Theorem 4.5], we see that

$$
\begin{aligned}
\widehat{K}(\zeta_1,\dots,\zeta_r) &= \mathbb{i}^{-rk} \Big(\prod_{i=1}^r \Big(\frac{\zeta_i}{|\zeta_i|} \Big)^{-k} \Big) \frac{2^{r+1}\pi^{2r}}{(r-1)!} \int_0^\infty \Big(\prod_{i=1}^r \int_0^\infty J_k(2\pi\rho t|\zeta_i|) e^{-\pi\rho^2} \rho\,\mathrm{d}\rho \Big) \frac{\mathrm{d}t}{t} \\
&\qquad\big[\text{change variable } t \to t/|\xi|, \text{ where } \xi = (\zeta_1,\dots,\zeta_r)\big] \\
&= \mathbb{i}^{-rk} \Big(\prod_{i=1}^r \Big(\frac{\zeta_i}{|\zeta_i|} \Big)^{-k} \Big) \frac{2^{r+1}\pi^{2r}}{(r-1)!} \int_0^\infty \Big(\prod_{i=1}^r \int_0^\infty J_k\Big(\frac{2\pi\rho t|\zeta_i|}{|\xi|} \Big) e^{-\pi\rho^2} \rho\,\mathrm{d}\rho \Big) \frac{\mathrm{d}t}{t},
\end{aligned}
$$

so comparing with (2.40) we obtain

$$
\begin{aligned}
u^{(k)}(\zeta_1,\dots,\zeta_r) &= \mathbb{i}^{rk} \Big(\prod_{i=1}^r \Big(\frac{\zeta_i}{|\zeta_i|} \Big)^{k} \Big) \frac{2^{r+1}\pi^{2r}}{(r-1)!} \int_0^\infty \Big(\prod_{i=1}^r \int_0^\infty J_k(2\pi\rho t|\zeta_i|) e^{-\pi\rho^2} \rho\,\mathrm{d}\rho \Big) \frac{\mathrm{d}t}{t} \\
&\qquad\big[\text{change variables } \rho \to \rho/\sqrt{2\pi k},\ t \to kt\sqrt{2/\pi}\big] \\
&= \frac{2\pi^r \mathbb{i}^{rk}}{(r-1)!} k^{-r} \Big(\prod_{i=1}^r \Big(\frac{\zeta_i}{|\zeta_i|} \Big)^{k} \Big) \int_0^\infty \Big(\prod_{i=1}^r \int_0^\infty J_k\big(2\sqrt{k}\rho t|\zeta_i|\big) e^{-\rho^2/2k} \rho\,\mathrm{d}\rho \Big) \frac{\mathrm{d}t}{t}
\end{aligned} \tag{2.42}
$$

for every $(\zeta_1,\dots,\zeta_r) \in \mathbb{S}^{2r-1}$.

Define $F_k\colon [0,\infty) \to \mathbb{R}$ by

$$
F_k(s) := \int_0^\infty J_k\big(2\sqrt{k}\rho s\big) e^{-\rho^2/2k} \rho\,\mathrm{d}\rho. \tag{2.43}
$$

A simple change of variable gives

$$
F_k(s) = \int_0^\infty J_k(\rho) \frac{\rho}{4ks^2} e^{-\rho^2/8k^2s^2}\,\mathrm{d}\rho \tag{2.44}
$$

for $s \in (0,\infty)$, while $F_k(0) = 0$. Let us show the following uniform bound in k,

$$0 \le F_k(s) \lesssim \min\{s^2, s^{-2}\} \quad \text{for } s \in (0,\infty),\ k \ge 3, \tag{2.45}$$

and the equality

$$\|F_k\|_{\mathrm{L}^1(0,\infty)} = \int_0^\infty F_k(s)\,\mathrm{d}s = \frac{1}{2}\sqrt{\frac{\pi}{2}} \quad \text{for } k \ge 1. \tag{2.46}$$

We first address (2.45). Using [17, Eq. 10.22.54] we can evaluate the integral (2.44) in terms of the confluent hypergeometric function $\mathbf{M}$ as

$$F_k(s) = 2\Gamma\Big(\frac{k}{2}+1\Big)(2k^2s^2)^{(k+2)/2}e^{-2k^2s^2}\mathbf{M}\Big(\frac{k}{2}, k+1, 2k^2s^2\Big).$$

Then we use the integral representation of $\mathbf{M}$ from [17, Eq. 13.4.1] and simplify to get a convenient formula

$$F_k(s) = \frac{2^{k/2}k^{k+1}s^k}{\Gamma(k/2)}\int_0^1 e^{-2k^2s^2\tau}\tau^{k/2}(1-\tau)^{k/2-1}\,\mathrm{d}\tau. \tag{2.47}$$

Using $1-\tau \le e^{-\tau}$ for $0 \le \tau \le 1$ we bound (2.47) for $k \ge 3$ as

$$\begin{aligned} F_k(s) &\le \frac{2^{k/2}k^{k+1}s^k}{\Gamma(k/2)}\int_0^\infty e^{-2k^2s^2\tau}\tau^{k/2}(e^{-\tau})^{k/2-1}\,\mathrm{d}\tau \\ &\qquad \big[\text{change variable } \tau \to 2\tau/(4k^2s^2+k-2)\big] \\ &= \frac{2^{k/2}k^{k+1}s^k}{\Gamma(k/2)}\Big(\frac{2}{4k^2s^2+k-2}\Big)^{k/2+1}\Gamma\Big(\frac{k}{2}+1\Big) \\ &= \frac{1}{4s^2}\Big(1+\frac{k-2}{4k^2s^2}\Big)^{-k/2-1}. \end{aligned}$$

Bernoulli's inequality gives

$$\Big(1+\frac{k-2}{4k^2s^2}\Big)^{k/4+1/2} \ge 1+\frac{k^2-4}{16k^2s^2} \ge 1+\frac{1}{40s^2},$$

which further controls $F_k(s)$ as

$$F_k(s) \le \frac{1}{4s^2}\Big(1+\frac{1}{40s^2}\Big)^{-2} \le \min\Big\{400s^2, \frac{1}{4s^2}\Big\}.$$

Next, in order to verify (2.46), we use (2.47) to write F_k as a superposition of the functions

$$(0,\infty) \ni s \mapsto s^k e^{-2k^2\tau s^2}.$$

Integrals of these functions easily compute to

$$\frac{\Gamma((k+1)/2)}{2^{(k+3)/2}k^{k+1}\tau^{(k+1)/2}}.$$

It remains to evaluate the integral in τ of this quantity multiplied with a weight appearing in (2.47), thus obtaining the right hand side of (2.46). This completes the proofs of the auxiliary claims (2.45) and (2.46). Moreover, by (2.45) we have

$$\int_{(0,\infty)\setminus[1/R,R]} F_k(s)\,\mathrm{d}s < \frac{1}{10}$$

for a sufficiently large number $R > 1$ that depends only on the implicit constant in (2.45). Combining this with (2.46) we also get

$$\int_{1/R}^{R} F_k(s)\,\mathrm{d}s \gtrsim 1 \quad \text{for } k \geq 1. \tag{2.48}$$

We have all elements to finalize the proof of the lemma. Note that (2.42) and the definition (2.43) give

$$\begin{aligned} \|u^{(k)}\|_{\mathrm{L}^\infty(\mathbb{S}^{2r-1})} &\lesssim_r k^{-r} \sup_{(\zeta_1,\ldots,\zeta_r)\in\mathbb{S}^{2r-1}} \int_0^\infty \prod_{i=1}^r F_k(t|\zeta_i|)\,\frac{\mathrm{d}t}{t} \\ &= k^{-r} \sup_{(\omega_1,\ldots,\omega_r)\in\mathbb{S}_+^{r-1}} \int_0^\infty \prod_{i=1}^r F_k(t\omega_i)\,\frac{\mathrm{d}t}{t}. \end{aligned} \tag{2.49}$$

Fix an arbitrary point $(\omega_1,\ldots,\omega_r)\in\mathbb{S}_+^{r-1}$. Let $\omega_{\max}$ be the largest number among $\omega_1,\ldots,\omega_r$. Clearly $\omega_{\max}\sim_r 1$. By (2.45) we have

$$\int_0^1 \prod_{i=1}^r F_k(t\omega_i)\,\frac{\mathrm{d}t}{t} \lesssim_r \int_0^1 F_k(t\omega_{\max})\,\frac{\mathrm{d}t}{t} \lesssim \int_0^1 t\,\frac{\mathrm{d}t}{t} \lesssim 1$$

and

$$\int_1^\infty \prod_{i=1}^r F_k(t\omega_i)\,\frac{\mathrm{d}t}{t} \lesssim_r \int_1^\infty F_k(t\omega_{\max})\,\frac{\mathrm{d}t}{t} \lesssim \int_1^\infty t^{-2}\,\frac{\mathrm{d}t}{t} \lesssim 1,$$

so that (2.49) guarantees

$$\|u^{(k)}\|_{\mathrm{L}^\infty(\mathbb{S}^{2r-1})} \lesssim_r k^{-r}. \tag{2.50}$$

Next, from (2.42) we also see

$$\|u^{(k)}\|_{\mathrm{L}^1(\mathbb{S}^{2r-1})} \sim_r k^{-r} \int_{\mathbb{S}^{2r-1}} \int_0^\infty \prod_{i=1}^r F_k(t|\zeta_i|)\,\frac{\mathrm{d}t}{t}\,\mathrm{d}\sigma_{2r-1}(\zeta_1,\ldots,\zeta_r)$$

$$\big[\text{use parametrization } \Psi \text{ and recall the surface element (2.35)}\big]$$

$$\begin{aligned}
&\sim_r k^{-r} \int_{\mathbb{S}_+^{r-1}} \int_0^\infty \prod_{i=1}^r F_k(t\omega_i)\,\frac{\mathrm{d}t}{t}\,\omega_1\cdots\omega_r\,\mathrm{d}\sigma_{r-1}(\omega) \\
&\qquad \big[\text{substitute } x_i = t\omega_i \text{ and denote } x=(x_1,\dots,x_r)\big] \\
&= k^{-r} \int_{(0,\infty)^r} \Big(\prod_{i=1}^r F_k(x_i)\Big) \frac{x_1\cdots x_r}{|x|^{2r}}\,\mathrm{d}x \\
&\geq k^{-r} \int_{[1/R,R]^r} \Big(\prod_{i=1}^r F_k(x_i)\Big) \frac{x_1\cdots x_r}{|x|^{2r}}\,\mathrm{d}x \\
&\gtrsim_{r,R} k^{-r} \Big(\int_{1/R}^R F_k(s)\,\mathrm{d}s\Big)^r,
\end{aligned}$$

remembering that we chose $R>1$ in the discussion preceding (2.48). Finally, estimate (2.48) gives

$$\|u^{(k)}\|_{\mathrm{L}^1(\mathbb{S}^{2r-1})} \gtrsim_r k^{-r}. \tag{2.51}$$

Combining (2.50) and (2.51) we clearly get (2.39). □

Now the proof of part (b) of Theorem 2.1.2 can be completed using Theorem 2.1.4 applied with $u=u^{(k)}$ introduced before and $v=v^{(k)}$ defined as

$$v^{(k)} := \sum_{j=rk}^\infty \mathbb{i}^{-j}\gamma_{n,j,n/p} Y_j^{(k)} = \sum_{j=rk}^\infty \gamma_{n,j,n/p}\gamma_{n,j,0}\widetilde{Y}_j^{(k)}.$$

Recall (2.36), i.e., we are taking the symbol m_Φ^k to be a smoothened version of $\widetilde{m}^{(k)}$ in a way that

$$\|\widetilde{m}^{(k)} - m_\Phi^k\|_{\mathrm{L}^2(\mathbb{S}^{n-1})} \leq \sigma_{n-1}\big(\mathbb{S}^{n-1}\setminus\Psi(D_\delta)\big)^{1/2} = o_n^{\delta\to 0+}(1). \tag{2.52}$$

Trivially,

$$\|\widetilde{m}^{(k)}\|_{\mathrm{L}^2(\mathbb{S}^{n-1})} \sim_n 1. \tag{2.53}$$

On the one hand,

$$\begin{aligned}
\|v^{(k)}\|^2_{\mathrm{L}^2(\mathbb{S}^{2r-1})} &= \sum_{j=rk}^\infty \gamma^2_{2r,j,2r/p}\gamma^2_{2r,j,0}\big\|\widetilde{Y}_j^{(k)}\big\|^2_{\mathrm{L}^2(\mathbb{S}^{2r-1})} \sim_r \sum_{j=rk}^\infty j^{4r/p-4r}\big\|\widetilde{Y}_j^{(k)}\big\|^2_{\mathrm{L}^2(\mathbb{S}^{2r-1})} \\
&\leq (rk)^{4r/p-4r} \sum_{j=rk}^\infty \big\|\widetilde{Y}_j^{(k)}\big\|^2_{\mathrm{L}^2(\mathbb{S}^{2r-1})} = (rk)^{4r/p-4r}\|\widetilde{m}^{(k)}\|^2_{\mathrm{L}^2(\mathbb{S}^{2r-1})},
\end{aligned}$$

so, in combination with (2.53),

$$\|v^{(k)}\|_{\mathrm{L}^2(\mathbb{S}^{n-1})} \lesssim_n k^{n/p-n}. \tag{2.54}$$

On the other hand,

$$\begin{aligned}\langle \widetilde{m}^{(k)}, v^{(k)} \rangle_{\mathrm{L}^2(\mathbb{S}^{2r-1})} &= \sum_{j=rk}^{\infty} \langle \mathbb{i}^{-j} \gamma_{2r,j,2r} Y_j^{(k)}, \mathbb{i}^{-j} \gamma_{2r,j,2r/p} Y_j^{(k)} \rangle_{\mathrm{L}^2(\mathbb{S}^{2r-1})} \\ &= \sum_{j=rk}^{\infty} \gamma_{2r,j,2r} \gamma_{2r,j,2r/p} \| Y_j^{(k)} \|^2_{\mathrm{L}^2(\mathbb{S}^{2r-1})} \sim_r \sum_{j=rk}^{\infty} j^{2r/p} \| Y_j^{(k)} \|^2_{\mathrm{L}^2(\mathbb{S}^{2r-1})} \\ &\geq (rk)^{2r/p} \sum_{j=rk}^{\infty} \| Y_j^{(k)} \|^2_{\mathrm{L}^2(\mathbb{S}^{2r-1})} = (rk)^{2r/p} \| u^{(k)} \|^2_{\mathrm{L}^2(\mathbb{S}^{2r-1})}.\end{aligned}$$

From this and (2.39) we get

$$\langle \widetilde{m}^{(k)}, v^{(k)} \rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})} \gtrsim_n k^{n/p-n}. \tag{2.55}$$

As a consequence of (2.52), (2.54), and (2.55) we can choose $\delta > 0$ sufficiently small depending on n only, to achieve

$$\begin{aligned}\langle m_\Phi^k, v^{(k)} \rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})} &\geq \langle \widetilde{m}^{(k)}, v^{(k)} \rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})} - \| \widetilde{m}^{(k)} - m_\Phi^k \|_{\mathrm{L}^2(\mathbb{S}^{n-1})} \| v^{(k)} \|_{\mathrm{L}^2(\mathbb{S}^{n-1})} \\ &\gtrsim_n k^{n/p-n}\end{aligned}$$

for every positive integer k. Combining this with (2.39), we can write

$$\frac{|\langle m_\Phi^k, v^{(k)} \rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})}|}{\| u^{(k)} \|_{\mathrm{L}^q(\mathbb{S}^{n-1})}} \gtrsim_n \frac{|\langle m_\Phi^k, v^{(k)} \rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})}|}{\| u^{(k)} \|_{\mathrm{L}^\infty(\mathbb{S}^{n-1})}} \gtrsim_n \frac{k^{n/p-n}}{k^{-n/2}} = k^{n(1/p-1/2)}.$$

Moreover,

$$\sum_{j=rk}^{\infty} j^n \| Y_j^{(k)} \|^2_{\mathrm{L}^2(\mathbb{S}^{n-1})} \sim_n \sum_{j=rk}^{\infty} \| \widetilde{Y}_j^{(k)} \|^2_{\mathrm{L}^2(\mathbb{S}^{n-1})} = \| \widetilde{m}^{(k)} \|^2_{\mathrm{L}^2(\mathbb{S}^{n-1})} < \infty,$$

so that Remark 2.3.2 applies and it guarantees L^q convergence of the series (2.38) on $\mathbb{S}^{n-1}$. We can now apply Theorem 2.1.4; estimates (2.18) and (2.19) respectively give (2.11) and (2.12) for every positive integer k.

2.5. Proof of Theorem 2.1.3

Take $\lambda \geq 1$. Decomposition of the symbol $m^{\lambda}_{\cos}$ into spherical harmonics is simply its Fourier series expansion,

$$m^{\lambda}_{\cos}(e^{\mathbb{i}\varphi}) = e^{\mathbb{i}\lambda\cos\varphi} = J_0(\lambda) + 2\sum_{j=1}^{\infty} \mathbb{i}^j J_j(\lambda)\cos j\varphi, \tag{2.56}$$

where J_j are again Bessel functions of the first kind and we used formula (2.41). Taking the real part of the identity (2.56) we get

$$\cos(\lambda\cos\varphi) = J_0(\lambda) + 2\sum_{l=1}^{\infty}(-1)^l J_{2l}(\lambda)\cos 2l\varphi \tag{2.57}$$

and then changing $\varphi \to \varphi - \pi/2$ we also obtain

$$\cos(\lambda\sin\varphi) = J_0(\lambda) + 2\sum_{l=1}^{\infty} J_{2l}(\lambda)\cos 2l\varphi. \tag{2.58}$$

The following technical lemma deals with sums of Bessel functions with even-integer orders, and it contains all information we need about the Fourier coefficients of $m^{\lambda}_{\cos}$.

Lemma 2.5.1. *We have*

$$\sum_{l=1}^{\infty} l^2 J_{2l}(\lambda)^2 \gtrsim \lambda^2, \tag{2.59}$$

$$\sum_{l=1}^{\infty} l^4 J_{2l}(\lambda)^2 \lesssim \lambda^4, \tag{2.60}$$

and for every $p \in [1,2]$ *we have*

$$\sum_{l=1}^{\infty} l^{2/p-1} J_{2l}(\lambda)^2 \gtrsim \lambda^{2/p-1}. \tag{2.61}$$

Proof. Since $m^{\lambda}_{\cos}$ is C^{∞} on $\mathbb{S}^1$, its coefficients $(J_j(\lambda))_{j=1}^{\infty}$ decay faster than $O_{\lambda}^{j\to\infty}(j^{-A})$ for every $A > 0$. We can differentiate the series in (2.57) term-by-term once or twice with respect to φ, which gives us two more identities:

$$\lambda\sin(\lambda\cos\varphi)\sin\varphi = 4\sum_{l=1}^{\infty}(-1)^{l-1} l J_{2l}(\lambda)\sin 2l\varphi, \tag{2.62}$$

$$-\lambda^2\cos(\lambda\cos\varphi)\sin^2\varphi + \lambda\sin(\lambda\cos\varphi)\cos\varphi = 8\sum_{l=1}^{\infty}(-1)^{l-1} l^2 J_{2l}(\lambda)\cos 2l\varphi. \tag{2.63}$$

On the one hand, an application of Parseval's formula to the function on the left hand side of (2.62) yields

$$\begin{aligned}\sum_{l=1}^{\infty} l^2 J_{2l}(\lambda)^2 &\sim \lambda^2 \int_{-\pi}^{\pi} \sin^2(\lambda\cos\varphi)\sin^2\varphi\,\mathrm{d}\varphi \\ &\geq 2\lambda^2 \int_{0}^{\pi} \sin^2(\lambda\cos\varphi)\sin^3\varphi\,\mathrm{d}\varphi \\ &\quad \big[\text{substitute } s=\cos\varphi\big] \\ &= 2\lambda^2 \int_{-1}^{1} (1-s^2)\sin^2\lambda s\,\mathrm{d}s = 2\lambda^2\Big(\frac{2}{3}+\frac{\cos 2\lambda}{2\lambda^2}-\frac{\sin 2\lambda}{4\lambda^3}\Big),\end{aligned}$$

which confirms (2.59). On the other hand, Parseval's formula applied to (2.63) clearly gives (2.60). Finally, for any fixed $p\in[1,2]$, by Hölder's inequality for sums we have

$$\Big(\sum_{l=1}^{\infty} l^2 J_{2l}(\lambda)^2\Big)^{5/2-1/p} \leq \Big(\sum_{l=1}^{\infty} l^{2/p-1} J_{2l}(\lambda)^2\Big)\Big(\sum_{l=1}^{\infty} l^4 J_{2l}(\lambda)^2\Big)^{3/2-1/p},$$

which, in combination with (2.59) and (2.60), gives (2.61). □

Let us return to the proof of Theorem 2.1.3. We are about to apply Theorem 2.1.4 with the function

$$u^{(\lambda)}(e^{\mathrm{i}\varphi}) := \cos(\lambda\sin\varphi) - J_0(\lambda),$$

recalling that its Fourier expansion can be seen from (2.58). The corresponding function $v^{(\lambda)}$ is then given by

$$v^{(\lambda)}(e^{\mathrm{i}\varphi}) := 2\sum_{l=1}^{\infty}(-1)^l \gamma_{2,2l,2/p} J_{2l}(\lambda)\cos 2l\varphi.$$

Clearly,

$$\|u^{(\lambda)}\|_{\mathrm{L}^q(\mathbb{S}^1)} \lesssim \|u^{(\lambda)}\|_{\mathrm{L}^\infty(\mathbb{S}^1)} \leq 2,$$

while (1.6) and (2.61) give

$$\langle m^{\lambda}_{\cos}, v^{(\lambda)}\rangle_{\mathrm{L}^2(\mathbb{S}^1)} = 4\pi\sum_{l=1}^{\infty}\gamma_{2,2l,2/p} J_{2l}(\lambda)^2 \gtrsim \lambda^{2/p-1}.$$

Since

$$\frac{|\langle m^{\lambda}_{\cos}, v^{(\lambda)}\rangle_{\mathrm{L}^2(\mathbb{S}^1)}|}{\|u^{(\lambda)}\|_{\mathrm{L}^q(\mathbb{S}^1)}} \gtrsim \lambda^{2/p-1} = \lambda^{2(1/p-1/2)},$$

estimates (2.18) and (2.19) give the lower bounds in (2.14) and (2.15), respectively, for every positive integer k.

The upper bounds in (2.14) and (2.15) are just special cases of (2.9) and (2.10) in the part (a) of Theorem 2.1.2.

2.6. Closing remarks

Remark 2.6.1. Note that the equality (2.36) only holds on the major cone in $\mathbb{R}^n = \mathbb{R}^{2r}$ and changing the symbol on a set of small measure can drastically change the Fourier multiplier norm. The fact that m_Φ^k does not exactly split into a tensor product of two-dimensional symbols prevents us from plugging in elementary tensors for "almost extremizing" functions.

Indeed, let us try to test our operator T_Φ^k on $f = f_{p,r,\varepsilon}$ and $g = g_{k,p,r,\varepsilon}$ given as r-fold elementary tensors

$$f(x_1,x_2,\ldots,x_{2r-1},x_{2r}) := f_2(x_1,x_2)\cdots f_2(x_{2r-1},x_{2r}),$$
$$g(x_1,x_2,\ldots,x_{2r-1},x_{2r}) := g_2(x_1,x_2)\cdots g_2(x_{2r-1},x_{2r}).$$

Here f_2 and g_2 depend on k,p,ε and they are chosen as in Lemma 2.3.1, i.e., such that

$$\Big\| f_2(x) - |x|^{-2/p}\mathbb{1}_{\{\varepsilon\le|x|\le 1/\varepsilon\}}(x)\Big\|_{\mathrm{L}^p_x(\mathbb{R}^2)} \lesssim_p 1,$$
$$\Big\| \widehat{f_2}(\xi) - \gamma_{2,0,2/q}|\xi|^{-2/q}\mathbb{1}_{\{\varepsilon\le|\xi|\le 1/\varepsilon\}}(\xi)\Big\|_{\mathrm{L}^q_\xi(\mathbb{R}^2)} \lesssim_p 1,$$
$$\Big\| g_2(x) - \Big(\frac{x_1+\mathbb{i}x_2}{|x_1+\mathbb{i}x_2|}\Big)^k |x|^{-2/q}\mathbb{1}_{\{\varepsilon\le|x|\le 1/\varepsilon\}}(x)\Big\|_{\mathrm{L}^q_x(\mathbb{R}^2)} \lesssim_{k,p} 1,$$
$$\Big\| \widehat{g_2}(\xi) - \mathbb{i}^{-k}\gamma_{2,k,2/p}\Big(\frac{\xi_1+\mathbb{i}\xi_2}{|\xi_1+\mathbb{i}\xi_2|}\Big)^k |\xi|^{-2/p}\mathbb{1}_{\{\varepsilon\le|\xi|\le 1/\varepsilon\}}(\xi)\Big\|_{\mathrm{L}^p_\xi(\mathbb{R}^2)} \lesssim_{k,p} 1.$$

Then we have

$$\|f\|_{\mathrm{L}^p(\mathbb{R}^n)}\|g\|_{\mathrm{L}^q(\mathbb{R}^n)} = \Big(4\pi\log\frac{1}{\varepsilon}\Big)^r + o^{\varepsilon\to 0+}_{k,p,r}\Big(\Big(\log\frac{1}{\varepsilon}\Big)^r\Big)$$

and

$$\begin{aligned}
&\langle T_\Phi^k f, g\rangle_{\mathrm{L}^2(\mathbb{R}^n)} = \big\langle m_\Phi^k \widehat{f}, \widehat{g}\big\rangle_{\mathrm{L}^2(\mathbb{R}^n)} \\
&= \big\langle m_\Phi^k(\xi)\,\widehat{f_2}(\xi_1,\xi_2)\cdots\widehat{f_2}(\xi_{2r-1},\xi_{2r}),\, \widehat{g_2}(\xi_1,\xi_2)\cdots\widehat{g_2}(\xi_{2r-1},\xi_{2r})\big\rangle_{\mathrm{L}^2_\xi(\mathbb{R}^n)} \\
&= \mathbb{i}^{rk}\gamma^r_{2,k,2/p}\int_{\mathbb{R}^n} m_\Phi^k(\xi)\Big(\prod_{j=1}^r \Big(\overline{\frac{\xi_{2j-1}+\mathbb{i}\xi_{2j}}{|\xi_{2j-1}+\mathbb{i}\xi_{2j}|}}\Big)^k |(\xi_{2j-1},\xi_{2j}))|^{-2} \\
&\qquad \mathbb{1}_{\{\varepsilon\le|(\xi_{2j-1},\xi_{2j})|\le 1/\varepsilon\}}(\xi_{2j-1},\xi_{2j})\,\mathrm{d}\xi_{2j-1}\,\mathrm{d}\xi_{2j}\Big) \qquad (2.64)\\
&\quad + o^{\varepsilon\to 0+}_{n,k,p}\Big(\Big(\log\frac{1}{\varepsilon}\Big)^r\Big)
\end{aligned}$$

The integral (2.64) restricted to the major cone M_δ equals

$$\int_{M_\delta} \prod_{j=1}^{r} |(\xi_{2j-1},\xi_{2j}))|^{-2} \mathbb{1}_{\{\varepsilon\le|(\xi_{2j-1},\xi_{2j})|\le 1/\varepsilon\}}(\xi_{2j-1},\xi_{2j})\,\mathrm{d}\xi_{2j-1}\,\mathrm{d}\xi_{2j},$$

but, for a fixed δ, this grows only like $\log(1/\varepsilon)$ and not like $(\log(1/\varepsilon))^r$, as it should. The major cone M_δ is not so much "major" in this matter.

Thus, we are better off sticking to functions f and g with more radial symmetry, just as we did before. This is also philosophically in line with the fact that the symbol m_Φ^k is homogeneous.

Remark 2.6.2. The auxiliary function $\widetilde{m}^{(k)}$ used in Subsection 2.4.2 has a quite complicated expansion into spherical harmonics despite its relatively simple defining formula. In $n=4$ dimensions this expansion can still be computed explicitly. For simplicity suppose that $k\ge 2$ is even. Then

$$\widetilde{m}^{(k)} = \sum_{\substack{j\ge 2k \\ j\text{ divisible by }4}} \widetilde{Y}_j^{(k)}, \tag{2.65}$$

where

$$\begin{aligned}\widetilde{Y}_j^{(k)}(\zeta_1,\zeta_2) = & \frac{\binom{j}{j/2}\binom{j/2}{j/4-k/2}(j+1)k}{\binom{j/2}{j/4}\binom{j}{j/2-k}(j/2)(j/2+1)} \\ & \times \zeta_1^k\zeta_2^k \sum_{l=0}^{j/2-k}(-1)^l \binom{j/2}{j/2-k-l}\binom{j/2}{l}|\zeta_1|^{j-2k-2l}|\zeta_2|^{2l}.\end{aligned}$$

Even though this formula is explicit, it still does not reveal how to compute the associated auxiliary function $u^{(k)}$, which is the work we were doing in the very technical proof of Lemma 2.4.1. However, one can argue that $u^{(k)}$ is "very close" to a constant multiple of

$$\mathbb{C}^2 \supset \mathbb{S}^3 \ni (\zeta_1,\zeta_2) \mapsto \Big(\frac{\zeta_1}{|\zeta_1|}\Big)^k\Big(\frac{\zeta_2}{|\zeta_2|}\Big)^k|\zeta_1|^2|\zeta_2|^2,$$

and the latter function could have been used for the same purpose, leading to a slightly shorter proof in four dimensions.

In higher dimensions computing the expansion of $\widetilde{m}^{(k)}$ into spherical harmonics seemed impossible to us, or at least not possible in any explicit or practical way. Also note a pleasant property of (2.65): only its every fourth term is nonzero. This property is not retained in higher dimensions, where only every other term in the corresponding expansion is equal to zero. Since the passage from u to v in Theorem 2.1.4 requires inserting the coefficients

$\mathbb{i}^{-j}\gamma_{n,j,n/p}$, we see that we are, in fact, also inserting $\pm$ signs into the series $\sum_{j=0}^{\infty} Y_j$, which is a very subtle operation.

3. NORM GROWTH OF POWERS OF UNIMODULAR MULTIPLIERS IS A GENERIC PROPERTY

The content of this chapter is based on the paper [6]. In this chapter we consider again the family of Fourier multiplier operators T^t_Φ associated with symbols $\xi \mapsto \exp(\mathbb{i}t\Phi(\xi))$ and prove that for a generic phase function Φ, one has the estimate $\|T^t_\Phi\|_{L^p\to L^p} \gtrsim_{d,p,\Phi} \langle t\rangle^{d|\frac{1}{p}-\frac{1}{2}|}$. That is the maximal possible order of growth in $t \to \pm\infty$, according to the previous chapter and the result shows that the special example of function Φ that induce the maximal growth, given in the previous chapter to prove sharpness of the estimate is just an example of the general phenomenon.

We emphasize again that in this chapter we use the following normalization for the Fourier transform:

$$\widehat{f}(\xi) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} f(x)e^{-\mathbb{i}x\cdot\xi}\mathrm{d}x$$

3.1. INTRODUCTION AND MAIN RESULTS

For $d \in \mathbb{N}$, $d \geq 2$ and $\Phi \in C^\infty(\mathbb{R}^d \setminus \{0\}, \mathbb{R})$, a homogeneous function of degree 0, we consider a family of Fourier multiplier operators indexed by $t \in \mathbb{R}$, defined on the set of Schwartz functions $\mathscr{S}(\mathbb{R}^d)$ with

$$T^t_\Phi f(x) := \int_{\mathbb{R}^d} e^{\mathbb{i}t\Phi(\xi)+\mathbb{i}\xi\cdot x}\widehat{f}(\xi)d\xi. \tag{3.1}$$

Observe the difference in the exponent, avoiding the 2π. By the Theorem 2.1.2 we know that such operators are bounded and we know that for every exponent $p \in (1,\infty)$ and $t \in \mathbb{R}$

we have

$$\left\|T_{\Phi}^{t}\right\|_{L^p\to L^p} \lesssim_{d,\Phi} (p^*-1)\,\langle t\rangle^{d\left|\frac{1}{2}-\frac{1}{p}\right|}.$$

Theorem 2.1.2 also proves that the estimate is asymptotically sharp in $t\to\infty$ in even dimensions by giving a concrete example of the function Φ_0 for which the following lower bound holds for all $k\in\mathbb{Z}$.

$$\left\|T_{\Phi_0}^{k}\right\|_{L^p\to L^p} \gtrsim_{d,\Phi_0} (p^*-1)|k|^{d\left|\frac{1}{2}-\frac{1}{p}\right|},$$

thus disproving the conjecture of Maz'ya from [48, §4.2] in even dimensions, where he asked whether the following estimate holds for all homogeneous functions $\Phi\in C^\infty(\mathbb{R}^d\setminus\{0\},\mathbb{R})$ of degree 0 and all $t\in\mathbb{R}$:

$$\left\|T_{\Phi}^{t}\right\|_{L^p\to L^p} \lesssim_{d,p,\Phi} \langle t\rangle^{(d-1)\left|\frac{1}{2}-\frac{1}{p}\right|}.$$

D. Stolyarov [68] independently, and using different techniques, proved the asymptotically sharp upper bound for $\|T_{\Phi}^{t}\|_{L^p\to L^p}$ and showed the existence of a function Φ_0 that proves the sharpness in all dimensions $d\geq 2$, but both without sharp dependence in $p\in(1,\infty)$.

Both proofs for the upper bound - the one in the previous chapter and the one in [68] are short. The author in [68] reduced the problem to the sharp version of the Hörmander–Mikhlin multiplier theorem.

However, regarding the sharpness of the estimate, both the proof from the previous chapter and the one from [68] give very specific functions for which the upper bound is sharp. Also, both proofs are relatively long and the cause of the exact worst asymptotics is not apparent. In this chapter we give a short proof of existence of a general phenomenon that drives the growth of powers of norms for a generic symbol, proving that all, when $d=2$, and "almost all", when $d\geq 3$, unimodular homogeneous symbols of degree 0 are counterexamples to the asymptotic order of growth conjectured by V. Maz'ya in [48, §4.2] and, in fact, exhibit the worst possible asymptotics. More precisely, we prove the following theorem.

Theorem 3.1.1. *Let $d\in\mathbb{N}$, $d\geq 2$, $p\in(1,\infty)$ and $t\in\mathbb{R}$.*

(a) *For $d=2$ and any nonconstant homogeneous function $\Phi \in C^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ of degree 0 there exists a constant $c_{d,p,\Phi} > 0$ such that*

$$\left\|T_\Phi^t\right\|_{L^p \to L^p} \geq c_{d,p,\Phi} \langle t \rangle^{2\left|\frac{1}{2}-\frac{1}{p}\right|}.$$

(b) *For $d \geq 3$, there is a dense open set $\mathscr{G}$ in the Whitney topology on $C^\infty(\mathbb{S}^{d-1}, \mathbb{R})$ such that for all $\phi \in \mathscr{G}$ there exists a constant $c_{d,p,\phi} > 0$ for which the 0-homogeneous extension $\Phi \in C^\infty(\mathbb{R}^d \setminus \{0\}, \mathbb{R})$ of ϕ, defined as $\Phi(\xi) := \phi(\frac{\xi}{|\xi|})$, satisfies*

$$\left\|T_\Phi^t\right\|_{L^p \to L^p} \geq c_{d,p,\phi} \langle t \rangle^{d\left|\frac{1}{2}-\frac{1}{p}\right|}.$$

In topology, the property that holds on a dense open set (or, more generally, on the complement of a countable union of nowhere dense sets) is called generic. Therefore, since a 0-homogeneous function on $\mathbb{R}^d \setminus \{0\}$ is uniquely defined by its restriction on the sphere $\mathbb{S}^{d-1}$, the previous theorem says that multipliers associated with powers of a generic 0-homogeneous unimodular symbol exhibit the asymptotically maximal possible order of growth of $L^p \to L^p$ norms.

The part (b) of 3.1.1 will be a consequence of the following theorem, which can be of its own interest, so we state it here. For the definition of nondegeneracy of a critical point see 1.4.

Theorem 3.1.2. *Let $d \in \mathbb{N}$, $d \geq 2$, $p \in (1,\infty)$ and $t \in \mathbb{R}$. For a homogeneous function $\Phi \in C^\infty(\mathbb{R}^d \setminus \{0\}, \mathbb{R})$ of degree 0 such that $\Phi|_{\mathbb{S}^{d-1}}$ has a nondegenerate local minimum or maximum, there exists a constant $c_{d,p,\Phi} > 0$ such that*

$$\|T_\Phi^t\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \geq c_{d,p,\Phi} \langle t \rangle^{d\left|\frac{1}{2}-\frac{1}{p}\right|}.$$

Choosing $\Phi(\xi) = \frac{\xi_1}{|\xi|}$ in Theorem 3.1.2, one gets the asymptotics for the so-called Riesz group (a.k.a. the Poincaré–Riesz–Sobolev group) that appears, when $d=3$, in the analysis of the Navier–Stokes equations in a rotating frame; see [27, §2], [28, §4], [29, Eq. (1.3)], or [24, Eq. (23)], while the asymptotics of the same symbol when $d=2$ was studied in [20]. In fact, the lower bound for $d \geq 3$ established by Stolyarov [68] is equivalent to the particular case of Theorem 3.1.2 for this particular choice of Φ.

Finally, for the sake of completeness, we repeat again the fact from the previous chapter that the asymptotics of $t \mapsto \|T_\Phi^t\|_{L^p \to L^p}$ is an uninteresting problem when $d=1$ because for any Φ as before and $t \in \mathbb{R}$, one can write T_{Φ}^{t} as a bounded linear combination of the identity and the Hilbert transform to get the bound $\|T_{\Phi}^{t}\|_{L^p \to L^p} \lesssim 1$ for all $t \in \mathbb{R}$.

3.2. Idea of the proof

Statements of the theorems are interesting only when $|t|$ is large, so the idea of the proof is to follow the approach from [74, Exercise 2.34], used to study the asymptotics of L^p behavior of the Schrödinger propagator (defined by (3.1) with $\Phi(\xi) = |\xi|^2$), that relies on the following observation. When $p \in (1,2]$, and $t > 0$, the correct asymptotics can be written as $t^{-\frac{d}{2}} \times t^{\frac{d}{p}}$, what can be interpreted as base $\times$ height approximation of the L^p norm of a function that resembles a bump of height $t^{-\frac{d}{2}}$ on the ball of radius t with a rapidly decreasing tail, both in x and t, outside of it. In the case of the Schrödinger propagator, Young's inequality for convolutions with an explicit calculation of the kernel, gives $\|T^t_{|\cdot|^2} f\|_{L^\infty} \lesssim t^{-\frac{d}{2}}$ and the method of nonstationary phase applied to (3.1) for x outside of the ball of radius $\gtrsim t$ proves that the function has a rapidly decreasing tail, both in x and t. The two estimates imply $\|T^t_{|\cdot|^2} f\|_{L^p} \lesssim t^{\frac{d}{p}-\frac{d}{2}}$ as $t \to \infty$ for any $p \in (1,\infty)$ and then log-convexity of L^p norms applied to p and $p' = \frac{p}{p-1}$, together with the fact that $T_{|\cdot|^2}$ is a unitary operator on L^2, transfers the upper bounds to lower bounds.

On the contrary, in the case of the homogeneous multipliers of degree 0, the kernel is singular, so one cannot use Young's convolution inequality to control the L^∞ size of $T^t_\Phi f$ and needs a different approach. The obvious method to try is the method of (non-)stationary phase, but since Φ is homogeneous of degree 0, it follows that for all $\xi \in \mathbb{R}^d \setminus \{0\}$

$$0 = \frac{d^2}{dh^2}(\Phi(\xi + h\xi))\Big|_{h=0} = \xi^\top H\Phi(\xi)\xi,$$

implying, together with the fact that the Hessian of the function $\xi \mapsto \langle \xi, x\rangle$ is equal to 0 for all $x, \xi \in \mathbb{R}^d$, that all stationary points of the phase are degenerate, so they don't fall under the scope of the classical method of stationary phase.

We are able to circumvent this problem and reduce the problem to the classical method of stationary phase by transforming the integral representation in the case of an appropriately localized function $\widehat{f}$ using the change of variables in the integral (see (3.3) below). The reason why the method works better when $d = 2$ is the fact that regularity of the Hessian of the phase in the transformed expression does not depend on the second derivative, contrary to the $d \geq 3$ case.

Applying the method of stationary phase to the modified representation (3.3) with a localized function $\widehat{f}$, it turns out again that the best L^∞ bound for $T_\Phi^t f$ is bigger than $t^{-\frac{d}{2}}$, so the function $T_\Phi^t f$ does not resemble a bump function of height $t^{-\frac{d}{2}}$ as it did in the case of the Schrödinger propagator.

However, using implicit and inverse function theorems we are able to show the existence of the set of x's of measure $\sim t^d$ on which the modified phase in (3.3) is stationary and nondegenerate, so the method of stationary phase gives the required asymptotics $|T_\Phi^t f(x)| \sim_{d,p,\Phi} t^{-\frac{d}{2}}$ on the given set. Using the base $\times$ height bound, this implies the required asymptotics, that is $\|T_\Phi^t f\|_p \gtrsim_{d,p,\Phi} t^{\frac{d}{p}-\frac{d}{2}}$.

3.3. PROOFS

Theorem 3.1.1 will essentially follow from the Theorem 3.1.2, but before we proceed to the proof of the Theorem 3.1.2, we prove the following technical lemma that is crucial for the proof of both theorems. It proves the existence of large set of x's for which the modified phase in (3.3) below is stationary and nondegenerate.

Lemma 3.3.1. *Let $d \in \mathbb{N}$, $d \geq 2$ and $\phi \in C^2(\mathbb{R}^{d-1}, \mathbb{R})$. Suppose that either $H\phi(0) > 0$ or $d = 2$ and $\phi'(0) \neq 0$. For $x, \xi \in \mathbb{R}^d$ define*

$$\Phi_x(\xi) := \phi(\xi_-) + \xi_d(\langle \xi_-, x_- \rangle + x_d). \tag{3.2}$$

There exist an open set $U \in \mathbb{R}^d$ and an open set $V \subset \mathbb{R}^{d-1} \times (\frac{1}{4}, 4)$ for which there is a unique function $g : U \to V$ such that for all $x \in U$ it holds $\nabla_\xi \Phi_x(g(x)) = 0$ and the matrix $H_\xi \Phi_x(g(x))$ is regular.

Proof. Suppose first that $H\phi(0) > 0$. Define $F : \mathbb{R}^{2d} \to \mathbb{R}^d$ with:

$$F(\xi, x) = \nabla_\xi \Phi_x(\xi) = \begin{bmatrix} \nabla\phi(\xi_-) + \xi_d x_-^\top & \xi_1 x_1 + \cdots + \xi_{d-1} x_{d-1} + x_d \end{bmatrix}.$$

We want to apply the implicit function theorem to prove the existence of function g such that $F(g(x), x) = 0$. First observe that

$$\nabla_\xi F(\xi, x) = \begin{bmatrix} H\phi(\xi_-) & x_- \\ x_-^\top & 0 \end{bmatrix}.$$

Applying determinant to both sides of the block-matrix identity

$$\begin{bmatrix} I_{d-1} & 0 \\ -x_-^\top (H\phi(\xi_-))^{-1} & 1 \end{bmatrix} \begin{bmatrix} H\phi(\xi_-) & x_- \\ x_-^\top & 0 \end{bmatrix} = \begin{bmatrix} H\phi(\xi_-) & x_- \\ 0 & -x_-^\top (H\phi(\xi_-))^{-1} x_- \end{bmatrix}$$

it follows that

$$\det \nabla_\xi F(\xi, x) = -\langle (H\phi(\xi_-))^{-1} x_-, x_- \rangle \det H\phi(\xi_-).$$

Since $H\phi(0) > 0$, there exists $\varepsilon > 0$ such that $H\phi(\xi_-) > 0$ for all $\xi_- \in B_{d-1}(0, \varepsilon)$. Furthermore, since the inverse of a positive definite matrix is positive definite, we conclude that $\nabla_\xi F(\xi, x)$ is regular whenever $\xi_- \in B_{d-1}(0, \varepsilon)$ and $x_- \neq 0$.

In order to apply the implicit function theorem, we need to show that there exist $\xi^0 \in B_d(0,\varepsilon) \times (\frac{1}{4},4)$ and $x^0 \in \mathbb{R}^d$ with $x^0_- \neq 0$ such that $F(\xi^0,x^0) = 0$.

Since $H\phi(0)$ is regular, because of the inverse function theorem there exist an open set $A \subset B_{d-1}(0,\varepsilon)$ and an open set $B \subset \mathbb{R}^{d-1}$, such that $\nabla\phi : A \to B$ is bijective. Taking $x^0_- \in (-B) \setminus \{0\}$ and ξ^0_- such that $\nabla\phi(\xi^0_-) = -x^0_-$ and defining:

$$\xi^0 = (\xi^0_-,1), \quad x^0 = \left(x^0_-, -\langle \xi^0_-, x^0_- \rangle\right),$$

we can see that $F(\xi^0,x^0) = 0$.

Therefore, the implicit function theorem implies the existence of open sets $U' \ni x^0$ and $V \subset B(0,\varepsilon) \times (\frac{1}{4},4)$ (the second inclusion follows from the fact that $\xi^0_d = 1$ by shrinking U' if necessary) and a unique function $g : U' \to V$ such that $F(g(x),x) = 0$ for every $x \in U'$. If we choose U to be a subset of U' such that all $x \in U$ satisfy $x_- \neq 0$, the regularity of $H_\xi \Phi_x$ follows from the fact that $H_\xi \Phi_y(\xi) = \nabla_\xi F(\xi,x)$ and the previous conclusion of regularity of $\nabla_\xi F(\xi,x)$.

When $d = 2$ and $\phi'(0) \neq 0$, solving

$$F(\xi,x) = \begin{bmatrix} \phi'(\xi_1) + \xi_2 x_1 & \xi_1 x_1 + x_2 \end{bmatrix} = 0,$$

one can see that the unique function $g : \mathbb{R}^2 \setminus (\{0\} \times \mathbb{R}) \to \mathbb{R}^2$ for which $F(g(x),x) = 0$ is given by

$$g(x_1,x_2) = \left(-\frac{x_2}{x_1}, -\frac{\phi'(-\frac{x_2}{x_1})}{x_1}\right).$$

Also, observe that the matrix

$$\nabla_\xi F(\xi,x) = \begin{bmatrix} \phi''(\xi_1) & x_1 \\ x_1 & 0 \end{bmatrix}$$

is regular whenever $x_1 \neq 0$, regardless of $\phi''(\xi_1)$. However, to satisfy the assumption that $V \subset \mathbb{R}^{d-1} \times (\frac{1}{4},4)$, one has to restrict x to a smaller set. Without loss of generality, we may assume that $c := \phi'(0) > 0$. By continuity, there exists $\delta > 0$ such that $\phi'(\xi_1) \in (\frac{c}{2},2c)$ for all $\xi_1 \in (-\delta,\delta)$. Therefore, if $x_1 \in (-2c,-\frac{c}{2})$ and $x_2 \in (-\frac{c\delta}{2},\frac{c\delta}{2})$, then one has $-\frac{x_2}{x_1} \in (-\delta,\delta)$ and $\frac{-\phi'(-\frac{x_2}{x_1})}{x_1} \in (\frac{1}{4},4)$, giving the proof of the theorem with $U = (-2c,-\frac{c}{2}) \times (-\frac{c\delta}{2},\frac{c\delta}{2})$ and $V = \mathbb{R} \times (\frac{1}{4},4)$. $\square$

Proof of Theorem 3.1.2. Using duality of L^p spaces and the fact that

$$\langle T_\Phi^t u, v\rangle = \langle u, T_\Phi^{-t} v\rangle = \langle T_\Phi^t \tilde{v}, \tilde{u}\rangle,$$

where $\tilde{u}(x) := \overline{u(-x)}$, we can, without loss of generality, assume that $t \geq 0$ and $p \in (1,2]$.

By composing the function Φ with the rotation, if necessary, we can assume that the function $\Phi|_{\mathbb{S}^{d-1}}$ has a local minimum at $e_d := (0,\ldots,0,1) \in \mathbb{R}^d$. From the fact that it is nondegenerate, we know that the function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ defined as restriction of Φ to the hyperplane $\langle \xi, e_d\rangle = 1$:

$$\phi(\xi_1,\ldots,\xi_{d-1}) := \Phi(\xi_1,\ldots,\xi_{d-1},1)$$

satisfies $H\phi(0) > 0$. Indeed, observing that

$$\phi(\xi_-) = \Phi(\xi_-,1) = \Phi|_{\mathbb{S}^{d-1}}\left(\frac{\xi_-}{\sqrt{1+|\xi_-|^2}}, \frac{1}{\sqrt{1+|\xi_-|^2}}\right),$$

the statement follows by the definition of nondegeneracy of $\Phi|_{\mathbb{S}^{d-1}}$ and the fact that the function $\xi_- \mapsto \frac{1}{\sqrt{1+|\xi_-|^2}}(\xi_-,1)$ is the inverse of the chart of $\mathbb{S}^{d-1}$ at e_d given as $\xi \mapsto \frac{1}{\xi_d}\xi_-$.

To reduce the integral to the correct form for application of the method of stationary phase, observe that

$$T_\Phi^t f(tx) = \int_{\mathbb{R}^d} e^{\mathrm{i}t(\Phi(\xi)+i\xi\cdot x)}\widehat{f}(\xi)d\xi.$$

Furthermore, observe that for any $\xi \in \mathbb{R}^{d-1}\times(0,\infty)$ one has $\Phi(\xi) = \phi(\frac{1}{\xi_d}\xi_-)$. Since the function $\Lambda(\xi) := \xi_d(\xi_-,1)$ is a C^∞ diffeomorphism from $\mathbb{R}^{d-1}\times\mathbb{R}_+$ onto itself, for any f such that $\operatorname{supp}\widehat{f} \subset \mathbb{R}^{d-1}\times\mathbb{R}_+$ the change of variables $\xi = \Lambda(\xi')$ gives

$$T_\Phi^t f(tx) = \int_{\mathbb{R}^d} e^{\mathrm{i}t(\phi(\xi_-)+\xi_d(\langle\xi_-,x_-\rangle+x_d))}\widehat{f}(\xi_d\xi_-,\xi_d)\xi_d^{d-1}d\xi. \tag{3.3}$$

Let U,V and g be as in Lemma 3.3.1. Since Λ a C^∞ diffeomorphism on V and V is open, there exist $\xi_0 \in \Lambda(g(U))$ and a ball $B_d(\xi_0,\varepsilon) \subset \Lambda(V)$. We choose a function $f \in \mathscr{S}(\mathbb{R}^d)$ such that $\operatorname{supp}\widehat{f} \subset B_d(\xi_0,\varepsilon)$ and $\widehat{f}(\xi) = 1$ for $\xi \in B_d(\xi_0,\frac{\varepsilon}{2})$. Denoting $F(\xi) := \widehat{f}(\Lambda(\xi))\xi_d^{d-1}$ and

$$U_1 = \{x \in U : \Lambda(g(x)) \in B_d(\xi_0,\frac{\varepsilon}{2})\},$$

the continuity of $\Lambda\circ g$ and the fact that $\xi_d \sim 1$ on V imply that U_1 is an open set such that $|F\circ g|_{U_1}| \gtrsim 1$. Denoting $\Phi_x(\xi)$ as in (3.2), the existence and uniqueness of the function g

in Lemma 3.3.1 imply that for any $x \in U$, the function Φ_x has a unique stationary point in $V \supset \operatorname{supp} F$. Therefore, from (3.3), Theorem 1.5.1 and the lower bound for $F \circ g$ on U_1, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |T_\Phi^t f(tx)|^p dx &\geq \int_U |T_\Phi^t f(tx)|^p dx \\
&= \int_U \left| \int_{\mathbb{R}^d} e^{\mathrm{i}t\Phi_x(\xi)} F(\xi) d\xi \right|^p dx \\
&= \int_U \left| t^{-\frac{d}{2}} (2\pi)^{\frac{d}{2}} F(g(x)) |\det H\Phi(g(x))|^{-\frac{1}{2}} + O_x(t^{-\frac{d}{2}-1}) \right|^p dx \\
&\gtrsim_{d,p} t^{-\frac{dp}{2}} \int_{U_1} \left| |\det H\Phi(g(x))|^{-\frac{1}{2}} + O_x(t^{-1}) \right|^p dx,
\end{aligned} \tag{3.4}$$

Taking the p-th root and applying Fatou's lemma, we have:

$$\begin{aligned}
\liminf_{t\to\infty} \frac{\|T_\Phi^t f\|_{L^p}}{t^{\frac{d}{p}-\frac{d}{2}}} &= \liminf_{t\to\infty} \frac{\|T_\Phi^t f(t\cdot)\|_{L^p}}{t^{-\frac{d}{2}}} \\
&\gtrsim_{d,p} \liminf_{t\to\infty} \left(\int_{U_1} \left| |\det H\Phi(g(x))|^{-\frac{1}{2}} + O_x(t^{-1}) \right|^p dx \right)^{\frac{1}{p}} \\
&\gtrsim_{d,p} \left(\int_{U_1} \left| |\det H\Phi(g(x))|^{-\frac{1}{2}} \right|^p dx \right)^{\frac{1}{p}} \\
&\gtrsim_{d,p,\Phi} 1.
\end{aligned}$$

Since f was fixed, one has $\|f\|_{L^p} \sim_p 1$, so the calculation implies that

$$\liminf_{t\to\infty} \frac{\|T_\Phi^t\|_{L^p\to L^p}}{t^{\frac{d}{p}-\frac{d}{2}}} \geq \liminf_{t\to\infty} \frac{\|T_\Phi^t f\|_{L^p}}{\|f\|_{L^p} t^{\frac{d}{p}-\frac{d}{2}}} \gtrsim_{d,p,\Phi} 1,$$

giving the proof of the theorem for $t \geq M$, where $M \in \mathbb{R}_+$ is an absolute constant.

The case $t \in [0, M]$, can be proved using soft methods. Fix any nonzero function $f \in \mathscr{S}(\mathbb{R}^d)$. The fact that the operator T_Φ^t is a unitary operator on L^2 for any $t \in \mathbb{R}$ implies that $T_\Phi^t f$ is not a zero function for any $t \in \mathbb{R}$. Therefore, for all $t \in \mathbb{R}$ one has $\|T_\Phi^t f\|_{L^p} > 0$. Furthermore, for any fixed $x \in \mathbb{R}^d$ and $t_0 \in \mathbb{R}$, from (3.1) and Lebesgue's dominated convergence theorem we have

$$\lim_{t\to t_0} T_\Phi^t f(x) = T_\Phi^{t_0} f(x).$$

Fatou's lemma then implies that

$$\liminf_{t\to t_0} \|T_\Phi^t f\|_{L^p} \geq \|T_\Phi^{t_0} f\|_{L^p},$$

meaning that the function $t \mapsto \|T_\Phi^t f\|_{L^p}$ is lower semicontinuous. Lower semicontinuous function attains a minimum on the compact interval $[0,M]$ and it must be positive by the previous observation. Finally, using the fact that $\langle t\rangle^{\frac{d}{p}-\frac{d}{2}} \sim_M 1$ for $t \in [0,M]$, one gets the required lower bound in the range $[0,M]$, giving the proof of the theorem. $\square$

Finally, we prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Let us prove the part (a). Since $\Phi|_{\mathbb{S}^{d-1}}$ is not constant, there exist a point $\xi_0 \in \mathbb{S}^{d-1}$ and a chart ψ at ξ_0 for which $\nabla(\Phi|_{\mathbb{S}^{d-1}} \circ \psi^{-1})(\psi(\xi_0)) \neq 0$. By composing the function Φ with rotation, if necessary, we can assume that $\xi_0 = e_2$ implying that $\partial_1\Phi(e_2) \neq 0$. Defining ϕ as in the proof of Theorem 3.1.2, Lemma 3.3.1 gives the same conclusion needed to repeat the the proof of the Theorem 3.1.2 verbatim, thus proving the part (a).

We continue to the proof of part (b). Let $\phi : \mathbb{S}^{d-1} \to \mathbb{R}$ be any Morse function on the sphere. Since sphere is a compact set, it has a minimum and the fact that function is Morse implies that the minimum is nondegenerate. Applying the Theorem 3.1.2 to the function $\Phi(\xi) := \phi(\frac{\xi}{|\xi|})$, we can see that all Morse functions satisfy the required asymptotics. From [36, §VI, Theorem 1.2] we know that the set of Morse functions is an open dense set in the standard topology on $C^\infty(\mathbb{S}^{d-1},\mathbb{R})$, so the statement follows. $\square$

Part II

Maximal Fourier restriction

4. Multi-parameter maximal Fourier restriction

The content of this chapter is based on the paper [7].

The main result of this chapter is the strengthening of a quite arbitrary a priori Fourier restriction estimate to a multi-parameter maximal estimate of the same type. This allows us to discuss a certain multi-parameter Lebesgue point property of Fourier transforms, which replaces Euclidean balls by ellipsoids. Along the lines of the same proof, we also establish a d-parameter Menshov–Paley–Zygmund-type theorem for the Fourier transform on $\mathbb{R}^d$. Such a result is interesting for $d \geq 2$ because, in a sharp contrast with the one-dimensional case, the corresponding endpoint L^2 estimate (i.e., a Carleson-type theorem) is known to fail since the work of C. Fefferman in 1970. Finally, we show that a Strichartz estimate for a given homogeneous constant-coefficient linear dispersive PDE can sometimes be strengthened to a certain pseudo-differential version.

4.1. Introduction and main results

A classical sub-branch of harmonic analysis, started in the late 1960s, asks to restrict meaningfully the Fourier transform $\widehat{f}$ of a certain non-integrable function f to certain curved lower-dimensional subsets of the Euclidean space; see Stein's book [66, §VIII.4]. A general setting is obtained by taking a σ-finite measure σ on Borel subsets of $\mathbb{R}^d$. Also, let $S \subseteq \mathbb{R}^d$ be a Borel set such that $\sigma(\mathbb{R}^d \setminus S) = 0$. Typically, S is a closed manifold in $\mathbb{R}^d$ and σ is an appropriately weighted surface measure on S. As soon as we have an a priori estimate

$$\big\|\widehat{f}\,\big|_S\big\|_{\mathrm{L}^q(S,\sigma)} \lesssim_{d,\sigma,p,q} \|f\|_{\mathrm{L}^p(\mathbb{R}^d)} \tag{4.1}$$

for some $p \in (1,\infty)$ and $q \in [1,\infty]$, we can define the *Fourier restriction operator* as the unique bounded linear operator

$$\mathscr{R}\colon \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^q(S,\sigma)$$

such that $\mathscr{R}f = \widehat{f}|_S$ for every function f in the Schwartz space $\mathscr{S}(\mathbb{R}^d)$.

Here and in what follows, we write $A \lesssim_P B$, when the estimate $A \le C_P B$ holds for some finite (but unimportant) constant C_P depending on a set of parameters P.

Let us agree to use the following normalization of the Fourier transform:

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi \mathbb{i} x\cdot\xi}\,\mathrm{d}x$$

for an integrable function f on $\mathbb{R}^d$ and for every $\xi \in \mathbb{R}^d$, so that the inverse Fourier transform is given by

$$\check{g}(x) := \int_{\mathbb{R}^d} g(\xi) e^{2\pi \mathbb{i} x\cdot\xi}\,\mathrm{d}\xi$$

for $g \in \mathrm{L}^1(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. We always have the trivial estimate

$$\big\|\widehat{f}\big|_S\big\|_{\mathrm{L}^\infty(S,\sigma)} \le \|f\|_{\mathrm{L}^1(\mathbb{R}^d)} \tag{4.2}$$

for every $f \in \mathrm{L}^1(\mathbb{R}^d)$, so restriction of the Fourier transform $f \mapsto \widehat{f}|_S$ also gives a bounded linear operator

$$\mathscr{R}\colon \mathrm{L}^1(\mathbb{R}^d) \to \mathrm{L}^\infty(S,\sigma).$$

Using the Riesz–Thorin theorem to interpolate between (4.1) and (4.2) then gives us a family of bounded linear operators

$$\mathscr{R}\colon \mathrm{L}^s(\mathbb{R}^d) \to \mathrm{L}^{qs'/p'}(S,\sigma)$$

for every $1 \le s \le p$, where p' denotes the conjugated exponent of p, i.e., $1/p + 1/p' = 1$. All these operators are mutually compatible on their intersections, so they are rightfully denoted by the same letter $\mathscr{R}$.

A novel route was taken recently by Müller, Ricci, and Wright [52], who initiated the program of justifying pointwise Fourier restriction,

$$\lim_{t\to 0+} \widehat{f} * \chi_t = \mathscr{R}f \quad \sigma\text{-a.e. on } S$$

for $f \in \mathrm{L}^p(\mathbb{R}^d)$, via maximal estimates

$$\Big\| \sup_{t\in(0,\infty)} \big|\widehat{f} * \chi_t\big| \Big\|_{\mathrm{L}^q(S,\sigma)} \lesssim_{d,\sigma,\chi,p,q} \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}. \tag{4.3}$$

Here, $\chi \in \mathscr{S}(\mathbb{R}^d)$ is a Schwartz function with integral 1 and we write $\chi_t(x) := t^{-d}\chi(t^{-1}x)$ for a given parameter $t \in (0,\infty)$. Note that the operator on left hand side of (4.3) cannot be understood as a composition of the Fourier transform with some maximal function of the Hardy–Littlewood type, since the measure σ can be (and typically is) singular with respect to the Lebesgue measure.

The authors of [52] achieved the aforementioned goal in two dimensions by imitating the proofs of (somewhat definite) two-dimensional restriction theorems of Carleson and Sjölin [13] and Sjölin [58]. This methodology was later followed by Ramos [54,55], Jesurum [41], and Fraccaroli [26] to obtain a few higher-dimensional or less smooth/regular results. The second approach to the maximal Fourier restriction was suggested by Vitturi [76], soon after the appearance of [52]. He deduced a nontrivial result for higher-dimensional compact hypersurfaces from ordinary restriction estimates (4.1) by inserting the iterated Hardy–Littlewood maximal function in a clever non-obvious way. The idea of using (4.1) as a black box was later also employed by Oliveira e Silva and one of the present authors [44], while the subsequent paper [43] built on this idea to show that the a priori estimate (4.1) implies the maximal estimate (4.3) in a general and abstract way, as soon as $p < q$. Each of these two approaches has its advantages and its limitations. The present paper builds further upon the second approach and it has been partially motivated by a post on Vitturi's blog [75]. In fact, Theorem 4.1.1 below answers one of the open questions that appeared in [75, §4].

For a given function $\chi\colon \mathbb{R}^d \to \mathbb{C}$ and arbitrary parameters $r_1,\dots,r_d \in (0,\infty)$ we define the *multi-parameter dilate* of χ as

$$\chi_{r_1,\dots,r_d}\colon \mathbb{R}^d \to \mathbb{C}, \quad \chi_{r_1,\dots,r_d}(x_1,\dots,x_d) := \frac{1}{r_1\cdots r_d}\chi\Big(\frac{x_1}{r_1},\dots,\frac{x_d}{r_d}\Big).$$

Also let

$$B_{r_1,\dots,r_d}(y_1,\dots,y_d) := \Big\{(x_1,\dots,x_d) \in \mathbb{R}^d : \frac{(x_1-y_1)^2}{r_1^2} + \cdots + \frac{(x_d-y_d)^2}{r_d^2} \le 1\Big\}$$

be the *ellipsoid* centered at $(y_1,\dots,y_d) \in \mathbb{R}^d$ with semi-axes of lengths $r_1,\dots,r_d$ in directions of the coordinate axes. Its volume will be written simply as $|B_{r_1,\dots,r_d}|$. The particular case $B_r(y) := B_{r,\ldots,r}(y)$ for $r \in (0,\infty)$ is simply the Euclidean ball.

Theorem 4.1.1. *Suppose that the measure space (S,σ) and the exponents $1 < p < q < \infty$ are such that the a priori restriction estimate (4.1) holds for every Schwartz function f.*

(a) Then for every $\chi \in \mathscr{S}(\mathbb{R}^d)$ and every $f \in \mathrm{L}^p(\mathbb{R}^d)$ one also has the multi-parameter maximal estimate

$$\Big\| \sup_{r_1,\ldots,r_d \in (0,\infty)} \big|\widehat{f} * \chi_{r_1,\ldots,r_d}\big| \Big\|_{\mathrm{L}^q(S,\sigma)} \lesssim_{d,\sigma,\chi,p,q} \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}. \tag{4.4}$$

(b) For every $\chi \in \mathscr{S}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \chi = 1$ and every $f \in \mathrm{L}^s(\mathbb{R}^d)$, $1 \le s \le p$, one also has the multi-parameter convergence result

$$\lim_{(0,\infty)^d \ni (r_1,\ldots,r_d) \to (0,\ldots,0)} \widehat{f} * \chi_{r_1,\ldots,r_d} = \mathscr{R} f \quad \sigma\text{-a.e. on } S. \tag{4.5}$$

(c) Moreover, if $f \in \mathrm{L}^s(\mathbb{R}^d)$, $1 \le s \le 2p/(p+1)$, then we also have the "multi-parameter Lebesgue point property"

$$\lim_{(0,\infty)^d \ni (r_1,\ldots,r_d) \to (0,\ldots,0)} \frac{1}{|B_{r_1,\ldots,r_d}|} \int_{B_{r_1,\ldots,r_d}(\xi)} \big|\widehat{f}(\eta) - (\mathscr{R} f)(\xi)\big| \,\mathrm{d}\eta = 0 \tag{4.6}$$

of σ-almost every point $\xi \in S$. In particular,

$$\lim_{(0,\infty)^d \ni (r_1,\ldots,r_d) \to (0,\ldots,0)} \frac{1}{|B_{r_1,\ldots,r_d}|} \int_{B_{r_1,\ldots,r_d}(\xi)} \widehat{f}(\eta) \,\mathrm{d}\eta = (\mathscr{R} f)(\xi) \tag{4.7}$$

for σ-a.e. $\xi \in S$.

Since (4.4) is a stronger maximal inequality than (4.3), Theorem 4.1.1 can be viewed as a multi-parameter generalization of [43, Theorem 1] suggested by Vitturi [75, §4]. For instance, by (4.5) now we are able to justify the existence of limits in various anisotropic scalings, such as

$$\lim_{t\to 0+} \widehat{f} * \chi_{t,t^2,\ldots,t^d}.$$

However, the required assumptions on χ are more restrictive here. The proof of Theorem 4.1.1 will only use that χ is a function satisfying

$$\big|\big(\partial_1 \cdots \partial_d \widehat{\chi}\big)(x)\big| \lesssim_{d,\delta} (1+|x|)^{-d-\delta} \tag{4.8}$$

for some $\delta > 0$ and every $x \in \mathbb{R}^d$. The last condition is different from

$$\big|(\nabla\widehat{\chi})(x)\big| \lesssim_{d,\delta} (1+|x|)^{-1-\delta},$$

used in [43], and (4.8) is not satisfied when χ is the normalized indicator function of the standard unit ball in $d \geq 2$ dimensions.

For similar reasons we conclude the convergence of the Fourier averages over shrinking ellipsoids, (4.6) and (4.7), only in the smaller range $1 \leq s \leq 2p/(p+1)$, and not in the full range $1 \leq s \leq p$, as it was the case with averages over balls [43]. This leads us to interesting open questions, like Problem 4.1.2 below. We will explain in Remark 4.3.1 after the proof of Theorem 4.1.1 that (4.6) and (4.7) could have been equally well formulated for axes-parallel rectangles as

$$\lim_{r_1\to 0+,\ldots,r_d\to 0+} \frac{1}{2^d r_1\cdots r_d} \int_{\xi+[-r_1,r_1]\times\cdots\times[-r_d,r_d]} \big|\widehat{f}(\eta) - (\mathscr{R}f)(\xi)\big| \,\mathrm{d}\eta = 0 \tag{4.9}$$

and

$$\lim_{r_1\to 0+,\ldots,r_d\to 0+} \frac{1}{2^d r_1\cdots r_d} \int_{\xi+[-r_1,r_1]\times\cdots\times[-r_d,r_d]} \widehat{f}(\eta) \,\mathrm{d}\eta = (\mathscr{R}f)(\xi), \tag{4.10}$$

respectively. This would have been a bit more standard. However, the same observation combined with a counterexample by Ramos [55, Proposition 4] reveals a limitation in obtaining the full range of exponents for (4.6) and (4.9) (see the comments in Remark 4.3.1 again), and thus also for (4.7) and (4.10), which are shown here as their consequences. On the other hand, it is still theoretically possible that (4.7) holds in the same range as (4.1). A supporting argument is that the proof of its one-parameter case in [43] actually depended on the geometry of Euclidean balls.

Problem 4.1.2. Prove or disprove that the assumptions of Theorem 4.1.1 imply (4.7) for every $f \in \mathrm{L}^p(\mathbb{R}^d)$ and for σ-a.e. $\xi \in S$.

Another question, related to property (4.6) and stated in Problem 4.1.3 below, remained open after [43] and its particular cases have already been studied by Ramos [54, 55] and Fraccaroli [26]. In words, we do not know how to extend the range $1 \leq s \leq 2p/(p+1)$ even when we only consider balls instead of arbitrary ellipsoids.

Problem 4.1.3. Prove or disprove that, for every $f \in \mathrm{L}^p(\mathbb{R}^d)$, the assumptions of Theorem 4.1.1 imply that σ-almost every point $\xi \in S$ is the Lebesgue point of $\widehat{f}$, in the sense that

$$\lim_{t\to 0+} \frac{1}{|B_t|} \int_{B_t(\xi)} \big|\widehat{f}(\eta) - (\mathscr{R}f)(\xi)\big| \,\mathrm{d}\eta = 0.$$

The general maximal principle from [43], concluding something about the Lebesgue sets of Fourier transforms $\widehat{f}$ from restriction estimates (4.1), has been used by Bilz [3]. It would be interesting to find similar applications of the stronger property (4.6).

The main new ingredient in the proof of Theorem 4.1.1 will be a multi-parameter variant of the Christ–Kiselev lemma [15]. Even if its generalization is somewhat straightforward, we will argue that it is substantial by using it to deduce the following result on the Fourier transform alone, with no restriction phenomena involved.

Theorem 4.1.4.

(a) For $p \in [1,2)$ and $f \in \mathrm{L}^p(\mathbb{R}^d)$ we have the maximal estimate

$$\Big\| \sup_{R_1,\ldots,R_d \in (0,\infty)} \big|\mathscr{F}\big(f \mathbb{1}_{[-R_1,R_1]\times\cdots\times[-R_d,R_d]}\big)\big| \Big\|_{\mathrm{L}^{p'}(\mathbb{R}^d)} \lesssim_{d,p} \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

and d-parameter convergence

$$\lim_{R_1\to\infty,\ldots,R_d\to\infty} \int_{[-R_1,R_1]\times\cdots\times[-R_d,R_d]} f(x) e^{-2\pi \mathrm{i} x\cdot\xi} \,\mathrm{d}x = \widehat{f}(\xi) \tag{4.11}$$

holds for a.e. $\xi \in \mathbb{R}^d$.

(b) If $d \geq 2$, then there exist a function $f \in \mathrm{L}^2(\mathbb{R}^d)$ and a set of positive measure $Q \subseteq \mathbb{R}^d$ such that

$$\limsup_{R_1\to\infty,\ldots,R_d\to\infty} \Big| \int_{[-R_1,R_1]\times\cdots\times[-R_d,R_d]} f(x) e^{-2\pi \mathrm{i} x\cdot\xi} \,\mathrm{d}x \Big| = \infty \quad \textit{for every } x \in Q. \tag{4.12}$$

In particular, even the weak L^2 estimate

$$\Big\| \sup_{R_1,\ldots,R_d \in (0,\infty)} \big|\mathscr{F}\big(f \mathbb{1}_{[-R_1,R_1]\times\cdots\times[-R_d,R_d]}\big)\big| \Big\|_{\mathrm{L}^{2,\infty}(\mathbb{R}^d)} \lesssim_{d} \|f\|_{\mathrm{L}^2(\mathbb{R}^d)}$$

does not hold.

Part (a) can be thought of as a *multi-parameter Menshov–Paley–Zygmund theorem*, while part (b) gives a counterexample to the corresponding multi-parameter analogue of Carleson's theorem [12]. The latter is not our original result, but a mere adaptation of the argument by Charles Fefferman [23] to the continuous setting. We include its detailed proof too for completeness of the exposition.

Finally, connections between the Fourier restriction problem and PDEs have been known since the work of Strichartz [69]. Let us comment on a certain reformulation of (4.4) in that direction. The following standard setting is taken from the textbook by Tao [74]; also see the lecture notes by Koch, Tataru, and Vişan [42]. Let $\phi\colon \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function. A self-adjoint operator $\phi(D) = \phi(\nabla/2\pi\mathbb{i})$ is defined to be the Fourier multiplier associated with the symbol ϕ, i.e.,

$$\widehat{(\phi(D)f)}(\xi) = \phi(\xi)\widehat{f}(\xi).$$

If ϕ happens to be a polynomial

$$\phi(\xi) = \sum_{|\alpha|\leq k} c_\alpha \xi^\alpha$$

in n variables $\xi = (\xi_1,\ldots,\xi_n)$ of degree k with real coefficients c_α, then $\phi(D)$ is just the self-adjoint differential operator acting on Schwartz functions,

$$\phi(D) = \sum_{|\alpha|\leq k} (2\pi\mathbb{i})^{-|\alpha|} c_\alpha \partial^\alpha.$$

The solution of a general scalar constant-coefficient linear dispersive initial value problem

$$\begin{cases} \partial_t u(x,t) = \mathbb{i}\phi(D)u(x,t) & \text{in } \mathbb{R}^n\times\mathbb{R}, \\ u(x,0) = f(x) & \text{in } \mathbb{R}^n \end{cases} \tag{4.13}$$

is given explicitly as

$$u(x,t) = (e^{\mathbb{i}t\phi(D)}f)(x) := \int_{\mathbb{R}^n} e^{\mathbb{i}t\phi(\xi)+2\pi\mathbb{i}x\cdot\xi}\widehat{f}(\xi)\,\mathrm{d}\xi$$

for $x\in\mathbb{R}^n$, $t\in\mathbb{R}$, and a Schwartz function $f\in\mathscr{S}(\mathbb{R}^d)$; see [74, Section 2.1].

Corollary 4.1.5. *Suppose that a Strichartz-type estimate for (4.13) of the form*

$$\big\|(e^{\mathbb{i}t\phi(D)}f)(x)\big\|_{\mathrm{L}^s_{(x,t)}(\mathbb{R}^n\times\mathbb{R})} \lesssim_{n,\phi} \|f\|_{\mathrm{L}^2(\mathbb{R}^n)} \tag{4.14}$$

holds for some exponent $s\in(2,\infty)$ and every Schwartz function $f\in\mathscr{S}(\mathbb{R}^n)$. Then for every $\psi\in\mathscr{S}(\mathbb{R}^{n+1})$ and any choice of measurable functions $r_1,\ldots,r_{n+1}\colon\mathbb{R}^n\to\mathbb{R}$ the pseudo-differential operator

$$(T_{\psi,r_1,\ldots,r_{n+1}}f)(x,t) := \int_{\mathbb{R}^n} \psi\big(r_1(\xi)x_1,\ldots,r_n(\xi)x_n,r_{n+1}(\xi)t\big)\,e^{\mathbb{i}t\phi(\xi)+2\pi\mathbb{i}x\cdot\xi}\widehat{f}(\xi)\,\mathrm{d}\xi$$

satisfies the same bound

$$\|T_{\psi,r_1,\ldots,r_{n+1}}f\|_{\mathrm{L}^s(\mathbb{R}^{n+1})} \lesssim_{n,\phi,\psi,s} \|f\|_{\mathrm{L}^2(\mathbb{R}^n)}, \tag{4.15}$$

with a constant that is independent of $r_1,\ldots,r_{n+1}$.

Note that (4.14) is a particular case of (4.15), as the former inequality can be easily recovered by taking $r_1,\ldots,r_{n+1}$ to be identically 0. Specifically for the Schrödinger equation, i.e., when $\phi(D) = \Delta$, the Strichartz estimate (4.14) holds with $s = 2+4/n$. A larger range of Strichartz estimates is available when one introduces the mixed norms [2], see [74, Theorem 2.3] or a review paper [16], but our proof of Corollary 4.1.5 is not well suited for this generalization.

While (4.15) is questionably interesting in the theory of PDEs, we merely wanted to present a restatement of (4.4) in that language. Note that in the definition of the above pseudo-differential operator it is only physically meaningful to scale the spatial variable x and the time variable t independently. In other words, just writing $\psi(r(\xi)(x,t))$ would make no sense. This also partly motivates the study of multiparameter maximal Fourier restriction estimates.

4.2. MULTI-PARAMETER CHRIST–KISELEV LEMMA

This section is devoted to a bound on rather general multi-parameter maximal operators, which generalizes a classical result of Christ and Kiselev [15].

Let $(\mathbb{X},\mathscr{X},\mu)$ and $(\mathbb{Y},\mathscr{Y},\nu)$ be measure spaces. Let d be a positive integer, which we interpret as the number of "parameters." For every $1\leq j\leq d$ we are also given a countable totally ordered set I_j and an increasing system $(E_j(i) : i\in I_j)$ of sets from $\mathscr{Y}$, i.e., an increasing function $E_j\colon I_j\to\mathscr{Y}$ with respect to the order on I_j and the set inclusion on $\mathscr{Y}$.

Lemma 4.2.1 (Multi-parameter Christ–Kiselev lemma)**.** *Take exponents* $1\leq p<q\leq\infty$ *and a bounded linear operator* $T\colon \mathrm{L}^p(\mathbb{Y},\mathscr{Y},\nu)\to\mathrm{L}^q(\mathbb{X},\mathscr{X},\mu)$*. The maximal operator*

$$(T_\star f)(x) := \sup_{(i_1,\ldots,i_d)\in I_1\times\cdots\times I_d} \left|T\left(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_d(i_d)}\right)(x)\right|$$

is also bounded from $\mathrm{L}^p(\mathbb{Y},\mathscr{Y},\nu)$ *to* $\mathrm{L}^q(\mathbb{X},\mathscr{X},\mu)$ *with the operator norm satisfying*

$$\|T_\star\|_{\mathrm{L}^p(\mathbb{Y})\to\mathrm{L}^q(\mathbb{X})} \leq \left(1-2^{1/q-1/p}\right)^{-d}\|T\|_{\mathrm{L}^p(\mathbb{Y})\to\mathrm{L}^q(\mathbb{X})}. \tag{4.16}$$

The particular case $d=1$ is precisely [15, Theorem 1.1]. The proof given below is a d-parameter modification of the approach from [15], incorporating a simplification due to Tao [70, Note #2], who used an induction on the cardinality of I_1 to immediately handle general measure spaces with atoms. We include all details, since we desire to have a self-contained exposition.

Proof. By the monotone convergence theorem it is sufficient to prove the claim when the ordered sets $I_1,\ldots,I_d$ are finite. Note that it is crucial that the desired bound does not depend anyhow on their sizes. Thus, the proof will only consider finite index sets I_j. The exponents p and q, the two measure spaces, and the operator T are fixed throughout the proof. We are using a nested mathematical induction, first on d and then on the cardinality of I_d, to prove (4.16) for all finite increasing systems of sets $(E_j(i) : i\in I_j)$, $1\leq j\leq d$. The induction basis $d=1=|I_1|$ is trivial, since then $T_\star$ satisfies the same bound as T.

We turn to the induction step. By renaming the indices we can achieve that $I_j = \{1,2,\dots,n_j\}$ for each $1 \le j \le d$ and some positive integers $n_1,\dots,n_d$. Denote

$$F(i) := E_1(n_1)\cap\cdots\cap E_{d-1}(n_{d-1})\cap E_d(i) \quad \text{for } 1\le i\le n_d.$$

Take a function $f \in \mathrm{L}^p(\mathbb{Y},\mathscr{Y},\nu)$. By the assumption that the system $(E_d(i) : i \in I_d)$ is increasing, we have

$$0 \le \|f\|_{\mathrm{L}^p(F(1))} \le \|f\|_{\mathrm{L}^p(F(2))} \le \cdots \le \|f\|_{\mathrm{L}^p(F(n_d))}.$$

Let $1 \le l \le n_d$ be the smallest integer such that

$$\|f\|^p_{\mathrm{L}^p(F(l))} \ge \frac{1}{2}\|f\|^p_{\mathrm{L}^p(F(n_d))}.$$

If $l \ge 2$, then

$$\|f\|^p_{\mathrm{L}^p(F(l-1))} < \frac{1}{2}\|f\|^p_{\mathrm{L}^p(F(n_d))} \le \frac{1}{2}\|f\|^p_{\mathrm{L}^p(\mathbb{Y})},$$

so applying the induction hypothesis with the last system of sets replaced with the subsystem

$$(E_d(i_d) : i_d \in \{1,\dots,l-1\}),$$

we get

$$\begin{aligned}
&\Big\| \max_{\substack{i_1,\dots,i_d\\ 1\le i_d\le l-1}} \big|T\big(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_d(i_d)}\big)\big| \Big\|_{\mathrm{L}^q(\mathbb{X})}\\
&= \Big\| \max_{\substack{i_1,\dots,i_d\\ 1\le i_d\le l-1}} \big|T\big(f\mathbb{1}_{F(l-1)}\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_d(i_d)}\big)\big| \Big\|_{\mathrm{L}^q(\mathbb{X})}\\
&\le \big(1-2^{1/q-1/p}\big)^{-d}\|T\|_{\mathrm{L}^p(\mathbb{Y})\to\mathrm{L}^q(\mathbb{X})}\|f\mathbb{1}_{F(l-1)}\|_{\mathrm{L}^p(\mathbb{Y})}\\
&\le 2^{-1/p}\big(1-2^{1/q-1/p}\big)^{-d}\|T\|_{\mathrm{L}^p(\mathbb{Y})\to\mathrm{L}^q(\mathbb{X})}\|f\|_{\mathrm{L}^p(\mathbb{Y})}.
\end{aligned} \tag{4.17}$$

Also,

$$\|f\|^p_{\mathrm{L}^p(F(n_d)\setminus F(l))} = \|f\|^p_{\mathrm{L}^p(F(n_d))} - \|f\|^p_{\mathrm{L}^p(F(l))} \le \frac{1}{2}\|f\|^p_{\mathrm{L}^p(F(n_d))} \le \frac{1}{2}\|f\|^p_{\mathrm{L}^p(\mathbb{Y})},$$

so, if $l \le n_d - 1$, then applying the induction hypothesis with the last system of sets replaced with the subsystem,

$$(E_d(i_d) : i_d \in \{l+1,\dots,n_d\}),$$

we obtain

$$\begin{aligned}
&\Big\| \max_{\substack{i_1,\ldots,i_d \\ l+1\le i_d\le n_d}} \big|T\big(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_{d-1}(i_{d-1})\cap(E_d(i_d)\setminus E_d(l))}\big)\big|\Big\|_{\mathrm{L}^q(\mathbb{X})} \\
&= \Big\| \max_{\substack{i_1,\ldots,i_d \\ l+1\le i_d\le n_d}} \big|T\big(f\mathbb{1}_{F(n_d)\setminus F(l)}\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_{d-1}(i_{d-1})\cap(E_d(i_d)\setminus E_d(l))}\big)\big|\Big\|_{\mathrm{L}^q(\mathbb{X})} \\
&\le \big(1-2^{1/q-1/p}\big)^{-d}\|T\|_{\mathrm{L}^p(\mathbb{Y})\to\mathrm{L}^q(\mathbb{X})}\|f\mathbb{1}_{F(n_d)\setminus F(l)}\|_{\mathrm{L}^p(\mathbb{Y})} \\
&\le 2^{-1/p}\big(1-2^{1/q-1/p}\big)^{-d}\|T\|_{\mathrm{L}^p(\mathbb{Y})\to\mathrm{L}^q(\mathbb{X})}\|f\|_{\mathrm{L}^p(\mathbb{Y})}. \qquad (4.18)
\end{aligned}$$

Finally, if $d\ge 2$, then we can also apply the induction hypothesis with the same first $d-1$ systems of sets, to conclude

$$\begin{aligned}
&\Big\| \max_{i_1,\ldots,i_{d-1}} \big|T\big(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_{d-1}(i_{d-1})\cap E_d(l)}\big)\big|\Big\|_{\mathrm{L}^q(\mathbb{X})} \\
&= \Big\| \max_{i_1,\ldots,i_{d-1}} \big|T\big(f\mathbb{1}_{E_d(l)}\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_{d-1}(i_{d-1})}\big)\big|\Big\|_{\mathrm{L}^q(\mathbb{X})} \\
&\le \big(1-2^{1/q-1/p}\big)^{-d+1}\|T\|_{\mathrm{L}^p(\mathbb{Y})\to\mathrm{L}^q(\mathbb{X})}\|f\mathbb{1}_{E_d(l)}\|_{\mathrm{L}^p(\mathbb{Y})} \\
&\le \big(1-2^{1/q-1/p}\big)^{-d+1}\|T\|_{\mathrm{L}^p(\mathbb{Y})\to\mathrm{L}^q(\mathbb{X})}\|f\|_{\mathrm{L}^p(\mathbb{Y})}. \qquad (4.19)
\end{aligned}$$

The last bound also holds in the case $d=1$, with the maximum disappearing from the left hand side, and it is a consequence of the mere boundedness of T.

Now denote

$$\begin{aligned}
S := \big\{x\in\mathbb{X} : (T_\star f)(x) = \big|\big(T\big(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_d(i_d)}\big)\big)(x)\big| \\
\text{for some } (i_1,\ldots,i_d)\in I_1\times\cdots\times I_d \text{ such that } i_d\le l-1\big\},
\end{aligned}$$

so, using linearity of T,

$$\begin{aligned}
T_\star f \le \mathbb{1}_S &\max_{\substack{i_1,\ldots,i_d \\ 1\le i_d\le l-1}} \big|T\big(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_{d-1}(i_{d-1})\cap E_d(i_d)}\big)\big| \\
&+ \mathbb{1}_{\mathbb{X}\setminus S} \max_{\substack{i_1,\ldots,i_d \\ l+1\le i_d\le n_d}} \big|T\big(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_{d-1}(i_{d-1})\cap(E_d(i_d)\setminus E_d(l))}\big)\big| \\
&+ \mathbb{1}_{\mathbb{X}\setminus S} \max_{i_1,\ldots,i_{d-1}} \big|T\big(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_{d-1}(i_{d-1})\cap E_d(l)}\big)\big|.
\end{aligned}$$

Here, maximum over an empty set is understood to be 0. When $q < \infty$ we conclude

$$\begin{aligned}\|T_\star f\|_{\mathrm{L}^q(\mathbb{X})} \leq \Bigg(& \Big\| \max_{\substack{i_1,\ldots,i_d \\ 1\leq i_d\leq l-1}} \big|T\big(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_d(i_d)}\big)\big| \Big\|_{\mathrm{L}^q(S)}^q \\ & + \Big\| \max_{\substack{i_1,\ldots,i_d \\ l+1\leq i_d\leq n_d}} \big|T\big(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_{d-1}(i_{d-1})\cap(E_d(i_d)\setminus E_d(l))}\big)\big| \Big\|_{\mathrm{L}^q(\mathbb{X}\setminus S)}^q \Bigg)^{1/q} \\ & + \Big\| \max_{i_1,\ldots,i_{d-1}} \big|T\big(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_{d-1}(i_{d-1})\cap E_d(l)}\big)\big| \Big\|_{\mathrm{L}^q(\mathbb{X}\setminus S)},\end{aligned}$$

while in the endpoint case $q = \infty$ we instead have

$$\begin{aligned}\|T_\star f\|_{\mathrm{L}^\infty(\mathbb{X})} \leq \max\Bigg\{ & \Big\| \max_{\substack{i_1,\ldots,i_d \\ 1\leq i_d\leq l-1}} \big|T\big(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_d(i_d)}\big)\big| \Big\|_{\mathrm{L}^\infty(S)}, \\ & \Big\| \max_{\substack{i_1,\ldots,i_d \\ l+1\leq i_d\leq n_d}} \big|T\big(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_{d-1}(i_{d-1})\cap(E_d(i_d)\setminus E_d(l))}\big)\big| \Big\|_{\mathrm{L}^\infty(\mathbb{X}\setminus S)} \Bigg\} \\ & + \Big\| \max_{i_1,\ldots,i_{d-1}} \big|T\big(f\mathbb{1}_{E_1(i_1)\cap\cdots\cap E_{d-1}(i_{d-1})\cap E_d(l)}\big)\big| \Big\|_{\mathrm{L}^\infty(\mathbb{X}\setminus S)}.\end{aligned}$$

Applying (4.17), (4.18), and (4.19) we complete the induction step. $\square$

Remark 4.2.2. An alternative proof of Lemma 4.2.1 can be obtained as follows. We can generalize the claim further to general sublinear operators T, i.e., operators satisfying

$$|T(\alpha f)| = |\alpha||Tf|, \quad |T(f+g)| \leq |Tf| + |Tg|$$

for all $\alpha \in \mathbb{C}$ and all $f,g \in \mathrm{L}^p(\mathbb{Y},\mathscr{Y},\nu)$. The advantage of doing this is that various maximal operators are always sublinear. Then we can write the operator $T_\star$ as a composition of d maximal truncations, each one with respect to a single increasing system $(E_j(i) : i \in I_j)$, namely

$$T_\star f = \sup_{i_1\in I_1}\sup_{i_2\in I_2}\cdots\sup_{i_d\in I_d}\Big|T\Big(\cdots\big((f\mathbb{1}_{E_1(i_1)})\mathbb{1}_{E_2(i_2)}\big)\cdots\mathbb{1}_{E_d(i_d)}\Big)\Big|,$$

so the claim is reduced merely to the one-parameter case. Finally, one can notice that the known proofs of the particular case $d = 1$, both the one by Christ and Kiselev [15, Theorem 1.1] and the one by Tao [70, Note #2], clearly remain valid for merely sublinear operators T. We leave the details to the reader.

Now assume that the second measurable space splits as a product

$$(\mathbb{Y},\mathscr{Y}) = (\mathbb{Y}_1\times\cdots\times\mathbb{Y}_d,\, \mathscr{Y}_1\otimes\cdots\otimes\mathscr{Y}_d)$$

of $d \geq 1$ measurable spaces $(\mathbb{Y}_j, \mathscr{Y}_j)$. Also suppose that for each $1 \leq j \leq d$ we have a countable totally ordered set I_j and an increasing system $(A_j^i : i \in I_j)$ of sets from $\mathscr{Y}_j$.

Corollary 4.2.3. *Take exponents $1 \leq p < q \leq \infty$ and a bounded linear operator $T \colon \mathrm{L}^p(\mathbb{Y}, \mathscr{Y}, \nu) \to \mathrm{L}^q(\mathbb{X}, \mathscr{X}, \mu)$. The maximal operator*

$$(T_\star f)(x) := \sup_{(i_1,\ldots,i_d) \in I_1 \times \cdots \times I_d} \left| T\left(f \mathbb{1}_{A_1^{i_1} \times \cdots \times A_d^{i_d}} \right)(x) \right| \tag{4.20}$$

is also bounded from $\mathrm{L}^p(\mathbb{Y}, \mathscr{Y}, \nu)$ to $\mathrm{L}^q(\mathbb{X}, \mathscr{X}, \mu)$ with the operator norm satisfying

$$\|T_\star\|_{\mathrm{L}^p(\mathbb{Y}) \to \mathrm{L}^q(\mathbb{X})} \leq \left(1 - 2^{1/q - 1/p}\right)^{-d} \|T\|_{\mathrm{L}^p(\mathbb{Y}) \to \mathrm{L}^q(\mathbb{X})}.$$

Proof. This result is an immediate consequence of Lemma 4.2.1, obtained by taking

$$E_j(i) = \mathbb{Y}_1 \times \cdots \times \mathbb{Y}_{j-1} \times A_j^i \times \mathbb{Y}_{j+1} \times \cdots \times \mathbb{Y}_d. \qquad \square$$

The constants blow up as q approaches p. An easy modification of the proof of Lemma 4.2.1 gives the following endpoint result with logarithmic losses when the sets I_j are finite.

Corollary 4.2.4. *Take an exponent $p \in [1, \infty]$ and a bounded linear operator $T \colon \mathrm{L}^p(\mathbb{Y}, \mathscr{Y}, \nu) \to \mathrm{L}^p(\mathbb{X}, \mathscr{X}, \mu)$. The maximal operator given by (4.20) satisfies*

$$\|T_\star\|_{\mathrm{L}^p(\mathbb{Y}) \to \mathrm{L}^p(\mathbb{X})} \leq \left(\lceil \log_2 |I_1| \rceil + 1\right) \cdots \left(\lceil \log_2 |I_d| \rceil + 1\right) \|T\|_{\mathrm{L}^p(\mathbb{Y}) \to \mathrm{L}^p(\mathbb{X})}.$$

Formulation of Corollary 4.2.4 is motivated by Tao's [70, Note #2, Q14]. The particular case when $p = 2$ and T is the Fourier transform could be called the *multi-parameter Rademacher–Menshov theorem*. We will not need Corollary 4.2.4 in the later text and we formulated it only for comparison with a very different method by Krause, Mirek, and Trojan [45, Section 3].

4.3. Proof of Theorem 4.1.1

Denote the maximal operator

$$\mathscr{M}f := \sup_{r_1,\ldots,r_d\in(0,\infty)} \big|\widehat{f}*\chi_{r_1,\ldots,r_d}\big|.$$

We begin with an observation that $\widehat{f}*\chi_{r_1,\ldots,r_d}$ is the Fourier transform of

$$(x_1,\ldots,x_d)\mapsto f(x_1,\ldots,x_d)\,\check{\chi}(r_1x_1,\ldots,r_dx_d).$$

Using (4.8) and the fundamental theorem of calculus we expand, for any $(x_1,\ldots,x_d)\in(\mathbb{R}\setminus\{0\})^d$ and $(r_1,\ldots,r_d)\in(0,\infty)^d$,

$$\begin{aligned}
&\check{\chi}(r_1x_1,\ldots,r_dx_d)\\
&= \sum_{\varepsilon\in\{-1,1\}^d} (-1)^{\#\varepsilon}\,\mathbb{1}_{Q(\varepsilon)}(x_1,\ldots,x_d) \int_{\{(t_1,\ldots,t_d)\in Q(\varepsilon):|t_j|\geq r_j|x_j| \text{ for } 1\leq j\leq d\}} \big(\partial_1\cdots\partial_d\check{\chi}\big)(t_1,\ldots,t_d)\,\mathrm{d}t_1\cdots\mathrm{d}t_d\\
&= \sum_{\varepsilon\in\{-1,1\}^d} (-1)^{\#\varepsilon} \int_{Q(\varepsilon)} \mathbb{1}_{R(\varepsilon;|t_1|/r_1,\ldots,|t_d|/r_d)}(x_1,\ldots,x_d)\,\big(\partial_1\cdots\partial_d\check{\chi}\big)(t_1,\ldots,t_d)\,\mathrm{d}t_1\cdots\mathrm{d}t_d.
\end{aligned}$$

Here $Q(\varepsilon)$ is the open coordinate "quadrant" determined by $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_d)\in\{-1,1\}^d$, i.e.,

$$Q(\varepsilon) := \{(x_1,\ldots,x_d)\in\mathbb{R}^d \,:\, \operatorname{sgn} x_j = \varepsilon_j \text{ for } 1\leq j\leq d\},$$

$\#\varepsilon$ denotes the number of 1's among the coordinates of ε, and we also denote

$$R(\varepsilon;s_1,\ldots,s_d) := Q(\varepsilon)\cap\big([-s_1,s_1]\times\cdots\times[-s_d,s_d]\big) \tag{4.21}$$

for any $s_1,\ldots,s_d\in(0,\infty)$. Multiplying by f and taking Fourier transforms we obtain the pointwise identity

$$\widehat{f}*\chi_{r_1,\ldots,r_d} = \sum_{\varepsilon\in\{-1,1\}^d} (-1)^{\#\varepsilon} \int_{Q(\varepsilon)} \mathscr{F}\big(f\,\mathbb{1}_{R(\varepsilon;|t_1|/r_1,\ldots,|t_d|/r_d)}\big)\,\big(\partial_1\cdots\partial_d\check{\chi}\big)(t_1,\ldots,t_d)\,\mathrm{d}t_1\cdots\mathrm{d}t_d,$$

so that

$$\mathscr{M}f \leq \int_{\mathbb{R}^d} \Big(\sup_{r_1,\ldots,r_d\in(0,\infty)} \big|\mathscr{F}\big(f\,\mathbb{1}_{R(\varepsilon;|t_1|/r_1,\ldots,|t_d|/r_d)}\big)\big|\Big)\,\big|\big(\partial_1\cdots\partial_d\check{\chi}\big)(t_1,\ldots,t_d)\big|\,\mathrm{d}t_1\cdots\mathrm{d}t_d.$$

Note that each of the sets (4.21) is a d-dimensional rectangle in $\mathbb{R}^d$, so invoking Corollary 4.2.3 with $T=\mathscr{F}$, which is known to satisfy (4.1), gives

$$\Big\|\sup_{r_1,\ldots,r_d\in(0,\infty)\cap\mathbb{Q}} \big|\mathscr{F}\big(f\,\mathbb{1}_{R(\varepsilon;|t_1|/r_1,\ldots,|t_d|/r_d)}\big)\big|\Big\|_{\mathrm{L}^q(S,\sigma)} \lesssim_{d,\sigma,p,q} \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}.$$

The last implicit constant is independent of $t_1,\dots,t_d$, so integrability of $\partial_1\cdots\partial_d\check{\chi}$, thanks to (4.8) again, establishes

$$\|\mathscr{M}f\|_{\mathrm{L}^q(S,\sigma)} \lesssim_{d,\sigma,\chi,p,q} \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}, \tag{4.22}$$

which is precisely (4.4).

The proof of (4.5) is now standard. The claim is clear for $f\in\mathrm{L}^1(\mathbb{R}^d)$. By

$$\mathrm{L}^s(\mathbb{R}^d)\subseteq\mathrm{L}^1(\mathbb{R}^d)+\mathrm{L}^p(\mathbb{R}^d)$$

it is sufficient to verify it when $f\in\mathrm{L}^p(\mathbb{R}^d)$. For any $\varepsilon>0$ denote the exceptional set

$$E_\varepsilon := \Big\{\xi\in\mathbb{R}^d \,:\, \inf_{r\in(0,\infty)} \sup_{r_1,\dots,r_d\in(0,r]} \big|\big(\widehat{f}*\chi_{r_1,\dots,r_d}\big)(\xi)-(\mathscr{R}f)(\xi)\big|\geq\varepsilon\Big\},$$

observing that (4.5) holds for every point outside of $\cup_{\varepsilon\in(0,\infty)}E_\varepsilon$. It is easy to see that for every $g\in\mathscr{S}(\mathbb{R}^d)$ by the mere continuity of $\widehat{g}$ we have

$$\lim_{(0,\infty)^d\ni(r_1,\dots,r_d)\to(0,\dots,0)} \widehat{g}*\chi_{r_1,\dots,r_d}=\mathscr{R}g$$

pointwise on S and, consequently,

$$E_\varepsilon\subseteq\Big\{\xi\in S\,:\,\mathscr{M}(f-g)(\xi)\geq\frac{\varepsilon}{2}\Big\}\cup\Big\{\xi\in S\,:\,\mathscr{R}(f-g)(\xi)\geq\frac{\varepsilon}{2}\Big\}.$$

Thus, Estimates (4.22), (4.1) and the Markov–Chebyshev inequality give

$$\sigma(E_\varepsilon)\lesssim\varepsilon^{-q}\|f-g\|_{\mathrm{L}^p(\mathbb{R}^d)}^q.$$

By the density of $\mathscr{S}(\mathbb{R}^d)$ in $\mathrm{L}^p(\mathbb{R}^d)$ we conclude $\sigma(E_\varepsilon)=0$ and nestedness of these sets also gives $\sigma(\cup_{\varepsilon\in(0,\infty)}E_\varepsilon)=0$. Thus, (4.5) really holds for σ-almost every $\xi\in S$.

Turning to (4.6), we define the ellipsoid maximal function of the Fourier transform as

$$\big(\widetilde{\mathscr{M}}f\big)(\xi):=\sup_{r_1,\dots,r_d\in(0,\infty)}\frac{1}{|B_{r_1,\dots,r_d}|}\int_{B_{r_1,\dots,r_d}(\xi)}\big|\widehat{f}(\eta)\big|\,\mathrm{d}\eta$$

and repeat a trick from [52]. It is again sufficient to verify the claim in the endpoint case $f\in\mathrm{L}^{2p/(p+1)}(\mathbb{R}^d)$. Define

$$g(x):=\int_{\mathbb{R}^d}f(y)\overline{f(y-x)}\,\mathrm{d}y,$$

so that $g \in \mathrm{L}^p(\mathbb{R}^d)$ and $\widehat{g}(\xi) = \big|\widehat{f}(\xi)\big|^2$. Choose any non-negative $\chi \in \mathscr{S}(\mathbb{R}^d)$ with integral 1 that is strictly positive on the closed unit ball $B_1(0,\ldots,0)$. Then, by the Cauchy–Schwartz inequality,

$$\begin{aligned} \frac{1}{|B_{r_1,\ldots,r_d}|}\int_{B_{r_1,\ldots,r_d}(\xi)} \big|\widehat{f}(\eta)\big|\,\mathrm{d}\eta &\leq \Big(\frac{1}{|B_{r_1,\ldots,r_d}|}\int_{B_{r_1,\ldots,r_d}(\xi)} \big|\widehat{f}(\eta)\big|^2\,\mathrm{d}\eta\Big)^{1/2} \\ &\lesssim_{\chi} \big(\widehat{g}*\chi_{r_1,\ldots,r_d}\big)(\xi)^{1/2}, \end{aligned}$$

so the bound (4.22) applied to g gives

$$\big\|\widetilde{\mathscr{M}}f\big\|_{\mathrm{L}^{2q}(S,\sigma)} \lesssim_{\chi} \|\mathscr{M}g\|_{\mathrm{L}^q(S,\sigma)}^{1/2} \lesssim_{d,\sigma,\chi,p,q} \|g\|_{\mathrm{L}^p(\mathbb{R}^d)}^{1/2} \leq \|f\|_{\mathrm{L}^{2p/(p+1)}(\mathbb{R}^d)}.$$

Now we can repeat exactly the same density argument as before to conclude that (4.6) holds for σ-almost every $\xi \in S$. Finally, (4.7) is an obvious consequence of (4.6) and the triangle inequality.

Remark 4.3.1. Note that (4.9) and (4.10) now also follow, only by observing that the maximal function $\widetilde{\mathscr{M}}$ is pointwise comparable to the rectangular maximal function,

$$\begin{aligned} \big(\mathscr{M}_{\mathrm{rect}}f\big)(\xi) &:= \sup_{r_1,\ldots,r_d\in(0,\infty)} \frac{1}{2^d r_1\cdots r_d}\int_{\xi+[-r_1,r_1]\times\cdots\times[-r_d,r_d]} \big|\widehat{f}(\eta)\big|\,\mathrm{d}\eta \\ &= \sup_{\substack{\mathscr{R}\text{ is an axes-parallel rectangle}\\ \mathscr{R}\ni\xi}} \frac{1}{|\mathscr{R}|}\int_{\mathscr{R}} \big|\widehat{f}(\eta)\big|\,\mathrm{d}\eta, \end{aligned}$$

so the latter one satisfies the same bound as before. In the other direction, Ramos [55, Proposition 4] showed that, in the case of spheres $S = \mathbb{S}^{d-1}$ in dimensions $d \geq 4$, the operator $\mathscr{M}_{\mathrm{rect}}$ does not satisfy estimates in the full conjectural range of (4.1).

4.4. Proof of Theorem 4.1.4

The maximal operator appearing in the part (a) is simply $T_\star$ from (4.20), where $\mathbb{X} = \mathbb{R}^d$, $\mathbb{Y}_j = \mathbb{R}$, $T = \mathscr{F}$, $q = p'$, $I_j = (0,\infty) \cap \mathbb{Q}$, and $A_j^R = [-R,R]$. Note that we use $p < 2$ in the condition $p < p' = q$, so that Corollary 4.2.3 applies and deduces the desired estimate from the well-known fact that the Fourier transform $\mathscr{F}$ is bounded from $\mathrm{L}^p(\mathbb{R}^d)$ to $\mathrm{L}^{p'}(\mathbb{R}^d)$. The convergence result is then proved via exactly the same density argument as the one used in the previous section.

We turn to the part (b). It will be merely an adaptation of Fefferman's argument [23] to the continuous case. We present the complete proof here because the construction was only outlined in the aforementioned paper and it is necessary for us to construct the function in $\mathrm{L}^2(\mathbb{R}^d)$ for which the limit (4.11) does not exist, instead of just disproving $\mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^{2,\infty}(\mathbb{R}^d)$ boundedness. Namely, Stein's maximal principle [63] does not apply in the case of non-compact groups, such as $\mathbb{R}^d$.

We define $D_R(t) := \sin(2\pi Rt)/\pi t$. The operator $S_{R_1,\ldots,R_d}$ is defined on $\mathrm{L}^2(\mathbb{R}^d)$ as

$$S_{R_1,\ldots,R_d} f := \mathscr{F}(\check{f}\mathbb{1}_{[-R_1,R_1]\times\cdots\times[-R_d,R_d]}) = f * (D_{R_1} \otimes \cdots \otimes D_{R_d}).$$

Here $u_1 \otimes \cdots \otimes u_d$ denotes the elementary tensor made of one-dimensional functions, defined as

$$(u_1 \otimes \cdots \otimes u_d)(x_1,\ldots,x_d) := u_1(x_1)\cdots u_d(x_d).$$

Observe that Young's convolution inequality implies

$$\|S_{R_1,\ldots,R_d}\|_{\mathrm{L}^2(\mathbb{R}^d)\to\mathrm{L}^\infty(\mathbb{R}^d)} \lesssim (R_1\cdots R_d)^{1/2}.$$

Following Fefferman's example, we use the following definition throughout the remainder of this section. For $\lambda \in \mathbb{R}$ we define

$$f_\lambda(x_1,x_2) := e^{2\pi\mathbb{i}\lambda x_1 x_2}\mathbb{1}_{[-2,2]^2}(x_1,x_2).$$

The next lemma gives bounds that are crucial for the proof.

Lemma 4.4.1.

(a) *There exists $C > 0$ such that for all $x_1, x_2 \in [2/3, 1]$ the following holds:*

$$|S_{\lambda x_2, \lambda x_1} f_\lambda(x_1, x_2)| \geq C \log \lambda$$

whenever λ is large enough.

(b) *There exists $C > 0$ such that for all $x_1, x_2 \in [2/3, 1]$ and $\lambda' \geq 3\lambda > 0$ the following holds:*

$$|S_{\lambda' x_2, \lambda' x_1} f_\lambda(x_1, x_2)| \leq C.$$

Before proving the lemma, we prove that it implies the part (b) of Theorem 4.1.4. We will prove that there exist a function $f \in \mathrm{L}^2(\mathbb{R}^d)$ and a number $\delta > 0$ such that

$$\limsup_{R_1 \to \infty, \ldots, R_d \to \infty} |S_{R_1, \ldots, R_d} f(x_1, \ldots, x_d)| = \infty \quad \text{for every } (x_1, \ldots, x_d) \in [2/3, 1]^2 \times [-\delta, \delta]^{d-2}, \tag{4.23}$$

so the function $\check{f} \in \mathrm{L}^2(\mathbb{R}^d)$ will be the one for which (4.12) holds.

Let $\psi \in \mathscr{S}(\mathbb{R})$ be a real-valued Schwartz function such that $\psi(0) > 0$ and $\operatorname{supp}(\check{\psi}) \subseteq [-1, 1]$. For the function $F(x_1, \ldots, x_d) := f(x_1, x_2) \prod_{j=3}^d \psi(x_j)$, because of the assumption on the support of $\check{\psi}$, we have

$$\limsup_{R_1 \to \infty, \ldots, R_d \to \infty} |S_{R_1, \ldots, R_d} F(x_1, \ldots, x_d)| = \limsup_{R_1 \to \infty, R_2 \to \infty} \Big| S_{R_1, R_2} f(x_1, x_2) \prod_{j=3}^d \psi(x_j) \Big|.$$

Furthermore, since $\psi(0) > 0$, there exists some $\delta > 0$ such that $\psi(x) > 0$ for all $x \in [-\delta, \delta]$, so it is enough to prove (4.23) for $d = 2$.

We define the sequence of positive real numbers $(a_k)_{k=1}^\infty$ recursively as $a_1 = 1$, $a_{k+1} = 2^{-k/a_k}$ and the sequence of positive real numbers $(\lambda_k)_{k=1}^\infty$ with $\lambda_k = a_{k+1}^{-1}$. Observing that $\sum_{k=1}^\infty a_k < \infty$, it follows that the function

$$f(x_1, x_2) := \sum_{k=0}^\infty a_k f_{\lambda_k}(x_1, x_2)$$

is well defined and in $\mathrm{L}^2(\mathbb{R}^2)$

We claim that there exist real numbers $C_i > 0$, $i = 1, 2, 3$ such that the following inequalities hold for all $x_1, x_2 \in [2/3, 1]$ and $n \in \mathbb{N}$:

(1) $|S_{\lambda_n x_2, \lambda_n x_1} f_{\lambda_n}(x_1, x_2)| \geq C_1 \log \lambda_n$,

(2) $|S_{\lambda_n x_2, \lambda_n x_1} f_{\lambda_k}(x_1, x_2)| \leq C_2$ when $k < n$,

(3) $|S_{\lambda_n x_2,\lambda_n x_1} f_{\lambda_k}(x_1,x_2)| \le C_3\lambda_n$ when $k>n$.

Indeed, since $\lambda_{k+1} \ge 4\lambda_k$ for all $k \in \mathbb{N}$, the first two inequalities follow from Lemma 4.4.1, while the third one follows from Young's convolution inequality. Therefore, observing that sequences satisfy $\lambda_n \sum_{k>n} a_k \lesssim 1$ for all $n \in \mathbb{N}$ and $a_n \log\lambda_n \sim n$, for $x_1, x_2 \in [2/3,1]$ and n large enough it follows that

$$\begin{aligned}|S_{\lambda_n x_2,\lambda_n x_1} f(x_1,x_2)| &\ge a_n |S_{\lambda_n x_2,\lambda_n x_1} f_{\lambda_n}(x_1,x_2)| - \sum_{k\neq n} a_k |S_{\lambda_n x_2,\lambda_n x_1} f_{\lambda_k}(x_1,x_2)| \\ &\ge C_1 a_n \log\lambda_n - C_2 \sum_{k<n} a_k - C_3\lambda_n \sum_{k>n} a_k \gtrsim n.\end{aligned}$$

Finally, noting that $\lambda_n x_2, \lambda_n x_1 \to \infty$ as $n\to\infty$ finishes the proof of (4.23) in the case $d=2$ and therefore also the part (b) of the theorem.

The following technical lemma will be needed in the proof of Lemma 4.4.1.

Lemma 4.4.2.

(a) There exist $C, \lambda_0 > 0$ such that

$$\left|\mathrm{p.v.}\int_{-1}^{1}\int_{-1}^{1} \frac{e^{2\pi \mathrm{i}\lambda x_1 x_2}}{x_1 x_2}\,\mathrm{d}x_1\,\mathrm{d}x_2\right| \ge C\log\lambda \quad \textit{for every } \lambda \ge \lambda_0.$$

(b) There exists $C>0$ such that for all $c_1, c_2 \in \mathbb{R}$ for which $\max\{|c_1|,|c_2|\} \ge 4/3$, the following holds:

$$\left|\mathrm{p.v.}\int_{-1}^{1}\int_{-1}^{1} \frac{e^{2\pi \mathrm{i}\lambda (x_1 x_2 + c_1 x_1 + c_2 x_2)}}{x_1 x_2}\,\mathrm{d}x_1\,\mathrm{d}x_2\right| \le C \quad \textit{for every } \lambda > 0.$$

Proof. (a) This was proved in [53], but we repeat the short proof for the completeness. Since $\int_0^\infty \sin t\,\mathrm{d}t/t = \pi/2$, there exists $\lambda_1 > 0$ such that $\int_0^x \sin t\,\mathrm{d}t/t \in [\pi/4, 3\pi/4]$, for all $x \ge \lambda_1$. Now, using symmetries of the integrand and change of variables, it follows

$$\begin{aligned}\mathrm{p.v.}\int_{-1}^{1}\int_{-1}^{1} \frac{e^{2\pi \mathrm{i}\lambda x_1 x_2}}{x_1 x_2}\,\mathrm{d}x_1\,\mathrm{d}x_2 &= 4\mathrm{i}\int_0^1\int_0^1 \frac{\sin(2\pi\lambda x_1 x_2)}{x_1 x_2}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= 4\mathrm{i}\int_0^{2\pi}\frac{1}{x_2}\int_0^{\lambda x_2}\frac{\sin t}{t}\,\mathrm{d}t\,\mathrm{d}x_2 \\ &= 4\mathrm{i}\int_0^{\lambda_1/\lambda}\frac{1}{x_2}\int_0^{\lambda x_2}\frac{\sin t}{t}\,\mathrm{d}t\,\mathrm{d}x_2 + 4\mathrm{i}\int_{\lambda_1/\lambda}^{2\pi}\int_0^{\lambda x_2}\frac{\sin t}{t}\,\mathrm{d}t\,\frac{\mathrm{d}x_2}{x_2}.\end{aligned}$$

For the first integral observe that $t \mapsto (\sin t)/t$ is absolutely bounded by 1, so the integral is absolutely bounded by λ_1. For the second integral we use fact that $\lambda x_2 \ge \lambda_1$ so:

$$\int_{\lambda_1/\lambda}^{2\pi}\int_0^{\lambda x_2}\frac{\sin t}{t}\,\mathrm{d}t\,\frac{\mathrm{d}x_2}{x_2} \gtrsim \int_{\lambda_1/\lambda}^{2\pi}\frac{\mathrm{d}x_2}{x_2} = \log\lambda + \log(2\pi) - \log\lambda_1.$$

Finally, adding the two integrals and choosing λ_0 large enough compared to λ_1, the statement holds.

(b) Assume, without loss of generality, that $c_1 \geq 4/3$. Using symmetries of the integrand, it follows that

$$\begin{aligned}\mathrm{p.v.}\int_{-1}^{1}\int_{-1}^{1}&\frac{e^{2\pi\lambda\mathbb{i}(x_1x_2+c_1x_1+c_2x_2)}}{x_1x_2}\,\mathrm{d}x_1\mathrm{d}x_2\\ &=\mathbb{i}\,\mathrm{p.v.}\int_{-1}^{1}\int_{-1}^{1}\frac{\sin(2\pi\lambda x_1(x_2+c_1))}{x_1x_2}e^{2\pi\mathbb{i}\lambda c_2x_2}\,\mathrm{d}x_1\mathrm{d}x_2.\end{aligned}$$

If we define

$$g_\varepsilon(x_2):=2\int_{\varepsilon}^{1}\frac{\sin(2\pi\lambda x_1(x_2+c_1))}{x_1}\,\mathrm{d}x_1,$$

from the assumption $c_1 \geq 4/3$, it follows that $|g_\varepsilon'(x_2)| \lesssim (x_2+c_1)^{-1} \lesssim 1$ for all $x_2 \in [-1,1]$, where the implicit constant is independent of both λ and ε. Therefore,

$$\begin{aligned}\left|\int_{([-1,1]\setminus[-\varepsilon,\varepsilon])^2}\frac{\sin(2\pi\lambda x_1(x_2+c_1))}{x_1}\frac{e^{2\pi\mathbb{i}\lambda c_2x_2}}{x_2}\,\mathrm{d}x_1\mathrm{d}x_2\right| &= \left|\int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} g_\varepsilon(x_2)\frac{e^{2\pi\mathbb{i}\lambda c_2x_2}}{x_2}\,\mathrm{d}x_2\right|\\ \leq \left|\int_{[-1,1]\setminus[-\varepsilon,\varepsilon]}\frac{g_\varepsilon(x_2)-g_\varepsilon(0)}{x_2}e^{2\pi\mathbb{i}\lambda c_2x_2}\,\mathrm{d}x_2\right| &+ \left|g_\varepsilon(0)\int_{[-1,1]\setminus[-\varepsilon,\varepsilon]}\frac{e^{2\pi\mathbb{i}\lambda c_2x_2}}{x_2}\,\mathrm{d}x_2\right|\\ \lesssim \int_{-1}^{1}\left|\sup_{t\in[-1,1]}g'(t)\right|\mathrm{d}x_2+\sup_{N>0}\left|\int_0^N\frac{\sin t}{t}dt\right|^2 &\lesssim 1.\end{aligned}$$

Letting $\varepsilon \to 0$, the statement follows. $\square$

We proceed to the proof of Lemma 4.4.1.

Proof of Lemma 4.4.1. Observe that:

$$S_{R_1,R_2}f_\lambda = T_{R_1,R_2}f_\lambda - T_{-R_1,R_2}f_\lambda - T_{R_1,-R_2}f_\lambda + T_{-R_1,-R_2}f_\lambda, \tag{4.24}$$

where

$$T_{r_1,r_2}f(x_1,x_2) = -\frac{1}{4\pi^2}\,\mathrm{p.v.}\int_{\mathbb{R}^2}\frac{e^{2\pi\mathbb{i}(r_1x_1'+r_2x_2')}}{x_1'x_2'}f(x_1-x_1',x_2-x_2')\,\mathrm{d}x_1'\mathrm{d}x_2'.$$

We prove the following two observations for the part (a) of the lemma.

(1) There exists $C>0$ such that for λ large enough and $x_1,x_2 \in [0,1]$:

$$|T_{\lambda x_2,\lambda x_1}f_\lambda(x_1,x_2)| \geq C\log\lambda.$$

(2) For $\lambda > 0$ and $x_1, x_2 \in [2/3, 1]$, all of the expressions

$$|T_{-\lambda x_2,\lambda x_1} f_\lambda(x_1,x_2)|, \quad |T_{\lambda x_2,-\lambda x_1} f_\lambda(x_1,x_2)|, \quad |T_{-\lambda x_2,-\lambda x_1} f_\lambda(x_1,x_2)|$$

are bounded by a constant independent of λ.

In order to prove the observation (1), we note that because

$$x_2 x_1' + x_1 x_2' + (x_1 - x_1')(x_2 - x_2') = x_1 x_2 + x_1' x_2',$$

the following holds

$$|T_{\lambda x_2,\lambda x_1} f_\lambda(x_1,x_2)| = \frac{1}{4\pi^2}\left|\mathrm{p.v.}\int_{[x_1-2,x_1+2]}\int_{[x_2-2,x_2+2]} \frac{e^{2\pi \mathrm{i}\lambda x_1' x_2'}}{x_1' x_2'}\,\mathrm{d}x_2'\mathrm{d}x_1'\right|.$$

We decompose the area of integration into 4 regions:

$$[-1,1]^2, \quad [-1,1]\times(\mathbb{R}\setminus[-1,1]), \quad (\mathbb{R}\setminus[-1,1])\times[-1,1], \quad (\mathbb{R}\setminus[-1,1])^2.$$

By the first part of Lemma 4.4.2, there exists $C > 0$ such that the integral over the first region is at least $C\log\lambda$ whenever λ is large enough. Integrals over the second and third regions are all $O(1)$ because of the following calculation:

$$\left|\int_1^{x_2+2}\int_{-1}^{1} \frac{\sin(2\pi\lambda x_1' x_2')}{x_1' x_2'}\,\mathrm{d}x_1'\mathrm{d}x_2'\right| = \int_1^{x_2+2} \frac{1}{x_2'}\left|\int_{-2\pi\lambda x_2'}^{2\pi\lambda x_2'} \frac{\sin t}{t}\,\mathrm{d}t\right|\mathrm{d}x_2' \lesssim \int_1^3 \frac{1}{x_2'}\,\mathrm{d}x_2' \lesssim 1.$$

Finally, the integral over the last region is bounded using the triangle inequality by:

$$\int_{x_1-2}^{x_1+2}\int_{x_2-2}^{x_2+2} \frac{1}{x_1' x_2'}\mathbb{1}_{\{|x_1'|,|x_2'|>1\}}\,\mathrm{d}x_1'\mathrm{d}x_2' \lesssim 1.$$

Summing all the bounds we prove the observation (1).

We turn to the proof of the observation (2). First note that for $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$,

$$\begin{aligned}&|T_{\varepsilon_1\lambda x_2,\varepsilon_2\lambda x_1} f_\lambda(x_1,x_2)| \\ &= \frac{1}{4\pi^2}\left|\mathrm{p.v.}\int_{[x_1-2,x_1+2]}\int_{[x_2-2,x_2+2]} \frac{e^{2\pi \mathrm{i}\lambda(x_1'x_2' + (\varepsilon_1-1)x_1'x_2 + (\varepsilon_2-1)x_2'x_1)}}{x_1' x_2'}\,\mathrm{d}x_2'\mathrm{d}x_1'\right|.\end{aligned}$$

Assume, without loss of generality that $\varepsilon_1 = -1$. From the assumption on x_2 it follows that $|(\varepsilon_1 - 1)x_2| \geq 4/3$, so using the second part of Lemma 4.4.2, the integral over the first region is bounded by a constant. Integral over the fourth region is bounded as in the observation (1). Integrals over the second and the third region can be bounded using the following calculation

$$\left|\int_{1}^{x_2+2}\int_{-1}^{1}\frac{\sin(2\pi\lambda x_1'(x_2'+(\varepsilon_1-1)x_2))}{x_1'}\,\mathrm{d}x_1'\,\frac{e^{2\pi\mathrm{i}\lambda(\varepsilon_2-1)x_2'x_1}}{x_2'}\,\mathrm{d}x_2'\right| \lesssim \int_1^3\frac{\mathrm{d}x_2'}{x_2'}\lesssim 1.$$

Combining observations (1) and (2) with (4.24), we conclude the proof of the part (a) of the lemma.

For the part (b), we observe that for $\varepsilon_1,\varepsilon_2\in\{-1,1\}$ the following holds:

$$\begin{aligned}
&|T_{\varepsilon_1\lambda' x_2,\varepsilon_2\lambda' x_1}f_\lambda(x_1,x_2)|\\
&=\frac{1}{4\pi^2}\left|\mathrm{p.v.}\int_{[x_1-2,x_1+2]}\int_{[x_2-2,x_2+2]}\frac{e^{2\pi\mathrm{i}\lambda(x_1'x_2'+(\varepsilon_1\lambda'/\lambda-1)x_1'x_2+(\varepsilon_2\lambda'/\lambda-1)x_2'x_1)}}{x_1'x_2'}\,\mathrm{d}x_2'\mathrm{d}x_1'\right|.
\end{aligned}$$

We then decompose the area of integration in the same four parts as before. For the first part, since $|(\varepsilon_1\lambda'/\lambda-1)x_2|\geq 4/3$, we use the second part of Lemma 4.4.2 to get the upper bound and we treat the other parts as in the part (a) of the lemma. $\square$

Remark 4.4.3. It is obvious that the function f in the proof of the part (b) is in $\mathrm{L}^1(\mathbb{R}^d)$, so the function $\check{f}$, for which the convergence (4.11) fails, is also continuous and therefore the counterexample exists in the class $C(\mathbb{R}^d)\cap\mathrm{L}^2(\mathbb{R}^d)$.

4.5. Proof of Corollary 4.1.5

Let

$$S := \Big\{\Big(\xi, \frac{\phi(\xi)}{2\pi}\Big) : \xi \in \mathbb{R}^n\Big\} \subseteq \mathbb{R}^{n+1}$$

be the hypersurface naturally associated with (4.13). Equip S with the projection measure $\mathrm{d}\sigma(\xi,\tau) = \mathrm{d}\xi$. For every $g \in \mathrm{L}^2(S,\sigma)$ there exist a unique $f \in \mathrm{L}^2(\mathbb{R}^n)$ such that

$$g\Big(\xi, \frac{\phi(\xi)}{2\pi}\Big) = \widehat{f}(\xi) \tag{4.25}$$

for a.e. $\xi \in \mathbb{R}^n$. By the assumption (4.14) and the Plancherel identity we then know that $\mathscr{E}$ given by the formula

$$(\mathscr{E}g)(x,t) := (e^{\mathbb{i}t\phi(D)} f)(x)$$

extends to a bounded linear operator $\mathscr{E}\colon \mathrm{L}^2(S,\sigma) \to \mathrm{L}^s(\mathbb{R}^{n+1})$. In the case when $f \in \mathscr{S}(\mathbb{R}^n)$, we can write

$$(\mathscr{E}g)(x,t) = \int_{\mathbb{R}^n} e^{\mathbb{i}t\phi(\xi)+2\pi\mathbb{i}x\cdot\xi} g\Big(\xi, \frac{\phi(\xi)}{2\pi}\Big)\,\mathrm{d}\xi = \int_S e^{2\pi\mathbb{i}(x,t)\cdot(\xi,\tau)} g(\xi,\tau)\,\mathrm{d}\sigma(\xi,\tau)$$

and, taking another Schwartz function $h \in \mathscr{S}(\mathbb{R}^{n+1})$,

$$\int_{\mathbb{R}^{n+1}} h(x,t)\,\overline{(\mathscr{E}g)(x,t)}\,\mathrm{d}x\,\mathrm{d}t = \int_S \widehat{h}(\xi,\tau)\,\overline{g(\xi,\tau)}\,\mathrm{d}\sigma(\xi,\tau).$$

By duality we now see that the a priori restriction estimate (4.1) holds with $d = n+1$, $p = s'$, $q = 2$. In fact, the Fourier restriction operator $\mathscr{R}\colon \mathrm{L}^{s'}(\mathbb{R}^{n+1}) \to \mathrm{L}^2(S,\sigma)$ is now known to be bounded and its adjoint is precisely $\mathscr{E}$, which is for this reason sometimes called the *Fourier extension operator*.

Note that $p = s' < 2 = q$. Now Theorem 4.1.1 applies, so that the maximal estimate (4.4) gives

$$\Big\| \sup_{r_1,\ldots,r_{n+1}\in(0,\infty)} \big|\widehat{h} * \chi_{r_1,\ldots,r_{n+1}}\big| \Big\|_{\mathrm{L}^2(S,\sigma)} \lesssim_{n,\phi,\chi,s} \|h\|_{\mathrm{L}^{s'}(\mathbb{R}^{n+1})} \tag{4.26}$$

for any given Schwartz function $\chi \in \mathscr{S}(\mathbb{R}^{n+1})$. If we extend the definition of dilates as

$$\chi_{r_1,\ldots,r_d}(x_1,\ldots,x_d) := \frac{1}{|r_1\cdots r_d|}\chi\Big(\frac{x_1}{r_1},\ldots,\frac{x_d}{r_d}\Big)$$

for $r_1,\ldots,r_d \in \mathbb{R}\setminus\{0\}$, then (4.26) implies

$$\Big\| \sup_{r_1,\ldots,r_{n+1}\in\mathbb{R}\setminus\{0\}} \big|\widehat{h} * \chi_{r_1,\ldots,r_{n+1}}\big| \Big\|_{\mathrm{L}^2(S,\sigma)} \lesssim_{n,\phi,\chi,s} \|h\|_{\mathrm{L}^{s'}(\mathbb{R}^{n+1})}, \tag{4.27}$$

by considering 2^{n+1} quadrants of $\mathbb{R}^{n+1}$, flipping χ as necessary, and increasing the implicit constant by the factor 2^{n+1}. Linearizing and dualizing (4.27) we obtain

$$\Big|\int_S \big(\widehat{h} * \chi_{r_1(\xi),\dots,r_{n+1}(\xi)}\big)(\xi,\tau)\,\overline{g(\xi,\tau)}\,\mathrm{d}\sigma(\xi,\tau)\Big| \lesssim_{n,\phi,\chi,s} \|g\|_{\mathrm{L}^2(S,\sigma)}\|h\|_{\mathrm{L}^{s'}(\mathbb{R}^{n+1})}$$

for any choice of measurable functions $r_1,\dots,r_{n+1}\colon \mathbb{R}^n \to \mathbb{R}\setminus\{0\}$. If we further substitute (4.25) and choose χ such that $\check{\chi} = \overline{\psi}$, then we can rewrite the last bilinear estimate as

$$\Big|\int_{\mathbb{R}^{n+1}} h(x,t)\,\overline{(T_{\psi,r_1,\dots,r_{n+1}}f)(x,t)}\,\mathrm{d}x\,\mathrm{d}t\Big| \lesssim_{n,\phi,\psi,s} \|f\|_{\mathrm{L}^2(\mathbb{R}^n)}\|h\|_{\mathrm{L}^{s'}(\mathbb{R}^{n+1})},$$

which is just the dualized formulation of the desired bound (4.15). The case of general measurable functions $r_1,\dots,r_{n+1}\colon \mathbb{R}^n \to \mathbb{R}$ now easily follows in the limit.

Conclusion

In the thesis, we studied the powers of multipliers associated with unimodular homogeneous symbols of degree 0 and multi-parameter maximal Fourier restriction.

In the first part of the thesis, we proved asymptotically sharp estimates for the norms of these multipliers and showed that the powers of a generic multiplier have an asymptotically maximal order of growth. Consequently, we disproved Maz'ya's conjecture regarding the asymptotically sharp estimates of such multipliers and solved the problem posed by Dragičević, Petermichl, and Volberg concerning the sharp lower estimate of a certain multiplier falling within the mentioned class. Two interesting questions, however, remain open. One is the question of whether the powers of all Fourier multipliers associated with unimodular 0-homogeneous symbols have asymptotically the same order of growth and the other one is the question of the possibility of extending the approach from chapter 2 to general multipliers that would give lower estimates that are sharp both in the power and in $p \in (1,\infty)$.

In the second part, we generalized the Christ–Kiselev lemma for maximal operators to a multi-parameter version. As a consequence, we solved the multi-parameter version of the maximal Fourier restriction problem, initiated by the work of Müller, Ricci and Wright, and we proved a multi-parameter version of the Menshov–Paley–Zygmund theorem for the multi-dimensional Fourier transform. The techniques that we used are pushed to the limit, so the questions regarding extension of the range in which certain estimates hold remain open.

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Curriculum Vitae

Aleksandar Bulj was born on the 20th of September 1994 in Rijeka, where he completed primary and secondary school. In 2013, he commenced his studies in mathematics at the Department of Mathematics, University of Zagreb.

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IZJAVA O IZVORNOSTI RADA

Ja, Aleksandar Bulj, doktorand na Matematičkom odsjeku Prirodoslovno–matematičkog fakulteta Sveučilišta u Zagrebu,

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ovim putem izjavljujem pod materijalnom i kaznenom odgovornošću da je moj doktorski rad pod naslovom: *Estimates for Homogeneous Fourier Multipliers and Multiparameter Maximal Fourier Restriction* (na hrvatskom: *Ocjene za homogene Fourierove multiplikatore i višeparametarsku maksimalnu Fourierovu restrikciju*), isključivo moje autorsko djelo, koje je u potpunosti samostalno napisano uz naznaku izvora drugih autora i dokumenata korištenih u radu.

U Zagrebu, 23. studenog 2023.

Aleksandar Bulj