Hawkingovo zračenje, W algebre i anomalije

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Contents

Temeljna dokumentacijska kartica

Sveučilište u Zagrebu
Prirodoslovno-matematički fakultet
Fizički odsjek

Doktorska disertacija

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Proširili smo analizu metode računanja bozonskog i fermionskog spektra Hawkingovog zračenja upotrebom struja višeg spina. Na početku pokažemo da je fiziku u blizini horizonta Kerrove crne rupe moguće približno opisati dvodimenzionalnom teorijom bozonskih ili fermionskih polja. Tada, upotrebom dvodimenzionalnih struja koje zadovoljavaju W_∞ algebru u bozonskom i $W_{1+\infty}$ algebru u fermionskom slučaju, konstruiramo beskonačan skup kovarijantnih struja od kojih svaka nosi pripadni moment Hawkingovog zračenja. Oni su dovoljni za rekonstrukciju spektra Hawkingovog zračenja, koji se slaže s onima izračunatima upotrebom drugih metoda. Još važnije, pokazali smo da uspješnost ove metode nije utemeljena na anomalijama struja višeg spina (s obzirom da su anomalije trivijalne), već na podilazećoj W_∞ ili $W_{1+\infty}$ strukturi. Naš rezultat ukazuje na postojanje simetrije u blizini horizonta, veće od one generirane Virasoro algebrom, a koja je vjerojatno povezana s nekom od W algebri.

(105 stranica, 79 literaturnih navoda, jezik izvornika engleski)

Ključne riječi: crne rupe, Hawkingovo zračenje, anomalije, W algebre

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Doctoral Thesis

Hawking radiation, W algebras and anomalies

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We have extended the analysis of the method used to calculate the bosonic and fermionic Hawking radiation spectrum via higher spin currents. We start by showing that the near-horizon physics for a Kerr black hole is approximated by an effective two-dimensional field theory of bosonic or fermionic fields. Then, using two-dimensional currents of any spin that form a W_∞ algebra in the bosonic and $W_{1+\infty}$ algebra in the fermionic case, we construct an infinite set of covariant currents, each of which carries the corresponding moment of the Hawking radiation. These are sufficient for the reconstruction of the Hawking radiation spectrum, which agrees with the result obtained by other methods. More importantly, we show that the predictive power of this method is based not on the anomalies of the higher-spin currents (which are trivial) but on the underlying W_∞ or $W_{1+\infty}$ structure. Our results point toward the existence of a symmetry larger than the Virasoro algebra, generated by some of the W algebras, in the near-horizon geometry.

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Prošireni sažetak

Uvod

Crne rupe bi se najkraće mogle opisati kao dijelovi prostor-vremena kauzalno odvojeni od svog komplementa. Fizikalno ovo znači da je unutar takvog dijela prostor-vremena gravitacija toliko snažna da otamo ništa ne može pobjeći, čak niti svjelost. Pojam *bijega* može se preciznije definirati upotrebom asimptotskog dijela prostor-vremena. Za asimptotski ravno prostor-vrijeme $(\mathscr{M}, g_{ab})$, s budućom svjetlosnom beskonačnosti $\mathscr{I}^+$, definiramo **crnu rupu** $\mathcal{B}$ preko

$$\mathcal{B} = \mathscr{M} - J^-(\mathscr{I}^+) \tag{1}$$

gdje J^- označava kauzalnu proošlost. Na sličan način moguće je definirati crne rupe i za druge tipove prostor-vremena s dobro definiranim asimptotskim područjima. **Horizont događaja** $\mathcal{H}$ crne rupe $\mathcal{B}$ je definiran kao njen rub,

$$\mathcal{H} = \mathscr{M} \cap \partial(J^-(\mathscr{I}^+)) \tag{2}$$

Vjeruje se da se crne rupe formiraju kada je neka materija stješnjena kolapsom unutar određenog kritičnog volumena. Danas obično govorimo o tri procesa koji bi mogli rezultirati formiranjem crnih rupa. Jedan od njih je gravitacijski kolaps zvijezde čija masa bi trebala biti negdje u području $2M_\odot \lesssim M \lesssim 100M_\odot$; ispod donje granice zvijezde izbjegavaju kolaps, dok iznad gornje granice zvijezde uopće niti ne postoje zbog pulsacijskih nestabilnosti. Drugi tip procesa događa se u središtima galaksija, gdje bi kolaps cijele jezgre gustog skupa zvijezda vodi do formiranja *supermasivnih* crnih rupa s masama do $\sim 10^{10} M_\odot$. U trećem, najspekulativnijem procesu, *primordijalne* crne rupe bi mogle biti formirane gravitacijskim kolapsom u područjima povećane gustoće u ranom svemiru.

Tijekom tzv. *zlatnog doba* opće teorije relativnosti, 1960ih i početkom 1970ih, niz teorijskih rezultata otkrio je neke važne osobine crnih rupa. Teorem *bez kose* tvrdi da su stacionarna, asimptotski ravna rješenja (nesingularna izvan horizonta događaja) Einsteinove gravitacijske jednadžbe s crnom rupom uz elektromagnetsko polje, u potpunosti opisana tek s tri fizikalna parametra: masom M, angularnim momentom J i električnim nabojem Q. Jednostavnost crnih rupa formiranih kolapsom kompleksnih objekata može se usporediti sa statističkim sustavima u termalnoj ravnoteži, koji su opisani malim skupom varijabli, u kontrastu s ogromnom količinom informacija potrebnih da za detaljan opis dinamike njihovog ponašanja. Štoviše, sličnost osnovnih mehaničkih zakona crnih rupa s osnovnim zakonima klasične termodinamike, prikazan u tablici ispod

zakon	termodinamika	crne rupe
nulti	T je konstantna po tijelu u termodinamičkoj ravnoteži	κ je konstantna po horizontu stacionarne crne rupe
prvi	$\delta E = T\delta S$ + radni članovi	$c^2\delta M = \frac{\kappa c^2}{8\pi G}\,\delta A + \Omega_{\mathsf{H}}\,\delta J$
drugi	$\delta S \geq 0$ u svakom procesu	$\delta A \geq 0$ u svakom procesu
treći	nemoguće je postići $T = 0$ fizikalnim procesom	nemoguće je postići $\kappa = 0$ fizikalnim procesom

ukazao je na postojanje dublje analogije između ovih fizikalnih pojava, gdje površinska gravitacija κ igra ulogu temperature T, površina horizonta A ulogu entropije S, a masa M ulogu unutrašnje energije E. Formalno, ovu korespodenciju možemo prikazati uvođenjem neku konstante α,

$$E \leftrightarrow Mc^2 \quad , \quad T \leftrightarrow \frac{c^2}{G}\,\alpha\kappa \quad , \quad S \leftrightarrow \frac{A}{8\pi\alpha}$$

Bekenstein [Bek73] je predložio poopćeni drugi zakon termodinamike: zbroj obične entropije materije van crne rupe i entropije pridružene crnoj rupi površinom horizonta je strogo rastuća funkcija. Pa ipak, u kontekstu klasične opće teorije relativnosti ovaj zakon bi mogao biti narušen jednostavnim pokusom; smještanje crne rupe u termalnu kupku na temperaturi nižoj od one koju smo formalno pridružili toj crnoj rupi rezultiralo bi tokom topline s hladnijeg (kupka) na toplije tijelo (crna rupa). To znači da bi na razini klasične fizike predložena analogija bila samo matematički kuriozitet.

Stephen Hawking je razriješio ova pitanja [Haw74, Haw75] otkrićem da semiklasične crne rupe mogu zračiti. Kvantni efekti u okolini horizonta rezultiraju stvaranjem čestica koje čine **Hawkingovo zračenje**, sa spektrom crnog tijela na temperaturi

$$T_{\mathsf{H}} = \frac{\hbar}{ck_{\mathsf{B}}}\,\frac{\kappa}{2\pi} \tag{3}$$

Ovaj rezultat je pokazao kako κ uistinu predstavlja termodinamičku temperaturu crne rupe, a ne tek neku veličinu koja igra ulogu matematički analognu temperaturi u mehaničkim zakonima crnih rupa. Neodređena konstanta je stoga $\alpha = G\hbar/(2\pi k_{\mathsf{B}} c^3)$, tako da je *fizikalna* entropija crne rupe dana formulom

$$S_{\mathsf{BH}} = \frac{k_{\mathsf{B}} c^3}{G\hbar}\,\frac{A}{4} \tag{4}$$

Valja istaknuti kako je Hawkingovo zračenje tipične makroskopske crne rupe sitan efekt. Primjerice, temperatura Schwarzschildove crne rupe mase M dana je formulom

$$T = \frac{\hbar c^3}{8\pi G M k_{\mathsf{B}}} = 6 \cdot 10^{-8} \, \frac{M_\odot}{M} \, K$$

Kako bi Hawkingovo zračenje realistične crne rupe bilo zasjenjeno čak i s kozmičkim mikrovalnim pozadinskim zračenjem na $2.73\,K$, njegova detekcija bila bi u principu iznimno teška. Ipak, posljednjih godina su se pojavili neki alternativni prijedlozi opažanja Hawkingovog zračenja. Isparavanje mini crnih rupa koje bi mogle biti producirane na Velikom hadronskom sudarivaču bilo lako detektirati zahvaljujući jasnom signalu [DL01]. Druga mogućnost pojavila se u kontekstu *akustičnih crnih rupa*, kauzalno odvojenih ("gluhih") područja u fluidu, čiji bi horizonti događaja bili izvori fononskog Hawkingovo zračenja [BLV05].

Metode računanja Hawkingovog zračenja

U orginalnom izvodu, Hawking se poslužio metodom Bogoljubljevih koeficijenata kako bi izračunao čestični sadržaj vakuuma definiranog na budućoj svjetlosnoj beskonačnosti. Uskoro nakon njega, Christensen i Fulling [CF77] predložili su upotrebu anomalije traga kako bi se izračunala ukupna izračena energija, odnosno najniži moment spektra Hawkingovog začnja. Interes za upotrebom anomalija u ovom kontekstu probudili su Robinson i Wilczek [RW05] pokazaši da se difeomorfna anomalija može upotrijebiti u istu svrhu. Iso, Morita i Umetsu su proširili [IMU07b] ove metode, pokazavši da je moguće izračunati i ostale detalje, više momente spektra Hawkingovog zračenja. Ovdje ćemo prikazati dva osnovna pristupa utemeljena na upotrebi anomalija.

Metoda anomalije traga temelji se na argumentu da je fizika u blizini horizonta događaja opisana dvodimenzionalnom konformnom teorijom polja. Klasično trag tenzora energije i impulsa isčezava na ljusci. Međutim, zahvaljujući anomaliji, on je općenito neisčezavajući na razini jedne petlje

$$\left\langle T^{\mu}{}_{\mu} \right\rangle = \frac{cR}{48\pi} \tag{5}$$

gdje je R Riccijev skalar pozadinske metrike, a c ukupni centralni naboj. Ovu informaciju, zajedno s kovarijantnim sačuvanjem tenzora energije i impulsa,

$$\nabla_\mu T^{\mu}{}_{\nu} = 0 \tag{6}$$

možemo upotrijebiti kako bi smo izračunali izlazni tok Hawkingovog zračenja u beskonačnosti.

Kao što će biti pokazano u ovoj tezi, (3+1)-dimenzionalna bozonska ili fermionska akcija u blizini horizonta crne rupe može se reducirati na (1+1)-dimenzionalnu efektivnu akciju s metrikom oblika

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)} \tag{7}$$

Kako je svaka dvodimenzionalna metrika konforomno ravna, pogodno je transformirati u manifestno konformnu formu. Ovo se može napraviti uvođenjem *kornjačine* koordinate r_* i svjetlosnih koordinata (u, v) definiranima preko

$$\frac{dr_*}{dr} = \frac{1}{f(r)} \quad ; \quad u = t - r_* \quad , \quad v = t + r_* \tag{8}$$

U ovim koordinatama metrika postaje

$$ds^2 = e^{\varphi(u,v)} du\, dv \quad , \quad g_{\mu\nu} = e^{\varphi}\eta_{\mu\nu} \tag{9}$$

gdje je $\varphi = \ln(f)$. Označimo, nadalje, s $T_{uu}(u, v)$ i $T_{vv}(u, v)$ klasično neisčezavajuće komponente tenzora energije i impulsa u novim koordinatama. Tenzor energije i impulsa možemo izračunati integriranjem gornjih jednadžbi sačuvanja i anomalije traga. Rezultat je

$$T_{uu}(u, v) = \frac{\hbar c_{\mathrm{R}}}{24\pi}(\partial_u^2\varphi - \frac{1}{2}(\partial_u\varphi)^2) + T_{uu}^{(\mathrm{hol})}(u) \tag{10}$$

gdje je $T_{uu}^{(\mathrm{hol})}(u)$ holomorfan dio, a $T_{uu}(u, v)$ konformno kovarijantan dio. Naime, pod konformnom transformacijom

$$u \;\rightarrow\; \tilde{u}(u) \quad , \quad v \;\rightarrow\; \tilde{v}(v)$$

imamo

$$T_{uu}(u, v) = (\tilde{u}'(u))^2\, \tilde{T}_{\tilde{u}\tilde{u}}(\tilde{u}, \tilde{v}) \tag{11}$$

Ukoliko upotrijebimo relaciju

$$\tilde{\varphi}(\tilde{u}, \tilde{v}) = \varphi(u, v) - \ln\left(\frac{d\tilde{u}}{du}\frac{d\tilde{v}}{dv}\right) \tag{12}$$

sređivanjem izraza slijedi

$$\begin{aligned} T_{uu}(u, v) = (\tilde{u}')^2\tilde{T}_{\tilde{u}\tilde{u}}(\tilde{u}, \tilde{v}) = \\ = \frac{\hbar c_{\mathrm{R}}}{24\pi}\left((\tilde{u}')^2\partial_{\tilde{u}}^2\tilde{\varphi} - \frac{1}{2}(\tilde{u}')^2(\partial_{\tilde{u}}\tilde{\varphi})^2 + \{\tilde{u}, u\}\right) + T_{uu}^{(\mathrm{hol})}(u) \end{aligned} \tag{13}$$

gdje smo koristili *Schwarzovu derivaciju*,

$$\{w, z\} \;\equiv\; \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 \tag{14}$$

Dobiveni rezultat može se pisati u obliku

$$\tilde{T}_{\tilde{u}\tilde{u}}(\tilde{u}, \tilde{v}) = \frac{\hbar c_{\mathrm{R}}}{24\pi}\left(\partial_{\tilde{u}}^2\tilde{\varphi} - \frac{1}{2}(\partial_{\tilde{u}}\tilde{\varphi})^2\right) + \frac{1}{(\tilde{u}')^2}\left(\frac{\hbar c_{\mathrm{R}}}{24\pi}\{\tilde{u}, u\} + T_{uu}^{(\mathrm{hol})}(u)\right)$$

odakle isčitavamo transformaciju holomorfnog dijela tenzora energije i impulsa,

$$\tilde{T}_{\tilde{u}\tilde{u}}^{(\mathrm{hol})}(\tilde{u}) = \frac{1}{(\tilde{u}')^2}\left(T_{uu}^{(\mathrm{hol})}(u) + \frac{\hbar c_{\mathrm{R}}}{24\pi}\{\tilde{u}, u\}\right) \tag{15}$$

Regularne koordinate u blizini horizonta su Kruskalove,

$$U = -e^{-\kappa u} \quad , \quad V = e^{\kappa v} \tag{16}$$

gdje je κ površinska gravitacija. Schwarzova derivacija u ovom slučaju iznosi

$$\{U, u\} = -\frac{\kappa^2}{2} \tag{17}$$

Upotrebom ove koordinatne transformacije imamo

$$\tilde{T}^{(\mathrm{hol})}_{UU}(U) = \frac{1}{(-\kappa U)^2}\left(\tilde{T}^{(\mathrm{hol})}_{uu}(u) + \frac{\hbar c_{\mathrm{R}}}{24\pi}\{U, u\}\right) = \frac{1}{(-\kappa U)^2}\left(\tilde{T}^{(\mathrm{hol})}_{uu}(u) + \frac{\hbar c_{\mathrm{R}}\kappa^2}{48\pi}\right)$$

Sada zahtijevamo da izlazni tok energije bude regularan na budućem horizontu ($U = 0$) u Kruskalovim koordinatama. Odmah vidimo da odave slijedi kako je $T^{(\mathrm{hol})}_{uu}(u)$ na horizontu jednak $\hbar c_{\mathrm{R}}\kappa^2/48\pi$. Također, kako je na horizontu

$$\partial_u^2\varphi - \frac{1}{2}(\partial_u\varphi)^2 = \frac{ff''}{4} - \frac{1}{2}\left(-\frac{f'}{2}\right)^2 = -\frac{\kappa^2}{2}$$

slijedi $T_{uu}(r_{\mathsf{H}}) = 0$. S obzirom da je pozadinska metrika statična, slijedi da $T^{(\mathrm{hol})}_{uu}(u)$ ne ovisi o koordinati t, a kako je neovisan i o v, također je neovisan o koordinati r. Pretpostavljamo da u beskonačnosti ($r = \infty$) nema ulaznog toka ($\langle T_{vv}\rangle_\infty = 0$) i da je pozadina trivijalna, tako da vakuumske očekivane vrijednosti $T^{(\mathrm{hol})}_{uu}(u)$ i $T_{uu}(u, v)$ asimptotski poklapaju. Asimptotski tok stoga iznosi

$$\langle T^r{}_t\rangle = \langle T_{uu}\rangle - \langle T_{vv}\rangle = \frac{\hbar\kappa^2}{48\pi}\, c_{\mathrm{R}} \tag{18}$$

Tok izlaznog zračenja poklapa se s konstantom a_o iz metode difeomorfne anomalije.

Metoda difeomorfne anomalije. Metoda koju su uveli Robinson i Wilczek temelji se na difeomorfnoj anomaliji u dvodimenzionalnoj efektivnoj teoriji polja u blizini horizonta crne rupe. Kako je horizont $\mathcal{H}$ hiperploha svjetlosnog tipa, modovi unutar horizonta ne mogu klasično utjecati na fizikalne procese izvan horizonta. Ti modovi mogu stoga biti izintegrirani, čime dolazimo do efektivne dvodimenzionalne kiralne teorije polja beskonačno mnogo polja sastavljenih isključivo od izlaznih modova. Uklanjanjem modova efektivna teorija postala je anomalna s obzirom na difeomorfizme. Pripadnu jednadžbu anomalije moguće je upotrijebiti za računanje toka izlaznog zračenja. Pozadinska teorija je, naravno, difeomorfno invarijantna i bez anomalija, pa one koje su se pojavile u efektivnoj teoriji moraju biti poništene kvantnim efektima klasično zanemarivih modova, kako bi povratili difeomorfnu invarijnatnost.

Ovdje ćemo izložiti metodu u ponešto pojednostavljenom obliku. Dio prostor-vremena izvan crne rupe podjeljen je duž radijalne koordinate u dva relevantna područja: područje $\mathcal{O}$ (definirano s $r > r_{\mathsf{H}} + \epsilon$) i područje $\mathcal{H}$ (definirano s $r_h < r < r_{\mathsf{H}} + \epsilon$). U području $\mathcal{H}$ ulazni modovi su izintegrirani, tako da je efektivna teorija polja anomalna, dok u $\mathcal{O}$ očekujemo u potpunosti invarijantnu teoriju. Ovo je iskazano iščezavanjem kovarijntne divergencije tenzora energije i impulsa,

$$\nabla_\mu T^\mu{}_{\nu(\mathcal{O})} = 0 \tag{19}$$

dok u području $\mathcal{H}$ imamo

$$\nabla_\mu T^\mu{}_{\nu(\mathcal{H})} = \frac{\hbar c_{\mathrm{R}}}{96\pi}\,\epsilon_{\nu\mu}\partial^\mu R \tag{20}$$

Ovo je kovarijantni oblik difeomorfne anomalije, s pripadnim koeficijentima za kiralnu (izlaznu, odnosno desnu) materiju s centralnim nabojem c_{R}. Upotrebom dvodimenzionalne metrike i statičnosti slijedi

$$\nabla_\mu T^\mu{}_t = \partial_r T^r{}_t + \Gamma^\mu_{\mu\sigma} T^\sigma{}_t - \Gamma^\sigma_{\mu t} T^\mu{}_\sigma = \partial_r T^r{}_t - \left(\Gamma^t_{rt} T^r{}_t + \Gamma^r_{tt} T^t{}_r\right) =$$

$$= \partial_r T^r{}_t - \left(\frac{f'}{2f} + \frac{ff'}{2} g^{tt} g_{rr}\right) T^r{}_t = \partial_r T^r{}_t$$

Kako je metrika stacionarna, gornje dvije jednadžbe poprimaju veoma jednostavan oblik (za $\nu = t$)

$$\partial_r T^r{}_{t(\mathcal{O})} = 0 \tag{21}$$

$$\partial_r T^r{}_{t(\mathcal{H})} = \frac{\hbar c_{\mathrm{R}}}{96\pi} \partial_r \left(ff'' - \frac{1}{2}(f')^2\right) \equiv \partial_r N^r{}_t \tag{22}$$

Ove jednadžbe je moguće integrirati,

$$T^r{}_{t(\mathcal{O})} = a_o \tag{23}$$

$$T^r{}_{t(\mathcal{H})}(r) = a_h + N^r{}_t(r) - N^r{}_t(r_{\mathsf{H}}) \tag{24}$$

gdje su a_o i a_h integracijske konstante. Valja istaknuti da a_o, s obzirom da je konstanta, zajedno s uvjetom da nema izlaznog toka zračenja iz beskonačnosti, daje tok izlazne energije. Sada definiramo ukupni tenzor energije i impulsa

$$T^\mu{}_\nu = T^\mu{}_{\nu(\mathcal{O})}\theta(r - r_{\mathsf{H}} - \epsilon) + T^\mu{}_{\nu(\mathcal{H})}(1 - \theta(r - r_h - \epsilon)) \tag{25}$$

Podrazumijeva se da je ϵ mali broj koji definira veličinu područja u kojem tenzor energije i impulsa nije sačuvan. Računanjem divergencije ukupnog tenzora energije i impulsa dobivamo za $\nu = t$,

$$\partial_r T^r{}_t = (a_o - a_h + N^r{}_t(r_h))\delta(r - r_{\mathsf{H}} - \epsilon) + \partial_r(N^r{}_t(r)H(r))$$

gdje je $H(r) = 1 - \theta(r - r_h - \epsilon)$. Sada uvodimo novi tenzor

$$\widehat{T}^r{}_t \equiv T^r{}_t - N^r{}_t(r)H(r) \tag{26}$$

koji je sačuvan

$$\partial_r \widehat{T}^r{}_t = 0 \tag{27}$$

ukoliko vrijedi

$$a_o - a_h + N^r{}_t(r_{\mathsf{H}}) = 0$$

Uvijet iščezavanja tenzora energije i impulsa na horizontu vodi do

$$0 \stackrel{!}{=} T^r{}_{t(\mathcal{H})}(r_{\mathsf{H}}) = a_h \tag{28}$$

Uzimajući ovo u obzir, kao i to da je $f'(r_{\mathsf{H}}) = 2\kappa$, u konačnici dobivamo

$$a_o = -N^r{}_t(r_h) = \frac{\hbar\kappa^2}{48\pi} c_{\mathrm{R}} \tag{29}$$

Ovo je izlazni tok u beskonačnosti i poklapa se s ukupnim Hawkingovim zračenjem emitiranim od crne rupe.

Zaključno možemo usporediti osnovne dijelove metoda baziranih na anomalijama:

a) kod prve metode integriramo zakon sačuvanja tenzora energije i impulsa uz prisutnost anomalije traga, dok kod druge integriramo anomalni i regularni zakon sačuvanja tenzora energije i impulsa;

b) u obje metode namećemo uvjet isčezavanja tenzora energije i impulsa na horizontu i odsustvo toka upadnog zračenja iz beskonačnosti.

Važnu razliku imamo u tome što kod metode anomalije traga ne moramo razdijeliti prostor-vrijeme na različita područja, već promatramo jedinstveno područje van horizonta.

U općenitom slučaju kiralne dvodimenzionalne teorije s centralnim nabojima c_{R} i c_{L} za, redom, holomorfni i antiholomorfni dio, prisutne su obje, difeomorfna i anomalija traga,

$$\nabla_\mu T^\mu{}_\nu = \frac{\hbar}{48\pi}\,\frac{c_{\mathrm{R}}-c_{\mathrm{L}}}{2}\,\epsilon_{\nu\mu}\partial^\mu R \tag{30}$$

$$T^\alpha{}_\alpha = \frac{\hbar}{48\pi}\,(c_{\mathrm{R}}+c_{\mathrm{L}})R \tag{31}$$

Upotrebom svjetlosnih koordinata (u,v) ove jednadžbe glase

$$\nabla_u T_{uv} + \nabla_v T_{uu} = \frac{\hbar}{48\pi}\,\frac{c_{\mathrm{R}}-c_{\mathrm{L}}}{2}\,\epsilon_{uv}\partial_u R \tag{32}$$

$$\nabla_u T_{vv} + \nabla_v T_{uv} = -\frac{\hbar}{48\pi}\,\frac{c_{\mathrm{R}}-c_{\mathrm{L}}}{2}\,\epsilon_{uv}\partial_v R \tag{33}$$

$$T_{uv} = -\frac{\hbar}{48\pi}\,\frac{c_{\mathrm{R}}+c_{\mathrm{L}}}{4}\,R\,e^\varphi \tag{34}$$

Uzevši u obzir elemente promatrane metrike iz Dodatka A imamo

$$R = 4e^{-\varphi}\varphi_{,uv} \quad , \quad \epsilon_{uv}\partial_u R = 2\partial_v \mathcal{T}_{uu} \quad , \quad \epsilon_{uv}\partial_v R = 2\partial_u \mathcal{T}_{vv}$$

gdje su

$$\mathcal{T}_{uu} = \partial_u^2\varphi - \frac{1}{2}\,(\partial_u\varphi)^2 \quad , \quad \mathcal{T}_{vv} = \partial_v^2\varphi - \frac{1}{2}\,(\partial_v\varphi)^2$$

Ove jednadžbe sada možemo jednostavno integrirati i rezultat su

$$T_{uu}(u,v) = \frac{\hbar c_{\mathrm{R}}}{24\pi}\,\mathcal{T}_{uu}(u,v) + T_{uu}^{(\mathrm{hol})}(u) \tag{35}$$

$$T_{vv}(u,v) = \frac{\hbar c_{\mathrm{L}}}{24\pi}\,\mathcal{T}_{vv}(u,v) + T_{vv}^{(\mathrm{a-hol})}(v) \tag{36}$$

Dva člana, $T_{uu}^{(\mathrm{hol})}(u)$ i $T_{vv}^{(\mathrm{a-hol})}(v)$, predstavljaju, redom, holomorfni i antiholomorfni dio tenzora energije i impulsa.

Kod metode anomalije traga upotrijebili smo $T_{uu}(u,v)$, zahtijevali da tenzor energije i impulsa bude kovarijantno sačuvan, te da su ispunjeni uvjeti pod b). Ovo odgovara odabiru $c_{\mathrm{R}} = c_{\mathrm{L}}$ u području van horizonta. Valja napomenuti kako je iz gornjih izvoda jasno da, radeći s dvodimenzionalnom teorijom, možemo integrirati anomaliju traga čak i kada vrijedi $c_{\mathrm{R}} \neq c_{\mathrm{L}}$.

Kod metode difeomorfne anomalije integrirali smo anomalnu jednadžbu kovarijantne divergencije tenzora energije i impulsa u blizini horizonta, te klasičnu jednadžbu u području daleko od horizonta. Potom smo nametnuli uvjet regularnosti na horizontu preko isčezavanja tenzora energije i impulsa, koji nam je omogućio dobivanje ukupne izračene energije Hawkingovog zračenja.

Viši momenti Hawkingovog zračenja

Termalni spektar zračenja Kerrove crne rupe dan je Bose-Einsteinovom (BE) ili Fermi-Diracovom (FD) raspodjelom

$$N_{\pm}(\omega) = \frac{g_*}{e^{\beta(\omega - m\Omega_{\mathsf{H}})} \pm 1} \tag{37}$$

gdje gornji predznak odgovara fermionskom, a donji bozonskom slučaju. Parametar β je inverzna Hawkingova temperatura crne rupe ($k_{\mathsf{B}}\beta\kappa = 2\pi$), ω je apsolutna vrijednost impulsa ($\omega = |k|$), Ω_{H} je kutna brzina horizonta, a m azimutalni kvantni broj. Parametar g_* predstavlja broj fizikalnih stupnjeva slobode u emitiranom zračenju. U dvodimenzionalnom slučaju viši momenti spektra Hawkingovog zračenja F_n definirani su prema

$$F_n^{\pm} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, k^{n-2}\omega N_{\pm}(\omega) = \frac{g_*}{4\pi}\int_{-\infty}^{\infty} dk\, \frac{k^{n-2}\omega}{e^{\beta(\omega - m\Omega_{\mathsf{H}})} \pm 1} \tag{38}$$

Dimenzionalna redukcija. Kako bi smo pojednostavili promatrani problem, upotrijebit ćemo aproksimaciju u blizini horizonta. Započinjemo s potpunom teorijom u (3+1)-dimenzionalnom prostor-vremenu, opisanu nekom akcijom, u kojoj separiramo kutni dio od ostatka. Kako se fizikalni procesi relevantni za promatrani fenomen događaju u blizini horizonta crne rupe, zanemarit ćemo efekte potencijala vezanog za cetrifugalnu barijeru. Upotrebom ove aproksimacije možemo identificirati pripadnu (1+1)-dimenzionalnu teoriju s nekim efektivnim stupnjevima slobode. Promotrit ćemo ovaj postupak u slučaju bozonskih i fermionskih polja u blizini Kerrove crne rupe.

Akcija za skalarno bezmaseno polje ϕ u općenitom zakrivljenom prostorvremenu definirana je s

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\, g^{\mu\nu}\nabla_\mu\phi^*\, \nabla_\nu\phi$$

Upotrebom komponenti Kerrove metrike eksplicitno raspisana metrika glasi

$$S = \frac{1}{2}\int dtdrd\theta d\varphi\, \sin\theta\,\Sigma\left(-\frac{\Xi}{\Delta\Sigma}(\partial_t\phi^*)(\partial_t\phi) - \frac{2Mra}{\Delta\Sigma}((\partial_t\phi^*)(\partial_\varphi\phi) + (\partial_\varphi\phi^*)(\partial_t\phi)) + \right.$$

$$+\frac{\Delta}{\Sigma}(\partial_r\phi^*)(\partial_r\phi)+\frac{1}{\Sigma}(\partial_\theta\phi^*)(\partial_\theta\phi)+\frac{\Delta-a^2\sin^2\theta}{\Delta\Sigma\sin^2\theta}(\partial_\varphi\phi^*)(\partial_\varphi\phi)\Big)$$

Parcijalnom integracijom akcija poprima sljedeći oblik

$$S=\frac{1}{2}\int dtdrd\theta d\varphi\,\sin\theta\;\phi^*\left(\frac{\Xi}{\Delta}\,\partial_t^2+\frac{4Mra}{\Delta}\,\partial_t\partial_\varphi-\partial_r(\Delta\partial_r)-\right.$$

$$\left.-\frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\partial_\theta)-\frac{\Delta-a^2\sin^2\theta}{\Delta\sin^2\theta}\,\partial_\varphi^2\right)\phi$$

Rastavom polja ϕ and ϕ^* preko sfernih harmonika,

$$\phi(t,r,\theta,\varphi)=\sum_{l,m}\phi_{lm}(t,r)Y_{lm}(\theta,\varphi) \tag{39}$$

dobivamo

$$S=\frac{1}{2}\int dtdrd\theta d\varphi\,\sin\theta\sum_{l',m'}\phi^*_{l'm'}Y^*_{l'm'}\left(\frac{(r^2+a^2)^2}{\Delta}\,\partial_t^2-a^2\sin^2\theta\,\partial_t^2+\right.$$

$$\left.+im\frac{4Mra}{\Delta}\,\partial_t-\partial_r(\Delta\partial_r)+l(l+1)-\frac{a^2m^2}{\Delta}\right)\sum_{l,m}\phi_{lm}Y_{lm}$$

Kako bi smo procijenili koji članovi su dominantni, a koji mogu biti zanemareni u akciji u blizini horizonta, prelazimo s radijalne koordinate r na kornjačinu koordinatu r_*, definiranu s

$$\frac{dr_*}{dr}=\frac{r^2+a^2}{\Delta}\equiv\frac{1}{f(r)}$$

Sada je

$$S=\frac{1}{2}\int dtdr_*d\theta d\varphi\,\sin\theta\sum_{l',m'}\phi^*_{l'm'}Y^*_{l'm'}\left((r^2+a^2)\,\partial_t^2-a^2f(r)\sin^2\theta\,\partial_t^2+\right.$$

$$\left.+im\frac{4Mra}{r^2+a^2}\,\partial_t-f(r)\partial_r(\Delta\partial_r)+f(r)l(l+1)-\frac{a^2m^2}{r^2+a^2}\right)\sum_{l,m}\phi_{lm}Y_{lm}$$

gdje preostalu radijalnu koordinatu r u zapisu treba promatrati kao funkciju $r(r_*)$. Zadržavanjem dominantnih članova akcija postaje

$$S\to S_{(\mathcal{H})}=\frac{1}{2}\int dtdr_*d\theta d\varphi\,\sin\theta\sum_{l',m'}\phi^*_{l'm'}Y^*_{l'm'}\left((r_+^2+a^2)\,\partial_t^2+\right.$$

$$\left.+im\frac{4Mr_+a}{r_+^2+a^2}\,\partial_t-f(r)\partial_r(\Delta\partial_r)-\frac{a^2m^2}{r_+^2+a^2}\right)\sum_{l,m}\phi_{lm}Y_{lm}$$

Povratkom na radijalnu koordinatu r i upotrebom ortogonalnosti sfernih harmonika, nakon sređivanja dobivamo

$$S_{(\mathcal{H})}=\frac{1}{2}\sum_{l,m}\int dtdr\,(r^2+a^2)\phi^*_{lm}\left(\frac{1}{f(r)}\left(\partial_t+\frac{iam}{r^2+a^2}\right)^2-\partial_r\left(f(r)\,\partial_r\right)\right)\phi_{lm} \tag{40}$$

Sada možemo predložiti interpretaciju ovako reduciranog oblika akcije, kao (1+1)-dimenzionalnu efektivnu teoriju beskonačno mnogo skalarnih polja ϕ_{lm} s

$$\text{dilatonom} \quad \overset{\flat}{\Phi} = r^2 + a^2 \qquad \text{i}$$

$$\text{baždarnim poljem} \quad \overset{\flat}{A}_t = -\tfrac{a}{r^2+a^2} \quad , \quad \overset{\flat}{A}_r = 0$$

unutar dvodimenzionalnog prostor-vremena s komponentama metrike

$$\overset{\flat}{g}_{tt} = -f(r) \quad , \quad \overset{\flat}{g}_{rr} = \frac{1}{f(r)} \quad , \quad \overset{\flat}{g}_{rt} = 0 \tag{41}$$

Pripadnu dvodimenzionalnu akciju moguće je stoga pisati u obliku

$$S = \frac{1}{2}\int dt dr\, \overset{\flat}{\Phi}\, \phi^* \overset{\flat}{D}_\mu \overset{\flat}{D}^\mu \phi \tag{42}$$

gdje je baždarno kovarijantna derivacija definirana s

$$\overset{\flat}{D}_\mu = \overset{\flat}{\nabla}_\mu - iq\overset{\flat}{A}_\mu \tag{43}$$

U daljnoj analizi ograničavamo se na područje u blizini horizonta, u kojem je dilatonsko polje približno konstantno, pa ćemo ga stoga i zanemariti.

Akcija za bezmaseno fermionsko polje ψ u općenitom prostor-vremenu je definirana s

$$S = \int d^4x\sqrt{-g}\,\bar{\psi}\, i\nabla\!\!\!/\,\psi \tag{44}$$

Tretiranje spinora u zakrivljenom prostor-vremenu moguće je pojednostavniti u tangentnom prostoru, uvođenjem lokalnih inercijalnih sustava. Transformacija iz općenitog u tangentni sustav u svakoj točki mnogostrukosti definirana je putem *vierbeina* ${e^a}_\mu$, definiranima s

$$\eta_{ab}{e^a}_\mu {e^b}_\nu = g_{\mu\nu} \tag{45}$$

Važno je naglasiti kako su vierbeini, poput lokalnih inercijalnih sustava, definirani do na lokalnu Lorenzovu transformaciju. Konvencija je upotrijebiti mala slova latinske abecede za ravne (Minkowski) koordinate, te mala slova grčke abecede za zakrivljene koordinate. Također, radi lakšeg razlikovanja između ova dva tipa koordinata, za ravne koristimo vrijednosti iz skupa $\mathbb{N}_0$, dok za zakrivljene neku od abeceda (na primjer, $a \in \{0,1,2,3\}$, $\mu \in \{t, r, \theta, \varphi\}$). Indeksi se podižu i spuštaju s pripadnom metrikom,

$${e_a}^\mu = \eta_{ab}\, g^{\mu\nu}\, {e^b}_\nu \tag{46}$$

Spin koneksija ${\omega^a}_{b\mu}$ je definirana s

$${\omega^a}_{b\mu} = {e^a}_\nu \nabla_\mu {e_b}^\nu \tag{47}$$

Ravne gama matrice definirane su antikomutatorom

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbf{1} \tag{48}$$

Adjungirani bispinor $\bar{\psi}$ definiran je pomoću ravne gama matrice γ^0, $\bar{\psi} \equiv \psi^\dagger\gamma^0$. Uzevši sve ovo u obzir, akcija glasi

$$S = \int d^4x\sqrt{-g}\,\psi^\dagger\gamma^0\gamma^a {e_a}^\mu \left(\partial_\mu - \frac{1}{8}\,\omega_{bc\mu}[\gamma^b, \gamma^c]\right)\psi \tag{49}$$

Odabiremo familiju lokalnih inercijalnih sustava putem vierbeina

$$\sqrt{\Delta\Sigma}\, {e_0}^{\mu}\partial_\mu = (r^2 + a^2)\partial_t + a\partial_\varphi \quad , \quad \sqrt{\Delta\Sigma}\, {e_1}^{\mu}\partial_\mu = \Delta\partial_r$$

$$\sqrt{\Delta\Sigma}\, {e_2}^{\mu}\partial_\mu = \sqrt{\Delta}\,\partial_\theta \quad , \quad \sqrt{\Delta\Sigma}\, {e_3}^{\mu}\partial_\mu = \sqrt{\Delta}\left(a\sin\theta\partial_t + \frac{1}{\sin\theta}\partial_\varphi\right)$$

Analizom analognom onoj u bozonskom slučaju, odabiremo dominantne članove u akciji; opet uvodimo separaciju kutnog dijela polja putem razvoja

$$\psi(t, r, \theta, \varphi) = \sum_{l,m} \psi_{lm}(t, r)\mathcal{Y}_{lm}(\theta, \varphi) \tag{50}$$

gdje su $\mathcal{Y}_{lm}$ modificirani sferni harmonici,

$$\mathcal{Y}_{lm}(\theta, \varphi) \;\equiv\; \frac{Y_{lm}(\theta, \varphi)}{\sqrt[4]{\Sigma}} \tag{51}$$

Nakon sređivanja akcija poprima oblik

$$S = 4\pi \int dtdr\, \frac{r_+^2 + a^2}{\sqrt{\Delta}} \sum_{l,m} \psi_{lm}^\dagger \left\{\gamma^0\gamma^0\left(\partial_t - \frac{iam}{r_+^2 + a^2}\right) + \gamma^0\gamma^1\left(\partial_{r_*} - \frac{r_+ - r_-}{4(r_+^2 + a^2)}\right)\right\}\psi_{lm}$$

Upotrebom 4- i 2-dimenzionalnih gama matrica, te rastavom bispinora

$$\psi_{lm} = \begin{pmatrix} \chi_{lm}^{(1)} \\ \chi_{lm}^{(2)} \end{pmatrix} \tag{52}$$

akcija poprima konačan oblik

$$S = 4\pi \int dtdr\, \frac{r_+^2 + a^2}{\sqrt{\Delta}} \sum_{s=1}^{2}\sum_{l,m} \chi_{lm}^{(s)\dagger} \left\{\sigma^0\sigma^0\left(\partial_t - \frac{iam}{r_+^2 + a^2}\right) + \right.$$

$$\left. + \sigma^0\sigma^1\left(\partial_{r_*} - \frac{r_+ - r_-}{4(r_+^2 + a^2)}\right)\right\}\chi_{lm}^{(s)} \tag{53}$$

Kao i kod bozonskog slučaja, sada možemo predložiti interpretaciju ovako reduciranog oblika akcije, kao (1+1)-dimenzionalnu efektivnu teoriju beskonačno mnogo fermionskih polja $\chi_{lm}^{(s)}$ s

$$\text{dilatonom} \quad \overset{\flat}{\Phi} = \sqrt{r^2 + a^2} \qquad \text{i}$$

$$\text{baždarnim poljem} \quad \overset{\flat}{A}_t = \tfrac{a}{r^2 + a^2} \quad , \quad \overset{\flat}{A}_r = 0$$

unutar dvodimenzionalnog prostor-vremena s komponentama metrike

$$\overset{\flat}{g}_{tt} = f(r) \quad , \quad \overset{\flat}{g}_{rr} = -\frac{1}{f(r)} \quad , \quad \overset{\flat}{g}_{rt} = 0 \tag{54}$$

Pripadnu dvodimenzionalnu akciju moguće je stoga pisati u obliku

$$S = \int dtdr\, \Phi\, \bar{\chi}_{lm}^{(s)}\, \overset{\flat}{\not{D}}\, \chi_{lm}^{(s)} \tag{55}$$

gdje je baždarno kovarijantna derivacija definirana s

$$\overset{\flat}{\not{D}} = \sigma^\mu \overset{\flat}{D}_\mu = \sigma^\mu \left(\overset{\flat}{\nabla}_\mu - iq\overset{\flat}{A}_\mu \right) \tag{56}$$

U daljnoj analizi ograničavamo se na područje u blizini horizonta, u kojem je dilatonsko polje približno konstantno, pa ćemo ga, kao i kod bozonskog slučaju nadalje zanemariti.

Holomorfne bozonske struje su definirane pomoću dvodimenzionalnog kompleksnog slobodnog polja ϕ. Upotrebljavamo Euklidski formalizam u kojem su (u, v) koordinate zamjenjene kompleksnima $(z, \bar{z})$. Korelacijske funkcije za ovo kompleksno bozonsko polje glase

$$\left\langle \phi(z_1)\bar{\phi}(z_2) \right\rangle = -\hbar \ln(z_1 - z_2) \tag{57}$$

$$\langle \phi(z_1)\phi(z_2) \rangle = \left\langle \bar{\phi}(z_1)\bar{\phi}(z_2) \right\rangle = 0 \tag{58}$$

Holomorfne bozonske struje koje zadovoljavaju W_∞ algebru su konstruirane u [BK90],

$$j^{(s)}_{z\ldots z}(z) = B(s) \sum_{k=1}^{s-1} (-1)^k A^s_k : \partial^k_z \phi(z) \partial^{s-k}_z \bar{\phi}(z) : \tag{59}$$

$$B(s) = q^{s-2} \frac{2^{s-3} s!}{(2s-3)!!} \quad , \quad A^s_k = \frac{1}{s-1} \binom{s-1}{k} \binom{s-1}{s-k}$$

gdje je q deformacijski parametar i $j^{(s)}_{z\ldots z}$ ima s donjih indeksa. Normalno uređenje je definirano upotrebom regularizacije razdvajanjem točke,

$$: \partial^m_z \phi\, \partial^n_z \bar{\phi} : \; = \lim_{z_2 \to z_1} \left(\partial^m_{z_1} \phi(z_1) \partial^n_{z_2} \bar{\phi}(z_2) - \partial^m_{z_1} \partial^n_{z_2} \left\langle \phi(z_1)\bar{\phi}(z_2) \right\rangle \right) \tag{60}$$

Kao što je uobičajeno u konformnoj teoriji polja, kod operatorskog produkta na desnoj strani jednadžbe pretpostavljamo da je radijalno uređen.

Struja $j^{(2)}_{zz}(z)$ je proporcionalna normaliziranom holomorfnom tenzoru energije i impulsa, te prilikom koordinatne transformacije $z \to w(z)$ mijenja se prema

$$: \partial_z \phi(z) \partial_z \bar{\phi}(z) : = (w')^2 : \partial_w \phi(w) \partial_w \bar{\phi}(w) : -\frac{\hbar}{6} \{w, z\} \tag{61}$$

Ovo možemo vidjeti na sljedeći način,

$$\begin{aligned} : \partial_{z_1} \phi(z_1) \partial_{z_2} \bar{\phi}(z_2) : &= \partial_{z_1} \phi(z_1) \partial_{z_2} \bar{\phi}(z_2) - \partial_{z_1} \partial_{z_2} \left\langle \phi(z_1)\bar{\phi}(z_2) \right\rangle = \\ &= w'(z_1) w'(z_2) \partial_{w_1} \phi(w_1) \partial_{w_2} \bar{\phi}(w_2) - \partial_{z_1} \partial_{z_2} \left\langle \phi(z_1)\bar{\phi}(z_2) \right\rangle = \\ &= w'(z_1) w'(z_2) : \partial_{w_1} \phi(w_1) \partial_{w_2} \bar{\phi}(w_2) : - G_{\mathrm{B}}(z_1, z_2) \end{aligned}$$

gdje je

$$\begin{aligned} G_{\mathrm{B}}(z_1, z_2) \;&\equiv\; -w'(z_1) w'(z_2) \partial_{w_1} \partial_{w_2} \left\langle \phi(w_1)\bar{\phi}(w_2) \right\rangle + \partial_{z_1} \partial_{z_2} \left\langle \phi(z_1)\bar{\phi}(z_2) \right\rangle = \\ &= -\partial_{z_1} \partial_{z_2} \left(\left\langle \phi(w_1(z_1))\bar{\phi}(w_2(z_2)) \right\rangle - \left\langle \phi(z_1)\bar{\phi}(z_2) \right\rangle \right) = \end{aligned}$$

$$= \hbar \, \frac{w'(z_1)w'(z_2)}{(w(z_1) - w(z_2))^2} - \frac{\hbar}{(z_1 - z_2)^2}$$

U limesu $z_2 \to z_1$ ovo teži k

$$\lim_{z_2 \to z_1} \frac{w'(z_1)w'(z_2)}{(w(z_1) - w(z_2))^2} - \frac{1}{(z_1 - z_2)^2} = \frac{1}{6}\{w, z_1\}$$

Mi smo zainteresirani za svojstva struja $j^{(s)}(u)$ povezana s prijelazom na Kruskalove koordinate,

$$w(z) = -e^{-\kappa z}$$

tako da je

$$G_{\mathrm{B}}(z_1, z_2) = G_{\mathrm{B}}(z_1 - z_2) = -\frac{\hbar}{(z_1 - z_2)^2} + \frac{\hbar\, \kappa^2}{4\, \mathrm{sh}^2(\kappa(z_1 - z_2)/2)}$$

Analogno, za holomorfne struje višeg spina možemo pisati

$$j^{(\mathrm{s})}_{z \ldots z}(z) = \lim_{z_1 \to z_2} \left(B(s) \sum_{k=1}^{s-1} (-1)^k A^s_k : \partial^k_{z_1} \phi(w(z_1))\, \partial^{s-k}_{z_2} \bar{\phi}(w(z_2)) : \right) + \langle X^{\mathsf{B}}_s \rangle$$

gdje je

$$\langle X^{\mathsf{B}}_s \rangle = B(s) \sum_{k=1}^{s-1} (-1)^k A^s_k \lim_{z_1 \to z_2} \left(\left\langle \partial^k_{z_1} \phi(w(z_1)) \partial^{s-k}_{z_2} \bar{\phi}(w(z_2)) \right\rangle - \left\langle \partial^k_{z_1} \phi(z_1) \partial^{s-k}_{z_2} \bar{\phi}(z_2) \right\rangle \right)$$

Sređivanjem i računanjem potrebnih suma, moguće je pokazati da vrijedi

$$\langle X^{\mathsf{B}}_s \rangle = \hbar\, (-1)^{s+1} (4q)^{s-2} \frac{B_s}{s}\, \kappa^s \tag{62}$$

Holomorfne fermionske struje su definirane pomoću dvodimenzionalnog kompleksnog fermionskog polja Ψ. Opet, radimo u Euklidskom formalizmu u kojem su (u, v) koordinate zamjenjene kompleksnima $(z, \bar{z})$. Korelacijske funkcije fermionskih holomorfnih polja dane su preko

$$\left\langle \Psi^\dagger(z_1) \Psi(z_2) \right\rangle = \frac{\hbar}{z_1 - z_2} \tag{63}$$

Holomorfne fermionske struje koje zadovoljavaju $W_{1+\infty}$ algebru su uvedene u [BPR+90],

$$j^{(\mathrm{s})}_{z \ldots z}(z) = -\frac{B(s)}{s} \sum_{k=1}^{s} (-1)^k \binom{s-1}{s-k}^2 : \partial^{s-k}_z \Psi^\dagger(z)\, \partial^{k-1}_z \Psi(z) : \tag{64}$$

$$B(s) = \frac{2^{s-3} s!}{(2s-3)!!}\, q^{s-2} \quad , \quad s \in \mathbb{N}$$

Uočite da upotrebljavamo konvenciju $(-1)!! = 1$. Struje spina s, $j^{(\mathrm{s})}_{z \ldots z}(z)$, su linearne kombinacije bilineara

$$j^{(\mathrm{m,n})}_{z \ldots z}(z) \equiv\, : \partial^m_z \Psi^\dagger \partial^n_z \Psi :\, = \lim_{z_2 \to z_1} \left(\partial^m_{z_1} \Psi^\dagger(z_1) \partial^n_{z_2} \Psi(z_2) - \partial^m_{z_1} \partial^n_{z_2} \left\langle \Psi^\dagger(z_1) \Psi(z_2) \right\rangle \right)$$

Mi ćemo promatrati transformacije struja povezane s promjenom koordinatnog sustava $z \to w(z)$. Holomorfne fermionske struje transformiraju se prema

$$\Psi(z) = (w'(z))^{\frac{1}{2}}\, \Psi(w) \tag{65}$$

Pomoću toga možemo dokazati da je

$$: \partial_{z_1}^m \Psi^\dagger(z_1)\, \partial_{z_2}^n \Psi(z_2) : = \partial_{z_1}^m\, \partial_{z_2}^n \Big((w_1'(z_1))^{\frac{1}{2}} (w_2'(z_2))^{\frac{1}{2}} : \Psi^\dagger(w_1)\, \Psi(w_2) : \Big) - \partial_{z_1}^m\, \partial_{z_2}^n G(z_1, z_2)$$

gdje smo uveli funkciju

$$G_{\mathrm{F}}(z_1, z_2) \equiv -(w_1'(z_1))^{\frac{1}{2}} (w_2'(z_2))^{\frac{1}{2}} \left\langle \Psi^\dagger(w_1)\, \Psi(w_2) \right\rangle + \left\langle \Psi^\dagger(z_1)\, \Psi(z_2) \right\rangle$$

Upotrebom fermionske korelacijske funkcije slijedi

$$G_{\mathrm{F}}(z_1, z_2) = -\hbar\, \frac{(w_1'(z_1))^{\frac{1}{2}} (w_2'(z_2))^{\frac{1}{2}}}{w_1 - w_2} + \frac{\hbar}{z_1 - z_2}$$

Kao i kod bozonskog slučaja zainteresirani smo za svojstva struja povezana s prijelazom na Kruskalove koordinate,

$$w(z) = -e^{-\kappa z}$$

pri kojem je

$$G_{\mathrm{F}}(z_1, z_2) = G_{\mathrm{F}}(z_1 - z_2) = -\frac{\hbar\,(\kappa/2)}{\mathrm{sh}\left(\frac{\kappa}{2}\,(z_1 - z_2)\right)} + \frac{\hbar}{z_1 - z_2}$$

Analognom procedurom za holomorfne struje višeg spina možemo pisati

$$j^{(\mathrm{s})}_{z\ldots z}(z) = -\frac{B(s)}{s} \sum_{k=1}^{s} (-1)^k \binom{s-1}{s-k}^2 \lim_{z_1 \to z_2} \partial_{z_1}^{s-k} \partial_{z_2}^{k-1}$$

$$\Big((w_1'(z_1))^{\frac{1}{2}} (w_2'(z_2))^{\frac{1}{2}} : \Psi^\dagger(w_1)\, \Psi(w_2) : \Big) + \langle X_s^{\mathsf{F}} \rangle$$

gdje je

$$\langle X_s^{\mathsf{F}} \rangle \equiv -\frac{B(s)}{s} \sum_{k=1}^{s} (-1)^{k+1} \binom{s-1}{s-k}^2 \lim_{z_1 \to z_2} \partial_{z_1}^{s-k} \partial_{z_2}^{k-1} G_{\mathrm{F}}(z_1, z_2)$$

Sređivanjem i računanjem potrebnih suma, moguće je pokazati da vrijedi

$$\langle X_s^{\mathrm{F}} \rangle = -\frac{\kappa^s B_s}{s}\, (1 - 2^{1-s})(4q)^{s-2}\, \hbar \tag{66}$$

Valja uočiti kako je $\langle X_s^{\mathsf{F}} \rangle = 0$ za neparan spin s. Za $s > 1$ to vrijedi zbog toga što je $B_s = 0$ za neparne $s > 1$. Za $s = 1$ to je zbog isčezavanja faktora $(1 - 2^{1-s})$ u izrazu za $\langle X_s^{\mathsf{F}} \rangle$.

Zaključak

U ovoj tezi pokazali smo kako u potpunosti rekonstruirati bozonski i fermionski spektar Hawkingovog zračenja pomoću struja višeg spina. Postupak je poopćenje metode anomalije traga, koju su uveli Christensen i Fulling [CF77]. Na početku promatramo dimenzionalnu redukciju bozonskih i fermionskih bezmasenih polja u blizini horizonta Kerrove crne rupe. Efektivna akcija pripada dvodimenzionalnim bozonskim ili fermionskim poljima vezanima za baždarno polje i dilaton (zanemariv u blizini horizonta) na pozadini (1+1)-dimenzionalne metrike oblika

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} \tag{67}$$

Na ovaj način je problem pojednostavljen zanemarivanjem efekata raspršenja na centrifugalnoj barijeri, a fokus se stavlja na efektivnu fiziku u blizini horizonta. Kako bi smo izračunali više momente spektra Hawkingovog zračenja slijedimo ideju iz [IMU07b]: postuliramo postojanje sačuvanih struja višeg spina. Ove su izgrađene iz bozonskih, odnosno fermionskih bilineara u dvodimenzionalnoj efektivnoj teoriji polja. Struje višeg spina igraju ulogu analognu ulozi tenzora energije i impulsa kod računanja ukupne izračene energije (najnižeg momenta). Prilikom konstrukcije ovih struja potreban nam je neki princip; naš rezultat sugerira da je krucijalna upotreba neke od W algebri.

Konkretno, ovo znači da započnemo od W_∞ algebre (u bozonskom slučaju), odnosno $W_{1+\infty}$ algebre (u fermionskim slučaju), definiranima u apstraktnoj kompleksnoj ravnini. Holomorfne struje koje zadovoljavaju ove algebre konstruirane su u [BK90] i [BPR+90]. Promatranjem transformacijskih svojstava ovih struja (koja odgovaraju prijelazu na Kruskalove koordinate) možemo izračunati njihovu vrijednost na horizontu zahtijevajući regularnost. Rezultati u bozonskom i fermionskom slučaju su, redom,

$$\langle j^{(s)}_{z\ldots z}\rangle_{\mathsf{H}} = \langle X^{\mathrm{B}}_s\rangle = \hbar\,(-1)^{s+1}(4q)^{s-2}\frac{B_s}{s}\,\kappa^s \tag{68}$$

$$\langle j^{(s)}_{z\ldots z}\rangle_{\mathsf{H}} = \langle X^{\mathrm{F}}_s\rangle = -\hbar\,\frac{\kappa^s B_s}{s}\,(1-2^{1-s})\,(4q)^{s-2} \tag{69}$$

Ukoliko identificiramo $j^{(s)}_{z\ldots z}(z)$ putem Wickove rotacije s $j^{(s)}_{u\ldots u}(u)$ možemo dobiti pripadnu vrijednost na horizontu, $\langle j^{(s)}_{u\ldots u}\rangle_{\mathsf{H}}$. S obzirom da je promatrano prostor-vrijeme stacionarno, te $j^{(s)}_{u\ldots u}(u)$ je kiralna veličina, slijedi da je ona konstantna u t i r. Stoga imamo

$$\left\langle J^{(s)r}{}_{t\ldots t}\right\rangle_\infty = \langle J^{(s)}_{u\ldots u}\rangle_\infty - \langle J^{(s)}_{v\ldots v}\rangle_\infty = \langle j^{(s)}_{u\ldots u}\rangle_\infty = \langle j^{(s)}_{u\ldots u}\rangle_{\mathsf{H}} \tag{70}$$

Nadalje, upotrebom regularizacije razdvajanjem točke moguće je iz holomorfnih konstruirati beskonačan skup kovarijantnih struja $J^{(s)}_{u\ldots u}$. Fizikalno, kovarijantne struje predstavljaju više momente spektra izlaznog zračenja u asimptotskoj beskonačnosti (područja u blizini horizonta). Kako se kovarijante struje poklapaju s holomorfnima u beskonačnosti, moguće je povezati veličine $\langle X^{\mathrm{B}}_s\rangle$ i $\langle X^{\mathrm{F}}_s\rangle$ s višim momentima spektra Hawkingovog zračenja; pretpostavljajući da nema ulaznog fluksa iz beskonačnosti dobijemo

$$\left\langle J^{(s)r}{}_{t\ldots t}\right\rangle_\infty = \langle J^{(s)}_{u\ldots u}\rangle_\infty - \langle J^{(s)}_{v\ldots v}\rangle_\infty = \langle j^{(s)}_{u\ldots u}\rangle_\infty = \langle j^{(s)}_{u\ldots u}\rangle_{\mathsf{H}} \tag{71}$$

Kako bi smo izvrijednili više momente, odabiremo $q = i/4$ i dijelimo struje faktorom -2π kako bi smo ispravno normalizirali (fizikalni) tenzor energije i impulsa. Momenti u bozonskom slučaju su

$$-\frac{1}{2\pi}\left\langle J^{(\mathrm{s})r}{}_{t\ldots t}\right\rangle_{\infty} = -\frac{1}{2\pi}\left\langle X_s^{\mathrm{B}}\right\rangle = i^{s-2}\,\frac{\kappa^s B_s}{2\pi s}\,\hbar \tag{72}$$

Desna strana jednakosti isčezava za neparne spinove s (osim za $s = 1$ koji je odsutan u našem slučaju) i poklapa se s višim momentima bozonskog termalnog spektra. U fermionskom slučaju bez elektromagnetskog polja ($m = 0$), momenti iznose

$$-\frac{1}{2\pi}\left\langle J^{(\mathrm{s})r}{}_{t\ldots t}\right\rangle_{\infty} = -\frac{1}{2\pi}\left\langle X_s^{\mathrm{F}}\right\rangle = i^{s-2}\frac{\kappa^s B_s}{2\pi s}\left(1-2^{1-s}\right)\hbar \tag{73}$$

Ovi su dovoljni za rekonstrukciju spektra Hawkingovog zračenja postupkom opisanim u poglavlju 3.5. Konačni rezultat se slaže s onim postignutim orginalnim Hawkingovim pristupom: Bose-Einsteinova raspodijela u bozonskom i Fermi-Diracova raspodijela u fermionskom slučaju. Izrazi za više momente fermionskog Hawkingovog zračenja u prisutnosti baždarnog polja su kompliciraniji; mi smo pokazali da se i ovi slažu s onima izračunatima putem metode Schwarzove derivacije.

Nadalje, promatranjem divergencija kovarijantnih struja, te putem kohomološke analize, pokazali smo odsustvo difeomorfnih anomalija u zakonima sačuvanja struja višeg spina. Eksplicitan primjer takvih struja, manifestno slobodnih od anomalija, su one koje zadovoljavaju W_∞ algebru (u bozonskom slučaju), odnosno $W_{1+\infty}$ algebru (u fermionskim slučaju). Ovaj zaključak je također plauzibilan iz perspektive klasičnog teorema o crnim rupama bez kose: netrivijalne anomalije struja višeg spina bi uvele novi tip “kose” vezan za različite pripadne centralne naboje višeg spina!

Zaključak je stoga kako su viši momenti vezani uz simetriju generiranu W algebrama, a *ne* anomalije, kako se to opetovano tvrdilo u nizu članaka. Motivirani ovim otkrićem predlažemo [BCPS08, BCPS09] hipotezu kako efektivnu teoriju u blizini horizonta događaja crne rupe prati neka skrivena simetrija vezana uz W algebre. Izvor ove simetrije ostaje u ovom trenutku nejasan, te će vjerojatno biti predmetom nekih budućih istraživanja.

Introduction

In this thesis we will show how the details of Hawking radiation spectrum can be calculated from higher spin currents. However, contrary to common belief, conservation laws of these currents are free of anomalies. This can be seen explicitly for those constructed from certain types of W-algebras, but we shall also provide a formal proof of the statement.

The thesis is organized as follows:

- In the first chapter we give a brief review of the black hole thermodynamics, anomalies in quantum field theories and algebras generating symmetries important for this work.

- In the second chapter we present several derivations of the Hawking radiation. Basic idea of the original Hawking's approach is explained on the case of bosonic massless field in somewhat simplified model of collapse. This is complemented by the analysis of thermality and some remarks on the backscattering effects. Two alternative methods are based on diffeomorphism and trace anomalies which are present in effective, near-horizon field theory. We conclude the chapter with comparison of the anomaly methods.

- In the third chapter, the central part of the thesis, we introduce the technique for the reconstruction of the complete of the Hawking radiation spectrum using higher spin currents. Also, by explicit calculation, we show that higher spin W-currents are anomalous-free. This is the evidence that the anomalies are relevant only for total flux and not for the other details of the Hawking radiation.

- The final, fourth chapter is a more technical one, where we present the formal proof of the absence of anomalies in higher spin currents using cohomological methods.

The main results of the thesis are the following

- complete reconstruction of the bosonic and fermionic Hawking radiation spectrum using higher spin currents with dimensional reduction (near-horizon approximation),

- discovery that the W-algebra and, contrary to the "folk theorem", **not** the anomalies underlie the higher moments of the Hawking radiation spectrum,

- conjecture that a certain symmetry, larger than that generated by Virasoro algebra, possibly some of the W-algebras, is somehow related to the effective two-dimensional description of the physics in the vicinity of the black hole event horizon, although the precise origin of such a symmetry remains obscure at this moment.

Chapter 1

Overture

Black holes can be succintly defined as portions of spacetime causaly disconnected from its complement. Physicaly this means that within such region of spacetime gravity is so strong that nothing, not even light, can escape from it. Notion of *escape* can be defined more precisely using asymptotic portion of spacetime. For asymptoticaly flat spacetime $(\mathscr{M}, g_{ab})$, with future null infinity $\mathscr{I}^+$, we define **black hole** region $\mathcal{B}$ as

$$\mathcal{B} = \mathscr{M} - J^-(\mathscr{I}^+) \tag{1.1}$$

where J^- denotes causal past. Similar definitions of black holes can be given for other spacetimes with well defined asymptotic region. The **event horizon** $\mathcal{H}$ of a black hole $\mathcal{B}$ is defined as its boundary,

$$\mathcal{H} = \mathscr{M} \cap \partial(J^-(\mathscr{I}^+)) \tag{1.2}$$

Technicaly, event horizon $\mathcal{H}$ is a null hypersurface which is composed of future inextendible null geodesics without caustics. Due to its "global" definition, event horizon has no local physical significance i.e. it is necessary to know the entire future history of the spacetime in order to determine the location of $\mathcal{H}$.

It is belived that the black holes form when some matter is confined by collapse within certain critical volume. Today we speak of three processes which might lead to a formation of a black hole. One of them is the gravitational collapse of a star with the mass which should be somewhere in the range $2M_\odot \lesssim M \lesssim 100M_\odot$; below the lower limit stars avoid collapse, while above the upper limit stars do not exist at all on account of pulsational instabilities. Second process occurs at the centers of the galaxies where the collapse of the entire central core of a dense cluster of stars lead to formation of *supermassive* black holes with masses up to $\sim 10^{10}M_\odot$. In the third, most speculative process, *primordial* black holes could be formed by gravitational collapse of regions of enhanced density in the early universe.

Given that the gravitational field of initial matter is, in principle, described by an infinite number of multipoles, one would expect that black holes come in multitude of species. Investigation of this question has begun during the *golden age* of general relativity, roughly stretching through 1960s and the first half of 1970s [Tho94]. Early perturbative analysis of a nonspherical collapse by [DZN65] preceded Israel's major breakthrough [Isr67]. Taking completely different strategy in treatment of general, large deviations from spherical collapse, Israel presented the first proof for the uniqueness of the Schwarzschild metric amongst all static vacuum black hole configurations. A series of subsequent results (see for example [Rob09]) culminated in a *no-hair theorem**, stating that stationary, asymptotically flat black hole solution in general relativity coupled to eletromagnetic field, nonsingular outside the event horizon, are completely described by merely a few physical parameters: mass M, angular momentum J and electric charge Q. The detailed analysis [Pri72a, Pri72b] unraveled the responsible physical mechanism for the no-hair theorem; in a gravitational collapse all multipole moments of the asymmetric body are radiated away in the form of gravitational and electromagnetic waves. This result is sometimes stated as *Price's theorem*: anything which can be radiated gets radiated away completely. Some further investigations have shown that no-hair conjecture fail to hold in more general setting [Heu98], for example when non-Abelian or dilatonic fields are present, although some of these either suffer from instability or do not lead to new conserved quantum numbers.

§ 1.1 Black hole thermodynamics

Simplicity of the black holes formed by collapse of complex objects can be compared with statistical systems in thermal equilibrium, which are described by a small set of state variables, in contrast to the enormous amount of information required to understand their dynamical behavior. At the beginning of the 1970s a series of theorems were proven suggesting that the black hole event horizons indeed satisfy relations which exibit analogy with the laws of thermodynamics. Prior to stating them we shall introduce several important concepts.

If an asymptotically flat spacetime $(\mathscr{M}, g_{ab})$ contains a black hole $\mathcal{B}$, then $\mathcal{B}$ is said to be *stationary* if there exists a one-parameter group of isometries generated by Killing field t^a, which is unit timelike at infinity. The black hole $\mathcal{B}$ is *static* if it is stationary and if, in addition, t^a is hypersurface orthogonal. The black hole $\mathcal{B}$ is said to be *axisymmetric* if there exists a one-parameter group of isometries, generated by Killing field ϕ^a, which correspond to rotations at infinity. A stationary, axisymmetric black hole is said to possess the "t-ϕ orthogonality property" if the 2-planes spanned by t^a and ϕ^a Killing fields are orthogonal to a family of 2-dimensional surfaces.

Killing horizon $\mathcal{K}$ is a null surface to which the Killing vector field ξ^a is normal. A priori, the notion of Killing horizon is completely independent of the

*first proposal of *no-hair conjecture* appeared in [RW71] notion of event horizon. However, there are two independent results [Car73, HE73], usually referred to as *rigidity theorems*, that under physicaly reasonable conditions, the event horizon $\mathcal{H}$ of stationary black hole must be the Killing horizon. For every Killing horizon $\mathcal{K}$ with adjecent Killing vector field ξ^a one can introduce the *surface gravity* κ, function defined on $\mathcal{K}$ with equation

$$\nabla^a(\xi^b\xi_b) = -2\kappa\,\xi^a \tag{1.3}$$

Suppose we do an gedankenexperiment in which a particle of mass m is kept stationary by a massless string, with the other side of the string held by stationary observer at infinity. It can be shown [Wal84] that the surface gravity κ is the limiting magnitude of the force that must exterted by this observer to hold a unit test mass in place at the horizon.

Most common technical assumptions, present in black hole theorems, are *cosmic censorship* (absence of naked singularities) or/and some of the energy conditions: *dominant* ($T_{ab}v^av^b \geq 0$ for all timelike vectors v^a and $T_{ab}v^a$ is nonspacelike vector), *weak* ($T_{ab}v^av^b \geq 0$ for all timelike vectors v^a), *null* ($T_{ab}l^al^b \geq 0$ for all null vectors l^a) and *strong* ($T_{ab}v^av^b \geq T^a{}_a\,v^bv_b/2$ for all timelike vectors v^a).

Four basic laws of black holes mechanics can be suggestively presented in the following order:

0$^{\text{th}}$ law

[BCH73] The surface gravity κ is constant on Killing horizon $\mathcal{K}$, provided that Einstein's equations hold with matter energy-momentum tensor satisfying the dominant energy condition.

[Car73] The surface gravity κ is constant on Killing horizon $\mathcal{K}$, provided that the black hole is static or is stationary-axisymmetric with the t-ϕ orthogonality property.

1$^{\text{st}}$ law [BCH73]

The variations of the parameters between two infinitesimally neighboring stationary (and axisymmetric) black hole solutions are related by

$$c^2\delta M = \frac{c^2}{G}\,\frac{\kappa}{8\pi}\,\delta A + \Omega_{\mathsf{H}}\,\delta J + \Phi_{\mathsf{H}}\,\delta Q \tag{1.4}$$

where Ω_{H} denotes the angular velocity of the horizon and Φ_{H} the electrostatic potential (both of which are constant on the horizon).

2$^{\text{rd}}$ law [Haw72] *Area law theorem*

Assuming the weak energy condition and cosmic censorship, the area A of the event horizon in an asymptotically flat spacetime is non-decreasing, $\delta A \geq 0$.

3$^{\text{rd}}$ law [Isr86]

If the energy-momentum tensor is bounded and satisfies the weak energy condition, then the surface gravity of a black hole cannot be reduced to zero within a finite advanced time.

These four laws of black hole mechanics bear a striking resemblance to the laws of thermodynamics, with κ playing the role of temperature T, A that of entropy S and M that of internal energy E. Formaly, this correspondence can be formulated as follows

$$E \leftrightarrow Mc^2 \quad , \quad T \leftrightarrow \frac{c^2}{G}\,\alpha\kappa \quad , \quad S \leftrightarrow \frac{A}{8\pi\alpha} \tag{1.5}$$

where α is some constant. A hint that the relation between black hole laws and thermodynamical laws might be more than just an analogy comes from the fact that E and M are not merely analogs in the formulas but represent the same physical quantity, total energy.

Situation with the 3$^{\text{rd}}$ law is more subtle and deserves an additional comment. There are two formulations of this law in thermodynamics: first one, stating that $T = 0$ cannot be reached by a finite number of steps, and the second one (due to Nernst), stating that the entropy S, as $T \to 0$, tends to a universal constant, which may be taken as zero. First version has its analogy in the black hole mechanics, whereas stronger, Nernst version is *not* satisfied in the case of black holes since, for example, area of Kerr-Newman black hole remains finite as $\kappa \to 0$ (see Appendix A). Nevertheless, it can be argued that the Nernst version of the 3$^{\text{rd}}$ law of thermodynamics should not be viewed as a fundamental law of thermodynamics, but rather as a (common) property of the density of states near the ground state in the thermodynamic limit (for a further discussion see e.g. [Wal97]).

Bekenstein [Bek73] has proposed generalized 2$^{\text{nd}}$ law of thermodynamics: the sum of the ordinary entropy of matter outside of a black hole plus a suitable multiple of the area of a black hole never decreases,

$$\delta\left(S + \frac{A}{8\pi\alpha}\right) \geq 0 \tag{1.6}$$

However, in the context of purely classical general relativity this law could be violated by putting a black hole in a thermal bath at a temperature lower than that formally assigned to the black hole, thereby producing heat flow from a cold body (the bath) to a hotter body (the black hole). Similary, there is a further discrepancy in the proposed analogy: since classical black holes cannot emit anything, their physical temperature is absolute zero! This implies that classicaly there is no physical relation between T and κ, hence thermodynamical analogy would be just mathematical curiosity and it would be inconsistent to assume a physical relationship between A, the area of a black hole, and S, its entropy.

Stephen Hawking resolved these questions [Haw74, Haw75] by discovering that semiclassical black holes can radiate. Quantum effect of particle creation in the vicinity of event horizon results in an effective emission of particles, **Hawking radiation** from a black hole with blackbody spectrum at temperature

$$T_{\mathsf{H}} = \frac{\hbar}{ck_{\mathsf{B}}} \frac{\kappa}{2\pi} \tag{1.7}$$

This result has proven that κ indeed represents the thermodynamic temperature of a black hole and not merely a quantity playing a role mathematically analogous to temperature in the laws of black hole mechanics. The undetermined constant is therefore set to $\alpha = G\hbar/(2\pi k_{\mathsf{B}} c^3)$, so that the *physical* entropy of a black hole is given by

$$S_{\mathsf{BH}} = \frac{k_{\mathsf{B}} c^3}{G\hbar} \frac{A}{4} \tag{1.8}$$

The generalized 2^{nd} law of thermodynamics can now be viewed as ordinary 2^{nd} law of thermodynamics applied to the system containing a black hole. As Hawking radiation escapes to infinity, carring some energy away from the black hole, the radiating black hole will loose its mass and shrink. This, however, doesn't violate the area theorem, since the quantum field energy-momentum tensor doesn't obey the weak energy condition near the horizon.

It should be emphasized that the Hawking radiation of a typical macroscopic black hole is a minute effect. For example, the temperature of a Schwarzschild black hole with mass M is given by

$$T = \frac{\hbar c^3}{8\pi G M k_{\mathsf{B}}} = 6 \cdot 10^{-8} \, \frac{M_{\odot}}{M} \, K \tag{1.9}$$

This means that the Hawking radiation from a realistic astrophysical black hole would be overshadowed even by $2.73\,K$ cosmic microwave background radiation, making it highly difficult to detect in principle. Nevertheless, some alternative proposals for the observation of the Hawking radiation have emerged in recent years. Evaporation of mini black holes, which could be produced at Large Hadron Collider, would be easy to detect due to clean signature with low background [DL01]. Another possibility appears in the context of *acoustic black holes*, causally disconected ("dumb") regions in a fluid, whose event horizons would be source of phononic Hawking radiation [BLV05].

§ 1.2 Witt, Virasoro and *W* algebras

Some physical systems exibit important type of behaviour, scale invariance, described by *conformal* symmetry. Generally our description of physical phenomena involves a number of *characteristic length scales*, so that the scale invariance is not an exact symmetry of Nature. However, important exceptions occur when these characteristic length scales are either zero or infinite. Examples are numerious, including classical electrodynamics, described by Maxwell equations, 2-dimensional systems undergoing second- or higher-order phase transitions, QCD in high-energy limit or effective quantum field theory in the vicinity of the black hole event horizons. What makes this field of research especially attractive is the fact that the 2-dimensional conformal field theories are examples of systems in which the symmetries are so powerful as to allow an exact solution of the problem (for an extensive overview of the subject see [dFMS97]).

Let $\mathscr{M}$ be an n-dimensional manifold with metric g_{ab} of any signature. If Λ is a smooth, strictly positive function, then the metric $\tilde{g}_{ab} = \Lambda\, g_{ab}$ is said to arise from g_{ab} via a *conformal transformation*. It should be emphasized that a conformal transformation is not, in general, associated with a diffeomorphism of $\mathscr{M}$ (see e.g. [Wal84], Appendix D). We shall, however, restrict our considerations to conformal coordinate transformations, that is invertible mapping $x \to x'$, such that

$$\widetilde{g}_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \tag{1.10}$$

If the $\Lambda(x)$ is a constant function we speak of *global* conformal transformations. These form a group which is identified with the noncompact group $SO(n+1,1)$. The group of global conformal transformations consists of translations, dilatations, rigid rotations and special conformal transformations (these are translations, preceded and followed by an inversion $x^\mu \to x^\mu/(x \cdot x)$). Obviously, Poincaré group is its subgroup, since it correspond to the special case $\Lambda(x) \equiv 1$. Now we turn our attention to a local conformal transformations in two dimensions.

1.2.1 Witt algebra

We introduce the (real) coordinates (z^0, z^1) on the plane. Under the coordinate change $z^\mu \to w^\mu$, the condition that defines conformal transformation is

$$g^{\alpha\beta}\,\frac{\partial w^\mu}{\partial z^\alpha}\frac{\partial w^\nu}{\partial z^\beta} = \Lambda^{-1}(z^0, z^1)\, g^{\mu\nu} \tag{1.11}$$

Using the fact that every 2-dimensional Riemannian metric is conformally flat, this condition implies

$$\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2$$

$$\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0}+\frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1}=0$$

These are equivalent to Cauchy-Riemann equations,

$$\frac{\partial w^1}{\partial z^0}=\pm\frac{\partial w^0}{\partial z^1}\quad,\quad\frac{\partial w^0}{\partial z^0}=\mp\frac{\partial w^1}{\partial z^1}\tag{1.12}$$

with the upper sign coresponding to holomorphic functions and the lower sign to antiholomorphic functions. This motivates the introduction of the complex coordinates

$$z=z^0+iz^1\quad,\quad\bar z=z^0-iz^1\tag{1.13}$$

It is important to emphasize that $\bar z$ is *not* the complex conjugate of z, but rather a distinct complex coordinate. Physical space, on the other hand, is the 2-dimensional submanifold (called the *real surface*) defined by $z^*=\bar z$.

We are particulary interested in the local conformal transformations. Any holomorphic infinitesimal transformation may be expressed as

$$z'=z+\epsilon(z)\quad,\quad\epsilon(z)=\sum_{n=-\infty}^{\infty}c_n\,z^{n+1}\tag{1.14}$$

where we have expanded $\epsilon(z)$ into Laurent series. Under this coordinate transformation spinless and dimensionless field $\phi(z,\bar z)$ transforms as

$$\phi'(z',\bar z')=\phi(z,\bar z)=\phi(z',\bar z')-\epsilon(z')\,\partial'\phi(z',\bar z')-\bar\epsilon(\bar z')\,\bar\partial'\phi(z',\bar z')$$

or

$$\delta\phi=-\epsilon(z)\partial\phi-\bar\epsilon(\bar z)\bar\partial\phi=\sum_n\left(c_n\,\ell_n\,\phi(z,\bar z)+\bar c_n\,\bar\ell_n\,\phi(z,\bar z)\right)\tag{1.15}$$

where we have introduced the generators of the local conformal transformation

$$\ell_m=-z^{m+1}\partial_z\qquad,\qquad\bar\ell_m=-\bar z^{m+1}\partial_{\bar z}\tag{1.16}$$

These generators obey the following commutation relations, which define the *Witt algebra*,

$$\begin{gathered}[\ell_m,\ell_n]=(m-n)\ell_{m+n}\\ \left[\bar\ell_m,\bar\ell_n\right]=(m-n)\bar\ell_{m+n}\\ \left[\ell_n,\bar\ell_m\right]=0\end{gathered}\tag{1.17}$$

We can emidiately see that Witt algebra is a direct product of two isomorphic algebras. Each of these is also a Lie algebra of the group of diffeomorphisms of the circle, Lie(Diff(S^1)). Subalgebra $\{\ell_{-1},\ell_0,\ell_1\}$ generates global conformal group; in particular, $\ell_{-1}=-\partial_z$ generates translations on a complex plane, $\ell_0=-z\partial_z$ generates scale transformations and rotations and $\ell_1=-z^2\partial_z$ generates special conformal transformations. The generators that preserve the real surface ($z_0,z_1\in\mathbb{R}$) are the linear combinations

$$\ell_n+\bar\ell_n\qquad\text{and}\qquad i(\ell_n-\bar\ell_n)$$

In particular, $\ell_0+\bar\ell_0$ generates dilations and $i(\ell_0-\bar\ell_0)$ generates rotations.

At this point we introduce several important notions. A *quasi-primary* field Φ transforms according to

$$\Phi'(w,\bar{w}) = (\partial_z w)^{-h}(\partial_{\bar{z}}\bar{w})^{-\bar{h}}\,\Phi(z,\bar{z}) \tag{1.18}$$

under a *global* conformal map $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$. Exponent h is refered as *holomorphic* and $\bar{h}$ as *antiholomorphic conformal dimension*. It can be said that quasi-primary field of conformal dimension $(h,\bar{h})$ transforms as component of a covariant tensor of rank $h+\bar{h}$ having h "z" indices and $\bar{h}$ "$\bar{z}$" indices. If a field Φ transforms according to the rule above under a *local* conformal map, then it is called a *primary* field. It is important emphesize that a primary field is, by definition, also a quasi-primary field, but the converse doesn't allways hold (example being energy-momentum tensor). Technicaly, operator product expansions (OPEs) of the holomorphic and antiholomorhic energy-momentum tensors with a primary field Φ have a form

$$T(z)\Phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\,\Phi(w,\bar{w}) + \frac{1}{z-w}\,\partial_w\Phi(w,\bar{w}) \tag{1.19}$$

$$\bar{T}(\bar{z})\Phi(w,\bar{w}) \sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\,\Phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\,\partial_{\bar{w}}\Phi(w,\bar{w}) \tag{1.20}$$

For example, OPEs of a scalar field ϕ and fermionic field ψ are

$$T(z)\phi(w) \sim \frac{\partial\phi(w)}{z-w} \quad , \quad T(z)\psi(w) \sim \frac{1}{2}\,\frac{\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}$$

Obviously, conformal dimension is 0 in bosonic and $\frac{1}{2}$ in fermionic case. On the other hand, OPE of the energy mometum tensor is

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

Due to a presence of a $(z-w)^{-4}$ term, energy mometum tensor is a quasi-primary field. The constant c is the *central charge*, equal to 1 for the free boson, $\frac{1}{2}$ for the free fermion, etc. The central charge cannot be determined from symmetry considerations; its value is determined by the short-distance behaviour of the theory.

Scaling dimension Δ and *planar spin* s are defined as

$$\Delta \equiv (h+\bar{h}) \quad , \quad s \equiv (h-\bar{h})$$

As h and $\bar{h}$ are eigenvalues of, respectfully, ℓ_0 and $\bar{\ell}_0$, the Δ and s are eigenvalues of $\ell_0+\bar{\ell}_0$ and $i(\ell_0-\bar{\ell}_0)$ (which justifies such nomenclature).

1.2.2 Virasoro algebra

The Virasoro algebra is a central extension of the Witt algebra. We will show (following [dAI95]) that this extension is in fact *unique* up to a choice of some parameter c, so that the name Virasoro algebra actually refers to a class of isomorphic Lie algebras. The general central extension of the Witt algebra can be written as

$$[L_m, L_n] = (m-n)L_{m+n} + c_{m,n} \tag{1.21}$$

defined by a set of constants $c_{m,n}$. From Jacobi identity

$$[[L_m, L_n], L_p] + [[L_n, L_p], L_m] + [[L_p, L_m], L_n] = 0 \tag{1.22}$$

we have

$$(m-n)c_{m+n,p} + (n-p)c_{n+p,m} + (p-m)c_{m+p,n} = 0 \tag{1.23}$$

The algebra doesn't change if one transforms generators by adding constant operators,

$$L_m \to L'_m = L_m + b(m) \tag{1.24}$$

This can be seen as follows,

$$[L'_m, L'_n] = [L_m + b(m), L_n + b(n)] = [L_m, L_n] = (m-n)L_{m+n} + c_{m,n} =$$

$$= (m-n)L'_{m+n} - (m-n)b(m+n) + c_{m,n} = (m-n)L'_{m+n} + c'_{m,n}$$

where

$$c'_{m,n} = c_{m,n} - (m-n)b(m+n)$$

Obviously, we have regained initial form of the algebra by redefining constants $c_{m,n}$. Now we choose

$$b(m) = \frac{c_{m,0}}{m} \quad (m \neq 0) \quad , \quad b(0) = \frac{c_{1,-1}}{2}$$

so that

$$c'_{0,n} = 0 \quad (n \neq 0) \quad , \quad c'_{-1,1} = 0$$

Thus, without loss of generality, we could choose $c_{0,n} = 0$ for $n \neq 0$ and $c_{-1,1} = 0$ from the start. Inserting $p = 0$ in (1.23) we get $(m+n)c_{m,n} = 0$, so that

$$c_{m,n} = c(m)\delta_{m,-n} \tag{1.25}$$

and, because of the antisymmetry of $c_{m,n}$, $c(-m) = -c(m)$. In order to find the expression for $c(m)$ we put $p = -(m+1)$ and $n = 1$ in (1.23):

$$(m-1)c(m+1)\delta_{m+1,n+1} + (m+2)c(-m)\delta_{-m,-m} = 0$$

$$(m-1)c(m+1) - (2+m)c(m) = 0$$

For $m = 1$ we get $c(1) = 0$ and from this, inserting $m = 0$, $c(0) = 0$. For all $m > 1$, we have

$$c(m+1) = \frac{m+2}{m-1}\,c(m)$$

Solution of this recursion is

$$c(m) = \frac{(m+1)!}{(m-2)!}\,\frac{c(2)}{3!} = \frac{1}{6}\,(m^3 - m)c(2)$$

If we define $c \equiv 2c(2)$ we finally have

$$\begin{aligned} c(m) &= \tfrac{1}{12}(m^3 - m)c \quad , \quad m \geq 2 \\ c(-m) &= -c(m) \quad , \quad c(0) = c(1) = c(-1) = 0 \end{aligned} \tag{1.26}$$

It remains to be shown that this extension is non-trivial, i.e. that there is no $b(m)$, such that

$$-(m-n)b(m+n) = \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

This is impossible since for $m + n = 0$ we would have condition

$$-2mb(0) = \frac{c}{12}(m^3 - m)$$

which, obviously, cannot be fulfilled for every m. Completely analogous procedure with $\bar{L}_n$ generators would lead to the set of central terms with the same properties, defined by some new constant $\bar{c}$. Altogether, we have proved that Virasoro algebra, the central extension of Witt algebra, has the following form

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}\, m(m^2 - 1)\delta_{m+n,0} \\ [\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12}\, m(m^2 - 1)\delta_{m+n,0} \\ [L_m, \bar{L}_n] &= 0 \end{aligned} \tag{1.27}$$

The generators L_m and $\bar{L}_m$ can be viewed as the coefficients in Laurent expansion of the holomorphic and antiholomorphic energy-momentum tensor in two dimensions (see e.g. [dFMS97], chapter 6). For this reason, Virasoro algebra is sometimes described as quantum version of the Witt algebra or, vice versa, Witt algebra as a classical limit of the Virasoro algebra.

1.2.3 *W* algebras

Since the Virasoro algebra can be viewed as the algebra of the quasi-primary spin-2 current $T(z)$, it is natural to seek for the extended algebras by the inclusion of currents of higher and, possibly, lower spins. The simplest higher-spin extension of the Virasoro algebra, W_3 algebra, was found by Zamolodchikov [Zam85]. It consists of two currents, the energy-momentum tensor $T(z)$ and a spin-3 primary current $W(z)$. Subsequently, this was generalized to the family of W_N algebras [FZ87, FL88], built from energy-momentum tensor $T(z)$ and higher-spin primary currents of each spin $3 \leq s \leq N$. A general feature of all the W_N algebras for $N \geq 3$ is that they are *non-linear*. This is a consequence of the fact that the commutator of generators of spin s and s' produces spin $s + s' - 2$. In algebra with highest spin N it is necessary to interpret terms with spin exceeding N as composite fields, built from multi-linear products of the basic fields.

This feature can be circumvented in the limit where $N \to \infty$, since then the commutators cannot produce spins larger that N. Indeed, there are linear Lie algebras which can be obtained by taking appropriate $N \to \infty$ limits of W_N. However, the process of taking such a limit is not a unique one, since many different N-dependent scalings of generators and structure constants can be chosen prior to setting $N = \infty$. One of those, the first to be discovered [Bak89] is known as w_∞ algebra. Compared to the W_N algebras, this has a much simpler structure since the only central term (allowed by Jacobi identities) is the one in the Virasoro subalgebra. There is an enlarged algebra, $w_{1+\infty}$, containing a spin-1 generator in addition to those of w_∞. Another limit, found in [PRS90a, PRS90c], is W_∞ algebra, generated by currents with conformal spins $2 \leq s \leq \infty$. As opposed to w_∞, this algebra admits central terms for *all* conformal spins. Again, there is related linear algebra, $W_{1+\infty}$, generated by currents with conformal spins $1 \leq s \leq \infty$ [PRS90b].

We shall adopt the usual convention of denoting the generators of spin s by V^i (where $s = i + 2$) and their m^{th} Laurent mode by V^i_m. We can make the following ansatz for the commutator

$$[V^i_m, V^j_n] = \sum_{\ell \geq 0} g^{ij}_{2\ell}(m,n) V^{i+j-2\ell}_{m+n} + c_i(m)\delta^{ij}\delta_{m+n,0} \tag{1.28}$$

where $g^{ij}_{2\ell}(m,n)$ are the structure constants of the algebra and $c_i(m)$ are the central terms. These can be determined by demanding the ansatz to be consistent with the Jacobi identities (explicit expressions can be found in e.g. [Pop91]). Both W_∞ and $W_{1+\infty}$ algebra have same form, (1.28), but with different structure constants and central terms. Another crucial difference is that in W_∞ algebra we have $i, j \geq 0$ (so that $s \geq 2$), while in $W_{1+\infty}$ we have $i, j \geq -1$ (so that $s \geq 1$).

There is a simple contraction of the W_∞ algebra, which can be obtained by rescaling the generators V^i_m with parameter q,

$$V^i_m \to q^{-i}\, V^i_m$$

changing the commutation relations to

$$[V^i_m, V^j_n] = \sum_{\ell \geq 0} q^{2\ell} g^{ij}_{2\ell}(m,n) V^{i+j-2\ell}_{m+n} + q^{2i} c_i(m)\delta^{ij}\delta_{m+n,0}$$

In the limit when $q \to 0$, only the highest-spin generator ($\ell = 0$) term on the RHS survives, together with the central term for $i = j = 0$ (the Virasoro subsector). If we denote the generators for this contraction of the algebra by v^i_m, then we have

$$[v^i_m, v^j_n] = ((j+1)m - (i+1)n)\, v^{i+j}_{m+n} + \frac{c}{12}\, m(m^2-1)\delta^{i,0}\delta^{j,0}\delta_{m+n,0}$$

known as w_∞ algebra. In some sence, w_∞ can be viewed as the classical limit of the quantum W_∞ algebra. Note that it admits a central extension only in the Virasoro subalgebra, generated by $L_m = v^0_m$. A contracton of $W_{1+\infty}$ analogous to that of W_∞, yeilds $w_{1+\infty}$ as a classical limit.

From a geometrical point of view, the w_∞ and $w_{1+\infty}$ algebras are related to area-preserving diffeomorphisms of a surface Σ, denoted by sDiff(Σ). In order to find the structure constants of sDiff algebras explicitly it is useful to expand their generators in a suitable basis. The choice of the basis depends on the topology and geometry of the surface Σ, as well on the type of functions we wish to expand on it. Usually, there is a lot of arbitrariness in this choice and the problem becomes more acute when one deals with noncompact surfaces, such as $\mathbb{R}^2$ or a cylinder $\mathbb{R} \times S^1$. In this case one usually works with basis functions which diverge at infinity. However, for a surface of cylindrical topology there is a particular choice of basis elements appropriate which, though divergent at infinity, yields a specific set of structure constants. An overview of this subject can be found in [Sez92].

§ 1.3 Anomalies in QFT

Symmetries and their corresponding conservation laws are the backbone of physical description of the fundamental forces of nature. They can be a guiding principle in construction of lagrangians, such are spacetime or local gauge symmetries, or a powerful tool in perturbative calculations (Ward and Slavnov-Taylor indentities). Some symmetries, however, are not protected when quantum effects are taken into account. When certain classical conservation law is violated in the quantized version we speak of **anomaly**. Historically, anomalies appeared in the late 1940s as a puzzle within pion-nucleon model treatment of the decay of neutral pion, $\pi^0 \to \gamma\gamma$. Initial perturbative calculations gave the prediction for the decay rate $\Gamma \sim m_\pi^7$ in collision with the observed $\Gamma \sim m_\pi^3$ (see [Wei96]). Explanation was found in anomaly violating the global chiral symmetry of the strong interactions. This came independently by [BJ69], working with σ-model and [Adl69] working in spinor electrodynamics. By calculating famous triangle Feynman diagram made up of one axial and two vector currents they found that while conservation of the vector current can be maintained, the conservation law of the axial current is broken,

$$\partial^\mu j_\mu = 0 \quad , \quad \partial^\mu j_\mu^5 = \mathcal{A} \tag{1.29}$$

where $\mathcal{A}$ represents ABJ anomaly, named after its discoverers,

$$\mathcal{A} = \frac{e^2}{16\pi^2}\,\varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \tag{1.30}$$

Another remarkable discovery came with Adler-Bardeen theorem [AB69]: the anomaly is completely determined at the 1-loop level and receives no contributions from radiative corrections!

Subsequent investigation of the subject provided several different explanations of the anomalies and, along with them, a deeper insight in the structure of the QFT. Originally, anomalies were observed as symmetry violation present in the regularization procedure after the regulator has been removed. By the end of the 1970s it was discovered that the singlet anomaly is determined by an index theorem (for an overview see [Ber96] or [FS04]). The reason is that the anomaly can be expressed by a sum of eigenfunctions φ_n of the Dirac operator $\not{D}$, where only the zero-modes of a given chirality survive,

$$\frac{1}{2i}\int dx\, \mathcal{A}(x) = \int dx \sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) = n_+ - n_- = \mathrm{index}(D_+) \tag{1.31}$$

$$D_+ = \not{D}P_+ \quad , \quad P_+ = \frac{1+\gamma_5}{2}$$

Using Atiyah-Singer index theorem one can express index in terms of characteristic classes; in 4 dimensional case,

$$\mathrm{index}(D_+) = -\frac{1}{8\pi^2}\int \mathrm{tr}\, FF \tag{1.32}$$

Fujikawa [Fuj79, Fuj80] approached the anomalies using path-integral framework for quantized fermions in an external gauge field A_μ,

$$Z[A_\mu] = \int \mathscr{D}\psi\mathscr{D}\bar{\psi}\, e^{iS[A_\mu]} \quad , \quad S = \int dx\, \bar{\psi}(i\not{D} - m)\psi \tag{1.33}$$

pointing out that, contrary to the action S, path-integral measure is *not* gauge invariant. Namely, under chiral transformation

$$\psi \to e^{i\beta(x)\gamma_5}\psi \quad , \quad \bar{\psi} \to \bar{\psi}e^{i\beta(x)\gamma_5}$$

path integral measure transforms as

$$\mathscr{D}\psi\mathscr{D}\bar{\psi} \to \mathscr{D}\psi\mathscr{D}\bar{\psi}\, \exp\left(-2i\int dx\, \beta(x)\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x)\right) \tag{1.34}$$

After appropriate regularization we get the anomaly contribution,

$$2\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) = -\frac{e^2}{16\pi^2}\,\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

inside of the Jacobian.

A modern differential geometric treatment of anomalies was initiated by Biedenharn [Bie] in 1972 and later developed by Bonora, Bregola, Cotta-Ramusino, Pasti, Stora, Zumino and others. The anomalies can be rewritten using language of differential forms and then defined as nontrivial cocycles with respect to BRS operator. Also, it is possible to contruct a chain of descent equations where several polynomials in ghost v and gauge field A, so-called *chain terms*, are linked together in different dimensions (see Appendix C). The amazing fact is that in this way one can relate singlet anomaly in $2n$ dimensions with the non-Abelian anomaly in $(2n-2)$ dimensions.

According to the type of the broken symmetry we speak of two general types of anomalies, *rigid* and *local* ones. From another point of view, anomalies can be written in *consistent* form, so that they satisfy WZ consistency condition, or *covariant* form, so that they behave covariantly under the symmetry transformations of the classical theory. Covariant and consistent anomalies are clearly distinguished by another fact: consistent anomalies appear as breakdowns of classical conservations of currents that are directly coupled in the action to the potentials of the theory; covariant anomalies, on the other hand, are linked to currents that do not couple to the potentials of the theory. This implies difference in their physical relevance. The presence of a consistent anomaly in a theory reveals a conflict between renormalizability and unitarity (anomalous Ward identities destroy the renormalizability); it is a generally accepted point of view that such a theory (in more that two dimensions) is inconsistent from a perturbative point of view. This provides a strong restriction to the physical content of a theory, a famous example being cancellation of anomalies in the Standard Model. Covariant anomalies may be of help in explanation of phenomena which takes place although they are forbidden by some classical invariance (e.g. $\pi^0 \to \gamma\gamma$, $K^+ \to \pi^+\pi^0\gamma$, $\gamma\gamma \to \pi^+\pi^-\pi^0$, etc.).

The consistent anomalies can be split into two subfamilies: those of *chiral type* (gauge, local Lorentz, diffeomorphism anomalies and their supersymmetric version), which appear only in chirally asymmetric theories, and those of *conformal type*, which exist in chiral and nonchiral theories as well.

1.3.1 BRS symmetry

Suppose we have massive fermionic matter field ψ coupled to the non-Abelian gauge field $A_\mu = A^a_\mu T^a$ where $\{T^a\}$ are the generators of the corresponding gauge group G. The total Lagrangian for this non-Abelian gauge theory consists of four parts,

$$\mathscr{L} = \mathscr{L}_\psi + \mathscr{L}_{\mathsf{YM}} + \mathscr{L}_{\mathsf{Gfix}} + \mathscr{L}_{\mathsf{FP}}$$

$$\mathscr{L}_\psi = \bar{\psi}\,(i\not{D} - m)\,\psi \quad , \quad \mathscr{L}_{\mathsf{YM}} = -\frac{1}{4}\,F^a_{\mu\nu}F^{a\mu\nu} \tag{1.35}$$

$$\mathscr{L}_{\mathsf{Gfix}} = -\frac{1}{2\zeta}\,(\partial_\mu A^{a\mu})^2 \quad , \quad \mathscr{L}_{\mathsf{FP}} = -\bar{v}^a \partial^\mu D^{ab}_\mu v^b$$

Gauge fixing term $\mathscr{L}_{\mathsf{Gfix}}$ allows us to choose some particular gauge via parameter ζ. The fourth term is a Lagrangian for Faddeev-Popov ghost, $v(x) = v^a(x)\,T^a$, and antighost fields $\bar{v}(x) = \bar{v}^a(x)\,T^a$.

Now, since the gauge fixing term manifestly breaks the gauge covariance of the total Lagrangian, one would like to provide some "enlarged" symmetry which confines the form of the Lagrangian to that of $\mathscr{L}$ and which, consequently, remains unbroken after the choice of the gauge. Such symmetry is generated by so called *Becchi-Rouet-Stora* or *BRS operator* s [BRS74, BRS75]. This operator is, by definition, nilpotent $s^2 = 0$ and anticommutes with ghost and antighost fields, $sv + vs = 0 = s\bar{v} + \bar{v}s$. We define its action upon the matter and gauge fields by turning the gauge parameter into a ghost one,

$$s\psi = -v\psi \quad , \quad s\bar{\psi} = \bar{\psi}v \quad , \quad sA_\mu = D_\mu v \tag{1.36}$$

Signs are chosen so that we immediately have

$$s\mathscr{L}_\psi = 0 \quad , \quad s\mathscr{L}_{\mathsf{YM}} = 0 \tag{1.37}$$

The transformation property of the ghost field follows from the nilpotency,

$$0 = s^2\psi = -s(v\psi) = -(sv)\psi + v(s\psi) = -(sv)\psi - v^2\psi \ ,$$

and therefore

$$sv = -v^2 \tag{1.38}$$

Gauge condition,

$$f[A_\mu] = \partial^\mu A_\mu = 0 \tag{1.39}$$

can be incorporated by introduction of the *auxiliary field* $b(x) = b^a(x)T^a$. We choose as the gauge fixing term

$$\mathscr{L}_{\mathsf{Gfix}} = b^a f^a[A_\mu] + \frac{\zeta}{2}\, b^a b^a \tag{1.40}$$

Corresponding equation of motion for the field b^a is

$$\frac{\delta\mathscr{L}_{\mathsf{Gfix}}}{\delta b^a} = f^a[A_\mu] + \frac{\zeta}{2}\, b^a = 0 \quad \Rightarrow \quad b^a = -\frac{2}{\zeta}\, f^a[A_\mu]$$

so that on-shell we recover gauge condition,

$$\mathscr{L}_{\mathsf{Gfix}} \approx -\frac{1}{2\zeta}\,(\partial^\mu A^a_\mu)^2$$

Due to canonical dimension 2 of auxiliary field b, it cannot aquire a kinetic term in renormalizable Lagrangian. In order to get remaining part of the total Lagrangian BRS invariant we choose the remaining transformations by

$$s\bar{v}^a = b^a \quad , \quad sb^a = 0 \tag{1.41}$$

$$\begin{aligned}\mathscr{L}_{\mathsf{Gfix}} + \mathscr{L}_{\mathsf{FP}} &= b^a f^a[A_\mu] + \frac{\zeta}{2}\, b^a b^a - \bar{v}^a s f^a[A_\mu] = \\ &= s(\bar{v}^a f^a[A_\mu]) + \frac{\zeta}{2}\, b^a b^a\end{aligned}$$

which is manifestly BRS invariant and consequently, $s\mathscr{L} = 0$.

It is customary to translate this formalism into a laguage of differential geometry. Usual, “minimalistic” notation keeps the form and gauge content of the objects, as well as wedge products of the forms implicit; for example

$$A = A_\mu dx^\mu = A^a_\mu\, T^a dx^\mu \quad , \quad F = \frac{1}{2}\, F_{\mu\nu} dx^\mu dx^\nu = \frac{1}{2}\, F^a_{\mu\nu}\, T^a dx^\mu dx^\nu$$

$$Dv = -D_\mu v\, dx^\mu = -D^{ab}_\mu v^b\, T^a dx^\mu$$

Note that in the last example an extra minus sign has been "produced" by pushing the form dx^μ, over the ghost v, to the far right side. Sometimes, in order to indicate that some Q is a p-form containing m ghost fields, we shall add corresponding upper and lower indices, Q_p^m. Now we construct extended, *graded algebra*, combining matter, gauge and ghost fields with exterior derivative d and BRS operator s, postulating

$$sd + ds = 0 \tag{1.42}$$

Assigning a ghost number to each of the fields ($+1$ to v, -1 to $\bar{v}$ and 0 to all the others), we can define the commutator between fields by

$$[P, Q] = PQ - (-1)^{\deg P \cdot \deg Q}\, QP \tag{1.43}$$

where $\deg P$ and $\deg Q$ denote the *total degree* (sum of form degree and ghost number) of, respectfully, P and Q. For example,

$$sA = -Dv \quad , \quad sF = [F, v] = Fv - vF$$

1.3.2 Algebraic approach to anomalies

The invariance of the quantum action $W = -i \ln Z$ can be succintly expressed using BRS operator s,

$$s\, W[A] = 0 \tag{1.44}$$

This is the *Ward identity*† whose validity is necessary to get a non-Abelian gauge theory renormalized. If it happens that the invariance is broken, the anomaly appears. Anomalous Ward identity is generally of form

$$s\, W[A] = G(v, A) \neq 0 \tag{1.45}$$

Anomaly can be expressed as integral over 4-dimensional manifold M,

$$G(v, A) = \int_M v^a G^a[A] \tag{1.46}$$

The 4-form $v^a G^a[A]$ is linear in ghost field v, local in both gauge A and ghost field v and determined up to an exact form. As a consequence of the nilpotency of the BRS operator s we get the *Wess-Zumino consistency condition* [WZ71],

$$s^2\, W[A] = s\, G(v, A) = s \int_M v^a G^a[A] = 0 \tag{1.47}$$

This equation always has a trivial solution of a form

$$G_{\text{triv}}(v, A) = s\, \widehat{G}[A] \tag{1.48}$$

†sometimes called *Slavnov-Taylor identity*, although the letter term is primarily used in the context of non-Abelian theories

where $\widehat{G}[A]$ represents a local polynomial in gauge field A. Such local functional $\widehat{G}[A]$ corresponds to an additional term in the quantum action which just redefines the regularization scheme and does not alter the anomalous content of the gauge theory. The *consistent anomaly* is defined as a nontrivial solution of the WZ consistency condition, local functional which is closed but not exact under BRS operator s.

Using Poincaré lemma we can rewrite the WZ consistency condition in a local way,

$$s(v^a G^a[A]) = -dQ_3^2(v, A) \tag{1.49}$$

The term Q_3^2 is a 3-form polynomial in v and A with ghost number 2. The systematic way to solve this equation (see Appendix C) is through the so called *chain of descent equations*,

$$\begin{aligned}
P_n(F) - dQ^0_{2n-1} &= 0 \\
sQ^0_{2n-1} + dQ^1_{2n-2} &= 0 \\
sQ^1_{2n-2} + dQ^2_{2n-3} &= 0 \\
&\vdots \\
sQ^{2n-2}_1 + dQ^{2n-1}_0 &= 0 \\
sQ^{2n-1}_0 &= 0
\end{aligned} \tag{1.50}$$

The symmetrized trace of the curvatures, $P(F^n)$ represents the singlet anomaly in $2n$ dimensions. The third equation in the chain of descent equations is a local form of WZ consistency condition. The term Q^1_{2n-2} represents the consistent form of the non-Abelian anomaly in $2n-2$ dimensions,

$$G(v, A) = N \int Q^1_{2n-2}(v, A) \quad , \quad N = \pm(-2\pi i)\frac{i^n}{(2\pi)^n n!} \tag{1.51}$$

where the sign $\pm$ stands for positive and negative chirality fields. The normalization N is not fixed by the chain of descent equations, but can be calculated by other methods; perturbatively, by Fujikawa path integral formalism or by topological analysis (Atiyah-Singer index theorem). It should be noted that the choice of initial invariant polynomial of the chain dependes on the physical theory (generally, it is a linear combination of trace products of different gauge forms F).

Anomalies may occur in two different forms, *consistent* or *covariant*, corresponding to the two different, consistent or covariant currents. The consistent current is defined as the variation of the vacuum functional. The covariant divergence of this current satisfies the WZ consistency condition and the consistent anomaly is a part of the chain of descendent equations. However, due to the anomaly, the consistent current does not transform covariantly under gauge transformation. Bardeen and Zumino discovered [BZ84] that it is possible to construct a gauge covariant current leading to the covariant type of non-Abelian anomaly. The trick is to add a polynomial to the consistent current which gauge transforms in opposite way in order to cancel the unwanted anomalous term. However, the resulting covariant current cannot be obtained from the variation of a vacuum functional and its covariant divergence (covariant anomaly) does not satisfy the WZ consistency condition. Consequently the covariant anomaly is not part of chain, but some more complicated algebraic equations. In the case of gauge symmetries it is possible to make a clear distinction between these two types of anomalies: consistent anomalies are anomalies of currents minimally coupled to the propagating gauge fields; covariant anomalies are anomalies of currents coupled to the external nonpropagating vector or axial fields. The letter can be used for phenomenological purposes, as in the case of the ABJ anomaly.

1.3.3 Gravitational anomalies

Throughout this thesis we shall mainly focus on the gravitational anomalies. As in the case of Yang-Mills fields, only the matter fields are quantized; the gravitation field, represented by metric $g_{\mu\nu}$ or vielbeins e^a_μ, is considered as external, nonquantized field. Generating functional Z is defined as

$$Z[{e^a}_\mu] = e^{-W[e^a_\mu]} = \int \mathscr{D}\psi \mathscr{D}\bar{\psi}\, e^{-S_\psi[e^a_\mu]}$$

Action for massless fermionic field ψ in curved spacetime

$$S_\psi = \int d^n x\, e\, \mathscr{L}_\psi = \int d^n x\, e\, \bar{\psi} i \not{D} \psi$$

with determinant $e \equiv |\det({e^a}_\mu)| = \sqrt{-g}$. The Dirac operator $\not{D}$ is defined as

$$\not{D} = \gamma^\mu(x) D_\mu = {e^a}_\mu(x)\gamma^a D_\mu$$

We shall overview the three important types of symmetries.

- infinitesimal local Lorentz transformation,

$$\delta^{\mathrm{L}}_\alpha\, {e^a}_\mu = -{\alpha^a}_b {e^b}_\mu \quad , \quad \delta^{\mathrm{L}}_\alpha\, {e_a}^\mu = {\alpha^b}_a {e_b}^\mu \quad , \quad \delta^{\mathrm{L}}_\alpha\, {\omega^a}_{b\mu} = D_\mu {\alpha^a}_b$$

$$\delta^{\mathrm{L}}_\alpha\, \psi = -\frac{1}{2}\,\alpha_{ab}\sigma^{ab}\psi \quad , \quad \delta^{\mathrm{L}}_\alpha\, \bar{\psi} = \frac{1}{2}\,\alpha_{ab}\bar{\psi}\sigma^{ab}$$

- infinitesimal diffeomorphism transformation

$$\delta^{c}_\xi\, {e^a}_\mu = \xi\cdot\partial {e^a}_\mu + {e^a}_\nu \partial_\mu \xi^\nu \quad , \quad \delta^{c}_\xi\, {e_a}^\mu = \xi\cdot\partial {e_a}^\mu - {e_a}^\nu \partial_\nu \xi^\mu$$

$$\delta^{c}_\xi\, {\omega^a}_{b\mu} = \xi\cdot\partial {\omega^a}_{b\mu} + {\omega^a}_{b\nu}\partial_\mu\xi^\nu \quad , \quad \delta^{c}_\xi\, \psi = \xi\cdot\partial\psi \quad , \quad \delta^{c}_\xi\, \bar{\psi} = \xi\cdot\partial\bar{\psi}$$

- infinitesimal conformal (Weyl) transformation,

$$\delta^{\mathrm{W}}_{\sigma}\, e^a{}_\mu = \sigma e^a{}_\mu \quad , \quad \delta^{\mathrm{W}}_{\sigma}\, e_a{}^\mu = -\sigma e_a{}^\mu \quad , \quad \delta^{\mathrm{W}}_{\sigma}\, \omega^a{}_{b\mu} = \partial_\nu \sigma \left(e^a{}_\mu e_b{}^\nu - e_{b\mu} e^{a\nu} \right)$$

$$\delta^{\mathrm{W}}_{\sigma}\, \psi = -r\sigma\psi \quad , \quad \delta^{\mathrm{W}}_{\sigma}\, \bar{\psi} = -r\sigma\bar{\psi}$$

where

$$r = \frac{m-1}{2} \quad , \quad m = \dim(\mathscr{M})$$

The **Lorentz anomaly** is defined by

$$\delta^{\mathrm{L}}_{\alpha}\, W[e^a{}_\mu] = \int dx\, e\, \alpha_{ab} \left\langle T^{ab} \right\rangle \equiv G^{\mathrm{L}}(\alpha) \tag{1.52}$$

Since α_{ab} is antisymmetric, the Lorentz anomaly is equivalent to the existence of an atisymmetric part of the energy-momentum tensor,

$$\left\langle T^{ab} \right\rangle = -\left\langle T^{ba} \right\rangle \tag{1.53}$$

The **diffeomorphism** (**Einstein**) **anomaly** is defined by

$$\delta^{\mathrm{c}}_{\xi}\, W[e^a{}_\mu] = -\int dx e\, \xi^\nu (\nabla_\mu \left\langle T^\mu{}_\nu \right\rangle - \omega_{ab\nu} \left\langle T^{ab} \right\rangle) \equiv G^{\mathrm{E}}(\xi) \tag{1.54}$$

This anomaly is related to nonconservation of energy-momentum tensor,

$$\nabla_\mu \langle T^{\mu\nu} \rangle \neq 0 \tag{1.55}$$

but even if diffeomorphism anomaly is absent $G^{\mathrm{E}}(\xi) = 0$, the nonconservation may hold due to a presence of the Lorentz anomaly. However, it is possible to make energy-momentum tensor symmetric, so that the second term in (1.54) vanishes and thus the diffeomorphism anomaly implies the nonconservation of energy-momentum tensor.

The **Weyl** (**trace**) **anomaly**, defined by

$$\delta^{\mathrm{W}}_{\sigma}\, W[e^a{}_\mu] = \int dx e\, \sigma \left\langle T^\mu{}_\mu \right\rangle \equiv G^{\mathrm{W}}(\sigma) \tag{1.56}$$

implies nonvanishing trace of energy-momentum tensor,

$$\left\langle T^\mu{}_\mu \right\rangle \neq 0 \tag{1.57}$$

Consistency conditions are simplified by introduction of ghosts (anticommuting Grassmann variables) and the BRS operator. We shall consider Einstein ghost $\xi^a(x)$, Lorentz ghost $\alpha_{ab}(x)$ and Weyl ghost $\sigma(x)$ corresponding to, respectfully, diffeomorphism, local Lorentz and Weyl (conformal) transformation.

Accordingly, the infinitesimal transformations become BRS operators, $\delta^{c}_{\xi} \to s_{\rm E}$, $\delta^{\rm L}_{\alpha} \to s_{\rm L}$ and $\delta_{\sigma} \to s_{\rm W}$. These operators act on the vielbein as follows

$$\begin{array}{rcl} s_{\rm E}\, e^{a}{}_{\mu} & = & \xi\cdot\partial e^{a}{}_{\mu} + e^{a}{}_{\nu}\partial_{\mu}\xi^{\nu} \\ s_{\rm L}\, e^{a}{}_{\mu} & = & -\alpha^{a}{}_{b}e^{b}{}_{\mu} \\ s_{\rm W}\, e^{a}{}_{\mu} & = & \sigma e^{a}{}_{\mu} \end{array} \tag{1.58}$$

The *total* BRS operator s is defined by

$$s = s_{\rm E} + s_{\rm L} + s_{\rm W} \tag{1.59}$$

BRS operator s is nilpotent and anticommutes with the exterior derivative,

$$s^2 = 0 \quad , \quad sd + ds = 0 \tag{1.60}$$

From condition $s^2 e^{a}{}_{\mu} = 0$ we get the transformation properties of the ghosts,

$$\begin{array}{rcl} s\,\xi^{a}(x) & = & \xi\cdot\partial\,\xi^{a} \\ s\,\alpha^{a}{}_{b} & = & \xi\cdot\partial\,\alpha^{a}{}_{b} - \alpha^{a}{}_{c}\alpha^{c}{}_{b} \\ s\,\sigma & = & \xi\cdot\partial\,\sigma \end{array} \tag{1.61}$$

The total gravitational anomaly is defined by

$$s\,W[e] = G^{\rm grav}_{\rm tot}(e,\xi,\alpha,\sigma) = G^{\rm E}(e,\xi) + G^{\rm L}(e,\alpha) + G^{\rm W}(e,\sigma) \tag{1.62}$$

The nilpotency of the total BRS operator s implies consistency condition for the gravitational anomaly,

$$s\,G^{\rm grav}_{\rm tot}(e,\xi,\alpha,\sigma) = 0 \tag{1.63}$$

The trivial solution is, of course, $G^{\rm grav}_{\rm tot} = s\widehat{G}[e]$, where $\widehat{G}[e]$ represents a local functional. In order to find true gravitational anomaly one has to find local functional in the vielbein and in the ghosts which are linear in the ghosts and closed under s but not exact. This can be done using the chain of descent equations, starting from invariant polynomial $P_n(R)$, where $R = d\Gamma + \Gamma^2$ or $R = d\omega + \omega^2$. The chain term Q^1_{2n-1} represents either the pure diffeomorphism anomaly,

$$G^{\rm E}(v_{\xi},\Gamma) = -\int Q^1_{2n-1}(v_{\xi},\Gamma,R) \tag{1.64}$$

where $v_{\xi} = \partial\xi$ in the matrix notation, or the Lorentz anomaly,

$$G^{\rm L}(\alpha,\omega) = -\int Q^1_{2n-1}(\alpha,\omega,R) \tag{1.65}$$

Here we are assuming that the normalization is included in Q^1_{2n-1}. For example, the 2-dimensional gravitational anomalies have a form

$$Q^1_2(v_{\xi},\Gamma) \sim {\rm tr}(v_{\xi}\Gamma) \quad , \quad Q^1_2(\alpha,\omega) \sim {\rm tr}(\alpha d\omega) \tag{1.66}$$

In fact, Bardeen and Zumino have shown [BZ84] that the Einstein anomalies and Lorentz anomalies are equivalent.

Due to antisymmetry of ${R^a}_b$, the invariant polynomial of odd degree vanishes,

$$P_{2k+1}(R) = 0 \tag{1.67}$$

Consequently, the gravitational anomaly can occur only in

$$2n - 2 = 4k + 2 = 2, 6, 10, \ldots \quad \text{dimensions!}$$

Chapter 2

Hawking radiation

Gravitational collapse cannot be described by the spacetime which is everywhere stationary. Consequently, the initial vacuum state will not be the same as the final vacuum state. Put in another words, time dependent metric will cause the creation of a certain number of particles. Since the exterior spacetime is stationary at late times, one might expect particle creation to be just a transient phenomenon determined by details of the collapse. However, infinite time dilation at the horizon of a black hole means that particles created in the collapse can take arbitrarily long time to escape. This suggests that a possible flux of particles at late times that is due to the existence of the horizon and independent of the details of the collapse.

§ 2.1 Original Hawking's approach

In the original derivation of the Hawking effect [Haw74, Haw75] the author uses a formalism for calculating particle creation in a curved spacetime that had been developed by Parker [Par69, Par71] and others. Hawking has considered a classical spacetime $(\mathscr{M}, g_{ab})$, describing gravitational collapse to a Schwarzschild black hole, and a free quantum field (initially in its vacuum state prior to the collapse) propagating in this background spacetime. Taking the positive frequency mode function corresponding to a particle state at late times, propagating it backwards in time, and determining its positive and negative frequency parts in the asymptotic past, it is possible to calculate the particle content of the field at infinity $\mathscr{I}^+$ at late times. At the end one gets that the expected number of particles at infinity corresponds to emission from a perfect black body at some specific temperature, defined by the properties of the black hole. Hawking has also shown how to generalize the results on thermal emission for different types of fields (bosonic and fermionic) and black holes (rotating and/or charged). Subsequent analysis in [Wal75, Par75] has shown complete absence of any correlation between different modes, proving that the Hawking radiation is indeed *thermal*. It should be noted that no infinities arise in the calculation of the Hawking effect for a free field, so the results are mathematically well defined, without any need for regularization or renormalization.

For the simplicity, we shall consider the case of massless noninteracting scalar field ϕ, satifying the Klein-Gordon equation in a curved background spacetime,

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0 \tag{2.1}$$

The Klein-Gordon inner product is defined by

$$(\phi_1, \phi_2) \ \equiv \ -i\int_\Sigma d\Sigma^\mu \left(\phi_1\nabla_\mu\phi_2^* - \phi_2^*\nabla_\mu\phi_1\right) \tag{2.2}$$

where Σ is an "initial data" Cauchy hypersurface and $d\Sigma^\mu = d\Sigma\, n^\mu$, with $d\Sigma$ being the volume element and n^μ a future directed unit normal vector to Σ. Using Stokes' theorem one can show that this definition of inner product is independent of the choice of hypersurface.

In order to be a properly defined inner product, the Klein-Gordon product (2.2) must be positive definite. This will hold as long as we restrict our consideration to positive frequency solutions. In a general curved spacetime, as oppose to Minkowski spacetime, there is no natural splitting of modes corresponding to positive and negative frequency solutions. Different choices lead, in general, to different definitions of the vacuum state and, consequently, to different definitions of the Fock space. However, if spacetime is stationary, we can use pertaining timelike Killing vector field ξ^a to define a natural notion of positive frequency modes $\tilde{u}_j$,

$$\xi^a\nabla_a\tilde{u}_j = -i\omega_j\tilde{u}_j \quad , \quad \omega_j > 0 \tag{2.3}$$

The spacetime we are particulary interested in, describing process of gravitational collapse, is *not* stationary and thereby one loses natural criterium to define positive frequency modes, as well the unique notion of vacuum state. Nevertheless, some spacetimes possess at least asymptotic stationary regions in the past (**in** region) and in the future (**out** region). Here we can construct two orthogonal set of modes, exact solutions of the wave equation in the whole spacetime: one having positive frequency with respect to the inertial time in the past (solutions $\tilde{u}_i^{\mathrm{in}}$) and the other having positive frequency with respect to the inertial time in the future (solutions $\tilde{u}_i^{\mathrm{out}}$). Each of these two sets of solutions satisfies following orthonormal relations

$$(\tilde{u}_i^{\mathrm{in}}, \tilde{u}_j^{\mathrm{in}}) = -(\tilde{u}_i^{\mathrm{in}*}, \tilde{u}_j^{\mathrm{in}*}) = \delta_{ij} \quad , \quad (\tilde{u}_i^{\mathrm{in}}, \tilde{u}_j^{\mathrm{in}*}) = 0 \tag{2.4}$$

$$(\tilde{u}_i^{\mathrm{out}}, \tilde{u}_j^{\mathrm{out}}) = -(\tilde{u}_i^{\mathrm{out}*}, \tilde{u}_j^{\mathrm{out}*}) = \delta_{ij} \quad , \quad (\tilde{u}_i^{\mathrm{out}}, \tilde{u}_j^{\mathrm{out}*}) = 0 \tag{2.5}$$

Using these, we can expand the field ϕ either in terms of the *in* modes,

$$\phi = \sum_i \left(a_i^{\mathrm{in}}\tilde{u}_i^{\mathrm{in}} + a_i^{\mathrm{in}\dagger}\tilde{u}_i^{\mathrm{in}*}\right) \tag{2.6}$$

or the *out* modes,

$$\phi = \sum_i \left(a_i^{\mathrm{out}}\tilde{u}_i^{\mathrm{out}} + a_i^{\mathrm{out}\dagger}\tilde{u}_i^{\mathrm{out}*}\right) \tag{2.7}$$

To canonically quantize this theory, we promote our classical variables (the fields and their conjugate momenta) to operators acting on a Hilbert space, and impose the canonical commutation relations on equal time hypersurfaces. Consequently, coefficients of the mode expansions become creation and annihilation operators satisfying canonical commutation relations,

$$[a_i^{\text{in}}, a_j^{\text{in}\,\dagger}] = \hbar\delta_{ij} \quad , \quad [a_i^{\text{in}}, a_j^{\text{in}}] = 0 = [a_i^{\text{in}\,\dagger}, a_j^{\text{in}\,\dagger}] \tag{2.8}$$

$$[a_i^{\text{out}}, a_j^{\text{out}\,\dagger}] = \hbar\delta_{ij} \quad , \quad [a_i^{\text{out}}, a_j^{\text{out}}] = 0 = [a_i^{\text{out}\,\dagger}, a_j^{\text{out}\,\dagger}] \tag{2.9}$$

With respect to these two sets operators we define corresponding vacuum states,

$$a_i^{\text{in}}\,|in\rangle = 0 \quad , \quad a_i^{\text{out}}\,|out\rangle = 0 \quad , \quad \forall\, i \tag{2.10}$$

Since both sets of modes are complete we can expand one in terms of the other, for example

$$\tilde{u}_j^{\text{out}} = \sum_i (\alpha_{ji}\tilde{u}_i^{\text{in}} + \beta_{ji}\tilde{u}_i^{\text{in}*})$$

These are the *Bogolubov transformations* and the matrix elements α_{ij} and β_{ij} are called the *Bogolubov coefficients*. The most important relations with Bogolubov transformations are presented in details through Appendix B.

If all the coefficients β_{ij} happen to vanish, the positive frequency modes, $\tilde{u}_i^{\text{in}}$ and $\tilde{u}_i^{\text{out}}$, will be related by unitary transformation, and therefore the definition of the vacuum remains unaltered, $|in\rangle = |out\rangle$. However, in any other case, if at least some of the coefficients β_{ij} do not vanish, the vacuum states $|in\rangle$ and $|out\rangle$ will be different. By computing the expectation value of the "out" particle number operator N_i^{out} for the i^{th} mode,

$$N_i^{\text{out}} \equiv \hbar^{-1} a_i^{\text{out}\,\dagger} a_i^{\text{out}} \tag{2.11}$$

in the $|in\rangle$ vacuum state,

$$\begin{aligned}\langle in|\, N_i^{\text{out}}\,|in\rangle = \hbar^{-1}\,\langle in|\, a_i^{\text{out}\,\dagger} a_i^{\text{out}}\,|in\rangle = \\ = \hbar^{-1}\,\langle in| \sum_j (-\beta_{ij} a_j^{\text{in}}) \sum_k (-\beta_{ik}^* a_k^{\text{in}\,\dagger})\,|in\rangle = \sum_j |\beta_{ij}|^2\end{aligned} \tag{2.12}$$

we see that the particle content of the $|in\rangle$ vacuum state, with respect to the *out* Fock space is non-trivial!

2.1.1 The model of collapse

In order to avoid most of the complications accompanying realistic formation of a black hole, we shall consider simplified model of gravitational collapse. The *Vaidya spacetime* is an exact solution of Einstein's field equations with a energy-momentum tensor

$$T_{vv} = \frac{1}{4\pi r^2}\,\frac{dM(v)}{dv} \tag{2.13}$$

describing a general purely ingoing radial flux of radiation. Its metric is given by

$$ds^2 = -\left(1 - \frac{2M(v)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \tag{2.14}$$

where v is the advanced Eddington-Finkelstein coordinate. Now, suppose a collapse consists of three regions of spacetime: Minkowski vacuum region ($v < v_i$), an intermediate collapse region ($v_i < v < v_f$) and the final Schwarzschild black hole configuration ($v > v_f$). Since the relevant regions for the Hawking radiation are the first and the third ones we shall furthermore narrow the intermediate region down to a single null surface. In other words, we have an ingoing shock wave located at some $v = v_0$, described by

$$M(v) = M\,\theta(v - v_0)$$

The resulting spacetime is therefore obtained by patching portions of Minkowski ("in" region) and Schwarzschild spacetimes ("out" region) along $v = v_0$. Relevant coordinate systems and corresonding metrics are summerized below,

- Minkowski region:

$$(t_m, r_m, \theta, \varphi) \quad , \quad ds^2 = -dt_m^2 + dr_m^2 + r_m^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

  or, introducing $u_m = t_m - r_m$ and $v = t_m + r_m$,

$$(u_m, v, \theta, \varphi) \quad , \quad ds^2 = -du_m dv + r_m^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Schwarzschild region:

$$(t_s, r_s, \theta, \varphi) \quad , \quad ds^2 = -\left(1 - \frac{r_s}{2M}\right) dt^2 + \frac{dr_s^2}{1 - \frac{r_s}{2M}} + r_s^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

  or, introducing $u_s = t_s - r_s^*$ and $v = t_s + r_s^*$, where

$$r_s^* = r_s + 2M \ln\left|\frac{r_s}{2M} - 1\right|$$

  we have

$$(u_s, v, \theta, \varphi) \quad , \quad ds^2 = -\left(1 - \frac{r_s}{2M}\right) du_s dv + r_s^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Since Vaidya spacetime is spherically symmetric it is natural to expand the scalar field ϕ using spherical harmonics,

$$\phi(t, r, \theta, \varphi) = \sum_{\ell, m} \frac{\phi_\ell(t, r)}{r}\, Y_{\ell m}(\theta, \varphi) \tag{2.15}$$

In this way the Klein-Gordon equation reduces to a 2-dimensional wave equation for $\phi_l(t, r)$; in the Minkowski region ($v < v_0$) it has a form

$$\left(-\partial_{t_m}^2 + \partial_{r_m}^2 - \frac{\ell(\ell+1)}{r_m}\right) \phi_\ell(t_m, r_m) = 0 \tag{2.16}$$

while in the Schwarzschild region ($v > v_0$)

$$\left(-\partial^2_{t_s} + \partial^2_{r_s^*} - V_\ell(r_s)\right) \phi_\ell(t_s, r_s) = 0 \tag{2.17}$$

where $V_\ell(r_s)$ is the effective potential given by

$$V_\ell(r_s) = \left(1 - \frac{2M}{r_s}\right) \left(\frac{\ell(\ell+1)}{r_s^2} + \frac{2M}{r_s^3}\right) \tag{2.18}$$

Since the important physics happens near the horizon, where the potential vanishes, we shall neglect the effective potential elsewhere.

$$\left(-\partial^2_{t_m} + \partial^2_{r_m}\right) \phi(t_m, r_m) = 0 \quad , \quad \left(-\partial^2_{t_s} + \partial^2_{r_s^*}\right) \phi(t_s, r_s) = 0$$

Among solutions we have *ingoing waves* $e^{-i\omega v}$ and *outgoing waves*, either $e^{-i\omega u_s}$ in the Schwarzschild region or $e^{-i\omega u_m}$ in the Minkowski region. We shall introduce two sets of orthonormal positive frequency modes,

$$\tilde{u}^{\text{in}}_\omega = \frac{1}{4\pi\sqrt{\omega}} \frac{e^{-i\omega v}}{r} \quad , \quad \tilde{u}^{\text{out}}_\omega = \frac{1}{4\pi\sqrt{\omega}} \frac{e^{-i\omega u_m}}{r} \tag{2.19}$$

associated to, respectfuly, natural time parameter v at $\mathscr{I}^-$ and u_m at $\mathscr{I}^+$. Normalization of the "in" modes is done over the past null infinity $\mathscr{I}^-$

$$(\tilde{u}^{\text{in}}_\omega, \tilde{u}^{\text{in}}_{\omega'}) = -i \int_{\mathscr{I}^-} r^2 dv d\Omega \, (\tilde{u}^{\text{in}}_\omega \partial_v \tilde{u}^{\text{in}*}_{\omega'} - \tilde{u}^{\text{in}*}_{\omega'} \partial_v \tilde{u}^{\text{in}}_\omega) = \delta(\omega - \omega')$$

Normalization of the "out" modes should be done over proper Cauchy surface, consisting of e.g. future null infinity and event horizon, $\mathscr{I}^+ \cup \mathcal{H}$. However, since the result is independent of the particular choice of the modes at the event horizon (see [FNS05]), we shall avoid unnecessary complications restricting normalization to the partial Cauchy surface at the future null infinity $\mathscr{I}^+$,

$$(\tilde{u}^{\text{out}}_\omega, \tilde{u}^{\text{out}}_{\omega'}) = -i \int_{\mathscr{I}^+} r^2 dv d\Omega \, (\tilde{u}^{\text{out}}_\omega \partial_v \tilde{u}^{\text{out}*}_{\omega'} - \tilde{u}^{\text{out}*}_{\omega'} \partial_v \tilde{u}^{\text{out}}_\omega) = \delta(\omega - \omega')$$

We can determine the form of the $\tilde{u}^{\text{out}}_\omega$ modes in the Minkowski region by imposing two conditions:

a) matching two regions of the spacetime along the shock wave at $v = v_0$, where we require equality of the metric on both sides, implies

$$r(v_0, u_m) = r(v_0, u_s)$$

so that

$$\frac{v_0 - u_m}{2} = \frac{v_0 - u_s}{2} - 2M \ln \left| \frac{v_0 - u_m}{4M} - 1 \right|$$

or

$$u_s(u_m) = u_m - 4M \ln \frac{|v_0 - 4M - u_m|}{4M} \tag{2.20}$$

The location of the event horizon $\mathcal{H}$ in Schwarzschild coordinates is defined by $u_s = +\infty$. Since at the origin, $r_m = 0$, we have $u_m = v$, the location of the event horizon in Minkowski coordinates is defined by $u_m(\mathcal{H}) = v_{\mathsf{H}}$. Inserting all this in the equation above we get

$$-4M \ln \frac{|v_0 - 4M - v_{\mathsf{H}}|}{4M} = +\infty$$

and consequently

$$v_{\mathsf{H}} = v_0 - 4M \tag{2.21}$$

b) regularity condition at the origin,

$$\phi(t_m, r_m = 0) \stackrel{!}{=} 0$$

forces the following form of the modes in the Minkowski region,

$$\tilde{u}_\omega^{\text{out}} = \frac{1}{4\pi\sqrt{\omega}} \left(\frac{e^{-i\omega u_s(u_m)}}{r} - \frac{e^{-i\omega u_s(v)}}{r} \theta(v_{\mathsf{H}} - v) \right) \tag{2.22}$$

where

$$u_s(v) = v - 4M \ln \frac{|v_0 - 4M - v|}{4M}$$

This mode is regular at the origin since $r = 0$ implies $u_m = v$.

We are particularly interested in behaviour of $\tilde{u}_\omega^{\text{out}}$ in two limiting regions. At early times $v \to -\infty$, so that $u_s(v) \approx v$. By construction, at the past null infinity $\mathscr{I}^-$ we have purely ingoing modes

$$\tilde{u}_\omega^{\text{out}} \approx -\frac{1}{4\pi\sqrt{\omega}} \frac{e^{-i\omega v}}{r} \tag{2.23}$$

At late times $u_s \to +\infty$ and $v \to v_{\mathsf{H}}$, so that

$$u_s(v) = v_{\mathsf{H}} - 4M \ln \frac{v_{\mathsf{H}} - v}{4M} \tag{2.24}$$

Again, by contruction, at the past null infinity $\mathscr{I}^-$ and close to v_{H} we have

$$\tilde{u}_\omega^{\text{out}} \approx -\frac{1}{4\pi\sqrt{\omega}} \frac{\exp(-i\omega(v_{\mathsf{H}} - 4M \ln \frac{v_{\mathsf{H}} - v}{4M}))}{r} \theta(v_{\mathsf{H}} - v) \tag{2.25}$$

2.1.2 Calculation of Hawking radiation spectrum

The quantity we want to determine is the expectation value of the number of particles of a given frequency ω emitted at $\mathscr{I}^+$,

$$\langle in | N_\omega^{\text{out}} | in \rangle = \int_0^\infty d\omega' \, |\beta_{\omega\omega'}|^2 \tag{2.26}$$

Usage of the states with a definite frequency implies absolute uncertainty in time, so that the quantity defined above provides the mean particle number with frequency ω emitted at *any* time u_s. Since we are mainly interested in its value at *late* retarted times ($u_s \to \infty$), when the black hole has setteled down to a stationary configuration, we have to replace completely delocalized plane wave modes by wave packets. An complete orthonormal set of wave packet modes at future null infinity $\mathscr{I}^+$ is given by

$$\tilde{u}^{\text{out}}_{jn} = \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} d\omega\, e^{2\pi i\omega n/\epsilon}\, \tilde{u}^{\text{out}}_{\omega} \tag{2.27}$$

where $j \geq 0$ and n are integers. Parameter ϵ serves localization of these wave packets, namely, they are peaked around $u_s = 2\pi n/\epsilon$ with width $2\pi/\epsilon$ and have frequency centered around $\omega \sim \omega_j = j\epsilon$. Using these wave packets, quantity $\langle in|\, N^{\text{out}}_{\omega}\, |in\rangle$ gives a mean particle number detected by a particle detector sensitive to frequency range $|\omega - \omega_j| \lesssim \epsilon/2$ during time interval $2\pi/\epsilon$ at time $u_s = 2\pi n/\epsilon$.

In order to determine the particle content of the Hawking radiation one has to evaluate the $\beta_{jn,\omega'}$ coefficients,

$$\beta_{jn,\omega'} = -(\tilde{u}^{\text{out}}_{jn}, \tilde{u}^{\text{in}*}_{\omega'}) = i \int_{\mathscr{I}^-} dv\, r^2 d\Omega\, (\tilde{u}^{\text{out}}_{jn} \partial_v \tilde{u}^{\text{in}}_{\omega'} - \tilde{u}^{\text{in}}_{\omega'} \partial_v \tilde{u}^{\text{out}}_{jn})$$

Performing a partial integration and discarding the boundary terms we get

$$\beta_{jn,\omega'} = 2i \int_{\mathscr{I}^-} dv\, r^2 d\Omega\, \tilde{u}^{\text{out}}_{jn} \partial_v \tilde{u}^{\text{in}}_{\omega'}$$

Late time particle production is obtained in the $n \to \infty$ limit. The wave packets at $\mathscr{I}^+$ with large n, propagated backwards in time, correspond to modes concentrated around v_{H} at $\mathscr{I}^-$. This means that we can use (2.25), so that

$$\beta_{jn,\omega'} = -\frac{1}{2\pi\sqrt{\epsilon}} \int_{-\infty}^{v_{\mathsf{H}}} dv \int_{j\epsilon}^{(j+1)\epsilon} d\omega\, e^{2\pi i\omega n/\epsilon} \sqrt{\frac{\omega'}{\omega}}\, \exp\left(-i\omega\left(v_{\mathsf{H}} - 4M\ln\frac{v_{\mathsf{H}} - v}{4M}\right) - i\omega' v\right)$$

and

$$\alpha_{jn,\omega'} = -\frac{1}{2\pi\sqrt{\epsilon}} \int_{-\infty}^{v_{\mathsf{H}}} dv \int_{j\epsilon}^{(j+1)\epsilon} d\omega\, e^{2\pi i\omega n/\epsilon} \sqrt{\frac{\omega'}{\omega}}\, \exp\left(-i\omega\left(v_{\mathsf{H}} - 4M\ln\frac{v_{\mathsf{H}} - v}{4M}\right) + i\omega' v\right)$$

It is convenient to ntroduce the variable $x = v_{\mathsf{H}} - v$ and auxiliary function

$$L(x) = \frac{2\pi n}{\epsilon} + 4M\ln\frac{x}{4M}$$

The integral over frequencies can be performed taking into account that ω varies in a small interval, that is $\omega_j = j\epsilon \approx (j + \frac{1}{2})\epsilon$. Using

$$\int_{j\epsilon}^{(j+1)\epsilon} d\omega\, e^{i\omega L} = \frac{2}{L}\, e^{i(j+\frac{1}{2})\epsilon L} \sin(\epsilon L/2) \approx \frac{\sin(\epsilon L/2)}{L/2}\, e^{iL\omega_j}$$

we get

$$\beta_{jn,\omega'} = -\frac{e^{-i(\omega_j+\omega')v_{\mathsf{H}}}}{\pi\sqrt{\epsilon}} \sqrt{\frac{\omega'}{\omega_j}} \int_0^\infty dx\, e^{i\omega' x}\, \frac{\sin(\epsilon L/2)}{L}\, e^{iL\omega_j} \tag{2.28}$$

and

$$\alpha_{jn,\omega'} = -\frac{e^{-i(\omega_j-\omega')v_{\mathsf{H}}}}{\pi\sqrt{\epsilon}}\sqrt{\frac{\omega'}{\omega_j}}\int_0^\infty dx\, e^{-i\omega' x}\,\frac{\sin(\epsilon L/2)}{L}\,e^{iL\omega_j} \tag{2.29}$$

The integral we need to evaluate is of type

$$I(\omega') = \int_0^\infty dx\, e^{-i\omega' x}\,\frac{\sin(\epsilon L/2)}{L}\,e^{iL\omega_j} \tag{2.30}$$

with $\omega' < 0$ for $\beta_{jn,\omega'}$ and $\omega' > 0$ for $\alpha_{jn,\omega'}$. In fact, all we need is to find the relation between these two integrals. This can be done using complex integral

$$\oint_{\mathcal{C}} dz\, e^{-i\omega' z}\,\frac{\sin(\epsilon L(z)/2)}{L(z)}\,e^{iL(z)\omega_j}$$

We place the branch cut along the negative real axis. The contour $\mathcal{C}$ goes counterclockwise along positive real and positive imaginary axis (joined by two circular arcs) in the $\omega' < 0$ case or clockwise along positive real and negative imaginary axis (joined by two circular arcs) in the $\omega' > 0$ case. Since there are no singularities inside any of these contours and the integral along the circular arcs vanishes in both cases, we can relate the real axis part with the imaginary axis part of the integral. This gives us

$$I(\omega' > 0) = -ie^{2\pi M\omega_j}e^{2\pi i n\omega_j/\epsilon}\int_0^\infty dy\, e^{-\omega' y}\,\frac{\sin(\epsilon L_-(y))}{L_-(y)}\,e^{i4M\omega_j \ln(y/4M)}$$

$$I(\omega' < 0) = ie^{-2\pi M\omega_j}e^{2\pi i n\omega_j/\epsilon}\int_0^\infty dy\, e^{\omega' y}\,\frac{\sin(\epsilon L_+(y))}{L_+(y)}\,e^{i4M\omega_j \ln(y/4M)}$$

where

$$L_\pm(y) \;\equiv\; \frac{2\pi n}{\epsilon} \pm 2\pi i M + 4M\ln\frac{y}{4M}$$

For very narrow wave packets ($\epsilon \ll 1$) centered around ω_j and at late times ($n/\epsilon \to \infty$) we have $\epsilon L_+ \approx \epsilon L_-$, so that

$$I(\omega' > 0) = -e^{4\pi M\omega_j}I(\omega' < 0)$$

Using this we can relate α and β coefficients,

$$\alpha_{jn,\omega'} = -e^{4\pi M\omega_j}e^{2i\omega' v_{\mathsf{H}}}\,\beta_{jn,\omega'} \tag{2.31}$$

which implies important result

$$|\alpha_{jn,\omega'}| = e^{4\pi M\omega_j}\,|\beta_{jn,\omega'}| \tag{2.32}$$

Furthermore, we shall imploy continuous version of the orthogonality condition,

$$\int_0^\infty d\omega'\left(\alpha_{jn,\omega'}\alpha^*_{j'n',\omega'} - \beta_{jn,\omega'}\beta^*_{j'n',\omega'}\right) = \delta_{jj'}\delta_{nn'}$$

Inserting particular values $j' = j$ and $n' = n$ we have

$$\int_0^\infty d\omega'\left(|\alpha_{jn,\omega'}|^2 - |\beta_{jn,\omega'}|^2\right) = 1 \tag{2.33}$$

Taking into account (2.32), we get

$$\left(e^{8\pi M\omega_j}-1\right)\int_0^\infty d\omega'\,|\beta_{jn,\omega'}|^2=1$$

This implies that, at late times ($n\to\infty$), the expectation value of the particle number operator

$$\langle in|\,N^{\text{out}}_{jn}\,|in\rangle=\int_0^\infty d\omega'|\beta_{jn,\omega'}|^2=\frac{1}{e^{8\pi M\omega_j}-1} \tag{2.34}$$

concides with the Bose-Einstein distribution with a temperature

$$T_{\mathsf{H}}=\frac{\hbar}{8\pi k_{\mathsf{B}}M} \tag{2.35}$$

called the **Hawking temperature** of the black hole. Completely analogous computation can be performed for fermions, resulting in the Fermi-Dirac distribution

$$\langle in|\,N^{\text{out}}_{jn}\,|in\rangle=\int_0^\infty d\omega'|\beta_{jn,\omega'}|^2=\frac{1}{e^{8\pi M\omega_j}+1} \tag{2.36}$$

2.1.3 Further details of Hawking radiation

In order to show that the Hawking radiation produced by a black hole is indeed *thermal*, that is, in complete agreement with black body thermal emission, one has to prove absence of any *correlations* between different modes, as was first point out by Wald [Wal75] and Parker [Par75]. For instance, using Bogoliubov coefficients and the commutation relations one can compute

$$\langle in|\,N^{\text{out}}_{jn}N^{\text{out}}_{jn}\,|in\rangle=\hbar^{-2}\,\langle in|\,a_i^{\text{out}\dagger}a_i^{\text{out}}a_i^{\text{out}\dagger}a_i^{\text{out}}\,|in\rangle=$$

$$=\int_0^\infty d\omega'\,|\beta_{jn,\omega'}|^2+2\left(\int_0^\infty d\omega'\,|\beta_{jn,\omega'}|^2\right)^2+\left|\int_0^\infty d\omega'\,\alpha_{jn,\omega'}\beta_{jn,\omega'}\right|^2$$

We already know the first two terms and the third can be show to be equal to zero. Inserting all this into the equation above, we get

$$\langle in|\,N^{\text{out}}_{jn}N^{\text{out}}_{jn}\,|in\rangle=\frac{e^{-8\pi M\omega_j}(1+e^{-8\pi M\omega_j})}{(1-e^{-8\pi M\omega_j})^2} \tag{2.37}$$

in agreement with expectation values related to a thermal probability

$$P(N_{jn})=(1-e^{-8\pi M\omega_j})e^{-8\pi N_{jn}M\omega_j} \tag{2.38}$$

Also, expectation value of particle number operators of different modes turns out to be

$$\begin{aligned}\langle in|\,N^{\text{out}}_{jn}N^{\text{out}}_{kn}\,|in\rangle&=\frac{1}{e^{8\pi M\omega_j}-1}\,\frac{1}{e^{8\pi M\omega_k}-1}=\\&=\langle in|\,N^{\text{out}}_{jn}\,|in\rangle\,\langle in|\,N^{\text{out}}_{kn}\,|in\rangle\end{aligned} \tag{2.39}$$

again, in agreement with the above thermal distribution. All higher correlation functions can be calculated in similar way and the result concides with the correlations of the thermal probability $P(N_{jn})$.

Up to this point we have ignored the effects of the potential barrier. By doing so, one would encounter a serious problem since each angular momentum component of the field contributes to the emitted energy flux with

$$L_\ell = \frac{(2\ell+1)}{2\pi}\int_0^\infty d\omega\, \frac{\hbar\omega}{e^{8\pi M\omega}-1} = \frac{(2\ell+1)}{2\pi}\,\frac{\hbar\pi^2}{6(8\pi M)^2} = \frac{(2\ell+1)\hbar}{768\pi M^2}$$

and summing over all angular momentum modes results in divergent result,

$$L = \sum_{\ell=0}^{\infty} L_\ell = \frac{\hbar}{768\pi M^2}\sum_{\ell=0}^{\infty}(2\ell+1) = \infty$$

Taking the effect of potential barrier, the *backscattering*, into account modifies the spectrum by *grey-body factor* $\Gamma_{\omega\ell}$ (see e.g. [FNS05]),

$$\langle in|\, N_{jn}^{\text{out}}\, |in\rangle = \frac{\Gamma_{\omega\ell}}{e^{8\pi M\omega}-1} \tag{2.40}$$

At low frequencies, $\omega M \ll 1$ the grey-body factor can be approximated as [Pag76b, Pag76a, Pag77],

$$\Gamma_{\omega\ell} \approx 16(\omega M)^{2\ell+2}\left(\frac{(\ell!)^3}{(2\ell)!(2\ell+1)!}\right)^2 \tag{2.41}$$

The main contribution is obtained for zero angular momenta, $\ell = 0$, where $\Gamma_{\omega\ell} \approx 16\omega^2 M^2$. Integration over the frequencies gives the approximate result

$$L_{\ell=0} \approx \frac{1}{2\pi}\int_0^\infty d\omega\, \frac{16\hbar\omega^3 M^2}{e^{8\pi M\omega}-1} = \frac{\hbar}{7680\pi M^2} \tag{2.42}$$

which is one tenth of the result one would obtain without the effect of the backscattering. Numerical calculation [BFF+01] corrects the above estimate by a factor of 1.62. To get an estimation of the total luminosity DeWitt [DeW75] has used the fact that at high frequencies ($\omega \gg M$) the black hole behaves as a "black sphere" of effective radius $3M\sqrt{3}$, giving the result

$$\sum_{\ell=0}^{\infty}(2\ell+1)\Gamma_{\omega\ell} \approx 27\pi M^2\omega^2$$

Integration over frequencies gives

$$L \approx \frac{1.69\,\hbar}{7680\pi M^2}$$

The above estimation can be again compared with the value obtained by numerical calculation [Els83],

$$L^{num} \approx \frac{1.79\hbar}{7680\pi M}$$

It is important to note that around 90% of the total luminostity of emitted Hawking radiation belongs to s-wave contribution. This distinguishing feature might signal that some highly spherical radiation captured at Large Hadron Collider (or some other future experiment) was emitted by a mini black hole [DL01].

§ 2.2 Anomaly approach

Another approach to the same phenomenon has been proposed by Christensen and Fulling soon after the Hawking's discovery [DFU76, CF77]. Their initial motivation was to extract any information about the energy-momentum tensor from general geometrical priciples without detailed calculations. It turns out that the knowledge of the trace is sufficient to restrict the form of energy-momentum tensor considerably. The Hawking radiation does not depend on the details of the collapse that gives rise to a black hole. Therefore, one expects that the methods to calculate it should have the same character of universality. The method of Christensen and Fulling makes use of the trace anomaly of the energy-momentum tensor in the near-horizon region to calculate the flux of radiation at infinity without specifing details of the collapse. More recently a renewed attention to the same problem has been pioneered by Robinson and Wilczek [RW05]. The method they have used was based on the diffeomorphism anomaly in a 2-dimensional effective field theory near the horizon of a radially symmetric static black hole. The basic argument is that, since just outside the horizon the ingoing modes cannot classically influence the physics outside the black hole, they can be integrated out, giving rise to an effective theory of purely outgoing modes. So the physics in that region can be described by an effective 2-dimensional chiral field theory (of infinite many fields). This implies an effective breakdown of the diffeomorphism invariance. The ensuing anomaly equation can be utilized to compute the outgoing flux of radiation. The latter appears as the quantum factor that restores the diffeomorphism symmetry.

We shall present here both methods and show that they can be reduced to the same basic elements. In other words, they are the spacial cases of the general anomaly approach to integrated Hawking radiation.

2.2.1 Trace anomaly method

This approach is based on the argument that the near-horizon physics is described by a 2-dimensional conformal field theory. Classically, the trace of the matter energy momentum tensor vanishes on shell. However, due to the anomaly it is generally nonvanishing at one loop level,

$$\langle T^{\mu}{}_{\mu} \rangle = \frac{cR}{48\pi} \tag{2.43}$$

where R is the background Ricci scalar and c is the total central charge of the matter system. The idea is to use this piece of information, along with the conservation of the energy-momentum tensor,

$$\nabla_{\mu} T^{\mu}{}_{\nu} = 0 \tag{2.44}$$

in order to compute the outgoing flux of the Hawking radiation at infinity.

As will be shown in chapter 3, (3+1)-dimensional bosonic or fermionic action near the black hole horizon can be reduced to a (1+1)-dimensional effective action with background metric of the form

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)} \tag{2.45}$$

As every 2-dimensional metric is conformally flat, it is convenient to transform the metric in manifestly conformal form. This can be done using *tortoise* coordinate r_* and then light-cone coordinates (u, v), defined by

$$\frac{dr_*}{dr} = \frac{1}{f(r)} \quad ; \quad u = t - r_* \quad , \quad v = t + r_* \tag{2.46}$$

$$ds^2 = e^{\varphi(u,v)} du\, dv \quad , \quad g_{\mu\nu} = e^{\varphi} \eta_{\mu\nu} \tag{2.47}$$

where $\varphi = \ln f$. Let us denote by $T_{uu}(u, v)$ and $T_{vv}(u, v)$ classically non vanishing components of the energy-momentum tensor in these new coordinates. Integrating the above equations, we get

$$T_{uu}(u, v) = \frac{\hbar c_{\mathrm{R}}}{24\pi} \left(\partial_u^2 \varphi - \frac{1}{2} (\partial_u \varphi)^2 \right) + T_{uu}^{(\mathrm{hol})}(u) \tag{2.48}$$

where $T_{uu}^{(\mathrm{hol})}(u)$ is holomorphic, while $T_{uu}(u, v)$ is conformally covariant. Namely, under a conformal transformation

$$u \;\rightarrow\; \tilde{u}(u) \quad , \quad v \;\rightarrow\; \tilde{v}(v)$$

one has

$$T_{uu}(u, v) = (\tilde{u}'(u))^2\, \tilde{T}_{\tilde{u}\tilde{u}}(\tilde{u}, \tilde{v})$$

Transformation of the holomorphic part of the energy-momentum tensor can be seen as follows: using

$$\partial_u = \tilde{u}'(u)\, \partial_{\tilde{u}} \qquad , \qquad \partial_{\tilde{u}}^2 = (\tilde{u}'(u))^2\, \partial_{\tilde{u}}^2 + \tilde{u}''(u)\, \partial_{\tilde{u}}$$

$$\tilde{\varphi}(\tilde{u}, \tilde{v}) = \varphi(u, v) - \ln \left(\frac{d\tilde{u}}{du} \frac{d\tilde{v}}{dv} \right)$$

after some elemntary algebra one gets

$$
\begin{aligned}
T_{uu}(u,v) = \frac{\hbar c_{\mathrm{R}}}{24\pi}\Big(&\tilde{u}''\partial_{\tilde{u}}\tilde{\varphi} + (\tilde{u}')^2\partial_{\tilde{u}}^2\tilde{\varphi} + \partial_u^2(\ln(\tilde{u}')) - \\
&-\frac{1}{2}\left(\tilde{u}'\partial_{\tilde{u}}\tilde{\varphi} + \partial_u \ln \tilde{u}'\right)^2\Big) + T_{uu}^{(\mathrm{hol})}(u) = \\
&= \frac{\hbar c_{\mathrm{R}}}{24\pi}\left(\tilde{u}''\partial_{\tilde{u}}\tilde{\varphi} + (\tilde{u}')^2\partial_{\tilde{u}}^2\tilde{\varphi} + \frac{\tilde{u}'''}{\tilde{u}'} - \frac{(\tilde{u}'')^2}{(\tilde{u}')^2} - \right. \\
&\left. -\frac{1}{2}(\tilde{u}')^2(\partial_{\tilde{u}}\tilde{\varphi})^2 - \tilde{u}''\partial_{\tilde{u}}\tilde{\varphi} - \frac{1}{2}\frac{(\tilde{u}'')^2}{(\tilde{u}')^2}\right) + T_{uu}^{(\mathrm{hol})}(u) = \\
= \frac{\hbar c_{\mathrm{R}}}{24\pi}&\left((\tilde{u}')^2\partial_{\tilde{u}}^2\tilde{\varphi} - \frac{1}{2}(\tilde{u}')^2(\partial_{\tilde{u}}\tilde{\varphi})^2 + \{\tilde{u},u\}\right) + T_{uu}^{(\mathrm{hol})}(u) = (\tilde{u}')^2\tilde{T}_{\tilde{u}\tilde{u}}(\tilde{u},\tilde{v})
\end{aligned}
$$

where we have used *Schwarzian derivative*,

$$
\{w,z\} \;\equiv\; \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 \tag{2.49}
$$

Using this result, we can write

$$
\tilde{T}_{\tilde{u}\tilde{u}}(\tilde{u},\tilde{v}) = \frac{\hbar c_{\mathrm{R}}}{24\pi}\left(\partial_{\tilde{u}}^2\tilde{\varphi} - \frac{1}{2}(\partial_{\tilde{u}}\tilde{\varphi})^2\right) + \frac{1}{(\tilde{u}')^2}\left(\frac{\hbar c_{\mathrm{R}}}{24\pi}\{\tilde{u},u\} + T_{uu}^{(\mathrm{hol})}(u)\right)
$$

or, finally

$$
\tilde{T}_{\tilde{u}\tilde{u}}^{(\mathrm{hol})}(\tilde{u}) = \frac{1}{(\tilde{u}')^2}\left(T_{uu}^{(\mathrm{hol})}(u) + \frac{\hbar c_{\mathrm{R}}}{24\pi}\{\tilde{u},u\}\right) \tag{2.50}
$$

Regular coordinates near the horizon are the Kruskal ones,

$$
U = -e^{-\kappa u} \quad , \quad V = e^{\kappa v}
$$

where the κ is surface gravity, equal to $\kappa = f'(r_{\mathsf{H}})/2$. The relevant Schwarzian derivative in this case is

$$
\{U,u\} = -\frac{\kappa^2}{2}
$$

Under this transformation we have

$$
\tilde{T}_{UU}^{(\mathrm{hol})}(U) = \frac{1}{(-\kappa U)^2}\left(T_{uu}^{(\mathrm{hol})}(u) + \frac{\hbar c_{\mathrm{R}}}{24\pi}\{U,u\}\right) = \frac{1}{(-\kappa U)^2}\left(T_{uu}^{(\mathrm{hol})}(u) - \frac{\hbar c_{\mathrm{R}}\kappa^2}{48\pi}\right)
$$

Now we require the outgoing energy flux to be regular at the future horizon $U = 0$ in the Kruskal coordinate. This implies that at that point $T_{uu}^{(\mathrm{hol})}(u)$ is given by $\hbar c_{\mathrm{R}}\kappa^2/48\pi$. Also, note that at the horizon,

$$
\partial_u^2\varphi - \frac{1}{2}\left(\partial_u\varphi\right)^2 = \frac{ff''}{4} - \frac{1}{2}\left(-\frac{f'}{2}\right)^2 = -\frac{\kappa^2}{2}
$$

so that in particular $T_{uu}(r_{\mathsf{H}}) = 0$. Since the background is stationary, $T_{uu}^{(\mathrm{hol})}(u)$ is constant in t and, since it is independent of coordinate v, it is also constant in r. We assume that at "infinity" of the near horizon region ($r = \infty$) there is no incoming flux ($\langle T_{vv}\rangle_\infty = 0$) and that the background is trivial, so that the vev of $T^{(\mathrm{hol})}_{uu}(u)$ and $T_{uu}(u,v)$ asymptotically coincide. Therefore, the asymptotic flux is

$$\langle T^r{}_t \rangle_\infty = \langle T_{uu} \rangle_\infty - \langle T_{vv} \rangle_\infty = \frac{\hbar\kappa^2}{48\pi}\, c_{\mathrm{R}} \tag{2.51}$$

As we shall see in the next section, the outgoing flux coincides with the constant a_o from the diffeomorphism anomaly method.

2.2.2 Diffeomorphism anomaly method

The method used in [RW05] was based on the diffeomorphism anomaly in a 2-dimensional effective field theory near the horizon of a spherically symmetric static black hole. The basic argument is that, since just outside the horizon $\mathcal{H}$ the ingoing modes cannot classically influence the physics outside the black hole, they can be integrated out, giving rise to an effective theory (2-dimensional chiral field theory of infinite many fields) of purely outgoing modes. If we formally remove modes to obtain the effective action in the exterior region, it becomes anomalous with respect to diffeomorphism (or gauge) symmetries. The ensuing anomaly equation can be utilized to compute the outgoing flux of radiation. The underlying theory is, of course, invariant. Therefore those anomalies must be cancelled by quantum effects of the modes that were irrelevant classically, in order to restore the diffeomorphism symmetry.

We shall present the method in somewhat simplified form. Part of the spacetime outside the black hole is divided along radial coordinate into two relevant regions: region $\mathcal{O}$ (defined by $r > r_{\mathsf{H}} + \epsilon$) and the region $\mathcal{H}$ (defined by $r_{\mathsf{H}} < r < r_{\mathsf{H}} + \epsilon$). In the region $\mathcal{H}$ the ingoing modes have been integrated out, so that the effective field theory there is anomalous, while in $\mathcal{O}$ we expect a fully symmetric theory. This is expressed by vanishing of the energy-momentum tensor covariant divergence,

$$\nabla_\mu T^\mu{}_{\nu(\mathcal{O})} = 0 \tag{2.52}$$

while the $\mathcal{H}$ region we have

$$\nabla_\mu T^\mu{}_{\nu(\mathcal{H})} = \frac{\hbar c_{\mathrm{R}}}{96\pi}\, \epsilon_{\nu\mu}\partial^\mu R \tag{2.53}$$

This is covariant form of the diffeomorphism anomaly, with a coefficient appropriate for chiral (outgoing or right) matter with central charge c_{R}. Using the elements of 2-dimensional metric (Appendix A), and the stationarity we have

$$\nabla_\mu T^\mu{}_t = \partial_r T^r{}_t + \Gamma^\mu_{\mu\sigma} T^\sigma{}_t - \Gamma^\sigma_{\mu t} T^\mu{}_\sigma = \partial_r T^r{}_t - \left(\Gamma^t_{rt} T^r{}_t + \Gamma^r_{tt} T^t{}_r\right) =$$

$$= \partial_r T^r{}_t - \left(\frac{f'}{2f} + \frac{ff'}{2}\, g^{tt} g_{rr}\right) T^r{}_t = \partial_r T^r{}_t$$

Since the metric is stationary, the two equations above take a very simple form; for the $\nu = t$

$$\partial_r T^r{}_{t(\mathcal{O})} = 0 \tag{2.54}$$

$$\partial_r T^r{}_{t(\mathcal{H})} = \frac{\hbar c_{\rm R}}{96\pi}\,\partial_r\left(ff'' - \frac{1}{2}\,(f')^2\right) \;\equiv\; \partial_r N^r{}_t \tag{2.55}$$

These can be integrated, giving

$$T^r{}_{t(\mathcal{O})} = a_o \tag{2.56}$$

$$T^r{}_{t(\mathcal{H})}(r) = a_h + N^r{}_t(r) - N^r{}_t(r_{\mathsf{H}}) \tag{2.57}$$

Where a_o and a_h are the integration constants. We remark that a_o, being constant, together with the condition that there is no ingoing flux from infinity, determines the outgoing energy flux. Now we define the overall energy-momentum tensor,

$$T^\mu{}_\nu = T^\mu{}_{\nu(\mathcal{O})}\,\theta(r - r_{\mathsf{H}} - \epsilon) + T^\mu{}_{\nu(\mathcal{H})}\,(1 - \theta(r - r_{\mathsf{H}} - \epsilon)) \tag{2.58}$$

It is understood that ϵ is a small number which specifies the size of the region where the energy-momentum tensor is not conserved. Taking the divergence of the overall energy-momentum tensor (for $\nu = t$), we get

$$\partial_r T^r{}_t = (a_o - a_h + N^r{}_t(r_{\mathsf{H}}))\delta(r - r_{\mathsf{H}} - \epsilon) + \partial_r(N^r{}_t(r)H(r))$$

where $H(r) = 1 - \theta(r - r_{\mathsf{H}} - \epsilon)$. We can now define a new overall tensor

$$\widehat{T}^r{}_t \;\equiv\; T^r{}_t - N^r{}_t(r)H(r) \tag{2.59}$$

which is conserved

$$\partial_r\,\widehat{T}^r{}_t = 0 \tag{2.60}$$

provided that

$$a_o - a_h + N^r{}_t(r_{\mathsf{H}}) = 0 \tag{2.61}$$

The condition that the energy-momentum tensor vanishes at the horizon leads to

$$0 \stackrel{!}{=} T^r{}_{t(\mathcal{H})}(r_{\mathsf{H}}) = a_h \tag{2.62}$$

Taking into account this, as well as $f'(r_{\mathsf{H}}) = 2\kappa$ we finally get

$$a_o = -N^r{}_t(r_{\mathsf{H}}) = \frac{\hbar\kappa^2}{48\pi}\,c_{\rm R} \tag{2.63}$$

This is the outgoing flux at infinity and coincides with the total Hawking radiation emitted by the black hole.

2.2.3 Comparison between anomaly methods

In summary we can say that the basic ingredients of the two anomaly methods are:

a) in the first case the integration of the energy-momentum conservation in the presence of a trace anomaly, in the second case the integration of the anomalous and non-anomalous conservation of the energy-momentum tensor;

b) in both cases we have the condition that the energy-momentum tensor vanishes at the horizon and there is no incoming energy flux from infinity.

One important difference is that in the trace anomaly method we do not have to split the space in different regions, but we consider a unique region outside the horizon.

The generic case of a chiral 2-dimensional theory with central charges c_{R} and c_{L} for the holomorphic and anti-holomorphic part, respectively, is characterized by the presence of both diffeomorphism and trace anomaly,

$$\nabla_\mu T^\mu{}_\nu = \frac{\hbar}{48\pi}\,\frac{c_{\mathrm{R}}-c_{\mathrm{L}}}{2}\,\epsilon_{\nu\mu}\partial^\mu R \tag{2.64}$$

$$T^\alpha{}_\alpha = \frac{\hbar}{48\pi}\,(c_{\mathrm{R}}+c_{\mathrm{L}})R \tag{2.65}$$

We can rewrite these equations in terms of the light-cone coordinates u and v (see Appendix A),

$$\nabla_u T_{uv} + \nabla_v T_{uu} = \frac{\hbar}{48\pi}\,\frac{c_{\mathrm{R}}-c_{\mathrm{L}}}{2}\,\epsilon_{uv}\partial_u R \tag{2.66}$$

$$\nabla_u T_{vv} + \nabla_v T_{uv} = -\frac{\hbar}{48\pi}\,\frac{c_{\mathrm{R}}-c_{\mathrm{L}}}{2}\,\epsilon_{uv}\partial_v R \tag{2.67}$$

$$T_{uv} = -\frac{\hbar}{48\pi}\,\frac{c_{\mathrm{R}}+c_{\mathrm{L}}}{4}\,R\,e^{\varphi} \tag{2.68}$$

Using elements of the metric from Appendix A, we have

$$R = 4e^{-\varphi}\varphi_{,uv} \quad , \quad \epsilon_{uv}\partial_u R = 2\partial_v \mathfrak{T}_{uu} \quad , \quad \epsilon_{uv}\partial_v R = 2\partial_u \mathfrak{T}_{vv}$$

where

$$\mathfrak{T}_{uu} = \partial_u^2\varphi - \frac{1}{2}\,(\partial_u\varphi)^2 \quad , \quad \mathfrak{T}_{vv} = \partial_v^2\varphi - \frac{1}{2}\,(\partial_v\varphi)^2$$

Inserting T_{uv} from (2.68) in equations (2.66) and (2.67) one gets

$$\partial_v T_{uu} = \frac{\hbar c_{\mathrm{R}}}{24\pi}\,\partial_v \mathfrak{T}_{uu}(u,v) \quad , \quad \partial_u T_{vv} = \frac{\hbar c_{\mathrm{L}}}{24\pi}\,\partial_u \mathfrak{T}_{vv}(u,v)$$

These can be now easily integrated, and the result is

$$T_{uu}(u,v) = \frac{\hbar c_{\mathrm{R}}}{24\pi}\,\mathfrak{T}_{uu}(u,v) + T^{(\mathrm{hol})}_{uu}(u) \tag{2.69}$$

$$T_{vv}(u,v) = \frac{\hbar c_{\mathrm{L}}}{24\pi}\,\mathfrak{T}_{vv}(u,v) + T^{(\mathrm{a-hol})}_{vv}(v) \tag{2.70}$$

Two terms, $T^{(\mathrm{hol})}_{uu}(u)$ and $T^{(\mathrm{a-hol})}_{vv}(v)$, represent respectfully holomorphic and antiholomorphic part of the energy momentum tensor.

In the trace anomaly method we have utilized $T_{uu}(u,v)$, required that the energy momentum tensor be conserved and imposed the conditions from b). This amounts to requiring $c_{\mathrm{R}} = c_{\mathrm{L}}$ in the region outside the horizon. Also, from the calculation above it is clear that, working with 2-dimensional theory, we can integrate the trace anomaly even if $c_{\mathrm{R}} \neq c_{\mathrm{L}}$.

In the diffeomorphism anomaly approach we integrated (2.64) in the near horizon region and the conserved energy-momentum divergence away from the horizon. Then we imposed vanishing of energy-momentum tensor at the horizon. It is obvious that we used again (2.69) and (2.70) in disguise. The tensor ${T^r}_t$ in the trace anomaly method corresponds to the tensor ${\widehat{T}^r}_t$ in diffeomorphism anomaly method. Also, the condition of vanishing energy-momentum tensor at the horizon in trace anomaly method corresponds to vanishing of $T_{uu}(u,v)$ tensor in diffeomorphism anomaly method.

It is important to stress the basic role of (2.69) and (2.70) because, as we will see, when we come to higher spin currents, it is not possible to describe the higher flux moments by means of anomalies (either trace or diff), but the analogues of (2.69) and (2.70) still hold and provide the desired description.

Chapter 3

Higher moments of Hawking radiation spectrum

The thermal spectrum of the Kerr black hole is given by either the Bose-Einstein (BE) or Fermi-Dirac (FD) distribution,

$$N_{\pm}(\omega) = \frac{g_*}{e^{\beta(\omega - m\Omega_{\mathsf{H}})} \pm 1} \tag{3.1}$$

where the upper sign corresponds to fermionic and the lower one to the bosonic case. Parameter β is reciprocal Hawking temperature of the black hole (with appropriate constants, $k_{\mathsf{B}}\beta\kappa = 2\pi$), ω is the absolute value of the momentum ($\omega = |k|$), Ω_{H} is the angular velocity of the horizon and m is the axial quantum number. The parameter g_* represents the number of physical degrees of freedom in the emitted radiation.

In two dimensions we can define the flux moments as follows

$$F_n^{\pm} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, k^{n-2}\omega N_{\pm}(\omega) = \frac{g_*}{4\pi}\int_{-\infty}^{\infty} dk\, \frac{k^{n-2}\omega}{e^{\beta(\omega - m\Omega_{\mathsf{H}})} \pm 1} \tag{3.2}$$

Normalization factor comes from the density of the states; in n spatial dimensions it is given by $d^n k/(2\pi)^n$. Our case has one spatial dimensional, hence we have $dk/(2\pi)$, but we have to divide this by 2 since half of the radiation goes back into the black hole and half of it escapes to infinity. Obviously, the odd moments vanish, while the even ones can be written as

$$F_{2n}^{\pm} = \frac{g_*}{2\pi}\int_0^{\infty} d\omega\, \frac{\omega^{2n-1}}{e^{\beta(\omega - m\Omega_{\mathsf{H}})} \pm 1} \tag{3.3}$$

Let us first consider the case $m = 0$. For the bosonic and fermionic case the fluxes are, respectfully,

$$F_{2n}^{-} = \frac{g_*}{8\pi n}(-1)^{n+1} B_{2n}\kappa^{2n} \tag{3.4}$$

$$F^+_{2n} = \frac{g_*}{8\pi n}(-1)^{n+1} B_{2n} \kappa^{2n} \left(1 - 2^{1-2n}\right) \tag{3.5}$$

where B_n are the Bernoulli numbers, defined in Appendix E. When $m \neq 0$ we don't have similar compact formulas; however, it makes sense to sum over the emission of a particle (with charge m) and the corresponding antiparticle (with charge $-m$). In this case the flux moments become

$$F^+_{n+1} = \frac{1}{2\pi}\left(\int_0^\infty \frac{\omega^n d\omega}{e^{\beta(\omega - m\Omega_{\mathsf{H}})} + 1} - (-1)^n \int_0^\infty \frac{\omega^n dx}{e^{\beta(\omega + m\Omega_{\mathsf{H}})} + 1}\right) =$$

$$= \frac{(m\Omega_{\mathsf{H}})^{n+1}}{2\pi(n+1)} - \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k \frac{n!(1-2^{1-2k})\kappa^{2k}}{2\pi(2k)!(n+1-2k)!} B_{2k} (m\Omega_{\mathsf{H}})^{n+1-2k} \tag{3.6}$$

The anomaly methods were devised for the calculation of the lowest moment F_2 (total flux, integrated distribution) of the Hawking radiation spectrum by means of an effective field theory. Natural question springing to one's mind is whether we can extend these methods to gain further details of the spectrum. Iso, Morita and Umetsu proposed [IMU07b] a generalization of the trace anomaly method. Their suggestion is to use higher tensorial currents, which play the role of the energy-momentum tensor for higher moments of the Hawking radiation spectrum. Through this chapter we shall make a thorough analysis of this method and finally show how to reconstruct Hawking radiation spectrum from its higher moments.

§ 3.1 Dimensional reduction

In order to simplify problem at hand as far as possible, we shall imploy *near horizon approximation.* We start from the full theory in (3+1)-spacetime dimensions, described by some action S, in which we separate angular part from the rest. As the relevant physics happens in the vicinity of the black hole event horizon, we discard the effects of potential barrier, contained in the angular part. Using this approximation we can identify reduced action to the corresponding (1+1)-dimensional theory with some effective degrees of freedom. We shall present this procedure for the cases of bosonic and fermionic fields in the vicinity of the Kerr black hole.

a) **Scalar massless field**

The action for scalar massless field ϕ in a general curved spacetime is given by

$$S = \frac{1}{2}\int d^4x \sqrt{-g}\, g^{\mu\nu} \nabla_\mu \phi^* \, \nabla_\nu \phi \tag{3.7}$$

Using the elements of the Kerr metric given in Appendix A, the action is explicetly written as

$$S = \frac{1}{2}\int dtdrd\theta d\varphi\,\sin\theta\,\Sigma\left(-\frac{\Xi}{\Delta\Sigma}(\partial_t\phi^*)(\partial_t\phi) - \frac{2Mra}{\Delta\Sigma}\Big((\partial_t\phi^*)(\partial_\varphi\phi)+\right.$$

$$\left.+(\partial_\varphi\phi^*)(\partial_t\phi)\Big) + \frac{\Delta}{\Sigma}(\partial_r\phi^*)(\partial_r\phi) + \frac{1}{\Sigma}(\partial_\theta\phi^*)(\partial_\theta\phi) + \frac{\Delta - a^2\sin^2\theta}{\Delta\Sigma\sin^2\theta}(\partial_\varphi\phi^*)(\partial_\varphi\phi)\right)$$

Performing a partial integration, this can be put in a following form,

$$S = \frac{1}{2}\int dtdrd\theta d\varphi\,\sin\theta\;\phi^*\left(\frac{\Xi}{\Delta}\,\partial_t^2 + \frac{4Mra}{\Delta}\,\partial_t\partial_\varphi - \partial_r(\Delta\partial_r)-\right.$$

$$\left.-\frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\partial_\theta) - \frac{\Delta - a^2\sin^2\theta}{\Delta\sin^2\theta}\,\partial_\varphi^2\right)\phi$$

We can decompose fields ϕ and ϕ^* using spherical harmonics,

$$\phi(t,r,\theta,\varphi) = \sum_{\ell,m}\phi_{\ell m}(t,r)Y_{\ell m}(\theta,\varphi) \tag{3.8}$$

$$S = \frac{1}{2}\int dtdrd\theta d\varphi\,\sin\theta\sum_{\ell',m'}\phi^*_{\ell'm'}Y^*_{\ell'm'}\left(\frac{(r^2+a^2)^2}{\Delta}\,\partial_t^2 - a^2\sin^2\theta\,\partial_t^2 + \frac{4Mra}{\Delta}\,\partial_t\partial_\varphi-\right.$$

$$\left.-\partial_r(\Delta\partial_r) - \frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\partial_\theta) - \frac{1}{\sin^2\theta}\,\partial_\varphi^2 + \frac{a^2}{\Delta}\,\partial_\varphi^2\right)\sum_{\ell,m}\phi_{\ell m}Y_{\ell m}$$

Also, we shall use properties of the angular momentum operator $\mathbf{L}$,

$$\mathbf{L}^2 = -\frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\partial_\theta) - \frac{1}{\sin^2\theta}\,\partial_\varphi^2 \quad , \quad \mathbf{L}_z = -i\partial_\varphi \tag{3.9}$$

$$\mathbf{L}^2 Y_{\ell m} = \ell(\ell+1)Y_{\ell m} \quad , \quad \mathbf{L}_z Y_{\ell m} = mY_{\ell m} \tag{3.10}$$

so that

$$S = \frac{1}{2}\int dtdrd\theta d\varphi\,\sin\theta\sum_{\ell',m'}\phi^*_{\ell'm'}Y^*_{\ell'm'}\left(\frac{(r^2+a^2)^2}{\Delta}\,\partial_t^2 - a^2\sin^2\theta\,\partial_t^2+\right.$$

$$\left.+\frac{4Mra}{\Delta}\,\partial_t\,i\mathbf{L}_z - \partial_r(\Delta\partial_r) + \mathbf{L}^2 - \frac{a^2}{\Delta}\,\mathbf{L}_z^2\right)\sum_{\ell,m}\phi_{\ell m}Y_{\ell m} =$$

$$= \frac{1}{2}\int dtdrd\theta d\varphi\,\sin\theta\sum_{l',m'}\phi^*_{l'm'}Y^*_{l'm'}\left(\frac{(r^2+a^2)^2}{\Delta}\,\partial_t^2 - a^2\sin^2\theta\,\partial_t^2+\right.$$

$$\left.+im\frac{4Mra}{\Delta}\,\partial_t - \partial_r(\Delta\partial_r) + \ell(\ell+1) - \frac{a^2m^2}{\Delta}\right)\sum_{\ell,m}\phi_{\ell m}Y_{\ell m}$$

In order to estimate which terms are dominant and which one can be neglected in the vicinity of the event horizon $\mathcal{H}$, we transform the radial coordinate into the *tortoise* coordinate r_*, defined by

$$\frac{dr_*}{dr} = \frac{r^2+a^2}{\Delta} \equiv \frac{1}{f(r)} \tag{3.11}$$

$$S=\frac{1}{2}\int dtdr_*d\theta d\varphi\,\sin\theta\sum_{\ell',m'}\phi^*_{\ell'm'}Y^*_{\ell'm'}\left((r^2+a^2)\,\partial_t^2-a^2f(r)\sin^2\theta\,\partial_t^2+\right.$$

$$\left.+im\frac{4Mra}{r^2+a^2}\,\partial_t-f(r)\partial_r(\Delta\partial_r)+f(r)\ell(\ell+1)-\frac{a^2m^2}{r^2+a^2}\right)\sum_{\ell,m}\phi_{\ell m}Y_{\ell m}$$

where the remaining r coordinate should be understood as the function $r(r_*)$. Notice that

$$f(r)\partial_r(\Delta\partial_r\phi)=f(r)\frac{dr_*}{dr}\,\partial_{r^*}\left(\Delta\frac{dr_*}{dr}\partial_{r^*}\phi\right)=\partial_{r_*}((r^2+a^2)\partial_{r^*}\phi)$$

The outer event horizon is located at $r=r_+$, defined by the equation $\Delta(r)=0$, so that

$$\lim_{r\to r_+}f(r)=0$$

This means that in the near horizon approximation two terms, proportional to $a^2f(r)\sin^2\theta\,\partial_t^2$ and $f(r)\ell(\ell+1)$ are supressed. Keeping the dominant terms, the action reduces to

$$S\to S_{(\mathcal{H})}=\frac{1}{2}\int dtdr_*d\theta d\varphi\,\sin\theta\sum_{\ell',m'}\phi^*_{\ell'm'}Y^*_{\ell'm'}\left((r_+^2+a^2)\,\partial_t^2+\right.$$

$$\left.+im\frac{4Mr_+a}{r_+^2+a^2}\,\partial_t-f(r)\partial_r(\Delta\partial_r)-\frac{a^2m^2}{r_+^2+a^2}\right)\sum_{l,m}\phi_{\ell m}Y_{\ell m}$$

Going back to the r coordinate and using orthogonality of the spherical harmonics,

$$\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin\theta\,Y^*_{\ell'm'}(\theta,\varphi)Y_{\ell m}(\theta,\varphi)=\delta_{\ell\ell'}\delta_{mm'}$$

we have

$$S_{(\mathcal{H})}=\frac{1}{2}\int dtdr\sum_{\ell,m}\phi^*_{\ell m}\left(\frac{(r^2+a^2)^2}{\Delta}\,\partial_t^2+ima\,\frac{4Mr}{\Delta}\,\partial_t+\frac{(iam)^2}{\Delta}-\right.$$

$$\left.-(r^2+a^2)\,\partial_r\left(\frac{\Delta}{r^2+a^2}\,\partial_r\right)\right)\phi_{\ell m}=$$

$$=\frac{1}{2}\sum_{\ell,m}\int dtdr\,\phi^*_{\ell m}\left(\frac{(r^2+a^2)^2}{\Delta}\left(\partial_t^2+ima\,\frac{4Mr}{(r^2+a^2)^2}\,\partial_t+\frac{(iam)^2}{(r^2+a^2)^2}\right)-\right.$$

$$\left.-(r^2+a^2)\,\partial_r\left(\frac{\Delta}{r^2+a^2}\partial_r\right)\right)\phi_{\ell m}$$

Now we use

$$ima\,\frac{4Mr}{(r^2+a^2)^2}=\frac{2iam}{r^2+a^2}\,\frac{2Mr}{r^2+a^2}=\frac{2iam}{r^2+a^2}\,\frac{r^2+a^2-\Delta}{r^2+a^2}=$$

$$=\frac{2iam}{r^2+a^2}\,(1-f(r))\to\frac{2iam}{r^2+a^2}$$

where the arrow $\to$ denotes limit $r \to r_+$. After some elementary algebra,

$$S_{(\mathcal{H})} = \frac{1}{2} \sum_{\ell,m} \int dtdr\, \phi^*_{\ell m} \left(\frac{(r^2+a^2)^2}{\Delta} \left(\partial_t^2 + \frac{2iam}{r^2+a^2} \partial_t + \frac{(iam)^2}{(r^2+a^2)^2} \right) - \right.$$

$$\left. -(r^2+a^2)\, \partial_r \left(\frac{\Delta}{r^2+a^2}\, \partial_r \right) \right) \phi_{\ell m} =$$

$$= \frac{1}{2} \sum_{\ell,m} \int dtdr\, (r^2{+}a^2) \phi^*_{\ell m} \left(\frac{r^2+a^2}{\Delta} \left(\partial_t + \frac{iam}{r^2+a^2} \right)^2 - \partial_r \left(\frac{\Delta}{r^2+a^2}\, \partial_r \right) \right) \phi_{\ell m}$$

action can be put in a form

$$S_{(\mathcal{H})} = \frac{1}{2} \sum_{\ell,m} \int dtdr\, (r^2+a^2) \phi^*_{\ell m} \left(\frac{1}{f(r)} \left(\partial_t + \frac{iam}{r^2+a^2} \right)^2 - \partial_r \left(f(r)\, \partial_r \right) \right) \phi_{\ell m}$$

At this point we can propose an interpretation of this reduced form of the action, as a (1+1)-dimensional effective field theory for an infinite number of scalar fields $\phi_{\ell m}$, propagating in the background given by

$$\text{dilaton} \quad \overset{\flat}{\Phi} = r^2 + a^2$$

$$\text{gauge field} \quad \overset{\flat}{A}_t = -\tfrac{a}{r^2+a^2} \quad , \quad \overset{\flat}{A}_r = 0$$

and 2-dimensional spacetime metric with the components

$$\overset{\flat}{g}_{tt} = -f(r) \quad , \quad \overset{\flat}{g}_{rr} = \frac{1}{f(r)} \quad , \quad \overset{\flat}{g}_{rt} = 0 \tag{3.12}$$

This can been seen as follows. Two dimensional action for a charged scalar massless field ϕ coupled to a gauge field A_μ and dilaton field Φ is given by

$$S = \frac{1}{2} \int dtdr\, \overset{\flat}{\Phi}\, \phi^* \overset{\flat}{D}_\mu \overset{\flat}{D}^\mu \phi \tag{3.13}$$

where the gauge covariant derivative is defined as

$$\overset{\flat}{D}_\mu = \overset{\flat}{\nabla}_\mu - iq\overset{\flat}{A}_\mu \tag{3.14}$$

It is usefull to notice that the following equality holds a general vector field V^μ,

$$D_\mu V^\mu = \frac{1}{\sqrt{-g}} (\partial_\mu - iqA_\mu)(\sqrt{-g}\, V^\mu)$$

Using proposed 2-dimensional metric (3.12), where $\sqrt{-g^\flat} = 1$, we have

$$D_\mu(D^\mu \phi) = (\partial_\mu - iqA_\mu)(g^{\mu\nu} D_\nu \phi) =$$

$$= (\partial_t - iqA_t)(g^{tt}(\partial_t - iqA_t)\phi) + (\partial_r - iqA_r)(g^{rr}(\partial_r - iqA_r)\phi) =$$

$$= \frac{1}{f(r)} (\partial_t - iqA_t)^2 \phi - (\partial_r - iqA_r)(f(r)(\partial_r - iqA_r)\phi)$$

One possible choice of componets of the gauge field $\overset{\flat}{A}_\mu$, reproducing the near horizon form of the bosonic action, is obviously one proposed above.

In the subsequent analysis we restrict our attention to a near-horizon region, where the dilaton is approximately constant. This means that we may disregard it: the equations of motion are those of free scalars in two dimensions, coupled to the metric and the gauge field (but not to the dilaton).

b) **Fermionic massless field**

The (3+1)-dimensional action for a massless fermionic field ψ is given by

$$S = \int d^4x\sqrt{-g}\,\bar{\psi}\, i\nabla\!\!\!\!/\,\psi \tag{3.15}$$

Treatment of the spinors in the curved spacetimes can be faciliated in tangent (Minkowski) space, introduced through local inertial frames. Transformation from general to a tangent frame at every point of the manifold is encoded in *vierbeins* $e^a{}_\mu$, defined by

$$\eta_{ab}e^a{}_\mu e^b{}_\nu = g_{\mu\nu} \tag{3.16}$$

It is important to emphasize that the vierbeins, just as local tangent frames, are defined up to local Lorentz transformation. Convention is to use lower case latin letters for the flat (Minkowski) coordinates and lower case greek letters for the curved coordinates. Also, in order to make clear distinction between these two types of coordinates, the flat ones shall take values from a set $\mathbb{N}_0$, while the curved ones from some of the alphabets (for example, $a \in \{0,1,2,3\}$, $\mu \in \{t,r,\theta,\varphi\}$). Indices are raised and lowered with associated metric,

$$e_a{}^\mu = \eta_{ab}\, g^{\mu\nu}\, e^b{}_\nu \tag{3.17}$$

The components of spin connection $\omega^a{}_{b\mu}$ are related to vielbeins by

$$\omega^a{}_{b\mu} = e^a{}_\nu \nabla_\mu e_b{}^\nu \tag{3.18}$$

Flat gamma matrices are defined by usual anticommutator

$$\{\gamma^a,\gamma^b\} = 2\eta^{ab}\mathbf{1} \tag{3.19}$$

The adjoint bispinor $\bar{\psi}$ is defined with flat gamma matrix γ^0,

$$\bar{\psi} \;\equiv\; \psi^\dagger\gamma^0 \tag{3.20}$$

Using all these elements, we can write fermionic action in the following form,

$$S = \int d^4x\sqrt{-g}\,\psi^\dagger\gamma^0\gamma^a e_a{}^\mu\left(\partial_\mu - \frac{1}{8}\,\omega_{bc\mu}[\gamma^b,\gamma^c]\right)\psi \tag{3.21}$$

We choose the following local Lorentz frame via vierbeins

$$\sqrt{\Delta\Sigma}\, e_0{}^\mu\partial_\mu = (r^2+a^2)\partial_t + a\partial_\varphi \quad , \quad \sqrt{\Delta\Sigma}\, e_1{}^\mu\partial_\mu = \Delta\partial_r$$

$$\sqrt{\Delta\Sigma}\, {e_2}^\mu \partial_\mu = \sqrt{\Delta}\, \partial_\theta \quad , \quad \sqrt{\Delta\Sigma}\, {e_3}^\mu \partial_\mu = \sqrt{\Delta} \left(a \sin\theta \partial_t + \frac{1}{\sin\theta} \partial_\varphi \right)$$

Near the horizon we have $r \to r_+$ and consequently $\Delta \to 0$. We can immediately see that the terms in the action proportional to $\gamma^2 {e_2}^\mu \partial_\mu$ and $\gamma^3 {e_3}^\mu \partial_\mu$ are suppressed a factor of $\sqrt{\Delta}$. Using again tortoise coordinate, defined by

$$\frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta} \equiv \frac{1}{f(r)} \tag{3.22}$$

we can see that the term proportional to

$$\sqrt{\Delta\Sigma}\, {e_1}^\mu \partial_\mu = (r^2 + a^2) \partial_{r_*}$$

is not supressed. Therefore, the leading order contribution from the term $\gamma^a {e_a}^\mu \partial_\mu$ in the action is $\gamma^0 {e_0}^t \partial_t + \gamma^0 {e_0}^\varphi \partial_\varphi + \gamma^1 {e_1}^r \partial_r$ and is of order $1/\sqrt{\Delta}$. Furhtermore, straightforward calculation (the spin coefficients are listed in Appendix A) shows that the leading contribution of the term ${e_a}^\mu \omega_{bc\mu}$ comes from

$${e_0}^\mu \omega_{01\mu} = \eta_{00} {e_0}^\mu {\omega^0}_{1\mu} = {\omega^0}_{10} = \frac{r_+ - r_-}{2\sqrt{\Delta\Sigma}}$$

and is also of order $1/\sqrt{\Delta}$. In summary, in the near horizon region we obtain

$$\nabla\!\!\!/ \psi = \left\{ \frac{\gamma^0}{\sqrt{\Delta\Sigma}} \left((r_+^2 + a^2) \partial_t + a \partial_\phi \right) + \frac{\gamma^1}{\sqrt{\Delta\Sigma}} \left((r_+^2 + a^2) \partial_{r_*} - \frac{1}{4}(r_+ - r_-) \right) \right\} \psi$$

To be able to integrate over angular coordinates θ and φ in the action, we expand ψ in the following way

$$\psi(t, r, \theta, \varphi) = \sum_{\ell, m} \psi_{\ell m}(t, r) \mathcal{Y}_{\ell m}(\theta, \varphi) \tag{3.23}$$

where $\mathcal{Y}_{\ell m}$ are modified spherical harmonics,

$$\mathcal{Y}_{\ell m}(\theta, \varphi) \;\equiv\; \frac{Y_{\ell m}(\theta, \varphi)}{\sqrt[4]{\Sigma}} \tag{3.24}$$

The normalization is given by

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \sqrt{\Sigma}\, \mathcal{Y}^*_{\ell' m'}\, \mathcal{Y}_{\ell m} = \delta_{\ell\ell'} \delta_{mm'}$$

Taking all this into account we have

$$S \to S_{(\mathcal{H})} = 4\pi \int dt dr\, \frac{r_+^2 + a^2}{\sqrt{\Delta}} \sum_{\ell, m} \psi^\dagger_{\ell m} \left\{ \gamma^0 \gamma^0 \left(\partial_t - \frac{iam}{r_+^2 + a^2} \right) + \right.$$

$$\left. + \gamma^0 \gamma^1 \left(\partial_{r_*} - \frac{r_+ - r_-}{4(r_+^2 + a^2)} \right) \right\} \psi_{\ell m} \tag{3.25}$$

Now we choose the following 4-dimensional gamma matrices,

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad , \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and the following 2-dimensional gamma matrices,

$$\sigma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

This particular choice ensures that $\gamma^0\gamma^1$ and $\sigma^0\sigma^1$ look very simple. Both are diagonal, and satisfy $\gamma^0\gamma^1 = \mathbf{1} \otimes \sigma^0\sigma^1$. We denote two upper components of bispinor $\psi_{\ell m}$ by $\chi^{(1)}_{\ell m}$ and two lower by $\chi^{(2)}_{\ell m}$,

$$\psi_{\ell m} = \begin{pmatrix} \chi^{(1)}_{\ell m} \\ \chi^{(2)}_{\ell m} \end{pmatrix} \tag{3.26}$$

In terms of $\chi^{(s)}_{\ell m}$ $(s = 1, 2)$ the action reads

$$S_{(\mathcal{H})} = 4\pi \int dtdr \, \frac{r_+^2 + a^2}{\sqrt{\Delta}} \sum_{s=1}^{2} \sum_{\ell,m} \chi^{(s)\dagger}_{\ell m} \left\{ \sigma^0\sigma^0 \left(\partial_t - \frac{iam}{r_+^2 + a^2} \right) + \right.$$

$$\left. + \sigma^0\sigma^1 \left(\partial_{r_*} - \frac{r_+ - r_-}{4(r_+^2 + a^2)} \right) \right\} \chi^{(s)}_{\ell m} \tag{3.27}$$

As in the bosonic case, we can propose an interpretation of this reduced form of the action, as a (1+1)-dimensional effective field theory for an infinite number of two component fermions $\chi^{(s)}_{\ell m}$, propagating in the background given by

$$\text{dilaton} \quad \overset{\flat}{\Phi} = \sqrt{r^2 + a^2}$$

$$\text{gauge field} \quad \overset{\flat}{A}_t = \tfrac{a}{r^2 + a^2} \quad , \quad \overset{\flat}{A}_r = 0$$

and 2-dimensional spacetime metric with the components

$$\overset{\flat}{g}_{tt} = f(r) \quad , \quad \overset{\flat}{g}_{rr} = -\frac{1}{f(r)} \quad , \quad \overset{\flat}{g}_{rt} = 0 \tag{3.28}$$

This can been seen as follows. Two dimensional action for a charged massless fermionic field χ coupled to a gauge field A_μ and dilaton field Φ is given by

$$S = \int dtdr \, \Phi \, \bar{\chi}^{(s)}_{lm} \, \overset{\flat}{\not{D}} \, \chi^{(s)}_{lm} \tag{3.29}$$

where the gauge covariant derivative is defined by

$$\overset{\flat}{\not{D}} = \sigma^\mu \overset{\flat}{D}_\mu = \sigma^\mu \left(\overset{\flat}{\nabla}_\mu - iq\overset{\flat}{A}_\mu \right) \tag{3.30}$$

Using zweibains $\overset{\flat}{e}_a{}^\mu$,

$$\overset{\flat}{e}_0{}^t = \frac{1}{\sqrt{f}} \quad , \quad \overset{\flat}{e}_1{}^r = \sqrt{f}$$

we can calculate the covariant derivative of Weyl spinor χ,

$$\overset{\flat}{\not\nabla}\chi = \sigma^a \, \overset{\flat}{e}_a{}^{\mu} \left(\overset{\flat}{\partial}_\mu - \frac{1}{8} \overset{\flat}{\omega}_{bc\mu} \left[\sigma^b, \sigma^c\right] \right) \chi \tag{3.31}$$

The first term is given by

$$\sigma^a \, \overset{\flat}{e}_a{}^{\mu} \overset{\flat}{\partial}_\mu = \frac{\sigma^0}{\sqrt{f}} \, \partial_t + \sqrt{f} \sigma^1 \partial_r = \frac{1}{\sqrt{f}} \left(\sigma^0 \partial_t + \sigma^1 \partial_{r_*} \right)$$

and the second by

$$\sigma^a \, \overset{\flat}{e}_a{}^{\mu} \, \overset{\flat}{\omega}_{bc\mu} \left[\sigma^b, \sigma^c\right] = \sigma^a \, \overset{\flat}{\omega}_{bca} \left[\sigma^b, \sigma^c\right] = 2 \frac{f'}{\sqrt{f}} \, \sigma^1$$

so that

$$\overset{\flat}{\not\nabla}\chi = \left(\frac{\sigma^0}{\sqrt{f(r)}} \, \partial_t + \frac{\sigma^1}{\sqrt{f(r)}} \left(\partial_{r_*} - \frac{f'(r)}{4} \right) \right) \chi$$

This means that

$$\bar{\chi} \, \overset{\flat}{\not D} \chi = \chi^\dagger \sigma^0 \left(\overset{\flat}{\not\nabla} - i q \overset{\flat}{\not A} \right) \chi =$$

$$= \chi^\dagger \left(\frac{\sigma^0 \sigma^0}{\sqrt{f(r)}} \left(\partial_t - i q \overset{\flat}{A}_t \right) + \frac{\sigma^0 \sigma^1}{\sqrt{f(r)}} \left(\partial_{r_*} - \frac{f'(r)}{4} - i q \overset{\flat}{A}_r \right) \right) \chi$$

One possible choice of componets of the gauge field $\overset{\flat}{A}_\mu$, reproducing the near horizon form of the fermionic action, is obviously the one proposed above. Notice that the value of the time component of the effective gauge field at the horizon corresponds to the angular velocity of the horizon,

$$\left\langle \overset{\flat}{A}_t \right\rangle_{\mathsf{H}} = \Omega_{\mathsf{H}} \tag{3.32}$$

In the subsequent analysis we restrict our attention to a near-horizon region, where the dilaton is approximately constant. This means that, as in the bosonic case, we may disregard it: the equations of motion are those of free fermions in two dimensions, coupled to the metric and the gauge field (but not to the dilaton).

§ 3.2 Holomorphic higher spin currents

In order to derive the higher Hawking fluxes the same way we derived above the integrated Hawking radiation, we *postulate* the existence of conserved spin currents consisting of fermionic bilinears in the 2-dimensional effective field theory near the horizon. They will play a role analogous to the energy-momentum tensor for the integrated radiation (the lowest moment).

3.2.1 Holomorphic bosonic W_∞ currents

Higher spin currents are expressed in terms of a 2-dimensional complex free bosonic field ϕ. We are using Euclidean formalism and the (u, v) coordinates are replaced by the complex ones $(z, \bar{z})$. Two point functions for this complex bosonic field are

$$\left\langle \phi(z_1)\bar{\phi}(z_2) \right\rangle = -\hbar \ln(z_1 - z_2) \tag{3.33}$$

$$\left\langle \phi(z_1)\phi(z_2) \right\rangle = \left\langle \bar{\phi}(z_1)\bar{\phi}(z_2) \right\rangle = 0 \tag{3.34}$$

Holomorphic bosonic currents obeying W_∞ algebra were constructed in [BK90],

$$j^{(s)}_{z\ldots z}(z) = B(s) \sum_{k=1}^{s-1} (-1)^k A^s_k : \partial_z^k \phi(z) \partial_z^{s-k} \bar{\phi}(z) : \tag{3.35}$$

$$B(s) = \frac{2^{s-3} s!}{(2s-3)!!} q^{s-2} \quad , \quad A^s_k = \frac{1}{s-1} \binom{s-1}{k} \binom{s-1}{s-k}$$

where q is a deformation parameter and $j^{(s)}_{z\ldots z}$ has s lower indices. The normal ordering is defined using point-splitting regularization,

$$: \partial_z^m \phi \, \partial_z^n \bar{\phi} : \; = \lim_{z_2 \to z_1} \left(\partial_{z_1}^m \phi(z_1) \partial_{z_2}^n \bar{\phi}(z_2) - \partial_{z_1}^m \partial_{z_2}^n \left\langle \phi(z_1)\bar{\phi}(z_2) \right\rangle \right) \tag{3.36}$$

As usual in the framework of conformal field theory, the operator product on the RHS is unterstood to be radially ordered.

Several first holomorphic bosonic currents are

$$j^{(2)}_{zz} = - : \partial_z \phi(z) \partial_z \bar{\phi}(z) : \tag{3.37}$$

$$j^{(3)}_{zzz} = -2q \left(: \partial_z \phi(z) \partial_z^2 \bar{\phi}(z) : - : \partial_z^2 \phi(z) \partial_z \bar{\phi}(z) : \right) \tag{3.38}$$

$$j^{(4)}_{zzzz} = -\frac{16q^2}{5} \left(: \partial_z \phi(z) \partial_z^3 \bar{\phi}(z) : -3 : \partial_z^2 \phi(z) \partial_z^2 \bar{\phi}(z) : + : \partial_z^3 \phi(z) \partial_z \bar{\phi}(z) : \right) \tag{3.39}$$

$$\begin{aligned} j^{(5)}_{z\ldots z} = -\frac{32q^3}{7} & \left(: \partial_z \phi(z) \partial_z^4 \bar{\phi}(z) : -6 : \partial_z^2 \phi(z) \partial_z^3 \bar{\phi}(z) : + \right. \\ & \left. +6 : \partial_z^3 \phi(z) \partial_z^2 \bar{\phi}(z) : - : \partial_z^4 \phi(z) \partial_z \bar{\phi}(z) : \right) \end{aligned} \tag{3.40}$$

The current $j^{(2)}_{zz}(z)$ is proportional to the normalized holomorphic energy-momentum tensor of the model and, upon change of coordinates $z \to w(z)$ transforms as

$$: \partial_z \phi(z) \partial_z \bar{\phi}(z) : = (w')^2 : \partial_w \phi(w) \partial_w \bar{\phi}(w) : -\frac{\hbar}{6} \{w, z\} \tag{3.41}$$

This can be proven as follows

$$: \partial_{z_1} \phi(z_1) \partial_{z_2} \bar{\phi}(z_2) : = \partial_{z_1} \phi(z_1) \partial_{z_2} \bar{\phi}(z_2) - \partial_{z_1} \partial_{z_2} \left\langle \phi(z_1)\bar{\phi}(z_2) \right\rangle =$$

$$= w'(z_1) w'(z_2) \partial_{w_1} \phi(w_1) \partial_{w_2} \bar{\phi}(w_2) - \partial_{z_1} \partial_{z_2} \left\langle \phi(z_1)\bar{\phi}(z_2) \right\rangle =$$

$$= w'(z_1) w'(z_2) : \partial_{w_1} \phi(w_1) \partial_{w_2} \bar{\phi}(w_2) : - G_{\mathrm{B}}(z_1, z_2)$$

where $\partial_{z_1}\phi(z_1)\partial_{z_2}\bar{\phi}(z_2)$ stands for the radial ordered product of two operators, and

$$\begin{aligned} G_{\text{B}}(z_1,z_2) \;\equiv\; & -w'(z_1)w'(z_2)\partial_{w_1}\partial_{w_2}\left\langle\phi(w_1)\bar{\phi}(w_2)\right\rangle + \partial_{z_1}\partial_{z_2}\left\langle\phi(z_1)\bar{\phi}(z_2)\right\rangle = \\ & = -\partial_{z_1}\partial_{z_2}\left(\left\langle\phi(w_1(z_1))\bar{\phi}(w_2(z_2))\right\rangle - \left\langle\phi(z_1)\bar{\phi}(z_2)\right\rangle\right) = \\ & = \hbar\,\frac{w'(z_1)w'(z_2)}{(w(z_1)-w(z_2))^2} - \frac{\hbar}{(z_1-z_2)^2} \end{aligned}$$

The limit $z_2 \to z_1$ results in

$$\lim_{z_2\to z_1}\frac{w'(z_1)w'(z_2)}{(w(z_1)-w(z_2))^2} - \frac{1}{(z_1-z_2)^2} = \frac{1}{6}\{w,z_1\}$$

We are interested in the transformation properties of currents $j^{(s)}(u)$, corresponding to transition to Kruskal coordinates, when $w(z)$ is

$$w(z) = -e^{-\kappa z}$$

so that

$$G_{\text{B}}(z_1,z_2) = G_{\text{B}}(z_1-z_2) = -\frac{\hbar}{(z_1-z_2)^2} + \frac{\hbar\,\kappa^2}{4\,\mathrm{sh}^2(\kappa(z_1-z_2)/2)}$$

Analogously to the analysis of spin-2 current from above, we have

$$j^{(\text{s})}_{z\ldots z}(z) = \lim_{z_1\to z_2}\left(B(s)\sum_{k=1}^{s-1}(-1)^k A^s_k :\partial^k_{z_1}\phi(w(z_1))\,\partial^{s-k}_{z_2}\bar{\phi}(w(z_2)):\right) + \langle X^{\text{B}}_s\rangle \tag{3.42}$$

where

$$\langle X^{\text{B}}_s\rangle = B(s)\sum_{k=1}^{s-1}(-1)^k A^s_k \lim_{z_1\to z_2}\left(\left\langle\partial^k_{z_1}\phi(w(z_1))\partial^{s-k}_{z_2}\bar{\phi}(w(z_2))\right\rangle - \left\langle\partial^k_{z_1}\phi(z_1)\partial^{s-k}_{z_2}\bar{\phi}(z_2)\right\rangle\right) =$$

$$= \lim_{z_1\to z_2} B(s)\sum_{k=1}^{s-1}(-1)^k A^s_k\partial^k_{z_1}\partial^{s-k}_{z_2}\left(\left\langle\phi(w_1(z_1))\bar{\phi}(w_2(z_2))\right\rangle - \left\langle\phi(z_1)\bar{\phi}(z_2)\right\rangle\right) =$$

$$= \lim_{z_1\to z_2} B(s)\sum_{k=0}^{s-2}(-1)^{k+1} A^s_{k+1}\partial^k_{z_1}\partial^{s-k-2}_{z_2}\partial_{z_1}\partial_{z_2}\left(\left\langle\phi(w_1(z_1))\bar{\phi}(w_2(z_2))\right\rangle - \left\langle\phi(z_1)\bar{\phi}(z_2)\right\rangle\right) =$$

$$= B(s)\sum_{k=0}^{s-2}(-1)^k A^s_{k+1}\lim_{z_1\to z_2}\partial^k_{z_1}\partial^{s-k-2}_{z_2}G_{\text{B}}(z_1,z_2)$$

It is useful to introduce the Taylor series

$$G_{\text{B}}(z_1,z_2) = G_{\text{B}}(z_1-z_2) = \sum_{n=0}^{\infty}\frac{1}{n!}\,\gamma_n(z_1-z_2)^n$$

so that

$$G^{\text{B}}_{m,n} \;\equiv\; \lim_{z_1\to z_2}\partial^m_{z_1}\partial^n_{z_2}\,G_{\text{B}}(z_1,z_2) = \lim_{z_1\to z_2}(-1)^n\,\partial^{m+n}_{z_1}\,G_{\text{B}}(z_1-z_2) =$$

$$= (-1)^n \, \partial_z^{m+n} \, G_{\rm B}(z)\Big|_{z=0} = (-1)^n \, \gamma_{m+n}$$

Using this notation we can write

$$\langle X_s^{\rm B} \rangle = B(s) \sum_{k=0}^{s-2} (-1)^k A_{k+1}^s G_{k,s-k-2} = (-1)^s B(s) \, \gamma_{s-2} \sum_{k=0}^{s-2} A_{k+1}^s \tag{3.43}$$

Details of the Taylor expansion for the function $G_{\rm B}(z)$ are presented in Appendix D. The result is

$$G_{\rm B}(z) = -\hbar \, \mathcal{S}_\kappa(z) = -\hbar \sum_{n=0}^{\infty} \frac{B_{n+2} \, \kappa^{n+2}}{n+2} \, \frac{z^n}{n!}$$

which gives us

$$\gamma_n = -\hbar \, \frac{B_{n+2}}{n+2} \, \kappa^{n+2}$$

Finally, using the value of the sum (see Appendix D)

$$\sum_{k=0}^{s-2} A_{k+1}^s = \frac{(2s-2)!}{(s-1)!s!}$$

we obtain

$$\langle X_s^{\rm B} \rangle = \hbar \, (-1)^{s+1} (4q)^{s-2} \, \frac{B_s}{s} \, \kappa^s \tag{3.44}$$

This is a higher order Schwarzian derivative evaluated for $w(z) = -e^{-\kappa z}$, which will play the crucial role in the it plays a role in the evaluation of the higher moments of the Hawking radiation spectrum. Finally we return to the question of the transformation properties of the holomorphic current. First, the derivative of the bosonic field ϕ transform according to

$$\partial_z \phi = w' \partial_w \phi \quad , \quad \partial_z^2 \phi = (w')^2 \partial_w^2 \phi + w'' \partial_w \phi$$

$$\partial_z^3 \phi = (w')^3 \partial_w^3 \phi + 3 w' w'' \partial_w^2 \phi + w''' \partial_w \phi$$

For the higher derivatives these expresions become complicated and are generally of form

$$\partial_z^n \phi = (w')^n \partial_w^n \phi + \sum_{k=1}^{n} \Delta_{n,k}(w) \partial_w^k \phi \tag{3.45}$$

where $\Delta_{n,k}(w)$ denotes some combination of derivatives of the function $w(z)$. Therefore we have

$$j^{(\rm s)}_{z \ldots z}(z) = B(s) \sum_{k=1}^{s-1} (-1)^k A_k^s : \left((w')^s \partial_z^k \phi(z) \partial_z^{s-k} \bar{\phi}(z) + S_{k,s-k}^{\rm B} \right) : + \langle X_s^{\rm B} \rangle \tag{3.46}$$

where

$$S_{k,s-k}^{\rm B} \equiv (w')^k \partial_z^k \phi(z) \left(\sum_{p=1}^{s-k} \Delta_{s-k,p}(w) \partial_w^p \bar{\phi} \right) +$$

$$+ \left(\sum_{q=1}^{k} \Delta_{k,q}(w) \partial_w^q \phi \right) (w')^{s-k} \partial_z^{s-k} \bar{\phi}(z) + \left(\sum_{q=1}^{k} \Delta_{k,q}(w) \partial_w^q \phi \right) \left(\sum_{p=1}^{s-k} \Delta_{s-k,p}(w) \partial_w^p \bar{\phi} \right)$$

Using all this we can write the transformation rule for the holomorphic current as

$$j^{(s)}_{z\dots z}(z) = (w')^s j^{(s)}_{w\dots w}(w) + B(s)\sum_{k=1}^{s-1}(-1)^k A^s_k : S^{\mathrm{B}}_{k,s-k} : + \langle X^{\mathrm{B}}_s\rangle$$

or

$$j^{(s)}_{w\dots w}(w) = \frac{1}{(w')^s}\left(j^{(s)}_{z\dots z}(z) - \langle X^{\mathrm{B}}_s\rangle - B(s)\sum_{k=1}^{s-1}(-1)^k A^s_k : S^{\mathrm{B}}_{k,s-k} :\right)$$

At this point we choose the transformation $w(z) = -e^{-\kappa z}$ in order to get the value of the holomorphic currents at the horizon. As we approach the horizon one has $w(z) \to 0$, just as $w^{(n)}(z) \to 0$ for all $n \in \mathbb{N}$. This also implies $\Delta_{i,j}(w) \to 0$ and thus $S^{\mathrm{B}}_{i,j} \to 0$. Requiering regularity of the current $j^{(s)}_{w\dots w}(w)$ at the horizon we get

$$j^{(s)}_{z\dots z} \to \langle j^{(s)}_{z\dots z}\rangle_{\mathsf{H}} = \langle X^{\mathrm{B}}_s\rangle \tag{3.47}$$

3.2.2 Holomorphic fermionic $W_{1+\infty}$ currents

Higher spin currents are expressed in terms of a 2-dimensional complex free fermionic field Ψ. Again, we are using Euclidean formalism and the (u, v) coordinates are replaced by the complex ones $(z, \bar{z})$. Two-point functions for the fermionic holomorphic fields is given by

$$\left\langle \Psi^\dagger(z_1)\Psi(z_2)\right\rangle = \frac{\hbar}{z_1 - z_2} \tag{3.48}$$

Holomorphic fermionic currents obeying $W_{1+\infty}$ algebra were introduced in [BPR+90],

$$j^{(s)}_{z\dots z}(z) = -\frac{B(s)}{s}\sum_{k=1}^{s}(-1)^k \binom{s-1}{s-k}^2 : \partial_z^{s-k}\Psi^\dagger(z)\,\partial_z^{k-1}\Psi(z) : \tag{3.49}$$

$$B(s) = \frac{2^{s-3}s!}{(2s-3)!!}\,q^{s-2} \quad , \quad s \in \mathbb{N}$$

Note that we are using convention $(-1)!! = 1$. The spin s currents $j^{(s)}_{z\dots z}(z)$ are linear combinations of bilinears

$$j^{(m,n)}_{z\dots z}(z) \equiv : \partial_z^m\Psi^\dagger\partial_z^n\Psi : = \lim_{z_2\to z_1}\left(\partial^m_{z_1}\Psi^\dagger(z_1)\partial^n_{z_2}\Psi(z_2) - \partial^m_{z_1}\partial^n_{z_2}\left\langle\Psi^\dagger(z_1)\Psi(z_2)\right\rangle\right)$$

Several first holomorphic bosonic currents are

$$j^{(1)}_z(z) = \frac{1}{4q} : \Psi^\dagger(z)\,\Psi(z) : \tag{3.50}$$

$$j^{(2)}_{zz}(z) = \frac{1}{2}\left(: \partial_z\Psi^\dagger(z)\,\Psi(z) : - : \Psi^\dagger(z)\,\partial_z\Psi(z) :\right) \tag{3.51}$$

$$j^{(3)}_{zzz}(z) = \frac{2q}{3}\left(: \partial_z^2\Psi^\dagger(z)\,\Psi(z) : -4 : \partial_z\Psi^\dagger(z)\,\partial_z\Psi(z) : + : \Psi^\dagger(z)\,\partial_z^2\Psi(z) :\right) \tag{3.52}$$

$$j^{(4)}_{zzzz}(z) = \frac{4q^2}{5}\Big(:\partial_z^3\Psi^\dagger(z)\,\Psi(z): -9:\partial_z^2\Psi^\dagger(z)\,\partial_z\Psi(z): +$$
$$+9:\partial_z\Psi^\dagger(z)\,\partial_z^2\Psi(z): - :\Psi^\dagger(z)\,\partial_z^3\Psi(z):\Big) \tag{3.53}$$

We want to relate the currents written in two different coordinate systems, connected by coordinate change $z \to w(z)$. Holomorphic fermionic field transforms according to

$$\Psi(z) = (w'(z))^{\frac{1}{2}}\,\Psi(w)$$

Using it we get

$$:\partial_{z_1}^m\Psi^\dagger(z_1)\,\partial_{z_2}^n\Psi(z_2): = \partial_{z_1}^m\,\partial_{z_2}^n :\Psi^\dagger(z_1)\,\Psi(z_2): =$$
$$= \partial_{z_1}^m\,\partial_{z_2}^n\left(\Psi^\dagger(z_1)\,\Psi(z_2) - \left\langle\Psi^\dagger(z_1)\,\Psi(z_2)\right\rangle\right) =$$
$$= \partial_{z_1}^m\,\partial_{z_2}^n\Big((w_1'(z_1))^{\frac{1}{2}}(w_2'(z_2))^{\frac{1}{2}}\,\Psi^\dagger(w_1)\,\Psi(w_2) - \left\langle\Psi^\dagger(z_1)\,\Psi(z_2)\right\rangle\Big) =$$
$$= \partial_{z_1}^m\,\partial_{z_2}^n\Big((w_1'(z_1))^{\frac{1}{2}}(w_2'(z_2))^{\frac{1}{2}} :\Psi^\dagger(w_1)\,\Psi(w_2):\Big) - \partial_{z_1}^m\,\partial_{z_2}^n G(z_1,z_2)$$

where we have introduced the function

$$G_{\mathrm{F}}(z_1,z_2) \equiv -(w_1'(z_1))^{\frac{1}{2}}(w_2'(z_2))^{\frac{1}{2}}\left\langle\Psi^\dagger(w_1)\,\Psi(w_2)\right\rangle + \left\langle\Psi^\dagger(z_1)\,\Psi(z_2)\right\rangle$$

Using the fermionic two point function, it is easy to show that

$$G_{\mathrm{F}}(z_1,z_2) = -\hbar\,\frac{(w_1'(z_1))^{\frac{1}{2}}(w_2'(z_2))^{\frac{1}{2}}}{w_1-w_2} + \frac{\hbar}{z_1-z_2}$$

As in the bosonic case we are interested in the Kruskal-type transformation,

$$w(z) = -e^{-\kappa z}$$

so that

$$G_{\mathrm{F}}(z_1,z_2) = G_{\mathrm{F}}(z_1-z_2) = -\frac{\hbar\,(\kappa/2)}{\mathrm{sh}\left(\frac{\kappa}{2}\,(z_1-z_2)\right)} + \frac{\hbar}{z_1-z_2}$$

Furthermore we proceed by procedure analogous to the bosonic one,

$$j^{(s)}_{z\ldots z}(z) = -\frac{B(s)}{s}\sum_{k=1}^{s}(-1)^k\binom{s-1}{s-k}^2\lim_{z_1\to z_2}\partial_{z_1}^{s-k}\partial_{z_2}^{k-1}$$
$$\Big((w_1'(z_1))^{\frac{1}{2}}(w_2'(z_2))^{\frac{1}{2}} :\Psi^\dagger(w_1)\,\Psi(w_2):\Big) + \langle X_s^{\mathrm{F}}\rangle \tag{3.54}$$

where

$$\langle X_s^{\mathrm{F}}\rangle \equiv -\frac{B(s)}{s}\sum_{k=1}^{s}(-1)^{k+1}\binom{s-1}{s-k}^2\lim_{z_1\to z_2}\partial_{z_1}^{s-k}\partial_{z_2}^{k-1}G_{\mathrm{F}}(z_1,z_2)$$

It is useful to introduce the Taylor series

$$G_{\mathrm{F}}(z_1,z_2) = G_{\mathrm{F}}(z_1-z_2) = \sum_{n=0}^{\infty}\frac{1}{n!}\,\widetilde{\gamma}_n(z_1-z_2)^n$$

so that

$$G^{\mathrm{F}}_{m,n} \equiv \lim_{z_1\to z_2} \partial^m_{z_1}\partial^n_{z_2}\, G_{\mathrm{F}}(z_1,z_2) = \lim_{z_1\to z_2} (-1)^n\, \partial^{m+n}_{z_1}\, G_{\mathrm{F}}(z_1-z_2) =$$

$$= (-1)^n\, \partial^{m+n}_{z}\, G_{\mathrm{F}}(z)\Big|_{z=0} = (-1)^n\, \widetilde{\gamma}_{m+n}$$

Using this notation we can write

$$\langle X^{\mathrm{F}}_s \rangle = -\frac{B(s)}{s}\, \widetilde{\gamma}_{s-1} \sum_{k=1}^{s} \binom{s-1}{s-k}^2$$

Details of the Taylor expansion for the function $G_{\mathrm{F}}(z)$ are presented in Appendix D. The result is

$$G(z_1,z_2) = \hbar \sum_{m=1}^{\infty} \frac{(\kappa/2)^{2m}(2^{2m-1}-1)B_{2m}}{m}\, \frac{(z_1-z_2)^{2m-1}}{(2m-1)!} =$$

$$= \hbar \sum_{m=1}^{\infty} \frac{\kappa^{2m}(1-2^{1-2m})B_{2m}}{2m}\, \frac{(z_1-z_2)^{2m-1}}{(2m-1)!}$$

From here it is easy to read off the coefficients

$$\widetilde{\gamma}_{2m-1} = \hbar\, \frac{\kappa^{2m}(1-2^{1-2m})B_{2m}}{2m} \tag{3.55}$$

Using the fact that odd Bernoulli numbers B_{2n+1} vanish for all $n\in\mathbb{N}$, this can be put in a simpler form,

$$\widetilde{\gamma}_n = \hbar\, \frac{\kappa^{n+1}(1-2^{-n})B_{n+1}}{n+1}$$

Using the value of the sum (see Appendix D)

$$\sum_{k=1}^{s} \binom{s-1}{s-k}^2 = 2^{s-1}\, \frac{(2s-3)!!}{(s-1)!}$$

we finally get

$$\langle X^{\mathrm{F}}_s \rangle = -\hbar\, \frac{\kappa^s B_s}{s}\, (1-2^{1-s})\, (4q)^{s-2} \tag{3.56}$$

Notice that $\langle X^{\mathrm{F}}_s \rangle = 0$ for an odd spin s. For $s>1$ this is because $B_s = 0$ for odd $s>1$. For $s=1$ it is because of the factor $(1-2^{1-s})$ in the expression for $\langle X^{\mathrm{F}}_s \rangle$. Finally, using the same argument as in the bosonic case, by requiering regularity at the horizon we get

$$j^{(\mathrm{s})}_{z\ldots z} \to \langle j^{(\mathrm{s})}_{z\ldots z} \rangle_{\mathsf{H}} = \langle X^{\mathrm{F}}_s \rangle \tag{3.57}$$

§ 3.3 Covariant higher spin currents

The holomorphic currents of the previous section refer to a background with a trivial Euclidean metric. In order to find a covariant expression of them we have to be able to incorporate the information of a non-trivial metric. In construction of covariant currents we follow the recipe described by Iso, Morita and Umetsu in [IMU07b] and [IMU08b]. This will be described in the sections bellow.

3.3.1 Covariant bosonic W_∞ currents

We begin the procedure by going back to light-cone coordinates (u, v). The uu component of the holomorphic spin-2 current $j^{(2)}_{uu}$ can be identified up to a constant with the holomorphic energy momentum tensor,

$$j^{(2)}_{uu} = -2\pi\, T^{(\mathrm{hol})}_{uu} \tag{3.58}$$

Similarly, we identify higher spin currents $j^{(\mathrm{s})}_{u\ldots u}$, with s lower indices, with an s^{th} order holomorphic tensor. In analogy with the energy-momentum tensor, we expect that there exist a conformally covariant version $J^{(\mathrm{s})}_{u\ldots u}$ of $j^{(\mathrm{s})}_{u\ldots u}$. It is natural to suppose that the covariant currents appear in an effective action S, where they are sourced by asymptotically trivial background fields $B^{(\mathrm{s})}_{\mu_1\ldots\mu_s}$, i.e.

$$J^{(\mathrm{s})}_{\mu_1\ldots\mu_s} = \frac{1}{\sqrt{g}}\,\frac{\delta S}{\delta B^{(\mathrm{s})}_{\mu_1\ldots\mu_s}} \tag{3.59}$$

In particular, $B^{(2)}_{\mu\nu} = g_{\mu\nu}/2$. We assume that all covariant currents $J^{(\mathrm{s})}_{\mu_1\ldots\mu_s}$ are completely symmetric and classicaly traceless, whose only other classically non-vanishing components are $J^{(\mathrm{s})}_{u\ldots u}$ and $J^{(\mathrm{s})}_{v\ldots v}$.

The W_∞ algebra is formulated in terms of a (complex, Euclidean) chiral bosonic field ϕ. For a conformally flat background the action of chiral scalar field boils down to that of a free chiral boson. In other words, the equation of motion of a chiral boson coupled to background conformal gravity is $\partial_v\phi = 0$, which simplifies the procedure.

To proceed with the covariantization program we now reduce the problem to a 1-dimensional one. We consider only the u dependence and keep v fixed. In one dimension a curved coordinate u is easily related to the corresponding normal coordinate x via the relation $\partial_x = e^{-\varphi(u)}\partial_u$. We view u as $u(x)$, assume that all currents $j^{(\mathrm{s})}_{u\ldots u}$ and their W_∞ relations refer in fact to the flat x coordinate, i.e. x corresponds to the Euclidean coordinate z used in the previous section, and by the above equivalence we extract the components in the new coordinate system. For instance, for a scalar field ϕ,

$$\partial^n_x\phi = e^{-n\varphi(u)}\,\nabla^n_u\phi \quad , \quad \partial^n_x\phi\,(dx)^n = \nabla^n_u\phi\,(du)^n$$

The W_∞ currents are constructed out of bilinears in ϕ and $\bar{\phi}$,

$$j^{(m,n)}_{u\ldots u} = :\partial^m_u \phi\, \partial^n_u \bar{\phi}:$$

We split the factors and evaluate one in $u_+ = u(x+\epsilon/2)$ and other in $u_- = u(x-\epsilon/2)$ for some real $\epsilon > 0$. Then we expand the bilinears in ϵ and take the limit $\epsilon \to 0$. Afterwards we restore the tensorial character of the product by multiplying it by a suitable $e^{n\varphi(u)}$ factor.

According to the recipe explained above, the covariant counterpart of $j^{(s)}_{u\ldots u}$ should be constructed using currents

$$J^{(m,n)}_{u\ldots u} = e^{(m+n)\varphi(u,v)} \lim_{\epsilon\to 0}\left(e^{-m\varphi(u_+,v)-n\varphi(u_-,v)} \nabla^m_u \phi(u_+) \nabla^n_u \bar{\phi}(u_-) - \frac{c^{\mathrm{B}}_{m,n}\hbar}{\epsilon^{m+n}} \right) \tag{3.60}$$

where $c^{\mathrm{B}}_{m,n} = (-1)^m (m+n-1)!$ are numerical constants determined in such a way that all singularities are canceled in the final expression for $J^{(m,n)}_{u\ldots u}$. Therefore, (3.60) defines the normal ordered current

$$J^{(m,n)}_{u\ldots u} = :\nabla^m_u \phi\, \nabla^n_u \bar{\phi}:$$

Finally, we define the covariant currents corresponding to the W_∞ bosonic currents,

$$J^{(s)}_{u\ldots u}(u) = B(s) \sum_{k=1}^{s} (-1)^k A^s_k\, J^{(s-k,k-1)}_{u\ldots u} \tag{3.61}$$

where

$$B(s) = \frac{2^{s-3} s!}{(2s-3)!!}\, q^{s-2} \quad , \quad A^s_k = \frac{1}{s-1} \binom{s-1}{k} \binom{s-1}{s-k}$$

After some algebra one gets several first currents,

$$J^{(2)}_{uu} = j^{(2)}_{uu} - \frac{\hbar}{6}\, \mathcal{T}_{uu} \tag{3.62}$$

$$J^{(3)}_{uuu} = j^{(3)}_{uuu} \tag{3.63}$$

$$J^{(4)}_{uuuu} = j^{(4)}_{uuuu} + \frac{\hbar}{30}\, \mathcal{T}^2_{uu} + \frac{2}{5}\, \mathcal{T}_{uu} J^{(2)}_{uu} \tag{3.64}$$

$$J^{(5)}_{uuuuu} = j^{(5)}_{uuuuu} + \frac{10}{7}\, \mathcal{T}_{uu} J^{(3)}_{uuu} \tag{3.65}$$

where

$$\mathcal{T}_{uu} = \partial^2_u \varphi - \frac{1}{2}\, (\partial_u \varphi)^2$$

The equations (3.62)-(3.65) are analogs of (2.69). The covariant divergences of these currents are

$$g^{uv} \nabla_v J^{(2)}_{uu} = \frac{\hbar}{12} \nabla_u R \tag{3.66}$$

$$g^{uv} \nabla_v J^{(3)}_{uuu} = 0 \tag{3.67}$$

$$g^{uv} \nabla_v J^{(4)}_{uuuu} + \frac{q^2}{5}\, (\nabla_u R) J^{(2)}_{uu} = 0 \tag{3.68}$$

$$g^{uv} \nabla_v J^{(5)}_{uuuuu} + \frac{5}{7}\, (\nabla_u R) J^{(3)}_{uuu} = 0 \tag{3.69}$$

The above equations tell us that all the higher spin equations are covariantly conserved. In the RHS of (3.67)-(3.69), unlike (3.66), there does not appear any terms proportional to $\hbar$. Any such term must be interpreted as the consequence of a trace anomaly (and possibly a diffeomorphism anomaly) as has been argued by [BC08]. In other words if there is a term proportional to $\hbar$ in $g^{uv}\nabla_v J_{u\dots u}$ this must be understood as related to the second term in the covariant divergence,

$$\nabla^\mu J_{\mu u\dots u} = g^{uv}\nabla_v J_{u\dots u} + g^{uv}\nabla_u J_{vu\dots u}$$

Such a term tells us that $J_{vu\dots u}$, which classically vanishes, takes on a nonzero value at one loop level, revealing the existence of a trace anomaly. This is precisely what happens for the spin-2 current (energy-momentum tensor) $J^{(2)}_{uu}$; its trace is $\mathrm{Tr}(J^{(2)}) = 2g^{vu}J^{(2)}_{vu}$. Thus, (3.66) reproduces the well known trace anomaly $\mathrm{Tr}(J^{(2)}) = -\hbar R/12$. Note that in our case the central charge is $c = 2$ and that there is relative factor -2π between spin-2 current and the energy-momentum tensor.

On the other hand, the terms that carry explicit factors of $\hbar$ have canceled out in the equations (3.67)-(3.69). This implies the absence of $\hbar$ terms in the trace, and consequently the absence of any trace anomaly as well as of any diffeomorphism anomaly. In [BC08] it was shown that, as far as trace anomalies are concerned, this result is to be expected, since via a cohomological analysis it can be seen that no true trace anomaly can exist in higher spin currents.

Of course we could repeat the same construction for antiholomorphic currents and find the corresponding covariant ones. We would find perfectly symmetric results with respect to the ones above.

3.3.2 Covariant fermionic $W_{1+\infty}$ currents

Prior to construction the covariant higher-spin currents from fermionic fields, first we recall some properties of fermions in two dimensions. The equation of motion for a right-handed fermion with unit charge is given by

$$\left(\partial_u - iA_u + \frac{1}{4}\,\partial_u\varphi\right)\psi(u,v) = 0$$

In the Lorentz gauge, the gauge field can be written locally as

$$A_u = \partial_u\eta(u,v) \quad , \quad A_v = -\partial_v\eta(u,v)$$

where $\eta(u,v)$ is a scalar field. Since gravitational and gauge fields are not generally holomorphic, $\psi(u,v)$ is not holomorphic either. In order to construct holomorphic quantities from a fermionic field, we define a new field Ψ by

$$\Psi \;\equiv\; \exp\left(\frac{1}{4}\varphi(u,v) + i\eta(u,v)\right)\psi(u,v)$$

It is easy to show that the equation of motion implies $\partial_v\Psi = 0$ and hence Ψ is holomorphic. Similary, we can define $\Psi^\dagger$ as

$$\Psi^\dagger \;\equiv\; \exp\left(\frac{1}{4}\varphi(u,v) - i\eta(u,v)\right)\psi^\dagger(u,v)$$

The equation of motion again guarantees that $\partial_v \Psi^\dagger = 0$, so that $\Psi^\dagger$ is also holomorphic. We will use Ψ and $\Psi^\dagger$ as basic chiral fields to construct $W_{1+\infty}$ algebra introduced above.

To covariantize the expressions of the currents we reduce the problem to one dimension by considering only the u dependence and keeping v fixed. In one dimension a curved coordinate u in the presence of a background metric

$$g_{\mu\nu} = e^{\varphi(u,v)} \eta_{\mu\nu}$$

is easly related to the corresponding normal coordinate x by the equation $\partial_x = e^{-\varphi(u,v)} \partial_u$. We wiew u as $u(x)$, and by the above equation, we extract the correspondence between $j^{(s)}_{z\ldots z}$ and $j^{(s)}_{u\ldots u}$ by identifying u with the complex coordinate z after Wick rotation. The expressions we get in this way are not yet components of the covariant currents. We have to remember the current conformal weights and introduce suitable factors in order to take them into account.

Under a holomorphic conformal transformation $u \to \tilde{u}$ the function $\varphi(u, v)$ and the field $\Psi(u)$ transform according to

$$\tilde{\varphi}(\tilde{u}, v) = \varphi(u, v) - \ln(du/d\tilde{u}) \quad , \quad \widetilde{\Psi}(\tilde{u}) = \left(\frac{d\tilde{u}}{du}\right)^{\frac{1}{2}} \Psi(u)$$

Therefore, $e^{-\varphi/2}\Psi(u)$ and analogously $e^{-\varphi/2}\Psi^\dagger(u)$ transform as a scalars with respect to a holomorphic coordinate transformation.

A remark is in order about the transformation property of the ferionic field Ψ under (holomorphic) gauge transformations; in the Lorentz gauge there remains a residual holomorphic gauge symmetry,

$$\psi'(u, v) = e^{i\Lambda(u)} \psi(u, v) \quad , \quad \eta'(u, v) = \eta(u, v) + \Lambda(u)$$

Under this transformation the field $\Psi(u)$ transforms as a field with twice the charge of ψ, i.e.

$$\Psi'(u) = e^{2i\Lambda(u)} \Psi(u)$$

As a consequence the covariant derivative of $\Psi(u)$ turns out to be

$$\nabla_u \Psi(u) = \left(\partial_u - \frac{1}{2}\,\partial_u \varphi - 2iA_u\right) \Psi(u)$$

$$\nabla_u \Psi^\dagger(u) = \left(\partial_u - \frac{1}{2}\,\partial_u \varphi + 2iA_u\right) \Psi^\dagger(u)$$

and for the higher covariant derivatives we have

$$\nabla_u^{m+1} \Psi(u) = \left(\partial_u - \left(m + \frac{1}{2}\right) \partial_u \varphi - 2iA_u\right) \nabla_u^m \Psi(u)$$

$$\nabla_u^{m+1} \Psi^\dagger(u) = \left(\partial_u - \left(m + \frac{1}{2}\right) \partial_u \varphi + 2iA_u\right) \nabla_u^m \Psi^\dagger(u)$$

It can be shown that $e^{-(m+\frac{1}{2})\varphi}\nabla_u^m \Psi(u)$ and $e^{-(m+\frac{1}{2})\varphi}\nabla_u^m \Psi^\dagger(u)$ transform as scalars under holomorphic coordinate transformation, for every $m \in \mathbb{N}$.

After this preliminaries the covariant currents are constructed using the following bricks:

$$J^{(\mathrm{m,n})}_{u\ldots u} = e^{(m+n+1)\varphi(u,v)} \lim_{\epsilon\to 0} \Bigg(e^{2i\int_{u_+}^{u_-} A_u(u',v)du'} \times$$

$$\times\, e^{-(m+1/2)\varphi(u_+,v)}\nabla_u^m \Psi^\dagger(u_+)\, e^{-(n+1/2)\varphi(u_+,v)}\nabla_u^n \Psi(u_-) - \frac{c^{\mathrm{F}}_{m,n}\hbar}{\epsilon^{m+n+1}} \Bigg) \tag{3.70}$$

where we have used abbrevations $u_\pm \equiv u(x \pm \epsilon/2)$. The gauge prefactor, known as *Wilson line*, was introduced in order to preserve gauge invariance of the expression (see e.g. [PS95], chapter 15). The numerical constants $c^{\mathrm{F}}_{m,n}$ defined by

$$c^{\mathrm{F}}_{m,n} = (-1)^m (m+n)!$$

are determined in such a way that all singularities are canceled in the final expressions for $J^{(\mathrm{m,n})}_{u\ldots u}$.

Finally, let us define the covariant currents corresponding to the $W_{1+\infty}$ fermionic currents,

$$J^{(\mathrm{s})}_{u\ldots u}(u) = -\frac{B(s)}{s}\sum_{k=1}^{s}(-1)^k \binom{s-1}{s-k}^2 J^{(\mathrm{s-k,k-1})}_{u\ldots u} \tag{3.71}$$

where

$$B(s) = \frac{2^{s-3}s!}{(2s-3)!!}\, q^{s-2}$$

The first few covariant $W_{1+\infty}$ fermionic currents can be written in pretty simple form,

$$J^{(1)}_u = j^{(1)}_u + \frac{i\hbar}{2q}\, A_u \tag{3.72}$$

$$J^{(2)}_{uu} = j^{(2)}_{uu} - 2A_u J^{(1)}_u + \hbar\left(2A_u^2 - \frac{\mathcal{T}_{uu}}{12}\right) \tag{3.73}$$

$$J^{(3)}_{uuu} = j^{(3)}_{uuu} - 4J^{(2)}_{uu}A_u + \left(\frac{\mathcal{T}_{uu}}{6} - 4A_u^2\right) J^{(1)}_u + \hbar\left(\frac{8A_u^3}{3} - \frac{A_u\mathcal{T}_{uu}}{3}\right) \tag{3.74}$$

$$\begin{aligned} J^{(4)}_{uuuu} = j^{(4)}_{uuuu} &+ \hbar\left(4A_u^4 - \frac{7\mathcal{T}_{uu}A_u^2}{5} - \frac{2}{5}(\nabla_u^2 A_u)A_u + \frac{7\mathcal{T}_{uu}^2}{240} + \frac{3}{5}\,(\nabla_u A_u)^2\right) - \\ &-8J^{(1)}_u A_u^3 - 12J^{(2)}_{uu}A_u^2 + \left(\frac{1}{5}\,\nabla_u^2 J^{(1)}_u + \frac{7\mathcal{T}_{uu}J^{(1)}_u}{5} - 6J^{(3)}_{uuu}\right) A_u - \\ &-\frac{3}{5}\,(\nabla_u A_u)(\nabla_u J^{(1)}_u) + \frac{1}{5}\,(\nabla_u^2 A_u)J^{(1)}_u + \frac{7\mathcal{T}_{uu}J^{(2)}_{uu}}{10} \end{aligned} \tag{3.75}$$

The covariant derivatives of the $W_{1+\infty}$ fermionic currents $J^{(\mathrm{s})}$ are given by

$$g^{uv}\nabla_v J^{(1)}_u = -\hbar F_u{}^u \tag{3.76}$$

$$g^{uv}\nabla_v J^{(2)}_{uu} = \frac{\hbar}{24}\nabla_u R + F_u{}^u J^{(1)}_u \tag{3.77}$$

$$g^{uv}\nabla_v J^{(3)}_{uuu} = 2F_u{}^u J^{(2)}_{uu} - \frac{1}{12}(\nabla_u R)J^{(1)}_u \tag{3.78}$$

$$g^{uv}\nabla_v J^{(4)}_{uuu} = \frac{3}{10}\,(\nabla_u F_u{}^u)(\nabla_u J^{(1)}_u) - \frac{1}{10}F_u{}^u(\nabla^2_u J^{(1)}_u) - \\ -\frac{1}{10}\,(\nabla^2_u F_u{}^u)J^{(1)}_u - \frac{7}{20}\,(\nabla_u R)J^{(2)}_{uu} + 3F_u{}^u J^{(3)}_{uuu} \tag{3.79}$$

In the case of lowest spin current, $J^{(1)}$, equation (3.76) gives rise to the gauge anomaly

$$g^{\mu\nu}\nabla_\mu J^{(1)}_\nu = -\frac{\hbar}{2}\,\epsilon^{\mu\nu}F_{\mu\nu} \tag{3.80}$$

Apart from the gauge anomaly in the spin-1 current, we are interested to check whether there are trace anomalies in the other currents. This is done as follows: after the RHS of the above equation in expressed in terms of covariant quantities, terms proportional to $\hbar$ are identifies as possible anomalies by proceeding in analogy to the energy-momentum tensor. One assumes that there is no anomaly in the conservation laws of covariant currents, that is, that the covariant derivatives of higher spin currents with the addition of suitable covariant terms (classical, not proportional to $\hbar$) vanish. Since

$$(\nabla\cdot J)_{u\ldots u} + \ldots = g^{uv}\nabla_v J_{u\ldots u} + g^{uv}\nabla_u J_{vu\ldots u} + \ldots = 0$$

where the dots denote the above mentioned classical covariant terms, one relates terms proportional to $\hbar$ in the u derivative of the trace ($vu\ldots u$ components) with the terms proportional to $\hbar$ in the v derivative of $u\ldots u$ components of the currents.

For the covariant energy-momentum tensor, $J^{(2)}$, we have

$$\mathrm{Tr}(J^{(2)}) = 2g^{vu}J^{(2)}_{vu} = -\frac{\hbar}{12}\,R$$

which is the well known trace anomaly. In the case of the $J^{(3)}$ current the terms that carry explicit factors of $\hbar$ cancel out in $g^{uv}\nabla_v J^{(3)}_{uuu}$ which implies absence of the trace anomaly. The same is true for $J^{(4)}$ and was checked up to spin-8 currents with *Mathematica* software.

§ 3.4 Identification of higher moments

Using the results from the previous section we would like now to identify higher moments of Hawking radiation spectrum through generalization of the procedure from the trace anomaly method. Since we have found the absence of the trace and diffeomorphism anomalies in the higher-spin currents, we propose a more apropriate name, **Schwarzian derivative method**.

By requiring regularity at the horizon we have previously found that the value of $\langle j^{(s)}_{z\ldots z}\rangle_{\mathsf{H}}$ (at the horizon) is given by either $\langle X^{\mathrm{B}}_s\rangle$ for the case of bosonic currents, or $\langle X_s^{\mathrm{F}} \rangle$ for the case of fermionic currents. If we identify $j^{(\mathrm{s})}_{z\ldots z}$ via a Wick rotation with $j^{(\mathrm{s})}_{u\ldots u}$, we get the corresponding value at the horizon $\langle j^{(\mathrm{s})}_{u\ldots u}\rangle_{\mathsf{H}}$. Since the problem we are considering is stationary and $j^{(\mathrm{s})}_{u\ldots u}$ is chiral, it follows that it is constant in t and r. Therefore

$$\langle j^{(\mathrm{s})}_{u\ldots u}\rangle_{\mathsf{H}} = \langle j^{(\mathrm{s})}_{u\ldots u}\rangle_{\infty}$$

Furthermore, since $j^{(\mathrm{s})}_{u\ldots u}$ and $J^{(\mathrm{s})}_{u\ldots u}$ asymptotically coincide, the asymptotic flux of these currents is

$$\left\langle J^{(\mathrm{s})r}{}_{t\ldots t}\right\rangle_{\infty} = \langle J^{(\mathrm{s})}_{u\ldots u}\rangle_{\infty} - \langle J^{(\mathrm{s})}_{v\ldots v}\rangle_{\infty} = \langle j^{(\mathrm{s})}_{u\ldots u}\rangle_{\infty} = \langle j^{(\mathrm{s})}_{u\ldots u}\rangle_{\mathsf{H}}$$

In order to evaluate higher moments we shall set $q = i/4$ and divide the currents by -2π in order to properly normalize the (physical) energy-momentum tensor.

In the bosonic case we get

$$-\frac{1}{2\pi}\left\langle J^{(\mathrm{s})r}{}_{t\ldots t}\right\rangle_{\infty} = -\frac{1}{2\pi}\langle X_s^{\mathrm{B}}\rangle = i^{s-2}\frac{\kappa^s B_s}{2\pi s}\,\hbar \tag{3.81}$$

The RHS vanishes for odd s (except for $s = 1$ which is not excited in our case) and coincides with the higher moments of the bosonic thermal spectrum (3.4) for the even $s = 2n$,

$$-\frac{1}{2\pi}\left\langle J^{(2\mathrm{n})r}{}_{t\ldots t}\right\rangle_{\infty} = -\frac{1}{2\pi}\langle X_{2\mathrm{n}}^{\mathrm{B}}\rangle = (-1)^{n+1}\frac{\kappa^{2n} B_{2n}}{4\pi n}\,\hbar \tag{3.82}$$

with $g_* = 2$ (because our bosonic currents carry both particle and antiparticle contributions).

Now we turn to fermionic case. We shall first consider the case in which the electromagnetic field is decoupled ($m = 0$)

$$-\frac{1}{2\pi}\left\langle J^{(\mathrm{s})r}{}_{t\ldots t}\right\rangle_{\infty} = -\frac{1}{2\pi}\langle X_s^{\mathrm{F}}\rangle = i^{s-2}\frac{\kappa^s B_s}{2\pi s}\,(1-2^{1-s})\,\hbar \tag{3.83}$$

The even moments are

$$-\frac{1}{2\pi}\left\langle J^{(2\mathrm{n})r}{}_{t\ldots t}\right\rangle_{\infty} = -\frac{1}{2\pi}\langle X_{2\mathrm{n}}^{\mathrm{F}}\rangle = (-1)^{n+1}\frac{\kappa^{2n} B_{2n}}{4\pi n}\,(1-2^{1-2n})\hbar \tag{3.84}$$

These values correspond precisely to the fluxes of the Hawking thermal spectrum defined by (3.5) with $g_* = 2$ (because our fermionic currents carry both particle and antiparticle contributions).

Next we wish to take into account the presence of the gauge field, which, in our case, vanishes at infinity but not at the horizon. This introduces a significant change in our method. Previously, our basic criterion was the regularity of $T^{(\mathrm{hol})}_{uu}$ (or its higher spin analogons) at the horizon. Now the presence of the electromagnetic field interferes with this regularity at the horizon. As a consequence we have to update our criterion.

Let us start with the first current $j^{(1)}_u$. For the simplicity we shall take for the moment the unit charge, $m = 1$. Introducing Kruskal coordinate $U = -e^{-\kappa u}$ we easily get (remember that $\langle X^{\mathrm{F}}_1\rangle$ vanishes)

$$J^{(1)}_U = j^{(1)}_U + \frac{i\hbar}{2q}\,A_U = \frac{1}{\partial_u U}\left(j^{(1)}_u + \frac{i\hbar}{2q}\,A_u\right) = \frac{1}{-\kappa U}\left(j^{(1)}_u + \frac{i\hbar}{2q}\,A_u\right)$$

It is evident that we have to require regularity at the horizon of the $j^{(1)}_U + i\hbar A_U/2q$ and not of $j^{(1)}_U$ alone. Therefore we get

$$\langle j^{(1)}_u\rangle_{\mathsf{H}} + \frac{i\hbar}{2q}\,\langle A_u\rangle_{\mathsf{H}} = 0$$

Using the fact that $j^{(1)}_u$ is constant in t and r, one can see that

$$-\frac{i\hbar}{2q}\,\langle A_u\rangle_{\mathsf{H}} = \langle j^{(1)}_u\rangle_{\mathsf{H}} = \langle j^{(1)}_u\rangle_{\infty}$$

Since $j^{(1)}_u(u)$ and $J^{(1)}_u(u)$ asymptotically coincide and $A_u(u)$ asymptotically vanishes, we get

$$-\frac{1}{2\pi}\,\langle J^{(1)r}\rangle = -\frac{1}{2\pi}\,\langle J^{(1)}_u\rangle + \frac{1}{2\pi}\,\langle J^{(1)}_v\rangle = \frac{i\hbar}{2\pi q}\,\langle A_u\rangle_{\mathsf{H}} = \frac{1}{2\pi}\,A_t = \frac{\Omega_{\mathsf{H}}}{2\pi}$$

where we have assumed that there is no incoming flux from infinity, $\langle J^{(1)}_v\rangle_\infty = 0$.

From this example we learn that we have to assume that the currents $J^{(\mathrm{s})}_{U\ldots U}$ are regular on the horizon in Kruskal coordinates U. Since these currents are covariant, we have

$$J^{(\mathrm{s})}_{U\ldots U} = \frac{1}{(-\kappa U)^s}\,J^{(\mathrm{s})}_{u\ldots u}(u)$$

It then follows that the currents $J^{(\mathrm{s})}_{u\ldots u}$ and their $n-1$ derivatives vanish. Using expressions for the covariant fermionic currents, we get

$$\langle j^{(2)}_{uu}\rangle_{\mathsf{H}} = -\hbar\left(2A^2_u - \frac{\mathcal{T}_{uu}}{12}\right)_{\mathsf{H}} \tag{3.85}$$

$$\langle j^{(3)}_{uuu}\rangle_{\mathsf{H}} = -\hbar\left(\frac{8A^3_u}{3} - \frac{A_u\mathcal{T}_{uu}}{3}\right)_{\mathsf{H}} \tag{3.86}$$

$$\langle j^{(4)}_{uuuu}\rangle_{\mathsf{H}} = -\hbar\left(4A^4_u - \frac{7TA^2_u}{5} - \frac{2}{5}(\nabla^2_u A_u)A_u + \frac{7T^2}{240} + \frac{3}{5}\,(\nabla_u A_u)^2\right)_{\mathsf{H}} \tag{3.87}$$

Evaluating the RHS expressions on the horizon we get at the infinity

$$-\frac{1}{2\pi}\left\langle J^{(2)r}{}_t\right\rangle_\infty = \frac{\kappa^2}{48\pi} + \frac{\Omega^2_{\mathsf{H}}}{4\pi} \tag{3.88}$$

$$-\frac{1}{2\pi}\left\langle J^{(3)r}{}_{tt}\right\rangle_\infty = \frac{\kappa^2\Omega_{\mathsf{H}}}{24\pi} + \frac{\Omega^3_{\mathsf{H}}}{6\pi} \tag{3.89}$$

$$-\frac{1}{2\pi}\left\langle J^{(4)r}{}_{ttt}\right\rangle_\infty = \frac{7\kappa^2}{1920\pi} + \frac{\kappa^2\Omega^2_{\mathsf{H}}}{16\pi} + \frac{\Omega^4_{\mathsf{H}}}{8\pi} \tag{3.90}$$

where we have used

$$\langle f'(r)\rangle_{\mathsf{H}} = 2\kappa \quad , \quad \langle \mathcal{T}_{uu}\rangle_{\mathsf{H}} = -\frac{\kappa^2}{2} \quad , \quad \langle A_t\rangle_{\mathsf{H}} = \Omega_{\mathsf{H}}$$

These results agree with the formula (3.6) after the replacement $A_t \to mA_t$.

§ 3.5 Reconstruction of the spectrum from higher moments

Having calculated higher moments of the Hawking radiation spectrum, one would like to know how to reconstruct the distribution and whether such procedure leads to an unique result. This question is a subject of several related mathematical problems, namely *Hausdorff*, *Stiltjes* and *Hamburger moment problem*, differing in domain of distribution function. For our particular problem we shall imploy the procedure explained in [Rei98], completed with some necessary technicalities.

The central idea is to connect the higher moments of spectrum with the derivatives of its Fourier transform. Higher moments F_n of the distribution $N(\omega)$ are defined as

$$F_n = \frac{1}{4\pi}\int_{-\infty}^{\infty} dk k^{n-2}\omega N(\omega) \tag{3.91}$$

where $\omega = |k|$. Introducing auxiliary function $Q(k)$,

$$Q(k) \;\equiv\; |k|N(|k|) = \omega N(\omega) \tag{3.92}$$

we can write

$$F_n = \frac{1}{4\pi}\int_{-\infty}^{\infty} dk k^{n-2} Q(k) \tag{3.93}$$

Nontrival moments are the even ones (the odds vanish),

$$F_{2n} = \frac{1}{4\pi}\int_{-\infty}^{\infty} dk k^{2n-2} Q(k) \tag{3.94}$$

The Fourier transform of the function $Q(k)$ is given by

$$\mathscr{Q}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, Q(k) e^{ikx} \tag{3.95}$$

The crucial observation is that the derivations of the Fourier transform $\mathscr{Q}(x)$, evaluated at the origin, are related to the higher moments,

$$\mathscr{Q}^{(2n-2)}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk (ik)^{2n-2} Q(k) = \frac{i^{2n-2}}{\sqrt{2\pi}}\, 4\pi F_{2n} = (-1)^{n-1}\frac{4\pi}{\sqrt{2\pi}}\, F_{2n}$$

Consistently with $F_{2n-1} = 0$ we have $\mathscr{Q}^{(2n-1)}(0) = 0$. Using Taylor series for the $\mathscr{Q}(x)$ we can write

$$\mathscr{Q}(x) = \sum_{j=0}^{\infty} \frac{\mathscr{Q}^{(j)}(0)}{j!} x^j = \sum_{m=1}^{\infty} \frac{\mathscr{Q}^{(2m-2)}(0)}{(2m-2)!} x^{2m-2} = \frac{4\pi}{\sqrt{2\pi}} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} F_{2m}}{(2m-2)!} x^{2m-2}$$

However, one must be cautious about the domain of analicity of this function.

For example, if we insert the moments for the bosonic case,

$$F_{2m} = g_* (-1)^{m+1} \frac{B_{2m} \kappa^{2m}}{8\pi m}$$

we get the Fourier transform

$$\mathscr{Q}(x) = \frac{g_*}{\sqrt{2\pi}} \sum_{m=1}^{\infty} \frac{B_{2m} \kappa^{2m}}{2m} \frac{x^{2m-2}}{(2m-2)!}$$

This series is evaluated in Appendix D and the result is

$$\mathscr{Q}(x) = \frac{g_*}{\sqrt{2\pi}} \mathcal{S}_\kappa(x) = \frac{g_*}{\sqrt{2\pi}} \left(\frac{1}{x^2} - \frac{\kappa^2}{4\,\mathrm{sh}^2(\kappa x/2)} \right) \tag{3.96}$$

Using analytic continuation this function can be extended over whole complex plain, except for the singularities (second order poles) on the imaginary axis ($z = 2m\pi i/\kappa$ for all $m \in \mathbb{Z}/\{0\}$). Next we perform inverse Fourier transform,

$$Q(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, \mathscr{Q}(x) e^{-ikx} = \frac{g_*}{2\pi} \int_{-\infty}^{\infty} dx \left(\frac{1}{x^2} - \frac{\kappa^2}{4\,\mathrm{sh}^2(\kappa x/2)} \right) e^{-ikx}$$

First half of the integral can be done by semicircle integration curve (avoiding the pole of second order at the origin of the complex plane), from which one gets

$$\int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2}\, dx = -\pi |k|$$

Second part is done integrating

$$\oint \frac{e^{-ikz}}{\mathrm{sh}^2(\kappa x/2)}\, dz$$

over rectangular integration curve, avoiding second order poles at $z = 0$ and $z = 2\pi i/\kappa$. From here one gets

$$\int_{-\infty}^{\infty} \frac{e^{-ikx}}{\mathrm{sh}^2(\kappa x/2)}\, dx = -\frac{4\pi k}{\kappa^2} \mathrm{cth}(\pi k/\kappa) = -\frac{4\pi |k|}{\kappa^2} \mathrm{cth}(\pi |k|/\kappa)$$

Using this, we have

$$Q(k) = \frac{g_*}{2} |k| \left(-1 + \frac{e^{|k|\pi/\kappa} + e^{-|k|\pi/\kappa}}{e^{|k|\pi/\kappa} - e^{-|k|\pi/\kappa}} \right) = g_* \frac{|k|}{e^{2\pi|k|/\kappa} - 1}$$

Finally,

$$N(\omega) = \frac{Q(\omega)}{\omega} = \frac{g_*}{e^{\beta\omega} - 1} \tag{3.97}$$

which is precisely the Bose-Einstein distribution (with the inverse temperature $\beta = 2\pi/\kappa$).

On the other hand, if we insert the moments for the fermionic case,

$$F_{2m} = g_*(-1)^{m+1}\frac{B_{2m}\kappa^{2m}}{8\pi m}\,(1-2^{1-2m})$$

we get

$$\begin{aligned}\mathscr{Q}(x) &= \frac{g_*}{2\sqrt{2\pi}}\,\kappa^2\sum_{m=1}^{\infty}\frac{B_{2m}}{m(2m-2)!}\,(\kappa x)^{2m-2}\,(1-2^{1-2m}) = \\ &= \frac{g_*}{\sqrt{2\pi}}\left(\mathcal{S}_\kappa(x) - \frac{1}{2}\,\mathcal{S}_\kappa(x/2)\right) = \\ &= \frac{g_*}{\sqrt{2\pi}}\left(-\frac{1}{x^2} - \frac{\kappa^2}{4\,\mathrm{sh}^2(\kappa x/2)} + \frac{\kappa^2}{8\,\mathrm{sh}^2(\kappa x/4)}\right)\end{aligned}$$

Fourier transform can be calculated using integrals from bosonic case,

$$Q(k) = \frac{g_*}{2}\,|k|\left(1 - 2\,\frac{e^{2|k|\pi/\kappa}+e^{-2|k|\pi/\kappa}}{e^{2|k|\pi/\kappa}-e^{-2|k|\pi/\kappa}} + \frac{e^{|k|\pi/\kappa}+e^{-|k|\pi/\kappa}}{e^{|k|\pi/\kappa}-e^{-|k|\pi/\kappa}}\right) = \frac{g_*|k|}{e^{2\pi|k|/\kappa}+1}$$

Finally, we have

$$N(\omega) = \frac{Q(\omega)}{\omega} = \frac{g_*}{e^{\beta\omega}+1} \tag{3.98}$$

which is precisely the Fermi-Dirac distribution, as expected.

§ 3.6 Hawking radiation via any currents?

At the end of this chapter we want to reevaluate the importance of the W algebras and the corresponding currents in the previosly described method. One could in priciple argue that *any* set of higher spin currents, normalized so that the coefficients satisfy the equality

$$B(s)\sum_{k=1}^{s-1} A_k^s = (4q)^{s-2} \tag{3.99}$$

shall reproduce the higher moments of the Hawking radiation. However, without some guiding principle, one generally needs to separately choose the normalization for every of the higher spin currents. In principle, there are multitude of choices of normalization: relative coefficients between binomial inside of each current, relative coefficients between the currents and the overall normalization. The choice of W currents emidiately solves this problem by telling us how to normalize the currents in such a way as to get an algebra; parameter q is fixed by one of the spin $s > 3$ currents in the bosonic or spin-1 current in the fermionic case.

Also, other currents have apparent anomaly in the trace, but we have shown that these are trivial, i.e. there are no trace (or diffeomorphism) anomalies in the conservation laws of the higher spin currents. The currents contructed with W algebra, on the other hand, explicitely exibit the absence of the anomalies. Because of this, by the principle of the Occam's razor, we propose the guiding principle for the Schwarzian derivative method:

The W algebra is the appropriate structure
underlying the thermal spectrum of the Hawking radiation.

Here we assume that the "W algebra" refers to W_∞ algebra in the bosonic and $W_{1+\infty}$ algebra in the fermionic case.

This result seems to imply that the 2-dimensional physics around the horizon is characterized by a symmetry larger than the Virasoro algebra, such as a W_∞ and/or $W_{1+\infty}$ algebra. One possible hint for the relation between W algebras and near-horizon symmetries might lie in the following fact: reparametrization of the q by some real parameter λ via $q \to \lambda q$ corresponds to change of temperature $\kappa \to \lambda\kappa$ and degrees of freedom $g_* \to g_*/\lambda^2$. Also, we note in passing that the limit $\lambda \to 0$ corresponds to the contraction $W_\infty \to w_\infty$.

Chapter 4

Cohomological analysis of anomalies

Existence of covariant anomalies is not easy to analyze in general, while general results can be obtained for consistent anomalies. Since absence of consistent anomalies implies absence of the corresponding covariant ones, we will show that for the conservation laws we are interested in, there are no consistent anomalies (except the well-known in the lowest spin currents). In this chapter we shall present the proof of absence of diffeomorphism anomalies for the spin-4 current and show that, under some resaonable assumptions, the argument can be extended to all higher-spin currents.

§ 4.1 Local field transformations

The conservation of the energy-momentum tensor corresponds to the symmetry of the theory under the diffeomorphism transformations $x^\mu \to x^\mu + \xi^\mu$,

$$\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{4.1}$$

The background fields transform in a covariant way under these transformation

$$\delta_\xi B^{(s)}_{\mu_1\ldots\mu_s} = \xi^\lambda \partial_\lambda B^{(s)}_{\mu_1\ldots\mu_s} + \partial_{\mu_1}\xi^\lambda B^{(s)}_{\lambda\mu_1\ldots\mu_s} + \ldots + \partial_{\mu_s}\xi^\lambda B^{(s)}_{\mu_1\ldots\mu_{s-1}\lambda} \tag{4.2}$$

Similary, the conservation of higher spin currents correspond to the symmetry under higher tensorial transformation. In particular, the conservation of the $J^{(4)}$ is due to invariance under

$$\delta_\tau B^{(4)}_{\mu_1\mu_2\mu_3\mu_4} = \nabla_{\mu_1}\tau_{\mu_2\mu_3\mu_4} + \text{cycl.} \tag{4.3}$$

where τ is a completely symmetric traceless tensor and "cycl." denotes the terms obtained by cyclic permutations of the indices.

To find the consistent anomalies of the energy-momentum tensor and higher spin currents with respect to the symmetry induced by the above transformations, we will analyze the solutions of the relevant WZ consistency conditions. An equivalent and simpler way is to transform the problem into a cohomological one. We promote the transformation parameters to anticommuting ghost fields and endow them with a suitable transformation law. This gives rise to a nilpotent operator acting on the local functionals of the fields and their derivatives. Local functionals (*cochains*) and nilpotent operator (*coboundary*) define a differential complex. Anomalies correspond to non-trivial cocycles.

The transformation laws are expressed in terms of symmetry parameters and basic background fields (in our case, metric $g_{\mu\nu}$). Their form is determined so that they form a Lie algebra and leave unchanged the terms in the effective action, in particular, the terms involving the matter fields. Also, we impose to δ_ξ and δ_τ transformations to be nilpotent and form a graded algebra,

$$\delta_\xi^2 = 0 \quad , \quad \delta_\tau^2 = 0 \quad , \quad \delta_\xi\delta_\tau + \delta_\tau\delta_\xi = 0 \tag{4.4}$$

Additional restriction is provided by the canonical dimension of the various fields. For the the ξ and τ parameters, metric g_{ab}, Ricci tensor R_{ab}, field B and covariant operator ∇, these are listed below

$$[\tau] = -3 \quad , \quad [B] = -2 \quad , \quad [\xi] = -1 \quad , \quad [g] = 0 \quad , \quad [\nabla] = +1 \quad , \quad [R] = +2$$

Taking all this into account, we have (see Appendix A of [BCPS08])

$$\delta_\tau g_{\mu\nu} = 0 \tag{4.5}$$

Consistency implies

$$\delta_\xi \xi^\mu = \xi^\lambda \partial_\lambda \xi^\mu \tag{4.6}$$

and

$$\delta_\xi \tau_{\mu\nu\rho} = \xi^\lambda \partial_\lambda \tau_{\mu\nu\rho} + \partial_\mu \xi^\lambda \tau_{\lambda\nu\rho} + \partial_\nu \xi^\lambda \tau_{\mu\lambda\rho} + \partial_\rho \xi^\lambda \tau_{\mu\nu\lambda} \tag{4.7}$$

In a similar manner, we assume that τ is Abelian parameter, that is,

$$\delta_\tau \tau_{\mu\nu\lambda} = 0 \tag{4.8}$$

Justification for this choice is explained in the Appendix A of [BCPS08].

In the following we will denote by δ_ξ and δ_τ the corresponding functional operators. It follows from (4.4) that the operator

$$\delta_{tr} = \delta_\xi + \delta_\tau \tag{4.9}$$

is also nilpotent. It is clear that δ_{tr} is not the total functional operator of our system, but rather a *truncated* one, since we are disregarding higher tensorial gauge transformations. Such a truncation is justified by the fact that our differential system is graded. This can be seen as follows.

Let us consider the nilpotent total differential operator

$$\delta_{tot} = \delta_\xi + \delta_\tau + \ldots \tag{4.10}$$

The integrated anomalies are defined by

$$\delta_{tot}\Gamma^{(1)} = \hbar\Delta \quad , \quad \delta_{tot}\Delta = 0 \tag{4.11}$$

where $\Gamma^{(1)}$ is the one-loop quantum action. The Δ, which is the integral of a local functional in the fields and their derivatives, splits naturally into $\Delta_\xi + \Delta_\tau + \ldots$; in turn, each addend splits into a sum of terms according to the degree of their integrand. The degree is defined by the number of derivatives of the integrand minus 1,

$$\Delta_\xi = \Delta_\xi^{(2)} + \Delta_\xi^{(4)} + \Delta_\xi^{(6)} + \ldots \quad , \quad \Delta_\tau = \Delta_\tau^{(4)} + \Delta_\tau^{(6)} + \ldots \tag{4.12}$$

As a consequence, $\delta_{tot}\Delta = 0$ splits into

$$\delta_\xi \Delta_\xi^{(2)} = 0 \quad , \quad \delta_\xi \Delta_\xi^{(4)} = 0 \quad , \quad \delta_\xi \Delta_\xi^{(6)} = 0 \quad , \quad \ldots \tag{4.13}$$

and (note that the action of the δ_τ on $\Delta_\xi^{(2)}$ is trivial)

$$\delta_\tau \Delta_\tau^{(4)} = 0 \quad , \quad \delta_\tau \Delta_\tau^{(6)} = 0 \quad , \quad \ldots \tag{4.14}$$

with the cross conditions

$$\delta_\tau \Delta_\xi^{(4)} + \delta_\xi \Delta_\tau^{(4)} = 0 \quad , \quad \delta_\tau \Delta_\xi^{(6)} + \delta_\xi \Delta_\tau^{(6)} = 0 \quad , \quad \ldots \tag{4.15}$$

Therefore, fortunately, our complex splits into subcomplexes and, for example it makes sense to truncate it at level 4, conditions are not affected by the higher order equations in the complex.

§ 4.2 The search for consistent anomalies

The solution to

$$\delta_\xi \Delta_\xi^{(2)} = 0$$

is well-known 2-dimensional diffeomorphism anomaly, so that we proceed to the higher cocycles, starting with $\Delta_\xi^{(4)}$. We begin our analysis with one important observation,

Theorem 1. Solutions of

$$\delta_\xi \Delta_\xi^{(4)} = 0 \tag{4.16}$$

are all trivial, that is, there exists a local function $C^{(4)}$ of the background fields, such that $\Delta_\xi^{(4)} = \delta_\xi C^{(4)}$.

We shall proof this theorem using a following result,

Theorem 2. (Bonora,Pasti,Tonin; [BPT86]) The general form of the solutions of equation (4.16) is

$$\Delta_\xi^{(4)} = \int d^2x \, (\partial_m \xi^m \, \mathfrak{b} + \partial_p \partial_q \xi^m \, \mathfrak{b}_m^{pq}) \tag{4.17}$$

where $\mathfrak{b}$ and $\mathfrak{b}_m^{pq}$ are polynomial expressions of the fields and their derivatives in which all the indices are saturated except for the explicitly shown ones, and $\mathfrak{b}$ is not itself a derivative.

Notice that $\mathfrak{b}$ and $\mathfrak{b}_m^{pq}$ are *not*, in general, covariant tensors. For future reference we will call **first type** and **second type** the cocycles having the form of the first and second term in the RHS of (4.17), respectively. We stress that we cannot, in general, use covariance as a classifying device. The first type cocycles were discussed in [BPT86]. Any such cocycle is a partner of a Weyl cocycle and can be eliminated in favor of the partner by substracting a suitable counterterm. Since it has been shown in [BC08] that, at order four, there are no non-trivial Weyl cocycles (trace anomalies), we will disregard these cocycles altogether and concentrate on cocycles of the second type in(4.17), i.e. on cocycles proportional to $\partial_p \partial_q \xi^m$. It is easy to realize that $\mathfrak{b}_m^{pq}$ can be synthetically written in the following general form

$$\mathfrak{b} = A_1 + \Gamma A_2 + \Gamma\Gamma A_3 + \Gamma\Gamma\Gamma A_3 + \partial\Gamma A_5 + \Gamma\partial\Gamma A_6 + \partial\partial\Gamma A_7 \tag{4.18}$$

where all the indices are implicitly understood (e.g. A_1 stands for $A_1{}_m^{pq}$) and $A_1, \ldots, A_7$ are all weight 1 covariant tensors. The symbol Γ represents the linear (not necessarily metric) connection Γ_{mn}^l.

The important remark is that, since $\Delta_\xi^{(4)}$ is degree 4, it follows that all expressions A_i can only be linear in the background field $B^{(4)}$ and contain $4-i$ derivatives for $i \in \{1,2,3,4\}$, one derivative for $i=5$ and no derivatives for $i = 6, 7$.

The fact that we cannot use covariance in expressing $\mathfrak{b}_m^{pq}$ is a tremendous complication, however, there are expedients one can use to symplify one's life. One such contrivance consists in splitting the functional operator δ_ξ into two parts,

$$\delta_\xi = \delta_\xi^c + \hat{\delta}_\xi \tag{4.19}$$

where δ_ξ^c acts on cochains as if they were covariant tensors, while $\hat{\delta}_\xi$ represents the non-covariant part of the δ_ξ action. For instance, we have

$$\hat{\delta}_\xi \, \Gamma_{mn}^l = \partial_m \partial_n \xi^l$$

$$\hat{\delta}_\xi \, \partial_k \Gamma_{mn}^l = \partial_k \partial_m \partial_n \xi^l + \partial_k \partial_m \xi^p \Gamma_{pn}^l + \partial_k \partial_n \xi^p \Gamma_{pm}^l - \partial_k \partial_p \xi^l \Gamma_{mn}^p$$

$$\hat{\delta}_\xi \, \xi^l = -\xi^m \partial_m \xi^l \quad , \quad \hat{\delta}_\xi \, \partial_n \xi^l = -\partial_n \xi^m \partial_m \xi^l$$

$$\hat{\delta}_\xi \, \partial_m \partial_n \xi^l = 0$$

$$\hat{\delta}_\xi \, \partial_m \partial_n \partial_p \xi^l = \partial_m \partial_n \xi^q \partial_q \partial_p \xi^l + \partial_p \partial_m \xi^q \partial_q \partial_n \xi^l + \partial_n \partial_p \xi^q \partial_q \partial_m \xi^l$$

It is easy to prove that $\hat{\delta}_\xi$ is nilpotent,

$$\hat{\delta}_\xi^2 = 0 \tag{4.20}$$

In general $\hat{\delta}_\xi$ does not commute with the operation of differentiation except when particular conditions are met, e.g. the cases when $\hat{\delta}_\xi$ acts on forms or on expressions without unsaturated indices.

It is convenient to write integrand in (4.17) as a 2-form,

$$\Delta_\xi = \int d^2x \, \partial_p \partial_q \xi^m \, \mathfrak{b}_m^{pq} \;\equiv\; \int Q_2^1 \tag{4.21}$$

The lower index in Q_s^r represents the form order and the upper index denotes the ghost number (number of ξ factors). It is important to notice that $dQ_2^1 = 0$ implies

$$\delta_\xi^c \Delta_\xi = \int \pounds_\xi Q_2^1 = \int (\imath_\xi d + d\imath_\xi) Q_2^1 = \int d(\imath_\xi Q_2^1) = 0 \tag{4.22}$$

Now, since $\delta_\xi \Delta_\xi = 0$ it follows that $\hat{\delta}_\xi \Delta_\xi = 0$. Therefore

$$\hat{\delta}_\xi Q_2^1 = dQ_1^2 \tag{4.23}$$

for some 1-form Q_1^2. Applying $\hat{\delta}_\xi$ to both sides of this equation and using

Local Poincaré lemma : if a p-form is a polynomial made of local fields and their derivatives, whose exterior derivative vanishes, either it is a top form, or it is a constant if it is a 0-form, or it is a total derivative*.

we get

$$\hat{\delta}_\xi Q_1^2 = dQ_0^3 \tag{4.24}$$

and finally

$$\hat{\delta} Q_0^3 = 0 \tag{4.25}$$

The reason why we introduce these descent equations is that the classification problem is easier on lower order forms (with higher ghost number) than on higher order forms. Briefly stated, the strategy consists in chopping off as many coboundaries and first type cocycles as possible, so as to be left with subset of possibilities which can be easily dealt with. The proof of the Theorem 1 relies mostly on ideas and results form [BPT86] and is presented in the Appendix B of [BCPS08]. Schematically, one first proves that the solutions to $\hat{\delta}_\xi(Q_2^1 - dP_1^1) = 0$, where Q_2^1 is defined by (4.21), correspond to first type cocycles or are trivial. As a consequence of this, one proves that solutions to $\hat{\delta}_\xi(Q_1^2 - dP_0^2) = 0$, where Q_1^2 is defined by (4.23), are trivial. Thus, possible non-trivial second type cocycles (that do not vanish up to a diffeomorphism transformation) are to be looked for among Q_0^3, defined by (4.24). We have found that none of such exists, which concludes the proof.

*This is an off-shell statement.

Using this theorem and anticommuting of δ_ξ and δ_τ operators (in graded algebra), we can write

$$0 = \delta_\tau \Delta_\xi^{(4)} + \delta_\xi \Delta_\tau^{(4)} = \delta_\tau \delta_\xi C^{(4)} + \delta_\xi \Delta_\tau^{(4)} = -\delta_\xi \delta_\tau C^{(4)} + \delta_\xi \Delta_\tau^{(4)}$$

The result we have obtained,

$$\delta_\xi \left(\Delta_\tau^{(4)} - \delta_\tau C^{(4)} \right) = 0 \tag{4.26}$$

tells us that any cocycle of δ_τ can be written in a diff-covariant form. This is a piece of very useful information because it strongly limits the forms of the cochains we have to analyze in order to find the solutions to $\delta_\tau \Delta_\tau^{(4)} = 0$. In particular, we can write the general form of cocycles,

$$\Delta_\tau^{(4)} = \int d^2x \sqrt{-g} \sum_{i=1}^{3} \left(a_i I_i^\tau + b_i K_i^\tau \right) \tag{4.27}$$

where a_i, b_i are some (real) constants and

$$I_1^\tau = \tau^{\mu\nu\lambda} \nabla_\mu \nabla_\nu \nabla_\lambda R \quad , \quad I_2^\tau = \tau^{\mu\nu}{}_\nu \,\Box\, \nabla_\mu R \quad , \quad I_3^\tau = \tau^{\mu\nu}{}_\nu \nabla_\mu R^2$$

$$K_1^\tau = \tau^{\mu\nu\sigma} \epsilon_{\mu\alpha} \nabla^\alpha \nabla_\nu \nabla_\sigma R \quad , \quad K_2^\tau = \tau^{\mu\nu}{}_\nu \epsilon_{\mu\alpha} \Box \nabla^\alpha R \quad , \quad K_3^\tau = \tau^{\mu\nu}{}_\nu \epsilon_{\mu\alpha} \nabla^\alpha R^2$$

It is important to stress that we have ignored tracelessness of τ. All these cochains are, trivially, cocycles of δ_τ and they are the only ones one can construct of this type.

Next we have to find out whether these cocycles are trivial or not. The only possible counterterms are of the form

$$C = \int d^2x \sqrt{-g} \sum_{j=1}^{3} c_j J_j^\tau \tag{4.28}$$

$$D = \int d^2x \sqrt{-g}\, B^{\mu\nu\sigma}{}_\sigma \epsilon_{\nu\alpha} \nabla^\alpha \nabla_\mu R \tag{4.29}$$

where

$$J_1 = B^{\mu\nu\sigma}{}_\sigma \nabla_\mu \nabla_\nu R \quad , \quad J_2 = B^{\mu\sigma}{}_{\mu\sigma} \nabla^\nu \nabla_\nu R \quad , \quad J_3 = B^{\mu\sigma}{}_{\mu\sigma} R^2$$

Prior to our consecutive analysis, we shall prove one useful relation,

Lemma. For 2-dimensional spacetime $(\mathscr{M}, g_{ab})$ the following is valid

$$[\,\nabla_\sigma, \Box\,]\, R = -\frac{1}{4} \nabla_\sigma R^2 \tag{4.30}$$

Proof:

$$\nabla_\sigma \Box R = g^{\alpha\beta}\nabla_\sigma\nabla_\alpha\nabla_\beta R = g^{\alpha\beta}\nabla_\alpha\nabla_\sigma\nabla_\beta R + g^{\alpha\beta}R_{\sigma\alpha\beta}{}^{\gamma}\nabla_\gamma R =$$
$$= g^{\alpha\beta}\nabla_\alpha\nabla_\beta\nabla_\sigma R + g^{\alpha\beta}\frac{R}{2}\left(g_{\sigma\beta}\, g_\alpha{}^{\gamma} - g_\sigma{}^{\gamma} g_{\alpha\beta}\right)\nabla_\gamma R =$$
$$= \Box\nabla_\sigma R + \frac{R}{2}\left(\nabla_\sigma R - 2\nabla_\sigma R\right) = \Box\nabla_\sigma R - \frac{R}{2}\nabla_\sigma R$$

□

We can now see whether the above counterterms indeed suffice for the cancellation:

$$\delta_\tau \int d^2x\sqrt{-g}\, J_1 = \int d^2x\sqrt{-g}\,\delta_\tau B^{\mu\nu\lambda}{}_{\lambda}\nabla_\mu\nabla_\nu R =$$
$$= \int d^2x\sqrt{-g}\left(\nabla^\mu\tau^{\nu\lambda}{}_{\lambda} + \nabla^\nu\tau^{\mu\lambda}{}_{\lambda} + 2\nabla^\lambda\tau^{\mu\nu}{}_{\lambda}\right)\nabla_\mu\nabla_\nu R =$$
$$= -\int d^2x\sqrt{-g}\left(\tau^{\nu\lambda}{}_{\lambda}\nabla^\mu\nabla_\mu\nabla_\nu R + \tau^{\mu\lambda}{}_{\lambda}\nabla^\nu\nabla_\mu\nabla_\nu R + 2\,\tau^{\mu\nu}{}_{\lambda}\nabla^\lambda\nabla_\mu\nabla_\nu R\right) =$$
$$= -\int d^2x\sqrt{-g}\left(I_2 + I_2 + 2I_1\right)$$

$$\delta_\tau \int d^2x\sqrt{-g}\, J_2 = \int d^2x\sqrt{-g}\,\delta_\tau B^{\mu\lambda}{}_{\mu\lambda}\nabla^\nu\nabla_\nu R =$$
$$= \int d^2x\sqrt{-g}\left(\nabla^\mu\tau^{\lambda}{}_{\mu\lambda} + \nabla^\lambda\tau^{\mu}{}_{\mu\lambda} + \nabla_\mu\tau^{\mu\lambda}{}_{\lambda} + \nabla_\lambda\tau^{\mu\lambda}{}_{\mu}\right)\Box R =$$
$$= \int d^2x\sqrt{-g}\, 4\nabla^\mu\tau^{\lambda}{}_{\mu\lambda}\,\Box R = -4\int d^2x\sqrt{-g}\,\tau^{\lambda}{}_{\mu\lambda}\nabla^\mu\Box R =$$
$$= -4\int d^2x\sqrt{-g}\,\tau^{\lambda}{}_{\mu\lambda}\left(\Box\nabla^\mu R - \frac{1}{4}\nabla^\mu R^2\right) = -\int d^2x\sqrt{-g}\left(4I_2 - I_3\right)$$

$$\delta_\tau \int d^2x\sqrt{-g}\, J_3 = \int d^2x\sqrt{-g}\,\delta_\tau B^{\mu\lambda}{}_{\mu\lambda} R^2 =$$
$$= \int d^2x\sqrt{-g}\, 4\left(\nabla^\mu\tau^{\lambda}{}_{\mu\lambda}\right) R^2 = -4\int d^2x\sqrt{-g}\,\tau^{\lambda}{}_{\mu\lambda}\nabla^\mu R^2 =$$
$$= -4\int d^2x\sqrt{-g}\, I_3$$

These results can be summarized in a following way,

$$\delta_\tau C = \int d^2x\sqrt{-g}\sum_{i,j=1}^{3} c_i M_{ij} I_j^\tau \tag{4.31}$$

where M_{ij} is the matrix

$$M_{ij} = -\begin{pmatrix} 2 & 2 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 4 \end{pmatrix} \tag{4.32}$$

Since the determinant of this matrix is nonvanishing we can always find c_i such that (4.31) reproduce I^τ terms from (4.27) for any choice of parameters a_i.

On the other hand, we have

$$\delta_\tau D = -\int d^2x\sqrt{-g}\,\Big(2\tau^{\mu\lambda\rho}\epsilon_{\rho\alpha}\nabla^\alpha\nabla_\lambda\nabla_\mu R +$$

$$+\,\tau^{\mu\sigma}{}_{\sigma}\,\epsilon_{\mu\alpha}\Box\nabla^\alpha R + 2\tau^{\mu\sigma}{}_{\sigma}\,\epsilon_{\mu\alpha}R\,\nabla^\alpha R\Big) \tag{4.33}$$

At this point we explain the demand for the tracelessness of parameter τ. This property is necessary because otherwise, the K_2^τ and K_3^τ terms (proportional to $\tau^{\mu\nu}{}_{\nu}$) would appear in conservation laws in which also the component $J^{(4)}_{uuvv}$ are 'excited'. This would bring us outside our system. Because of this, only K_1^τ survives among the cocycles, and only the first term survives in the RHS of (4.33). The letter precisely cancels the only possible nontrivial cocycle.

To conclude, all the cocycles $\Delta_\tau^{(4)}$ are trivial and there are no non-trivial consistent anomalies in the divergence of the spin-4 current!

Finally, we shall show how to generalize the above argument to higher spin currents, provide we assume that all the chains can be written in a covariant form.

Theorem 3. Let $\omega^{\mu_1\cdots\mu_{2n-1}}$ be a totally symmetric, traceless anticommuting parameter (the generalization of $\tau_{\mu\nu\lambda}$) of higher tensorial transformation. Assuming that

$$\delta_\omega g_{\mu\nu} = 0 \tag{4.34}$$

and that any cocycle of δ_ω can be written in a diff-covariant form, all solutions of the equation

$$\delta_\omega\Delta_\omega^{(2\mathrm{n})} = 0 \tag{4.35}$$

are trivial, i.e. there exist a local functional $C^{(2\mathrm{n})}$ of the background fields, such that

$$\Delta_\omega^{(2\mathrm{n})} = \delta_\omega C^{(2\mathrm{n})} \tag{4.36}$$

Proof: Taking into account the assumptions of the theorem, the most general cocycles can be written in a following form,

$$\Delta_\omega^{(2\mathrm{n})} = \int d^2x\sqrt{-g}\,(aI^\omega + bK^\omega) \tag{4.37}$$

where a and b are constants and I^ω and K^ω are the *only* possible terms we can construct in 2-dimensional theory. Their explicit form is

$$I^\omega = \omega^{\mu_1\cdots\mu_{2n-1}}\nabla_{\mu_1}\cdots\nabla_{\mu_{2n-1}}R \tag{4.38}$$

$$K^\omega = \omega^{\mu_1\cdots\mu_{2n-1}}\,\epsilon_{\mu_1\alpha}\,\nabla^\alpha\nabla_{\mu_2}\ldots\nabla_{\mu_{2n-1}}R \tag{4.39}$$

The terms of the form

$$\omega^{\mu_1\cdots\mu_{2n-1}}\,\epsilon_{\mu_1\alpha_1}\,\epsilon_{\mu_2\alpha_2}\ldots\epsilon_{\mu_k\alpha_k}\,\nabla^{\alpha_1}\nabla^{\alpha_2}\ldots\nabla^{\alpha_k}\nabla_{\mu_{k+1}}\ldots\nabla_{\mu_{2n-1}}R \tag{4.40}$$

are in fact *equivalent* to either I^ω or K^ω. Using the formula valid in 2-dimensional spacetimes,

$$\epsilon_{\alpha\beta}\epsilon_{\mu\nu} = g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu} \tag{4.41}$$

we can eliminate ϵ-tensors in (4.40) two by two. In every step we produce two terms, out of which first is zero because of the tracelessness of ω and the second contracts additional two indices of ω with the indices of the covariant derivatives on the right side. Form of the final result will depend on parity of k; in the case of even k we get $(-1)^{k/2}\,I^\omega$ and in the case of odd k, $(-1)^{(k-1)/2}\,K^\omega$. Therefore, all such terms are already included in the general form of Δ_ω, written above.

Now we claim that the corresponding counterterm is the following:

$$C^{(2n)} = -\frac{1}{2}\int d^2x\sqrt{-g}\,(aJ^\omega + bL^\omega) \tag{4.42}$$

where

$$J^\omega = B^{\mu_1\cdots\mu_{2n-2}\sigma}{}_\sigma\,\nabla_{\mu_1}\ldots\nabla_{\mu_{2n-2}}R$$

$$L^\omega = B^{\mu_1\cdots\mu_{2n-2}\sigma}{}_\sigma\,\epsilon_{\mu_1\alpha}\nabla^\alpha\nabla_{\mu_2}\ldots\nabla_{\mu_{2n-2}}R$$

and B is the corresponding background field. Using the formulae

$$\delta_\omega g_{\mu\nu} = 0 \quad , \quad \delta_\omega B^{\mu_1\cdots\mu_{2n}} = \nabla^{\mu_1}\omega^{\mu_2\cdots\mu_{2n}} + \text{cycl.}$$

and, again, the fact that ω is traceless we get, after integration by parts

$$\delta_\omega C^{(2n)} = \int d^2x\sqrt{-g}\,\big(a\,\omega^{\mu_1\cdots\mu_{2n-2}\sigma}\nabla_\sigma\nabla_{\mu_1}\ldots\nabla_{\mu_{2n-2}}R +$$

$$+b\,\omega^{\mu_1\cdots\mu_{2n-2}\sigma}\epsilon_{\mu_1\alpha}\nabla_\sigma\nabla^\alpha\nabla_{\mu_2}\ldots\nabla_{\mu_{2n-2}}R\big)$$

Second term under the integral has to be rearranged by reversing the order of the first two covariant derivatives ($\nabla_\sigma\nabla^\alpha$). Using the general form of 2-dimensional Riemann tensor,

$$R_{\alpha\beta\mu\nu} = \frac{R}{2}\,(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}) \tag{4.43}$$

we have

$$\omega^{\mu_1\cdots\mu_{2n-2}\sigma}\epsilon_{\mu_1\alpha}\nabla_\sigma\nabla^\alpha\nabla_{\mu_2}\ldots\nabla_{\mu_{2n-2}}R =$$

$$= \omega^{\mu_1\cdots\mu_{2n-2}\sigma}\epsilon_{\mu_1\alpha}\nabla^\alpha\nabla_\sigma\nabla_{\mu_2}\ldots\nabla_{\mu_{2n-2}}R +$$

$$+\,\omega^{\mu_1\cdots\mu_{2n-2}\sigma}\,\epsilon_{\mu_1\alpha}\,R_\sigma{}^\alpha{}_{\mu_2}{}^\lambda\nabla_\lambda\nabla_{\mu_3}\ldots\nabla_{\mu_{2n-2}}R + \ldots$$

So, a typical additional term has a form

$$\frac{R}{2}\,\omega^{\mu_1\cdots\mu_{2n-2}\sigma}\,\epsilon_{\mu_1\alpha}\,(g_{\sigma\mu_i}\,g^{\alpha\lambda} - g_\sigma^\lambda\,g_{\mu_i}^\alpha)\nabla_{\mu_2}\ldots\nabla_{\mu_{i-1}}\nabla_\lambda\nabla_{\mu_{i+1}}\ldots\nabla_{\mu_{2n-2}}R$$

First part of this term vanishes because of the tracelessness of ω and the second because it leads to contraction of antisymmetric ϵ tensor and symmetric indices in ω. Therefore, we have proven that

$$\delta_\omega C^{(2n)} = \int d^2x\sqrt{-g}\,(aI^\omega + bK^\omega)$$

This means that there are no non-trivial anomalies in any higher spin currents. Therefore, a properly chosen regularization should not produce any covariant anomaly either.

* * *

Now we want to reexamine the cohomology problem when the gauge field is present in the theory. This gives rise to well-known gauge anomaly in the covariant derivative of the $J^{(1)}$ current. Apart from this, we shall show that the conclusion of [BC08] on the absence of trace anomalies in the higher-spin currents still holds.

In addition to the series of $B^{(s)}$ fields, there must be other background fields with the same characteristics (that is, maximally symmetric and asymptotically trivial). Their function is to explain the presence of the additional covariant terms in the conservation equations of the higher currents (the terms on the RHS of (3.77)-(3.79)). Let us call these additional fields $C^{(s)}$, $D^{(s)}$, ... As an example let us consider the conservation of $J^{(3)}$

$$\nabla^\mu J^{(3)}_{\mu\nu\lambda} = 2F^\rho{}_\nu\, J^{(2)}_{\rho\lambda} - \frac{1}{6}\,(\nabla_\nu R)J^{(1)}_\lambda \tag{4.44}$$

where symmetrization over the indices ν and λ is understood on the RHS. The LHS is due to assumed invariance of the effective action under

$$\delta_\xi B^{(3)}_{\mu_1\mu_2\mu_3} = \nabla_{\mu_1}\xi_{\mu_2\mu_3} + \text{cycl.} \tag{4.45}$$

where ξ is a symmetric traceless tensor. In order to explain the presence of the RHS terms, we assume that there exist, in the effective action, other background potentials $C^{(s)}$, coupled to the two terms in the RHS of (4.44), which transform like

$$\delta_\xi C^{(3)} = \xi_{\mu\nu} \tag{4.46}$$

while all the other fields in the game are invariant under ξ transformations. These fields must have transformation properties that guarantee the invariance of the terms they are involved in.

In an analogous way we can deal with the other conservation laws. We remark that the transformations of the $C^{(s)}$ potentials are intrinsically Abelian. Unfortunately we do not know how to derive the transformation (4.46) from the first principles, but we can use consistency to conclude that these two equations represent the only possibility. In principle, one could envisage possible non-Abelian transformations in order to account for the $J^{(3)}$ conservation law. These, however, must form a Lie algebra and such a condition strongly restricts the form of the transformations and, consequently, that of the effective action. For example, the presence of the terms in the RHS of (4.44) could be formally explained by different transformation laws of other fields; in particular, for the spin-3 current, one could have transformation of the type

$$\delta_\tau g_{\mu\nu} \sim \tau^\lambda{}_\mu F_{\lambda\nu} \quad , \quad \delta_\tau A_\mu \sim \tau^\lambda{}_\mu \nabla_\lambda R$$

But these are not good symmetry transformation, for they do not form an algebra. This can be seen by promoting ξ and τ to anticommuting parameters and verifying that such transformations are not nilpotent.

One can indeed verify that the higher potentials transformation laws are so strongly restricted that it is generically impossible to avoid the conclusion that they must be Abelian (see also [BCPS08]). Under these circumstances (4.46) represents the generic case for higher-spin quantities. The presence of these additional background fields, which were not considered in [BC08] and [BCPS08], may complicate the anomaly analysis. However, to simplify it, one can remark that these potentials can increase the number of cocycles only if they explicitly appear in the cocycles themselves. Since eventually these potentials are set to zero, the corresponding cocycles vanish. As a consequence they cannot give rise to the anomalies we are interested in, and their study is just of academic interest. For this reason, for the sake of simplicity, we choose to dispense of it. Thus henceforth we will ignore the additional potentials.

Taking into account these preliminary remarks, we proceed with the analysis of spin-3 current. Setting $B^{(3)}_{\mu\nu\lambda} \equiv B_{\mu\nu\lambda}$ the Weyl transformation of the various fields involved is (see [BC08])

$$\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu} \quad , \quad \delta_\sigma B_{\mu\nu\lambda} = x\sigma B_{\mu\nu\lambda} \quad , \quad \delta_\sigma A_\mu = 0 \tag{4.47}$$

which induces the trace of the energy-momentum tensor, and

$$\delta_\tau g_{\mu\nu} = 0 \quad , \quad \delta_\tau B_{\mu\nu\lambda} = \tau_\mu g_{\nu\lambda} + \text{cycl.} \quad , \quad \delta_\tau A_\mu = 0 \tag{4.48}$$

which induces the trace of $J^{(3)}$. Moreover, for consistency with (4.47) we must have

$$\delta_\sigma \tau_\mu = (x-2)\sigma\tau_\mu \tag{4.49}$$

where x is an arbitrary number. The transformations (4.47)-(4.48) are determined using symmetry parameters and basic background fields ($g_{\mu\nu}$ and A_μ), so that they form a Lie algebra and leave unchanged the terms in the effective action (in partuicular, the terms involving the matter fields). Final restriction is provided by the canonical dimension of the various fields ($2-s$ for $B^{(s)}$ and $1-s$ for $C^{(s)}$). Using these we can repeat the analysis done in [BC08]. We promote the σ and τ_μ to anticommuting fields, so that they form a graded algebra,

$$\delta_\sigma^2 = 0 \quad , \quad \delta_\tau^2 = 0 \quad , \quad \delta_\sigma\delta_\tau + \delta_\tau\delta_\sigma = 0 \tag{4.50}$$

Integrated anomalies are defined by

$$\delta_\sigma \Gamma^{(1)} = \hbar\Delta_\sigma \quad , \quad \delta_\tau \Gamma^{(1)} = \hbar\Delta_\tau \tag{4.51}$$

where $\Gamma^{(1)}$ is the one-loop quantum action and Δ_σ, Δ_τ local functionals linear in σ and τ, respectively. The unintegrated anomalies, i.e. the traces $T^\mu{}_\mu$ and $J^{(3)\mu}{}_{\mu\lambda}$ are obtained by functionally differentiating with respect to σ and τ_λ, respectively. By applying δ_σ and δ_τ to (4.51) we see that candidates for anomalies Δ_σ and Δ_τ must satisfy the consistency conditions

$$\delta_\sigma \Delta_\sigma = 0 \quad , \quad \delta_\tau \Delta_\sigma + \delta_\sigma \Delta_\tau = 0 \quad , \quad \delta_\tau \Delta_\tau = 0 \tag{4.52}$$

Once we have determined these cocycles ,we have to make sure that they are true anomalies, that is, that they are nontrivial. In other words, there must ot exist local counterterm C in the action such that

$$\Delta_\sigma = \delta_\sigma \int d^2x \sqrt{-g}\, C \quad , \quad \Delta_\tau = \delta_\tau \int d^2x \sqrt{-g}\, C \tag{4.53}$$

If such a C existed we could redefine the quantum action by substracting these counterterms and get rid of the (trivial) anomalies.

First we consider the problem of the trace $J^{(3)\mu}{}_{\mu\lambda}$. Suppose we find cocycle $\Delta_\tau^{(3)}$,

$$\Delta_\tau^{(3)} = \int d^2x \sqrt{-g}\, \tau^\mu I_\mu^{(3)} \tag{4.54}$$

where $I^{(3)}$ is a canonical dimension 3 tensor made of the metric, the gauge field and their derivatives, such as $\nabla_\mu R$, $\nabla_\nu F_\mu{}^\nu$ or even a non-gauge-invariant tensor, such as $A_\mu R$. Then it is straightforward to write down a counterterm

$$C^{(3)} \sim B^\mu{}_\lambda{}^\lambda I_\mu^{(3)} \tag{4.55}$$

which cancels (4.54).

As for the trace $J^{(4)\mu}{}_{\mu\lambda\rho}$, we can proceed analogously. Setting $B^{(4)}_{\mu\nu\lambda\rho} \equiv B_{\mu\nu\lambda}$ the Weyl transformation of the various fields is

$$\delta_\tau g_{\mu\nu} = 0 \quad , \quad \delta_\tau B_{\mu\nu\lambda\rho} = \tau_{\mu\nu} g_{\lambda\rho} + \text{cycl.} \quad , \quad \delta_\tau A_\mu = 0 \tag{4.56}$$

while the variations with respect to the ordinary Weyl parameter σ are

$$\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu} \quad , \quad \delta_\sigma \tau_{\mu\nu} = (x-2)\sigma \tau_{\mu\nu} \quad , \quad \delta_\sigma B_{\mu\nu\lambda\rho} = x\sigma B_{\mu\nu\lambda\rho} \tag{4.57}$$

where, again, x is an arbitrary number. Now we can repeat the previous argument. Let a cocycle have the form

$$\Delta_\tau^{(4)} = \int d^2x \sqrt{-g}\, \tau^{\mu\nu} I_{\mu\nu}^{(4)} \tag{4.58}$$

where $I^{(4)}$ is a dimension 4 tensor made out of the metric, th gauge field and their derivatives. The counterterm of the type

$$C^{(4)} \sim B^{\mu\nu}{}_\lambda{}^\lambda I_{\mu\nu}^{(4)} \tag{4.59}$$

cancels (4.58).

This procedure can be generalized to higher-spin currents in a straightforward way. We believe that all these results, together with those of [BC08, BCPS08] suffice as evidence that anomalies may not rise in the higher-spin currents under any condition.

Chapter 5

Discussion and conclusion

In this thesis we have shown how to completely reconstruct the bosonic and fermionic Hawking radiation spectrum using higher spin currents. The method is an extension of the trace anomaly method, proposed by Christensen and Fulling [CF77]. We start by examining the dimension reduction of the action for the bosonic and fermionic massless fields in the near-horizon region of the Kerr black hole. The effective action corresponds to 2-dimensional bosonic or fermionic field coupled to gauge field and dilaton (which can be neglected in the near-horizon region) propagating in the background of (1+1)-dimensional metric of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} \tag{5.1}$$

This procedure allows us to simplify the problem by ignoring the effects of backscattering and focus on the relevant effective physics near the horizon. In order to derive the higher moments of the Hawking radiation spectrum we follow the proposition of Iso, Morita and Umetsu [IMU07b], who postulated the existence of conserved higher-spin currents. These are built out of bosonic or fermionic bilinears in the 2-dimensional effective field theory. The higher-spin currents play a role analogous to the energy-momentum for the integrated radiation (the lowest moment). In order to construct these currents one needs a guiding priciple and we argue that it is crucial to use some of the W algebras.

This means that we start from the W_∞ algebra (in the bosonic case) or $W_{1+\infty}$ algebra (in the fermionic case) defined in the abstract flat complex plane. The holomorphic currents $j^{(s)}_{z\dots z}(z)$ satisfying these algebras were constructed in [BK90] and [BPR+90]. Examining the transformation properties of these currents (corresponding to transition to Kruskal coordinates) we can obtain their value at the horizon by requiring the regularity. The result in the bosonic and fermionic case are given, respectfuly, by

$$\langle j^{(s)}_{z\dots z}\rangle_{\mathsf{H}} = \langle X^{\mathrm{B}}_s\rangle = \hbar\,(-1)^{s+1}(4q)^{s-2}\frac{B_s}{s}\,\kappa^s \tag{5.2}$$

$$\langle j^{(s)}_{z\dots z}\rangle_{\mathsf{H}} = \langle X^{\mathrm{F}}_s\rangle = -\hbar\,\frac{\kappa^s B_s}{s}\,(1-2^{1-s})\,(4q)^{s-2} \tag{5.3}$$

If we identify $j^{(s)}_{z\ldots z}(z)$ via a Wick rotation with $j^{(s)}_{u\ldots u}(u)$, we get the corresponding value at the horizon, $\langle j^{(s)}_{u\ldots u}\rangle_{\mathsf{H}}$. Since the problem we are considering is stationary and $j^{(s)}_{u\ldots u}(u)$ is chiral, it follows that it is constant in t and r. Therefore, we have

$$\langle j^{(s)}_{u\ldots u}\rangle_{\mathsf{H}} = \langle j^{(s)}_{u\ldots u}\rangle_{\infty} \tag{5.4}$$

Next, using point-splitting regularization scheme it is possible to construct an infinite set of covariant currents $J^{(s)}_{u\ldots u}$ out of the holomorphic ones. Physically, covariant currents represent higher moments of the spectrum of outgoing radiation at the asymptotic infinity (of the near-horizon region). Since the covariant currents coincide with the holomorphic currents at the infinity, it is possible to relate $\langle X^{\mathrm{B}}_s\rangle$ and $\langle X^{\mathrm{F}}_s\rangle$ to higher moments of the Hawking radiation spectrum; assuming that there is no incoming fluxes from infinity we get

$$\left\langle J^{(s)r}{}_{t\ldots t}\right\rangle_{\infty} = \langle J^{(s)}_{u\ldots u}\rangle_{\infty} - \langle J^{(s)}_{v\ldots v}\rangle_{\infty} = \langle j^{(s)}_{u\ldots u}\rangle_{\infty} = \langle j^{(s)}_{u\ldots u}\rangle_{\mathsf{H}} \tag{5.5}$$

In order to evaluate higher moments we shall set $q = i/4$ and divide the currents by -2π in order to properly normalize the (physical) energy-momentum tensor. The moments for the bosonic case are

$$-\frac{1}{2\pi}\left\langle J^{(s)r}{}_{t\ldots t}\right\rangle_{\infty} = -\frac{1}{2\pi}\langle X^{\mathrm{B}}_s\rangle = i^{s-2}\,\frac{\kappa^s B_s}{2\pi s}\,\hbar \tag{5.6}$$

The RHS vanishes for odd s (except for $s = 1$ which is not excited in our case) and coincides with the higher moments of the bosonic thermal spectrum (3.4). In the fermionic case without the electromagnetic field ($m = 0$), the moments are given by

$$-\frac{1}{2\pi}\left\langle J^{(s)r}{}_{t\ldots t}\right\rangle_{\infty} = -\frac{1}{2\pi}\langle X^{\mathrm{F}}_s\rangle = i^{s-2}\frac{\kappa^s B_s}{2\pi s}\left(1-2^{1-s}\right)\hbar \tag{5.7}$$

These are sufficient for the reconstruction of the Hawking radiation spectrum through a procedure described in the section 3.5. The final result agrees with those obtained by the original Hawking's approach: Bose-Einstein distribution in the bosonic and the Fermi-Dirac distribution in the fermionic case. The expressions for the higher moments of the fermionic Hawking radiation spectrum in the presence of the gauge field are more complicated; we have shown that these also agree with the higher moments obtained by the Schwarzian derivative method.

Next, by examining the divergences of the covariant currents and through cohomological analysis, we have shown the absence of the diffeomorphism anomalies in the laws of conservation of the higher spin currents. The explicit example of such, manifestly anomalous free currents, are those obeying W_∞ algebra (in the bosonic case) and $W_{1+\infty}$ algebra (in the fermionic case). This result is also plausible from the perspective of classical no hair theorem: non-trivial higher spin anomalies would give rise to a new kind of hairs corresponding different higher-spin central charges!

The conclusion is, therefore, that higher moments are related to a symmetry generated by some of W algebras, and *not* to anomalies, as oppose to a recent claims. Motivated by this discovery, we propose [BCPS08, BCPS09] a conjecture that the effective theory in the near horizon region may be underlined by some hidden symmetry related to W algebras. The source and the precise formulation of this symmetry remains obscure at this moment and shall be a subject of some future investigations.

Appendix A

Elements of stationary spacetimes

We are frequently using a 2-dimensional metric of the form

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} \tag{A.1}$$

The nonvanishing components of the Christoffel symbols are

$$\Gamma^t_{tr} = \frac{f'}{2f} \quad , \quad \Gamma^r_{tt} = \frac{ff'}{2} \quad , \quad \Gamma^r_{rr} = -\frac{f'}{2f} \tag{A.2}$$

It is useful to notice the the following equality holds

$$\Gamma^\mu_{\mu t} = \Gamma^\mu_{\mu r} = 0$$

The nonvanishing components of the Riemann and Ricci tensor are

$$R^t{}_{rtr} = -\frac{f''}{2f} = R_{rr} \quad , \quad R^r{}_{trt} = \frac{ff''}{2} = R_{tt} \tag{A.3}$$

The Ricci scalar is equal to

$$R = -f''(r) \tag{A.4}$$

Note that the change in convention of $\eta_{\mu\nu}$ is obtained by changing $f \to -f$ in the above equations.

Transformation to light-like (u, v) coordinates is defined by introduction of the *tortoise* coordinate r_*,

$$\frac{dr_*}{dr} = \frac{1}{f(r)} \quad , \quad u = t - r_* \quad , \quad v = t + r_* \tag{A.5}$$

A useful set of relations is listed below,

$$\frac{du}{dt} = \frac{dv}{dt} = 1 \quad , \quad \frac{du}{dr} = -\frac{1}{f(r)} \quad , \quad \frac{dv}{dr} = \frac{1}{f(r)}$$

$$\partial_t = \partial_v + \partial_u \quad , \quad \partial_r = e^{-\varphi}(\partial_v - \partial_u)$$

$$\partial_u = \frac{1}{2}\left(\partial_t - f(r)\partial_r\right) \quad , \quad \partial_v = \frac{1}{2}\left(\partial_t + f(r)\partial_r\right)$$

$$\partial_t^2 = \partial_v^2 + 2\partial_{vu} + \partial_u^2$$

In these coordinates the metric has a form

$$ds^2 = -f(r)dudv = -e^{\varphi(u,v)}dudv \tag{A.6}$$

$$g_{uv} = -\frac{1}{2}e^{\varphi(u,v)} = -\epsilon_{uv} \quad , \quad g^{uv} = -2e^{-\varphi(u,v)} = +\epsilon^{uv}$$

The nonvanishing components of the Christoffel symbols, Riemann and Ricci tensor are now

$$\Gamma^u_{uu} = \varphi_{,u} \quad , \quad \Gamma^v_{vv} = \varphi_{,v} \tag{A.7}$$

$$R^u{}_{uvu} = R^v{}_{vuv} = \varphi_{,uv} \quad , \quad R_{uv} = R_{vu} = -R^u{}_{uvu} = -\varphi_{,uv} \tag{A.8}$$

The Ricci scalar is

$$R = 4e^{-\varphi}\varphi_{,uv} = -2g^{uv}v \tag{A.9}$$

Introducing abbrevations

$$\Gamma_u \;\equiv\; \Gamma^u_{uu} = \varphi_{,u} \quad , \quad \Gamma_v \;\equiv\; \Gamma^v_{vv} = \varphi_{,v}$$

$$\mathcal{T}_{uu} \;\equiv\; \partial_u^2\varphi - \frac{1}{2}\left(\partial_u\varphi\right)^2 \quad , \quad \mathcal{T}_{vv} \;\equiv\; \partial_v^2\varphi - \frac{1}{2}\left(\partial_v\varphi\right)^2$$

we can write

$$\nabla_u R = 4e^{-\varphi}\partial_v\mathcal{T}_{uu} = -2g^{uv}\partial_v\mathcal{T}_{uu}$$

$$\nabla_u^2 R = 2g^{uv}(2\Gamma_u\partial_v\mathcal{T}_{uu} - \partial_u\partial_v\mathcal{T}_{uu})$$

$$\nabla_u^3 R = 2g^{uv}\left((2\mathcal{T}_{uu} - 5\Gamma_u^2)\partial_v\mathcal{T}_{uu} + 5\Gamma_u\partial_u\partial_v\mathcal{T}_{uu} - \partial_u^2\partial_v\mathcal{T}_{uu}\right)$$

The same set of equation remains valid upon interchange $u \leftrightarrow v$.

The surface gravity is given by

$$\kappa = \frac{f'(r_{\rm H})}{2} \tag{A.10}$$

Zweinbeins (vielbeins) for this (1+1)-dimensional metric are defined by

$$\overset{\flat}{e}{}^a{}_\mu\, \overset{\flat}{e}{}^b{}_\nu\, \eta_{ab} = g_{\mu\nu} \quad , \quad \overset{\flat}{e}{}_a{}^\mu = \eta_{ab}\, g^{\mu\nu}\, \overset{\flat}{e}{}^b{}_\nu \tag{A.11}$$

where $\mu = r, t$, $a = 0, 1$.

$$\overset{\flat}{e}{}^0{}_t = \sqrt{f} \quad , \quad \overset{\flat}{e}{}^1{}_r = \frac{1}{\sqrt{f}}$$

$$\overset{\flat}{e}{}_0{}^t = \frac{1}{\sqrt{f}} \quad , \quad \overset{\flat}{e}{}_1{}^r = \sqrt{f}$$

The components of Christoffel symbols with all flat indices is obtained via

$$\Gamma^a_{bc} = \overset{\flat}{e}{}^a{}_\mu\, \overset{\flat}{e}{}_b{}^\sigma\, \overset{\flat}{e}{}_c{}^\tau\, \Gamma^\mu_{\sigma\tau} \tag{A.12}$$

$$\Gamma^0_{01} = \Gamma^1_{00} = -\Gamma^1_{11} = \frac{f'}{2\sqrt{f}}$$

The components of spin connection with all flat indices can be calculated from

$$\overset{\flat}{\omega}{}^a{}_{bc} = \overset{\flat}{e}{}^a{}_\nu \, \overset{\flat}{e}{}_c{}^\mu \partial_\mu \overset{\flat}{e}{}_b{}^\nu + \Gamma^a_{bc} \tag{A.13}$$

Because of the symmetry, $\overset{\flat}{\omega}{}^a{}_{ab} = 0$ (no summation over index a) for every b. Also,

$$\overset{\flat}{\omega}{}^0{}_{10} = \overset{\flat}{\omega}{}^1{}_{00} = \frac{f'}{2\sqrt{f}}$$

Kerr-Newman black hole

The *Kerr-Newman metric* with mass M, angular momentum J and electric charge Q, written in *Boyer-Lindquist coordinates* $x^\mu = (t, r, \theta, \varphi)$

$$ds^2 = -\left(1 - \frac{2Mr - Q^2}{\Sigma}\right) dt^2 - \frac{(2Mr - Q^2)2a\sin^2\theta}{\Sigma}\, dt d\varphi +$$

$$+\frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2 + \frac{\Xi \sin^2\theta}{\Sigma}\, d\varphi^2 \tag{A.14}$$

where we have used notations

$$a = \frac{J}{M} \quad , \quad \Delta = r^2 - 2Mr + a^2 + Q^2 \quad , \quad \Sigma = r^2 + a^2\cos^2\theta$$

$$\Xi = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta$$

The non-vanishing contravariant components $g^{\mu\nu}$ of the metric are

$$g^{tt} = -\frac{\Xi}{\Delta\Sigma} \quad , \quad g^{t\varphi} = -\frac{a(2Mr - Q^2)}{\Delta\Sigma} \quad , \quad g^{\varphi\varphi} = \frac{\Delta - a^2\sin^2\theta}{\Delta\Sigma\sin^2\theta}$$

$$g^{rr} = \frac{\Delta}{\Sigma} \quad , \quad g^{\theta\theta} = \frac{1}{\Sigma}$$

Covariant volume element is given by

$$\sqrt{-g}\, dV = \Sigma \sin\theta \, dt dr d\theta d\phi \tag{A.15}$$

The outer event horizon (defined by the equation $\Delta = 0$) is located at

$$r_+ = M + \sqrt{M^2 - a^2 - Q^2}$$

Area of the outer event horizon is given by

$$A = 4\pi(r_+^2 + a^2) \tag{A.16}$$

The surface gravity κ, the angular velocity Ω_{H} and the electric potential Φ_{H} for the Kerr-Newman black hole are

$$\kappa = \frac{r_+ - M}{r_+^2 + a^2} \quad , \quad \Omega_{\mathsf{H}} = \frac{a}{r_+^2 + a^2} \quad , \quad \Phi_{\mathsf{H}} = \frac{Q r_+}{r_+^2 + a^2} \tag{A.17}$$

Special cases of this metric are *Kerr* ($Q = 0$), *Reissner-Nordstrøm* ($a = 0$) and *Schwarzschild* ($Q = 0 = a$) black holes.

Spin connections for the Kerr metric (and our choice of vierbeins),

$$\omega^a{}_{bc} = e_c{}^\mu \, \omega^a{}_{b\mu}$$

$$\omega^0{}_{10} = \frac{(r-M)\Sigma - r\Delta}{\sqrt{\Delta\Sigma^3}} \quad , \quad \omega^0{}_{20} = -\frac{a^2\cos\theta\sin\theta}{\Sigma^{3/2}}$$

$$\omega^0{}_{31} = \omega^0{}_{13} = -\frac{ar\sin\theta}{\Sigma^{3/2}} \quad , \quad \omega^0{}_{32} = -\omega^0{}_{23} = \frac{a\cos\theta\sqrt{\Delta}}{\Sigma^{3/2}}$$

$$\omega^1{}_{21} = -\frac{a^2\cos\theta\sin\theta}{\Sigma^{3/2}} \quad , \quad \omega^1{}_{30} = -\frac{ar\sin\theta}{\Sigma^{3/2}} \quad , \quad \omega^1{}_{22} = \omega^1{}_{33} = -\frac{r\sqrt{\Delta}}{\Sigma^{3/2}}$$

$$\omega^2{}_{30} = -\frac{a\cos\theta\sqrt{\Delta}}{\Sigma^{3/2}} \quad , \quad \omega^2{}_{33} = -\frac{(2Mr+\Delta)\cos\theta}{\Sigma^{3/2}\sin\theta}$$

Appendix B

Bogoliubov transformations

The Klein-Gordon scalar product in curved spacetime is defined as

$$(\phi_1, \phi_2) \;\equiv\; -i \int_\Sigma d\Sigma^\mu \left(\phi_1 \nabla_\mu \phi_2^* - \phi_2^* \nabla_\mu \phi_1\right) \tag{B.1}$$

where Σ is an "initial data" Cauchy hypersurface and $d\Sigma^\mu = d\Sigma\, n^\mu$, with $d\Sigma$ being the volume element and n^μ a future directed unit normal vector to Σ. The scalar product has the folowing property,

$$(\alpha\phi_1, \phi_2) = \alpha(\phi_1, \phi_2) \quad , \quad (\phi_1, \alpha\phi_2) = \alpha^*(\phi_1, \phi_2) \tag{B.2}$$

for all $\alpha \in \mathbb{C}$. We choose the orthonormal base $\{\tilde{u}_i\}$ among the solutions of Klein-Gordon equation,

$$\Box\tilde{u}_i - m^2\tilde{u}_i = 0 \tag{B.3}$$

such that

$$(\tilde{u}_i, \tilde{u}_j) = \delta_{ij} \quad , \quad (\tilde{u}_i^*, \tilde{u}_j^*) = -\delta_{ij} \quad , \quad (\tilde{u}_i, \tilde{u}_j^*) = 0 \tag{B.4}$$

Now, suppose we take another basis $\{\tilde{v}_i\}$, related to the first one via

$$\tilde{v}_i = \sum_j \left(\alpha_{ij}\tilde{u}_j + \beta_{ij}\tilde{u}_j^*\right) \tag{B.5}$$

or, inversely

$$\tilde{u}_i = \sum_j \left(\alpha'_{ij}\tilde{v}_j + \beta'_{ij}\tilde{v}_j^*\right) \tag{B.6}$$

It is conventional to treat the coefficients α_{ij} and β_{ij} as elements of, respectfully, matrices $\mathbf{A}$ and $\mathbf{B}$. Also, we can put elements of the basis $\{\tilde{u}_i\}$ and $\{\tilde{v}_i\}$ in column matrices $\mathbf{u}$ and $\mathbf{v}$, in order to write the change of basis as

$$\mathbf{v} = \mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{u}^* \quad , \quad \mathbf{u} = \mathbf{A}'\,\mathbf{v} + \mathbf{B}'\mathbf{v}^* \tag{B.7}$$

Orthonormalization of the new basis imposes the following relation,

$$(\tilde{v}_i, \tilde{v}_j) = \Big(\sum_k (\alpha_{ik}\tilde{u}_k + \beta_{ik}\tilde{u}_k^*)\,, \sum_l (\alpha_{jl}\tilde{u}_l + \beta_{jl}\tilde{u}_l^*)\Big) =$$

$$= \sum_{k,l} (\alpha_{ik}\alpha^*_{jl} - \beta_{ik}\beta^*_{jl})\delta_{kl} = \sum_k (\alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk}) = (\mathbf{A}\mathbf{A}^\dagger - \mathbf{B}\mathbf{B}^\dagger)_{ij} \stackrel{!}{=} \delta_{ij}$$

$$(\tilde{v}_i, \tilde{v}^*_j) = \Big(\sum_k (\alpha_{ik}\tilde{u}_k + \beta_{ik}\tilde{u}^*_k) \,, \sum_l (\alpha^*_{jl}\tilde{u}^*_l + \beta^*_{jl}\tilde{u}_l) \Big) =$$

$$= \sum_{k,l} (\alpha_{ik}\beta_{jl} - \beta_{ik}\alpha_{jl})\delta_{kl} = \sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = (\mathbf{A}\mathbf{B}^\mathrm{T} - \mathbf{B}\mathbf{A}^\mathrm{T})_{ij} \stackrel{!}{=} 0$$

In a summary, we have

$$\mathbf{A}\mathbf{A}^\dagger - \mathbf{B}\mathbf{B}^\dagger = \mathbf{1} \tag{B.8}$$

$$\mathbf{A}\mathbf{B}^\mathrm{T} - \mathbf{B}\mathbf{A}^\mathrm{T} = \mathbf{0} \tag{B.9}$$

From this properties also, as a consequence, follows $(\tilde{v}^*_i, \tilde{v}^*_j) = -\delta_{ij}$.

Furthermore,

$$\mathbf{u} = \mathbf{A}' \,(\mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{u}^*) + \mathbf{B}'(\mathbf{A}^*\mathbf{u}^* + \mathbf{B}^*\mathbf{u}) =$$
$$= (\mathbf{A}'\mathbf{A} + \mathbf{B}'\mathbf{B}^*)\mathbf{u} + (\mathbf{A}'\,\mathbf{B} + \mathbf{B}'\mathbf{A}^*)\mathbf{u}^*$$

Therefore

$$\mathbf{A}'\mathbf{A} + \mathbf{B}'\mathbf{B}^* = \mathbf{1} \tag{B.10}$$

$$\mathbf{A}'\,\mathbf{B} + \mathbf{B}'\mathbf{A}^* = \mathbf{0} \tag{B.11}$$

Acting on equation (B.10) with $\mathbf{B}^\mathrm{T}$ from the right, on equation (B.11) with $\mathbf{A}^\mathrm{T}$ from the right, substracting them and using (B.9) we get

$$\mathbf{B}'\mathbf{B}^*\mathbf{B}^\mathrm{T} - \mathbf{B}'\mathbf{A}^*\mathbf{A}^\mathrm{T} = \mathbf{B}^\mathrm{T}$$

This can be written as

$$\mathbf{B}'(\mathbf{B}\mathbf{B}^\dagger - \mathbf{A}\mathbf{A}^\dagger)^* = \mathbf{B}^\mathrm{T}$$

which, via (B.8), implies

$$\mathbf{B}' = -\mathbf{B}^\mathrm{T} \tag{B.12}$$

Acting on equation (B.10) with $\mathbf{A}^\dagger$ from the right, on equation (B.11) with $\mathbf{B}^\dagger$ from the right, substracting them and using (B.8) we get

$$\mathbf{A}' + \mathbf{B}'\mathbf{B}^*\mathbf{A}^\dagger - \mathbf{B}'\mathbf{A}^*\mathbf{B}^\dagger = \mathbf{A}^\dagger$$

Using (B.12) this can be written as

$$\mathbf{A}' + (-\mathbf{B}^\mathrm{T})(\mathbf{B}^*\mathbf{A}^\dagger - \mathbf{A}^*\mathbf{B}^\dagger) = \mathbf{A}^\dagger$$

or

$$\mathbf{A}' + \mathbf{B}^\mathrm{T}(\mathbf{A}\mathbf{B}^\mathrm{T} - \mathbf{B}\mathbf{A}^\mathrm{T})^* = \mathbf{A}^\dagger$$

The equation (B.9) now implies

$$\mathbf{A}' = \mathbf{A}^\dagger \tag{B.13}$$

Using the same type of reasoning which led to the equations (B.8) and (B.9), one can prove that the matrices $\mathbf{A}'$ and $\mathbf{B}'$ have to satisfy analogous relations,

$$\mathbf{A}'\mathbf{A}'^\dagger - \mathbf{B}'\mathbf{B}'^\dagger = \mathbf{1} \quad , \quad \mathbf{A}'\,\mathbf{B}'^\mathrm{T} - \mathbf{B}'\mathbf{A}'^\mathrm{T} = \mathbf{0}$$

or, because of (B.12) and (B.13),

$$\mathbf{A}^\dagger\mathbf{A} - \mathbf{B}^{\mathrm{T}}\mathbf{B}^* = \mathbf{1} \tag{B.14}$$

$$\mathbf{A}^\dagger\mathbf{B} - \mathbf{B}^{\mathrm{T}}\mathbf{A}^* = \mathbf{0} \tag{B.15}$$

These conditions doesn't follow from (B.8) and (B.9); new information stems from the invertibility of the change of basis.

For the special case $\mathbf{B} = \mathbf{0}$ we have $\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}\mathbf{A}^\dagger = \mathbf{1}$. Change of basis in this case is just an unitary transformation which permutes the annihilation operators but doesn't change the definition of the vacuum!

Now, suppose we write the field ϕ in either $\{\tilde{u}_i\}$ or $\{\tilde{v}_i\}$ basis,

$$\phi = \sum_i (\mathbf{a}_i\,\tilde{u}_i + \mathbf{a}_i^\dagger\,\tilde{u}_i^*) = \sum_i (\mathbf{a}'_i\,\tilde{v}_i + {\mathbf{a}'_i}^\dagger\,\tilde{v}_i^*) \tag{B.16}$$

$$\mathbf{a}_i = (\phi, \tilde{u}_i) \quad , \quad \mathbf{a}'_i = (\phi, \tilde{v}_i) \quad , \quad \mathbf{a}_i^\dagger = -(\phi, \tilde{u}_i^*) \quad , \quad {\mathbf{a}'_i}^\dagger = -(\phi, \tilde{v}_i^*)$$

$$\mathbf{a}'_i = \Big(\sum_j (\mathbf{a}_j\tilde{u}_j + \mathbf{a}_j^\dagger\tilde{u}_j^*), \tilde{v}_i \Big) = \sum_j \mathbf{a}_j(\tilde{u}_j, \tilde{v}_i) + \mathbf{a}_j^\dagger(\tilde{u}_j^*, \tilde{v}_i)$$

Using

$$(\tilde{u}_j, \tilde{v}_i) = \Big(\sum_k (\alpha_{kj}^*\tilde{v}_k - \beta_{kj}\tilde{v}_k^*), v_i \Big) = \alpha_{ij}^*$$

and

$$(\tilde{u}_j^*, \tilde{v}_i) = \Big(\sum_k (\alpha_{kj}\tilde{v}_k^* - \beta_{kj}^*\tilde{v}_k), v_i \Big) = -\beta_{ij}^*$$

we have

$$\mathbf{a}'_i = \sum_j (\alpha_{ij}^*\mathbf{a}_j - \beta_{ij}^*\mathbf{a}_j^\dagger) \tag{B.17}$$

Analogously,

$${\mathbf{a}'_i}^\dagger = -\Big(\sum_j (\mathbf{a}_j\tilde{u}_j + \mathbf{a}_j^\dagger\tilde{u}_j^*), \tilde{v}_i^* \Big) = -\sum_j \mathbf{a}_j(\tilde{u}_j, \tilde{v}_i^*) + \mathbf{a}_j^\dagger(\tilde{u}_j^*, \tilde{v}_i^*)$$

Using

$$(\tilde{u}_j, \tilde{v}_i^*) = \Big(\sum_k (\alpha_{kj}^*\tilde{v}_k - \beta_{kj}\tilde{v}_k^*), \tilde{v}_i^* \Big) = \beta_{ij}$$

and

$$(\tilde{u}_j^*, \tilde{v}_i^*) = \Big(\sum_k (\alpha_{kj}\tilde{v}_k^* - \beta_{kj}^*\tilde{v}_k), \tilde{v}_i^* \Big) = -\alpha_{ij}$$

we have

$${\mathbf{a}'_i}^\dagger = \sum_j (\alpha_{ij}\mathbf{a}_j^\dagger - \beta_{ij}\mathbf{a}_j) \tag{B.18}$$

We can check commutation relations,

$$[\mathbf{a}'_i, \mathbf{a}'_j] = \Big[\sum_k (\alpha_{ik}^*\mathbf{a}_k - \beta_{ik}^*\mathbf{a}_k^\dagger)\,, \sum_l (\alpha_{jl}^*\mathbf{a}_l - \beta_{jl}^*\mathbf{a}_l^\dagger) \Big] =$$

$$= \sum_{kl}(-\alpha^*_{ik}\beta^*_{jl} + \beta^*_{ik}\alpha^*_{jl})\delta_{kl} = \sum_{k}(-\alpha^*_{ik}\beta^*_{jk} + \beta^*_{ik}\alpha^*_{jk}) =$$

$$= (-\mathbf{A}^*\mathbf{B}^\dagger + \mathbf{B}^*\mathbf{A}^\dagger)_{ij} = -(\mathbf{A}\mathbf{B}^\mathrm{T} - \mathbf{B}\mathbf{A}^\mathrm{T})^*_{ij} = 0$$

$$[\mathbf{a}'_i, \mathbf{a}'^\dagger_j] = \Big[\sum_k(\alpha^*_{ik}\mathbf{a}_k - \beta^*_{ik}\mathbf{a}^\dagger_k)\,,\sum_l(\alpha_{jl}\mathbf{a}^\dagger_l - \beta_{jl}\mathbf{a}_l)\Big] =$$

$$= \sum_{kl}(\alpha^*_{ik}\alpha_{jl} - \beta^*_{ik}\beta_{jl})\delta_{kl} = (\mathbf{A}^*\mathbf{A}^\dagger - \mathbf{B}^*\mathbf{B}^\mathrm{T})_{ij} =$$

$$= (\mathbf{A}\mathbf{A}^\dagger - \mathbf{B}\mathbf{B}^\dagger)^*_{ij} = (\mathbf{1})^*_{ij} = \delta_{ij}$$

Inverse relations can be obtained through analogous calculation,

$$\mathbf{a}_i = (\phi, \tilde{u}_i) = \Big(\sum_j(\mathbf{a}'_j\tilde{v}_j + \mathbf{a}'^\dagger_j\tilde{v}^*_j), \tilde{u}_i\Big) = \sum_j \mathbf{a}'_j(\tilde{v}_j, \tilde{u}_i) + \mathbf{a}'^\dagger_j(\tilde{v}^*_j, \tilde{u}_i)$$

$$(\tilde{v}_j, \tilde{u}_i) = \Big(\sum_k(\alpha_{jk}\tilde{u}_k + \beta_{jk}\tilde{u}^*_k), \tilde{u}_i\Big) = \alpha_{ji}$$

$$(\tilde{v}^*_j, \tilde{u}_i) = \Big(\sum_k(\alpha^*_{jk}\tilde{u}^*_k + \beta^*_{jk}\tilde{u}_k), u_i\Big) = \beta^*_{ji}$$

$$\mathbf{a}_i = \sum_j(\alpha_{ji}\mathbf{a}'_j + \beta^*_{ji}\mathbf{a}'^\dagger_j) \tag{B.19}$$

$$\mathbf{a}^\dagger_i = -(\phi, \tilde{u}^*_i) = -\Big(\sum_j(\mathbf{a}'_j\tilde{v}_j + \mathbf{a}'^\dagger_j\tilde{v}^*_j), \tilde{u}^*_i\Big) = -\sum_j \mathbf{a}'_j(\tilde{v}_j, \tilde{u}^*_i) + \mathbf{a}'^\dagger_j(\tilde{v}^*_j, \tilde{u}^*_i)$$

$$(\tilde{v}_j, \tilde{u}^*_i) = \Big(\sum_k(\alpha_{jk}\tilde{u}_k + \beta_{jk}\tilde{u}^*_k), \tilde{u}^*_i\Big) = -\beta_{ji}$$

$$(\tilde{v}^*_j, \tilde{u}^*_i) = \Big(\sum_k(\alpha^*_{jk}\tilde{u}^*_k + \beta^*_{jk}\tilde{u}_k), \tilde{u}^*_i\Big) = -\alpha^*_{ji}$$

$$\mathbf{a}^\dagger_i = \sum_j(\alpha^*_{ji}\mathbf{a}'^\dagger_j + \beta_{ji}\mathbf{a}'_j) \tag{B.20}$$

Finally, we introduce particle number operators $N_i = \mathbf{a}^\dagger_i\mathbf{a}_i$ and $N'_i = \mathbf{a}'^\dagger_i\mathbf{a}'_i$. Using vacuum $|0\rangle$, such that $\mathbf{a}_i|0\rangle = 0$ for all i, we have

$$\langle 0|\, N'_i\, |0\rangle = \langle 0|\, \mathbf{a}'^\dagger_i\mathbf{a}'_i\, |0\rangle = \sum_{jk}\langle 0|\,(\alpha_{ij}\mathbf{a}^\dagger_j - \beta_{ij}\mathbf{a}_j)(\alpha^*_{ik}\mathbf{a}_k - \beta^*_{ik}\mathbf{a}^\dagger_k)\,|0\rangle =$$

$$= \sum_{jk}(-\beta_{ij})(-\beta^*_{ik})\,\langle 0|\,\mathbf{a}_j\mathbf{a}^\dagger_k\,|0\rangle = \sum_{jk}\beta_{ij}\beta^*_{ik}\delta_{jk} =$$

$$= \sum_j |\beta_{ij}|^2 = \mathrm{Tr}(\mathbf{B}\mathbf{B}^\dagger) \tag{B.21}$$

Appendix C

Chain of descent equations

Let $M(k,\mathbb{C})$ be a set of complex $k \times k$ matrices, and $P : \otimes^n M(k,\mathbb{C}) \to \mathbb{C}$ n-linear functions, which are

a) symmetric,

$$P(A_1,\dots,A_i,\dots,A_j,\dots,A_n) = P(A_1,\dots,A_j,\dots,A_i,\dots,A_n) \qquad \text{(C.1)}$$

for all $A_r \in M(k,\mathbb{C})$, $1 \le r \le n$, and

b) adjoint invariant,

$$P(g^{-1}A_1 g,\dots,g^{-1}A_n g) = P(A_1,\dots,A_n) \qquad \text{(C.2)}$$

for all so $g \in G$ and $A_r \in \mathfrak{g} = \mathrm{Lie}\, G$. Function P_n is called *invariant polynomial.*

Also, we shall use following abbrevations

$$P_n(A) \equiv P(\underbrace{A,\dots,A}_{n}) \quad , \quad P_n(A,B^{n-1}) \equiv P(A,\underbrace{B,\dots,B}_{n-1}) \quad , \quad \text{etc.}$$

An example of invariant polynomial is *symmetrized trace*,

$$P(A_1,\dots,A_n) = \mathrm{str}\,(A_1,\dots,A_n) \equiv \frac{1}{n!}\sum_{\pi} \mathrm{tr}\left(A_{\pi(1)}\cdots A_{\pi(n)}\right)$$

where π denotes the permutations of the indices $\{1,\dots,n\}$. This function is symmetric by construction and manifestly gauge invariant.

It is easy to extend the domain of invariant polynomials from $\mathfrak{g}$ to $\mathfrak{g}$-valued p-forms on M, by

$$P(A_1\eta_1,\dots,A_n\eta_n) \equiv P(A_1,\dots,A_n)\,\eta_1 \wedge \ldots \wedge \eta_n \qquad , \qquad \eta_i \in \Lambda^{p_i}(M)$$

Lemma C1. Let P be an invariant polynomial and $\beta = \beta^a_\mu T^a dx^\mu \in \mathfrak{g} \otimes \Lambda^1(M)$. Then

$$dP(A_1\eta_1,\dots,A_n\eta_n) = \sum_{i=1}^{n} (-1)^{p_1+\ldots+p_{i-1}}\, P(A_1\eta_1,\dots,D(A_i\eta_i),\dots,A_n) \quad \text{(C.3)}$$

where

$$D(A_i\eta_i) = A_i d\eta_i + [\beta, A_i\eta_i] = A_i d\eta_i + \beta^a_\mu A^b_i\,[T^a, T^b]\,dx^\mu \wedge \eta_i$$

Proof: First we insert $g = g_t \equiv e^{tJ}$ in (C.2), with $J \in \mathfrak{g}$ and $t \in \mathbb{R}$, and then differentiate it with respect to t,

$$P(g_t^{-1}(-JA_1)g_t, \ldots, g_t^{-1}A_n g_t) + P(g_t^{-1}(A_1 J)g_t, \ldots, g_t^{-1}A_n g_t) + \ldots$$

$$\ldots + P(g_t^{-1}A_1 g_t, \ldots, g_t^{-1}(-JA_n)g_t) + P(g_t^{-1}A_1 g_t, \ldots, g_t^{-1}(A_n J)g_t) = 0$$

For $t = 0$ this gives

$$\sum_{i=1}^{n} P(A_1, \ldots, A_{i-1}, [J, A_i], A_{i+1}, \ldots, A_n) = 0 \tag{C.4}$$

For $\mathfrak{g}$-valued 1-form β one gets

$$P(A_1\eta_1, \ldots, [\beta, A_i\eta_i], \ldots, A_n\eta_n) =$$

$$= P(A_1, \ldots, [\beta_\mu, A_n], \ldots, A_n)(-1)^{p_1+\ldots+p_{i-1}} dx^\mu \wedge \eta_1 \wedge \ldots \wedge \eta_n$$

Using (C.4) we have

$$\sum_{i=1}^{n} (-1)^{p_1+\ldots+p_{i-1}}\, P(A_1\eta_1, \ldots, [\beta, A_i\eta_i], \ldots, A_n\eta_n) = 0 \tag{C.5}$$

Applying the exterior derivative to invariant polynomial we have

$$dP(A_1\eta_1, \ldots, A_n\eta_n) = P(A_1, \ldots, A_n)\, d(\eta_1 \wedge \ldots \wedge \eta_n) =$$

$$= \sum_{i=1}^{n} P(A_1, \ldots, A_n)\,(-1)^{p_1+\ldots+p_{i-1}}\, \eta_1 \wedge \ldots \wedge d\eta_i \wedge \ldots \wedge \eta_n =$$

$$= \sum_{i=1}^{n} (-1)^{p_1+\ldots+p_{i-1}}\, P(A_1\eta_1, \ldots, A_i d\eta_i, \ldots, A_n\eta_n) \tag{C.6}$$

Summing (C.5) and (C.6) we get the result.

□

Theorem. (Chern-Weil) Let P_n be an invariant polynomial, $\omega \in \mathfrak{g} \otimes \Lambda^1(M)$ connection and $\Omega \in \mathfrak{g} \otimes \Lambda^2(M)$ curvature,

$$\Omega = d\omega + \omega^2$$

Furthermore, let $\Omega' = d\omega' + \omega'^2$ be a curvature corresponding to a different connection ω'. Then $P_n(\Omega)$ is closed form, $dP_n(\Omega) = 0$, and $P_n(\Omega') - P_n(\Omega)$ is an exact form.

Comment: For example, in a case of Yang-Mills theories curvature Ω is field strength 2-form F; in a gravitational case Ω is Riemann 2-form R.

Proof: First equality is an immediate consequence of the above Lemma and Bianchi identity, $D\Omega = 0$. In order to prove second part of the theorem it is convenient to introduce following notation

$$\omega_t = \omega_0 + t\beta \quad , \quad \beta = \omega_1 - \omega_0 \quad , \quad t \in [0,1]$$

$$\Omega_t = d\omega_t + \omega_t^2 = \Omega + tD\beta + t^2\beta^2$$

so that $\omega' = \omega_1$, $\omega = \omega_0$, $\Omega' = \Omega_1$ and $\Omega = \Omega_0$. The covariant derivative is defined with respet to connection ω_0,

$$D\beta = d\beta + [\omega_0, \beta]$$

It is important to notice that the derivative of Ω_t by parameter t can be rewritten in a following manner,

$$\partial_t \Omega_t = D\beta + 2t\beta^2 = d\beta + [\omega_0, \beta] + 2t\beta^2 = d\beta + [\omega_t, \beta] \equiv D_t\beta$$

Using this relation and Bianchi identity $D_t\Omega_t = 0$, we have

$$\partial_t P_n(\Omega_t) = nP_n\left(\partial_t\Omega_t, \Omega_t^{n-1}\right) = nP_n(D_t\beta, \Omega_t^{n-1}) = ndP(\beta, \Omega_t^{n-1})$$

Integrating both sides we get

$$P_n(\Omega_1) - P_n(\Omega_0) = nd\int_0^1 dt\, P_n(\omega_1 - \omega_0, \Omega_t^{n-1}) \equiv dQ_{2n-1}(\omega_1, \omega_0) \tag{C.7}$$

□

The relation (C.7) is sometimes referred to as *transgression*. The $(2n-1)$-form Q_{2n-1} is known as *Chern-Simons form*. Locally, or in a case of trivial bundle, we can choose $\omega_0 = 0$ and consequently $\Omega_0 = 0$, so that

$$P_n(\Omega) = dQ_{2n-1}(\omega, \Omega) = nd\int_0^1 dt\, P_n(\omega, \Omega_t^{n-1})$$

$$\omega_t = t\omega \quad , \quad \Omega_t = d\omega_t + \omega_t^2 = t\Omega + (t^2 - t)\omega$$

For example, for 4-dimensional ($n = 2$) Yang-Mills we get

$$P_2(F) = \mathrm{tr}(F^2) = dQ_3 \quad , \quad Q_3 = \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right)$$

In a 6-dimensional ($n = 3$) case we get

$$P_3(F) = \mathrm{tr}(F^3) = dQ_5 \quad , \quad Q_5 = \mathrm{tr}\left(A(dA)^2 + \frac{3}{2}A^3dA + \frac{3}{5}A^5\right)$$

In order to derive the descent equations, we introduce the idea of *shifting*,

$$A \to \widehat{A} = A + v \quad , \quad d \to \Delta = d + s$$

Notice that the shifted exterior derivative is nilpotent, $\Delta^2 = 0$.

Lemma C2. (Russian formula) Shifting leaves the field strength 2-form unchanged,

$$\widehat{F}(\widehat{A}) \equiv \Delta\widehat{A} + \widehat{A}^2 = dA + A^2 = F(A)$$

Proof:

$$\Delta\widehat{A} = (d+s)(A+v) = dA + dv + sA + sv =$$

$$= dA + dv - (dv + [A,v]) - v^2 = dA + A^2 - A^2 - Av - vA - v^2 =$$

$$= F(A) - (A+v)^2 = F(A) - \widehat{A}^2$$

□

Introducing the whole set of shifted companions of objects and operations in the Chern-Weil theorem,

$$\widehat{A}_t = t\widehat{A} \quad , \quad \widehat{F}_t(\widehat{A}_t) = \Delta\widehat{A}_t + \widehat{A}_t^2$$

$$\widehat{D} = \Delta + [\widehat{A}\ ,\ ] \quad , \quad \widehat{D}_t = \Delta + [\widehat{A}_t\ ,\ ]$$

it easy to show that the shifted versions of Bianchi identity holds,

$$\widehat{D}\widehat{F} = \widehat{D}_t\widehat{F}_t = 0$$

and, hence, we have the shifted transgression formula,

$$P_n(\widehat{F}) = \Delta Q_{2n-1}(\widehat{A},\widehat{F}) \quad , \quad Q_{2n-1} = n\int_0^1 dt\, P_n(\widehat{A},\widehat{F}_t^{n-1}) \tag{C.8}$$

Using this and the Russian formula, one gets

$$\Delta Q_{2n-1}(A+v,F) = P_n(\widehat{F}) = P_n(F) = dQ_{2n-1}(A,F)$$

We can expand Chern-Simons form Q_{2n-1} in powers of ghosts v,

$$Q_{2n-1}(A+v,F) = Q^0_{2n-1}(A,F) + Q^1_{2n-2}(v,A,F) + \ldots + Q_0^{2n-1}(v)$$

Comparing the terms of the same form degree and same power in v on both sides of equation (C.8) we obtain the **chain of descent equations**,

$$\begin{aligned}
P_n(F) - dQ^0_{2n-1} &= 0\\
sQ^0_{2n-1} + dQ^1_{2n-2} &= 0\\
sQ^1_{2n-2} + dQ^2_{2n-3} &= 0\\
&\ \vdots\\
sQ_1^{2n-2} + dQ_0^{2n-1} &= 0\\
sQ_0^{2n-1} &= 0
\end{aligned} \tag{C.9}$$

Notice that the third equation in the chain represents the local form of WZ consistency condition. Therefore, the term Q^1_{2n-2} can be identified with the anomaly $v^a G^a[A]$.

It is important to emphasize that the chain terms Q^k_{2n-1-k} are not uniquely determined. Zumino [Zum85] has used this ambiguity to simplify the explicit solution. His choice is characterized by having a) the smallest power in the gauge connection A and b) the largest power in the number of derivatives dv. Following his idea, we shall first introduce shifting described by

$$\widetilde{A}_t = tA + v \quad , \quad \widetilde{F}_t = \Delta\widetilde{A}_t + \widetilde{A}_t^2 = F_t + (1-t)dv$$

$$\widetilde{F}_1 = F \quad , \quad \widetilde{F}_0 = dv$$

$$\partial_t\widetilde{F}_t = \Delta\partial_t\widetilde{A}_t + [\widetilde{A}_t, \partial_t\widetilde{A}_t] \equiv \widetilde{D}_t\partial_t\widetilde{A}_t$$

Bianchi identity is still valid, $\widetilde{D}_t\widetilde{F}_t = 0$, so that

$$\partial_t P_n(\widetilde{F}_t) = nP_n(\partial_t\widetilde{F}_t, \widetilde{F}_t^{n-1}) = nP_n(\widetilde{D}_t\partial_t\widetilde{A}_t, \widetilde{F}_t^{n-1}) = n\Delta P_n(\partial_t\widetilde{A}_t, \widetilde{F}_t^{n-1})$$

Integrating both sides we get

$$P_n(F) - P_n(dv) = \Delta Q_{2n-1} \tag{C.10}$$

where

$$Q_{2n-1} = n\int_0^1 dt\, P_n(A, \widetilde{F}_t^{n-1}) = \sum_{k=0}^{n-1} Q^k_{2n-1-k}$$

Comparing the terms of the same form degree and same power in v in equation (C.10) we get the following chain of descent equations,

$$\begin{aligned} P_n(F) - dQ^0_{2n-1} &= 0 \\ sQ^0_{2n-1} + dQ^1_{2n-2} &= 0 \\ &\vdots \\ sQ^{n-1}_n + P_n(dv) &= 0 \end{aligned} \tag{C.11}$$

On the other hand, we can expand invariant polynomial,

$$P_n(A, \widetilde{F}_t^{n-1}) = \sum_{k=0}^{n-1}\binom{n-1}{k}(1-t)^k P_n((dv)^k, A, F_t^{n-1-k})$$

where we have used the fact that dv has a total degree 2, so it commutes with form content of the arguments of P_n. Now, comparing the terms of the same form degree and same power in v in the expansions of Q_{2n-1} and $P_n(A, \widetilde{F}_t^{n-1})$ we obtain formula for each chain term Q^k_{2n-1-k},

$$Q^k_{2n-1-k} = n\binom{n-1}{k}\int_0^1 dt\,(1-t)^k P_n((dv)^k, A, F_t^{n-1-k}) \tag{C.12}$$

Another type of shifting is described by

$$\widetilde{A}_t = tv \quad , \quad \widetilde{F}_t = \Delta \widetilde{A}_t + \widetilde{A}_t^2 = tdv + (t^2 - t)v$$

$$\widetilde{F}_1 = dv \quad , \quad \widetilde{F}_0 = 0$$

$$\partial_t \widetilde{F}_t = \Delta \partial_t \widetilde{A}_t + [\widetilde{A}_t, \partial_t \widetilde{A}_t] \equiv \widetilde{D}_t \partial_t \widetilde{A}_t$$

Bianchi identity is still valid, $\widetilde{D}_t \widetilde{F}_t = 0$, so that the transgression gives us

$$P_n(dv) = \Delta Q_{2n-1} \tag{C.13}$$

with

$$Q_{2n-1} = n \int_0^1 dt\, P_n(v, \widetilde{F}_t^{n-1}) = \sum_{k=0}^{n-1} Q^k_{2n-1-k}$$

Comparing the terms of the same form degree and same power in v in equation (C.13) we get another chain of descent equations,

$$\begin{aligned} P_n(dv) - dQ^n_{n-1} &= 0 \\ sQ^n_{n-1} + dQ^{n+1}_{n-2} &= 0 \\ &\vdots \\ sQ^{2n-2}_1 + dQ^{2n-1}_0 &= 0 \\ sQ^{2n-1}_0 &= 0 \end{aligned} \tag{C.14}$$

Expanding in powers of dv and v^2, we have

$$P_n(v, \widetilde{F}_t^{n-1}) = \sum_{k=0}^{n-1} \binom{n-1}{k} t^{n-1-k} (t^2 - t)^k P_n((dv)^{n-1-k}, v, (v^2)^k) =$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} t^{n-1} (t-1)^k P_n((dv)^{n-1-k}, v, (v^2)^k)$$

Again, comparing the terms of the same form degree and same power in v in the expansions of Q_{2n-1} and $P_n(v, \widetilde{F}_t^{n-1})$ and using the properties of the beta function $B(p,q)$,

$$\int_0^1 dt\, t^{n-1}(t-1)^k = (-1)^k B(n, k+1) = (-1)^k \frac{\Gamma(n)\Gamma(k+1)}{\Gamma(n+k+1)} = (-1)^k \frac{(n-1)!k!}{(n+k)!}$$

one can write the explicit, integrated expression for the chain term Q^{n+k}_{n-1-k},

$$Q^{n+k}_{n-1-k} = (-1)^k \frac{n!(n-1)!}{(n-1-k)!(n+k!)} P_n((dv)^{n-1-k}, v, (v^2)^k) \tag{C.15}$$

Appendix D

Some useful sums

The basic sum is (E.3); using the fact that odd Bernoulli numbers B_{2n+1} vanish for $n \in \mathbb{N}$, we can write

$$\frac{x}{e^x - 1} \;=\; \sum_{m=0}^{\infty} \frac{B_m}{m!}\, x^m \;=\; 1 + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\, x^{2k} \tag{D.1}$$

We define 1-parametric family of frequently used functions,

$$\mathcal{S}_a(x) \;\equiv\; \frac{1}{x^2} - \frac{a^2}{4\,\mathrm{sh}^2(ax/2)} \tag{D.2}$$

The Taylor series for $\mathcal{S}_a(x)$ can be found using following observation

$$\mathcal{S}_a(x) = -\frac{d}{dx}\left(\frac{1}{x}\left(1 - \frac{ax}{e^{ax}-1}\right)\right) = \frac{d}{dx}\sum_{m=1}^{\infty} \frac{B_m a^m}{m!}\, x^{m-1} =$$

$$= \sum_{m=2}^{\infty} \frac{B_m a^m}{m}\, \frac{x^{m-2}}{(m-2)!} = \sum_{k=1}^{\infty} \frac{B_{2k} a^{2k}}{2k}\, \frac{x^{2k-2}}{(2k-2)!}$$

or, equivalently

$$\mathcal{S}_a(x) = a^2 \sum_{n=0}^{\infty} \frac{B_{n+2}}{n+2}\, \frac{(ax)^n}{n!} \tag{D.3}$$

These results allow us to expand the following functions, occuring through the calculations above:

$$\frac{1}{\mathrm{sh}(ax)} = \frac{2e^{ax}}{e^{2ax}-1} = \frac{2}{e^{ax}-1}\,\frac{e^{ax}+1-1}{e^{ax}+1} =$$

$$= \frac{2}{e^{ax}-1}\left(1 - \frac{1}{e^{ax}+1}\right) = \frac{2}{e^{ax}-1} - \frac{2}{e^{2ax}-1}$$

$$\frac{a}{\mathrm{sh}(ax)} - \frac{1}{x} = \frac{1}{x}\left(\frac{ax}{\mathrm{sh}(ax)} - 1\right) = \frac{2}{x}\left(\frac{ax}{e^{ax}-1} - \frac{ax}{e^{2ax}-1} - \frac{1}{2}\right) =$$

$$= \frac{2}{x}\left(\left(\frac{ax}{e^{ax}-1}-1\right)-\frac{1}{2}\left(\frac{2ax}{e^{2ax}-1}-1\right)\right) =$$

$$= \frac{2}{x}\left(\sum_{k=1}^{\infty}\frac{(ax)^{2k}}{(2k)!}B_{2k}-\frac{1}{2}\sum_{l=1}^{\infty}\frac{(2ax)^{2l}}{(2l)!}B_{2l}\right) =$$

$$= 2\sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}a^{2m}\left(1-2^{2m-1}\right)x^{2m-1}$$

Now we shall evaluate the sums involving binomial coefficients. First, we prove *Vandermonde's identity*

$$\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}=\binom{m+n}{r} \tag{D.4}$$

where $m, n, r \in \mathbb{N}$. Namely, if we evaluate both sides of the equation

$$(1+x)^m(1+x)^n=(1+x)^{m+n}$$

using binomial theorem, the identity (D.4) is obtained by comparing coefficients of x^r on both sides.

The sum appearing in the context of the bosonic holomorphic currents can be written as

$$\sum_{k=0}^{s-2}A^s_{k+1}=\sum_{k=0}^{s-2}\frac{1}{s-1}\binom{s-1}{k+1}\binom{s-1}{s-k-1}=\sum_{p=1}^{s-1}\frac{1}{s-1}\binom{s-1}{p}\binom{s-1}{s-p}$$

where we have shifted the index, $p=k+1$. Furthermore, using the fact that

$$\binom{s-1}{s}=0$$

and (D.4), we have

$$\sum_{k=0}^{s-2}A^s_{k+1}=\sum_{p=0}^{s-1}\frac{1}{s-1}\binom{s-1}{p}\binom{s-1}{s-p}=\frac{1}{s-1}\binom{2s-2}{s}=\frac{(2s-2)!}{(s-1)!s!}$$

The sum appearing in the context of the fermionic holomorphic currents is

$$\sum_{k=1}^{s}\binom{s-1}{k-1}^2=\sum_{p=0}^{s-1}\binom{s-1}{p}^2=\sum_{p=0}^{s-1}\binom{s-1}{p}\binom{s-1}{s-p-1}=\binom{2s-2}{s-1}=$$

$$=\frac{(2s-2)!}{((s-1)!)^2}=\frac{(2s-2)!!(2s-3)!!}{((s-1)!)^2}=\frac{2^{s-1}(s-1)!(2s-3)!!}{((s-1)!)^2}=\frac{2^{s-1}(2s-3)!!}{(s-1)!}$$

Appendix E

Conventions

List of symbols

$\pounds$	Lie derivative	∇	covariant derivative
$Y_{\ell m}$	spherical harmonics	$\mathcal{Y}_{\ell m}$	modified spherical harmonics
ℓ_m, $\bar{\ell}_m$	generators of Witt algebra	L_m, $\bar{L}_m$	generators of Virasoro algebra
$\imath_\xi$	contraction by the vector ξ^a	$\mathscr{T}$	time ordering
$\epsilon_{abc...}$	Levi-Civita tensor	$\varepsilon_{abc...}$	Levi-Civita tensor density
tr	trace	str	symmetrized trace

2-dimensional objects are denoted by the $\flat$ symbol over the object, as in

$$\overset{\flat}{\nabla} \quad , \quad \overset{\flat}{e}_a{}^\mu \quad , \quad \overset{\flat}{\omega}_{ab} \quad , \quad \overset{\flat}{g}_{\mu\nu} \quad , \quad \ldots$$

List of abbrevations

QFT	quantum field theory	CFT	conformal field theory
WZ	Wess-Zumino	BRS	Becchi-Rouet-Stora
BE	Bose-Einstein	FD	Fermi-Dirac
LHS	left hand side	RHS	right hand side
vev	vacuum expectation value	*cycl.*	cyclic permutation

Throughout most of the thesis we follow the conventions for the metric and the curvature used in [MTW70]. The metric signature is $(-,+,+,+)$, apart from sections involving calculation with fermions and gamma matrices, where we imploy the opposite convention. Riemann tensor is defined by

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\beta\mu} \tag{E.1}$$

Ricci tensor and scalar are defined through contractions,

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \qquad , \qquad R = g^{\mu\nu} R_{\mu\nu} \tag{E.2}$$

Tensors and tensorial equations are written using abstract index notation, described in [Wal84]. Notice, however, that the vielbein indices (lower case latin and greek letters) denote different frames and not "tensor type vs coordinate component" distinction.

Levi-Civita tensor density is normalized as $\varepsilon^{012\ldots n} = +1$.

We use Bernoulli numbers B_m defined via series

$$\frac{x}{e^x - 1} \;=\; \sum_{n=0}^{\infty} \frac{B_n}{n!}\, x^n \tag{E.3}$$

convergent for $|x| < 2\pi$. Several first Bernoulli numbers are

$$B_0 = 1 \quad , \quad B_1 = -\frac{1}{2} \quad , \quad B_2 = \frac{1}{6} \quad , \quad B_4 = -\frac{1}{30} \quad , \quad \ldots$$

Apart from B_1, all other odd Bernoulli numbers are zero, $B_{2n+1} = 0$ for $n \in \mathbb{N}$.

Schwarzian derivative is defined as

$$\{y, x\} \;\equiv\; \frac{y'''(x)}{y'(x)} - \frac{3}{2} \left(\frac{y''(x)}{y'(x)} \right)^2 \tag{E.4}$$

For the correlation function of several operators $(A,\, B,\, \ldots,\, Z)$ we use notation

$$\langle AB \ldots Z \rangle \;\equiv\; \langle 0 | \mathscr{T}(AB \ldots Z) | 0 \rangle \tag{E.5}$$

where $\mathscr{T}$ denotes time-ordering or, equivalently, radial-ordering.

Feynman's *slash* notation is defined as

$$\not{A} \;\equiv\; \gamma^\mu A_\mu \quad , \quad \not{\nabla} \;\equiv\; \gamma^\mu \nabla_\mu \tag{E.6}$$

Matrices are written in capital bold letters $(\mathbf{A},\, \mathbf{B},\, \mathbf{C},\, \ldots)$. Unit matrix is written as $\mathbf{1}$. Elementary operations with matrices, transposition, complex and hermitian conjugation, are written and defined as

$$[\mathbf{A}^{\mathrm{T}}]_{ij} = [\mathbf{A}]_{ji} \quad , \quad [\mathbf{A}^*]_{ij} = ([\mathbf{A}]_{ij})^* \quad , \quad [\mathbf{A}^\dagger]_{ij} = ([\mathbf{A}]_{ji})^*$$

where $[\mathbf{A}]_{ij}$ is a matrix element at i^{th} raw and j^{th} column.

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