Homogenization of the elastic plate equation

Jankov, Jelena

University of Zagreb
Faculty of Science
Department of Mathematics

Jelena Jankov

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Supervisors:
Assoc. Prof. Krešimir Burazin
Assoc. Prof. Marko Vrdoljak

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Jelena Jankov

Homogenizacija jednadžbe elastične ploče

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Mentori:
Izv. prof. dr. sc. Krešimir Burazin
Izv. prof. dr. sc. Marko Vrdoljak

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Summary

The main goal of this thesis is to study homogenization of the Kirchhoff-Love model for pure bending of a thin symmetric elastic plate, which is described by the fourth order elliptic equation. Homogenization theory is one of the most successful approaches for dealing with optimal design problems (in conductivity or linearized elasticity), which consists of arranging given materials such that obtained body satisfies some optimality criteria, typically expressed mathematically as the minimization of some (integral) functional under some (PDE) constraints. The key role in homogenization theory has H-convergence.

After a brief introduction, in Chapter 1 we prove a number of properties of H-convergence, such as locality, independence of boundary conditions, metrizability of H-topology, convergence of energies and a corrector result. We also discuss smooth dependence of H-limit on a parameter and calculate the H-limit of a periodic sequence of tensors. Moreover, we give special emphasis to calculating the first correction in the small-amplitude homogenization limit of a sequence of periodic tensors.

Using this newly developed theory, in Chapter 2 we put our focus on the composite elastic plate. We show the local character of the set of all possible composites, also called the G-closure, and prove that the set of composites obtained by periodic homogenization is dense in that set. Additionally, we derive explicit expressions for elastic coefficients of composite plate obtained by mixing two materials in thin layers (known as laminated material), and for mixing two materials in the low-contrast regime. Moreover, we derive optimal bounds on the effective energy of a composite material, known as Hashin-Shtrikman bounds. In the case of two-phase isotropic materials, explicit optimal Hashin-Shtrikman bounds are calculated. We show that an analogous results can be derived for the complementary energy of a composite material.

Keywords: Kirchhoff-Love model of elastic plate, composite material, G-closure, Hashin-Shtrikman bounds, homogenization, H-convergence, laminated material, small-amplitude homogenization;

Sažetak

Teorija homogenizacije razvijena je za eliptičku jednadžbu drugog reda, a glavni cilj ove disertacije je razvoj teorije homogenizacije za Kirchhoff-Loveovu jednadžbu tanke simetrične elastične ploče, koja je eliptička jednadžba četvrtog reda. Teorija homogenizacije jedan je od najuspješnijih pristupa rješavanju problema optimalnog dizajna (u vodljivosti i lineariziranoj elastičnosti), gdje je cilj odrediti raspored danih materijala (ili samo jednog materijala) u danom univerzalnom skupu. Pri tome se optimalnost rasporeda (distribucije) materijala mjeri funkcionalom koji je obično integralni funkcional koji ovisi o distribuciji materijala, ali i rješenju pripadne parcijalne diferencijalne jednadžbe.

Osnovni pojam teorije homogenizacije predstavlja H-konvergencija. Spagnolo je 1968. godine uveo pojam G-konvergencije za simetrične koeficijente, a zatim su taj pojam generalizirali Tartar 1975. godine, te Murat i Tartar za nesimetrične koeficijente, pod imenom H-konvergencija 1978. godine. Teorija je najprije razvijena za jednadžbu stacionarne difuzije, a kasnije proširena na sustav linearizirane elastičnosti. Postoje i rezultati za eliptičke jednadžbe višeg reda, a također i opsežna literatura od strane ruskih autora koji često koriste termin *jaka G-konvergencija*. Motivirani mogućim primjenama u optimalnom dizajnu, 1999. godine Antonić i Balenović definirali su H-konvergenciju u kontekstu jednadžbe elastične ploče te su pokazali da vrijedi teorem kompaktnosti.

Nakon uvoda, u Poglavlju 1 dokazuju se novi rezultati o svojstvima H-konvergencije promatrane jednadžbe, poput lokalnosti, neovisnosti o rubnim uvjetima, metrizabilnosti H-topologije i konvergencije energija. Izvode se rezultati o korektorima, te se komentira njihova jedinstvenost. Pri izvođenju ovih rezultata, koristi se Tartarova metoda oscilirajućih test funkcija i rezultat kompaktnosti kompenzacijom, čija je varijanta dokazana za jednadžbu elastične ploče. Analizira se glatka ovisnost H-limesa o parametru i računa H-limes periodičkog niza tenzora. Općenito, H-limes je nemoguće eksplicitno izračunati, osim u nekim posebnim slučajevima, među kojima je i proces periodičke homogenizacije. Proučava se i homogenizacija malih amplituda u periodičkom slučaju, čiji je cilj izračunati H-limes niza koeficijenata koji imaju slična elastična svojstva.

Koristeći prethodno dokazane rezultate, u Poglavlju 2 poseban naglasak stavljen je na kompozitne materijale, tj. na mješavinu materijala na mikroskali. Ovdje se prirodno pojavljuje problem određivanja skupa svih mogućih mješavina dobivenih postupkom homogenizacije, koji je poznat pod nazivom Problem G-zatvarača. Općenito, za jednadžbu elastične ploče G-zatvarač nije poznat, čak ni za mješavine dvaju izotropnih faza. Pokazuje se lokalni karakter G-zatvarača, te da je skup svih mješavina dobivenih procesom periodičke homogenizacije gust podskup G-zatvarača. Nadalje, izvode se efektivni koeficijenti elastične ploče nastale miješanjem dva materijala u tankim slojevima (ovako proizvedeni materijali nazivaju se lamine), te efektivni koeficijenti ploče napravljene od dva materijala sa sličnim elastičnim svojstvima, odnosno pod pretpostavkom malog kontrasta ili malih amplituda. Izvode se i optimalne ocjene na efektivnu energiju kompozitnog materijala, poznate kao Hashin-Shtrikmanove ocjene. Za primjenu u optimalnom dizajnu potrebno ih je eksplicitno izračunati, kao i odgovarajuće (nizovne) lamine koje ih saturiraju, stoga se u slučaju mješavine dva izotropna materijala, računaju eksplicitne Hashin-Shtrikmanove ocjene. Također, analogni rezultati izvode se i za komplementarnu eneriju kompozitnog materijala.

Očekuje se da će dobiveni rezultati utrti put k novim rezultatima vezanim za optimalni dizajn tankih elastičnih ploča.

Ključne riječi: Kirchhoff-Loveov model elastične ploče, kompozitni materijal, G-zatvarač, Hashin-Shtrikmanove ocjene, homogenizacija, H-konvergencija, laminirani materijal, homogenizacija malih amplituda;

Contents

Introduction

Historical roots and motivation

The theory of homogenization is interesting from both the theoretical and the practical perspective. In order to use its full potential, one first has to develop theoretical results, which might have important applications, for example in optimal design problems. Commonly, optimal design problems do not have solutions (if they exist, such solutions are usually called classical). Therefore, one needs to consider a proper relaxation of the original problem. A relaxation by the homogenization method was introduced in [58], and it consists in introducing generalized composite materials, which are mixtures of original phases on a microscopic scale. Such relaxed problems have solutions, and we call them relaxed or generalized solutions.

The physical idea of homogenization is to average heterogeneous media in order to derive effective properties: we have a fine mixture of some materials and we want to approximate it by a new homogeneous one. Justification for this procedure is that we are not interested in what is happening at every point of the problem domain but rather what is happening on a macroscopic scale. For example, in the model problem of conductivity we are not interested in the pointwise temperature, but in average temperature in some (small) region. The outcomes of this approach are very important, since from a numerical point of view, solving equations will require too much effort if the length scale of heterogeneity is very small.

Therefore, rather than considering a simple heterogeneous media with a fixed length scale $\varepsilon(n)$, such that $\varepsilon(n) \to 0$ as $n \to +\infty$, and studying a single problem, we observe a sequence of similar problems:

$$\begin{cases} A_n u_n = f \quad \text{in} \quad \Omega \\ \text{initial/boundary condition}\,, \end{cases}$$

where A_n, $n \in \mathbf{N}$, are partial differential operators and Ω some highly heterogeneous domain. Information about the heterogeneity of Ω is usually contained in coefficients of the corresponding PDE. One can let the length scale go to zero: if $u_n \to u$ and $A_n \to A$ (in some sense), as $n \to \infty$, the following initial/boundary value problem is called the limit (effective, homogenized) problem:

$$\begin{cases} Au = f \quad \text{in} \quad \Omega \\ \text{initial/boundary condition} . \end{cases}$$

Clearly, the mathematical difficulty is to define an adequate topology for this notion of convergence of problems, as $n \to \infty$; the most important concept in the theory of homogenization is that of H-convergence. It was introduced by Spagnolo through the concept of G-convergence for symmetric coefficients [65], and further generalized by Tartar [71] and Murat and Tartar for non-symmetric coefficients under the name H-convergence [58]. The theory was first developed for the stationary diffusion equation and later extended to a linearized elasticity system (see [2] and references therein). There is also a quite extensive literature by Russian authors who often use the term *strong G-convergence* [61, 76]. The compactness of H-convergence and many properties such as metrizability, locality, irrelevance of boundary conditions and energy convergence are proved. Also, corrector results are derived. We can say that theory is well developed for second order elliptic partial differential equations, and there are also some results for higher order elliptic equations [77]. Motivated by a possible applications in optimal design, Antonić and Balenović defined H-convergence in the context of elastic plate equation and established the compactness of H-convergence [9, 10].

Let us remark that H-convergence is not the only approach in the theory of homogenization, although it is probably the most general. There are also a stochastic theory of homogenization [43], and variational theory of homogenization, known as the Γ-convergence method [29]. It is interesting to note that the mathematical theory of homogenization started in at least three directions. The oldest one is concerned with a general theory for the convergence of operators already mentioned as the G-convergence or H-convergence. The second direction is the asymptotic study of perforated domains which contain many small holes [52, 63], and third is the study of periodic homogenization problems [16]. Since then, the mathematical theory of homogenization has been significantly developed, and has numerous applications.

The goal of this thesis is to develop homogenization theory for the Kirchhoff-Love equation of an elastic, thin, symmetric plate, which is a fourth order elliptic equation. This model can be formally justified by taking a limit in $3D$ elasticity equations with a variant of the H-convergence method [28] or by using Γ-convergence [18]. An assumption that plate is symmetric with respect to its midplane simplifies the theory, since it is equivalent to consider G-convergence [35, 51] instead of H-convergence. However, in this thesis the general theory shall be presented, ignoring this symmetry assumption. For the general theory of elastic plates see [23].

The homogenization method appears to be a physically justified tool for the modelling of composite materials, i.e. mixtures of two or more materials on a microscopic scale. It shows that such mixtures (e.g. steel, carbon fibers) can have much better properties than the components it is made of, so these materials are intensively studied by physicists, engineers and mathematicians [2, 22, 35, 51, 54, 72, 74]. The natural problem is to describe the composite material obtained by the homogenization process. Describing the set of all composite materials obtained by the homogenization process is known as the G-closure problem. Characterization of the G-closure is known for the mixture of two isotropic conductors [48, 49], but it is unknown for linearized elasticity system, even for the mixture of two isotropic materials. In the case of an elastic plate, G-closure is known only in some special regimes [50]. It is possible to obtain approximations of the G-closure in the small-amplitude or low contrast regime in the setting of stationary diffusion equation [72], when we mix two materials with similar properties.

By using H-convergence and H-measures as a tool, the small-amplitude homogenization for stationary diffusion equation is developed [68], i.e. the explicit formula for coefficients up to the second order term is derived. In this way, a small-amplitude homogenization result for the periodic case [16] is extended. Using similar techniques, Antonić and Vrdoljak developed the small-amplitude homogenization result for the parabolic equation [12, 13]. For the elastic plate equation, the low contrast regime was not studied up to date, and that is one of the goals of this thesis.

In order to derive some effective properties of composite materials, Hashin-Shtrikman bounds are calculated, i.e. bounds on the effective energy of a composite material, which are well known for stationary diffusion equation and elasticity [2]. However, to obtain effective properties and for application in optimal design, it is necessary to calculate them explicitly, as well as the corresponding (sequential) laminates that saturate them [2]. In the case of two-dimensional linearized elasticity this is done in [5], but for the plate equation that is an open problem, which is one of the topics of this thesis.

Overview

In Chapter 1 we prove a number of properties of H-convergence, discuss smooth dependence of H-limit on a parameter and calculate the H-limit of a sequence of periodic tensors. Moreover, we give special emphasis to calculating the first non-vanishing (usually second-order) term in the small-amplitude homogenization limit of a sequence of periodic tensors.

In Chapter 2 we establish the local character of the G-closure, and prove the density of the set of composites obtained by periodic homogenization in that set. We describe the sequential laminates, a particularly interesting class of composite materials, and derive optimal Hashin-Shtrikman bounds on the primal and complementary energy. Moreover, we derive expressions for elastic coefficients of a composite plate obtained by mixing two materials in low-contrast regime. In the case of two-phase isotropic materials, explicit

Hashin-Shtrikman bounds on the primal and complementary energy are calculated.

Before reading this thesis, the reader may wish to view the Appendix, since it contains some basic notation and elementary results.

Chapter 1

General homogenization theory for elastic plate equation

In this chapter we prove the main properties of the H-convergence, which correspond to the similar properties obtained for the stationary diffusion equation, including *locality*, *independence of boundary conditions*, *metrizability of H-topology*, *convergence of energies* and *corrector result*. The proofs are commonly based on Tartar's method of oscillating test functions. We also discuss smooth dependence of the H-limit on a parameter and calculate the H-limit of a periodic sequence of tensors. Moreover, we explicitly calculate the first correction in the small-amplitude homogenization limit of a sequence of periodic tensors describing material properties in the Kirchhoff model for pure bending of a thin solid symmetric plate under a transverse load. The majority of the results of this chapter can be found in [20, 21].

1.1 Introduction

We consider a homogeneous Dirichlet boundary value problem for a general fourth-order partial differential equation

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla u) = f \text{ in } \Omega \\ u \in \mathrm{H}^2_0(\Omega) \end{cases}, \tag{1.1}$$

where $\Omega \subseteq \mathbf{R}^d$ is an open and bounded set, and $\mathbf{M}$ is a tensor valued function, which can be understood as a linear operator on the space of all symmetric $d \times d$ real matrices, denoted by Sym.

The weak solution u of (1.1) is defined as a function $u \in \mathrm{H}^2_0(\Omega)$ satisfying

$$(\forall\, v \in \mathrm{H}^2_0(\Omega)) \qquad \int_\Omega \mathbf{M}\nabla\nabla u : \nabla\nabla v \, d\mathbf{x} = {}_{\mathrm{H}^{-2}(\Omega)}\langle\, f, v\,\rangle_{\mathrm{H}^2_0(\Omega)} \,.$$

The problem is elliptic, if we assume that $\mathbf{M}$ is bounded (almost everywhere) and coercive.

More precisely, we assume that

$$\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega) := \Big\{ \mathbf{N} \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall \mathbf{S} \in \mathrm{Sym})\, \mathbf{N}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \alpha \mathbf{S} : \mathbf{S} \ \& \ \mathbf{N}(\mathbf{x})^{-1}\mathbf{S} : \mathbf{S} \geq \frac{1}{\beta} \mathbf{S} : \mathbf{S} \ a.\ e.\ \mathbf{x} \in \Omega \Big\},$$

where $\beta > \alpha > 0$ are given, and : stands for the scalar product on the space Sym. The bounds are chosen in this form to ensure their preservation during the homogenization process, as it was shown in the case of stationary diffusion equation [58].

The well-posedness follows by a standard application of the Lax-Milgram lemma. To be precise, the differential operator $\mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla\cdot) : \mathrm{H}^2_0(\Omega) \longrightarrow \mathrm{H}^{-2}(\Omega)$ is an isomorphism, i.e. a linear and continuous operator with bounded inverse (the bound depending only on Ω and α).

In the two-dimensional case, boundary value problem (1.1) describes the Kirchhoff (also known as Kirchhoff-Love) model for pure bending of a thin, solid symmetric plate clamped at the boundary, under a transverse load f. This model can be derived by taking a limit in 3d elasticity equations with a technique similar to H-convergence [28], or by means of Gamma-convergence [18] (for classical reference see [23]). The plate is assumed to be symmetric with respect to its midplane Ω and a tensor valued function $\mathbf{M}$ describes its elastic properties (depending on the material properties and the thickness of the plate). In this model, additional symmetry is present, making the tensor valued function $\mathbf{M}$ self-adjoint. This assumption simplifies the theory, since it is equivalent to consider G-convergence [35, 51] instead of H-convergence. However, in this chapter we shall present the general theory (in arbitrary space dimension), ignoring this symmetry assumption.

We are interested in the general (non-periodic) homogenization theory for this equation. This theory is well developed for second-order elliptic problems, such as the stationary diffusion equation or the system of linearized elasticity, for which the notion of H- (or G-) convergence has been studied and properties, such as *compactness, locality, independence of boundary conditions* and *convergence of energies*, have been established (see [2, 72] and references therein). In [77], a homogenization of a general elliptic system of partial differential equations has been considered, and some of the above mentioned properties have been shown in such full generality. However, due to this generality, some of the important properties are missing, while proofs end up being rather complicated.

The results concerning homogenization of the elastic plate equation have already been initiated by Antonić and Balenović [9, 10], where, prompted by possible applications in optimal design problems, a more direct approach to the homogenization of the stationary plate equation was considered, and an appropriate variant of H-convergence was defined. Additionally, compactness of H-convergence was established.

Definition 1 A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha,\beta;\Omega)$ is said to H-converge to $\mathbf{M}\in\mathfrak{M}_2(\alpha',\beta';\Omega)$ if for any $f\in \mathrm{H}^{-2}(\Omega)$ the sequence of solutions (u_n) of problems

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n\nabla\nabla u_n) = f \\ u_n\in \mathrm{H}^2_0(\Omega)\end{cases}$$

converges weakly to a limit u in $\mathrm{H}^2_0(\Omega)$, while the sequence $(\mathbf{M}^n\nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $\mathrm{L}^2(\Omega;\mathrm{Sym})$. If this is the case, then $\mathbf{M}$ is called H-limit of the sequence $(\mathbf{M}^n)$; note that u solves the boundary value problem

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \\ u\in \mathrm{H}^2_0(\Omega)\end{cases}.$$

The sequences (u_n) and $(\mathbf{M}^n\nabla\nabla u_n)$ in the above definition are bounded in $\mathrm{H}^2_0(\Omega)$ and $\mathrm{L}^2(\Omega;\mathrm{Sym})$, respectively, and thus converge (on a subsequence). Therefore, H-convergence just makes a connection between their limits. Since the existence of the H-limit $\mathbf{M}$ is doubtful, the following compactness theorem justifies the previous definition. Moreover, it shows that the bounds in definition of $\mathfrak{M}_2(\alpha,\beta;\Omega)$, which could also be written in many equivalent ways, are chosen in such a way that in the previous definition one actually has $\alpha'=\alpha$ and $\beta'=\beta$.

Theorem 1 (Compactness theorem for H-convergence) Let $(\mathbf{M}^n)$ be a sequence in $\mathfrak{M}_2(\alpha,\beta;\Omega)$. Then there is a subsequence $(\mathbf{M}^{n_k})$ and a tensor function $\mathbf{M}\in\mathfrak{M}_2(\alpha,\beta;\Omega)$ such that $(\mathbf{M}^{n_k})$ H-converges to $\mathbf{M}$.

In order to proceed with the proof of Theorem 1, we need the following two lemmas [9, 10]. The first of them presents the *compactness by compensation result* and has the key role in proving properties of H-convergence for elastic plate equation. This lemma plays the same role as the div-rot lemma in the theory of homogenization for second-order operators [72].

Lemma 1 (Compactness by compensation result) Let the following convergences be valid:

$$\begin{aligned} w^n &\rightharpoonup w^\infty \quad \text{in} \quad \mathrm{H}^2_{\mathrm{loc}}(\Omega)\,, \\ \mathbf{D}^n &\rightharpoonup \mathbf{D}^\infty \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathrm{Sym})\,, \end{aligned}$$

with an additional assumption that the sequence $(\operatorname{div}\operatorname{div}\mathbf{D}^n)$ is contained in a precompact (for the strong topology) set of the space $\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$. Then we have

$$\mathbf{E}^n:\mathbf{D}^n \overset{*}{\rightharpoonup} \mathbf{E}^\infty:\mathbf{D}^\infty$$

in the space of Radon measures on Ω, denoted by $\mathcal{M}(\Omega)$, where $\mathbf{E}^n := \nabla\nabla w^n$, for $n\in\mathbf{N}\cup\{\infty\}$.

Proof. Since the sequence $(\operatorname{div}\operatorname{div}\mathbf{D}^n)$ is contained in a precompact (for the strong topology) set of the space $\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$, and $\operatorname{div}\operatorname{div}\mathbf{D}^n \rightharpoonup \operatorname{div}\operatorname{div}\mathbf{D}^\infty$ weakly in $\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$, there is a subsequence $(\operatorname{div}\operatorname{div}\mathbf{D}^{n_k})$ converging to $\operatorname{div}\operatorname{div}\mathbf{D}^\infty$ in $\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$ strongly. On the other hand, for $\varphi \in \mathrm{C}^\infty_c(\Omega)$, the sequence (φw^n) converges weakly to φw^∞ in $H^2_c(\Omega)$, therefore we have

$$ {}_{\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)}\langle \operatorname{div}\operatorname{div}\mathbf{D}^{n_k}, \varphi w^{n_k} \rangle_{H^2_c(\Omega)} \to {}_{\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)}\langle \operatorname{div}\operatorname{div}\mathbf{D}^{\infty}, \varphi w^{\infty} \rangle_{H^2_c(\Omega)} = \int_\Omega \mathbf{D}^\infty : \nabla\nabla(\varphi w^\infty)\, d\mathbf{x}. \tag{1.2} $$

Integration by parts of the term on the left-hand side of (1.2) yields

$$ \begin{aligned} {}_{\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)}\langle \operatorname{div}\operatorname{div}\mathbf{D}^{n_k}, \varphi w^{n_k} \rangle_{H^2_c(\Omega)} &= \int_\Omega \mathbf{D}^{n_k} : \nabla\nabla(\varphi w^{n_k})\, d\mathbf{x} \\ = \int_\Omega \mathbf{D}^{n_k} : (\nabla\nabla\varphi) w^{n_k}\, d\mathbf{x} &+ 2\int_\Omega \mathbf{D}^{n_k} : (\nabla\varphi \otimes \nabla w^{n_k})\, d\mathbf{x} + \int_\Omega \mathbf{D}^{n_k} : \varphi\nabla\nabla w^{n_k}\, d\mathbf{x}. \end{aligned} $$

By using the compactness argument for Sobolev imbeddings, we have $\nabla w^{n_k} \longrightarrow \nabla w^\infty$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{R}^d)$ and $w^{n_k} \longrightarrow w^\infty$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$. Therefore, we can pass to the limit in the first two terms of the above equality. On the other hand, a comparison argument shows that the term $\int_\Omega \mathbf{D}^{n_k} : \varphi\nabla\nabla w^{n_k}\, d\mathbf{x}$ converges to the limit

$$ \begin{aligned} &\int_\Omega \mathbf{D}^\infty : \nabla\nabla(\varphi w^\infty)\, d\mathbf{x} - \int_\Omega \mathbf{D}^\infty : (\nabla\nabla\varphi) w^\infty\, d\mathbf{x} - 2\int_\Omega \mathbf{D}^\infty : (\nabla\varphi \otimes \nabla w^\infty)\, d\mathbf{x} \\ &= \int_\Omega \mathbf{D}^\infty : \varphi\nabla\nabla w^\infty\, d\mathbf{x}. \end{aligned} $$

This gives the statement of the lemma for a subsequence. However, one can easily see that the same holds for any subsequence, with the same limit, and thus for the entire sequence itself. ■

Lemma 2 Let $(\mathbf{M}^n)$ be a sequence of tensor functions in $\mathfrak{M}_2(\alpha,\beta;\Omega)$ and $A_n : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$ defined with:

$$ A_n v := \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla v), \;\; v \in \mathrm{H}^2_0(\Omega). $$

Then there is a subsequence $(\mathbf{M}^{n_k})$, and operators $A_\infty \in \mathcal{L}(\mathrm{H}^2_0(\Omega);\mathrm{H}^{-2}(\Omega))$, $R \in \mathcal{L}(\mathrm{H}^{-2}(\Omega);\mathrm{L}^2(\Omega;\mathrm{Sym}))$, such that $A^{-1}_{n_k} \rightharpoonup A^{-1}_\infty$ weakly in the sense of operators, and that for arbitrary $f \in \mathrm{H}^{-2}(\Omega)$ we have $\mathbf{M}^{n_k}\nabla\nabla u_{nk} \rightharpoonup Rf$ in $\mathrm{L}^2(\Omega;\mathrm{Sym})$, where (u_{n_k}) is the sequence of solutions of problems

$$ \begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^{n_k}\nabla\nabla u_{n_k}) = f \\ u_{n_k} \in \mathrm{H}^2_0(\Omega) \end{cases} . \tag{1.3} $$

Proof. Let $\mathcal{G} = \{f_1, f_2, \dots\}$ be a countable dense subset of $\mathrm{H}^{-2}(\Omega)$. In the sequel, by using a diagonal procedure, we shall construct operators B and R which are well defined on $\mathcal{G}$, and then extend those operators by continuity to linear operators on $\mathrm{H}^{-2}(\Omega)$.

More precisely, since $\|A_n^{-1}\|_{\mathcal{L}(\mathrm{H}^{-2}(\Omega),\mathrm{H}_0^2(\Omega))} \le \frac{1}{\alpha}$, the sequence $(A_n^{-1}f_1)$ is bounded in $\mathrm{H}_0^2(\Omega)$, and has a weakly convergent subsequence which converges to Bf_1. We repeat the same procedure with that subsequence for f_2 and denote the cluster point by Bf_2; analogously we do for f_3, etc. Finally, we take a diagonal subsequence (A_{n_k}) so that the following holds:

$$(\forall m \in \mathbf{N})\ A_{n_k}^{-1} f_m \rightharpoonup B f_m \text{ in } \mathrm{H}_0^2(\Omega).$$

Let us now extend the operator $B : \mathcal{G} \to \mathrm{H}_0^2(\Omega)$ to linear operator in $\mathrm{H}^{-2}(\Omega)$. For arbitrary $f \in \mathrm{H}^{-2}(\Omega)$, we take a sequence (f_m) in $\mathcal{G}$ such that $f_m \to f$, as $m \to \infty$, in $\mathrm{H}^{-2}(\Omega)$, and define $Bf := \lim_{m\to\infty} Bf_m$. From this construction one can easily conclude that

$$(\forall f \in \mathrm{H}^{-2}(\Omega))\ A_{n_k}^{-1} f \rightharpoonup Bf \text{ in } \mathrm{H}_0^2(\Omega),$$

which yields a well defined linear operator $B : \mathrm{H}^{-2}(\Omega) \to \mathrm{H}_0^2(\Omega)$. This operator is bounded by $\frac{1}{\alpha}$, since

$$_{\mathrm{H}^{-2}(\Omega)}\langle f, A_{n_k}^{-1} f \rangle_{\mathrm{H}_0^2(\Omega)} = {}_{\mathrm{H}^{-2}(\Omega)}\langle A_{n_k} u_{n_k}, u_{n_k} \rangle_{\mathrm{H}_0^2(\Omega)} \ge \alpha \|A_{n_k}^{-1} f\|^2_{\mathrm{H}_0^2(\Omega)},$$

and by taking the limit inferior in k we have:

$$\|Bf\|_{\mathrm{H}_0^2(\Omega)} \|f\|_{\mathrm{H}^{-2}(\Omega)} \ge {}_{\mathrm{H}^{-2}(\Omega)}\langle f, Bf \rangle_{\mathrm{H}_0^2(\Omega)} \ge \alpha \liminf_k \|A_{n_k}^{-1} f\|^2_{\mathrm{H}_0^2(\Omega)} \ge \alpha \|Bf\|^2_{\mathrm{H}_0^2(\Omega)}.$$

Moreover, it is easy to show that B is coercive with $\frac{\alpha}{\beta^2}$:

$$_{\mathrm{H}^{-2}(\Omega)}\langle f, Bf \rangle_{\mathrm{H}_0^2(\Omega)} \ge \alpha \liminf_k \|A_{n_k}^{-1} f\|^2_{\mathrm{H}_0^2(\Omega)} \ge \alpha \liminf_k \frac{1}{\beta^2} \|f\|^2_{\mathrm{H}^{-2}(\Omega)} = \frac{\alpha}{\beta^2} \|f\|^2_{\mathrm{H}^{-2}(\Omega)}.$$

By Lax-Milgram lemma B is invertible, and after denoting $A_\infty := B^{-1}$ it follows that $A_{n_k}^{-1} \rightharpoonup A_\infty^{-1}$ weakly in the sense of operators.

If (u_{n_k}) is a sequence of solutions to (1.3), then

$$u_{n_k} \rightharpoonup u_\infty = Bf \text{ in } \mathrm{H}_0^2(\Omega).$$

The sequence $(\mathbf{M}^{n_k} \nabla\nabla u_{n_k})$ is bounded in $\mathrm{L}^2(\Omega; \mathrm{Sym})$, therefore, by using a diagonal procedure once more, we can construct a subsequence $(\mathbf{M}^{n_k})$ (still denoted by n_k) such that for $f \in \mathcal{G}$ we have

$$\mathbf{M}^{n_k} \nabla\nabla u_{n_k} \rightharpoonup Rf \text{ in } \mathrm{L}^2(\Omega; \mathrm{Sym}),$$

where u_{n_k} are solutions to (1.3) for that f. This defines an operator $R : \mathcal{G} \to \mathrm{L}^2(\Omega; \mathrm{Sym})$, which is clearly bounded: since $\mathbf{M}^{n_k}\nabla\nabla u_{n_k} = \mathbf{M}^{n_k}\nabla\nabla(A_{n_k}^{-1} f)$, we have

$$\|\mathbf{M}^{n_k}\nabla\nabla u_{n_k}\|_{\mathrm{L}^2(\Omega;\mathrm{Sym})} \le \beta\|\nabla\nabla(A_{n_k}^{-1} f)\|_{\mathrm{L}^2(\Omega;\mathrm{Sym})} \le \beta\|A_{n_k}^{-1} f\|_{\mathrm{H}_0^2(\Omega)} \le \frac{\beta}{\alpha}\|f\|_{\mathrm{H}^{-2}(\Omega)}.$$

Finally, after taking the limit inferior in k, we conclude that $\|R\| \le \dfrac{\beta}{\alpha}$. An analogous construction as in the first part of the proof yields a linear operator $R : \mathrm{H}^{-2}(\Omega) \to \mathrm{L}^2(\Omega; \mathrm{Sym})$, which completes the proof. ■

Proof of Theorem 1. Let (A_n) and A_∞ as in Lemma 2. First we prove that the operator A_∞ is of the same form as operators A_n, in the sense that there is a tensor $\mathbf{M}^\infty$ such that $A_\infty u = \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^\infty\nabla\nabla u)$. This can be shown by using the method of oscillating test functions [66]. This method consists of constructing a sequence of functions (v_n) in $\mathrm{H}^2(\Omega)$ such that

$$\begin{aligned} v_n &\rightharpoonup v_\infty \text{ in } \mathrm{H}^2(\Omega),\\ \mathsf{div}\,\mathsf{div}\,\Big((\mathbf{M}^n)^T\nabla\nabla v_n\Big) &\longrightarrow g_\infty \text{ in } \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega),\\ (\mathbf{M}^n)^T\nabla\nabla v_n &\rightharpoonup \mathbf{W}^\infty \text{ in } \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym}). \end{aligned} \tag{1.4}$$

In order to construct the sequence of oscillating test functions, we choose an open set Ω' which contains the closure of Ω. For $\mathbf{x} \in \Omega'\backslash\Omega$ we define the extension of tensor $\mathbf{M}^n(\mathbf{x}) := \alpha\mathbf{I}_4$, and for a given $g \in \mathrm{H}^{-2}(\Omega')$ define (v_n) to be the sequence of solutions to boundary value problems

$$\begin{cases} \tilde{A}_n v_n := \mathsf{div}\,\mathsf{div}\,((\mathbf{M}^n)^T\nabla\nabla v_n) = g \\ v_n \in \mathrm{H}_0^2(\Omega') \end{cases}.$$

Since we obviously have $(\mathbf{M}^n)^T \in \mathfrak{M}_2(\alpha, \beta; \Omega')$, the sequence (v_n) is bounded in $\mathrm{H}_0^2(\Omega')$, and therefore in $\mathrm{H}^2(\Omega)$, hence it has a subsequence satisfying (1.4). Finally, the associated operator $\tilde{A}_\infty$ is an isomorphism between spaces $\mathrm{H}_0^2(\Omega')$ and $\mathrm{H}^{-2}(\Omega')$; therefore, by choosing an arbitrary function g we can get any $v_\infty \in \mathrm{H}_0^2(\Omega')$ and vice versa.

Let (u_n) be the sequence of solutions of problems (1.3). By Lemma 1, we can pass to the limit on both sides of the equality

$$\mathbf{M}^n\nabla\nabla u_n : \nabla\nabla v_n = \nabla\nabla u_n : (\mathbf{M}^n)^T\nabla\nabla v_n,$$

which gives us $Cu_\infty : \nabla\nabla v_\infty = \nabla\nabla u_\infty : \mathbf{W}_\infty$, where $C = RA_\infty$ (see Lemma 2). By choosing $v_\infty(\mathbf{x}) := \frac{1}{2}x_i x_j$ in Ω, we have $(Cu_\infty)_{ij} = \nabla\nabla u_\infty : \mathbf{W}_\infty^{ij}$, or, in other words, there is a tensor $\mathbf{M}^\infty$ such that $Cu_\infty = \mathbf{M}^\infty\nabla\nabla u_\infty$. The above construction yields $\mathbf{M}^\infty \in \mathrm{L}^2(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.

It remains to show that $\mathbf{M}^\infty \in \mathfrak{M}_2(\alpha,\beta;\Omega)$, i.e. we shall show the equivalent claim that $(\mathbf{M}^\infty)^T \in \mathfrak{M}_2(\alpha,\beta;\Omega)$: let $\varphi \in C_c^\infty(\Omega)$ and $v_\infty(\mathbf{x}) := \frac{1}{2}\mathbf{N}\mathbf{x}\cdot\mathbf{x}$, $\mathbf{N} \in \mathrm{Sym}$. Since $(\mathbf{M}^n)^T \in \mathfrak{M}_2(\alpha,\beta;\Omega)$, we have:

$$\int_\Omega \varphi^2 (\mathbf{M}^n)^T \nabla\nabla v_n : \nabla\nabla v_n \, d\mathbf{x} \geq \alpha \int_\Omega \varphi^2 |\nabla\nabla v_n|^2 \, d\mathbf{x}. \tag{1.5}$$

Applying the Lemma 1 to the left-hand side of (1.5), gives

$$\int_\Omega \varphi^2 (\mathbf{M}^\infty)^T \mathbf{N} : \mathbf{N} \, d\mathbf{x} \geq \alpha \liminf_n \int_\Omega \varphi^2 |\nabla\nabla v_n|^2 \, d\mathbf{x} \geq \alpha \int_\Omega \varphi^2 \mathbf{N} : \mathbf{N} \, d\mathbf{x}.$$

This implies the coercivity of $(\mathbf{M}^\infty)^T$ a. e. in Ω. Since $(\mathbf{M}^n)^T$ belongs to $\mathfrak{M}_2(\alpha,\beta;\Omega)$, it also satisfies

$$\int_\Omega \varphi^2 ((\mathbf{M}^n)^T)^{-1} (\mathbf{M}^n)^T \nabla\nabla v_n : (\mathbf{M}^n)^T \nabla\nabla v_n \, d\mathbf{x} \geq \frac{1}{\beta} \int_\Omega \varphi^2 |(\mathbf{M}^n)^T \nabla\nabla v_n|^2 \, d\mathbf{x}. \tag{1.6}$$

Analogously as when showing coercivity of $(\mathbf{M}^n)^T$, from (1.6) we obtain:

$$\int_\Omega \varphi^2 (\mathbf{M}^\infty)^T \mathbf{N} : \mathbf{N} \, d\mathbf{x} \geq \frac{1}{\beta} \int_\Omega \varphi^2 |(\mathbf{M}^\infty)^T \mathbf{N}|^2 \, d\mathbf{x}.$$

This implies the boundedness of $(\mathbf{M}^\infty)^T$ a. e. in Ω, which completes the proof, i.e. $(\mathbf{M}^\infty)^T \in \mathfrak{M}_2(\alpha,\beta;\Omega)$. ■

As it is already said, in this chapter we are also interested in the small-amplitude homogenization limit of a sequence of periodic tensors. The small-amplitude homogenization procedure of Tartar [72] consists in computing the first correction in the H-limit of a sequence of coefficients, whose difference is proportional to a small parameter. More precisely, after making an asymptotic expansion of the H-limit in terms of the small-amplitude parameter, one wishes to explicitly characterize its first non-vanishing (usually second-order) term. Its physical relevance is in deriving (approximate) effective properties of (conducting or elastic) material that is made by mixing two materials under the so called small-amplitude, small-contrast or small aspect ratio assumption, i.e. that original materials have *close* coefficients or material properties (for some applications see for example [3, 4, 39]).

The explicit formula for the correction in the case of second-order elliptic [68] (or parabolic [12]) equation can in general be obtained by using H-measures [69, 70] (or their variants [11]). However, in the case of periodic coefficients, the same can be done by using Fourier expansions [13]. In this thesis we use the second approach and explicitly calculate the first correction in the small-amplitude homogenization process for the periodic sequence of tensors.

We are interested in the following expansion of the H-limit:

$$\mathbf{A}_p := \mathbf{A}_0 + p\mathbf{B}_0 + p^2\mathbf{C}_0 + o(p^2) \quad \text{in} \quad \Omega,$$

where p is some positive real number, and thus we shall use a variant of Taylor's theorem which is appropriate for Banach spaces. We first recall the notion of Frechet differentiable function [7], which is a natural extension of the usual definition the differential of a map in Euclidean spaces to Banach spaces.

Let X and Y be Banach spaces, U an open subset of X, and we denote

$$\mathrm{Inv}(X, Y) = \{A \in L(X, Y) : A \text{ is invertible}\},$$

where $L(X, Y)$ is the space of linear continuous maps $A : X \to Y$.

Definition 2 Let $\mathbf{x}_0 \in U$. We say that F is Frechet differentiable at $\mathbf{x}_0$ if there exists $A \in L(X, Y)$, such that

$$R(\mathbf{h}) := F(\mathbf{x}_0 + \mathbf{h}) - F(\mathbf{x}_0) - A(\mathbf{h})$$

satisfies

$$R(\mathbf{h}) = o(\|\mathbf{h}\|_X).$$

Such an operator A is uniquely determined (if it exists) and will be called the (Frechet) differential of F at $\mathbf{x}_0$, with notation $A = F'(\mathbf{x}_0)$. If F is differentiable at all $\mathbf{x}_0 \in U$ we say that F is differentiable in U.

To define the n-th differential ($n \geq 2$) we can proceed by induction. The n-th differential at a point $\mathbf{x}_0 \in U$ will be identified with a continuous n-linear map from $X \times X \times \cdots \times X$ (n times) to Y, and denoted by $F^{(n)}(\mathbf{x}_0)$.

Proposition 1 [7, p. 31, Proposition 1.1]

(i) $\mathrm{Inv}(X, Y)$ is an open subset of $L(X, Y)$. More precisely, if $A \in \mathrm{Inv}(X, Y)$ then any $T \in L(X, Y)$ such that

$$\|T - A\|_{L(X,Y)} < \frac{1}{\|A^{-1}\|_{L(Y,X)}}$$

is invertible.

(ii) The map $J : \mathrm{Inv}(X, Y) \to L(Y, X)$ defined by $J(A) = A^{-1}$ is C^k for all $k \geq 1$ (i.e. C^∞). Additionally, $J'(A)(B) = -A^{-1} \circ B \circ A^{-1}$, $B \in L(X, Y)$.

For $\mathbf{u}, \mathbf{v} \in U$ let $[\mathbf{u}, \mathbf{v}] := \{t\mathbf{u} + (1 - t)\mathbf{v} : t \in [0, 1]\}$.

Theorem 2 [7, p. 13, Theorem 1.8] Let $F : U \subseteq X \to Y$ be Frechet differentiable at every point of U. Given $\mathbf{u}$, $\mathbf{v} \in U$ such that $[\mathbf{u}, \mathbf{v}] \subseteq U$, it follows

$$\|F(\mathbf{u}) - F(\mathbf{v})\|_Y \leq \sup\{\|F'(\mathbf{w})\|_{L(X,Y)} : \mathbf{w} \in [\mathbf{u}, \mathbf{v}]\}\|\mathbf{u} - \mathbf{v}\|_X.$$

Theorem 3 [59, p. 187, Theorem 6.1](Taylor's theorem) Let $F : U \subseteq \mathbf{R} \to Y$ be n times Frechet differentiable at a point $x_0 \in U$. Then

$$F(x_0 + h) = F(x_0) + F'(x_0)h + \cdots + \frac{1}{n!}F^{(n)}(x_0)h^n + r(x_0; h),$$

where $r(x_0; h) \in o(|h|^n)$.

In order to state the small-amplitude homogenization results precisely, we need to show that the H-limit of a sequence depending smoothly on a parameter is also smooth. Since continuity is preserved by uniform convergence, we shall use the Arzelà-Ascoli theorem for the purpose of constructing a uniformly converging subsequence.

Definition 3 Let (X, d_1) and (Y, d_2) be two metric spaces. A family $\mathcal{F}$ of functions defined on a set E in a metric space X, with codomain Y, is said to be equicontinuous on E if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in E)(\forall f \in \mathcal{F})\, d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon.$$

Theorem 4 [64, p. 158, Theorem 7.25](Arzelà-Ascoli) If K is compact, $f_n \in C(K)$, $n \in \mathbf{N}$, and if $\{f_n : n \in \mathbf{N}\}$ is pointwise bounded and equicontinuous on K, then

(i) (f_n) is bounded in $C(K)$,

(ii) (f_n) has a uniformly converging subsequence.

When dealing with periodic homogenization, we need the notion of a quotient space [44, 53].

Let M be a subspace of a vector space X over a field K. We define an equivalence relation on X such that for $\mathbf{x}$, $\boldsymbol{y} \in X$, $\mathbf{x} \sim \boldsymbol{y}$ if and only if $\mathbf{x} - \boldsymbol{y} \in M$. For $\mathbf{x} \in X$, an equivalence class is defined with

$$[\mathbf{x}] := \mathbf{x} + M = \{\mathbf{x} + \mathbf{m} : \mathbf{m} \in M\}.$$

On the quotient set

$$X/M := \{\mathbf{x} + M : \mathbf{x} \in X\}$$

the following operations are well defined:

$$(\mathbf{x} + M) + (\boldsymbol{y} + M) := (\mathbf{x} + \boldsymbol{y}) + M, \quad \mathbf{x},\ \boldsymbol{y} \in X,$$

and

$$\alpha(\mathbf{x}+M) := (\alpha\mathbf{x}) + M, \;\; \mathbf{x} \in X,\, \alpha \in K.$$

The vector space X/M over a field K, with the vector space operations given above, is called the quotient space.

Theorem 5 [53, p. 51-53] Let M be a closed subspace of a normed space X. The quotient norm of X/M is given by the formula

$$\|\mathbf{x}+M\|_{X/M} := \inf\{\|\mathbf{x}+\mathbf{m}\|_X : \mathbf{m} \in M\}, \;\; \mathbf{x} \in X,$$

and it is a norm on X/M. Additionally, if X is a Banach space, then X/M is also a Banach space.

We are also interested in duals of quotient spaces.

Definition 4 Let X be a normed space and M a subspace of X. We define its annihilator by

$$M^0 := \{f \in X' : {}_{X'}\langle\, f, \mathbf{x}\,\rangle_X = 0, \;\; \mathbf{x} \in M\}.$$

Obviously, M^0 is a subspace of X'.

Theorem 6 [75, p. 85, Theorem 4.4.3] Let X be a normed space and M a closed subspace of X. Then $(X/M)'$ is isometrically isomorphic to M^0.

Theorem 7 [60, p. 108, Theorem 7.2] Assume that Ω is a bounded, open subset of $\mathbf{R}^d$ with Lipschitz boundary, and let $\mathcal{P}_{k-1}$ be the space of polynomials of degree $\leq k-1$. Then there exist $c_1, c_2 \in \mathbf{R}^+$ such that

$$c_1\|[u]\|_{W^{k,p}(\Omega)/\mathcal{P}_{k-1}} \leq \left[\sum_{|\alpha|=k} \|D^\alpha u\|^p_{\mathrm{L}^p(\Omega)}\right]^{\frac{1}{p}} \leq c_2\|[u]\|_{W^{k,p}(\Omega)/\mathcal{P}_{k-1}}.$$

If $p=2$, $H^k(\Omega)/\mathcal{P}_{k-1}$ is a Hilbert space with the scalar product

$$([v],[u]) := \sum_{|\alpha|=k} \int_\Omega D^\alpha v D^\alpha \bar{u}\, d\mathbf{x}.$$

1.2 Properties of H-convergence

Using Tartar's method of oscillating test functions, we give proofs for the above mentioned properties of H-convergence for the stationary plate equation, and additionally prove a number of results, such as *the metrizability* and *the corrector result*. The relationship between H-convergence and some other types of convergence is studied in the following theorem.

Theorem 8 Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that either converges strongly to a limit tensor $\mathbf{M}$ in $L^1(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, or converges to $\mathbf{M}$ almost everywhere in Ω. Then, $(\mathbf{M}^n)$ also H-converges to $\mathbf{M}$.

Proof. The sequence $(\mathbf{M}^n)$ belongs to $\mathfrak{M}_2(\alpha, \beta; \Omega)$ and therefore it is bounded in $L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$. By the Lebesgue dominated convergence theorem $(\mathbf{M}^n)$ converges strongly to $\mathbf{M}$ in $L^p(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, for any $1 \leq p < \infty$. If u_n is the solution of

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n \nabla\nabla u_n) = f \\ u_n \in H^2_0(\Omega) \end{cases},$$

then the sequence (u_n) is bounded in $H^2_0(\Omega)$, and therefore (up to a subsequence) it converges weakly to $u \in H^2_0(\Omega)$.

Since $(\mathbf{M}^n)$ converges strongly to $\mathbf{M}$ in $L^2(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$ and $(\nabla\nabla u_n)$ converges to $\nabla\nabla u$ weakly in $L^2(\Omega; \mathrm{Sym})$, we conclude that $\boldsymbol{\sigma}_n := \mathbf{M}^n \nabla\nabla u_n$ converges weakly to $\boldsymbol{\sigma} = \mathbf{M}\nabla\nabla u$ in $L^1(\Omega; \mathrm{Sym})$, and thus also in $L^2(\Omega; \mathrm{Sym})$, as the sequence $(\boldsymbol{\sigma}_n)$ is bounded in this space.

The homogenized equation in Definition 1 has a unique solution $u \in H^2_0(\Omega)$, so each subsequence of (u_n) converges to the same limit u and this implies that the entire sequence (u_n) converges to u. Since $f \in H^{-2}(\Omega)$ is arbitrary, it follows that $(\mathbf{M}^n)$ H-converges to $\mathbf{M}$.

■

H-convergence is related to the material properties of an elastic plate and it would be desirable that properties of a given material do not depend on boundary conditions, e.g. that it is not important whether the plate is clamped at the boundary or not. The next theorem implies that the notion of H-convergence is not tied to the prescribed boundary conditions: instead of homogeneous Dirichlet boundary conditions in Definition 1 we can take any boundary conditions which ensure well posedness of the boundary value problem.

Theorem 9 (Irrelevance of boundary conditions) Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to $\mathbf{M}$. For any sequence (z_n) such that

$$\begin{aligned} z_n &\rightharpoonup z \quad \text{in } H^2_{\mathrm{loc}}(\Omega) \\ \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n \nabla\nabla z_n) &\longrightarrow f \quad \text{in } H^{-2}_{\mathrm{loc}}(\Omega), \end{aligned}$$

the weak convergence $\mathbf{M}^n \nabla\nabla z_n \rightharpoonup \mathbf{M}\nabla\nabla z$ in $L^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})$ holds.

Proof. Let ω be an open set compactly embedded in Ω. The sequence (z_n) is bounded in $H^2(\omega)$, implying that $(\mathbf{M}^n \nabla\nabla z_n)$ is bounded in $L^2(\omega; \mathrm{Sym})$. If we denote $\boldsymbol{\sigma}_n := \mathbf{M}^n \nabla\nabla z_n$, we can extract a weakly convergent subsequence such that $\boldsymbol{\sigma}_n \rightharpoonup \boldsymbol{\sigma}$ in $L^2(\omega; \mathrm{Sym})$.

Since $\omega \Subset \Omega$, there exists $\varphi \in C^\infty_c(\Omega)$ such that $\varphi|_\omega = 1$. For arbitrary $\mathbf{N} \in \mathrm{Sym}$, we define

$$w(\mathbf{x}) := \frac{1}{2}\varphi(\mathbf{x})\mathbf{N}\mathbf{x} \cdot \mathbf{x}\,,$$

$$g := \mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla w) \in \mathrm{H}^{-2}(\Omega).$$

Let (w_n) be a sequence of solutions to

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n\nabla\nabla w_n) = g \\ w_n \in \mathrm{H}_0^2(\Omega) \end{cases}.$$

Since $(\mathbf{M}^n)$ H-converges to $\mathbf{M}$, the following holds:

$$w_n \rightharpoonup w \quad \text{in} \quad \mathrm{H}_0^2(\Omega),$$

$$\mathbf{M}^n\nabla\nabla w_n \rightharpoonup \mathbf{M}\nabla\nabla w \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym}).$$

By coercivity of $\mathbf{M}^n$ we have

$$(\mathbf{M}^n\nabla\nabla z_n - \mathbf{M}^n\nabla\nabla w_n) : (\nabla\nabla z_n - \nabla\nabla w_n) \geq 0 \quad \text{a. e. in } \Omega\,,$$

which, after passing to the limit and using the compactness by compensation result, becomes

$$(\boldsymbol{\sigma} - \mathbf{M}\nabla\nabla w) : (\nabla\nabla z - \nabla\nabla w) \geq 0 \quad \text{a. e. in } \Omega.$$

If we consider the previous inequality only in ω, we have:

$$(\boldsymbol{\sigma} - \mathbf{M}\mathbf{N}) : (\nabla\nabla z - \mathbf{N}) \geq 0 \quad \text{a. e. in } \omega. \tag{1.7}$$

For any joint Lebesgue point $\mathbf{x}_0 \in \omega$ of $\nabla\nabla z$, $\boldsymbol{\sigma}$ and $\mathbf{M}$, let $\mathbf{N} = \nabla\nabla z(\mathbf{x}_0) + t\mathbf{O}$, where $\mathbf{O} \in \mathrm{Sym}$ and $t \in \mathbf{R}^+$ are arbitrary. Now (1.7) yields

$$\Big(\boldsymbol{\sigma}(\mathbf{x}_0) - \mathbf{M}(\mathbf{x}_0)\nabla\nabla z(\mathbf{x}_0) - t\mathbf{M}(\mathbf{x}_0)\mathbf{O}\Big) : (-t\mathbf{O}) \geq 0\,,$$

and after dividing this inequality by $-t$ and taking the limit $t \to 0^+$, it follows

$$\Big(\boldsymbol{\sigma}(\mathbf{x}_0) - \mathbf{M}(\mathbf{x}_0)\nabla\nabla z(\mathbf{x}_0)\Big) : \mathbf{O} \leq 0.$$

By arbitrariness of $\mathbf{O} \in \mathrm{Sym}$, the equality $\boldsymbol{\sigma}(\mathbf{x}_0) = \mathbf{M}(\mathbf{x}_0)\nabla\nabla z(\mathbf{x}_0)$ easily follows. Due to uniqueness of the limit $\boldsymbol{\sigma}$, the entire sequence $\mathbf{M}^n\nabla\nabla z_n$ converges weakly to $\mathbf{M}\nabla\nabla z$ in $\mathrm{L}^2(\omega; \mathrm{Sym})$, which completes the proof.

■

Remark 1 If we change the assumptions of Theorem 9, such that

$$\begin{aligned} z_n &\rightharpoonup z \quad && \text{in } \mathrm{H}^2(\Omega) \\ \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n\nabla\nabla z_n) &\longrightarrow f \quad && \text{in } \mathrm{H}^{-2}(\Omega), \end{aligned}$$

the weak convergence $\mathbf{M}^n\nabla\nabla z_n \rightharpoonup \mathbf{M}\nabla\nabla z$ in $\mathrm{L}^2(\Omega;\mathrm{Sym})$ holds.

H-convergence also implies the convergence of energies, as stated in the sequel.

Theorem 10 (Energy convergence) Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha,\beta;\Omega)$ that H-converges to $\mathbf{M}$. For any $f \in \mathrm{H}^{-2}(\Omega)$, the sequence (u_n) of solutions to

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n\nabla\nabla u_n) = f \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}$$

satisfies

$$\mathbf{M}^n\nabla\nabla u_n : \nabla\nabla u_n \xrightarrow{*} \mathbf{M}\nabla\nabla u : \nabla\nabla u$$

in $\mathcal{M}(\Omega)$, and

$$\int_\Omega \mathbf{M}^n\nabla\nabla u_n : \nabla\nabla u_n \, d\mathbf{x} \longrightarrow \int_\Omega \mathbf{M}\nabla\nabla u : \nabla\nabla u \, d\mathbf{x}\,,$$

where u is the solution of the homogenized equation

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla u) = f \\ u \in \mathrm{H}^2_0(\Omega) \end{cases} .$$

Proof. If we apply Lemma 1, it can easily be seen that

$$\mathbf{M}^n\nabla\nabla u_n : \nabla\nabla u_n \xrightarrow{*} \mathbf{M}\nabla\nabla u : \nabla\nabla u$$

in the space of Radon measures, which proves the first statement.

From the weak formulation of given homogeneous Dirichlet boundary value problems we get

$$\begin{aligned} \int_\Omega \mathbf{M}^n\nabla\nabla u_n : \nabla\nabla u_n \, d\mathbf{x} &= {}_{\mathrm{H}^{-2}(\Omega)}\langle\, f, u_n \,\rangle_{\mathrm{H}^2_0(\Omega)}\,, \\ \int_\Omega \mathbf{M}\nabla\nabla u : \nabla\nabla u \, d\mathbf{x} &= {}_{\mathrm{H}^{-2}(\Omega)}\langle\, f, u \,\rangle_{\mathrm{H}^2_0(\Omega)}\,, \end{aligned}$$

and since (u_n) converges weakly to u in $\mathrm{H}^2_0(\Omega)$, we have

$${}_{\mathrm{H}^{-2}(\Omega)}\langle\, f, u_n \,\rangle_{\mathrm{H}^2_0(\Omega)} \longrightarrow {}_{\mathrm{H}^{-2}(\Omega)}\langle\, f, u \,\rangle_{\mathrm{H}^2_0(\Omega)}\,,$$

which concludes the proof.

■

Theorem 11 (Locality of H-convergence) Let $(\mathbf{M}^n)$ and $(\mathbf{O}^n)$ be two sequences of tensors in $\mathfrak{M}_2(\alpha,\beta;\Omega)$, which H-converge to $\mathbf{M}$ and $\mathbf{O}$, respectively. Let ω be an open subset compactly embedded in Ω. If $\mathbf{M}^n(\mathbf{x}) = \mathbf{O}^n(\mathbf{x})$, $\mathbf{x} \in \omega$, then $\mathbf{M}(\mathbf{x}) = \mathbf{O}(\mathbf{x})$, $\mathbf{x} \in \omega$.

Proof. The proof goes along the same lines as the proof of Theorem 9: since ω is compactly embedded in Ω, there exists $\varphi \in \mathrm{C}_c^\infty(\Omega)$ such that $\varphi|_\omega = 1$. For arbitrary $\mathbf{N} \in \mathrm{Sym}$, let us define

$$w(\mathbf{x}) := \frac{1}{2}\varphi(\mathbf{x})\mathbf{N}\mathbf{x}\cdot\mathbf{x}\,,$$

$$g := \mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla w) \in \mathrm{H}^{-2}(\Omega)\,,$$

and let w_n be a sequence of solutions to

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n\nabla\nabla w_n) = g \\ w_n \in \mathrm{H}_0^2(\Omega) \end{cases}.$$

Since $(\mathbf{M}^n)$ H-converges to $\mathbf{M}$, it follows that

$$w_n \rightharpoonup w \quad \text{in} \quad \mathrm{H}_0^2(\Omega),$$

$$\mathbf{M}^n\nabla\nabla w_n \rightharpoonup \mathbf{M}\nabla\nabla w \quad \text{in} \quad \mathrm{L}^2(\Omega;\mathrm{Sym}).$$

For sequence $(\mathbf{O}^n)$ we can proceed similarly: for any $\mathbf{S} \in \mathrm{Sym}$ we introduce

$$v(\mathbf{x}) := \frac{1}{2}\varphi(\mathbf{x})\mathbf{S}\mathbf{x}\cdot\mathbf{x}\,,$$

$$f := \mathsf{div}\,\mathsf{div}\,(\mathbf{O}\nabla\nabla v) \in \mathrm{H}^{-2}(\Omega)\,,$$

and let (v_n) be a sequence of solutions to

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{O}^n\nabla\nabla v_n) = f \\ v_n \in \mathrm{H}_0^2(\Omega) \end{cases},$$

thus obtaining

$$v_n \rightharpoonup v \quad \text{in} \quad \mathrm{H}_0^2(\Omega),$$

$$\mathbf{O}^n\nabla\nabla v_n \rightharpoonup \mathbf{O}\nabla\nabla v \quad \text{in} \quad \mathrm{L}^2(\Omega;\mathrm{Sym}).$$

By applying the compactness by compensation result, we get

$$(\mathbf{M}^n\nabla\nabla w_n - \mathbf{O}^n\nabla\nabla v_n) : (\nabla\nabla w_n - \nabla\nabla v_n) \xrightarrow{\ast} (\mathbf{M}\nabla\nabla w - \mathbf{O}\nabla\nabla v) : (\nabla\nabla w - \nabla\nabla v) \tag{1.8}$$

in the space of Radon measures. On ω we have $\nabla\nabla v = \mathbf{S}$ and $\nabla\nabla w = \mathbf{N}$, so by assumption $\mathbf{O}^n = \mathbf{M}^n$ in ω, the sequence in (1.8) equals

$$\mathbf{O}^n(\nabla\nabla w_n - \nabla\nabla v_n) : (\nabla\nabla w_n - \nabla\nabla v_n)\,,$$

which is nonnegative because of the coercivity of $\mathbf{O}^n$. Therefore, the limit in (1.8) is also nonnegative, i.e. $(\mathbf{MN} - \mathbf{OS}) : (\mathbf{N} - \mathbf{S}) \geq 0$ a. e. in ω. If we choose $\mathbf{S} = \mathbf{N} + t\mathbf{Z}$, $t \in \mathbf{R}^+$, $\mathbf{Z} \in \mathrm{Sym}$, we obtain

$$(\mathbf{MN} - \mathbf{ON} - t\mathbf{OZ}) : (-t\mathbf{Z}) \geq 0 \quad \text{in } \omega\,,$$

and after dividing this inequality by $-t$ and letting $t \to 0^+$, we achieve $(\mathbf{M} - \mathbf{O})\mathrm{N} : \mathbf{Z} \leq 0$. Since $\mathbf{Z}$ and $\mathbf{N}$ are arbitrary, this implies $\mathbf{M} = \mathbf{O}$ a. e. in ω.

■

We can rephrase the previous theorem by stating that values of the homogenized tensor $\mathbf{M}$ in a region ω do not depend on values of the sequence $(\mathbf{M}^n)$ outside of this region.

The next theorem implies that H-convergence preserves the order of the tensors. Recall that tensors describe the material properties of the given plate.

Theorem 12 (Ordering property) Let $(\mathbf{M}^n)$ and $(\mathbf{O}^n)$ be two sequences of symmetric tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converge to the homogenized tensors $\mathbf{M}$ and $\mathbf{O}$, respectively. Assume that $(\mathbf{M}^n)$ and $(\mathbf{O}^n)$ are ordered: for each $n \in \mathbf{N}$

$$\mathbf{M}^n\boldsymbol{\xi} : \boldsymbol{\xi} \leq \mathbf{O}^n\boldsymbol{\xi} : \boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \mathrm{Sym}\,.$$

Then the homogenized coefficients are also ordered:

$$\mathbf{M}\boldsymbol{\xi} : \boldsymbol{\xi} \leq \mathbf{O}\boldsymbol{\xi} : \boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \mathrm{Sym}\,.$$

Proof. Let us define a sequence (v_n) of oscillating test functions satisfying

$$\begin{aligned} v_n &\rightharpoonup \frac{1}{2}\mathbf{Nx}\cdot\mathbf{x} \quad \text{in} \quad \mathrm{H}^2(\Omega)\,, \\ \mathsf{div}\,\mathsf{div}\,(\mathbf{O}^n\nabla\nabla v_n) &\longrightarrow g^{\mathbf{O}} \quad \text{in} \quad \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)\,, \end{aligned}$$

where $\mathbf{N} \in \mathrm{Sym}$ is arbitrary. Existence of such a sequence is established in the proof of Theorem 1. Note that $\nabla\nabla v_n \rightharpoonup \mathbf{N}$, and additionally we have $\mathbf{O}^n\nabla\nabla v_n \rightharpoonup \mathbf{ON}$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})$, by Theorem 9.

Similarly, let us take a sequence (w_n) of oscillating test functions satisfying

$$\begin{aligned} w_n &\rightharpoonup \frac{1}{2}\mathbf{Nx}\cdot\mathbf{x} \quad \text{in} \quad \mathrm{H}^2(\Omega)\,, \\ \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n\nabla\nabla w_n) &\longrightarrow g^{\mathbf{M}} \quad \text{in} \quad \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)\,, \\ \mathbf{M}^n\nabla\nabla w_n &\rightharpoonup \mathbf{MN} \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})\,. \end{aligned}$$

Due to the coercivity of $\mathbf{M}^n$, we have

$$\mathbf{M}^n\nabla\nabla w_n : \nabla\nabla w_n - \mathbf{M}^n\nabla\nabla w_n : \nabla\nabla v_n - \mathbf{M}^n\nabla\nabla v_n : \nabla\nabla w_n + \mathbf{M}^n\nabla\nabla v_n : \nabla\nabla v_n$$

$$= \mathbf{M}^n(\nabla\nabla w_n - \nabla\nabla v_n) : (\nabla\nabla w_n - \nabla\nabla v_n) \geq 0$$

in Ω. Since $\mathbf{M}^n \leq \mathbf{O}^n$, it follows that

$$\mathbf{M}^n\nabla\nabla w_n : \nabla\nabla w_n - \mathbf{M}^n\nabla\nabla w_n : \nabla\nabla v_n - \mathbf{M}^n\nabla\nabla v_n : \nabla\nabla w_n + \mathbf{O}^n\nabla\nabla v_n : \nabla\nabla v_n \geq 0 \quad \text{in} \quad \Omega.$$

By applying the compactness by compensation result, we can pass to the limit in each term of the above expression and get

$$\mathbf{MN} : \mathbf{N} - \mathbf{MN} : \mathbf{N} - \mathbf{MN} : \mathbf{N} + \mathbf{ON} : \mathbf{N} = (\mathbf{O} - \mathbf{M})\mathbf{N} : \mathbf{N} \geq 0\,.$$

Since $\mathbf{N}$ is arbitrary, it follows that $\mathbf{O} \geq \mathbf{M}$.

■

In the following theorem we introduce bounds on homogenized tensor, in the sense of the standard order on symmetric tensors. The bounds are given in terms of weak-$*$ limits, representing the harmonic and arithmetic mean of the corresponding sequence.

Theorem 13 Let $(\mathbf{M}^n)$ be a sequence of symmetric tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to $\mathbf{M}$. Assume that

$$\begin{aligned} \mathbf{M}^n &\overset{*}{\rightharpoonup} \overline{\mathbf{M}} \quad \text{in} \quad \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})), \\ (\mathbf{M}^n)^{-1} &\overset{*}{\rightharpoonup} \underline{\mathbf{M}}^{-1} \quad \text{in} \quad \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})). \end{aligned}$$

Then the homogenized tensor satisfies

$$\underline{\mathbf{M}}\boldsymbol{\xi} : \boldsymbol{\xi} \leq \mathbf{M}\boldsymbol{\xi} : \boldsymbol{\xi} \leq \overline{\mathbf{M}}\boldsymbol{\xi} : \boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \mathrm{Sym}.$$

Proof. As before, let us take a sequence (w_n) of oscillating test functions satisfying

$$\begin{aligned} w_n &\rightharpoonup \frac{1}{2}\mathbf{Nx}\cdot\mathbf{x} \quad \text{in} \quad \mathrm{H}^2(\Omega)\,, \\ \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n\nabla\nabla w_n) &\longrightarrow g^{\mathbf{M}} \quad \text{in} \quad \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)\,, \\ \mathbf{M}^n\nabla\nabla w_n &\rightharpoonup \mathbf{MN} \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})\,, \end{aligned}$$

where $\mathbf{N} \in \mathrm{Sym}$ is an arbitrary matrix. Since $\mathbf{M}^n$ is coercive it follows

$$\mathbf{M}^n(\nabla\nabla w_n - \mathbf{N}) : (\nabla\nabla w_n - \mathbf{N}) \geq 0,$$

which, by symmetry of $\mathbf{M}^n$, is equivalent to

$$\mathbf{M}^n\nabla\nabla w_n : \nabla\nabla w_n - 2\mathbf{M}^n\nabla\nabla w_n : \mathbf{N} + \mathbf{M}^n\mathbf{N} : \mathbf{N} \geq 0.$$

By the compactness by compensation result, passing to the limit gives

$$\mathbf{M}\mathrm{N} : \mathrm{N} - 2\mathbf{M}\mathrm{N} : \mathrm{N} + \overline{\mathbf{M}}\mathrm{N} : \mathrm{N} \geq 0\,,$$

thus proving inequality $\overline{\mathbf{M}} \geq \mathbf{M}$, by arbitrariness of $\mathbf{N}$.

Similarly, for $\boldsymbol{\sigma} \in \mathrm{Sym}$, the coercivity of $(\mathbf{M}^n)^{-1}$ implies

$$(\mathbf{M}^n)^{-1}(\mathbf{M}^n\nabla\nabla w_n - \boldsymbol{\sigma}) : (\mathbf{M}^n\nabla\nabla w_n - \boldsymbol{\sigma}) \geq 0,$$

which is equivalent to

$$\mathbf{M}^n\nabla\nabla w_n : \nabla\nabla w_n - 2\nabla\nabla w_n : \boldsymbol{\sigma} + (\mathbf{M}^n)^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq 0.$$

Passing to the limit as before gives

$$\mathbf{M}\mathrm{N} : \mathrm{N} - 2\mathrm{N} : \boldsymbol{\sigma} + \underline{\mathbf{M}}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq 0\,,$$

which for $\boldsymbol{\sigma} = \underline{\mathbf{M}}\mathrm{N}$ becomes

$$\mathbf{M}\mathrm{N} : \mathrm{N} - 2\underline{\mathbf{M}}\mathrm{N} : \mathrm{N} + \underline{\mathbf{M}}\mathrm{N} : \mathrm{N} \geq 0,$$

i.e.

$$(\mathbf{M} - \underline{\mathbf{M}})\mathrm{N} : \mathrm{N} \geq 0\,.$$

This proves the second inequality, and concludes the proof.

■

Theorem 14 Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$. If $(\mathbf{M}^n)$ H-converges to $\mathbf{M}$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$, then the sequence $((\mathbf{M}^n)^T)$ H-converges to $\mathbf{M}^T$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$.

Proof. For $f \in \mathrm{H}^{-2}(\Omega)$, let (u_n) be the sequence of solutions to

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,((\mathbf{M}^n)^T\nabla\nabla u_n) = f \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}.$$

As sequences (u_n) and $((\mathbf{M}^n)^T\nabla\nabla u_n)$ are bounded in $\mathrm{H}^2_0(\Omega)$ and $\mathrm{L}^2(\Omega; \mathrm{Sym})$, respectively, we can extract a weakly convergent subsequence such that

$$u_n \rightharpoonup u \quad \text{in} \quad \mathrm{H}^2_0(\Omega),$$

$$(\mathbf{M}^n)^T\nabla\nabla u_n \rightharpoonup \boldsymbol{\sigma} \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym}).$$

On the other hand, since $(\mathbf{M}^n)$ H-converges to $\mathbf{M}$, for $g \in \mathrm{H}^{-2}(\Omega)$ the sequence (v_n) of solutions to

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n \nabla\nabla v_n) = g \\ v_n \in \mathrm{H}_0^2(\Omega) \end{cases}$$

satisfies

$$v_n \rightharpoonup v \quad \text{in} \quad \mathrm{H}_0^2(\Omega),$$

$$\mathbf{M}^n \nabla\nabla v_n \rightharpoonup \mathbf{M}\nabla\nabla v \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym}),$$

where v is the solution of the homogenized equation

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M} \nabla\nabla v) = g \\ v \in \mathrm{H}_0^2(\Omega) \end{cases}.$$

Applying the compactness by compensation result to

$$\mathbf{M}^n \nabla\nabla v_n : \nabla\nabla u_n = \nabla\nabla v_n : (\mathbf{M}^n)^T \nabla\nabla u_n$$

leads to

$$\mathbf{M}\nabla\nabla v : \nabla\nabla u = \nabla\nabla v : \boldsymbol{\sigma}. \tag{1.9}$$

For an arbitrary open set $\omega \Subset \Omega$, there exists $\varphi \in \mathrm{C}_c^\infty(\Omega)$ such that $\varphi|_\omega = 1$. Choosing $g := \mathsf{div}\,\mathsf{div}\,\left(\mathbf{M}\nabla\nabla\left(\frac{1}{2}\varphi(\mathbf{x})\mathbf{N}\mathbf{x}\cdot\mathbf{x}\right)\right)$, $\mathbf{N} \in \mathrm{Sym}$, implies that $v(\mathbf{x}) = \dfrac{1}{2}\mathbf{N}\mathbf{x}\cdot\mathbf{x}$ in ω. Using this, (1.9) becomes

$$\mathbf{M}\mathbf{N} : \nabla\nabla u = \mathbf{N} : \boldsymbol{\sigma} \quad \text{a. e. in } \omega\,,$$

which implies that $\boldsymbol{\sigma} = \mathbf{M}^T \nabla\nabla u$ almost everywhere in Ω, by arbitrariness of ω and $\mathbf{N}$. Due to uniqueness of the limit $\boldsymbol{\sigma}$, the entire sequence $((\mathbf{M}^n)^T \nabla\nabla u_n)$ converges weakly to $\mathbf{M}^T\nabla\nabla u$ in $\mathrm{L}^2(\Omega; \mathrm{Sym})$, which gives the claim of the theorem. ■

The following result states that H-convergence defines a metrizable topology on the set $\mathfrak{M}_2(\alpha, \beta; \Omega)$.

Theorem 15 Let $F = \{f_n : n \in \mathbf{N}\}$ be a dense countable family in $\mathrm{H}^{-2}(\Omega)$, $\mathbf{M}$ and $\mathbf{O}$ tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$, and (u_n), (v_n) sequences of solutions to

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M} \nabla\nabla u_n) = f_n \\ u_n \in \mathrm{H}_0^2(\Omega) \end{cases}$$

and

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{O} \nabla\nabla v_n) = f_n \\ v_n \in \mathrm{H}_0^2(\Omega) \end{cases},$$

respectively. Then,

$$d(\mathbf{M}, \mathbf{O}) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|u_n - v_n\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{M}\nabla\nabla u_n - \mathbf{O}\nabla\nabla v_n\|_{\mathrm{H}^{-1}(\Omega;\mathrm{Sym})}}{\|f_n\|_{\mathrm{H}^{-2}(\Omega)}} \tag{1.10}$$

is a metric function on $\mathfrak{M}_2(\alpha,\beta;\Omega)$ and H-convergence is equivalent to the convergence with respect to d.

Proof. Since $\mathfrak{M}_2(\alpha,\beta;\Omega)$ is bounded in $\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$ and $\mathrm{L}^2(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$ is continuously imbedded in $\mathrm{H}^{-1}(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$, there exists a constant $c>0$ such that

$$(\forall\,\mathbf{M}\in\mathfrak{M}_2(\alpha,\beta;\Omega))\quad \|u_n\|_{\mathrm{L}^2(\Omega)}+\|\mathbf{M}\nabla\nabla u_n\|_{\mathrm{H}^{-1}(\Omega;\mathrm{Sym})}\le c\|f_n\|_{\mathrm{H}^{-2}(\Omega)}\,.$$

Clearly, the same is true if we replace $\mathbf{M}$ and (u_n) with tensor $\mathbf{O}$ and the corresponding sequence (v_n), which implies that the series in the definition of d converges. In order to verify that d is a metric, we shall only prove that $d(\mathbf{M},\mathbf{O})=0$ implies $\mathbf{M}=\mathbf{O}$, as other properties are straightforward. The equality $d(\mathbf{M},\mathbf{O})=0$ implies that for any $f\in\mathrm{H}^{-2}(\Omega)$, the solutions u and v of

$$\begin{cases}\mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla u)=f\\ u\in\mathrm{H}^2_0(\Omega)\end{cases}$$

and

$$\begin{cases}\mathsf{div}\,\mathsf{div}\,(\mathbf{O}\nabla\nabla v)=f\\ v\in\mathrm{H}^2_0(\Omega)\end{cases}$$

satisfy $u=v$ and $\mathbf{M}\nabla\nabla u=\mathbf{O}\nabla\nabla v$ in Ω. Indeed, by definition of d, this immediately follows for $f\in F$, and then for any $f\in\mathrm{H}^{-2}(\Omega)$ by the density of F in $\mathrm{H}^{-2}(\Omega)$ and continuity of the linear mappings $f\mapsto u$ and $f\mapsto v$ from $\mathrm{H}^{-2}(\Omega)$ to $\mathrm{H}^2_0(\Omega)$. For a set ω compactly embedded in Ω let us take $\varphi\in\mathrm{C}^\infty_c(\Omega)$ such that $\varphi|_\omega=1$. If we take $f=\mathsf{div}\,\mathsf{div}\,\left(\mathbf{M}\nabla\nabla\left(\frac{1}{2}\varphi(\mathbf{x})\mathbf{S}\mathbf{x}\cdot\mathbf{x}\right)\right)$, for arbitrary $\mathbf{S}\in\mathrm{Sym}$, this yields $\nabla\nabla u=\nabla\nabla v=\mathbf{S}$ in ω, implying $\mathbf{MS}=\mathbf{OS}$ in ω, and finally $\mathbf{M}=\mathbf{O}$, by arbitrariness of $\mathbf{S}$ and ω.

It remains to prove that H-convergence is equivalent to the convergence in this metric space. Assume that sequence $(\mathbf{M}^m)$ in $\mathfrak{M}_2(\alpha,\beta;\Omega)$ H-converges to $\mathbf{M}$ in $\mathfrak{M}_2(\alpha,\beta;\Omega)$, and let (u^m_n), (u_n) be the sequences of solutions of

$$\begin{cases}\mathsf{div}\,\mathsf{div}\,(\mathbf{M}^m\nabla\nabla u^m_n)=f_n\\ u^m_n\in\mathrm{H}^2_0(\Omega)\end{cases}$$

and

$$\begin{cases}\mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla u_n)=f_n\\ u_n\in\mathrm{H}^2_0(\Omega)\end{cases},$$

respectively. Since $(\mathbf{M}^m)$ H-converges to $\mathbf{M}$ it follows

$$\begin{aligned} u^m_n &\rightharpoonup u_n \quad\text{in}\quad \mathrm{H}^2_0(\Omega)\,,\\ \mathbf{M}^m\nabla\nabla u^m_n &\rightharpoonup \mathbf{M}\nabla\nabla u_n \quad\text{in}\quad \mathrm{L}^2(\Omega;\mathrm{Sym})\,,\end{aligned}$$

and by the Rellich compactness theorem we have strong convergences $u^m_n\to u_n$ in $\mathrm{L}^2(\Omega)$ and $\mathbf{M}^m\nabla\nabla u^m_n\to\mathbf{M}\nabla\nabla u_n$ in $\mathrm{H}^{-1}(\Omega;\mathrm{Sym})$, which imply $d(\mathbf{M}^m,\mathbf{M})\to 0$.

In order to prove the converse statement, let a sequence $(\mathbf{M}^m)$ and $\mathbf{M}$ belong to $\mathfrak{M}_2(\alpha,\beta;\Omega)$ and $d(\mathbf{M}^m,\mathbf{M}) \to 0$. We take an arbitrary $f \in \mathrm{H}^{-2}(\Omega)$ and a sequence $(f_{n'}) \subseteq F$ strongly converging to f in $\mathrm{H}^{-2}(\Omega)$. Let u, u^m, $u_{n'}$ and $u^m_{n'}$ be solutions of

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla u) = f \\ u \in \mathrm{H}^2_0(\Omega) \end{cases},$$

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^m\nabla\nabla u^m) = f \\ u^m \in \mathrm{H}^2_0(\Omega) \end{cases},$$

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla u_{n'}) = f_{n'} \\ u_{n'} \in \mathrm{H}^2_0(\Omega) \end{cases},$$

and

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^m\nabla\nabla u^m_{n'}) = f_{n'} \\ u^m_{n'} \in \mathrm{H}^2_0(\Omega) \end{cases},$$

respectively. For any $n' \in \mathbf{N}$ the sequences $(u^m_{n'})_m$ and $(\mathbf{M}^m\nabla\nabla u^m_{n'})_m$ are bounded in $\mathrm{H}^2_0(\Omega)$ and $\mathrm{L}^2(\Omega;\mathrm{Sym})$, respectively, and therefore converge weakly on a subsequence. However, from $d(\mathbf{M}^m,\mathbf{M}) \to 0$ it follows that, for every $n' \in \mathbf{N}$, $u^m_{n'} \to u_{n'}$ in $\mathrm{L}^2(\Omega)$ and $\mathbf{M}^m\nabla\nabla u^m_{n'} \to \mathbf{M}\nabla\nabla u_{n'}$ in $\mathrm{H}^{-1}(\Omega;\mathrm{Sym})$, which implies the convergence of whole sequences:

$$\begin{aligned} &u^m_{n'} \rightharpoonup u_{n'} \quad \text{in} \quad \mathrm{H}^2_0(\Omega), \\ &\mathbf{M}^m\nabla\nabla u^m_{n'} \rightharpoonup \mathbf{M}\nabla\nabla u_{n'} \quad \text{in} \quad \mathrm{L}^2(\Omega;\mathrm{Sym}), \end{aligned} \tag{1.11}$$

as $m \to \infty$.

If we subtract the equations for u and $u_{n'}$, we get

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla(u - u_{n'})) = f - f_{n'} \\ u - u_{n'} \in \mathrm{H}^2_0(\Omega) \end{cases},$$

and similarly for u^m and $u^m_{n'}$:

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^m\nabla\nabla(u^m - u^m_{n'})) = f - f_{n'} \\ u^m - u^m_{n'} \in \mathrm{H}^2_0(\Omega) \end{cases}.$$

Since $(f_{n'})$ strongly converges to f, the well-posedness result for these problems ensures that $u_{n'} \to u$ in $\mathrm{H}^2_0(\Omega)$ and thus $\mathbf{M}\nabla\nabla u_{n'} \to \mathbf{M}\nabla\nabla u$ in $\mathrm{L}^2(\Omega;\mathrm{Sym})$, as well as $u^m_{n'} \to u^m$ in $\mathrm{H}^2_0(\Omega)$ and thus $\mathbf{M}^m\nabla\nabla u^m_{n'} \to \mathbf{M}^m\nabla\nabla u^m$ in $\mathrm{L}^2(\Omega;\mathrm{Sym})$, uniformly in m as $n' \to \infty$. Here, for the last convergence we have also used boundedness of the sequence $(\mathbf{M}^m)$ in $\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$.

Together with (1.11) this implies

$$
\begin{aligned}
u^m &\rightharpoonup u \quad \text{in} \quad \mathrm{H}^2_0(\Omega) \\
\mathbf{M}^m \nabla\nabla u_m &\rightharpoonup \mathbf{M}\nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym})\,,
\end{aligned} \tag{1.12}
$$

i.e. $(\mathbf{M}^m)$ H-converges to $\mathbf{M}$, by arbitrariness of f. Indeed, for an arbitrary $f \in \mathrm{H}^{-2}(\Omega)$ and $\varepsilon > 0$, the above (uniform) convergences imply that first and third term on the right-hand side of the inequality

$$
\begin{aligned}
\Big| {}_{\mathrm{H}^{-2}(\Omega)}\langle f, u^m - u \rangle_{\mathrm{H}^2_0(\Omega)} \Big| \le{}& \Big| {}_{\mathrm{H}^{-2}(\Omega)}\langle f, u^m - u^m_{n'} \rangle_{\mathrm{H}^2_0(\Omega)} \Big| + \\
&+ \Big| {}_{\mathrm{H}^{-2}(\Omega)}\langle f, u^m_{n'} - u_{n'} \rangle_{\mathrm{H}^2_0(\Omega)} \Big| + \\
&+ \Big| {}_{\mathrm{H}^{-2}(\Omega)}\langle f, u_{n'} - u \rangle_{\mathrm{H}^2_0(\Omega)} \Big|
\end{aligned}
$$

can be made ε small for n' large enough, i.e.

$$
\Big| {}_{\mathrm{H}^{-2}(\Omega)}\langle f, u^m - u \rangle_{\mathrm{H}^2_0(\Omega)} \Big| \le 2\varepsilon + \Big| {}_{\mathrm{H}^{-2}(\Omega)}\langle f, u^m_{n'} - u_{n'} \rangle_{\mathrm{H}^2_0(\Omega)} \Big|
$$

is valid for every m and n' large enough. Taking the limit as $m \to \infty$, from (1.11) and arbitrariness of ε and f we get the first convergence in (1.12), while the second one can be derived similarly. ∎

1.3 Corrector result

This section is devoted to the corrector result in dimension $d = 2$. Its goal is to improve convergence of $\nabla\nabla u_n$ by adding correctors, and ending up with strong convergence, instead of the weak one given by the definition of H-convergence.

Definition 5 Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a limit $\mathbf{M}$. For $1 \le i, j \le 2$ let $(w^{ij}_n)_n$ be a sequence of oscillating test functions satisfying

$$
\begin{aligned}
w^{ij}_n &\rightharpoonup \frac{1}{2} x_i x_j \quad \text{in} \quad \mathrm{H}^2(\Omega)\,, \\
\operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla w^{ij}_n) &\longrightarrow g_{ij} \quad \text{in} \quad \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)\,,
\end{aligned} \tag{1.13}
$$

where g_{ij} are some elements of $\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$. The tensor $\mathbf{W}^n$ with components $W^n_{ijkm} := [\nabla\nabla w^{km}_n]_{ij}$ is called the corrector.

It is important to note that functions $(w^{ij}_n)_{1 \le i,j \le 2}$ are not uniquely defined. However, for any other family of such functions, it is easy to see that their difference converges strongly to zero in $\mathrm{H}^2(\Omega)$, and similar holds true for the corrector tensors.

Lemma 3 Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a tensor $\mathbf{M}$. A sequence of correctors $(\mathbf{W}^n)$ is unique in the sense that, for any two sequences of correctors $(\mathbf{W}^n)$ and $(\tilde{\mathbf{W}}^n)$, their difference $(\mathbf{W}^n - \tilde{\mathbf{W}}^n)$ converges strongly to zero in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.

Proof. For $1 \le i, j \le 2$, let $(w_n^{ij})_n$ and $(\widetilde{w}_n^{ij})_n$ be two sequence satisfying (1.13) and let $\varphi \in \mathrm{C}_c^\infty(\Omega)$. Using coercivity of $\mathbf{M}^n$, and integrating by parts two times we obtain:

$$\begin{aligned}
\alpha \|\varphi(\nabla\nabla w_n^{ij} - \nabla\nabla \widetilde{w}_n^{ij})\|^2_{\mathrm{L}^2(\Omega;\mathrm{Sym})} \le & \int_\Omega \varphi^2 \mathbf{M}^n \nabla\nabla(w_n^{ij} - \widetilde{w}_n^{ij}) : \nabla\nabla(w_n^{ij} - \widetilde{w}_n^{ij})\, d\mathbf{x} \\
= & \ {}_{\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)}\langle\, \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n \nabla\nabla(w_n^{ij} - \widetilde{w}_n^{ij}), \varphi^2(w_n^{ij} - \widetilde{w}_n^{ij})\,\rangle_{H^2_c(\Omega)} + \\
& + {}_{\mathrm{H}^{-1}_{\mathrm{loc}}(\Omega)}\langle\, \mathsf{div}\,(\mathbf{M}^n \nabla\nabla(w_n^{ij} - \widetilde{w}_n^{ij})), \nabla(\varphi^2)(w_n^{ij} - \widetilde{w}_n^{ij})\,\rangle_{H^1_c(\Omega)} - \\
& - {}_{\mathrm{L}^2_{\mathrm{loc}}(\Omega)}\langle\, \mathbf{M}^n \nabla\nabla(w_n^{ij} - \widetilde{w}_n^{ij}), \nabla(w_n^{ij} - \widetilde{w}_n^{ij})\nabla(\varphi^2))\,\rangle_{L^2_c(\Omega)}\,.
\end{aligned}$$

Each term on the right hand side tends to zero when $n \to \infty$, the first one because of the assumption (1.13), while the second one and the third one converge to zero by the Rellich compactness theorem. Thus, we deduce that $\nabla\nabla(w_n^{ij} - \widetilde{w}_n^{ij})$ converges strongly to zero in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})$, which proves the statement.

■

Lemma 4 Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a limit $\mathbf{M}$, and $(\mathbf{W}^n)$ the corresponding sequence of correctors. Then

$$\begin{aligned}
\mathbf{W}^n &\rightharpoonup \mathbf{I}_4 \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))\,, \\
\mathbf{M}^n\mathbf{W}^n &\rightharpoonup \mathbf{M} \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))\,, \\
(\mathbf{W}^n)^T\mathbf{M}^n\mathbf{W}^n &\rightharpoonup \mathbf{M} \quad \text{in} \quad \mathrm{D}'(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))\,.
\end{aligned}$$

Proof. The first convergence is a consequence of the definition of correctors. The second one follows from the definition of H-convergence, and the third one from the compactness by compensation result applied to the components of $(\mathbf{W}^n)^T$ and $\mathbf{M}^n\mathbf{W}^n$. ■

In the next theorem we clarify in what sense correctors *transform a weak convergence into the strong one.*

Theorem 16 Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ which H-converges to $\mathbf{M}$. For $f \in \mathrm{H}^{-2}(\Omega)$, let (u_n) be the sequence of solutions to

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n \nabla\nabla u_n) = f \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}.$$

Let u be the weak limit of (u_n) in $\mathrm{H}^2_0(\Omega)$, i.e., the solution of the homogenized equation

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M} \nabla\nabla u) = f \\ u \in \mathrm{H}^2_0(\Omega) \end{cases}.$$

Then, if we denote $r_n := \nabla\nabla u_n - \mathbf{W}^n \nabla\nabla u$, where $\mathbf{W}^n$ is a corrector, it holds that (r_n) converges strongly to zero in $\mathrm{L}^1_{\mathrm{loc}}(\Omega; \mathrm{Sym})$.

Proof. Let $\varphi \in C_c^\infty(\Omega)$, and let (v_m) be a sequence in $C_c^\infty(\Omega)$ such that $v_m \to u$ in $\mathrm{H}^2_0(\Omega)$. Since $\mathbf{M}^n$ is coercive we have

$$
\begin{aligned}
&\alpha \|\varphi(\nabla\nabla u_n - \mathbf{W}^n \nabla\nabla v_m)\|^2_{\mathrm{L}^2(\Omega;\mathrm{Sym})} \\
&\qquad \le \int_\Omega \varphi^2 \mathbf{M}^n (\nabla\nabla u_n - \mathbf{W}^n \nabla\nabla v_m) : (\nabla\nabla u_n - \mathbf{W}^n \nabla\nabla v_m)\, d\mathbf{x} \\
&\qquad = \int_\Omega \varphi^2 \mathbf{M}^n \nabla\nabla u_n : \nabla\nabla u_n \, d\mathbf{x} - \int_\Omega \varphi^2 \mathbf{M}^n \nabla\nabla u_n : \mathbf{W}^n \nabla\nabla v_m \, d\mathbf{x} - \\
&\qquad - \int_\Omega \varphi^2 \mathbf{M}^n \mathbf{W}^n \nabla\nabla v_m : \nabla\nabla u_n \, d\mathbf{x} + \int_\Omega \varphi^2 (\mathbf{W}^n)^T \mathbf{M}^n \mathbf{W}^n \nabla\nabla v_m : \nabla\nabla v_m \, d\mathbf{x}.
\end{aligned}
$$

As $n \to +\infty$, the first term on the right hand side converges by Theorem 10, while the second and the third term converge by the compensated compactness result. The last term converges by Lemma 4, leading to

$$
\limsup_{n\to\infty} \|\varphi(\nabla\nabla u_n - \mathbf{W}^n \nabla\nabla v_m)\|^2_{\mathrm{L}^2(\Omega;\mathrm{Sym})} \le \frac{1}{\alpha} \int_\Omega \varphi^2 \mathbf{M} \nabla\nabla(u - v_m) : \nabla\nabla(u - v_m)\, d\mathbf{x}\,.
$$

If u is smooth (in that case we can choose $v_m = u$), the proof is finished. If u is not smooth, than after taking limit as $n \to +\infty$ in the estimate (c is a generic constant below)

$$
\begin{aligned}
&\|\varphi(\nabla\nabla u_n - \mathbf{W}^n \nabla\nabla u)\|_{\mathrm{L}^1(\Omega;\mathrm{Sym})} \\
&\quad \le \|\varphi(\nabla\nabla u_n - \mathbf{W}^n \nabla\nabla v_m)\|_{\mathrm{L}^1(\Omega;\mathrm{Sym})} + \|\varphi \mathbf{W}^n (\nabla\nabla v_m - \nabla\nabla u)\|_{\mathrm{L}^1(\Omega;\mathrm{Sym})} \\
&\quad \le \|\varphi(\nabla\nabla u_n - \mathbf{W}^n \nabla\nabla v_m)\|_{\mathrm{L}^2(\Omega;\mathrm{Sym})} + \|\varphi \mathbf{W}^n\|_{\mathrm{L}^2(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))} \|\nabla\nabla v_m - \nabla\nabla u\|_{\mathrm{L}^2(\Omega;\mathrm{Sym})}\,,
\end{aligned}
$$

we get

$$
\begin{aligned}
&\limsup_{n\to\infty} \|\varphi(\nabla\nabla u_n - \mathbf{W}^n \nabla\nabla u)\|_{\mathrm{L}^1(\Omega;\mathrm{Sym})} \\
&\qquad \le c \Big(\|\varphi(\nabla\nabla u - \nabla\nabla v_m)\|_{\mathrm{L}^2(\Omega;\mathrm{Sym})} + \|\nabla\nabla v_m - \nabla\nabla u\|_{\mathrm{L}^2(\Omega;\mathrm{Sym})} \Big)\,,
\end{aligned}
$$

and finally

$$
\begin{aligned}
&\limsup_{n\to\infty} \|\varphi(\nabla\nabla u_n - \mathbf{W}^n \nabla\nabla u)\|_{\mathrm{L}^1(\Omega;\mathrm{Sym})} \\
&\qquad \le c \limsup_{m\to\infty} (\|\varphi(\nabla\nabla u - \nabla\nabla v_m)\|_{\mathrm{L}^2(\Omega;\mathrm{Sym})} + \|\nabla\nabla u - \nabla\nabla v_m\|_{\mathrm{L}^2(\Omega;\mathrm{Sym})}) = 0\,,
\end{aligned}
$$

which finishes the proof, by arbitrariness of φ.

■

1.4 H-convergent sequence depending on a parameter

A prerequisite for the small-amplitude homogenization is that H-convergence preserves a smooth (or analytic) dependence with respect to a parameter. More precisely, we shall prove that, if a sequence $\mathbf{M}^n(\cdot, p)$ depends smoothly on a parameter p, so does the H-limit $\mathbf{M}(\cdot, p)$. We shall begin with this simple lemma, whose proof mimics the one in the case of the stationary diffusion equation [12, 72] but we present it here for completeness.

Lemma 5 If $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ and $\mathbf{O} \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$ such that $\|\mathbf{O}\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))} \leq \delta < \alpha$, then

$$\mathbf{M} + \mathbf{O} \in \mathfrak{M}_2\left(\alpha - \delta, \frac{\alpha\beta - \delta^2}{\alpha - \delta}; \Omega\right).$$

Proof. One can easily see that

$$(\mathbf{M} + \mathbf{O})\boldsymbol{\xi} : \boldsymbol{\xi} \geq \alpha|\boldsymbol{\xi}|^2 - \delta|\boldsymbol{\xi}|^2, \tag{1.14}$$

where $|\cdot|$ on the right-hand side of (1.14) denotes the Frobenius norm. This proves the coercivity of $\mathbf{M} + \mathbf{O}$.

The other bound $(\mathbf{M} + \mathbf{O})\boldsymbol{\xi} : \boldsymbol{\xi} \leq \dfrac{\alpha\beta - \delta^2}{\alpha - \delta}\boldsymbol{\xi} : \boldsymbol{\xi}$ can be written in two equivalent forms, as

$$(\mathbf{M} + \mathbf{O})\boldsymbol{\xi} : \boldsymbol{\xi} \geq \frac{1}{2L}|(\mathbf{M} + \mathbf{O})\boldsymbol{\xi}|^2 \quad \text{or} \quad |(\mathbf{M} + \mathbf{O})\boldsymbol{\xi} - L\boldsymbol{\xi}| \leq L|\boldsymbol{\xi}|,$$

where $L := \dfrac{\alpha\beta - \delta^2}{2(\alpha - \delta)}$. Note that $L > \dfrac{1}{2}\beta$, therefore we have

$$(-2L\delta + \delta^2)|\boldsymbol{\xi}|^2 = (\beta - 2L)\alpha|\boldsymbol{\xi}|^2 \geq (\beta - 2L)\mathbf{M}\boldsymbol{\xi} : \boldsymbol{\xi} \geq |\mathbf{M}\boldsymbol{\xi}|^2 - 2L\mathbf{M}\boldsymbol{\xi} : \boldsymbol{\xi},$$

and the obtained inequality can be rewritten as

$$|\mathbf{M}\boldsymbol{\xi} - L\boldsymbol{\xi}|^2 \leq (L - \delta)^2|\boldsymbol{\xi}|^2.$$

Finally,

$$|(\mathbf{M} + \mathbf{O})\boldsymbol{\xi} - L\boldsymbol{\xi}| \leq |\mathbf{M}\boldsymbol{\xi} - L\boldsymbol{\xi}| + |\mathbf{O}\boldsymbol{\xi}| \leq (L - \delta)|\boldsymbol{\xi}| + \delta|\boldsymbol{\xi}| = L|\boldsymbol{\xi}|,$$

or equivalently

$$(\mathbf{M} + \mathbf{O})\boldsymbol{\xi} : \boldsymbol{\xi} \geq \frac{1}{2L}|(\mathbf{M} + \mathbf{O})\boldsymbol{\xi}|^2.$$

■

Let us now describe a bound for the L^∞-distance between the H-limits of two sequences $\mathbf{M}^n \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ and $\mathbf{O}^n \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$, that are nearby in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.

Lemma 6 Let $\mathbf{M}^n \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ and $\mathbf{O}^n \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$ be two sequences of tensors that H-converge to the homogenized tensors $\mathbf{M}$ and $\mathbf{O}$, respectively. Assume that,

$$(\exists \varepsilon > 0)\ (\forall n \in \mathbf{N})\ \|\mathbf{O}^n - \mathbf{M}^n\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))} \le \varepsilon.$$

Then

$$\|\mathbf{O} - \mathbf{M}\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))} \le \varepsilon\sqrt{\frac{\beta\beta'}{\alpha\alpha'}}.$$

Proof. For $f, g \in \mathrm{H}^{-2}(\Omega)$, let (u_n) and (v_n) be sequences of solutions to

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n \nabla\nabla u_n) = f \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}$$

and

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,((\mathbf{O}^n)^T \nabla\nabla v_n) = g \\ v_n \in \mathrm{H}^2_0(\Omega) \end{cases}.$$

Since $(\mathbf{M}^n)$ H-converges to $\mathbf{M}$, and $(\mathbf{O}^n)^T$ H-converges to $\mathbf{O}^T$, it follows

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in} \quad \mathrm{H}^2_0(\Omega), \\ \mathbf{M}^n \nabla\nabla u_n &\rightharpoonup \mathbf{M}\nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym}), \\ v_n &\rightharpoonup v \quad \text{in} \quad \mathrm{H}^2_0(\Omega), \\ (\mathbf{O}^n)^T \nabla\nabla v_n &\rightharpoonup \mathbf{O}^T\nabla\nabla v \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym}). \end{aligned}$$

By Lemma 1, we have that $\mathbf{M}^n\nabla\nabla u_n : \nabla\nabla v_n$ and $\nabla\nabla u_n : (\mathbf{O}^n)^T\nabla\nabla v_n$ converge vaguely to $\mathbf{M}\nabla\nabla u : \nabla\nabla v$ and $\nabla\nabla u : \mathbf{O}^T\nabla\nabla v$, respectively. Therefore, for every $\varphi \in \mathrm{C}^\infty_c(\Omega)$ one has

$$\lim_n \int_\Omega \varphi(\mathbf{O}^n - \mathbf{M}^n)\nabla\nabla u_n : \nabla\nabla v_n \, d\mathbf{x} = \int_\Omega \varphi(\mathbf{O} - \mathbf{M})\nabla\nabla u : \nabla\nabla v \, d\mathbf{x}.$$

For every $\varphi \ge 0$ one can conclude

$$\left| \int_\Omega \varphi(\mathbf{O} - \mathbf{M})\nabla\nabla u : \nabla\nabla v \, d\mathbf{x} \right| \le \varepsilon \limsup_n \int_\Omega \varphi |\nabla\nabla u_n||\nabla\nabla v_n| \, d\mathbf{x}. \tag{1.15}$$

Since for arbitrary $a, b \in \mathbf{R}^+$ such that $4ab\alpha\alpha' \ge 1$ it holds

$$|\nabla\nabla u_n||\nabla\nabla v_n| \le a\alpha|\nabla\nabla u_n|^2 + b\alpha'|\nabla\nabla v_n|^2,$$

and $\mathbf{M}^n \in \mathfrak{M}_2(\alpha, \beta; \Omega)$, $\mathbf{O}^n \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$, we have

$$\left| \int_\Omega \varphi(\mathbf{O} - \mathbf{M})\nabla\nabla u : \nabla\nabla v \, d\mathbf{x} \right| \le \varepsilon \limsup_n \int_\Omega \varphi[a(\mathbf{M}^n\nabla\nabla u_n : \nabla\nabla u_n) + b(\mathbf{O}^n\nabla\nabla v_n : \nabla\nabla v_n)] \, d\mathbf{x}$$

$$= \varepsilon \int_\Omega \varphi[a(\mathbf{M}\nabla\nabla u : \nabla\nabla u) + b(\mathbf{O}\nabla\nabla v : \nabla\nabla v)]\, d\mathbf{x}$$
$$\leq \varepsilon a\beta \int_\Omega \varphi|\nabla\nabla u|^2\, d\mathbf{x} + \varepsilon b\beta' \int_\Omega \varphi|\nabla\nabla v|^2\, d\mathbf{x}.$$

Since this inequality is true for every $\varphi \in \mathrm{C}_c^\infty(\Omega)$, $\varphi \geq 0$, it follows

$$|(\mathbf{O} - \mathbf{M})\nabla\nabla u : \nabla\nabla v| \leq \varepsilon(a\beta|\nabla\nabla u|^2 + b\beta'|\nabla\nabla v|^2) \quad \text{a. e. in } \Omega. \tag{1.16}$$

After minimizing the right-hand side of the previous inequality over all a, b satisfying the condition $4ab\alpha\alpha' \geq 1$ we get

$$|(\mathbf{O} - \mathbf{M})\nabla\nabla u : \nabla\nabla v| \leq \varepsilon\sqrt{\frac{\beta\beta'}{\alpha\alpha'}}|\nabla\nabla u||\nabla\nabla v| \quad \text{a. e. in } \Omega\,.$$

An alternative way to obtain the minimum (over a and b) for the right-hand side of (1.16) is to use the arithmetic-geometric mean inequality and $ab \geq \frac{1}{4\alpha\alpha'}$. Hence, the desired inequality follows by arbitrariness of u and v, by using an analogous arguments as was done in the proof of Theorem 15. ■

Let us now prove that when passing to the H-limit in a sequence depending on a parameter, the smoothness is preserved. This result appears to be very important since we want to calculate first correction in the small-amplitude limit.

Theorem 17 Let $\mathbf{M}^n : \Omega \times P \to \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ be a sequence of tensors, such that $\mathbf{M}^n(\cdot, p) \in \mathfrak{M}_2(\alpha, \beta; \Omega)$, for $p \in P$, where $P \subseteq \mathbf{R}$ is an open set. Assume that (for some $k \in \mathbf{N}_0$) a mapping $p \mapsto \mathbf{M}^n(\cdot, p)$ is of class C^k from P to $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, with all derivatives up to order k being equicontinuous on every compact set $K \subseteq P$:

$$\begin{aligned}&(\forall K \in \mathcal{K}(P))\,(\forall\varepsilon > 0)(\exists\delta > 0)(\forall p, q \in K)(\forall n \in \mathbf{N})(\forall i \in \{0, \ldots, k\})\\ &\qquad |p - q| < \delta \Rightarrow \|(\mathbf{M}^n)^{(i)}(\cdot, p) - (\mathbf{M}^n)^{(i)}(\cdot, q)\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))} < \varepsilon.\end{aligned} \tag{1.17}$$

Then there is a subsequence $(\mathbf{M}^{n_k})$ such that for every $p \in P$

$$\mathbf{M}^{n_k}(\cdot, p) \xrightarrow{H} \mathbf{M}(\cdot, p) \quad \text{in} \quad \mathfrak{M}_2(\alpha, \beta; \Omega) \tag{1.18}$$

and $p \mapsto \mathbf{M}(\cdot, p)$ is a C^k mapping from P to $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.

Proof. For a countable dense subset Π of P, by the Cantor diagonal method and compactness of $\mathfrak{M}_2(\alpha, \beta; \Omega)$ there exists a subsequence $(\mathbf{M}^{n_k})$ such that for every $p \in \Pi$ $\mathbf{M}^{n_k}(\cdot, p) \xrightarrow{H} \mathbf{M}(\cdot, p)$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$. For arbitrary compact $K \subseteq P$, by (1.17), it follows that

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall p, q \in K)(\forall n \in \mathbf{N})$$

$$|p-q|<\delta \Rightarrow \|\mathbf{M}^n(\cdot,p)-\mathbf{M}^n(\cdot,q)\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))}<\varepsilon.$$

Since for $p,q\in K\cap\Pi$ we have $\mathbf{M}^{n_k}(\cdot,p)\xrightarrow{H}\mathbf{M}(\cdot,p)$ and $\mathbf{M}^{n_k}(\cdot,q)\xrightarrow{H}\mathbf{M}(\cdot,q)$, by Lemma 6 it follows

$$\|\mathbf{M}(\cdot,p)-\mathbf{M}(\cdot,q)\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))}<\varepsilon\frac{\beta}{\alpha},$$

which implies uniform continuity of $p\mapsto\mathbf{M}(\cdot,p)$ on $K\cap\Pi$, and thus it can be extended by continuity to the entire set K. In order to prove that (1.18) holds for every $p\in K$, let us suppose the opposite, i.e. that this H-convergence fails for some $p\in K$. Due to the compactness of H-convergence, there exists a subsequence $(\mathbf{M}^{n_{k_r}})$ and $\mathbf{N}\in\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$ such that $\mathbf{M}^{n_{k_r}}(\cdot,p)\xrightarrow{H}\mathbf{M}(\cdot,p)+\mathbf{N}$, where $\varepsilon:=\|\mathbf{N}\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))}>0$. Using the equicontinuity of $(\mathbf{M}^{n_{k_r}})$ and uniform continuity of $\mathbf{M}$ over K one can conclude that there exists a $\delta>0$ such that for every $q\in K$ such that $|p-q|<\delta$, it follows

$$\|\mathbf{M}^{n_{k_r}}(\cdot,p)-\mathbf{M}^{n_{k_r}}(\cdot,q)\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))}<\varepsilon\frac{\alpha}{2\beta},\quad\text{and}$$
$$\|\mathbf{M}(\cdot,p)-\mathbf{M}(\cdot,q)\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))}<\frac{\varepsilon}{2}.$$

From the second inequality, it easily follows

$$\begin{aligned}&\|\mathbf{M}(\cdot,p)+\mathbf{N}-\mathbf{M}(\cdot,q)\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))}\\&\qquad\geq\left|\|\mathbf{N}\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))}-\|\mathbf{M}(\cdot,p)-\mathbf{M}(\cdot,q)\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))}\right|>\frac{\varepsilon}{2},\end{aligned}$$

while from the first one and Lemma 6 we have

$$\|\mathbf{M}(\cdot,p)+\mathbf{N}-\mathbf{M}(\cdot,q)\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))}<\frac{\varepsilon}{2},$$

for $q\in\Pi\cap K$ and $|p-q|<\delta$, which is a contradiction. Therefore, (1.18) holds for every $p\in K$ and, by arbitrariness of K, the mapping $p\mapsto\mathbf{M}(\cdot,p)$ is well defined and continuous on P.

In order to prove that $p\mapsto\mathbf{M}(\cdot,p)$ is a C^k mapping from P to $\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$ let us define a family of operators $\tau_n(p):\mathrm{H}^2_0(\Omega)\to\mathrm{H}^{-2}(\Omega)$, for $n\in\mathbf{N}$ and $p\in P$, with

$$\tau_n(p)v:=\operatorname{div}\operatorname{div}(\mathbf{M}^n(\cdot,p)\nabla\nabla v),\quad v\in\mathrm{H}^2_0(\Omega).$$

Note that $\tau_n(p)$ may be written as a composition $\mathcal{P}\circ\mathbf{M}^n(\cdot,p)$, where $\mathcal{P}:\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))\to\mathcal{L}(\mathrm{H}^2_0(\Omega),\mathrm{H}^{-2}(\Omega))$ is defined as $\mathcal{P}(\mathbf{N}):=\operatorname{div}\operatorname{div}(\mathbf{N}\nabla\nabla\cdot)$, for $\mathbf{N}\in\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$. Since $\mathcal{P}$ is linear and $p\mapsto\mathbf{M}^n(\cdot,p)$ is of class C^k one can conclude that $\tau_n:P\to\mathcal{L}(\mathrm{H}^2_0(\Omega),\mathrm{H}^{-2}(\Omega))$ is a sequence of C^k mappings. Additionally, (τ_n) satisfies the same equicontinuity property as the sequence $(\mathbf{M}^n)$.

Since $\tau_n(p)$ is an isomorphism (for every p and n), and, by Proposition 1, taking inverse is a C^∞ mapping, it follows that the mapping $p \mapsto (\tau_n(p))^{-1}$ is also a C^k mapping. Moreover, one can conclude the following:

$$\begin{aligned}&(\forall K \in \mathcal{K}(P))\,(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p, q \in K)(\forall n \in \mathbf{N})(\forall i \in \{0, \dots, k\})\\ &\qquad |p-q| < \delta \Rightarrow \|((\tau_n(p))^{-1})^{(i)} - ((\tau_n(q))^{-1})^{(i)}\|_{\mathcal{L}(\mathrm{H}^{-2}(\Omega), \mathrm{H}^2_0(\Omega))} < \varepsilon. \end{aligned} \tag{1.19}$$

We shall prove this only for $i = 0$, since for $i \in \{1, \dots, k\}$ it can be shown analogously. By Theorem 2 and using the notation as in Proposition 1, with $X := \mathrm{H}^2_0(\Omega)$, $Y := \mathrm{H}^{-2}(\Omega)$, we have:

$$\begin{aligned}&(\forall K \in \mathcal{K}(P))\,(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p, q \in K)(\forall n \in \mathbf{N})|p-q| < \delta \Rightarrow\\ &\quad \|(\tau_n(p))^{-1} - (\tau_n(q))^{-1}\|_{L(Y,X)} = \|J(\tau_n(p)) - J(\tau_n(q))\|_{L(Y,X)}\\ &\quad \le \sup\Big\{\|J'(T)\|_{L(L(X,Y),L(Y,X))} : T \in [\tau_n(p), \tau_n(q)]\Big\} \cdot \|\tau_n(p) - \tau_n(q)\|_{L(X,Y)}\\ &\quad < \sup\Big\{\|J'(T)\|_{L(L(X,Y),L(Y,X))} : T \in [\tau_n(p), \tau_n(q)]\Big\} \cdot \varepsilon.\end{aligned}$$

Due to equicontinuity property of the sequence (τ_n), one only has to check that $\sup\Big\{\|J'(T)\|_{L(L(X,Y),L(Y,X))} : T \in [\tau_n(p), \tau_n(q)]\Big\}$ is finite:

$$\begin{aligned}\|J'(T)\|_{L(L(X,Y),L(Y,X))} &= \sup_{\|B\|_{L(X,Y)}=1} \|J'(T)(B)\|_{L(Y,X)}\\ &= \sup_{\|B\|_{L(X,Y)}=1} \|-T^{-1} \circ B \circ T^{-1}\|_{L(Y,X)}\\ &\le \sup_{\|B\|_{L(X,Y)}=1} \|T^{-1}\|_{L(Y,X)} \|B\|_{L(X,Y)} \|T^{-1}\|_{L(Y,X)}\\ &= \|T^{-1}\|^2_{L(Y,X)}.\end{aligned}$$

As $T \in [\tau_n(p), \tau_n(q)]$, $\tau_n(p) = \mathcal{P} \circ \mathbf{M}^n(\cdot, p)$, and by using convexity of the set $\mathfrak{M}_2(\alpha, \beta; \Omega)$, for some $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ we have $T = \mathcal{P} \circ \mathbf{M} \in \mathcal{L}(\mathrm{H}^2_0(\Omega), \mathrm{H}^{-2}(\Omega))$, which is a bounded and coercive operator with constants independent of p, q and n. Thus, it follows that $\|T^{-1}\|^2_{L(Y,X)} < \infty$, i.e.

$$\sup\Big\{\|J'(T)\|_{L(L(X,Y),L(Y,X))} : T \in [\tau_n(p), \tau_n(q)]\Big\} < \infty.$$

Since the subsequence $(\mathbf{M}^{n_k}(\cdot, p))$ H-converges to $\mathbf{M}(\cdot, p)$ for every $p \in P$, it follows that

$$(\forall p \in P)\,(\forall f \in \mathrm{H}^{-2}(\Omega))\;\; (\tau_{n_k}(p))^{-1} f \xrightarrow{\mathrm{H}^2_0(\Omega)} (\tau(p))^{-1} f, \tag{1.20}$$

where $\tau(p) = \mathcal{P}(\mathbf{M}(\cdot, p))$. Let us define a family of functions $\Psi^{f,g}_n : P \to \mathbf{R}$, $n \in \mathbf{N}$, and $\Psi^{f,g} : P \to \mathbf{R}$ as

$$\Psi^{f,g}_n(p) = \frac{{}_{\mathrm{H}^2_0(\Omega)}\langle\, (\tau_n(p))^{-1} f, g\,\rangle_{\mathrm{H}^{-2}(\Omega)}}{\|f\|_{\mathrm{H}^{-2}(\Omega)} \|g\|_{\mathrm{H}^{-2}(\Omega)}},$$

$$\Psi^{f,g}(p) = \frac{{}_{\mathrm{H}_0^2(\Omega)}\langle\, (\tau(p))^{-1} f, g \,\rangle_{\mathrm{H}^{-2}(\Omega)}}{\|f\|_{\mathrm{H}^{-2}(\Omega)}\|g\|_{\mathrm{H}^{-2}(\Omega)}},$$

where $f, g \in \mathrm{H}^{-2}(\Omega)$ are arbitrary nonzero functions. Since $p \mapsto (\tau_n(p))^{-1}$ is a C^k mapping from P to $\mathcal{L}(\mathrm{H}^{-2}(\Omega), \mathrm{H}_0^2(\Omega))$, we have that $p \mapsto \Psi_n^{f,g}(p)$ is of class C^k from P to $\mathbf{R}$. Note that due to equicontinuity properties of a sequence $((\tau_n(p))^{-1})$ it follows that

$$\begin{aligned}
&(\forall K \in \mathcal{K}(P))\,(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p, q \in K)(\forall n \in \mathbf{N})(\forall i \in \{0, \dots, k\})\, |p - q| < \delta \Rightarrow \\
&|(\Psi_n^{f,g})^{(i)}(p) - (\Psi_n^{f,g})^{(i)}(q)| \qquad (1.21)\\
&= \left| \frac{{}_{\mathrm{H}_0^2(\Omega)}\Big\langle \left((\tau_n(p))^{-1}\right)^{(i)} f, g \Big\rangle_{\mathrm{H}^{-2}(\Omega)}}{\|f\|_{\mathrm{H}^{-2}(\Omega)}\|g\|_{\mathrm{H}^{-2}(\Omega)}} - \frac{{}_{\mathrm{H}_0^2(\Omega)}\Big\langle \left((\tau_n(q))^{-1}\right)^{(i)} f, g \Big\rangle_{\mathrm{H}^{-2}(\Omega)}}{\|f\|_{\mathrm{H}^{-2}(\Omega)}\|g\|_{\mathrm{H}^{-2}(\Omega)}} \right| \\
&= \frac{1}{\|f\|_{\mathrm{H}^{-2}(\Omega)}\|g\|_{\mathrm{H}^{-2}(\Omega)}} \left| {}_{\mathrm{H}_0^2(\Omega)}\Big\langle \left(\left((\tau_n(p))^{-1}\right)^{(i)} - \left((\tau_n(q))^{-1}\right)^{(i)} \right) f, g \Big\rangle_{\mathrm{H}^{-2}(\Omega)} \right| < \varepsilon.
\end{aligned}$$

Additionally, from (1.20) we have that

$$(\forall p \in P) \quad \Psi_{n_k}^{f,g}(p) \longrightarrow \Psi^{f,g}(p)\,. \qquad (1.22)$$

By the Arzelà-Ascoli theorem it follows that the sequence $(\Psi_{n_k}^{f,g})$, with all its derivatives, is bounded in $C(K)$, where $K \subseteq P$ is an arbitrary compact set, and pointwise convergence in (1.22) is actually uniform, thus $p \to \Psi^{f,g}(p)$ is of class C^k from P to $\mathbf{R}$. After passing to the limit in (1.21), one can conclude that $\Psi^{f,g}$ has the same equicontinuity properties as the sequence $(\Psi_{n_k}^{f,g})$. It follows that $p \mapsto (\tau(p))^{-1}$ is a C^k mapping, and using the same reasoning as before, the mapping $p \mapsto \tau(p)$ is also of class C^k.

Let us now consider a sequence $Z_n : P \to \mathcal{L}(\mathrm{H}_0^2(\Omega); \mathrm{L}^2(\Omega; \mathrm{Sym}))$ defined by

$$Z_n(p)v := \mathbf{M}^n(\cdot, p)\nabla\nabla((\tau_n(p))^{-1}\tau(p)v),\ v \in \mathrm{H}_0^2(\Omega).$$

Note that, with $v_n \in \mathrm{H}_0^2(\Omega)$ defined as

$$\tau_n(p)v_n = \tau(p)v,$$

one has

$$Z_n(p)v = \mathbf{M}^n(\cdot, p)\nabla\nabla v_n.$$

As it is a composition of C^k mappings it follows that each Z_n is also of class C^k. By H-convergence of a subsequence $(\mathbf{M}^{n_k}(\cdot, p))$ to $\mathbf{M}(\cdot, p)$, one can easily see that $Z_{n_k}(p)v$ converges weakly in $\mathrm{L}^2(\Omega; \mathrm{Sym})$ to $Z(p)v := \mathbf{M}(\cdot, p)\nabla\nabla v$, for arbitrary $v \in \mathrm{H}_0^2(\Omega)$. Similarly as for $(\tau(p))^{-1}$ above, one can easily show that $p \mapsto Z(p)$ belongs to the class $\mathrm{C}^k(P; \mathcal{L}(\mathrm{H}_0^2(\Omega); \mathrm{L}^2(\Omega; \mathrm{Sym})))$.

The Lebesgue measure λ is inner regular on $\mathbf{R}^d$, hence for every $\varepsilon > 0$ there exists a compact set $K \subseteq \Omega$, such that $\lambda(\Omega \backslash K) < \varepsilon$. For $O := K \backslash \partial K \Subset \Omega$, and $\mathbf{S} \in \mathrm{Sym}$, let us take $v \in \mathrm{H}^2_0(\Omega)$ such that $v(\mathbf{x}) = \frac{1}{2}\mathbf{S}\mathbf{x} \cdot \mathbf{x}$ on O. We easily conclude that $p \mapsto \mathbf{M}(\cdot, p)\mathbf{S}$ belongs to the class $\mathrm{C}^k(P; \mathrm{L}^\infty(O; \mathrm{Sym}))$, and moreover, since $\lambda(\Omega \backslash K) < \varepsilon$, to $\mathrm{C}^k(P; \mathrm{L}^\infty(\Omega; \mathrm{Sym}))$. Due to arbitrariness of $\mathbf{S}$, it follows that $p \mapsto \mathbf{M}(\cdot, p)$ is of class C^k from P to $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, which concludes the proof.

■

Remark 2 It is easy to see that the above theorem is valid if we take $P \subseteq \mathbf{R}^d$ an open set. Furthermore, it can be shown that H-convergence also preserves an analytic dependence with respect to a parameter. To be precise, if we assume in the previous theorem that every $\mathbf{M}^n$ is analytic mapping $P \longrightarrow \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, then the corresponding H-limit $\mathbf{M}$ (after extracting a subsequence) is also analytic. This can be proved using the fact that any weakly converging sequence of analytic functions of operators has a limit which is analytic as well [42], and by following the same technique as in the proof of Theorem 17.

1.5 Periodic homogenization

When studying homogenization theory, periodic homogenization [16] appears to be the simplest case. There is a wide range of applications of periodic homogenization, for example in mechanics, physics, chemistry and engineering, in the study of crystalline or polymer structures, nuclear reactor design, etc. If the period of the observed structure is small compared to size of a region in which the system is studied, then asymptotic analysis is used. An example of a periodic domain is given in Figure 1.1, where x denotes the width of Ω.

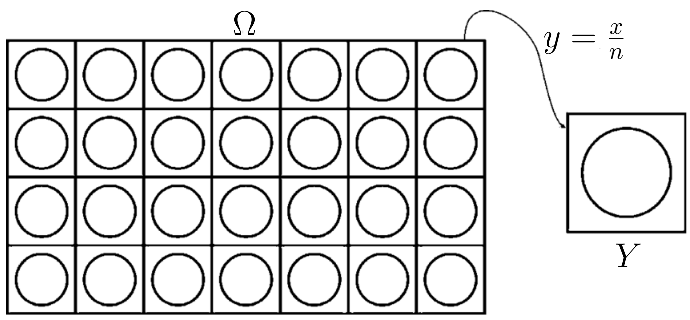


Figure 1.1: Periodic domain.

For the unit cube $Y = [0,1]^d$ in $\mathbf{R}^d$ and $p \in [1, \infty]$, let us consider the following normed spaces of Y-periodic functions [2]: $\mathrm{L}^p_\#(Y) := \{f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ such that f is $\mathrm{Y-periodic}\}$, equipped with the norm $\|\cdot\|_{\mathrm{L}^p(Y)}$, $\mathrm{H}^2_\#(Y) := \{f \in \mathrm{H}^2_{\mathrm{loc}}(\mathbf{R}^d)$ such that f is $\mathrm{Y-periodic}\}$ with the norm $\|\cdot\|_{\mathrm{H}^2(Y)}$, and the quotient space $\mathrm{H}^2_\#(Y)/\mathbf{R}$, equipped with the norm $\|\nabla\nabla \cdot\|_{\mathrm{L}^2(Y)}$. For simplicity of notation, the class $[u]$ in the quotient space will usually be identified with a representative of the class $u \in [u]$. If we identify Y with the d-dimensional torus T (by *gluing together* the opposite faces of Y), which is a smooth compact manifold without boundary, these spaces are isomorphic to $\mathrm{L}^2(T)$, $\mathrm{H}^2(T)$ and $\mathrm{H}^2(T)/\mathbf{R}$, respectively. They can also be defined for vector, matrix or tensor valued functions.

Furthermore, let $\mathbf{E}_{ij}$, for $1 \le i, j \le d$, be $d \times d$ matrices, with entries

$$[\mathbf{E}_{ij}]_{kl} = \begin{cases} 1, & \text{if } i = j = k = l \\ \frac{1}{2}, & \text{if } i \neq j, (k,l) \in \{(i,j),(j,i)\} \\ 0, & \text{otherwise.} \end{cases}$$

We are interested in what happens in the limit of the periodic case, i.e., we want to derive the explicit formula for the homogenization limit of a periodic sequence of tensors. To be precise, for a Y-periodic tensor function $\mathsf{M} \in \mathrm{L}^\infty_\#(Y; \mathcal{L}(\mathrm{Sym},\mathrm{Sym})) \cap \mathfrak{M}_2(\alpha, \beta; Y)$ and a bounded, open set $\Omega \subseteq \mathbf{R}^d$, we are interested in the H-limit of the sequence $\mathsf{M}^n \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ defined by

$$\mathsf{M}^n(\mathbf{x}) := \mathsf{M}(n\mathbf{x}),\ \mathbf{x} \in \Omega. \tag{1.23}$$

Let us first remark that, for $f \in (\mathrm{H}^2_\#(Y)/\mathbf{R})'$, the boundary value problem

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathsf{M}(\boldsymbol{y})\nabla\nabla w(\boldsymbol{y})) = f(\boldsymbol{y}) & \text{in } Y \\ \boldsymbol{y} \mapsto w(\boldsymbol{y}) & \text{is } Y\text{-periodic} \end{cases}, \tag{1.24}$$

has a unique solution in $\mathrm{H}^2_\#(Y)/\mathbf{R}$.

In order to prove that there exists a unique solution of (1.24), we shall check the assumptions of the Lax-Milgram lemma. Obviously,

$$B(w,\varphi) = \int_Y \mathsf{M}\nabla\nabla w : \nabla\nabla\varphi \, d\boldsymbol{y},\ \ w,\, \varphi \in \mathrm{H}^2_\#(Y)/\mathbf{R}, \tag{1.25}$$

is a bilinear form and it doesn't depend on the choice of representatives of the equivalence classes. Since $\mathsf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ and $\|\nabla\nabla\cdot\|_{\mathrm{L}^2(Y)}$ is norm on the space $\mathrm{H}^2_\#(Y)/\mathbf{R}$, one can easily conclude that bilinear form (1.25) is bounded and coercive:

$$B(w,\varphi) = \int_Y \mathsf{M}\nabla\nabla w : \nabla\nabla\varphi \, d\boldsymbol{y} \le \beta \|\nabla\nabla w\|_{\mathrm{L}^2(Y)} \|\nabla\nabla\varphi\|_{\mathrm{L}^2(Y)}$$

and

$$B(w,w) = \int_Y \mathsf{M}\nabla\nabla w : \nabla\nabla w \, d\boldsymbol{y} \ge \alpha \int_Y \|\nabla\nabla w\|^2_{\mathrm{L}^2(Y)} \, d\boldsymbol{y} = \alpha \|\nabla\nabla w\|^2_{\mathrm{L}^2(Y)}.$$

Now, by the Lax-Milgram lemma there exists a unique solution in $\mathrm{H}^2_\#(Y)/\mathbf{R}$ of boundary value problem (1.24).

Remark 3 Analogously as in [2, p. 58, Lemma 1.3.21], it can easily be shown that $f \in \mathrm{L}^2_{\#}(Y)$ belongs to $(\mathrm{H}^2_{\#}(Y)/\mathbf{R})'$ if and only if $\int_Y f(y)\,dy = 0$.

Theorem 18 Let (M^n) be a sequence of tensors defined by (1.23). Then (M^n) H-converges to a constant tensor $\widehat{\mathsf{M}} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ with entries

$$\hat{m}_{klij} = \int_Y \mathsf{M}(\boldsymbol{y})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\boldsymbol{y})) : (\mathbf{E}_{kl} + \nabla\nabla w_{kl}(\boldsymbol{y}))\,d\boldsymbol{y}, \tag{1.26}$$

where $(w_{ij})_{1\le i,j\le d}$ is the family of unique solutions in $\mathrm{H}^2_{\#}(Y)/\mathbf{R}$ of boundary value problems

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathsf{M}(\boldsymbol{y})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\boldsymbol{y}))) = 0 \quad \text{in } Y, \\ \boldsymbol{y} \mapsto w_{ij}(\boldsymbol{y}) \quad \text{is } Y\text{-periodic.} \end{cases} \qquad i, j = 1, \ldots, d \tag{1.27}$$

Proof. The solution of (1.27), with right-hand side $f \in (\mathrm{H}^2_{\#}(Y)/\mathbf{R})'$ instead of zero, is any function $w_{ij} \in \mathrm{H}^2_{\#}(Y)/\mathbf{R}$ satisfying

$$\int_Y M(\boldsymbol{y})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\boldsymbol{y})) : \nabla\nabla\varphi(\boldsymbol{y})\,d\boldsymbol{y} = \int_Y f(\boldsymbol{y})\varphi(\boldsymbol{y})\,d\boldsymbol{y}, \tag{1.28}$$

for arbitrary function $\varphi \in \mathrm{H}^2_{\#}(Y)/\mathbf{R}$.

By Theorem 1 there is a subsequence (M^{n_k}) of (M^n) and a tensor valued function $\widehat{\mathsf{M}}$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ such that (M^{n_k}) H-converges to $\widehat{\mathsf{M}}$. Let us define

$$w_n^{ij}(\mathbf{x}) := \frac{1}{2}x_i x_j + \frac{1}{n^2} w_{ij}(n\mathbf{x})\,,$$

where $w_{ij} \in \mathrm{H}^2_{\#}(Y)/\mathbf{R}$ are unique solutions of (1.27). Since $w_{ij}(n\cdot)$ converges weakly to the average of w_{ij} in $H^2(\Omega)$, we easily conclude the convergence $w_n^{ij} \rightharpoonup \frac{1}{2}x_i x_j$ in $H^2(\Omega)$. Since

$$\mathsf{div}\,\mathsf{div}\,(\mathsf{M}^{n_k}(\mathbf{x})\nabla\nabla w_{n_k}^{ij}(\mathbf{x})) = \mathsf{div}\,\mathsf{div}\,(\mathsf{M}^{n_k}(\mathbf{x})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(n_k\mathbf{x}))) = 0 \text{ in } \Omega\,,$$

Theorem 9 implies

$$\mathsf{M}^{n_k}\nabla\nabla w_{n_k}^{ij} \rightharpoonup \widehat{\mathsf{M}}\mathbf{E}_{ij} \quad \text{in } \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})\,.$$

However, due to periodicity we have

$$\mathsf{M}^{n_k}\nabla\nabla w_{n_k}^{ij} = \mathsf{M}(n_k\cdot)(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(n_k\cdot)) \rightharpoonup \int_Y \mathsf{M}(\boldsymbol{y})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\boldsymbol{y}))\,d\boldsymbol{y}\,,$$

which implies that for components of $\widehat{\mathbf{M}}$ we can conclude

$$\begin{aligned}
\hat{m}_{klij} &= \widehat{\mathbf{M}}\mathbf{E}_{ij} : \mathbf{E}_{kl} \\
&= \int_Y \mathbf{M}(\boldsymbol{y})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\boldsymbol{y})) : \mathbf{E}_{kl}\, d\boldsymbol{y} \\
&= \int_Y \mathbf{M}(\boldsymbol{y})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\boldsymbol{y})) : (\mathbf{E}_{kl} + \nabla\nabla w_{kl}(\boldsymbol{y}))\, d\boldsymbol{y},
\end{aligned}$$

where we used that w_{ij} is a weak solution of (1.27) and took w_{kl} as test function in (1.28). Since every H-converging subsequence of $(\mathbf{M}^n)$ has the same limit, it follows that the entire sequence $(\mathbf{M}^n)$ H-converges to $\widehat{\mathbf{M}}$. ■

1.6 Small-amplitude homogenization in the periodic case

Let us consider a sequence of small perturbations of a constant coercive tensor $\mathbf{A}_0 \in \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$:

$$\mathbf{A}_p^n(\boldsymbol{y}) = \mathbf{A}_0 + p\mathbf{B}^n(\boldsymbol{y}),$$

where $\mathbf{B}^n(\boldsymbol{y}) := \mathbf{B}(n\boldsymbol{y})$, $\boldsymbol{y} \in \Omega$, $\Omega \subseteq \mathbf{R}^d$ is a bounded, open set and $\mathbf{B} \in \mathrm{L}^\infty_\#(Y; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$ such that $\int_Y \mathbf{B}(\boldsymbol{y})\, d\boldsymbol{y} = \mathbf{0}$. Note that $p \mapsto \mathbf{A}_p^n$ is a C^k mapping from $P \subseteq \mathbf{R}$ an open set, such that $0 \in \mathrm{Cl}\, P$, to $\mathrm{L}^\infty_\#(Y; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, for every $k \in \mathbf{N}$, thus we have a smooth dependence with respect to a parameter p.

Theorem 17 implies that there is a subsequence $(\mathbf{A}_p^{n_k})$ such that for every $p \in P$, $\mathbf{A}_p^{n_k} \xrightarrow{H} \widehat{\mathbf{A}}_p$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$, and $p \mapsto \widehat{\mathbf{A}}_p$ is a C^k mapping from P to $\mathrm{L}^\infty_\#(Y; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$. By Theorem 18, every H-converging subsequence of $(\mathbf{A}_p^n)$ has the same limit, thus the entire sequence $(\mathbf{A}_p^n)$ H-converges to $\widehat{\mathbf{A}}_p$. Since $p \mapsto \widehat{\mathbf{A}}_p$ is a C^k mapping from P to $\mathrm{L}^\infty_\#(Y; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, by using Taylor's theorem it follows that

$$\widehat{\mathbf{A}}_p := \mathbf{A}_0 + p\mathbf{B}_0 + p^2\mathbf{C}_0 + o(p^2) \quad \text{in} \quad \Omega. \tag{1.29}$$

The goal of small-amplitude homogenization is to obtain the explicit formula for the leading terms $\mathbf{B}_0$ and $\mathbf{C}_0$ in the expansion of the homogenization limit. Theorem 18 implies that

$$\widehat{\mathbf{A}}_p\mathbf{E}_{mn} : \mathbf{E}_{rs} = \int_Y (\mathbf{A}_0 + p\mathbf{B}(\boldsymbol{y}))(\mathbf{E}_{mn} + \nabla\nabla w^p_{mn}(\boldsymbol{y})) : (\mathbf{E}_{rs} + \nabla\nabla w^p_{rs}(\boldsymbol{y}))\, d\boldsymbol{y}\,, \tag{1.30}$$

for $m, n, r, s \in \{1, 2, \cdots, d\}$, where $w^p_{mn} \in \mathrm{H}^2_\#(Y)/\mathbf{R}$ are solutions of (1.27) with $\mathbf{A}_0 + p\mathbf{B}$

instead of $\mathbf{M}$, i.e. of

$$\begin{cases} \operatorname{div}\operatorname{div}\left((\mathbf{A}_0 + p\mathbf{B}(\boldsymbol{y}))(\mathbf{E}_{mn} + \nabla\nabla w^p_{mn}(\boldsymbol{y}))\right) = 0 \quad \text{in } Y, \\ \boldsymbol{y} \mapsto w^p_{mn}(\boldsymbol{y}) \quad \text{is } Y\text{-periodic.} \end{cases} \qquad m,n = 1,\dots,d \tag{1.31}$$

By using the integration by parts in (1.30), one easily gets

$$\widehat{\mathbf{A}}_p\mathbf{E}_{mn} : \mathbf{E}_{rs} = \mathbf{A}_0\mathbf{E}_{mn} : \mathbf{E}_{rs} + p\int_Y \mathbf{B}\nabla\nabla w^p_{mn} : \mathbf{E}_{rs}\, d\boldsymbol{y} + \int_Y \mathbf{A}_0\nabla\nabla w^p_{mn} : \nabla\nabla w^p_{rs}\, d\boldsymbol{y} + \\ + p\int_Y \mathbf{B}\mathbf{E}_{mn} : \nabla\nabla w^p_{rs}\, d\boldsymbol{y} + p\int_Y \mathbf{B}\nabla\nabla w^p_{mn} : \nabla\nabla w^p_{rs}\, d\boldsymbol{y}. \tag{1.32}$$

Let us define

$$T(p)v := \operatorname{div}\operatorname{div}\left((\mathbf{A}_0 + p\mathbf{B})\nabla\nabla v\right),\ v \in \mathrm{H}^2_\#(Y)/\mathbf{R}.$$

Note that $T(p)$ may be written as a composition $\mathcal{P} \circ (\mathbf{A}_0 + p\mathbf{B})$, where $\mathcal{P} : \mathrm{L}^\infty_\#(Y;\mathcal{L}(\mathrm{Sym},\mathrm{Sym})) \to \mathcal{L}(\mathrm{H}^2_\#(Y)/\mathbf{R}, (\mathrm{H}^2_\#(Y)/\mathbf{R})')$ is defined with $\mathcal{P}(\mathbf{N}) := \operatorname{div}\operatorname{div}(\mathbf{N}\nabla\nabla\cdot)$, $\mathbf{N} \in \mathrm{L}^\infty_\#(Y;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$. Furthermore, going along the same lines as in the proof of Theorem 17, it follows that the mapping $p \mapsto (T(p))^{-1}$ is also a C^k mapping from $P \subseteq \mathbf{R}$ to $\mathcal{L}((\mathrm{H}^2_\#(Y)/\mathbf{R})', \mathrm{H}^2_\#(Y)/\mathbf{R})$. By using this and the definition of w^p_{mn}, we conclude that $p \mapsto w^p_{mn}$ is a C^k mapping from $P \subseteq \mathbf{R}$ to $\mathrm{H}^2_\#(Y)/\mathbf{R}$, for any $k \in \mathbf{N}$. Hence, one can write w^p_{mn} as

$$w^p_{mn} = w^{mn}_0 + p w^{mn}_1 + o(p).$$

Due to the given boundary value problem (1.31), after comparing expressions corresponding to the same powers of p, it is easy to conclude that $w^{mn}_0 = 0$: first we insert w^p_{mn} in the corresponding boundary value problem

$$\begin{cases} \operatorname{div}\operatorname{div}\left((\mathbf{A}_0 + p\mathbf{B}(\boldsymbol{y}))(\mathbf{E}_{mn} + \nabla\nabla w^p_{mn}(\boldsymbol{y}))\right) = 0 \quad \text{in } Y, \\ \boldsymbol{y} \mapsto w^p_{mn}(\boldsymbol{y}) \quad \text{is } Y\text{-periodic.} \end{cases} \qquad m,n = 1,\dots,d$$

By comparing expressions corresponding to the same powers of p, we obtain

$$\begin{cases} \operatorname{div}\operatorname{div}\left(\mathbf{A}_0(\mathbf{E}_{mn} + \nabla\nabla w^{mn}_0)\right) = 0 \quad \text{in } Y, \\ \boldsymbol{y} \mapsto w^{mn}_0 \quad \text{is } Y\text{-periodic.} \end{cases} \qquad m,n = 1,\dots,d$$

Uniqueness of the solution of this boundary value problem implies $w^{mn}_0 = 0$.

If we insert the expression for w^p_{mn} in formula (1.32), we have

$$\widehat{\mathbf{A}}_p\mathbf{E}_{mn} : \mathbf{E}_{rs} = \mathbf{A}_0\mathbf{E}_{mn} : \mathbf{E}_{rs} + p^2\int_Y \mathbf{B}\nabla\nabla w^{mn}_1 : \mathbf{E}_{rs}\, d\boldsymbol{y} + \\ + p^2\int_Y \mathbf{A}_0\nabla\nabla w^{mn}_1 : \nabla\nabla w^{rs}_1\, d\boldsymbol{y} + \tag{1.33}$$

$$+ p^2 \int_Y \mathbf{B}\mathbf{E}_{mn} : \nabla\nabla w_1^{rs}\, d\boldsymbol{y} + o(p^2).$$

By comparing (1.29) and (1.33) we easily conclude that $\mathsf{B}_0 = \mathbf{0}$ and

$$\mathsf{C}_0\mathbf{E}_{mn} : \mathbf{E}_{rs} = \int_Y \mathsf{B}\nabla\nabla w_1^{mn} : \mathbf{E}_{rs}\, d\boldsymbol{y} + \int_Y \mathsf{A}_0\nabla\nabla w_1^{mn} : \nabla\nabla w_1^{rs}\, d\boldsymbol{y} + \int_Y \mathsf{B}\mathbf{E}_{mn} : \nabla\nabla w_1^{rs}\, d\boldsymbol{y}. \tag{1.34}$$

In order to fully describe C_0 it remains to calculate w_1^{mn}, which we shall do in terms of its Fourier coefficients: if we insert $pw_1^{mn} + o(p)$ instead of w_{mn}^p in the corresponding boundary value problem (1.31), by equating powers of p we can easily see that w_1^{mn} solves

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathsf{A}_0\nabla\nabla w_1^{mn}) = -\operatorname{div}\operatorname{div}(\mathsf{B}\mathbf{E}_{mn}) & \text{in } Y \\ w_1^{mn} \in \mathrm{H}^2_\#(Y)/\mathbf{R} \end{cases} \qquad m,n = 1,\dots,d. \tag{1.35}$$

If we choose a representative of this solution satisfying $\int_Y w_1^{mn}(\boldsymbol{y})\, d\boldsymbol{y} = 0$, its Fourier series expansion takes the form

$$w_1^{mn}(\boldsymbol{y}) = \sum_{\mathbf{k}\in J} a_{\mathbf{k}}^{mn} e^{2\pi i\mathbf{k}\cdot\boldsymbol{y}},$$

where $a_{\mathbf{k}}^{mn}$, $\mathbf{k} \in J := \mathbf{Z}^d\backslash\{\mathbf{0}\}$ are its Fourier coefficients. A simple calculation shows that

$$\nabla\nabla w_1^{mn} = \sum_{\mathbf{k}\in J} (2\pi i)^2 a_{\mathbf{k}}^{mn} e^{2\pi i\mathbf{k}\cdot\boldsymbol{y}} \mathbf{k}\otimes\mathbf{k}$$

and

$$\operatorname{div}\operatorname{div}(\mathsf{A}_0\nabla\nabla w_1^{mn}) = \sum_{\mathbf{k}\in J} (2\pi i)^4 a_{\mathbf{k}}^{mn} e^{2\pi i\mathbf{k}\cdot\boldsymbol{y}} \mathsf{A}_0(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k}). \tag{1.36}$$

If $\mathsf{B}(\boldsymbol{y}) = \sum_{\mathbf{k}\in J} \mathsf{B}_{\mathbf{k}} e^{2\pi i\mathbf{k}\cdot\boldsymbol{y}}$ is the Fourier series expansion of function B, similarly as above we can calculate the right-hand side of (1.35):

$$\operatorname{div}\operatorname{div}(\mathsf{B}\mathbf{E}_{mn}) = \sum_{\mathbf{k}\in J} (2\pi i)^2 e^{2\pi i\mathbf{k}\cdot\boldsymbol{y}} (\mathsf{B}_{\mathbf{k}}\mathbf{E}_{mn}\mathbf{k})\cdot\mathbf{k}. \tag{1.37}$$

From (1.35), (1.36) and (1.37) it follows that

$$\sum_{\mathbf{k}\in J} (2\pi i)^4 e^{2\pi i\mathbf{k}\cdot\boldsymbol{y}} a_{\mathbf{k}}^{mn} \mathsf{A}_0(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k}) = -\sum_{\mathbf{k}\in J} (2\pi i)^2 e^{2\pi i\mathbf{k}\cdot\boldsymbol{y}} (\mathsf{B}_{\mathbf{k}}\mathbf{E}_{mn}\mathbf{k})\cdot\mathbf{k}, \tag{1.38}$$

and consequently for Fourier coefficients we conclude

$$a_{\mathbf{k}}^{mn} = -\frac{\mathsf{B}_{\mathbf{k}}\mathbf{E}_{mn}\mathbf{k}\cdot\mathbf{k}}{(2\pi i)^2\mathsf{A}_0(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})}, \quad \mathbf{k} \in J\,. \tag{1.39}$$

If we insert the Fourier expansions of $\mathbf{B}$ and w_1^{mn} in (1.34) we get

$$\begin{aligned}\mathbf{C}_0\mathbf{E}_{mn} : \mathbf{E}_{rs} = &\int_Y \sum_{\mathbf{k}\in J} \mathbf{B}_{\mathbf{k}} e^{2\pi i \mathbf{k}\cdot\boldsymbol{y}} \sum_{\mathbf{k}'\in J} (2\pi i)^2 a_{\mathbf{k}'}^{mn} e^{2\pi i\mathbf{k}'\cdot\boldsymbol{y}} \mathbf{k}'\otimes\mathbf{k}' : \mathbf{E}_{rs}\, d\boldsymbol{y} + \\ &+ \int_Y \mathbf{A}_0 \sum_{\mathbf{k}\in J} (2\pi i)^2 a_{\mathbf{k}}^{mn} e^{2\pi i\mathbf{k}\cdot\boldsymbol{y}} \mathbf{k}\otimes\mathbf{k} : \sum_{\mathbf{k}'\in J} (2\pi i)^2 a_{\mathbf{k}'}^{rs} e^{2\pi i\mathbf{k}'\cdot\boldsymbol{y}} \mathbf{k}'\otimes\mathbf{k}'\, d\boldsymbol{y} + \\ &+ \int_Y \sum_{\mathbf{k}\in J} e^{2\pi i\mathbf{k}\cdot\boldsymbol{y}} \mathbf{B}_k \mathbf{E}_{mn} : \sum_{\mathbf{k}'\in J} (2\pi i)^2 a_{\mathbf{k}'}^{rs} e^{2\pi i\mathbf{k}'\cdot\boldsymbol{y}} \mathbf{k}'\otimes\mathbf{k}'\, d\boldsymbol{y},\end{aligned}$$

and by the orthogonality of Fourier basis, we finally conclude

$$\begin{aligned}\mathbf{C}_0\mathbf{E}_{mn} : \mathbf{E}_{rs} = &(2\pi i)^2 \sum_{\mathbf{k}\in J} a_{-\mathbf{k}}^{mn} \mathbf{B}_{\mathbf{k}} (\mathbf{k}\otimes\mathbf{k}) : \mathbf{E}_{rs} + \\ &+ (2\pi i)^4 \sum_{\mathbf{k}\in J} a_{\mathbf{k}}^{mn} \mathbf{A}_0 (\mathbf{k}\otimes\mathbf{k}) : a_{-\mathbf{k}}^{rs} \mathbf{k}\otimes\mathbf{k} + \\ &+ (2\pi i)^2 \sum_{\mathbf{k}\in J} \mathbf{B}_{\mathbf{k}} \mathbf{E}_{mn} : a_{-\mathbf{k}}^{rs} \mathbf{k}\otimes\mathbf{k}\,.\end{aligned} \tag{1.40}$$

The following theorem summarizes the previous results.

Theorem 19 Let $\mathbf{A}_0 \in \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ be a constant coercive tensor, $\Omega \subseteq \mathbf{R}^d$ a bounded, open set, $\mathbf{B} \in \mathrm{L}^\infty_\#(Y; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, such that $\int_Y \mathbf{B}(\boldsymbol{y})\, d\boldsymbol{y} = \mathbf{0}$ and $\mathbf{B}^n(\boldsymbol{y}) := \mathbf{B}(n\boldsymbol{y})$, $\boldsymbol{y} \in \Omega$. Additionally, let $p \in P$, where $P \subseteq \mathbf{R}$ is an open set such that $0 \in \mathrm{Cl}\, P$, and

$$\mathbf{A}_p^n(\boldsymbol{y}) = \mathbf{A}_0 + p\mathbf{B}^n(\boldsymbol{y}).$$

Then, $\mathbf{A}_p^n$ H-converges to

$$\widehat{\mathbf{A}}_p := \mathbf{A}_0 + p^2\mathbf{C}_0 + o(p^2) \quad \text{in} \quad \Omega$$

with coefficient $\mathbf{C}_0$ being (a constant tensor) given by (1.40), for $m, n, r, s \in \{1, 2, \cdots, d\}$, where

$$a_{\mathbf{k}}^{mn} = -\frac{\mathbf{B}_{\mathbf{k}}\mathbf{E}_{mn}\mathbf{k}\cdot\mathbf{k}}{(2\pi i)^2 \mathbf{A}_0(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})}, \quad \mathbf{k} \in J, \quad m, n \in \{1, 2, \cdots, d\}$$

and $\mathbf{B}_{\mathbf{k}}$, $\mathbf{k} \in J$, are the Fourier coefficients of $\mathbf{B}$.

Chapter 2

On the effective properties of composite elastic plate

In this chapter we are interested in the application of the homogenization theory to the modeling of composite elastic plate. After an introductory section, the rest of the chapter is organized as follows: in the second section we show the local character of the set of all possible composites, and prove that the set of composites obtained by periodic homogenization is dense in that set. In the third section we describe the sequential laminates, a particularly interesting class of composite materials. In the fourth section we derive Hashin-Shtrikman bounds on primal energy, which are optimal bounds on the effective energy of a composite material, and, in the next section, we consider an analogous results for the complementary energy of a composite material. After that, we give a characterisation of the G-closure for the Kirchhoff-Love plate in the low contrast or small-amplitude regime. Finally, in the last two sections, we calculate explicit Hashin-Shtrikman bounds on primal and complementary energy for mixtures of two isotropic materials in dimension $d = 2$.

2.1 Introduction

In this chapter we apply previous results to the modeling of a composite elastic plate. Composite materials are heterogeneous materials obtained by mixing several materials on a very fine scale. It is impossible to imagine everyday life without composite materials. They are generally used for buildings, bridges, structures such as boat hulls, storage tanks, swimming pool panels, etc. Some widely used composite materials are wood, which is a natural composite of cellulose fibers in a lignin matrix, and concrete, which is a composite of aggregate (rock, sand or gravel), cement and water.

The main problem with composite materials is to determine their effective (macroscopic) properties. However, homogenization theory allows one to define a composite material as an H-limit. There is an extensive literature on this topic, we refer the interested reader to [2, 6, 22, 46, 48, 54]. We shall mainly put the focus on composites obtained by mixing only two different materials, i.e. two-phase composite materials. From the physical point of view, these materials are determined with its two phases $\mathbf{A}$ and $\mathbf{B}$, their proportions θ and $1-\theta$, respectively, and by their microstructure, i.e. by their geometric arrangement in the mixture.

Although it will be defined later for m materials and in a more general setting, let us now define a two-phase composite material [2].

Definition 6 Let $\chi^n \in \mathrm{L}^\infty(\Omega;\{0,1\})$ be a sequence of characteristic functions and $(\mathbf{M}^n)$ be a sequence of tensors defined by

$$\mathbf{M}^n(\mathbf{x}) := \chi^n(\mathbf{x})\mathbf{A} + (1-\chi^n(\mathbf{x}))\mathbf{B},$$

where $\mathbf{A}$ and $\mathbf{B}$ are assumed to be positive definite fourth-order tensors. Assume that there exist $\theta \in \mathrm{L}^\infty(\Omega;[0,1])$ and $\mathbf{M} \in \mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$ such that

$$\begin{aligned} \chi^n &\xrightarrow{*} \theta \text{ in } \mathrm{L}^\infty(\Omega;[0,1]), \\ \mathbf{M}^n &\xrightarrow{H} \mathbf{M} \text{ in } \mathfrak{M}_2(\alpha,\beta;\Omega), \end{aligned}$$

where $\beta > \alpha > 0$ are given. The H-limit $\mathbf{M}$ is said to be the homogenized tensor of a two-phase composite material obtained by mixing $\mathbf{A}$ and $\mathbf{B}$ in proportions θ and $(1-\theta)$, respectively, with a microstructure defined by the sequence (χ^n).

Before we proceed further with composite materials, let us introduce some definitions and results which are necessary for better understanding this chapter.
Let $F : X \to \mathbf{R}$, where X is a Banach space.

Definition 7 Let F be Lipschitz near a given point $\mathbf{x} \in X$ and let $\mathbf{v} \in X$. The upper generalized directional derivative of F at $\mathbf{x}$ in the direction $\mathbf{v}$, denoted $F^0(\mathbf{x};\mathbf{v})$, is defined as follows:

$$F^0(\mathbf{x};\mathbf{v}) := \limsup_{\substack{\boldsymbol{y}\to\mathbf{x}\\ t\to 0^+}} \frac{F(\boldsymbol{y}+t\mathbf{v}) - F(\boldsymbol{y})}{t},$$

where $\boldsymbol{y} \in X$ and $t \in \mathbf{R}^+$.

Definition 8 The generalized gradient of F at $\mathbf{x}$, denoted $\partial F(\mathbf{x})$, is the subset of X' given by

$$\{\xi \in X' : F^0(\mathbf{x};\mathbf{v}) \geq {}_{X'}\langle \xi, \mathbf{v}\rangle_X, \ \ \mathbf{v} \in X\}.$$

Remark 4 [25, p. 36, Proposition 2.2.7] If F is convex on U and Lipshitz near $\mathbf{x}$, then $\partial F(\mathbf{x})$ coincides with the subdifferential at $\mathbf{x}$ in the sense of convex analysis, and $F^0(\mathbf{x};\mathbf{v})$ coincides with the directional derivative, for each $\mathbf{v}$.

Proposition 2 [25, p. 38, Proposition 2.3.2] If F attains a local minimum or maximum at $\mathbf{x}$, then $0 \in \partial F(\mathbf{x})$.

The following theory can be found in [25]. One may find it helpful in the fourth section, when showing the optimality of Hashin-Shtrikman bounds.

Let $f_{\mathbf{t}}$ be a family of functions on a Banach space X, with codomain $\mathbf{R}$, parametrized by $\mathbf{t} \in T$, where T is a topological space. Suppose that for some point $\mathbf{x} \in X$, each function $f_{\mathbf{t}}$ is Lipshitz near $\mathbf{x}$. First, we make the following hypotheses:

(i) T is a sequentially compact space.

(ii) For some neighborhood U of $\mathbf{x}$, the map $\mathbf{t} \mapsto f_{\mathbf{t}}(\boldsymbol{y})$ is upper semicontinuous for each $\boldsymbol{y} \in U$.

(iii) Each $f_{\mathbf{t}}$, $\mathbf{t} \in T$, is Lipschitz on U, and $\{f_{\mathbf{t}}(\mathbf{x}) : \mathbf{t} \in T\}$ is bounded.

Let us define a function $f : X \to \mathbf{R}$ via

$$f(\boldsymbol{y}) := \max\{f_{\mathbf{t}}(\boldsymbol{y}) : \mathbf{t} \in T\}.$$

By $M(\boldsymbol{y})$ we denote the set $\{\mathbf{t} \in T : f_{\mathbf{t}}(\boldsymbol{y}) = f(\boldsymbol{y})\}$, and for any subset S of T, $P[S]$ signifies the collection of probability Radon measures supported on S.

Theorem 20 [25, p. 86-87] In addition to the hypotheses given above, suppose that X is a vector space of finite dimension. Additionally, assume that each $f_{\mathbf{t}}$ is Frechet differentiable on U, and that $f'_{\mathbf{t}}(\mathbf{x})$ is continuous as a function of $(\mathbf{t}, \mathbf{x})$. Then, for each $\mathbf{x} \in U$ one has

$$\partial f(\mathbf{x}) = \left\{ \int_T f'_{\mathbf{t}}(\mathbf{x})\mu(d\mathbf{t}) : \mu \in P[M(\mathbf{x})] \right\}.$$

Theorem 21 [1, p. 513, Theorem 15.11] If X is a compact metric space, the set $P(X)$ of probability measures on X is compact in the weak-$*$ topology.

The notion of conjugate function appears to be a very useful tool in this chapter.

Definition 9 Let E be a normed vector space and $\varphi : E \to (-\infty, +\infty]$ a function such that $\varphi \not\equiv +\infty$. We define the conjugate function $\varphi^* : E' \to (-\infty, +\infty]$:

$$\varphi^*(f) := \sup_{\mathbf{x} \in E} [{}_{E'}\langle f, \mathbf{x} \rangle_E - \varphi(\mathbf{x})], \quad f \in E'.$$

The function φ^* is called the Legendre transform of φ.

Remark 5 [17, p. 11-13] Note that φ^* is convex and lower semicontinuous on E'. If we iterate the operation *, we could obtain a function φ^{**} defined on E''. Instead of that, we restrict φ^{**} to E, i.e. we define

$$\varphi^{**}(\mathbf{x}) := \sup_{f \in E'} [{}_{E'}\langle f, \mathbf{x} \rangle_E - \varphi^*(f)], \quad \mathbf{x} \in E.$$

Theorem 22 [17, p. 13, Theorem 1.11](Fenchel-Moreau) Assume that $\varphi : E \to (-\infty, +\infty]$ is convex, lower semicontinuous and $\varphi \not\equiv +\infty$. Then $\varphi^{**} = \varphi$.

Remark 6 If $\varphi_1, \varphi_2 : E \to (-\infty, +\infty]$ are convex, lower semicontinuous and $\varphi_1, \varphi_2 \not\equiv +\infty$, by using the Fenchel-Moreau theorem one can show that $\varphi_1^* = \varphi_2^* \Rightarrow \varphi_1 = \varphi_2$.

Furthermore, we are going to state some well-known and simple, but also crucial facts which shall be used for calculating explicit Hashin-Shtrikman bounds for mixtures of two isotropic materials in dimension $d = 2$.

Theorem 23 [56, Theorem 1] If $\mathbf{A}$ and $\mathbf{B}$ are hermitian $d \times d$ matrices with eigenvalues

$$\kappa_1 \geq \cdots \geq \kappa_d, \;\; \lambda_1 \geq \cdots \geq \lambda_d$$

respectively, then

$$\sum_{i=1}^{d} \kappa_i \lambda_{d-i+1} \leq \mathrm{tr}(\mathbf{AB}) \leq \sum_{i=1}^{d} \kappa_i \lambda_i.$$

Definition 10 A simultaneously diagonalizable family $\mathcal{F} \subseteq M_d(\mathbf{R})$ is a family for which there is a single nonsingular matrix $\mathbf{S} \in M_d(\mathbf{R})$ such that $\mathbf{S}^{-1}\mathbf{AS}$ is diagonal for every $\mathbf{A} \in \mathcal{F}$.

Definition 11 Let K be a closed, convex set. A point $\mathbf{e}$ of K is called an extreme point of K if it is not the interior point of a line segment in K. Equivalently, $\mathbf{x}$ is not an extreme point of K if there exist $\boldsymbol{y}$, $\mathbf{z} \in K$, $\boldsymbol{y} \neq \mathbf{z}$, such that

$$\mathbf{x} = \frac{\boldsymbol{y} + \mathbf{z}}{2}.$$

The set of extreme points of K is denoted by Ext$K$.

Theorem 24 [45, p. 195, Theorem 10](Carathéodory's theorem) Let K be a nonempty closed bounded convex set in a vector space X, dim$X = n$. Then every point of K can be represented as a convex combination of at most $(n + 1)$ extreme points of K.

Theorem 25 [26, p. 147, Theorem 8.4], [36, p. 87, Proposition 5.22] Let X be a compact Hausdorff topological space. Then $P(X)$ is a convex set and

$$\mathrm{Ext}(P(X)) = \{\delta_{\mathbf{x}} : \mathbf{x} \in X\}.$$

Lemma 7 Let

$$C = \left\{ \int_{S^{d-1}} \frac{(\mathbf{e} \otimes \mathbf{e}) \otimes (\mathbf{e} \otimes \mathbf{e})}{\mathsf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})} \, d\nu(\mathbf{e}) : \nu \in P[M(\boldsymbol{\eta})] \right\},$$

where $M(\boldsymbol{\eta})$ is a closed subset of S^{d-1}. Extreme points of C are exactly the points

$$\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})},$$

where $\mathbf{e}\in M(\boldsymbol{\eta})$.

Proof. The idea is to prove that extreme points of the set C correspond to a measure ν being a Dirac mass. Suppose that

$$\mathbf{M} = \int_{S^{d-1}} \frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\, d\nu(\mathbf{e})$$

is an extreme point of the set C. Let us take a linear functional L on Sym^4, and suppose that

$$L\left(\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\right)$$

is not a constant function on the support of ν. Following ideas from [2], we define the essential infimum L^- and essential supremum L^+ with

$$L^- = \sup\left[\alpha\in\mathbf{R} : \nu\left\{\mathbf{e}\in S^{d-1} : L\left(\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\right)\le\alpha\right\} = 0\right]$$

and

$$L^+ = \inf\left[\beta\in\mathbf{R} : \nu\left\{\mathbf{e}\in S^{d-1} : L\left(\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\right)\ge\beta\right\} = 0\right].$$

Our assumption that $L\left(\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\right)$ is not a constant function on the support of ν implies that there exists some value of the functional L, denoted by $\bar{L}$, such that $L^- < \bar{L} < L^+$. With this we can define sets

$$E^- = \left\{\mathbf{e}\in S^{d-1} : L\left(\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\right)\le\bar{L}\right\}$$

and

$$E^+ = \left\{\mathbf{e}\in S^{d-1} : L\left(\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\right)>\bar{L}\right\},$$

which are nonempty. Furthermore, $\rho^- = \nu(E^-) > 0$, $\rho^+ = \nu(E^+) > 0$ and $\rho^- + \rho^+ = 1$. Now, we can decompose the measure ν as a convex combination of probability measures:

$$\nu = \rho^-\frac{\nu|_{E^-}}{\rho^-} + \rho^+\frac{\nu|_{E^+}}{\rho^+}.$$

Using this, the tensor $\mathbf{M}$ can also be decomposed as a convex combination of two points in C:

$$\mathbf{M} = \rho^- \mathbf{M}^- + \rho^+ \mathbf{M}^+,$$

where

$$\mathbf{M}^- = \frac{1}{\rho^-} \int_{S^{d-1}} \frac{(\mathbf{e} \otimes \mathbf{e}) \otimes (\mathbf{e} \otimes \mathbf{e})}{\mathbf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})} \, d\nu|_{E^-}$$

and

$$\mathbf{M}^+ = \frac{1}{\rho^+} \int_{S^{d-1}} \frac{(\mathbf{e} \otimes \mathbf{e}) \otimes (\mathbf{e} \otimes \mathbf{e})}{\mathbf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})} \, d\nu|_{E^+} .$$

Since $\mathbf{M}^- \neq \mathbf{M}^+$ this is in contradiction with the fact that $\mathbf{M}$ is an extreme point of C. Thus, the assumption was wrong, i.e. $L\left(\frac{(\mathbf{e} \otimes \mathbf{e}) \otimes (\mathbf{e} \otimes \mathbf{e})}{\mathbf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})}\right)$ is constant on the support of ν and it can be replaced by a Dirac mass. It follows that every extreme point of C is given as

$$\frac{(\mathbf{e} \otimes \mathbf{e}) \otimes (\mathbf{e} \otimes \mathbf{e})}{\mathbf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})},$$

where $\mathbf{e} \in M(\boldsymbol{\eta})$. ■

Definition 12 We say that the sequence (u_n) in $\mathrm{L}^2(\Omega)$ does not oscillate in x_1, if

(i) $u_n \rightharpoonup u$ in $\mathrm{L}^2(\Omega)$,

(ii) for every sequence of functions (f_n) in $\mathrm{L}^\infty(\Omega)$ depending only upon x_1, such that $f_n \overset{*}{\rightharpoonup} f$ in $\mathrm{L}^\infty(\Omega)$, it follows that $f_n u_n \rightharpoonup fu$ in $\mathrm{L}^2(\Omega)$.

Lemma 8 [10, Lemma 6] Let $(\mathbf{D}^n)$ be a sequence in $\mathrm{L}^2(\Omega; \mathrm{Sym})$ that converges weakly to $\mathbf{D}$. If the sequence $(\mathsf{div}\,\mathsf{div}\,\mathbf{D}^n)$ is contained in a precompact set of the space $\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$, then D^n_{11} does not oscillate in x_1.

2.2 The G-closure problem

Suppose that we want to fill a bounded and open set $\Omega \subseteq \mathbf{R}^d$ with m different phases $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_m \in \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ which are assumed to be positive definite fourth-order tensors. Let $\mathcal{R} : SO(\mathbf{R}^d) \times \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}) \to \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ be the operator of rotations acting on the space $\mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$, as is defined in the Appendix, $\mathbf{R}_n \in \mathrm{L}^\infty(\Omega; SO(\mathbf{R}^d))$, and $\boldsymbol{\chi}^n = (\chi^n_1, \ldots, \chi^n_m) \in \mathrm{L}^\infty(\Omega; T)$ a sequence of characteristic functions, where

$$T := \left\{ \boldsymbol{\theta} = (\theta_1, \ldots, \theta_m) \in \{0,1\}^m : \sum_{i=1}^m \theta_i = 1 \right\}.$$

By $(\mathbf{M}^n)$ we denote a sequence defined by

$$\mathbf{M}^n(\mathbf{x}) := \sum_{i=1}^m \chi^n_i(\mathbf{x}) \mathcal{R}\left(\mathbf{R}_n(\mathbf{x}), \mathbf{M}_i\right). \tag{2.1}$$

If

$$\begin{aligned}\boldsymbol{\chi}^n &\xrightarrow{*} \boldsymbol{\vartheta} \text{ in } \mathrm{L}^\infty(\Omega;\mathbf{R}^m),\\ \mathbf{M}^n &\xrightarrow{H} \mathbf{M} \text{ in } \mathfrak{M}_2(\alpha,\beta;\Omega),\end{aligned}$$

we say that $\mathbf{M}$ is a homogenized tensor of a m-phase composite material obtained by mixing $\mathbf{M}_1,\mathbf{M}_2,\dots,\mathbf{M}_m$ with proportions ϑ_i, and microstructure defined by the sequence $(\boldsymbol{\chi}^n)$ and the sequence of rotations $(\mathbf{R}_n)$. Due to the characterisation of the weak-$*$ closure [73, p. 6, Theorem 6], note that function $\boldsymbol{\vartheta}$ belongs to the set $\mathrm{L}^\infty(\Omega;\overline{T})$, where

$$\overline{T} := \mathrm{Cl}\,\mathrm{conv}T = \left\{\boldsymbol{\theta} \in [0,1]^m : \sum_{i=1}^m \theta_i = 1\right\}.$$

For given materials $\mathbf{M}_1,\mathbf{M}_2,\dots,\mathbf{M}_m$ and fixed proportions $\vartheta_i \in \mathrm{L}^\infty(\Omega;[0,1])$ of each material $\mathbf{M}_i$, respectively, it is of interest to find all possible homogenized tensors $\mathbf{M}$, which can be obtained in this way. This problem is well known under the name *G-closure problem* and it appears to be quite difficult to solve. It was done in the conductivity setting for mixtures of two isotropic conductors [48, 72], while in the elasticity setting for two isotropic phases it is still an open problem (for partial results we refer to [55] and references therein), even for the elastic plates [35, 51]. One can only obtain bounds that must be satisfied by the effective properties, called Hashin-Shtrikman bounds in their most general form [40]. However, results can be pushed much further under a simplifying assumption of a small-amplitude or low contrast regime [2, 3, 4, 33, 38, 39, 69].

Due to its local character (see Theorem 26 below), the G-closure problem reduces to describing the set $G_{\boldsymbol{\theta}}$ of all possible composite materials obtained by mixing $\mathbf{M}_1,\mathbf{M}_2,\dots,\mathbf{M}_m$ in constant proportion $\boldsymbol{\theta} \in \overline{T}$:

$$\begin{aligned}G_{\boldsymbol{\theta}} := \Big\{&\mathbf{M} \in \mathcal{L}(\mathrm{Sym},\mathrm{Sym}) : \Big(\exists\,(\boldsymbol{\chi}^n)\text{ in } \mathrm{L}^\infty(\Omega;T)\Big)\Big(\exists\,(\mathbf{R}_n)\text{ in } \mathrm{L}^\infty(\Omega;SO(\mathbf{R}^d))\Big)\\ &\boldsymbol{\chi}^n \xrightarrow{*} \boldsymbol{\theta} \text{ in } \mathrm{L}^\infty(\Omega;\mathbf{R}^m)\;\&\;\mathbf{M}^n := \sum_{i=1}^m \chi_i^n\,\mathcal{R}\left(\mathbf{R}_n,\mathbf{M}_i\right) \xrightarrow{H} \mathbf{M} \text{ in } \mathfrak{M}_2(\alpha,\beta;\Omega)\Big\}.\end{aligned} \tag{2.2}$$

An interesting example of composite materials are periodic mixtures, which are obtained in the following way: let $Y = [0,1]^d$, $\boldsymbol{\chi} \in \mathrm{L}^\infty(Y;T)$, $\mathbf{R} \in \mathrm{L}^\infty(Y;SO(\mathbf{R}^d))$, $\mathbf{M}(\mathbf{x}) := \sum_{i=1}^m \chi_i(\mathbf{x})\mathcal{R}\left(\mathbf{R}(\mathbf{x}),\mathbf{M}_i\right)$ and let us extend this functions periodically to $\mathbf{R}^d$. Then, for sequences defined by $\boldsymbol{\chi}^n(\mathbf{x}) := \boldsymbol{\chi}(n\mathbf{x})$, $\mathbf{R}_n(\mathbf{x}) := \mathbf{R}(n\mathbf{x})$ and $\mathbf{M}^n(\mathbf{x}) := \mathbf{M}(n\mathbf{x}) = \sum_{i=1}^m \chi_i^n(\mathbf{x})\mathcal{R}\left(\mathbf{R}_n(\mathbf{x}),\mathbf{M}_i\right)$, $\mathbf{x} \in \mathbf{R}^d$, by Lemma 18 and Theorem 18 the following convergences hold (for any open and bounded set $\Omega \subseteq \mathbf{R}^d$):

$$\boldsymbol{\chi}^n \xrightarrow{*} \boldsymbol{\theta} := \int_Y \boldsymbol{\chi}\,d\mathbf{x}, \text{ in } \mathrm{L}^\infty(\Omega;\mathbf{R}^m)$$

$$\mathbf{M}^n \xrightarrow{H} \widehat{\mathbf{M}}, \text{ in } \mathfrak{M}_2(\alpha, \beta; \Omega)$$

where the entries of $\widehat{\mathbf{M}} \in G_{\boldsymbol{\theta}}$ are given by (1.26).

For fixed $\boldsymbol{\theta} \in \overline{T}$, by $P_{\boldsymbol{\theta}} \subseteq G_{\boldsymbol{\theta}}$ we denote the set of all constant homogenized tensors obtained by periodic homogenization, as described above, i.e. by mixing $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_m$ in the sense of (2.2), for some Y-periodic functions $\boldsymbol{\chi} \in \mathrm{L}^\infty(\mathbf{R}^d; T)$ and $\mathbf{R} \in \mathrm{L}^\infty(\mathbf{R}^d; SO(\mathbf{R}^d))$, such that $\int_Y \boldsymbol{\chi}\, d\boldsymbol{y} = \boldsymbol{\theta}$.

The following theorem asserts the local character of the set of all possible composites, and together with the last theorem in this section, it implies that the set of composites obtained by periodic homogenization is dense in the set of all possible composites. Actually, the statements and the proofs of these two theorems mimic the ones in the case of stationary diffusion equation (see [74] and references therein).

Theorem 26 Let $(\boldsymbol{\chi}^n)$, $(\mathbf{R}_n)$ be a sequences in $\mathrm{L}^\infty(\Omega; T)$ and $\mathrm{L}^\infty(\Omega; SO(\mathbf{R}^d))$, respectively, and $(\mathbf{M}^n)$ defined with (2.1) such that

$$\begin{aligned} \boldsymbol{\chi}^n &\xrightarrow{*} \boldsymbol{\vartheta} \text{ in } \mathrm{L}^\infty(\Omega; \mathbf{R}^m), \\ \mathbf{M}^n &\xrightarrow{H} \mathbf{M} \text{ in } \mathfrak{M}_2(\alpha, \beta; \Omega). \end{aligned} \tag{2.3}$$

Then

$$\mathbf{M}(\mathbf{x}) \in G_{\boldsymbol{\vartheta}(\mathbf{x})} \text{ a. e. in } \Omega. \tag{2.4}$$

On the other hand, if for $\boldsymbol{\vartheta} \in \mathrm{L}^\infty(\Omega; \overline{T})$ and $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ (2.4) holds, then there exist sequences $(\boldsymbol{\chi}^n)$ in $\mathrm{L}^\infty(\Omega; T)$ and $(\mathbf{R}_n)$ in $\mathrm{L}^\infty(\Omega; SO(\mathbf{R}^d))$ such that, for sequence $(\mathbf{M}^n)$ given with (2.1), the convergences in (2.3) hold.

Proof. Assume that $(\boldsymbol{\chi}^n)$, $(\mathbf{R}_n)$ are sequences in $\mathrm{L}^\infty(\Omega; T)$ and $\mathrm{L}^\infty(\Omega; SO(\mathbf{R}^d))$, respectively, and $(\mathbf{M}^n)$ is defined with (2.1) such that (2.3) holds. Let $\mathbf{x}_0 \in \Omega$ be any joint Lebesgue point of $\boldsymbol{\vartheta}$ and $\mathbf{M}$. For $r > 0$ small enough, an open cube Q_r with center at $\mathbf{x}_0$ and side length r is contained in Ω. An affine transformation $\mathbf{x} \mapsto (\mathbf{x} - \mathbf{x}_0)/r$ maps Q_r to a unit cube Y centered at the origin. The function f defined on Q_r induces function f_r defined on Y in the following way:

$$f_r(\boldsymbol{y}) = f(\mathbf{x}_0 + r\boldsymbol{y}), \ \boldsymbol{y} \in Y.$$

Moreover, using this notation, by (2.3) we have

$$\begin{aligned} \boldsymbol{\chi}_r^n &\xrightarrow{*} \boldsymbol{\vartheta}_r \text{ in } \mathrm{L}^\infty(Y; \mathbf{R}^m), \\ \mathbf{M}_r^n &\xrightarrow{H} \mathbf{M}_r \text{ in } \mathfrak{M}_2(\alpha, \beta; Y), \end{aligned} \tag{2.5}$$

as $n \to \infty$. Since $\mathbf{x}_0$ is a Lebesgue point of $\boldsymbol{\vartheta}$ and $\mathbf{M}$, it follows that

$$\begin{aligned} \boldsymbol{\vartheta}_r &\longrightarrow \boldsymbol{\vartheta}(\mathbf{x}_0) \text{ in } \mathrm{L}^1(Y;\mathbf{R}^m), \\ \mathbf{M}_r &\longrightarrow \mathbf{M}(\mathbf{x}_0) \text{ in } \mathrm{L}^1(Y;\mathcal{L}(\mathrm{Sym},\mathrm{Sym})), \end{aligned} \tag{2.6}$$

as $r \to 0$. Hence, by Theorem 8, $\mathbf{M}_r$ also H-converges to $\mathbf{M}(\mathbf{x}_0)$. Using (2.5) and (2.6), by the Cantor diagonal method we conclude that there exists a sequence (r_n), such that $r_n \to 0$ and

$$\begin{aligned} \boldsymbol{\chi}^n_{r_n} &\xrightarrow{\;*\;} \boldsymbol{\vartheta}(\mathbf{x}_0) \text{ in } \mathrm{L}^\infty(Y;\mathbf{R}^m), \\ \mathbf{M}^n_{r_n} &\xrightarrow{\;H\;} \mathbf{M}(\mathbf{x}_0) \text{ in } \mathfrak{M}_2(\alpha,\beta;Y). \end{aligned}$$

The definition of $G_{\boldsymbol{\theta}}$ yields $\mathbf{M}(\mathbf{x}_0) \in G_{\boldsymbol{\vartheta}(\mathbf{x}_0)}$. Since almost every point $\mathbf{x}_0 \in \Omega$ is a Lebesgue point of $\boldsymbol{\vartheta}$ and $\mathbf{M}$, we get the claim.

On the other hand, suppose that for $\boldsymbol{\vartheta} \in \mathrm{L}^\infty(\Omega;\overline{T})$ and $\mathbf{M} \in \mathfrak{M}_2(\alpha,\beta;\Omega)$ (2.4) holds. For $\varepsilon > 0$, let us divide the image of the mapping $(\boldsymbol{\vartheta},\mathbf{M})$ into measurable sets Q_i, $i = 1,\dots,p$, whose diameter is less than ε. Note that the image of the mapping $(\boldsymbol{\vartheta},\mathbf{M})$ is a bounded set in $\mathbf{R}^m \times \mathcal{L}(\mathrm{Sym},\mathrm{Sym})$. Clearly, measurable sets $E_i := (\boldsymbol{\vartheta},\mathbf{M})^{-1}(Q_i)$, $i = 1,\dots,p$, form a partition of Ω. Furthermore, we choose $\mathbf{x}_i \in E_i$ such that

$$\mathbf{M}(\mathbf{x}_i) \in G_{\boldsymbol{\vartheta}(\mathbf{x}_i)},\ i = 1,\dots,p,$$

and define on Ω a piecewise constant function

$$(\boldsymbol{\vartheta}^\varepsilon,\mathbf{M}^\varepsilon)\big|_{E_i} = (\boldsymbol{\vartheta}(\mathbf{x}_i),\mathbf{M}(\mathbf{x}_i)),\ i = 1,\dots,p.$$

We observe

$$\|\boldsymbol{\vartheta}^\varepsilon - \boldsymbol{\vartheta}\|_{\mathrm{L}^\infty(\Omega;\mathbf{R}^m)} < \varepsilon,\ \|\mathbf{M}^\varepsilon - \mathbf{M}\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))} < \varepsilon. \tag{2.7}$$

Our goal is to replace measurable sets E_i with open sets U_i, such that (2.7) still holds, eventually in some weaker norm. Let $C_1,\dots,C_p$ be a compact sets in $\mathbf{R}^d$ such that

$$C_i \subseteq E_i \quad \text{and} \quad \lambda(E_i \backslash C_i) < \varepsilon/p,\ i = 1,\dots,p.$$

Since C_i, $i = 1,\dots,p$, are pairwise disjoint, there exist pairwise disjoint open sets $U_i \supseteq C_i$, $i = 1,\dots,p$, which cover Ω, up to a set U^ε whose measure is less than ε. Next, we define a piecewise constant function

$$(\tilde{\boldsymbol{\vartheta}}^\varepsilon,\widetilde{\mathbf{M}}^\varepsilon)\Big|_{U_i} = (\boldsymbol{\vartheta}(\mathbf{x}_i),\mathbf{M}(\mathbf{x}_i)),\ i = 1,\dots,p,$$

and extend it to Ω arbitrarily, such that $\tilde{\boldsymbol{\vartheta}}^\varepsilon\Big|_{U^\varepsilon} \in \mathrm{L}^\infty(\Omega;\mathbf{R}^m)$ and $\widetilde{\mathbf{M}}^\varepsilon\Big|_{U^\varepsilon} \in \mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$.

Let us notice that

$$\|\tilde{\boldsymbol{\vartheta}}^\varepsilon - \boldsymbol{\vartheta}^\varepsilon\|_{\mathrm{L}^1(\Omega;\mathbf{R}^m)} < 2\varepsilon\mu(\Omega), \ \ \|\widetilde{\mathbf{M}}^\varepsilon - \mathbf{M}^\varepsilon\|_{\mathrm{L}^1(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))} < 2\varepsilon\mu(\Omega), \tag{2.8}$$

which together with (2.7) gives

$$\begin{aligned} &\tilde{\boldsymbol{\vartheta}}^\varepsilon \longrightarrow \boldsymbol{\vartheta} \text{ in } \mathrm{L}^1(\Omega;\mathbf{R}^m), \\ &\widetilde{\mathbf{M}}^\varepsilon \longrightarrow \mathbf{M} \text{ in } \mathrm{L}^1(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym})), \end{aligned} \tag{2.9}$$

as $\varepsilon \to 0$. According to the choice of points $\mathbf{x}_i$, $i = 1, \dots, p$, on $\Omega \backslash U^\varepsilon$ we have

$$\widetilde{\mathbf{M}}^\varepsilon \in G_{\tilde{\boldsymbol{\vartheta}}^\varepsilon}.$$

Consequently, on every set U_i, $i = 1, \dots, p$, there exist sequences $(\boldsymbol{\chi}_n^{\varepsilon,i})$ and $(\mathbf{R}_n^{\varepsilon,i})$ in $\mathrm{L}^\infty(U_i;T)$ and $\mathrm{L}^\infty(U_i;SO(\mathbf{R}^d))$, respectively, and a sequence $(\mathbf{M}_n^{\varepsilon,i})$ defined with (2.1) such that

$$\begin{aligned} &\boldsymbol{\chi}_n^{\varepsilon,i} \xrightarrow{*} \tilde{\boldsymbol{\vartheta}}^\varepsilon \text{ in } \mathrm{L}^\infty(U_i;\mathbf{R}^m), \\ &\mathbf{M}_n^{\varepsilon,i} \xrightarrow{H} \widetilde{\mathbf{M}}^\varepsilon \text{ in } \mathfrak{M}_2(\alpha,\beta;U_i), \end{aligned}$$

as $n \to \infty$. Let us denote by $\boldsymbol{\chi}_n^\varepsilon$ and $\mathbf{M}_n^\varepsilon$ functions on Ω whose restrictions on U_i are equal to $\boldsymbol{\chi}_n^{\varepsilon,i}$ and $\mathbf{M}_n^{\varepsilon,i}$, respectively. On the set U^ε we can define them arbitrarily (in the permissible set of values). The locality of H-convergence, Banach-Alaoglu theorem [53] and compactness of $\mathfrak{M}_2(\alpha,\beta;\Omega)$, imply that on a subsequence one has

$$\begin{aligned} &\boldsymbol{\chi}_{n'}^{\varepsilon} \xrightarrow{*} \tilde{\boldsymbol{\vartheta}}^\varepsilon \text{ in } \mathrm{L}^\infty(\Omega;\mathbf{R}^m), \\ &\mathbf{M}_{n'}^{\varepsilon} \xrightarrow{H} \widetilde{\mathbf{M}}^\varepsilon \text{ in } \mathfrak{M}_2(\alpha,\beta;\Omega), \end{aligned} \tag{2.10}$$

as $n' \to \infty$, where we have predefined functions $\tilde{\boldsymbol{\vartheta}}^\varepsilon$ and $\widetilde{\mathbf{M}}^\varepsilon$ on the set U^ε. Since the measure of U^ε is less than ε, (2.8) and (2.9) still hold. By using (2.9), (2.10) and the Cantor diagonal method, we can extract a subsequence $(\varepsilon(n'))$ which converges to zero, such that

$$\begin{aligned} &\boldsymbol{\chi}_{n'}^{\varepsilon(n')} \xrightarrow{*} \boldsymbol{\vartheta} \text{ in } \mathrm{L}^\infty(\Omega;\mathbf{R}^m), \\ &\mathbf{M}_{n'}^{\varepsilon(n')} \xrightarrow{H} \mathbf{M} \text{ in } \mathfrak{M}_2(\alpha,\beta;\Omega), \end{aligned}$$

which gives the claim of the theorem. ∎

In the following lemma we use notation of Section 1.5.

Lemma 9 Let $(\mathbf{M}^n)$ be a sequence of Y-periodic tensor functions in $\mathfrak{M}_2(\alpha,\beta;\mathbf{R}^d)$ such that

$$\mathbf{M}^n \xrightharpoonup{H} \mathbf{M}^\infty \text{ in } \mathfrak{M}_2(\alpha, \beta; Y).$$

Then

$$\widehat{\mathbf{M}^n} \longrightarrow \widehat{\mathbf{M}^\infty} \text{ in } \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}).$$

Proof. For $\mathbf{E} \in \mathrm{Sym}$ and $n \in \mathbf{N} \cup \{\infty\}$, by $w^n_\mathbf{E} \in \mathrm{H}^2_\#(Y)/\mathbf{R}$ we denote the solution of boundary value problem

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n(\mathbf{x})(\mathbf{E} + \nabla\nabla w^n_\mathbf{E}(\mathbf{x}))) = 0 & \text{in } Y \\ \mathbf{x} \mapsto w^n_\mathbf{E}(\mathbf{x}) & \text{is } Y\text{-periodic} \end{cases} . \tag{2.11}$$

The sequence $(w^n_\mathbf{E})$ is bounded in $\mathrm{H}^2_{\mathrm{loc}}(\mathbf{R}^d)$, which implies that there is a subsequence

$$w^n_\mathbf{E} \rightharpoonup v \text{ in } \mathrm{H}^2_{\mathrm{loc}}(\mathbf{R}^d).$$

Thus, we deduce that

$$u_n := \frac{1}{2}\mathbf{E}\mathbf{x} \cdot \mathbf{x} + w^n_\mathbf{E} \rightharpoonup u_\infty := \frac{1}{2}\mathbf{E}\mathbf{x} \cdot \mathbf{x} + v \text{ in } \mathrm{H}^2_{\mathrm{loc}}(\mathbf{R}^d).$$

By Theorem 9 it follows

$$\mathbf{M}^n\nabla\nabla u_n \rightharpoonup \mathbf{M}^\infty\nabla\nabla u_\infty \text{ in } \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym}). \tag{2.12}$$

Furthermore, v is a solution of boundary value problem (2.11) for $n = \infty$ and it coincides with $w^\infty_\mathbf{E}$. Therefore, we have

$$u_\infty = \frac{1}{2}\mathbf{E}\mathbf{x} \cdot \mathbf{x} + w^\infty_\mathbf{E}. \tag{2.13}$$

By (2.12) and (2.13), we can pass to the limit in the following integral:

$$\widehat{\mathbf{M}^n}\mathbf{E} = \int_Y \mathbf{M}^n(\mathbf{x})(\mathbf{E} + \nabla\nabla w^n_\mathbf{E}(\mathbf{x}))\,d\mathbf{x} \longrightarrow \int_Y \mathbf{M}^\infty(\mathbf{x})(\mathbf{E} + \nabla\nabla w^\infty_\mathbf{E}(\mathbf{x}))\,d\mathbf{x} = \widehat{\mathbf{M}^\infty}\mathbf{E}.$$

This finishes the proof by arbitrariness of $\mathbf{E} \in \mathrm{Sym}$. ■

Theorem 27 For every $\boldsymbol{\theta} \in \overline{T}$

$$G_{\boldsymbol{\theta}} = \mathrm{Cl}P_{\boldsymbol{\theta}}.$$

Proof. In order to show that $\mathrm{Cl}P_{\boldsymbol{\theta}} \subseteq G_{\boldsymbol{\theta}}$, it is enough to prove that $G_{\boldsymbol{\theta}}$ is closed (obviously $P_{\boldsymbol{\theta}} \subseteq G_{\boldsymbol{\theta}}$). Let $(\mathbf{M}^n)$ be a sequence in $G_{\boldsymbol{\theta}}$ such that

$$\mathbf{M}^n \longrightarrow \mathbf{M} \text{ in } \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}).$$

Hence, for every $n \in \mathbf{N}$, there exist sequences $(\boldsymbol{\chi}^n_k)$ in $\mathrm{L}^\infty(\Omega; T)$ and $(\mathbf{R}^n_k)$ in $\mathrm{L}^\infty(\Omega; SO(\mathbf{R}^d))$

such that
$$\mathbf{M}_k^n(\mathbf{x}) = \sum_{i=1}^{m} (\boldsymbol{\chi}_k^n)_i(\mathbf{x}) \mathcal{R}\left(\mathbf{R}_k^n(\mathbf{x}), \mathbf{M}_i\right)$$

and

$$\boldsymbol{\chi}_k^n \xrightarrow{*} \boldsymbol{\theta} \text{ in } \mathrm{L}^\infty(\Omega; \mathbf{R}^m),$$
$$\mathbf{M}_k^n \xrightarrow{H} \mathbf{M}^n \text{ in } \mathfrak{M}_2(\alpha, \beta; \Omega),$$

as $k \to \infty$. Since convergence of constant tensors implies H-convergence on Ω, by the Cantor diagonal method we have $\mathbf{M} \in G_{\boldsymbol{\theta}}$.

To prove that $G_{\boldsymbol{\theta}} \subseteq \mathrm{Cl}P_{\boldsymbol{\theta}}$, let us take $\mathbf{M} \in G_{\boldsymbol{\theta}}$. There exist sequences $(\boldsymbol{\chi}^n)$ in $\mathrm{L}^\infty(Y;T)$ and $(\mathbf{R}_n)$ in $\mathrm{L}^\infty(Y; SO(\mathbf{R}^d))$, such that for $\mathbf{M}^n$ defined with (2.1), it follows

$$\boldsymbol{\chi}^n \xrightarrow{*} \boldsymbol{\theta} \text{ in } \mathrm{L}^\infty(Y; \mathbf{R}^m),$$
$$\mathbf{M}^n \xrightarrow{H} \mathbf{M} \text{ in } \mathfrak{M}_2(\alpha, \beta; Y).$$

Furthermore,
$$\boldsymbol{\theta}^n := \int_Y \boldsymbol{\chi}^n \, d\boldsymbol{y} \to \boldsymbol{\theta}.$$

For $n \in \mathbf{N}$, we predefine $\boldsymbol{\chi}^n$ to $\tilde{\boldsymbol{\chi}}^n \in \mathrm{L}^\infty(Y;T)$ such that

$$\int_Y \tilde{\boldsymbol{\chi}}^n \, d\boldsymbol{y} = \boldsymbol{\theta} \quad \text{and} \quad \|\boldsymbol{\chi}^n - \tilde{\boldsymbol{\chi}}^n\|_{\mathrm{L}^1(Y;T)} = \|\boldsymbol{\theta} - \boldsymbol{\theta}^n\|_{\mathbf{R}^m},$$

and we extend it Y-periodically to $\mathbf{R}^d$. Next, let us denote

$$\widetilde{\mathbf{M}}^n(\mathbf{x}) = \sum_{i=1}^{m} \tilde{\chi}_i^n(\mathbf{x}) \mathcal{R}\left(\mathbf{R}_n(\mathbf{x}), \mathbf{M}_i\right).$$

From here we conclude that $\|\mathbf{M}^n - \widetilde{\mathbf{M}}^n\|_{L^1} \to 0$, which implies that $\widetilde{\mathbf{M}}^n \xrightarrow{H} \mathbf{M}$ (on an arbitrary bounded, open set $\Omega \subseteq \mathbf{R}^d$). By Lemma 9 we have

$$\widehat{\widetilde{\mathbf{M}}^n} \longrightarrow \widehat{\mathbf{M}} = \mathbf{M}.$$

Since $\widehat{\widetilde{\mathbf{M}}^n} \in P_{\boldsymbol{\theta}}$, the claim of the theorem follows. ■

Remark 7 In this section, the characterisation of the G-closure, which is the set of all homogenized tensors obtained by mixing m materials in fixed volume fractions such that rotations of materials are allowed, has been given. Since the assumption that rotations of materials are allowed makes further results highly nontrivial to derive (or sometimes even impossible), from now on we shall consider only mixtures with no rotations. Actually, this is a standard setting when dealing with composite materials. It is easy to check that all results of this section are valid if we replace the sequence $(\mathbf{M}^n)$ from (2.1), with a similar one, but with rotations excluded:

$$\mathbf{M}^n(\mathbf{x}) = \sum_{i=1}^{m} \chi_i^n(\mathbf{x})\mathbf{M}_i,$$

for $\boldsymbol{\chi}^n = (\chi_1^n, \dots, \chi_m^n) \in \mathrm{L}^\infty(\Omega; T)$.

2.3 Homogenization of laminated materials

In this section we shall study laminated composite materials which are homogeneous in all directions orthogonal to some fixed unit vector $\mathbf{e}$, which is called the direction of lamination. For simplicity, let us take for the direction of lamination the first canonical basis vector. In this case, the sequence of tensors $(\mathbf{M}^n)$ depends only on the first variable x_1. We are interested in elastic properties of laminated materials, which can be derived by using the following theorem. Its proof goes along the same lines as proof of two-dimensional result [9, 10, 57]

Theorem 28 Let $\Omega \subseteq \mathbb{R}^d$ be an open and bounded set, and $(\mathbf{M}^n)$ a sequence of tensor valued functions in $\mathfrak{M}_2(\alpha, \beta; \Omega)$, such that for each n, $\mathbf{M}^n$ depends on x_1 only. Then $(\mathbf{M}^n)$ H-converges to $\mathbf{M}$ if and only if its components m_{ijkl} satisfy the following:

$$\begin{aligned}
\frac{1}{m_{1111}^n} &\overset{*}{\longrightarrow} \frac{1}{m_{1111}}, \\
\frac{m_{11jk}^n}{m_{1111}^n} &\overset{*}{\longrightarrow} \frac{m_{11jk}}{m_{1111}}, \quad 1 \le j \le d,\ 2 \le k \le d, \\
\frac{m_{ik11}^n}{m_{1111}^n} &\overset{*}{\longrightarrow} \frac{m_{ik11}}{m_{1111}}, \quad 1 \le i \le d,\ 2 \le k \le d, \\
m_{ikjl}^n - \frac{m_{ik11}^n m_{11jl}^n}{m_{1111}^n} &\overset{*}{\longrightarrow} m_{ikjl} - \frac{m_{ik11} m_{11jl}}{m_{1111}}, \quad 1 \le i, j \le d,\ 2 \le k, l \le d
\end{aligned} \tag{2.14}$$

in $\mathrm{L}^\infty(\Omega)$.

Proof. Assume that the sequence $(\mathbf{M}^n)$ H-converges to $\mathbf{M}$, and, for an arbitrary function $f \in \mathrm{H}^{-2}(\Omega)$, let (u_n) be the sequence of solutions to

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n \nabla\nabla u_n) = f \\ u_n \in \mathrm{H}_0^2(\Omega) \end{cases}.$$

Since $(\mathbf{M}^n)$ H-converges to $\mathbf{M}$, the following holds:

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in} \quad \mathrm{H}^2_0(\Omega)\,, \\ \mathbf{M}^n\nabla\nabla u_n &\rightharpoonup \mathbf{M}\nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega;\mathrm{Sym})\,. \end{aligned}$$

For $n \in \mathbf{N}$, we denote

$$\mathbf{E}^n := \nabla\nabla u_n,\, \mathbf{E} := \nabla\nabla u \quad \text{and} \quad \mathbf{D}^n := \mathbf{M}^n\mathbf{E}^n.$$

Since $\mathsf{div}\,\mathsf{div}\,\mathbf{D}^n = f$, by Lemma 8 D^n_{11} does not oscillate in x_1. Furthermore, let us define $\tilde{\mathbf{D}}^n \in \mathrm{L}^\infty(\Omega;\mathrm{Sym})$ with

$$\tilde{\mathbf{D}}^n_{ij} := \begin{cases} 0, & \text{for } i=j=1 \\ g^n_{ij}, & \text{otherwise,} \end{cases}$$

where functions $g^n_{ij} \in \mathrm{L}^\infty(\Omega)$, such that $g^n_{ij} = g^n_{ji}$, depend only upon x_1. Assume that, for every pair of indices (i,j),

$$g^n_{ij} \xrightarrow{*} g_{ij} \text{ in } \mathrm{L}^\infty(\Omega).$$

After deducing that $\mathsf{div}\,\mathsf{div}\,\tilde{\mathbf{D}}^n = 0$, by Lemma 1 it follows

$$\tilde{\mathbf{D}}^n : \mathbf{E}^n \xrightarrow{*} \tilde{\mathbf{D}} : \mathbf{E}$$

in the space of Radon measures on Ω. On the other hand, the sequence $(\tilde{\mathbf{D}}^n : \mathbf{E}^n)$ is bounded in $\mathrm{L}^2(\Omega)$ and thus converges weakly to the same limit in that space. Convenient choices of functions g^n_{ij} lead to conclusion that for each pair of indices $(i,j) \neq (1,1)$, E^n_{ij} does not oscillate in x_1. This allows us to define matrices $\mathbf{G}^n, \mathbf{O}^n \in \mathrm{L}^2(\Omega;\mathrm{Sym})$, respectively made from *good* (nonoscillating) and *bad* (oscillating) components of $\mathbf{E}^n$ and $\mathbf{D}^n$:

$$G^n_{ij} := \begin{cases} D^n_{11}, & \text{for } i=j=1 \\ E^n_{ij}, & \text{otherwise} \end{cases}$$

and

$$O^n_{ij} := \begin{cases} E^n_{11}, & \text{for } i=j=1 \\ D^n_{ij}, & \text{otherwise} \end{cases}.$$

Additionally, we define a nonlinear mapping $\Psi : \mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym})) \to \mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$, and denote $\mathbf{K}^n := \Psi(\mathbf{M}^n)$, where

$$\begin{aligned} K^n_{1111} &= \frac{1}{m^n_{1111}}, \\ K^n_{11jk} &= -\frac{m^n_{11jk}}{m^n_{1111}}, \quad 1 \le j \le d,\, 2 \le k \le d, \\ K^n_{ik11} &= \frac{m^n_{ik11}}{m^n_{1111}}, \quad 1 \le i \le d,\, 2 \le k \le d, \end{aligned}$$

$$K^n_{ikjl} = m^n_{ikjl} - \frac{m^n_{ik11} m^n_{11jl}}{m^n_{1111}}, \ \ 1 \le i, j \le d, \, 2 \le k, l \le d.$$

Now, we replace the relation $\mathbf{D}^n = \mathbf{M}^n \mathbf{E}^n$ by the equivalent relation

$$\mathbf{O}^n = \mathbf{K}^n \mathbf{G}^n.$$

Since $(\mathbf{M}^n)$ belongs to the space $\mathfrak{M}_2(\alpha, \beta; \Omega)$, we conclude that

$$\alpha \le m^n_{1111} \le \beta, \ \text{for a. e. } \mathbf{x} \in \Omega,$$

hence $(\mathbf{K}^n)$ is bounded in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$ and has an accumulation point $\mathbf{K}$ in the weak-$*$ topology. Using that $(\mathbf{G}^n)$ does not oscillate in x_1 and that $(\mathbf{K}^n)$ depends only upon x_1, after passing to a subsequence, we obtain

$$\mathbf{O} = \mathbf{K}\mathbf{G},$$

where $\mathbf{O}$, $\mathbf{G} \in \mathrm{L}^2(\Omega; \mathrm{Sym})$ are weak limits of the sequences $(\mathbf{O}^n)$ and $(\mathbf{G}^n)$, respectively. From here we conclude $\mathbf{K} = \Psi(\mathbf{M})$, i.e.

$$\Psi(\mathbf{M}^{n_k}) \overset{*}{\rightharpoonup} \Psi(\mathbf{M}) \text{ in } \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})).$$

Since this argument holds for every accumulation point, it follows that the entire sequence $\Psi(\mathbf{M}^n)$ converges to $\Psi(\mathbf{M})$.

In order to prove the reverse implication, by using notation introduced in the first part of the proof, assume that

$$\Psi(\mathbf{M}^n) \overset{*}{\rightharpoonup} \Psi(\mathbf{M}) \text{ in } \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})).$$

Similarly as before, let (u_n) be the sequence of solutions to

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n \nabla\nabla u_n) = f \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}$$

for an arbitrary function $f \in \mathrm{H}^{-2}(\Omega)$. The sequences (u_n) and $(\mathbf{M}^n \nabla\nabla u_n)$ are bounded in $\mathrm{H}^2_0(\Omega)$ and $\mathrm{L}^2(\Omega; \mathrm{Sym})$, respectively, and therefore converge weakly on a subsequence:

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } \mathrm{H}^2_0(\Omega), \\ &\mathbf{M}^n \nabla\nabla u_n \rightharpoonup \boldsymbol{\sigma} \text{ in } \mathrm{L}^2(\Omega; \mathrm{Sym}). \end{aligned}$$

In order to prove that $\mathsf{M}^n \xrightarrow{H} \mathsf{M}$, we only need to show that

$$\boldsymbol{\sigma} = \mathsf{M}\nabla\nabla u.$$

In the same way as in the first part of the proof, we define matrices $\mathbf{G}^n, \mathbf{O}^n \in \mathrm{L}^2(\Omega; \mathrm{Sym})$, respectively made from *good* (nonoscillating) and *bad* (oscillating) components of $\mathbf{E}^n$ and $\mathbf{D}^n$, such that

$$\mathbf{O}^n = \mathsf{K}^n\mathbf{G}^n, \tag{2.15}$$

with $\mathbf{K}^n := \Psi(\mathsf{M}^n)$. By using an analogous arguments, we can pass to a subsequence in (2.15) and obtain

$$\mathbf{O} = \mathsf{K}\mathbf{G}, \tag{2.16}$$

where $\mathsf{K} = \Psi(\mathsf{M})$. After some calculation, and using also that $\mathbf{E}^n = \nabla\nabla u_n \rightharpoonup \nabla\nabla u$ in $\mathrm{L}^2(\Omega; \mathrm{Sym})$, from (2.16) it follows that $\boldsymbol{\sigma} = \mathsf{M}\nabla\nabla u$. ■

The previous theorem shows that H-convergence can be reduced to the weak-∗ convergence of some combination of entries of the tensors M^n, $n \in \mathbf{N}$, in the case when they depend only on one variable. Its special case of particular interest is the following lamination formula, which gives the elastic properties of a simple laminate of two materials.

Remark 8 In order to indicate dependence of a function f, defined on Ω, only on $\mathbf{x} \cdot \mathbf{e}$, for some $\mathsf{e} \in \mathbf{R}^d$, in the sequel we shall abuse the notation $f(\mathbf{x} \cdot \mathbf{e})$ for both, the mapping $\mathbf{x} \mapsto f(\mathbf{x} \cdot \mathbf{e})$, as well as the value of this mapping at the point $\mathbf{x} \in \Omega$.

Corollary 1 Let A and B be two constant tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ and $\chi_n(x_1)$ be a sequence of characteristic functions that converges to $\theta(x_1)$ in $\mathrm{L}^\infty(\Omega; [0,1])$ weakly-∗. Then, a sequence (M^n) of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$, defined as

$$\mathsf{M}^n(x_1) := \chi_n(x_1)\mathsf{A} + (1 - \chi_n(x_1))\mathsf{B}$$

H-converges to

$$\mathsf{M} = \theta\mathsf{A} + (1-\theta)\mathsf{B} - \frac{\theta(1-\theta)(\mathsf{A}-\mathsf{B})(\mathbf{e}_1 \otimes \mathbf{e}_1) \otimes (\mathsf{A}-\mathsf{B})^T(\mathbf{e}_1 \otimes \mathbf{e}_1)}{(1-\theta)\mathsf{A}(\mathbf{e}_1 \otimes \mathbf{e}_1) : (\mathbf{e}_1 \otimes \mathbf{e}_1) + \theta\mathsf{B}(\mathbf{e}_1 \otimes \mathbf{e}_1) : (\mathbf{e}_1 \otimes \mathbf{e}_1)}, \tag{2.17}$$

which also depends only on x_1.

Proof. Since $\chi_n(x_1)$ is a sequence of characteristic functions (i.e. it is always equal to 0 or 1), the above formula can be derived by using Theorem 28:

$$\begin{aligned}\frac{1}{m^n_{1111}} &= \frac{1}{\chi_n(x_1)a_{1111} + (1-\chi_n(x_1))b_{1111}} \\ &= \frac{\chi_n(x_1)}{a_{1111}} + \frac{1-\chi_n(x_1)}{b_{1111}} \xrightarrow{\ast} \frac{\theta}{a_{1111}} + \frac{1-\theta}{b_{1111}} = \frac{1}{m_{1111}}.\end{aligned} \tag{2.18}$$

Furthermore, for $1 \leq j \leq d$, $2 \leq k \leq d$,

$$\begin{aligned}\frac{m^n_{11jk}}{m^n_{1111}} &= \frac{\chi_n(x_1)a_{11jk} + (1-\chi_n(x_1))b_{11jk}}{\chi_n(x_1)a_{1111} + (1-\chi_n(x_1))b_{1111}} \\ &= \chi_n(x_1)\frac{a_{11jk}}{a_{1111}} + (1-\chi_n(x_1))\frac{b_{11jk}}{b_{1111}} \xrightarrow{\ *\ } \theta\frac{a_{11jk}}{a_{1111}} + (1-\theta)\frac{b_{11jk}}{b_{1111}} = \frac{m_{11jk}}{m_{1111}}.\end{aligned} \tag{2.19}$$

Similarly, for $1 \leq i \leq d$, $2 \leq k \leq d$,

$$\begin{aligned}\frac{m^n_{ik11}}{m^n_{1111}} &= \frac{\chi_n(x_1)a_{ik11} + (1-\chi_n(x_1))b_{ik11}}{\chi_n(x_1)a_{1111} + (1-\chi_n(x_1))b_{1111}} \\ &= \chi_n(x_1)\frac{a_{ik11}}{a_{1111}} + (1-\chi_n(x_1))\frac{b_{ik11}}{b_{1111}} \xrightarrow{\ *\ } \theta\frac{a_{ik11}}{a_{1111}} + (1-\theta)\frac{b_{ik11}}{b_{1111}} = \frac{m_{ik11}}{m_{1111}}.\end{aligned} \tag{2.20}$$

Finally, for $1 \leq i, j \leq d$, $2 \leq k, l \leq d$,

$$\begin{aligned}& m^n_{ikjl} - \frac{m^n_{ik11}m^n_{11jl}}{m^n_{1111}} \\ &= \chi_n(x_1)a_{ikjl} + (1-\chi_n(x_1))b_{ikjl} - \\ &\quad - \frac{(\chi_n(x_1)a_{ik11} + (1-\chi_n(x_1))b_{ik11})(\chi_n(x_1)a_{11jl} + (1-\chi_n(x_1))b_{11jl})}{\chi_n(x_1)a_{1111} + (1-\chi_n(x_1))b_{1111}} = \\ &= \chi_n(x_1)\left(a_{ikjl} - \frac{a_{ik11}a_{11jl}}{a_{1111}}\right) + (1-\chi_n(x_1))\left(b_{ikjl} - \frac{b_{ik11}b_{11jl}}{b_{1111}}\right) \xrightarrow{\ *\ } \\ & \theta\left(a_{ikjl} - \frac{a_{ik11}a_{11jl}}{a_{1111}}\right) + (1-\theta)\left(b_{ikjl} - \frac{b_{ik11}b_{11jl}}{b_{1111}}\right) = m_{ikjl} - \frac{m_{ik11}m_{11jl}}{m_{1111}}.\end{aligned} \tag{2.21}$$

It remains to validate that entries of the H-limit $\mathbf{M}$, calculated above, coincide with those given by formula (2.17). This requires some simple, but rather technical computation: from (2.18) we have that $m_{1111} = \dfrac{a_{1111}b_{1111}}{(1-\theta)a_{1111} + \theta b_{1111}}$, and by using this, from (2.19), (2.20) and (2.21) we obtain other entries of the tensor $\mathbf{M}$. ∎

Remark 9 If we take some other unit vector $\mathbf{e} \in \mathbf{R}^d$ for the lamination direction, and let $\theta(\mathbf{x}\cdot\mathbf{e})$ be the weak limit of the sequence $\chi_n(\mathbf{x}\cdot\mathbf{e})$, then the formula (2.17) is still valid, in an analogous form:

$$\mathbf{M} = \theta\mathbf{A} + (1-\theta)\mathbf{B} - \frac{\theta(1-\theta)(\mathbf{A}-\mathbf{B})(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{A}-\mathbf{B})^T(\mathbf{e}\otimes\mathbf{e})}{(1-\theta)\mathbf{A}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e}) + \theta\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}. \tag{2.22}$$

This can be derived by a simple change of variables in the result of Theorem 28. The composite $\mathbf{M}$ is called a simple laminate in the direction $\mathbf{e}$, obtained by mixing the phases $\mathbf{A}$ and $\mathbf{B}$ in proportions θ and $1-\theta$, respectively.

This approach for deriving the lamination formula was used by Tartar (see e.g. [72]) in the conductivity setting. A different approach was considered in [51] (see also [47]), by using the average values of strain and moment over the given area, which is assumed

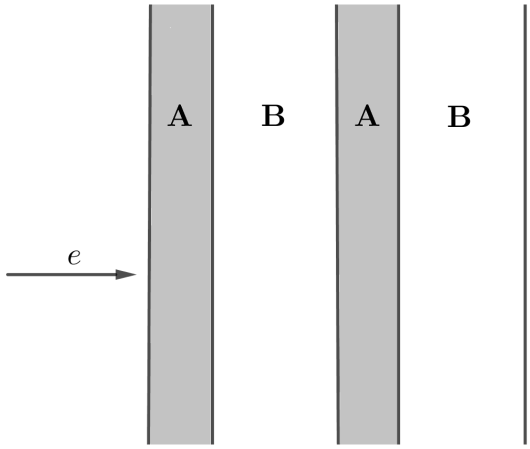


Figure 2.1: Layers orthogonal to the vector $\mathbf{e}$.

to be *physically small*. Then, the lamination formula was derived under the additional assumption that along the boundaries dividing the layers, the conditions of continuity for the normal and tangential component of strain hold.

The process of lamination can be repeated in an iterative way. It is of interest to consider a particular subset of laminated materials, obtained by an iterative process of lamination, where the previous laminate is laminated again with a single pure phase (always the same one). The composite material obtained by this process is called a sequential laminate. Since it is quite difficult to iterate formula (2.22), we shall express it in a different form, by using the following lemma, whose idea is due to Tartar [67].

Lemma 10 If N is an invertible fourth-order tensor, $\mathbf{E} \in \mathrm{Sym}$ and $c \in \mathbf{R}$ such that $1 + c(\mathsf{N}\mathbf{E} : \mathbf{E}) \neq 0$, then the inverse of $\mathsf{N} + c(\mathsf{N}\mathbf{E}) \otimes (\mathsf{N}^T\mathbf{E})$ is

$$\mathsf{N}^{-1} - \frac{c}{1 + c(\mathsf{N}\mathbf{E} : \mathbf{E})}\mathbf{E} \otimes \mathbf{E}.$$

Proof. This can be proved by using a variation of the well-known Sherman-Morrison formula [27], but it can also be done straightforwardly: by denoting $\mathsf{A} := \mathsf{N} + c(\mathsf{N}\mathbf{E}) \otimes (\mathsf{N}^T\mathbf{E})$ and $\mathsf{B} := \mathsf{N}^{-1} - \frac{c}{1 + c(\mathsf{N}\mathbf{E} : \mathbf{E})}\mathbf{E} \otimes \mathbf{E}$, one easily checks that $\mathsf{AB} = \mathsf{BA} = \mathsf{I}_4$. ■

Corollary 2 If $(\mathsf{A} - \mathsf{B})$ is an invertible fourth-order tensor, then formula (2.22) for M is equivalent to

$$\theta(\mathsf{M} - \mathsf{B})^{-1} = (\mathsf{A} - \mathsf{B})^{-1} + \frac{1 - \theta}{\mathsf{B}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})}(\mathbf{e} \otimes \mathbf{e}) \otimes (\mathbf{e} \otimes \mathbf{e}). \tag{2.23}$$

Proof. First, let us write formula (2.22) in the following way:

$$\mathsf{M} - \mathsf{B} = \theta(\mathsf{A} - \mathsf{B}) + \frac{\theta(\theta - 1)(\mathsf{A} - \mathsf{B})(\mathbf{e} \otimes \mathbf{e}) \otimes (\mathsf{A} - \mathsf{B})^T(\mathbf{e} \otimes \mathbf{e})}{(1 - \theta)\mathsf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e}) + \theta\mathsf{B}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})}.$$

Denoting
$$c = \frac{\theta - 1}{(1-\theta)\mathbf{A}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e}) + \theta\mathbf{B}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})},$$
as a consequence of Lemma 10, with $\mathbf{N} = \mathbf{A} - \mathbf{B}$ and $\mathbf{E} = \mathbf{e}\otimes\mathbf{e}$, we have
$$\begin{aligned}\theta(\mathbf{M} - \mathbf{B})^{-1} &= (\mathbf{A} - \mathbf{B})^{-1} - \frac{c}{1 + c(\mathbf{A} - \mathbf{B})(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e}) \\ &= (\mathbf{A} - \mathbf{B})^{-1} + \frac{1-\theta}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e}),\end{aligned}$$

which is the required formula. ■

Let $\mathbf{A}_1^*$ be a simple laminate, obtained from the materials $\mathbf{A}$ and $\mathbf{B}$, in proportions θ_1 and $(1-\theta_1)$, respectively, in the direction $\mathbf{e}_1$ of lamination:
$$\theta_1(\mathbf{A}_1^* - \mathbf{B})^{-1} = (\mathbf{A} - \mathbf{B})^{-1} + \frac{1-\theta_1}{\mathbf{B}(\mathbf{e}_1\otimes\mathbf{e}_1) : (\mathbf{e}_1\otimes\mathbf{e}_1)}(\mathbf{e}_1\otimes\mathbf{e}_1)\otimes(\mathbf{e}_1\otimes\mathbf{e}_1).$$

$\mathbf{A}_1^*$ can be laminated again with the material $\mathbf{B}$, in proportion θ_2 of material $\mathbf{A}_1^*$ and $1-\theta_2$ of material $\mathbf{B}$, in the direction of lamination $\mathbf{e}_2$, to obtain a new laminate $\mathbf{A}_2^*$:
$$\theta_2(\mathbf{A}_2^* - \mathbf{B})^{-1} = (\mathbf{A}_1^* - \mathbf{B})^{-1} + \frac{1-\theta_2}{\mathbf{B}(\mathbf{e}_2\otimes\mathbf{e}_2) : (\mathbf{e}_2\otimes\mathbf{e}_2)}(\mathbf{e}_2\otimes\mathbf{e}_2)\otimes(\mathbf{e}_2\otimes\mathbf{e}_2),$$

i.e.
$$\begin{aligned}\theta_1\theta_2(\mathbf{A}_2^* - \mathbf{B})^{-1} &= (\mathbf{A} - \mathbf{B})^{-1} + (1-\theta_1)\frac{(\mathbf{e}_1\otimes\mathbf{e}_1)\otimes(\mathbf{e}_1\otimes\mathbf{e}_1)}{\mathbf{B}(\mathbf{e}_1\otimes\mathbf{e}_1) : (\mathbf{e}_1\otimes\mathbf{e}_1)} + \\ &+ \theta_1(1-\theta_2)\frac{(\mathbf{e}_2\otimes\mathbf{e}_2)\otimes(\mathbf{e}_2\otimes\mathbf{e}_2)}{\mathbf{B}(\mathbf{e}_2\otimes\mathbf{e}_2) : (\mathbf{e}_2\otimes\mathbf{e}_2)}.\end{aligned}$$

If we continue this iterative process and in p-th step laminate the previously obtained laminate $\mathbf{A}_{p-1}^*$ with material $\mathbf{B}$ in the lamination direction $\mathbf{e}_p$ and proportion θ_p of $\mathbf{A}_{p-1}^*$, we obtain a composite material called a *rank-p sequential laminate* with matrix $\mathbf{B}$ and core $\mathbf{A}$, which is determined by the following formula:
$$\left(\prod_{j=1}^{p}\theta_j\right)(\mathbf{A}_p^* - \mathbf{B})^{-1} = (\mathbf{A} - \mathbf{B})^{-1} + \sum_{i=1}^{p}\left((1-\theta_i)\prod_{j=1}^{i-1}\theta_j\right)\frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i) : (\mathbf{e}_i\otimes\mathbf{e}_i)}. \quad (2.24)$$

The overall volume fraction of material $\mathbf{A}$ in this rank-p sequential laminate is $\theta = \prod_{i=1}^{p}\theta_i$. The following result is a simple consequence of (2.24) (for an analogous result in the context of stationary diffusion see [2]).

Lemma 11 Let θ be a volume fraction in $[0,1]$, $(\mathbf{e}_i)_{1\le i\le p}$ unit vectors in $\mathbf{R}^d$, and $(m_i)_{1\le i\le p}$ nonnegative real numbers such that $\sum_{i=1}^{p} m_i = 1$. Then, there exists a rank-p sequential laminate $\mathbf{A}_p^*$ with core $\mathbf{A}$ and matrix $\mathbf{B}$, and with lamination directions $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_p$, such that

$$\theta(\mathbf{A}_p^* - \mathbf{B})^{-1} = (\mathbf{A} - \mathbf{B})^{-1} + (1-\theta)\sum_{i=1}^{p} m_i \frac{(\mathbf{e}_i \otimes \mathbf{e}_i) \otimes (\mathbf{e}_i \otimes \mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i \otimes \mathbf{e}_i) : (\mathbf{e}_i \otimes \mathbf{e}_i)}. \tag{2.25}$$

Proof. One should compare formulas (2.24) and (2.25), which gives the following equality:

$$(1-\theta)m_i = (1-\theta_i)\prod_{j=1}^{i-1}\theta_j, \tag{2.26}$$

for $1 \le i \le p$. Knowing the parameters $(m_i)_{1\le i\le p}$ and θ, it is easy to compute proportions $(\theta_i)_{1\le i\le p}$ such that $\prod_{i=1}^{p}\theta_i = \theta$, since $\sum_{i=1}^{p} m_i = 1$. On the contrary, if $(\theta_i)_{1\le i\le p}$ are known, parameters $(m_i)_{1\le i\le p}$ can easily be computed from (2.26) upon defining $\theta = \prod_{i=1}^{p}\theta_i$. ■

One could interchange the roles of $\mathbf{A}$ and $\mathbf{B}$ and obtain a *symmetric* class of sequential laminates, i.e. if we repeat the iterative lamination process p times, in lamination directions $(\mathbf{e}_i)_{1\le i\le p}$ and proportions $(\theta_i)_{1\le i\le p}$ of material $\mathbf{A}$, we obtain a rank-p sequential laminate with matrix $\mathbf{A}$ and core $\mathbf{B}$, which is defined by the following formula:

$$\left(\prod_{j=1}^{p}(1-\theta_j)\right)(\mathbf{A}_p^* - \mathbf{A})^{-1} = (\mathbf{B} - \mathbf{A})^{-1} + \sum_{i=1}^{p}\left(\theta_i \prod_{j=1}^{i-1}(1-\theta_j)\right)\frac{(\mathbf{e}_i \otimes \mathbf{e}_i) \otimes (\mathbf{e}_i \otimes \mathbf{e}_i)}{\mathbf{A}(\mathbf{e}_i \otimes \mathbf{e}_i) : (\mathbf{e}_i \otimes \mathbf{e}_i)}. \tag{2.27}$$

In this case an analogous of Lemma 11 holds true:

Lemma 12 Let θ be a volume fraction in $[0,1]$, $(\mathbf{e}_i)_{1\le i\le p}$ unit vectors in $\mathbf{R}^d$, $(m_i)_{1\le i\le p}$ nonnegative real numbers such that $\sum_{i=1}^{p} m_i = 1$. Then, there exists a rank-p sequential laminate $\mathbf{A}_p^*$ with core $\mathbf{B}$ and matrix $\mathbf{A}$, and with lamination directions $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_p$, such that

$$(1-\theta)(\mathbf{A}_p^* - \mathbf{A})^{-1} = (\mathbf{B} - \mathbf{A})^{-1} + \theta\sum_{i=1}^{p} m_i \frac{(\mathbf{e}_i \otimes \mathbf{e}_i) \otimes (\mathbf{e}_i \otimes \mathbf{e}_i)}{\mathbf{A}(\mathbf{e}_i \otimes \mathbf{e}_i) : (\mathbf{e}_i \otimes \mathbf{e}_i)}. \tag{2.28}$$

2.4 Hashin-Shtrikman bounds on the primal energy

In the sequel we want to derive optimal bounds on the effective energy of a two-phase composite material, obtained by mixing $\mathbf{A}, \mathbf{B} \in \mathrm{Sym}^4$ in proportions θ and $1-\theta$, respectively. These bounds are based on the *Hashin-Shtrikman variational principle* [40] which applies only when materials are well-ordered, i.e. we assume that

$$\mathbf{A}\boldsymbol{\xi} : \boldsymbol{\xi} \le \mathbf{B}\boldsymbol{\xi} : \boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \mathrm{Sym}.$$

Otherwise, when materials are not well-ordered, one could apply the translation method as was done in [5]. The notion of an optimal bound is introduced in the following definition.

Definition 13 The function $f^-(\theta, \mathbf{A}, \mathbf{B}; \cdot) : \mathrm{Sym} \to \mathbf{R}$ (respectively, $f^+(\theta, \mathbf{A}, \mathbf{B}; \cdot) : \mathrm{Sym} \to \mathbf{R}$), is said to be a lower bound (respectively, an upper bound) if for any $\mathbf{A}^* \in G_\theta$ it holds

$$(\forall \boldsymbol{\xi} \in \mathrm{Sym}) \;\; \mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \geq f^-(\theta, \mathbf{A}, \mathbf{B}; \boldsymbol{\xi}), \quad \text{(respectively, } (\forall \boldsymbol{\xi} \in \mathrm{Sym}) \;\; \mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \leq f^+(\theta, \mathbf{A}, \mathbf{B}; \boldsymbol{\xi})).$$

A lower bound $f^-(\theta, \mathbf{A}, \mathbf{B}; \cdot)$ (respectively, the upper bound $f^+(\theta, \mathbf{A}, \mathbf{B}; \cdot)$) is said to be optimal if for any $\boldsymbol{\xi} \in \mathrm{Sym}$ there exists $\mathbf{A}^* \in G_\theta$ such that

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} = f^-(\theta, \mathbf{A}, \mathbf{B}; \boldsymbol{\xi}) \quad \text{(respectively, } \mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} = f^+(\theta, \mathbf{A}, \mathbf{B}; \boldsymbol{\xi})).$$

Theorem 29 The effective energy of a composite material $\mathbf{A}^* \in G_\theta$ satisfies the following bounds:

$$(\forall \boldsymbol{\xi} \in \mathrm{Sym}) \;\; \mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \geq \mathbf{A}\boldsymbol{\xi} : \boldsymbol{\xi} + (1-\theta) \max_{\boldsymbol{\eta} \in \mathrm{Sym}} [2\boldsymbol{\xi} : \boldsymbol{\eta} - (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - \theta g(\boldsymbol{\eta})], \tag{2.29}$$

where $g(\boldsymbol{\eta})$ is defined by

$$g(\boldsymbol{\eta}) := \sup_{\mathbf{k} \in \mathbf{Z}^d, \mathbf{k} \neq 0} \frac{|(\mathbf{k} \otimes \mathbf{k}) : \boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{k} \otimes \mathbf{k}) : (\mathbf{k} \otimes \mathbf{k})} \tag{2.30}$$

and

$$(\forall \boldsymbol{\xi} \in \mathrm{Sym}) \;\; \mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \leq \mathbf{B}\boldsymbol{\xi} : \boldsymbol{\xi} + \theta \min_{\boldsymbol{\eta} \in \mathrm{Sym}} [2\boldsymbol{\xi} : \boldsymbol{\eta} + (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - (1-\theta) h(\boldsymbol{\eta})], \tag{2.31}$$

where $h(\boldsymbol{\eta})$ is defined by

$$h(\boldsymbol{\eta}) := \inf_{\mathbf{k} \in \mathbf{Z}^d, \mathbf{k} \neq 0} \frac{|(\mathbf{k} \otimes \mathbf{k}) : \boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{k} \otimes \mathbf{k}) : (\mathbf{k} \otimes \mathbf{k})}. \tag{2.32}$$

These bounds are called the Hashin-Shtrikman bounds. Moreover, (2.29) and (2.31) are optimal in the sense of Definition 13 and optimality can be achieved by a rank-p sequential laminate, where $p = \dfrac{(d+3)(d+2)(d+1)d}{24} + 1$.

Remark 10 The supremum and infimum in formulas (2.30) and (2.32) can also be taken over $\mathbf{R}^d \backslash \{0\}$. By using the fact that $\mathbf{Q} \cdot \mathbf{Z}^d$ is dense in $\mathbf{R}^d$ and continuity of functions

$$\mathbf{k} \mapsto \frac{|(\mathbf{k} \otimes \mathbf{k}) : \boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{k} \otimes \mathbf{k}) : (\mathbf{k} \otimes \mathbf{k})} \quad \text{and} \quad \mathbf{k} \mapsto \frac{|(\mathbf{k} \otimes \mathbf{k}) : \boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{k} \otimes \mathbf{k}) : (\mathbf{k} \otimes \mathbf{k})}, \;\; \mathbf{k} \in \mathbf{R}^d \backslash \{0\},$$

we obtain

$$h(\boldsymbol{\eta}) = \inf_{\mathbf{k} \in \mathbf{Z}^d, \mathbf{k} \neq 0} \frac{|(\mathbf{k} \otimes \mathbf{k}) : \boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{k} \otimes \mathbf{k}) : (\mathbf{k} \otimes \mathbf{k})}$$

$$= \inf_{\substack{\mathbf{k}\in\mathbf{Z}^d,\mathbf{k}\neq 0\\ \lambda\in\mathbf{Q}}} \frac{|(\lambda\mathbf{k}\otimes\lambda\mathbf{k}):\boldsymbol{\eta}|^2}{\mathbf{B}(\lambda\mathbf{k}\otimes\lambda\mathbf{k}):(\lambda\mathbf{k}\otimes\lambda\mathbf{k})}$$
$$= \inf_{\mathbf{k}\in\mathbf{R}^d,\mathbf{k}\neq 0} \frac{|(\mathbf{k}\otimes\mathbf{k}):\boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{k}\otimes\mathbf{k}):(\mathbf{k}\otimes\mathbf{k})}.$$

An analogous result can be shown for function g.

Remark 11 It follows straightforwardly from the definition of a convex function that the function under the supremum in the definition of g is convex on Sym. This, together with the fact that the maximum of convex functions is also a convex function [8], implies that g is convex, so the lower Hashin-Shtrikman bound is given as the result of a finite-dimensional concave maximization.

On the other hand, it can be shown that $h(\boldsymbol{\eta}) \leq \mathbf{B}^{-1}\boldsymbol{\eta}:\boldsymbol{\eta}$: for arbitrary $\mathbf{k}\in\mathbf{R}^d\backslash\{0\}$, by substituting $\mathbf{k}\otimes\mathbf{k} = \mathbf{B}^{-\frac{1}{2}}\mathbf{X}$, $\mathbf{X}\in$ Sym, and by using the Cauchy–Bunyakovsky–Schwarz inequality, we obtain that

$$\frac{|(\mathbf{k}\otimes\mathbf{k}):\boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{k}\otimes\mathbf{k}):(\mathbf{k}\otimes\mathbf{k})} \leq \mathbf{B}^{-1}\boldsymbol{\eta}:\boldsymbol{\eta},$$

hence $h(\boldsymbol{\eta}) \leq \mathbf{B}^{-1}\boldsymbol{\eta}:\boldsymbol{\eta}$. Together with $(\mathbf{B}-\mathbf{A})^{-1} - (1-\theta)\mathbf{B}^{-1} \geq 0$, this implies that the upper Hashin-Shtrikman bound is given as the result of a finite-dimensional convex minimization.

Proof of Theorem 29. Since $\mathrm{Cl}P_\theta = G_\theta$, it is enough to prove that these bounds hold for $\mathbf{A}^*$ obtained by periodic homogenization. We begin with the lower Hashin-Shtrikman bound. By using the weak formulation of the periodic boundary value problem (1.27) introduced in Theorem 18, with $\mathbf{M}(\boldsymbol{y}) = \chi(\boldsymbol{y})\mathbf{A} + (1-\chi(\boldsymbol{y}))\mathbf{B}$, we have that

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi} = \min_{w\in \mathrm{H}^2_\#(Y)} \int_Y (\chi(\boldsymbol{y})\mathbf{A} + (1-\chi(\boldsymbol{y}))\mathbf{B})(\boldsymbol{\xi} + \nabla\nabla w(\boldsymbol{y})):(\boldsymbol{\xi} + \nabla\nabla w(\boldsymbol{y}))\,d\boldsymbol{y}, \tag{2.33}$$

where $\boldsymbol{\xi}\in$ Sym and $\chi\in \mathrm{L}^\infty_\#(Y;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$ is a characteristic function such that $\int_Y \chi(\boldsymbol{y})\,d\boldsymbol{y} = \theta$. If we subtract and add from $\chi\mathbf{A} + (1-\chi)\mathbf{B}$ the tensor $\mathbf{A}$, we obtain

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi} = \min_{w\in \mathrm{H}^2_\#(Y)} \Bigg(\int_Y (1-\chi(\boldsymbol{y}))(\mathbf{B}-\mathbf{A})(\boldsymbol{\xi}+\nabla\nabla w(\boldsymbol{y})):(\boldsymbol{\xi}+\nabla\nabla w(\boldsymbol{y}))\,d\boldsymbol{y} + \\ + \int_Y \mathbf{A}(\boldsymbol{\xi}+\nabla\nabla w(\boldsymbol{y})):(\boldsymbol{\xi}+\nabla\nabla w(\boldsymbol{y}))\,d\boldsymbol{y}\Bigg). \tag{2.34}$$

We denote $\boldsymbol{\zeta} := \boldsymbol{\xi} + \nabla\nabla w \in L^2_\#(Y; \mathrm{Sym})$, and

$$\varphi(\boldsymbol{\zeta}) := \int_Y (1-\chi(\boldsymbol{y}))(\mathbf{B}-\mathbf{A})\boldsymbol{\zeta}(\boldsymbol{y}) : \boldsymbol{\zeta}(\boldsymbol{y})\, d\boldsymbol{y}.$$

Since $\mathbf{B}-\mathbf{A} \geq 0$, by using the Legendre transform, one can easily conclude that

$$\begin{aligned}\varphi^*(\boldsymbol{\eta}) &= \sup_{\boldsymbol{\zeta}\in L^2_\#(Y;\mathrm{Sym})} \left[\int_Y \boldsymbol{\zeta}(\boldsymbol{y}) : \boldsymbol{\eta}(\boldsymbol{y})\, d\boldsymbol{y} - \varphi(\boldsymbol{\zeta})\right] \\ &= \frac{1}{4}\int_Y (1-\chi(\boldsymbol{y}))(\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}(\boldsymbol{y}) : \boldsymbol{\eta}(\boldsymbol{y})\, d\boldsymbol{y}.\end{aligned}$$

Next, by the Fenchel-Moreau theorem

$$\begin{aligned}\varphi(\boldsymbol{\zeta}) &= (\varphi^*)^*(\boldsymbol{\zeta}) \\ &= \sup_{\boldsymbol{\eta}\in L^2_\#(Y;\mathrm{Sym})} \left[\int_Y \boldsymbol{\zeta}(\boldsymbol{y}) : \boldsymbol{\eta}(\boldsymbol{y})\, d\boldsymbol{y} - \varphi^*(\boldsymbol{\eta})\right] \\ &= \max_{\boldsymbol{\eta}\in L^2_\#(Y;\mathrm{Sym})} \int_Y (1-\chi(\boldsymbol{y}))\left[2(\boldsymbol{\xi}+\nabla\nabla w(\boldsymbol{y})) : \boldsymbol{\eta}(\boldsymbol{y}) - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}(\boldsymbol{y}) : \boldsymbol{\eta}(\boldsymbol{y})\right] d\boldsymbol{y} \\ &\geq \max_{\boldsymbol{\eta}\in\mathrm{Sym}} \int_Y (1-\chi(\boldsymbol{y}))\left[2(\boldsymbol{\xi}+\nabla\nabla w(\boldsymbol{y})) : \boldsymbol{\eta} - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta}\right] d\boldsymbol{y} \\ &\geq \int_Y (1-\chi(\boldsymbol{y}))\left[2(\boldsymbol{\xi}+\nabla\nabla w(\boldsymbol{y})) : \boldsymbol{\eta} - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta}\right] d\boldsymbol{y},\end{aligned}$$

hence

$$\varphi(\boldsymbol{\zeta}) \geq (1-\theta)\left[2\boldsymbol{\xi} : \boldsymbol{\eta} - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta}\right] - 2\int_Y \chi(\boldsymbol{y})\nabla\nabla w(\boldsymbol{y}) : \boldsymbol{\eta}\, d\boldsymbol{y}, \quad \boldsymbol{\eta}\in\mathrm{Sym}. \tag{2.35}$$

Furthermore, by periodicity of $w \in \mathrm{H}^2_\#(Y)$ it follows:

$$\int_Y \mathbf{A}(\boldsymbol{\xi}+\nabla\nabla w(\boldsymbol{y})) : (\boldsymbol{\xi}+\nabla\nabla w(\boldsymbol{y}))\, d\boldsymbol{y} = \mathbf{A}\boldsymbol{\xi} : \boldsymbol{\xi} + \int_Y \mathbf{A}\nabla\nabla w(\boldsymbol{y}) : \nabla\nabla w(\boldsymbol{y})\, d\boldsymbol{y}. \tag{2.36}$$

By using (2.34), (2.35) and (2.36), similarly as in [2, Proposition 2.2.6], we conclude that

$$\begin{aligned}\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \geq \min_{w\in\mathrm{H}^2_\#(Y)} \Bigg((1-\theta)\left(2\boldsymbol{\xi} : \boldsymbol{\eta} - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta}\right) - 2\int_Y \chi(\boldsymbol{y})\nabla\nabla w(\boldsymbol{y}) : \boldsymbol{\eta}\, d\boldsymbol{y} + \\ + \mathbf{A}\boldsymbol{\xi} : \boldsymbol{\xi} + \int_Y \mathbf{A}\nabla\nabla w(\boldsymbol{y}) : \nabla\nabla w(\boldsymbol{y})\, d\boldsymbol{y}\Bigg) = \\ = \mathbf{A}\boldsymbol{\xi} : \boldsymbol{\xi} + (1-\theta)\left(2\boldsymbol{\xi} : \boldsymbol{\eta} - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta}\right) - g(\chi, \boldsymbol{\eta}),\end{aligned}$$

where

$$g(\chi,\boldsymbol{\eta}) = -\min_{w\in \mathrm{H}^2_{\#}(Y)} \left(\int_Y \mathbf{A}\nabla\nabla w(\boldsymbol{y}) : \nabla\nabla w(\boldsymbol{y})\, d\boldsymbol{y} - 2\int_Y \chi(\boldsymbol{y})\nabla\nabla w(\boldsymbol{y}) : \boldsymbol{\eta}\, d\boldsymbol{y} \right). \tag{2.37}$$

To obtain an explicit formula for g, one can use Fourier analysis since (2.37) represents minimization over periodic functions. Note that the function to be minimized in (2.37) involves only bilinear terms depending on $\boldsymbol{y}$. Thus, by periodicity, one can write χ and w as

$$\chi(\boldsymbol{y}) = \sum_{\mathbf{k}\in\mathbf{Z}^d} \hat{\chi}(\mathbf{k}) e^{2\pi i \mathbf{k}\cdot\boldsymbol{y}},$$
$$w(\boldsymbol{y}) = \sum_{\mathbf{k}\in\mathbf{Z}^d} \hat{w}(\mathbf{k}) e^{2\pi i \mathbf{k}\cdot\boldsymbol{y}}.$$

These are complex Fourier series [31, 37], and since functions χ and w are real valued, their Fourier coefficients must satisfy $\overline{\hat{\chi}(\mathbf{k})} = \hat{\chi}(-\mathbf{k})$ and $\overline{\hat{w}(\mathbf{k})} = \hat{w}(-\mathbf{k})$. The Parseval's relation gives:

$$\begin{aligned}
&\int_Y \mathbf{A}\nabla\nabla w(\boldsymbol{y}) : \nabla\nabla w(\boldsymbol{y})\, d\boldsymbol{y} - 2\int_Y \chi(\boldsymbol{y})\nabla\nabla w(\boldsymbol{y}) : \boldsymbol{\eta}\, d\boldsymbol{y} \\
&= \sum_{\mathbf{k}\in\mathbf{Z}^d} \left(16\pi^4 |\hat{w}(\mathbf{k})|^2 \mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k}) + 8\pi^2 \overline{\hat{\chi}(\mathbf{k})}\hat{w}(\mathbf{k})\mathbf{k}\otimes\mathbf{k} : \boldsymbol{\eta}\right) \\
&= \sum_{\mathbf{k}\in\mathbf{Z}^d} \left(16\pi^4 |\hat{w}(\mathbf{k})|^2 \mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k}) + 8\pi^2 \mathrm{Re}(\overline{\hat{\chi}(\mathbf{k})}\hat{w}(\mathbf{k})\mathbf{k}\otimes\mathbf{k} : \boldsymbol{\eta})\right).
\end{aligned} \tag{2.38}$$

The minimization over $w \in \mathrm{H}^2_{\#}(Y)$ in (2.37) is equivalent to the minimization over $\hat{w}(\mathbf{k}) \in \mathbf{C}$, for each $\mathbf{k} \in \mathbf{Z}^d$, and the minimization can be performed on each component of the sum (2.38) independently. Note that $\mathbf{k} = 0$ contributes nothing to the sum (2.38). Now, it is easily seen that the above minimum is attained when

$$\hat{w}(\mathbf{k}) = \frac{-\hat{\chi}(\mathbf{k})(\mathbf{k}\otimes\mathbf{k}) : \boldsymbol{\eta}}{4\pi^2 \mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})}, \quad \mathbf{k} \in \mathbf{Z}^d \backslash \{0\}.$$

Therefore

$$\begin{aligned}
g(\chi,\boldsymbol{\eta}) &= \sum_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq 0} \frac{|\hat{\chi}(\mathbf{k})|^2 |(\mathbf{k}\otimes\mathbf{k}) : \boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})} \\
&\leq \sup_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq 0} \frac{|(\mathbf{k}\otimes\mathbf{k}) : \boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})} \sum_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq 0} |\hat{\chi}(\mathbf{k})|^2 \\
&= g(\boldsymbol{\eta}) \sum_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq 0} |\hat{\chi}(\mathbf{k})|^2,
\end{aligned}$$

where $g(\boldsymbol{\eta})$ is given by (2.30). Now, the Parseval's relation implies

$$\sum_{\mathbf{k}\in\mathbf{Z}^d,\mathbf{k}\neq 0} |\hat{\chi}(\mathbf{k})|^2 = \int_Y \chi^2(\boldsymbol{y})\,d\boldsymbol{y} - \theta^2 = \int_Y \chi(\boldsymbol{y})\,d\boldsymbol{y} - \theta^2 = \theta(1-\theta),$$

which yields the lower Hashin-Shtrikman bound. The upper Hashin-Shtrikman bound can be derived analogously.

Let us now prove that the lower bound is optimal in the sense of Definition 13: first, we denote

$$\phi(\boldsymbol{\eta}) := 2\boldsymbol{\xi} : \boldsymbol{\eta} - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - \theta g(\boldsymbol{\eta}). \tag{2.39}$$

As ϕ is strictly concave in $\boldsymbol{\eta}$ (see Remark 11) and $-\phi$ is coercive, there exists a unique maximum point $\boldsymbol{\eta}^*$ of ϕ. Since ϕ is usually not a smooth function, the first-order (both necessary and sufficient) optimality condition for (2.39) reads $\mathsf{0} \in \partial\phi(\boldsymbol{\eta}^*)$, where $\partial\phi(\boldsymbol{\eta}^*)$ is the subdifferential of function ϕ at $\boldsymbol{\eta}^*$. To calculate the subdifferential of ϕ, one should calculate the subdifferential of g.

By simple change $\mathbf{k} = \|\mathbf{k}\|\mathbf{e}$, $\mathbf{e} \in S^{d-1}$, we have

$$g(\boldsymbol{\eta}) = \sup_{\mathbf{k}\in\mathbf{Z}^d,\mathbf{k}\neq 0} \frac{|(\mathbf{k}\otimes\mathbf{k}) : \boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})} = \max_{\mathbf{e}\in S^{d-1}} \frac{|(\mathbf{e}\otimes\mathbf{e}) : \boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}. \tag{2.40}$$

We denote

$$M(\boldsymbol{\eta}) = \left\{\mathbf{e} \in S^{d-1} : g(\boldsymbol{\eta}) = \frac{|(\mathbf{e}\otimes\mathbf{e}) : \boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}\right\}$$

and let $P[M(\boldsymbol{\eta})]$ be the collection of probability Radon measures on the unit sphere, supported on $M(\boldsymbol{\eta})$. By Theorem 20 it follows that

$$\begin{aligned}\partial g(\boldsymbol{\eta}) &= \left\{2\int_{S^{d-1}} \frac{(\mathbf{e}\otimes\mathbf{e}) : \boldsymbol{\eta}}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}\mathbf{e}\otimes\mathbf{e}\,d\nu(\mathbf{e}) : \nu \in P[M(\boldsymbol{\eta})]\right\} \\ &= \left\{2\left(\int_{S^{d-1}} \frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}\,d\nu(\mathbf{e})\right)\boldsymbol{\eta} : \nu \in P[M(\boldsymbol{\eta})]\right\}.\end{aligned}$$

Now, let X be the space of fully symmetric fourth-order tensors $\mathbf{M}$ satisfying

$$M_{ijkl} = M_{kjil} = M_{klij} = M_{jikl},\;\; 1 \le i,j,k,l \le d$$

with dimension $n = \dfrac{(d+3)(d+2)(d+1)d}{24}$, and we denote

$$K := \left\{\int_{S^{d-1}} \frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}\,d\nu(\mathbf{e}) : \nu \in P[M(\boldsymbol{\eta})]\right\}.$$

Since K is a nonempty, closed, bounded, convex subset of a vector space X of finite dimension n, by Carathéodory's theorem (see Theorem 24) and Lemma 7, every point from K can be represented as a convex combination of at most $n+1$ extreme points of K, i.e. as

$$\sum_{i=1}^{n+1} m_i \frac{(\mathbf{e}_i \otimes \mathbf{e}_i) \otimes (\mathbf{e}_i \otimes \mathbf{e}_i)}{\mathbf{A}(\mathbf{e}_i \otimes \mathbf{e}_i) : (\mathbf{e}_i \otimes \mathbf{e}_i)},$$

for some $\mathbf{e}_i \in M(\boldsymbol{\eta})$ and $m_i \geq 0$, such that $\sum_{i=1}^{n+1} m_i = 1$. Now, the first-order optimality condition for maximizing (2.39): $\boldsymbol{\eta}^*$ is optimal if and only if

$$2\boldsymbol{\xi} - 2(\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta}^* \in \theta \partial g(\boldsymbol{\eta}^*), \tag{2.41}$$

which is equivalent to

$$\boldsymbol{\xi} - (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta}^* - \theta \sum_{i=1}^{n+1} m_i \frac{(\mathbf{e}_i \otimes \mathbf{e}_i) \otimes (\mathbf{e}_i \otimes \mathbf{e}_i)}{\mathbf{A}(\mathbf{e}_i \otimes \mathbf{e}_i) : (\mathbf{e}_i \otimes \mathbf{e}_i)} \boldsymbol{\eta}^* = 0, \tag{2.42}$$

for some $\mathbf{e}_i \in M(\boldsymbol{\eta}^*)$ and $m_i \geq 0$, such that $\sum_{i=1}^{n+1} m_i = 1$. Note that in deriving (2.41), the symmetry of $\mathbf{A}$ and $\mathbf{B}$ is used, otherwise the optimality condition would be given as

$$2\boldsymbol{\xi} - (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta}^* - \left((\mathbf{B} - \mathbf{A})^{-1}\right)^T \boldsymbol{\eta}^* \in \theta \partial g(\boldsymbol{\eta}^*),$$

which would significantly complicate further algebraic calculations.

Taking the inner product of (2.42) with $\boldsymbol{\eta}^*$ gives:

$$\begin{aligned} \boldsymbol{\xi} : \boldsymbol{\eta}^* &= (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta}^* : \boldsymbol{\eta}^* + \theta \sum_{i=1}^{n+1} m_i \frac{(\mathbf{e}_i \otimes \mathbf{e}_i) \otimes (\mathbf{e}_i \otimes \mathbf{e}_i)}{\mathbf{A}(\mathbf{e}_i \otimes \mathbf{e}_i) : (\mathbf{e}_i \otimes \mathbf{e}_i)} \boldsymbol{\eta}^* : \boldsymbol{\eta}^* \\ &= (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta}^* : \boldsymbol{\eta}^* + \theta g(\boldsymbol{\eta}^*). \end{aligned} \tag{2.43}$$

By (2.39) and (2.43) we have

$$\begin{aligned} \phi(\boldsymbol{\eta}^*) &= \boldsymbol{\xi} : \boldsymbol{\eta}^* + \boldsymbol{\xi} : \boldsymbol{\eta}^* - (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta}^* : \boldsymbol{\eta}^* - \theta g(\boldsymbol{\eta}^*) \\ &= \boldsymbol{\xi} : \boldsymbol{\eta}^*. \end{aligned}$$

Thus, the lower bound (2.29) can be expressed as

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \geq \mathbf{A}\boldsymbol{\xi} : \boldsymbol{\xi} + (1 - \theta)\boldsymbol{\xi} : \boldsymbol{\eta}^*. \tag{2.44}$$

In order to achieve equality in (2.44), let us consider the sequential laminate given by formula (2.28):

$$(1-\theta)(\mathbf{A}^*_{n+1}-\mathbf{A})^{-1} = (\mathbf{B}-\mathbf{A})^{-1} + \theta \sum_{i=1}^{n+1} m_i \frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{A}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)}. \tag{2.45}$$

Multiplying (2.45) by $\boldsymbol{\eta}^*$ and using (2.42) gives

$$(1-\theta)(\mathbf{A}^*_{n+1}-\mathbf{A})^{-1}\boldsymbol{\eta}^* = \boldsymbol{\xi}, \tag{2.46}$$

which equals

$$\mathbf{A}^*_{n+1}\boldsymbol{\xi} = (1-\theta)\boldsymbol{\eta}^* + \mathbf{A}\boldsymbol{\xi}. \tag{2.47}$$

After taking the inner product of (2.47) with $\boldsymbol{\xi}$, we obtain

$$\mathbf{A}^*_{n+1}\boldsymbol{\xi}:\boldsymbol{\xi} = (1-\theta)\boldsymbol{\eta}^*:\boldsymbol{\xi} + \mathbf{A}\boldsymbol{\xi}:\boldsymbol{\xi} = (1-\theta)\phi(\boldsymbol{\eta}^*) + \mathbf{A}\boldsymbol{\xi}:\boldsymbol{\xi},$$

thus optimality is achieved by a sequential laminate of rank $n+1$ given with (2.28), where $\mathbf{e}_i \in M(\boldsymbol{\eta}^*)$ and $m_i \geq 0$, such that $\sum_{i=1}^{n+1} m_i = 1$. For the upper Hashin-Shtrikman bound a similar conclusion holds, i.e. the optimality is achieved by a finite-rank sequential laminate (2.25). ■

The previous theorem extends easily to the sum of energies.

Theorem 30 For any $\boldsymbol{\xi}_1, \ldots \boldsymbol{\xi}_p \in \mathrm{Sym}$, a homogenized tensor $\mathbf{A}^* \in G_\theta$ satisfies the following bounds:

$$\sum_{i=1}^{p} \mathbf{A}^*\boldsymbol{\xi}_i:\boldsymbol{\xi}_i \geq \sum_{i=1}^{p} \mathbf{A}\boldsymbol{\xi}_i:\boldsymbol{\xi}_i + (1-\theta) \max_{\substack{\boldsymbol{\eta}_i\in\mathrm{Sym}\\ i=1,\ldots,p}} \left[\sum_{i=1}^{p}\left(2\boldsymbol{\xi}_i:\boldsymbol{\eta}_i - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}_i:\boldsymbol{\eta}_i\right) - \theta g(\boldsymbol{\eta}_1,\ldots,\boldsymbol{\eta}_p)\right], \tag{2.48}$$

where $g(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_p)$ is defined by

$$g(\boldsymbol{\eta}_1,\ldots,\boldsymbol{\eta}_p) := \sup_{\mathbf{k}\in\mathbf{Z}^d,\mathbf{k}\neq\mathbf{0}} \sum_{i=1}^{p} \frac{|(\mathbf{k}\otimes\mathbf{k}):\boldsymbol{\eta}_i|^2}{\mathbf{A}(\mathbf{k}\otimes\mathbf{k}):(\mathbf{k}\otimes\mathbf{k})} \tag{2.49}$$

and

$$\sum_{i=1}^{p} \mathbf{A}^*\boldsymbol{\xi}_i:\boldsymbol{\xi}_i \leq \sum_{i=1}^{p} \mathbf{B}\boldsymbol{\xi}_i:\boldsymbol{\xi}_i + \theta \min_{\substack{\boldsymbol{\eta}_i\in\mathrm{Sym}\\ i=1,\ldots,p}} \left[\sum_{i=1}^{p}\left(2\boldsymbol{\xi}_i:\boldsymbol{\eta}_i + (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}_i:\boldsymbol{\eta}_i\right) - (1-\theta) h(\boldsymbol{\eta}_1,\ldots,\boldsymbol{\eta}_p)\right], \tag{2.50}$$

where $h(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_p)$ is defined by

$$h(\boldsymbol{\eta}_1,\ldots,\boldsymbol{\eta}_p) := \inf_{\mathbf{k}\in\mathbf{Z}^d,\mathbf{k}\neq\mathbf{0}} \sum_{i=1}^{p} \frac{|(\mathbf{k}\otimes\mathbf{k}):\boldsymbol{\eta}_i|^2}{\mathbf{B}(\mathbf{k}\otimes\mathbf{k}):(\mathbf{k}\otimes\mathbf{k})}.$$

Moreover, (2.48) and (2.50) are optimal in the sense of Definition 13 and optimality can be achieved by a rank-p sequential laminate, where $p = \dfrac{(d+3)(d+2)(d+1)d}{24} + 1$.

Proof. The proof goes along the same lines as the proof for a single effective energy. Let us just highlight the main differences (only for the lower Hashin-Shtrikman bound, as it was done in the proof of Theorem 29). We have:

$$\sum_{i=1}^{p} \mathbf{A}^*\boldsymbol{\xi}_i : \boldsymbol{\xi}_i = \sum_{i=1}^{p} \left(\min_{w_i \in \mathrm{H}^2_{\#}(Y)} \int_Y (\chi(\boldsymbol{y})\mathbf{A} + (1-\chi(\boldsymbol{y}))\mathbf{B})(\boldsymbol{\xi}_i + \nabla\nabla w_i(\boldsymbol{y})) : (\boldsymbol{\xi}_i + \nabla\nabla w_i(\boldsymbol{y}))\, d\boldsymbol{y} \right)$$
$$\geq \sum_{i=1}^{p} \Big(\mathbf{A}\boldsymbol{\xi}_i : \boldsymbol{\xi}_i + (1-\theta)\left(2\boldsymbol{\xi}_i : \boldsymbol{\eta}_i - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}_i : \boldsymbol{\eta}_i\right) - g(\chi, \boldsymbol{\eta}_i) \Big),$$

where

$$g(\chi, \boldsymbol{\eta}_i) = - \min_{w_i \in \mathrm{H}^2_{\#}(Y)} \left(\int_Y \mathbf{A}\nabla\nabla w_i(\boldsymbol{y}) : \nabla\nabla w_i(\boldsymbol{y})\, d\boldsymbol{y} - 2\int_Y \chi(\boldsymbol{y})\nabla\nabla w_i(\boldsymbol{y}) : \boldsymbol{\eta}_i\, d\boldsymbol{y} \right).$$

Again, by using Fourier analysis, we obtain

$$\begin{aligned}
\sum_{i=1}^{p} g(\chi, \boldsymbol{\eta}_i) &= \sum_{i=1}^{p} \left(\sum_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq 0} \frac{|\hat{\chi}(\mathbf{k})|^2 |(\mathbf{k}\otimes\mathbf{k}) : \boldsymbol{\eta}_i|^2}{\mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})} \right) \\
&= \sum_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq 0} \left(\sum_{i=1}^{p} \frac{|\hat{\chi}(\mathbf{k})|^2 |(\mathbf{k}\otimes\mathbf{k}) : \boldsymbol{\eta}_i|^2}{\mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})} \right) \\
&= \sum_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq 0} |\hat{\chi}(\mathbf{k})|^2 \sum_{i=1}^{p} \frac{|(\mathbf{k}\otimes\mathbf{k}) : \boldsymbol{\eta}_i|^2}{\mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})} \\
&\leq \sup_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq 0} \sum_{i=1}^{p} \frac{|(\mathbf{k}\otimes\mathbf{k}) : \boldsymbol{\eta}_i|^2}{\mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})} \sum_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq 0} |\hat{\chi}(\mathbf{k})|^2 \\
&= \theta(1-\theta) g(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_p),
\end{aligned}$$

where $g(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_p)$ is given with (2.49). This yields the lower Hashin-Shtrikman bound. The upper Hashin-Shtrikman bound can be derived analogously.

Let us now prove the optimality of the lower Hashin-Shtrikman bound on the sum of energies. Similarly as before, we denote:

$$\phi(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_p) := \sum_{i=1}^{p} \Big(2\boldsymbol{\xi}_i : \boldsymbol{\eta}_i - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}_i : \boldsymbol{\eta}_i \Big) - \theta g(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_p),$$

which is a strictly concave function in $(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_p)$, such that $-\phi$ is coercive, and consequently there exists a unique maximum point $(\boldsymbol{\eta}_1^*, \ldots, \boldsymbol{\eta}_p^*)$ of ϕ. Furthermore, the same arguments as in the proof of Theorem 29 show that the first-order optimality condition

$$\sum_{i=1}^{p} (2\boldsymbol{\xi}_i - 2(\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}_i^*) \in \theta\partial g(\boldsymbol{\eta}_1^*, \ldots, \boldsymbol{\eta}_p^*)$$

is equivalent to

$$\sum_{i=1}^{p}(\boldsymbol{\xi}_i - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}_i^*) - \theta\sum_{i=1}^{p}\sum_{k=1}^{n+1} m_k \frac{(\mathbf{e}_k\otimes\mathbf{e}_k)\otimes(\mathbf{e}_k\otimes\mathbf{e}_k)}{\mathbf{A}(\mathbf{e}_k\otimes\mathbf{e}_k):(\mathbf{e}_k\otimes\mathbf{e}_k)}\boldsymbol{\eta}_i^* = 0,$$

for some $\mathbf{e}_k \in M(\boldsymbol{\eta}_1^*, \ldots, \boldsymbol{\eta}_p^*)$ and $m_k \geq 0$, such that $\sum_{k=1}^{n+1} m_k = 1$. By using this, an analogous reasoning as for the single effective energy implies the claim. ∎

The following corollary is an immediate consequence of the Theorem 29.

Corollary 3 Let $\mathbf{A}^* \in G_\theta$. A lower Hashin-Shtrikman bound (2.29) is equivalent to

$$(\forall\boldsymbol{\eta}\in \mathrm{Sym})\ \ (1-\theta)(\mathbf{A}^*-\mathbf{A})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} \leq (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} + \theta g(\boldsymbol{\eta}).$$

Similarly, an upper Hashin-Shtrikman bound (2.31) is equivalent to

$$(\forall\boldsymbol{\eta}\in \mathrm{Sym})\ \ \theta(\mathbf{B}-\mathbf{A}^*)^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} \leq (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} - (1-\theta)h(\boldsymbol{\eta}).$$

Proof. Let us start with the upper bound. First, we denote

$$\varphi_1(\boldsymbol{\xi}) := \frac{1}{\theta}(\mathbf{B}-\mathbf{A}^*)\boldsymbol{\xi}:\boldsymbol{\xi},$$

and

$$\begin{aligned}\varphi_2(\boldsymbol{\xi}) &= -\min_{\boldsymbol{\eta}\in\mathrm{Sym}}[2\boldsymbol{\xi}:\boldsymbol{\eta} + (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} - (1-\theta)h(\boldsymbol{\eta})] \\ &= \max_{\boldsymbol{\eta}\in\mathrm{Sym}}[2\boldsymbol{\xi}:\boldsymbol{\eta} - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} + (1-\theta)h(\boldsymbol{\eta})].\end{aligned}$$

Obviously, (2.31) is equivalent to $\varphi_1(\boldsymbol{\xi}) \geq \varphi_2(\boldsymbol{\xi})$, $\boldsymbol{\xi}\in\mathrm{Sym}$, and φ_1, φ_2 are convex functions. By Fenchel-Moreau theorem, the Legendre transform of φ_2^* is again φ_2:

$$\begin{aligned}(\varphi_2^*)^*(\boldsymbol{\xi}) = \varphi_2(\boldsymbol{\xi}) &= \max_{\boldsymbol{\eta}\in\mathrm{Sym}}\left[2\boldsymbol{\xi}:\boldsymbol{\eta} - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} + (1-\theta)h(\boldsymbol{\eta})\right] \\ &= \max_{\boldsymbol{\eta}\in\mathrm{Sym}}\left[\boldsymbol{\xi}:\boldsymbol{\eta} - \frac{1}{4}\Big((\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} - (1-\theta)h(\boldsymbol{\eta})\Big)\right] \\ &= \sup_{\boldsymbol{\eta}\in\mathrm{Sym}}\left[\boldsymbol{\xi}:\boldsymbol{\eta} - \varphi_2^*(\boldsymbol{\eta})\right],\end{aligned} \tag{2.51}$$

where the third equality in (2.51) follows after replacing $\boldsymbol{\eta}$ by $\frac{\boldsymbol{\eta}}{2}$. Thus, by Remark 6

$$\varphi_2^*(\boldsymbol{\eta}) = \frac{1}{4}\Big((\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} - (1-\theta)h(\boldsymbol{\eta})\Big).$$

Similarly,

$$\begin{aligned}\varphi_1^*(\boldsymbol{\eta}) &= \sup_{\boldsymbol{\xi}\in\mathrm{Sym}}\left[\boldsymbol{\eta}:\boldsymbol{\xi} - \frac{1}{\theta}(\mathbf{B}-\mathbf{A}^*)\boldsymbol{\xi}:\boldsymbol{\xi}\right] \\ &= \frac{\theta}{4}(\mathbf{B}-\mathbf{A}^*)^{-1}\boldsymbol{\eta}:\boldsymbol{\eta}.\end{aligned}$$

Since φ_1 and φ_2 are convex and continuous, the inequality

$$\varphi_1(\boldsymbol{\xi}) \geq \varphi_2(\boldsymbol{\xi}), \;\; \boldsymbol{\xi} \in \mathrm{Sym}$$

is equivalent to

$$\varphi_1^*(\boldsymbol{\eta}) \leq \varphi_2^*(\boldsymbol{\eta}), \;\; \boldsymbol{\eta} \in \mathrm{Sym}.$$

In other words, an upper Hashin-Shtrikman bound is equivalent to

$$\theta(\mathbf{B}-\mathbf{A}^*)^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} \leq (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} - (1-\theta)h(\boldsymbol{\eta}), \;\; \boldsymbol{\eta}\in\mathrm{Sym}.$$

The assertion for the lower bound can be proved analogously. ■

2.5 Hashin-Shtrikman bounds on the complementary energy and lamination formulas

In this section we want to derive bounds on the complementary energy of a two-phase composite material, obtained by mixing $\mathbf{A}, \mathbf{B} \in \mathrm{Sym}^4$ in proportions θ and $1-\theta$, respectively. Furthermore, we assume that

$$\mathbf{A}\boldsymbol{\xi}:\boldsymbol{\xi} \leq \mathbf{B}\boldsymbol{\xi}:\boldsymbol{\xi}, \tag{2.52}$$

for any $\boldsymbol{\xi} \in \mathrm{Sym}$.

Recall that, in the case of the primal energy, when considering bounds on $\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi}$, we concluded that optimality of Hashin-Shtrikman bounds is achieved by a finite-rank sequential laminate. Now, when our focus is on ${\mathbf{A}^*}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}$, first we want to derive lamination formulas in terms of $\mathbf{A}_p^{*-1}$, where $\mathbf{A}_p^*$ is a rank-p sequential laminate with core $\mathbf{A}$ and matrix $\mathbf{B}$, and with lamination directions $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_p$.

Let us briefly recall the following result: if $\mathbf{A}$ and $\mathbf{B}$ are two constant tensors in $\mathfrak{M}_2(\alpha,\beta;\Omega)$ and $\chi_n(\mathbf{x}\cdot\mathbf{e})$ a sequence of characteristic functions that converges to $\theta(\mathbf{x}\cdot\mathbf{e})$ in $\mathrm{L}^\infty(\Omega;[0,1])$ weakly-$*$, then the sequence $(\mathbf{M}^n)$ of tensors in $\mathfrak{M}_2(\alpha,\beta;\Omega)$, defined as

$$\mathbf{M}^n(\mathbf{x}\cdot\mathbf{e}) = \chi_n(\mathbf{x}\cdot\mathbf{e})\mathbf{A} + (1-\chi_n(\mathbf{x}\cdot\mathbf{e}))\mathbf{B}$$

H-converges to

$$\mathbf{M} = \theta\mathbf{A} + (1-\theta)\mathbf{B} - \frac{\theta(1-\theta)(\mathbf{A}-\mathbf{B})(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{A}-\mathbf{B})^T(\mathbf{e}\otimes\mathbf{e})}{(1-\theta)\mathbf{A}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e}) + \theta\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}, \tag{2.53}$$

which also depends only on $\mathbf{x}\cdot\mathbf{e}$. By Corollary 2, if $(\mathbf{A}-\mathbf{B})$ is an invertible fourth-order tensor, then formula (2.53) for $\mathbf{M}$ is equivalent to

$$\theta(\mathbf{M}-\mathbf{B})^{-1} = (\mathbf{A}-\mathbf{B})^{-1} + \frac{1-\theta}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e}). \tag{2.54}$$

Before we proceed further, let us mention that the basic tool for deriving desired lamination formulas will be the following identity:

$$(\mathbf{B}-\mathbf{A})^{-1} = \mathbf{A}^{-1}(\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}\mathbf{A}^{-1} - \mathbf{A}^{-1}, \tag{2.55}$$

which holds for two invertible fourth-order tensors $\mathbf{A}$ and $\mathbf{B}$, such that $\mathbf{B}-\mathbf{A}$ and $\mathbf{A}^{-1}-\mathbf{B}^{-1}$ are also invertible. Note that this is satisfied by constant tensors $\mathbf{A}, \mathbf{B} \in \mathfrak{M}_2(\alpha,\beta;\Omega)$ defined with (2.52).

By applying identitity (2.55) to terms $(\mathbf{M}-\mathbf{B})^{-1}$ and $(\mathbf{A}-\mathbf{B})^{-1}$, we obtain that (2.54) is equivalent to

$$\theta(\mathbf{M}^{-1}-\mathbf{B}^{-1})^{-1} = (\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1} + (1-\theta)\left[\mathbf{B} - \mathbf{B}\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\mathbf{B}\right]. \tag{2.56}$$

For the purpose of deriving the desired formula in the terms of $\mathbf{A}_p^{*-1}$, let us recall the following statement:

Lemma 13 Let θ be a volume fraction in $[0,1]$, $(\mathbf{e}_i)_{1\le i\le p}$ unit vectors in $\mathbf{R}^d$, $(m_i)_{1\le i\le p}$ nonnegative real numbers such that $\sum_{i=1}^{p} m_i = 1$. Then, there exists a rank-p sequential laminate $\mathbf{A}_p^*$ with core $\mathbf{A}$ and matrix $\mathbf{B}$, and with lamination directions $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_p$, such that

$$\theta(\mathbf{A}_p^*-\mathbf{B})^{-1} = (\mathbf{A}-\mathbf{B})^{-1} + (1-\theta)\sum_{i=1}^{p} m_i \frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)}. \tag{2.57}$$

Now, after applying the identity (2.55) to terms $(\mathbf{A}_p^*-\mathbf{B})^{-1}$ and $(\mathbf{A}-\mathbf{B})^{-1}$, it follows that (2.57) is equivalent to

$$\theta(\mathbf{A}_p^{*-1}-\mathbf{B}^{-1})^{-1} = (\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1} + (1-\theta)\left[\mathbf{B} - \sum_{i=1}^{p} m_i\mathbf{B}\frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)}\mathbf{B}\right]. \tag{2.58}$$

One could interchange the roles of $\mathbf{A}$ and $\mathbf{B}$ and obtain a symmetric class of sequential laminates written in terms of $\mathbf{A}_p^{*-1}$:

Lemma 14 Let θ be a volume fraction in $[0,1]$, $(\mathbf{e}_i)_{1\le i\le p}$ unit vectors in $\mathbf{R}^d$, $(m_i)_{1\le i\le p}$

nonnegative real numbers such that $\sum_{i=1}^{p} m_i = 1$. Then, there exists a rank-p sequential laminate $\mathbf{A}_p^*$ with core $\mathbf{B}$ and matrix $\mathbf{A}$, and with lamination directions $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_p$, such that

$$(1-\theta)(\mathbf{A}_p^{*-1}-\mathbf{A}^{-1})^{-1} = (\mathbf{B}^{-1}-\mathbf{A}^{-1})^{-1}+\theta\left[\mathbf{A}-\sum_{i=1}^{p} m_i \mathbf{A}\frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{A}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)}\mathbf{A}\right]. \quad (2.59)$$

Our next step is to derive bounds for the effective energy written in terms of stress, i.e. bounds on the complementary or dual energy. These bounds, as well as bounds on the primal energy, will pave the way for new results concerning optimal design of a thin, elastic plates.

Theorem 31 The homogenized tensor $\mathbf{A}^* \in G_\theta$ satisfies

$$(\forall\boldsymbol{\sigma}\in \mathrm{Sym})\;\; \mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \geq \mathbf{B}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}+\theta \max_{\boldsymbol{\eta}\in \mathrm{Sym}}\left[2\boldsymbol{\sigma}:\boldsymbol{\eta}-(\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta}-(1-\theta)g^c(\boldsymbol{\eta})\right], \quad (2.60)$$

where

$$g^c(\boldsymbol{\eta}) := \mathbf{B}\boldsymbol{\eta}:\boldsymbol{\eta}-h_{\mathbf{B}}(\boldsymbol{\eta}), \quad (2.61)$$

while $h_{\mathbf{B}}(\boldsymbol{\eta})$ is defined with

$$h_{\mathbf{B}}(\boldsymbol{\eta}) := \min_{\mathbf{e}\in S^{d-1}} \frac{|(\mathbf{e}\otimes\mathbf{e}):\mathbf{B}\boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}. \quad (2.62)$$

Moreover,

$$(\forall\boldsymbol{\sigma}\in \mathrm{Sym})\;\; \mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \leq \mathbf{A}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}+(1-\theta) \min_{\boldsymbol{\eta}\in \mathrm{Sym}}\left[2\boldsymbol{\sigma}:\boldsymbol{\eta}+(\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta}-\theta h^c(\boldsymbol{\eta})\right], \quad (2.63)$$

where

$$h^c(\boldsymbol{\eta}) := \mathbf{A}\boldsymbol{\eta}:\boldsymbol{\eta}-g_{\mathbf{A}}(\boldsymbol{\eta}), \quad (2.64)$$

while $g_{\mathbf{A}}(\boldsymbol{\eta})$ is defined with

$$g_{\mathbf{A}}(\boldsymbol{\eta}) := \max_{\mathbf{e}\in S^{d-1}} \frac{|(\mathbf{e}\otimes\mathbf{e}):\mathbf{A}\boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}. \quad (2.65)$$

Additionally, (2.60) and (2.63) are optimal in the sense of Definition 13 and optimality is achieved by a rank-p sequential laminate, where $p = \frac{(d+3)(d+2)(d+1)d}{24}+1$.

Remark 12 Analogously as for the Hashin-Shtrikman bounds on the primal energy, one can show that the function

$$F_1(\boldsymbol{\eta}) := 2\boldsymbol{\sigma}:\boldsymbol{\eta}-(\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta}-(1-\theta)g^c(\boldsymbol{\eta})$$

is concave, hence the bound (2.60) is given as the result of a finite-dimensional concave maximization, and that the function

$$F_2(\boldsymbol{\eta}) := 2\boldsymbol{\sigma} : \boldsymbol{\eta} + (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - \theta h^c(\boldsymbol{\eta})$$

is convex, which implies that the bound (2.63) is given as the result of a finite-dimensional convex minimization.

Proof of Theorem 31. The idea of the proof is to show that (2.60) is equivalent to the upper Hashin-Shtrikman bound given with (2.31) and that (2.63) is equivalent to the lower Hashin-Shtrikman bound, given with (2.29).

Let us start with the lower bound (2.60). By Corollary 3 we have that (2.31) is equivalent to

$$\theta(\mathbf{B} - \mathbf{A}^*)^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} \leq (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - (1-\theta)h(\boldsymbol{\eta}), \quad \boldsymbol{\eta} \in \mathrm{Sym}. \tag{2.66}$$

Next, we show that (2.60) is equivalent to

$$\theta(\mathbf{A}^{*-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} \leq (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} + (1-\theta)g^c(\boldsymbol{\eta}), \quad \boldsymbol{\eta} \in \mathrm{Sym}. \tag{2.67}$$

Let us first denote

$$\varphi_1(\boldsymbol{\sigma}) = \max_{\boldsymbol{\eta}\in\mathrm{Sym}} \left[2\boldsymbol{\sigma} : \boldsymbol{\eta} - (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - (1-\theta)g^c(\boldsymbol{\eta})\right]$$

and

$$\varphi_2(\boldsymbol{\sigma}) = \frac{1}{\theta}\left(\mathbf{A}^{*-1} - \mathbf{B}^{-1}\right)\boldsymbol{\sigma} : \boldsymbol{\sigma}.$$

Obviously, (2.60) is equivalent to $\varphi_1(\boldsymbol{\sigma}) \leq \varphi_2(\boldsymbol{\sigma})$, $\boldsymbol{\sigma} \in \mathrm{Sym}$, and $\varphi_1, \varphi_2 : \mathrm{Sym} \to \mathbf{R}$ are convex functions. By Fenchel-Moreau theorem, since φ_1 is continuous, convex and $\varphi_1 \not\equiv +\infty$, the Legendre transform of φ_1^* is again φ_1:

$$\begin{aligned}
(\varphi_1^*)^*(\boldsymbol{\sigma}) = \varphi_1(\boldsymbol{\sigma}) &= \max_{\boldsymbol{\eta}\in\mathrm{Sym}} \left[2\boldsymbol{\sigma} : \boldsymbol{\eta} - (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - (1-\theta)g^c(\boldsymbol{\eta})\right] \\
&= \max_{\boldsymbol{\eta}\in\mathrm{Sym}} \left[\boldsymbol{\sigma} : \boldsymbol{\eta} - \frac{1}{4}\left((\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} + (1-\theta)g^c(\boldsymbol{\eta})\right)\right] \\
&= \sup_{\boldsymbol{\eta}\in\mathrm{Sym}} \left[\boldsymbol{\sigma} : \boldsymbol{\eta} - \varphi_1^*(\boldsymbol{\eta})\right].
\end{aligned}$$

By Remark 6, it easily follows that

$$\varphi_1^*(\boldsymbol{\eta}) = \frac{1}{4}\left((\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} + (1-\theta)g^c(\boldsymbol{\eta})\right).$$

Furthermore,

$$\begin{aligned}\varphi_2^*(\boldsymbol{\eta}) &= \sup_{\boldsymbol{\sigma}\in\mathrm{Sym}}\left[\boldsymbol{\sigma}:\boldsymbol{\eta}-\frac{1}{\theta}(\mathbf{A}^{*-1}-\mathbf{B}^{-1})\boldsymbol{\sigma}:\boldsymbol{\sigma}\right]\\ &= \frac{\theta}{4}(\mathbf{A}^{*-1}-\mathbf{B}^{-1})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta}.\end{aligned}$$

Since φ_1 and φ_2 are both convex and continuous,

$$\varphi_1(\boldsymbol{\sigma}) \le \varphi_2(\boldsymbol{\sigma}),\ \ \boldsymbol{\sigma}\in\mathrm{Sym}$$

is equivalent to

$$\varphi_1^*(\boldsymbol{\eta}) \ge \varphi_2^*(\boldsymbol{\eta}),\ \ \boldsymbol{\eta}\in\mathrm{Sym}.$$

Thus, we have that (2.31)⇔(2.66) and (2.60)⇔(2.67). It remains to show that (2.66) is equivalent to (2.67), which will complete the proof. Replacing $\boldsymbol{\eta}$ by $\mathbf{B}^{-1}\boldsymbol{\eta}$ in (2.67) gives

$$\theta(\mathbf{A}^{*-1}-\mathbf{B}^{-1})^{-1}\mathbf{B}^{-1}\boldsymbol{\eta}:\mathbf{B}^{-1}\boldsymbol{\eta}\le(\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}\mathbf{B}^{-1}\boldsymbol{\eta}:\mathbf{B}^{-1}\boldsymbol{\eta}+(1-\theta)g^c(\mathbf{B}^{-1}\boldsymbol{\eta}),$$

which, by symmetry of $\mathbf{B}^{-1}$, can be written as

$$\theta\mathbf{B}^{-1}(\mathbf{A}^{*-1}-\mathbf{B}^{-1})^{-1}\mathbf{B}^{-1}\boldsymbol{\eta}:\boldsymbol{\eta}\le\mathbf{B}^{-1}(\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}\mathbf{B}^{-1}\boldsymbol{\eta}:\boldsymbol{\eta}+(1-\theta)\mathbf{B}^{-1}\boldsymbol{\eta}:\boldsymbol{\eta}-(1-\theta)h(\boldsymbol{\eta}).$$

By using (2.55) and after some calculation, one easily verifies that (2.67) is equivalent to (2.66), which completes this part of the proof. The upper Hashin-Shtrikman bound on the complementary energy can be derived analogously.

Let us now prove that the lower bound (2.60) is optimal in the sense of Definition 13. The proof is analogous as for the optimality of Hashin-Shtrikman bounds on primal energy, hence, we shall just highlight the main parts. We denote

$$\phi(\boldsymbol{\eta}) = 2\boldsymbol{\sigma}:\boldsymbol{\eta}-(\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta}-(1-\theta)g^c(\boldsymbol{\eta}). \tag{2.68}$$

It is easy to see that

$$\begin{aligned}h_{\mathbf{B}}(\boldsymbol{\eta}) &= \min_{\mathbf{e}\in S^{d-1}}\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\mathbf{B}\boldsymbol{\eta}:\mathbf{B}\boldsymbol{\eta}\\ &= \min_{\mathbf{e}\in S^{d-1}}\mathbf{B}\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\mathbf{B}\boldsymbol{\eta}:\boldsymbol{\eta}.\end{aligned}$$

By Theorem 20, the subdifferential of $h_{\mathbf{B}}$ is given with

$$\partial h_{\mathbf{B}}(\boldsymbol{\eta}) = \left\{2\int_{S^{d-1}}\mathbf{B}\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\mathbf{B}\boldsymbol{\eta}\,d\nu(\mathbf{e}):\nu\in P[M(\boldsymbol{\eta})]\right\}$$

$$= \left\{ 2\mathbf{B} \left(\int_{S^{d-1}} \frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\, d\nu(\mathbf{e}) \right) \mathbf{B}\boldsymbol{\eta} : \nu \in P[M(\boldsymbol{\eta})] \right\},$$

where

$$M(\boldsymbol{\eta}) = \left\{ \mathbf{e} \in S^{d-1} : h_{\mathbf{B}}(\boldsymbol{\eta}) = \frac{|(\mathbf{e}\otimes\mathbf{e}):\mathbf{B}\boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})} \right\}$$

and $P[M(\boldsymbol{\eta})]$ is the collection of probability Radon measures on the unit sphere, supported on $M(\boldsymbol{\eta})$. By using Carathéodory's theorem, similarly as in the proof of the Theorem 29, the subdifferential of $h_{\mathbf{B}}(\boldsymbol{\eta})$ is equivalently defined by

$$\partial h_{\mathbf{B}}(\boldsymbol{\eta}) = \left\{ 2\mathbf{B} \left(\sum_{i=1}^{p} m_i \frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)} \right) \mathbf{B}\boldsymbol{\eta} \right\},$$

for some $\mathbf{e}_i \in M(\boldsymbol{\eta})$ and $m_i \geq 0$, such that $\sum_{i=1}^{p} m_i = 1$, where $p = \frac{(d+3)(d+2)(d+1)d}{24} + 1$.

Furthermore, the optimality condition $\mathbf{0} \in \partial\phi(\boldsymbol{\eta}^*)$ is equivalent to

$$\boldsymbol{\sigma} - (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta}^* - (1-\theta)\left[\mathbf{B}\boldsymbol{\eta}^* - \sum_{i=1}^{p} m_i \mathbf{B} \frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)} \mathbf{B}\boldsymbol{\eta}^* \right] = 0. \quad (2.69)$$

Taking the inner product of (2.69) with $\boldsymbol{\eta}^*$ gives

$$\begin{aligned} &\boldsymbol{\sigma} : \boldsymbol{\eta}^* - (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta}^* : \boldsymbol{\eta}^* - (1-\theta)\left[\mathbf{B}\boldsymbol{\eta}^* : \boldsymbol{\eta}^* - \sum_{i=1}^{p} m_i \mathbf{B} \frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)} \mathbf{B}\boldsymbol{\eta}^* : \boldsymbol{\eta}^* \right] \\ &= \boldsymbol{\sigma} : \boldsymbol{\eta}^* - (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta}^* : \boldsymbol{\eta}^* - (1-\theta)\left[\mathbf{B}\boldsymbol{\eta}^* : \boldsymbol{\eta}^* - h_{\mathbf{B}}(\boldsymbol{\eta}^*) \right] \quad (2.70) \\ &= \boldsymbol{\sigma} : \boldsymbol{\eta}^* - (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta}^* : \boldsymbol{\eta}^* - (1-\theta) g^c(\boldsymbol{\eta}^*) = 0. \end{aligned}$$

To achieve equality in the lower Hashin-Shtrikman bound on the complementary energy, let us consider the sequential laminate provided by formula (2.58):

$$\theta(\mathbf{A}_p^{*-1} - \mathbf{B}^{-1})^{-1} = (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1} + (1-\theta)\left[\mathbf{B} - \sum_{i=1}^{p} m_i \mathbf{B} \frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)} \mathbf{B} \right]. \quad (2.71)$$

By using (2.69) and (2.71), we have

$$\mathbf{A}_p^{*-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} = \mathbf{B}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + \theta\boldsymbol{\sigma} : \boldsymbol{\eta}^*.$$

In order to conclude that optimality is achieved by a finite-rank sequential laminate given with (2.58), it remains to show that $\phi(\boldsymbol{\eta}^*) = \boldsymbol{\sigma} : \boldsymbol{\eta}^*$, which easily follows from (2.70). For the upper Hashin-Shtrikman bound on the complementary energy a similar conclusion holds, i.e. the optimality is achieved by a finite-rank sequential laminate (2.59). ■

2.6 G-closure in the small-amplitude regime

In this section we shall give a characterisation of the G-closure for the Kirchhoff-Love plate in one simplified regime, namely the low-contrast or small-amplitude regime: we mix two materials $\mathbf{A}, \mathbf{B} \in \mathrm{Sym}^4$, in proportions θ and $1-\theta$, respectively, with a microstructure defined by the sequence (χ^n), and additionally we assume that $\mathbf{A}$ and $\mathbf{B}$ have close properties. More precisely, we assume that there exists a small positive parameter γ and a coercive, symmetric fourth order tensor $\mathbf{D}$ such that

$$\mathbf{B} - \mathbf{A} = \gamma \mathbf{D}.$$

In order to describe the G-closure in the low-contrast regime, we shall use H-measures. They were introduced independently for weakly convergent sequences in $\mathrm{L}^2(\mathbf{R}^d)$ by Tartar [69] and Gérard [34] (under the name *microlocal defect measures*). An idea is to associate a measure to a weakly converging (sub)sequence, and such measure is called the H-measure associated with that (sub)sequence. The main result which assures existence of such measure is the existence theorem for H-measures [69, Theorem 1.1]. In the case when the sequence is defined with $\chi^n(\mathbf{x}) := \chi(n\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^d$, for a Y-periodic function $\chi \in \mathrm{L}^\infty(\mathbf{R}^d; \{0,1\})$ such that $\int_Y \chi(\boldsymbol{y})\, d\boldsymbol{y} = \theta$, this theorem implies that there exists a probability measure ν on S^{d-1} such that

$$\theta(1-\theta) \int_{S^{d-1}} f(\mathbf{e})\, d\nu(\mathbf{e}) = \sum_{\mathbf{k} \in \mathbf{Z}^d, \mathbf{k} \neq \mathbf{0}} |\hat{\chi}(\mathbf{k})|^2 f\left(\frac{\mathbf{k}}{\|\mathbf{k}\|}\right)$$

for any $f \in C(S^{d-1})$, where $\hat{\chi}(\mathbf{k})$ are Fourier coefficients of the function χ. Since the sequence (χ^n) is uniquely described with function χ (in this periodic case), we shall say that probability measure ν is the H-measure of χ [2, 3].

In the following theorem we give a formula for the composite $\mathbf{M}^*$ obtained by periodic homogenization, up to a second order in γ, in terms of the H-measure ν of characteristic function χ.

Theorem 32 Let $\mathbf{M}^*$ be a homogenized tensor in P_θ, associated to a characteristic function χ. For any $\boldsymbol{\xi} \in \mathrm{Sym}$ it follows that

$$\mathbf{M}^* \boldsymbol{\xi} : \boldsymbol{\xi} = (\theta \mathbf{A} + (1-\theta)\mathbf{B})\boldsymbol{\xi} : \boldsymbol{\xi} - \gamma^2 \theta(1-\theta) \mathbf{D} \left(\int_{S^{d-1}} f_{\mathbf{A}}(\mathbf{e}) d\nu(\mathbf{e}) \right) \mathbf{D}\boldsymbol{\xi} : \boldsymbol{\xi} + \mathcal{O}(\gamma^3),$$

where

$$f_{\mathbf{A}}(\mathbf{e}) = \frac{(\mathbf{e} \otimes \mathbf{e}) \otimes (\mathbf{e} \otimes \mathbf{e})}{\mathbf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})}.$$

Additionally, the remainder is uniform with respect to $\mathbf{M}^*$ in the sense that there exists a positive constant c, independent of γ and χ, such that

$$|\mathcal{O}(\gamma^3)| \leq c\gamma^3.$$

Proof. The theorem will be proved in two steps.

I In the periodic case we have an explicit formula for the homogenization limit, i.e. by Theorem 18, for any $\boldsymbol{\xi} \in \mathrm{Sym}$

$$\mathbf{M}^*\boldsymbol{\xi} : \boldsymbol{\xi} = \int_Y \mathbf{M}(\boldsymbol{y})(\boldsymbol{\xi} + \nabla\nabla w(\boldsymbol{y})) : \boldsymbol{\xi}\, d\boldsymbol{y}, \tag{2.72}$$

where $\mathbf{M}(\boldsymbol{y}) = \chi(\boldsymbol{y})\mathbf{A} + (1-\chi(\boldsymbol{y}))\mathbf{B}$, $\int_Y \chi(\boldsymbol{y})\, d\boldsymbol{y} = \theta$ and $w \in \mathrm{H}^2_\#(Y)/\mathbf{R}$ is the unique solution of the boundary value problem

$$\begin{cases} \mathsf{div\,div}\,(\mathbf{M}(\boldsymbol{y})(\boldsymbol{\xi} + \nabla\nabla w(\boldsymbol{y}))) = 0 & \text{in } Y \\ \boldsymbol{y} \mapsto w(\boldsymbol{y}) & \text{is } Y\text{-periodic} \end{cases}.$$

Substituting $\mathbf{B} - \mathbf{A} = \gamma\mathbf{D}$, we have

$$\mathbf{M}(\boldsymbol{y}) = \mathbf{A} + \gamma(1 - \chi(\boldsymbol{y}))\mathbf{D}, \tag{2.73}$$

and therefore the solution w depends analytically on the small parameter γ, although this fact shall not be used. Let us rather introduce a, $b \in \mathrm{H}^2_\#(Y)/\mathbf{R}$ such that

$$w(\boldsymbol{y}) = \gamma a(\boldsymbol{y}) + \gamma^2 b(\boldsymbol{y}),$$

where $a \in \mathrm{H}^2_\#(Y)/\mathbf{R}$ is the unique solution of

$$\begin{cases} \mathsf{div\,div}\,(\mathbf{A}\nabla\nabla a(\boldsymbol{y}) + (1 - \chi(\boldsymbol{y}))\mathbf{D}\boldsymbol{\xi}) = 0 & \text{in } Y \\ \boldsymbol{y} \mapsto a(\boldsymbol{y}) & \text{is } Y\text{-periodic} \end{cases}, \tag{2.74}$$

and $b \in \mathrm{H}^2_\#(Y)/\mathbf{R}$ is the unique solution of

$$\begin{cases} \mathsf{div\,div}\,(\mathbf{M}(\boldsymbol{y})\nabla\nabla b(\boldsymbol{y}) + (1 - \chi(\boldsymbol{y}))\mathbf{D}\nabla\nabla a(\boldsymbol{y})) = 0 & \text{in } Y \\ \boldsymbol{y} \mapsto b(\boldsymbol{y}) & \text{is } Y\text{-periodic} \end{cases}. \tag{2.75}$$

If we insert expressions for $\mathbf{M}$ and w in formula (2.72), we have

$$\mathbf{M}^*\boldsymbol{\xi} : \boldsymbol{\xi} = \int_Y (\mathbf{A} + \gamma(1 - \chi(\boldsymbol{y}))\mathbf{D})(\boldsymbol{\xi} + \gamma\nabla\nabla a(\boldsymbol{y}) + \gamma^2\nabla\nabla b(\boldsymbol{y})) : \boldsymbol{\xi}\, d\boldsymbol{y}. \tag{2.76}$$

After applying the integration by parts in (2.76), one easily gets

$$\begin{aligned}
\mathbf{M}^*\boldsymbol{\xi} : \boldsymbol{\xi} &= \int_Y (\mathbf{A} + \gamma(1-\chi(\boldsymbol{y}))\mathbf{D})\boldsymbol{\xi} : \boldsymbol{\xi}\, d\boldsymbol{y} + \gamma^2 \int_Y (1-\chi(\boldsymbol{y}))\mathbf{D}\nabla\nabla a(\boldsymbol{y}) : \boldsymbol{\xi}\, d\boldsymbol{y} + \\
&\quad + \gamma^3 \int_Y (1-\chi(\boldsymbol{y}))\mathbf{D}\nabla\nabla b(\boldsymbol{y}) : \boldsymbol{\xi}\, d\boldsymbol{y} = \\
&= \int_Y \Big(\chi(\boldsymbol{y})\mathbf{A} + (1-\chi(\boldsymbol{y}))\mathbf{B}\Big)\boldsymbol{\xi} : \boldsymbol{\xi}\, d\boldsymbol{y} + \gamma^2 \int_Y (1-\chi(\boldsymbol{y}))\mathbf{D}\nabla\nabla a(\boldsymbol{y}) : \boldsymbol{\xi}\, d\boldsymbol{y} + \\
&\quad + \gamma^3 \int_Y (1-\chi(\boldsymbol{y}))\mathbf{D}\nabla\nabla b(\boldsymbol{y}) : \boldsymbol{\xi}\, d\boldsymbol{y} = \\
&= (\theta\mathbf{A} + (1-\theta)\mathbf{B})\boldsymbol{\xi} : \boldsymbol{\xi} + \gamma^2 \int_Y (1-\chi(\boldsymbol{y}))\mathbf{D}\nabla\nabla a(\boldsymbol{y}) : \boldsymbol{\xi}\, d\boldsymbol{y} + \\
&\quad + \gamma^3 \int_Y (1-\chi(\boldsymbol{y}))\mathbf{D}\nabla\nabla b(\boldsymbol{y}) : \boldsymbol{\xi}\, d\boldsymbol{y}.
\end{aligned}$$

In order to compute the term of order γ^2, we use the weak formulation of the boundary value problem (2.74) with $a \in \mathrm{H}^2_\#(Y)/\mathbf{R}$ as a test function, which yields:

$$\int_Y (1-\chi(\boldsymbol{y}))\mathbf{D}\nabla\nabla a(\boldsymbol{y}) : \boldsymbol{\xi}\, d\boldsymbol{y} = -\int_Y \mathbf{A}\nabla\nabla a(\boldsymbol{y}) : \nabla\nabla a(\boldsymbol{y})\, d\boldsymbol{y}.$$

Moreover, using the fact that the solution of (2.74) is also a unique solution of the corresponding minimization problem, we obtain

$$\begin{aligned}
&-\min_{w\in \mathrm{H}^2_\#(Y)} \left(\int_Y \mathbf{A}\nabla\nabla w(\boldsymbol{y}) : \nabla\nabla w(\boldsymbol{y})\, d\boldsymbol{y} - 2\int_Y \chi(\boldsymbol{y})\mathbf{D}\boldsymbol{\xi} : \nabla\nabla w(\boldsymbol{y})\, d\boldsymbol{y} \right) \\
&\qquad = \int_Y \mathbf{A}\nabla\nabla a(\boldsymbol{y}) : \nabla\nabla a(\boldsymbol{y})\, d\boldsymbol{y}.
\end{aligned} \tag{2.77}$$

The minimum in (2.77) can be computed by using Fourier analysis, in an analogous way as it was done in the proof of Theorem 29, when calculating the function g given by (2.37):

$$\begin{aligned}
&-\min_{w\in \mathrm{H}^2_\#(Y)} \left(\int_Y \mathbf{A}\nabla\nabla w(\boldsymbol{y}) : \nabla\nabla w(\boldsymbol{y})\, d\boldsymbol{y} - 2\int_Y \chi(\boldsymbol{y})\mathbf{D}\boldsymbol{\xi} : \nabla\nabla w(\boldsymbol{y})\, d\boldsymbol{y} \right) \\
&\quad = \sum_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq\mathbf{0}} \frac{|\hat{\chi}(\mathbf{k})|^2 |(\mathbf{k}\otimes\mathbf{k}) : \mathbf{D}\boldsymbol{\xi}|^2}{\mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})} = \sum_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq\mathbf{0}} |\hat{\chi}(\mathbf{k})|^2 \frac{(\mathbf{k}\otimes\mathbf{k})\otimes(\mathbf{k}\otimes\mathbf{k})}{\mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})} (\mathbf{D}\boldsymbol{\xi}) : (\mathbf{D}\boldsymbol{\xi}) \\
&\quad = \sum_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq\mathbf{0}} |\hat{\chi}(\mathbf{k})|^2 f_{\mathbf{A}}\left(\frac{\mathbf{k}}{\|\mathbf{k}\|}\right) (\mathbf{D}\boldsymbol{\xi}) : (\mathbf{D}\boldsymbol{\xi}),
\end{aligned}$$

where

$$f_{\mathbf{A}}(\mathbf{e}) = \frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}, \quad \mathbf{e} \in S^{d-1}.$$

Introducing the H-measure ν of characteristic function χ and using that $\mathbf{D}$ is a symmetric tensor, yields

$$\begin{aligned}
&\sum_{\mathbf{k}\in\mathbf{Z}^d,\mathbf{k}\neq\mathbf{0}} |\hat{\chi}(\mathbf{k})|^2 f_{\mathbf{A}}\left(\frac{\mathbf{k}}{\|\mathbf{k}\|}\right)(\mathbf{D}\boldsymbol{\xi}):(\mathbf{D}\boldsymbol{\xi}) \\
&\quad= \theta(1-\theta)\left(\int_{S^{d-1}} f_{\mathbf{A}}(\mathbf{e})d\nu(\mathbf{e})\right)(\mathbf{D}\boldsymbol{\xi}):(\mathbf{D}\boldsymbol{\xi}) \\
&\quad= \theta(1-\theta)\mathbf{D}\left(\int_{S^{d-1}} f_{\mathbf{A}}(\mathbf{e})d\nu(\mathbf{e})\right)\mathbf{D}\boldsymbol{\xi}:\boldsymbol{\xi}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{M}^*\boldsymbol{\xi}:\boldsymbol{\xi} =&(\theta\mathbf{A}+(1-\theta)\mathbf{B})\boldsymbol{\xi}:\boldsymbol{\xi}-\gamma^2\theta(1-\theta)\mathbf{D}\left(\int_{S^{d-1}} f_{\mathbf{A}}(\mathbf{e})d\nu(\mathbf{e})\right)\mathbf{D}\boldsymbol{\xi}:\boldsymbol{\xi}+ \\
&+\gamma^3\int_Y (1-\chi(\boldsymbol{y}))\mathbf{D}\nabla\nabla b(\boldsymbol{y}):\boldsymbol{\xi}\,d\boldsymbol{y}.
\end{aligned}$$

II To conclude the proof, one only has to show that the term of order γ^3 can be estimated with a positive constant c, independent of γ and χ. Note that the solution a of boundary value problem (2.74) is independent of γ, but that is not the case for the solution b of (2.75). However, since

$$\mathbf{M}(\boldsymbol{y})\boldsymbol{\xi}:\boldsymbol{\xi}\geq\mathbf{A}\boldsymbol{\xi}:\boldsymbol{\xi}\geq\alpha|\boldsymbol{\xi}|^2,\ \boldsymbol{\xi}\in\mathrm{Sym},$$

one can show that b satisfies an a priori estimate

$$\|b\|_{\mathrm{H}^2_{\#}(Y)/\mathbf{R}}=\|\nabla\nabla b\|_{\mathrm{L}^2(Y;\mathrm{Sym})}\leq C|\boldsymbol{\xi}|,\ C\in\mathbf{R}^+, \tag{2.78}$$

as will be proved in the sequel.

For simplicity, we identify the unit cube Y with the unit d-dimensional torus T, which can be done by *gluing* together opposite faces of Y, and in the sequel, due to this identification, a periodic function in Y is actually defined as a function on the unit torus [37]. Moreover, equations (2.74) and (2.75) can be seen as posed in the unit torus, which is a smooth compact manifold without boundary, and thus $\mathrm{H}^2_0(T)=H^2(T)$ [15, 41].

From the boundary value problem (2.75), and by using an a priori estimate based on Lax-Milgram lemma, we have:

$$\|b\|_{H^2(T)/\mathbf{R}}\leq\frac{1}{\alpha}\|\mathsf{div}\,\mathsf{div}\,((1-\chi)\mathbf{D}\nabla\nabla a)\|_{(H^2(T)/\mathbf{R})'}$$

$$= \frac{1}{\alpha}\|\mathsf{div}\,\mathsf{div}\,((1-\chi)\mathbf{D}\nabla\nabla a)\|_{H^{-2}(T)}, \tag{2.79}$$

where the last equality in (2.79) follows by Theorem 6. Next, since $\mathsf{div}\,\mathsf{div} : \mathrm{L}^2(T;\mathrm{Sym}) \to H^{-2}(T)$ is linear and continuous, and by definition of the norm on the quotient space $H^2(T)/\mathbf{R}$, it follows (C is a generic constant below):

$$\begin{aligned} \frac{1}{\alpha}\|\mathsf{div}\,\mathsf{div}\,((1-\chi)\mathbf{D}\nabla\nabla a)\|_{H^{-2}(T)} &\le C\|(1-\chi)\mathbf{D}\nabla\nabla a\|_{\mathrm{L}^2(T;\mathrm{Sym})} \\ &\le C\|\nabla\nabla a\|_{\mathrm{L}^2(T;\mathrm{Sym})} \le C\|a\|_{H^2(T)/\mathbf{R}}. \end{aligned} \tag{2.80}$$

By using an analogous arguments, from the boundary value problem (2.74), we conclude

$$\begin{aligned} C\|a\|_{H^2(T)/\mathbf{R}} &\le C\|\mathsf{div}\,\mathsf{div}\,(1-\chi)\mathbf{D}\boldsymbol{\xi}\|_{(H^2(T)/\mathbf{R})'} \\ &= C\|\mathsf{div}\,\mathsf{div}\,(1-\chi)\mathbf{D}\boldsymbol{\xi}\|_{H^{-2}(T)} \\ &\le C\|(1-\chi)\mathbf{D}\boldsymbol{\xi}\|_{\mathrm{L}^2(T;\mathrm{Sym})} \le C|\boldsymbol{\xi}|,\ C \in \mathbf{R}^+. \end{aligned} \tag{2.81}$$

Therefore, by (2.79), (2.80) and (2.81),

$$\|b\|_{H^2(T)/\mathbf{R}} \le C|\boldsymbol{\xi}|,\ C \in \mathbf{R}^+. \tag{2.82}$$

Using (2.82), it is easy to conclude that the remainder is uniform with respect to $\mathbf{M}^*$, i.e.

$$\left|\int_Y (1-\chi(\boldsymbol{y}))\mathbf{D}\nabla\nabla b(\boldsymbol{y}) : \boldsymbol{\xi}\, d\boldsymbol{y}\right| \le C|\boldsymbol{\xi}|^2,\ C \in \mathbf{R}^+,$$

which completes the proof.

■

Theorem 32 can be extended to any composite tensor $\mathbf{M}^* \in G_\theta$.

Theorem 33 Let $\mathbf{M}^* \in G_\theta$ be the homogenized tensor obtained by mixing two materials $\mathbf{A}, \mathbf{B} \in \mathrm{Sym}^4$, in proportions θ and $1-\theta$, respectively, with a microstructure defined by the sequence χ^n, such that there exists a small positive parameter γ and a coercive, symmetric fourth order tensor $\mathbf{D}$ so that

$$\mathbf{B} - \mathbf{A} = \gamma\mathbf{D}.$$

We take $\mathbf{M}^n := \tilde{\chi}^n\mathbf{A} + (1-\tilde{\chi}^n)\mathbf{B}$, $n \in \mathbf{N}$, to be a sequence in P_θ such that $\mathbf{M}^n \to \mathbf{M}^*$ (($\mathbf{M}^n$) exists by Theorem 27), and for $n \in \mathbf{N}$, let ν_n be the H-measure associated to a characteristic function $\tilde{\chi}^n$. Then $\nu_n \rightharpoonup \nu$ weakly-$*$ in $P(S^{d-1})$ (on a subsequence), and, for any $\boldsymbol{\xi} \in \mathrm{Sym}$,

$$\mathbf{M}^* \boldsymbol{\xi} : \boldsymbol{\xi} = (\theta \mathbf{A} + (1-\theta)\mathbf{B})\boldsymbol{\xi} : \boldsymbol{\xi} - \gamma^2 \theta(1-\theta)\mathbf{D} \left(\int\limits_{S^{d-1}} f_{\mathbf{A}}(\mathbf{e}) d\nu(\mathbf{e}) \right) \mathbf{D}\boldsymbol{\xi} : \boldsymbol{\xi} + \mathcal{O}(\gamma^3),$$

where

$$f_{\mathbf{A}}(\mathbf{e}) = \frac{(\mathbf{e} \otimes \mathbf{e}) \otimes (\mathbf{e} \otimes \mathbf{e})}{\mathbf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})}.$$

Additionally, the remainder is uniform with respect to $\mathbf{M}^*$ in the sense that there exists a positive constant c, independent of γ and χ, such that

$$|\mathcal{O}(\gamma^3)| \leq c\gamma^3.$$

Proof. By Theorem 32, for every $n \in \mathbf{N}$, $\mathbf{M}^n$ is given by

$$\mathbf{M}^n \boldsymbol{\xi} : \boldsymbol{\xi} = (\theta \mathbf{A} + (1-\theta)\mathbf{B})\boldsymbol{\xi} : \boldsymbol{\xi} - \gamma^2 \theta(1-\theta)\mathbf{D} \left(\int\limits_{S^{d-1}} f_{\mathbf{A}}(\mathbf{e}) d\nu_n(\mathbf{e}) \right) \mathbf{D}\boldsymbol{\xi} : \boldsymbol{\xi} + \mathcal{O}(\gamma^3),$$

for arbitrary $\boldsymbol{\xi} \in \mathrm{Sym}$.

From the compactness of the set $P(S^{d-1})$ in the weak-$*$ topology, the claim of the theorem directly follows, i.e. the small amplitude formula of Theorem 32 can be extended to any composite tensor $\mathbf{M}^* \in G_\theta$. ■

2.7 Explicit Hashin-Shtrikman bounds on the primal energy for mixtures of two isotropic materials in dimension $d = 2$

In the sequel we consider elastic composite materials obtained by mixing two isotropic phases $\mathbf{A}$ and $\mathbf{B}$ in proportions $\theta_1 := \theta$ and $\theta_2 := 1 - \theta$, respectively. The following results will be stated in dimension $d = 2$, due to the fact that they all refer to the Kirchhoff model for pure bending of a thin, solid symmetric plate under a transverse load. Isotropic phases $\mathbf{A}$ and $\mathbf{B}$ are defined by

$$\begin{aligned} \mathbf{A} &= 2\mu_1 \mathbf{I}_4 + (\kappa_1 - \mu_1)\mathbf{I}_2 \otimes \mathbf{I}_2 \\ \mathbf{B} &= 2\mu_2 \mathbf{I}_4 + (\kappa_2 - \mu_2)\mathbf{I}_2 \otimes \mathbf{I}_2, \end{aligned}$$

where κ_1, κ_2 are the bulk moduli, while μ_1, μ_2 are the shear moduli. Since $\mathbf{A}$ and $\mathbf{B}$ are assumed to be well-ordered, the following holds for bulk and shear moduli:

$$0 < \kappa_1 \leq \kappa_2,\ 0 < \mu_1 \leq \mu_2.$$

In order to explicitly calculate the Hashin-Shtrikman bounds, first we have to evaluate functions g and h from Theorem 29.

Lemma 15 If we label the eigenvalues of $\boldsymbol{\eta}$ by η_1 and η_2, the function g defined by (2.30) equals

$$g(\boldsymbol{\eta}) = \frac{1}{\mu_1 + \kappa_1} \begin{cases} \eta_1^2, \text{ if } & |\eta_1| \geq |\eta_2| \\ \eta_2^2, \text{ if } & |\eta_2| \geq |\eta_1| \end{cases}, \tag{2.83}$$

and the maximum in (2.30) is achieved when $\mathbf{k}$ is an eigenvector of $\boldsymbol{\eta}$ associated with an eigenvalue of the largest absolute value.

The function h defined by (2.32) equals

$$h(\boldsymbol{\eta}) = \frac{1}{\mu_2 + \kappa_2} \begin{cases} \eta_2^2, \text{ if } & \eta_1 \leq \eta_2 \leq 0 \text{ or } 0 \leq \eta_2 \leq \eta_1 \\ 0, \text{ if } & \eta_1 < 0 < \eta_2 \text{ or } \eta_2 < 0 < \eta_1 \\ \eta_1^2, \text{ if } & \eta_2 \leq \eta_1 \leq 0 \text{ or } 0 \leq \eta_1 \leq \eta_2 \end{cases}, \tag{2.84}$$

and the minimum in (2.32), in any case except for $\eta_1 < 0 < \eta_2$ and $\eta_2 < 0 < \eta_1$, is achieved when $\mathbf{k}$ is an eigenvector of $\boldsymbol{\eta}$ associated with an eigenvalue of the least absolute value.

Proof. Let us first express the functions g and h in the terms of bulk and shear moduli of isotropic materials A and B:

$$\begin{aligned} g(\boldsymbol{\eta}) &= \max_{\mathbf{e} \in S^1} \frac{|(\mathbf{e} \otimes \mathbf{e}) : \boldsymbol{\eta}|^2}{(2\mu_1 \mathbf{I}_4 + (\kappa_1 - \mu_1)\mathbf{I}_2 \otimes \mathbf{I}_2)(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})} \\ &= \max_{\mathbf{e} \in S^1} \frac{(\boldsymbol{\eta}\mathbf{e} \cdot \mathbf{e})^2}{(\mu_1 + \kappa_1)(\mathbf{e} \cdot \mathbf{e})^2}. \end{aligned}$$

Since $\mathbf{e} \in S^1$ we have

$$g(\boldsymbol{\eta}) = \frac{1}{\mu_1 + \kappa_1} \max_{\mathbf{e} \in S^1} (\boldsymbol{\eta}\mathbf{e} \cdot \mathbf{e})^2, \tag{2.85}$$

and analogously,

$$h(\boldsymbol{\eta}) = \frac{1}{\mu_2 + \kappa_2} \min_{\mathbf{e} \in S^1} (\boldsymbol{\eta}\mathbf{e} \cdot \mathbf{e})^2. \tag{2.86}$$

It is easy to notice that functions under the maximum in (2.85) and minimum in (2.86) are actually the same:

$$f(\mathbf{e}) = (\boldsymbol{\eta}\mathbf{e} \cdot \mathbf{e})^2.$$

The assertion now follows from the fact that $\eta_i \leq \eta_j$ implies

$$\eta_i \leq \boldsymbol{\eta}\mathbf{e} \cdot \mathbf{e} \leq \eta_j, \;\; \mathbf{e} \in S^1,$$

where equalities are achieved by the corresponding eigenvectors of the matrix $\boldsymbol{\eta}$. ■

Theorem 34 After denoting by ξ_1 and ξ_2 the eigenvalues of $\boldsymbol{\xi}$, and $\theta_1 := \theta$, $\theta_2 := 1 - \theta$ as before, the explicit formula for the lower Hashin-Shtrikman bound (2.29) is given as follows:

(i) if

$$\begin{aligned}&\theta_1(\kappa_2-\kappa_1)|\xi_1+\xi_2| < (\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)|\xi_1-\xi_2| \;\; \& \\ &(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)|\xi_1+\xi_2| > \theta_1(\mu_2-\mu_1)|\xi_1-\xi_2|,\end{aligned} \tag{2.87}$$

then

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi} \geq (\theta_1\mathbf{A}+\theta_2\mathbf{B})\boldsymbol{\xi}:\boldsymbol{\xi} - \theta_1\theta_2\frac{[(\kappa_2-\kappa_1)|\xi_1+\xi_2|+(\mu_2-\mu_1)|\xi_1-\xi_2|]^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)};$$

(ii) if

$$\theta_1(\kappa_2-\kappa_1)|\xi_1+\xi_2| \geq (\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)|\xi_1-\xi_2|, \tag{2.88}$$

then

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi} \geq \mu_1(\xi_1-\xi_2)^2 + \frac{\kappa_1\kappa_2+\mu_1(\theta_1\kappa_1+\theta_2\kappa_2)}{\theta_1\kappa_2+\theta_2\kappa_1+\mu_1}(\xi_1+\xi_2)^2;$$

(iii) if

$$(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)|\xi_1+\xi_2| \leq \theta_1(\mu_2-\mu_1)|\xi_1-\xi_2|, \tag{2.89}$$

then

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi} \geq \kappa_1(\xi_1+\xi_2)^2 + \frac{\mu_1\mu_2+\kappa_1(\theta_1\mu_1+\theta_2\mu_2)}{\theta_1\mu_2+\theta_2\mu_1+\kappa_1}(\xi_1-\xi_2)^2.$$

Cases $(i)-(iii)$ are disjoint, and the union of all $(\xi_1,\xi_2)\in\mathbf{R}^2$ which satisfy one of the conditions (2.87), (2.88) and (2.89), equals $\mathbf{R}^2$.

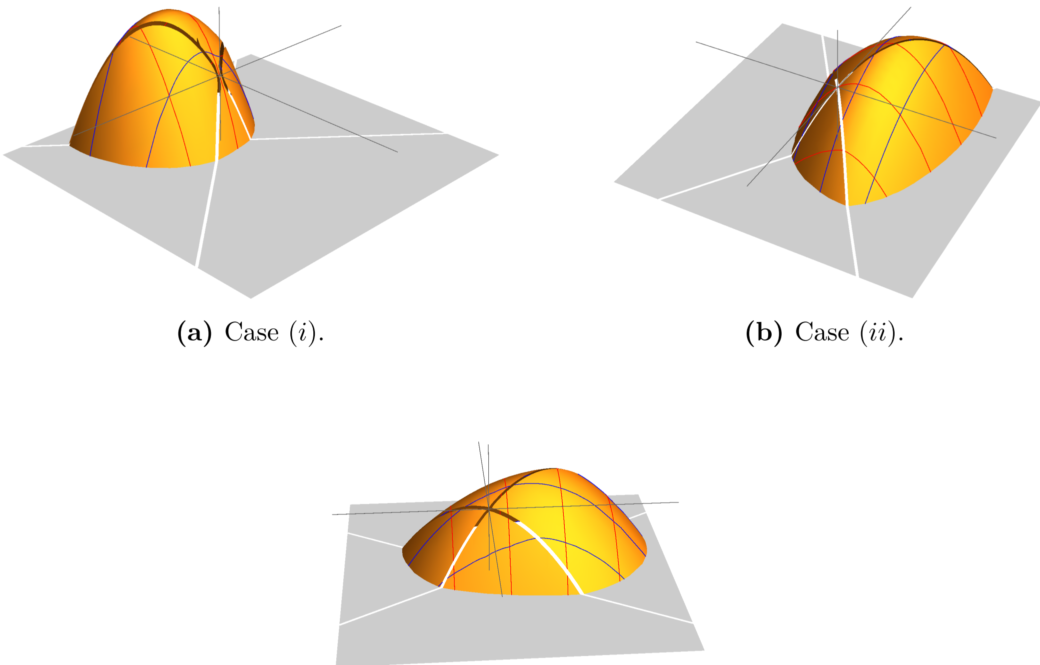
Proof. Firstly, note that the expression

$$(\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} = \frac{1}{2(\mu_2-\mu_1)}(\eta_1^2+\eta_2^2)+\frac{1}{4}\left(\frac{1}{\kappa_2-\kappa_1}-\frac{1}{\mu_2-\mu_1}\right)(\eta_1+\eta_2)^2,$$

as well as the function $g(\boldsymbol{\eta}) = g(\eta_1,\eta_2)$, depend only on the eigenvalues η_1 and η_2 of the matrix $\boldsymbol{\eta}$. Therefore, in order to explicitly compute the lower Hashin-Shtrikman bound, we shall use the classical von Neumann result (see Theorem 23), which implies that the maximum of $\boldsymbol{\xi}:\boldsymbol{\eta}$ equals $\sum_{i=1}^{2}\eta_i\xi_i$, and it is obtained when $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ are simultaneously diagonalizable. This reduces the problem of computing the maximum on the right-hand side of (2.29) to maximization of the concave function

$$\begin{aligned}F(\eta_1,\eta_2) = 2(\xi_1\eta_1+\xi_2\eta_2) - \frac{1}{2(\mu_2-\mu_1)}(\eta_1^2+\eta_2^2) -\\ -\frac{1}{4}\left(\frac{1}{\kappa_2-\kappa_1}-\frac{1}{\mu_2-\mu_1}\right)(\eta_1+\eta_2)^2-\theta_1 g(\eta_1,\eta_2),\end{aligned}$$

over all real numbers η_1 and η_2. Note that the function F is quadratic by parts. An analogous argument was used in [5], for computing the explicit Hashin-Shtrikman bounds in the context of 2D linearized elasticity. In each of the cases $(i)-(iii)$, and for $\theta_1 = 0.6$, $\mu_1 = 1, \mu_2 = 3, \kappa_1 = 2, \kappa_2 = 4$, the graph of the function F is given in Figure 2.2.



(a) Case (i). **(b)** Case (ii). **(c)** Case (iii).

Figure 2.2: Graphs of the function F.

Due to expression (2.83) for g, we shall consider several cases (note that g is differentiable everywhere except on the lines $\eta_1 = \eta_2$ and $\eta_1 = -\eta_2$).

I If $|\eta_1| > |\eta_2|$, then the first-order optimality conditions for F are

$$\frac{\partial F}{\partial \eta_1} = \xi_1 - \frac{\eta_1}{2(\mu_2-\mu_1)} - \frac{1}{4}\left(\frac{1}{\kappa_2-\kappa_1} - \frac{1}{\mu_2-\mu_1}\right)(\eta_1+\eta_2) - \frac{\theta_1\eta_1}{\mu_1+\kappa_1} = 0,$$
$$\frac{\partial F}{\partial \eta_2} = \xi_2 - \frac{\eta_2}{2(\mu_2-\mu_1)} - \frac{1}{4}\left(\frac{1}{\kappa_2-\kappa_1} - \frac{1}{\mu_2-\mu_1}\right)(\eta_1+\eta_2) = 0.$$

This linear system has a unique solution:

$$\eta_1 = \frac{(\mu_1+\kappa_1)((\mu_2-\mu_1)(\xi_1-\xi_2)+(\kappa_2-\kappa_1)(\xi_1+\xi_2))}{\theta_2(\mu_1+\kappa_1)+\theta_1(\mu_2+\kappa_2)} \tag{2.90}$$
$$\eta_2 = \frac{[(\mu_1+\kappa_1)(\kappa_2-\kappa_1)+2\theta_1(\kappa_2-\kappa_1)(\mu_2-\mu_1)](\xi_1+\xi_2)}{\theta_2(\mu_1+\kappa_1)+\theta_1(\mu_2+\kappa_2)} - \frac{[(\mu_1+\kappa_1)(\mu_2-\mu_1)+2\theta_1(\kappa_2-\kappa_1)(\mu_2-\mu_1)](\xi_1-\xi_2)}{\theta_2(\mu_1+\kappa_1)+\theta_1(\mu_2+\kappa_2)}.$$

If this solution fits the case $|\eta_1| > |\eta_2|$ then the maximum of F is

$$\max F(\eta_1,\eta_2) = \frac{(\kappa_2-\kappa_1)(\mu_1+\kappa_1+\theta_1(\mu_2-\mu_1))(\xi_1+\xi_2)^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)} + \\ + \frac{(\mu_2-\mu_1)(\mu_1+\kappa_1+\theta_1(\kappa_2-\kappa_1))(\xi_1-\xi_2)^2 - 2\theta_1(\kappa_2-\kappa_1)(\mu_2-\mu_1)(\xi_1^2-\xi_2^2)}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)},$$

and the lower bound is given with

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi} \geq (\theta_1\mathbf{A}+\theta_2\mathbf{B})\boldsymbol{\xi}:\boldsymbol{\xi} - \theta_1\theta_2\frac{[(\kappa_2-\kappa_1)(\xi_1+\xi_2)+(\mu_2-\mu_1)(\xi_1-\xi_2)]^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}. \tag{2.91}$$

The requirement that the solution (2.90) satisfies $|\eta_1| > |\eta_2|$ is fulfilled if and only if

$$\begin{aligned} &|(\mu_1+\kappa_1)((\mu_2-\mu_1)(\xi_1-\xi_2)+(\kappa_2-\kappa_1)(\xi_1+\xi_2))| > \\ &\Big|[(\mu_1+\kappa_1)(\kappa_2-\kappa_1)+2\theta_1(\kappa_2-\kappa_1)(\mu_2-\mu_1)](\xi_1+\xi_2)- \\ &-[(\mu_1+\kappa_1)(\mu_2-\mu_1)+2\theta_1(\kappa_2-\kappa_1)(\mu_2-\mu_1)](\xi_1-\xi_2)\Big|. \end{aligned} \tag{2.92}$$

Taking the square and factorizing the inequality (2.92) yields

$$\begin{aligned} &[-\theta_1(\kappa_2-\kappa_1)(\xi_1+\xi_2)+(\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)(\xi_1-\xi_2)] \\ &\cdot[(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)(\xi_1+\xi_2)-\theta_1(\mu_2-\mu_1)(\xi_1-\xi_2)] > 0. \end{aligned} \tag{2.93}$$

Thus, (2.93) is equivalent to either

$$\begin{aligned} &\theta_1(\kappa_2-\kappa_1)(\xi_1+\xi_2)-(\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)(\xi_1-\xi_2) < 0, \\ &(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)(\xi_1+\xi_2)-\theta_1(\mu_2-\mu_1)(\xi_1-\xi_2) > 0, \\ &\xi_1-\xi_2 > 0 \ \& \ \xi_1+\xi_2 > 0 \end{aligned} \tag{2.94}$$

or

$$\begin{aligned} &\theta_1(\kappa_2-\kappa_1)(\xi_1+\xi_2)-(\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)(\xi_1-\xi_2) > 0, \\ &(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)(\xi_1+\xi_2)-\theta_1(\mu_2-\mu_1)(\xi_1-\xi_2) < 0, \\ &\xi_1-\xi_2 < 0 \ \& \ \xi_1+\xi_2 < 0. \end{aligned} \tag{2.95}$$

II The case $|\eta_1| < |\eta_2|$ is symmetric to the previous one, and one only has to change the roles of ξ_1 and ξ_2 in (2.91), (2.94) and (2.95):

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi} \geq (\theta_1\mathbf{A}+\theta_2\mathbf{B})\boldsymbol{\xi}:\boldsymbol{\xi} - \theta_1\theta_2\frac{[(\kappa_2-\kappa_1)(\xi_1+\xi_2)+(\mu_2-\mu_1)(\xi_2-\xi_1)]^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}$$

if and only if ξ_1, ξ_2 satisfy

$$\begin{aligned}
&\theta_1(\kappa_2-\kappa_1)(\xi_1+\xi_2)-(\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)(\xi_2-\xi_1)<0,\\
&(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)(\xi_1+\xi_2)-\theta_1(\mu_2-\mu_1)(\xi_2-\xi_1)>0,\\
&\xi_2-\xi_1>0\ \&\ \xi_1+\xi_2>0
\end{aligned} \tag{2.96}$$

or

$$\begin{aligned}
&\theta_1(\kappa_2-\kappa_1)(\xi_1+\xi_2)-(\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)(\xi_2-\xi_1)>0,\\
&(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)(\xi_1+\xi_2)-\theta_1(\mu_2-\mu_1)(\xi_2-\xi_1)<0,\\
&\xi_2-\xi_1<0\ \&\ \xi_1+\xi_2<0.
\end{aligned} \tag{2.97}$$

Cases **I** and **II**, using some standard algebraic calculations, can jointly be written as follows: the bound (2.29) is equivalent to

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi}\geq(\theta_1\mathbf{A}+\theta_2\mathbf{B})\boldsymbol{\xi}:\boldsymbol{\xi}-\theta_1\theta_2\frac{[(\kappa_2-\kappa_1)|\xi_1+\xi_2|+(\mu_2-\mu_1)|\xi_1-\xi_2|]^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)} \tag{2.98}$$

if and only if ξ_1 and ξ_2 satisfy

$$\begin{aligned}
&\theta_1(\kappa_2-\kappa_1)|\xi_1+\xi_2|-(\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)|\xi_1-\xi_2|<0\ \&\\
&(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)|\xi_1+\xi_2|-\theta_1(\mu_2-\mu_1)|\xi_1-\xi_2|>0.
\end{aligned} \tag{2.99}$$

Note that this corresponds to the case (**i**) in the statement of the theorem.

III If condition (2.99) is not satisfied, then the maximum of F is attained on one of the lines $\eta_1=\eta_2$ or $\eta_1=-\eta_2$.

(ii) If $\eta:=\eta_1=\eta_2$, an easy calculation gives us that the maximum of F is reached for

$$\eta=\frac{(\kappa_2-\kappa_1)(\mu_1+\kappa_1)(\xi_1+\xi_2)}{\mu_1+\theta_1\kappa_2+\theta_2\kappa_1}. \tag{2.100}$$

If the maximum is attained in this case, the corresponding bound is

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi}\geq\mu_1(\xi_1-\xi_2)^2+\frac{\kappa_1\kappa_2+\mu_1(\theta_1\kappa_1+\theta_2\kappa_2)}{\theta_1\kappa_2+\theta_2\kappa_1+\mu_1}(\xi_1+\xi_2)^2. \tag{2.101}$$

(iii) If $\eta:=\eta_1=-\eta_2$, the maximum of F is attained when

$$\eta=\frac{(\mu_2-\mu_1)(\mu_1+\kappa_1)(\xi_1-\xi_2)}{\theta_1\mu_2+\theta_2\mu_1+\kappa_1}, \tag{2.102}$$

and the explicit Hashin-Shtrikman bound is given with

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \geq \kappa_1(\xi_1 + \xi_2)^2 + \frac{\mu_1\mu_2 + \kappa_1(\theta_1\mu_1 + \theta_2\mu_2)}{\theta_1\mu_2 + \theta_2\mu_1 + \kappa_1}(\xi_1 - \xi_2)^2. \tag{2.103}$$

By the following inequality, which can easily be derived by using elementary algebraic operations,

$$\theta_1^2(\kappa_2 - \kappa_1)(\mu_2 - \mu_1) \leq (\theta_1\mu_2 + \theta_2\mu_1 + \kappa_1)(\theta_1\kappa_2 + \theta_2\kappa_1 + \mu_1),$$

it is easy to check that the maximum is attained on the line $\eta_1 = \eta_2$ if and only if

$$\theta_1(\kappa_2 - \kappa_1)|\xi_1 + \xi_2| - (\theta_1\kappa_2 + \theta_2\kappa_1 + \mu_1)|\xi_1 - \xi_2| \geq 0.$$

Conversely, the maximum is attained on the line $\eta_1 = -\eta_2$ if and only if

$$(\theta_1\mu_2 + \theta_2\mu_1 + \kappa_1)|\xi_1 + \xi_2| - \theta_1(\mu_2 - \mu_1)|\xi_1 - \xi_2| \leq 0.$$

■

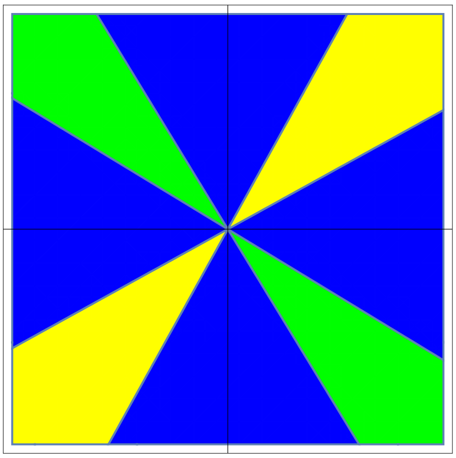


Figure 2.3: Domain division for eigenvalues of $\boldsymbol{\xi}$ in the case of the lower HS bound.

It is interesting to see, for some arbitrary parameters $\theta_1, \mu_1, \mu_2, \kappa_1, \kappa_2$, how the division of $\mathbf{R}^2$ in conditions (2.87), (2.88) and (2.89) looks like. For $\mu_1 = 26, \kappa_1 = 40$ (parameters of the first material, i.e. glass), $\mu_2 = 79, \kappa_2 = 160$ (steel) and $\theta_1 = 0.4$, $\xi_1, \xi_2 \in [-1000, 1000]$, in Figure 2.3 one can see that if (ξ_1, ξ_2) belongs to the blue area then condition (2.87) is satisfied and maximum is attained in one of the cases $|\eta_2| < |\eta_1|$ or $|\eta_1| < |\eta_2|$. If (ξ_1, ξ_2) belongs to the yellow area, then maximum is attained on the line $\eta_1 = -\eta_2$, and the green area represents (ξ_1, ξ_2) such that maximum is attained on the line $\eta_1 = \eta_2$.

It remains to explicitly describe the optimal microstructures which saturate the lower Hashin-Shtrikman bound. Theorem 29 assures that the lower Hashin-Shtrikman bound is optimal and that optimality is achieved by a finite-rank sequential laminate. An optimality condition for the maximization problem on the right-hand side of (2.29) is given by (2.42), i.e.

$$\boldsymbol{\xi} - (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta} = \theta_1 \sum_{i=1}^{p} m_i \frac{(\mathbf{e}_i \otimes \mathbf{e}_i) \otimes (\mathbf{e}_i \otimes \mathbf{e}_i)}{\mathbf{A}(\mathbf{e}_i \otimes \mathbf{e}_i) : (\mathbf{e}_i \otimes \mathbf{e}_i)}\boldsymbol{\eta}, \tag{2.104}$$

where $m_i \geq 0$, $\sum_{i=1}^{p} m_i = 1$ and each $\mathbf{e}_i$ is extremal for

$$g(\boldsymbol{\eta}) = \max_{\mathbf{e} \in S^1} \frac{|(\mathbf{e} \otimes \mathbf{e}) : \boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})}. \tag{2.105}$$

If g is differentiable at the optimal $\boldsymbol{\eta}$, then the optimality condition simplifies to

$$\boldsymbol{\xi} - (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta} = \theta_1 \frac{(\mathbf{e} \otimes \mathbf{e}) \otimes (\mathbf{e} \otimes \mathbf{e})}{\mathbf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})}\boldsymbol{\eta}, \tag{2.106}$$

where $\mathbf{e}$ is an extremal for (2.105). By using an analogous algebraic calculations as in the proof of Theorem 29, we show that the lower bound (2.29) can be expressed as

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \geq \mathbf{A}\boldsymbol{\xi} : \boldsymbol{\xi} + \theta_2 \boldsymbol{\xi} : \boldsymbol{\eta}^*, \tag{2.107}$$

where $\boldsymbol{\eta}^*$ is the optimal point for the maximization problem on the right-hand side of (2.29). One can conclude that the equality in (2.107) is obtained with laminate $\mathbf{A}_1^*$ defined by the formula

$$\theta_2(\mathbf{A}_1^* - \mathbf{A})^{-1} = (\mathbf{B} - \mathbf{A})^{-1} + \theta_1 \frac{(\mathbf{e} \otimes \mathbf{e}) \otimes (\mathbf{e} \otimes \mathbf{e})}{\mathbf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})}. \tag{2.108}$$

To be precise, after multiplying (2.108) by $\boldsymbol{\eta}^*$ and using (2.106), we obtain

$$\theta_2(\mathbf{A}_1^* - \mathbf{A})^{-1}\boldsymbol{\eta}^* = \boldsymbol{\xi}, \tag{2.109}$$

which after taking the inner product with $\boldsymbol{\xi}$ gives the claim. Clearly, in this case the optimality is achieved by a rank-one laminate with the lamination direction $\mathbf{e}$, where $\mathbf{e}$ is an extremal for $g(\boldsymbol{\eta}^*)$ given with (2.105). This corresponds to the first case in the Theorem 34, i.e. when the maximum in (2.29) is achieved for $|\eta_1| \neq |\eta_2|$.

It remains to specify the optimal microstructure in the other two cases, i.e. when $\eta_1 = \eta_2$ and $\eta_1 = -\eta_2$.

a) Let $\eta := \eta_1 = \eta_2$, $p = 2$ and note that we can write $\boldsymbol{\eta} = \eta \mathbf{I}_2$, where η is given with (2.100). Obviously, every unit vector is an eigenvector of $\boldsymbol{\eta}$ and, by Lemma 15, it is also an extremal vector for function (2.105). Thus, for the direction of lamination we can arbitrarily choose unit eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\boldsymbol{\xi}$, since $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ are simultaneously diagonalizable.

Additionally, using the fact that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are simultaneously diagonalizable and symmetric, it follows that there exists orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q}^T\boldsymbol{\xi}\mathbf{Q}$ and $\mathbf{Q}^T\boldsymbol{\eta}\mathbf{Q}$ are diagonal matrices with eigenvalues of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ as the diagonal entries, respectively. If we multiply (2.104) from the left by $\mathbf{Q}^T$ and from the right by $\mathbf{Q}$, we conclude that (2.104) is equivalent to

$$\begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} - \frac{\eta}{2(\kappa_2-\kappa_1)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{\theta_1\eta}{\mu_1+\kappa_1}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}.$$

This determines m_1 and m_2:

$$m_1 = \frac{2\xi_1(\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)-(\mu_1+\kappa_1)(\xi_1+\xi_2)}{2\theta_1(\kappa_2-\kappa_1)(\xi_1+\xi_2)},$$
$$m_2 = \frac{2\xi_2(\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)-(\mu_1+\kappa_1)(\xi_1+\xi_2)}{2\theta_1(\kappa_2-\kappa_1)(\xi_1+\xi_2)},$$

and it is easy to check that $m_1 + m_2 = 1$ and m_1, $m_2 \geq 0$, as a consequence of the condition

$$\theta_1(\kappa_2-\kappa_1)|\xi_1+\xi_2| \geq (\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)|\xi_1-\xi_2|$$

which defines this regime. In this case, the bound is obviously achieved by a second rank laminate in the following way: we first layer **B** with **A** in volume fractions $\rho = 1-\theta_1 m_1$ and $1-\rho$ respectively, in the direction of lamination $\mathbf{v}_1$, to get composite **C**. After that, we layer **C** with **A** in volume fractions $\rho' = \dfrac{\theta_2}{1-\theta_1 m_1}$ and $1-\rho'$, respectively, in the direction of lamination $\mathbf{v}_2$, and obtain a composite $\mathbf{A}^*$ which saturates the lower Hashin-Shtrikman bound.

b) Let $\eta_1 = -\eta_2$, $p = 2$ and let $\mathbf{v}_1$, $\mathbf{v}_2$ be the associated unit eigenvectors of $\boldsymbol{\eta}$, such that they are also eigenvectors of $\boldsymbol{\xi}$. Additionally, by Lemma 15, these vectors are extremal for function (2.105). By denoting $\eta := \eta_1 = -\eta_2$, we have that η is given with (2.102), and using the fact that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are simultaneously diagonalizable, we conclude that (2.104) is equivalent to

$$\begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} - \frac{\eta}{2(\mu_2-\mu_1)}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{\theta_1\eta}{\mu_1+\kappa_1}\begin{bmatrix} m_1 & 0 \\ 0 & -m_2 \end{bmatrix}.$$

This determines m_1 and m_2:

$$m_1 = \frac{2\xi_1(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)-(\mu_1+\kappa_1)(\xi_1-\xi_2)}{2\theta_1(\mu_2-\mu_1)(\xi_1-\xi_2)},$$
$$m_2 = \frac{-2\xi_2(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)-(\mu_1+\kappa_1)(\xi_1-\xi_2)}{2\theta_1(\mu_2-\mu_1)(\xi_1-\xi_2)},$$

and it is easy to check that $m_1 + m_2 = 1$ and m_1, $m_2 \geq 0$, as a consequence of the condition

$$(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)|\xi_1+\xi_2|-\theta_1(\mu_2-\mu_1)|\xi_1-\xi_2|\le 0,$$

which defines this regime. Now, we conclude that the bound is achieved by a second rank laminate in an analogous way as for $\eta_1=\eta_2$.

The following theorem summarizes the previous results.

Theorem 35 Let ξ_1 and ξ_2 be the eigenvalues of $\boldsymbol{\xi}$, and $\theta_1:=\theta$, $\theta_2:=1-\theta$.

(i) If

$$\begin{aligned}&\theta_1(\kappa_2-\kappa_1)|\xi_1+\xi_2|<(\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)|\xi_1-\xi_2| \ \ \&\\ &(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)|\xi_1+\xi_2|>\theta_1(\mu_2-\mu_1)|\xi_1-\xi_2|,\end{aligned}$$

then the optimal microstructure for which the bound (2.29) is saturated is a simple laminate with layers orthogonal to the eigenvector associated with an eigenvalue of the largest absolute value, of the extremal $\boldsymbol{\eta}$ in (2.29).

(ii) If

$$\theta_1(\kappa_2-\kappa_1)|\xi_1+\xi_2|\ge(\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)|\xi_1-\xi_2|,$$

then the optimal microstructure for which the bound (2.29) is saturated is a rank-2 laminate with directions of lamination given with eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\boldsymbol{\xi}$, and corresponding lamination parameters

$$\begin{aligned}m_1&=\frac{2\xi_1(\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)-(\mu_1+\kappa_1)(\xi_1+\xi_2)}{2\theta_1(\kappa_2-\kappa_1)(\xi_1+\xi_2)},\\ m_2&=\frac{2\xi_2(\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)-(\mu_1+\kappa_1)(\xi_1+\xi_2)}{2\theta_1(\kappa_2-\kappa_1)(\xi_1+\xi_2)}.\end{aligned}$$

(iii) If

$$(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)|\xi_1+\xi_2|\le\theta_1(\mu_2-\mu_1)|\xi_1-\xi_2|,$$

then the optimal microstructure for which the bound (2.29) is saturated is a rank-2 laminate with directions of lamination given with eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the extremal $\boldsymbol{\eta}$ in (2.29) (which are also eigenvectors of $\boldsymbol{\xi}$), and corresponding lamination parameters

$$\begin{aligned}m_1&=\frac{2\xi_1(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)-(\mu_1+\kappa_1)(\xi_1-\xi_2)}{2\theta_1(\mu_2-\mu_1)(\xi_1-\xi_2)},\\ m_2&=\frac{-2\xi_2(\theta_1\mu_2+\theta_2\mu_1+\kappa_1)-(\mu_1+\kappa_1)(\xi_1-\xi_2)}{2\theta_1(\mu_2-\mu_1)(\xi_1-\xi_2)}.\end{aligned}$$

Theorem 36 After denoting by ξ_1 and ξ_2 the eigenvalues of $\boldsymbol{\xi}$, and $\theta_1:=\theta$, $\theta_2:=1-\theta$ as before, the explicit formula for the bound (2.31) is given as follows:

(i) if

$$\theta_2(\kappa_2-\kappa_1)|\xi_1+\xi_2| < (\theta_1\kappa_2+\theta_2\kappa_1+\mu_2)|\xi_1-\xi_2| \ \ \& \tag{2.110}$$
$$(\kappa_2-\kappa_1)|\xi_1+\xi_2| \geq (\mu_2-\mu_1)|\xi_2-\xi_1|,$$

then

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi} \leq (\theta_1\mathbf{A}+\theta_2\mathbf{B})\boldsymbol{\xi}:\boldsymbol{\xi} - \theta_1\theta_2\frac{[(\kappa_2-\kappa_1)|\xi_1+\xi_2|-(\mu_2-\mu_1)|\xi_1-\xi_2|]^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)};$$

(ii) if

$$(\kappa_2-\kappa_1)|\xi_1+\xi_2| < (\mu_2-\mu_1)|\xi_2-\xi_1|, \tag{2.111}$$

then

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi} \leq (\theta_1\mathbf{A}+\theta_2\mathbf{B})\boldsymbol{\xi}:\boldsymbol{\xi};$$

(iii) if

$$\theta_2(\kappa_2-\kappa_1)|\xi_1+\xi_2| \geq (\theta_1\kappa_2+\theta_2\kappa_1+\mu_2)|\xi_1-\xi_2|, \tag{2.112}$$

then

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi} \leq \mu_2(\xi_1-\xi_2)^2 + \frac{\kappa_1\kappa_2+\mu_2(\theta_2\kappa_2+\theta_1\kappa_1)}{\mu_2+\theta_1\kappa_2+\theta_2\kappa_1}(\xi_1+\xi_2)^2.$$

Cases $(i)-(iii)$ are disjoint, and the union of all $(\xi_1,\xi_2)\in\mathbf{R}^2$ which satisfy one of the conditions (2.110), (2.111) and (2.112), equals $\mathbf{R}^2$.

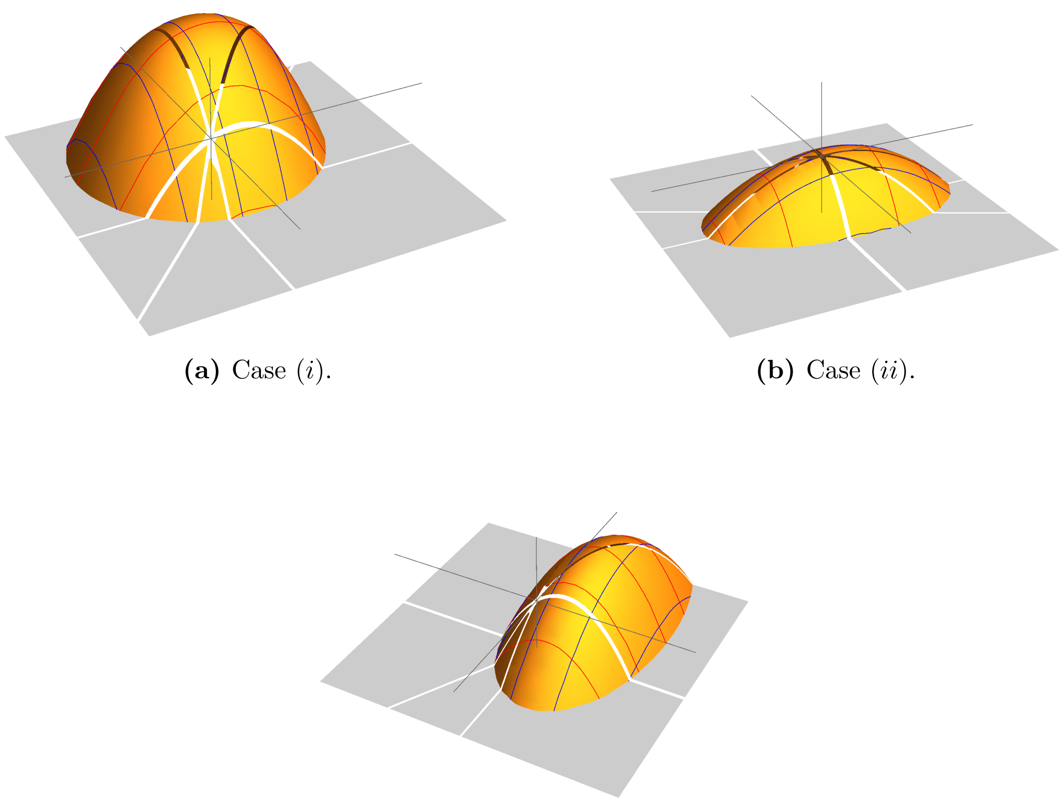
Proof. It is easy to see that (2.31) is equivalent to

$$(\forall\boldsymbol{\xi}\in\mathrm{Sym})\ \ \mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi} \leq \mathbf{B}\boldsymbol{\xi}:\boldsymbol{\xi} - \theta_1\max_{\boldsymbol{\eta}\in\mathrm{Sym}}[2\boldsymbol{\xi}:\boldsymbol{\eta}-(\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta}+\theta_2 h(\boldsymbol{\eta})]. \tag{2.113}$$

Similarly as for the lower Hashin-Shtrikman bound, the problem is equivalent to maximizing the concave function

$$F(\eta_1,\eta_2) = 2(\xi_1\eta_1+\xi_2\eta_2) - \frac{1}{2(\mu_2-\mu_1)}(\eta_1^2+\eta_2^2) - \frac{1}{4}\left(\frac{1}{\kappa_2-\kappa_1}-\frac{1}{\mu_2-\mu_1}\right)(\eta_1+\eta_2)^2 + \theta_2 h(\eta_1,\eta_2)$$

over all real numbers η_1 and η_2. In each of the cases $(i)-(iii)$, and for $\theta_1 = 0.6$, $\mu_1 = 1, \mu_2 = 3, \kappa_1 = 2, \kappa_2 = 4$, the graph of the function F is given in Figure 2.4.



(a) Case (i).

(b) Case (ii).

(c) Case (iii).

Figure 2.4: Graphs of the function F.

Obviously, the function h defined by (2.84) is differentiable everywhere except on the line $\eta_1 = \eta_2$.

I Assume $\eta_1 < \eta_2 \leq 0$. After solving the system of equations

$$\frac{\partial F}{\partial \eta_1} = 0 \quad \& \quad \frac{\partial F}{\partial \eta_2} = 0,$$

we obtain that

$$\begin{aligned}\eta_1 =& \frac{[(\mu_2+\kappa_2)(\kappa_2-\kappa_1) - 2\theta_2(\mu_2-\mu_1)(\kappa_2-\kappa_1)](\xi_1+\xi_2)}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)} \\ &- \frac{[(\mu_2+\kappa_2)(\mu_2-\mu_1) - 2\theta_2(\mu_2-\mu_1)(\kappa_2-\kappa_1)](\xi_2-\xi_1)}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)} \\ \eta_2 =& \frac{(\mu_2+\kappa_2)((\kappa_2-\kappa_1)(\xi_1+\xi_2)+(\mu_2-\mu_1)(\xi_2-\xi_1))}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)},\end{aligned}$$

and therefore

$$\max F(\eta_1,\eta_2) = \frac{(\kappa_2-\kappa_1)(\kappa_2+\theta_1\mu_2+\theta_2\mu_1)(\xi_1+\xi_2)^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)} +$$

$$+\frac{(\mu_2-\mu_1)(\mu_2+\theta_1\kappa_2+\theta_2\kappa_1)(\xi_1-\xi_2)^2+2\theta_2(\kappa_2-\kappa_1)(\mu_2-\mu_1)(\xi_2^2-\xi_1^2)}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}.$$

We see that the bound (2.31) is given with

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi}\leq(\theta_1\mathbf{A}+\theta_2\mathbf{B})\boldsymbol{\xi}:\boldsymbol{\xi}-\theta_1\theta_2\frac{[(\kappa_2-\kappa_1)(\xi_1+\xi_2)+(\mu_2-\mu_1)(\xi_2-\xi_1)]^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}\tag{2.114}$$

when ξ_1 and ξ_2 satisfy

$$\begin{aligned}&\theta_2(\kappa_2-\kappa_1)(\xi_1+\xi_2)>(\mu_2+\theta_1\kappa_2+\theta_2\kappa_1)(\xi_1-\xi_2),\\&(\kappa_2-\kappa_1)(\xi_1+\xi_2)\leq(\mu_2-\mu_1)(\xi_1-\xi_2),\\&\xi_1-\xi_2<0\ \&\ \xi_1+\xi_2<0.\end{aligned}$$

II Assume $0\leq\eta_1<\eta_2$. Similarly as above we obtain optimal

$$\begin{aligned}\eta_1=&\frac{(\mu_2+\kappa_2)((\kappa_2-\kappa_1)(\xi_1+\xi_2)+(\mu_2-\mu_1)(\xi_1-\xi_2))}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}\\\eta_2=&\frac{[(\mu_2+\kappa_2)(\kappa_2-\kappa_1)-2\theta_2(\mu_2-\mu_1)(\kappa_2-\kappa_1)](\xi_1+\xi_2)}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}\\&-\frac{[(\mu_2+\kappa_2)(\mu_2-\mu_1)-2\theta_2(\mu_2-\mu_1)(\kappa_2-\kappa_1)](\xi_1-\xi_2)}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}.\end{aligned}$$

After an easy calculation, we get

$$\begin{aligned}\max F(\eta_1,\eta_2)=&\frac{(\kappa_2-\kappa_1)(\kappa_2+\theta_1\mu_2+\theta_2\mu_1)(\xi_1+\xi_2)^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}+\\+&\frac{(\mu_2-\mu_1)(\mu_2+\theta_1\kappa_2+\theta_2\kappa_1)(\xi_1-\xi_2)^2+2\theta_2(\kappa_2-\kappa_1)(\mu_2-\mu_1)(\xi_1^2-\xi_2^2)}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)},\end{aligned}$$

and the bound (2.31) in this case is

$$\mathbf{A}^*\boldsymbol{\xi}:\boldsymbol{\xi}\leq(\theta_1\mathbf{A}+\theta_2\mathbf{B})\boldsymbol{\xi}:\boldsymbol{\xi}-\theta_1\theta_2\frac{[(\kappa_2-\kappa_1)(\xi_1+\xi_2)+(\mu_2-\mu_1)(\xi_1-\xi_2)]^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}\tag{2.115}$$

when ξ_1 and ξ_2 satisfy

$$\begin{aligned}&\theta_2(\kappa_2-\kappa_1)(\xi_1+\xi_2)<(\mu_2+\theta_1\kappa_2+\theta_2\kappa_1)(\xi_2-\xi_1),\\&(\kappa_2-\kappa_1)(\xi_1+\xi_2)\geq(\mu_2-\mu_1)(\xi_2-\xi_1),\\&\xi_2-\xi_1>0\ \&\ \xi_1+\xi_2>0.\end{aligned}\tag{2.116}$$

The case $0\leq\eta_2<\eta_1$ is symmetric to $0\leq\eta_1<\eta_2$, and one only has to change the roles of ξ_1 and ξ_2 in (2.115) and (2.116). A similar statement holds for the cases $\eta_2<\eta_1\leq 0$ and $\eta_1<\eta_2\leq 0$. Cases **I** and **II** can jointly be written as: the bound

(2.31) is equivalent to

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \leq (\theta_1\mathbf{A} + \theta_2\mathbf{B})\boldsymbol{\xi} : \boldsymbol{\xi} - \theta_1\theta_2 \frac{[(\kappa_2-\kappa_1)|\xi_1+\xi_2| - (\mu_2-\mu_1)|\xi_1-\xi_2|]^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)} \tag{2.117}$$

if and only if ξ_1 and ξ_2 satisfy

$$\begin{aligned} &\theta_2(\kappa_2-\kappa_1)|\xi_1+\xi_2| < (\theta_1\kappa_2+\theta_2\kappa_1+\mu_2)|\xi_1-\xi_2| \ \ \& \\ &(\kappa_2-\kappa_2)|\xi_1+\xi_2| \geq (\mu_2-\mu_1)|\xi_2-\xi_1|. \end{aligned} \tag{2.118}$$

Note that cases **I** and **II** written jointly, correspond to case (**i**) given above.

III Assume $\eta_1 < 0 < \eta_2$. Similar computation as before gives us

$$\begin{aligned} \eta_1 &= (\kappa_2-\kappa_1)(\xi_1+\xi_2) - (\mu_2-\mu_1)(\xi_2-\xi_1) \\ \eta_2 &= (\kappa_2-\kappa_1)(\xi_1+\xi_2) - (\mu_2-\mu_1)(\xi_1-\xi_2). \end{aligned}$$

Furthermore,

$$\max F(\eta_1,\eta_2) = (\mu_2-\mu_1)(\xi_1-\xi_2)^2 + (\kappa_2-\kappa_1)(\xi_1+\xi_2)^2,$$

which yields

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \leq (\theta_1\mathbf{A} + \theta_2\mathbf{B})\boldsymbol{\xi} : \boldsymbol{\xi} \tag{2.119}$$

when ξ_1 and ξ_2 satisfy

$$\begin{aligned} &(\mu_1-\mu_2)(\xi_2-\xi_1) + (\kappa_2-\kappa_1)(\xi_1+\xi_2) < 0 \ \ \& \\ &(\mu_2-\mu_1)(\xi_2-\xi_1) + (\kappa_2-\kappa_1)(\xi_1+\xi_2) > 0. \end{aligned} \tag{2.120}$$

In the case $\eta_2 < 0 < \eta_1$, function F is the same as for $\eta_1 < 0 < \eta_2$, as well as the obtained bound. An easy computation shows that the bound (2.31) is equivalent to (2.119) if and only if ξ_1 and ξ_2 satisfy

$$(\kappa_2-\kappa_1)|\xi_1+\xi_2| < (\mu_2-\mu_1)|\xi_2-\xi_1|. \tag{2.121}$$

This case corresponds to the case (**ii**) in the statement of the theorem.

IV If ξ_1 and ξ_2 satisfy neither of the conditions (2.118) and (2.121), then the maximum of F is attained on the line $\eta_1 = \eta_2$. In this case the maximum is reached for

$$\eta_1 = \eta_2 = \frac{(\mu_2+\kappa_2)(\kappa_2-\kappa_1)(\xi_1+\xi_2)}{\theta_1\kappa_2+\theta_2\kappa_1+\mu_2}, \tag{2.122}$$

and the upper Hashin-Shtrikman bound is given by

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \leq \mu_2(\xi_1 - \xi_2)^2 + \frac{\kappa_1\kappa_2 + \mu_2(\theta_2\kappa_2 + \theta_1\kappa_1)}{\mu_2 + \theta_1\kappa_2 + \theta_2\kappa_1}(\xi_1 + \xi_2)^2.$$

This case corresponds to the case (**iii**) in the statement of the theorem.

■

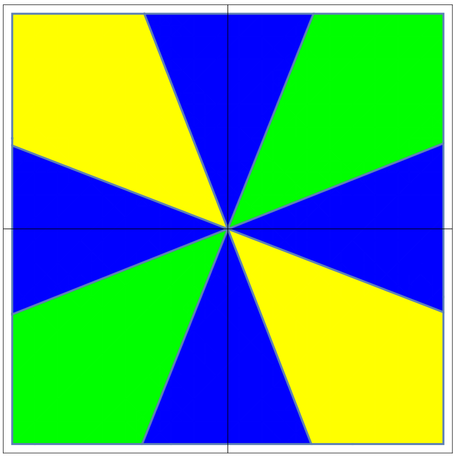


Figure 2.5: Domain division for eigenvalues of $\boldsymbol{\xi}$ in the case of the upper HS bound.

Let us check how the division of $\mathbf{R}^2$ in conditions (2.110), (2.111) and (2.112) looks like, for some given parameters $\theta_1, \mu_1, \mu_2, \kappa_1, \kappa_2$, as for the lower Hashin-Shtrikman bound. For $\mu_1 = 26, \kappa_1 = 40$ (parameters of glass), $\mu_2 = 79, \kappa_2 = 160$ (steel) and $\theta_1 = 0.4$, $\xi_1, \xi_2 \in [-1000, 1000]$, in Figure 2.5 one can see that if (ξ_1, ξ_2) belongs to the blue area then condition (2.110) is satisfied and the maximum is attained in one of the cases $0 \leq \eta_1 < \eta_2$, $0 \leq \eta_2 < \eta_1$, $\eta_1 < \eta_2 \leq 0$ or $\eta_2 < \eta_1 \leq 0$. If (ξ_1, ξ_2) belongs to the yellow area, then the maximum is attained in one of the cases $\eta_2 < 0 < \eta_1$ or $\eta_1 < 0 < \eta_2$, and the green area represents (ξ_1, ξ_2) such that the maximum is attained on the line $\eta_1 = \eta_2$.

It remains to explicitly describe the optimal microstructures for which the upper Hashin-Shtrikman bound is saturated. Theorem 29 assures that the upper Hashin-Shtrikman bound is optimal and that it is saturated by a sequentially laminated microstructures. The optimality condition for the maximization problem on the right-hand side of (2.113) can be derived analogously as for (2.29):

$$-\boldsymbol{\xi} + (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta} = \theta_2 \sum_{i=1}^{p} m_i \frac{(\mathbf{e}_i \otimes \mathbf{e}_i) \otimes (\mathbf{e}_i \otimes \mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i \otimes \mathbf{e}_i) : (\mathbf{e}_i \otimes \mathbf{e}_i)}\boldsymbol{\eta}, \tag{2.123}$$

where $m_i \geq 0$, $\sum_{i=1}^{p} m_i = 1$ and each $\mathbf{e}_i$ is extremal for

$$h(\boldsymbol{\eta}) = \min_{\mathbf{e} \in S^1} \frac{|(\mathbf{e} \otimes \mathbf{e}) : \boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})}. \tag{2.124}$$

If h is differentiable at the optimal $\boldsymbol{\eta}$ then the optimality condition simplifies to

$$-\boldsymbol{\xi} + (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta} = \theta_2 \frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}\boldsymbol{\eta},$$

where $\mathbf{e}$ is an extremal for $h(\boldsymbol{\eta})$ given with (2.124), for this optimal $\boldsymbol{\eta}$. Analogously as for the lower Hashin-Shtrikman bound, one can conclude that in this case the optimality is achieved by a rank-one laminate with the lamination direction $\mathbf{e}$. This is true for every case, except for $\eta_1 = \eta_2$. It remains to specify the optimal microstructure in that case.

Let $\eta := \eta_1 = \eta_2$, $p = 2$ and note that $\boldsymbol{\eta} = \eta\mathbf{I}_2$, where η is given with (2.122). Obviously, every unit vector is an eigenvector of $\boldsymbol{\eta}$ and, by Lemma 15, it is also an extremal vector for function (2.124). Thus, for the direction of lamination we can arbitrarily choose unit eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\boldsymbol{\xi}$, since $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ are simultaneously diagonalizable, and we conclude that (2.123) is equivalent to

$$\begin{bmatrix} -\xi_1 & 0 \\ 0 & -\xi_2 \end{bmatrix} + \frac{\eta}{2(\kappa_2 - \kappa_1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{\theta_2\eta}{\mu_2 + \kappa_2} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}.$$

This determines m_1 and m_2:

$$\begin{aligned} m_1 &= \frac{-2\xi_1(\theta_1\kappa_2 + \theta_2\kappa_1 + \mu_2) + (\mu_2 + \kappa_2)(\xi_1 + \xi_2)}{2\theta_2(\kappa_2 - \kappa_1)(\xi_1 + \xi_2)}, \\ m_2 &= \frac{-2\xi_2(\theta_1\kappa_2 + \theta_2\kappa_1 + \mu_2) + (\mu_2 + \kappa_2)(\xi_1 + \xi_2)}{2\theta_2(\kappa_2 - \kappa_1)(\xi_1 + \xi_2)}, \end{aligned}$$

and it is easy to check that $m_1 + m_2 = 1$ and $m_1,\ m_2 \geq 0$, as a consequence of the condition

$$\theta_2(\kappa_2 - \kappa_1)|\xi_1 + \xi_2| \geq (\theta_1\kappa_2 + \theta_2\kappa_1 + \mu_2)|\xi_1 - \xi_2|$$

which defines this regime. It follows that the bound is achieved by a second rank laminate in the following way: we first layer $\mathbf{A}$ with $\mathbf{B}$ in volume fractions $\rho = 1 - \theta_2 m_1$ and $1 - \rho$ respectively, using layers orthogonal to $\mathbf{v}_1$, to get composite $\mathbf{C}$. After that, we layer $\mathbf{C}$ with $\mathbf{B}$ in volume fractions $\rho' = \dfrac{\theta_1}{1 - \theta_2 m_1}$ and $1 - \rho'$, respectively, using layers orthogonal to $\mathbf{v}_2$, and obtain a composite $\mathbf{A}^*$, which saturates the upper Hashin-Shtrikman bound.

The following theorem summarizes the previous results.

Theorem 37 Let ξ_1 and ξ_2 be the eigenvalues of $\boldsymbol{\xi}$, and $\theta_1 := \theta$, $\theta_2 := 1 - \theta$.

(i) If

$$\begin{aligned} &\theta_2(\kappa_2 - \kappa_1)|\xi_1 + \xi_2| < (\theta_1\kappa_2 + \theta_2\kappa_1 + \mu_2)|\xi_1 - \xi_2| \ \ \& \\ &(\kappa_2 - \kappa_1)|\xi_1 + \xi_2| \geq (\mu_2 - \mu_1)|\xi_2 - \xi_1|, \end{aligned}$$

then the optimal microstructure for which the bound (2.31) is saturated is a simple laminate with layers orthogonal to the eigenvector associated with an eigenvalue of the least absolute value, of the extremal $\boldsymbol{\eta}$ in (2.31).

(ii) If

$$(\kappa_2 - \kappa_1)|\xi_1 + \xi_2| < (\mu_2 - \mu_1)|\xi_2 - \xi_1|,$$

then the optimal microstructure for which the bound (2.31) is saturated is a simple laminate with layers orthogonal to $\mathbf{e}$, such that $\mathbf{e}$ is an extremal for (2.124), where h is function of extremal $\boldsymbol{\eta}$ in (2.31).

(iii) If

$$\theta_2(\kappa_2 - \kappa_1)|\xi_1 + \xi_2| \geq (\theta_1\kappa_2 + \theta_2\kappa_1 + \mu_2)|\xi_1 - \xi_2|,$$

then the optimal microstructure for which the bound (2.31) is saturated is a rank-2 laminate with directions of lamination given with eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\boldsymbol{\xi}$, and corresponding lamination parameters

$$m_1 = \frac{-2\xi_1(\theta_1\kappa_2 + \theta_2\kappa_1 + \mu_2) + (\mu_2 + \kappa_2)(\xi_1 + \xi_2)}{2\theta_2(\kappa_2 - \kappa_1)(\xi_1 + \xi_2)},$$
$$m_2 = \frac{-2\xi_2(\theta_1\kappa_2 + \theta_2\kappa_1 + \mu_2) + (\mu_2 + \kappa_2)(\xi_1 + \xi_2)}{2\theta_2(\kappa_2 - \kappa_1)(\xi_1 + \xi_2)}.$$

2.8 Explicit Hashin-Shtrikman bounds on the complementary energy for mixtures of two isotropic materials in dimension $d = 2$

In the sequel we consider elastic composite materials obtained by mixing two well-ordered isotropic phases $\mathbf{A}$ and $\mathbf{B}$ in proportions $\theta_1 := \theta$ and $\theta_2 := 1 - \theta$, respectively, in an analogous way as it was done when calculating explicit Hashin-Shtrikman bounds on primal energy. The following results will be stated in dimension $d = 2$. In order to explicitly calculate the Hashin-Shtrikman bounds on the complementary energy, first we have to evaluate functions $g_{\mathbf{A}}(\boldsymbol{\eta})$ and $h_{\mathbf{B}}(\boldsymbol{\eta})$. By using that

$$\mathbf{A} = 2\mu_1\mathbf{I}_4 + (\kappa_1 - \mu_1)\mathbf{I}_2 \otimes \mathbf{I}_2,$$
$$\mathbf{B} = 2\mu_2\mathbf{I}_4 + (\kappa_2 - \mu_2)\mathbf{I}_2 \otimes \mathbf{I}_2,$$

where κ_1, κ_2 are the bulk moduli, while μ_1, μ_2 are the shear moduli such that

$$0 < \kappa_1 \leq \kappa_2,\ 0 < \mu_1 \leq \mu_2,$$

we have

$$\mathbf{A}\boldsymbol{\eta} = 2\mu_1\boldsymbol{\eta} + (\kappa_1 - \mu_1)\mathrm{tr}\boldsymbol{\eta}\,\mathbf{I}_2,$$
$$\mathbf{B}\boldsymbol{\eta} = 2\mu_2\boldsymbol{\eta} + (\kappa_2 - \mu_2)\mathrm{tr}\boldsymbol{\eta}\,\mathbf{I}_2.$$

If we label the eigenvalues of $\boldsymbol{\eta}$ by η_1 and η_2, it easily follows that the eigenvalues of $\mathbf{A}\boldsymbol{\eta}$ are equal to

$$\begin{aligned} \lambda_1 &= \eta_1(\mu_1 + \kappa_1) + \eta_2(\kappa_1 - \mu_1), \\ \lambda_2 &= \eta_1(\kappa_1 - \mu_1) + \eta_2(\mu_1 + \kappa_1), \end{aligned} \tag{2.125}$$

while eigenvalues of $\mathbf{B}\boldsymbol{\eta}$ are given in the similar way with

$$\begin{aligned} \nu_1 &= \eta_1(\mu_2 + \kappa_2) + \eta_2(\kappa_2 - \mu_2), \\ \nu_2 &= \eta_1(\kappa_2 - \mu_2) + \eta_2(\mu_2 + \kappa_2). \end{aligned} \tag{2.126}$$

By using Lemma 15, for the isotropic phase $\mathbf{A}$, the function $g_{\mathbf{A}}(\boldsymbol{\eta})$ defined by

$$g_{\mathbf{A}}(\boldsymbol{\eta}) = \max_{\mathbf{e}\in S^1} \frac{|(\mathbf{e}\otimes\mathbf{e}) : \mathbf{A}\boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})} \tag{2.127}$$

equals

$$g_{\mathbf{A}}(\boldsymbol{\eta}) = \frac{1}{\mu_1 + \kappa_1}\begin{cases} \lambda_1^2, \text{ if } & |\lambda_1| \geq |\lambda_2| \\ \lambda_2^2, \text{ if } & |\lambda_2| \geq |\lambda_1| \end{cases}, \tag{2.128}$$

where λ_1, λ_2 are given by (2.125), while for the isotropic phase $\mathbf{B}$ the function $h_{\mathbf{B}}(\boldsymbol{\eta})$ defined by

$$h_{\mathbf{B}}(\boldsymbol{\eta}) = \min_{\mathbf{e}\in S^1} \frac{|(\mathbf{e}\otimes\mathbf{e}) : \mathbf{B}\boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})} \tag{2.129}$$

equals

$$h_{\mathbf{B}}(\boldsymbol{\eta}) = \frac{1}{\mu_2 + \kappa_2}\begin{cases} \nu_2^2, \text{ if } & \nu_1 \leq \nu_2 \leq 0 \text{ or } 0 \leq \nu_2 \leq \nu_1 \\ 0, \text{ if } & \nu_1 < 0 < \nu_2 \text{ or } \nu_2 < 0 < \nu_1 \\ \nu_1^2, \text{ if } & \nu_2 \leq \nu_1 \leq 0 \text{ or } 0 \leq \nu_1 \leq \nu_2 \end{cases}, \tag{2.130}$$

where ν_1, ν_2 are given by (2.126).

Theorem 38 After denoting by σ_1 and σ_2 the eigenvalues of $\boldsymbol{\sigma}$, and $\theta_1 := \theta$, $\theta_2 := 1 - \theta$ as before, the explicit formula for the bound (2.60) is given as follows:

(i) if

$$\begin{aligned} &\theta_2\mu_2(\kappa_2 - \kappa_1)|\sigma_1 + \sigma_2| < (\theta_1\mu_2\kappa_1 + \theta_2\mu_2\kappa_2 + \kappa_1\kappa_2)|\sigma_1 - \sigma_2| \;\; \& \\ &(\kappa_2 - \kappa_1)(\theta_1\mu_1 + \theta_2\mu_2)|\sigma_1 + \sigma_2| \geq (\mu_2 - \mu_1)(\theta_1\kappa_1 + \theta_2\kappa_2)|\sigma_1 - \sigma_2|, \end{aligned} \tag{2.131}$$

then

$$\mathbf{A}^{*-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq (\theta_1\mathbf{A}^{-1} + \theta_2\mathbf{B}^{-1})\boldsymbol{\sigma} : \boldsymbol{\sigma} - \\ - \theta_1\theta_2\frac{[\mu_1\mu_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| + \kappa_1\kappa_2(\mu_2-\mu_1)|\sigma_1-\sigma_2|]^2}{4\mu_1\mu_2\kappa_1\kappa_2[\mu_1\kappa_1\theta_1(\mu_2+\kappa_2)+\mu_2\kappa_2\theta_2(\kappa_1+\mu_1)]};$$

(ii) if

$$(\kappa_2-\kappa_1)(\theta_1\mu_1+\theta_2\mu_2)|\sigma_1+\sigma_2| < (\mu_2-\mu_1)(\theta_1\kappa_1+\theta_2\kappa_2)|\sigma_1-\sigma_2|, \tag{2.132}$$

then

$$\mathbf{A}^{*-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq \mathbf{B}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + \frac{\theta_1}{4}\left[\frac{(\mu_2-\mu_1)(\sigma_1-\sigma_2)^2}{\mu_2(\theta_1\mu_1+\theta_2\mu_2)} + \frac{(\kappa_2-\kappa_1)(\sigma_1+\sigma_2)^2}{\kappa_2(\theta_1\kappa_1+\theta_2\kappa_2)}\right];$$

(iii) if

$$\mu_2\theta_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| \geq (\theta_1\mu_2\kappa_1+\theta_2\mu_2\kappa_2+\kappa_1\kappa_2)|\sigma_1-\sigma_2|, \tag{2.133}$$

then

$$\mathbf{A}^{*-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq \mathbf{B}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + \frac{\theta_1(\kappa_2-\kappa_1)(\mu_2+\kappa_2)(\sigma_1+\sigma_2)^2}{4\kappa_2[\kappa_1(\mu_2+\kappa_2)+\mu_2(\kappa_2-\kappa_1)\theta_2]}.$$

Cases $(i) - (iii)$ are disjoint, and the union of all $(\sigma_1, \sigma_2) \in \mathbf{R}^2$ which satisfy one of the conditions (2.131), (2.132) and (2.133), equals $\mathbf{R}^2$.

Proof. Firstly, note that the expression

$$(\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} = \frac{2\mu_1\mu_2}{\mu_2-\mu_1}(\eta_1^2+\eta_2^2) + \left(\frac{\mu_1\mu_2}{\mu_1-\mu_2} - \frac{\kappa_1\kappa_2}{\kappa_1-\kappa_2}\right)(\eta_1+\eta_2)^2,$$

as well as the function $g^c(\boldsymbol{\eta}) = g^c(\eta_1, \eta_2)$, depend only on the eigenvalues η_1 and η_2 of the matrix $\boldsymbol{\eta}$. Accordingly, in order to explicitly compute the lower Hashin-Shtrikman bound on the complementary energy, we shall use the von Neumann result, which implies that the maximum of $\boldsymbol{\sigma} : \boldsymbol{\eta}$ is obtained when $\boldsymbol{\eta}$ and $\boldsymbol{\sigma}$ are simultaneously diagonalizable and therefore equals $\sum_{i=1}^{2}\eta_i\sigma_i$.

This simplifies the problem, which is now equivalent to maximizing the concave function

$$F(\eta_1,\eta_2) = 2(\sigma_1\eta_1+\sigma_2\eta_2) - \frac{2\mu_1\mu_2}{\mu_2-\mu_1}(\eta_1^2+\eta_2^2) - \left(\frac{\mu_1\mu_2}{\mu_1-\mu_2} - \frac{\kappa_1\kappa_2}{\kappa_1-\kappa_2}\right)(\eta_1+\eta_2)^2 \\ - 2\theta_2\mu_2(\eta_1^2+\eta_2^2) - \theta_2(\kappa_2-\mu_2)(\eta_1+\eta_2)^2 + \theta_2 h_{\mathbf{B}}(\eta_1,\eta_2),$$

in $\mathbf{R}^2$. Note that the function F is quadratic by parts, analogously as in the case of explicit Hashin-Shtrikman bounds on primal energy.

Due to the expression (2.130) for $h_{\mathbf{B}}$, we shall consider several cases (note that $h_{\mathbf{B}}$ is differentiable everywhere except on the line $\nu_1 = \nu_2$).

I Assume $\nu_1 < \nu_2 \leq 0$. After solving the system of equations

$$\frac{\partial F}{\partial \eta_1} = 0 \quad \& \quad \frac{\partial F}{\partial \eta_2} = 0,$$

we obtain that

$$\begin{aligned}
\eta_1 =& \frac{(\mu_2+\kappa_2)[(\kappa_2-\kappa_1)\mu_1\mu_2(\sigma_1+\sigma_2)-(\mu_2-\mu_1)\kappa_1\kappa_2(\sigma_2-\sigma_1)]}{4\mu_2\kappa_2(\mu_1\mu_2\kappa_1\theta_1+\mu_2\kappa_1\kappa_2\theta_2+\mu_1\mu_2\kappa_2\theta_2+\mu_1\kappa_1\kappa_2\theta_1)} \\
\eta_2 =& \frac{(\kappa_2-\kappa_1)\mu_2[\mu_1(\mu_2+\kappa_2)+2\theta_2(\mu_2-\mu_1)\kappa_2](\sigma_1+\sigma_2)}{4\mu_2\kappa_2(\mu_1\mu_2\kappa_1\theta_1+\mu_2\kappa_1\kappa_2\theta_2+\mu_1\mu_2\kappa_2\theta_2+\mu_1\kappa_1\kappa_2\theta_1)} + \\
&+ \frac{(\mu_2-\mu_1)\kappa_2[\kappa_1(\mu_2+\kappa_2)+2\theta_2(\kappa_2-\kappa_1)\mu_2](\sigma_2-\sigma_1)}{4\mu_2\kappa_2(\mu_1\mu_2\kappa_1\theta_1+\mu_2\kappa_1\kappa_2\theta_2+\mu_1\mu_2\kappa_2\theta_2+\mu_1\kappa_1\kappa_2\theta_1)}.
\end{aligned}$$

One can easily calculate $\max F(\eta_1, \eta_2)$, which yields that the bound (2.60) is given by

$$\begin{aligned}
\mathbf{A}^{*-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq & (\theta_1\mathbf{A}^{-1} + \theta_2\mathbf{B}^{-1})\boldsymbol{\sigma} : \boldsymbol{\sigma} - \\
& - \theta_1\theta_2 \frac{[\kappa_1\kappa_2(\mu_1-\mu_2)(\sigma_1-\sigma_2)+\mu_1\mu_2(\kappa_1-\kappa_2)(\sigma_1+\sigma_2)]^2}{4\mu_1\mu_2\kappa_1\kappa_2[\mu_1\kappa_1\theta_1(\mu_2+\kappa_2)+\mu_2\kappa_2\theta_2(\kappa_1+\mu_1)]}
\end{aligned}$$

and, due to $\nu_1 < \nu_2 \leq 0$, it follows that σ_1 and σ_2 satisfy

$$\begin{aligned}
&\theta_2(\kappa_2-\kappa_1)\mu_2(\sigma_1+\sigma_2) > (\theta_1\mu_2\kappa_1+\theta_2\mu_2\kappa_2+\kappa_1\kappa_2)(\sigma_1-\sigma_2), \\
&(\kappa_2-\kappa_1)(\theta_1\mu_1+\theta_2\mu_2)(\sigma_1+\sigma_2) \leq (\mu_2-\mu_1)(\theta_1\kappa_1+\theta_2\kappa_2)(\sigma_1-\sigma_2), \\
&\sigma_1-\sigma_2 < 0 \ \& \ \sigma_1+\sigma_2 < 0.
\end{aligned}$$

II Assume $0 \leq \nu_1 < \nu_2$. After solving the linear system $\nabla F(\eta_1, \eta_2) = 0$ we obtain that

$$\begin{aligned}
\eta_1 =& \frac{(\kappa_2-\kappa_1)\mu_2[\mu_1(\mu_2+\kappa_2)+2\theta_2(\mu_2-\mu_1)\kappa_2](\sigma_1+\sigma_2)}{4\mu_2\kappa_2(\mu_1\mu_2\kappa_1\theta_1+\mu_2\kappa_1\kappa_2\theta_2+\mu_1\mu_2\kappa_2\theta_2+\mu_1\kappa_1\kappa_2\theta_1)} + \\
&+ \frac{(\mu_2-\mu_1)\kappa_2[\kappa_1(\mu_2+\kappa_2)+2\theta_2(\kappa_2-\kappa_1)\mu_2](\sigma_1-\sigma_2)}{4\mu_2\kappa_2(\mu_1\mu_2\kappa_1\theta_1+\mu_2\kappa_1\kappa_2\theta_2+\mu_1\mu_2\kappa_2\theta_2+\mu_1\kappa_1\kappa_2\theta_1)} \\
\eta_2 =& \frac{(\mu_2+\kappa_2)[(\kappa_2-\kappa_1)\mu_1\mu_2(\sigma_1+\sigma_2)-(\mu_2-\mu_1)\kappa_1\kappa_2(\sigma_1-\sigma_2)]}{4\mu_2\kappa_2(\mu_1\mu_2\kappa_1\theta_1+\mu_2\kappa_1\kappa_2\theta_2+\mu_1\mu_2\kappa_2\theta_2+\mu_1\kappa_1\kappa_2\theta_1)}.
\end{aligned}$$

The bound (2.60) in this case equals

$$\begin{aligned}
\mathbf{A}^{*-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq & (\theta_1\mathbf{A}^{-1} + \theta_2\mathbf{B}^{-1})\boldsymbol{\sigma} : \boldsymbol{\sigma} - \\
& - \theta_1\theta_2 \frac{[\kappa_1\kappa_2(\mu_2-\mu_1)(\sigma_1-\sigma_2)+\mu_1\mu_2(\kappa_1-\kappa_2)(\sigma_1+\sigma_2)]^2}{4\mu_1\mu_2\kappa_1\kappa_2[\mu_1\kappa_1\theta_1(\mu_2+\kappa_2)+\mu_2\kappa_2\theta_2(\kappa_1+\mu_1)]}
\end{aligned} \tag{2.134}$$

when σ_1 and σ_2 satisfy

$$\begin{aligned}
&\theta_2(\kappa_2-\kappa_1)\mu_2(\sigma_1+\sigma_2) < (\theta_1\mu_2\kappa_1+\theta_2\mu_2\kappa_2+\kappa_1\kappa_2)(\sigma_2-\sigma_1),\\
&(\kappa_2-\kappa_1)(\theta_1\mu_1+\theta_2\mu_2)(\sigma_1+\sigma_2) \geq (\mu_2-\mu_1)(\theta_1\kappa_1+\theta_2\kappa_2)(\sigma_2-\sigma_1), \qquad (2.135)\\
&\sigma_1-\sigma_2<0 \ \&\ \sigma_1+\sigma_2>0.
\end{aligned}$$

The case $0 \leq \nu_2 < \nu_1$ is symmetric to $0 \leq \nu_1 < \nu_2$, and one only has to change the roles of σ_1 and σ_2 in (2.134) and (2.135). Similar holds for the cases $\nu_2 < \nu_1 \leq 0$ and $\nu_1 < \nu_2 \leq 0$.

Cases **I** and **II**, using some standard, but rather technical algebraic calculations, can jointly be written as: the bound (2.60) is equivalent to

$$\begin{aligned}
\mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \geq (\theta_1\mathbf{A}^{-1}+\theta_2\mathbf{B}^{-1})\boldsymbol{\sigma}:\boldsymbol{\sigma}-\\
-\theta_1\theta_2\frac{[\mu_1\mu_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2|+\kappa_1\kappa_2(\mu_2-\mu_1)|\sigma_1-\sigma_2|]^2}{4\mu_1\mu_2\kappa_1\kappa_2(\mu_1\kappa_1\theta_1(\mu_2+\kappa_2)+\mu_2\kappa_2\theta_2(\kappa_1+\mu_1))}
\end{aligned}$$

if and only if σ_1 and σ_2 satisfy

$$\begin{aligned}
&\theta_2\mu_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| < (\theta_1\mu_2\kappa_1+\theta_2\kappa_2\mu_2+\kappa_1\kappa_2)|\sigma_1-\sigma_2| \ \& \qquad (2.136)\\
&(\kappa_2-\kappa_1)(\theta_1\mu_1+\theta_2\mu_2)|\sigma_1+\sigma_2| \geq (\mu_2-\mu_1)(\theta_1\kappa_1+\theta_2\kappa_2)|\sigma_1-\sigma_2|.
\end{aligned}$$

Note that cases **I** and **II** written jointly, correspond to the case (**i**) given above.

III Assume $\nu_1 < 0 < \nu_2$. Similar computation as before gives

$$\begin{aligned}
\eta_1 &= \frac{1}{4}\left[\frac{(\mu_2-\mu_1)(\sigma_1-\sigma_2)}{\mu_2(\theta_1\mu_1+\theta_2\mu_2)}+\frac{(\kappa_2-\kappa_1)(\sigma_1+\sigma_2)}{\kappa_2(\theta_1\kappa_1+\theta_2\kappa_2)}\right]\\
\eta_2 &= \frac{1}{4}\left[\frac{(\mu_2-\mu_1)(\sigma_2-\sigma_1)}{\mu_2(\theta_1\mu_1+\theta_2\mu_2)}+\frac{(\kappa_2-\kappa_1)(\sigma_1+\sigma_2}{\kappa_2(\theta_1\kappa_1+\theta_2\kappa_2)}\right].
\end{aligned}$$

Furthermore, this yields that

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \geq \mathbf{B}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}+\frac{\theta_1}{4}\left[\frac{(\mu_2-\mu_1)(\sigma_1-\sigma_2)^2}{\mu_2(\theta_1\mu_1+\theta_2\mu_2)}+\frac{(\kappa_2-\kappa_1)(\sigma_1+\sigma_2)^2}{\kappa_2(\theta_1\kappa_1+\theta_2\kappa_2)}\right] \qquad (2.137)$$

when σ_1 and σ_2 satisfy

$$\begin{aligned}
&(\kappa_2-\kappa_1)(\theta_1\mu_1+\theta_2\mu_2)(\sigma_1+\sigma_2) < (\mu_2-\mu_1)(\theta_1\kappa_1+\theta_2\kappa_2)(\sigma_2-\sigma_1) \ \& \qquad (2.138)\\
&(\mu_2-\mu_1)(\theta_1\kappa_1+\theta_2\kappa_2)(\sigma_1-\sigma_2) < (\kappa_2-\kappa_1)(\theta_1\mu_1+\theta_2\mu_2)(\sigma_1+\sigma_2).
\end{aligned}$$

In the case $\nu_2 < 0 < \nu_1$, the function F is the same as for $\nu_1 < 0 < \nu_2$, as well as the obtained bound. An easy computation shows that the bound (2.60) is equivalent to

(2.137) if and only if σ_1 and σ_2 satisfy

$$(\kappa_2-\kappa_1)(\theta_1\mu_1+\theta_2\mu_2)|\sigma_1+\sigma_2| < (\mu_2-\mu_1)(\theta_1\kappa_1+\theta_2\kappa_2)|\sigma_2-\sigma_1|. \tag{2.139}$$

This case corresponds to case (**ii**) in the statement of the theorem.

IV If σ_1 and σ_2 satisfy neither of the conditions (2.136) and (2.139), then the maximum of F is attained on the line $\nu_1=\nu_2$, i.e. $\eta_1=\eta_2$. In this case the maximum is reached for

$$\eta_1=\eta_2=\frac{(\mu_2+\kappa_2)(\kappa_2-\kappa_1)(\sigma_1+\sigma_2)}{4\kappa_2[\kappa_1(\mu_2+\kappa_2)+\theta_2\mu_2(\kappa_2-\kappa_1)]}, \tag{2.140}$$

and the lower Hashin-Shtrikman bound on the complementary energy is given by

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}\geq\mathbf{B}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}+\theta_1\frac{(\mu_2+\kappa_2)(\kappa_2-\kappa_1)(\sigma_1+\sigma_2)^2}{4\kappa_2[\kappa_1(\mu_2+\kappa_2)+\theta_2\mu_2(\kappa_2-\kappa_1)]}.$$

This case corresponds to the case (**iii**) in the statement of the theorem.

■

It remains to explicitly describe the optimal microstructures for which the lower Hashin-Shtrikman bound on the complementary energy is saturated. Theorem 31 assures that the lower Hashin-Shtrikman bound on the complementary energy is optimal and that it is saturated by a sequentially laminated microstructure. The optimality condition for the maximization problem on the right-hand side of (2.60) is given by

$$\boldsymbol{\sigma}-(\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}\boldsymbol{\eta}-\theta_2\left[\mathbf{B}\boldsymbol{\eta}-\sum_{i=1}^{p}m_i\mathbf{B}\frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)}\mathbf{B}\boldsymbol{\eta}\right]=0, \tag{2.141}$$

where $m_i\geq 0$, $\sum_{i=1}^{p}m_i=1$ and each $\mathbf{e}_i$ is extremal for (2.129). If $h_{\mathbf{B}}$ is differentiable at the optimal $\boldsymbol{\eta}$ then the optimality condition simplifies to

$$\boldsymbol{\sigma}-(\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}\boldsymbol{\eta}-\theta_2\left[\mathbf{B}\boldsymbol{\eta}-\mathbf{B}\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\mathbf{B}\boldsymbol{\eta}\right]=0,$$

where $\mathbf{e}$ is an extremal for (2.129). In this case optimality is achieved by a rank-one laminate with the lamination direction $\mathbf{e}$. It remains to specify the microstructures in the case $\nu_1=\nu_2$.

Let $\eta:=\eta_1=\eta_2$, $p=2$ and note that $\boldsymbol{\eta}=\eta\mathbf{I}_2$, where η is given by (2.140). Obviously, every unit vector is an eigenvector of $\boldsymbol{\eta}$ and it is also an extremal vector for function (2.129). Thus, for the direction of lamination we can arbitrarily choose unit eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\boldsymbol{\sigma}$, since $\boldsymbol{\eta}$ and $\boldsymbol{\sigma}$ are simultaneously diagonalizable, and we conclude that

(2.141) is equivalent to

$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} - \left(2\theta_2\kappa_2\eta - \frac{2\eta\kappa_1\kappa_2}{\kappa_1 - \kappa_2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$- \frac{2\theta_2\kappa_2\eta m_1}{\mu_2 + \kappa_2} \begin{bmatrix} \mu_2 + \kappa_2 & 0 \\ 0 & \kappa_2 - \mu_2 \end{bmatrix} - \frac{2\theta_2\kappa_2\eta m_2}{\mu_2 + \kappa_2} \begin{bmatrix} \kappa_2 - \mu_2 & 0 \\ 0 & \mu_2 + \kappa_2 \end{bmatrix}.$$

This determines m_1 and m_2:

$$m_1 = \frac{2\theta_2\mu_2(\kappa_1 - \kappa_2)\sigma_2 + \kappa_1(\mu_2 + \kappa_2)(\sigma_1 - \sigma_2)}{2\theta_2\mu_2(\kappa_1 - \kappa_2)(\sigma_1 + \sigma_2)},$$
$$m_2 = \frac{2\theta_2\mu_2(\kappa_1 - \kappa_2)\sigma_1 - \kappa_1(\mu_2 + \kappa_2)(\sigma_1 - \sigma_2)}{2\theta_2\mu_2(\kappa_1 - \kappa_2)(\sigma_1 + \sigma_2)},$$

and it is easy to check that $m_1 + m_2 = 1$ and m_1, $m_2 \geq 0$, as a consequence of the condition

$$\mu_2\theta_2(\kappa_2 - \kappa_1)|\sigma_1 + \sigma_2| \geq (\theta_1\mu_2\kappa_1 + \theta_2\mu_2\kappa_2 + \kappa_1\kappa_2)|\sigma_1 - \sigma_2|$$

which defines this regime. It follows that the bound is achieved by a second rank laminate in the following way: we first layer $\mathbf{A}$ with $\mathbf{B}$ in volume fractions $\rho = 1 - \theta_2 m_1$ and $1 - \rho$ respectively, using layers orthogonal to $\mathbf{v}_1$, to get composite $\mathbf{C}$. After that, we layer $\mathbf{C}$ with $\mathbf{B}$ in volume fractions $\rho' = \dfrac{\theta_1}{1 - \theta_2 m_1}$ and $1 - \rho'$, respectively, using layers orthogonal to $\mathbf{v}_2$, to get composite $\mathbf{A}^*$ which achieves equality in the lower Hashin-Shtrikman bound on the complementary energy.

The previous results are summarized in the following theorem.

Theorem 39 Let σ_1 and σ_2 be the eigenvalues of $\boldsymbol{\sigma}$, and $\theta_1 := \theta$, $\theta_2 := 1 - \theta$.

(i) If

$$\theta_2\mu_2(\kappa_2 - \kappa_1)|\sigma_1 + \sigma_2| < (\theta_1\mu_2\kappa_1 + \theta_2\mu_2\kappa_2 + \kappa_1\kappa_2)|\sigma_1 - \sigma_2| \;\; \&$$
$$(\kappa_2 - \kappa_1)(\theta_1\mu_1 + \theta_2\mu_2)|\sigma_1 + \sigma_2| \geq (\mu_2 - \mu_1)(\theta_1\kappa_1 + \theta_2\kappa_2)|\sigma_1 - \sigma_2|,$$

then the optimal microstructure for which the bound (2.60) is saturated is a simple laminate with layers orthogonal to the eigenvector associated with an eigenvalue of the least absolute value, of the extremal $\boldsymbol{\eta}$ in (2.60).

(ii) If

$$(\kappa_2 - \kappa_1)(\theta_1\mu_1 + \theta_2\mu_2)|\sigma_1 + \sigma_2| < (\mu_2 - \mu_1)(\theta_1\kappa_1 + \theta_2\kappa_2)|\sigma_1 - \sigma_2|,$$

then the optimal microstructure for which the bound (2.60) is saturated is a simple laminate with layers orthogonal to $\mathbf{e}$, such that $\mathbf{e}$ is an extremal for (2.129), where $h_{\mathbf{B}}$ is function of extremal $\boldsymbol{\eta}$ in (2.60).

(iii) If

$$\mu_2\theta_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| \geq (\theta_1\mu_2\kappa_1+\theta_2\mu_2\kappa_2+\kappa_1\kappa_2)|\sigma_1-\sigma_2|,$$

then the optimal microstructure for which the bound (2.60) is saturated is a rank-2 laminate with directions of lamination given by eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\boldsymbol{\sigma}$, and corresponding lamination parameters

$$m_1 = \frac{2\theta_2\mu_2(\kappa_1-\kappa_2)\sigma_2+\kappa_1(\mu_2+\kappa_2)(\sigma_1-\sigma_2)}{2\theta_2\mu_2(\kappa_1-\kappa_2)(\sigma_1+\sigma_2)},$$
$$m_2 = \frac{2\theta_2\mu_2(\kappa_1-\kappa_2)\sigma_1-\kappa_1(\mu_2+\kappa_2)(\sigma_1-\sigma_2)}{2\theta_2\mu_2(\kappa_1-\kappa_2)(\sigma_1+\sigma_2)}.$$

Theorem 40 After denoting by σ_1 and σ_2 the eigenvalues of $\boldsymbol{\sigma}$, and $\theta_1 := \theta$, $\theta_2 := 1-\theta$ as before, the explicit formula for the bound (2.63) is given as follows:

(i) if

$$\begin{aligned}&\theta_1\mu_1(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| < (\theta_2\mu_1\kappa_2+\theta_1\mu_1\kappa_1+\kappa_1\kappa_2)|\sigma_2-\sigma_1| \ \ \& \\ &\theta_1\kappa_1(\mu_2-\mu_1)|\sigma_2-\sigma_1| < (\theta_1\mu_1\kappa_1+\theta_2\mu_2\kappa_1+\mu_1\mu_2)|\sigma_1+\sigma_2|,\end{aligned} \tag{2.142}$$

then

$$\begin{aligned}\mathsf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \leq {} & (\theta_1\mathsf{A}^{-1}+\theta_2\mathsf{B}^{-1})\boldsymbol{\sigma}:\boldsymbol{\sigma} - \\ & -\theta_1\theta_2\frac{[\mu_1\mu_2(\kappa_1-\kappa_2)|\sigma_1+\sigma_2|+\kappa_1\kappa_2(\mu_2-\mu_1)|\sigma_1-\sigma_2|]^2}{4\mu_1\mu_2\kappa_1\kappa_2[\mu_1\kappa_1\theta_1(\mu_2+\kappa_2)+\mu_2\kappa_2\theta_2(\kappa_1+\mu_1)]};\end{aligned}$$

(ii) if

$$\theta_1\mu_1(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| \geq (\theta_2\mu_1\kappa_2+\theta_1\mu_1\kappa_1+\kappa_1\kappa_2)|\sigma_2-\sigma_1|, \tag{2.143}$$

then

$$\mathsf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \leq \mathsf{A}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} + \frac{\theta_2(\mu_1+\kappa_1)(\kappa_1-\kappa_2)(\sigma_1+\sigma_2)^2}{4\kappa_1[\kappa_2(\mu_1+\kappa_1)+\theta_1\mu_1(\kappa_1-\kappa_2)]};$$

(iii) if

$$\theta_1\kappa_1(\mu_2-\mu_1)|\sigma_2-\sigma_1| \geq (\theta_1\mu_1\kappa_1+\theta_2\mu_2\kappa_1+\mu_1\mu_2)|\sigma_1+\sigma_2|, \tag{2.144}$$

then

$$\mathsf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \leq \mathsf{A}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} + \frac{\theta_2(\mu_1-\mu_2)(\mu_1+\kappa_1)(\sigma_1-\sigma_2)^2}{4\mu_1[\mu_2(\mu_1+\kappa_1)+\theta_1\kappa_1(\mu_1-\mu_2)]}.$$

Cases $(i)-(iii)$ are disjoint, and the union of all $(\sigma_1,\sigma_2) \in \mathbf{R}^2$ which satisfy one of the conditions (2.142), (2.143) and (2.144), equals $\mathbf{R}^2$.

Proof. It is easy to see that (2.63) is equivalent to

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \le \mathbf{A}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} - \theta_2 \max_{\boldsymbol{\eta}\in\mathrm{Sym}}[2\boldsymbol{\sigma}:\boldsymbol{\eta} - (\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} + \theta_1\mathbf{A}\boldsymbol{\eta}:\boldsymbol{\eta} - \theta_1 g_{\mathbf{A}}(\boldsymbol{\eta})]. \tag{2.145}$$

Similarly as for the lower Hashin-Shtrikman bound on the complementary energy, the problem is equivalent to maximizing the concave function

$$\begin{aligned} F(\eta_1,\eta_2) =& 2(\sigma_1\eta_1+\sigma_2\eta_2) - \frac{2\mu_1\mu_2}{\mu_2-\mu_1}(\eta_1^2+\eta_2^2) - \left(\frac{\mu_1\mu_2}{\mu_1-\mu_2} - \frac{\kappa_1\kappa_2}{\kappa_1-\kappa_2}\right)(\eta_1+\eta_2)^2 + \\ &+ 2\theta_1\mu_1(\eta_1^2+\eta_2^2) + \theta_1(\kappa_1-\mu_1)(\eta_1+\eta_2)^2 - \theta_1 g_{\mathbf{A}}(\eta_1,\eta_2), \end{aligned}$$

over all real numbers η_1 and η_2. Note that the function F is quadratic by parts, with minus sign.

Due to the expression (2.128) for $g_{\mathbf{A}}$, we shall consider several cases (note that $g_{\mathbf{A}}$ is differentiable everywhere except on the lines $\lambda_1 = \lambda_2$ and $\lambda_1 = -\lambda_2$, i.e. $\eta_1 = \eta_2$ and $\eta_1 = -\eta_2$, respectively).

I If $|\lambda_1| > |\lambda_2|$, after solving the system of equations

$$\frac{\partial F}{\partial \eta_1} = 0 \quad \& \quad \frac{\partial F}{\partial \eta_2} = 0,$$

we obtain that

$$\begin{aligned} \eta_1 =& \frac{(\kappa_2-\kappa_1)\mu_1[\mu_2(\mu_1+\kappa_1) - 2\theta_1(\mu_2-\mu_1)\kappa_1](\sigma_1+\sigma_2)}{4\mu_1\kappa_1(\mu_1\mu_2\kappa_1\theta_1 + \mu_2\kappa_1\kappa_2\theta_2 + \mu_1\mu_2\kappa_2\theta_2 + \mu_1\kappa_1\kappa_2\theta_1)} + \\ &+ \frac{(\mu_2-\mu_1)\kappa_1[\kappa_2(\mu_1+\kappa_1) - 2\theta_1(\kappa_2-\kappa_1)\mu_1](\sigma_1-\sigma_2)}{4\mu_1\kappa_1(\mu_1\mu_2\kappa_1\theta_1 + \mu_2\kappa_1\kappa_2\theta_2 + \mu_1\mu_2\kappa_2\theta_2 + \mu_1\kappa_1\kappa_2\theta_1)} \\ \eta_2 =& \frac{(\mu_1+\kappa_1)[(\kappa_2-\kappa_1)\mu_1\mu_2(\sigma_1+\sigma_2) - (\mu_2-\mu_1)\kappa_1\kappa_2(\sigma_1-\sigma_2)]}{4\mu_1\kappa_1(\mu_1\mu_2\kappa_1\theta_1 + \mu_2\kappa_1\kappa_2\theta_2 + \mu_1\mu_2\kappa_2\theta_2 + \mu_1\kappa_1\kappa_2\theta_1)}. \end{aligned} \tag{2.146}$$

If maximum is attained in this case, the upper bound is given by

$$\begin{aligned} \mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \le& (\theta_1\mathbf{A}^{-1} + \theta_2\mathbf{B}^{-1})\boldsymbol{\sigma}:\boldsymbol{\sigma} - \\ &- \theta_1\theta_2 \frac{[\mu_1\mu_2(\kappa_1-\kappa_2)(\sigma_1+\sigma_2) + \kappa_1\kappa_2(\mu_2-\mu_1)(\sigma_1-\sigma_2)]^2}{4\mu_1\mu_2\kappa_1\kappa_2[\theta_2\mu_2\kappa_1\kappa_2 + \mu_1(\theta_1\kappa_1\kappa_2 + \mu_2(\theta_2\kappa_2+\theta_1\kappa_1))]}. \end{aligned} \tag{2.147}$$

This bound is asserted if and only if the solution (2.146) satisfies $|\lambda_1| > |\lambda_2|$, i.e. if

$$\begin{aligned} &\theta_1\mu_1(\kappa_2-\kappa_1)(\sigma_1+\sigma_2) > (\theta_1\mu_1\kappa_1 + \theta_2\mu_1\kappa_2 + \kappa_1\kappa_2)(\sigma_1-\sigma_2), \\ &\theta_1\kappa_1(\mu_2-\mu_1)(\sigma_1-\sigma_2) > (\theta_1\mu_1\kappa_1 + \theta_2\mu_2\kappa_1 + \mu_1\mu_2)(\sigma_1+\sigma_2), \\ &\sigma_1-\sigma_2 < 0 \;\&\; \sigma_1+\sigma_2 < 0 \end{aligned} \tag{2.148}$$

or

$$\begin{aligned} &\theta_1\mu_1(\kappa_2-\kappa_1)(\sigma_1+\sigma_2) < (\theta_1\mu_1\kappa_1+\theta_2\mu_1\kappa_2+\kappa_1\kappa_2)(\sigma_1-\sigma_2),\\ &\theta_1\kappa_1(\mu_2-\mu_1)(\sigma_1-\sigma_2) < (\theta_1\mu_1\kappa_1+\theta_2\mu_2\kappa_1+\mu_1\mu_2)(\sigma_1+\sigma_2),\\ &\sigma_1-\sigma_2>0 \ \&\ \sigma_1+\sigma_2>0. \end{aligned} \tag{2.149}$$

II The case $|\lambda_1| < |\lambda_2|$ is symmetric to the previous one, and one only has to change the roles of σ_1 and σ_2 in (2.147), (2.148) and (2.149).

Cases **I** and **II** can jointly be written as: the bound (2.63) is equivalent to

$$\begin{aligned} \mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \le (\theta_1\mathbf{A}^{-1}+\theta_2\mathbf{B}^{-1})\boldsymbol{\sigma}:\boldsymbol{\sigma}-\\ -\theta_1\theta_2\frac{[\mu_1\mu_2(\kappa_1-\kappa_2)|\sigma_1+\sigma_2|+\kappa_1\kappa_2(\mu_2-\mu_1)|\sigma_1-\sigma_2|]^2}{4\mu_1\mu_2\kappa_1\kappa_2[\theta_2\mu_2\kappa_1\kappa_2+\mu_1(\theta_1\kappa_1\kappa_2+\mu_2(\theta_2\kappa_2+\theta_1\kappa_1))]} \end{aligned}$$

if and only if σ_1 and σ_2 satisfy

$$\begin{aligned} &\theta_1\mu_1(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| < (\theta_1\mu_1\kappa_1+\theta_2\mu_1\kappa_2+\kappa_1\kappa_2)|\sigma_2-\sigma_1| \ \& \\ &\theta_1\kappa_1(\mu_2-\mu_1)|\sigma_2-\sigma_1| < (\theta_1\mu_1\kappa_1+\theta_2\mu_2\kappa_1+\mu_1\mu_2)|\sigma_1+\sigma_2|. \end{aligned} \tag{2.150}$$

Note that this corresponds to the case (**i**) in the statement of the theorem.

III If condition (2.150) is not satisfied, then the maximum of F is attained on one of the lines $\lambda_1=\lambda_2$ or $\lambda_1=-\lambda_2$, which is equivalent to $\eta_1=\eta_2$ or $\eta_1=-\eta_2$, respectively.

(ii) If $\eta:=\eta_1=\eta_2$, an easy calculation gives us that the maximum of F is reached for

$$\eta=\frac{(\kappa_2-\kappa_1)(\mu_1+\kappa_1)(\sigma_1+\sigma_2)}{4\kappa_1[\kappa_2(\mu_1+\kappa_1)+\theta_1\mu_1(\kappa_1-\kappa_2)]}. \tag{2.151}$$

If maximum is attained in this case, the corresponding bound is

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}\le\mathbf{A}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}+\frac{\theta_2(\mu_1+\kappa_1)(\kappa_1-\kappa_2)(\sigma_1+\sigma_2)^2}{4\kappa_1[\kappa_2(\mu_1+\kappa_1)+\theta_1\mu_1(\kappa_1-\kappa_2)]}. \tag{2.152}$$

(iii) If $\eta:=\eta_1=-\eta_2$, the maximum of F is attained when

$$\eta=\frac{(\mu_2-\mu_1)(\mu_1+\kappa_1)(\sigma_1-\sigma_2)}{4\mu_1[\mu_2(\mu_1+\kappa_1)+\theta_1\kappa_1(\mu_1-\mu_2)]}. \tag{2.153}$$

If F attains its maximum on the line $\eta_1=-\eta_2$, the explicit Hashin-Shtrikman bound is given by

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}\le\mathbf{A}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}+\frac{\theta_2(\mu_1-\mu_2)(\mu_1+\kappa_1)(\sigma_1-\sigma_2)^2}{4\mu_1[\mu_2(\mu_1+\kappa_1)+\theta_1\kappa_1(\mu_1-\mu_2)]}. \tag{2.154}$$

By using the inequality

$$\theta_1^2\mu_1\kappa_1(\mu_2-\mu_1)(\kappa_2-\kappa_1) < (\theta_2\kappa_2\mu_1+\theta_1\mu_1\kappa_1+\kappa_1\kappa_2)\cdot(\theta_2\mu_2\kappa_1+\theta_1\mu_1\kappa_1+\mu_1\mu_2),$$

it is easy to check that the maximum is attained on the line $\eta_1 = \eta_2$ if and only if

$$\theta_1\mu_1(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| \geq (\theta_2\mu_1\kappa_2+\theta_1\mu_1\kappa_1+\kappa_1\kappa_2)|\sigma_2-\sigma_1|.$$

Conversely, the maximum is attained on the line $\eta_1 = -\eta_2$ if and only if

$$\theta_1\kappa_1(\mu_2-\mu_1)|\sigma_2-\sigma_1| \geq (\theta_1\mu_1\kappa_1+\theta_2\mu_2\kappa_1+\mu_1\mu_2)|\sigma_1+\sigma_2|.$$

■

It remains to explicitly describe the optimal microstructures for which the upper Hashin-Shtrikman bound on the complementary energy is saturated. Theorem 31 assures that the upper Hashin-Shtrikman bound on the complementary energy is optimal and that optimality is achieved by a finite-rank sequential laminate. The optimality condition for the maximization problem on the right-hand side of (2.145) can be derived analogously as for (2.60):

$$\boldsymbol{\sigma} - (\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}\boldsymbol{\eta} + \theta_1\left[\mathbf{A}\boldsymbol{\eta} - \sum_{i=1}^{p} m_i \mathbf{A}\frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{A}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)}\mathbf{A}\boldsymbol{\eta}\right] = 0, \tag{2.155}$$

where $m_i \geq 0$, $\sum_{i=1}^{p} m_i = 1$ and each $\mathbf{e}_i$ is extremal for (2.127). If $g_{\mathbf{A}}$ is differentiable at the optimal $\boldsymbol{\eta}$ then the optimality condition simplifies to

$$\boldsymbol{\sigma} - (\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}\boldsymbol{\eta} + \theta_1\left[\mathbf{A}\boldsymbol{\eta} - \mathbf{A}\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\mathbf{A}\boldsymbol{\eta}\right] = 0,$$

where $\mathbf{e}$ is an extremal for (2.127). In this case optimality is achieved by a rank-one laminate with the lamination direction $\mathbf{e}$. This corresponds to the first case in Theorem 40, i.e. when the maximum in (2.63) is achieved for $|\lambda_1| \neq |\lambda_2|$. It remains to specify the microstructures in the cases $\lambda_1 = \lambda_2$ and $\lambda_1 = -\lambda_2$, i.e. $\eta_1 = \eta_2$ and $\eta_1 = -\eta_2$, respectively.

a) Let $\eta := \eta_1 = \eta_2$, $p = 2$ and note that $\boldsymbol{\eta} = \eta\mathbf{I}_2$, where η is given by (2.151). Every unit vector is an eigenvector of $\boldsymbol{\eta}$ and it is also an extremal vector for function (2.127). Thus, for the direction of lamination we can arbitrarily choose unit eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\boldsymbol{\sigma}$, since $\boldsymbol{\eta}$ and $\boldsymbol{\sigma}$ are simultaneously diagonalizable, and we conclude that

(2.155) is equivalent to

$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} + \left(2\theta_1\kappa_1\eta + \frac{2\eta\kappa_1\kappa_2}{\kappa_1-\kappa_2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$\frac{2\theta_1\kappa_1\eta m_1}{\mu_1+\kappa_1}\begin{bmatrix} \mu_1+\kappa_1 & 0 \\ 0 & \kappa_1-\mu_1 \end{bmatrix} + \frac{2\theta_1\kappa_1\eta m_2}{\mu_1+\kappa_1}\begin{bmatrix} \kappa_1-\mu_1 & 0 \\ 0 & \mu_1+\kappa_1 \end{bmatrix}.$$

This determines m_1 and m_2:

$$m_1 = \frac{2\theta_1\mu_1(\kappa_2-\kappa_1)\sigma_2 + \kappa_2(\mu_1+\kappa_1)(\sigma_1-\sigma_2)}{2\theta_1\mu_1(\kappa_2-\kappa_1)(\sigma_1+\sigma_2)},$$
$$m_2 = \frac{2\theta_1\mu_1(\kappa_2-\kappa_1)\sigma_1 + \kappa_2(\mu_1+\kappa_1)(\sigma_2-\sigma_1)}{2\theta_1\mu_1(\kappa_2-\kappa_1)(\sigma_1+\sigma_2)},$$

and it is easy to check that $m_1+m_2=1$ and $m_1,\, m_2 \geq 0$, as a consequence of the condition

$$\theta_1\mu_1(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| \geq (\theta_2\mu_1\kappa_2+\theta_1\mu_1\kappa_1+\kappa_1\kappa_2)|\sigma_2-\sigma_1|$$

which defines this regime. In this case, the bound is obviously achieved by a second rank laminate in the following way: we first layer **B** with **A** in volume fractions $\rho = 1-\theta_1 m_1$ and $1-\rho$ respectively, in direction of lamination $\mathbf{v}_1$, to get composite **C**. After that, we layer **C** with **A** in volume fractions $\rho' = \dfrac{\theta_2}{1-\theta_1 m_1}$ and $1-\rho'$, respectively, in direction of lamination $\mathbf{v}_2$, to get composite $\mathbf{A}^*$ which achieves equality in the upper Hashin-Shtrikman bound on the complementary energy.

b) Let $\eta_1 = -\eta_2$, $p=2$ and $\mathbf{v}_1$, $\mathbf{v}_2$ the associated unit eigenvectors of $\boldsymbol{\eta}$, such that they are also eigenvectors of $\boldsymbol{\sigma}$. Additionally, these vectors are extremal for function (2.127). Denoting $\eta := \eta_1 = -\eta_2$, we have that η is given by (2.153).

Using the fact that $\boldsymbol{\sigma}$ and $\boldsymbol{\eta}$ are simultaneously diagonalizable, we conclude that (2.155) is equivalent to

$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} + \left(2\theta_1\mu_1\eta - \frac{2\eta\mu_1\mu_2}{\mu_2-\mu_1}\right)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} =$$

$$\frac{2\theta_1\mu_1\eta m_1}{\mu_1+\kappa_1}\begin{bmatrix} \mu_1+\kappa_1 & 0 \\ 0 & \kappa_1-\mu_1 \end{bmatrix} - \frac{2\theta_1\mu_1\eta m_2}{\mu_1+\kappa_1}\begin{bmatrix} \kappa_1-\mu_1 & 0 \\ 0 & \mu_1+\kappa_1 \end{bmatrix}.$$

This determines m_1 and m_2:

$$m_1 = \frac{2\kappa_1\theta_1(\mu_2-\mu_1)\sigma_2 - \mu_2(\mu_1+\kappa_1)(\sigma_1+\sigma_2)}{2\kappa_1\theta_1(\mu_2-\mu_1)(\sigma_2-\sigma_1)},$$

$$m_2 = \frac{-2\kappa_1\theta_1(\mu_2 - \mu_1)\sigma_1 + \mu_2(\mu_1 + \kappa_1)(\sigma_1 + \sigma_2)}{2\kappa_1\theta_1(\mu_2 - \mu_1)(\sigma_2 - \sigma_1)}.$$

It is easy to check that $m_1 + m_2 = 1$ and m_1, $m_2 \geq 0$, as a consequence of the condition

$$\theta_1\kappa_1(\mu_2 - \mu_1)|\sigma_2 - \sigma_1| \geq (\theta_1\mu_1\kappa_1 + \theta_2\mu_2\kappa_1 + \mu_1\mu_2)|\sigma_1 + \sigma_2|$$

which defines this regime. Now, one can conclude that the bound is achieved by a second rank laminate in an analogous way as for $\eta_1 = \eta_2$.

The following theorem summarizes the previous results.

Theorem 41 Let σ_1 and σ_2 be the eigenvalues of $\boldsymbol{\sigma}$, and $\theta_1 := \theta$, $\theta_2 := 1 - \theta$.

(i) If

$$\theta_1\mu_1(\kappa_2 - \kappa_1)|\sigma_1 + \sigma_2| < (\theta_2\mu_1\kappa_2 + \theta_1\mu_1\kappa_1 + \kappa_1\kappa_2)|\sigma_2 - \sigma_1| \ \ \& \\ \theta_1\kappa_1(\mu_2 - \mu_1)|\sigma_2 - \sigma_1| < (\theta_1\mu_1\kappa_1 + \theta_2\mu_2\kappa_1 + \mu_1\mu_2)|\sigma_1 + \sigma_2|,$$

then the optimal microstructure for which the bound (2.63) is saturated is a simple laminate with layers orthogonal to the eigenvector associated with an eigenvalue of the largest absolute value, of the extremal $\boldsymbol{\eta}$ in (2.63).

(ii) If

$$\theta_1\mu_1(\kappa_2 - \kappa_1)|\sigma_1 + \sigma_2| \geq (\theta_2\mu_1\kappa_2 + \theta_1\mu_1\kappa_1 + \kappa_1\kappa_2)|\sigma_2 - \sigma_1|,$$

then the optimal microstructure for which the bound (2.63) is saturated is a rank-2 laminate with directions of lamination given by eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\boldsymbol{\sigma}$, and corresponding lamination parameters

$$m_1 = \frac{2\theta_1\mu_1(\kappa_2 - \kappa_1)\sigma_2 + \kappa_2(\mu_1 + \kappa_1)(\sigma_1 - \sigma_2)}{2\theta_1\mu_1(\kappa_2 - \kappa_1)(\sigma_1 + \sigma_2)},$$
$$m_2 = \frac{2\theta_1\mu_1(\kappa_2 - \kappa_1)\sigma_1 + \kappa_2(\mu_1 + \kappa_1)(\sigma_2 - \sigma_1)}{2\theta_1\mu_1(\kappa_2 - \kappa_1)(\sigma_1 + \sigma_2)}.$$

(iii) If

$$\theta_1\kappa_1(\mu_2 - \mu_1)|\sigma_2 - \sigma_1| \geq (\theta_1\mu_1\kappa_1 + \theta_2\mu_2\kappa_1 + \mu_1\mu_2)|\sigma_1 + \sigma_2|,$$

then the optimal microstructure for which the bound (2.63) is saturated is a rank-2 laminate with directions of lamination given with eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the extremal $\boldsymbol{\eta}$ in (2.63) (which are also eigenvectors of $\boldsymbol{\xi}$), and corresponding lamination parameters

$$m_1 = \frac{2\kappa_1\theta_1(\mu_2 - \mu_1)\sigma_2 - \mu_2(\mu_1 + \kappa_1)(\sigma_1 + \sigma_2)}{2\kappa_1\theta_1(\mu_2 - \mu_1)(\sigma_2 - \sigma_1)},$$

$$m_2 = \frac{-2\kappa_1\theta_1(\mu_2 - \mu_1)\sigma_1 + \mu_2(\mu_1 + \kappa_1)(\sigma_1 + \sigma_2)}{2\kappa_1\theta_1(\mu_2 - \mu_1)(\sigma_2 - \sigma_1)}.$$

Appendix

Let us first introduce some general notation used in the thesis. In the sequel, by U, W we denote open subsets of $\mathbf{R}^d$.

We say that W is compactly embedded in U and write $W \Subset U$, if $W \subseteq \operatorname{Cl} W \subseteq U$ and $\operatorname{Cl} W$ is compact.

- $C(U)$ denotes vector space of continuous functions defined on U, while $C^k(U)$ denotes vector space of k-times continuously differentiable functions defined on U.

- $C_c(U)$, $C_c^k(U)$, $L_c^p(U)$, etc. denote vector spaces of functions in $C(U)$, $C^k(U)$, $L^p(U)$, etc. with compact support.

- $\mathrm{L}^p_{\mathrm{loc}}(U) := \{f : U \to \mathbf{R} : (\forall W \Subset U)\ f \in \mathrm{L}^p(W)\}$.

Analogously, one could define functions $H^k_{\mathrm{loc}}(U)$, $H^k_c(U)$, etc. We refer the interested reader to [19].

Let $R : X \to Y$, where X, Y are Banach spaces. We write $R(\mathbf{h}) := o(\|\mathbf{h}\|_X)$, if

$$\lim_{\mathbf{h}\to 0} \frac{\|R(\mathbf{h})\|_Y}{\|\mathbf{h}\|_X} = 0.$$

In the following two definitions [32], by X we denote a locally compact Hausdorff space (a topological space is called locally compact if every point has a compact neighbourhood).

Definition 14 Let μ be a Borel measure on X and E a Borel subset of X. The measure μ is called outer regular on E if

$$\mu(E) = \inf\{\mu(U) : E \subset U,\, U \text{ open}\}$$

and inner regular on E if

$$\mu(E) = \sup\{\mu(K) : K \subset E,\, K \text{ compact}\}.$$

If μ is outer and inner regular on all Borel sets, μ is called regular. For example, the Lebesgue measure λ on $\mathbf{R}^d$ is regular.

Definition 15 A Radon measure on X is a Borel measure that is finite on all compact sets, outer regular on all Borel sets and inner regular on all open sets. By $\mathcal{M}(X)$ we denote the space of Radon measures on X.

The connection between strong and pointwise convergence is given in the following lemma.

Lemma 16 [2, p. 17, Lemma 1.2.3] Let Ω be a bounded open set in $\mathbf{R}^d$. For $1 < p \leq +\infty$, let (u_n) be a bounded sequence in $\mathrm{L}^p(\Omega)$ such that

$$u_n(\mathbf{x}) \longrightarrow u(\mathbf{x}) \text{ a. e. in } \Omega.$$

Then the sequence (u_n) converges strongly to u in any $\mathrm{L}^q(\Omega)$ with $1 \leq q < p$.

Lemma 17 [2, p. 18, Lemma 1.2.6] (Rellich theorem) Let Ω be a bounded open set in $\mathbf{R}^d$, and (u_n) a bounded sequence in $W^{1,p}(\Omega)$, $1 \leq p < \infty$. Then, there exists a subsequence, still denoted by n, and a limit $u \in W^{1,p}(\Omega)$, such that, for this subsequence, (u_n) converges strongly to u in $\mathrm{L}^p(\Omega)$.

Theorem 42 [30, p. 732, Theorem 6] Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d)$.

(i) Then for a. e. point $\mathbf{x}_0 \in \mathbf{R}^d$

$$\fint_{B(\mathbf{x}_0,r)} f(\mathbf{x})\, d\mathbf{x} \longrightarrow f(\mathbf{x}_0) \text{ as } r \longrightarrow 0.$$

(ii) For a. e. point $\mathbf{x}_0 \in \mathbf{R}^d$

$$\fint_{B(\mathbf{x}_0,r)} |f(\mathbf{x}) - f(\mathbf{x}_0)|\, d\mathbf{x} \longrightarrow 0 \text{ as } r \longrightarrow 0. \tag{2.156}$$

A point $\mathbf{x}_0$ at which (2.156) holds is called a Lebesgue point of f.

Remark 13 [30, p. 733] If $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ for some $1 \leq p < \infty$, then for a. e. point $\mathbf{x}_0 \in \mathbf{R}^d$ we have

$$\fint_{B(\mathbf{x}_0,r)} |f(\mathbf{x}) - f(\mathbf{x}_0)|^p\, d\mathbf{x} \longrightarrow 0 \text{ as } r \longrightarrow 0.$$

The following lemma is widely used in the case of periodic homogenization.

Lemma 18 [24, p. 33, Theorem 2.6] Let $f \in \mathrm{L}^p_{\#}(Y)$, $1 \leq p \leq +\infty$. The sequence (f_n), defined by

$$f_n(\mathbf{x}) := f(n\mathbf{x}),$$

converges weakly in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ to the average $\int_Y f(\boldsymbol{y})d\boldsymbol{y}$ (weakly-$*$ if $p = +\infty$).

Let us recall the following well known, but also elementary results.

Proposition 3 [14, p. 37, Proposition 2.3.1] Let V be a vector space, $L : V \to \mathbf{R}$ a linear form and $a : V \times V \to \mathbf{R}$ a bilinear, symmetric, coercive form. Then the following two statements are equivalent for $\mathbf{u} \in V$:

(i) $a(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}),\ \ \mathbf{v} \in V,$

(ii) $J(\mathbf{u}) \leq J(\mathbf{v}),\ \ \mathbf{v} \in V,$

where $J(\mathbf{v}) := \dfrac{1}{2}a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v})$.

Theorem 43 [14, p. 67, Theorem 3.1.1](Riesz) Let V be a Hilbert space and $L \in V'$ a linear continuous form on V. Then

$$(\exists!\, \mathbf{f} \in V)\ \ L(\mathbf{v}) = (\mathbf{f}, \mathbf{v}),\ \ \mathbf{v} \in V.$$

Theorem 44 [14, p. 69, Theorem 3.1.2](Lax-Milgram) Let V be a Hilbert space with the scalar product $(\cdot, \cdot)$ and $\|\cdot\|_V = \sqrt{(\cdot, \cdot)}$ the associated norm. Let $a : V \times V \to \mathbf{R}$ be a bilinear form which satisfies:

(i) a is continuous:

$$(\exists M \in \mathbf{R}^+)\quad |a(\mathbf{u}, \mathbf{v})| \leq M\|\mathbf{u}\|_V \cdot \|\mathbf{v}\|_V,\ \ \mathbf{u}, \mathbf{v} \in V;$$

(ii) a is coercive:

$$(\exists \alpha > 0)\quad a(\mathbf{v}, \mathbf{v}) \geq \alpha\|\mathbf{v}\|_V^2,\ \ \mathbf{v} \in V.$$

Then for any $L \in V'$ there exists a unique $\mathbf{u} \in V$ such that

$$a(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}),\ \ \mathbf{v} \in V.$$

A useful tool for dealing with periodic functions is the Fourier series. Therefore, we introduce some basic facts of Fourier analysis on the torus T. Recall that the d-dimensional torus T is the cube $[0, 1]^d$ with opposite sides identified. Functions on T are defined as functions f on $\mathbf{R}^d$ that satisfy $f(\mathbf{x} + \mathbf{m}) = f(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^d$, $\mathbf{m} \in \mathbf{Z}^d$ [37, p. 162-163].

Definition 16 For a complex-valued function $f \in \mathrm{L}^1(T)$ and $\mathbf{k} \in \mathbf{Z}^d$, we define

$$\hat{f}(\mathbf{k}) := \int_T f(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}}\, d\mathbf{x}.$$

We call $\hat{f}(\mathbf{k})$ the k-th Fourier coefficient of f. The Fourier series of f at $\mathbf{x} \in T$ is the series

$$\sum_{\mathbf{k} \in \mathbf{Z}^d} \hat{f}(\mathbf{k}) e^{2\pi i \mathbf{k} \cdot \mathbf{x}}.$$

Let us denote by $\overline{f}$ the complex conjugate of the function f, and by $\tilde{f}$ the function $\tilde{f}(\mathbf{x}) := f(-\mathbf{x})$, $\mathbf{x} \in T$.

Proposition 4 [37, p. 164, Proposition 3.1.2] Let $f, g \in \mathrm{L}^1(T)$. Then for all $\mathbf{k}, \mathbf{m} \in \mathbf{Z}^d$, $\lambda \in \mathbf{C}$ and $\boldsymbol{y} \in T$ we have:

(i) $\widehat{f+g}\,(\mathbf{k}) = \hat{f}(\mathbf{k}) + \hat{g}(\mathbf{k})$,

(ii) $\widehat{\lambda f}(\mathbf{k}) = \lambda \hat{f}(\mathbf{k})$,

(iii) $\widehat{\overline{f}}(\mathbf{k}) = \overline{\hat{f}(-\mathbf{k})}$,

(iv) $\widehat{\tilde{f}}(\mathbf{k}) = \hat{f}(-\mathbf{k})$,

(v) $\hat{f}(0) = \int\limits_T f(\mathbf{x})\, d\mathbf{x}$,

(vi) $\sup\limits_{\mathbf{k} \in \mathbf{Z}^d} |\hat{f}(\mathbf{k})| \leq \|f\|_{\mathrm{L}^1(T)}$.

Proposition 5 [37, p. 170, Proposition 3.1.16] The following are valid for $f, g \in \mathrm{L}^2(T)$:

(i) (Plancherel's identity)

$$\|f\|^2_{\mathrm{L}^2(T)} = \sum_{\mathbf{k} \in \mathbf{Z}^d} |\hat{f}(\mathbf{k})|^2.$$

(ii) (Parseval's relation)

$$\int\limits_T f(\mathbf{x})\overline{g(\mathbf{x})}\, d\mathbf{x} = \sum_{\mathbf{k} \in \mathbf{Z}^d} \hat{f}(\mathbf{k})\overline{\hat{g}(\mathbf{k})}.$$

Now, we summarise some elementary facts about fourth-order tensors, which are necessary for better understanding this thesis.

Let $\mathrm{Sym} := \{\mathbf{A} = \mathbf{A}^T \in M_d(\mathbf{R})\}$ be the set of real symmetric matrices, and by $\mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ we denote the space of linear operators $\mathsf{A} : \mathrm{Sym} \to \mathrm{Sym}$. The product of A, $\mathsf{B} \in \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ is defined by the composition:

$$(\mathsf{AB})\boldsymbol{\eta} = \mathsf{A}(\mathsf{B}\boldsymbol{\eta}), \quad \boldsymbol{\eta} \in \mathrm{Sym}.$$

It is easy to see that $\mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ can be identified with the space of fourth-order tensors whose entries satisfy the symmetry condition: $a_{ijkl} = a_{jikl} = a_{ijlk}$, $1 \leq i, j, k, l \leq d$, with

$$[\mathsf{A}\boldsymbol{\eta}]_{ij} = \sum_{1 \leq k,l \leq d} a_{ijkl}\eta_{kl}, \quad \boldsymbol{\eta} \in \mathrm{Sym}$$

and

$$[\mathsf{AB}]_{ijkl} = \sum_{1 \leq m,n \leq d} a_{ijmn} b_{mnkl}.$$

Furthermore,

$$\mathrm{Sym}^4 := \{\mathbf{A} \in \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}) : a_{ijkl} = a_{klij}\}$$

is the set of real symmetric fourth-order tensors acting on symmetric matrices. We also consider *fully symmetric* fourth-order tensors, which are symmetric tensors with additional symmetry $a_{ijkl} = a_{kjil}$. Note that every permutation of the indices $\{i, j, k, l\}$ gives the same entry a_{ijkl} of fully symmetric fourth-order tensor.

The transpose of $\mathbf{A} \in \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$, defined with $\mathbf{A}\boldsymbol{\eta} : \boldsymbol{\xi} = \boldsymbol{\eta} : \mathbf{A}^T\boldsymbol{\xi}$, $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathrm{Sym}$, has entries which satisfy

$$[\mathbf{A}^T]_{ijkl} = [\mathbf{A}]_{klij}.$$

The identity operator $\mathbf{I}_4 \in \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$, defined by $\mathbf{I}_4\boldsymbol{\eta} = \boldsymbol{\eta}$, $\boldsymbol{\eta} \in \mathrm{Sym}$, has entries given by

$$[\mathbf{I}_4]_{ijkl} = \begin{cases} 1, & \text{if } i = j = k = l \\ \frac{1}{2}, & \text{if } i \neq j, (k,l) \in \{(i,j),(j,i)\} \\ 0, & \text{otherwise,} \end{cases}$$

while $\mathbf{I}_2$ denotes identity matrix in $M_d(\mathbf{R})$.

For $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathrm{Sym}$ the standard inner product is defined as

$$\boldsymbol{\xi} : \boldsymbol{\eta} = \sum_{1 \leq i,j \leq d} \xi_{ij}\eta_{ij}.$$

If $\mathbf{a}, \mathbf{b} \in \mathbf{R}^d$, their tensor product is defined as $d \times d$ matrix whose entries are given by

$$[\mathbf{a} \otimes \mathbf{b}]_{ij} = a_i b_j.$$

The tensor product of two matrices $\mathbf{A}, \mathbf{B} \in M_d(\mathbf{R})$ is the fourth-order tensor with entries

$$[\mathbf{A} \otimes \mathbf{B}]_{ijkl} = a_{ij} b_{kl}.$$

For $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbf{R}^d$, $\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\rho}, \boldsymbol{\sigma} \in \mathrm{Sym}$ and $\mathbf{A} \in \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$, the following identities hold:

$$[(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d})]_{ij} = (\mathbf{b} \cdot \mathbf{c})\, \mathbf{a}_i \mathbf{d}_j,$$

$$\mathbf{A}(\boldsymbol{\xi} \otimes \boldsymbol{\eta}) = (\mathbf{A}\boldsymbol{\xi}) \otimes \boldsymbol{\eta},$$

$$(\boldsymbol{\xi} \otimes \boldsymbol{\eta})\mathbf{A} = \boldsymbol{\xi} \otimes \mathbf{A}^T\boldsymbol{\eta},$$

$$(\boldsymbol{\xi} \otimes \boldsymbol{\eta})(\boldsymbol{\rho} \otimes \boldsymbol{\sigma}) = (\boldsymbol{\eta} : \boldsymbol{\rho})\boldsymbol{\xi} \otimes \boldsymbol{\sigma}.$$

If the columns of $\mathbf{S} \in M_d(\mathbf{R})$ are denoted by $\mathbf{S}^j$, $j = 1, \ldots, d$, then $\operatorname{div} \mathbf{S}$ is a vector with entries

$$(\operatorname{div} \mathbf{S})_j = \operatorname{div} \mathbf{S}^j, \; j = 1, \ldots, d.$$

Let $\mathbf{R} \in SO(\mathbf{R}^d)$ be a rotation matrix. The general rule for applying the rotation to a fourth-order tensor $\mathbf{C}$ is the following: we obtain a tensor $\mathbf{C}'$ with components

$$c'_{mnop} = \sum_{1 \le i,j,k,l \le d} r_{mi} r_{nj} r_{ok} r_{pl} c_{ijkl}.$$

For simplicity of notation, we denote $\mathbf{C}' = \mathcal{R}(\mathbf{R}, \mathbf{C})$, where $\mathcal{R} : SO(\mathbf{R}^d) \times \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}) \to \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ (see [62] for details).

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Curriculum vitae

Jelena Jankov was born on March 26, 1991 in Vukovar, Croatia. After finishing high school in Vukovar, she enroled in the undergraduate program in Mathematics at the Department of Mathematics, J. J. Strossmayer University of Osijek. In July 2014, she obtained master's degree Financial Mathematics and Statistics at the Department of Mathematics, University of Osijek. In November 2014, she started Croatian doctoral program in Mathematics at the Department of Mathematics, Faculty of Science at the University of Zagreb. From October 2014 she has been employed as a teaching assistant at the Department of Mathematics, University of Osijek.

During hers career she participated in many programmes of professional development; she attended five international schools and twelve international conferences and workshops so far, giving a talk at seven of them and poster presentation at one. She was one of the collaborators in two scientific projects, "Damping optimization in mechanical systems excited with external force" supported by University of Osijek in 2015/16, under the leadership of assoc. prof. Z. Tomljanović and bilateral project with Serbia, "Calculus of variations, optimisation and applications" supported by Serbian Ministry of Science in 2016/17 under the leadership of assoc. prof. K. Burazin from University of Osijek (croatian side) and prof. N. Krejić from University of Novi Sad (serbian side). Currently, she is the collaborator in scientific project "Homogenization, dimension reduction and structural optimization in continuum mechanics" funded by Croatian Science Foundation since 2018, under the leadership of assoc. prof. I. Velčić.

List of publications

Journal Publications:

1. K. Burazin, J. Jankov, M. Vrdoljak, Homogenization of elastic plate equation, *Mathematical Modelling and Analysis*, 23/2, 190-204, 2018.

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