Entropy of black holes: quantum aspects

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Master Thesis

Zagreb, 2023

SVEUČILIŠTE U ZAGREBU
PRIRODOSLOVNO-MATEMATIČKI FAKULTET
FIZIČKI ODSJEK

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Entropija crnih rupa: kvantni aspekti

Diplomski rad

Zagreb, 2023

UNIVERSITY OF ZAGREB
FACULTY OF SCIENCE
DEPARTMENT OF PHYSICS

INTEGRATED UNDERGRADUATE AND GRADUATE UNIVERSITY PROGRAMME IN PHYSICS

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Master Thesis

The entropy of black holes: quantum aspects

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Master Thesis grade: ______________________

Committee: 1. ________________________

2. ________________________

3. ________________________

Master Thesis defence date: ________________

Zagreb, 2023

I would like to express my gratitude to my supervisor, dr. sc. Tajron Jurić, for providing me with the opportunity and pleasure of working with him and giving me invaluable knowledge. I also want to thank my friends Ivan, Klara, and Mateo for making my student years fun and enjoyable. Finally, my deepest appreciation goes to my parents, Slavica and Damir, for their unwavering support and love, even during the toughest of times.

Entropija crnih rupa: kvantni aspekti

Sažetak

Cilj ovog diplomskog rada je bio istražiti i analizirati literaturu koja se bavi kvantnim aspektima entropije crnih rupa, s naglaskom na kvantnim korekcijama. U prvom poglavlju generaliziramo ’t Hooftovu *brick wall* metodu na više dimenzionalne crne rupe, i te rezultate primjenjujemo na Schwarzschildovu te Reissner-Nordströmovu crnu rupu. U drugom poglavlju promatramo ispreplitanje stupnjeva slobode unutra i izvan crne rupe, tretirajući njen horizont kao ispreplitajuću plohu te interpretirajući dobivenu entropiju kao entropiju crne rupe. Predstavljamo dvije metode koje olakšavaju računanje isprepletene entropije: *replica trick* u kojem smo zadatak računanja isprepletene entropije sveli na računanje particijske funkcije na repliciranoj mnogostrukosti i *heat kernel* metodu gdje računamo entropiju putem traga tzv. *heat kernela*. U trećem poglavlju promatramo jednopetljenu renormalizaciju gravitacijske akcije putem Pauli-Villarsovog regulatora, te pokazujemo da to vodi na renormalizaciju Bekenstein-Hawkingove entropije. U zadnjem poglavlju zaključujemo raspravom o mogućnosti primjene ovih metoda na crnim rupama nižih dimenzija.

Ključne riječi: Opća teorija relativnosti, kvantna mehanika, statistička mehanika, crne rupe, termodinamika crnih rupa, entropija, kvantna teorija polja na zakrivljenim prostorima

The entropy of black holes: quantum aspects

Abstract

The goal of this thesis was to research and analyze the literature that deals with quantum aspects of black hole entropy, with a heavy emphasis on quantum corrections. In the first chapter, we generalize 'Hooft's brick wall method to higher-dimensional black holes, and we apply those results to the Schwarzschild and Reissner-Nordström black holes. In the second chapter, we look at the entanglement between the degrees of freedom inside and outside a black hole, while considering the horizon as the entanglement surface and interpreting the obtained entropy as the entropy of the black hole. We present two methods that simplify the calculations of the entanglement entropy: the replica trick, where we turn the task of calculating the entanglement entropy to calculating the partition function on a replicated manifold, and the heat kernel method, where we calculate the entropy by taking the trace of the so-called heat kernel. In the third chapter, we consider the one-loop renormalization of the gravitational action, and using the Pauli-Villars regulator, we show that it leads to the renormalization of the Bekenstein-Hawking entropy. In the final chapter, we conclude with a discussion on the applicability of these methods to lower-dimensional black holes.

Keywords: General theory of relativity, quantum mechanics, statistical mechanics, black holes, black hole thermodynamics, entropy, quantum field theory on curved spacetimes

Contents

1 Introduction

A black hole is a region of spacetime where gravity is so strong that nothing, including light, can escape it. This boundary of no escape is called the event horizon. Mathematically, black holes emerge as the solutions to the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.1}$$

Physically, they arise from the gravitational collapse of massive stars, giving rise to a system with a very strong gravitational field. This property makes black holes an ideal arena, both theoretically and phenomenologically, to observe the quantum effects of gravity and, specifically, any possible quantum corrections that a system may obtain. In the stationary case, black holes can be described by three parameters: mass M, charge Q, and angular momentum J, which is a consequence of the no-hair theorem [1]. To illustrate the need for applying thermodynamic laws to black holes, let us imagine a scenario where some matter falls into a black hole. Prior to crossing the horizon, this matter possessed a certain entropy S. After crossing the horizon, an outside observer cannot determine what occurred with this entropy. Nevertheless, we do know that before the matter fell into the black hole, the overall entropy was S. After the matter entered the black hole and after a sufficiently long time, all we observed was the steady state of the black hole, as described by its mass, charge, and angular momentum. If there were only a single state for a black hole, characterized by these three quantities, the entropy of each such black hole would be zero. This implies that when matter enters a black hole, the total entropy diminishes, conflicting with the second law of thermodynamics. This issue was resolved by Bekenstein [2], who postulated that black holes also possess entropy. Building upon these facts and observations, we can draw parallels between the mass M, charge Q, and angular momentum J of the black hole and thermodynamic variables, leading us to the four laws of the mechanics of black holes [3–4].

The zeroth law states that stationary black holes have a constant surface gravity κ at the event horizon.

The first law is expressed as

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ, \tag{1.2}$$

where Ω represents the angular velocity of the horizon, and A signifies the surface area of the horizon.

The second law states that the surface area of the horizon is non-decreasing

$$\delta A \geq 0. \tag{1.3}$$

The third law claims that the surface gravity κ cannot be reduced to zero in a finite number of steps.

Comparing the first law with the one from thermodynamics, $dE = TdS - pdV$, and matching ΩdJ with the work term, we have a correspondence of TdS with $\kappa dA/8\pi G$. Hawking [5] calculated that black holes radiate like black bodies and have a temperature of

$$T_{\mathrm{H}} = \frac{\hbar\kappa}{2\pi}. \tag{1.4}$$

Utilizing the aforementioned results, we arrive at the formula for the entropy of black holes

$$S_{\mathrm{BH}} = \frac{A}{4\ell_{\mathrm{Pl}}^2}, \tag{1.5}$$

where $\ell_{\mathrm{Pl}} = \sqrt{G\hbar}$ denotes the Planck length. This formula is called the Bekenstein-Hawking entropy. The unique feature of the Bekenstein-Hawking entropy is that it is proportional to the area of the black hole's surface, known as the horizon, which differs from typical entropy calculations where the entropy is usually proportional to volume. During the 80s of the previous century, researchers explored different approaches to understanding black hole entropy. 't Hooft [6] calculated the entropy of Hawking particles just outside the black hole horizon, treating them like a thermal gas. Although this calculation yielded an entropy proportional to the horizon area, it required a "brick wall" boundary near the horizon to regulate certain divergences. Bombelli, Koul, Lee, and Sorkin [7] considered a reduced density matrix obtained by tracing over quantum field degrees of freedom inside a black hole's horizon. This procedure seemed natural for black holes because their horizon acts as a causal boundary, making events inside inaccessible to observers outside. Srednicki [8] calculated the entropy directly in flat spacetime by tracing over degrees of freedom inside an imaginary surface. This entropy, known as entanglement entropy, was also found to be proportional to the entanglement surface. The entropy arises from short-distance correlations in the quantum field system near the surface and is thus sensitive to the size of the region near the surface. This means that only modes located in a region near the surface contribute to the entropy, signifying that the size of this region plays the role of a UV regulator. An interesting thing to note is that the entanglement entropy of a quantum field in flat spacetime already establishes the area law without the need for a black hole. Furthermore, it was realized that the entropy obtained by the "brick wall" model and the entanglement entropy are related [9]. Another method to calculate the entanglement entropy had been developed by Susskind [10], introducing a small conical singularity at the entangling surface, then evaluating the effective action of a quantum field on the background metric with a conical singularity, and then differentiating the action with respect to the deficit angle. This method is the so-called replica trick. Using this method, systematic calculations of the UV divergent terms in the entanglement entropy of black holes have been made [11], and particularly, logarithmic UV divergent terms have been found [12]. These logarithmic correction terms have also been found using several different methods, including string theory [13], higher loop corrections to the gravitational action [14], the heat kernel method [15], and using noncommutative geometry [16]. The important question of whether the UV divergence in entanglement entropy can be properly renormalized was explored by Susskind and Uglum [17], which revealed that the standard renormalization of Newton's constant produces a finite entropy if we consider the entanglement entropy as a quantum contribution to the Bekenstein-Hawking entropy. Later on, Ryu and Takayanagi [18] proposed a holographic interpretation of entanglement entropy, linking it to the area of a minimal surface in anti-de Sitter spacetime through the AdS/CFT correspondence. Ongoing research suggests that entanglement entropy holds promise for a better understanding of black holes and Quantum Gravity, as several reviews have covered its role for black holes [19, 20], its calculation in quantum field theory in flat spacetime [21], and its holographic aspects [22].

This thesis will provide a detailed analysis of the original methods of calculating black hole entropy and a review of the newer methods of calculation, such as the replica trick method, the heat kernel method, and the one-loop renormalization of the gravitational action. Thus, the structure of this thesis will be as follows: In the second chapter, we will generalize the brick wall method of calculating the entropy to higher-dimensional black holes and apply it to certain black hole solutions. The third chapter will cover the entanglement entropy calculation, the replica trick, and the heat kernel method. In the fourth chapter, we will go in detail over the one-loop renormalization of the gravitational action. The final chapter includes concluding remarks. The units we will be using are $c = k_B = 1$, unless otherwise stated.

2 The brick wall model

One of the very first ways of calculating the entropy of black holes was using the brick wall method that was suggested by 't Hooft [6]. In this section, we will go over the method, generalize it for higher-dimensional black holes, and find that the quantum corrections to the entropy that arise due to considering a quantum theory are logarithmic in black hole area [23].

The brick wall model is a semi-classical approach to understanding the microscopic origin of black hole entropy. The black hole geometry is assumed to be a fixed classical background on which matter fields propagate, and the entropy of black holes arises due to the canonical entropy of matter fields outside the black hole event horizon, evaluated at the Hawking temperature. Considering the number of energy levels that a particle can occupy in the vicinity of a black hole, we find that the density of states diverges [6]. This, in turn, would suggest that the entropy of a black hole diverges and can be used as an argument for claiming that a black hole is an infinite sink of information. To remedy this infinity, we introduce a so-called brick wall near the horizon. That is, we assume that all fields must vanish within some fixed distance h from the horizon,

$$\Phi(r) = 0, \quad r = r_{\rm H} + h, \tag{2.1}$$

where $r_{\rm H}$ is the position of the horizon of the black hole. To further simplify our calculations, we shall take Φ to be a scalar wave function, that is, we will consider a scalar field theory. We shall consider a $(D+2)$-dimensional, spherically symmetric black hole spacetime with the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_D^2, \tag{2.2}$$

where $d\Omega_D^2$ is the metric of a $D-$sphere. The surface gravity κ defined as

$$k^a \nabla_a k^b = \kappa k^b, \tag{2.3}$$

where k^a and k^b are Killing vectors defined by the Killing equation

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0. \tag{2.4}$$

Since a given Killing vector is normal to its Killing horizon, we can use the following property [24]

$$k_a \nabla_b k_c + k_b \nabla_c k_a + k_c \nabla_a k_b = 0, \tag{2.5}$$

Combining (2.4) and (2.5), we arrive at

$$k_c \nabla_a k_b = -k_a \nabla_b k_c + k_b \nabla_a k_c. \tag{2.6}$$

Contracting from the left with $\nabla^a k^b$,

$$\begin{aligned}(\nabla^a k^b)(\nabla_a k_b)k_c &= -k_a \nabla^a k^b \nabla_b k_c - k_b \nabla^b k^a \nabla_a k_c \\ &= -\kappa k_a \nabla^a k_c - \kappa k_b \nabla^b k_c = -2\kappa k^a \nabla_a k_c = -2\kappa^2 k_c.\end{aligned} \tag{2.7}$$

We arrive at a formula for the surface gravity

$$\kappa = \sqrt{-\frac{1}{2}(\nabla^a k^b)(\nabla_a k_b)}\bigg|_{r=r_{\mathrm{H}}} = \sqrt{-\frac{1}{2}g^{aa'}g^{bb'}(\nabla_{a'}k_{b'})(\nabla_a k_b)}\bigg|_{r=r_{\mathrm{H}}}. \tag{2.8}$$

In the case of a spherically symmetric spacetime, there are four Killing vectors, one time-like and three space-like ones. Evaluating (2.8) for the simplest of these vectors, the time-like one, which is of the form

$$k^\mu = (1,0,0,0), \quad k_\mu = (-f(r),0,0,0), \tag{2.9}$$

we find the surface gravity to be given by

$$\kappa = \sqrt{-\frac{1}{2}g^{rr}g^{tt}(\nabla_r k_t)(\nabla_r k_t)}\bigg|_{r=r_{\mathrm{H}}} = \left[\sqrt{\frac{g(r)}{f(r)}}\left(\frac{f'(r)}{2}\right)\right]\bigg|_{r=r_{\mathrm{H}}}, \tag{2.10}$$

where r_{H} is the location of the event horizon of a given black hole. Since we will consider the behavior of the field near the horizon, it is useful to expand the metric functions $g(r)$, and $f(r)$ near the horizon up to second order,

$$\begin{aligned}f(r) &= f'(r_{\mathrm{H}})(r-r_{\mathrm{H}}) + \frac{1}{2}f''(r_{\mathrm{H}})(r-r_{\mathrm{H}})^2 + \dots, \\ g(r) &= g'(r_{\mathrm{H}})(r-r_{\mathrm{H}}) + \frac{1}{2}g''(r_{\mathrm{H}})(r-r_{\mathrm{H}})^2 + \dots,\end{aligned} \tag{2.11}$$

where the horizon is defined as the value at which they vanish

$$g(r_{\rm H}) = f(r_{\rm H}) = 0. \tag{2.12}$$

Another useful quantity that we will use is the proper radial distance from the event horizon of the black hole to the brick wall. We will name it h_c, and it is given by

$$h_c = \int_{r_{\rm H}}^{r_{\rm H}+h} \frac{dr}{\sqrt{g(r)}}. \tag{2.13}$$

Expanding $g(r)$ up to first order and using the substitution $x = r - r_{\rm H}$ we find

$$h_c = \int_0^h \frac{dx}{\sqrt{g'(r_{\rm H})x}} = \sqrt{\frac{4h}{g'(r_{\rm H})}}. \tag{2.14}$$

Since we are working in a scalar field theory, our field Φ satisfies the Klein-Gordon equation:

$$\left(\Box - \frac{m^2}{\hbar^2}\right)\Phi = 0, \tag{2.15}$$

where $\Box = \nabla^\mu \nabla_\mu$ represents the d'Alembertian operator, and m is the mass of the scalar field.

Since the metric that we are using is diagonal, its determinant g has the form

$$g = -\frac{f(r)}{g(r)} r^{2D} H(\theta, \phi, ...), \tag{2.16}$$

where H is a function of all angle variables of the $D-$sphere. The d'Alembertian can then be rewritten as

$$\begin{aligned} \Box = \nabla^\mu \nabla_\mu &= \frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu}\sqrt{-g}\partial_\nu) \qquad (2.17) \\ &= \frac{1}{\sqrt{-g}} \partial_t (g^{tt}\sqrt{-g}\partial_t) + \frac{1}{\sqrt{-g}} \partial_i (g^{ij}\sqrt{-g}\partial_j) = \frac{1}{\sqrt{-g}} \partial_t (g^{tt}\sqrt{-g}\partial_t) + \nabla^2_{D+1}, \end{aligned}$$

where ∇^2_{D+1} is the $(D+1)$-dimensional Laplace operator in curved spacetime, which in the case of a diagonal metric has the form

$$\nabla^2_{D+1} = \frac{1}{\sqrt{-g}} \partial_i (g^{ij}\sqrt{-g}\partial_j) = \frac{1}{\sqrt{-g}} \sum_{i=1}^{D+1} \partial_i (g^{ii}\sqrt{-g}\partial_i). \tag{2.18}$$

For a diagonal metric, the elements of the inverse metric, $g^{\mu\nu}$, can be written as $g^{\mu\nu} = 1/g_{\mu\nu}$.

Since g^{tt} i $\sqrt{-g}$ are both independent of t, the first term in the d'Alembertian can be rewritten as

$$g^{tt}\partial_{tt} = -\frac{1}{f(r)}\partial_{tt}, \tag{2.19}$$

and the Klein-Gordon equation is now given by

$$\left(-\frac{1}{f(r)}\partial_{tt} + \nabla^2_{D+1} - \frac{m^2}{\hbar^2}\right)\Phi = 0. \tag{2.20}$$

Due to the rotational symmetry of the spacetime (2.2) we can use the ansatz [25]

$$\Phi = e^{-iEt/\hbar}\frac{R(r)}{r^{D/2}G(r)^{1/2}}Y_{\ell m_i}(\theta, \phi_i), \tag{2.21}$$

where $G(r) = \sqrt{f(r)g(r)}$, $i \in \{1, ..., (D-1)\}$ and $Y_{\ell m_i}(\theta, \phi_i)$ denote the hyperspherical harmonics. By plugging in (2.21) into (2.20), we show in Appendix A that the radial equation for the scalar wave function is given by

$$R''(r) + \left[\frac{V^2(r)}{\hbar^2} - \Delta(r)\right]R(r) = 0, \tag{2.22}$$

where $V^2(r)$ is defined in (A.9), and where $\Delta(r)$ is given by (A.18). We can notice that the $V^2(r)$ term plays the role of the effective potential [23]. Since it is difficult to find an exact analytical solution for the function $R(r)$, we resort to the WKB approximation. The WKB ansatz we are going to use is

$$R(r) = \frac{1}{\sqrt{P(r)}}\exp\left[\frac{i}{\hbar}\int^r P(r')dr'\right]. \tag{2.23}$$

Using

$$R'(r) = -\frac{1}{2}\frac{1}{P(r)}P'(r)R(r) + \frac{i}{\hbar}P(r)R(r) = \left(-\frac{1}{2}\frac{P'(r)}{P(r)} + \frac{i}{\hbar}P(r)\right)R(r), \tag{2.24}$$

and

$$
\begin{aligned}
R''(r) &= \left(-\frac{1}{2}\frac{P''(r)}{P(r)} + \frac{1}{2}\left(\frac{P'(r)}{P(r)}\right)^2 + \frac{i}{\hbar}P'(r)\right) R(r) + \left(-\frac{1}{2}\frac{P'(r)}{P(r)} + \frac{i}{\hbar}P(r)\right)^2 R'(r) \\
&= \left(-\frac{1}{2}\frac{P''(r)}{P(r)} + \frac{1}{2}\left(\frac{P'(r)}{P(r)}\right)^2 + \frac{i}{\hbar}P'(r) + \frac{1}{4}\left(\frac{P'(r)}{P(r)}\right)^2 - \frac{i}{\hbar}P'(r) - \frac{1}{\hbar^2}P(r)^2\right) R(r) \\
&= \left(-\frac{1}{2}\frac{P''(r)}{P(r)} + \frac{3}{4}\left(\frac{P'(r)}{P(r)}\right)^2 - \frac{1}{\hbar^2}P(r)^2\right) R(r), \qquad (2.25)
\end{aligned}
$$

and plugging this into (2.22), we arrive at the following equation

$$
P^2(r)[P^2(r) - V^2(r)] = \hbar^2\left[\frac{3}{4}P'(r)^2 - \frac{1}{2}P''(r)P(r) - \Delta(r)P(r)^2\right]. \qquad (2.26)
$$

Originally, 't Hooft [6] stopped at the leading order WKB solution for $R(r)$, and used that to evaluate the number of states $N(E)$, which was then used to calculate the free energy F and entropy S of the quantum field. Here we will extend the analysis by going to higher orders of $\hbar$ in the WKB aproximation. Now, writing $P(r)$ as a series in $\hbar^2$

$$
P(r) = \sum_{n=0}^{\infty} \hbar^{2n} P_{2n}(r), \qquad (2.27)
$$

and inserting this series into (2.26), and grouping the terms by powers of $\hbar^2$, we arrive at the equations for $P_{2n}(r)$, up to $n = 2$

$$
\begin{aligned}
P_0(r) &= \pm V(r), \qquad (2.28)\\
P_2(r) &= \left(\frac{3}{8P_0(r)}\right)\left(\frac{P_0'(r)}{P_0(r)}\right)^2 - \left(\frac{P_0''(r)}{4P_0(r)^2}\right) - \left(\frac{\Delta(r)}{2P_0(r)}\right),\\
P_4(r) &= -\left(\frac{5P_2(r)^2}{2V(r)}\right) - \left(\frac{4P_2(r)\Delta(r) + P_2''(r)}{4V^2(r)}\right) + \left(\frac{3P_2'(r)V'(r) - P_2(r)V''(r)}{4V^3(r)}\right).
\end{aligned}
$$

Notice how higher order terms P_{2n} for $n > 1$ can be recursively written as functions of $P_0(r)$. Furthermore, $P(r)$ was written as a series in $\hbar^2$ instead of $\hbar$ because the terms proportional to odd powers of $\hbar$ are zero. This can be easily shown by plugging an expansion that includes odd powers into the equation (2.26) and grouping the terms in powers of $\hbar$.

Since our WKB ansatz is of the form $R \sim e^{i\int P dr}$, we follow the standard quantization procedure [6], imposing a quantization condition similar to $\int P dr = \hbar\pi n$, for each mode, P_{2n}, where the total number of states of the field is given by $N = \sum n$.

More specifically, we impose

$$\int_{r_{\rm H}+h}^{L} dr \int_{0}^{\ell_{max}} d\ell (2\ell + D - 1)\mathcal{W}(\ell) P_{2n}(r) = \pi\hbar^{1-2n} N_{2n}(E), \tag{2.29}$$

where we have transitioned from the sum over the angular quantum number ℓ to the integral, since the difference between any two angular momenta is smaller than ℓ_{max}, i.e $\ell_{max} \gg 1$, where

$$\mathcal{W}(\ell) = \frac{(\ell + D - 2)!}{(D-1)!\ell!} \tag{2.30}$$

is the degeneracy factor of the angular momentum, which becomes important at spacetime dimensions that differ from $D = 2$, and where $N_{2n}(E)$ is the contribution of the n-th mode to the total number of states of the field with energy less than E,

$$N(E) = \sum_{n=0}^{\infty} N_{2n}(E). \tag{2.31}$$

We should note that the upper bound of the radial integral (2.29), L, signifies the infrared cutoff, imposed to guarantee the finiteness of the entropy at large distances. ℓ_{max} is given in such a way that the functions P_{2n} are real, or equivalently, that $P_0(r)$ is real. The condition on ℓ_{max} is then

$$\ell_{max}(\ell_{max} + D - 1) = \frac{r^2}{\hbar^2}\left(\frac{E^2}{f(r)} - m^2\right). \tag{2.32}$$

We decompose the entropy and the free energy in the same manner

$$S = \sum_{n=0}^{\infty} S_{2n}, \quad F = \sum_{n=0}^{\infty} F_{2n}, \tag{2.33}$$

where

$$F_{2n} = -\int_0^{\infty} \frac{N_{2n}(E)}{e^{\beta E} - 1} dE, \quad S_{2n} = \beta^2 \frac{\partial F_{2n}}{\partial \beta}. \tag{2.34}$$

To derive the left relation in (2.34), we use statistical mechanics.

$$e^{-\beta F} = \mathcal{Z} = \sum_i e^{-\beta E_i} = \prod_{n,l,m} \sum_N \left(e^{-\beta E}\right)^N = \prod_{n,l,m} \frac{1}{1 - e^{-\beta E}}, \tag{2.35}$$

where $\mathcal{Z}$ is the partition function, and where it is summed over all the allowed states indexed by i, where some states might have the same energy. Since the energy spectrum of the given particles is determined by n, l and m, and using the assumption that the particles are independent, we can switch from the sum to the integral over those quantum numbers, summing over all the particles in that particular mode. The free energy is then given by

$$F = \frac{1}{\beta}\sum_N \ln(1 - e^{-\beta E}) = \frac{1}{\beta}\int dN \ln(1 - e^{-\beta E}) = -\int dE \frac{N(E)}{e^{\beta E} - 1}, \tag{2.36}$$

where we transitioned from the sum to the integral since we are assuming that we can have arbitrary many particles in the vicinity of the black hole, and where we partially integrated in the last step.

2.1 *Electrically neutral black holes in* 4 *dimensions*

In this subsection, we will calculate the entropy up to the second order for the case of a four-dimensional black hole ($D = 2$), with $f(r)$=$g(r)$ and $m = 0$. Equations (A.9) and (A.18) are now given by

$$V^2(r) = \frac{1}{g^2(r)}\left(E^2 - g(r)\frac{\ell(\ell+1)\hbar^2}{r^2}\right), \tag{2.37}$$

and

$$\Delta(r) = \left(\frac{g''(r)}{2g(r)}\right) - \left(\frac{g'(r)^2}{4g(r)^2}\right) + \left(\frac{1}{r}\right)\left(\frac{g'(r)}{g(r)}\right). \tag{2.38}$$

For the zeroth order, $P_0(r)$ can be written as

$$P_0(r) = \pm\frac{1}{g(r)}\left[E^2 - g(r)\frac{\hbar^2\ell(\ell+1)}{r^2}\right]^{1/2}, \tag{2.39}$$

and after plugging it into $N_0(E)$

$$N_0(E) = \frac{1}{\hbar\pi}\int_{r_{\rm H}+h}^{L} dr \int_0^{\ell_{max}} d\ell(2\ell+1)P_0(r), \tag{2.40}$$

using the substitution $\lambda = \ell(\ell+1)$ we are left with

$$N_0(E) = \frac{1}{\hbar\pi}\int_{r_{\rm H}+h}^{L} dr \int_0^{r^2E^2/\hbar^2 g(r)} d\lambda \sqrt{\frac{E^2}{g^2(r)} - \frac{\hbar^2\lambda}{g(r)r^2}}, \tag{2.41}$$

and we finally obtain

$$N_0(E) = \frac{2E^3}{3\pi\hbar^3}\int_{r_{\rm H}+h}^{L} \frac{r^2}{g^2(r)} dr. \tag{2.42}$$

The free energy is given by

$$F_0 = -\frac{2\pi^3}{45\hbar^3}\frac{1}{\beta^4}\int_{r_{\rm H}+h}^{L} \frac{r^2}{g^2(r)} dr, \tag{2.43}$$

and the entropy by

$$S_0 = \frac{8\pi^3}{45\hbar^3}\frac{1}{\beta^3}\int_{r_{\rm H}+h}^{L} \frac{r^2}{g^2(r)} dr. \tag{2.44}$$

Before introducing the brick wall, the entropy diverged at the horizon due to the diverging density of states. We use this fact to conclude that the dominant contribution to the entropy will be from the terms that, after the r integration, diverge as $h \to 0$. To that end, we introduce the near-horizon variable $x = r - r_{\rm H}$. After expanding the metric function to higher orders near the horizon

$$g(r) = (r - r_{\rm H})g'(r_{\rm H}) + \frac{1}{2}(r - r_{\rm H})^2 g''(r_{\rm H}), \quad g'(r) = g'(r_{\rm H}) + (r - r_{\rm H}), \tag{2.45}$$
$$g''(r_{\rm H}), \quad g''(r) = g''(r_{\rm H}),$$

and, instead of $g'(r_{\rm H})$ and $g''(r_{\rm H})$ using g' i g'', we can write the entropy as

$$S_0 = \frac{\kappa^3}{45}\int_h^{L-r_{\rm H}} dx \frac{(x + r_{\rm H})^2}{(g'x + x^2 g''/2)^2}. \tag{2.46}$$

The dominant terms will now be those of the form x^{-k}, $k \geq 1$. We can now use the following expansions to calculate the entropy

$$\frac{1}{xg' + x^2g''/2} = \frac{1}{xg'(1 + xg''/(2g'))} = \frac{1}{xg'}\left(1 - \frac{xg''}{2g'}\right), \tag{2.47}$$
$$\frac{1}{(xg' + x^2g''/2)^2} = \frac{1}{x^2g'^2 + x^3g''g'} = \frac{1}{x^2g'^2(1 + xg''/g')} = \frac{1}{x^2g'^2}\left(1 - \frac{xg''}{g'}\right).$$

Furthermore, we will ignore all constant and linear terms of x under the integral since after the integration, they will be proportional to h i h^2, and hence will not be divergent as $h \to 0$, and neither contribute significantly to the entropy.

$$
\begin{aligned}
S_0 &= \frac{\kappa^3}{45}\int_h^{L-r_{\rm H}} dx \frac{(x+r_{\rm H})^2}{(g'x+x^2g''/2)^2} \\
&= \frac{\kappa^3}{45}\int_h^{L-r_{\rm H}} dx \left(\frac{1}{g'^2}+\frac{2r_{\rm H}}{xg'^2}+\frac{r_{\rm H}^2}{x^2g'^2}\right)\left(1-\frac{xg''}{g'}\right) \\
&= \frac{\kappa^3}{45}\int_h^{L-r_{\rm H}} dx \left(\frac{2r_{\rm H}}{xg'^2}+\frac{r_{\rm H}^2}{x^2g'^2}-\frac{r_{\rm H}^2g''}{xg'^3}\right) \\
&= \frac{g'^3}{8\cdot 45}\frac{r_{\rm H}^2}{hg'^2}+\left(\frac{2}{45}\frac{\kappa g'^2}{4}\frac{r_{\rm H}}{g'^2}-\frac{g'^3}{8\cdot 45}\frac{r_{\rm H}^2g''}{g'^3}\right)\ln\left(\frac{L-h}{h}\right) \\
&= \frac{r_{\rm H}^2}{90h_c^2}-\left(\frac{g''(r_{\rm H})r_{\rm H}^2}{360}-\frac{\kappa r_{\rm H}}{90}\right)\ln\left(\frac{\alpha}{h}\right)
\end{aligned}
\tag{2.48}
$$

The previous integration leads to a correction term in the form of

$$
\ln\left(\frac{L-r_{\rm H}}{h}\right)=\ln\left(\frac{L-r_{\rm H}}{h}\frac{\alpha}{\alpha}\right)=\ln\left(\frac{\alpha}{h}\right)+\ln\left(\frac{L-h}{\alpha}\right)\approx\ln\left(\frac{\alpha}{h}\right),
\tag{2.49}
$$

where we have ignored the part proportional to L, since it signifies the infrared divergence and can thus always be removed. We also interpret this term as the vacuum contribution to the entropy. Essentially, this way we have introduced a free constant, α, which we can choose as we like. The way that we will choose it is that after we obtain the equation for the entropy, we will equate the most divergent part of the entropy, with regards to the cutoff h_c, with $S_{\rm BH}$. Using the relation that connects h_c and h, (2.14), and plugging that h back in the logarithm term, we will determine α in such a way that the argument of the logaritham has the form $\left(\frac{\mathcal{A}}{\ell_{\rm Pl}^2}\right)$.

$$
S_0=\frac{r_{\rm H}^2}{90h_c^2}+\left[\frac{\kappa r_{\rm H}}{90}-\frac{g''(r_{\rm H})r_{\rm H}^2}{360}\right]\ln\left(\frac{\alpha}{h}\right).
\tag{2.50}
$$

We notice that even in the leading order we have a correction term that is logarithmic in nature. If we were to stop at this order, we would have to equate the most divergent part of the entropy with $S_{\rm BH}$, to obtain h_c^2.

$$
(S_0)_{div}=\frac{r_{\rm H}^2}{90h_c^2}=S_{\rm BH}=\frac{\mathcal{A}}{4\ell_{\rm Pl}^2},
\tag{2.51}
$$

using $\mathcal{A}=4r_{\rm H}^2\pi$, we have

$$
h_c^2=\frac{\ell_{\rm Pl}^2}{90\pi}.
\tag{2.52}
$$

So to ensure that the logarithmic part has the form $\mathcal{A}/\ell_{\rm Pl}^2$, we have to choose $\alpha = r_{\rm H}^2 g'(r_{\rm H})/90$. The entropy at the zeroth order would be given by

$$S = S_0 = \frac{r_{\rm H}^2}{90h_c^2} + \left[\frac{\kappa r_{\rm H}}{90} - \frac{g''(r_{\rm H})r_{\rm H}^2}{360}\right] \ln\left(\frac{4r_{\rm H}^2}{90h_c^2}\right), \tag{2.53}$$

and after plugging in (2.52), we are left with

$$S = S_{\rm BH} + \mathcal{F}_1(\mathcal{A}) \ln\left(\frac{\mathcal{A}}{\ell_{\rm Pl}^2}\right), \tag{2.54}$$

where

$$\mathcal{F}_1(\mathcal{A}) = \frac{\kappa r_{\rm H}}{90} - \frac{g''(r_{\rm H})r_{\rm H}^2}{360}. \tag{2.55}$$

To calculate the second order correction to the entropy, notice that we can rewrite $P_2(r)$ as

$$P_2(r) = \left(\frac{P_2^{(0)}(r)}{\mathcal{G}(\mathcal{E},r)}\right) + \lambda(r)\left(\frac{P_2^{(1)}(r)}{\mathcal{G}^3(\mathcal{E},r)}\right) + \lambda^2(r)\left(\frac{P_2^{(2)}(r)}{\mathcal{G}^5(\mathcal{E},r)}\right), \tag{2.56}$$

where we defined

$$\mathcal{G}(\mathcal{E},r) = [\mathcal{E} - \lambda(r)]^{1/2}, \tag{2.57}$$

with $\mathcal{E} = E^2$ and

$$\lambda(r) = \ell(\ell+1)\hbar^2\frac{g(r)}{r^2}, \tag{2.58}$$

and where $P_2^{(0)}(r), P_2^{(1)}(r), P_2^{(2)}(r)$ are given by

$$P_2^{(0)}(r) = -\frac{g'(r)}{2r}, \tag{2.59}$$

$$P_2^{(1)}(r) = \frac{3g(r)}{4r^2} - \frac{3g'(r)}{4r} + \frac{g''(r)}{8} + \frac{g'(r)^2}{8g}, \tag{2.60}$$

$$P_2^{(2)}(r) = \frac{5g}{8r^2} - \frac{5g'(r)}{8r} + \frac{5g'(r)^2}{32g}. \tag{2.61}$$

Before we turn to calculating $N_2(r)$, notice that the following relations for the function (2.57) hold

$$\frac{1}{\mathcal{G}(\mathcal{E},r)} = 2\frac{\partial\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}}, \quad \frac{1}{\mathcal{G}^3(\mathcal{E},r)} = -4\frac{\partial^2\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}^2}, \quad \frac{1}{\mathcal{G}^5(\mathcal{E},r)} = \frac{8}{3}\frac{\partial^3\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}^3}. \tag{2.62}$$

They will prove useful when applying the Leibniz rule

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f[x,t]dt = f[x,b(x)]\left(\frac{db(x)}{dx}\right) \tag{2.63}$$
$$-f[x,a(x)]\left(\frac{da(x)}{dx}\right)+\int_{a(x)}^{b(x)}\left[\frac{\partial f(x,t)}{\partial x}\right]dt,$$

to calculate $N_2(E)$. $N_2(E)$ is defined as

$$N_2(E) = \frac{\hbar}{\pi}\int_{r_{\rm H}+h}^{L} dr \int_0^{\ell_{max}} d\ell(2\ell+1)P_2(r). \tag{2.64}$$

Substituting (2.58) into (2.64), we have

$$\hbar N_2(E) = \frac{1}{\pi}\int_{r_{\rm H}+h}^{L}\frac{r^2}{g(r)}dr\int_0^{\mathcal{E}}\left[2\frac{\partial\mathcal{G}(\mathcal{E},\lambda)}{\partial\mathcal{E}}P_2^{(0)}(r)d\lambda\right. \tag{2.65}$$
$$\left.-4\lambda\frac{\partial^2\mathcal{G}(\mathcal{E},\lambda)}{\partial\mathcal{E}^2}P_2^{(1)}(r)d\lambda+\frac{8}{3}\lambda^2\frac{\partial^3\mathcal{G}(\mathcal{E},\lambda)}{\partial\mathcal{E}^3}P_2^{(2)}(r)d\lambda\right].$$

We now use the Leibniz rule to extract the divergences that would arise if we were to execute the λ integration on the right-hand side. The first integral does not lead to divergent terms. Using (2.63) with $a(\mathcal{E}) = 0$ and $b(\mathcal{E}) = \mathcal{E}$ we have

$$\int_0^{\mathcal{E}} P_2^{(0)}(r)\frac{\partial\mathcal{G}(\mathcal{E},r)}{\partial\mathcal{E}}d\lambda = \frac{\partial}{\partial\mathcal{E}}\int_0^{\mathcal{E}} P_2^{(0)}(r)\mathcal{G}(\mathcal{E},r)d\lambda. \tag{2.66}$$

Applying (2.63) to the second integral and third integral in (2.65), we arrive at

$$\int_0^{\mathcal{E}}\lambda\frac{\partial^2\mathcal{G}(\mathcal{E},\lambda)}{\partial\mathcal{E}^2}d\lambda = \frac{\partial}{\partial\mathcal{E}}\int_0^{\mathcal{E}}\lambda\frac{\partial\mathcal{G}(\mathcal{E},\lambda)}{\partial\mathcal{E}}d\lambda - \mathcal{E}\frac{\partial\mathcal{G}(\mathcal{E},\lambda)}{\partial\mathcal{E}}\bigg|_{\mathcal{E}=\lambda} \tag{2.67}$$
$$= \frac{\partial^2}{\partial\mathcal{E}^2}\int_0^{\mathcal{E}}\lambda\mathcal{G}(\mathcal{E},\lambda)d\lambda - \frac{\mathcal{E}}{2(\mathcal{E}-\lambda)^{1/2}}\bigg|_{\mathcal{E}=\lambda},$$
$$\int_0^{\mathcal{E}}\lambda^2\frac{\partial^3\mathcal{G}(\mathcal{E},\lambda)}{\partial\mathcal{E}^3}d\lambda = \frac{\partial^3}{\partial\mathcal{E}^3}\int_0^{\mathcal{E}}\lambda^2\mathcal{G}(\mathcal{E},\lambda)d\lambda - \left[\frac{\partial}{\partial\mathcal{E}}\left[\frac{\mathcal{E}^2}{2\mathcal{G}(\mathcal{E},\lambda)}\right]-\frac{\mathcal{E}^2}{4\mathcal{G}^3(\mathcal{E},\lambda)}\right]\bigg|_{\mathcal{E}=\lambda}.$$

From the upper two equations, we see that both integrals have a finite and a divergent part. The divergence happens at the turning point $\mathcal{E} = \lambda$. This is a non-physical divergence that arises due to the fact that the WKB approximation is not viable near the turning points of the effective potential [26]. After discarding the non-physical divergences, $N_2(E)$ is given by

$$\hbar N_2(E) = \frac{2}{\pi}\int_{r_{\mathrm{H}}+h}^{L} dr \frac{r^2}{g(r)} P_2^{(0)}(r) \frac{\partial}{\partial \mathcal{E}} \int_0^{\mathcal{E}} \mathcal{G}(\mathcal{E},\lambda) d\lambda \qquad (2.68)$$
$$- \frac{4}{\pi}\int_{r_{\mathrm{H}}+h}^{L} dr \frac{r^2}{g(r)} P_2^{(1)}(r) \frac{\partial^2}{\partial \mathcal{E}^2} \int_0^{\mathcal{E}} \lambda\mathcal{G}(\mathcal{E},\lambda) d\lambda$$
$$+ \frac{8}{3\pi}\int_{r_{\mathrm{H}}+h}^{L} dr \frac{r^2}{g(r)} P_2^{(2)}(r) \frac{\partial^3}{\partial \mathcal{E}^3} \int_0^{\mathcal{E}} \lambda^2\mathcal{G}(\mathcal{E},\lambda) d\lambda.$$

Using the integrals

$$\int_0^{\mathcal{E}} \mathcal{G}(\mathcal{E},\lambda) d\lambda = \frac{2}{3}\mathcal{E}^{3/2}, \quad \int_0^{\mathcal{E}} \lambda\mathcal{G}(\mathcal{E},\lambda) d\lambda = \frac{4}{15}\mathcal{E}^{5/2}, \qquad (2.69)$$
$$\int_0^{\mathcal{E}} \lambda^2\mathcal{G}(\mathcal{E},\lambda) d\lambda = \frac{16}{105}\mathcal{E}^{7/2},$$

and after differentiating by $\mathcal{E}$ and plugging $\mathcal{E} = E^2$, $N_2(E)$ is given by

$$N_2(E) = \frac{E}{\pi\hbar}\int_{r_{\mathrm{H}}+h}^{L} \frac{r^2}{g(r)} \left(2P_2^{(0)} - 4P_2^{(1)} + \frac{16}{3}P_2^{(2)}\right) dr. \qquad (2.70)$$

Plugging in (2.59-2.61), the final expression for $N_2(E)$ is given by

$$N_2(E) = \frac{E}{\hbar\pi}\int_{r_{\mathrm{H}}+h}^{L} dr \left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left(\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right)\right]. \qquad (2.71)$$

Applying the formula for the free energy (2.34), we first evaluate the energy integral

$$\int_0^\infty \frac{EdE}{e^{\beta E}-1} = \{u = \beta E, \quad du = \beta dE\} = \frac{1}{\beta^2}\int_0^\infty \frac{udu}{e^u - 1} \qquad (2.72)$$
$$= \left\{\int_0^\infty \frac{x}{e^x - 1} = \frac{\pi^2}{6}\right\} = \frac{\pi^2}{6\beta}.$$

The free energy is given by

$$F_2 = -\frac{\pi}{6\hbar\beta^2}\int_{r_{\mathrm{H}}+h}^{L} dr \left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left(\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right)\right], \qquad (2.73)$$

and the entropy

$$S_2 = \frac{\pi}{3\hbar\beta}\int_{r_{\mathrm{H}}+h}^{L} dr \left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left(\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right)\right]. \qquad (2.74)$$

After applying the same procedure for calculating S_0, and using the substitution $x =$

$r - r_{\rm H}$, the entropy has the following form

$$S_2 = \frac{\pi}{3\hbar\beta}\int_h^{L-r_{\rm H}} dx\Bigg[-\frac{4(x+r_{\rm H})(g'+xg'')}{3(xg'+x^2g''/2)} \tag{2.75}$$
$$+(x+r_{\rm H})^2\left(\frac{(g'+xg'')^2}{3(xg'+x^2g''/2)^2}-\frac{g''}{2(xg'+x^2g''/2)}\right)\Bigg],$$

The first term in the integrand can be rewritten as

$$\begin{aligned}-\frac{4(x+r_{\rm H})(g'+xg'')}{3(xg'+x^2g''/2)} &= -\frac{4}{3}\frac{(x+r_{\rm H})(g'+xg'')}{3xg'}\left(1-\frac{xg''}{2g'}\right) \\ &= -\frac{4}{3g'}\left(1+\frac{r_{\rm H}}{x}\right)(g'+xg'')\left(1-\frac{xg''}{2g'}\right) \\ &= -\frac{4}{3g'}\frac{g'r_{\rm H}}{x}\left(1-\frac{xg''}{2g'}\right) = -\frac{4r_{\rm H}}{3x},\end{aligned} \tag{2.76}$$

where we kept terms proportional to x^{-k}, $k \geq 1$, as done previously. The second factor in the second term in the integral can be written as

$$\begin{aligned}&\left(\frac{g'^2}{3x^2g'^2}+\frac{2xg'g''}{3x^2g'^2}+\frac{x^2g''^2}{3x^2g'^2}\right)\left(1-\frac{xg''}{g'}\right)-\frac{g''}{2xg'}\left(1-\frac{xg''}{2g'}\right) \\ &= \frac{1}{3x^2}+\frac{2g''}{3xg'}-\frac{g''}{2xg'} = \frac{1}{3x^2}+\frac{g''r_{\rm H}^2}{6g'}.\end{aligned} \tag{2.77}$$

The entire second term is thus given by

$$(x^2+2xr_{\rm H}+r_{\rm H}^2)\left(\frac{1}{3x^2}+\frac{g''r_{\rm H}^2}{6}\right) = \left(\frac{2r_{\rm H}}{3x}+\frac{r_{\rm H}^2}{3x^2}+\frac{g''r_{\rm H}^2}{6g'}\right). \tag{2.78}$$

At Hawking temperature, the factor in front of the entire integral assumes the following form

$$\frac{\pi}{3\hbar\beta} = \{1/\beta = T_{\rm H} = \hbar\kappa/(2\pi)\} = \frac{\kappa}{6}. \tag{2.79}$$

The entropy is now

$$\begin{aligned}S_2 &= \frac{\kappa}{6}\int_h^{L-r_{\rm H}} dx\left(\frac{r_{\rm H}^2}{3x^2}-\frac{2r_{\rm H}}{3x}+\frac{g''r_{\rm H}^2}{6}\right) = \frac{g'}{12}\frac{r_{\rm H}^2}{3h}+\left(\frac{g'}{12}\frac{g''r_{\rm H}^2}{6g'}-\frac{\kappa r_{\rm H}}{9}\right)\ln\left(\frac{\alpha}{h}\right) \\ &= \{h_c^2 = 4h/g'\} = \frac{r_{\rm H}^2}{9h_c^2}-\left(\frac{g''(r_{\rm H})r_{\rm H}^2}{72}+\frac{\kappa r_{\rm H}}{9}\right)\ln\left(\frac{\alpha}{h}\right).\end{aligned} \tag{2.80}$$

The entropy up to second order $S = S_0 + S_2$ has the form

$$S = \frac{11r_{\rm H}^2}{90h_c^2}-\left(\frac{\kappa r_{\rm H}}{10}+\frac{g''(r_{\rm H})r_{\rm H}^2}{60}\right)\ln\left(\frac{\alpha}{h}\right). \tag{2.81}$$

Equating the most divergent term with the Bekenstein-Hawking entropy, the relation for h_c is given by

$$h_c^2 = \frac{11\ell_{\mathrm{Pl}}^2}{90\pi}. \tag{2.82}$$

Choosing α following the same procedure as before, $\alpha = 11r_{\mathrm{H}}^2 g'(r_{\mathrm{H}})/90$, the entropy up to second order can now be written in the form

$$S = S_{\mathrm{BH}} + \mathcal{F}(\mathcal{A}) \ln\left(\frac{\mathcal{A}}{\ell_{\mathrm{Pl}}^2}\right), \tag{2.83}$$

where

$$\mathcal{F}(\mathcal{A}) = -\frac{\kappa r_{\mathrm{H}}}{10} - \frac{g''(r_{\mathrm{H}})r_{\mathrm{H}}^2}{60}. \tag{2.84}$$

2.1.1 Schwarzschild black hole

Applying this result to the Schwarzschild black hole,

$$g(r) = 1 - \frac{2GM}{r}, \quad r_{\mathrm{H}} = 2GM, \quad \kappa = \frac{1}{4GM} = \frac{1}{2r_{\mathrm{H}}}, \quad g''(r_{\mathrm{H}}) = -\frac{1}{2G^2M^2} = -\frac{2}{r_{\mathrm{H}}^2},$$

we obtain

$$S = S_{\mathrm{BH}} - \frac{1}{60} \ln\left(\frac{\mathcal{A}}{\ell_{\mathrm{Pl}}^2}\right). \tag{2.85}$$

The Schwarzschild black hole receives a correction to its entropy in the form of a logarithm of the horizon area of the Schwarzschild black hole. An important thing to note is that the factor that multiplies the logarithm is a constant and not dependent on the horizon area.

2.2 Charged black holes and charged probes

To expand our consideration to charged black holes and charged probes, we will solve the Klein-Gordon equation using the minimal substitution $\partial_\mu \to \partial_\mu + \frac{i}{\hbar}qA_\mu$, where $A_\mu = (A_0(r), 0, 0, 0)$ describes the 4-vector potential of the charged spacetime, and q is the charge of the scalar field we will be probing the spacetime with. Furthermore, we will assume $f(r) = g(r)$. The Klein-Gordon equation now becomes

$$\left(\frac{1}{\sqrt{-g}}\left(\partial_\mu + \frac{i}{\hbar}qA_\mu\right)\left(\sqrt{-g}g^{\mu\nu}\left(\partial_\nu + \frac{i}{\hbar}qA_\nu\right)\right) - \frac{m^2}{\hbar^2}\right)\Phi = 0. \tag{2.86}$$

After multiplying out, we are left with

$$\begin{aligned}\left(\frac{1}{\sqrt{-g}}\left(\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) + \partial_\mu\left(\sqrt{-g}g^{\mu\nu}\frac{i}{\hbar}qA_\nu\right)\right.\right. \\ \left.\left. + \frac{i}{\hbar}qA_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\right) - \frac{1}{\hbar^2}g^{\mu\nu}q^2A_\mu A_\nu - \frac{m^2}{\hbar^2}\right)\Phi = 0.\end{aligned} \tag{2.87}$$

In Appendix B, we show that, after using the same ansatz as in the neutral case, the radial equation is given by

$$R''(r) + \left(\frac{W^2(r)}{\hbar^2} - \Delta(r)\right)R(r) = 0. \tag{2.88}$$

where we defined a new quantity

$$W^2(r) = \frac{1}{G^2(r)}\left((E - qA_0(r))^2 - f(r)\left[m^2 + \left(\frac{\ell(\ell + D - 1)\hbar^2}{r^2}\right)\right]\right), \tag{2.89}$$

for which the radial equation then has the same structure as before (2.22), with $V^2(r) \to W^2(r)$, i.e., $E \to E - qA_0(r)$. This also means that, in order to calculate the zeroth and second order corrections to the entropy for a charged black hole that is probed with a charged scalar field, we just have to apply the transformation $E \to E - qA_0(r)$ in the defining equations for N_0 (2.42) and N_2 (2.71) and evaluate those integrals.

2.2.1 Reissner-Nordström black hole

In this subsection, we calculate the entropy of a Reissner-Nordström black hole probed with a charged masless scalar field Φ, where [16]

$$\begin{gathered}A_0(r) = \frac{Q}{r}, \quad g(r) = 1 - \frac{2GM}{r} + \frac{Q^2G}{r^2}, \quad D = 2, \quad m = 0, \\ r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2}, \quad r_{\mathrm{H}} = r_+.\end{gathered} \tag{2.90}$$

The number of modes in the zeroth order is

$$
\begin{aligned}
N_0(E) &= \frac{2}{3\pi\hbar^3}\int_{r_++h}^{L} dr \frac{\left(E-q\frac{Q}{r}\right)^3 r^2 dr}{g^2(r)} = \frac{2}{3\pi\hbar}\int_{r_++h}^{L} dr \frac{\left(E-q\frac{Q}{r}\right)^3 r^6}{(r-r_+)^2(r-r_-)^2} \\
&= \frac{2}{3\pi\hbar^3}\int_{h}^{L-r_+} dx \frac{\left(E-q\frac{Q}{x+r_+}\right)^3 (x+r_+)^6}{x^2(x+r_+-r_-)^2} \\
&= \frac{2}{3\pi\hbar^3}\int_{h}^{L-r_+} dx \frac{\left(E(x+r_+)^2-qQ(x+r_+)\right)^3}{x^2(x+r_+-r_-)^2} \\
&= \frac{2}{3\pi\hbar^3}\frac{r_+^6(E-\frac{qQ}{r_+})^3}{(r_+-r_-)^2}\frac{1}{h} \\
&+ \frac{2}{3\pi\hbar^3}\left(\frac{3r_+^5\left(2E-\frac{qQ}{r_+}\right)\left(E-\frac{qQ}{r_+}\right)^2}{(r_+-r_-)^2} - \frac{2r_+^6\left(E-\frac{qQ}{r_+}\right)^3}{(r_+-r_-)^3}\right)\ln\left(\frac{\alpha}{h}\right).
\end{aligned}
\tag{2.91}
$$

The first term has the same structure as the number of modes in [16], while the second is the logarithmic correction. We can write the free energy as

$$
\begin{aligned}
F_0 = &-\frac{2}{3\pi\hbar^3}\frac{r_+^6}{(r_+-r_-)^2}\frac{1}{h}K(\beta) \\
&-\frac{2}{3\pi\hbar^3}\left(\frac{3r_+^5}{(r_+-r_-)^2}K'(\beta) - \frac{2r_+^6}{(r_+-r_-)^3}K(\beta)\right)\ln\left(\frac{\alpha}{h}\right),
\end{aligned}
\tag{2.92}
$$

where the functions $K(\beta)$ and $K'(\beta)$ are given by

$$
\begin{aligned}
K(\beta) &= \int_0^\infty dE \frac{\left(E-\frac{qQ}{r_+}\right)^3}{e^{\beta E}-1} \\
&= \frac{\Gamma(4)\zeta(4)}{\beta^4} - \frac{3qQ\Gamma(3)\zeta(3)}{r_+\beta^3} + \frac{3q^2Q^2\Gamma(2)\zeta(2)}{r_+^2\beta^2} - \frac{q^3Q^3\Gamma(1)\zeta(1)}{r_+^3\beta},
\end{aligned}
\tag{2.93}
$$

and

$$
\begin{aligned}
K'(\beta) &= \int_0^\infty \frac{\left(2E-\frac{qQ}{r_+}\right)\left(E-\frac{qQ}{r_+}\right)^2}{e^{\beta E}-1} \\
&= \frac{2\Gamma(4)\zeta(4)}{\beta^4} - \frac{5qQ\Gamma(3)\zeta(3)}{r_+\beta^3} + \frac{4q^2Q^2\Gamma(2)\zeta(2)}{r_+^2\beta^2} - \frac{q^3Q^3\Gamma(1)\zeta(1)}{r_+^3\beta}.
\end{aligned}
\tag{2.94}
$$

The $K(\beta)$ and $K'(\beta)$ functions have an infinite contribution from the electrostatic self-energy of the charge q of the scalar particle, which is contained in the $\zeta(1)$ term, which we can regulize by rescaling $S_0^{reg} = S_0(\beta) - S_0(\beta=\infty)$. This way, we ensure that $S = 0$ when $T = 1/\beta = 0$. The entropy is then given by

$$
\begin{aligned}
S_0 = &- \frac{2}{3\pi\hbar^3}\frac{r_+^6}{(r_+ - r_-)^2}\frac{1}{h}\left(-\frac{4\Gamma(4)\zeta(4)}{\beta^3} + \frac{9qQ\Gamma(3)\zeta(3)}{r_+\beta^2} - \frac{6q^2Q^2\Gamma(2)\zeta(2)}{r_+^2\beta}\right) \\
&- \frac{2}{3\pi\hbar^3}\left(\frac{3r_+^5}{(r_+ - r_-)^2}\left(-\frac{8\Gamma(4)\zeta(4)}{\beta^3} + \frac{15qQ\Gamma(3)\zeta(3)}{r_+\beta^2} - \frac{8q^2Q^2\Gamma(2)\zeta(2)}{r_+^2\beta}\right)\right. \\
&\left. - \frac{2r_+^6}{(r_+ - r_-)^3}\left(-\frac{4\Gamma(4)\zeta(4)}{\beta^3} + \frac{9qQ\Gamma(3)\zeta(3)}{r_+\beta^2} - \frac{6q^2Q^2\Gamma(2)\zeta(2)}{r_+^2\beta}\right)\right)\ln\left(\frac{\alpha}{h}\right),
\end{aligned}
\tag{2.95}
$$

Evaluating the entropy at the Hawking temperature, $T_{\mathrm{H}} = 1/\beta = \kappa\hbar/(2\pi) = g'(r_{\mathrm{H}})\hbar/(4\pi)$, and using $\Gamma(n) = (n-1)!$, $\zeta(4) = \pi^4/90$, $\zeta(2) = \pi^2/6$ we have

$$
\begin{aligned}
S_0 =& \frac{r_+^6}{(r_+ - r_-)^2}\frac{1}{h}\left(\frac{(g'(r_+))^3}{360} - \frac{3qQ\zeta(3)(g'(r_+))^2}{4r_+\hbar} + \frac{q^2Q^2g'(r_+)}{6r_+^2\hbar^2}\right) \\
&+ \left(\frac{r_+^5}{(r_+ - r_-)^2}\left(\frac{(g'(r_+))^3}{60} - \frac{15qQ\zeta(3)(g'(r_+))^2}{4\pi^3r_+\hbar} + \frac{2q^2Q^2g'(r_+)}{3r_+^2\hbar^2}\right)\right. \\
&\left. - \frac{r_+^6}{(r_+ - r_-)^3}\left(\frac{(g'(r_+))^3}{180} - \frac{3qQ\zeta(3)(g'(r_+))^2}{2\pi^3r_+\hbar} + \frac{q^2Q^2g'(r_+)}{3r_+^2\hbar^2}\right)\right)\ln\left(\frac{\alpha}{h}\right).
\end{aligned}
\tag{2.96}
$$

Using

$$
g(r) = 1 - \frac{2GM}{r} + \frac{Q^2G}{r^2},
\tag{2.97}
$$

and

$$
\begin{aligned}
g'(r_+) &= \frac{2}{r_+^3}(r_+GM - Q^2G) = 2\frac{GM(GM + \sqrt{G^2M^2 - Q^2G}) - Q^2G}{r_+^3} \\
&= 2\frac{G^2M^2 - Q^2G + GM\sqrt{G^2M^2 - Q^2G}}{r_+^3} \\
&= 2\sqrt{G^2M^2 - Q^2G}\frac{\sqrt{G^2M^2 - Q^2G} + GM}{r_+^3} = \frac{r_+ - r_-}{r_+^2},
\end{aligned}
\tag{2.98}
$$

we are now left with

$$
\begin{aligned}
S_0 =& \frac{1}{h}\left(\frac{(r_+ - r_-)}{360} - \frac{3qQr_+\zeta(3)}{4\pi^3\hbar} + \frac{q^2Q^2r_+^2}{6(r_+ - r_-)\hbar^2}\right) \\
&+ \left(\left(\frac{(r_+ - r_-)}{60r_+} - \frac{15qQ\zeta(3)}{4\pi^3\hbar} + \frac{2q^2Q^2r_+}{3(r_+ - r_-)\hbar^2}\right)\right. \\
&\left. - \left(\frac{1}{180} - \frac{3qQr_+\zeta(3)}{2\pi^3(r_+ - r_-)\hbar} + \frac{q^2Q^2r_+^2}{3(r_+ - r_-)^2\hbar^2}\right)\right)\ln\left(\frac{\alpha}{h}\right).
\end{aligned}
\tag{2.99}
$$

After rearranging, we finally have

$$
\begin{aligned}
S_0 =& \frac{1}{h}\left(\frac{r_+ - r_-}{360} - \frac{3qQr_+\zeta(3)}{4\pi^3\hbar} + \frac{q^2Q^2r_+^2}{6(r_+ - r_-)\hbar^2}\right) \\
&+ \left(\frac{2r_+ - 3r_-}{180r_+} - \frac{3qQ\zeta(3)(3r_+ - 5r_-)}{4\pi^3(r_+ - r_-)\hbar} + \frac{q^2Q^2r_+(r_+ - 2r_-)}{3(r_+ - r_-)^2\hbar^2}\right)\ln\left(\frac{\alpha}{h}\right).
\end{aligned} \tag{2.100}
$$

With $Q \to 0$, i.e. $r_+ = r_\mathrm{H} = 2GM$ and $r_- = 0$,

$$
\lim_{Q\to 0} S_0 = \frac{GM}{180h} + \frac{1}{90}\ln\left(\frac{\alpha}{h}\right), \tag{2.101}
$$

Which completely coincides with S_0 for the result obtained for the Schwarzschild black hole (2.54)

$$
\begin{aligned}
S_{0,\text{SCHW}} &= \frac{r_\mathrm{H}^2}{90h_c^2} - \left(\frac{g''(r_\mathrm{H})r_\mathrm{H}^2}{360} - \frac{\kappa r_\mathrm{H}}{90}\right)\ln\left(\frac{\alpha}{h}\right) \\
&= \left\{h_c^2 = \frac{4h}{g'(r_\mathrm{H})}, \quad g'(r_\mathrm{H}) = \frac{1}{r_\mathrm{H}}, \quad \kappa = \frac{1}{2r_\mathrm{H}}, \quad g''(r_\mathrm{H}) = -\frac{2}{r_\mathrm{H}^2}\right\} \\
&= \frac{r_\mathrm{H}}{360h} - \left(-\frac{1}{180} - \frac{1}{180}\right)\ln\left(\frac{\alpha}{h}\right) = \frac{GM}{180h} + \frac{1}{90}\ln\left(\frac{\alpha}{h}\right).
\end{aligned} \tag{2.102}
$$

For $q \to 0$, (2.100) obtains the form

$$
S_0 = \frac{r_+ - r_-}{360h} + \frac{2r_+ - 3r_-}{180r_+}\ln\left(\frac{\alpha}{h}\right). \tag{2.103}
$$

Applying $Q \to 0$ to this result, we reproduce (2.102). Furthermore, the leading contribution to both cases is in agreement with the solution given in [16]. For the second order, we apply $E \to E - qA_0(r)$ to $N_2(E)$

$$
N_2(E) = \frac{1}{\hbar\pi}\int_{r_\mathrm{H}+h}^{L} dr\left(E - \frac{qQ}{r}\right)\left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left(\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right)\right]. \tag{2.104}
$$

which leads to the following free energy

$$
F_2 = -\frac{1}{\hbar\pi}\int_{r_\mathrm{H}+h}^{L} dr\left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left(\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right)\right]K''(\beta) \tag{2.105}
$$

with

$$
K''(\beta) = \int_0^\infty dE\frac{\left(E - \frac{qQ}{r}\right)}{e^{\beta E} - 1}, \quad K''(\beta) = \frac{\Gamma(2)\zeta(2)}{\beta^2} - \frac{qQ\Gamma(1)\zeta(1)}{r\beta}. \tag{2.106}
$$

We once again discard the $\zeta(1)$ term, interpreting it as the infinite contribution from the electrostatic self-energy. We are left with

$$F_2 = -\frac{\pi}{6\hbar\beta^2}\int_{r_{\rm H}+h}^{L} dr\left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left(\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right)\right]. \tag{2.107}$$

The entropy evaluated at Hawking temperature is now

$$\begin{aligned} S_2 &= \frac{\pi}{3\hbar\beta}\int_{r_{\rm H}+h}^{L} dr\left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left(\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right)\right] \\ &= \frac{\kappa}{6}\int_{r_{\rm H}+h}^{L} dr\left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left(\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right)\right]. \end{aligned} \tag{2.108}$$

To calculate this, we introduce some shorthand notation,

$$g(r) = 1 - \frac{2GM}{r} + \frac{Q^2G}{r^2} = \frac{(r-r_-)(r-r_+)}{r^2}, \quad r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2} \tag{2.109}$$

$$g'(r) = \frac{2GMr - 2Q^2G}{r^3} \equiv \frac{ar+b}{r^3}, \quad a = 2GM, \quad b = -2Q^2G, \tag{2.110}$$

$$g''(r) = \frac{6Q^2G - 4GMr}{r^4} \equiv \frac{c+dr}{r^4}, \quad c = 6Q^2G, \quad d = -4GM \tag{2.111}$$

and now we can write the entropy as

$$\begin{aligned} S_2 &= \frac{\kappa}{6}\int_{r_{\rm H}+h}^{L} dr\left[-\frac{4}{3}\frac{ar+b}{(r-r_-)(r-r_+)} + \frac{1}{3}\frac{(ar+b)^2}{(r-r_-)^2(r-r_+)^2} - \frac{1}{2}\frac{c+dr}{(r-r_+)(r-r_-)}\right] \\ &= \frac{\kappa}{6}\int_{h}^{L-r_{\rm H}} dx\left[-\frac{\frac{4}{3}(a(x+r_+)+b) + \frac{1}{2}(c+d(x+r_+))}{x(x+r_+-r_-)} + \frac{1}{3}\frac{(a(x+r_+)+b)^2}{x^2(x+r_+-r_-)^2}\right], \end{aligned} \tag{2.112}$$

where we ignored the constant term in the integral since it would be proportional to h after the integration. Using the following series expansions, with $D = r_+ - r_-$

$$\frac{1}{x+D} = \frac{1}{D}\left(1 - \frac{x}{D}\right), \quad \frac{1}{(x+D)^2} = \frac{1}{D^2}\left(1 - \frac{2x}{D}\right), \tag{2.113}$$

the entropy is now

$$
\begin{aligned}
S_2 &= \frac{\kappa}{6}\int_h^{L-r_{\rm H}} dx\left[-\frac{Ax+B}{xD}\left(1-\frac{x}{D}\right)+\frac{1}{3}\frac{(Cx+E)^2}{x^2D^2}\left(1-\frac{2x}{D}\right)\right] \\
&= \frac{\kappa}{6}\int_h^{L-r_{\rm H}} dx\left[-\frac{B}{Dx}+\frac{1}{3}\left(\frac{2EC}{xD^2}+\frac{E^2}{x^2D^2}-\frac{2E^2}{xD^3}\right)\right] \\
&= \frac{\kappa}{6}\left(\frac{E^2}{3D^2}\frac{1}{h}+\left(\frac{2EC}{3D^2}-\frac{2E^2}{3D^3}-\frac{B}{D}\right)\ln\left(\frac{\alpha}{h}\right)\right),
\end{aligned}
\tag{2.114}
$$

where the constants A, B, C, D, E are defined as

$$
\begin{gathered}
E = ar_+ + b, \quad D = r_+ - r_-, \quad C = a, \\
B = \frac{4}{3}ar_+ + \frac{4}{3}b + \frac{1}{2}c + \frac{1}{2}dr_+, \quad A = \frac{4}{3}a + \frac{1}{2}d
\end{gathered}
\tag{2.115}
$$

Using

$$
\frac{\kappa}{6} = \frac{g'(r_+)}{12} = \frac{1}{12}\frac{r_+ - r_-}{r_+^2},
\tag{2.116}
$$

we arrive at

$$
S_2 = \frac{r_+ - r_-}{36h} - \frac{1}{36}\ln\left(\frac{\alpha}{h}\right).
\tag{2.117}
$$

We have arrived at a result that does not depend on the charge of the probe, q as we discarded the only part that had q, due to its appearance in the divergent contribution to the entropy. For $Q \to 0$

$$
S_2 = \frac{r_{\rm H}}{36h} - \frac{1}{36}\ln\left(\frac{\alpha}{h}\right) = \frac{GM}{18h} - \frac{1}{36}\ln\left(\frac{\alpha}{h}\right),
\tag{2.118}
$$

we recover the Schwarzschild result for S_2 from [23]

$$
\begin{aligned}
S_2 &= \frac{r_{\rm H}^2}{9h_c^2} - \left(\frac{g''(r_{\rm H})r_{\rm H}^2}{72} + \frac{\kappa r_{\rm H}}{9}\right)\ln\left(\frac{\alpha}{h}\right) \\
&= \{g'(r_{\rm H}) = 1/r_{\rm H}, g''(r_{\rm H}) = -2/r_{\rm H}^2\} = \frac{GM}{18h} - \frac{1}{36}\ln\left(\frac{\alpha}{h}\right).
\end{aligned}
\tag{2.119}
$$

The total entropy for the Reissner-Nordström black hole up to second order is given by

$$
\begin{aligned}
S &= S_0 + S_2 \qquad (2.120)\\
&= \frac{1}{h}\left(\frac{11(r_+ - r_-)}{360} - \frac{3qQr_+\zeta(3)}{4\pi^3\hbar} + \frac{q^2Q^2r_+^2}{6(r_+ - r_-)\hbar^2}\right)\\
&+ \left(-\frac{(r_+ + r_-)}{60r_+} - \frac{3qQ\zeta(3)(3r_+ - 5r_-)}{4\pi^3(r_+ - r_-)\hbar} + \frac{q^2Q^2r_+(r_+ - 2r_-)}{3(r_+ - r_-)^2\hbar^2}\right)\ln\left(\frac{\alpha}{h}\right).
\end{aligned}
$$

For the limit $Q \to 0$ we have shown that both S_0 and S_2 recover the Schwarzschild entropy, and so their total contribution also coincides with the total entropy up to second order for the Schwarzschild black hole. For $q \to 0$ we have

$$
S = \frac{11(r_+ - r_-)}{360h} - \frac{(r_+ + r_-)}{60r_+}\ln\left(\frac{\alpha}{h}\right), \qquad (2.121)
$$

which is the same entropy as the one obtained in [23], where only a charged space-time was considered and not a charged probe.

3 Entanglement entropy

Entanglement entropy is a measure of the quantum entanglement between two subsystems, usually obtained by tracing out one of the subsystems from the total quantum state. In the context of black hole physics, the subsystems of interest are the degrees of freedom inside and outside the event horizon. The calculation of entanglement entropy involves dividing the spacetime into two regions: the black hole interior and the exterior. The entanglement entropy is then defined as the von Neumann entropy of the reduced density matrix corresponding to the interior region.

The calculation of entanglement entropy has provided a microscopic understanding of black hole entropy. It has also led to an understanding of some of the underlying microscopic degrees of freedom responsible for the entropy and established a connection between quantum entanglement and gravity. In this section of the thesis, we will calculate the entanglement entropy of a system by modeling a scalar field on $\mathbb{R}^3$ as a collection of coupled oscillators. We will primarily follow [7-9], and we will also assume $\hbar = 1$.

Let us consider a pure vacuum state $|\psi\rangle$ of a quantum system defined within a spacelike section $\mathcal{O}$, and let us assume that the degrees of freedom are localized in some regions of $\mathcal{O}$. If we investigate an arbitrary surface Σ, which partitions the space $\mathcal{O}$ onto two disjoint subspaces A i B, then the given quantum system can be represented as a union of two subsystems. The wave function of the entire system is then given by the linear combination of the product of the quantum states of every subsystem,

$$|\psi\rangle = \sum_{i,a} \psi_{i,a} \left|A\right\rangle_i \left|B\right\rangle_a , \tag{3.1}$$

where the states $\left|A\right\rangle_i$ are constructed from degrees of freedom localized in the region A, while states $\left|B\right\rangle_a$ from the degrees of freedom in region B.

The density matrix, which corresponds to the pure quantum of the system $|\psi\rangle$ is given by

$$\rho_0(A, B) = |\psi\rangle \langle\psi| , \tag{3.2}$$

and has zero von Neumann entropy, since the pure state is one without uncertainty. If we trace over the degrees of freedom in region A, i.e., the partial trace Tr_A for the density matrix of the pure state, we obtain the reduced density matrix for the subsystem B

$$\rho_B = Tr_A \rho_0(A, B). \tag{3.3}$$

The statistical entropy for some density matrix is called the von Neumann entropy and is given by

$$S = -Tr(\rho \ln \rho). \tag{3.4}$$

This means that the entropy for the subsystem B is given by

$$S_B = -Tr(\rho_B \ln \rho_B), \tag{3.5}$$

which coincides with the entanglement entropy, which is connected with the surface Σ. Applying the same procedure, we can obtain the entanglement entropy S_A. We can now show that the following equality holds true

$$S_A = S_B. \tag{3.6}$$

Let A be the region inside the horizon and B the one outside. We have [8] $(\rho_A)_{ij} = (\psi\psi^\dagger)_{ij}$, and $(\rho_B)_{ij} = (\psi^T\psi^*)_{ij}$. We can see that, for $k \in \mathbb{N}$,

$$\begin{aligned} Tr(\rho_A^k) &= Tr((\psi\psi^\dagger) \underbrace{\ldots}_{k-2 \text{ terms}} (\psi\psi^\dagger)) = \{\text{cyclicity}\} = Tr(\psi^\dagger\psi \ldots \psi^\dagger\psi) \\ &= Tr((\psi^T\psi^*)^T \ldots (\psi^T\psi^*)^T) = Tr((\rho_B^k)^T) = Tr(\rho_B^k). \end{aligned} \tag{3.7}$$

Writing $\ln(\rho_A)$ as a power series,

$$\ln(\rho_A) = -\sum_{k=1}^{\infty} \frac{(-)^k(\rho_A - 1)^k}{k} = -\sum_{k=1}^{\infty}\sum_{j=0}^{k} \binom{k}{j} (-)^j \rho_A^k, \tag{3.8}$$

we can show that the entropies are indeed equal

$$\begin{aligned} S_A = -Tr(\rho_A \ln \rho_A) &= -\sum_{k=1}^{\infty}\sum_{j=0}^{k} \binom{k}{j} (-)^j Tr(\rho_A^{k+1}) \\ &= \sum_{k=1}^{\infty}\sum_{j=0}^{k} \binom{k}{j} (-)^j Tr(\rho_B^{k+1}) = S_B. \end{aligned} \tag{3.9}$$

This shows that the entanglement entropy for a system in a pure state is not an extensive quantity, i.e., it does not depend on the size of neither region A nor region B, and thus is determined by the geometry of the surface that partitions the space, Σ. Applying this to black holes, we can conclude that the black hole entropy can be calculated by calculating the entropy outside the black hole. This is in accordance with the brick wall method, where we calculated the entropy by observing the behavior of scalar fields outside the black hole, and where we also concluded that the leading contribution to the entropy comes from the black hole horizon.

3.1 Entropy of a collection of coupled harmonic oscillators

To showcase how entanglement entropy works, let us consider a simple case where we model a scalar field on $\mathbb{R}^3$ as a collection of coupled oscillators on a lattice of spaced points, labeled by capital Latin indices, and the displacement at each point gives the value of the scalar field at that point. The Lagrangian of such a system is given by

$$L = \frac{1}{2}G_{MN}\dot{q}^M\dot{q}^N - \frac{1}{2}V_{MN}q^Mq^N, \tag{3.10}$$

where q^M gives the displacement of the Mth oscillator, and $\dot{q}^M$ its generalized velocity. The symmetric tensor G_{MN} is positive definite and has the following property

$$G^{MP}G_{PN} = \delta^M_N, \tag{3.11}$$

where G^{MN} is its inverse. This matrix can be considered a metric on this configuration space of the coupled harmonic oscillators. The matrix V_{MN} is also symmetric and positive definite. We introduce the conjugate momentum to q^M,

$$P_M = G_{MN}\dot{q}^N \tag{3.12}$$

And the Hamiltonian for this system now has the form of

$$H = \frac{1}{2}G_{MN}\dot{q}^M\dot{q}^N + \frac{1}{2}V_{MN}q^Mq^N \tag{3.13}$$

Now we define a new symmetric matrix W_{MN} as

$$G^{AB}W_{MA}W_{BN} = W_{MA}W^A{}_N = V_{MN}, \tag{3.14}$$

where the metric G was used to raise indices. Essentially, the W matrix can be considered the square root of the matrix V in the scalar product induced by G.

We can rewrite the Hamiltonian in the following way

$$H = \frac{1}{2}G^{MN}(P_M - iW_{MA}q^A)^*(P_N - iW_{NB}q^B) + \frac{1}{2}Tr(W), \tag{3.15}$$

where $Tr(W)$ term is the zero-point energy. To prove this, we multiply the terms in the brackets and use the definition of the conjugate momentum

$$\begin{aligned} H &= \frac{1}{2}G^{MN}P_M P_N + \frac{i}{2}(G^{MN}W_{MA}q^A P_N - G^{MN}P_M W_{NB}q^B) \\ &+ \frac{1}{2}G^{MN}W_{MA}W_{NB}q^A q^B + \frac{1}{2}W^A{}_A \\ &= \frac{1}{2}G^{MN}G_{MA}\dot{q}^A G_{NB}\dot{q}^B + \frac{i}{2}(G^{MN}W_{MA}q^A G_{NB}\dot{q}^B - G^{MN}G_{MA}\dot{q}^A W_{NB}q^B) \\ &+ \frac{1}{2}V_{AB}q^A q^B \frac{1}{2} + W^A{}_A \\ &= \frac{1}{2}G_{AB}\dot{q}^A\dot{q}^B + \frac{1}{2}V_{AB}q^A q^B + \frac{1}{2}W^A{}_A \end{aligned} \tag{3.16}$$

This form looks like the familiar Hamiltonian of a quantum harmonic oscillator, if we interpret the following two operators as the creation and annihilation operators

$$\begin{aligned} a_M^* &= \left(P_M - iW_{MA}q^A\right)^* = P_M + iW_{MA}q^A, \\ a_M &= P_M - iW_{MA}q^A. \end{aligned} \tag{3.17}$$

Imposing the standard commutation relations for P and q,

$$[q^M, P_N] = i\delta_N^M, \quad [q^M, q^N] = 0, \quad [P_N, P_M] = 0, \tag{3.18}$$

we can notice that the creation and annihilation operators do not correspond to normal modes since they do not satisfy the normal mode commutation relation

$$[a_M, a_N^*] = i\delta_{MN}, \tag{3.19}$$

and instead satisfy

$$[a_M, a_N^*] = [P_M - iW_{MA}q^A, P_N + iW_{NB}q^B] = [P_M, iW_{NB}q^B] - [iW_{MA}q^A, P_N] \quad (3.20)$$
$$= -i \cdot i\delta_M^B W_{NB} - i \cdot i\delta_N^A W_{MA} = 2W_{MN},$$

as the W matrix will, in general, not be given by

$$W_{MN} = \frac{i}{2}\delta_{MN}, \quad (3.21)$$

as we expect it to for normal modes. The Hamiltonian of our system now has the form

$$H = \frac{1}{2}G^{MN}a_M^* a_N + \frac{1}{2}Tr(W). \quad (3.22)$$

To construct the ground state $|\psi_0\rangle$ of our system, it must be cancelled by the annihilation operator for all modes. That is, the following condition must hold

$$(P_M - iW_{MA}q^A)\,|\psi_0\rangle = 0, \quad \forall M \in \mathbb{R}^3 \quad (3.23)$$

In the position representation, where $P_M = -i\partial/\partial q^M \equiv -i\partial_M$, we have

$$\left(\partial_M + W_{MB}q^B\right)\psi_0(\{q^A\}) = 0, \quad \forall M \in \mathbb{R}^3 \quad (3.24)$$

where $\psi_0(\{q^A\}) = \langle\{q^A\}|\psi_0\rangle$ is the wave function of the ground state in the position representation, and it depends on the displacement for each oscillator. The wave function that solves this equation has the following form

$$\psi_0(\{q^A\}) = N\exp\left[-\frac{1}{2}W_{AB}q^Aq^B\right], \quad (3.25)$$

where N is the normalization constant that we have yet to compute. To prove that this function does indeed solve equation (3.24), we just need to plug it in.

$$(\partial_M + W_{MN}q^N)\exp\left[-\frac{1}{2}W_{AB}q^Aq^B\right] \quad (3.26)$$
$$= \left(-\frac{1}{2}W_{AB}\delta_M^A q^B - \frac{1}{2}W_{AB}q^A\delta_M^B + W_{MN}q^N\right)\exp\left[-\frac{1}{2}W_{AB}q^Aq^B\right]$$
$$= \left(-\frac{1}{2}W_{MN}q^N - \frac{1}{2}W_{MN}q^N + W_{MN}q^N\right)\exp\left[-\frac{1}{2}W_{AB}q^Aq^B\right] = 0,$$

where in the third row we used the symmetry of the W matrix, as well as relabel the repeated A and B indices of the first two terms to N, showing that all the terms cancel. To completely determine the ground state wave function, we just need to find the normalization constant N. To do this, we need to find N such that the following condition holds

$$\int dq^1 \dots dq^N |\psi_0|^2 \overset{!}{=} 1. \tag{3.27}$$

The integral we need to evaluate is then the following

$$\begin{aligned}
&\int dq^1 \dots dq^N \exp\left[-q^A W_{AB} q^B\right] \qquad (3.28)\\
&= \{q^i = U_{ij} q'^j, U \text{ is unitary, such that } U^T W U \equiv D \text{ is diagonal}\}\\
&= \int dq'^1 \dots dq'^N \left|\frac{\partial q}{\partial q'}\right| \exp\left[-U_{Ai} q'^i W_{AB} U_{Bj} q'^j\right]\\
&= \int dq'^1 \dots dq'^N |\det U| \exp\left[-q'^i (U^T)_{iA} W_{AB} U_{Bj} q'^j\right]\\
&= \int dq'^1 \dots dq'^N \exp\left[-q'^i D_{ij} q'^j\right] = \int dq'^1 \dots dq'^N \exp\left[-\sum_{i=1}^{N} D_{ii} (q'^i)^2\right]\\
&= \left\{\int_{\mathbb{R}} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}}\right\} = \sqrt{\frac{\pi^N}{\prod_{i=1}^N D_{ii}}} = \sqrt{\frac{\pi \dots \pi}{D_{11} \dots D_{NN}}} = \sqrt{\frac{1}{\det\left(\frac{D}{\pi}\right)}},
\end{aligned}$$

where we have unitarily transformed the displacement vectors that we are integrating over in such a way that we diagonalize W. This is possible due to the Autonne-Takagi factorization theorem, which states that if we have a complex symmetric matrix (in our case, W), then a unitary matrix U exists such that $U^T W U = D$, where D is a real diagonal matrix with nonnegative entries. Since U is a unitary matrix, the determinant that we obtain from the Jacobian has a value of one. Finally, since the matrix D is N dimensional, we can allocate each π to every element and compactly write the final expression as one over the square root of the determinant of a single matrix.

The normalization constant is then given by

$$N^2 \sqrt{\frac{1}{\det\left(\frac{D}{\pi}\right)}} = 1 \implies N = \left(\det\left(\frac{D}{\pi}\right)\right)^{1/4}. \tag{3.29}$$

The ground state wave function finally looks like

$$\psi_0(\{q^A\}) = \left(\det\left(\frac{D}{\pi}\right)\right)^{1/4} \exp\left[-\frac{1}{2}W_{AB}q^Aq^B\right]. \tag{3.30}$$

Which agrees with Srednicki's calculation as well [8]. The density matrix for the vacuum state is given by

$$\rho = |\psi_0\rangle\langle\psi_0|\,, \tag{3.31}$$

which in the position representation has the form

$$\begin{aligned}\rho(\{q^A\},\{q'^B\}) &\equiv \langle\{q^A\}|\psi_0\rangle\langle\psi_0|\{q'^B\}\rangle = \psi_0(\{q^A\})\psi_0(\{q'^B\})^* \\ &= \left(\det\left(\frac{D}{\pi}\right)\right)^{1/2} \exp\left[-\frac{1}{2}W_{AB}(q^Aq^B + q'^Aq'^B)\right]\end{aligned} \tag{3.32}$$

Let us consider a region Ω of $\mathbb{R}^3$. We will decompose the vector space of displacements of the oscillaters into two subspaces, one within Ω, and one outside. We will label the oscillators within Ω with Greek indices and the oscillators outside the region with lower case Latin indices,

$$q^A = \begin{pmatrix} q^a \\ q^\alpha \end{pmatrix}. \tag{3.33}$$

Similarly we will decompose the matrix W into a block form

$$W_{AB} = \begin{pmatrix} W_{ab} & W_{a\alpha} \\ W_{\alpha b} & W_{\alpha\beta} \end{pmatrix}, \tag{3.34}$$

where W_{ab} couples two oscillators within Ω, $W_{\alpha\beta}$ couples two oscillators outside Ω, and where the other two matrices describe the coupling of oscillators between the two regions. If we consider that the information of the displacement about the oscillators inside Ω is unavailable (as it is when applied to black holes), we can obtain a reduced density matrix ρ_{red} for the oscillators outside Ω, integrating over $\mathbb{R}$ for each of the oscillators in the region Ω,

$$\rho_{red}(\{q^a\},\{q^b\}) \equiv \langle\{q^a\}|\rho|\{q^b\}\rangle = \int\prod_\alpha dq^\alpha\,\langle\{q^a,q^\alpha\}|\rho|\{q'^b,q^\alpha\}\rangle\,. \tag{3.35}$$

To calculate this, let us first look at the product in the exponent of (3.32), which we can rewrite and simplify as follows

$$q^A W_{AB} q^B = \begin{pmatrix} q^a & q^\alpha \end{pmatrix} \begin{pmatrix} W_{ab} & W_{a\beta} \\ W_{\alpha b} & W_{\alpha\beta} \end{pmatrix} \begin{pmatrix} q^b \\ q^\beta \end{pmatrix} = \begin{pmatrix} q^a & q^\alpha \end{pmatrix} \begin{pmatrix} W_{ab}q^b + W_{a\beta}q^\beta \\ W_{\alpha b}q^b + W_{\alpha\beta}q^\beta \end{pmatrix} \tag{3.36}$$
$$= q^a W_{ab} q^b + q^a W_{a\beta} q^\beta + q^\alpha W_{\alpha b} q^b + q^\alpha W_{\alpha\beta} q^\beta.$$

And similarly for the primed oscillators,

$$q'^A W_{AB} q'^B = q'^a W_{ab} q'^b + q'^a W_{a\beta} q'^\beta + q'^\alpha W_{\alpha b} q'^b + q'^\alpha W_{\alpha\beta} q'^\beta. \tag{3.37}$$

While calculating the reduced density matrix, since we are integrating out over the inner region, we should note that we will have $q'^\sigma = q^\sigma$ for all the primed displacements in the region Ω.
The exponential part of (3.35) under the integral then has the form of

$$-\frac{1}{2}\left(W_{ab}(q^a q^b + q'^a q'^b) + q^a W_{a\beta} q^\beta + q'^a W_{a\beta} q^\beta + q^\alpha W_{\alpha b} q^b + q^\alpha W_{\alpha b} q'^b + 2q^\alpha W_{\alpha\beta} q^\beta\right)$$
$$= -\frac{1}{2} W_{ab}(q^a q^b + q'^a q'^b) - W_{\alpha\beta} q^\alpha q^\beta - W_{a\alpha}(q^a + q'^a) q^\alpha, \tag{3.38}$$

where we have renamed some indices to have a more compact form. The reduced density matrix now looks like

$$\rho_{red}(\{q^a\}, \{q'^b\}) = \left(\det\left(\frac{D_{AB}}{\pi}\right)\right)^{1/2} \exp\left[-\frac{1}{2} W_{ab}(q^a q^b + q'^a q'^b)\right] \tag{3.39}$$
$$\times \int \prod_\alpha dq^\alpha \exp\left[-W_{\alpha\beta} q^\alpha q^\beta - W_{a\alpha}(q^a + q'^a) q^\alpha\right]$$

In our further discussions, we shall also use the inverse matrix of W_{AB},

$$W^{AB} = \begin{pmatrix} W^{ab} & W^{a\alpha} \\ W^{\alpha b} & W^{\alpha\beta} \end{pmatrix} \tag{3.40}$$

where we should note that the raised indices were not obtained by raising indices using G^{AB}. Furthermore, we will denote the inverses of the submatrices of W_{AB} using an overhead tilde. For example, $\widetilde{W}^{ab}$ is the inverse of W_{ab} and so on. To evaluate the density matrix completely, we still need to evaluate the integral. To do that, we will complete the square in the exponent, and then it will be reduced to the form where we will just have to use the Gaussian integral, just as we did in (3.28). The exponent of (3.39) can now be written as

$$\begin{aligned}
&- W_{\alpha\beta}(q^\alpha q^\beta + \widetilde{W}^{\alpha\beta} W_{a\alpha}(q^a + q'^a)q^\alpha \pm \frac{1}{4}(\widetilde{W}^{\alpha\beta} W_{a\alpha}(q^a + q'^a))^2) \\
&- W_{\alpha\beta}\left(q^\beta + \frac{1}{2} W_{a\alpha}\widetilde{W}^{\alpha\beta}(q^a + q'^a)\right)^2 + \frac{1}{4} W_{\alpha\beta}\widetilde{W}^{\alpha\beta} W_{a\alpha}\widetilde{W}^{\alpha\beta} W_{b\beta}(q^a + q'^a)(q^b + q'^b) \\
&= -W_{\alpha\beta}\left(q^\alpha + \frac{1}{2} W_{a\beta}\widetilde{W}^{\alpha\beta}(q^a + q'^a)\right)^2 + \frac{1}{4}\widetilde{W}^{\alpha\beta} W_{a\alpha} W_{b\beta}(q + q')^a (q + q')^b.
\end{aligned} \tag{3.41}$$

Using the Gaussian integral from (3.28), the reduced density matrix can now be written as

$$\begin{aligned}
\rho_{red}(\{q^a\}, \{q'^b\}) = &\left(\det\left(\frac{D_{AB}}{\pi}\right)\right)^{1/2} \left(\det\left(\frac{D_{\alpha\beta}}{\pi}\right)\right)^{-1/2} \\
&\times \exp\left[-\frac{1}{2} W_{ab}(q^a q^b + q'^a q'^b)\right] \exp\left[\frac{1}{4}\widetilde{W}^{\alpha\beta} W_{a\alpha} W_{b\beta}(q + q')^a (q + q')^b\right],
\end{aligned} \tag{3.42}$$

where $D_{\alpha\beta}$ is the diagonal submatrix of W, corresponding to the coupling between oscillators in the interior of Ω. We can further symplify this by using the identity

$$\det(W_{AB}) \equiv \det(\widetilde{W_{ab}}) \det(W_{\alpha\beta}), \tag{3.43}$$

which can be obtained by taking the determinant of the following equation,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & 1 \end{pmatrix}. \tag{3.44}$$

After decomposing (3.34) in such a way and taking the determinant, we arrive at

$$\det(W_{AB}) = \det(W_{\alpha\beta}) \det(W_{ab} - W_{a\alpha}\widetilde{W}^{\alpha\beta} W_{\alpha b}). \tag{3.45}$$

This means, that to to verify (3.43), we need to show that

$$\widetilde{W}_{ab} = W_{ab} - W_{a\alpha}\widetilde{W}^{\alpha\beta} W_{\alpha b}. \tag{3.46}$$

Which means we need to show that the RHS is the inverse of W^{ab}.

$$\begin{aligned} & W^{ca}(W_{ab} - W_{a\alpha}\tilde{W}^{\alpha\beta}W_{\beta b}) = \{W^{cA}W_{Ab} = W^{ca}W_{ab} + W^{c\gamma}W_{\gamma b}\} \\ & = W^{cA}W_{Ab} - W^{c\gamma}W_{\gamma b} - (W^{cA}W_{A\alpha} - W^{c\gamma}W_{\gamma\alpha})\widetilde{W}^{\alpha\beta}W_{\beta b} \\ & = \{W^{cA}W_{Ab} = \delta^c_b, W^{c\gamma}W_{\gamma b} = 0 \ (c \text{ and } b \text{ are not in } \Omega), W_{A\alpha}\widetilde{W}^{\alpha\beta} = \delta^\beta_A, W_{\gamma\alpha}\widetilde{W}^{\alpha\beta} = \delta^\beta_\gamma\} \\ & = \delta^c_b - W^{cA}\delta^\beta_A W_{\beta b} + W^{c\gamma}\delta^\beta_\gamma W_{\beta b} = \delta^c_b. \end{aligned} \tag{3.47}$$

We have now verified (3.46). Using the fact that unitary matrices have a determinant of one, we can trivially find the diagonal form for (3.43),

$$\det(D_{AB}) = \det(\widetilde{D}_{ab})\det(D_{\alpha\beta}). \tag{3.48}$$

When applying this to (3.42), the determinant factors simplify to

$$\left(\det\left(\frac{\widetilde{D}_{ab}}{\pi}\right)\right)^{1/2}. \tag{3.49}$$

Introducing

$$M_{ab} \equiv \widetilde{W}_{ab}, \quad N_{ab} = W_{a\alpha}\widetilde{W}^{\alpha\beta}W_{\beta b}, \tag{3.50}$$

we can write the exponential part of the reduced density matrix (3.42) as

$$\begin{aligned} & -\frac{1}{2}(\widetilde{W}_{ab} + W_{a\alpha}\widetilde{W}^{\alpha\beta}W_{\alpha b})(q^a q^b + q'^a q'^b) + \frac{1}{4}\widetilde{W}^{\alpha\beta}W_{a\alpha}W_{b\beta}(q+q')^a(q+q')^b \\ & = -\frac{1}{2}(M_{ab} + N_{ab})(q^a q^b + q'^a q'^b) + \frac{1}{4}N_{ab}(q^a q^b + q'^a q^b + q^b q'^a + q'^a q'^b) \\ & = -\frac{1}{2}M_{ab}(q^a q^b + q'^a q'^b) + \frac{1}{4}N_{ab}(-q^a q^b + q'^a q^b + q^b q'^a - q'^a q'^b) \\ & = -\frac{1}{2}M_{ab}(q^a q^b + q'^a q'^b) - \frac{1}{4}N_{ab}(q-q')^a(q-q')^b. \end{aligned} \tag{3.51}$$

Finally, the reduced density matrix is given by the following Gaussian matrix

$$\begin{aligned} \rho_{red}(\{q^a\},\{q'^b\}) = \left(\det\left(\frac{M_{ab}}{\pi}\right)\right)^{1/2} \exp\left[-\frac{1}{2}M_{ab}(q^a q^b + q'^a q'^b)\right] \\ \times \exp\left[-\frac{1}{4}N_{ab}(q-q')^a(q-q')^b\right]. \end{aligned} \tag{3.52}$$

To find the entropy of such a density matrix, we will study the entropy of a Gaussian density matrix obtained for a coherent state of two oscillators and then extend this result to a general Gaussian density matrix.

3.1.1 Entropy of a coherent state

When we have a system of two oscillators, each one has its own degree of freedom. Let us say that a and b are annihilation operators for the two oscillators. Let us consider the coherent state

$$
\begin{aligned}
|\psi\rangle &= Ce^{\gamma a^* b^*} |0\rangle_a \otimes |0\rangle_b \\
&= C\sum_{n=0}^{\infty} \frac{\gamma^n}{n!}(\sqrt{n!}\,|n\rangle_a) \otimes (\sqrt{n!}\,|n\rangle_b) = C\sum_{n=0}^{\infty} \gamma^n |n\rangle_a \otimes |n\rangle_b\,,
\end{aligned}
\tag{3.53}
$$

where γ is a real number and C is the normalization constant

$$
\begin{aligned}
\langle\psi|\psi\rangle = |C|^2 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \gamma^{n+m}{}_a\langle m|n\rangle_a \otimes {}_b\langle m|n\rangle_b = |C|^2 \sum_{n=0}^{\infty} \gamma^{2n} \overset{!}{=} 1 \\
\implies |C|^2 \frac{1}{1-\gamma^2} = 1 \implies C = (1-\gamma^2)^{1/2},
\end{aligned}
\tag{3.54}
$$

defined up to a global phase. Note that since we must have a finite normalization constant, we obtain a restriction on γ, as to ensure the convergence of the geometric sum in the above equation, $\gamma^2 < 1$.

Forming the density matrix $\rho = |\psi\rangle\langle\psi|$, and tracing out over the oscillator b, we obtain the following reduced density matrix

$$
\rho_{red} = \sum_{m=0}^{\infty} {}_b\langle m|\psi\rangle\langle\psi|m\rangle_b = \sum_{m=0}^{\infty} C^2\gamma^{2m} |m\rangle_{a\,a}\langle m|\,. \tag{3.55}
$$

Since the matrix is diagonal, it is easy to evaluate the entropy associated with this density matrix.

$$
\begin{aligned}
S &= -Tr(\rho_{red}\ln\rho_{red}) = -\sum_{m=0}^{\infty}(1-\gamma^2)\gamma^{2m}\ln\left[(1-\gamma^2)\gamma^{2m}\right] \\
&= -(1-\gamma^2)\frac{\ln(1-\gamma^2)}{1-\gamma^2} - (1-\gamma^2)\ln\gamma^2 \sum_{m=0}^{\infty} m\gamma^{2m}
\end{aligned}
\tag{3.56}
$$

To calculate the second sum, we use

$$\sum_{m=0}^{\infty} \gamma^{2m} = \frac{1}{1-\gamma^2}, \tag{3.57}$$

and differentiating with respect to γ, we obtain

$$\sum_{m=0}^{\infty} 2m\gamma^{2m-1} = -\frac{(-2\gamma)}{(1-\gamma^2)^2} \implies \sum_{m=0}^{\infty} m\gamma^{2m} = \frac{\gamma^2}{(1-\gamma^2)^2}. \tag{3.58}$$

The entropy is then given by

$$S = -\ln{(1-\gamma^2)} - \frac{\gamma^2}{1-\gamma^2}\ln\gamma^2 \tag{3.59}$$

In order to relate this entropy to that of a Gaussian density matrix, we will, as in the previous parts of this section, find the density matrix in the position representation. Acting with the annihilation operators a and b onto the state vector (3.53), we find that the following equalities hold

$$\begin{aligned} a\left|\psi\right\rangle &= C\sum_{n=1}^{\infty}\gamma^n\sqrt{n}\left|n-1\right\rangle_a\otimes\left|n\right\rangle_b = C\sum_{n=0}^{\infty}\gamma^{n+1}\sqrt{n+1}\left|n\right\rangle_a\otimes\left|n+1\right\rangle_b = \gamma b^*\left|\psi\right\rangle \\ b\left|\psi\right\rangle &= C\sum_{n=1}^{\infty}\gamma^n\sqrt{n}\left|n\right\rangle_a\otimes\left|n-1\right\rangle_b = C\sum_{n=0}^{\infty}\gamma^{n+1}\sqrt{n+1}\left|n+1\right\rangle_a\otimes\left|n\right\rangle_b = \gamma a^*\left|\psi\right\rangle. \end{aligned} \tag{3.60}$$

If we denote that the displacement of the two oscillators is given by x and y, with corresponding conjugate momenta p and q, we can write the annihilation and creation operators as

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(p-ix), \quad a^* = \frac{1}{\sqrt{2}}(p+ix), \\ b &= \frac{1}{\sqrt{2}}(q-iy), \quad b^* = \frac{1}{\sqrt{2}}(q+iy). \end{aligned} \tag{3.61}$$

The set of equations (3.60) can now be rewritten as

$$\begin{aligned} &[(p-\gamma q) - i(x+\gamma y)]\left|\psi\right\rangle = 0, \\ &[(p-\gamma p) - i(y+\gamma x)]\left|\psi\right\rangle = 0. \end{aligned} \tag{3.62}$$

In order to solve (3.62), we introduce new oscillators by defining their displacements, u and v as

$$u \equiv x + \gamma y, \quad v \equiv y + \gamma x. \tag{3.63}$$

Since the displacement of the "new" u and v oscillators is a combination of the displacements of the "old" x and y oscillators, we can suppose that the momentum operators for the u and v oscillators will also only be a combination of the momentum operators of the x and y oscillators,

$$P_u = ap + bq, \quad P_v = cp + dq. \tag{3.64}$$

Now, after we impose the same commutation relations for the displacement and the momentum of the "new" oscillators as we did for "old" oscillators,

$$[u, P_u] = i, \quad [v, P_v] = i, \quad [v, P_u] = 0, \quad [u, P_v] = 0, \tag{3.65}$$

while also retaining the previous comutation relations

$$[x, p] = i, \quad [y, q] = i, \quad [x, q] = 0, \quad [y, p] = 0, \tag{3.66}$$

we can obtain the constants a, b, c, d. For the oscillator u

$$i = [x + \gamma y, ap + bq] = a[x, p] + b\gamma[y, q] \implies a + b\gamma = 1 \implies a = 1 - b\gamma, \tag{3.67}$$

$$\begin{aligned}
P_u &= (1 - b\gamma)p + bq, \\
0 &= [v, P_u] = [y + \gamma x, (1 - b\gamma)p + bq] = (1 - b\gamma)\gamma[x, p] + b[y, q] \\
&= (1 - b\gamma)\gamma + b \implies b\gamma^2 - b = \gamma \implies b = \frac{\gamma}{\gamma^2 - 1} \\
P_u &= \left(1 - \frac{\gamma^2}{\gamma^2 - 1}\right) p + \frac{\gamma}{\gamma^2 - 1} q = -\frac{1}{\gamma^2 - 1} p + \frac{\gamma}{\gamma^2 - 1} q = \frac{1}{1 - \gamma^2}(p - \gamma q).
\end{aligned}$$

Following the same procedure for the oscillator v, we obtain

$$P_v = \frac{1}{1 - \gamma^2}(q - \gamma p). \tag{3.68}$$

The pair of differential equations (3.62) can now be rewritten as

$$
\begin{aligned}
\left[(1-\gamma^2)P_u - iu\right]|\psi\rangle &= 0, \\
\left[(1-\gamma^2)P_v - iv\right]|\psi\rangle &= 0.
\end{aligned}
\tag{3.69}
$$

In the position representation we have

$$
P_u = -i\partial_u, \quad P_v = -i\partial_v, \tag{3.70}
$$

and now the pair of equations have a similar form as (3.24),

$$
\begin{aligned}
\left[\partial_u + \frac{1}{1-\gamma^2}u\right]\psi(u,v) &= 0, \\
\left[\partial_v + \frac{1}{1-\gamma^2}v\right]\psi(u,v) &= 0
\end{aligned}
\tag{3.71}
$$

where W_{AB} can be written as a diagonal 2×2 matrix with the same diagonal entries $(1-\gamma^2)^{-1}$. The solution then has the same structure (3.25)

$$
\psi(u,v) = K \exp\left(-\frac{1}{2}\frac{1}{1-\gamma^2}(u^2+v^2)\right), \tag{3.72}
$$

and K is the normalization constant. Plugging the definitions of u and v (3.63) back into (3.72), we can obtain $\psi(x,y)$. Using

$$
\begin{aligned}
u^2+v^2 = (x+\gamma y)^2 + (y+\gamma x)^2 &= x^2 + 4\gamma xy + \gamma^2 x^2 + y^2 + \gamma^2 y^2 \\
&= (x^2+y^2)(1+\gamma^2) + 4\gamma xy,
\end{aligned}
\tag{3.73}
$$

we have

$$
\psi(x,y) = K \exp\left[-\frac{1+\gamma^2}{1-\gamma^2}\frac{x^2+y^2}{2} - \frac{2\gamma}{1-\gamma^2}xy\right]. \tag{3.74}
$$

Using the normalization condition we obtain the constant K

$$
\begin{aligned}
&\int dxdy\psi(x,y)\psi^*(x,y) = K^2\int dxdy\exp\left[-\frac{1+\gamma^2}{1-\gamma^2}(x^2+y^2)-\frac{4\gamma}{1-\gamma^2}xy\right]\\
&= K^2\int_0^{2\pi}d\phi\int_0^{\infty}rdr\exp\left[-r^2\left(\frac{1+\gamma^2}{1-\gamma^2}r^2+\frac{4\gamma}{1-\gamma^2}\sin\phi\cos\phi\right)\right]\\
&= \left\{x=r^2, dx=2rdr, A(\phi)\equiv\frac{1+\gamma^2}{1-\gamma^2}r^2+\frac{4\gamma}{1-\gamma^2}\sin\phi\cos\phi\right\}\\
&= \frac{K^2}{2}\int_0^{2\pi}d\phi\int_0^{\infty}dxe^{-xA(\phi)} = \frac{K^2}{2}\int_0^{2\pi}\frac{d\phi}{-A(\phi)}e^{-ax}\Big|_0^{\infty}\\
&= \frac{K^2}{2}\int_0^{2\pi}\frac{d\phi}{\frac{1+\gamma^2}{1-\gamma^2}+\frac{2\gamma}{1-\gamma^2}\sin(2\phi)} = \frac{K^2}{2}\left(-2\pi\frac{|y-1||y+1|}{y^2-1}\right)\\
&= \{|y|<1\} = K^2\pi \stackrel{!}{=} 1 \implies K^2=\frac{1}{\pi}
\end{aligned}
\tag{3.75}
$$

The density matrix can now be constructed in the same manner as before (3.32)

$$
\rho[(x,y),(x',y')] = \psi(x,y)\psi^*(x',y') \tag{3.76}
$$

After tracing over one of the oscillators, we obtain the following reduced density matrix

$$
\begin{aligned}
&\rho_{red}(x,x') = \int dy\rho[(x,y),(x',y)] \qquad (3.77)\\
&= \frac{1}{\pi}\int dy\exp\left(-\frac{1+\gamma^2}{1-\gamma^2}\frac{x^2+y^2}{2}-\frac{2\gamma}{1-\gamma^2}xy-\frac{1+\gamma^2}{1-\gamma^2}\frac{x'^2+y^2}{2}-\frac{2\gamma}{1-\gamma^2}x'y\right)\\
&= \frac{1}{\pi}\exp\left[-\frac{1+\gamma^2}{1-\gamma^2}\frac{x^2+x'^2}{2}\right]\int dy\exp\left(-\frac{1+\gamma^2}{1-\gamma^2}y^2-\frac{2\gamma}{1-\gamma^2}(x+x')y\right)\\
&= \frac{1}{\pi}\exp\left[-\frac{1+\gamma^2}{1-\gamma^2}\frac{x^2+x'^2}{2}\right]\int dy\exp\left[-\frac{1+\gamma^2}{1-\gamma^2}\left(y^2-\frac{2\gamma}{1+\gamma^2}(x+x')y\pm\frac{\gamma^2}{(1+\gamma^2)^2}(x+x')^2\right)\right]\\
&= \frac{1}{\pi}\exp\left[-\frac{1+\gamma^2}{1-\gamma^2}\frac{x^2+x'^2}{2}\right]\int dy\exp\left[-\frac{1+\gamma^2}{1-\gamma^2}\left(y-\frac{2\gamma}{1+\gamma^2}(x+x')\right)^2+\frac{\gamma^2(x+x')^2}{(1+\gamma^2)(1-\gamma^2)}\right]\\
&= \frac{1}{\pi}\exp\left[-\frac{1+\gamma^2}{1-\gamma^2}\frac{x^2+x'^2}{2}\right]\sqrt{\frac{\pi(1-\gamma^2)}{(1+\gamma^2)}}\exp\left[\frac{\gamma^2}{(1-\gamma^4)}(x+x')^2\right]\\
&= \sqrt{\frac{(1-\gamma^2)}{\pi(1+\gamma^2)}}\exp\left[-\frac{1+\gamma^2}{1-\gamma^2}\frac{x^2+x'^2}{2}+\frac{\gamma^2}{(1-\gamma^4)}(x+x')^2\right].
\end{aligned}
$$

To rewrite in the same form as (3.52), we introduce

$$
\mu\equiv\gamma^2,\quad M\equiv\frac{1-\mu}{1+\mu},\quad \text{and } N\equiv\frac{4\mu}{1-\mu^2}, \tag{3.78}
$$

and we now have

$$\rho_{red}(x, x') = \sqrt{\frac{M}{\pi}} \exp\left[-\frac{1}{2}M(x^2 + x'^2) - \frac{1}{4}N(x - x')^2\right]. \tag{3.79}$$

Which is a Gaussian density matrix of the form (3.52). Now we finally know the connection between the entropy obtained previously (3.56) and the family of Gaussian density matrices, parametrized by one degree of freedom, γ. Relabelling (3.56) with $\mu = \gamma^2$, the entropy has the following form

$$S = -\frac{\mu}{1-\mu}\ln\mu - \ln(1-\mu) = -\frac{\mu\ln\mu + (1-\mu)\ln(1-\mu)}{1-\mu}. \tag{3.80}$$

To ensure the finiteness of the normalization constant C of our starting state vector (3.54), we arrived at the constraint on γ^2

$$\gamma^2 < 1 \implies \mu < 1. \tag{3.81}$$

With this constraint, we can find the unique connection between μ and N and M, using their definitions (3.78)

$$\mu = 1 + \frac{2M}{N} - 2\sqrt{\frac{M}{N}\left(1 + \frac{M}{N}\right)}. \tag{3.82}$$

Since we wish to find the entropy of the density matrix (3.79) for arbitrary M and N, we shall consider a density matrix of the same form, but with freely specified M and N. Using the fact that the entropy must be dimensionless, we know that it can only depend on the dimensionless ratio $\lambda \equiv N/M$ of the dimensional parameters in the density matrix. If we define

$$\mu \equiv 1 + 2\lambda^{-1} - 2[\lambda^{-1}(1 + \lambda^{-1})]^{1/2}, \tag{3.83}$$

we can generalize our result to state that the entropy of any density matrix that has the form (3.79), with M and N freely specified and μ given by (3.75), is

$$S = -\frac{\mu\ln\mu + (1-\mu)\ln(1-\mu)}{1-\mu}. \tag{3.84}$$

We can show that the density matrix associated with this entropy can be rewritten as

$$\rho_{red}(x,x') = \pi^{-1/2}\exp\left[-\frac{1}{2}(x^2+x'^2) - \frac{1}{4}\lambda(x-x')^2\right]. \tag{3.85}$$

3.1.2 Entropy of a general Gaussian density matrix

Since we wanted to calculate the entropy of (3.52) in the first place, which is the generalization of (3.79), we want to write it in a form to which we can apply the previous result as a product of density matrices given by (3.85). To achieve this and justify (3.85), we construct a basis in which both M and N are diagonal. For these reasons, we will consider M to be a metric on configuration space (similar to what we have considered G before) and choose as a basis a complete orthonormal set of vectors with respect to it. This means our basis is fixed up to an M-orthogonal transformation, since any transformation that is in the orthogonal direction with respect to any basis vector will be in the direction of some other basis vector and will thus not change M. This will be used to diagonalize N. Applying this to (3.52), we arrive at the following expression

$$\rho_{red}(\{q^a\},\{q'^b\}) = \prod_n \left\{\pi^{-1/2}\exp\left[-\frac{1}{2}(q_n q^n + q'_n q'^n) - \frac{1}{4}\lambda_n (q-q')_n (q-q')^n\right]\right\}, \tag{3.86}$$

where we are not summing over the repeated indices, and where λ_n are the diagonal elements of N in the M-orthogonal basis. These diagonal elements correspond to the eigenvalues of the operator

$$\Lambda^a{}_b \equiv (M^{-1})^{ac} N_{cb}. \tag{3.87}$$

ρ_{red} has the form

$$\rho_{red} = \bigotimes_n \rho(\lambda_n), \tag{3.88}$$

which means that the entropy is given by

$$S = \sum_n S[\rho(\lambda_n)], \tag{3.89}$$

where each λ_n corresponds to a different μ in (3.78), and thus a different (3.80) is obtained for each distinct n. To conclude, the entropy associated with the Gaussian density matrix of the form (3.52) for a system with many degrees of freedom can be calculated as

$$S = -\sum_n \frac{\mu_n \ln \mu_n + (1-\mu_n)\ln(1-\mu_n)}{1-\mu_n}, \tag{3.90}$$

where

$$\mu_n = 1 + 2\lambda_n^{-1} - 2\sqrt{\lambda_n^{-1}(1+\lambda_n^{-1})}, \tag{3.91}$$

and $\{\lambda_n\}$ are the eigenvalues of

$$\Lambda^a{}_b \equiv (M^{-1})^{ac} N_{cb}. \tag{3.92}$$

Plugging in the original definitions of M and N through W (3.50) into (3.92), we can obtain the form of Λ through W.

$$\begin{aligned}\Lambda^a{}_b &= (M^{-1})^{ac} N_{cb} = W^{ac} W_{c\alpha} \widetilde{W}^{\alpha\beta} W_{\beta b} \\ &= \left\{ W^{aC} W_{C\alpha} = 0 \implies W^{ac} W_{c\alpha} = -W^{a\gamma} W_{\gamma\alpha} \implies W^{a\beta} = -W^{ac} W_{c\alpha} \widetilde{W}^{\alpha\beta} \right\} \\ &= -W^{a\beta} W_{\beta b}.\end{aligned} \tag{3.93}$$

This shows how Λ depends on the choice of unavailable oscillators (the ones in Ω), since it takes values associated with Ω through the Greek indices in W. If we consider W_{ab} as a metric for the configuration space of the oscillators outside of Ω, we can lower or raise either index on Λ (both indices are lower case Latin latters, so they take the values from oscillators in the space outside of Ω). Acting with such a lowering operator on Λ, we have the following identity

$$W_{am} \Lambda^m{}_b = -W_{am} W^{m\alpha} W_{\alpha b} = W_{a\mu} W^{\mu\alpha} W_{\alpha b}. \tag{3.94}$$

But since W_{AB}, and $W^{\alpha\beta}$ are positive definite, we conclude that $\Lambda^a{}_b$ is also a positive definite operator.

Finally, to evaluate the entropy of a system, we must consider the dynamics of the field we want to study, the specific set of oscillators that we will ignore (that is, the region Ω we will integrate over), write down the form of the operator Λ, determine its eigenvalues, and then plug them into (3.91) and (3.92).

Srednicki [8] independently used a variation of this method to calculate the entropy inside a sphere of radius R, where he traced the ground state of a massless scalar field over the degrees of freedom inside a radial lattice. The entropy he obtained numerically is given by

$$S = 0.3M^2R^2, \tag{3.95}$$

where M is the inverse length scale and is given by $M = 1/a$, where a is the distance between lattice points. This result is quite similar to the one obtained for the Bekenstein-Hawking entropy, considering that $R^2 \sim \mathcal{A}$. Both papers [7,8] also claim that this entropy should not be considered the sole contributor to the entropy of a black hole.

3.1.3 Continuum case

Let us now consider a real scalar field and the problem of calculating the entropy associated with this field in the presence of a black hole. To apply our formalism, we will imagine a black hole to be simulated as a region, Ω, of flat spacetime. In the continuum limit of our formalism, we have

$$\frac{1}{2}V_{AB}q^Aq^B \to \frac{1}{2}\langle\phi|\nabla^2 + m^2|\psi\rangle = \int \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right] d^3x. \tag{3.96}$$

With this, we can try to apply the formalism that we have developed so far. That is, we want to construct a similar operator that corresponds to ${\Lambda^a}_b$, evaluate its eigenvalues, and calculate the entropy for some appropriate region Ω. To make our discussion easier, we shall work in momentum representation. The matrix V in this representation is given by

$$V(x, y) = \int \frac{d^3k}{(2\pi)^3}(k^2 + m^2)e^{ik\cdot(x-y)}, \tag{3.97}$$

the "square root" matrix W will have the following continuum form in the momentum representation

$$W(x,y) = \int \frac{d^3k}{(2\pi)^3}(k^2+m^2)^{1/2}e^{ik\cdot(x-y)}, \tag{3.98}$$

and the inverse of the square root of the same matrix is given by

$$W^{-1}(x,y) = \int \frac{d^3k}{(2\pi)^3}(k^2+m^2)^{-1/2}e^{ik\cdot(x-y)}, \tag{3.99}$$

where now, instead of discrete indices denoted by capital letters A, B, we have continuous indices over $\mathbb{R}^3$. Following the matrix definition of Λ, as a multiplication of W matrices (3.93), the continuous case will be given by

$$\Lambda(x,y) = -\int_\Omega d^3z W^{-1}(x,z)W(z,y), \tag{3.100}$$

$$\Lambda(x,y) = -\int_\Omega d^3z \left(\int \frac{d^3k}{(2\pi)^3}(k^2+m^2)^{-1/2}e^{ik\cdot(x-z)}\right)\left(\int \frac{d^3k'}{(2\pi)^3}(k'^2+m^2)^{1/2}e^{ik'\cdot(z-y)}\right)$$

And to determine the eigenvalues, we have to solve the continuous eigenvalue equation

$$\int d^3y\Lambda(x,y)f(y) = \lambda f(x). \tag{3.101}$$

We should also note that the upper integral includes the entire space, not just Ω. After solving this equation, and finding the eigenvalues, we can plug them into (3.91) and (3.92) to properly calculate the entropy, where instead of a finite sum, we will have an infinite one.

By looking at the dimension of the entropy, we can conclude that there has to be a length cutoff in the integrals above that ensures the finiteness of the entropy. To prove this, let us look at the case of a massless field $m = 0$. Since the entropy is a dimensionless quantity, S has to be invariant under a rescaling of the region Ω, and the only answer we can expect to get for it is either 0 or ∞. If $m \neq 0$, S could dimensionally be a function of mR, where R is some characteristic size of Ω (in the case of black holes, this would be its radius). If the entropy was not infinite, we would expect it to vanish in the limit as the size $R \to 0$ (since then the black hole would disappear), but this limit is equivalent to $m \to 0$, which gives an infinite entropy from the previous argument. Physically, this divergence of the entropy is ultraviolet in origin; since it is not removed by a nonvanishing mass, it means that it arises from modes of arbitrarily small wavelengths. This means that there has to be a fundamental length in the theory. We introduce a dimensional parameter l that will act as a cutoff (in the brick wall model, this fundamental length cutoff is the brick wall, h). Now the entropy can be a function of R/l. The way that we take into account this cutoff into our calculations is by using a position cutoff near the boundary of Ω (similarly as what was done in the brick wall method), which means the integration over Ω will be restricted to points at a distance of at least l from the boundary. Furthermore, we consider the cutoff to be of Plank length order. This way, we disable the correlations between points inside and outside Ω if they are less than a distance of l apart. Another thing to note is that the high-frequency modes that we expect to contribute most to S are localized near the boundary (we can see the parallels with the Bekenstein-Hawking entropy for black holes here as well, as the degrees of freedom that contribute the most to the entropy of black holes are the ones near the horizon). For these types of calculations, we usually choose the space we are integrating over Ω to have some sort of symmetries, so we can reduce a three-dimensional problem to an effective one-dimensional one. For example, one might use a sphere of radius R for the region Ω, and to reduce this problem to an effective one-dimensional one, we have to assume the following ansatz for the eigenfunctions of the operator Λ, $f(r,\theta,\phi) = Y_{lm}(\theta,\phi)f(r)$. This ansatz leads to numerous calculation difficulties and is outside the scope of this thesis, and will thus not be explored further. The main point of this subsubsection was to outline a procedure for systematically calculating the eigenvalues of the operator Λ, while also explaining the need for a cutoff scale. As a final note and for future developments, it would be useful to try and extend this method to curved spacetimes.

3.2 The replica trick

As we have shown in the previous section, the entanglement entropy can be obtained if we know how to diagonalize the reduced density matrix and obtain its eigenvalues. However, the calculation is troublesome in a continuum quantum field theory with infinite degrees of freedom. Thus, it is more convenient to derive the entanglement entropy by using the replica trick [27, 28]. The idea is to first consider a Rényi entropy, which is a one-parameter generalization of the entanglement entropy equipped with an additional Rényi index q, which is initially an integer, and recover the entanglement entropy after we analytically continue q to the real numbers, along with taking the limit $q \to 1$. This way, our task is reduced to calculating the partition function of a replicated manifold $\hat{\mathcal{M}}_q$ using the path integral, which is simpler for QFT. We shall use the replica trick to explicitly calculate the entanglement entropy of static and spherically-symmetric black hole spacetimes. We consider entanglement between two timelike regions in the maximally extended black hole spacetime that are connected by a wormhole with a radius equal to the black hole radius. Let ρ_A be the reduced density matrix obtained by tracing out the degrees of freedom in the unobservable region B, leaving only the ones in the accessible region A. We consider the Rényi entropy, given by

$$S_{Ren}^{(q)} = \frac{1}{1-q} \log(Tr(\rho_A^q)), \tag{3.102}$$

where q is an interger-valued Rényi index. When we analytically continue q to a continuous variable and take the limit $q \to 1$, using $\log Tr(q_A) = 0$

$$\begin{aligned} \lim_{q\to 1} S_{Ren}^{(q)} &= \lim_{q\to 1} \frac{1}{1-q} \log(Tr(q_A^q)) = -\lim_{q\to 1} \frac{\log(Tr(q_A^q)) - \log(Tr(q_A))}{q-1} \\ &= -\partial_q \log(Tr(\rho_A^q)|_{q=1} = S_A, \end{aligned} \tag{3.103}$$

we see that the Rényi entropy reduces to the entanglement entropy. Without loss of generality, we will suppose that the subsystems are at a constant-time slice where $x > 0$ for subsystem A, and $x < 0$ for subsystem B. Hence, the boundary $\partial A = \partial B$ is located at $x = 0$. We can calculate the trace $Tr(\rho_A^q)$ by considering a partition function $Z[\mathcal{M}_q]$, which is represented by a quantum path integral in a replicated manifold $\mathcal{M}_q$, which contains a q-copy of the manifold of the original QFT, $\mathcal{M}$, which are sewn together cyclically along the subregion A by following the cyclicity of the trace $Tr(\rho_A^q)$. Since we are considering a normalized reduced density matrix ρ_A, the trace $Tr(\rho_A^q)$ is given by

$$Tr(\rho_A^q) = \frac{Z[M_q]}{(Z[M])^q}, \tag{3.104}$$

where $\mathcal{M}_1 \equiv \mathcal{M}$ is the original manifold of the original QFT. In the manifold $\mathcal{M}_q$, we need to circulate the boundary ∂A q-times before finally arriving at the starting position. This means that a complete cycling of ∂A takes $2\pi q$ instead of 2π. To calculate the partition function of $\mathcal{M}_q$, we set $\phi \sim \phi + 2\pi q$ everywhere, which introduces a conical singularity at the origin ($r = 0$) with an excess angle. It is possible to simplify the calculation of the entropy from apparent conical singularities by replacing $\mathcal{M}_q$ with a single-sheeted manifold $\hat{\mathcal{M}}_q$, which has a conical singularity located at ∂A with deficit angle $\Delta\phi = 2\pi(1 - 1/q)$. This conical singularity appears because the manifold $\hat{\mathcal{M}}_q$ is smooth if the periodicity is $\phi \sim \phi + 2\pi q$, but we identify the angle with $\phi \sim \phi + 2\pi$. These calculations essentially lead to the same conclusion, but the latter identification simplifies the entropy calculation for multi-horizon black holes. To write a metric for the manifold $\hat{\mathcal{M}}_q$, let us assume that r is the distance between any point from ∂A to ∂A, and ϕ is the angle between r and A, which is shown in Figure 3.1.

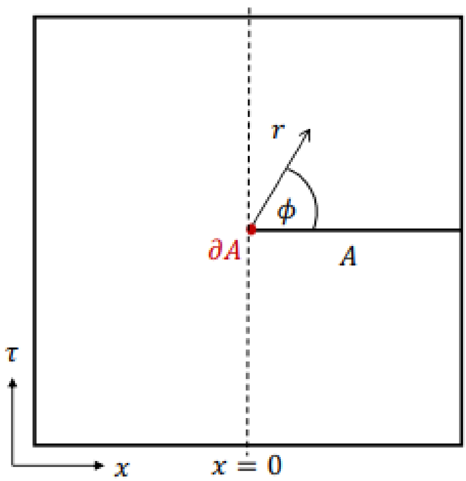

Figure 3.1: A parameterization of the replica spacetime with a conical singularity located at $r = 0$ in flat spacetime. The region A in the $\tau = 0$ slice is defined as $x > 0$ [28].

In a flat background, we parametrize the metric of $\hat{\mathcal{M}}_q$ with polar coordinates in the following way

$$ds^2_{\hat{\mathcal{M}}_q} = \frac{r^2}{q^2} d\phi^2 + dr^2 + \dots, \tag{3.105}$$

where the remainder represents the transverse coordinates. Identifying $\phi \sim \phi + 2\pi$ gives $\hat{\mathcal{M}}_q$ a conical singularity at $r = 0$, that is, at ∂A, with a deficit angle $\Delta\phi =$

$2\pi(1-1/q)$. Since the action is local, the relation between the partition functions $Z[\mathcal{M}_q]$ and $Z[\hat{\mathcal{M}}_q]$ is as follows

$$Z[\mathcal{M}_q] = (Z[\hat{\mathcal{M}}_q])^q. \tag{3.106}$$

The entanglement entropy can now be calculated as

$$S_A = -\partial_q \log Tr(\rho_A^q)|_{q=1} = -\partial_q \log \frac{Z[\mathcal{M}_q]}{(Z[\mathcal{M}])^q} = -q\partial_q \log Z[\hat{\mathcal{M}}_q]|_{q=1} \tag{3.107}$$

Using the formula above, we will calculate the entanglement entropy in Schwarzschild spacetime.

$$ds^2 = -\left(1-\frac{r_S}{r}\right)dt^2 + \left(1-\frac{r_S}{r}\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{3.108}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and $r_S = 2G_N M$ is the Schwarzschild radius of a black hole with mass M, and where G_N is the 4-dimensional Newton's constant. The metric can be obtained by solving the classical equation of motion $\delta I = 0$ from the Einstein-Hilbert action

$$I[\mathcal{M}] = \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x\sqrt{-g}R, \tag{3.109}$$

where, as before, g is the determinant of the metric, and R is the Ricci scalar. Since the Ricci scalar is zero for Schwarzschild spacetime, the contribution to the entropy comes from the surface term when varying the action. The Schwarzschild spacetime can be maximally extended into two asymptotically flat spacetimes sperated by a boundary that is located at the horizon $r = r_S$. The Penrose diagram of a maximally extended Schwarzschild spacetime is depicted in Figure 3.2.

Each point on Figure 3.2 corresponds to a 2-sphere, since we have suppressed these two dimensions to be able to graphically show a 4-dimensional space in $2D$. We can imagine that there is a codimension-one $t = 0$ slice Σ that is constructed from two subregions such that $\Sigma = \Sigma_A \cup \Sigma_B$. The boundary separating Σ_A and Σ_B is given by a codimension-two surface $\mathbf{S}^2$ with radius $r = r_S$. To calculate the entanglement entropy, we will only consider the boundary that separates the two subregions A and B, as we have done previously. We will suppose that all points in a maximally

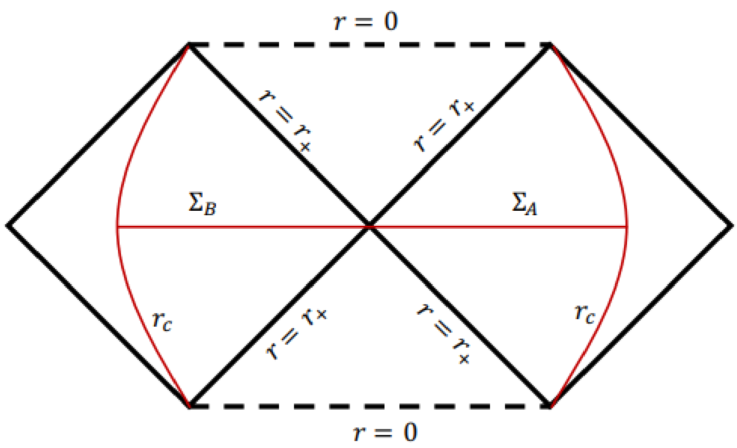

Figure 3.2: Penrose diagram of a maximally extended Schwarzschild spacetime [28].

extended Schwarzschild spacetime $\mathcal{M}$ represent quantum gravitational degrees of freedom in the background $\mathcal{M}$. Following the action integral approach to quantum gravity [29], we assume that in the low energy or classical limit, the gravitational (Euclidean) partition function is given by

$$Z[\mathcal{M}] = \int \mathcal{D}g e^{-I_E[g]} \approx e^{-I_E[\mathcal{M}]}, \tag{3.110}$$

where we used the saddle point approximation, and $I_E[\mathcal{M}]$ is the on-shell Euclidean action with the metric (3.113). The Euclidean action $I_E[\mathcal{M}]$ is obtained by Wick rotating the time coordinate $t \to i\tau$ of the Einstein-Hilbert action (3.109). Since in Schwarzschild spacetime we do not have a proper field theoretical description of the quantum degrees of freedom, we are only considering the entanglement of the degrees of freedom in the semiclassical approach, where we have assumed that the quantum gravitational path integral of the theory can be described by the saddle-point approximation of the Einstein-Hilbert action in the classical limit. The Euclidean Schwatzschild metric is given by

$$ds^2 = \left(1 - \frac{r_S}{r}\right) d\tau^2 + \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{3.111}$$

To avoid the conical singularity at $r = r_S$, the Euclidean time coordinate needs to be periodic with $\tau \sim \tau + \beta$, where $\beta = 8\pi M$ is the inverse temperature of the black hole. Introducing the Kruskal coordinates in the same way as one does in Lorentzian

Schwarzschild, the spacetime has the following form

$$ds^2 = \frac{4r_S^3}{r}e^{-r/r_S}(dT_E^2 + dR^2) + r^2 d\Omega^2, \tag{3.112}$$

where

$$T_E^2 + R^2 = \left(\frac{r}{r_S} - 1\right) e^{r/r_S}. \tag{3.113}$$

The positivity of the left-hand side of the upper equations restricts us to the region where $r > r_S$. With this and the periodicity of τ, the Euclidean Schwarzschild can be described by a "cigar" geometry, which is depicted in the following figure.

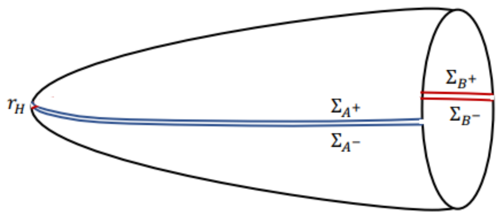

Figure 3.3: Illustration of the total (unnormalized) density matrix ρ. The slice Σ divides the cigar into two time-symmetric parts. To normalize ρ, we divide the calculation by $Tr(\rho)$ which is given by a full cigar [28].

In Figure 3.3, Σ is a slice in the time-reflection symmetry axis that equally separates the upper and lower part at $\tau = 0, \beta/2$. This way, the subregions Σ_A and Σ_B are the lines that start from $r = r_S$ to $r \to \infty$ at $\tau = 0$ and $\tau = \beta/2$ respectively. The cigar geometry with a cut in Σ gives us the pure density matrix ρ of the quantum gravitational degrees of freedom in $\mathcal{M}$. The reduced density matrix ρ_A can be obtained by identifying points from $r = r_S$ to $r \to \infty$ at $\tau = \beta/2$, which leaves a cut from $r = r_S$ to $r \to \infty$ at $\tau = 0$. To calculate the entanglement enttropy of the region Σ_A, we perform the replica trick by computing $Tr(\rho_A^q)$. The trace $Tr(\rho_A^q)$ is described by q-sheeted cigars, which are cyclically identified through the cuts following the cyclicity of the trace $Tr(\rho_A^q)$, which can be shown in the following Figure.

This is the $\mathcal{M}_q$ manifold for the Euclidean Schwarzschild. Following the previously outlined procedure, we now replace $\mathcal{M}_q$ with $\hat{\mathcal{M}}_q$, where $\hat{\mathcal{M}}_q$ is a single manifold with a conical singularity located at the origin $r = r_S$, with deficit angle $\Delta\phi = 2\pi(1 - 1/q)$, given in Figure 3.5, with analytically continued q into non-integer values.

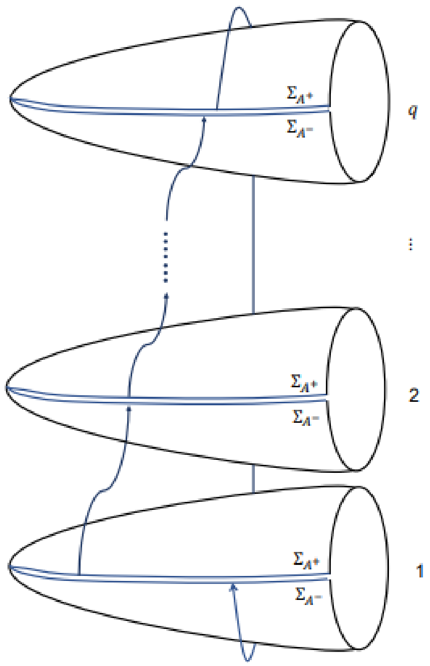

Figure 3.4: Illustration of the partition function $Z[\mathcal{M}_q]$ of the manifold $\mathcal{M}_q$. Blue lines connected by the arrow are identified [28].

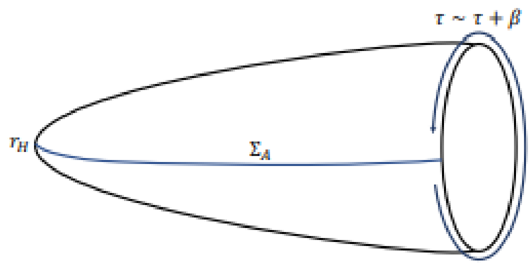

Figure 3.5: The one sheeted manifold $\hat{\mathcal{M}}_q$ which now has a conical singularity at the fixed point $r = r_S$ with deficit angle $\Delta\phi = 2\pi(1 - 1/q)$ [28].

We need to parametrize the Euclidean Schwarzschild with a new coordinate that describes the distance to the horizon $r = r_S$ and the angle. If $\xi \equiv r - r_S$ is the distance to the horizon while the angle is the Euclidean time τ, and replacing r with ξ and Taylor expanding around $\xi = 0$, we obtain the near-horizon geometry for a Euclidean Schwarzschild (notice that this is a similar procedure to what we have done in the brick wall method, where the main contribution to the entropy was the near-horizon contribution). Using

$$1 - r_S/r = \frac{r - r_S}{r} = \{\xi = r - r_S, \xi \approx 0\} \approx \frac{\xi}{r_S}, \tag{3.114}$$

we have

$$ds^2 = \frac{\xi}{r_S} d\tau^2 + r_S \frac{d\xi^2}{\xi} + r_S^2 d\Omega^2 + \dots, \tag{3.115}$$

where the remainder represents terms of higher order in ξ. Reparametrizing with $y^2 = \xi$, $\tilde{y} = 2\sqrt{r_S} y$, $\tilde{\tau} = \tau/2r_S$, we have

$$\frac{\xi}{r_S} d\tau^2 = y^2 4 r_S d\tilde{\tau}^2 = \tilde{y}^2 d\tilde{\tau}^2 \tag{3.116}$$

and

$$r_S \frac{d\xi^2}{\xi} = r_S \frac{4y^2 dy^2}{y^2} = 4 r_S dy^2 = d\tilde{y}^2. \tag{3.117}$$

In total,

$$ds^2 = \tilde{y}^2 d\tilde{\tau}^2 + d\tilde{y}^2 + r_S^2 d\Omega^2 + \dots, \tag{3.118}$$

where the remaining terms represent the higher order terms in $\tilde{y}$. The metric near $\tilde{y} = 0$ is given by $\mathbb{R}^2 \times \mathbf{S}^2$. To avoid the conical singularity at $\tilde{y} = 0$, $\tilde{\tau}$ needs to be periodic with $\tilde{\tau} \sim \tilde{\tau} + 2\pi$. To build the metric for $\hat{\mathcal{M}}_q$, we add the q-dependence to the metric in (3.118) in the same way as in (3.105),

$$ds^2 = \frac{\tilde{y}^2}{q^2} d\tilde{\tau}^2 + d\tilde{y}^2 + r_S^2 d\Omega^2 + \dots, \tag{3.119}$$

with $\tilde{\tau} \sim \tilde{\tau} + 2\pi$. Now we have a conical singularity at the $\tilde{y} = 0$, with a deficit angle of $\Delta\phi = 2\pi(1 - 1/q)$. This metric describes the geometry of $\hat{\mathcal{M}}_q$ near the tip of the $\hat{\mathcal{M}}_q$ cigar. Now we can finally calculate the entanglement entropy.

$$S_A = q\partial_q I_E[\hat{\mathcal{M}}_q]|_{q=1}. \tag{3.120}$$

To obtain $\partial_q I_E[\hat{\mathcal{M}}_q]$ we perform variations of the action $I_E[\hat{\mathcal{M}}_q]$ with respect to the

Rényi index q, and replace the variation with partial derivatives [30],

$$\partial_q I_E[\hat{\mathcal{M}}_q] = \int_{\hat{\mathcal{M}}_q} \frac{d^4x}{16\pi G_N} \sqrt{-g} G_{\mu\nu} \partial_q g^{(q)\mu\nu} \tag{3.121}$$
$$+ \int_{\hat{\Omega}_\varepsilon} \frac{d^3x}{16\pi G_N} \sqrt{\gamma} \hat{n}_\rho (g^{(q)}_{\mu\nu} \nabla^\rho \partial_q g^{(q)\mu\nu} - \nabla_\mu \partial_q g^{(q)\rho\mu}),$$

where $g^{(q)}_{\mu\nu}$ is the component of the metric tensor given in (3.119). The first term vanishes since $G_{\mu\nu} = 0$. The second term is the surface term evaluated at a hypersurface $\hat{\Omega}_\varepsilon$ of constant and arbitrarily small radius ε centered at $\tilde{y} = 0$. Notice how this ε is reminiscent of the brick wall cutoff h, introduced previously. This hypersurface is necessary to isolate the conical singularity from our calculations, and we take the limit of $\varepsilon \to 0$ after the calculations. γ is the determinant of the induced metric in $\hat{\Omega}_\varepsilon$, and $\hat{n}^\mu$ is a unit normal vector of $\hat{\Omega}_\varepsilon$ point away from $\tilde{y} = 0$. After plugging the metric into the above equation and performing the $\tilde{\tau}$ integration from 0 to 2π, it can be shown that

$$\partial_q I_E[\hat{\mathcal{M}}_q] = \frac{4\pi r_S^2}{4G_N q^2} + \mathcal{O}(\tilde{y})\Big|_{\tilde{y}=\varepsilon}. \tag{3.122}$$

As $\varepsilon \to 0$, the higher order terms in $\tilde{y}$ all vanish. Multiplying this by q and evaluating at $q = 1$, the entanglement entropy recovers the area law,

$$S_A = \frac{4\pi r_S^2}{4G_N} = \frac{\mathcal{A}}{4G_N}. \tag{3.123}$$

Since the result is independent of Euclidean time τ, it is valid for Lorentzian spacetime as well. Another way of calculating the entanglement entropy is by introducing an arbitrarily large but finite cutoff r_c such that the regions Σ_A and Σ_B range from $r = r_S$ to $r = r_c$ at $\tau = 0$ and $\tau = \beta/2$ respectively. Adding such a cutoff does not affect the final result of the entanglement entropy since we only need the near-horizon metric (that is, the metric near the tip of the cigar) to calculate the entanglement entropy. If the cutoff is large enough, we may then impose the boundary condition at the cutoff such that the variation there is vanishing. Meaning we obtain a result independent of the cutoff r_c as long as it is large enough. This fact is also useful for calculating the entanglement entropy for multihorizon black holes. Also note that this cutoff plays a similar role to the infrared cutoff in the brick wall method. The entanglement entropy calculated here describes the correlations that are strong near the horizon, and thus the leading term of the entanglement entropy is proportional to the area of the entangling surface $\partial\Sigma$. Any contribution from long-range correlation, i.e., entanglement between degrees of freedom that are relatively far from the entangling surface, only contributes as the sub-leading term of the entanglement entropy. This physical interpretation helps us to be sure that any entangled degrees of freedom outside the cutoff region $r > r_c$ does not contribute to the entanglement entropy, and the entropy is still well described by the area law (3.123). Now we can ask ourselves if it is possible to find a full metric that reduces to (3.119) near $r = r_S$. To do this, we replace $r_S \to r_S/q$ in the Euclidean Schwazschild metric,

$$ds^2 = \left(1 - \frac{r_S}{rq}\right) d\tau^2 + \left(1 - \frac{r_S}{rq}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{3.124}$$

This metric can be interpreted as a Schwarzschild metric with a horizon located at $r = r_S/q$ and a temperature of $T = q/4\pi r_S$. Expanding near $\xi = 0$, for $\xi = r - r_S$ we have

$$1 - \frac{r_S}{rq} = \frac{rq - r_S}{rq} = \frac{(\xi + r_S)q - r_S}{r_S q} = \left(1 - \frac{1}{q}\right) + \frac{\xi}{r_S q} \equiv \varepsilon_q + \frac{1}{r_S q}\xi. \tag{3.125}$$

Here, we have an extra term ε_q, which is of order $\mathcal{O}(1 - 1/q)$ and vanishes when $q = 1$, i.e., $\varepsilon_1 = 0$. For $1 < q < \infty$, we have $0 < \varepsilon_q < 1$. Keeping ξ fixed and expanding the metric near $q = 1$, we have

$$ds^2 = \frac{\xi}{r_S q} d\tau^2 + \frac{r_S q}{\xi} d\xi^2 + r_S^2 d\Omega^2 + \mathcal{O}(\varepsilon_q), \tag{3.126}$$

which, using a similar parametrization as before, $y^2 = \xi$, $\tilde{y}^2 = 4 r_S q y^2$, and $\tilde{\tau} = \tau/2r_S$, reduces to

$$ds^2 = \frac{\tilde{y}^2}{q^2} d\tilde{\tau}^2 + d\tilde{y}^2 + r_S^2 d\Omega^2 + \mathcal{O}(\varepsilon_q), \tag{3.127}$$

which looks very similar to the q-dependent near-horizon metric (3.119), meaning that the metric (3.124) seems to be a good candidate for $\hat{\mathcal{M}}_q$, the only difference from the near-horizon metric being the subleading terms in $\mathcal{O}(\varepsilon_q)$. To calculate the entanglement entropy, we first ignore those terms since we will take the $q \to 1$ limit in the end, and then calculate the surface term (3.121), since directly plugging in the full spacetime would lead to divergences. Instead of directly calculating the full spacetime, we first consider the metric perturbatively by keeping only the subleading terms of order ε_q. The metric coefficients can be rewritten as

$$1-\frac{r_S}{rq}=\{\varepsilon_q=1-1/q, 1/q=1-\varepsilon_q\}=1-\frac{r_S}{r}(1-\varepsilon_q)=1-\frac{r_S}{r}+\frac{r_S}{r}\varepsilon_q, \tag{3.128}$$

and

$$\begin{aligned}\left(1-\frac{r_S}{rq}\right)^{-1}&=\frac{1}{1-r_S/r+r_S/r\varepsilon}=\frac{1}{1-r_S/r}\left(\frac{1}{1+\frac{r_S}{r(1-r_S/r)}\varepsilon_q}\right)\\&=\frac{1}{1-r_S/r}\left(1-\frac{r_S}{r(1-r_S/r)}\varepsilon_q\right)=\frac{1}{1-r_S/r}-\frac{r_S}{r(1-r_S/r)^2}\varepsilon_q\end{aligned} \tag{3.129}$$

The full Euclidean metric is then given by

$$ds^2=\left(1-\frac{r_S}{r}+\frac{r_S}{r}\varepsilon_q\right)d\tau^2+\left[\frac{1}{1-r_S/r}-\frac{r_S}{r(1-r_S/r)}\varepsilon_q\right]dr^2+r^2d\Omega^2. \tag{3.130}$$

The metric can also be written as $g^{(q)}_{\mu\nu}=\bar{g}_{\mu\nu}+\delta g^{(q)}_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the background metric when $q=1$, and the perturbation vanishes at $q=1$, $\delta g^{(1)}_{\mu\nu}=0$. We can find the inverse of the metric perturbation by using $\delta g^{\mu\nu(q)}=\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\delta g^{(q)}_{\alpha\beta}$. They are given by

$$\begin{aligned}\delta g^{(q)}_{\tau\tau}&=\frac{r_S}{r}\varepsilon_q, \quad \delta g^{(q)}_{rr}=-\frac{r_S}{r(1-r_S/r)}\varepsilon_q,\\ \delta g^{\tau\tau(q)}&=\frac{r_S}{r(1-r_S/r)^2}\varepsilon_q, \quad \delta g^{rr(q)}=-\frac{r_S}{r}\varepsilon_q.\end{aligned} \tag{3.131}$$

Now we have all the components needed to calculate $\partial_q I_E[\hat{\mathcal{M}}_q]$ to leading order of ε_q,

$$\partial_q I_E[\hat{\mathcal{M}}_q]=\int_{r=r_S+\varepsilon}\frac{d^3x}{16\pi G_N}\sqrt{\gamma}\hat{n}_\rho(\bar{g}_{\mu\nu}\bar{\nabla}^\rho\partial_q\delta g^{\mu\nu(q)}-\bar{\nabla}_\mu\partial_q\delta g^{\mu\rho(q)}), \tag{3.132}$$

where γ is the induced metric of a surface with $r=r_S+\varepsilon$, $\hat{n}_\rho$ is its unit normal vector, and all components with overbars correspond to the background metric coefficients. The Euclidean time coordinate τ is integrated from 0 to β, where $\beta=4\pi r_S$, to introduce the coordinate singularity at the origin. A direct calculation gives

$$\partial_q I_E[\hat{\mathcal{M}}_q] = \frac{4\pi r_S^2}{4G_N q^2}, \tag{3.133}$$

and reproduces the earlier result. This is an alternative way to calculate the entanglement entropy without relying on the near-horizon geometry.

3.3 The heat kernel method

In this subsection, we will outline an entropy calculation method called the heat kernel method, and we will calculate the entanglement entropy for a $(D-2)$-dimensional plane Σ in D-dimensional spacetime. We will predominantly follow [9]. If we consider a quantum bosonic field that is described by a field operator $\mathcal{D}$ so that the partition function is $Z = \det^{-1/2}\mathcal{D}$, then the effective action is defined as

$$W = -\frac{1}{2}\int_{\epsilon^2}^{\infty}\frac{ds}{s}TrK(s). \tag{3.134}$$

where ϵ is the UV cutoff. The effective action is expressed in terms of the trace of the so-called heat kernel

$$K(s, X, X') = \langle X|e^{-s\mathcal{D}}|X'\rangle\,, \tag{3.135}$$

which is defined as a solution to the heat equation

$$\begin{aligned}(\partial_s + \mathcal{D})K(s, X, X') &= 0, \\ K(s = 0, X, X') &= \delta(X, X').\end{aligned} \tag{3.136}$$

In order to calculate the effective action $W(\alpha)$, where α has the same functionality as the Rényi index q in the previous subsection, indicating that the ϕ angle in a given spacetime with a conical singularity is $2\pi\alpha$ periodic. If we are working in a Lorentz invariant theory, we can use the invariance under abelian symmetry $\phi \to \phi + w$ to connect the heat kernel on a space with a conical singularity with the heat kernel on flat spacetime, which is 2π periodic with respect to the difference in angles, by applying the Sommerfeld formula [31]

$$K_\alpha(s,\phi,\phi') = K(s,\phi-\phi') + \frac{i}{4\pi\alpha}\int_\Gamma \cot\frac{w}{2\alpha} K(s,\phi-\phi'+w)dw, \tag{3.137}$$

where we have suppressed all other coordinates besides ϕ, and where the contour Γ has two lines, one from $(-\pi+i\infty)$ to $(-\pi-i\infty)$ and from $(\pi-i\infty)$ to $(\pi+i\infty)$. $\cot(w/2\alpha)$ will in general generate first order poles: $2\pi\alpha k$, where $k\in\mathbb{Z}$. In the case of a plane in flat spacetime, if the operator $\mathcal{D}$ is the Laplace operator,

$$\mathcal{D} = -\nabla^2, \tag{3.138}$$

the heat kernel function in D spacetime dimensions that satisfies (3.136) can be written in Fourier space as

$$K(s,X,X') = \frac{1}{(2\pi)^d}\int d^d p e^{ip_\mu(X^\mu - X'^\mu)} e^{-sp^2}. \tag{3.139}$$

Evaluating the integral in a hyperspherical coordinate system, such that $z^i = z'^i$ for $i=1,...,D-2$, and choosing the system to be oriented so that $\phi=\phi'+w$, we have

$$p_\mu(X-X')^\mu = 2pr\sin\frac{w}{2}\cos\theta, \quad p=\sqrt{p^\mu p_\mu}, \tag{3.140}$$

and where θ is the angle between p^μ and $(X-X')^\mu$.

The integral now has the following form

$$K(s,w,r) = \frac{\Omega_{D-1}}{(2\pi)^d}\int_0^\infty dp\, p^{D-1}\int_0^\pi d\theta \sin^{d-2}(\theta) e^{2ipr\sin\frac{w}{2}\cos(\theta)} e^{-sp^2}, \tag{3.141}$$

where

$$\Omega_{D-1} = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)} \tag{3.142}$$

is the area of the unit sphere in $D-1$ dimension. Using the substitution $x=\cos(\theta)$, we can simplify our integral

$$K(s,w,r) = \frac{\Omega_{D-1}}{(2\pi)^D}\int_0^\infty dp\, p^{D-1} e^{-sp^2}\int_{-1}^1 dx (1-x^2)^{\frac{D-3}{2}} e^{2ipr\sin\frac{w}{2}x}. \tag{3.143}$$

Using

$$\int_{-1}^{1} dx(1-x^2)^{\frac{D-3}{2}} e^{2ipr\sin\frac{w}{2\alpha}x} = 2^{\frac{D-2}{2}}\pi^{\frac{1}{2}}\Gamma\left(\frac{D-1}{2}\right)\left(2pr\sin\frac{w}{2}\right)^{\frac{2-D}{2}} J_{\frac{D-2}{2}}\left(2pr\sin\frac{w}{2}\right), \tag{3.144}$$

where $J_n(x)$ is the n-th Bessel function of the first kind, we are left with

$$K(s,w,r) = \frac{\Omega_{D-1}\sqrt{\pi}}{(2\pi)^D}\frac{\Gamma\left(\frac{D-1}{2}\right)}{\left(r\sin\frac{w}{2}\right)^{\frac{D-2}{2}}}\int_0^\infty dp\, p^{\frac{D}{2}} J_{\frac{D-2}{2}}\left(2pr\sin\frac{w}{2}\right)e^{-sp^2} \tag{3.145}$$

The trace can be found to be

$$TrK(s,w) = \frac{s}{(4\pi s)^{D/2}}\frac{\pi\alpha}{\sin^2\frac{w}{2}}A(\Sigma), \tag{3.146}$$

where $A(\Sigma) = \int d^{D-2}x$ is the area of the surface Σ. Evaluating the contour integral via residues [32], the trace of the heat kernel on a space with a conical singularity can be found to be

$$TrK_\alpha(s) = \frac{1}{(4\pi s)^{D/2}}\left(\alpha V + \frac{\pi s}{3\alpha}(1-\alpha)^2 A(\Sigma)\right), \tag{3.147}$$

where $V = \int d\tau d^{D-1}x$ is the volume of spacetime. The entropy has a similar form as (3.107), where the density matrix is not normalized, with $q \to \alpha$, and $W(\alpha) = -\log Z(\alpha)$

$$S = (\alpha\partial_\alpha - 1)W(\alpha)|_{\alpha=1}. \tag{3.148}$$

Substituting (3.147) into (3.134), we find that the effective action has the following form

$$W(\alpha) = -\frac{1}{2(4\pi)^D}\int_{\epsilon^2}^\infty \left(\frac{\alpha V}{s^{D/2+1}} + \frac{\pi}{3\alpha s^{D/2}}(1-\alpha)^2 A(\Sigma)\right). \tag{3.149}$$

The first term, proportional to the volume V, reproduces the vacuum energy in the effective action, and the second term, proportional to the area, is responsible for the entropy. Making use of (3.148), the entanglement entropy of an infinite plane Σ in

D spacetime dimensions can be shown to have the following form

$$S = \frac{A(\Sigma)}{(D-2)(4\pi)^{(D-2)/2}\epsilon^{D-2}}. \tag{3.150}$$

Since any surface can locally look like a plane, and since curved spacetime is locally Minkowski, this result gives the leading contribution to the entanglement entropy of any surface Σ in both flat and curved spacetime. If we are considering a massive field, we would have $-\nabla^2 \to -\nabla^2 + m^2$, we just need to add a factor of $e^{-m^2 s}$ to the massless case (3.139), which plays a role when we need to evaluate the following integral

$$S = \frac{A(\Sigma)}{12(4\pi)^{(D-2)/2}} \int_{\epsilon^2}^{\infty} \frac{ds}{s^{d/2}} e^{-m^2 s}. \tag{3.151}$$

In the case of $D = 4$ this integral leads to a logarithmic divergence in the UV cutoff,

$$S = \frac{A(\Sigma)}{48\pi} \left(\frac{1}{\epsilon^2} + 2m^2 \ln \epsilon + m^2 \ln m^2 + m^2(\gamma - 1) \right), \tag{3.152}$$

similarly to the result obtained using the brick wall model (2.81).

4 One-loop renormalization of the gravitational action

In this chapter, we will connect the divergences in the entropy with the renormalization of the coupling constants in the theory by examining a scalar field propagating in a 4-dimensional, non-extremal Reissner-Nordström black hole background. We will consider the renormalization of the coupling constants in the gravitational action by a quantum scalar field theory. We will regulate the scalar field loops by using a Pauli-Villars renormalization scheme and determine the renormalization of Newton's constant. We will apply this regularization scheme in the brick wall model, and the Pauli-Villars regulator will be shown to implement a cutoff for the entropy calculation, removing the need for the brick wall. This way, we can directly compare the divergences appearing in the entropy and in the effective action. This chapter will primarily follow [9, 33], and we will use $\hbar = 1$.

Let us consider the gravitational action

$$I_g = \int d^4x\sqrt{-g}\left[-\frac{\Lambda_B}{8\pi 8G_B} + \frac{R}{16\pi G_B} + \frac{\alpha_B}{4\pi}R^2 + \frac{\beta_B}{4\pi}R_{ab}R^{ab} + \frac{\gamma_B}{4\pi}R_{abcd}R^{abcd} + \dots\right], \tag{4.1}$$

where Λ_B is the cosmological constant, G_B Newton's constant, and $\alpha_B, \beta_B, \gamma_B$ the dimensionless coupling constants for the quadratic curvature interactions. The B index just denotes that they are the bare constants, and the remaining terms denote higher order curvature terms, along with derivative interactions. The action for a minimally couple scalar field is

$$I_m = -\frac{1}{2}\int d^4x\sqrt{-g}\left[g^{ab}\nabla_a\phi\nabla_b\phi + m^2\phi^2\right]. \tag{4.2}$$

We wish to determine the effective action for the metric. To do so, we first obtain the partition function after we integrate out the scalar field in the path integral.

$$Z(g) = \int \mathcal{D}\phi e^{-iI_m[\phi,g]} \tag{4.3}$$

Performing this integration [34], we find the effective action to be

$$W(g) = -\frac{i}{2} Tr(\log[-G_F(g, m^2)], \tag{4.4}$$

where $G_F(g, m^2)$ is the Feynmann propagator of a field with mass m, and where the background metric is g. This expression is divergent, and we must regulate it for it to be properly defined. We have a representation of the scalar one-loop action as an asymptotic series [35,36]

$$W(g) = -\frac{1}{32\pi^2} \int d^4x \sqrt{-g} \int_0^\infty \frac{ds}{s^3} \sum_{n=0}^{\infty} a_n(x)(is)^n e^{-im^2 s}, \tag{4.5}$$

where the $a_n(x)$ coefficients are functionals of the local geometry at x, e.g.

$$a_0(x) = 1, \quad a_1(x) = \frac{1}{6}R, \quad a_2(x) = \frac{1}{180}R^{abcd}R_{abcd} - \frac{1}{180}R^{ab}R_{ab} + \frac{1}{30}\Box R + \frac{1}{72}R^2. \tag{4.6}$$

The ultraviolet divergences rise as $s \to 0$ in the first three terms of the above series. To regulate this effective action, we use a Pauli-Villars regularization scheme, where we introduce fictious fields with large masses that are set by a certain ultraviolet cutoff scale. Some of these regulator fields are quantized using incorrect statistics, so that their contributions in loops cancel those of the remaining fields. All of these features are chosen so that all the ultraviolet divergences become finite. Specifically, we introduce five regulator fields: ϕ_1 and ϕ_2 which are two anticommuting fields with the same mass, $m_{1,2} = \sqrt{\mu^2 + m^2}$; ϕ_3 and ϕ_4 which are two commuting fields with mass, $m_{3,4} = \sqrt{3\mu^2 + m^2}$; and ϕ_5 which is an anticommuting field with mass, $m_5 = \sqrt{4\mu^2 + m^2}$. The total action for the matter fields is then given as

$$I_m = -\frac{1}{2} \sum_{i=0}^{5} \int d^4x \sqrt{-g} [g^{ab} \nabla_a \phi_i \nabla_b \phi_i + m_i^2 \phi_i^2], \tag{4.7}$$

where ϕ_0 is the original field ϕ with mass m. Each field makes a contribution to the effective action, where the commuting fields are contributing as (4.4) and the anticommuting fields as

$$W(g) = +\frac{i}{2} Tr(\log[-G_F(g, m^2)]). \tag{4.8}$$

The divergent terms in the effective action are then

$$
\begin{aligned}
W_{div} = &-\frac{1}{32\pi^2}\int d^4x\sqrt{-g}\int_0^\infty \frac{ds}{s^3}[a_0(x)+isa_1(x)+(is)^2a_2(x)] \\
&\times\left[e^{-im^2s}-2e^{-i(\mu^2+m^2)s}+2e^{-i(3\mu^2+m^2)s}-e^{-i(4\mu^2+m^2)s}\right] \\
&=\frac{1}{32\pi^2}\int d^4x\sqrt{-g}[-Ca_0(x)+Ba_1(x)+Aa_2(x)].
\end{aligned}
\tag{4.9}
$$

Where we have defined A, B, C as the s integrals in front of $a_2(x)$, $a_1(x)$ and $a_0(x)$ respectively,

$$
A=\int_0^\infty \frac{ds}{s}\left[e^{-im^2s}-2e^{-i(\mu^2+m^2)s}+2e^{-i(3\mu^2+m^2)s}-e^{-i(4\mu^2+m^2)s}\right], \tag{4.10}
$$

$$
B=-i\int_0^\infty \frac{ds}{s^2}\left[e^{-im^2s}-2e^{-i(\mu^2+m^2)s}+2e^{-i(3\mu^2+m^2)s}-e^{-i(4\mu^2+m^2)s}\right], \tag{4.11}
$$

$$
C=\int_0^\infty \frac{ds}{s^3}\left[e^{-im^2s}-2e^{-i(\mu^2+m^2)s}+2e^{-i(3\mu^2+m^2)s}-e^{-i(4\mu^2+m^2)s}\right]. \tag{4.12}
$$

We can notice that we have three classes of integrals that appear in the calculation of the constants A, B, C:

$$
\int_0^\infty \frac{e^{-i(a\mu^2+m^2)s}}{s^n}, \quad n=1,2,3 \tag{4.13}
$$

They can be evaluated as indefinite integrals, and then, by looking at the asymptotic behavior at $s=0$ and $s=\infty$, we can keep the dominant terms. In the case of $n=1$, we have to look at the following type of integral

$$
\int \frac{e^{-i(a\mu^2+m^2)x}}{x}dx = E_i(-ix(a\mu^2+m^2))+C, \tag{4.14}
$$

where $E_i(x)$ is the exponential integral

$$
E_i(x) = -\int_{-x}^\infty \frac{e^{-t}}{t}dt. \tag{4.15}
$$

The integral (4.14) has the following asymptotic behavior at $x=0$

$$
\lim_{x\to 0} E_i(-ix(a\mu^2+m^2)) = \frac{1}{2}\left(2\ln(x)+\ln\left(-(a\mu^2+m^2)^2\right)+2\gamma\right), \tag{4.16}
$$

where γ is the Euler-Mascheroni constant. And the following behavior for $x = \infty$

$$\frac{1}{2}e^{-ix(a\mu^2+m^2)}\left(\frac{2i}{(a\mu^2+m^2)x} + \mathcal{O}\left(\left(\frac{1}{x}\right)^2\right)\right), \tag{4.17}$$

which obviously vanishes at infinity. Adding together the contributions for $a = 0, 1, 3, 4$ in (4.10), the diverging $\ln(x)$ contributions in (4.16) all cancel out, and the formula for A is given by

$$A = -\frac{1}{2}\ln(-m^4) + \ln(-(\mu^2+m^2)^2) - \ln(-(3\mu^2+m^2)^2) + \frac{1}{2}\ln(-(4\mu^2+m^2))^2), \tag{4.18}$$

and can be rewritten as

$$A = \ln\frac{4\mu^2+m^2}{m^2} + 2\ln\frac{\mu^2+m^2}{3\mu^2+m^2}. \tag{4.19}$$

To evaluate B, we look at the case of $n = 2$ in (4.13), which is the following type of integral

$$\int \frac{e^{-i(a\mu^2+m^2)x}}{x^2}dx = -i(a\mu^2+m^2)E_i(-i(a\mu^2+m^2)x) - \frac{e^{-i(a\mu^2+m^2)x}}{x} + C. \tag{4.20}$$

The term $\frac{e^{-i(a\mu^2+m^2)x}}{x}$ goes to zero as $x \to \infty$, and has the following asymptotic behavior at $x = 0$

$$\lim_{x\to 0}\frac{e^{-i(a\mu^2+m^2)x}}{x} = \frac{1}{x} - i(a\mu^2+m^2). \tag{4.21}$$

Thus, the main contribution to the integral (4.20) when plugging in the bounds are

$$\begin{aligned}\int_0^\infty \frac{e^{-i(a\mu^2+m^2)x}}{x^2}dx = &\frac{i}{2}(a\mu^2+m^2)(2\ln(x) + \ln(-(a\mu^2+m^2)^2) + 2\gamma)\\ &-\frac{1}{x} + i(a\mu^2+m^2)\end{aligned} \tag{4.22}$$

Adding up this type of integral for $a = 0, 1, 3, 4$ in (4.11), the diverging terms $\ln(x)$ and $1/x$ all cancel out, and the formula for B is given by

$$B = \frac{1}{2}(m^2 \ln(-m^4) - 2(\mu^2 + m^2)\ln(-(\mu^2 + m^2)^2) \\ + 2(3\mu^2 + m^2)\ln(-(3\mu^2 + m^2)^2) - (4\mu^2 + m^2)\ln(-(4\mu^2 + m^2)^2)). \tag{4.23}$$

After grouping it in terms of μ^2 and m^2, we are finally left with

$$B = \mu^2 \left[2\ln\frac{3\mu^2 + m^2}{\mu^2 + m^2} + 4\ln\frac{3\mu^2 + m^2}{4\mu^2 + m^2}\right] + m^2 \left[\ln\frac{m^2}{4\mu^2 + m^2} + 2\ln\frac{3\mu^2 + m^2}{\mu^2 + m^2}\right]. \tag{4.24}$$

We can apply the same principles for the $n = 3$ case,

$$\int \frac{e^{-i(a\mu^2+m^2)x}}{x^3} = -\frac{1}{2}(a\mu^2 + m^2)^2 E_i(-i(a\mu^2 + m^2)) + \frac{ie^{-i(a\mu^2+m^2)x}((a\mu^2 + m^2)x + i)}{2x^2}. \tag{4.25}$$

Using the following asymptotic behavior,

$$\lim_{x\to 0} \frac{e^{-i(a\mu^2+m^2)x}}{x^2} = \frac{1}{x^2} - \frac{ia}{x} - \frac{a^2}{2}, \tag{4.26}$$

we can write the main contribution to the integral (4.25) when plugging in the bounds

$$\int_0^\infty \frac{e^{-i(a\mu^2+m^2)x}}{x^3} = \frac{1}{4}(a\mu^2 + m^2)^2(2\ln(x) + \ln(-(a\mu^2 + m^2)^2) + 2\gamma) \\ -\frac{i}{2}(a\mu^2 + m^2)\left(\frac{1}{x} - i(a\mu^2 + m^2)\right) + \frac{1}{2}\left(\frac{1}{x^2} - \frac{i(a\mu^2 + m^2)}{x} - \frac{(a\mu^2 + m^2)^2}{2}\right) \tag{4.27}$$

Finally, it can be shown that, after adding all the different terms for $a = 0, 1, 3, 4$ in (4.12), the diverging terms $\ln(x)$, $1/x$, and $1/x^2$ all cancel out, and C has the following form

$$C = \mu^4 \left[8\ln\frac{3\mu^2 + m^2}{4\mu^2 + m^2} + \ln\frac{3\mu^2 + m^2}{\mu^2 + m^2}\right] + 2m^2\mu^2 \left[\ln\frac{3\mu^2 + m^2}{\mu^2 + m^2} + \ln\frac{3\mu^2 + m^2}{4\mu^2 + m^2}\right] \\ + \frac{m^4}{2}\left[\ln\frac{m^2}{4\mu^2 + m^2} + 2\ln\frac{3\mu^2 + m^2}{\mu^2 + m^2}\right]. \tag{4.28}$$

Combining the scalar one-loop action with the bare action, we obtain the renormalized coupling constants in the effective gravitational action

$$
\begin{aligned}
I_{eff} &= I_g + W \qquad (4.29)\\
&= \int d^4x\sqrt{-g}\Big[-\frac{1}{8\pi}\left(\frac{\Lambda_B}{G_B}+\frac{C}{4\pi}\right)+\frac{R}{16\pi}\left(\frac{1}{G_B}+\frac{B}{12\pi}\right)+\frac{R^2}{4\pi}\left(\alpha_B+\frac{A}{576\pi}\right)\\
&\quad+\frac{1}{4\pi}R_{ab}R^{ab}\left(\beta_B-\frac{A}{1440\pi}\right)+\frac{1}{4\pi}R_{abcd}R^{abcd}\left(\gamma_B+\frac{A}{1440\pi}\right)+\dots\Big].
\end{aligned}
$$

We have discarded the total derivative term of $\Box R$ in $a_2(x)$, since it vanishes at the boundary. We can also identify the renormalized Newton's constant

$$\frac{1}{G_R}=\frac{1}{G_B}+\frac{B}{12\pi}. \qquad (4.30)$$

Note that for large values of μ, the constants A, B, C have leading order terms of the form $\ln\mu/m$, μ^2 and μ^4 respectively, as well as subleading contributions (those of lower powers in μ), and finite contributions (those without μ). Another thing to note is that, since we will be considering the entropy of a Reissner-Nordström black hole, we should expect a $U(1)$ gauge potential as well as a coupling between the gauge field and the metric. However, since we are only considering a neutral scalar field, in the effective action, the gauge field interaction will be unaffected by the scalar one-loop contribution. We can apply this procedure of introducing Pauli-Villars fields to calculate the entropy by looking at the calculation for the entropy of a Reissner-Nordström black hole using the brick wall method for a massive field. Using (2.39), (2.40), and (2.90),

$$N(E)=\frac{1}{\pi}\int_{r_{\rm H}+h}^{L}dr\int_0^{\ell_{max}}d\ell(2\ell+1)P(r), \qquad (4.31)$$

$$P(r)=\frac{1}{g(r)}\left[E^2-g(r)\left(\frac{\ell(\ell+1)}{r^2}+m^2\right)\right]^{1/2}, \qquad (4.32)$$

$$g(r)=1-\frac{2GM}{r}+\frac{Q^2G}{r^2}=\left(1-\frac{r_-}{r}\right)\left(1-\frac{r_+}{r}\right), \qquad (4.33)$$

$$r_\pm=GM\pm\sqrt{G^2M^2-GQ^2}, \quad r_{\rm H}=r_+, \quad m\neq 0,$$

for the free energy we have

$$F=-\frac{1}{\pi}\int_0^\infty\frac{dE}{e^{\beta E}-1}\int_{r_++h}^{L}dr\frac{1}{g(r)}\int_0^{\ell_{max}}d\ell(2\ell+1)\left[E^2-g(r)\left(\frac{\ell(\ell+1)}{r^2}+m^2\right)\right]^{1/2} \tag{4.34}$$

where $\ell_{\max}$ has the value such that $P(r)$ is real,

$$\ell_{max}(\ell_{max}+1)=r^2\left(\frac{E^2}{g(r)}-m^2\right) \tag{4.35}$$

We can execute the ℓ integration by using the substitution $\Lambda=\ell(\ell+1)$,

$$\int_0^{r^2\left(\frac{E^2}{g(r)}-m^2\right)}d\Lambda\left[E^2-g(r)\left(\frac{\Lambda}{r^2}+m^2\right)\right]^{1/2}=\frac{2r^2}{3g(r)}(E^2-g(r)m^2)^{3/2}. \tag{4.36}$$

The free energy can then be rewritten as

$$F=-\frac{2}{3\pi}\int_0^\infty\frac{dE}{e^{\beta E}-1}\int_{r_++h}^{L}dr\frac{1}{g(r)^2}r^2\left[E^2-g(r)m^2\right]^{3/2}. \tag{4.37}$$

The integral has a double pole at $r=r_+$, which is regulated with the brick wall, h. By introducing a new variable, $s=1-r_+/r$, our integral assumes the following form

$$F=-\frac{2r_+^3}{3\pi}\int_0^\infty\frac{dE}{e^{\beta E}-1}\int_{h'}^{L'}\frac{ds}{s^2(1-s)^4(1-u+us)^2}[E^2-s(1-u+us)m^2]^{3/2}, \tag{4.38}$$

where $u=r_-/r_+$, $L'=1-r_+/L$ and $h'=h/(r_++h)\simeq h/r_+$. For small values of s, we have $\int_{h'}ds/s^2\simeq 1/h'$, which diverges as we pull the brick wall back to the horizon $h'\to 0$ as expected. We are repeating the 't Hooft calculation for the Pauli-Villars regulated field theory, where each field makes a contribution to the free energy. The total free energy is now

$$\bar{F}=-\frac{2r_+^3}{3\pi}\sum_{i=0}^{5}\Delta_i\int_0^\infty\frac{dE}{e^{\beta E}-1}\int_{h'}^{L'}\frac{ds}{s^2(1-s)^4(1-u+us)^2}[E^2-s(1-u+us)m_i^2]^{3/2}, \tag{4.39}$$

where $\Delta_0=\Delta_3=\Delta_4=+1$, for the commuting fields, and $\Delta_1=\Delta_2=\Delta_5=-1$ for the anticommuting fields. The free energy of the anticommuting regulator fields comes with a minus sign with respect to the commuting fields, as is required since the role of these fields is to cancel the contribution of very high energy modes in the regulated theory. For small s, we now have $\sum_{i=0}^{5} \Delta_i \int_{h'} ds/s^2 = 0$, since there are 3 commuting and 3 anticommuting fields. The sub-leading logarithmic divergence at small s is also cancelled since $\sum_{i=0}^{5} \Delta_i m_i^2 = 0$. Meaning, in the Pauli-Villars regulated theory, we are free to remove the brick wall, setting $h' = 0$, since it would not appear in the near-horizon (small s) integration. The total free energy is then

$$\bar{F} = -\frac{2r_+^3}{3\pi} \int_0^\infty \frac{dE}{e^{\beta E} - 1} \int_0^{L'} \frac{ds}{s^2(1-s)^4(1-u+us)^2} \sum_{i=0}^{5} \Delta_i [E^2 - s(1-u+us)m_i^2]^{3/2}. \tag{4.40}$$

For small s, we can use the following approximations,

$$\begin{gathered}
\frac{1}{(1-s)^4} \simeq 1+4s, \quad 1-u+us \simeq 1-u, \quad \frac{1}{(1-u+us)^2} \simeq \frac{1}{(1-u)^2}\left(1 - \frac{2us}{1-u}\right), \\
\frac{1}{s^2(1-s)^4(1-u+us)^2} \simeq \frac{(1+4s)\left(1-\frac{2us}{1-u}\right)}{s^2(1-u)^2} \simeq \frac{1+4s-\frac{2us}{1-u}}{s^2(1-u)^2} = \frac{1}{s^2(1-u)^2} + \frac{2(2-3u)}{s(1-u)^3}, \\
\left(1 - \frac{s(1-u)m_i^2}{E^2}\right)^{3/2} \simeq 1 - \frac{3s(1-u)m_i^2}{2E^2}.
\end{gathered} \tag{4.41}$$

The total free energy is now

$$\begin{aligned}
\bar{F} &= -\frac{2r_+^3}{3\pi} \int_0^\infty \frac{dE E^3}{e^{\beta E} - 1} \int_0^{L'} ds \left(\frac{1}{s^2(1-u)^2} + \frac{2(2-3u)}{s(1-u)^3}\right) \sum_{i=0}^{5} \Delta_i \left(1 - \frac{3s(1-u)m_i^2}{2E^2}\right) \\
&\equiv \frac{2r_+^3}{3\pi} \int_0^\infty \frac{dE E^3}{e^{\beta E} - 1} \left(\frac{1}{(1-u)^2}\tilde{B} + \frac{2(2-3u)}{(1-u)^3}\tilde{A}\right),
\end{aligned} \tag{4.42}$$

where

$$\tilde{B} = -\sum_{i=0}^{5} \Delta_i \int_0^{L'} \frac{ds}{s^2} \left(1 - \frac{3s(1-u)m_i^2}{2E^2}\right), \tag{4.43}$$

$$\tilde{A} = -\sum_{i=0}^{5} \Delta_i \int_0^{L'} \frac{ds}{s} \left(1 - \frac{3s(1-u)m_i^2}{2E^2}\right). \tag{4.44}$$

We can connect these quantities to B and A from (4.24) and (4.19), respectively. The definition of B is

$$B = -i \sum_{i=0}^{5} \Delta_i \int_0^\infty \frac{ds}{s^2} e^{-im_i^2 s} = \{\text{small } s\} \simeq -i \sum_{i=0}^{5} \Delta_i \int_0^\infty (1 - im_i^2 s). \tag{4.45}$$

Using the substitution

$$ix = -\frac{3s(1-u)}{2E^2}, \quad ds = \frac{2iE^2}{3(1-u)} dx, \quad s^2 = -x^2 \frac{4}{9} E^4 \frac{1}{(1-u)^2}, \tag{4.46}$$

$\tilde{B}$ can be written as

$$\tilde{B} = -\frac{3}{2}\frac{(1-u)}{E^2}(-i) \sum_{i=0}^{5} \int_0^\infty \frac{dx}{x^2}(1 - ixm_i^2) = -\frac{3}{2}\frac{(1-u)}{E^2} B. \tag{4.47}$$

Using similar arguments, we find the connection between A and $\tilde{A}$ to be $\tilde{A} = -A$. The total free energy can be expressed as

$$\bar{F} = -\frac{2r_+^3}{3\pi} \int_0^\infty \frac{dEE^3}{e^{\beta E} - 1} \left(\frac{1}{(1-u)^2} \frac{3(1-u)}{2E^2} B + \frac{2(2-3u)}{(1-u)^3} A \right), \tag{4.48}$$

Using

$$\int_0^\infty \frac{dEE}{e^{\beta E} - 1} = \frac{\pi^2}{6\beta^2}, \quad \int_0^\infty \frac{dEE^3}{e^{\beta E} - 1} = \frac{\pi^4}{15\beta^4} \tag{4.49}$$

The total free energy is then

$$\bar{F} = -r_+^3 \left[\frac{\pi}{6(1-u)\beta^2} B + \frac{4\pi^3(2-3u)}{45(1-u)^3\beta^4} A \right]. \tag{4.50}$$

The entropy can now be finally calculated as

$$S = \beta^2 \frac{\partial \bar{F}}{\partial \beta} = r_+^3 \left[\frac{\pi}{3(1-u)\beta} B + \frac{16(2-3u)\pi^3}{45(1-u)^3\beta^3} A \right] \tag{4.51}$$

If we evaluate this at the inverse Hawking temperature,

$$\beta_H = \frac{4\pi r_+}{1-u}, \tag{4.52}$$

the entropy is

$$S = \frac{\mathcal{A}}{4}\frac{B}{12\pi} + \frac{(2-3u)A}{180}, \tag{4.53}$$

where $\mathcal{A} = 4\pi r_+^2$ is the surface area of the event horizon. We see that the entropy has the constants A and B, which give the dependence on the regulator mass μ appearing in the renormalization of Newton's constant. Adding this entropy to the standard Bekenstein-Hawking entropy, we have

$$S_{BH} + S = \frac{\mathcal{A}}{4}\left(\frac{1}{G_B} + \frac{B}{12\pi}\right) + \frac{(2-3u)A}{180} = \frac{\mathcal{A}}{4G_R} + \frac{(2-3u)A}{180}. \tag{4.54}$$

We find that the first contribution proportional to B in the scalar field entropy provides the one-loop renormalization of the Bekenstein-Hawking entropy. To account for the constant term, we expect that this contribution to the entropy is related to the quadratic curvate interaction in the action. This part of the total action can be written as

$$I_2 = \int d^4x\sqrt{-g}\left[\frac{\alpha_B}{4\pi}R^2 + \frac{\beta_B}{4\pi}R_{ab}R^{ab} + \frac{\gamma_B}{4\pi}R_{abcd}R^{abcd}\right], \tag{4.55}$$

and it was found that the entropy contribution has the form [37]

$$\Delta S = -8\pi u\beta_B + 16\pi(1-2u)\gamma_B. \tag{4.56}$$

Including this contribution, the total entropy is

$$\begin{aligned} S_{total} &= S_{BH} + \Delta S + S \qquad (4.57)\\ &= \frac{\mathcal{A}}{4}\left(\frac{1}{G_B} + \frac{B}{12\pi}\right) - 8\pi u\left(\beta_B - \frac{A}{1440\pi}\right) + 16\pi(1-2u)\left(\gamma_B + \frac{A}{1440\pi}\right)\\ &= \frac{\mathcal{A}}{4G_R} - 8\pi u\beta_R + 16\pi(1-2u)\gamma_R \end{aligned}$$

Both terms in the scalar field entropy account for the scalar one-loop renormalization of the full black hole entropy. Furthermore, the divergences appearing in 't Hooft's statistical-mechanical calculation of black hole entropy are precisely the quantum field theory divergences associated with the renormalization of the coupling constants appearing in the expressions of the entropy.

So far in this discussion, we have considered a minimally coupled scalar field. We would like to know whether the obtained results hold for arbitrary field theories coupled to gravity. If we consider a non-minimally couple scalar field, the matter action has the form

$$I'_m = -\frac{1}{2}\int d^4x\sqrt{-g}[g^{ab}\nabla_a\phi\nabla_b\phi + m^2\phi^2 + \xi R\phi^2]. \tag{4.58}$$

This additional coupling to the curvature modifies the adiabatic expansion coefficients (4.6) [34], which in turn affects the renormalization of the bare coupling constants. The renormalized Newton's constant now looks like

$$\frac{1}{G_R} = \frac{1}{G_B} + \frac{B}{2\pi}\left(\frac{1}{6} - \xi\right), \tag{4.59}$$

and the new equation of motion for the scalar field has the form

$$(\Box - m^2 - \xi R)\phi = 0. \tag{4.60}$$

However, since $R = 0$ for the RN metric, the equation of motion for the scalar field has the same form as in the minimally coupled case, meaning the calculations end up the same. Since Newton's constant is renormalized as (4.59), and the entropy is independent of ξ, it means that the entropy does not properly account for the renormalization of the Bekenstein-Hawking formula. This shows the limitation in 't Hooft's brick wall model, as it does not capture the full physics of the problem.

5 Concluding remarks

In this thesis, we researched and analyzed some methods of calculating the black hole entropy and its corrections.

We systematically generalized 't Hooft's brick wall method to black holes of arbitrary dimension and derived the entropy formula for an electrically neutral black hole in 4 dimensions up to the second order of the WKB approximation by including only the dominant near-horizon terms. We found that the quantum correction to the Bekenstein-Hawking entropy is proportional to the logarithm of the black hole area. We evaluated this entropy for the Schwarzschild black hole, and we found that the factor in front of the logarithm is constant. We expanded the consideration to a charged spacetime with a charged probe, and using the minimal substitution, we found that the radial equation is the same as in the non-charged case, up to a transformation of energy. We then evaluated the entropy up to the second order for the Reissner-Nordström black hole, and by taking the chargeless limits, we confirmed that the entropy reduces to that of Schwarzschild.

We explained in detail a method for calculating the entanglement entropy of a black hole by modeling a scalar field as a collection of coupled oscillators on a lattice of spaced points. After finding the wave function of the ground state for such a system, we constructed its density matrix. By tracing this density matrix outside the black hole, we constructed the reduced density matrix and found it to be given by a Gaussian matrix. We rewrote this reduced density matrix as a multiplication of density matrices. The entropy of such a density matrix was then given as a sum of the entropies of each matrix in the multiplication. When applying this formalism to a sphere of radius R, the entropy was found to be proportional to the area, similar to the Bekenstein-Hawking law. We inspected the continuum case and sketched the method for calculating the eigenvalues of the coupling operator between two oscillators, which appear in the formula for the entropy. Since these types of calculations are difficult to perform, we outlined two other methods. In the replica trick method, we considered a Rényi entropy, which is a one-parameter generalization of the entanglement entropy. We obtained the entanglement entropy after the parameter q was analytically continued to the real numbers and the limit where it goes to unity was applied. This way, we reduced our problem of finding the entropy to calculating the partition function of a replicated manifold by using the path integral. We considered a q-copy of the original manifold. After setting a periodicity condition, we introduced a conical singularity at the origin. By using the saddle point approximation and Wick rotating to Euclidean time, we obtained the entanglement entropy for the Schwarzschild black hole. This result reproduces the Bekenstein-Hawking formula and does not depend on Euclidean time, meaning it is also valid for Lorentzian spacetime. In the heat kernel method, we outlined an entropy calculation for the entanglement entropy of a plane embedded in a higher-dimensional spacetime. The effective action was defined by the trace of the so-called heat kernel, which is defined as a solution to the heat equation for a given operator. In the massless case, we showed that the entanglement entropy reproduced the area law and that it is quadratically divergent in the UV cutoff. In the massive case, we obtained, alongside a quadratically divergent term, a logarithmically divergent term, similarly to what we obtained with the brick wall model before.

The last method of calculating the black hole entropy that we considered was the one-loop renormalization of the gravitational action. We considered the gravitational action, which had terms up to the second order in curvature. We found the effective action for a minimally coupled field, which we represent as an asymptotic series. Since this series is divergent, we use a Pauli-Villars regularization scheme, where we introduce five fictitious fields with different masses and statistics. Adding this regulized effective one-loop action with the gravitational action, we obtain the renormalized coupling constants in the effective gravitational action. We apply this procedure of introducing Pauli-Villars fields to calculate the entropy of a Reissner-Nordström black hole using the brick wall method for a massive field. We find that in this regulized theory, we can remove the brick wall. After we add the brick wall entropy to the Bekenstein-Hawking entropy and the entropy of the quadratic curvature interaction, we see that all terms in the scalar field entropy account for the scalar one-loop renormalization of the full black hole entropy. We also notice that in the non-minimal coupling case, the entropy does not account for the renormalization of the Bekenstein-Hawking formula.

Now let us list out some potential future applications of the methods that we described in this thesis. Using the brick wall method, we can calculate entropies for lower-dimensional black holes, such as the BTZ, QBTZ (charged BTZ), and NCQBTZ (noncommutative charged BTZ) black holes and see how they behave. Using the replica trick, we could extend our calculation to include charge and apply it to a Reissner-Nordström black hole. Finally, using the one-loop renormalization method, we could include higher order terms in curvature and see if they could remedy the failure of the entropy to remormalize the Bekenstein-Hawking formula.

Appendices

Appendix A Radial equation for a free scalar field

We are solving the massive Klein-Gordon equation in $(D+2)$-dimensional spacetime,

$$\left(-\frac{1}{f(r)}\partial_{tt} + \nabla^2_{D+1} - \frac{m^2}{\hbar^2}\right)\Phi = 0. \tag{A.1}$$

Due to the rotational symmetry of the spacetime (2.2), we can use the ansatz [24]

$$\Phi = e^{-iEt/\hbar}\frac{R(r)}{r^{D/2}G(r)^{1/2}}Y_{\ell m_i}(\theta,\phi_i), \tag{A.2}$$

where $G(r) = \sqrt{f(r)g(r)}$, $i \in \{1, ..., (D-1)\}$ and $Y_{\ell m_i}(\theta,\phi_i)$ denote the hyperspherical harmonics. By plugging in (A.2) into (A.1), we have

$$\begin{aligned} -\frac{1}{f(r)}\frac{R(r)}{r^{D/2}G(r)^{1/2}}Y_{\ell m_i}(\theta,\phi_i)\partial_{tt}\left(e^{-iEt/\hbar}\right) + e^{-iEt/\hbar}\nabla^2_{D+1}\left(\frac{R(r)}{r^{D/2}G(r)^{1/2}}Y_{\ell m_i}(\theta,\phi_i)\right) \\ -\frac{m^2}{\hbar^2}\left(e^{-iEt/\hbar}\frac{R(r)}{r^{D/2}G(r)^{1/2}}Y_{\ell m_i}(\theta,\phi_i)\right) = 0. \end{aligned} \tag{A.3}$$

After differentiating twice with respect to t and dividing the entire equation with $e^{-iEt/\hbar}$, we are left with

$$\begin{aligned} \frac{1}{f(r)}\frac{E^2}{\hbar^2}\frac{R(r)}{r^{D/2}G(r)^{1/2}}Y_{\ell m_i}(\theta,\phi_i) + Y_{\ell m_i}(\theta,\phi_i)\nabla^2_{D+1}\left(\frac{R(r)}{r^{D/2}G(r)^{1/2}}\right) \\ +\frac{R(r)}{r^{D/2}G(r)^{1/2}}\nabla^2_{D+1}\left(Y_{\ell m_i}(\theta,\phi_i)\right) - \frac{m^2}{\hbar^2}\left(\frac{R(r)}{r^{D/2}G(r)^{1/2}}Y_{\ell m_i}(\theta,\phi_i)\right) = 0. \end{aligned} \tag{A.4}$$

Using [24]

$$\nabla^2_{D+1}Y_{\ell m_i}(\theta,\phi_i) = -\frac{\ell(\ell+D-1)}{r^2}Y_{\ell m_i}(\theta,\phi_i), \tag{A.5}$$

and dividing by $Y_{\ell m_i}(\theta,\phi_i)$, (A.4) simplifies to

$$\begin{aligned} \frac{1}{f(r)}\frac{E^2}{\hbar^2}\frac{R(r)}{r^{D/2}G(r)^{1/2}} + \nabla^2_{D+1}\left(\frac{R(r)}{r^{D/2}G(r)^{1/2}}\right) \\ +\frac{R(r)}{r^{D/2}G(r)^{1/2}}\left(-\frac{\ell(\ell+D-1)}{r^2}\right) - \frac{m^2}{\hbar^2}\left(\frac{R(r)}{r^{D/2}G(r)^{1/2}}\right) = 0. \end{aligned} \tag{A.6}$$

Regrouping and using

$$\nabla^2_{D+1}\left(\frac{R(r)}{r^{D/2}G(r)^{1/2}}\right)=\frac{1}{\sqrt{-g}}\partial_r\left(g^{rr}\sqrt{-g}\partial_r\left(\frac{R(r)}{r^{D/2}G(r)^{1/2}}\right)\right), \tag{A.7}$$

plugging $g^{rr}=g(r)$ and (2.16) into (A.7), (A.6) becomes

$$\begin{aligned}&\left(\frac{1}{f(r)}\frac{E^2}{\hbar^2}-\frac{\ell(\ell+D-1)}{r^2}-\frac{m^2}{\hbar^2}\right)\frac{R(r)}{r^{D/2}G(r)^{1/2}}\\ &+\sqrt{\frac{g(r)}{f(r)r^{2D}}}\partial_r\left(g(r)\sqrt{\frac{f(r)r^{2D}}{g(r)}}\partial_r\left(\frac{R(r)}{r^{D/2}G(r)^{1/2}}\right)\right)=0.\end{aligned} \tag{A.8}$$

We introduce a new quantity, which is interpreted in the main body of the text

$$V^2(r)=\frac{1}{G^2(r)}\left(E^2-f(r)\left[m^2+\left(\frac{\ell(\ell+D-1)\hbar^2}{r^2}\right)\right]\right). \tag{A.9}$$

The first term in (A.8), after factoring out $1/f(r)\hbar^2$ reduces to

$$\frac{G^2(r)V^2(r)}{f(r)\hbar^2}\frac{R(r)}{r^{D/2}G(r)^{1/2}}=\frac{g(r)V^2(r)}{\hbar^2}\frac{R(r)}{r^{D/2}G(r)^{1/2}}, \tag{A.10}$$

after multiplying (A.8) with $r^{D/2}G(r)^{1/2}/g(r)$, its first term is then given by

$$\frac{V^2(r)}{\hbar^2}R(r), \tag{A.11}$$

while the factor in front of the second term is given by

$$\frac{r^{D/2}G(r)^{1/2}}{g(r)}\sqrt{\frac{g(r)}{f(r)r^{2D}}}=\frac{r^{D/2}G(r)^{1/2}}{\sqrt{g(r)f(r)}r^D}=\{G(r)=\sqrt{f(r)g(r)}\}=\frac{1}{G(r)^{1/2}r^{D/2}}. \tag{A.12}$$

We can now rewrite the second term as

$$\begin{aligned}&\frac{1}{G(r)^{1/2}r^{D/2}}\partial_r\left(G(r)r^D\partial_r\left(\frac{R(r)}{r^{D/2}G(r)^{1/2}}\right)\right)\\ &=\frac{1}{G(r)^{1/2}r^{D/2}}\partial_r\left(G(r)^{1/2}r^{D/2}R'(r)-\frac{D}{2}r^D\frac{G(r)^{1/2}R(r)}{r^{D/2+1}}-\frac{1}{2}G(r)\frac{r^{D/2}R(r)}{G(r)^{3/2}}G'(r)\right)\\ &=\frac{1}{G(r)^{1/2}r^{D/2}}\partial_r\left(G(r)^{1/2}r^{D/2}R'(r)-\frac{D}{2}r^{D/2-1}G(r)^{1/2}R(r)-\frac{1}{2}\frac{r^{D/2}R(r)}{G(r)^{1/2}}G'(r)\right).\end{aligned} \tag{A.13}$$

The first term in the upper equation is then

$$\frac{1}{G(r)^{1/2}r^{D/2}}\partial_r\left(R'(r)G(r)^{1/2}r^{D/2}\right)=R''(r)+\frac{1}{2}\frac{R'(r)}{G(r)}G'(r)+\frac{D}{2}\frac{R'(r)}{r}, \tag{A.14}$$

the second term is then

$$\begin{aligned}\frac{1}{G(r)^{1/2}r^{D/2}}\partial_r\left(-\frac{D}{2}r^{D/2-1}G(r)^{1/2}R(r)\right)&=-\frac{D}{2}\left(\frac{D}{2}-1\right)\frac{R(r)}{r^2}\\&-\frac{D}{4}\frac{R(r)}{G(r)r}G'(r)-\frac{D}{2}\frac{R'(r)}{r},\end{aligned} \tag{A.15}$$

and the third term is

$$\begin{aligned}\frac{1}{G(r)^{1/2}r^{D/2}}\partial_r\left(-\frac{1}{2}\frac{r^{D/2}R(r)}{G(r)^{1/2}}G'(r)\right)&=-\frac{D}{4}\frac{R(r)}{G(r)r}G'(r)-\frac{1}{2}\frac{R'(r)}{G(r)}G'(r)\\&+\frac{1}{4}\frac{R(r)}{G^2(r)}(G'(r))^2-\frac{1}{2}\frac{R(r)}{G(r)}G''(r).\end{aligned} \tag{A.16}$$

Adding the upper three equations (A.13) is ultimatively given by

$$R''(r)+\left(-\frac{D(D-2)}{4r^2}-\frac{D}{2r}\frac{G'(r)}{G(r)}+\frac{(G'(r))^2}{4G^2(r)}-\frac{G''(r)}{2G(r)}\right)R(r). \tag{A.17}$$

Definiting a new quantity, $\Delta(r)$, as

$$\Delta(r)=\left(\frac{D(D-2)}{4r^2}+\frac{D}{2r}\frac{G'(r)}{G(r)}-\frac{(G'(r))^2}{4G^2(r)}+\frac{G''(r)}{2G(r)}\right), \tag{A.18}$$

(A.17) is reduced to

$$R''(r)-\Delta(r)R(r). \tag{A.19}$$

Adding (A.11) and (A.19), we obtain the radial equation for our scalar wave function

$$R''(r)+\left[\frac{V^2(r)}{\hbar^2}-\Delta(r)\right]R(r)=0. \tag{A.20}$$

Appendix B Radial equation for a free charged scalar field in charged spacetimes

We are solving the massive charged Klein-Gordon equation in 4-dimensional charged spacetime. Starting from (2.86),

$$\begin{aligned}\Bigg(\frac{1}{\sqrt{-g}}\Bigg(\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)+\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\frac{i}{\hbar}qA_\nu\right)\\ +\frac{i}{\hbar}qA_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\Bigg)-\frac{1}{\hbar^2}g^{\mu\nu}q^2A_\mu A_\nu-\frac{m^2}{\hbar^2}\Bigg)\Phi=0,\end{aligned} \tag{B.1}$$

since only the zeroth component of A_μ is non-vanishing, the mixed terms of ∂ and A can be written as

$$\begin{aligned}\frac{1}{\sqrt{-g}}\left(\partial_t\left(\sqrt{-g}\left(\frac{-1}{f(r)}\right)\frac{i}{\hbar}qA_0(r)\Phi\right)+\frac{i}{\hbar}qA_0(r)\left(\sqrt{-g}\left(\frac{-1}{f(r)}\right)\partial_t\Phi\right)\right)\\ =-2\frac{1}{f(r)}\frac{i}{\hbar}qA_0(r)\partial_t\Phi.\end{aligned} \tag{B.2}$$

(B.1) now has the form

$$\left(\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)-2\frac{1}{f(r)}\frac{i}{\hbar}qA_0(r)\partial_t-\frac{1}{\hbar^2}g^{\mu\nu}q^2A_\mu A_\nu-\frac{m^2}{\hbar^2}\right)\Phi=0, \tag{B.3}$$

Using the same ansatz as before (A.2), we see that the first and fourth terms reproduce (A.20). (B.3) is now

$$R''(r)+\left(\frac{V^2(r)}{\hbar^2}-\Delta(r)\right)R(r)+\left(-\frac{2}{G^2(r)}\frac{1}{\hbar^2}qA_0(r)+\frac{1}{G^2(r)}\frac{1}{\hbar^2}q^2A_0^2(r)\right)R(r)=0. \tag{B.4}$$

Since

$$V^2(r)=\frac{1}{G^2(r)}\left(E^2-f(r)\left[m^2+\left(\frac{\ell(\ell+D-1)\hbar^2}{r^2}\right)\right]\right), \tag{B.5}$$

we can define a new quantity

$$W^2(r)=\frac{1}{G^2(r)}\left((E-qA_0(r))^2-f(r)\left[m^2+\left(\frac{\ell(\ell+D-1)\hbar^2}{r^2}\right)\right]\right), \tag{B.6}$$

for which the radial equation then has the same structure as before, with $V^2(r)\rightarrow$

$W^2(r)$, i.e., $E \to E - qA_0(r)$,

$$R''(r) + \left(\frac{W^2(r)}{\hbar^2} - \Delta(r)\right) R = 0. \tag{B.7}$$

6 Prošireni sažetak

6.1 Uvod

Crne rupe su područje prostorvremena gdje je gravitacija toliko snažna da ništa, uključujući svjetlost, ne može pobjeći iz njih. Ova granica se naziva horizont događaja. Matematički, crne rupe se pojavljuju kao rješenja Einsteinovih jednadžbi

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{6.8}$$

Fizički, one nastaju gravitacijskim urušavanjem masivnih zvijezda, stvarajući sustav s izuzetno jakim gravitacijskim poljem. Ovo svojstvo čine crne rupe idealnom arenom, kako teorijski, tako i fenomenološki, za promatranje kvantnih efekta gravitacije, a posebice mogućih kvantnih korekcija koje sustav može dobiti. U stacionarnom slučaju, crne rupe se mogu opisati s tri parametra: masa M, naboj Q i angularni moment J, što proizlazi iz *no-hair* teorema [1]. Kako bismo ilustrirali potrebu primjene termodinamičkih zakona na crne rupe, zamislimo scenarij u kojem neka tvar pada u crnu rupu. Prije prijelaza horizonta, ova tvar posjeduje određenu entropiju S. Nakon prijelaza horizonta, promatrač izvana ne može odrediti što se dogodilo s ovom entropijom. Ipak, znamo da je prije nego što je tvar upala u crnu rupu, ukupna entropija bila S. Nakon što tvar uđe u crnu rupu i nakon dovoljno dugog vremena, sve što vidimo je stabilno stanje crne rupe opisano njezinom masom, naboja i angularnim momentom. Ako bi postojalo samo jedno stanje za crnu rupu, karakterizirano s ova tri parametra, entropija svake takve crne rupe bi bila nula. To implicira da kada tvar uđe u crnu rupu, ukupna entropija pada, što je u sukobu s drugim zakonom termodinamike. Ovaj problem riješio je Bekenstein [2], postulirajući da crne rupe također posjeduju entropiju. Nadovezujući se na ove činjenice i promatranja, možemo povući paralele između mase M, naboja Q i angularnog momenta J crne rupe i termodinamičkih varijabli, što nas dovodi do četiri zakona mehanike crnih rupa [3-4]. Nulti zakon govori da stacionarne crne rupe imaju konstantnu površinsku gravitaciju κ na horizontu događaja. Prvi zakon je prikazan kao

$$dM = \frac{\kappa}{8\pi}dA + \Omega dJ, \tag{6.9}$$

gdje Ω označava kutnu brzinu horizonta, a A njegovu površinu. Drugi zakon kaže da se površina horizonta ne smanjuje

$$\delta A \geq 0. \tag{6.10}$$

Treći zakon kaže da se površinska gravitacija κ ne može smanjiti na nulu u konačno mnogo koraka. Uspoređujući prvi zakon s onim iz termodinamike, $dE = TdS - pdV$, i izjednačavajući ΩdJ član s radom, dobivamo jednakost $TdS = \kappa dA/8\pi G$. Hawking [5] je izračunao da crne rupe zrače poput crnih tijela i imaju temperaturu

$$T_{\mathrm{H}} = \frac{\hbar\kappa}{2\pi}. \tag{6.11}$$

Koristeći navedene rezultate, dolazimo do formule za entropiju crnih rupa

$$S_{\mathrm{BH}} = \frac{A}{4\ell_{\mathrm{Pl}}^2}, \tag{6.12}$$

gdje $\ell_{\mathrm{Pl}} = \sqrt{G\hbar}$ označava Planckovu duljinu. Ova formula naziva se Bekenstein-Hawkingova entropija. Jedinstvena značajka Bekenstein-Hawkingove entropije je da je proporcionalna površini crne rupe, poznatoj kao horizont, što se razlikuje od tipičnih računa entropije gdje je entropija obično proporcionalna volumenu. Tijekom osamdesetih godina prošlog stoljeća, istraživači su istraživali različite pristupe razumijevanju entropije crnih rupa. 't Hooft [6] je izračunao entropiju Hawkingovih čestica neposredno izvan horizonta crne rupe, tretirajući ih kao termalni plin. Iako je ovaj izračun rezultirao entropijom proporcionalnom površini horizonta, zahtijevao je tzv. *brick wall* blizu horizonta kako bi regulirao određene divergencije. Bombelli, Koul, Lee i Sorkin [7] razmatrali su reduciranu matricu gustoće, dobivenu uzimanjem traga preko kvantnih stupnjeva slobode unutar horizonta crne rupe. Ovaj postupak činio se prirodnim za crne rupe jer njihov horizont djeluje kao kauzalna granica, čineći događaje unutar crne rupe nedostupnima promatračima izvana. Srednicki [8] izračunao je entropiju izravno u ravnom prostoru uzimanjem traga preko stupnjeva slobode unutar zamišljene površine. Ova entropija, poznata kao entropija isprepletenosti ili sprege, također je bila proporcionalna površini preko koje se odvijalo preplitanje. Entropija proizlazi iz kratkodosežnih korelacija u kvantnom polju blizu površine, pa je osjetljiva na veličinu područja blizu površine. To znači da samo modovi smješteni u području blizu površine doprinose entropiji, što implicira da veličina tog područja igra ulogu UV regulatora. Zanimljivo je primijetiti da entropija isprepletenosti kvantnog polja u ravnom prostoru već vodi na činjenicu da je entropija procionalna površini, bez postojanje ikakve crne rupe. Osim toga, shvaćeno je da su entropija dobivena *brick wall* modelom i entropija isprepletenosti povezane [9]. Druga metoda za izračun entropije isprepletenosti razvijena je od strane Susskinda [10], uvodeći malu koničnu singularnost na površinu ispreplitanja, zatim izvrijednjujući efektivnu akciju kvantnog polja na pozadinsku metriku s koničnom singularnošću te zatim derivirajući akciju s obzirom na kutni defekt. Ova metoda naziva se tzv. *replica trick*. Koristeći ovu metodu, napravljeni su sistematski izračuni UV divergentnih članova u entropiji isprepletenosti crnih rupa [11], a posebno su pronađeni logaritamski UV divergentni članovi [12]. Ti logaritamski korekcijski članovi pronađeni su i s pomoću nekoliko različitih metoda, uključujući teoriju struna [13], korekcijama viših petlji gravitacijske akcije [14], *heat kernel* metodom [15] i upotrebom nekomutativne geometrije [16]. Pitanje može li UV divergencija u entropiji isprepletenosti biti pravilno renormalizirana istraživali su Susskind i Uglum [17]. Utvrdili su da standardna renormalizacija Newtonove konstante proizvodi konačnu entropiju, ako razmatramo entropiju isprepletenosti kao kvantni doprinos Bekenstein-Hawkingovoj entropiji. Kasnije su Ryu i Takayanagi [18] predložili holografsko tumačenje entropije isprepletenosti, povezujući je s površinom minimalne plohe u anti-de Sitterovom prostorvremenu putem AdS/CFT korespondencije. Trenutna istraživanja sugeriraju da entropija isprepletenosti obećava bolje razumijevanje crnih rupa i kvantne gravitacije, budući da su se nekoliko revija bavili njezinom ulogom u crnim rupama [19, 20], njezinim izračunom u kvantnoj teoriji polja u ravnom prostorvremenu [21] i njezinim holografskim aspektima [22].

6.2 *Brick wall metoda*

U ovom poglavlju smo razvili *brick wall* metodu, koja je semiklasična metoda za razumijevanje mikroskopskog uzroka entropije crnih rupa. Pretpostavlja se fiksna pozadina na kojoj polja propagiraju, a entropija se računa pomoću kanonske entropija od polja materije izvan horizonta crne rupe, izvrijednjenoj na Hawkingovoj temperaturi. Zbog velikog broja energijskih stanja koja čestica može okupirati, gustoća stanja divergira [6]. Ovo sugerira onda da entropija crne rupe divergira, te služi kao argument da su crne rupe beskonačni ponori informacija. Kako bismo se riješili ove beskonačnosti, uvodimo tzv. *brick wall* blizu horizonta. Drugim riječima, pretpostavlja se da sve polja iščezavaju unutar neke fiksne udaljenosti od horizonta.

$$\Phi(r) = 0, \quad r = r_{\rm H} + h, \tag{6.13}$$

Razmatramo skalarnu valnu funkciju Φ na $(D{+}2)$-dimenzionalnom, sfernosimetričnom prostorvremenu crne rupe, koja zadovoljava Klein-Gordonovu jednadžbu

$$\left(\Box - \frac{m^2}{\hbar^2}\right)\Phi = 0, \tag{6.14}$$

gdje je $\Box = \nabla^\mu \nabla_\mu$ d'Alembertov operator, a m is masa skalarnog polja. Možemo riješiti ovu jednadžbu koristeći ansatz [24].

$$\Phi = e^{-iEt/\hbar}\frac{R(r)}{r^{D/2}G(r)^{1/2}}Y_{\ell m_i}(\theta, \phi_i), \tag{6.15}$$

gdje su $G(r) = \sqrt{f(r)g(r)}$, $i \in \{1, \ldots, (D-1)\}$, a Y_{ℓ, m_i} hipersferni harmonici. Koristeći to dobijemo radijalnu jednadžbu,

$$R''(r) + \left[\frac{V^2(r)}{\hbar^2} - \Delta(r)\right]R(r) = 0, \tag{6.16}$$

gdje je $\Delta(r)$ pokrata, a $V^2(r)$ ima ulogu efektivnog potencijala. Da riješimo ovu jednadžbu, koristimo ansatz

$$R(r) = \frac{1}{\sqrt{P(r)}}\exp\frac{i}{\hbar}\int^r P(r')dr'. \tag{6.17}$$

Kako bismo išli u više redove WKB aproksimacije, $P(r)$ pišemo poput

$$P(r) = \sum_0^\infty \hbar^{2n}P_{2n}(r). \tag{6.18}$$

Zbog oblika našeg WKB ansatza, pratimo standardnu kvantizacijsku proceduru, zadajući naredni kvantizacijski uvjet

$$\int_{r_{\mathrm{H}}+h}^{L} dr \int_{0}^{\ell_{max}} d\ell (2\ell + D - 1) \mathcal{W}(\ell) P_{2n}(r) = \pi \hbar^{1-2n} N_{2n}(E), \tag{6.19}$$

gdje $\mathcal{W}$ označava degeneracijski faktor za angularni moment, a N_{2n} je doprinos n-tog moda ukupnom broju stanja skalarnog polja s energijom nižom od E,

$$N(E) = \sum_{n=0}^{\infty} N_{2n}(E). \tag{6.20}$$

Uvjet na $\ell_{\max}$ je dan tako da zahtijevamo da su funkcije P_{2n} realne. Entropiju i slobodnu energiju za dani mod računamo preko

$$F_{2n} = -\int_{0}^{\infty} \frac{N_{2n}}{e^{\beta E} - 1} dE, \quad S_{2n} = \beta^2 \frac{\partial F_{2n}}{\partial \beta}. \tag{6.21}$$

Entropija do drugog reda za slučaj četverodimenzionalne crne rupe ($D = 2$) sa $f(r) = g(r)$ i $m = 0$ iznosi

$$S = S_{\mathrm{BH}} - \left(\frac{\kappa r_{\mathrm{H}}}{10} + \frac{g''(r_{\mathrm{H}}) r_{\mathrm{H}}^2}{60} \right) \ln \left(\frac{\mathcal{A}}{\ell_{\mathrm{Pl}}^2} \right), \tag{6.22}$$

gdje su nakon integracije sadržani najdominantniji članovi u limesu $h \to 0$, budući da očekujemo da ti članovi najviše utječu na entropiju, jer su oni upravo ti koji su vodili do divergencije zbog divergentne gustoće stanja. Primjenjujući ovu formulu na primjeru najjednostavnije crne rupe, Schwarzschildove,

$$g(r) = 1 - \frac{2GM}{r}, \quad r_{\mathrm{H}} = 2GM, \kappa = \frac{1}{4GM} = \frac{1}{2r_{\mathrm{H}}}, \quad g''(r_{\mathrm{H}}) = -\frac{2}{r_{\mathrm{H}}^2}, \tag{6.23}$$

dobivamo

$$S = S_{\mathrm{BH}} - \frac{1}{60} \ln \left(\frac{\mathcal{A}}{\ell_{\mathrm{Pl}}^2} \right). \tag{6.24}$$

Schwarzschildova crna rupa prima korekciju na njenu entropiju u obliku logaritma površine horizonta. Primijetimo kako je faktor ispred logaritma konstanta, neovisna o površini horizonta.

Sada možemo proširiti razmatranje na nabijene crne rupe i nabijene probe. U tom slučaju moramo napraviti minimalnu supstituciju u Klein-Gordonovu jednadžbu, $\partial\mu \to \partial_\mu + \frac{i}{\hbar} q A_\mu$, gdje $A_\mu = (A_0(r), 0, 0, 0)$ opisuje 4-potencijal nabijenog prostorvremena, a q je naboj skalarnog polja s kojim ispitujemo prostorvrijeme. Uz to, koristimo pretpostavku da je $f(r) = g(r)$. Korištenjem istog ansatza kao i prije, možemo jednostavno doći do radijalne jednadžbe koja ima sličnu strukturu kao i prijašnja

$$R''(r) + \left(\frac{W^2(r)}{\hbar^2} - \Delta(r)\right) R(r) = 0, \tag{6.25}$$

gdje je $W^2(r)$ funkcija koja ima isti oblik kao i $V^2(r)$ uz $E \to E - qA_0(r)$. Ovo znači da, kako bismo izračunali nulti i drugi red korekcije, moramo samo napraviti zamjenu $E \to E - qA_0(r)$ u definicijskim jednadžbama za N_0 i N_2. Napravimo ovo na primjeru Reissner-Nordströmove crne rupe

$$\begin{gathered} A_0(r) = \frac{Q}{r}, \quad g(r) = 1 - \frac{2GM}{r} + \frac{Q^2 G}{r^2}, \quad D = 2, \quad m = 0, \\ r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2}, \quad r_{\mathrm{H}} = r_+ . \end{gathered} \tag{6.26}$$

Primjenjujući istu proceduru kao i prije, dolazimo do entropije za Reissner-Nordströmovu crnu rupu do drugog reda

$$\begin{gathered} S = S_0 + S_2 = \\ \frac{1}{h}\left(\frac{11(r_+ - r_-)}{360} - \frac{3qQr_+\zeta(3)}{4\pi^3\hbar} + \frac{q^2Q^2r_+^2}{6(r_+ - r_-)\hbar^2}\right) + \\ \left(-\frac{(r_+ + r_-)}{60r_+} - \frac{3qQ\zeta(3)(3r_+ - 5r_-)}{4\pi^3(r_+ - r_-)\hbar} + \frac{q^2Q^2r_+(r_+ - 2r_-)}{3(r_+ - r_-)^2\hbar^2}\right) \ln\left(\frac{\alpha}{h}\right). \end{gathered} \tag{6.27}$$

U limesu $Q \to 0$ reproduciramo Schwarzschildovu entropiju. Dok za $q \to 0$ dobijemo

$$S = \frac{11(r_+ - r_-)}{360h} - \frac{(r_+ + r_-)}{60r_+} \ln\left(\frac{\alpha}{h}\right), \tag{6.28}$$

što je isti rezultat za entropiju kao u [23] koji je dobiven razmatranjem samo nabijenog prostorvremena, a ne nabijene probe.

6.3 *Isprepletena entropija*

Isprepletena entropija je mjera kvantnog isprepletanja između dva podsistema, dobivena uzimanjem traga po jednom od podsistema ukupnog sistema. U kontekstu crnih rupa, podsistemi od interesa su stupnjevi slobodi unutra i izvan horizonta događaja. Računi prepletene entropije uključuju dijeljenje prostorvremena u dvije regije: unutrašnjost crne rupe i vanjski dio. Entropija ispreplitanja je definirana kao von Neumannova entropija reducirane matrice gustoće koja odgovara unutarnjem području. Računi isprepletene entropije daju mikroskopsko razumijevanje entropije crnih rupa. To vodi do razumijevanja mikroskopskih stupnjeva slobode koji su zaslužni za entropiju, te utemeljuje poveznicu između kvantnog preplitanja i gravitacije. U ovom dijelu diplomskog rada računat ćemo isprepletenu entropiju sistem, modelirajući skalarno polje na $\mathbb{R}^3$ kao skup vezanih oscilatora. Pratit ćemo [7-9] the ćemo pretpostaviti $\hbar = 1$.

Razmotrimo čisto vakuumsko stanje $|\psi\rangle$ kvantnog sustava definiranog unutar prostornolike regije $\mathcal{O}$, i pretpostavimo da su stupnjevi slobode lokalizirani unutar određenih regija $\mathcal{O}$. Valna funkcija cijelog sustava $\mathcal{O}$, kojeg ploha Σ dijeli na dva podsustava A i B tada je dana linearnom kombinacijom produkta kvantnih stanja svakog podsustava,

$$|\psi\rangle = \sum_{i,a} \psi_{i,a} |A\rangle_i |B\rangle_a \,, \tag{6.29}$$

Matricu gustoće koja odgovara čistom kvantnom stanju sustava $|\psi\rangle$ dana je izrazom

$$\rho_0(A, B) = |\psi\rangle \langle\psi| \,, \tag{6.30}$$

i ima von Neumannovu entropiju jednaku nuli, budući da je čisto stanje ono bez neodređenosti. Ako uzmemo trag po stupnjevima slobode u području A, tj. parcijalni trag Tr_A za matricu gustoće čistog stanja, dobivamo reduciranu matricu gustoće za podsustav B

$$\rho_B = Tr_A \rho_0(A, B). \tag{6.31}$$

Statistička entropija za neku matricu gustoće naziva se von Neumannova entropija i dana je izrazom

$$S = -Tr(\rho \ln \rho). \tag{6.32}$$

To znači da je entropija za podsustav B dana izrazom

$$S_B = -Tr(\rho_B \ln \rho_B), \tag{6.33}$$

što se poklapa s entropijom isprepletenosti koja je povezana s plohom Σ. Primjenjujući isti postupak, možemo dobiti entropiju isprepletenosti S_A. Može se pokazati da vrijedi

$$S_A = S_B. \tag{6.34}$$

Ovo znači da isprepletena entropija nekog sistema u čistom stanju nije ekstenzivna veličina, tj. određena je geometrijom plohe koja dijeli prostor, Σ.

Prvo promotrimo entropiju skupa vezanih harmoničkih oscilatora u $\mathbb{R}^3$ opisani s Lagrangianom

$$L = \frac{1}{2} G_{MN} \dot{q}^M \dot{q}^N - \frac{1}{2} V_{MN} q^M q^N, \tag{6.35}$$

gdje su q i $\dot{q}$ otklon i generalizirana brzina respektivno. Simetrična matrica G ima ulogu metrike, dok je V simetrična matrica. Uvođenjem matrice "korijena" od V, $W_{MA} W_N^A = V_{MN}$, možemo uvesti operatore stvaranja i poništenja,

$$a_M^* = P_M + i W_{MA} q^A, \quad a_M = P_M - i W_{MA} q^A, \tag{6.36}$$

gdje je P generalizirani impuls od q. Zahtijevajući da operator poništenja poništava osnovno stanje našeg sistema, dolazimo do izraza za valnu funkciju osnovno stanja

$$\phi_0(\{q^A\}) = \left(\det\left(\frac{D}{\pi}\right)\right)^{1/4} \exp\left[-\frac{1}{2} W_{AB} q^A q^B\right], \tag{6.37}$$

gdje je D dijagonalizirana matrica od W. Matrica gustoće onda ima oblik

$$\begin{aligned} \rho(\{q^A\}, \{q'^B\}) &\equiv \langle \{q^A\} | \psi_0 \rangle \langle \psi_0 | \{q'^B\} \rangle = \psi_0(\{q^A\}) \psi_0(\{q'^B\})^* \\ &= \left(\det\left(\frac{D}{\pi}\right)\right)^{1/2} \exp\left[-\frac{1}{2} W_{AB} (q^A q^B + q'^A q'^B)\right]. \end{aligned} \tag{6.38}$$

Ako napravimo trag po nekom potprostoru Ω od $\mathbb{R}^3$, dobivamo reduciranu matricu gustoće

$$\begin{aligned} \rho_{red}(\{q^a\}, \{q'^b\}) = \left(\det\left(\frac{M_{ab}}{\pi}\right)\right)^{1/2} \exp\left[-\frac{1}{2} M_{ab} (q^a q^b + q'^a q'^b)\right] \\ \times \exp\left[-\frac{1}{4} N_{ab} (q - q')^a (q - q')^b\right]. \end{aligned} \tag{6.39}$$

Entropija ovakve reducirane matrice gustoće za sustav s mnogo stupnjeva slobode je dana sa

$$S = -\sum_n \frac{\mu_n \ln \mu_n + (1-\mu_n)\ln(1-\mu_n)}{1-\mu_n}, \tag{6.40}$$

gdje je

$$\mu_n = 1 + 2\lambda_n^{-1} - 2\sqrt{\lambda_n^{-1}(1+\lambda_n^{-1})}, \tag{6.41}$$

a λ_n su svojstvene vrijednosti od $\Lambda^a_b = -W^{a\beta}W_{\beta b}$. Srednicki [8] je ovakvom metodom izračunao entropiju za sferu radijusa R uzimanjem traga osnovnog stanja bezmasenog skalarnog polja preko stupnjevima slobode unutar radijalne rešetke. Za entropiju je numerički dobio

$$S = 0.3M^2R^2. \tag{6.42}$$

Drugi način računanja isprepletene entropije je replika trik gdje gledamo Rényi entropiju, kao jedno parametarsku generalizaciju prepletene entropije s indeksom q koji analitički produljimo na realne brojeve, te napravimo limes $q \to 1$. Ovom metodom entropiju možemo izračunati tako da izračunamo particijsku funkciju replicirane mnogostrukosti $\hat{\mathcal{M}}_q$ koristeći integrale po putevima, što je puno jednostavnije za kvantnu teoriju polja. Promatramo ispreplitanje između dvije vremenolike regije koje su povezane s crvotočinom koja ima isti radijus kao i Schwarzschildova crna rupa. Formula za entropiju glasi

$$S = -q\partial_q \log Z[\hat{\mathcal{M}}_q]|_{q=1}. \tag{6.43}$$

Gledajući niskoenergijski limes, Euklidska gravitacijska particijska funkcija je dana s

$$Z[\mathcal{M}] = \int \mathcal{D}g e^{-I_E[g]} \approx e^{-I_E[M]}, \tag{6.44}$$

gdje je $I_E[\mathcal{M}]$ on-shell Euklidska akcija sa Schwarzschildovom metrikom. Entropija se može napisati preko derivacije te akcije, i reproducira se Bekenstein-Hawkingov zakon

$$S = q\partial_q I_E[\hat{\mathcal{M}}_q]_{q=1} = \frac{4\pi r_S^2}{4G_N} = \frac{\mathcal{A}}{4G_N} \tag{6.45}$$

Treći način računanja je heat kernel metoda, i ovdje računamo entropiju ispreplitanja za $(D-2)$-dimenzionalnu ravninu u D-dimenzionalnom prostorvremenu. Efektivna akcija je dana s

$$W = -\frac{1}{2}\int_{\epsilon^2}^{\infty} \frac{ds}{s} TrK(s), \tag{6.46}$$

gdje je $K(s, X, X') = \langle X|e^{-s\mathcal{D}}|X'\rangle$ heat kernel definiran kao rješenje toplinske jednadžbe za operator $\mathcal{D}$

$$(\partial_s + \mathcal{D})K(s, X, X') = 0, \quad K(s = 0, X, X') = \delta(X, X'). \tag{6.47}$$

Za ravninu u ravnom prostorvremenu, uz operator $\mathcal{D} = -\nabla^2$, trag heat kernela iznosi

$$TrK(s, w) = \frac{s}{(4\pi s)^{D/2}} \frac{\pi\alpha}{\sin^2 \frac{w}{2}} A(\Sigma), \tag{6.48}$$

gdje je $A(\Sigma) = \int d^{D-2}x$ površina ravnine Σ. Trag se uvrsti u efektivnu akciju i onda se entropija se računa istom formulom kao i kod replica trika, uz $q \to \alpha$, i $W(\alpha) = -\log Z(\alpha)$. Entropija onda glasi

$$S = \frac{A(\Sigma)}{(D-2)(4\pi)^{(D-2)/2}\epsilon^{D-2}} \tag{6.49}$$

Ako dodamo masu u operator $\mathcal{D}$, danog s

$$\mathcal{D} = -\nabla^2 + m^2, \tag{6.50}$$

entropija poprima sličan oblik kao i ona dobivena u brick wall metodi, tj. ima logaritamsku divergenciju u UV regulatoru, ϵ

$$S = \frac{A(\Sigma)}{12(4\pi)} \left(\frac{1}{\epsilon^2} + 2m^2 \ln \epsilon + m^2 \ln m^2 + m^2(\gamma - 1) \right). \tag{6.51}$$

6.4 Jednopetljena renormalizacija gravitacijske akcije

U ovom poglavlju povezujemo divergencije u entropiji s renormalizacijom konstanti vezanja u teoriji, promatrajući propagiranje skalarnog polja u 4-dimenzionalnoj neekstremalnoj Reissner-Nordströmovoj pozadini. Korištenjem Pauli-Villarsove regularizacijske sheme, regularizirati ćemo jednopetljene doprinose skalarnih polja te odrediti renormalizaciju Newtonove konstante. Pokazujemo da Pauli-Villarsov regulator sam po sebi daje UV cutoff, zamijenjujući tako potrebu za *brick wallom*. Uz gravitacijsku akciju

$$I_g = \int d^4x\sqrt{-g}\left[-\frac{\Lambda_B}{8\pi 8G_B} + \frac{R}{16\pi G_B} + \frac{\alpha_B}{4\pi}R^2 + \frac{\beta_B}{4\pi}R_{ab}R^{ab} + \frac{\gamma_B}{4\pi}R_{abcd}R^{abcd} + \dots\right], \tag{6.52}$$

i akciju minimalno vezanog skalarnog polja,

$$I_m = -\frac{1}{2}\int d^4x\sqrt{-g}\left[g^{ab}\nabla_a\phi\nabla_b\phi + m^2\phi^2\right], \tag{6.53}$$

želimo odrediti efektivnu akciju za metriku. Može se pokazati da ona iznosi

$$W(g) = -\frac{i}{2}Tr(\log[-G_F(g,m^2)], \tag{6.54}$$

kako je ovaj izraz divergentan, moramo ga regulirati da bude ispravno definiran. Možemo ga zapisati u obliku

$$W(g) = -\frac{1}{32\pi^2}\int d^4x\sqrt{-g}\int_0^\infty \frac{ds}{s^3}\sum_{n=0}^\infty a_n(x)(is)^n e^{-im^2 s}, \tag{6.55}$$

i onda uvesti 5 regulatorskih polja, tako da ukupna akcija za polja materije iznosi

$$I_m = -\frac{1}{2}\sum_{i=0}^{5}\int d^4x\sqrt{-g}[g^{ab}\nabla_a\phi_i\nabla_b\phi_i + m_i^2\phi_i^2]. \tag{6.56}$$

Najdivergentniji članovi u efektivnoj akciji su onda dani s

$$
\begin{aligned}
W_{div} = -\frac{1}{32\pi^2}\int d^4x\sqrt{-g}\int_0^\infty \frac{ds}{s^3}[a_0(x) + isa_1(x) + (is)^2a_2(x)] \\
\times[e^{-im^2s} - 2e^{-i(\mu^2+m^2)s} + 2e^{-i(3\mu^2+m^2)s} - e^{-i(4\mu^2+m^2)s}] \\
= \frac{1}{32\pi^2}\int d^4x\sqrt{-g}[-Ca_0(x) + Ba_1(x) + Aa_2(x)].
\end{aligned}
\tag{6.57}
$$

Spajajući skalarnu jednopetljenu akciju s golom akcijom, dobivamo renormalizirane konstante vezanja u efektivnoj gravitacijskoj akciji,

$$
\begin{aligned}
I_{eff} = I_g + W \\
= \int d^4x\sqrt{-g}\Big[-\frac{1}{8\pi}\left(\frac{\Lambda_B}{G_B} + \frac{C}{4\pi}\right) + \frac{R}{16\pi}\left(\frac{1}{G_B} + \frac{B}{12\pi}\right) + \frac{R^2}{4\pi}\left(\alpha_B + \frac{A}{576\pi}\right) \\
+\frac{1}{4\pi}R_{ab}R^{ab}\left(\beta_B - \frac{A}{1440\pi}\right) + \frac{1}{4\pi}R_{abcd}R^{abcd}\left(\gamma_B + \frac{A}{1440\pi}\right) + \ldots\Big].
\end{aligned}
\tag{6.58}
$$

Možemo identificirati renormaliziranu Newtonovu konstantu

$$
\frac{1}{G_R} = \frac{1}{G_B} + \frac{B}{12\pi}. \tag{6.59}
$$

Sada možemo primijeniti ovu metodu uvođenja Pauli-Villarsovih polja da izračunamo entropiju Reissner-Nordströmove crne rupe koristeći brick wall metodu. Možemo pokazati da se pomoću tih polja makne potreba za brick wall, h, i tada ukupna slobodna energija za sva polja glasi

$$
\bar{F} = -\frac{2r_+^3}{3\pi}\int_0^\infty \frac{dE}{e^{\beta E}-1}\int_0^{L'} \frac{ds}{s^2(1-s)^4(1-u+us)^2}\sum_{i=0}^{5}\Delta_i[E^2 - s(1-u+us)m_i^2]^{3/2}. \tag{6.60}
$$

Ukupna entropija, izvrijednjena na inverznoj Hawkingovoj temperaturi $\beta_H = \frac{4\pi r_+}{1-u}$ je onda

$$
S = \frac{\mathcal{A}}{4}\frac{B}{12\pi} + \frac{(2-3u)A}{180}. \tag{6.61}
$$

Dodavanje ove entropije Bekenstein-Hawkingovoj, dobivamo da doprinos proporcionalan s B daje jednopetljenu renormalizaciju Bekenstein-Hawkingove entropije, ako uključimo i članove koji su kvadratni u zakrivljenosti, dobivamo sljedeću formulu za totalnu entropiju, gdje se pojavljuju i članovi s ostalim renormaliziranim konstantama vezanja

$$S_{total} = \frac{\mathcal{A}}{4G_R} - 8\pi u\beta_R + 16\pi(1-2u)\gamma_R. \tag{6.62}$$

Ako bismo imali neminimalno vezano skalarno polje, onda entropija više ne bi uključivala potpunu renormalizaciju Bekenstein-Hawkingove formule, zato što Riccijev skalar iščezava za Reissner-Nordströmovu crnu rupu. Ova nekonzistentnost ukazuje na ograničenja u brick wall modelu.

6.5 Završne opaske

U zaključku smo dali pregled svih rezultata dobivenih u ovom radu, te spomenuli da se metode mogu proširiti primjenjujući ih na crne rupe nižih dimenzija poput: BTZ, QBTZ te NCQBTZ.

6.6 HR nazivi slika i tablica

Slika 3.1: Parametrizacija repliciranog prostorvremena s koničnim singularitetom lociranom na $r = 0$ u ravnom prostorvremenu. Regija A u $\tau = 0$ vremenu je definirana kao $x > 0$ [28].

Slika 3.2: Penroseov dijagram maksimalno-proširenog Schwarzschildovog prostorvremena [28].

Slika 3.3: Ilustracija ukupne (nenormalizirane) matrice gustoće ρ. Rez Σ dijeli cigaru na dva vremenski simetrična dijela. Da bismo normalizirali ρ, dijelimo račun s $Tr(\rho)$ koje opisuje cijelu cigaru [28].

Slika 3.4: Ilustracija particijske funkcije $Z[\mathcal{M}_q$, Plave linije povezane sa strelicom su identificirane [28].

Slika 3.5: Jednolistna mnogostrukost $\hat{\mathcal{M}}$ koja sada ima konični singularitet na fiksnoj točci $r = r_S$ s angularnim defektom od $\Delta\phi = 2\pi(1 - 1/q)$ [28].

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