Perturbative Stability of Topologically Ordered Systems with Local Defects

Kurečić, Ivana

UNIVERSITY OF ZAGREB
FACULTY OF SCIENCE
DEPARTMENT OF PHYSICS

Ivana Kurečić

PERTURBATIVE STABILITY OF TOPOLOGICALLY ORDERED SYSTEMS WITH LOCAL DEFECTS

Master Thesis

Zagreb, 2016

SVEUČILIŠTE U ZAGREBU
PRIRODOSLOVNO-MATEMATIČKI FAKULTET
FIZIČKI ODSJEK

Ivana Kurečić

PERTURBATIVNA STABILNOST TOPOLOŠKI UREĐENIH SUSTAVA S LOKALNIM DEFEKTIMA

Diplomski rad

Zagreb, 2016.

UNIVERSITY OF ZAGREB
FACULTY OF SCIENCE
DEPARTMENT OF PHYSICS

STUDY PROGRAM: RESEARCH ORIENTED STUDY OF PHYSICS

Ivana Kurečić

Master Thesis

Perturbative Stability of Topologically Ordered Systems with Local Defects

Advisor: Professor Tobias J. Osborne, PhD
Co-Advisor: Assistant Professor Davor Horvatić, PhD

Master Thesis grade: _____

Committee: 1. _____

2. _____

3. _____

Master Thesis defense date: _____

Zagreb, 2016

This project would not have been possible without the support and guidance of my mentor and supervisor, Dr. Courtney Brell and Prof. Dr. Tobias Osborne, as well as all of the members of the Quantum Information Group of the Institute for Theoretical Physics at the University of Hannover, where this work was done through the Erasmus+ Traineeship program, and where I had received an immensely warm welcome.
I would like to thank the secretary of the group, Wiebke Möller, and the staff of the Students' Office of the Department of Physics of the Faculty of Science at the University of Zagreb, who have been successfully dealing with all of the bureaucratic maneuvering that I have been going through to realize this exchange, as well as Asst. Prof. Dr. Davor Horvatić of the University of Zagreb, who had graciously accepted to co-advise my thesis.

Completing this task at one point felt as an insurmountable challenge, and I was intimidated by the vast empty pages I needed to perfectly fill. At that time I always had somebody to turn to, and it was more help than I would have hoped for. I would like to thank Leander Fiedler for the discussions that repeatedly helped me move forward when faced with a challenge, Christopher Cedzich for the company and conversation at strange hours, Prof. Dr. Tobias Osborne for giving me structure, invaluable advice and a firm footing, Dr. Courtney Brell for the cheer and confidence, and for showing me all of these shiny things, directing me sensibly, and putting up with my quirks, as well as my colleague and partner, Marius Lewerenz, for holding my hand through this, sharing advice, and keeping me steady. You guys had me push through my self-doubt and come out the other end.

Finally, I would like to express special gratitude to my high school astronomy teacher, Prof. Ljiljana Nemet, without whose unfaltering optimism and encouragement I never would have chosen to study physics.

Perturbativna stabilnost topološki uređenih sustava s lokalnim defektima

Sažetak

Kvantni spinski sustavi donose uvijek nove izazove u modernoj fizici proteklih nekoliko desetljeća; teorijski opis topološkog kvantnog uređenja je otkrio novo poglavlje u razumijevanju faznih prijelaza u kvantnim sustavima.

Novoopažena svojstva topološki uređenih sustava zaokupila su pozornost i znanstvenika iz polja kvantne teorije informacija, čime je započelo istraživanje njihovoga potencijala u kvantnom računarstvu. Od nedavno, korisnost topološki uređenih sustava detaljno se istražuje u području topološkog kvantnog računarstva.

Sustavi u kojima se pojavljuje topološko uređenje imaju vrijedne mogućnosti za kvantno računarstvo – korištenjem globalnih opservabli koje ostaju robustne kada se na sustav djeluje lokalnom perturbacijom. Pod ovim uvjetima, informacije pohranjene u sustavu ostaju stabilne, što pruža jedinstven način zaobilaženja problema koji zaokupljaju standardne sheme korištene u kvantnom računarstvu.

U svrhu bolje primjenjivosti ovih sustava u kvantnom računarstvu otpornom na pogreške, potrebno je razumjeti točno ponašanje pohranjenih informacija pod utjecajem vanjske perturbacije. Ovaj rad predstavlja važne matematičke metode koje se koriste pri istraživanju dinamike lokalnih kvantnih sustava — kvazi-adijabatsko produljenje, Lieb-Robinsonove granice, i razvoj vlastite energije — te prikazuje njihovu primjenu pri unaprjeđivanju razumijevanja utjecaja slabih lokalnih perturbacija na informacije pohranjene u topološki uređenim sustavima.

Specifično, istražen je utjecaj perturbacije na topološki uređene sustave s lokalnim defektima tipa rupe te su izvedene opće granice za stabilnost topološkog uređenja u takvim sustavima.

Perturbative Stability of Topologically Ordered Systems with Local Defects

Abstract

Quantum spin systems have been providing modern physics with ever new challenges to tackle in the past several decades; from the theoretical postulation of topological quantum order, a new chapter was revealed in the understanding of phase transitions in quantum systems.

The novel properties of topologically ordered systems had caught the interest of quantum information scientists, and research into their potential in the field of quantum computing soon began. Nowadays, this utility of topologically ordered systems is thoroughly researched in the field of topological quantum computing.

Systems that exhibit topological order present a valuable resource for quantum computing – by use of the global observables that remain robust when such a system is acted on by a local perturbation. Under these conditions, the information stored in the system remains stable, providing a unique modus of circumventing the problems that plague standard quantum computing schemes.

To improve on the applicability of such systems in fault-tolerant quantum computation, there exists a need to understand the exact behavior of the encoded information under perturbation. To this end, this thesis presents critical mathematical methods used to research the dynamics of local quantum systems — the quasi-adiabatic continuation, the Lieb-Robinson bounds, and the self-energy expansion — and utilizes them to further the understanding of the effect of weak local perturbations on the information stored in a topologically ordered system.

In particular, the influence of perturbation on topologically ordered systems with local puncture defects is examined, and general bounds on the stability of topological order in such systems are derived.

Contents

1 Introduction

Quantum information and *quantum computation* are terms that denote the study of the information processing tasks that can be accomplished using quantum mechanical systems [1] – and they are quantum mechanical counterparts of the classical fields of the same name. The story of the birth of these fields is long and it follows the discovery of the underlying quantum structure of the world. The property of superposition of quantum states enables a view of information that is vastly different to the one of classical information theory. While the states of classical bits are defined as $\mathbb{Z}_2 = \{0, 1\}$, quantum bits — or *qubits* — are represented by $\mathbb{C}^2 = \mathbb{C}[\mathbb{Z}_2]$, which implies that classical bit states can be seen as mere basis vectors for qubits [2].

In 1982, Richard Feynman, a reputed physicist, had shown that a standard computing machine that does not rely on quantum mechanics for its operation, would not be able to efficiently simulate quantum phenomena [3]. In fact, it would be expected to experience an exponential slowdown, while a universal *quantum simulator* that he proposed in the same publication – would not. Following this leading thought, research into quantum simulators and quantum computers truly began. In 1996, Seth Lloyd proved Feynman's conjecture to be correct [4].

Ever since its modest beginnings, classical computing has experienced an unbridled expansion in the sense that its capabilities and availability have soared beyond any initial prediction. However, practical limitations exist to the actual computing power of classical computers, and they cannot be expected to reach the capabilities of quantum computers. In 1965, Gordon Moore presented a hypothesis (later known as *Moore's law*) based on the observation of the development of integrated elecronics. He claimed that the power of computer hardware (the number of components on integrated circuits) would double for constant cost roughly every year in the foreseeable future – and has later been reported to increase that period to two years [5]. Time has shown that he was approximately right with this statement. However, as the computing power rises, the size of manufactured hardware components diminishes – as processors grow smaller, quantum effects gain a non-negligible role in their physical properties. Moore's law is expected to hit a fundamental wall and fail in the near future. One way of addressing the interfering quantum effects in classical computing hardware is to accept the fall of its advancement and turn to quantum computing – a flagship project for many scientific and technological research institutions that focus on the frontier of science (for examples of experimental achievements in topological quantum computing, see Subsection 2.5).

It is not only classical hardware that is expected to fold next to quantum computing, many core classical algorithms (such as the famed RSA cryptosystem – see [6]) are outshined by innovative quantum algorithms, most notably Peter Shor's factoring algorithm [7] and Lov Grover's search algorithm [8]. Some of the invented quantum algorithms outperform their classical equivalents, possibly making previously unfea-

sible calculations simplified and approachable, given that there exists a *quantum computer* that could run them (for example, a new field of *post-quantum cryptography* [9] is currently in the making).

Analogously to the way a classical computer is built to contain electrical circuits that contain logic gates used to perform computational tasks, a quantum computer is built of *quantum circuits* that contain *quantum gates*. These schemes can be utilized to build *quantum algorithms*.

An important facet of quantum computing is the possibility to store quantum information in a system, and use it at a later time for computing protocols – this involves creating reliable *quantum memories* (for example, see [10]). A troublesome issue with any quantum systems is the prospect of *decoherence*, which would destroy the information stored in the system and render any computational process useless. Additionally, as quantum systems are much more sensitive to outside disturbances than is the case with classical computing components, the *stability* of such systems comes into question. A good overviev of the general requirements for a physical implementation of quantum computation can be found in [11].

One of the most reliable ways of combating the problems inherent to quantum computing systems is *quantum error correction*, in a quest for *fault-tolerant quantum computing* (see Subsection 3.1). The methods developed under these denominators focus on the development of setups and procedures in quantum computing that would allow for the correction of any errors that may arise in such systems, allowing for a confident use of quantum computing protocols. The *threshold theorem* [12], formulated by Dorit Aharonov and Michael Ben-Or in 1999, confirms that quantum computation can be made robust against errors and inaccuracies when the error rate is smaller than some threshold value, by use of quantum error correcting methods. A comprehensive introduction to quantum error correction can be found in [13] and [1], and a short overview is presented in [14].

An innovative approach to the problem of instability of quantum computing setups is the use of systems that exhibit topological quantum order. The research on such systems focuses on the global invariants that are robust against local perturbation. This means that discovering and tracking errors in *topological quantum computers* becomes a much easier task than that which is faced by other venues of quantum computing [15].

Topologically ordered systems are most famously observed in *fractional quantum Hall liquids* [16] (see other experimental realizations in Subsection 2.5), and they are not yet well understood at temperatures above absolute zero – a topological phase of matter is a state of matter whose low-energy effective theory is a topological quantum field theory [2].

An interesting characteristic of topologically ordered systems is that they support

the existence and creation of *anyons* – indistinguishable quasiparticles that may only occur in two-dimensional systems, and that do not follow bosonic, or fermionic exchange statistics. They were first mathematically described by Jon Magne Leinaas and Jan Myrheim in 1977 [17], and named by Frank Wilczek in 1982 [18]. Anyons may be observed that obey exchange statistics of the form $|\Psi_1\Psi_2\rangle = e^{i\theta}|\Psi_2\Psi_1\rangle$, where Ψ_1 and Ψ_2 are the quantum states of the two anyons, and θ has *any* value. This equation is exactly valid only in one direction of exchange (i.e. either counterclockwise or clockwise), because of the existence of *nonabelian anyons*, for which the direction of exchange matters.

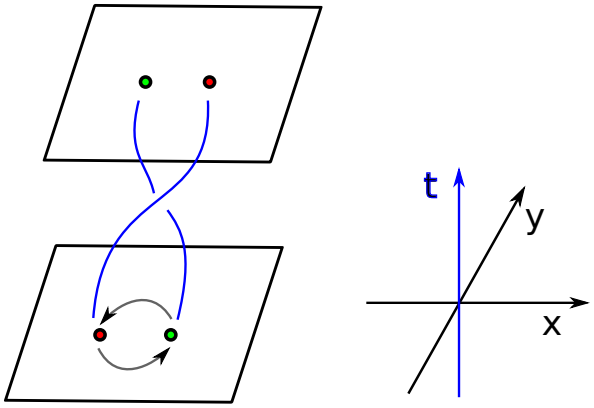


Figure 1.1: A representation of anyon braiding in a 2-dimensional x-y system. The anyons are represented as points on a 2-dimensional sheet, and the vertical axis is the time axis, t. They are braided in time-space by rotating one around the other, as denoted by arrows and blue lines.

Logical gates in topological quantum computation are constructed by *braiding* anyons in $(2+1)$-dimensional space-time (see Figure 1.1 for a visual representation). Separating the strings of the braid is non-trivial, which makes the manipulation of the observables in such systems appropriate for topological quantum computation. Much more on the topic of anyons and the braiding of anyons can be found in [19].

In this thesis, the focus was given to the perturbative stability of topologically ordered systems. In 2010, Sergey Bravyi, Matthew Hastings, and Spyridon Michalakis had proven that the zero-temperature topological phases of matter remain robust against small and local time-independent perturbations [20]. This is a very important result for topological quantum computation, as it indicates that topological invariants of such systems can be considered as such, under the conditions of little noise. To understand precisely how a topologically ordered system is affected by perturbation, it is necessary to examine the exact consequences of the introduction of perturbation to such a system. This thesis presents results on the perturbative stability of topologically ordered systems, and additionally topologically ordered systems with *local puncture defects* (for a description, see Subsection 5.2. This work concludes with the derivation and interpretation of general bounds on the stability of topological order

in systems with local puncture defects.

The contents of this thesis are as follows.

In Section 2 the basics of topologically ordered quantum spin systems are presented, covering an introduction to the mathematical description and basic properties of quantum spin systems (Subsection 2.1) and the concept of quantum phase transitions (Subsection 2.2). The section provides several equivalent descriptions of topological quantum order (Subsection 2.3, Subsection 2.4), and a short overview of the experimental realizations of topologically ordered systems, from the viewpoint of quantum computing (Subsection 2.5).

Section 3 is an introduction to fault-tolerant quantum computation, covering the relevant topics in quantum error correction (Subsection 3.1), presenting the stabilizer formalism (Subsection 3.2), defining the toric code (Subsection 3.3) and considering the perturbative stability of topologically ordered systems in general terms (Subsection 3.4).

The mathematical tools used for the analysis of the setups discussed in this work are described in Section 4. Subsection 4.1 provides a description and a characterization of the properties of the quasi-adiabatic continuation, Subsection 4.2 serves as an introduction to the Lieb-Robinson bound, and Subsection 4.3 presents an overview of Kitaev's self-energy expansion.

The main analysis and results of this work are contained in Section 5. The goal of this section is the investigation of the stability of qubits encoded in the punctured toric code. Subsection 5.1 examines the application of a small perturbation onto a system described by the toric code, and provides a relation that bounds the truncation error for the evolved logical operator in such a system. Subsection 5.2 introduces local defects — punctures — in the toric code, and provides an overview of the properties of a punctured system. The following subsections focus on the calculations of the effect of a small perturbation applied to a punctured toric code setup – Subsection 5.3 approaches the punctured toric code by use of Kitaev's self-energy expansion, while Subsection 5.4 analyses such a system by use of quasi-adiabatic continuation and the Lieb-Robinson bounds. A summary and interpretation of the results is found in Subsection 5.5.

Section 6 is a compilation of the properties of topologically ordered quantum spin systems and tools used in the analysis of the related problems in quantum information theory that were presented in this thesis, as well as the accomplished results. It is an overview of the topics that were discussed in this work, and it provides a short summary and interpretation of the calculated bounds. This section finalizes the thesis by presenting a summary of the introduced topic, and providing an outlook for the direction of future work.

An expanded Croatian abstract, of the length of a minimum of 20% of the textual content in the thesis, can be found in Section 7.

2 Quantum Spin Systems and Topological Order

A *quantum spin system* [21] is a term denoting a toy model that is most commonly used to examine a nonrelativistic quantum system with some number of degrees of freedom that each have a finite-dimensional state space. Quantum spin systems are one of the types of systems that are investigated under the denominator of the theory of quantum many-body systems, and they may exhibit macroscopically unusual properties that challenge the understanding of the underlying mechanisms that are observed in such systems, providing a relatively fresh challenge for many areas of modern physics.

This section serves as an introduction to the topic of topological order. The first part of the section (Subsection 2.1) explains the concept of quantum many-body systems in the context of quantum spin systems, followed by an introduction to the topic of quantum phase transitions (Subsection 2.2). In the second part of the section, topological order is described by a definition relating to a topologically ordered system's Hamiltonian (Subsection 2.3), as well as two alternative and equivalent approaches relating to the system's phases and its states (Subsection 2.4), and finally, some examples on the experimental realization of topologically ordered systems are given (Subsection 2.5) to round off the introduction to the underlying setup for the remainder of the thesis.

2.1 Quantum Many-Body Systems

In general, the field of *quantum many-body systems* [22] is concerned with models that can be used to investigate problems relating to the states and dynamics of systems that consist of a large number of interacting particles, in which the principles of quantum mechanics are necessarily used to reach sufficiently accurate results. Its classical equivalent is known as N-body systems, and the approach to solving these is straightforward in the statistical limit. The particles in quantum many-body systems are usually very small so that quantum effects play a relevant (and commonly complicated) role in the dealing with the system.

One of the types of systems that quantum many-body systems deal with are quantum spin systems. These systems feature some number of degrees of freedom spread out on a graph, with each having a finite-dimensional state space, commonly associated with spin. Additionally, there exists a notion of distance between the spin particles in such graphs, providing a concept of *locality* [23, 24].

This subsection provides a short introduction to quantum spin systems, as they are approached in the theory of quantum many-body systems.

Instead of dealing with the properties and states of the individual elements of

the system, the research on quantum many-body systems deals with the states and observables on a macroscopic level – commonly expanding, also, the systems to the thermodynamic limit, where they consist of an infinite number of subsystems [25]. An important area of interest in quantum many-body systems are the phase transitions that may occur; an introduction to quantum phases is given in Subsection 2.2.

The Hilbert space of a quantum spin system Λ can be described using a tensor product,

$$\mathcal{H} = \bigotimes_{u \in \Lambda} \mathcal{H}_u, \qquad \dim \mathcal{H}_u = \mathcal{O}(1), \tag{2.1}$$

where $\mathcal{H}_u$ refers to the Hilbert spaces of all the individual degrees of freedom in the system, u. The time evolution of such a system is uniquely described by its Hamiltonian, H_t, and the Schrödinger equation, and it is commonly — as is true for this thesis — approached in the Heisenberg picture, such that

$$\frac{d}{dt} Y(t) = \frac{i}{\hbar}[H, Y(t)] + \left(\frac{\partial Y(t)}{\partial t} \right)_H, \tag{2.2}$$

where Y_t is an operator or observable in the system, H is the system's Hamiltonian, and t stands for time. That is, the state vectors of the examined system are handled as time-independent, while all of the time-dependency is transferred to the operators acting in the system:

$$Y(t) = e^{iHt/\hbar} Y e^{-iHt/\hbar}, \tag{2.3}$$

for simplicity of calculation.

The most common mathematical setup when it comes to exploring the properties of these systems is a distribution of spin degrees of freedom on an integer lattice, with *local Hamiltonians*, which means that the full Hamiltonian of the system, H, can be written as the sum of Hamiltonians describing interactions between geometrically local elements in the system, such that

$$H = \sum_{r < r_0} \sum_{i \in \Lambda} H_{\mathcal{B}(i,r)}, \tag{2.4}$$

where $\mathcal{B}(i, r)$ denotes a ball of radius r that is smaller than some r_0, around the point i on the lattice describing the system, and $H_{\mathcal{B}(i,r)}$ is an operator that has support only on this region. Most commonly, the systems in which $\|H_{\mathcal{B}(i,r)}\|$ decays rapidly with r are considered. The quantum spin system models are a mathematical simplification of complicated many-particle systems with high utility – they provide a strictly defined enviroment for the calculation of the systems' spectra, eigenstates, stationary states, dynamics, and much more.

For a concise overview of the historical approach to quantum spin systems and the important properties that such systems exhibit, see [21]. The quantum information approach to quantum many-body systems is comprehensibly presented in a recently composed manuscript [26].

2.2 Quantum Phase Transitions

For a very long time, the generally accepted description of phase transitions [27] in condensed matter physics was given by Lev Landau's symmetry-breaking theory (and later expanded on in collaboration with Vitaly Lazarevich Ginzburg, dubbing it the Ginzburg-Landau theory), and it described with satisfactory accuracy the phase transitions in thus far observed orders of matter. However, the observation of the *fractional quantum Hall effect* [28,29] changed the absolute prevalence of this paradigm. In systems that exhibit the fractional quantum Hall effect several quantum Hall states emerge, which are associated to certain filling factors, ν, and characterized by different physical properties, which can be regarded as different phases in a phase diagram – but cannot be classified via the use of a local order parameter, as in the classical Ginzburg-Landau phase transition theory. At temperatures sufficiently close to zero, these properties can be described by the notion of *topological quantum order* (for an overview, see [30] and [31]).

This subsection presents the basic ideas and concepts of quantum phases and quantum phase transitions. Topological quantum order is described in more detail in the following subsections – Subsection 2.3 and Subsection 2.4.

In the Ginzburg-Landau theory of phase transitions, the phase transitions between different orders in matter are explained by a change of the physical symmetries in the system. The occurrence of such symmetries in matter is described by a parameter of order, which depends on temperature – as, for example, is commonly observed in the transition between ferromagnetic and paramagnetic states in a material. The basic notion of this theory is that a phase transition in a system occurs when it is in the minimum of its free energy, which depends on the parameter of order.

Phase transitions in the systems that exhibit the fractional quantum Hall effect cannot be accurately described by use of the Ginzburg-Landau symmetry-breaking theory, as the same classically recognized symmetry may be observed for two differing phases. The observations of such phases marked the discovery of a new type of order in matter – *topological order* (or *topological quantum order*), named after topological quantum field theory, which was primarily used to describe chiral spin states, also found to exhibit this novel type of order [31]. Nowadays, topological order is described and characterized by several approaches (see Subsection 2.3 and Subsection 2.4), but the systems that exhibit topological order are not yet well understood at temperatures above absolute zero. For quantum computing and quantum information theory, one of the most interesting properties of these systems is their robustness against perturbations [20], as they are described by non-local observables – *topological invariants*.

Quantum phases are phases of matter at zero temperature [32] – they correspond to the degenerate ground states of the Hamiltonians of topologically ordered systems.

Let a quantum spin system on a lattice be described by a Hamiltonian $H(g)$ with an energy gap above the ground state space of the system, where g is a dimensionless coupling parameter, such that g couples only to a conserved quantity – for example, have $H(g) = H_0 + gH_1$, where $[H_0, H_1] = 0$, so that H_0 and H_1 can be diagonalized simultaneously. The eigenfunctions of the Hamiltonian do not depend on the parameter g, even though the corresponding eigenvalues do. This means that there can exist a value $g = g_c$ for which the ground state energy is non-analytic, such that an excited state becomes the ground state (see Figure 2.1). The point of non-analyticity

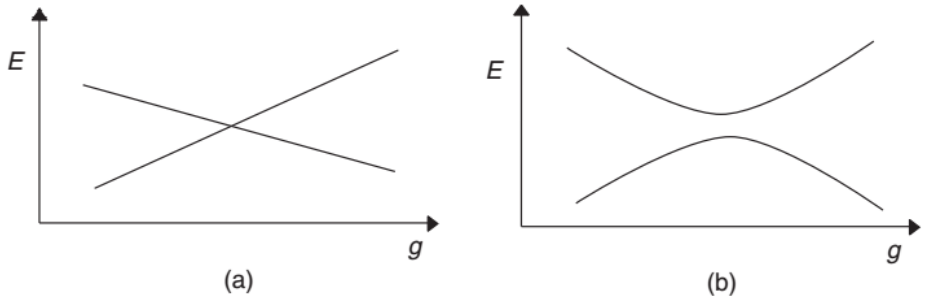


Figure 2.1: A representation of the eigenvalues of the ground state and first excited state of a Hamiltonian $H(g) = H_0 + gH_1$ which depends on a dimensionless coupling constant g, and where H_0 and H_1 commute and are independent of g. Figure *(a)* shows a level-crossing, Figure *(b)* shows an avoided level-crossing. (Image taken from [32].)

in a finite sized lattice presents a level-crossing, where such a point in an infinite lattice may represent either a level-crossing or an avoided level-crossing (see Figure 2.1) in the infinite lattice limit. Any non-analytic point in the ground state energy of an infinite lattice system is identified as a *quantum phase transition* [32]. In other words, for a quantum phase transition to occur, the spectral gap above the ground state of a quantum spin system must close. Such phase transitions may happen naturally at the temperature of absolute zero, where quantum effects that warrant such transitions become predominant. When a system goes through a quantum phase transition, the nature of the correlations in its ground state usually changes.

The quantum phases in which a quantum spin system can be found represent specific *long-range quantum entanglement patterns* – or, topological order. The following subsections provide concise descriptions of topological order.

2.3 Topological Quantum Order

Topologically ordered phases in quantum spin systems are a phenomenon most commonly described by comparison with the classical Ginzburg-Landau theory of ordered phases in materials. Whereas the parameters of order that govern the symmetry states of materials that can be accurately described by the Ginzburg-Landau theory

have been studied in detail and are fairly well understood (such as its polarization, magnetization, crystal lattice deformation, or the wave function of electron pairs in superconductors [33]), the theoretical approach to topologically ordered phases is still not sufficiently conclusive. The concept of topological order is tied to the existence of global variables which are robust to local perturbations of the system's Hamiltonian. This ordering is not yet well understood at temperatures above absolute zero, and it is currently approached from a number of different angles. In this subsection, the description of topological order found in the recent works of Sergey Bravyi, Matthew Hastings, and Spyridon Michalakis [20,34] will be presented in conjunction with the general notions and implications of topological order.

We observe a system of quantum particles distributed on the sites of an N-dimensional lattice Λ of linear dimension L, with periodic boundary conditions, for which its Hilbert space is represented as stated in (2.1). If $\mathcal{S}(r)$ is defined as the set of all blocks $A \subseteq \Lambda$ with linear size r, where $r > 0$ (see Figure 2.2), it is valid that $\mathcal{S}(L) = \Lambda$, and $\mathcal{S}(r) = \emptyset$ for $r > L$. Considering a coarse-grained lattice, such that the

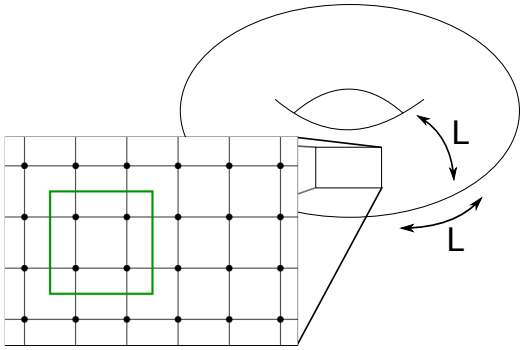


Figure 2.2: An example of particles (shown as dots) distributed on the sites of a 2-dimensional lattice of linear dimension L, with periodic boundary conditions, on a torus. The green square on the enlarged image represents an example of a block A with linear size 2, as described in the paragraph preceeding equation (2.5).

unperturbed Hamiltonian of the system, H_0, involves only interactions between particles inside of blocks $A \in \mathcal{S}(2)$ (an example is shown in Figure 2.2), the Hamiltonian of the system can be written as:

$$H_0 = \sum_{A \in \mathcal{S}(2)} Q_A, \tag{2.5}$$

where Q_A is an interaction with support on A, and with the following properties:

$$Q_A^2 = Q_A, \qquad Q_A Q_B = Q_B Q_A, \qquad \text{for all } A, B \in \mathcal{S}(2). \tag{2.6}$$

The commuting property of the Hamiltonian, as stated in (2.6), imposes a strong restriction on it, and even though this holds as a requirement for the following def-

inition, it is possible to describe topological order in more general terms, as in Subsection 2.4. Defining the Hamiltonian H_0 to have zero ground state energy, and a finite spectral gap between the ground state energy and the energy of the first excited state, the projectors onto the ground subspace and the excited subspace of H_0 can be defined as P and Q, respectively, as:

$$P = \prod_{A \in \mathcal{S}(2)} (I - Q_A), \qquad Q = I - P, \tag{2.7}$$

where I is the identity operator. Similarly, the local versions of these operators, for any block $B \in \mathcal{S}(r \geq 2)$ are:

$$P_B = \prod_{\substack{A \in \mathcal{S}(2) \\ A \subseteq B}} (I - Q_A), \qquad Q_B = I - P_B. \tag{2.8}$$

To define the existence of topological order, the following two properties described in detail in [20] (commonly known as *TQO-1* and *TQO-2*) need to hold true, assuming that there exists an integer $L^\star \geq \alpha L$ for some constant $\alpha > 0$ and sufficiently large L:

1. **TQO-1:** For any block $A \in \mathcal{S}(r)$ with $r \leq L^\star$,

$$PO_AP = cP, \qquad c \in \mathbb{C}, \tag{2.9}$$

   for any operator O_A acting on A.

2. **TQO-2:** For blocks $A \in \mathcal{S}(r)$ with $r \leq L^\star$ and $B \in \mathcal{S}(r+2)$, where B is the block that contains A and all nearest neighbors of the sites in A, define reduced density matrices $\rho_A = \mathrm{Tr}_{A^c}(P)$ and $\rho_A^{(B)} = \mathrm{Tr}_{A^c}(P_B)$, where $A^c = \Lambda \setminus A$. Then

$$\ker \rho_A = \ker \rho_A^{(B)}. \tag{2.10}$$

The integer $L^\star$ is chosen to depend on the size of the lattice, L, in a linear fashion so that it defines a length scale for local operations in the system. Its dependence on L ensures that the stated properties of a system with topological order do not refer only to strictly local subsystems, but describe global properties of the system instead.

The first condition (TQO-1) is commonly thought of as the chief definition of topological order, and it states that it is impossible for a local operator to induce a transition between orthogonal ground states of the system, or to distinguish between two such states. Consequently, it is coloquially known as the condition of local indistinguishability of ground states of a topologically ordered system. From the statement of the condition, it straightforwardly follows that for a system with an orthonormal basis of ground states $\{|\psi_i\rangle\}$, the local operator O_A acts in the following manner:

$$\langle \psi_i | \, O_A \, | \psi_j \rangle = \begin{cases} \text{const.} & \text{if } \; i = j, \\ 0 & \text{if } \; i \neq j. \end{cases} \tag{2.11}$$

Therefore, it can be said that any information encoded in the ground state space of a topologically ordered system is not affected by local perturbations.

The second condition defining the existence of topological order (TQO-2) states that the projectors P_B and P must act equivalently on the subset $A \subset B$, certifying that the local ground subspace of the system P_B will be consistent with its global ground subspace P, on subsets which are sufficiently far from the boundary of B. This consistency may be violated in the cases where the observed region possesses a non-trivial topology, e.g. a hole – then, the local ground subspace may include areas with a non-trivial topological charge inside the hole, whereas this will not be the case for the global ground subspace.

2.4 Other Definitions of Topological Quantum Order

In Subsection 2.3, one approach to describing a system exhibiting topological order was presented. Generally speaking, topological order describes equivalence classes in a system with long-range entanglement. However, the efforts to define topological order are by no means complete; many different methods are used to tackle this problem. In this section, additional prominent approaches will be concisely described, as also compiled in [30].

Topological quantum order can be described through the phases of Hamiltonians of topologically ordered systems.

For a local quantum system whose Hamiltonian has a spectral gap above the ground state space, and a smooth dependence on a parameter g, $H(g)$, let $|\Phi(g_i)\rangle$ be a ground state of $H(g_i)$. For the ground state average of any local operator O, $\langle O\rangle(g)$ (local in the same sense as the local operator O_A in TQO-1 (2.9)), to be a smooth function, the Hamiltonian of the system must remain gapped for all g. If the gap closes for some g_c, there exists a local operator such that its ground state average reaches a singularity at g_c. Defining $|\Phi(0)\rangle$ as the ground state of $H(0)$ and $|\Phi(1)\rangle$ as the ground state of $H(1)$, it is possible to describe a quantum phase as an equivalence class. If there exists a smooth path connecting the Hamiltonians $H(0)$ and $H(1)$, $H(0 \leq g \leq 1)$, such that no phase transition exists in the system along the path, then the two ground states $|\Psi(0)\rangle$ and $|\Psi(1)\rangle$ belong to the same phase. If all excitations above $|\Psi(0)\rangle$ have a gap, then for small enough g, the systems described by $H(g)$ is also gapped, and $|\Psi(g)\rangle$ is in the same phase as $|\Psi(0)\rangle$. Therefore, for gapped systems, a quantum phase transition can happen if and only if the energy gap closes through an adiabatic evolution. The reverse is also true: if two ground states of a system that has a spectral gap above the ground state energy, $|\Psi(0)\rangle$ and $|\Psi(1)\rangle$, are in the same phase, there always exists a family of Hamiltonians $H(g)$, such that the energy gap is finite for all $g \in [0, 1]$, and $|\Psi(0)\rangle$ and $|\Psi(1)\rangle$ are ground states of $H(0)$ and $H(1)$, respectively.

A system with non-trivial topological order is one that can be described with a Hamiltonian that doesn't lie in the same phase as the Hamiltonian with a product state as its ground state.

An alternate, equivalent way to describe topological order found in quantum states can be reached by employing a relation between quantum phases and quantum circuits with finite depth.

A piece-wise local unitary operator with range l is defined as $U_{pwl} = \prod_i U_i$, where U_i is a set of unitary operators that act on disjunct regions with size smaller than some finite number l. In this case, a quantum circuit with depth M is the product of M piece-wise local unitary operators, such that $U_{circ}^{M} = \prod_i U_{pwl}^{(i)}$. A visual representation of a quantum circuit can be found in Figure 2.3. Two ground states are said to belong to the same phase if and only if one can be transformed into the other by use of a quantum circuit with finite depth. A phase is topologically trivial if it contains a product state.

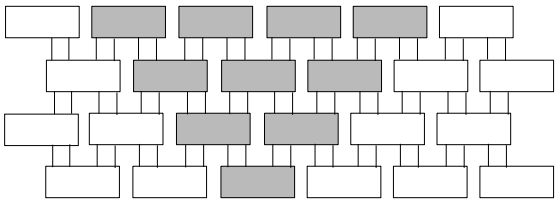


Figure 2.3: A visual representation of a quantum circuit. The rectangles represent unitary operators on regions of finite size l, and the lines represent their multiplication. The shading explains the causality in the system.

In [30] it is shown that the two definitions of topological quantum phases explained in this subsection are related in the following manner. For two gapped Hamiltonians that are in the same phase, $H(0)$ and $H(1)$, it follows from quasi-adiabatic continuation (see Subsection 4.1) that their ground states, respectively $|\Psi(0)\rangle$ and $|\Psi(1)\rangle$, belong to the same phase if and only if they are related by a local unitary evolution. A local unitary evolution is an unitary operation generated by the finite time evolution of a local Hamiltonian. Stated clearly,

$$|\Psi(1)\rangle \sim |\Psi(0)\rangle \text{ iff } |\Psi(1)\rangle = \mathcal{T}\left[e^{-i\int_0^1 \mathrm{d}g\tilde{H}(g)}\right]|\Psi(0)\rangle\,, \tag{2.12}$$

where $\mathcal{T}$ is the path-ordering operator, and $\tilde{H}(g) = \sum_i O_i(g)$ is a sum of local Hermitian operators. As any local unitary evolution can be approximately simulated with a constant depth quantum circuit, it follows that:

$$|\Psi(1)\rangle \sim |\Psi(0)\rangle \text{ iff } |\Psi(1)\rangle = U_{circ}^{M}|\Psi(0)\rangle\,, \tag{2.13}$$

where M is a constant that is independent of the size of the system.

For a quantitative measure of the global entanglement in the ground state of a topologically ordered system, topological entanglement entropy is commonly used [35]. For a system exhibiting topological order, topological entanglement entropy is a constant that provides a measure of the long-range entanglement and the von Neumann entropy of the system, which quantifies the entanglement of a bipartite pure state. For more on this topic, see [35] and [36].

2.5 Experimental Realizations of Topologically Ordered Systems

The prospects of possible experimental realizations of topologically ordered systems have slowly become a promising lead toward the construction of fault-tolerant quantum computers. In this subsection, examples of the current reaches and prospects of the experimental achievements in the field of topologically ordered systems will be presented, with an accent on their potential application in topological quantum computing.

Firstly, the relevant systems that naturally exhibit topological quantum order will be presented, followed by an overview of the experimental approaches to the contruction of topologically ordered systems.

Historically, the most relevant type of systems that naturally exhibit topological order is that in which the *fractional quantum Hall effect* can be observed (for example, see [37], or the original publications [28, 29]). In the fractional quantum Hall effect experiments, a two-dimensional electron gas may be created on the interface of two different semiconductors. Under strong magnetic fields, the Hall conductance of the two-dimensional system shows quantized plateaus that can be found at specific fractional values of e^2/h, where e is the elementary electron charge, and h is the Planck constant. The discovery and partial interpretation of these plateaus earned Robert Laughlin, Horst Störmer, and Daniel Tsui the Nobel Prize in Physics in 1998 (see [16]), and the full explanation of the mechanism by which the fractional quatum Hall effect comes to be contiues to be an unsolved problem.

Other notable examples include Majorana wires [38], $(p_x + ip_y)$-type superconductors [39, 40] and superfluids [41]. An overview can be found in [42] and [43].

On the other hand, there exists a wave of different approaches to specifically engineer microscopic interactions in systems that would then exhibit controlled topological order that would enable quantum computation, such as with *optical lattices* (a comprehensive introduction to these can be found in [44]). Several academic and technological research groups have been tackling this challenge; the following paragraphs give a short introduction to some of these groups and their successes.

The first example is the group of John Martinis (currently affiliated with the University of California, Santa Barbara, and Google Inc.), where so-called Xmon qubits

are created as part of a surface code (for an introduction to surface codes, see Subsection 3.3 and Subsection 5.2). Their recent relevant publications include [45] and [46].

The group of Jerry Chow of the IBM Corporation has built a surface code that consists of one sole plaquette, by use of Josephson junctions [47].

At the University of Innsbruck, the group of Rainer Blatt has constructed qubits via ion traps and performed quantum error correction (for an introduction to quantum error correction, see Subsection 3.1) on the smallest possible example of the two-dimensional color code [48] (the color code is a stabilizer code — see Subsection 3.2 — not unlike the toric code — Subsection 3.3 — the introduction to which can be found in [49]).

Many groups centered around the Delft University of Technology (also funded by the Intel Corporation) are working on the experimental realization of surface codes; the group of Leonardo DiCarlo has recently reported on the use of stabilizer measurements on superconducting qubits [50].

3 Fault-Tolerant Quantum Computation

Inasmuch as quantum mechanics has become an invaluable resource in the problem-solving of modern physics, the theory of quantum information processing does have its practical limitations, with ones of the most notable problems being those of quantum decoherence and the instability of the used systems to perturbation. Because of its quantum nature, the information that is encoded in quantum spin systems or transferred through quantum circuits suffers from a relevant loss of reliability when compared to its classical equivalent. For this reason, it is of immense use to develop methods of fault-tolerant quantum computation. These methods provide a measure to the extent of information lost, as well as techniques to increase its accurate preservation when transferred or changed.

In this section, the basic principles of quantum error correction are presented (Subsection 3.1), with the accent on the manner in which they apply to the stabilizer formalism (Subsection 3.2). Finally, the toric code is introduced (Subsection 3.3), and its stability under perturbation is discussed (Subsection 3.4).

3.1 Quantum Error Correction

Error correction is a concept in information processing that describes the methods used to reliably protect some information being transferred or stored in the presence of noise. Unlike classical information, quantum information suffers a great threat of becoming corrupted because of the inherent properties owing to the principles of quantum mechanics. For example, there exists the risk stemming from decoherence, but also, potential leakage errors — as quantum information tends to be encoded in systems that can attain more than the two levels of the binary system used for classical information — run the risk of becoming relevant. This subsection covers the basics of the theory of *quantum error correction*, its relevance and utility. Unless otherwise stated, the chief literature source used in this section will be the comprehensive textbook on quantum computation and quantum information by Michael A. Nielsen and Isaac L. Chuang [1].

In the theory of error correction, the main challenge consists of two steps – encoding some state containing the desired information, followed by decoding it successfully. In classical computation this does not present a great practical problem, as the failure rate of modern binary systems is negligibly small (less than one error in 10^7 operations [1]). Because the probability of noise affecting classically stored or transferred information is so small, the methods of preserving it are standardly quite simple. A modest example is the *repetition code*. In this coding scheme, the information is encoded by increasing the number of bits in the string by repeating

the original bit (at least two times), as can be seem in Table 3.1. These bit strings

Original bit	Encoded bit
0	000
1	111

Table 3.1: The standard three bit repetition code.

are commonly called the *logical* 0 and *logical* 1. If a bit in the encoded message gets changed (for example, from 000 to 100), the message will get decoded following the principle of majority voting – choosing the decoded bit to be the one whose copies appear most often in the encoded message, given that the probability of a bit flip is not too high, which is to be expected of any reliable information processing scheme (for the case of 100 that would be 0, as it appears twice, compared to the one appearance of the bit with the error, 1). Naturally, if the majority of the bits in the encoded string experience a bit flip due to the presence of noise, the decoded bit will be wrong, but for classical computation this is not a likely outcome, and it is easily tended to by increasing the number of copies of the original bit in the encoding scheme. As for quantum information, the complexity and relevance of error correcting codes increases because of the principles of quantum mechanics that govern the processing mechanisms of quantum information.

Compared to the straightforward classical error correction, which relies on measurement to decode information, quantum error correction suffers the issue of destroying the quantum state upon observation, which makes the direct recovery of the original state from the encoded state impossible. In addition, simple repetition codes cannot be implemented in quantum computation because of the *no-cloning theorem* [51], which testifies to the impossibility of duplicating a quantum bit. Finally, the errors encountered in quantum computation are continuous, which makes an error correcting procedure much more difficult than the one used on classical information. A simple example of a quantum error correction code is the three qubit bit flip code, which will be presented in short in the following paragraph.

The *bit flip code* may be used in a situation where there is a desire to protect the information carried by qubits that are being sent through a bit flip channel, which performs a bit flip on them with a probability p – i.e. $|\psi\rangle \rightarrow \sigma^x |\psi\rangle$, where $\sigma^x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the Pauli X operator, while leaving them unchanged with the probability of $1-p$. If the basis states of a qubit are defined as in Table 3.2, where $|0_L\rangle$ and $|1_L\rangle$ denote the *logical* $|0\rangle$ *and* $|1\rangle$ *states* (as opposed to the physical ones!), the state of a single qubit $|\psi\rangle = a|0\rangle + b|1\rangle$ can be encoded as $a|000\rangle + b|111\rangle$. Following this, the encoded state is passed through the bit flip channel, after which the error correcting decoding procedure can be implemented. This is done in two steps: first an error-detection (also called *syndrome diagnosis*) is performed, and after that the state can go through re-

Original state	Encoded state
$\|0\rangle$	$\|0_L\rangle \equiv \|000\rangle$
$\|1\rangle$	$\|1_L\rangle \equiv \|111\rangle$

Table 3.2: The three qubit flip code.

covery. Similarly to the classical majority voting implemented on the repetition code, what is considered here is a state in which an error — the bit flip — has occurred on at most one of the qubits. A measurement is performed via four projection operators, the result of which gives a specific *error syndrome*, as seen in Table 3.3. For example,

Projection operator	Error syndrome
$P_0 \equiv \|000\rangle\langle 000\| + \|111\rangle\langle 111\|$	no error
$P_1 \equiv \|100\rangle\langle 100\| + \|011\rangle\langle 011\|$	bit flip on the first qubit
$P_2 \equiv \|010\rangle\langle 010\| + \|101\rangle\langle 101\|$	bit flip on the second qubit
$P_3 \equiv \|001\rangle\langle 001\| + \|110\rangle\langle 110\|$	bit flip on the third qubit

Table 3.3: The three qubit flip code error-detection projectors, with corresponding error syndromes.

if the error occurred on the third qubit, the corrupted state is $a\,|001\rangle + b\,|110\rangle$, and $\langle\psi|\,P_3\,|\psi\rangle = 1$, with the corrupted state not being changed. The projective measurement gives merely the syndrome — the information on which qubit experienced a bit flip — but does not divulge anything about the original state. The principle of the recovery process involves simply flipping the qubit that is recognized as having the error, which gives the original state again: $a\,|000\rangle + b\,|111\rangle$.

Many kinds of errors can occur to qubits, so this procedure is far from uniform, but it is a clear example of a quantum error correcting code, and others process the state information in a similar manner.

In general, any quantum error correction procedure on a quantum state can be summarized as follows:

1. Encoding the quantum state into a quantum error correcting code, defined as a subspace C of some larger Hilbert space, by a unitary operation, and commonly referred to as the *code space*, or *code subspace*.

2. The system is exposed to noise.

3. A syndrome measurement is performed.

4. A recovery procedure is performed depending on the error syndrome, returning the system to the original state of the code.

To ensure a reliable measurement of the error syndrome, the chosen subspaces of the total Hilbert space need to be orthogonal, and they must be defined as the undeformed versions of the original code space. This is crucial so that the errors that occur on the system map the encoded states to orthogonal states, thus making the recovery procedure more successful.

As the possible errors that may occur in quantum systems are many and varied, and the same holds true to the employed error correcting codes, the full quantum error correction procedures as such are commonly investigated in general terms. For a system influenced by some noise that is described by a quantum operation $\mathcal{E}$, and whose recovery error correction operation can be written as a trace-preserving quantum operation $\mathcal{R}$, the error correction is considered successful if for any state ρ with support in the code subspace C, the following is true:

$$(\mathcal{R} \circ \mathcal{E})\, \rho \approx \rho. \tag{3.1}$$

The presented expression would be a trivial equality if the noise operator $\mathcal{E}$ were a unitary operator – this may not always be the case.

Additionally, the conditions that show whether a quantum error correcting code protects against some noise $\mathcal{E}$ can be summarized in the following theorem.

Theorem 3.1. *Let C be a quantum code, P the projector onto C, and $\mathcal{E}$ a quantum operation with operation elements — errors — $\{E_i\}$. An error correction operation $\mathcal{R}$ that corrects $\mathcal{E}$ on C exists if and only if the following equality is true for some Hermitian matrix γ of complex numbers:*

$$P E_i^\dagger E_j P = \gamma_{ij} P. \tag{3.2}$$

Then, the set $\{E_i\}$ is called a correctable set of errors.

A proof of this theorem and further details on specific quantum error correcting codes can be found in [1].

A notable example of quantum error correcting codes is the widely used CSS-type (Calderbank-Shor-Steane) quantum code family (see [52] and [53]).

3.2 Stabilizer Formalism

The *stabilizer formalism* is a widely used and very powerful setting utilized to investigate various quantum error correcting codes in quantum mechanics. As the name implies, it relies on the property of the stability of certain quantum states when some operations are performed on them. In simple terms, for a state $|\psi\rangle$, if by acting on it with an operator P it is left unchanged, $P|\psi\rangle = |\psi\rangle$, it can be said that the state $|\psi\rangle$ is *stabilized* by the operator P. The approach of working with operators that act in a quantum state space instead of the states themselves often comes out as more

convenient, presenting problems that are otherwise difficult to grasp in a clear and concise manner. This subsection covers the basic definitions and descriptions of the facets of the stabilizer code theory, as well as the conditions set on the construction of stabilizer codes, laying out the base to enable the introduction of the toric code in Subsection 3.3. Unless otherwise noted, the main reference for this subsection is [1].

Stabilizer codes are quantum error correcting codes which utilize the stabilizer formalism in order to reach conclusions about the effect of noise in quantum information processing.

The key idea of stabilizer codes lies in the utilization of the Pauli groups G under matrix multiplication, defined as

$$G \equiv \{\pm\mathbb{I}, \pm i\mathbb{I}, \pm\sigma^x, \pm i\sigma^x, \pm\sigma^y, \pm i\sigma^y, \pm\sigma^z, \pm i\sigma^z\} \tag{3.3}$$

for a single qubit, where $\sigma^x, \sigma^y, \sigma^z$ are the Pauli operators. The Pauli group for n qubits consists simply of all n-fold tensor products of the elements of the Pauli groups G of the considered qubits.

Definition 3.1. *Let S be a subgroup of the Pauli group for n qubits, G_n. For a vector subspace V_S of all n-qubit states for which it is valid that $(\forall\psi \in V_s)\wedge(\forall S_\xi \in S) : S_\xi |\psi\rangle = |\psi\rangle$, the subgroup S is called the stabilizer of the space V_S.*

The motivation of this definition is easy to see – every element of the group V_S is stable when acted on by the elements of S. Therefore, it can be said that V_S is the vector space stabilized by the group S.

The choice of subgroup of the Pauli group is not arbitrary, as not all subgroups can stabilize a non-trivial vector space – e.g. $(-\mathbb{I})|\Psi\rangle = |\Psi\rangle$ gives only $|\Psi\rangle = 0$. Two trivial conditions arise: first, the subgroup cannot contain $-\mathbb{I}$, and second, the elements of the stabilizer group must commute. It follows that, for a stabilizer group S that is a subgroup of the Pauli group for an n-qubit system, G_n, all of the generators of the subgroup, $g_1, \ldots, g_n$, will commute. The generators are chosen so that they exhibit mutual independence – removing one generator would make the generated group smaller.

The final important property of stabilizer codes touches upon the dimensionality of the stabilized vector spaces, and is covered by the following proposition, stated in part in [1], and expanded here.

Proposition 3.1. *Let $S = \langle g_1, \ldots, g_{n-k}\rangle$ be a stabilizer group generated by $n-k$ independent and commuting $g_i \in G_n$, such that $-\mathbb{I} \notin S$. Then the vector space it stabilizes, denoted as V_S, is 2^k-dimensional.*

Proof. Let $x = (x_1, \ldots, x_{n-k})$ be a $(n-k)$-dimensional vector, with $x_i \in \mathbb{Z}_2$. Define

$$P_S^x \equiv \frac{\prod_{j=1}^{n-k}(\mathbb{I} + (-1)^{x_j} g_j)}{2^{n-k}}. \tag{3.4}$$

If a state is decomposed into a sum of vectors on the $+1$ and -1 eigenstate of a generator g_i, such that $|\Psi\rangle = |\Psi_+\rangle + |\Psi_-\rangle$ and $g_i |\Psi_\pm\rangle = \pm |\Psi_\pm\rangle$, it follows directly that $(\mathbb{I}+g_i)/2$ is a projector onto the $+1$ eigenspace of the generator g_i. Then, $P_S^{(0,\,\ldots,\,0)}$ is the projector onto V_S.

Proposition 10.4 in [1] shows that for any $g_j \in G_n$ there exists $g \in G_n$ such that $g g_j g^\dagger = -g_j$, and $g g_i g^\dagger = g_i$ for all $i \neq j$. It can be shown that for each x there exists $g_x \in G_n$ such that

$$g_x P_S^{(0,\,\ldots,\,0)} g_x^\dagger = P_S^x. \tag{3.5}$$

If g_x is defined as

$$g_x = \prod_{j:\ x_j=1} \tilde{g}_j, \quad \tilde{g}_j \in G_n, \tag{3.6}$$

then

$$\begin{aligned} g_x P_S^{(0,\,\ldots,\,0)} g_x^\dagger &= \frac{\prod_{i=1}^{n-k} g_x \left(\mathbb{I} + g_i\right) g_x^\dagger}{2^{n-k}} \\ &= \frac{\prod_{i=1}^{n-k} \left(\mathbb{I} + g_x g_i g_x^\dagger\right)}{2^{n-k}} \end{aligned} \tag{3.7}$$

In the case in which $x_i = 0$, it is easy to see that $g_x g_i g_x^\dagger = g_i$. If $x_i = 1$, it follows:

$$\begin{aligned} g_x g_i g_x^\dagger &= \Big(\prod_{j_1<i} \tilde{g}_{j_1}\Big) \tilde{g}_i \Big(\prod_{j_2>i} \tilde{g}_{j_2}\Big) g_i \Big(\prod_{j_3>i} \tilde{g}_{j_3}^\dagger\Big) \tilde{g}_i \Big(\prod_{j_4<i} \tilde{g}_{j_4}^\dagger\Big) \\ &= \Big(\prod_{j_1<i} \tilde{g}_{j_1}\Big) \tilde{g}_i g_i \tilde{g}_i^\dagger \Big(\prod_{j_4<i} \tilde{g}_{j_4}^\dagger\Big) \\ &= -\Big(\prod_{j_1<i} \tilde{g}_{j_1}\Big) g_i \Big(\prod_{j_4<i} \tilde{g}_{j_4}^\dagger\Big) \\ &= -g_j, \end{aligned} \tag{3.8}$$

where $g_x^\dagger g_x = \mathbb{I}$ has been taken into consideration.

As (3.5) is valid, $\dim(P_S^x) = \dim(V_S)$. Additionally, for distinct x, P_S^x are mutually orthogonal. Then,

$$\begin{aligned} \sum_x P_S^x &= \sum_{x_1} \sum_{x_2} \cdots \sum_{x_{n-k}} P_S^x \\ &= \sum_{x_2} \cdots \sum_{x_{n-k}} \left(\frac{\mathbb{I}+g_1}{2} + \frac{\mathbb{I}-g_1}{2}\right) \prod_{l=2}^{n-k} \frac{\mathbb{I} + (-1)^{x_l} g_l}{2} \\ &= \sum_{x_2} \cdots \sum_{x_{n-k}} \prod_{l=2}^{n-k} \frac{\mathbb{I} + (-1)^{x_l} g_l}{2} \\ &= \mathbb{I}, \end{aligned} \tag{3.9}$$

where $\mathbb{I}$ is a projector onto a 2^n-dimensional space.

As the projectors P_S^x are orthogonal and of the same dimension as V_S, it follows that the dimension of V_S is 2^k. □

The dimension of the stabilized vector space V_S is equal to 2 to the power of the difference between the number of qubits in the system and the number of independent generators of the stabilizer subgroup, and 2^k orthonormal vectors in V_S may act as logical computational basis states.

The remainder of this subsection will cover the practical dealings with stabilizer codes themselves, and provide an example for their use.

Definition 3.2. *An [n,k] stabilizer code is the vector space V_S that is stabilized by a subgroup S of G_n such that $-\mathbb{I} \notin S$, and S has $n-k$ independent and commuting generators, $S = \langle g_1, \ldots, g_{n-k} \rangle$. This code is denoted by $C(S)$.*

If there exists an error $E \in G_n$ acting on a stabilizer code $C(S)$ and if E anticommutes with an element of the stabilizer, the code is transformed into an orthogonal subspace, and the error may be able to be detected and corrected.

Definition 3.3. *If an error $E \in G_n \setminus S$ acting on a stabilizer code $C(S)$ commutes with all of the elements of S, the set of $E \in G_n$ such that $Eg = gE$ for all $g \in S$ is called the centralizer of S in G_n, and is denoted by $Z(S)$.*

The following theorem provides a statement on the error-correction conditions for stabilizer codes, and a proof for it can be found in [1].

Theorem 3.2. *Let S be the stabilizer for a stabilizer code $C(S)$. Suppose $\{E_j\}$ is a set of operators in G_n such that $E_j^\dagger E_k \notin Z(S) - S$ for all j and k. Then $\{E_j\}$ is a correctable set of errors for the code $C(S)$.*

This introduction to correctability of errors in stabilizer codes is analogous to the conclusion in Theorem 3.1, and it leads to a definition of distance for a quantum code. The *weight* of an error $E \in G_n$ is the number of terms in the tensor product which are not equal to the indentity operator – then, the *distance* of a stabilizer code $C(S)$ is defined as the minimum weight of an element of $Z(S) - S$.

Definition 3.4. *If $C(S)$ is an $[n,k]$ stabilizer code with distance d, then it is denoted as an $[n,k,d]$ code.*

As an example of a stabilizer code, the five qubit code is a valuable example – a five qubit code is the smallest possible size for a setup in which any error on a single qubit that is encoded in the system can be detected and corrected. The generators of the five qubit code are given in Table 3.4.

3.3 Toric Code

The *toric code* is an example of a stabilizer code and a toy model for an interacting quantum spin system, which is widely popular for the simplicity of derivation

Generator	Operator
g_1	$XZZXI$
g_2	$IXZZX$
g_3	$XIXZZ$
g_4	$ZXIXZ$
$\bar{Z}$	$ZZZZZ$
$\bar{X}$	$XXXXX$

Table 3.4: The four generators and logical Z and X operators for the five qubit code. The logical operators act as the operator equivalent of logical basis states, as described for classical codes in Subsection 3.1.

of the values of various relevant observables, stemming from the commutativity of its stabilizer operators, and its intuitive representation of a system of qubits. It was developed by Alexei Kitaev in the late 1990's (a summary can be found in [54]), and it quickly became an indispensable tool in the investigation and utilization of topologically ordered quantum spin systems.

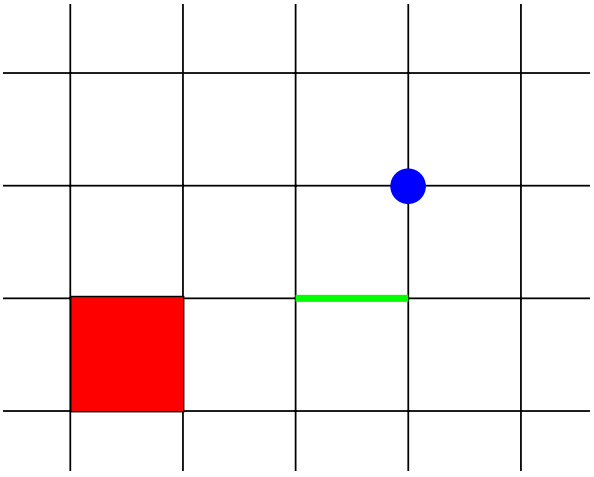


Figure 3.1: The elements of the toric code; it consist of a square lattice with periodic boundary conditions. A plaquette is colored red, a vertex is colored blue, and one of the edges is colored green.

A $k \times k$ toric code TOR(k) is an $[k^2, 2, k]$ stabilizer code defined on a two-dimensional square lattice on a torus, Λ. Each intersection of the lines on the lattice is called a *vertex* (or a *star*), and each face is called a *plaquette*, whereas the connecting lines between the vertices are called the *edges* – see Figure 3.1. Each edge of the lattice contains one half-spin degree of freedom. The stabilizer operators of the toric code are associated with its vertices v and plaquettes p, and they are of the following form:

$$A_v = \prod_{j \in \text{vertex}(v)} \sigma_j^x, \qquad B_p = \prod_{j \in \text{boundary}(p)} \sigma_j^z, \tag{3.10}$$

where vertex(v) and boundary(p) each represent the edges that are connected to the vertex v and plaquette p, respectively. The operators σ_j^x and σ_j^z are Pauli operators acting on an edge j, such that $\sigma^x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \sigma^z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. See Figure 3.2 for a visual representation on the lattice.

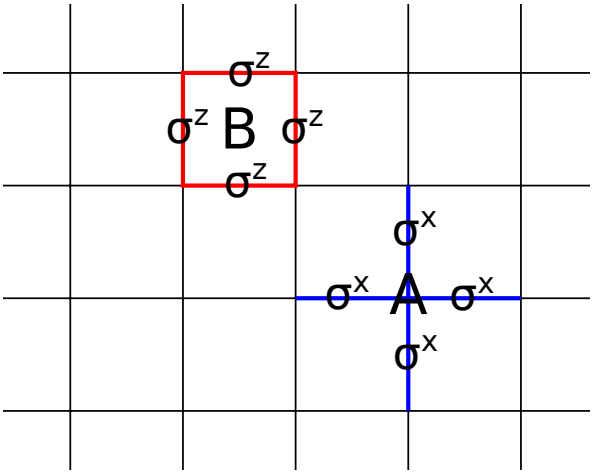


Figure 3.2: A visual representation of the stabilizer operators A and B in the toric code (see equation (3.10)); σ^x and σ^z are Pauli operators acting on the spin-½ degrees of freedom located on the edges. The colored edges denote the edges that are related to the corresponding stabilizer.

In a $k \times k$ lattice, there exist $n = 2k^2$ edges. Defining $\mathcal{N}$ to be the Hilbert space of all qubits, its protected code subspace $\mathcal{L} \subseteq \mathcal{N}$ is then:

$$\mathcal{L} = \{|\xi\rangle \in \mathcal{N} : \quad A_v |\xi\rangle = |\xi\rangle\,,\ B_p |\xi\rangle = |\xi\rangle\,, \quad \forall\ v, p\}\,. \tag{3.11}$$

The stabilizer operators A_v and B_p commute with each other, because each vertex and boundary have either 0 or 2 common edges (follows directly from the Pauli operator anticommutation relations); these operators are Hermitian, and have eigenvalues $+1$ and -1.

Additionally, the toric code can be used to represent an interacting quantum spin system, which makes it an indispensable tool for quantum information theory in general. This system exhibits topological order, as defined in Subsection 2.3 (particularly, note (2.5) and (2.6) – the stabilizers of the toric code act locally on the spins on the edges of the lattice, and the spins are involved with a finite number of stabilizers).

To find the dimensionality of the subspace $\mathcal{L}$, it is possible to merely observe the conditions on the stabilizer operators,

$$\prod_{v \in \Lambda} A_v = 1, \qquad \prod_{p \in \Lambda} B_p = 1, \tag{3.12}$$

which constrain the number of existing independent stabilizer operators from $n = 2k^2$ to $m = 2k^2 - 2$. From [55] or the proven Proposition 3.1 it follows that the

dimensionality of $\mathcal{L}$ is $2^{n-m} = 4$. This means that two qubits of quantum information can be stored in a standard toric code. In a more intuitive approach, defining the algebra of all linear operators on the subspace $\mathcal{L}$ as $\mathbf{L}(\mathcal{L})$, and the algebra of all operators generated by the stabilizer operators as $\mathcal{F} \subseteq \mathbf{L}(\mathcal{N})$, it can be stated that, for the algebra $\mathcal{G}$ of all the operators that commute with the stabilizer operators, it is valid that $\mathbf{L}(\mathcal{L}) \cong \mathcal{G}/\mathcal{I}$, where $\mathcal{I} \subset \mathcal{G}$ is the ideal generated by $A_v - 1$ and $B_p - 1$. The algebra $\mathcal{G}$ is generated by the operators that can be represented as:

$$Z = \prod_{j \in c} \sigma_j^z, \qquad X = \prod_{j \in c'} \sigma_j^x, \tag{3.13}$$

where the operator Z is a product of Pauli σ^z operators acting on qubits on a "loop" of the lattice, c, and the operator X is a product of Pauli σ^x operators acting on qubits on a path on the dual lattice, c', i.e. a "cut". If a loop or a cut is contractible (i.e. if it can be smoothly deformed into a point), its operator can be trivially written as a product of stabilizer operators. However, if it is non-contractible, i.e. a loop or cut that wraps around the torus, it can be considered as non-trivial.

In the standard toric code (defined on a surface with genus 1 and with no defects), there exist four non-trivial loops, X_1, X_2, Z_1, and Z_2 (see Figure 3.3). These act as *logical operators* (a definition equivalent to that of logical basis states – see Subsection 3.1) and encode a total of two qubits of information in the system. Each qubit is encoded by two non-commuting operators – X_1 and Z_2, and X_2 and Z_1. These operators anticommute and form a Pauli algebra on the code space of the system.

The Hamiltonian of the toric code can be written as:

$$H_0 = -\sum_{v \in \Lambda} A_v - \sum_{p \in \Lambda} B_p, \tag{3.14}$$

where the stabilizers are summed over all vertices and plaquettes of the system. The ground state of the Hamiltonian acts as the protected subspace of the system, $\mathcal{L}$, and the Hamiltonian is easy to diagonalize, since all of the stabilizers commute with each other. The ground state space of the Hamiltonian corresponds to the $+1$ eigenspace of all of the stabilizers in the system. The excited states of the system are separated from the ground state by an energy gap ΔE, where $\Delta E \geq 2$, because of the difference between the eigenvalues of the stabilizers ($+1$ and -1), although in practice, this quantity becomes doubled because there cannot exist only one quasiparticle excitation in an otherwise standard toric code.

The errors in the toric code are induced by acting on the spin-½ degrees of freedom that are on the edges of the lattice with Pauli operators – in further text, this will be referred to simply as acting on an edge with Pauli operators. Acting on an edge with a Pauli σ^x operator will create excitations on the neighboring plaquettes,

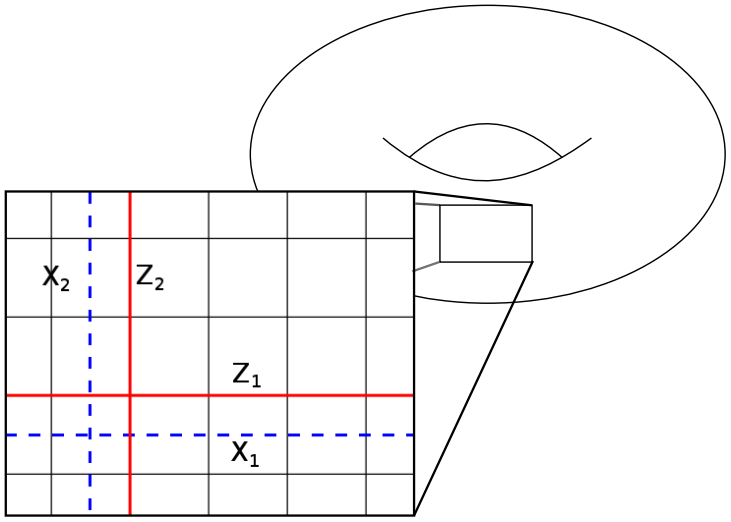
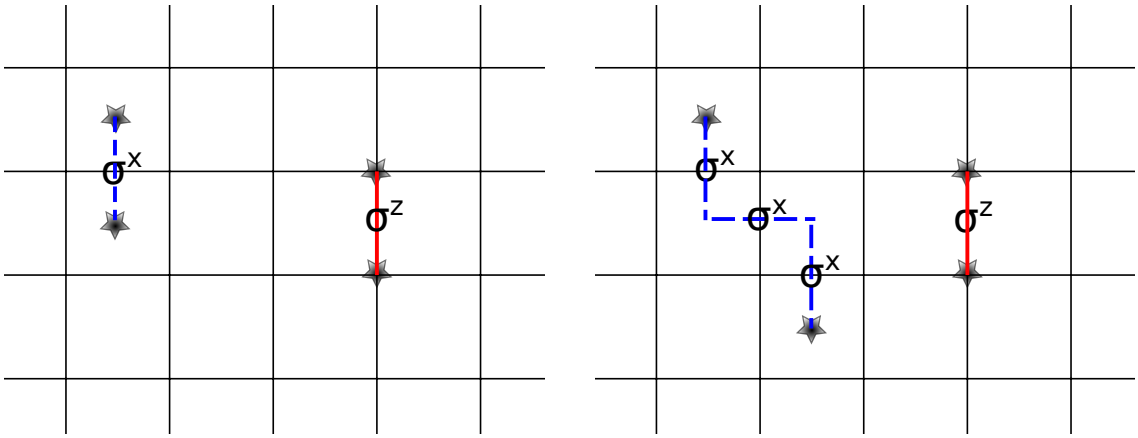


Figure 3.3: A view of the logical operators in the toric code, as seen on a torus. The full red lines represent the edges that support the Pauli σ^z operators that are part of the Z logical operators, the edges that are crossed by the dashed blue lines on the dual lattice represent those that support the Pauli σ^x operators that are part of the X logical operators (see equations (3.13)).

whereas acting on an edge with a Pauli σ^z operator will do the same for the connecting vertices. These excitations change the eigenvalues of corresponding stabilizers, from $+1$ to -1, and they exhibit behavior equivalent to that of toric code *anyons* (see the Introduction for an overview). As acting on an edge adjacent to an excitation with an operator of the same type as the one that created the excitation returns the excited plaquette or vertex to its starting state (because their product commutes with the stabilizer), these anyons can effectively travel through the toric code, connected by a string of Pauli operators (see an example in Figure 3.4).

Anyons on vertices are better known as those of the electric, or *e* type, while those on plaquettes are referred to as the magnetic, flux, or *m* type. Trivial loops of strings of operators that connect two excitations that may have moved on the lattice cause the two excitations to annihilate into vacuum, but non-trivial ones — such as those whose path includes a crossing of a σ^x and a σ^z operator, which is known as *braiding* — may cause the excitations to exhibit anyonic statistics, acquiring a phase to their fusion result.

A good overview of the mathematical structure of anyonic systems and the anyonic fusion rules can be found in the Quantum Computation notes by John Preskill [19].



(a) The Pauli σ^x operator creates excitations (shown here as stars) on the adjacent plaquettes, and the σ^z on the adjacent vertices.

(b) The excitations can travel from one plaquette or vertex to the neighboring one by use of the appropriate Pauli operator. A string of three σ^x operators is shown.

Figure 3.4: The creation and moving of anyons in the toric code.

3.4 *Perturbative Stability of Topologically Ordered Systems*

An important facet of the properties of the toric code is the study of its stability under perturbation. Relevant conclusions and interpretations with regard to the use of the perturbative stability of the toric code in quantum computation will be presented in this subsection.

The most important result on the perturbative stability of topologically ordered systems was presented in [20] (and with a similar proof in a shorter article [34]), and it proves the the zero-temperature stability of topological phases of matter under weak and local time-independent perturbations. The proven theorem shows the following:

Theorem 3.3. *Suppose that the unperturbed Hamiltonian H_0 obeys the properties defining a topologically ordered system (TQO-1,2 –* (2.9)*,* (2.10)*). Let there be a perturbation V acting on the system, such that it can be written as a sum of geometrically local interactions with with a bounded norm:*

$$V = \sum_{r \geq 1} \sum_{A \in \mathcal{S}(r)} V_{r,A}, \tag{3.15}$$

where $\mathcal{S}(r)$ is a set of cubes of linear size r, and $V_{r,A}$ is an operator acting only on the sites of A. It is assumed that the magnitude of the interactions decays exponentially in r:

$$\max_{A \in \mathcal{S}(r)} \|V_{r,A}\| \leq J e^{-\mu r}, \tag{3.16}$$

where $J, \mu > 0$ are constants independent of the linear size of the system.

Then there exist constants $J_0, c_1, c_2 > 0$ depending only on μ and the spatial dimension D such that for all $J \leq J_0$ the spectrum of $H = H_0 + V$ is contained (up to an

overall energy shift) in the union of intervals $\bigcup_{k\leq 0} I_k$*, where* k *runs over the spectrum of* H_0 *and*

$$I_k = \{\lambda \in \mathbb{R} \ : \ k(1 - c_1 J) - \delta \leq \lambda \leq k(1 + c_1 J) + \lambda\}, \tag{3.17}$$

and

$$\delta = \text{poly}(L) \exp\left(-c_2 L^{3/8}\right), \tag{3.18}$$

where L *is the linear size of the system.*

This theorem testifies to the stability of the Hamiltonians of topologically ordered quantum spin systems that can be written as a sum of geometrically local commuting projectors (see (2.5), (2.6)), when acted on by a weak and local perturbation. In [20] it was proven that when such a Hamiltonian is acted on by any local perturbation, the eigenvalues of its excited states change at most by a constant factor $(1 \pm c_1 J)$ with an exponentially small correction (δ), whereas the eigenvalue of its ground state $(k = 0)$ becomes a band of exponentially small width (2δ). For a visual representation, see Figure 3.5. Additionally, for a weak enough perturbation, such that $J < \frac{1}{c_1(4k+2)}$,

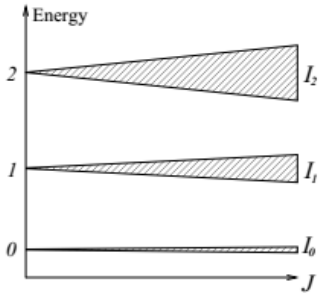


Figure 3.5: The energy bands I_k describing the spectrum of a perturbed Hamiltonian $H = H_0 + V$ for three distinct eigenstates, with respect to an increasing perturbation V. Figure taken from [20].

all I_k bands are separated from each other by a gap of at least $1/2$. This stability result allows for the construction of a quasi-adiabatic continuation (see Subsection 4.1) for any relevant operator associated with the original system, which can be used to demonstrate that all topological invariants of the original system stay constant in such a situation.

An important thing to note is that, as the ordering number of the eigenvalues of the Hamiltonian rises, the corresponding perturbed band will widen more, linearly in k. Therefore, a situation in which the energy bands overlap and the gaps between the excited states of the system close becomes easier to achieve as more states are taken into consideration. For such a system, Theorem 3.3 will hold if J is chosen appropriately, such that the gap remains open for all system sizes.

An equivalent result that deals in the entanglement entropy of the quantum states of topologically ordered systems can be found in [56], and an alternative approach

to the exponential stability of the ground state degeneracy of a topologically ordered system via its logical operators is shown in Section 5 (most generally in Subsection 5.1).

4 Dynamics of Local Quantum Systems

The investigation of the effects of small perturbations on the Hamiltonians of topologically ordered quantum spin systems relies heavily on the locality properties of these systems, opting to employ bounds on the emerging discrepancies from the original ground state space of the system. Not many of these methods exist; they are constantly being improved upon and used in the examination of the problems found in the fields of fault-tolerant quantum computation, quantum information, and others.

In this section, the most important methods used in error-bounding on local stabilizer codes are presented: quasi-adiabatic continuation (Subsection 4.1), Lieb-Robinson bounds (Subsection 4.2), and Kitaev's self-energy expansion (Subsection 4.3).

Kitaev's self-energy expansion is a perturbation theory approach to a system with with a degenerate ground state and a finite spectral gap above the ground state space. It is a relatively rough method of calculating the eigenvalues of the perturbed Hamiltonian of such a system, and it can be used to gauge the effect of a specific perturbation on a system described by a stabilizer code, such as the toric code, because the calculation involves the multiplication of the exact operators acting in the system, enabling the utilization of their commutativity properties.

On the other hand, *quasi-adiabatic continuation* is a powerful method used to examine a gapped quantum spin system that is acted on with a small perturbation, by parametrizing the perturbation and then slowly and smoothly evolving the original system's local operators to their final form. Unlike Kitaev's self-energy expansion, the calculations involved in this approach can be performed on a generic perturbation.

The use of *Lieb-Robinson bounds* is a powerful tool that ties in to quasi-adiabatic continuation. For quantum systems, there exists a theoretical finite upper limit to the velocity with which information can propagate through the system — the Lieb-Robinson velocity — giving a theoretical limit to the locality properties in such systems. This limit is known as the Lieb-Robinson bound, and it is commonly applied in various problems involving the notion of locality in quantum systems.

Following the introduction of these tools in this section, they will be applied to the perturbed toric code in Section 5.

4.1 Quasi-Adiabatic Continuation

One of the most important tools used to understand the effects of perturbation on quantum many-body systems is the principle of quasi-adiabatic continuation, first developed by Matthew Hastings in his work published in the early 2000's [57, 58].

By applying this method it is possible to clearly investigate the effects of a small perturbation on a quantum spin system's Hamiltonian. Since the first publication

of Hastings' proposal, quasi-adiabatic continuation has been further developed by many authors and it has subsequently been applied to a variety of problems in quantum information (for example, see [23, 56, 59–62]). The chief principle of quasi-adiabatic continuation is the possibility to — by slowly increasing the magnitude of the perturbation to its final value — smoothly transform local operators in the starting condition to local operators in the final state of the system. This continuation can be applied to the Hamiltonian paths of gapped quantum many-body systems. In this subsection, the method of quasi-adiabatic continuation will be presented as described in [61], [56] and [20].

If a quantum system with a known Hamiltonian is acted on by a small perturbation, the adiabatic evolution of this system can be constructed by adding a parameter-dependency on the Hamiltonian of this system and varying it slowly in the scale of the gap. We observe the local Hamiltonians of the system ($H_Z(s)$) change, varied adiabatically from $s = 0$ to $s = 1$, where $s = 0$ presents the starting setup, and $s = 1$ the final state of the system. Under the assumption that the complete Hamiltonian of the system can be written as $H_s = \sum_Z H_Z(s)$, where the local Hamiltonians are differentiable in s, and the spectral gap of the system has a lower bound that is uniform in s, it is possible to define the Hermitian *quasi-adiabatic continuation generator* $\mathcal{D}_s$, such that:

$$\partial_s |\psi_0(s)\rangle = i\mathcal{D}_s |\psi_0(s)\rangle , \tag{4.1}$$

where $|\psi_0(s)\rangle$ is the ground state of the system dependent on the parameter s. To describe the quasi-adiabatic continuation operator, $\mathcal{D}_s$, first it is necessary to define a filter function dependent on the spectral gap of the system $\gamma > 0$, as $F(t) \in L^1(\mathbb{R})$, with the following properties:

- it is odd, so that $\mathcal{D}_s$ is anti Hermitian,
- its Fourier transform is equal to $\bar{F}(\omega) = -1/\omega$ for $|\omega| \geq 1/2$ (normalized value related to the spectral gap),
- it decays faster than any negative power of time for large $|t|$.

Examples of these functions can be found in [56] and [61], and they are generally quite difficult to work with analytically.

The generator of the quasi-adiabatic continuation is then of the form:

$$\mathcal{D}_s \equiv i \int \mathrm{d}t F(t) \exp(iH_s t) \left(\partial_s H_s\right) \exp(-iH_s t). \tag{4.2}$$

The unitary evolution operator used to perform the quasi-adiabatic continuation is defined through $\partial_s U_s = i\mathcal{D}_s U_s,\ U_0 = \mathbb{1}$, and becomes:

$$U_s \equiv \mathcal{S}' \exp\left(i \int_0^s \mathrm{d}s' \mathcal{D}'_s\right), \tag{4.3}$$

where $\mathcal{S}'$ is used to denote a path-ordered exponential.

For a system with a gapped spectrum, a projector P onto an eigenspace of $H(s)$ will evolve as:

$$P(s) = U_s P(0) U_s^\dagger \tag{4.4}$$

under the perturbation (a proof can be found in [20]). Similarly, *dressed operators* can be defined as the perturbed forms of operators acting on the system, and they are expressed as

$$O_i(s) = U_s O_i U_s^\dagger, \tag{4.5}$$

where O_i is an operator acting on the system, and U_s is as previously defined in (4.3). These dressed operators transform from local operators to quasi-local operators, keep the notion of their locality, as well as the commutation and anti-commutation relations of the original operators.

Additionally, the generator of the quasi-adiabatic continuation can be expressed as:

$$\mathcal{D}_s = \sum_{u \in \Lambda} \sum_{r \geq r_0} \mathcal{D}_s(u; r), \tag{4.6}$$

where $\mathcal{D}_s(u; r)$ represents the generator with support on a ball of radius r, centered on u, with the property of decaying sub-exponentially with r [56]. Following from the validity of this relation, it can be shown that U_s satisfies the Lieb-Robinson bounds modified for sub-exponential decay [61].

In general, the construction of this method implies that if for a system there exist two local gapped Hamiltonians that can be connected by a parametrized path — i.e. that are in the same topological phase (see the first part of the Subsection 2.4) — their ground states can be evolved into each other by use of the unitary operator U_s.

4.2 Lieb-Robinson Bounds

In the study of quantum systems, there exists a theoretical finite upper limit to the velocity with which information can propagate, which was discovered and reported on by Elliot H. Lieb and Derek W. Robinson in their 1972 seminal paper [63]. Through the application of the presented theorem, the limits on the locality properties of investigated physical systems can be described – these limits are generally known as Lieb-Robinson bounds. After the publication of this result, Lieb-Robinson bounds have been found to have application in the treatment of many problems involving locality conditions in quantum systems, and as such, they have been built upon and improved by various authors in relevant articles, such as [23, 56, 57, 61, 64–66]. In this subsection, the essence and overview of the Lieb-Robinson bounds appropriate to the purpose of this thesis will be presented, as shown in [23] and [67].

The core of the employment of Lieb-Robinson bounds is the existence of the notion of locality in quantum many-body systems. For example, putting such an N-particle system on a lattice Λ with N sites, it is straightforward to introduce a graph distance between the points on the lattice. The distance between two sets, A and B, will be defined as

$$\mathrm{dist}(A,B) = \min_{i\in A, j\in B}\mathrm{dist}(i,j), \tag{4.7}$$

where $\mathrm{dist}(i,j)$ is simply defined as the distance between sites i and j, which may be differently established for differing systems – for a one-dimensional system with open boundary conditions, it is naturally equal to $|i-j|$, whereas for a one-dimensional system with periodic boundary conditions, it is more natural to define it as $\min_n|i-j+nN|$, where n is an integer. The diameter of a set A is defined as:

$$\mathrm{diam}(A) = \max_{i,j\in A}\mathrm{dist}(i,j). \tag{4.8}$$

The Hamiltonians that are considered in this approach are of the type

$$H = \sum_Z H_Z, \;\; \mathrm{supp}(H_Z) = Z, \tag{4.9}$$

such that $\|H_Z\|$ decays rapidly with $\mathrm{diam}(Z)$.

Then, the following can be stated, as proven in [23].

Theorem 4.1. *Suppose that the following holds for all sites i:*

$$\sum_{X\ni i}\|H_X\||X|\exp[\mu\,\mathrm{diam}(X)] \le s < \infty, \quad \mu, s > 0. \tag{4.10}$$

Let A_X and B_Y be operators supported on sets X and Y, respectively. Let t stand for time, and define $A_X(t)$ to be the time evolution of the operator A_X, given the Hamiltonian of the system. Then, if $\mathrm{dist(X,Y)} > 0$*,*

$$\begin{aligned}\|[A_X(t), B_Y]\| &\le 2\|A_X\|\|B_Y\|\sum_{i\in X}\exp[-\mu\ \cdot \mathrm{dist}(i,Y)]\left[e^{2s|t|}-1\right] \\ &\le 2\|A_X\|\|B_Y\||X|\exp[-\mu\ \cdot \mathrm{dist}(X,Y)]\left[e^{2s|t|}-1\right].\end{aligned} \tag{4.11}$$

What is stated in this theorem are the bounds that describe the time evolution of operators under a local Hamiltonian. Let A_X be some operator, and $B_l(X)$ the set of sites i, such that $\mathrm{dist}(i,X) \le l$. Then, $A_X(t)$ is the time evolution of the operator A_X. The operator $A_X^l(t)$ is defined as

$$A_X^l(t) = \int \mathrm{d}U U A_X(t) U^\dagger, \tag{4.12}$$

where the integral is over all unitary operators supported on the set of sites $\Lambda\setminus B_l(X)$, with the Haar measure – then, A_X^l is supported on the ball $B_l(X)$. It can be written that

$$UA_X(t)U^\dagger = A_X(t) + U[A_X(t), U^\dagger], \tag{4.13}$$

so it follows:

$$\|A_X^l(t) - A_X(t)\| \leq \int \mathrm{d}U \|[A_X(t), U]\|, \tag{4.14}$$

to which the Lieb-Robinson bounds in (4.11) can be applied. By bounding the right-hand side of the equation (4.14), it is trivial to see that the time evolved operator $A_X^l(t)$ is exponentially close to $A_X(t)$ if the distance l is sufficiently large when compared to $2st/\mu$, or

$$\|A_X^l(t) - A_X(t)\| \leq \text{const.} \cdot e^{-l/\xi}, \quad \xi > 0. \tag{4.15}$$

That is, the truncation error for the time evolution of an operator limited to the region $\Lambda \setminus B_l$ can be considered negligible for large l.

The core theorem (Theorem 4.1) can be resolved more intuitively by stating that

$$\|[A_X(t), B_Y]\| \leq c \cdot \exp(-a(\mathrm{dist}(X, Y) - v_{LR}|t|)), \quad c, a > 0, \tag{4.16}$$

where v_{LR} is known as the *Lieb-Robinson velocity*. Put in this form, the Lieb-Robinson velocity demonstrates the finite velocity of the propagation of information in non-relativistic quantum many-body systems, up to an error that is exponentially small in the distance from the original set of support of the relevant operator. In most practical applications of the Lieb-Robinson bounds, this error term — the propagation of information outside of the Lieb-Robinson light cone — is negligible.

4.3 Kitaev's Self-Energy Expansion

In the research on the effects of small perturbations on the Hamiltonians of quantum spin systems, an intuitive approach can be found in degenerate perturbation theory. In this subsection, the self-energy expansion — also known as Kitaev's self-energy expansion — given in a 2004 paper by Kempe, Kitaev, and Regev [68] will be presented.

For a topologically ordered system with a degenerate ground state and a spectral gap above it, the effect of a small perturbation to its Hamiltonian can be derived from the approximation the state of the system, by use of a type of generalization of Green's function – the *resolvent* of H. This operator-valued function is of the form:

$$G(z) = (z\mathbb{I} - H)^{-1}, \tag{4.17}$$

where H is the perturbed Hamiltonian. This is a meromorphic function – it is holomorphic on all except for a discrete subset of $z \in \mathbb{C}$. These singularities are poles of the function, and they can be preserved under projections. The following definition is taken from [68].

Definition 4.1. *Let $\mathcal{H} = \mathcal{L}_+ \oplus \mathcal{L}_-$, where $\mathcal{L}_+$ is the space spanned by eigenvectors of H_0 with eigenvalues $\lambda \geq \lambda_\star$ and $\mathcal{L}_-$ is spanned by eigenvectors of H of eigenvalue $\lambda < \lambda_\star$.*

Let $\Pi_\pm$ be the corresponding projection onto $\mathcal{L}_\pm$. For an operator X on $\mathcal{H}$ define the operator $X_{++} = X|_{\mathcal{L}_+} = \Pi_+ X \Pi_+$ on $\mathcal{L}_+$ and similarly $X_{--} = X|_{\mathcal{L}_-}$. We also define $X_{+-} = \Pi_+ X \Pi_-$ as an operator from $\mathcal{L}_-$ to $\mathcal{L}_+$, and similarly X_{-+}.

The expression

$$\Sigma_-(z) = z\mathbb{I}_- - G_{--}^{-1}(z) \tag{4.18}$$

is defined as the *self-energy* of the system. In some sense, the self-energy acts as an analogue of the equation (4.17), behaving as the Hamiltonian for the projected resolvent $G_{--}(z)$ – therefore, it is possible to find an effective Hamiltonian H_{eff}, which approximates $\Sigma_-(z)$ in a certain range of z. Then, the following theorem holds true.

Theorem 4.2. *Assume H_0 has a spectral gap γ around a cutoff eigevalue λ_s, i.e. all of its eigenvalues are in $(-\infty, \lambda_-] \cup [\lambda_+, +\infty)$, where $\lambda_+ = \lambda_s + \gamma/2$ and $\lambda_- = \lambda_s - \gamma/2$. Assume that $\|V\| < \gamma/2$, and let $\epsilon > 0$. Assume an operator H_{eff} exists such that $\text{Spec}(H_{\text{eff}}) \subseteq [c, d]$ for some $c < d < \lambda_s - \epsilon$, and the inequality*

$$\|\Sigma_-(z) - H_{\text{eff}}\| \leq \epsilon \tag{4.19}$$

holds for all $z \in [c - \epsilon, d + \epsilon]$. Then, each eigenvalue of the perturbed Hamiltonian H is ϵ close to the corresponding eigenvalue of the unperturbed Hamiltonian H_0.

A short proof of this theorem can be found in [68]. Additionally, the self-energy can be expressed via a natural series expansion, giving:

$$\begin{aligned}\Sigma_-(z) = H_- + V_{--} + V_{-+}G_+V_{+-} + V_{-+}G_+V_{++}G_+V_{+-} \\ + V_{-+}G_+V_{++}G_+V_{++}G_+V_{++}G_+V_{+-} + \dots\ .\end{aligned} \tag{4.20}$$

Thus, by equating $\mathcal{L}_-$ with the ground state space, and $\mathcal{L}_+$ with the space of all excited states of the system's Hamiltonian, Π_- is the projector operator for the ground state space, and we can name it P_0 ($P_0 \equiv \Pi_-$). For the toric code, it is defined as:

$$P_0 = \prod_{v \in \Lambda} \left(\frac{1 - A_v}{2} \right) \prod_{p \in \Lambda} \left(\frac{1 - B_p}{2} \right), \tag{4.21}$$

over all vertices v and plaquettes p in the toric code lattice Λ, where A_v and B_p are the vertex and plaquette stabilizer operators. From the projection operator's hermiticity, and directly because these operators commute with each other, and as $\sigma_i^\dagger = \sigma_i$ is valid for all Pauli operators σ_i (see Subsection 3.3 and equations (3.10)), it follows that

$$P_0^\dagger = P_0. \tag{4.22}$$

This operator is the product of all the individual projectors to the $+1$ eigenspaces of the vertices and plaquettes of the system.

The eigenvalues of the effective Hamiltonian can be found approximated by the poles of the projector's resolvent, that is $G(z) = P_0(z - H)^{-1}P_0$. By renaming the parameter z to E, for energy, the bracketed expression can be more clearly expressed as $(E - E_0 - \Sigma(E))^{-1}$, where $\Sigma(E)$ is the self-energy of the system – the eigenvalues of the effective Hamiltonian are then the values for which the operator in the brackets is degenerate. For $E \approx E_0$, the effective Hamiltonian can be approximated as $H_{\text{eff}} = E_0 + \Sigma(E_0)$, where the self-energy can be expressed as:

$$\Sigma(E) = P_0 \Sigma_{n=1}^{\infty} \left(V G_0'(E)\right)^{n-1} V P_0, \tag{4.23}$$

with $G_0'(E) = ((E - H_0)^{-1})'$ denoting the unperturbed Green's function for the excited state space of H_0 – this function vanishes when acting on ground states. Therefore, by setting $E = E_0$, the self-energy for H_{eff} can be calculated to an arbitrary order.

If this method is applied to the toric code with a sufficiently localized perturbation, it can be shown that, in the sense of the persistence of the gap and the degeneracy of the ground space as in the topological stability theorem (3.3), the original behavior of the system survives up to an order of the perturbation expansion that is as large as the linear size of the system.

Some examples for the use of this method can be found in [69].

5 Perturbations of Punctured Topological Systems

This section presents the results achieved when considering the stability properties of topological systems under local perturbation.

In particular, the work shown in this section considers the stability of topological systems with local puncture defects, under local perturbation. The perturbed systems are approached via the toric code, and the states of logical operators in the considered models are inspected in order to explicitly deduce on the stability of the information encoded in such systems when acted on by a local perturbation.

In this section a study of the stability of logical operators in topological systems is presented. First, the influence of a local perturbation on a logical operator is inspected by use of the quasi-adiabatic continuation and the Lieb-Robinson bounds (Subsection 5.1), after which the stability of logical operators in a punctured toric code is examined. Subsection 5.2 explains the formalism and properties of the punctured toric code, and it is followed by the investigation of a punctured toric code by the Kitaev's self-energy expansion perturbative method (Subsection 5.3). Finally, the influence of a local perturbation on such a system is derived exactly (Subsection 5.4).

The section ends with a discussion of the results and their implications (Subsection 5.5).

5.1 Logical Operator under Local Perturbation

This subsection presents a relation relevant to the norm of the difference of a logical operator in a topologically ordered system with a local perturbation and the same operator in the system without the perturbation. This relation is completely general for all topologically ordered systems that can be described by a model with string-like logical operators, and is not specific to the toric code. The result is achieved by use of the quasi-adiabatic continuation (Subsection 4.1) and the Lieb-Robinson bound (Subsection 4.2), and is used in the following subsections to reach conclusions on the stability of the encoded qubits in a punctured toric code (see Subsection 5.4).

A topologically ordered system described by the Hamiltonian H_0 is subjected to a small perturbation, so that the new Hamiltonian of the system is of the form

$$H(s) = H_0 + sV, \tag{5.1}$$

and the effect of the perturbation on the logical operator is examined using quasi-adiabatic continuation. Following that, the truncation error of the perturbed logical operator is calculated.

By letting the system evolve via quasi-adiabatic continuation, a logical operator L with support on the set A will take the form as in (4.5): $L_s = U_s L U_s^\dagger$, where U_s is

the quasi-adiabatic continuation evolution operator. L_C is defined to be the integral of the evolved logical operator over all unitary operators with the Haar measure on the set $A^C \setminus R'$, which is equivalent to the evolved logical operator traced out over the region R', or:

$$L_C = \int_{A^C \setminus R'} \mathrm{d}U U L_s U^\dagger = \operatorname*{Tr}_{A^C \setminus R'}(L_s), \tag{5.2}$$

where $A^C \setminus R'$ denotes the set complementary to the set A, but without the strip of thickness R on both sides around the logical operator L, as can be seen in Figure 5.1. The norm of the difference between L_C and L_s can be calculated:

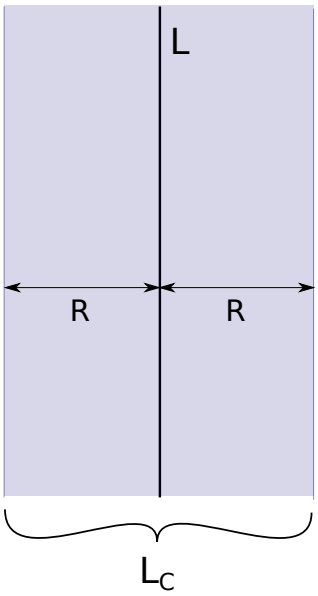


Figure 5.1: A representation of a logical operator L, with the denoted operator L_C being the original logical operator evolved via quasi-adiabatic continuation and truncated to a region of thickness R on both sides of the operator L (see equation (5.2)).

$$\begin{aligned} \|L_C - L_s\| &\leq \| \int_{A^C \setminus R'} \mathrm{d}U U \left[L_s, U^\dagger\right] \| \\ &\leq \int_{A^C \setminus R'} \mathrm{d}U \| \left[L_s, U^\dagger\right] \| \\ &\leq \sup_U \left(\| \left[L_s, U^\dagger\right] \| \right), \end{aligned} \tag{5.3}$$

where the operator U is defined on the set $A^C \setminus R'$, as in equation (5.2). For this calculation, the relation $U L_s U^\dagger = L_s + U \left[L_s, U^\dagger\right]$ and the value of the norm of the integral over an operator with the Haar measure have been taken into account. As quasi-adiabatic continuation obeys the Lieb-Robinson bounds (see Subsection 4.1),

it follows (from [67]):

$$\begin{aligned}\|L_C - L_s\| &\leq \sup_U \left(2\|L_s\|\|U^\dagger\| \cdot \text{const.} \cdot e^{-\mu\left(d(A,A^C\setminus R') - v_{LR}|s|\right)}\right) \\ &\leq \sup_U \left(\|L\| \cdot \text{const.} \cdot e^{-\mu\left(d(A,A^C\setminus R') - v_{LR}|s|\right)}\right), \end{aligned} \tag{5.4}$$

where v_{LR} is the Lieb-Robinson velocity, $\mu > 0$, $|s|$ is the parameter of the perturbation, playing the role of an interaction time, and $d(A, A^C \setminus R')$ is the distance between the sets on which L_s and U have support. In the final line, unitarity and the properties of the Haar measure have been taken into consideration.

Therefore, the difference between the two operators decays exponentially, and becomes arbitrarily small as the size of the set R around A is increased. In conclusion, the difference between the evolved logical operator and its partial trace over the region $A^C \setminus R'$ falls exponentially to zero with enough distance between the sets A and $A^C \setminus R'$. This means that a logical operator will continue to behave as a logical operator even in a system with a small perturbation, if examined from a large enough distance.

5.2 Properties of the Punctured Toric Code

In a realistic setting, various types of defects may arise in a topologically ordered system, depending greatly on the experimental realization of such a system (for an overview, see [70]). To thoroughly reconstruct the effect that these defects may have on a quantum system and the information stored in it in a non-perfect environment, it is imperative to investigate the consequences of applying a perturbation to the system. In this thesis, a view of the puncture type defects is given in a simple setup, and the effects of local perturbations on logical operators are calculated and interpreted – this subsection features a mathematical description of punctures in the toric code, followed by an analysis of the emergent properties of such a system.

A single-stabilizer *puncture* in the toric code is, mathematically, simply the absence of a stabilizer operator – there can exist plaquette or vertex punctures, depending on the relevant stabilizer operators, as can be seen in Figure 5.2. The effect of the existence of this type of local defect in the toric code is not analogous to the effect that comes about when the unperturbed toric code supports a small local perturbation, because punctures change the Hamiltonian of the system starkly, and the change cannot be varied to lead to a sensible analysis. It is important to note that removing only one plaquette or vertex stabilizer operator from the toric code will not affect the code space of the system, as the relations for the product of either type of stabilizer operators give one dependent stabilizer operator per type (see (3.12)). In other words, as every stabilizer operator of a certain type can be written as the

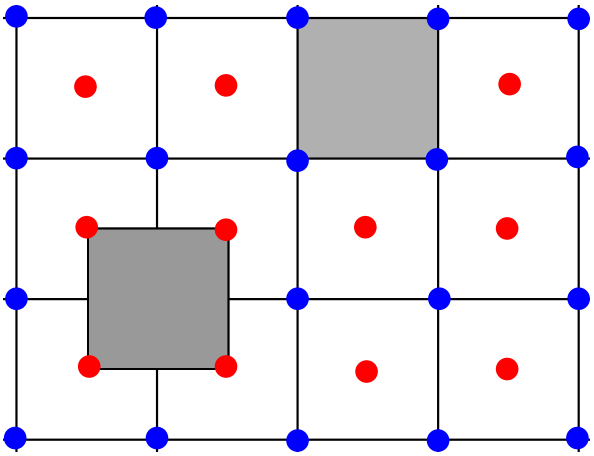


Figure 5.2: A representation of single-stabilizer punctures in the toric code. The shaded squares represent the punctures, while the blue dots denote the existence of vertex, and the red ones the existence of plaquette stabilizer operators. The topmost puncture is a plaquette puncture, removing the corresponding plaquette stabilizer, and the lower puncture is a vertex puncture, removing a vertex stabilizer.

product of all other stabilizer operators of the same type, the number of generators in the toric code is equal to the number of plaquettes or vertices reduced by 1 for each type, or $2k^2 - 2$ for a $k \times k$ lattice Λ. Therefore, only a toric code with two or more punctures of the same type can be expected to exhibit large-scale behavior differing from the unpunctured toric code.

For example, by removing two plaquette stabilizers from the toric code, B_{p_1} and B_{p_2}, the Hamiltonian of the system can be written as

$$H' = -\sum_{\substack{p\in\Lambda \\ p\neq p_1,p_2}} B_p - \sum_{v\in\Lambda} A_v. \tag{5.5}$$

By removing two stabilizer operators of the same type, the number of generators decreases, and the dimension of the code space of the system increases two-fold ($2^{2+1} = 8$), with two additional logical operators emerging in the system. The new logical operators in this case can be simply chosen as the string of Pauli σ^x operators connecting the bordering edges of the two missing plaquettes, and one of the missing plaquette stabilizer operators (see Figure 5.3). These new logical operators naturally anticommute with each other and commute with the system's Hamiltonian. For a greater number of punctures in the toric code, the number of the logical qubits encoded in the system, δ, can be summarized as:

$$\delta = 2 + \max(n'_p - 1, 0) + \max(n'_v - 1, 0), \tag{5.6}$$

where n'_p and n'_v are the number of plaquette and the number of vertex punctures, respectively, with the ground state degeneracy of the code being 2^δ, but the precise definitions of the logical operators in systems with many punctures may become cum-

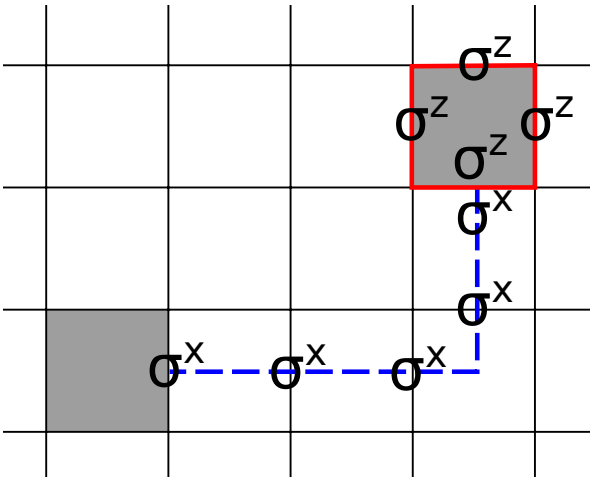


Figure 5.3: Two logical operators that may arise in the toric code with two plaquette punctures. The blue line represents a string of Pauli σ^x operators as the X logical operator, and the red line represents a string of Pauli σ^z operators, as the Z logical operator.

bersome to display.

The use of punctures as a tool for quantum computation tends to focus on the punctured finite sized Bravyi-Kitaev *surface code* [49], as shown in Figure 5.4. In this

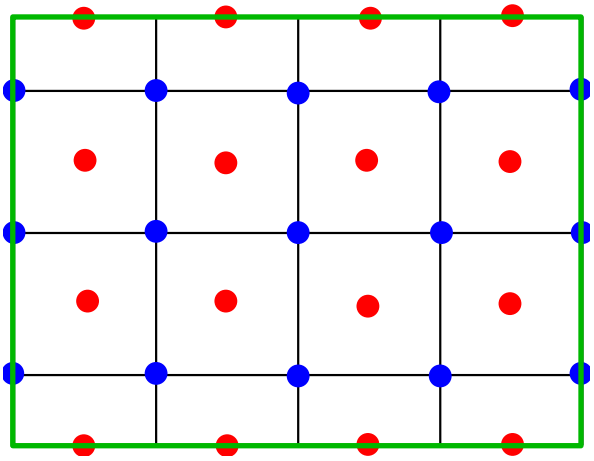


Figure 5.4: An example of a Bravyi-Kitaev surface code. The vertex stabilizer operators are shown with blue dots, while the plaquette stabilizer operators are shown with red dots. The boundary of the system is denoted by a green outline. The boundary on the top and bottom of the image is a rough one, and the ones on the sides are smooth ones.

surface code it is important to consider the nature of the boundary of the system. The *boundary* of the code can be either smooth or rough, i.e. it can consist of vertices or of plaquettes. The logical operators that arise in this system depend on these boundary conditions; the logical operators that span between the smooth boundaries are of the X type, while those that span between rough ones are the Z-type logical

operators. In some sense, the smooth boundaries are similar to plaquette puncture defects, while the rough ones are similar to vertex punctures. For such a system, its properties may change substantially with the dimensions of the system, compared to the ones found in the toric code with periodic boundary conditions, because some sequences of edge operators interact with the boundaries in non-trivial ways.

In addition to single-stabilizer punctures, larger punctures can be defined in the toric code, as well, and they play in with the finite sized toric code setups (for an overview of surface codes, see [71], and for an introduction to topological quantum memory via surface codes, see [10]). The details on these are not covered in this work, but recent results on the effectiveness of various setups of punctures in the toric code can be found in [72].

5.3 Kitaev's Self-Energy Expansion Applied to the Punctured Toric Code

In this subsection, a setup similar to the one in Subsection 5.1 is considered – differing in the fact that the toric code contains punctures. In this case, a setup with two single-stabilizer punctures equally distanced from a logical operator is considered (as shown in Figure 5.5), and the result can easily be generalized to some more complicated cases.

To examine such a punctured system, a perturbative method – Kitaev's self-energy expansion is used, as described in Subsection 4.3. To conclude on the effect of a small local perturbation in such a system on the logical operator, the commutator of the logical operator and the system's Hamiltonian is calculated and discussed for the perturbed case. Additionally, the stability of the punctured toric code will be examined by the influence of a small local perturbation on the logical operators associated with the existence of the punctures in the code.

To determine the effect of a small local perturbation on the punctured toric code, this subsection will first feature a calculation showing the effect of a small local perturbation on an unpunctured toric code, generalizing the result to the punctured toric code subsequently. As an alternative approach, the direct calculation of Kitaev's self-energy expansion and the effect of its summands on the logical operators in the toric code with two punctures will be presented.

Let the investigated system be described by a two-dimensional toric code on a $k \times k$ lattice, with the corresponding Hamiltonian H_0, and let it be perturbed by a local perturbation of a norm small when compared to that of H_0, such that:

$$V = \alpha \sum_i \left(\sigma_i^x + \sigma_i^z\right), \quad \alpha \in \mathbb{R}, \tag{5.7}$$

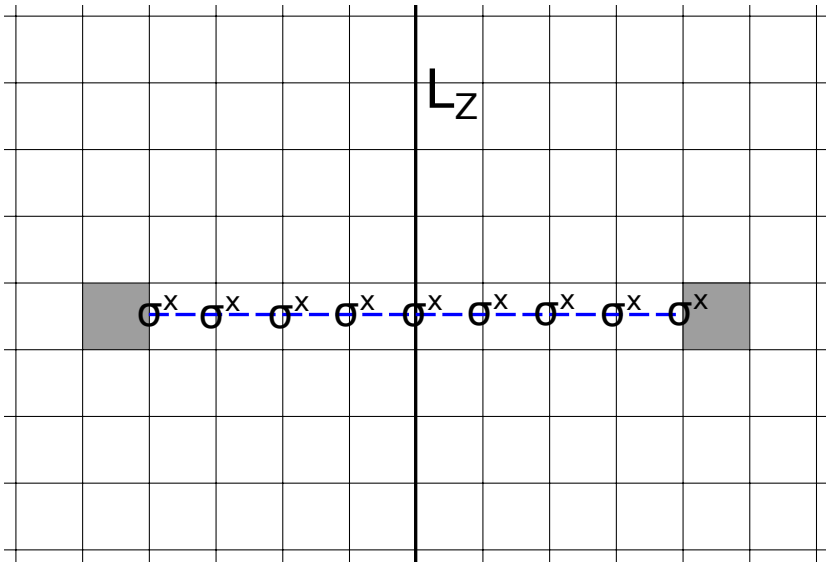


Figure 5.5: A representation of a Z-type logical operator on the toric code (bolded string of edges) with a one-stabilizer plaquette puncture on both sides (gray shading). The X-type logical operator that connects the two punctures and arises as a result of their existence is shown as a blue string crossing the edges on which its Pauli σ^x operators act.

with the sum going over all of the edges of the lattice. This indicates that, disregarding the norming factor α, every edge of the lattice is acted on with both a Pauli σ^x and a Pauli σ^z operator.

An important thing to note is the conjunction of the Pauli operators with the projection operators in the toric code. Because the toric code is a stabilizer code (see Subsection 3.2), the following property holds true for all stabilizer operators A_v and B_p:

$$A_v P_0 = P_0, \qquad B_p P_0 = P_0, \tag{5.8}$$

where P_0 is the projection operator onto the ground state of the unperturbed system. Every contractible loop of Pauli σ^x or σ^z operators can be written as a product of stabilizer operators (see Subsection 3.3), so it follows that the expression

$$L_{\text{closed}} P_0 = P_0 \tag{5.9}$$

holds for any contractible loop of Pauli operators L_{closed} – σ^z on the lattice, and σ^x on the dual lattice, as shown in Figure 3.3. For every open string of Pauli operators, there exist two quasi-particle excitations, on either end of the string; therefore, any setup that contains an open string does not place the system into the ground state. The following is valid for all open strings of Pauli operators L_{open}:

$$P_0 L_{\text{open}} P_0 = 0. \tag{5.10}$$

Then, it follows from (4.23) that the perturbed Hamiltonian can be calculated as

a series via Kitaev's self-energy expansion:

$$H \equiv H_0 + \sum_{n=1}^{\infty} H_{\text{eff}}^{(n)} = H_0 + P_0 \sum_{n=1}^{\infty} \left(V G_0'(E)\right)^{n-1} V P_0, \tag{5.11}$$

where $G_0'(E) = \left((E - H_0)^{-1}\right)'$ is the unperturbed Green's function for the excited states of H_0 – the prime indicates that it acts as expected on the excited state of the unperturbed Hamiltonian, but vanishes on the ground states. In the first order, the perturbation expansion for the system's Hamiltonian is:

$$H_{\text{eff}}^{(1)} = \alpha P_0 V P_0 = \alpha \sum_i P_0 \left(\sigma_i^x + \sigma_i^z\right) P_0 = 0, \tag{5.12}$$

which can be proven by considering the anticommutation relations of Pauli operators with the stabilizers, or trivially, following (5.10). In the second order,

$$H_{\text{eff}}^{(2)} = \alpha^2 \sum_i \sum_j P_0 \left(\sigma_j^x + \sigma_j^z\right) G_0'(E) \left(\sigma_i^x + \sigma_i^z\right) P_0. \tag{5.13}$$

As the resolvent in the above expression acts on states that contain exactly two quasiparticle excitations, each of energy two (see equation (3.14) and the following paragraph), the definition of the resolvent in (4.17) gives $G_0'(E) = -1/4$. Thus,

$$\begin{aligned} H_{\text{eff}}^{(2)} =& \frac{-\alpha^2}{4} \left(\sum_{i \neq j} P_0 \left(\sigma_j^x + \sigma_j^z\right) \left(\sigma_i^x + \sigma_i^z\right) P_0 + \sum_l P_0 \left(\sigma_l^x + \sigma_l^z\right)^2 P_0 \right) \\ =& 0 + \frac{-\alpha^2}{4} \sum_l 2\mathbb{I} P_0 P_0 \\ =& -\alpha^2 k^2 P_0. \end{aligned} \tag{5.14}$$

The first part of the sum is equal to zero because the Pauli operators in the expression form open strings, and in the final line of the calculation, the number of edges in the lattice was inserted ($2k^2$).

The third order of the self-energy expansion is equal to zero, because there does not exist a way to arrange three Pauli operators in order to create a closed loop ($H_{\text{eff}}^{(3)} = 0$), and already with the fourth order, the expressions in the calculation become very lengthy, and the derivation of all the combinatorial factors tends to be very cumbersome from the fourth order onwards. For this reason and because of the assumed small norm of the potential, only the powers of α are advisable to keep track of. In general, up to the length of a logical operator, k, the elements of the expansion are:

$$H_{\text{eff}}^{(n)} = \begin{cases} \alpha^n \zeta P_0 & \text{if } n \pmod 2 = 0 \text{ and } n < k, \\ 0 & \text{if } n \pmod 2 = 1 \text{ and } n < k. \end{cases} \tag{5.15}$$

The ζ in the first line of the expression (5.15) represents a value that is constant with regard to the variation of the parameter α, but it, naturally (as can be seen

in equation (5.14)), changes as the size of the system (and with it, k) is changed. At $n = k$, there exist topologically non-trivial loops of Pauli operators – namely, the logical operators of the code. For a toric code on a $k \times k$ lattice, there exist four logical operators of length (minimal number of operators in the non-trivial string) k. Thus,

$$H_{\text{eff}}^{(k)} = \begin{cases} \alpha^k \left(\zeta P_0 + \eta P_0 \left(L_{X_1} + L_{X_2} + L_{Z_1} + L_{Z_2}\right) P_0\right) & \text{if } k \pmod 2 = 0, \\ \alpha^k \eta P_0 \left(L_{X_1} + L_{X_2} + L_{Z_1} + L_{Z_2}\right) P_0 & \text{if } k \pmod 2 = 1, \end{cases} \tag{5.16}$$

where ζ and η are constants with respect to α. Therefore, the following can be stated for any logical operator L:

$$\|[H_{\text{eff}}, L]\| = \mathcal{O}(\alpha^k), \tag{5.17}$$

which means that the information encoded in the toric code is exponentially stable against local perturbations in the system size k.

This relation is not completely precise, as the factors ζ and η do depend on the size of the system, k; however, the result reached via this calculation does present the behavior of the system in general terms. Use of a more rigorous perturbation formalism would allow for a more precise result.

An analogous conclusion intuitively holds for the punctured toric code. As two new logical operators are added to the system with two single-stabilizer punctures of the same type (adding just one doesn't change the number of logical operators, because of the product conditions on the stabilizer operators – see (3.12)), for a $k \times k$ lattice, this new logical operator would also be the shortest.

As an example, let there be two plaquette punctures in the toric code, at a distance R, and examine the effect of a small local perturbation (5.7) on the logical operator L_Z that passes naturally between the punctures. In this setup, the a new X-type logical operator connects one puncture to the other by a string of Pauli σ^x operators (see Figure 5.5). The choice of the puncture type in this case is arbitrary, as plaquette punctures are dual to the vertex punctures, and their behavior is essentially the same. Then, trivially, for small $|\alpha|$ and $R < k$, by using Kitaev's self-energy expansion, the norm of the commutator of the effective Hamiltonian and L_Z will be exponentially stable in the distance between the two punctures, or:

$$\|[H_{\text{eff}}, L_Z]\| = \mathcal{O}(\alpha^R). \tag{5.18}$$

This result suggests the perturbative stability of the qubit degrees of freedom in the toric code upon the addition of two single-stabilizer punctures of the same type into the code. However, a direct analysis of the influence of a small local perturbation on the degrees of freedom that are introduced into the system by the implementation

of these punctures is still necessary to make a complete statement about the stability of the code space of the punctured toric code.

Let the used setup be the same as earlier described, with a pertubation potential V (5.7), and with two plaquette punctures at a distance R from each other, and let $R > 4$. Then, the first four orders of Kitaev's self-energy expansion are calculated through (5.11). In the first order, there is no difference between the punctured and the unpunctured toric code, so $H_{\text{eff}}^{(1)} = 0$. However, in the second order, four cases need to be considered for strings of Pauli σ^x operators – a case in which both Pauli operators of the same type in the sum act on the same edge of a puncture boundary, denoted as $\beta(p)$, a case in which only one of those Pauli operators acts on an edge from $\beta(p)$, and two cases in which neither of the Pauli operators acts on an edge in $\beta(p)$: both of the Pauli operators of the same type in the sum can either act on the same edge, or they can create an open string (see Figure 5.6). Of these four cases, two of them which create open (or *half*-open) strings in the final state — with their product creating an excited state — create nil sums because the final state is acted on by P_0 in the sum. It follows:

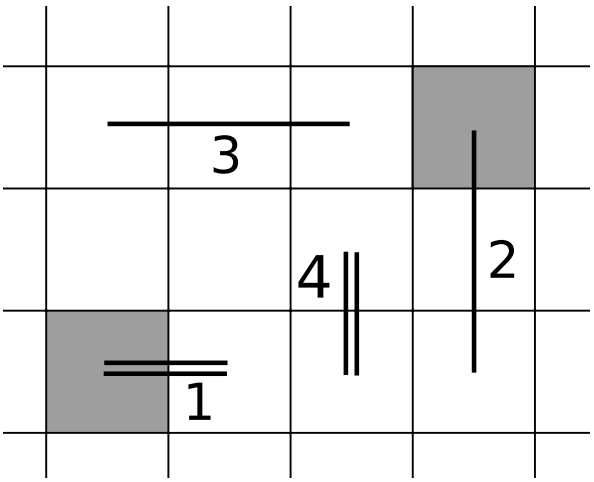


Figure 5.6: A representation of the four different types of Pauli σ^x operator strings formed by two Pauli operators in a toric code with plaquette punctures; the grey shading represents the punctures.

$$
\begin{aligned}
H_{\text{eff}}^{(2)} &= \alpha^2 \cdot \sum_{i \in \beta(p)} P_0 \left(\sigma_i^x + \sigma_i^z\right) \left(\frac{-1}{2}\sigma_i^x + \frac{-1}{4}\sigma_i^z\right) P_0 \\
&\quad + \alpha^2 \cdot \sum_{j \notin \beta(p)} P_0 \left(\sigma_j^x + \sigma_j^z\right) \frac{-1}{4} \left(\sigma_j^x + \sigma_j^z\right) P_0 \\
&= -\alpha^2 \cdot \left(\frac{3}{4} \sum_{i \in \beta(p)} P_0 + \frac{1}{2} \sum_{j \notin \beta(p)} P_0\right) \\
&= -\alpha^2 \cdot \left(6 + (k^2 - 4)\right) P_0 = -\alpha^2 (k^2 + 2) P_0, \qquad (5.19)
\end{aligned}
$$

which is analogous to the result obtained for the unpunctured toric code, up to a constant (5.14). The third, and all further odd orders of the expansion up to length R are equal to zero, as no closed string of an odd number of Pauli operators can be created in this situation. In the fourth order, an interesting factor arises, bound to the stabilizer operators of the missing plaquettes, which is not proportional to P_0, and is of the form:

$$H_{\text{eff}}^{(4)} = \alpha^4 \left(\zeta P_0 + \sum_{\substack{i_1,i_2,i_3,i_4 \in \beta(p) \\ \text{plaquette loop}}} \text{const.} \cdot P_0 B_{p \in \{p_1,p_2\}} P_0 \right), \tag{5.20}$$

where $i_1, \dots i_4$ are the edges on which the Pauli operators are acting, p_1 and p_2 are the missing plaquettes, and the sum goes over closed loops of operators acting on the edges that define a plaquette.

Thus, the information encoded in the qubit related to the logical operators that depend on the existence of the punctures in the system is only stable up to the fourth order in Kitaev's self-energy expansion. It is not exponentially suppressed by the distance between the two punctures, and therefore not sufficiently perturbatively stable. In other words, if X is the logical operator connecting the two punctures (shown with a blue line in Figure 5.5),

$$\|[H, X]\| = \mathcal{O}(\alpha^4). \tag{5.21}$$

The ground space degeneracy of the corresponding encoded qubit splits by an amount of $\mathcal{O}(\alpha^4)$. In the R-th order of the expansion, the string of Pauli σ^x operators connecting the two punctures (the logical operator X) comes into play:

$$H_{\text{eff}}^{(R)} = \alpha^R \cdot (\zeta + \text{const.} \cdot P_0 X P_0), \tag{5.22}$$

$$\zeta = \begin{cases} \text{const.}'' \cdot P_0 + \text{const.}' \cdot (P_0(B_{p_1} + B_{p_2})P_0) & \text{if } R \pmod 2 = 0, \\ 0 & \text{if } R \pmod 2 = 1. \end{cases} \tag{5.23}$$

As L_Z anticommutes with X, it follows:

$$\|[H, L_Z]\| = \mathcal{O}(\alpha^R). \tag{5.24}$$

From this, it can be deduced that the addition of the two punctures to the toric code influences the perturbative stability of the information encoded in the degrees of freedom of the system related to its topology – instead of being exponentially small in the system size (5.17), the corrections to the encoded information are only exponentially small in the distance between the two punctures.

5.4 Punctured Toric Code under Local Perturbation

It is possible to examine the effect of punctures in addition to a small perturbation on the stability of a string-like logical operator in a more general fashion.

To do this, the norm of the difference between the perturbed logical operator in the original system, L_s, and the one in a system with two punctures, L'_s, should be found. The punctures are located on opposite sides of the logical operator, and they are separated from each other by a distance $2d$, each at a distance of d from the logical operator (see Figure 5.7). The two systems — one without the punctures (as in Subsection 5.1), and one with them — are subject to a small perturbation, and the logical operators are evolved via the quasi-adiabatic continuation (Subsection 4.1), and the Lieb-Robinson bound is employed (Subsection 4.2). The result reached in Subsection 5.1 is utilized in this calculation.

The effect of the existence of punctures at a fixed distance from the logical operator is examined by comparing the influence of the local Hamiltonians of different zones of the lattice on the quasi-adiabatically evolved logical operator when there exists a small (and local) perturbation acting on the state of the system, using a relevant lemma from [56].

Spyridon Michalakis has shown (in his paper from 2012, [56]) that the error between the quasi-adiabatic continuation operator, U_s, and an approximation that splits the action of the quasi-adiabatic continuation operator into the product of operators acting on two disjunct subsets and their boundary decays sub-exponentially in the thickness of the chosen boundary, R:

$$\|U_s(A) \otimes U_s(A^C) U_s(\partial A(R)) - U_s\| \leq \epsilon_s(R). \tag{5.25}$$

In the shown equation (5.25), the error, $\epsilon_s(R)$, is of the form:

$$\epsilon_s(R) = c_1 \left(e^{c_2(J_2/\gamma)|s|} - 1\right) |\partial A| f_\gamma(c_3 R), \tag{5.26}$$

where $c_1, c_2, c_3, J_2 > 0$, $|\partial A|$ is the cardinality of the boundary between the two subsets, γ is the system's spectral gap, s is the evolution parameter for quasi-adiabatic continuation, and f_γ is a function that decays sub-exponentially in R.

If we define the quasi-adiabatic continuation operator as $\xi_s(R)$ for clarity, from equation (5.25) it follows that it is equal to:

$$\xi_s(R) = U_s(A) \otimes U_s(A^C) U_s(\partial A(R)) + \hat{\epsilon}_s(R), \tag{5.27}$$

where $\hat{\epsilon}_s(R)$ is an operator for which $\|\hat{\epsilon}_s(R)\| = \epsilon_s(R)$. The quasi-adiabatic evolution of a logical operator L can be expressed as:

$$L_s = \xi_s(R) L \xi_s^\dagger(R). \tag{5.28}$$

Furthermore, the set A from (5.25) is chosen to contain the set on which L acts.

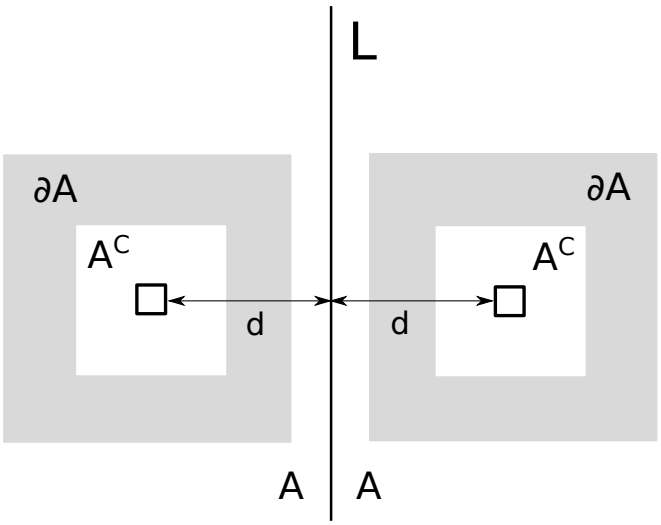


Figure 5.7: A representation of the setup used in Subsection 5.4. L is a string-like logical operator, and the squares denote punctures that are distanced by d from the logical operator. The logical operator acts on the set A, the punctures are defined to be in the set A^C, and their boundary set, ∂A, envelops the punctures, and doesn't contain either of the two main sets.

Then the set A^C contains both the punctures, and the boundary spans on both sides of the logical operator, not containing the operator, nor the punctures, but enveloping the punctures (see Figure 5.7). The quasi-adiabatic continuation operators acting on different regions of the lattice are generated by the local Hamiltonians in those regions (4.6), which allows for the evolution operator acting on the logical operator in the system with two punctures, ξ'_s, to differ from the one for the unpunctured system only in the region contained in the set A^C. By defining $\xi_s = \xi + \hat{\epsilon}_s$ and $\xi'_s = \xi' + \hat{\epsilon}'_s$, omitting the thickness of the boundary in the arguments of the operators, it follows:

$$\begin{aligned}\|L'_s - L_s\| &= \|(\xi' + \hat{\epsilon}'_s)L(\xi' + \hat{\epsilon}'_s)^\dagger - (\xi + \hat{\epsilon}_s)L(\xi + \hat{\epsilon}_s)^\dagger\| \\ &= \|\xi' L\xi'^\dagger + \xi' L\hat{\epsilon}'^\dagger_s + \hat{\epsilon}'_s L\xi'^\dagger + \hat{\epsilon}'_s L\hat{\epsilon}'^\dagger_s - \big(\xi L\xi^\dagger + \xi L\hat{\epsilon}^\dagger_s + \hat{\epsilon}_s L\xi^\dagger + \hat{\epsilon}_s L\hat{\epsilon}^\dagger_s\big)\|.\end{aligned} \tag{5.29}$$

The terms in the expression (5.29) that contain only the logical operator and ϵ_s terms will give a norm of $\mathcal{O}(\epsilon_s^2)$, as the ϵ_s operators commute with the logical operator. For the expression $\xi' L\xi'^\dagger - \xi L\xi^\dagger$, given the definition in equation (5.27),

the following is valid:

$$
\begin{aligned}
&\xi' L \xi'^{\dagger} - \xi L \xi^{\dagger} = \\
&\quad = \left(U'_s(A) \otimes U'_s(A^C) U'_s(\partial A(R))\right) L \left(U'^{\dagger}_s(\partial A(R)) U'^{\dagger}_s(A) \otimes U'^{\dagger}_s(A^C)\right) - \\
&\quad\quad \left(U_s(A) \otimes U_s(A^C) U_s(\partial A(R))\right) L \left(U^{\dagger}_s(\partial A(R)) U^{\dagger}_s(A) \otimes U^{\dagger}_s(A^C)\right) \\
&\quad = \left(U'_s(A) \otimes U'_s(A^C)\right) L U'^{\dagger}_s(\partial A(R)) U'_s(\partial A(R)) \left(U'^{\dagger}_s(A) \otimes U'^{\dagger}_s(A^C)\right) - \\
&\quad\quad \left(U_s(A) \otimes U_s(A^C)\right) L U^{\dagger}_s(\partial A(R)) U_s(\partial A(R)) \left(U^{\dagger}_s(A) \otimes U^{\dagger}_s(A^C)\right) \qquad (5.30) \\
&\quad = \left(U'_s(A) \otimes U'_s(A^C)\right) L \left(U'^{\dagger}_s(A) \otimes U'^{\dagger}_s(A^C)\right) - \\
&\quad\quad \left(U_s(A) \otimes U_s(A^C)\right) L \left(U^{\dagger}_s(A) \otimes U^{\dagger}_s(A^C)\right) \qquad (5.31) \\
&\quad = \left(U'_s(A) \otimes U'_s(A^C)\right) \left(\mathbb{I} \otimes U'^{\dagger}_s(A^C)\right) L \left(U'^{\dagger}_s(A) \otimes \mathbb{I}\right) - \\
&\quad\quad \left(U_s(A) \otimes U_s(A^C)\right) \left(\mathbb{I} \otimes U^{\dagger}_s(A^C)\right) L \left(U^{\dagger}_s(A) \otimes \mathbb{I}\right) \qquad (5.32) \\
&\quad = \left(U'_s(A) \otimes \mathbb{I}\right) L \left(U'^{\dagger}_s(A) \otimes \mathbb{I}\right) - \left(U_s(A) \otimes \mathbb{I}\right) L \left(U^{\dagger}_s(A) \otimes \mathbb{I}\right) \qquad (5.33) \\
&\quad = 0. \qquad (5.34)
\end{aligned}
$$

In line (5.30) the commutativity of the logical operator with the evolution operators acting on the boundary, $U_s(\partial A(R))$, has been taken into consideration. The logical operator is defined locally and at a finite distance from the boundary subset of the lattice. The logical operator acts on a different subset and because of that it commutes with the evolution operators that act on the boundary. The same commutativity argument that is given for the logical operator and the evolution operator on the boundary is valid for the evolution operator that acts on the subset A^C (see line (5.32)). And lastly, because punctures only are present in the region A^C, it follows that

$$U'_s(A) = U_s(A), \qquad (5.35)$$

because no difference in the two systems exists in the A subset. Then, the conclusion in line (5.34) follows trivially.

For the $\xi L \hat{\epsilon}^{\dagger}_s$ term in the equation (5.29), by use of the Cauchy-Schwarz and the triangle inequality, it follows:

$$
\begin{aligned}
\|\xi L \hat{\epsilon}^{\dagger}_s\| &\leq \|\xi\| \cdot \|L\| \cdot \|\hat{\epsilon}^{\dagger}_s\| \\
&= \|\hat{\epsilon}^{\dagger}_s\| \\
&\leq \mathcal{O}(\epsilon_s), \qquad (5.36)
\end{aligned}
$$

because of the unitarity properties of the operators. A similar argument can be used for the terms of the form $\hat{\epsilon}_s L \hat{\epsilon}^{\dagger}_s$.

Then, the bound for the equation (5.29) is:

$$\|L'_s - L_s\| \leq \mathcal{O}(\epsilon_s). \qquad (5.37)$$

An important thing to note is the cardinality of the set ∂A, as it influences the value of ϵ_s (see equation (5.26)) – if the region A^C is defined as in Figure 5.7, with the separation of the two punctures being equal to $2d$, and the width of the boundary ∂A being equal to R, the following inequality is valid:

$$|\partial A| < \text{const.}' \cdot d^2 < \text{const.} \cdot R^2. \tag{5.38}$$

It follows that the expression $\|L'_s - L_s\|$ will decay sub-exponentially in the width of the boundary, R, as ϵ_s will certainly be a function decaying sub-exponentially in R, i.e. the logical operator for the system without punctures approximates the one for the system with punctures sub-exponentially well. This means that the logical operator for the punctured system is sub-exponentially close to the logical operator for the unpunctured system in the distance of separation of the two punctures.

5.5 Discussion of the Results

The results that have been presented in this section have dealt with the perturbative stability of topologically ordered systems – in particular, those with local puncture defects. This subsection presents a summary of the shown results, provides an interpretation and explains their significance.

In Subsection 3.4 the theorem on the perturbative stability of topological order by Sergey Bravyi, Matthew Hastings, and Spyridon Michalakis [20] was presented in short. It proves the exponential stability of topological invariants in topologically ordered systems when they are subjected to a weak local perturbation. It is then trivial to postulate that the information stored in a system that can be accurately described by use of the toric code (the toric code exhibits topological order, see Subsection 3.3) will stay sufficiently protected when the system is exposed to some weak perturbation that can be decomposed into a sum of geometrically local terms. In the preceeding subsections, this claim was examined more explicitly, through the stability of the degeneracy of the ground state space — code space — of a toric code system.

First, the stability of a logical operator in such a system was considered in Subsection 5.1, and the result reached stated that the norm of the difference between the perturbed logical operator limited to some band around the set on which the original logical operator had support, and the original logical operator will decay exponentially in the size of this band (for details see (5.4)). This result is completely in agreement with the perturbative stability theorem, as it simply states that the information stored in the system will be exponentially stable to a weak local perturbation, in the linear size of the system.

This result comes from an explicit calculation that employs the quasi-adiabatic continuation and the Lieb-Robinson bounds, providing detailed confirmation of the

perturbative stability theorem for the specific case of the logical operators of the system.

The results presented in this section go further than the examination of the perturbative stability of the logical operators in topologically ordered systems – Subsection 5.2 presents an introduction to the properties of a type of system with local defects, the punctured toric code, defining punctures in the toric code, discussing their effect on the topological invariants of the toric code, and providing an overview of the use of such mathematical forms in the development of procedures in quantum computing. After this overview, Subsections 5.3 and 5.4 showcase the obtained results.

Subsection 5.3 presents the results of the application of a degenerate perturbation theory, Kitaev's self-energy expansion, on the unpunctured toric code, as well as the toric code with two punctures. For any logical operator, an expression for the norm of its commutator with the effective Hamiltonian is achieved (see (5.17)). As is expected, it follows that the information stored in the toric code is exponentially stable against local perturbations in the system size – the minimal length of the string of the edges that define the logical operator. Even though this is a relatively crude, perturbative method, the achieved result is somewhat in line with the perturbative stability theorem.

Following this calculation, the toric code with two punctures of the same type, that have been set at a fixed distance, is examined by the same method. This method requires calculations of ever-growing combinatorial complexity to be performed, so the results have been derived up to constant factors, for the most relevant orders of the expansion. For two punctures in this setup, the perturbative stability of the logical operators that emerge in the system because of their existence is not satisfactory – as a new logical operator of length four exists, that is equivalent to one of the missing stabilizer operators. The stability of the information encoded in the logical operators associated with the two punctures is not stable in the system size. Additionally, the information encoded in the logical operators and the code space associated with the unperturbed toric code is in this case not exponentially perturbatively stable in the system size, but in the difference between the two punctures! This is a fascinating result that serves as a motivator for the exploration of the field of surface codes in quantum information (an interesting recent result can be found in [72]).

In Subsection 5.4 an innovative approach enabled for a general solution of the problem of the preturbative stability of the logical operators in the toric code with two punctures of the same type. Using a lemma from [56], this problem is tackled by use of the quasi-adiabatic continuation and the Lieb-Robinson bounds. In conclusion, for punctures that are set far enough apart from each other and the logical operator, the norm of the tail of the evolved logical operator in the system with the punctures

outside of some band defined around the set on which the original logical operator has support, will decay sub-exponentially in the linear size of the band when compared to the logical operator evolved in the same system – but without the punctures (see (5.37)).

This result implies that the topological invariants of a system may remain stable up to a negligible factor with the addition of a finite number of local puncture defects, given that the linear size of the system becomes large enough.

It is interesting to note that the same would not hold true for the (infinite) toric code with an infinite number of punctures – in this case, it depends on how the distance between punctures scales with the system size. For example, for the case of a constant density of punctures, the average distance between punctures is expected to remain constant and independent of the size of the system. Then, the system would not be perturbatively stable, as going to a larger size of the system would not improve on the error terms in the dressed logical operators.

6 Summary and Outlook

In this thesis, topological order in quantum spin systems with local puncture defects, under a small local perturbation was investigated. This section serves as a short summary of the topic presented in this thesis, and provides a conclusion and outlook for the achieved results.

The task that was solved in this thesis was the derivation and interpretation of general bounds on the perturbative stability of topologically ordered systems with local puncture defects. Topological order is a yet not well understood phase of matter, and it provides a fresh challenge for many fields of physics. Its perturbative stability and the emergence of anyonic excitations make it a highly anticipated solution to the problem of instability of quantum computers, via the denominator of topological quantum computation. However, to fully understand how topological order is affected by outside perturbation — for further manipulation — it is necessary to investigate the precise changes in a topologically ordered system when it is subjected to perturbation. In addition to that, to use topologically ordered systems in quantum computing, the possibility of defects cannot be neglected. For this purpose, the culmination of the results presented in this thesis shows an analysis of topological order in the toric code with local puncture defects.

The achieved results begin from a precise analysis of the perturbative stability of the degeneracy of the ground state in the toric code, led by the main result of the perturbative stability theorem, but expand on this notion, focusing on attempting to conclude whether an analogous property applies to the toric code with local puncture defects, as well.

What was shown points to an affirmative answer. Further analysis of specific setups relevant to the procedures used in quantum information and quantum error correction is needed for the application of the reached conclusions to concrete problems. In conjuction to that, the achieved results invite to an analysis of similar problems in topological quantum computation.

For example, a natural expansion of the presented setup would include puncture defects that span more than one stabilizer operator, as well as punctures with mixed boundary conditions (for example, see [72]). The changes that the addition of such punctures imposes on a toric code system are non-trivial, and a further look into this problem would illuminate the question of the stability of topological order in systems with puncture defects – however, it is beyond the scope of this thesis.

Some of the work presented here may also be used as an aid in the examination of the perturbative stability of surface codes with complicated boundary conditions – see [49] and [72].

Another interesting generalization of the results presented here involves the study

of the perturbative stability of the punctured toric code in higher dimensions, including fractal codes (for example, see [10] and [73]).

Some other models that describe topological order may also be considered – and the results compared to those that are achievable for the two-dimensional punctured toric code. These include the color codes [74], the cubic code [75], as well as homological codes [76].

It might also be interesting to investigate the stability of quasi-topological phases, as described in [77]. This analysis would present an additional challenge, as quasi-adiabatic continuation cannot be applied in gapless systems.

In conclusion, this thesis served as a medium to present the bounds on the stability of topological order in the toric code with local puncture defects – it approached the problem via the self-energy expansion, as well as by employing the quasi-adiabatic continuation and the Lieb-Robinson bounds. Bounds on the stability of such topologically ordered systems were presented and analyzed. They agree with the previously derived result for the non-punctured toric code [20], and extend it, suggesting that a system that can be described by the toric code with local punctures will remain stable under weak and local perturbations, given that the distance between the punctures remains large enough.

7 Prošireni sažetak

Prošireni sažetak na hrvatskom jeziku sastoji se od povezanog pregleda svih poglavlja predstavljenih u ovom radu i kratkog opisa sadržaja predstavljenog u njima. Više detalja može se pronaći u izvornom radu na engleskom jeziku.

7.1 Uvod

Kvantna teorija informacija i *kvantno računarstvo* pojmovi su koji se odnose na kvantne ekvivalente klasičnoj teoriji informacija i klasičnog računarstva. Za razliku od informacije u klasičnom smislu, kvantna informacija podrazumijeva iskorištavanje kvantnih učinaka pri skladištenju ili mijenjaju informacije. Umjesto klasičnih bitova koji koriste diskretne vrijednosti, $\mathbb{Z}_2 = \{0, 1\}$, kvantne jedinice informacije — *qubiti* — sačinjene su od $\mathbb{C}^2 = \mathbb{C}[\mathbb{Z}_2]$, pri čemu se koristi koncept kvantne superpozicije.

Od prijedloga *kvantnih simulatora* Richarda Feynmana [3], do približavanja klasičnoj granici *Mooreovog zakona* [5], te iznenađujućih mogućnosti *kvantnih algoritama* [7,8], kvantno računarstvo iznimno je perspektivno područje istraživanja u modernoj fizici. Sveobuhvatni uvod u kvantnu teoriju informacija i kvantno računarstvo moguće je pronaći u [1].

Međutim, važni problemi kvantnog računarstva su dekoherencija i nestabilnost na smetnje kvantnih sustava – uobičajeno rješenje jest istraživanje mogućnosti *ispravljanja grešaka* koje se pojavljuju pri čuvanju *kvantnih memorija* i promjeni kvantne informacije u kvantnim algoritmima, u sklopu teorije *kvantnog računarstva otpornog na pogreške*. *Teorem o pragu* [12] potvrđuje da se kvantno računarstvo može učiniti pouzdanim u prisustvu smetnji i grešaka koristeći metode kvantnog računarstva otpornog na pogreške, uz uvjet da se pogreške u korištenom sustavu javljaju rjeđe od neke granične vrijednosti.

S druge strane, ovim problemima se može pristupiti koristeći svojstva sustava u kojima se javlja *topološko uređenje*, u funkciji *topološkog kvantnog računarstva* [2]. Topološko uređenje je faza na temperaturi apsolutne nule, koju odlikuju fazni prijelazi koji se ne mogu opisati klasičnom Ginzburg-Landau teorijom faznih prijelaza, degenerirano osnovno stanje, procjep u spektru iznad najniže svojstvene vrijednosti, te globalne, *topološke invarijante*, koje omogućuju korištenje pojednostavljenih metoda otkrivanja i praćenja grešaka koje se javljaju u kvantnom računarstvu. Topološko uređenje pouzdano je otkriveno u razlomljenom kvantnom Hallovom efektu [16], a potpuni teorijski opis sustava u kojima se javlja još uvijek nije postignut.

Pobuđenja u topološki uređenim sustavima lokalizirana su i nastaju kao *anyoni* (engl. *anyons*) – čestice koje se ne mogu razlikovati, ali koje ne prate fermionsku ili bozonsku statistiku izmjene [17, 18], već njihove valne funkcije mogu poprimiti *bilo koji* faktor (engl. *any*) pri zamjeni mjesta (ukratko, $|\Psi_1\Psi_2\rangle = e^{i\theta}\,|\Psi_2\Psi_1\rangle$, gdje su $|\Psi_1\rangle$ i $|\Psi_2\rangle$ kvantna stanja dva anyona, a θ bilo koja vrijednost). *Kvantna logička vrata* u

ovim sustavima tvore se *pletenjem* anyona u $(2+1)$-dimenzionalnom prostor-vremenu (vidi sliku 1.1). Ovako stvorene anyonske niti ne mogu se trivijalno razdvojiti [19], te tvore dobru osnovu za teoriju topološkog kvantnog računarstva.

Cilj ovog rada jest istraživanje perturbativne stabilnosti topološki uređenih sustava. Dokazano je da topološko uređenje ostaje stabilno pri malim i lokalnim perturbacijama na temperaturi apsolutne nule [20], no za donošenje zaključaka o praktičnoj uporabi postupaka iz topološkog kvantnog računarstva, nužno je eksplicitno razumjeti utjecaj perturbacije na informaciju spremljenu u topološki uređenom sustavu. U tu svrhu, iskorištene su važne matematičke metode — kvazi-adijabatsko produljenje, Lieb-Robinsonove granice, razvoj vlastite energije — i izvedene su granice stabilnosti topološkog uređenja. Uz to, promotreni su i topološki uređeni sustavi s lokalnim defektima tipa rupe te ovaj diplomski rad završava zaključkom o općim granicama stabilnosti topološki uređenih sustava s lokalnim defektima ovog tipa, nagovještavajući buduću primjenu ovih i sličnih rezultata u daljnjem razvoju metoda kvantnog računarstva otpornog na pogreške, kao i praktičnu primjenu izvedenih vrijednosti u eksperimentalnim postavima.

Pregled sadržaja proširenog sažetka je sljedeći: u Potpoglavlju 7.2 uvedeni su pojmovi vezani uz topološko uređenje, te je ono matematički definirano, u Potpoglavlju 7.3 nalazi se kratak pregled kvantnog računarstva otpornog na pogreške, te su uvedeni stabilizacijski kodovi i posebno torusni kod. U Potpoglavlju 7.4 izložene su korištene matematičke metode, a Potpoglavlje 7.5 predstavlja pregled proučavanih sustava i ostvarenih rezultata. Potpoglavlje 7.6 služi kao pregled i zaključak tema predstavljenih u ovom radu, a u Potpoglavlju 7.7 nalazi se popis korištenih slika i tablica, s hrvatskim prijevodima opisa.

7.2 Kvantni spinski sustavi i topološko uređenje

Kvantni višečestični sustavi [22] pojednostavljeni su modeli koji se koriste za proučavanje nerelativističkih kvantnih sustava koji se sastoje od određenog broja stupnjeva slobode, od kojih pak svaki ima konačno-dimenzionalan prostor stanja. U ovakvim sustavima mogu se pojavljivati makroskopski neobična svojstva, koja nastaju zbog njihove kvantne prirode. Jedna od vrsta sustava koji pripadaju kategoriji kvantnih višečestičnih sustava su i *kvantni spinski sustavi*, u kojima se stupnjevima slobode u sustavu dodjeljuje svojstvo spina. U ovakvim sustavima, čestice su raspoređene na grafu na kojemu je definiran pojam udaljenosti, a time i *lokalnosti* [23, 24].

Hilbertov prostor kvantnog spinskog sustava Λ može se opisati tenzorskim produktom Hilbertovih prostora koji pripadaju pojedinačnim stupnjevima slobode u sustavu, $\mathcal{H}_u$, kao u jednadžbi (2.1). Vremenska evolucija sustava jedinstveno je određena njegovim Hamiltonijanom i Schrödingerovom jednadžbom te joj je u ovom radu pristupljeno u Heisenbergovoj slici, tako da stanja sustava ostaju nepromije-

njena u vremenu, a sva vremenska ovisnost nalazi se u operatorima koji djeluju u sustavu. Također, Hamiltonijan koji opisuje kvantni spinski sustav moguće je zapisati kao sumu *lokalnih Hamiltonijana,* koji opisuju interakcije geometrijski lokalnih elemenata sustava, kao u jednadžbi (2.4), gdje je $\mathcal{B}(i,r)$ kugla radijusa r koji je manji od određenog r_0, a $H_{\mathcal{B}(i,r)}$ označava operator koji je nošen samo u području te kugle. U najčešćem slučaju proučavaju se sustavi u kojima $\|H_{\mathcal{B}(i,r)}\|$ ima brzi pad u ovisnosti o r.

Više o kvantnim spinskim sustavima i njihovom pristupu iz perspektive kvantne teorije informacija moguće je pronaći u [21, 25, 26].

Fazni prijelazi su vrlo dugo bili uspješno opisivani Ginzburg-Landauovom teorijom faznih prijelaza slamanjem simetrije, no prije nekoliko desetljeća otkriveni su sustavi čiji se fazni prijelazi ne mogu opisati ovom teorijom — najznačajnije, tekućine koje iskazuju razlomljeni kvantni Hallov efekt — te je razvijena teorija *topološkog kvantnog uređenja* (ili *topološkog uređenja*), koja opisuje ove fazne prijelaze na temperaturama blizu apsolutne nule [30, 31]. Najzanimljivije je svojstvo topološki uređenih sustava za područje kvantnog računarstva njihova otpornost na smetnje, budući da ih se opisuje globalnim opservablama – *topološkim invarijantama*.

Jednostavan opis *kvantnih faznih prijelaza* koji postoje u topološki uređenim sustavima dan je u [32]. Promatra se kvantni spinski sustav sa česticama raspoređenima na rešetki, koji je opisan Hamiltonijanom $H(g) = H_0 + gH_1$ s energetskim procjepom iznad prostora osnovnog stanja sustava, gdje je g bezdimenzijski parametar povezivanja, a $[H_0, H_1] = 0$ i H_0 i H_1 ne ovise o g. Tada se komponente Hamiltonijana mogu dijagonalizirati istovremeno i može postojati vrijednost $g = g_c$ za koju je osnovno stanje sustava neanalitičko, tako da pobuđeno stanje postaje osnovno stanje, kao što je prikazano na slici 2.1. Kvantni fazni prijelaz odgovara neanalitičkoj točki energije osnovnog stanja beskonačno-dimenzionalnog kvantnog spinskog sustava – dakle, procjep iznad prostora osnovnog stanja mora se zatvoriti. Prijelazom iz jedne faze u drugu mijenja se *dugodosežna kvantna sprega* u sustavu, a različiti uzorci isprepletenosti odgovaraju topološkim uređenjima sustava.

Topološki uređene faze na temperaturi apsolutne nule matematički se opisuje na mnogo međusobno ekvivalentnih načina, no još uvijek ne postoji odgovarajući opis na temperaturama iznad apsolutne nule. U [20, 34] topološko uređenje definirano je pomoću dva uvjeta — TQO-1 i TQO-2 — koji će biti uvedeni u nastavku ovog ulomka. Promatran je N-dimenzionalni kvantni spinski sustav Λ linearne dimenzije L, s periodičnim rubnim uvjetima, s Hilbertovim prostorom kao u (2.1). Skup svih blokova $A \subseteq \Lambda$ linearne veličine $r > 0$ definiran je kao $\mathcal{S}(r)$ (vidi sliku 2.2). Za sustav u kojem se u obzir uzimaju samo interakcije u skupovima $A \in \mathcal{S}(2)$, Hamiltonijan sustava, H_0, može biti zapisan kao u jednadžbi (2.5), gdje je Q_A međudjelovanje s podrškom na A i svojstvima kao u jednadžbi (2.6). Ako je projektor na potprostor osnovnog stanja označen s P, a projektor na potprostor pobuđenih stanja označen s

Q u jednadžbama (2.7), gdje je I operator identiteta, a lokalne verzije operatora za blok $B \in \mathcal{S}(r \geq 2)$, kao u jednadžbama (2.8), moguće je iskazati uvjete topološkog uređenja;

1. **TQO-1:** Za blok $A \in \mathcal{S}(r)$ s $r \leq L^\star$,

$$PO_AP = cP, \qquad c \in \mathbb{C}, \tag{7.1}$$

   za bilo koji operator O_A koji djeluje na A.

2. **TQO-2:** Za blokove $A \in \mathcal{S}(r)$ s $r \leq L^\star$ i $B \in \mathcal{S}(r+2)$, gdje je B blok koji sadrži A i sve najbliže susjede stupnjeva slobode u A, neka su reducirane matrice gustoće definirane kao $\rho_A = \mathrm{Tr}_{A^c}(P)$ i $\rho_A^{(B)} = \mathrm{Tr}_{A^c}(P_B)$, gdje je $A^c = \Lambda \setminus A$. Tada vrijedi

$$\ker \rho_A = \ker \rho_A^{(B)}. \tag{7.2}$$

Prvi uvjet često se smatra najvažnijim uvjetom koji definira topološko uređenje i naziva se uvjetom lokalne neraspoznatljivosti osnovnih stanja sustava. Drugi uvjet jamči da projektori P i P_B djeluju na jednak način na podskupu $A \subset B$, te da je lokalni prostor osnovnog stanja dosljedan globalnom na podskupovima koji su dovoljno daleko od B.

Topološko uređenje moguće je matematički opisati s naglaskom na faze Hamiltonijana te pomoću relacije između kvantnih faza i *kvantnih sklopova* konačne dubine (za vizualni prikaz vidi sliku 2.3). Pregled ovih opisa nalazi se u [30].

Eksperimentalna ostvarenja topološki uređenih sustava razvijaju se ponajprije za uspješnu realizaciju topoloških kvantnih računala te uključuju dva pristupa – sustavi koji prirodno iskazuju topološko uređenje i sustavi koji se planirano sastavljaju kako bi se u njima stvorilo topološko uređenje. Povijesno, najvažnija vrsta sustava u kojima prirodno postoje topološke faze je ona u kojoj se javlja razlomljeni kvantni topološki efekt [16, 37], a također i $(p_x + ip_y)$ supravodiči [39, 40] i supratekućine [41], te Majoranine žice [38]. Pregled ovih sustava može se naći u [42, 43].

Općenito, eksperimentalne istraživačke grupe najčešće usmjeravaju pozornost prema mikroskopskoj izgradnji sustava koji iskazuju topološko uređenje, koristeći optičke rešetke, ionske klopke, Josephsonove spojeve, supravodljive qubite i slično. Detaljniji opis nalazi se u Potpoglavlju 2.5.

7.3 Kvantno računarstvo otporno na pogreške

Kao što je opisano u Potpoglavlju 7.1, kvantno računarstvo more problemi dekoherencije i nestabilnosti na smetnje, što uvodi greške u korišteni sustav. Zbog toga je razvijena teorija *kvantnog računarstva otpornog na pogreške*, u sklopu kojeg se proučavaju metode *kvantnog ispravljanja grešaka*. Klasična teorija računarstva otpornog na pogreške nema istu važnost kao ona kvantnog računarstva otpornog na

pogreške budući da je klasična informacija iznimno robustna, a jednostavan primjer metode korištene u klasičnom računarstvu otpornom na pogreške jest *kod s ponavljanjem*. Ovom metodom vrijednosti nule i jedinice kodiraju se u *logička stanja* u kojima se njihove vrijednosti ponavljaju nekoliko puta (vidi tablicu 3.1) – u slučaju pogreške i izmjene vrijednosti jednog od broja u kodiranom obliku, originalni bit povraća se težinskim glasanjem, tako da se odabire onaj bit koji se više puta pojavljuje u kodiranom obliku. Zbog *teorema o nemogućnosti kloniranja* [51], isti postupak nije moguće primijeniti na kvantnu informaciju te korišteni postupci postaju mnogo kompliciraniji. Primjerice, za ispravljanje grešaka koje obrću qubite (iz vrijednosti $|0\rangle$ u vrijednost $|1\rangle$ i obratno), može se koristiti *kod za obrtanje qubita*. Ako se logička stanja kodiraju kao u tablici 3.2, greške je moguće otkriti primjenom projektora, kao u tablici 3.3. Ovaj postupak kao povratnu informaciju daje *sindrom greške*, odnosno ukazuje na to kakav tip pogreške se dogodio – u ovom slučaju na kojem qubitu.

Općenito, postupak kvantnog ispravljanja grešaka svodi se na sljedeće korake:

1. Kodiranje kvantnog stanja u kvantni kod za ispravljanje grešaka, definiran kao potprostor C nekog većeg Hilbertovog prostora, koristeći unitarnu operaciju. Ovaj potprostor naziva se *kodnim prostorom* ili *kodnim potprostorom*.

2. Izlaganje sustava uzroku grešaka.

3. Mjerenje sindroma greške.

4. Izvođenje postupka povrata originalne informacije, ovisno o sindromu.

Za najbolje rezultate pri mjerenju sindroma, odabrani potprostori originalnog kodnog prostora moraju biti međusobno ortogonalni.

Postupak ispravljanja grešaka, $\mathcal{R}$, smatra se uspješnim ako, za sustav na kojeg djeluje izvor grešaka $\mathcal{E}$, vrijedi relacija (3.1) – radi se o jednakosti u slučaju u kojem je operator $\mathcal{E}$ unitaran. Detaljnije informacije o kvantnom ispravljanju grešaka moguće je pronaći u [1], a opis često korištenih kodova koji pripadaju CSS-tipu (Calderbank-Shor-Steane) nalazi se u [52] i [53].

Stabilizacijski kodovi predstavljaju važan oblik kodova za kvantno ispravljanje grešaka, i jednostavno se definiraju na sljedeći način, s time da Paulijeva tenzorska grupa predstavlja tenzorski produkt Paulijevih grupa G kao u jednadžbi (3.3), s grupnom operacijom množenja (σ operatori su odgovarajući Paulijevi operatori).

Definicija 7.1. *Neka je S podgrupa Paulijeve tenzorske grupe za n qubita, G_n. Za vektorski potprostor V_S svih n-qubitnih stanja za koja vrijedi $(\forall\psi \in V_S) \wedge (\forall S_\xi \in S) : S_\xi |\psi\rangle = |\psi\rangle$, podgrupa S se naziva stabilizacijskom grupom prostora V_S; prostor V_S naziva se stabilizacijskim prostorom.*

Generatori podgrupe S komutiraju, a za sustav od n qubita i stabilizacijsku grupu s $n-k$ nezavisnih generatora, odgovarajući stabilizacijski prostor V_S je 2^k-dimenzionalan (dokaz ovog svojstva nalazi se u Potpoglavlju 3.2). Posljedično, u stabilizacijskom prostoru 2^k ortogonalnih vektora može služiti kao baza za kodiranje logičkih stanja.

Primjer generatora za stabilizacijski kod za pet qubita nalazi se u tablici 3.4.

Najčešće korišten stabilizacijski kod je *torusni kod* [54], koji služi i kao pojednostavljeni model kvantnih spinskih sustava. Definiran je na dvodimenzionalnoj pravokutnoj rešetci na torusu, gdje su spinski stupnjevi slobode na rubovima rešetke, a elementi rešetke su prikazani na slici 3.1. Stabilizacijski operatori torusnog koda Λ povezani su s vrhovima i pločicama rešetke, i definirani u jednadžbi (3.10), te prikazani na slici 3.2. Oni međusobno komutiraju, a njihove svojstvene vrijednosti su $+1$ i -1, a ako je Hilbertov prostor svih qubita u sustavu označen s $\mathcal{N}$, kodni prostor jednak je relaciji u jednadžbi (3.11). Zbog rubnih uvjeta na stabilizacijske operatore (jednadžbe (3.12)), dimenzija kodnog prostora je jednaka $2^2 = 4$. *Logički operatori* odgovaraju značenju logičkih stanja te im je uloga čuvanje informacije u kodu, a u torusnom kodu su predstavljeni netrivijalnim zatvorenim petljama Paulijevih operatora, kao što je prikazano na slici 3.3, a definirani su jednadžbama (3.13). Očito, u torusnom kodu postoje četiri logička operatora — dva σ^x tipa i dva σ^z tipa — što znači da se u sustav mogu kodirati dva qubita, koristeći po dva nekomutirajuća logička operatora. Ovi logički operatori tvore Paulijevu algebru na kodnom prostoru sustava.

Hamiltonijan torusnog koda predstavlja $+1$ svojstveni prostor stabilizatora, i zapisan je u jednadžbi (3.14). Osnovna stanja sustava lokalno se ne mogu razlikovati, a pobuđena stanja od prostora osnovnih stanja odvojena su za konačnu vrijednost energijskog procjepa, te torusni kod odgovara definiciji topološkog uređenja.

Greške u torusnom kodu nastaju djelovanjem Paulijevih operatora na spinske stupnjeve slobode na rubovima rešetke, što stvara pobuđenja anyonskog tipa na susjednim elementima rešetke – za σ^x operator to su pločice, a za σ^z operator vrhovi. Ova pobuđenja mogu se pomicati po odgovarajućim elementima rešetke korištenjem Paulijevih operatora, kao što je prikazano na slici 3.4. Njihovim netrivijalnim pletenjem tvore se *kvantna vrata* u sustavu (za detaljan opis, vidi [19]).

Topološki uređeni sustavi stabilni su pri utjecaju lokalnih perturbacija, što je dokazano u [20] i [34] sljedećim teoremom.

Teorem 7.1. *Neka postoji Hamiltonijan bez smetnje, H_0, koji odgovara svojstvima koja definiraju topološki uređeni sustav (TQO-1,2 –* (7.1)*,* (7.2)*), a na sustav neka djeluje perturbacija V, koja se može zapisati kao suma geometrijski lokalnih interakcija s ograničenom normom:*

$$V = \sum_{r \geq 1} \sum_{A \in \mathcal{S}(r)} V_{r,A}, \tag{7.3}$$

gdje je $\mathcal{S}(r)$ skup kocaka linearne veličine r, a $V_{r,A}$ je operator koji djeluje samo na stupnjeve slobode u A. Neka veličina interakcije ima eksponencijalni pad u r:

$$\max_{\mathcal{A}\in\mathcal{S}(r)}\|V_{r,A}\| \leq Je^{-\mu r}, \tag{7.4}$$

gdje su $J, \mu > 0$ konstante neovisne o linearnoj veličini sustava.

Tada postoje konstante $J_0, c_1, c_2 > 0$ koje ovise samo o μ i prostornoj dimenziji D, tako da je za sve $J \leq J_0$ spektar $H = H_0 + V$ sadržan (do općenitog linearnog pomaka u iznosu energije) u uniji intervala $\bigcup_{k\leq 0} I_k$, gdje k ima vrijednosti u spektru H_0 tako da

$$I_k = \{\lambda \in \mathbb{R} \ : \ k(1 - c_1 J) - \delta \leq \lambda \leq k(1 + c_1 J) + \lambda\}, \tag{7.5}$$

i

$$\delta = \mathrm{poly}(L) \exp\left(-c_2 L^{3/8}\right), \tag{7.6}$$

gdje je L linearna veličina sustava.

Prikaz ovog rezultata je na slici 3.5. Za dovoljno malu perturbaciju, tako da je $J < \frac{1}{c_1(4k+2)}$, sve spektralne vrpce razdvojene su za bar $1/2$.

7.4 Dinamika lokalnih kvantnih sustava

Dinamika lokalnih kvantnih sustava istražuje se pomoću nekoliko moćnih matematičkih metoda koje su razvijane od strane velikog broja autora i dosad primijenjene na mnogim problemima. U ovom radu korištene su tri metode, koje će u ovom potpoglavlju biti okvirno predstavljene; radi se o *kvazi-adijabatskom produljenju*, *Lieb-Robinsonovoj granici* i *Kitaevljevom razvoju vlastite energije*.

Kvazi-adijabatsko produljenje (engl. *quasi-adiabatic continuation*) metoda je koju je po prvi put predstavio Hastings tijekom prošlog desetljeća [57,58], a mnogi autori potom razvili ju i prilagodili specifičnim problemima (primjerice, [23, 56, 59–62]). Koristi se za proučavanje dinamike kvantnih spinskih sustava sa spektralnim procjepom iznad energije osnovnog stanja, kada se na taj sustav djeluje smetnjom.

Ovom metodom Hamiltonijanu sustava dodaje se ovisnost o parametru, te se pomiče sporo na skali spektralnog procjepa. Promatraju se promjene lokalnih Hamiltonijana, $H_Z(s)$, od $s = 0$ do $s = 1$, ili od početnog stanja do punog djelovanja perturbacije. Uz pretpostavku da za Hamiltonijan sustava vrijedi $H_s = \sum_Z H_Z(s)$, tako da su lokalni Hamiltonijalni diferencijabilni u s, i da se spektralni procjep ne zatvara ($\gamma > 0$), definira se *generator kvazi-adijabatskog produljenja*, $\mathcal{D}_s$, kao u jednadžbi (4.1), gdje je $|\psi_0(s)\rangle$ osnovno stanje sustava. Definira se filtar-funkcija koja ovisi o spektralnom procjepu, $F(t) \in L^1(\mathbb{R})$ [56, 61], sa sljedećim svojstvima:

- neparna je, tako da je $\mathcal{D}_s$ antihermitska,
- njena Fourierova transformacija je $\bar{F}(\omega) = -1/\omega$ za $|\omega| \geq 1/2$ (normirana vrijednost vezana uz spektralni procjep),

- ima brži pad od bilo koje negativne potencije vremena za velike $|t|$.

Tada je generator definiran jednadžbom (4.2), a unitarni operator produljenja je definiran izrazom $\partial_s U_s = i\mathcal{D}_s U_s,\ U_0 = \mathbb{1}$ i ima oblik kao u jednadžbi (4.3), gdje $\mathcal{S}'$ označava da se radi o eksponentu uređenom po putu (engl. *path-ordered*).

Tada za projektor P u sustavu vrijedi (4.4), a *obučeni operatori* (engl. *dressed operators*) u sustavu definirani su kao u jednadžbi (4.5).

Također, generator $\mathcal{D}_s$ može se izraziti kao u jednadžbi (4.6), gdje je $\mathcal{D}_s(u;r)$ generator nošen kuglom radijusa r, sa središtem u u, koji ima eksponencijalni pad s r [56]. Može se pokazati da U_s zadovoljava Lieb-Robinsonove granice za subeksponencijalni pad.

Ukratko, ako za sustav postoje dva Hamiltonijana u istoj topološkoj fazi, njihova osnovna stanja se mogu pretvoriti jedno u drugo koristeći operator U_s.

Lieb-Robinsonova granica [63] predstavlja teorijsku gornju granicu na brzinu širenja informacije kroz kvantne višečestične sustave. Ta granica koristi se za uvođenje geometrijskih granica utjecaja smetnje u sustavima i za opis svojstva lokalnosti, a u ovom diplomskom radu predstavljena je modifikacija koja je korištena u Poglavlju 5, iz definicija u [23] i [67].

Za N-čestični sustav na rešetci Λ, uvodi se udaljenost na grafu između dva skupa, A i B, kao u jednadžbi (4.7), gdje je $\mathrm{dist}(i,j)$ udaljenost između točaka i i j. Promjer skupa A definiran je u jednadžbi (4.8). Hamiltonijan sustava se može zapisati kao u jednadžbi (4.9), tako da iznos $\|H_Z\|$ ima brzi pad s $\mathrm{diam}(Z)$. Vrijedi sljedeći teorem.

Teorem 7.2. *Neka za sve čestice na točkama i vrijedi:*

$$\sum_{X\ni i}\|H_X\||X|\exp[\mu\ \mathrm{diam}(X)] \le s < \infty, \quad \mu, s > 0. \tag{7.7}$$

Neka su A_X i B_Y operatori s podrškom na skupovima X i Y, a $A_X(t)$ neka je vremenska evolucija operatora A_X. Tada, ako je $\mathrm{dist(X, Y)} > 0$*,*

$$\begin{aligned}\|[A_X(t), B_Y]\| &\le 2\|A_X\|\|B_Y\|\sum_{i\in X}\exp[-\mu\ \cdot \mathrm{dist}(i,Y)]\left[e^{2s|t|}-1\right] \\ &\le 2\|A_X\|\|B_Y\||X|\exp[-\mu\ \cdot \mathrm{dist}(X,Y)]\left[e^{2s|t|}-1\right].\end{aligned} \tag{7.8}$$

Intuitivnije, ovaj se teorem može izreći kao u jednadžbi (4.16), gdje je v_{LR} *Lieb-Robinsonova brzina* – konačna brzina širenja informacije kroz sustav, greške koja je eksponencijalno mala u udaljenosti od originalnog skupa podrške operatora.

Razvoj vlastite energije (ili Kitaevljev razvoj vlastite energije) degeneracijska je teorija perturbacije koja se može primijeniti na sustavima sa spektralnim procjepom iznad prostora osnovnog stanja, a koristi se za ocjenu učinka specifične perturbacije na sustav opisan stabilizacijskim kodom [68].

Pri izvodu ove metode koristi se generalizacija Greenove funkcije – *rezolventa Hamiltonijana*, kao u jednadžbi (4.17). *Svojstvena energija sustava* može se zapisati kao u jednadžbi (4.23), gdje je P_0 projektor na prostor osnovnog stanja sustava, V perturbacija koja djeluje na sustav, a $G_0'(E) = ((E - H_0)^{-1})'$ neperturbirana Greenova funkcija za pobuđeno stanje originalnog Hamiltonijana, H_0, uz energiju sustava zapisanu kao E.

Primjenom ove metode u sustavu na koji djeluje dovoljno lokalna perturbacija, može se pokazati da originalno ponašanje sustava ostaje jednako, u smislu degeneracije prostora osnovnog stanja i postojanja spektralnog procjepa iznad istog, do reda razvoja vlastite energije koji odgovara linearnoj veličini sustava. Više detalja nalazi se u [69] i [68].

7.5 Rezultati

U ovom potpoglavlju predstavljeni su rezultati analize granica utjecaja perturbacije u topološki uređenom sustavu s lokalnim rupama.

Kad se na topološki uređeni sustav opisan Hamiltonijanom H_0 djeluje perturbacijom sV, perturbirani Hamiltonijan dan jednadžbom (5.1), a utjecaj smetnje na logički operator L nošen skupom A u torusnom kodu za ovaj sustav može se proučavati koristeći kvazi-adijabatsko produljenje i Lieb-Robinsonove granice.

Logički operator razvijen kvazi-adijabatskim produljenjem zapisan je u jednadžbi (5.2), gdje je U_s operator evolucije, a $A^C \setminus R'$ skup komplementaran skupu A, ali bez trake debljine R s obje strane logičkog operatora, što je i prikazano na slici 5.1. Korištenjem kvazi-adijabatskog produljenja dobiven je obučeni logički operator, L_s, te je izveden izraz u jednadžbi (5.4), gdje je v_{LR} Lieb-Robinsonova brzina, $\mu > 0$, a $d(A, A^C \setminus R')$ udaljenost je između dva skupa.

Dakle, razlika između ograničenog i neograničenog razvijenog logičkog operatora opada eksponencijalno s R, što govori o lokalnosti utjecaja perturbacije u sustavu.

Jednostabilizatorski defekti tipa rupe u torusnom su kodu definirani kao nedostajući stabilizacijski operatori – vizualni prikaz se nalazi na slici 5.2. Uklanjanje stabilizacijskih operatora utječe na svojstva koda; broj logičkih qubita u sustavu može se zapisati izrazom (5.6), gdje je n_p' broj rupa na pločicama, a n_v' broj rupa na vrhovima. Primjer logičkih operatora koji nastaju dodavanjem jednostabilizatorskih rupa prikazan je na slici 5.3. Sličan oblik modifikacije torusnog koda površinski su kodovi [10, 49, 71], kao što je prikazano na slici 5.4, a u njima postoje logički operatori koji ovise o vrsti granica.

U Potpoglavlju 5.3 problemu postojanja jednostabilizatorskih rupa u torusnom kodu pristupljeno je Kitaevljevim razvojem vlastite energije (slika 5.5), te su izve-

dene veličine komutatora za logički operator torusnog koda i efektivni Hamiltonijan u sustavu sa smetnjom (jednadžba (5.7)). Bitan rezultat iskazan je u jednadžbi (5.24), gdje je L_Z promatrani logički operator, H Hamiltonijan sustava sa smetnjom, a R udaljenost između rupa. Dakle, kodirana informacija nije eksponencijalno stabilna u linearnoj veličini sustava, već tek u udaljenosti između rupa. Međutim, ovom problemu se može mnogo detaljnije pristupiti koristeći kvazi-adijabatsko produljenje i Lieb-Robinsonove granice.

Koristeći lemu iz [56], koja uvodi relaciju kao u jednadžbi (5.25), gdje je U_s operator razvoja kvazi-adijabatskog produljenja, $U_s(A)$ isti operator ograničen na skup A, a $\epsilon_s(R)$ parametar koji ima subeksponencijalni pad u debljini granice između skupova A i A^C — $\partial A(R)$ — kao u jednadžbi (5.26).

Sustav je podijeljen kao na slici 5.7, te je izvedena relacija koja govori o odnosu presvučenog logičkog operatora u sustavu bez rupa, L_s, i obučenog logičkog operatora u sustavu kao na slici 5.7, L'_s, koja je prikazana u izrazu (5.37).

Dobiveni rezultat znači da je obučeni logički operator u sustavu bez rupa subeksponencijalno u R dobra aproksimacija presvučenog logičkog operatora u sustavu s rupama.

7.6 Zaključak

U ovom diplomskom radu izvedene su i objašnjene granice za perturbativnu stabilnost topološki uređenih sustava s lokalnim rupama. Ideja teorema o perturbativnoj stabilnosti degeneracije osnovnog stanja u topološki uređenom sustavu [20, 34] proširena je izravnom analizom utjecaja perturbacije na torusni kod s lokalnim defektima. Zaključuje se da, za dovoljno udaljene rupe, ova izjava vrijedi, što topološke invarijante u sustavu ostavlja stabilnima, čuvajući kodiranu informaciju. Daljnja analiza specifičnih postava koji su važni za procese u kvantnoj informaciji i kvantnom računarstvu nužna je kako bi mogli biti donijeti zaključci o konkretnim problemima koji se javljaju u topološkom kvantnom računarstvu otpornom na pogreške.

Također, prirodna proširenja dobivenih rezultata javljaju se kao analiza ponašanja drugih stabilizacijskih kodova u sličnim postavima [10, 73–76], u svrhu provjere međusobne konzistentnosti ovih modela. Važno je proučiti djelovanje perturbacije na površinske kodove, za što bi se ovi rezultati mogli pokazati korisnima, a zanimljivo je i pitanje utjecaja perturbacije na površinske kodove s neuobičajenim rubnim uvjetima, kao i na kodove s većim rupama, te rupama s miješanim rubnim uvjetima [49, 72]. Poseban izazov javlja se u sustavima koji iskazuju kvazi-topološko uređenje [77], zbog toga što se u njima ne može primijeniti kvazi-adijabatsko produljenje.

U zaključku, ovaj diplomski rad predstavio je granice stabilnosti topološkog uređenja u torusnom kodu s lokalnim defektima tipa rupe, koristeći kvazi-adijabatsko produljenje, Lieb-Robinsonove granice i Kitaevljev razvoj vlastite energije – u ovom radu su te granice predstavljene i analizirane. Rezultati se slažu s prethodno dokazanim rezultatom za torusni kod bez defekata [20], te ga proširuju, uz indikaciju da sustav koji je opisan torusnim kodom s lokalnim rupama ostaje stabilan pod slabim lokalnim smetnjama, uz uvjet da je udaljenost između rupa dovoljno velika.

7.7 Hrvatski nazivi slika i tablica

Popis slika

Popis tablica

References

[1] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge; New York, 10th anniversary edition, 2010.

[2] Z. Wang. *Topological Quantum Computation*. Number 112. American Mathematical Soc., 2010.

[3] R. P. Feynman. Simulating physics with computers. *International Journal of Theoretical Physics*, 21(6-7):467–488, 1982.

[4] S. Lloyd. Universal Quantum Simulators. *Science*, 273(5278):1073–1078, August 1996.

[5] G. Moore. Cramming More Components Onto Integrated Circuits. *Proceedings of the IEEE*, 86(1):82–85, January 1998.

[6] B. Schneier. *Applied Cryptography: Protocols, Algorithms, and Source Code in C*. Wiley, New York, 2nd ed edition, 1996.

[7] P. Shor. Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer. *SIAM Review*, 41(2):303–332, January 1999.

[8] L. K. Grover. A fast quantum mechanical algorithm for database search. *arXiv:quant-ph/9605043*, May 1996.

[9] D. J. Bernstein, J. Buchmann, and E. Dahmen, editors. *Post-Quantum Cryptography*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.

[10] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill. Topological quantum memory. *Journal of Mathematical Physics*, 43(9):4452, 2002.

[11] D. P. DiVincenzo and IBM. The Physical Implementation of Quantum Computation. *Fortschritte der Physik*, 48(9-11):771–783, September 2000.

[12] D. Aharonov and M. Ben-Or. Fault-Tolerant Quantum Computation With Constant Error Rate. *arXiv:quant-ph/9906129*, June 1999.

[13] S. J. Devitt, K. Nemoto, and W. J. Munro. Quantum Error Correction for Beginners. *Reports on Progress in Physics*, 76(7):076001, July 2013.

[14] J. Preskill. Reliable Quantum Computers. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 454(1969):385–410, January 1998.

[15] M. H. Freedman, A. Kitaev, M. J. Larsen, and Z. Wang. Topological Quantum Computation. *arXiv:quant-ph/0101025*, January 2001.

[16] The Nobel Prize in Physics 1998 - Advanced Information. http://www.nobelprize.org/nobel_prizes/physics/laureates/1998/advanced.html, 1998.

[17] J. M. Leinaas and J. Myrheim. On the Theory of Identical Particles. *Il Nuovo Cimento*, 37 B(1):1–23, 1977.

[18] F. Wilczek. Quantum mechanics of fractional-spin particles. *Physical review letters*, 49(14):957, 1982.

[19] J. Preskill. Lecture notes for Physics 219: Quantum computation. *Caltech Lecture Notes*, 1999.

[20] S. Bravyi, M. Hastings, and S. Michalakis. Topological quantum order: Stability under local perturbations. *Journal of Mathematical Physics*, 51(9):093512, 2010.

[21] B. Nachtergaele. Quantum Spin Systems. *arXiv:math-ph/0409006*, September 2004.

[22] W. Nolting and W. D. Brewer. *Fundamentals of Many-Body Physics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.

[23] M. B. Hastings. Locality in Quantum Systems. *arXiv:1008.5137 [math-ph, physics:quant-ph]*, August 2010.

[24] B. Nachtergaele and R. Sims. Locality Estimates for Quantum Spin Systems. *arXiv:0712.3318 [cond-mat, physics:math-ph, physics:quant-ph]*, December 2007.

[25] P. Naaijkens. Quantum spin systems on infinite lattices. *arXiv:1311.2717 [math-ph, physics:quant-ph]*, November 2013.

[26] B. Zeng, X. Chen, D.-L. Zhou, and X.-G. Wen. Quantum Information Meets Quantum Matter – From Quantum Entanglement to Topological Phase in Many-Body Systems. *arXiv:1508.02595 [cond-mat, physics:quant-ph]*, August 2015.

[27] L. D. Landau and E. M. Lifshitz. *Statistical Physics*. Elsevier, October 2013.

[28] D. C. Tsui, H. L. Stormer, and A. C. Gossard. Two-Dimensional Magnetotransport in the Extreme Quantum Limit. *Physical Review Letters*, 48(22):1559–1562, May 1982.

[29] R. B. Laughlin. Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations. *Physical Review Letters*, 50(18):1395–1398, May 1983.

[30] X. Chen, Z.-C. Gu, and X.-G. Wen. Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order. *Physical Review B*, 82(15), October 2010.

[31] X.-G. Wen. An introduction of topological orders. *http://web. mit. edu/physics/people/faculty/docs/wen_intro_topological_orders.pdf*, August 2016.

[32] S. Sachdev. *Quantum Phase Transitions*. Cambridge University Press, Cambridge; New York, second edition, 2011.

[33] C. Kittel. *Introduction to Solid State Physics*. John Wiley & Sons, Inc., 7th edition, 1996.

[34] S. Bravyi and M. B. Hastings. A short proof of stability of topological order under local perturbations. *Communications in mathematical physics*, 307(3):609–627, 2011.

[35] A. Kitaev and J. Preskill. Topological entanglement entropy. *Physical Review Letters*, 96(11), March 2006.

[36] M. Levin and X.-G. Wen. Detecting topological order in a ground state wave function. *Physical Review Letters*, 96(11), March 2006.

[37] X. G. Wen and Q. Niu. Ground-state degeneracy of the fractional quantum Hall states in the presence of a random potential and on high-genus Riemann surfaces. *Physical Review B*, 41(13):9377–9396, May 1990.

[38] A. Kitaev. Unpaired Majorana fermions in quantum wires. *Physics-Uspekhi*, 44(10S):131–136, October 2001.

[39] Y. Peng, F. Pientka, E. Berg, Y. Oreg, and F. von Oppen. Signatures of topological Josephson junctions. *Physical Review B*, 94(8), August 2016.

[40] S. D. Sarma, C. Nayak, and S. Tewari. Proposal to stabilize and detect half-quantum vortices in strontium ruthenate thin films: Non-Abelian braiding statistics of vortex matter in a $p_x + ip_y$ superconductor. *Physical Review B*, 73(22), June 2006.

[41] V. Gurarie, L. Radzihovsky, and A. V. Andreev. Quantum phase transitions across p-wave Feshbach resonance. *Physical Review Letters*, 94(23), June 2005.

[42] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma. Non-Abelian anyons and topological quantum computation. *Reviews of Modern Physics*, 80(3):1083–1159, September 2008.

[43] X.-G. Wen. Topological order: From long-range entangled quantum matter to an unification of light and electrons. *ISRN Condensed Matter Physics*, 2013:1–20, 2013.

[44] A. Micheli, G. K. Brennen, and P. Zoller. A toolbox for lattice-spin models with polar molecules. *Nature Physics*, 2(5):341–347, May 2006.

[45] R. Barends, J. Kelly, A. Megrant, A. Veitia, D. Sank, E. Jeffrey, T. C. White, J. Mutus, A. G. Fowler, B. Campbell, Y. Chen, Z. Chen, B. Chiaro, A. Dunsworth, C. Neill, P. O'Malley, P. Roushan, A. Vainsencher, J. Wenner, A. N. Korotkov, A. N. Cleland, and J. M. Martinis. Superconducting quantum circuits at the surface code threshold for fault tolerance. *Nature*, 508(7497):500–503, April 2014.

[46] J. Kelly, R. Barends, A. G. Fowler, A. Megrant, E. Jeffrey, T. C. White, D. Sank, J. Y. Mutus, B. Campbell, Y. Chen, Z. Chen, B. Chiaro, A. Dunsworth, I.-C. Hoi, C. Neill, P. J. J. O'Malley, C. Quintana, P. Roushan, A. Vainsencher, J. Wenner, A. N. Cleland, and J. M. Martinis. State preservation by repetitive error detection in a superconducting quantum circuit. *Nature*, 519(7541):66–69, March 2015.

[47] A. Córcoles, E. Magesan, S. J. Srinivasan, A. W. Cross, M. Steffen, J. M. Gambetta, and J. M. Chow. Demonstration of a quantum error detection code using a square lattice of four superconducting qubits. *Nature Communications*, 6:6979, April 2015.

[48] D. Nigg, M. Mueller, E. A. Martinez, P. Schindler, M. Hennrich, T. Monz, M. A. Martin-Delgado, and R. Blatt. Experimental Quantum Computations on a Topologically Encoded Qubit. *Science*, 345(6194):302–305, July 2014.

[49] S. B. Bravyi and A. Y. Kitaev. Quantum codes on a lattice with boundary. *arXiv:quant-ph/9811052*, November 1998.

[50] D. Ristè, S. Poletto, M.-Z. Huang, A. Bruno, V. Vesterinen, O.-P. Saira, and L. DiCarlo. Detecting bit-flip errors in a logical qubit using stabilizer measurements. *Nature Communications*, 6:6983, April 2015.

[51] W. K. Wootters and W. H. Zurek. A single quantum cannot be cloned. *Nature*, 299:802–803, 1982.

[52] A. R. Calderbank and P. W. Shor. Good Quantum Error-Correcting Codes Exist. *Physical Review A*, 54(2):1098–1105, August 1996.

[53] A. M. Steane. Efficient fault-tolerant quantum computing. *Nature*, 399(6732):124–126, May 1999.

[54] A. Y. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2–30, January 2003.

[55] D. Gottesman. An introduction to quantum error correction and fault-tolerant quantum computation. In *Quantum Information Science and Its Contributions to Mathematics, Proceedings of Symposia in Applied Mathematics*, volume 68, pages 13–58, 2009.

[56] S. Michalakis. Stability of the Area Law for the Entropy of Entanglement. *arXiv:1206.6900 [quant-ph]*, June 2012.

[57] M. B. Hastings. Lieb-Schultz-Mattis in higher dimensions. *Physical review B*, 69(10):104431, 2004.

[58] M. B. Hastings. Locality in quantum and Markov dynamics on lattices and networks. *Physical review letters*, 93(14):140402, 2004.

[59] M. B. Hastings. Quasi-adiabatic Continuation for Disordered Systems: Applications to Correlations, Lieb-Schultz-Mattis, and Hall Conductance. *arXiv:1001.5280 [math-ph]*, January 2010.

[60] T. J. Osborne. Simulating adiabatic evolution of gapped spin systems. *Physical Review A*, 75(3):032321, 2007.

[61] S. Bachmann, S. Michalakis, B. Nachtergaele, and R. Sims. Automorphic Equivalence within Gapped Phases of Quantum Lattice Systems. *Communications in Mathematical Physics*, 309(3):835–871, February 2012.

[62] T. J. Osborne. Ground state of a class of noncritical one-dimensional quantum spin systems can be approximated efficiently. *Physical Review A*, 75(4):042306, April 2007.

[63] E. H. Lieb and D. W. Robinson. The Finite Group Velocity of Quantum Spin Systems. *Communications in mathematical physics*, 28:251–257, 1972.

[64] B. Nachtergaele and R. Sims. Lieb-Robinson bounds and the exponential clustering theorem. *Communications in mathematical physics*, 265(1):119–130, 2006.

[65] S. Bravyi, M. B. Hastings, and F. Verstraete. Lieb-Robinson Bounds and the Generation of Correlations and Topological Quantum Order. *Physical Review Letters*, 97(5), July 2006.

[66] T. Matsuta, T. Koma, and S. Nakamura. Improving the Lieb-Robinson bound for long-range interactions. *arXiv:1604.05809 [math-ph]*, April 2016.

[67] B. Nachtergaele and R. Sims. Much Ado About Something: Why Lieb-Robinson bounds are useful. *arXiv:1102.0835 [math-ph]*, February 2011.

[68] J. Kempe, A. Kitaev, and O. Regev. The complexity of the local Hamiltonian problem. In *FSTTCS 2004: Foundations of Software Technology and Theoretical Computer Science*, pages 372–383. Springer, 2004.

[69] A. Kitaev. Anyons in an exactly solved model and beyond. *Annals of Physics*, 321(1):2–111, 2006.

[70] B. J. Brown. *Defects in Topologically Ordered Lattice Models*. PhD thesis, Imperial College London, Department of Physics, September 2013.

[71] A. G. Fowler, M. Mariantoni, J. M. Martinis, and A. N. Cleland. Surface codes: Towards practical large-scale quantum computation. *Physical Review A*, 86(3), September 2012.

[72] N. Delfosse, P. Iyer, and D. Poulin. Generalized surface codes and packing of logical qubits. *arXiv:1606.07116 [quant-ph]*, June 2016.

[73] C. G. Brell. A proposal for self-correcting stabilizer quantum memories in 3 dimensions (or slightly less). *New Journal of Physics*, 18(1):013050, January 2016.

[74] H. Bombin and M. A. Martin-Delgado. Topological Quantum Distillation. *Physical Review Letters*, 97(18), October 2006.

[75] S. Bravyi and J. Haah. Analytic and numerical demonstration of quantum self-correction in the 3D Cubic Code. *Physical Review Letters*, 111(20), November 2013.

[76] H. Bombin and M. A. Martin-Delgado. Homological Error Correction: Classical and Quantum Codes. *Journal of Mathematical Physics*, 48(5):052105, 2007.

[77] P. Bonderson and C. Nayak. Quasi-Topological Phases of Matter and Topological Protection. *Physical Review B*, 87(19), May 2013.