Recurrences and graphs of recurrences

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DOCTORAL DISSERTATION

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PRIRODOSLOVNO–MATEMATIČKI FAKULTET

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SUMMARY

In the first part of the dissertation, we count tilings of narrow strips in the hexagonal grid with a given number and type of tiles and investigate identities involving the obtained enumerating sequences. On the same substrate, we also study the number of tilings in which both the number and the shape of tiles are free. In both cases, the enumerating sequences satisfy linear recurrences with constant coefficients. In contrast to the first part where linear recurrences are obtained as a solution to the problem, the central part of the work uses them as a starting point. Namely, it is dedicated to the family of graphs that generalize Fibonacci cubes and whose number of vertices satisfies the recursion $s_n = as_{n-1} + s_{n-2}$. We determine basic enumerative and metric properties and show that the family retains many desirable properties of Fibonacci cubes. Canonical decomposition and distribution of degrees will be investigated, as well as the existence of Hamiltonian paths and cycles. In the last part, we explore the possibility of extending the family of graphs to the Horadam recursion $s_n = as_{n-1} + bs_{n-2}$.

Sažetak

U prvom dijelu rada prebrojavamo popločenja uskih traka šesterokutne mreže s unaprijed poznatim brojem pločica određenog tipa i istražujemo mogućnost dobivanja identiteta o nizovima koje tako dobivamo. Preciznije, istražujemo na koliko se načina može popločati uska šesterokutna mreža koristeći samo monomere i dimere, a s unaprijed određenim brojem dimera. Nadalje, proučavamo sličan problem, ali umjesto horizontalnih dimera koristimo trimere, dobivene od tri uzastopna šesterokuta na mreži. Na istom supstratu promatramo i ukupni broj podjela promatrane šesterokutne mreže, tj. broj popločenja u kojima su i broj i oblik pločica proizvoljni. U ovom slučaju dobivamo jednostavan odgovor, broj načina na koji možemo podjeliti usku šesterokutnu mrežu odgovara Fibonaccijevim brojevima neparnog indexa. U svim slučajevima dobivaju se nizovi zadani linearnim rekurzijama.

Središnji dio rada posvećen je familiji grafova koja poopćuje Fibonaccijeve kocke. Broj vrhova s_n u grafovima takve familije također zadovoljava linearnu rekurziju, $s_n = as_{n-1} + s_{n-2}$. Određujemo njihova osnovna enumerativna i metrička svojstava te pokazujemo da familija zadržava mnoga poželjna svojstva Fibonaccijevih kocaka. Određivanjem kanonske dekompozicije pokazujemo da se grafovi mogu dekomponirati u više grafova iste vrste, a manje dimenzije. Također, dekomponirajući ih u rešetke, pronalazimo još jednu vezu promatrane familije i Fibonaccijevih kocki. Određujemo i raspodjelu stupnjeva vrhova i dajemo nužne uvjete za postojanje Hamiltonovih putova i ciklusa. U posljednjem dijelu rada istražujemo mogućnost proširenja familije grafova na Horadamovu rekurziju $s_n = as_{n-1} + bs_{n-2}$, i pokazujemo da se mnoga svojstva Fibonaccijevih i metalnih kocaka nasljeđuju za grafove te familije.

CONTENTS

INTRODUCTION

Tilings or tessellations appear as natural solutions to many practical problems and their aesthetic appeal motivates the interest that goes way beyond the limits of their practical relevance. In mathematics, tiling-related problems appear in almost all areas, ranging from purely recreational settings of plane geometry all the way to the deep questions of eigenvalue count asymptotics for boundary-value problems in higher-dimensional spaces [44,45]. Many of those problems, formulated in simple and intuitive terms and seemingly innocuous, quickly turn out to be quite intractable in their generality. That motivates interest in their restricted versions that might be more accessible. In Chapter 1, we look at several such restricted problems when the tiled area has a given structure and the allowed tiles belong to a small set of given shapes. In particular, we consider the problems of tiling a narrow strip of the hexagonal lattice in the plane with several types of tiles made of regular hexagons. Similar problems for strips in square and triangular lattices have been considered in several recent papers [3, 14, 15, 29]. In the last part of Chapter 1, we lift all restrictions on the shape of a tile. All these problems can be naturally formulated in terms of linear recurrences with constant coefficients. Problems considered in Chapter 1 can be formulated as problems of packings in graphs leading to linear recurrences.

In Chapter 2, we reverse the direction. To a given linear second-order linear recurrence with constant coefficients we assign a family of graphs whose number of vertices satisfies the recurrences, while the adjacency pattern is modeled after the Fibonacci and Lucas cubes. Namely, we define and investigate a family of graphs that generalizes Fibonacci cubes. Here we list some basic definitions. The graph G is an ordered pair $G = (V(G), E(G))$ where $V(G)$ denotes the set whose elements are called *vertices*, and $E(G)$ is the set whose elements are unordered pairs of elements of $V(G)$. The elements of the set $E(G)$ are called *edges*. For vertices $v_1, v_2 \in E(G)$ we say they are *adjacent* if the pair $\{v_1, v_2\} \in E(G)$. As the number of vertices of the graph obtained at the n^{th} step of construction for a fixed value of a satisfies the three-term linear recurrence $s_n^a = as_{n-1}^a + s_{n-1}^a$, we call the resulting graphs the *metallic cubes*. We find the name informative, reflecting both their partial-cube nature and the asymptotic behavior of the number of vertices, expressed via roots of the characteristic equation $x^2 - ax - 1 = 0$, known as the *metallic means* [7]. We present a recursive decomposition as well as the decomposition into grids. We determine the distribution of degrees and provide some necessary conditions for this family of graphs to be Hamiltonian. Also, recurrence and formulas for the number of edges are obtained. Furthermore, we determine metric properties such as radius, diameter, center, and periphery.

In Chapter 3, we explore the possibility of further generalization. We define and investigate the family of graphs whose number of vertices is given by the sequence that satisfies the recurrence $s_n^{a,b} = as_{n-1}^{a,b} + bs_{n-2}^{a,b}$, thus going beyond the Fibonacci and metallic cubes. Although such sequences were studied already at the beginning of the 20th century [48], and probably even earlier, they got their name after A. F. Horadam, who popularized them in a series of papers in the early sixties [24–26]. So, we call the new family of graphs the *Horadam cubes*. We explore and establish their basic structural and enumerative properties. In particular, we show that they inherit their recursive decompositions and decompositions into grids from the metallic cubes. Furthermore, the Horadam cubes are induced subgraphs of hypercubes and bipartite median graphs. All Horadam cubes contain a Hamiltonian path, and a Horadam cube is Hamiltonian if the number of vertices is even. In this chapter, we also explore the cube coefficients, cube numbers, and cube polynomials, thus, by setting $b = 1$, extend a portrait of the metallic cubes.

1. Tilings of a honeycomb strip

1.1. Tilings of a honeycomb strip and tetranacci numbers

The substrate (i.e., the area to be tiled) is a honeycomb strip H_n composed of n regular hexagons arranged in two rows in which the hexagons are numbered starting from the bottom left corner, as shown in Figure 1.2 for $n = 12$. The number of hexagons in the strip will be called its length. The choice of the substrate might seem arbitrary, but it provides a neat visual model for a linear array of locally interacting units with additional longer-range connections: the inner dual of a strip of length n is, in fact, P_n^2, the path on n vertices with edges between all vertices at a distance less than or equal to 2. Another way to look at it is as the ladder graph with descending diagonals, another familiar structure. The graph P_n^2 and ladder graph are presented in Figure 1.1. Here it is worth mentioning that *dual graph* of a plane graph G is a graph G' whose vertices are faces of G, and an edge between two faces x and y in G corresponds to the edge e' between x and y in G' [49]. Removing the external face in G and all corresponding edges yields the *inner dual*.

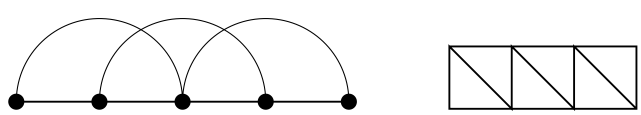

Figure 1.1: The graph P_n^2 and ladder graph.

Tilings with monomers and dimers in the strip correspond to matchings in its inner dual, thus enabling us to transfer known results about matchings directly into our context.

We start by examining the tilings of such strips by monomers (i.e., single hexagons)

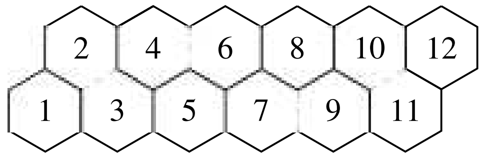

Figure 1.2: A tiling of a honeycomb strip of length 12 using 4 dimers.

and dimers made of two hexagons joined along an edge. Such tilings have been considered recently by Dresden and Jin [14], who found that the total number of such tilings is given by the tetranacci numbers. We refine their results in several ways. First, in Section 1.1.1, we obtain the formula for the number of such tilings with a specified number of dimers. Then we consider tilings with colored monomers and dimers in Section 1.1.2. Along the way, we obtain combinatorial proofs for generalizations of several identities involving tetranacci numbers from the paper by Dresden and Jin [14]; we present them in Section 1.1.3. Section 1.2 is devoted to another type of restricted tilings of the honeycomb strip. There we prohibit horizontal dimers but allow trimers of consecutive hexagons. By prohibiting any other type of trimers, we avoid long-range interaction. In terms of graphs, here we investigate the packings of P_n^2 with path graphs P_2 and P_3. The total number of such tilings (packings) is given in terms of tribonacci numbers, and Padovan and Narayana's cow numbers appear as special cases. Combinatorial proofs of some related identities are presented in Section 1.2.1.

1.1.1. Tiling a honeycomb strip with exactly k dimers

At the beginning, we define the sequences appearing in this section. Fibonacci numbers are the sequence of integers with initial values $F_0 = 0$ and $F_1 = 1$ satisfying the recurrence

$$F_n = F_{n-1} + F_{n-2}.$$

They appear as the sequence A000045 in The On-Line Encyclopedia of Integer Sequences [46]. Tetranacci numbers (sequence A000078 in OEIS) have initial values $Q_0 = Q_1 = Q_2 = 0$ and $Q_3 = 1$ and are defined by recurrence

$$Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3} + Q_{n-4}.$$

In this section, we consider a honeycomb strip of length n and its tilings by hexagonal monomers and dimers shown in Figure 1.3. We are interested in the number of such tilings with a given number of dimers. The dimers can be in any position; in Figure 1.3 we see a *descending*, a *horizontal*, and an *ascending* dimer, from left to right, respectively. The ascending and descending dimers will be both called *slanted* when their exact orientation is not important. We denote the number of all possible tilings of a honeycomb strip of length n using exactly k dimers by $c_{n,k}$.

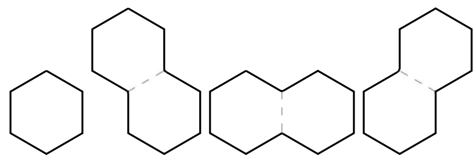

Figure 1.3: Monomer and three possible positions of a dimer tile.

Dresden and Jin [14] proved that the total number of all possible ways to tile a strip with monomers and dimers h_n satisfies the recurrence

$$h_n = h_{n-1} + h_{n-2} + h_{n-3} + h_{n-4}$$

with initial values $h_1 = 1$, $h_2 = 2$, and $h_3 = 4$. It makes sense to define $h_0 = 1$, accounting for the only possible tiling, the empty one, of the empty honeycomb strip. Their recurrence is the same as the recurrence for the tetranacci numbers Q_n with shifted initial values. Hence, $h_n = Q_{n+3}$. We wish to determine $c_{n,k}$, the number of such tilings using exactly k dimers, and hence $n-2k$ monomers. It is easy to see that $c_{n,k} = 0$ for $k > \lfloor \frac{n}{2} \rfloor$, since the strip with n hexagons can contain at most $\lfloor \frac{n}{2} \rfloor$ dimers. On the lower end, there is only one tiling without dimers, so $c_{n,0} = 1$ for all n. By stacking k dimers at the beginning of the strip, it is always possible to tile the remainder by monomers, so it follows that all $c_{n,k}$ for k between 1 and $\lfloor \frac{n}{2} \rfloor$ will be strictly positive. Hence the numbers $c_{n,k}$ will be arranged in a triangular array without internal zeros. In Table 1.1 we give the list of initial values that can be easily verified.

Table 1.1: Initial values of $c_{n,k}$.

$c_{0,0} = 1$	
$c_{1,0} = 1$	
$c_{2,0} = 1$	$c_{2,1} = 1$
$c_{3,0} = 1$	$c_{3,1} = 3$

The next theorem gives a recurrence relation for $c_{n,k}$.

Theorem 1.1.1. Let $n \geq 4$ be an integer and $c_{n,k}$ be the number of ways to tile a honeycomb strip of a length n by using exactly k dimers and $n-2k$ monomers. Then the numbers $c_{n,k}$ satisfy the recurrence relation

$$c_{n,k} = c_{n-1,k} + c_{n-2,k-1} + c_{n-3,k-1} + c_{n-4,k-2} \tag{1.1}$$

with the initial conditions given in Table 1.1.

Proof. We consider an arbitrary tiling which uses k dimers and note that the n^{th} hexagon can be tiled by a dimer or monomer. The number of all such tilings with the last hexagon tiled by a monomer is $c_{n-1,k}$, since the number of dimers k remains the same. If the last hexagon is part of a dimer, then we distinguish two possible situations: either the dimer is slanted or it is horizontal. The number of tilings ending in a slanted dimer is $c_{n-2,k-1}$ since the last dimer increases the length of a strip by two and the number of dimers by one. If the dimer is horizontal, it means that it must cover the $(n-2)^{\text{nd}}$ and the n^{th} hexagon. In that case, we have two subcases: either the $(n-1)^{\text{st}}$ hexagon is tiled by monomer, and the rest of the strip can be tiled in $c_{n-3,k-1}$ ways, or $(n-1)^{\text{st}}$ hexagon forms a dimer with $(n-3)^{\text{rd}}$ hexagon, and rest of the strip can be tiled in $c_{n-4,k-2}$ ways. Described cases are illustrated in Figure 1.4 from left to right, respectively.

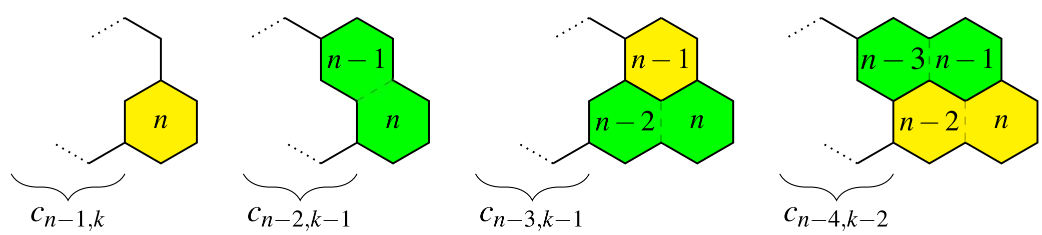

Figure 1.4: All possible endings of a tiled honeycomb strip of length n.

Since the listed cases and subcases are disjoint and describe all possible situations, the total number of tilings is the sum of the respective counting numbers i.e., $c_{n,k} = c_{n-1,k} + c_{n-2,k-1} + c_{n-3,k-1} + c_{n-4,k-2}$, which proves our theorem. ■

We are now able to list the initial rows of the triangle of $c_{n,k}$, which we do in Table 1.2 below.

Table 1.2: The initial values of $c_{n,k}$.

n/k	0	1	2	3	4	5	6	$\cdots$	Q_n
0	1								1
1	1								1
2	1	1							2
3	1	3							4
4	1	5	2						8
5	1	7	7						15
6	1	9	16	3					29
7	1	11	29	15					56
8	1	13	46	43	5				108

The triangle of Table 1.2 appears as the sequence A101350 in [46]. As mentioned before, the tilings of this kind can be represented as matching in the inner dual. But besides counting the tilings, the triangle $c_{n,k}$ also represents some coefficients in the number of matchings (Hosoya index) of some triangle chain graphs [11]. Its leftmost column consists of all 1's, counting the unique tilings without dimers. The second column seems to be given by $c_{n,1} = 2n-3$. Indeed, the only dimer in the tiling can cover either hexagons i and $i+1$ for $1 \leq i \leq n-1$, or hexagons i and $i+2$ for $1 \leq i \leq n-2$, resulting in $2n-3$ possible tilings. As expected, the rows of the triangle sum to the (shifted) tetranacci numbers, $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,k} = Q_{n+3}$, since by disregarding values k, the recurrence (1.1) becomes the defining recurrence for the tetranacci numbers. The appearance of the Fibonacci numbers as the rightmost diagonal, $c_{2n,n} = F_{n+1}$, can be readily explained by looking at the inner dual of the strip. As mentioned before, it is the ladder graph with the descending diagonal in each square, as shown in Figure 1.5. Tilings with n dimers correspond to perfect matchings in the inner dual. A simple parity argument dictates that no diagonal can participate in such a perfect matching. By omitting the diagonals we are left with a ladder graph and it is a well-known folklore result that perfect matchings in ladder graphs are counted by Fibonacci numbers. Somewhat less obvious is the appearance of the convolution of Fibonacci numbers and shifted Fibonacci numbers as the first descending subdiagonal, $c_{2n+1,n} = \sum_{k=0}^{n} F_{k+1}F_{n+2-k} = \text{A023610}(n)$, but it follows by observing that

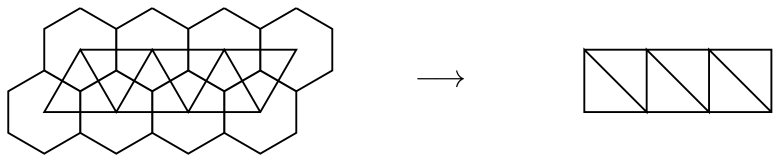

Figure 1.5: Hexagonal strip of a length $2n$ and its inner dual

the only monomer breaks the strip into two pieces each of which can be tiled by dimers only, and the number of such tilings is obtained by summing the corresponding products, hence leading to convolution. There are no formulas in the OEIS for other columns or diagonals. In the rest of this section, we determine formulas for all elements of the triangle $c_{n,k}$.

It is well known that for the Fibonacci numbers, one has $c_{2n,n} = \sum_{m=0}^{n} \binom{n-m}{m}$. By writing this as $c_{2n,n} = \sum_{m=0}^{n} \binom{n-m}{m}\binom{n-m}{0}$ and by noting that a similar formula $c_{2n+1,n} = \sum_{m=0}^{n-1} \binom{n-m}{m}\binom{n-m}{1}$ can be readily verified by induction, it becomes natural to consider $\sum_{m=0}^{n-k} \binom{n-m}{m}\binom{n-m}{k}$ as the formula for the elements on descending diagonals. By shifting the indices $n \to n-k$ and $k \to n-2k$ one arrives at expression for $c_{n,k}$.

Theorem 1.1.2. The number of ways to tile a honeycomb strip of length n using k dimers and $n-2k$ monomers is equal to

$$c_{n,k} = \sum_{m=0}^{k} \binom{n-k-m}{m}\binom{n-k-m}{n-2k}.$$

Theorem 1.1.2 can be proven by induction, but we prefer to present a combinatorial proof. To do that we need some new terms and one lemma.

We say that a tiling of a honeycomb strip is *breakable* at the position k if a given tiling can be divided into two tiled strips, the first strip of length k and the second of length $n-k$. Note that breaking the strip is only allowed along the edge of the tile. If no such k exists, we say tiling is *unbreakable*.

For example, if the first two hexagons form a dimer, the tiling is unbreakable at position 1, since it is not allowed to break a tiling through the dimer. As an example, Figure 1.6 illustrates all breakable positions of a given tiling.

Lemma 1.1.3. For $n > 4$, every tiled strip of length n is breakable into four types of unbreakable tiled strips: length-one strip tiled with a single monomer, length-two strip

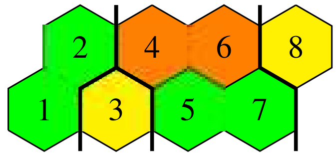

Figure 1.6: Tiling of a honeycomb strip that is breakable at positions 2, 3 and 7.

tiled with a single dimer, length-three strip tiled with a horizontal dimer and a monomer, and length-four strip tiled with two horizontal dimers.

Proof. Every left or right-slanted dimer forms a strip of length two. When removed, we are left with smaller strips, each of them tiled with hexagons and horizontal dimers. Every horizontal dimer occupies positions in the form $\{i, i+2\}$. If position $i+1$ is occupied by a monomer, hexagons in positions $i, i+1$ and $i+2$ form a length-three tiled strip. If position $i+1$ is occupied by another horizontal dimer, that dimer can occupy positions $i-1$ and $i+1$ or $i+1$ and $i+3$. Either way, those two horizontal dimers form a length-four tiled strip. After they are removed, we are left with only monomers, where each monomer forms a simple tiled strip of length one. Those are the only four types of unbreakable tilings. They are illustrated in Figure 1.7. ■

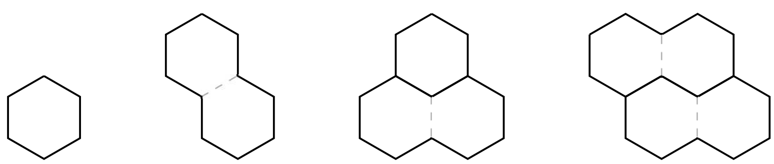

Figure 1.7: All unbreakable types of the tiled strip. The second and the fourth strip can be left or right-slanted, and the third one can be upside down, depending on the parity of the position.

Proof of the Theorem 1.1.2. We denote the types of tiled strip from Figure 1.7 by M, D, T and V, from left to right, respectively. By Lemma 1.1.3, an arbitrary tiling of a strip H_n of length $n > 4$ can be broken into those four types of unbreakable tilings. Breaking a given tiled strip into unbreakable strips produces a unique number of tiled strips of each type. So, let k_1 denote the number of strips of type D, k_2 the number of strips of type T, k_3 the number of strips of type V, and since the strip has length n, what remains are $n-2k_1-3k_2-4k_3$ strips of type M. Now we establish 1-1 correspondence between two sets: the first set, one that contains all tilings of a strip H_n which by breaking produce k_1 strips of type D, k_2 strips of type T, k_3 strips of type V and $n-2k_1-3k_2-4k_3$ strips of type M, and the second set that contains all permutations with repetition of a set with $n-k_1-2k_2-3k_3$ elements, where there are k_1 elements d, k_2 elements t, k_3 elements v and $n-2k_1-3k_2-4k_3$ elements m. From an arbitrary permutation, we obtain the corresponding tiling as follows: we replace elements v, t, d, and m with the tiled strips of type V, T, D, and M, respectively. For example, permutation $dmdvt$ yields the tiling shown in Figure 1.8. The way to obtain a permutation from a given tiling is obvious.

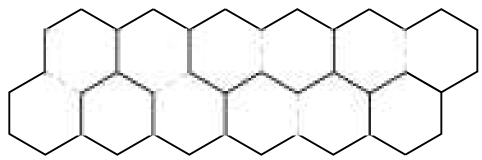

Figure 1.8: Tiling that corresponds to permutation $dmdvt$.

The number of all permutations in the second set is

$$\frac{(n-k_1-2k_2-3k_3)!}{k_1!k_2!k_3!(n-2k_1-3k_2-4k_3)!}.$$

One can easily verify that this expression can be written as

$$\binom{n-k_1-2k_2-3k_3}{k_1}\binom{n-2k_1-2k_2-3k_3}{k_3}\binom{n-2k_1-2k_2-4k_3}{k_2}.$$

Since we are interested in the number $c_{n,k}$ which denotes the number of ways to tile a length-n strip that contains exactly k dimers, note that $k_1+k_2+2k_3$ must be equal to k. Hence, we have

$$c_{n,k}=\sum_{k_1+k_2+2k_3=k}\binom{n-k_1-2k_2-3k_3}{k_1}\binom{n-2k_1-2k_2-3k_3}{k_3}\binom{n-2k_1-2k_2-4k_3}{k_2}.$$

By introducing a new index of summation $m=k_2+k_3$ and by substitutions $k_2=m-k_3$ and $k_1=k-k_2-2k_3=k-m-k_3$ we obtain:

$$\begin{aligned}
c_{n,k} &= \sum_{m=0}^{k}\sum_{k_3=0}^{m}\binom{n-k-m}{k-m-k_3}\binom{n-2k+k_3}{k_3}\binom{n-2k}{m-k_3}\\
&= \sum_{m=0}^{k}\sum_{k_3=0}^{m}\binom{n-k-m}{n-2k+k_3}\binom{n-2k+k_3}{n-2k}\binom{n-2k}{m-k_3}.
\end{aligned}$$

Finally, by using identity $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$ on the first two binomial coefficients and Vandermonde's convolution $\sum_k \binom{n}{k}\binom{r}{m-k} = \binom{n+r}{m}$ [21], we arrive to

$$\begin{aligned} c_{n,k} &= \sum_{m=0}^{k} \binom{n-k-m}{n-2k} \left(\sum_{k_3=0}^{m} \binom{k-m}{k_3} \binom{n-2k}{m-k_3} \right) \\ &= \sum_{m=0}^{k} \binom{n-k-m}{n-2k} \binom{n-k-m}{m}, \end{aligned}$$

which concludes our proof. ■

Since the row sums in Table 1.2 are tetranacci numbers, Theorem 1.1.2 gives us identity

$$Q_{n+3} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} \binom{n-k-m}{m} \binom{n-k-m}{n-2k}.$$

1.1.2. Tilings of honeycomb strip with colored dimers and monomers

Katz and Stenson [29] used colored squares and dominos to tile $(2 \times n)$-rectangular board and obtained a recursive relation for the number of all ways to tile a board. They also proved some combinatorial identities involving the number of such tilings. In this section, we do a honeycomb strip analog. We continue to count tilings of a hexagon strip with dimers and monomers, but we allow a different colors for monomers and b different colors for dimers. Let $h_n^{a,b}$ denotes the number of all different tilings of a strip with n hexagons. It is convenient to define $h_0^{a,b} = 1$. We start with initial values illustrated in Figure 1.9. One can easily see that $h_1^{a,b} = a$ since we have a colors to choose from for a monomer. Similarly, $h_2^{a,b} = a^2 + b$, since we can tile a strip with two monomers in a^2 ways or with one dimer in b ways. For $n = 3$, note that if we use only monomers, we can choose colors in a^3 ways, and if we use one dimer and one monomer, we can put the dimer in 3 different positions, and for each of those positions we can choose colors for tiles in ab ways. Hence, $h_3^{a,b} = a^3 + 3ab$.

The next theorem gives a recursive relation for $h_n^{a,b}$.

Theorem 1.1.4. For $n \geq 4$, the number of all possible tilings of the honeycomb strip containing n hexagons with a different kinds of monomers and b different kinds of dimers

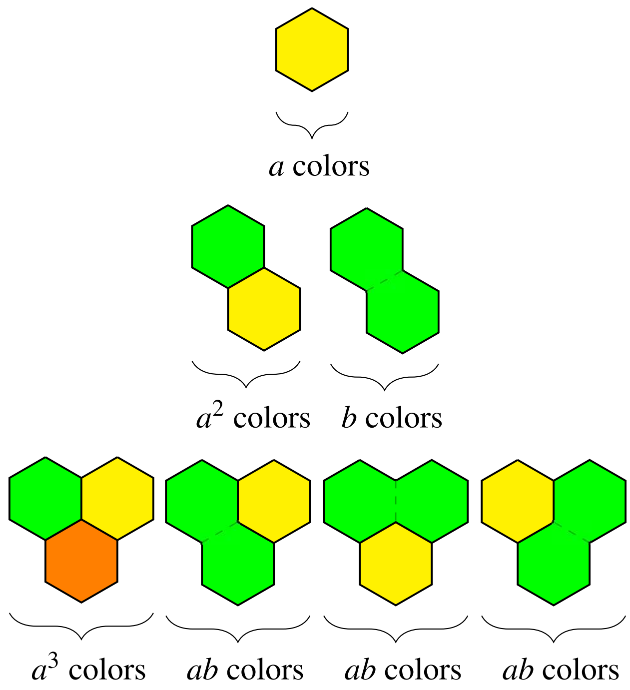

Figure 1.9: All possible tilings for $n = 1,2,3$.

satisfies the recursive relation

$$h_n^{a,b} = a \cdot h_{n-1}^{a,b} + b \cdot h_{n-2}^{a,b} + ab \cdot h_{n-3}^{a,b} + b^2 \cdot h_{n-4}^{a,b}$$

with the initial conditions $h_0^{a,b} = 1$, $h_1^{a,b} = a$, $h_2^{a,b} = a^2 + b$, and $h_3^{a,b} = a^3 + 3ab$.

Proof. The proof is similar to the proof of Theorem 1.1.1, but here we must also pay attention to the colors. We consider an arbitrary tiling and note that n^{th} hexagon can either be tiled by a monomer or a dimer. In the case when n^{th} hexagon is tiled by a monomer, the rest of the strip can be tiled in $h_{n-1}^{a,b}$ ways, but the monomer can be colored in a different ways, which gives us the total of $a \cdot h_n^{a,b}$ possible ways. If the last hexagon is part of a dimer, then we distinguish two possible situations: either the dimer is slanted, or the dimer is horizontal. The number of tilings ending in a slanted dimer is $h_{n-2}^{a,b}$, and the last dimer can be colored in b ways. So there are $b \cdot h_{n-2}^{a,b}$ such tilings. As in the proof of Theorem 1.1.1, if the dimer is horizontal, it means that it covers the $(n-2)^{\text{nd}}$ and the n^{th} hexagon. In that case, the $(n-1)^{\text{st}}$ hexagon can be tiled by monomer, we can choose colors in ab ways, and the rest of the strip can be tiled in $h_{n-3}^{a,b}$ ways. This gives us the $ab \cdot h_{n-3}^{a,b}$ possible tiling in this case. The last case is if the $(n-1)^{\text{st}}$ hexagon forms a dimer with $(n-3)^{\text{rd}}$ hexagon. There are $b^2 \cdot h_{n-4}^{a,b}$ such tilings. All cases are illustrated in Figure

1.10. This gives us a relation $h_n^{a,b} = a \cdot h_{n-1}^{a,b} + b \cdot h_{n-2}^{a,b} + ab \cdot h_{n-3}^{a,b} + b^2 \cdot h_{n-4}^{a,b}$, which proves

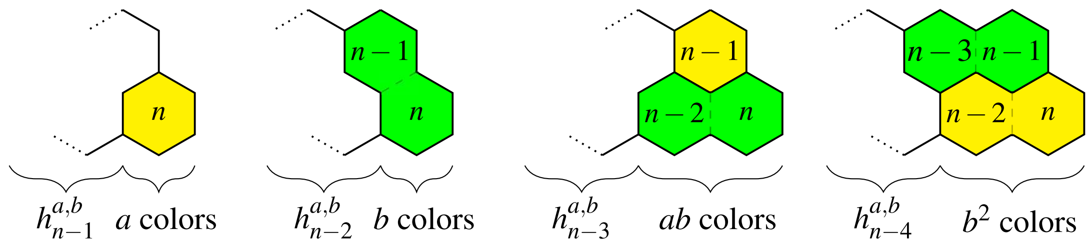

Figure 1.10: All possible endings of a colored tiling of a strip with n hexagons.

our theorem. ■

We can now list some first values of $h_n^{a,b}$ in Table 1.3. We can notice that the values

Table 1.3: Some first values of $h_n^{a,b}$.

n	$h_n^{a,b}$
0	1
1	a
2	a^2+b
3	a^3+3ab
4	$a^4+5a^2b+2b^2$
5	$a^5+7a^3b+7ab^2$
6	$a^6+9a^4b+16a^2b^2+3b^3$

$c_{n,k}$ from the last section appear in every row as coefficients of a bivariate polynomial. The connection between these values is given in the next theorem.

Theorem 1.1.5. The number $h_n^{a,b}$ of all possible tilings of the honeycomb strip of length n with monomers of a different colors and dimers of b different colors is given by

$$h_n^{a,b} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,k} a^{n-2k} b^k = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{k} \binom{n-k-m}{m} \binom{n-k-m}{k-m} a^{n-2k} b^k.$$

Proof. We could prove the theorem by induction, but again we present a simple combinatorial proof. The number $h_n^{a,b}$ denotes the number of all possible tilings of the strip of length n. For a fixed $0 \le k \le \lfloor \frac{n}{2} \rfloor$, there are $c_{n,k}$ possible ways to tile a strip with exactly k dimers, and since this tiling has k dimers and $n-2k$ monomers, the colors can be selected in $a^{n-2k}b^k$ ways which gives total of $c_{n,k}a^{n-2k}b^k$ possible tilings. Since every tiling of the strip can contain 0, 1, ..., $\lfloor \frac{n}{2} \rfloor - 1$ or $\lfloor \frac{n}{2} \rfloor$ dimers, the overall number of tilings is the sum of these cases, that is $h_n^{a,b} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,k}a^{n-2k}b^k$. ∎

1.1.3. Some (generalized) combinatorial identities involving tetranacci numbers

In this section, we generalize several identities obtained by Dresden and Jin [14] to the case of colored tilings of a honeycomb strip. All of the following identities are reduced to the mentioned identities by setting $a = b = 1$.

Theorem 1.1.6. For every $m, n \geq 0$

$$\begin{aligned} h_{m+n}^{a,b} =& h_m^{a,b}h_n^{a,b} + h_{m-1}^{a,b}\left(bh_{n-1}^{a,b} + abh_{n-2}^{a,b} + b^2h_{n-3}^{a,b}\right) + \\ & h_{m-2}^{a,b}\left(abh_{n-1}^{a,b} + b^2h_{n-2}^{a,b}\right) + b^2h_{n-1}^{a,b}h_{m-3}^{a,b}. \end{aligned}$$

Proof. We consider a tiling of a honeycomb strip containing $m+n$ hexagons. We have $h_{m+n}^{a,b}$ such tilings. On the other hand, there are $h_m^{a,b} \cdot h_n^{a,b}$ tilings that are breakable at position m, as shown in the Figure 1.11. All other tilings are unbreakable at position m.

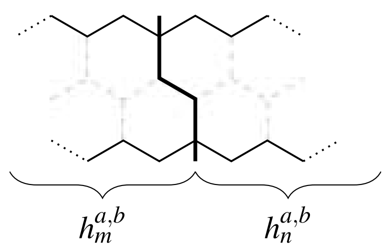

Figure 1.11: Breakable tiling at position m.

If that is the case, unbreakability can occur because of the right-inclined, left-inclined or horizontal dimer crossing the line of the break. Figure 1.12 shows all possible situations that can occur if tiling is not breakable at position m. Note that any tiling of a honeycomb strip is breakable if $n > 4$. Summing all these cases gives us proof of the theorem. ∎

Our second identity counts tilings of the strip containing at least one dimer.

Theorem 1.1.7. For every integer $n \geq 1$,

$$h_n^{a,b} - a^n = bh_{n-2}^{a,b} + 2b\sum_{k=3}^{n} a^{k-2}h_{n-k}^{a,b} + b^2\sum_{k=3}^{n} a^{k-3}h_{n-k-1}^{a,b}.$$

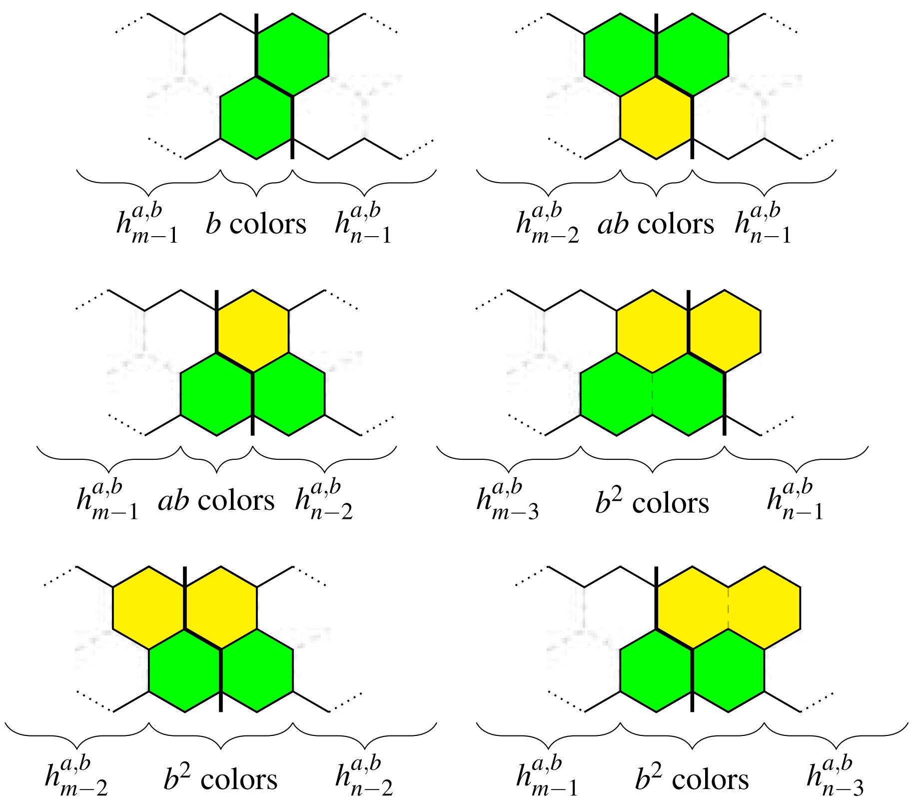

Figure 1.12: Layouts that can occur if tiling is not breakable at position m.

Proof. We prove the result by double-counting all the ways to tile a strip using at least one dimer. On one hand, there are $h_n^{a,b} - a^n$ such tilings, since the only tiling without dimers uses only monomers, and we can choose colors in a^n ways. The other way to count such tilings is to keep a trace of the position where the first dimer occurs. Since the dimer covers two positions in the strip, we use the larger number to determine its position. For example, dimer occupying hexagons 1 (or 2) and 3 has position 3. First, we start with slanted dimers. If the position of the first dimer is k for $k \geq 2$, the first part of the strip consists of $k-2$ monomers and the rest of the strip can be tiled in $h_{n-k}^{a,b}$ ways, which gives us total of $b\sum_{k=2}^{n} a^{k-2}h_{n-k}^{a,b}$ ways. Now we consider the horizontal dimer case. Note that the horizontal dimer cannot have positions 1 and 2. If the position of the dimer is k for $k \geq 3$, then the dimer occupies hexagons $k-2$ and k. We have two subcases, depending on whether the $(k-1)^{\text{st}}$ hexagon is tiled by a monomer or by a dimer. In the second case, it must be paired with $(k+1)^{\text{st}}$ hexagon, since position k is first to occur. In the first subcase, dimer and monomer can be colored in ab ways, the first part of the strip consisting of $k-3$ monomers can be colored in a^{k-3} ways, and the rest of the strip can be tiled in $h_{n-k}^{a,b}$ ways, which gives us the total of $b\sum_{k=3}^{n} a^{k-2}h_{n-k}^{a,b}$ ways. The latter subcase involves two dimers, the first occupying hexagons $k-2$ and k, and the second covering $k-1$ and $k+1$. These dimers can be colored in b^2 ways, the first part of the strip consisting of $k-3$ monomers can be colored in a^{k-3} ways, and the rest can be tiled in $h_{n-k-1}^{a,b}$ ways. Since all the cases are disjoint, the overall number is the sum of the respective counting numbers, which proves our theorem. ■

We conclude this section with a pair of identities counting tilings of the strip that contain at least one monomer.

Theorem 1.1.8. For every integer $n \geq 0$ we have

$$h_{2n}^{a,b} - b^n F_{n+1} = a\sum_{k=0}^{n} b^k h_{2n-2k-1}^{a,b} F_{k+2}$$

and

$$h_{2n-1}^{a,b} = a\sum_{k=0}^{n} b^k h_{n-2k-1}^{a,b} F_{k+2}.$$

Proof. The number of tilings of the $2n$-strip by dimers only is $b^n F_{n+1}$. Hence, the number of tilings containing at least one monomer is $h_n^{a,b} - b^n F_{n+1}$. On the other hand, we can count such tilings based on the position of the first monomer. First, we consider the odd positions in the strip. If the first monomer occurs at position $2k+1$, for some $0 \leq k \leq n-1$, the first part of the strip is tiled by dimers only, and that can be done in $b^k F_{k+1}$ ways, the monomer can be colored in a ways, and the rest of the strip can be tiled in $h_{2n-2k-1}^{a,b}$ ways. Figure 1.13 illustrates this case. Since the monomer can occur at any position

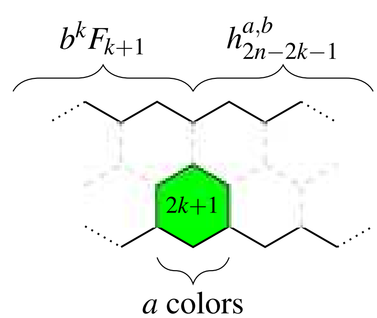

Figure 1.13: The hexagon occurs at position $2k+1$.

$2k+1$ for $0 \leq k \leq n-1$, the total number of ways that monomer occurs at odd position is $a\sum_{k=0}^{n-1} b^k h_{2n-2k-1}^{a,b} F_{k+1}$.

Now we consider the even positions. The case is similar but with some different details. If the first monomer occurs at position $2k$ for $1 \leq k \leq n$, then all $2k-1$ hexagons must be tiled with dimers. For this to be possible, the $(2k+1)^{\text{st}}$ hexagon must be tiled by the same dimer as $(2k-1)^{\text{st}}$. This dimer and monomer can be colored in ab ways. The first part of the strip containing $2k-2$ hexagons can be tiled by dimers and in $b^{k-1}F_k$ ways, and the rest of the strip in $h^{a,b}_{2n-2k-1}$ ways. This case is illustrated in Figure 1.14.

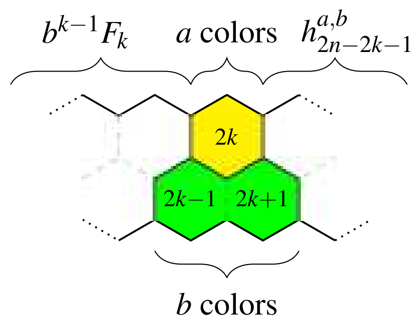

Figure 1.14: The hexagon occurs at position $2k$.

The overall number of ways in which a monomer occurs at an even position is $ab\sum_{k=0}^{n-1} b^{k-1}h^{a,b}_{2n-2k-1}F_k$, and the total number of tilings is the sum of these two cases. Since $h^{a,b}_{-1} = 0$ and $F_0 = 0$, the first sum can be extended to $k = n$ and the second to $k = 0$.

$$\begin{aligned}
h^{a,b}_{2n} - b^n F_{n+1} &= a\sum_{k=0}^{n-1} b^k h^{a,b}_{2n-2k-1}F_{k+1} + ab\sum_{k=1}^{n} b^{k-1}h^{a,b}_{2n-2k-1}F_k \\
&= a\sum_{k=0}^{n} b^k h^{a,b}_{2n-2k-1}F_{k+1} + a\sum_{k=0}^{n} b^k h^{a,b}_{2n-2k-1}F_k \\
&= a\sum_{k=0}^{n} b^k h^{a,b}_{2n-2k-1}F_{k+2}
\end{aligned}$$

The proof of the second identity is similar. In the case of the odd length of the strip, i.e. $2n-1$, the left-hand side is $h^{a,b}_n$ since it cannot be tiled by dimers only, and the proof for the right-hand side is the same. The theorem follows.

■

1.2. Tiling of a honeycomb strip and tribonacci numbers

The tribonacci numbers (sequence A000073 in OEIS [46]) are the sequence of integers starting with $T_0 = 0$, $T_1 = 0$ and $T_2 = 1$ and defined by recursive relation

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \text{ for } n \geq 3.$$

We list the first few values of the sequence in Table 1.4.

Table 1.4: The first few values of tribonacci numbers.

n	0	1	2	3	4	5	6	7	8	9	10
T_n	0	0	1	1	2	4	7	13	27	44	81

In this section we are interested in counting all tilings of a honeycomb strip of a given length, but now by using different types of tiles. We still allow monomers and slanted dimers, but we prohibit horizontal dimers. In addition, we allow trimers that cover consecutively numbered hexagons. By prohibiting horizontal dimers we effectively suppress longer-range connections represented by horizontal edges in the inner dual. Also, by allowing trimers of the form $\{i-1, i, i+1\}$ we abandon the context of matchings and instead work with packings in the inner dual. The allowed tiles are illustrated in Figure 1.15.

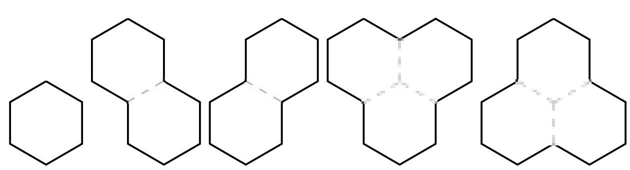

Figure 1.15: The allowed types of tiles.

Let g denote the number of ways to tile a hexagonal strip of length n by using only the allowed tiles. It is convenient to define $g_0 = 1$, and it is immediately clear that $g_1 = 1$, $g_2 = 2$.

Theorem 1.2.1. Let g_n denote the number of all ways to tile a honeycomb strip of length by using only the allowed tiles. Then

$$g_n = T_{n+2},$$

where T_n denotes n^{th} tribonacci number.

Proof. We start with an arbitrary tiling of a strip. Three disjoint cases involve the n^{th} hexagon. If the hexagon is tiled by a monomer, then the rest of the strip can be tiled in g_{n-1} ways. If it is covered by a dimer, there are g_{n-2} such tilings, and finally, if the rightmost hexagon is covered by a trimer, the are g_{n-3} such tilings. By summing the respective numbers we obtain a recurrence that is the same as the defining recurrence for the tribonacci numbers, and the initial values determine the value of the shift. ■

In the next part, we refine our result by counting the number of tilings where the number of trimers, dimers, or monomers is fixed. We denote these numbers by $t_{n,k}$, $u_{n,k}$ and $v_{n,k}$, respectively, and here n, as usual, denotes the length of a strip, and k the number of tiles of a certain kind. We can also fix the number of all types of tiles. Let $g_n^{k,l}$ denotes the number of all ways to tile a strip of a length n by using exactly k trimers, l dimers, and $n-3k-2l$ monomers. We list some first values in Table 1.5. From the definition it is clear that $t_{n,k} = 0$ for $k > \lfloor \frac{n}{3} \rfloor$, $u_{n,k} = 0$ for $k > \lfloor \frac{n}{2} \rfloor$ and $v_{n,k} = 0$ for $k > n$. It is also convenient to define $t_{0,0} = u_{0,0} = v_{0,0} = 1$. For these sequences, we can obtain recursive relations in the obvious way, by considering the state of the last hexagon to see whether it is covered by a trimer, a dimer, or a monomer. The recursive relations are

$$t_{n,k} = t_{n-1,k} + t_{n-2,k} + t_{n-3,k-1},$$

$$u_{n,k} = u_{n-1,k} + u_{n-2,k-1} + u_{n-3,k}$$

and

$$v_{n,k} = v_{n-1,k-1} + v_{n-2,k} + v_{n-3,k}.$$

We can now list some first values of the corresponding triangles. The first and the second triangle of Table 1.5 are not in the OEIS, while the third one appears as entry A104578, the Padovan convolution triangle. The same arguments as the ones used on $c_{n,k}$ show that the rows of those triangles do not have internal zeros, with the obvious exception of the zeros appearing in the first descending subdiagonal of $v_{n,k}$.

Table 1.5: Initial values for $t_{n,k}$, $u_{n,k}$ and $v_{n,k}$.

n/k	0	1	2
0	1		
1	1		
2	2		
3	3	1	
4	5	2	
5	8	5	
6	13	10	1
7	21	20	3
8	34	38	9

$t_{n,k}$

n/k	0	1	2	3	4
0	1				
1	1				
2	1	1			
3	2	2			
4	3	3	1		
5	4	6	3		
6	6	11	6	1	
7	9	18	13	4	
8	13	30	27	10	1

$u_{n,k}$

n/k	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	1	0	1						
3	1	2	0	1					
4	1	2	3	0	1				
5	2	3	3	4	0	1			
6	2	6	6	4	5	0	1		
7	3	7	12	10	5	6	0	1	
8	4	12	16	20	15	6	7	0	1

$v_{n,k}$

Before we go any further, we introduce two closely related sequences defined by Fibonacci-like recurrences of length three, the Narayana's cows sequence (A000930) and the Padovan sequence (A000931). We denote the n^{th} element of these sequences by N_n and P_n, respectively. The initial values are $N_0 = N_1 = N_2 = 1$ and $P_0 = 1$, $P_1 = P_2 = 0$, and for $n \geq 3$ we have recursive relations $N_n = N_{n-1} + N_{n-3}$ and $P_n = P_{n-2} + P_{n-3}$. In particular, there are several other sequences referred to as the Narayana's numbers, for example, A001263, a very important triangle of numbers refining the Catalan numbers and appearing in many different contexts. In the rest of this section, when we refer to Narayana's numbers, we always mean A000930.

We now take a closer look at sequences $t_{n,0}$, $u_{n,0}$ and $v_{n,0}$, i.e., at the number of tilings where one type of tile is omitted. The sequence $t_{n,0} = F_{n+1}$, since such tilings contain only slanted dimers and monomers; since such tilings correspond to matchings in the path on n vertices, they are counted by Fibonacci numbers.

The sequence $u_{n,0}$ counts the number of all ways to tile a length-n strip by using monomers and trimers, hence its elements satisfy the defining recurrence for the Narayana's cow sequence. Similarly, since the elements of the sequence $v_{n,0}$ are the numbers of all different tilings where monomers are omitted, they satisfy Padovan's recursion. Taking into account the initial values, we have $u_{n,0} = N_n$ and $v_{n,0} = P_{n+3}$. In the next three theorems, we present connections between the number of tilings and the above-listed sequences. It turns out that the elements of the three triangles of Table 1.5 can be expressed by convolution-like formulas involving the Fibonacci, the Narayana's, and the Padovan numbers. Such formulas could have been anticipated from the second column of triangle $t_{n,k}$ which seems to be the (shifted) self-convolution of Fibonacci numbers and also from the name of the entry A104578 in OEIS.

Theorem 1.2.2. For $n \geq 0$, the number of ways to tile a strip with n hexagons using exactly k trimers is

$$t_{n,k} = \sum_{\substack{i_0,\dots,i_k \geq 0 \\ i_0+\cdots+i_k = n-2k+1}} F_{i_0} \cdots F_{i_k}.$$

Proof. If there are no trimers in the tiling, one can only use dimers or monomers to tile a strip and the number of ways to do that is $t_{n,0} = F_{n+1}$. If we use exactly k trimers, those trimers divide our strip into $k+1$ smaller strips. In this context we allow the strip to be of length 0 if two trimers are adjacent. The substrips of length 0 can also appear at the beginning or the end of a strip. We have a strip with n hexagons which is tiled with k trimers, so there are $n-3k$ hexagons left to tile. Since the position of each trimer is arbitrary, the lengths of strips between and around them can vary from 0 to $n-3k$, but the sum of the lengths must be constant, that is $i_0 + i_1 + \cdots + i_k = n-3k$. Each of those smaller strips is tiled by dimers or monomers, hence in $t_{i_j,0}$ ways, where $0 \leq j \leq k$.

Summing this over all positions of the trimers we have

$$\begin{aligned} t_{n,k} &= \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k=n-3k}} t_{i_0,0}\cdots t_{i_k,0} \\ &= \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k=n-3k}} F_{i_0+1}\cdots F_{i_k+1} \\ &= \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k=n-2k+1}} F_{i_0}\cdots F_{i_k}. \end{aligned}$$

■

Note that Theorem 1.2.2 allows us to express the tribonacci numbers as a double sum

$$T_{n+2} = \sum_{k=0}^{n} t_{n,k} = \sum_{k=0}^{n} \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k=n-2k+1}} F_{i_0}\cdots F_{i_k}.$$

Theorem 1.2.3. For $n \geq 0$, the number of ways to tile a strip with n hexagons using exactly k dimers is

$$u_{n,k} = \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k=n-2k}} N_{i_0}\cdots N_{i_k}.$$

Proof. We already know that the number of tilings with no dimers is $u_{n,0} = N_n$. Now we look at the tilings of the strip with n hexagons that have exactly k dimers. That leaves us with $n-2k$ hexagons to be tiled by monomers and trimers. As in the proof of Theorem 1.2.2, we note that k dimers divide the strip into $k+1$ smaller strips, each of the length $0 \leq i_j \leq n-2k$. Each smaller strip can be tiled in N_{i_j} ways, and after summing over all possible positions of k dimers we have

$$u_{n,k} = \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k=n-2k}} N_{i_0}\cdots N_{i_k}.$$

■

The next result gives a new combinatorial interpretation of the sequence A104578.

Theorem 1.2.4. For $n \geq 0$, the number of ways to tile a strip with n hexagons using exactly k monomers is

$$v_{n,k} = \sum_{\substack{i_0,\ldots,i_k > 0 \\ i_0+\cdots+i_k=n+2k+3}} P_{i_0}\cdots P_{i_k}.$$

Proof. The proof will be analogous to the two previous proofs. The number of tilings with no monomers is $v_{n,0} = P_{n+3}$. A monomer does not divide our strip, but if it first appears in position i, we will consider strips left and right from it. We count the number of tilings of the strip H_n that have exactly k monomers. That leaves $n-k$ untiled hexagons. After omitting k hexagons what remains are $k+1$ smaller strips, each of the length $0 \leq i_j \leq n-k$. Each smaller strip can be tiled in P_{i_j+3} ways, and after summing over all possible positions of k monomers we have

$$v_{n,k} = \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k = n-k}} P_{i_0+3} \cdots P_{i_k+3}$$
$$= \sum_{\substack{i_0,\ldots,i_k > 0 \\ i_0+\cdots+i_k = n+2k+3}} P_{i_0} \cdots P_{i_k}.$$

■

Similar as before, Theorems 1.2.3 and 1.2.4 allow us to express the tribonacci numbers as double sums

$$T_{n+2} = \sum_{k=0}^{n} u_{n,k} = \sum_{k=0}^{n} \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k = n-2k}} N_{i_0} \cdots N_{i_k}.$$

and

$$T_{n+2} = \sum_{k=0}^{n} v_{n,k} = \sum_{k=0}^{n} \sum_{\substack{i_0,\ldots,i_k > 0 \\ i_0+\cdots+i_k = n+2k+3}} P_{i_0} \cdots P_{i_k}.$$

Now we turn our attention to the number of tilings of a strip H_n with numbers of all types of tiles fixed. Recall that the number of tilings consisting of k trimers, l dimers, and $n-3k-2l$ monomers is denoted by $g_n^{k,l}$. In the next theorem, we give a closed-form formula for $g_n^{k,l}$.

Theorem 1.2.5. For $n \geq 0$, the number of ways to tile a strip with n hexagons using exactly k trimers, l dimers and $n-2k-l$ monomers is

$$g_n^{k,l} = \binom{n-3k-l}{l}\binom{n-2k-l}{k}.$$

Proof. Consider a set consisting of all arbitrary tilings of a strip H_n that have exactly k trimers, l dimers, and $n-3k-2l$ monomers. To prove this theorem, we establish a 1-1 correspondence between that set and the set of all permutations of $n-2k-l$ elements where we have k elements t, l elements d and $n-3k-2l$ elements m. From an arbitrary permutation, we obtain the corresponding tiling as follows: we replace each element t with a trimer, each element d with a dimer, and each element m with a monomer. In this manner, we obtained a tiling of a strip of length n with the prescribed number of tiles of each type. For example, the permutation $tmdmt$ corresponds to the tiling shown in the Figure 1.16. We can also obtain a permutation from a given tiling in an obvious way.

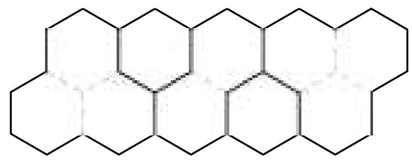

Figure 1.16: Tiling corresponding to the permutation $tmdmt$.

Since the total number of permutations of this set is $\frac{(n-2k-l)!}{k!l!(n-3k-2l)!}$, we arrive to:

$$\begin{aligned} g_n^{k,l} &= \frac{(n-2k-l)!}{k!l!(n-3k-2l)!} \cdot \frac{(n-3k-l)!}{(n-3k-l)!} \\ &= \frac{(n-3k-l)!}{l!(n-3k-2l)!} \cdot \frac{(n-2k-l)!}{k!(n-3k-l)!} \\ &= \binom{n-3k-l}{l}\binom{n-2k-l}{k}. \end{aligned}$$

■

From Theorem 1.2.5 we arrive to yet another identity for tribonacci numbers:

$$T_{n+2} = \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \sum_{l=0}^{\lfloor \frac{n-3k}{2} \rfloor} \binom{n-3k-l}{l}\binom{n-2k-l}{k}.$$

Specially, if we set $k=0$ in the first equation we have

$$\begin{aligned} t_{n,0} &= \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l}\binom{n-l}{0} \\ &= \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l} \\ &= F_{n+1}. \end{aligned}$$

Since Theorem 1.2.5 gives us the number of all tilings using the prescribed number of tiles of each type, we can express values $t_{n,k}$ and $u_{n,k}$ in a new way by summing over l and k, respectively.

Corollary 1.2.6. For $n \geq 0$,

$$t_{n,k} = \sum_{l=0}^{\lfloor \frac{n-3k}{2} \rfloor} \binom{n-3k-l}{l}\binom{n-2k-l}{k},$$

and

$$u_{n,l} = \sum_{k=0}^{\lfloor \frac{n-2l}{3} \rfloor} \binom{n-3k-l}{l}\binom{n-2k-l}{k}.$$

1.2.1. Some combinatorial identities involving tribonacci numbers

In this section, we prove, in a combinatorial way, several identities involving the tribonacci, Narayana, Padovan and Fibonacci numbers. We begin with a well-known identity for tribonacci numbers and we give it a new combinatorial interpretation:

Theorem 1.2.7. For $n \geq 4$,

$$T_n + T_{n-4} = 2T_{n-1}.$$

Proof. Let $\mathscr{G}_n$ denotes the set of all tilings of the strip H_n, $\mathscr{M}_n$, $\mathscr{D}_n$ and $\mathscr{T}_n$ the tilings ending with a monomer, dimer or trimer, respectively. As before, the cardinal number of the set $\mathscr{G}_n$ is g_n. It is clear that $\mathscr{T}_n = \mathscr{M}_n \dot\cup \mathscr{D}_n \dot\cup \mathscr{T}_n$. To prove the theorem we have to establish 1-1 correspondence between sets $\mathscr{G}_{n-2} \cup \mathscr{G}_{n-6}$ and $\mathscr{G}_{n-3} \times \{0,1\}$.

To each tiling from the set $\mathscr{G}_{n-3}$ we add a monomer at the end to obtain an element of $\mathscr{M}_{n-2}$. Thus, we obtained bijection between the sets $\mathscr{G}_{n-3}$ and $\mathscr{M}_{n-2}$. In this way, we have used all the tilings of the set $\mathscr{G}_{n-3}$ once. Now we take the tilings from the set $\mathscr{G}_{n-3}$ again, and if it is an element of $\mathscr{T}_{n-3}$, i.e., if it ends with a trimer, we remove it to obtain a tiling of length $n-6$, i.e., an element of a set $\mathscr{G}_{n-6}$. If it ends with a dimer (an element of $\mathscr{D}_{n-3}$), we remove it and replace it with a trimer to obtain an element from $\mathscr{T}_{n-2}$. Finally, if the tiling is an element of $\mathscr{M}_{n-3}$, we replace the last monomer with a dimer, to obtain an element of $\mathscr{D}_{n-2}$. In this way, we have used every tiling of a length $n-3$ twice and obtained all tilings of a length $n-2$ and $n-6$ exactly once. A diagram that visualizes 1-1 correspondence between the two sets is shown in Figure 1.17.

It follows that $g_{n-2} + g_{n-6} = 2g_{n-3}$, and since $g_n = T_{n+2}$, the theorem follows. ■

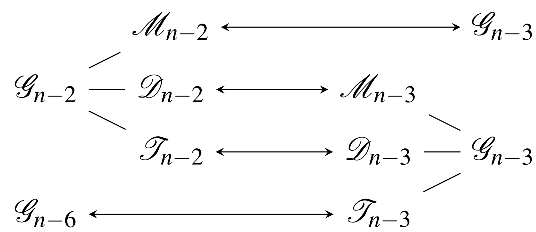

Figure 1.17: 1-1 correspondence between sets $\mathscr{G}_{n-2} \cup \mathscr{G}_{n-6}$ and $\mathscr{G}_{n-3} \times \{0,1\}$.

For the next few identities, it is useful to recall the definition of breakability. We say that the tiling of a honeycomb strip is breakable at the position k if given tiling can be divided into two tiled strips, the first one containing the leftmost k hexagons and the second one containing the rest.

Our next identity differentiates tilings based on their breakability.

Theorem 1.2.8. For any integers $m,n \geq 1$ we have the identity

$$T_{m+n} = T_m T_n + T_{m+1} T_{n+1} + T_{m-1} T_n + T_m T_{n-1}.$$

Proof. We consider an arbitrary tiling of a strip of length $m+n-2$. If the tiling is breakable at position $m-1$, we divide it into two strips of a length $m-1$ and $n-1$. Hence, the total number of tiling in this case is $g_{m-1} g_{n-1}$. If the tiling is not breakable at position $m-1$, that means that either a dimer or a trimer is blocking it. If the dimer is preventing the tiling from breaking, there are strips of lengths $m-2$ and $n-2$ on each side, so the total number of tilings in this case is $g_{m-2} g_{n-2}$. If the trimer is blocking it, it can reduce the length of the left or the right strip by two. So the total number of tilings in this case is $g_{m-3} g_{n-2} + g_{m-2} g_{n-3}$. By summing the contributions of all these cases we obtain $g_{m+n-2} = g_{m-1} g_{n-1} + g_{m-2} g_{n-2} + g_{m-3} g_{n-2} + g_{m-2} g_{n-3}$, and by using the equality $g_n = T_{n+2}$ we have $T_{m+n} = T_m T_n + T_{m+1} T_{n+1} + T_{m-1} T_n + T_m T_{n-1}$. ■

The next identity was proved by Frontczak [20] by using generating functions. Here we provide a combinatorial interpretation.

Theorem 1.2.9. For any integer $n \geq 0$ we have the identity

$$T_{n+2} = \sum_{k=0}^{n+1} F_k T_{n-k}.$$

Proof. We prove this theorem by counting all ways to tile a strip by using at least one trimer. The total number of ways to tile a length-n strip without trimers is $t_{n,0} = F_{n+1}$, hence the number of tilings having at least one trimer is $T_{n+2} - F_{n+1}$. On the other hand, we can count the same tilings by observing where the first trimer appears. If the leftmost trimer occupies hexagons $\{i, i+1, i+2\}$, we say that the position of the trimer is i. So, all possible positions range from 1 to $n-2$. If a trimer first appears at position k, the leftmost $k-1$ hexagons are tiled only by monomers and dimers, and the number of all ways to do that is F_k. The rest of the strip, of length $n-k-2$, can be tiled in T_{n-k} ways. By summing over all possible positions of the leftmost trimer, we have $T_{n+2} - F_{n+1} = \sum_{k=1}^{n-2} F_n T_{n-k}$. By using $T_{n-3} = T_n - T_{n-1} - T_{n-2}$, one can extend the tribonacci numbers to negative integers and obtain $T_{-1} = 1$. Since $T_1 = T_0 = 0$, the sum above can be extended to obtain $T_{n+2} = \sum_{k=1}^{n+1} F_k T_{n-k}$, which concludes our proof. ■

In their recent paper, Dresden and Tulskikh [15] proved a generalized formula for the collection of convolution formulas involving sequences that satisfy similar recurrences. Our last two identities can be derived from that formula but again here we present a combinatorial interpretation.

Theorem 1.2.10. For any integer $n \geq 0$ we have the identity

$$T_{n+2} = \sum_{k=0}^{n} N_k T_{n-k} + N_n.$$

Proof. We prove this theorem by counting all ways to tile a strip by using at least one dimer. The proof is analogous to the previous one. The total number of ways to tile a strip of length n without dimers is $u_{n,0} = N_n$, hence the number of tilings having at least one dimer is $T_{n+2} - N_n$. Similarly, as before, we can count the same thing by observing where the leftmost dimer appears. If the leftmost dimer occupies hexagons $\{i, i+1\}$, we say that its position is i. So, all possible positions range from 1 to $n-1$. If a dimer first appears at position k, the leftmost $k-1$ hexagons can be tiled in N_{k-1} ways. The rest of the strip is of length $n-k-1$ and it can be tiled in T_{n-k+1} ways. By summing over all possible positions of the leftmost dimer we have $T_{n+2} - N_n = \sum_{k=1}^{n-1} N_{k-1} T_{n-k+1}$. Some rearranging of indexes and the fact that $T_0 = T_1 = 0$ bring us to $T_{n+2} - N_n = \sum_{k=0}^{n} N_k T_{n-k}$ and our proof is over. ■

Theorem 1.2.11. For any integer $n \geq 0$ we have the identity

$$T_{n+2} = \sum_{k=1}^{n} P_{k+2} T_{n-k+2} + P_{n+3}.$$

Proof. Analogously as in two previous theorems, we prove this theorem by counting all ways to tile a strip by using at least one monomer. The number of ways to tile a strip of length n with at least one monomer is $g_n - v_{n,0} = T_{n+2} - P_{n+3}$. Now we can count the same thing by observing the position of the leftmost monomer. All possible positions for the first monomer range from 1 to n. If it first appears at position k, the first part of the strip, i.e., the leftmost $k-1$ hexagons can be tiled in P_{k+2} ways. The rest of the strip is of length $n-k$ and can be tiled in T_{n-k+2} ways. By summing over all possible positions of the leftmost monomer we have $T_{n+2} - P_{n+3} = \sum_{k=1}^{n} P_{k+2} T_{n-k+2}$, which concludes our proof. ■

1.2.2. Some (generalized) combinatorial identities involving full-history Horadam sequences

In this section, we generalize some of the identities from the previous section. But first, we introduce some new notation. Let $m > 0$ be an integer and let $F_n^{(m)}$ denote the n^{th} m-nacci number, i.e., the n^{th} element of a sequence that satisfies the recursive relation

$$F_n^{(m)} = F_{n-1}^{(m)} + \cdots + F_{n-m}^{(m)} \tag{1.2}$$

with initial values $F_0^{(m)} = 1$ and $F_n^{(m)} = 0$ for $n < 0$. From these sequences one can obtain the usually indexed m-nacci numbers by shifting indices by $m-1$, but here we use the unshifted indices to simplify our expressions.

Now we take a step further and generalize the sequence $F_n^{(m)}$ by defining a new sequence S_n with recursive relation

$$S_n = a_1 S_{n-1} + \cdots + a_{n-1} S_1 + a_n S_0 \tag{1.3}$$

with initial values

$$S_0 = 1 \text{ and } S_k = 0 \text{ for } k < 0,$$

where $n, a_1, a_2, \dots, a_n \geq 0$ are non-negative integers. If $a_n \neq 0$, we say that S_n satisfies a *full-history recurrence*.

First, we consider a honeycomb strip of length n, and tiles where each tile covers k consecutive hexagons of the strip and where k is an integer between 1 and m. We are interested in tiling the strip using that set of tiles. We denote the number of all possible such tilings by $g_n^{(m)}$. The next theorem by Benjamin and Quinn [2, Section 3.1] provides the number of tilings for rectangular strips, but since we consider tiles that cover only consecutive hexagons, the number of tilings is the same as in the rectangular case.

Theorem 1.2.12. Let $a_1, a_2, \dots, a_m$ be non-negative integers, and let A_n be the sequence of numbers defined by the recurrence $A_n = a_1 A_{n-1} + \cdots + a_m A_{n-m}$ with initial conditions $A_0 = 1$ and $A_n = 0$ for $n \leq 0$. Then the n^{th} element of this sequence equals the number of all possible colored tilings of a board of length n, where the tiles are of length k for $1 \leq k \leq m$ and each tile of a length k admits a_k colors.

Hence, for a special case of a previous theorem where $a_1 = a_2 = \cdots = a_m = 1$ we have $g_n^{(m)} = F_n^{(m)}$. For $m = 2$, we have the classical Fibonacci numbers, for $m = 3$, the tribonacci numbers, and in general the m-nacci numbers.

In this section, we first provide a combinatorial interpretation of the generalization of Theorem 1.2.7 to all sequences that satisfy recurrence (1.2). The identity itself,

$$F_n^{(m)} + F_{n-m-1}^{(m)} = 2F_{n-1}^{(m)}$$

follows directly from the defining recurrence. Namely,

$$\begin{aligned} F_n^{(m)} + F_{n-m-1}^{(m)} &= F_{n-1}^{(m)} + F_{n-2}^{(m)} + \cdots + F_{n-m}^{(m)} + F_{n-m-1}^{(m)} \\ &= 2F_{n-1}^{(m)}. \end{aligned}$$

Theorem 1.2.13. For $n \geq m+1$, we have $F_n^{(m)} + F_{n-m-1}^{(m)} = 2F_{n-1}^{(m)}$.

Proof. Let $\mathscr{G}_n$ denote the set of all tilings of a strip of length n using tiles where each tile covers at most m consecutive hexagons. Let $\mathscr{T}_n^1, \mathscr{T}_n^2, \dots, \mathscr{T}_n^m$ denote the set of all tilings that end with a monomer, a dimer, … or an m-mer, respectively. It is easy to see that the cardinality of the set $\mathscr{G}_n$ is $g_n = F_n^{(m)}$. It is clear that $\mathscr{G}_n = \mathscr{T}_n^1 \cup \mathscr{T}_n^2 \cup \cdots \cup \mathscr{T}_n^m$, where all the sets $\mathscr{T}_n^i, 1 \leq i \leq m$ are disjoint. To prove the theorem we have to establish a one-to-one correspondence between the sets $\mathscr{G}_n \cup \mathscr{G}_{n-m-1}$ and $\mathscr{G}_{n-1} \times \{0,1\}$.

To each tiling from the set $\mathscr{G}_{n-1}$ we add a monomer at the end to obtain an element of $\mathscr{T}_n^1$. Thus, we obtained bijection between the sets $\mathscr{G}_{n-1}$ and $\mathscr{T}_n^1$. In this way, we have used all the tilings of the set $\mathscr{G}_{n-1}$ once. Now we take the tilings from the set $\mathscr{T}_{n-1}^m$ and remove the last m-mer to obtain a tiling from a set $\mathscr{G}_{n-m-1}$. This shows a one-to-one correspondence between those sets. For an arbitrary $2 \leq i \leq m$, we consider the sets $\mathscr{T}_n^i$ and $\mathscr{T}_{n-1}^{i-1}$. Each tiling from $\mathscr{T}_n^i$ can be obtained from a tiling in the set $\mathscr{T}_{n-1}^{i-1}$ as follows: remove the last $(i-1)$-mer and replace it with an i-mer.

In this way, we have used each tiling of length $n-1$ twice and obtained each tiling of length n and each tiling of length $n-m-1$ exactly once. A diagram that visualizes the described one-to-one correspondence between the two sets is shown in Figure 1.18.

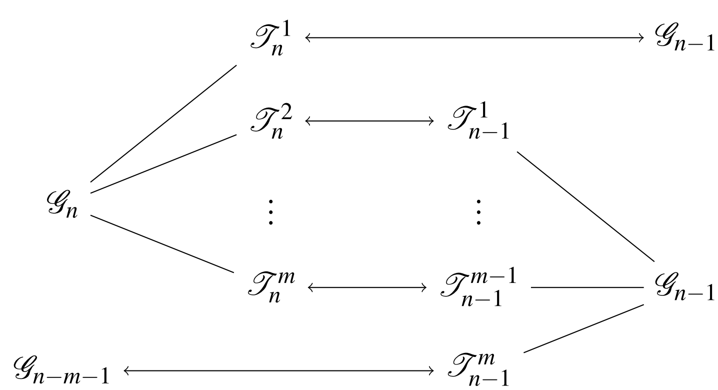

Figure 1.18: One-to-one correspondence between sets $\mathscr{G}_n \cup \mathscr{G}_{n-m-1}$ and $\mathscr{G}_{n-1} \times \{0,1\}$.

■

In the next theorem, we provide a generalization of the Theorem 1.2.8 to all sequences that satisfy recurrence (1.3). To achieve that, we consider a honeycomb strip of length n and a set of tiles that covers at most n consecutive hexagons, but in this case, the tiles are colored. More precisely, if each tile that covers i consecutive hexagons admits a_i colors, then the number of such tilings g_n satisfy recurrence (1.3), so we have $g_n = S_n$. Note that we do not need to use all tiles from the tile set; if we want to consider tilings where some size of a tile is omitted, say j, we set $a_j = 0$. Let $\mathscr{T}$ denote the set of all tile lengths considered in a tiling, i.e., $\mathscr{T} = \{i \in \{1,2,\dots,n\} : a_i > 0\}$. For example if we want to consider the Fibonacci sequence then $\mathscr{T} = \{1,2\}$, but if we consider the Pell numbers, the set $\mathscr{T}$ is the same because we care only whether the number a_k is 0 or not.

In our next theorem, we obtain an identity that differentiates tilings based on their breakability.

Theorem 1.2.14. For any integers $n,k \geq 1$ we have the identity

$$S_{n+k} = S_n S_k + \sum_{\substack{i \in \mathscr{T} \\ i \neq 1}} a_i \sum_{j=1}^{i-1} S_{n-j} S_{k-i+j}. \tag{1.4}$$

Proof. There are S_{n+k} tilings of a honeycomb strip of length $n+k$. We consider an arbitrary tiling from this set. If the tiling is breakable at position n, we divide it into two tiled strips of lengths n and k, respectively. Hence, the total number of tilings in this case is $S_n S_k$. If the tiling is not breakable at position n, that means that the tile of length i is blocking it for some $i \in \mathscr{T}$. Note that there is only one type of tile that can not block breaking, and that is the tile of length 1. Hence, $i > 1$. Every such tile occupies $j-1$ positions left of the n^{th} hexagon and $i-j$ positions right of the n^{th} hexagon. The number of such tilings is $a_i S_{n-j} S_{k-i+j}$ since the tile that blocks the breaking can be colored in a_i ways, the strip left of the tile can be colored in S_{n-j} ways, and the strip right in S_{k-i+j} ways. The n^{th} hexagon can be first in the tile that blocks breaking but can not be last, otherwise, the tiling would be breakable at position n. Hence, index j can vary from 1 to $i-1$. The total number of tilings in this case is $a_i \sum_{j=1}^{i-1} S_{n-j} S_{k-i+j}$. Since i can be any tile from a set of tiles $\mathscr{T}$ except the tile of length 1, we have the total number $\sum_{i \in \mathscr{T}, i \neq 1} a_i \sum_{j=1}^{i-1} S_{n-j} S_{k-i+j}$. Now we sum up all the cases to obtain

$$S_{n+k} = S_n S_k + \sum_{\substack{i \in \mathscr{T} \\ i \neq 1}} a_i \sum_{j=1}^{i-1} S_{n-j} S_{k-i+j},$$

which concludes our proof. ■

Although many of the following identities are known, we list them as a corollary of our main theorem. We obtain sequences from the recursive relation (1.3) by defining numbers a_k and by shifting indices if needed.

As stated in the book *Fibonacci and Lucas Numbers with Applications* [35, p. 79, p. 89], the next well-known identities that concern Fibonacci numbers were first proven by Lucas in 1876 and Mana in 1969, but here we derive them as a corollary of Theorem 1.2.14.

Corollary 1.2.15 (Fibonacci numbers)**.** For the Fibonacci numbers we have the following identities: $F_{n+k} = F_{n+1}F_k + F_nF_{k-1}$, $F_{2n+1} = F_{n+1}^2 + F_n^2$ and $F_{2n} = (F_{n+1} + F_{n-1})F_n$.

Proof. Since the recursive relation for Fibonacci numbers is $F_n = F_{n-1} + F_{n-2}$, we set $a_1 = a_2 = 1$ and $a_k = 0$ otherwise. Thus we have $\mathscr{T} = \{1,2\}$. Now the recursive relation (1.3) reduces to the recurrence for Fibonacci numbers; the initial values need some adjusting which we will do later. By Theorem 1.2.14 we have

$$\begin{aligned} S_{n+k} &= S_nS_k + \sum_{\substack{i\in\{1,2\} \\ i\neq 1}} \sum_{j=1}^{i-1} S_{n-j}S_{k-i+j} \\ &= S_nS_k + S_{n-1}S_{k-1}. \end{aligned}$$

Since it is usual to set $F_0 = 0$ for Fibonacci numbers, we shift the index in the array S_n by 1 to obtain F_n. Thus, we have $F_{n+k+1} = F_{n+1}F_{k+1} + F_nF_k$ and, after replacing k with $k-1$, we obtain a more suitable expression

$$F_{n+k} = F_{n+1}F_k + F_nF_{k-1}.$$

Now by setting $n = k$, we obtain two famous identities:

$$F_{2n+1} = F_{n+1}^2 + F_n^2$$

and

$$F_{2n} = (F_{n+1} + F_{n-1})F_n.$$

■

The next few identities first appeared in the book *Proofs that really count* [2], but can also be derived as Corollaries of Theorem 1.2.14. Since all corollaries have similar proofs, some are omitted.

The Pell numbers are another well-known sequence defined by recursive relation. Their defining recurrence is

$$P_n = 2P_{n-1} + P_{n-2},$$

with the initial values $P_0 = 0$ and $P_1 = 1$.

Corollary 1.2.16 (Pell numbers)**.** For the Pell numbers, we have the same identities as in Corollary 1.2.15.

Proof. Since the Pell numbers satisfy the relation $P_n = 2P_{n-1} + P_{n-2}$, we derive new identities for Pell numbers from recursion (1.3) by setting $a_1 = 2$ and $a_2 = 1$. Since identity (1.4) does not depend on number a_1, all identities for Fibonacci numbers are also valid for Pell numbers. Hence,

$$\begin{aligned} P_{n+k} &= P_{n+1}P_k + P_nP_{k-1}, \\ P_{2n+1} &= P_{n+1}^2 + P_n^2, \\ P_{2n} &= (P_{n+1} + P_{n-1})P_n \end{aligned}$$

■

As mentioned before, since the identity (1.4) does not depend on the number a_1, all Fibonacci and Pell identities from these corollaries can be extended to all sequences S_n that satisfy recursion

$$S_n = aS_{n-1} + S_{n-2}, \quad S_0 = 0, S_1 = 1.$$

The Jacobsthal numbers J_n are defined by recursion

$$J_n = J_{n-1} + 2J_{n-2} \tag{1.5}$$

with the initial values $J_0 = 0$ and $J_1 = 1$. The n^{th} element of this sequence can be expressed as the nearest integer of $\frac{2^n}{3}$.

Corollary 1.2.17 (Jacobsthal numbers)**.** Jacobsthal numbers satisfy the following identities: $J_{n+k} = J_{n+1}J_k + 2J_nJ_{k-1}$, $J_{2n} = (J_{n+1} + 2J_{n-1})J_n$ and $J_{2n+1} = J_{n+1}^2 + 2J_n^2$.

The result from Corollary 1.2.17 can be generalized for all Horadam sequences defined with recursion

$$S_n = aS_{n-1} + bS_{n-2}, \quad S_0 = 0, S_1 = 1,$$

for arbitrary integers a and b. One can easily verify that

$$S_{n+k} = S_{n+1}S_k + bS_nS_{k-1}.$$

The identity for tribonacci numbers was established in Theorem 1.2.8, but here we show how can it be derived from Theorem 1.2.14.

Corollary 1.2.18 (Tribonacci numbers)**.** Tribonacci numbers satisfy the identity

$$T_{n+k} = T_{n+1}T_{k+1} + T_nT_k + T_nT_{k-1} + T_{n-1}T_k.$$

Proof. We derive recursive relation for tribonacci numbers from recursion (1.3) by setting $a_1 = a_2 = a_3 = 1$ and $a_k = 0$ otherwise. This yields $\mathscr{T} = \{1,2,3\}$ and we have

$$\begin{aligned} S_{n+k} &= S_nS_k + \sum_{i=2}^{3}\sum_{j=1}^{i-1} S_{n-j}S_{k-i+j} \\ &= S_nS_k + \sum_{j=1}^{1} S_{n-j}S_{k+j-2} + \sum_{j=1}^{2} S_{n-j}S_{k+j-3} \\ &= S_nS_k + S_{n-1}S_{k-1} + S_{n-1}S_{k-2} + S_{n-2}S_{k-1}. \end{aligned}$$

Since $S_n = T_{n+2}$, we shift indices by 2 and after replacing n and k with $n-1$ and $k-1$, respectively, we obtain expression

$$T_{n+k} = T_{n+1}T_{k+1} + T_nT_k + T_nT_{k-1} + T_{n-1}T_k.$$

that concludes the proof. ■

Dresden and Jin already proved several tetranacci identities in their recent paper [14]. Here we provide yet another tetranacci identity.

Corollary 1.2.19 (Tetranacci numbers)**.** The Tetranacci numbers satisfy

$$Q_{n+k} = Q_{n+2}Q_{k+1} + Q_{n+1}(Q_{k+2} - Q_{k+1}) + Q_n(Q_{k-1} + Q_k) + Q_{n-1}Q_k.$$

Proof. We derive recursive relation for tetranacci numbers from recursion (1.3) by setting $a_1 = a_2 = a_3 = a_4 = 1$ as the only non-zero coefficients. This yields $\mathscr{T} = \{1,2,3,4\}$ and we have

$$S_{n+k} = S_nS_k + \sum_{i=2}^{4}\sum_{j=1}^{i-1} S_{n-j}S_{k-i+j}$$

$$= S_nS_k + \sum_{j=1}^{1} S_{n-j}S_{k-2+j} + \sum_{j=1}^{2} S_{n-j}S_{k-3+j} + \sum_{j=1}^{3} S_{n-j}S_{k-4+j}$$

$$= S_nS_k + S_{n-1}S_{k-1} + \sum_{j=1}^{2} S_{n-j}S_{k-3+j} + \sum_{j=1}^{3} S_{n-j}S_{k-4+j}$$

$$= S_nS_k + S_{n-1}\left(S_{k+1} - S_k\right) + S_{n-2}S_{k-1} + S_{n-2}S_{k-2} + S_{n-3}S_{k-1}.$$

By shifting indices by 3 and replacing n with $n-1$ and k with $k-2$ we arrive at

$$Q_{n+k} = Q_{n+2}Q_{k+1} + Q_{n+1}(Q_{k+2} - Q_{k+1}) + Q_n(Q_{k-1} + Q_k) + Q_{n-1}Q_k. \quad \blacksquare$$

Our last example involves a full-history recursion. The result can be also obtained from Corollary 1.2.15, but here we want to present an application of Theorem 1.2.14 in the case where $\mathscr{T} = \mathbb{N}$. We tile a honeycomb strip of length n with tiles of all possible sizes that vary from 1 to n where each tile of length k covers k consecutive hexagons in the strip and admits k colors. So, let $a_k = k$ for $k \geq 0$. We obtain a recursive relation

$$S_n = 1 \cdot S_{n-1} + 2 \cdot S_{n-2} + \cdots + n \cdot S_0 = \sum_{i=0}^{n-1} (n-i)S_i.$$

Thus we have $\mathscr{T} = \mathbb{N}$. It is not hard to see that $S_1 = 1$, $S_2 = 3$, $S_3 = 8$ and in fact, it was shown that $S_n = F_{2n}$, where F_n, as before, denotes n^{th} Fibonacci number [16]. By Theorem 1.2.14 we have

$$S_{n+k} = S_nS_k + \sum_{i=2}^{\infty} i \sum_{j=1}^{i-1} S_{n-j}S_{k-i+j},$$

and since $S_k = 0$ for $k < 0$, we can restrict our range of summation. Hence,

$$S_{n+k} = S_nS_k + \sum_{i=2}^{n+k} i \sum_{j=1}^{i-1} S_{n-j}S_{k-i+j}$$

$$= S_nS_k + \sum_{j=1}^{n+k-1} S_{n-j} \sum_{i=j+1}^{n+k} iS_{k-i+j}$$

$$= S_nS_k + \sum_{j=1}^{n} S_{n-j} \sum_{i=j+1}^{j+k} iS_{k-i+j}.$$

Since

$$\sum_{j=1}^{n} S_{n-j} \sum_{i=j+1}^{j+k} iS_{k-i+j} = \sum_{j=1}^{n} S_{n-j}\left((j+1)S_{k-1} + \cdots + (j+k)S_0\right)$$

$$= \sum_{j=1}^{n} S_{n-j}\left(j \sum_{i=0}^{k-1} S_i + 1 \cdot S_{k-1} + \cdots + k \cdot S_0\right)$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} S_{n-j}\left(j\sum_{i=0}^{k-1} S_i + S_k\right) \\
&= S_n \sum_{i=0}^{k-1} S_i + S_k \sum_{j=0}^{n-1} S_j \\
&= S_n \sum_{j=1}^{k-1} S_j + S_k \sum_{j=1}^{n-1} S_j + S_n + S_k,
\end{aligned}
$$

we finally have

$$S_{n+k} = S_n S_k + S_n \sum_{j=1}^{k-1} S_j + S_k \sum_{j=1}^{n-1} S_j + S_n + S_k.$$

All indices above are greater than 0, so we can replace S_n with F_{2n} and, after using identity $\sum_{j=1}^{n} F_{2j} = F_{2n+1} - 1$, we obtain

$$
\begin{aligned}
F_{2n+2k} &= F_{2n}F_{2k} + F_{2n}\sum_{j=1}^{k-1} F_{2j} + F_{2k}\sum_{j=1}^{n-1} F_{2j} + F_{2n} + F_{2k} \\
&= F_{2n}F_{2k} + F_{2n}(F_{2k-1} - 1) + F_{2k}(F_{2n-1} - 1) + F_{2n} + F_{2k} \\
&= F_{2n}F_{2k} + F_{2n}F_{2k-1} + F_{2k}F_{2n-1} \\
&= F_{2n}F_{2k+1} + F_{2k}F_{2n-1}.
\end{aligned}
$$

Finally, and after setting $n = k$,

$$
\begin{aligned}
F_{4n} &= F_{2n}(F_{2n+1} + F_{2n-1}) \\
&= F_n(F_{n+1} + F_{n-1})(F_{n+1}^2 + 2F_n^2 + F_{n-1}^2).
\end{aligned}
$$

1.3. DIVISIONS OF A HONEYCOMB STRIP

Similar to the previous sections, this section is also devoted to counting the number of all possible tilings of a honeycomb strip but since here we lift all restrictions on the shape of a tile, it makes more sense to call it a *division* of the honeycomb strip. In a recent paper, Brown [5] showed that the number of divisions of $2 \times m$ rectangular grid satisfies the recursive relations

$$r(m+1) = 6r(m) + r(m-1).$$

As an example, Figure 1.19 shows a division of 2×5 rectangular grid into 4 parts.

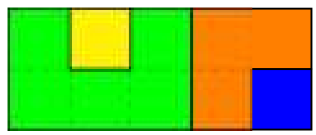

Figure 1.19: One division of 2×5 rectangular strip into 4 parts.

Here we consider a honeycomb strip analog. We are interested in finding the number of divisions into exactly k pieces. Hence we consider k to be any integer between (and including) 1 and n. Only the divisions along the edges of hexagons are considered. The hexagons are labeled in the order as it is shown in Figure 1.20.

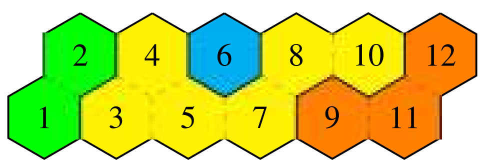

Figure 1.20: Honeycomb strip with 12 hexagons divided into 4 pieces.

Let $D_k(n)$ denote the set of all possible divisions of the honeycomb strip with n hexagons into k pieces, and $d_k(n) = |D_k(n)|$ the number of elements of the set $D_k(n)$. Clearly, $d_1(n) = 1$ for every non-negative integer n, since there is only one way to obtain one piece, and $d_n(n) = 1$, since there is only one way to obtain n pieces, that is to let each hexagon form a piece on its own. Furthermore, we assume $d_k(n) = 0$ for $k < 1$ and for $k > n$. It is convenient to set $d_1(0) = 1$. As an example, we list all possible divisions of the strip containing 4 hexagons as the first non-trivial case.

$d_1(4) = 1 \quad \{1234\}$

$d_2(4) = 6 \quad \{1,234\},\{134,2\},\{124,3\},\{123,4\},\{12,34\},\{13,24\}$

$d_3(4) = 5 \quad \{12,3,4\},\{13,2,4\},\{1,23,4\},\{1,24,3\},\{1,2,34\}$

$d_4(4) = 1 \quad \{1,2,3,4\}$

Note that the division $\{14,23\}$ is not included, since hexagons 1 and 4 are not adjacent, as shown in Figure 1.21, thus cannot form a part.

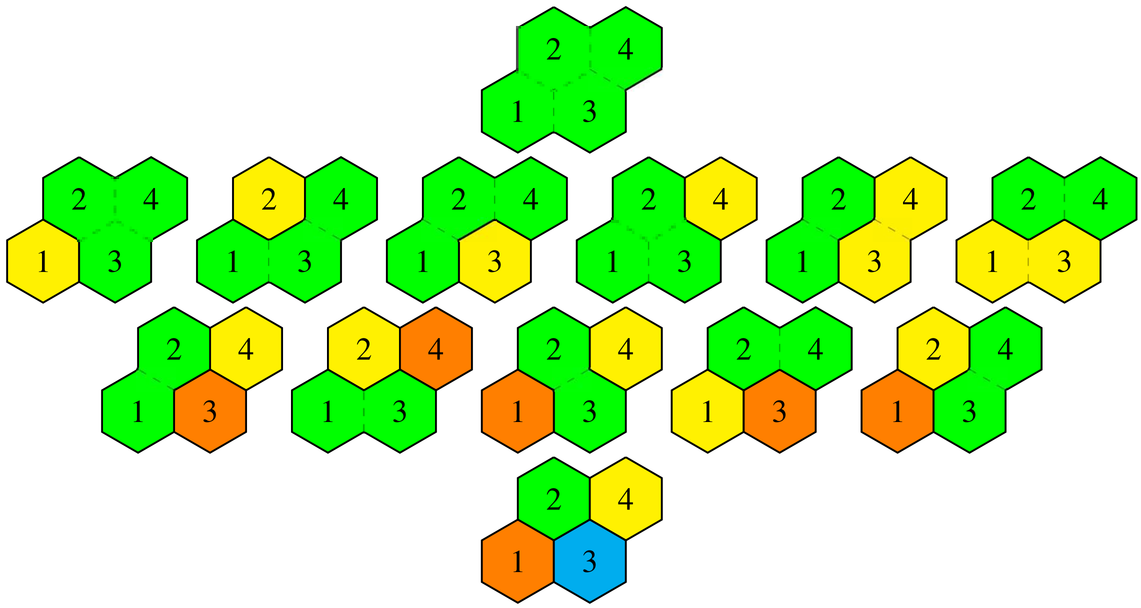

Figure 1.21: All possible divisions of the strip with 4 hexagons.

1.3.1. Recurrences and explicit formulas

Recall that $D_k(n)$ denotes the set of all possible divisions of the honeycomb strip with n hexagons into k pieces and $d_k(n) = |D_k(n)|$ the number of elements of the set $D_k(n)$. To count the divisions correctly, special attention must be paid to the rightmost two hexagons, since the new $(n+1)^{\text{st}}$ cell can interact only with them. Whether these hexagons are in the same piece or not plays a crucial role in how the new hexagon can be added. We denote by $S_k(n)$ the set of all possible divisions of the honeycomb strip with n hexagons into k pieces with two last hexagons in different pieces. Similarly, let $T_k(n)$ denote the set of all possible divisions of the strip into k pieces with two rightmost hexagons belonging to the same piece. Let $s_k(n) = |S_k(n)|$ and $t_k(n) = |T_k(n)|$. Since the last two hexagons can either be together or separated, we have divided the set $D_k(n)$ into two disjoint sets, $D_k(n) = S_k(n) \cup T_k(n)$, hence $d_k(n) = t_k(n) + s_k(n)$.

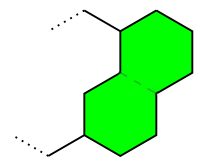

Figure 1.22: A honeycomb strip with two rightmost hexagons in the same piece.

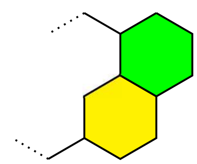

Figure 1.23: A honeycomb strip with two rightmost hexagons in different pieces.

We first establish an auxiliary result.

Theorem 1.3.1. For $n \geq 1$, the sequence $s_k(n)$ that counts all possible divisions of the honeycomb strip with n hexagons into k pieces, with the two last hexagons being in different pieces satisfies the recurrence

$$s_k(n+1) = s_{k-1}(n) + 2s_k(n) - s_k(n-1). \tag{1.6}$$

Proof. We start with a strip of length n and add one new hexagon to obtain a strip of length $n+1$. The new hexagon can either increase the number of parts in the division by 1 or not increase this number. To obtain a division with k pieces, we can only start with the division with $k-1$ or k pieces. These are two disjoint sets, so the number of all divisions will be the sum of their cardinalities.

When starting with a division consisting of $k-1$ pieces, we can obtain k pieces by adding the new hexagon as a single piece. Since there is only one way to do that, the number of divisions that can be obtained this way is $d_{k-1}(n)$. Note that the condition that the rightmost two hexagons belong to different pieces is satisfied, as shown in Figure 1.24.

It remains to consider the other case. We start with a strip divided into k pieces and we add the $(n+1)^{\text{st}}$ hexagon. If the last two hexagons in the division are together, we

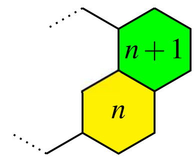

Figure 1.24: The element of $S_k(n+1)$ obtained from the element of $D_{k-1}(n)$ by adding the new hexagon as a separate piece.

cannot add the new hexagon so that the number of pieces remains the same and that the two rightmost hexagons are in different pieces. So, the last two hexagons in the division must be separated. There is only one way to add the new hexagon to the existing strip, and it is to put the $(n+1)^{\text{st}}$ hexagon together with $(n-1)^{\text{st}}$ as shown in Figure 1.25. Every other layout would be in contradiction with either the number of pieces or the fact that the two last hexagons should be separated, since putting $(n+1)^{\text{st}}$ hexagon together with n^{th} hexagon would produce the element of $T_k(n)$. So in this case we have $s_k(n)$ ways to obtain a desired division.

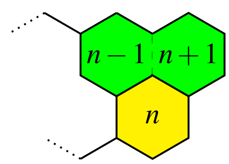

Figure 1.25: The element of $S_k(n+1)$ obtained from the element of $S_k(n)$ by joining the new hexagon with $(n-1)^{\text{st}}$ hexagon.

By summing these two cases, we obtain the recursive relation

$$s_k(n+1) = d_{k-1}(n) + s_k(n). \tag{1.7}$$

To eliminate $d_{k-1}(n)$ from relation (1.7), we use the fact that $d_{k-1}(n) = t_{k-1}(n) + s_{k-1}(n)$. By removing the last hexagon from the strip, we establish a 1-1 correspondence between all divisions of a strip with $n-1$ hexagons and divisions of a strip with n hexagons where two last hexagons are in the same part. Hence, $t_k(n) = d_k(n-1)$. Then we have

$$\begin{aligned} s_k(n+1) &= s_{k-1}(n) + t_{k-1}(n) + s_k(n) \\ &= s_{k-1}(n) + d_{k-1}(n-1) + s_k(n), \end{aligned}$$

hence $d_{k-1}(n-1) = s_k(n+1) - s_{k-1}(n) - s_k(n)$, which combined with relation (1.7) and after shifting some indices yields

$$s_k(n+1) = s_{k-1}(n) + 2s_k(n) - s_k(n-1),$$

and we proved the theorem. ∎

By disregarding values of k in recursive relation (1.6) we obtain

$$s(n+1) = 3s(n) - s(n-1),$$

where $s(n)$ represents the number of all divisions of a honeycomb strip of length n with two last hexagons in different parts. Since we obtained the same recursive relation as for the bisection of the Fibonacci sequence with $s(1) = 0$ and $s(2) = 1$ we have

$$s(n) = F_{2n-2}.$$

Our main result of this section now follows from much the same reasoning, as $d_k(n)$ satisfy the same recurrence as $s_k(n)$.

Theorem 1.3.2. For $n \geq 1$, the number of all possible divisions $d_k(n)$ of n honeycomb strip into k pieces satisfies the following relation:

$$d_k(n+1) = d_{k-1}(n) + 2d_k(n) - d_k(n-1). \tag{1.8}$$

Proof. We start with a strip with n hexagons and add one extra hexagon at the end to obtain a strip with $n+1$ hexagons. The new hexagon can either increase the number of parts in the division by 1 or not increase the number at all. Similar as in Theorem 1.3.1, the division with k pieces can only occur if we start from a division with $k-1$ or k pieces. The final number of divisions will be the sum of these cases.

First, let us consider division with $k-1$ pieces. Obtaining k pieces by adding a new hexagon can be achieved in only one way, if the new hexagon forms a piece on its own. There are $d_{k-1}(n)$ ways to obtain such division.

We move towards the divisions with k pieces. The added hexagon must be joined with an existing piece since we cannot increase the number of pieces. If the observed division is an element of $T_k(n)$, the two last added hexagons of the strip belong to the same piece.

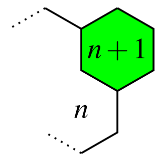

Figure 1.26: The element of $D_k(n+1)$ obtained from the element of $D_k(n)$ by adding a new hexagon as its own piece.

The Figure 1.27 shows there is only one way to add a new hexagon to the strip, by joining them all together in one piece. So the number of all divisions involving this situation is $t_k(n)$.

If the division belongs to $S_k(n)$, the two last-added hexagons belong to different parts, and there are two ways to add two new hexagons; by joining $(n+1)^{\text{st}}$ hexagon with n^{th} or $(n-1)^{\text{st}}$ hexagon. The number of all divisions involving this situation is $2s_k(n)$. This case is presented in Figure 1.28.

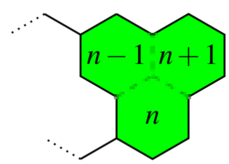

Figure 1.27: The element of $D_k(n+1)$ obtained from the element of $T_k(n)$ by joining new hexagon together with the last two hexagons.

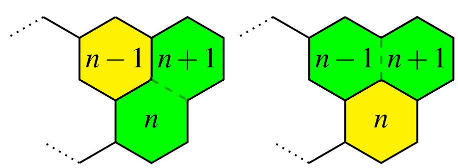

Figure 1.28: The element of $D(n+1,k)$ obtained from the element of $S(n,k)$ in two ways.

Finally, by using the fact that $t_k(n)+s_k(n)=d_k(n)$, we obtain the recursive relation

$$\begin{aligned} d_k(n+1) &= d_{k-1}(n)+t_k(n)+2s_k(n) \\ &= d_{k-1}(n)+d_k(n)+s_k(n). \end{aligned}$$

What is left to complete the proof is to eliminate term $s_k(n)$ and we achieve that using relation (1.6) and the above result which yields a system of recursive relations

$$s_k(n+1) = d_{k-1}(n)+s_k(n)$$

$$d_k(n+1) = d_{k-1}(n) + d_k(n) + s_k(n),$$

and after some re-indexing we have

$$d_k(n+1) = d_{k-1}(n) + 2d_k(n) - d_k(n-1)$$

which proves our theorem. ■

Again, by grouping together terms of recurrence (1.8) with respect to n, we obtain the recurrence satisfied by the sequence $d(n)$ counting the total number of subdivisions of a honeycomb strip of length n as

$$d(n+1) = 3d(n) - d(n-1).$$

Taking into account the initial conditions $d(1) = 1$ and $d(2) = 2$ yields a very simple answer.

Theorem 1.3.3. The total number of divisions of a honeycomb strip of length n is given by $d(n) = F_{2n-1}$, where F_n denotes the n^{th} Fibonacci number.

Note that, since $d_k(n) = t_k(n-1)$, we have also $t_k(n-1) = F_{2n-3}$. In other words, Theorem 1.3.2 provides the alternative proof that the numbers F_n satisfy the recursive relation for Fibonacci numbers.

The above theorem yields a nice combinatorial interpretation of the odd-indexed Fibonacci numbers which seems to be absent from the entry A001519 in the OEIS [46].

Using the results from Theorem 1.3.2, we list some first values of the sequence $d_k(n)$ in Table 1.6.

With the above result at hand, it is not too difficult to guess the explicit formulas for $d_k(n)$ and $s_k(n)$. The following theorem is easily proved by simply verifying that the proposed expressions satisfy the respective recurrences and initial values, and we present the details only for $d_k(n)$.

Theorem 1.3.4. The number of all divisions $d_k(n)$ of the honeycomb strip with n hexagons into exactly k pieces is

$$d_k(n) = \binom{n+k-2}{n-k}. \tag{1.9}$$

Table 1.6: Some first values of $d_k(n)$.

n/k	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	1	3	1					
4	1	6	5	1				
5	1	10	15	7	1			
6	1	15	35	28	9	1		
7	1	21	70	84	45	11	1	
8	1	28	126	210	165	66	13	1

The number $s_k(n)$ of all divisions of the honeycomb strip with n hexagons into k pieces such that two rightmost hexagons belong to different pieces is equal to zero if $n = 1$ and for $n \geq 2$ it is given as

$$s_k(n) = \binom{n+k-3}{n-k}.$$

Proof. It is trivial to check that the formula (1.9) satisfies initial values. Now we must show that the formula (1.9) satisfies the recurrence (1.8). To do that we use the identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

repeatedly.

$$\begin{aligned}
d_k(n+1) &= d_{k-1}(n) + 2d_k(n) - d_k(n-1) \\
&= \binom{n+k-3}{n-k+1} + 2\binom{n+k-2}{n-k} - \binom{n+k-3}{n-k-1} \\
&= \binom{n+k-3}{n-k+1} + \binom{n+k-2}{n-k} + \binom{n+k-3}{n-k} \\
&= \binom{n+k-2}{n-k+1} + \binom{n+k-2}{n-k} \\
&= \binom{n+k-1}{n-k+1}
\end{aligned}$$

■

By double counting the set $D(n)$, we gave a new meaning to the well-known identity

$$\sum_{k=1}^{n}\binom{n+k-2}{n-k}=F_{2n-1}.$$

1.3.2. Combinatorial approach

In this section, we offer three combinatorial proofs of Theorems 1.3.3 and 1.3.4. The first one is based on a counting argument, while the other two are bijective. First, we introduce some terms which will be useful in those proofs.

Any side shared by two hexagons in the strip is called an *internal edge*. Similarly, the sides of a hexagon in the strip belonging to one hexagon only will be referred to as the *external edges*. Any vertex shared by three hexagons is called an *internal vertex*. A vertex shared by exactly two hexagons is called a *cut vertex*. There are n cut vertices and we label them with numbers from 1 to n as in Figure 1.29. For each pair of cut vertices n_1 and n_2, where $1 \le n_1 < n_2 \le n$, we define the cut $\{n_1, n_2\}$ as the shortest path from n_1 to n_2 that does not contain any external edge. For example, Figure 1.29 shows the strip with 12 hexagons. The cuts $\{3,6\}$ and $\{8,10\}$ are colored red.

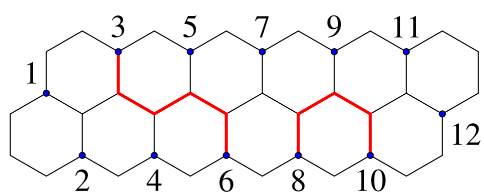

Figure 1.29: Labeling the cut vertices and cuts $\{3,6\}$ and $\{8,10\}$ in the strip with 12 hexagons.

Theorem 1.3.5. The number of all divisions $d_k(n)$ of the honeycomb strip with n hexagons into exactly k pieces is

$$d_k(n)=\binom{n+k-2}{n-k}.$$

Proof. There is only one way to obtain one part, so the theorem holds for $k=1$. To obtain two parts, we choose two cut vertices that produce one cut that can be achieved in $\binom{n}{2}$ ways. As an example, Figure 1.30 shows the cut $\{2,6\}$ which yields two parts.

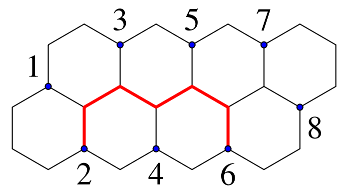

Figure 1.30: Obtaining two parts by choosing two cut vertices and the corresponding cut.

Three parts can be obtained by choosing three cut vertices $n_1 < n_2 < n_3$ and their corresponding cuts $\{n_1, n_2\}$ and $\{n_2, n_3\}$. But three parts can also be obtained by choosing four vertices $n_1 < n_2 < n_3 < n_4$ and the cuts $\{n_1, n_2\}$ and $\{n_3, n_4\}$. This is shown in Figure 1.31.

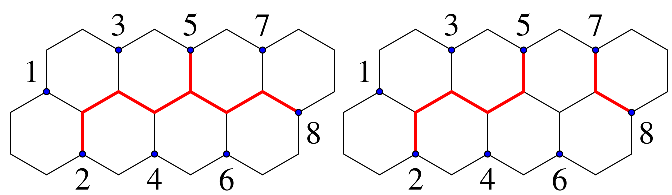

Figure 1.31: Obtaining three parts by choosing three or four cut vertices.

Since obtaining k parts in a division requires $k-1$ cuts, to divide the strip into four parts, we need three cuts, and three cuts can be obtained from four, five, or six cut vertices. But when we choose five or six cut vertices, we have some extra cuts to remove. The left part of Figure 1.32 shows an example where we choose cut vertices $1,3,5$ and 6 that produce three cuts $\{1,3\}$, $\{3,5\}$ and $\{5,6\}$, and hence four parts. In the central part of Figure 1.32, we choose five cut vertices $1,3,5,6$ and 8 which yield four cuts $\{1,3\}$, $\{3,5\}$, $\{5,6\}$, and $\{6,8\}$. This would produce five parts in the division, so we have to remove one cut, for example, the cut $\{5,6\}$. Instead of the cut $\{5,6\}$, we could also remove the cut $\{3,5\}$, but not $\{1,3\}$ or $\{6,8\}$, because that would lead to a division with only four cut vertices, and that was counted elsewhere. Similarly, the right part of Figure 1.32 demonstrates obtaining four parts by choosing six cut vertices $1,2,3,5,6$ and 8. In this case, we have to remove two cuts, and the only cuts that can be removed are $\{2,3\}$ and $\{5,6\}$. Choosing any other pair of cuts would decrease the number of cut vertices.

More generally, dividing a strip into k parts requires choosing the minimum of k cut vertices, where each two neighboring cuts share an endpoint, and a maximum of $2k-2$

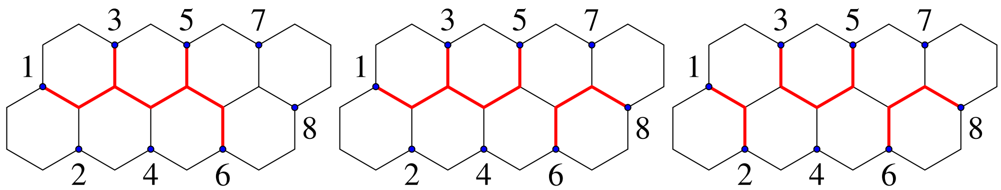

Figure 1.32: Obtaining four parts by choosing four, five, or six cut vertices.

cut vertices, where no two cuts share an endpoint. Since k cut vertices produce $k-1$ cuts and k parts, to obtain a division with exactly k parts by using $k+t$ cut vertices, we need to remove t cuts. Let $n_1, n_2, \ldots, n_{k+t-1}$ and n_{k+t} denote the chosen cut vertices. They generate $k+t-1$ cuts, $\{n_1, n_2\}, \{n_2, n_3\}, \ldots, \{n_{k+t-2}, n_{k+t-1}\}$ and $\{n_{k+t-1}, n_{k+t}\}$. Removing the first and the last cut is not an option, since we want to generate a division using exactly $k+t$ cut vertices. Also, we can not remove any two consecutive cuts $\{n_{l-1}, n_l\}$ and $\{n_l, n_{l+1}\}$ because it produces a division without cut vertex n_l. Hence, from $k+t-3$ cuts, we need to remove t cuts where no two cuts are consecutive. Recall that, from p objects arranged in an array, there are $\binom{p-s+1}{s}$ ways to choose s objects where no two objects are consecutive. So, there are $\binom{n}{k+t}$ ways to choose $k+t$ cut vertices and $\binom{k+t-3-t+1}{t} = \binom{k-2}{t}$ ways to remove the extra t cuts, where each of the $k+t$ cut vertices is an endpoint of at least one remaining cut. Finally, using the identity $\binom{n}{k} = \binom{n}{n-k}$ and Vandermonde's convolution [21], for $k \geq 2$, the number of divisions of the strip into k parts is

$$\begin{aligned} d_k(n) &= \sum_{t=0}^{k-2} \binom{n}{k+t} \binom{k-2}{t} \\ &= \sum_{t=0}^{k-2} \binom{n}{n-k-t} \binom{k-2}{t} \\ &= \binom{n+k-2}{n-k}. \end{aligned}$$

■

Note that the above theorem is just the first half of Theorem 1.3.4. The other half of Theorem 1.3.4, the formula for the numbers $s_k(n)$, can be proved combinatorially in the same manner. The only difference is the following: to obtain a division with the last two hexagons in separate pieces, choosing the cut vertex n is mandatory. Also, notice that by summing over all possible values of k, the above counting argument also yields a proof of Theorem 1.3.3.

The next theorem presents an alternative proof that the number of divisions satisfies the recurrence for the odd-indexed Fibonacci number.

Theorem 1.3.6. For $n \geq 2$, the number of all possible divisions $d(n)$ of the honeycomb strip with n hexagons satisfies the following relation:

$$d(n+1) = 3d(n) - d(n-1). \tag{1.10}$$

Proof. We use the technique of Benjamin and Quinn [2, Identity 7] to show that $3d(n) = d(n+1) + d(n-1)$. To be precise, we establish a one-to-one correspondence between the three copies of the divisions of a honeycomb strip with n hexagons and the divisions of honeycomb strips with $n+1$ and $n-1$ hexagons, respectively. In total, there are $3d(n)$ such divisions on one side, and $d(n+1)+d(n-1)$ divisions on the other side. Recall that the set of all divisions of the honeycomb strip with n hexagons can be organized into two disjoint subsets based on the status of the last hexagon. In the first case, the n^{th} hexagon is in the same part as the $(n-1)^{\text{st}}$ hexagon, and in the second case it is not. In the second case, we distinguish two more subcases. In the first subcase, we have all divisions in which the last hexagon forms a part on its own, and the second subcase is one where the n^{th} hexagon is in the same part as the $(n-2)^{\text{nd}}$ hexagon but not in the same part with the $(n-1)^{\text{st}}$ hexagon. All the cases and subcases are presented in Figure 1.33.

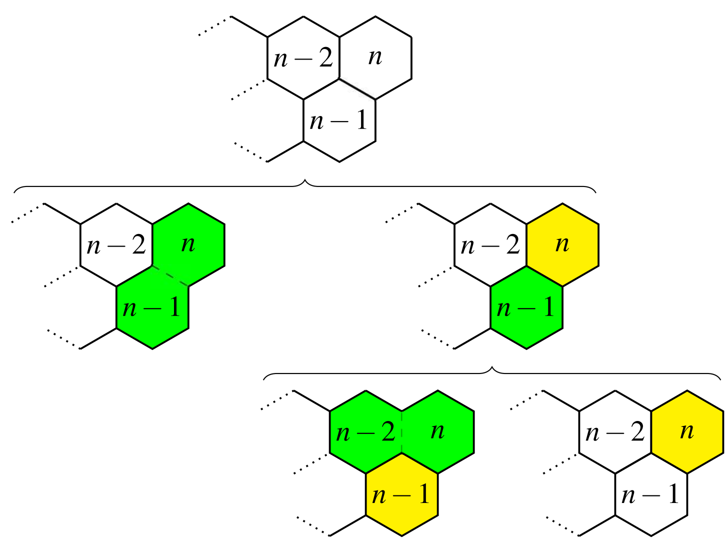

Figure 1.33: All divisions of the honeycomb strip are organized into cases and subcases, based on the status of the last hexagon.

For the first copy of the divisions of a honeycomb strip with n hexagons, we add a single $(n+1)^{\text{st}}$ hexagon in the same part with the n^{th} hexagon. Thus, we obtained all divisions of a honeycomb strip with $n+1$ hexagons, where the two last hexagons are in the same part. For the second copy of the divisions of a honeycomb strip with n hexagons, we add a single $(n+1)^{\text{st}}$ hexagon forming a part on its own. In the third copy, we distinguish two cases. If the n^{th} and $(n-1)^{\text{st}}$ hexagons are not in the same part, we add on a single $(n+1)^{\text{st}}$ hexagon to the same part with the $(n-1)^{\text{st}}$ hexagon. If the n^{th} and $(n-1)^{\text{st}}$ hexagons are in the same part, we remove the n^{th} hexagon which yields a honeycomb strip with $n-1$ hexagons. In this case, we obtained all divisions of the honeycomb strip with $n-1$ hexagons. Since the $(n+1)^{\text{st}}$ hexagon can only form a part on its own, be in the same part with the n^{th} hexagon or be in the same part with the $(n-1)^{\text{st}}$ hexagon but not with the n^{th} hexagon, we also obtained all division of the honeycomb strip with $n+1$ hexagons. Figure 1.34 shows the 1-1 correspondence between the three copies of the divisions of a honeycomb strip with n hexagons on one side, and the divisions of a honeycomb strip with $n-1$ and $n+1$ hexagons, on the other side.

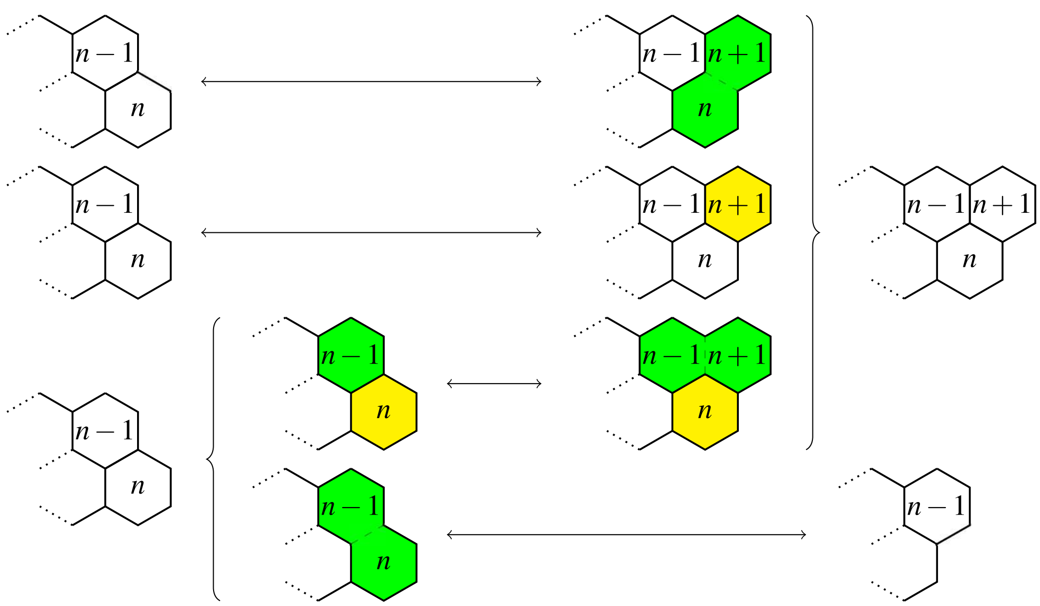

Figure 1.34: 1-1 correspondence between the three copies of divisions of the honeycomb strip with n hexagons and divisions of the honeycomb strip with $n-1$ and $n+1$ hexagons.

■

In the next part, we want to prove Theorem 1.3.3 by obtaining a bijective correspondence between the number of divisions of a narrow honeycomb strips and matchings in a path of a suitable length. Recall that a *matching* in a graph G is a collection M of edges of G such that no two edges in M share a vertex. In other words, a matching is a collection of isolated edges. The study of matchings is an important and well-developed branch of graph theory [37].

It is a simple exercise to show that matchings in a path P_n on n vertices are enumerated by the Fibonacci numbers F_{n+1}. Indeed, if we denote by p_n the number of matchings in P_n, we can obtain a simple recurrence for p_n by taking $n \geq 3$ and considering the terminal edge in P_n. Let us denote this edge by e. Now, the matchings in P_n which do not contain e are counted by p_{n-1}, while those containing e are counted by P_{n-2} since the edge that shares a vertex with e cannot participate. As every matching in P_n either contains e or does not contain it, we obtain that p_n satisfy the recurrence $p_n = p_{n-1} + p_{n-2}$. The claim now follows by taking into account the initial values $p_1 = 1$ and $p_2 = 2$.

Theorem 1.3.7. Let $n \geq 2$ be an integer. Then there is a bijection between the set of all matchings in P_{2n-2} and the set of all divisions of a honeycomb strip of length n.

Proof. There are $2n-3$ internal edges in the honeycomb strip containing n hexagons. They are labeled with numbers from 1 to $2n-3$ using the pattern shown in Figure 1.35. Every division can be expressed as the set of deleted edges D, but not in a unique way. For

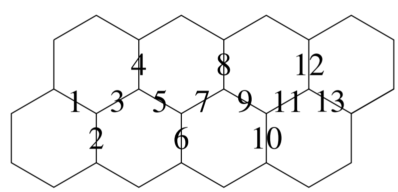

Figure 1.35: Labeling the internal edges in the honeycomb strip.

example, deleting internal edges 3 and 5 yields the same division as deleting the edges 3, 4 and 5 or just 3 and 4. To avoid ambiguity, we proceed as follows. If the hexagons i and $i+1$ are in the same piece, there is only one way to express that information: deleting the edge $2i-1$. Similarly, if the hexagons i and $i+2$ are in the same piece, but not together with the hexagon $i+1$, there is only one way to indicate that situation: deleting the edge $2i$. In the last case, where all three hexagons i, $i+1$, and $i+2$ are in the same piece, we only add the edges $2i-1$ and $2i+1$ to the set D. Since the edges $2i-1$ and $2i+1$ are already deleted, deleting the edge $2i$ produces no effect, so we do not include it in the set D. This means that every division has a unique expression as the set of deleted edges D where no two edges are labeled with consecutive numbers. This gives us clear 1-1 correspondence with the set of all matchings of the path graph with $2n-3$ edges which concludes the proof. ■

Figure 1.36 shows all matchings of the path graph P_6 and corresponding divisions of the honeycomb strip of length 4. The edges in the path graph are labeled naturally, and the edges participating in the matching are red.

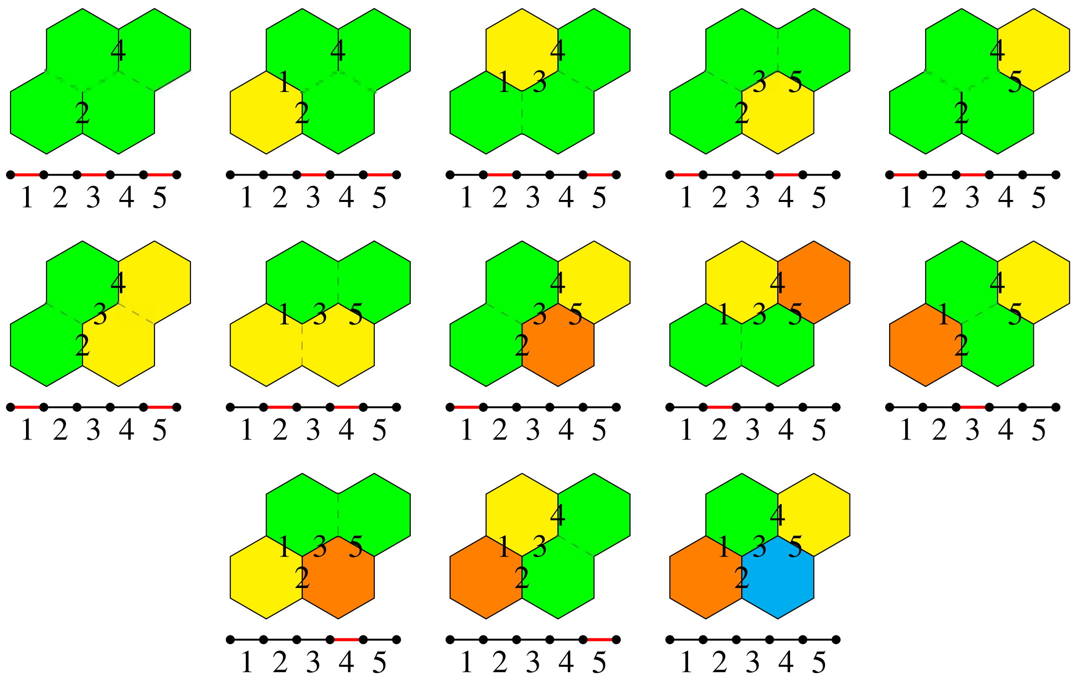

Figure 1.36: 1-1 correspondence between the divisions of the honeycomb strip with 4 hexagons and all the matchings in the path with 5 edges.

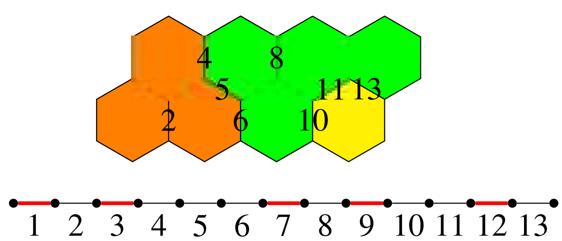

Figure 1.37: The honeycomb strip divided into 3 pieces and the corresponding matching of the path graph P_{14}.

Another example with the honeycomb strip of length 8 is given in Figure 1.37. The division can be uniquely expressed as $D = \{1,3,7,9,12\}$ which corresponds to the matching of the path P_{14} shown below the honeycomb strip.

1.3.3. Transfer matrix method

In this section, we present another approach to obtain an overall number of divisions, the one based on transfer matrices [47]. It might seem less natural than recurrence relations and combinatorial approach, but it often turns out to be suitable when recurrence relations are complicated or unknown.

We again consider a honeycomb strip such as the one shown in the Figure 1.20, and look at its rightmost column, i.e., at the hexagons labeled by $n-1$ and n. There are two possible situations regarding these hexagons. They can be in the same piece of a subdivision, or they can belong to two different pieces. We denote a strip with the last two hexagons together as a type T strip and a strip with the last two hexagons separated as a type S strip. Adding the $(n+1)^{\text{st}}$ hexagon might result again in a type S strip or a type T strip. There are altogether four possibilities, each of them producing certain effects on the number of pieces in the resulting strip. For example, if we start with a strip of type S and we want to obtain a strip of type S, we can either add the new hexagon to the part which contains the $(n-1)^{\text{st}}$ hexagon, or we can let the $(n+1)^{\text{st}}$ hexagon to form its own part. In the first case, the number of parts will remain the same, in the second case it will increase by one. Figure 1.3.3 shows this case. The main idea of the transfer matrix method is

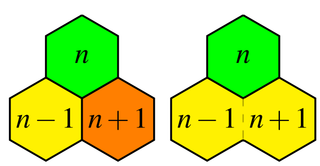

Figure 1.38: Both cases result with a strip of type S.

to arrange the effects of adding a single hexagon into a 2×2 matrix whose entries will keep track of the number of pieces via a formal variable, say, y. The rows and columns of such a matrix are indexed by possible states, in our case T and S, and the element at the position S,S in our example will be $1+y$. That captures the fact that transfer from S to S results either in the same number of pieces or the number of pieces increases by one.

The other three possible transitions, $T \to T$, $S \to T$ and $T \to S$ are described by matrix elements $1, 1$, and y, respectively. Indeed, it is clear that adding a hexagon to obtain the rightmost column together cannot increase the number of pieces, hence the two ones, and that starting from T and arriving at S is possible only by the last hexagon forming a new piece, hence increasing the number of pieces by one, hence y. If we denote our matrix by H, we can write it as

$$H(y) = \begin{bmatrix} 1 & 1 \\ y & 1+y \end{bmatrix}.$$

By construction, it is clear that adding a new hexagon will be well described by multiplying some vector of states by our matrix $H(y)$, and that repeated addition of hexagons will correspond to multiplication by powers of $H(y)$. It remains to account for the initial values.

For $n = 1$, we have a trivial case, one hexagon forms one part. For $n = 2$ we have two possibilities, hexagons are in the same part or separated. Hence this case is represented by a vector

$$\overrightarrow{h_2} = x^2 \begin{bmatrix} y \\ y^2 \end{bmatrix}.$$

By introducing another formal variable, say x, to keep track of the length, the above procedure will produce a sequence of bivariate polynomials whose coefficients are our numbers $d_k(n)$. The first few polynomials are shown in Table 1.3.3 after the theorem which summarizes the described procedure.

Theorem 1.3.8. The number of divisions of a honeycomb strip of a length n into k parts is the coefficient by $x^n y^k$ in the expression

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ y & 1+y \end{bmatrix}^{n-2} \begin{bmatrix} y \\ y^2 \end{bmatrix} x^n. \tag{1.11}$$

The coefficients by $x^n y^k$ in expression (1.11) could be determined by studying the powers of the transfer matrix. By looking at the first few cases,

$$H(y)^2 = \begin{bmatrix} 1+y & 2+y \\ 2y+y^2 & 1+3y+y^2 \end{bmatrix} \quad \text{and} \quad H(y)^3 = \begin{bmatrix} 1+3y+y^2 & 3+4y+y^2 \\ 3y+4y^2+y^3 & 1+6y+5y^2+y^3 \end{bmatrix},$$

we could guess the entries in the general case and then verify them by induction.

n	
1	xy
2	$x^2(y+y^2)$
3	$x^3(y+3y^2+y^3)$
4	$x^4(y+6y^2+5y^3+y^4)$
5	$x^5(y+10y^2+15y^3+7y^4+y^5)$
6	$x^6(y+15y^2+35y^3+28y^4+9y^5+y^6)$

Table 1.7: The first few bivariate polynomials from the transfer matrix method.

Lemma 1.3.9. Matrix

$$H(y)^n = \begin{bmatrix} p(n) & s(n) \\ ys(n) & p(n+1) \end{bmatrix}$$

with $p(n) = \sum_{k=1}^{n} \binom{n+k-2}{n-k} y^{k-1}$ and $s(n) = \sum_{k=1}^{n} \binom{n+k-1}{n-k} y^{k-1}$.

Proof. Proof is by induction. For $n=1$ we have $p(1)=s(1)=1$ and $p(2)=1+y$. To finish to proof we need to show that

$$\begin{bmatrix} p(n) & s(n) \\ ys(n) & p(n+1) \end{bmatrix} \begin{bmatrix} p(1) & s(1) \\ ys(1) & p(2) \end{bmatrix} = \begin{bmatrix} p(n)+ys(n) & p(n)+(1+y)s(n) \\ ys(n)+yp(n+1) & ys(n)+(1+y)p(n+1) \end{bmatrix} = \begin{bmatrix} p(n+1) & s(n+1) \\ ys(n+1) & p(n+2) \end{bmatrix}.$$

We have

$$\begin{aligned} p(n-1)p(1)+s(n-1)ys(1) &= p(n-1)+ys(n-1) \\ &= \sum_{k=1}^{n} \binom{n+k-2}{n-k} y^{k-1} + \sum_{k=1}^{n} \binom{n+k-1}{n-k} y^{k} \\ &= 1 + \sum_{k=1}^{n-1} \binom{n+k-1}{n-k-1} y^{k} + \sum_{k=1}^{n-1} \binom{n+k-1}{n-k} y^{k} + y^n \\ &= 1 + \sum_{k=2}^{n} \binom{n+k-1}{n-k+1} y^{k-1} + y^n \\ &= \sum_{k=1}^{n+1} \binom{n+k-1}{n-k+1} y^{k-1} \\ &= p(n+1). \end{aligned}$$

The rest of the proof can be easily done similarly but here we omit that part. ■

Lemma 1.3.9 allows us to simplify the expression (1.11) to have

$$\begin{aligned}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ y & 1+y\end{bmatrix}^{n-2}\begin{bmatrix}y\\ y^2\end{bmatrix}x^n &= \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}p(n-2) & s(n-2)\\ ys(n-2) & p(n-1)\end{bmatrix}\begin{bmatrix}y\\ y^2\end{bmatrix}x^n\\ &= \left(yp(n-2)+ys(n-2)+y^2s(n-2)+y^2p(n-1)\right)x^n\\ &= (yp(n-1)+y\,(s(n-2)+yp(n-1)))\,x^n\\ &= p(n)x^n y\\ &= \sum_{k=1}^{n}\binom{n+k-2}{n-k}y^k x^n\end{aligned}$$

By Theorem 1.3.8 we have

$$d(n,k)=\binom{n+k-2}{n-k}.$$

Now we turn our attention to the number of all possible divisions, i.e. we wish to determine the number $d(n)$. To do that, we again use a matrix $H(y)$ and Theorem 1.3.8. By setting $y=1$, we have $H(1)=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}=\begin{bmatrix}F_1 & F_2\\ F_2 & F_3\end{bmatrix}$. Again, the following claim is easily guessed and verified by induction.

Lemma 1.3.10. $H(1)^n=\begin{bmatrix}F_{2n-1} & F_{2n}\\ F_{2n} & F_{2n+1}\end{bmatrix}$.

Finally, by Lemma 1.3.10 we can simplify expression (1.11) to have

$$\begin{aligned}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}F_{2n-5} & F_{2n-4}\\ F_{2n-4} & F_{2n-3}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} &= \begin{bmatrix}F_{2n-5}+F_{2n-4} & F_{2n-4}+F_{2n-3}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}\\ &= \begin{bmatrix}F_{2n-3} & F_{2n-2}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}\\ &= F_{2n-1}.\end{aligned}$$

By Theorem 1.3.8 we have $d(n)=F_{2n-1}$.

1.4. Further possibilities of research

In this chapter, we have considered various ways of tiling a narrow honeycomb strip of a given length with different types of tiles. We have refined some previously known results for the total number of tilings of a given type by deriving formulas for the number of such tilings with the prescribed number of tiles of a given type. We have also considered tilings with colored tiles and obtained the corresponding formulas. Along the way, we have provided combinatorial interpretations for some known identities and established several new ones. Also, we have provided closed-form expressions for several triangles of numbers appearing in the OEIS.

But this work opens a possibility for many directions of further research. In particular, we have not considered any jamming-related scenarios, i.e., the tilings that are suboptimal with respect to the number of large(r) tiles. The existence of connections between our tilings and such problems is indicated by the appearance of Padovan numbers in both contexts [13]. Further, we have not examined statistical properties such as the expected number of tiles in a random tiling of a strip of a given length in the way done in ref. [9]. We have not looked at the asymptotic behavior of the counting sequences. Each of the mentioned omissions could be an interesting topic for further research.

Another interesting direction would be to look in more detail at triangles $c_{n,k}$, $t_{n,k}$, $u_{n,k}$, and $v_{n,k}$. We have shown that their rows (with one trivial exception) do not have internal zeros. By inspection of the first few rows of $c_{n,k}$, $t_{n,k}$, and $u_{n,k}$ one can observe that the rows seem to be also unimodal and even log-concave. It would be interesting to investigate whether those properties hold for the whole triangles. Both properties are violated in rows of $v_{n,k}$, but the violations seem to be restricted to the right end. What happens if the rightmost three elements are omitted? Also, the position of the maximum presents an interesting challenge. Finally, more interesting identities could be derived by looking at ascending and descending diagonals of different slopes in those triangles.

As for the divisions of a honeycomb strip, our results could be further exploited in several directions. From the recurrences established in Section 1.3.1, one could compute the bivariate generating functions for the numbers $d_k(n)$, which could then be used to compute the expected number of parts in a randomly chosen division. Another possibly fruitful direction would be to consider wider strips. Especially, the approach in the proof of Theorem 1.3.5 could be modified to encompass wider strips. At first glance, it is obvious that such an approach should include some paths along the internal edges that are not the shortest. Alternatively, one could keep the strip-width low, but arrange the hexagons in the zig-zag pattern. Finally, one could consider division problems on other lattices.

2. METALLIC CUBES

2.1. DEFINITIONS AND ILLUSTRATIONS

The remaining two chapters of the dissertation justify the second part of the title. Here we use recurrence as a tool to define graphs. In this chapter, we define and investigate a family of graphs whose number of vertices satisfies the recurrence $s_n^a = as_{n-1}^a + s_{n-2}^a$. We start with the definition of hypercubes and Fibonacci cubes.

The *hypercube* Q_n is defined as follows: the set of vertices $V(Q_n)$ consists of all binary strings of length n and two vertices are adjacent if and only if they differ in a single bit, i.e., if one vertex can be obtained from the other by replacing 0 with 1 or vice versa, only once. Let $\square$ operator denote the Cartesian product of graphs. Recall that for two graphs G and H, their Cartesian product is the graph denoted by $G\square H$ with $V(G\square H) = V(G) \times V(H)$ and $(u_1,v_1)(u_2,v_2) \in E(G\square H)$ if $u_1 = u_2$ and v_1 and v_2 are adjacent in H, or $v_1 = v_2$ and u_1 and u_2 are adjacent in G [23]. Then alternatively, the hypercubes can be defined as the Cartesian product of K_2 with itself n times, so $Q_n = K_2\square\cdots\square K_2 = K_2^n$. Here K_2 denotes the complete graph with two vertices. *Fibonacci cubes* Γ_n are special subgraphs of hypercubes where the set of vertices $\mathscr{F}_n$ consists of *Fibonacci strings*, a binary strings of length n without consecutive ones. The number of vertices in Fibonacci cubes is counted by the Fibonacci numbers, $|\mathscr{F}_n| = F_{n+2}$. Fibonacci cubes have attracted much attention and spawned several generalizations, most of them leading to the graphs with the number of vertices counted by various types of higher-order Fibonacci numbers [17,27]. The *Pell graphs* Π_n, introduced by Munarini [40] in 2019, are graphs whose vertices are strings of length n over the alphabet $\{0,1,2\}$ with property that 2 comes only in blocks of even length. Two vertices are adjacent if one can be obtained from the other by replacing 0 with 1 or by replacing block 11 with block 22. The vertices in this family are counted by the Pell numbers. Munarini's paper inspired us to try to generalize this idea by obtaining a family of graphs whose numbers of vertices satisfy recurrence $s_n = a \cdot s_{n-1} + s_{n-2}$ for arbitrary non-negative integer a, while preserving the main properties of Pell graphs.

Let a be a non-negative integer and let $\mathscr{S}^a$ denote the free monoid containing $a+1$ generators $\{0,1,2,\ldots,a-1,0a\}$. By a *string* we mean an element of monoid $\mathscr{S}^a$, i.e., a word from alphabet $\{0,1,2,\ldots,a-1,a\}$ with property that letter a can only appear in the block $0a$. Other letters can appear arbitrarily. For the strings $\alpha = \alpha_1 \cdots \alpha_n$ and $\beta = \beta_1 \cdots \beta_m$, we define their *concatenation* in the usual way, as $\alpha\beta = \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m$.

If $\mathscr{S}_n^a$ denotes the set of all elements from the monoid $\mathscr{S}^a$ of length n, one can easily obtain a recurrence for the cardinal number $s_n^a = |\mathscr{S}_n^a|$. A string of length n can end with any of the letters $0,1,\ldots,a-1$, and the rest of the string can be formed in s_{n-1}^a ways. If a string ends with the letter a, that necessarily means that there is at least one zero preceding a, and the rest of the string can be formed in s_{n-2}^a ways. Hence, s_n^a satisfies the recurrence

$$s_n^a = a s_{n-1}^a + s_{n-2}^a \tag{2.1}$$

with initial values $s_0^a = 1$ and $s_1^a = a$.

The *metallic cube* of dimension n, denoted by Π_n^a, is a graph whose vertices are elements of the set $\mathscr{S}_n^a$, i.e., $V(\Pi_n^a) = \mathscr{S}_n^a$ and for any $v_1, v_2 \in V(\Pi_n^a)$ we have $v_1 v_2 \in E(\Pi_n^a)$ if and only if one vertex can be obtained from the other by replacing a single letter k with $k+1$ for $0 \le k \le a-1$. An alternative definition of adjacency can be given via modified Hamming distance. For $v_1 = \alpha_1 \cdots \alpha_n$ and $v_2 = \beta_1 \cdots \beta_n$ we define

$$\overline{h}(v_1, v_2) = \sum_{k=1}^{n} |\alpha_k - \beta_k|.$$

Then v_1 and v_2 are adjacent if and only if $\overline{h}(v_1, v_2) = 1$. As an example, Figure 2.1 shows graphs Π_n^a for $a = 3$ and $n = 1,2,3$.

From the definition of Π_n^a, it is immediately clear that $\Pi_n^1 = \Gamma_{n-1}$. Namely, removing the first 0 in vertices of Π_n^1 establishes a graph-isomorphism between the metallic cube Π_n^1 and the Fibonacci cube Γ_{n-1}. However, Π_n^2 are not isomorphic to Pell graphs. One possible generalization could be obtained using the alphabet $\{0,1,\ldots,a\}$ where a appears

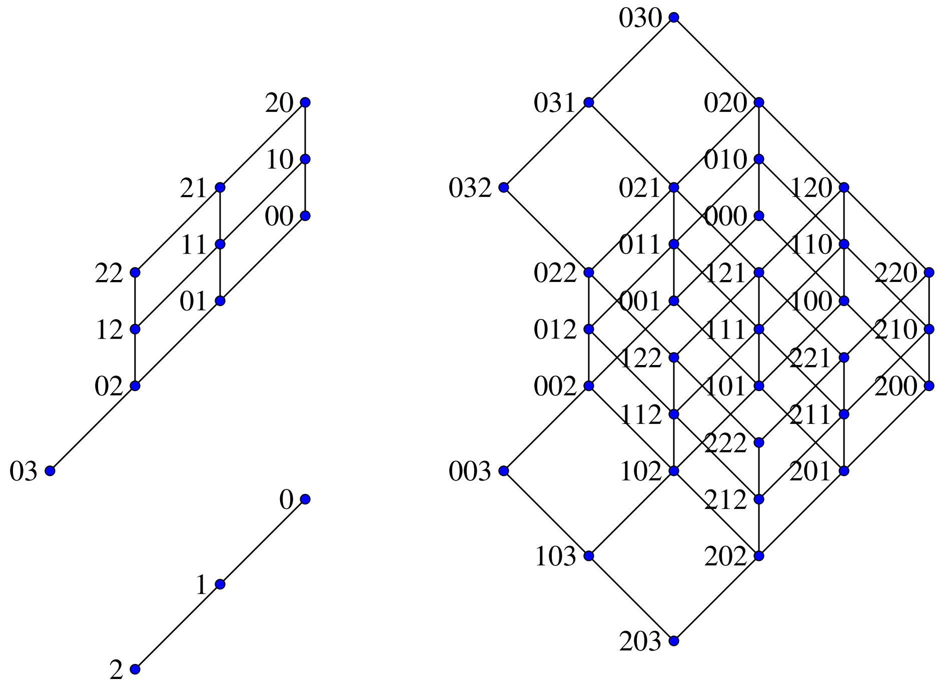

Figure 2.1: Graph Π_n^3 for $n = 1$ (lower left), 2 (upper left), 3 (right).

only in the blocks of even length and two vertices are adjacent if one can be obtained from the other by replacing the single letter i with $i+1$ for $0 \leq i \leq a-2$ or by replacing a block $(a-1)(a-1)$ with aa. This was independently done by Iršič, Klavžar, and Tan [28]. This family of graphs is a straightforward generalization of Pell graphs as defined by Munarini and inherits many of their properties. Instead of pursuing this approach, we opted for a generalization better suited to higher values of a yielding along the way an alternative definition of Pell graphs and offering numerous possibilities for comparisons. The difference is illustrated in Figure 2.2, where it can be seen that, although similar, the graphs obtained under the two generalizations are not isomorphic, for vertex 111 of Pell graph shown in the left panel of Figure 2.2 has degree 5, while the maximum degree in Π_3^2 is 4.

In Table 2.1 we show the number of vertices for a few initial values of a and n. The Pell numbers appear in the second column, while the (shifted) Fibonacci numbers appear in the first one. It can be observed that the columns of Table 2.1 grow exponentially, as the n-th powers of the corresponding metallic means, while the elements in rows grow

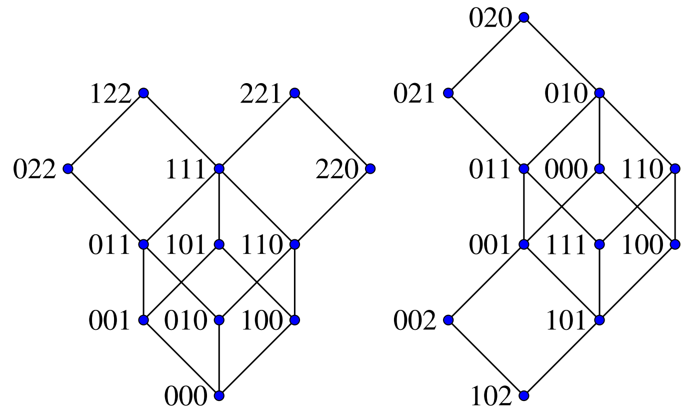

Figure 2.2: The Pell graph Π_3 (left) and the corresponding metallic cube Π_3^2 (right).

Table 2.1: The number of vertices in Π_n^a.

$n \backslash a$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	5	10	17	26	37
3	3	12	33	72	135	228
4	5	29	109	305	701	1405
5	8	70	360	1292	3640	8658
6	13	169	1189	5473	18901	53353
7	21	408	3927	23184	98145	328776
8	34	985	12970	98209	509626	2026009

polynomially. In fact, the number s_n^a of vertices of Π_n^a is equal to

$$s_n^a = F_{n+1}(a) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} a^{n-2k}, \tag{2.2}$$

where $F_n(x)$ are the Fibonacci polynomials. So, for a fixed n, we obtain that the growth of the number of vertices is polynomial with degree n. It is also worth mentioning that

$$s_n^a = \frac{1}{\sqrt{a^2+4}} \left(\left(\frac{a+\sqrt{a^2+4}}{2} \right)^{n+1} - \left(\frac{a-\sqrt{a^2+4}}{2} \right)^{n+1} \right).$$

2.2. BASIC STRUCTURAL PROPERTIES

2.2.1. Canonical decompositions and bipartivity

It is well-known that the Fibonacci cubes and the Pell graphs admit recursive decomposition [40, 41]. In this section, we show that such decompositions naturally extend to the metallic cubes.

We start with the observation that the set of vertices $\mathscr{S}_n^a$ can be divided into disjoint sets based on the starting letter. One set contains the vertices starting with block $0a$, and the remaining a sets contain the vertices that start with $0, 1, \ldots, a-2$ and $a-1$, respectively. Assuming $\alpha \in \mathscr{S}_{n-1}^a$ and $\beta \in \mathscr{S}_{n-2}^a$, the vertices 0α, 1α, …, $(a-1)\alpha$ generate a copies of a graph Π_{n-1}^a and vertices $0a\beta$ generate the only copy of Π_{n-2}^a. We call this the *canonical decomposition*, and we use the $\oplus$ operator to indicate the decomposition. That brings us to our first result.

Theorem 2.2.1. For $n \geq 2$, the metallic cube Π_n^a admits the decomposition

$$\Pi_n^a = \Pi_{n-1}^a \oplus \cdots \oplus \Pi_{n-1}^a \oplus \Pi_{n-2}^a$$

in which the first factor Π_{n-1}^a is repeated a times.

The decomposition of Theorem 2.2.1 is called the *canonical decomposition* of Π_n^a.

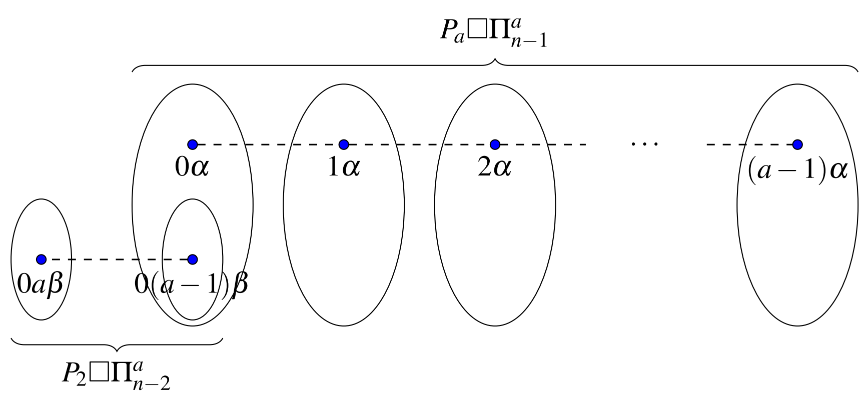

Figure 2.3: Canonical decomposition $\Pi_n^a = \Pi_{n-1}^a \oplus \cdots \oplus \Pi_{n-1}^a \oplus \Pi_{n-2}^a$.

In Figure 2.3 we show a schematic representation of the described canonical decomposition. As an example, Figure 2.4 shows the canonical decomposition of graph Π_3^3 into three copies of graph Π_2^3 and one copy of graph Π_1^3.

It is worth noting that $\Pi^a_{n-1} \oplus \cdots \oplus \Pi^a_{n-1} = P_a \square \Pi^a_{n-1}$, where P_a is the path graph on a vertices. Here the $\square$ operator denotes the Cartesian product of graphs.

A map $\chi : V(G) \to \{0,1\}$ is a *proper* 2*-coloring* of a graph G if $\chi(v_1) \neq \chi(v_2)$ for every two adjacent vertices $v_1, v_2 \in V(G)$. A graph G is *bipartite* if its set of vertices $V(G)$ can be decomposed into two disjoint subsets A and B such that no two vertices of the same subset are adjacent. Equivalently, a graph G is bipartite if it admits a proper 2-coloring.

Theorem 2.2.2. All metallic cubes are bipartite.

Proof. The proof is by induction on n. We observe that Π^a_1 is isomorphic to a path graph on a vertices and, thus, bipartite. Since Π^a_2 is an $a \times a$ grid with one additional vertex $0a$, it is an easy exercise to see that it also admits a proper 2-coloring. Now we suppose that Π^a_k is bipartite for every $k < n$. By the inductive hypothesis, it admits a proper 2-coloring $\chi : \Pi^a_{n-1} \to \{0,1\}$. Consider the map $\chi' : \Pi^a_{n-1} \to \{0,1\}$, where $\chi'(v) = 1 - \chi(v)$. Then the map χ' is a complementary proper 2-coloring for the graph Π^a_{n-1}. Since $\Pi^a_n = \Pi^a_{n-1} \oplus \cdots \oplus \Pi^a_{n-1} \oplus \Pi^a_{n-2}$, we can choose a coloring χ for the subgraph Π^a_{n-1} in Π^a_n if the subgraph is induced by vertices starting with even k, and χ' if k is odd. Finally, for the only copy of Π^a_{n-2} in the canonical decomposition, we can choose χ' restricted to Π^a_{n-2}. Thus we obtained a proper 2-coloring of graph Π^a_n. ■

Observe that the proof of Theorem 2.2.2 implies that the sets A and B have the same cardinality whenever the number of vertices is even. Otherwise, the cardinality differs by one. Figure 2.4 shows a proper 2-coloring of the metallic cube Π^3_3. Since the number of vertices in that graph is 33, we have 17 vertices in one set and 16 vertices in the other.

We now present another decomposition of the metallic cubes. To this end, recall that P_a denotes a path graph with a vertices, and P^k_a denotes the Cartesian product of path P_a with itself k times, that is $P_a \square \cdots \square P_a$. Also, note that $|V(P^k_a)| = a^k$. Graphs P^k_a are called *grids* or *lattices*. The following theorem illustrates the combinatorial meaning of the Fibonacci polynomials of identity (2.2).

Theorem 2.2.3. The metallic cube Π^a_n can be decomposed into F_{n+1} grids, where F_n denotes n-th Fibonacci number.

Proof. Consider the subset of strings $S \subset \mathscr{S}^a_n$ where each string $\alpha \in S$ has k blocks $0a$ in the same position. Then α has $n - 2k$ letters that are not part of a $0a$-block. There

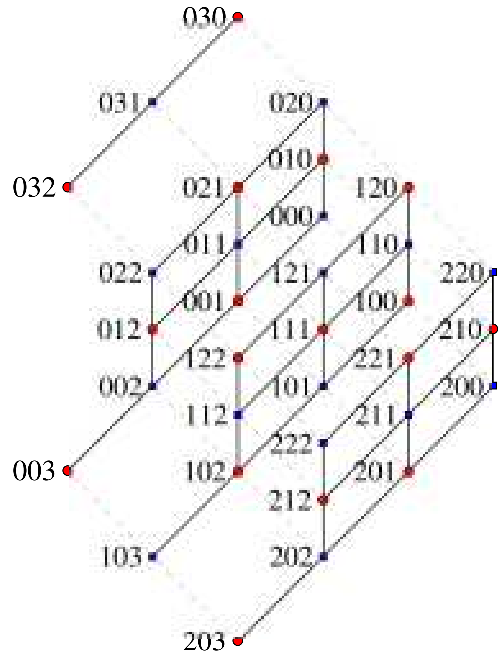

Figure 2.4: The canonical decomposition and a proper 2-coloring of $\Pi_3^3 = \Pi_2^3 \oplus \Pi_2^3 \oplus \Pi_2^3 \oplus \Pi_1^3$.

are a^{n-2k} such strings and the subset $S \subset \mathscr{S}_n^a$ induces a subgraph of Π_n^a isomorphic to a grid P_a^{n-2k} with a^{n-2k} vertices. Also, note that different locations of blocks $0a$ produce different strings. Hence, for different alignments, the induced grids are vertex-disjoint. To finish our proof, we just need to determine the number of strings with exactly k blocks $0a$. We can identify each such block as a single letter, and reduce our problem to a subset of $n-k$ positions, where we need to choose k positions for those blocks. Thus, we have $\binom{n-k}{k}$ different alignments for k blocks $0a$, and each alignment induces disjoint subgraphs P_a^{n-2k}. We obtained a decomposition

$$\Pi_n^a = \bigoplus_{k\geq 0} \binom{n-k}{k} P_a^{n-2k}.$$

Then the number of grids is $\sum_{k\geq 0} \binom{n-k}{k} = F_{n+1}$, and this completes our proof. ■

To obtain our last result in this subsection, we consider the map $\rho : V(\Pi_n^a) \to \mathscr{F}_n$ defined on the alphabet $\{0,1,\ldots,a\}$ and extended to $V(\Pi_n^a)$ by concatenation, as

$$\rho(\alpha) = \begin{cases} 0, & 0 \leq \alpha \leq a-1, \\ 1, & \alpha = a. \end{cases}$$

For a Fibonacci string w, let $\rho^{-1}(w)$ denote the subgraph of Π_n^a induced by vertices $\{v \in V(\Pi_n^a) | \rho(v) = w\}$.

Theorem 2.2.4. Let Π_n^a/ρ be the quotient graph of Π_n^a obtained by identifying all vertices which are identified by ρ, and two vertices w_1 and w_2 in Π_n^a/ρ are adjacent if there is at least one edge in Π_n^a connecting blocks $\rho^{-1}(w_1)$ and $\rho^{-1}(w_2)$. Then Π_n^a/ρ is isomorphic to the Fibonacci cube Γ_{n-1}.

Proof. The map ρ identifies two vertices $v_1, v_2 \in \Pi_n^a$ if they have the same number and the same positions of the blocks $0a$. But all vertices having an equal number of k blocks $0a$ in the same positions induce a grid subgraph P_a^{n-2k}. So, each such grid is mapped into a single vertex in Π_n^a/ρ. By Theorem 2.2.3, $|V(\Pi_n^a/\rho)| = F_{n+1}$. Two grids have at least one edge connecting them if the number of the blocks $0a$ between them differs by exactly one, with the grid having one block less, having all blocks $0a$ in the same positions as the other grid. For example, for $a = 5$ and $n = 5$, one grid P_5^1 is induced by vertices $\alpha 0505$ and another grid P_5^3 is induced by vertices $\beta_1 05 \beta_2 \beta_3$, where $\alpha, \beta_1, \beta_2, \beta_3 \in \{0,1,2,3,4\}$. Then those grids must have an edge connecting them. For example, the edge connecting vertices 00504 and 00505. From the definition of the map ρ it is clear that ρ maps two grids in the neighboring vertices in Π_n^a/ρ only if their image by ρ differs in a single bit. Hence, Π_n^a/ρ is isomorphic to Γ_{n-1}. ■

As an example, Figure 2.5 shows the metallic cube Π_4^2 decomposed into grids. Observe that Π_4^2/ρ is isomorphic to the Fibonacci cube Γ_3.

2.2.2. Embedding into hypercubes

Since hypercubes Q_n have binary strings as vertices, and all binary strings of length n are vertices in Π_n^a for $a \geq 2$, we have a natural inclusion $Q_n \subset \Pi_n^a$. Furthermore, for $a \geq 1$, since $\mathscr{S}_n^1 \subset \mathscr{S}_n^a$, for every a, we have $\Gamma_{n-1} = \Pi_n^1 \subseteq \Pi_n^a$. The following theorem shows another direction. Every metallic cube is an induced subgraph of some Fibonacci cube.

Theorem 2.2.5. For all $a \geq 1$ and $n \geq 1$, the metallic cubes are induced subgraphs of Fibonacci cubes and, hence, hypercubes.

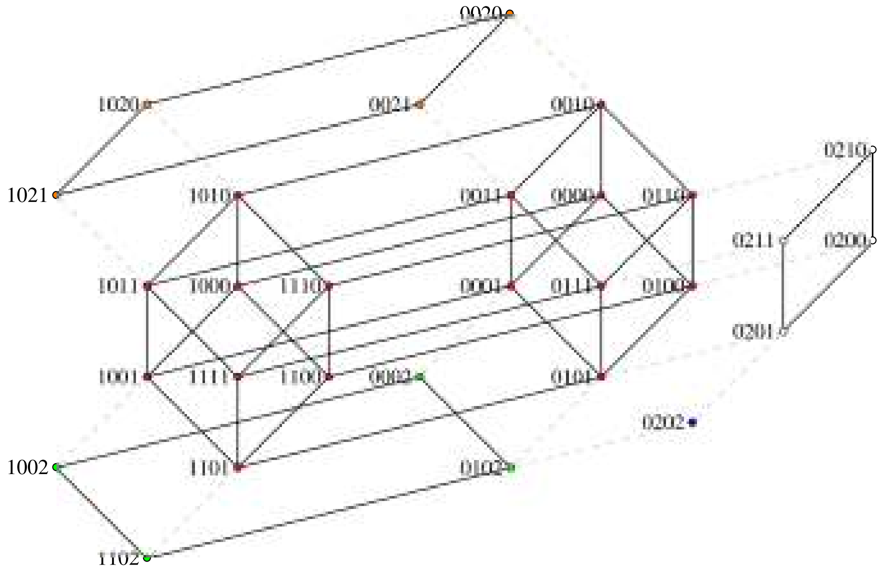

Figure 2.5: Decomposition of $\Pi_4^2 = P_2^4 \oplus P_2^2 \oplus P_2^2 \oplus P_2^2 \oplus P_2^0$.

Proof. For $a = 2$, we define a map σ on the primitive blocks, i.e., on the blocks that every string from the set $\mathscr{S}_n^2$ can be uniquely decomposed into, 0, 1, and 02 as follows

$$\sigma(0) = 001$$
$$\sigma(1) = 000$$
$$\sigma(02) = 001010.$$

For $a > 2$, the primitive blocks, are $0, 1, \ldots, a-1$ and $0a$. So, we set

$$\begin{aligned}
\sigma(0) &= 010101 \cdots 010101 \\
\sigma(1) &= 000101 \cdots 010101 \\
\sigma(2) &= 000001 \cdots 010101 \\
&\vdots \\
\sigma(a-2) &= 000000 \cdots 000001 \\
\sigma(a-1) &= 000000 \cdots 000000 \\
\sigma(0a) &= 010101 \cdots 01010100100000 \cdots 000000,
\end{aligned}$$

where strings $\sigma(k)$ have length $2a-2$, for $0 \le k \le a-1$ and string $\sigma(0a)$ has length $4a-4$. The first half of the string $\sigma(0a)$ contains $a-1$ 1's and the second half contains 1 only once.

In case of $a = 2$, the map σ, defined on the primitive blocks, extends to $\sigma : \Pi_n^2 \to \Pi_{3n}^1$ by concatenation. In case of $a \geq 3$, it extends to $\sigma : \Pi_n^a \to \Pi_{(2a-2)n}^1$. Since the map is injective and preserves adjacency, we obtain an induced subgraph of the Fibonacci cube isomorphic to the graph Π_n^a. ■

Next, we show that metallic cubes are median graphs. A *median* of three vertices is a vertex that lies on a shortest path between every two of three vertices. We say that a graph G is a *median graph* if every three vertices of G have a unique median.

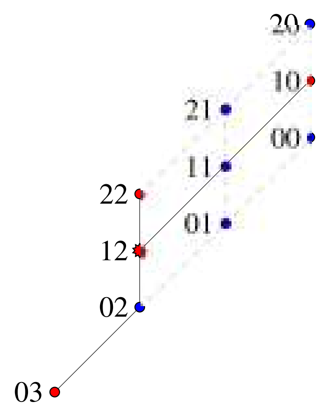

Figure 2.6: The median of the vertices 10, 22 and 03 is the vertex 12 .

Figure 2.6 shows that the unique median of the vertices 10, 22 and 03 is vertex 12 because that is the only vertex that lies in a shortest path between every two red vertices.

Klavžar [30] proved that Fibonacci and Lucas cubes are median graphs, and Munarini [40] proved that Pell graphs are median graphs. They both used the following theorem by Mulder [39].

Theorem 2.2.6 (Mulder)**.** A graph G is a median graph if and only if G is a connected induced subgraph of an n-cube such that with any three vertices of G their median in n-cube is also a vertex of G.

Theorem 2.2.7. For any $a \geq 1$ and $n \geq 1$, the metallic cube Π_n^a is a median graph.

Proof. It is well-known that the median of the triple in Q_n is obtained by the majority rule. So let $\alpha = \varepsilon_1 \cdots \varepsilon_n$, $\beta = \delta_1 \cdots \delta_n$ and $\gamma = \rho_1 \cdots \rho_n$ be binary strings, i.e., $\varepsilon_i, \delta_i, \rho_i \in \{0, 1\}$. Then their median is $m = \zeta_i \cdots \zeta_n$, where ζ_i is equal to the number that appears at least twice among the numbers ε_i, δ_i and ρ_i.

Let $a \geq 3$ and let σ be the map defined in the proof of Theorem 2.2.5. By the same theorem, Π_n^a is a connected induced subgraph of a $(2a-2)n$-cube. To finish our proof, we just need to verify that the subgraph induced by the set $\sigma(\mathscr{S}_n^a)$ is median closed, i.e., that for every three vertices in $\sigma(\mathscr{S}_n^a)$, their median is also a vertex in $\sigma(\mathscr{S}_n^a)$. Every string from the set $\sigma(\mathscr{S}_n^a)$ can be uniquely decomposed into blocks $\sigma(j)$ with length $2a-2$, for $0 \leq j \leq a-1$, and a block $\sigma(0a)$ with length $4a-4$. Note that the median of three blocks, where two blocks are the same, is that block that appears at least twice. Hence, we only need to consider cases where all three blocks are different. First consider three blocks $\sigma(i)$, $\sigma(j)$ and $\sigma(k)$. Without loss of generality, we can assume that $0 \leq i < j < k \leq a-1$. Then their median is $\sigma(j)$. In the second case, we have $\sigma(i)$ and $\sigma(j)$ for $0 \leq i \leq j \leq a-1$, and the second half of the block $\sigma(0a)$, i.e. of the string $0010\cdots0$ with length $(2a-2)n$. But, since one comes only in even positions in strings $\sigma(i)$ and $\sigma(j)$, their median is $\sigma(j)$. Also, note that we do not have to consider the case with the first half of the block $\sigma(0a)$ because it is equal to the block $\sigma(a-1)$. So, we conclude that the subgraph induced by the set $\sigma(\mathscr{S}_n^a)$ is median closed. The proof for $a = 2$ is similar, so we omit the details. ■

2.3. The number of edges and distribution of degrees

Now that we have elucidated the recursive structure of metallic cubes via the canonical decomposition, the recurrences for the number of edges and the details of degree distributions can be simply read off from our structural results. We start with counting the edges of Π_n^a.

Let e_n^a denote the number of edges in Π_n^a, i.e., $e_n^a = |E(\Pi_n^a)|$. Since Π_0^a is an empty graph, $e_0^a = 0$. The graph Π_1^a is the path graph with a vertices, so, we have $e_1^a = a-1$. The graph Π_2^a is an $a \times a$ grid with the addition of the vertex $0a$ being adjacent to the vertex $0(a-1)$. Hence, $e_2^a = 2a^2 - 2a + 1$. For larger values of n, the graph Π_n^a consists of a copies of Π_{n-1}^a and a single copy of Π_{n-2}^a, and those subgraphs contribute with $a \cdot e_{n-1}^a + e_{n-2}^a$ edges. Furthermore, there are $(a-1) \cdot s_{n-1}^a + s_{n-2}^a$ edges connecting a subgraphs Π_{n-1}^a and one subgraph Π_{n-2}^a. Since $s_n^a - s_{n-1}^a = (a-1)s_{n-1}^a + s_{n-2}^a$, for $n > 0$, the overall number of edges is

$$e_n^a = a \cdot e_{n-1}^a + e_{n-2}^a + s_n^a - s_{n-1}^a. \tag{2.3}$$

Hence, the number of edges satisfies a non-homogeneous linear recurrence of order 2 with the same coefficients in the homogeneous part as the recurrence for the number of vertices. Table 2.2 shows some first few values of e_n^a. With some care, patterns of the coefficients of the polynomials appearing there can be analyzed, suggesting the explicit formula established in our following theorem.

Theorem 2.3.1. The number of edges in the metallic cube Π_n^a is

$$e_n^a = \sum_{k=0}^{n} (-1)^{n-k} \left\lceil \frac{n+k}{2} \right\rceil \binom{\lfloor \frac{n+k}{2} \rfloor}{k} a^k.$$

Proof. We proceed by induction on n. For $n = 1$ and $n = 2$, the statement holds. Furthermore, since the number of vertices s_n^a satisfy the identity (2.2), we have

$$s_n^a - s_{n-1}^a = \sum_{k \geq 0} \binom{n-k}{k} a^{n-2k} - \sum_{k \geq 0} \binom{n-k-1}{k} a^{n-2k-1}$$

Table 2.2: The number of edges in Π_n^a.

n	e_n^a
1	$a-1$
2	$2a^2-2a+1$
3	$3a^3-3a^2+4a-2$
4	$4a^4-4a^3+9a^2-6a+2$
5	$5a^5-5a^4+16a^3-12a^2+9a-3$
6	$6a^6-6a^5+25a^4-20a^3+24a^2-12a+3$

$$
\begin{aligned}
&= \sum_{k=0}^{n}(-1)^k\binom{n-\left\lceil\frac{k}{2}\right\rceil}{\left\lfloor\frac{k}{2}\right\rfloor}a^{n-k}\\
&= \sum_{k=0}^{n}(-1)^k\binom{n-\left\lceil\frac{k}{2}\right\rceil}{n-k}a^{n-k}\\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{n-\left\lceil\frac{n-k}{2}\right\rceil}{k}a^{k}\\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{\left\lfloor\frac{n+k}{2}\right\rfloor}{k}a^{k}.
\end{aligned}
$$

By using the inductive hypothesis, after adjusting indices and expanding the range of summation, we obtain

$$
\begin{aligned}
a\cdot e_{n-1}^a+e_{n-2}^a &= \sum_{k=0}^{n-1}(-1)^{n-k-1}\left\lceil\frac{n+k-1}{2}\right\rceil\binom{\left\lfloor\frac{n+k-1}{2}\right\rfloor}{k}a^{k+1}+\\
&\quad+\sum_{k=0}^{n-2}(-1)^{n-k}\left\lceil\frac{n+k-2}{2}\right\rceil\binom{\left\lfloor\frac{n+k-2}{2}\right\rfloor}{k}a^{k}\\
&=\sum_{k=1}^{n}(-1)^{n-k}\left\lceil\frac{n+k-2}{2}\right\rceil\binom{\left\lfloor\frac{n+k-2}{2}\right\rfloor}{k-1}a^{k}+\\
&\quad+\sum_{k=0}^{n}(-1)^{n-k}\left\lceil\frac{n+k-2}{2}\right\rceil\binom{\left\lfloor\frac{n+k-2}{2}\right\rfloor}{k}a^{k}\\
&=\sum_{k=0}^{n}(-1)^{n-k}\left\lceil\frac{n+k-2}{2}\right\rceil\binom{\left\lfloor\frac{n+k}{2}\right\rfloor}{k}a^{k}.
\end{aligned}
$$

Now, by using expressions for $s_n^a-s_{n-1}^a$ and $a\cdot e_{n-1}^a+e_{n-2}^a$ our claim follows at once. ■

For the Fibonacci cubes, Klavžar [30] proved that $|E(\Gamma_n)| = F_{n+1}+\sum_{k=1}^{n-2}F_kF_{n+1-k}$. By plugging in $a=1$ into our result and recalling that $\Pi_n^1=\Gamma_{n-1}$, we have obtained a combinatorial proof of the following identity.

Corollary 2.3.2.

$$\sum_{k=0}^{n}(-1)^{n-k}\left\lceil\frac{n+k}{2}\right\rceil\binom{\left\lfloor\frac{n+k}{2}\right\rfloor}{k}=\sum_{k=0}^{n}F_kF_{n-k}.$$

Starting from the recurrence (2.3), it is easy to find the generating function

$$\sum_{n\geq 0}e_n^a t^n=\frac{(a-1)t+t^2}{(1-at-t^2)^2}.$$

Setting $a=1$ yields a self-convolution of Fibonacci numbers which appears as the sequence A001629 in OEIS [46]. For $a=2$ we have the sequence A364553 that corresponds to the number of vertices in n-Pell graphs [28]. Although the n-Pell graphs and metallic cubes are not isomorphic, they have the same number of vertices and edges. Sequences e_n^a for the higher values of a do not appear in the OEIS.

Now we turn our attention to the degrees of vertices of the metallic cubes and their distribution. We consider first the case when $a\geq 3$. Hence, let $a\geq 3$ and $v=\alpha_1\cdots\alpha_n\in\Pi_n^a$ be any vertex. If $0<\alpha_i<a-1$ for all i, then v has the maximum possible degree, which is $2n$. Each $0a$-block reduces the degree by 3, each $0(a-1)$-block reduces the degree by 1, as well as each letter 0 or $a-1$ which is not part of some $0(a-1)$-block. We are interested in the number of vertices with degree $2n-m$. So, let v be a vertex containing l $0a$-blocks, h $0(a-1)$-blocks and k letters 0 or $a-1$, where none of k letters 0 and $a-1$ are part of $0(a-1)$-block. Then v has degree $2n-3l-h-k$. Now we want to count the number of such vertices. There are $\binom{n-h-l}{h}\binom{n-2h-l}{l}\binom{n-2h-2l}{k}$ ways to choose the positions of $0a$-blocks, $0(a-1)$-blocks and letters 0 or $a-1$. The remaining $n-2h-2l-k$ positions can be filled in $(a-2)^{n-2h-2l-k}$ ways. If we set that there are 2^k ways to fill k positions with 0 or $a-1$, we possibly obtained more than l $0(a-1)$-blocks and counted some vertices more than once. Let $q(n,l,h,k)=q(l,k)=\binom{n-h-l}{h}\binom{n-2h-l}{l}\binom{n-2h-2l}{k}2^k(a-2)^{n-2h-2l-k}$ denote the number of vertices where all vertices with exactly l blocks are counted once, and vertices with more than l blocks are counted more than once. More precisely, beside vertices of length l, the number $q(l)$ counts all vertices with $l+1,l+2,\ldots,l+\left\lfloor\frac{k}{2}\right\rfloor$ blocks $0(a-1)$, where each vertex with exactly $l+j$ blocks is counted $\binom{l+j}{l}$ times. If $p(n,l,h,k)=p(l,k)$ denotes the number of vertices with precisely l blocks $0(a-1)$, we have

$$q(l,k)=\sum_{j=0}^{\left\lfloor\frac{k}{2}\right\rfloor}\binom{l+j}{l}p(l+j,k-2j).$$

Now we retrieve the numbers $p(l,k)$ by direct counting. To obtain the number of vertices with exactly l blocks, we have to subtract the number of vertices with more than l blocks. The vertices having $l+1$ blocks were counted $\binom{l+1}{l}$ times in $q(l,k)$ and subtracting $q(l,k)-q(l+1,k-2)\binom{l+1}{l}$ corrects that error. But then the vertices containing $l+2$ blocks were counted $\binom{l+2}{l}-\binom{l+2}{l+1}\binom{l+1}{l}=\binom{l+2}{l}$ times so far, so we need to add $\binom{l+2}{l}q(l+2,k-4)$ to compensate for that error. Inductively, a vertex with $l+j$ blocks is counted

$$\binom{l+j}{j}-\sum_{s=0}^{j-1}\binom{l+j}{l+s}\binom{l+s}{l}=\binom{l+j}{j}-\binom{l+j}{l}\sum_{s=0}^{j-1}\binom{s}{j}$$
$$=\begin{cases}0 & \text{for } j \text{ odd}\\ -2\binom{l+j}{j} & \text{for } j \text{ even}\end{cases}$$

times, and there are $q(l+j,k-2j)$ such vertices. So we conclude that the number of vertices that have exactly l blocks $0(a-1)$ is

$$p(l,k)=\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^j\binom{l+j}{j}q(l+j,k-2j).$$

Let $\Delta_{n,m}$ denote the overall number of vertices in Π_n^a of degree m. Then we have

$$\Delta_{n,2n-m}=\sum_{\substack{3h+l+k=m\\ 2h+2l+k\leq n\\ l,h,k\geq 0}}p(n,h,l,k). \tag{2.4}$$

The maximum degree is achieved for $m=0$. The formula (2.4) reduces to $\Delta_{n,2n}=(a-2)^n$. The minimum degree is achieved for the largest possible number of $0a$-blocks, which reduces the degree the most, hence $h=\lfloor\frac{n}{2}\rfloor$ and $k=n-2\lfloor\frac{n}{2}\rfloor$. Thus, $m=n+\lfloor\frac{n}{2}\rfloor$. The formula for $\Delta_{n,\lceil\frac{n}{2}\rceil}$ reduces to

$$\Delta_{n,\lceil\frac{n}{2}\rceil}=q\left(n,\left\lfloor\frac{n}{2}\right\rfloor,0,n-2\left\lfloor\frac{n}{2}\right\rfloor\right)$$
$$=\left(\binom{n-\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{n}{2}\rfloor}2^{n-2\lfloor\frac{n}{2}\rfloor}\right)(a-2)^0$$
$$=\begin{cases}1 & \text{for } n \text{ even;}\\ n+1 & \text{for } n \text{ odd.}\end{cases}$$

Figure 2.7 shows the degree of each vertex in the graphs Π_1^3, Π_2^3 and Π_3^3.

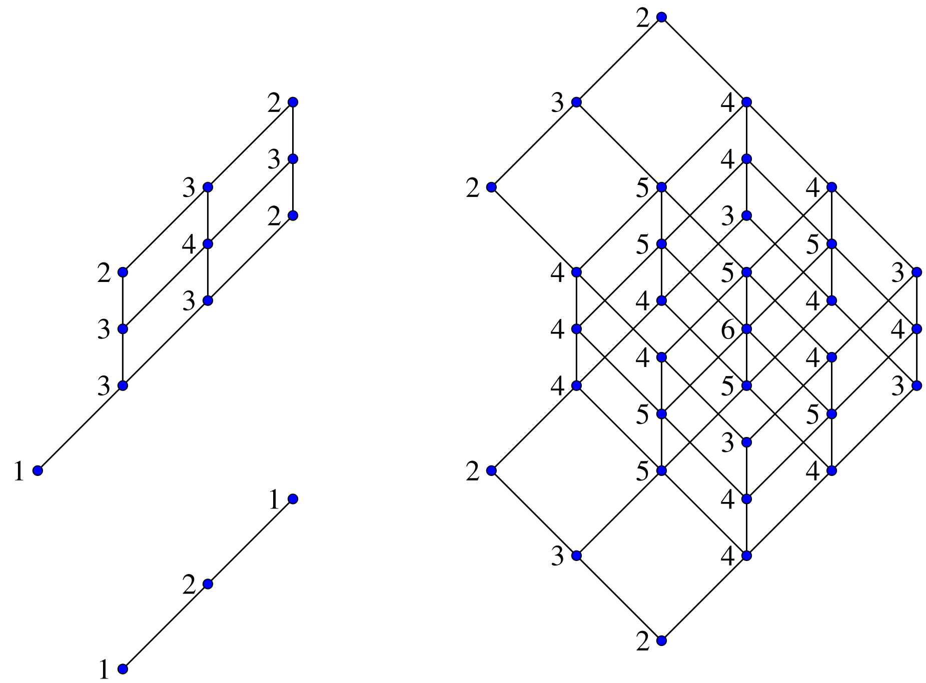

Figure 2.7: The graph Π_n^3 for $n = 1,2,3$ with the degree of each vertex.

It is now time to consider the cases $a = 1$ and $a = 2$. Their separate treatment is necessary since, for $a = 2$, there are no letters beside $0, a-1$ and a, and for $a = 1$ we only have 0 and a. For $a = 2$ we have $k = n-2h-2l$ and $q(n,h,l,k)$ becomes

$$q(n,h,l) = \binom{n-h-l}{h}\binom{n-2h-l}{l}2^{n-2h-2l},$$

and the rest of the argument is the same. The maximum degree is now obtained by setting $l = \lfloor \frac{n}{2} \rfloor$ and $h = 0$. The minimum degree is obtained by setting $h = \lfloor \frac{n}{2} \rfloor$ and $l = 0$. Since both cases produce the same number of vertices, we have

$$\begin{aligned}
\Delta_{n,\lceil \frac{n}{2} \rceil} = \Delta_{n,n+\lfloor \frac{n}{2} \rfloor} &= q\left(n, 0, \left\lfloor \frac{n}{2} \right\rfloor\right) \\
&= \binom{n-\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor} 2^{n-2\lfloor \frac{n}{2} \rfloor} \\
&= \begin{cases} 1 & \text{for } n \text{ even;} \\ n+1 & \text{for } n \text{ odd.} \end{cases}
\end{aligned}$$

For $a = 1$, we would need to modify our approach, but we omit the details since the distribution of degrees for Fibonacci cubes is already available in the literature [33].

We conclude this section by computing the bivariate generating functions for two-indexed sequences $\Delta_{n,k}$ counting the number of vertices in Π_n^a with exactly k neighbors. We suppress a to simplify the notation. Take $a \geq 2$. Recall that $\mathscr{S}^a$ denotes the set of all words from the alphabet $\{0,1,2,\ldots,a-1,a\}$ with the property that the letter a can only appear immediately after 0. Every letter starts with $0,\ldots,a-1$ or with the block $0a$. Hence, the set $\mathscr{S}^a$ can be decomposed into disjoint subsets based on starting letters. We write

$$\mathscr{S}^a = \varepsilon + 0\mathscr{S}^a + 1\mathscr{S}^a + \cdots + (a-1)\mathscr{S}^a + 0a\mathscr{S}^a,$$

where ε denotes the empty word. Let $\Delta(x,y) = \sum\limits_{n,k} \Delta_{n,k}x^n y^k$ be the formal power series where $\Delta_{n,k}$ is the number of vertices in Π_n^a having k neighbors. Consider the set that contains all vertices that start with 0, but not with the $0a$-block and let $\Delta^0(x,y)$ denote the corresponding formal series. Then we have

$$\begin{aligned}\Delta(x,y) &= 1 + \Delta^0(x,y) + (a-2)xy^2\Delta(x,y) + xy\Delta(x,y) + x^2y\Delta(x,y)\\ \Delta^0(x,y) &= xy\left(1 + \Delta^0(x,y) + (a-1)xy^2\Delta(x,y) + x^2y\Delta(x,y)\right).\end{aligned}$$

Note that adding 0 at the beginning of a vertex increases the number of neighbors by one, except when the vertex starts with $a-1$, in which case adding 0 increases the number of neighbors by two. By solving the above system of equations we readily obtain the generating function:

$$\Delta(x,y) = \frac{1}{1-(2y+(a-2)y^2)x-(y-y^2+y^3)x^2}.$$

Here we list some first few values of $\Delta_{n,k}$ in Tables 2.3 and 2.4. It is interesting to observe

Table 2.3: Some first values of $\Delta_{n,k}$ for $a = 2$.

$n\backslash k$	1	2	3	4	5	6	7
1	2	0	0	0	0	0	0
2	1	3	1	0	0	0	0
3	0	4	4	4	0	0	0
4	0	1	10	7	10	1	0
5	0	0	6	20	18	20	6

Table 2.4: Some first values of $\Delta_{n,k}$ for $a = 3$.

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	2	1	0	0	0	0	0	0	0	0
2	1	3	5	1	0	0	0	0	0	0
3	0	4	6	14	8	1	0	0	0	0
4	0	1	10	19	33	34	11	1	0	0
5	0	0	6	23	60	85	108	63	14	1

that their rows are not unimodal for $a = 2$, but for $a = 3$ they appear to be unimodal. Maybe it would be interesting to explore for which values of a unimodal rows occur. Also, they do not (yet) appear in the *On-Line Encyclopedia of Integer Sequences* [46].

2.4. METRIC PROPERTIES

In this section, we look at some metric-related properties of the metallic cubes. In particular, for a given metallic cube Π_n^a, we compute its radius and diameter and determine its center and periphery as functions of a and n.

We recall some basic definitions [6]. A path of length n in a graph is a sequence of n edges $e_1, e_2, \ldots, e_n$ such that e_i and e_{i+1} have a common vertex for every $1 \leq i \leq n-1$. The distance between any two vertices $v_1, v_2 \in V(G)$, denoted by $d(v_1, v_2)$, is the length of any shortest path between them. By *eccentricity* $e(v)$ of a vertex $v \in V(G)$, we mean the distance from v to any vertex farthest from it. More precisely, $e(v) = \max_{w \in V(G)} d(v, w)$. The *radius* of G, denoted by $r(G)$, is the minimum eccentricity of the vertices, and *diameter* of G, denoted by $d(G)$, is the maximum eccentricity over all vertices of G. A vertex v is *central* if $e(v) = r(G)$. The subset $Z(G) = \{v \in V(G) : e(v) = r(G)\} \subseteq V(G)$ is called the *center* of G, while the subset $P(G) = \{v \in V(G) : e(v) = d(G)\} \subseteq V(G)$ is called the *periphery* of G. In the next few theorems, we determine the radius, center, diameter, and periphery of the metallic cubes.

Remark 2.4.1. Let $v = \alpha_1 \cdots \alpha_n \in V(\Pi_n^a)$ be an arbitrary vertex. To reach a vertex most distanced from v, every letter in v greater than $\left\lfloor \frac{a}{2} \right\rfloor$ must be changed into 0. Any letter smaller than $\left\lfloor \frac{a}{2} \right\rfloor$ must be changed into $a-1$ or, if the letter before it was changed into 0, into a. A letter equal to $\left\lfloor \frac{a}{2} \right\rfloor$ must be changed into 0 or into a.

Theorem 2.4.2. For $a \geq 1$ and $n \geq 1$ we have

$$r(\Pi_n^a) = \left\lfloor \frac{a}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{a}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor. \tag{2.5}$$

Proof. Let $\varepsilon = \left\lfloor \frac{a}{2} \right\rfloor$ and $\hat{\varepsilon} = \varepsilon\varepsilon \cdots \varepsilon \in V(\Pi_n^a)$. We want to compute the eccentricity $e(\hat{\varepsilon})$. It is not hard to see that any longest path from $\hat{\varepsilon}$ leads to the vertex $0a0a \cdots 0a0$ if n is odd, or to $0a0a \cdots 0a$ if n is even. Note that the most distant vertex of $\hat{\varepsilon}$ does not have to be unique. Single letter ε can be changed into 0 in $\left\lfloor \frac{a}{2} \right\rfloor$ steps, and to a in $a - \left\lfloor \frac{a}{2} \right\rfloor = \left\lceil \frac{a}{2} \right\rceil$ steps. Since $\hat{\varepsilon}$ has length n, there are $\left\lceil \frac{n}{2} \right\rceil$ odd positions and $\left\lfloor \frac{n}{2} \right\rfloor$ even positions. That brings us to the total of $\left\lfloor \frac{a}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{a}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor$ steps. Thus, $e(\varepsilon\varepsilon \cdots \varepsilon) = \left\lfloor \frac{a}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{a}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor$.

As the next step, we want to show that every vertex has the eccentricity at least $e(\hat{\varepsilon})$. Let $v \in V(\Pi_n^a)$ be arbitrary. If v has k blocks $0a$, then it has $n-2k$ letters in the alphabet

$\{0,1,\ldots,a-1\}$. The most distant vertex is obtained if we change every single letter $0,1,\ldots,a-1$ in string v at least $\lfloor \frac{a}{2} \rfloor$ times, and if we change each block $0a$ into $(a-1)0$. The latter transition requires $k(2a-1)$ steps.

To prove that $e(v) \geq \lfloor \frac{a}{2} \rfloor \lceil \frac{n}{2} \rceil + \lceil \frac{a}{2} \rceil \lfloor \frac{n}{2} \rfloor$, we consider cases of even and odd a separately. If a is even, then the formula (2.5) becomes $r(\hat{\varepsilon}) = n \cdot \frac{a}{2}$. Hence,

$$e(v) \geq (n-2k) \cdot \frac{a}{2} + k(2a-1) = n \cdot \frac{a}{2} + k(a-1) \geq n \cdot \frac{a}{2} = e(\hat{\varepsilon}).$$

For a odd, the formula (2.5) reduces to $r(\hat{\varepsilon}) = n \cdot \frac{a}{2} - \frac{1}{2}\left(\lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor\right)$. We have to prove that changing an arbitrary number of letters in $\hat{\varepsilon}$ does not decrease the eccentricity. So, let k denote the number of letters that differ from ε. That leaves us with s blocks of ε of lengths $h_1, h_2 \ldots, h_s$. Note that $n = k + \sum h_i$ and $s \leq k+1$. According to the first part of the proof, each ε-block contributes to the eccentricity with $h_i \cdot \frac{a}{2} - \frac{1}{2}\left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor\right)$, and each $\alpha \neq \varepsilon$ contributes at least with $\frac{a+1}{2}$. Note that block $0a$ contributes more than any other two letters, so we can assume v does not contain $0a$-blocks. We have

$$\begin{aligned} e(v) &\geq \sum \left(h_i \cdot \frac{a}{2} - \frac{1}{2}\left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor\right)\right) + k \cdot \frac{a+1}{2} \\ &= n \cdot \frac{a}{2} - \frac{1}{2} \sum \left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor\right) + \frac{k}{2} \\ &\geq n \cdot \frac{a}{2} + \frac{k-s}{2} \\ &\geq n \cdot \frac{a}{2} - \frac{1}{2}. \end{aligned}$$

Hence, for n odd, we have $e(v) \geq e(\hat{\varepsilon})$. For n even, we have to show that $\sum\left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor\right) \leq k$. If $\sum\left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor\right) = k+1$, each h_i is odd, and s and k have different parity. If s is odd, then $\sum h_i$ is odd, and since $n = \sum h_i + k$, k is odd too. It follows that s is even. Then $\sum h_i$ is even and k is even too. Thus, we reached a contradiction with $\sum\left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor\right) = k+1$. It follows that $\sum\left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor\right) \leq k$, and

$$\begin{aligned} e(v) &\geq n \cdot \frac{a}{2} - \frac{1}{2} \sum \left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor\right) + \frac{k}{2} \\ &\geq n \cdot \frac{a}{2} \\ &= e(\hat{\varepsilon}). \end{aligned}$$

■

Corollary 2.4.3. For a and n odd, the center of Π_n^a consists of a single vertex, $Z(\Pi_n^a) = \{\hat{\varepsilon}\}$.

Proof. Let a and n be odd and let $\varepsilon = \left\lfloor \frac{a}{2} \right\rfloor = \frac{a-1}{2}$. By Theorem 2.4.2, we have $r(\Pi_n^a) = n \cdot \frac{a}{2} - \frac{1}{2}$ and $\hat{\varepsilon} \in Z(\Pi_n^a)$. Suppose $v \in Z(\Pi_n^a)$. We want to prove that $v = \hat{\varepsilon}$. As in the proof of Theorem 2.4.2, let $h_1, \ldots, h_s$ denote the lengths of ε-blocks in v, and let k denote the number of letters $\alpha \neq \varepsilon$. Since $v \in Z(\Pi_n^a)$, the contributions of ε-blocks and letters between them must be minimal. A minimal contribution of each letter $\alpha \neq \varepsilon$ is $\frac{a+1}{2}$, which can be achieved only by changing $\varepsilon + 1$ into 0 or $\varepsilon - 1$ into $a-1$. Hence, $\alpha = \varepsilon \pm 1$. The minimal contribution of each ε-block is $h_i \cdot \frac{a}{2} - \frac{1}{2}\left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor\right)$. We have:

$$\begin{aligned} e(v) &= \sum \left(h_i \cdot \frac{a}{2} - \frac{1}{2}\left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor\right)\right) + k \cdot \frac{a+1}{2} \\ &= n \cdot \frac{a}{2} - \frac{1}{2} \sum \left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor\right) + \frac{k}{2} \\ &= n \cdot \frac{a}{2} - \frac{1}{2}. \end{aligned}$$

It follows that $\frac{1}{2}\sum\left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor\right) - \frac{k}{2} = \frac{1}{2}$, so we conclude that $\sum\left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor\right) = s = k+1$. That means that the length of ε-blocks must be odd and between every two blocks there is a single letter α_i different from ε. So v has the form $\hat{\varepsilon}_1 \alpha_1 \cdots \hat{\varepsilon}_k \alpha_k \hat{\varepsilon}_{k+1}$. The first ε-block must be changed into $0a0 \cdots 0a0$, hence α_1 must be $\varepsilon + 1$, which, by changing into 0 contributes with $\frac{a+1}{2}$. Otherwise, $\alpha_1 = \varepsilon - 1$ could be changed into a, and contribution would be increased by one. But then, to achieve the maximal distance, the block $\hat{\varepsilon}_2$ must be changed into $a0a0a \cdots 0a$, which is a contradiction with the minimality of the contribution. That means $k = 0$, and $v = \hat{\varepsilon}$. ■

Let $\varepsilon = \left\lfloor \frac{a}{2} \right\rfloor$ and $\hat{\varepsilon} = \varepsilon \cdots \varepsilon \in V(\Pi_n^a)$, as before. We define $U_n^a \subseteq V(\Pi_n^a)$ to be the set containing all vertices such that

1. they differ from $\hat{\varepsilon}$ in at most one position, i.e., they have form $\varepsilon \cdots \varepsilon \alpha \varepsilon \cdots \varepsilon$, where $\alpha = \varepsilon$ or $\alpha = \varepsilon \pm 1$,

2. the letter $\varepsilon + 1$ can only appear in even positions,

3. the letter $\varepsilon - 1$ can only appear in odd positions.

Note that for an even n, every vertex $v \in U_n^a$ not equal to $\hat{\varepsilon}$ has exactly one ε-block of odd length.

Corollary 2.4.4. For a odd and n even, $Z(\Pi_n^a) = U_n^a$.

Proof. By Theorem 2.4.2, $\hat{\varepsilon} \in Z(\Pi_n^a)$, so let $v \in U_n^a$ be arbitrary and different from $\hat{\varepsilon}$. Let h_1 denote the length of an odd ε-block and, since n is even, the other ε-block has even length h_2. If v has one letter $\varepsilon + 1$, then $\varepsilon + 1$ is immediately after the ε-block of length h_1. So, by the proof of Theorem 2.4.2, the ε-block of length h_1, by changing into block $0a0 \cdots 0a0$, contributes with $h_1 \cdot \frac{a}{2} - \frac{1}{2}$, the letter $\varepsilon + 1$ contributes with $\frac{a+1}{2}$ by changing into 0, and the ε-block of length h_2 contributes with $h_2 \cdot \frac{a}{2}$ by changing into block $a0 \cdots a0$. So,

$$e(v) = h_1 \cdot \frac{a}{2} - \frac{1}{2} + \frac{a+1}{2} + h_2 \cdot \frac{a}{2} = n \cdot \frac{a}{2},$$

hence $v \in Z(\Pi_n^a)$. In the other case, where v contains a letter $\varepsilon - 1$ just before the ε-block of length h_1, we have similar situation. If the letter $\varepsilon - 1$ is the first letter, it contributes with $\frac{a+1}{2}$ by changing into $a-1$. The ε-block of length h_2 contributes with $h_2 \cdot \frac{a}{2}$ by changing into $0a \cdots 0a$. If $\varepsilon - 1$ appears after the ε-block of length h_2, the last ε in that block can be changed into 0 and $\varepsilon - 1$ into a, or the last ε can be changed into a and $\varepsilon - 1$ into 0. In both cases, the sum of the contribution is the same, $h_2 \cdot \frac{a}{2} + \frac{a+1}{2}$. The ε-block of length h_1 contributes with $h_1 \cdot \frac{a}{2} - \frac{1}{2}$. Thus, we proved $U_n^a \subseteq Z(\Pi_n^a)$.

It remains to prove that $Z(\Pi_n^a) \subseteq U_n^a$. Let $v \in Z(\Pi_n^a)$. Then $e(v) = n \cdot \frac{a}{2}$. On the other hand, let k denote the number of letters different from ε. Those letters must be $\varepsilon \pm 1$ since every other letter increases the eccentricity. If $h_1, \ldots, h_s$ denote the lengths of the ε-blocks, then

$$\begin{aligned} e(v) &= \sum \left(h_i \cdot \frac{a}{2} - \frac{1}{2} \left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor \right) \right) + k \cdot \frac{a+1}{2} \\ &= n \cdot \frac{a}{2} - \frac{1}{2} \sum \left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor \right) + \frac{k}{2}. \end{aligned}$$

It follows that $\sum \left(\left\lceil \frac{h_i}{2} \right\rceil - \left\lfloor \frac{h_i}{2} \right\rfloor \right) = k$, which means that the number of ε-blocks of odd length is k. If $k = 0$, then $v = \hat{\varepsilon}$. If $k = 1$, then there are at most two ε-blocks with lengths h_1 and h_2. Since $\varepsilon + 1$ coming before an ε-block of odd length increases its eccentricity, we conclude that $\varepsilon + 1$ must come after ε-block. In the other case, for the same reason, the letter $\varepsilon - 1$ must come before an ε-block of odd length. Since n is even, the other ε-block has an even length. So, $v \in U_n^a$. If $k > 1$, then $v = \hat{\varepsilon}_1 \alpha_1 \cdots \hat{\varepsilon}_k \alpha_k \hat{\varepsilon}_{k+1}$, where k blocks have odd lengths and one block has even length. Since $k > 1$, we have at least one substring $\hat{\varepsilon}_i \alpha_i \hat{\varepsilon}_j$ with both ε-blocks of odd length. Since α_i must be $\varepsilon + 1$, it increases the eccentricity of block $\hat{\varepsilon}_j$. Hence, $k \leq 1$ and $v \in U_n^a$. ■

Let $\varepsilon = \frac{a}{2}$ and let $V_n^a \subseteq Z(\Pi_n^a)$ be the set containing all vertices with letters ε or $\varepsilon - 1$ such that $\varepsilon - 1$ can not appear after an ε.

Corollary 2.4.5. For a even, $Z(\Pi_n^a) = V_n^a$.

Proof. Let a be even and let $\varepsilon = \frac{a}{2}$. By Theorem 2.4.2, we have $r(\Pi_n^a) = n \cdot \frac{a}{2}$ and $\hat{\varepsilon} \in Z(\Pi_n^a)$. Let $v \in Z(\Pi_n^a)$. The minimal contribution of each letter is $\frac{a}{2}$. For the minimal contribution to be achieved, one can only change the letter $\frac{a}{2}$ into 0 or a, and $\frac{a}{2} - 1$ into $a - 1$. For the letter $\frac{a}{2} - 1$ to be changed into $a - 1$, the letter just before it must be changed into $a - 1$ or a. So, letter $\frac{a}{2} - 1$ can be the first one or come after another letter $\frac{a}{2} - 1$. If it comes after $\frac{a}{2}$ which can always be changed into 0, the contribution of $\frac{a}{2} - 1$ would increase by one.

Conversely, from the definition of V_n^a, it is easy to see that $e(v) = n \cdot \frac{a}{2}$ whenever $v \in V_n^a$, because every $\varepsilon - 1$ contributes with $\frac{a}{2}$ by changing into $a - 1$, and every letter ε contributes with $\frac{a}{2}$ by changing either in 0 or a. Hence, $V_n^a \subseteq Z(\Pi_n^a)$. ■

For example, $Z(\Pi_6^5) = \{122222, 221222, 222212, 222222, 222223, 222322, 232222\}$, $Z(\Pi_7^5) = \{2222222\}$ and $Z(\Pi_4^4) = \{1111, 1112, 1122, 1222, 2222\}$.

Remark 2.4.6. $|Z(\Pi_n^a)| = \begin{cases} 1, & \text{for } a \text{ and } n \text{ odd,} \\ n+1, & \text{otherwise.} \end{cases}$

Our final result in this section is about the diameter of the metallic cubes.

Theorem 2.4.7. For $a \geq 1$ and $n \geq 1$ we have

$$d(\Pi_n^a) = an - 1.$$

The periphery consists of two vertices, $0a0a \cdots 0a(0)$ and $(a-1)0a \cdots a0(a)$.

Proof. Consider the vertex $v = 0a0a\cdots 0a0$ if n is odd or $v = 0a0a\cdots 0a$ if n is even. The first letter contributes with $a-1$ by changing into $a-1$, and every other letter with a by changing into 0 or a. We have $e(v) = an-1$. It is clear that no other vertex has eccentricity greater than $an-1$, because a string can not begin with the letter a. Thus the maximal contribution of the first letter is also achieved. So for n even we have $P(\Pi_n^a) = \{0a\cdots 0a, (a-1)0a\cdots a0\}$, and for n odd $P(\Pi_n^a) = \{0a\cdots a0, (a-1)0a\cdots 0a\}$. ■

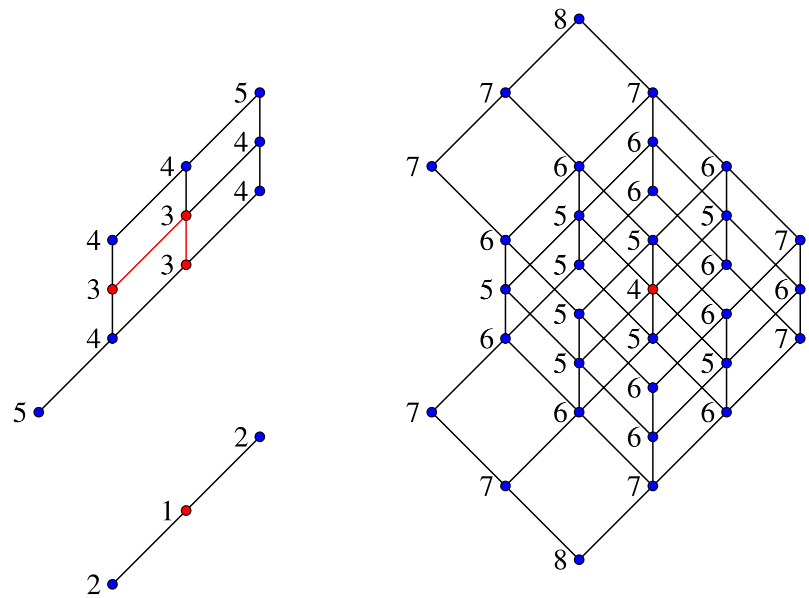

Figure 2.8: The metallic cube Π_n^3 for $n = 1, 2, 3$ with the eccentricity of every vertex.

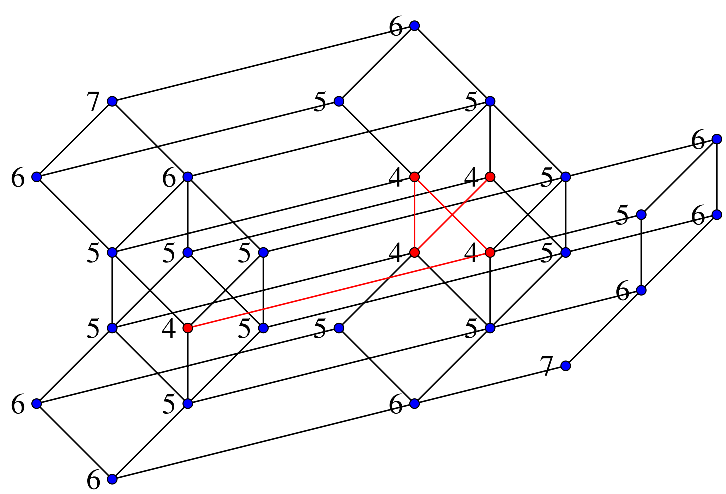

Figure 2.9: The metallic cube Π_4^2 with the eccentricity of every vertex.

2.5. HAMILTONICITY

A *Hamiltonian path* in a graph G is a path that visits each vertex of G exactly once. A *Hamiltonian cycle* in G is a cycle that visits each vertex exactly once. A graph containing a Hamiltonian cycle is called *Hamiltonian graph*. In this section, we examine some Hamiltonicity-related properties of the metallic cubes.

In their paper [36], Liu et al. showed that all (generalized) Fibonacci cubes contain a Hamiltonian path. In the next theorem, we show that every metallic cube contains a Hamiltonian path.

Theorem 2.5.1. For all $a, n \geq 1$, the metallic cube Π_n^a contains a Hamiltonian path.

Proof. The proof is by induction on n. We first consider the case of a even. We wish to prove that, not only the graph Π_n^a contains a Hamiltonian path, but it contains a path starting at $0a\cdots 0a(0)$ and ending at $(a-1)\cdots(a-1)$. For $n=1$, graph Π_1^a is a path on a vertices and the claim is valid. For $n=2$, graph Π_2^a is an $a \times a$ grid with addition of one vertex $0a$ being adjacent to the vertex $0(a-1)$. So, for even a, a Hamiltonian path is $0a \to \cdots \to 00 \to 10 \to \cdots \to 1(a-1) \to \cdots \to (a-1)(a-1)$. Figure 2.10 shows a Hamiltonian path in Π_2^2 and in Π_2^3. Now suppose that Π_k^a contains a Hamiltonian path for all $k < n$. By the assumption, let H_{n-1} and H_{n-2} denote the Hamiltonian paths in Π_{n-1}^a and Π_{n-2}^a, respectively. The paths H_{n-1} and H_{n-2} start at $0a0a\cdots 0a(0)$ and end at $(a-1)\cdots(a-1)$. Let $0H_{n-1}, \ldots, (a-1)H_{n-1}$ and $0aH_{n-2}$ denote the paths is Π_n^a obtained from H_{n-1} and H_{n-2} by appending $0, 1, \ldots, a-1$ or $0a$ on each vertex and let $0\overline{H}_{n-1}, \ldots, (a-1)\overline{H}_{n-1}$ and $0a\overline{H}_{n-2}$ denote their reverse paths. The path $0aH_{n-2}$ ends at $0a(a-1)\cdots(a-1)$ and the path $0\overline{H}_{n-1}$ begins at $0(a-1)\cdots(a-1)$. Since the vertices $0a(a-1)\cdots(a-1)$ and $0(a-1)\cdots(a-1)$ are adjacent in Π_n^a, those two paths can be concatenated. Hence, by concatenating $0aH_{n-2}, 0\overline{H}_{n-1}, 1H_{n-1}, \cdots, (a-2)\overline{H}_{n-1}$ and $(a-1)H_{n-1}$, we obtain a Hamiltonian path in Π_n^a that begins at $0a\cdots 0a(0)$ and ends at $(a-1)\cdots(a-1)$.

Let a now be odd. We wish to prove that Π_n^a contains a Hamiltonian path with endpoints at $0a(a-1)\cdots 0a(a-1)$ and $(a-1)0a\cdots(a-1)0a$. For example, if $a=3$ and $n=1,2,3,4,5$, the endpoints are 0, 2, 03, 20, 032, 203, 0320, 2032, 03203 and

20320, respectively. The rest of the proof is carried out for n where $n \equiv 0 \pmod 3$. The cases for $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$ are similar, so we omit the details. The base of induction is easy to check, similar to the case of a even. By the inductive assumption, let H_{n-1} and H_{n-2} denote Hamiltonian paths in Π^a_{n-1} and Π^a_{n-2}, respectively. The path H_{n-1} has the starting point $0a(a-1)0a(a-1)\cdots 0a$ and the ending point $(a-1)0a(a-1)0a\cdots(a-1)0$, and the path H_{n-2} starts at $0a(a-1)0a(a-1)\cdots 0$ and ends at $(a-1)0a(a-1)0a\cdots(a-1)$. Let $0H_{n-1},\ldots,(a-1)H_{n-1}$ and $0aH_{n-2}$ denote the paths in Π^a_n obtained from H_{n-1} and H_{n-2} by appending $0,1,\ldots,a-1$ and $0a$, respectively, on each vertex, and let $0\overline{H}_{n-1},\ldots,(a-1)\overline{H}_{n-1}$ and $0a\overline{H}_{n-2}$ denote their reverse paths. The path $0aH_{n-2}$ ends at $0a(a-1)\cdots 0a(a-1)$ and the path $0\overline{H}_{n-1}$ begins at $0(a-1)0a\cdots(a-1)0$. Since the vertices $0a(a-1)\cdots 0a(a-1)$ and $0(a-1)0a\cdots(a-1)0$ are adjacent in Π^a_n, those two paths can be concatenated. Hence, by concatenating $0aH_{n-2}, 0\overline{H}_{n-1}, 1H_{n-1}, 2\overline{H}_{n-1}\cdots,(a-2)H_{n-1}$ and $(a-1)\overline{H}_{n-1}$, we obtain a Hamiltonian path in Π^a_n that begins at $0a(a-1)\cdots 0a(a-1)$ and ends at $(a-1)0a\cdots(a-1)0a$. ■

As an example, Figure 2.11 shows Hamiltonian paths in Π^2_4 and in Π^3_4.

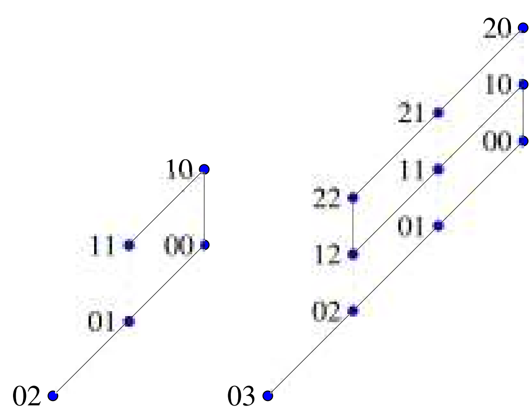

Figure 2.10: Hamiltonian path of Π^2_2 and Π^3_2.

For a even, the numbers s^a_{2n+1} are even and s^a_{2n} are odd. On the other hand, for a odd, the numbers s^a_{3n+1} are even, but s^a_{3n} and s^a_{3n+2} are odd. A simple consequence of Theorem 2.5.1 is the following result on the existence of perfect and semi-perfect matchings in metallic cubes.

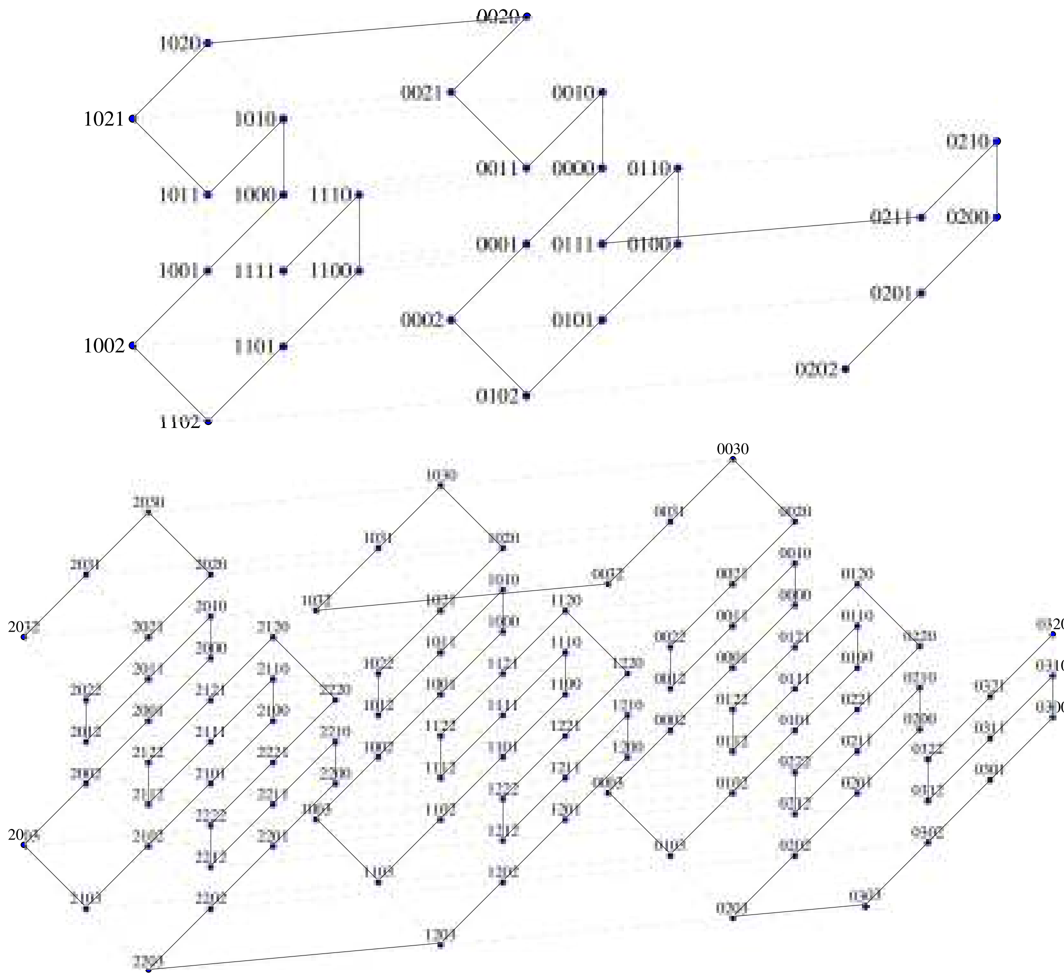

Figure 2.11: Hamiltonian paths in Π_4^2 (above) and in Π_4^3 (below).

Corollary 2.5.2. Graph Π_n^a admits a perfect matching if the number of vertices is even, otherwise, it admits a semi-perfect matching.

Now we examine the existence of Hamiltonian cycles in the metallic cubes.

Theorem 2.5.3. For a even and $n > 1$ odd, the metallic cube Π_n^a is Hamiltonian.

Proof. Let a be even. Then Π_2^a is an $a \times a$ grid with additional vertex $0a$. It is easy to see that $a \times a$ grids contain a Hamiltonian cycle, which implies that graph Π_2^a contains a cycle that visits all vertices but one, namely, the vertex $0a$. Now we wish to construct a Hamiltonian cycle in Π_3^a. The graph Π_3^a can be decomposed into a copies of Π_2^a and one copy of Π_1^a. By Theorem 2.5.1, there exists a Hamiltonian path in each subgraph Π_2^a and Π_1^a. Let $0H_2, 1H_2, \ldots, (a-1)H_2$ and $0aH_1$ denote those paths. Connecting starting points in $0H_2$ and $1H_2$, as well as the ending points in $0H_2$ and $1H_2$ yields a cycle in $0\Pi_2^a \oplus 1\Pi_2^a$.

Since the number of copies of graph Π_2^a is even, we can pair up those copies to obtain $\frac{a}{2}$ cycles in the same manner. Now choose the vertices $1v, 1w \in 1\Pi_2^a$ so that the edge $(1v)(1w)$ is in the Hamiltonian cycle in $1\Pi_2^a$, and the corresponding vertices $2v$ and $2w$ in Π_2^a. By removing edges $(1v)(1w)$ and $(2v)(2w)$ from the cycles in $1\Pi_2^a$ and $2\Pi_2^a$ and by adding the edges $(1v)(2v)$ and $(1w)(2w)$, we obtain a Hamiltonian cycle for the subgraph $0\Pi_2^a \oplus 1\Pi_2^a \oplus 2\Pi_2^a \oplus 3\Pi_2^a$. This method is applicable between every two (neighboring) of the $\frac{a}{2}$ cycles. Thus, we obtained a Hamiltonian cycle in $P_a \square \Pi_2^a$. Since Π_1^a is a path graph with an even number of vertices and, by Theorem 2.5.1 and construction so far, the path $0(a-1)H_1^a \subset 0H_2^a$ is a part of the Hamiltonian cycle in $P_a \square \Pi_2^a$. Hence, we can extend a Hamiltonian cycle in $P_a \square \Pi_2^a$ to Π_3^a by adding a zig-zag shape path to include all vertices of Π_1^a. Figure 2.12 shows a Hamiltonian cycle for $a = 2$ and $a = 4$, but it is clear that construction can be done whenever a is even.

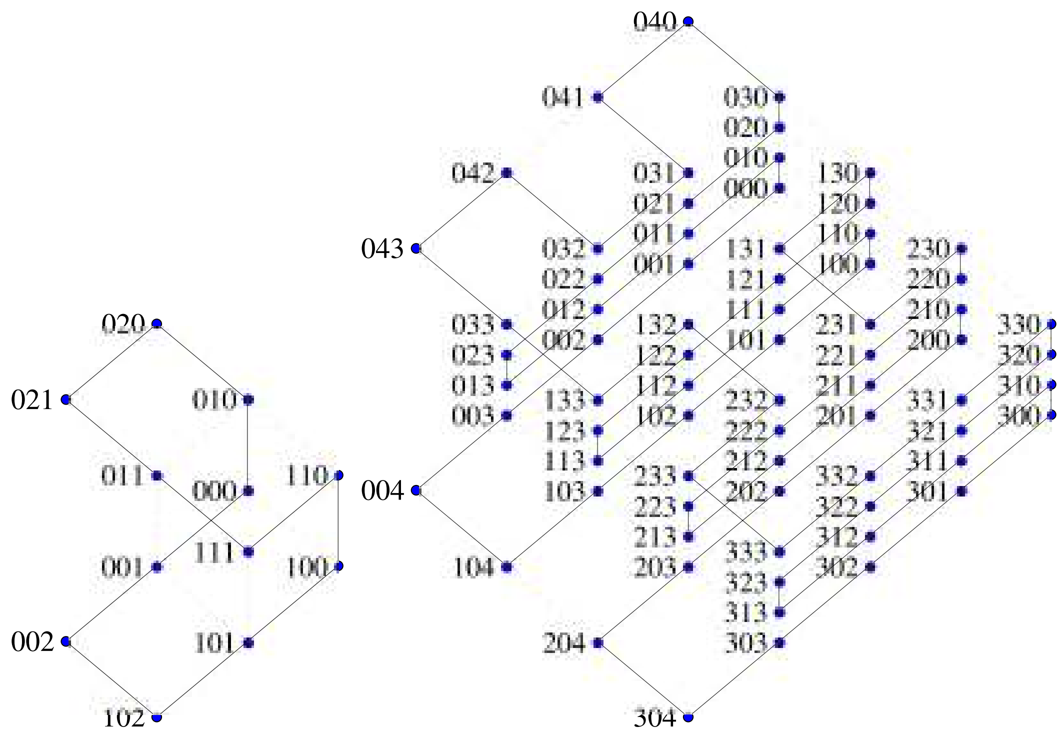

Figure 2.12: Hamiltonian cycle in Π_3^2 and in Π_3^4.

Now we wish to prove that every metallic cube contains a Hamiltonian cycle when a is even and n is odd. In the case of both a and n even, it contains a cycle that visits all vertices but one. The proofs for n even and n odd are similar, so we present just the proof for n odd. Since $\Pi_n^a = P_a \square \Pi_{n-1}^a \oplus \Pi_{n-2}^a$, by the assumption of induction, there is a cycle in each copy Π^a_{n-1} that visits all vertices but one. Let $k\Pi^a_{n-1}$ denote the copy of graph Π^a_{n-1} induced by the vertices that start with k, where $0 \le k \le a-1$. Let $\alpha \in \mathscr{S}^a_{n-1}$ and let $k\alpha$ denote those vertices omitted by the Hamiltonian cycle in each subgraph $k\Pi^a_{n-1}$, $0 \le k \le a-1$. Furthermore, by the assumption, there is a Hamiltonian cycle in Π^a_{n-2}. Now we choose $\beta, \gamma \in \mathscr{S}^a_{n-1}$ so that the vertices $k\alpha$ and $k\beta$ are adjacent in $k\Pi^a_{n-1}$ and that the edge $(k\beta)(k\gamma)$ is a part of the constructed Hamiltonian cycle in $k\Pi^a_{n-1}$. For $k = 0,2,4,\dots,a-2$, we remove the edge $(k\beta)(k\gamma)$ and add the edges $(k\beta)(k\alpha)$, $(k\alpha)((k+1)\alpha)$, $((k+1)\alpha)(k+1)\beta$ and $(k\gamma)(k+1)\gamma$. Thus we obtained a Hamiltonian cycles in $k\Pi^a_{n-1} \oplus (k+1)\Pi^a_{n-1}$. To merge those $\frac{a}{2}$ cycles and one cycle in Π^a_{n-2} together, we can choose any two corresponding edges that are part of the Hamiltonian cycles in the neighboring subgraphs and apply a similar method to merge all cycles into a Hamiltonian cycle of the graph Π^a_n. Choosing corresponding edges is possible, for the construction of the Hamiltonian cycle in this manner is inductive, hence, if some edge is a part of the Hamiltonian cycle in one copy of Π^a_{n-1}, then it is a part of the Hamiltonian cycle in all copies of Π^a_{n-1}. Similar observation holds for the subgraphs $0\Pi^a_{n-1}$ and $0a\Pi^a_{n-2}$. Figure 2.13 schematically shows the described method of the construction of the Hamiltonian cycles in Π^a_n.

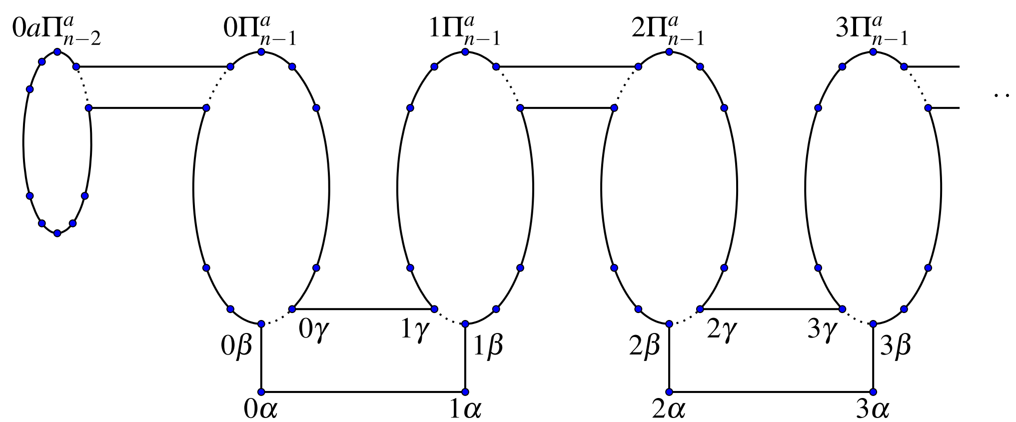

Figure 2.13: The construction of a Hamiltonian cycle.

■

2.6. WIENER INDEX

A topological index is a graph invariant used in the study of the connection between the structure and some property. One of the oldest and most investigated topological indices is the Wiener index [50,51]. The Wiener index attracted a lot of attention and it was studied extensively not only in the mathematical [8,22,43], but also in the chemical literature [42]. It is defined as the sum of the distances between each unordered pair of vertices. More precisely, for a graph $G = (V(G), E(G))$,

$$W(G) = \frac{1}{2} \sum_{uv \in E(G)} d(u,v).$$

In this section, we are interested in computing the Wiener index of metallic cubes.

Recall that the hypercube $Q_n = K_2^n$ is a graph whose vertices are binary strings of length n, and two vertices are adjacent if they differ in only one position. A subgraph H of the graph G is said to be *isometric* subgraph if the distance between each pair of vertices in H is equal to the distance between the same vertices in G. An isometric subgraph of a hypercube is called *partial cube*. In particular, Fibonacci cubes are partial cubes [30]. Let G be a graph isometrically embedded into a hypercube Q_m. By n we denote the number of vertices in G, and by n_i we denote the number of vertices of G that have 1 on the i-th position. To obtain the Wiener index of the metallic cubes, we use the following theorem [31]:

Theorem 2.6.1 (Klavžar, Gutman, Mohar)**.** Let G be a graph isometrically embedded into a hypercube Q_m. Then

$$W(G) = \sum_{k=1}^{m} n_i(n - n_i).$$

Here it is worth mentioning that, if G is embeddable, its embedding into a hypercube is unique up to permutation of coordinates, positions, and unused coordinates [52].

The next theorem provides the Wiener index of the metallic cubes.

Theorem 2.6.2. For any $n \geq 0$,

$$W(\Pi_n^a) = \sum_{k=1}^{n} s_{k-1}^a s_{n-k}^a \sum_{t=1}^{a} \left(t s_{n-k}^a + s_{n-k-1}^a\right)\left(s_k^a - t s_{k-1}^a\right).$$

Proof. The metallic cubes are, of course, partial cubes. One such isometrically embedding in the hypercube can be defined on the letters $\{0,1,2\ldots a\}$ and extended to an embedding $\sigma : \Pi_n^a \to Q_{an}$ as

$$\sigma(t) = \underbrace{111\cdots111}_{t}\underbrace{000\cdots000}_{a-t}$$

for $0 \le t \le a$. For example, if $a = 3$ then $\sigma(0) = 000$, $\sigma(1) = 100$, $\sigma(2) = 110$ and $\sigma(3) = 111$. Then, for each vertex v in Π_n^a, we have that $\sigma(v)$ is composed of n blocks, where each block is of length a. Now consider the vertices with 0 on the first position in the k-th block. That means that there is 0 on the k-th position in v. We are interested in the number of vertices with that property. Since 0 can stand alone or be part of the $0a$-block, first $k-1$ positions can be filled in s_{k-1}^a ways and the remaining $n-k$ positions can be filled in $s_{n-k-1}^a + s_{n-k-2}^a$ ways. Hence, the number of vertices having 0 on the first position in k-th block is $s_{k-1}^a\left(s_{n-k}^a + s_{n-k-1}^a\right)$. Similar, the number of vertices having 1 on the first position in k-th block is $(a-1)s_{k-1}^a s_{n-k}^a + s_{k-2}^a s_{n-k}^a$, since 1 on the first position in k-th block means that there is $1,2,3,\ldots,(a-1)$ or a on k-th position in v. More generally, the number of vertices having 0 on t-th position in k-th block is $ts_{k-1}^a s_{n-k}^a + s_{k-1}^a s_{n-k-1}^a$ and the number of vertices having 1 on t-th position in k-th block $(a-t)s_{k-1}^a s_{n-k}^a + s_{k-2}^a s_{n-k}^a$. So, by Theorem 2.6.1,

$$\begin{aligned}
W(\Pi_n^a) &= \sum_{k=1}^{n}\sum_{t=1}^{a}\left(ts_{k-1}^a s_{n-k}^a + s_{k-1}^a s_{n-k-1}^a\right)\left((a-t)s_{k-1}^a s_{n-k}^a + s_{k-2}^a s_{n-k}^a\right)\\
&= \sum_{k=1}^{n} s_{k-1}^a s_{n-k}^a \sum_{t=1}^{a}\left(ts_{n-k}^a + s_{n-k-1}^a\right)\left((a-t)s_{k-1}^a + s_{k-2}^a\right)\\
&= \sum_{k=1}^{n} s_{k-1}^a s_{n-k}^a \sum_{t=1}^{a}\left(ts_{n-k}^a + s_{n-k-1}^a\right)\left(s_k^a - ts_{k-1}^a\right)
\end{aligned}$$

■

For $a = 1$ one has $s_n^1 = F_{n+1}$ and

$$\begin{aligned}
W(\Pi_n^1) &= \sum_{k=1}^{n} s_{k-1}^a s_{n-k}^a\left(s_{n-k}^a + s_{n-k-1}^a\right)\left(s_k^a - s_{k-1}^a\right)\\
&= \sum_{k=1}^{n} s_{k-1}^a s_{n-k}^a s_{n-k+1}^a s_{k-2}^a\\
&= \sum_{k=1}^{n} F_k F_{n-k+1} F_{n-k+2} F_{k-1}
\end{aligned}$$

which immediately retrieves the result on the Wiener index of the Fibonacci cubes [32]. For $a = 2$ we have the sequence $1, 16, 146, 1168, ...$, and for $a = 3$ we obtain $4, 99, 1664, 24552, ...$ None of the sequences for $a > 1$ appears in OEIS.

2.7. Further possibilities of research

In this chapter, we have introduced and studied a family of graphs that both generalize the Fibonacci cubes and offer an alternative definition of the Pell graphs introduced recently by Munarini [40]. Our results, however, are far from presenting a comprehensive portrait of the new family. For example, we calculated the Wiener index of the metallic cubes, but our results on degree distribution open the possibility of computing many other degree-based topological invariants, while our results on distances do the same for some other distance-based invariants such as, e.g., the Mostar index [10]. Our results on degrees could be further developed by looking at irregularity, in the way it was done for Fibonacci and Lucas cubes [1, 18]. Similarly, apart from a short comment about perfect and semi-perfect matchings, we have not exploited our decomposition results to investigate any matching-, independence- and dominance-related problems that have been studied for hypercubes and the Fibonacci and Lucas cubes [38].

3. HORADAM CUBES

3.1. DEFINITIONS

In this chapter, we define and investigate a family of graphs whose number of vertices is given by the sequence that satisfies the recurrence $s_n^{a,b} = as_{n-1}^{a,b} + bs_{n-2}^{a,b}$, and their adjacency is defined in the same way as for the Fibonacci and metallic cubes. Thus the new family is a natural generalization of the Fibonacci and metallic cubes.

Let a, b be non-negative integers, and let $\mathscr{S}^{a,b}$ denote the free monoid containing $a+b$ generators $\{0, 1, 2, \ldots, 0a, 0(a+1), \ldots, 0(a+b-1)\}$. By a *string* we mean an element of monoid $\mathscr{S}^{a,b}$, i.e., a word from the alphabet $\{0, 1, 2, \ldots, a+b-1\}$ with the property that letters $a, a+1, \ldots, a+b-2$, and $a+b-1$ can appear only immediately after 0. Other letters can appear without any restrictions. For two strings, $\alpha = \alpha_1 \cdots \alpha_n$ and $\beta = \beta_1 \cdots \beta_m$, we define their *concatenation* as $\alpha\beta = \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m$. If $\mathscr{S}_n^{a,b}$ denotes the set of all elements from the monoid $\mathscr{S}^{a,b}$ of length n, we can easily obtain that the cardinal number $s_n^{a,b} = \left|\mathscr{S}_n^{a,b}\right|$ satisfies the Horadam's recursive relation

$$s_n^{a,b} = as_{n-1}^{a,b} + bs_{n-2}^{a,b}, \tag{3.1}$$

with initial values $s_0^{a,b} = 1$, $s_1^{a,b} = a$ and $s_n^{a,b} = 0$ for $n < 0$. In fact, the numbers $s_n^{a,b}$ satisfy the well-known identity

$$s_n^{a,b} = F_{n+1}(a,b) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} a^{n-2k} b^k, \tag{3.2}$$

where $F_n(x,y)$ are the modified Fibonacci polynomials defined recursively as

$$F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y)$$

with initial values $F_0(x,y) = 1$ and $F_1(x,y) = x$.

Horadam's recursive relation generalizes many well-known sequences. For example, by setting $a = b = 1$, one obtains the recursive relation for the (shifted) Fibonacci numbers

$$F_n = F_{n-1} + F_{n-2}.$$

By setting $a = 2$ and $b = 1$, we have a recursive relation for the (shifted) Pell numbers

$$P_n = 2P_{n-1} + P_{n-2},$$

and $a = 1$ and $b = 2$ yields the (shifted) Jacobsthal numbers

$$J_n = J_{n-1} + 2J_{n-2}.$$

Starting from the recursive relation (3.1), one can readily obtain the generating function $S(x) = \sum_{n \geq 0} s_n^{a,b} x^n$ for the sequence $s_n^{a,b}$ as

$$S(x) = \frac{1}{1 - ax - bx^2}. \tag{3.3}$$

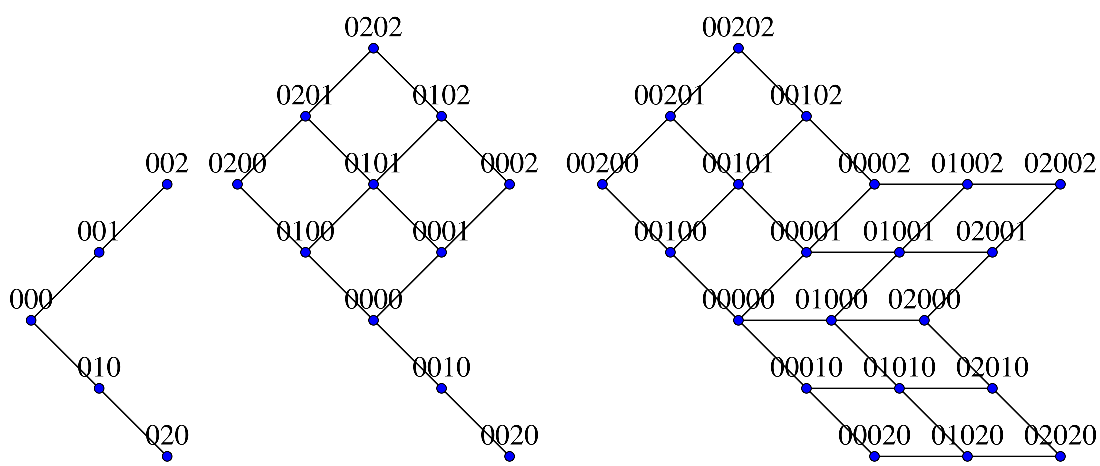

Figure 3.1: The Jacobsthal cubes for $n = 3, 4, 5$.

Now we move toward the definition of the Horadam cubes. Let $\Pi_n^{a,b}$ denote the graph such that $V\left(\Pi_n^{a,b}\right) = \mathscr{S}_n^{a,b}$ and for any $v_1, v_2 \in V\left(\Pi_n^{a,b}\right)$, we have that v_1 and v_2 are adjacent, i.e., $v_1 v_2 \in E\left(\Pi_n^{a,b}\right)$ if and only if one vertex can be obtained from the other by replacing a single letter k with $k+1$ for $0 \leq k \leq a+b-2$. Alternatively, $v_1 = \alpha_1 \cdots \alpha_n$ and $v_2 = \beta_1 \cdots \beta_n$ are adjacent if and only if

$$\sum_{k=1}^{n} |\alpha_i - \beta_i| = 1.$$

The graph $\Pi_0^{a,b}$ is trivial, i.e., it contains a single vertex. Note that for $b = 1$, one obtains the metallic cubes, and for $a = 1$ and $b = 1$, the Fibonacci cubes. As an example, Figure 3.1 shows some small Horadam cubes for $a = 1$ and $b = 2$, which can also be referred to as *Jacobsthal cubes*, since their vertices are counted by Jacobsthal numbers J_n. Another example for $a = 3$ and $b = 2$ is presented in Figure 3.2.

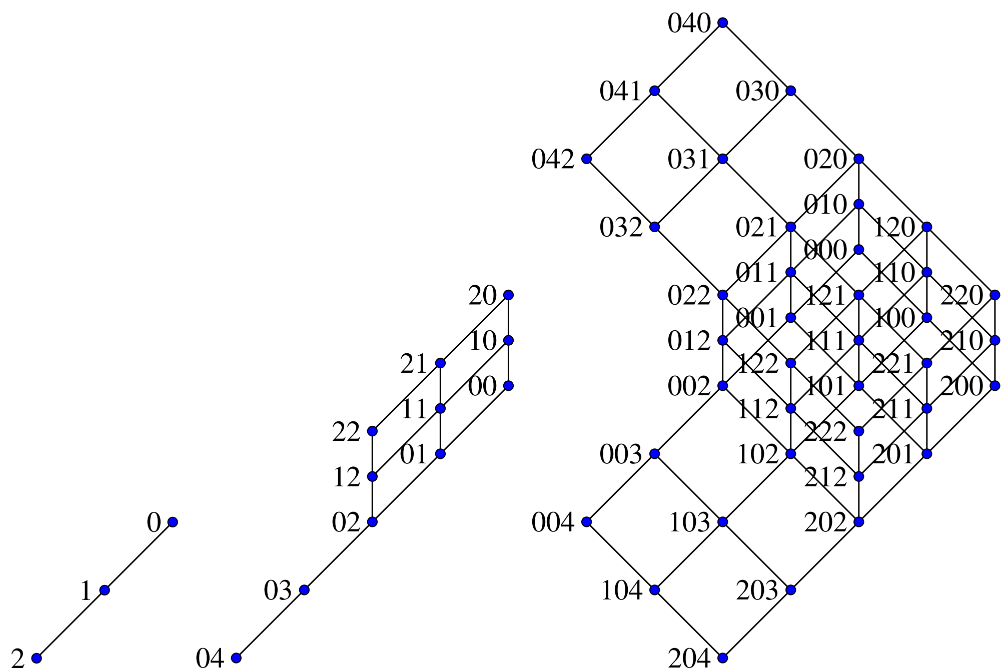

Figure 3.2: The Horadam cubes $\Pi_n^{3,2}$ for $n = 1, 2, 3$.

3.2. Canonical decomposition and bipartivity

The canonical decomposition of the Fibonacci cubes and metallic cubes [12,41] naturally extends to the Horadam cubes. Similar to those cases, the set of vertices $\mathscr{S}_n^{a,b}$ can be arranged into disjoint subsets based on the starting letter. Namely, there are b subsets that contain the vertices starting with block $0a, 0(a+1), \ldots, 0(a+b-2)$ and $0(a+b-1)$, and the remaining a subsets contain the vertices that start with $0, 1, \ldots, a-2$ and $a-1$, respectively. Assuming $\alpha \in \mathscr{S}_{n-1}^{a,b}$ and $\beta \in \mathscr{S}_{n-2}^{a,b}$, the vertices 0α, 1α, $\ldots$, $(a-1)\alpha$ generate a copies of $\Pi_{n-1}^{a,b}$, and vertices $0a\beta, 0(a+1)\beta, \ldots 0(a+b-1)\beta$ generate b copies of $\Pi_{n-2}^{a,b}$. That yields the following theorem:

Theorem 3.2.1. For $n \geq 2$, the Horadam cube $\Pi_n^{a,b}$ admits the decomposition

$$\Pi_n^{a,b} = \Pi_{n-1}^{a,b} \oplus \cdots \oplus \Pi_{n-1}^{a,b} \oplus \Pi_{n-2}^{a,b} \oplus \cdots \oplus \Pi_{n-2}^{a,b} = P_a \square \Pi_{n-1}^{a,b} \oplus P_b \square \Pi_{n-2}^{a,b},$$

where P_a and P_b denote the path graphs of lengths a and b, respectively, and $\square$ operator denotes the Cartesian product of graphs.

The decomposition of Theorem 3.2.1 is called the *canonical decomposition* of $\Pi_n^{a,b}$. A copy of the subgraph $\Pi_{n-1}^{a,b}$ induced by vertices in $\Pi_n^{a,b}$ starting with the letter k where $0 \leq k \leq a-1$ is denoted by $k\Pi_{n-1}^{a,b}$, and a copy of the subgraph $\Pi_{n-2}^{a,b}$ induced by vertices in $\Pi_n^{a,b}$ starting with the block $0l$ where $a \leq l \leq a+b-1$ is denoted by $0l\,\Pi_{n-2}^{a,b}$. This notation is useful if it is important to distinguish those copies. For example, $1\Pi_{n-1}^{a,b}$ denotes the induced subgraph of $\Pi_n^{a,b}$ isomorphic to $\Pi_{n-1}^{a,b}$ induced by vertices in $\Pi_n^{a,b}$ that start with 1.

In Figure 3.3 we present a schematic representation of the described canonical decomposition and Figure 3.4 shows the canonical decomposition of the Horadam cube $\Pi_4^{2,2}$.

Recall that the map $\chi : V(G) \to \{0,1\}$ is a proper 2-coloring of a graph G if $\chi(v_1) \neq \chi(v_2)$ for every two adjacent vertices $v_1, v_2 \in V(G)$. A graph G is bipartite if its set of vertices $V(G)$ can be decomposed into two disjoint subsets, A and B, such that an edge connects no two vertices of the same subset. Equivalently, a graph G is bipartite if and only if it admits a proper 2-coloring.

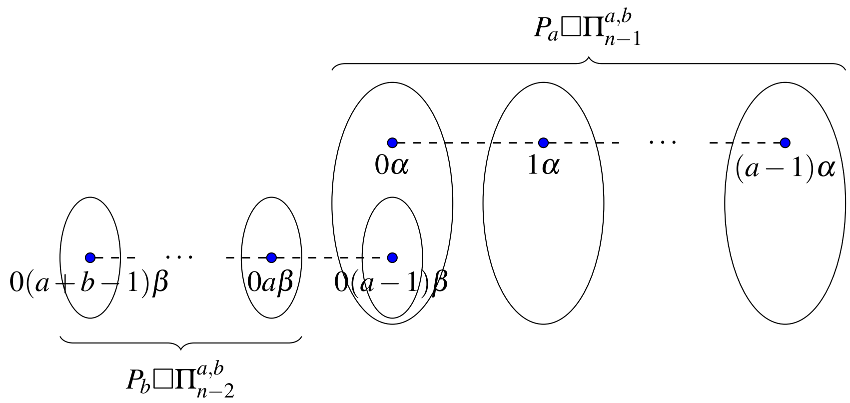

Figure 3.3: Canonical decomposition of $\Pi_n^{a,b} = P_a\square\Pi_{n-1}^{a,b} \oplus P_b\square\Pi_{n-2}^{a,b}$.

Theorem 3.2.2. All Horadam cubes are bipartite.

Proof. The proof is by induction on n. We observe that $\Pi_1^{a,b}$ is isomorphic to a path graph on a vertices and, thus, bipartite. Since $\Pi_2^{a,b}$ is an $a\times a$ grid with the path graph containing b vertices appended to the vertex $0(a-1)$, it also admits a proper 2-coloring. Now, we suppose that $\Pi_k^{a,b}$ is bipartite for every $k<n$. By the inductive hypothesis, graph $\Pi_{n-1}^{a,b}$ admits a proper 2-coloring $\chi : \Pi_{n-1}^{a,b} \to \{0,1\}$. Consider the map $\chi' : \Pi_{n-1}^{a,b} \to \{0,1\}$, where $\chi'(v) = 1-\chi(v)$. Then the map χ' is a complementary proper 2-coloring for the graph $\Pi_{n-1}^{a,b}$. Since $\Pi_n^{a,b} = \Pi_{n-1}^{a,b} \oplus\cdots\oplus \Pi_{n-1}^{a,b} \oplus \Pi_{n-2}^{a,b} \oplus\cdots\oplus \Pi_{n-2}^{a,b}$, we can choose a coloring χ for the subgraph $k\Pi_{n-1}^{a,b}$ in $\Pi_n^{a,b}$ if the subgraph is induced by vertices starting with even k, and χ' if k is odd. Finally, for the b copies of $\Pi_{n-2}^{a,b}$ in the canonical decomposition, we consider χ restricted to the subgraph $0(a-1)\Pi_{n-2}^{a,b}$, and choose complementary coloring χ' restricted to $\Pi_{n-2}^{a,b}$ for the graph induced by the block $0a$. For the remaining $b-1$ copies, we can alternate between χ and χ' to obtained a proper 2-coloring for $\Pi_n^{a,b}$. ■

Similar to the case of the metallic cubes, the proof of Theorem 3.2.2 implies that the sets A and B have the same cardinality whenever the number of vertices is even. Otherwise, the cardinality differs by one. Figure 3.4 shows a proper 2-coloring of the Horadam cube $\Pi_4^{2,2}$. Since the graph $\Pi_4^{2,2}$ has 44 vertices, we have 22 vertices in each set.

We now present another decomposition of the Horadam cubes. To this end, recall that P_a^k denotes the Cartesian product of path P_a with itself k times, that is, $P_a\square P_a\cdots\square P_a$, and $P_a\square P_b$ denotes the Cartesian product of paths P_a and P_b. Also, note that $|V(P_a^k\square P_b^m)| = a^k b^m$. The Cartesian products of the path graphs are called *grids* or *lattices* [23]. The

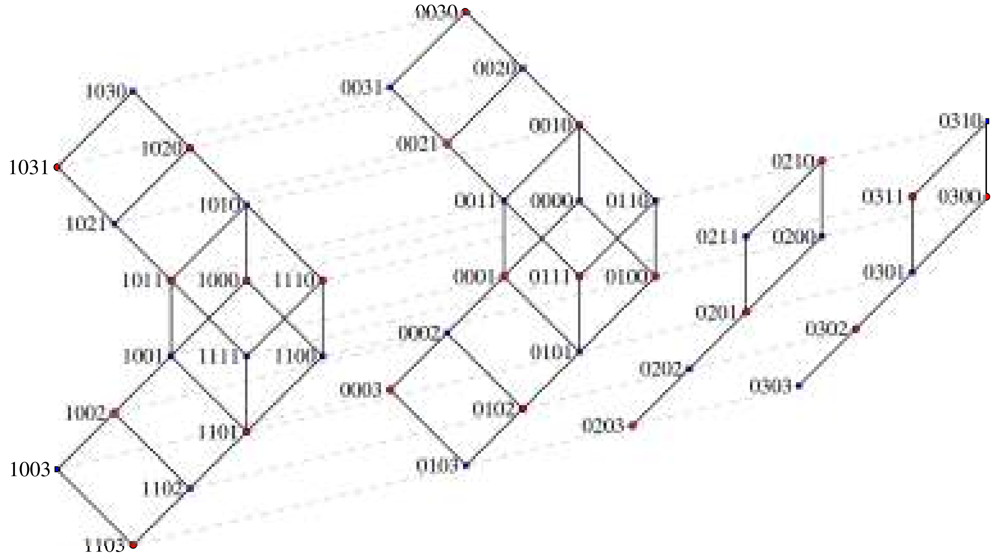

Figure 3.4: The canonical decomposition and a proper 2-coloring of $\Pi_4^{2,2} = \Pi_3^{2,2} \oplus \Pi_3^{2,2} \oplus \Pi_2^{2,2} \oplus \Pi_2^{2,2}$.

following theorem illustrates how the combinatorial meaning of the Fibonacci polynomials of the identity (3.2) extends to the Horadam cubes. Before we state the next theorem, we need to define a map ρ which will be used as a connection between the Horadam and Fibonacci cubes. Let $\mathscr{F}_n$ be the set that contains all Fibonacci strings of length n and recall that $|\mathscr{F}_n| = F_{n+2}$. We define $\rho : V(\Pi_n^{a,b}) \to \mathscr{F}_n$ to be the map on the alphabet $\{0,1,\dots,a+b-1\}$ and then extended to $V(\Pi_n^{a,b})$ by concatenation, as follows

$$\rho(\alpha) = \begin{cases} 0, & 0 \le \alpha \le a-1, \\ 1, & \alpha \ge a. \end{cases}$$

For a Fibonacci string w, let $\rho^{-1}(w)$ denote the subgraph of $\Pi_n^{a,b}$ induced by vertices that ρ maps into w, i.e, the subgraph induced by vertices $\left\{v \in V(\Pi_n^{a,b}) | \rho(v) = w\right\}$.

Theorem 3.2.3. The Horadam cube $\Pi_n^{a,b}$ can be decomposed into F_{n+1} grids, where F_n denotes the n^{th} Fibonacci number. If $\Pi_n^{a,b}/\rho$ denotes the quotient graph of $\Pi_n^{a,b}$ obtained by identifying all vertices which are identified by ρ, and two vertices w_1 and w_2 in $\Pi_n^{a,b}/\rho$ are adjacent if there is at least one edge in $\Pi_n^{a,b}$ connecting the blocks $\rho^{-1}(w_1)$ and $\rho^{-1}(w_2)$. Then $\Pi_n^{a,b}/\rho$ is isomorphic to the Fibonacci cube Γ_{n-1}.

Proof. Let w be any Fibonacci string that contains k ones. By definition of the map ρ, the induced subgraph $\rho^{-1}(w)$ of $\Pi_b^{a,b}$ is isomorphic to $P_a^{n-2k} P_b^k$. Also note that ρ maps

$V(\Pi_n^{a,b})$ into Fibonacci strings that start with 0, hence $|\rho(V(\Pi_n^{a,b}))| = F_{n+1}$. Furthermore, for every two different $w, v \in \mathscr{F}_n$, subgraphs $\rho^{-1}(w)$ and $\rho^{-1}(v)$ are vertex disjoint. That yields a decomposition into F_{n+1} grids, as for each Fibonacci string w, we have the corresponding grid $\rho^{-1}(w)$. If two Fibonacci strings w_1 and w_2 differ in only one position, then there is at least one edge connecting the grids $\rho^{-1}(w_1)$ and $\rho^{-1}(w_2)$. Indeed, without loss of generality, let w_1 be the Fibonacci string with 1 on some position l, and w_2 be the same string except for 0 on position l. Now, let v_1 be the vertex in $\Pi_n^{a,b}$ that has the letter a at the same positions where w_1 has ones, and v_2 be the same string as v_1 except for $a-1$ on position l. Then $\rho(v_1) = w_1$, $\rho(v_2) = w_2$, and the edge between the grids $\rho^{-1}(w_1)$ and $\rho^{-1}(w_2)$ is the edge connecting v_1 and v_2. On the other side, if two grids $\rho^{-1}(w_1)$ and $\rho^{-1}(w_2)$ have an edge connecting them, then it is clear that w_1 and w_2 must differ in only one position. The claim follows. ■

Figure 3.5 shows the decomposition of $\Pi_3^{3,2}$ and the Jacobsthal cube $\Pi_5^{1,2}$. Note that $\Pi_3^{3,2}/\rho = \Gamma_2$ and $\Pi_5^{1,2}/\rho = \Gamma_4$.

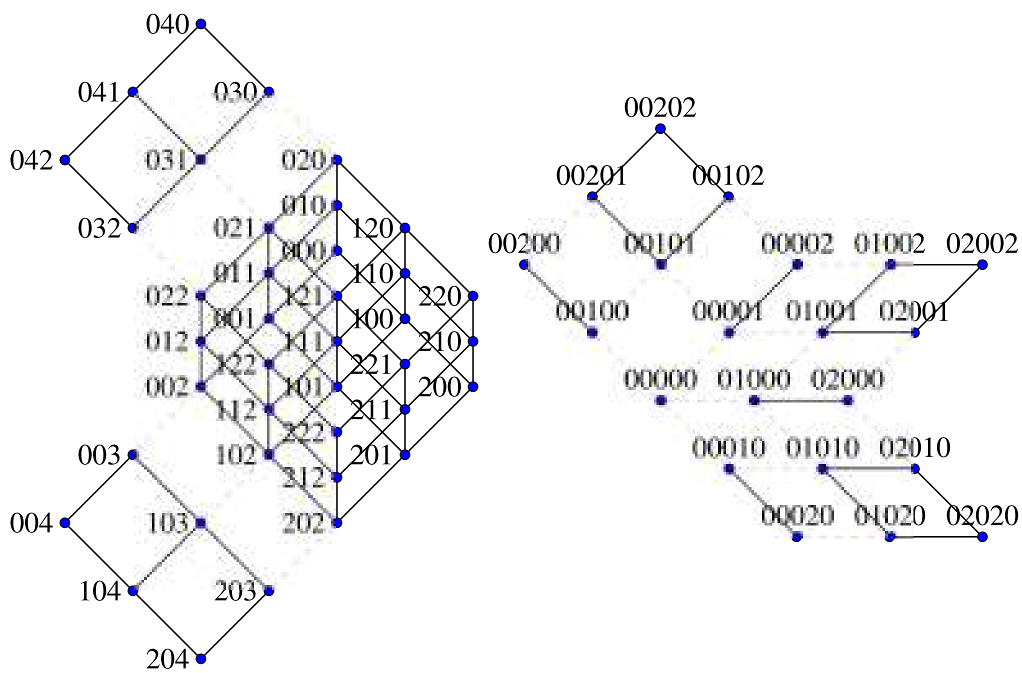

Figure 3.5: The decomposition of $\Pi_3^{3,2} = P_3^3 \oplus 2P_2 \Box P_3$ and Jacobsthal cube $\Pi_5^{1,2} = P_1^5 \oplus 4P_1^3 \Box P_2 \oplus 3P_2^2$.

3.3. NUMBER OF EDGES

The canonical decomposition from Theorem 3.2.1 is a useful tool to obtain the recursive relation for the number of edges. Let $e_n^{a,b}$ denote the number of edges in the Horadam cube $\Pi_n^{a,b}$. Since $\Pi_1^{a,b} = P_a$ and $\Pi_2^{a,b}$ is $a \times a$ grid with the path P_b appended to the vertex $0(a-1)$, we have $e_1^{a,b} = a-1$ and $e_2^{a,b} = 2a^2 - 2a + b$. Since the graph $\Pi_n^{a,b}$ contains a copies of the graph $\Pi_{n-1}^{a,b}$ and b copies of the graph $\Pi_{n-2}^{a,b}$, the contribution of those subgraphs to the number of edges is $ae_{n-1}^{a,b} + be_{n-2}^{a,b}$. Furthermore, there are $(a-1)s_{n-1}^{a,b}$ edges between the a copies of the graph $\Pi_{n-1}^{a,b}$ and $bs_{n-2}^{a,b}$ edges between the b copies of the graph $\Pi_{n-2}^{a,b}$. This analysis shows that the recursive relation for the number of edges of the metallic cubes naturally expands to the number of edges of the Horadam cubes. We have

$$e^{a,b} = ae_{n-1}^{a,b} + be_{n-2}^{a,b} + s_n^{a,b} - s_{n-1}^{a,b}. \tag{3.4}$$

Recurrence (3.4) immediately yields the generating function $E(x) = \sum_{n \geq 0} e_n^{a,b} x^n$ for the number of edges in Horadam cubes as

$$E(x) = \frac{(a-1)x + bx^2}{(1-ax-bx^2)^2}. \tag{3.5}$$

Using recurrence (3.4), we list some first values of the sequence $e_n^{a,b}$ in Table 3.1.

Table 3.1: The number of edges in the Horadam cubes.

n	e_n^a
1	$a-1$
2	$2a^2-2a+b$
3	$3a^3-3a^2+4ab-2b$
4	$4a^4-4a^3+9a^2b-6ab+2b^2$
5	$5a^5-5a^4+16a^3b-12a^2b+9ab^2-3b^2$
6	$6a^6-6a^5+25a^4b-20a^3b+24a^2b^2-12ab^2+3b^3$

Now we can state the theorem that expresses the numbers $e_n^{a,b}$ in terms of the number of vertices in the Horadam cubes.

Theorem 3.3.1. The number of edges in the Horadam cube $\Pi_n^{a,b}$ is

$$e_n^{a,b} = \sum_{k=0}^{n-1} s_k^{a,b}\left(s_{n-k}^{a,b} - s_{n-1-k}^{a,b}\right).$$

Proof. By using generating functions (3.3) and (3.5, and the fact that

$$S^2(x) = \sum_{n\geq 0}\sum_{k=0}^{n} s_k^{a,b} s_{n-k}^{a,b} x^n,$$

we have

$$\begin{aligned}
E(x) &= \frac{(a-1)x+bx^2}{(1-ax-bx^2)^2}\\
&= ((a-1)x+bx^2)S^2(x)\\
&= a\sum_{n\geq 0}\sum_{k=0}^{n} s_k^{a,b}s_{n-k}^{a,b}x^{n+1} + b\sum_{n\geq 0}\sum_{k=0}^{n} s_k^{a,b}s_{n-k}^{a,b}x^{n+2} - \sum_{n\geq 0}\sum_{k=0}^{n} s_k^{a,b}s_{n-k}^{a,b}x^{n+1}\\
&= a\sum_{n\geq 1}\sum_{k=0}^{n-1} s_k^{a,b}s_{n-1-k}^{a,b}x^{n} + b\sum_{n\geq 2}\sum_{k=0}^{n-2} s_k^{a,b}s_{n-2-k}^{a,b}x^{n} - \sum_{n\geq 1}\sum_{k=0}^{n-1} s_k^{a,b}s_{n-1-k}^{a,b}x^{n}\\
&= \sum_{n\geq 0}\sum_{k=0}^{n-1} s_k^{a,b}\left(as_{n-1-k}^{a,b} + bs_{n-2-k}^{a,b} - s_{n-1-k}^{a,b}\right)x^n\\
&= \sum_{n\geq 0}\sum_{k=0}^{n-1} s_k^{a,b}\left(s_{n-k}^{a,b} - s_{n-1-k}^{a,b}\right)x^n.
\end{aligned}$$

∎

For $a = b = 1$ we have $s_n^{1,1} = F_{n+1}$ and

$$\begin{aligned}
e_n^{1,1} &= \sum_{k=0}^{n-1} s_k^{1,1}\left(s_{n-k}^{1,1} - s_{n-1-k}^{1,1}\right)\\
&= \sum_{k=0}^{n-1} s_k^{1,1}s_{n-k-2}^{1,1}\\
&= \sum_{k=1}^{n} s_{k-1}^{1,1}s_{n-k-1}^{1,1}\\
&= \sum_{k=0}^{n} F_kF_{n-k},
\end{aligned}$$

which immediately retrieves the result on the number of edges in the Fibonacci cubes [30].

Theorem 3.3.1 allows us to extend this result for $a = 1$ and $b \geq 1$ and we obtain

$$e_n^{1,b} = b\sum_{k=0}^{n-1} s_k^{1,b}s_{n-k-2}^{1,b}.$$

In particular, for the Jacobsthal cubes, since $s_n^{1,2} = J_{n+1}$, we obtained the sequence A095977:

$$e_n^{1,2} = 2\sum_{k=0}^{n} J_k J_{n-k}.$$

This provides a new combinatorial interpretation for the sequence A095977.

Another way to express the number of edges is by using binomial coefficients. We do that in the next theorem.

Theorem 3.3.2. The number of edges in the Horadam cube $\Pi_n^{a,b}$ is

$$e_n^{a,b} = \sum_{k=0}^{n} (-1)^{n-k} \left\lceil \frac{n+k}{2} \right\rceil \binom{\left\lfloor \frac{n+k}{2} \right\rfloor}{k} a^k b^{\left\lfloor \frac{n-k}{2} \right\rfloor}.$$

Proof. It is easy to verify that the statement holds for $n = 1$ and $n = 2$. We proceed by induction. Since the number of vertices $s_n^{a,b}$ satisfies the identity (3.2), we have

$$\begin{aligned}
s_n^{a,b} - s_{n-1}^{a,b} &= \sum_{k\geq 0} \binom{n-k}{k} a^{n-2k} b^k - \sum_{k\geq 0} \binom{n-k-1}{k} a^{n-2k-1} b^k \\
&= \sum_{k=0}^{n} (-1)^k \binom{n-\left\lceil \frac{k}{2} \right\rceil}{\left\lfloor \frac{k}{2} \right\rfloor} a^{n-k} b^{\left\lfloor \frac{k}{2} \right\rfloor} \\
&= \sum_{k=0}^{n} (-1)^k \binom{n-\left\lceil \frac{k}{2} \right\rceil}{n-k} a^{n-k} b^{\left\lfloor \frac{k}{2} \right\rfloor} \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{n-\left\lceil \frac{n-k}{2} \right\rceil}{k} a^{k} b^{\left\lfloor \frac{n-k}{2} \right\rfloor} \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{\left\lfloor \frac{n+k}{2} \right\rfloor}{k} a^{k} b^{\left\lfloor \frac{n-k}{2} \right\rfloor}.
\end{aligned}$$

By using the inductive hypothesis, after adjusting indices and expanding the range of summation, we obtain

$$\begin{aligned}
a\cdot e_{n-1}^{a,b} + b\cdot e_{n-2}^{a,b} &= \sum_{k=0}^{n-1} (-1)^{n-k-1} \left\lceil \frac{n+k-1}{2} \right\rceil \binom{\left\lfloor \frac{n+k-1}{2} \right\rfloor}{k} a^{k+1} b^{\left\lfloor \frac{n-k-1}{2} \right\rfloor} + \\
&\quad + \sum_{k=0}^{n-2} (-1)^{n-k} \left\lceil \frac{n+k-2}{2} \right\rceil \binom{\left\lfloor \frac{n+k-2}{2} \right\rfloor}{k} a^{k} b^{\left\lfloor \frac{n-k}{2} \right\rfloor} \\
&= \sum_{k=1}^{n} (-1)^{n-k} \left\lceil \frac{n+k-2}{2} \right\rceil \binom{\left\lfloor \frac{n+k-2}{2} \right\rfloor}{k-1} a^{k} b^{\left\lfloor \frac{n-k}{2} \right\rfloor} + \\
&\quad + \sum_{k=0}^{n} (-1)^{n-k} \left\lceil \frac{n+k-2}{2} \right\rceil \binom{\left\lfloor \frac{n+k-2}{2} \right\rfloor}{k} a^{k} b^{\left\lfloor \frac{n-k}{2} \right\rfloor}
\end{aligned}$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \left\lceil \frac{n+k-2}{2} \right\rceil \binom{\lfloor \frac{n+k}{2} \rfloor}{k} a^k b^{\lfloor \frac{n-k}{2} \rfloor}.$$

Now, by using expressions for $s_n^a - s_{n-1}^a$ and $a \cdot e_{n-1}^a + b \cdot e_{n-2}^a$ our claim follows at once.

■

Note that setting $a = 2$ in the generating function (3.3) yields $S'(x) = \dfrac{2+2bx}{(1-2x-bx^2)^2}$ and

$$E(x) = \frac{1}{2} x S'(x).$$

Hence, we can expand the result obtained by Munarini for all $b \geq 1$, to obtain

$$e_n^{2,b} = \frac{n}{2} s_n^{2,b}.$$

Here, it is worth mentioning that, although the Pell graphs and Horadam cubes $\Pi_n^{2,1}$ are not isomorphic, they share the same number of vertices and edges. In particular, we have

$$\sum_{k=0}^{n} (-1)^{n-k} \left\lceil \frac{n+k}{2} \right\rceil \binom{\lfloor \frac{n+k}{2} \rfloor}{k} 2^k = \frac{nP_{n+1}}{2}.$$

3.4. Embedding into hypercubes and median graphs

In this section, we justify the "cube" part of the name by showing that Horadam cubes are, indeed, induced subgraphs of hypercubes.

Theorem 3.4.1. For any $a, b \geq 1$, the Horadam cube $\Pi_n^{a,b}$ is an induced subgraph of the hypercube $Q_{(a+b-1)n}$.

Proof. We define a map $\sigma : \Pi_n^{a,b} \to Q_{(a+b-1)n}$ on the primitive blocks, i.e., the blocks every string from the set $\mathscr{S}_n^{a,b}$ can be uniquely decomposed into. The map ρ transfers the vertices of the Horadam cube into binary strings. Those blocks are $0, 1, \ldots, a-1, 0a, 0(a+1), \ldots, 0(a+b-2)$ and $0(a+b-1)$. We define σ as follows:

$$\sigma(0) = \underbrace{011\cdots111}_{a}\underbrace{000\cdots000}_{b-1}$$

$$\sigma(1) = \underbrace{001\cdots111}_{a}\underbrace{000\cdots000}_{b-1}$$

$$\sigma(2) = \underbrace{000\cdots111}_{a}\underbrace{000\cdots000}_{b-1}$$

$$\vdots$$

$$\sigma(a-2) = \underbrace{000\cdots001}_{a}\underbrace{000\cdots000}_{b-1}$$

$$\sigma(a-1) = \underbrace{000\cdots000}_{a}\underbrace{000\cdots000}_{b-1}$$

$$\sigma(0a) = \underbrace{111\cdots111}_{a}\underbrace{000\cdots000}_{b-1}\underbrace{000\cdots000}_{a}\underbrace{000\cdots000}_{b-1}$$

$$\sigma(0(a+1)) = \underbrace{111\cdots111}_{a}\underbrace{000\cdots000}_{b-1}\underbrace{000\cdots000}_{a}\underbrace{100\cdots000}_{b-1}$$

$$\sigma(0(a+2)) = \underbrace{111\cdots111}_{a}\underbrace{000\cdots000}_{b-1}\underbrace{000\cdots000}_{a}\underbrace{110\cdots000}_{b-1}$$

$$\vdots$$

$$\sigma(0(a+b-1)) = \underbrace{111\cdots111}_{a}\underbrace{000\cdots000}_{b-1}\underbrace{000\cdots000}_{a}\underbrace{111\cdots111}_{b-1}$$

The string $\sigma(k)$ for $0 \leq k \leq a-1$ has length a and starts with $k+1$ zeros followed by $a-1-k$ ones. The remaining $b-1$ letters are zeroes. On the other hand, the string $\sigma(0(a+l))$ for $0 \leq l \leq b-1$ has length $2(a+b-1)$ and contains a ones followed by $a+b-1$ zeroes, l ones and $b-1-l$ zeroes. The map σ, defined on the primitive blocks, extends by concatenation to $\sigma : \Pi_n^{a,b} \to Q_{(a+b-1)n}$.

Since the map σ is injective and preserves adjacency, we obtained a subgraph of $Q_{(a+b-1)n}$ isomorphic to the graph $\Pi_n^{a,b}$. ■

Remark 3.4.2. For $a=3$ and $b=2$, we have $\sigma(0)=0110$, $\sigma(1)=0010$, $\sigma(2)=0000$, $\sigma(03)=11100000$ and $\sigma(04)=1110001$. For the Jacobsthal cube, where $a=1$ and $b=2$, we have $\sigma(0)=00$, $\sigma(01)=1000$ and $\sigma(02)=1001$.

Remark 3.4.3. The dimension of the hypercube from Theorem 3.4.1 is not the smallest possible. For example, the number of vertices in Jacobsthal cubes grows slower than the number of vertices in hypercubes. More precisely, $|V(\Pi_n^{1,2})| = \frac{2}{3}2^n + \frac{1}{3}(-1)^n < 2^n = |V(Q_n)|$. Thus, we can define $\sigma(0)=0$, $\sigma(01)=10$, and $\sigma(02)=11$ to obtain an embedding of the Jacobsthal cube into a hypercube of the same dimension. But this embedding does not yield an induced and median-closed subgraph of the hypercube Q_n, so, in Theorem 3.4.1, we constructed an embedding into a larger hypercube to obtain an induced and median-closed subgraph. As an example, Figure 3.6 shows an embedding of the Jacobsthal cube $\Pi_4^{1,2}$ into the hypercube Q_4. But vertices 0001 and 0020, not adjacent in $\Pi_4^{1,2}$, are mapped into adjacent vertices in Q_4. The same situation occurs with the vertices 0200 and 0010. Thus, although we obtained an embedding, we did not obtain an induced subgraph of Q_4. Maybe it would be interesting to determine the smallest dimension of the hypercube for which the embedding of a Horadam cube is possible.

Recall that a *median* of three vertices is a vertex that lies on the shortest path between every two of three vertices. We say that a graph G is a *median graph* if every three vertices of G have a unique median. Since the Fibonacci cubes, Pell graphs, and metallic cubes are median graphs [12, 30, 40], the next natural step is to check whether the Horadam cubes are also median graphs. We found the answer positive, and we present that in the next theorem. To do that, we use Mulder's theorem 2.2.6.

Theorem 3.4.4. For any $a,b \geq 1$ and $n \geq 1$, the Horadam cube $\Pi_n^{a,b}$ is a median graph.

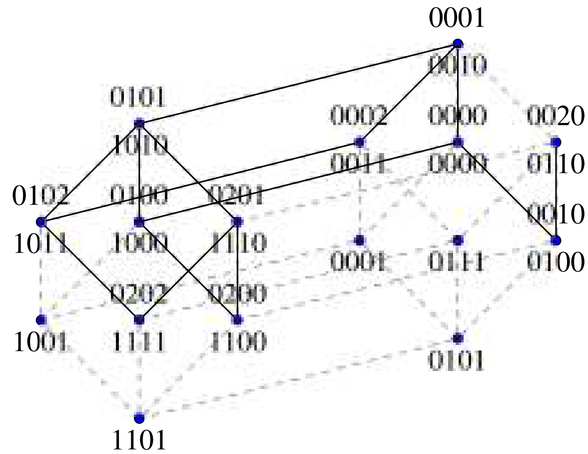

Figure 3.6: Embedding of the Jacobsthal cube $\Pi_4^{1,2}$ into the hypercube Q_4.

Proof. The median of the triple in the hypercube Q_n is obtained by the majority rule. If $\alpha = \varepsilon_1 \cdots \varepsilon_n$, $\beta = \delta_1 \cdots \delta_n$ and $\gamma = \rho_1 \cdots \rho_n$ are binary strings, i.e., $\varepsilon_i, \delta_i, \rho_i \in \{0,1\}$, then their median is $m = \zeta_i \cdots \zeta_n$, where ζ_i is equal to the number that appears at least twice among the numbers ε_i, δ_i and ρ_i.

Let σ be the map defined as in the proof of Theorem 3.4.1. The graph $\Pi_n^{a,b}$ is a connected induced subgraph of a hypercube. To finish our proof, we just need to verify that the subgraph induced by the set $\sigma\left(\mathscr{S}_n^{a,b}\right)$ is median closed. Every string from the set $\sigma\left(\mathscr{S}_n^{a,b}\right)$ can be uniquely decomposed into blocks $\sigma(k), 0 \leq k \leq a-1$ of length $a+b-1$, and $\sigma(0(a+l)), 0 \leq l \leq b-1$ of length $2(a+b-1)$. First we consider three blocks, $\sigma(i)$, $\sigma(j)$, and $\sigma(k)$. Without loss of generality, we can assume that $0 \leq i \leq j \leq k \leq a-1$. Then their median is $\sigma(j)$. In the second case, we have $\sigma(i)$ and $\sigma(j)$ for $0 \leq i \leq j \leq a-1$, and either the first or second half of the block $\sigma(0(a+l))$. Their median is now $\sigma(i)$. The remaining cases include all combinations of blocks; since they are done similarly, we omit the details. ■

For example, when $a=3$ and $b=2$, we have $\sigma(0)=0110$, $\sigma(1)=0010$, $\sigma(2)=0000$, $\sigma(03)=11100000$, and $\sigma(04)=11100001$. Then the median of vertices 042, 204 and 110 is 112.

3.5. DISTRIBUTION OF DEGREES

The distribution of degrees in Fibonacci and Lucas cubes was determined by Klavžar et al. [33]. The distribution of degrees in the metallic cubes was determined in Section 2.3 and also in [12]. Here we generalize those results to all Horadam cubes. More precisely, in this section, we investigate the distribution of degrees by determining the recurrence relations and by computing the bivariate generating functions for two-indexed sequences $\Delta_{n,k}$ which count the number of vertices in $\Pi_n^{a,b}$ of degree k. We suppress a and b to simplify the notation. The distribution of degrees in Horadam cubes is considered in two separate cases, for $a = 1$ and $a \geq 2$. The two following theorems provide recursive relations for $\Delta_{n,k}$.

Theorem 3.5.1. Let $a \geq 2$ and $\Delta_{n,k}$ denote the number of vertices in the Horadam cube $\Pi_n^{a,b}$ with exactly k neighbors. Then the sequence $\Delta_{n,k}$ satisfies the recurrence

$$\Delta_{n,k} = 2\Delta_{n-1,k-1} + (a-2)\Delta_{n-1,k-2} + \Delta_{n-2,k-1} + (b-2)\Delta_{n-2,k-2} + \Delta_{n-2,k-3},$$

with initial values $\Delta_{0,0} = 1$, $\Delta_{1,1} = 2$ and $\Delta_{1,2} = a-2$.

Proof. Since the graph $\Pi_0^{a,b}$ is an empty graph, and $\Pi_1^{a,b}$ is a path graph P_a, one can easily verify the initial values. Let $n \geq 2$. By Theorem 3.2.1, the Horadam cube $\Pi_n^{a,b}$ contains a copies of the graph $\Pi_{n-1}^{a,b}$. Recall that the copy of the subgraph $\Pi_{n-1}^{a,b}$ induced by the vertices in $\Pi_n^{a,b}$ starting with the letter k for $0 \leq k \leq a-1$, is denoted by $k\Pi_{n-1}^{a,b}$ and the copy of the subgraph $\Pi_{n-2}^{a,b}$ induced by vertices in $\Pi_n^{a,b}$ starting with the block $0l$ for $a \leq l \leq a+b-1$, is denoted by $0l\,\Pi_{n-2}^{a,b}$. Each vertex in subgraphs $k\Pi_{n-1}^{a,b}$, where $1 \leq k \leq a-2$, has exactly two new neighbors, namely the corresponding vertices in adjacent subgraphs, $(k-1)\Pi_{n-1}^{a,b}$ and $(k+1)\Pi_{n-1}^{a,b}$. Thus the degree of each vertex in those $a-2$ subgraphs is increased by two and they contribute to $\Delta_{n,k}$ with $(a-2)\Delta_{n-1,k-2}$. Furthermore, the vertices of the copy of $\Pi_{n-1}^{a,b}$ induced by vertices starting with $a-1$, denoted by $(a-1)\Pi_{n-1}^{a,b}$, have exactly one new neighbor, the corresponding vertex in the graph $(a-2)\Pi_{n-1}^{a,b}$. So, the contribution of this subgraph is $\Delta_{n-1,k-1}$. Similar analysis for b copies of the graph $\Pi_{n-2}^{a,b}$ yields the contribution of $\Delta_{n-2,k-1} + (b-1)\Delta_{n-2,k-2}$. The subgraph $0\Pi_{n-1}^{a,b}$ needs to be treated differently, because it is the only copy neighboring two copies of different dimensions. Namely, it neighbors the copy $1\Pi_{n-1}^{a,b}$ and the copy $0a\Pi_{n-2}^{a,b}$. This implies that some vertices in $0\Pi_{n-1}^{a,b}$ have two new neighbors and others have only one. More precisely, subgraph $0(a-1)\Pi_{n-2}^{a,b}$ of the subgraph $0\Pi_{n-1}^{a,b}$ contains vertices with two new neighbors. The remaining vertices in $0\Pi_{n-1}^{a,b}$ have one new neighbor. Hence, the contribution of the subgraph $0\Pi_n^{a,b}$ to $\Delta_{n,k}$ is $\Delta_{n-1,k-1} - \Delta_{n-2,k-2} + \Delta_{n-2,k-3}$, and the claim follows. ■

Figure 3.7 shows the recursive nature of the degrees of vertices in the Horadam cube $\Pi_n^{3,2}$ for $n = 1,2,3$.

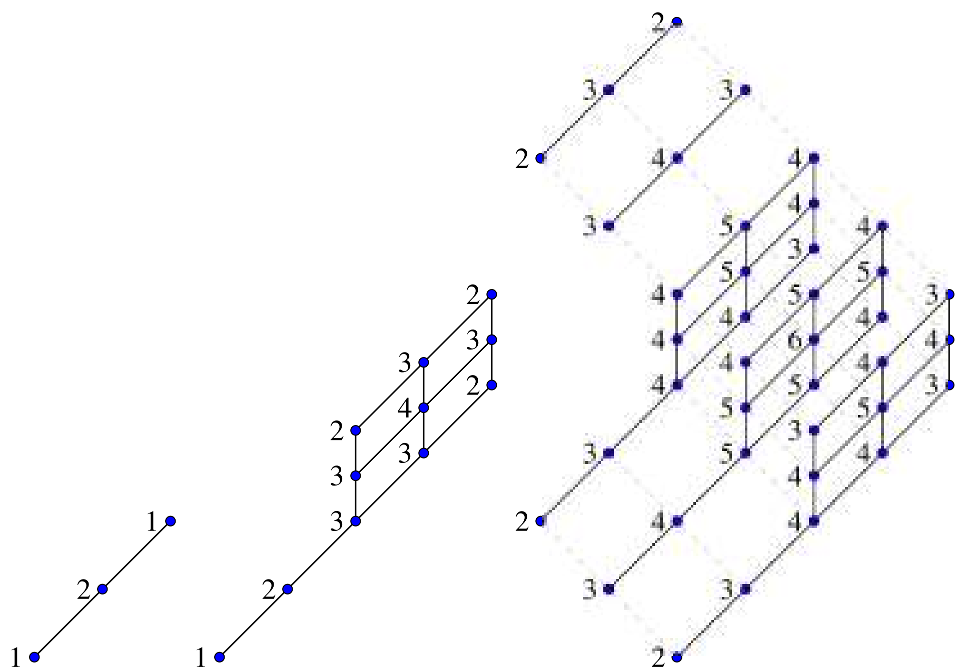

Figure 3.7: Recursive nature of the degrees in Horadam cube $\Pi_n^{3,2}$ for $n = 1,2,3$.

Theorem 3.5.2. For $a = 1$, the sequence $\Delta_{n,k}$ satisfies recursive relation

$$\begin{aligned}\Delta_{n,k} = {}& \Delta_{n-1,k-1} + \Delta_{n-2,k-1} + (b-1)\Delta_{n-2,k-2} \\ & + \Delta_{n-3,k-1} + (b-2)\Delta_{n-3,k-2} - (b-1)\Delta_{n-3,k-3},\end{aligned}$$

with initial conditions $\Delta_{0,0} = 1$, $\Delta_{1,0} = 1$, $\Delta_{2,1} = 2$ and $\Delta_{2,2} = b-2$.

Proof. Since $\Pi_0^{1,b}$ and $\Pi_1^{1,b}$ are empty graphs and $\Pi_2^{1,b}$ is a path graph with $b+1$ vertices, the initial values hold. Let $n \geq 3$. By Theorem 3.2.1, the graph $\Pi_n^{1,b}$ contains one copy of the graph $\Pi^{1,b}_{n-1}$ and b copies of the graph $\Pi^{1,b}_{n-2}$. Furthermore, by the same theorem, $\Pi^{1,b}_{n-1}$ can be further decomposed into one copy of $\Pi^{1,b}_{n-2}$ and b copies of $\Pi^{1,b}_{n-3}$. Some vertices in the subgraph $\Pi^{1,b}_{n-1} = \Pi^{1,b}_{n-2} \oplus P_b \Box \Pi^{1,b}_{n-3}$ have one new neighbor in $\Pi^{1,b}_{n}$, while others have no new neighbors. More precisely, the vertices in a subgraph $00l\,\Pi^{1,b}_{n-3} \subset \Pi^{1,b}_{n-1}$, where $1 \leq l \leq b$, have no new neighbors, while the vertices $00\Pi^{1,b}_{n-2} \subset \Pi^{1,b}_{n-1}$ have one new neighbor. Vertices in subgraph $00b\Pi^{1,b}_{n-3} \subset \Pi^{1,b}_{n}$ have a degree for one greater than the corresponding vertices in the graph $\Pi^{1,b}_{n-3}$. Similarly, vertices in subgraph $00l\,\Pi^{1,b}_{n-3} \subset \Pi^{1,b}_{n}$, for $1 \leq l \leq b-1$, have a degree for two greater than the corresponding vertices in the graph $\Pi^{1,b}_{n-3}$. So, the contribution to $\Delta_{n,k}$ is $\Delta_{n-1,k-1} - \Delta_{n-3,k-2} + \Delta_{n-3,k-1} + (b-1)(\Delta_{n-3,k-2} - \Delta_{n-3,k-3})$. What remains are b copies of $\Pi^{1,b}_{n-2}$. Vertices in $0l\,\Pi^{1,b}_{n-2}$ contribute with $(b-1)\Delta_{n-2,k-2}$ for $1 \leq l \leq b-1$, and with $\Delta_{n-2,k-1}$ for $l = b$. This completes the proof. ■

Figure 3.8 shows the recursive nature of the degrees of vertices in the Jacobsthal cube for $n = 2,3,4,5$.

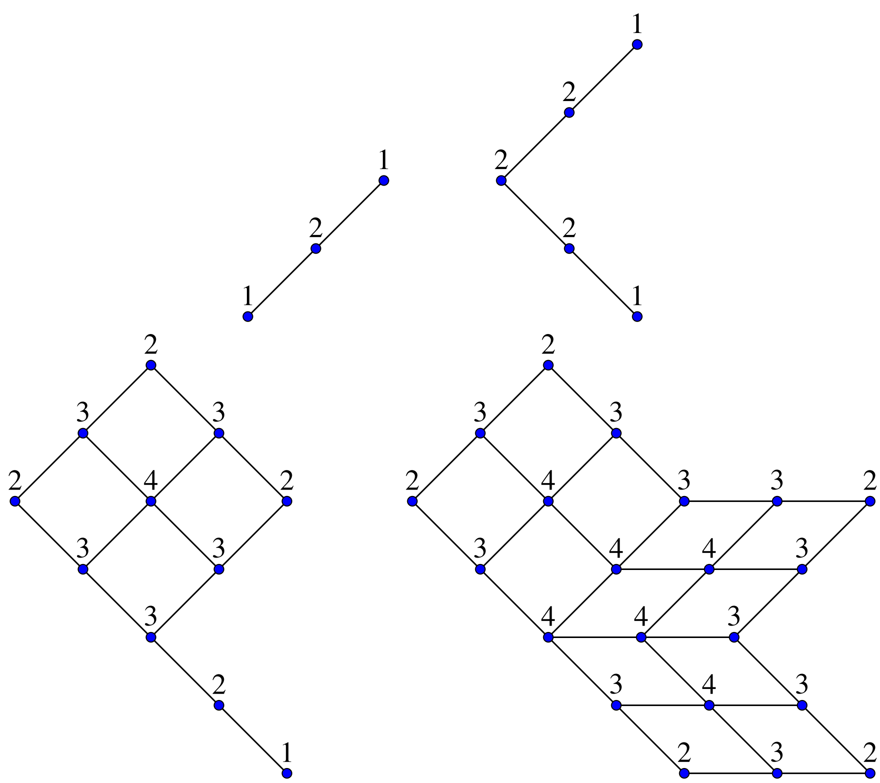

Figure 3.8: Recursive nature of the degrees in Jacobsthal cube for $n = 2,3,4,5$.

From Theorems 3.5.1 and 3.5.2, we can readily obtain bivariate generating functions.

The generating function for the sequence $\Delta_{n,k}$ when $a \geq 2$ is

$$\Delta(x,y) = \frac{1}{1-(2y+(a-2)y^2)x-(y+(b-2)y^2+y^3)x^2}, \tag{3.6}$$

and for $a = 1$ we have

$$\Delta(x,y) = \frac{1-(y-1)x}{1-xy-(y+(b-1)y^2)x^2-(y+(b-2)y^2-(b-1)y^3)x^3}. \tag{3.7}$$

Now we present another way to derive the generating functions. Recall that $\mathscr{S}^{a,b}$ denotes the set of all words from the alphabet $\{0,1,\ldots,a+b-1\}$ with the property that the letters greater than or equal to a can only appear immediately after 0. Every word either starts with $0,\ldots,a-1$, or with the block $0(a+l)$ for some $0 \leq l \leq b-1$. Hence,

$$\mathscr{S}^a = \varepsilon + 0\mathscr{S}^{a,b} + 1\mathscr{S}^{a,b} + \cdots + (a-1)\mathscr{S}^{a,b} + 0a\mathscr{S}^{a,b} + \cdots + 0(a+b-1)\mathscr{S}^{a,b},$$

where ε denotes the empty word. Let $\Delta(x,y) = \sum_{n,k} \Delta_{n,k} x^n y^k$ be a formal power series, where $\Delta_{n,k}$ is the number of vertices of length n having k neighbors, as before. Consider the set that contains all vertices that start with 0, but not with the $0l$-block for $a \leq l \leq a+b-1$, and let $\Delta^0(x,y)$ denote the corresponding formal power series. Then we have

$$\begin{aligned}\Delta(x,y) &= 1 + \Delta^0(x,y) + \left((a-2)xy^2 + xy + (b-1)x^2y^2 + x^2y\right)\Delta(x,y)\\ \Delta^0(x,y) &= xy\left(1 + \Delta^0(x,y) + \left((a-1)xy^2 + (b-1)x^2y^2 + x^2y\right)\Delta(x,y)\right).\end{aligned}$$

Note that adding 0 at the beginning of a vertex increases the number of neighbors by one, except when the vertex starts with $a-1$, in which case adding 0 increases the number of neighbors by two. By solving the above system of equations, we obtain the generating function (3.6).

In the case of $a = 1$, adding 0 at the beginning of a vertex increases the number of neighbors only if the vertex does not start with a $0l$ block. So we have

$$\begin{aligned}\Delta(x,y) &= 1 + \Delta^0(x,y) + \left((b-1)x^2y^2 + x^2y\right)\Delta(x,y)\\ \Delta^0(x,y) &= x\left(1 + y\Delta^0(x,y) + \left((b-1)x^2y^2 + x^2y\right)\Delta(x,y)\right),\end{aligned}$$

which yields the generating function (3.7). We list some first few values of $\Delta_{n,k}$ in Table 3.2.

In Section 2.3, we showed that their rows are not unimodal for $a = 2$ and $b = 1$, but for any other values of a and b they appear to be unimodal. They do not (yet) appear in the *On-Line Encyclopedia of Integer Sequences* [46].

Table 3.2: Some first values of $\Delta_{n,k}$ for $\Pi_n^{1,2}$, $\Pi_n^{2,2}$ and $\Pi_n^{3,2}$.

$a=1, b=2$

$n\backslash k$	1	2	3	4	5
1	1	0	0	0	0
2	2	1	0	0	0
3	2	3	0	0	0
4	1	4	5	1	0
5	0	5	10	6	0

$a=2, b=2$

$n\backslash k$	1	2	3	4	5	6	7	8
1	2	0	0	0	0	0	0	0
2	1	4	1	0	0	0	0	0
3	0	4	8	4	0	0	0	0
4	0	1	12	18	12	1	0	0
5	0	0	6	32	44	32	6	0

$a=3, b=2$

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	2	1	0	0	0	0	0	0	0	0
2	1	4	5	1	0	0	0	0	0	0
3	0	4	10	16	8	5	1	0	0	0
4	0	1	12	30	47	37	11	1	0	0
5	0	0	6	35	92	142	138	67	14	1

3.6. SUBHYPERCUBES

In this section, any subgraph isomorphic to the hypercube is called *subhypercube*. The *cube coefficient* of the graph G, denoted by $\alpha_k(G)$, is the number of induced subhypercubes Q_k in G. The *cube number* $\alpha(G)$ is total number of induced subhypercubes in G, i.e., $\alpha(G) = \sum_{k\geq 0} \alpha_k(G)$. Observe that $a_0(G) = |V(G)|$ and $a_1(G) = |E(G)|$. Here we determine the cube coefficients α_k for the Horadam cube $\Pi_n^{a,b}$.

Theorem 3.6.1. Cube coefficients $\alpha_k(\Pi_n^{a,b})$ for the Horadam cube $\Pi_n^{a,b}$ satisfy recursive relation

$$\alpha_k(\Pi_n^{a,b}) = a\alpha_k(\Pi_{n-1}^{a,b}) + b\alpha_k(\Pi_{n-2}^{a,b}) + (a-1)\alpha_{k-1}(\Pi_{n-1}^{a,b}) + b\alpha_{k-1}(\Pi_{n-2}^{a,b})$$

with initial values $\alpha_0(\Pi_n^{a,b}) = s_n^{a,b}$ and $\alpha_1(\Pi_n^{a,b}) = a-1$. Furthermore, if

$$A(x,y) = \sum_{n,k\geq 0} \alpha_k(\Pi_n^{a,b})x^n y^k$$

denotes their bivariate generating function, we have

$$A(x,y) = \frac{1}{1-ax-bx^2-(a-1)xy-bx^2y}.$$

Proof. By Theorem 3.2.1, the Horadam cube $\Pi_n^{a,b}$ contains a copies of the graph $\Pi_{n-1}^{a,b}$ and b copies of the graph $\Pi_{n-2}^{a,b}$. Then all induced subhypercubes in those copies contribute to the number of hypercubes of dimension k in $\Pi_n^{a,b}$ with $a\alpha_k(\Pi_{n-1}^{a,b}) + b\alpha_k(\Pi_{n-2}^{a,b})$. Moreover, the hypercube $Q_k = P_2 \Box Q_{k-1}$ can be part of the two adjacent copies of $\Pi_{n-1}^{a,b}$. Let $Q_{k-1} \subseteq \Pi_{n-1}^{a,b}$ be induced subhypercube. Then, for $0 \leq m \leq a-2$, appending letters m and $m+1$ to the vertices of $Q_{k-1} \subseteq \Pi_{n-1}^{a,b}$ yields a hypercube Q_k in $\Pi_n^{a,b}$. Similar construction can be done for $Q_{k-1} \subseteq \Pi_{n-2}^{a,b}$. Hypercubes constructed as described contribute with $(a-1)\alpha_{k-1}(\Pi_{n-1}^{a,b}) + b\alpha_{k-1}(\Pi_{n-2}^{a,b})$. From the recursive relation, one can easily obtain the generating function. The claim follows. ■

Disregarding the dimension of the subhypercube, Theorem 3.6.1 gives a simple consequence.

Corollary 3.6.2. Cube numbers $\alpha(\Pi_n^{a,b})$ for the Horadam cube $\Pi_n^{a,b}$ satisfy recursive relation

$$\alpha(\Pi_n^{a,b}) = (2a-1)\alpha(\Pi_{n-1}^{a,b}) + 2b\alpha(\Pi_{n-2}^{a,b})$$

with initial conditions $\alpha(\Pi_0^{a,b}) = 1$ and $\alpha(\Pi_1^{a,b}) = 2a-1$. If

$$A(x) = \sum_{n\geq 0} \alpha(\Pi_n^{a,b})x^n$$

denote the generating function, we have

$$A(x) = \frac{1}{1-(2a-1)x-2bx^2}.$$

The cube polynomial was first introduced by Brešar, Klavžar, and Škrekovski [4]. For a graph G we define the *cube polynomial* as

$$c(G,x) = \sum_{k\geq 0} \alpha_k(G)x^k.$$

The next theorem provides cube polynomials for the Horadam cubes. It follows from a similar argument as the proof of the Theorem 3.6.1, so we omit the proof.

Theorem 3.6.3. The cube polynomials $c(\Pi_n^{a,b},x)$ satisfy the recursive relation

$$c(\Pi_n^{a,b},x) = (a+(a-1)x)c(\Pi_{n-1}^{a,b},x) + (b+bx)c(\Pi_{n-2}^{a,b},x)$$

with initial conditions $c(\Pi_0^{a,b},x) = 1$ and $c(\Pi_1^{a,b},x) = a+(a-1)x$.

Table 3.3 shows some first values of the cube polynomials for the Jacobsthal cube and $\Pi_n^{3,2}$.

Table 3.3: Cube polynomials $c(\Pi_n^{a,b},x)$ of some Horadam cubes.

n	$c(\Pi_n^{1,2},x)$	$c(\Pi_n^{3,2},x)$
0	1	1
1	1	$2x+3$
2	$2x+3$	$4x^2+14x+11$
3	$4x+5$	$8x^3+44x^2+74x+39$
4	$4x^2+14x+11$	$16x^4+120x^3+316x^2+350x+139$
5	$12x^2+32x+21$	$32x^5+304x^4+1096x^3+1884x^2+1554x+495$

3.7. HORADAM CUBES ARE (MOSTLY) HAMILTONIAN

At the beginning of this section, it is convenient to recall that any path that visits all vertices in the graph is called a *Hamiltonian path*, while a cycle with the same property is called a *Hamiltonian cycle*. A graph that contains a Hamiltonian cycle is *Hamiltonian*. In this section, we explore the possibility of extending the results on the existence of the Hamiltonian paths and Hamiltonian cycles in metallic cubes to the Horadam cubes. The answer to the question of the existence of Hamiltonian paths turns out to be positive. In fact, since we obtained the same result as for the Fibonacci cubes [36], the title of this section is borrowed from that paper. The following theorem extends the established result regarding the existence of Hamiltonian paths in the metallic cubes to encompass all Horadam cubes.

Theorem 3.7.1. Let $a,b,n \geq 1$. Then the Horadam cube $\Pi_n^{a,b}$ contains a Hamiltonian path.

Proof. Since, by Theorem 3.2.1, Horadam cube admits canonical decomposition, i.e., $\Pi_n^{a,b} = P_a \square \Pi_{n-1}^{a,b} \oplus P_b \square \Pi_{n-2}^{a,b}$, it is only natural to construct a Hamiltonian path inductively. First, consider the case of a odd and b even. We claim that $\Pi_n^{a,b}$ contains Hamiltonian path with endpoints $0(a+b-1)0(a+b-1)\cdots 0(a+b-1)$ and $(a-1)0(a+b-1)0(a+b-1)\cdots 0(a+b-1)$ for n even, and $0(a+b-1)0(a+b-1)\cdots 0(a+b-1)0$ and $(a-1)0(a+b-1)0(a+b-1)\cdots 0(a+b-1)0$ for n odd. Similar to the case of metallic cubes, for $n=1$, graph $\Pi_1^{a,b}$ is a path on a vertices and the claim is true. For $n=2$, graph $\Pi_2^{a,b}$ is an $a \times a$ grid with the path that contains b vertices appended to the vertex $0(a-1)$. Hence, we can construct a Hamiltonian path from the vertex $0(a+b-1)$ to the vertex $(a-1)0$. Suppose that $\Pi_k^{a,b}$ contains a Hamiltonian path with endpoints $0(a+b-1)0(a+b-1)\cdots 0(a+b-1)(0)$ and $(a-1)0(a+b-1)(a-1)0(a+b-1)\cdots(a-1)0(a+b-1)$ for all $k<n$. Then we can construct a Hamiltonian path in $P_a \square \Pi_{n-1}^{a,b} \subseteq \Pi_n^{a,b}$ by adding the edges between corresponding endpoints of the Hamiltonian paths in the neighboring graphs $\Pi_{n-1}^{a,b}$. Since a is odd, the path in $P_a \square \Pi_{n-1}^{a,b} \subseteq \Pi_n^{a,b}$ has endpoints $(a-1)0(a+b-1)0(a+b-1)\cdots 0(a+b-1)(0)$ and

$0(a-1)0(a+b-1)0(a+b-1)\cdots0(a+b-1)(0)$. In a similar way we construct a path in $P_b\square\Pi_{n-2}^{a,b}\subseteq\Pi_n^{a,b}$ with endpoints $0a0(a+b-1)0(a+b-1)\cdots0(a+b-1)(0)$ and $0(a+b-1)0(a+b-1)\cdots0(a+b-1)(0)$. Since the vertices $0(a-1)0(a+b-1)0(a+b-1)\cdots0(a+b-1)(0)$ and $0a0(a+b-1)0(a+b-1)\cdots0(a+b-1)(0)$ are adjacent in $\Pi_n^{a,b}$, we add the edge connecting them to obtain a Hamiltonian path in $\Pi_n^{a,b}$. In the case of a odd and b odd, $\Pi_n^{a,b}$ contains Hamiltonian path with endpoints $0(a+b-1)(a-1)0(a+b-1)(a-1)\cdots0(a+b-1)(a-1)$ and $(a-1)0(a+b-1)\cdots(a-1)0(a+b-1)$. For a and b even endpoints are $(a-1)(a-1)\cdots(a-1)$ and $0(a+b-1)(a-1)\cdots(a-1)$. In the last case, where a is even a b is odd we have endpoints $(a-1)(a-1)\cdots(a-1)$ and $0(a+b-1)\cdots0(a+b-1)$. Since all those cases are similar, we omit the details. ■

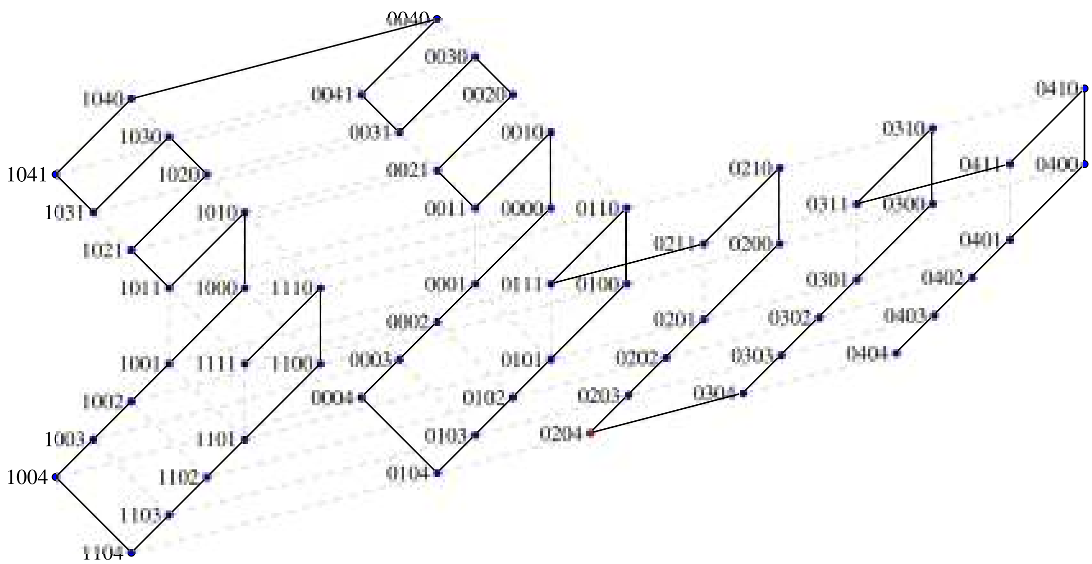

Figure 3.9: The Hamiltonian path in the Horadam cube $\Pi_4^{2,3}$.

Metallic cubes are Hamiltonian when a is even and n is odd. The next few theorems provide the necessary conditions for the Horadam cube to be Hamiltonian. In particular, the next theorem improves the results for the metallic cubes, because for some cases of the Horadam cubes, there is a Hamiltonian cycle for every $n\geq3$, which is not the case with the metallic cubes.

Theorem 3.7.2. Let a and b be even. Then the Horadam cube $\Pi_n^{a,b}$ is Hamiltonian for every $n\geq3$.

Proof. By Theorem 3.2.1, the Horadam cube $\Pi_n^{a,b}$ admit canonical decomposition , i.e., $\Pi_n^{a,b} = P_a \square \Pi_{n-1}^{a,b} \oplus P_b \square \Pi_{n-2}^{a,b}$. There are even number of copies of the graphs $\Pi_{n-1}^{a,b}$ and $\Pi_{n-2}^{a,b}$ in $\Pi_n^{a,b}$, and, by Theorem 3.7.1, each copy contains a Hamiltonian path. Consider the subgraphs $0\Pi_{n-1}^{a,b}$ and $1\Pi_{n-1}^{a,b}$. Similar to the case of metallic cubes, we join the endpoints of Hamiltonian cycles in those subgraphs to obtain a Hamiltonian cycle in the subgraph $0\Pi_{n-1}^{a,b} \oplus 1\Pi_{n-1}^{a,b}$. A similar construction can be done for the remaining copies of subgraphs $\Pi_{n-1}^{a,b}$ as well as for the b copies of the subgraph $\Pi_{n-2}^{a,b}$. Note that if some edge is part of a Hamiltonian cycle in one copy of the subgraph $\Pi_{n-1}^{a,b}$, then it is part of a Hamiltonian cycle in every copy of $\Pi_{n-1}^{a,b}$. The same observation holds for the subgraphs $\Pi_{n-2}^{a,b}$. Also, note that if the edge $(0a\gamma)(0a\delta)$ lies in the cycle in $0a\Pi_{n-2}^{a,b}$ for some $\gamma, \delta \in \mathscr{S}_{n-2}^{a,b}$, then the edge $(0(a-1)\gamma)(0(a-1)\delta)$ belongs to the cycle in $0\Pi_{n-1}^{a,b}$. We construct a Hamiltonian cycle in $\Pi_n^{a,b}$ in following way: if the edge $(1\alpha)(1\beta)$ is part of a Hamiltonian cycle of the subgraph $1\Pi_{n-1}^{a,b}$ for some $\alpha, \beta \in \mathscr{S}_{n-1}^{a,b}$, then the edge $(2\alpha)(2\beta)$ is part of a Hamiltonian cycle of the subgraph $2\Pi_{n-1}^{a,b}$. Removing the edges $(1\alpha)(1\beta)$ and $(2\alpha)(2\beta)$, and adding new edges $(0\alpha)(1\alpha)$ and $(0\beta)(1\beta)$ yields a Hamiltonian cycle in $0\Pi_{n-1}^{a,b} \oplus 1\Pi_{n-1}^{a,b} \oplus 2\Pi_{n-1}^{a,b} \oplus 3\Pi_{n-1}^{a,b}$. Since the numbers a and b are even, it is clear that this method can be further extended to obtain a Hamiltonian cycle of the $\Pi_n^{a,b}$. ■

As an example, Figure 3.10 shows the Hamiltonian cycle in the Horadam cube $\Pi_4^{2,2}$.

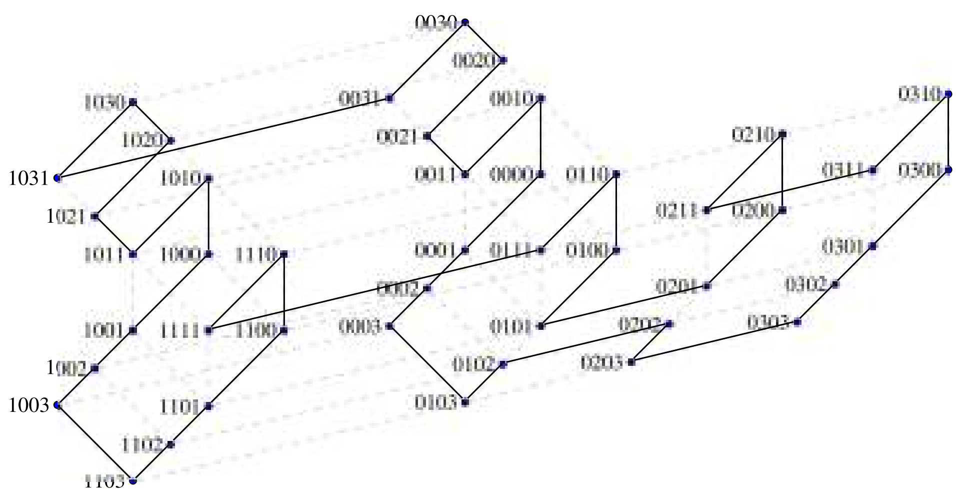

Figure 3.10: The Hamiltonian cycle in the Horadam cube $\Pi_4^{2,2}$.

The next theorem explores the existence of the Hamiltonian cycle in cases of a even and b odd. Setting $b = 1$ immediately yields the result for the metallic cubes.

Theorem 3.7.3. Let a be even, b odd and $n > 1$ odd. Then the Horadam cube $\Pi_n^{a,b}$ is Hamiltonian.

Proof. Because a is even and $\Pi_n^{a,b} = P_a \square \Pi_{n-1}^{a,b} \oplus P_b \square \Pi_{n-2}^{a,b}$, we can construct Hamiltonian cycle in the subgraph $P_a \square \Pi_{n-1}^{a,b} \oplus P_{b-1} \square \Pi_{n-2}^{a,b}$ in the same manner as in the proof of Theorem 3.7.2. Since $\Pi_1^{a,b} = P_a$, the Hamiltonian cycle in $P_a \square \Pi_2^{a,b} \oplus P_{b-1} \square \Pi_1^{a,b}$ can be extended to the cycle in the Horadam cube $\Pi_3^{a,b}$. Furthermore, $\Pi_2^{a,b}$ is $a \times a$ grid with a path P_b appended to the vertex $0(a-1)$. More precisely, $\Pi_2^{a,b} = P_a^2 \oplus P_b$. By the same argument, we can construct a Hamiltonian cycle in $P_a \square \Pi_3^{a,b} \oplus P_{b-1} \square \Pi_2^{a,b}$ and in $P_a^2 \subseteq \Pi_2^{a,b}$. It is easy to verify that those two cycles can be merged into a single cycle in $P_a \square \Pi_3^{a,b} \oplus P_{b-1} \square \Pi_2^{a,b} \oplus P_a^2$. By Theorem 3.7.1, the path subgraph induced by vertices that start with $0(a+b-2)$ is part of that cycle. Since b is odd, we can extend the cycle to visit all vertices of the neighboring path induced by vertices that start with $0(a+b-1)$ but one. Thus, we obtained a cycle that visits all vertices but one in $\Pi_4^{a,b}$.

Now we proceed inductively. Regardless of the parity of n, the subgraph $P_a \square \Pi_{n-1}^{a,b} \oplus P_{b-1} \square \Pi_{n-2}^{a,b}$ in $\Pi_n^{a,b}$ always contains a Hamiltonian cycle. If n is odd, then $n-2$ is odd too, and the last copy of $\Pi_{n-2}^{a,b}$ contains a Hamiltonian cycle which can be merged to obtain a cycle in $\Pi_n^{a,b}$. If n is even, a similar construction yields a cycle that visits all vertices but one. This completes the proof. ■

Figure 3.11 shows the Hamiltonian cycle in the Horadam cube $\Pi_3^{2,3}$, and Figure 3.12 a cycle that visits all vertices but the vertex 0404.

In the next theorem, we consider the last remaining case, where a and b are both odd.

Theorem 3.7.4. If a and b are odd and $n = 3k-1, k \geq 2$, then the Horadam cube $\Pi_n^{a,b}$ is Hamiltonian.

Proof. By Theorem 3.2.1, the Horadam cube $\Pi_5^{a,b} = P_{a-1} \square \Pi_4^{a,b} \oplus \Pi_4^{a,b} \oplus \Pi_3^{a,b} \oplus P_{b-1} \square \Pi_3^{a,b}$. By Theorems 3.7.1 and 3.7.2, whenever the number of copies in the decomposition of the Horadam cube is even, there is a Hamiltonian cycle in that subgraph. Since $a-1$ and $b-1$ are even, there is a cycle that visits all vertices in the subgraphs $P_{a-1} \square \Pi_4^{a,b}$ and $P_{b-1} \square \Pi_3^{a,b}$. More precisely, there is a cycle in the subgraph $1\Pi_4^{a,b} \oplus \cdots \oplus (a-1)\Pi_4^{a,b}$, and a cycle in the subgraph $0(a+1)\Pi_3^{a,b} \oplus \cdots \oplus 0(a+b-1)\Pi_3^{a,b}$. Moreover, the remaining copies $0\Pi_4^{a,b}$ and $0a\Pi_3^{a,b}$ can be further decomposed as

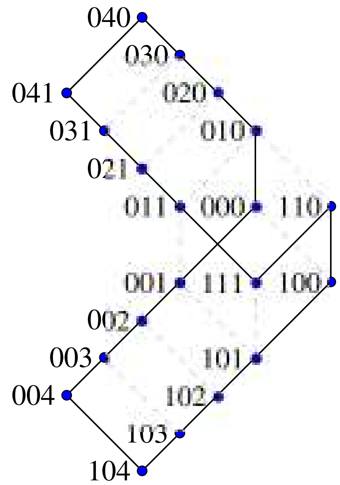

Figure 3.11: The Hamiltonian cycle in the Horadam cube $\Pi_3^{2,3}$.

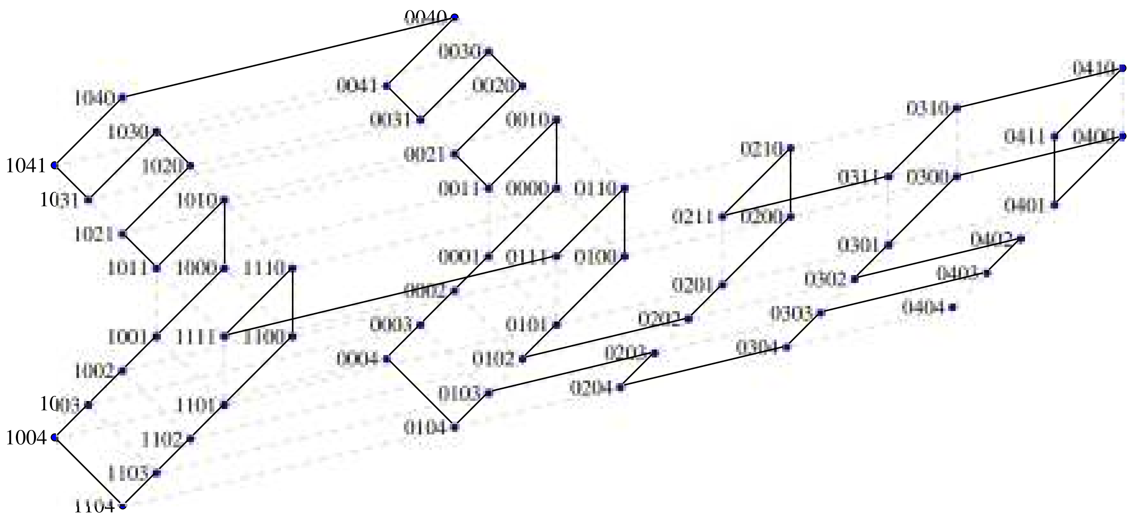

Figure 3.12: A cycle in the Horadam cube $\Pi_4^{2,3}$ that visits all vertices but one.

$\Pi_4^{a,b} \oplus \Pi_3^{a,b} = P_{a+1} \square \Pi_3^{a,b} \oplus P_b \square \Pi_2^{a,b}$, where $a+1$ is even. Thus we have a Hamiltonian cycle in the subgraph $0(a-1)\Pi_3^{a,b} \oplus 0a\Pi_3^{a,b} \oplus \cdots \oplus 0(a+b-1)\Pi_3^{a,b}$. To obtain a Hamiltonian cycle in $\Pi_5^{a,b}$ we just need to verify that a cycle constructed so far can be extended to the subgraph $P_b \square \Pi_2^{a,b}$ induced by vertices that start with $00l$ for $a \leq l \leq a+b-1$. Since $\Pi_2^{a,b}$ is $a \times a$ grid with the path on b vertices appended to the vertex $0(a-1)$, the subgraph $P_b \square \Pi_2^{a,b}$ can be decomposed into two grids, $P_{a+b} \square P_b$, induced by vertices $00l0k$ for $0 \leq k \leq a+b-1$, and $a \leq l \leq a+b-1$, and $P_{a-1} \square P_a$. Both grids have an even number of vertices. The construction of the cycle so far is inductive, hence the path subgraph $10a0k$ for $0 \leq k \leq a+b-1$ is part of the constructed cycle and it can be extended to encompass the grid $P_{a+b} \square P_b$. A similar argument holds for the grid $P_{a-1} \square P_a$. Thus, the cycle constructed so far can be extended to the full Hamiltonian cycle in $\Pi_5^{a,b}$. As an example, Figure 3.13 shows an extension of the cycle to the subgraph $P_3 \square \Pi_2^{3,3} \subset \Pi_5^{3,3}$, and it is clear that the extension can be done whenever a and b are odd. The full line denotes the constructed Hamiltonian cycle and the dashed line indicates the extension.

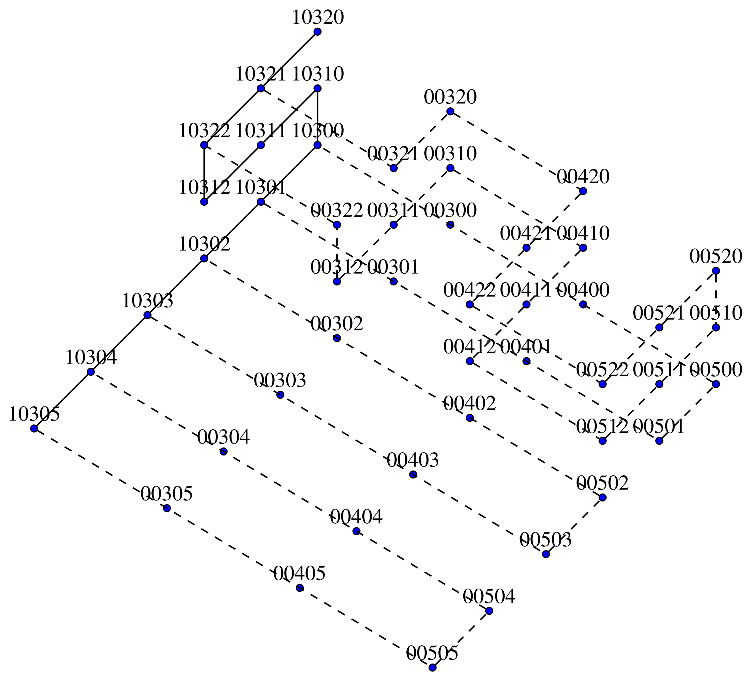

Figure 3.13: A fragment of the Hamiltonian cycle in the Horadam cube $\Pi_5^{3,3}$.

Now let $n = 3k-1$ and $k \geq 3$. Then $\Pi_{3k-1}^{a,b} = P_{a-1} \square \Pi_{3k-2}^{a,b} \oplus \Pi_{3k-2}^{a,b} \oplus \Pi_{3k-3}^{a,b} \oplus P_{b-1} \square \Pi_{3k-3}^{a,b}$. Furthermore, $\Pi_{3k-2}^{a,b} \oplus \Pi_{3k-3}^{a,b} = P_{a+1} \Pi_{3k-3}^{a,b} \oplus P_b \Pi_{3k-4}^{a,b}$. The numbers $a-1$, $b-1$, and $a+1$ are even, and, by induction, there is a Hamiltonian cycle in the subgraph $\Pi_{3k-4}^{a,b}$. Since the construction of the cycles is inductive, they can be merged into a single cycle of the Horadam cube $\Pi_{3k-1}^{a,b}$. ■

As an example, Figure 3.14 shows a construction of a Hamiltonian cycle in the Horadam cube $\Pi_6^{1,3}$.

In their paper [36], the authors proved that every Fibonacci cube contains a Hamiltonian path and that the Fibonacci cube is Hamiltonian whenever the number of vertices is even. It follows that the generalization of the Fibonacci cubes to the Horadam cubes preserves all those properties. Namely, summing the results of Theorems 3.7.1, 3.7.2,

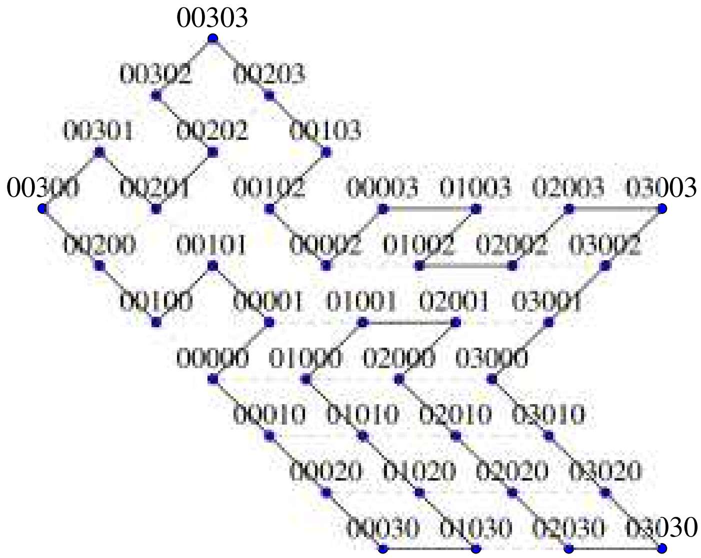

Figure 3.14: The Hamiltonian cycle in the Horadam cube $\Pi_6^{1,3}$.

3.7.3 and 3.7.4, we can state all results in one simple theorem:

Theorem 3.7.5. Every Horadam cube contains a Hamiltonian path. If $n \geq 3$ and the number of vertices in a Horadam cube is even, then the Horadam cube is Hamiltonian.

In particular, for $b = 1$, Theorem 3.7.4 extends the results for the Hamiltonicity of the metallic cubes for a odd.

3.8. Further possibilities of research

In this chapter, we defined and investigated the Horadam cubes, a new family of graphs that generalizes Fibonacci and metallic cubes. While our results offer some insight into the characteristics of the new graph family, they fall short of presenting a complete overview. For example, in the context of the edges, one can investigate edge general position set [34], and in the context of the degree distribution, our results pave the way for the computation of some degree-based topological invariants. Similarly, our results on distances provide a framework for the derivation of distance-based invariants, including notable examples such as the Wiener and Mostar indices [10], computed for the Fibonacci and Lucas cubes [19, 32]. Extending our results on the distribution of degrees, a thorough exploration of irregularities, modeled after the calculations for the Fibonacci and Lucas cubes [1, 18], would contribute to a more comprehensive portrait of the Horadam cubes. Another direction of research would include an investigation of the existence of the perfect codes in the Horadam cubes, similar to what was done for the Lucas cubes [38], or determining the smallest dimension of a hypercube that contains a Horadam cube of dimension n as a subgraph.

CONCLUSION

In the first part of the dissertation, we considered three different types of tilings of a honeycomb strip. In the first case, when we considered the tilings using monomers and dimers, we obtained a formula for the number of tilings when the number of dimers is fixed. Thus, we refined the already-known results and determined the number of tilings where monomers and dimers admit colors. In the second case, where the horizontal dimer is omitted, but instead, the trimer is introduced, we presented formulas that count the number of tilings with or without the fixed number of tiles of a certain type. Using those formulas, we were able to obtain several identities including sequences such as Fibonacci, tribonacci, tetranacci, Narayana, Padovan numbers, and others. Some of the obtained identities were generalized to the sequences that satisfy full-history linear recurrence with constant coefficients. Finally, in the last case, where we investigated the number of divisions of a honeycomb strip, we found that the answer is, simply, odd-indexed Fibonacci numbers. This result was retrieved in several ways, by determining the recurrences, by counting arguments, and in a combinatorial way.

Chapters 2 and 3 were dedicated to the graphs of recurrences. In Chapter 2, we have introduced and studied metallic cube, a family of graphs that both, generalizes the Fibonacci cubes, and, offers an alternative definition of the Pell graphs introduced recently by Munarini [40]. Their name reflects the fact that they are induced subgraphs of hypercubes and that their number of vertices satisfies two-term recurrences whose (larger) characteristic roots are known as the metallic means. We have investigated their basic structural, enumeration, and metric properties and settled some Hamiltonicity-related questions. Our results show that the new generalization preserves many interesting properties of the Fibonacci and Lucas cubes, and the Pell graphs, indicating their potential applicability in all related settings. In Chapter 3, we expand the generalization to encompass all graphs that satisfy Horadam's recurrence. As it turned out, the generalized family of graphs retains many appealing and useful properties of the Fibonacci and metallic cubes. In particular, it was shown that they admit recursive decomposition, as well as the decomposition into grids, similar to the metallic cubes. Furthermore, the Horadam cubes are induced subgraphs of hypercubes and bipartite median graphs. All Horadam cubes contain a Hamiltonian path, and a Horadam cube is Hamiltonian if the number of vertices is even. We obtained recurrent sequences that describe the distribution of the degrees as well as the number of induced subhypercubes of a given dimension.

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Curriculum Vitae

Luka Podrug was born 13.6.1985 in Split. In 2008, he finished his undergraduate studies at the Faculty of Science and Mathematics at the University of Zagreb. The same year, he enrolled Graduate University Program in Theoretical Mathematics where he graduated *cum laude* in 2010. After eight years as an elementary school teacher, in 2019 he started to work as a teaching and research assistant at the Faculty of Civil Engineering University of Zagreb. The same year he enrolled in doctoral studies in mathematics at the University's Faculty of Science and Mathematics under the supervision of Professor Tomislav Došlić. He participates in the work of the Seminar for Discrete Mathematics and Combinatorics. He published five scientific papers, one of which received the Editors' Choice Award in 2024. He participated at four international conferences with a contributed talk or a poster. From 2021 to 2023, he participated in the Croatian-Hungarian bilateral project as a collaborator.

Publication list:

- Tomislav Došlić, Luka Podrug. Solving the dimer problem on Apollonian gasket. *Proceedings of the* 4^{th} *Croatian Combinatorial Days* (2023), 9-19.
- Luka Podrug, Dragutin Svrtan. Some refinements of formulae involving floor and ceiling functions. *Mathematical Communications* 28 (2023), no. 2; 303–316.
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- Tomislav Došlić, Luka Podrug. Sweet Division Problems: From Chocolate Bars to Honeycomb Strips. *The American Mathematical Monthly.* Accepted for publication.

IZJAVA O IZVORNOSTI RADA

Ja, Luka Podrug, student Prirodoslovno-matematičkog fakulteta Sveučilišta u Zagrebu, s prebivalištem na adresi [redacted] [redacted], ovim putem izjavljujem pod materijalnom i kaznenom odgovornošću da je moj doktorski rad pod naslovom:

Recurrences and graphs of recurrences,

isključivo moje autorsko djelo, koje je u potpunosti samostalno napisano uz naznaku izvora drugih autora i dokumenata korištenih u radu.

U Zagrebu, 9. svibnja 2024.

Potpis