H-distributions and compactness by compensation

Mišur, Marin

University of Zagreb
Faculty of Science
Department of Mathematics

Marin Mišur

H-distributions and compactness by compensation

Doctoral thesis

Supervisor:
Prof. dr. sc. Nenad Antonić

Zagreb, 2017

Sveučilište u Zagrebu
Prirodoslovno-matematički fakultet
Matematički odsjek

Marin Mišur

H-distribucije i kompaktnost kompenzacijom

Doktorski rad

Mentor:
Prof. dr. sc. Nenad Antonić

Zagreb, 2017.

In memory of Ante (1981–2016) and Marko (1969–2014)

Foreword

Most of the problems of continuum physics, which are modelled by partial differential equations, consists of two types of laws:

(1) linear differential relations (conservation laws)
(2) nonlinear pointwise connections (laws of constitution).

Systematic approach to the laws of constitution is possible by using Young (parametrised) measures, and the connection to the conservation laws is done by compactness by compensation. This approach allows the systematic treatment of many problems of continuum mechanics, first of which was non-periodic homogenisation. A special case of linear differential relations is irrotationality of a vector field (which is then locally potential field), and on that simple, yet so common case in applications, there has been a myriad of results, more known under the joint name of variational theory of microstructures. More complex example is an application to hyperbolic conservation laws. However, an important limitation of this theory is that conservation laws are restricted only to linear relations with constant coefficients.

For evolution equations Young measures, as objects which can not see the direction of propagation, are not particularly suitable, and one needs to replace them with new objects. One of such new objects are H-measures, non-negative Radon measures on co-spherical tangent bundle over the given domain, which depend on both the physical and the phase variable (which describes the direction of propagation). They have been introduced independently by Luc Tartar and Patrick Gérard (under the name microlocal defect measures). In applications the so called localisation principle of H-measures is of utter importance because it enables us to localise the support of H-measure by using a partial differential equations which is satisfied by the generating sequence of functions. An application of the localisation principle of H-measures is a form of compactness by compensation with linear differential relations, but with variable coefficients which are continuous functions. That generalisation proved to be useful in applications to the small amplitude homogenisation (in other words, to problem of mixing materials with low density contrast) and to optimal design, where important industrial problems are considered: how to mix available materials with the goal of obtaining a mixture with optimal physical properties.

Over the course of time, it became evident that classical H-measures, although successful in some areas, are not entirely suitable for studying problems with different order of derivatives with respect to different variables, i.e. heat equation, Schödinger equation, Navier-Stokes equations and vibrating plate equation. Last eight years saw development of parabolic H-measures, ultra-parabolic H-measures, fractional H-measures, one-scale H-measures and microlocal compactness forms.

Even tough localisation principle was shown in the above mentioned cases, obtaining an appropriate form of compactness by compensation is not always simple. For example, localisation principle for parabolic H-measures includes fractional derivatives (which are non-local operators), so it is not entirely clear how to get a corresponding compactness by compensation result. On the other hand, H-measures are by definition tied to the Hilbert space L^2, which is suitable for linear problems. But by that, the success of their application to nonlinear problems with solutions in L^p spaces is severely restricted. A step forward in that direction was done by Nenad Antonić and Darko Mitrović by introduction of H-distributions as a generalisation of H-measures to the L^p spaces. In the L^2 case, Fourier transform together with Plancherel's theorem proved to be a very efficient tool. Unfortunately, in the L^p case it is necessary to use more complex results and tools of pseudodifferential calculus and analysis, i.e. the Hörmander-Mihlin and the Schwartz kernel theorem. Additional disadvantage with H-distributions is that they are not Radon measures anymore, but distributions in the Schwartz sense. By that, some of the standard tools of the measure theory are not at our disposal. Namely, the Radon-Nikodym theorem gave a relatively simple representation of H-measures which proved useful in obtaining compactness by compensation result from the localisation principle. In the case of distributions, there is no analogue of Radon-Nikodym theorem, which makes obtaining compactness by compensation from the localisation principle highly nontrivial.

The goal of this thesis is better understanding of H-distributions and its variants, namely, their localisation principles and compactness by compensation result.

In the first chapter we generalise the known results of the so called First commutation lemma in the L^p case using a result of Krasnoselskij type.

Then in the second chapter, in order to give a more precise description of H-distributions, we refine the notion of distributions by introducing a notion of anisotropic distributions of finite order. We prove the Schwartz kernel theorem for anisotropic distributions and with its help, we show that H-distributions are anisotropic distributions of finite order. This leads to the improvement of the existing localisation principle.

In the third chapter, we give a variant of H-distributions and a compactness by compensation result with variable discontinuous coefficients. An application to the nonlinear equation of parabolic type is given.

I would like to use this opportunity to thank my advisor Nenad Antonić for all the generous help and moral support during my PhD research, and Darko Mitrović for interesting problems and joyful moments which followed their investigation. They became not only my mentors, but my friends as well.

I thank my friends and co-authors Marko Erceg and Andrej Novak. Nowadays I gladly remember the frustrating afternoons Marko and I spent studying the Schwartz kernel theorem and nuclear spaces. His enthusiastic suggestions and help shaped the exposition given in the chapter on the Schwartz kernel theorem.

I would like to thank senior members of our research group: Krešimir Burazin, Ivan Ivec, Martin Lazar and Marko Vrdoljak on useful discussions and constructive remarks which lead to the final version of results presented in this thesis.

I would also like to thank junior members of the research group and members of the Seminar for differential equations and numerical analysis who patiently listened to my presentations and gave useful comments.

During my travels and talks, I have met kind people whose unselfish help, comments and suggestions helped me and made me improve. Just to name a few: Debora Amadori, Christian Klingenberg, Michael Kunzinger, Evgenij Jurjevič Panov, Stevan Pilipović, Luc Tartar, and Enrique Zuazua. Thank You!

I thank my friends Josip Saratlija, Saša Stanko, Petar Sirković, Luka Rimanić, Vedran Novaković and Neven Krajina for listening to my ravings and advice they gave.

I thank Marlies for the chocolate.

Finally, I would like to thank my parents and my brother who have always been my support during my studies.

Contents

I. The First commutation lemma

In this chapter we review and generalise results on compactness of commutators of multiplication and Fourier multiplier operators by H. O. Cordes in several directions with respect to the smoothness of multiplication function and by replacing the Fourier multiplier operator by a more general pseudodifferential operator. We review and improve the known results both in the standard L^2 setting, as well as for general L^p, with $1 < p < \infty$. Furthermore, we extend these results to less regular symbols.

Majority of results of this chapter can be found in [6].

1. Overview

In their seminal paper [31], Joseph John Kohn and Louis Nirenberg introduced pseudodifferential operators as sums of elementary operators on L^2 (or the corresponding Sobolev spaces H^s), which were of the form

$$Au = \bar{\mathcal{F}}(\psi\mathcal{F}(bu)) ,$$

where $\mathcal{F}u(\boldsymbol{\xi}) = \hat{u}(\boldsymbol{\xi}) = \int e^{-2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}u(\mathbf{x})d\mathbf{x}$ denotes the Fourier transform, with the inverse $\bar{\mathcal{F}}v(\mathbf{x}) = \check{v}(\mathbf{x}) = \int e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}v(\boldsymbol{\xi})d\boldsymbol{\xi}$. The above elementary operator is a composition $Au = \mathcal{A}_\psi M_b u$, with $M_b u = bu$. For good enough ψ, the Fourier multiplier operator $\mathcal{A}_\psi u := (\psi\hat{u})^\vee$ is a bounded operator on L^p, the sufficient conditions being provided by two celebrated results: the Marcinkiewicz and the Hörmander-Mihlin theorem [24, Chapter 5.2]. However, for $p = 2$ a simpler result characterises the space of good ψ as L^∞ [24, Chapter 2.5].

In order to prove that pseudodifferential operators form an algebra (modulo smoothing operators), it was important to know that their commutator

$$[\mathcal{A}_\psi, M_b] := \mathcal{A}_\psi M_b - M_b\mathcal{A}_\psi$$

is an operator of lower order (i.e. that it maps L^2 to H^1 continuously). In fact, this means that different quantisations (like Kohn-Nirenberg's, adjoint or Weyl's), to a given function (symbol) associate operators which are equal modulo lower order operators.

A decade later, Heinz Otto Cordes [12] investigated the properties of such commutators, both in the L^2 case, improving the earlier boundedness result of Alberto Pedro Calderón and Rémi Vaillancourt, as well as in the L^p case, for $p \in \langle 1, \infty\rangle$.

The L^2 result [12, Theorem C] reads:

Theorem 1. *If bounded continuous functions b and ψ satisfy*

$$(1)\qquad \lim_{|\mathbf{x}|\to\infty}\sup_{|\mathbf{h}|\leq 1}\Big|b(\mathbf{x}+\mathbf{h}) - b(\mathbf{x})\Big| = 0 \quad \textit{and} \quad \lim_{|\boldsymbol{\xi}|\to\infty}\sup_{|\mathbf{h}|\leq 1}\Big|\psi(\boldsymbol{\xi}+\mathbf{h}) - \psi(\boldsymbol{\xi})\Big| = 0 ,$$

then the commutator $[\mathcal{A}_\psi, M_b]$ is a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$. ∎

Theorem 1 in this form appears to be quite useful as it can be seen in several recent papers. In [39], it was essentially used in the proof of linear instability of 2D water waves modelled by the Euler equations. In [38], similar results relaying on the same tools were obtained for nonlinear solitary waves which emerge from several classes of equations (BBM, KdV, and Boussinesq type equations). In [52], Theorem 1 was applied in order to derive new formulae for the wave operators for a Friedrichs scattering system with a rank-one perturbation. We shall pay special attention to the H-measures [60, 23] where a variant of Theorem 1 was substantially used.

As for the L^p-compactness, if we additionally assume that ψ satisfies conditions of either the Hörmander-Mihlin or the Marzinkiewicz multiplier theorem, then the commutator $[\mathcal{A}_\psi, M_b]$ is L^p-compact. Such a result is used in [51] for deriving Fredholm property of Douglis–Nirenberg elliptic systems as well as in [13] where the same property was shown for a class of operators with homogeneous kernels of compact type.

Despite their obvious importance, up to our knowledge, there are no substantial extensions of the Cordes results. While it seems that one cannot say much more in the L^2-case, than it was said above in Theorem 1, it is unclear what are the precise properties of commutators when the well known Marcinkiewicz (7) or Hörmander-Mihlin (5) sufficient conditions for L^p-continuity of the Fourier multiplier operator are fulfilled.

The Marcinkiewicz conditions easily imply compactness of the commutator (we actually significantly simplify the proof of the less general statement from [12]; see Corollary 1). On the other hand, the Hörmander-Mihlin conditions require a substantially new and non-trivial proof based on the techniques used in the proof of the Hörmander-Mihlin theorem.

Finally we analyse a natural extension, when the Fourier multiplier operator is replaced by a general pseudodifferential operator in the commutator, and provide appropriate sufficient conditions for compactness.

The chapter is organised as follows: in the next section we briefly describe our motivation and compare various L^2 results. The third section is devoted to the L^p case, for $p \in \langle 1, \infty\rangle$, while in the fourth section we provide certain conditions under which the regularity of b can be relaxed (see the conditions of Lemma 5 and Corollary 4). Although Fourier multipliers do not act between local Lebesgue spaces, the result of Theorem 4 could be understood in that sense. It also represents the single most important result of the fourth section. Finally, in the last section we consider the commutators of pseudodifferential operators and multiplication and provide a new result when symbol $a(\mathbf{x}, \boldsymbol{\xi})$ satisfies some suitable additional conditions.

2. H-measures and the L^2 case

H-measures, independently introduced by Luc Tartar [60] and Patrick Gérard [23], are defined as Radon measures on the co-spherical bundle over the domain Ω; for a single parametrisation ($\Omega \subseteq \mathbf{R}^d$) they are measures on the product $\Omega \times S^{d-1}$. We often refer to Ω as the *physical space*, while $\boldsymbol{\xi} \in S^{d-1}$ is the dual variable to $\mathbf{x} \in \Omega$.

Having some practical applications in mind, where the lower regularity of functions is important, Tartar defined the symbols as continuous functions on the phase space, in variables $\mathbf{x} \in \Omega$ and $\boldsymbol{\xi} \in \mathbf{R}^d_* := \mathbf{R}^d \setminus \{0\}$, satisfying certain additional properties; in the simplest case they are products $p(\mathbf{x}, \boldsymbol{\xi}) = b(\mathbf{x})\psi(\boldsymbol{\xi})$. In particular, it is assumed that ψ is defined on S^{d-1}, and then extended to $\mathbf{R}^d_*$ by homogeneity:

$$\psi(\boldsymbol{\xi}) = \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) .$$

This introduces the projection $\boldsymbol{\xi} \mapsto \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$, of $\mathbf{R}^d_*$ onto S^{d-1}.

In some applications, e.g. when dealing with the heat equation $u_t - u_{xx} = 0$, we are led to consider some variants [5; 61, Chapter 28]. The symbol of the heat operator is $i\tau + \xi^2$, and the natural scaling is no longer along the rays through the origin, but along the parabolas $\tau = c\xi^2$ in the dual space.

Function ψ should be constant along these parabolas, and we can choose a set of representative points for each parabola. One choice could be the unit sphere S^{d-1}; however, the coordinate expression for the projection is not convenient in this case. A smooth compact hypersurface (in fact, a rotational ellipsoid) we chose in [5] is implicitly given by:

$$\mathrm{P}^{d-1} \;\ldots\; \rho^4(\tau,\boldsymbol{\xi}) := \tau^2 + \frac{|\boldsymbol{\xi}|^2}{2} = 1 \;.$$

For any given point $(\tau,\boldsymbol{\xi}) \in \mathbf{R}^d_*$, its projection to P^{d-1} is given by (as $\rho^4 > 0$ on $\mathbf{R}^d_*$, by choosing positive determination of roots, the projection is uniquely defined)

$$\pi(\tau,\boldsymbol{\xi}) = \left(\frac{\tau}{\rho^2(\tau,\boldsymbol{\xi})}, \frac{\boldsymbol{\xi}}{\rho(\tau,\boldsymbol{\xi})}\right) \;.$$

In particular, from this formula it is clear that we indeed have a projection on P^{d-1}. Some further variants were studied in [49, 43, 19].

In general, we shall be able to define a variant H-measure [61, loc. cit.] as long as we have a smooth compact hypersurface Σ in $\mathbf{R}^d$, and a smooth projection $\pi : \mathbf{R}^d_* \longrightarrow \Sigma$. For given $M, \varrho \in \mathbf{R}^+$ we denote the set

$$Y(M,\varrho) = \{(\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathbf{R}^{2d} : |\boldsymbol{\xi}|, |\boldsymbol{\eta}| \geq M \;\&\; |\boldsymbol{\xi}-\boldsymbol{\eta}| \leq \varrho\} \;,$$

which is depicted in Figure 1.

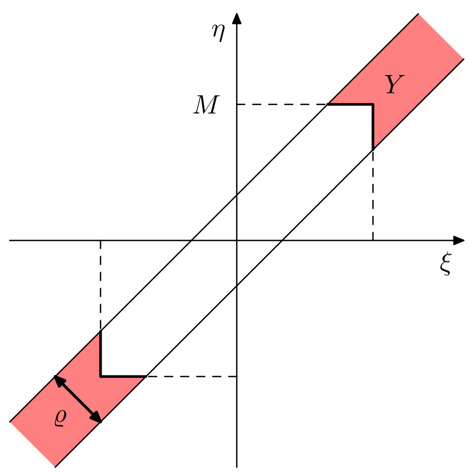

Figure 1. Set $Y(M,\varrho)$.

Tartar's First commutation lemma [61, Lemma 28.2] (in its general form) reads

Lemma 1. *If $b \in \mathrm{C}_0(\mathbf{R}^d)$, while $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ satisfies the condition*

$$(2) \qquad (\forall\, \varrho, \varepsilon \in \mathbf{R}^+)(\exists\, M \in \mathbf{R}^+) \quad |\psi(\boldsymbol{\xi}) - \psi(\boldsymbol{\eta})| \leq \varepsilon \;\; (\text{ss } (\boldsymbol{\xi},\boldsymbol{\eta}) \in Y(M,\varrho)) \;,$$

then $[\mathcal{A}_\psi, M_b]$ is a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$. ∎

A set of sufficient conditions for (2), which covers both the classical H-measures and their parabolic variants, is given in the following lemma:

Lemma 2. *Let $\pi : \mathbf{R}^d_* \longrightarrow \Sigma$ be a smooth projection to a smooth compact hypersurface Σ, such that $|\nabla\pi(\boldsymbol{\xi})| \longrightarrow 0$ for $|\boldsymbol{\xi}| \longrightarrow \infty$, and let $\psi \in \mathrm{C}(\Sigma)$. Then $\psi \circ \pi$ (ψ extended by homogeneity of order 0) satisfies (2).*

Dem. Clearly, taking a constant C resulting from uniform continuity of ψ on the compact Σ we have:

$$\Big|\psi(\boldsymbol{\xi}) - \psi(\boldsymbol{\eta})\Big| = \Big|\psi(\pi(\boldsymbol{\xi})) - \psi(\pi(\boldsymbol{\eta}))\Big| \leq C\Big|\pi(\boldsymbol{\xi}) - \pi(\boldsymbol{\eta})\Big| \leq |\boldsymbol{\xi} - \boldsymbol{\eta}| \sup_{\boldsymbol{\zeta}\in[\boldsymbol{\xi},\boldsymbol{\eta}]} |\nabla\pi(\boldsymbol{\zeta})| \ ,$$

where for the last inequality we applied the Mean value theorem to the projection π.

For $|\boldsymbol{\xi} - \boldsymbol{\eta}| \leq \rho$ and $\varepsilon > 0$ given, we can find M large enough such that for $|\boldsymbol{\xi}|, |\boldsymbol{\eta}| \geq M > \rho$ the above is bounded by $|\varepsilon|$.

Q.E.D.

Let us check that the last result applies to the two cases of our particular interest indeed. In both cases we have a continuous function ψ defined on a smooth compact surface Σ (S^{d-1} or P^{d-1}), and then it is extended to $\mathbf{R}^d_*$ taking constant values along certain curves, which transversally intersect Σ and cover the whole space (rays from the origin, or parts of quadratic parabolas in the parabolic case). It only remains to be shown that the projections satisfy $|\nabla\pi(\boldsymbol{\xi})| \longrightarrow 0$ for $|\boldsymbol{\xi}| \longrightarrow \infty$. However, it is a matter of straightforward calculation to check that $|\nabla\pi| \leq 1/|\boldsymbol{\xi}|$ in the first case, and $|\nabla\pi| \leq c\rho^{-2}$ in the second (c being some positive constant).

This result allows for introduction of a class of *pseudodifferential* operators and corresponding symbols, which, in contrast to the classical theory, are not required to be smooth. For the sake of completeness, and in order to stress the importance of the preceding lemmata, we briefly sketch the procedure bellow. A reader interested into further details and some applications should consult [60; 61, Chapter 28; 5].

An *admissible symbol* is a function $p \in \mathrm{C}(\mathbf{R}^d \times \Sigma)$ which can be written in the form $p(\mathbf{x}, \boldsymbol{\xi}) = \sum_k b_k(\mathbf{x})\psi_k(\boldsymbol{\xi})$, with $b_k \in \mathrm{C}_0(\mathbf{R}^d)$ and $\psi_k \in \mathrm{C}(\Sigma)$, such that it satisfies the rapidly converging condition

$$\sum_k \|b_k\|_{\mathrm{L}^\infty} \|\psi_k\|_{\mathrm{L}^\infty} < \infty \ . \tag{3}$$

We say that an operator $L \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$ has an admissible symbol p if that operator can be written as a sum: $L = \sum_k \mathcal{A}_{\psi_k} M_{b_k} (\mathrm{mod}\ \mathcal{K}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d)))$ (i.e. up to a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$); where the elementary operators M_{b_k} and $\mathcal{A}_{\psi_k}$ were defined in the Introduction. Here we implicitly assume that ψ_k has already been extended to $\mathbf{R}^d_*$, i.e. we identify it with $\psi_k \circ \pi$.

Among all such operators corresponding to a given symbol p we can choose the *standard* one: $L_0 := \sum_k \mathcal{A}_{\psi_k} M_{b_k}$. It satisfies (for $u \in \mathrm{L}^2(\mathbf{R}^d) \cap \mathrm{L}^1(\mathbf{R}^d)$)

$$\mathcal{F}(L_0 u)(\boldsymbol{\xi}) = \sum_k \psi_k(\pi(\boldsymbol{\xi})) \int_{\mathbf{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} b_k(\mathbf{x}) u(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} p(\mathbf{x}, \pi(\boldsymbol{\xi})) u(\mathbf{x})\, d\mathbf{x} \ .$$

Thus, L_0 is well defined—it does not depend on the choice of a representation for p.

If we consider operator $L := \sum_k M_{b_k}\mathcal{A}_{\psi_k}$, where $\mathcal{A}_{\psi_k}$ and M_{b_k} are as in the decomposition of standard operator L_0, we have for $u \in \mathrm{L}^2(\mathbf{R}^d) \cap \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d))$

$$Lu(\mathbf{x}) = \sum_k b_k(\mathbf{x}) \int_{\mathbf{R}^d} e^{2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} \psi_k(\pi(\boldsymbol{\xi}))\hat{u}(\boldsymbol{\xi}) d\boldsymbol{\xi} = \int_{\mathbf{R}^d} e^{2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} p(\mathbf{x}, \pi(\boldsymbol{\xi}))\hat{u}(\boldsymbol{\xi}) d\boldsymbol{\xi} \ ,$$

and this is exactly the operator with symbol p in the framework of classical linear theory.

Remark 1. Note that in classical theory the symbols are additionally assumed to be smooth; more precisely, we say that $a \in \mathrm{C}^\infty(\mathbf{R}^d \times \mathbf{R}^d; \mathbf{C})$ is a *symbol* if

$$(\forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{N}_0^d)\ (\exists C_{\boldsymbol{\alpha},\boldsymbol{\beta}} > 0)(\forall \mathbf{x} \in \mathbf{R}^d)(\forall \boldsymbol{\xi} \in \mathbf{R}^d) \quad |\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\boldsymbol{\xi}}^{\boldsymbol{\beta}} a(\mathbf{x}, \boldsymbol{\xi})| \leq C_{\boldsymbol{\alpha},\boldsymbol{\beta}} \langle \boldsymbol{\xi} \rangle^{m-\rho|\boldsymbol{\beta}|+\delta|\alpha|} ,$$

where $0 \leq \delta \leq \rho \leq 1$, $\langle \boldsymbol{\xi} \rangle = \sqrt{1 + 4\pi^2 |\boldsymbol{\xi}|^2}$, while $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ are constants depending only on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The space of these symbols we denote by $S^m_{\rho,\delta}$, and with a family of seminorms (the best constants $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ above are the values of corresponding seminorms) it becomes a Fréchet space. The symbols we defined are global; if for any compact $K \subseteq \mathbf{R}^d$ there are constants such that the above is valid only for $\mathbf{x} \in K$, then we get a wider class of symbols.

The pseudodifferential operator $a(\mathbf{x}, D)$ associated to symbol $a(\mathbf{x}, \boldsymbol{\xi})$ is defined by

$$[a(\mathbf{x}, D)u](\mathbf{x}) := (a\hat{u})^\vee(\mathbf{x}) .$$

For such operators a similar result to Lemma 1 is valid, even though they do not necessarily satisfy the rapidly converging condition (3), nor they can be expressed as sums of elementary operators. We shall return to such operators in the last section.

Let us stress that L and L_0 differ only by a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$:

$$L - L_0 = \sum_k (M_{b_k} \mathcal{A}_{\psi_k} - \mathcal{A}_{\psi_k} M_{b_k}) = \sum_k [M_{b_k}, \mathcal{A}_{\psi_k}] ,$$

because by the First commutation lemma each of the commutators is compact, with the norm less than $2\|b_k\|_{\mathrm{L}^\infty} \|\psi_k\|_{\mathrm{L}^\infty}$. Here we have used the well known fact that the uniform limit of compact operators is compact.

For two operators L and L' with symbols $s = \sum b_m \psi_m$ and $s' = \sum b'_n \psi'_n$, it is immediate that $s + s'$ is also a symbol, and it is the symbol of operator $L + L'$. Slightly more complicated is to check that ss' is also a symbol (of operator $L \circ L'$).

Indeed, $ss' = \sum_{m,n} (b_m b'_n)(\psi_m \psi'_n)$, and the partial sums of this series are bounded:

$$\sum_{m=1}^M \sum_{n=1}^N \|b_m b'_n\|_{\mathrm{L}^\infty} \|\psi_m \psi'_n\|_{\mathrm{L}^\infty} \leq \left(\sum_{m=1}^M \|b_m\|_{\mathrm{L}^\infty} \|\psi_m\|_{\mathrm{L}^\infty} \right) \left(\sum_{n=1}^N \|b'_n\|_{\mathrm{L}^\infty} \|\psi'_n\|_{\mathrm{L}^\infty} \right) .$$

The remaining arguments are straightforward now.

Remark 2. The results proven in Lemma 1 can be extended in an obvious way to the case where function b has a limit as $|\mathbf{x}| \to \infty$ (i.e. it can be extended to a continuous function on the one-point Aleksandrov compactification of $\mathbf{R}^d$, which we shall denote by $\mathbf{R}^d_\infty$). Indeed, if we denote the limit by $\tilde{b}$, then the commutator $[\mathcal{A}_\psi, M_{b-\tilde{b}}]$ satisfies the conditions of Lemma 1. Since $[\mathcal{A}_\psi, M_{\tilde{b}}] = 0$ (keep in mind that $\tilde{b}$ is a constant), we conclude that $[\mathcal{A}_\psi, M_b]$ is a compact operator on L^2 as well. This has already been noticed by Tartar [61, p. 342].

Comparing Lemma 1 to Theorem 1, we see that the former allows ψ to be in $\mathrm{L}^\infty(\mathbf{R}^d)$, while the latter requires the corresponding symbol to be continuous and bounded on $\mathbf{R}^d$. Otherwise, the assumption from Theorem 1 on ψ and the ones from Lemma 1 are readily equivalent.

3. A generalisation to $\mathrm{L}^p(\mathbf{R}^d)$

In [7], H-distributions were introduced, a generalisation of H-measures to the L^p-setting, for $p > 1$ (see also [4, 36, 42]). One of the crucial parts in their construction was compactness of the commutator $C := [\mathcal{A}_\psi, M_b]$ on L^p.

Let us first state the Hörmander-Mihlin theorem [24, Theorem 5.2.7.] which gives sufficient conditions on the symbol of multiplier operator for it to be continuous on all L^p, $p \in \langle 1, \infty\rangle$.

Theorem 2. *Let $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ have partial derivatives of order less than or equal to κ - the least integer strictly greater than $d/2$ (i.e. $\kappa = [d/2] + 1$). If for some $A > 0$*

$$(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \quad |\boldsymbol{\alpha}| \leq \kappa \qquad \Longrightarrow \qquad |\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})| \leq A |\boldsymbol{\xi}|^{-|\boldsymbol{\alpha}|}\,, \tag{4}$$

or

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \quad |\boldsymbol{\alpha}| \leq \kappa \quad \Longrightarrow \quad \int_{r<|\boldsymbol{\xi}|<2r} |\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \leq A^2 r^{d-2|\boldsymbol{\alpha}|}, \tag{5}$$

then for any $p \in \langle 1, \infty\rangle$ and the associated multiplier operator $\mathcal{A}_\psi$ there exists a constant C_d such that

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \leq C_d \max\{p, 1/(p-1)\}(A + \|\psi\|_{\mathrm{L}^\infty(\mathbf{R}^d)}).$$

∎

The condition (4) is called the Mihlin condition, while (5) is called the Hörmander condition. Let us just remark that while the Hörmander condition is more general than the Mihlin condition, the latter is considerably easier to check.

One can find a variant of the commutator compactness result in [12, Theorem C_p] under the following conditions:

$$\Big(\forall \psi \in \mathrm{C}^{2\kappa}(\mathbf{R}^d)\Big)\, (\exists c > 0)(\forall \boldsymbol{\xi} \in \mathbf{R}^d) \qquad \Big|(1 + |\boldsymbol{\xi}|)^{|\boldsymbol{\alpha}|} D^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})\Big| \leq c\,, \tag{6}$$

where $\boldsymbol{\alpha} \in \mathbf{N}_0^d$, $|\boldsymbol{\alpha}| \leq 2\kappa$, while b satisfies the same conditions as in Theorem 1. A similar result was stated in [7, Lemma 3.1] for $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ and $b \in \mathrm{C}_0(\mathbf{R}^d)$:

Lemma 3. *Let (v_n) be a bounded sequence, both in $\mathrm{L}^2(\mathbf{R}^d)$ and in $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 2, \infty]$, and such that $v_n \rightharpoonup 0$ in the sense of distributions. Then $Cv_n \longrightarrow 0$ strongly in $\mathrm{L}^q(\mathbf{R}^d)$, for any $q \in [2, r] \setminus \{\infty\}$.*

∎

The proof was based on the simple interpolation inequality of L^p spaces, namely that $\|f\|_{\mathrm{L}^q} \leq \|f\|_{\mathrm{L}^2}^\theta \|f\|_{\mathrm{L}^r}^{1-\theta}$, where $1/q = \theta/2 + (1-\theta)/r$, but for $q = r$ the proof was incomplete.

In this section we shall show that commutator C is compact on each $\mathrm{L}^p(\mathbf{R}^d)$, $p \in \langle 1, \infty\rangle$. For that we use a result of Mark Aleksandrovič Krasnosel'skij [34], which was proven only for a bounded domain there. As we were not able to find a proof for the unbounded domain in the literature, we provide the complete proof below.

Lemma 4. *Assume that linear operator A is compact on $\mathrm{L}^2(\mathbf{R}^d)$ and bounded on $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 1, \infty\rangle$. Then A is also compact on $\mathrm{L}^p(\mathbf{R}^d)$, for any p between 2 and r (i.e. such that $1/p = \theta/2 + (1-\theta)/r$, for some $\theta \in \langle 0, 1\rangle$).*

Dem. We shall construct a sequence of finite-rank operators A_k on $\mathrm{L}^2(\mathbf{R}^d)$, approximating A in the operator norm. Notice that the image of the unit ball in $\mathrm{L}^2(\mathbf{R}^d)$ by A, which we denote by S, is relatively compact, thus precompact in the norm topology on $\mathrm{L}^2(\mathbf{R}^d)$, so completely bounded, i.e. for any size $\frac{1}{2k}$ there is an $m \in \mathbf{N}$ and points $h_1, \dots, h_m \in S$ such that the balls of radius $\frac{1}{2k}$ centred around these points cover S. As $\mathrm{L}^2(\mathbf{R}^d) \cap \mathrm{L}^\infty(\mathbf{R}^d)$ is dense in $\mathrm{L}^2(\mathbf{R}^d)$, by allowing for a larger mesh size $\varepsilon := 1/k$, we can replace the h_i-s by $g_1, \dots, g_m \in \mathrm{L}^2(\mathbf{R}^d) \cap \mathrm{L}^\infty(\mathbf{R}^d)$.

Choose a bounded measurable set $E \subseteq \mathbf{R}^d$, of positive Lebesgue measure $\mathsf{vol}(E) > 0$, such that for each $i \in 1..m$

$$\|g_i(1-\chi_E)\|_{\mathrm{L}^2(\mathbf{R}^d)} \leq \varepsilon.$$

As $g_1\chi_E, \dots, g_m\chi_E$ are bounded functions on a set of finite measure, they can be approximated in L^∞ norm by step functions. In particular, for step functions we can choose the values on each step to be the average of the function over the corresponding part. More precisely, for given $\varepsilon' = \varepsilon/\sqrt{\mathsf{vol}(E)}$, we can find a finite disjoint decomposition of $E = \bigcup_{i=1}^K E_i$, where each E_i is of positive measure, such that the *averaging operator*

$$\mathcal{E}g = \sum_{i=1}^{K} \Big(\fint_{E_i} g \Big) \chi_{E_i}$$

satisfies

$$(\forall i \in 1..m) \qquad \|\mathcal{E}g_i - g_i\chi_E\|_{\mathrm{L}^\infty(\mathbf{R}^d)} \leq \varepsilon' \,.$$

Let us examine more closely the properties of the averaging operator $\mathcal{E}$.

From its definition it is clear that $\mathcal{E}g \equiv 0$ outside E. Furthermore, for a given i, denote $m := \fint_{E_i} g$. Now we have, by Jensen's inequality, that

$$\fint_{E_i} |m|^2 = \Big| \fint_{E_i} g \Big|^2 \leq \fint_{E_i} |g|^2 \,,$$

so on each E_i the operator $\mathcal{E}$ does not increase the L^2 norm. By the additivity of integral over the domain, we see that $\mathcal{E}$ is a non-expansive map on $\mathrm{L}^2(E)$.

For an arbitrary $g \in S$, let g_i be such that $\|g - g_i\|_{\mathrm{L}^2(\mathbf{R}^d)} \leq \varepsilon$; we have the following

$$\begin{aligned}
\|\mathcal{E}g - g\|_{\mathrm{L}^2(\mathbf{R}^d)} &\leq \|\mathcal{E}(g-g_i)\|_{\mathrm{L}^2(\mathbf{R}^d)} + \|\mathcal{E}g_i - g_i\|_{\mathrm{L}^2(\mathbf{R}^d)} + \|g_i - g\|_{\mathrm{L}^2(\mathbf{R}^d)} \\
&\leq 2\|g - g_i\|_{\mathrm{L}^2(\mathbf{R}^d)} + \|\mathcal{E}g_i - g_i\|_{\mathrm{L}^2(\mathbf{R}^d)} \\
&\leq 2\varepsilon + \|\mathcal{E}g_i - g_i\|_{\mathrm{L}^2(\mathbf{R}^d)} \,,
\end{aligned}$$

where we have used the fact that $\mathcal{E}$ is a non-expansive map on $\mathrm{L}^2(E)$. Concerning the last term, recalling the choice of E, it follows that

$$\|\mathcal{E}g_i - g_i\|_{\mathrm{L}^2(\mathbf{R}^d)} \leq \|g_i(1-\chi_E)\|_{\mathrm{L}^2(\mathbf{R}^d)} + \sqrt{\mathsf{vol} E}\, \|\mathcal{E}g_i - g_i\chi_E\|_{\mathrm{L}^\infty(\mathbf{R}^d)} \leq 2\varepsilon.$$

Thus, for an arbitrary $g \in S$, we have that $\|\mathcal{E}g - g\|_{\mathrm{L}^2(\mathbf{R}^d)} \leq 4\varepsilon$.

By defining $A_k := \mathcal{E} \circ A$, we obtain a finite-rank operator satisfying

$$\|A - A_k\|_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} = \sup_{\|f\|_{\mathrm{L}^2(\mathbf{R}^d)} \leq 1} \|Af - \mathcal{E}Af\|_{\mathrm{L}^2(\mathbf{R}^d)} = \sup_{g \in S} \|g - \mathcal{E}g\|_{\mathrm{L}^2(\mathbf{R}^d)} \leq 4\varepsilon.$$

To finish the proof, we have to use a simple interpolation argument. Take an arbitrary p between 2 and r, for which there is a $\theta \in \langle 0,1\rangle$ such that

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{r}.$$

By an application of the Riesz-Thorin interpolation theorem, we have

$$\begin{aligned}\|A - A_k\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d);\mathrm{L}^p(\mathbf{R}^d))} &\leq \|A - A_k\|^{1-\theta}_{\mathcal{L}(\mathrm{L}^r(\mathbf{R}^d);\mathrm{L}^r(\mathbf{R}^d))} \|A - A_k\|^{\theta}_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \\ &\leq \left(2\|A\|_{\mathcal{L}(\mathrm{L}^r(\mathbf{R}^d);\mathrm{L}^r(\mathbf{R}^d))}\right)^{1-\theta} (4\varepsilon)^{\theta},\end{aligned}$$

which concludes the proof.

Q.E.D.

Let us notice that, if we knew that A were compact on $\mathrm{L}^q(\mathbf{R}^d)$ for some $q \in \langle 1,\infty\rangle$ (not necessarily equal to 2), then the same proof as above would have remained valid with 2 replaced by q.

Remark 3. We can avoid using the averaging operator $\mathcal{E}$. Instead, we can use a cut-off operator $M_{\chi_E} f := \chi_E f$. It is obvious that $M_{\chi_E} : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ is a non-expansive map for each $p \in \langle 1,\infty\rangle$. Then we have the following bounds $\|M_{\chi_E} g_i - g_i\|_{\mathrm{L}^2(\mathbf{R}^d)} \leq \varepsilon$ for every i and $\|M_{\chi_E} g - g\|_{\mathrm{L}^2(\mathbf{R}^d)} \leq 3\varepsilon$ for every $g \in S$. Defining $\tilde{A}_k := M_{\chi_E} A$, we have $\|A - \tilde{A}_k\|_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \leq 3\varepsilon$. In this case $\tilde{A}_k$ is only a compact operator, while A_k from the previous lemma is of finite rank. This remark is due to Evgenij Jurjevič Panov.

Using the preceding lemma, we can significantly simplify the proof of Theorem C_p in [12].

Corollary 1. *Let the function b be bounded and continuous over $\mathbf{R}^d$ and satisfy (1). Furthermore, assume that function ψ is in $\mathrm{C}^\kappa(\mathbf{R}^d)$ and that it satisfies the boundedness condition in (6) for $\boldsymbol{\alpha}$ such that $|\boldsymbol{\alpha}| \leq \kappa$.*

Then the commutator C is a compact operator on $\mathrm{L}^p(\mathbf{R}^d)$, for any $p \in \langle 1,\infty\rangle$.

Dem. First notice that conditions (6) imply condition (1), i.e. we have immediately the $\mathrm{L}^2 \to \mathrm{L}^2$ compactness of commutator C where $b \in \mathrm{C}_b(\mathbf{R}^d)$ and ψ satisfies (6). Indeed, fix an arbitrary $\mathbf{h} \in \mathbf{R}^d$. Keeping in mind the Mean value theorem, we know that for some $\mathbf{z} = \eta\mathbf{x} + (1-\eta)(\mathbf{x}+\mathbf{h})$, where $\eta \in [0,1]$:

$$|\psi(\mathbf{x}+\mathbf{h}) - \psi(\mathbf{x})| \leq |\nabla\psi(\mathbf{z})|\,|\mathbf{h}| \leq \frac{c}{(1+|\mathbf{x}|)}|\mathbf{h}| \to 0 \text{ as } \mathbf{x} \to \infty.$$

We have used (6) in the second inequality. Moreover, as noticed in [12], condition in (6) implies the Hörmander condition (5) of Theorem 2. Since the Hörmander-Mihlin theorem provides the $\mathrm{L}^p \to \mathrm{L}^p$ boundedness of the Fourier multiplier operator $\mathcal{A}_\psi$, we have the $\mathrm{L}^p \to \mathrm{L}^p$ boundedness of commutator C, and this together with the $\mathrm{L}^2 \to \mathrm{L}^2$ compactness and Lemma 4 implies the $\mathrm{L}^p \to \mathrm{L}^p$ compactness of commutator C.

Q.E.D.

We can reach the same conclusion if we replace conditions (6) on ψ by less restrictive ones required by the following corollary of the Marcinkiewicz multiplier theorem (compare [24, Corollary 5.2.5] and [57, Theorem IV.6.6']):

Corollary 2. *Suppose that $\psi \in \mathrm{C}^d(\mathbf{R}^d \setminus \cup_{j=1}^d \{\xi_j = 0\})$ is a bounded function such that for some constant $A > 0$ it holds*

$$|\boldsymbol{\xi}^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})| \leq A, \qquad \boldsymbol{\xi} \in \mathbf{R}^d \setminus \cup_{j=1}^d \{\xi_j = 0\} \tag{7}$$

for every multi-index $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}_0^d$ such that $|\boldsymbol{\alpha}| \leq d$. Then, the function ψ is a symbol of L^p-multiplier for $p \in \langle 1, \infty\rangle$, and there exists a constant C_d such that

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \leq C_d (A + \|\psi\|_{\mathrm{L}^\infty}) \max\{p, (p-1)^{-1}\}^{6d}\,.$$

∎

We can easily prove the following improvement of Theorem C_p in [12]:

Corollary 3. *Assume that $\psi \in \mathrm{C}^d(\mathbf{R}^d)$ satisfies (1) and (7). Then commutator C is compact on $\mathrm{L}^p(\mathbf{R}^d)$.*

Dem. As it has already been mentioned, condition (7) implies the $\mathrm{L}^p \to \mathrm{L}^p$ boundedness, while (1) gives the $\mathrm{L}^2 \to \mathrm{L}^2$ compactness of commutator C. Therefore, according to Lemma 4, we have the $\mathrm{L}^p \to \mathrm{L}^p$ compactness of commutator C.

Q.E.D.

Let us remark in passing that the Hörmander-Mihlin and the Marcinkiewicz theorem are not related in the sense that one is a generalisation of the other. Namely, there exist functions satisfying the Marcinkiewicz theorem, but not the Hörmander-Mihlin (e.g. the function $\frac{\xi_1^2}{\xi_1^2+\xi_2^4}$) while the opposite is clear since the symbol satisfying (5) needs to be only C^κ-times differentiable.

Remark 4. A function defined on the sphere S^{d-1} and extended by homogeneity to the whole $\mathbf{R}^d_*$ satisfies the Hörmander-Mihlin and the Marcinkiewicz conditions (provided the symbol is of required smoothness), as it has been remarked in [57, Ex. 2, p. 96] and [24, Ex. 5.2.6]. In these cases, constant A in bounds (5) and (7) can be taken to be equal to $\|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})}$ and $\|\psi\|_{\mathrm{C}^d(\mathrm{S}^{d-1})}$, respectively. For a symbol defined on a more general smooth manifold Σ, the condition of the Marcinkiewicz theorem is satisfied [35, Lemma 5] (see Lemma 1 of the third chapter). Now, Lemma 4 combined with Lemma 2 gives the $\mathrm{L}^p \to \mathrm{L}^p$ compactness of commutator C.

It might be worth noting that this proof can be carried over to the case of mixed-norm Lebesgue spaces introduced in the sixties by Agnes Ilona Benedek and Rafael Panzone (for more details v. [4] and references therein).

By $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ we denote (with identification of almost everywhere equal functions) the space of all measurable complex functions f on $\mathbf{R}^d$ for which we have

$$\|f\|_{\mathbf{p}} = \Big(\int_{\mathbf{R}} \cdots \Big(\int_{\mathbf{R}} \Big(\int_{\mathbf{R}} |f(x_1, \dots, x_d)|^{p_1} dx_1\Big)^{p_2/p_1} dx_2\Big)^{p_3/p_2} \cdots dx_d\Big)^{1/p_d} < \infty\,.$$

In other words, for $i = 1, \dots, d$ we compute (in that order) the (quasi)norms $\|\cdot\|_{\mathrm{L}^{p_i}}$ in variable x_i. Analogously for some $p_i = \infty$, with obvious changes.

In the above proof of Lemma 4, in the last line one should use the version of Riesz-Thorin theorem for mixed-norm spaces [4, Theorem 5], together with the corresponding version of Hörmander-Mihlin theorem [4, Theorem 7]. In this way we get the result of Lemma 3, with assumption that (v_n) is bounded in $\mathrm{L}^{\mathbf{r}}(\mathbf{R}^d)$ for some $\mathbf{r} \in \langle 1, \infty\rangle^d \setminus \{\mathbf{2}\}$, implying $Cv_n \to 0$ strongly in $\mathrm{L}^{\mathbf{q}}(\mathbf{R}^d)$ for any $\mathbf{q} \in \langle 1, \infty\rangle^d$ such that there exists $\theta \in [0,1]$ satisfying

$$\frac{1}{q_i} = \frac{\theta}{2} + \frac{1-\theta}{p_i}, \quad i \in 1..d\,.$$

This, in turn, allows us to improve the result on existence of H-distributions removing the previous requirements $\mathbf{q} \in [2,\infty]^d$ and $\mathbf{q} > \mathbf{p}'$, where $1/p_i + 1/p'_i = 1$:

Theorem 3. *Let $\kappa = [d/2]+1$ and $\mathbf{p} \in \langle 1, \infty\rangle^d$. If $u_n \rightharpoonup 0$ weakly in $\mathrm{L}^{\mathbf{p}}_{loc}(\mathbf{R}^d)$, while $v_n \overset{*}{\rightharpoonup} v$ in $\mathrm{L}^{\mathbf{q}}_{loc}(\mathbf{R}^d)$, for some $\mathbf{q} \geq \mathbf{p}'$, then there exist subsequences (u'_n) and (v'_n) and a complex valued distribution $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$, such that for every $\phi_1, \phi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, one has*

$$\begin{aligned}\lim_{n'} {}_{\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)}\langle\, \mathcal{A}_\psi(\phi_1 u_{n'}), \phi_2 v_{n'}\,\rangle_{\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)} &= \lim_{n'} {}_{\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)}\langle\, \phi_1 u_{n'}, \mathcal{A}_{\overline{\psi}}(\phi_2 v_{n'})\,\rangle_{\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)}\\ &= \left\langle \mu, \overline{\phi_1}\phi_2 \boxtimes \overline{\psi}\right\rangle.\end{aligned}$$

∎

This is an improvement of the existing results on H-distributions, both on Lebesgue spaces [7, Theorem 2.1] and mixed-norm Lebesgue spaces [4, Theorem 9].

4. A result under lower regularity of b and the Hörmander condition

A natural question is what we can say about commutator $C = [\mathcal{A}_\psi, M_b]$ if merely the Hörmander condition is satisfied. The following holds.

Theorem 4. *Let $\psi \in \mathrm{C}^\kappa(\mathbf{R}^d_*)$ be bounded and satisfy Hörmander's condition (5), while $b \in \mathrm{C}_c(\mathbf{R}^d)$. Then for any $u_n \overset{*}{\longrightarrow} 0$ in $\mathrm{L}^\infty(\mathbf{R}^d)$ and $p \in \langle 1, \infty\rangle$ one has:*

$$(8)\qquad (\forall \varphi, \phi \in \mathrm{C}^\infty_c(\mathbf{R}^d)) \qquad \phi C(\varphi u_n) \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^p(\mathbf{R}^d)\,.$$

Dem. We need a number of subtle arguments, partly included in the proof of the Hörmander-Mihlin theorem, for which we refer to [24, Theorem 5.2.7] (we shall follow the notation of that proof as far as feasible, regarding the notation already used in previous sections).

As ψ is a bounded function, it is a Fourier multiplier on $\mathrm{L}^2(\mathbf{R}^d)$, so $\mathcal{A}_\psi$ can be expressed as a convolution with temperate distribution $W := \check{\psi}$ [24, Theorem 2.5.10]. Take $\hat{\zeta} \in \mathrm{C}^\infty_c(\mathbf{R}^d)$ such that $\operatorname{supp}\hat{\zeta} \subseteq \mathrm{K}[0;2] \setminus \mathrm{K}(0;1/2)$, and such that

$$\sum_{j\in\mathbf{Z}} \hat{\zeta}(2^{-j}\boldsymbol{\xi}) = 1 \quad \text{for} \quad \boldsymbol{\xi} \neq 0\,.$$

This allows us to decompose ψ as a sum of $m_j := \psi\hat{\zeta}(2^{-j}\cdot)$ and define $K_j := \check{m}_j$. Notice that the support of m_j is contained in $\left\{\boldsymbol{\xi} \in \mathbf{R}^d : 2^{j-1} \leq |\boldsymbol{\xi}| \leq 2^{j+1}\right\}$. Applying Hörmander's condition with $r = 2^{j-1}$ and $r = 2^j$, it follows that for each $\boldsymbol{\alpha} \in \mathbf{N}^d_0$

$$(9)\qquad \int_{\mathbf{R}^d} |D^{\boldsymbol{\alpha}}_{\boldsymbol{\xi}} m_j(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} = \int_{2^{j-1}\leq|\boldsymbol{\xi}|\leq 2^{j+1}} |D^{\boldsymbol{\alpha}}_{\boldsymbol{\xi}} m_j(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \leq C_0 A^2 2^{j(d-2|\boldsymbol{\alpha}|)}\,,$$

where constant $C_0 = 1 + 2^{-d+2|\boldsymbol{\alpha}|}$ does not depend on j. In [24, pp. 367–9] the following was proven:

1. $\sum_{j=-m}^{m} K_j \xrightarrow{*} W$ in $\mathcal{S}'$ (note that both K_j and $\sum_{j=-m}^{m} K_j$ are functions).
2. There is a constant $\tilde{C}_d > 0$ such that the following holds:

$$\sup_j \int_{\mathbf{R}^d} |K_j(\mathbf{x})| \left(1 + 2^j|\mathbf{x}|\right)^{1/4} d\mathbf{x} \le \tilde{C}_d \tilde{A} \,,$$

$$\sup_j 2^{-j} \int_{\mathbf{R}^d} |\nabla K_j(\mathbf{x})| \left(1 + 2^j|\mathbf{x}|\right)^{1/4} d\mathbf{x} \le \tilde{C}_d \tilde{A} \,.$$

   In particular, this means that both K_j and $\sum_{j=-m}^{m} K_j$ are in $\mathrm{L}^1(\mathbf{R}^d)$.
3. For each $\mathbf{x} \in \mathbf{R}^d$, we have

$$|K_j(\mathbf{x})| \le c_d 2^{jd} \|\psi\|_{\mathrm{L}^\infty(\mathbf{R}^d_*)} \,,$$

   which shows that $\sum_{j\le 0} |K_j(\mathbf{x})|$ is bounded on $\mathbf{R}^d$, independently of $\mathbf{x}$.

Moreover, by the first estimate in point 2, for any $\delta > 0$ and $j \ge 0$ we have

$$\left(1 + 2^j\delta\right)^{1/4} \int_{|\mathbf{x}|\ge\delta} |K_j(\mathbf{x})| d\mathbf{x} \le \tilde{C}_d \tilde{A} \,,$$

so $\sum_{j\ge 0} |K_j(\mathbf{x})|$ is summable away from the origin, thus finite almost everywhere. Therefore, $\sum_{j\in\mathbf{Z}} K_j$ represents a well-defined function K on $\mathbf{R}^d_*$, which coincides with the distribution $W = \check{\psi}$. In particular,

$$\sum_{j=-m}^{m} K_j \longrightarrow K \quad \text{in} \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d_*) \,.$$

Now we depart from the proof of the Hörmander-Mihlin theorem as it is presented in [24].

Let us first note that $A^- := \sum_{j=-\infty}^{-1} K_j$ is a bounded function on $\mathbf{R}^d$ by estimate in point 3 above, and is therefore also locally summable. The operator defined by $\mathcal{A}^- u := A^- * u = \left(\widehat{A^-}\, \hat{u}\right)^{\vee}$ is a bounded Fourier multiplier operator on $\mathrm{L}^2(\mathbf{R}^d)$. One just needs to notice that

$$\widehat{A^-} = \sum_{j=-\infty}^{-1} \widehat{K}_j = \sum_{j=-\infty}^{-1} m_j = \sum_{j=-\infty}^{-1} \psi\, \hat{\zeta}(2^{-j}\cdot) \,.$$

Since the right hand side is bounded by a bounded function ψ, we conclude that $\widehat{A^-}$ is a bounded function as well. Now we can define $C^- := \left[\mathcal{A}^-, M_b\right]$. For fixed $\mathbf{x} \in \mathbf{R}^d$ we have that

$$C^-(\varphi u_n)(\mathbf{x}) = \int_{\mathbf{R}^d} \left(b(\mathbf{y}) - b(\mathbf{x})\right) A^-(\mathbf{x}-\mathbf{y})(\varphi u_n)(\mathbf{y}) d\mathbf{y} \longrightarrow 0 \,.$$

Indeed, $A^-(\mathbf{x} - .)$ is summable over the compact $\operatorname{supp}\varphi$, the L^1 norm depending only on L^∞ norm of A^- and the volume of $\operatorname{supp}\varphi$, b is bounded, and the convergence of integrals follows from the weak-$*$ convergence of u_n. The above argument gives us also the bound

$$|C^-(\varphi u_n)(\mathbf{x})| \le c_1 \,,$$

independent of $\mathbf{x}$ and n.

After multiplication by $\phi \in \mathrm{C}_c^\infty(\mathbf{R}^d)$, $\phi C^-(\varphi u_n)$ is compactly supported in $\mathbf{x}$ and bounded, thus in any L^p space, while an application of the Lebesgue dominated convergence theorem gives us that

$$\int_{\mathbf{R}^d} \left|\phi C^-(\varphi u_n)\right|^p d\mathbf{x} \longrightarrow 0 \ .$$

Similarly, for a fixed $\delta > 0$, $A_\delta^+ := \sum_{j=0}^\infty K_j|_{\mathbf{R}^d \setminus \mathrm{K}[0;\delta]}$ is a summable function, and by the Riemann-Lebesgue lemma $\widehat{A_\delta^+}$ is bounded. For a fixed $\mathbf{x} \in \mathbf{R}^d$ we get

$$C_\delta^+(\varphi u_n)(\mathbf{x}) = \int_{|\mathbf{x}-\mathbf{y}|\geq\delta} (b(\mathbf{y}) - b(\mathbf{x}))A_\delta^+(\mathbf{x}-\mathbf{y})(\varphi u_n)(\mathbf{y})d\mathbf{y} \longrightarrow 0 \ .$$

Furthermore, the same arguments as above show that $C_\delta^+(\varphi u_n)(\mathbf{x})$ is bounded independently of $\mathbf{x}$ and n (the bound depends, of course, on δ).

After multiplication by $\phi \in \mathrm{C}_c^\infty(\mathbf{R}^d)$, $\phi C_\delta^+(\varphi u_n)$ is compactly supported in $\mathbf{x}$ and bounded, thus in any L^p space, while an application of the Lebesgue dominated convergence theorem gives the convergence to zero.

In a similar manner as we have just done, one can check that the same results a valid for the operator $C_{m,\delta}^+$ which corresponds to $A_{m,\delta}^+ = \sum_{j=0}^m K_j|_{\mathbf{R}^d \setminus \mathrm{K}[0;\delta]}$.

Finally, set $A_{m,\delta} := \sum_{j=0}^m K_j|_{\mathrm{K}[0,\delta]}$. From previous considerations, it follows that $A_{m,\delta} \longrightarrow K - A^- - A_\delta^+$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d_*)$. Since $\widehat{A_{m,\delta}}$ is a bounded function on $\mathbf{R}^d_*$ (just notice that K_j restricted to a bounded set belongs to L^1), the corresponding Fourier multiplier operator $\mathcal{A}_{m,\delta} := A_{m,\delta} *$ is bounded on $\mathrm{L}^2(\mathbf{R}^d)$.

Next we shall obtain the bounds for $A_{m,\delta}$, whose proof will differ slightly depending whether d is odd or even. For odd d, we have $d - 2\kappa = -1$, and by using the Hölder inequality, we get

$$\begin{aligned}
\int_{\mathrm{K}[0,\delta]} |\mathbf{y}||K_j(\mathbf{y})|d\mathbf{y} &= \int_{\mathrm{K}[0,\delta]} |\mathbf{y}|^{1-\kappa}|\mathbf{y}|^\kappa|K_j(\mathbf{y})|d\mathbf{y} \\
&\leq \left(\int_{\mathrm{K}[0,\delta]} |\mathbf{y}|^{2-2\kappa}d\mathbf{y}\right)^{1/2} \left(\int_{\mathrm{K}[0,\delta]} |\mathbf{y}|^{2\kappa}|K_j(\mathbf{y})|^2d\mathbf{y}\right)^{1/2} \\
&\leq C\left(\int_0^\delta r^{2-2\kappa+d-1}dr\right)^{1/2} \left(2^{j(d-2\kappa)}\right)^{1/2} \quad = \quad C\left(2^j\right)^{-1/2}\delta^{1/2}.
\end{aligned}$$

In the third line, we rewrote the first integral in the polar coordinates and for the second integral we used Plancherel's theorem and estimate (9).

For even d, $d - 2\kappa = -2$ and after applying the generalised Hölder inequality we have:

$$\int_{\mathrm{K}[0,\delta]} |\mathbf{y}||K_j(\mathbf{y})|d\mathbf{y} \le \sqrt{\int_{\mathrm{K}[0,\delta]} |\mathbf{y}|^{3-2\kappa} d\mathbf{y}} \sqrt[4]{\int_{\mathrm{K}[0,\delta]} |\mathbf{y}|^{2\kappa-2}|K_j(\mathbf{y})|^2 d\mathbf{y}} \quad \times$$
$$\times \quad \sqrt[4]{\int_{\mathrm{K}[0,\delta]} |\mathbf{y}|^{2\kappa}|K_j(\mathbf{y})|^2 d\mathbf{y}}$$
$$\le C \left(\int_0^\delta r^{3-2\kappa+d-1} dr \right)^{1/2} \left(2^{j(d-2\kappa+2)}\right)^{1/4} \left(2^{j(d-2\kappa)}\right)^{1/4}$$
$$= \quad C\left(2^j\right)^{-1/2} \delta^{1/2}.$$

If we additionally assume that $b \in \mathrm{C}^1_c(\mathbf{R}^d)$, then for fixed $\mathbf{x} \in \mathbf{R}^d$ we have that

$$\begin{aligned} C_{m,\delta}(\varphi u_n)(\mathbf{x}) &= \int_{\mathrm{K}[\mathbf{x},\delta]} (b(\mathbf{y}) - b(\mathbf{x})) A_{m,\delta}(\mathbf{x}-\mathbf{y})(\varphi u_n)(\mathbf{y}) d\mathbf{y} \\ &= \int_{\mathrm{K}[\mathbf{x},\delta]} |\mathbf{y}-\mathbf{x}| A_{m,\delta}(\mathbf{x}-\mathbf{y}) \frac{b(\mathbf{y}) - b(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} (\varphi u_n)(\mathbf{y}) d\mathbf{y} \\ &\le \|\nabla b\|_{\mathrm{L}^\infty(\mathbf{R}^d)} \|\varphi\|_{\mathrm{L}^\infty(\mathbf{R}^d)} \|u_n\|_{\mathrm{L}^\infty(\mathbf{R}^d)} \int_{\mathrm{K}[\mathbf{x},\delta]} |\mathbf{y}-\mathbf{x}| A_{m,\delta}(\mathbf{y}-\mathbf{x}) d\mathbf{y} \\ &\le \|\nabla b\|_{\mathrm{L}^\infty(\mathbf{R}^d)} \|\varphi\|_{\mathrm{L}^\infty(\mathbf{R}^d)} \left(\sup_n \|u_n\|_{\mathrm{L}^\infty(\mathbf{R}^d)} \right) C_0 \delta^{1/2}, \end{aligned}$$

uniformly in $\mathbf{x}, m$ and n. In the second step we have used the fact that $b \in \mathrm{C}^1_c(\mathbf{R}^d)$; in the last step we have used one of the preceding two estimates (depending on the parity of d). Let us briefly comment that constant C_0 does not depend on m, n and δ, since geometric series $\sum_j (1/\sqrt{2})^j$ is finite and equal to $2 + \sqrt{2}$.

After multiplication by a test function ϕ, taking the power p and integrating in $\mathbf{x}$, we obtain that $\|\phi C_{m,\delta}(\varphi u_n)\|_{\mathrm{L}^p(\mathbf{R}^d)}$ is of order δ independently of m and n.

In order to finish, we shall have to use some tools from measure theory. To this end, define operator $\widetilde{C}_m = C^- + C^+_{m,\delta} + C_{m,\delta}$. Then for every fixed $u \in \mathrm{L}^2(\mathbf{R}^d)$, $\widetilde{C}_m(u) \to C(u)$ pointwise almost everywhere. Indeed, using the definition of a commutator and Plancherel's theorem, we obtain the following bound

$$\begin{aligned} \|\widetilde{C}_m(u) - C(u)\|_{\mathrm{L}^2(\mathbf{R}^d)} &= \|(C^- + C^+_{m,\delta} + C_{m,\delta})(u) - C(u)\|_{\mathrm{L}^2(\mathbf{R}^d)} \\ &\le \left\| \left(\widehat{A^-} + \widehat{A^+_{m,\delta}} + \widehat{A_{m,\delta}} - \psi \right) \widehat{bu} \right\|_{\mathrm{L}^2(\mathbf{R}^d)} \quad + \\ &\quad + \|b\|_{\mathrm{L}^\infty(\mathbf{R}^d)} \left\| \left(\widehat{A^-} + \widehat{A^+_{m,\delta}} + \widehat{A_{m,\delta}} - \psi \right) \widehat{u} \right\|_{\mathrm{L}^2(\mathbf{R}^d)}. \end{aligned}$$

Before we proceed, let us notice that

$$\begin{aligned} \widehat{A^-} + \widehat{A^+_{m,\delta}} + \widehat{A_{m,\delta}} - \psi &= \left(A^- + A^+_{m,\delta} + A_{m,\delta} - \check{\psi} \right)^\wedge \\ &= \left(\sum_{j=-\infty}^{-1} K_j + \sum_{j=0}^{m} K_j|_{\mathbf{R}^d \setminus \mathrm{K}[0;\delta]} + \sum_{j=0}^{m} K_j|_{\mathrm{K}[0;\delta]} - \check{\psi} \right)^\wedge \\ &= \left(- \sum_{j>m} \check{m}_j \right)^\wedge = - \sum_{j>m} \psi \hat{\zeta}(\cdot/2^j) \,, \end{aligned}$$

where we have used $K_j = \check{m}_j$ and the decomposition of ψ from the beginning of this proof. Remembering that the support of $\hat{\zeta}(\cdot/2^j)$ is contained in $\{\boldsymbol{\xi} \in \mathbf{R}^d : 2^{j-1} \leq |\boldsymbol{\xi}| \leq 2^{j+1}\}$, we get the following

$$\begin{aligned}\left\|\left(\widehat{A^-} + \widehat{A^+_{m,\delta}} + \widehat{A_{m,\delta}} - \psi\right)\widehat{u}\right\|^2_{\mathrm{L}^2(\mathbf{R}^d)} &= \int_{\mathbf{R}^d}\Big|\sum_{j>m}\psi(\boldsymbol{\xi})\hat{\zeta}(\boldsymbol{\xi}/2^j)\widehat{u}(\boldsymbol{\xi})\Big|^2 d\boldsymbol{\xi} \leq \\ &\leq \|\psi\|^2_{\mathrm{L}^\infty(\mathbf{R}^d)}\int_{|\boldsymbol{\xi}|>2^m}|\widehat{u}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \longrightarrow 0 \quad \text{as} \quad m \to 0\,.\end{aligned}$$

Thus, we have shown that $\widetilde{C}_m(u) \longrightarrow C(u)$ in $\mathrm{L}^2(\mathbf{R}^d)$, which implies pointwise convergence almost everywhere on a subsequence (which we do not relabel). Using this result, for any compact set $K \subset \mathbf{R}^d$ and $q \in \langle 1, \infty\rangle$, we get

$$\begin{aligned}\int_K |C(u)|^q d\mathbf{x} &= \lim_m \int_K |\widetilde{C}_m(u)|^q d\mathbf{x} = \limsup_m \int_K |\widetilde{C}_m(u)|^q d\mathbf{x} \leq \\ &\leq \limsup_m \int_K |(C^- + C^+_{m,\delta} + C_{m,\delta})(u)|^q d\mathbf{x} \leq \\ &\leq 3^q \limsup_m \int_K |C^-(u)|^q + |C^+_{m,\delta}(u)|^q + |C_{m,\delta}(u)|^q d\mathbf{x} \leq \\ &\leq 3^q\left(\int_K |C^-(u)|^q d\mathbf{x} + \int_K |C^+_\delta(u)|^q d\mathbf{x} + \limsup_m \int_K |C_{m,\delta}(u)|^q d\mathbf{x}\right)\ ,\end{aligned}$$

where we have used Fatou's lemma for the middle integral.

Now we have set up almost everything to conclude the proof. We shall show that for arbitrary $\varepsilon > 0$, we can find $n_0 = n_0(\varepsilon)$ such that for every $n \geq n_0$ it holds $\|\phi C(\varphi u_n)\|^q_{\mathrm{L}^q(\mathbf{R}^d)} \leq \varepsilon$. Indeed, take an arbitrary $\varepsilon > 0$ and define

$$\delta = \left(\frac{\varepsilon}{3^{q+1}C_0\|\varphi\|_{\mathrm{L}^\infty(\mathbf{R}^d)}\|\phi\|_{\mathrm{L}^\infty(\mathbf{R}^d)}\|\nabla b\|_{\mathrm{L}^\infty(\mathbf{R}^d)}\sup_n\|u_n\|_{\mathrm{L}^q(\mathbf{R}^d)}}\right)^2 .$$

Using the convergence results obtained in the first part of the proof, there are $n_1 = n_1(\varepsilon)$ such that $n \geq n_1$ implies $\int_{\mathbf{R}^d}|\phi C^-(\varphi u_n)|^q d\mathbf{x} \leq \varepsilon/3^{q+1}$ and $n_2 = n_2(\delta(\varepsilon))$ such that $n \geq n_2$ implies $\int_{\mathbf{R}^d}|\phi C^+_\delta(\varphi u_n)|^q d\mathbf{x} \leq \varepsilon/3^{q+1}$. From the bound on $C_{m,\delta}(\varphi u_n)(\mathbf{x})$ and the special form of δ, we get the following bound for every m and n

$$\int_{\mathbf{R}^d}|\phi C_{m,\delta}(\varphi u_n)|^q d\mathbf{x} \leq \varepsilon/3^{q+1}\ .$$

Now we conclude that for every $n \geq \max\{n_1, n_2\}$, we get $\|\phi C(\varphi u_n)\|^q_{\mathrm{L}^q(\mathbf{R}^d)} \leq \varepsilon$.

It still remains to be show that we can use $b \in \mathrm{C}_c(\mathbf{R}^d)$. First, approximate function b by a sequence (b_n) in $\mathrm{C}^1_c(\mathbf{R}^d)$ in the topology of space $\mathrm{C}_c(\mathbf{R}^d)$. The corresponding sequence of commutators $\widehat{C}_n := [\mathcal{A}_\psi, M_{b_n}]$ converges in the operator norm towards C, and by the preceding result, each $\widehat{C}_n$ satisfies condition (8).

Q.E.D.

Finally, it is of interest to know whether we can relax the regularity of function b appearing in the previous theorem. Actually, we have the following lemma.

Lemma 5. *Let (u_n) be a bounded, uniformly compactly supported sequence in $\mathrm{L}^\infty(\mathbf{R}^d)$, converging to 0 in the sense of distributions. Assume that $\psi \in \mathrm{C}^\kappa(\mathbf{R}^d_*)$ satisfies conditions (1) and (5).*

Then for any $s > 1$, each $b \in \mathrm{L}^s(\mathbf{R}^d)$ satisfies

$$\lim_{n\to\infty} \|b\mathcal{A}_\psi(u_n) - \mathcal{A}_\psi(bu_n)\|_{\mathrm{L}^r(\mathbf{R}^d)} = 0, \quad r \in \langle 1, s\rangle.$$

Dem. The difference between the conditions of this lemma and those in Lemma 1 is in the regularity of function b. Namely, in Lemma 1 it is assumed that $b \in \mathrm{C}_0(\mathbf{R}^d)$ and it is restricted to the L^2 setting. However, we have better assumptions on sequence (u_n) (in our case, it belongs to $\mathrm{L}^p(\mathbf{R}^d)$ for every $p \geq 1$).

Indeed, let (b_ε) by a family of smooth functions with compact support such that $\|b_\varepsilon - b\|_{\mathrm{L}^s(\mathbf{R}^d)} \to 0$ as $\varepsilon \to 0$. Then it holds

$$\begin{aligned}\|b\mathcal{A}_\psi(u_n) - \mathcal{A}_\psi(bu_n)\|_{\mathrm{L}^r(\mathbf{R}^d)} &\leq \|b\mathcal{A}_\psi(u_n) - b_\varepsilon\mathcal{A}_\psi(u_n)\|_{\mathrm{L}^r(\mathbf{R}^d)} \\ &+ \|b_\varepsilon\mathcal{A}_\psi(u_n) - \mathcal{A}_\psi(b_\varepsilon u_n)\|_{\mathrm{L}^r(\mathbf{R}^d)} + \|\mathcal{A}_\psi(b_\varepsilon u_n) - \mathcal{A}_\psi(bu_n)\|_{\mathrm{L}^r(\mathbf{R}^d)}.\end{aligned}$$

The middle term on the right-hand side here tends to zero as $n \to \infty$ according to Lemma 4 with Theorem 1 and the Hörmander-Mihlin theorem. Estimating the other terms on the right-hand side above by the Hölder inequality and using the Hörmander-Mihlin theorem, we get

$$\lim_{n\to\infty} \|b\mathcal{A}_\psi(u_n) - \mathcal{A}_\psi(bu_n)\|_{\mathrm{L}^r(\mathbf{R}^d)} \leq C\|b_\varepsilon - b\|_{\mathrm{L}^s(\mathbf{R}^d)} .$$

Letting $\varepsilon \to 0$, we conclude the proof.

Q.E.D.

Remark 5. We could have used ψ satisfying conditions (1) and (7) to get the same result.

The conclusion of Lemma 5 remains valid even if $b \in \mathrm{L}^\infty(\mathbf{R}^d)$. Indeed, denote by $K \subset \mathbf{R}^d$ a compact set containing the supports of all u_n, and by $U \subset \mathbf{R}^d$ an open bounded set containing K. To show compactness of the commutator in $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 1, \infty\rangle$, choose $q \in \langle r, \infty\rangle$ and approximate b by test functions in $\mathrm{L}^q(U)$. The rest of the proof goes along the same lines as in the proof of Lemma 5.

A worthy observation is that this result allows us to consider commutators when b is a characteristic function of a measurable set. This observation proved useful in [20].

Before showing that we can lower the regularity of function b even further, let us define functions of *bounded mean oscillation*, in other words, the $\mathrm{BMO}(\mathbf{R}^d)$ functions.

A locally integrable function f is said to belong to $\mathrm{BMO}(\mathbf{R}^d)$ if there exists a constant $A > 0$ such that the following inequality holds for all balls $B \subseteq \mathbf{R}^d$:

$$⨍_B |f - f_B| d\mathbf{x} \leq A ,$$

where f_B is the mean value of f over the ball B. Since the mean oscillation of every constant function is zero, we identify any two functions in $\mathrm{BMO}(\mathbf{R}^d)$ that differ by a constant almost everywhere. The smallest A such that the above inequality holds is taken to be the norm of f in the resulting quotient space which we still denote by $\mathrm{BMO}(\mathbf{R}^d)$.

We are particularly interested in its closed subspace of *functions of vanishing mean oscillation*, the space being denoted by $\mathrm{VMO}(\mathbf{R}^d)$. It is defined as the closure of $\mathrm{C}_c(\mathbf{R}^d)$ functions in the $\mathrm{BMO}(\mathbf{R}^d)$ norm (we use the notation of Coifman and Weiss [11]). For further properties of $\mathrm{BMO}(\mathbf{R}^d)$ and $\mathrm{VMO}(\mathbf{R}^d)$ spaces, the interested reader should consult Stein's book [58] and aforementioned article [11].

Now, we shall show that the commutator C remains compact on $\mathrm{L}^p(\mathbf{R}^d)$ when function b belongs to the $\mathrm{VMO}(\mathbf{R}^d)$ space. We shall use the following well-known result of Uchiyama on the commutator of multiplication and Riesz transform $R_j := \mathcal{A}_{i\xi_j/|\boldsymbol{\xi}|}$, for $j \in \{1, \cdots, d\}$ (cf. [64, Theorem 2]):

Theorem 5. *Let $b \in \cup_{q>1}\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$. Then the commutator $[M_b, R_j]$ is a compact operator on $\mathrm{L}^p(\mathbf{R}^d)$, for any $p \in \langle 1, \infty\rangle$, if and only if $b \in \mathrm{VMO}(\mathbf{R}^d)$.* ∎

Before we proceed, let us show the following lemma

Lemma 6. *Let a be a function which is a polynomial in $\boldsymbol{\xi}/|\boldsymbol{\xi}|$ and $b \in \mathrm{VMO}(\mathbf{R}^d)$. Then the commutator $[M_b, \mathcal{A}_a]$ is a compact operator on $\mathrm{L}^p(\mathbf{R}^d)$, for any $p \in \langle 1, \infty\rangle$.*

Dem. The proof is by induction on the degree of polynomial a. The basis case $a = i\xi_j/|\boldsymbol{\xi}|$ is the Uchiyama result. Let us assume that the lemma is valid for all symbols a of degree smaller or equal to n. Take $\tilde{a}$ to be a monomial in $\boldsymbol{\xi}/|\boldsymbol{\xi}|$ of order $n+1$. We can write it in the form $\tilde{a} = i(\xi_j/|\boldsymbol{\xi}|)a$, for some $j \in \{1, \cdots, d\}$, where a is a polynomial in $\boldsymbol{\xi}/|\boldsymbol{\xi}|$ of order n. The following identity holds:

$$[M_b, \mathcal{A}_{\tilde{a}}] = [M_b, \mathcal{A}_a R_j] = [M_b, \mathcal{A}_a]R_j + \mathcal{A}_a[M_b, R_j]\ .$$

Let us remark that $\mathcal{A}_a$ is a bounded operator on $\mathrm{L}^p(\mathbf{R}^d)$ since a is a smooth bounded function satisfying the conditions of Hörmander-Mihlin's (and Marcinkiewicz's) theorem; also, the Riesz transform R_j is bounded on $\mathrm{L}^p(\mathbf{R}^d)$ ([24, Corollary 4.2.8]). It remains to notice that the commutators on the right hand side of the above's identity are compact operators on $\mathrm{L}^p(\mathbf{R}^d)$. Indeed, Uchiyama's result gives us the compactness of $[M_b, R_j]$, while the compactness of $[M_b, \mathcal{A}_a]$ follows from the assumption of the induction.

Q.E.D.

Using the preceding lemma and the Weierstrass's theorem, we have the following result for the $p = 2$ case (cf. [61, p. 336]):

Corollary 4. *Let $b \in \mathrm{L}^\infty(\mathbf{R}^d) \cap \mathrm{VMO}(\mathbf{R}^d)$ and $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$. Then the commutator $[M_b, \mathcal{A}_\psi]$ is compact on $\mathrm{L}^2(\mathbf{R}^d)$.* ∎

By now the standard argument (combining L^2-compactness and L^p-boundedness of the commutator with Lemma 4), we have the following result for $\mathrm{L}^p(\mathbf{R}^d)$:

Corollary 5. *Let $b \in \mathrm{L}^\infty(\mathbf{R}^d) \cap \mathrm{VMO}(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$. Then the commutator $[M_b, \mathcal{A}_\psi]$ is compact on $\mathrm{L}^p(\mathbf{R}^d)$, $p \in \langle 1, \infty\rangle$.* ∎

Notice that, in contrast to Theorem 5 and Lemma 6, we had to assume boundedness of function b in the last two corollaries in order to secure the boundedness of commutators on the desired spaces.

At the end of this section, let us mention that Uchiyama's result is stronger than we stated in Theorem 5. Namely, the result is valid if we replace Riesz transform operator with more general Calderon-Zygmund singular integral operator K with smooth kernel. In other words, if Ω is nonzero smooth function homogeneous of order zero, which satisfies

$$\int_{\mathrm{S}^{d-1}} \Omega = 0 \qquad \text{and} \qquad |\Omega(\mathbf{x}) - \Omega(\mathbf{y})| \leq |\mathbf{x} - \mathbf{y}| \quad \text{for all} \quad \mathbf{x}, \mathbf{y} \in \mathrm{S}^{d-1}\,,$$

then a Calderon-Zygmund singular integral operator K with smooth kernel Ω is given by (for $f \in \mathcal{S}(\mathbf{R}^d)$):

$$(Kf)(\mathbf{x}) = \text{p.v.} \int_{\mathbf{R}^d} \frac{\Omega(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^d} f(\mathbf{y})\, d\mathbf{y}\,, \qquad f \in \mathcal{S}(\mathbf{R}^d)\,.$$

Concerning the question of boundedness, a celebrated result of Coifman-Rochberg-Weiss [10] and Janson [30] states that commutator of M_b and K is bounded on all L^p if and only if b belongs to the BMO space.

5. Commutators of pseudodifferential operators and multiplication

It is natural to ask ourselves what can be said when, instead of the Fourier multiplier operator, we have a general pseudodifferential operator $a(\mathbf{x}, D)$; that is the one whose symbol $a(\mathbf{x}, \boldsymbol{\xi})$ cannot be written as a product of a function solely in $\mathbf{x}$ and a function solely in $\boldsymbol{\xi}$. For a symbol $a(\mathbf{x}, \boldsymbol{\xi})$, they are defined as (see Remark 1):

$$[a(\mathbf{x}, D)u] = (a\hat{u})^\vee\,.$$

Observe that if $a(\mathbf{x}, \boldsymbol{\xi}) = \psi(\boldsymbol{\xi})$ is independent of $\mathbf{x}$, then $a(\mathbf{x}, D) = \mathcal{A}_\psi$ is the Fourier multiplier operator associated with the symbol ψ, while if $a(\mathbf{x}, \boldsymbol{\xi}) = b(\mathbf{x})$ is independent of $\boldsymbol{\xi}$, then $a(\mathbf{x}, D) = M_b$ is the pointwise multiplication operator by b.

Boundedness (and, as a consequence, compactness as well) is not an easy question for such operators. For illustration, the L^2-boundedness for Fourier multiplier operators and multiplication operators have been nicely characterised: such operator will be bounded if and only if ψ or b are bounded functions. Take $\psi, b \in \mathrm{L}^2$ and consider $a(\mathbf{x}, \boldsymbol{\xi}) = b(\mathbf{x})\psi(\boldsymbol{\xi})$. In this case a could be unbounded function, but a simple application of the Hölder inequality shows that $a(\mathbf{x}, D)$ is a bounded operator on L^2.

In [28, Theorem 1] the following result on L^2-boundedness was shown:

Theorem 6. *Let $\Omega = \Pi_{i=1}^d \langle l_i, r_r\rangle$, $l_i < r_i$, be a bounded open box in $\mathbf{R}^d$ and $a : \Omega \times \mathbf{R}^d \to \mathrm{C}$ a measurable function whose derivatives $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} a$ in the distributional sense satisfy the condition:*

$$(\exists C > 0)(\forall \boldsymbol{\alpha} \in \{0,1\}^d) \quad \|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} a\|_{\mathrm{L}^\infty(\Omega\times\mathbf{R}^d)} \leq C\,.$$

(The smallest constant $C > 0$ satisfying the above condition on partial derivatives of a we denote by $\|a\|_*$, which is actually the norm on the anisotropic Sobolev space with dominant mixed derivative, consisting of functions with bounded derivatives up to order one in each x_j. Such spaces have first been studied by S. M. Nikol'skij.) Then $a(\mathbf{x}, D)$ is bounded from $\mathrm{L}^2(\mathbf{R}^d)$ to $\mathrm{L}^2(\Omega)$ with its norm bounded by $C_{d,\Omega}\|a\|_*$. ∎

For a smooth compact manifold $\Sigma \subseteq \mathbf{R}^d$, let us denote by $\mathrm{C}_b^{(d,0)}(\Omega \times \Sigma)$ the space of functions which have bounded and continuous derivatives of order up to d with respect to the first d variables $x_1, \dots, x_d$, and are bounded and continuous with respect to the remaining variables $\xi_1, \dots, \xi_d$, endowed with the natural norm inherited from anisotropic Sobolev space $\mathrm{W}^{(d,0);\infty}(\Omega \times \Sigma)$. Take a symbol $a \in \mathrm{C}_b^{(d,0)}(\Omega \times \Sigma)$ and notice that one of the constants that satisfy the above theorem's condition is $C = \|a\|_{\mathrm{W}^{(d,0);\infty}(\Omega\times\mathbf{R}^d)}$. Note that we implicitly assume that a has been extended from Σ to the whole $\mathbf{R}^d_*$ by homogeneity of order zero.

Recall that we have denoted by $\mathbf{R}^d_\infty$ the one-point Aleksandrov compactification of $\mathbf{R}^d$, and by $\mathrm{C}(\mathbf{R}^d_\infty)$ we shall denote the space of all continuous functions on $\mathbf{R}^d$ with finite limit at infinity.

Theorem 7. *Let $\Omega = \Pi_{i=1}^d \langle l_i, r_r\rangle$, $l_i < r_i$, be a bounded open box in $\mathbf{R}^d$. Then for any $a \in \mathrm{C}_b^{(d,0)}(\Omega\times\Sigma)$ and $b \in \mathrm{C}(\mathbf{R}^d_\infty)$, the commutator $[a(\mathbf{x},D), M_b]$ is compact from $\mathrm{L}^2(\mathbf{R}^d)$ to $\mathrm{L}^2(\Omega)$.*

Dem. As the first step, let us approximate a by a sequence of tensor products from $\mathrm{C}^d(\Omega)\boxtimes\mathrm{C}(\Sigma)$ [63, Chapter 39], i.e. let us have:

$$(\forall\varepsilon>0)(\exists m\in\mathbf{N})\left(\exists\varphi_1,\dots,\varphi_m\in\mathrm{C}_b^d(\Omega)\right)(\exists\psi_1,\dots\psi_m\in\mathrm{C}(\Sigma))$$
$$\Big\|a-\sum_{i=1}^m\varphi_i\boxtimes\psi_i\Big\|_{\mathrm{W}^{(d,0);\infty}(\Omega\times\Sigma)}\leq\varepsilon\,.$$

Next, take an arbitrary sequence $u_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, and estimate the commutator as follows:

$$\begin{aligned}
&\|[a(\mathbf{x},D),M_b](u_n)\|_{\mathrm{L}^2(\Omega)} = \\
&= \Big\|a(\mathbf{x},D)(bu_n)\pm\Big[\sum_{i=1}^m\varphi_i(\mathbf{x})\psi_i(D)\Big](bu_n)\pm b\Big[\sum_{i=1}^m\varphi_i(\mathbf{x})\psi_i(D)\Big](u_n)-ba(\mathbf{x},D)(u_n)\Big\|_{\mathrm{L}^2(\Omega)}\\
&\quad\leq\Big\|a(\mathbf{x},D)(bu_n)-\Big[\sum_{i=1}^m\varphi_i(\mathbf{x})\psi_i(D)\Big](bu_n)\Big\|_{\mathrm{L}^2(\Omega)}\\
&\qquad+\Big\|\Big[\sum_{i=1}^m\varphi_i(\mathbf{x})\psi_i(D)\Big](bu_n)-b\Big[\sum_{i=1}^m\varphi_i(\mathbf{x})\psi_i(D)\Big](u_n)\Big\|_{\mathrm{L}^2(\Omega)}\\
&\qquad+\Big\|b\Big[\sum_{i=1}^m\varphi_i(\mathbf{x})\psi_i(D)\Big](u_n)-ba(\mathbf{x},D)(u_n)\Big\|_{\mathrm{L}^2(\Omega)}\qquad=\qquad I_1+I_2+I_3\,.
\end{aligned}$$

The terms I_1 and I_3 can be bounded using Theorem 6:

$$I_1,I_3\leq C_{d,\Omega}\|b\|_{\mathrm{L}^\infty(\mathbf{R}^d)}\|u_n\|_{\mathrm{L}^2(\mathbf{R}^d)}\Big\|a-\sum_{i=1}^m\varphi_i\psi_i\Big\|_{\mathrm{W}^{(d,0);\infty}(\Omega\times\mathbf{R}^d)}\leq\tilde{C}\,\varepsilon\,,$$

where we have used the L^2-boundedness of (u_n).

Concerning the middle term I_2, it is enough to bound the L^2-norm of each commutator $[\varphi_i(\mathbf{x})\psi_i(D), M_b](u_n)$. To do that, we shall employ the following identity:

$$(\varphi(\mathbf{x})\psi(D))(u)(\mathbf{x})=\int_{\mathbf{R}^d}e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}\varphi(\mathbf{x})\psi(\boldsymbol{\xi})\hat{u}(\boldsymbol{\xi})d\boldsymbol{\xi}=\varphi(\mathbf{x})\left(\psi\hat{u}\right)^\vee(\mathbf{x})=\left(\varphi\mathcal{A}_\psi(u)\right)(\mathbf{x}).$$

Thus we have

$$\begin{aligned}
\|[\varphi_i(\mathbf{x})\psi_i(D),M_b](u_n)\|_{\mathrm{L}^2(\Omega)}&=\|\varphi_i\left(\mathcal{A}_{\psi_i}(bu_n)-b\mathcal{A}_{\psi_i}(u_n)\right)\|_{\mathrm{L}^2(\Omega)}\\
&\leq\|\varphi_i\|_{\mathrm{L}^\infty(\Omega)}\left\|[\mathcal{A}_{\psi_i},M_b](u_n)\right\|_{\mathrm{L}^2(\Omega)}\,,
\end{aligned}$$

which goes to zero by Lemma 2, since $[\mathcal{A}_{\psi_i}, M_b]$ is a compact operator.

Therefore

$$(\forall\,\varepsilon>0)(\exists\,n_0\in\mathbf{N})(\forall\,n\geq n_0)\qquad\|[a(\mathbf{x},D),M_b](u_n)\|_{\mathrm{L}^2(\Omega)}\leq\varepsilon\,,$$

which implies that $[a(\mathbf{x},D), M_b]$ is compact from $\mathrm{L}^2(\mathbf{R}^d)$ to $\mathrm{L}^2(\Omega)$.

Q.E.D.

Since any compact subset of Ω can be covered by a union of finite open boxes in Ω, we get the following result:

Corollary 6. *Let Ω be a bounded open subset of $\mathbf{R}^d$, $a \in \mathrm{C}_b^{(d,0)}(\Omega \times \Sigma)$ such that there is a compact $K \subseteq \Omega$ such that $\operatorname{supp} a \subseteq K \times \Sigma$ and $b \in \mathrm{C}(\mathbf{R}^d_\infty)$, then the commutator $[a(\mathbf{x}, D), M_b]$ is compact from $\mathrm{L}^2(\mathbf{R}^d)$ to $\mathrm{L}^2(K)$.* ∎

Another condition from which we can obtain compactness on L^2 is shown in [29]. For $m \in \mathbf{R}$ and $k \in \mathbf{R} \setminus \mathbf{N}$, Hwang and Lee defined $\Lambda_k^m(\mathbf{R}^d \times \mathbf{R}^d)$ to be the space of continuous functions $a : \mathbf{R}^d \times \mathbf{R}^d \to \mathrm{C}$ whose derivatives $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} a$ in the distributional sense satisfy:

There is a constant $C > 0$ such that for any $\boldsymbol{\alpha} \in \mathbf{N}^d$ and all $\mathbf{x}, \boldsymbol{\xi}, \mathbf{h} \in \mathbf{R}^d$, we have

1) if $|\boldsymbol{\alpha}| \leq [k]$ then $|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} a(\mathbf{x}, \boldsymbol{\xi})| \leq C\langle\boldsymbol{\xi}\rangle^m$,
2) if $|\boldsymbol{\alpha}| = [k]$ and $|\mathbf{h}| \leq 1$ then $|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} a(\mathbf{x}+\mathbf{h}, \boldsymbol{\xi}) - \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} a(\mathbf{x}, \boldsymbol{\xi})| \leq C\langle\boldsymbol{\xi}\rangle^m |\mathbf{h}|^{k-[k]}$.

The smallest constant C satisfying the above conditions we denote by $\|a\|_*$. Define $m(p) = -d|1/p - 1/2|$, and let $K \subseteq \mathbf{R}^d$ be a compact such that $\operatorname{supp} a \subseteq K \times \mathbf{R}^d$. Note that since $m(p) \leq 0$ for all p, the first condition implies $\|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} a\|_{\mathrm{L}^\infty(K \times \mathbf{R}^d)} \leq C$, while the second condition implies $(k - [k])$-Hölder continuity on the unit ball of the highest order derivatives in $\mathbf{x}$ variable. Also, notice that if we take $k > d$ and $a \in \Lambda_k^0(\mathbf{R}^d \times \mathbf{R}^d) \cap \mathrm{C}^{d,0}(\mathbf{R}^d \times \mathbf{R}^d)$, we are in the framework of Corollary 6.

The following result is known [29, Theorem 3.1 and Theorem 3.2].

Theorem 8. *Let $k \in \mathbf{R}^+ \setminus \mathbf{N}$, K as above, $a \in \Lambda_k^{m(p)}(\mathbf{R}^d \times \mathbf{R}^d)$ and define $\Omega_1 = \{\mathbf{x} \in \mathbf{R}^d : \ d(\mathbf{x}, K) \leq 1\}$. Then it holds:*

a) If $p \in \langle 1, 2]$ and $k > d/2$, then $a(\mathbf{x}, D)$ is continuous from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^p(K)$ with the norm bounded by $C_{K,d,p,k} \mathsf{vol}(\Omega_1)^{1/p} \|a\|_*$.
b) If $p \in \langle 2, \infty\rangle$ and $k > d/p$, then $a(\mathbf{x}, D)$ is continuous from $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ to $\mathrm{L}^p(K)$. ∎

In the following, by $\mathrm{C}^{[k],0;k-[k],0}(\mathbf{R}^d \times \mathbf{R}^d)$ we denote the space of functions which have continuous derivatives in $\mathbf{x}$ of order less than or equal to $[k]$ and whose $[k]$-th derivative in $\mathbf{x}$ is $(k - [k])$-Hölder continuous, while they are just continuous with respect to the $\boldsymbol{\xi}$ variable.

Theorem 9. *Let $k \in \langle \frac{d}{2}, \infty\rangle \setminus \mathbf{N}$, with K as above, $b \in \mathrm{C}(\mathbf{R}^d_\infty)$, and $a \in \Lambda_k^{m(p)}(\mathbf{R}^d \times \mathbf{R}^d) \cap \mathrm{C}^{[k],0;k-[k],0}(\mathbf{R}^d \times \mathbf{R}^d)$. Then the following holds:*

a) The commutator $[a(\mathbf{x}, D), M_b]$ is compact from $\mathrm{L}^2(\mathbf{R}^d)$ to $\mathrm{L}^2(K)$.
b) For $p \in \langle 1, 2\rangle$, the commutator $[a(\mathbf{x}, D), M_b]$ is compact from $\mathrm{L}^q(\mathbf{R}^d)$ to $\mathrm{L}^q(K)$, for any $q \in \langle p, 2]$.
c) For $p \in \langle 2, \infty\rangle$, the commutator $[a(\mathbf{x}, D), M_b]$ is compact from $\mathrm{L}^q(\mathbf{R}^d)$ to $\mathrm{L}^q(K)$, for any $q \in [2, p\rangle$.

Dem. **a)** Take a symbol $a \in \Lambda_k^0(\mathbf{R}^d \times \mathbf{R}^d) \cap \mathrm{C}^{[k],0;k-[k],0}(\mathbf{R}^d \times \mathbf{R}^d)$ such that $\operatorname{supp} a \subseteq K \times \mathbf{R}^d$.

For each $\varepsilon > 0$, we can approximate a by $a_\varepsilon \in \mathrm{C}^\infty(\mathbf{R}^d \times \mathbf{R}^d)$ in the $\mathrm{C}^{[k]}(\mathbf{R}^d \times \mathbf{R}^d)$ norm:

$$\|a - a_\varepsilon\|_{\mathrm{C}^{[k]}(\mathbf{R}^d \times \mathbf{R}^d)} \leq \varepsilon \,.$$

Indeed, the convolution with standard mollifier and multiplication with appropriate cut-off function suffices. Notice that a_ε is bounded (since a is) and that it has compact support in $\mathbf{x}$ larger than K, but it can be made smaller than Ω_1.

Take an arbitrary sequence $u_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, and estimate the commutator:

$$\begin{aligned}\|[a(\mathbf{x},D),M_b](u_n)\|_{\mathrm{L}^2(K)} &=\\ &= \|a(\mathbf{x},D)(bu_n) \pm a_\varepsilon(\mathbf{x},D)(bu_n) \pm ba_\varepsilon(\mathbf{x},D)(u_n) - ba(\mathbf{x},D)(u_n)\|_{\mathrm{L}^2(K)}\\ &\leq \|a(\mathbf{x},D)(bu_n) - a_\varepsilon(\mathbf{x},D)(bu_n)\|_{\mathrm{L}^2(K)} + \|a_\varepsilon(\mathbf{x},D)(bu_n) - ba_\varepsilon(\mathbf{x},D)(u_n)\|_{\mathrm{L}^2(K)}\\ &\quad + \|ba_\varepsilon(\mathbf{x},D)(u_n) - ba(\mathbf{x},D)(u_n)\|_{\mathrm{L}^2(K)} \quad = \quad I_1 + I_2 + I_3\ .\end{aligned}$$

As in the proof of Theorem 7, we can bound the terms I_1 and I_3 in the following way:

$$I_1, I_3 \leq C_{K,d,p,k}\|b\|_{\mathrm{L}^\infty(\mathbf{R}^d)}\sqrt{\mathsf{vol}\Omega_1}\|u_n\|_{\mathrm{L}^2(\mathbf{R}^d)}\|a-a_\varepsilon\|_{\mathrm{C}^{[k]}(\mathbf{R}^d\times\mathbf{R}^d)} \leq \tilde{C}\varepsilon\ ,$$

where we have used the bound from part a) of Theorem 8 after noticing that $\|a\|_*$ can be bounded by $2\|a\|_{\mathrm{C}^{[k]}(\mathbf{R}^d\times\mathbf{R}^d)}$ since the second condition needs to be valid only for $|\mathbf{h}|\leq 1$. The middle term I_2 goes to zero by applying Corollary 6 to the commutator $[a_\varepsilon(\mathbf{x},D),M_b]$. Thus, we have proved the claim.

b) Let us examine the relation between classes $\Lambda_k^{m(p)}(\mathbf{R}^d\times\mathbf{R}^d)$ and $\Lambda_k^{m(q)}(\mathbf{R}^d\times\mathbf{R}^d)$ for $p,q\in\langle 1,2\rangle$ and $k\in\mathbf{R}^+\backslash\mathbf{N}$. It is straightforward to see that $p<q$ implies $m(p)<m(q)<m(2)=0$, and that we have the following chain of inclusions

$$\Lambda_k^{m(p)}(\mathbf{R}^d\times\mathbf{R}^d) \subseteq \Lambda_k^{m(q)}(\mathbf{R}^d\times\mathbf{R}^d) \subseteq \Lambda_k^{0}(\mathbf{R}^d\times\mathbf{R}^d)\ .$$

This means that for any $p\in\langle 1,2\rangle$ and $a\in\Lambda_k^{m(p)}(\mathbf{R}^d\times\mathbf{R}^d)\cap\mathrm{C}^{[k],0;k-[k],0}(\mathbf{R}^d\times\mathbf{R}^d)$ such that $\operatorname{supp} a\subseteq K\times\mathbf{R}^d$, where $K\subseteq\mathbf{R}^d$ is compact, the commutator will be a bounded operator from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^p(K)$ (result of part (a) of Theorem 8) and compact from $\mathrm{L}^2(\mathbf{R}^d)$ to $\mathrm{L}^2(K)$ (result of part (a) of this corollary). Lemma 4 now gives the claim.

c) Let us remark that for $p>2$ one has the following chain of inequalities: $k>\frac{d}{2}>\frac{d}{p}$, which, together with the inclusions

$$\Lambda_k^{m(q)}(\mathbf{R}^d\times\mathbf{R}^d) \subseteq \Lambda_k^{m(p)}(\mathbf{R}^d\times\mathbf{R}^d) \subseteq \Lambda_k^{0}(\mathbf{R}^d\times\mathbf{R}^d) \quad \text{for } p<q\ ,$$

and the reasoning as above, implies the claim.

Q.E.D.

Notice that it might be natural to impose conditions on $\boldsymbol{\xi}$ symmetric to those we already have on $\mathbf{x}$, in the sense that the results on boundedness and compactness in L^2 would be valid for both $a(\mathbf{x},D)$ and $a(D,\mathbf{x})$. In fact, having conditions on derivatives with respect to $\boldsymbol{\xi}$ as well, relinquishes the requirement of having a compact support in $\mathbf{x}$. In order to show that, we shall use a class of symbols defined in [29]. Let $m\in\mathbf{R}$ and $k,k'\in\mathbf{R}^+\setminus\mathbf{N}$. Hwang and Lee defined $\Lambda^m_{k,k'}(\mathbf{R}^d\times\mathbf{R}^d)$ to be the space of continuous functions $a:\mathbf{R}^d\times\mathbf{R}^d\to\mathrm{C}$ whose derivatives $\partial^{\boldsymbol{\alpha}}_{\mathbf{x}}\partial^{\boldsymbol{\beta}}_{\boldsymbol{\xi}}a$ in the distributional sense satisfy the following:

There is a constant $C>0$ such that for any $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbf{N}^d$ and $\mathbf{x},\boldsymbol{\xi},\mathbf{h},\boldsymbol{\eta}\in\mathbf{R}^d$, we have

1) If $|\boldsymbol{\alpha}|\leq[k]$ and $|\boldsymbol{\beta}|\leq[k']$ then $|\partial^{\boldsymbol{\alpha}}_{\mathbf{x}}\partial^{\boldsymbol{\beta}}_{\boldsymbol{\xi}}a(\mathbf{x},\boldsymbol{\xi})|\leq C\langle\boldsymbol{\xi}\rangle^m$.
2) If $|\boldsymbol{\alpha}|=[k]$, $|\boldsymbol{\beta}|\leq[k']$ and $|\mathbf{h}|\leq 1$ then $|\partial^{\boldsymbol{\alpha}}_{\mathbf{x}}\partial^{\boldsymbol{\beta}}_{\boldsymbol{\xi}}a(\mathbf{x}+\mathbf{h},\boldsymbol{\xi})-\partial^{\boldsymbol{\alpha}}_{\mathbf{x}}\partial^{\boldsymbol{\beta}}_{\boldsymbol{\xi}}a(\mathbf{x},\boldsymbol{\xi})|\leq C\langle\boldsymbol{\xi}\rangle^m|\mathbf{h}|^{k-[k]}$.

3) If $|\boldsymbol{\alpha}| \leq [k]$, $|\boldsymbol{\beta}| = [k']$ and $|\boldsymbol{\eta}| \leq 1$ then $|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\boldsymbol{\xi}}^{\boldsymbol{\beta}} a(\mathbf{x}, \boldsymbol{\xi} + \boldsymbol{\eta}) - \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\boldsymbol{\xi}}^{\boldsymbol{\beta}} a(\mathbf{x}, \boldsymbol{\xi})| \leq C\langle\boldsymbol{\xi}\rangle^m |\boldsymbol{\eta}|^{k'-[k']}$.
4) If $|\boldsymbol{\alpha}| = [k]$, $|\boldsymbol{\beta}| = [k']$, $|\mathbf{h}| \leq 1$ and $|\boldsymbol{\eta}| \leq 1$ then

$$|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\boldsymbol{\xi}}^{\boldsymbol{\beta}} a(\mathbf{x} + \mathbf{h}, \boldsymbol{\xi} + \boldsymbol{\eta}) - \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\boldsymbol{\xi}}^{\boldsymbol{\beta}} a(\mathbf{x} + \mathbf{h}, \boldsymbol{\xi}) - \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\boldsymbol{\xi}}^{\boldsymbol{\beta}} a(\mathbf{x}, \boldsymbol{\xi} + \boldsymbol{\eta}) + \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\boldsymbol{\xi}}^{\boldsymbol{\beta}} a(\mathbf{x}, \boldsymbol{\xi})| \leq$$
$$\leq C\langle\boldsymbol{\xi}\rangle^m |\mathbf{h}|^{k-[k]} |\boldsymbol{\eta}|^{k'-[k']}.$$

The smallest constant C such that the above condition holds, we denote by $\|a\|_*$.

We have the following result [29, Theorem 3.3 and Theorem 3.4]:

Theorem 10. *Let $k, k' \in \mathbf{R}^+ \setminus \mathbf{N}$ and $a \in \Lambda_{k,k'}^{m(p)}(\mathbf{R}^d \times \mathbf{R}^d)$. The following are valid:*

a) If $p \in \langle 1, 2], k > d/2, k' > d/p$, then $a(\mathbf{x}, D)$ is continuous from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^p(\mathbf{R}^d)$ with its norm bounded by $C_{d,p,k,k'}\|a\|_*$.

b) If $p \in \langle 2, \infty\rangle, k > d/p, k' > d/2$, then $a(\mathbf{x}, D)$ is continuous from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^p(\mathbf{R}^d)$ with its norm bounded by $C_{d,p,k,k'}\|a\|_*$.

∎

The corresponding compactness results are:

Corollary 7. *Let $k, k' \in \mathbf{R}^+ \setminus \mathbf{N}$, $b \in \mathrm{C}(\mathbf{R}^d_\infty)$, and $a \in \Lambda_{k,k'}^{m(p)}(\mathbf{R}^d \times \mathbf{R}^d) \cap \mathrm{C}^{[k],[k'];k-[k],k'-[k']}(\mathbf{R}^d \times \mathbf{R}^d)$. The following holds:*

a) If $k, k' > d/2$, then the commutator $[a(\mathbf{x}, D), M_b]$ is compact from $\mathrm{L}^2(\mathbf{R}^d)$ to $\mathrm{L}^2(\mathbf{R}^d)$.

b) If $p \in \langle 1, 2\rangle, k > d/2, k' > d/p$, then the commutator $[a(\mathbf{x}, D), M_b]$ is compact from $\mathrm{L}^q(\mathbf{R}^d)$ to $\mathrm{L}^q(\mathbf{R}^d)$, for any $q \in \langle p, 2]$.

c) If $p \in \langle 2, \infty\rangle, k, k' > d/2$, then the commutator $[a(\mathbf{x}, D), M_b]$ is compact from $\mathrm{L}^q(\mathbf{R}^d)$ to $\mathrm{L}^q(\mathbf{R}^d)$, for any $q \in [2, p\rangle$.

Dem.

a) We apply the same procedure as in part (a) of Theorem 9 and get the claim.

b) Parts (a) of this corollary and of the preceding theorem, together with Lemma 4 give the claim.

c) Combining part (a) of this corollary together with the part (b) of the preceding theorem and Lemma 4, we get the claim.

Q.E.D.

II. Anisotropic distributions

We define distributions of anisotropic order, and establish their immediate properties. The central result is the Schwartz kernel theorem for such distributions, which represents continuous operators from $\mathrm{C}^l_c(X)$ to $\mathcal{D}'_m(Y)$ by kernels, which are distributions of order l in x, but higher, though still finite order in y.

Our main motivation for introducing these distributions is the fact that the H-distributions, recently introduced generalisation of H-measures are, in fact, distributions of order 0 (i.e. Radon measures) in $\mathbf{x} \in \mathbf{R}^d$, and of finite order in $\boldsymbol{\xi} \in \mathrm{S}^{d-1}$. This allows us to obtain some more precise results on H-distributions, with further applications to partial differential equations.

1. Introduction

H-measures were introduced by Luc Tartar [60] and independently by Patrick Gérard [23] to study oscillation and concentration effects in partial differential equations (for a variant tailored to parabolic problems see [5], and for further generalisations [19]). However, they are suitable only for problems expressed in the L^2 framework. In order to overcome that limitation, Nenad Antonić and Darko Mitrović introduced H-distributions [7], as an extension of H-measures to the $\mathrm{L}^p - \mathrm{L}^q$ context. Some further variants and existing applications are related to the mixed-norm Lebesgue spaces [4], the velocity averaging [35] and $\mathrm{L}^p - \mathrm{L}^q$ compactness by compensation [42] (also see the next chapter).

Before we proceed further, let us introduce the notation which we shall use in the paper. In what follows, we shall need Fourier multiplier operators defined by functions on the unit sphere S^{d-1} in $\mathbf{R}^d$. Whenever we say that a function $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ is a symbol of the Fourier multiplier operator (and write $\mathcal{A}_\psi$), we shall actually mean that the symbol is $\psi \circ \pi$, a function homogeneous of order zero on $\mathbf{R}^d_* := \mathbf{R}^d \setminus \{0\}$, where $\pi : \mathbf{R}^d_* \longrightarrow \mathrm{S}^{d-1}$ is the projection onto unit sphere along rays. For boundedness of such operator, please see Remark 4 of the previous chapter.

For $p \in [1, \infty]$, by $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ we denote the space of all distributions u such that the following holds

$$(\forall \varphi \in \mathrm{C}^\infty_c(\mathbf{R}^d)) \quad \varphi u \in \mathrm{L}^p(\mathbf{R}^d)\,.$$

Actually, $\mathrm{C}^\infty_c(\mathbf{R}^d)$ can be reduced to $\mathcal{G}$, its subset such that

$$(\forall \mathbf{x} \in \mathbf{R}^d)(\exists \varphi \in \mathcal{G})\ \mathrm{Re}\,\varphi(\mathbf{x}) > 0,$$

which can be chosen to be countable. We endow $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ with the locally convex topology induced by a family of seminorms $|\cdot|_{\varphi,p}$ (for $\varphi \in \mathcal{G}$)

$$|u|_{\varphi,p} := \|\varphi u\|_{\mathrm{L}^p(\mathbf{R}^d)}\,.$$

It can be shown that the definition of $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ and its topology does not depend on the choice of family $\mathcal{G}$. This definition is equivalent to a definition where one requires that $\mathrm{L}^p_{\mathrm{loc}}$ functions have finite L^p norms over every compact subset of $\mathbf{R}^d$, which can be covered by the above one, if we take $\mathcal{G}$ to consist of all characteristic functions χ_K of compacts $K \subseteq \mathbf{R}^d$ (notice that smoothness of functions in $\mathcal{G}$ is actually not needed for the definition of $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ space).

We say that a sequence (u_n) is bounded in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ if for every seminorm $|\cdot|_{\varphi,p}$ there exists $C_{\varphi,p} > 0$ such that $|u_n|_{\varphi,p} < C_{\varphi,p}$ uniformly in n. By choosing a countable $\mathcal{G} = \{\vartheta_l \ :\ l \in \mathbf{N}\}$ such that $0 \le \vartheta_l \le 1$ and $\chi_{K_l} \le \vartheta_l \le \chi_{K_{l+1}}$, where $K_l \subseteq \mathbf{R}^d$ is a closed ball of radius l centred around the origin, we can define a metric d_p on $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ by

$$d_p(u,v) := \sup_{l\in\mathbf{N}} 2^{-l}\frac{|u-v|_{\vartheta_l,p}}{1+|u-v|_{\vartheta_l,p}}\ .$$

With this metric $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ is a Fréchet space for each $p\in[1,\infty]$, separable for $p\in[1,\infty\rangle$ and reflexive for $p\in\langle 1,\infty\rangle$. For $p\in[1,\infty\rangle$, it is also valid that $\mathrm{C}^\infty_c(\mathbf{R}^d)$ is dense in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$, while $\mathrm{L}^{p'}_c(\mathbf{R}^d)$, a subspace of $\mathrm{L}^{p'}(\mathbf{R}^d)$ consisting of all functions in that space having a compact support, equipped with the topology of strict inductive limit $\left(\mathrm{L}^{p'}_c(\mathbf{R}^d)=\bigcup_l \mathrm{L}^{p'}(K_l)\right)$, is the dual of $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$. Let us just remark that we could have replaced $\mathbf{R}^d$ by any open set $\Omega\subseteq\mathbf{R}^d$ and all of the above definitions and conclusions would remain valid. For omitted proofs and further references, we refer the interested reader to [2].

Regarding the notation, by $\langle\cdot,\cdot\rangle$ we shall denote various duality products, always assuming that it is antilinear in the first argument, and linear in the second. In particular, this will allow us to identify it with the L^2 scalar product

$$\langle\, f \mid g\,\rangle_{\mathrm{L}^2} = \int \overline{f(\mathbf{x})}g(\mathbf{x})d\mathbf{x} = \langle f,g\rangle\ .$$

In order to precisely state the theorem on H-distributions, we need the notion of distributions of anisotropic (finite) order, which we introduce in the next section, together with some immediate properties. The third section is devoted to the proof of the Schwartz kernel theorem for distributions of anisotropic order, which is the necessary prerequisite for a precise version of the existence of H-distributions presented in the fourth section, which basic properties, like the criterion for strong convergence and the connection to defect measures are covered in the fifth section. Here, the crucial role is played by the Nemyckiĭ operator $\Phi_p(u)=|u|^{p-2}u$, which maps L^p to $\mathrm{L}^{p'}$.

An important example of a weakly converging sequence is concentration, which is treated in the sixth section, followed by the investigation how an approximation of sequences can preserve the H-distribution in the next section. In the last section we present some applications.

2. Distributions of anisotropic order

Functions differentiable of order l in one variable $\mathbf{x}$, and differentiable of order m in the other variable $\mathbf{y}$, can easily be defined. This notion has been extended to Sobolev functions (see e.g. [46]). However, in the theory of distributions of finite order such distinction, up to our knowledge, has never been made.

As the objects we study will be distributions of order zero in $\mathbf{x}$ variable, and distributions of order $\kappa\in\mathbf{N}$ in the other variable $\boldsymbol{\xi}$ (actually, $\boldsymbol{\xi}\in\mathrm{S}^{d-1}$), in this section we shall sketch such a definition, and extend the classical proofs (cf. [8, 15, 27, 55]) to this situation.

Every differentiable manifold X is locally diffeomorphic to $\mathbf{R}^d$, which means that there is a local homeomorphism $\pi : X\to\mathbf{R}^d$, and that the structure of $\mathbf{R}^d$ as a differential manifold is the same as the one obtained by transporting the structure of X by means of π. For more details, please see [15, Section 16.2.6].

Thus, by charts and partitions of unity, we will always transfer our situation to the case where X is a subset of some Euclidean space. Then, using the local nature of distributions (see [15, Chapter 17.4.2]) and the identification of distributions on manifolds with distributions on subsets of Euclidean spaces (see [15, Chapters 17.3.7 & 17.4.4]), we will transfer the obtained results back to the case where X is a manifold.

Let X and Y be open sets in $\mathbf{R}^d$ and $\mathbf{R}^r$, and $\Omega \subseteq X \times Y$ an open set. By $\mathrm{C}^{l,m}(\Omega)$ we denote the space of functions f on Ω, such that for any $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ and $\boldsymbol{\beta} \in \mathbf{N}_0^r$, if $|\boldsymbol{\alpha}| \le l$ and $|\boldsymbol{\beta}| \le m$, $\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f = \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} f \in \mathrm{C}(\Omega)$. Of course, the order in which derivatives are taken is not important.

By choosing a sequence of compact sets K_n in Ω, such that $\Omega = \bigcup_{n\in\mathbf{N}} K_n$ and $K_n \subseteq \mathrm{Int}K_{n+1}$, We can use the definition of seminorms from the Euclidean setting to define the following increasing sequence of seminorms

$$p_{K_n}^{l,m}(f) := \max_{|\boldsymbol{\alpha}|\le l, |\boldsymbol{\beta}|\le m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{\mathrm{L}^\infty(K_n)} ,$$

and $\mathrm{C}^{l,m}(\Omega)$ becomes a Fréchet space. This space has the topology of uniform convergence on compact sets of functions and their derivatives up to order l in $\mathbf{x}$ and m in $\mathbf{y}$.

In the case where X and Y are differentiable manifolds of dimensions d and r, respectively, that is separable metrizable topological spaces equipped with an equivalence class of smooth atlases (see [15, Chapter 16.1]), we consider smooth sections of vector bundles instead of functions. We will only consider trivial complex line bundles over the manifolds.

Notice that if (U,φ) is a chart on X and (V,ψ) a chart on Y, then $(U\times V, \varphi\boxtimes\psi)$ is a chart on $X\times Y$. For $\Omega \subseteq X\times Y$ an open set, take at most countable family of charts $(U_i\times V_j, \varphi_i\boxtimes\psi_j)$ such that $(U_i\times V_j)$ is a locally finite open covering of Ω. Now, $\mathrm{C}^{l,m}(\Omega)$ is a space of all sections f of the trivial complex line bundle $(X\times Y)\times \mathrm{C}$ over Ω such that the mapping $(\mathbf{x},\mathbf{y}) \mapsto f|_{U_i\times V_j} \circ (\varphi_i^{-1}\boxtimes\psi_j^{-1})$ belongs to $\mathrm{C}^{l,m}(\varphi_i(U_i)\times\psi_j(V_j))$.

For fixed i and j, let us choose a sequence of compact sets $K_n = K_n(i,j)$ in $\varphi_i(U_i)\times\psi_j(V_j)$, such that $\varphi_i(U_i)\times\psi_j(V_j) = \bigcup_{n\in\mathbf{N}} K_n$ and $K_n \subseteq \mathrm{Int}K_{n+1}$. We define a corresponding increasing sequence of seminorms:

$$p_{K_n(i,j)}^{l,m}(f) := p_{K_n}^{l,m}\left(f|_{U_i\times V_j} \circ (\varphi_i^{-1}\boxtimes\psi_j^{-1})\right) ,$$

where $p_{K_n}^{l,m}$ is a seminorm on $\mathrm{C}^{l,m}(\varphi_i(U_i)\times\psi_j(V_j))$, as defined in the Euclidean setting.

For a compact set $K \subseteq \Omega$ we can consider only those functions which are supported in K, and define a subspace of $\mathrm{C}^{l,m}(\Omega)$

$$\mathrm{C}_K^{l,m}(\Omega) := \left\{ f \in \mathrm{C}^{l,m}(\Omega) : \ \mathrm{supp}\, f \subseteq K \right\} .$$

This subspace inherits the topology from $\mathrm{C}^{l,m}(\Omega)$, which is, when considered only on the subspace, a norm topology determined by

$$\|f\|_{l,m,K} := p_K^{l,m}(f) ,$$

and $\mathrm{C}_K^{l,m}(\Omega)$ is a Banach space (it can be identified with a proper subspace of $\mathrm{C}^{l,m}(K)$). However, if $l=\infty$ or $m=\infty$ (in order to keep the notation simple, we assume that $m=\infty$), then we shall not get a Banach space, but a Fréchet space. As it was the case in the isotropic situation, note that an increasing sequence of seminorms that makes $\mathrm{C}^{l,\infty}_{K_n}(\Omega)$ a Fréchet space is given by $(p^{l,k}_{K_n}), k \in \mathbf{N}_0$. Throughout the rest of this section, we shall consider $m \in \mathbf{N}_0 \cup \{\infty\}$, unless explicitly stated otherwise.

We can also consider the space

$$\mathrm{C}^{l,m}_c(\Omega) := \bigcup_{n\in\mathbf{N}} \mathrm{C}^{l,m}_{K_n}(\Omega) \ ,$$

of all functions with compact support in $\mathrm{C}^{l,m}(\Omega)$, and equip it by a stronger topology than the one induced from $\mathrm{C}^{l,m}(\Omega)$: by the topology of *strict inductive limit*. More precisely, it can easily be checked that

$$\mathrm{C}^{l,m}_{K_n}(\Omega) \hookrightarrow \mathrm{C}^{l,m}_{K_{n+1}}(\Omega) \ ,$$

the inclusion being continuous. Also, the topology induced on $\mathrm{C}^{l,m}_{K_n}(\Omega)$ by that of $\mathrm{C}^{l,m}_{K_{n+1}}(\Omega)$ coincides with the original one, and $\mathrm{C}^{l,m}_{K_n}(\Omega)$ (as a Banach space in that topology, or a Fréchet space for $m = \infty$) is a closed subspace of $\mathrm{C}^{l,m}_{K_{n+1}}(\Omega)$. Then we have that the inductive limit topology on $\mathrm{C}^{l,m}_c(\Omega)$ induces on each $\mathrm{C}^{l,m}_{K_n}(\Omega)$ the original topology, while a subset of $\mathrm{C}^{l,m}_c(\Omega)$ is bounded if and only if it is contained in one $\mathrm{C}^{l,m}_{K_n}(\Omega)$, and bounded there [8, Theorem 1.3].

Of course, $\mathrm{C}^{\infty}_c(\Omega) \hookrightarrow \mathrm{C}^{l,m}_c(\Omega)$ is a continuous and dense imbedding. In particular, by taking $l = m = 0$, we obtain $\mathrm{C}_c(\Omega)$, the space of continuous functions with compact support, its dual being the space of Radon measures (the distributions of order zero).

It is now straightforward to define distributions of order l in $\mathbf{x}$, and order m in $\mathbf{y}$.

Definition. A *distribution of order l in $\mathbf{x}$ and order m in $\mathbf{y}$* is any linear functional on $\mathrm{C}^{l,m}_c(\Omega)$, continuous in the strict inductive limit topology. We denote the space of such functionals by $\mathcal{D}'_{l,m}(\Omega)$.

Clearly, in the case of orientable manifolds it holds $\mathrm{C}^{\infty}_c(\Omega) \hookrightarrow \mathrm{C}^{l,m}_c(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$, with continuous and dense imbeddings, thus $\mathrm{C}^{l,m}_c(\Omega)$ is a normal space of distributions, hence its dual $\mathcal{D}'_{l,m}(\Omega)$ forms a subspace of $\mathcal{D}'(\Omega)$. If we equip it with a strong topology, it is even continuously imbedded in $\mathcal{D}'(\Omega)$.

In order to better understand the properties of elements of $\mathcal{D}'_{l,m}(\Omega)$, we shall relate them to tensor products. The first step is to consider the algebraic tensor product $\mathrm{C}^l_c(X)\boxtimes \mathrm{C}^m_c(Y)$, the vector space of all (finite) linear combinations of functions of the form $(\phi \boxtimes \psi)(\mathbf{x},\mathbf{y}) := \phi(\mathbf{x})\psi(\mathbf{y})$. This is a vector subspace of $\mathrm{C}^{l,m}_c(X \times Y)$. By a slight modification of the proof, we can get an analogous result to [15, 17.10.2]:

Lemma 1. *Let X and Y be C^∞ manifolds, $K, K' \subseteq X$ compacts such that $K \subseteq \mathrm{Int}K'$, and similarly $L, L' \subseteq Y$ with $L \subseteq \mathrm{Int}L'$. Then any $u \in \mathrm{C}^{l,m}_{K\times L}(X \times Y)$ can be approximated by a sequence of functions from $\mathrm{C}^\infty_{K'}(X) \boxtimes \mathrm{C}^\infty_{L'}(Y)$ in the topology of $C^{l,m}_{K'\times L'}(X \times Y)$.*

Dem. Let us just briefly comment the case where our both compacts are contained in the domains of single charts on X and Y, respectively; the general case follows by usage of the partition of unity.

The main idea is that for a given $u \in \mathrm{C}^{l,m}_{K\times L}(X \times Y)$, we can find a sequence of polynomials (p_n) which approximates u such that its derivatives up to order l in $\mathbf{x}$

and order m in $\mathbf{y}$ approximate corresponding derivatives of u uniformly on each compact. This can be achieved by a convolution with suitable polynomial approximation of identity. Taking σ to be a C^∞ function on X equal to one on K with support in K' and, analogously, ρ to be C^∞ function on Y equal to one on L with support in L', we get a sequence $(\sigma\rho p_n)$ approximating u in the topology of $C^{l,m}_{K'\times L'}(X\times Y)$. Since every monomial is a tensor product, we have the claim.

Q.E.D.

The above lemma has an important consequence: a distribution $u \in \mathcal{D}'_{l,m}(X\times Y)$ is uniquely determined by its values on tensor products.

Lemma 2. *Let X and Y be C^∞ manifolds. For a linear functional u on $\mathrm{C}^{l,m}_c(X\times Y)$, the following statements are equivalent*

a) $u \in \mathcal{D}'_{l,m}(X\times Y)$,

b) $(\forall K \in \mathcal{K}(X\times Y))(\exists C>0)(\exists n\in\mathbf{N})(\exists I$ *a finite set of indices*$)(\forall \Psi \in \mathrm{C}^{l,m}_K(X\times Y))$

$$|\langle u,\Psi\rangle| \le C \max_{(i,j)\in I} p^{l,m}_{K_n(i,j)}(\Psi).$$

Dem. Let $u\in\mathcal{D}'_{l,m}(X\times Y)$. First we will consider the case where the compact K belongs to the domain of a single chart $(U_{i_0}\times V_{j_0}, \psi_{i_0}\boxtimes\psi_{j_0})$ on $X\times Y$. The continuity in the strict inductive limit topology means that for any compact $K\subseteq X\times Y$ (it is enough to consider only an increasing sequence of compacts $(K_n(i_0,j_0))$ as above), the restriction of u to $\mathrm{C}^{l,m}_K(X\times Y)$ is continuous. This implies that there is a neighbourhood of zero

$$V_\varepsilon = \Big\{\Psi\in\mathrm{C}^{l,m}_K(X\times Y):\ p^{l,m}_K(\Psi)\leqslant\varepsilon\Big\},$$

such that $|\langle u,\Psi\rangle|\leqslant 1$ for $\Psi\in V_\varepsilon$. On the other hand, for all non-zero $\Psi\in\mathrm{C}^{l,m}_K(X\times Y)$

$$\frac{\varepsilon\Psi}{p^{l,m}_K(\Psi)}\in V_\varepsilon,$$

from which (b) follows with $C=1/\varepsilon, I=\{(i_0,j_0)\}$ and n big enough such that $K\subset \operatorname{Int} K_n$.

Now, let us consider the case when K does not belong only to the domain of a single chart. As before, we can find a sequence of charts $(U_i\times V_j,\varphi_i\boxtimes\psi_j)$ of $X\times Y$ such that $(U_i\times V_j)$ forms a locally finite open covering of $X\times Y$. Since K is compact, there exists its finite covering by the sets $U_i\times V_j$, and the indices (i,j) of such sets form I. Let (f_{ij}) be a smooth partiction of unity subordinate to the open covering $(U_i\times V_j)$. For $\Psi\in\mathrm{C}^{l,m}_K(X\times Y)$, we can write $\Psi=\sum_{(i,j)\in I}\Psi f_{ij}$, where every Ψf_{ij} has support contained in the domain of a single chart. The triangle inequality and the case we considered at the beginning of the proof give us (b).

For the reverse implication, notice that (b) implies that u is a continuous linear functional on $\mathrm{C}^{l,m}_{K_n}(X\times Y)$ for every K_n as above. Hence, u is a continuous linear functional on $\mathrm{C}^{l,m}_c(X\times Y)$.

Q.E.D.

In the rest of the proofs in this section, we will, without loss of generality, assume that we are working on open subsets of Euclidean spaces.

It is straightforward to see that for a linear functional u on $\mathrm{C}_c^{l,m}(X \times Y)$, statement (b) of previous lemma implies:

$$(\forall K \in \mathcal{K}(X))(\forall L \in \mathcal{K}(Y)(\exists C > 0)(\exists n, \widetilde{n} \in \mathbf{N})(\exists I_K, I_L \text{ finite sets of indices})$$
$$(\forall \varphi \in \mathrm{C}_K^l(X))(\forall \psi \in \mathrm{C}_L^m(Y)) \quad |\langle u, \varphi \boxtimes \psi \rangle| \leq C \max_{i \in I_K} p_{K_n(i)}^l(\varphi) \max_{j \in I_L} p_{K_{\widetilde{n}}(j)}^m(\psi) .$$

The reverse implication would have significantly greater practical use, but, as we will see at the end of this section, it is not true in general.

We continue with the following result:

Corollary 1. *A linear functional u defined on $\mathrm{C}_c^{l,m}(X \times Y)$ is in $\mathcal{D}'_{l,m}(X \times Y)$ if and only if for every sequence (φ_k) converging to zero in $\mathrm{C}_c^{l,m}(X \times Y)$ the scalar sequence $(\langle u, \varphi_k \rangle)$ converges to zero.*

Dem. Assume $u \in \mathcal{D}'_{l,m}(X \times Y)$ and let (φ_k) be a sequence converging to zero in $\mathrm{C}_c^{l,m}(X \times Y)$. Then there exists a compact $K_n \subseteq X \times Y$ such that $\varphi_k \in \mathrm{C}_{K_n}^{l,m}(X \times Y)$ for every k. Since u is continuous on $\mathrm{C}_{K_n}^{l,m}(X \times Y)$, the claim follows.

Conversely, it is enough to show that u is continuous on every Fréchet space $\mathrm{C}_{K_n}^{l,m}(X \times Y)$ (recall that for $l, m < \infty$ these spaces are even Banach spaces). We argue by contradiction: suppose that there exists n_0 such that u is not continuous on $\mathrm{C}_{K_{n_0}}^{l,m}(X \times Y)$. Then, for some sequence (φ_k) converging to zero in $\mathrm{C}_{K_{n_0}}^{l,m}(X \times Y)$, the sequence $(\langle u, \varphi_k \rangle)$ does not converge to zero. Since the imbedding $\mathrm{C}_{K_{n_0}}^{l,m}(X \times Y) \hookrightarrow \mathrm{C}_c^{l,m}(X \times Y)$ is continuous, (φ_k) must converge to zero in $\mathrm{C}_c^{l,m}(X \times Y)$, hence $(\langle u, \varphi_k \rangle)$ must converge to zero by assumption, which is a contradiction.

Q.E.D.

Now we can define the tensor product of two distributions, as the unique distribution given by the following theorem (cf. [15, 17.10.3]).

Theorem 1. *Let X and Y be C^∞ manifolds, $u \in \mathcal{D}'_l(X)$ and $v \in \mathcal{D}'_m(Y)$. Then*

$$\Big(\exists! w \in \mathcal{D}'_{l,m}(X \times Y)\Big)\Big(\forall \varphi \in \mathrm{C}_c^l(X)\Big)\Big(\forall \psi \in \mathrm{C}_c^m(Y)\Big) \quad \langle w, \varphi \boxtimes \psi \rangle = \langle u, \varphi \rangle \langle v, \psi \rangle.$$

Furthermore, for any $\Phi \in \mathrm{C}_c^{l,m}(X \times Y)$, function $V: \mathbf{x} \mapsto \langle v, \Phi(\mathbf{x}, \cdot) \rangle$ is in $\mathrm{C}_c^l(X)$, while $U: \mathbf{y} \mapsto \langle u, \Phi(\cdot, \mathbf{y}) \rangle$ is in $\mathrm{C}_c^m(Y)$, and we have that

$$\langle w, \Phi \rangle = \langle u, V \rangle = \langle v, U \rangle.$$

Dem. The uniqueness is clear from Lemma 1. The rest of the proof follows along the same lines as in the aforementioned proof in [15]. Let us highlight the main parts. First of all, for Φ of the form $f \boxtimes g$ where $f \in \mathrm{C}_c^l(X)$ and $g \in \mathrm{C}_c^m(Y)$, we get $V(\mathbf{x}) = f(\mathbf{x}) \langle v, g \rangle$ and $\langle u, V \rangle = \langle u, f \rangle \langle v, g \rangle$, which equals $\langle w, \Phi \rangle$. We need to prove that the linear mapping $\Phi \mapsto \langle u, V \rangle$ is continuous on $\mathrm{C}_M^{l,m}(X \times Y)$ for each compact set M in $X \times Y$. Take a sequence (Φ_j) converging to zero in $\mathrm{C}_M^{l,m}(X \times Y)$. Since v is a distribution, the corresponding (V_j) converges to zero uniformly on $\pi_M(X)$. Noticing that $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} V_j(\mathbf{x}) = \langle v, \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \Phi_j(\mathbf{x}, \cdot) \rangle$, for $|\boldsymbol{\alpha}| \leqslant l$, we conclude that derivatives of V_j up to order l converge to zero uniformly on $\pi_X(M)$ as well. From this, we get that $\langle u, V_j \rangle$ converges to zero.

Q.E.D.

It is straightforward to check the following claim:

Lemma 3. *If $u \in \mathcal{D}'_{l,m}(X \times Y)$ then, for any $\psi \in \mathrm{C}^{l,m}(X \times Y)$, ψu is a well defined distribution of order at most (l, m).*

∎

For $u \in \mathcal{D}'_{l,m}(X \times Y)$, we define the support of u (and denote it $\operatorname{supp} u$) as the complement of union of all open sets on which u vanishes.

Theorem 2. *Let $u \in \mathcal{D}'_{l,m}(X\times Y)$ and take $F \subseteq X\times Y$ relatively compact set such that $\operatorname{supp} u \subseteq F$. Then there exists unique linear functional $\tilde{u}$ on $\mathcal{Q} := \{\varphi \in \mathrm{C}^{l,m}(X \times Y) : F \cap \operatorname{supp} \varphi \subset X \times Y$ compactly $\}$ such that*

a) $(\forall \varphi \in \mathrm{C}^{l,m}_c(X \times Y)) \quad \langle \tilde{u}, \varphi\rangle = \langle u, \varphi\rangle$,

b) $(\forall \varphi \in \mathrm{C}^{l,m}(X \times Y)) \quad F \cap \operatorname{supp} \varphi = \emptyset \quad \Longrightarrow \quad \langle \tilde{u}, \varphi\rangle = 0.$

The domain of $\tilde{u}$ is largest for $F = \operatorname{supp} u$.

Dem. Take $\varphi \in \mathcal{Q}$ and denote $K := F \cap \operatorname{supp} \varphi$. According to Lemma 1, there exists $\psi \in \mathrm{C}^\infty_c(X \times Y)$ such that $\psi \equiv 1$ on some neighbourhood of K. Decompose $\varphi = \varphi_0 + \varphi_1$ where $\varphi_0 = \psi\varphi \in \mathrm{C}^{l,m}_c(X \times Y)$, $\varphi_1 = (1 - \psi)\varphi$ and $F \cap \operatorname{supp} \varphi_1 = \emptyset$. Define $\tilde{u}$ by

$$\langle \tilde{u}, \varphi\rangle = \langle \tilde{u}, \varphi_0\rangle + \langle \tilde{u}, \varphi_1\rangle = \langle u, \varphi_0\rangle.$$

Obviously, conditions (a) and (b) are satisfied. Let us next check that the extension does not depend on the decomposition of φ: assume that $\varphi = \varphi'_0+\varphi'_1$ is another decomposition as above. It holds

$$\varphi_0 - \varphi'_0 = \varphi'_1 - \varphi_1 \in \mathrm{C}^{l,m}_c(X \times Y), \qquad F \cap \operatorname{supp} (\varphi_0 - \varphi'_0) = F \cap \operatorname{supp} (\varphi'_1 - \varphi_1),$$

which implies $0 = \langle u, \varphi_0 - \varphi'_0\rangle = \langle u, \varphi_0\rangle - \langle u, \varphi'_0\rangle$. Thus, $\tilde{T}$ is well-defined.

Q.E.D.

The following conjecture, if valid, would give us conditions under which the reverse implication would hold:

Conjecture. *Let X, Y be C^∞ manifolds and let u be a linear functional on $\mathrm{C}^{l,m}_c(X\times Y)$. If $u \in \mathcal{D}'(X \times Y)$ and satisfies*

$$\begin{gathered}(\forall K \in \mathcal{K}(X))(\forall L \in \mathcal{K}(Y)(\exists C > 0)(\exists n, \widetilde{n} \in \mathbf{N})(\exists I_K, I_L \text{ finite sets of indices})\\ (\forall \varphi \in \mathrm{C}^\infty_K(X))(\forall \psi \in \mathrm{C}^\infty_L(Y)) \quad |\langle u, \varphi \boxtimes \psi\rangle| \le C \max_{i\in I_K} p^l_{K_n(i)}(\varphi) \max_{j\in I_L} p^m_{K_{\tilde{n}}(j)}(\psi),\end{gathered}$$

then u can be uniquely extended to $\mathcal{D}'_{l,m}(X \times Y)$.

Unfortunately, this conjecture, in general, fails. Let us first see where the standard straightforward approach to the proof would fail, and after that we will provide a counterexample which was kindly communicated to us by Evgenij Panov. We will only consider the case where X and Y are open subsets of Euclidean spaces.

First, one would take an arbitrary compact $M \subseteq X \times Y$. Let L and K be its projections to X and Y, which are compact. Then replacing them by larger compacts K' and L', as it was done in Lemma 1, one would approximate any $\Psi \in \mathrm{C}^{l,m}_{K\times L}(X \times Y)$ by a sequence $(\varphi_k \boxtimes \psi_k)$ of functions from $\mathrm{C}^\infty_{K'}(X) \boxtimes \mathrm{C}^\infty_{L'}(Y)$, and one would be tempted to define

$$\langle u, \Psi\rangle = \lim_k \langle u, \varphi_k \boxtimes \psi_k\rangle.$$

However, the problem with this approach is in obtaining an appropriate bound for $\langle u, \Psi\rangle$. Naturally, one would like to use the already available bound for the tensor product from conjecture's assumption. Since the number of elements in tensor approximation can be unbounded (tough, for each element it is finite), the constant for each seminorm from the definition of anisotropic distributions would be unbounded. Thus, this approach fails to yield the desired bound.

For the counterexample, let us assume $l = m = 0$ and $d = r = 1$. First notice that $\ln|x-y| \in \mathrm{L}^1_{loc}(\mathbf{R}_x \times \mathbf{R}_y)$, so it can be identified with an elelment of $\mathcal{D}'(\mathbf{R}^2)$. Consider a distribution $u = -\frac{1}{\pi}\partial_y \ln|x-y|$. For $g \in \mathrm{C}^{0,1}(\mathbf{R}\times\mathbf{R})$, we get

$$\langle u, g\rangle = \frac{1}{\pi}\int_{\mathbf{R}^2} \ln|x-y|\partial_y g(x,y)\,dxdy\,.$$

It follows that $u \in \mathcal{D}'_{0,1}(\mathbf{R}\times\mathbf{R})$, and we can extend it to a linear (but not necessarily continuous) functional on $\mathrm{C}_c(\mathbf{R}\times\mathbf{R})$ (for example, by an application of the Hahn-Banach theorem). Take g to be of the form $\varphi(x)\boxtimes\psi(y)$, for $\varphi \in \mathrm{C}_c(\mathbf{R})$ and $\psi \in \mathrm{C}^1_c(\mathbf{R})$. It holds:

$$\langle u, \varphi\boxtimes\psi\rangle = \frac{1}{\pi}\int_{\mathbf{R}}\varphi(x)\int_{\mathbf{R}}\ln|x-y|\psi'(y)\,dydx\,,$$

and the inner integral, after integration by parts, becomes

$$\begin{aligned}\int_{\mathbf{R}}\ln|x-y|\psi'(y)\,dy &= \lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon}\ln|y-x|\psi'(y)\,dy\\ &= \lim_{\varepsilon\to 0}\left((\psi(x-\varepsilon)-\psi(x+\varepsilon))\ln\varepsilon + \int_{|y-x|>\varepsilon}\frac{\psi(y)}{x-y}\,dy\right)\\ &= \mathrm{V.P.}\int_{\mathbf{R}}\frac{\psi(y)}{x-y}\,dy = \pi H\psi(x)\,,\end{aligned}$$

where $H\psi$ denotes the Hilbert transform of function $\psi \in \mathrm{C}^1_c(\mathbf{R})$. Since it is an isometry on $\mathrm{L}^2(\mathbf{R})$ (see [24, Chapter 5.1.1]), we have the following bound

$$|\langle u, \varphi\boxtimes\psi\rangle| = \left|\int_{\mathbf{R}}\varphi(x)H\psi(x)\,dx\right| \le \|\varphi\|_{\mathrm{L}^2(\mathbf{R})}\|\psi\|_{\mathrm{L}^2(\mathbf{R})} \le |K|\|\varphi\|_{\mathrm{L}^\infty(\mathbf{R})}\|\psi\|_{\mathrm{L}^\infty(\mathbf{R})}\,,$$

for smooth functions φ and ψ whose both supports are contained in a compact set $K \subset \mathbf{R}$. Thus, all the assumptions of the conjecture are satisfied, but $u \notin \mathcal{D}'_{0,0}(\mathbf{R}\times\mathbf{R})$.

To demonstrate that, assume the contrary, that u is a distribution of order 0 on $\mathbf{R}^2$. Take a test-function g whose support does not intersect the diagonal of $\mathbf{R}^2$. After integration by parts, we get the identity

$$\langle u, g\rangle = \frac{1}{\pi}\int_{\mathbf{R}^2}\frac{g(x,y)}{x-y}\,dxdy\,.$$

Now, take a compact set $K = [0,1]\times[0,1]$, and since u is a distribution of order 0, there exists a constant $C_K > 0$ such that for any test-function g whose support in in K, we get

$$|\langle u, g\rangle| \le C_K\|g\|_{\mathrm{L}^\infty(\mathbf{R}^2)}\,.$$

Take a sequence of non-negative test-functions (g_ε) such that $\|g_\varepsilon\|_{\mathrm{L}^\infty(\mathbf{R}^2)} = 1$ and whose supports are contained in the triangle with vertices $(0,0), (1,1)$ and $(0,1)$. Furthermore, assume that the distance of $\operatorname{supp} g_\varepsilon$ to the diagonal of $\mathbf{R}^2$ is ε. Clearly, supports of all g_ε and contained in K and they do not intersect the diagonal of $\mathbf{R}^2$. Thus, we can write

$$\begin{aligned}\langle u, g_\varepsilon\rangle &= \frac{1}{\pi}\int_{\mathbf{R}^2}\frac{g_\varepsilon(x,y)}{x-y}\,dxdy = \frac{1}{\pi}\int_0^1\int_{y+\varepsilon}^1\frac{g_\varepsilon(x,y)}{x-y}\,dxdy\\ &\geq \frac{1}{\pi}\int_0^1\int_{y+\varepsilon}^1\frac{1}{x-y}\,dxdy = \frac{1}{\pi}\int_0^1\ln(1-y)\,dy - \frac{\ln\varepsilon}{\pi}\\ &\geq \frac{-1}{\pi} - \frac{\ln\varepsilon}{\pi}\,.\end{aligned}$$

On one hand, we have the uniform bound $\langle u, g_\varepsilon\rangle = |\langle u, g_\varepsilon\rangle| \leq C_K$, while on the other hand, the above bound implies $\langle u, g_\varepsilon\rangle \to \infty$ as $\varepsilon \to 0$. A contradiction.

The lack of the above result is the main reason why we need to consider a variant of the Schwartz kernel theorem for anisotropic distributions.

Remark 1. It might be interesting to see which results from this section remain valid if we take X and Y to be C^l and C^m manifolds, respectively. If this were the case, then $\mathcal{D}'_{l,m}(X\times Y)$ would not be a subspace of the classical distributions $\mathcal{D}'(X\times Y)$, which could only be defined on C^∞ manifolds. However, in the rest of the paper, we shall consider only $\mathbf{R}^d$ and the unit sphere in $\mathbf{R}^d$, which are C^∞ manifolds.

3. The Schwartz kernel theorem for distributions of anisotropic order

In this section we shall prove a version of the Schwartz kernel theorem for distributions of anisotropic order. While doing that, we shall follow the proof in [15, 23.9.2] and carefully take note of the order of distributions appearing. For other possible approaches, see the remarks after the proof of the following theorem.

Theorem 3. *Let X and Y be two C^∞ differentiable manifolds. Then the following statements hold:*

a) *Let $K \in \mathcal{D}'_{l,m}(X\times Y)$. Then for each $\varphi \in \mathrm{C}^l_c(X)$ the linear form K_φ, defined by $\psi \mapsto \langle K, \varphi\boxtimes\psi\rangle$, is a distribution of order not more than m on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathrm{C}^l_c(X)$ with its inductive limit topology to $\mathcal{D}'_m(Y)$ with weak $*$ topology, is linear and continuous.*
b) *Let $A : \mathrm{C}^l_c(X) \to \mathcal{D}'_m(Y)$ be a continuous linear operator, in the pair of topologies as above. Then there exists unique distribution $K \in \mathcal{D}'(X\times Y)$ such that for any $\varphi \in \mathrm{C}^\infty_c(X)$ and $\psi \in \mathrm{C}^\infty_c(Y)$*

$$\langle K, \varphi\boxtimes\psi\rangle = \langle K_\varphi, \psi\rangle = \langle A\varphi, \psi\rangle.$$

Furthermore, $K \in \mathcal{D}'_{l,r(m+2)}(X\times Y)$.

Dem. a) Let $\varphi \in \mathrm{C}^l_c(X)$ be an arbitrary function. In order to prove that K_φ is continuous on $\mathrm{C}^m_c(Y)$ with strict inductive limit topology, we have to show that for any $H \in \mathcal{K}(Y)$, the mapping $\psi \mapsto \langle K_\varphi, \psi\rangle$ is a continuous functional on $\mathrm{C}^m_H(Y)$. The mapping is clearly linear since the tensor product is bilinear, while K is linear.

Let us notice that we can assume that X and Y are open subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$. Indeed, first we can take some open covering of Y consisting of chart domains and a partition of unity (f_α) subordinate to that covering such that $\sum_\alpha f_\alpha(\mathbf{y}) = 1$ for every

$\mathbf{y} \in H$ (note that the sum is finite). Similarly, we can break up φ into a finite number of functions with supports small enough such that each lies in the domain of some chart of X. In this way, we limit ourselves to considering the domains of a pair of charts for X and Y. By using the results of [15, Chapter 17.4] (see also [25, Chapter 3.1.4]), we can identify distributions localised on chart domains with distributions on subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$. Thus, in what follows we shall assume that X and Y are open subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$.

We shall therefore show that there exists a constant $C > 0$ such that for any $\psi \in \mathrm{C}^m_H(Y)$ it holds

$$|\langle K_\varphi, \psi\rangle| \leqslant C \max_{|\boldsymbol{\beta}|\leqslant m} \|\partial^{\boldsymbol{\beta}}\psi\|_{\mathrm{L}^\infty(H)} \ ,$$

for m finite, while for $m = \infty$ we should modify the above to

$$(\exists\, m' \in \mathbf{N})(\exists\, C > 0)(\forall\, \psi \in \mathrm{C}^\infty_H(Y)) \quad |\langle K_\varphi, \psi\rangle| \leqslant C \max_{|\boldsymbol{\beta}|\leqslant m'} \|\partial^{\boldsymbol{\beta}}\psi\|_{\mathrm{L}^\infty(H)} \ ,$$

To this end, let us remark that since K is a distribution of anisotropic order on $X \times Y$, which means

$$\begin{aligned}(\forall\, M \in \mathcal{K}(X\times Y))(\exists \tilde{C} > 0)&(\forall \Psi \in \mathrm{C}^{l,m}_c(X\times Y)) \\ &\operatorname{supp}\Psi \subseteq M \Longrightarrow |\langle K, \Psi\rangle| \leqslant \tilde{C} \max_{|\boldsymbol{\alpha}|\leqslant l, |\boldsymbol{\beta}|\leqslant m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}}\Psi\|_{\mathrm{L}^\infty(M)} \ ,\end{aligned}$$

with obvious modifications if either l or m is infinite, by taking M to be of the form $L \times H$, with $L \in \mathcal{K}(X)$, and $\Psi = \varphi \boxtimes \psi$ such that $\operatorname{supp}\varphi \subseteq L$, we have that

$$\begin{aligned}|\langle K_\varphi, \psi\rangle| = |\langle K, \varphi\boxtimes\psi\rangle| &\leqslant \tilde{C} \max_{|\boldsymbol{\alpha}|\leqslant l, |\boldsymbol{\beta}|\leqslant m} \|\partial^{\boldsymbol{\alpha}}\varphi \boxtimes \partial^{\boldsymbol{\beta}}\psi\|_{\mathrm{L}^\infty(L\times H)} \\ &\leqslant \tilde{C} \max_{|\boldsymbol{\alpha}|\leqslant l} \|\partial^{\boldsymbol{\alpha}}\varphi\|_{\mathrm{L}^\infty(L)} \max_{|\boldsymbol{\beta}|\leqslant m} \|\partial^{\boldsymbol{\beta}}\psi\|_{\mathrm{L}^\infty(H)} \\ &\leqslant C \max_{|\boldsymbol{\beta}|\leqslant m} \|\partial^{\boldsymbol{\beta}}\psi\|_{\mathrm{L}^\infty(H)} \ ,\end{aligned}$$

and therefore $K_\varphi \in \mathcal{D}'_m(Y)$.

The linearity of mapping $\varphi \mapsto K_\varphi$ readily follows from the bilinearity of tensor product and the linearity of K. For continuity, take an arbitrary $L \in \mathcal{K}(X)$ and an arbitrary $\psi \in \mathrm{C}^m_c(Y)$. We need to show the existence of $\bar{C} > 0$ such that

$$|\langle K_\varphi, \psi\rangle| \leqslant \bar{C} \max_{|\boldsymbol{\alpha}|\leqslant l} \|\partial^{\boldsymbol{\alpha}}\varphi\|_{\mathrm{L}^\infty(L)} \ .$$

However, we have already shown that above: just take $\bar{C} = \tilde{C} \max_{|\boldsymbol{\beta}|\leqslant m} \|\partial^{\boldsymbol{\beta}}\psi\|_{\mathrm{L}^\infty(H)}$. Therefore, the mapping $\varphi \mapsto K_\varphi$, from $\mathrm{C}^l_c(X)$ to $\mathcal{D}'_m(Y)$ is linear and continuous.

b) Let us first prove the uniqueness. By formula

$$\langle K, \varphi\boxtimes\psi\rangle = \langle K_\varphi, \psi\rangle = \langle A\varphi, \psi\rangle \ ,$$

a continuous functional K on $\mathrm{C}^\infty_c(X) \boxtimes \mathrm{C}^\infty_c(Y)$ is defined. As it is defined on a dense subset of $\mathrm{C}^\infty_c(X\times Y)$, such K is uniquely determined on the whole $\mathrm{C}^\infty_c(X\times Y)$.

The proof of existence will be divided into two steps. In the first step we assume that X and Y are open subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$, and additionally, that the range of A is

$\mathrm{C}(Y) \subseteq \mathcal{D}'(m)Y$ (understood as distributions which can be identified with continuous functions). This will allow us to write explicitly the action of $A\varphi$ on a test function $\psi \in \mathrm{C}_c^m(Y)$, which will finally enable us to define the kernel K. In the second step, we shall use a partition of unity and the structure theorem of distributions to reduce the problem to the first step. Let us begin.

Step I. Assume that X and Y are open and bounded subsets of euclidean spaces, and that for each $\varphi \in \mathrm{C}_c^l(X)$, $A\varphi \in \mathrm{C}(Y)$. Its action on a test function $\psi \in \mathrm{C}_c^m(Y)$ is given by

$$\langle A\varphi, \psi \rangle = \int_Y \overline{(A\varphi)(\mathbf{y})}\psi(\mathbf{y})d\mathbf{y} \ .$$

The continuity assumption on A implies that $A : \mathrm{C}_c^l(X) \longrightarrow \mathrm{C}(Y)$ is continuous when the range is equipped with the weak $*$ topology inherited from $\mathcal{D}'(m)Y$.

As the latter is a Hausdorff space, that operator has a closed graph, but this remains true even when we replace the topology on $\mathrm{C}(Y)$ by its standard Fréchet topology [45, Exercise 14.101(a)], which is stronger. Now we can apply the Closed graph theorem [45, Theorem 14.3.4(b)] (the proof of this form is essentially the same as the classical Banach's proof), as $\mathrm{C}_c^l(X)$ is barrelled, as a strict inductive limit of barrelled spaces, to conclude that $A : \mathrm{C}_c^l(X) \longrightarrow \mathrm{C}(Y)$ is continuous with usual strong topologies on its domain and range.

For $\mathbf{y} \in Y$ consider a linear functional $F_{\mathbf{y}} : \mathrm{C}_c^l(X) \longrightarrow \mathbf{C}$ defined by

$$F_{\mathbf{y}}(\varphi) = (A\varphi)(\mathbf{y}) \ .$$

Since $A\varphi$ is a continuous function, $F_{\mathbf{y}}$ is well-defined and clearly it is continuous as a composition of continuous mappings, thus a distribution in $\mathcal{D}'(l)X$.

Let us take a test function $\Psi \in \mathrm{C}_c^{l,0}(X \times Y)$. If we fix its second variable, we can consider it as a function from $\mathrm{C}_c^l(X)$ and apply $F_{\mathbf{y}}$; we are interested in the properties of this mapping:

$$\mathbf{y} \mapsto F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y})) = \Big(A\Psi(\cdot, \mathbf{y})\Big)(\mathbf{y}) \ .$$

Clearly, it is well defined on Y, with a compact support contained in the projection $\pi_Y(\operatorname{supp}\Psi)$. Furthermore, we have the following bounds:

$$\begin{aligned}\Big|F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y}))\Big| = \Big|\Big(A\Psi(\cdot, \mathbf{y})\Big)(\mathbf{y})\Big| &\leqslant \|A\Psi(\cdot, \mathbf{y})\|_{\mathrm{L}^\infty(\pi_Y(\operatorname{supp}\Psi))} \\ &\leqslant C\|\Psi(\cdot, \mathbf{y})\|_{\mathrm{C}^l_{\pi_Y(\operatorname{supp}\Psi)}(X)} \leqslant C\|\Psi\|_{\mathrm{C}^{l,0}_{\operatorname{supp}\Psi}(X\times Y)} \ .\end{aligned}$$

The proof of continuity is a bit more involved; we shall show sequential continuity: take a sequence $\mathbf{y}_n \to \mathbf{y}$ in Y. Denote $H = \pi_X(\operatorname{supp}\Psi)$ and let $L \subseteq Y$ be a compact such that $\mathbf{y}_n, \mathbf{y} \in L$; Ψ is uniformly continuous on compact $H \times L$. This is also valid for $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\Psi$, where $|\boldsymbol{\alpha}| \leqslant l$, which results in $\Psi(\cdot, \mathbf{y}_n) \longrightarrow \Psi(\cdot, \mathbf{y})$ in $\mathrm{C}_c^l(X)$. As A is continuous, thus the convergence is carried to $\mathrm{C}(Y)$, i.e. to uniform convergence on compacts of a sequence of functions $A\Psi(\cdot, \mathbf{y}_n)$ to $A\Psi(\cdot, \mathbf{y})$. In particular, this gives that $(A\Psi(\cdot, \mathbf{y}_n))(\bar{\mathbf{y}}) - (A\Psi(\cdot, \mathbf{y}))(\bar{\mathbf{y}})$ is arbitrary small independently of $\bar{\mathbf{y}} \in L$, for large enough n On the other hand, $A\Psi(\cdot, \mathbf{y})$ is uniformly continuous, thus $(A\Psi(\cdot, \mathbf{y}))(\bar{\mathbf{y}}) - (A\Psi(\cdot, \mathbf{y}))(\mathbf{y})$ is small for large n, independently of $\bar{\mathbf{y}} \in L$. In other terms, we have the required convergence

$$F_{\mathbf{y}_n}(\Psi(\cdot, \mathbf{y}_n)) \longrightarrow F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y})) \ .$$

Any continuous function with compact support is summable, so we can define functional K on $\mathrm{C}_c^{l,0}(X \times Y)$:

$$\langle K, \Psi \rangle = \int_Y F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y}))\, d\mathbf{y} \ ,$$

which is obviously linear in Ψ, as $F_{\mathbf{y}}$ is.

In order to show that it is continuous, we cannot follow [15, 23.9.2], as our spaces are not Montel spaces. However, we can check that K is continuous at zero (with obvious modifications for $l = \infty$):

$$(\forall\, H \in \mathcal{K}(X))(\forall\, L \in \mathcal{K}(Y))(\exists\, C > 0)(\forall\, \Psi \in \mathrm{C}_c^{l,0}(X \times Y))$$
$$\operatorname{supp} \Psi \subseteq H \times L \quad \Longrightarrow \quad |\langle K, \Psi \rangle| \leqslant C \|\Psi\|_{\mathrm{C}^{l,0}_{K \times L}(X \times Y)} \ .$$

However, the continuity of $A : \mathrm{C}_c^l(X) \longrightarrow \mathrm{C}(Y)$, for Ψ supported in $H \times L$ and the fact that the support of $A\Psi(\cdot, \mathbf{y})$ is contained in L gives us the estimate

$$\Big| \int_Y F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y}))\, d\mathbf{y} \Big| \leqslant (\mathsf{vol} L) C \|\Psi\|_{\mathrm{C}^{l,0}_{K \times L}(X \times Y)} \ ,$$

as needed.

Finally, it is easy to check that for $\varphi \in \mathrm{C}_c^\infty(X)$ and $\psi \in \mathrm{C}_c^\infty(Y)$, we have:

$$\langle K, \varphi \boxtimes \psi \rangle = \int_Y F_{\mathbf{y}}(\varphi \boxtimes \psi(\mathbf{y})) d\mathbf{y} = \int_Y F_{\mathbf{y}}(\varphi) \psi(\mathbf{y}) d\mathbf{y} = \int_Y (A\varphi)(\mathbf{y}) \psi(\mathbf{y}) d\mathbf{y} = \langle A\varphi, \psi \rangle \ .$$

Step II. Let (U_α) and (V_β) be covers of X and Y consisting of relatively compact open sets. It is sufficient to show existence of distributions $K_{\alpha\beta}$ on $U_\alpha \times V_\beta$ which satisfy $\langle A\varphi, \psi \rangle = \langle K_{\alpha\beta}, \varphi \boxtimes \psi \rangle$ for all $\varphi \in \mathcal{D}(U_\alpha)$ and $\psi \in \mathcal{D}(V_\beta)$. Indeed, the uniqueness of distribution $K \in \mathcal{D}'(X \times Y)$ then follows from the fact that two distributions $K_{\alpha\beta}$ and $K_{\gamma\delta}$ will coincide on open sets $(U_\alpha \cap U\gamma) \times (V_\beta \cap V_\delta)$ of $X \times Y$, while the existence of K will be a result of localisation theorem [15, 17.4.2]. Furthermore, if we assume that U_α and V_β lie within domains of some charts of X and Y, in the light of results of [15, Chapter 17.4] (see also [25, Chapter 3.1.4]), we can identify the distributions localised to these chart domains with distributions on open subsets of $\mathbf{R}^d$. Thus, without loss of generality, we assume that U and V are relatively compact open subsets of $\mathbf{R}^d$.

So let us consider an operator $\tilde{A} : \mathrm{C}_c^l(U) \to \mathcal{D}'_m(V)$ defined in the following way: for $\varphi \in \mathrm{C}_c^l(U)$ and $\psi \in \mathrm{C}_c^m(V)$

$$\langle \tilde{A}\varphi, \psi \rangle = \langle A\varphi, \psi \rangle \ .$$

It is clear that $\tilde{A}$ is well-defined and we proceed by observing that by the assumption of the theorem it is also continuous. Thus, its image is a subset of distributions of order at most m. Take a relatively compact open neighbourhood W of the closure of V in Y and pick a smooth cut-off function ρ which is equal to one on the closure of V and whose support is contained in W. Multiplying a distribution of finite order with ρ does not change its order. Thus, for $\varphi \in \mathrm{C}_c^l(U)$, $\rho\tilde{A}\varphi$ is an element of the space $\mathcal{D}'_m(W)$ and has a compact support. The next step is to use the so called structure theorem for distributions: from the proof of the Theorem 5.4.1 of [22], it follows that we can write

$$\rho\tilde{A}\varphi = \left(\partial_1^{m+2} \dots \partial_r^{m+2}\right) \Big(E_{m+2} * (\rho\tilde{A}\varphi)\Big) \ ,$$

where E_{m+2} is the fundamental solution of the differential operator $\partial_1^{m+2}\dots\partial_r^{m+2}$ (we take partial derivatives with respect to the $\mathbf{y}$ variable), i.e. it satisfies in the sense of distributions the following equation $\left(\partial_1^{m+2}\dots\partial_r^{m+2}\right)E_{m+2}=\delta_0$. For the explicit formula for E_{m+2}, see [22, Chapter 5.4]. Furthermore, in the proof of [22, Theorem 5.4.1], it was shown that $E_{m+2}*(\rho\tilde{A}\varphi)$ is a continuous function. Denoting by $\widetilde{E}_{m+2}*$ transpose of the operator $E_{m+2}*$, we write for $\varphi\in \mathrm{C}_c^l(U)$ and $\psi\in\mathrm{C}_c^m(W)$

$$\left\langle E_{m+2}*(\rho\tilde{A}\varphi),\psi\right\rangle=\left\langle \tilde{A}\varphi,\rho\widetilde{E}_{m+2}*\psi\right\rangle \ ,$$

from which we conclude that the mapping $\varphi\mapsto E_{m+2}*(\rho\tilde{A}\varphi)$ is continuous from $\mathrm{C}_c^l(U)$ to $\mathcal{D}'_m(W)$. Now we can apply Step I and find a distribution $R\in\mathcal{D}'_{l,0}(U\times W)$ such that for all $\varphi\in\mathrm{C}_c^\infty(U)$ and $\psi\in\mathrm{C}_c^\infty(W)$ it holds

$$\left\langle E_{m+2}*(\rho\tilde{A}\varphi),\psi\right\rangle=\langle R,\varphi\boxtimes\psi\rangle \ .$$

Taking $\varphi\in\mathrm{C}_c^\infty(U)$ and $\psi\in\mathrm{C}_c^\infty(V)$, we have

$$\begin{aligned}\left\langle R,\varphi\boxtimes\left(\partial_1^{m+2}\dots\partial_r^{m+2}\right)\psi\right\rangle&=\left\langle E_{m+2}*(\rho\tilde{A}\varphi),\left(\partial_1^{m+2}\dots\partial_r^{m+2}\right)\psi\right\rangle\\&=(-1)^{d(m+2)}\left\langle\left(\partial_1^{m+2}\dots\partial_r^{m+2}\right)\Big(E_{m+2}*(\rho\tilde{A}\varphi)\Big),\psi\right\rangle\\&=(-1)^{d(m+2)}\left\langle\rho\tilde{A}\varphi,\psi\right\rangle\\&=(-1)^{d(m+2)}\left\langle\tilde{A}\varphi,\rho\psi\right\rangle\\&=(-1)^{d(m+2)}\langle A\varphi,\psi\rangle,\end{aligned}$$

which gives $\langle A\varphi,\psi\rangle=(-1)^{d(m+2)}\left\langle\left(\partial_1^{m+2}\dots\partial_r^{m+2}\right)R,\varphi\boxtimes\psi\right\rangle$, where the derivatives are taken with respect to the variable $\mathbf{y}$. Since R was an element of $\mathcal{D}'_{l,0}(U\times W)$, we conclude that $A\in\mathcal{D}'_{l,r(m+2)}(U\times V)$.

Q.E.D.

Remark 2. Note that in part *ii*) of the theorem, we did not get that $K\in\mathcal{D}'_{l,m}(X\times Y)$, as one would wish to get while observing the statement in *i*) part: the order with respect to $\mathbf{x}$ variable remained the same, but the order with respect to the $\mathbf{y}$ variable increased from m to $d(m+2)$.

Now, we can have a different look at our Conjecture. From the preceding theorem we have $u\in\mathcal{D}'_{l,r(m+2)}(X\times Y)$. Interchanging the roles of X and Y, the same proof as above would give us $u\in\mathcal{D}'_{d(l+2),m}(X\times Y)$, where order with respect to $\mathbf{y}$ remained the same, but order with respect to the $\mathbf{x}$ variable increased from l to $d(l+2)$. Since uniqueness of $u\in\mathcal{D}'(X\times Y)$ has already been determined, we conclude that $u\in\mathcal{D}'_{l,r(m+2)}(X\times Y)\cap\mathcal{D}'_{d(l+2),m}(X\times Y)$. It might be interesting to see some additional properties of that intersection. Note that order up to which we got the increase is determined by the structure theorem for distributions we used in the proof.

Remark 3. If one used a more constructive proof of Schwartz kernel theorem, for example [56, Theorem 1.3.4], one would end up increasing the order with respect to both variables $\mathbf{x}$ and $\mathbf{y}$. In this case, increasing the order with respect to both variables occurs naturally because one needs to secure the integrability of the function which is used to define the kernel function.

Remark 4. A remark in passing that one interesting approach to the kernel theorem is given in [63, Chapter 51]. This approach is based on deep results of functional analysis on tensor products of nuclear spaces of Alexander Grothendieck. The authors are convinced that this approach might result in further improvements of the preceding theorem.

The structure theorem of distributions used in the second step proved to be pivotal in reducing our general situation to the specific one resolved in the first step. Let us consider the following two results from [63]:

Corollary 2. *Let Ω be an open subset of $\mathbf{R}^d$. Then following holds:*

a) [Corollary 1, p. 261] A distribution in Ω is of order less than or equal to m if and only if it is equal to a finite sum of derivatives of order less than or equal to m of Radon measures in Ω.

b) [Corollary 1, p. 263] Every Radon measure in Ω is a finite sum of derivatives of order less than or equal to $2d$ of continuous functions. ∎

It is clear that combining these two results one can represent a distribution of order less than or equal to m as a finite sum of derivatives of order less than or equal to $2d+m$ of continuous functions. We could have used this in the second step of the proof of Theorem 3: for each element of the sum we would have obtained a corresponding distribution of order less than or equal to $2d+m$ in the $\boldsymbol{\xi}$ variable. However, one can easily see that that this representation is not unique.

In the end, let us notice that the kernel K identified in part (b) of Theorem 3 would belong to the space $\mathcal{D}'_{l,2r+m}(X \times Y)$, which is a slight improvement of estimated order in the $\boldsymbol{\xi}$ variable which Theorem 3 gives.

Let us now see two easy examples (compare with [27, Chapter V]).

In the first example, let $A : \mathrm{C}^l_c(X) \to \mathrm{C}^l_c(X)$ be the identity mapping $(A\varphi)(\mathbf{x}) = \varphi(\mathbf{x})$, where $X \subset \mathbf{R}^d$ is an open set. Following the construction given in Step I of Theorem 3, its kernel K has support contained in the diagonal $\{(\mathbf{x},\mathbf{x}) : \mathbf{x} \in X\} \subset X \times X$ and is given by

$$\langle K, \Phi \rangle = \int_X \Phi(\mathbf{x},\mathbf{x})\, d\mathbf{x}, \qquad \Phi \in \mathrm{C}^{l,d(l+2)}_c(X \times X)\,.$$

In the second example, let $f : X \to Y$ be a continuous function between two subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$, and take $A : \mathrm{C}^l_c(Y) \to \mathrm{C}^l_c(X)$ defined by $A\varphi = \varphi \circ f$. Its kernel K has support contained in the graph of f and is given by

$$\langle K, \Phi \rangle = \int_X \Phi(\mathbf{x}, f(\mathbf{x}))\, d\mathbf{x}, \qquad \Phi \in \mathrm{C}^{d(l+2),l}_c(X \times Y)\,.$$

4. H-distributions

An important result that was used in the proof of existence of H-measures was the First commutation lemma [60], which stated that the commutator of multiplication and Fourier multiplier operator was compact on $\mathrm{L}^2(\mathbf{R}^d)$. We will need a variant of this result for the $\mathrm{L}^p(\mathbf{R}^d)$ spaces, which was shown in [6]. It is a consequence of the following Krasnoselskij type lemma (for details and proofs, see [6]):

Lemma 4. *Assume that linear operator A is compact on $\mathrm{L}^2(\mathbf{R}^d)$ and bounded on $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 1, \infty\rangle \setminus \{2\}$. Then A is also compact on $\mathrm{L}^p(\mathbf{R}^d)$, for any p between 2 and r (i.e. such that $1/p = \theta/2 + (1-\theta)/r$, for some $\theta \in \langle 0, 1\rangle$).* ∎

With this result in hand, one just needs to use Tartar's First commutation lemma on $\mathrm{L}^2(\mathbf{R}^d)$ for compactness, and the Hörmander-Mihlin theorem for boundedness on $\mathrm{L}^p(\mathbf{R}^d)$, for all $p \in \langle 1, \infty\rangle$, to conclude the following:

Corollary 3. *If $b \in \mathrm{C}_0(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, then the commutator $\mathcal{A}_\psi b - b\mathcal{A}_\psi$ is compact on $\mathrm{L}^p(\mathbf{R}^d)$, for all $p \in \langle 1, \infty\rangle$.* ∎

We are now ready to reprove the theorem on existence of H-distributions [7], showing that they are actually distributions of order 0 (Radon measures) in $\mathbf{x}$, and of order κ in $\boldsymbol{\xi}$.

Theorem 4. *If $u_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ and $v_n \overset{*}{\rightharpoonup} v$ in $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ for some $p \in \langle 1, \infty\rangle$ and $q \geq p'$, then there exist subsequences $(u_{n'})$, $(v_{n'})$ and a complex valued distribution $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$, such that, for every $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, for $\kappa = [d/2]+1$, one has:*

$$\begin{aligned}\lim_{n'\to\infty}\int_{\mathbf{R}^d}\mathcal{A}_\psi(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})}d\mathbf{x} &= \lim_{n'\to\infty}\int_{\mathbf{R}^d}(\varphi_1 u_{n'})(\mathbf{x})\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})(\mathbf{x})}d\mathbf{x}\\ &= \langle \mu, \varphi_1\overline{\varphi}_2 \boxtimes \psi\rangle,\end{aligned}$$

where $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ is the Fourier multiplier operator with symbol $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$. Moreover, μ belongs to the space $\mathcal{D}'_{0,d(\kappa+2)}(\mathbf{R}^d \times \mathrm{S}^{d-1})$, i.e. it is a distribution of order 0 in $\mathbf{x}$ and of order not more than $d(\kappa+2)$ in $\boldsymbol{\xi}$. ∎

Before we proceed with the proof, let us just give a remark concerning the dual product we wrote above. In the proof, it will become evident that the bilinear mapping $(\varphi, \psi) \mapsto \langle \mu, \varphi\boxtimes\psi\rangle$ is continuous in the product topology of $\mathrm{C}_c(\mathbf{R}^d) \times \mathrm{C}^\kappa(\mathrm{S}^{d-1})$. In turn, we can extend it by continuity to the whole product space.

Dem. The first equality above is clear, as the adjoint of $\mathcal{A}_\psi$ is $\mathcal{A}_{\overline{\psi}}$. Without loss of generality, we may assume that $p \leq 2$ (if $p > 2$, we would use the first equality in the statement of the theorem and proceed as in the case $p \leq 2$).

The rest of the proof follows along the same lines as in [7], after noting that

$$\lim_n {}_{\mathrm{L}^{p'}}\Big\langle \mathcal{A}_{\overline{\psi}}(\varphi_2 v), \varphi_1 u_n \Big\rangle_{\mathrm{L}^p} = 0\ .$$

Indeed, as $q \geq p'$, we have that $\varphi_2 v \in \mathrm{L}^{p'}(\mathbf{R}^d)$, thus $\mathcal{A}_{\overline{\psi}}(\varphi_2 v) \in \mathrm{L}^{p'}(\mathbf{R}^d)$ as well, and we can pass to the limit in the product.

Take ϑ_l and K_l as in the definition of metric d_p on $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ in the introduction; therefore $\operatorname{supp}\varphi_2 \subseteq K_l \subseteq \operatorname{supp}\vartheta_l$ for some $l \in \mathbf{N}$, and we have:

$$\begin{aligned}\lim_n {}_{\mathrm{L}^{p'}}\Big\langle \mathcal{A}_{\overline{\psi}}(\varphi_2 v_n), \varphi_1 u_n \Big\rangle_{\mathrm{L}^p} &= \lim_n {}_{\mathrm{L}^{p'}}\Big\langle \mathcal{A}_{\overline{\psi}}\Big(\varphi_2\vartheta_l(v_n - v)\Big), \varphi_1 u_n \Big\rangle_{\mathrm{L}^p}\\ &= \lim_n {}_{\mathrm{L}^{p'}}\Big\langle \mathcal{A}_{\overline{\psi}}(\vartheta_l(v_n - v)), \varphi_1\overline{\varphi}_2\vartheta_l u_n \Big\rangle_{\mathrm{L}^p}\\ &= \lim_n {}_{\mathrm{L}^{p'}}\Big\langle \mathcal{A}_{\overline{\psi}}(\vartheta_l v_n), \varphi_1\overline{\varphi}_2\vartheta_l u_n \Big\rangle_{\mathrm{L}^p} \quad =: \lim_n \mu_{n,l}(\varphi_1\overline{\varphi}_2, \psi)\ .\end{aligned}$$

In the second equality we have used the version of First commutation lemma for $\mathrm{L}^p(\mathbf{R}^d)$ spaces. The final expression shows that each integral is indeed a bilinear functional depending on $\varphi = \varphi_1\overline{\varphi}_2$ and ψ.

Furthermore, by the Hölder inequality and the continuity of Fourier multiplier operator, we have

$$|\mu_{n,l}(\varphi,\psi)| \leq \|\varphi\vartheta_l u_n\|_{\mathrm{L}^p}\|\mathcal{A}_{\overline{\psi}}(\vartheta_l v_n)\|_{\mathrm{L}^{p'}} \leq \tilde{C}\|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})}\|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} ,$$

where the constant $\tilde{C}$ is given by $\tilde{C} = C|u_n|_{\vartheta_l,p}|v_n|_{\vartheta_l,p'}$, C depending only on d and p and comes from continuity of Fourier multiplier $\mathcal{A}_\psi$.

For $l \in \mathbf{N}$, we can bound $|u_n|_{\vartheta_l,p}$ and $|v_n|_{\vartheta_l,p'}$ by constants independent of n and apply Lemma 3.2 from [7], obtaining operators $D^l \in \mathcal{L}(\mathrm{C}_{K_l}(\mathbf{R}^d);(\mathrm{C}^\kappa(\mathrm{S}^{d-1}))')$, such that D^l is an extension of D^{l-1}.

This allows us to define the operator D on $\mathrm{C}_c(\mathbf{R}^d)$: for $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$ we take $l \in \mathbf{N}$ such that $\operatorname{supp}\varphi \subseteq K_l$, and set $D\varphi := D^l\varphi$, which satisfies:

$$\|D\varphi\|_{(\mathrm{C}^\kappa(\mathrm{S}^{d-1}))'} \leq C_{K_l}\|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} .$$

As this operator D is continuous when restricted to each $\mathrm{C}_{K_l}(\mathbf{R}^d)$, D is continuous on the strict inductive limit of these spaces as well, i.e. on $\mathrm{C}_c(\mathbf{R}^d)$.

Finally define $\mu(\varphi,\psi) := \langle D\varphi,\psi\rangle$. D can be restricted to an operator $\tilde{D}$ defined only on $\mathrm{C}^\infty_c(\mathbf{R}^d)$, remaining continuous (as $\mathrm{C}_{K_l}(\mathbf{R}^d)$ norm is one of the seminorms defining the topology on $\mathrm{C}^\infty_{K_l}(\mathbf{R}^d)$). Moreover, the space $(\mathrm{C}^\kappa(\mathrm{S}^{d-1}))'$ of distributions of order κ is a subspace of $\mathcal{D}'(\mathrm{S}^{d-1})$. Thus we have a continuous operator from $\mathrm{C}^\infty_c(\mathbf{R}^d)$ to $\mathcal{D}'(\mathrm{S}^{d-1})$, which by the Schwartz kernel theorem can be identified to a distribution from $\mathcal{D}'(\mathbf{R}^d\times\mathrm{S}^{d-1})$; thus $\mu \in \mathcal{D}'(\mathbf{R}^d\times\mathrm{S}^{d-1})$.

We want to show that μ is actually in the space $\mathcal{D}'_{0,d(\kappa+2)}(\mathbf{R}^d\times\mathrm{S}^{d-1})$, i.e. that it could be extended to a continuous linear functional on $\mathrm{C}^{0,d(\kappa+2)}_c(\mathbf{R}^d\times\mathrm{S}^{d-1})$. In the same way as above, we obtain the following bound with $\varphi := \varphi_1\overline{\varphi_2}$:

$$|\langle\mu,\varphi\boxtimes\psi\rangle| \leq C\|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})}\|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} ,$$

where C does not depend on φ and ψ. Now we just need to apply the Schwartz kernel theorem to conclude that μ is a continuous linear functional on $\mathrm{C}^{0,d(\kappa+2)}_c(\mathbf{R}^d\times\mathrm{S}^{d-1})$.

Q.E.D.

In fact, taking into account the discussion after Theorem 3, we conclude that μ is a distribution of order not more than $2d+\kappa$ in the $\boldsymbol{\xi}$ variable.

Remark 5. For $q \in \langle 1,\infty\rangle$, weak and weak-$*$ convergence coincide since $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ is reflexive.

Remark 6. The preceding theorem only gives us an upper bound of the order in $\boldsymbol{\xi}$. To illustrate, in the case $p=q=2$, H-distribution is actually an H-measure, which is of order 0 in $\boldsymbol{\xi}$.

We shall say that (u_n) and (v_n) form a pure pair of sequences if the associated H-distribution is unique for all subsequences.

If (u_n) and (v_n) are L^p and L^q sequences, respectively, defined on an open set $\Omega \subseteq \mathbf{R}^d$, extending them by zero to the whole space, we would still retain weak and weak-$*$ convergence of corresponding sequences to corresponding limits. Then, applying the preceding theorem, we get that the corresponding H-distribution is supported on

$\mathrm{Cl}\Omega \times \mathrm{S}^{d-1}$. Indeed, the claim follows easily if one takes test functions supported within the complement of the closure $\mathrm{Cl}\Omega$. The analogous statement holds for $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$ and $\mathrm{L}^q_{\mathrm{loc}}(\Omega)$ sequences, if we can extend them by zero to the whole space, which is not always possible (for example, take an $\mathrm{L}^1_{\mathrm{loc}}(\langle 0,1\rangle)$ function $\mathbf{x} \mapsto \frac{1}{|\mathbf{x}|}$, which can not be extended to an $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d)$ function).

Similar reasoning leads to the following result:

Corollary 4. *Let (u_n) and (v_n) be sequences from the preceding theorem. If there exist closed sets F_1 and F_2 of $\mathbf{R}^d$ such that u_n keep their support in F_1 and v_n in F_2, then the support of any H-distribution corresponding to subsequences of (u_n) and (v_n) is included in $(F_1 \cap F_2) \times \mathrm{S}^{d-1}$.* ∎

5. Basic properties

One of the useful features of H-measures is that they can determine if a weakly converging $\mathrm{L}^2_{\mathrm{loc}}$ sequence, converges strongly in the same space. Namely, an $\mathrm{L}^2_{\mathrm{loc}}$ sequence will converge to zero strongly if and only if the corresponding H-measure is zero. In this section we prove an analogous property for H-distributions.

Canonical choice of $\mathrm{L}^{p'}$ sequence corresponding to an L^p sequence (u_n) is given by $v_n = \Phi_p(u_n)$, where Φ_p is an operator from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ defined by $\Phi_p(u) = |u|^{p-2}u$.

Before we proceed, let us state some properties of operator Φ_p, $p \in \langle 1, \infty\rangle$, which we will need later. First of all, Φ_p is a nonlinear Nemytskii operator, thus it is continuous from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ and additionally we have the following bound

$$\|\Phi_p(u)\|_{\mathrm{L}^{p'}(\mathbf{R}^d)} \leq \|u\|^{p/p'}_{\mathrm{L}^p(\mathbf{R}^d)}.$$

Moreover, it is continuous from $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ to $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$. Indeed, take an arbitrary seminorm $|\cdot|_{\vartheta_k,p'}$ from the definition of the metric $d_{p'}$ and any $u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ to get

$$\begin{aligned}|\Phi_p(u)|^{p'}_{\vartheta_k,p'} &= \|\vartheta_k \Phi_p(u)\|^{p'}_{\mathrm{L}^{p'}(\mathbf{R}^d)} = \int_{\mathbf{R}^d} |\vartheta_k(\mathbf{x})|^{p'} |u(\mathbf{x})|^{(p-1)p'} d\mathbf{x} = \int_{\mathbf{R}^d} |\vartheta_k(\mathbf{x})|^{p'} |u(\mathbf{x})|^{p} d\mathbf{x}\\ &\leq \int_{\mathbf{R}^d} |\vartheta_{k+1}(\mathbf{x})|^{p} |u(\mathbf{x})|^{p} d\mathbf{x} = \|\vartheta_{k+1} u\|^p_{\mathrm{L}^p(\mathbf{R}^d)} = |u|^p_{\vartheta_{k+1},p}.\end{aligned}$$

From here we conclude that Φ_p maps bounded sets in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ topology to bounded sets in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$ topology. Moreover, if (u_n) is a bounded sequence in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$, then from the above bound we get

$$|\Phi_p(u_n)|_{\vartheta_k,p'} \leq |u_n|^{p/p'}_{\vartheta_{k+1},p} < C^{p/p'}_{\vartheta_{k+1},p}\ ,$$

from which follows that $(\Phi_p(u_n))$ is a bounded sequence in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$, which is (semi-)reflexive space. This implies that $(\Phi_p(u_n))$ is weakly precompact in $\mathrm{L}^{p'}_{loc}(\mathbf{R}^d)$ (see [45, Theorem 15.2.4]).

To conclude, take arbitrary $\varepsilon > 0$, $u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ and $k \in \mathbf{N}$. The continuity of Φ_p guarantees the existence of $\delta > 0$ such that

$$(\forall v \in \mathrm{L}^p(\mathbf{R}^d)) \Big(\|v - \vartheta_{k+1}u\|_{\mathrm{L}^p(\mathbf{R}^d)} < \delta \Longrightarrow \|\Phi_p(v) - \Phi_p(\vartheta_{k+1}u)\|_{\mathrm{L}^{p'}(\mathbf{R}^d)} < \varepsilon\Big).$$

Take $v \in \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ such that $|v-u|_{\vartheta_{k+1},p} < \delta$. We have the following:

$$\begin{aligned}
|\Phi_p(v) - \Phi_p(u)|_{\vartheta_k,p'} &= \left\|\vartheta_k\Big(\Phi_p(v) - \Phi_p(u)\Big)\right\|_{\mathrm{L}^{p'}(\mathbf{R}^d)}\\
&= \left\|\vartheta_k\vartheta_{k+1}\Big(\Phi_p(v) - \Phi_p(u)\Big)\right\|_{\mathrm{L}^{p'}(\mathbf{R}^d)}\\
&= \left\|\vartheta_k\Big(\Phi_p(\vartheta_{k+1}v) - \Phi_p(\vartheta_{k+1}u)\Big)\right\|_{\mathrm{L}^{p'}(\mathbf{R}^d)}\\
&\leq \|\Phi_p(\vartheta_{k+1}v) - \Phi_p(\vartheta_{k+1}u)\|_{\mathrm{L}^{p'}(\mathbf{R}^d)} \qquad < \varepsilon.
\end{aligned}$$

Since k was arbitrary, we conclude that $\Phi_p : \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d) \longrightarrow \mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$ is continuous.

Now we can state the main result of this section:

Lemma 5. *For a sequence* (u_n) *in* $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$, $p \in \langle 1, \infty\rangle$, *the following are equivalent*

a) $u_n \to 0$ *in* $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$,

b) *for every sequence* (v_n) *satisfying conditions of the existence theorem,* (u_n) *and* (v_n) *form a pure pair and the corresponding H-distribution is zero.*

Dem. For the first implication, it is enough to notice that due to the compact support of test function φ_1 and boundedness properties of the Fourier multiplier operator $\mathcal{A}_\psi$, we get that $\mathcal{A}_\psi(\varphi_1 u_n) \to 0$ in $\mathrm{L}^p(\mathbf{R}^d)$, thus

$$\lim_n \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_n)(\mathbf{x})\overline{\varphi_2 v_n}(\mathbf{x})d\mathbf{x} = 0.$$

Take a sequence $(\Phi_p(u_n))$. We have already concluded that it is bounded and weakly precompact in $\mathrm{L}^{p'}_{loc}(\mathbf{R}^d)$. Taking symbol ψ to be equal to one (so that $\mathcal{A}_\psi$ is the identity) and test functions φ_1 and φ_2, we get

$$0 = \lim_{n'} \int_{\mathbf{R}^d} (\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})}d\mathbf{x} = \lim_{n'} \int_{\mathbf{R}^d} \varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})}|u_{n'}(\mathbf{x})|^p d\mathbf{x},$$

which implies $u_{n'} \to 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$. From here we conclude that the whole sequence (u_n) converges to zero strongly in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ as well. If this were not the case, we could find a subsequence which either converges to some nontrivial limit, which is impossible due to the weak convergence to zero, or diverges, which is impossible due to boundedness of the sequence.

Q.E.D.

Remark 7. Let us notice that in the previous Lemma, we have proved that the two claims in (a) and (b) are equivalent to

b') u_n and $(\Phi_p(u_n))$ form a pure pair and the corresponding H-distribution is zero.

Remark 8. It is easy to see that claim in (b) does not imply strong convergence to zero in $\mathrm{L}^p(\mathbf{R}^d)$ of the sequence (u_n) in $\mathrm{L}^p(\mathbf{R}^d)$. Indeed, take a nontrivial $u \in \mathrm{L}^p_c(\mathbf{R}^d)$ and unit vector $\mathsf{e} \in \mathrm{S}^{d-1}$. Define a sequence $u_n(\mathbf{x}) = u(\mathbf{x} - n\mathsf{e})$ which weakly converges to zero in $\mathrm{L}^p(\mathbf{R}^d)$. Support of u_n goes to infinity so the corresponding H-distribution is zero, while u_n does not converge to zero strongly in $\mathrm{L}^p(\mathbf{R}^d)$.

Let (u_n) be a sequence weakly converging to 0 in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$. Then the sequence $(|u_n|^p)$ is bounded in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d)$, so $|u_n|^p \xrightarrow{*} \nu$ in $\mathcal{D}'(\mathbf{R}^d)$ (up to a subsequence). Since all the elements of the sequence $(|u_n|^p)$ are positive (in terms of distributions), the limit ν is a positive distributions, hence (unbounded) Radon measure.
On the other hand, let μ be any H-distribution corresponding to the above chosen subsequence of (u_n) and $(\Phi_p(u_n))$. Taking ψ to be equal to one and test functions φ_1, φ_2 such that φ_2 is equal to one on the support of φ_1, we get the following connection between μ and ν:

$$\langle \mu, \varphi_1 \boxtimes 1 \rangle = \lim_n \int_{\mathbf{R}^d} \varphi_1 |u_n|^p d\mathbf{x} = \langle \nu, \varphi_1 \rangle.$$

We sum this observation in the following corollary:

Corollary 5. *Let (u_n) converge weakly to zero in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$, for $p \in \langle 1, \infty \rangle$, and let $(|u_n|^p)$ converge weakly-$*$ to a measure ν in the space of unbounded Radon measures $\mathcal{M}(\mathbf{R}^d)$. Then for all $\varphi \in \mathrm{C}^\infty_c(\mathbf{R}^d)$, it holds*

$$\langle \mu, \varphi \boxtimes 1 \rangle = \lim_n \int_{\mathbf{R}^d} \varphi |u_n|^p d\mathbf{x} = \langle \nu, \varphi \rangle,$$

where μ is any H-distribution corresponding to some subsequences of (u_n) and $(\Phi_p(u_n))$.

∎

Let us see what happens when we change positions of sequences (u_n) and (v_n):

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathbf{R}^d} (\varphi_1 v_n)(\mathbf{x}) \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 u_n)(\mathbf{x})} d\mathbf{x} &= \lim_{n\to\infty} \int_{\mathbf{R}^d} \overline{(\varphi_2 u_n)(\mathbf{x})} \mathcal{A}_\psi(\varphi_1 v_n)(\mathbf{x}) d\mathbf{x} = \\
&= \lim_{n\to\infty} \overline{\int_{\mathbf{R}^d} (\varphi_2 u_n)(\mathbf{x}) \overline{\mathcal{A}_\psi(\varphi_1 v_n)(\mathbf{x})} d\mathbf{x}} = \\
&= \overline{\left\langle \mu, \overline{\varphi_1}\varphi_2 \boxtimes \overline{\psi} \right\rangle} = \left\langle \overline{\mu}, \varphi_1 \overline{\varphi_2} \boxtimes \psi \right\rangle,
\end{aligned}$$

where μ is a H-distribution corresponding to sequences (u_n) and (v_n).

Next, we turn our attention to the relation between H-distributions corresponding to conjugated sequences. First we will prove some auxiliary results. It is easy to see that for every $v \in \mathcal{S}(\mathbf{R}^d)$ it holds: $\hat{\overline{v}}(\boldsymbol{\xi}) = \overline{\overline{\int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} \overline{v}(\mathbf{x}) d\mathbf{x}}} = \overline{\int_{\mathbf{R}^d} e^{2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} v(\mathbf{x}) d\mathbf{x}} = \overline{\check{v}}(\boldsymbol{\xi})$ and analogously $\check{\overline{v}} = \overline{\hat{v}}$. Using these, we arrive at the following chain of equalities valid for any $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$

$$\mathcal{A}_\psi(\overline{v}) = (\psi \hat{\overline{v}})^\vee = (\psi \overline{\check{v}})^\vee = \left(\overline{\overline{\psi} \check{v}}\right)^\vee = \overline{\widehat{\overline{\psi}\check{v}}}.$$

Let us rewrite the last term above:

$$\begin{aligned}
\widehat{\overline{\psi}\check{v}}(\mathbf{x}) &= \int_{\mathbf{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} \overline{\psi}(\boldsymbol{\xi}) \check{v}(\boldsymbol{\xi}) d\boldsymbol{\xi} = \int_{\mathbf{R}^d} e^{2\pi i \boldsymbol{\eta}\cdot\mathbf{x}} \overline{\psi}(-\boldsymbol{\eta}) \check{v}(-\boldsymbol{\eta}) d\boldsymbol{\eta} = \int_{\mathbf{R}^d} e^{2\pi i \boldsymbol{\eta}\cdot\mathbf{x}} \overline{\tilde{\psi}}(\boldsymbol{\eta}) \hat{v}(\boldsymbol{\eta}) d\boldsymbol{\eta} \\
&= \left(\overline{\tilde{\psi}} \hat{v}\right)^\vee(\mathbf{x}) = \mathcal{A}_{\overline{\tilde{\psi}}}(v)(\mathbf{x}),
\end{aligned}$$

where we have used the change of variables $\boldsymbol{\eta} = -\boldsymbol{\xi}$ and the notation $\tilde{v}(\mathbf{x}) = v(-\mathbf{x})$. Since $\mathcal{A}_\psi$ (and $\mathcal{A}_{\overline{\tilde{\psi}}}$) are continuous on $\mathrm{L}^p(\mathbf{R}^d)$, while $\mathcal{S}(\mathbf{R}^d)$ is dense in $\mathrm{L}^p(\mathbf{R}^d)$, we have showed that for every $v \in \mathrm{L}^p(\mathbf{R}^d)$ it holds $\mathcal{A}_\psi(\overline{v}) = \overline{\mathcal{A}_{\overline{\tilde{\psi}}}(v)}$. Now, we can write

$$\begin{aligned}\lim_n \langle \mathcal{A}_\psi(\varphi_1 \overline{u_n}), \varphi_2 \overline{v_n}\rangle &= \lim_n \langle \mathcal{A}_\psi(\overline{\overline{\varphi_1} u_n}), \overline{\overline{\varphi_2} v_n}\rangle = \lim_n \left\langle \overline{\mathcal{A}_{\overline{\widetilde{\psi}}}(\overline{\varphi_1} u_n), \overline{\varphi_2} v_n} \right\rangle \\ &= \lim_n \overline{\langle \mathcal{A}_{\overline{\widetilde{\psi}}}(\overline{\varphi_1} u_n), \overline{\varphi_2} v_n\rangle} = \overline{\left\langle \mu, \overline{\varphi_1} \varphi_2 \boxtimes \overline{\widetilde{\psi}} \right\rangle} \\ &= \langle \overline{\mu}, \varphi_1 \overline{\varphi_2} \boxtimes \widetilde{\psi}\rangle = \langle \widetilde{\overline{\mu}}, \varphi_1 \overline{\varphi_2} \boxtimes \psi\rangle,\end{aligned}$$

where μ is the H-distribution corresponding to subsequences of (u_n) and (v_n) and in the last step the tilde operation is taken only with respect to the $\boldsymbol{\xi}$ variable. We have proved the following lemma:

Lemma 6. *Let (u_n) and (v_n) form a pure pair of sequences and let μ be the corresponding H-distribution. The following holds*

a) *The H-distribution corresponding to (v_n) and (u_n) is $\overline{\mu}$,*

b) *$(\overline{u_n})$ and $(\overline{v_n})$ is a pure pair and the corresponding H-distribution is $\widetilde{\overline{\mu}}$, where the tilde operation is taken only with respect to the dual variable.*

∎

6. Example with concentrations

Vitali's convergence theorem gives sufficient and necessary conditions under which a sequence of L^p functions will converge strongly to a measurable function in L^p. One of them is uniform integrability which implies that there are no concentration effects in the sequence. Hence, it is of interest to consider concentration effects in weakly converging sequences.

Take u from $\mathrm{L}^p_c(\mathbf{R}^d)$, for some $p \in \langle 1, \infty\rangle$ and define a sequence $u_n(\mathbf{x}) = n^{\frac{d}{p}} u(n(\mathbf{x} - \mathbf{z}))$ for some $\mathbf{z} \in \mathbf{R}^d$. A simple change of variables shows that $\|u_n\|_{\mathrm{L}^p(\mathbf{R}^d)} = \|u\|_{\mathrm{L}^p(\mathbf{R}^d)}$ and that it weakly converges to 0 in $\mathrm{L}^p(\mathbf{R}^d)$. Indeed, the sequence is bounded, while by taking a continuous test function φ with compact support we get

$$\begin{aligned}\int_{\mathbf{R}^d} u_n(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} &= \int_{\mathbf{R}^d} n^{d/p} u(n(\mathbf{x}-\mathbf{z}))\varphi(\mathbf{x})d\mathbf{x} = \int_{\mathbf{R}^d} n^{d/p-d} u(\mathbf{y})\varphi(\mathbf{y}/n+\mathbf{z})d\mathbf{y} = \\ &= \frac{1}{n^{d/p'}} \int_{\mathbf{R}^d} u(\mathbf{y})\chi_{\operatorname{supp} u}(\mathbf{y})\varphi(\mathbf{y}/n+\mathbf{z})d\mathbf{y} \leq \\ &\leqslant \left(\frac{\mathsf{vol}(\operatorname{supp} u)}{n^d}\right)^{1/p'} \|u\|_{\mathrm{L}^p(\mathbf{R}^d)} \max_{\mathbf{R}^d} |\varphi|,\end{aligned}$$

where we have used the change of variables in the second equality and the Hölder inequality in the last step. Passing to the limit, we get our claim.

We will show that the H-distribution corresponding to sequences (u_n) and $(\Phi_p(u_n))$ is given by $\delta_{\mathbf{z}} \boxtimes \nu$, where ν is a distribution on $\mathrm{C}^\kappa(\mathrm{S}^{d-1})$ defined for $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ by

$$\langle \nu, \psi\rangle = \int_{\mathbf{R}^d} u(\mathbf{x}) \overline{\mathcal{A}_{\bar{\psi}}(|u|^{p-2}u)(\mathbf{x})} d\mathbf{x}.$$

Before we proceed, we will need the following two lemmata:

Lemma 7. *Let $u \in \mathrm{L}^p(\mathbf{R}^d)$ and $u_n(\mathbf{x}) = n^{d/p}u(n(\mathbf{x}-\mathbf{z}))$. Then, for every $\varphi \in \mathrm{C}(\mathbf{R}^d)$ it holds*

$$\varphi u_n - \varphi(\mathbf{z})u_n \longrightarrow 0 \textit{ in } \mathrm{L}^p(\mathbf{R}^d).$$

Dem. Using the change of variables $\mathbf{y} = n(\mathbf{x}-\mathbf{z})$, we get

$$\begin{aligned}\int_{\mathbf{R}^d} |\varphi(\mathbf{x})u_n(\mathbf{x}) - \varphi(\mathbf{z})u_n(\mathbf{x})|^p d\mathbf{x} &= \int_{\mathbf{R}^d} |\varphi(\mathbf{x})n^{\frac{d}{p}}u(n(\mathbf{x}-\mathbf{z})) - \varphi(\mathbf{z})n^{\frac{d}{p}}u(n(\mathbf{x}-\mathbf{z}))|^p d\mathbf{x} \\ &= \int_{\mathbf{R}^d} |\varphi(\mathbf{y}/n+\mathbf{z}) - \varphi(\mathbf{z})|^p |u(\mathbf{y})|^p d\mathbf{y},\end{aligned}$$

which goes to 0, by the Lebesgue dominated convergence theorem.

Q.E.D.

Lemma 8. *For any $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, $v \in \mathrm{L}^p(\mathbf{R}^d)$, $p \in \langle 1,\infty\rangle$, $\mathbf{z} \in \mathbf{R}^d$ and $n \in \mathbf{N}$ it holds*

$$\Big(\mathcal{A}_\psi v(n(\cdot - \mathbf{z}))\Big)(\mathbf{x}) = (\mathcal{A}_\psi v)(n(\mathbf{x}-\mathbf{z})).$$

Dem. For $v \in \mathcal{S}(\mathbf{R}^d)$, we have

$$\begin{aligned}\Big(\mathcal{A}_\psi v(n(\cdot - \mathbf{z}))\Big)(\mathbf{x}) &= \bar{\mathcal{F}}\Big(\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\int_{\mathbf{R}^d} e^{-2\pi i\mathbf{y}\cdot\boldsymbol{\xi}} v(n(\mathbf{y}-\mathbf{z}))d\mathbf{y}\Big)(\mathbf{x}) = \\ &= \bar{\mathcal{F}}\Big(n^{-d}e^{-2\pi i\mathbf{z}\cdot\boldsymbol{\xi}}\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\int_{\mathbf{R}^d} e^{-2\pi i\frac{\boldsymbol{\zeta}}{n}\cdot\boldsymbol{\xi}} v(\boldsymbol{\zeta})d\boldsymbol{\zeta}\Big)(\mathbf{x}) = \\ &= n^{-d}\bar{\mathcal{F}}\Big(e^{-2\pi i\mathbf{z}\cdot\boldsymbol{\xi}}\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\hat{v}(\boldsymbol{\xi}/n)\Big)(\mathbf{x}) = \\ &= n^{-d}\int_{\mathbf{R}^d} e^{2\pi i(\mathbf{x}-\mathbf{z})\cdot\boldsymbol{\xi}}\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\hat{v}(\boldsymbol{\xi}/n)d\boldsymbol{\xi} = \\ &= \int_{\mathbf{R}^d} e^{2\pi i\boldsymbol{\eta}\cdot(n(\mathbf{x}-\mathbf{z}))}\psi(\boldsymbol{\eta}/|\boldsymbol{\eta}|)\hat{v}(\boldsymbol{\eta})d\boldsymbol{\eta} = \\ &= (\mathcal{A}_\psi v)(n(\mathbf{x}-\mathbf{z})),\end{aligned}$$

where we have used the change of variables $n(\mathbf{y}-\mathbf{z}) = \boldsymbol{\zeta}$ in the second equality and $n\boldsymbol{\eta} = \boldsymbol{\xi}$ in the penultimate one. Since $\mathcal{S}(\mathbf{R}^d)$ is dense in $\mathrm{L}^p(\mathbf{R}^d)$, while $\mathcal{A}_\psi$ is continuous on $\mathrm{L}^p(\mathbf{R}^d)$, we get the claim.

Q.E.D.

Taking test functions φ_1 and φ_2 to be continuous with compact supports and φ_2 to be equal to one on the set $\operatorname{supp} u \cup (\mathbf{z} + \operatorname{supp} u)$, we get the following

$$
\begin{aligned}
\langle\mu,\varphi_1\overline{\varphi_2}\boxtimes\psi\rangle &= \lim_n \int\limits_{\mathbf{R}^d} \varphi_1(\mathbf{x})u_n(\mathbf{x})\overline{\mathcal{A}_{\bar\psi}(\varphi_2|u_n|^{p-2}u_n)(\mathbf{x})}d\mathbf{x}\\
&= \lim_n \int\limits_{\mathbf{R}^d} \varphi_1(\mathbf{x})n^d u(n(\mathbf{x}-\mathbf{z}))\overline{\mathcal{A}_{\bar\psi}\Big(\varphi_2|u|^{p-2}(n(\cdot-\mathbf{z}))u(n(\cdot-\mathbf{z}))\Big)(\mathbf{x})}d\mathbf{x}\\
&= \lim_n \int\limits_{\mathbf{R}^d} \varphi_1(\mathbf{z})n^d u(n(\mathbf{x}-\mathbf{z}))\overline{\mathcal{A}_{\bar\psi}\Big(\varphi_2|u|^{p-2}(n(\cdot-\mathbf{z}))u(n(\cdot-\mathbf{z}))\Big)(\mathbf{x})}d\mathbf{x}\\
&= \lim_n \varphi_1(\mathbf{z}) \int\limits_{\mathbf{R}^d} u(\mathbf{y})\overline{\mathcal{A}_{\bar\psi}\Big(\chi_{\operatorname{supp} u}\varphi_2(\cdot/n+\mathbf{z})|u|^{p-2}u\Big)(\mathbf{y})}d\mathbf{y}\\
&= \lim_n \varphi_1(\mathbf{z}) \int\limits_{\mathbf{R}^d} u(\mathbf{y})\overline{\mathcal{A}_{\bar\psi}(|u|^{p-2}u)(\mathbf{y})}d\mathbf{y}\\
&= \varphi_1(\mathbf{z})\Big\langle u, \mathcal{A}_{\bar\psi}(|u|^{p-2}u)\Big\rangle,
\end{aligned}
$$

where μ denotes the H-distribution corresponding to subsequences of (u_n) and $(\Phi_p(u_n))$. We have used the preceding lemmata in the third and fourth equalities, and the change of variables $\mathbf{y} = n(\mathbf{x}-\mathbf{z})$ in the fourth one. In the last step we have noticed that the expression on the right hand side does not depend on n anymore. Using the corollary on the support of the H-distribution, we finally arrive at

$$
\langle\mu,\varphi_1\boxtimes\psi\rangle = \varphi_1(\mathbf{z})\Big\langle u, \mathcal{A}_{\bar\psi}(|u|^{p-2}u)\Big\rangle.
$$

Remark 9. If we had chosen u and p such that $u\in \mathrm{L}^r_c(\mathbf{R}^d)$ where $r\geq\max\{2,2p-2\}$ (case $r=\infty$ included), we would have been able to use Plancherel's theorem and rewrite the integral in polar coordinates to get

$$
\begin{aligned}
\langle\mu,\varphi_1\boxtimes\psi\rangle &= \varphi_1(\mathbf{z})\int_{\mathbf{R}^d}\hat u(\boldsymbol\xi)\psi(\boldsymbol\xi/|\boldsymbol\xi|)(\overline{|u|^{p-2}u})^\wedge(\boldsymbol\xi)d\boldsymbol\xi\\
&= \varphi_1(\mathbf{z})\int_{\mathrm{S}^{d-1}}\int_0^\infty \hat u(t\boldsymbol\eta)\psi(\boldsymbol\eta)(\overline{|u|^{p-2}u})^\wedge(t\boldsymbol\eta)t^{d-1}dt\; d\boldsymbol\eta,
\end{aligned}
$$

since the given bound on r implies $r\geq 2$ and $r\geq p$.

7. Perturbations

Let (u_n) be a sequence weakly converging to 0 in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ for some $p\in\langle 1,\infty\rangle$ and $h_n \xrightharpoonup{*} h$ in $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ for $q\geq p'$. Consider a sequence (d_n) strongly converging to zero in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$. Then $u_n+d_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ and we may ask ourselves if there exists a connection between the H-distribution μ corresponding to subsequences of (u_n) and (h_n) and the H-distribution μ_d corresponding to subsequences of (u_n+d_n) and (h_n). It is easy to see that these two H-distributions are the same:

$$
\int_{\mathbf{R}^d}\varphi_1(u_n+d_n)\overline{\mathcal{A}_{\bar\psi}(\varphi_2 v_n)}d\mathbf{x} = \int_{\mathbf{R}^d}\varphi_1 u_n\overline{\mathcal{A}_{\bar\psi}(\varphi_2 v_n)}d\mathbf{x} + \int_{\mathbf{R}^d}\varphi_1 d_n\overline{\mathcal{A}_{\bar\psi}(\varphi_2 v_n)}d\mathbf{x}.
$$

The left hand side goes to $\langle \mu_d, \varphi_1\bar{\varphi}_2 \boxtimes \psi\rangle$, the second term on the right hand side goes to 0 according to one of the preceding results, while the first term goes to $\langle \mu, \varphi_1\bar{\varphi}_2 \boxtimes \psi\rangle$.

In the above, we could have perturbed sequence (h_n) with a sequence strongly converging to zero in $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ as well. We would still get the same conclusion. In particular, for $p = q = 2$ we obtain a similar result for H-measures.

Now, let us turn our attention to generating sequences. We would like to know if we could use a smoother sequence to obtain the same H-distribution. Assume that we are given a family of sequences (u_n^m) in $\mathrm{C}^\infty_c(\mathbf{R}^d)$ such that $u_n^m \longrightarrow u_n$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$, as $m \to \infty$. We can always find such approximating sequences since the space $\mathrm{C}^\infty_c(\mathbf{R}^d)$ is dense in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ for $p < \infty$. Using the Cantor diagonal procedure, we can extract a subsequence $v_k = u_k^{m(k)}$ such that $d_p(v_k, u_k) \leq 1/k$. It is straightforward to see that v_k weakly converges to zero in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$: for every $\varphi \in \mathrm{L}^{p'}_c(\mathbf{R}^d)$ it holds

$$\int_{\mathbf{R}^d} v_k\varphi = \int_{\mathbf{R}^d} (v_k - u_k)\varphi + \int_{\mathbf{R}^d} u_k\varphi.$$

The claim follows from the strong convergence of $(v_k - u_k)$ and the weak convergence of (u_k).

Lemma 9. *Sequences (v_n) and (u_n) generate the same H-distribution. In other words, for all $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d)$, $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ and any sequence $h_n \xrightarrow{*} h$ in $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ for $q \geq p'$, it holds*

$$\lim_n \int_{\mathbf{R}^d} \varphi_1 v_n \overline{\mathcal{A}_{\bar{\psi}}(\varphi_2 h_n)} d\mathbf{x} = \lim_n \int_{\mathbf{R}^d} \varphi_1 u_n \overline{\mathcal{A}_{\bar{\psi}}(\varphi_2 h_n)} d\mathbf{x}.$$

Dem. By the Hölder inequality, we get

$$\begin{aligned}\lim_n \int_{\mathbf{R}^d} |\varphi_1(v_n - u_n)\overline{\mathcal{A}_{\bar{\psi}}(\varphi_2 h_n)} d\mathbf{x}| &\leq \lim_n ||\varphi_1(v_n - u_n)||_{\mathrm{L}^p(\mathbf{R}^d)} ||\mathcal{A}_{\bar{\psi}}(\varphi_2 h_n)||_{\mathrm{L}^{p'}(\mathbf{R}^d)} \\ &\leq C_\psi \lim_n ||\varphi_1(v_n - u_n)||_{\mathrm{L}^p(\mathbf{R}^d)} ||\varphi_2 h_n||_{\mathrm{L}^{p'}(\mathbf{R}^d)},\end{aligned}$$

where we have used boundedness of the Fourier multiplier operator $\mathcal{A}_{\bar{\psi}}$ in the second inequality. Since the sequence $(\varphi_2 h_n)$ is bounded in $\mathrm{L}^{p'}$, we get that the right hand side goes to zero.

Q.E.D.

If $q < \infty$, we can approximate h_n with smooth sequences in $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ as well. Analogously as we did with (u_n), we would arrive at a smooth sequence (g_n) such that $d_q(g_k, h_k) \leq 1/k$. We have the following:

Corollary 6. *If $q \in [p', \infty\rangle$, pairs (u_n), (h_n) and (v_n), (g_n) generate the same H-distribution. In other words, for all $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, it holds:*

$$\lim_n \int_{\mathbf{R}^d} \varphi_1 v_n \overline{\mathcal{A}_{\bar{\psi}}(\varphi_2 g_n)} d\mathbf{x} = \lim_n \int_{\mathbf{R}^d} \varphi_1 u_n \overline{\mathcal{A}_{\bar{\psi}}(\varphi_2 h_n)} d\mathbf{x}.$$

∎

As a consequence, we get a similar statement in the case of H-measures:

Corollary 7. *If the sequence (u_n) in $\mathrm{L}^2(\mathbf{R}^d)$ generates an H-measure, then there exists a smooth sequence which generates the same H-measure.*

∎

The corollary above covers the case when we use $(\Phi_p(u_n))$, the canonical $\mathrm{L}^{p'}_{\mathrm{loc}}$-sequence associated with (u_n). We could approximate it with some other sequence in $\mathrm{L}^{p'}_{\mathrm{loc}}$. However, it would be convenient if we could choose smooth sequence (v_n) approximating (u_n) such that $(\Phi_p(v_n))$ approximates $(\Phi_p(u_n))$. This can be achieved because $\Phi_p : \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d) \longrightarrow \mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$ is continuous: repeating the construction from the beginning of this section, we need to choose $m(k) \in \mathbf{N}$ such that for $v_k = u_k^{m(k)}$ it holds both $d_p(v_k, u_k) \leq 1/k$ and $d_{p'}(\Phi_p(v_k), \Phi_p(u_k)) \leq 1/k$.

Corollary 8. *Two pairs of sequences (u_n), $(\Phi_p(u_n))$ and (v_n), $\Phi_p(v_n))$, where (v_n) is chosen as above, generate the same H-distribution. In other words, for any $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, it holds:*

$$\lim_n \int_{\mathbf{R}^d} \varphi_1 v_n \overline{\mathcal{A}_{\bar{\psi}}(\varphi_2 \Phi_p(v_n))} d\mathbf{x} = \lim_n \int_{\mathbf{R}^d} \varphi_1 u_n \overline{\mathcal{A}_{\bar{\psi}}(\varphi_2 \Phi_p(u_n))} d\mathbf{x}.$$

∎

Our next step is to show that we can improve the regularity of the symbol $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$. Since unit sphere S^{d-1} is a compact $(d-1)$-dimensional surface in $\mathbf{R}^d$, there exists an at most denumerable smooth partition of unity (f_j) on S^{d-1} with corresponding one-to-one local parametrisations (Ψ_j), where $\Psi_j : U_j \longrightarrow \mathrm{S}^{d-1}$, for some open set $U_j \subseteq \mathbf{R}^d$ (cf. [15, Chapter 16, Section 4]). Since S^{d-1} is a smooth surface, we can choose smooth Ψ_j such that Ψ_j^{-1} are smooth as well (this follows from the Inverse function theorem). Locally, on each U_j, we can approximate $(f_j\psi) \circ \Psi_j : U_j \longrightarrow \mathbf{R}$ with C^∞ functions $(\psi^j_k)_k$ (for example, using convolution with the standard mollifier) in the topology of the space $\mathrm{C}^\kappa(U_j)$. Thus, $(\psi^j_k \circ \Psi_j^{-1})_k$ are smooth approximations of $f_j\psi$ in the space $\mathrm{C}^\kappa(\mathrm{S}^{d-1})$. Now, for every $k \in \mathbf{N}$ we can choose $\psi_k \in \mathrm{C}^\infty(\mathrm{S}^{d-1})$ such that $\|\psi - \psi_k\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})} < 1/k$. We have the following lemma:

Lemma 10. *Let $u_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ for some $p \in \langle 1, \infty\rangle$ and $h_n \xrightharpoonup{*} h$ in $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ for $q \geq p'$. Let $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ and $\psi_k \in \mathrm{C}^\infty(\mathrm{S}^{d-1})$ be as above. Then for every $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d)$ it holds*

$$\lim_n \int_{\mathbf{R}^d} \varphi_1 u_n \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 h_n)} d\mathbf{x} = \lim_k \lim_n \int_{\mathbf{R}^d} \varphi_1 u_n \overline{\mathcal{A}_{\overline{\psi_k}}(\varphi_2 h_n)} d\mathbf{x}.$$

Dem. Similarly as we did for the bound of $\mu_{n,l}$ in the proof of the existence of H-distributions, we arrive at the following

$$\begin{aligned}\Big|\lim_n \int_{\mathbf{R}^d} \varphi_1 u_n \overline{\mathcal{A}_{\overline{\psi-\psi_k}}(\varphi_2 h_n)} d\mathbf{x}\Big| &\leq C_{d,p} \|\varphi_1\varphi_2\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} \|\psi - \psi_k\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})} \\ &\leq \frac{C_{d,p}}{k} \|\varphi_1\varphi_2\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)}.\end{aligned}$$

Letting $k \to \infty$, we get the conclusion.

Q.E.D.

Remark 10. Throughout this article we have used symbols associated to functions $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ by composing ψ with projection π from $\mathbf{R}^d \setminus \{0\}$ to S^{d-1}. As it is well known in the theory of pseudodifferential calculus, we can replace such symbols with $\widetilde{\psi} \in \mathrm{C}^\kappa(\mathbf{R}^d)$ functions which are identically equal to $\psi \circ \pi$ only for large $|\boldsymbol{\xi}|$. Indeed, one needs to notice that $\eta(\boldsymbol{\xi}) := \psi(\boldsymbol{\xi}) - (\psi \circ \pi)(\boldsymbol{\xi})$ is a bounded $\mathrm{C}^\kappa(\mathbf{R}^d \setminus \{0\})$ function with compact support. Thus, $\mathcal{A}_\eta$ is a compact operator on $\mathrm{L}^p(\mathbf{R}^d)$ which, for $u_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$, implies that

$$0 = \lim_n \int_{\mathbf{R}^d} \mathcal{A}_\eta(\varphi_1 u_n)(\mathbf{x})\overline{(\varphi_2 v_n)(\mathbf{x})} d\mathbf{x} = \lim_n \int_{\mathbf{R}^d} \mathcal{A}_{\widetilde{\psi}}(\varphi_1 u_n)(\mathbf{x})\overline{(\varphi_2 v_n)(\mathbf{x})} d\mathbf{x} - \\ - \lim_n \int_{\mathbf{R}^d} \mathcal{A}_{\psi\circ\pi}(\varphi_1 u_n)(\mathbf{x})\overline{(\varphi_2 v_n)(\mathbf{x})} d\mathbf{x}.$$

Hence,

$$\lim_n \int_{\mathbf{R}^d} \mathcal{A}_{\widetilde{\psi}}(\varphi_1 u_n)(\mathbf{x})\overline{(\varphi_2 v_n)(\mathbf{x})} d\mathbf{x} = \Big\langle \mu,\ \varphi_1\overline{\varphi_2} \boxtimes \psi \Big\rangle,$$

where we understand the right hand side in the sense of existence theorem on H-distributions.
A similar observation for H-measures is due to Panov.

8. Localisation principle and compactness by compensation

Next, we present a localisation principle for H-distributions. The proof of the following result follows along the same lines as proof of Theorem 4.1. in [7].

Theorem 5. *Assume that $u_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ and $f_n \longrightarrow 0$ in $\mathrm{W}^{-1,q}_{loc}(\mathbf{R}^d)$ for some $p \in \langle 1, \infty\rangle$ and $q \in \langle 1, d\rangle$, such that they satisfy*

$$\sum_{i=1}^d \partial_i(a_i(\mathbf{x})u_n(\mathbf{x})) = f_n(\mathbf{x}),$$

where $a_i \in \mathrm{C}_c(\mathbf{R}^d)$. Take an arbitrary sequence (v_n) bounded in $\mathrm{L}^\infty_{loc}(\mathbf{R}^d)$, and by μ denote the H-distribution corresponding to some subsequences of sequences (u_n) and (v_n). Then,

$$\sum_{i=1}^d a_i(\mathbf{x})\xi_i \mu(\mathbf{x}, \boldsymbol{\xi}) = 0$$

in the sense of distributions on $\mathbf{R}^d \times \mathrm{S}^{d-1}$. ∎

One can use Marcinkiewicz's theorem instead of Hörmander-Mihlin's for continuity of the Fourier multipliers. After using Lemma 5 from [35], one obtains H-distributions which require a higher regularity of the symbol $\psi \in \mathrm{C}^d(\mathrm{S}^{d-1})$. This approach was used in [35, 42] where they had a variant of H-measures and H-distributions on manifolds different than unit sphere. Application of the Marcinkiewicz theorem allowed them to have a different, yet simpler proof of the corresponding localisation principle. Mimicking the proof of [42, Proposition 11], we obtain the following version of localisation principle:

Theorem 6. *Assume that* $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p_{\rm loc}(\mathbf{R}^d)$ *and* $v_n \rightharpoonup 0$ *in* $\mathrm{L}^{p'}_{\rm loc}(\mathbf{R}^d)$, *where* $p \in \langle 1, \infty\rangle$ *and* $q \in [p', \infty\rangle$. *Furthermore, assume that* (u_n) *satisfies:*

$$G_n := \mathsf{div}\,(\mathbf{a}u_n) \longrightarrow 0 \text{ in } \mathrm{W}^{-1;p}(\mathbf{R}^d)\,,$$

where $\mathbf{a} = (a_1, \ldots, a_d) \in \mathrm{C}_c(\mathbf{R}^d; \mathbf{R}^d)$. *Then,*

$$\sum_{i=1}^d a_i(\mathbf{x})\xi_i\mu(\mathbf{x}, \boldsymbol{\xi}) = 0 \tag{1}$$

in the sense of distributions on $\mathbf{R}^d \times \mathrm{S}^{d-1}$, *where* μ *is the H-distribution corresponding to some subsequences* (u_n) *and* (v_n).

∎

A result of this type was already announced in Remark 12 in [42] (see also Remark 4 of the next chapter). Remark that the function $(\mathbf{x}, \boldsymbol{\xi}) \mapsto \sum_{i=1}^d a_i(\mathbf{x})\xi_i$ belongs to $\mathrm{C}_c(\mathbf{R}^d) \otimes \mathrm{C}^\infty(\mathrm{S}^{d-1})$ since we have a polynomial in $\boldsymbol{\xi}$ in the expression.

The most general variant of compactness by compensation was given in [42], where authors generalised result given for L^2 in [60, 49] to the $\mathrm{L}^p - \mathrm{L}^q$ setting. We will present those results in the next chapter. We will comment on the case when $q = p'$ in several remarks throughout the chapter. It will correspond to the case we have here. For the convenience of the reader and completeness, we will briefly state the main result here:

First, let us define a variable bilinear form $q(\mathbf{x}; \lambda, \eta) := Q(\mathbf{x})\lambda\eta$, where $Q \in \mathrm{C}(\mathbf{R}^d)$ is a real function, and introduce the set:

$$\Lambda_{\mathcal{D}} = \left\{\mu \in \mathcal{D}'_{0,d(d+2)}(\mathbf{R}^d \times \mathrm{S}^{d-1}) : \sum_{i=1}^d a_i\xi_i\mu = 0\right\}.$$

Let us introduce a definition (compare it with the Definition in the next chapter):

Definition. We say that the set $\Lambda_{\mathcal{D}}$, bilinear form q, and H-distribution $\mu \in \mathcal{D}'_{0,d(d+2)}(\mathbf{R}^d \times \mathrm{S}^{d-1})$ satisfy *the strong consistency condition* if μ belongs to $\Lambda_{\mathcal{D}}$ and for every non-negative $\phi \in \mathrm{C}_c(\mathbf{R}^d)$ it holds:

$$\langle Q\phi \otimes 1, \mu\rangle \geq 0\,.$$

To formulate the compactness by compensation result, we will need the following truncation operator for $l \in \mathbf{N}$:

$$T_l(u) = \begin{cases} u, & |u| < l \\ 0, & |u| \geq l \end{cases}.$$

The compactness by compensation result now reads (for proof see Corollary 1 of the next chapter):

Theorem 7. *Assume that sequences* (u_n) *and* (v_n) *are bounded in* $\mathrm{L}^p(\mathbf{R}^d)$ *and* $\mathrm{L}^{p'}(\mathbf{R}^d)$ *and converge toward* u *and* v *in the sense of distributions. Furthermore, assume that for every* $l \in \mathbf{N}$, *the sequences* $(T_l(v_n))$ *converge weakly in* $\mathrm{L}^{p'}(\mathbf{R}^d)$ *toward* h_l, *where truncation operator is understood pointwisely. In addition, assume that there exists* $V \in \mathrm{L}^{p'}(\mathbf{R}^d)$ *such that* $|v_n| \leq V$, *and that (1) holds.*

Let $q(\mathbf{x}; u_n, v_n) \rightharpoonup \omega$ *in the sense of distributions. If for every* $l \in \mathbf{N}$, *the set* $\Lambda_{\mathcal{D}}$, *the bilinear form* q, *and the H-distributions* μ_l *corresponding to the sequences* $(u_n - u)$ *and* $(T_l(v_n) - h_l)$ *satisfy the strong consistency condition, then it holds*

$$q(\mathbf{x}; u, v) \leq \omega \qquad \text{in} \quad \mathcal{D}'(\mathbf{R}^d)\,.$$

If we have equality in the strong consistency condition, then we have equality in the above conclusion as well.

∎

As noticed in Remark 6 of the next chapter, the assumption on existence of V is necessary.

III. Compactness by compensation

In this chapter we investigate conditions under which, for two sequences $(\mathbf{u}_r)$ and $(\mathbf{v}_r)$ weakly converging to $\mathbf{u}$ and $\mathbf{v}$ in $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^N)$ and $\mathrm{L}^q(\mathbf{R}^d;\mathbf{R}^N)$, respectively, $1/p + 1/q \leq 1$, a quadratic form $q(\mathbf{x};\mathbf{u}_r,\mathbf{v}_r) = \sum\limits_{j,m=1}^{N} q_{jm}(\mathbf{x}) u_{jr} v_{mr}$ converges toward $q(\mathbf{x};\mathbf{u},\mathbf{v})$ in the sense of distributions. The conditions involve fractional derivatives and variable coefficients, and they represent a generalisation of the known compactness by compensation theory. The proofs are accomplished using an appropriate variant of H-distributions. We apply the developed techniques to a nonlinear (degenerate) parabolic equation.

The results of this chapter are mostly contained in [42].

1. Introduction

The compactness by compensation theory proved to be a very useful tool in investigating problems involving partial differential equations (both linear and nonlinear). Suppose, for instance, that we aim to solve a nonlinear partial differential equation which we write symbolically as $A[u] = f$, where A denotes a given nonlinear operator. One of usual approaches is to approximate it by a collection of *nicer* problems $A_r[u_r] = f_r$, where (A_r) is a sequence of operators which is somehow close to A. Then we try to prove that the sequence (u_r) converges toward a solution to the original problem $A[u] = f$. In general, it is relatively easy to obtain weak convergence on a subsequence of (u_r) towards some function u. Due to the nonlinear nature of A, this does not mean that u will represent a solution to the original problem $A[u] = f$. However, in some cases, the nonlinearity of A can be *compensated* by certain properties of the sequence (u_r) (see [7, 16, 40] and references therein). The theory which investigates such phenomena is called compactness by compensation and it was introduced in the works of F. Murat and L. Tartar [44,60].

The most general version of the classical result of compensated compactness theory has been recently proved in [49]. Let us briefly recall it. First, we introduce anisotropic Sobolev spaces $\mathrm{W}^{-1,-2;p}(\mathbf{R}^d)$, where -1 is with respect to $x_1, \dots, x_\nu$ and -2 is with respect to $x_{\nu+1}, \dots, x_d$, as a subset of tempered distributions

$$\{u \in \mathcal{S}' : \ \exists v \in \mathrm{L}^p(\mathbf{R}^d), \ k\hat{u} = \hat{v}\},$$

where $k(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) = \sqrt{1 + (2\pi|\boldsymbol{\xi}_1|)^2 + (2\pi|\boldsymbol{\xi}_2|)^4}$, $\boldsymbol{\xi}_1 \in \mathbf{R}^\nu$, $\boldsymbol{\xi}_2 \in \mathbf{R}^{d-\nu}$. It is Hörmader's class $B_{p,k}$ and the Banach space with dual $B_{p',1/k}$ (see chapter 10 of [27]).

Assume that the sequence $(\mathbf{u}_r) = (u_{1r}, \dots, u_{Nr})$ is bounded in $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^N)$, $2 \leq p \leq \infty$, and converges in $\mathcal{D}'(\mathbf{R}^d)$ to a vector function $\mathbf{u}$. Let $q = \frac{p}{p-1}$ if $p < \infty$, and $q > 1$ if $p = \infty$. Assume that the sequences

$$\sum_{j=1}^{N}\sum_{k=1}^{\nu} \partial_{x_k}(a_{sjk} u_{jr}) + \sum_{j=1}^{N}\sum_{k,l=\nu+1}^{d} \partial_{x_k x_l}(b_{sjkl} u_{jr}), \tag{1}$$

for $s = 1, \dots, m$, are precompact in the anisotropic Sobolev space $\mathrm{W}^{-1,-2;q}_{loc}(\mathbf{R}^d)$. The (variable) coefficients a_{sjk} and b_{sjkl} belong to $\mathrm{L}^{2\bar{q}}(\mathbf{R}^d)$, $\bar{q} = \frac{p}{p-2}$ if $p > 2$, and to the space $\mathrm{C}(\mathbf{R}^d)$ if $p = 2$.

Next, introduce the set

$$
\begin{aligned}
\Lambda(\mathbf{x}) = \Big\{ & \boldsymbol{\lambda} \in \mathbf{R}^N \,|\, (\exists \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{0\})(\forall s = 1, \dots, m) \\
& \sum_{j=1}^{N} \Big(i \sum_{k=1}^{\nu} a_{sjk}(\mathbf{x})\xi_k - \sum_{k,l=\nu+1}^{d} b_{sjkl}(\mathbf{x})\xi_k \xi_l \Big) \lambda_j = 0 \Big\}.
\end{aligned} \tag{2}
$$

Consider the bilinear form on $\mathbf{R}^N$

$$
q(\mathbf{x}; \boldsymbol{\lambda}, \boldsymbol{\eta}) = Q(\mathbf{x}) \boldsymbol{\lambda} \cdot \boldsymbol{\eta}, \tag{3}
$$

where Q is a symmetric matrix with coefficients

$$
q_{jm} \in \begin{cases} \mathrm{L}^{\bar{q}}_{loc}(\mathbf{R}^d), & p > 2 \\ \mathrm{C}(\mathbf{R}^d), & p = 2 \end{cases}, \quad j, m = 1, \dots, N.
$$

Finally, let $q(\mathbf{x}; \mathbf{u}_r, \mathbf{u}_r) \rightharpoonup \omega$ weakly-$*$ in the space of Radon measures.

The following theorem (Theorem 1 of [49]) holds

Theorem 1. *Assume that $q(\mathbf{x}; \boldsymbol{\lambda}, \boldsymbol{\lambda}) \geq 0$ for all $\boldsymbol{\lambda} \in \Lambda(\mathbf{x})$, a.e. $\mathbf{x} \in \mathbf{R}^d$. Then $q(\mathbf{x}; \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{x})) \leq \omega$ in the sense of measures. If $q(\mathbf{x}; \boldsymbol{\lambda}, \boldsymbol{\lambda}) = 0$ for all $\boldsymbol{\lambda} \in \Lambda(\mathbf{x})$, a.e. $\mathbf{x} \in \mathbf{R}^d$, then $q(\mathbf{x}; \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{x})) = \omega$.*

∎

The connection between q and Λ given in the previous theorem, we shall call *the consistency condition.*

We would like to formulate and extend the results from Theorem 1 to the $\mathrm{L}^p - \mathrm{L}^q$ framework for appropriate (greater than one) indices p and q where $p < 2$. More precisely, we want to find conditions on two vector-valued sequences $(\mathbf{u}_r)$ and $(\mathbf{v}_r)$ weakly converging to $\mathbf{u}$ and $\mathbf{v}$ in $\mathrm{L}^p(\mathbf{R}^d)$ and $\mathrm{L}^q(\mathbf{R}^d)$, respectively, to ensure that the sequence $(q(\mathbf{x}; \mathbf{u}_r, \mathbf{v}_r))$, where q is the bilinear form from (3), satisfies

$$
\lim_{r \to \infty} q(\mathbf{x}; \mathbf{u}_r, \mathbf{v}_r) = q(\mathbf{x}; \mathbf{u}, \mathbf{v}) \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d). \tag{4}
$$

Ideally, it should be $1/p + 1/q = 1$. Due to technical obstacles (see Remark 3), we are able to prove (4) only when $1/p + 1/q < 1$. However, under additional assumptions on the sequences $(\mathbf{u}_r)$ and $(\mathbf{v}_r)$, we are also able to obtain the optimal $\mathrm{L}^p - \mathrm{L}^{p'}$-variant of the compactness by compensation. Here and in the sequel, $1/p + 1/p' = 1$.

This extension will be done in the next section. In the last section we shall show how to apply this result to a (nonlinear) parabolic type equation.

2. The main result

In order to formulate the $\mathrm{L}^p - \mathrm{L}^q$ variant of the compactness by compensation, we need H-distributions.

Let us recall that H-measures describe the loss of strong precompactness for sequences belonging to L^p for $p \geq 2$, and they were the basic tool in the mentioned work on compactness by compensation [49]. The variant of H-distributions that we are basically going to use is formulated in [36, 37]. Let us recall its definition.

We need multiplier operators with symbols defined on a manifold P determined by an d-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbf{R}^d_+$, where $\alpha_k \in \mathbf{N}$ or $\alpha_k \geq d$

$$\mathrm{P} = \Big\{ \boldsymbol{\xi} \in \mathbf{R}^d : \sum_{k=1}^{d} |\xi_k|^{2\alpha_k} = 1 \Big\}.$$

The manifold P is smooth enough and we are able to associate an L^p multiplier to a function defined on P as follows. We define the projection from $\mathbf{R}^d \setminus \{0\}$ to P by means of the mapping

$$(\pi_{\mathrm{P}}(\boldsymbol{\xi}))_j = \xi_j \left(|\xi_1|^{2\alpha_1} + \ldots + |\xi_d|^{2\alpha_d} \right)^{-1/2\alpha_j}, \quad j = 1, \ldots, d, \quad \boldsymbol{\xi} \in \mathbf{R}^d \backslash \{0\}.$$

That composition $\psi \circ \pi_{\mathrm{P}}$ is indeed an L^p multiplier was shown in Lemma 5 of [35]. Because of its importance in definition and applications of a variant of H-distributions which we are going to use in this chapter, we state it here and briefly comment on its proof:

Lemma 1. *For any $\psi \in \mathrm{C}^d(\mathrm{P})$, the composition $\psi \circ \pi_{\mathrm{P}}$ is an L^p-multiplier, $p \in \langle 1, \infty \rangle$, and the operator norm of the corresponding Fourier multiplier operator depends only on p, d and $\|\psi\|_{\mathrm{C}^d(\mathrm{P})}$.*

Dem. The first step is to use the Faa di Bruno formula (see [26]) to reduce the proof to showing that each coordinate function $(\pi_{\mathrm{P}})_j$ satisfies conditions of Corollary 2 of the first chapter. Then one proceeds by the induction argument. It is not hard to see that for the first derivatives one has:

$$\xi_k \partial_k (\pi_{\mathrm{P}}(\boldsymbol{\xi}))_j = -\frac{\alpha_k}{\alpha_j} \pi_j(\boldsymbol{\xi}) \pi_k^{2\alpha_k}(\boldsymbol{\xi}) \qquad \text{if} \quad k \neq j\,,$$

$$\xi_j \partial_j (\pi_{\mathrm{P}}(\boldsymbol{\xi}))_j = -\pi_j(\boldsymbol{\xi}) \left(1 - \pi_j^{2\alpha_j}(\boldsymbol{\xi}) \right) .$$

Hypothesis is that for $\boldsymbol{\beta} \in \mathbf{N}_0^d$ such that $|\boldsymbol{\beta}| = m$ it holds

$$\boldsymbol{\xi}^{\boldsymbol{\beta}} \partial^{\boldsymbol{\beta}} (\pi_{\mathrm{P}}(\boldsymbol{\xi}))_j = P_{\boldsymbol{\beta}} \left((\pi_{\mathrm{P}}(\boldsymbol{\xi}))_1, \cdot, (\pi_{\mathrm{P}}(\boldsymbol{\xi}))_d \right) ,$$

for some polynomial $P_{\boldsymbol{\beta}}$. For $\boldsymbol{\gamma} \in \mathbf{N}_0^d$ such that $|\boldsymbol{\gamma}| = m+1$, notice that we can write $\boldsymbol{\gamma} = \mathrm{e}_i + \boldsymbol{\beta}$ for some $|\boldsymbol{\beta}| = m$ and some e_i canonical vector. We have:

$$\partial^{\boldsymbol{\gamma}} (\pi_{\mathrm{P}}(\boldsymbol{\xi}))_j = \partial_i \partial^{\boldsymbol{\gamma}} (\pi_{\mathrm{P}}(\boldsymbol{\xi}))_j = \partial_i \left(\frac{1}{\boldsymbol{\xi}^{\boldsymbol{\beta}}} P_{\boldsymbol{\beta}} \left((\pi_{\mathrm{P}}(\boldsymbol{\xi}))_1, \cdot, (\pi_{\mathrm{P}}(\boldsymbol{\xi}))_d \right) \right) ,$$

and we proceed as in the base step. Now, let us just notice that in each step we had a finite number of continuous functions defined on a compact manifold.

Q.E.D.

The following statement holds [36]:

Theorem 2. *Let (u_n) be a bounded sequence in $\mathrm{L}^p(\mathbf{R}^d)$, $p > 1$, and let (v_n) be a bounded sequence of uniformly compactly supported functions in $\mathrm{L}^\infty(\mathbf{R}^d)$ weakly converging to 0 in the sense of distributions. Then, after passing to a subsequence (not relabelled), for any $\bar{p} \in \langle 1, p \rangle$ there exists a continuous bilinear functional B on $\mathrm{L}^{\bar{p}'}(\mathbf{R}^d) \otimes \mathrm{C}^d(\mathrm{P})$ such that for every $\varphi \in \mathrm{L}^{\bar{p}'}(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^d(\mathrm{P})$ it holds*

$$B(\varphi, \psi) = \lim_{n \to \infty} \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_n(\mathbf{x}) \overline{(\mathcal{A}_{\psi_{\mathrm{P}}} v_n)(\mathbf{x})} d\mathbf{x}\,, \tag{5}$$

where $\mathcal{A}_{\psi_{\mathrm{P}}}$ is the (Fourier) multiplier operator on $\mathbf{R}^d$ associated to $\psi \circ \pi_{\mathrm{P}}$ and $\frac{1}{\bar{p}} + \frac{1}{\bar{p}'} = 1$.

The bound of the functional B is equal to $C_u\, C_v\, C_{d,q}$, where C_u is the $\mathrm{L}^{\bar{p}}$-bound of the sequence (u_n); C_v is the L^q-bound of the sequence (v_n) where $\frac{1}{p} + \frac{1}{\bar{p}'} + \frac{1}{q} = 1$; and $C_{d,q}$ is the constant from Corollary 2 of the first chapter. ∎

We shall now prove that we can extend the bilinear functional B from the previous theorem to a functional on $\mathrm{L}^{\bar{p}'}(\mathbf{R}^d; \mathrm{C}^d(\mathrm{P}))$. We shall need the following theorem a proof of which in the case of real functionals can be found in [36].

Theorem 3. *Let B be a (complex valued) continuous bilinear functional on $\mathrm{L}^p(\mathbf{R}^d) \otimes E$, where E is a separable Banach space and $p \in \langle 1, \infty\rangle$. Then B can be extended as a (complex valued) continuous functional on $\mathrm{L}^p(\mathbf{R}^d; E)$ if and only if there exists a (non-negative) function $b \in \mathrm{L}^{p'}(\mathbf{R}^d)$ such that for every $\psi \in E$ and almost every $\mathbf{x} \in \mathbf{R}^d$, it holds*

$$|\tilde{B}\psi(\mathbf{x})| \leq b(\mathbf{x})\|\psi\|_E\,, \tag{6}$$

where $\tilde{B}$ is a bounded linear operator $E \to \mathrm{L}^{p'}(\mathbf{R}^d)$ defined by $\langle \tilde{B}\psi, \varphi\rangle = B(\varphi, \psi)$, $\varphi \in \mathrm{L}^p(\mathbf{R}^d)$.

Dem. The proof goes along the lines of the proof of Theorem 2.1 of [36] when we separately consider real (Re) and imaginary (Im) parts of the functional B and the operator $\tilde{B}$. Let us briefly recall it.

Let us assume that (6) holds. In order to prove that B can be extended as a linear functional on $\mathrm{L}^p(\mathbf{R}^d; E)$, it is enough to obtain an appropriate bound on the following dense subspace of $\mathrm{L}^p(\mathbf{R}^d; E)$:

$$\Big\{ \sum_{j=1}^N \psi_j \chi_j(\mathbf{x}) :\ \psi_j \in E, N \in \mathbf{N} \Big\}\,, \tag{7}$$

where χ_i are characteristic functions associated to mutually disjoint, finite measure sets.

For an arbitrary function $g = \sum\limits_{i=1}^N \psi_i \chi_i$ from (7), the bound follows easily once we notice that

$$\begin{aligned}\Big|B\Big(\sum_{j=1}^N \psi_j\chi_j\Big)\Big| &:= \Big|\sum_{j=1}^N B(\chi_j, \psi_j)\Big| = \Big|\int_{\mathbf{R}^d} \sum_{j=1}^N \tilde{B}\psi_j(\mathbf{x})\chi_j(\mathbf{x})d\mathbf{x}\Big| \\ &\leq \int_{\mathbf{R}^d} b(\mathbf{x}) \sum_{j=1}^N \chi_j(\mathbf{x})\|\psi_j\|_E d\mathbf{x} \leq \|b\|_{\mathrm{L}^{p'}(\mathbf{R}^d)}\|g\|_{\mathrm{L}^p(\mathbf{R}^d;E)}.\end{aligned}$$

In order to prove the converse, take a countable dense set of functions from the unit ball of E, and denote them by ψ_j, $j \in \mathbf{N}$. Assume that the functions $\psi_{-j} := -\psi_j$ are also in E. For each function $\tilde{B}\psi_j \in \mathrm{L}^{p'}(\mathbf{R}^d)$ denote by D_j the corresponding set of Lebesgue points, and their intersection by $D = \cap_j D_j$.

For any $\mathbf{x} \in D$ and $k \in \mathbf{N}$ denote

$$b_k(\mathbf{x}) = \max_{|j|\leq k} \mathsf{Re}\,(\tilde{B}\psi_j)(\mathbf{x}) = \sum_{|j|=1}^k \mathsf{Re}\,(\tilde{B}\psi_j)(\mathbf{x})\chi_j^k(\mathbf{x})$$

where $\chi_{j_0}^k$ is the characteristic function of set $X_{j_0}^k$ of all points $\mathbf{x} \in D$ for which the above maximum is achieved for $j = j_0$. Furthermore, we can assume that for each k the sets X_j^k are mutually disjoint. The sequence (b_k) is clearly monotonic sequence of positive functions, bounded in $\mathrm{L}^{p'}(\mathbf{R}^d)$, whose limit (in the same space) we denote by b^{Re}. Indeed, choose $\varphi \in \mathrm{L}^p(\mathbf{R}^d)$, $g = \sum_{|j|=1}^{k} \varphi(\mathbf{x})\chi_j^k(\mathbf{x})\psi_j \in \mathrm{L}^p(\mathbf{R}^d; E)$, and consider:

$$\int_{\mathbf{R}^d} b_k(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} = \mathsf{Re}\left(\int_{\mathbf{R}^d} \tilde{B}\sum_{|j|=1}^{k} \psi_j\chi_j^k(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}\right)$$
$$= \mathsf{Re}\left(\sum_{|j|=1}^{k} B(\chi_j^k\varphi, \psi_j)\right) = \mathsf{Re}\ (B(g)) \leq C\|g\|_{\mathrm{L}^p(\mathbf{R}^d;E)} \leq C\|\varphi\|_{\mathrm{L}^p(\mathbf{R}^d)},$$

where C is the norm of B on $(\mathrm{L}^p(\mathbf{R}^d; E))'$. Since $\varphi \in \mathrm{L}^p(\mathbf{R}^d)$ is arbitrary, we get that (b_k) is bounded in $\mathrm{L}^{p'}(\mathbf{R}^d)$.

As D is a set of full measure, for every ψ_j we have

$$|\mathsf{Re}\,(\tilde{B}\psi_j)(\mathbf{x})| \leq b^{\mathsf{Re}}(\mathbf{x}), \qquad (\text{a.e. } \mathbf{x} \in \mathbf{R}^d).$$

We are able to obtain a similar bound for the imaginary part of $\tilde{B}\psi_j$. In other words, there exists $b^{\mathsf{Im}} \in \mathrm{L}^{p'}(\mathbf{R}^d)$ such that

$$|\mathsf{Im}\,(\tilde{B}\psi_j)(\mathbf{x})| \leq b^{\mathsf{Im}}(\mathbf{x}), \qquad (\text{a.e. } \mathbf{x} \in \mathbf{R}^d).$$

The assertion now follows since (6) holds for $b = b^{\mathsf{Re}} + b^{\mathsf{Im}}$ on the dense set of functions ψ_j, $j \in \mathbf{N}$. For details see (12) below.

Q.E.D.

We need the following lemma which will also be used in the last section.

Lemma 2. *If the real symbol $\psi \in \mathrm{C}^d(\mathrm{P})$ of the multiplier operator $\mathcal{A}_\psi$ is an even function ($\psi(\boldsymbol{\xi}) = \psi(-\boldsymbol{\xi})$), then for every real $u \in \mathrm{L}^p(\mathbf{R}^d)$, $p > 1$, $\mathcal{A}_\psi(u)$ is a real function for a.e. $\mathbf{x} \in \mathbf{R}^d$.*

If the real symbol $\psi \in \mathrm{C}^d(\mathrm{P})$ of the multiplier operator $\mathcal{A}_\psi$ is an odd function ($\psi(\boldsymbol{\xi}) = -\psi(-\boldsymbol{\xi})$), then for every real $u \in \mathrm{L}^p(\mathbf{R}^d)$, $p > 1$, $\mathcal{A}_\psi(u)$ is a purely imaginary function for a.e. $\mathbf{x} \in \mathbf{R}^d$.

Dem. Assume first that the symbol ψ is an even function. It is enough to prove that, for arbitrary real functions $u, v \in \mathrm{L}^2(\mathbf{R}^d) \cap \mathrm{L}^p(\mathbf{R}^d)$, it holds

$$\int v\mathcal{A}_\psi(u)d\mathbf{x} = \int v\overline{\mathcal{A}_\psi(u)}d\mathbf{x}.$$

This follows from the Plancherel theorem, and the change of variables $\boldsymbol{\xi} \mapsto -\boldsymbol{\xi}$. Indeed,

$$\int v\mathcal{A}_\psi(u)d\mathbf{x} = \int \overline{v}\mathcal{A}_\psi(u)d\mathbf{x} = \int \psi(\boldsymbol{\xi})\overline{\hat{v}}(\boldsymbol{\xi})\hat{u}(\boldsymbol{\xi})d\boldsymbol{\xi} = (\boldsymbol{\xi} \mapsto -\boldsymbol{\xi})$$
$$= \int \psi(\boldsymbol{\xi})\hat{v}(\boldsymbol{\xi})\overline{\hat{u}}(\boldsymbol{\xi})d\boldsymbol{\xi} = \int v\overline{\mathcal{A}_\psi(u)}d\mathbf{x}.$$

The proof is the same when the symbol is odd.

Q.E.D.

Now, we can prove the following result, a proof of which can be found in [37]. For the sake of completeness, we give a slightly different proof here.

Lemma 3. *The bilinear functional B defined in Theorem 2 can be extended by continuity to a functional on $\mathrm{L}^{\bar{p}'}(\mathbf{R}^d;\mathrm{C}^d(\mathrm{P}))$. The bound of the extension is equal to $2^{p'+1}C_u\,C_v\,C_{d,p'}$, $1/p+1/\bar{p}'+1/q=1$ (with the notation of Theorem 2). In fact, we can remove the factor $2^{p'}$ from the bound (see Remark 2).* ∎

Dem. We will show that B satisfies conditions of Theorem 3, namely, that there exists a function $b\in\mathrm{L}^{\bar{p}}(\mathbf{R}^d)$ such that for every $\psi\in\mathrm{C}^d(\mathrm{P})$, $\|\psi\|_{\mathrm{C}^d(\mathrm{P})}\le 1$ and almost every $\mathbf{x}\in\mathbf{R}^d$ it holds

$$|(\tilde{B}\psi)(\mathbf{x})|\le b(\mathbf{x})\|\psi\|_{\mathrm{C}^d(\mathrm{P})}, \tag{8}$$

where $\tilde{B}:\mathrm{C}^d(\mathrm{P})\to\mathrm{L}^{\bar{p}}(\mathbf{R}^d)$ is a bounded linear operator defined by $\langle\tilde{B}\psi,\varphi\rangle=B(\varphi,\psi)$, $\varphi\in\mathrm{L}^{\bar{p}'}(\mathbf{R}^d)$.

We proceed as follows: choose a dense countable set E of functions ψ_j, $j\in\mathbf{N}$, from the set $\{\psi\in\mathrm{C}^d(\mathrm{P})\colon\ \|\psi\|_{\mathrm{C}^d(\mathrm{P})}\le 1\}$. Define functions $\psi_{-j}(\boldsymbol{\xi})=-\psi_j(\boldsymbol{\xi})$ and add them to E. Moreover, add the linear combinations of the form $\psi_j^e(\boldsymbol{\xi})=\frac{1}{2}(\psi_j(\boldsymbol{\xi})+\psi_j(-\boldsymbol{\xi}))$ and $\psi_j^o(\boldsymbol{\xi})=\frac{1}{2}(\psi_j(\boldsymbol{\xi})-\psi_j(-\boldsymbol{\xi}))$ for $j\in\mathbf{Z}\setminus\{0\}$ to E as well. Remark that functions ψ_j^e are even, while ψ_j^o are odd (in the sense of Lemma 2) and that the set E is still countable and dense.

For each j choose a function $\tilde{B}\psi_j$ from $\mathrm{L}^{\bar{p}}(\mathbf{R}^d)$ and denote by D_j the corresponding set of Lebesgue points (for definiteness, we can take $\tilde{B}\psi_j$ to be the precise representative of the class (see Chapter 1.7. of [21]). The set D_j is of full measure, and thus the set $D=\cap_j D_j$ as well.

For any $\mathbf{x}\in D$ and $k\in\mathbf{N}$ denote ($i=\sqrt{-1}$ below)

$$b_k^e(\mathbf{x}):=\max_{|j|\le k}\tilde{B}\psi_j^e(\mathbf{x})=\sum_{|j|=1}^{k}\tilde{B}\psi_j^e(\mathbf{x})\chi_j^k(\mathbf{x})\in\mathbf{R}^+, \tag{9}$$

$$b_k^o(\mathbf{x}):=\max_{|j|\le k}i\tilde{B}\psi_j^o(\mathbf{x})=\sum_{|j|=1}^{k}i\tilde{B}\psi_j^o(\mathbf{x})\tilde{\chi}_j^k(\mathbf{x})\in\mathbf{R}^+, \tag{10}$$

where $\chi_{j_0}^k$ ($\tilde{\chi}_{j_0}^k$ respectively) is a characteristic function of the set of all points for which the above maximum is achieved for $\psi_{j_0}^e$ ($\psi_{j_0}^o$ respectively) and it has not been achieved for ψ_j^e (ψ_j^o respectively), $-k\le j<j_0$.

First, note that we can make sure that χ_j^k have disjoint supports for fixed k: define χ_j^k to be equal to one on the set

$$\Big\{\mathbf{x}\in D:(\tilde{B}\psi_j^e)(\mathbf{x})=b_k^e(\mathbf{x})\ \&\ (\forall l<j)(\tilde{B}\psi_l^e)(\mathbf{x})<b_k^e(\mathbf{x})\Big\},$$

and extend it with zero to the whole $\mathbf{R}^d$.

Next, we shall prove that the sequence of functions (b_k^e) is bounded in $\mathrm{L}^{\bar{p}}(\mathbf{R}^d)$. To this effect, take an arbitrary $\phi\in\mathrm{C}_c(\mathbf{R}^d)$, and denote $K=\operatorname{supp}\phi$. Since (v_n) is a bounded sequence of uniformly compactly supported functions in $\mathrm{L}^\infty(\mathbf{R}^d)$, it belongs to $\mathrm{L}^q(\mathbf{R}^d)$ for every $q \in \langle 1, \infty\rangle$. Since $\bar{p} < p$, we can find $q > 1$ such that $1/q + 1/\bar{p}' = 1/p'$. Fix such q. Choose $r > 1$ such that $q = r'p'$. Denote by $\chi_j^{k,\varepsilon} \in \mathrm{C}_c(\mathbf{R}^d)$, $j = 1, \dots, k$ smooth approximations of characteristic functions from (9) on K such that (note that $\|\chi_j^k\|_{\mathrm{L}^\infty} \leq 1$)

$$\|\chi_j^{k,\varepsilon} - \chi_j^k\|_{\mathrm{L}^{\max\{p',r\}}(K)} \leq \frac{\varepsilon}{2k}.$$

As before, denote by C_u an L^p bound of (u_n) and by C_v an L^q bound of (v_n).

According to (9) and the definition of operator $\tilde{B}$, we have

$$\begin{aligned}
&\big|{}_{\mathrm{L}^{\bar{p}}(\mathbf{R}^d)}\langle b_k^e, \phi\rangle_{\mathrm{L}^{\bar{p}'}(\mathbf{R}^d)}\big| = \Big|\lim_{n\to\infty}\int_{\mathbf{R}^d}\sum_{|j|=1}^k (\phi u_n \chi_j^k)(\mathbf{x})\overline{(\mathcal{A}_{\psi_j^e} v_n)(\mathbf{x})}d\mathbf{x}\Big| \\
&\leq \limsup_{n\to\infty}\int_{\mathbf{R}^d}\left(\sum_{|j|=1}^k |u_n|^p\chi_j^k(\mathbf{x})\right)^{1/p}\left(\sum_{|j|=1}^k \chi_j^k|\phi\,\mathcal{A}_{\psi_j^e} v_n|^{p'}(\mathbf{x})\right)^{1/p'} d\mathbf{x} \\
&\leq \limsup_{n\to\infty}\Big\|\sum_{|j|=1}^k |u_n|^p\chi_j^k\Big\|_{\mathrm{L}^1(\mathbf{R}^d)}^{1/p}\Big\|\sum_{|j|=1}^k \chi_j^k|\phi\,\mathcal{A}_{\psi_j^e} v_n|^{p'}\Big\|_{\mathrm{L}^1(\mathbf{R}^d)}^{1/p'} \\
&\leq \limsup_{n\to\infty}\|u_n\|_{\mathrm{L}^p(\mathbf{R}^d)}\left(\sum_{|j|=1}^k \Big\||\chi_j^k\phi\,\mathcal{A}_{\psi_j^e} v_n|^{p'}\Big\|_{\mathrm{L}^1(\mathbf{R}^d)}\right)^{1/p'} \\
&\leq 2^{p'}\limsup_{n\to\infty}\|u_n\|_{\mathrm{L}^p(\mathbf{R}^d)}\Bigg(\sum_{|j|=1}^k \Big\||(\chi_j^k - \chi_j^{k,\varepsilon})\phi\,\mathcal{A}_{\psi_j^e} v_n|^{p'}\Big\|_{\mathrm{L}^1(\mathbf{R}^d)} \\
&\qquad\qquad + \sum_{|j|=1}^k \Big\||\chi_j^{k,\varepsilon}\phi\,\mathcal{A}_{\psi_j^e} v_n|^{p'}\Big\|_{\mathrm{L}^1(\mathbf{R}^d)}\Bigg)^{1/p'} \\
&\leq 2^{p'}\limsup_{n\to\infty}\|u_n\|_{\mathrm{L}^p(\mathbf{R}^d)}\Bigg(\sum_{|j|=1}^k \Big\||(\chi_j^k - \chi_j^{k,\varepsilon})||\phi\,\mathcal{A}_{\psi_j^e} v_n|^{p'}\Big\|_{\mathrm{L}^1(\mathbf{R}^d)} \\
&\qquad\qquad + \sum_{|j|=1}^k \Big\|\chi_j^{k,\varepsilon}\phi\,\mathcal{A}_{\psi_j^e} v_n\Big\|_{\mathrm{L}^{p'}(\mathbf{R}^d)}^{p'}\Bigg)^{1/p'} \\
&\leq 2^{p'}C_u\limsup_{n\to\infty}\Bigg(\sum_{|j|=1}^k \Big\|\chi_j^k - \chi_j^{k,\varepsilon}\Big\|_{\mathrm{L}^r(K)}\Big\|\mathcal{A}_{\psi_j^e}(\phi v_n)\Big\|_{\mathrm{L}^q(\mathbf{R}^d)}^{p'} \\
&\qquad\qquad + \sum_{|j|=1}^k \Big\|\mathcal{A}_{\psi_j^e}(\chi_j^{k,\varepsilon}\phi v_n)\Big\|_{\mathrm{L}^{p'}(\mathbf{R}^d)}^{p'}\Bigg)^{1/p'},
\end{aligned}$$

where in the second step we have used discrete version of Hölder inequality and the fact that $|\lim_n a_n| \leq \limsup_n |a_n|$; in the third we have used Hölder inequality and in the fifth an inequality $(|a| + |b|)^{p'} \leq 2^{p'}(|a|^{p'} + |b|^{p'})$; in the sixth step to every term of the first sum we applied the inequality $|a|^{p'} \leq |a|$ which is valid for $|a| \leq 1$, after noticing that

$0 \le \chi_j^{k,\varepsilon} \le 1$ gives $\|\chi_j^k - \chi_j^{k,\varepsilon}\|_{\mathrm{L}^\infty} \le 1$ (we can obtain such functions by convolution with the standard Friderichs mollifier). Finally, in the last step we have used a version of the First commutation lemma (see Remark 4 and Lemma 2 of the first chapter) and Hölder inequality with $1/r+1/r'=1$ remembering that $r'p'=q$. By means of the Marcinkiewicz multiplier theorem (Corollary 2 of the first chapter) and properties of the functions $\chi_j^{k,\varepsilon}$ it follows

$$|\langle b_k^e, \phi\rangle| \le 2^{p'} C_u \limsup_{n\to\infty} \left(\varepsilon\, C_\phi\, C_{q,d}^{p'} \|v_n\|^{p'}_{\mathrm{L}^q(\mathbf{R}^d)} + C^{p'}_{p',d} \sum_{|j|=1}^{k} \|\chi_j^{k,\varepsilon} \phi v_n\|^{p'}_{\mathrm{L}^{p'}(\mathbf{R}^d)} \right)^{1/p'},$$

where $C_{p',d}$ is the constant from Corollary 2 from the first chapter (recall that $\|\psi_j^e\|_{\mathrm{C}^d(\mathrm{P})} \le 1$), while $C_\phi = \|\phi\|^{p'}_{\mathrm{L}^\infty(\mathbf{R}^d)}$. By letting $\varepsilon \to 0$, we conclude

$$|\langle b_k^e, \phi\rangle| \le 2^{p'} C_u C_{p',d} \limsup_{n\to\infty} \left(\sum_{|j|=1}^{k} \|\chi_j^{k} \phi v_n\|^{p'}_{\mathrm{L}^{p'}(\mathbf{R}^d)} \right)^{1/p'}$$

since $\chi_j^{k,\varepsilon} \to \chi_j^k$ in $\mathrm{L}^{p'}(K)$. Since supports of functions χ_j^k are disjoint and remembering the choice of q, we get

$$\sum_{|j|=1}^{k} \|\chi_j^k \phi v_n\|^{p'}_{\mathrm{L}^{p'}(\mathbf{R}^d)} \le \|\phi v_n\|^{p'}_{\mathrm{L}^{p'}(\mathbf{R}^d)} \le \Big(\|\phi\|_{\mathrm{L}^{\bar p'}(\mathbf{R}^d)} \|v_n\|_{\mathrm{L}^q(\mathbf{R}^d)} \Big)^{p'},$$

since $\sum\limits_{|j|=1}^{k} (\chi_j^k)^{p'} = \sum\limits_{|j|=1}^{k} \chi_j^k \le 1$. From this, it follows

$$|\langle b_k^e, \phi\rangle| \le 2^{p'} C_u C_{d,p'} C_v \|\phi\|_{\mathrm{L}^{\bar p'}(\mathbf{R}^d)}, \tag{11}$$

where all the constants on the right hand side do not depend on k. Since $\mathrm{C}_c(\mathbf{R}^d)$ is dense in $\mathrm{L}^{\bar p'}(\mathbf{R}^d)$ we conclude that the sequence (b_k^e) is bounded in $\mathrm{L}^{\bar p}(\mathbf{R}^d)$. Noticing that (b_k^e) is a non-decreasing sequence of positive functions, it follows from Beppo-Levi's theorem on monotone convergence that its (pointwise) limit b^e is an $\mathrm{L}^{\bar p}(\mathbf{R}^d)$ function.

In the completely same way, we conclude that (b_k^o) converges toward $b^o \in \mathrm{L}^{\bar p}(\mathbf{R}^d)$.

The function $b = b^e + b^o$ satisfies (8) for $\tilde B\psi$ when $\psi = \psi_j^e + \psi_{j'}^o$ for some $j, j' \in \mathbf{Z}\backslash\{0\}$. On the other hand, every $\psi \in \mathrm{C}^d(\mathrm{P})$ can be represented as a sum of odd and even functions as follows $\psi(\boldsymbol{\xi}) = \frac12(\psi(\boldsymbol{\xi}) + \psi(-\boldsymbol{\xi})) + \frac12(\psi(\boldsymbol{\xi}) - \psi(-\boldsymbol{\xi}))$ and we conclude that (8) holds for any $\psi \in E$. By continuity, the statement can be generalised to an arbitrary $\psi \in \mathrm{C}^d(\mathrm{P})$: take a sequence $(\psi_n) \subseteq E$ such that $\psi_n \to \psi$ in $\mathrm{C}^d(\mathrm{P})$ and write

$$\begin{aligned} \int_{\mathbf{R}^d} |(\tilde B\psi)(\mathbf{x})|\varphi(\mathbf{x})d\mathbf{x} &\le \int_{\mathbf{R}^d} |(\tilde B\psi - \tilde B\psi_n)(\mathbf{x})|\varphi(\mathbf{x})d\mathbf{x} + \int_{\mathbf{R}^d} |(\tilde B\psi_n)(\mathbf{x})|\varphi(\mathbf{x})d\mathbf{x} \\ &\le o_n(1) + \int_{\mathbf{R}^d} b(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}, \end{aligned} \tag{12}$$

for arbitrary $\varphi \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}_0^+)$ where we have used continuity of $\tilde B$. Due to arbitrariness of the function φ, the result follows from Theorem 3. Remark finally that from (11) and the equality $b = b^e + b^o$, it follows that the bound of the extension is equal to $2^{p'+1} C_u C_v C_{d,p'}$.

Q.E.D.

Remark 1. Note that if the set $L := \{\psi \in \mathrm{C}^d(\mathrm{P})\colon \|\psi\|_{\mathrm{C}^d(\mathrm{P})} \leq 1\}$ were at most countable, we could have defined $b \in \mathrm{L}^{\bar{p}}(\mathbf{R}^d)$ in the following straightforward way

$$b(\mathbf{x}) = \sup_{\psi\in L}|(\tilde{B}\psi)(\mathbf{x})|.$$

However, L is uncountable, so this definition does not necessarily result in a measurable function. Taking supremum over a countable dense subset of L would result in a measurable function which may not be $\mathrm{L}^{\bar{p}}$-function.

Remark 2. In the fourth step of calculation to get desired bound on b_k^e of the preceding lemma, we have a sum of expressions of the form $\|\chi\phi\mathcal{A}_\psi v_n\|_{\mathrm{L}^{p'}}$. Instead of regularising characteristic functions χ, we could have proceeded in a different manner. First, we have $\chi\phi \in \mathrm{L}^{\bar{p}'}$ and $p' < \bar{p}'$, and so by an application of Lemma 5 of the first chapter, we get

$$\|\chi\phi\mathcal{A}_\psi v_n - \mathcal{A}_\psi(\chi\phi v_n)\|_{\mathrm{L}^{p'}} \to 0\,,$$

thus, we do not get the factor of $2^{p'}$ in the bound on B. In the same calculation in [42], we made a small oversight and here we gave a detailed correction of our argument, which resulted in an additional factor $2^{p'}$ in the bound. However, as this remark shows, the bound on B given in [42] is still correct, but it requires Lemma 5 from the first chapter, which was not known at the time of writing article [42].

Now, we are ready to prove a variant of compactness by compensation in the $\mathrm{L}^p - \mathrm{L}^q$ framework. Before we proceed, we recall that the dual of the space $\mathrm{L}^p(\mathbf{R}^d; \mathrm{C}^d(\mathrm{P}))$ is the space $\mathrm{L}^{p'}_{w*}(\mathbf{R}^d; \mathrm{C}^d(\mathrm{P})')$ of weakly-$*$ measurable functions $B : \mathbf{R}^d \to \mathrm{C}^d(\mathrm{P})'$ such that $\int_{\mathbf{R}^d} \|B(\mathbf{x})\|^{p'}_{\mathrm{C}^d(\mathrm{P})'} d\mathbf{x}$ is finite (for details see p.606 of [17]).

We first need to extend the notion of H-distributions from Theorem 2 as follows.

Theorem 4. *Let (u_r) be a sequence of uniformly compactly supported functions weakly converging to zero in $\mathrm{L}^p(\mathbf{R}^d)$, $p > 1$, and let (v_r) be a bounded sequence of uniformly compactly supported functions in $\mathrm{L}^q(\mathbf{R}^d)$, $1/q + 1/p < 1$, weakly converging to 0 in the sense of distributions. Then, after passing to a subsequence (not relabelled), for any $\bar{p} \in \langle 1, \frac{pq}{p+q}\rangle$ there exists a continuous bilinear functional B on $\mathrm{L}^{\bar{p}'}(\mathbf{R}^d) \otimes \mathrm{C}^d(\mathrm{P})$ such that for every $\varphi \in \mathrm{L}^{\bar{p}'}(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^d(\mathrm{P})$, it holds*

$$B(\varphi, \psi) = \lim_{r\to\infty} \int_{\mathbf{R}^d} \varphi(\mathbf{x})u_r(\mathbf{x})\overline{(\mathcal{A}_{\psi_{\mathrm{P}}} v_r)(\mathbf{x})}d\mathbf{x}\,, \tag{13}$$

where $\mathcal{A}_{\psi_{\mathrm{P}}}$ is the (Fourier) multiplier operator on $\mathbf{R}^d$ associated to $\psi \circ \pi_{\mathrm{P}}$.

The bilinear functional B can be continuously extended to a linear functional on $\mathrm{L}^{\bar{p}'}(\mathbf{R}^d; \mathrm{C}^d(\mathrm{P}))$.

Dem. Introduce the truncation operator

$$T_l(v) = \begin{cases} v, & |v| < l \\ 0, & |v| \geq l \end{cases}, \quad l \in \mathbf{N}, \tag{14}$$

and rewrite v_r in the form

$$v_r(\mathbf{x}) = T_l(v_r)(\mathbf{x}) + (v_r - T_l(v_r))(\mathbf{x}),$$

where $T_l(v_r)$ is understood pointwisely. Notice that

$$\limsup_{l,r\to\infty} \|v_r - T_l(v_r)\|_{L^1(K)} = 0 \tag{15}$$

for any relatively compact measurable $K \subseteq \mathbf{R}^d$. Indeed, denote by

$$\Omega_r^l = \{\mathbf{x} \in \mathbf{R}^d : |v_r(\mathbf{x})| > l\}.$$

It holds

$$\lim_{l\to\infty} \sup_{r\in\mathbf{N}} \mathrm{meas}(\Omega_r^l) = 0. \tag{16}$$

The latter follows since (v_r) is bounded in $L^q(\mathbf{R}^d)$ and

$$\sup_{r\in\mathbf{N}} \int_{\mathbf{R}^d} |v_r(\mathbf{x})|^q d\mathbf{x} \geq \sup_{r\in\mathbf{N}} \int_{\Omega_r^l} l^q d\mathbf{x} \geq l^q \sup_{r\in\mathbf{N}} \mathrm{meas}(\Omega_r^l).$$

Now, we simply use the Hölder inequality

$$\int_K |v_r - T_l(v_r)| dx = \int_{K\cap\Omega_r^l} |v_r| dx \leq \mathrm{meas}(K\cap\Omega_r^l)^{1/q'} \|v_r\|_{L^q(K)}$$

and this tends to zero uniformly with respect to r and l according to (16) and the boundedness of (v_r) in $L^q(\mathbf{R}^d)$. Thus, (15) is proved. Since (v_r), and therefore $(T_l(v_r))$ are bounded in $L^q(\mathbf{R}^d)$, (15) and interpolation inequalities imply that for any $\bar{q} \in [1, q\rangle$

$$\limsup_{l,r\to\infty} \|v_r - T_l(v_r)\|_{L^{\bar{q}}(K)} = 0. \tag{17}$$

Next, denote by μ_l the H-distribution corresponding to (u_r) and $(T_l(v_r))$ in the sense of Theorem 2. From here and (15), we conclude that we can rewrite the right-hand side of (13) in the form

$$\begin{aligned}
&\lim_{r\to\infty} \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_r(\mathbf{x}) \overline{(\mathcal{A}_{\psi_P} v_r)(\mathbf{x})} d\mathbf{x} \\
&= \lim_{r\to\infty} \left(\int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_r(\mathbf{x}) \overline{\mathcal{A}_{\psi_P}(T_l(v_r))(\mathbf{x})} d\mathbf{x} + \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_r(\mathbf{x}) \overline{\mathcal{A}_{\psi_P}(v_r - T_l(v_r))(\mathbf{x})} d\mathbf{x} \right) \\
&= \langle \mu_l, \varphi\psi \rangle + o_l(1),
\end{aligned} \tag{18}$$

where $o_l(1) \to 0$ as $l \to \infty$ follows from (17) and the application of the Hölder inequality as follows:

$$\begin{aligned}
&\Big| \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_r(\mathbf{x}) \overline{\mathcal{A}_{\psi_P}(v_r - T_l(v_r))(\mathbf{x})} d\mathbf{x} \Big| \\
&\leq C_{d,\bar{q}} \|\varphi\|_{L^{\bar{p}'}(\mathbf{R}^d)} \|\psi\|_{C^d(P)} \sup_r \|u_r\|_{L^p(\mathbf{R}^d)} \sup_r \|v_r - T_l(v_r)\|_{L^{\bar{q}}(\mathbf{R}^d)},
\end{aligned}$$

where $1/\bar{p}' + 1/p + 1/\bar{q} = 1$ (and obviously $\bar{q} < q$ implying that we can apply (17)).

Since $\psi \circ \pi_P$ is an $L^{\bar{q}}$-multiplier (see Lemma 1), by the Hölder inequality used with the exponents $\bar{p}'$, p, and $\bar{q} < q$, we get

$$\begin{aligned}
\Big| \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_r(\mathbf{x}) \overline{(\mathcal{A}_{\psi_P} T_l(v_r))(\mathbf{x})} d\mathbf{x} \Big| &\leq C_{d,\bar{q}} \|\varphi\|_{L^{\bar{p}'}(\mathbf{R}^d)} \|u_r\|_{L^p(\mathbf{R}^d)} \|\psi\|_{C^d(P)} \|T_l(v_r)\|_{L^{\bar{q}}(\mathbf{R}^d)} \\
&\leq C_u\, C_v\, C_{d,\bar{q}} \|\varphi\|_{L^{\bar{p}'}(\mathbf{R}^d)} \|\psi\|_{C^d(P)}
\end{aligned}$$

From here, after passing to the limit $r \to \infty$ and using the continuity of extension from Lemma 3, we conclude that (μ_l) is bounded sequence in $(L^{\bar{p}'}(\mathbf{R}^d; C^d(P))' = L^{\bar{p}}_{w*}(\mathbf{R}^d; C^d(P)')$ (remark that the bound of (μ_l) is $2C_u\, C_v\, C_{d,p'}$). Since $L^{\bar{p}}_{w*}(\mathbf{R}^d; C^d(P)')$ is dual of the Banach space, according to the Banach-Alaoglu theorem, (μ_l) admits a weak-$*$ limit $\mu \in L^{\bar{p}}_{w*}(\mathbf{R}^d; C^d(P)')$ along a subsequence. The functional μ satisfies (13).

Q.E.D.

Remark 3. In the case $1/p+1/q=1$, the same proof gives us continuous bilinear functional on $\mathrm{C}(\mathbf{R}^d)\otimes \mathrm{C}^d(\mathrm{P})$. We cannot use Lemma 3 anymore, but using the Schwartz kernel theorem, we can (only) extend it to a distribution from $\mathcal{D}'_{0,d(d+2)}(\mathbf{R}^d\times \mathrm{P})$. Therefore, our variant of the compactness by compensation is confined on $\mathrm{L}^p-\mathrm{L}^q$ framework for $1/p+1/q<1$. However, under additional assumptions, we are able to prove the result in the optimal case $1/p+1/q=1$ (Corollary 1).

Before we proceed, let us recall the definition of fractional derivatives. For $\alpha\in\mathbf{R}^+$, we define $\partial^\alpha_{x_k}$ to be a pseudodifferential operator with a polyhomogeneous symbol $(2\pi i\xi_k)^\alpha$, i.e.

$$\partial^\alpha_{x_k}u=((2\pi i\xi_k)^\alpha\hat{u}(\boldsymbol{\xi}))\check{}\,.$$

In the sequel, we shall assume that sequences $(\mathbf{u}_r)$ and $(\mathbf{v}_r)$ are uniformly compactly supported. This assumption can be removed if the orders of derivatives $(\alpha_1,\dots,\alpha_d)$ are natural numbers. Otherwise, since the Leibnitz rule does not hold for fractional derivatives, the former assumption seems necessary.

Let us now introduce the localisation principle corresponding to an H-distribution.

Lemma 4. *Assume that sequences $(\mathbf{u}_r)$ and $(\mathbf{v}_r)$ are bounded in $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^N)$ and $\mathrm{L}^q(\mathbf{R}^d;\mathbf{R}^N)$, where $1/p+1/q<1$, and converge toward 0 and $\mathbf{v}=(v_1,\dots,v_N)$ in the sense of distributions.*

Furthermore, assume that the sequence $(\mathbf{u}_r)$ satisfies, for every $s=1,\dots,M$:

$$G_{rs}:=\sum_{j=1}^N\sum_{k=1}^d\partial^{\alpha_k}_{x_k}(a_{sjk}u_{jr})\to 0 \text{ in } \mathrm{W}^{-\alpha_1,\dots,-\alpha_d;p}(\mathbf{R}^d), \tag{19}$$

where $\alpha_k\in\mathbf{N}$ or $\alpha_k>d$, $k=1,\dots,d$, and $a_{sjk}\in\mathrm{L}^{\bar{s}'}(\mathbf{R}^d)$, $\bar{s}\in\langle 1,\frac{pq}{p+q}\rangle$.

Finally, by μ_{jm} denote the H-distribution (Theorem 4) corresponding to a pair of subsequences of (u_{jr}) and $(v_{mr}-v_m)$. Then the following relations hold in the sense of distributions for $m=1,\dots,N$, $s=1,\dots,M$ ($i=\sqrt{-1}$ below)

$$\sum_{j=1}^N\sum_{k=1}^n a_{sjk}(2\pi i\xi_k)^{\alpha_k}\mu_{jm}=0. \tag{20}$$

Dem. Assume, without loosing any generality, that $\mathbf{v}=0$. Denote by $\mathcal{B}_\psi$ the Fourier multiplier operator with the symbol

$$(\psi\circ\pi_{\mathrm{P}})(\boldsymbol{\xi})\frac{(1-\theta(\boldsymbol{\xi}))}{(|\xi_1|^{2\alpha_1}+\ldots+|\xi_d|^{2\alpha_d})^{1/2}},$$

where θ is a cutoff function equal to one in a neighbourhood of zero.

According to Lemma 1, for any $\psi\in\mathrm{C}^d(\mathrm{P})$ and any $\hat{s}>1$, the multiplier operator $\mathcal{B}_\psi:\mathrm{L}^2(\mathbf{R}^d)\cap\mathrm{L}^{\hat{s}}(\mathbf{R}^d)\to\mathrm{W}^{\alpha_1,\dots,\alpha_d;\hat{s}}(\mathbf{R}^d)$ is bounded (with $\mathrm{L}^{\hat{s}}$ norm considered on the domain of $\mathcal{B}_\psi$); indeed, one just needs to notice that the symbol of $\partial^{\alpha_k}_{x_k}\circ\mathcal{B}_\psi$ given by

$$(\psi\circ\pi_{\mathrm{P}})(\boldsymbol{\xi})\frac{(1-\theta(\boldsymbol{\xi}))(2\pi i\xi_k)^{\alpha_k}}{(|\xi_1|^{2\alpha_1}+\ldots+|\xi_d|^{2\alpha_d})^{1/2}},$$

is a smooth, bounded function satisfying conditions of Marcinkiewicz's multiplier theorem (Theorem IV.6.6' of [57] or Corollary 2 of the first chapter).

Insert in (19) the test function g_{rm} given by:

$$g_{rm}(\mathbf{x}) = \mathcal{B}_\psi(\phi v_{mr})(\mathbf{x}), \quad m \in \{1, \dots, N\} \tag{21}$$

where $\psi \in \mathrm{C}^d(\mathrm{P})$ and $\phi \in \mathrm{C}_c^\infty(\mathbf{R}^d)$. We get

$$\begin{aligned}\int_{\mathbf{R}^d} G_{rs}\overline{g_{rm}}d\mathbf{x} &= \int_{\mathbf{R}^d}\sum_{j=1}^N\sum_{k=1}^n a_{sjk}u_{jr}\overline{\mathcal{A}_{(\psi\circ\pi_{\mathrm{P}})(\boldsymbol{\xi})\frac{(1-\theta(\boldsymbol{\xi}))(2\pi i\xi_k)^{\alpha_k}}{\left(|\xi_1|^{2\alpha_1}+\dots+|\xi_d|^{2\alpha_d}\right)^{1/2}}}(\phi v_{mr})}d\mathbf{x}\\ &= \int_{\mathbf{R}^d}\sum_{j=1}^N\sum_{k=1}^n a_{sjk}u_{jr}\overline{\mathcal{A}_{(\psi\circ\pi_{\mathrm{P}})(\boldsymbol{\xi})\frac{\overline{(2\pi i\xi_k)^{\alpha_k}}}{\left(|\xi_1|^{2\alpha_1}+\dots+|\xi_d|^{2\alpha_d}\right)^{1/2}}}(\phi v_{mr})}d\mathbf{x}\\ &\quad - \int_{\mathbf{R}^d}\sum_{j=1}^N\sum_{k=1}^n a_{sjk}u_{jr}\overline{\mathcal{A}_{(\psi\circ\pi_{\mathrm{P}})(\boldsymbol{\xi})\frac{\theta(\boldsymbol{\xi})\overline{(2\pi i\xi_k)^{\alpha_k}}}{\left(|\xi_1|^{2\alpha_1}+\dots+|\xi_d|^{2\alpha_d}\right)^{1/2}}}(\phi v_{mr})}d\mathbf{x}.\end{aligned} \tag{22}$$

Due to the boundedness properties of operator $\mathcal{B}_\psi$ mentioned above and the compact support of ϕ, the sequence (g_{rm}) is bounded in $\mathrm{W}^{\alpha_1,\dots,\alpha_d;t}(\mathbf{R}^d)$ for $t \in \langle 1, q]$. Letting $r \to \infty$ in (22), we get (20) after taking into account Theorem 4 and the strong convergence of (G_{rs}). Note that the second summand in the above identity goes to 0 because of the compact support of the function θ.

Q.E.D.

Remark 4. In the case $1/p + 1/q = 1$, taking into account Remark 3 and coefficients a_{sjk} from the space $\mathrm{C}_0(\mathbf{R}^d)$, we get the same result as in (20) for distributions μ_{jm} from $\mathcal{D}'(\mathbf{R}^d \times \mathrm{P})$.

We can now formulate conditions under which (4) holds. We call them the strong consistency conditions. They represent a generalisation of the standard consistency conditions given above.

As before, let $\bar{s} \in \langle 1, \frac{pq}{p+q}\rangle$ be a fixed number for given $p, q > 1$. Introduce the set

$$\begin{aligned}\Lambda_{\mathcal{D}} = \Big\{\boldsymbol{\mu} = (\mu_1, \dots, \mu_N) \in &\mathrm{L}^{\bar{s}}_{w*}(\mathbf{R}^d; (\mathrm{C}^d(\mathrm{P}))')^N :\\ &\sum_{j=1}^N\sum_{k=1}^d (2\pi i\xi_k)^{\alpha_k} a_{sjk}\mu_j = 0,\ s = 1, \dots, M\Big\},\end{aligned} \tag{23}$$

where the given equality is understood in the sense of $\mathrm{L}^{\bar{s}}_{w*}(\mathbf{R}^d; (\mathrm{C}^d(\mathrm{P}))')$.

Remark 5. Let us notice the difference between the $\Lambda_{\mathcal{D}}$ and set Λ from (2). In the case of H-measures, an application of the Radon-Nikodym theorem yields a nice representation of the measure, which in turn simplifies the definition of the set Λ. There is no known variant of Radon-Nikodym result for distributions.

Although at the moment we do not have the definitive answer, we are inclined to believe that it is also negative for variant of H-distributions we are using in this chapter. Namely, in situations when we are dealing with Bôchner spaces $\mathrm{L}^p(\mathbf{R}^d; E)$, existence of some version of Radon-Nikodym result will mostly depend on the properties of the dual

E' (see results of [14]). In some cases, one of the sufficient conditions that is easy to check is separability: if E' were separable, then it would have Radon-Nikodym property. Unfortunately, while $\mathrm{C}^d(\mathrm{P})$ is separable Banach space, its dual $(\mathrm{C}^d(\mathrm{P}))'$ is not (in much simpler situation, remember that Dirac distributions form a discrete uncountable subset of the space of Radon measures). This is also a reason why the dual of $\mathrm{L}^p(\mathbf{R}^d;\mathrm{C}^d(\mathrm{P}))$ includes some non-Bôchner-measurable functions. Thus, the space $\left(\mathrm{C}^d(\mathrm{P})\right)'$ does not posses Radon-Nikodym property (also compare characterisation of Theorem IV.1 in [14]).

Let us assume that

(24)

coefficients of the bilinear form q from (3) belong to the space $\mathrm{L}^t(\mathbf{R}^d)$, where $t \geq \bar{s}'$.

Remark that since $\bar{s} \in \langle 1, \frac{pq}{p+q}\rangle$ and $t \geq \bar{s}'$, it also must be $1/t + 1/p + 1/q < 1$.

Definition. We say that the set $\Lambda_{\mathcal{D}}$, bilinear form q from (3) satisfying (24), and the matrix $\boldsymbol{\mu} = [\mu_{jm}]_{j,m=1,\dots,N}$, $\mu_{jm} \in \mathrm{L}^{\bar{s}}_{w\star}(\mathbf{R}^d;(\mathrm{C}^d(\mathrm{P}))')$ satisfy *the strong consistency condition* if for every fixed $m \in \{1,\dots,N\}$, the N-tuple $(\mu_{1m},\dots,\mu_{Nm})$ belongs to $\Lambda_{\mathcal{D}}$, and it holds

$$\sum_{j,m=1}^{N} \langle \phi q_{jm} \otimes 1, \mu_{jm}\rangle \geq 0, \quad \phi \in \mathrm{C}^\infty_c(\mathbf{R}^d;\mathbf{R}^+_0). \tag{25}$$

Under the given strong consistency condition, we have the following theorem:

Theorem 5. *Assume that sequences* $(\mathbf{u}_r)$ *and* $(\mathbf{v}_r)$ *are bounded in* $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^N)$ *and* $\mathrm{L}^q(\mathbf{R}^d;\mathbf{R}^N)$*, where* $1/p+1/q<1$*, and converge toward* $\mathbf{u}$ *and* $\mathbf{v}$ *in the sense of distributions. Assume that (19) holds.*

Assume that

$$q(\mathbf{x};\mathbf{u}_r,\mathbf{v}_r) \rightharpoonup \omega \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d)$$

for the bilinear form q from (3) satisfying (24).

If the set $\Lambda_{\mathcal{D}}$*, the bilinear form (3), and the (matrix of) H-distributions* $\boldsymbol{\mu}$ *corresponding to the sequences* $(\mathbf{u}_r-\mathbf{u})$ *and* $(\mathbf{v}_r-\mathbf{v})$ *satisfy the strong consistency condition, then it holds*

$$q(\mathbf{x};\mathbf{u},\mathbf{v}) \leq \omega \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d). \tag{26}$$

If in (25) stands equality, then we have equality in (26) as well.

Dem. Let us abuse the notation by denoting $\mathbf{u}_r = \mathbf{u}_r - \mathbf{u} \rightharpoonup 0$ and $\mathbf{v}_r = \mathbf{v}_r - \mathbf{v} \rightharpoonup 0$ as $r \to \infty$.

Remark that, according to Theorem 4, for any non-negative $\phi \in \mathcal{D}(\mathbf{R}^d)$

$$\lim_{r\to\infty} \int_{\mathbf{R}^d} \sum_{j,m=1}^{N} q_{jm} u_{jr} v_{mr} \phi \, d\mathbf{x} = \Big\langle \phi \sum_{j,m=1}^{N} q_{jm} \otimes 1, \mu_{jm}\Big\rangle, \tag{27}$$

where μ_{jm} is a H-distribution corresponding to sequences $u_{jr}, v_{mr} \rightharpoonup 0$. Since, according to the localisation principle (20), for every fixed $m \in \{1,\dots,N\}$, the N-tuple $(\mu_{1m},\dots,\mu_{Nm})$ belongs to $\Lambda_{\mathcal{D}}$, we conclude from the strong consistency condition that

$$\Big\langle \phi \sum_{j,m=1}^{N} q_{jm} \otimes 1, \mu_{jm}\Big\rangle \geq 0.$$

From here, (27), and the fact that (since q is bilinear)

$$q(\mathbf{x};\mathbf{u}_r,\mathbf{v}_r) \rightharpoonup \omega - q(\mathbf{x};\mathbf{u},\mathbf{v}) \geq 0 \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d),$$

the statement of the theorem follows.

Q.E.D.

If we assume that the sequence $(\mathbf{v}_n)$ is bounded in $\mathrm{L}^{p'}(\mathbf{R}^d;\mathbf{R}^N)$ and additionally assume that it can be well approximated by the truncated sequence $(T_l(\mathbf{v}_n))$, $l\in\mathbf{N}$, we can state the optimal variant of the compensated compactness as follows.

Corollary 1. *Assume that*

- *sequences $(\mathbf{u}_r)$ and $(\mathbf{v}_r)$ are bounded in $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^N)$ and $\mathrm{L}^{p'}(\mathbf{R}^d;\mathbf{R}^N)$, where $1/p+1/p'=1$, and converge toward $\mathbf{u}$ and $\mathbf{v}$ in the sense of distributions;*
- *for every $l\in\mathbf{N}$, the sequences $(T_l(\mathbf{v}_r))$ converge weakly in $\mathrm{L}^{p'}(\mathbf{R}^d;\mathbf{R}^N)$ toward $\mathbf{h}^l$, where the truncation operator T_l from (14) is understood coordinatewise;*
- *there exists a vector valued function $\mathbf{V}\in\mathrm{L}^{p'}(\mathbf{R}^d;\mathbf{R}^N)$ such that $|\mathbf{v}_r|\leq\mathbf{V}$ holds coordinatewise for every $r\in\mathbf{N}$;*
- *(19) holds with $a_{skl}\in\mathrm{C}_0(\mathbf{R}^d)$ and $q_{jm}\in\mathrm{C}(\mathbf{R}^d)$.*
  Assume that

$$q(\mathbf{x};\mathbf{u}_r,\mathbf{v}_r) \rightharpoonup \omega \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d).$$

If for every $l\in\mathbf{N}$, the set $\Lambda_{\mathcal{D}}$, the bilinear form (3), and the (matrix of) H-distributions $\boldsymbol{\mu}_l$ corresponding to the sequences $(\mathbf{u}_r-\mathbf{u})$ and $(T_l(\mathbf{v}_r)-\mathbf{h}^l)_r$ satisfy the strong consistency condition, then it holds

$$q(\mathbf{x};\mathbf{u},\mathbf{v}) \leq \omega \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d). \tag{28}$$

If in (25) stands equality, then we have equality in (28) as well.

Dem. For every $l\in\mathbf{N}$, notice that $(q(\mathbf{x};\mathbf{u}_r,T_l(\mathbf{v}_r)))_r$ is bounded in $\mathrm{L}^p(\mathbf{R}^d)$:

$$\begin{aligned}\int_{\mathbf{R}^d}|q(\mathbf{x};\mathbf{u}_r,T_l(\mathbf{v}_r))|^p d\mathbf{x} &\leq N^{2(p-1)}\sum_{j,m=1}^N\int_{\mathbf{R}^d}|q_{jm}|^p|u_{jr}|^p|T_l(v_{mr})|^p d\mathbf{x}\\ &\leq C_{N,l,p}\max_{j,m}(\|q_{jm}\|^p_{\mathrm{L}^\infty(K)}\|u_{jr}\|^p_{\mathrm{L}^p(K)}),\end{aligned}$$

where $K\subseteq\mathbf{R}^d$ is a compact set (remember that sequences $(\mathbf{u}_r)$, $(\mathbf{v}_r)$ are uniformly compactly supported). Therefore, the sequence $(q(\mathbf{x};\mathbf{u}_r,T_l(\mathbf{v}_r)))$ (we remind that l is fixed) admits a weak limit in $\mathrm{L}^p(\mathbf{R}^d)$ (and thus in $\mathcal{D}'(\mathbf{R}^d)$) along a subsequence. Using a diagonal procedure, we can extract a subsequence (not relabelled) such that for every $l\in\mathbf{N}$ it holds

$$q(\mathbf{x};\mathbf{u}_r,T_l(\mathbf{v}_r)) \rightharpoonup \omega_l \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d).$$

where ω_l is a weak limit of $(q(\mathbf{x};\mathbf{u}_r,T_l(\mathbf{v}_r)))_r$. According to the assumptions of the corollary on the strong consistency conditions involving $\boldsymbol{\mu}_l$ and the sequences $(\mathbf{u}_r-\mathbf{u})$ and $(T_l(\mathbf{v}_r)-\mathbf{h}^l)_r$, and Theorem 5 (remark that $(T_l(\mathbf{v}_r))_r$ is bounded), it holds

$$q(\mathbf{x};\mathbf{u},\mathbf{h}^l) \leq \omega_l \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d). \tag{29}$$

We will finish the corollary if we show that for every non-negative function $\varphi \in C_c^\infty(\mathbf{R}^d)$ it holds $\int_{\mathbf{R}^d}(\omega - q(\mathbf{x};\mathbf{u},\mathbf{v}))\varphi d\mathbf{x} \geq 0$. It holds

(30)

$$\begin{aligned}\int_{\mathbf{R}^d}(\omega - q(\mathbf{x};\mathbf{u},\mathbf{v}))\varphi d\mathbf{x} &= \int_{\mathbf{R}^d}(\omega - q(\mathbf{x};\mathbf{u}_r,\mathbf{v}_r))\varphi d\mathbf{x}\\ &+ \int_{\mathbf{R}^d}(q(\mathbf{x};\mathbf{u}_r,\mathbf{v}_r) - q(\mathbf{x};\mathbf{u}_r,T_l(\mathbf{v}_r)))\varphi d\mathbf{x}\\ &+ \int_{\mathbf{R}^d}(q(\mathbf{x};\mathbf{u}_r,T_l(\mathbf{v}_r)) - \omega_l)\varphi d\mathbf{x} + \int_{\mathbf{R}^d}(\omega_l - q(\mathbf{x};\mathbf{u},\mathbf{h}^l))\varphi d\mathbf{x}\\ &+ \int_{\mathbf{R}^d}(q(\mathbf{x};\mathbf{u},\mathbf{h}^l) - q(\mathbf{x};\mathbf{u},\mathbf{v}))\varphi d\mathbf{x}.\end{aligned}$$

Since the left hand side of (30) does not depend on r and l, we can take $\limsup\limits_{l\to\infty}\lim\limits_{r\to\infty}$ there. The first summand on the right hand side of the expression goes to zero according to the assumptions of the corollary; the third summand goes to zero according to the definition of ω_l; we have established in (29) that the fourth summand is non-negative. Let us show that the second summand in (30) goes to zero:

$$\begin{aligned}\Big|\int_{\mathbf{R}^d}(q(\mathbf{x};\mathbf{u}_r,\mathbf{v}_r) - q(\mathbf{x};\mathbf{u}_r,T_l(\mathbf{v}_r)))\varphi d\mathbf{x}\Big| &\leq \int_{\mathbf{R}^d}|\varphi\, \mathbf{Q}\mathbf{u}_r\cdot(\mathbf{v}_r - T_l(\mathbf{v}_r))|d\mathbf{x}\\ &\leq \|\mathbf{Q}\mathbf{u}_r\|_{\mathrm{L}^p}\|\varphi\,(\mathbf{v}_r - T_l(\mathbf{v}_r))\|_{\mathrm{L}^{p'}},\end{aligned}$$

where we have used the Hölder inequality. Since $\mathbf{v}_r - T_l(\mathbf{v}_r) \to 0$ pointwise, according to the assumption $|\mathbf{v}_r| \leq \mathbf{V}$ and the Lebesgue dominated convergence theorem, we conclude that $\|\varphi\,(\mathbf{v}_r - T_l(\mathbf{v}_r))\|_{\mathrm{L}^{p'}} \to 0$ as $l, r \to \infty$ (or as $l \to \infty$ uniformly with respect to r). For the last summand, we will proceed in a similar manner. Let us notice that we can write

$$\begin{aligned}q(\mathbf{x};\mathbf{u},\mathbf{h}^l) - q(\mathbf{x};\mathbf{u},\mathbf{v}) &= \mathbf{Q}\mathbf{u}\cdot(\mathbf{h}^l - \mathbf{v})\\ &= \mathbf{Q}\mathbf{u}\cdot((\mathbf{h}^l - T_l(\mathbf{v}_r)) + (T_l(\mathbf{v}_r) - \mathbf{v}_r) + (\mathbf{v}_r - \mathbf{v})).\end{aligned}$$

The first and the last summand on the right hand side of the last expression will go to zero according to the assumptions of the corollary. Concerning the second summand, from the Lebesgue dominated convergence theorem as before, we conclude $\limsup\limits_{l\to\infty}\lim\limits_{r\to\infty}\|(T_l(\mathbf{v}_r) - \mathbf{v}_r)\varphi\|_{\mathrm{L}^1(\mathbf{R}^d)} = 0$. This concludes the proof.

Q.E.D.

Remark 6. The condition concerning existence of the dominating function $\mathbf{V}$ from the previous theorem might look superfluous. However, as the following example shows, we cannot avoid it. Indeed, consider the case $d = N = 1$, $a = a_{111} = 0$. Let

$$u_r(\mathbf{x}) = v_r(\mathbf{x}) = \begin{cases} r, & |x| < r^{-2}\\ 0, & |x| \geq r^{-2}\end{cases}.$$

Then, $\|u_r\|_2 = 2$ for all $r \in \mathbf{N}$. Clearly, $u_r = v_r \rightharpoonup 0$ weakly as $r \to \infty$, while $T_l(u_r) \to 0$ as $r \to \infty$ strongly in $\mathrm{L}^2(\mathbf{R})$ for every $l \in \mathbf{N}$. Therefore, the H-distributions μ_l corresponding to the sequences (u_r) and $(T_l(v_r))$ are trivial: $\mu_l \equiv 0$. Thus, the strong consistency condition is satisfied with the equality sign, but $q(u_r, v_r) = u_r^2 \rightharpoonup 2\delta(\mathbf{x}) \neq 0 = q(0,0)$.

This remark is thanks to Evgenij Jurjevič Panov.

In a conclusion of the section, we would like to make a comment concerning a connection between the standard consistency condition and, at least at first sight stronger, the strong consistency condition. To this end, note that we can rewrite the consistency condition (2) in the following form (we shall omit the second order derivatives since they have no influence on the reasoning below):

$$\Lambda_{\mathcal{F}} = \left\{ \boldsymbol{\lambda} : \mathbf{R}^d \times \mathrm{S}^{d-1} \to \mathbf{R}^N : \ \sum_{j=1}^{N} \sum_{k=1}^{\nu} a_{sjk}(\mathbf{x})\xi_k \lambda_j(\mathbf{x}, \boldsymbol{\xi}) = 0, \ s = 1, \dots, M \right\}$$

and

$$q(\mathbf{x}; \boldsymbol{\lambda}(\mathbf{x}, \boldsymbol{\xi}), \boldsymbol{\lambda}(\mathbf{x}, \boldsymbol{\xi})) \geq 0 \quad \text{for all } \boldsymbol{\lambda} \in \Lambda_{\mathcal{F}} \text{ and all } (\mathbf{x}, \boldsymbol{\xi}) \in \mathbf{R}^d \times \mathrm{S}^{d-1}.$$

Having such a representation of the consistency condition, it seems reasonable to ask whether $\Lambda_{\mathcal{D}}$ is a closure of $\Lambda_{\mathcal{F}}$ in the sense of distributions. If this is the case, the generalisation presented here holds under the standard consistency condition. At this moment, we do not have any answer to this question.

However, we shall present an example showing that our approach can be used.

3. Application to nonlinear parabolic equation

Let us consider the nonlinear parabolic type equation

$$L(u) = \partial_t u - \sum_{k,l=1}^{d} \partial_{x_l x_k}(a_{kl}(t, \mathbf{x}) g(t, \mathbf{x}, u)) \tag{31}$$

on $\Omega = \langle 0, \infty \rangle \times V$, where V is an open subset of $\mathbf{R}^d$. We assume that

$$u \in \mathrm{L}^p(\Omega), \quad g(t, \mathbf{x}, u) \in \mathrm{L}^q(\Omega), \quad 1 < p, q,$$
$$a_{kl} \in \mathrm{L}^s_{loc}(\Omega), \quad \text{where} \quad 1/p + 1/q + 1/s < 1,$$

and that the matrix function $\mathbf{A} = [a_{kl}]_{k,l=1,\dots,d}$ is strictly positive definite on Ω, i.e.

$$\mathbf{A}\boldsymbol{\xi} \cdot \boldsymbol{\xi} > 0, \quad \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{0\}, \quad \text{a.e. } (t, \mathbf{x}) \in \Omega.$$

Furthermore, assume that g is a Carathèodory function and non-decreasing with respect to the third variable.

The following theorem holds.

Theorem 6. *Assume that sequences*

- (u_r) *and* $g(\cdot, u_r)$ *are such that* $u_r, g(u_r) \in \mathrm{L}^2(\mathbf{R}^+ \times \mathbf{R}^d)$ *for every* $r \in \mathbf{N}$*;*
- *that they are bounded in* $\mathrm{L}^p(\mathbf{R}^+ \times \mathbf{R}^d)$, $p \in \langle 1, 2]$, *and* $\mathrm{L}^q(\mathbf{R}^+ \times \mathbf{R}^d)$, $q > 2$, *respectively, where* $1/p + 1/q < 1$*;*
- $u_r \rightharpoonup u$ *and, for some,* $f \in \mathrm{W}^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d)$, *the sequence*

$$L(u_r) = f_r \to f \quad \text{strongly in } \mathrm{W}^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d).$$

Under the assumptions given above, it holds

$$L(u) = f \ \text{ in } \ \mathcal{D}'(\mathbf{R}^+ \times \mathbf{R}^d).$$

Dem. Let us first define all functions on $\mathbf{R}\times\mathbf{R}^d$ by extending them with zero out of $\mathbf{R}^+\times\mathbf{R}^d$. Denote by w a distributional limit of $g(\cdot,u_r)$ along not relabelled subsequence. Our first step is to show that the product of u_r and $g(\cdot,u_r)$ converges to uw in the sense of distributions. To do that, denote

$$u_{1r}=u_r-u,\quad u_{2r}=g(\cdot,u_r)-w. \tag{32}$$

Note that the following sequence of equations is satisfied

$$\partial_t u_{1r}-\sum_{k,l=1}^d \partial_{x_l x_k}(a_{kl}u_{2r})=f_r-f, \tag{33}$$

and that f_r-f tends to zero strongly in $\mathrm{W}^{-1,-2;p}(\mathbf{R}^+\times\mathbf{R}^d)$. Introduce
(34)

$$\Lambda_{\mathcal{D}}=\Big\{\boldsymbol{\mu}=(\mu_1,\mu_2)\in\mathrm{L}^{s'}_{w\star}(\mathbf{R}^+\times\mathbf{R}^d;\mathrm{C}^{d+1}(\mathrm{P})')^2:\ -2i\pi\xi_0\mu_1+4\pi^2\sum_{k,l=1}^d\xi_k\xi_l a_{kl}\mu_2=0\Big\},$$

and remark that, according to the localisation principle given in Lemma 4,

$$(\mu_{12},\mu_{22})\in\Lambda_{\mathcal{D}} \tag{35}$$

for H-distributions μ_{12} and μ_{22}, corresponding to sequences (ϕu_{1r}) and (ϕu_{2r}), and (ϕu_{2r}) and (ϕu_{2r}), respectively. Above, $\phi\in\mathrm{C}^2_c(\mathbf{R}^+\times\mathbf{R}^d)$ is fixed.

From the localisation principle, for $\psi\in\mathrm{C}^{d+1}(\mathrm{P})$ (here and in the sequel, symbols are real functions) and $\varphi\in\mathrm{C}^2_c(\mathbf{R}^d)$, it holds

$$i\langle -2\pi\xi_0\psi\varphi,\mu_{12}\rangle+\Big\langle 4\pi^2\sum_{k,l=1}^d\xi_k\xi_l a_{kl}(\cdot,\cdot)\psi\varphi,\mu_{22}\Big\rangle=0. \tag{36}$$

Remark that for any $\psi\in\mathrm{C}^{d+1}(\mathrm{P})$ the function $f_\psi=\langle\psi,\mu_{j2}\rangle$ is in $\mathrm{L}^{s'}(\mathbf{R}^+\times\mathbf{R}^d)$, $j=1,2$. For the functions f_ψ, where ψ belongs to a dense countable subset E of $\mathrm{C}^{d+1}(\mathrm{P})$ containing a dense subset of odd and even functions (which we may choose since $\mathrm{C}^{d+1}(\mathrm{P})$ is separable and we can represent every function as a sum of even and odd functions $\psi(\boldsymbol{\xi})=\frac12(\psi(\boldsymbol{\xi})+\psi(-\boldsymbol{\xi}))+\frac12(\psi(\boldsymbol{\xi})-\psi(-\boldsymbol{\xi}))$), and the functions a_{kl}, $k,l=1,\dots,d$, denote by $D\subseteq\mathbf{R}^+\times\mathbf{R}^d$ the set of their common Lebesgue points (which is of full measure).

Now, fix $(t_0,\mathbf{x}_0)\in D$. According to the Plancherel theorem, we get

$$\int\overline{\varphi v}\mathcal{A}_\psi(\varphi v)=\int\widehat{\overline{\varphi v}}\,\psi\widehat{\varphi v}\in\mathbf{R} \tag{37}$$

for all $v\in\mathrm{L}^2(\mathbf{R}^+\times\mathbf{R}^d)$, real bounded multipliers ψ, and $\varphi\in\mathrm{C}^2_c(\mathbf{R}^d)$. From here we conclude that

$$\Big\langle 4\pi^2\sum_{k,l=1}^d\xi_k\xi_l a_{kl}(t_0,\mathbf{x}_0)\psi\varphi,\mu_{22}\Big\rangle\in\mathbf{R} \tag{38}$$

for any real multiplier ψ. Indeed, for a scalar matrix $\mathbf{A}(t_0,\mathbf{x}_0)$, taking into account that $4\pi^2\mathbf{A}(t_0,\mathbf{x}_0)\boldsymbol{\xi}\cdot\boldsymbol{\xi}\geq 0$, we notice that $4\pi^2\mathbf{A}(t_0,\mathbf{x}_0)\boldsymbol{\xi}\cdot\boldsymbol{\xi}\psi\otimes\varphi$ is a real function in $\boldsymbol{\xi}$ (where φ is constant with respect to $\boldsymbol{\xi}$). Insert symbol $4\pi^2(\mathbf{A}(t_0,\mathbf{x}_0)\boldsymbol{\xi}\cdot\boldsymbol{\xi}\psi/\rho_{\mathrm{P}})\otimes\varphi$ and sequences $u_r=v_r=\phi u_{2r}$ into definition (13) of H-distributions where

$$\rho_{\mathrm{P}}=\Big(\xi_0^2+\sum_{j=1}^d\xi_j^4\Big)^{1/2}.$$

Now, the claim follows once we notice that, due to equation (37), equation (13) gives us a limit of real numbers.

On the other hand, from Lemma 2, we conclude that for any odd ψ, the function

$$\Big\langle 4\pi^2\sum_{k,l=1}^d\xi_k\xi_l a_{kl}(t_0,\mathbf{x}_0)\psi\varphi,\mu_{22}\Big\rangle\in i\mathbf{R}. \tag{39}$$

Thus, from (38) and (39), we conclude that for any odd function ψ it must be

$$\Big\langle 4\pi^2\sum_{k,l=1}^d\xi_k\xi_l a_{kl}(t_0,\mathbf{x}_0)\psi\varphi,\mu_{22}\Big\rangle=0. \tag{40}$$

Taking into account (40), assuming $\psi\in E$, and inserting $(t,\mathbf{x})=(t_0,\mathbf{x}_0)$ into (36), we conclude that for all points from D, it holds

$$\langle -2\pi\xi_0\psi,\mu_{12}(t_0,\mathbf{x}_0,\cdot)\rangle=0. \tag{41}$$

Now, since $u_r\in\mathrm{L}^2(\mathbf{R}^+\times\mathbf{R}^d)$ for every $r\in\mathbf{N}$, we can test (33) by $\varphi\overline{\mathcal{A}_{(1-\theta)\psi_{\mathrm{P}}/\rho_{\mathrm{P}}}(\varphi u_{1r})}$ where θ is a compactly supported even smooth function equal to one in a neighbourhood of zero. Then, we let $r\to\infty$ and use the Plancherel theorem to obtain a relation similar to (36) (remark that $\mathcal{A}_{(1-\theta)\psi_{\mathrm{P}}/\rho_{\mathrm{P}}}$ is a compact $\mathrm{L}^p\to\mathrm{L}^p$ operator for any $p>1$):

$$\begin{aligned}\lim_{r\to\infty}\int_{\mathbf{R}^{d+1}}-2\pi i\frac{(1-\theta(\boldsymbol{\xi}))\xi_0}{\rho_{\mathrm{P}}(\boldsymbol{\xi})}\psi_{\mathrm{P}}(\boldsymbol{\xi})\mathcal{F}(\varphi u_{1r})\overline{\mathcal{F}(\varphi u_{1r})}d\boldsymbol{\xi}&\\ +\Big\langle 4\pi^2\sum_{k,l=1}^d\xi_k\xi_l a_{kl}(\cdot,\cdot)\psi\varphi,\mu_{12}\Big\rangle&=0,\end{aligned} \tag{42}$$

where, as usual, $\psi_P=\psi\circ\rho_{\mathrm{P}}$. Denote by

$$\begin{aligned}I_r(\psi_{\mathrm{P}})&=\int_{\mathbf{R}^{d+1}}-2\pi i\frac{(1-\theta(\boldsymbol{\xi}))\xi_0}{\rho_{\mathrm{P}}(\boldsymbol{\xi})}\psi_{\mathrm{P}}(\boldsymbol{\xi})\mathcal{F}(\varphi u_{1r})\overline{\mathcal{F}(\varphi u_{1r})}d\boldsymbol{\xi}\\&=\int_{\mathbf{R}^{d+1}}-2\pi i\frac{(1-\theta(\boldsymbol{\xi}))\xi_0}{\rho_{\mathrm{P}}(\boldsymbol{\xi})}\psi_{\mathrm{P}}(\boldsymbol{\xi})|\mathcal{F}(\varphi u_{1r})|^2d\boldsymbol{\xi}.\end{aligned} \tag{43}$$

We shall prove that for every even ψ

$$I_r(\psi_{\mathrm{P}})=0. \tag{44}$$

Clearly, for any real ψ, it holds (see (43))

$$I_r(\psi_{\mathrm{P}}) \in i\mathbf{R}. \tag{45}$$

However, from Lemma 2, we conclude that for any even ψ, it holds

$$\begin{aligned} I_r(\psi_{\mathrm{P}}) &= \int_{\mathbf{R}^+\times\mathbf{R}^d} \varphi(\mathbf{x}) u_{1r}(t,\mathbf{x}) \partial_t \overline{(\mathcal{A}_{(1-\theta)\psi_{\mathrm{P}}/\rho_{\mathrm{P}}}(\varphi u_{1r}))(t,\mathbf{x})} dt d\mathbf{x} \\ &= \int_{\mathbf{R}^+\times\mathbf{R}^d} \varphi(\mathbf{x}) u_{1r}(t,\mathbf{x}) \partial_t (\mathcal{A}_{(1-\theta)\psi_{\mathrm{P}}/\rho_{\mathrm{P}}}(\varphi u_{1r}))(t,\mathbf{x}) dt d\mathbf{x} \in \mathbf{R}. \end{aligned}$$

Being both purely real for any even ψ and purely imaginary for any ψ (see (45)), it follows that $I_r(\psi_{\mathrm{P}})$ must be zero for any even ψ. From here, (44) follows.

Now, since the function $\varphi \in \mathrm{C}^2_c(\mathbf{R}^{d+1})$ is arbitrary, from (42) we get the following relation for every (Lebesgue) point $(t,\mathbf{x}) \in D$ and even $\psi_2 \in E$:

$$\Big\langle 4\pi^2 \sum_{k,l=1}^{d} \xi_k\xi_l a_{kl}(t,\mathbf{x})\psi_2, \mu_{12}(t,\mathbf{x},\cdot)\Big\rangle = 0. \tag{46}$$

Since the set D is of full measure, summing the results from (41) and (46), we conclude that for any odd symbol $\psi_1 \in E$ and even symbol $\psi_2 \in E$, we have

$$\langle 2\pi\xi_0\psi_1\varphi, \mu_{12}\rangle + \Big\langle 4\pi^2 \sum_{k,l=1}^{d} \xi_k\xi_l a_{kl}(t,\mathbf{x})\psi_2\varphi, \mu_{12}\Big\rangle = 0.$$

Thus, by taking $\psi_1 = \xi_0\psi$ and $\psi_2 = \psi$ for an even symbol $\psi \in E$, we conclude:

$$\Big\langle \Big(2\pi\xi_0^2 + 4\pi^2 \sum_{k,l=1}^{d} \xi_k\xi_l a_{kl}(t,\mathbf{x})\Big)\psi\varphi, \mu_{12}\Big\rangle = 0. \tag{47}$$

Since μ_{12} is continuous on $\mathrm{L}^s(\mathbf{R}^{d+1}; C^{d+1}(\mathrm{P}))$, we conclude that (47) holds for any even $\psi \in \mathrm{C}^{d+1}(\mathrm{P})$.

Since the function

$$f(t,\mathbf{x},\boldsymbol{\xi}) = \frac{\varphi}{2\pi\xi_0^2 + 4\pi^2 \sum\limits_{k,l=1}^{d} \xi_k\xi_l a_{kl}} \in \mathrm{L}^s(\mathbf{R}^+\times\mathbf{R}^d; \mathrm{C}^{d+1}(\mathrm{P}))$$

is even with respect to the variable $\boldsymbol{\xi}$, we conclude from (47) (we can put f instead $\varphi\psi$ there) that

$$\langle 1\otimes\varphi, \mu_{12}\rangle = 0. \tag{48}$$

From (35) and (48), we conclude that the following bilinear form

$$q(\mathbf{x};\boldsymbol{\lambda},\boldsymbol{\eta}) = \lambda_1\eta_2, \quad \boldsymbol{\lambda} = (\lambda_1,\lambda_2),\ \boldsymbol{\eta} = (\eta_1,\eta_2),$$

satisfies the strong consistency condition with the set $\Lambda_{\mathcal{D}}$ introduced in (34). Now we can apply Theorem 5 to conclude that

$$q(\mathbf{x};(u_{1r},u_{2r}),(u_{2r},u_{2r})) = u_{1r}u_{2r} \rightharpoonup 0 = q(\mathbf{x};(0,0),(0,0)) \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^+\times\mathbf{R}^d) \tag{49}$$

since both $u_{1r} = u_r - u$ and $u_{2r} = g(\cdot, u_r) - w$ weakly converge to 0. Using the bilinearity of q, we conclude

$$u_r g(\cdot, u_r) \rightharpoonup uw \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^+ \times \mathbf{R}^d). \tag{50}$$

Our next step is to identify $g(\cdot, u)$ as a weak limit of $g(\cdot, u_r)$. To do that we will employ the theory of Young measures. Up to this moment we didn't need any assumption on the function g itself, only on the sequence $g(\cdot, u_r)$.
Denote by $\eta_{t,\mathbf{x}}$ the Young measure associated to a subsequence of the sequence (u_r). Since g is a Carathèodory function, from (32) and (50), it holds [50]:

$$\begin{cases} u(t,\mathbf{x}) = \int \lambda d\eta_{t,\mathbf{x}}(\lambda), \\ w(t,\mathbf{x}) = \int g(t,\mathbf{x},\lambda) d\eta_{t,\mathbf{x}}(\lambda), \end{cases} \tag{51}$$

and

$$u(t,\mathbf{x}) \int g(t,\mathbf{x},\lambda) d\eta_{t,\mathbf{x}}(\lambda) = u(t,\mathbf{x}) w(t,\mathbf{x}) = \int \lambda g(t,\mathbf{x},\lambda) d\eta_{t,\mathbf{x}}(\lambda).$$

The latter equality implies

$$\begin{aligned} &\int (\lambda - u(t,\mathbf{x})) g(t,\mathbf{x},\lambda) d\eta_{t,\mathbf{x}}(\lambda) = \\ &\qquad \int \Big(\lambda - u(t,\mathbf{x})\Big)\Big(g(t,\mathbf{x},\lambda) - g(t,\mathbf{x},u(t,\mathbf{x}))\Big) d\eta_{t,\mathbf{x}}(\lambda) = 0, \end{aligned} \tag{52}$$

because

$$\begin{aligned} \int (\lambda - u) g(t,\mathbf{x},u) d\eta_{t,\mathbf{x}}(\lambda) &= g(t,\mathbf{x},u) \int \lambda d\eta_{t,\mathbf{x}}(\lambda) - g(t,\mathbf{x},u) u \int d\eta_{t,\mathbf{x}}(\lambda) \\ &= g(t,\mathbf{x},u)u - g(t,\mathbf{x},u)u \\ &= 0, \end{aligned}$$

where function u does not depend on λ and we have used first equality in (51) and the fact that $\eta_{t,\mathbf{x}}$ is a probability measure.

Since g is non-decreasing with respect to λ, we conclude from (52)

$$g(t,\mathbf{x},\lambda) = g(t,\mathbf{x},u(t,\mathbf{x})) \quad \text{on} \quad \operatorname{supp} \eta_{t,\mathbf{x}},$$

which implies

$$w(t,\mathbf{x}) = \int g(t,\mathbf{x},\lambda) d\eta_{t,\mathbf{x}}(t,\mathbf{x}) = g(t,\mathbf{x},u(t,\mathbf{x})).$$

From here, we finally conclude that

$$L(u_r) \rightharpoonup L(u) = f \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^+ \times \mathbf{R}^d).$$

Q.E.D.

Appendix

In this chapter we review the notion of distributions on manifolds and recall classical results of compactness by compensation theory, namely the div-rot lemma and the Quadratic theorem.

In the Euclidean case $\mathbf{R}^d$, one of the conceptually simplest examples of distributions are locally integrable functions. Namely, for $f \in \mathrm{L}^1_{loc}(\mathbf{R}^d)$, one defines its associated distribution T_f on $\mathcal{D}(\mathbf{R}^d)$ by the following mapping rule:

$$\varphi \mapsto \int_{\mathbf{R}^d} f(\mathbf{x})\varphi(\mathbf{x})\, d\mathbf{x}\,,$$

i.e. we identify f with $f d\mathbf{x}$. In the case of a manifold X, there is no invariant way of integrating the product $f\varphi$ in order to identify f with a linear form T_f. This is the place where currents come into play. In this chapter, we will shortly describe notions needed to define distributions on manifolds as currents.

An alternative approach would be to define distributions on manifolds simply as distributions in the local coordinates which behave in a prescribed way with the change of coordinates. For that one still needs the theory of densities, a special case of currents of de Rham. For this alternative approach and the connection between the one we will present here, see [27, Chapter VI] and books [55, Chapter IX] and [25, Chapter 3].

Compactness by compensation theory has been applied to many problems of calculus of variations, homogenisation theory, fluid mechanics, nonlinear elasticity and conservation laws. The first and most well-known result of the theory is the the div-rot lemma, shown by Tartar and Murat. Later, Tartar generalised the div-rot lemma to a setting that includes general differential operators with constant coefficients. This is the result known as the Quadratic theorem. A variant with variable continuous coefficients was given in [60] with the help of H-measures. A further generalisation to the case of discontinuous coefficients was shown by Panov in [49]. For further historical remarks and applications, please see [61, 44, 59].

To end the introduction, let us remark that there are two essentially different approaches to prove the div-rot lemma. One uses harmonic analysis and it was the approach Murat and Tartar used, while the second one uses Hodge decomposition theory and it was used by Robin, Rogers and Temple (see [54, 41, 61, 32, 33]). For a short overview of compactness by compensation theory, the reader is referred to [62].

1. Distributions on manifolds

Material of this section will follow the exposition and use the notation of Volume 3 of [15]. In the following numbers in the square brackets will denote the corresponding chapter, section and subsection in [15].

[16.1.3] A *differentiable manifold* is a separable metrizable topological space X on which is given an equivalence class of atlases (or equivalently, a saturated atlas).

[16.12.1] A *differential fibration* is a triple $\lambda = (X, B, \pi)$ in which X and B are differential manifolds and π is a C^∞-mapping of X into B which is surjective and satisfies the following condition of local triviality:

(LT) for each $b \in B$ there exists an open neighbourhood $U \in B$ of b, a differential manifold F and a diffeomorphism $\varphi : U \times F \to \pi^{-1}(U)$ such that $\pi(\varphi(y,t)) = y$ for all $y \in U$ and $t \in F$.

It is easy to see that π is a submersion. X is called the space of fibration λ, manifold B its base, and the mapping π its projection. For $b \in B$, the preimage $X_b = \pi^{-1}(b)$ is called the *fiber* of λ over b. Sometimes, we will say that X is a differential fiber bundle with base B and projection π, and that the submanifold $X_{\pi(x)}$ is the fiber through the point $x \in X$.

If f is a mapping of a set E into B, the mapping $f' : E \to X$ is called a *lifting* of f if $\pi(f'(z)) = f(z)$ for all $z \in E$.

[16.12.6] A *section* of a fibration (X, B, π) (or a section of the differential fiber bundle X) is any mapping $s : B \to X$ (not necessarily continuous) such that $\pi \circ s = 1_B$ (in other words, s is a lifting of 1_B).

[16.15.1] Let (E, B, π) be a differential fibration such that for each $b \in B$ the fiber $E_b = \pi^{-1}(b)$ is endowed with the structure of a finite-dimensional complex (resp. real) vector space. Then E, endowed with the structure defined by the fibration (E, B, π) and the vector space structures on the fibers E_b, is said to be a complex (resp. real) *vector bundle* if the following condition holds:

(VB) for each $b \in B$, there exists an open neighbourhood $U \subset B$ of b, a finite-dimensional complex (resp. real) vector space F, and a diffeomorphism $\varphi : U \times F \to \pi^{-1}(U)$ such that $\pi(\varphi(y, t)) = y$ for all $y \in U$ and $t \in F$, and such that for each $y \in U$ the partial mapping $\varphi(y, \cdot)$ is $\mathbf{C}$-linear (resp. $\mathbf{R}$-linear) bijection of the vector space F onto the vector space E_y.

Condition (VB) is equivalent to the following:

(VB') for each $b \in B$, there exists an open neighbourhood $U \subset B$ of b, an integer $n = n(b)$ and n mappings $s_i : U \to E$ of class C^∞ such that $\pi \circ s_i = 1_U$ for each i and such that the mapping $\varphi : (y, \xi^1, \cdots, \xi^n) \mapsto \xi^1 s_1(y) + \cdots + \xi^n s_n(y)$ is a diffeomorphism of $U \times \mathbf{C}^n$ (resp. $U \times \mathbf{R}^n$) onto $\pi^{-1}(U)$.

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[d, "\pi"'] & U \times F \arrow[l, "\varphi"'] \arrow[d, "\pi_1"'] \\
U \arrow[r, "1_U"] & U
\end{tikzcd}$$

The dimension of the vector space E_b over $\mathbf{C}$ (resp. over $\mathbf{R}$) is called the *rank of E at b* and denoted by $\mathrm{rk}_b(E)$. Condition (VB) implies that $\mathrm{rk}_b(E)$ is constant on each connected component of B. When $\mathrm{rk}_b(E)$ is constant its value is called the *rank of E*. A vector bundle E of rank 1 is called a *line-bundle*.

[16.15.1] Now, for differential fibration (E, B, π), assume that $\dim F = \dim \pi^{-1}(x) = \bar{d}$ for every x and that (U, φ) satisfies condition (VB). Denote by $\rho : F \to \mathbf{C}^{\bar{d}}$ (resp. $\rho : F \to \mathbf{R}^{\bar{d}}$) an isomorphism. For $V \subset U$, let (V, ψ) be a chart on B. Then $(\pi^{-1}(V), (\psi \times \rho) \circ \varphi^{-1})$ is a chart on E at each point of the fiber $\pi^{-1}(x)$. This chart on E is called a *fibered chart*.

$$E \supset \pi^{-1}(V) \quad \ni \quad z \overset{(\psi\times\rho)\circ\varphi^{-1}}{\longmapsto} \Big(\psi(\pi(z)),\, \rho\left(\pi_2(\varphi^{-1}(z))\right)\Big) \quad \in \quad \psi(V) \times \mathbf{C}^{\bar{d}},$$

$$\pi^{-1}(V) \xleftarrow{\ \varphi\ } V \times F \xrightarrow{\psi\times\rho} \psi(V) \times \mathbf{C}^{\bar{d}}\ .$$

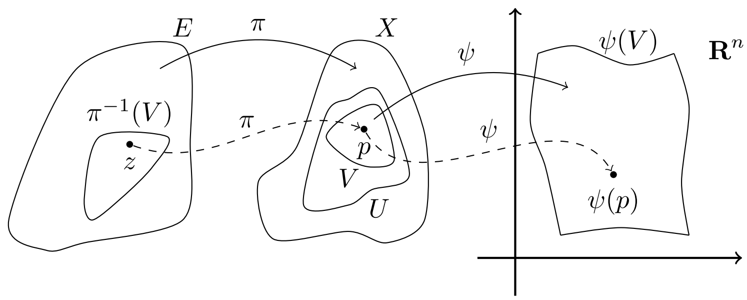

Figure 1. Diagram of the mappings from [16.15.1].

[17.2] For E complex (resp. real) vector bundle with base B and projection π, we denote by $\mathcal{E}(B;\mathbf{C})$ (resp. $\mathcal{E}(B;\mathbf{R})$) a set of C^∞ mappings of B into $\mathbf{C}$ (resp. $\mathbf{R}$), and by $\Gamma(B,E)$ (or just $\Gamma(E)$) a set of all C^∞-sections of E. In case where U is open in B, we could consider the vector bundle induced on $\pi^{-1}(U)$ and analogously define the set $\Gamma(U,E)$ of C^∞-sections of E over U. For $r \in \mathbf{N}_0$, by $\Gamma^r(U,E)$ we denote the vector space of C^r-sections of E over U. For notational conveniences, set $\Gamma^\infty(U,E) = \Gamma(U,E)$.

[16.15.4] For a differential manifold M, let $T(M)$ be the union of all the pairwise disjoint tangent spaces $T_x(M)$, $x \in M$. Denote by $o_M : T(M) \to M$ the mapping such that for $h_x \in T_x(M)$, $o_M(h_x) = x$. There exists a unique structure of differential manifold on $T(M)$ such that $(T(M), M, o_M)$ is a fibration (the tangent spaces $T_x(M)$ being the fibers) and such that the following condition holds:

(TB) for each chart $c = (U, \varphi, n)$ on M, the mapping $\phi_c : (x,h) \mapsto (d_x\varphi)^{-1}h$ of $U \times \mathbf{R}^n$ onto $o_M^{-1}(U)$ is a diffeomorphism. Here $d_x\varphi : T_x(M) \to \mathbf{R}^n$ is a differential of φ at the point x.

The vector space structures of $T_x(M)$ and the fibration $(T(M), M, o_M)$ define on $T(M)$ a structure of the real vector bundle called the *tangent bundle* of the differential manifold M. The dual $T(M)^*$ of the tangent bundle $T(M)$ is called the *cotangent bundle*.

[17.2] Let X be a pure manifold of dimension n and let (E, X, π) be a complex vector bundle of rank N over X. For U an open subset of X, we can endow the space $\Gamma^r(U,E)$, $r \in \mathbf{N}_0 \cup \{\infty\}$, with the structure of a Hausdorff locally convex topological space. Indeed, let $(V_\alpha, \varphi_\alpha, n)$ be at most denumerable family of charts of U such that V_α form a locally finite open covering of U, and such that E is trivializable over each V_α. For each α, let $z \mapsto (\varphi_\alpha(\pi(z)), v_{1\alpha}(z), \cdots, v_{N\alpha}(z))$ be a diffeomorphism of $\pi^{-1}(V_\alpha)$ onto $\varphi(V_\alpha) \times \mathbf{C}^N$, the $v_{j\alpha}$ being linear on each fiber $\pi^{-1}(x)$. Let $p'_{s,m,\alpha}$ be a family of seminorms on the standard space $\mathcal{E}^{(r)}(\varphi_\alpha(V_\alpha)) = C^r(\varphi_\alpha(V_\alpha))$. For each section $u \in \Gamma^r(U,E)$, define

$$p_{s,m,\alpha}(u) = \sum_{j=1}^{N} p'_{s,m,\alpha}(v_{j\alpha} \circ u|_{V_\alpha} \circ \varphi_\alpha^{-1}) .$$

It can be checked that $p_{s,m,\alpha}$ are seminorms which distinguish points and that the following property holds:

(SN) a sequence (u_k) of section of $\Gamma^r(U,E)$ converges to zero if and only if, for each chart (V, φ, n) of X over which E is trivialisable, each diffeomorphism

$$z \mapsto (\varphi(\pi(z)), v_1(z), \cdots, v_N(z))$$

of $\pi^{-1}(V)$ onto $\varphi(V)\times\mathbf{C}^N$, where the v_j are linear on each fiber $\pi^{-1}(x)$, each compact $K\subset\varphi(V)$ and each multi-index $\boldsymbol{\nu}$ such that $|\boldsymbol{\nu}|\le r$, the sequence $((D^{\boldsymbol{\nu}}w_{jk})|_K))_k$ converges uniformly to zero for $j\in 1..N$, where $w_{jk}(t)=v_j(u_k(\varphi^{-1}(t)))$ for $t\in\varphi(V)$.

It is equivalent to:

(SN') a sequence (u_k) of section of $\Gamma^r(U,E)$ converges to zero if and only if, for each α, the sequence of restrictions $(u_k|_{V_\alpha})$ converges to zero in $\Gamma^r(V_\alpha,E)$; and $\Gamma^r(V_\alpha,E)$ is isomorphic to $\left(\mathcal{E}^{(r)}(\varphi_\alpha(V_\alpha))\right)^N$.

[17.2.2] The spaces $\Gamma^r(U,E)$, for $r\in\mathbf{N}_0\cup\{\infty\}$, are separable Fréchet spaces. In the case when $X=\mathbf{R}^n$ and $E=X\times\mathbf{C}$ is the trivial complex line bundle over X, $\Gamma(U,E)$ is the standard space $\mathcal{E}(U)=\mathrm{C}^\infty(U)$, while $\Gamma^r(U,E)=\mathrm{C}^r(U)$.

[17.3.1] For $p\in\mathbf{N}$, a section over A of the complex bundle $\left(\overset{p}{\bigwedge}T(X)^*\right)_{(\mathbf{C})}$ of tangent p-covectors is called a complex-valued *differential p-form* on A. For $p=0$ we define these to be complex-valued functions. The set $\Gamma^r\left(X,(\overset{p}{\bigwedge}T(X)^*)_{(\mathbf{C})}\right)$, $r\in\mathbf{N}_0\cup\{\infty\}$, of differential p-forms of class C^r on X is usually denoted by $\mathcal{E}_p^{(r)}(X)$.

For a compact set $K\subset X$, denote by $\mathcal{D}_p^{(r)}(X;K)$ the vector subspace of $\mathcal{E}_p^{(r)}(X)$ consisting of the complex differential p-forms of class C^r with support contained in K, and by $\mathcal{D}_p^{(r)}(X)$ the space of all complex differential p-forms of class C^r with compact support. When $p=0$ or $r=\infty$, we drop them from notation.

A complex *p-current* (or a current of dimension p) on X is a linear form T on $\mathcal{D}_p(X)$ whose restriction to each Fréchet space $\mathcal{D}_p(X;K)$ is continuous.

For a linear form T on $\mathcal{D}_p(X)$ the following are equivalent:

i) T is a p-current,
ii) for every sequence (u_k) of C^∞ differential p-forms, with supports contained in the same compact set K and which converge to zero in $\mathcal{E}_p(X)$, the sequence $(T(u_k))$ tends to zero in $\mathbf{C}$,
iii) for each compact set K of X, there exist integers s,m and a finite number of indices $\alpha_1,\cdots,\alpha_r$, together with a constant $C_K\ge 0$, such that for each C^∞ p-form u with support contained in K, we have

$$|T(u)|\le C_K\sup_i p_{s,m,\alpha_i}(u)\,.$$

[17.3.2] A p-current T is said to be of order smaller or equal to r if the restriction of T to $\mathcal{D}_p(X;K)$ is continuous with respect to the topology induced by that of $\mathcal{D}_p^{(r)}(X;K)$, for every compact $K\subset X$. The order of a current is the smallest integer r with such property.

A 0-current on X is called a *distribution on X*.

[17.6.1] A vector bundle $\overset{p}{\bigwedge}T(X)^*$ can be identified with a subbundle of the real vector bundle $\left(\overset{p}{\bigwedge}T(X)^*\right)_{(\mathbf{C})}$ which is the direct sum $\overset{p}{\bigwedge}T(X)^*\oplus i\overset{p}{\bigwedge}T(X)^*$. Then $\mathcal{E}_p^{(r)}(X)$ is equal to $\mathcal{E}_{p,\mathbf{R}}^{(r)}(X)+i\mathcal{E}_{p,\mathbf{R}}^{(r)}(X)$, where $\mathcal{E}_p^{(r)}(X)$ is the space of real differential p-forms of class C^r on X.

A p-current on X is said to be real if its restriction to $\mathcal{E}_{p,\mathbf{R}}(X)$ is real-valued.

[17.4.4] Let X and X' be two pure manifolds of the same dimension n, and let $\pi : X' \to X$ be a local diffeomorphism. Then for each current on T on X there exists a unique current T' on X' with the following property: for each open subset $U' \subset X'$ such that the restriction $\pi_{U'} : U' \to \pi(U')$ is a diffeomorphism, we have $\pi_{U'}(T'_{U'}) = T_{\pi(U')}$, where by T_U we denote restriction of a current T to a set U. The current T' is called the inverse image of T by π.

[17.5.1] Let β be a locally integrable differential $(n-p)$-form on X, $0 \le p \le n$. For each p-form $\alpha \in \mathcal{D}_p^{(0)}(X)$ with compact support, the linear form $\alpha \mapsto \int \beta \wedge \alpha$ is a p-current (of order 0). We denote it by T_β. Restricting the mapping $\beta \mapsto T_\beta$ to the space $\mathcal{E}_{n-p}^{(0)}(X)$ of continuous differential $(n-p)$-forms, it becomes injective. This allows us to identify the continuous differential $(n-p)$-form β with the p-current T_β (of order 0).

Let us consider two examples:

1) If ν is locally integrable n-form, then the mapping $f \mapsto \int f\nu$ is a distribution (0-current) on X denoted by T_ν (it is positive if and only if $\nu(\mathbf{x}) \ge 0$ almost everywhere with respect to the orientation of X).
2) If f is locally integrable complex function on X, then the mapping $\nu \mapsto \int f\nu$ is an n-current on X (of order 0) denoted by T_f.

In the end, we would like to comment how we identify functions with currents in the case when X is an oriented differential pure manifold of dimension n (that is, to say a bit more about the second example above).

[17.5.3] First, let us fix a smooth differential n-form ν_0 which belongs to the orientation of X. Every differential n-form can be uniquely written in the form $f\nu_0$, where f is a complex-valued function on X. The form $f\nu_0$ is locally integrable if and only if function f is locally integrable. The linear mapping $f \mapsto f\nu_0$ is a bijection of locally integrable complex-valued functions on X onto the space of locally integrable differential n-forms on X. We normally write T_f instead of $T_{f\nu_0}$ and identify the function f with the corresponding distribution T_f (that is $T_{f\nu_0}$). To illustrate in the case $\mathbf{R}^d$ with the canonical orientation, we could take ν_0 to be the canonical n-form $d\xi^1 \wedge d\xi^2 \wedge \cdots \wedge d\xi^n$.

Furthermore, on orientable manifolds, by fixing ν_0, we can identify n-currents with distributions since $g \mapsto g\nu_0$ is an isomorphism of the spaces $\mathcal{D}_0(X)$ and $\mathcal{D}_n(X)$. Every n-current can be expressed as $g\nu_0 \mapsto T(g)$, where T is a distribution. We denote this n-current by $T_{|\nu_0}$. With this notation, for every locally integrable complex-valued function f on X, we have $(T_f)_{|\nu_0} = T_{f\nu_0}$.

2. Compactness by compensation

The material of this section heavily follows the exposition in [41, 9].

For $\Omega \subset \mathbf{R}^3$ open set, let us introduce the following function spaces:

$$\mathrm{L}^2_{\mathsf{div}}(\Omega) = \{\mathbf{v} \in \mathrm{L}^2(\Omega; \mathbf{C}^3) : \mathsf{div}\,\mathbf{v} \in \mathrm{L}^2(\Omega)\},$$

$$\mathrm{L}^2_{\mathsf{rot}}(\Omega) = \{\mathbf{v} \in \mathrm{L}^2(\Omega; \mathbf{C}^3) : \mathsf{rot}\,\mathbf{v} \in \mathrm{L}^2(\Omega; \mathbf{C}^3)\}.$$

These spaces are Banach spaces when equipped with the corresponding graph norms.

Lemma 1. *Let Ω be an open subset of $\mathbf{R}^3$ and assume:*

$$\mathsf{E}^n \rightharpoonup \mathsf{E} \quad \textit{in} \quad \mathrm{L}^2(\Omega; \mathbf{R}^3),$$

$$\mathsf{D}^n \rightharpoonup \mathsf{D} \quad \textit{in} \quad \mathrm{L}^2(\Omega;\mathbf{R}^3),$$
$$\operatorname{div}\mathsf{D}^n \quad \textit{bounded in} \quad \mathrm{L}^2(\Omega),$$
$$\operatorname{rot}\mathsf{E}^n \quad \textit{bounded in} \quad \mathrm{L}^2(\Omega;\mathbf{R}^3).$$

Then

$$\mathsf{D}^n\cdot\mathsf{E}^n \rightharpoonup \mathsf{D}\cdot\mathsf{E}$$

in the sense of distributions.

Dem. Multiplying by $\phi\in\mathcal{D}(\Omega)$ and $\psi\in\mathcal{D}(\Omega)$ such that $\psi=1$ on $\operatorname{supp}\phi$, and extending by zero to the whole $\mathbf{R}^3$, we have the following functions:

$$\mathsf{e}_n=\phi\mathsf{E}^n,\quad \mathsf{d}_n=\psi\mathsf{D}^n,$$

$$\mathsf{e}=\phi\mathsf{E},\quad \mathsf{d}=\psi\mathsf{D}.$$

From the conditions of the lemma, we have that (e_n) is bounded in $\mathrm{L}^2_{\mathsf{rot}}(\mathbf{R}^3)$, while (d_n) is bounded in $\mathrm{L}^2_{\mathsf{div}}(\mathbf{R}^3)$, and

$$\mathsf{e}_n\rightharpoonup\mathsf{e},\quad \mathsf{d}_n\rightharpoonup\mathsf{d}\quad \text{in}\quad \mathrm{L}^2(\mathbf{R}^3;\mathbf{R}^3).$$

Since supports are compact, we have the following two equalities:

$$\int_\Omega \mathsf{D}^n\cdot\mathsf{E}^n\phi\, d\mathbf{x}=\int_{\mathbf{R}^3}\mathsf{d}_n\cdot\mathsf{e}_n\, d\mathbf{x}, \tag{53}$$

$$\int_\Omega \mathsf{D}\cdot\mathsf{E}\phi\, d\mathbf{x}=\int_{\mathbf{R}^3}\mathsf{d}\cdot\mathsf{e}\, d\mathbf{x}. \tag{54}$$

Let us apply the Fourier transform to d_n: since d_n weakly converges to D in $\mathrm{L}^2(\mathbf{R}^3;\mathbf{R}^3)$ we obtain boundedness of the sequence $(\hat{\mathsf{d}}_n)$ in $\mathrm{L}^2(\mathbf{R}^3)$, and convergence $\hat{\mathsf{d}}_n\rightharpoonup\hat{\mathsf{d}}$ in the space of tempered distributions $\mathcal{S}'$. This gives us uniqueness of the accumulation point: $\hat{\mathsf{d}}_n\rightharpoonup\hat{\mathsf{d}}$ in L^2. Since (d_n) is bounded in $\mathrm{L}^2_{\mathsf{div}}(\mathbf{R}^3)$, after applying the Fourier transform, we obtain boundedness of the sequence $(\boldsymbol{\xi}\cdot\hat{\mathsf{d}}_n)$ in L^2.

Completely analogously, we obtain convergence $\hat{\mathsf{e}}_n\rightharpoonup\mathsf{e}$ in $\mathrm{L}^2(\mathbf{R}^3)$, and boundedness of sequence $(\boldsymbol{\xi}\times\hat{\mathsf{e}}_n)$ in L^2. Similarly, on the limit we get that $\boldsymbol{\xi}\cdot\hat{\mathsf{d}}$ and $\boldsymbol{\xi}\times\hat{\mathsf{e}}$ are bounded in L^2.

Let us decompose $\hat{\mathsf{d}}$ and $\hat{\mathsf{e}}$ on two components: one in the direction of $\boldsymbol{\xi}$: $\hat{\mathsf{d}}_\top,\hat{\mathsf{e}}_\top$ and the other perpendicular to $\boldsymbol{\xi}$: $\hat{\mathsf{d}}_\perp,\hat{\mathsf{e}}_\perp$

$$\hat{\mathsf{d}}=\hat{\mathsf{d}}_\top+\hat{\mathsf{d}}_\perp,\quad \hat{\mathsf{e}}=\hat{\mathsf{e}}_\top+\hat{\mathsf{e}}_\perp.$$

Our previous considerations imply that $|\boldsymbol{\xi}|\hat{\mathsf{d}}_\top$ and $|\boldsymbol{\xi}|\hat{\mathsf{e}}_\perp$ are bounded in L^2, and since $\|\hat{\mathsf{d}}_\top\|_{\mathrm{L}^2}\le\|\hat{\mathsf{d}}\|_{\mathrm{L}^2}$ and $\|\hat{\mathsf{e}}_\perp\|_{\mathrm{L}^2}\le\|\hat{\mathsf{e}}\|_{\mathrm{L}^2}$, after applying Cauchy-Schwarz inequality, we get:

$$|\boldsymbol{\xi}|\ \hat{\mathsf{d}}_\top\cdot\hat{\mathsf{e}}_\top\in\mathrm{L}^1,$$

$$|\boldsymbol{\xi}|\ \hat{\mathsf{d}}_\perp\cdot\hat{\mathsf{e}}_\perp\in\mathrm{L}^1,$$

which implies $|\boldsymbol{\xi}|\ \hat{\mathsf{d}}\cdot\hat{\mathsf{e}}\in\mathrm{L}^1$ after summing up.

Similarly, because L^2-bounds are uniform in n, we boundedness of $(|\boldsymbol{\xi}|\ \hat{\mathsf{d}}_n\cdot\hat{\mathsf{e}}_n)$ in L^1.

Additionally, we have:

$$\begin{aligned}\|\hat{\mathsf{d}}_n\|_{\mathrm{L}^\infty(\mathbf{R}^3;\mathbf{R}^3)} &\leq \|\mathsf{d}_n\|_{\mathrm{L}^1(\mathbf{R}^3;\mathbf{R}^3)} = \|\mathsf{d}_n\|_{\mathrm{L}^1(\omega;\mathbf{R}^3)}\\ &\leq c_1\|\mathsf{d}_n\|_{\mathrm{L}^2(\omega;\mathbf{R}^3)} \leq c_1\|\mathsf{d}_n\|_{\mathrm{L}^2(\omega;\mathbf{R}^3)} \leq C,\end{aligned}$$

where $\omega = \operatorname{supp}\psi$, and the last inequality follows from the weak L^2-convergence of sequence (d_n). The same weak L^2-convergence of (d_n) gives:

$$\hat{\mathsf{d}}_n(\boldsymbol{\xi}) = \int_\omega e^{-2\pi i x\cdot\boldsymbol{\xi}}\mathsf{d}_n(\mathbf{x})\,d\mathbf{x} \to \int_\omega e^{-2\pi i x\cdot\boldsymbol{\xi}}\mathsf{d}(\mathbf{x})\,d\mathbf{x} = \hat{\mathsf{d}}(\boldsymbol{\xi}).$$

In the same manner we get $\|\hat{\mathsf{e}}_n\|_{\mathrm{L}^\infty(\mathbf{R}^3;\mathbf{R}^3)} \leq C$ and $\hat{\mathsf{e}}_n(\boldsymbol{\xi}) \to \hat{\mathsf{e}}(\boldsymbol{\xi})$. An application of the Lebesgue theorem on dominated convergence, for every bounded set $B \subseteq \mathbf{R}^3$, gives:

$$\hat{\mathsf{d}}_n\cdot\hat{\mathsf{e}}_n \to \hat{\mathsf{d}}\cdot\hat{\mathsf{e}} \text{ strongly u } \mathrm{L}^1(B).$$

Since $|\boldsymbol{\xi}|\ \hat{\mathsf{d}}_n\cdot\hat{\mathsf{e}}_n$ and $|\boldsymbol{\xi}|\ \hat{\mathsf{d}}\cdot\mathsf{e}$ are bounded, we get:

$$\hat{\mathsf{d}}_n\cdot\mathsf{e}_n \to \hat{\mathsf{d}}\cdot\hat{\mathsf{e}} \text{ strongly u } \mathrm{L}^1(\mathbf{R}^3),$$

that is,

$$\int_{\mathbf{R}^3}\hat{\mathsf{d}}_n\cdot\hat{\mathsf{e}}_n\,\mathsf{d}\boldsymbol{\xi} \to \int_{\mathbf{R}^3}\hat{\mathsf{d}}\cdot\hat{\mathsf{e}}\,d\boldsymbol{\xi},$$

from where, by the Plancherel theorem, we have

$$\int_{\mathbf{R}^3}\mathsf{d}_n\cdot\mathsf{e}_n\,d\mathbf{x} \to \int_{\mathbf{R}^3}\mathsf{d}\cdot\mathsf{e}\,d\mathbf{x}.$$

The claim of the lemma now follows from (53) and (54).

Q.E.D.

Let $\Omega \subset \mathbf{R}^d$ be open. For a sequence of functions (u_n) satisfying the following constraint:

$$\left(\sum_{k=1}^d \mathbf{A}^k\partial_k\mathsf{u}^n\right) \qquad \text{bounded in} \quad \mathrm{L}^2_{loc}(\Omega;\mathbf{R}^q)\,,$$

where $\mathbf{A}^k \in M_{q,r}(\mathbf{R})$ are constant real matrices, we define the following sets:

$$\mathcal{V} = \left\{(\boldsymbol{\lambda},\boldsymbol{\xi}) \in \mathbf{R}^r\times\mathbf{R}^d : \sum_{k=1}^d \xi_k\mathbf{A}^k\boldsymbol{\lambda} = \mathsf{0}\right\},$$

$$\Lambda = \left\{\boldsymbol{\lambda}\in\mathbf{R}^r : (\exists\boldsymbol{\xi}\in\mathbf{R}^d\setminus\{\mathsf{0}\})\quad (\boldsymbol{\lambda},\boldsymbol{\xi})\in\mathcal{V}\right\}.$$

Let us look at the example where for $\mathbf{v}^n, \mathsf{w}^n \in \mathrm{L}^\infty(\Omega;\mathbf{R}^d)$ it holds:

$$\mathbf{v}^n \rightharpoonup \mathbf{v} \quad \text{weakly}-* \text{ in} \quad \mathrm{L}^\infty(\Omega;\mathbf{R}^d)\,,$$

$$\mathsf{w}^n \rightharpoonup \mathsf{w} \quad \text{weakly}-* \text{ in} \quad \mathrm{L}^\infty(\Omega;\mathbf{R}^d)\,,$$

$$(\operatorname{div}\mathbf{v}^n) \quad \text{is bounded in} \quad \mathrm{L}^2_{loc}(\Omega)\,,$$

$$(\mathsf{rot}\,\mathsf{w}^n) \quad \text{is bounded in} \quad \mathrm{L}^2_{loc}(\Omega;\mathbf{R}^d)\,.$$

Conditions on the derivatives imply that the set $\mathcal{V}$ has the following form:

$$\mathcal{V} = \Big\{((\boldsymbol{\lambda},\boldsymbol{\mu}),\boldsymbol{\xi}) \in \mathbf{R}^{2d}\times\mathbf{R}^d : \boldsymbol{\lambda}\cdot\boldsymbol{\xi} = 0 \quad \& \quad \boldsymbol{\mu}\otimes\boldsymbol{\xi} - \boldsymbol{\xi}\otimes\boldsymbol{\mu} = \mathsf{0}\Big\}\,,$$

which under convention that nul-vector is parallel to every vector, reads

$$\mathcal{V} = \Big\{((\boldsymbol{\lambda},\boldsymbol{\mu}),\boldsymbol{\xi}) \in \mathbf{R}^{2d}\times\mathbf{R}^d : \boldsymbol{\lambda}\perp\boldsymbol{\xi} \quad \& \quad \boldsymbol{\mu}\parallel\boldsymbol{\xi}\Big\}\,.$$

This gives the set Λ:

$$\Lambda = \Big\{(\boldsymbol{\lambda},\boldsymbol{\mu}) \in \mathbf{R}^d\times\mathbf{R}^d : \boldsymbol{\lambda}\cdot\boldsymbol{\mu} = 0\Big\}\,.$$

Before showing Tartar's Quadratic theorem, let us show the following lemma:

Lemma 2. *Let set Λ be given:*

$$\Lambda := \Big\{\boldsymbol{\lambda}\in\mathbf{R}^r \;:\; (\exists\boldsymbol{\xi}\in\mathbf{R}^d\setminus\{\mathsf{0}\}) \quad \sum_{k=1}^d \xi_k\mathbf{A}^k\boldsymbol{\lambda} = \mathsf{0}\Big\}\,,$$

for fixed matrices $\mathbf{A}^k\in M_{q,r}(\mathbf{R})$, $k\in 1..d$, and assume $\mathbf{Q}\in M_r(\mathbf{R})$ is a constant matrix such that it holds

$$(\forall\boldsymbol{\lambda}\in\Lambda)\quad Q(\boldsymbol{\lambda}) := \mathbf{Q}\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}\geqslant 0\,.$$

Denote by $\tilde{Q}$ hermitian extension of quadratic form Q to the whole $\mathbf{C}^r$ defined by

$$\tilde{Q}(\boldsymbol{\lambda}) := \mathbf{Q}\boldsymbol{\lambda}\cdot\bar{\boldsymbol{\lambda}}\,.$$

It holds:

$$(\forall\varepsilon>0)(\exists C_\varepsilon\in\mathbf{R})(\forall\boldsymbol{\lambda}\in\mathbf{C}^r)(\forall\boldsymbol{\eta}\in\mathbf{R}^d)$$

$$|\boldsymbol{\eta}| = 1 \quad\Longrightarrow\quad \mathsf{Re}\,\tilde{Q}(\boldsymbol{\lambda}) \geqslant -\,\varepsilon|\boldsymbol{\lambda}|^2 - C_\varepsilon\sum_{i=1}^q\Big|\sum_{j,k} a_{ijk}\lambda_j\eta_k\Big|^2\,.$$

$\mathsf{Dem.}$ Assume the opposite, that there exists $\varepsilon>0$ such that for every $C_\varepsilon = n\in\mathbf{N}$ exist $\boldsymbol{\lambda}^n\in\mathbf{C}^r$ and $\boldsymbol{\eta}^n\in\mathbf{R}^r$, such that $|\boldsymbol{\eta}^n| = 1$ and

$$\mathsf{Re}\,\tilde{Q}(\boldsymbol{\lambda}^n) < -\varepsilon|\boldsymbol{\lambda}^n|^2 - n\sum_{i=1}^q\Big|\sum_{j,k} a_{ijk}\lambda^n_j\eta^n_k\Big|^2\,. \tag{55}$$

Scaling the above inequality, we could have chosen $\boldsymbol{\lambda}^n$ such that $|\boldsymbol{\lambda}^n| = 1$. Such sequence $(\boldsymbol{\lambda}^n)$ is relatively compact, and on the subsequence we have convergence:

$$\begin{aligned}\boldsymbol{\lambda}^n &\longrightarrow \boldsymbol{\lambda}\\ \boldsymbol{\eta}^n &\longrightarrow \boldsymbol{\eta}\,.\end{aligned}$$

Let us define a bilinear form β on $\mathbf{R}^r\times\mathbf{R}^r$ by $\beta(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2) := \mathbf{Q}\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2$. For $\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2\in\Lambda$, it holds

$$\tilde{Q}(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2 i) = Q(\boldsymbol{\lambda}_1) + Q(\boldsymbol{\lambda}_2) + (\beta(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)+\beta(\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_1))i\,.$$

And for $\boldsymbol{\lambda} = \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 i \in \Lambda + i\Lambda$, we get

$$\mathsf{Re}\,\tilde{Q}(\boldsymbol{\lambda}) = Q(\boldsymbol{\lambda}_1) + Q(\boldsymbol{\lambda}_2) \geqslant 0\,.$$

$\mathsf{Re}\,\tilde{Q}$ is a continuous function on unit sphere, thus (55) implies

$$\sum_{i=1}^{q} \Big|\sum_{j,k} a_{ijk}\lambda_j^n \eta_k^n\Big|^2 \leqslant \frac{-\varepsilon - \mathsf{Re}\,\tilde{Q}(\boldsymbol{\lambda}^n)}{n} \leqslant \frac{C}{n}\,.$$

On the limit we get

$$\sum_{i=1}^{q} \Big|\sum_{j,k} a_{ijk}\lambda_j \eta_k\Big|^2 = 0\,,$$

which in turn gives $\boldsymbol{\lambda} \in \Lambda + i\Lambda$, so it implies

$$\mathsf{Re}\,\tilde{Q}(\boldsymbol{\lambda}) \geqslant 0\,.$$

From (55) we get

$$\mathsf{Re}\,\tilde{Q}(\boldsymbol{\lambda}) = \lim_n \mathsf{Re}\,\tilde{Q}(\boldsymbol{\lambda}^n) \leqslant -\varepsilon < 0\,,$$

which is a contradiction.

Q.E.D.

Now we state and prove the Quadratic theorem:

Theorem 1. *Let $\Omega \subseteq \mathbf{R}^d$ be open, and $\Lambda \subseteq \mathbf{R}^r$ be defined as in Lemma 2. Let Q be a real quadratic form on $\mathbf{R}^r$ non-negative on Λ:*

$$(\forall\, \boldsymbol{\lambda} \in \Lambda) \quad Q(\boldsymbol{\lambda}) \geqslant 0\,.$$

Furthermore, assume that sequence (u^n) satisfies

(*P1*) $$\mathsf{u}^n \rightharpoonup \mathsf{u} \quad \textit{weakly in} \quad \mathrm{L}^2_{loc}(\Omega; \mathbf{R}^r)\,,$$

(*P2*) $$\Big(\sum_k \mathbf{A}^k \partial_k \mathsf{u}^n\Big) \quad \textit{relatively compact in} \quad \mathrm{H}^{-1}_{loc}(\Omega; \mathbf{R}^q)\,.$$

Then every subsequence of $(Q \circ \mathsf{u}^n)$ converging to its accumulation point L, satisfies

$$L \geqslant Q \circ \mathsf{u}$$

in the sense of distributions.

Dem.
Step I: Assume that (after passing to a subsequence) $Q \circ \mathsf{u}^n \rightharpoonup L$ weakly in $\mathcal{D}'(\Omega)$. From (P1) and (P2) for $\mathbf{v}^n := \mathsf{u}^n - \mathsf{u}$, we get

(56) $$\mathbf{v}^n \rightharpoonup 0 \quad \text{weakly in} \quad \mathrm{L}^2_{loc}(\Omega; \mathbf{R}^r)\,.$$

(57) $$\Big(\sum_{k=1}^{d} \mathbf{A}^k \partial_k \mathbf{v}^n\Big) \quad \text{is relatively compact in} \quad \mathrm{H}^{-1}_{loc}(\Omega; \mathbf{R}^q)\,.$$

Let $\beta : \mathbf{R}^r \times \mathbf{R}^r \longrightarrow \mathbf{R}$ be a bilinear form defined as in Lemma 2, such that for any $\mathbf{a} \in \mathbf{R}^r$, it holds

$$Q(\mathbf{a}) = \beta(\mathbf{a}, \mathbf{a})\,.$$

Now, the following

$$Q(\mathbf{v}^n) = \beta(\mathsf{u}^n - \mathsf{u}, \mathsf{u}^n - \mathsf{u}) = Q(\mathsf{u}^n) - \beta(\mathsf{u}^n, \mathsf{u}) - \beta(\mathsf{u}, \mathsf{u}^n) + Q(\mathsf{u})\,,$$

together with

$$\begin{aligned} Q(\mathsf{u}^n) &\rightharpoonup L \\ \beta(\mathsf{u}^n, \mathsf{u}) &\rightharpoonup \beta(\mathsf{u}, \mathsf{u}) = Q(\mathsf{u}) \\ \beta(\mathsf{u}, \mathsf{u}^n) &\rightharpoonup \beta(\mathsf{u}, \mathsf{u}) = Q(\mathsf{u})\,, \end{aligned}$$

in the space of distributions, gives

$$Q \circ \mathbf{v}^n \rightharpoonup L - Q \circ \mathsf{u} =: m\,.$$

The claim will follow once we show $m \geqslant 0$.

Step II: Take a real test function φ on Ω, and define functions with compact support $\mathsf{w}^n := \varphi \mathbf{v}^n$. It holds

$$\mathsf{w}^n \rightharpoonup \mathsf{0} \quad \text{weakly in} \quad \mathrm{L}^2(\Omega; \mathbf{R}^r) \tag{58}$$

and

$$\sum_k \mathbf{A}^k \partial_k \mathsf{w}^n = \varphi \sum_k \mathbf{A}^k \partial_k \mathbf{v}^n + \sum_k \partial_k \varphi\, \mathbf{A}^k \partial_k \mathbf{v}^n\,.$$

Both sequences in the above sum are relatively compact in $\mathrm{H}^{-1}(\Omega; \mathbf{R}^q)$ (the second one is bounded in $\mathrm{L}^2(\Omega; \mathbf{R}^q)$ and has a uniform support in a compact, thus it is relatively compact in $\mathrm{H}^{-1}(\Omega; \mathbf{R}^q)$). For the left hand side of equality now follows:

$$\Big(\sum_k \mathbf{A}^k \partial_k \mathsf{w}^n\Big) \quad \text{is relatively compact in} \quad \mathrm{H}^{-1}(\Omega; \mathbf{R}^q)\,,$$

and on the subsequence (not relabelled), it holds

$$\sum_k \mathbf{A}^k \partial_k \mathsf{w}^n \longrightarrow \tilde{L} \quad \text{strongly in} \quad \mathrm{H}^{-1}(\Omega; \mathbf{R}^q)\,.$$

After imbedding to the space of distributions, from (58), since derivation is continuous operator, we get

$$(\forall\, k \in 1..d) \quad \partial_k \mathsf{w}^n \rightharpoonup \mathsf{0}$$

in the space of distributions, and this implies $\tilde{L} = \mathsf{0}$, i.e.

$$\sum_k \mathbf{A}^k \partial_k \mathsf{w}^n \longrightarrow 0 \quad \text{strongly in} \quad \mathrm{H}^{-1}(\Omega; \mathbf{R}^q)\,. \tag{59}$$

Step III: Notice that the following holds in the space of distributions

$$Q \circ \mathsf{w}^n = \varphi^2 Q \circ \mathbf{v}^n \rightharpoonup \varphi^2 m,$$

i.e. for arbitrary $\psi \in \mathcal{D}(\Omega)$

$$\int_{\mathbf{R}^d} Q \circ \mathsf{w}^n \psi \longrightarrow {}_{\mathcal{D}'(\Omega)}\langle\, \varphi^2 m, \psi \,\rangle_{\mathcal{D}(\Omega)} \,.$$

Since

$${}_{\mathcal{D}'(\Omega)}\langle\, \varphi^2 m, \psi \,\rangle_{\mathcal{D}(\Omega)} = {}_{\mathcal{D}'(\Omega)}\langle\, m, \psi\varphi^2 \,\rangle_{\mathcal{D}(\Omega)} \,,$$

and

$$\mathrm{supp}\,(Q \circ \mathsf{w}^n) \subseteq \mathrm{supp}\,\varphi \,,$$

choosing ψ such that $\psi_{|\mathrm{supp}\,\varphi} = 1$, we get

$$\int_{\mathbf{R}^d} Q \circ \mathsf{w}^n \longrightarrow {}_{\mathcal{D}'(\Omega)}\langle\, m, \varphi^2 \,\rangle_{\mathcal{D}(\Omega)} \,.$$

Notice that in order to show that $m \geqslant 0$ in the sense of distributions, it is enough to show

$$\lim_n \int_{\mathbf{R}^d} Q \circ \mathsf{w}^n \geqslant 0 \,. \tag{60}$$

Step IV: Let $\tilde{Q}$ be the hermitian extension of the quadratic form Q as in Lemma 2. Extend w^n by zero outside Ω, and apply the Fourier transform to get

$$\hat{\mathsf{w}}^n(\boldsymbol{\xi}) := \int_{\mathbf{R}^d} \mathsf{w}^n(\mathbf{x}) e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} d\mathbf{x}$$

The Plancherel theorem now gives:

$$\int_{\mathbf{R}^d} Q \circ \mathsf{w}^n = \int_{\mathbf{R}^d} \tilde{Q} \circ \mathsf{w}^n = \int_{\mathbf{R}^d} \tilde{Q} \circ \hat{\mathsf{w}}^n = \int_{\mathbf{R}^d} \mathsf{Re}\,\tilde{Q} \circ \hat{\mathsf{w}}^n \,,$$

so (60) equivalent to

$$\lim_n \int_{\mathbf{R}^d} \mathsf{Re}\,\tilde{Q} \circ \hat{\mathsf{w}}^n \geqslant 0 \,. \tag{61}$$

Step V: Since $\mathrm{supp}\,\mathsf{w}^n \subseteq \mathrm{supp}\,\varphi =: K \in \mathcal{K}(\Omega)$, we can integrate only over K in the expression for the Fourier transform:

$$\hat{\mathsf{w}}^n(\boldsymbol{\xi}) := \int_K \mathsf{w}^n(\mathbf{x}) e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} d\mathbf{x} \,.$$

For fixed $\boldsymbol{\xi}$, the function $\mathbf{x} \mapsto e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}}$ belongs to $\mathrm{L}^2(K; \mathbf{C})$. This together with $\mathsf{w}^n \rightharpoonup \mathsf{0}$ u $\mathrm{L}^2(\Omega; \mathbf{R}^r)$ implies

$$\hat{\mathsf{w}}^n \longrightarrow \mathsf{0} \quad \text{(ss)} \,.$$

From boundedness of (w^n) in $\mathrm{L}^2(\Omega; \mathbf{R}^r)$ and $\mathrm{supp}\,\mathsf{w}^n \subseteq K$, we get the boundedness of the same sequence in $\mathrm{L}^1(K; \mathbf{R}^r)$. Since it holds

$$|\hat{\mathsf{w}}^n(\boldsymbol{\xi})| \leqslant \int_K |\mathsf{w}^n(\mathbf{x})| |e^{-2\pi i \boldsymbol{\xi}\cdot(\mathbf{x})}| d\mathbf{x} = \int_K |\mathsf{w}^n(\mathbf{x})| d\mathbf{x} \,,$$

we get that the sequence

$$(\hat{\mathsf{w}}^n) \quad \text{is bounded in} \quad \mathrm{L}^\infty(\mathbf{R}^d;\mathbf{C}^r)\,.$$

Lebesgue theorem on dominated convergence now implies

$$\hat{\mathsf{w}}^n \longrightarrow \mathsf{0} \quad \text{strongly in} \quad \mathrm{L}^2_{loc}(\mathbf{R}^d;\mathbf{C}^r)\,,$$

while quadratic nature of form $\tilde{Q}$ gives

$$\int_{K(\mathsf{0},R)} \mathsf{Re}\,\tilde{Q}\circ\hat{\mathsf{w}}^n \longrightarrow 0\,, \tag{62}$$

where R is positive arbitrary number.
Taking Fourier transform of (59), we get

$$\frac{2\pi i}{\sqrt{1+|\boldsymbol{\xi}|^2}}\sum_k \xi_k \mathbf{A}^k \hat{\mathsf{w}}^n \longrightarrow \mathsf{0} \quad \text{strongly in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^q)\,.$$

Inequality $\frac{1}{|\boldsymbol{\xi}|}\leqslant\frac{2}{\sqrt{1+|\boldsymbol{\xi}|^2}}$, valid for $|\boldsymbol{\xi}|\geqslant 1$, gives:

$$\frac{1}{|\boldsymbol{\xi}|}\sum_k \xi_k \mathbf{A}^k \hat{\mathsf{w}}^n \longrightarrow \mathsf{0} \quad \text{strongly u} \quad \mathrm{L}^2(S;\mathbf{C}^q)\,, \tag{63}$$

where we have denoted $S := \mathbf{R}^d \setminus K(\mathsf{0},1)$. Now, Lemma 2 with notation

$$\boldsymbol{\lambda} = \hat{\mathsf{w}}^n(\boldsymbol{\xi})\,, \quad \boldsymbol{\eta} = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\,,$$

implies

$$\mathsf{Re}\,\tilde{Q}\circ\hat{\mathsf{w}}^n(\boldsymbol{\xi}) \geqslant -\varepsilon|\hat{\mathsf{w}}^n(\boldsymbol{\xi})|^2 - C_\varepsilon\sum_{i=1}^q\Big|\sum_{j,k} a_{ijk}\frac{\hat{w}^n_j(\boldsymbol{\xi})\xi_k}{|\boldsymbol{\xi}|}\Big|^2\,,$$

and after integration we arrive to

$$\int_S \mathsf{Re}\,\tilde{Q}\circ\hat{\mathsf{w}}^n(\boldsymbol{\xi})d\boldsymbol{\xi} \geqslant -\varepsilon\int_S|\hat{\mathsf{w}}^n(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} - C_\varepsilon\sum_{i=1}^q\int_S\Big|\sum_{j,k} a_{ijk}\frac{\hat{w}^n_j(\boldsymbol{\xi})\xi_k}{|\boldsymbol{\xi}|}\Big|^2 d\boldsymbol{\xi}\,. \tag{64}$$

Since Fourier transform is an isometry on $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$, and weak convergence of (w^n) implies its boundedness in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, there exists constant $C\geqslant 0$, such that

$$\int_{\mathbf{R}^d}|\hat{\mathsf{w}}^n|^2 \leqslant C\,, n\in\mathbf{N}\,.$$

Passing to the limit in (64), from (63) we get

$$\lim_n \int_S \mathsf{Re}\,\tilde{Q}\circ\hat{\mathsf{w}}^n \geqslant -\varepsilon C\,.$$

This implies

$$\lim_n \int_S \mathsf{Re}\,\tilde{Q}\circ\hat{\mathsf{w}}^n \geqslant 0\,,$$

which together with (62) gives the claim.

Q.E.D.

Remark 1. Notice that sequence $(Q \circ \mathsf{u}^n)$ always has a convergent subsequence in the space of distributions. Indeed, since Q is a quadratic form and (u^n) is bounded in $\mathrm{L}^2_{loc}(\Omega;\mathbf{R}^r)$, it implies boundedness of $v^n := Q \circ \mathsf{u}^n$ in $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$. Thus, for any test function φ, the sequence (φv^n) is bounded in $\mathrm{L}^1(\Omega)$, and consequently in $\mathcal{M}_b(\Omega)$ as well. This implies the existence of a subsequence weakly-$*$ converging in $\mathcal{M}_b(\Omega)$, and in turn in the space of distributions as well. Moreover, if $\Phi \subseteq \mathcal{D}(\Omega)$ is countable, by Cantor's diagonal procedure we can extract a subsequence of the sequence (v^n) (not relabelled), such that for every $\varphi \in \Phi$, (φv^n) converges in the space of distributions. Let Φ be a set of functions (φ_n), such that φ_n is identically equal to one on the set

$$\left\{\mathbf{x} \in \Omega \,:\, d(\mathbf{x}, \mathsf{Fr}\,\Omega) > \frac{1}{n}\right\} \cap K(\mathsf{0}, n)\,.$$

Then

$$(\forall\, \psi \in \mathcal{D}(\Omega))(\exists\, n_0 \in \mathbf{N}) \quad n \geqslant n_0 \quad \Longrightarrow \quad \varphi_n\big|_{\mathrm{supp}\,\psi} = 1\,.$$

For fixed $\psi \in \mathcal{D}(\Omega)$ denote by n_ψ the smallest n_0 satisfying the above property. Now for every $n \geqslant n_\psi$ it holds

$${}_{\mathcal{D}'(\Omega)}\langle\, v^n, \psi\,\rangle_{\mathcal{D}(\Omega)} = {}_{\mathcal{D}'(\Omega)}\langle\, \varphi_{n_\psi} v^n, \psi\,\rangle_{\mathcal{D}(\Omega)} \longrightarrow l_\psi\,.$$

Taking

$${}_{\mathcal{D}'(\Omega)}\langle\, v, \psi\,\rangle_{\mathcal{D}(\Omega)} := l_\psi\,,$$

we get $v^n \rightharpoonup v$ in the space of distributions.

Corollary 1. *Under assumptions of Theorem 2, assume that the quadratic form Q additionally satisfies*

$$(\forall\, \boldsymbol{\lambda} \in \Lambda) \quad Q(\boldsymbol{\lambda}) = 0\,.$$

Then for every sequence (u^n) satisfying (P1) and (P2), it holds

$$Q \circ \mathsf{u}^n \rightharpoonup Q \circ \mathsf{u} \quad u \quad \mathcal{D}'(\Omega)\,.$$

Dem. Applying Theorem 2 to forms Q and $-Q$, we conclude that every subsequence of $(Q \circ \mathsf{u}^n)$ converging in the distributions, must converge to $Q \circ \mathsf{u}$. Preceding remark implies that every subsequence of $(Q \circ \mathsf{u}^n)$ has a weakly converging subsequence in the distributions, which implies that the whole sequence converges to $Q \circ \mathsf{u}$.

Q.E.D.

Using the calculation of the set Λ from the example we considered before Lemma 2, we can prove a improved version of the div-rot lemma than the one given in Lemma 1:

Corollary 2. *Assume that the sequences $(\mathbf{v}^n)$ and (w^n) satisfy*

$$\begin{aligned}
\mathbf{v}^n \rightharpoonup \mathbf{v} \quad &\textit{weakly in} \quad \mathrm{L}^2_{loc}(\Omega;\mathbf{R}^d)\,,\\
\mathsf{w}^n \rightharpoonup \mathsf{w} \quad &\textit{weakly in} \quad \mathrm{L}^2_{loc}(\Omega;\mathbf{R}^d)\,,\\
(\mathsf{div}\,\mathbf{v}^n) \quad &\textit{is relatively compact in} \quad \mathrm{H}^{-1}_{loc}(\Omega)\,,\\
(\mathsf{rot}\,\mathsf{w}^n) \quad &\textit{is relatively compact in} \quad \mathrm{H}^{-1}_{loc}(\Omega;\mathbf{R}^d)\,.
\end{aligned}$$

Then it holds

$$\mathbf{v}^n \cdot \mathsf{w}^n \rightharpoonup \mathbf{v} \cdot \mathsf{w} \quad \textit{weakly in} \quad \mathcal{D}'(\Omega)\,.$$

Dem. We have already shown

$$\Lambda = \Big\{ (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbf{R}^d \times \mathbf{R}^d \ : \ \boldsymbol{\lambda} \cdot \boldsymbol{\mu} = 0 \Big\},$$

and since $Q(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \boldsymbol{\lambda} \cdot \boldsymbol{\nu}$, the conditions of the Corollary 1 are satisfied. Set $\mathsf{u}^n = (\mathbf{v}^n, \mathsf{w}^n)$ and the claim follows from its application.

Q.E.D.

Remark 2. In fact, it can be shown that $Q(\boldsymbol{\lambda}, \boldsymbol{\nu}) = a\boldsymbol{\lambda} \cdot \boldsymbol{\nu}$, $a \in \mathbf{R}$, is the only non-affine function (up to an affine function) which is sequentially weakly$-*$ continuous on sequences satisfying the conditions of Corollary 2.

Literature

[1] Jelena Aleksić, Darko Mitrović: *On the compactness for two dimensional scalar conservation law with discontinuous flux*, *Comm. Math. Sci.* **7** (2009) 963–971.

[2] Nenad Antonić, Krešimir Burazin: *On certain properties of spaces of locally Sobolev functions*, in *Proceedings of the Conference on applied mathematics and scientific computing*, Zlatko Drmač et al. (eds.), Springer, 2005, pp. 109–120.

[3] Nenad Antonić, Marko Erceg, Marin Mišur: *On H-distributions*, in preparation.

[4] Nenad Antonić, Ivan Ivec: *On the Hörmander-Mihlin theorem for mixed-norm Lebesgue spaces*, *J. Math. Analysis Appl.* **433** (2016) 176–199.

[5] Nenad Antonić, Martin Lazar: *Parabolic H-measures*, *J. Funct. Analysis* **265** (2013) 1190–1239.

[6] Nenad Antonić, Marin Mišur, Darko Mitrović: *On the First commutation lemma*, submitted.

[7] Nenad Antonić, Darko Mitrović: *H-distributions: an extension of H-measures to an $\mathrm{L}^p - \mathrm{L}^q$ setting*, *Abs. Appl. Analysis* **2011** Article ID 901084 (2011) 12 pp.

[8] Jose Barros-Neto: *An introduction to the theory of distributions*, Dekker, 1973.

[9] Krešimir Burazin: *Application of compensated compactness in the theory of hyperbolic systems*, magister thesis (in Croatian), Zagreb, 2004.

[10] Ronald Raphael Coifman, Richard Rochberg, Guido Leopold Weiss: *Factorization theorems for Hardy spaces in several variables, Ann. of Math.* **103** (1976) 611–635.

[11] Ronald Raphael Coifman, Guido Leopold Weiss: *Extensions of Hardy spaces and their use in analysis*, *Bull. Amer. Math. Soc.* **84** (1977) 569–645.

[12] Heinz Otto Cordes: *On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators*, *J. Funct. Analysis* **18** (1975) 115–131.

[13] Vladimir Mihailovič Deundyak: *Multidimensional integral operators with homogeneous kernels of compact type and multiplicatively weakly oscillating coefficients*, *Math. Notes* **87** (2010) 672–686.

[14] Joseph Diestel, John Jerry Uhl: *Vector measures*, American Mathematical Society, 1977.

[15] Jean Alexandre Dieudonné: *Treatise on analysis I-VIII*, Academic Press, 1969–1993.

[16] Ronald J. DiPerna: *Convergence of the viscosity method for isentropic gas dynamics*, *Comm. Math. Phys.* **91** (1983) 1–30.

[17] Robert Edmund Edwards: *Functional Analysis*, Holt, Rinehart and Winston, 1965.

[18] Marko Erceg: *One-scale H-measurs and variants*, doctoral dissertation (in Croatian), Zagreb, 2016.

[19] Marko Erceg, Ivan Ivec: *On a generalisation of H-measures*, to appear in *Filomat*, 19 pp.

[20] Marko Erceg, Marin Mišur, Darko Mitrović: *Existence of solutions for a degenerate parabolic equation with a rough flux*, in preparation.

[21] Lawrence Craig Evans, Ronald F. Gariepy: *Measure Theory and Fine Properties of Functions*, CRC Press, 1992.

[22] Friedrich Gerard Friedlander, Mark Suresh Joshi: *Introduction to the theory of distributions*, Cambridge University Press, 1998.

[23] Patrick Gérard: *Microlocal defect measures*, *Comm. Partial Diff. Eq.* **16** (1991) 1761–1794.

[24] Loukas Grafakos: *Classical Fourier Analysis*, Springer, 2008.

[25] Michael Grösser, Michael Kunzinger, Michael Oberguggenberger, Roland Steinbauer: *Geometric theory of generalized functions with applications to general relativity*, Springer, 2001.

[26] Michael Hardy: *Combinatorics of Partial Derivatives*, *The Electronic Journal of Combinatorics* **13** (2006) #R1 13 pp.

[27] Lars Hörmander: *The analysis of linear partial differential operators I–IV*, Springer, 1983–1985.

[28] Ing-Lung Hwang: *The* L^2*-boundedness of pseudo-differential operators*, *Trans. Amer. Math. Soc.* **302** (1987) 55–76.

[29] Ing-Lung Hwang, R. B. Lee: *The* L^p*-boundedness of pseudo-differential operators of class* $S_{0,0}$, *Trans. Amer. Math. Soc.* **346** (1994) 489–510.

[30] Svante Janson: *Mean oscillation and commutators of singular integral operators*, *Ark. Math.* **16** (1978) 263–270.

[31] Joseph John Kohn, Louis Nirenberg: *An algebra of pseudo-differential operators*, *Comm. Pure Appl. Math.* **18** (1965) 269–305.

[32] Hideo Kozono, Taku Yanagisawa: *Global Div-Curl lemma on bounded domains in* $\mathbf{R}^3$, *J. Funct. Analysis* **256** (2009) 3847–3859.

[33] Hideo Kozono, Taku Yanagisawa: *Global Compensated Compactness Theorem for General Differential Operators of First Order*, *Archive for Rational Mechanics and Analysis* **207** (2013) 879–905.

[34] Mark Aleksandrovič Krasnosel'skij: *On a theorem of M. Riesz*, *Dokl. Akad. Nauk SSSR* **131** (1960) 246–248 (in russian); translated as *Soviet Math. Dokl.* **1** (1960) 229–231.

[35] Martin Lazar, Darko Mitrović: *Velocity averaging — a general framework*, *Dyn. Partial Diff. Eq.* **9** (2012) 239–260.

[36] Martin Lazar, Darko Mitrović: *On an extension of a bilinear functional on* $\mathrm{L}^p(\mathbf{R}^d) \times \mathrm{E}$ *to a Bôchner space with an application to velocity averaging*, *C. R. Acad. Sci. Paris, Ser. I* **351** (2013) 261–267.

[37] Martin Lazar, Darko Mitrović: *Optimal velocity averaging in a degenerate elliptic setting*, preprint, arXiv:1310.4285v1 [math.AP].

[38] Zhiwu Lin: *Instability of nonlinear dispersive solitary waves*, *J. Funct. Analysis* **255** (2008) 1191–1224.

[39] Zhiwu Lin: *On linear instability of 2D solitary water waves*, *Internat. Math. Research Notices* **2009** (2009) 1247–1303.

[40] Yunguang Lu: *Hyperbolic Conservation Laws and the Compensated Compactness Method*, Chapman and Hall/CRC, 2002.

[41] Marin Mišur: *Compactness by compensation*, diploma thesis (in Croatian), Zagreb, 2012.

[42] Marin Mišur, Darko Mitrović: *On a generalisation of compensated compactness in the* $\mathrm{L}^p - \mathrm{L}^q$ *setting*, *J. Funct. Analysis* **268** (2015) 1904–1927.

[43] Darko Mitrović, Ivan Ivec: *A generalization of H-measures and application on purely fractional scalar conservation laws*, *Comm. Pure Appl. Analysis* **10** (2011) 1617–1627.

[44] François Murat: *A survey on compensated compactness* in Contributions to modern calculus of variations (Bologna, 1985), 145–183, Pitman Res. Notes Math. Ser. 148, Longman Sci. Tech., Harlow, 1987.

[45] Lawrence Narici, Edward Beckenstein: *Topological vector spaces*, CRC Press, 2011.

[46] Sergej Mihajlovič Nikol'skij : *Approximation of functions of several variables and imbedding theorems*, Springer, 1975.

[47] Evgenij Jurjevič Panov: *On sequences of measure-valued solutions of a first-order quasilinear equation*, *Russian Acad. Sci. Sb. Math.* **81** (1995) 211–227.

[48] Evgenij Jurjevič Panov: *Ultra-parabolic equations with rough coefficients. Entropy solutions and strong pre-compactness property*, *J. Math. Sci.* **159** (2009) 180–228.

[49] Evgenij Jurjevič Panov: *Ultra-parabolic H-measures and compensated compactness*, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **28** (2011) 47–62.

[50] Pablo Pedregal: *Parametrized measures and variational principles*, Birkhäuser, 1997.

[51] Patrick J. Rabier: *Fredholm and regularity theory of Douglis–Nirenberg elliptic systems on* $\mathbf{R}^n$, *Math. Z.* **270** (2012) 369–393.

[52] Serge Richard, Rafael Tiedra de Aldecoa: *New formulae for the wave operators for a rank one interaction*, *Integr. Eq. Oper. Theory* **66** (2010) 283–292.

[53] Filip Rindler: *Directional oscillations, concentrations, and compensated compactness via microlocal compactness forms*, *Arch. Ration. Mech. Anal.* **215** (2015) 1–63.

[54] Joel William Robin, Robert Charles Rogers, J. Blake Temple: *On weak continuity and the Hodge decomposition*, *Trans. Amer. Math. Soc.* **303** (1987) 609–618.

[55] Laurent Schwartz: *Théorie des distributions*, Hermann, 1966.

[56] Santiago R. Simanca: *Pseudo-differential operators*, Longman, 1990.

[57] Elias Menachem Stein: *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.

[58] Elias Menachem Stein: *Harmonic analysis*, Princeton University Press, 1993.

[59] Luc Tartar: *Compensated compactness and applications to partial differential equations*, in Knops (ed.) Nonlinear Analysis and Mechanics: Heriot Watt Symposium, Vol IV, Res. Notes in Math. 39, Pitman (1979) 136–212.

[60] Luc Tartar: *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations*, *Proc. Roy. Soc. Edinburgh* **115A** (1990) 193–230.

[61] Luc Tartar: *The general theory of homogenization: A personalized introduction,* Springer, 2009.

[62] Luc Tartar: *Compensated Compactness with More Geometry*, in Chen, Grinfeld, Knops (eds) Differential Geometry and Continuum Mechanics. Springer Proceedings

in Mathematics & Statistics, vol 137. Springer (2015) 3–26.

[63] François Tréves: *Topological vector spaces, distributions and kernels*, Dover, 2006.

[64] Akihito Uchiyama: *On the compactness of operators of Hankel type, Tohoku Math. Journ.* **30** (1978) 163–171.

Summary

H-measures are matrix Radon measures describing the behaviour of weak limits of quadratic quantities. They proved to be very successful tool in investigations of asymptotic limits of quadratic quantities. However, they turned insufficient for nonlinear problems. Recent investigations resulted with the introduction of variant H-measures, so called H-distributions, which surmount some of the noted inadequacies, and allow the treatment of terms involving sequences of L^p functions.

Basic tools for the construction of aforementioned microlocal objects are pseudodifferential and singular integral operators. In the L^2 case, Fourier transform together with Plancherel's theorem proved to be a very efficient tool. However, in the L^p case one needs to use the theory of Fourier multipliers (Marcinkiewicz theorem, Hörmander-Mihlin theorem) which requires a higher regularity of the space of test functions, together with corresponding bounds on derivatives.

There are two crucial steps in the construction of the above mentioned microlocal objects. First one is an application of the First commutation lemma to pass from trilinear functional to a bilinear one, while the second one is an application of the Schwartz kernel theorem to identify the obtained bilinear functional with an element from the dual of smooth functions on the product domain.

We showed the Krasnoselskij type of result for unbounded domains, and with its help we lowered the regularity of the symbol needed for a variant of the First commutation lemma for the L^p spaces. We also showed how the same idea can be used to improve the results on existence of H-distributions on the Lebesgue spaces with mixed norm. Furthermore, we we studied how further we can lower the regularity of the Sobolev multiplier under the assumption that the symbol of the Fourier multiplier operator satisfies only the Hörmander condition. Whats more, in the case when the symbol of the Fourier multiplier is defined on the unit sphere, Luc Tartar showed that the result remains valid in the L^2 case even when we have coefficients from the VMO space (the space of functions of vanishing mean oscillations). We arrived at the same conclusion in the L^p space. In the end we showed a variant of the First commutation lemma in the case when we have general pseudodifferential operator instead of the Fourier multiplier operator. For that we used the bounds from Hwang's results on boundedness of pseudodifferential operators.

To give a better description of H-distributions, we refined the notion of distributions by introducing a notion of anisotropic distributions of finite order. Those are distributions which have different order in different coordinate directions. The main obstacle was adjusting the Schwartz kernel theorem to this new notion. We used Dieudonne's approach which used the structure theorem of distributions. An advantage of this approach is that the order of distribution increases only with respect to one variable, while it remains unchanged with respect to the other. This allowed us to consider partial differential equations with continuous coefficients in the localisation principle of H-distributions. Up

to now, we could only consider the smooth case. Let us emphasise that continuous coefficients were optimal in the L^2 case.

Motivated by Panov's approach in the article on ultra-parabolic H-measures, we showed a variant of compactness by compensation. For that we used a variant of H-distributions, which we obtained from a result on the extension of the bilinear functionals to Bôchner spaces. This variant of H-distributions allowed us to consider variable discontinuous coefficients in differential restrictions and quadratic form. What is more, the derivatives in differential restrictions could be of fractional order. Because of that, we do not have symbols defined on the unit sphere, but on a more general manifold. For this reason we had to use the Marcinkiewicz multiplier theorem for continuity of the Fourier multiplier operators. We applied this new variant of compactness by compensation to a nonlinear degenerate equation of parabolic type, for which the known theory of H-measures was not adequate.

Keywords: H-measures, H-distributions, Fourier multiplier, kernel theorem, compactness by compensation.
Mathematics subject classification 2010: 35B99, 35D30, 35S05, 42B15, 46F05.

Sažetak

H-mjere su matrične Radonove mjere koje opisuju slabi limes kvadratičnih izraza. Pokazale su se kao vrlo uspješan alat za proučavanje asimptotičkog ponašanja kvadratičnih izraza. No, nisu dovoljno dobre za promatranje nelinearnih zadaća. Nedavna istraživanja su rezultirala uvođenjem inačica H-mjera, nazvanih H-distribucijama, koje uklanjaju neke od uočenih nedostataka, i omogućuju proučavanje izraza koji sadrže nizove L^p funkcija.

Osnovni alati za konstrukciju navedenih mikrolokalnih objekata su pseudodiferencijalni i singularni integralni operatori. U slučaju L^2 funkcija, Fourierova pretvorba je preko Plancherelovog teorema vrlo efikasan alat. Medjutim, u L^p teoriji moramo koristiti teoriju Fourierovih množitelja (posebno Marcinkiewiczev ili Hörmander-Mihlinov teorem) koja zahtijevaja i veću regularnost prostora probnih funkcija, te odgovarajuće ocjene na derivacije.

Dva su ključna koraka u dokazu egzistencije ovih mikrolokalnih objakata. Prvi je korištenje prve komutacijske leme da bi se iz trilinearnog funkcionala dobio bilinearni funckional, dok je drugi primjena Schwartzovog teorema o jezgri kako bi se dobiveni bilinearni funkcional poistovjetio s elementom duala glatkih funkcija na produktnoj domeni.

Pokazali smo Krasnosel'skijev tip rezultata za neograničene domene, te pomoću njega smanjili regularnost simbola potrebnu za varijantu prve komutacijske leme za L^p prostore. Vidjeli smo da se ista ideja može iskorititi i za poboljšanje rezultata na Lebesgueove prostore s mješovitom normom. Nadalje, proučili smo koliko možemo smanjiti regularnost Soboljevljevog množitelja uz pretpostavku da simbol Fourierovog množitelja zadovoljava samo Hörmanderov uvjet. Štoviše, u slučaju kad je simbol Fourierovog množitelja definiran na sferi, Tartar je pokazao da rezultat u L^2 ostaje valjan i za koeficijente iz prostora VMO (prostora funkcija s iščezavajućim srednjim oscilacijama). Mi smo došli do istog zaključka i za L^p slučaj. Na kraju smo pokazali varijantu prve komutacijske leme za slučaj kad umjesto Fourierovog množitelja imamo općeniti pseudodiferencijalni operator. Za to smo koristili ocjene iz Hwangovih rezultata o neprekidnosti pseudodiferencijalnih operatora.

Za bolji opis H-distribucija, profinili smo pojam distribucije uvođenjem pojma anizotropnih distribucija konačnog reda. To su distribucije koje imaju različit red u različitim koordinatnim smjerovima. Glavna prepreka u tom smjeru je bila prilagodba Schwartzovog teorema o jezgri. Koristili smo Dieudonneov dokaz koji koristi strukturni teorem za distribucije. Prednost ovog pristupa u odnosnu na ostale leži u činjenici da se red jezgre povećava samo po jednoj varijabli, dok po drugoj ostaje nepromijenjen. Ovo nam je omogućilo da u lokalizacijskom svojstvu H-distribucija promatramo jednadžbe čiji koeficijenti više nisu glatke funkcije, već su samo neprekidne. Naglasimo da su neprekidni koeficijenti bili optimalni u L^2 slučaju.

Motivirani Panovljevim pristupom ultraparaboličkim H-mjerama, pokazali smo varijantu kompaktnosti kompenzacijom. Za to nam je bila potrebna varijanta H-distribucija koju smo dobili koristeći rezultat o proširenju bilinearnih funkcionala na Bôchnerove prostore. Ova varijanta H-distribucija nam je omogućila korištenje koeficijenata u kvadratnoj formi i u diferencijalnim ograničenjima koji su varijabilne prekidne funkcije. Štoviše, derivacije u diferencijalnim ograničenjima mogu biti i razlomljenog reda. Iz tog razloga, nemamo više simbol definiran na sferi, vec na općenitijoj mnogostrukosti, za što smo trebali koristiti Marcinkiewiczev teorem za neprekidnost Fourierovih množitelja. Dobiveni rezultat kompaktnosti kompenzacijom smo primijenili na nelinearnu degeneriranu jednadžbu paraboličkog tipa za koju poznata L^2 teorija nije bila dostatna.

Ključne riječi: H-mjere, H-distribucije, Fourierov množitelj, teorem o jezgri, kompaktnost kompenzacijom.
Mathematics subject classification 2010: 35B99, 35D30, 35S05, 42B15, 46F05.

Curriculum vitae

Marin Mišur was born 18th of December 1988 in Split. He lived in Metković, where he finished his primary education and gymnasium. In autumn 2012 he started undergraduate studies at University of Zagreb. In October 2010 he started graduate studies in applied mathematics at the same university. He graduated *summa cum laude* on 11th of July 2012 with diploma thesis *Compactness by compensation* under supervision of prof. Nenad Antonić.

In October 2012 he started doctoral studies at University of Zagreb under supervision of prof. Nenad Antonić. During the course of the studies he passed the following exams: *Real and functional analysis*, *Partial differential equations*, *Convex and non-smooth analysis*, *Vector spaces and matrix algorithms*, *Selected topics of numerical linear algebra*, and *Microlocal defect functionals and applications*. Additionally, he audited the following classes: *Homogenisation theory and applications in optimal design*, *Contemporary topics in conservation laws and evolution equations*, and *Convex integration and non-regular solutions of partial differential equations*. He is a member of the Seminar for differential equations and numerical analysis.

He was a teaching assistant for the following courses at the Faculty of Science: Introduction to mathematics, Introduction to optimisation, Methods of mathematical physics, Mathematical analysis II and Linear algebra I.

He attended a number of conferences and summer schools at which he gave 11 talks and 8 poster presentations. He had 6 study visits (University of Dubrovnik, University of Montenegro, Basque Center for Applied Mathematics), and one internship (Basque Center for Applied Mathematics).

He is a team member of the HRZZ project *Weak convergence methods and applications* (PI: Nenad Antonić), and he was a collaborator on two bilateral projects *Microlocal analysis, partial differential equations and applications to heterogeneous materials* (with Serbia; croatian PI: Nenad Antonić), and *Multiscale methods and calculus of variations* (with Montenegro; croatian PI: Nenad Antonić). He participated in the realisation of a one-year project *Evolutionary Friedrichs systems* (PI: Krešimir Burazin). His stays at Basque Center for Applied Mathematics were within the frame of the ERC project FP7-246775 *NUMERIWAVES* (PI: Enrique Zuazua).

He co-authored 2 published scientific works and 2 more that are under review.