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Generalized Symmetries and Tensor Gauge Theories

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Generalizirane simetrije i tenzorske baždarne teorije

Sažetak

Rad istražuje složene odnose između generaliziranih globalnih simetrija i tenzorskih baždarnih teorija, posebno se fokusirajući na interpretaciju gravitona kao Nambu-Goldstoneovog bozona za spontano slomljenu generaliziranu globalnu simetriju.

Prvi dijelovi predstavljaju pregled običnih simetrija te simetrija viših formi, vođeni primjerom slobodne Maxwellove teorije. Središnji cilj je razumjeti kako višedimenzionalni objekti prirodno proširuju principe simetrije te kako se one manifestiraju u raznim fizikalnim kontekstima. Uvodimo poopćenje globalnih simetrija kroz poznate obične simetrije. Simetrije viših formi su simetrije koje dovode do sačuvanih struja viših formi. O njima se raspravlja kroz primjer slobodne Maxwellove teorije i njezine elektromagnetske dualnosti. Također su navedeni topološki aspekti simetrije viših formi. Nadalje, kratko su predstavljeni i drugi razvoji u generalizaciji principa simetrije, kao što su simetrije više-grupe (poput -grupne simetrije) i neinvertibilne simetrije.

-grupne simetrije, tj. simetrije koje omogućuju miješanje pozadinskih baždarnih polja pod njihovim odgovarajućim baždarnim transformacijama, razmatrane su kroz najjednostavniji abelovski slučaj gdje smo pokazali kako one proizlaze iz obične produktne simetrije okusa baždarenjem. Druga značajna generalizacija koja se raspravlja u ovom radu su neinvertibilne simetrije. Za razliku od tradicionalnih simetrija, koje su invertibilne (tj. mogu se poništiti primjenom inverzne transformacije), operatori neinvertibilnih simetrija nemaju inverze. Ove simetrije nastaju u sustavima gdje uobičajena grupna struktura simetrija propada. Umjesto standardnih pravila množenja generatora, operatori koji implementiraju takvu simetriju poštuju neka druga pravila *fuzije*.

Za teorije koje pokazuju dualnost, globalne simetrije moraju se podudarati - to vrijedi i za simetrije viših formi. Postupak dualizacije detaljno je objašnjen za Maxwellovu teoriju i kasnije je skraćen na recept koji djeluje za lagranžijane slične Maxwellovom, a prikazan je, ne samo za slobodnu Maxwellovu teoriju, nego i za neke nelinearne elektrodinamičke teorije.

Teorija linearizirane gravitacije je tenzorska baždarna teorija ključna za ovaj rad. Tenzorske baždarne teorije generaliziraju konvencionalne baždane teorije koristeći tenzore višeg ranga. Cilj rada leži u poglavlju "Graviton kao Nambu-Goldstoneov bozon", gdje se istražuje ideja da se graviton, kvant gravitacijskog polja, može promatrati kao Nambu-Goldstoneov bozon. Kroz prizmu linearizirane gravitacije, bezmaseni mod perturbacije metrike reinterpretira se kao rezultat spontano slomljene simetrije više forme, konkretno biformne simetrije. Ova perspektiva ne samo da pruža nov pogled na prirodu gravitona, već također ističe ulogu narušenja simetrije u razumijevanju bezmasenih spin- čestica u baždarnoj teoriji. Zaključujući s nedavnim napretcima, također smo predstavili potencijalni put za buduće istraživanje gravitona kao Nambu-Goldstoneova bozona.

Ključne riječi: generalizirane simetrije, simetrije viših formi, slobodna Maxwellova teorija, linearizirana gravitacija, graviton, baždarne teorije, dualnost, ’t Hooftove anomalije

Generalized Symmetries and Tensor Gauge Theories

Abstract

This work investigates the intricate relationships between generalized global symmetries and tensor gauge theories, particularly focusing on the interpretation of the graviton as a Nambu-Goldstone boson for a spontaneously broken generalized global symmetry.

The first sections present a review of ordinary and higher-form symmetries, guided by the example of the free Maxwell theory. The central aim is to understand how higher-dimensional objects, naturally extend symmetry principles and manifest in various physical contexts. We introduce the generalization of global symmetries through familiar ordinary symmetries. Higher-form symmetries are symmetries that lead to higher-form conserved currents. They are discussed within the examples of free Maxwell theory and its electric-magnetic duality. The topological aspects of higher-form symmetries are provided, too. Furthermore, other developments in the generalization of symmetry principles are briefly presented, such as higher-group symmetries (like -group symmetries) and non-invertible symmetries.

-group symmetries, i.e. symmetries that allow for the mixing of background gauge fields under their respective gauge transformations are considered through the simplest Abelian case where we have shown how it is derived from ordinary product flavor symmetry by gauging. Another notable generalization discussed in this work is non-invertible symmetries. Unlike traditional symmetries, which are invertible (i.e., they can be undone by applying the inverse transformation), non-invertible symmetry operators do not have inverses. These symmetries arise in systems where the usual group structure of symmetries breaks down. Instead of standard group multiplication rules, the operators that implement such symmetry obey some other fusion rules.

For theories exhibiting some duality, the global symmetries must match - this is also true for higher-form symmetries. The dualization procedure is done in detail for the Maxwell theory and is later abbreviated to a recipe that works for Maxwell-like Lagrangians and is shown for, not only the free Maxwell theory but also for some non-linear theories of electrodynamics.

The theory of linearized gravity is a tensor gauge theory pivotal to this work. Tensor gauge theories generalize conventional gauge theories by employing higher-rank tensors. The core of the thesis lies in the chapter "Graviton as a Nambu-Goldstone boson." This section explores the idea that the graviton, the quantum of the gravitational field, can be seen as a Nambu-

Goldstone boson. Through the lens of linearized gravity, the massless mode of the metric perturbation is reinterpreted as arising from a spontaneously broken higher-form symmetry, specifically a biform symmetry. This perspective not only provides a fresh outlook on the nature of the graviton but also highlights the role of symmetry breaking in understanding massless spin-2 particles in gauge theory. Concluding with the recent developments, we have also presented potential avenues for future exploration regarding the graviton as a Nambu-Goldstone boson.

Keywords: generalized symmetries, higher-form symmetries, free Maxwell theory, linearized gravity, graviton, gauge theories, dualities, 't Hooft anomalies

Contents

1 Introduction

1.1 Historical context

The emergence of symmetry as *the property of a physical system that is preserved under some transformation* had its greatest turning point with the occurrence of Noether's theorem in 1918 [1]. Here, a relation between a specific type of symmetry and the laws of conservation was mathematically shown which became one of the greatest reasons for the importance of symmetries in physics. Throughout classical mechanics, spatial and temporal invariances were known and used, as well as global spacetime symmetries for electrodynamics that were derived before Einstein's special theory of relativity. Nevertheless, the latter represents a new approach, or even another breakthrough in the application of symmetry in physics since, unlike those before him, Einstein derived the laws from the invariances. Even today, physical theories are often built on desired symmetries. The significance of symmetries in physics was quickly made clear in quantum mechanics where applying the theory of groups and their representations played a crucial role. This is how symmetry evolved from an aesthetic principle rooted in geometry to a scientific tool necessary for understanding modern-day physics. Needless to say, new applications and approaches still appear, making the symmetry principles a relevant subject amongst physicists and mathematicians.

It is clear that there are many applications of the concept: some purely aesthetic, some as obvious and as practical as bilateral symmetry of airplane design, and some as crucial as the applications in crystallography [2]. Besides that, the concept of deviating from perfect symmetry has become crucial in physics where symmetry in basic forces is simultaneously central and yet at times enigmatically broken. This is where the interplay of simplicity and subtlety starts making the concept of symmetry not only a plainly visible property of objects seen in everyday life but also a profound characteristic deeply rooted in theories. The main difference between the two can be seen as the distinction between the symmetries of an object and the invariances of physical phenomena to certain transformations.

1.2 Why generalize?

As said earlier, symmetry is the property of a physical system that is preserved when the system undergoes some transformations. A family of such transformations can be described using groups - Lie groups for *continuous* symmetries and finite groups for *discrete* symmetries.

Continuous and discrete symmetries correspond to continuous and discrete transformations, respectively. Amongst many other divisions of symmetries, we should mention the difference between external and internal symmetries where external refers to the symmetries of spacetime, and internal symmetries correspond to the internal degrees of freedom of the theory. However, for our further observations, it will be most important to distinguish local from global symmetries. **Global symmetries** keep a property invariant for a transformation that is applied simultaneously at all points of spacetime, whereas **local symmetries** are features invariant to transformations parametrized by spacetime coordinates. Local symmetries are the foundation of gauge field theories, i.e. gauge theory is presented with a Lagrangian density invariant to a smooth family of operations. Because gauge fields (which take values in the Lie algebra of the gauge group) are included in the Lagrangian density to ensure its gauge invariance, gauge theories have additional, i.e. redundant degrees of freedom. For example, the photon has two physical polarizations, but the gauge field that we use to describe it in a relativistic manner has four components. The Standard Model, one of the most successful and accurate physical theories, is based on gauge symmetries.

The discussion above is well-known for what one could call *ordinary* symmetries that have been very successful in explaining phenomena across many physical theories: mechanics, electromagnetism, and even quantum mechanics. As said in the previous chapter, in the 20th century, the use of symmetries became much more prominent than before and they even gave rise to new theories, therefore, it seems natural to wonder how the generalization of the symmetry principles became so important. Although greatly applied and still very much in use, ordinary symmetries have some limitations since they, by Noether's first theorem that will be revisited in 2.1.1, provide a conserved current that is a vector and corresponds to a point-like conserved charge. Moreso, they generally apply to point particles, specific spacetime transformations, etc., and are not suitable for a description of more complex systems, particularly in quantum field theory and topological phases of matter. The need to develop a universal tool for the application of symmetries became noticeable in quantum field theory as the study of higher-form[1] gauge fields became standard in mathematics and physics. Roughly speaking, generalizing global symmetries is applying the concept to objects of higher dimensions. Such generalized global symmetries [3] have shown to have applications within string theory and condensed matter physics, as well as in the study of extended operators and defects and of the anomaly structure in quantum field theory. They have recently been a subject of

[1]Higher-forms refer to differential -forms with . The basics are covered in Appendix A.

discussion in various fields of theoretical physics as they provide a new and organized language for thinking about symmetry principles. So, the behavior of modern physical systems will sometimes include interactions involving branes or topological charges, making ordinary symmetries insufficient. It turns out, however, that, when generalized, symmetries can truly provide charges of "higher dimension" than point-like charges.

Some other cases for the generalization of symmetry principles can be made, and among them is the phenomenon of *topological phases of matter*. These states of matter exhibit behavior that cannot be explained via ordinary symmetries. Topological phases are resistant to local perturbations and are stable in spite of disturbances such as impurities and often behave differently at their boundaries. A simple example would be the quantum Hall effect in which a sheet of a two-dimensional material has a quantized Hall resistance whose values are invariant under change of shape or smoothness of the material. The quantization is a consequence of symmetry breaking that differs from conventional symmetry breaking in the following sense: the broken symmetry is not local, but rather a global topological property of the material. Such materials are best understood through higher-form symmetries that explain the stability of topological phases despite the local perturbations. Great applications of generalized symmetries are, therefore, met in condensed matter physics where, besides the given example, the occurrence of some complex excitations (that exhibit neither fermionic nor bosonic behavior) are protected by higher-form symmetries.

Apart from the motivation for the generalization of symmetries, the reasoning behind combining generalized symmetries and tensor gauge theories into a single discussion should be provided. It is well-known that vector fields play a crucial role in physics – for example, in electromagnetism. In more advanced theories involving higher-dimensional objects, a suitable generalization is often used: tensor fields, that allow for more complex interactions. By generalizing a scalar or a vector field, an object of "higher dimension" is encountered – a tensor field, and when such objects are used in a theory, ordinary symmetries are no longer sufficient to describe some of the theory's properties. Therefore, it is natural to think of generalized symmetries and theories containing tensor fields as intertwined.

2 From Ordinary to Higher-form: Symmetry Upgraded

2.1 Ordinary Global Symmetries

As was said in chapter 1.2, one of the most important classifications of symmetries is that of global and local. Global symmetries correspond to transformations simultaneously applied to all points of spacetime, whereas local symmetries depend on the spacetime coordinates. Local symmetries require the introduction of gauge fields into the Lagrangian - they ensure its invariance under the local transformation. Such gauge fields lead to the formulation of fundamental forces and can be linked to conservation laws using Noether's theorem (discussed in detail in 2.1.1). However, the conserved quantities associated with local symmetries are not generally physical (not generally non-trivial), as opposed to the conserved quantities associated with global symmetries. Local symmetries can, nevertheless, lead to quantization of the global conserved charges. Since the focus of this chapter is to introduce the generalization of the symmetry principles that lead to conserved quantities of higher dimensions, only global symmetries will be discussed. To show the concept, ordinary global symmetries will first be revisited.

2.1.1 Noether's First Theorem

Noether's theorem shows that for every *continuous global symmetry*, there is a corresponding *conserved current* given with:

(2.1)

or, using a bit different language (see A):

d (2.2)

where is a -form. If Hodge dual is once again applied to equation (2.2), the final expression is obtained:

d

Since only ordinary symmetries are revisited in this section, only the first expression (2.1) will be used for further observation.

The *conserved charge* is defined as:

(2.3)

or, once again, in a bit different manner:

To show Noether's theorem in the language of quantum field theory, only expressions using indices 2.3 and (2.1) will be necessary.

To start with, an action is to be considered:

as well as a transformation:

(2.4)

In the equations above, is a field (for example, a fermionic field that will later be coupled to the electromagnetic background field), is an infinitesimal parameter and all of the other notations are standard. The Lagrangian density also transforms as shown below.

(2.5)

Note that is not the conserved current. Variation of the Lagrangian density is, of course, equal to the second term in (2.5) which gives the following equation:

Including and regrouping terms results with:

where the LHS of the Euler-Lagrange equation can be recognized in the square brackets, meaning it can be replaced with zero (on-shell):

(2.6)

By comparison with (2.1), expression in the brackets in equation (2.6) is the conserved current :

(2.7)

To show that the charge defined with (2.3) is conserved, the equation of continuity (2.1) will be used in the third step:

Finally, by applying Gauss' theorem,

(2.8)

equation (2.8) is obtained, showing how the charge is conserved.

2.1.2 Example: Abelian Global Symmetries and the Electric Charge

To be able to understand the generalization of global symmetries, revisiting a familiar global symmetry of the Standard Model will be useful. There are multiple laws of conservation within the Standard Model that come from global symmetries. To see how conserved charges arise, the example of the electric charge will be shown.

The idea is to apply the discussion shown in section 2.1.1 to a free fermionic field coupled to the electromagnetic field, as given with action :

where

with standard notation. Note that the dynamical term is not of interest here, hence is used simply as a background gauge field. To demonstrate that the Lagrangian density is invariant under a phase transformation of :

characterized by group of transformations, the transformation should first be expanded. Using the Taylor series to the first order, and comparing with (2.4) the following result is obtained:

as well as:

meaning that the two behave differently only with the respect to the sign. Due to the latter, when deriving the change in the Lagrangian density:

the terms cancel:

providing the announced result: the Lagrangian density is truly invariant to such transformations. It immediately follows from (2.5):

(2.9)

To complete the example and find the conserved current, obtained results can be plugged in equation (2.7):

Altogether, the conserved current is given with:

(2.10)

The result for the conserved charge follows by including (2.10) in (2.3):

These principles are what we want to now generalize so that both the current and the associated charge can be of "higher dimension".

2.2 Higher-form Symmetries

Notion of symmetry has broadened in various directions [4], including higher-form, higher-group, non-invertible, fractal symmetry, and many more. Although they all have many applications, this work will consider only the most fruitful ones, with the higher-form symmetries as the core generalization.

A -*form global symmetry* [3] is a global symmetry for which the conserved current is a -form and the conserved charges are of dimension . In this language, -form global symmetries are the ordinary global symmetries that were previously described and many of the properties of -form global symmetries can be applied. These generalized global symmetries are not some exotic generalizations in complicated theories but rather appear naturally in gauge theories. To make the layout more general, there will be no reliance on a specific Lagrangian density but rather a characterization of the charged objects as abstract operators.

If there is a symmetry that arises from a conserved -form current that satisfies

d

the conserved charges are given with:

The charged objects for these symmetries are -dimensional. For instance, in the simplest case of , the charged objects can be line operators [2]. This justifies that these symmetries are not something strange but are essentially present in any theory that has extended observables[3].

The classical source for the current is an abelian -form gauge field , so the action must contain the following term:

Under transformation, the gauge field should transform as follows:

d

where is a -form gauge parameter. The next step is to look at an example of free Maxwell theory to get a better understanding of the generalization.

2.2.1 Example: Free Maxwell Theory

We consider a gauge theory with gauge field and the corresponding field strength with two -form global symmetries: "electric" and "magnetic" with respective background fields and :

[2]such as the Wilson and the 't Hooft lines that are discussed in 2.2.2

[3]like Wilson's loops

For currents defined as:

— (2.11)

— (2.12)

corresponding conservation laws are obtained if the source-free Maxwell equations are used:

d d (2.13)

The source-free Maxwell equations given with (2.13), of course, relate to the familiar layout of Maxwell's equations: d is associated with source-free Gauss' law and source-free Ampère's law, whereas d corresponds to Gauss' law for magnetism and Faraday's law. Applying the exterior derivative to (2.11) and (2.12), and plugging Maxwell's equations (2.13) in, trivially yields the conservation of currents and . This is the process of formulation of a theory of a dynamical gauge field in the environment of the two background (non-dynamical) gauge fields which, due to their symmetries undergo gauge transformations given with:

d

To be able to write the action for this theory, i.e. to couple the dynamical field to the two background gauge fields, two approaches can be taken: "electric" and "magnetic". Surely, the two approaches must be equivalent, meaning that they show the *duality* of the theory which might be useful. Here, the electric approach will be taken, and the duality discussion will be revisited in 2.2.3.

In the "electric" approach, the gauge field shifts under background transformation, but remains the same under background transformation. Since d , the same is true for the field strength :

d

If the gauge field is coupled to the background gauge fields, the action is expected to have typical terms , which translates to:

$$(2.14)$$

In the equation above, the fact that the theory is taken to be invariant under electric background transformation is used. The first term ensures this, since d is added to both and after the background transformation. It is easy to check that, after the transformation, d derived from the transformation of and d obtained from the transformation of , cancel each other out, as they come with opposite signs. The first term is often referred to as kinetic, and the second term is referred to as magnetic. As neither nor shift under a background transformation, the first term is also invariant to magnetic background transformations. What remains is to ensure the invariance of the second term. The second term in (2.14) is invariant under magnetic background transformations, which can be easily verified using source-free Maxwell equations (2.13), the derivative of an exterior product (see A) and Stokes' theorem:

d d

d d

However, this (magnetic) term experiences a shift under background transformation - we use mathematical tools (from A) and Stokes' theorem to see exactly how:

d (2.15)

d (2.16)

d d (2.17)

d (2.18)

The term in (2.18) given with d constitutes a **'t Hooft anomaly** between and . To understand the importance and the meaning of this anomaly, we should first revisit some important notions that arise in the understanding of the generalization of the symmetry principles.

2.2.2 Anomalies and Loop Operators

Upon attempting to quantize a theory with a global symmetry, an *anomaly* can occur. Roughly speaking, an anomaly is a *classical symmetry that does not remain when theory is quantized* [5]. Some anomalies can be canceled by adding terms to the action. The most important here will be the *'t Hooft anomalies* which present an obstruction to gauging a global symmetry. A global symmetry with a 't Hooft anomaly remains a symmetry in the quantum theory, but when the symmetry is coupled to a background gauge field, the charges that were previously conserved are then not.

An anomaly is a term within the (effective[1]) action that shifts the action and corresponds to a non-conservation law. Notation stands for the background gauge fields. An anomaly is usually summarized by a -form gauge invariant anomaly polynomial , meaning that the background gauge fields and their gauge transformations are extended to dimensions. The relation between and , as well as the relation between the polynomials that present a procedure for the extension of to are given as:

(2.19)

d (2.20)

d (2.21)

The procedure can be used in both directions.

Furthermore, a notion of *Wilson loops* and *'t Hooft loops* as observables will be made, so let's take a look at their definitions. The Wilson line is an object that tells us how a complex vector carried by a particle moves around the manifold with connection (a Lie-algebra valued gauge field):

[1] Anomalies are recognized in action , but refers to the shift in the effective action , where is the partition function.

Here, stands for the path ordering, while and are the initial and final points of the particle's movement, respectively. In mathematics, this notion is called holonomy. The *Wilson loop* is a gauge invariant object, an observable, defined as the trace of the Wilson line on a closed path :

$$\text{tr} \tag{2.22}$$

The *'t Hooft loop* is also an observable, similar to the Wilson loop, and related to it as shown below.

$$\tag{2.23}$$

In the expression (2.23) stands for an element in the center of the gauge group, and is the Gaussian linking number between the two spatial loops. Since they are observables, these objects are of great importance, particularly in non-abelian theories (such as Yang-Mills), where electric and magnetic fields are not observables.

2.2.3 Example: Electric-magnetic Duality

To continue the discussion on the previous example 2.2.1 of the free Maxwell theory, an emphasis on a previously mentioned notion of 't Hooft anomalies should be made: if there are *no 't Hooft anomalies, the theory can be gauged.* As discussed, anomalies are usually shown using a -form, i.e. a -form polynomial . Using the procedure explained in chapter 2.2.2 and given with (2.19)-(2.21), as well as, once again, the nature of the exterior derivative of an exterior product and Stokes' theorem as before, the following expression is obtained from the anomalous term in (2.18):

$$\frac{\quad}{\quad} \quad \mathrm{d} \tag{2.24}$$

presenting an obstruction to gauging the electric and the magnetic symmetry simultaneously. One might change the presentation of the anomaly by adding local counterterms, but this would only switch the action to being invariant under and would still give rise to a 't Hooft anomaly under background transformations [6]. This is better reflected through inspection of the "electric-magnetic" duality of the theory. It just goes to show that higher-form symmetries play an important role in the context of duality, where more different Lagrangians describe the same theory. In such cases, global symmetries must match, whether the ordinary ones or the higher-form ones. The various dual descriptions should have corresponding charged operators that match, too [3].

To show the "electric-magnetic" duality of the theory, and all of the mentioned principles through the example, we will derive the dual "magnetic" representation starting from the "electric" presentation of the theory. We should mention that ’t Hooft anomalies are reproduced in any possible description of the theory, meaning that the magnetic formulation of free Maxwell theory should reproduce the same ’t Hooft anomaly as (2.24). Although gauge symmetries may differ in the dual descriptions, the global symmetries of the theory must be the same in the dual formulations. This is an important fact to remember about any kind of duality in physics: *the global symmetries must always match, even if the gauge symmetries might not.*

The dualization is done by considering an extended theory with action that includes a Lagrange multiplier which is also a -form gauge field associated with its own gauge symmetry.

— d (2.25)

Note that Bianchi’s identity for is still satisfied. The appropriate shift of under background gauge transformations:

ensures the invariance under background gauge transformations up to the ’t Hooft anomaly obtained earlier. We now want to find the appropriate equation of motion for for action to depend only on , and the new gauge field . In other words, we want to "lose" the dependence of action on and replace it with dependence on . The equation of motion for is obtained by varying over , as shown in the next equation.

— —

(2.26)

— d

Due to symmetry of , the second term in (2.26) can be written as:

—

and due to graded commutativity, the last term in (2.26) can be replaced with:

— d

The latter means that can be extracted from all of the terms in (2.26) as follows:

— — d

and the equation of motion for is obtained when the expression in the square brackets is set to zero.

— d (2.27)

As is now easily expressed from the previous equation (2.27), that expression can be included in (2.25). The dual presentation of the theory with action , now depending only on , and , is, therefore, derived:

— d d

— d

This expression shows that the duality generates a counterterm proportional to which will reproduce the same ’t Hooft anomaly as before with (2.24). The conserved currents within this theory are given with the following equations and are related to the currents defined in the "electric" presentation, as shown here:

— d —d

Holonomies of and around a closed -cycle are Wilson's loops and 't Hooft's loops , defined with (2.22) and (2.23), given as:

where are charges of the Wilson and 't Hooft loops respectively. When moving from one formulation to another, is exchanged, and so are the loops .

2.2.4 Topological Aspects

Higher-form symmetries can also naturally be introduced through topological operators, as follows. A -form symmetry in dimensions is implemented by an operator associated with a codimension[4]- closed manifold :

where is an element of the symmetry group . The meaning of it being a symmetry [3] is that the manifold can be slightly deformed without affecting the correlation functions - they are purely dependent on the topology of . Even though the use of higher-form symmetries was implemented in the previous sections without mentioning the corresponding topological operator, it is very useful to look at the principles from this perspective - mostly due to other generalizations that require this formalism, such as *non-invertible symmetries* discussed in chapter 3.2.

To argue this formalism, it is good to first consider the relation between topology and conservation. It is well-known that, in quantum mechanics, for a transformation to be symmetry, its operator has to commute with .

(2.28)

When it comes to quantum field theory, one might intuitively try to generalize this condition (2.28) by requiring the commutator between the symmetry operator and the stress-energy tensor to be zero, but that would be insufficient, as other constraints are necessary for a proper formulation of the symmetry condition. Precisely, in relativistic quantum field theory (QFT) with Euclidean signature, conservation under time evolution, as given by (2.28), must

[4]Codimension refers to the difference between the dimension of a space and the dimension of a subspace or an object within that space.

be further extended to invariance under any deformation in spacetime. It will be discussed here how this extension requires that the operator U be *topological* beginning with a fairly simple example that will better describe the concept: an n-dimensional theory with a $U(1)$ symmetry. Although this was considered in section 2.1, a different approach will be used here.

The said symmetry is associated with a conserved current $j_\mu(x)$ satisfying:

$$\partial^\mu j_\mu = -\partial_0 j_0 + \partial_i j_i = 0 \; ,$$

in Lorentzian signature. The only difference from (2.1) is that i spans all the spatial coordinates, and the problem is not limited to 4 dimensions. The charge operator is similarly given with:

$$Q = \oint j_0 d^{n-1}x \; . \tag{2.29}$$

Assuming the space has no boundary or that appropriate boundary conditions are imposed, the conservation of the charge can be shown using reasoning similar to what is shown in 2.1:

$$\partial_0 Q = 0 \; .$$

The symmetry operator that embodies the symmetry of a $U(1)$ rotation with angle θ is:

$$U_\theta = e^{i\theta Q} \; . \tag{2.30}$$

Incorporating equation (2.29) into expression (2.30), one gets:

$$U_\theta = e^{i\theta \oint j_0 d^{n-1}x} \; .$$

The operator is unitary and conserved and is referred to as the symmetry operator, whereas Q is known as the charge operator. The topological aspect arises upon generalizing this principle in a covariant manner using the mathematical framework introduced previously:

$$U_\theta\left(M^{(n-1)}\right) = \exp\left(i\theta \oint_{M^{(n-1)}} *j\right) ,$$

where $M^{(n-1)}$ is a closed $(n-1)$-dimensional manifold (and, by definition, with no boundary) in n-dimensional Euclidian spacetime. The correlation functions containing $U_\theta(M^{(n-1)})$ are independent of small deformations of $M^{(n-1)}$ by Stokes theorem since $*j$ is a closed form due to the conservation. This illustrates how a conserved current operator in a relativistic QFT generalizes to a topological object $U_\theta\left(M^{(n-1)}\right)$ on a codimension-1 manifold in spacetime. The correlation functions of the symmetry operator are independent of small deformations of the manifold, confirming the operator's topological nature.

2.2.5 Higher-form Symmetries Revisited

For more intuition, all of the concepts introduced so far (ordinary and higher-form symmetries) can also be presented in this language. In fact, an ordinary global symmetry is considered to be an invertible[5] 0-form symmetry, and the concept can just be generalized from there. A q-form global symmetry [3] is associated with a $(n-q-1)$-dimensional topological operator $U_g\left(M^{(n-q-1)}\right)$ in n spacetime dimensions. This symmetry then acts on a q-dimensional object D as:

$$U_g\left(M^{(n-q-1)}\right) \cdot D(N^{(q)}) = g(D)D(N^{(q)}) \,,$$

where $M^{(n-q-1)}$ and $N^{(q)}$ are linked in spacetime and $g(D)$ is a representation of g.

A good illustration would be recalling the free Maxwell theory with no charged matter from section 2.2.1. There, a topological operator is defined with:

$$U_\theta\left(M^{(n-2)}\right) = \exp\left(-\frac{\theta}{e^2}\oint_{M^{(n-2)}} *F\right) .$$

The exponent is the electric flux and, due to the field strength F being closed, the operator U_θ is closed. The conserved charges are non-topological Wilson lines W:

$$W = \exp\left(in\oint_{N^{(1)}} A\right) ,$$

and the topological operator acts on the Wilson line as:

$$U_\theta\left(M^{(n-2)}\right) \cdot \exp\left(in\oint_{N^{(1)}} A\right) = e^{i\theta}\exp\left(in\oint_{N^{(1)}} A\right) ,$$

or simply put, adds a phase. This completes the basics of the higher-form symmetry use and provides a better insight in the principles through the free Maxwell example.

[5]as opposed to non-invertible symmetries discussed in 3.2

2.2.6 Spontaneous -form Symmetry Breaking

It is well-known that an ordinary global symmetry can be spontaneously broken [7] [8], but, according to recent developments [3], so can higher-form symmetries. As said before, higher-form symmetries adopt many of the properties of ordinary symmetries and one of them is spontaneous breaking. To diagnose the spontaneous symmetry breaking, an inspection of the behavior of the large loops of the theory is used.

In quantum field theory (QFT) and symmetry discussions, area law, perimeter law, and Coulomb law refer to different scaling behaviors of the potential energy between particles, such as quarks, under various interactions, especially in gauge theories like QCD (Quantum Chromodynamics) and electrodynamics. These laws often arise when discussing confinement, deconfinement, or screening of charges.
Area law refers to a regime where the expectation value of a Wilson loop (that measures the potential between two sources), scales with the area enclosed by the loop:

area

Here, is a parameter that stands for the string tension. The area law is usually associated with confinement, meaning that the potential between the sources (particles, for example, quarks) grows linearly with the distance between them. In the QCD example, this relates to the quark-antiquark potential that increases linearly with . On the other hand, *perimeter law* describes the expectation value of the Wilson loop when it scales with the perimeter of the loop rather than the area:

perimeter

where is a constant related to a mechanism often associated with this behavior - screening. In such cases, particles could be separated without an infinite energy cost. Another possibility of the Wilson loop behavior is the Coulomb law that describes the potential energy between particles as a function of distance with its exponent being , as in the classical Coulomb law:

–

which typically occurs in QED. These behaviors give insight into symmetry breaking [3].

An area law for a charged loop operator means that a corresponding -form symmetry is unbroken. This is because the expectation value of the loop vanishes as its size goes to infinity. On the contrary, perimeter law and Coulomb behavior mean that the loop has a nonzero expectation value when it is large and the symmetry is spontaneously broken.

In other words, a -form global symmetry can break to a subgroup . If that happens, the charged operators under the subgroup exhibit area law. However, the loops that are charged under , but not under transformations exhibit perimeter or Coulomb law. Even though we are discussing -form symmetries here, further generalization is made by employing the key principle: inspecting whether charged objects have a non-zero vacuum expectation value when they are large. Of course, as in the case of ordinary global symmetries, when a continuous -form symmetry is spontaneously broken, a Nambu-Goldstone boson arises in the system. This **Nambu-Goldstone boson is a massless photon** meaning that photon can be seen as a Nambu-Goldstone boson of a higher-form symmetry breaking. A similar principle will be applied to describe the graviton as a Nambu-Goldstone boson 4.2.3.

The spontaneous symmetry breaking can be noticed from the matrix element of the -form Noether current between a photon with polarization and momentum and the vacuum:

(2.31)

which is nonzero. The possibility of the spontaneous symmetry breaking is generalized [3] to the following: A continuous -form symmetry is always unbroken for and a discrete for .

In the example of the free Maxwell theory which is a pure gauge theory with an electric and magnetic -form global symmetries, in four dimensions, the -form symmetry does not satisfy the condition for remaining unbroken:

but this does not ensure its breaking. To see how the symmetry gets spontaneously broken, we should consider the Wilson line given with (2.2.5). The Coulomb behavior of the line can be reasoned through the static point-like charge that would satisfy the Coulomb gauge – making the exponential in the Wilson line Coulomb-like. The gauge field would be a generalization of this, taking all of the space-time configurations.

Because the behavior of the Wilson lines is Coulomb-like, global symmetries are spontaneously broken and the massless photon is a Nambu-Goldstone boson. This can be further proved using equation (2.31) for both the electric and the magnetic current.

3 Generalized Global Symmetries: Further Developments

3.1 2-group Symmetries

Although the higher-form symmetries present the pillar of the generalization of symmetry principles, there are other generalizations, one of them being higher-group symmetries which we will present through the basics of -group symmetries with the example of Maxwell's theory, again. The concept can be understood as a generalization of a product symmetry.

A quantum field theory has a -group symmetry [6], if it can be coupled to a -form background gauge field (here denoted) that undergoes a -group shift in addition to its own background gauge transformations. In other words, these are global symmetries where t*he mixing of background gauge fields under their respective gauge transformations is allowed*. The -groups themselves will not be explored, but rather the -group background gauge fields. To keep it straightforward, the simplest example where the mixing of a background gauge field for a -form flavor symmetry is taken, i.e. and a -form background gauge field for previously mentioned -form symmetry is involved. Such -group symmetry is said to be abelian and denoted as shown below.

(3.1)

Here, [6] is a -group structure constant that characterizes the -group symmetry. To see what means, we should consider the transformation rules for gauge fields and . The transformation rule for remains standard:

d

but, as said before, undergoes an additional shift.

d — (3.2)

In the previous expression, d is the field strength. The consistency of the transformation rule given with (3.2) is ensured with being quantized. It ought to be mentioned that characterizes the -group symmetry because it does not change with the rescaling of the gauge fields. This is visible in equation (3.2) where there is an additional shift proportional to the field strength . The latter shows that we cannot turn a non-trivial profile of the gauge field on without it affecting . It should be recognized that, for , the -group shift in (3.2) disappears. Therefore, the -group symmetry dissolves into ordinary product symmetry:

Many quantum field theories possess the -group symmetry given with (3.1) such as QED with many flavors [6].

It can be shown that the -group symmetry described with (3.1) arises from a "parent" theory with

(3.3)

flavor symmetry, with being the corresponding background gauge field. Because of this, it will be useful to take a look at parent theories with such abelian -form flavor symmetry. In fact, gauging leads to a new -form global symmetry. The latter can be shown by gauging from (3.3) by promoting and its field strength to dynamical fields. The change is denoted with .

The action should contain:

(3.4)

In the previous expression (3.4) the theta-term[2] is added.

To ensure that can be gauged, the theory must be checked for anomalies. The most general anomaly -form polynomial [6], constructed of the field strengths and is:

(3.5)

[2]the gauge invariant term that can be added to a -dimensional action, quadratic in field strength

For further analysis, additional context is required. An anomaly polynomial is called *reducible* if it can be written as a product of closed, gauge invariant polynomials and of lower degree.

d

When trying to obtain , through the reversed procedure (described earlier with (2.19)-(2.21)), an ambiguity gets involved since the exterior derivative can be removed from either factor in (3.5). This can be described using a real parameter :

d

where

d d

Yet another similar ambiguity arises when a similar procedure is used further [6] to get to the -anomaly. Altogether, for dimensions, it follows:

d d

where is a real parameter introduced in the same manner as . Using the procedure described in (2.19)-(2.21), the expression for anomaly is computed.

For to be gauged, new gauge transformations must be anomaly-free, i.e. is imposed. This is satisfied when:

The previous conclusion leads to an anomaly that appears under gauge transformations and is of the form presented here [6]:

$$(3.6)$$

In its most general form, the -dimensional anomaly includes terms with and , but when those are set to zero (for the purpose of removing any anomaly under background transformation that we want to gauge), the anomaly is fixed as shown with (3.6). Also as a consequence of the parameters and being fixed, the following non-conservation law is obtained:

d (3.7)

When the gauge field is promoted to the dynamical gauge field , together with its field strength, these anomalous shifts become operator-valued and need to be accounted for. We can firstly examine the mixed anomaly in the non-conservation law shown above (3.7). So, we are interested in the following term:

d (3.8)

The term shown in (3.8) violates the conservation of the current but can be accounted for if some transformation rules are changed. However, this will not affect the dynamics of the theory and is different than cancellations of anomalies which involve coupling the theory to additional fields. The anomaly is resolved if in (3.4) is promoted to a background field that shifts under background gauge transformations as:

(3.9)

which shows that, since is not dynamical, but rather a background field, is explicitly broken and no profile of would stay the same after transformation (3.9).

Therefore, if we demand that the symmetry of the theory cannot be explicitly broken, the only resolution is to impose the following condition:

This analysis should be applied once again, this time for term in (3.7). This can be done by observing how this term appears in the non-conservation law, once the field is promoted (gauged) to the dynamical :

$$\mathrm{d} \qquad \text{——} \tag{3.10}$$

The current is obviously not conserved, unless is trivial. But, there is an appropriate source (field) for that will cancel the anomaly and will do so if it undergoes a -group shift when transformation is applied. This shows that a symmetry, when gauged, gives birth to a theory with a -group symmetry.

To see exactly how, let's set up a symmetry with a -form background gauge field that is associated with the gauge field strength as the current :

$$\text{—}$$

is conserved due to the Bianchi identity that is earlier said to satisfy. Therefore, if the current is defined as above, its classical source should be a background gauge field with its own background gauge transformation:

$$\mathrm{d}$$

which includes the Bianchi identity for . The action should, then, contain:

$$\text{—}$$

The resolution of the anomaly given in (3.10) is the following: one can impose that undergoes a -group shift under background gauge transformation, and, since is an appropriate source for , if the transformation is of the following form:

$$\text{—}$$

where

–

then the operator-valued shift given in (3.10) is canceled. Although the transformation rule for given with (3.9) and the transformation rule for might seem to take a similar form, there is a difference: transforms only if is non-trivial. This is why, in the first case, the symmetry was explicitly broken and had to be set to zero, unlike here, where the current is conserved if the field strength .

In conclusion, we have shown how a theory with flavor symmetry and a anomaly gives rise to a theory with abelian -group symmetry when is gauged. An interesting property is worth mentioning: an inverse construction is possible. Therefore, upon gauging a in a theory with -group symmetry, a parent theory is recovered with flavor symmetry and ’t Hooft anomaly [6].

3.2 Non-invertible Symmetries

As was discussed earlier, other interesting phenomena occur when symmetry principles are generalized. One of the interesting, yet applicable generalized symmetries are *non-invertible symmetries* [4]. In short, when it comes to generalized symmetries, they do not necessarily have to be implemented by an operator that has an inverse. This means there are symmetries that cannot be reversed when applied.

Global symmetries in quantum mechanics are described with (anti-)unitary operators, by Wigner’s theorem, which have inverses. But, when it comes to relativistic QFTs, symmetries can be non-invertible, as described by operators that do not possess inverses. This comes as a consequence of the requirement that the symmetry operator must be topological, as discussed in 2.2.4 without restrictions on the possession of an inverse. Non-invertible symmetries lead to new conservation laws and selection rules and are also not to be seen as overly exotic since they do have a history in physics: an infinite number of conserved charges that do not lead to unitary operators in integrable systems. It has been pointed out in recent years [9] [10] that these non-invertible symmetries related to the topological defects are to be seen as a form of generalization of global symmetries, hence, they will be considered here. It is worth mentioning that they are often found to be intertwined with higher-form symmetries in general spacetime dimensions and that they exist in realistic quantum field theories, such as four-dimensional free Maxwell theory. To provide the proper analysis of this concept, a topological aspect, such as that of section 2.2.4 is to be taken. Note that many of the interesting applications and concepts provided by the non-invertible symmetries will not be discussed here.

3.2.1 Defects

Some key concepts and notions must first be introduced to start the analysis. A *topological defect* is a localized disruption in the order of a system. The latter often refers to fields or materials and it arises from constraints of the system's topology. If not encountered by another defect, a topological defect will remain stable. Formally, such defects occur if it is impossible to deform the field configuration into a trivial configuration due to the nontrivial elements of the *homotopy*[6] group , with being the manifold describing the order parameter space of the system, and is the number of dimensions. These, however, mostly refer to solitons or other extended objects that are a consequence of a nontrivial topology of the field space and typically have nonzero tension[7]. Here, "topological defects" correspond to defects whose infinitesimal deformations in spacetime do not change any physical observables - the correlation functions will not depend on the detailed shape and location, but only on the topology, and they have *zero tension*. In short, the use of the term "topological" is somewhat like that in "topological field theories". To make the distinction, the traditional defects are sometimes referred to as "homotopy defects" [4].

To better elucidate the role of the symmetry operator , again a familiar example of symmetry will be used. The idea is to show that, if is the whole space at a fixed time, is conserved and unitary and acts on the corresponding Hilbert space . But, when is extended in the time direction and localized in one spatial direction, such as , becomes a defect that modifies the quantization resulting with *twisted Hilbert space* , labeled by the rotation angle .
For instance, consider a free complex scalar field in -dimensions (i.e.,):

[6]Homotopy refers to a continuous deformation of one shape or function to another, that can be done smoothly. example: Any larger loop can be shrunk to a smaller loop smoothly. Moreover, any loop can be continuously contracted to a point making all of the loops homotopic to each other. The homotopy group is since there are no nontrivial loops.

[7]Tension describes how far a map is from being in the state of minimal energy. Having nonzero tension means that these objects would require energy to be moved in spacetime.

The theory possesses a global symmetry depicted with the transformation:

and the corresponding conserved current:

If the space is considered to be a circle parametrized by , the corresponding Hilbert space is obtained by the canonical quantization of the free scalar field subject to the periodic boundary condition:

The conserved current yields a unitary operator:

acting on the said Hilbert space . What is interesting, is that an alternative, but equivalent approach can be taken - inserting a defect:

along the Euclidean time direction at changing thereat the boundary condition of the scalar field to:

(3.11)

Canonical quantization with the twisted boundary condition (3.11) now yields a twisted Hilbert space , as announced, denoted with the group element . In other words, we can relate the current to a unitary operator when is at a fixed time, or insert a topological defect along the Euclidean time direction at .

In the case of discrete symmetries , there is no conserved current or charge operator, but the symmetry can still (in general) be formulated by the existence of a conserved unitary operator for each element of the group . Moving to relativistic quantum field theories, the conserved operators will again be generalized to topological operators, i.e. defects in Euclidean spacetime. Given the previous discussion, a general principle can be extracted for global symmetries in relativistic quantum field theories: every global symmetry leads to a conserved *operator* acting on the Hilbert space and to a topological *defect* that modifies the quantization and yields a twisted Hilbert space. The latter ensures the consistency of Euclidean correlation functions by giving some strong constraints on symmetries. Both operator and defect are captured in the same object and the correlation functions involving it are invariant under infinitesimal deformation of . And when is the entire space, the invariance under time evolution is in place.

When it comes to relativistic QFT, an ordinary global symmetry is associated with a topological operator and a topological defect, but there are topological defects/operators that do not correspond to invertible symmetries. They do not obey a group multiplication rule, but a *fusion rule* of a different kind. In -dimensions, their fusion rule takes the following form:

where . They generally do not have an inverse and are, therefore, called non-invertible operators/defects. Higher-form symmetries have to be commutative but are not necessarily invertible.

So, non-invertible symmetries are transformations that will leave the Lagrangian invariant and are associated with conserved quantities, but an inverse transformation cannot be applied to return to the original state. Even though this process is irreversible, it is not to be mistaken or held responsible for other irreversible processes in physics such as hysteresis. The underlying principles might sound the same, but they arise in very different contexts. Hysteresis happens if the response of the system is dependent on its history, which might include an applied field or force, and not a symmetry transformation.

The interpretation of the non-invertible symmetries using the topological operator and the codimension- topological defect described with the same object is the following. Since the operator has no inverse, the symmetry transformation is non-invertible. From a different, but equal perspective, the transformation that "crosses" the defect is non-invertible, whereas symmetry transformations along "one side" of the defect behave more conventionally. Therefore, because the symmetries can be imposed by implementing a defect, we can simply say that the crossing of that topological defect is not invertible, in the case of the non-invertible symmetries.

The central part of this subsection will be the example of non-invertible symmetries [4]: -dimensional Maxwell theory, but to get into that, condensation defects are to be understood.

3.2.2 Higher Gauging and Condensation Defects

The simplest way to construct a non-invertible symmetry is to take a linear combination of two invertible ones. For example, defining an operator as:

where is the identity operator and , i. e. it has a symmetry. Operator can then be called a "twice the projection". Although a very easy way to construct a non-invertible symmetry, it is not very interesting since it can be written (as it is constructed) as a linear combination of operators of the same dimensionality. There are yet more creative ways such as summing over a finite set of topological defects of lower dimensions along the nontrivial cycles of a higher dimensional manifold. The procedure effectively creates a topological defect on the higher dimensional manifold - *condensation defect*. Such defects cannot be written as a linear combination of other defects with the same dimension, but are a mixture of topological defects of lower dimensions. The term *condensation defects* was introduced because the idea of higher gauging raised from anyon condensation along a one-dimensional line in the -dimensional space [11], and it is only later recognized as gauging a higher-form global symmetry along a higher dimensional manifold. The principle was also referred to as the categorical generalization of the projection operator [4] and is the basic part of the non-invertible fusion algebra in general spacetime dimensions.

If there are no ’t Hooft anomalies for a given symmetry, it can be gauged in various ways, however sometimes, a discrete -form global symmetry can be gauged, not in the entire spacetime, but only along a codimension submanifold in spacetime. This is the concept of *higher gauging*, i. e. -gauging a -form symmetry. An ordinary gauging would be simply -gauging of a -form symmetry. In essence, the process of gauging is, thus, also generalized. As mentioned before, the obstructions to gauging are not excluded: a -form global symmetry can be not only but also . To be more precise, a symmetry is -gaugable if it can be gauged for all , and if not, it is -anomalous. Furthermore, because a -form symmetry is related to -defects, it is easy to see that -gauging it is possible only if:

(3.12)

If only the equal part of expression (3.12) is considered, higher gauging coincides with taking a linear combination of the symmetry's topological defects. One should remember that condensation defects can be both invertible and non-invertible.

3.2.3 Topological Defects in -dimensional Free Maxwell Theory

To provide an example of the occurrence of non-invertible symmetries, a free Maxwell theory [12] can once again be considered:

(3.13)

In Lagrangian (3.13), is again the field strength, a -form, and the theta-term is included. All of the properties considered so far (such as those in subsections 2.2.1 and 2.2.3), are still valid, of course. What can be added and recalled, are the electric and magnetic -form symmetries that are responsible for the electric-magnetic duality of the theory. The said symmetries provide electric and magnetic charges as:

(3.14)

where is the dual field strength, and is a -dimensional space. The observables are Wilson and 't Hooft lines:

Without getting into the discussion on duality itself, that exchanges and given as:

topological defects that arise from the related transformation will be studied. The electric-magnetic duality exchanges both and and the coupling – which is one of the reasons making this duality interesting[8]. What will be of interest here is the recognition of the self-dual point $\tau = i$ since the coupling τ is given with:

$$\tau = \frac{\theta}{2\pi} + \frac{i}{e^2} .$$

At the self-dual point, the duality becomes a global symmetry that exchanges F and $*F$. In [13] , the topological defects for integer values of $i\tau$ were constructed, but here a more general approach will be taken for topological defects implementing the said transformation away from the self-dual point. The goal is to understand the topological defects in Maxwell's theory that implement even more general transformations between F and $*F$. As was announced, the duality itself will not be considered yet, so only the defects within the same theory are considered and not the relation between the dual theories. Therefore, the coupling τ is the same on both "sides" of the defect. This amounts to writing the Lagrangian $\mathcal{L}$ in terms of the defect along a surface S that splits the spacetime M into two regions, such that

$$M = S^+ \cup S^- ,$$

and:

$$\partial S^- = -\partial S^+ = S .$$

To make the matter simple, S can be chosen to be an infinite flat surface at $x = 0$ and the fields on the sides of the defects can collectively be denoted as left and right: A_L, A_R. The Lagrangian $\mathcal{L}$ is the same on both sides of the defect because, as said earlier, only one theory is considered and not the interface between the dual theories. In such terms, the full action in the presence of a defect can be written as:

$$S = \int_{S} \mathcal{L}(A_L, e^2, \theta) + \int_{S} \mathcal{L}_S(A_L, A_R, b) + \int_{S+} \mathcal{L}(A_R, e^2, \theta) . \tag{3.15}$$

In the expression (3.15) above, b stands for any additional dynamical fields that live on the defect.

[8]As explained in section 2.2.3, the exchange allows for better understanding of different regimes. This is because strong coupling in one theory turns to weak coupling in the dual theory. Although it is not strictly so in the *free* Maxwell theory, the principle sheds light on more general situations.

The next important thing to consider is the most general transformation that would act linearly on and , so the following transformation can be considered:

and analogously:

where and are real parameters. With the assumption of the defect Lagrangian as not dependent on the metric, for the defect to be topological, the energy-momentum tensor must obey:

(3.16)

In other words, for the energy-momentum tensor to be conserved, the previous expression must hold. In (3.16), is the normal vector, perpendicular to the surface . The condition is not generally satisfied, i. e. only specific relations between and will allow (3.16), meaning that some constraints are in order. Specifically, equations of motions for , , and derived from the defect Lagrangian will provide them. The equations of motion are given as follows.

and

The form degrees of and are marked with and , respectively. So, the strategy is not to start from a given defect Lagrangian, but to first determine the relations between and on the defect, such that (3.16) is satisfied. The relations will give a suitable candidate for a topological defect and the defect Lagrangian can be built.

The energy-momentum tensor is explicitly given as:

and, since the satisfaction of (3.16) is imposed on it, the most general manner for the transformations to be generated by a topological defect is an rotation [12]:

The theory also enjoys a charge conjugation symmetry and the corresponding topological defect is obtained when is set to . The topological defects that implement the transformation of to away from the self-dual point are obtained with $-$. Some other interesting cases [14] have been considered using higher gauging.

For , a simple example of a condensation defect for arbitrary values of and is realized by the defect Lagrangian:

where is an auxiliary gauge field that is non-trivial only on the defect. Factor is an integer that can be taken to be positive without loss of generality and serves to ensure the gauge invariance. The equation of motion for will provide a condition:

(3.17)

and the other equations of motions give:

(3.18)

When condition (3.17) is included in equation (3.18), the following constraint is obtained:

After appropriate integration on the defect, a condition on the electric charges is yielded:

where is the integer measuring the flux of , and and are the -form electric charges defined by (3.14). If , the defect is trivial, but if , the charges are to be multiples of and, although the defect acts trivially on local operators, the non-trivial transformation of line operators is induced, showing the basic property of a *topological* defect as discussed in the previous sections.

As announced, the case of also includes the charge conjugation. It is obtained with the following defect Lagrangian:

— (3.19)

as the non-trivial element of the . In the case of , the well-known invertible charge conjugation symmetry is implemented. However, for , the condensation defect occurs.

The condensation defect is best seen when the previously discussed defect (3.19) and the charge conjugation defect are considered simultaneously. Then, one can formulate the action as:

— — (3.20)

where corresponds to the field living only on the condensation defect, and is the field living only on the charge conjugation defect surface. The action (3.20) consists of two terms: the first one corresponds to a trivial decoupled topological field theory and can be dropped, and the second is the product of the usual charge conjugation and the condensation defect given earlier (3.19). To put it simply, the defect Lagrangian (3.20) provides the familiar invertible charge conjugation symmetry for , but for , it provides a product of the usual charge conjugation and the condensation defect given with (3.19).

3.3 Non-linear electrodynamics

When two seemingly different physical theories are shown to be equivalent, they are said to be each others' *dual theories*. One thing to remember about duality in physics, as was mentioned in 2.2.3 is that the *global symmetries of dual theories must coincide*, whereas the same does not have to be the case for the gauge symmetries. This is one of the ways in which the generalization of the symmetry principles provides a whole new frontier. If the symmetry algebras of the two theories are the same, it may point to a duality and the degrees of freedom in these theories do not need to be the same. It is also worth mentioning that dual theories typically interchange weak and strong coupling making it easier to find explanations for both regimes, as was seen in the example of "electric-magnetic" duality in 2.2.3. Even though there are many types of duality, it is sufficient to know that they provide new insights into both theories.

Here, we will consider a specific class of Lagrangians and show how to find their dual theories, as an extension of what was considered with discussion around free Maxwell theory in 2.2.3. The procedure is essentially the same but depicted through a recipe that shortens the calculations [15]. The procedure works for Maxwell-like Lagrangians and will be shown for the case of Maxwell's free theory and for more complex, non-linear cases.

When considering a -form dynamical field with an Abelian gauge transformation:

d

and the corresponding -form field strength , a class of Lagrangians of interest can be formulated as a function of two independent Lorentz invariant scalars :

– –

and :

– –

Because of the elegant formalism, a class of Lagrangians that can be written in these terms, as an analytic algebraic function of and , the equations of motion can also be written simply in terms of , , , and . Because these theories typically enjoy -form symmetries, conserved currents (again, "electric" and "magnetic") can also be expressed this way. The sources of the currents, and , can be minimally coupled to the action that would be written in general as:

— (3.21)

It is now clear that a new formulation of and is required to reflect the introduction of the background -form fields and :

–

–

When the background field goes to zero, both and turn to their old forms and , respectively. What follows is the off-shell dualization recipe.
Using a *parent* action:

—

one simply needs to solve equations for a given Lagrangian and include the results in (3.21) to obtain the dual theory. The final equations that are needed to solve are [15]:

$$– \qquad (3.22)$$

$$(3.23)$$

$$(3.24)$$

where , and are given as follows:

–

–

The parent action can be also simplified to:

— —

To show the concept, it is useful to, yet again, consider the Maxwell theory in dimensions, as announced.

3.3.1 Maxwell's Electromagnetism

Maxwell's action can now be written using the previously introduced abbreviations:

Because there is no explicit dependence on , equations (3.22)-(3.24) simplify to:

To specify, equation (3.23) leads to:

making the first equation (3.22) lead to:

Plugging this in the parent action (3.21):

yields the dual action:

Although the coupling was not considered, it would have been properly converted using this procedure, too [15].

3.3.2 Born-Infeld Electromagnetism

A famous example of non-linear electromagnetism is the Born-Infeld electromagnetism developed in the 1930s [16]. The motivation for the establishment of this theory was to regularise some divergences associated with point-like charges. The inspiration was taken from the translation of the Newtonian free-particle Lagrangian to the relativistic case .

In a similar manner, Maxwell's Lagrangian was modified firstly by Born [17] so that the divergences from the Coulomb potential are regularised due to the upper bound in the theory [18]:

(3.25)

In (3.25), represents the upper limit of possible electric fields. Because such action is purely motivated by the upper limit, it had some other shortcomings and, together with Infeld, Born searched for a theoretically more appealing guiding principle. They concluded that, just like the Lorentz group reduces to the Galilean transformations in the limit of small velocities, Maxwell electromagnetism should be the limit of some other theory with a larger group of symmetries: the group of general coordinate transformations. The action at which they have arrived is given with [16]:

Here, the action coupled to the background fields is considered, in terms of previously introduced scalar combinations:

Following the procedure given with (3.22) - (3.24), from equation (3.23), we obtain:

which is the usual case for theories of type that have electric-magnetic duality rotations.

The solutions of the remaining two equations are:

The dual theory is found by plugging these results in (3.21):

Besides the Born-Infeld theory, there is yet another extension of Maxwell's theory to a non-linear regime that recently spiked the interests of physicists, called ModMax theory.

3.3.3 ModMax Theory

A non-linear extension of free -dimensional Maxwell's electrodynamics called ModMax was recently discovered [19]. What is particularly interesting about this theory is that it admits classical conformal invariance, as well as all the symmetries of Maxwell's electrodynamics, including the electric-magnetic duality. It arises as a weak limit of the generalized Born-Infeld theory. The Lagrangian density of ModMax has the following form [20]:

and provides two -form currents as in the previous cases, making it suitable for the procedure. The coupling constant is dimensionless and non-negative and Maxwell's theory is recovered when . When coupled to the appropriate sources (the two -form background fields), the ModMax action can be written as:

Due to the nature of the coupling constant, only will be considered, since the case is the Maxwell theory that was already discussed. As was said before, the theory admits the electric-magnetic invariance and, as in section 3.3.2, equation (3.23) reduces to:

and the solutions of equation (3.24) are:

both leading to different in the equation (3.22) and, consequently to a different dual action. The final result is:

Of course, the distinction is not alone in providing different actions, since the action itself is different for different values of the coupling constant . A noticeable property, however, is that of:

This implies a negative angle in which is forbidden by the theory. In conclusion, the dual theory is given with . Similarly to the two previous examples, if we set the background fields to zero, the dual theory is the same as the original ModMax theory, but with different parameters. Simply put, the dual theory is again of the "ModMax type".

The discussion around the non-linear extension of Maxwell's theory offers a better insight into the use of higher-form symmetries: the dual theories are easily found using the principles grounded in the discussion 2.2.3 in an abbreviated manner.

4 Generalized Global Symmetries in Tensor Gauge Theories

4.1 Tensor Gauge Theories

Gauge theories are the ones that admit *gauge invariance*, previously (chapter 1.2) described as local symmetries - invariance under transformations that can vary from point to point in spacetime. These transformations are implemented in the Lagrangian with the use of a gauge field that undergoes these transformations. Gauge invariance results in redundancy since the gauge transformations do not change the system's physical state, making the gauge invariance a somewhat odd principle on which to build some of the best theories of physics. It is, therefore, not a nature's property, but rather a property of the choice of description of nature. Removing the redundancy, by fixing some specific gauge, causes some of the theory's properties to be removed, too. As odd as redundancy may seem, this principle provides a description of three of the fundamental forces that are responsible for the success of the Standard Model. In the words of David Tong [5]: *"Although the use of gauge choice was commonplace among classical physicists, it was viewed as a trick for finding solutions to the equations of electromagnetism. It took a surprisingly long time for physicists to appreciate the idea of gauge invariance as an important principle in its own right. Fock was the first to realize, in 1926, that the action of gauge symmetry is intricately tied to the phase of the wavefunction in quantum mechanics."*

A classic example of gauge theory that was considered throughout this work is *electromagnetism*. In the language of field theory, it is described by the gauge group with being the gauge field - the vector electromagnetic potential. The gauge symmetry is local: the phase of the wavefunction of a charged particle can be arbitrarily changed at every point in spacetime. The gauge field is introduced in the Lagrangian and it transforms so that it cancels the effect of the local phase change. The field strength is invariant under gauge transformations. The -vector potential is then recognized as the photon, the force carrier of the electromagnetic force. Similarly, to describe the weak nuclear force, and bosons arise as massive spin-1 mediators, the consequence of the gauge group , where is the isospin and is the hypercharge. And again, in quantum chromodynamics, the gauge fields of the non-Abelian symmetry are implemented via gauge fields and correspond to gluons. The symmetry is that of color , and spans from to , indicating the colors of the carriers - gluons. They are again, spin- particles, massless and, unlike photons, interact with each other.

As was discussed in 2.2.6, the photon can be viewed as a Nambu-Goldstone boson - a massless particle that arises from spontaneous symmetry breaking, and in the following chapter, we will build a case for the graviton to be considered one in linearized gravity, too. The symmetry that we hold accountable for the rise of the graviton as a Nambu-Goldstone boson will be higher-form - more so, a *biform symmetry*, as will be explained in 4.2.2. The argument will be made in the theory of linearized gravity which is a **tensor gauge theory**, and the biform symmetries, as another generalization of symmetry principles, need to be properly introduced.

We should first build some intuition around tensor gauge theories. Unlike the usual gauge theories that are based on vector fields, for example, the -form in electromagnetism, tensor gauge theories employ higher-rank tensor fields, meaning that the gauge fields transform as tensors under the gauge transformations. The motivation lies in describing higher-spin fields (spin greater than) [21] as well as in attempts to formulate consistent theories in the study of gravity and string theory.

To show an example of how tensor gauge theories can be viewed as a generalization of gauge theories, we can take a glance at one of the most famous examples of tensor gauge field - the antisymmetric field called *Kalb-Ramond field* [22] that appears in string theory as a generalization of the electromagnetic field. The gauge transformation is:

where is a -form. The corresponding field strength is given with:

which is considered to be an analogue of the field strength in electromagnetism. The free field Lagrangian is also analogous to the familiar electromagnetism gauge theory.

—

Both the field and the field strength are totally antisymmetric, i.e. they are differential forms, but of higher rank, making this theory a tensor gauge theory.

This example might serve as an introduction to how tensor gauge theories might look like, but our focus will be on linearized gravity as the main goal is to see how the graviton can arise as a Nambu-Goldstone boson. Although the following discussion is fairly recent[23] [24], the closely related insights in tensor gauge theories, their symmetries and dualities were explored before [25] [26], intertwined with [27] [28]. A unified approach to standard and exotic dualizations of gauge theories was also considered through the language of graded geometry [29].

4.2 Graviton as a Nambu-Goldstone boson

4.2.1 Linearized Gravity

As was said before, since we are to present the graviton as a Nambu-Goldstone boson in linearized gravity, we should first cover the basics of the theory. **Linearized gravity** assumes that the metric tensor is given by:

where is the flat metric and is a small perturbation. Since follows as a deviation from the flat metric (that is always symmetric), it is a symmetric rank- tensor. An action for the theory containing propagating in the flat metric will, therefore, be a tensor gauge theory. The gauge symmetry will follow from the diffeomorphisms of the full Einstein theory. Expanding the Einstein equations to linear order in the small perturbation leads to thinking of gravity as a symmetric spin- field propagating in Minkowski space . In this regime, indices are lowered and raised with the flat metric, for example:

The various curvature tensors are constructed from the metric. When we only keep the linear order in , the Christoffel symbols, the Riemann tensor the Ricci tensor, and the Ricci scalar are obtained with structures [30] that lead to the Einstein tensor of the following form:

$$- \qquad (4.1)$$

The Bianchi identity for the full Einstein tensor[9] reduces to:

Even though Einstein's equations become linear, they are still somewhat complicated and present a set of partial differential equations where the stress-energy tensor also has to be taken for small, to stay consistent. The linearized equations of motion are derived by varying the **Fierz-Pauli** action:

$$ \text{—— } \quad - \quad - \quad - \quad - \qquad (4.2) $$

Truly, when varied, after some integration by parts, the Fierz-Pauli action yields:

which amounts to the vacuum Einstein equations and matter can be coupled by adding term to the action.

Linearized gravity inherits a gauge symmetry from the diffeomorphisms of the full theory. In other words, the action is invariant under:

In the following discussion, instead of (4.2), we will refer to the Fierz-Pauli action written in the following convention (4.3):

$$ - \qquad (4.3) $$

where the Einstein tensor is given with (4.1) but without the – factor, namely:

which is often slightly abbreviated using the symmetrization notation (with unit weight):

[9] The usual Bianchi identity for the full Einstein tensor is

From the perspective given with (4.3), it is also clear that the Einstein vacuum equations are given with:

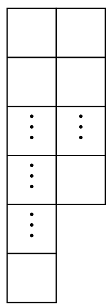
One should keep in mind that the linearized Einstein tensor is also gauge invariant. For further development, we will also state the linearized Riemann tensor in the same convention:

(4.4)

We should notice that the Einstein tensor is not the most general gauge-invariant local operator in linearized gravity - the full Riemann tensor is also gauge invariant.

4.2.2 Biform Symmetries

To show some symmetries of tensor gauge theories, and in particular, linearized gravity, a concept of *biform* should be introduced. A biform is a mixed-symmetry tensor and can be denoted with indices, meaning that its depiction through the Young diagram consists of two columns where is the length of the left column and of the right. For example, for , the Young diagram could be sketched as follows.



The adopted convention is the following: the mixed-symmetry tensors are antisymmetric in the indices associated with a column while antisymmetrizing all the indices from one of the columns with any index from the second column causes the tensor to vanish [23]. A theory that adopts an electric current of the nature is said to enjoy a biform symmetry [24]. In more detail, a biform is a tensor which is an element of , where is the space of - forms on an -dimensional manifold and analogously for . The index symmetries can be written as:

and for $p > q$:

$$A_{[\mu_1 \dots \mu_p | \nu_1] \nu_2 \dots \nu_q} = 0 \ .$$

Due to having two columns, exterior derivatives can act on either, therefore, to differentiate them, one can introduce left and right exterior derivatives, d_L and d_R that act on the left and the right column, respectively:

$$d_L : \Omega^p \otimes \Omega^q \to \Omega^{p+1} \otimes \Omega^q \ ,$$

$$d_R : \Omega^p \otimes \Omega^q \to \Omega^p \otimes \Omega^{q+1} \ .$$

Although the notion is familiar, it is good to see the definition of the operators explicitly:

$$(d_L A)_{\mu_1 \dots \mu_{p+1} | \nu_1 \dots \nu_q} = \partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}] | \nu_1 \dots \nu_q} \ ,$$

$$(d_R A)_{\mu_1 \dots \mu_p | \nu_1 \dots \nu_{q+1}} = A_{\mu_1 \dots \mu_{p+1} | [\nu_1 \dots \nu_q, \nu_{q+1}]} \ .$$

In the previous equation, the comma stands for the partial derivative. The operators obey:

$$d_L^2 = d_R^2 = 0 \, , (d_L + d_R)^3 = 0 \ ,$$

and commute. In a similar manner, left and right Hodge dual is introduced: $*_L$ and $*_R$.

$$(*_L A)_{\mu_1 \dots \mu_{n \ p} | \nu_1 \dots \nu_q} = \frac{1}{p!} \epsilon_{\mu_1 \dots \mu_{n \ p} \alpha_1 \dots \alpha_p} A^{\alpha_1 \dots \alpha_p}{}_{| \nu_1 \dots \nu_q}$$
$$(*_L A)_{\mu_1 \dots \mu_p | \nu_1 \dots \nu_{n \ q}} = \frac{1}{q!} \epsilon_{\mu_1 \dots \mu_{n \ q} \alpha_1 \dots \alpha_q} A_{\mu_1 \dots \mu_p |}{}^{\alpha_1 \dots \alpha_q}$$

All of the other operations necessary to work with these mixed-symmetry representations (that we call biforms) are done in the similar manner, by employing the operations done on differential forms, but separately for each set of indices. We can finally explain how a graviton can be seen as a Nambu-Goldstone boson through these principles.

4.2.3 Graviton as a Nambu-Goldstone boson

The Fierz-Pauli action given with (4.3) is not only invariant to linearized diffeomorphisms but also under a global -biform symmetry where shifts as follows:

(4.1)

where is a constant symmetric tensor. The corresponding Noether current is a -biform[10] [23]:

$-$ (4.2)

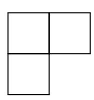
that, although conserved on-shell[11], is *not gauge invariant*. So, in the search for the symmetry that would yield the graviton as a Nambu-Goldstone boson, we have to look at a somewhat different but related transformation. If we choose to write as:

(4.3)

where is a -biform. The consequence of this choice is that *the associated Noether current is the Riemann tensor*. Since the Riemann tensor is gauge invariant, such a choice has led us to a both conserved and gauge-invaru’0bgiant current. We can further build the picture of the graviton as the Nambu-Goldstone mode for this biform symmetry. Of course, the choice given with (4.3) is not consistent with being a constant symmetric tensor since we have allowed for some spatial dependence. This is not an issue if the shift is chosen so that the Riemann tensor built from it vanishes, or, at the free level, that the Einstein tensor built from it vanishes.

Even though the proper symmetry that will allow us to obtain the graviton as a Nambu-Goldstone mode is established, there are still some points to be made. The first instinct is to find the appropriate background field that would act as a source for this symmetry.

[10]The depiction of a -biform current with a Young diagram is:



[11]The divergence of the -current is the linearized Einstein tensor which is zero in the absence of matter.

This is more easily done if the action for linearized gravity is written in an analogous Palatini-like formulation [23]:

(4.4)

In the expression (4.4), is a symmetric tensor and is the linearized analogue of the Christoffel symbol, and is symmetric in its first two indices, whereas the last index has no specific symmetry. This action, as it is analogous to the Fierz-Pauli action, also has the gauge symmetry, but the gauge transformations are now combined:

where is an arbitrary -dimensional vector. As a representation of its symmetry group, is reducible and can be decomposed into a totally symmetric tensor and a -biform [23]. An interesting property occurs where only the latter contributes to the Riemann tensor, whereas the totally symmetric part of the linearized Christoffel connection is unnecessary. Therefore, after the action is written in terms of the discussed decomposition, we can integrate the symmetric tensor out using its equations of motion. When that is included back in the action, linearized Einstein-Hilbert action is obtained, but the quadratic dependence on is present, which we would like to remove. There is a certain amount of freedom of adding terms to this action that would not affect the equations of motion. There is, in fact, a choice [23] that removes the quadratic dependence on which is unique up to an overall rescaling parametrized with .

This particular formulation is especially convenient because it decouples the symmetry whose Noether current is the gauge non-invariant (4.2) from the symmetry whose current is the Riemann tensor.

We can see that by reviewing all of the symmetries that this theory enjoys starting with the gauge transformations:

the biform symmetry that shifts :

and an additional symmetry of the form (4.3):

(4.5)

(4.6)

Now that the two symmetries are decoupled, we can find the proper background gauge field for the symmetry transformation given with (4.6) and promote the transformation to a symmetry for an arbitrary function of , removing the condition that the Einstein tensor built out of vanishes. The gauging is done by introducing a gauge field (which has the same symmetries as the Riemann tensor) and the corresponding couplings into the action and ensuring its gauge invariance. This improved action is then used to derive the gauge-improved current [23]. The said current is conserved on-shell but does not satisfy all of the properties that the previous current (that coincides with the linearized Riemann tensor). That -biform current satisfies all of the following:

(4.7)

(4.8)

(4.9)

while the gauge-improved current satisfies only the on-shell conservation like (4.7), but the other two conditions are violated. Upon trying to enforce as many of the conditions as possible to the improved current (via choices of some free parameters), we come across a *nontrivial mixed 't Hooft anomaly*. It arises because not all of the conservation conditions

(4.7)-(4.9) can be satisfied. Moreover, *the case of dimensions is special* due to even fewer possible conditions [23].

The presented theory has a conserved -biform current and the appropriate dual -current in dimensions[12], that is obtained by acting on the with both the "left" and the "right" Hodge dual. For such theories, an equivalent of the Goldstone theorem can be made [23]: there must be a gapless spin- excitation in the spectrum. Any theory with the conserved current of the discussed form with a mixed anomaly will be in a gapless phase where the massless degree of freedom has spin . The Goldstone-like theorem is developed by decomposing the current two-point function using the Kähllén-Lehmann spectral representation. The gapless spectrum of the theory is read off by matching the spectral density to the correlator of the two currents which is completely fixed by the symmetry structure and the anomalies. The conclusion is that, as a consequence of the symmetry (4.3), there is a massless spin- particle in the spectrum - the graviton.

4.2.4 The Most Recent Developments

Fairly recently, there has been some further development in the discussion around the graviton as a Nambu-Goldstone boson. In [24] linearized gravity with a global symmetry under which the graviton is shifted is again considered, however, the symmetry transformation that was used was the one given with (4.1) and (4.3) was interpreted just as a special case. It is, therefore, argued that the symmetry given by (4.3) is not the most general symmetry. Also, a mixed 't Hooft anomaly is obtained in the discussion of gauging the global symmetry simultaneously with gauging the dual symmetry - associated with a global shift of the dual graviton. What follows is a brief look into the basic concepts of this work.

As was discussed in 4.2.1, the Fierz-Pauli action (4.3) with the mass term set to zero can be written as:

$$-$$

where is the linearised Einstein tensor. The action (4.3) is, as said earlier, invariant under the gauge symmetries of the graviton:

[12] In -dimensions, this would be a -current.

Given the language from chapter 4.2.2 and the fact that graviton can be viewed as a -biform, the connection can be written as [24]:

Analogously, the Riemann tensor from (4.4) can be rewritten:

The variation of the action yields the following equation of motion:

as expected. When considering a shift (4.1) transformation of the graviton:

where is a -biform, we look into the variation of the action under the transformation. The term with vanishes due to the equation of motion, and what is left is:

For the action to be invariant under (4.1), must satisfy:

(4.10)

i.e., for the transformation (4.1) to be a global symmetry of the theory. Using an appropriate extension of Noether's arguments [24], a -biform current can be constructed:

that is left-conserved on-shell:

and right-conserved off-shell:

$$d_R^\dagger J(h) = 0 \,.$$

The current is considered to be Noether-like and is associated with the continuous global symmetry given with (4.1) and is analogous to the electric 1-form symmetry in Maxwell theory that shifts the photon, but it is not gauge-invariant (it is not a well-defined local operator). However, a gauge-invariant object can be constructed using this knowledge - the on-shell conservation of the Noether gauge-invariant current R should follow from that of $J(h)$. Noether current R is given as:

$$R(h)_{\mu\nu}{}^{\rho\sigma} = 2\left(\partial^{[\rho} J(h)_{\mu\nu|}{}^{\sigma]} - \frac{2}{n-2}\delta^{\rho}_{[\mu} J(h)_{\nu]\alpha|}{}^{\alpha}\right) \,.$$

In the case of a conserved p-form current, its integration over a p-cycle gives a topological operator, but the procedure is more subtle when it comes to biform currents. It must first be contracted with a suitable tensor to provide a conserved p-form current that can lead to a topological operator. In the case of a codimension-1 topological operator that generates the 0-form symmetry (4.1) where α satisfies (4.10), the associated 1-form Noether current $j(b)$ is of the following form:

$$j(b)_\alpha = -3\delta^{\nu\gamma\delta}_{\mu\alpha\beta}\left(b^\mu{}_\nu \partial_\gamma h^\beta{}_\delta - \partial_\gamma b^\mu{}_\nu h^\beta{}_\delta\right)$$

The codimension-1 topological operator (charge) which generates this shift is:

$$Q(b) = \int_{\Sigma_{n-1}} *j(b) \,.$$

In the case of $b_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)}$, the associated charges $Q(b)$ vanish on-shell, as is expected for the gauge transformation. If we go back to the particular shift (4.3), it necessarily satisfies:

$$\partial^\mu b_{\mu\nu} = 0 \,,$$

and produces the gauge-invariant Noether current (that coincides with the Riemann tensor). However, that is not the most general global shift symmetry - for the developments laid out here, a much weaker constraint is necessary:

$$G(b)_{\mu\nu} = 0 \,.$$

Although the gauge-invariant current had to be further developed, the symmetry considered here is more general. Furthermore, in [24] the dual graviton is considered and a mixed 't Hooft anomaly is obtained. The graviton is successfully presented as a Nambu-Goldstone mode of this, more general shift symmetry.

5 Conclusion and Outlook

Because they lead to conservation laws, continuous global symmetries play an important role in physics. As research of higher-form gauge fields became usual in physics and mathematics, a generalization of symmetry principles to objects of higher dimensions was needed. A -form global symmetry leads to a -form conserved current and a conserved charge that has spatial dimensions, meaning it obeys Noether's theorem. Higher-form symmetries present a new organized way to think about symmetry principles and open possibilities to further developments.

Having discussed generalized global symmetries through an example of the free Maxwell theory - a free gauge theory with two -form global symmetries , we have also found the dual formulation of the theory with a matching 't Hooft anomaly which presents an obstruction to gauging. The obtained charges are the Wilson and the 't Hooft loops that interchange when we move from one theory to its dual. The process of finding the dual theory is later extended to non-linear electrodynamics. The higher-form symmetries can also be understood through a topological aspect: they are implemented using a topological operator and are represented in section 2.2.5 on the example of free Maxwell theory. Furthermore, we have revisited spontaneous symmetry breaking in the case of the -form symmetry that leads to the photon as a Nambu-Goldstone boson.

Besides the higher-form symmetries, other developments have been made in recent years, including higher-group symmetries that were discussed here through the -group symmetries. A -group symmetry is a global symmetry where the mixing of background gauge fields under their respective gauge transformations is involved. The notion of such symmetries was described here using the simplest case: a theory with abelian - group symmetry . We have presented how such symmetry arises from an ordinary product symmetry by promoting the background gauge field to a dynamical one. In our attempt to do so, we have analyzed the anomalies that occurred. Non-invertible symmetries present another generalization of the symmetry principles where the topological operator that implements the symmetry does not have an inverse.

The main part of the work focuses on implementing the generalized symmetries in linearized gravity. The treatment of the graviton as a Nambu-Goldstone boson is key because it situates the graviton, a massless spin-2 particle, within the framework of spontaneous symmetry breaking. Traditional Goldstone theorems, well-established for lower spin particles, are generalized here to include spin-2 particles, fundamentally altering the way gravitational interactions can be understood at the quantum level. The notion that a massless particle like the graviton arises due to a broken biform symmetry is a bold and insightful interpretation. Tensor gauge fields are higher-rank fields that generalize the structure of ordinary gauge theories, and through the use of linearized gravity, it is shown how these fields naturally support the graviton's interpretation as a Nambu-Goldstone boson. Importantly, the diffeomorphisms of Einstein's theory, when linearized, lead to a gauge theory that fits neatly within this framework, indicating that gravity might be another example of generalized symmetry principles. The core of this interpretation lies in the introduction of biform symmetries. These symmetries, which extend the concept of higher-form symmetries, become critical in explaining the graviton's massless nature. In particular, the shift symmetries acting on the graviton field are identified with a specific biform symmetry. We have shown how Fierz-Pauli action, a well-known framework for spin- particles, is not only invariant under linearized diffeomorphisms but also displays an additional symmetry that can be associated with a conserved Noether current. In chapter 4.2.3, we have shown this for a particular shift, whereas the most recent developments reflect how the graviton can be viewed as a Nambu-Goldstone boson for a more general shift symmetry. Furthermore, the notion of anomalies is crucial in this discussion. The work examines the role of 't Hooft anomalies in constraining the possible symmetry structures in theories involving the graviton. The existence of mixed anomalies between the gravitational and dual symmetries is discussed, offering insight into how symmetries may be gauged and the implications for gravitational theories. The 't Hooft anomaly is crucial in determining the graviton as a Nambu-Goldstone mode of the theory since it fixes the correlation function of the dual currents.

Further development can be made in making a case for the graviton being the Nambu-Goldstone mode of the more general shift symmetry. One idea is to build a theory from "the bottom". This could potentially be done by starting from a massive mixed-symmetry tensor [31] that is decomposed using the Stueckelberg [32] procedure to a massless -form a massless -form, massless -form, and a massless -form. When such decomposition is included in the Curtright action [33], it transforms it to a sum of the massless Curtright action, the massless Fierz-Pauli action, an additional term with the massless -form field, and some mixing terms. Considering certain transformations (including the shift of the -form) of these fields might lead to a new and more general approach to finding the graviton to be a Nambu-Goldstone boson.

Appendices

Appendix A Mathematical tools

This work involves the application of differential geometry. For the reader's convenience, the essential concepts and foundational principles are reviewed in this Appendix, following [34] and [30]. A -form (differential form) is a totally antisymmetric tensor field. For a more intuitive approach, a -form is a function and a -form is a covector. Generally, the antisymmetry of -forms has the following consequence: there cannot be any form of degree higher than the dimension of the manifold on which the form is defined. Note that an -form is often referred to as *top form*, or *volume form*.

Let be a -form and a -form. The *exterior product* or *wedge product* is a construction of a -tensor:

———

via the tensor product that is antisymmetrized to ensure some properties, such as if is an odd-degree form. It can also be shown that:

To build some intuition, we can take a look at a special case where , meaning we take and to both be -forms and . It is easy to show what their exterior product is in terms of the tensor product :

The *exterior derivative* (is a map that) in local coordinates, acts as:

d — ——— (A.1)

where is a -form. The exterior derivative defined by (A.1) returns a -form. One can think of the exterior derivative as an antisymmetric covariant derivative.

It is important to note that, due to the antisymmetry, if we act on equation (A.1) with the exterior derivative again, we obtain:

$$\mathrm{d}\left(\mathrm{d}\omega\right) = 0 \; ,$$

which is true for every q-form and often written in the following form:

$$\mathrm{d}^2 = 0 \; .$$

The exterior derivative of a wedge product for ω and η of degrees as given before, is:

$$\mathrm{d}\left(\omega \wedge \eta\right) = \left(\mathrm{d}\omega\right) \wedge \eta + (-1)^q \, \omega \wedge \left(\mathrm{d}\eta\right) \quad .$$

The *Hodge dual* is a map that takes a q-form ω to an $(n-q)$-form, denoted $*\omega$ as follows:

$$(*\omega)_{\mu_1 \ldots \mu_{n-q}} = \frac{1}{q!}\sqrt{|g|}\epsilon_{\mu_1 \ldots \mu_{n-q}\nu_1 \ldots \nu_q}\omega^{\nu_1 \ldots \nu_q} \; .$$

Note that the Hodge dual is independent of the choice of coordinates. Using the Hodge dual, an inner product of two q-forms ω and η is defined:

$$\langle \eta, \omega \rangle = \int_M \eta \wedge *\omega \; ,$$

which shows how the dimension of $\eta \wedge *\omega$ is equal to the dimension n of the manifold M. To get a better picture of the Hodge star operator, one should keep in mind that it gives the part of the manifold orthogonal to the differential form that it is acting on. A common 3D example follows:

$$*dx = dy \wedge dz \; ,$$

which provides a better insight into the way that Hodge dual acts.

6 Prošireni sažetak

6.1 Uvod: Kontekst i motivacija

Jedan od središnjih koncepata moderne teorijske fizike zasigurno je simetrija - *svojstvo fizikalnog sustava koje ostaje nepromijenjeno primjenom neke transformacije*. Simetrije su od velike važnosti u fizici, ponajviše zbog Noetherinog teorema (koji će biti proučen kasnije) koji pokazuje kako, za svaku kontinuiranu globalnu simetriju postoji odgovarajući zakon očuvanja. Invarijantnosti na prostorne i vremenske transformacije bile su poznate i u klasičnoj mehanici, kao što su i globalne simetrije prostorvremena izvedene za elektrodinamiku prije Einsteinove teorije relativnosti. Ipak, potonje predstavlja novi pristup primjeni simetrija u fizici budući da je, suprotno onima prije njega, Einstein izveo zakone iz simetrija. Značaj simetrija u fizici postao je brzo jasan u kvantnoj mehanici gdje je primjena teorije grupa i njihovih reprezentacija imala ključnu ulogu.

Potreba za razvojem univerzalnog alata za primjenu simetrija postala je primjetna u kvantnoj teoriji polja kako je izučavanje baždarnih polja viših formi postalo standardno u matematici i fizici. Generaliziranje globalnih simetrija je, grubo govoreći, primjena koncepta na objekte viših dimenzija. Takve generalizirane globalne simetrije[**2**] pokazale su se korisnima u teoriji struna i fizici čvrstog stanja te imaju primjenu u proučavanju proširenih operatora, defekata i struktura anomalija u kvantnoj teoriji polja. Nedavno su postale tema rasprava i suradnji različitih područja teorijske fizike budući da daju nov i organiziran jezik za koncepte simetrija.

Kako je rečeno, simetrija je svojstvo fizikalnog sustava koje je očuvano pod nekim transformacijama. Familija takvih transformacija može se opisati upotrebom grupa - Lijevih grupa za kontinuirane simetrije i konačnim grupama za diskretne simetrije. Kontinuirane i diskretne simetrije odgovaraju kontinuiranim i diskretnim transformacijama, respektivno. Među mnogim ostalim podjelama simetrija, valja napomenuti razlikovanje eksternih i internih simetrija gdje se eksterne odnose na simetrije prostorvremena, a interne simetrije odgovaraju unutarnjim stupnjevima slobode teorije. Svakako, za naše daljnje razmatranje, od najveće će važnosti biti razlučivanje lokalnih od globalnih simetrija. Globalne simetrije ostavljaju svojstvo invarijantnim za transformacije koje su primijenjene podudarno u svim točkama prostorvremena, za razliku od lokalnih simetrija koje predstavljaju svojstva invarijantna na transformacije parametrizirane koordinatama prostorvremena. Lokalne simetrije temelj su baždarnih teorija polja, to jest, baždarna teorija predstavljena je gustoćom lagranžijana invarijantnom na glatku familiju operacija. Budući da su baždarna polja (čije su vrijednosti u Lievoj algebri baždarne grupe) uključena u gustoću lagranžijana kako bi osigurali njenu baždarnu invarijantnost, baždarne teorije imaju dodatne, odnosno suvišne stupnjeve slobode. Primjerice, foton, koji ima dvije fizikalne polarizacije, prikazan je baždarnim poljem koje u relativističkoj formulaciji ima četiri komponente. Standardni model, jedna od najuspješnijih fizikalnih teorija, temeljena je na baždarnim simetrijama.

Izuzev svrhovitosti generalizacije simetrija, valja motivirati i kombiniranje istih s tenzorskim baždarnim teorijiama u jedinstvenu raspravu. Važna uloga vektorskih polja u fizici, npr. u elektromagnetizmu, dobro je poznata. U nešto "naprednijim" teorijama koje zahtijevaju objekte viših dimenzija, odgovarajuća generalizacija je često u upotrebi: tenzorska polja, koja dopuštaju kompleksnije interakcije. Ako su takvi objekti korišteni u teoriji, obične simetrije više nisu dostatne za opis svih svojstava teorije. Prirodno je, stoga, razmišljati o generaliziranim simetrijama i teorijama koje sadrže tenzore kao međusobno poveznaim. Ovo se posebno ističe u centralnom dijelu i cilju rada: prikaz gravitona u lineariziranoj gravitaciji (tenzorskoj baždarnoj teoriji) kao posljedica spontanog loma generalizirane globalne simetrije.

6.2 Generalizirane globalne simetrije: simetrije viših formi

6.2.1 Obične globalne simetrije

Noetherin teorem[1] pokazuje da, za svaku kontinuiranu globalnu simetriju, postoji odgovarajuća *očuvana struja* dana s:

koja daje pripadajući očuvani naboj:

Noetherin teorem među glavnim je razlozima važnosti simetrija u fizici.

6.2.2 Simetrije viših formi

Generalizacija simetrija otvara mnogo mogućnosti pa su tako neke od generalizacija simetrijskih principa: simetrije viših formi, simetrije više-grupe i neinvertibilne simetrije. Premda su temeljni koncepti navedenih primjera obrađeni u radu, poseban naglasak je upravo an simetrijama viših formi, budući da će se u konačnici graviton prikazati kao posljedica spontanog loma takve simetrije.

Generalizirana globalna simetrija -forme [3] globalna je simetrija za koju je očuvana struja -forma, a očuvani su naboji dimenzije . U ovom kontekstu, globalne simetrije -forme su obične globalne simetrije opisane ranije. Mnoga svojstva globalnih simetrija -forme mogu se primijeniti i ovdje. Ove generalizirane globalne simetrije nisu neke egzotične generalizacije u kompliciranim teorijama, već se pojavljuju vrlo prirodno u baždarnim teorijama. Da bismo predstavili generalizirane globalne simetrije, nećemo koristiti konkretnu gustoću lagranžijana, nego okarakterizirati objekte kao apstraktne operatore, čineći pritom izlaganje općenitim.

Ako promotrimo simetriju proizašlu iz očuvane struje koja je -forma i zadovoljava:

d

očuvani naboji su dani, onda, s:

Nabijeni objekti su za ove simetrije -dimenzionalni. Na primjer, u najjednostavnijem slučaju , nabijeni objekti linijski su operatori poput ’t Hooftovih i Wilsonovih linija. Ovo obrazlaže kako ove simetrije nisu nešto neobično, već su prisutne u bilo kojoj teoriji koja ima proširene opservable poput Wilsonovih petlji.

Klasičan izvor za struju je abelovsko polje koje je -forma. Akcija treba sadržavati sljedeći član:

Pod transformacijom, baždarno polje se treba tansformirati kako slijedi:

d

gdje je baždarni parametar -forma. U svrhu boljeg razumijevanja ovakve generalizacije, slijedi primjer slobodne Maxwellove teorije.

6.2.3 Slobodna Maxwellova teorija

Uzmimo baždarnu teoriju s baždarnim poljem i odgovarajućom jakosti polja te dvjema globalnim simetrijama -forme: "električna" i "magnetska" s pripadnim pozadinskim poljima i [6].

Za struje definirane kao:

—

—

dobivamo odgovarajuće zakone očuvanja koristimo li Maxwellove jednadžbe bez izvora:

d d (6.1)

Maxwellove jednadžbe opisane sa (6.1), svakako su povezane s poznatim oblikom Maxwellovih jednadžbi: d je pridružena Gaussovom zakonu i Ampèreovom zakonu bez izvora, dok d odgovara Gaussovom zakonu za magnetizam i Faradayevom zakonu. Primijenimo li vanjsku derivaciju na definirane struje i uvrstimo li Maxwellove jednadžbe, lako se pokaže da su struje i očuvane. Dakle, gradimo teoriju dinamičkog baždarnog polja u okruženju dvaju (nedinamičkih) baždarnih polja koja se, zbog svojih simetrija podvrgavaju baždarnim transformacijama:

d

Pisanju akcije za ovu teoriju, odnosno za vezanje dinamičkog polja za dva pozadinska baždarna polja, možemo pristupiti na dva načina: "električna" formulacija i "magnetska" formulacija. Ta dva pristupa zasigurno moraju biti ekvivalentna, to jest, pokazivati dualnost teorije, što može biti korisno. Promotrit ćemo prvo "električni" pristup: baždarno polje transformira se pod pozadinskom transformacijom, ali ostaje isto pod pozadinskom transformacijom. Budući da je d , isto vrijedi i za jakost polja :

d

Vežemo li baždarno polje za pozadinska baždarna polja, očekujemo da će akcija imati tipične članove , odnosno:

(6.2)

U prethodnoj jednadžbi, također smo koristili činjenicu da želimo teoriju invarijantnu na električne pozadinske transformacije. Prvi član to osigurava jer se, pri pozadinskoj transformaciji, d dodaje i jakosti polja i polju .

Lako je provjeriti da se, nakon transformacije, d dobiven iz transformacije jakosti polja i d koji se pojavljuje pri transformaciji pozadinskog polja poništavaju, zato što ulaze s obrnutim predznakom u akciju. Prvi član se često naziva kinetičkim, a drugi član magnetskim. Kako se ni ni ne transformiraju pod pozadinskim transformacijama, prvi je član također invarijantan na magnetske pozadinske transformacije. Preostalo je samo osigurati invarijantnost drugog člana. Drugi član u (6.2) invarijantan je na magnetske pozadinske transformacije, što se može lako utvrditi korištenjem Maxwellovih jednadžbi danih sa (6.1), svojstva vanjske derivacije [34] i vanjskog produkta te Stokesovog teorema:

d d

d d

Ipak, ovaj (magnetski) član ima otklon prilikom pozadinske transformacije -upotrijebimo ponovno matematičke manipulacije i Stokesov teorem kako bismo vidjeli točno kakav.

d

d

d d

d

Član dan s d reproducira *'t Hooftovu anomaliju* [5] između i . Treba naglasiti da se, ako nema 't Hooftovih anomalija, teorija može baždariti što u suprotnom nije moguće. Poradi boljeg razumijevanja daljnjih implikacija slijedi pregled povezanih pojmova.

6.2.4 Anomalije, Wilsonove i 't Hooftove petlje

Prilikom pokušaja kvantizacije teorije s globalnom simetrijom, može se pojaviti *anomalija*. Za sljedeća poglavlja, razumijevanje anomalija u kvantnoj teoriji polja bit će nužno. Grubo rečeno, anomalija je klasična simetrija koja ne opstaje kad je teorija kvantizirana. Neke se anomalije mogu poništiti dodavanjem članova u akciju. Nama će najvažnije biti 't Hooftove anomalije koje predstavljaju prepreku baždarenju globalne simetrije. Globalna simetrija s 't Hooftovom anomalijom ostaje simetrija u kvantnoj teoriji, ali, kad se simetrija veže za pozadinsko baždarno polje, naboji koji su ranije bili očuvani više to nisu.

Anomalija [6] član je (efektivne [1]) akcije koji se u njoj pojavljuje kao otklon i koji odgovara zakonu neočuvanja. Ovdje smo koristili notaciju za pozadinska baždarna polja. Anomalija je obično prikazana polinomom anomalije koji je baždarno invarijantna -forma, gdje je broj dimenzija. Dakle, pozadinska baždarna polja i njihove baždarne transformacije prošireni su na dimenzije. Relacija i , kao i odnosi ostalih

[1] Anomalije se prepoznaju u akciji , ali se odnosi na promjenu u efektivnoj akciji definiranoj s , gdje je particijska funckija.

polinoma koji predstavljaju postupak za ekstenziju do dani su s:

(6.3)

d (6.4)

d (6.5)

Procedura, naravno, može biti korištena u obama smjerovima.

Nadalje, pogledajmo definiciju *Wilsonovih petlji* i *'t Hooftovih petlji* [5] budući da ćemo ih u idućem poglavlju koristiti. Wilsonova *linija* objekt je koji govori kako se kompleksni vektor (kojega nosi čestica) pomiče po mnogostrukosti s konekcijom (baždarnim poljem izvrijednjenim u Liejevoj algebri):

Ovdje, predstavlja uređenje po putevima, dok su i početna i konačna točka gibanja čestice. U matematici se ovaj pojam naziva holonomijom.

Wilsonova petlja je baždarno invarijantni objekt, opservabla, definirana kao trag Wilsonove linije po zatvorenoj putanji :

tr (6.6)

't Hooftova petlja također je opservabla, slična Wilsonovoj petlji i s njom povezana kako je pokazano niže.

(6.7)

U izrazu (6.7), je element središta baždarne grupe, a je Gaussov vezni broj između dviju prostornih petlji. Budući da su obje petlje opservable, ovi objekti su od velike važnosti, osobito u neabelovskim teorijama (poput Yang-Millsove teorije) gdje električno i magnetsko polje nisu opservable.

6.2.5 "Električno-magnetska" dualnost

Kako je najavljeno ranije, anomalije se obično prikazuju polinomom koji je -forma, to jest, polinomom koji je -forma. Koristeći postupak opisan sa (6.3) - (6.5), kao i, još jednom, svojstva vanjske derivacije i vanjskog produkta, te Stokesov teorem, izveden je sljedeći izraz za anomaliju dobivenu od drugog člana u (6.2):

$$— \quad d \tag{6.8}$$

Dualizaciju postižemo promatranjem proširene teorije s akcijom koja uključuje Lagrangeov multiplikator koji je također baždarno polje (-forma) pridruženo vlastitoj baždarnoj simetriji.

$$— \quad d \tag{6.9}$$

Treba primijetiti da je Bianchijev identitet za i dalje zadovoljen. Prikladna transformacija polja pri pozadinskim baždarnim transformacijama:

osigurava invarijantnost na pozadinske baždarne transformacije do na ’t Hooftovu anomaliju dobivenu ranije.

Sada želimo naći jednadžbu gibanja za kako bi akcija ovisila samo o , i novom polju . Drugim riječima, želimo "izgubiti" ovisnost akcije o i zamijeniti je s ovisnosti o . Jednadžba gibanja za dobivena je variranjem po , kao što je pokazano sljedećom jednadžbom:

$$— \qquad —$$

$$— \quad d$$

Zbog simetrije djelovanja , drugi član može se zamijeniti s — , a zbog stupnjevane komutativnosti, posljednji se član može zapisati kao — d .

Prethodno rečeno znači da se može izvući iz svih članova kako slijedi:

— — d (6.10)

a jednadžba gibanja za dobiva se kad je izraz u uglatim zagradama jednak nuli:

— d (6.11)

Budući da je sada lako izraziti iz prethodne jednadžbe (6.11), taj se izraz može uključiti u (6.9). Dvojna formulacija teorije s akcijom , zapisane preko , i je, stoga, izvedena.

— d d

— d (6.12)

Jednadžba (6.12) pokazuje da dualna formulacija generira član proporcionalan s koji će reproducirati istu ’t Hooftovu anomaliju kao (6.8). Očuvane struje unutar ove teorije dane su sljedećim jednadžbama i povezane s očuvanima strujama definiranim u "električnoj" formulaciji, kako je pokazano ovdje.

— d —d (6.13)

Holonomije polja i polja po zatvorenoj krivulji su Wilsonove petlje i ’t Hooftove petlje , definirane s (6.6) i (6.7) te se za njih dobiva:

(6.14)

gdje su naboji Wilsonovih, to jest, ’t Hooftovih petlji. Prijelazom iz jedne u drugu formulaciju, izmjenjujemo polja , ali i petlje . Ovime su temeljna svojstva i upotreba koju simetrije viših formi nude na primjeru slobodne Maxwellove teorije zaključena.

6.2.6 Topološki aspekti

Simetrije viših formi mogu se uvesti putem topoloških operatora. Simetrija -forme u dimenzija implementira se operatorom povezanim s zatvorenom mnogostrukošću :

gdje je element simetrijske grupe. Simetrija je transformacija prilikom koje se mnogostrukost može deformirati bez promjene korelacijskih funkcija, jer ovise samo o topologiji [3]. Iako se simetrije viših formi mogu uvesti bez spominjanja topoloških operatora, ovaj formalizam je koristan, posebno kod drugih generalizacija poput neinvertibilnih simetrija.

U kvantnoj mehanici operator je operator simetrije ako komutira s hamiltonijanom :

Prelaskom u kvantnu teoriju polja, intuitivno je proširiti ovaj uvjet na komutiranje s tenzorom energije i impulsa , no to je nedovoljno. U relativističkoj teoriji polja s euklidskom signaturom očuvanje podrazumijeva invarijantnost pod deformacijama prostorvremena. To znači da operator mora biti *topološki*. Dobar primjer je teorija s simetrijom, gdje struja zadovoljava:

a operator naboja je:

Pod pretpostavkom odgovarajućih rubnih uvjeta, očuvanje naboja daje:

Operator simetrije za rotaciju s kutom dan je s:

a uključivanjem izraza za Q, dobivamo:

$$U_\theta = e^{i\theta \oint j_0, d^{n-1}x} .$$

Generalizacija ovog principa je:

$$U_\theta \left(M^{(n-1)} \right) = \exp \left(i\theta \oint_{M^{(n-1)}} *j \right) ,$$

gdje je $M^{(n-1)}$ zatvorena $(n-1)$-dimenzionalna mnogostrukost. Funkcije korelacije koje sadrže $U_\theta(M^{(n-1)})$ ne ovise o malim deformacijama $M^{(n-1)}$ zbog Stokeosovog teorema, jer je $*j$ zatvorena forma.

Simetrija više forme može se općenito prikazati $(n-q-1)$-dimenzionalnim topološkim operatorom:

$$U_g \left(M^{(n-q-1)} \right) \cdot D(N^q) = g(D) D(N^q) .$$

Primjer iz slobodne Maxwellove teorije pokazuje topološki operator:

$$U_\theta \left(M^{(n-2)} \right) = \exp \left(-\frac{\theta}{e^2} \oint_{M^{(n-2)}} *F \right) ,$$

koji generira očuvane naboje - netopološke Wilsonove operatore:

$$W = \exp \left(in \oint_{N^1} A \right) .$$

Topološki operator djeluje na Wilsonovu liniju kao:

$$U_\theta \left(M^{(n-2)} \right) \cdot \exp \left(in \oint_{N^1} A \right) = e^{i\theta} \exp \left(in \oint_{N^1} A \right) .$$

Generalizacija, dakle, topološkog aspekta simetrija vrlo je intuitivna.

6.2.7 Spontani lom generalizirane globalne simetrije

Spontano narušavanje simetrije -forme može se dijagnosticirati ponašanjem Wilsonovih petlji (operatora) teorije. U tom kontekstu, *zakon površine* [13] opisuje skaliranje vrijednosti Wilsonove petlje na sljedeći način:

area

a zakonom "opsega" [14] nazivamo ponašanje dano s:

perimeter

Još je jedno tipično ponašanje koje nazivamo *Coulombovim zakonom*:

–

Ako je u danoj teoriji sa simetrijom povezana Wilsonova petlja koja poštuje "zakon površine", onda ona ostaje neslomljena. Ovo vrijedi zbog toga što očekivana vrijednost petlje nestaje kako njena veličina ide prema beskonačnosti. Suprotno, kada veličina petlje ide u beskonačnost, petlje koje poštuju "zakon opsega" ili "Coulombov zakon" imaju vrijednost različitu od nule. Zbog toga je u takvim slučajevima *simetrija spontano slomljena*.

U slobodnoj Maxwellovoj teoriji, i simetrije su spontano narušene, s fotonom kao Nambu-Goldstoneovim[8] [7] bozonom.

6.3 Generalizirane globalne simetrije: Daljnji razvoj

6.3.1 Simetrije -grupe

Premda su simetrije viših formi temelj generalizacije načela simetrije, postoje i druge generalizacije, među kojima su simetrije više-grupa koje ćemo predstaviti kroz osnove -grupnih simetrija na primjeru Maxwellove teorije. Koncept se može smatrati generalizacijom simetrije opisane produktom grupa.

Kvantna teorija polja ima -grupnu simetriju [6] ako se može povezati s pozadinskim baždarnim poljem koje je -forma (označeno kao), a koje doživljava -grupni otklon uz vlastite pozadinske baždarne transformacije. Drugim riječima, to su globalne

[13] eng., "area law"
[14] eng., "perimeter law"

simetrije gdje je dopušteno miješanje pozadinskih baždarnih polja pod njihovim odgovarajućim baždarnim transformacijama. Same -grupe se neće istraživati, već pozadinska baždarna polja -grupe. Najjednostavniji primjer uzima miješanje pozadinskog baždarnog polja za -formnu flavor simetriju, tj. , i -formno pozadinsko baždarno polje za ranije spomenutu simetriju -forme.

Takva se -grupna simetrija naziva Abelovom i označava se kako slijedi:

(6.15)

Ovdje, [6] je strukturna konstanta -grupe koja karakterizira -grupnu simetriju. Kako bismo vidjeli što znači, trebamo razmotriti pravila transformacije za baždarna polja i . Pravilo transformacije za ostaje standardno:

d

ali, kao što je rečeno, podliježe dodatnom otklonu:

d — (6.16)

U prethodnom izrazu, d je jakost polja. Dosljednost pravila transformacije danog sa (6.16) osigurana je time da je kvantiziran. Treba spomenuti da karakterizira -grupnu simetriju jer se ne mijenja reskaliranjem baždarnih polja. To je vidljivo u jednadžbi (6.16), gdje postoji dodatni otklon proporcionalan jakosti polja . Ovo pokazuje da ne možemo trivijalno postaviti profil baždarnog polja bez da to utječe na . Treba prepoznati da za , -grupni otklon u (6.16) nestaje. Stoga, -grupna simetrija prelazi u običnu produktnu simetriju:

Mnoge kvantne teorije polja posjeduju -grupnu simetriju danu sa (6.15), poput QED-a s više okusa[6]. Može se pokazati da -grupna simetrija opisana s (6.15) proizlazi iz "matične" teorije s:

(6.17)

okusnom simetrijom, pri čemu je odgovarajuće pozadinsko baždarno polje čijim baždarenjem se "rađa" simetrija dana sa (6.15) [6].

6.3.2 Neinvertibilne simetrije

Kao što je ranije raspravljano, zanimljivi se fenomeni pojavljuju kada se principi simetrije poopće, a među njima su i *neinvertibilne simetrije* [4]. Globalne simetrije u kvantnoj mehanici opisuju se (anti-)unitarnim operatorima koji imaju inverze, dok u relativističkim kvantnim teorijama polja simetrije mogu biti neinvertibilne, odnosno implementirane operatorima koji nemaju inverze. Ove simetrije dovode do novih zakona očuvanja i pravila "fuzije"[15]

Među osnovnim konceptima koje vežemo uz neinvertibilne simetrije je *topološki defekt* - lokalizirana smetnja u redu sustava koja se javlja uslijed topoloških ograničenja. Topološki defekti ostaju stabilni ako nisu ometani drugim defektima. Fromalno, javljaju se kada nije moguće deformirati konfiguraciju polja u trivijalnu zbog netrivijalnih elemenata grupe *homotopije* , pri čemu je mnogostrukost. Topološki defekti su defekti čije infinitezimalne deformacije u vremenu ne mijenjaju fizičke opservable.

Ulogu simetrijskog operatora i odgovarajućeg topološkog defekta zgodno je upoznati na primjeru simetrije. Ako je cijeli prostor u fiksnom vremenu, je očuvan i djeluje na odgovarajući Hilbertov prostor . Kada se proširi u vremenskom smjeru i lokalizira u jednoj prostornoj koordinati, kao što je , postaje defekt koji modificira kvantizaciju rezultirajući s *"uvijenim"* [16] *Hilbertovim prostorom* , označenim kutnom rotacijom .

Razmotrimo slobodno kompleksno skalarno polje u -dimenzijama (tj.):

Teorija posjeduje globalnu simetriju:

s pripadajućom očuvanom strujom:

[15]Operatori ovakvih simetrija ne poštuju grupno pravilo umnoška, već je ono modificirano npr. u -dimenzionalnom prostorvremenu na sljedeći način: .

[16]eng., *twisted*

Ako se za prostor uzme kružnica parametrizirana s , tada se odgovarajući Hilbertov prostor dobiva kanoničkom kvantizacijom slobodnog skalarnog polja pod uvjetom periodičnosti:

Očuvana struja vodi na simetrijski unitarni operator:

koji djeluje na navedeni Hilbertov prostor .

Ono što je zanimljivo jest da se može uzeti alternativan, ali ekvivalentan pristup - umetanje defekta:

po liniji euklidskog vremena u mijenjajući pritom rubni uvjet u:

(6.18)

Kanonska kvantizacija s "uvijenim" rubnim uvjetom (6.18) sada daje "uvijeni" Hilbertov prostor , kako je najavljeno, označen s . Drugim riječima, očuvanu struju možemo povezati s unitarnim operatorom kad se odnosi na fiksirano vrijeme ili unijeti topološki defekt po euklidskom vremenu u .

Kod diskretnih simetrija , ne postoji očuvana struja ili operator naboja, ali simetrija se može formulirati postojanjem očuvanog unitarnog operatora za svaki element grupe . U relativističkim kvantnim teorijama polja, očuvani operatori se generaliziraju na topološke operatore, tj. defekte u euklidskom prostoru. Svaka globalna simetrija dovodi do očuvanog *operatora* koji djeluje na Hilbertov prostor i do topološkog *defekta* koji modificira kvantizaciju. Operator i defekt obuhvaćeni su istim objektom . Neinvertibilne su simetrije, dakle, transformacije koje će ostaviti lagranžijan nepromijenjen i povezane su s očuvanim veličinama, ali inverzna transformacija se ne može primijeniti kako bi se sustav vratio u prvotno stanje - prvotnu topološku fazu. Iako je proces ireverzibilan, ne treba ga miješati s drugim ireverzibilnim procesima u fizici poput onih opisanim histerezom.

Zaključno, kako operator simetrije nema inverza, simetrija je neinvertibilna. Iz druge, ali ekvivalentne perspektive, transformacija koja "prelazi" defekt je neinvertibilna, dok simetrijske transformacije s "jedne strane" defekta imaju konvencionalnije ponašanje. U slučaju neinvertibilnih simetrija, "prelazak" defekta je neinvertibilan.

6.3.3 Više-baždarenje i kondenzacijski defekti

Najjednostavniji način za konstrukciju neinvertibilne simetrije jest uzeti linearnu kombinaciju dviju invertibilnih simetrija. Primjerice, definirajmo operator kao:

gdje je operator identiteta, a , što znači da ima simetriju. Iako je ovo jednostavan način konstrukcije neinvertibilne simetrije, nije previše zanimljiv jer se može napisati kao linearna kombinacija operatora iste dimenzionalnosti. Kreativniji pristupi uključuju zbrajanje topoloških defekata niže dimenzije duž netrivijalnih ciklusa višedimenzionalne mnogostrukosti. Ovaj postupak stvara *defekt kondenzacije*, koji je mješavina topoloških defekata nižih dimenzija i ne može se napisati kao linearni spoj drugih defekata iste dimenzije.

Pojam *defekt kondenzacije* dolazi iz kondenzacije aniona duž jedne dimenzije u -dimenzionalnom prostoru [11], a kasnije se prepoznaje kao baždarenje simetrija viših formi duž mnogostrukosti više dimenzije. Ako nema 't Hooftovih anomalija za danu simetriju, ona se može baždariti na različite načine. Ponekad se diskretna globalna simetrija -forme može baždariti, ne u cijelom prostorvremenu, već samo duž podmnogostrukosti kodimenzije , . Ovo se naziva *više baždarenje*[17] (-baždarenje globalne simetrije -forme). Obično baždarenje je, stoga, -baždarenje -formne simetrije. Time je i proces baždarenja generaliziran.

6.4 Graviton kao Nambu-Goldstoneov bozon

6.4.1 Linearizirana gravitacija

Kako bismo prikazali graviton kao Nambu-Goldstoneov bozon u lineariziranoj gravitaciji, prvo trebamo promotriti osnove teorije. **Linearizirana gravitacija** pretpostavlja da je metrički tenzor zadan kao:

gdje je ravna metrika, a mala perturbacija. Tada je simetrični tenzor ranga

[17]eng., *higher gauging*

. Akcija za teoriju s u ravnom prostoru je tenzorska baždarna teorija. Baždarna simetrija naslijeđena je iz difeomorfizama u Einsteinovoj teoriji. Lineariziranjem Einsteinovih jednadžbi, gravitacija se opisuje kao simetrično polje spina , , koje propagira u prostoru Minkowskog. Indeksi se podižu i spuštaju ravnim metrikom:

Za Einsteinov tenzor u lineariziranoj gravitaciji dobiva se:

$$- \quad (6.19)$$

Bianchijev identitet se reducira na:

Iako su Einsteinove jednadžbe linearizirane, još uvijek su složene parcijalne diferencijalne jednadžbe. Jednadžbe gibanja dobivaju se iz varijacije **Fierz-Paulijeve**[30] akcije:

$$\frac{}{} \quad - \quad - \quad - \quad - \quad (6.20)$$

Akcija je invarijantna na transformaciju danu s:

Za daljnje razmatranje koristit će se Fierz-Paulijeva akcija napisana u nešto drukčijoj konvenciji:

$$- \quad (6.21)$$

gdje je Einsteinov tenzor dan s (6.19), no bez predfaktora. Linearizirani Riemannov tenzor prikazan je niže.

Svi ostali bitni rezultati dobivaju se iz navedenih.

6.4.2 Uvod u bifromne simetrije

Za prikaz simetrija tenzorskih teorija, uključujući linearaziranu gravitaciju, uvodi se pojam *biforme*. Biforma je tenzor miješane simetrije , što znači da je antisimetrična unutar svakog skupa indeksa, dok antisimetrizacija između setova rezultira nulom. Teorija sa sačuvanim strujama koje su -biforme posjeduje simetriju biforme[18].
Vanjske derivacije djeluju na svaki skup indekasa posebno, kao i :

Analogno se uvode i lijevi i desni Hodgeov dual, i i sve ostale operacije - tako da zasebno djeluju na svaki set indekasa. Graviton se može interpretirati kao Nambu-Goldstoneov bozon, tj. kao posljedica spontanog loma ovakve simetrije.

6.4.3 Graviton kao Nambu-Goldstoneov bozon

Fierz-Paulijva akcija iz (6.21) nije samo invarijantna na linearizirane difeomorfizme, već i na globalnu simetriju -biforme gdje se transformira na sljedeći način:

(6.22)

a je konstantan simetričan tenzor. Odgovarajuća Noetherina struja je -forma:

– (6.23)

koja je očuvana na ljusci mase[19], ali nije baždarno-invarijantna. Kako bismo interpretirali graviton kao Nambu-Goldstoneov bozon, trebamo razmotriti drugačiju transformaciju [23]. Ako napišemo kao:

(6.24)

Noetherina struja postaje Riemannov tenzor, koji je i baždarno-invarijantan i očuvan. Time gradimo sliku gravitona kao Nambu-Goldstoneovog bozona za ovu biformnu simetriju. Iako

[18]U tekstu se povremeno koristi i izraz "biformna simetrija" odnoseći se, pritom, na simetriju biforme.
[19]Divergencija -struje je linearizirani Einsteinov tenzor koji je jednak ako materija nije uključena

nije konstantan tenzor, to nije problem ako pripadajući Riemannov tenzor iščezava.

Kako bi se lakše ostvarila interpretacija gravitona kao Nambu-Goldstoneovog bozona za ovu simetriju (6.22), zgodno je zapisati akciju linearizirane gravitacije u "Palatini-stilu" [23]:

Ova akcija je analogna (6.21) pa također ima baždarnu simetriju koja je sada opisana sljedećim transformacijama:

Potonji zapis je posebno prikladan zbog toga što se u ovom formalizmu odvajaju transformacije (6.22) i (6.24) (zapisane preko drugih varijabli - linearizirane konekcije i polja). Odvajanje se dobiva razdvajanjem konekcije na sasvim simetrični dio i dio koji je - biforma i koji doprinosi Riemannovom tenzoru.

Baždarenjem simetrije (6.24) i uvođenjem odgovarajućeg baždarnog polja dolazi se do poboljšane struje koja je očuvana na ljusci mase, no ne zadovoljava sve uvjete koje zadovoljava struja :

(6.25)

(6.26)

(6.27)

Nezadovoljeni uvjeti vode do miješane 't Hooftove anomalije, a u dimenzije situacija je posebna zbog dodatnih ograničenja. Teorija s očuvanom -strujom i miješanom anomalijom mora imati bezmaseno pobuđenje sa spinom , što prema Goldstoneovom teoremu implicira postojanje gravitona kao bezmasene čestice. Ovaj analogon Goldstoneovom teoremu dobiva se promatranjem dekompozicije korelacijske funkcije struje i njzinog duala gdje se nalazi bezmaseni mod u potpunosti fiksiran uvjetima simetrije i 't Hooftovom anomalijom. Zaključak je da, kao posljedica simetrije (6.24), u spektru postoji bezmasena čestica spina - graviton.

Najnoviji radovi[24] pokazuju i općenitiji pristup, gdje se i za općenitu transformaciju (6.22) u lineariziranoj gravitaciji može konstruirati baždarno-invarijantan objekt počevši od Noetherine struje te pronaći bezmaseni mod. Drugim riječima, i u takvom slučaju postoji način za interpretaciju gravitona kao Nambu-Goldstoneovog bozona. Tada transformacija (6.24) predstavlja samo poseban slučaj u kojem je očuvana struja jednaka Riemannovom tenzoru.

6.5 Zaključak

Budući da vode na zakone očuvanja, kontinuirane globalne simetrije igraju važnu ulogu u fizici. Kako je istraživanje polja koja su više forme (diferencijalne -forme gdje je) postalo uobičajeno u fizici i matematici, tako je i potreba za poopćenjem načela simetrije na objekte viših dimenzija postala vidljiva. Globalna simetrija -forme dovodi do očuvane struje koja je -forma. Ovakvo poopćenje nudi organizirani način razmišljanja o principima simetrije te otvara mogućnosti za daljnji razvoj. Navedeno je raspravljeno na primjeru slobodne Maxwellove teorije s dvije globalne simetrije -formi i pokazano je formuliranje dualne teorije s ekvivalentnom ’t Hooftovom anomalijom. Simetrije viših formi mogu se razumjeti i kroz topološki aspekt: one se primjenjuju uvođenjem topološkog operatora. Kako on ne mora imati inverz, ovakav pristup omogućava i definiciju neinvertibilih simetrija.

Simetrije više-grupe mogu se shvatiti kao generalizacija produkta simetrija u kojem je dopušteno miješanje baždarnih polja prilikom njihovih baždarnih transformacija. Potonje je također prikazano na primjeru slobodne Maxwellove teorije.

Izuzev ovakvih generalizacija, i ostali koncepti vezani uz simetrije mogu se lako generalizirati, poput spontanog loma simetrije. On se prepoznaje na temelju ponašanja velikih Wilsonovih petlji dane teorije te je uočen u slobodnoj Maxwellovoj teoriji gdje je pripadajući Nambu-Goldstoneov bozon foton.

Cilj je rada prikazati implementaciju generaliziranih simetrija u lineariziranoj gravitaciji. Tretiranje gravitona kao Nambu-Goldstoneove čestice ključno je jer smješta graviton, česticu bez mase s spinom , unutar okvira spontanog narušenja simetrije. Tradicionalni Goldstoneovi teoremi, dobro uspostavljeni za čestice nižeg spina, ovdje su generalizirani kako bi uključili čestice spina , što fundamentalno mijenja način na koji se gravitacijske interakcije mogu razumjeti na kvantnoj razini. Pojam da čestica bez mase poput gravitona nastaje zbog narušene biformne simetrije smjela je i pronicljiva interpretacija. Tenzorska baždarna polja su polja višeg reda koja generaliziraju strukturu običnih baždarnih teorija a kroz korištenje linearizirane gravitacije pokazuje se kako ta polja prirodno podržavaju interpretaciju gravitona kao Goldstoneove čestice. Srž ove interpretacije leži u uvođenju biformnih simetrija. Te simetrije, koje proširuju koncept simetrija viših formi, postaju ključne u objašnjavanju bezmasene prirode gravitona. Konkretno, simetrije koje otklanjaju graviton identificirane su sa specifičnom simetrijom biforme. Pokazali smo kako je Fierz-Paulijeva akcija, dobro poznat okvir za čestice sa spinom , ne samo invarijantna pod lineariziranim difeomorfizmima već također prikazuje dodatnu simetriju koja se može povezati s Noetherinom očuvanom strujom. Ovo je prikazano za specifičan otklon [23], dok najnovija saznanja[24] odražavaju kako se graviton može smatrati Nambu-Goldstoneovim bozonom za općenitiju simetriju. Nadalje, pojam anomalija ključan je u ovoj raspravi - rad ispituje ulogu ’t Hooftovih anomalija u ograničavanju mogućih struktura simetrije u teorijama koje uključuju graviton. Raspravlja se o postojanju miješanih anomalija između gravitacijskih i dualnih simetrija, što nudi uvid u to kako se simetrije mogu mjeriti i koje su implikacije za gravitacijske teorije. ’t Hooftova anomalija ključna je za određivanje gravitona kao Nambu-Goldstoneova moda teorije jer fiksira korelacijsku funkciju dualnih struja.

Daljnji razvoj može se ostvariti u iznošenju argumenta da je graviton Nambu-Goldstoneov mod općenitijeg otklona. Jedna ideja je izgraditi teoriju "od temelja". To bi se potencijalno moglo postići krećući od masivnog tenzora s miješanom simetrijom [31] koji se dekomponira pomoću Stueckelbergovog[32] postupka u sumu bezmasenih polja: -biforma, -biforma, -forma i -forma. Kada se takva dekompozicija uključi u Curtrightovu akciju [33], ona se transformira u zbroj bezmasene Curtrightove akcije, bezmasene Fierz-Paulijeve akcije, dodatnog člana s bezmasenim -formnim poljem i nekim mješovitim članovima. Razmatranje određenih transformacija (uključujući -biformnni otklon) moglo bi dovesti do novog i općenitijeg pristupa interpretaciji gravitona kao Nambu-Goldstoneovog bozona.

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