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η and η′ mesons and dimension 2 gluon condensate ⟨A²⟩

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The study of light pseudoscalar quark-antiquark bound states in the Dyson-Schwinger approach with the *effective* QCD coupling enhanced by the interplay of the dimension 2 gluon condensate ⟨A²⟩ and dimension 4 gluon condensate ⟨F²⟩, is extended to the η−η′ complex. We include the effects of the gluon axial anomaly into the Dyson-Schwinger approach to mesons. The calculated masses, mixing and two-photon decay widths of η and η′ mesons are in agreement with experiment. Also, in a model-independent way, we give the modification of the Gell-Mann−Okubo and Schwinger nonet relations due to the interplay of the gluon anomaly and SU(3) flavor symmetry breaking.

I. INTRODUCTION

The dimension-2 gluon condensate $\langle A^a_\mu A^{a\mu} \rangle \equiv \langle A^2 \rangle$ attracted the attention of some researchers well over a decade ago; e.g., see Refs. [1–5]. However, there was a widespread opinion that, since this condensate is not gauge invariant, it cannot have observable consequences and cannot play an important role in QCD. In contrast to that, the gauge-invariant, dimension-4 gluon condensate $\langle F^a_{\mu\nu} F^{a\mu\nu} \rangle \equiv \langle F^2 \rangle$ [6] has, over 25 years now, been a subject of studies discussing its value (e.g., see Ref. [7]) and implications for QCD.

After it turned out more recently that the Landau-gauge value of ⟨A²⟩ corresponds to a more general gauge-invariant quantity, it attracted a lot of theoretical attention [8–18], to quote just several of many papers offering evidence that ⟨A²⟩ condensate may be important for the nonperturbative regime of Yang-Mills theories. In Ref. [15], we argued that ⟨A²⟩ condensate may be relevant for the Dyson-Schwinger (DS) approach to QCD. Namely, in order that this approach [19–22] leads to a successful hadronic phenomenology [which has so far been treated *widely* only in the rainbow-ladder approximation (RLA)], an enhancement of the effective quark-gluon interaction seems to be needed at intermediate spacelike momenta, $Q^2 \sim 0.5$ GeV². Reference [15] showed that the interplay of the dimension-2 condensate ⟨A²⟩ with the dimension-4 condensate ⟨F²⟩ can provide such an enhancement. It also showed that the resulting effective strong running coupling leads to the sufficiently strong dynamical chiral symmetry breaking (DχSB) and successful phenomenology in the sector of light pseudoscalar mesons. In addition, the issues such as quark propagator solutions, p^2-dependent dressed ("constituent") quark masses and a more detailed discussion of the parameter dependence of the results were addressed in Refs. [23,24].

In the present paper, we extend the treatment of the nonzero-isospin light pseudoscalar mesons of Ref. [15] to the η−η′ complex. First, in the next section, the key result of Ref. [15], namely, its gluon-condensate-enhanced interaction (15), is briefly rederived in another, less rigorous and heuristic way. In the third section, we review how the DS approach employing such an interaction, can give the successful light (i.e., involving quark flavors $q = u, d, s$) meson phenomenology, as this is also needed for the good description of the presently interesting η and η′ mesons. Nevertheless, for η and η′ this is not enough because of the influence of the gluon anomaly and thus we explain how its effects are included in the manner of Refs. [25,26]. The implementation of the anomaly and the SU(3) flavor symmetry breaking, as well as their interplay, are presented independently of any concrete dynamics in subsection III A, and in detail, because of some renewed interest in the flavor dependence of the mixing in the η−η′ complex (e.g., Ref. [27] and references therein). After that, the masses, mixing angle and two-photon (γγ) decay widths of η and η′ are calculated. A discussion and conclusion are in Sec. IV.

II. STRONG COUPLING ENHANCED BY GLUON CONDENSATES

Reference [15] showed how the interaction (15), phenomenologically successful in DS studies in the Landau gauge and RLA, resulted from combining the form [Eq. (3) below] which the running coupling has in the Landau-gauge DS studies [20,28–31] and the ideas on the possible relevance of the ⟨A²⟩ gluon condensate [2–5,8–14]. In the present paper, we give a simplified and more intuitive derivation thereof as follows.

The full gluon propagator $D^{ab}_{\mu\nu}(k)$ in the Landau gauge is defined through the free gluon propagator $D^{ab}_{\mu\nu}(k)_0$ and the gluon renormalization function $Z(-k^2)$ like this:

$$D^{ab}_{\mu\nu}(k) = Z(-k^2) D^{ab}_{\mu\nu}(k)_0 \equiv Z(-k^2) \frac{\delta^{ab}}{k^2} \left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \right).$$

$$(1)$$

The full ghost propagator $D_G(k)$ is similarly defined by the

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ghost renormalization function $G(-k^2)$:

$$D_G(k) = \frac{G(-k^2)}{k^2}. \qquad (2)$$

The strong running coupling $\alpha_s(Q^2)$ used in the Landau-gauge DS studies [20,28–31] is defined as

$$\alpha_s(Q^2) = \alpha_s(\mu^2)Z(Q^2)G(Q^2)^2, \qquad (3)$$

where $\alpha_s(\mu^2) = g^2/4\pi$ and $Z(\mu^2)G(\mu^2)^2 = 1$ at the renormalization point $Q^2 = \mu^2$. Our convention is $k^2 = -Q^2 < 0$ for spacelike momenta k.

The functions Z and G can be expressed through the corresponding gluon (A) and ghost (G) polarization functions $\Pi_A(Q^2)$ and $\Pi_G(Q^2)$:

$$Z(Q^2) = \frac{1}{1 + \frac{\Pi_A(Q^2)}{Q^2}}, \qquad G(Q^2) = \frac{1}{1 + \frac{\Pi_G(Q^2)}{Q^2}}. \qquad (4)$$

Almost two decades ago, it was noted that in the operator product expansion (OPE) the gluon condensate $\langle A^2 \rangle$ can contribute to QCD propagators; e.g., see Refs. [2–5]. Their $\langle A^2 \rangle$-contributions to the OPE-improved gluon (A) and ghost (G) polarization functions were more recently confirmed by Kondo [11]. For the Landau gauge adopted throughout this paper, three QCD colors ($N_c = 3$) and four space-time dimensions ($D = 4$), their expressions for the polarizations become

$$\Pi_i(Q^2) = m_i^2 + \mathcal{O}_i(1/Q^2), \qquad (i = A, G), \qquad (5)$$

$$m_A^2 = \frac{3}{32} g^2 \langle A^2 \rangle = -m_G^2, \qquad (6)$$

where m_A and m_G are, respectively, dynamically generated effective gluon and ghost mass. Reference [15] estimated $m_A = 0.845$ GeV (and found that it was, phenomenologically, a remarkably successful initial estimate) by using in Eq. (6) the lattice result $g^2 \langle A^2 \rangle = 2.76$ GeV^2 of Ref. [8], a value compatible with the bound resulting from the discussions of Gubarev *et al.* [9,10] on the physical meaning of $\langle A^2 \rangle$ and its possible importance for confinement.

As for the contributions $\mathcal{O}_i(1/Q^2)$ ($i = A, G$) in Eq. (5), it turned out [3–5] that they contain many kinds of mostly unknown condensates [e.g., gauge-dependent gluon, ghost and mixed ones, where terms $\propto (1/Q^2)^n$ ($n > 1$) were not considered at all]. The only practical approach at this point is therefore that these complicated contributions are approximated by the terms $\propto 1/Q^2$ and parameterized, i.e.,

$$\mathcal{O}_A(1/Q^2) \approx \frac{C_A}{Q^2}, \qquad \mathcal{O}_G(1/Q^2) \approx \frac{C_G}{Q^2}. \qquad (7)$$

Thus, C_A and C_G would in principle be free parameters to be fixed by phenomenology. However, as noted in Ref. [15], in the effective gluon propagator proposed by Lavelle [32], the $\mathcal{O}_A(1/Q^2)$ polarization is (for the Landau gauge and $D = 4$) given by the dimension-4 gluon condensate $\langle F^2 \rangle$ as

$$\Pi_A^{\langle F^2 \rangle}(Q^2) = \frac{34 N_c \pi \alpha_s \langle F^2 \rangle}{9(N_c^2 - 1)Q^2} = \frac{(0.640 \text{ GeV})^4}{Q^2}. \qquad (8)$$

Since Lavelle's [32] propagator misses some unknown three- and four-gluon contributions [4,5] and since the precise value of $\alpha_s \langle F^2 \rangle$ is still not certain, we regard the value $C_A = (0.640 \text{ GeV})^4$ just as an inspired initial estimate. Still, together with the assumption $C_G = C_A$, it was a very useful starting guess in our Refs. [15,23,24], leading to very good phenomenological fits.

We are now prepared to give a general, although heuristic argument why the contribution (6) of the dimension-2 $\langle A^2 \rangle$ condensate to the gluon and ghost polarization functions (5), should indeed lead to the form of $\alpha_{\text{eff}}(Q^2)$ already found through a more detailed argument in Ref. [15]. Our first step is to assume that in the gluon and ghost polarizations (5), Π_A and Π_G, one can disentangle the perturbative (pert) from nonperturbative (Npert) parts, $\Pi_i = \Pi_i^{\text{pert}} + \Pi_i^{\text{Npert}}$ ($i = A, G$). At least for high momenta Q^2, it is then possible to approximately factor away the perturbative from nonperturbative contributions; for $i = A$,

$$Z(Q^2) \approx \frac{1}{1 + \frac{\Pi_A^{\text{pert}}(Q^2)}{Q^2}} \frac{1}{1 + \frac{\Pi_A^{\text{Npert}}(Q^2)}{Q^2}} \equiv Z^{\text{pert}}(Q^2)Z^{\text{Npert}}(Q^2), \qquad (9)$$

where the approximation means neglecting the contribution of the term

$$\frac{\Pi_A^{\text{pert}}(Q^2)\Pi_A^{\text{Npert}}(Q^2)}{Q^4}. \qquad (10)$$

Analogously,

$$G(Q^2) \approx \frac{1}{1 + \frac{\Pi_G^{\text{pert}}(Q^2)}{Q^2}} \frac{1}{1 + \frac{\Pi_G^{\text{Npert}}(Q^2)}{Q^2}} \equiv G^{\text{pert}}(Q^2)G^{\text{Npert}}(Q^2). \qquad (11)$$

Since the general QCD coupling $\alpha_s(Q^2)$ must reduce to the *perturbative* QCD coupling $\alpha_{\text{pert}}(Q^2)$ for so very high Q^2 that nonperturbative contributions are negligible, Eq. (3) implies that

$$\alpha_s(\mu^2)Z^{\text{pert}}(Q^2)G^{\text{pert}}(Q^2)^2 = \alpha_{\text{pert}}(Q^2). \qquad (12)$$

We can also assume, for high Q^2, that nonperturbative parts are given by the OPE-based results of Refs. [2–5,11]. In our present case, they amount to Eqs. (5) and (6) for the gluon and ghost polarizations, and to the parameterization (7). This then gives

$$Z^{\text{Npert}}(Q^2) = \frac{1}{1 + \frac{m_A^2}{Q^2} + \frac{C_A}{Q^4}}, \qquad (13)$$

$$G^{\mathrm{Npert}}(Q^2) = \frac{1}{1 - \frac{m_A^2}{Q^2} + \frac{C_G}{Q^4}}. \tag{14}$$

Equations (3), (9), (11), (13), and (14) considered together, then suggest an effective coupling $\alpha_{\mathrm{eff}}(Q^2)$ of the form

$$\alpha_{\mathrm{eff}}(Q^2) = \alpha_{\mathrm{pert}}(Q^2) Z^{\mathrm{Npert}}(Q^2) G^{\mathrm{Npert}}(Q^2)^2. \tag{15}$$

Obviously, the above derivation of the coupling (15) is only heuristic, but we have already presented its more rigorous derivation in Ref. [15].

In Refs. [15,23,24], we discussed why, how and when the form (15) was sufficiently enhanced at intermediate Q^2 to lead to the successful pion and kaon phenomenology when used in the DS approach in RLA.

III. DS APPROACH AND ITS EXTENSION TO η–η' COMPLEX

In the DS approach to QCD, one solves the gap equation, i.e., the DS equation for quark two-point functions, namely, dressed quark propagators

$$S_q(p) = \frac{1}{\not{p}\mathcal{A}_q(p^2) - \mathcal{B}_q(p^2)} \equiv \frac{\mathcal{A}_q(p^2)^{-1}}{\not{p} - \mathcal{M}_q(p^2)} \tag{16}$$

of various flavors q, and so explicitly constructs constituent quarks characterized by the dynamical masses $\mathcal{M}_q(p^2) \equiv \mathcal{B}_q(p^2)/\mathcal{A}_q(p^2)$. The constituent quarks and antiquarks of respective flavors q and q' build meson bound states, which are solutions of the Bethe-Salpeter (BS) equation for the bound-state vertex $\Gamma_{q\bar{q}'}$:

$$[\Gamma_{q\bar{q}'}(k, P)]_{ef} = \int \frac{d^4\ell}{(2\pi)^4} \left[S_q\left(\ell + \frac{P}{2}\right) \Gamma_{q\bar{q}'}(\ell, P) \right. $$
$$\left. \times S_{q'}\left(\ell - \frac{P}{2}\right) \right]_{gh} [K(k-\ell)]_{ef}^{hg}, \tag{17}$$

where e, f, g, h *schematically* represent spinor, color and flavor indices. Solving Eq. (17) for $\Gamma_{q\bar{q}'}$ also yields $M_{q\bar{q}'}$, the mass eigenvalue of the $q\bar{q}'$ bound state. Unfortunately, the *full* interaction kernel $K(k, \ell, P)$ for the BS equation is not known. Also, the full kernels of the gap equations [which supply the quark propagators $S_q(\ell)$ (16) to Eq. (17)] involve the full gluon propagator and the full quark-gluon vertex, which satisfy their own DS equations. They in turn involve higher n-point functions and their DS equations, etc. This infinite tower of the integral DS equations must be truncated to make the problem tractable [19–22]. The approximations employed in the gap equation and the BS equation must be mutually consistent in order to preserve the important characteristics of the full theory. In the low-energy sector of QCD, the nonperturbative phenomenon of DχSB is the most important feature. Phenomenological DS studies have therefore mostly been relying on the *consistently* used RLA, where DχSB is well understood [20,22,33–35]. The consistent RLA implies

that for interactions between quarks one uses Ansätze of the form

$$[K(k)]_{ef}^{hg} = \mathrm{i}4\pi\alpha_{\mathrm{eff}}(-k^2) D_{\mu\nu}^{ab}(k)_0 \left[\frac{\lambda^a}{2}\gamma^\mu\right]_{eg} \left[\frac{\lambda^a}{2}\gamma^\nu\right]_{hf} \tag{18}$$

both in the BS Eq. (17) and the gap Eq. (19) for the dressed quark propagators (16),

$$S_q^{-1}(p) = \not{p} - \tilde{m}_q - \mathrm{i}4\pi \int \frac{d^4\ell}{(2\pi)^4} \alpha_{\mathrm{eff}}[-(p-\ell)^2]$$
$$\times D_{\mu\nu}^{ab}(p-\ell)_0 \frac{\lambda^a}{2}\gamma^\mu S_q(\ell) \frac{\lambda^b}{2}\gamma^\nu. \tag{19}$$

Here, $\tilde{m}_q$ is the bare mass of the quark flavor q breaking the chiral symmetry explicitly. The case $\tilde{m}_q = 0$ corresponds to the chiral limit where the current quark mass $m_q = 0$.

One solves the DS Eq. (19) for dressed propagators of the light quarks ($q = u, d, s$) using a, hopefully, phenomenologically successful interaction [in the present paper, it is given by Eq. (15)]. These light-quark propagators are then employed in BS equations for quark-antiquark ($q\bar{q}$) relativistic bound states.

The most important advantage of adopting RLA in the context of low-energy QCD is that DχSB and, consequently, the appearance of light pseudoscalar mesons as (almost-)Goldstone bosons in (the vicinity of) the chiral limit, is well understood and under control in this approximation scheme [20,22,33–35]. Solving the BS equation in the chiral limit gives the vanishing pion mass, $M_\pi = 0$. More generally and precisely, the light pseudoscalar masses $M_{q\bar{q}} \propto \sqrt{\tilde{m}_q}$, as required by QCD through Gell-Mann-Oakes-Renner (GMOR) relation. All this is a manifestation of the correct chiral QCD behavior in the DS approach, in which all light pseudoscalar mesons ($\pi^{0,\pm}, K^{0,\pm}, \bar{K}^0, \eta$) manifest themselves *both* as $q\bar{q}$ bound states *and* (almost-)Goldstone bosons of dynamically broken chiral symmetry. This resolution of the dichotomy "$q\bar{q}$ bound state vs Goldstone boson," enables one to work with the mesons as explicit $q\bar{q}$ bound states while reproducing (even analytically in the chiral limit) the famous results of the axial anomaly for the light pseudoscalar mesons, most notably the $\pi^0 \to \gamma\gamma$ decay amplitude $T_{\pi^0}^{\gamma\gamma} = 1/(4\pi^2 f_\pi)$ in the chiral limit [36,37]. This is unique among the bound state approaches—e.g., see Refs. [38,39] and references therein for discussion. Nevertheless, one keeps the advantage of bound state approaches that from the $q\bar{q}$ substructure one can calculate many important quantities (such as the pion decay constant f_π) which are just parameters in most of other chiral approaches to the light-quark sector.

In the DS approach, the DχSB is obtained in an, essentially, Nambu–Jona-Lasinio fashion, but the DS model interaction, encoded in various popular forms of $\alpha_{\mathrm{eff}}(p^2)$, is usually less schematic. Typically, as in

Refs. [21,25,33,39–41] for example, it combines the modeled nonperturbative component (which is strong enough to obtain required DχSB) and the known perturbative QCD contribution. Our gluon condensate-induced $\alpha_{\text{eff}}(Q^2)$ (15) also behaves nonperturbatively at low Q^2 and perturbatively at very high Q^2, as seen both in its earlier derivation [15] and the alternative derivation in Sec. II. Thus, as shown in Ref. [15], our interaction (15) leads to the deep Euclidean behaviors of quark propagators consistent with the asymptotic freedom of QCD [39]. However, what enables the successful description of pions and kaons in Ref. [15] and what is also crucial in the present paper, are the contributions to the gap and BS equations at low and intermediate momenta. The crucial result is the behavior of the gap equation solutions for the mass functions $\mathcal{M}_q(Q^2)$ at $Q^2 = 0$ to $Q^2 \approx (500 \text{ MeV})^2$, where $\mathcal{M}_q(Q^2)$ ($q = u, d, s$), due to DχSB, have values consistent with typical values of the constituent mass parameters in constituent quark models. [When we need to be specific, we can, for definiteness, choose to call the $Q^2 = 0$ value $\mathcal{M}_q(0)$ the constituent mass.] In the chiral limit, where, as usual in the consistent DS approach [19–22], we get the correct chiral Goldstone-pion behavior, our $\alpha_{\text{eff}}(Q^2)$ (15) with the parameters from Ref. [15], namely

$$C_A = (0.6060 \text{ GeV})^4 = C_G, \qquad m_A = 0.8402 \text{ GeV}, \tag{20}$$

gives us $\mathcal{M}_{u,d}(0) = 369$ MeV, and empirically already quite acceptable values for the pion decay constant and the $\langle \bar{q}q \rangle$ condensate. Away from the chiral limit (but still for isosymmetric u- and d-quarks), the same interaction yields $\mathcal{M}_{u,d}(0) = 384$ MeV [just 4% above $\mathcal{M}_{u,d}(0)$ in the chiral limit] with the bare mass $\tilde{m}_{u,d} = 3.046$ MeV, for which value our model reproduces the experimental values of π^0 mass, f_{π^+} decay constant as well as $\pi^0 \to \gamma\gamma$ decay amplitude, and respects GMOR relation [15]. The empirical values of the kaon mass and decay constant were also reproduced very well when one in addition takes $\tilde{m}_s = 67.70$ MeV [15]. The s quark constituent, dynamical mass is then $\mathcal{M}_s(0) = 592$ MeV. (In our earlier papers [15,23,24] we also found that the results for $\tilde{m}_{u,d}$ and $\tilde{m}_s$ were rather robust. For example, the values quoted here because they are preferred when the interaction (15) is employed in the gap and BS equations, (19) and (17), are quite close to the values of $\tilde{m}_{u,d}$ and $\tilde{m}_s$ preferred when the Jain-Munczek effective interaction [33] is used instead.)

Up to accounting for the gluon anomaly, the results in Table I are the input from the well-described pion and kaon sector [15] which enables, without any refitting of the model parameters $\tilde{m}_{u,d}$, $\tilde{m}_s$, and (20), the good description of the η–η' complex.

TABLE I. Masses M_P, decay constants f_P and $\gamma\gamma$ decay amplitudes $T_P^{\gamma\gamma}$ of the pseudoscalar $q\bar{q}'$ bound states $P = \pi, K$ and $s\bar{s}$ resulting from our gluon condensate-induced $\alpha_{\text{eff}}(Q^2)$ (15), along with the parameter values fixed in Ref. [15] by fitting the pion and kaon properties—see Eq. (20) and the text below it. These masses and decay constants are the input for the description of the η–η' complex. The last column is the constituent quark mass $\mathcal{M}_q(0)$ pertinent to the corresponding $q\bar{q}$ meson, i.e., $\mathcal{M}_u(0) = \mathcal{M}_d(0)$ for $P = \pi$ and $\mathcal{M}_s(0)$ for $P = s\bar{s}$. [Later, in Eq. (40), we will name the unphysical $s\bar{s}$ pseudoscalar meson η_S, but note that the mass M_{η_S} (51), introduced in the NS–S mass matrix (41), includes the contribution from the gluon anomaly, whereas $M_{s\bar{s}}$ does not.]

P	M_P	f_P	$1/(4\pi^2 f_P)$	$T_P^{\gamma\gamma}$	$\mathcal{M}_q(0)$
π	0.1350	0.09293	0.2726	0.2560	0.3842
K	0.4949	0.1115			
$s\bar{s}$	0.7221	0.1329	0.1905	0.08599	0.5922

A. Masses in the η–η' complex

The description [25,26,42] of η and η' is especially noteworthy, as it is successful in spite of the limitations of the DS approach in the ladder approximation. For this description, the crucial issues are the meson mixing and construction of physical meson states. For the DS approach, they are formulated in Refs. [25,26], where solving of appropriate BS Eqs. (17) yields the eigenvalues of the squared masses, $M_{u\bar{u}}^2$, $M_{d\bar{d}}^2$, $M_{s\bar{s}}^2$, and $M_{u\bar{s}}^2$, of the respective quark-antiquark bound states $|u\bar{u}\rangle$, $|d\bar{d}\rangle$, $|s\bar{s}\rangle$, and $|u\bar{s}\rangle$. The last one is simply the kaon, and $M_{u\bar{s}}$ is its mass M_K. Nevertheless, the first three do not correspond to any physical pseudoscalar mesons. Thus, $M_{u\bar{u}}^2$, $M_{d\bar{d}}^2$, $M_{s\bar{s}}^2$ do not automatically represent any physical masses, although the mass matrix (to be precise, its *nonanomalous part*, which vanishes in the chiral limit) is simply

$$\hat{M}_{NA}^2 = \begin{bmatrix} M_{u\bar{u}}^2 & 0 & 0 \\ 0 & M_{d\bar{d}}^2 & 0 \\ 0 & 0 & M_{s\bar{s}}^2 \end{bmatrix} \tag{21}$$

in the basis $|q\bar{q}\rangle$, ($q = u, d, s$). However, the flavor SU(3) quark model, and especially the almost exact isospin symmetry, leads one to recouple these states into the familiar SU(3) octet-singlet basis of the zero-charge subspace of the light unflavored pseudoscalar mesons of well-defined isospin quantum numbers:

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle), \tag{22}$$

$$|\eta_8\rangle = \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle), \tag{23}$$

$$|\eta_0\rangle = \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle). \tag{24}$$

With $|u\bar{u}\rangle$ and $|d\bar{d}\rangle$ being practically chiral states as opposed to a significantly heavier $|s\bar{s}\rangle$, Eqs. (22), (23), and (34) do not define the octet and singlet states of the exact SU(3) flavor symmetry, but the *effective* octet and singlet states. However, in spite of this flavor symmetry breaking by the s quark, these equations still implicitly assume nonet symmetry in the sense [43] that the same states $|q\bar{q}\rangle$ ($q = u, d, s$) appear in both the octet member η_8 (23) and the singlet η_0 (24). Nevertheless, in order to avoid the $U_A(1)$ problem, this symmetry must ultimately be broken by gluon anomaly at least at the level of the masses of pseudoscalar mesons.

In the basis (22)–(24), the nonanomalous part of the mass matrix of π^0 and etas is

$$\hat{M}^2_{NA} = \begin{bmatrix} M^2_\pi & 0 & 0 \\ 0 & M^2_{88} & M^2_{80} \\ 0 & M^2_{08} & M^2_{00} \end{bmatrix}, \tag{25}$$

showing that the isospin $I = 1$ state π^0 does not mix with the $I = 0$ states η_8 and η_0, thanks to our working in the isospin limit, where $M_{u\bar{u}} = M_{d\bar{d}}$, which we then can strictly identify with our model pion mass M_π. Since in this model we can also calculate $M^2_{s\bar{s}} = \langle s\bar{s}|\hat{M}^2_{NA}|s\bar{s}\rangle$, this gives us our calculated entries in the mass matrix:

$$M^2_{88} \equiv \langle \eta_8|\hat{M}^2_{NA}|\eta_8\rangle = \frac{2}{3}\left(M^2_{s\bar{s}} + \frac{1}{2}M^2_\pi\right)$$
$$= (594.72 \text{ MeV})^2, \tag{26}$$

$$M^2_{80} \equiv \langle \eta_8|\hat{M}^2_{NA}|\eta_0\rangle = M^2_{08} = \frac{\sqrt{2}}{3}(M^2_\pi - M^2_{s\bar{s}})$$
$$= -(487.05 \text{ MeV})^2, \tag{27}$$

$$M^2_{00} \equiv \langle \eta_0|\hat{M}^2_{NA}|\eta_0\rangle = \frac{2}{3}\left(\frac{1}{2}M^2_{s\bar{s}} + M^2_\pi\right)$$
$$= (431.22 \text{ MeV})^2. \tag{28}$$

The values on the far right of these equations were calculated from $M_\pi = M_{u\bar{u}}$ and $M_{s\bar{s}}$ from Table I, i.e., they result from the parameters fixed in Ref. [15]. These contributions are substantial, but if we take the chiral limit, all of them would tend to zero according to the GMOR relation, as required by the chiral symmetry of QCD. Thanks to this relation, even for the realistically large strange mass our approach has

$$M^2_{s\bar{s}} \approx 2M^2_K - M^2_\pi \tag{29}$$

in a reasonably good approximation, at the 10% level, whereby

$$M^2_{88} \approx \frac{4}{3}M^2_K - \frac{1}{3}M^2_\pi. \tag{30}$$

Equation (26) is thus revealed as a variant of the standard η_8 Gell-Mann–Okubo relation (30) featuring only pion

and kaon masses, which are not affected by the anomaly[1]. This is not surprising, as the role of M^2_{88} in Eq. (25) is the nonanomalous η_8 "mass." Similarly, even M_{00} (28), which by Eq. (29) gives the "η_0 Gell-Mann–Okubo relation"

$$M^2_{00} \approx \frac{2}{3}M^2_K + \frac{1}{3}M^2_\pi, \tag{31}$$

is just the *nonanomalous part* of the η_0 mass M_{η_0}. It however requires the anomalous, chiral-limit-nonvanishing part to avoid the $U_A(1)$ problem.

In our DS approach, all the model masses $M_{q\bar{q}'}$ ($q, q' = u, d, s$) and corresponding $q\bar{q}'$ bound state amplitudes are obtained in the ladder approximation. Thus, regardless of any concrete model, they are obtained with an interaction kernel which cannot possibly capture the effects of gluon anomaly. Fortunately, the large N_c expansion indicates that the leading approximation in that expansion describes the bulk of main features of QCD. The gluon anomaly is suppressed as $1/N_c$ and is viewed as a perturbation in the large N_c expansion. It is thus a meaningful approximation [25] to consider the gluon anomaly effect only at the level of mass shifts and neglect its effects on the bound-state solutions.

In the chiral limit and, as it will turn out, the SU(3) flavor limit, the gluon anomaly is coupled *only* to the singlet combination η_0 (24). Only the η_0 mass receives, from the gluon anomaly, a contribution which, unlike quasi-Goldstone masses $M_{q\bar{q}'}$'s comprising $\hat{M}^2_{NA}$, does *not* vanish in the chiral limit. As discussed in detail in Ref. [25], in the present bound-state context it is most convenient to adopt the standard way (see, e.g., Refs. [44,45]) to *parameterize* the anomaly effect. We thus break the $U_A(1)$ symmetry by shifting the η_0 (squared) mass by an amount denoted by $3\beta \equiv \lambda_\eta$ (in the respective notations of Refs. [25,26,46]). The complete mass matrix is then $\hat{M}^2 = \hat{M}^2_{NA} + \hat{M}^2_A$, where

$$\hat{M}^2_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3\beta \end{bmatrix}. \tag{32}$$

The value of the anomalous η_0 mass shift 3β is related to the topological susceptibility of the vacuum, but in the present approach must be treated as a parameter to be determined outside of our bound-state model, i.e., fixed by phenomenology or taken from the lattice calculations such as Refs. [47–50].

We now want to incorporate the effects of the realistic breaking of the SU(3) flavor symmetry into the description of the gluon anomaly. At this point it is customary to go straight to the nonstrange-strange (NS–S) basis (39) and (40), but before doing this, it is instructive to rewrite for a

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[1]The kaon is protected from mixing not only by isospin, but also by strangeness.

moment the matrix (32) in the flavor, $|q\bar{q}\rangle$ basis, where it has the pairing form,

$$\hat{M}_A^2 = \beta \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad (33)$$

since this may be the most transparent place to introduce the effect of flavor symmetry breaking into the anomalous mass shift. Namely, Eq. (33) tells us that due to the gluon anomaly, there are transitions $|q\bar{q}\rangle \rightarrow |q'\bar{q}'\rangle$; $q, q' = u, d, s$. However, the amplitudes for the transition from, and into, light $u\bar{u}$ and $d\bar{d}$ pairs are expected to be different, namely, larger, than those for the significantly more massive $s\bar{s}$. To allow for the effects of the breaking of the SU(3) flavor symmetry, we can write

$$\langle q\bar{q}|\hat{M}_A^2|q'\bar{q}'\rangle = b_q b_{q'}, \qquad (34)$$

where $b_q = \sqrt{\beta}$ for $q = u, d$ and $b_q = X\sqrt{\beta}$ for $q = s$. The anomalous mass matrix (33) is, in the flavor-broken case, thereby modified to

$$\hat{M}_A^2 = \beta \begin{bmatrix} 1 & 1 & X \\ 1 & 1 & X \\ X & X & X^2 \end{bmatrix}$$

$$\rightarrow \beta \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{3}(1-X)^2 & \frac{\sqrt{2}}{3}(2-X-X^2) \\ 0 & \frac{\sqrt{2}}{3}(2-X-X^2) & \frac{1}{3}(2+X)^2 \end{bmatrix}, \qquad (35)$$

where the arrow denotes rewriting $\hat{M}_A^2$ in the octet-singlet basis (22)–(24). Comparison with Eqs. (25) and (32) shows that incorporating into the anomaly the flavor symmetry breaking, $X \neq 1$, leads to the following. First, the anomaly influences the $\eta_8 \leftrightarrow \eta_0$ transitions, reducing the negative M_{80}^2 (27) by $\beta\frac{\sqrt{2}}{3}(2-X-X^2)$. More notably, the η_8 and η_0 masses including *both* nonanomalous and anomalous contributions are given by Eqs. (26), (28), and (35) as

$$M_{\eta_8}^2 = M_{88}^2 + \frac{2}{3}(1-X)^2\beta, \qquad (36)$$

$$M_{\eta_0}^2 = M_{00}^2 + \frac{1}{3}(2+X)^2\beta. \qquad (37)$$

Not only M_{η_0} is modified, but the interplay of the gluon anomaly and flavor breaking modified the Gell-Mann–Okubo relation (30) as the anomaly becomes coupled also to η_8 and influences its mass M_{η_8}.

The Schwinger nonet formula, derived from the condition that the trace and determinant of $\hat{M}^2 = \hat{M}_{NA}^2 + \hat{M}_A^2$ be equal to those of the same matrix in the basis of mass eigenstates, here $\hat{M}^2 = \mathrm{diag}(M_\pi^2, M_\eta^2, M_{\eta'}^2)$, now acquires the new term on the right-hand side:

$$(4M_K^2 - 3M_\eta^2 - M_\pi^2)(3M_{\eta'}^2 + M_\pi^2 - 3M_K^2) - 8(M_K^2 - M_\pi^2)$$

$$= -4(M_K^2 - M_\pi^2)3\beta(1-X^2), \qquad (38)$$

where we also used Eq. (29). The usual Schwinger formula is known to be satisfied well for the vector and tensor nonets, but *not* for the pseudoscalar nonet [51]. Equation (38) reduces to the usual Schwinger pseudoscalar-meson relation for the limit of no anomaly, $3\beta \rightarrow 0$, but also for just $X \rightarrow 1$, the limit of no influence of the flavor symmetry breaking on the anomalous mass shifts. Thus, introducing only the anomalous shift of the η_0 mass still yields the usual Schwinger relation, as noted by Ref. [51] in a different approach.

The pion remains decoupled from the etas as long as one stays in the isospin limit; i.e., after one adds the anomalous contribution $\hat{M}_A^2$ (35) to Eq. (25), one still can restrict oneself to 2×2 submatrix in the subspace of etas. The calculationally convenient basis for that subspace is the so-called *NS–S* basis:

$$|\eta_{NS}\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) = \frac{1}{\sqrt{3}}|\eta_8\rangle + \sqrt{\frac{2}{3}}|\eta_0\rangle, \qquad (39)$$

$$|\eta_S\rangle = |s\bar{s}\rangle = -\sqrt{\frac{2}{3}}|\eta_8\rangle + \frac{1}{\sqrt{3}}|\eta_0\rangle. \qquad (40)$$

In this basis, the η–η' mass matrix is

$$\hat{M}^2 = \begin{bmatrix} M_{\eta_{NS}}^2 & M_{\eta_S\eta_{NS}}^2 \\ M_{\eta_{NS}\eta_S}^2 & M_{\eta_S}^2 \end{bmatrix}$$

$$= \begin{bmatrix} M_{u\bar{u}}^2 + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M_{s\bar{s}}^2 + \beta X^2 \end{bmatrix} \xrightarrow{\phi} \begin{bmatrix} M_\eta^2 & 0 \\ 0 & M_{\eta'}^2 \end{bmatrix}, \qquad (41)$$

where the indicated diagonalization is given by the *NS–S* mixing relations[2]

$$|\eta\rangle = \cos\phi|\eta_{NS}\rangle - \sin\phi|\eta_S\rangle,$$
$$|\eta'\rangle = \sin\phi|\eta_{NS}\rangle + \cos\phi|\eta_S\rangle, \qquad (42)$$

rotating η_{NS}, η_S to the mass eigenstates η, η'. (In the last section we will use the effective-singlet-octet mixing angle θ, defined by analogous mixing relations where $\eta_{NS} \rightarrow \eta_8$, $\eta_S \rightarrow \eta_0$, $\phi \rightarrow \theta$. It is related to the completely equivalent *NS–S* mixing angle ϕ as $\theta = \phi - \arctan\sqrt{2} = \phi - 54.74°$.)

Now the *NS–S* mass matrix (41) tells us that due to the gluon anomaly, there are transitions $|\eta_{NS}\rangle \leftrightarrow |\eta_S\rangle$. As in the argument above Eq. (35), the amplitude for the transition from, and into, η_{NS} need not be the same as those for the more massive η_S. The role of the flavor-symmetry-

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[2]The relation between the present approach and the two-mixing-angle scheme is clarified in the Appendix of Ref. [26].

breaking factor X is to allow for that possibility. There are arguments [26], supported by phenomenology, that the transition suppression is estimated well by the nonstrange-to-strange ratio of respective quark constituent masses, $\mathcal{M}_u$ and $\mathcal{M}_s$. Because of the Goldberger-Treiman relation, this ratio must be close [25,26] to the ratio of η_{NS} and η_S pseudoscalar decay constants $f_{\eta_{NS}} = f_\pi$ and $f_{\eta_S} = f_{s\bar{s}}$. In other words, we can estimate the flavor-symmetry-breaking suppression factor as $X \approx \mathcal{M}_u/\mathcal{M}_s$ or $X \approx f_\pi/f_{s\bar{s}}$. (Yet another, but again closely related way of estimating X, is from the ratios of $\gamma\gamma$ amplitudes, as in Refs. [26,46].) In the present paper, we use $X \approx f_\pi/f_{s\bar{s}}$, because these decay constants are not only calculable in the DS approach, but also (in contrast to "constituent quark masses") defined without any arbitrariness, and in the case of f_π even experimentally measurable. Our present model result $f_\pi/f_{s\bar{s}} = 0.6991$ (see Table I) is reasonably close to $X_{exp} \approx 0.78$ extracted phenomenologically [26] from the *empirical* mass matrix $\hat{m}_{exp}^2$ featuring experimental pion and kaon masses, or, after diagonalization, experimental η and η' masses—see Eq. (43) below.

B. Mixing angle and other results from η–η' mass matrix

In this subsection, let us first see what hints we get from phenomenology. In our present notation, capital M_a's denote the calculated, model pseudoscalar masses, whereas lowercase m_a's denote the corresponding empirical masses.

From our calculated, model mass matrix (41) we form its empirical counterpart (43) by (i) obvious substitutions $M_{u\bar{u}} \equiv M_\pi \rightarrow m_\pi$ and $M_{s\bar{s}} \rightarrow m_{s\bar{s}}$, and (ii) by noting that $m_{s\bar{s}}$, the "empirical" mass of the unphysical $s\bar{s}$ pseudoscalar bound state, is given in terms of masses of physical particles as $m_{s\bar{s}}^2 \approx 2m_K^2 - m_\pi^2$ due to GMOR. Then,

$$\hat{m}_{exp}^2 = \begin{bmatrix} m_\pi^2 + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & 2m_K^2 - m_\pi^2 + \beta X^2 \end{bmatrix} \xrightarrow[\phi_{exp}]{} \begin{bmatrix} m_\eta^2 & 0 \\ 0 & m_{\eta'}^2 \end{bmatrix},$$
(43)

where the arrow indicates the diagonalization (42) for the angle value ϕ_{exp}.

Since $M_{u\bar{u}}$, obtained by solving the BS equation, is identical to our model pion mass M_π, it was fitted to the empirical pion mass m_π in Ref. [15]. Similarly, $M_{u\bar{s}} \equiv M_K$ is fitted to the empirical kaon mass m_K, so that Eq. (29) implies $M_{s\bar{s}}^2 \approx 2m_K^2 - m_\pi^2$. We thus see that in our model mass matrix, the parts stemming from its *nonanomalous* part $\hat{M}_{NA}^2$ (25) are already close to the corresponding parts in $\hat{m}_{exp}^2$. We can thus expect a good overall description of the masses in η and η' complex. We now proceed to verify this expectation.

The *anomalous* entry β, along with X (and then necessarily also ϕ), is fixed phenomenologically if we require that they fit the masses m_η and $m_{\eta'}$ in the empirical matrix

(43) to their experimental values. This is achieved by requiring that trace and determinant of $\hat{m}_{exp}^2$ have their experimental values, which leads (e.g., see [52,53]) to the relations

$$\beta_{exp} = \frac{(m_{\eta'}^2 - m_\pi^2)(m_\eta^2 - m_\pi^2)}{4(m_K^2 - m_\pi^2)} = 0.2785 \text{ GeV}^2, \quad (44)$$

$$X_{exp} = \sqrt{2\frac{(m_{\eta'}^2 - 2m_K^2 + m_\pi^2)(2m_K^2 - m_\pi^2 - m_\eta^2)}{(m_{\eta'}^2 - m_\pi^2)(m_\eta^2 - m_\pi^2)}}$$
$$= 0.7791, \quad (45)$$

$$\phi_{exp} = \arctan\left[\frac{(m_{\eta'}^2 - 2m_K^2 + m_\pi^2)(m_\eta^2 - m_\pi^2)}{(2m_K^2 - m_\pi^2 - m_\eta^2)(m_{\eta'}^2 - m_\pi^2)}\right]^{1/2}$$
$$= 41.88°. \quad (46)$$

Now we want the analogous results from our theoretical mass matrix (41), where only β is not a calculated quantity. For example, in this subsection we use $X = f_\pi/f_{s\bar{s}} \approx 0.6991$ from Table I. We thus fix only β by requiring that the experimental value of the trace $m_\eta^2 + m_{\eta'}^2 \approx 1.22$ GeV^2 be fitted by the theoretical mass matrix (41). This yields

$$\beta = \frac{1}{2 + X^2}[(m_\eta^2 + m_{\eta'}^2)_{exp} - (M_{u\bar{u}}^2 + M_{s\bar{s}}^2)], \quad (47)$$

whereby X, M_{π^0}, and $M_{s\bar{s}}$ from Table I give us $\beta = 0.2723$ GeV^2, in good agreement with β_{exp} (44) obtained from the empirical mass matrix (43).

Since $M_{u\bar{u}}^2 + M_{s\bar{s}}^2 = 2m_K^2$ holds to a very good approximation due to GMOR, our approach satisfies well the first equality [from Eq. (47)] in

$$\beta(2 + X^2) = m_\eta^2 + m_{\eta'}^2 - 2m_K^2 = \frac{2N_f}{f_\pi^2}\chi, \quad (48)$$

where the second equality is the Witten-Veneziano (WV) formula [54], with χ being the topological susceptibility of the pure Yang-Mills gauge theory. The WV formula with experimental masses and f_π yields $\chi \approx (179 \text{ MeV})^4$, which is in good agreement with our value $\chi = (177 \text{ MeV})^4$, implied by Eq. (48) and our model values of $X = f_\pi/f_{s\bar{s}}$ and β from Eq. (47). Our prediction for χ is also in reasonable agreement with the recent lattice results [47–49] considered in Ref. [50]. The central value of Lucini *et al.* $\chi_1 = (177 \pm 7 \text{ MeV})^4$ [47] agrees precisely with our value, while their value obtained by a different method, $\chi_2 = (184 \pm 7 \text{ MeV})^4$, is higher but still consistent with our χ. However, $\chi_3 = (191 \pm 5 \text{ MeV})^4$ of Ref. [48] is too high for that. On the other hand, our result is only marginally too high to be consistent with the most precise lattice topological susceptibility so far, $\chi_4 = (174.3 \pm 0.5 \pm 1.2^{+1.1}_{-0.2} \text{ MeV})^4$ [49]. In summary, our χ is

consistent with

$$\chi = (175.7 \pm 1.5 \text{ MeV})^4, \qquad (49)$$

the weighted average of the recent lattice results [47–49]. Let us pause briefly to note that β does not have to be treated as the parameter to be fixed by fitting the masses, since Eq. (48) enables one to determine β from the lattice results on the topological susceptibility. In fact, this is how we get the results in the column C of Table II, as detailed in the last, concluding section.

However, now we continue with β from Eq. (47) fitting $(m_\eta^2 + m_{\eta'}^2)_{\text{exp}}$. For this value, $\beta = 0.2723 \text{ GeV}^2$, the values of our *calculated* η_{NS} and η_S masses are

$$M_{\eta_{NS}}^2 = M_{u\bar{u}}^2 + 2\beta = M_\pi^2 + 2\beta = 0.5628 \text{ GeV}^2$$
$$= (750.2 \text{ MeV})^2 \qquad (50)$$

and

$$M_{\eta_S}^2 = M_{s\bar{s}}^2 + \beta X^2 = 0.6545 \text{ GeV}^2 = (809.0 \text{ MeV})^2. \qquad (51)$$

These are reasonable values, in good agreement with, e.g., η_{NS} and η_S masses calculated in the dynamical SU(3) linear σ model [46].

TABLE II.   Various theoretical results on η and η' and comparison with their experimental masses and $\gamma\gamma$ decay widths. All calculated quantities were obtained with parameters which gave the very good description of pions and kaons in Ref. [15], i.e., Eq. (20) and $\tilde{m}_{u,d} = 3.046 \text{ MeV}$, $\tilde{m}_s = 67.70 \text{ MeV}$. The contents of the columns is this: $\mathcal{A}$) Summarized predictions of subsecs. III A and III B, i.e., with $X = f_\pi/f_{s\bar{s}}$. $\mathcal{B}$) X estimated from $P \to \gamma\gamma$ amplitudes (see text). In both $\mathcal{A}$ and $\mathcal{B}$, the parameter β is fixed by fitting the experimental value of the mass matrix trace, Eq. (47). C) $X = f_\pi/f_{s\bar{s}}$ again, while β is not a fitting parameter but obtained from $\chi = (175.7 \text{ MeV})^4$, the weighted average (49) of the recent lattice topological susceptibilities [47–49]. The results of this column are thus obtained without any free parameters. $\mathcal{D}$) Starting from the calculated $\gamma\gamma$ amplitudes $T_{\pi^0}^{\gamma\gamma}$ and $T_{s\bar{s}}^{\gamma\gamma}$ in Table I, the mixing angle value is obtained as the fitting parameter in the best χ^2 fit to the experimental amplitudes (62) of η and η'. $\mathcal{E}$) experimental values. The widths $\Gamma(\eta, \eta' \to \gamma\gamma)$ are calculated using the experimental masses. The $\gamma\gamma$ decay widths are in units of keV, masses are in units of MeV, 3β in units of MeV^2, while X and the mixing angles are dimensionless.

	$\mathcal{A}$	$\mathcal{B}$	C	$\mathcal{D}$	$\mathcal{E}$
θ	$-14.57°$	$-14.93°$	$-15.18°$	$-14.27°$	
M_η	579.3	576.1	577.1		547.75 ± 0.12
$M_{\eta'}$	939.0	941.0	932.0		957.78 ± 0.14
X	0.6991	0.7124	0.6991		
3β	816951	810835	798060		
$\Gamma(\eta \to \gamma\gamma)$	0.5034	0.5114	0.5170	0.4968	0.510 ± 0.026
$\Gamma(\eta' \to \gamma\gamma)$	4.272	4.229	4.199	4.308	4.29 ± 0.15

The mixing angle is then determined to be $\phi = 40.17°$ (or equivalently, $\theta = -14.57°$), for example, through the convenient relation

$$\tan 2\phi = \frac{2\sqrt{2}\beta X}{M_{\eta_S}^2 - M_{\eta_{NS}}^2}. \qquad (52)$$

The diagonalization of the NS–S mass matrix gives us the η and η' masses:

$$M_\eta^2 = \cos^2\phi M_{\eta_{NS}}^2 - \sqrt{2}\beta X \sin 2\phi + \sin^2\phi M_{\eta_S}^2, \qquad (53)$$

$$M_{\eta'}^2 = \sin^2\phi M_{\eta_{NS}}^2 + \sqrt{2}\beta X \sin 2\phi + \cos^2\phi M_{\eta_S}^2. \qquad (54)$$

Plugging in the above predictions for β, X, $M_{\eta_{NS}}$, and M_{η_S}, our model η and η' masses then turn out to be $M_\eta = 579 \text{ MeV}$ and $M_{\eta'} = 939 \text{ MeV}$, in good agreement with the respective empirical values of 547 MeV and 958 MeV.

The model values predicted in this and in the next subsection are summarized in column $\mathcal{A}$ of Table II. Columns $\mathcal{B}$ and C give analogously obtained results, but with differently chosen either flavor breaking parameter X or the topologically susceptibility, i.e., β. The comparison of our predictions shows they are robust under these variations.

C. Two-photon decays of η and η'

Having obtained the predictions for the mixing in the η–η' complex, we can get the predictions for the $\gamma\gamma$ decays of η and η' from the decay amplitudes $T_{\pi^0}^{\gamma\gamma}$ and $T_{s\bar{s}}^{\gamma\gamma}$ already given in Table I. However, for the sake of completeness, let us first briefly review how these amplitudes are obtained.

The transition between the neutral pseudoscalar meson P and two photons $\gamma(k)$ and $\gamma(k')$ with momenta k and k' can be described by a scalar amplitude we denote $T_P(k^2, k'^2)$ [39]. The special case of the decay $P \to \gamma\gamma$ into two real, on shell photons corresponds to the $k^2 = k'^2 = 0$ amplitude

$$T_P^{\gamma\gamma} \equiv T_P(0, 0) = \text{const.}, \qquad (55)$$

so that by integrating over the phase space and summing over the photon polarizations one gets the decay width

$$\Gamma(P \to \gamma\gamma) = \frac{\pi\alpha_{\text{em}}^2 m_P^3}{4} |T_P^{\gamma\gamma}|^2, \qquad (P = \pi^0, \eta, \eta'). \qquad (56)$$

The calculation of the electromagnetic transition amplitudes proceeds in the same way as in our earlier papers such as Refs. [15,25,26,39–42,55], since the incorporation of the quark-photon interactions is the same as adopted there through the scheme of a generalized impulse approximation, where all propagators, bound-state vertices, and quark-photon vertices are dressed. (In the present application, this impulse approximation is illustrated by the $q\bar{q}$
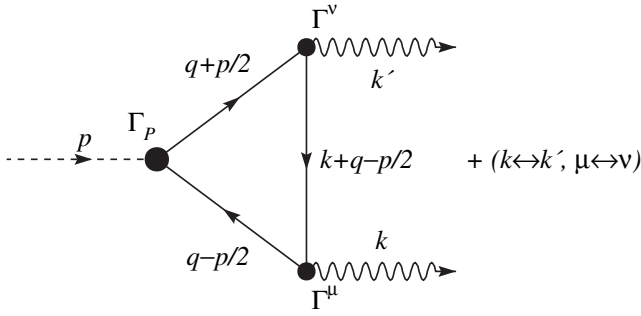
FIG. 1. The triangle graph (and its crossed graph) relevant for the interaction of two photons of momenta k and k' with the neutral pseudoscalar meson P of momentum p represented by the quark-antiquark bound-state vertex $\Gamma_P(q, p)$. The quark-photon coupling is in general given by dressed vector vertices $\Gamma_\mu(q_1, q_2)$, which in the free limit reduce to $\hat{Q}\gamma_\mu$, where $\hat{Q} = \mathrm{diag}(Q_u, Q_d, Q_s) = e\,\mathrm{diag}(+2/3, -1/3, -1/3)$ is the flavor SU(3) quark charge matrix and e is the electromagnetic charge, $e = \sqrt{4\pi\alpha_{\mathrm{em}}}$.

pseudoscalar-$\gamma\gamma$ triangle graph in Fig. 1.) They are all dressed mutually consistently, so that the pertinent Ward-Takahashi identities are respected (e.g., see Refs. [36,37,56]). This is necessary for reproducing exactly and analytically anomalous $\gamma\gamma$ (on shell) amplitudes[3] in the chiral limit, and requires the usage of a dressed quark-photon vertex satisfying the vector Ward-Takahashi identity. We employ the Ball-Chiu vertex [60], which is widely used (e.g., see Refs. [20–22,25,39–42,55] and references therein).

If one works in the NS–S basis (39) and (40), the diagonalization (42) and the amplitudes

$$T^{\gamma\gamma}_{\eta_{NS}} = \frac{1}{\sqrt{2}}[T^{\gamma\gamma}_{u\bar{u}} + T^{\gamma\gamma}_{d\bar{d}}] = \frac{5}{3}T^{\gamma\gamma}_{\pi^0}, \qquad (57)$$

$$T^{\gamma\gamma}_{\eta_S} = T^{\gamma\gamma}_{s\bar{s}}, \qquad (58)$$

give the (mixing-dependent) amplitudes of the physical particles η and η':

$$T^{\gamma\gamma}_\eta = \cos\phi\, T^{\gamma\gamma}_{\eta_{NS}} - \sin\phi\, T^{\gamma\gamma}_{\eta_S}, \qquad (59)$$

$$T^{\gamma\gamma}_{\eta'} = \sin\phi\, T^{\gamma\gamma}_{\eta_{NS}} + \cos\phi\, T^{\gamma\gamma}_{\eta_S}. \qquad (60)$$

In the preceding subsection, describing the breaking of the SU(3) flavor symmetry by $X = f_\pi/f_{s\bar{s}}$ led to $\phi = 40.17°$. This mixing angle value yields the physical amplitude values

$$T^{\gamma\gamma}_\eta = 0.2706 \text{ GeV}^{-1}, \qquad T^{\gamma\gamma}_{\eta'} = 0.3410 \text{ GeV}^{-1}, \quad (61)$$

which agree very well with the experimental amplitudes

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[3]And others, notably the "box anomaly" process $\gamma\pi^+ \rightarrow \pi^+\pi^0$; see Refs. [55,57–59].

$$(T^{\gamma\gamma}_\eta)_{\mathrm{exp}} = 0.2724 \pm 0.0069 \text{ GeV}^{-1},$$
$$(T^{\gamma\gamma}_{\eta'})_{\mathrm{exp}} = 0.3417 \pm 0.0060 \text{ GeV}^{-1}. \qquad (62)$$

The corresponding calculated and experimental widths $\Gamma(P \rightarrow \gamma\gamma)$ ($P = \eta, \eta'$) are, respectively, displayed in the columns $\mathcal{A}$ and $\mathcal{E}$ of Table II. As already mentioned, other columns give our predictions based on somewhat different choices for either the flavor-symmetry-breaking parameter X or the topological susceptibility. Similarly to the results of the previous subsection, our results on $\gamma\gamma$ decays turn out to be robust under these variations, since the resulting changes in the mixing angle are not excessive.

It is, anyway, very satisfying to note that the *mixing-independent* combination of our theoretical $\gamma\gamma$ decay amplitudes

$$|T^{\gamma\gamma}_\eta|^2 + |T^{\gamma\gamma}_{\eta'}|^2 = |T^{\gamma\gamma}_{\eta_{NS}}|^2 + |T^{\gamma\gamma}_{\eta_S}|^2 = 0.1895 \text{ GeV}^{-2} \qquad (63)$$

matches very well the corresponding experimental value

$$|(T^{\gamma\gamma}_\eta)_{\mathrm{exp}}|^2 + |(T^{\gamma\gamma}_{\eta'})_{\mathrm{exp}}|^2 = 0.1909 \pm 0.0056 \text{ GeV}^{-2}, \qquad (64)$$

because this match does not depend on the mixing angle at all.

IV. DISCUSSION OF RESULTS AND CONCLUSION

A. On modification of analytic mass relations

The interplay of the gluon anomaly and flavor symmetry breaking modifies some flavor SU(3) mass relations. The details of forming the mass matrix in subsection III A revealed the corrected η_8 Gell-Mann–Okubo relation (36), its η_0 analogue (37), and Schwinger formula (38). It is obvious that such results are independent of the DS approach and even of any concrete dynamics, and depend only on the way the gluon anomaly and the SU(3) flavor symmetry breaking are implemented. It is thus not surprising that our Eqs. (36)–(38) are, respectively, equivalent to Eqs. (13), (15), and (38) of, e.g., Ref. [27] which does not address any dynamics.

Our primary aim concerning these mass relations is to point out that all of them have been implicitly contained in our DS approach to the η–η' complex and its results ever since Ref. [26] incorporated into the DS approach the effects of the flavor symmetry breaking on the anomalous mass shift. However, it is instructive to consider these relations explicitly. Already Eq. (38) showed that the usual Schwinger formula, which is badly violated for pseudo-scalars, acquires the correction proportional to $3\beta(1 - X^2)$. Note that the corrected Schwinger formula (38) is identically satisfied if β and X are, respectively, given by Eqs. (44) and (45), since these equations stem from the determinant and trace conditions, just like Eq. (38) itself. As for the η_8 Gell-Mann–Okubo formula, the original one

(30) was modified in Eq. (36) by the term

$$\delta \equiv \frac{2}{3}(1 - X)^2 \beta = \chi \frac{4N_f}{3f_\pi^2} \frac{(1 - X)^2}{2 + X^2}, \qquad (65)$$

where the last equality comes from expressing β by the topological susceptibility χ through the WV relation (48). This result is independent of our concrete model, or even of the DS approach in general, as essentially the only strong simplifying assumption was the one of the (flavor-broken) nonet symmetry. If $X = f_\pi/f_{s\bar{s}}$ is chosen, the first-order flavor-symmetry-breaking estimate of $f_{s\bar{s}}$ (which is in our model satisfied better than 0.5%—see Table I), $f_{s\bar{s}} \approx 2f_K - f_\pi$, permits expressing the correction (65) exclusively through experimental and lattice values. So, the experimental f_π, f_K and the central value of the weighted average (49) of the recent lattice results [47–49] on χ gives $\delta \approx (127 \text{ MeV})^2$ independently of any model—but in good agreement with our model result $\delta = (128 \text{ MeV})^2$ [from our model values of $X = f_\pi/f_{s\bar{s}}$ and β from Eq. (47)]. Equation (36) then yields $M_{\eta_8} = 608$ MeV, in excellent agreement with the chiral perturbation theory result of some 610 MeV [61]. It is satisfying that regardless of whether we take the lattice results for χ and empirical results for decay constants, or our model values for β and X, our correction (65) is in reasonable agreement with Eq. (13.9) of Hagiwara *et al.* [62] (where our $\delta/M_{88}^2 = \Delta$ in their notation). It gives $\theta = -14.1°$, in good agreement with our model results discussed in the next, main subsection of the conclusion.

B. Model results

The bulk of our results on the η–η' complex are summarized in Table II, namely, our model predictions for the masses, the flavor-symmetry-breaking parameter X, then 3β, the anomalous squared mass of η_0 in the chiral or SU(3) flavor limit (where $\eta' = \eta_0$), the mixing angle $\theta = \phi - \arctan\sqrt{2}$ and the η, $\eta' \to \gamma\gamma$ decays. The results in the column $\mathcal{A}$ were also given in the text in the course of explaining our procedure. This column was obtained by assuming that the flavor-symmetry-breaking parameter X is given by our model value of $f_\pi/f_{s\bar{s}}$. The requirement (47) that the inclusion of the anomalous mass contribution fit the sum of the (squared) experimental masses, $(m_\eta^2 + m_{\eta'}^2)_{\text{exp}}$, then fixes the mass matrix and the mixing angle which diagonalizes it. This angle is then used to calculate the $\gamma\gamma$ decay amplitudes and, equivalently, the corresponding widths. The values in the column $\mathcal{B}$ were obtained in the same way except that the flavor-symmetry-breaking parameter X is estimated from the ratios of our calculated $\gamma\gamma$ amplitudes, $X = (T_{u\bar{u}}^{\gamma\gamma}/\mathcal{Q}_u^2)/(T_{s\bar{s}}^{\gamma\gamma}/\mathcal{Q}_s^2)$, as explained in Ref. [26]. In the column $\mathcal{C}$ we again use $X = f_\pi/f_{s\bar{s}}$ as in the column $\mathcal{A}$, but instead of fixing β by fitting $(m_\eta^2 + m_{\eta'}^2)_{\text{exp}}$, we obtain β through the WV relation (48) from the central value of the weighted average (49) of the recent

results on the topological susceptibility calculated on the lattice [47–49]. Thus, the column $\mathcal{C}$ gives the masses, mixing and $\gamma\gamma$ decay widths without any parameter fitting whatsoever. In spite of this, the results in the column $\mathcal{C}$ are just as consistent with the experimental masses and decay widths as the results in the columns in which β was used for fitting.

In the column $\mathcal{D}$ we give the best χ^2-fit to the experiment which our theoretical $\gamma\gamma$ decay amplitudes $T_{\pi^0}^{\gamma\gamma}$ and $T_{s\bar{s}}^{\gamma\gamma}$ can give using the mixing angle as a free fitting parameter, regardless of the results on the mixing angle from the η–η' mass matrices. Nevertheless, the comparison with other columns in Table II shows that what we find from η, $\eta' \to \gamma\gamma$ processes is actually close to what we find from the mass matrix, which is of course very satisfying. Actually, the comparison of all results in Table II shows generally that our theoretical results in all columns are similar among themselves, exhibiting robustness under input variations, and all are in reasonable agreement with the experimental results in column $\mathcal{E}$.

Let us now compare the present results with our earlier work [25,26] on the η–η' complex. It also employed the consistently coupled DS approach but using the Jain–Munczek effective interaction [33] instead of our Eq. (15). The treatment of the η–η' complex used here was largely formulated already in Ref. [25], except that it did not consider the interplay of the SU(3) flavor symmetry breaking and the gluon anomaly, which was taken into account in Ref. [26] and found important for the successful meson phenomenology. The present paper, employing a different interaction, confirms its importance and further clarifies it by displaying explicitly the interplay of β and X in the mass relations (36)–(38). While the reproduction of the η and η' masses in Ref. [26] was similarly successful, we find that our η, $\eta' \to \gamma\gamma$ decay widths agree with the experiment much better now, although the error bars shrunk as the precision of the experimental widths substantially increased in the meantime [63].

Concerning the agreement with other approaches, we may point out that the η–η' mass matrix obtained on lattice by UKQCD collaboration [64] agrees reasonably well with our model mass matrix if we insert in it our values of β, X, $M_{u\bar{u}}$, and $M_{s\bar{s}}$. We should also recall that already Ref. [26] clearly showed that our DS approach and results are not in conflict, but in fact agree very well with results in the two-mixing-angle scheme (reviewed and discussed in, e.g, Ref. [65]). Actually, Ref. [26] showed that our results can also be given in the two-mixing-angle scheme, but it is defined with respect to the mixing of decay constants, and therefore in our case, as in the DS approach in general, the scheme with one angle defining the mixing of the states is more convenient.

Another important general feature of the consistent DS approach which also holds in the present paper, is the correct chiral QCD behavior, so that all light pseudoscalar

mesons constructed in our approach are both $q\bar{q}$ bound states and (almost-)Goldstone bosons of DχSB. In the chiral limit for all quark flavors, the anomalous mass matrix $\hat{M}_A^2$ [with $X \to 1$, Eq. (32)] is the only nonvanishing contribution as $\hat{M}_{NA}^2 \to 0$. Thus, in this limit $\eta \to \eta_8$ with vanishing mass, and $\eta' \to \eta_0$ with the mass $\sqrt{3}\beta$ (= 0.904 GeV in column $\mathcal{A}$ and similarly in the other columns), which is the only nonvanishing pseudoscalar mass in that limit, being induced purely by gluon anomaly. This chiral-limit mass $\sqrt{3}\beta$ is thus only 6% below the experimental η' mass. Even with the realistic breaking of the chiral symmetry (as in Table II) which rises the η by more than 0.5 GeV, our mixing angles show that η and η' are much closer to, respectively, the (almost-)Goldstone octet η_8 and non-Goldstone singlet η_0 than to η_{NS} and η_S (which is why in this section we switched from ϕ to θ). For example, column $\mathcal{A}$, with $\theta = -14.57°$, implies

$$
\begin{aligned}
|\eta\rangle &= 0.968|\eta_8\rangle + 0.257|\eta_0\rangle, \\
|\eta'\rangle &= -0.257|\eta_8\rangle + 0.968|\eta_0\rangle,
\end{aligned}
\tag{66}
$$

where squaring of the coefficients $\cos\theta = 0.968$ and $\sin\theta = -0.257$ says that η is 93.4% η_8 and 6.6% η_0, and reverse for η'.

In conclusion, the consistently coupled DS approach with the effective interaction (15) enhanced by the gluon condensates gives a very good description of the η–η' complex. This was achieved along the lines formulated in

Refs. [25,26], where the consistently coupled DS approach was extended by assuming the mass shift of the singlet η_0 due to the gluon anomaly. After Ref. [15] found the model parameters for which the gluon-condensate-enhanced interaction (15) leads to a sufficiently strong DχSB, pions and kaons as (quasi-)Goldstone bosons of QCD, and their successful DS phenomenology, this minimal extension was the only new element in the otherwise fixed model. This was enough to successfully model η and η' mesons without any parameter refitting, which is not very surprising after the success with pions and kaons in Ref. [15]. In Refs. [25] and especially [26] we have already given good descriptions of η and η' employing the Jain-Munczek effective interaction [33], which is however purely modeled at the low and intermediate energies. In contrast, the important intermediate-momentum behavior of the presently used interaction (15) may be actually understood in terms of gluon condensates, instead of just modeled. In addition, this interaction has presently given the description of η and η' which is on the whole somewhat better than in Refs. [25,26], especially in view of the increased precision of the η, $\eta' \to \gamma\gamma$ measurements [63].

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