Spectral analysis of thin heterogeneous elastic structures

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DOCTORAL DISSERTATION

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Summary

This thesis consists of two parts. In the first part of the thesis, we analyse the behaviour of thin composite plates whose material properties vary periodically in-plane and possess a high degree of contrast between the individual components. Starting from the resolvent equations of three-dimensional linear elasticity that describe soft inclusions embedded in a relatively stiff thin-plate matrix, we derive the corresponding asymptotically equivalent two-dimensional plate equations. Our approach is based on recent results concerning decomposition of deformations with bounded scaled symmetrised gradients. Using an operator-theoretic approach, first we calculate the limit resolvent and analyse the associated limit spectrum and effective evolution equations. We obtain our results under various asymptotic relations between the size of the soft inclusions (equivalently, the period) and the plate thickness as well as under various scaling combinations between the contrast, spectrum, and time. In particular, we demonstrate significant qualitative differences between the asymptotic models obtained in different regimes.

In the second part of the thesis, we provide resolvent asymptotics as well as various operator-norm estimates for the system of linear partial differential equations describing the thin infinite elastic rod with material coefficients which periodically highly oscillate along the rod. The resolvent asymptotics is derived simultaneously with respect to the thickness of the rod and the period of material oscillations. These two parameters are taken to be of the same order. The analysis is carried out separately on two invariant subspaces pertaining to the out-of-line and in-line displacements, under some additional assumptions, as well as in the general case where these two sorts of displacements intertwine inseparably.

Keywords Homogenisation · Dimension reduction · Two-scale convergence · High-contrast · Resolvent asymptotics · Elastic heterogeneous rods and plates

Sažetak

Ovaj rad sastoji se od dva dijela. U prvom dijelu rada analiziramo ponašanje tankih kompozitnih ploča čija svojstva materijala periodično variraju u ravnini i posjeduju visok stupanj kontrasta između pojedinih komponenti. Polazeći od rezolventnih jednadžbi trodimenzionalne linearne elastičnosti koje opisuju meke inkluzije ugrađene u relativno krutu matricu tanke ploče, izvodimo odgovarajuće asimptotski ekvivalentne jednadžbe dvodimenzionalne ploče. Naš pristup temelji se na nedavnim rezultatima o dekompoziciji deformacija s ograničenim simetriziranim gradijentima. Koristeći pristup teorije operatora, najprije izračunavamo limes rezolventu te analiziramo pridruženi limes spektar i efektivne evolucijske jednadžbe. Naše rezultate dobivamo pod različitim asimptotičkim odnosima između veličine mekih inkluzija (perioda oscilacija) i debljine ploče, kao i pod različitim kombinacijama skaliranja između kontrasta, spektra i vremena. Također pokazujemo značajne kvalitativne razlike između asimptotskih modela dobivenih u različitim režimima.

U drugom dijelu rada izvodimo asimptotiku rezolventi kao i razne ocjene u operatorskim normama za sustav linearnih parcijalnih diferencijalnih jednadžbi koje opisuju tanki beskonačni elastični štap s materijalnim koeficijentima koji periodično jako osciliraju duž štapa. Rezolventnu asimptotiku izvodimo simultano s obzirom na debljinu štapa i period oscilacija materijala. Uzimamo da su ova dva parametra istog reda. Analizu provodimo zasebno na dva invarijantna podprostora koji se odnose na pomake duž prostiranja štapa i pomake okomite na prostiranje štapa, pri čemu pretpostavljamo neke dodatne pretpostavke. Također provodimo analizu i u općem slučaju kada se ove dvije vrste pomaka neraskidivo isprepliću.

Ključne riječi Homogenizacija · Redukcija dimenzije · Dvoskalna konvergencija · Visoki kontrast · Rezolventna asimptotika · Elastični heterogeni štapovi i ploče

Contents

1. Introduction

1.1. Motivation

The main objective of this thesis is to provide rigorous derivation of the lower dimensional homogeneous models for thin elastic structures and establish the approximation properties of various interesting objects such as the spectra of underlying operators, their resolvents and associated semigroups. We employ the methods of simultaneous homogenization and dimension reduction in order to answer some of the pending questions regarding the behavior of the spectrum of composite thin elastic structures, when the physical parameters related to the thickness of the material and the period of material oscillations are fairly small.

The models for thin heterogeneous structures, where the heterogeneity is of periodic nature, are accompanied by the two parameters $\varepsilon > 0$, $h > 0$, where the former represents the period of material oscillations, while the latter plays the role of the thickness of the material in one or several directions. Depending on the mutual relation between the orders of magnitude of these parameters, namely:

$$h \ll \varepsilon, \quad h \sim \varepsilon, \quad h \gg \varepsilon,$$

one obtains quite different effective models for these structures. The effective models, which we are interested in, are the models of lower dimensional homogeneous structures, which in a certain way, to a certain degree approximate the behaviour of the starting structure. Homogenization and dimension reduction is performed simultaneously in order to derive physically relevant models.

In material sciences, it is of great importance to understand the behavior of thin elastic structures. When developing lower dimensional models, one has to recognize the appropriate scaling of variables in the corresponding system of partial differential equations, which also leads to the scaling of spectrum of the underlying operator and external forces. This theory is well understood, with many developed techniques for its analysis. Composite elastic structures are often modeled with partial differential equations with rapidly oscillating coefficients. It is known that composite materials often exhibit properties which are very different from their constitutive parts. Homogenization theory has extensively been used in order to derive effective models which approximate the original oscillating models in some sense. Various methods have been developed for this cause including the method of two-scale convergence introduced by Nguetseng [50], Allaire [1] as well as the theory of Gamma convergence introduced by De-Giorgi.

The analysis and characterization of spectrum of the underlying differential operators is of much importance for engineering and material sciences. Therefore, it is important to answer a question in which sense the spectrum of the effective operators approximates the spectrum of operators which model oscillating composite structures. The question of spectral convergence and the approximation of evolution equations can be answered by observing the convergence of the corresponding resolvents. In some cases, these results can be quantified by providing the exact estimates on the distance of the associated resolvens to their effective counterparts in various operator norms. Such results yield even greater benefits for understanding the matter, such as the bounds on the spectral gaps and the estimates on the associated semigroups.

Dimension reduction in elasticity always requires a special treatment, due to the degeneracy of the problem as a consequence of the fact that the constant in the Korn's inequality blows up as the domain thickness goes to zero. From the point of view of spectral analysis, the operator of the associated problem on a rescaled domain of finite thickness has spectrum of order h^2, with the associated eigenfunctions which describe the so-called bending deformations. A standard physical interpretation is that bending deformations carry very small energy in comparison with their magnitude, while on the other hand, the magnitude of the so-called stretching deformations is comparable to their energy. Thus for bounded thin elastic structures there are two distinct orders of eigenvalues/characteristic frequencies: ones of order h^2 and the rest of order one. (On the infinite plates or rods there is no natural way to scale the spectrum, see [20].) As a result, in the evolution analysis, one would scale time (or mass density) accordingly, in order to capture the motion occurring on different time-scales.

Particularly interesting phenomena occur when the coefficients of materials which constitute the heterogeneous material are in high contrast. This means that their material coefficients have values on entirely different scales from one another. Since these materials exhibit peculiar, somewhat nonphysical properties, they are widely called metamaterials. Such composites possess macroscopic, or ”effective”, material properties not commonly found in nature, such as time the non-locality (leading to ”memory” effects) or negative refraction, which motivates their use in the context of electromagnetic or acoustic wave propagation for the development of novel devices with cloaking and superlensing properties.

1.2. Literature overview

Derivation of limit models for thin structures in linear and non-linear elasticity is a well-established topic (for example, for the approach via formal asymptotics, see [23, 24] and references therein). As part of recent related activity, there appeared a number of works that derive models of (highly) heterogeneous thin structures by simultaneous homogenisation and dimension reduction, see [10, 33, 48, 49, 69]; for the older work see also [11]. In this thesis, we continue in this direction with the derivation of effective models for thin plates with high-contrast inclusions in the context of spectral and evolution analysis. Simultaneously with the above activity in relation to the analysis of thin structures, the past two decades have seen a growing interest to the analysis of materials with high-contrast inclusions (for early papers on this subject, see [9,70,71]) that exhibit frequency-dependent material properties (equivalently, time-nonlocal evolution), which is representative of what one may refer to as "metamaterial" behaviour [12]. Furthermore, as was recently discussed in [18], high contrast in material parameters corresponds to regimes of length-scale interactions, when parts of the medium exhibit resonant response to an external field. Due to the dependence of the effective parameters on frequency, the wave propagation spectrum of these materials has a characteristic band-gap structure (i.e.waves of some frequencies do not propagate through the material, see also [3, 61]).

There have been several works dealing with high-contrast inclusions in the context of elasticity: spectral analysis on bounded domains is given in [3], in the whole space in [73], see also [61] for treating partial degeneracy (when "directional localisation" takes place), for different models of high-contrast plates (where the starting equations are two-dimensional equations for an "infinitely thin" elastic plate), see [56, 57]. In subsequent developments, [25] deals with high-contrast inclusions with partial degeneracy, when only one of several material constants (namely, the shear modulus) is relatively small, [22] discusses the limit spectrum of planar elastic frameworks made of rods and filled with a soft material, and [15] derives an effective model for the case of of high-contrast inclusions in the stiff matrix in the context of non-linear elasticity, under an assumption of small loads. In the more recent push towards a quantitative description of metamaterials, elliptic differential equations with high contrast have been analysed in the sense of approximating the associated resolvent with respect to the operator norm (see [18], [21]). In the related papers, using the Gelfand transform as a starting point, a new operator family was constructed that approximates the resolvent of the original one and that cannot be obtained directly from the standard limit operator inferred from the earlier qualitative analysis. However, these results are by now obtained only for the whole-space setting and for the particular case of the diffusion operator. In relation to quantifying the resolvent behaviour with respect to the operator norm, we should also mention [40], where the dimension reduction for a class of differential operators is carried out in the abstract setting (on a finite domain) and [20], where thin infinite elastic plates in moderate contrast are analysed.

In terms of understanding the structure of two-scale limits of partial differential operators with high contrast, we refer to [38], where an approach to spectral analysis and its consequences for materials with high-contrast inclusions (including partial degeneracies) on bounded domains is presented, via two-scale convergence. While addressing the description of the limit spectrum only partially, [38] provided a general framework for the analysis of the limit resolvent, on which new results concerning elasticity and other physically relevant setups could subsequently build. Finally, the subject of homogenisation of stochastic high-contrast media, which naturally follows the analysis of periodic setups, was recently initiated in [16] and further developed in [17].

In this thesis, namely in Chapter 2, we assume that all elastic moduli of the soft component are of the same order (unlike in [25, 61]). While we do not apply any additional scaling to either elastic moduli or the mass densities, we do discuss models obtained on different time scales. Note that this kind of time scaling is sometimes interpreted as a scaling of the mass density (see [24, 54]).

The theory of operator type estimates in homogenisation is studied in the series of papers [5], [6], [7] in which the authors use the spectral approach to the derivation of the estimates. Firstly it is done in whole space setting and later these estimates were used to obtain the estimates on the finite domain in the works [65] and [67]. The approach initiated by Birman and Suslina has proven to be fruitful in obtaining operator-norm and energy estimates for a number of related problems: boundary-value operators [65], [67], parabolic semigroups [64], [62], [43], hyperbolic groups [6], [45], [44], perforated domains [66]. The key technical milestones for this progress are boundary-layer analysis for bounded domains (as in [65], [67]) and two-parametric operator-norm estimates [63]. It seems natural to conjecture that similar developments could be pursued in the context of thin plates and rods, both infinite and bounded, by taking either the spectral germ approach or the one which was used in [20] and in this thesis, namely Chapter 3 (see, however, Section 3.1.6 for comparison).

An overview of the existing approaches to obtaining operator-norm estimates would not be complete without mentioning also the works [32], [72], [39], whose methods could also be considered in the context of thin structures.

The rigorous study of thin elastic rod is quite an old topic, see [24] and references therein for the linear theory. An overview of the derivation of various rod models in the static and evolution case can be found in the works [37], [35], [36] and [68]. Spectral analysis for the case of finite plates, together with estimates on eigenvalues, is done in [27], where the considered material is homogeneous and isotropic. The derivation of different models of rods, starting from 3D non-linear elasticity is done in [47], [46], [58] and [59] by means of Γ-convergence.

In this thesis, in Chapter 3, we assume that the elastic material is heterogeneous with the coefficients being in moderate contrast, while the oscillations of the material are of the same order as the thickness, and for this setup we carry out simultaneous homogenisation and dimension reduction. In [11] the author derives limit plate model by doing simultaneous homogenisation and dimension reduction, only for the case of isotropic material. In [26], the authors also perform the simultaneous homogenisation and dimension reduction in the case of plates without the assumption on periodicity and using material (planar) symmetries of the elasticity tensor, by introducing the notion of H-convergence adapted to dimension reduction. Derivation of the non-linear plate model in von Kármán regime by simultaneous homogenisation and dimension reduction is obtained in [49]. In [10] the authors derive the limit plate models by doing simultaneous homogenisation and dimension reduction in the general case by means of Γ-convergence (the analysis presented there also covers some non-linear models). The derivation of the model of the non-linear rod in the bending regime by doing simultaneous homogenisation and dimension reduction and without the assumption on periodicity is given in [42].

For an extensive overview of models of composite structures, one can consider the book by Panasenko [52] in which one can find thorough exposure of asymptotic expansions for the models of thin heterogeneous elastic structures, where the full asymptotics with error estimates and boundary layer analysis is given. However, the constants in the error estimates obtained there in the case of heterogeneous plates and rods with oscillating material depend non-linearly on the loads, which makes these estimates not useful for the spectral analysis.

1.3. Thesis overview

The thesis consists of two parts. The first part is related to establishing rigorous qualitative approximation properties for the models of thin heterogeneous plates in various regimes by means of two-scale resolvent convergence. The heterogeneity of the analysed material is of high contrast and therefore ”metamaterial” effects are present in the limit model. This part is covered in the Chapter 2.

Adopting the operator-theoretic perspective, we start by deriving the limit resolvent in different scaling regimes. To that end, we combine suitable decompositions of deformations that have bounded symmetrised gradients with some special properties of two-scale convergence (see Appendix and the references therein). Here we obtain different models depending on the effective parameter $\delta \in [0, \infty]$, which is the limit ratio between the thickness of the domain h and the period ε, where ε tends to zero simultaneously with h. In order to obtain high-contrast effects for “small” spectrum, we also treat a non-standard scaling of the coefficients of high-contrast inclusions (”higher” contrast).

In order to derive the limit spectrum, we employ elements of the approach of [70, 71]. Surprisingly, in the regime $\delta = \infty$, the limit spectrum does not coincide with the spectrum of the limit operator, which necessitates additional analysis (see Section 2.2.3.5 and Remark 2.3.3). This, however, is not specific for elasticity and would also happen if one carried out simultaneous high-contrast homogenisation and dimension reduction for the diffusion equation.

Suitably adapting the approach of [53] to dimension reduction in linear elasticity (see Appendix for details), we use our results on resolvent convergence to derive appropriate limit evolution equations. To infer weak convergence of solutions from the weak convergence of initial conditions and loads, we use the fact that the resolvent is the Laplace transform of the evolution operator, while for deriving strong convergence of solutions for all times t (from the strong convergence of initial conditions and loads), one needs to show the strong convergence of exponential functions on the basis of the strong convergence of resolvents. Both these implications are analysed in [53] in an abstract form, which guides our study in the specific context of dimension reduction.

In Chapter 2, we first present the results (effective tensors, limit resolvent, limit spectrum, limit evolution equations in different regimes), see Sections 2.2, and then, in Section 2.3, we provide the proofs of all statements.

The second part of the thesis is centered around deriving the precise sharp estimates on the distance between the resolvents of ε problems and homogenised resolvents in operator norms. Here, we analyse infinite heterogeneous elastic rods, where the heterogeneity is of moderate contrast, namely, the tensor of material coefficients is uniformly positive definite. We use the approach started in [20] and adapt it to the case of rods. A fairly large part of the analysis is the development of the asymptotic procedure for calculating the corrector operators which contribute to the approximation of the resolvent operator in stronger operator norms. This part of the thesis is covered in Chapter 3. We assume that the heterogeneity of the rod appears in a periodic manner along the rod. The norm-resolvent asymptotics is performed with respect to a small parameter that simultaneously plays the role of the rod thickness as well as the period of material oscillations. We first focus on the case when material symmetries are assumed. This yields a separation of the problem into the two mutually orthogonal problems, from which we draw the motivation for tackling the general case. These two orthogonal problems pertain to describing the in-line and out-of-line displacements, which in the general case intertwine. The norm-resolvent estimates are obtained in various operator norms, from where one can see interesting new nonstandard corrector terms appearing in the approximation.

In Section 3.1 we introduce the problem and the methods and state the main results. In Section 3.2 we provide apriori estimates necessary for the asymptotic expansions of the resolvents, as well as spectral estimates which serve as the motivation for different problem scalings. In Section 3.3 we establish the resolvent asymptotics with respect to the parameter of quasimomentum in the case of additional assumptions on the material symmetries. In Section 3.4 we combine the obtained results into the norm-resolvent estimates in the real domain, but only in the case of additional material symmetries. In Section 3.5 we finally are able to repeat the procedure and derive the norm-resolvent estimates for the case of general tensor.

The third part of the thesis, namely Chapter 4, is the Appendix in which we collected auxiliary results which we use in the proofs throughout the thesis. These results consist of useful claims about decomposition of displacements with bounded scaled symmetric gradients, two-scale convergence, extension operators and operator theoretical approach to high-contrast.

2. Spectral and evolution analysis of elastic plates in the high-contrast regime

2.1. Notation and setup

In this section we introduce the notaion which we will use throughout Chapter 2. Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and consider the open interval $I = (-1/2, 1/2) \subset \mathbb{R}$. Given a small positive number $h > 0$, we define a three-dimensional plate

$$\Omega^h := \omega \times (hI),$$

whose boundary consists of the lateral surface $\Gamma^h := \partial\omega \times (hI)$ and the transverse boundary $\omega \times \partial(hI)$. We assume that the part of the boundary of Ω^h on which the Dirichlet (zero-displacement) boundary condition is set has the form $\Gamma^h_{\mathrm{D}} := \gamma_{\mathrm{D}} \times (hI) \subset \Gamma^h$, where $\gamma_{\mathrm{D}} \subset \omega$ has positive (one-dimensional) measure.

For a vector $\boldsymbol{a} \in \mathbb{R}^k$, we denote by a_j, $j = 1, \dots, k$, its components, so $\boldsymbol{a} = (a_1, \dots, a_k)$. Similarly, the entries of a matrix $\boldsymbol{A} \in \mathbb{R}^{k\times k}$, are referred to as A_{ij}, $i, j = 1, \dots, k$. We denote by $x = (x_1, x_2, x_3) =: (\hat{x}, x_3)$ the standard Euclidean coordinates in $\mathbb{R}^3$. (Note that we reserve the boldface for vectors and matrices representing elastic displacements and their gradients and regular type for coordinate vectors in the corresponding reference domains.) The unit basis vectors in $\mathbb{R}^k$ are denoted by $\boldsymbol{e}_i$, $i = 1, \dots, k$. Furthermore, for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^k$ we denote by $\boldsymbol{a} \otimes \boldsymbol{b} \in \mathbb{R}^{k\times k}$, the matrix whose ij-entry is $a_i b_j$:

$$\boldsymbol{a} \otimes \boldsymbol{b} = \{a_i b_j\}_{ij=1}^k.$$

For $\boldsymbol{A} \in \mathbb{R}^{k\times l}$, by $\boldsymbol{A}^\top$ we denote its transpose and for the case $k = l$ we denote by $\operatorname{sym}\boldsymbol{A} = (\boldsymbol{A}+\boldsymbol{A}^\top)/2$ the "symmetrisation" of $\boldsymbol{A}$.

For an operator $\mathcal{A}$ (or a bilinear form a) the domain of $\mathcal{A}$ (respectively a) is denoted by $\mathcal{D}(\mathcal{A})$ (respectively $\mathcal{D}(a)$).

Throughout the chapter, we use the notation ε_h interchangeably with ε, to emphasize the fact that ε goes to zero simultaneously with h.

Furthermore, when indicating a function space X in the notation for a norm $\|\cdot\|_X$, we omit the physical domain on which functions in X are defined whenever it is clear from the context. For example, we often write $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^1}$ instead of $\|\cdot\|_{L^2(\Omega;\mathbb{R}^k)}$, $\|\cdot\|_{H^1(\Omega;\mathbb{R}^k)}$, $k = 2,3$.

Finally, we use the label C for all constants present in estimates for functions in various sets. In such cases C can be shown to admit some positive value independent of the function being estimated.

2.1.1. Differential operators of linear elasticity

Consider the reference cell $Y := [0,1)^2$. Let $Y_0 \subset Y$ be an open set with Lipschitz boundary (unless otherwise stated) such that its closure is a subset of the interior of Y, and set $Y_1 = Y\setminus Y_0$. We denote by χ_{Y_0} the characteristic function of Y_0 and by χ_{Y_1} the characteristic function of Y_1. For any subset of $A \subset \mathbb{R}^k$, we denote by χ_A the characteristic function of the set A. The domain Ω^h is then divided into two subdomains $\Omega_0^{h,\varepsilon_h}$ and $\Omega_1^{h,\varepsilon_h}$:

$$\Omega_0^{h,\varepsilon_h} := \bigcup_{z\in\mathbb{Z}^2:\varepsilon_h(Y+z)\subset\omega} \left\{\varepsilon_h(Y_0+z)\times hI\right\}, \qquad \Omega_1^{h,\varepsilon_h} := \Omega^h \backslash \Omega_0^{h,\varepsilon_h}.$$

Furthermore, we denote

$$\Omega_0^{\varepsilon_h} := \Omega_0^{1,\varepsilon_h}, \qquad \Omega_1^{\varepsilon_h} := \Omega_1^{1,\varepsilon_h}.$$

By ρ^{h,ε_h} we denote function representing the mass density of the medium. We then define

$$\rho^{h,\varepsilon_h}(x) = \rho_0\left(\hat{x}/\varepsilon_h\right)\chi_{\Omega_0^{h,\varepsilon_h}} + \rho_1\left(\hat{x}/\varepsilon_h\right)\chi_{\Omega_1^{h,\varepsilon_h}}, \qquad x \in \Omega^h,$$

where ρ_0,ρ_1 are periodic positive bounded functions, defined on Y_0 and Y_1 respectively and extended via periodicity. Namely, there exist $c_1, c_2 > 0$ such that

$$c_1 < \rho_0(y) < c_2 \quad \forall y \in Y_0, \qquad c_1 < \rho_1(y) < c_2 \quad \forall y \in Y_1.$$

We also denote $\rho := \rho_0\chi_{Y_0} + \rho_1\chi_{Y_1}$, $\rho^{\varepsilon_h} := \rho^{1,\varepsilon_h}$. We make use of the variational space with zero Dirichlet boundary conditions, defined as:

$$H^1_{\Gamma^h_{\mathrm{D}}}(\Omega^h,\mathbb{R}^3) := \left\{ \boldsymbol{v} \in H^1(\Omega^h;\mathbb{R}^3) : \boldsymbol{v} = 0 \text{ on } \Gamma^h_{\mathrm{D}} \right\}.$$

The elastic properties of periodically heterogeneous material are stored in the elasticity tensor $\mathbb{C}^{\mu_h}$, which is assumed to be of the form:

$$\mathbb{C}^{\mu_h}(y) = \begin{cases} \mathbb{C}_1(y), & y \in Y_1, \\ \mu_h^2\mathbb{C}_0(y), & y \in Y_0. \end{cases}$$

where μ_h is a parameter that goes to zero simultaneously with h, ε_h. The tensor $\mathbb{C}^{\mu_h}$ is then extended to $\mathbb{R}^2$ via Y-periodicity. The tensors $\mathbb{C}_0$ and $\mathbb{C}_1$ are assumed to be uniformly positive definite on symmetric matrices, namely there exists $\nu > 0$ such that

$$\nu|\xi|^2 \leq \mathbb{C}_{0,1}(y)\xi : \xi \leq \nu^{-1}|\xi|^2 \qquad \forall \xi \in \mathbb{R}^{3\times 3},\ \xi^\top = \xi. \tag{2.1}$$

It is well known that for a hyperelastic material the following symmetries hold, which we assume henceforth:

$$\mathbb{C}_{\alpha,ijkl} = \mathbb{C}_{\alpha,jikl} = \mathbb{C}_{\alpha,klij}, \quad i,j,k,l \in \{1,2,3\}, \qquad \alpha \in \{0,1\}.$$

The focus of our analysis is the differential operator of linear elasticity $\mathcal{A}^h_{\varepsilon_h}$ corresponding to the differential expression

$$-\left(\rho^{h,\varepsilon_h}\right)^{-1} \operatorname{div}\left(\mathbb{C}^{\mu_h}(\hat{x}/\varepsilon_h)\operatorname{sym}\nabla\right),$$

It is defined as an unbounded operator in $L^2(\Omega^h,\mathbb{R}^3)$ (where the inner product is weighted[1] by the mass density function ρ^{h,ε_h}) with domain

$$\mathcal{D}(\mathcal{A}^h_{\varepsilon_h}) \subset H^1_{\Gamma^h_{\mathrm{D}}}(\Omega^h;\mathbb{R}^3),$$

via the bilinear form

$$a^h_{\varepsilon_h}(\boldsymbol{U},\boldsymbol{V}) := \int_{\Omega^h} \mathbb{C}^{\varepsilon_h}\left(\frac{\hat{x}}{\varepsilon_h}\right)\operatorname{sym}\nabla\boldsymbol{U}(x) : \operatorname{sym}\nabla\boldsymbol{V}(x)\,dx,$$

$$\boldsymbol{U},\boldsymbol{V} \in \mathcal{D}(a^h_{\varepsilon_h}) = H^1_{\Gamma^h_{\mathrm{D}}}(\Omega^h;\mathbb{R}^3) = \mathcal{D}\big((\mathcal{A}^h_{\varepsilon_h})^{1/2}\big),$$

[1] $(\boldsymbol{u},\boldsymbol{v})_{\varepsilon_h} := \int_{\Omega^h}\rho^{h,\varepsilon_h}\boldsymbol{u}\boldsymbol{v}$.

with zero Dirichlet boundary condition on the part Γ^h_{D} of the boundary, which corresponds to the partially clamped case. For a given pair (h, ε_h) we denote by $\boldsymbol{U}^{\varepsilon_h}$ any "deformation field" on Ω^h, i.e. the solution to the integral identity

$$a^h_{\varepsilon_h}(\boldsymbol{U}^{\varepsilon_h}, \boldsymbol{V}) = \int_{\Omega^h} \boldsymbol{F}(x) \cdot \boldsymbol{V}(x) dx \qquad \forall \boldsymbol{V} \in H^1_{\Gamma^h_{\mathrm{D}}}(\Omega^h; \mathbb{R}^3),$$

for some $\boldsymbol{F} \in L^2(\Omega^h; \mathbb{R}^3)$.

We assume that the following limits for the ratio of the period ε_h and the thickness h exist:

$$\lim_{h \to 0} \frac{h}{\varepsilon_h} =: \delta \in [0, \infty], \qquad \lim_{h \to 0} \frac{h}{\varepsilon_h^2} =: \kappa \in [0, \infty]$$

and will discuss different asymptotic regimes in terms of the values of δ, κ.

The asymptotic regime $\mu_h = O(1)$ corresponds to the standard case of moderate-contrast (i.e. uniformly elliptic) homogenisation. However, in the present chapter we are interested in the "critical" case $\mu_h = \varepsilon_h$, which corresponds to high contrast in material coefficients. In addition to this, due to the the dimension reduction in elasticity, higher orders of contrast will also be of interest, namely $\mu_h = \varepsilon_h h$ for $\delta > 0$ and $\mu_h = \varepsilon_h^2$ for $\delta = 0$, see the table in Section 2.1.3.

Parts of the following assumption will be used occasionally to showcase special situations.

Assumption 2.1.1. (1) The elasticity tensor is planar symmetric:

$$\mathbb{C}_{\alpha,ijk3} = 0, \mathbb{C}_{\alpha,i333} = 0, \quad i,j,k \in \{1,2\}, \qquad \alpha \in \{0,1\}.$$

(2) The inclusion set Y_0 has a "centre point" $y^0 = (y^0_1, y^0_2) \in Y_0$, such that Y_0 is symmetric with respect to the lines $y_1 = y^0_1$, $y_2 = y^0_2$. We also assume that the elasticity tensor $y \mapsto \mathbb{C}_0(y)$ and density $y \mapsto \rho_0(y)$ are invariant under the corresponding symmetry transformations.

(3) The inclusion set Y_0 is invariant under the rotations with respect to the angle $\pi/2$ around the point (y^0_1, y^0_2). Additionally, assume that the following material symmetries hold:

$$\mathbb{C}_{0,11ij} = \mathbb{C}_{0,22ij}, \quad \mathbb{C}_{0,12kk} = 0, \quad i,j,k \in \{1,2,3\},$$

and that the function $y \mapsto \rho_0(y)$ is symmetric with respect to the rotation through $\pi/2$ around the point (y^0_1, y^0_2).

We define the following subspaces of $L^2(\Omega^h;\mathbb{R}^3)$:

$L^{2,\mathrm{bend}}(\Omega^h;\mathbb{R}^3) := \left\{ \boldsymbol{V} = (V_1,V_2,V_3) \in L^2(\Omega^h;\mathbb{R}^3);\ V_1,V_2 \text{ are odd w.r.t. } x_3,\ V_3 \text{ is even w.r.t. } x_3 \right\}$,

$L^{2,\mathrm{memb}}(\Omega^h;\mathbb{R}^3) := \left\{ \boldsymbol{V} = (V_1,V_2,V_3) \in L^2(\Omega^h;\mathbb{R}^3);\ V_1,V_2 \text{ are even w.r.t. } x_3,\ V_3 \text{ is odd w.r.t. } x_3 \right\}$.

Similarly, we define $L^{2,\mathrm{bend}}(\Omega\times Y;\mathbb{R}^3)$, $L^{2,\mathrm{memb}}(\Omega\times Y;\mathbb{R}^3)$, $L^{2,\mathrm{bend}}(I\times Y_0;\mathbb{R}^3)$, $L^{2,\mathrm{memb}}(I\times Y_0;\mathbb{R}^3)$.

Remark 2.1.1. Part (1) of Assumption 2.1.1 is needed to infer that the spaces $L^{2,\mathrm{bend}}(\Omega^h;\mathbb{R}^3)$, $L^{2,\mathrm{memb}}$ $(\Omega^h;\mathbb{R}^3)$ are invariant for the operator $\mathcal{A}^h_{\varepsilon_h}$. Part (2) of the same assumption will additionally be used when we want to infer that the values of the Zhikov function β, see (2.17), are diagonal matrices, and part (3) will be used in combination with parts (1) and (2) when we want to infer that the $(1,1)$ and $(2,2)$ entries of the Zhikov function are equal. Although we do not assume the dependence on the x_3 variable, our analysis can be easily extended to this case (at the expense of obtaining more complex limit equations in some cases). In the case of planar symmetries, a natural assumption would be that the elasticity tensor is even in the x_3 variable.

In order to work in a fixed domain $\Omega := \Omega^1$, $\Gamma := \Gamma^1$, $\Gamma_{\mathrm{D}} := \Gamma^1_{\mathrm{D}}$, we apply the change of variables

$$(x_1,x_2,x_3) := (x_1^h, x_2^h, h^{-1}x_3^h), \quad (x_1^h,x_2^h,x_3^h) \in \Omega^h,$$

and define $\boldsymbol{u}^{\varepsilon_h}(x) := \boldsymbol{U}^{\varepsilon_h}(x^h)$. In the new variables, we will be dealing with a scaled symmetrized gradient and scaled divergence, given by

$$\operatorname{sym}\nabla \boldsymbol{U}^{\varepsilon_h}(x^h) = \operatorname{sym}\nabla_h \boldsymbol{u}^{\varepsilon_h}(x), \qquad \operatorname{div}\boldsymbol{U}^{\varepsilon_h}(x^h) = \operatorname{tr}\nabla_h \boldsymbol{u}^{\varepsilon_h}(x) =: \operatorname{div}_h \boldsymbol{u}^{\varepsilon_h}(x),$$

where for a given function $\boldsymbol{u}$ we use the notation $\nabla_h \boldsymbol{u} := \left(\nabla_{\hat{x}}\boldsymbol{u} | h^{-1}\partial_{x_3}\boldsymbol{u}\right)$ for the gradient scaled "transversally", and tr denotes the trace of a matrix. Thus, we are dealing with an operator $\mathcal{A}_{\varepsilon_h}$ in $L^2(\Omega;\mathbb{R}^3)$ (where the inner product is defined with the weight function ρ^{ε_h}) whose differential expression and domain are given by

$$-\left(\rho^{\varepsilon_h}\right)^{-1}\operatorname{div}_h\left(\mathbb{C}^{\varepsilon_h}(\hat{x}/\varepsilon_h)\operatorname{sym}\nabla_h\right), \qquad \mathcal{D}(\mathcal{A}_{\varepsilon_h}) \subset H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3),$$

respectively. The operator $\mathcal{A}_{\varepsilon_h}$ is defined by the form

$$a_{\varepsilon_h}(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \mathbb{C}^{\varepsilon_h}\left(\frac{\hat{x}}{\varepsilon_h}\right)\operatorname{sym}\nabla_h \boldsymbol{u}(x) : \operatorname{sym}\nabla_h \boldsymbol{v}(x)\,dx, \quad \boldsymbol{u},\boldsymbol{v} \in \mathcal{D}(a_{\varepsilon_h}) = H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3) = \mathcal{D}\big(\mathcal{A}^{1/2}_{\varepsilon_h}\big).$$

As in Remark 2.1.1, under Assumption 2.1.1 (1) the spaces $L^{2,\mathrm{bend}}(\Omega;\mathbb{R}^3)$, $L^{2,\mathrm{memb}}(\Omega;\mathbb{R}^3)$ are invariant for the operator $\mathcal{A}_{\varepsilon_h}$. We will also say that the operator $\mathcal{A}_{\varepsilon_h}$ represents the bilinear form a_{ε_h} (a symmetric bilinear form defines a self-adjoint densely defined unbounded operator, see e.g. [60]). In connection with $\mathcal{A}_{\varepsilon_h}$ we define the operator $\tilde{\mathcal{A}}_{\varepsilon_h}$ as the restriction of $\mathcal{A}_{\varepsilon_h}$ onto the space $L^{2,\mathrm{memb}}(\Omega;\mathbb{R}^3)$. Additionally, we define the self-adjoint operators $\mathring{\mathcal{A}}_{\varepsilon_h}$ in $L^2(I\times Y_0;\mathbb{R}^3)$ whose differential expression and domain are given by

$$-\rho_0^{-1}\operatorname{div}_{\frac{h}{\varepsilon_h}}\left(\mathbb{C}_0(y)\operatorname{sym}\nabla_{\frac{h}{\varepsilon_h}}\right), \qquad \mathcal{D}(\mathring{\mathcal{A}}_{\varepsilon_h})\subset H^1_{00}(I\times Y_0;\mathbb{R}^3),$$

as the operators represented by the respective bilinear forms

$$\mathring{a}_{\varepsilon_h}(\boldsymbol{u},\boldsymbol{v}) = \int_{I\times Y_0}\mathbb{C}_0(y)\operatorname{sym}\nabla_{\frac{h}{\varepsilon_h}}\boldsymbol{u} : \operatorname{sym}\nabla_{\frac{h}{\varepsilon_h}}\boldsymbol{v}\,dx_3dy, \qquad \boldsymbol{u},\boldsymbol{v}\in\mathcal{D}(\mathring{a}_{\varepsilon_h}) = \left(H^1_{00}(I\times Y_0;\mathbb{R}^3)\right)^2,$$

where $H^1_{00}(I\times Y_0;\mathbb{R}^k)$ stands for the subspace of $H^1(I\times Y_0;\mathbb{R}^k)$ consisting of functions with zero trace on $I\times\partial Y_0$. Finally, we define $\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h}$ as the operator corresponding to the same differential expression as $\mathring{\mathcal{A}}_{\varepsilon_h}$ but acting in the space $L^{2,\mathrm{memb}}(I\times Y_0;\mathbb{R}^3)$, hence representing an appropriate bilinear form

$$\mathring{\tilde{a}}_{\varepsilon_h} : \left(H^1_{00}(I\times Y_0;\mathbb{R}^3)\right)^2 \cap \left(L^{2,\mathrm{memb}}(I\times Y_0;\mathbb{R}^3)\right)^2 \to \mathbb{R}.$$

2.1.2. Additional notation

The inner product of $x,y\in\mathbb{R}^n$ is denoted by $(x,y) := \sum_{i=1}^n x_iy_i$. For a function $f\in L^1(A)$ (and similarly for $\boldsymbol{f}\in L^1(A;\mathbb{R}^3)$), we denote by

$$\fint_A f := \frac{1}{|A|}\int_A f,$$

its mean over A. We will also use the shorthand notation

$$\overline{f} := \int_I f(x_3)dx_3, \qquad \langle f\rangle := \int_Y f(y)dy, \qquad \boldsymbol{f}_* := \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{pmatrix},$$

where in the last expression it is assumed that $\boldsymbol{f}$ is a (three-component) vector-valued function. In line with (2.1.2), the notation $\overline{f}$ and $\langle f\rangle$ is naturally extended to vector-valued functions.

Next, denote by ι the "embedding" operator

$$\iota : \mathbb{R}^{2\times 2} \to \mathbb{R}^{3\times 3}, \qquad \iota \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Similarly, we define an operator $\iota : \mathbb{R}^{3\times 2} \to \mathbb{R}^{3\times 3}$. We use the same notation for this operator and the operator defined in (2.1.2), as it will be clear from the context which of the two embeddings is used in each particular case. For $\boldsymbol{a} \in \mathbb{R}^3$ we denote by ι_1 the mapping

$$\iota_1 : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}, \qquad \iota_1(\boldsymbol{a}) = \begin{pmatrix} & 0 & a_1 \\ & & a_2 \\ a_1 & a_2 & a_3 \end{pmatrix}.$$

Furthermore, for $l > 0$ we define the "scaling" matrix

$$\pi_l := \begin{pmatrix} l & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We also define the space $H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)$ of $\mathbb{R}^2$-valued H^1 functions vanishing on γ_{D} and the space $H^2_{\gamma_{\mathrm{D}}}(\omega)$ of scalar H^2 functions vanishing on γ_{D} together with their first derivatives.

In what follows, we denote by $\mathcal{Y}$ the flat unit torus in $\mathbb{R}^2$, by $\mathcal{Y}_1$ the flat unit torus in $\mathbb{R}^2$ with a hole corresponding to the set Y_1, by $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ the space of symmetric matrices, $\mathbb{R}^{n\times n}_{\mathrm{skew}}$ the space of skew-symmetric matrices, by $\boldsymbol{I}_{n\times n}$ the unit matrix in $\mathbb{R}^{n\times n}$, and by $\delta_{\alpha\beta}$ the Kronecker delta function. Furthermore, $H^1(\mathcal{Y})$, $H^2(\mathcal{Y})$ denote the spaces of periodic functions in $H^1(Y)$, $H^2(Y)$. Similarly, we denote by $H^1(I\times\mathcal{Y})$ the space of functions in $H^1(I\times Y)$ that are periodic in $y \in Y$. The spaces $\dot{H}^1(\mathcal{Y})$, $\dot{H}^1(I\times\mathcal{Y})$ are defined to consist of functions in $H^1(\mathcal{Y})$, $H^1(I\times\mathcal{Y})$ whose mean value is zero. Similarly, we define the spaces $H^k(\mathcal{Y}_1)$ for $k = 1,2$. Note that every function in $H^1_{00}(I\times Y_0;\mathbb{R}^k)$ can be naturally extended by zero to a function in $H^1(I\times\mathcal{Y};\mathbb{R}^k)$.

The space $C^k(\mathcal{Y})$ denotes the space of smooth functions on the torus $\mathcal{Y}$ that have continuous derivatives up to order k. In a similar way we define the space $C^k(I\times\mathcal{Y})$. Furthermore, $C^k_{00}(I\times Y_0)$ denotes the space of k-differentiable functions on $I\times Y_0$ whose derivatives up to order k are zero on $I\times\partial Y_0$. For $A\subset\mathbb{R}^n$, the space $C^k_{\mathrm{c}}(A)$ consists of functions with compact support in A that have continuous derivatives up to order k.

For a function $\boldsymbol{u}\in H^1(I\times Y;\mathbb{R}^3)$, we use the notation $\widetilde{\nabla}_\delta$ for the "anisotropically scaled" gradient whose third column is obtained from the usual gradient by scaling with δ^{-1} :

$$\widetilde{\nabla}_\delta\boldsymbol{u}:=\left(\nabla_y\boldsymbol{u}|\delta^{-1}\partial_{x_3}\boldsymbol{u}\right).$$

Next, for $\boldsymbol{\varphi}\in L^2(\omega;H^1(I\times\mathcal{Y};\mathbb{R}^3))$, we denote

$$C_\delta(\boldsymbol{\varphi})=\operatorname{sym}\widetilde{\nabla}_\delta\boldsymbol{\varphi},$$

and for $\boldsymbol{\varphi}_1\in L^2(\omega;H^1(\mathcal{Y};\mathbb{R}^2))$, $\varphi_2\in L^2(\omega;H^2(\mathcal{Y}))$, $\boldsymbol{g}\in L^2(\Omega\times Y;\mathbb{R}^3)$, we use the notation

$$C_0(\boldsymbol{\varphi}_1,\varphi_2,\boldsymbol{g})(x,y):=\begin{pmatrix}\begin{matrix}\operatorname{sym}\nabla_y\boldsymbol{\varphi}_1(\hat{x},y)-x_3\nabla_y^2\varphi_2(\hat{x},y)\end{matrix} & \begin{matrix}g_1(x,y)\\ g_2(x,y)\end{matrix}\\ \begin{matrix}g_1(x,y) & g_2(x,y)\end{matrix} & g_3(x,y)\end{pmatrix}.$$

Furthermore, for $\boldsymbol{w}\in L^2(\Omega;\dot{H}^1(\mathcal{Y};\mathbb{R}^3))$, $\boldsymbol{g}\in L^2(\Omega;\mathbb{R}^3)$, we define

$$C_\infty(\boldsymbol{w},\boldsymbol{g})(x,y):=\begin{pmatrix}\operatorname{sym}\nabla_y\boldsymbol{w}_*(x,y) & \begin{matrix}g_1(x)+\partial_{y_1}w_3(x,y)\\ g_2(x)+\partial_{y_2}w_3(x,y)\end{matrix}\\ \begin{matrix}g_1(x)+\partial_{y_1}w_3(x,y) & g_2(x)+\partial_{y_2}w_3(x,y)\end{matrix} & g_3(x)\end{pmatrix},$$

where $\boldsymbol{w}_*$ is defined via (2.1.2).

For different values of δ,κ, we introduce the spaces

$$\mathfrak{C}_\delta(\Omega\times Y):=\begin{cases}\left\{C_\delta(\boldsymbol{\varphi}):\boldsymbol{\varphi}\in L^2(\omega;H^1(I\times\mathcal{Y};\mathbb{R}^3))\right\}, & \delta\in(0,\infty),\\ \Big\{C_0(\boldsymbol{\varphi}_1,\varphi_2,\boldsymbol{g}):\boldsymbol{\varphi}_1\in L^2(\omega;H^1(\mathcal{Y};\mathbb{R}^2)),\varphi_2\in L^2(\omega;H^2(\mathcal{Y})),\boldsymbol{g}\in L^2(\Omega\times Y;\mathbb{R}^3)\Big\},\\ \qquad\delta=0,\\ \left\{C_\infty(\boldsymbol{w},\boldsymbol{g}):\boldsymbol{w}\in L^2(\Omega;\dot{H}^1(\mathcal{Y};\mathbb{R}^3)),\boldsymbol{g}\in L^2(\Omega\times Y;\mathbb{R}^3)\right\}, & \delta=\infty;\end{cases}$$

$$V_{1,\delta,\kappa}(\omega\times Y):=\begin{cases}H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)\times L^2(\omega), & \delta\in[0,\infty],\ \kappa=\infty,\\ H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)\times L^2(\omega;H^2(\mathcal{Y}_1)\times L^2(Y_0)), & \delta=0,\ \ \kappa\in(0,\infty),\\ H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)\times L^2(\omega\times Y), & \delta=0,\ \ \kappa=0;\end{cases}$$

$$V_{2,\delta}(\Omega\times Y_0) := \begin{cases} L^2(\omega;H^1_{00}(I\times Y_0;\mathbb{R}^3)), & \delta\in(0,\infty), \\ L^2(\Omega;H^1_0(Y_0;\mathbb{R}^3)), & \delta=\infty, \\ L^2(\omega;H^1_0(Y_0;\mathbb{R}^2))\times L^2(\omega\times Y_0), & \delta=0; \end{cases}$$

$$H_{\delta,\kappa}(\Omega\times Y) := \begin{cases} L^2(\omega;\mathbb{R}^3)+L^2(\Omega\times Y_0;\mathbb{R}^3), & \delta\in(0,\infty],\ \ \kappa=\infty, \\ L^2(\omega;\mathbb{R}^3)+L^2(\omega\times Y_0;\mathbb{R}^3), & \delta=0,\ \ \kappa=\infty, \\ \big(L^2(\omega;\mathbb{R}^2)+L^2(\omega\times Y_0;\mathbb{R}^2)\big)\times L^2(\omega\times Y), & \delta=0,\ \ \kappa\in[0,\infty), \end{cases}$$

where functions defined on ω are assumed to be constant across the plate whenever they are considered in Ω. (In other words, $L^2(\omega\times Y)$ is treated as naturally embedded in $L^2(\Omega\times Y)$.) We denote by $P_{\delta,\kappa}$ and P^0 the orthogonal projections $P_{\delta,\kappa}: L^2(\Omega\times Y;\mathbb{R}^3)\to H_{\delta,\kappa}(\Omega\times Y)$ and $P^0: L^2(\Omega\times Y)\to L^2(\omega)+L^2(\omega\times Y_0)$, respectively. The mappings

$$L^2(\Omega)+L^2(\Omega\times Y_0)\ni u(x)+\mathring{u}(x,y)\mapsto u(x)\in L^2(\Omega)$$

and

$$L^2(\Omega)+L^2(\Omega\times Y_0)\ni u(x)+\mathring{u}(x,y)\mapsto \mathring{u}(x,y)\in L^2(\Omega\times Y_0)$$

are labelled by S_1 and S_2, respectively. For Hilbert spaces V, W and a linear operator $\mathcal{A}: V\to W$, we denote by $\mathcal{R}(\mathcal{A})\subset W$ its range, and for a linear operator $\mathcal{A}: V\to V$, we denote by $\sigma(\mathcal{A})$ its spectrum. Furthermore, $\sigma_{\rm ess}(\mathcal{A})$ and $\sigma_{\rm disc}(\mathcal{A})$ denote the essential and discrete spectrum of $\mathcal{A}$, respectively. Throughout, we denote by $\mathcal{I}$ the identity operator on the appropriate ambient space.

For the definition of two-scale convergence, the related notation and properties of importance for our analysis, we refer the reader to Appendix (for the basic properties and introduction, see also [1]). Finally, for a Hilbert space V, we denote by V^* its dual, and $\rightharpoonup$, $\to$ denote, respectively, the weak and strong convergence.

2.1.3. Section guide for different scaling regimes

The table below shows the different scalings considered in this chapter for the period of oscillations ε_h with respect to the thickness h as well as appropriate scalings of the contrast, time, and spectrum.

	Time	$h \ll \varepsilon_h$ ($\delta = 0$)	Spec	Time	$h \sim \varepsilon_h$ ($0 < \delta < \infty$)	Spec	Time	$h \gg \varepsilon_h$ ($\delta = \infty$)	Spec
$\mu_h = \varepsilon_h$	Non-scaled: 2.2.4.2	$\tau = 0:$ 2.2.2.2.A	2.2.3.3	Long: 2.2.4.1 ——— Non-scaled: 2.2.4.2	$\tau = 2:$ 2.2.2.1.A ——— $\tau = 0:$ 2.2.2.1.B	$\tau = 2:$ 2.2.3.2 ——— $\tau = 0:$ 2.2.3.3	Non-scaled: 2.2.4.2	$\tau = 0:$ 2.2.2.3.A	2.2.3.54
$\mu_h = \varepsilon_h h$	*****	*****	****	Long: 2.2.4.3	$\tau = 2:$ 2.2.2.1.C	2.2.3.4	Long: 2.2.4.3	$\tau = 2:$ 2.2.2.3.B	2.2.3.5
$\mu_h = \varepsilon_h^2$	Long: 2.2.4.4	$\tau = 2:$ 2.2.2.2.B	2.2.3.4	*****	*****	*****	*****	*****	****

Table 2.1: Overview of sections and results in Chapter 2

2.2. Main results

2.2.1. Effective elasticity tensors

In this section we will define limit elasticity tensors that will appear in various regimes. For $\delta \in (0,\infty)$, we define a symmetric tensor $\mathbb{C}_\delta^{\mathrm{hom}}$ via

$$\begin{aligned}
&\mathbb{C}_\delta^{\mathrm{hom}}(\boldsymbol{A},\boldsymbol{B}) : (\boldsymbol{A},\boldsymbol{B}) := \\
&\qquad = \min_{\boldsymbol{\varphi}\in H^1(I\times\mathcal{Y};\mathbb{R}^3)} \int_I \int_{Y_1} \mathbb{C}_1(y)\left[\iota(\boldsymbol{A}-x_3\boldsymbol{B}) + \operatorname{sym}\widetilde{\nabla}_\delta\boldsymbol{\varphi}\right] : \left[\iota(\boldsymbol{A}-x_3\boldsymbol{B}) + \operatorname{sym}\widetilde{\nabla}_\delta\boldsymbol{\varphi}\right] dy\, dx_3, \\
&\qquad\qquad \boldsymbol{A},\boldsymbol{B}\in\mathbb{R}^{2\times 2}_{\mathrm{sym}},
\end{aligned} \tag{2.2}$$

as well as tensors $\mathbb{C}_\delta^{\mathrm{memb}}$, $\mathbb{C}_\delta^{\mathrm{bend}}$ via

$$\begin{aligned}
\mathbb{C}_\delta^{\mathrm{memb}}\boldsymbol{A} : \boldsymbol{A} &:= \mathbb{C}_\delta^{\mathrm{hom}}(\boldsymbol{A},\boldsymbol{0}) : (\boldsymbol{A},\boldsymbol{0}), \quad \boldsymbol{A}\in\mathbb{R}^{2\times 2}_{\mathrm{sym}}, \\
\mathbb{C}_\delta^{\mathrm{bend}}\boldsymbol{B} : \boldsymbol{B} &:= \mathbb{C}_\delta^{\mathrm{hom}}(\boldsymbol{0},\boldsymbol{B}) : (\boldsymbol{0},\boldsymbol{B}), \quad \boldsymbol{B}\in\mathbb{R}^{2\times 2}_{\mathrm{sym}}.
\end{aligned}$$

Remark 2.2.1. Under an additional assumption on the material symmetries, namely Assumption 2.1.1 (1), the tensor $\mathbb{C}_\delta^{\mathrm{hom}}$ can be written as the orthogonal direct sum

$$\mathbb{C}_\delta^{\mathrm{hom}} = \mathbb{C}_\delta^{\mathrm{memb}} \oplus \mathbb{C}_\delta^{\mathrm{bend}},$$

in the sense that

$$\mathbb{C}_\delta^{\mathrm{hom}} = \begin{bmatrix} \mathbb{C}_\delta^{\mathrm{memb}} & 0 \\ 0 & \mathbb{C}_\delta^{\mathrm{bend}} \end{bmatrix},$$

i.e.

$$\mathbb{C}_\delta^{\mathrm{hom}}(\boldsymbol{A},\boldsymbol{B}) : (\boldsymbol{A},\boldsymbol{B}) = \mathbb{C}_\delta^{\mathrm{memb}}\boldsymbol{A} : \boldsymbol{A} + \mathbb{C}_\delta^{\mathrm{bend}}\boldsymbol{B} : \boldsymbol{B}, \qquad \boldsymbol{A},\boldsymbol{B}\in\mathbb{R}^{2\times 2}_{\mathrm{sym}}.$$

For the case $\delta = 0$ the following tensor $\mathbb{C}^{\mathrm{hom,r}}$ will be important (in this case we assume that Y_0 is of class $C^{1,1}$):

$$\begin{aligned}
&\mathbb{C}^{\mathrm{hom,r}}(\boldsymbol{A},\boldsymbol{B}) : (\boldsymbol{A},\boldsymbol{B}) := \\
&= \min \int_I \int_{Y_1} \mathbb{C}_1(y)\left[\iota(\boldsymbol{A}-x_3\boldsymbol{B}) + C_0(\boldsymbol{\varphi}_1,\varphi_2,\boldsymbol{g})(x_3,y)\right] : \left[\iota(\boldsymbol{A}-x_3\boldsymbol{B}) + C_0(\boldsymbol{\varphi}_1,\varphi_2,\boldsymbol{g})(x_3,y)\right] dy\, dx_3, \\
&\qquad\qquad \boldsymbol{A},\boldsymbol{B}\in\mathbb{R}^{2\times 2}_{\mathrm{sym}},
\end{aligned} \tag{2.3}$$

where the minimum is taken over $\boldsymbol{\varphi}_1 \in \dot{H}^1(\mathcal{Y};\mathbb{R}^2)$, $\varphi_2 \in \dot{H}^2(\mathcal{Y})$, $\boldsymbol{g} \in L^2(I\times Y,\mathbb{R}^3)$. Note that in (2.3) the definition (2.1.2) of C_0 is used with $\boldsymbol{\varphi}_1$, φ_2, $\boldsymbol{g}$ independent of $\hat{x}$. Furthermore, we define a tensor function $\mathbb{C}_0^{\mathrm{red}}(y)$, $y \in Y_0$, by the formula

$$\mathbb{C}_0^{\mathrm{red}}(y)(\boldsymbol{A},\boldsymbol{B}) : (\boldsymbol{A},\boldsymbol{B}) :=$$
$$= \min_{\boldsymbol{g}\in L^2(I;\mathbb{R}^3)} \int_I \mathbb{C}_0(y)\left[\iota(\boldsymbol{A}-x_3\boldsymbol{B})+\iota_1(\boldsymbol{g}(x_3))\right] : \left[\iota(\boldsymbol{A}-x_3\boldsymbol{B})+\iota_1(\boldsymbol{g}(x_3))\right] dx_3, \tag{2.4}$$
$$\boldsymbol{A},\boldsymbol{B} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}.$$

In addition, for $\alpha = 0,1$ we define a tensor-valued function $\mathbb{C}^{\mathrm{r}}_\alpha(y)$, $y \in Y$, via the formula

$$\mathbb{C}^{\mathrm{r}}_\alpha(y)\boldsymbol{A} : \boldsymbol{A} = \min_{\boldsymbol{d}\in\mathbb{R}^3} \mathbb{C}_\alpha(y)[\iota(\boldsymbol{A})+\iota_1(\boldsymbol{d})] : [\iota(\boldsymbol{A})+\iota_1(\boldsymbol{d})], \qquad \boldsymbol{A} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \quad y \in Y_\alpha.$$

Remark 2.2.2. It is easily seen that for a $\boldsymbol{\varphi}_1$, φ_2, $\boldsymbol{g}$ on which the minimum in (2.3) is attained, one has $\boldsymbol{g}(x_3,y) = \boldsymbol{g}_0(y) + x_3\boldsymbol{g}_1(y)$, for some $\boldsymbol{g}_0$, $\boldsymbol{g}_1 \in L^2(Y,\mathbb{R}^3)$. It follows that

$$\mathbb{C}^{\mathrm{hom,r}}(\boldsymbol{A},\boldsymbol{B}) : (\boldsymbol{A},\boldsymbol{B}) = \mathbb{C}_1^{\mathrm{memb,r}}\boldsymbol{A} : \boldsymbol{A} + \mathbb{C}_1^{\mathrm{bend,r}}\boldsymbol{B} : \boldsymbol{B}, \qquad \boldsymbol{A},\boldsymbol{B} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}},$$

where

$$\begin{aligned}
\mathbb{C}_1^{\mathrm{memb,r}}\boldsymbol{A} : \boldsymbol{A} &:= \mathbb{C}^{\mathrm{hom,r}}(\boldsymbol{A},\boldsymbol{0}) : (\boldsymbol{A},\boldsymbol{0}) \\
&= \min_{\boldsymbol{\varphi}_1\in\dot{H}^1(\mathcal{Y};\mathbb{R}^2)} \int_{Y_1} \mathbb{C}^r_1(y)[\boldsymbol{A}+\nabla_y\boldsymbol{\varphi}_1(y)] : [\boldsymbol{A}+\nabla_y\boldsymbol{\varphi}_1(y)]\,dy, \qquad \boldsymbol{A} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \\
\mathbb{C}_1^{\mathrm{bend,r}}\boldsymbol{B} : \boldsymbol{B} &:= \mathbb{C}^{\mathrm{hom,r}}(\boldsymbol{0},\boldsymbol{B}) : (\boldsymbol{0},\boldsymbol{B}) \\
&= \min_{\varphi\in\dot{H}^2(\mathcal{Y})} \frac{1}{12}\int_{Y_1} \mathbb{C}^r_1(y)[\boldsymbol{B}+\nabla_y^2\varphi(y)] : [\boldsymbol{B}+\nabla_y^2\varphi(y)]\,dy, \qquad \boldsymbol{B} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}.
\end{aligned}$$

Similarly to the above, it is seen that the minimum in (2.4) is attained on the vector fields of the form $\boldsymbol{g}(x_3) = \boldsymbol{g}_0 + x_3\boldsymbol{g}_1$, where $\boldsymbol{g}_0$, $\boldsymbol{g}_1 \in \mathbb{R}^3$. Furthermore, we have the following decomposition:

$$\mathbb{C}_0^{\mathrm{red}}(y)(\boldsymbol{A},\boldsymbol{B}) : (\boldsymbol{A},\boldsymbol{B}) = \mathbb{C}_0^{\mathrm{memb,r}}(y)\boldsymbol{A} : \boldsymbol{A} + \mathbb{C}_0^{\mathrm{bend,r}}(y)\boldsymbol{B} : \boldsymbol{B}, \qquad \boldsymbol{A},\boldsymbol{B} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \qquad y \in Y_0,$$

where

$$\mathbb{C}_0^{\mathrm{memb,r}}(y)\boldsymbol{A} : \boldsymbol{A} := \mathbb{C}^r_0(y)\boldsymbol{A} : \boldsymbol{A}, \quad \mathbb{C}_0^{\mathrm{bend,r}}(y)\boldsymbol{B} : \boldsymbol{B} := \frac{1}{12}\mathbb{C}^{\mathrm{r}}_0(y)\boldsymbol{B} : \boldsymbol{B}, \quad \boldsymbol{A},\boldsymbol{B} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \quad y \in Y_0.$$

For the case $\delta = \infty$, a tensor $\mathbb{C}^{\mathrm{hom,h}}$ will be important, which is defined by

$$
\begin{aligned}
&\mathbb{C}^{\mathrm{hom,h}}(\boldsymbol{A},\boldsymbol{B}) : (\boldsymbol{A},\boldsymbol{B}) := \\
&= \min \int_I \int_{Y_1} \mathbb{C}(y) \left[\iota(\boldsymbol{A} - x_3\boldsymbol{B}) + C_\infty(\boldsymbol{w},\boldsymbol{g})(x_3,y)\right] : \left[\iota(\boldsymbol{A} - x_3\boldsymbol{B}) + C_\infty(\boldsymbol{w},\boldsymbol{g})(x_3,y)\right] dy\, dx_3, \\
&\qquad\qquad \boldsymbol{A},\boldsymbol{B} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}},
\end{aligned}
\tag{2.5}
$$

where the minimum is taken over $\boldsymbol{w} \in L^2(I;\dot{H}^1(\mathcal{Y};\mathbb{R}^3))$, $\boldsymbol{g} \in L^2(I;\mathbb{R}^3)$. (As in the case of the expression C_0 entering (2.3), for the expression C_∞ in (2.5) we take the functions $\boldsymbol{w}$, $\boldsymbol{g}$ to be independent of $\hat{x}$.)

Remark 2.2.3. It is easily seen that the minimum in (2.5) is attained on $\boldsymbol{g} = \boldsymbol{g}_0 + x_3\boldsymbol{g}_1$, $\boldsymbol{w} = \boldsymbol{w}_0(y) + x_3\boldsymbol{w}_1(y)$, for some $\boldsymbol{g}_0$, $\boldsymbol{g}_1 \in \mathbb{R}^3$, $\boldsymbol{w}_0$, $\boldsymbol{w}_1 \in L^2(Y;\mathbb{R}^3)$. It follows that

$$
\mathbb{C}^{\mathrm{hom,h}}(\boldsymbol{A},\boldsymbol{B}) : (\boldsymbol{A},\boldsymbol{B}) = \mathbb{C}^{\mathrm{memb,h}}\boldsymbol{A} : \boldsymbol{A} + \mathbb{C}^{\mathrm{bend,h}}\boldsymbol{B} : \boldsymbol{B}, \qquad \boldsymbol{A},\boldsymbol{B} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}},
$$

where

$$
\begin{aligned}
\mathbb{C}^{\mathrm{memb,h}}\boldsymbol{A} : \boldsymbol{A} &:= \mathbb{C}^{\mathrm{hom,h}}(\boldsymbol{A},\boldsymbol{0}) : (\boldsymbol{A},\boldsymbol{0}) \\
&= \min_{\boldsymbol{w}\in H^1(\mathcal{Y},\mathbb{R}^3),\boldsymbol{g}\in\mathbb{R}^3} \int_{Y_1} \mathbb{C}(y)[\boldsymbol{A} + C_\infty(\boldsymbol{w},\boldsymbol{g})] : [\boldsymbol{A} + C_\infty(\boldsymbol{w},\boldsymbol{g})]\, dy, \quad \boldsymbol{A} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \\
\mathbb{C}^{\mathrm{bend,h}}\boldsymbol{B} : \boldsymbol{B} &:= \mathbb{C}^{\mathrm{hom,h}}(\boldsymbol{0},\boldsymbol{B}) : (\boldsymbol{0},\boldsymbol{B}) \\
&= \min_{\boldsymbol{w}\in H^1(\mathcal{Y},\mathbb{R}^3),\boldsymbol{g}\in\mathbb{R}^3} \frac{1}{12} \int_{Y_1} \mathbb{C}(y)[\boldsymbol{B} + C_\infty(\boldsymbol{w},\boldsymbol{g})] : [\boldsymbol{B} + C_\infty(\boldsymbol{w},\boldsymbol{g})]\, dy, \quad \boldsymbol{B} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}.
\end{aligned}
$$

The following proposition is proved in Section 2.3.1.

Proposition 2.2.4. *The tensor $\mathbb{C}^{\mathrm{hom}}_\delta$ (and consequently the tensors $\mathbb{C}^{\mathrm{memb}}_\delta$, $\mathbb{C}^{\mathrm{bend}}_\delta$ as well) is bounded and coercive, i.e., there exists $\nu > 0$ such that*

$$
\nu\left(|\boldsymbol{A}|^2 + |\boldsymbol{B}|^2\right) \le \mathbb{C}^{\mathrm{hom}}_\delta(\boldsymbol{A},\boldsymbol{B}) : (\boldsymbol{A},\boldsymbol{B}) \le \nu^{-1}\left(|\boldsymbol{A}|^2 + |\boldsymbol{B}|^2\right) \qquad \forall \boldsymbol{A},\boldsymbol{B} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}.
$$

Analogous claims are valid for tensors $\mathbb{C}^{\mathrm{hom,r}}$, $\mathbb{C}^{\mathrm{hom,h}}$, $\mathbb{C}^{\mathrm{red}}_0$ (and consequently tensors $\mathbb{C}^{\mathrm{memb,r}}_1$, $\mathbb{C}^{\mathrm{bend,r}}_1$, $\mathbb{C}^{\mathrm{memb,h}}$, $\mathbb{C}^{\mathrm{bend,h}}$, $\mathbb{C}^{\mathrm{memb,r}}_0$, $\mathbb{C}^{\mathrm{bend,r}}_0$).

2.2.2. Limit resolvent equations

Our starting point is the following resolvent formulation. For $\tau, \lambda > 0$ and a given $\boldsymbol{f}^{\varepsilon_h} \in L^2(\Omega;\mathbb{R}^3)$, find $\boldsymbol{u}^{\varepsilon_h} \in H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)$ such that the following variational formulation holds:

$$\frac{1}{h^\tau}\int_\Omega \mathbb{C}^{\mu_h}\left(\frac{\hat{x}}{\varepsilon_h}\right)\operatorname{sym}\nabla \boldsymbol{u}^{\varepsilon_h} : \operatorname{sym}\nabla_h \boldsymbol{v}\,dx + \lambda\int_\Omega \rho^{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h}\cdot\boldsymbol{v}\,dx = \int_\Omega \boldsymbol{f}^{\varepsilon_h}\cdot\boldsymbol{v}\,dx \qquad \forall \boldsymbol{v}\in H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3). \tag{2.6}$$

We derive the limit resolvent equation, as $h \to 0$, depending on various assumptions about the parameter $\delta = h/\varepsilon_h$, the exponent τ, and the scaling of the load density $\boldsymbol{f}^{\varepsilon_h}$. In Section 2.2.3 we discuss implications of these results for the limit spectrum and evolution equations. Different scalings of the operator will, in particular, yield different scalings of the spectrum and the time variable (or mass density) in the evolution problems. Note that the load density scaling will also depend on the asymptotic regime considered.

It is standard in the theory of plates that one discusses limit equations (both static and dynamic) depending on an appropriate scaling of the external loads. Furthermore, we will see that the limit resolvent equation is always degenerate in some sense. From the mathematical point of view, this is a consequence of the fact that for thin domains the constant in Korn's inequality blows up and by further analysis one can see that this implies that the out-of-plane and in-plane components of the solution are scaled differently in the limit problem. From the physical point of view, it is much easier (i.e. energetically more convenient) for the plate to bend then to stretch. As a result, bending and membrane waves propagate through the medium on different time scales. The effect of high-contrast is also non-negligible, yielding different behaviour depending on the asymptotic regime: the small elastic inclusions behave like three-dimensional objects (regime $h \sim \varepsilon_h$) or like small thin plates (regime $h \ll \varepsilon_h$). We next present our main results.

2.2.2.1 Asymptotic regime $h \sim \varepsilon_h$

A. "Bending" scaling: $\mu_h = \varepsilon_h, \tau = 2$

The following proposition provides an appropriate compactness result, namely a bound on the sequence of energies for a fixed value of δ, see (2.1.1), and its consequences in terms of two-scale convergence.

Proposition 2.2.5. *Consider a sequence $\{(h, \varepsilon_h)\}$ such that $\delta = \lim_{h\to 0} h/\varepsilon_h \in (0,\infty)$, and suppose that $\mu_h = \varepsilon_h$, $\tau = 2$. The following statements hold:*

1. *There exists $C > 0$, independent of h, such that for any sequence $(\boldsymbol{f}^{\varepsilon_h})_{h>0} \subset L^2(\Omega;\mathbb{R}^3)$ of load densities and the corresponding solutions $\boldsymbol{u}^{\varepsilon_h}$ to the resolvent problem (2.6) one has*
$$h^{-2} a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2} < C \big\|\pi_h \boldsymbol{f}^{\varepsilon_h}\big\|^2_{L^2}.$$

2. *If*
$$\limsup_{h\to 0} \Big(h^{-2} a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2}\Big) < \infty, \qquad (\boldsymbol{u}^{\varepsilon_h})_{h>0} \subset H^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^3),$$
then there exist functions $\mathfrak{a} \in H^1_{\gamma_{\rm D}}(\omega;\mathbb{R}^2)$, $\mathfrak{b} \in H^2_{\gamma_{\rm D}}(\omega)$, $C \in \mathfrak{C}_\delta(\Omega\times Y)$, $\mathring{\boldsymbol{u}} \in V_{2,\delta}(\Omega\times Y_0)$, such that for a subsequence, which we continue labelling with ε_h, one has
$$\begin{aligned} \boldsymbol{u}^{\varepsilon_h} &= \tilde{\boldsymbol{u}}^{\varepsilon_h} + \mathring{\boldsymbol{u}}^{\varepsilon_h}, \quad \tilde{\boldsymbol{u}}^{\varepsilon_h}, \mathring{\boldsymbol{u}} \in H^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^3), \quad \mathring{\boldsymbol{u}}^{\varepsilon_h}|_{\Omega_1^{\varepsilon_h}} = 0,\\ \pi_{1/h}\tilde{\boldsymbol{u}}^{\varepsilon_h} &\xrightarrow{L^2} \big(\mathfrak{a}_1(\hat{x}) - x_3\partial_1\mathfrak{b}(\hat{x}), \mathfrak{a}_2(\hat{x}) - x_3\partial_2\mathfrak{b}(\hat{x}), \mathfrak{b}(\hat{x})\big)^\top,\\ h^{-1}\mathring{\boldsymbol{u}}^{\varepsilon_h}(x) &\xrightarrow{dr-2} \mathring{\boldsymbol{u}}(x,y),\\ h^{-1}\operatorname{sym}\nabla_h\tilde{\boldsymbol{u}}^{\varepsilon_h}(x) &\xrightarrow{dr-2} \iota\big(\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x}) - x_3\nabla^2_{\hat{x}}\mathfrak{b}(\hat{x})\big) + C(x,y),\\ \varepsilon_h h^{-1}\operatorname{sym}\nabla_h\mathring{\boldsymbol{u}}^{\varepsilon_h}(x) &\xrightarrow{dr-2} \operatorname{sym}\widetilde{\nabla}_\delta\mathring{\boldsymbol{u}}(x,y), \end{aligned} \tag{2.7}$$
where $\xrightarrow{dr-2}$ stands for the "dimension-reduction two-scale convergence" defined in Appendix 4.3.

3. *If, additionally to 2, one assumes that*
$$\lim_{h\to 0}\Big(h^{-2} a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2}\Big) = a^{\mathfrak{b}}_\delta(\mathfrak{b},\mathfrak{b}) + \|\mathfrak{b}\|^2_{L^2},$$
where the form $a^{\mathfrak{b}}_\delta$ is defined in (2.11), then one has the strong two-scale convergence (cf. Appendix 4.3)
$$\pi_{1/h}\boldsymbol{u}^{\varepsilon_h} \xrightarrow{dr-2} \big(\mathfrak{a}_1(\hat{x}) - x_3\partial_1\mathfrak{b}(\hat{x}), \mathfrak{a}_2(\hat{x}) - x_3\partial_2\mathfrak{b}(\hat{x}), \mathfrak{b}(\hat{x})\big)^\top,$$
with $\mathfrak{a} = \mathfrak{a}^{\mathfrak{b}}$ (for the definition of $\mathfrak{a}^{\mathfrak{b}}$ see (2.10) below).

Remark 2.2.6. It can be seen from the proof of Proposition 2.2.5 that the assumption in its third statement is equivalent to the convergence

$$h^{-1}\operatorname{sym}\nabla_h\boldsymbol{u}^{\varepsilon_h}\chi_{\Omega^1_{\varepsilon_h}} \xrightarrow{dr-2} \iota\big(\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x}) - x_3\nabla^2_{\hat{x}}\mathfrak{b}(\hat{x})\big)\chi_{I\times Y_1} + C(x,y)\chi_{I\times Y_1},$$

$$\varepsilon_h h^{-1} \operatorname{sym}\nabla_h \mathring{\boldsymbol{u}}^{\varepsilon_h} \xrightarrow{dr-2} 0,$$

$$\pi_{1/h}\boldsymbol{u}^{\varepsilon_h} \xrightarrow{dr-2} \left(\mathfrak{a}_1(\hat{x}) - x_3\partial_1\mathfrak{b}(\hat{x}), \mathfrak{a}_2(\hat{x}) - x_3\partial_2\mathfrak{b}(\hat{x}), \mathfrak{b}(\hat{x})\right)^\top.$$

Here $\mathfrak{a} = \mathfrak{a}^{\mathrm{b}}$ and $C(x,\cdot)$ solves the minimization problem (2.2) with $\boldsymbol{A} = \operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x})$ and $\boldsymbol{B} = \nabla^2_{\hat{x}}\mathfrak{b}(\hat{x})$. The analogous claim is valid in all other regimes. As we do not explicitly use it in what follows, we shall omit it.

The following theorem provides the limit resolvent equation. It can be seen that the limit equations do not couple $(\mathfrak{a},\mathfrak{b})$ and $\mathring{\boldsymbol{u}}$. This is not usual in high-contrast analysis and is a consequence of setting $\tau = 2$. This case is thus less interesting and we shall omit its analysis in other regimes ($\delta = 0$ and $\delta = \infty$). However, we will study it here, as it resembles the standard model of a moderate-contrast plate (and so the corresponding evolution is obtained on a longer time scale).

Theorem 2.2.7. *Under the notation of Proposition 2.2.5, suppose that* $\delta \in (0,\infty)$, $\mu_h = \varepsilon_h$, $\tau = 2$, *and consider a sequence* $(\boldsymbol{f}^{\varepsilon_h})_{h>0}$ *of load densities such that*

$$\pi_h \boldsymbol{f}^{\varepsilon_h} \xrightarrow{dr-2} \boldsymbol{f}(x,y) \in L^2(\Omega\times Y;\mathbb{R}^3). \tag{2.8}$$

Then the sequence of solutions to the resolvent problem (2.6) *converges in the sense of* (2.7) *to the unique solution of the following problem: Determine* $\mathfrak{a} \in H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)$, $\mathfrak{b} \in H^2_{\gamma_{\mathrm{D}}}(\omega)$, $\mathring{\boldsymbol{u}} \in V_{2,\delta}(\Omega\times Y_0)$, *such that*

$$\begin{aligned}
&\int_\omega \mathbb{C}^{\mathrm{hom}}_\delta \left(\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x}), \nabla^2_{\hat{x}}\mathfrak{b}(\hat{x})\right) : \left(\operatorname{sym}\nabla_{\hat{x}}\boldsymbol{\theta}_*(\hat{x}), \nabla^2_{\hat{x}}\theta_3(\hat{x})\right) d\hat{x} + \lambda \int_\omega \langle\rho\rangle \mathfrak{b}(\hat{x})\theta_3(\hat{x})\, d\hat{x} \\
&\quad = \int_\omega \langle\overline{\boldsymbol{f}}\rangle(\hat{x})\cdot\left(\boldsymbol{\theta}_*(\hat{x}),\theta_3(\hat{x})\right)^\top d\hat{x} - \int_\omega \langle\overline{x_3\boldsymbol{f}_*}\rangle(\hat{x})\cdot\nabla_{\hat{x}}\theta_3(\hat{x})\, d\hat{x} \quad \forall\boldsymbol{\theta}_* \in H^1_{\gamma_{\mathrm{D}}}(\omega,\mathbb{R}^2),\ \theta_3 \in H^2_{\gamma_{\mathrm{D}}}(\omega), \\
&\int_I \int_{Y_0} \mathbb{C}_0(y) \operatorname{sym}\widetilde{\nabla}_\delta \mathring{\boldsymbol{u}}(x,y) : \operatorname{sym}\widetilde{\nabla}_\delta \mathring{\boldsymbol{\xi}}(x_3,y)\, dy\, dx_3 \\
&\quad = \int_I \int_{Y_0} \boldsymbol{f}(x,y)\cdot\left(\mathring{\xi}_1(x_3,y),\mathring{\xi}_2(x_3,y),0\right)^\top dy\, dx_3 \qquad \forall\mathring{\boldsymbol{\xi}} \in H^1_{00}(I\times Y_0;\mathbb{R}^3),\ \text{a.e. } \hat{x}\in\omega.
\end{aligned} \tag{2.9}$$

If additionally one assumes the strong two-scale convergence in (2.8)*, then one has*

$$\pi_{1/h}\boldsymbol{u}^{\varepsilon_h} \xrightarrow{dr-2} \left(\mathfrak{a}_1(\hat{x}) - x_3\partial_1\mathfrak{b}(\hat{x}) + \mathring{u}_1, \mathfrak{a}_2(\hat{x}) - x_3\partial_2\mathfrak{b}(\hat{x}) + \mathring{u}_2, \mathfrak{b}(\hat{x})\right)^\top.$$

Remark 2.2.8. Under Assumption 2.1.1 (1) the first identity in (2.9) uncouples into two independent identities (see Remark 2.2.1)

$$\int_\omega \mathbb{C}_\delta^{\mathrm{memb}} \operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x}) : \operatorname{sym}\nabla_{\hat{x}}\boldsymbol{\theta}_*(\hat{x})\,d\hat{x} = \int_\omega \langle\overline{\boldsymbol{f}_*}\rangle(\hat{x})\cdot\boldsymbol{\theta}_*(\hat{x})\,d\hat{x} \qquad \forall\boldsymbol{\theta}_*\in H^1_{\gamma_{\mathrm{D}}}(\omega,\mathbb{R}^2),$$

$$\int_\omega \mathbb{C}_\delta^{\mathrm{bend}}\nabla^2_{\hat{x}}\mathfrak{b}(\hat{x}) : \nabla^2_{\hat{x}}\theta_3(\hat{x})\,d\hat{x} + \lambda\int_\Omega \langle\rho\rangle\mathfrak{b}(\hat{x})\theta_3(\hat{x})\,d\hat{x}$$

$$= \int_\omega \langle\overline{\boldsymbol{f}_3}\rangle(\hat{x})\theta_3(\hat{x})\,d\hat{x} - \int_\omega \langle\overline{x_3\boldsymbol{f}_*}\rangle(\hat{x})\cdot\nabla_{\hat{x}}\theta_3(\hat{x})\,d\hat{x} \qquad \forall\theta_3\in H^2_{\gamma_{\mathrm{D}}}(\omega).$$

In connection with the limit problem, we consider a self-adjoint operator $\mathcal{A}_\delta^{\mathrm{hom}}$ defined on the $\langle\rho\rangle$-weighted space $L^2(\omega;\mathbb{R}^2)\times L^2(\omega)$ and corresponding to the differential expression[2]

$$\langle\rho\rangle^{-1}\left(-\operatorname{div}_{\hat{x}},\operatorname{div}_{\hat{x}}\operatorname{div}_{\hat{x}}\right)\mathbb{C}_\delta^{\mathrm{hom}}\left(\operatorname{sym}\nabla_{\hat{x}},\nabla^2_{\hat{x}}\right).$$

More precisely, the operator $\mathcal{A}_\delta^{\mathrm{hom}}$ is defined via the bilinear form

$$a_\delta^{\mathrm{hom}}\big((\boldsymbol{u},v),(\boldsymbol{\zeta},\xi)\big) := \int_\omega \mathbb{C}_\delta^{\mathrm{hom}}\left(\operatorname{sym}\nabla_{\hat{x}}\boldsymbol{u}(\hat{x}),\nabla^2_{\hat{x}}v(\hat{x})\right) : \left(\operatorname{sym}\nabla_{\hat{x}}\boldsymbol{\zeta}(\hat{x}),\nabla^2_{\hat{x}}\xi(\hat{x})\right)\,d\hat{x},$$

$$a_\delta^{\mathrm{hom}} : \left(H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)\times H^2_{\gamma_{\mathrm{D}}}(\omega)\right)^2 \to \mathbb{R}.$$

We also make use of the following observation. Plugging $\boldsymbol{\theta}_3 = 0$ into the first equation in (2.9) and using linearity, we decompose $\mathfrak{a} = \mathfrak{a}^{\mathrm{b}} + \mathfrak{a}^{\boldsymbol{f}_*}$, where $\mathfrak{a}^{\mathrm{b}},\mathfrak{a}^{\boldsymbol{f}_*}\in H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)$ are solutions to the integral identities

$$\int_\omega \mathbb{C}_\delta^{\mathrm{memb}}\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}^{\mathrm{b}}(\hat{x}) : \operatorname{sym}\nabla_{\hat{x}}\boldsymbol{\theta}_*(\hat{x})\,d\hat{x}$$

$$= -\int_\omega \mathbb{C}_\delta^{\mathrm{hom}}\left(0,\nabla^2_{\hat{x}}\mathfrak{b}(\hat{x})\right) : \left(\operatorname{sym}\nabla_{\hat{x}}\boldsymbol{\theta}_*(\hat{x}),0\right)\,d\hat{x} \qquad \forall\boldsymbol{\theta}_*\in H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2),$$

$$\int_\omega \mathbb{C}_\delta^{\mathrm{memb}}\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}^{\boldsymbol{f}_*}(\hat{x}) : \operatorname{sym}\nabla_{\hat{x}}\boldsymbol{\theta}_*(\hat{x})\,d\hat{x} = \int_\omega \langle\overline{\boldsymbol{f}_*}\rangle(\hat{x})\cdot\boldsymbol{\theta}_*(\hat{x})\,d\hat{x} \qquad \forall\boldsymbol{\theta}_*\in H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2). \tag{2.10}$$

Notice that the in-plane deformation $\mathfrak{a}^{\mathrm{b}}$ can be calculated from the out-of-plane deformation $\mathfrak{b}$ by solving the first identity alone. It is easily seen that the solutions $\mathfrak{a}^{\mathrm{b}}$, $\mathfrak{a}^{\boldsymbol{f}_*}$ satisfy the estimates

$$\left\|\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}^{\mathrm{b}}\right\|_{L^2(\omega;\mathbb{R}^2)} \le C\left\|\nabla^2_{\hat{x}}\mathfrak{b}\right\|_{L^2(\omega)}, \quad \left\|\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}^{\boldsymbol{f}_*}\right\|_{L^2(\omega;\mathbb{R}^2)} \le C\left\|\overline{\boldsymbol{f}_*}\right\|_{L^2(\omega)}.$$

[2]The repeated divergence $\operatorname{div}_{\hat{x}}\operatorname{div}_{\hat{x}}$ is applied to matrices and corresponds to the usual divergence applied row-wise and column-wise sequentially.

The first identity in (2.10) defines a bounded linear operator $\mathcal{A}_\delta^{\mathfrak{a};\mathfrak{b}} : H^2_{\gamma_\mathrm{D}}(\omega) \to H^1_{\gamma_\mathrm{D}}(\omega;\mathbb{R}^2)$ by the formula $\mathcal{A}_\delta^{\mathfrak{a};\mathfrak{b}}\mathfrak{b} := \mathfrak{a}^\mathfrak{b}$. Furthermore, the bilinear form $a_\delta^\mathfrak{b} : \left(H^2_{\gamma_\mathrm{D}}(\omega)\right)^2 \to \mathbb{R}$ given by

$$\begin{aligned} a_\delta^\mathfrak{b}(\mathfrak{b},\theta) &:= \int_\omega \mathbb{C}_\delta^{\mathrm{hom}}\left(\operatorname{sym}\nabla_{\hat{x}}\mathcal{A}_\delta^{\mathfrak{a};\mathfrak{b}}\mathfrak{b}(\hat{x}), \nabla^2_{\hat{x}}\mathfrak{b}(\hat{x})\right) : \left(0, \nabla^2_{\hat{x}}\theta(\hat{x})\right) d\hat{x} \\ &= \int_\omega \mathbb{C}_\delta^{\mathrm{hom}}\left(\operatorname{sym}\nabla_{\hat{x}}\mathcal{A}_\delta^{\mathfrak{a};\mathfrak{b}}\mathfrak{b}(\hat{x}), \nabla^2_{\hat{x}}\mathfrak{b}(\hat{x})\right) : \left(\operatorname{sym}\nabla_{\hat{x}}\mathcal{A}_\delta^{\mathfrak{a};\theta}\theta, \nabla^2_{\hat{x}}\theta(\hat{x})\right) d\hat{x}, \end{aligned} \tag{2.11}$$

defines positive definite (as a consequence of Proposition 2.2.4) self-adjoint operator on $L^2(\omega)$, which we denote by $\mathcal{A}_\delta^{\mathfrak{b},\mathrm{hom}}$. The first identity in (2.9) can now be written as follows:

$$\begin{aligned} a_\delta^\mathfrak{b}(\mathfrak{b},\theta) + \lambda\int_\omega \langle\rho\rangle \mathfrak{b}(\hat{x})\theta(\hat{x})\,d\hat{x} &= \int_\omega \mathbb{C}_\delta^{\mathrm{hom}}\left(\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}^{\boldsymbol{f}_*}, 0\right) : \left(0, \nabla^2_{\hat{x}}\theta(\hat{x})\right) d\hat{x} \\ &\quad + \int_\omega \langle\overline{f_3}\rangle(\hat{x})\theta(\hat{x})\,d\hat{x} - \int_\omega \langle\overline{x_3\boldsymbol{f}_*}\rangle(\hat{x})\cdot\nabla_{\hat{x}}\theta(\hat{x})\,d\hat{x} =: \mathcal{F}_\delta(\boldsymbol{f})(\theta), \\ &\qquad \forall\theta \in H^2_{\gamma_\mathrm{D}}(\omega). \end{aligned} \tag{2.12}$$

Notice that for $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^3)$ the right-hand side of (2.12) can be interpreted as an element of $(H^2_{\gamma_\mathrm{D}}(\omega))^*$, which we denoted by $\mathcal{F}_\delta(\boldsymbol{f})$. This reveals the resolvent structure of the limit problem (2.9).

B. "Membrane" scaling: $\mu_h = \varepsilon_h, \tau = 0$

To formulate the convergence result for the present scaling, we consider a non-negative self-adjoint operator $\mathcal{A}_{\delta,\infty}$ defined by the bilinear form

$$\begin{aligned} a_{\delta,\infty}((\mathfrak{a},\mathfrak{b}) + \mathring{\boldsymbol{u}}, (\boldsymbol{\theta},\varphi) + \mathring{\boldsymbol{\xi}}) &:= \int_\omega \mathbb{C}_\delta^{\mathrm{memb}} \operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x}) : \operatorname{sym}\nabla_{\hat{x}}\boldsymbol{\theta}(\hat{x})\,d\hat{x} \\ &\quad + \int_\Omega\int_{Y_0} \mathbb{C}_0(y)\operatorname{sym}\widetilde{\nabla}_\delta\mathring{\boldsymbol{u}}(x,y) : \operatorname{sym}\widetilde{\nabla}_\delta\mathring{\boldsymbol{\xi}}(x,y)\,dy\,dx, \end{aligned} \tag{2.13}$$

$$a_{\delta,\infty} : \left(V_{1,\delta,\infty}(\omega\times Y) + V_{2,\delta}(\Omega\times Y_0)\right)^2 \to \mathbb{R}.$$

Notice that $\mathcal{A}_{\delta,\infty}$ is degenerate with an infinite-dimensional kernel:

$$\mathcal{A}_{\delta,\infty}(0,0,v) = 0 \quad \forall v \in L^2(\omega).$$

However, the restriction of $\mathcal{A}_{\delta,\infty}$ on the space $H_{\delta,\infty}(\Omega\times Y) \cap L^{2,\mathrm{memb}}(\Omega\times Y;\mathbb{R}^3)$ does not exhibit such degeneracies (under Assumption 2.1.1 (1)).

The following proposition gives a suitable compactness result, similar to Proposition 2.2.5.

Proposition 2.2.9. *Consider a sequence $\{(h,\varepsilon_h)\}$ such that $\delta = \lim_{h\to 0} h/\varepsilon_h \in (0,\infty)$, and suppose that $\mu_h = \varepsilon_h$, $\tau = 0$. The following statements hold:*

1. *There exists $C>0$, independent of h, such that for any sequence $(\boldsymbol{f}^{\varepsilon_h})_{h>0} \subset L^2(\Omega;\mathbb{R}^3)$ and the corresponding solutions $\boldsymbol{u}^{\varepsilon_h}$ to the problem (2.6) one has*

$$a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|_{L^2}^2 \le C\|\boldsymbol{f}^{\varepsilon_h}\|_{L^2}^2.$$

2. *If a sequence $(\boldsymbol{u}^{\varepsilon_h})_{h>0} \subset H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)$ is such that*

$$\limsup_{h\to 0}\left(a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|_{L^2}^2\right) < \infty,$$

   then there exist functions $(\mathfrak{a},\mathfrak{b})^\top \in V_{1,\delta,\infty}(\omega\times Y)$, $C \in \mathfrak{C}_\delta(\Omega\times Y)$, $\mathring{\boldsymbol{u}} \in V_{2,\delta}(\Omega\times Y_0)$ such that, up to extracting a subsequence, one has

$$\begin{aligned}
\boldsymbol{u}^{\varepsilon_h} &= \tilde{\boldsymbol{u}}^{\varepsilon_h} + \mathring{\boldsymbol{u}}^{\varepsilon_h}, \quad \tilde{\boldsymbol{u}}^{\varepsilon_h}, \mathring{\boldsymbol{u}} \in H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3), \quad \mathring{\boldsymbol{u}}^{\varepsilon_h}|_{\Omega_1^{\varepsilon_h}} = 0,\\
\tilde{\boldsymbol{u}}_*^{\varepsilon_h} &\xrightarrow{L^2} \mathfrak{a},\\
\tilde{u}_3^{\varepsilon_h} &\xrightarrow{dr-2} \mathfrak{b}(\hat{x}),\\
\mathring{\boldsymbol{u}}^{\varepsilon_h}(x) &\xrightarrow{dr-2} \mathring{\boldsymbol{u}}(x,y),\\
\operatorname{sym}\nabla_h \tilde{\boldsymbol{u}}^{\varepsilon_h}(x) &\xrightarrow{dr-2} \iota\left(\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x})\right) + C(x,y),\\
\varepsilon_h \operatorname{sym}\nabla_h \mathring{\boldsymbol{u}}^{\varepsilon_h}(x) &\xrightarrow{dr-2} \operatorname{sym}\widetilde{\nabla}_\delta \mathring{\boldsymbol{u}}(x,y).
\end{aligned} \tag{2.14}$$

3. *If, additionally to 2, one has*

$$\lim_{h\to 0}\left(a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|_{L^2}^2\right) = a_{\delta,\infty}\left((\mathfrak{a},0)^\top + \mathring{\boldsymbol{u}}, (\mathfrak{a},0)^\top + \mathring{\boldsymbol{u}}\right) + \left\|(\mathfrak{a},\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\right\|_{L^2}^2,$$

   where $a_{\delta,\infty}$ is defined in (2.13), then the strong two-scale convergence

$$\boldsymbol{u}^{\varepsilon_h} \xrightarrow{dr-2} (\mathfrak{a},\mathfrak{b})^\top + \mathring{\boldsymbol{u}}$$

   holds.

The following theorem provides the limit resolvent equation.

Theorem 2.2.10. *Under the notation of Proposition 2.2.9, suppose that $\delta \in (0,\infty)$, $\mu_h = \varepsilon_h$, $\tau = 0$, and consider a sequence $(\boldsymbol{f}^{\varepsilon_h})_{h>0}$ of load densities such that*

$$\boldsymbol{f}^{\varepsilon_h}(x) \xrightharpoonup{dr-2} \boldsymbol{f}(x,y) \in L^2(\Omega \times Y; \mathbb{R}^3). \tag{2.15}$$

Then the sequence of solutions $(\boldsymbol{u}^{\varepsilon_h})_{h>0}$ to the resolvent problems (2.6) converges in the sense of (2.14) to the unique solution of the following problem: Determine $(\mathfrak{a},\mathfrak{b})^\top \in V_{1,\delta,\infty}(\omega \times Y)$, $\mathring{\boldsymbol{u}} \in V_{2,\delta}(\Omega \times Y_0)$, such that

$$\int_\omega \mathbb{C}_\delta^{\mathrm{memb}} \operatorname{sym} \nabla_{\hat{x}} \mathfrak{a}(\hat{x}) : \operatorname{sym} \nabla_{\hat{x}} \boldsymbol{\theta}(\hat{x}) + \lambda \int_\omega \langle \rho \rangle \mathfrak{a}(\hat{x}) \cdot \boldsymbol{\theta}(\hat{x})\, d\hat{x} + \lambda \int_\omega \langle \rho_0 \overline{\mathring{\boldsymbol{u}}_*} \rangle(\hat{x}) \cdot \boldsymbol{\theta}(\hat{x})\, d\hat{x}$$

$$= \int_\omega \langle \overline{\boldsymbol{f}_*} \rangle(\hat{x}) \cdot \boldsymbol{\theta}(\hat{x})\, d\hat{x} \qquad \forall \boldsymbol{\theta} \in H^1_{\gamma_{\mathrm{D}}}(\omega; \mathbb{R}^2),$$

$$\langle \rho \rangle \mathfrak{b}(\hat{x}) + \langle \rho_0 \overline{\mathring{u}_3} \rangle(\hat{x}) = \lambda^{-1} \langle \overline{f_3} \rangle(\hat{x}) \quad \text{a.e. } \hat{x} \in \omega.$$

$$\begin{aligned} &\int_I \int_{Y_0} \mathbb{C}_0(y) \operatorname{sym} \widetilde{\nabla}_\delta \mathring{\boldsymbol{u}}(x,y) : \operatorname{sym} \widetilde{\nabla}_\delta \mathring{\boldsymbol{\xi}}(x_3,y)\, dy\, dx_3 \\ &\qquad + \lambda \int_I \int_{Y_0} \rho_0(y) \big(\mathfrak{a}_1(\hat{x}), \mathfrak{a}_2(\hat{x}), \mathfrak{b}(\hat{x})\big)^\top \cdot \mathring{\boldsymbol{\xi}}(x_3,y)\, dy\, dx_3 \\ &\qquad + \lambda \int_I \int_{Y_0} \rho_0(y) \mathring{\boldsymbol{u}}(x,y) \cdot \mathring{\boldsymbol{\xi}}(x_3,y)\, dy\, dx_3 \\ &\qquad = \int_I \int_{Y_0} \boldsymbol{f}(x,y) \cdot \mathring{\boldsymbol{\xi}}(x_3,y)\, dy\, dx_3 \qquad \forall \mathring{\boldsymbol{\xi}} \in H^1_{00}(I \times Y; \mathbb{R}^3), \quad \text{a.e. } \hat{x} \in \omega. \end{aligned} \tag{2.16}$$

If, additionally, one assumes the strong two-scale convergence in (2.15), then one has

$$\boldsymbol{u}^{\varepsilon_h}(x) \xrightarrow{dr-2} (\mathfrak{a},\mathfrak{b})^\top + \mathring{\boldsymbol{u}}(x,y).$$

Corollary 2.2.11. *Under Assumption 2.1.1 (1) and provided $(\boldsymbol{f}^{\varepsilon_h})_{h>0} \subset L^{2,\mathrm{memb}}(\Omega; \mathbb{R}^3)$, in addition to the convergences in Proposition 2.2.9 one has*

$$\tilde{u}_3^{\varepsilon_h} \xrightharpoonup{H^1} 0,$$

and thus $\mathfrak{b} = 0$ in the limit equations (2.16).

Remark 2.2.12. The limit system (2.16) can be written as a resolvent problem on the space $H_{\delta,\infty}(\Omega \times Y)$, as follows:[3]

$$\left(\mathcal{A}_{\delta,\infty} + \lambda \mathcal{I}\right) \boldsymbol{u} = P_{\delta,\infty} \boldsymbol{f}, \quad \boldsymbol{u} = (\mathfrak{a},\mathfrak{b})^\top + \mathring{\boldsymbol{u}}$$

[3]Notice that this requires to take the inner product with the weight function $\langle \rho \rangle \chi_{Y_1} + \rho_0 \chi_{Y_0}$.

which is the usual structure for the limit problem in the high-contrast setting (see [53] and Section 4.5).

Next, the operator $\tilde{\mathcal{A}}_\delta$ on the space $H_{\delta,\infty}(\Omega\times Y)\cap L^{2,\mathrm{memb}}(\Omega\times Y;\mathbb{R}^3)$ is defined via the form $\tilde{a}_\delta$ given by the expression in (2.13) with a different domain:

$$\tilde{a}_\delta : \left(H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)\times\{0\}+V_{2,\delta}(\Omega\times Y_0)\right)^2\cap\left(L^{2,\mathrm{memb}}(\Omega\times Y;\mathbb{R}^3)\right)^2\to\mathbb{R}.$$

This operator can only be defined under Assumption 2.1.1 (1).

In relation to the limit problem, we also define the following operators. The operator $\mathcal{A}_{00,\delta}$, referred to as the Bloch operator, corresponds to the differential expression[4]

$$-(\rho_0)^{-1}\widetilde{\operatorname{div}}_{2,\delta}\left(\operatorname{sym}\widetilde{\nabla}_\delta\right),$$

and is defined via the bilinear form

$$a_{00,\delta}(\boldsymbol{u},\boldsymbol{v}):=\int_{I\times Y_0}\mathbb{C}_0(y)\operatorname{sym}\widetilde{\nabla}_\delta\boldsymbol{u}(x_3,y):\operatorname{sym}\widetilde{\nabla}_\delta\boldsymbol{v}(x_3,y)\,dx_3\,dy,\quad a_{00,\delta}:\left(H^1_{00}(I\times Y_0;\mathbb{R}^3)\right)^2\to\mathbb{R}.$$

Similarly to the way the form $\tilde{a}_\delta$ and the associated operator $\tilde{\mathcal{A}}_\delta$ were defined by restricting the form $a_{\delta,\infty}$, we define a form

$$\tilde{a}_{00,\delta}:\left(H^1_{00}(I\times Y_0;\mathbb{R}^3)\right)^2\cap\left(L^{2,\mathrm{memb}}(I\times Y_0;\mathbb{R}^3)\right)^2\to\mathbb{R},$$

and the associated operator $\tilde{\mathcal{A}}_{00,\delta}$ by restricting the form $a_{00,\delta}$. We also define a positive self-adjoint operator $\mathcal{A}^{\mathrm{memb}}_\delta$ on $L^2(\omega;\mathbb{R}^2)$ corresponding to the differential expression

$$-\langle\rho\rangle^{-1}\operatorname{div}_{\hat{x}}\left(\mathbb{C}^{\mathrm{memb}}_\delta\operatorname{sym}\nabla_{\hat{x}}\right),$$

as the one defined (on an appropriate weighted L^2 space) by the bilinear form

$$a^{\mathrm{memb}}_\delta(\boldsymbol{u},\boldsymbol{v}):=\int_\omega\mathbb{C}^{\mathrm{memb}}_\delta\operatorname{sym}\nabla_{\hat{x}}\boldsymbol{u}(\hat{x}):\operatorname{sym}\nabla_{\hat{x}}\boldsymbol{v}(\hat{x})d\hat{x},\qquad a^{\mathrm{memb}}_\delta:\left(H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)\right)^2\to\mathbb{R}.$$

In order to simplify the system (2.16), one is led to first solve the equation (where we replace λ with $-\lambda$)

$$(\mathcal{A}_{00,\delta}-\lambda\mathcal{I})\mathring{\boldsymbol{u}}=\lambda\left(\mathfrak{a}(\hat{x}),\mathfrak{b}(\hat{x})\right)^\top+(\rho_0)^{-1}\boldsymbol{f}(\hat{x},\cdot),$$

[4]The differential expression $\widetilde{\operatorname{div}}_{2,\delta}$ stands for $\widetilde{\nabla}_\delta\cdot$ (applied row-wise), *i.e.* it is in the same relation to the gradient $\widetilde{\nabla}_\delta$ as the standard divergence is in the relation to the standard gradient.

where the variable $\hat{x}$ is treated as a parameter (see e.g. [73]). When $\boldsymbol{f}|_{Y_0} = 0$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}_+$, the equation (2.2.2.1) can be solved via the more basic problems

$$(\mathcal{A}_{00,\delta} - \lambda \mathcal{I})\boldsymbol{b}_i^\lambda = \boldsymbol{e}_i, \qquad i = 1,2,3.$$

The following matrix-valued function β_δ taking values in $\mathbb{R}^{3\times 3}$ will be important for characterizing the spectrum of the limit operator:

$$\big(\beta_\delta(\lambda)\big)_{ij} = \lambda\langle\rho\rangle\delta_{ij} + \lambda^2\big\langle\rho_0\overline{(b_i^\lambda)_j}\big\rangle, \qquad i,j = 1,2,3.$$

We refer to β_δ as "Zhikov function", to acknowledge its scalar version appearing in [71]. Its significance will be clear in the next section. We can obtain an alternative representation of the Zhikov function as follows.

First, separate the spectrum of $\mathcal{A}_{00,\delta}$ into two parts:

$$\sigma(\mathcal{A}_{00,\delta}) = \{\eta_1,\eta_2,...\}\cup\{\alpha_1,\alpha_2,...\},$$

where the second subset corresponds to eigenvalues with all associated eigenfunctions having zero ρ_0-weighted mean in all components. In each of the two subsets the eigenvalues are assumed to be arranged in the ascending order. Next, denote by $(\boldsymbol{\varphi}_n)_{n\in\mathbb{N}}$ the set of orthonormal eigenfunctions corresponding to the eigenvalues from the set $\{\eta_1,\eta_2,...\}$ in (2.2.2.1), repeating every eigenvalue according to its multiplicity. Using the spectral decomposition, one obtains

$$\big(\beta_\delta(\lambda)\big)_{ij} = \lambda\langle\rho\rangle\delta_{ij} + \sum_{n=1}^{\infty}\frac{\lambda^2}{\eta_n-\lambda}\big\langle\rho_0\overline{(\varphi_n)_i}\big\rangle\cdot\big\langle\rho_0\overline{(\varphi_n)_j}\big\rangle, \qquad i,j = 1,2,3. \tag{2.17}$$

Under Assumption 2.1.1 (1), one is actually only interested in the operator $\tilde{\mathcal{A}}_{00,\delta}$. We can then define a version of the Zhikov function, denoted by $\tilde{\beta}_\delta^{\mathrm{memb}}$ and taking values in $\mathbb{R}^{2\times 2}$ (dropping the third row and the third column of β_δ, which necessarily vanish as a consequence of symmetries) as the one associated with this operator. Eliminating those values η_n and α_n in (2.2.2.1) whose eigenfunctions belong to the subspace $L^{2,\mathrm{bend}}(I\times Y_0,\mathbb{R}^3)$, we write

$$\sigma(\tilde{\mathcal{A}}_{00,\delta}) = \{\tilde{\eta}_1,\tilde{\eta}_2,\dots\}\cup\{\tilde{\alpha}_1,\tilde{\alpha}_2,\dots\}.$$

Here, similarly to the above, the eigenvalues in the second set are those whose all eigenfunctions have zero weighted mean in all of their components. (Note that due to symmetry the third component has zero weighted mean automatically.) We use the notation

$\sigma(\tilde{\mathcal{A}}_{00,\delta})'$ for the set of such eigenvalues:

$$\sigma(\tilde{\mathcal{A}}_{00,\delta})' = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \dots\}.$$

Analogously to (2.17), we can write a formula for the function $\tilde{\beta}_\delta^{\mathrm{memb}}$. Notice, in particular, that

$$\big(\tilde{\beta}_\delta^{\mathrm{memb}}\big)_{\alpha\beta} = \big(\beta_\delta\big)_{\alpha\beta}, \quad \alpha,\beta = 1,2.$$

C. "Strong high-contrast bending" scaling: $\mu_h = \varepsilon_h h$, $\tau = 2$

As was shown above, in the case $\mu_h = \varepsilon_h$, $\tau = 2$ one does not see effects of high-contrast inclusions in the limit equations (i.e the limit equations are not coupled). Here we consider an asymptotic regime of higher contrast, where the limit equations are coupled. Proposition 2.2.13 below provides the relevant compactness result. Before proceeding to its statement, we introduce some auxiliary objects.

In order to analyse the spectral problem, we will require a positive self-adjoint operator $\hat{\mathcal{A}}_\delta$ on the Hilbert space $\{0\}^2 \times L^2(\omega) + L^2(\Omega \times Y_0, \mathbb{R}^3)$, as the one defined by the bilinear form

$$\hat{a}_\delta\big((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}, (0,0,\theta)^\top + \mathring{\boldsymbol{\xi}}\big) = a_\delta^{\mathrm{b}}(\mathfrak{b},\theta) + \int_\omega a_{00,\delta}(\mathring{\boldsymbol{u}}, \mathring{\boldsymbol{\xi}})\, d\hat{x},$$

$$\hat{a}_\delta : \Big(\{0\}^2 \times H^2_{\gamma_{\mathrm{D}}}(\omega) + V_{2,\delta}(\omega \times Y_0)\Big)^2 \to \mathbb{R}. \quad (2.18)$$

We also define a scalar Zhikov function $\hat{\beta}_\delta$ associated with this problem. Namely, we eliminate the eigenvalues of $\mathcal{A}_{00,\delta}$ all of whose eigenfunctions have zero weighted mean in the third component and set

$$\hat{\beta}_\delta := \beta_{\delta,33}.$$

We also define $\hat{\sigma}(\mathcal{A}_{00,\delta})$ as the set of the eigenvalues of $\mathcal{A}_{00,\delta}$ all of whose eigenfunctions have zero weighted mean in the third component.

Proposition 2.2.13. *Consider a sequence $\{(h, \varepsilon_h)\}$ such that $\delta = \lim_{h\to 0} h/\varepsilon_h \in (0,\infty)$, and suppose that $\mu_h = \varepsilon_h h$, $\tau = 2$. The following statements hold:*

1. *There exists $C > 0$, independent of h, such that for any sequence $(\boldsymbol{f}^{\varepsilon_h})_{h>0} \subset L^2(\Omega;\mathbb{R}^3)$ of load densities and the corresponding solutions $\boldsymbol{u}^{\varepsilon_h}$ to the problem (2.6) one has*

$$h^{-2} a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2} \le C \|\boldsymbol{f}^{\varepsilon_h}\|^2_{L^2}.$$

2. *If*

$$\limsup_{h\to 0}\left(h^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h})+\|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2}\right)<\infty,\qquad (\boldsymbol{u}^{\varepsilon_h})_{h>0}\subset H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3),$$

then there exist functions $\mathfrak{a}\in H_{\gamma_D}(\omega;\mathbb{R}^2)$, $\mathfrak{b}\in H^2_{\gamma_{\mathrm{D}}}(\omega)$, $C\in\mathfrak{C}_\delta(\Omega\times Y)$, $\mathring{\boldsymbol{u}}\in V_{2,\delta}(\Omega\times Y_0)$, *such that (up to a subsequence):*

$$\begin{aligned}
\boldsymbol{u}^{\varepsilon_h}&=\tilde{\boldsymbol{u}}^{\varepsilon_h}+\mathring{\boldsymbol{u}}^{\varepsilon_h},\qquad \tilde{\boldsymbol{u}}^{\varepsilon_h},\mathring{\boldsymbol{u}}\in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3),\qquad \mathring{\boldsymbol{u}}^{\varepsilon_h}|_{\Omega_1^{\varepsilon_h}}=0,\\
\pi_{1/h}\tilde{\boldsymbol{u}}^{\varepsilon_h}&\xrightarrow{L^2}\left(\mathfrak{a}_1(\hat{x})-x_3\partial_1\mathfrak{b}(\hat{x}),\mathfrak{a}_2(\hat{x})-x_3\partial_2\mathfrak{b}(\hat{x}),\mathfrak{b}(\hat{x})\right)^\top,\\
\mathring{\boldsymbol{u}}^{\varepsilon_h}(x)&\xrightarrow{dr-2}\mathring{\boldsymbol{u}}(x,y), \qquad\qquad (2.19)\\
h^{-1}\operatorname{sym}\nabla_h\tilde{\boldsymbol{u}}^{\varepsilon_h}(x)&\xrightarrow{dr-2}\iota\left(\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x})-x_3\nabla^2_{\hat{x}}\mathfrak{b}(\hat{x})\right)+C(x,y),\\
\varepsilon_h\operatorname{sym}\nabla_h\mathring{\boldsymbol{u}}^{\varepsilon_h}(x)&\xrightarrow{dr-2}\operatorname{sym}\widetilde{\nabla}_\delta\mathring{\boldsymbol{u}}(x,y).
\end{aligned}$$

3. *If, additionally to 2, one assumes that*

$$\lim_{h\to0}\left(h^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h})+\|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2}\right)=\hat{a}_\delta((0,0,\mathfrak{b})^\top+\mathring{\boldsymbol{u}},(0,0,\mathfrak{b})^\top+\mathring{\boldsymbol{u}})+\|(0,0,\mathfrak{b})^\top+\mathring{\boldsymbol{u}}\|^2_{L^2},$$

where $\hat{a}_\delta$ is defined in (2.18)*, then one has the strong two-scale convergence*

$$\boldsymbol{u}^{\varepsilon_h}\xrightarrow{dr-2}(0,0,\mathfrak{b})^\top+\mathring{\boldsymbol{u}}.$$

The following theorem describes the limit resolvent equation.

Theorem 2.2.14. *Suppose that $\delta\in(0,\infty)$, $\mu_h=\varepsilon_h h$, and $\tau=2$, and consider a sequence $(\boldsymbol{f}^{\varepsilon_h})_{h>0}$ of load densities such that*

$$\boldsymbol{f}^{\varepsilon_h}\xrightarrow{dr-2}\boldsymbol{f}\in L^2(\Omega\times Y;\mathbb{R}^3). \tag{2.20}$$

Then the sequence of solutions to the resolvent problem (2.6) *converges in the sense of* (2.19) *to the unique solution of the following problem: Determine* $\mathfrak{a}\in H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)$, $\mathfrak{b}\in$

$H^2_{\gamma_D}(\omega)$, $\mathring{\boldsymbol{u}} \in V_{2,\delta}(\Omega \times Y_0)$ *such that*

$$\int_\omega \mathbb{C}^{\hom}_\delta \left(\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x}), \nabla^2_{\hat{x}}\mathfrak{b}(\hat{x})\right) : \left(\operatorname{sym}\nabla_{\hat{x}}\boldsymbol{\theta}_*(\hat{x}), \nabla^2_{\hat{x}}\theta_3(\hat{x})\right) d\hat{x} + \lambda \int_\omega \left(\langle\rho\rangle\mathfrak{b}(\hat{x}) + \langle\rho_0\overline{\mathring{u}_3}\rangle(\hat{x})\right)\theta_3(\hat{x})\,d\hat{x}$$
$$= \int_\omega \langle\overline{\boldsymbol{f}_3}\rangle(\hat{x})\theta_3(\hat{x})\,d\hat{x} \qquad \forall \theta_3 \in H^2_{\gamma_D}(\omega),$$

$$\int_I\int_{Y_0} \mathbb{C}_0(y)\operatorname{sym}\widetilde{\nabla}_\delta\mathring{\boldsymbol{u}}(x,y) : \operatorname{sym}\widetilde{\nabla}_\delta\boldsymbol{\xi}(x_3,y)\,dy\,dx_3 + \lambda\int_I\int_{Y_0}\rho_0(y)\mathfrak{b}(\hat{x})\cdot\mathring{\boldsymbol{\xi}}_3(x_3,y)\,dy\,dx_3$$
$$+\lambda\int_I\int_{Y_0}\rho_0(y)\mathring{\boldsymbol{u}}(x,y)\cdot\mathring{\boldsymbol{\xi}}(x_3,y)\,dy\,dx_3$$
$$= \int_I\int_{Y_0}\boldsymbol{f}(x,y)\cdot\mathring{\boldsymbol{\xi}}(x_3,y)\,dy\,dx_3 \qquad \forall\mathring{\boldsymbol{\xi}} \in H^1_{00}(I\times Y_0;\mathbb{R}^3),\ \ a.e.\ \hat{x}\in\omega.$$
$$(2.21)$$

In the case when the strong two-scale convergence holds in (2.20)*, one has*

$$\boldsymbol{u}^{\varepsilon_h} \xrightarrow{dr-2} (0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}.$$

Remark 2.2.15. The limit problem (2.21) can again be written as the resolvent problem on $\{0\}^2 \times L^2(\omega) + L^2(\Omega\times Y_0;\mathbb{R}^3)$:

$$(\hat{\mathcal{A}}_\delta + \lambda\mathcal{I})\boldsymbol{u} = \left(S_2(P_{\delta,\infty}\boldsymbol{f})_1, S_2(P_{\delta,\infty}\boldsymbol{f})_2, (P_{\delta,\infty}\boldsymbol{f})_3\right)^\top, \quad \boldsymbol{u} = (0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}$$

which is again the general desired structure.

Remark 2.2.16. Under Assumption 2.1.1 (1), the first equation in (2.21) decouples from the second (see Remark 2.2.1) and one has

$\mathfrak{a} = 0,$

$$\int_\omega \mathbb{C}^{\mathrm{bend}}_\delta\nabla^2_{\hat{x}}\mathfrak{b}(\hat{x}) : \nabla^2_{\hat{x}}\theta_3(\hat{x})\,d\hat{x} + \lambda\int_\omega\left(\langle\rho\rangle\mathfrak{b}(\hat{x}) + \langle\rho_0\overline{\mathring{u}_3}\rangle(\hat{x})\right)\theta_3(\hat{x})\,d\hat{x} = \int_\omega\langle\overline{\boldsymbol{f}_3}\rangle(\hat{x})\theta_3(\hat{x})\,d\hat{x} \quad \forall\theta_3\in H^2_{\gamma_D}(\omega).$$

In the following sections we will analyse only those two cases for each regime when there is a coupling between the deformations inside and outside the inclusions.

2.2.2.2 Asymptotic regime $h \ll \varepsilon_h$: "very thin" plate

Throughout this section, we additionally assume that the set Y_0 has $C^{1,1}$ boundary, to ensure the validity of some auxiliary extension results, see Appendix 4.4.

A. "Membrane" scaling: $\mu_h = \varepsilon_h, \tau = 0$

Similarly to Part B of Section 2.2.2.1, where the membrane scaling is discussed for the regime $h \sim \varepsilon_h$, we define the following objects using the limit resolvent in Theorem 2.2.19 below (expression (2.24) for the limit resolvent) :

- For each $\kappa \in [0,\infty]$, a form $a_{0,\kappa} : \left(V_{1,0,\kappa}(\omega \times Y) + V_{2,0}(\Omega \times Y_0)\right)^2 \to \mathbb{R}$ and the associated operator $\mathcal{A}_{0,\kappa}$ on the space $V_{0,\kappa}(\Omega \times Y)$, analogous to $a_{\delta,\infty}$ and $\mathcal{A}_{\delta,\infty}$ of Part B, Section 2.2.2.1. In this way the limit problem (2.24) can be written in the form

$$(\mathcal{A}_{0,\kappa} + \lambda \mathcal{I})\boldsymbol{u} = P_{\delta,\kappa}\boldsymbol{f}, \quad \boldsymbol{u} = ((\mathfrak{a}, \mathfrak{b})^\top + \mathring{\boldsymbol{u}});$$

- A form
$$\tilde{a}_0 : \left(H^1_{\gamma_\mathrm{D}}(\omega;\mathbb{R}^2) + L^2(\omega; H^1_0(Y_0;\mathbb{R}^2))\right)^2 \to \mathbb{R}$$
and the associated operator $\tilde{\mathcal{A}}_0$ on $L^2(\omega;\mathbb{R}^2) + L^2(\omega \times Y_0;\mathbb{R}^2)$ (analogous to $\tilde{a}_\delta$ and $\tilde{\mathcal{A}}_\delta$) — these are correctly defined under Assumption 2.1.1(1);

- A bilinear form $a_0^{\mathrm{memb}} : \left(H^1_{\gamma_\mathrm{D}}(\omega;\mathbb{R}^2)\right)^2 \to \mathbb{R}$ and the associated operator $\mathcal{A}_0^{\mathrm{memb}}$ on $L^2(\omega;\mathbb{R}^2)$ (analogous to a_δ^{memb} and $\mathcal{A}_\delta^{\mathrm{memb}}$);

- A bilinear form $\tilde{a}_{00,0} : \left(H^1_0(Y_0,;\mathbb{R}^2)\right)^2 \to \mathbb{R}$ and the associated operator $\tilde{\mathcal{A}}_{00,0}$ on $L^2(Y_0;\mathbb{R}^2)$ (analogous to $\tilde{a}_{00,\delta}$ and $\tilde{\mathcal{A}}_{00,\delta}$);

- Functions $\beta_0, \tilde{\beta}_0^{\mathrm{memb}}$, by analogy with $\beta_\delta, \tilde{\beta}_\delta^{\mathrm{memb}}$;

- A set $\sigma(\tilde{\mathcal{A}}_{00,0})'$, by analogy with $\sigma(\tilde{\mathcal{A}}_{00,\delta})'$.

We do not write these definitions explicitely, since we assume their definition is natural. The following proposition provides a compactness result for solutions to (2.6).

Proposition 2.2.17. *Suppose that $\delta = 0$, $\mu_h = \varepsilon_h$, $\tau = 0$. The following statements hold:*

1. *There exists $C > 0$, independent of h, such that for a sequence $(\boldsymbol{f}^{\varepsilon_h})_{h>0} \subset L^2(\Omega;\mathbb{R}^3)$ of load densities and the corresponding solutions $\boldsymbol{u}^{\varepsilon_h}$ to the problem (2.6) one has*

$$a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2} \le C\|\boldsymbol{f}^{\varepsilon_h}\|^2_{L^2}.$$

2. *If*

$$\limsup_{h\to 0}\left(a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h})+\|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2}\right)<\infty,\quad (\boldsymbol{u}^{\varepsilon_h})_{h>0}\subset H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3),$$

then there exist $(\mathfrak{a},\mathfrak{b})^\top\in V_{1,0,\kappa}(\omega\times Y)$, $\boldsymbol{\varphi}_1\in L^2(\omega;\dot{H}^1(\mathcal{Y};\mathbb{R}^2))$, $\varphi_2\in L^2(\omega;\dot{H}^2(\mathcal{Y}))$, $\mathring{\boldsymbol{u}}\in V_{2,0}(\Omega\times Y_0)$, $\mathring{\boldsymbol{g}}\in L^2(\Omega\times Y;\mathbb{R}^3)$, $\mathring{\boldsymbol{g}}_{|\Omega\times Y_1}=0$ *such that (up to a subsequence)*

$$\begin{aligned}
&\boldsymbol{u}^{\varepsilon_h}=\tilde{\boldsymbol{u}}^{\varepsilon_h}+\mathring{\boldsymbol{u}}^{\varepsilon_h},\quad \tilde{\boldsymbol{u}}^{\varepsilon_h},\mathring{\boldsymbol{u}}\in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3),\quad \mathring{\boldsymbol{u}}^{\varepsilon_h}|_{\Omega_1^{\varepsilon_h}}=0,\\
&\tilde{\boldsymbol{u}}^{\varepsilon_h}_*\xrightarrow{L^2}\mathfrak{a},\\
&\tilde{\boldsymbol{u}}^{\varepsilon_h}_3\xrightarrow{dr-2}\begin{cases}\mathfrak{b}(\hat{x}), & \kappa=\infty,\\ \mathfrak{b}(\hat{x},y), & \kappa\in[0,\infty),\end{cases}\\
\operatorname{sym}\nabla_h\tilde{\boldsymbol{u}}^{\varepsilon_h}(x)&\xrightarrow{dr-2}\begin{cases}\iota\left(\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x})\right)+C_0(\boldsymbol{\varphi}_1,\varphi_2,\boldsymbol{g})(x,y), & \kappa=\infty,\\ \iota\left(\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x})\right)+C_0(\boldsymbol{\varphi}_1,\kappa\mathfrak{b},\boldsymbol{g})(x,y), & \kappa\in(0,\infty),\\ \iota\left(\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x})\right)+C_0(\boldsymbol{\varphi}_1,0,\boldsymbol{g})(x,y), & \kappa=0,\end{cases}\\
&\mathring{\boldsymbol{u}}^{\varepsilon_h}\xrightarrow{dr-2}\mathring{\boldsymbol{u}}(\hat{x},y),\\
\varepsilon_h\operatorname{sym}\nabla_h\mathring{\boldsymbol{u}}^{\varepsilon_h}&\xrightarrow{dr-2}C_0(\mathring{\boldsymbol{u}}_*,0,\mathring{\boldsymbol{g}})(x,y).
\end{aligned}\tag{2.22}$$

3. *If, additionally to 2, one assumes that*

$$\lim_{h\to 0}\left(a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h})+\|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2}\right)=a_{0,\kappa}((\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}},(\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}})+\|(\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}}\|^2_{L^2},$$

where the form $a_{0,\kappa}$ *is defined above, then one has*

$$\boldsymbol{u}^{\varepsilon_h}\xrightarrow{dr-2}(\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}}.$$

Remark 2.2.18. In the regimes $h\sim\varepsilon_h^2$ and $h\ll\varepsilon_h^2$ we are not able to identify the functions $\mathfrak{b}(\hat{x},y)$ and $\mathring{u}_3$ separately on $\omega\times Y_0$ (in the following theorem). However, we are able to identify their sum, which is the only relevant object, since the third component of solution converges to their sum. Thus we artificially set $\mathfrak{b}(\hat{x},y)=0$ on $\omega\times Y_0$, to have uniqueness of the solution of the limit problem. In the case when $\mathfrak{b}$ is a function of $\hat{x}$ only, the decomposition $\mathfrak{b}(\hat{x})+\mathring{u}_3(\hat{x},y)$ is unique in $L^2(\omega\times Y)$, since we know that $\mathring{u}_{3|\omega\times Y_1}=0$.

The limit resolvent problem for the model of homogenized plate is given by the following theorem.

Theorem 2.2.19. *Let $\delta = 0$, $\mu_h = \varepsilon_h$, $\tau = 0$ and suppose that the sequence of load densities converge as follows:*

$$\boldsymbol{f}^{\varepsilon_h} \xrightarrow{dr-2} \boldsymbol{f} \in L^2(\Omega \times Y; \mathbb{R}^3). \tag{2.23}$$

Then the sequence of solutions to the resolvent problem (2.6) converges in the sense of (2.22) to the unique solution (see also Remark 2.2.18) of the following problems: Find $(\mathfrak{a}, \mathfrak{b})^\top \in V_{1,0,\kappa}(\omega \times Y)$, $\mathring{\boldsymbol{u}} \in V_{2,0}(\Omega \times Y_0)$ such that

$$\int_\omega \mathbb{C}_1^{\mathrm{memb,r}} \operatorname{sym} \nabla_{\hat{x}} \mathfrak{a}(\hat{x}) : \operatorname{sym} \nabla_{\hat{x}} \boldsymbol{\theta}_*(\hat{x})\, d\hat{x} + \lambda \int_\omega \big(\langle \overline{\rho} \rangle \mathfrak{a}(\hat{x}) + \langle \rho_0 \mathring{\boldsymbol{u}}_* \rangle \big) \cdot \boldsymbol{\theta}_*(\hat{x})\, d\hat{x}$$

$$= \int_\omega \langle \overline{\boldsymbol{f}_*} \rangle (\hat{x}) \cdot \boldsymbol{\theta}_*(\hat{x})\, d\hat{x} \qquad \forall \boldsymbol{\theta}_* \in H^1_{\gamma_{\mathrm{D}}}(\omega; \mathbb{R}^2),$$

$$\int_{Y_0} \mathbb{C}_0^{\mathrm{memb,r}}(y) \operatorname{sym} \nabla_y \mathring{\boldsymbol{u}}_*(\hat{x}, y) : \operatorname{sym} \nabla_y \mathring{\boldsymbol{\xi}}_*(y)\, dy + \lambda \int_{Y_0} \rho_0(y) \left(\mathfrak{a}(\hat{x}) + \mathring{\boldsymbol{u}}_*(\hat{x}, y) \right) \cdot \mathring{\boldsymbol{\xi}}_*(y)\, dy,$$

$$= \int_{Y_0} \overline{\boldsymbol{f}_*}(\hat{x}, y) \cdot \mathring{\boldsymbol{\xi}}_*(y)\, dy \qquad \forall \mathring{\boldsymbol{\xi}}_* \in H^1_0(Y_0; \mathbb{R}^2), \ \text{a.e. } \hat{x} \in \omega,$$

$$\langle \rho \rangle \mathfrak{b}(\hat{x}) + \rho_0(y) \mathring{u}_3(\hat{x}, y) = \lambda^{-1} P^0 \overline{f}_3(\hat{x}, y), \qquad \kappa = \infty,$$

$$\left. \begin{aligned} &\frac{\kappa^2}{12} \int_{Y_1} \mathbb{C}_1^r(y) \nabla_y^2 \mathfrak{b}(\hat{x}, y) : \nabla_y^2 v(y)\, dy + \lambda \int_{Y_1} \rho_1(y) \mathfrak{b}(\hat{x}, y) v(y)\, dy \\ &\qquad = \int_{Y_1} \overline{f}_3(\hat{x}, y) v(y)\, dy \qquad \forall v \in H^2(\mathcal{Y}), \ \text{a.e. } \hat{x} \in \omega. \\ &\rho_0(y) \mathring{u}_3(\hat{x}, y) = \lambda^{-1} \overline{f}_3(\hat{x}, y), \quad \mathfrak{b}(\hat{x}, y) = 0, \qquad y \in Y_0 \end{aligned} \right\} \quad \kappa \in (0, \infty),$$

$$\rho_1(y) \mathfrak{b}(\hat{x}, y) + \rho_0(y) \mathring{u}_3(\hat{x}, y) = \lambda^{-1} \overline{f}_3(\hat{x}, y); \quad \mathfrak{b}(\hat{x}, y) = 0, \quad y \in Y_0, \qquad \kappa = 0. \tag{2.24}$$

If additionally we assume the strong two-scale convergence in (2.23), then we additionally have the strong two-scale convergence

$$\boldsymbol{u}^{\varepsilon_h}(x) \xrightarrow{dr-2} (\mathfrak{a}, \mathfrak{b})^\top + \mathring{\boldsymbol{u}}(x, y).$$

Corollary 2.2.20. *Under the Assumption 2.1.1 (1) and provided $(\boldsymbol{f}^{\varepsilon_h})_{h>0} \subset L^{2,\mathrm{memb}}(\Omega^h, \mathbb{R}^3)$, in addition to the convergences stated in Proposition 2.2.17, we have*

$$\tilde{u}_3^{\varepsilon_h} \xrightarrow{H^1} 0, \quad \mathring{u}_3^{\varepsilon_h} \xrightarrow{L^2} 0,$$

and thus we also have that $\mathfrak{b} = \mathring{u}_3 = 0$ in the limit equations (2.24).

B. "Bending" scaling: $\mu_h = \varepsilon_h^2, \tau = 2$

In the regime $h \ll \varepsilon_h$ the band gap structure of the spectrum for the spectrum of order h^2 appears when we scale the coefficients in inclusions with ε_h^4. This can be seen from the a priori estimates obtained in Appendix.

We define, for every $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^3)$,

$$\mathcal{F}_0(\boldsymbol{f})\big((0,0,\theta)^\top + \mathring{\boldsymbol{\xi}}\big) := \int_\omega \langle \overline{f}_3\rangle \theta \, d\hat{x} + \int_\omega \int_{Y_0} \overline{\boldsymbol{f}_*}(\hat{x},y)\cdot \mathring{\boldsymbol{\xi}}_*(x,y)\, dy\, d\hat{x},$$

$$+ \int_\omega \int_{Y_0} \overline{f}_3(\hat{x},y)\mathring{\xi}_3(\hat{x},y)\, dy\, d\hat{x} - \int_\omega \int_{Y_0} \overline{x_3 \boldsymbol{f}_*}(\hat{x},y)\cdot \nabla_y \mathring{\xi}_3(\hat{x},y)\, dy\, d\hat{x}, \quad (2.25)$$

$$[0.35em]\theta \in L^2(\omega), \quad \mathring{\boldsymbol{\xi}} = (\mathring{\boldsymbol{\xi}}_*, \xi_3) \in L^2\big(\omega; H_0^1(Y_0;\mathbb{R}^2)\times H_0^2(Y_0)\big).$$

Furthermore, in connection with the limit problem described in Theorem 2.2.22 (expression (2.28) below), we introduce several objects:

- A bilinear form

$$a_0^{\mathrm{hom}} : \big(H^2_{\gamma_{\mathrm{D}}}(\omega)\big)^2 \to \mathbb{R}$$

  and the associated operator $\mathcal{A}_0^{\mathrm{hom}}$ on $L^2(\omega)$, analogous to a_δ^{hom} and $\mathcal{A}_\delta^{\mathrm{hom}}$ of Part A, Section 2.2.2.1 (notice that here the situation is simpler since necessarily $\mathfrak{a} = 0$);

- The bilinear form

$$\hat{a}_{00,0}(\mathring{u},\mathring{\xi}) := \frac{1}{12}\int_{Y_0} \mathbb{C}_0^{\mathrm{bend,r}}(y)\, \mathrm{sym}\, \nabla_y^2 \mathring{u} : \mathrm{sym}\, \nabla^2 \mathring{\xi}\, dy, \qquad \hat{a}_{00,0} : \big(H_0^2(Y_0)\big)^2 \to \mathbb{R}$$

  and the associated "Bloch operator" $\hat{\mathcal{A}}_{00,0}$ on $L^2(Y_0)$.

- A scalar Zhikov function $\hat{\beta}_0$ defined via the operator $\hat{\mathcal{A}}_{00,0}$ (analogous to $\hat{\beta}_\delta$ defined via the operator $\mathcal{A}_{00,\delta}$, see Part C, Section 2.2.2.1);

- A set $\hat{\sigma}(\hat{\mathcal{A}}_{00,0})$ (analogous to $\hat{\sigma}(\mathcal{A}_{00,\delta})$);

- The bilinear form

$$\hat{a}_0\big(\mathfrak{b}+\mathring{u}, \theta+\mathring{\xi}\big) = a_0^{\mathrm{hom}}(\mathfrak{b},\theta) + \int_\omega \hat{a}_{00,0}(\mathring{u},\mathring{\xi}), \qquad \hat{a}_0 : \Big(H^2_{\gamma_{\mathrm{D}}}(\omega) + L^2(\omega; H_0^2(Y_0))\Big)^2 \to \mathbb{R}.$$

  and the corresponding operator $\hat{\mathcal{A}}_0$ on $L^2(\omega) + L^2(\omega; H_0^2(Y_0))$.

The following proposition gives a suitable compactness result for the regime $h \ll \varepsilon_h$.

Proposition 2.2.21. *Let $\delta = 0$, $\mu_h = \varepsilon_h^2$, $\tau = 2$. The following statements hold:*

1. *There exists $C > 0$, independent of h, such that for any sequence $(\boldsymbol{f}^{\varepsilon_h})_{h>0} \subset L^2(\Omega;\mathbb{R}^3)$ of load densities and the corresponding solutions $\boldsymbol{u}^{\varepsilon_h}$ to the problem (2.6) one has*

$$h^{-2} a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|_{L^2}^2 \leq C \left\| \pi_{h/\varepsilon_h} \boldsymbol{f}^{\varepsilon_h} \right\|_{L^2(\Omega;\mathbb{R}^3)}^2.$$

2. *If*

$$\limsup_{h\to 0} \left(h^{-2} a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|_{L^2}^2 \right) < \infty, \quad (\boldsymbol{u}^{\varepsilon_h}) \subset H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3),$$

then there exist $\mathfrak{a} \in H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)$, $\mathfrak{b} \in H^2_{\gamma_{\mathrm{D}}}(\omega)$, $C \in \mathfrak{C}_0(\Omega \times Y)$, $\mathring{u}_\alpha \in L^2(\omega; H^1_0(Y_0))$, $\alpha = 1,2$, $\mathring{u}_3 \in L^2(\omega; H^2_0(Y_0))$, $\mathring{\boldsymbol{g}} \in L^2(\Omega \times Y;\mathbb{R}^3)$, $\mathring{\boldsymbol{g}}_{|\Omega\times Y_1} = 0$ such that (up to subsequence)

$$\begin{aligned}
&\boldsymbol{u}^{\varepsilon_h} = \tilde{\boldsymbol{u}}^{\varepsilon_h} + \mathring{\boldsymbol{u}}^{\varepsilon_h}, \quad \tilde{\boldsymbol{u}}^{\varepsilon_h}, \mathring{\boldsymbol{u}}^{\varepsilon_h} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3), \quad \mathring{\boldsymbol{u}}^{\varepsilon_h}|_{\Omega_1^{\varepsilon_h}} = 0,\\
&\pi_{1/h} \tilde{\boldsymbol{u}}^{\varepsilon_h} \xrightarrow{L^2} \left(\mathfrak{a}_1(\hat{x}) - x_3 \partial_1 \mathfrak{b}(\hat{x}), \mathfrak{a}_2(\hat{x}) - x_3 \partial_2 \mathfrak{b}(\hat{x}), \mathfrak{b}(\hat{x}) \right)^\top,\\
&\pi_{\varepsilon_h/h} \mathring{\boldsymbol{u}}^{\varepsilon_h} \xrightarrow{dr-2} \left(\mathring{u}_1(\hat{x},y) - x_3 \partial_{y_1} \mathring{u}_3(\hat{x},y), \mathring{u}_2(\hat{x},y) - x_3 \partial_{y_2} \mathring{u}_3(\hat{x},y), \mathring{u}_3(\hat{x},y) \right)^\top\\
&h^{-1} \operatorname{sym} \nabla_h \tilde{\boldsymbol{u}}^{\varepsilon_h}(x) \xrightarrow{dr-2} \iota\left(\operatorname{sym} \nabla_{\hat{x}} \mathfrak{a}(\hat{x}) - x_3 \nabla^2_{\hat{x}} \mathfrak{b}(\hat{x}) \right) + C(x,y),\\
&\varepsilon_h^2 h^{-1} \operatorname{sym} \nabla_h \mathring{\boldsymbol{u}}^{\varepsilon_h} \xrightarrow{dr-2} C_0(\mathring{\boldsymbol{u}}_*, \mathring{u}_3, \mathring{\boldsymbol{g}})(x,y).
\end{aligned} \tag{2.26}$$

3. *If additionally to 2, one has*

$$\lim_{h\to 0} \left(h^{-2} a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|_{L^2}^2 \right) = \hat{a}_0(\mathfrak{b} + \mathring{u}_3, \mathfrak{b} + \mathring{u}_3) + \|\mathfrak{b} + \mathring{u}_3\|_{L^2}^2,$$

where $\hat{a}_0$ is defined below, then the strong two-scale convergence holds:

$$\pi_{\varepsilon_h/h} \boldsymbol{u}^{\varepsilon_h} \xrightarrow{dr-2} (0, 0, \mathfrak{b} + \mathring{u}_3)^\top.$$

The following theorem provides the limit resolvent equation.

Theorem 2.2.22. *Let $\delta = 0$, $\mu_h = \varepsilon_h^2$, $\tau = 2$ and let the sequence of load densities satisfy*

$$\pi_{h/\varepsilon_h} \boldsymbol{f}^{\varepsilon_h} \xrightarrow{dr-2} \boldsymbol{f} \in L^2(\Omega \times Y;\mathbb{R}^3). \tag{2.27}$$

Then the sequence of solutions to the resolvent problem (2.6) *converges in the sense of* (2.26) *to the unique solution of the following problem: Determine* $\mathfrak{a} \in H^1_{\gamma_D}(\omega;\mathbb{R}^2)$, $\mathfrak{b} \in H^2_{\gamma_D}(\omega)$, $\mathring{u}_\alpha \in L^2(\omega, H^1_0(Y_0))$, $\alpha = 1,2$, $\mathring{u}_3 \in L^2(\omega, H^2_0(Y_0))$ *such that*

$$
\begin{aligned}
&\mathfrak{a} = 0,\\
&\frac{1}{12}\int_\omega \mathbb{C}_1^{\mathrm{bend,r}} \nabla^2_{\hat{x}}\mathfrak{b}(\hat{x}) : \nabla^2_{\hat{x}}\theta_3(\hat{x})\,d\hat{x} + \lambda \int_\omega \big(\langle\rho\rangle \mathfrak{b}(\hat{x}) + \langle \rho_0 \mathring{u}_3\rangle(\hat{x})\big)\,\theta_3(\hat{x})\,d\hat{x}\\
&\qquad\qquad = \int_\omega \langle \overline{f}_3\rangle(\hat{x})\theta_3(\hat{x})\,d\hat{x} \qquad \forall \theta_3 \in H^2_{\gamma_D}(\omega),\\
&\int_{Y_0} \mathbb{C}_0^{\mathrm{memb,r}}(y)\,\mathrm{sym}\,\nabla_y \mathring{\boldsymbol{u}}_*(\hat{x},y) : \nabla_y \mathring{\boldsymbol{\xi}}_*(y)\,dy = \int_{Y_0} \overline{\boldsymbol{f}_*}(\hat{x},y)\cdot \mathring{\boldsymbol{\xi}}_*(y)\,dy \quad \forall \mathring{\boldsymbol{\xi}}_* \in H^1_0(Y_0;\mathbb{R}^2),\ a.e.\ \hat{x}\in\omega.\\
&\frac{1}{12}\int_{Y_0} \mathbb{C}_0^{\mathrm{bend,r}}(y)\nabla^2_y \mathring{u}_3(\hat{x},y) : \nabla^2_y \mathring{\xi}_3(y)\,dy + \lambda \int_{Y_0} \rho_0(y)\big(\mathfrak{b}(\hat{x}) + \mathring{u}_3(\hat{x},y)\big)\mathring{\xi}_3(y)\,dy\\
&\qquad\qquad = \int_{Y_0} \overline{f}_3(\hat{x},y)\mathring{\xi}_3(y)\,dy - \int_{Y_0} \overline{x_3 \boldsymbol{f}_*}(\hat{x},y)\cdot \nabla_y \mathring{\xi}_3(y)\,dy \quad \forall \mathring{\xi}_3 \in H^2_0(Y_0),\ a.e.\ \hat{x}\in\omega.
\end{aligned}
\tag{2.28}
$$

If the strong two-scale convergence in (2.27) *holds, then additionally one has*

$$
\pi_{\varepsilon_h/h}\mathring{\boldsymbol{u}}^{\varepsilon_h} \xrightarrow{dr-2} \big(\mathring{u}_1(\hat{x},y) - x_3\partial_{y_1}\mathring{u}_3(\hat{x},y), \mathring{u}_2(\hat{x},y) - x_3\partial_{y_2}\mathring{u}_3(\hat{x},y), \mathring{u}_3(\hat{x},y)\big)^\top.
$$

The right-hand side of (2.28) can be interpreted as the element of dual of

$$
\{0\}^2 \times L^2(\omega) + L^2(\omega; H^1_0(Y_0;\mathbb{R}^2) \times H^2_0(Y_0)).
$$

Notice that the second equation in (2.28) is completely separated from the rest of the system.

2.2.2.3 Asymptotic regime $\varepsilon_h \ll h$: "moderately thin" plate

A. "Membrane" scaling: $\mu_h = \varepsilon_h$, $\tau = 0$

Similarly to Section 2.2.2.1, we define the following objects using Theorem 2.2.24 (the expression for the limit resolvent (2.31)) :

- A bilinear form

$$
a_{\infty,\infty} : \big(V_{1,\infty,\infty}(\omega\times Y) + V_{2,\infty}(\Omega\times Y_0)\big)^2 \to \mathbb{R}
$$

and the associated operator $\mathcal{A}_{\infty,\infty}$ on the space $H_{\infty,\infty}(\Omega\times Y)$, analogous to $a_{\delta,\infty}$ and $\mathcal{A}_{\delta,\infty}$ of Part B, Section 2.2.2.1. In this way the limit problem (2.31) can be written in the form

$$(\mathcal{A}_{\infty,\infty}+\lambda\mathcal{I})\boldsymbol{u}=P_{\infty,\infty}\boldsymbol{f},\quad \boldsymbol{u}=(\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}};$$

- A bilinear form

$$\tilde{a}_\infty:\left(H^1_{\gamma_{\rm D}}(\omega;\mathbb{R}^2)\times\{0\}+V_{2,\infty}(\Omega\times Y_0)\right)^2\cap\left(L^{2,\rm memb}(\Omega\times Y,\mathbb{R}^3)\right)^2\to\mathbb{R},$$

and the associated operator $\tilde{\mathcal{A}}_\infty$ on the space

$$\left(L^2(\omega;\mathbb{R}^2)\times\{0\}+L^2(\Omega\times Y_0;\mathbb{R}^3)\right)\cap L^{2,\rm memb}(\Omega\times Y,\mathbb{R}^3)$$

(analogous to $\tilde{a}_\delta$ and $\tilde{\mathcal{A}}_\delta$) — these are correctly defined under Assumption 2.1.1 (1);

- A bilinear form $a^{\rm memb}_\infty:(H^1_{\gamma_{\rm D}}(\omega;\mathbb{R}^2))^2\to\mathbb{R}$ and the associated operator $\mathcal{A}^{\rm memb}_\infty$ on $L^2(\omega;\mathbb{R}^2)$ (analogous to $a^{\rm memb}_\delta$ and $\mathcal{A}^{\rm memb}_\delta$);
- A bilinear form $\tilde{a}_{00,\infty}:(H^1_0(Y_0;\mathbb{R}^3))^2\to\mathbb{R}$ and the associated operator $\tilde{\mathcal{A}}_{00,\infty}$ on $L^2(Y_0;\mathbb{R}^3)$ (analogous to $\tilde{a}_{00,\delta}$ and $\tilde{\mathcal{A}}_{00,\delta}$);
- Functions β_∞, $\tilde{\beta}^{\rm memb}_\infty$, by analogy with β_δ, $\tilde{\beta}^{\rm memb}_\delta$;
- A set $\sigma(\tilde{\mathcal{A}}_{00,\infty})'$, by analogy with $\sigma(\tilde{\mathcal{A}}_{00,\delta})'$.

As in the case of other regimes, we first prove an appropriate compactness result, as follows.

Proposition 2.2.23. *Let $\delta=\infty$, $\mu_h=\varepsilon_h,\tau=0$. The following statements hold:*

1. *There exists $C>0$, independent of h, such that for any sequence $(\boldsymbol{f}^{\varepsilon_h})_{h>0}\subset L^2(\Omega;\mathbb{R}^3)$ of load densities and the corresponding solutions $\boldsymbol{u}^{\varepsilon_h}$ to the problem (2.6) one has*

$$a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h})+\|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2}\leq C\|\boldsymbol{f}^{\varepsilon_h}\|^2_{L^2}.$$

2. *If*

$$\limsup_{h\to 0}\left(a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h})+\|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2}\right)<\infty,\quad (\boldsymbol{u}^{\varepsilon_h})_{h>0}\subset H^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^3),$$

there exist $(\mathfrak{a},\mathfrak{b})^\top \in V_{1,\infty,\infty}(\omega\times Y)$, $\mathring{\boldsymbol{u}} \in H_{\infty,\infty}(\Omega\times Y)$, $C \in \mathfrak{C}_\infty(\Omega\times Y)$ *such that (up to subsequence)*

$$
\begin{aligned}
\boldsymbol{u}^{\varepsilon_h} &= \tilde{\boldsymbol{u}}^{\varepsilon_h} + \mathring{\boldsymbol{u}}^{\varepsilon_h}, \quad \tilde{\boldsymbol{u}}^{\varepsilon_h}, \mathring{\boldsymbol{u}} \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^3), \quad \mathring{\boldsymbol{u}}^{\varepsilon_h}|_{\Omega_1^{\varepsilon_h}} = 0,\\
\tilde{\boldsymbol{u}}_*^{\varepsilon_h} &\xrightarrow{L^2} \mathfrak{a},\\
\tilde{\boldsymbol{u}}_3^{\varepsilon_h} &\xrightharpoonup{dr-2} \mathfrak{b}(\hat{x}),\\
\mathring{\boldsymbol{u}}^{\varepsilon_h} &\xrightharpoonup{dr-2} \mathring{\boldsymbol{u}},\\
\operatorname{sym}\nabla_h \tilde{\boldsymbol{u}}^{\varepsilon_h}(x) &\xrightharpoonup{dr-2} \iota\left(\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x})\right) + C(x,y),\\
\varepsilon_h \operatorname{sym}\nabla_h \mathring{\boldsymbol{u}}^{\varepsilon_h} &\xrightharpoonup{dr-2} \operatorname{sym}\iota(\nabla_y \mathring{\boldsymbol{u}}).
\end{aligned}
\tag{2.29}
$$

3. *If, additionally to 2, one has:*

$$
\lim_{h\to 0}\left(a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2}\right) = a_{\infty,\infty}\left((\mathfrak{a},0)^\top + \mathring{\boldsymbol{u}}, (\mathfrak{a},0)^\top + \mathring{\boldsymbol{u}}\right) + \left\|(\mathfrak{a},\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\right\|^2_{L^2},
$$

where the form $a_{\infty,\infty}$ *is defined above, then we have strong two-scale convergence*

$$
\boldsymbol{u}^{\varepsilon_h} \xrightarrow{dr-2} (\mathfrak{a},\mathfrak{b})^\top + \mathring{\boldsymbol{u}}.
$$

The following theorem provides the limit resolvent equation.

Theorem 2.2.24. *Let* $\delta = \infty$, $\mu_h = \varepsilon_h, \tau = 0$ *and let the sequence of load densities satisfy the following convergence:*

$$
\boldsymbol{f}^{\varepsilon_h} \xrightharpoonup{dr-2} \boldsymbol{f} \in L^2(\Omega\times Y;\mathbb{R}^3). \tag{2.30}
$$

The sequence of solutions to the resolvent problem (2.6) *converges in the sense of* (2.29) *to the unique solution of the following problem: Find* $(\mathfrak{a},\mathfrak{b})^\top \in V_{1,\infty,\infty}(\omega\times Y)$, $\mathring{\boldsymbol{u}} \in V_{2,\infty}(\Omega\times$

Y_0) *such that*

$$\int_\omega \mathbb{C}^{\mathrm{memb,h}} \operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x}) : \operatorname{sym}\nabla_{\hat{x}}\boldsymbol{\theta}_*(\hat{x})\,d\hat{x} + \lambda \int_\omega \Big(\langle\rho\rangle\mathfrak{a}(\hat{x}) + \langle\rho_0\overline{\mathring{\boldsymbol{u}}_*}\rangle(\hat{x})\Big)\cdot\boldsymbol{\theta}_*(\hat{x})\,d\hat{x}$$

$$= \int_\omega \langle\overline{\boldsymbol{f}_*}\rangle(\hat{x})\cdot\boldsymbol{\theta}_*(\hat{x})\,d\hat{x} \qquad \forall\boldsymbol{\theta}_* \in H^1_{\gamma_D}(\omega;\mathbb{R}^2),$$

$$\langle\rho\rangle\mathfrak{b}(\hat{x}) + \big\langle\rho_0\overline{\mathring{u}}_3\big\rangle(\hat{x}) = \lambda^{-1}\langle\overline{f}_3\rangle(\hat{x}),$$

$$\int_{Y_0} \mathbb{C}_0(y)\operatorname{sym}\iota\big(\nabla_y\mathring{\boldsymbol{u}}(x,y)\big) : \operatorname{sym}\iota\big(\nabla_y\mathring{\boldsymbol{\xi}}(y)\big)\,dy$$

$$+\lambda\int_{Y_0}\rho_0(y)\big\{(\mathfrak{a}_1(\hat{x}),\mathfrak{a}_2(\hat{x}),\mathfrak{b}(\hat{x}))^\top + \mathring{\boldsymbol{u}}(x,y)\big\}\cdot\mathring{\boldsymbol{\xi}}(y)\,dy$$

$$= \int_{Y_0}\boldsymbol{f}(x,y)\cdot\mathring{\boldsymbol{\xi}}(y)\,dy \qquad \mathring{\boldsymbol{\xi}} \in H^1_0(Y_0;\mathbb{R}^3),\ \text{a.e. } x\in\Omega. \tag{2.31}$$

If we assume the strong two-scale convergence in (2.30)*, then the strong two-scale convergence*

$$\boldsymbol{u}^{\varepsilon_h}(x) \xrightarrow{dr-2} (\mathfrak{a},\mathfrak{b})^\top + \mathring{\boldsymbol{u}}(x,y)$$

holds.

Corollary 2.2.25. *Under Assumption 2.1.1 (1) and provided* $(\boldsymbol{f}^{\varepsilon_h})_{h>0} \subset L^{2,\mathrm{memb}}(\Omega;\mathbb{R}^3)$*, in addition to the convergences in Proposition* (2.2.23) *we have*

$$\tilde{u}_3^{\varepsilon_h} \xrightarrow{L^2} 0,$$

and thus $\mathfrak{b} = 0$ *in the limit equations* (2.31).

Notice that the variable x_3 is also just the parameter in the last equation in (2.31).

B. "Strong high-contrast bending" scaling $\mu_h = \varepsilon_h h$, $\tau = 2$

Here we define the following objects using Theorem 2.2.27 (the expression (2.34) for the limit resolvent):

- A bilinear form $a_\infty^{\mathrm{hom}} : (H^2_{\gamma_D}(\omega))^2 \to \mathbb{R}$ and the associated operator $\mathcal{A}_\infty^{\mathrm{hom}}$ on $L^2(\omega)$, analogous to a_δ^{hom} and $\mathcal{A}_\delta^{\mathrm{hom}}$ of Part A, Section 2.2.2.1 (notice that here the situation is simpler since necessarily $\mathfrak{a} = 0$);

- A scalar Zhikov function $\hat{\beta}_\infty$, analogous to $\hat{\beta}_\delta$ of Part C, Section 2.2.2.1, so similarly to (2.2.2.1) we have
$$\hat{\beta}_\infty = \beta_{\infty,33};$$
- A set $\hat{\sigma}(\mathcal{A}_{00,\infty})$, analogous to $\hat{\sigma}(\mathcal{A}_{00,\delta})$;
- The operator $\hat{\mathcal{A}}_\infty$ on $\{0\}^2 \times L^2(\omega) + L^2(\Omega \times Y_0;\mathbb{R}^3)$ defined via the bilinear form
$$\hat{a}_\infty\big((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}, (0,0,\theta)^\top + \mathring{\boldsymbol{\xi}}\big) = a_\infty^{\mathrm{hom}}(\mathfrak{b},\theta) + \int_\Omega \tilde{a}_{00,\infty}(\mathring{\boldsymbol{u}}, \mathring{\boldsymbol{\xi}}),$$
$$\hat{a}_\infty : \Big(\{0\}^2 \times H^2_{\gamma_\mathrm{D}}(\omega) + V_{2,\infty}(\Omega \times Y_0)\Big)^2 \to \mathbb{R}.$$

Similarly to the regimes discussed above, a suitable compactness result is proved.

Proposition 2.2.26. *Let $\delta = \infty$, $\mu_h = \varepsilon_h h$, $\tau = 2$. The following statements hold:*

1. *There exists $C > 0$, independent of h, such that for a sequence $(\boldsymbol{f}^{\varepsilon_h})_{h>0} \subset L^2(\Omega;\mathbb{R}^3)$ of load densities and solutions $\boldsymbol{u}^{\varepsilon_h}$ to the problem (2.6) one has*
$$h^{-2} a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2} \le C \|\boldsymbol{f}^{\varepsilon_h}\|^2_{L^2}.$$

2. *If*
$$\limsup_{h\to 0} \Big(h^{-2} a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2}\Big) < \infty, \quad (\boldsymbol{u}^{\varepsilon_h})_{h>0} \subset H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3),$$
then there exist functions $\mathfrak{a} \in H^1_{\gamma_\mathrm{D}}(\omega;\mathbb{R}^2)$, $\mathfrak{b} \in H^2_{\gamma_\mathrm{D}}(\omega)$, $\mathring{\boldsymbol{u}} \in V_{2,\infty}(\Omega \times Y_0)$, $C \in C_\infty(\Omega \times Y)$ such that (up to subsequence)
$$\begin{aligned} &\boldsymbol{u}^{\varepsilon_h} = \tilde{\boldsymbol{u}}^{\varepsilon_h} + \mathring{\boldsymbol{u}}^{\varepsilon_h}, \quad \tilde{\boldsymbol{u}}^{\varepsilon_h}, \mathring{\boldsymbol{u}}^{\varepsilon_h} \in H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3), \quad \mathring{\boldsymbol{u}}^{\varepsilon_h}|_{\Omega_1^{\varepsilon_h}} = 0, \\ &\pi_{1/h}\tilde{\boldsymbol{u}}^{\varepsilon_h} \xrightarrow{L^2} \big(\mathfrak{a}_1(\hat{x}) - x_3\partial_1\mathfrak{b}(\hat{x}), \mathfrak{a}_2(\hat{x}) - x_3\partial_2\mathfrak{b}(\hat{x}), \mathfrak{b}(\hat{x})\big)^\top, \\ &\mathring{\boldsymbol{u}}^{\varepsilon_h}(x) \xrightarrow{dr-2} \mathring{\boldsymbol{u}}(x,y), \\ &h^{-1}\,\mathrm{sym}\,\nabla_h \tilde{\boldsymbol{u}}^{\varepsilon_h}(x) \xrightarrow{dr-2} \iota\Big(\mathrm{sym}\,\nabla_{\hat{x}}\mathfrak{a}(\hat{x}) - x_3 \nabla^2_{\hat{x}}\mathfrak{b}(\hat{x})\Big) + C(x,y), \\ &\varepsilon_h\,\mathrm{sym}\,\nabla_h \mathring{\boldsymbol{u}}^{\varepsilon_h}(x) \xrightarrow{dr-2} \mathrm{sym}\,\iota\big(\nabla_y \mathring{\boldsymbol{u}}(x,y)\big). \end{aligned} \tag{2.32}$$

3. *If, additionally to 2, one has*
$$\lim_{h\to 0}\Big(h^{-2} a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2}\Big) = \hat{a}_\infty((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}, (0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}) + \|(0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\|^2_{L^2},$$

where $\hat{\mathfrak{a}}_\infty$ is defined in (2.2.2.3), then the strong two-scale convergence

$$\boldsymbol{u}^{\varepsilon_h} \xrightarrow{dr-2} (0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}$$

holds.

The following theorem provides the limit resolvent equation.

Theorem 2.2.27. *Let $\delta = \infty$, $\mu_h = \varepsilon_h h$, $\tau = 2$ and let the sequence of load densities satisfy the following convergence:*

$$\boldsymbol{f}^{\varepsilon_h} \xrightarrow{dr-2} \boldsymbol{f} \in L^2(\Omega \times Y; \mathbb{R}^3). \tag{2.33}$$

Then the sequence of solutions to the resolvent problem (2.6) *converges in the sense of* (2.32) *to the unique solution of the following problem: Determine $\mathfrak{a} \in H^1_{\gamma_D}(\omega;\mathbb{R}^2)$, $\mathfrak{b} \in H^2_{\gamma_D}(\omega)$, $\mathring{\boldsymbol{u}} \in V_{2,\infty}(\Omega \times Y_0)$ such that*

$$\begin{aligned}
&\mathfrak{a} = 0,\\
&\frac{1}{12}\int_\omega \mathbb{C}^{\mathrm{bend,h}} \nabla^2_{\hat{x}} \mathfrak{b}(\hat{x}) : \nabla^2_{\hat{x}} \theta_3(\hat{x})\, d\hat{x} + \lambda \int_\omega \big(\rho_0(y)\mathfrak{b}(\hat{x}) + \langle \rho_0 \overline{\mathring{\boldsymbol{u}}}_3\rangle(\hat{x})\big)\, \theta_3(\hat{x})\, d\hat{x}\\
&\qquad\qquad = \int_\omega \langle \overline{f}_3\rangle(\hat{x})\, \theta_3(\hat{x})\, d\hat{x} \qquad \forall \theta_3 \in H^2_{\gamma_D}(\omega),\\
&\int_{Y_0} \mathbb{C}_0(y)\, \mathrm{sym}\,\iota\big(\nabla_y \mathring{\boldsymbol{u}}(x,y)\big) : \mathrm{sym}\,\iota\big(\nabla_y \mathring{\boldsymbol{\xi}}(y)\big)\, dy + \lambda \int_{Y_0} \rho_0(y)\mathfrak{b}(\hat{x})\mathring{\xi}_3(y)\, dy + \lambda \int_{Y_0} \rho_0(y)\mathring{\boldsymbol{u}}(x,y)\cdot \mathring{\boldsymbol{\xi}}(y)\, dy\\
&\qquad\qquad = \int_{Y_0} \boldsymbol{f}(x,y)\cdot \mathring{\boldsymbol{\xi}}(y)\, dy \qquad \forall \mathring{\boldsymbol{\xi}} \in H^1_0(Y_0;\mathbb{R}^3),\ a.e.\ x \in \Omega.
\end{aligned} \tag{2.34}$$

If strong two-scale convergence takes place in (2.33)*, then one additionally has*

$$\boldsymbol{u}^{\varepsilon_h}(x) \xrightarrow{dr-2} (0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}(x,y).$$

Remark 2.2.28. The limit resolvent equations exhibit several differences between the regimes discussed: beside different effective tensors (this also happens in the moderate-contrast setting, see, e.g., [49] in the case of nonlinear von Kármán plate theory), one has different kinds of behaviour on the inclusions: in the regime $h \sim \varepsilon_h$ the inclusions behave like three-dimensional objects, while for $\delta = 0$ they can be seen as small plates.

Furthermore, different scalings of load densities are required in different regimes, which does not happen in the case moderate contrast. Finally, the case $\delta = 0$ has an additional effective parameter κ; when $\kappa \in (0, \infty)$ the elastic energy only resists to oscillatory (spatial) motion (i.e. oscillations on the level of periodicity cells) in the out-of-plane direction.

2.2.3. Limit spectrum

In this section we will use the above resolvent convergence results to infer convergence of spectra of the operators $\mathcal{A}_{\varepsilon_h}$. As we shall see below in the proofs of the spectral convergence, one does not need to apply different scalings to different components of external loads, and thus only simplified versions of the limit resolvent equations will be necessary. Also, the presence of a spectrum of order h^2 implies that any other scaling will cause the limit set to be the whole positive real line (see [14]). Thus, for the case when $\mu_h = \varepsilon_h$ in (2.1.1), in order for the limit spectrum to have a "band-gap" structure we are forced to restrict ourselves to the "membrane" subspace $L^{2,\mathrm{memb}}$, which is possible under Assumption 2.1.1 (1) concerning material symmetries. Otherwise, for the same case, the limit resolvent captures only the order-one part of the limit spectrum. This is consistent with the standard result that the strong resolvent convergence only implies that the spectrum of the limit operator is contained in limit spectrum for $\mathcal{A}_{\varepsilon_h}$, while an additional compactness argument is necessary for the opposite inclusion (see, e.g, [71]). In our setting, compactness of eigenfunctions is lost when passing from the spectrum of order h^2 (or order-one spectrum for the restriction to $L^{2,\mathrm{memb}}$) to the order-one spectrum for the full operator, as the transversal component of an eigenfunction would converge only weakly two-scale.

Under Assumption 2.1.1, for the membrane scalings of Part B of Section 2.2.2.1 and Parts A of Sections 2.2.2.2, 2.2.2.3, the resolvent equation can be restricted to the invariant subspace $L^{2,\mathrm{memb}}$, where the solutions happen to be compact in the strong topology, see Corollaries 2.2.11, 2.2.20, 2.2.25. This compactness property enables one to prove the convergence of spectra of order one for this restriction. Notice that in the regime $h \ll \varepsilon_h$ there are different types of limit resolvents (distinguished by different values of the parameter κ) when Assumption 2.1.1 is not satisfied. In this regime, the convergence of the third component of the displacements is only weak two-scale, which is the reason why we do not invoke different resolvent limits in the analysis of the convergence of spectra in the mentioned regime. However, we will use this information in our study of the limit evolution equations.

Remark 2.2.29. For the case of spectra of order h^2, in order to be able to obtain limit spectra with band gaps, one needs to consider different scalings of the coefficients on high-contrast inclusions. This motivated us for the analysis of this situation in Part C of 2.2.2.1 and Parts B of Sections 2.2.2.2, 2.2.2.3.

2.2.3.1 Preliminaries on spectral convergence

The Lax-Milgram theorem (see [51]) implies that for each $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^3)$ the equation

$$\mathcal{A}_{\varepsilon_h}\boldsymbol{u} = \boldsymbol{f}$$

has a unique solution $\boldsymbol{u} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)$ understood in the weak sense. The operator

$$\mathcal{T}_{\varepsilon_h} : L^2(\Omega;\mathbb{R}^3) \to H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3), \quad \mathcal{T}_{\varepsilon_h}\boldsymbol{f} := \boldsymbol{u},$$

is compact due to the compact embedding $H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3) \hookrightarrow L^2(\Omega;\mathbb{R}^3)$ (this compactness will be lost in the limit problem, except for the first case analysed). Therefore, $\mathcal{T}_{\varepsilon_h}$ has countably many eigenvalues forming a non-increasing sequence of positive numbers converging to zero, the only remaining element of the spectrum $\mathcal{T}_{\varepsilon_h}$. Therefore, the spectrum of $\mathcal{A}_{\varepsilon_h}$ consists of eigenvalues ordered in a non-decreasing positive sequence $\lambda_n^{\varepsilon_h}$ that tends to infinity.

In what follows, we are interested in understanding the relationship between the spectra of $\mathcal{A}_{\varepsilon_h}$ as $h \to 0$ and eigenvalues of the limit operators discussed in Section 2.2.2. To this end, the following standard notion of convergence will be referred to throughout.

Definition 2.2.1. We say that a sequence of sets S_h (e.g. $S_h = \sigma(\mathcal{A}_{\varepsilon_h})$) converges in the Hausdorff sense to the set S if:

- (H_1) For any $\lambda \in S$, there exists a sequence of $\lambda^h \in S_h$ convergent to λ (as $h \to 0$.)
- (H_2) The limit of any convergent sequence of $\lambda^h \in S_h$ is an element of S.

For various scalings of Section 2.2.2, we will discuss the convergence in the Hausdorff sense of $\sigma(\mathcal{A}_{\varepsilon_h})$ to the spectrum of the corresponding limit operator.

The first property of Hausdorff convergence of spectra is normally a direct consequence of the strong resolvent convergence, while the second property requires the compactness of the sequence of eigenfunctions in an appropriate topology.

2.2.3.2 Asymptotic regime $\delta \in (0,\infty)$, scaling $\tau = 2$

In this section we will analyse the limit spectrum of order h^2 for ε_h^2-scaling of the coefficients in the inclusions. We will show that the high-contrast has no effect on the limit, in that the (scaled) limit spectrum is of the same type as for the ordinary plate (i.e. homogeneous or with moderate contrast), in particular the limit operator has compact resolvent. This is precisely the reason why we analyse this combination of scalings of the spectrum and the coefficients only for the asymptotic regime $h \sim \varepsilon_h$ (i.e. $\delta \in (0,\infty)$).

On the one hand we would like to show that in the case of an ordinary plate the resolvent approach can also provide information about the convergence of spectra (alternatively to, say, using Rayleigh quotients), and on the other hand we aim at demonstrating that in the mentioned case the limit problem does not exhibit spectral gaps and thus a different scaling of the coefficients is required for them to appear.

The following theorem provides the relevant result concerning spectral convergence.

Theorem 2.2.30. *Let* $\lim_{h\to 0} h/\varepsilon_h = \delta \in (0,\infty)$, $\mu_h = \varepsilon_h$. *The spectra* $\sigma\big(h^{-2}\mathcal{A}_{\varepsilon_h}\big)$ *converge in the Hausdorff sense to the spectrum of* $\mathcal{A}_\delta^{\mathrm{b,hom}}$, *as* $h \to 0$, *which is an increasing sequence of positive eigenvalues* $(\lambda_{\delta,n})_{n\in\mathbb{N}}$ *that tend to infinity, each of finite multiplicity. More precisely, if by* $\lambda_n^{\varepsilon_h}$ *we denote the n-th eigenvalue of* $\mathcal{A}_{\varepsilon_h}$ *(by repeating each eigenvalue according to its multiplicity), then*

$$h^{-2}\lambda_n^{\varepsilon_h} \to \lambda_{\delta,n}, \quad h \to 0,$$

where $\lambda_{\delta,n}$ *is n-th eigenvalue of* $\mathcal{A}_\delta^{\mathrm{b,hom}}$ *(again repeated in accordance with multiplicity). Furthermore, for any fixed n and any choice of normalised eigenfunctions with eigenvalues* $\lambda_n^{\varepsilon_h}$, *there is a (h-indexed) subsequence such that the corresponding eigenfunctions converge, as* $h \to 0$, *to an eigenfunction with the eigenvalue* $\lambda_{\delta,n}$ *of the limit problem.*

2.2.3.3 Asymptotic regime $\delta \in [0,\infty)$, scaling $\mu_h = \varepsilon_h$, $\tau = 0$

In this section we analyse the operator $\mathcal{A}_{\varepsilon_h}$ in the space $L^{2,\mathrm{memb}}(\Omega;\mathbb{R}^3)$. In the regime $\delta = 0$ we require that the component Y_0 has $C^{1,1}$ boundary. We define the following generalized eigenvalue problem: Find $\lambda > 0$ and $\mathfrak{a} \in H^1_{\gamma_{\mathrm{D}}}(\omega,\mathbb{R}^2)$ such that

$$\int_\omega \mathbb{C}^{\mathrm{memb}}_\delta \operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x}) : \operatorname{sym}\nabla_{\hat{x}}\boldsymbol{\varphi}(\hat{x})d\hat{x} = \int_\omega \tilde{\beta}^{\mathrm{memb}}_\delta(\lambda)\mathfrak{a}(\hat{x})\cdot\boldsymbol{\varphi}(\hat{x})d\hat{x}, \quad \forall\boldsymbol{\varphi} \in H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2). \tag{2.35}$$

(In the case $\delta = 0$ we put $\mathbb{C}^{\mathrm{memb,r}}_1$ instead of $\mathbb{C}^{\mathrm{memb}}_\delta$, in the case when $\delta = \infty$ we put $\mathbb{C}^{\mathrm{memb},h}$ instead of $\mathbb{C}^{\mathrm{memb}}_\delta$, see Section 2.2.1 for the relevant definitions.) The following theorem contains the spectral convergence result for the regime considered here.

Theorem 2.2.31. *Suppose* $\lim_{h\to 0} h/\varepsilon_h = \delta \in [0,\infty)$*,* $\mu_h = \varepsilon_h$ *and let Assumption 2.1.1 (1) be valid. The set of all* $\lambda > 0$ *for which the problem* (2.35) *has a non-trivial solution* $\mathfrak{a} \in H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)$ *is at most countable. The spectra of the operators* $\tilde{\mathcal{A}}_{\varepsilon_h}$ *converge in the Hausdorff sense to the spectrum of* $\tilde{\mathcal{A}}_\delta$*, and one has*

$$\sigma(\tilde{\mathcal{A}}_\delta) = \sigma(\tilde{\mathcal{A}}_{00,\delta})' \cup \overline{\{\lambda > 0 : \text{ The generalized eigenvalue problem (2.35) is solvable.}\}}. \tag{2.36}$$

Additionally, under Assumption 2.1.1 (2,3), the matrix $\tilde{\beta}^{\mathrm{memb}}_\delta(\lambda)$ *is scalar and*

$$\sigma(\tilde{\mathcal{A}}_\delta) = \sigma(\tilde{\mathcal{A}}_{00,\delta})' \cup \overline{\left\{\lambda > 0 : \tilde{\beta}^{\mathrm{memb}}_{\delta,11}(\lambda) \in \sigma(\mathcal{A}^{\mathrm{memb}}_\delta)\right\}}.$$

Remark 2.2.32. It was shown in [31] that each non-empty interval of the form $(\tilde{\omega}_n,\tilde{\omega}_{n+1})$, $n\in\mathbb{N}$, contains a subinterval $(\tilde{\omega}_n,\alpha)$, $\tilde{\omega}_n \le \alpha < \tilde{\omega}_{n+1}$ in which both eigenvalues of the matrix $\tilde{\beta}^{\mathrm{memb}}_\delta$ are negative, a subinterval (α,β), $\alpha < \beta \le \tilde{\omega}_{n+1}$ in which one of its eigenvalues is negative while the other is positive, and the interval $(\beta,\tilde{\omega}_{n+1})$ in which both its eigenvalues are positive. It follows, as is explained in [31], that in the interval $(\tilde{\omega}_n,\alpha)$ there is no wave propagation in any direction, while in the interval (α,β) one has evanescent solutions in the direction of the negative eigenvectors, and finally in the intervals $(\beta,\tilde{\omega}_{n+1})$ all directions allow wave propagation.

Under Assumption 2.1.1 (1–3), the above spectral structure can be quantified in a straightforward way and $\tilde{\omega}_n < \alpha = \beta < \tilde{\omega}_{n+1}$, see [70,71]. In this case the matrix $\tilde{\beta}^{\mathrm{memb}}_\delta$ is scalar and

$$\lim_{\lambda\to\tilde{\omega}_n^+} \tilde{\beta}_{\delta,11}^{\mathrm{memb}}(\lambda) = -\infty, \qquad \lim_{\lambda\to\tilde{\omega}_{n+1}^-} \tilde{\beta}_{\delta,11}^{\mathrm{memb}}(\lambda) = +\infty,$$

where $\tilde{\beta}_{\delta,11}^{\mathrm{memb}}$ is one of the two equal diagonal elements, and the limits are taken as λ approaches $\tilde{\omega}_n$ on the right and on the left, respectively.

The above properties of the limit spectrum are relevant in a variety of applications, such as noise suppression. Being peculiar to wave propagation in high-contrast media, they are often referred to as "high-contrast effects".

2.2.3.4 Asymptotic regime $\delta \in (0,\infty)$, scaling $\mu_h = \varepsilon_h h$, $\tau = 2$ and asymptotic regime $\delta = 0$, scaling $\mu_h = \varepsilon_h^2$, $\tau = 2$

For the regimes considered here, we show that high-contrast effects occur in the limit as $h \to 0$. As before, when $\delta = 0$ we assume that Y_0 has $C^{1,1}$ boundary. We have the following theorem.

Theorem 2.2.33. *Let* $\lim_{h\to 0} h/\varepsilon_h = \delta \in [0,\infty)$. *In the cases* $\delta = 0$, $\delta > 0$ *we assume that* $\mu_h = \varepsilon_h^2$ *and* $\mu_h = \varepsilon_h h$, *respectively. The spectrum of the operator* $h^{-2}\mathcal{A}_{\varepsilon_h}$ *converges in the Hausdorff sense to the spectrum of the operator* $\hat{\mathcal{A}}_\delta$, *given by*

$$\sigma(\hat{\mathcal{A}}_\delta) = \begin{cases} \hat{\sigma}(\mathcal{A}_{00,\delta}) \cup \overline{\left\{\lambda > 0 : \hat{\beta}_\delta(\lambda) \in \sigma(\mathcal{A}_\delta^{\mathrm{b,hom}})\right\}}, & \delta \in (0,\infty), \\ \hat{\sigma}(\hat{\mathcal{A}}_{00,0}) \cup \overline{\left\{\lambda > 0 : \hat{\beta}_0(\lambda) \in \sigma(\mathcal{A}_0^{\mathrm{hom}})\right\}}, & \delta = 0. \end{cases}$$

Remark 2.2.34. The operator $\mathcal{A}_\delta^{\mathrm{b,hom}}$ is non-local when Assumption 2.1.1 is not satisfied. It is not known to us whether this has been commented on in the existing literature, even in the case of a homogeneous plate.

2.2.3.5 Asymptotic regime $\delta = \infty$

As we see below, in the case $\delta = \infty$, the limit spectrum has points outside spectrum of the limit operator. From the intuitive point of view, the effective behaviour is similar to that of a cuboid with disjoint soft inclusions in the shape of long thin rods arranged parallel to each other and connecting two opposite sides of the body.

In order to formulate the result of this section, we define:

- An operator as the one defined via the bilinear form

$$\mathring{a}_{\mathrm{strip}} = \int_{\mathbb{R}\times Y_0} \mathbb{C}_0(y)\,\mathrm{sym}\nabla\boldsymbol{u} : \mathrm{sym}\nabla\boldsymbol{v}\,dx_3dy, \qquad \mathring{a}_{\mathrm{strip}} : \left(H^1_{00}(\mathbb{R}\times Y_0;\mathbb{R}^3)\right)^2 \to \mathbb{R};$$

- An operator $\mathring{\mathcal{A}}^+_{\mathrm{strip}}$ on $L^2(\mathbb{R}^+_0 \times Y_0;\mathbb{R}^3)$ as the one defined via the form

$$\mathring{a}^+_{\mathrm{strip}} = \int_{\mathbb{R}^+_0\times Y_0} \mathbb{C}_0(y)\,\mathrm{sym}\nabla\boldsymbol{u} : \mathrm{sym}\nabla\boldsymbol{v}\,dx_3dy, \qquad \mathring{a}^+_{\mathrm{strip}} : \left(H^1_{00}(\mathbb{R}^+_0\times Y_0;\mathbb{R}^3)\right)^2 \to \mathbb{R};$$

- An operator $\mathring{\mathcal{A}}^-_{\mathrm{strip}}$ on $L^2(\mathbb{R}^-_0 \times Y_0;\mathbb{R}^3)$ as the one defined via the form

$$\mathring{a}^-_{\mathrm{strip}} = \int_{\mathbb{R}^-_0\times Y_0} \mathbb{C}_0(y)\,\mathrm{sym}\nabla\boldsymbol{u} : \mathrm{sym}\nabla\boldsymbol{v}\,dx_3dy, \qquad \mathring{a}^-_{\mathrm{strip}} : \left(H^1_{00}(\mathbb{R}^-_0\times Y_0;\mathbb{R}^3)\right)^2 \to \mathbb{R}.$$

- The restriction $\mathring{\mathring{\mathcal{A}}}_{\mathrm{strip}}$ of the operator $\mathring{\mathcal{A}}_{\mathrm{strip}}$ to the membrane subspace $L^{2,\mathrm{memb}}(\mathbb{R}\times Y_0;\mathbb{R}^3)$, whenever Assumption 2.1.1 (1) holds.

First, we give characterisations of the limit spectra of $\mathring{\mathcal{A}}_{\varepsilon_h}$ and $\mathring{\mathring{\mathcal{A}}}_{\varepsilon_h}$, which in this regime play significant roles.

Theorem 2.2.35. *Suppose that $\varepsilon_h \ll h$. Then one has*

$$\lim_{h\to 0}\sigma(\mathring{\mathcal{A}}_{\varepsilon_h}) = \lim_{h\to 0}\sigma(\mathring{\mathring{\mathcal{A}}}_{\varepsilon_h}) = \sigma(\mathring{\mathcal{A}}_{\mathrm{strip}})\cup\sigma(\mathring{\mathcal{A}}^+_{\mathrm{strip}})\cup\sigma(\mathring{\mathcal{A}}^-_{\mathrm{strip}}). \tag{2.37}$$

Moreover, one has

$$\sigma_{\mathrm{ess}}(\mathring{\mathcal{A}}^\pm_{\mathrm{strip}}) = \sigma(\mathring{\mathcal{A}}_{\mathrm{strip}}), \tag{2.38}$$

and there exists $m_0 > 0$ such that

$$\sigma(\mathring{\mathcal{A}}_{\mathrm{strip}}) = \sigma_{\mathrm{ess}}(\mathring{\mathcal{A}}_{\mathrm{strip}}) = [m_0, +\infty).$$

Under Assumption 2.1.1 (1), one additionally has

$$\sigma(\mathring{\mathcal{A}}^+_{\mathrm{strip}}) = \sigma(\mathring{\mathcal{A}}^-_{\mathrm{strip}}) \supset \sigma(\mathring{\mathring{\mathcal{A}}}_{\mathrm{strip}}) = \sigma_{\mathrm{ess}}(\mathring{\mathring{\mathcal{A}}}_{\mathrm{strip}}) = \sigma\left(\mathring{\mathcal{A}}_{\mathrm{strip}}\right). \tag{2.39}$$

Next we provide a characterisation of the limit spectrum for $\mathcal{A}_{\varepsilon_h}$.

Theorem 2.2.36. *Let $\varepsilon_h \ll h$, $\mu_h = \varepsilon_h$ and $\tau = 0$. The set of all $\lambda > 0$ for which the problem* (2.35) *obtains a nontrivial solution $\mathfrak{a} \in H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)$ is at most countable. The spectra of $\tilde{\mathcal{A}}_{\varepsilon_h}$ converge in the Hausdorff sense to $\sigma(\mathring{\mathcal{A}}^+_{\mathrm{strip}})\cup\sigma(\tilde{\mathcal{A}}_\infty)$, where*

$$\sigma(\tilde{\mathcal{A}}_\infty) = \sigma(\tilde{\mathcal{A}}_{00,\infty})' \cup \overline{\{\lambda > 0 : \textit{The generalized eigenvalue problem (2.35) is solvable.}\}}. \tag{2.40}$$

Under Assumption 2.1.1 (2, 3), the matrix $\tilde{\beta}^{\rm memb}_{\infty}(\lambda)$ *is scalar and*

$$\sigma(\tilde{\mathcal{A}}_{\infty}) = \sigma(\tilde{\mathcal{A}}_{00,\infty})' \cup \overline{\left\{\lambda > 0 : \tilde{\beta}^{\rm memb}_{\infty,11}(\lambda) \in \sigma(\mathcal{A}^{\rm memb}_{\delta})\right\}}. \tag{2.41}$$

Furthermore, one has $\sigma(\tilde{\mathcal{A}}_{00,\infty}) \subset \sigma(\mathring{\mathcal{A}}^{+}_{\rm strip})$.

Theorem 2.2.37. *Suppose that* $\varepsilon_h \ll h$, $\mu_h = \varepsilon_h h$, $\tau = 2$. *The spectra of* $h^{-2}\mathcal{A}_{\varepsilon_h}$ *converge in the Hausdorff sense to* $\sigma(\mathring{\mathcal{A}}^{+}_{\rm strip}) \cup \sigma(\mathring{\mathcal{A}}^{-}_{\rm strip}) \cup \sigma(\hat{\mathcal{A}}_{\infty})$ *and*

$$\sigma(\hat{\mathcal{A}}_{\infty}) = \hat{\sigma}(\mathcal{A}_{00,\infty}) \cup \overline{\left\{\lambda > 0 : \hat{\beta}_{\infty}(\lambda) \in \sigma(\mathcal{A}^{\rm hom}_{\infty})\right\}}.$$

Remark 2.2.38. As is shown in Lemma 2.3.2, the set $\lim_{h\to 0}\sigma(\mathring{\mathcal{A}}_{\varepsilon_h})$ (appropriately scaled) is always a subset of the limit spectrum. In the regime $\delta = \infty$, the operator has a scaling factor ε_h/h in front of the derivative in x_3. This allows eigenfunctions to oscillate in the out-of-plane direction (and thus weakly converge to zero). This is the reason for so-called "spectral pollution" (see e.g. [2]).

2.2.4. Limit evolution equations

It is expected from the results of Section 2.2.2 concerning the resolvent convergence for the operators $\mathcal{A}_{\varepsilon_h}$ that the limit evolution equations have the form of a system that links the behaviour on the stiff matrix and the soft inclusions by means of coupled solution components, which can be viewed as macroscopic and microscopic variables. Representing the system in terms of the macroscopic component only leads to a non-trivial effective description exhibiting memory effects. This is one of the reasons what makes high-contrast materials interesting in applications.

The present section aims at providing a detailed study of the consequences of the form of the limit resolvent equations obtained for different asymptotic regimes in Section 2.2.2 on the limit evolution equations in the corresponding regimes. On the abstract level, this connection has been analysed in [53]. A key fact used in that paper is that the resolvent is the Laplace transform of the exponential function of the operator of the wave equation, obtained from an equivalent system of equations of first order in time. In what follows, we adjust our analysis to these general results, in order to account for the particular features of our setup due to dimension reduction in linear elasticity. As we see below, in this context different scalings of spectra imply different scalings of time (i.e. bending waves propagate on a slower time scale than in-plane, “membrane”, waves). As far as we know, the effect of considering different time scalings has not been addressed in the literature; see [54] for the analysis of limit evolution of isotropic homogeneous plates for the commonly considered “long” time scaling of order h^{-1}.

It should also be noted that some load density scalings prevent us from using the results of [53], in which case separate analysis is necessary to show weak convergence of solutions (see e.g. the proof of Theorem 2.2.39). This happens for the case $\tau = 2$ (i.e. for long times of order h^{-1}) in the regimes $\delta \in (0,\infty)$, $\mu_h = \varepsilon_h$ and $\delta = 0$, $\mu_h = \varepsilon_h^2$. For these, to prove weak convergence we use the Laplace transform directly, following the same overall strategy as the one adopted in [53] in the abstract setting, see Apprendix. However, due to the said load density scalings, a modification of the results of [53] is required, in order to account for the specific structure of the right-hand side of the limit problem; this is also discussed in Appendix, see in particular Theorems 4.5.4, 4.5.15.

The starting point of this section is the family of Cauchy problems ($h > 0$)

$$\begin{cases} \partial_{tt}\boldsymbol{u}^{\varepsilon_h}(t) + h^{-\tau}\mathcal{A}_{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h}(t) = \boldsymbol{f}^{\varepsilon_h}(t), \\ \boldsymbol{u}^{\varepsilon_h}(0) = \boldsymbol{u}_0^{\varepsilon_h}, \quad \partial_t\boldsymbol{u}^{\varepsilon_h}(0) = \boldsymbol{u}_1^{\varepsilon_h}, \end{cases} \tag{2.42}$$

understood in the weak sense. The term $\boldsymbol{f}^{\varepsilon_h}(t)$ represents the load density at time $t > 0$. For each h, we suppose that $\boldsymbol{f}^{\varepsilon_h}$ is provided on the time interval $[0, T_h]$, $T_h > 0$. The functions $\boldsymbol{u}_0^{\varepsilon_h}, \boldsymbol{u}_1^{\varepsilon_h}$ are the initial data for the displacement and velocity fields, respectively. We make the following assumptions:

$$\boldsymbol{u}_0^{\varepsilon_h} \in \mathcal{D}(\mathcal{A}_{\varepsilon_h}^{1/2}) = H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3), \quad \boldsymbol{u}_1^{\varepsilon_h} \in L^2(\Omega;\mathbb{R}^3), \quad \boldsymbol{f}^{\varepsilon_h} \in L^2(0,T;L^2(\Omega;\mathbb{R}^3)),$$

In what follows, we shall analyse the “critical” cases $\tau = 2$ and $\tau = 0$ for the time scaling. Conditions for well-posedness of the problem (2.42) can be found in Appendix, see Section 4.5.

In conclusion of this section, we reiterate that there are two ways to interpret the scaling $h^{-\tau}$ of the differential expression in (2.42): by scaling the density of the material (with h^τ) or by introducing the new time scale $\tilde{t} = t/h^{\tau/2}$. We adopt the latter interpretation throughout. Multiplying (2.42) by h^τ and replacing t by $\tilde{t}$, we obtain the family of problems ($h>0$)

$$\begin{cases} \partial_{\tilde{t}\tilde{t}} \boldsymbol{u}^{\varepsilon_h}(\tilde{t}) + \mathcal{A}_{\varepsilon_h} \boldsymbol{u}^{\varepsilon_h}(\tilde{t}) = \tilde{\boldsymbol{f}}^{\varepsilon_h}(\tilde{t}), \\ \boldsymbol{u}^{\varepsilon_h}(0) = \boldsymbol{u}_0^{\varepsilon_h}, \quad \partial_{\tilde{t}} \boldsymbol{u}^{\varepsilon_h}(0) = \tilde{\boldsymbol{u}}_1^{\varepsilon_h}, \end{cases} \tag{2.43}$$

where $\tilde{\boldsymbol{f}}^{\varepsilon_h}(\tilde{t}) := h^\tau \boldsymbol{f}^{\varepsilon_h}(h^\tau \tilde{t})$, $\tilde{\boldsymbol{u}}_1^{\varepsilon_h} := h^{\tau/2} \boldsymbol{u}_1^{\varepsilon_h}$. Thus discussing the solution of (2.42) on a time interval $[0,T]$ (with an appropriate scaling of the load density) is equivalent to discussing the solution of (2.43) on the time interval $[0, T/h^{\tau/2}]$ (with the corresponding scaling of the loads). While from now on we shall work in the framework of the equation (2.42), which is convenient from the mathematical point of view, it is the equation (2.43) that represents the actual physical wave motion, which thereby takes place on an appropriate time scale of order $h^{-\tau/2}$.

2.2.4.1 Long-time behaviour for the regime $\delta \in (0,\infty)$, $\mu_h = \varepsilon_h$, $\tau = 2$

The case analysed here resembles the standard (moderate-contrast) plate model. The following convergence statement holds for the evolution problem.

Theorem 2.2.39. *Suppose that $\delta \in (0,\infty)$, $\mu_h = \varepsilon_h$, $\tau = 2$. Let $(\boldsymbol{u}^{\varepsilon_h})_{h>0}$ be a sequence of solutions to (2.42) and assume that*

$$\left(h \partial_t \boldsymbol{f}_\alpha^{\varepsilon_h}\right)_{h>0} \subset L^2([0,T]; L^2(\Omega \times Y)) \ \textit{bounded}, \ \ \alpha = 1,2, \tag{2.44}$$

$$\pi_h \boldsymbol{f}^{\varepsilon_h} \xrightarrow{\text{t,dr}-2} \boldsymbol{f} \in L^2([0,T]; L^2(\Omega \times Y; \mathbb{R}^3)), \tag{2.45}$$

$$\boldsymbol{u}_0^{\varepsilon_h} \xrightarrow{dr-2} \boldsymbol{u}_0(\hat{x}) \in \{0\}^2 \times H^2_{\gamma_{\rm D}}(\omega), \tag{2.46}$$

$$\boldsymbol{u}_1^{\varepsilon_h} \xrightarrow{dr-2} \boldsymbol{u}_1(x,y) \in L^2(\Omega \times Y; \mathbb{R}^3), \tag{2.47}$$

and assume additionally that

$$\limsup_{h\to 0} \left(h^{-2} a_{\varepsilon_h}(\boldsymbol{u}_0^{\varepsilon_h}, \boldsymbol{u}_0^{\varepsilon_h}) + \|\boldsymbol{u}_0^{\varepsilon_h}\|^2_{L^2} \right) < \infty.$$

Then one has

$$\pi_{1/h} \boldsymbol{u}^{\varepsilon_h} \xrightarrow{\text{t,dr}-2} \begin{pmatrix} \mathfrak{a}_1(t,\hat{x}) - x_3 \partial_1 \mathfrak{b}(t,\hat{x}) + \mathring{u}_1(t,x,y) \\ \mathfrak{a}_2(t,\hat{x}) - x_3 \partial_2 \mathfrak{b}(t,\hat{x}) + \mathring{u}_2(t,x,y) \\ \mathfrak{b}(t,\hat{x}) \end{pmatrix},$$

$$\partial_t \boldsymbol{u}^{\varepsilon_h} \xrightarrow{\text{t,dr}-2} \left(0,0,\partial_t \mathfrak{b}(t,\hat{x})\right)^\top,$$

where $\mathfrak{a} \in C([0,T]; H^1_{\gamma_D}(\omega;\mathbb{R}^2))$, $\mathfrak{b} \in C([0,T]; H^2_{\gamma_D}(\omega))$, $\mathring{\boldsymbol{u}} \in C([0,T]; L^2(\omega; H^1_{00}(I \times Y_0;\mathbb{R}^3)))$ *are determined uniquely by solving the problem*

$$\partial_{tt}\mathfrak{b}(t) + \mathcal{A}^{\mathfrak{b},\mathrm{hom}}_{\delta}\mathfrak{b}(t) = \mathcal{F}_{\delta}\big(\boldsymbol{f}(t)\big), \quad \textit{(see (2.10))} \tag{2.48}$$

$$\mathfrak{b}(0) = u_{0,3} \in H^2_{\gamma_D}(\omega), \qquad \partial_t\mathfrak{b}(0) = S_1(P_{\delta,\infty}\boldsymbol{u}_1)_3 \in L^2(\omega),$$

$$\mathfrak{a}(t) = \mathfrak{a}^{\mathfrak{b}(t)} + \mathfrak{a}^{\boldsymbol{f}_*(t)}, \quad \textit{(see (2.12))} \tag{2.49}$$

$$\mathcal{A}_{00,\delta}\mathring{\boldsymbol{u}}(t,\hat{x},\cdot) = (\boldsymbol{f}_*(t,\hat{x},\cdot),0)^{\top}, \tag{2.50}$$

so that $\partial_t\mathfrak{b} \in C([0,T]; L^2(\omega))$. *One also has*

$$\limsup_{h\to 0}\int_0^T \Big(h^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}(t),\boldsymbol{u}^{\varepsilon_h}(t)) + \|\boldsymbol{u}^{\varepsilon_h}(t)\|^2_{L^2}\Big)\,dt < \infty.$$

If one assumes strong two-scale convergence of load densities

$$\begin{aligned} &\pi_h\boldsymbol{f}^{\varepsilon_h} \xrightarrow{\mathrm{t,dr-2}} (0,0,\boldsymbol{f})^{\top} \in L^2\Big([0,T]; L^2(\omega;\mathbb{R}^3)\Big), \\ &h\partial_t\boldsymbol{f}^{\varepsilon_h}_{\alpha} \longrightarrow 0 \ \textit{ strongly in } L^2\Big([0,T]; L^2(\Omega)\Big), \ \ \alpha = 1,2, \end{aligned} \tag{2.51}$$

strong two-scale convergence of the initial data in (2.46), (2.47)*, where* $(\boldsymbol{u}_1)_* = 0$, $u_{1,3} \in L^2(\omega)$*, and the condition*

$$\lim_{h\to 0}\Big(h^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}_0,\boldsymbol{u}^{\varepsilon_h}_0) + \|\boldsymbol{u}^{\varepsilon_h}_0\|^2\Big) = a^{\mathfrak{b}}_{\delta}\big(\mathfrak{b}(0),\mathfrak{b}(0)\big) + \|\mathfrak{b}(0)\|^2_{L^2},$$

then one has

$$\pi_{1/h}\boldsymbol{u}^{\varepsilon_h} \xrightarrow{\mathrm{t,dr-2}} (\mathfrak{a}^{\mathfrak{b}}_1 - x_3\partial_1\mathfrak{b}, \mathfrak{a}^{\mathfrak{b}}_2 - x_3\partial_2\mathfrak{b}, \mathfrak{b})^{\top}, \qquad \partial_t\boldsymbol{u}^{\varepsilon_h} \xrightarrow{\mathrm{t,dr-2}} (0,0,\partial_t\mathfrak{b})^{\top}. \tag{2.52}$$

Moreover, the following convergence of energy sequences holds for all $t \in [0,T]$:

$$\lim_{h\to 0}\Big(h^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}(t),\boldsymbol{u}^{\varepsilon_h}(t)) + \|\boldsymbol{u}^{\varepsilon_h}(t)\|^2_{L^2}\Big) = a^{\mathfrak{b}}_{\delta}\big(\mathfrak{b}(t),\mathfrak{b}(t)\big) + \|\mathfrak{b}(t)\|^2_{L^2}.$$

Corollary 2.2.40. *Suppose that for each* $h > 0$*, a surface load density* $\mathcal{G}^{\varepsilon_h} \in L^2([0,T]; (H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)^*)$ *is added to the right-hand side of* (2.42)*. We assume that* $\mathcal{G}^{\varepsilon_h}$ *is generated by an* L^2*-function* $\boldsymbol{g}^{\varepsilon_h}$ *(representing the "true" surface load) so that*

$$\mathcal{G}^{\varepsilon_h}(\boldsymbol{g}^{\varepsilon_h})(\boldsymbol{\theta}) = \int_{\omega\times\{-1/2,1/2\}} \boldsymbol{g}^{\varepsilon_h}\boldsymbol{\theta}\,d\hat{x}, \qquad \boldsymbol{\theta} \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^3),$$

where an obvious shorthand for a sum of two integrals over ω *is used, and make the following additional assumptions on* $\boldsymbol{g}^{\varepsilon_h}$:

$$\pi_h\partial_t\boldsymbol{g}^{\varepsilon_h}_{\alpha} \subset L^2\Big([0,T]; L^2\big(\omega\times\{-1/2,1/2\}\times Y;\mathbb{R}^3\big)\Big) \textit{ is bounded },$$

$$\pi_h \boldsymbol{g}^{\varepsilon_h} \xrightarrow{\text{t,dr}-2} \boldsymbol{g} \in L^2\Big([0,T]; L^2\big(\omega \times \{-1/2, 1/2\} \times Y; \mathbb{R}^3\big)\Big).$$

Then a variant of Theorem 2.2.39 holds, where in the limit equations (2.48) the right-hand side has an additional term $\mathcal{G}_\delta^1(\boldsymbol{g}) \in L^2([0,T]; (H^2_{\gamma_D}(\omega))^*)$, represented by a limiting surface load L^2 vector function $\boldsymbol{g} = (g_1, g_2, g_3)$ so that

$$\begin{aligned}
\mathcal{G}_\delta^1(\boldsymbol{g})(t)(\theta) &= \int_\omega \Big(\big\langle g_3(t,\hat{x},-1/2,\cdot) + g_3(t,\hat{x},1/2,\cdot)\big\rangle\Big)\, \theta(\hat{x})\, d\hat{x} \\
&\quad + \frac{1}{2}\int_\omega \Big(\big\langle g_1(t,\hat{x},-1/2,\cdot) - g_1(t,\hat{x},1/2,\cdot)\big\rangle\Big)\, \partial_1\theta(\hat{x})\, d\hat{x} \\
&\quad + \frac{1}{2}\int_\omega \Big(\big\langle g_2(t,\hat{x},-1/2,\cdot) - g_2(t,\hat{x},1/2,\cdot)\big\rangle\Big)\, \partial_2\theta(\hat{x})\, d\hat{x} \\
&\quad + \int_\omega \mathbb{C}_\delta^{\text{hom}}\big(\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}^{\boldsymbol{g}_*(t)}, 0\big) : \big(0, \nabla^2_{\hat{x}}\theta(\hat{x})\big)\, d\hat{x}, \qquad \theta \in \big(H^2_{\gamma_D}(\omega)\big)^*.
\end{aligned}$$

In the above formula, for every $t \in [0,T]$, the function $\mathfrak{a}^{\boldsymbol{g}_*(t)} \in H^1_{\gamma_D}(\omega; \mathbb{R}^2)$ is the solution to the problem

$$\int_\omega \mathbb{C}_\delta^{\text{memb}} \operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}^{\boldsymbol{g}_*(t)}(\hat{x}) : \operatorname{sym}\nabla_{\hat{x}}\boldsymbol{\theta}_*(\hat{x})\, d\hat{x} = \mathcal{G}_\delta^2(\boldsymbol{g}_*)(t)(\boldsymbol{\theta}_*) \qquad \forall \boldsymbol{\theta}_* \in H^1_{\gamma_D}(\omega; \mathbb{R}^2),$$

where the functional $\mathcal{G}_\delta^2(\boldsymbol{g}_*)(t)$ is defined by the formula

$$\begin{aligned}
\mathcal{G}_\delta^2(\boldsymbol{g}_*)(t)(\boldsymbol{\theta}_*) &= \int_\omega \Big(\big\langle g_1(t,\hat{x},-1/2,\cdot) + g_1(t,\hat{x},1/2,\cdot)\big\rangle\Big)\, \boldsymbol{\theta}_1\, d\hat{x} \\
&\quad + \int_\omega \Big(\big\langle g_2(t,\hat{x},-1/2,\cdot) + g_2(t,\hat{x},1/2,\cdot)\big\rangle\Big)\, \boldsymbol{\theta}_2\, d\hat{x}, \qquad \boldsymbol{\theta}_* \in H^1_{\gamma_D}(\omega; \mathbb{R}^2).
\end{aligned}$$

Also, the right-hand side of (2.49) contains $\mathfrak{a}^{\boldsymbol{g}_*(t)} \in L^2([0,T]; H^1_{\gamma_D}(\omega, \mathbb{R}^2))$ as an additional term, while on the right-hand side of (2.50) one additionally has $\mathcal{G}^3(\boldsymbol{g}_*) \in L^2([0,T]; (H^1_{00}(I \times Y_0; \mathbb{R}^3)^*)$ defined by

$$\mathcal{G}^3(\boldsymbol{g}_*)(t,\hat{x})(\boldsymbol{\xi}) = \int_{\{-1/2,1/2\}\times Y_0} \boldsymbol{g}_*(t,\hat{x},\cdot)\cdot\boldsymbol{\xi}_*(\cdot), \qquad \boldsymbol{\xi} \in H^1_{00}(I \times Y_0; \mathbb{R}^3), \qquad t \in [0,T], \quad \hat{x} \in \omega.$$

Remark 2.2.41. For each of the other regimes studied, a statement analogous to Corollary 2.2.40 is valid.

Remark 2.2.42. The statement of Theorem 2.2.39 can be strengthened as follows. The boundedness and convergence conditions (2.44) and (2.45) can be replaced by the requirement of boundedness and convergence, respectively, of the sequences $(\pi_h \boldsymbol{f}^{\varepsilon_h})_{h>0}$

and $(h\partial_t \boldsymbol{f}_\alpha^{\varepsilon_h})_{h>0}$ in the corresponding spaces of L^1 functions on $[0,T]$. Under this weaker assumption, a still stronger version of (2.2.39), (2.2.39) holds, where the weak convergence in L^2 spaces on $[0,T]$ is replaced by a weak* convergence in the corresponding L^∞ spaces on $[0,T]$, see the comment following Definition 4.3.2.

Similarly, the L^2 convergences (2.51) can be replaced by the weaker conditions

$$\pi_h \boldsymbol{f}^{\varepsilon_h} \xrightarrow{t,1,\mathrm{dr}-2} (0,0,\boldsymbol{f})^\top \in L^1\left([0,T];L^2(\omega;\mathbb{R}^3)\right)$$

$$h\partial_t \boldsymbol{f}_\alpha^{\varepsilon_h} \longrightarrow 0 \;\text{ strongly in } L^1\left([0,T];L^2(\Omega)\right), \;\; \alpha = 1,2,$$

to obtain a strong two-scale convergence $\xrightarrow{t,\infty,\mathrm{dr}-2}$ for both sequences in (2.52); see the same comment at the end of Section 4.3 for the definition of $\xrightarrow{t,\infty,\mathrm{dr}-2}$.

These stronger versions of the claims in Theorem 2.2.39 follow immediately from a priori estimates, see also Remark 4.5.16, Remark 4.5.17, however we choose to remain in the L^2 setting.

A version of the discussion within this remark applies also to Theorem 2.2.47, Theorem 2.2.51, and Theorem 2.2.53.

Remark 2.2.43. The limit equations (2.48)–(2.50) are obtained on a long time scale. The stiff component behaves like a perforated domain, and there is no coupling between its deformation and the deformation of the inclusions. The deformation of the inclusions and the even part of the in-plane deformation of the stiff component behave quasi-statically (i.e. without an inertia term), as a consequence of small forces slowly varying in time. (Recall that the physical equation is (2.43) with the right-hand side $\tilde{\boldsymbol{f}}^{\varepsilon_h}$ subject to an appropriate version of the condition (2.44).) Since there is no coupling in the limit between the inclusions and the stiff component, there are no memory effects in the time evolution. However, it is expected that high-contrast effects would be seen in higher-order terms ("correctors") of the deformation, which we do not pursue here.

Without making additional symmetry assumptions about the material properties, the limit operator for the evolution of the out-of-plane component is spatially non-local, due the coupling between the in-plane and out-of-plane components.

Remark 2.2.44. We are not able to obtain pointwise in time convergence without additional assumptions on the load density. This is expected (replacing weak two-scale convergence with strong two-scale convergence) also as a consequence of the analysis presented in [53].

Remark 2.2.45. The influence of in-plane forces on the limit model is seen through their mean value across the plate, represented above by an integral over the interval $I = [-1/2, 1/2]$, as well as through the mean value of their moments over the same interval I. In the case of planar symmetries, see Assumption 2.1.1 (1), moments of in-plane forces have the same effect on the limit deformation as out-of-plane forces, i.e., they produce out-of-plane displacements. This is expected from the physical point of view and is standard for plate theories (see, e.g., [23]).

Remark 2.2.46. Considering whether different components of the load density should be scaled differently is important from the modelling perspective. Indeed, if its in-plane and out-of-plane components had the same magnitude, one would not see the effects of the in-plane components in the (leading order of the) deformation. On the other hand, it is expected that sufficiently large in-plane loads do influence the limit deformation. However, for some of the asymptotic regimes analysed here the effects on the in-plane and out-of-plane loads on the limit deformation are similar, in which case these loads are set to have the same magnitude in the equations. This kind of situation also occurs in the context of linear elastic shells, see [24] for shells as compared to the case of linear elastic plates [23].

2.2.4.2 Real-time behaviour for $\mu_h = \varepsilon_h$, $\tau = 0$ in different regimes

Here we discuss a class of evolution problems with "non-standard" effective behaviour, which manifests itself, in particular, through time non-locality.

Theorem 2.2.47. *Suppose that $\mu_h = \varepsilon_h$, $\tau = 0$, $\delta, \kappa \in [0, \infty]$, and consider the sequence $(\boldsymbol{u}^{\varepsilon_h})_{h>0}$ of solutions to the problem (2.42), assuming that*

$$\boldsymbol{f}^{\varepsilon_h} \xrightarrow{\text{t,dr}-2} \boldsymbol{f} \in L^2([0,T]; L^2(\Omega \times Y; \mathbb{R}^3)), \tag{2.53}$$

$$\boldsymbol{u}_0^{\varepsilon_h} \xrightarrow{dr-2} \boldsymbol{u}_0(x,y) \in V_{1,\delta,\kappa}(\omega \times Y) + V_{2,\delta}(\Omega \times Y_0),$$

$$\boldsymbol{u}_1^{\varepsilon_h} \xrightarrow{dr-2} \boldsymbol{u}_1(x,y) \in L^2(\Omega \times Y; \mathbb{R}^3). \tag{2.54}$$

Assume also that

$$\limsup_{h\to 0}\left(a_{\varepsilon_h}(\boldsymbol{u}_0^{\varepsilon_h},\boldsymbol{u}_0^{\varepsilon_h})+\|\boldsymbol{u}_0^{\varepsilon_h}\|_{L^2}\right)<\infty.$$

Then one has

$$\boldsymbol{u}^{\varepsilon_h}\xrightarrow{\text{t,dr}-2}(\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}},\tag{2.55}$$

$$\partial_t\boldsymbol{u}^{\varepsilon_h}\xrightarrow{\text{t,dr}-2}\partial_t\big((\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}}\big),\tag{2.56}$$

where $(\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}}\in C([0,T];V_{1,\delta,\kappa}(\omega\times Y)+V_{2,\delta}(\Omega\times Y_0))$ *is the unique weak solution of the problem*

$$\partial_{tt}\big((\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}}\big)(t)+\mathcal{A}_{\delta,\kappa}\big((\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}}\big)(t)=P_{\delta,\kappa}\boldsymbol{f}(t),$$

$$\big((\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}}\big)(0)=\boldsymbol{u}_0(x,y),\qquad \partial_t\big((\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}}\big)(0)=P_{\delta,\kappa}\boldsymbol{u}_1(x,y),$$

such that $\partial_t\left((\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}}\right)\in C([0,T];H_{\delta,\kappa}(\Omega\times Y))$*. Furthermore, the following limit energy bound holds:*

$$\limsup_{h\to 0}\int_0^T\left(a_{\varepsilon_h}\big(\boldsymbol{u}^{\varepsilon_h}(t),\boldsymbol{u}^{\varepsilon_h}(t)\big)+\|\boldsymbol{u}^{\varepsilon_h}(t)\|_{L^2}^2\right)<\infty.$$

If strong two-scale convergence holds in (2.53)–(2.54) *with* $\boldsymbol{f}\in L^2([0,T];H_{\delta,\kappa}(\Omega\times Y))$*,* $\boldsymbol{u}_1\in H_{\delta,\kappa}(\Omega\times Y)$*, and*

$$\lim_{h\to 0}\left(a_{\varepsilon_h}\big(\boldsymbol{u}_0^{\varepsilon_h},\boldsymbol{u}_0^{\varepsilon_h}\big)+\|\boldsymbol{u}_0^{\varepsilon_h}\|_{L^2}^2\right)=a_{\delta,\kappa}\big(((\mathfrak{a},0)^\top+\mathring{\boldsymbol{u}})(0),((\mathfrak{a},0)^\top+\mathring{\boldsymbol{u}})(0)\big)+\left\|\big((\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}}\big)(0)\right\|_{L^2}^2,$$

then strong two-scale convergence holds in (2.55)–(2.56)*. Moreover, one has*

$$\lim_{h\to 0}\Big(a_{\varepsilon_h}\big(\boldsymbol{u}^{\varepsilon_h}(t),\boldsymbol{u}^{\varepsilon_h}(t)\big)+\|\boldsymbol{u}^{\varepsilon_h}(t)\|_{L^2}\Big)=a_{\delta,\kappa}\big(((\mathfrak{a},0)^\top+\mathring{\boldsymbol{u}})(t),((\mathfrak{a},0)^\top+\mathring{\boldsymbol{u}})(t)\big)+\left\|\big((\mathfrak{a},\mathfrak{b})^\top+\mathring{\boldsymbol{u}}\big)(t)\right\|_{L^2}^2,$$

for every $t\in[0,T]$*.*

Remark 2.2.48. The models obtained here are degenerate with respect to the out-of-plane component of the displacement. Indeed, in the static case it is substantially easier for the plate to bend than to extend in-plane; however, in the dynamic case in real time, for the forces of magnitude one, there is no elastic resistance to out-of-plane motions, which are therefore entirely due to external loads.

It is also worthwhile noting that in the high-contrast setting out-of-plane loads $\boldsymbol{f}=(0,0,f_3)$ for which $\overline{f}_3=0$ do produce some in-plane motion in the case when $\delta\in(0,\infty]$, as a consequence of the coupling between the deformations on the stiff component and on the inclusions, which is not possible in the setting of homogenisation with moderate contrast. (In the regime $\delta = 0$, inclusions behave like small plates and thus only the effects of the average loads $\overline{\boldsymbol{f}}$ in the variable x_3 are seen in the limit.) On a related note, from the point of view of quantitative analysis, it is not expected that the effect elastic resistance to out-of-plane motions disappears entirely, as it may manifest itself in lower-order terms, see [20] for a quantitative analysis of the resolvent equation for a thin infinite plate in moderate contrast.

Remark 2.2.49. To the best of our knowledge, dynamic models representing "real time behaviour" have not been discussed in the literature, even in the case of an ordinary plate. Certainly, these models are not as physically relevant as those in which elastic resistance to out-of-plane motions is observed. This might be due to the fact that for most materials mass density is much smaller than Lamé constants (in dimensionless terms). However, since these models exhibit high-contrast effects, which does not happen when the time is scaled (unless the coefficients on the inclusions are scaled in a non-standard way in relation to the coefficients on the stiff component), we find it is important to discuss them also.

Remark 2.2.50. In the limit problem, due to the coupling of the deformation on the stiff component, given by $(\mathfrak{a}, \mathfrak{b})^\top$, and the oscillatory part of the deformation on the soft component, given by $\mathring{\boldsymbol{u}}$, there are memory effects (under the assumption that the micro-variable $\mathring{\boldsymbol{u}}$ is unknown). The emergence of these memory effects can be seen as follows. If one would like to know the deformation on the stiff component at time T, given by $(\mathfrak{a}, \mathfrak{b})^\top(T)$, one would not only require the initial data (deformation and speed) on the stiff component at an "initial" time $t_0 < T$ and loads $\boldsymbol{f}$ on the time interval $[t_0, T]$, but also the value of the micro-variable $\mathring{\boldsymbol{u}}$ and its speed at time t_0. It one cannot measure this micro-variable (which is a physically meaningful scenario), then the corresponding degree of freedom becomes "hidden" internally, which results in a non-local time dependence macroscopically.

2.2.4.3 Long-time behaviour for $\delta \in (0,\infty]$, $\mu_h = \varepsilon_h h$, $\tau = 2$

Here we demonstrate that by varying the contrast between material properties of the two components ("stiff" and "soft"), the evolution problem may be shown to exhibit time non-locality also in the regime of long times.

Theorem 2.2.51. *Suppose that $\delta \in (0,\infty]$, $\mu_h = \varepsilon_h h$, $\tau = 2$, and let $(\boldsymbol{u}^{\varepsilon_h})_{h>0}$ be the sequence of solutions of the problem (2.42), assuming that*

$$\boldsymbol{f}^{\varepsilon_h} \xrightarrow{\text{t,dr}-2} \boldsymbol{f} \in L^2\big([0,T]; L^2(\Omega\times Y;\mathbb{R}^3)\big), \tag{2.57}$$

$$\boldsymbol{u}_0^{\varepsilon_h} \xrightarrow{dr-2} \boldsymbol{u}_0(\hat{x}) + \mathring{\boldsymbol{u}}_0(x,y) \in \{0\}^2 \times H^2_{\gamma_{\rm D}}(\omega) + V_{2,\delta}(\Omega\times Y_0),$$

$$\boldsymbol{u}_1^{\varepsilon_h} \xrightarrow{dr-2} \boldsymbol{u}_1(x,y) \in L^2(\Omega\times Y;\mathbb{R}^3). \tag{2.58}$$

Assume also that

$$\limsup_{h\to 0} \big(a_{\varepsilon_h}(\boldsymbol{u}_0^{\varepsilon_h}, \boldsymbol{u}_0^{\varepsilon_h}) + \|\boldsymbol{u}_0^{\varepsilon_h}\|_{L^2}\big) < \infty.$$

Then one has

$$\boldsymbol{u}^{\varepsilon_h} \xrightarrow{\text{t,dr}-2} (0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}, \tag{2.59}$$

$$\partial_t \boldsymbol{u}^{\varepsilon_h} \xrightarrow{\text{t,dr}-2} \partial_t\big((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big), \tag{2.60}$$

where $(0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}} \in C([0,T];\{0\}^2\times H^2_{\gamma_{\rm D}}(\omega) + V_{2,\delta}(\Omega\times Y_0))$ is the unique weak solution to the problem

$$\partial_{tt}\big((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big)(t) + \hat{\mathcal{A}}_\delta\big((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big)(t) = \big(S_2(P_{\delta,\infty}\boldsymbol{f}(t))_1, S_2(P_{\delta,\infty}\boldsymbol{f}(t))_2, (P_{\delta,\infty}\boldsymbol{f}(t))_3\big)^\top,$$

$$\big((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big)(0) = \boldsymbol{u}_0(x,y), \quad \partial_t\big((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big)(0) = \big(S_2(P_{\delta,\infty}\boldsymbol{u}_1)_1, S_2(P_{\delta,\infty}\boldsymbol{u}_1)_2, (P_{\delta,\infty}\boldsymbol{u}_1)_3\big)^\top(x,y),$$

such that $\partial_t\big((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big) \in C([0,T]; H_{\delta,\infty}(\Omega\times Y))$. Furthermore, the following limit energy bound holds:

$$\limsup_{h\to 0} \int_0^T \Big(h^{-2} a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}(t), \boldsymbol{u}^{\varepsilon_h}(t)) + \|\boldsymbol{u}^{\varepsilon_h}(t)\|^2_{L^2}\Big) < \infty.$$

If strong two-scale convergence holds in (2.57)–(2.58) with $\boldsymbol{f} \in L^2([0,T]; H_{\delta,\infty}(\Omega\times Y))$, $\boldsymbol{u}_1 \in H_{\delta,\infty}(\Omega\times Y)$, and

$$\lim_{h\to 0}\Big(h^{-2} a_{\varepsilon_h}(\boldsymbol{u}_0^{\varepsilon_h}, \boldsymbol{u}_0^{\varepsilon_h}) + \|\boldsymbol{u}_0^{\varepsilon_h}\|^2_{L^2}\Big) = \hat{a}_\delta\big(((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}})(0), ((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}})(0)\big) + \Big\|((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}})(0)\Big\|^2_{L^2},$$

then strong two-scale convergence holds in (2.59)–(2.60). *Moreover, for every* $t \in [0,T]$ *one has*

$$\lim_{h\to 0}\left(h^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}(t),\boldsymbol{u}^{\varepsilon_h}(t))+\|\boldsymbol{u}^{\varepsilon_h}(t)\|^2_{L^2}\right)$$
$$=\hat{a}_\delta\big(((0,0,\mathfrak{b})^\top+\mathring{\boldsymbol{u}})(t),((0,0,\mathfrak{b})^\top+\mathring{\boldsymbol{u}})(t)\big)+\left\|\big((0,0,\mathfrak{b})^\top+\mathring{\boldsymbol{u}})(t)\right\|^2_{L^2}.$$

Remark 2.2.52. The above limit model exhibits memory effects, due to the coupling of the deformations on the stiff component and on the inclusions, similarly to what happened in Section 2.2.4.2. As before, see Remark 2.2.43, in the case when $\delta \in (0,\infty)$ and no additional symmetries are imposed on the material properties, the limit macro-operator $\hat{\mathcal{A}}_\delta$ is spatially non-local.

2.2.4.4 Long-time behaviour for $\delta = 0$, $\mu_h = \varepsilon_h^2$, $\tau = 2$

Here we discuss an analogue of the result of the previous section for the case $\delta = 0$, in which we need to apply different scalings to the in-plane and out-of-plane loads. As already emphasized in Sections 2.2.2.2 (resolvent convergence), 2.2.3.4 (limit spectrum), in this regime we require that Y_0 have $C^{1,1}$ boundary.

Theorem 2.2.53. *Suppose that* $\delta = 0$, $\mu_h = \varepsilon_h^2$, $\tau = 2$, *and let* $(\boldsymbol{u}^{\varepsilon_h})_{h>0}$ *be the sequence of solutions to the problem* (2.42), *assuming that*

$$\big((h/\varepsilon_h)\partial_t \boldsymbol{f}^{\varepsilon_h}_\alpha\big)_{h>0} \subset L^2([0,T];L^2(\Omega\times Y)) \text{ is bounded}, \qquad \alpha = 1,2,$$
$$\pi_{h/\varepsilon_h}\boldsymbol{f}^{\varepsilon_h} \xrightarrow{\text{t,dr}-2} \boldsymbol{f} \in L^2([0,T];L^2(\Omega\times Y;\mathbb{R}^3)),$$
$$\boldsymbol{u}_0^{\varepsilon_h} \xrightarrow{dr-2} \boldsymbol{u}_0(\hat{x},y) \in \{0\}^2\times H^2_{\gamma_{\rm D}}(\omega)+\{0\}^2\times L^2(\omega\times Y_0), \tag{2.61}$$
$$\boldsymbol{u}_1^{\varepsilon_h}(x) \xrightarrow{dr-2} \boldsymbol{u}_1(x,y) \in L^2(\Omega\times Y;\mathbb{R}^3), \tag{2.62}$$

and assume additionally that

$$\limsup_{h\to 0}\left(h^{-2}a_{\varepsilon_h}(\boldsymbol{u}_0^{\varepsilon_h},\boldsymbol{u}_0^{\varepsilon_h})+\|\boldsymbol{u}_0^{\varepsilon_h}\|_{L^2}\right)<\infty.$$

Then one has

$$\pi_{\varepsilon_h/h}\boldsymbol{u}^{\varepsilon_h} \xrightarrow{\text{t,dr}-2} \begin{pmatrix} \mathring{u}_1(t,\hat{x},y)-x_3\partial_{y_1}\mathring{u}_3(t,\hat{x},y) \\ \mathring{u}_2(t,\hat{x},y)-x_3\partial_{y_2}\mathring{u}_3(t,\hat{x},y) \\ \mathfrak{b}(t,\hat{x})+\mathring{u}_3(t,\hat{x},y) \end{pmatrix}$$

$$\partial_t \boldsymbol{u}^{\varepsilon_h} \xrightarrow{\text{t,dr}-2} \Big(0,0,\partial_t\left(\mathfrak{b}(t,\hat{x})+\mathring{u}_3(t,\hat{x},y)\right)\Big)^\top,$$

where the pair $\mathfrak{b}\in C([0,T];H^2_{\gamma_{\rm D}}(\omega))$, $\mathring{\boldsymbol{u}}\in C([0,T];L^2(\omega;H^1_{00}(I\times Y_0;\mathbb{R}^3)))$ *form the unique weak solution of the problem*

$$\partial_{tt}\left(\mathfrak{b}+\mathring{u}_3\right)(t)+\hat{\mathcal{A}}_0\left(\mathfrak{b}+\mathring{u}_3\right)(t)=\mathcal{F}_0(\boldsymbol{f}), \quad \textit{(see (2.25))}$$

$$\left(\mathfrak{b}+\mathring{u}_3\right)(0)=u_{0,3}\in H^2_{\gamma_{\rm D}}(\omega)+L^2(\omega\times Y_0), \qquad \partial_t(\mathfrak{b}+\mathring{u}_3)(0)=P^0u_{1,3}\in L^2(\omega)+L^2(\omega\times Y_0),$$

$$\tilde{\mathcal{A}}_{00,0}\mathring{\boldsymbol{u}}_*(t,\hat{x},\cdot)=\boldsymbol{f}_*(t,\hat{x},\cdot), \tag{2.63}$$

such that $\partial_t(\mathfrak{b}(t,\hat{x})+\mathring{u}_3(t,\hat{x},y))\in C([0,T];L^2(\omega)+L^2(\omega\times Y_0))$. *Furthermore, the following limit energy bound holds:*

$$\limsup_{h\to 0}\int_0^T\left(h^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}(t),\boldsymbol{u}^{\varepsilon_h}(t))+\|\boldsymbol{u}^{\varepsilon_h}(t)\|^2_{L^2}\right)dt<\infty.$$

If one additionally assumes that

$$\pi_{h/\varepsilon_h}\boldsymbol{f}^{\varepsilon_h}\xrightarrow{\text{t,dr}-2}(0,0,\boldsymbol{f})^\top\in L^2\left([0,T];L^2(\omega;\mathbb{R}^3)+L^2(\omega\times Y_0;\mathbb{R}^3)\right),$$

$$(h/\varepsilon_h)\partial_t\boldsymbol{f}^{\varepsilon_h}_\alpha\longrightarrow 0 \;\; \textit{strongly in } L^2\left([0,T];L^2(\Omega)\right), \qquad \alpha=1,2,$$

the two-scale convergence in (2.61) *and* (2.62) *holds in the strong sense with* $(\boldsymbol{u}_1)_*=0$, $\boldsymbol{u}_{1,3}\in L^2(\omega)+L^2(\omega\times Y_0)$, *and that*

$$\lim_{h\to 0}\left(h^{-2}a_{\varepsilon_h}(\boldsymbol{u}_0^{\varepsilon_h},\boldsymbol{u}_0^{\varepsilon_h})+\|\boldsymbol{u}_0^{\varepsilon_h}\|^2\right)=a^{\mathfrak{b}}_\delta\big((\mathfrak{b}+\mathring{u}_3)(0),(\mathfrak{b}+\mathring{u}_3)(0)\big)+\left\|(\mathfrak{b}+\mathring{u}_3)(0)\right\|^2_{L^2},$$

then one has

$$\boldsymbol{u}^{\varepsilon_h}\xrightarrow{\text{t,dr}-2}(0,0,\mathfrak{b}(t,\hat{x})+\mathring{u}_3(t,\hat{x},y))^\top, \quad \partial_t\boldsymbol{u}^{\varepsilon_h}\xrightarrow{\text{t,dr}-2}(0,0,\partial_t(\mathfrak{b}(t,\hat{x})++\mathring{u}_3(t,\hat{x},y)))^\top.$$

Moreover, for every $t\in[0,T]$ *the convergence*

$$\lim_{h\to 0}\left(h^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}(t),\boldsymbol{u}^{\varepsilon_h}(t))+\|\boldsymbol{u}^{\varepsilon_h}(t)\|^2_{L^2}\right)=a^{\mathfrak{b}}_\delta\big((\mathfrak{b}+\mathring{u}_3)(t),(\mathfrak{b}+\mathring{u}_3)(t)\big)+\left\|(\mathfrak{b}+\mathring{u}_3)(t)\right\|^2_{L^2}$$

holds.

Remark 2.2.54. In the regime $\delta=0$ inclusions behave like small plates and thus the corresponding deformation satisfies a version of the classical Kirchhoff-Love ansatz. Using the rationale discussed in Remark 2.2.46, we argue that in order to see the effects of both in-plane and out-of-plane components of loads in the limit model, we should scale them differently to one another.

Similarly to the regime $\delta \in (0, \infty)$, $\mu_h = \varepsilon_h$, $\tau = 2$, we impose a restriction on the time derivatives of in-plane forces, see (2.2.53), which in terms of the "physical" time corresponds to slowly acting loads. This results in a (partial) quasi-static evolution in the limit, see (2.63). Furthermore, in order to obtain strong two-scale convergence of solutions, akin to (2.52), we impose a further restriction that properly scaled in-plane forces together with their time derivatives, in the spirit to (2.51), go to zero as $h \to 0$.

2.3. Proofs

2.3.1. Proof of Proposition 2.2.4

Proof. We provide the proof for the case $\delta \in (0,\infty)$; the other cases are dealt in a similar fashion, bearing in mind Remark 2.2.2 and Remark 2.2.3.

Consider the minimiser $\boldsymbol{\varphi} \in H^1(I \times \mathcal{Y};\mathbb{R}^3)$ in the variational formulation (2.2). Then for arbitrary symmetric matrices $\boldsymbol{A},\boldsymbol{B} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ one has a lower bound for elastic stored energy density, as follows:

$$\begin{aligned}
\mathbb{C}^{\mathrm{hom}}_{\delta}(\boldsymbol{A},\boldsymbol{B}) : (\boldsymbol{A},\boldsymbol{B}) &\geq C \int_I \left\| \iota(\boldsymbol{A} - x_3\boldsymbol{B}) + \operatorname{sym}\widetilde{\nabla}_{2,\gamma}\boldsymbol{\varphi}(x_3,\cdot)\right\|^2_{L^2(Y_1;\mathbb{R}^{3\times 3})} dx_3 \\
&\geq C \int_I \left\| \boldsymbol{A} - x_3\boldsymbol{B} + \operatorname{sym}\nabla_y \boldsymbol{\varphi}_*(x_3,\cdot)\right\|^2_{L^2(Y_1;\mathbb{R}^{2\times 2})} dx_3,
\end{aligned} \tag{2.64}$$

due to the coercivity of the tensor $\mathbb{C}_1$ representing the elastic properties on the stiff component. In order to eliminate the corrector $\boldsymbol{\varphi}_*$ from the bound (2.64), we first construct an extension for it from Y_1 to the whole cell Y for each $x_3 \in I$. To this end, we first define the symmetric affine part of an arbitrary H^1 function, as follows. For $\boldsymbol{\xi} = (\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)^\top \in H^1(Y_1;\mathbb{R}^2)$, we consider the function $\hat{\boldsymbol{\xi}} \in H^1(Y;\mathbb{R}^2)$ defined by

$$\hat{\boldsymbol{\xi}}(y) := \fint_{Y_1} \boldsymbol{\xi}(y)dy + \fint_{Y_1} \operatorname{sym}\nabla_y\boldsymbol{\xi}(y)dy \left(y - \fint_{Y_1} y dy \right).$$

Notice that the operator $\hat{\cdot}$ is linear and satisfies the following properties:

$$\nabla_y\hat{\boldsymbol{\xi}} = \operatorname{sym}\nabla_y\hat{\boldsymbol{\xi}} = \fint_{Y_1}\operatorname{sym}\nabla_y\boldsymbol{\xi}(y)dy, \qquad \fint_{Y}\hat{\boldsymbol{\xi}}(y)dy = \fint_{Y_1}\boldsymbol{\xi}(y)dy,$$
$$\left\|\operatorname{sym}\nabla_y\hat{\boldsymbol{\xi}}\right\|_{L^2(Y;\mathbb{R}^{2\times 2})} \leq |Y|/|Y_1| \left\|\operatorname{sym}\nabla_y\boldsymbol{\xi}\right\|_{L^2(Y_1;\mathbb{R}^{2\times 2})}.$$

Now we define the extension operator $\hat{E} : H^1(Y_1;\mathbb{R}^2) \to H^1(Y;\mathbb{R}^2)$, via

$$\hat{E}\boldsymbol{\xi} := E(\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}) + \hat{\boldsymbol{\xi}},$$

where E is the extension operator from [51, Lemma 4.1], which satisfies the estimate

$$\left\|\operatorname{sym}\nabla_y(E\boldsymbol{\xi})\right\|_{L^2(Y;\mathbb{R}^{2\times 2})} \leq C \left\|\operatorname{sym}\nabla_y\boldsymbol{\xi}\right\|_{L^2(Y_1;\mathbb{R}^{2\times 2})}.$$

It is easy to see that

$$\left\|\operatorname{sym}\nabla_y(\hat{E}\boldsymbol{\xi})\right\|^2_{L^2(Y;\mathbb{R}^{2\times 2})} \le C\left\|\operatorname{sym}\nabla_y\boldsymbol{\xi}\right\|^2_{L^2(Y_1;\mathbb{R}^{2\times 2})}. \tag{2.65}$$

Next, consider the function

$$\boldsymbol{\psi}(y) := (\boldsymbol{A} - x_3\boldsymbol{B})\,y + \boldsymbol{\varphi}_*(y).$$

Clearly, one has

$$\hat{E}\boldsymbol{\psi}(y) = E(\boldsymbol{\varphi}_* - \hat{\boldsymbol{\varphi}}_*)(y) + (\boldsymbol{A} - x_3\boldsymbol{B})y + \hat{\boldsymbol{\varphi}}_*(y) = \hat{E}\boldsymbol{\varphi}_*(y) + (\boldsymbol{A} - x_3\boldsymbol{B})y.$$

Furthermore, from (2.65) one has

$$\left\|\operatorname{sym}\nabla_y(\hat{E}\boldsymbol{\psi})\right\|^2_{L^2(Y;\mathbb{R}^{2\times 2})} \le C\left\|\operatorname{sym}\nabla_y\boldsymbol{\psi}\right\|^2_{L^2(Y_1;\mathbb{R}^{2\times 2})} = C\left\|\boldsymbol{A} - x_3\boldsymbol{B} + \operatorname{sym}\nabla_y\boldsymbol{\varphi}_*\right\|^2_{L^2(Y_1;\mathbb{R}^{2\times 2})}. \tag{2.66}$$

At the same time, the following bound holds:

$$\begin{aligned}\left\|\operatorname{sym}\nabla_y(\hat{E}\boldsymbol{\psi})\right\|^2_{L^2(Y;\mathbb{R}^{2\times 2})} &= \left\|\operatorname{sym}\nabla_y(\hat{E}\boldsymbol{\varphi}_*) + (\boldsymbol{A} - x_3\boldsymbol{B})\right\|^2_{L^2(Y;\mathbb{R}^{2\times 2})} \\ &= \left\|\operatorname{sym}\nabla_y(\hat{E}\boldsymbol{\varphi}_*)\right\|^2_{L^2(Y;\mathbb{R}^{2\times 2})} + \|\boldsymbol{A} - x_3\boldsymbol{B}\|^2_{L^2(Y;\mathbb{R}^{2\times 2})} \ge |\boldsymbol{A} - x_3\boldsymbol{B}|^2.\end{aligned} \tag{2.67}$$

Integrating (2.67) over I and taking into account (2.66) and then (2.64), the claim follows.

■

2.3.2. Proofs for Section 2.2.2

A. Proof of Proposition 2.2.5

Proof. Notice first that using $\boldsymbol{u}^{\varepsilon_h}$ as a test function in (2.6) immediately yields

$$h^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2(\Omega;\mathbb{R}^3)} \le C\|\pi_h\boldsymbol{f}^{\varepsilon_h}\|_{L^2(\Omega;\mathbb{R}^3)}\|\pi_{1/h}\boldsymbol{u}^{\varepsilon_h}\|_{L^2(\Omega;\mathbb{R}^3)}. \tag{2.68}$$

Next, we define $\tilde{\boldsymbol{u}}^{\varepsilon_h}$ by applying Theorem 4.4.1 to extend $\boldsymbol{u}^{\varepsilon_h}|_{\Omega_1^{\varepsilon_h}}$ to the whole domain Ω and set

$$\mathring{\boldsymbol{u}}^{\varepsilon_h} := \tilde{\boldsymbol{u}}^{\varepsilon_h} - \boldsymbol{u}^{\varepsilon_h}.$$

Furthermore, Theorem 4.4.1 and Lemma 4.4.4 imply

$$\|\operatorname{sym}\nabla_h\tilde{\boldsymbol{u}}^{\varepsilon_h}\|^2_{L^2} \le Ca_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}), \qquad h^{-2}\|\mathring{\boldsymbol{u}}^{\varepsilon_h}\|^2_{L^2} + \|\nabla_h\mathring{\boldsymbol{u}}^{\varepsilon_h}\|^2_{L^2} \le Ch^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}). \tag{2.69}$$

Combining Corollary 4.2.5 with (2.69), we obtain

$$\|\pi_{1/h}\boldsymbol{u}^{\varepsilon_h}\|_{L^2}^2 \le 2\|\pi_{1/h}\tilde{\boldsymbol{u}}^{\varepsilon_h}\|_{L^2}^2 + 2\|\pi_{1/h}\mathring{\boldsymbol{u}}^{\varepsilon_h}\|_{L^2}^2 \le h^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h}). \tag{2.70}$$

The claim in part 1 now follows directly from (2.68) and (2.70).

Proceeding to the proof of part 2, we notice that the fourth convergence in (2.7) is a direct consequence of (2.69) and Theorem 4.3.1 (1b). To prove the first and second convergence in (2.7), we use Lemma 4.2.8 and (2.69). Lemma 4.2.8 (3) now yields the following decomposition of the sequence $\tilde{\boldsymbol{u}}^{\varepsilon_h}$:

$$\frac{1}{h}\tilde{\boldsymbol{u}}^{\varepsilon_h}(x) = \begin{pmatrix} -x_3\partial_1\mathfrak{b} \\ -x_3\partial_2\mathfrak{b} \\ h^{-1}\mathfrak{b} \end{pmatrix} + \begin{pmatrix} \mathfrak{a}_1 \\ \mathfrak{a}_2 \\ 0 \end{pmatrix} + \boldsymbol{\psi}^{\varepsilon_h},$$

$$\frac{1}{h}\operatorname{sym}\nabla_h\tilde{\boldsymbol{u}}^{\varepsilon_h} = \iota\left(\operatorname{sym}\nabla_{\hat{x}}\boldsymbol{\mathfrak{a}} - x_3\nabla_{\hat{x}}^2\mathfrak{b}\right) + \operatorname{sym}\nabla_h\boldsymbol{\psi}^{\varepsilon_h},$$

where $\mathfrak{b} \in H^2_{\gamma_{\mathrm{D}}}(\omega)$, $\boldsymbol{\mathfrak{a}} \in H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)$, and $(\boldsymbol{\psi}^{\varepsilon_h})_{h>0} \subset H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)$ is such that $h\pi_{1/h}\boldsymbol{\psi}^{\varepsilon_h} \longrightarrow 0$ in L^2.

To prove the third convergence in (2.7), we first assume that ω has $C^{1,1}$ boundary. By virtue of Lemma 4.2.10 (3), there are sequences $(\varphi^{\varepsilon_h})_{h>0} \subset H^2_{\gamma_{\mathrm{D}}}(\omega)$, $(\tilde{\boldsymbol{\psi}}^{\varepsilon_h})_{h>0} \subset H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)$, $(\boldsymbol{o}^{\varepsilon_h})_{h>0} \subset L^2(\Omega;\mathbb{R}^{3\times 3})$ such that

$$\operatorname{sym}\nabla_h\boldsymbol{\psi}^{\varepsilon_h} = -x_3\iota\left(\nabla_{\hat{x}}^2\varphi^{\varepsilon_h}\right) + \operatorname{sym}\nabla_h\tilde{\boldsymbol{\psi}}^{\varepsilon_h} + \boldsymbol{o}^{\varepsilon_h},$$

where

$$\varphi^{\varepsilon_h} \xrightarrow{L^2} 0, \quad \nabla_{\hat{x}}\varphi^{\varepsilon_h} \xrightarrow{L^2} 0, \quad \|\nabla_{\hat{x}}^2\varphi^{\varepsilon_h}\|_{L^2} \le C,$$

$$\tilde{\boldsymbol{\psi}}^{\varepsilon_h} \xrightarrow{L^2} 0, \quad \|\nabla_h\tilde{\boldsymbol{\psi}}^{\varepsilon_h}\|_{L^2} \le C,$$

$$\boldsymbol{o}^{\varepsilon_h} \xrightarrow{L^2} 0.$$

In view of Lemma 4.3.3 (1) and Theorem 4.3.1 (1a), there exist $z \in L^2(\omega;H^2(\mathcal{Y}))$ and $\tilde{\boldsymbol{\psi}} \in L^2(\omega;H^1(I\times\mathcal{Y};\mathbb{R}^3))$ such that (up to extracting a subsequence)

$$\nabla_{\hat{x}}^2\varphi^{\varepsilon_h}(\hat{x}) \xrightarrow{dr-2} \nabla_y^2 z(\hat{x},y),$$

$$\operatorname{sym}\nabla_h\tilde{\boldsymbol{\psi}}^{\varepsilon_h}(x) \xrightarrow{dr-2} \operatorname{sym}\widetilde{\nabla}_\delta\tilde{\boldsymbol{\psi}}(x,y).$$

Introducing the function

$$\boldsymbol{\varphi}(x,y) := \begin{pmatrix} -x_3\partial_{y_1} z(\hat{x},y) \\ -x_3\partial_{y_2} z(\hat{x},y) \\ \delta^{-1} z(\hat{x},y) \end{pmatrix} + \tilde{\boldsymbol{\psi}}(x,y),$$

we have

$$\operatorname{sym}\widetilde{\nabla}_\delta \boldsymbol{\varphi}(x,y) = -x_3\iota\big(\nabla_y^2 z(\hat{x},y)\big) + \operatorname{sym}\widetilde{\nabla}_\delta \tilde{\boldsymbol{\psi}}(x,y),$$

from which the third convergence in (2.7) follows.

Next we can extend this result to the case of an arbitrary Lipschitz domain. In the general case we can only conclude that since $h^{-1}\operatorname{sym}\nabla_h \tilde{\boldsymbol{u}}(x)$ is bounded in $L^2(\Omega;\mathbb{R}^3)$ there exists $C \in L^2(\Omega\times Y;\mathbb{R}^{3\times 3})$ such that

$$h^{-1}\operatorname{sym}\nabla_h \tilde{\boldsymbol{u}}^{\varepsilon_h}(x) \xrightarrow{dr-2} \iota\big(\operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x}) - x_3\nabla_{\hat{x}}^2\mathfrak{b}(\hat{x})\big) + C(x,y).$$

Take a sequence $(\omega_n)_{n\in\mathbb{N}}$ of increasing domains with $C^{1,1}$ boundary such that $\omega_n \subset \omega$, $\cup_{n\in\mathbb{N}}\omega_n = \omega$. By the preceding analysis we conclude that for every $n \in \mathbb{N}$ there exists $\boldsymbol{\varphi}^n \in L^2(\omega_n; H^1(I\times\mathcal{Y};\mathbb{R}^3))$ such that

$$C(x,y) = \operatorname{sym}\widetilde{\nabla}_{2,\delta}\boldsymbol{\varphi}^n(x,y) \quad \text{a.e. } \hat{x}\in\omega_n,\ (x_3,y)\in I\times\mathcal{Y}.$$

Furthermore, notice that

$$\big\|\operatorname{sym}\widetilde{\nabla}_{2,\delta}\boldsymbol{\varphi}^n\big\|_{L^2(\omega^n\times I\times Y;\mathbb{R}^{3\times3})} \le \|C\|_{L^2(\Omega\times Y;\mathbb{R}^{3\times3})}, \quad \forall n\in\mathbb{N}.$$

Finally, we extend $\boldsymbol{\varphi}^n$ by zero outside $\omega_n\times I$. The claim follows from the fact that $C_\delta(\Omega\times I)$ is weakly closed, which in turn is a consequence of Korn's inequality for functions in $\dot{H}^1(I\times\mathcal{Y};\mathbb{R}^3)$ (see [48, Theorem 6.3.8]).

To prove part 3, we first notice that

$$\lim_{h\to 0} h^{-2} a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h}) = a_\delta^{\mathrm{b}}(\mathfrak{b},\mathfrak{b}).$$

Using lower semicontinuity of convex functionals with respect to weak two-scale convergence and the definition of a_δ^{b}, we conclude that $\operatorname{sym}\widetilde{\nabla}_\delta\mathring{\boldsymbol{u}}(x,y) = 0$, $\mathfrak{a} = \mathfrak{a}^{\mathrm{b}}$ and that $C(x,\cdot)$ solves the minimisation problem (2.2) with $\boldsymbol{A} = \operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}(\hat{x})$ and $\boldsymbol{B} = \nabla_{\hat{x}}^2\mathfrak{b}(\hat{x})$.

The strong two-scale convergence claim of part 3 as well as Remark 2.2.6 follow from the strict convexity of the tensors $\mathbb{C}_\alpha$, $\alpha = 1,2$, viewed as quadratic forms on symmetric matrices. ■

B. Proof of Theorem 2.2.7

Proof. We choose the test function $\boldsymbol{v}$ in (2.6) to be of the form

$$\boldsymbol{v}^{\varepsilon_h}(x) = \begin{pmatrix} h\theta_1(\hat{x}) - hx_3\partial_1\theta_3(\hat{x}) \\ h\theta_2(\hat{x}) - hx_3\partial_2\theta_3(\hat{x}) \\ \theta_3(\hat{x}) \end{pmatrix} + h\varepsilon_h \boldsymbol{\zeta}\left(x, \frac{\hat{x}}{\varepsilon_h}\right) + h\mathring{\boldsymbol{\xi}}\left(x, \frac{\hat{x}}{\varepsilon_h}\right),$$

where $\boldsymbol{\theta}_* \in C^1_{\rm c}(\omega;\mathbb{R}^2)$, $\theta_3 \in C^2_{\rm c}(\omega)$, $\boldsymbol{\zeta} \in C^1_{\rm c}(\Omega; C^1(I\times\mathcal{Y};\mathbb{R}^3))$, $\mathring{\boldsymbol{\xi}} \in C^1_{\rm c}(\omega; C^1_{00}(I\times Y_0;\mathbb{R}^3))$. The arbitrary choice of $\boldsymbol{\zeta}$ and a density argument imply

$$\int_I\int_{Y_1} \mathbb{C}_1(y)\left[\iota\left(\nabla_{\hat{x}}\mathfrak{a} - x_3\nabla^2_{\hat{x}}\mathfrak{b}\right) + C(\hat{x},\cdot)\right] : \operatorname{sym}\widetilde{\nabla}_\delta\boldsymbol{\zeta}\, dy dx_3 = 0 \qquad \text{a.e.}\, \hat{x}\in\omega,$$

from which the effective tensor $\mathbb{C}^{\rm hom}_\delta$ is then obtained. Another density argument and Proposition 2.2.5 now provide the validity of the equations (2.9). The uniqueness of the solution to (2.9) follows from Lax-Milgram and Proposition 2.2.4, while the last claim follows by energy considerations or by duality arguments [70, Proposition 2.8], see also the proof of Theorem 2.2.33. ∎

Remark 2.3.1. It is not difficult to incorporate surface loads into the statement of Theorem 2.2.7. Namely, if one adds to the right-hand side of (2.6) the term

$$\int_{\omega\times\{-1/2,1/2\}} \boldsymbol{g}^{\varepsilon_h}\boldsymbol{\theta}\, d\hat{x}, \qquad \boldsymbol{\theta}\in H^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^3),$$

where $\boldsymbol{g}^{\varepsilon_h} \in L^2(\omega\times\{-1/2,1/2\};\mathbb{R}^3)$ and the integral over $\omega\times\{-1/2,1/2\}$ represents a sum of two integrals over ω, and assumes that

$$\pi_h\boldsymbol{g}^{\varepsilon_h} \xrightarrow{\text{t,dr}-2} \boldsymbol{g} \in L^2(\omega\times\{-1/2,1/2\}\times Y;\mathbb{R}^3),$$

then using the proof of Theorem 2.2.7 and Remark 4.4.5, one concludes that the limit equations (2.9) have an additional term

$$\int_{\omega\times\{-1/2,1/2\}} \langle\boldsymbol{g}\rangle(\hat{x})\cdot\boldsymbol{\theta}(\hat{x})\, d\hat{x} - \int_\omega \left(\langle\boldsymbol{g}_*(\hat{x},1/2,y)\rangle - \langle\boldsymbol{g}_*(\hat{x},-1/2,y)\rangle\right)\cdot\nabla_{\hat{x}}\boldsymbol{\theta}_3(\hat{x})\, d\hat{x},$$

in the first equation and

$$\int_{Y_0} \boldsymbol{g}(\hat{x},-1/2,y)\cdot\left(\mathring{\xi}_1(-1/2,y),\mathring{\xi}_2(-1/2,y),0\right)^\top + \int_{Y_0} \boldsymbol{g}(\hat{x},1/2,y)\cdot\left(\mathring{\xi}_1(1/2,y),\mathring{\xi}_2(1/2,y),0\right)^\top dy,$$

in the second equation.

C. Proof of Proposition 2.2.9 and Corollary 2.2.11

Proof. The proof partially follows the proof of Proposition 2.2.5. Part 1 is obtained immediately by plugging $\boldsymbol{v}^{\varepsilon_h} = \boldsymbol{u}^{\varepsilon_h}$ in (2.6).

Proceeding to part 2, we perform an extension procedure similar to that undertaken in the proof of Proposition 2.2.5. Using Theorem 4.4.1, we define $\tilde{\boldsymbol{u}}^{\varepsilon_h}$ as the extension of $\boldsymbol{u}^{\varepsilon_h}|_{\Omega_1^{\varepsilon_h}}$ to the whole domain Ω and then set

$$\mathring{\boldsymbol{u}}^{\varepsilon_h} := \tilde{\boldsymbol{u}}^{\varepsilon_h} - \boldsymbol{u}^{\varepsilon_h}.$$

Theorem 4.4.1 and Lemma 4.4.4 now imply the estimates (2.69).

Next, we characterise the behaviour of the sequence $\tilde{\boldsymbol{u}}^{\varepsilon_h}$. To this end, notice that Lemma 4.2.8 yields the following decomposition of the sequence $\tilde{\boldsymbol{u}}^{\varepsilon_h}$:

$$\tilde{\boldsymbol{u}}^{\varepsilon_h}(x) = \begin{pmatrix} -x_3\partial_1\tilde{\mathfrak{b}} \\ -x_3\partial_2\tilde{\mathfrak{b}} \\ h^{-1}\tilde{\mathfrak{b}} \end{pmatrix} + \begin{pmatrix} \mathfrak{a}_1 \\ \mathfrak{a}_2 \\ 0 \end{pmatrix} + \boldsymbol{\psi}^{\varepsilon_h}, \quad \operatorname{sym}\nabla_h\tilde{\boldsymbol{u}}^{\varepsilon_h} = \iota\left(-x_3\nabla_{\hat{x}}\tilde{\mathfrak{b}} + \operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}\right) + \operatorname{sym}\nabla_h\boldsymbol{\psi}^{\varepsilon_h},$$

where $\tilde{\mathfrak{b}} \in H^2_{\gamma_{\rm D}}(\omega)$, $\mathfrak{a} \in H^1_{\gamma_{\rm D}}(\omega;\mathbb{R}^2)$, $(\boldsymbol{\psi}^{\varepsilon_h})_{h>0} \subset H^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^3)$, and $h\pi_{1/h}\boldsymbol{\psi}^{\varepsilon_h} \longrightarrow 0$ in L^2. Since $u_3^{\varepsilon_h}$, and hence $\tilde{u}_3^{\varepsilon_h}$ as well, is bounded in $L^2(\Omega;\mathbb{R}^3)$ (see Lemma 4.4.4), we infer that

$$\tilde{\mathfrak{b}} = h\tilde{u}_3^{\varepsilon_h} - h\psi_3^{\varepsilon_h} \xrightarrow{L^2} 0,$$

so consequently $\tilde{\mathfrak{b}} = 0$. By Theorem 4.2.3, we can decompose the third component as $\tilde{u}_3^{\varepsilon_h} = \hat{\psi}_3^{\varepsilon_h} + \bar{\psi}_3^{\varepsilon_h}$, where $\hat{\psi}_3^{\varepsilon_h} = \int_I \tilde{u}_3^{\varepsilon_h}$ and $\|\bar{\psi}_3^{\varepsilon_h}\|_{L^2(\Omega)} \leq Ch$. Thus, by two-scale compactness, we conclude that there exists $\mathfrak{b} \in L^2(\omega\times Y;\mathbb{R}^3)$ such that

$$\tilde{u}_3^{\varepsilon_h}(x) = \psi_3^{\varepsilon_h} \xrightarrow{dr-2} \mathfrak{b}(\hat{x},y).$$

Furthermore, by invoking Remark 4.2.7 and applying Lemma 4.3.2 (1), we note that $\mathfrak{b}(\hat{x},y) = \mathfrak{b}(\hat{x})$. The rest of the proof is analogous to that of Proposition 2.2.5.

To prove Corollary 2.2.11, we invoke Remark 4.2.6, Remark 4.4.3, as well as the symmetries of the solution due to the assumption concerning the symmetries of the elasticity tensor. ■

D. Proof of Theorem 2.2.10

Proof. We begin by plugging in (2.6) test functions of the form

$$\boldsymbol{v}^{\varepsilon_h}(x) = \begin{pmatrix} \theta_1(\hat{x}) - hx_3\partial_1\theta_3(\hat{x})) \\ \theta_2(\hat{x}) - hx_3\partial_2\theta_3(\hat{x}) \\ \theta_3(\hat{x}) \end{pmatrix} + \varepsilon_h \boldsymbol{\zeta}\left(x, \frac{\hat{x}}{\varepsilon_h}\right) + \mathring{\boldsymbol{\xi}}\left(x, \frac{\hat{x}}{\varepsilon_h}\right),$$

where $\boldsymbol{\theta}_* \in C^1_{\mathrm{c}}(\omega;\mathbb{R}^2)$, $\theta_3 \in C^2_{\mathrm{c}}(\omega)$, $\boldsymbol{\zeta} \in C^1_{\mathrm{c}}(\Omega; C^1(I\times\mathcal{Y};\mathbb{R}^3))$, $\mathring{\boldsymbol{\xi}} \in C^1_{\mathrm{c}}(\omega; C^1_{00}(I\times Y_0;\mathbb{R}^3))$, and using the compactness result from Proposition 2.2.9. The rest of the argument follows the proof of Theorem 2.2.7. ∎

E. Proof of Proposition 2.2.13 and Theorem 2.2.14

Proof. To obtain part 1 of Proposition 2.2.13, we plug $\boldsymbol{u}^{\varepsilon_h}$ in (2.6). The rest of the proof of Proposition 2.2.13 and the proof of Theorem 2.2.14 follow the steps of the proofs of Proposition 2.2.5 and Theorem 2.2.7, respectively. ∎

F. Proof of Proposition 2.2.17

Proof. To prove part 1, we first plug in $\boldsymbol{v}^{\varepsilon_h} = \boldsymbol{u}^{\varepsilon_h}$ in (2.6). Next, using Theorem 4.4.6, Corollary 4.2.5, and Remark 4.4.8, we obtain the following a priori bounds:

$$\boldsymbol{u}^{\varepsilon_h} = \tilde{\boldsymbol{u}}^{\varepsilon_h} + \mathring{\boldsymbol{u}}^{\varepsilon_h}, \quad \tilde{\boldsymbol{u}}^{\varepsilon_h} = E^{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h},$$

$$\left\|\operatorname{sym}\nabla_h \tilde{\boldsymbol{u}}^{\varepsilon_h}\right\|_{L^2} + h^2\left\|\pi_{1/h}\tilde{\boldsymbol{u}}^{\varepsilon_h}\right\|^2_{H^1} + \|\tilde{\boldsymbol{u}}^{\varepsilon_h}\|^2_{L^2} \le C\left(a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|^2_{L^2}\right),$$

$$\mathring{\boldsymbol{u}}^{\varepsilon_h} = \begin{pmatrix} -\varepsilon_h x_3\partial_1 \mathring{v}^{\varepsilon_h} \\ -\varepsilon_h x_3\partial_2 \mathring{v}^{\varepsilon_h} \\ h^{-1}\varepsilon_h \mathring{v}^{\varepsilon_h} \end{pmatrix} + \mathring{\boldsymbol{\psi}}^{\varepsilon_h},$$

$$\|\mathring{v}^{\varepsilon_h}\|^2_{L^2} + \varepsilon_h^2\left\|\nabla \mathring{v}^{\varepsilon_h}\right\|^2_{L^2} + \varepsilon_h^4\left\|\nabla^2 \mathring{v}^{\varepsilon_h}\right\|^2_{L^2} + \left\|\mathring{\boldsymbol{\psi}}^{\varepsilon_h}\right\|^2_{L^2} + \varepsilon_h^2\|\nabla_h\mathring{\boldsymbol{\psi}}^{\varepsilon_h}\|^2_{L^2} \le C\varepsilon_h^2\|\operatorname{sym}\nabla_h\mathring{\boldsymbol{u}}^{\varepsilon_h}\|^2_{L^2} \le Ca_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h}),$$

$$h^{-1}\varepsilon_h\|\mathring{v}^{\varepsilon_h}\|^2_{L^2} \le C\left(a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h}) + \|\boldsymbol{u}^{\varepsilon_h}\|_{L^2}\right),$$

where $\mathring{v}^{\varepsilon_h} \in H^2(\omega)$, $\mathring{\boldsymbol{\psi}} \in H^1(\Omega;\mathbb{R}^3)$, $\mathring{v}^{\varepsilon_h} = \mathring{\boldsymbol{\psi}} = 0$ on $\Omega_1^{\varepsilon_h}$.

Proceeding to part 2, we note that the first convergence in (2.22) follows directly from Theorem 4.2.3 and Remark 4.2.7. To prove the remaining convergence statements, by analogy with the argument of Proposition 2.2.5 we first assume that ω has $C^{1,1}$ boundary. Using Lemma 4.2.8 and Lemma 4.2.10, 3 we have

$$\tilde{u}_3^{\varepsilon_h} = h^{-1}\varphi^{\varepsilon_h} + w^{\varepsilon_h} + \tilde{\psi}_3^{\varepsilon_h},$$

where $(\varphi^{\varepsilon_h})_{h>0}$ is bounded in $H^2(\omega)$, $(w^{\varepsilon_h})_{h>0}$ is bounded in $H^1(\omega)$ and $\tilde{\psi}_3^{\varepsilon_h} \xrightarrow{L^2} 0$. Since $h^{-1}\varphi^{\varepsilon_h}$ is bounded in L^2, the first part of the second convergence in (2.22) follows from Lemma 4.3.2 (2). Furthermore, the first part of the third convergence in (2.22) follows from Remark 4.2.7, Remark 4.2.9, Theorem 4.3.1 (2) and Lemma 4.3.3 (1) (in addition to the more standard Lemma 4.2.8 and Lemma 4.2.10 (3).) The second and third parts of the third convergence statement in (2.22) need to be additionally combined with the second convergence statement in (2.22) through Lemma 4.3.3 (2). Finally, the fourth and fifth convergence in (2.22) follow from Lemma 4.3.3 (3) and Lemma 4.3.4 (1) by noticing that $\mathring{v}^{\varepsilon_h} \longrightarrow 0$ in L^2 as a consequence of the fact $h \ll \varepsilon_h$. This concludes the proof of part 2 for the case when ω has $C^{1,1}$ boundary. For the general case of ω with Lipschitz boundary, we now use Lemma 4.2.10 (3), Theorem 4.3.1 (2) and Lemma 4.3.3 (1) in combination with the approach of the proof of Proposition 2.2.5.

The argument for part 3 is analogous to that for Proposition 2.2.5. ■

G. Proof of Theorem 2.2.19

Proof. The proof is carried out by taking appropriate vest functions $\boldsymbol{v} = \boldsymbol{v}^{\varepsilon_h}$ in 2.6 and then passing to the limit as $h \to 0$, for which we invoke a combination of Proposition 2.2.17, Remark 2.2.2, and a density argument.

Different equations in (2.24) are obtained by using different kinds of test functions. For the first equation, we use test functions of the form

$$\begin{aligned}\boldsymbol{v}^{\varepsilon_h}(\hat{x}) = {} & \big(\theta_1(\hat{x}), \theta_2(\hat{x}), 0\big)^\top + \varepsilon_h \left(\zeta_1\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), \zeta_2\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), 0\right)^\top \\ & + \varepsilon_h \left(-x_3\partial_{y_1}\psi\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), -x_3\partial_{y_2}\psi\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), \frac{1}{h}\psi\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right)\right) \\ & + h\int_0^{x_3} \boldsymbol{r}\left(x, \frac{\hat{x}}{\varepsilon_h}\right) dx_3,\end{aligned}$$

where $\boldsymbol{\theta} \in C_c^1(\omega;\mathbb{R}^2)$, $\boldsymbol{\zeta} \in C_c^1(\omega; C^2(\mathcal{Y};\mathbb{R}^2))$, $\psi \in C_c^1(\omega; C^1(\mathcal{Y}))$, $\boldsymbol{r} \in C_c^1(\Omega; C^1(\mathcal{Y};\mathbb{R}^3))$. Next, for the second equation we use test function of the form

$$v^{\varepsilon_h}(x) = \left(\mathring{\boldsymbol{\xi}}_1\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), \mathring{\boldsymbol{\xi}}_2\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), 0\right)^\top + \frac{h}{\varepsilon_h}\int_0^{x_3} \mathring{\boldsymbol{r}}\left(x, \frac{\hat{x}}{\varepsilon_h}\right) dx_3,$$

where $\mathring{\boldsymbol{\xi}} \in C^1_{\mathrm{c}}(\omega; C^1_{\mathrm{c}}(Y_0;\mathbb{R}^2))$, $\mathring{\boldsymbol{r}} \in C^1_{\mathrm{c}}(\Omega; C^1_{\mathrm{c}}(Y_0;\mathbb{R}^3))$. Further, for the third equation ($\kappa = \infty$) we use test equation of the form

$$\boldsymbol{v}^{\varepsilon_h}(x) = (-hx_3\partial_1\theta(\hat{x}), -hx_3\partial_2\theta(\hat{x}), \theta(\hat{x}))^\top + \left(-\frac{h}{\varepsilon_h}x_3\partial_{y_1}\mathring{\xi}\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), -\frac{h}{\varepsilon_h}x_3\partial_{y_1}\mathring{\xi}\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), \mathring{\xi}\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right)\right)^\top,$$

where $\theta \in C^2_{\mathrm{c}}(\omega)$, $\mathring{\xi} \in C^1_{\mathrm{c}}(\omega; C^2_{\mathrm{c}}(Y_0))$. For the fourth equation ($\kappa \in (0,\infty)$) we use test functions of the form

$$\boldsymbol{v}^{\varepsilon_h}(x) = \left(-\frac{h}{\varepsilon_h}x_3\partial_{y_1}v\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), -\frac{h}{\varepsilon_h}x_3\partial_{y_2}v\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), v\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right)\right)^\top,$$

where $v \in C^1_{\mathrm{c}}(\omega, C^2(\mathcal{Y}))$. For the fifth equation ($\kappa \in (0,\infty)$) we use test functions of the form

$$\boldsymbol{v}^{\varepsilon_h}(x) = \left(-\frac{h}{\varepsilon_h}x_3\partial_{y_1}\mathring{\xi}\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), -\frac{h}{\varepsilon_h}x_3\partial_{y_1}\mathring{\xi}\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), \mathring{\xi}\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right)\right)^\top,$$

$\mathring{\xi} \in C^1_{\mathrm{c}}(\omega, C^2_{\mathrm{c}}(Y_0))$. Finally, for the sixth equation ($\kappa = 0$) we use test functions of the form

$$\boldsymbol{v}^{\varepsilon_h}(x) = \left(-\frac{h}{\varepsilon_h}x_3\partial_{y_1}v\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), -\frac{h}{\varepsilon_h}x_3\partial_{y_1}v\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), v\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right)\right)^\top + \left(-\frac{h}{\varepsilon_h}x_3\partial_{y_1}\mathring{\xi}\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), -\frac{h}{\varepsilon_h}x_3\partial_{y_1}\mathring{\xi}\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right), \mathring{\xi}\left(\hat{x}, \frac{\hat{x}}{\varepsilon_h}\right)\right)^\top,$$

where $v \in C^1_{\mathrm{c}}(\omega; C^2(\mathcal{Y}))$, $\mathring{\xi} \in C^1_{\mathrm{c}}(\omega; C^2_{\mathrm{c}}(Y_0))$. The proof of the remaining claims follow an analogous part of the proof of Theorem 2.2.7. ∎

H. Proof of Corollary 2.2.20

Proof. The proof follows easily from Remark 4.2.6, Remark 4.2.11, and Remark 4.4.9. ∎

I. Proof of Proposition 2.2.21

Proof. Part 1 follows easily from Theorem 4.4.6 (in particular, (4.16)–(4.18)) and Corollary 4.2.5, after plugging $\boldsymbol{v}^{\varepsilon_h} = \boldsymbol{u}^{\varepsilon_h}$ into (2.6). To justify the scaling, notice that as a consequence of the above mentioned statements we have

$$\|\pi_{\varepsilon_h/h}\boldsymbol{u}^{\varepsilon_h}\|_{L^2} \leq Ch^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h}, \boldsymbol{u}^{\varepsilon_h}),$$

see also the last expression in (2.71) below. To prove part 2 we use Theorem 4.4.6 and Corollary 4.2.5 again and obtain

$$\boldsymbol{u}^{\varepsilon_h} = \tilde{\boldsymbol{u}}^{\varepsilon_h} + \mathring{\boldsymbol{u}}^{\varepsilon_h}, \quad \tilde{\boldsymbol{u}}^{\varepsilon_h} = E^{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h},$$

$$\left\|\pi_{1/h}\tilde{\boldsymbol{u}}^{\varepsilon_h}\right\|^2_{H^1(\omega;\mathbb{R}^3)} \leq Ch^{-2}\left\|\operatorname{sym}\tilde{\boldsymbol{u}}^{\varepsilon_h}\right\|^2_{L^2(\Omega;\mathbb{R}^3)} \leq Ch^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h}),$$

$$\mathring{\boldsymbol{u}}^{\varepsilon_h} = \begin{pmatrix} -\varepsilon_h x_3\partial_1\mathring{v}^{\varepsilon_h} \\ -\varepsilon_h x_3\partial_2\mathring{v}^{\varepsilon_h} \\ h^{-1}\varepsilon_h\mathring{v}^{\varepsilon_h}\end{pmatrix} + \mathring{\boldsymbol{\psi}}^{\varepsilon_h},$$

$$\begin{aligned}\left\|h^{-1}\varepsilon_h\mathring{v}^{\varepsilon_h}\right\|^2_{L^2(\omega)} + \varepsilon_h^2\left\|h^{-1}\varepsilon_h\nabla\mathring{v}^{\varepsilon_h}\right\|^2_{L^2(\omega;\mathbb{R}^2)} + \varepsilon_h^4\left\|h^{-1}\varepsilon_h\nabla^2\mathring{v}^{\varepsilon_h}\right\|^2_{L^2(\omega;\mathbb{R}^2)} + \left\|h^{-1}\varepsilon_h\mathring{\boldsymbol{\psi}}^{\varepsilon_h}\right\|^2_{L^2(\Omega;\mathbb{R}^3)} \\ + \varepsilon_h^2\left\|h^{-1}\varepsilon_h\nabla_h\mathring{\boldsymbol{\psi}}^{\varepsilon_h}\right\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq Ch^{-2}\varepsilon_h^4\left\|\operatorname{sym}\nabla_h\mathring{\boldsymbol{u}}^{\varepsilon_h}\right\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq Ch^{-2}a_{\varepsilon_h}(\boldsymbol{u}^{\varepsilon_h},\boldsymbol{u}^{\varepsilon_h}),\end{aligned} \tag{2.71}$$

where $\mathring{v}^{\varepsilon_h} \in H^2(\omega)$, $\mathring{\boldsymbol{\psi}} \in H^1(\Omega;\mathbb{R}^3)$, $\mathring{v}^{\varepsilon_h} = \mathring{\boldsymbol{\psi}} = 0$ on $\Omega_1^{\varepsilon_h}$. Assuming first that ω has $C^{1,1}$ boundary, part 2 follows by using Lemma 4.2.8, Lemma 4.2.10 (3), Theorem 4.3.1 (2), Lemma 4.3.3 (1,3), Lemma 4.3.4 (1), and Theorem 4.4.6. For general Lipschitz domains we follow the approach of Proposition 2.2.5 and Proposition 2.2.17. Finally, part 3 is obtained the same way as part 3 of Proposition 2.2.5. ■

J. Proof of Theorem 2.2.22

Proof. Proof follows the approach of the proof of Theorem 2.2.7, by using Proposition 2.2.21, Remark 2.2.2, and test functions of the form

$$\begin{aligned}\boldsymbol{v}^{\varepsilon_h}(x) = &\left(h\theta_1(\hat{x}) - hx_3\partial_1\theta_3(\hat{x}), h\theta_2(\hat{x}) - hx_3\partial_2\theta_3(\hat{x}), \boldsymbol{\theta}_3(\hat{x})\right)^\top + h\varepsilon_h\left(\zeta_1\left(\hat{x},\frac{\hat{x}}{\varepsilon_h}\right),\zeta_2\left(\hat{x},\frac{\hat{x}}{\varepsilon_h}\right),0\right)^\top \\ &+\varepsilon_h\left(-hx_3\partial_{y_1}\psi\left(\hat{x},\frac{\hat{x}}{\varepsilon_h}\right),-hx_3\partial_{y_2}\psi\left(\hat{x},\frac{\hat{x}}{\varepsilon_h}\right),\psi\left(\hat{x},\frac{\hat{x}}{\varepsilon_h}\right)\right)^\top + h^2\int_0^{x_3}\boldsymbol{r}\left(x,\frac{\hat{x}}{\varepsilon_h}\right)dx_3 \\ &+\left(\frac{h}{\varepsilon_h}\left(\mathring{\xi}_1 - x_3\partial_{y_1}\mathring{\xi}_3\left(\hat{x},\frac{\hat{x}}{\varepsilon_h}\right)\right),\frac{h}{\varepsilon_h}\left(\mathring{\xi}_2 - x_3\partial_{y_2}\mathring{\xi}_3\left(\hat{x},\frac{\hat{x}}{\varepsilon_h}\right)\right),\mathring{\boldsymbol{\xi}}_3\left(\hat{x},\frac{\hat{x}}{\varepsilon_h}\right)\right)^\top \\ &+\frac{h}{\varepsilon_h}\int_0^{x_3}\mathring{\boldsymbol{r}}\left(x,\frac{\hat{x}}{\varepsilon_h}\right)dx_3,\end{aligned}$$

where $\boldsymbol{\theta}_* \in C_c^1(\omega;\mathbb{R}^2)$, $\boldsymbol{\theta}_3 \in C_c^2(\omega)$, $\boldsymbol{\zeta} \in C_c^1(\omega;C^1(\mathcal{Y};\mathbb{R}^2))$, $\psi \in C_c^1(\omega;C^1(\mathcal{Y}))$, $\boldsymbol{r} \in C_c^1(\omega;C^1(\mathcal{Y};\mathbb{R}^3))$, $\mathring{\boldsymbol{\xi}}_* \in C_c^1(\omega;C_c^1(Y_0;\mathbb{R}^2))$, $\mathring{\boldsymbol{\xi}}_3 \in C_c^1(\omega;C_c^2(Y_0))$, $\mathring{\boldsymbol{r}} \in C_c^1(\Omega;C_c^1(Y_0;\mathbb{R}^3))$. ■

K. Proof of Proposition 2.2.23 and Corollary 2.2.25

Proof. The proof proceeds in the same way as the proofs of Proposition 2.2.9 or Proposition 2.2.5 by invoking additionally Theorem 4.3.1 (3), Theorem 4.4.10, and Lemma 4.3.4 (2). In order to conclude the form of C_∞ from Lemma 4.2.8, Lemma 4.2.10 (3), and Theorem 4.3.1 (3), it is also important to see that the following simple identity holds:

$$x_3 \nabla_y^2 \varphi(\hat{x}, y) = \nabla_y (x_3 \partial_{y_1} \varphi, x_3 \partial_{y_2} \varphi, 0)^\top, \quad \forall \varphi \in L^2(\omega; H^2(\mathcal{Y})).$$

The proof of Corollary 2.2.25 uses Remark 4.2.6, Remark 4.4.11, and symmetries of the solution, as a consequence of the assumption on symmetries of the elasticity tensor. ■

L. Proof of Theorem 2.2.24

Proof. The proof is similar to the proof of Theorem 2.2.10, by invoking Remark 2.2.3 and plugging in (2.6) test functions of the form

$$\boldsymbol{v}^{\varepsilon_h}(x) = \begin{pmatrix} \theta_1(\hat{x}) - h x_3 \partial_1 \theta_3(\hat{x}) \\ \theta_2(\hat{x}) - h x_3 \partial_1 \theta_3(\hat{x}) \\ \theta_3(\hat{x}) \end{pmatrix} + \varepsilon_h \boldsymbol{\zeta}\left(x, \frac{\hat{x}}{\varepsilon_h}\right) + h \int_0^{x_3} \boldsymbol{r}(x)\, dx_3 + \mathring{\boldsymbol{\xi}}\left(x, \frac{\hat{x}}{\varepsilon_h}\right),$$

where $\boldsymbol{\theta} \in C_c^1(\omega; \mathbb{R}^2)$, $\theta_3 \in C_c^2(\omega)$, $\boldsymbol{\zeta} \in C_c^1(\Omega; C^1(I \times \mathcal{Y}; \mathbb{R}^3))$, $\boldsymbol{r} \in C_c^1(\Omega)$, $\mathring{\boldsymbol{\xi}} \in C_c^1(\omega; C_{00}^1(I \times Y_0; \mathbb{R}^3))$. ■

M. Proof of Proposition 2.2.26

Proof. The proof is carried out similarly to the proof of Proposition 2.2.9 and Proposition 2.2.5, where we additionally use Theorem 4.3.1 (3), Theorem 4.4.10, and Lemma 4.3.4 (2). ■

N. Proof of Theorem 2.2.27

Proof. The proof follows the proof of Theorem 2.2.10, by using Remark 2.2.3 and by plugging in (2.6) test functions of the form

$$\boldsymbol{v}^{\varepsilon_h}(x) = \begin{pmatrix} h\theta_1(\hat{x}) - h x_3 \partial_1 \theta_3(\hat{x}) \\ h\theta_2(\hat{x}) - h x_3 \partial_1 \theta_3(\hat{x}) \\ \theta_3(\hat{x}) \end{pmatrix} + h\varepsilon_h \boldsymbol{\zeta}\left(x, \frac{\hat{x}}{\varepsilon_h}\right) + h^2 \int_0^{x_3} \boldsymbol{r}(x)\, dx_3 + \mathring{\boldsymbol{\xi}}\left(x, \frac{\hat{x}}{\varepsilon_h}\right),$$

where $\boldsymbol{\theta} \in C_c^1(\omega; \mathbb{R}^2)$, $\boldsymbol{\theta}_3 \in C_c^2(\omega)$, $\boldsymbol{\zeta} \in C_c^1(\Omega; C^1(I \times \mathcal{Y}; \mathbb{R}^3))$, $\boldsymbol{r} \in C_c^1(\Omega)$, $\mathring{\boldsymbol{\xi}} \in C_c^1(\omega; C_{00}^1(I \times Y_0; \mathbb{R}^3))$. ■

2.3.3. Proofs for Section 2.2.3

A. Proof of Theorem 2.2.30

Proof. It is easy to see from Proposition 2.2.4 that the operator $\mathcal{A}_\delta^{\mathfrak{b},\mathrm{hom}}$ is positive definite, coercive, and has compact inverse. This, in particular, allows one to obtain immediately a characterization of its spectrum, which we omit.

From Proposition 2.2.5 and Theorem 2.2.7 we infer that if $\boldsymbol{f}^{\varepsilon_h} \longrightarrow \boldsymbol{f}$ in L^2, then the sequence of solutions $(\boldsymbol{u}^{\varepsilon_h})_{h>0}$ of (2.6) for $\lambda = 1$ satisfies $\boldsymbol{u}^{\varepsilon_h} \to (0,0,\mathfrak{b})^\top$, where $\mathfrak{b} \in H^2_{\gamma_{\mathrm{D}}}(\omega)$ solves

$$\left(\mathcal{A}_\delta^{\mathfrak{b},\mathrm{hom}} + \mathcal{I}\right)\mathfrak{b} = \langle\rho\rangle^{-1}\boldsymbol{f}_3.$$

Using the proof of [70, Proposition 2.2], we show that the property (H_1) in Definition 2.2.1 holds. To prove the property (H_2), we take a sequence λ^{ε_h} of eigenvalues of the operator $h^{-2}\mathcal{A}_{\varepsilon_h}$ converging to $\lambda > 0$. Next, consider the sequence $(\boldsymbol{u}^{\varepsilon_h})_{h>0}$ of the corresponding eigenfunctions

$$h^{-2}\mathcal{A}_{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h} = \lambda^{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h}, \quad \|\boldsymbol{u}^{\varepsilon_h}\|_{L^2} = 1.$$

Multiplying the above equation by $\boldsymbol{u}^{\varepsilon_h}$, using the compactness result from Proposition 2.2.5, and invoking an argument similar to that of Theorem 2.2.7, we conclude that $\boldsymbol{u}^{\varepsilon_h} \longrightarrow (0,0,\mathfrak{b})^\top$ in L^2, where $\mathfrak{b} \in H^2_{\gamma_{\mathrm{D}}}(\omega)$ solves

$$\mathcal{A}_\delta^{\mathfrak{b},\mathrm{hom}}\mathfrak{b} = \lambda\mathfrak{b}, \quad \|\mathfrak{b}\|_{L^2} = 1,$$

which completes the proof of (H_2). This also proves the convergence of eigenfunctions.

To prove a refined version of the Hausdorff convergence concerning the convergence of eigenvalues ordered in the increasing order, we take an arbitrary closed curve $\Gamma \subset \mathbb{C}$, intersecting an interval in $(0,\infty)$ and not passing through any of the eigenvalues $\lambda_{\delta,n}$ and define the following projection operators:

$$P_\Gamma^{\varepsilon_h} = -\frac{1}{2\pi \mathrm{i}}\oint_\Gamma \left(\frac{1}{h^2}\mathcal{A}_{\varepsilon_h} - z\mathcal{I}\right)^{-1} dz, \quad P_\Gamma = -\frac{1}{2\pi \mathrm{i}}\oint_\Gamma \left(\mathcal{A}_\delta^{\mathfrak{b},\mathrm{hom}} - z\mathcal{I}\right)^{-1} dz.$$

We claim that for small enough $\varepsilon_h > 0$ the dimensions of the ranges $\mathcal{R}(P_\Gamma^{\varepsilon_h})$ and $\mathcal{R}(P_\Gamma)$ coincide. (Note that they are finite by the compactness of the resolvent.) Indeed, from the compactness result in Proposition 2.2.5 and Lebesgue theorem on dominated convergence it follows that if $(\boldsymbol{f}^{\varepsilon_h})_{h>0} \subset L^2(\Omega;\mathbb{R}^3)$, $(\boldsymbol{v}^{\varepsilon_h})_{h>0} \subset L^2(\Omega;\mathbb{R}^3)$ are such that $\boldsymbol{f}^{\varepsilon_h} \rightharpoonup \boldsymbol{f}$, $\boldsymbol{v}^{\varepsilon_h} \rightharpoonup \boldsymbol{v}$ weakly in L^2, then one has

$$\left(P_\Gamma^{\varepsilon_h}\boldsymbol{f}^{\varepsilon_h}, \boldsymbol{v}^{\varepsilon_h}\right) \to (P_\Gamma \boldsymbol{f}, \boldsymbol{v}).$$

It follows that $P_\Gamma^{\varepsilon_h}\boldsymbol{f}^{\varepsilon_h} \longrightarrow P_\Gamma \boldsymbol{f}$ in L^2. This immediately implies that the dimensions of $\mathcal{R}(P_\Gamma^{\varepsilon_h})$ and $\mathcal{R}(P_\Gamma)$ coincide for sufficiently small ε_h.

Next, fix a closed curve $\Gamma_{\delta,n} \subset \mathbb{C}$ containing in its interior the eigenvalue $\lambda_{\delta,n}$ and no other eigenvalues, intersecting the real line at w_1 and w_2 such that $\lambda_{\delta,n-1} < w_1 < \lambda_{\delta,n} < w_2 < \lambda_{\delta,n+1}$, where we set $\lambda_{\delta,0} = 0$. The multiplicity $k_{\delta,n}$ of this eigenvalue equals $\dim \mathcal{R}(P_{\Gamma_{\delta,n}})$. By using the above claim, we know that for small enough ε_h exactly $k_{\delta,n}$ eigenvalues of $h^{-2}\mathcal{A}_{\varepsilon_h}$ (including their multiplicities) are contained in the interval (w_1, w_2). ■

Before giving the rest of the proofs we will state and prove one helpful lemma:

Lemma 2.3.2. *1. If $\mu_h = \varepsilon_h$, one has*

$$\lim_{h\to 0} \sigma(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h}) \subset \lim_{h\to 0} \sigma(\tilde{\mathcal{A}}_{\varepsilon_h}).$$

2. If $\delta \in (0,\infty]$, $\mu_h = \varepsilon_h h$, one has

$$\lim_{h\to 0} \sigma(\mathring{\mathcal{A}}_{\varepsilon_h}) \subset \lim_{h\to 0} h^{-2}\sigma(\mathcal{A}_{\varepsilon_h}).$$

3. If $\delta = 0$, $\mu_h = \varepsilon_h^2$, one has

$$\lim_{h\to 0} h^{-2}\varepsilon_h^2\sigma(\mathring{\mathcal{A}}_{\varepsilon_h}) \subset \lim_{h\to 0} h^{-2}\sigma(\mathcal{A}_{\varepsilon_h}).$$

Proof. We prove part 1 for the case $\delta \in (0,\infty)$ only, as the cases $\delta = 0$ and $\delta = \infty$ are dealt with by similar arguments. We take $\lambda^{\varepsilon_h} \in \sigma(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h})$ such that $\lambda^{\varepsilon_h} \to \lambda$ and $\mathring{\boldsymbol{u}}_r^{\varepsilon_h} \in H^1_{00}(I \times Y_0;\mathbb{R}^3)$ such that $\|\mathring{\boldsymbol{u}}_r^{\varepsilon_h}\|_{L^2} = 1$ and $\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h}\mathring{\boldsymbol{u}}_r^{\varepsilon_h} = \lambda^{\varepsilon_h}\mathring{\boldsymbol{u}}_r^{\varepsilon_h}$. The convergence properties of λ^{ε_h} and $\mathring{\boldsymbol{u}}_r^{\varepsilon_h}$ immediately imply that the sequence

$$\left(\|\operatorname{sym}\nabla_{\frac{h}{\varepsilon_h}}\mathring{\boldsymbol{u}}_r^{\varepsilon_h}\|_{L^2}\right)_{h>0}$$

is bounded.[5] For each h we take a cube $Q^h = q^h \times I$ such that $q^h \subset \omega$ has vertices in $\varepsilon_h\mathbb{Z}^2$ and side length $2n^h\varepsilon_h$, where n^h is an integer. Furthermore, we assume that $n^h\varepsilon_h$

[5] Actually it can be concluded that the sequence $(\|\mathring{\boldsymbol{u}}_r^{\varepsilon_h}\|_{H^1})_{h>0}$ is bounded, see [20, Section 7].

converge to some positive number as $h \to 0$. We define $\boldsymbol{u}^{\varepsilon_h}$ as follows. Consider $z \in \mathbb{Z}^2$ such that the cube of size ε_h whose left corner is at $\varepsilon_h z$ is contained in q^h. On the inclusion $\varepsilon_h(Y_0 + z) \times I$, we set $\boldsymbol{u}^{\varepsilon_h}$ to be equal to $\mathring{\boldsymbol{u}}_r^{\varepsilon_h}(\hat{x}/\varepsilon_h - z, x_3)$ if z_1 (the first coordinate of z) is even and to $-\mathring{\boldsymbol{u}}_r^{\varepsilon_h}(\hat{x}/\varepsilon_h - z, x_3)$ if z_1 is odd. We then extend $\boldsymbol{u}^{\varepsilon_h}$ by zero outside $\varepsilon_h(Y_0+z) \times I$. This procedure is repeated for all $z \in \mathbb{Z}^2$ with the above property, and finally $\boldsymbol{u}^{\varepsilon_h}$ is set to zero on $\Omega \setminus Q^h$.

It can be easily checked that for $\boldsymbol{\xi} \in H^1_{\gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3) \cap L^{2,\mathrm{memb}}(\Omega;\mathbb{R}^3)$ one has

$$\int_\Omega \mathbb{C}^{\mu_h}\left(\frac{\hat{x}}{\varepsilon_h}\right) \operatorname{sym}\nabla_h \boldsymbol{u}^{\varepsilon_h}(x) : \operatorname{sym}\nabla_h \boldsymbol{\xi}(x)\, dx - \lambda^{\varepsilon_h} \int_\Omega \rho \boldsymbol{u}^{\varepsilon_h} \cdot \boldsymbol{\xi} \tag{2.72}$$
$$= \int_{Q^h \cap \Omega_0^{\varepsilon_h}} \mathbb{C}^{\mu_h}\left(\frac{\hat{x}}{\varepsilon_h}\right) \operatorname{sym}\nabla_h \boldsymbol{u}^{\varepsilon_h}(x) : \operatorname{sym}\nabla_h \tilde{\boldsymbol{\xi}}(x)\, dx - \lambda^{\varepsilon_h} \int_{Q^h \cap \Omega_0^{\varepsilon_h}} \rho_0 \boldsymbol{u}^{\varepsilon_h} \cdot \tilde{\boldsymbol{\xi}}\, dx,$$

where $\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}} + \mathring{\boldsymbol{\xi}}$, with $\tilde{\boldsymbol{\xi}}$ being the extension provided by Theorem 4.4.1. Recall that, as a consequence of Corollary 4.2.5,

$$\|\tilde{\boldsymbol{\xi}}\|_{H^1} \leq C \|\operatorname{sym}\nabla_h \tilde{\boldsymbol{\xi}}\|_{L^2},$$

where $C > 0$ does not depend on h. Using this fact and the definition of $\boldsymbol{u}^{\varepsilon_h}$ (noting that the mean value of $\boldsymbol{u}^{\varepsilon_h}$ is zero on each two neighbouring small cubes of size ε_h in the x_1 direction) it can be easily seen that the right hand side of (2.72) can be written in the form

$$\int_\Omega \boldsymbol{f}_1^{\varepsilon_h} : \operatorname{sym}\nabla_h \tilde{\boldsymbol{\xi}}\, dx + \int_\Omega \boldsymbol{f}_2^{\varepsilon_h} \cdot \tilde{\boldsymbol{\xi}}\, dx,$$

where $\boldsymbol{f}_1^{\varepsilon_h} \in L^2(Q^h;\mathbb{R}^{3\times3})$, $\boldsymbol{f}_2^{\varepsilon_h} \in L^2(Q^h;\mathbb{R}^3)$, and $\|\boldsymbol{f}_1^{\varepsilon_h}\|_{L^2} \to 0$, $\|\boldsymbol{f}_2^{\varepsilon_h}\|_{L^2} \to 0$ as $h \to 0$. To see this, we divide the domain into small rectangles containing two neighbouring cubes, where the first coordinate of the left corner is even and odd respectively, and apply the Poincaré inequality. This yields an estimate for the right-hand side of (2.72) by the expression $C\varepsilon_h(\|\nabla_h \tilde{\boldsymbol{\xi}}\|_{L^2} + \|\tilde{\boldsymbol{\xi}}\|_{L^2})$, where $C > 0$ is h-independent. By using the Riesz representation theorem (applied first on the physical domain and then moved on the canonical domain) and the fact that on Q^h the norm $\|\cdot\|_{L^2} + \|\nabla_h(\cdot)\|_{L^2}$ is equivalent to the norm $\|\cdot\|_{L^2} + \|\operatorname{sym}\nabla_h(\cdot)\|_{L^2}$, we conclude that the right-hand side of (2.72) can be written in the form

$$\varepsilon_h \left(\int_{Q^h} \operatorname{sym}\nabla_h \boldsymbol{r}^{\varepsilon_h}(x) : \operatorname{sym}\nabla_h \tilde{\boldsymbol{\xi}}(x)\, dx + \int_{Q^h} \boldsymbol{r}^{\varepsilon_h} \cdot \tilde{\boldsymbol{\xi}}\, dx \right),$$

where $\|\operatorname{sym}\nabla_h \boldsymbol{r}^{\varepsilon_h}\|_{L^2} + \|\boldsymbol{r}^{\varepsilon_h}\|_{L^2}$ is bounded independently of h. The claim follows by taking $\boldsymbol{f}_1^{\varepsilon_h} = \varepsilon_h \operatorname{sym}\nabla_h \boldsymbol{r}^{\varepsilon_h}$ and $\boldsymbol{f}_2^{\varepsilon_h} = \varepsilon_h \boldsymbol{r}^{\varepsilon_h}$ in (2.3.3).

To conclude the proof of part 1, we note that there exists $C > 0$ such that $\|\boldsymbol{u}^{\varepsilon_h}\|_{L^2} \geq C$ and hence, by applying a suitable version of Lemma 4.6.4 (see also Remark 4.6.5), one has

$$\operatorname{dist}\big(\lambda^{\varepsilon_h}, \sigma(\tilde{\mathcal{A}}_{\varepsilon_h})\big) \to 0 \ \text{ as } \ h \to 0.$$

The proof of part 2 proceeds in a similar way. Part 3 requires an additional explanation while following the same kind of argument. We again take $\lambda^{\varepsilon_h} \in \sigma(\varepsilon_h^2 h^{-2} \mathring{\tilde{\mathcal{A}}}_{\varepsilon_h})$ such that $\lambda^{\varepsilon_h} \to \lambda$ and $\mathring{\boldsymbol{u}}_r^{\varepsilon_h} \in H^1_{00}(I \times Y_0; \mathbb{R}^3)$ such that $\|\mathring{\boldsymbol{u}}_r^{\varepsilon_h}\|_{L^2} = 1$ and $\varepsilon_h^2 h^{-2} \mathring{\tilde{\mathcal{A}}}_{\varepsilon_h} \mathring{\boldsymbol{u}}_r^{\varepsilon_h} = \lambda^{\varepsilon_h} \mathring{\boldsymbol{u}}_r^{\varepsilon_h}$. Using the same argument as in the proof of Theorem 2.2.30 (notice that here the n-th eigenvalue is of order $h^{-2}\varepsilon_h^2$), we infer immediately that $\big(\varepsilon_h h^{-1} \|\operatorname{sym}\nabla_{\frac{h}{\varepsilon_h}} \mathring{\boldsymbol{u}}_r^{\varepsilon_h}\|_{L^2}\big)_{h>0}$ is bounded. Furthermore, invoking Corollary 4.2.5, we obtain

$$\left\| \left(\frac{\varepsilon_h}{h} \mathring{u}^{\varepsilon_h}_{r,1}, \frac{\varepsilon_h}{h} \mathring{u}^{\varepsilon_h}_{r,2}, \mathring{u}^{\varepsilon_h}_{r,3} \right) \right\|_{H^1} \leq C \frac{\varepsilon_h}{h} \left\| \operatorname{sym}\nabla_{\frac{h}{\varepsilon_h}} \mathring{\boldsymbol{u}}_r^{\varepsilon_h} \right\|_{L^2}$$

for some $C > 0$ independent of h. The rest of the proof follows the proof of part 1. ■

Remark 2.3.3. Using a standard approach (resolvent convergence and compactness of eigenfunctions), it can be easily shown that when $\delta \in (0, \infty)$ one has $\lim_{h\to 0} \sigma(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h}) = \sigma(\tilde{\mathcal{A}}_{00,\delta})$ and $\lim_{h\to 0} \sigma(\mathring{\mathcal{A}}_{\varepsilon_h}) = \sigma(\mathcal{A}_{00,\delta})$. To obtain this result one needs to use uniform (in δ) Korn inequality, see e.g. [20, Section 7].

In the case $\delta = 0$ one can prove (similarly to the proof of Theorem 2.2.30) that $\lim_{h\to 0} \sigma(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h}) = \sigma(\tilde{\mathcal{A}}_{00,0})$ and $\lim_{h\to 0} \varepsilon_h^2 h^{-2} \sigma(\mathring{\mathcal{A}}_{\varepsilon_h}) = \sigma(\mathcal{A}_{00,0})$.

The analogous claim is not valid for $\delta = \infty$. This is the main reason why in this regime the limiting spectrum is different than the spectrum of the limit operator. Here, due to the fact that only resolvent convergence (as in Theorem 2.2.24) holds and no compactness of eigenfunctions is available, one only has $\sigma(\tilde{\mathcal{A}}_{00,\infty}) \subset \lim_{h\to 0} \sigma(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h})$, $\sigma(\mathcal{A}_{00,\infty}) \subset \lim_{h\to 0} \sigma(\mathring{\mathcal{A}}_{\varepsilon_h})$.

B. Proof of Theorem 2.2.31

Proof. The countability of the solutions of (2.35) is proved in Proposition 4.6.3.

The equality (2.36) is proved in the same way as in [71, Section 8], by analysing the resolvent equation for the limit operator.

The proof of the Hausdorff convergence consists of two parts: the statement (H_1) is the direct consequence of the strong resolvent convergence established in Theorem 2.2.10

and Theorem 2.2.19. The statement (H_2) is proved by following the strategy of Theorem 2.2.30: taking the sequence of the solutions to

$$\tilde{\mathcal{A}}_{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h} = \lambda^{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h}, \quad \|\boldsymbol{u}^{\varepsilon_h}\|_{L^2} = 1,$$

where $\lambda^{\varepsilon_h} \to \lambda$. One only needs to establish that the L^2 weak limit of $\boldsymbol{u}^{\varepsilon_h}$ is not zero, so (H_2) then follows by letting $\varepsilon_h \to 0$ in (2.3.3). This claim is verified by proving that the sequence $(\boldsymbol{u}^{\varepsilon_h})_{h>0}$ converges strongly two-scale to the limit $\boldsymbol{u}$, i.e.

$$\boldsymbol{u}^{\varepsilon_h} \xrightarrow{dr-2} \boldsymbol{u}.$$

Note that, due to Lemma 2.3.2, one can assume without loss of generality that $\lambda \notin \lim_{h\to 0}\sigma(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h}) = \sigma(\tilde{\mathcal{A}}_{00,\delta})$. One can then prove (2.3.3) in the same way as in [71, Lemma 8.2], see also [16, Theorem 6.2] for an analogous proof in the stochastic setting as well as the proof of Theorem 2.2.33 below. It is important to emphasize that the proof requires strong convergence in L^2 of the sequence of extensions $(\tilde{\boldsymbol{u}}^{\varepsilon_h})_{h>0}$, which can be ensured by imposing Assumption 2.1.1 (1) and using Corollary 2.2.11 and Corollary 2.2.20.

The claim about the symmetry of $\tilde{\beta}^{\mathrm{memb}}_{\delta}$ is a direct consequence of Assumption 2.1.1.

■

C. Proof of Theorem 2.2.33

Proof. The proof follows the lines of the proof of Theorem 2.2.31. The analysis of the spectrum of limit operator is carried out as in [71, Section 8], by studying the limit resolvent equations in Theorem 2.2.14 and Theorem 2.2.22. Furthermore, in Theorem 2.2.22 we take $\boldsymbol{f}_* = 0$, which implies $\mathring{\boldsymbol{u}}_* = 0$. Strong resolvent convergence is then obtained as the last statement in the mentioned theorem, and compactness of an appropriate sequence of eigenfunctions can be proved by invoking [71, Lemma 8.2]. The only fact we will additionally comment on is the strong two-scale convergence of the eigenfunctions in the regime $\delta = 0$. We take $\lambda^{\varepsilon_h} \in \sigma(h^{-2}\mathcal{A}_{\varepsilon_h})$ such that $\liminf_{h\to 0}\operatorname{dist}(\lambda^{\varepsilon_h}, h^{-2}\varepsilon_h^2\sigma(\mathring{\mathcal{A}}_{\varepsilon_h})) > 0$ (this is again the only situation that requires special analysis, due to Lemma 2.3.2) and $\lambda^{\varepsilon_h} \to \lambda$. Next, we take $\boldsymbol{u}^{\varepsilon_h} \in \mathcal{D}(\mathcal{A}_{\varepsilon_h})$, such that $\|\boldsymbol{u}^{\varepsilon_h}\|_{L^2} = 1$ and $h^{-2}\mathcal{A}_{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h} = \lambda^{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h}$. In order to prove that λ is in the spectrum of the limit operator, we show that the sequence $\boldsymbol{u}^{\varepsilon_h}$ is compact in the sense of strong two-scale convergence. We decompose $\boldsymbol{u}^{\varepsilon_h} = \tilde{\boldsymbol{u}}^{\varepsilon_h} + \mathring{\boldsymbol{u}}^{\varepsilon_h}$, where $\tilde{\boldsymbol{u}}^{\varepsilon_h} = E^{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h}$, where E^{ε_h} is an extension given in Theorem 4.4.6. In the same way as in Proposition 2.2.21, we infer that (2.71) holds. Taking test function $\mathring{\boldsymbol{\xi}} \in H^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^3)$ that vanish on $\Omega_1^{\varepsilon_h}$, we conclude that

$$\frac{1}{h^2}\int_{\Omega_0^{\varepsilon_h}} \mathbb{C}^{\mu_h}\left(\frac{\hat{x}}{\varepsilon_h}\right) \operatorname{sym}\nabla_h \mathring{\boldsymbol{u}}^{\varepsilon_h}(x) : \operatorname{sym}\nabla_h \mathring{\boldsymbol{\xi}}^{\varepsilon_h}(x)\,dx - \lambda^{\varepsilon_h}\int_{\Omega_0^{\varepsilon_h}} \rho \mathring{\boldsymbol{u}}^{\varepsilon_h}\cdot\mathring{\boldsymbol{\xi}}\,dx =$$
$$\frac{1}{h^2}\int_{\Omega_0^{\varepsilon_h}} \mathbb{C}^{\mu_h}\left(\frac{\hat{x}}{\varepsilon_h}\right) \operatorname{sym}\nabla_h \tilde{\boldsymbol{u}}^{\varepsilon_h}(x) : \operatorname{sym}\nabla_h \mathring{\boldsymbol{\xi}}^{\varepsilon_h}(x)\,dx - \lambda^{\varepsilon_h}\int_{\Omega_0^{\varepsilon_h}} \rho \tilde{\boldsymbol{u}}^{\varepsilon_h}\cdot\mathring{\boldsymbol{\xi}}\,dx. \tag{2.73}$$

To prove the strong two-scale convergence, we shall use a duality argument. To this end, consider the identity

$$\frac{1}{h^2}\int_{\Omega_0^{\varepsilon_h}} \mathbb{C}^{\mu_h}\left(\frac{\hat{x}}{\varepsilon_h}\right) \operatorname{sym}\nabla_h \mathring{\boldsymbol{z}}^{\varepsilon_h}(x) : \operatorname{sym}\nabla_h \mathring{\boldsymbol{\xi}}(x)\,dx - \lambda^{\varepsilon_h}\int_{\Omega_0^{\varepsilon_h}} \rho \mathring{\boldsymbol{z}}^{\varepsilon_h}\cdot\mathring{\boldsymbol{\xi}}\,dx = \int_{\Omega_0^{\varepsilon_h}} \mathring{\boldsymbol{f}}^{\varepsilon_h}\cdot\mathring{\boldsymbol{\xi}}\,dx,$$
$$\forall \mathring{\boldsymbol{\xi}} \in H^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^3), \quad \mathring{\boldsymbol{\xi}} = 0 \text{ on } \Omega_1^{\varepsilon_h}, \tag{2.74}$$

where $\mathring{\boldsymbol{f}}^{\varepsilon_h} \in L^2(\Omega;\mathbb{R}^3)$ and $\mathring{\boldsymbol{z}}^{\varepsilon_h} \in H^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^3)$, $\mathring{\boldsymbol{z}}^{\varepsilon_h} = 0$ on $\Omega_1^{\varepsilon_h}$. Denoting by $\mathring{\boldsymbol{u}}_{\rm c}^{\varepsilon_h}$ the solution of (2.74) with $\mathring{\boldsymbol{f}}^{\varepsilon_h} = -\lambda^{\varepsilon_h}\rho\tilde{\boldsymbol{u}}^{\varepsilon_h}$, subtracting (2.74) from (2.73), and using an appropriate version of Lemma 4.6.4 (see Remark 4.6.5), we obtain that

$$\varepsilon_h\left\|\operatorname{sym}\nabla_h(\mathring{\boldsymbol{u}}^{\varepsilon_h} - \mathring{\boldsymbol{u}}_c^{\varepsilon_h})\right\|_{L^2} + \left\|\mathring{\boldsymbol{u}}^{\varepsilon_h} - \mathring{\boldsymbol{u}}_{\rm c}^{\varepsilon_h}\right\|_{L^2} \to 0 \quad \text{as} \quad h \to 0.$$

Notice also that $\mathring{\boldsymbol{u}}_*^{\varepsilon_h} \xrightarrow{L^2} 0$ as the consequence of apriori estimates, see also (2.71). We now take $\mathring{\boldsymbol{g}}^{\varepsilon_h} \in L^2(\Omega;\mathbb{R}^3)$ such that $\mathring{\boldsymbol{g}}^{\varepsilon_h} \xrightarrow{dr-2} \mathring{\boldsymbol{g}} \in L^2(\Omega\times Y;\mathbb{R}^3)$. Furthermore, we take $\mathring{\boldsymbol{s}}^{\varepsilon_h}$ as the solution of (2.74) with $\mathring{\boldsymbol{f}}^{\varepsilon_h} = \mathring{\boldsymbol{g}}^{\varepsilon_h}$. Substituting $\mathring{\boldsymbol{s}}^{\varepsilon_h}$ as a test function in the equation for $\mathring{\boldsymbol{u}}_{\rm c}^{\varepsilon_h}$ and $\mathring{\boldsymbol{u}}_{\rm c}^{\varepsilon_h}$ as a test function in the equation for $\boldsymbol{s}^{\varepsilon_h}$, we obtain by the same argument as in the proof of Theorem 2.2.22 that

$$\frac{1}{12}\int_{\omega\times Y_0} \mathbb{C}_0^{\rm bend,r}(y)\nabla_y^2\mathring{u}_3(\hat{x},y) : \nabla_y^2\mathring{s}_3(\hat{x},y)\,d\hat{x}dy - \lambda\int_{\omega\times Y_0}\rho_0(y)\mathring{u}_3(\hat{x},y)\cdot\mathring{s}_3(\hat{x},y)\,d\hat{x}dy$$
$$= -\lambda\int_{\omega\times Y_0}\rho_0(y)\tilde{u}_3(\hat{x},y)\cdot\mathring{s}_3(\hat{x},y)\,d\hat{x}dy = \int_{\omega\times Y_0}\overline{\mathring{g}}_3(\hat{x},y)\cdot\mathring{u}_3(\hat{x},y)\,d\hat{x}dy,$$

where $\mathring{u}_3$, $\mathring{s}_3 \in L^2(\omega;H_0^2(Y_0))$ are weak two-scale limits of $\mathring{u}_3^{\varepsilon_h}$, $\mathring{s}_3^{\varepsilon_h}$ while $\mathring{\boldsymbol{s}}_*^{\varepsilon_h} \longrightarrow 0$ in L^2, and $\tilde{u}_3 \in H^2_{\gamma_D}(\omega)$ is the strong limit of $\tilde{u}_3^{\varepsilon_h}$ while $\tilde{\boldsymbol{u}}_*^{\varepsilon_h} \longrightarrow 0$ in L^2. It follows that

$$\lim_{h\to0}\int_\Omega \mathring{\boldsymbol{g}}^{\varepsilon_h}\cdot\mathring{\boldsymbol{u}}^{\varepsilon_h}\,dx = -\lambda\lim_{h\to0}\int_\Omega \rho\tilde{\boldsymbol{u}}^{\varepsilon_h}\cdot\mathring{\boldsymbol{s}}^{\varepsilon_h}\,dx = -\lambda\int_{\Omega\times Y}\rho\tilde{u}_3(\hat{x},y)\cdot\mathring{s}_3 = \int_{\Omega\times Y} g_3(x,y)\mathring{u}_3(\hat{x},y)d\hat{x}dy.$$

Therefore, the sequence $\mathring{\boldsymbol{u}}^{\varepsilon_h}$, and consequently $\boldsymbol{u}^{\varepsilon_h}$, converges strongly two-scale. Passing to the limit in the (weak formulation of the) equation $h^{-2}\mathcal{A}_{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h} = \lambda^{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h}$, we immediately obtain $\hat{\mathcal{A}}_0\boldsymbol{u} = \lambda\boldsymbol{u}$, where $\boldsymbol{u} \neq 0$ is the two-scale limit of $\boldsymbol{u}^{\varepsilon_h}$. ■

D. Proof of Theorem 2.2.35

Proof. The proof uses some ideas given in [2] adapted to the present, simpler, setup.

We start by characterising the sets $\sigma(\mathring{\mathcal{A}}_{\text{strip}})$, $\sigma_{\text{ess}}(\mathring{\mathcal{A}}^{+}_{\text{strip}})$, and $\sigma_{\text{ess}}(\mathring{\mathcal{A}}^{-}_{\text{strip}})$. By applying the Fourier transform, it is easily seen that the generalised eigenfunctions of $\mathring{\mathcal{A}}_{\text{strip}}$ are of the form

$$\boldsymbol{u}^{\eta}_{\text{strip}}(y_1,y_2,x_3) = e^{i\eta x_3}\boldsymbol{u}^{\eta}(y_1,y_2), \quad \eta \in \mathbb{R},$$

where $\boldsymbol{u}^{\eta} \in H^1_0(Y_0;\mathbb{C}^3)$ is an eigenfunction of the self-adjoint operator $\mathring{\mathcal{A}}^{\eta}_{\text{strip}}$ on $L^2(Y_0;\mathbb{C}^3)$ defined via the bilinear form

$$\mathring{a}^{\eta}_{\text{strip}}(\boldsymbol{u},\boldsymbol{v}) = \int_{Y_0} \mathbb{C}_0(y)\,\mathrm{sym}\big(\partial_{y_1}\boldsymbol{u}\,|\,\partial_{y_2}\boldsymbol{u}\,|\,\mathrm{i}\eta\boldsymbol{u}\big) : \mathrm{sym}\big(\partial_{y_1}\overline{\boldsymbol{v}}\,|\,\partial_{y_2}\overline{\boldsymbol{v}}\,|\,\overline{\mathrm{i}\eta\boldsymbol{v}}\big)\,dy,$$
$$\mathring{a}^{\eta}_{\text{strip}} : H^1_0(Y_0;\mathbb{C}^3)\times H^1_0(Y_0;\mathbb{C}^3) \to \mathbb{C}.$$

It is easily seen that for each $\eta \in \mathbb{R}$ the operator $\mathring{\mathcal{A}}^{\eta}_{\text{strip}}$ is positive definite and has compact resolvent, and thus it has an increasing sequence of eigenvalues $\{\alpha^{\eta}_1, \alpha^{\eta}_2, \dots\}$ diverging to $+\infty$. It follows that

$$\sigma(\mathring{\mathcal{A}}_{\text{strip}}) = \bigcup_{\eta\in\mathbb{R}}\big\{\alpha^{\eta}_1, \alpha^{\eta}_2, \dots\big\}.$$

By using a suitable Korn's inequality on the on $I\times Y_0$ (applied to the function $(x_3,y_1,y_2) \mapsto e^{i\eta x_3}\boldsymbol{u}(y_1,y_2)$) and (2.1), we obtain that there exists a constant $C > 0$, which is independent of η, such that

$$\|\boldsymbol{u}\|^2_{L^2} + \big\|\big(\partial_{y_1}\boldsymbol{u}\,|\,\partial_{y_2}\boldsymbol{u}\,|\,\mathrm{i}\eta\boldsymbol{u}\big)\big\|^2_{L^2} \le C\mathring{a}^{\eta}_{\text{strip}}(\boldsymbol{u},\boldsymbol{u}) \qquad \forall \boldsymbol{u} \in H^1_0(Y_0;\mathbb{C}^3).$$

Furthermore, using the characterisation of eigenvalues through a Rayleigh quotient, we obtain

$$\alpha^{\eta}_1 = \min_{\boldsymbol{u}\in H^1_0(Y_0;\mathbb{C}^3)} \frac{\mathring{a}^{\eta}_{\text{strip}}(\boldsymbol{u},\boldsymbol{u})}{\|\boldsymbol{u}\|^2_{L^2}}.$$

Combining this with (2.3.3), we infer that there exists $c > 0$, independent of η, such that

$$\alpha^{\eta}_1 \ge c \min_{\boldsymbol{u}\in H^1_0(Y_0;\mathbb{C}^3)} \frac{\big\|\big(\partial_{y_1}\boldsymbol{u}\,|\,\partial_{y_2}\boldsymbol{u}\,|\,\mathrm{i}\eta\boldsymbol{u}\big)\big\|^2_{L^2}}{\|\boldsymbol{u}\|^2_{L^2}}.$$

Finally, using Poincaré's inequality on Y_0, we obtain the existence of $c > 0$ such that $\alpha_1^\eta \geq c + \eta^2$. The continuity of α_1^η with respect to η (which can also be inferred from (2.3.3)) implies that the range of the mapping $\eta \mapsto \alpha_1^\eta$ is $[m_0, +\infty)$ for some $m_0 > 0$. This concludes the characterisation of the set $\sigma(\mathring{\mathcal{A}}_{\text{strip}})$, provided by (2.2.35).

Proceeding to the discussion of the sets $\sigma_{\text{ess}}(\mathring{\mathcal{A}}^{\pm}_{\text{strip}})$, we show that they in fact coincide with $\sigma(\mathring{\mathcal{A}}_{\text{strip}})$. The proof of this claim, for which we just provide a sketch, is similar to the argument of [2, Proposition 7.5]. Consider a Weyl sequence associated to $\lambda \in \sigma_{\text{ess}}(\mathring{\mathcal{A}}^{+}_{\text{strip}})$, i.e., $(\boldsymbol{u}^{+,n})_{n\in\mathbb{N}} \in \mathcal{D}(\mathring{\mathcal{A}}^{+}_{\text{strip}})$ such that

$$\|\boldsymbol{u}^{+,n}\|_{L^2} = 1, \quad \boldsymbol{u}^{+,n} \xrightarrow{L^2} 0, \quad \left\|\mathring{\mathcal{A}}^{+}_{\text{strip}}\boldsymbol{u}^{+,n} - \lambda\boldsymbol{u}^{+,n}\right\|_{L^2} \to 0. \tag{2.75}$$

The properties (2.75) imply that $(\boldsymbol{u}^{+,n})_{n\in\mathbb{N}}$ is bounded in H^1. Next, take a smooth positive function $\psi : \mathbb{R}_0^+ \to \mathbb{R}$ that takes zero values on $(-\infty, 1]$ and is equal to unity on $[2, +\infty)$ and show that for all $v \in H^1_{00}(\mathbb{R}_0^+ \times Y_0; \mathbb{R})$ one has

$$\begin{aligned}
&\int_{\mathbb{R}_0^+\times Y_0} \mathbb{C}_0(y)\nabla(\psi\boldsymbol{u}^{+,n}) : \nabla\boldsymbol{v}dx_3dy - \lambda\int_{\mathbb{R}_0^+\times Y_0} \rho_0(\psi\boldsymbol{u}^{+,n})\cdot\boldsymbol{v}dx_3dy \\
&\qquad = \int_{\mathbb{R}\times Y_0} \mathbb{C}_0(y)\nabla(\psi\boldsymbol{u}^{+,n}) : \nabla\boldsymbol{v}dx_3dy - \lambda\int_{\mathbb{R}\times Y_0} \rho_0(\psi\boldsymbol{u}^{+,n})\cdot\boldsymbol{v}dx_3dy \\
&\qquad = \int_{[1,2]\times Y_0} \mathbb{C}_0(y)\,\mathrm{sym}\big(0|0|\partial_{x_3}\psi\boldsymbol{u}^{+,n}\big) : \mathrm{sym}\,\nabla\boldsymbol{v}dx_3dy \\
&\qquad\quad + \int_{[1,2]\times Y_0} \mathbb{C}_0(y)\,\mathrm{sym}\,\nabla\boldsymbol{u}^{+,n} : \mathrm{sym}\big(0|0|\partial_{x_3}\psi\boldsymbol{v}\big)\,dx_3dy.
\end{aligned} \tag{2.76}$$

Combining (2.75) with compact embedding of H^1 into L^2 on bounded domains, we conclude that for all bounded sets A one has $\|\boldsymbol{u}^{+,n}\|_{L^2(A)} \to 0$. Furthermore, considering a smooth non-negative compactly supported function ψ_A that is equal to one on A and noting that by virtue of (2.75) one has

$$\int_{\mathbb{R}_0^+\times Y_0} \big(\mathring{\mathcal{A}}^{+}_{\text{strip}}\boldsymbol{u}^{+,n} - \lambda\boldsymbol{u}^{+,n}\big)\psi_A\boldsymbol{u}^{+,n} \to 0,$$

we obtain that actually $\|\boldsymbol{u}^{+,n}\|_{H^1(A)} \to 0$.

Thus we conclude that the right-hand side of (2.76) can be written in the form

$$\int_{[1,2]\times Y_0} \boldsymbol{f}_1^n : \mathrm{sym}\,\nabla\boldsymbol{v}dx_3dy + \int_{[1,2]\times Y_0} \boldsymbol{f}_2^n\cdot\boldsymbol{v}dx_3dy,$$

where $\|\boldsymbol{f}_1^n\|_{L^2} \to 0$ and $\|\boldsymbol{f}_2^n\|_{L^2} \to 0$ as $n \to \infty$. By combining a suitable version of Lemma 4.6.4 with (2.76), we conclude that $\lambda \in \sigma(\mathring{\mathcal{A}}_{\text{strip}})$. In a similar fashion, starting from the generalised eigenfunction (2.3.3), we conclude that $\sigma(\mathring{\mathcal{A}}_{\text{strip}}) \subset \sigma(\mathring{\mathcal{A}}^+_{\text{strip}})$.

By repeating the above argument for $\mathcal{A}^-_{\text{strip}}$, we also obtain

$$\sigma_{\text{ess}}\big(\mathring{\mathcal{A}}^-_{\text{strip}}\big) = \sigma\big(\mathring{\mathcal{A}}_{\text{strip}}\big).$$

This establishes the property (2.38). We now proceed to proving (2.39).

First, by virtue of the symmetries of the elastic tensor (and considering appropriate Weyl sequences), we easily obtain the equality $\sigma(\mathring{\mathcal{A}}^+_{\text{strip}}) = \sigma(\mathring{\mathcal{A}}^-_{\text{strip}})$. Next we show that

$$\sigma_{\text{ess}}\big(\mathring{\mathcal{A}}^+_{\text{strip}}\big) \subset \sigma\big(\mathring{\mathring{\mathcal{A}}}_{\text{strip}}\big), \qquad \sigma\big(\mathring{\mathring{\mathcal{A}}}_{\text{strip}}\big) = \sigma\big(\mathring{\mathcal{A}}_{\text{strip}}\big) \tag{2.77}$$

To show the first inclusion in (2.77), we take a Weyl sequence associated to the $\lambda \in \sigma_{\text{ess}}(\mathring{\mathcal{A}}^+_{\text{strip}})$, i.e. $(\boldsymbol{u}^{+,n})_{n\in\mathbb{N}} \subset \mathcal{D}(\mathring{\mathcal{A}}^+_{\text{strip}})$ such that

$$\|\boldsymbol{u}^{+,n}\|_{L^2} = 1, \qquad \boldsymbol{u}^{+,n} \xrightarrow{L^2} 0, \qquad \big\|\mathring{\mathcal{A}}^+_{\text{strip}}\boldsymbol{u}^{+,n} - \lambda\boldsymbol{u}^{+,n}\big\|_{L^2} \to 0.$$

Using the elastic symmetries once again, we infer that for the functions

$$\boldsymbol{u}_*^{-,n}(x_3,y) := \boldsymbol{u}_*^{+,n}(-x_3,y), \qquad u_3^{-,n}(x_3,y) := -u_3^{+,n}(-x_3,y), \qquad (x_3,y) \in \mathbb{R}_0^+ \times Y_0,$$

one has

$$\|\boldsymbol{u}^{-,n}\|_{L^2} = 1, \qquad \boldsymbol{u}^{-,n} \xrightarrow{L^2} 0, \qquad \big\|\mathring{\mathcal{A}}^-_{\text{strip}}\boldsymbol{u}^{-,n} - \lambda\boldsymbol{u}^{-,n}\big\|_{L^2} \to 0.$$

We also note that the sequences $(\boldsymbol{u}^{\pm,n})_{n\in\mathbb{N}}$ are bounded in H^1. We now define

$$\boldsymbol{u}^n(x_3,y) := \psi(x_3)\boldsymbol{u}^{+,n}(x_3,y) + \psi(-x_3)\boldsymbol{u}^{-,n}(x_3,y), \qquad (x_3,y) \in \mathbb{R} \times Y_0.$$

In the same way as in (2.76), we conclude that for every $v \in H^1_{00}(\mathbb{R}_0^+ \times Y_0;\mathbb{R})$ one has

$$\int_{\mathbb{R}\times Y_0} \mathbb{C}_0(y)\nabla\boldsymbol{u}^n : \nabla\boldsymbol{v}\,dx_3dy - \lambda\int_{\mathbb{R}\times Y_0} \rho_0\boldsymbol{u}^n\boldsymbol{v}\,dx_3dy$$
$$= \int_{([1,2]\cup[-2,-1])\times Y_0} \boldsymbol{f}_1^n : \operatorname{sym}\nabla\boldsymbol{v}\,dx_3dy + \int_{([1,2]\cup[-2,-1])\times Y_0} \boldsymbol{f}_2^n\boldsymbol{v}\,dx_3dy,$$

where $\|\boldsymbol{f}_1^n\|_{L^2} \to 0$ and $\|\boldsymbol{f}_2^n\|_{L^2} \to 0$ as $n \to \infty$, from which it follows that $\lambda \in \sigma(\mathring{\mathring{\mathcal{A}}}_{\text{strip}})$.

For the last equality in (2.77) it suffices to argue that $\sigma(\mathring{\mathcal{A}}_{\text{strip}}) \subset \sigma(\mathring{\mathring{\mathcal{A}}}_{\text{strip}})$. To this end, we apply the Fourier transform and for $\eta \in \mathbb{R}$ we consider generalised eigenfunctions of the operator $\mathring{\mathcal{A}}_{\text{strip}}$ of the form

$$\boldsymbol{u}^\eta_{\text{strip}}(x_3,y) = e^{i\eta x_3}\boldsymbol{u}^\eta(y), \qquad (x_3,y) \in \mathbb{R} \times Y_0,$$

where $\boldsymbol{u}^\eta \in H^1_{00}(Y_0;\mathbb{C}^3)$ is an eigenfunction of the operator $\mathring{\mathcal{A}}^\eta_{\rm strip}$, i.e., $\mathring{\mathcal{A}}^\eta_{\rm strip}\boldsymbol{u}^\eta(y_1,y_2) = \alpha_i^\eta \boldsymbol{u}^\eta(y_1,y_2)$, $\boldsymbol{u}^\eta \neq 0$, for some $i \in \mathbb{N}$. Invoking the symmetries, we infer that for each $\eta \in \mathbb{R}$

$$(\boldsymbol{u}^{-\eta}_{\rm strip})_*(x_3,y) := (\boldsymbol{u}^{\eta}_{\rm strip})_*(-x_3,y), \qquad u^{-\eta}_{{\rm strip},3}(x_3,y) := -u^{\eta}_{{\rm strip},3}(-x_3,y), \qquad (x_3,y) \in \mathbb{R}\times Y_0,$$

is also a generalised eigenfunction of the operator $\mathring{\mathcal{A}}_{\rm strip}$ associated with the same eigenvalue α_i^η. Therefore, the function $\big(\boldsymbol{u}^{\eta}_{\rm strip} + \boldsymbol{u}^{-\eta}_{\rm strip}\big)/2$ is a generalised eigenfunction of the operator $\mathring{\mathring{\mathcal{A}}}_{\rm strip}$ (and hence the operator $\mathring{\mathcal{A}}_{\rm strip}$) associated with the same eigenvalue. Since every element of the spectrum of the operator $\mathring{\mathcal{A}}_{\rm strip}$ coincides with α_i^η for some $\eta \in \mathbb{R}$ and $i \in \mathbb{N}$, we conclude that $\sigma(\mathring{\mathcal{A}}_{\rm strip}) \subset \sigma(\mathring{\mathring{\mathcal{A}}}_{\rm strip})$. This construction also proves that $\sigma_{\rm ess}(\mathring{\mathring{\mathcal{A}}}_{\rm strip}) = \sigma\big(\mathring{\mathcal{A}}_{\rm strip}\big)$. Since we have already established that $\sigma\big(\mathring{\mathcal{A}}_{\rm strip}\big) \subset \sigma\big(\mathring{\mathcal{A}}^+_{\rm strip}\big)$, the property (2.39) follows.

Next we establish the property (2.37). We start by proving the inclusion

$$\lim_{h\to 0} \sigma(\mathring{\mathcal{A}}_{\varepsilon_h}) \subset \sigma(\mathring{\mathcal{A}}_{\rm strip}) \cup \sigma(\mathcal{A}^+_{\rm strip}) \cup \sigma(\mathcal{A}^-_{\rm strip}). \tag{2.78}$$

Let us take $\lambda^{\varepsilon_h} \in \sigma(\mathring{\mathcal{A}}_{\varepsilon_h})$ and $\boldsymbol{u}^{\varepsilon_h} \in \mathcal{D}(\mathring{\mathcal{A}}_{\varepsilon_h})$ such that $\lambda^{\varepsilon_h} \to \lambda$ and

$$\mathring{\mathcal{A}}^{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h} = \lambda^{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h}, \quad \|\boldsymbol{u}^{\varepsilon_h}\|_{L^2} = 1.$$

Consider smooth positive functions ψ_i, $i=1,2,3$ on $\mathbb{R}$ such that $\psi_1+\psi_2+\psi_3 = 1$, $\operatorname{supp}\psi_2 \subset [-1/4,1/4]$, $\operatorname{supp}\psi_1 \subset (-\infty,-1/8]$, $\operatorname{supp}\psi_3 \subset [1/8,\infty)$, and $\psi_3(x_3) = \psi_1(-x_3)$. Then there exists $i \in \{1,2,3\}$ such that (up to a subsequence)

$$\left\|\psi_i \boldsymbol{u}^{\varepsilon_h}\right\|_{L^2} \geq \frac{1}{3} \quad \forall h.$$

If $i=2$, we extend $\psi_2\boldsymbol{u}^{\varepsilon_h}$ by zero on $\mathbb{R}\times Y_0$ and, by scaling the variable x_3, define

$$\boldsymbol{u}^{\varepsilon_h}_{\rm strip}(x_3,y) = \sqrt{\frac{h}{\varepsilon_h}}\,\psi_2\left(\frac{\varepsilon_h}{h}x_3\right)\boldsymbol{u}^{\varepsilon_h}\left(\frac{\varepsilon_h}{h}x_3,y\right), \qquad (x_3,y) \in \mathbb{R}\times Y_0.$$

It is straightforward to see that

$$\|\boldsymbol{u}^{\varepsilon_h}_{\rm strip}\|_{L^2} \geq \frac{1}{3}$$

and that for all $\boldsymbol{v} \in H^1_{00}(\mathbb{R}\times Y_0;\mathbb{R}^3)$ one has

$$\int_{\mathbb{R}\times Y_0} \mathbb{C}_0(y)\nabla \boldsymbol{u}^{\varepsilon_h}_{\rm strip} : \nabla \boldsymbol{v}\,dx_3dy - \lambda^{\varepsilon_h}\int_{\mathbb{R}\times Y_0}\rho_0 \boldsymbol{u}^{\varepsilon_h}_{\rm strip}\boldsymbol{v}\,dx_3dy = \int_{\mathbb{R}\times Y_0}\boldsymbol{f}_1^{\varepsilon_h} : \operatorname{sym}\nabla\boldsymbol{v}\,dx_3dy + \int_{\mathbb{R}\times Y_0}\boldsymbol{f}_2^{\varepsilon_h}\boldsymbol{v},$$

where $\|\boldsymbol{f}_1^{\varepsilon_h}\|_{L^2} \to 0$, $\|\boldsymbol{f}_2^{\varepsilon_h}\|_{L^2} \to 0$ as $h \to 0$. By using an appropriate analogue of Lemma 4.6.4 (see also Remark 4.6.5) adapted to the operator $\mathring{\mathcal{A}}_{\varepsilon_h}$, we conclude that

$$\lambda \in \sigma\big(\mathring{\mathcal{A}}_{\mathrm{strip}}\big).$$

If $i = 1$ or $i = 3$ we argue similarly that $\lambda \in \mathring{\mathcal{A}}^{+}_{\mathrm{strip}}$, i.e. $\lambda \in \mathring{\mathcal{A}}^{-}_{\mathrm{strip}}$ respectively.

Next, we prove that $\sigma(\mathring{\mathcal{A}}_{\mathrm{strip}}) \subset \lim_{h\to 0}\sigma(\mathring{\mathcal{A}}_{\varepsilon_h})$. Considering α_i^η and $\boldsymbol{u}^\eta \in \mathcal{D}(\mathring{\mathcal{A}}^{\eta}_{\mathrm{strip}})$ such that

$$\mathring{\mathcal{A}}^{\eta}_{\mathrm{strip}}\boldsymbol{u}^\eta = \alpha_i^\eta \boldsymbol{u}^\eta, \qquad \|\boldsymbol{u}^\eta\|_{L^2} = 1$$

we set

$$\boldsymbol{u}^{\eta}_{\mathrm{strip}}(x_3, y) = e^{i\eta x_3}\boldsymbol{u}^\eta(y), \qquad (x_3, y) \in \mathbb{R}\times Y_0.$$

It is easily seen that $\mathring{\mathcal{A}}_{\mathrm{strip}}\boldsymbol{u}^{\eta}_{\mathrm{strip}} = \alpha_i^\eta \boldsymbol{u}^{\eta}_{\mathrm{strip}}$. We define

$$\boldsymbol{u}^{\varepsilon_h}(x_3, y) = \left\|\psi_2(x_3)\boldsymbol{u}^{\eta}_{\mathrm{strip}}\left(\frac{h}{\varepsilon_h}x_3, y\right)\right\|_{L^2}^{-1}\psi_2(x_3)\boldsymbol{u}^{\eta}_{\mathrm{strip}}\left(\frac{h}{\varepsilon_h}x_3, y\right), \qquad (x_3, y) \in I\times Y_0.$$

It then follows easily that for every $h > 0$ and $\boldsymbol{v} \in H^1_{00}(I\times Y_0; \mathbb{C}^3)$ one has

$$\begin{aligned}\int_{I\times Y_0}\mathbb{C}_0(y)\nabla_{\frac{h}{\varepsilon_h}}\boldsymbol{u}^{\varepsilon_h} : \nabla_{\frac{h}{\varepsilon_h}}\boldsymbol{v}\,dx_3dy - \alpha_i^\eta\int_{I\times Y_0}\rho_0\boldsymbol{u}^{\varepsilon_h}\cdot\boldsymbol{v}\,dx_3dy \\ = \int_{I\times Y_0}\boldsymbol{f}_1^{\varepsilon_h} : \operatorname{sym}\nabla_{\frac{h}{\varepsilon_h}}\boldsymbol{v}\,dx_3dy + \int_{I\times Y_0}\boldsymbol{f}_2^{\varepsilon_h}\cdot\boldsymbol{v},\end{aligned} \tag{2.79}$$

where $\|\boldsymbol{f}_1^{\varepsilon_h}\|_{L^2} \to 0$, $\|\boldsymbol{f}_2^{\varepsilon_h}\|_{L^2} \to 0$ as $h \to 0$. By using a result analogous to Lemma 4.6.4 (see also Remark 4.6.5) we obtain

$$\operatorname{dist}\big(\alpha_i^\eta, \sigma(\mathring{\mathcal{A}}_{\varepsilon_h})\big) \to 0 \quad \text{as} \quad h \to 0.$$

It can be also easily deduced that $\sigma_{\mathrm{disc}}(\mathring{\mathcal{A}}^{+}_{\mathrm{strip}}) \subset \lim_{h\to0}\sigma(\mathring{\mathcal{A}}_{\varepsilon_h})$. Namely, for an eigenvalue $\alpha^{+}_{\mathrm{strip}}$ of $\mathring{\mathcal{A}}^{+}_{\mathrm{strip}}$ and associated eigenfunction $\boldsymbol{u}^{\alpha^+}_{\mathrm{strip}} \in \mathcal{D}(\mathring{\mathcal{A}}^{+}_{\mathrm{strip}})$, $\|\boldsymbol{u}^{\alpha^+}_{\mathrm{strip}}\|_{L^2} = 1$, i.e. $\mathring{\mathcal{A}}^{+}_{\mathrm{strip}}\boldsymbol{u}^{\alpha^+}_{\mathrm{strip}} = \alpha^{+}_{\mathrm{strip}}\boldsymbol{u}^{\alpha^+}_{\mathrm{strip}}$, it can be easily shown that the sequence

$$\boldsymbol{u}^{\varepsilon_h}(x_3, y) = \left\|\psi_1(x_3)\boldsymbol{u}^{\alpha^+}_{\mathrm{strip}}\left(\frac{h}{\varepsilon_h}\left(x_3 + \frac{1}{2}\right), y\right)\right\|_{L^2}^{-1}\psi_1(x_3)\boldsymbol{u}^{\alpha^+}_{\mathrm{strip}}\left(\frac{h}{\varepsilon_h}\left(x_3 + \frac{1}{2}\right), y\right), \qquad (x_3, y) \in I\times Y_0,$$

satisfies (2.79) with $\|\boldsymbol{f}_1^{\varepsilon_h}\|_{L^2} \to 0$, $\|\boldsymbol{f}_2^{\varepsilon_h}\|_{L^2} \to 0$ as $h \to 0$ and with α_i^η replaced by λ. It follows that $\lambda \in \lim_{h\to0}\sigma(\mathring{\mathcal{A}}_{\varepsilon_h})$. In view of (2.38), we obtain the opposite inclusion in (2.78).

It remains to prove, under the Assumption 2.1.1 (1), the characterisation of $\lim_{h\to 0}\sigma(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h})$ provided by (2.37). By the same argument as in the case without planar symmetries, we obtain

$$\lim_{h\to 0}\sigma\big(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h}\big)\subset\sigma\big(\mathring{\tilde{\mathcal{A}}}_{\rm strip}\big)\cup\sigma\big(\mathring{\mathcal{A}}^{+}_{\rm strip}\big)\cup\sigma\big(\mathring{\mathcal{A}}^{-}_{\rm strip}\big),\tag{2.80}$$

and $\sigma(\mathring{\tilde{\mathcal{A}}}_{\rm strip})\subset\lim_{h\to 0}\sigma(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h})$. By virtue of (2.38) and (2.39), it remains to prove the inclusion

$$\sigma_{\rm disc}\big(\mathring{\mathcal{A}}^{\pm}_{\rm strip}\big)\subset\lim_{h\to 0}\sigma\big(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h}\big).$$

This will be done by a slightly different argument, as follows. For $\alpha\in\sigma_{\rm disc}(\mathring{\mathcal{A}}^{+}_{\rm strip})$ we take the associated eigenfunction $\boldsymbol{u}^{\alpha,+}_{\rm strip}\in\mathcal{D}(\mathring{\mathcal{A}}^{+}_{\rm strip})$, $\|\boldsymbol{u}^{\alpha,+}_{\rm strip}\|_{L^2}=1$, of the operator $\mathring{\mathcal{A}}^{+}_{\rm strip}$ i.e. $\mathring{\mathcal{A}}^{+}_{\rm strip}\boldsymbol{u}^{\alpha,+}_{\rm strip}=\alpha\boldsymbol{u}^{\alpha,+}_{\rm strip}$. Using the elastic symmetries, we infer that the functions $\boldsymbol{u}^{\alpha,-}_{\rm strip}$ defined by

$$\boldsymbol{u}^{\alpha,-}_{*}(x_3,y):=\boldsymbol{u}^{\alpha,+}_{*}(-x_3,y),\qquad \boldsymbol{u}^{\alpha,-}_{3}(x_3,y):=-\boldsymbol{u}^{\alpha,+}_{3}(-x_3,y),\qquad (x_3,y)\in\mathbb{R}^{+}_{0}\times Y_0,$$

satisfy $\boldsymbol{u}^{\alpha,-}_{\rm strip}\in\mathcal{D}(\mathring{\mathcal{A}}^{-}_{\rm strip})$, $\|\boldsymbol{u}^{\alpha,-}_{\rm strip}\|_{L^2}=1$ and $\mathring{\mathcal{A}}^{-}_{\rm strip}\boldsymbol{u}^{\alpha,-}_{\rm strip}=\alpha\boldsymbol{u}^{\alpha,-}_{\rm strip}$. Finally, we define

$$\boldsymbol{u}^{\varepsilon_h}(x_3,y)=\frac{\psi_1(x_3)\boldsymbol{u}^{\alpha,+}_{\rm strip}\left(\dfrac{h}{\varepsilon_h}\left(x_3+\dfrac{1}{2}\right),y\right)+\psi_3(x_3)\boldsymbol{u}^{\alpha,-}_{\rm strip}\left(\dfrac{h}{\varepsilon_h}\left(x_3-\dfrac{1}{2}\right),y\right)}{\left\|\psi_1(x_3)\boldsymbol{u}^{\alpha,+}_{\rm strip}\left(\dfrac{h}{\varepsilon_h}\left(x_3+\dfrac{1}{2}\right),y\right)+\psi_3(x_3)\boldsymbol{u}^{\alpha,-}_{\rm strip}\left(\dfrac{h}{\varepsilon_h}\left(x_3-\dfrac{1}{2}\right),y\right)\right\|_{L^2}},\quad (x_3,y)\in I\times Y_0,$$

and use an argument similar to that employed for showing that $\alpha\in\lim_{h\to 0}\sigma(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h})$ under no symmetry assumptions.

Similarly, we demonstrate that

$$\sigma_{\rm disc}(\mathring{\mathcal{A}}^{-}_{\rm strip})\subset\lim_{h\to 0}\sigma(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h}),$$

which concludes the proof of the opposite inclusion in (2.80). ∎

Remark 2.3.4. In the same way as in [2, Proposition 7.5], it can be shown that eigenfunctions associated with eigenvalues in $\sigma(\mathring{\mathcal{A}}^{\pm}_{\rm strip})$ have exponential decay at infinity.

E. Proof of Theorem 2.2.36 and Theorem 2.2.37

Proof. The equality (2.40) is proved in the same way as in [71, Section 8]. The inclusion $\sigma(\tilde{\mathcal{A}}_\infty)\subset\lim_{h\to 0}\sigma(\tilde{\mathcal{A}}_{\varepsilon_h})$ follows from resolvent convergence provided by Theorem 2.2.24

and Corollary 2.2.25, while the inclusion $\sigma(\mathring{\mathcal{A}}^{+}_{\rm strip}) = \lim_{h\to 0}\sigma(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h}) \subset \lim_{h\to 0}\sigma(\tilde{\mathcal{A}}_{\varepsilon_h})$ follows from Theorem 2.2.35 and Lemma 2.3.2.

It remains to show that $\lim_{h\to 0}\sigma(\tilde{\mathcal{A}}_{\varepsilon_h}) \subset \sigma(\mathring{\mathcal{A}}^{+}_{\rm strip}) \cup \sigma(\tilde{\mathcal{A}}_{\infty})$. To this end, consider $\lambda^{\varepsilon_h} \in \sigma(\tilde{\mathcal{A}}_{\varepsilon_h})$ such that

$$\liminf_{h\to 0} \operatorname{dist}\left(\lambda^{\varepsilon_h}, \sigma(\mathring{\tilde{\mathcal{A}}}_{\varepsilon_h})\right) > 0$$

(which is the only case that requires analysis, due to Lemma 2.3.2) and $\lambda^{\varepsilon_h} \to \lambda$. Furthermore, consider $\boldsymbol{u}^{\varepsilon_h} \in \mathcal{D}(\tilde{\mathcal{A}}_{\varepsilon_h})$ such that $\|\boldsymbol{u}^{\varepsilon_h}\|_{L^2} = 1$ and $\tilde{\mathcal{A}}_{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h} = \lambda^{\varepsilon_h}\boldsymbol{u}^{\varepsilon_h}$. The strong two-scale compactness of $\boldsymbol{u}^{\varepsilon_h}$ is proved in the same way as in the proof of Theorem 2.2.33 by combining (2.3.3) with Lemma 4.6.4, see also Remark 4.6.5. The equation (2.41) is a direct consequence of the symmetry assumptions.

The proof of Theorem 2.2.37 is carried out in a similar fashion. ■

2.3.4. Proofs for Section 2.2.4

A. Proof of Theorem 2.2.39

Proof. It is not possible to put the first claim in the framework of Theorem 4.5.13 or Theorem 4.5.15 directly (i.e. using Proposition 2.2.5 and Theorem 2.2.7) and we will provide a direct proof instead, using Laplace transform similarly to how it was done in the proofs of these theorems. The reason why we cannot put the first claim in the framework of Theorem 4.5.13 or Theorem 4.5.15 directly comes from the fact that $\boldsymbol{f}_* \neq 0$ and they influence the (quasistatic) behavior of the part of in-plane deformation.

For every $\varepsilon_h > 0$, we write the system (2.42) for $\mu_h = \varepsilon_h$, $\tau = 2$, using the formula (4.32), where $\mathbb{A} = \mathbb{A}_{\varepsilon_h}$ is given by formula (4.26) and the associated operator $\mathcal{A}$ is given by $h^{-2}\mathcal{A}_{\varepsilon_h}$. Furthermore, we set $H_{\varepsilon_h} = L^2(\Omega;\mathbb{R}^3)$, $V_{\varepsilon_h} = \mathcal{D}(\mathcal{A}_{\varepsilon_h}^{1/2}) = H^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^3)$, $H = L^2(\Omega\times Y;\mathbb{R}^3)$, $H_0 = \{0\}^2\times L^2(\omega)$, $V = \{0\}^2\times\mathcal{D}((\mathcal{A}_\delta^{\rm b,hom})^{1/2}) = \{0\}^2\times H^2_{\gamma_{\rm D}}(\omega)$. The space H_{ε_h} is equipped with the L^2 inner product with weight ρ^h, while the space H is equipped with the L^2 inner product with weight ρ.

In accordance with the abstract approach of Section 4.5, for $\boldsymbol{v} \in V_{\varepsilon_h}$ we set $\|\boldsymbol{v}\|_{V_{\varepsilon_h}} := \|(h^{-2}\mathcal{A}_{\varepsilon_h} + \mathcal{I})^{1/2}\boldsymbol{v}\|_{L^2}$ and, similarly, for $\boldsymbol{v} \in V$ we set $\|\boldsymbol{v}\|_V := \|(\mathcal{A}_\delta^{\rm b,hom} + \mathcal{I})^{1/2}\boldsymbol{v}\|_{L^2}$. Furthermore, the convergence $\xrightarrow{H_{\varepsilon_h}}$ is given by two-scale convergence. Next, for $\boldsymbol{f} \in \mathbb{R}^3$, we define the vectors $\boldsymbol{f}_v := (0,0,f_3)^\top$, $\boldsymbol{f}_h := (\boldsymbol{f}_*, 0)^\top$. We apply the estimate (4.34) to the case of the loads $\boldsymbol{f}_v^{\varepsilon_h}$ and initial conditions $\boldsymbol{u}_0^{\varepsilon_h}$, $\boldsymbol{u}_1^{\varepsilon_h}$ and the estimate (4.37) to the case of the loads $\boldsymbol{f}_h^{\varepsilon_h}$ and zero initial conditions. This yields

$$
\begin{aligned}
&\left\|(h^{-2}\mathcal{A}_{\varepsilon_h}+\mathcal{I})^{1/2}\boldsymbol{u}^{\varepsilon_h}\right\|_{L^\infty([0,T];H_{\varepsilon_h})}+\|\partial_t\boldsymbol{u}^{\varepsilon_h}\|_{L^\infty([0,T];H_{\varepsilon_h})}\leq\\
&\qquad Ce^{\top}\Big(\left\|(h^{-2}\mathcal{A}_{\varepsilon_h}+\mathcal{I})^{1/2}\boldsymbol{u}_0^{\varepsilon_h}\right\|_{H_{\varepsilon_h}}+\|\boldsymbol{u}_1^{\varepsilon_h}\|_{H_{\varepsilon_h}}+\|\boldsymbol{f}_v^{\varepsilon_h}\|_{L^1([0,T];H_{\varepsilon_h})}\\
&\qquad\quad+\left\|(h^{-2}\mathcal{A}_{\varepsilon_h}+\mathcal{I})^{-1/2}\boldsymbol{f}_h^{\varepsilon_h}(0)\right\|_{H_{\varepsilon_h}}+\left\|(h^{-2}\mathcal{A}_{\varepsilon_h}+\mathcal{I})^{-1/2}\partial_t\boldsymbol{f}_h^{\varepsilon_h}\right\|_{L^1([0,T];H_{\varepsilon_h})}\Big).
\end{aligned}
\tag{2.81}
$$

In order to obtain the boundedness of the last two terms in (2.81), notice that for $\boldsymbol{l}^{\varepsilon_h}\in V_{\varepsilon_h}^*$ one has

$$
\left\|(h^{-2}\mathcal{A}_{\varepsilon_h}+\mathcal{I})^{-1/2}\boldsymbol{l}^{\varepsilon_h}\right\|_{H_{\varepsilon_h}}^2=h^{-2}a_{\varepsilon_h}(\boldsymbol{s}^{\varepsilon_h},\boldsymbol{s}^{\varepsilon_h})+(\boldsymbol{s}^{\varepsilon_h},\boldsymbol{s}^{\varepsilon_h}),\tag{2.82}
$$

where $\boldsymbol{s}^{\varepsilon_h}\in\mathcal{D}(\mathcal{A}_{\varepsilon_h}^{1/2})$ is the solution of the problem

$$
h^{-2}a_{\varepsilon_h}(\boldsymbol{s}^{\varepsilon_h},\boldsymbol{v})+(\boldsymbol{s}^{\varepsilon_h},\boldsymbol{v})_{H_{\varepsilon_h}}=\boldsymbol{l}^{\varepsilon_h}(\boldsymbol{v}),\quad\forall\boldsymbol{v}\in V_{\varepsilon_h}.\tag{2.83}
$$

Combining the result of Proposition 2.2.5 (1) with (2.82) and (2.83), we obtain the existence of $C>0$, independent of h, such that

$$
\left\|(h^{-2}\mathcal{A}_{\varepsilon_h}+\mathcal{I})^{-1/2}\boldsymbol{l}^{\varepsilon_h}\right\|_{H_{\varepsilon_h}}^2\leq C\|\pi_h\boldsymbol{l}^{\varepsilon_h}\|_{H_\varepsilon}^2,\qquad\boldsymbol{l}^{\varepsilon_h}\in H_{\varepsilon_h}.\tag{2.84}
$$

Taking into account (2.44) and (2.45), this implies the stated boundedness property. Also, a consequence of (2.44) and (2.45), we have

$$
\pi_h\boldsymbol{f}_h^{\varepsilon_h}\xrightarrow{t,\infty,\mathrm{dr}-2}\boldsymbol{f}_h,\quad\pi_h\partial_t\boldsymbol{f}^{\varepsilon_h}\xrightarrow{t,\mathrm{dr}-2}\partial_t\boldsymbol{f}_h.
$$

From (2.81) and Corollary 4.2.5 we conclude that $\pi_{1/h}\boldsymbol{u}^{\varepsilon_h}$ is bounded in $L^\infty([0,T];V_{\varepsilon_h})$ and $\partial_t\boldsymbol{u}^{\varepsilon_h}$ is bounded in $L^\infty([0,T];H_{\varepsilon_h})$, and hence there exists $\boldsymbol{u}_l\in L^\infty([0,T];V)$, $\partial_t\boldsymbol{u}_l\in L^\infty([0,T];H)$ such that

$$
\pi_{1/h}\boldsymbol{u}^{\varepsilon_h}\xrightarrow{t,\infty,\mathrm{dr}-2}\boldsymbol{u}_l,\qquad\partial_t\boldsymbol{u}^{\varepsilon_h}\xrightarrow{t,\infty,\mathrm{dr}-2}\partial_t\boldsymbol{u}_l.
$$

As in Section 4.5, we use the notation $\vec{\boldsymbol{u}}^{\varepsilon_h}:=(\boldsymbol{u}^{\varepsilon_h},\partial_t\boldsymbol{u}^{\varepsilon_h})$. Similarly, we introduce $\vec{\boldsymbol{u}}_0^{\varepsilon_h}$, $\vec{\boldsymbol{u}}_0$, $\vec{\boldsymbol{u}}_l$, $\vec{\boldsymbol{u}}$, as well as

$$
\begin{aligned}
\pi_{1/h}\vec{\boldsymbol{u}}^{\varepsilon_h}&:=\big(\pi_{1/h}\boldsymbol{u}^{\varepsilon_h},\partial_t\boldsymbol{u}^{\varepsilon_h}\big)^\top,\quad\vec{\boldsymbol{f}}^{\varepsilon_h}:=\big(0,0,0,(\boldsymbol{f}^{\varepsilon_h})^\top\big)^\top,\\
\vec{\boldsymbol{f}}_v^{\varepsilon_h}&:=\big(0,0,0,(\boldsymbol{f}_v^{\varepsilon_h})^\top\big)^\top,\quad\vec{\boldsymbol{f}}_h^{\varepsilon_h}:=\big(0,0,0,\boldsymbol{f}_h^{\varepsilon_h}\big)^\top.
\end{aligned}
$$

We then follow the proof of Theorem 4.5.13 or Theorem 4.5.15. On the one hand, for every $\lambda > 1$, we have

$$\pi_{1/h}\mathcal{L}(\vec{\boldsymbol{u}}^{\varepsilon_h})(\lambda) = \mathcal{L}(\pi_{1/h}\vec{\boldsymbol{u}}^{\varepsilon_h})(\lambda) \xrightarrow{dr-2} \mathcal{L}(\vec{\boldsymbol{u}}_l)(\lambda) \quad \text{as} \quad h \to 0,$$

where $\mathcal{L}$ denotes the Laplace transform. On the other hand, by combining

$$\mathcal{L}(\vec{\boldsymbol{u}}^{\varepsilon_h})(\lambda) = (\mathbb{A}_{\varepsilon_h} + \lambda\mathbb{I})^{-1}\mathcal{L}(\vec{\boldsymbol{f}}^{\varepsilon_h})(\lambda) + (\mathbb{A}_{\varepsilon_h} + \lambda\mathbb{I})^{-1}\vec{\boldsymbol{u}}_0^{\varepsilon} \qquad \forall\lambda > 1,$$

the representation (4.29), and Theorem 2.2.7, we obtain

$$\pi_{1/h}\mathcal{L}(\vec{\boldsymbol{u}}^{\varepsilon_h})(\lambda) = \mathcal{L}(\pi_{1/h}\vec{\boldsymbol{u}}^{\varepsilon_h})(\lambda) \xrightarrow{dr-2} \mathcal{L}(\vec{\boldsymbol{u}})(\lambda) \qquad \forall\lambda > 1,$$

where $\vec{\boldsymbol{u}} = (\mathfrak{a}_1 - x_3\partial_1\mathfrak{b} + \mathring{u}_1, \mathfrak{a}_2 - x_3\partial_2\mathfrak{b} + \mathring{u}_2, \mathfrak{b}, 0, 0, \partial_t\mathfrak{b})^\top$, with the functions $\boldsymbol{\mathfrak{a}}$, $\mathfrak{b}$, $\mathring{\boldsymbol{u}}$ being the solutions of the equations (2.48)–(2.50) for the loads $\boldsymbol{f}$. It follows that $\vec{\boldsymbol{u}}_l = \vec{\boldsymbol{u}}$.

The existence and uniqueness of the solution of the limit problem follows from Theorem 4.5.1 and Theorem 4.5.4. Note that one can split the limit problem into two: the one with initial conditions $u_{0,3}$, $S_1 P_{\delta,\infty} u_{1,3}$ and out-of-plane loads, given by the part of $\mathcal{F}_\delta(\boldsymbol{f})$ depending on $\boldsymbol{f}_3$ (where we apply Theorem 4.5.1), and the one with zero initial conditions and in-plane loads, given by the part of $\mathcal{F}_\delta(\boldsymbol{f})$ depending on $\boldsymbol{f}_*$ (where we apply Theorem 4.5.4.) The last claim of the theorem follows by combining Theorem 4.5.14 applied to initial conditions $\boldsymbol{u}_0^{\varepsilon_h}$, $\boldsymbol{u}_1^{\varepsilon_h}$ and loads $\boldsymbol{f}_v^{\varepsilon_h}$ and the second claim of Theorem 4.5.15 applied to initial conditions equal to zero and loads $\boldsymbol{f}_h^{\varepsilon_h}$ (using the resolvent compactness and convergence proved in Proposition 2.2.5 and Theorem 2.2.7). The conditions (4.48) follow by applying (2.84) to $\boldsymbol{l}^{\varepsilon_h} = \boldsymbol{f}_h^{\varepsilon_h}(0)$ and $\boldsymbol{l}^{\varepsilon_h}(t) = \partial_t \boldsymbol{f}_h^{\varepsilon_h}(t)$ and integrating over the interval $[0, T]$.

■

B. Proof of Corollary 2.2.40

Proof. The proof follows from the first part of Theorem 4.5.15 for the weak convergence and from the second part of the same theorem for the strong two-scale convergence. We will just briefly outline the proof of the weak convergence. From (4.37) we obtain the estimate

$$\left\|(h^{-2}\mathcal{A}_{\varepsilon_h} + \mathcal{I})^{1/2}\boldsymbol{u}^{\varepsilon_h}\right\|_{L^\infty([0,T];H_{\varepsilon_h})} + \|\partial_t \boldsymbol{u}^{\varepsilon_h}\|_{L^\infty([0,T];H_{\varepsilon_h})}$$

$$\leq Ce^{T}\Big(\big\|(h^{-2}\mathcal{A}_{\varepsilon_h}+\mathcal{I})^{-1/2}\mathcal{G}^{\varepsilon_h}(\boldsymbol{g}^{\varepsilon_h})(0)\big\|_{H_{\varepsilon_h}}+\big\|(h^{-2}\mathcal{A}_{\varepsilon_h}+\mathcal{I})^{-1/2}\partial_t\mathcal{G}(\boldsymbol{g}^{\varepsilon_h})\big\|_{L^1([0,T];H_{\varepsilon_h})}\Big).$$

Similarly to the argument of Section A above (see (2.3.4)), we have

$$\pi_h\boldsymbol{g}^{\varepsilon_h}\xrightarrow{t,\infty,\mathrm{dr}-2}\boldsymbol{g},\quad \pi_h\partial_t\boldsymbol{g}^{\varepsilon_h}\xrightarrow{t,\mathrm{dr}-2}\partial_t\boldsymbol{g}.$$

Furthermore, using (2.82) and (2.83) we infer by Theorem 4.4.1, Remark 4.4.5, and Corollary 4.2.5 that for $\boldsymbol{l}^{\varepsilon_h}\in L^2(\omega\times\{-1/2,1/2\};\mathbb{R}^3)$ one has

$$\big\|(h^{-2}\mathcal{A}_{\varepsilon_h}+\mathcal{I})^{-1/2}\boldsymbol{l}^{\varepsilon_h}\big\|^2_{H_{\varepsilon_h}}\leq\|\pi_h\boldsymbol{l}^{\varepsilon_h}\|^2_{L^2(\omega\times\{-1/2,1/2\};\mathbb{R}^3)}.$$

The remainder of the argument follows the proof of Theorem 2.2.39, using Remark 2.3.1.

■

C. Proof of Theorem 2.2.47 and Theorem 2.2.51

Proof. The claims are established directly by applying Theorem 4.5.13, Theorem 4.5.14, and the results of Section 2.2.2 concerning resolvent convergence. For example, in the case $\delta\in(0,\infty)$, $\mu_h=\varepsilon_h$, $\tau=0$ we set $H_{\varepsilon_h}=L^2(\Omega;\mathbb{R}^3)$, $\mathcal{A}_\varepsilon=\mathcal{A}_{\varepsilon_h}$, $\mathcal{A}=\mathcal{A}_{\delta,\infty}$, $H=L^2(\Omega\times Y;\mathbb{R}^3)$, $H_0=V_{\delta,\infty}(\Omega\times Y)$, and the convergence $\xrightarrow{H_{\varepsilon_h}}$ is the two-scale convergence. ■

D. Proof of Theorem 2.2.53

Proof. The argument follows the proof of Theorem 2.2.39. The first part of the statement, which concerns weak two-scale convergence, is proved separately, by using the Laplace transform, Proposition 2.2.21, and Theorem 2.2.22 while separating out-of-plane and horizontal forces. The proof of the second part is carried out using Theorem 4.5.14 and the second part of Theorem 4.5.15. We leave the details to the interested reader. ■

3. Operator-norm resolvent estimates for thin elastic periodically heterogeneous rods in moderate contrast

3.1. Setting and main results

In this section we state the main results of this chapter along with the setup for studying the matter of elastic heterogeneous rods.

3.1.1. Elastic heterogeneous rod

We state the definition of the domain representing the infinite thin rod. Fix $h > 0$ (the width of the rod), $\omega \subset \mathbb{R}^2$ a bounded Lipschitz domain and denote with ω^h the contraction of ω such that $|\omega^h| = h^2|\omega|$ (For example: $\omega = I \times I, I \subset \mathbb{R}$ interval ,$\omega^h = hI \times hI$). We take ω to be central symmetric with respect to the origin, which can be neatly expressed with the following central symmetry operator:

$$\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}^2, \quad \mathcal{S}(x_1, x_2) := (-x_1, -x_2),$$

and stating:

$$\mathcal{S}(\omega) = \omega.$$

The consequence of this is the following:

$$\int_\omega x_1 = 0, \quad \int_\omega x_2 = 0.$$

Additionally, one can choose the rotation of the coordinate system $(x_1, x_2) = (\tilde{x}_1 \cos\varphi + \tilde{x}_2 \sin\varphi, -\tilde{x}_1 \sin\varphi + \tilde{x}_2 \cos\varphi)$ as to achieve

$$\int_\omega x_1 x_2 = 0.$$

This is done by choosing

$$\varphi = \frac{1}{2}\arctan\left(-2\int_\omega x_1 x_2 \Big/ \int_\omega (x_1^2 - x_2^2)\right).$$

We also take $|\omega| = 1$, and define the following constants:

$$c_1(\omega) := \int_\omega x_1^2, \quad c_2(\omega) := \int_\omega x_2^2.$$

The thin infinite rod is represented with $\Omega^h := \omega^h \times \mathbb{R}$. The heterogeneity of the rod is introduced in the following way: fix $\varepsilon > 0$ (the period of material oscillation) and let $Y := [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$ be a "unit cell". The elastic properties of the heterogeneous material are given with the elasticity tensor

$$\mathbb{C} : Y \to \mathbb{R}^{3\times3\times3\times3}, \quad \mathbb{C} \in L^\infty(Y; \mathbb{R}^{3\times3\times3\times3}),$$

defined on the unit cell and then extended via $Y-$periodicity. We assume that $\mathbb{C}$ is uniformly positive definite on symmetric matrices, namely: $\exists \nu > 0$ such that

$$\nu|\xi|^2 \le \mathbb{C}(y)\xi : \xi \le \frac{1}{\nu}|\xi|^2, \quad \forall \xi \in \mathbb{R}^{3\times3}, \xi^T = \xi. \tag{3.1}$$

In addition, we require the following restrictions on the material coefficients:

$$\mathbb{C}_{ijkl}(y) = \mathbb{C}_{jikl}(y) = \mathbb{C}_{klij}(y), \quad \forall y \in Y, \quad i,j,k,l \in \{1,2,3\}.$$

For any point $(x_1, x_2, x_3/\varepsilon) \in \Omega^h$, the elasticity tensor is given with $\mathbb{C}(x_1, x_2, x_3/\varepsilon) := \mathbb{C}(x_3/\varepsilon)$. The following assumption yields significant simplification in the analysis, as we will see later. However, the assumption is physically relevant as it covers materials such as isentropic materials and more. Still, we carry out the analysis with and without this assumption as to showcase the different phenomena occurring in the rod dynamics.

Assumption 3.1.1. The elasticity tensor satisfies the following material symmetries:

$$\mathbb{C}_{ijk3}(y) = 0, \mathbb{C}_{i333}(y) = 0, \quad \forall y \in Y, \quad i,j,k \in \{1,2\}.$$

In this chapter, we are interested only in the regime where the period of material oscillations is of the same order as the thickness of the rod. Therefore we assume $\varepsilon = h$. We study the system of resolvent equations for the operator of 3D linear elasticity on the domain Ω^ε defined with the bilinear form:

$$H^1(\omega^\varepsilon \times \mathbb{R}; \mathbb{R}^3) \times H^1(\omega^\varepsilon \times \mathbb{R}; \mathbb{R}^3) \ni (\boldsymbol{u}, \boldsymbol{v}) \to \int_{\Omega^\varepsilon} \mathbb{C}\left(\tfrac{x_3}{\varepsilon}\right) \operatorname{sym} \nabla \boldsymbol{u} : \operatorname{sym} \nabla \boldsymbol{v} dx.$$

As it is standard in the dimension reduction, we transform the problem onto the canonical domain:

$$\omega^\varepsilon \times \mathbb{R} \ni (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) = x^\varepsilon \to x = (x_1, x_2, x_3) = (\tfrac{1}{\varepsilon} x_1^\varepsilon, \tfrac{1}{\varepsilon} x_2^\varepsilon, x_3^\varepsilon) \in \omega \times \mathbb{R}.$$

This change of coordinates allows us to work on the fixed domain. With these new coordinates, we define the following bilinear form:

$$a_\varepsilon : H^1(\omega \times \mathbb{R}; \mathbb{R}^3) \times H^1(\omega \times \mathbb{R}; \mathbb{R}^3) \to \mathbb{R}, \quad a_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) = \int_{\omega \times \mathbb{R}} \mathbb{C}\left(\tfrac{x_3}{\varepsilon}\right) \operatorname{sym} \nabla_\varepsilon \boldsymbol{u} : \operatorname{sym} \nabla_\varepsilon \boldsymbol{v} dx, \tag{3.2}$$

where the scaled gradient ∇_ε is defined with:

$$\nabla_\varepsilon \boldsymbol{u}(x) := \begin{bmatrix} \frac{1}{\varepsilon}\partial_1 \boldsymbol{u}_1 & \frac{1}{\varepsilon}\partial_2 \boldsymbol{u}_1 & \partial_3 \boldsymbol{u}_1 \\ \frac{1}{\varepsilon}\partial_1 \boldsymbol{u}_2 & \frac{1}{\varepsilon}\partial_2 \boldsymbol{u}_2 & \partial_3 \boldsymbol{u}_2 \\ \frac{1}{\varepsilon}\partial_1 \boldsymbol{u}_3 & \frac{1}{\varepsilon}\partial_2 \boldsymbol{u}_3 & \partial_3 \boldsymbol{u}_3 \end{bmatrix}.$$

The associated operator $\mathcal{A}_\varepsilon : \mathcal{D}(\mathcal{A}_\varepsilon) \to L^2(\omega \times \mathbb{R}; \mathbb{R}^3)$ is closed, densely defined in $L^2(\omega \times \mathbb{R}; \mathbb{R}^3)$ and self-adjoint.

In our analysis, we will make use of the orthogonal decomposition of the space $L^2(\omega \times \mathbb{R}; \mathbb{R}^3)$ into two spaces L^2_{bend} and L^2_{stretch}, defined with:

$$L^2_{\text{bend}} = \left\{\boldsymbol{u} \in L^2(\omega \times \mathbb{R}; \mathbb{R}^3), \quad \boldsymbol{u}_\alpha(\mathcal{S}(x_1, x_2)) = \boldsymbol{u}_\alpha(x_1, x_2), \alpha = 1, 2, \quad \boldsymbol{u}_3(\mathcal{S}(x_1, x_2)) = -\boldsymbol{u}_3(x_1, x_2)\right\}$$
$$L^2_{\text{stretch}} = \left\{\boldsymbol{u} \in L^2(\omega \times \mathbb{R}; \mathbb{R}^3), \quad \boldsymbol{u}_\alpha(\mathcal{S}(x_1, x_2)) = -u_\alpha(x_1, x_2), \alpha = 1, 2, \quad \boldsymbol{u}_3(\mathcal{S}(x_1, x_2)) = \boldsymbol{u}_3(x_1, x_2)\right\}$$

Functions belonging to these two spaces play the role of in-line forces and out-of-line forces, which, under some additional assumptions on the symmetry of the material response tensor, cause the rod to deform in the same way. We refer to these deformations as stretching and bending deformations, respectively.

3.1.2. Homogenised operators

In order to define homogenized limit operators, we make use of the following inclusion matrices:

$$\mathcal{J}^{\text{bend}}_{m_1,m_2}(\hat{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -x_1m_1 - x_2m_2 \end{bmatrix}, \quad \mathcal{J}^{\text{stretch}}_{m_3,m_4}(\hat{x}) = \begin{bmatrix} 0 & 0 & \frac{x_2m_3}{2} \\ 0 & 0 & \frac{-x_1m_3}{2} \\ \frac{x_2m_3}{2} & \frac{-x_1m_3}{2} & m_4 \end{bmatrix}$$

$$\mathcal{J}^{\text{rod}}_{m}(\hat{x}) = \begin{bmatrix} 0 & 0 & \frac{x_2m_3}{2} \\ 0 & 0 & \frac{-x_1m_3}{2} \\ \frac{x_2m_3}{2} & \frac{-x_1m_3}{2} & m_4 - x_1m_1 - x_2m_2 \end{bmatrix}, \quad m = (m_1, m_2, m_3, m_4) \in \mathbb{R}^4, \quad \hat{x} \in \omega.$$

The homogenized tensor $\mathbb{C}^{\text{rod}}$ containing the material properties of homogeneous rod is defined in the following way: For $m, d \in \mathbb{R}^4$ we define the form:

$$c^{\text{rod}}(m,d) := \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}^{\text{rod}}_m(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^m\right) : \mathcal{J}^{\text{rod}}_d(\hat{x})d\hat{x}dy,$$

where $\boldsymbol{u}^m \in H^1_\#(Y; H^1(\omega;\mathbb{R}^3))$ is the unique solution of:

$$\int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}^{\text{rod}}_m(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^m\right) : \operatorname{sym}\nabla \boldsymbol{v} d\hat{x}dy = 0, \quad \forall \boldsymbol{v} \in H^1_\#(Y; H^1(\omega;\mathbb{R}^3)). \tag{3.3}$$

Proposition 3.1.1. *The form c^{rod} is a positive bilinear form on $\mathbb{R}^4\times\mathbb{R}^4$, uniquely represented with a positive definite tensor $\mathbb{C}^{\text{rod}} \in \mathbb{R}^{4\times4}$, namely, there exists $\eta > 0$ such that*

$$c^{\text{rod}}(m,d) = \mathbb{C}^{\text{rod}} m\cdot d, \quad \mathbb{C}^{\text{rod}} m\cdot m \geq \eta|m|^2.$$

Proof. First we show that $c^{\text{rod}}(\cdot,\cdot)$ is bilinear. For that we consider:

$$c^{\text{rod}}(m+\alpha n,d) = \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}^{\text{rod}}_{m+\alpha n}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^{m+\alpha n}\right) : \mathcal{J}^{\text{rod}}_d(\hat{x})d\hat{x}dy, \quad m,n \in R^4, \alpha \in \mathbb{R},$$

$$\int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}^{\text{rod}}_{m+\alpha n}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^m\right) : \operatorname{sym}\nabla \boldsymbol{v} d\hat{x}dy = 0, \quad \forall \boldsymbol{v} \in H^1_\#(Y; H^1(\omega;\mathbb{R}^3)).$$

But $J^{\text{rod}}_{m+\alpha n}(x) = J^{\text{rod}}_m(x) + \alpha J^{\text{rod}}_n(x)$, and by the uniqueness of the solution of (3.3), we conclude that

$$u^{m+\alpha n} = u^m + \alpha u^n$$

and thus:

$$c^{\rm rod}(m+\alpha n,d) = \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}_m^{\rm rod}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^m\right) : \mathcal{J}_d^{\rm rod}(\hat{x}) d\hat{x}dy$$
$$+\alpha \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}_n^{\rm rod}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^n\right) : \mathcal{J}_d^{\rm rod}(\hat{x}) d\hat{x}dy = c^{\rm rod}(m,d) + \alpha c^{\rm rod}(n,d).$$

Therefore

$$c^{\rm rod}(m,d) = \mathbb{C}^{\rm rod} m\cdot d, \quad \mathbb{C}^{\rm rod} \in \mathbb{R}^{4\times 4}$$

and the entries of the tensor $\mathbb{C}^{\rm rod} m\cdot d$ can be represented with $[\mathbb{C}^{\rm rod} m\cdot d]_{i,j} = \mathbb{C}^{\rm rod} e_i \cdot e_j$. Notice that, since $\mathbb{C}(y)$ is symmetric, $\mathbb{C}^{\rm rod}$ is symmetric as well. Namely:

$$\mathbb{C}^{\rm rod} m\cdot d = \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}_m^{\rm rod}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^m\right) : \left(\mathcal{J}_d^{\rm rod}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^d\right) d\hat{x}dy$$
$$= \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}_d^{\rm rod}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^d\right) : \left(\mathcal{J}_m^{\rm rod}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^m\right) d\hat{x}dy = \mathbb{C}^{\rm rod} d\cdot m.$$

Now we see that $\mathbb{C}^{\rm rod} m\cdot d$ is actually uniquely defined with expressions of type $\mathbb{C}^{\rm rod} m\cdot m$, $m\in\mathbb{R}^4$. This is because:

$$\mathbb{C}^{\rm rod} m\cdot d = \frac{1}{2}\left[\mathbb{C}^{\rm rod} m\cdot m + \mathbb{C}^{\rm rod} d\cdot d + \mathbb{C}^{\rm rod}(m-d)\cdot(m-d)\right].$$

Notice now that the expression

$$\int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}_m^{\rm rod}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{\phi}\right) : \operatorname{sym}\nabla \boldsymbol{v} d\hat{x}dy$$

is a first variation of the quadratic functional:

$$\boldsymbol{\phi} \mapsto \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}_m^{\rm rod}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{\phi}\right) : \left(\mathcal{J}_m^{\rm rod}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{\phi}\right) d\hat{x}dy, \quad \boldsymbol{\phi}\in H^1_\#(Y;H^1(\omega;\mathbb{R}^3)).$$

Thus we have that:

$$\mathbb{C}^{\rm rod} m\cdot m = \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}_m^{\rm rod}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^m\right) : \left(\mathcal{J}_m^{\rm rod}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^m\right) d\hat{x}dy$$
$$= \inf_{\boldsymbol{\phi}\in H^1_\#(Y;H^1(\omega;\mathbb{R}^3))} \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}_m^{\rm rod}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{\phi}\right) : \left(\mathcal{J}_m^{\rm rod}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{\phi}\right), \quad m\in\mathbb{R}^4. \tag{3.4}$$

Therefore the expression (3.4) defines the tensor $\mathbb{C}^{\rm rod}$ uniquely. It is straight forward to show that the tensor $\mathbb{C}^{\rm rod}$ is positive definite. To see this, we use the pointwise coercivity estimate (3.1):

$$\begin{aligned}
\mathbb{C}^{\text{rod}} m\cdot m =& \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}_m^{\text{rod}}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^m\right) : \left(\mathcal{J}_m^{\text{rod}}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^m\right) d\hat{x}dy \\
\geq& C\left\|\mathcal{J}_m^{\text{rod}}(\hat{x}) + \operatorname{sym}\nabla \boldsymbol{u}^m\right\|^2_{L^2(\omega\times Y;\mathbb{R}^{3\times 3})} \\
\geq& C\left(\left\|x_2 m_3 - \left(\partial_1 \boldsymbol{u}_3^m + \partial_3 \boldsymbol{u}_1^m\right)\right\|^2_{L^2(\omega\times Y)} + \left\|-x_1 m_3 - \left(\partial_2 \boldsymbol{u}_3^m + \partial_3 \boldsymbol{u}_2^m\right)\right\|^2_{L^2(\omega\times Y)}\right) \\
&+ C\left\|m_4 - x_1 m_1 - x_2 m_2 - \partial_3 \boldsymbol{u}_3^m\right\|^2_{L^2(\omega\times Y)}.
\end{aligned}$$

It is clear that $\partial_3 \boldsymbol{u}_3^m \perp m_4 - x_1 m_1 - x_2 m_2$ in $L^2(\omega\times Y)$, so we have:

$$\left\|m_4 - x_1 m_1 - x_2 m_2 - \partial_3 \boldsymbol{u}_3^m\right\|^2_{L^2(\omega\times Y)} \geq \|m_4 - x_1 m_1 - x_2 m_2\|^2_{L^2(\omega\times Y)} \geq C\left(|m_1|^2 + |m_2|^2 + |m_4|^2\right).$$

On the other hand, we have:

$$\begin{aligned}
&\left\|x_2 m_3 - \left(\partial_1 \boldsymbol{u}_3^m + \partial_3 \boldsymbol{u}_1^m\right)\right\|^2_{L^2(\omega\times Y)} + \left\|-x_1 m_3 - \left(\partial_2 \boldsymbol{u}_3^m + \partial_3 \boldsymbol{u}_2^m\right)\right\|^2_{L^2(\omega\times Y)} = \\
&\qquad \left\| m_3 \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} - \partial_3 \begin{bmatrix} \boldsymbol{u}_1^m \\ \boldsymbol{u}_2^m \end{bmatrix} - \nabla_{\hat{x}} \boldsymbol{u}_3 \right\|^2_{L^2(\omega\times Y;\mathbb{R}^2)}.
\end{aligned}$$

Consider the projection operator P_G on $L^2(\omega;\mathbb{R}^2)$ onto the set $G := \left\{\nabla v, v \in H^1(\omega)\right\}$. The operator $I - P_G$ is bounded since $G^\perp$ is closed. We have:

$$\begin{aligned}
\left\| m_3 \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} - \partial_3 \begin{bmatrix} \boldsymbol{u}_1^m \\ \boldsymbol{u}_2^m \end{bmatrix} - \nabla_{\hat{x}} \boldsymbol{u}_3 \right\|^2_{L^2(\omega\times Y;\mathbb{R}^2)} &\geq C\left\|(I - P_G)\left(m_3 \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} - \partial_3 \begin{bmatrix} \boldsymbol{u}_1^m \\ \boldsymbol{u}_2^m \end{bmatrix}\right)\right\|^2_{L^2(\omega\times Y;\mathbb{R}^2)} \\
&= C\left(m_3 \left\|(I - P_G)\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}\right\|^2_{L^2(\omega\times Y;\mathbb{R}^2)} + \left\|(I - P_G)\partial_3 \begin{bmatrix} \boldsymbol{u}_1^m \\ \boldsymbol{u}_2^m \end{bmatrix}\right\|^2_{L^2(\omega\times Y;\mathbb{R}^2)}\right) \\
&\quad + C\int_Y \left\langle m_3 (I - P_G)\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, (I - P_G)\partial_3 \begin{bmatrix} \boldsymbol{u}_1^m \\ \boldsymbol{u}_2^m \end{bmatrix}\right\rangle_{L^2(\omega;\mathbb{R}^2)}.
\end{aligned}$$

But,

$$\begin{aligned}
\int_Y \left\langle m_3 (I - P_G)\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, (I - P_G)\partial_3 \begin{bmatrix} \boldsymbol{u}_1^m \\ \boldsymbol{u}_2^m \end{bmatrix}\right\rangle_{L^2(\omega;\mathbb{R}^2)} &= \int_Y \left\langle m_3 (I - P_G)\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, \partial_3 \begin{bmatrix} \boldsymbol{u}_1^m \\ \boldsymbol{u}_2^m \end{bmatrix}\right\rangle_{L^2(\omega;\mathbb{R}^2)} \\
&= m_3 \int_\omega \left\langle (I - P_G)\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, \partial_3 \begin{bmatrix} \boldsymbol{u}_1^m \\ \boldsymbol{u}_2^m \end{bmatrix}\right\rangle_{L^2(Y;\mathbb{R}^2)} = 0.
\end{aligned}$$

So:
$$\left\| m_3 \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} - \partial_3 \begin{bmatrix} \boldsymbol{u}_1^m \\ \boldsymbol{u}_2^m \end{bmatrix} - \nabla_{\hat{x}} \boldsymbol{u}_3 \right\|^2_{L^2(\omega\times Y;\mathbb{R}^2)} \geq C|m_3|^2.$$

■

In the case of material symmetries 3.1.1 we make use of the following matrices of order 2:

$$\begin{aligned}
&\mathbb{C}^{\text{bend}}(m_1,m_2)^T \cdot (m_1,m_2)^T \\
&\quad := \inf_{\phi\in H^1_\#(Y;H^1(\omega;\mathbb{R}^3))} \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}^{\text{bend}}_{m_1,m_2}(\hat{x}) + \operatorname{sym}\nabla\phi\right) : \left(\mathcal{J}^{\text{bend}}_{m_1,m_2}(\hat{x}) + \operatorname{sym}\nabla\phi\right), \quad (3.5) \\
&\quad (m_1,m_2)\in\mathbb{R}^2.
\end{aligned}$$

$$\begin{aligned}
&\mathbb{C}^{\text{stretch}}(m_3,m_4)^T \cdot (m_3,m_4)^T \\
&\quad := \inf_{\phi\in H^1_\#(Y;H^1(\omega;\mathbb{R}^3))} \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}^{\text{stretch}}_{m_3,m_4}(\hat{x}) + \operatorname{sym}\nabla\phi\right) : \left(\mathcal{J}^{\text{stretch}}_{m_3,m_4}(\hat{x}) + \operatorname{sym}\nabla\phi\right) (3.6) \\
&\quad (m_3,m_4)\in\mathbb{R}^2.
\end{aligned}$$

We have an easy consequence:

Corollary 3.1.2. *There exists a constant* $\nu > 0$ *such that*

$$\mathbb{C}^{\text{bend}}(m_1,m_2)^T \cdot (m_1,m_2)^T \geq \nu|(m_1,m_2)|^2, \quad \mathbb{C}^{\text{stretch}}(m_3,m_4)^T \cdot (m_3,m_4)^T \geq \nu|(m_3,m_4)|^2.$$

The homogenised matrices (3.5) and (3.6) can equivalently be defined with:

$$\mathbb{C}^{\text{bend}}(m_1,m_2)^T \cdot (d_1,d_2)^T := \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}^{\text{bend}}_{m_1,m_2}(\hat{x}) + \operatorname{sym}\nabla\boldsymbol{u}_{m_1,m_2}\right) : \mathcal{J}^{\text{bend}}_{m_1,m_2}(\hat{x}),$$
$$(m_1,m_2),(d_1,d_2)\in\mathbb{R}^2,$$

$$\mathbb{C}^{\text{stretch}}(m_3,m_4)^T \cdot (m_3,m_4)^T := \int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}^{\text{stretch}}_{m_3,m_4}(\hat{x}) + \operatorname{sym}\nabla\boldsymbol{u}_{m_3,m_4}\right) : \mathcal{J}^{\text{stretch}}_{m_3,m_4}(\hat{x}),$$
$$(m_3,m_4),(d_3,d_4)\in\mathbb{R}^2,$$

where the functions $\boldsymbol{u}_{m_1,m_2}$ and $\boldsymbol{u}_{m_3,m_4}$ are the solutions to the cell problems:

$$\int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}^{\text{bend}}_{m_1,m_2}(\hat{x}) + \operatorname{sym}\nabla\boldsymbol{u}_{m_1,m_2}\right) : \operatorname{sym}\nabla\boldsymbol{v}\, d\hat{x}dy = 0, \quad \forall\boldsymbol{v}\in H^1_\#(Y;H^1(\omega;\mathbb{R}^3)).$$

$$\int_{\omega\times Y} \mathbb{C}(y)\left(\mathcal{J}^{\text{stretch}}_{m_3,m_4}(\hat{x}) + \operatorname{sym}\nabla\boldsymbol{u}_{m_3,m_4}\right) : \operatorname{sym}\nabla\boldsymbol{v}\, d\hat{x}dy = 0, \quad \forall\boldsymbol{v}\in H^1_\#(Y;H^1(\omega;\mathbb{R}^3)).$$

It is clear that we have the following decompositions:

$$\mathbb{C}^{\mathrm{rod}}(m_1,m_2,0,0)^T\cdot(d_1,d_2,0,0)^T=\mathbb{C}^{\mathrm{bend}}(m_1,m_2)^T\cdot(d_1,d_2)^T,$$
$$\mathbb{C}^{\mathrm{rod}}(0,0,m_3,m_4)^T\cdot(0,0,d_3,d_4)^T=\mathbb{C}^{\mathrm{stretch}}(m_3,m_4)^T\cdot(d_3,d_4)^T,$$

while in the case of the material symmetries 3.1.1:

$$\mathbb{C}^{\mathrm{rod}}m\cdot d=\mathbb{C}^{\mathrm{bend}}(m_1,m_2)^T\cdot(d_1,d_2)^T+\mathbb{C}^{\mathrm{stretch}}(m_3,m_4)^T\cdot(d_3,d_4)^T.$$

Next we define the homogenized limit differential operators given with the following differential expressions:

$$\mathcal{A}^{\mathrm{bend}}=\frac{d^2}{dx_3^2}\mathbb{C}^{\mathrm{bend}}\frac{d^2}{dx_3^2},\quad \mathcal{A}^{\mathrm{stretch}}=\frac{d}{dx_3}\mathbb{C}^{\mathrm{stretch}}\frac{d}{dx_3},$$
$$\mathcal{A}^{\mathrm{rod}}_\varepsilon=\left(\varepsilon\frac{d^2}{dx_3^2},\varepsilon\frac{d^2}{dx_3^2},\frac{d}{dx_3},\frac{d}{dx_3}\right)^T\mathbb{C}^{\mathrm{rod}}\left(\varepsilon\frac{d^2}{dx_3^2},\varepsilon\frac{d^2}{dx_3^2},\frac{d}{dx_3},\frac{d}{dx_3}\right)^T.$$

with the domains:

$$\mathcal{D}(\mathcal{A}^{\mathrm{bend}}):=H^4(\mathbb{R};\mathbb{R}^2),\quad \mathcal{D}(\mathcal{A}^{\mathrm{stretch}}):=H^2(\mathbb{R};\mathbb{R}^2),\quad \mathcal{D}(\mathcal{A}^{\mathrm{rod}}_\varepsilon):=H^4(\mathbb{R};\mathbb{R}^2)\times H^2(\mathbb{R};\mathbb{R}^2).$$

3.1.3. Gelfand transform and periodic decomposition

We denote the Sobolev space of $Y-$periodic functions in third variable with $H^1_\#(Y;H^1(\omega;\mathbb{C}^3))$. For every parameter $\chi\in[-\pi,\pi]$, we define the parametrized family of Sobolev spaces of $Y-$quasiperiodic functions:

$$H^1_\chi(Y;H^1(\omega;\mathbb{C}^3)):=\left\{e^{i\chi y}\boldsymbol{u}(x_1,x_2,y),\quad \boldsymbol{u}\in H^1_\#(Y;H^1(\omega;\mathbb{C}^3))\right\},\quad \chi\in[-\pi,\pi].$$

Analogously, we define spaces $L^2_\#(Y;L^2(\omega;\mathbb{C}^3))$ and $L^2_\chi(Y;L^2(\omega;\mathbb{C}^3))$.
For fixed $\varepsilon>0$ we define the operator $\mathcal{G}_\varepsilon$ on $L^2(\omega\times\mathbb{R};\mathbb{R}^3)$ with the formula:

$$(\mathcal{G}_\varepsilon\boldsymbol{u})(x_1,x_2,y,\chi):=\sqrt{\frac{\varepsilon}{2\pi}}\sum_{n\in\mathbb{Z}}e^{-i\chi(y+n)}\boldsymbol{u}(x_1,x_2,\varepsilon(y+n)),\quad (x_1,x_2,y)\in\omega\times\mathbb{R},\quad \chi\in[-\pi,\pi].$$

We refer to this operator as the scaled Gelfand transform. Note that the scaled Gelfand transform $\mathcal{G}_\varepsilon$ transforms functions into $Y-$periodic functions in variable y, namely:

$$(\mathcal{G}_\varepsilon\boldsymbol{u})(x_1,x_2,y+1,\chi)=(\mathcal{G}_\varepsilon\boldsymbol{u})(x_1,x_2,y,\chi),\quad a.e.$$

The scaled Gelfand transform is an isometry

$$\mathcal{G}_\varepsilon : L^2(\omega\times\mathbb{R};\mathbb{R}^3) \to L^2([-\pi,\pi];L^2_\#(Y;L^2(\omega;\mathbb{C}^3))) = \int^{\oplus}_{[-\pi,\pi]} L^2_\#(Y,\chi;L^2(\omega;\mathbb{C}^3))d\chi.$$

in the sense that:

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)} = \int_{-\pi}^{\pi} \langle \mathcal{G}_\varepsilon\boldsymbol{u},\mathcal{G}_\varepsilon\boldsymbol{v}\rangle_{L^2_\#(\omega\times Y;\mathbb{C}^3)}d\chi, \quad \forall \boldsymbol{u},\boldsymbol{v}\in L^2(\omega\times\mathbb{R};\mathbb{R}^3).$$

The original can be reconstructed with the following formula:

$$\boldsymbol{u}(x_1,x_2,x_3) = \frac{1}{\sqrt{2\pi\varepsilon}}\int_{-\pi}^{\pi} e^{i\chi x_3/\varepsilon}(\mathcal{G}_\varepsilon\boldsymbol{u})(x_1,x_2,x_3/\varepsilon,\chi)d\chi.$$

This formula can be interpreted as decomposing $L^2(\omega\times\mathbb{R},\mathbb{R}^3)$ into a direct integral

$$L^2(\omega\times\mathbb{R};\mathbb{R}^3) = \int^{\oplus}_{[-\pi,\pi]} L^2_\#(Y,\chi;L^2(\omega;\mathbb{C}^3))d\chi.$$

Also, by noting that the scaled Gelfand transform commutes with derivatives in the following way:

$$\mathcal{G}_\varepsilon(\partial_{x_\alpha})\boldsymbol{u} = \partial_{x_\alpha}(\mathcal{G}_\varepsilon\boldsymbol{u}), \quad \mathcal{G}_\varepsilon(\partial_{x_3}\boldsymbol{u}) = \frac{1}{\varepsilon}(\partial_y(\mathcal{G}_\varepsilon\boldsymbol{u}) + i\chi\mathcal{G}_\varepsilon\boldsymbol{u}),$$

we can see that

$$a_\varepsilon(\boldsymbol{u},\boldsymbol{v}) = \frac{1}{\varepsilon^2}a_\chi(\mathcal{G}_\varepsilon\boldsymbol{u},\mathcal{G}_\varepsilon\boldsymbol{v}), \quad \forall\chi\in[-\pi,\pi],$$

where

$$a_\chi(\boldsymbol{u},\boldsymbol{v}) := \int_{\omega\times Y}\mathbb{C}(y)(\operatorname{sym}\nabla + iX_\chi)\boldsymbol{u} : \overline{(\operatorname{sym}\nabla + iX_\chi)\boldsymbol{v}}, \quad \boldsymbol{u},\boldsymbol{v}\in H^1_\#(Y;H^1(\omega;\mathbb{C}^3)).$$

The operator X_χ, acting on the space $L^2(\omega\times Y;\mathbb{C}^3)$ is defined with:

$$X_\chi\boldsymbol{u} = \begin{bmatrix} 0 & 0 & \frac{1}{2}\chi\boldsymbol{u}_1 \\ 0 & 0 & \frac{1}{2}\chi\boldsymbol{u}_2 \\ \frac{1}{2}\chi\boldsymbol{u}_1 & \frac{1}{2}\chi\boldsymbol{u}_2 & \chi\boldsymbol{u}_3 \end{bmatrix}.$$

For a fixed $\chi\in[-\pi,\pi]$, we define the operator

$$\mathcal{A}_\chi := (\operatorname{sym}\nabla + iX_\chi)^*\mathbb{C}(y)(\operatorname{sym}\nabla + iX_\chi) : \mathcal{D}(\mathcal{A}_\chi)\subset H^1_\#(Y;H^1(\omega;\mathbb{C}^3)) \to L^2(\omega\times Y,\mathbb{C}^3)$$

associated with this form.

The scaled Gelfand transform, applied to the resolvent problem, can be depicted with the following equality:

$$(\mathcal{A}_\varepsilon + I)^{-1} = \mathcal{G}_\varepsilon^{-1}\left(\int^{\oplus}_{[-\pi,\pi]}\left(\frac{1}{\varepsilon^2}\mathcal{A}_\chi + I\right)^{-1}d\chi\right)\mathcal{G}_\varepsilon. \tag{3.7}$$

We interpret this in the following way: by using Gelfand transform for transforming the problem, we have decomposed the resolvent operator $(\mathcal{A}_\varepsilon + I)^{-1}$ into the continuous family of resolvent operators $\left(\frac{1}{\varepsilon^2}\mathcal{A}_\chi + I\right)^{-1}$ indexed by $\chi \in [-\pi,\pi]$. As we will see, in contrast to the original resolvent operator, this family consists of compact operators with discrete spectrum.

Closely related to the scaled Gelfand transform is the scaled Floquet transform defined with:

$$(\mathcal{F}_\varepsilon \boldsymbol{u})(x_1,x_2,y,\chi) := \sqrt{\frac{\varepsilon}{2\pi}} \sum_{n\in\mathbb{Z}} e^{-i\chi n} \boldsymbol{u}(x_1,x_2,\varepsilon(y+n)), \quad (x_1,x_2,y) \in \omega\times Y, \quad \chi \in [-\pi,\pi].$$

For every $\chi \in [-\pi,\pi]$, the function $(\mathcal{F}_\varepsilon \boldsymbol{u})(x_1,x_2,y,\chi)$ belongs to the space of quasiperiodic functions $H^1_\chi(Y;H^1(\omega;\mathbb{C}^3))$. The link between Floquet and Gelfand transform is the following identity:

$$\mathcal{F}_\varepsilon \boldsymbol{u} = e^{i\chi y} \mathcal{G}_\varepsilon \boldsymbol{u}, \quad \forall \boldsymbol{u} \in L^2(\omega\times\mathbb{R};\mathbb{R}^3).$$

The scaled Floquet transform is an isometry as well:

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)} = \int_{-\pi}^{\pi} \langle \mathcal{F}_\varepsilon \boldsymbol{u}, \mathcal{F}_\varepsilon \boldsymbol{v}\rangle_{L^2_{\chi,y}(\omega\times Y;\mathbb{C}^3)}\, d\chi.$$

Similarily we have:

$$a_\varepsilon(\boldsymbol{u},\boldsymbol{v}) = \frac{1}{\varepsilon^2} a(\mathcal{F}_\varepsilon \boldsymbol{u}, \mathcal{F}_\varepsilon \boldsymbol{v}),$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) := \int_{\omega\times Y} \mathbb{C}(y)\,\mathrm{sym}\,\nabla \boldsymbol{u} : \overline{\mathrm{sym}\,\nabla \boldsymbol{v}}, \quad \boldsymbol{u},\boldsymbol{v} \in H^1_\chi(Y;H^1(\omega;\mathbb{C}^3)).$$

This is due to the following formulae:

$$\mathcal{F}_\varepsilon(\partial_{x_\alpha})\boldsymbol{u} = \partial_{x_\alpha}(\mathcal{F}_\varepsilon \boldsymbol{u}), \quad \mathcal{F}_\varepsilon(\partial_{x_3}\boldsymbol{u}) = \frac{1}{\varepsilon}\partial_y(\mathcal{F}_\varepsilon \boldsymbol{u}).$$

The importance of the operator X_χ lies in the fact that for quasiperiodic functions $\boldsymbol{w} \in H^1_\chi(Y;H^1(\omega;\mathbb{C}^3))$, namely $\boldsymbol{w} = e^{i\chi y}\boldsymbol{u}(x_1,x_2,y)$, where $\boldsymbol{u} \in H^1_\#(Y;H^1(\omega;\mathbb{C}^3))$ we have:

$$\mathrm{sym}\,\nabla \boldsymbol{w} = e^{i\chi y}(\mathrm{sym}\,\nabla \boldsymbol{u} + iX_\chi \boldsymbol{u}).$$

Also note that for all $\boldsymbol{u} \in L^2(\omega\times Y;\mathbb{C}^3)$ we have:

$$C_1|\chi|\|\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^3)} \le \|X_\chi \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^3)} \le C_2|\chi|\|\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

For an overview of the use of Gelfand and Floquet transform one can consult the works [30], [41].

3.1.3.1 Formulation in terms of scaled quasimomenta

Here we note the alternative definition of Gelfand transform in terms of scaled quasimomentum $\theta := \chi/\varepsilon$:

$$(\mathcal{G}_\varepsilon \boldsymbol{u})(x_1,x_2,y,\theta) := \sqrt{\frac{\varepsilon}{2\pi}}\sum_{n\in\mathbb{Z}} e^{-i\theta\varepsilon(y+n)}\boldsymbol{u}(x_1,x_2,\varepsilon(y+n)), \quad (x_1,x_2,y)\in\omega\times\mathbb{R}, \quad \theta\in[-\pi/\varepsilon,\pi/\varepsilon],$$

where the inverse is given with:

$$\left(\mathcal{G}_\varepsilon^{-1}\boldsymbol{U}\right)(x_1,x_2,x_3) = \sqrt{\frac{\varepsilon}{2\pi}}\int_{-\pi/\varepsilon}^{\pi/\varepsilon} e^{i\theta x_3}\boldsymbol{U}(x_1,x_2,x_3/\varepsilon,\theta)d\theta.$$

It is straightforward to link the Gelfand transform with the Fourier transform:

$$\begin{aligned}
\int_Y (\mathcal{G}_\varepsilon \boldsymbol{u})(x_1,x_2,y,\theta)dy &= \int_Y \sqrt{\frac{\varepsilon}{2\pi}}\sum_{n\in\mathbb{Z}} e^{-i\theta\varepsilon(y+n)}\boldsymbol{u}(x_1,x_2,\varepsilon(y+n))dy \\
&= \sqrt{\frac{\varepsilon}{2\pi}}\sum_{n\in\mathbb{Z}}\int_Y e^{-i\theta\varepsilon(y+n)}\boldsymbol{u}(x_1,x_2,\varepsilon(y+n))dy \\
&= \frac{1}{\sqrt{2\pi\varepsilon}}\sum_{n\in\mathbb{Z}}\int_{\varepsilon(Y+n)} e^{-i\theta y}\boldsymbol{u}(x_1,x_2,y)dy \\
&= \frac{1}{\sqrt{2\pi\varepsilon}}\int_{\mathbb{R}} e^{-i\theta y}\boldsymbol{u}(x_1,x_2,y)dy = \frac{1}{\sqrt{2\pi\varepsilon}}\widehat{\boldsymbol{u}}(x_1,x_2,\theta/2\pi),
\end{aligned}$$

where $\hat{\boldsymbol{u}}$ denotes the Fourier transform of $\boldsymbol{u}$.

3.1.4. Smoothing operator

We define the following smoothing operator $\Xi_\varepsilon : L^2(\omega\times\mathbb{R}) \to L^2(\omega\times\mathbb{R})$:

$$\Xi_\varepsilon f := \mathcal{G}_\varepsilon^{-1}\int_Y (\mathcal{G}_\varepsilon f)(y)dy,$$

which appears in the approximative problem definition. The purpose of Ξ_ε is cutting of the high frequencies in a function, namely frequencies higher than $\frac{1}{2\varepsilon}$. To see this, we calculate:

$$\begin{aligned}
\Xi_\varepsilon f &:= \mathcal{G}_\varepsilon^{-1}\int_Y (\mathcal{G}_\varepsilon f)(y)dy = \mathcal{G}_\varepsilon^{-1}\left(\frac{1}{\sqrt{2\pi\varepsilon}}\widehat{f}(\theta/2\pi)\right) \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{\varepsilon}}^{\frac{\pi}{\varepsilon}} e^{i\theta x}\widehat{f}(\theta/2\pi)d\theta = \int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}} e^{2\pi i\theta x}\widehat{f}(\theta)d\theta \\
&= \widecheck{\mathbb{1}_{\left[\frac{-1}{2\varepsilon},\frac{1}{2\varepsilon}\right]}} * f,
\end{aligned}$$

where $\check{\ }$ denotes the inverse Fourier transform.

3.1.5. Main results of Chapter 3

In order to provide the link between the ε problem and the homogenized limit problem, we define the following force momentum operators:

$$\mathcal{M}_\varepsilon^{\rm bend}\begin{bmatrix}\boldsymbol{f}_1\\ \boldsymbol{f}_2\\ \boldsymbol{f}_3\end{bmatrix} := \int_\omega \begin{bmatrix}\boldsymbol{f}_1-\varepsilon x_1\frac{d}{dx_3}\boldsymbol{f}_3\\ \boldsymbol{f}_2-\varepsilon x_2\frac{d}{dx_3}\boldsymbol{f}_3\end{bmatrix},\quad \mathcal{M}_\varepsilon^{\rm stretch}\begin{bmatrix}\boldsymbol{f}_1\\ \boldsymbol{f}_2\\ \boldsymbol{f}_3\end{bmatrix} := \int_\omega \begin{bmatrix}x_2\boldsymbol{f}_1-x_1\boldsymbol{f}_2\\ \boldsymbol{f}_3\end{bmatrix},$$

$$\mathcal{M}_\varepsilon^{\rm rod}\begin{bmatrix}\boldsymbol{f}_1\\ \boldsymbol{f}_2\\ \boldsymbol{f}_3\end{bmatrix} := \begin{bmatrix}\mathcal{M}_\varepsilon^{\rm bend}\boldsymbol{f}\\ \mathcal{M}_\varepsilon^{\rm stretch}\boldsymbol{f}\end{bmatrix} = \int_\omega \begin{bmatrix}\boldsymbol{f}_1-\varepsilon x_1\frac{d}{dx_3}\boldsymbol{f}_3\\ \boldsymbol{f}_2-\varepsilon x_2\frac{d}{dx_3}\boldsymbol{f}_3\\ x_2\boldsymbol{f}_1-x_1\boldsymbol{f}_2\\ \boldsymbol{f}_3\end{bmatrix}.$$

Note that for $f_3\in L^2(\omega\times\mathbb{R};\mathbb{R})$, $\mathcal{M}_\varepsilon^{\rm bend}\boldsymbol{f}$ takes values in $H^{-1}(\omega\times\mathbb{R};\mathbb{R}^2)$.

In order to display the possibility of obtaining the full physical model by allowing the heterogeneity in the order of force terms, we make use of the following scaling matrix:

$$S_\varepsilon := \begin{bmatrix}1&0&0\\0&1&0\\0&0&1/\varepsilon\end{bmatrix},\quad \varepsilon>0.$$

We label with $P_i:\mathbb{R}^3\to\mathbb{R}$ the projection on the $i-th$ coordinate, as well as $\pi_i:\mathbb{R}^2\to\mathbb{R}$. We are able to prove the following results:

Theorem 3.1.3 ($L^2\to L^2$ norm-resolvent estimate)**.** *Let $\gamma>-2$ be the parameter of spectral scaling. Let $\delta\geq 0$ be the parameter of force term scaling. There exists $C>0$ such that for every $\varepsilon>0$ we have:*

$$\left\|P_i\left(\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon+I\right)^{-1}-(\mathcal{M}_\varepsilon^{\rm rod})^*\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon^{\rm rod}+C^{\rm rod}(\omega)\right)^{-1}\mathcal{M}_\varepsilon^{\rm rod}\Xi_\varepsilon\right)\right\|_{L^2\to L^2}\leq\begin{cases}C\varepsilon^{\frac{\gamma+2}{4}}, & i=1,2;\\ C\varepsilon^{\frac{\gamma+2}{2}}, & i=3.\end{cases}$$

Under the additional assumption of the material symmetries 3.1.1 we have:

$$\left\| \left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon + I\right)^{-1}|_{L^2_{\text{stretch}}} - (\mathcal{M}^{\text{stretch}})^*\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}} + C^{\text{stretch}}(\omega)\right)^{-1}\mathcal{M}^{\text{stretch}}\Xi_\varepsilon\right\|_{L^2\to L^2} \le C\varepsilon^{\frac{\gamma+2}{2}},$$

$$\left\| P_i\left(\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon + I\right)^{-1}|_{L^2_{\text{bend}}} - (\mathcal{M}^{\text{bend}}_\varepsilon)^*\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}} + I\right)^{-1}\mathcal{M}^{\text{bend}}_\varepsilon\Xi_\varepsilon\right)S_{\varepsilon^\delta}\right\|_{L^2\to L^2}$$

$$\le \begin{cases} C\varepsilon^{\frac{\gamma+2}{4}}\max\left\{\varepsilon^{\frac{\gamma+2}{4}-\delta}, 1\right\}, & i = 1,2; \\ C\varepsilon^{\frac{\gamma+2}{2}}\max\left\{\varepsilon^{\frac{\gamma+2}{4}-\delta}, 1\right\}, & i = 3. \end{cases}$$

Remark 3.1.4. In the Theorem 3.1.3, the operators $(\mathcal{M}^{\text{rod}}_\varepsilon)^*$, $(\mathcal{M}^{\text{bend}}_\varepsilon)^*$ and $(\mathcal{M}^{\text{stretch}})^*$ denote the adjoints of force momentum operators. The link between the Kirchoff-Love expression for rods and the associated force moments is given with the following operators:

$$\mathcal{I}^{\text{bend}}_\varepsilon : H^1(\mathbb{R};\mathbb{R}^2) \to L^2_{\text{bend}}(\omega\times\mathbb{R};\mathbb{R}^3), \quad \mathcal{I}^{\text{bend}}_\varepsilon \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ -\varepsilon\frac{d}{dx_3}(x_1\boldsymbol{u}_1 + x_2\boldsymbol{u}_2) \end{bmatrix},$$

$$\mathcal{I}^{\text{stretch}} : H^1(\mathbb{R};\mathbb{R}^2) \to L^2_{\text{stretch}}(\omega\times\mathbb{R};\mathbb{R}^3), \quad \mathcal{I}^{\text{stretch}} \begin{bmatrix} \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \end{bmatrix} = \begin{bmatrix} x_2\boldsymbol{u}_3 \\ -x_1\boldsymbol{u}_3 \\ \boldsymbol{u}_4 \end{bmatrix}.$$

Note that we have the following duality relations:

$$\left\langle \mathcal{I}^{\text{bend}}_\varepsilon \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{bmatrix} \right\rangle = \int_{\omega\times\mathbb{R}} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ -\varepsilon\frac{d}{dx_3}(x_1\boldsymbol{u}_1 + x_2\boldsymbol{u}_2) \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{bmatrix}$$

$$= \int_{\mathbb{R}} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix} \cdot \int_\omega \begin{bmatrix} \boldsymbol{f}_1 - \varepsilon x_1\frac{d}{dx_3}\boldsymbol{f}_3 \\ \boldsymbol{f}_2 - \varepsilon x_2\frac{d}{dx_3}\boldsymbol{f}_3 \end{bmatrix} = \left\langle \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}, \mathcal{M}^{\text{bend}}_\varepsilon \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{bmatrix} \right\rangle.$$

$$\left\langle \mathcal{I}^{\text{stretch}} \begin{bmatrix} \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \end{bmatrix}, \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{bmatrix} \right\rangle = \int_{\omega\times\mathbb{R}} \begin{bmatrix} x_2\boldsymbol{u}_3 \\ -x_1\boldsymbol{u}_3 \\ \boldsymbol{u}_4 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{bmatrix}$$

$$= \int_{\mathbb{R}} \begin{bmatrix} \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \end{bmatrix} \cdot \int_{\omega} \begin{bmatrix} x_2 \boldsymbol{f}_1 - x_1 \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{bmatrix} = \left\langle \begin{bmatrix} \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \end{bmatrix}, \mathcal{M}^{\mathrm{stretch}} \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{bmatrix} \right\rangle .$$

These relations show that, in the suitable sense, we have the following duality:

$$\mathcal{I}^{\mathrm{stretch}} = (\mathcal{M}^{\mathrm{stretch}})^*, \quad \mathcal{I}_\varepsilon^{\mathrm{bend}} = (\mathcal{M}_\varepsilon^{\mathrm{bend}})^*.$$

Remark 3.1.5. The matrices $C^{\mathrm{stretch}}(\omega)$ and $C^{\mathrm{rod}}(\omega)$ represent additional effects coming from the torsion deformations and the shape of the domain ω, and will be formally defined in the continuation of the text, namely (3.11).

Remark 3.1.6. Notice that in the Theorem 3.1.3, the operator $\mathcal{M}_\varepsilon^{\mathrm{bend}}$ is composed with the smoothing operator Ξ_ε. Thus, for $\boldsymbol{f} \in L^2(\omega\times\mathbb{R};\mathbb{R}^3)$, the resulting loads $\mathcal{M}_\varepsilon^{\mathrm{bend}}\Xi_\varepsilon \boldsymbol{f}$ belong to $L^2(\omega\times\mathbb{R};\mathbb{R}^2)$ (instead of $H^{-1}(\omega\times\mathbb{R};\mathbb{R}^2)$).

Remark 3.1.7. The role of the parameter of force scaling δ is to enrich the model. It is known in the analysis of thin structures (plates and rods), as a consequence of anisotropy, one needs to incorporate different scalings of loads depending on the direction, which yields the richer structure of the limiting model, see [23,52,68]. One could argue that the most interesting cases of parameters γ, δ would be $\gamma = 0$, $\delta = 0$ and $\gamma = 2$, $\delta = 1$, since these are the standard regimes which emerge when studying thin structures on a finite domain, see [68]. However, in the case of infinite rod (similarly like in the case of infinite plate [20]), there is no natural spectral scaling. The role of the parameter γ becomes clear in the case of evolution, where it serves to obtain the models in different time scales (the evolution models of plates and rods are usually analyzed with $\gamma = 2$, see [54], [68])

Remark 3.1.8. Actually, the smoothing operator Ξ_ε can be removed from the estimates in Theorem 3.1.3 while preserving the order of the estimates. This is shown in the Corollary 3.4.5. One easy consequence of the Theorem 3.1.3 are the estimates on the band gaps in the spectrum. See Corollary 3.4.8.

Remark 3.1.9. It is possible to obtain the the same estimates as in the Theorem 3.1.3, even in the case when the ratio $\frac{h}{\varepsilon}$ belongs to the fixed interval $[\alpha,\beta]$, but the constant in the estimates will depend on the α and β. See Section 8 in [20]. The same is true for Theorem 3.1.10 and Theorem 3.1.11.

3.1.5.1 Higher precision estimates

We also find the resolvent estimates in the $L^2 \to H^1$ operator norm, as follows:

Theorem 3.1.10 ($L^2 \to H^1$ norm-resolvent estimate)**.** *Let $\gamma > -2$ be the parameter of spectral scaling. Let $\delta \geq 0$ be the parameter of force term scaling. There exists $C > 0$ such that for every $\varepsilon > 0$ we have:*

$$\left\| P_i \left(\left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} - (\mathcal{M}_\varepsilon^{\mathrm{rod}})^* \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon^{\mathrm{rod}} + C^{\mathrm{rod}}(\omega) \right)^{-1} \mathcal{M}_\varepsilon^{\mathrm{rod}} \Xi_\varepsilon - \mathcal{A}_{\mathrm{rod}}^{\mathrm{corr}}(\varepsilon) \right) \right\|_{L^2 \to H^1} \leq \begin{cases} C\varepsilon^{\frac{\gamma+2}{4}}, & i = 1,2; \\ C\varepsilon^{\frac{\gamma+2}{2}}, & i = 3. \end{cases}$$

Under the additional assumption of the material symmetries 3.1.1 we have:

$$\left\| \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} |_{L^2_{\mathrm{stretch}}} - (\mathcal{M}^{\mathrm{stretch}})^* \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}^{\mathrm{stretch}} + C^{\mathrm{stretch}}(\omega) \right)^{-1} \mathcal{M}^{\mathrm{stretch}} \Xi_\varepsilon - \mathcal{A}_{\mathrm{stretch}}^{\mathrm{corr}}(\varepsilon) \right\|_{L^2 \to H^1} \leq C \max \left\{ \varepsilon^{\gamma+1}, \varepsilon^{\frac{\gamma+2}{2}} \right\},$$

$$\left\| P_i \left(\left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} |_{L^2_{\mathrm{bend}}} - (\mathcal{M}_\varepsilon^{\mathrm{bend}})^* \left(\frac{1}{\varepsilon^{\gamma-2}} \mathcal{A}^{\mathrm{bend}} + I \right)^{-1} \mathcal{M}_\varepsilon^{\mathrm{bend}} \Xi_\varepsilon - \mathcal{A}_{\mathrm{bend}}^{\mathrm{corr}}(\varepsilon) \right) S_{\varepsilon^\delta} \right\|_{L^2 \to H^1} \leq \begin{cases} C \max \left\{ \varepsilon^{\frac{\gamma+2}{4}}, \varepsilon^{\frac{\gamma}{2}} \right\} \max \left\{ \varepsilon^{\frac{\gamma+2}{4} - \delta}, 1 \right\}, & \text{if } i = 1,2; \\ C \max \left\{ \varepsilon^{\frac{\gamma+2}{2}}, \varepsilon^{\frac{3\gamma+2}{4}} \right\} \max \left\{ \varepsilon^{\frac{\gamma+2}{4} - \delta}, 1 \right\}, & \text{if } i = 3. \end{cases}$$

The operators $\mathcal{A}_{\mathrm{bend}}^{\mathrm{corr}}(\varepsilon)$ and $\mathcal{A}_{\mathrm{stretch}}^{\mathrm{corr}}(\varepsilon)$ are standard first order correctors in the theory of homogenisation, defined in the following text with the expression (3.37).

Our asymptotic analysis resulted in obtaining correctors which allow us to calculate $L^2 \to L^2$ norm resolvent estimates with even higher precision. These corrector terms were previously unknown in the theory of heterogeneous elastic rods, however they resemble the higher order correctors which appear in the works of Birman and Suslina. For the precise definition of these corrector operators consider (3.39).

Theorem 3.1.11 (Higher order $L^2 \to L^2$ norm-resolvent estimate)**.** *Let $\gamma > -2$ be the parameter of spectral scaling. Let $\delta \geq 0$ be the parameter of force term scaling. There* *exists $C > 0$ such that for every $\varepsilon > 0$ we have:*

$$\left\| P_i \left(\left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} - (\mathcal{M}_\varepsilon^{\mathrm{rod}})^* \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}^{\mathrm{hom}}_{\mathrm{rod},\varepsilon} + C^{\mathrm{rod}}(\omega) \right)^{-1} \mathcal{M}_\varepsilon^{\mathrm{rod}} \Xi_\varepsilon - \mathcal{A}^{\mathrm{corr}}_{\mathrm{rod}}(\varepsilon) - \widetilde{\mathcal{A}}^{\mathrm{corr}}_{\mathrm{rod}}(\varepsilon) \right) \right\|_{L^2 \to L^2} \leq \begin{cases} C\varepsilon^{\frac{\gamma+2}{2}}, & i = 1,2; \\ C\varepsilon^{\frac{3(\gamma+2)}{4}}, & i = 3. \end{cases}$$

Under the additional assumption of the material symmetries 3.1.1 we have:

$$\left\| \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} |_{L^2_{\mathrm{stretch}}} - (\mathcal{M}^{\mathrm{stretch}})^* \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}^{\mathrm{stretch}} + C^{\mathrm{stretch}}(\omega) \right)^{-1} \mathcal{M}^{\mathrm{stretch}} \Xi_\varepsilon - \mathcal{A}^{\mathrm{corr}}_{\mathrm{stretch}}(\varepsilon) - \widetilde{\mathcal{A}}^{\mathrm{corr}}_{\mathrm{stretch}}(\varepsilon) \right\|_{L^2 \to L^2} \leq C\varepsilon^{\gamma+2},$$

$$\left\| P_i \left(\left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} |_{L^2_{\mathrm{bend}}} - (\mathcal{M}_\varepsilon^{\mathrm{bend}})^* \left(\frac{1}{\varepsilon^{\gamma-2}} \mathcal{A}^{\mathrm{bend}} + I \right)^{-1} \mathcal{M}_\varepsilon^{\mathrm{bend}} \Xi_\varepsilon - \mathcal{A}^{\mathrm{corr}}_{\mathrm{bend}}(\varepsilon) - \widetilde{\mathcal{A}}^{\mathrm{corr}}_{\mathrm{bend}}(\varepsilon) \right) S_{\varepsilon^\delta} \right\|_{L^2 \to L^2} \leq \begin{cases} C\varepsilon^{\frac{\gamma+2}{2}} \max\left\{ \varepsilon^{\frac{\gamma+2}{4} - \delta}, 1 \right\}, & i = 1,2; \\ C\varepsilon^{\frac{3(\gamma+2)}{4}} \max\left\{ \varepsilon^{\frac{\gamma+2}{4} - \delta}, 1 \right\}, & i = 3. \end{cases}$$

3.1.6. The methodology and the strategy of the proofs

Out approach begins with the application of the Gelfand transform in order to decompose the resolvent problem for the operator defined with (3.2) into the continuous family of resolvent problems posed in the space of periodic functions on a unit cell, in the sense of (3.7). The next step would be to analyse the problem for each fibre χ in order to provide the approximation of resolvent operators with a homogenised resolvents. all while estimating the difference between the two in the operator norm topology.

Even though the results which we aim to obtain are the estimates with respect to the physical parameter ε (playing the role of the thickness of the rod as well as the period of material oscillations), the general agenda is to first provide the estimates with respect to the quasimomentum variable χ, for each fibre $\chi \in [-\pi, \pi]$. These estimates are then translated into the desired norm-resolvent estimates by the means of Cauchy integral formula.

The approximation with respect to the quasimomentum χ is done by performing carefully devised asymptotic procedure adapted for handling the particularities arising from the the fact that the rod is thin and oscillating in one direction only.

All of this is first carried out in the case of the assumption 3.1.1, under which the problem separates into two relatively simpler problems to handle. For example, the asymptotic procedure can be carried out separately for these two problems and is much simpler.

In order to gain insight into the spectral properties of the underlying operators and also to be able to prove the desired estimates, we first derive needed Korn-type inequalities depending on the quasimomentum χ.

The structure of the chapter is the following:

- In Section 3.2 we derive the Korn-type inequalities in the general case, but also for the bending and stretching deformations in the case of material symmetries. With the use of these inequalities, we estimate the spectrum by the means of Rayleigh quotients. The outcome of the analysis is the conclusion on the orders of magnitudes of eigenvalues and the revelation of the appropriate scalings of the operators in the next step. The difference with respect to [5], [7] comes from the fact that the lowest eigenvalues appear with different order in $|\chi|$ (two of them of order $|\chi|^4$ and two of them of order $|\chi|^2$). Due to this, the approach by spectral germ is not directly applicable, since the usual assumption on regularity of spectral germ implies that all the eigenvalues of the lowest order have the same order, namely $|\chi|^2$.

- In Section 3.3 we approximate the resolvent operators by performing the iterative asymptotic procedure used for gradually defining the approximations of the solution to the original problem, where the bounds depend only on the norm of the loads and the quasimomentum χ (with increasing order of magnitude). Here the analysis is done only in the case of Assumption 3.1.1, separately for the two invariant spaces. The case of stretching deformations resembles in a way to the bulk case, primarily with regard to the order of the operator scaling. Like in [20] we use χ-dependent asympotics, which is a natural choice as a consequence of apriori bounds and enables us to perform the asymptotics up to any order in $|\chi|$. This makes these approaches different with respect to the approach in [18], where the ε-dependent asymptotics is done.

- In Section 3.4 we dissect the obtained estimates and combine them with the Cauchy integral formula in order to make the estimates depend only on the physical parameter ε. We use different parts of the obtained asymptotic expansions in order to define the corrector operators appearing in the Theorem 3.1.10 and the Theorem 3.1.11. Then we combine all the fiberwise estimates back onto the physical domains. Here we also deal only with invariant subspaces under the Assumption 3.1.1.

- In the last section, namely Section 3.5, we repeat the procedure but only this time under no additional assumptions. This requires us to perform two simultaneous asymptotic procedures, with different scalings, and then combine them together in the end. The last step is again the Cauchy integral formula.

3.2. Auxiliary results and apriori estimates

In this section we provide Korn-type inequalities in before mentioned spaces in order to deduce apriori estimates on the solutions to relevant resolvent problems. The following is the expression for rigid motions in $\omega\times Y$:

$$\boldsymbol{v}(x_1,x_2,x_3) := Ax+\boldsymbol{c} = \begin{bmatrix} 0 & d & a \\ -d & 0 & b \\ -a & -b & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} dx_2+ax_3+c_1 \\ -dx_1+bx_3+c_2 \\ -ax_1-bx_2+c_3 \end{bmatrix}, \quad a,b,c_1,c_2,c_3,d\in\mathbb{C}. \tag{3.8}$$

Rigid motions are deformations consisting of only rotations (skew symmetric matrix A) and translations (vector c). These deformations belong to the kernel of the operator $\operatorname{sym}\nabla$. We state the second Korn inequality in the following form:

Proposition 3.2.1. *For every $\boldsymbol{u}\in H^1(\omega\times Y;\mathbb{C}^3)$ we have the following estimate:*

$$\|\boldsymbol{u}(x)-(Ax+\boldsymbol{c})\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C\|\operatorname{sym}\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})},$$

where

$$A = \begin{bmatrix} 0 & d & a \\ -d & 0 & b \\ -a & -b & 0 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad c_j = \int_{\omega\times Y}(\boldsymbol{u}_j), \quad j=1,2,3,$$

$$a = \int_{\omega\times Y}(\partial_3\boldsymbol{u}_1-\partial_1\boldsymbol{u}_3), \quad b = \int_{\omega\times Y}(\partial_3\boldsymbol{u}_2-\partial_2\boldsymbol{u}_3), \quad d = \int_{\omega\times Y}(\partial_2\boldsymbol{u}_1-\partial_1\boldsymbol{u}_2),$$

The constant C depends only on the domain $\omega\times Y$.

In order to make proofs in this section more elegant, we have decided to present the results with respect to quasiperiodic functions, namely the image of Floquet transform. This is obviously equivalent to the approach with periodic functions via:

$$H^1_\chi(Y;H^1(\omega;\mathbb{C}^3)) \longleftrightarrow H^1_\#(Y;H^1(\omega;\mathbb{C}^3)), \quad \mathcal{F}_\varepsilon \longleftrightarrow \mathcal{G}_\varepsilon, \quad \operatorname{sym}\nabla \longleftrightarrow \operatorname{sym}\nabla + iX_\chi.$$

The following lemma provides estimates for the approximating rigid motions of quasiperiodic functions in y variable with respect to the norm of the symmetrized gradient.

Lemma 3.2.2. *There is a constant $C > 0$ such that $\forall \chi \in [-\pi,\pi] \setminus \{0\}$, $\forall \boldsymbol{u} \in H^1_\chi(Y; H^1(\omega;\mathbb{C}^3))$ we have that*

$$\|\boldsymbol{u} - \boldsymbol{v}\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C\|\mathrm{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})},$$

where $\boldsymbol{v}$ is a rigid motion defined with (3.8) with the coefficients satisfying the following estimates:

$$\begin{aligned}
\max\{|a|,|b|,|d|,|c_3|\} &\leq \frac{1}{|\chi|}C\|\mathrm{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}\\
|(e^{i\chi}-1)c_2 - b| &\leq C\|\mathrm{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}\\
|(e^{i\chi}-1)c_1 - a| &\leq C\|\mathrm{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}\\
\max\{|c_1|,|c_2|\} &\leq \frac{1}{|\chi|^2}C\|\mathrm{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}.
\end{aligned}$$

Proof. By using the trace theorem for functions in $H^1_\chi(Y; H^1(\omega;\mathbb{C}^3))$, as well as Korn's inequality, we conclude that for $\boldsymbol{w}$ supplied by the Proposition 3.2.1 we have:

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{w}\|_{L^2(\omega\times\{y=1\})} &\leq C\|\mathrm{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})},\\
\|\boldsymbol{u} - \boldsymbol{w}\|_{L^2(\omega\times\{y=0\})} &\leq C\|\mathrm{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}.
\end{aligned}$$

Furthermore, for smooth quasiperiodic $\boldsymbol{u}$ we have:

$$\boldsymbol{u}(x_1,x_2,1) = e^{i\chi}\boldsymbol{u}(x_1,x_2,0), \quad \forall (x_1,x_2) \in \omega,$$

thus:

$$|\boldsymbol{w}(x_1,x_2,1) - e^{i\chi}\boldsymbol{w}(x_1,x_2,0)| \leq |\boldsymbol{u}(x_1,x_2,1) - \boldsymbol{w}(x_1,x_2,1)| + |e^{i\chi}(\boldsymbol{u}(x_1,x_2,0) - \boldsymbol{w}(x_1,x_2,0))|,$$

for every $x_1, x_2 \in \omega$. Thus, extending this to all $\boldsymbol{u} \in H^1_\chi(Y; H^1(\omega;\mathbb{C}^3))$ gives:

$$\|\boldsymbol{w}(x_1,x_2,1) - e^{i\chi}\boldsymbol{w}(x_1,x_2,0)\|_{L^2(\omega;\mathbb{C}^3)} \leq \|\boldsymbol{u} - \boldsymbol{w}\|_{L^2(\omega\times\{y=1\};\mathbb{C}^3)} + \|\boldsymbol{u} - \boldsymbol{w}\|_{L^2(\omega\times\{y=0\};\mathbb{C}^3)}$$

and therefore

$$\|\boldsymbol{w}(x_1,x_2,1) - e^{i\chi}\boldsymbol{w}(x_1,x_2,0)\|_{L^2(\omega;\mathbb{C}^3)} \leq C\|\mathrm{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

Componentwise, this means that

$$\begin{aligned}
\int_\omega |(e^{i\chi}-1)(c_1 + dx_2) - a|^2 dx_1 dx_2 &\leq C\|\mathrm{sym}\,\nabla\boldsymbol{u}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}\\
\int_\omega |(e^{i\chi}-1)(c_2 - dx_1) - b|^2 dx_1 dx_2 &\leq C\|\mathrm{sym}\,\nabla\boldsymbol{u}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}
\end{aligned}$$

$$\int_\omega |(e^{i\chi}-1)(c_3-ax_1-bx_2)|^2 dx_1dx_2 \quad \le \quad C\|\mathrm{sym}\,\nabla \boldsymbol{u}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}.$$

Recall that we are using the coordinate system with the following symmetries:

$$\int_\omega x_1 = 0, \quad \int_\omega x_2 = 0, \quad \int_\omega x_1x_2 = 0.$$

Also, by using Taylor expansion we note that there exist constants $C_1, C_2 > 0$ such that for all $\chi \in [-\pi,\pi]$ we have

$$C_1|\chi| \le |e^{i\chi}-1| \le C_2|\chi|.$$

These remarks allow us to deduce:

$$\begin{aligned}
|(e^{i\chi}-1)c_1-a|^2 + C|\chi|^2|d|^2 &\le C\int_\omega |(e^{i\chi}-1)(c_1+dx_2)-a|^2 dx_1dx_2,\\
|(e^{i\chi}-1)c_2-b|^2 + C|\chi|^2|d|^2 &\le C\int_\omega |(e^{i\chi}-1)(c_2-dx_1)-b|^2 dx_1dx_2\\
|\chi|^2(|c_3|^2+|a|^2+|b|^2) &\le C\int_\omega |(e^{i\chi}-1)(c_3-ax_1-bx_2)|^2 dx_1dx_2.
\end{aligned}$$

which yield the final estimates.

■

3.2.1. The leading order term

Proposition 3.2.3. *There is a constant $C > 0$ such that for every $\chi \in [-\pi,\pi]\setminus\{0\}$, $\boldsymbol{u} \in H^1_\chi(Y; H^1(\omega;\mathbb{C}^3))$ there exist a function $\boldsymbol{w} \in H^1_\chi(Y;H^1(\omega;\mathbb{C}^3))$,*

$$\boldsymbol{w}(x_1,x_2,y) = e^{i\chi y}\left(\begin{bmatrix} dx_2 \\ -dx_1 \\ c_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ -i\chi(c_1x_1+c_2x_2)\end{bmatrix}\right), \quad c_1,c_2,c_3,d \in \mathbb{C},$$

such that

$$\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{w}\|_{H^1(\omega\times Y;\mathbb{C}^3)} &\le C\|\mathrm{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})},\\
\max\{|d|,|c_3|\} &\le \frac{1}{|\chi|}C\|\mathrm{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})},\\
\max\{|c_1|,|c_2|\} &\le \frac{1}{|\chi|^2}C\|\mathrm{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}.
\end{aligned}$$

Proof. By using the estimations of error in Taylor expansion:

$$|e^{i\chi}-(1+i\chi)| = O(\chi^2), \quad |e^{i\chi}-1| = O(\chi),$$

we can deduce the following:

$$
\begin{aligned}
|i\chi c_1 - a| &\le \left\|(e^{i\chi}-1)c_1 - a\right\| + \left\|(e^{i\chi}-(1+i\chi))c_1\right\| \\
&\le C_1 \|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})} + C_2|\chi|^2\frac{1}{|\chi|^2}\|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})} \\
&\le C\|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})}, \\
|i\chi c_2 - b| &\le C\|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})}
\end{aligned}
\tag{3.9}
$$

In other words, we have that $\boldsymbol{u}$ can be approximated in H^1 norm with the rigid motion whose coefficients satisfy the above estimates, namely:

$$
\begin{aligned}
\|\boldsymbol{u}(x_1,x_2,y) - \boldsymbol{v}(x_1,x_2,y)\|_{H^1(\omega\times Y;\mathbb{C}^3)} &= \left\| \boldsymbol{u}(x_1,x_2,y) - \left(\begin{bmatrix} 0 & d & a \\ -d & 0 & b \\ -a & -b & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right) \right\|_{H^1(\omega\times Y;\mathbb{C}^3)} \\
&= \left\| \begin{bmatrix} \boldsymbol{u}_1(x_1,x_2,y) - dx_2 - ay - c_1 \\ \boldsymbol{u}_2(x_1,x_2,y) + dx_1 - by - c_2 \\ \boldsymbol{u}_3(x_1,x_2,y) + ax_1 + bx_2 - c_3 \end{bmatrix} \right\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C\|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})}.
\end{aligned}
$$

But, the estimates (3.9) allow us to eliminate coefficients a and b in the above estimate with respect to c_1, c_2 in $H^1(\omega\times Y;\mathbb{C}^3)$ norm:

The following calculation proves this:

$$
\begin{aligned}
\left\|ay + c_1 - e^{i\chi y}c_1\right\|_{L^2(\omega\times Y;\mathbb{C}^3)} &= \left\|ay - \left(e^{i\chi y}-1\right)c_1\right\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
&\le \|ay - i\chi y c_1\|_{L^2(\omega\times Y;\mathbb{C}^3)} + \left\|i\chi y c_1 - \left(e^{i\chi y}-1\right)c_1\right\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
&= \|y(a - i\chi c_1)\|_{L^2(\omega\times Y;\mathbb{C}^3)} + \left\|\left(i\chi y - \left(e^{i\chi y}-1\right)\right)c_1\right\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
&\le C_1\|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})} + C_2|\chi|^2\frac{1}{|\chi|^2}\|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})} \\
&\le C\|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})}.
\end{aligned}
$$

$$
\begin{aligned}
\left\|\partial_y\left(ay + c_1 - e^{i\chi y}c_1\right)\right\|_{L^2(\omega\times Y;\mathbb{C}^3)} &= \left\|a - i\chi e^{i\chi y}c_1\right\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
&\le \|a - i\chi c_1\|_{L^2(\omega\times Y;\mathbb{C}^3)} + \left\|i\chi\left(1 - e^{i\chi y}c_1\right)\right\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
&\le C_1\|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})} + C_2|\chi|^2\frac{1}{|\chi|^2}\|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})} \\
&\le C\|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})}.
\end{aligned}
$$

Therefore:

$$\left\| ay + c_1 - e^{i\chi y} c_1 \right\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C \|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}.$$

Similarly:

$$\left\| by + c_2 - e^{i\chi y} c_2 \right\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C \|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}.$$

Next, we calculate:

$$\begin{aligned}
& \left\| ax_1 + bx_2 - e^{i\chi y} i\chi (x_1 c_1 + x_2 c_2) \right\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
\leq\ & \left\| x_1 \left(a - e^{i\chi y} i\chi c_1 \right) \right\|_{L^2(\omega\times Y;\mathbb{C}^3)} + \left\| x_2 \left(b - e^{i\chi y} i\chi c_2 \right) \right\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
\leq\ & C_1 \|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})} + C_2 \|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
\leq\ & C \|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})},
\end{aligned}$$

$$\begin{aligned}
\left\| \partial_{x_1} \left(ax_1 + bx_2 - e^{i\chi y} i\chi (x_1 c_1 + x_2 c_2) \right) \right\|_{L^2(\omega\times Y;\mathbb{C}^3)} &\leq \left\| a - e^{i\chi y} i\chi c_1 \right\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
&\leq C \|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}, \\
\left\| \partial_{x_2} \left(ax_1 + bx_2 - e^{i\chi y} i\chi (x_1 c_1 + x_2 c_2) \right) \right\|_{L^2(\omega\times Y;\mathbb{C}^3)} &\leq \left\| b - e^{i\chi y} i\chi c_2 \right\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
&\leq C \|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}, \\
\left\| \partial_{y} \left(ax_1 + bx_2 - e^{i\chi y} i\chi (x_1 c_1 + x_2 c_2) \right) \right\|_{L^2(\omega\times Y;\mathbb{C}^3)} &\leq \left\| \chi^2 e^{i\chi y} (x_1 c_1 + x_2 c_2) \right\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
&\leq C |\chi|^2 \frac{1}{|\chi|^2} \|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})} \\
&\leq C \|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}.
\end{aligned}$$

Therefore, we have:

$$\left\| ax_1 + bx_2 - e^{i\chi y} i\chi (x_1 c_1 + x_2 c_2) \right\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C \|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}.$$

■

Remark 3.2.4. We will denote this approximation of $\boldsymbol{u} \in H^1_\chi(Y; H^1(\omega;\mathbb{C}^3))$ as: $\mathrm{rod}(\boldsymbol{u})$, namely:

$$\mathrm{rod}(\boldsymbol{u}) := \mathrm{e}^{\mathrm{i}\chi \mathrm{y}} (\mathrm{dx}_2, -\mathrm{dx}_1, \mathrm{c}_3)^{\mathrm{T}} + \mathrm{e}^{\mathrm{i}\chi \mathrm{y}} (\mathrm{c}_1, \mathrm{c}_2, -\mathrm{i}\chi(\mathrm{c}_1 \mathrm{x}_1 + \mathrm{c}_2 \mathrm{x}_2))^{\mathrm{T}}.$$

We interpret the previous proposition in the following fashion:

$$\boldsymbol{u} = \mathrm{rod}(\boldsymbol{u}) + O(\|\mathrm{sym}\,\nabla \boldsymbol{u}\|_{\mathrm{L}^2(\omega\times \mathrm{Y};\mathbb{C}^{3\times 3})}).$$

3.2.2. Invariant subspaces

In the case of certain symmetries of the elastic properties of the material 3.1.1 one can reduce the problem to two simpler problems. We are able to isolate two orthogonal subspaces which then turn out to be invariant under the operator of elasticity. These subspaces consist of ”out of line” displacements, which we refer to as bending displacements, and ”in line” displacements, which we refer to as stretching displacements. We begin with decomposing the space $L^2(\omega\times Y;\mathbb{C}^3)$ into two orthogonal subspaces $L^2_{\rm bend}$ and $L^2_{\rm stretch}$ defined as follows:

$$L^2_{\rm bend} = \left\{\boldsymbol{u}\in L^2(\omega\times Y;\mathbb{C}^3), \quad \boldsymbol{u}_\alpha(\mathcal{S}(x_1,x_2)) = \boldsymbol{u}_\alpha(x_1,x_2), \alpha=1,2, \quad \boldsymbol{u}_3(\mathcal{S}(x_1,x_2)) = -\boldsymbol{u}_3(x_1,x_2)\right\}$$

$$L^2_{\rm stretch} = \left\{\boldsymbol{u}\in L^2(\omega\times Y;\mathbb{C}^3), \quad \boldsymbol{u}_\alpha(\mathcal{S}(x_1,x_2)) = -u_\alpha(x_1,x_2), \alpha=1,2, \quad \boldsymbol{u}_3(\mathcal{S}(x_1,x_2)) = \boldsymbol{u}_3(x_1,x_2)\right\}$$

Here we make a small remark noting that these spaces are mutually orthogonal also in $H^1_\chi(Y;H^1(\omega;\mathbb{C}^3))$:

$$\left(H^1_\chi(Y;H^1(\omega;\mathbb{C}^3))\cap L^2_{\rm bend}\right)^\perp = H^1_\chi(Y;H^1(\omega;\mathbb{C}^3))\cap L^2_{\rm stretch},$$

Notice that, for $\boldsymbol{u}\in H^1_\chi(Y;H^1(\omega;\mathbb{C}^3))$, we have $\mathrm{rod}(\boldsymbol{u}) = \boldsymbol{u}_{\rm bend}+\boldsymbol{u}_{\rm stretch}$, where

$$\boldsymbol{u}_{\rm stretch} := e^{i\chi y}(dx_2,-dx_1,c_3)^T\in L^2_{\rm stretch}, \quad \boldsymbol{u}_{\rm bend} = e^{i\chi y}(c_1,c_2,-i\chi(c_1x_1+c_2x_2))^T\in L^2_{\rm bend}.$$

By simple calculation we find that

$$\nabla\boldsymbol{u}_{\rm bend} = e^{i\chi y}\begin{bmatrix} 0 & 0 & i\chi c_1 \\ 0 & 0 & i\chi c_2 \\ -i\chi c_1 & -i\chi c_2 & -(i\chi)^2(c_1x_1+c_2x_2)\end{bmatrix}$$

$$\mathrm{sym}\,\nabla\boldsymbol{u}_{\rm bend} = e^{i\chi y}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -(i\chi)^2(c_1x_1+c_2x_2)\end{bmatrix}$$

$$\nabla\boldsymbol{u}_{\rm stretch} = e^{i\chi y}\begin{bmatrix} 0 & d & i\chi dx_2 \\ -d & 0 & -i\chi dx_1 \\ 0 & 0 & i\chi c_3\end{bmatrix}$$

$$\operatorname{sym}\nabla \boldsymbol{u}_{\text{stretch}} = e^{i\chi y}\begin{bmatrix} 0 & 0 & \frac{1}{2}i\chi d x_2 \\ 0 & 0 & -\frac{1}{2}i\chi d x_1 \\ \frac{1}{2}i\chi d x_2 & -\frac{1}{2}i\chi d x_1 & i\chi c_3 \end{bmatrix}.$$

Here we make the observation that $\operatorname{sym}\nabla \boldsymbol{u}_{\text{stretch}} = iX_\chi \boldsymbol{u}_{\text{stretch}}$. We can easily calculate the following L^2 estimates:

$$\|\boldsymbol{u}_{\text{bend}}\|_{L^2(\omega\times Y,\mathbb{C}^3)} \le C\max\{|c_1|,|c_2|\} \le C\frac{1}{|\chi|^2}\|\operatorname{sym}\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})},$$

$$\|\boldsymbol{u}_{\text{stretch}}\|_{L^2(\omega\times Y,\mathbb{C}^3)} \le C\max\{|d|,|c_3|\} \le C\frac{1}{|\chi|}\|\operatorname{sym}\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})},$$

$$\|\operatorname{sym}\nabla \boldsymbol{u}_{\text{bend}}\|_{L^2(\omega\times Y,\mathbb{C}^{3\times 3})} \le |\chi|^2\max\{|c_1|,|c_2|\} \le C\|\operatorname{sym}\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}$$

$$\|\operatorname{sym}\nabla \boldsymbol{u}_{\text{stretch}}\|_{L^2(\omega\times Y,\mathbb{C}^{3\times 3})} \le |\chi|\max\{|d|,|c_3|\} \le C\|\operatorname{sym}\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}.$$

Next, we define the following subspaces:

$$V_\chi^{\text{bend}} := \left\{(c_1,c_2,-i\chi(c_1x_1+c_2x_2))^T e^{i\chi y}, c_1,c_2\in\mathbb{C}\right\} \le H^1_\chi(Y;H^1(\omega;\mathbb{C}^3)),$$

$$V_\chi^{\text{stretch}} := \left\{(dx_2,-dx_1,c_3)^T e^{i\chi y}, d,c_3\in\mathbb{C}\right\} \le H^1_\chi(Y;H^1(\omega;\mathbb{C}^3)).$$

Note the following facts:

$$\dim V_\chi^{\text{bend}} = \dim V_\chi^{\text{stretch}} = 2, \quad V_\chi^{\text{bend}} \perp V_\chi^{\text{stretch}},$$

$$V_\chi^{\text{bend}} < L^2_{\text{bend}}, \quad V_\chi^{\text{stretch}} < L^2_{\text{stretch}}.$$

The following estimates are crucial for the spectral analysis.

Proposition 3.2.5. *There exists a constant $C>0$ such that:*

- *For every $\boldsymbol{u}\in H^1_\chi(Y;H^1(\omega;\mathbb{C}^3))$ we have:*

$$\|\boldsymbol{u}\|_{L^2(\omega\times Y,\mathbb{C}^3)} \le \frac{C}{|\chi|^2}\|\operatorname{sym}\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}, \tag{3.10}$$

- *For every $\boldsymbol{u}\in (V_\chi^{\text{bend}})^\perp$:*

$$\|\boldsymbol{u}\|_{L^2(\omega\times Y,\mathbb{C}^3)} \le \frac{C}{|\chi|}\|\operatorname{sym}\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})},$$

- *For every $\boldsymbol{u}\in (V_\chi^{\text{bend}}\cup V_\chi^{\text{stretch}})^\perp$:*

$$\|\boldsymbol{u}\|_{L^2(\omega\times Y,\mathbb{C}^3)} \le C\|\operatorname{sym}\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})},$$

Proof. The proof relies on the following reasoning:

$$\boldsymbol{u} \in H^1_\chi(Y; H^1(\omega; \mathbb{C}^3)) \Rightarrow \|\boldsymbol{u} - \boldsymbol{u}_{\text{stretch}} - \boldsymbol{u}_{\text{bend}}\|_{L^2(\omega\times Y,\mathbb{C}^3)} \leq C\|\text{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})}.$$

We see that

$$\begin{aligned}\|\boldsymbol{u}\|_{L^2(\omega\times Y,\mathbb{C}^3)} &\leq \|\boldsymbol{u} - \boldsymbol{u}_{\text{stretch}} - \boldsymbol{u}_{\text{bend}}\|_{L^2(\omega\times Y,\mathbb{C}^3)} + \|\boldsymbol{u}_{\text{bend}}\|_{L^2(\omega\times Y,\mathbb{C}^3)} + \|\boldsymbol{u}_{\text{stretch}}\|_{L^2(\omega\times Y,\mathbb{C}^3)} \\ &\leq C\frac{1}{|\chi|^2}\|\text{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})}.\end{aligned}$$

For $\boldsymbol{u} \in (V^{\text{bend}}_\chi)^\perp$ we have

$$\|\boldsymbol{u} - \boldsymbol{u}_{\text{stretch}}\|^2_{L^2} + \|\boldsymbol{u}_{\text{bend}}\|^2_{L^2} = \|\boldsymbol{u} - \boldsymbol{u}_{\text{stretch}} - \boldsymbol{u}_{\text{bend}}\|^2_{L^2(\omega\times Y,\mathbb{C}^3)} \leq C\|\text{sym}\,\nabla\boldsymbol{u}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times3})},$$

Therefore

$$\|\boldsymbol{u} - \boldsymbol{u}_{\text{stretch}}\|_{L^2(\omega\times Y,\mathbb{C}^3)} \leq C\|\text{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})},$$

so

$$\|\boldsymbol{u}\|_{L^2(\omega\times Y,\mathbb{C}^3)} \leq \|\boldsymbol{u} - \boldsymbol{u}_{\text{stretch}}\|_{L^2(\omega\times Y,\mathbb{C}^3)} + \|\boldsymbol{u}_{\text{stretch}}\|_{L^2(\omega\times Y,\mathbb{C}^3)} \leq C\frac{1}{|\chi|}\|\text{sym}\,\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})}.$$

Last, for $\boldsymbol{u} \in (V^{\text{bend}}_\chi \cup V^{\text{stretch}}_\chi)^\perp$:

$$\begin{aligned}\|\boldsymbol{u}\|^2_{L^2(\omega\times Y,\mathbb{C}^3)} + \|\boldsymbol{u}_{\text{stretch}}\|^2_{L^2(\omega\times Y,\mathbb{C}^3)} + \|\boldsymbol{u}_{\text{bend}}\|^2_{L^2(\omega\times Y,\mathbb{C}^3)} &= \|\boldsymbol{u} - \boldsymbol{u}_{\text{stretch}} - \boldsymbol{u}_{\text{bend}}\|^2_{L^2(\omega\times Y,\mathbb{C}^3)} \\ &\leq C\|\text{sym}\,\nabla\boldsymbol{u}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times3})}.\end{aligned}$$

■

Remark 3.2.6. Under additional assumptions on the symmetries of the elasticity tensor $\mathbb{C}(y)$ 3.1.1, we can separately analyze the problem on each of these two orthogonal subspaces. Actually, under the assumptions 3.1.1, the spaces L^2_{bend} and L^2_{stretch} are invariant for the operator $\mathcal{A}_\chi$. Let $S : L^2(\omega \times Y, \mathbb{C}^3) \to L^2(\omega \times Y, \mathbb{C}^3)$ be the symmetry operator defined with

$$(S\boldsymbol{u})(x_1, x_2, y) := (-\boldsymbol{u}_1(\mathcal{S}(x_1, x_2), y), -\boldsymbol{u}_2(\mathcal{S}(x_1, x_2), y), \boldsymbol{u}_3(\mathcal{S}(x_1, x_2), y)).$$

It is clear that

$$L^2_{\text{bend}} = \left\{\boldsymbol{u} \in L^2(\omega \times Y, \mathbb{C}^3); S\boldsymbol{u} = -\boldsymbol{u}\right\}, \quad L^2_{\text{stretch}} = \left\{\boldsymbol{u} \in L^2(\omega \times Y, \mathbb{C}^3); S\boldsymbol{u} = \boldsymbol{u}\right\}.$$

The invariance for these spaces means that

$$a_\chi(S\boldsymbol{u},\boldsymbol{v}) = a_\chi(\boldsymbol{u},S\boldsymbol{v}), \quad a_\chi(S\boldsymbol{u},S\boldsymbol{v}) = a_\chi(\boldsymbol{u},\boldsymbol{v}), \quad \forall \boldsymbol{u},\boldsymbol{v} \in H^1_\chi(Y;H^1(\omega;\mathbb{C}^3)).$$

Note that the operator S is continuous operator on $L^2(\Omega)$ and that $S^2 = I$. Spaces L^2_{bend} and L^2_{stretch} are mutually orthogonal also with respect to the form a_χ. As a consequence, the resolvent problem splits into two separated problems on each of these invariant subspaces.

3.2.3. Spectral estimates

By Rellich-Kondrachev we have that $H^1_\chi(Y;H^1(\omega;\mathbb{C}^3))$ is compactly embedded into $L^2_\chi(Y;L^2(\omega;\mathbb{C}^3))$. Thus, by the theorem of spectrum of compact operators, we deduce that the spectrum of $\mathcal{A}_\chi$ consists of nondecreasing sequence of eigenvalues $(\lambda^\chi_n)_n$, which tends to infinity.

Here we can state some results on the structure of the spectrum and its scaling. Recall the definition of the Rayleigh quotient associated with the bilinear form a_χ, namely:

$$\mathcal{R}_\chi(\boldsymbol{u}) = \frac{a_\chi(\boldsymbol{u},\boldsymbol{u})}{\|\boldsymbol{u}\|^2_{L^2(\omega\times Y,\mathbb{C}^3)}}, \quad \boldsymbol{u} \in H^1_\chi(Y;H^1(\omega;\mathbb{C}^3)).$$

The Rayleigh quotient is closely related with the spectrum via the following characterizations:

$$\lambda^\chi_n = \min_{V\in L^n} \max_{\boldsymbol{v}\in V} \mathcal{R}_\chi(\boldsymbol{v}),$$
$$\lambda^\chi_n = \min\left\{\mathcal{R}_\chi(v); \quad v \perp v_i \text{ in } L^2(\omega\times Y;\mathbb{C}^3), \quad 1 \le i \le n-1\right\},$$

where L^n denotes the family of $n-$dimensional subspaces of $H^1_\chi(Y;H^1(\omega;\mathbb{C}^3))$.

Remark 3.2.7. Here we note that the function $\chi \to \lambda^\chi_n$ is continuous for all $n \in \mathbb{N}$. For that reason, we conclude that the spectrum of the operator $\mathcal{A}_\varepsilon$ is a union of intervals

$$[\underline{\lambda}_n, \overline{\lambda}_n] := \bigcup_{\chi\in[-\pi,\pi]} \lambda^\chi_n, \quad n \in \mathbb{N}.$$

The presence of the previously defined subspaces V^{bend}_χ, V^{stretch}_χ yields inhomogeneity in the order of magnitude of eigenvalues with respect to the quasimomentum χ. In particular, we have the following proposition:

Proposition 3.2.8. *There exist constants $C_1 > C_2 > 0$ such that:*

- $\forall \boldsymbol{u} \in H^1_\chi(Y; H^1(\omega; \mathbb{C}^3)), \quad \mathcal{R}_\chi(u) \geq C_2 |\chi|^4,$

- $\forall \boldsymbol{u} \in V_\chi^{\text{bend}}, \quad \mathcal{R}_\chi(u) \leq C_1 |\chi|^4,$

- $\forall \boldsymbol{u} \in (V_\chi^{\text{bend}})^\perp, \quad \mathcal{R}_\chi(u) \geq C_2 |\chi|^2,$

- $\forall \boldsymbol{u} \in V_\chi^{\text{stretch}}, \quad \mathcal{R}_\chi(u) \leq C_1 |\chi|^2,$

- $\forall \boldsymbol{u} \in (V_\chi^{\text{bend}} \cup V_\chi^{\text{stretch}})^\perp, \quad \mathcal{R}_\chi(u) \geq C_2,$

Proof. By using the uniform positive definiteness of the tensor $\mathbb{C}$ together with the estimates (3.10) we get the following:

$$\mathcal{R}_\chi(\boldsymbol{u}) = \frac{a_\chi(\boldsymbol{u},\boldsymbol{u})}{\|\boldsymbol{u}\|^2_{L^2(\omega\times Y,\mathbb{C}^3)}} \geq \nu \frac{\|\operatorname{sym}\nabla \boldsymbol{u}\|^2_{L^2(\omega\times Y,\mathbb{C}^3)}}{\|\boldsymbol{u}\|^2_{L^2(\omega\times Y,\mathbb{C}^3)}} \geq C\nu |\chi|^4.$$

For arbitrary $\boldsymbol{u} = (c_1, c_2, -i\chi(c_1x_1 + c_2x_2))^\perp e^{i\chi y} \in V_\chi^{\text{bend}}$ we calculate:

$$\|\operatorname{sym}\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})} \leq \max\left\{ \sqrt{c_1(\omega)},\, \sqrt{c_2(\omega)} \right\} |\chi|^2 \sqrt{c_1^2 + c_2^2}$$

$$\|\boldsymbol{v}\|_{L^2(\omega\times Y,\mathbb{C}^3)} \geq |c_1| + |c_2| + \min\{c_1(\omega), c_2(\omega)\} |\chi| \sqrt{c_1^2 + c_2^2} \geq |c_1| + |c_2| \geq \sqrt{c_1^2 + c_2^2}.$$

Combined, we see that

$$\mathcal{R}_\chi(\boldsymbol{u}) = \frac{a_\chi(\boldsymbol{u},\boldsymbol{u})}{\|\boldsymbol{u}\|^2_{L^2(\omega\times Y,\mathbb{C}^3)}} \leq \frac{1}{\nu} \frac{\|\operatorname{sym}\nabla \boldsymbol{u}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}}{\|\boldsymbol{u}\|^2_{L^2(\omega\times Y,\mathbb{C}^3)}} \leq \frac{C(\omega)}{\nu} |\chi|^4$$

Now take arbitrary $\boldsymbol{v} = (dx_2, -dx_1, c_3)^\perp e^{i\chi y} \in V_\chi^{\text{stretch}}$. We have:

$$\|\operatorname{sym}\nabla \boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})} \leq |\chi| \max\left\{ \sqrt{c_1(\omega) + c_2(\omega)}, 1 \right\} \sqrt{d^2 + c_3^2},$$

while

$$\|\boldsymbol{v}\|_{L^2(\omega\times Y,\mathbb{C}^3)} \geq \min\left\{ \sqrt{c_1(\omega) + c_2(\omega)}, 1 \right\} \sqrt{d^2 + c_3^2}.$$

Combined, we have that

$$\mathcal{R}_\chi(\boldsymbol{v}) \leq \frac{C(\omega)}{\nu} |\chi|^2.$$

These calculations, together with results from Proposition 3.2.5 finish the proof. ∎

The previous proposition allows us to deduce the following result on the structure and the scaling of the spectrum:

Theorem 3.2.9. *The spectrum $\sigma(\mathcal{A}_\chi)$ consists of two eigenvalues of order $O(|\chi|^4)$, two eigenvalues of order $O(|\chi|^2)$ and the rest of order $O(1)$.*
Under the additional assumptions on the material symmetries 3.1.1, the spectrum $\sigma\left\{\mathcal{A}_\chi\right\}$ is a disjoint union of spectra $\sigma\left\{\mathcal{A}_\chi|_{L^2_{\text{bend}}}\right\}$ and $\sigma\left\{\mathcal{A}_\chi|_{L^2_{\text{stretch}}}\right\}$. The spectrum $\sigma\left\{\mathcal{A}_\chi|_{L^2_{\text{bend}}}\right\}$ contains two eigenvalues of order $O(|\chi|^4)$ and the rest of order $O(1)$. On the other hand, the spectrum $\sigma\left\{\mathcal{A}_\chi|_{L^2_{\text{stretch}}}\right\}$ consists of two eigenvalues of order $O(|\chi|^2)$ and the rest of order $O(1)$.

Proof. The proof relies on the estimates on Rayleigh quotients and the characterization of eigenvalues via min-max principle. ■

Under the symmetry assumptions, the spectrum of the operator $\mathcal{A}_\chi$ can be decomposed into two sets $\sigma(\mathcal{A}_\chi|_{L^2_{\text{bend}}})$ and $\sigma(\mathcal{A}_\chi|_{L^2_{\text{stretch}}})$ so it is of interest to perform the asymptotic analysis of the resolvent problems for the scaled operators $\frac{1}{|\chi|^4}\mathcal{A}_\chi|_{L^2_{\text{bend}}}$ and $\frac{1}{|\chi|^2}\mathcal{A}_\chi|_{L^2_{\text{stretch}}}$. The following proposition provides us with Korn type inequalities which are crucial for calculating apriori estimates for the resolvent problems.

Proposition 3.2.10. *There exists $C > 0$ such that for every $\chi \in [-\pi,\pi]\setminus\{0\}$ we have:*
For every $u \in H^1_\chi(Y;H^1(\omega;\mathbb{C}^3)) \cap L^2_{\text{stretch}}$:

$$\|\boldsymbol{u}\|_{H^1(\omega\times Y,\mathbb{C}^3)} \leq \frac{C}{|\chi|}\|\operatorname{sym}\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})}.$$

For every $\boldsymbol{u} \in H^1_\chi(Y;H^1(\omega;\mathbb{C}^3)) \cap L^2_{\text{bend}}$:

$$\|\boldsymbol{u}_1\|_{H^1(\omega\times Y,\mathbb{C})} \leq \frac{C}{|\chi|^2}\|\operatorname{sym}\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})}, \quad \|\boldsymbol{u}_2\|_{H^1(\omega\times Y,\mathbb{C})} \leq \frac{C}{|\chi|^2}\|\operatorname{sym}\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})},$$
$$\|\boldsymbol{u}_3\|_{H^1(\omega\times Y,\mathbb{C})} \leq \frac{C}{|\chi|}\|\operatorname{sym}\nabla\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})}.$$

Proof. The proof relies on the orthogonality of spaces $H^1_\chi(Y;H^1(\omega;\mathbb{C}^3)) \cap L^2_{\text{stretch}}$ and $H^1_\chi(Y;H^1(\omega;\mathbb{C}^3)) \cap L^2_{\text{bend}}$ in $H^1(\omega\times Y;\mathbb{C}^3)$ scalar product. ■

Remark 3.2.11. This heterogeneity in componentwise estimates allows the scaling of the third component of the force terms in the bending case.

3.3. Asymptotic analysis of resolvents

3.3.1. Helpful definitions

The purpose of this section is to establish the estimates on the distance of the solution of the resolvent problem $\boldsymbol{u}$ to the leading order term, which is the solution to the homogenized problem. The distance is estimated with respect to the quasimomentum χ and the norm of the force term. This will, in return, have as a consequence the estimates on the resolvent operators in the operator norm topology. In order to do this, we will perform the asymptotic expansion of $\boldsymbol{u}$ with respect to the quasimomentum χ, starting with the solution to the homogenized problem. Therefore, we proceed with the definition of the homogenized material response matrices. Fix $m = (m_1, m_2, m_3, m_4) \in \mathbb{C}^4$. We define the following embedding operators:

$$\mathcal{I}_\chi^{\text{bend}} : \mathbb{C}^2 \to L^2_{\text{bend}}(\omega \times Y; \mathbb{C}^3), \quad \mathcal{I}_\chi^{\text{bend}} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ -i\chi(x_1 m_1 + x_2 m_2) \end{bmatrix},$$

$$\mathcal{I}^{\text{stretch}} : \mathbb{C}^2 \to L^2_{\text{stretch}}(\omega \times Y; \mathbb{C}^3), \quad \mathcal{I}^{\text{stretch}} \begin{bmatrix} m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} x_2 m_3 \\ -x_1 m_3 \\ m_4 \end{bmatrix},$$

$$\mathcal{I}_\chi^{\text{rod}} : \mathbb{C}^4 \to L^2(\omega \times Y; \mathbb{C}^3), \quad \mathcal{I}_\chi^{\text{rod}} m = \mathcal{I}_\chi^{\text{bend}} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \mathcal{I}^{\text{stretch}} \begin{bmatrix} m_3 \\ m_4 \end{bmatrix}.$$

These operators serve as a link between the appropriate Euclidean spaces and the finite dimensional subspaces of rod displacement approximations. We define the complex force momentum operators in the following way: for each $\chi \in [-\pi, \pi]$, $\boldsymbol{f} \in L^2(\omega \times Y; \mathbb{C}^3)$ we have:

$$\mathcal{M}^{\text{bend}}_{\chi}\begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{bmatrix} := \int_{\omega\times Y}\begin{bmatrix} \boldsymbol{f}_1 + i\chi \boldsymbol{f}_3 x_1 \\ \boldsymbol{f}_2 + i\chi \boldsymbol{f}_3 x_2 \end{bmatrix} \in \mathbb{C}^2, \quad \mathcal{M}^{\text{stretch}}_{\chi}\begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{bmatrix} := \int_{\omega\times Y}\begin{bmatrix} x_2\boldsymbol{f}_1 - x_1\boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{bmatrix} \in \mathbb{C}^2,$$

$$\mathcal{M}^{\text{rod}}_{\chi}\begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{bmatrix} := \begin{bmatrix} \mathcal{M}^{\text{bend}}_{\chi}\boldsymbol{f} \\ \mathcal{M}^{\text{stretch}}_{\chi}\boldsymbol{f} \end{bmatrix} = \int_{\omega\times Y}\begin{bmatrix} \boldsymbol{f}_1 + i\chi \boldsymbol{f}_3 x_1 \\ \boldsymbol{f}_2 + i\chi \boldsymbol{f}_3 x_2 \\ x_2\boldsymbol{f}_1 - x_1\boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{bmatrix} \in \mathbb{C}^4.$$

These momentum operators satisfy the following estimates:

$$\left\|\mathcal{M}^{\text{bend}}_{\chi}\boldsymbol{f}\right\| \leq \left\|\boldsymbol{f}_1\right\|_{L^2(\omega\times Y)} + \left\|\boldsymbol{f}_2\right\|_{L^2(\omega\times Y)} + |\chi|\left\|\boldsymbol{f}_3\right\|_{L^2(\omega\times Y)},$$
$$\left\|\mathcal{M}^{\text{stretch}}_{\chi}\boldsymbol{f}\right\| \leq \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, \quad \left\|\mathcal{M}^{\text{rod}}_{\chi}\boldsymbol{f}\right\| \leq \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

We have the following:

$$\mathcal{G}_\varepsilon^{-1}\mathcal{M}^{\text{bend}}_{\chi}\mathcal{G}_\varepsilon\boldsymbol{f} = \mathcal{G}_\varepsilon^{-1}\int_{\omega\times Y}\begin{bmatrix} \mathcal{G}_\varepsilon\boldsymbol{f}_1 + i\chi\mathcal{G}_\varepsilon\boldsymbol{f}_3 x_1 \\ \mathcal{G}_\varepsilon\boldsymbol{f}_2 + i\chi\mathcal{G}_\varepsilon\boldsymbol{f}_3 x_2 \end{bmatrix} = \int_\omega\left(\begin{bmatrix} \mathcal{G}_\varepsilon^{-1}\int_Y\mathcal{G}_\varepsilon\boldsymbol{f}_1 + x_1\mathcal{G}_\varepsilon^{-1}\left(i\chi\int_Y\mathcal{G}_\varepsilon\boldsymbol{f}_3\right) \\ \mathcal{G}_\varepsilon^{-1}\int_Y\mathcal{G}_\varepsilon\boldsymbol{f}_2 + x_2\mathcal{G}_\varepsilon^{-1}\left(i\chi\int_Y\mathcal{G}_\varepsilon\boldsymbol{f}_3\right) \end{bmatrix}\right)$$
$$= \int_\omega\left(\begin{bmatrix} \mathcal{G}_\varepsilon^{-1}\int_Y\mathcal{G}_\varepsilon\boldsymbol{f}_1 - \varepsilon x_1\frac{d}{dx_3}\mathcal{G}_\varepsilon^{-1}\int_Y\mathcal{G}_\varepsilon\boldsymbol{f}_3 \\ \mathcal{G}_\varepsilon^{-1}\int_Y\mathcal{G}_\varepsilon\boldsymbol{f}_2 - \varepsilon x_2\frac{d}{dx_3}\mathcal{G}_\varepsilon^{-1}\int_Y\mathcal{G}_\varepsilon\boldsymbol{f}_3 \end{bmatrix}\right) = \mathcal{M}^{\text{bend}}_{\varepsilon}\Xi_\varepsilon\boldsymbol{f}.$$

Similarly we have:

$$\mathcal{G}_\varepsilon^{-1}\mathcal{M}^{\text{stretch}}_{\chi}\mathcal{G}_\varepsilon\boldsymbol{f} = \mathcal{M}^{\text{stretch}}_{\varepsilon}\Xi_\varepsilon\boldsymbol{f}, \quad \mathcal{G}_\varepsilon^{-1}\mathcal{M}^{\text{rod}}_{\chi}\mathcal{G}_\varepsilon\boldsymbol{f} = \mathcal{M}^{\text{rod}}_{\varepsilon}\Xi_\varepsilon\boldsymbol{f}.$$

We also define the following matrices, which appear in the calculations and contain information on the cross section of the domain ω.

$$C^{\text{stretch}} = \begin{bmatrix} c_1(\omega) + c_2(\omega) & 0 \\ 0 & 1 \end{bmatrix}, \quad C^{\text{bend}}_{\chi}(\omega) = \begin{bmatrix} 1 + |\chi|^2 c_1(\omega) & 0 \\ 0 & 1 + |\chi|^2 c_2(\omega) \end{bmatrix},$$

$$C^{\text{rod}}_{\chi}(\omega) := \begin{bmatrix} C^{\text{bend}}_{\chi}(\omega) & 0 \\ 0 & C^{\text{stretch}} \end{bmatrix} = \begin{bmatrix} 1 + |\chi|^2 c_1(\omega) & 0 & 0 & 0 \\ 0 & 1 + |\chi|^2 c_2(\omega) & 0 & 0 \\ 0 & 0 & c_1(\omega) + c_2(\omega) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.11}$$

We will also make use of the notation $C^{\mathrm{rod}}(\omega) := C_0^{\mathrm{rod}}(\omega)$.

For every $m = (m_1, m_2, m_3, m_4)^T$, $d = (d_1, d_2, d_3, d_4)^T \in \mathbb{C}^4$, we have:

$$\int_{\omega\times Y} \mathcal{I}_\chi^{\mathrm{bend}} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \cdot \overline{\mathcal{I}_\chi^{\mathrm{bend}} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}} dxdy = C_\chi^{\mathrm{bend}}(\omega) \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \cdot \overline{\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}},$$

$$\int_{\omega\times Y} \mathcal{I}^{\mathrm{stretch}} \begin{bmatrix} m_3 \\ m_4 \end{bmatrix} \cdot \overline{\mathcal{I}^{\mathrm{stretch}} \begin{bmatrix} d_3 \\ d_4 \end{bmatrix}} dxdy = C^{\mathrm{stretch}} \begin{bmatrix} m_3 \\ m_4 \end{bmatrix} \cdot \overline{\begin{bmatrix} d_3 \\ d_4 \end{bmatrix}},$$

$$\int_{\omega\times Y} \mathcal{I}_\chi^{\mathrm{rod}} m \cdot \overline{\mathcal{I}_\chi^{\mathrm{rod}} d}\, dxdy = C_\chi^{\mathrm{rod}}(\omega) m \cdot \overline{d}.$$

Notice that, for every $\boldsymbol{f} \in L^2(\omega \times Y; \mathbb{C}^3)$, $d = (d_1, d_2, d_3, d_4)^T \in \mathbb{C}^4$, we have the following:

$$\int_{\omega\times Y} \boldsymbol{f} \cdot \overline{\mathcal{I}_\chi^{\mathrm{bend}} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}} dxdy = \mathcal{M}_\chi^{\mathrm{bend}} \boldsymbol{f} \cdot \overline{\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}},$$

$$\int_{\omega\times Y} \boldsymbol{f} \cdot \overline{\mathcal{I}^{\mathrm{stretch}} \begin{bmatrix} d_3 \\ d_4 \end{bmatrix}} dxdy = \mathcal{M}_\chi^{\mathrm{stretch}} \boldsymbol{f} \cdot \overline{\begin{bmatrix} d_3 \\ d_4 \end{bmatrix}},$$

$$\int_{\omega\times Y} \boldsymbol{f} \cdot \overline{\mathcal{I}_\chi^{\mathrm{rod}} d}\, dxdy = \mathcal{M}_\chi^{\mathrm{rod}} \boldsymbol{f} \cdot \overline{d}.$$

Note that these are, in fact, the following dualities:

$$\left\langle \mathcal{I}_\chi^{\mathrm{bend}} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \boldsymbol{f} \right\rangle_{L^2} = \left\langle \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \mathcal{M}_\chi^{\mathrm{bend}} \boldsymbol{f} \right\rangle_{\mathbb{C}^2},$$

$$\left\langle \mathcal{I}^{\mathrm{stretch}} \begin{bmatrix} m_3 \\ m_4 \end{bmatrix}, \boldsymbol{f} \right\rangle_{L^2} = \left\langle \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \mathcal{M}^{\mathrm{stretch}} \boldsymbol{f} \right\rangle_{\mathbb{C}^2},$$

$$\left\langle \mathcal{I}_\chi^{\mathrm{rod}} m, \boldsymbol{f} \right\rangle_{L^2} = \left\langle m, \mathcal{M}_\chi^{\mathrm{rod}} \boldsymbol{f} \right\rangle_{\mathbb{C}^4}.$$

Due to these relations, we conclude:

$$\mathcal{I}^{\mathrm{stretch}} = (\mathcal{M}^{\mathrm{stretch}})^*, \quad \mathcal{I}_\chi^{\mathrm{bend}} = (\mathcal{M}_\chi^{\mathrm{bend}})^*, \quad \mathcal{I}_\chi^{\mathrm{rod}} = (\mathcal{M}_\chi^{\mathrm{rod}})^*.$$

We also make use of the following matrices which contain the information on the symmetrized gradient of Kirchoff-Love deformations:

$$\Lambda^{\text{stretch}}_{\chi,m_3,m_4}(x) := (i\chi)\begin{bmatrix} 0 & 0 & \frac{x_2 m_3}{2} \\ 0 & 0 & \frac{-x_1 m_3}{2} \\ \frac{x_2 m_3}{2} & \frac{-x_1 m_3}{2} & m_4 \end{bmatrix}, \quad \Lambda^{\text{bend}}_{\chi,m_1,m_2}(x) := (i\chi)^2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m_1x_1 - m_2x_2 \end{bmatrix},$$

$$\Lambda^{\text{rod}}_{\chi,m}(x) := \Lambda^{\text{bend}}_{\chi,m_1,m_2}(x) + \Lambda^{\text{stretch}}_{\chi,m_3,m_4}(x).$$

It is clear that:

$$\Lambda^{\text{bend}}_{\chi,m_1,m_2}(x) = (i\chi)^2\mathcal{J}^{\text{bend}}_{\chi,m_1,m_2}, \quad \Lambda^{\text{stretch}}_{\chi,m_3,m_4}(x) = (i\chi)\mathcal{J}^{\text{stretch}}_{\chi,m_3,m_4}.$$

Here we note that

$$\begin{aligned} &\operatorname{sym}\nabla\begin{bmatrix} m_3x_2 \\ -m_3x_1 \\ m_4 \end{bmatrix} + iX_\chi\begin{bmatrix} m_3x_2 \\ -m_3x_1 \\ m_4 \end{bmatrix} \\ &= \operatorname{sym}\begin{bmatrix} 0 & m_3 & 0 \\ -m_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{1}{2}i\chi m_3x_2 \\ 0 & 0 & -\frac{1}{2}i\chi m_3x_1 \\ \frac{1}{2}i\chi m_3x_2 & -\frac{1}{2}i\chi m_3x_1 & i\chi m_4 \end{bmatrix} = \Lambda^{\text{stretch}}_{\chi,m_3,m_4}(x), \end{aligned}$$

also,

$$\begin{aligned} &\operatorname{sym}\nabla\begin{bmatrix} m_1 \\ m_2 \\ -i\chi(m_1x_1 + m_2x_2) \end{bmatrix} + iX_\chi\begin{bmatrix} m_1 \\ m_2 \\ -i\chi(m_1x_1 + m_2x_2) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -\frac{1}{2}i\chi m_1 \\ 0 & 0 & -\frac{1}{2}i\chi m_2 \\ -\frac{1}{2}i\chi m_1 & -\frac{1}{2}i\chi m_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{1}{2}i\chi m_1 \\ 0 & 0 & \frac{1}{2}i\chi m_2 \\ \frac{1}{2}i\chi m_1 & \frac{1}{2}i\chi m_2 & \chi^2(m_1x_1 + m_2x_2) \end{bmatrix} = \Lambda^{\text{bend}}_{\chi,m_1,m_2}(x). \end{aligned}$$

Easily, we can calculate the following estimates

$$C_1(\omega)|\chi|^2|(m_1,m_2)| \le \|\Lambda^{\text{bend}}_{\chi,m_1,m_2}(x)\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})} \le C_2(\omega)|\chi|^2|(m_1,m_2)|,$$

$$C_1(\omega)|\chi||(m_3,m_4)| \le \|\Lambda^{\text{stretch}}_{\chi,m_3,m_4}(x)\|_{L^2(\omega\times Y;\mathbb{C}^{3\times3})} \le C_2(\omega)|\chi||(m_3,m_4)|,$$

Now, consider the problem of finding $\boldsymbol{u} \in H^1_\#(Y;H^1(\omega;\mathbb{C}^3))$ such that:

$$\int_{\omega\times Y}\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u} : \overline{\operatorname{sym}\nabla\boldsymbol{v}} = \int_{\omega\times Y}\boldsymbol{f}\boldsymbol{v}, \quad \forall\boldsymbol{v} \in H^1_\#(Y;H^1(\omega;\mathbb{C}^3)).$$

In order for this problem to be well posed, the right hand side $\boldsymbol{f}$ must be orthogonal to the kernel of the operator $\operatorname{sym}\nabla$. The kernel of this operator consists of rigid motions, but since we are dealing with functions periodic in y, we have:

$$H^1_\#(Y;H^1(\omega;\mathbb{C}^3))\cap\ker(\operatorname{sym}\nabla)=\left\{\begin{bmatrix} dx_2+c_1\\ -dx_1+c_2\\ c_3\end{bmatrix},c_1,c_2,c_3,d\in\mathbb{C}\right\}<H^1_\#(Y;H^1(\omega;\mathbb{C}^3)).$$

We define the following space:

$$\begin{aligned} H &:= \left[H^1_\#(Y;H^1(\omega;\mathbb{C}^3))\cap\ker(\operatorname{sym}\nabla)\right]^\perp\\ &= \left\{\boldsymbol{u}\in H^1_\#(Y;H^1(\omega;\mathbb{C}^3));\int_{\omega\times Y}\boldsymbol{u}=0,\int_{\omega\times Y}x_2\boldsymbol{u}_1-x_1\boldsymbol{u}_2=0\right\}.\end{aligned}$$

Note that H is closed subspace of a Hilbert space $H^1_\#(Y;H^1(\omega;\mathbb{C}^3))$ and that Korn's inequality, as well as Poincare inequality hold on H. For $\boldsymbol{f}\in H$, Lax-Milgram theorem yields the existence of unique solution $\boldsymbol{u}\in H$ of the above problem, with test functions taken in the space H. But, since, $\boldsymbol{f}\in H$, this is equivalent to allowing arbitrary test functions $\boldsymbol{v}\in H^1_\#(Y;H^1(\omega;\mathbb{C}^3))$.

This argumentation allows us to consider the following well posed problem: Find $\boldsymbol{u}_{\chi,m}\in H$, such that:

$$\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_{\chi,m}+\Lambda^{\mathrm{rod}}_{\chi,m}(x)\right):\overline{\operatorname{sym}\nabla\boldsymbol{v}}dxdy=0,\quad\forall\boldsymbol{v}\in H.$$

The above problem has a unique solution and can be equivalently rewritten by using the adjoint of the operator $\operatorname{sym}\nabla$, namely

$$\operatorname{sym}\nabla^*:L^2_\#(Y;H^1(\omega;\mathbb{C}^3))\to H,$$

as follows:

$$\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_{\chi,m}=-\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda^{\mathrm{rod}}_{\chi,m}(x),\quad\boldsymbol{u}_{\chi,m}\in H.$$

Next, we define the matrix $\mathbb{C}^{\mathrm{rod}}_\chi\in\mathbb{C}^{4\times4}$ with the formula for the bilinear form:

$$\mathbb{C}^{\mathrm{rod}}_\chi m\cdot\overline{d}=\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_{\chi,m}+\Lambda^{\mathrm{rod}}_{\chi,m}(x)\right):\overline{\Lambda^{\mathrm{rod}}_{\chi,d}(x)}dxdy$$

In order to analyze the problem separately on two invariant subspaces we also make use of the following spaces:

$$H^{\mathrm{stretch}} := \left[H^1_\#(Y;H^1(\omega;\mathbb{C}^3))\cap\ker(\operatorname{sym}\nabla)\cap L^2_{\mathrm{stretch}}\right]^\perp$$

$$
\begin{aligned}
&= \left\{ \boldsymbol{u} \in H^1_\#(Y; H^1(\omega;\mathbb{C}^3)); \int_{\omega\times Y} \boldsymbol{u}_3 = 0, \int_{\omega\times Y} x_1\boldsymbol{u}_2 - x_2\boldsymbol{u}_1 = 0 \right\},\\
H^{\text{bend}} &:= \left[H^1_\#(Y; H^1(\omega;\mathbb{C}^3)) \cap \ker(\operatorname{sym}\nabla) \cap L^2_{\text{bend}} \right]^\perp\\
&= \left\{ \boldsymbol{u} \in H^1_\#(Y; H^1(\omega;\mathbb{C}^3)); \int_{\omega\times Y} \boldsymbol{u}_1 = 0, \int_{\omega\times Y} \boldsymbol{u}_2 = 0 \right\}.
\end{aligned}
$$

Notice that

$$
H^1_\#(Y; H^1(\omega;\mathbb{C}^3)) \cap \ker(\operatorname{sym}\nabla) \cap L^2_{\text{bend}} = \left\{ \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix}, c_1, c_2 \in \mathbb{C} \right\} \leq H^1_\#(Y; H^1(\omega;\mathbb{C}^3)).
$$

$$
H^1_\#(Y; H^1(\omega;\mathbb{C}^3)) \cap \ker(\operatorname{sym}\nabla) \cap L^2_{\text{stretch}} = \left\{ \begin{bmatrix} dx_2 \\ -dx_1 \\ c_3 \end{bmatrix}, c_3, d \in \mathbb{C} \right\} \leq H^1_\#(Y; H^1(\omega;\mathbb{C}^3)).
$$

We consider the solutions of the following equations:

$$
\begin{aligned}
\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}^{\text{bend}}_{\chi,m_1,m_2} &= -\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda^{\text{bend}}_{\chi,m_1,m_2}(x), \quad \boldsymbol{u}^{\text{bend}}_{\chi,m_1,m_2} \in H^{\text{bend}},\\
\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}^{\text{stretch}}_{\chi,m_3,m_4} &= -\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda^{\text{stretch}}_{\chi,m_3,m_4}(x), \quad \boldsymbol{u}^{\text{stretch}}_{\chi,m_3,m_4} \in H^{\text{stretch}}.
\end{aligned}
$$

Note that the well posedness of the above problems requires invoking material symmetries in order to see that nontrivial unique solutions exist. Thus, by defining the matrices $\mathbb{C}^{\text{bend}}_\chi, \mathbb{C}^{\text{stretch}}_\chi \in \mathbb{C}^{2\times2}$ with the following bilinear forms:

$$
\begin{aligned}
\mathbb{C}^{\text{bend}}_\chi(m_1,m_2)^T \cdot \overline{(d_1,d_2)^T} &= \int_{\omega\times Y} \mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}^{\text{bend}}_{\chi,m_1,m_2} + \Lambda^{\text{bend}}_{\chi,m_1,m_2}(x)\right) : \overline{\Lambda^{\text{bend}}_{\chi,d_1,d_2}}(x)dxdy\\
\mathbb{C}^{\text{stretch}}_\chi(m_3,m_4)^T \cdot \overline{(d_3,d_4)^T} &= \int_{\omega\times Y} \mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}^{\text{stretch}}_{\chi,m_3,m_4} + \Lambda^{\text{stretch}}_{\chi,m_3,m_4}(x)\right) : \overline{\Lambda^{\text{stretch}}_{\chi,d_3,d_4}}(x)dxdy,
\end{aligned}
$$

under the assumptions 3.1.1, we have the following decomposition :

$$
\mathbb{C}^{\text{rod}}_\chi m \cdot \overline{d} = \mathbb{C}^{\text{bend}}_\chi(m_1,m_2)^T \cdot \overline{(d_1,d_2)^T} + \mathbb{C}^{\text{stretch}}_\chi(m_3,m_4)^T \cdot \overline{(d_3,d_4)^T}.
$$

Remark 3.3.1. We make a small remark on noting the following properties of these matrices: First, we can easily see that the matrix $\mathbb{C}^{\text{rod}}_\chi$ is hermitian.

$$
\begin{aligned}
&\mathbb{C}^{\text{rod}}_\chi m \cdot \overline{d} = \int_{\omega\times Y} \mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_{\chi,m} + \Lambda^{\text{rod}}_{\chi,m}(x)\right) : \overline{\Lambda^{\text{rod}}_{\chi,d}(x)}dxdy\\
&= \int_{\omega\times Y} \mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_{\chi,m} + \Lambda^{\text{rod}}_{\chi,m}(x)\right) : \overline{\operatorname{sym}\nabla\boldsymbol{u}^d + \Lambda^{\text{rod}}_{\chi,d}(x)}dxdy\\
&= \overline{\int_{\omega\times Y} \mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}^d + \Lambda^{\text{rod}}_{\chi,d}(x)\right) : \overline{\operatorname{sym}\nabla\boldsymbol{u}_{\chi,m} + \Lambda^{\text{rod}}_{\chi,m}(x)}dxdy}\\
&= \overline{\int_{\omega\times Y} \mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}^d + \Lambda^{\text{rod}}_{\chi,d}(x)\right) : \overline{\Lambda^{\text{rod}}_{\chi,m}(x)}dxdy} = \overline{\mathbb{C}^{\text{rod}}_\chi d \cdot \overline{m}}.
\end{aligned}
$$

The same is true for the matrices $\mathbb{C}_\chi^{\text{stretch}}$, $\mathbb{C}_\chi^{\text{bend}}$. Second, from the structure of matrices $\Lambda^{\text{bend}}_{\chi,m_1,m_2}(x) and \Lambda^{\text{stretch}}_{\chi,m_3,m_4}(x)$ we see the following:

$$\mathbb{C}_\chi^{\text{rod}}(0,0,m_3,m_4)^T \cdot \overline{(0,0,d_3,d_4)^T} = (i\chi)^2 \mathbb{C}^{\text{rod}}(0,0,m_3,m_4)^T \cdot \overline{(0,0,d_3,d_4)^T},$$
$$\mathbb{C}_\chi^{\text{rod}}(m_1,m_2,0,0)^T \cdot \overline{(d_1,d_2,0,0)^T} = (i\chi)^4 \mathbb{C}^{\text{rod}}(m_1,m_2,0,0)^T \cdot \overline{(d_1,d_2,0,0)^T},$$

For that reason it is clear that:

$$C_1(\omega)|\chi|^4|(m_1,m_2)|^2 \le \mathbb{C}_\chi^{\text{bend}}(m_1,m_2)\cdot\overline{(m_1,m_2)^T} \le C_2(\omega)|\chi|^4|(m_1,m_2)|^2,$$
$$C_1(\omega)|\chi|^2|(m_3,m_4)|^2 \le \mathbb{C}_\chi^{\text{stretch}}(m_3,m_4)\cdot\overline{(m_3,m_4)^T} \le C_2(\omega)|\chi|^2|(m_3,m_4)|^2.$$

We conclude this remark with the fact that these two matrices both have two real eigenvalues, where these eigenvalues are of the order $|\chi|^4$, for the bending case, and $|\chi|^2$ for the stretching case.

We have the following estimate:

Proposition 3.3.2. *There exist a constant $\mu > 0$ such that $\forall m = (m_1,m_2,m_3,m_4)^T \in \mathbb{C}^3$ we have:*

$$\mu\left(|\chi|^4|(m_1,m_2)|^2 + |\chi|^2|(m_3,m_4)|^2\right) \le \mathbb{C}_\chi^{\text{rod}} m\cdot\overline{m} \le \frac{1}{\mu}\left(|\chi|^4|(m_1,m_2)|^2 + |\chi|^2|(m_3,m_4)|^2\right).$$

Proof.

$$\begin{aligned}
\mathbb{C}_\chi^{\text{rod}} m\cdot\overline{m} &= \int_{\omega\times Y} \mathbb{C}(y)\left(\operatorname{sym}\nabla \boldsymbol{u}_{\chi,m} + \Lambda^{\text{rod}}_{\chi,m}(x)\right) : \overline{\operatorname{sym}\nabla \boldsymbol{u}^m + \Lambda^{\text{rod}}_{\chi,m}(x)}\,dxdy \\
\ge& C\left\|\operatorname{sym}\nabla \boldsymbol{u}_{\chi,m} + \Lambda^{\text{rod}}_{\chi,m}(x)\right\|^2_{L^2(\omega\times Y;\mathbb{R}^{3\times3})} \\
\ge& C\left(\left\|i\chi x_2 m_3 + \left(\partial_1(\boldsymbol{u}_{\chi,m})_3 + \partial_3(\boldsymbol{u}_{\chi,m})_1\right)\right\|^2_{L^2(\omega\times Y)} + \left\|-i\chi x_1 m_3 + \left(\partial_2(\boldsymbol{u}_{\chi,m})_3 + \partial_3(\boldsymbol{u}_{\chi,m})_2\right)\right\|^2_{L^2(\omega\times Y)}\right) \\
&+ C\left\|i\chi m_4 - (i\chi)^2(x_1m_1 - x_2m_2) - \partial_3(\boldsymbol{u}_{\chi,m})_3\right\|^2_{L^2(\omega\times Y)}.
\end{aligned}$$

Due to orthogonality, it is clear that

$$\left\|i\chi m_4 - (i\chi)^2(x_1m_1 - x_2m_2) - \partial_3(\boldsymbol{u}_{\chi,m})_3\right\|^2_{L^2(\omega\times Y)} \ge C\left(|\chi|^2|m_4|^2 + |\chi|^4|(m_1,m_2)|^2\right).$$

By using similar trick as before, one can obtain the following inequality as well:

$$\left\|i\chi x_2 m_3 + \left(\partial_1(\boldsymbol{u}_{\chi,m})_3 + \partial_3(\boldsymbol{u}_{\chi,m})_1\right)\right\|^2_{L^2(\omega\times Y)} + \left\|-i\chi x_1 m_3 + \left(\partial_2(\boldsymbol{u}_{\chi,m})_3 + \partial_3(\boldsymbol{u}_{\chi,m})_2\right)\right\|^2_{L^2(\omega\times Y)}$$
$$\ge C|\chi|^2|m_3|^2.$$

■

Notice that by applying the scaled Gelfand transform to the homogenized operators we get the following formulae:

$$\mathcal{G}_\varepsilon \mathcal{A}^{\mathrm{bend}} \Xi_\varepsilon \mathcal{G}_\varepsilon^{-1} \boldsymbol{u} = \frac{1}{\varepsilon^4} \mathbb{C}_\chi^{\mathrm{bend}} \int_Y \boldsymbol{u},$$
$$\mathcal{G}_\varepsilon \mathcal{A}^{\mathrm{stretch}} \Xi_\varepsilon \mathcal{G}_\varepsilon^{-1} \boldsymbol{u} = \frac{1}{\varepsilon^2} \mathbb{C}_\chi^{\mathrm{stretch}} \int_Y \boldsymbol{u}.$$

This is seen also by examining formulae:

$$P_{\mathbb{C}^2} \mathcal{G}_\varepsilon \mathcal{A}^{\mathrm{bend}} \mathcal{G}_\varepsilon^{-1} P_{\mathbb{C}^2} = P_{\mathbb{C}^2} \mathcal{G}_\varepsilon \left(\left(\frac{d^2}{dx_3^2} \right) \mathbb{C}^{\mathrm{bend}} \left(\frac{d^2}{dx_3^2} \right) \right) \mathcal{G}_\varepsilon^{-1} P_{\mathbb{C}^2} = \frac{1}{\varepsilon^4} P_{\mathbb{C}^2} \mathbb{C}_\chi^{\mathrm{bend}} P_{\mathbb{C}^2},$$
$$P_{\mathbb{C}^2} \mathcal{G}_\varepsilon \mathcal{A}^{\mathrm{stretch}} \mathcal{G}_\varepsilon^{-1} P_{\mathbb{C}^2} = P_{\mathbb{C}^2} \mathcal{G}_\varepsilon \left(\left(\frac{d}{dx_3} \right) \mathbb{C}^{\mathrm{stretch}} \left(\frac{d}{dx_3} \right) \right) \mathcal{G}_\varepsilon^{-1} P_{\mathbb{C}^2} = \frac{1}{\varepsilon^2} P_{\mathbb{C}^2} \mathbb{C}_\chi^{\mathrm{stretch}} P_{\mathbb{C}^2},$$
$$P_{\mathbb{C}^4} \mathcal{G}_\varepsilon \mathcal{A}^{\mathrm{rod}} \mathcal{G}_\varepsilon^{-1} P_{\mathbb{C}^4} = \left(\frac{1}{\varepsilon^2} P_{\mathbb{C}^2} \mathbb{C}_\chi^{\mathrm{bend}} P_{\mathbb{C}^2}, \frac{1}{\varepsilon^2} P_{\mathbb{C}^2} \mathbb{C}_\chi^{\mathrm{stretch}} P_{\mathbb{C}^2} \right)^T, \quad \text{(under the Assumption 3.1.1)}.$$

3.3.2. The asymptotics in the stretching space

Here we provide the estimates for the error in the approximation of the solution to the following resolvent equation posed in the space of stretching deformations:

Find $\boldsymbol{u} \in H^1_\#(Y; H^1(\omega; \mathbb{C}^3)) \cap L^2_{\mathrm{stretch}}$

$$\frac{1}{|\chi|^2} \int_{\omega \times Y} \mathbb{C}(\operatorname{sym} \nabla \boldsymbol{u} + iX_\chi \boldsymbol{u}) : \overline{(\operatorname{sym} \nabla \boldsymbol{v} + iX_\chi \boldsymbol{v})} + \int_{\omega \times Y} \boldsymbol{u} \bar{\boldsymbol{v}} = \int_{\omega \times Y} \boldsymbol{f} \bar{\boldsymbol{v}}, \tag{3.14}$$

For all $\boldsymbol{v} \in H^1_\#(Y; H^1(\omega; \mathbb{C}^3)) \cap L^2_{\mathrm{stretch}}$.
This can be rewritten, using the adjoint of the operator $\operatorname{sym} \nabla + iX_\chi$, as a problem of finding $\boldsymbol{u} \in H^1_\#(Y; H^1(\omega; \mathbb{C}^3)) \cap L^2_{\mathrm{stretch}}$ such that

$$\frac{1}{|\chi|^2} \left(\operatorname{sym} \nabla^* + \left(iX_\chi \right)^* \right) \mathbb{C}(y) \left(\operatorname{sym} \nabla + iX_\chi \right) \boldsymbol{u} + \boldsymbol{u} = \boldsymbol{f}.$$

Remark 3.3.3. We test the above equation with the solution $\boldsymbol{u}$ and employ Proposition 3.2.10 to obtain:

$$\begin{aligned} \frac{1}{|\chi|^2} \| \left(\operatorname{sym} \nabla + iX_\chi \right) \boldsymbol{u} \|^2_{L^2(\omega \times Y; \mathbb{C}^{3\times 3})} + \|\boldsymbol{u}\|^2_{L^2(\omega \times Y, \mathbb{C}^3)} &\le C \|\boldsymbol{f}\|_{L^2(\omega \times Y, \mathbb{C}^3)} \|\boldsymbol{u}\|_{L^2(\omega \times Y, \mathbb{C}^3)} \\ &\le \frac{C}{|\chi|} \|\boldsymbol{f}\|_{L^2(\omega \times Y, \mathbb{C}^3)} \| \left(\operatorname{sym} \nabla + iX_\chi \right) \boldsymbol{u} \|_{L^2(\omega \times Y; \mathbb{C}^{3\times 3})}, \end{aligned}$$

so:

$$\|\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})} \leq C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)}.$$

Again, by Proposition 3.2.10 we deduce the following apriori estimate:

$$\|\boldsymbol{u}\|_{H^1(\omega\times Y,\mathbb{C}^3)} \leq C\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)}$$

We want to construct a function which approximates the solution to the given problem with the error of the order $|\chi|^2$. This can be done by several steps, introducing corrector terms which can be calculated with the procedure explained below. The leading order term in the approximation is given with the expression for the stretching rigid motion.

1) The leading order term and first order corrector

Consider the solution $(m_3, m_4)^T$ to the following equation:

$$\left(\frac{1}{|\chi|^2}\mathbb{C}_\chi^{\text{stretch}} + C^{\text{stretch}}\right)\begin{bmatrix} m_3 \\ m_4 \end{bmatrix} = \mathcal{M}^{\text{stretch}}\boldsymbol{f}.$$

By testing this equation against $(m_3, m_4)^T$, it is clear that we have the estimate:

$$|(m_3, m_4)^T| \leq C\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

Set $\boldsymbol{u}_0 := \mathcal{I}^{\text{stretch}}\begin{bmatrix} m_3 \\ m_4 \end{bmatrix}$ and note that it can be estimated

$$\|\boldsymbol{u}_0\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

Define the first order corrector term $\boldsymbol{u}_1 \in H^{\text{stretch}}$ as the solution to the well posed problem

$$\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_1 = -\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda^{\text{stretch}}_{\chi,m_3,m_4}(x), \quad \boldsymbol{u}_1 \in H^{\text{stretch}}.$$

The well posedness of the above problem comes from the fact that the range of the operator $\operatorname{sym}\nabla^*$ is orthogonal to the kernel of $\operatorname{sym}\nabla$. Also, it is clear from the definition of

$\mathbb{C}_\chi^{\text{stretch}}$ that the following holds:

$$\frac{1}{|\chi|^2}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla \boldsymbol{u}_1+\Lambda^{\text{stretch}}_{\chi,m_3,m_4}(x)\right):\overline{\Lambda^{\text{stretch}}_{\chi,d_3,d_4}(x)}dxdy+\int_{\omega\times Y}\begin{bmatrix} m_3x_2\\ -m_3x_1\\ m_4\end{bmatrix}\cdot\overline{\begin{bmatrix} d_3x_2\\ -d_3x_1\\ d_4\end{bmatrix}}dxdy$$
$$=\int_{\omega\times Y}\begin{bmatrix}\boldsymbol{f}_1\\ \boldsymbol{f}_2\\ \boldsymbol{f}_3\end{bmatrix}\cdot\overline{\begin{bmatrix} d_3x_2\\ -d_3x_1\\ d_4\end{bmatrix}}dxdy \tag{3.15}$$

The corrector term $\boldsymbol{u}_1$ belongs to the space L^2_{stretch} due to the structure of the elasticity tensor $\mathbb{C}$. Elliptic estimates yield the following:

$$\|\boldsymbol{u}_1\|_{H^1(\omega\times Y;\mathbb{C}^3)}\leq C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

Define the functional

$$\tilde{\boldsymbol{f}}_1=-\frac{1}{|\chi|^2}\left(iX_\chi\right)^*\mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_1+\Lambda^{\text{stretch}}_{\chi,m_3,m_4}(x)\right)-\boldsymbol{u}_0+\boldsymbol{f},\quad \tilde{\boldsymbol{f}}_1:=L^2(\omega\times Y;\mathbb{C}^3)\to\mathbb{C}.$$

It is clear that $\tilde{\boldsymbol{f}}_1$ vanishes when tested against stretching rigid motions, which follows directly from (3.15).

2) Second order corrector

In view of the equation (3.14) we define the second order corrector term $\boldsymbol{u}_2\in H^{\text{stretch}}$ with the following equation:

$$\begin{aligned}&\frac{1}{|\chi|^2}\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_2\\ &=-\frac{1}{|\chi|^2}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_1+\operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_1+\left(iX_\chi\right)^*\mathbb{C}(y)\Lambda^{\text{stretch}}_{\chi,m_3,m_4}(x)\right)-\boldsymbol{u}_0+\boldsymbol{f},\\ &\boldsymbol{u}_2\in H^{\text{stretch}}.\end{aligned} \tag{3.16}$$

The right hand side is equal to the functional $\tilde{\boldsymbol{f}}_1-\frac{1}{|\chi|^2}\left(\operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_1\right)$ which clearly vanishes when tested against functions in H^{stretch}.

Therefore this problem is well posed and the corrector term $\boldsymbol{u}_2$ satisfies the following estimate:

$$\|\boldsymbol{u}_2\|_{H^1(\omega\times Y;\mathbb{C}^3)}\leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

Remark 3.3.4. The definition of the leading order term might seem like it was introduced blindly. However it is very well motivated, which can be justified with the following reasoning: The first requirement for $\boldsymbol{u}_0$ is to solve:

$$\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_0 = 0.$$

From this we see the structure $\boldsymbol{u}_0 = (c_1 + dx_2, c_2 - dx_1, c_3)^T$ since these are the only rigid motions which are periodic in y. Next, since $\boldsymbol{u}_0$ belongs to $L^2_{\text{bend}}(\omega\times Y;\mathbb{C}^3)$, we conclude that $\boldsymbol{u}_0 = (dx_2, -dx_1, c_3)^T$. Here we are left with two degrees of freedom in determining the precise values of the constants d and c_3. But it turns out that these two degrees of freedom are enough to fulfill the well posedness condition for the problem (3.16). Moreover, the well posedness condition for (3.16) uniquely determines the values of d and c_3 via the equation (3.15), which directly translates to (3.3.2). It is somewhat evident that, since we have depleted our spare degrees of freedom, there would be issues with continuation of this procedure.

The total approximation built up so far $\boldsymbol{u}_{\text{approx}} := \boldsymbol{u}_0 + \boldsymbol{u}_1 + \boldsymbol{u}_2$ satisfies the following equation:

$$\frac{1}{|\chi|^2}\left(\operatorname{sym}\nabla^* + \left(iX_\chi\right)^*\right)\mathbb{C}(y)\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u}_{\text{approx}} + \boldsymbol{u}_{\text{approx}} - \boldsymbol{f} = R_\chi.$$

The residual R_χ can be calculated and is given with the following expression:

$$R_\chi = \frac{1}{|\chi|^2}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_2 + \operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2 + \left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_1 + \left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2\right)$$
$$+\boldsymbol{u}_1 + \boldsymbol{u}_2.$$

It also satisfies the estimate

$$\|\tilde{R}_\chi\|_{H^{-1}_{\#}(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

These approximation is not satisfactory since the order of the error is, for our purposes, not large enough with respect to χ. We have to proceed further with the approximation calculation in order to reduce the error. Unfortunately, the problem

$$\frac{1}{|\chi|^2}\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_3$$
$$= -\frac{1}{|\chi|^2}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_2 + \operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2 + \left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_1\right) - \boldsymbol{u}_1, \quad \boldsymbol{u}_3 \in H^{\text{stretch}}.$$

is ill posed and the procedure must be terminated here. In order to define better approximation we must restart the procedure.

Remark 3.3.5. Here we note that the continuation of the procedure is only necessary if one wishes to calculate higher precision norm-resolvent estimates. The approximation defined so far is enough for the $L^2 \to L^2$ norm-resolvent estimates displayed in the Theorem 3.1.3.

3) Reseting the procedure

We proceed with the correction of the leading order term in the following way:
Set $\boldsymbol{u}_0^{(1)} := \mathcal{I}^{\text{stretch}} \begin{bmatrix} m_3^{(1)} \\ m_4^{(1)} \end{bmatrix}$, where $\begin{bmatrix} m_3^{(1)} \\ m_4^{(1)} \end{bmatrix}$ satisfies:

$$\left(\frac{1}{|\chi|^2} \mathbb{C}_\chi^{\text{stretch}} + C^{\text{stretch}} \right) \begin{bmatrix} m_3^{(1)} \\ m_4^{(1)} \end{bmatrix} \cdot \overline{\begin{bmatrix} d_3 \\ d_4 \end{bmatrix}} = -\frac{1}{|\chi|^2} \int_{\omega \times Y} \mathbb{C}(y) \left(\operatorname{sym} \nabla \boldsymbol{u}_2 + iX_\chi \boldsymbol{u}_1 \right) : \overline{\Lambda_{\chi, d_3, d_4}^{\text{stretch}}(x)} dx dy, \tag{3.17}$$

$\forall (d_3, d_4)^T \in \mathbb{C}^2$.

It is easy to see that $\boldsymbol{u}_0^{(1)}$ defined in this way satisfy the following:

$$\|\boldsymbol{u}_0^{(1)}\|_{H^1(\omega \times Y; \mathbb{C}^3)} \leq C|\chi| \|\boldsymbol{f}\|_{L^2(\omega \times Y; \mathbb{C}^3)}.$$

The next correction is the function $\boldsymbol{u}_1^{(1)}$ which is the solution to the problem:

$$\operatorname{sym} \nabla^* \mathbb{C}(y) \operatorname{sym} \nabla \boldsymbol{u}_1^{(1)} = -\operatorname{sym} \nabla^* \mathbb{C}(y) \Lambda_{\chi, m_3^{(1)}, m_4^{(1)}}^{\text{stretch}}(x), \quad \boldsymbol{u}_1^{(1)} \in H^{\text{stretch}}.$$

It satisfies

$$\|\boldsymbol{u}_1^{(1)}\|_{H^1(\omega \times Y; \mathbb{C}^3)} \leq C|\chi|^2 \|\boldsymbol{f}\|_{L^2(\omega \times Y; \mathbb{C}^3)}.$$

It is easy to check that these two equations yield

$$\frac{1}{|\chi|^2} \int_{\omega \times Y} \mathbb{C}(y) \left(\operatorname{sym} \nabla \boldsymbol{u}_1^{(1)} + \Lambda_{\chi, m_3^{(1)}, m_4^{(1)}}^{\text{stretch}}(x) \right) : \overline{\Lambda_{\chi, d_3, d_4}^{\text{stretch}}(x)} dx dy + \int_{\omega \times Y} \begin{bmatrix} m_3^{(1)} x_2 \\ -m_3^{(1)} x_1 \\ m_4^{(1)} \end{bmatrix} \cdot \overline{\begin{bmatrix} d_3 x_2 \\ -d_3 x_1 \\ d_4 \end{bmatrix}} dx dy$$

$$= -\frac{1}{|\chi|^2} \int_{\omega \times Y} \mathbb{C}(y) \left(\operatorname{sym} \nabla \boldsymbol{u}_2 + iX_\chi \boldsymbol{u}_1 \right) : \overline{\Lambda_{\chi, d_3, d_4}^{\text{stretch}}(x)} dx dy, \quad \forall (d_3, d_4)^T \in \mathbb{C}^2,$$

which means that the functional $\tilde{\boldsymbol{f}}_2 := L^2(\omega\times Y;\mathbb{C}^3)\to\mathbb{C}$, defined with:

$$\tilde{\boldsymbol{f}}_2 := -\frac{1}{|\chi|^2}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla(\boldsymbol{u}_2+\boldsymbol{u}_1^{(1)})+\left(iX_\chi\right)^*\mathbb{C}(y)\Lambda^{\text{stretch}}_{\chi,m_3^{(1)},m_4^{(1)}}(x)+\left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_1\right)$$
$$-\boldsymbol{u}_0^{(1)}-\boldsymbol{u}_1,$$

vanishes on stretching rigid motions. This fact allows us to pose the following problem:

$$\frac{1}{|\chi|^2}\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_2^{(1)} = -\frac{1}{|\chi|^2}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla(\boldsymbol{u}_2+\boldsymbol{u}_1^{(1)})+\operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi(\boldsymbol{u}_2+\boldsymbol{u}_1^{(1)})\right)$$
$$-\frac{1}{|\chi|^2}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\Lambda^{\text{stretch}}_{\chi,m_3^{(1)},m_4^{(1)}}(x)+\left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_1\right)-\boldsymbol{u}_0^{(1)}-\boldsymbol{u}_1,\quad \boldsymbol{u}_2^{(1)}\in H^{\text{stretch}}.$$

The problem is well posed and the solution satisfies:

$$\|\boldsymbol{u}_2^{(1)}\|_{H^1(\omega\times Y;\mathbb{C}^3)}\le C|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

4) Final approximation

With these corrections defined, we are done with the approximation procedure. We define the function

$$\tilde{\boldsymbol{u}}_{\text{approx}} := \boldsymbol{u}_0+\boldsymbol{u}_0^{(1)}+\boldsymbol{u}_1+\boldsymbol{u}_1^{(1)}+\boldsymbol{u}_2+\boldsymbol{u}_2^{(1)}.$$

This approximation $\tilde{\boldsymbol{u}}_{\text{approx}}$ now satisfies

$$\frac{1}{|\chi|^2}\left(\operatorname{sym}\nabla^*+\left(iX_\chi\right)^*\right)\mathbb{C}(y)\left(\operatorname{sym}\nabla+iX_\chi\right)\tilde{\boldsymbol{u}}_{\text{approx}}+\tilde{\boldsymbol{u}}_{\text{approx}}-\boldsymbol{f}=\tilde{R}_\chi.$$

The residual $\tilde{R}_\chi$ is now given with the following expression:

$$\tilde{R}_\chi = \frac{1}{|\chi|^2}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_2^{(1)}+\operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2^{(1)}+\left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2^{(1)}+\left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2\right)$$
$$+\frac{1}{|\chi|^2}\left(\left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2+\left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_1^{(1)}\right)+\boldsymbol{u}_1^{(1)}+\boldsymbol{u}_2+\boldsymbol{u}_2^{(1)}.$$

It also satisfies the estimate

$$\|\tilde{R}_\chi\|_{H^{-1}_\#(\omega\times Y;\mathbb{C}^3)}\le C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

Define the error of the approximation

$$\boldsymbol{u}_{\text{error}} := \boldsymbol{u}-\tilde{\boldsymbol{u}}_{\text{approx}}.$$

The function $\boldsymbol{u}_{\text{error}}$ satisfies

$$\frac{1}{|\chi|^2}\left(\operatorname{sym}\nabla^* + \left(iX_\chi\right)^*\right)\mathbb{C}(y)\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u}_{\text{error}} + \boldsymbol{u}_{\text{error}} = -\tilde{R}_\chi.$$

Remark 3.3.6. Notice that by testing the previous equation with the solution $\boldsymbol{u}_{\text{error}}$, we obtain:

$$\begin{aligned}\|\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})} + \|\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^3)} \\ \leq \frac{1}{|\chi|^2}\|\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})} + \|\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^3)} \\ \leq C\|\tilde{R}_\chi\|_{H^{-1}_\#(\omega\times Y;\mathbb{C}^3)}\|\boldsymbol{u}_{\text{error}}\|_{H^1(\omega\times Y;\mathbb{C}^3)}.\end{aligned}$$

On the other hand

$$\|\operatorname{sym}\nabla\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})} \leq C\left(\|\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})} + |\chi|^2\|\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^3)}\right),$$

which yields

$$\|\operatorname{sym}\nabla\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})} - C|\chi|^2\|\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^3)} \leq C\left(\|\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}\right).$$

Therefore

$$\begin{aligned}\|\operatorname{sym}\nabla\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})} + \left(1 - C|\chi|^2\right)\|\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^3)} \\ \leq C\|\tilde{R}_\chi\|_{H^{-1}_\#(\omega\times Y;\mathbb{C}^3)}\|\boldsymbol{u}_{\text{error}}\|_{H^1(\omega\times Y;\mathbb{C}^3)}.\end{aligned}$$

Since for $\boldsymbol{u}_{\text{error}} \in H^1_\#(Y; H^1(\omega;\mathbb{C}^3))$ we have Korn's inequality

$$\|\boldsymbol{u}_{\text{error}}\|^2_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C\left(\|\operatorname{sym}\nabla\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})} + \|\boldsymbol{u}_{\text{error}}\|^2_{L^2(\omega\times Y;\mathbb{C}^3)}\right),$$

it is clear that, for $|\chi| < \eta$, where η is a fixed small constant, we obtain

$$\|\boldsymbol{u}_{\text{error}}\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C\|\tilde{R}_\chi\|_{H^{-1}_\#(\omega\times Y;\mathbb{C}^3)}. \tag{3.18}$$

where the constant $C > 0$ depends on $\eta > 0$.

We can now employ the estimate (3.18) to deduce

$$\|\boldsymbol{u}_{\text{error}}\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

By leaving out higher order terms, we can estimate the error in the approximation by lower order terms:

Proposition 3.3.7. *Let $\boldsymbol{u} \in H^1_\#(Y; H^1(\omega;\mathbb{C}^3))$ be the solution of problem (3.14). Then, the following estimates are valid:*

$$\begin{aligned} \left\| \boldsymbol{u} - \mathcal{I}^{\text{stretch}} \begin{bmatrix} m_3 \\ m_4 \end{bmatrix} \right\|_{H^1(\omega\times Y,\mathbb{C}^3)} &\leq C|\chi| \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, \\ \left\| \boldsymbol{u} - \mathcal{I}^{\text{stretch}} \begin{bmatrix} m_3 + m_3^{(1)} \\ m_4 + m_4^{(1)} \end{bmatrix} - \boldsymbol{u}_1 \right\|_{H^1(\omega\times Y,\mathbb{C}^3)} &\leq C|\chi|^2 \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, \end{aligned} \tag{3.19}$$

where $m_3, m_4, m_3^{(1)}, m_4^{(1)}, \boldsymbol{u}_1$ are defined with the approximation procedure above.

Remark 3.3.8. The first estimate in (3.19) can be rewritten as:

$$\left\| \left(\frac{1}{|\chi|^2} \mathcal{A}_\chi + I \right)^{-1} |_{L^2_{\text{stretch}}} - \left(\mathcal{M}_\chi^{\text{stretch}} \right)^* \left(\frac{1}{|\chi|^2} \mathbb{C}_\chi^{\text{stretch}} + C^{\text{stretch}} \right)^{-1} \mathcal{M}_\chi^{\text{stretch}} \right\|_{L^2\to H^1} \leq C|\chi|. \tag{3.20}$$

The second estimate can be rewritten as:

$$\left\| \left(\frac{1}{|\chi|^2} \mathcal{A}_\chi + I \right)^{-1} |_{L^2_{\text{stretch}}} - \left(\mathcal{M}_\chi^{\text{stretch}} \right)^* \left(\frac{1}{|\chi|^2} \mathbb{C}_\chi^{\text{stretch}} + C^{\text{stretch}} \right)^{-1} \mathcal{M}_\chi^{\text{stretch}} - \mathcal{A}_{\chi,\text{corr}}^{\text{stretch}} - \widetilde{\mathcal{A}}_{\chi,\text{corr}}^{\text{stretch}} \right\|_{L^2\to H^1} \leq C|\chi|^2,$$

where the bounded operators $\mathcal{A}_{\chi,\text{corr}}^{\text{stretch}}$ and $\widetilde{\mathcal{A}}_{\chi,\text{corr}}^{\text{stretch}}$ are defined with the asymptotic procedure above with:

$$\mathcal{A}_{\chi,\text{corr}}^{\text{stretch}} \boldsymbol{f} := \boldsymbol{u}_1, \quad \widetilde{\mathcal{A}}_{\chi,\text{corr}}^{\text{stretch}} \boldsymbol{f} := \boldsymbol{u}_0^{(1)}.$$

3.3.3. The analysis in the bending space

We analyze the following problem: find $\boldsymbol{u} \in H^1_\#(Y; H^1(\omega;\mathbb{C}^3))$ such that

$$\frac{1}{|\chi|^4} \int_{\omega\times Y} \mathbb{C}(\operatorname{sym}\nabla \boldsymbol{u} + iX_\chi \boldsymbol{u}) : \overline{(\operatorname{sym}\nabla \boldsymbol{u} + iX_\chi \boldsymbol{u})} + \int_{\omega\times Y} \boldsymbol{u}\overline{\boldsymbol{v}} = \int_{\omega\times Y} S_{|\chi|} \boldsymbol{f}\overline{\boldsymbol{v}},$$

for $\boldsymbol{v} \in H^1_\#(Y; H^1(\omega;\mathbb{C}^3)) \cap L^2_{\text{bend}}$, where $\boldsymbol{f} \in L^2_{\text{bend}}$. We denote the scaled forces with $S_{|\chi|}\boldsymbol{f} := (\boldsymbol{f}_1, \boldsymbol{f}_2, \frac{1}{|\chi|}\boldsymbol{f}_3)^T$. The problem can be rewritten as before: find $\boldsymbol{u} \in H^1_\#(Y; H^1(\omega;\mathbb{C}^3)) \cap L^2_{\text{bend}}$ such that

$$\frac{1}{|\chi|^4} \left(\operatorname{sym}\nabla^* + \left(iX_\chi \right)^* \right) \mathbb{C}(y) \left(\operatorname{sym}\nabla + iX_\chi \right) \boldsymbol{u} + \boldsymbol{u} = S_{|\chi|}\boldsymbol{f}. \tag{3.21}$$

It is clear from the material properties that the solution $\boldsymbol{u}$ posses the same symmetry properties as the force term $\boldsymbol{f}$. In bending space we want to construct a function which approximates the solution to the given problem with the error in H^1 norm of the order $|\chi|^4$. We are repeating the approximation heuristic. The leading order term in the approximation is given with the expression for the bending rigid motion.

Remark 3.3.9. By testing (3.21) against the solution $\boldsymbol{u}$ and applying Proposition 3.2.10, we arrive at:

$$\begin{aligned}
&\frac{1}{|\chi|^4}\|\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u}\|^2_{L^2(\omega\times Y,\mathbb{C}^3)} + \|\boldsymbol{u}\|^2_{L^2(\omega\times Y,\mathbb{C}^3)} \\
\leq & C\left(\|\boldsymbol{f}_1\|_{L^2(\omega\times Y,\mathbb{C})}\|\boldsymbol{u}_1\|_{L^2(\omega\times Y,\mathbb{C})} + \|\boldsymbol{f}_2\|_{L^2(\omega\times Y,\mathbb{C})}\|\boldsymbol{u}_2\|_{L^2(\omega\times Y,\mathbb{C})} + \frac{1}{|\chi|}\|\boldsymbol{f}_3\|_{L^2(\omega\times Y,\mathbb{C})}\|\boldsymbol{u}_3\|_{L^2(\omega\times Y,\mathbb{C})}\right) \\
\leq & \frac{C}{|\chi|^2}\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)}\|\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})}.
\end{aligned}$$

Therefore:

$$\|\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u}\|_{L^2(\omega\times Y;\mathbb{C}^{3\times 3})} \leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)}.$$

Finally, again by Proposition 3.2.10, we have:

$$\begin{gathered}
\|\boldsymbol{u}_1\|_{H^1(\omega\times Y,\mathbb{C})} \leq C\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)}, \quad \|\boldsymbol{u}_2\|_{H^1(\omega\times Y,\mathbb{C})} \leq C\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)}, \\
\|\boldsymbol{u}_3\|_{H^1(\omega\times Y,\mathbb{C})} \leq C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)}.
\end{gathered} \tag{3.22}$$

1) The leading order term and second order corrector

The leading order term is defined with the following equation:
Find all $(m_1, m_2)^T \in \mathbb{C}^2$ which satisfy:

$$\left(\frac{1}{|\chi|^4}\mathbb{C}^{\text{bend}}_\chi + C^{\text{bend}}_\chi(\omega)\right)\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \mathcal{M}^{\text{bend}}_\chi S_{|\chi|}\boldsymbol{f}.$$

The solution satisfies the estimate

$$|(m_1, m_2)^T| \leq C\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)}$$

By setting $\boldsymbol{u}_0 = \mathcal{I}^{\text{bend}}_\chi \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$, it is clear that we have the estimates:

$$\|(\boldsymbol{u}_0)_\alpha\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, \quad \|(\boldsymbol{u}_0)_3\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

We define the first order corrector as the solution to the following:

$$\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_1 = -\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda^{\mathrm{bend}}_{\chi,m_1,m_2}(x), \quad \boldsymbol{u}_1 \in H_{\mathrm{bend}}.$$

This problem is well posed and the solution satisfies

$$\|\boldsymbol{u}_1\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

It is clear from the definition of $\mathbb{C}^{\mathrm{bend}}_\chi$ that the following holds:

$$\frac{1}{|\chi|^4}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_1 + \Lambda^{\mathrm{bend}}_{\chi,m_1,m_2}(x)\right) : \overline{\Lambda^{\mathrm{bend}}_{\chi,d_1,d_2}(x)}dxdy$$
$$+\int_{\omega\times Y}\begin{bmatrix} m_1 \\ m_2 \\ -i\chi(m_1x_1+m_2x_2)\end{bmatrix}\cdot\overline{\begin{bmatrix} d_1 \\ d_2 \\ -i\chi(d_1x_1+d_2x_2)\end{bmatrix}}dxdy = \int_{\omega\times Y}\begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \frac{1}{|\chi|}\boldsymbol{f}_3\end{bmatrix}\cdot\overline{\begin{bmatrix} d_1 \\ d_2 \\ -i\chi(d_1x_1+d_2x_2)\end{bmatrix}}dxdy,$$

for every $(d_1,d_2)^T \in \mathbb{C}^2$. Here, we note that

$$iX_\chi(C_1,C_2,0)^T = \begin{bmatrix} 0 & 0 & \frac{1}{2}i\chi C_1 \\ 0 & 0 & \frac{1}{2}i\chi C_2 \\ \frac{1}{2}i\chi C_1 & \frac{1}{2}i\chi C_1 & 0\end{bmatrix} = -\operatorname{sym}\nabla(0,0,-i\chi(C_1x_1+C_2x_2))^T.$$

From this we conclude that:

$$\Lambda^{\mathrm{bend}}_{\chi,C_1,C_2}(x) = iX_\chi(0,0,-i\chi(C_1x_1+C_2x_2))^T.$$

2) Higher order correctors

In view of the equation (3.21) we define the third order corrector term $\boldsymbol{u}_2$ in the following way: Define the functional

$$\tilde{\boldsymbol{f}}_2 = -\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_1 + \operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_1 + \left(iX_\chi\right)^*\mathbb{C}(y)\Lambda^{\mathrm{bend}}_{\chi,m_1,m_2}(x)\right)$$
$$-(0,0,-i\chi(m_1x_1+m_2x_2))^T + (0,0,\frac{1}{|\chi|}\boldsymbol{f}_3)^T, \quad \tilde{\boldsymbol{f}}_2 := L^2(\Omega)\to\mathbb{C}.$$

This functional vanishes when tested against constant functions of type $(C_1, C_2, 0)^T$. To see this, we do the following calculation:

$$\begin{aligned}\Big(\big(iX_\chi\big)^* \mathbb{C}(y)\,\mathrm{sym}\,\nabla \boldsymbol{u}_1 + \big(iX_\chi\big)^* \mathbb{C}(y)\Lambda^{\mathrm{bend}}_{\chi,m_1,m_2}(x)\Big)\,(C_1, C_2, 0)^T \\ = \int_{\omega\times Y} \mathbb{C}(y)\left(\mathrm{sym}\,\nabla \boldsymbol{u}_1 + \Lambda^{\mathrm{bend}}_{\chi,m_1,m_2}(x)\right) : \overline{iX_\chi (C_1, C_2, 0)^T}\,dxdy \\ = -\int_{\omega\times Y} \mathbb{C}(y)\left(\mathrm{sym}\,\nabla \boldsymbol{u}_1 + \Lambda^{\mathrm{bend}}_{\chi,m_1,m_2}(x)\right) : \overline{\mathrm{sym}\,\nabla(0, 0, -i\chi(C_1x_1 + C_2x_2))^T}\,dxdy \\ = -\left(\mathrm{sym}\,\nabla^*\mathbb{C}(y)\,\mathrm{sym}\,\nabla \boldsymbol{u}_1 + \mathrm{sym}\,\nabla^*\mathbb{C}(y)\Lambda^{\mathrm{bend}}_{\chi,m_1,m_2}(x)\right)(0, 0, -i\chi(C_1x_1 + C_2x_2))^T = 0.\end{aligned}$$

The last equality follows from the definition of the corrector term $\boldsymbol{u}_1$ and the fact that $(0, 0, i\chi(C_1x_1 + C_2x_2))^T \in H^{\mathrm{bend}}$. Next we define the corrector $\boldsymbol{u}_2$ as the solution to the following well posed problem:

$$\frac{1}{|\chi|^4}\,\mathrm{sym}\,\nabla^*\mathbb{C}(y)\,\mathrm{sym}\,\nabla \boldsymbol{u}_2 = \tilde{\boldsymbol{f}}_2, \quad \boldsymbol{u}_2 \in H^{\mathrm{bend}}.$$

The corrector term $\boldsymbol{u}_2$ satisfies the following estimate:

$$\|\boldsymbol{u}_2\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C|\chi|^3 \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

We are able to define even higher order corrector term in order to cancel out more of the residual parts and to decrease the error. The corrector term $\boldsymbol{u}_3$ is defined with the following equation:

$$\frac{1}{|\chi|^4}\,\mathrm{sym}\,\nabla^*\mathbb{C}(y)\,\mathrm{sym}\,\nabla \boldsymbol{u}_3 = \tilde{\boldsymbol{f}}_3, \quad \boldsymbol{u}_3 \in H^{\mathrm{bend}},$$

where the right hand side $\tilde{\boldsymbol{f}}_3$ is defined with:

$$\begin{aligned}\tilde{\boldsymbol{f}}_3 = -\frac{1}{|\chi|^4}\Big(\big(iX_\chi\big)^* \mathbb{C}(y)\,\mathrm{sym}\,\nabla \boldsymbol{u}_2 + \mathrm{sym}\,\nabla^*\mathbb{C}(y) iX_\chi \boldsymbol{u}_2 + \big(iX_\chi\big)^* \mathbb{C}(y) iX_\chi \boldsymbol{u}_1\Big) \\ -(m_1, m_2, 0)^T + (\boldsymbol{f}_1, \boldsymbol{f}_2, 0)^T.\end{aligned}$$

In order to show

$$\left\langle \tilde{\boldsymbol{f}}_3, (C_1, C_2, 0)^T \right\rangle = 0,$$

we do the following calculation:

$$
\begin{aligned}
&\frac{1}{|\chi|^4}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_2+iX_\chi\boldsymbol{u}_1\right):iX_\chi(C_1,C_2,0)^T\\
=&-\frac{1}{|\chi|^4}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_2+iX_\chi\boldsymbol{u}_1\right):\operatorname{sym}\nabla(0,0,-i\chi(C_1x_1+C_2x_2))^T\\
=&\frac{1}{|\chi|^4}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_1+\Lambda^{\text{bend}}_{\chi,m_1,m_2}(x)\right):iX_\chi(0,0,-i\chi(C_1x_1+C_2x_2))^T\\
&+\int_{\omega\times Y}(0,0,-i\chi(m_1x_1+m_2x_2))^T\cdot\overline{(0,0,-i\chi(C_1x_1+C_2x_2))^T}\\
&-\int_{\omega\times Y}(0,0,\frac{1}{|\chi|}\boldsymbol{f}_3)^T\cdot\overline{(0,0,-i\chi(C_1x_1+C_2x_2))^T}\\
=&\frac{1}{|\chi|^4}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_1+\Lambda^{\text{bend}}_{\chi,m_1,m_2}(x)\right):\overline{\Lambda^{\text{bend}}_{\chi,C_1,C_2}(x)}dxdy\\
&+\int_{\omega\times Y}(0,0,-i\chi(m_1x_1+m_2x_2))^T\cdot\overline{(0,0,-i\chi(C_1x_1+C_2x_2))^T}\\
&-\int_{\omega\times Y}(0,0,\frac{1}{|\chi|}\boldsymbol{f}_3)^T\cdot\overline{(0,0,-i\chi(C_1x_1+C_2x_2))^T}\\
=&-\int_{\omega\times Y}(m_1,m_2,0)^T\cdot\overline{(C_1,C_2,0)^T}+\int_{\omega\times Y}(\boldsymbol{f}_1,\boldsymbol{f}_2,0)^T\cdot\overline{(C_1,C_2,0)^T}.
\end{aligned}
$$

This proves that the problem is well posed and the solution satisfies the following estimate:

$$\|\boldsymbol{u}_3\|_{H^1(\omega\times Y;\mathbb{C}^3)}\le C|\chi|^4\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

The total approximation built up so far $\boldsymbol{u}_{\text{approx}}:=\boldsymbol{u}_0+\boldsymbol{u}_1+\boldsymbol{u}_2+\boldsymbol{u}_3$ satisfies the the following equation:

$$\frac{1}{|\chi|^4}\left(\operatorname{sym}\nabla^*+\left(iX_\chi\right)^*\right)\mathbb{C}(y)\left(\operatorname{sym}\nabla+iX_\chi\right)\boldsymbol{u}_{\text{approx}}+\boldsymbol{u}_{\text{approx}}-S_{|\chi|}\boldsymbol{f}=R_\chi,$$

where the residual is given with the following expression:

$$
\begin{aligned}
R_\chi=&\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2+\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_3+\operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_3+\left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_3\right)\\
&+\boldsymbol{u}_1+\boldsymbol{u}_2+\boldsymbol{u}_3.
\end{aligned}
$$

It satisfies the estimate:

$$\|R_\chi\|_{H^{-1}_\#(\omega\times Y;\mathbb{C}^3)}\le C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)}.$$

Here, we must restart the procedure in order to further develop the approximation.

3) The first restart of the approximation procedure

In order to eliminate low order terms in the residual, we gradually define better approximation by updating the leading order term. We do this by defining $\boldsymbol{u}_0^{(1)} = \mathcal{I}_\chi^{\text{bend}} \begin{bmatrix} m_1^{(1)} \\ m_2^{(1)} \end{bmatrix}$, where $\begin{bmatrix} m_1^{(1)} \\ m_2^{(1)} \end{bmatrix}$ is the solution of the following problem:

$$\left(\frac{1}{|\chi|^4}\mathbb{C}_\chi^{\text{bend}} + C_\chi^{\text{bend}}(\omega)\right) \begin{bmatrix} m_1^{(1)} \\ m_2^{(1)} \end{bmatrix} \cdot \overline{\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}} = -\frac{1}{|\chi|^4}\int_{\omega\times Y} \mathbb{C}(y)\left(\operatorname{sym}\nabla \boldsymbol{u}_3 + iX_\chi \boldsymbol{u}_2\right) : iX_\chi(d_1,d_2,0)^T, \tag{3.23}$$

$\forall (d_1,d_2)^T \in \mathbb{C}^2$. We have

$$|m_1^{(1)}, m_2^{(1)}| \leq C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)},$$
$$\|(\boldsymbol{u}_0^{(1)})_\alpha\|_{H^1(\omega\times Y;\mathbb{C})} \leq C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, \quad \alpha = 0,1,$$
$$\|(\boldsymbol{u}_0^{(1)})_3\|_{H^1(\omega\times Y;\mathbb{C})} \leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

We define the next corrector $\boldsymbol{u}_1^{(1)}$ with the relation:

$$\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla \boldsymbol{u}_1^{(1)} = -\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda^{\text{bend}}_{\chi,m_1^{(1)},m_2^{(1)}}(x), \quad \boldsymbol{u}_1^{(1)} \in H^{\text{bend}}.$$

This yields the estimate

$$\|\boldsymbol{u}_1^{(1)}\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

These correctors satisfy the following:

$$\begin{aligned}
&\frac{1}{|\chi|^4}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla \boldsymbol{u}_1^{(1)} + \Lambda^{\text{bend}}_{\chi,m_1^{(1)},m_2^{(1)}}(x)\right) : \overline{\Lambda^{\text{bend}}_{\chi,d_1,d_2}(x)}\,dxdy \\
&\quad + \int_{\omega\times Y} \begin{bmatrix} m_1^{(1)} \\ m_2^{(1)} \\ -i\chi(m_1^{(1)}x_1 + m_2^{(1)}x_2) \end{bmatrix} \cdot \overline{\begin{bmatrix} d_1 \\ d_2 \\ -i\chi(d_1x_1 + d_2x_2) \end{bmatrix}}\,dxdy \\
&= -\frac{1}{|\chi|^4}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla \boldsymbol{u}_3 + iX_\chi \boldsymbol{u}_2\right) : \overline{iX_\chi(d_1,d_2,0)^T}\,dxdy, \quad \forall (d_1,d_2)^T \in \mathbb{C}^2.
\end{aligned}$$

The continuation of the approximation procedure is similar as before. We progressively define two more corrections $\boldsymbol{u}_2^{(1)}, \boldsymbol{u}_3^{(1)} \in H^1_\#(Y; H^1(\omega;\mathbb{C}^3))$ as solutions of a well posed problems:

$$\frac{1}{|\chi|^4}\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_2^{(1)}$$
$$= -\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_1^{(1)} + \operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_1^{(1)} + \left(iX_\chi\right)^*\mathbb{C}(y)\Lambda^{\mathrm{bend}}_{\chi,m_1^{(1)},m_2^{(1)}}(x)\right)$$
$$-(0,0,-i\chi(m_1^{(1)}x_1+m_2^{(1)}x_2))^T,\quad \boldsymbol{u}_2^{(1)}\in H^{\mathrm{bend}}.$$

$$\frac{1}{|\chi|^4}\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_3^{(1)} = -\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla(\boldsymbol{u}_2^{(1)}+\boldsymbol{u}_3) + \operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi(\boldsymbol{u}_2^{(1)}+\boldsymbol{u}_3)\right)$$
$$-\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi(\boldsymbol{u}_1^{(1)}+\boldsymbol{u}_2)\right) - (m_1^{(1)},m_2^{(1)},0)^T,\quad \boldsymbol{u}_3^{(1)}\in H^{\mathrm{bend}}.$$

The solutions satisfy the following estimates:

$$\|\boldsymbol{u}_2^{(1)}\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C|\chi|^4\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$
$$\|\boldsymbol{u}_3^{(1)}\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C|\chi|^5\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

For our purposes it is necessary to further decrease the error for which we again need to restart the procedure:

4) The second restart of the approximation procedure

Here, we provide the definitions for the correctors which eliminate the remaining low order terms in the residual, thus achieving the desired error in the approximation.

The correctors $\boldsymbol{u}_0^{(2)} = \mathcal{I}_\chi^{\mathrm{bend}}\begin{bmatrix} m_1^{(2)} \\ m_2^{(2)}\end{bmatrix}, \boldsymbol{u}_1^{(2)}, \boldsymbol{u}_2^{(2)}, \boldsymbol{u}_3^{(2)} \in H^{\mathrm{bend}}$ are gradually built with the following relations:

$$\left(\frac{1}{|\chi|^4}\mathbb{C}_\chi^{\mathrm{bend}} + C_\chi^{\mathrm{bend}}(\omega)\right)\begin{bmatrix} m_1^{(2)} \\ m_2^{(2)}\end{bmatrix}\cdot\overline{\begin{bmatrix} d_1 \\ d_2\end{bmatrix}} = -\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_3^{(1)} + iX_\chi\boldsymbol{u}_2^{(1)} + iX_\chi\boldsymbol{u}_3\right) : iX_\chi(d_1,d_2,0)^T,$$

$\forall (d_1,d_2)^T\in\mathbb{C}^2.$

$$\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_1^{(2)} = -\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda^{\mathrm{bend}}_{\chi,m_1^{(2)},m_2^{(2)}}(x),\quad \boldsymbol{u}_1^{(2)}\in H^{\mathrm{bend}}.$$

$$\frac{1}{|\chi|^4}\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_2^{(2)}$$
$$= -\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_1^{(2)} + \operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_1^{(2)} + \left(iX_\chi\right)^*\mathbb{C}(y)\Lambda^{\mathrm{bend}}_{\chi,m_1^{(2)},m_2^{(2)}}(x)\right)$$
$$-(0,0,-i\chi(m_1^{(2)}x_1+m_2^{(2)}x_2))^T,\quad \boldsymbol{u}_2^{(2)}\in H^{\mathrm{bend}}.$$

$$
\begin{aligned}
&\frac{1}{|\chi|^4}\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_3^{(2)}\\
&=-\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla(\boldsymbol{u}_2^{(2)}+\boldsymbol{u}_3^{(1)})+\operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi(\boldsymbol{u}_2^{(2)}+\boldsymbol{u}_3^{(1)})\right)\\
&-\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi(\boldsymbol{u}_1^{(2)}+\boldsymbol{u}_2^{(1)}+\boldsymbol{u}_3)\right)-(m_1^{(2)},m_2^{(2)},0)^T-\boldsymbol{u}_1,\quad \boldsymbol{u}_3^{(2)}\in H^{\text{bend}}.
\end{aligned}
$$

All of these problems are well posed which can be seen by reviewing the relations throughout the process, thus concluding that the right hand sides vanish when tested against functions in H^{bend}. These approximations satisfy the following estimates:

$$
\|(\boldsymbol{u}_0^{(2)})_\alpha\|_{H^1(\omega\times Y;\mathbb{C}^3)}\le C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},\quad \|(\boldsymbol{u}_0^{(2)})_3\|_{H^1(\omega\times Y;\mathbb{C}^3)}\le C|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}
$$

$$
\|\boldsymbol{u}_1^{(2)}\|_{H^1(\omega\times Y;\mathbb{C}^3)}\le C|\chi|^4\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.
$$

$$
\|\boldsymbol{u}_2^{(2)}\|_{H^1(\omega\times Y;\mathbb{C}^3)}\le C|\chi|^5\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.
$$

$$
\|\boldsymbol{u}_3^{(2)}\|_{H^1(\omega\times Y;\mathbb{C}^3)}\le C|\chi|^6\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.
$$

5) Final approximation

We have finished the approximation procedure and define the function

$$
\tilde{\boldsymbol{u}}_{\text{approx}}:=\boldsymbol{u}_0+\boldsymbol{u}_0^{(1)}+\boldsymbol{u}_0^{(2)}+\boldsymbol{u}_1+\boldsymbol{u}_1^{(1)}+\boldsymbol{u}_1^{(2)}+\boldsymbol{u}_2+\boldsymbol{u}_2^{(1)}+\boldsymbol{u}_2^{(2)}.
$$

This approximation $\tilde{\boldsymbol{u}}_{\text{approx}}$ satisfies

$$
\frac{1}{|\chi|^4}\left(\operatorname{sym}\nabla^*+\left(iX_\chi\right)^*\right)\mathbb{C}(y)\left(\operatorname{sym}\nabla+iX_\chi\right)\tilde{\boldsymbol{u}}_{\text{approx}}+\tilde{\boldsymbol{u}}_{\text{approx}}-S_{|\chi|}\boldsymbol{f}=\tilde{R}_\chi,
$$

where the residual $\tilde{R}_\chi$ is given with the following expression:

$$
\begin{aligned}
\tilde{R}_\chi=\frac{1}{|\chi|^4}&\left(\left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi(\boldsymbol{u}_3^{(1)}+\boldsymbol{u}_2^{(2)}+\boldsymbol{u}_3^{(2)})+\operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_3^{(2)}+\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_3^{(2)}\right)\\
&+\boldsymbol{u}_2+\boldsymbol{u}_3+\boldsymbol{u}_1^{(1)}+\boldsymbol{u}_2^{(1)}+\boldsymbol{u}_3^{(1)}+\boldsymbol{u}_1^{(2)}+\boldsymbol{u}_2^{(2)}+\boldsymbol{u}_3^{(2)},
\end{aligned}
$$

and can be estimated with:

$$
\|\tilde{R}_\chi\|_{H_\#^{-1}(\omega\times Y;\mathbb{C}^3)}\le C|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)}.
$$

The error of the approximation

$$
\boldsymbol{u}_{\text{error}}:=\boldsymbol{u}-\tilde{\boldsymbol{u}}_{\text{approx}}.
$$

satisfies

$$\frac{1}{|\chi|^4}\left(\operatorname{sym}\nabla^* + \left(iX_\chi\right)^*\right)\mathbb{C}(y)\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u}_{\text{error}} + \boldsymbol{u}_{\text{error}} = -\tilde{R}_\chi.$$

Here, like in (3.18), one can easily derive estimate on the approximation error:

$$\|\boldsymbol{u}_{\text{error}}\|_{H^1(\omega\times Y,\mathbb{C})} \leq C|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)}.$$

The following proposition concludes our work as it gives final estimates on the approximation.

Proposition 3.3.10. *Let $\boldsymbol{u} \in H^1_\#(Y;H^1(\omega;\mathbb{C}^3))$ be the solution of problem (3.21). Then, the following estimates are valid:*

$$\left\|P_i\left(\boldsymbol{u} - \mathcal{I}^{\text{bend}}_\chi\begin{bmatrix} m_1 \\ m_2\end{bmatrix}\right)\right\|_{H^1(\omega\times Y,\mathbb{C}^2)} \leq \begin{cases} C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, & i=1,2; \\ C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, & i=3.\end{cases}$$

$$\left\|P_i\left(\boldsymbol{u} - \mathcal{I}^{\text{bend}}_\chi\begin{bmatrix} m_1 \\ m_2\end{bmatrix} - \mathcal{I}^{\text{bend}}_\chi\begin{bmatrix} m_1^{(1)} \\ m_2^{(1)}\end{bmatrix} - \boldsymbol{u}_1\right)\right\|_{H^1(\omega\times Y,\mathbb{C}^2)} \leq \begin{cases} C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, & i=1,2; \\ C|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, & i=3.\end{cases} \tag{3.24}$$

where $m_1, m_2, m_1^{(1)}, m_2^{(1)}, \boldsymbol{u}_1$ are defined with the approximation procedure above.

Remark 3.3.11. Note that, due to the scaling of the operator, the previous estimates are valid also if we replace $m_1, m_2, m_1^{(1)}, m_2^{(1)}$ with solutions $\tilde{m}_1, \tilde{m}_2, \tilde{m}_1^{(1)}, \tilde{m}_2^{(1)}$ to the analogous homogenized problem with $C^{\text{bend}}_\chi(\omega) = I$.

Plugging $(m_1 - \tilde{m}_1, m_2 - \tilde{m}_2)^T$ as a test function we get:

$$\frac{1}{|\chi|^4}\mathbb{C}^{\text{bend}}_\chi\begin{bmatrix} m_1-\tilde{m}_1 \\ m_2-\tilde{m}_2\end{bmatrix}\cdot\begin{bmatrix} m_1-\tilde{m}_1 \\ m_2-\tilde{m}_2\end{bmatrix} + |\chi|^2\begin{bmatrix} c_1(\omega) & 0 \\ 0 & c_2(\omega)\end{bmatrix}\begin{bmatrix} m_1 \\ m_2\end{bmatrix}\cdot\begin{bmatrix} m_1-\tilde{m}_1 \\ m_2-\tilde{m}_2\end{bmatrix} = 0.$$

Therefore:

$$\left\|\begin{bmatrix} m_1-\tilde{m}_1 \\ m_2-\tilde{m}_2\end{bmatrix}\right\| \leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}$$

Remark 3.3.12. The first estimate in (3.24) can be rewritten as:

$$\left\|P_i\left(\left(\frac{1}{|\chi|^4}\mathcal{A}_\chi + I\right)^{-1}|_{L^2_{\text{bend}}} - \left(\mathcal{M}^{\text{bend}}_\chi\right)^*\left(\frac{1}{|\chi|^4}\mathbb{C}^{\text{bend}}_\chi + I\right)^{-1}\mathcal{M}^{\text{bend}}_\chi\right)S_{|\chi|}\right\|_{L^2\to H^1} \leq \begin{cases} C|\chi|, & i=1,2; \\ C|\chi|^2, & i=3.\end{cases} \tag{3.25}$$

The second estimate can be rewritten as:

$$\left\| P_i \left(\left(\frac{1}{|\chi|^4} \mathcal{A}_\chi + I \right)^{-1} |_{L^2_{\rm bend}} - \left(\mathcal{M}_\chi^{\rm bend} \right)^* \left(\frac{1}{|\chi|^4} \mathbb{C}_\chi^{\rm bend} + I \right)^{-1} \mathcal{M}_\chi^{\rm bend} - \mathcal{A}_{\chi,{\rm corr}}^{\rm bend} - \widetilde{\mathcal{A}}_{\chi,{\rm corr}}^{\rm bend} \right) S_{|\chi|} \right\|_{L^2 \to H^1}$$
$$\leq \begin{cases} C|\chi|^2, & i = 1,2; \\ C|\chi|^3, & i = 3, \end{cases}$$

where the bounded operators $\mathcal{A}_{\chi,{\rm corr}}^{\rm bend}$ and $\widetilde{\mathcal{A}}_{\chi,{\rm corr}}^{\rm bend}$ are defined with the asymptotic procedure above with:

$$\mathcal{A}_{\chi,{\rm corr}}^{\rm bend} \boldsymbol{f} := \boldsymbol{u}_1, \quad \widetilde{\mathcal{A}}_{\chi,{\rm corr}}^{\rm bend} \boldsymbol{f} := \boldsymbol{u}_0^{(1)}.$$

Remark 3.3.13. We note that the absence of the scaling term $S_{|\chi|}$ is the same as the absence of the out-of-line force term, when we consider the H^1 estimate for the distance between the unhomogenised and homogenised resolvents. Consider the following two problems:

$$\left(\frac{1}{|\chi|^4} \mathbb{C}_\chi^{\rm bend} + I \right) \begin{bmatrix} \hat{m}_1 \\ \hat{m}_2 \end{bmatrix} = \mathcal{M}_\chi^{\rm bend} \boldsymbol{f}, \quad \left(\frac{1}{|\chi|^4} \mathbb{C}_\chi^{\rm bend} + I \right) \begin{bmatrix} \tilde{m}_1 \\ \tilde{m}_2 \end{bmatrix} = \mathcal{M}_\chi^{\rm bend} S_\infty \boldsymbol{f} = \int_{\omega \times Y} \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix},$$

where

$$S_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The difference $(\hat{m}_1 - \tilde{m}_1, \hat{m}_2 - \tilde{m}_2)^T$ satisfies the following:

$$\left(\frac{1}{|\chi|^4} \mathbb{C}_\chi^{\rm bend} + I \right) \begin{bmatrix} \hat{m}_1 - \tilde{m}_1 \\ \hat{m}_2 - \tilde{m}_2 \end{bmatrix} = \int_{\omega \times Y} \begin{bmatrix} i\chi \boldsymbol{f}_3 x_1 \\ i\chi \boldsymbol{f}_3 x_2 \end{bmatrix},$$

so

$$|(\hat{m}_1 - \tilde{m}_1, \hat{m}_2 - \tilde{m}_2)^T| \leq C|\chi| \, \|\boldsymbol{f}\|_{L^2(\omega \times Y; \mathbb{C}^3)}.$$

In other words, we have:

$$\left\| P_i \left(\mathcal{I}_\chi^{\rm bend} \begin{bmatrix} \hat{m}_1 \\ \hat{m}_2 \end{bmatrix} - \mathcal{I}_\chi^{\rm bend} \begin{bmatrix} \tilde{m}_1 \\ \tilde{m}_2 \end{bmatrix} \right) \right\|_{H^1(\omega \times Y; \mathbb{C}^3)} \leq \begin{cases} C|\chi| \|\boldsymbol{f}\|_{L^2(\omega \times Y; \mathbb{C}^3)}, & i = 1,2; \\ C|\chi|^2 \|\boldsymbol{f}\|_{L^2(\omega \times Y; \mathbb{C}^3)}, & i = 3. \end{cases}$$

Therefore

$$\left\| P_i \left(\mathcal{M}_\chi^{\text{bend}} \right)^* \left(\frac{1}{|\chi|^4} \mathbb{C}_\chi^{\text{bend}} + I \right)^{-1} \mathcal{M}_\chi^{\text{bend}} (I - S_\infty) \right\|_{L^2 \to H^1} \leq \begin{cases} C|\chi|, & i = 1,2; \\ C|\chi|^2, & i = 3. \end{cases}$$

Similarly, due to (3.22) we have:

$$\left\| P_i \left(\frac{1}{|\chi|^4} \mathcal{A}_\chi + I \right)^{-1} (I - S_\infty) \right\|_{L^2 \to H^1} \leq \begin{cases} C|\chi|, & i = 1,2; \\ C|\chi|^2, & i = 3. \end{cases} \tag{3.26}$$

Remark 3.3.14. We emphasise here that the asymptotic procedure, performed in this section, eventually stabilises and can be extended to an approximation of arbitrary order in $|\chi|$, thus generating the series of approximations. One can write down explicit recurrence relations which define these correctors up to any order in $|\chi|$. This leads to approximation in arbitrary precision, however with physically unambiguous terms.

3.4. Norm-resolvent estimates under material symmetries

In this section, we provide the norm-resolvent estimates for the operators on the original domain $\omega\times\mathbb{R}$. This is being done by chosing optimal estimates with respect to ε for each small χ, and then applying the Gelfand pullback in order to rephrase the estimates in the original physical setting.

3.4.1. $L^2 \to L^2$ norm-resolvent estimates

We start with the stretching case.

Theorem 3.4.1. *Suppose that the assumptions on the material symmetries 3.1.1 hold and that the forces $\boldsymbol{f}$ belong to L^2_{stretch}. Let $\gamma > -2$ be the parameter of spectral scaling. Then we have the following estimate:*

$$\begin{aligned}
&\left\| P_1 \left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon + I\right)^{-1} \boldsymbol{f} - x_2\pi_1 \left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}} + C^{\text{stretch}}\right)^{-1} \mathcal{M}^{\text{stretch}}\Xi_\varepsilon \boldsymbol{f}\right\|_{L^2(\omega\times\mathbb{R})} \leq \varepsilon^{\frac{\gamma+2}{2}} \|\boldsymbol{f}\|_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)},\\
&\left\| P_2 \left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon + I\right)^{-1} \boldsymbol{f} + x_1\pi_1 \left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}} + C^{\text{stretch}}\right)^{-1} \mathcal{M}^{\text{stretch}}\Xi_\varepsilon \boldsymbol{f}\right\|_{L^2(\omega\times\mathbb{R})} \leq \varepsilon^{\frac{\gamma+2}{2}} \|\boldsymbol{f}\|_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)},\\
&\left\| P_3 \left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon + I\right)^{-1} \boldsymbol{f} - \pi_2 \left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}} + C^{\text{stretch}}\right)^{-1} \mathcal{M}^{\text{stretch}}\Xi_\varepsilon \boldsymbol{f}\right\|_{L^2(\omega\times\mathbb{R})} \leq \varepsilon^{\frac{\gamma+2}{2}} \|\boldsymbol{f}\|_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)}
\end{aligned} \tag{3.27}$$

Proof. We recall the estimates (3.19) for the weak solutions of the resolvent equations:

$$\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi + I\right)\boldsymbol{u} = \boldsymbol{f};$$

$$\left(\frac{1}{|\chi|^2}\mathbb{C}_\chi^{\text{stretch}} + C^{\text{stretch}}\right)(m_1, m_2)^T = \mathcal{M}_\chi^{\text{stretch}}\boldsymbol{f}.$$

Now, from (3.20) we see that we have the following norm-resolvent estimates:

$$\begin{aligned}
&\left\| P_1 \left(\frac{1}{|\chi|^2}\mathcal{A}_\chi + I\right)^{-1} \boldsymbol{f} - x_2\pi_1 \left(\frac{1}{|\chi|^2}\mathbb{C}_\chi^{\text{stretch}} + C^{\text{stretch}}\right)^{-1} \mathcal{M}_\chi^{\text{stretch}} \boldsymbol{f}\right\|_{L^2(\omega\times Y)} \leq |\chi| \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},\\
&\left\| P_2 \left(\frac{1}{|\chi|^2}\mathcal{A}_\chi + I\right)^{-1} \boldsymbol{f} + x_1\pi_1 \left(\frac{1}{|\chi|^2}\mathbb{C}_\chi^{\text{stretch}} + C^{\text{stretch}}\right)^{-1} \mathcal{M}_\chi^{\text{stretch}} \boldsymbol{f}\right\|_{L^2(\omega\times Y)} \leq |\chi| \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},
\end{aligned}$$

$$\left\| P_3\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi + I\right)^{-1} \boldsymbol{f} - \pi_2\left(\frac{1}{|\chi|^2}\mathbb{C}^{\text{stretch}}_\chi + C^{\text{stretch}}\right)^{-1} \mathcal{M}^{\text{stretch}}_\chi \boldsymbol{f}\right\|_{L^2(\omega\times Y)} \leq |\chi| \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

Recall that the operator $\mathcal{A}_\chi$ is selfadjoint, positive, with compact resolvent, so its spectrum consists of real nonnegative eigenvalues, all of which are of order $O(1)$ except for the two smallest, $\lambda_1^\chi, \lambda_2^\chi$, which are of order $O(|\chi|^2)$. (We know the precise fixed interval in which $\lambda_1^\chi/|\chi|^2$ and $\lambda_2^\chi/|\chi|^2$ can be found, uniformly on $|\chi|$).

For every fixed $\varepsilon > 0$, $\chi \neq 0$ we define the function

$$g_{\varepsilon,\chi}(z) := \left(\frac{|\chi|^2}{\varepsilon^{\gamma+2}}z + 1\right)^{-1}, \ \Re(z) > 0 \tag{3.28}$$

for which we have the following: For every fixed $\eta > 0$, function $g_{\varepsilon,\chi}$ is bounded on the halfplane: $\left\{z \in \mathbb{C}, \Re(z) \geq \eta\right\}$,

$$|g_{\varepsilon,\chi}(z)| \leq C\left(\max\left\{\frac{|\chi|^2}{\varepsilon^{\gamma+2}}, 1\right\}\right)^{-1}.$$

This is due to the following calculation

$$|g_{\varepsilon,\chi}(z)|^{-1} = |\frac{|\chi|^2}{\varepsilon^{\gamma+2}}z + 1| \geq \frac{|\chi|^2}{\varepsilon^{\gamma+2}}\eta + 1 \geq \begin{cases} \frac{|\chi|^2}{\varepsilon^{\gamma+2}} & \text{if } |\chi|^2 \gg \varepsilon^{\gamma+2}; \\ 1 & \text{if } \varepsilon^{\gamma+2} \approx |\chi|^2. \end{cases}$$

Due to the bounds on the both eigenvalues of $\mathcal{A}_\chi$ of order $|\chi|^2$, we deduce that two smallest eigenvalues of the operator $\frac{1}{|\chi|^2}\mathcal{A}_\chi$ are uniformly positioned in the fixed interval in the right halfplane, with the interval not depending on χ nor ε. The same is true for the two eigenvalues $\overline{\lambda}_1^\chi$, $\overline{\lambda}_2^\chi$ of the matrix $\mathbb{C}^{\text{stretch}}_\chi$. The uniform bounds on these eigenvalues allow us to deduce that there exists a closed contour $\Gamma \subset \left\{z \in \mathbb{C}, \Re(z) > 0\right\}$ and a constant $\mu > 0$, such that for every $\chi \in [-\mu,\mu] \setminus \{0\}$ one has the following properties:

- Γ encloses the two smallest eigenvalues of both the operators $\frac{1}{|\chi|^2}\mathcal{A}_\chi$ and $\frac{1}{|\chi|^2}\mathbb{C}^{\text{stretch}}_\chi$.

- Γ does not enclose any other eigenvalue (of higher order).

- $\exists \rho_0 > 0$, $\inf_{z\in\Gamma}|z - \lambda_i^\chi| \geq 0$, $\inf_{z\in\Gamma}|z - \overline{\lambda}_i^\chi| \geq \rho_0$, $i = 1,2$.

Notice that $g_{\chi,\varepsilon}$ is analytic on the right half plane, thus for $a \in \mathbb{C}$ inside Γ the Cauchy integral formula gives us:

$$g_{\chi,\varepsilon}(a) := \frac{1}{2\pi i} \oint_\Gamma \frac{g_{\chi,\varepsilon}(z)}{z-a} dz.$$

The similar is valid for $g_{\chi,\varepsilon}$ taken as the function of operator:

$$g_{\chi,\varepsilon}(\mathcal{A}) P_\Gamma := \frac{1}{2\pi i} \oint_\Gamma g_{\chi,\varepsilon}(z)(zI - \mathcal{A})^{-1} dz,$$

where P_Γ is the projection operator onto the eigenspace spanned with eigenfunctions corresponding with those eigenvalues enclosed with Γ. Note that

$$\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I = \frac{|\chi|^2}{\varepsilon^{\gamma+2}} \left(\frac{1}{|\chi|^2} \mathcal{A}_\chi \right) + I.$$

Thus we can write

$$P_\Gamma \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} P_\Gamma \boldsymbol{f} = \frac{1}{2\pi i} \oint_\Gamma g_{\chi,\varepsilon}(z) \left(zI - \frac{1}{|\chi|^2} \mathcal{A}_\chi \right)^{-1} \boldsymbol{f} dz.$$

Due to the uniform estimates on the spectrum of $\mathcal{A}_\chi$, we have:

$$\left\| \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} - P_\Gamma \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} P_\Gamma \right\|_{L^2 \to H^1} = \left\| (I - P_\Gamma) \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} (I - P_\Gamma) \right\|_{L^2 \to H^1} \leq C\varepsilon^{\gamma+2},$$

so we will omit the projection P_Γ since the resolvent estimate is valid for the whole operator. Notice that the integral formula can also be applied to the resolvent $\left(\frac{1}{|\chi|^2} \mathbb{C}_\chi^{\text{stretch}} + C^{\text{stretch}} \right)^{-1}$ despite its not standard structure. We have the following result:

$$\begin{aligned} &\left\| P_1 \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} \boldsymbol{f} - x_2 \pi_1 \left(\frac{1}{\varepsilon^{\gamma+2}} \mathbb{C}_\chi^{\text{stretch}} + C^{\text{stretch}} \right)^{-1} \mathcal{M}_\chi^{\text{stretch}} \boldsymbol{f} \right\|_{L^2(\omega \times Y)} \\ &\leq \frac{1}{2\pi} \oint_\Gamma |g_{\chi,\varepsilon}(z)| \left\| P_1 \left(zI - \frac{1}{|\chi|^2} \mathcal{A}_\chi \right)^{-1} \boldsymbol{f} - x_2 \pi_1 \left(zC^{\text{stretch}} - \frac{1}{|\chi|^2} \mathbb{C}_\chi^{\text{stretch}} \right)^{-1} \mathcal{M}_\chi^{\text{stretch}} \boldsymbol{f} \right\|_{L^2(\omega \times Y)} dz \\ &\leq C|\chi| \left(\max \left\{ \frac{|\chi|^2}{\varepsilon^{\gamma+2}}, 1 \right\} \right)^{-1} \|\boldsymbol{f}\|_{L^2(\omega \times Y; \mathbb{C}^3)} \leq C\varepsilon^{\frac{\gamma+2}{2}} \|\boldsymbol{f}\|_{L^2(\omega \times Y; \mathbb{C}^3)}, \end{aligned}$$

where the bound is the sharpest when $|\chi|^2 \approx \varepsilon^{\gamma+2}$. For the second and third component we have an analogous result:

$$\left\| P_2 \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} \boldsymbol{f} + x_1 \pi_1 \left(\frac{1}{\varepsilon^{\gamma+2}} \mathbb{C}_\chi^{\text{stretch}} + C^{\text{stretch}} \right)^{-1} \mathcal{M}_\chi^{\text{stretch}} \boldsymbol{f} \right\|_{L^2(\omega \times Y)}$$

$$\leq C\varepsilon^{\frac{\gamma+2}{2}}\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},$$

$$\left\|P_3\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi+I\right)^{-1}\boldsymbol{f}-\pi_2\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}_\chi^{\text{stretch}}\boldsymbol{f}\right\|_{L^2(\omega\times Y)}$$

$$\leq C\varepsilon^{\frac{\gamma+2}{2}}\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},$$

The statement follows from

$$\begin{aligned}\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}&=\mathcal{G}_\varepsilon^{-1}\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{G}_\varepsilon,\\ \left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon+I\right)^{-1}&=\mathcal{G}_\varepsilon^{-1}\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi+I\right)^{-1}\mathcal{G}_\varepsilon\end{aligned}\tag{3.29}$$

and consequently

$$\begin{aligned}&\mathcal{G}_\varepsilon^{-1}\left(P_1\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi+I\right)^{-1}-x_2\pi_1\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}_\chi^{\text{stretch}}\right)\mathcal{G}_\varepsilon\\ &\quad=P_1\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon+I\right)^{-1}-x_2\pi_1\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}^{\text{stretch}}\Xi_\varepsilon,\\ &\mathcal{G}_\varepsilon^{-1}\left(P_2\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi+I\right)^{-1}+x_1\pi_1\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}_\chi^{\text{stretch}}\right)\mathcal{G}_\varepsilon\\ &\quad=P_2\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon+I\right)^{-1}+x_1\pi_1\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}^{\text{stretch}}\Xi_\varepsilon,\\ &\mathcal{G}_\varepsilon^{-1}\left(P_3\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi+I\right)^{-1}-\pi_2\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}_\chi^{\text{stretch}}\right)\mathcal{G}_\varepsilon\\ &\quad=P_3\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon+I\right)^{-1}-\pi_2\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}^{\text{stretch}}\Xi_\varepsilon,\end{aligned}$$

as well as the fact that the Gelfand transform is an isometry. ■

Next, we analyse the bending case.

Theorem 3.4.2. *Suppose that the assumptions on the material symmetries 3.1.1 hold. Let the forces $\boldsymbol{f}$ belong to L^2_{bend}. Let $\gamma > -2$ be the parameter of spectral scaling. Let $\delta \geq 0$*

be the parameter of force scaling. Then we have the following estimates:

$$\left\| P_i \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} S_{\varepsilon^\delta} \boldsymbol{f} - \pi_i \left(\frac{1}{\varepsilon^{\gamma-2}} \mathcal{A}^{\mathrm{bend}} + I \right)^{-1} \mathcal{M}_\varepsilon^{\mathrm{bend}} \Xi_\varepsilon S_{\varepsilon^\delta} \boldsymbol{f} \right\|_{L^2(\omega\times\mathbb{R})}$$
$$\leq C \varepsilon^{\frac{\gamma+2}{4}} \max\left\{ \varepsilon^{\frac{\gamma+2}{4}-\delta}, 1 \right\} \|\boldsymbol{f}\|_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)}, \quad i = 1,2,$$
$$\left\| P_3 \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} S_{\varepsilon^\delta} \boldsymbol{f} + \varepsilon \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \cdot \frac{d}{dx_3} \left(\frac{1}{\varepsilon^{\gamma-2}} \mathcal{A}^{\mathrm{bend}} + I \right)^{-1} \mathcal{M}_\varepsilon^{\mathrm{bend}} \Xi_\varepsilon S_{\varepsilon^\delta} \boldsymbol{f} \right\|_{L^2(\omega\times\mathbb{R})}$$
$$\leq C \varepsilon^{\frac{\gamma+2}{2}} \max\left\{ \varepsilon^{\frac{\gamma+2}{4}-\delta}, 1 \right\} \|\boldsymbol{f}\|_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)}. \tag{3.30}$$

Proof. It is clear from (3.25) that we have the following norm-resolvent estimates:

$$\left\| P_i \left(\frac{1}{|\chi|^4} \mathcal{A}_\chi + I \right)^{-1} S_{|\chi|} \boldsymbol{f} - \pi_i \left(\frac{1}{|\chi|^4} \mathbb{C}_\chi^{\mathrm{bend}} + I \right)^{-1} \mathcal{M}_\chi^{\mathrm{bend}} S_{|\chi|} \boldsymbol{f} \right\|_{L^2\to L^2} \leq C|\chi| \, \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},$$
$$i = 1,2,$$
$$\left\| P_3 \left(\frac{1}{|\chi|^4} \mathcal{A}_\chi + I \right)^{-1} S_{|\chi|} \boldsymbol{f} + i\chi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \left(\frac{1}{|\chi|^4} \mathbb{C}_\chi^{\mathrm{bend}} + I \right)^{-1} \mathcal{M}_\chi^{\mathrm{bend}} S_{|\chi|} \boldsymbol{f} \right\|_{L^2\to L^2} \leq C|\chi|^2 \, \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

For each fixed $\varepsilon > 0$, $\chi \neq 0$ we define the function

$$f_{\varepsilon,\chi}(z) := \left(\frac{|\chi|^4}{\varepsilon^{\gamma+2}} z + 1 \right)^{-1}, \ \Re(z) > 0. \tag{3.31}$$

Similarly as before, we have that for every fixed $\eta > 0$, function $f_{\varepsilon,\chi}$ is bounded on the halfplane $\left\{ z \in \mathbb{C}, \Re(z) \geq \eta \right\}$,

$$|f_{\varepsilon,\chi}(z)| \leq C \left(\max\left\{ \frac{|\chi|^4}{\varepsilon^{\gamma+2}}, 1 \right\} \right)^{-1}.$$

Due to the bounds on both the eigenvalues λ_1^χ, λ_2^χ of $\mathcal{A}_\chi$ of the order $|\chi|^4$, and the eigenvalues $\overline{\lambda}_1^\chi$, $\overline{\lambda}_2^\chi$ of the matrix $\mathbb{C}_\chi^{\mathrm{bend}}$, there exists a closed contour $\Gamma \subset \left\{ z \in \mathbb{C}, \Re(z) > 0 \right\}$ and a constant $\mu > 0$ such that for every $\chi \in [-\mu, \mu] \setminus \{0\}$ one has the following properties:

- Γ encloses the two smallest eigenvalues of both the operators $\frac{1}{|\chi|^4} \mathcal{A}_\chi$ and $\frac{1}{|\chi|^4} \mathbb{C}_\chi^{\mathrm{bend}}$.

- Γ does not enclose any other eigenvalue (of higher order).

- $\exists \rho_0 > 0$, $\inf_{z\in\Gamma} |z-\lambda_i^\chi| \geq \rho_0$, $\inf_{z\in\Gamma} |z-\overline{\lambda}_i^\chi| \geq \rho_0$, $i=1,2$.

Due to the fact that $f_{\chi,\varepsilon}$ is analytic on the right halfplane, the Cauchy integral formula gives us:

$$P_\Gamma f_{\chi,\varepsilon}(\mathcal{A}) P_\Gamma := \frac{1}{2\pi i}\oint_\Gamma f_{\chi,\varepsilon}(z)(zI-\mathcal{A})^{-1}dz.$$

We make the following observations:

$$\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi + I = \frac{|\chi|^4}{\varepsilon^{\gamma+2}}\left(\frac{1}{|\chi|^4}\mathcal{A}_\chi\right) + I, \qquad S_{\varepsilon^\delta}\boldsymbol{f} = S_{|\chi|}S_{\varepsilon^\delta/|\chi|}\boldsymbol{f}.$$

Note that:

$$\|S_{\varepsilon^\delta/|\chi|}\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)} \leq \max\left\{1, \frac{|\chi|}{\varepsilon^\delta}\right\}\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

Thus we can write

$$P_\Gamma\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi + I\right)^{-1}P_\Gamma S_{\varepsilon^\delta}\boldsymbol{f} = \frac{1}{2\pi i}\oint_\Gamma f_{\chi,\varepsilon}(z)\left(zI - \frac{1}{|\chi|^4}\mathcal{A}_\chi\right)^{-1}S_{|\chi|}S_{\varepsilon^\delta/|\chi|}\boldsymbol{f}dz.$$

In the same fashion as before, we omit writing the projection operator since the estimates are valid for the whole operator.

For $i=1,2$ we calculate:

$$\begin{aligned}
&\left\|P_i\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi + I\right)^{-1}S_{\varepsilon^\delta}\boldsymbol{f} - \pi_i\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\mathrm{bend}} + I\right)^{-1}\mathcal{M}_\chi^{\mathrm{bend}}S_{\varepsilon^\delta}\boldsymbol{f}\right\|_{L^2(\omega\times Y)} \\
\leq&\frac{1}{2\pi}\oint_\Gamma |f_{\varepsilon,\chi}(z)|\left\|P_i\left(zI - \frac{1}{|\chi|^4}\mathcal{A}_\chi\right)^{-1}S_{|\chi|}S_{\varepsilon^\delta/|\chi|}\boldsymbol{f}\right. \\
&\left.-\pi_i\left(zI - \frac{1}{|\chi|^4}\mathbb{C}_\chi^{\mathrm{bend}}\right)^{-1}\mathcal{M}_\chi^{\mathrm{bend}}S_{|\chi|}S_{\varepsilon^\delta/|\chi|}\boldsymbol{f}\right\|_{L^2(\omega\times Y)}dz \\
\leq&C|\chi|\left(\max\left\{\frac{|\chi|^4}{\varepsilon^{\gamma+2}}, 1\right\}\right)^{-1}\|S_{\varepsilon^\delta/|\chi|}\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
\leq&C|\chi|\left(\max\left\{\frac{|\chi|^4}{\varepsilon^{\gamma+2}}, 1\right\}\right)^{-1}\max\left\{\frac{|\chi|}{\varepsilon^\delta}, 1\right\}\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
\leq&C\varepsilon^{\frac{\gamma+2}{4}}\max\left\{\varepsilon^{\frac{\gamma+2}{4}-\delta}, 1\right\}\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},
\end{aligned}$$

where the bound is the sharpest when $|\chi|^4 \approx \varepsilon^{\gamma+2}$. For the third component we have:

$$
\begin{aligned}
&\left\| P_3 \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} S_{\varepsilon^\delta} \boldsymbol{f} + i\chi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \left(\frac{1}{\varepsilon^{\gamma+2}} \mathbb{C}_\chi^{\text{bend}} + I \right)^{-1} \mathcal{M}_\chi^{\text{bend}} S_{\varepsilon^\delta} \boldsymbol{f} \right\|_{L^2(\omega\times Y)} \\
&\leq \frac{1}{2\pi} \oint_\Gamma |f_{\varepsilon,\chi}(z)| \left\| P_3 \left(zI - \frac{1}{|\chi|^4} \mathcal{A}_\chi \right)^{-1} S_{|\chi|} S_{\varepsilon^\delta/|\chi|} \boldsymbol{f} \right. \\
&\left. + i\chi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \left(zI - \frac{1}{|\chi|^4} \mathbb{C}_\chi^{\text{bend}} \right)^{-1} \mathcal{M}_\chi^{\text{bend}} S_{|\chi|} S_{\varepsilon^\delta/|\chi|} \boldsymbol{f} \right\|_{L^2(\omega\times Y)} dz \\
&\leq C|\chi|^2 \left(\max\left\{ \frac{|\chi|^4}{\varepsilon^4}, 1 \right\} \right)^{-1} \max\left\{ \frac{|\chi|}{\varepsilon^\delta}, 1 \right\} \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)} \\
&\leq C\varepsilon^{\frac{\gamma+2}{2}} \max\left\{ \varepsilon^{\frac{\gamma+2}{4}-\delta}, 1 \right\} \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.
\end{aligned}
$$

Since

$$
\left(\frac{1}{\varepsilon^{\gamma-2}} \mathcal{A}^{\text{bend}} + I \right)^{-1} = \mathcal{G}_\varepsilon^{-1} \left(\frac{1}{\varepsilon^{\gamma+2}} \mathbb{C}_\chi^{\text{bend}} + I \right)^{-1} \mathcal{G}_\varepsilon,
$$

$$
\left(\frac{1}{\varepsilon^{\gamma}} \mathcal{A}_\varepsilon + I \right)^{-1} = \mathcal{G}_\varepsilon^{-1} \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} \mathcal{G}_\varepsilon,
$$

we also have:

$$
\begin{aligned}
&\mathcal{G}_\varepsilon^{-1} \left(P_i \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} - \pi_i \left(\frac{1}{\varepsilon^{\gamma+2}} \mathbb{C}_\chi^{\text{bend}} + I \right)^{-1} \mathcal{M}_\chi^{\text{bend}} \right) \mathcal{G}_\varepsilon \\
&\qquad = P_i \left(\frac{1}{\varepsilon^{\gamma}} \mathcal{A}_\varepsilon + I \right)^{-1} - \pi_i \left(\frac{1}{\varepsilon^{\gamma-2}} \mathcal{A}^{\text{bend}} + I \right)^{-1} \mathcal{M}_\varepsilon^{\text{bend}} \Xi_\varepsilon,
\end{aligned}
\tag{3.32}
$$

$$
\begin{aligned}
&\mathcal{G}_\varepsilon^{-1} \left(P_3 \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} + i\chi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \left(\frac{1}{\varepsilon^{\gamma+2}} \mathbb{C}_\chi^{\text{bend}} + I \right)^{-1} \mathcal{M}_\chi^{\text{bend}} \right) \mathcal{G}_\varepsilon \\
&\qquad = P_3 \left(\frac{1}{\varepsilon^{\gamma}} \mathcal{A}_\varepsilon + I \right)^{-1} + \varepsilon \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \frac{d}{dx_3} \left(\frac{1}{\varepsilon^{\gamma-2}} \mathcal{A}^{\text{bend}} + I \right)^{-1} \mathcal{M}_\varepsilon^{\text{bend}} \Xi_\varepsilon.
\end{aligned}
\tag{3.33}
$$

In order to finish the proof we use the fact that the Gelfand transform is an isometry. ■

Remark 3.4.3. By using the notation introduced in the Remark 3.1.4, we can rephrase the estimates (3.27) and (3.30) in a more compact way as follows:

$$
\left\| \left(\frac{1}{\varepsilon^{\gamma}} \mathcal{A}_\varepsilon + I \right)^{-1} |_{L^2_{\text{stretch}}} - (\mathcal{M}^{\text{stretch}})^* \left(\frac{1}{\varepsilon^{\gamma}} \mathcal{A}^{\text{stretch}} + C^{\text{stretch}} \right)^{-1} \mathcal{M}^{\text{stretch}} \Xi_\varepsilon \right\|_{L^2\to L^2} \leq C\varepsilon^{\frac{\gamma+2}{2}},
\tag{3.34}
$$

$$\left\| P_i \left(\left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} |_{L^2_{\mathrm{bend}}} - (\mathcal{M}_\varepsilon^{\mathrm{bend}})^* \left(\frac{1}{\varepsilon^{\gamma-2}} \mathcal{A}^{\mathrm{bend}} + I \right)^{-1} \mathcal{M}_\varepsilon^{\mathrm{bend}} \Xi_\varepsilon \right) S_{\varepsilon^\delta} \right\|_{L^2 \to L^2}$$
$$\leq \begin{cases} C\varepsilon^{\frac{\gamma+2}{4}} \max\left\{ \varepsilon^{\frac{\gamma+2}{4} - \delta}, 1 \right\}, & i = 1, 2; \\ C\varepsilon^{\frac{\gamma+2}{2}} \max\left\{ \varepsilon^{\frac{\gamma+2}{4} - \delta}, 1 \right\}, & i = 3. \end{cases}$$

Remark 3.4.4. It is clear that, due to (3.26), in the case $\delta = 0$, in addition to the estimate:

$$\left\| P_i \left(\left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} |_{L^2_{\mathrm{bend}}} - (\mathcal{M}_\varepsilon^{\mathrm{bend}})^* \left(\frac{1}{\varepsilon^{\gamma-2}} \mathcal{A}^{\mathrm{bend}} + I \right)^{-1} \mathcal{M}_\varepsilon^{\mathrm{bend}} \Xi_\varepsilon \right) \right\|_{L^2 \to L^2}$$
$$\leq \begin{cases} C\varepsilon^{\frac{\gamma+2}{4}}, & i = 1, 2; \\ C\varepsilon^{\frac{\gamma+2}{2}}, & i = 3, \end{cases}$$

one also has:

$$\left\| P_i \left(\left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} |_{L^2_{\mathrm{bend}}} - (\mathcal{M}_\varepsilon^{\mathrm{bend}})^* \left(\frac{1}{\varepsilon^{\gamma-2}} \mathcal{A}^{\mathrm{bend}} + I \right)^{-1} \mathcal{M}_\varepsilon^{\mathrm{bend}} \Xi_\varepsilon S_\infty \right) \right\|_{L^2 \to L^2}$$
$$\leq \begin{cases} C\varepsilon^{\frac{\gamma+2}{4}}, & i = 1, 2; \\ C\varepsilon^{\frac{\gamma+2}{2}}, & i = 3, \end{cases}$$

Namely, this is the norm-resolvent estimate in the abscence of out-of-line force terms.

We show that the smoothing operator Ξ_ε appearing in the above norm-resolvent estimates can be neglected with the estimates still being valid.

Corollary 3.4.5. *Suppose that the assumptions on the material symmetries 3.1.1 hold. Let $\gamma > -2$ be the parameter of spectral scaling. Then there exists $C > 0$ such that for every $\varepsilon > 0$ we have:*

$$\left\| \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} |_{L^2_{\mathrm{stretch}}} - (\mathcal{M}^{\mathrm{stretch}})^* \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}^{\mathrm{stretch}} + \mathcal{C}^{\mathrm{stretch}} \right)^{-1} \mathcal{M}^{\mathrm{stretch}} \right\|_{L^2 \to L^2} \leq C\varepsilon^{\frac{\gamma+2}{2}},$$

$$\left\| P_i \left(\left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} |_{L^2_{\mathrm{bend}}} - (\mathcal{M}_\varepsilon^{\mathrm{bend}})^* \left(\frac{1}{\varepsilon^{\gamma-2}} \mathcal{A}^{\mathrm{bend}} + I \right)^{-1} \mathcal{M}_\varepsilon^{\mathrm{bend}} S_\infty \right) \right\|_{L^2 \to L^2}$$
$$\leq \begin{cases} C\varepsilon^{\frac{\gamma+2}{4}}, & i = 1, 2; \\ C\varepsilon^{\frac{\gamma+2}{2}}, & i = 3, \end{cases} \qquad (3.35)$$

Proof. The application of the Fourier transform to the limit resolvent in the stretching case yields:

$$\left((\mathcal{M}^{\text{stretch}})^*\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}^{\text{stretch}}(I-\Xi_\varepsilon)\boldsymbol{f}\right)^\wedge(\xi)$$
$$=(\mathcal{M}^{\text{stretch}})^*\left(\frac{\xi^2}{\varepsilon^\gamma}\mathbb{C}^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}^{\text{stretch}}\widehat{\boldsymbol{f}}(\xi)\mathbb{1}_{\left\langle-\infty,-\frac{1}{2\varepsilon}\right]\cup\left[\frac{1}{2\varepsilon},\infty\right\rangle}(\xi).$$

But, for $|\xi|>\frac{1}{2\varepsilon}$ and $\gamma>-2$, we have:

$$\left(\frac{\xi^2}{\varepsilon^\gamma}\mathbb{C}^{\text{stretch}}+C^{\text{stretch}}\right)m\cdot m^T\geq\frac{\xi^2}{\varepsilon^\gamma}C|m|^2\geq\frac{C}{\varepsilon^{\gamma+2}}|m|^2,$$

and hence:

$$\left|\left(\frac{\xi^2}{\varepsilon^\gamma}\mathbb{C}^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\right|\leq C\varepsilon^{\gamma+2}.$$

With all of this combined we have:

$$\left\|\left((\mathcal{M}^{\text{stretch}})^*\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}^{\text{stretch}}(I-\Xi_\varepsilon)\boldsymbol{f}\right)^\wedge\right\|_{L^2}\leq C\varepsilon^{\gamma+2}\left\|\widehat{\boldsymbol{f}}\right\|_{L^2},$$

so Ξ_ε can be removed from (3.34). Similarly as in the stretching case, one can eliminate the smoothing operator from the norm-resolvent estimate in the case of absence of out-of-line force terms. For $i=1,2$ one has:

$$P_i\left((\mathcal{M}_\varepsilon^{\text{bend}})^*\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}}+I\right)^{-1}\mathcal{M}_\varepsilon^{\text{bend}}(I-\Xi_\varepsilon)S_\infty\right)=\pi_i\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}}+I\right)^{-1}(I-\Xi_\varepsilon).$$

Now:

$$\left(\pi_i\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}}+I\right)^{-1}(I-\Xi_\varepsilon)\boldsymbol{f}\right)^\wedge(\xi)=\left(\frac{\xi^4}{\varepsilon^{\gamma-2}}\mathbb{C}^{\text{bend}}+I\right)^{-1}\widehat{\boldsymbol{f}}(\xi)\mathbb{1}_{\left\langle-\infty,-\frac{1}{2\varepsilon}\right]\cup\left[\frac{1}{2\varepsilon},\infty\right\rangle}(\xi).$$

But,for $|\xi|>\frac{1}{2\varepsilon}$ and $\gamma>-2$, we have:

$$\left(\frac{\xi^4}{\varepsilon^{\gamma-2}}\mathbb{C}^{\text{bend}}+I\right)m\cdot m^T\geq\frac{\xi^4}{\varepsilon^{\gamma-2}}C|m|^2\geq\frac{C}{\varepsilon^{\gamma+2}}|m|^2.$$

Finally, we obtain:

$$\left\|P_i\left((\mathcal{M}_\varepsilon^{\text{bend}})^*\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}}+I\right)^{-1}\mathcal{M}_\varepsilon^{\text{bend}}(I-\Xi_\varepsilon)S_\infty\boldsymbol{f}\right)^\wedge\right\|_{L^2}\leq C\varepsilon^{\gamma+2}\left\|\widehat{\boldsymbol{f}}\right\|_{L^2},i=1,2.$$

Same can be done in the case $i = 3$,

$$P_3\left((\mathcal{M}_\varepsilon^{\text{bend}})^*\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}}+I\right)^{-1}\mathcal{M}_\varepsilon^{\text{bend}}(I-\Xi_\varepsilon)S_\infty\right)$$
$$=\varepsilon\begin{bmatrix}x_1\\x_2\end{bmatrix}^T\cdot\frac{d}{dx_3}\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}}+I\right)^{-1}(I-\Xi_\varepsilon),$$

where we have:

$$\left(\varepsilon\begin{bmatrix}x_1\\x_2\end{bmatrix}^T\cdot\frac{d}{dx_3}\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}}+I\right)^{-1}(I-\Xi_\varepsilon)\boldsymbol{f}\right)^\wedge(\xi)$$
$$=\varepsilon\xi\begin{bmatrix}x_1\\x_2\end{bmatrix}^T\cdot\left(\frac{\xi^4}{\varepsilon^{\gamma-2}}\mathbb{C}^{\text{bend}}+I\right)^{-1}\widehat{\boldsymbol{f}}(\xi)\mathbb{1}_{\left\langle-\infty,-\frac{1}{2\varepsilon}\right]\cup\left[\frac{1}{2\varepsilon},\infty\right\rangle}(\xi).$$

■

The following result is an easy consequence:

Corollary 3.4.6. *Suppose that the assumptions on the material symmetries 3.1.1 hold. Let $\gamma > -2$ be the parameter of spectral scaling. Then there exists C such that for every $\varepsilon > 0$ we have:*

$$\left\|\left(\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon+I\right)^{-1}|_{L^2_{\text{bend}}}-(\mathcal{M}_0^{\text{bend}})^*\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}}+I\right)^{-1}\mathcal{M}_0^{\text{bend}}\right)\right\|_{L^2\to L^2}\le C\varepsilon^{\frac{\gamma+2}{4}}. \quad (3.36)$$

Proof. The ellipticity of the operator $\mathcal{A}^{\text{bend}}$ yields that the solution $\boldsymbol{b}\in H^2(\mathbb{R};\mathbb{R}^2)$ of

$$\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}}+I\right)\boldsymbol{b}=\boldsymbol{g},$$

satisfies:

$$\|\boldsymbol{b}\|_{L^2(\mathbb{R};\mathbb{R}^2)}\le C\|\boldsymbol{g}\|_{L^2(\mathbb{R};\mathbb{R}^2)},\quad \|\nabla^2\boldsymbol{b}\|_{L^2(\mathbb{R};\mathbb{R}^{2\times2\times2})}\le C\varepsilon^{\frac{\gamma-2}{2}}\|\boldsymbol{g}\|_{L^2(\mathbb{R};\mathbb{R}^2)}.$$

By using the interpolation inequality

$$\|\nabla\boldsymbol{b}\|^2_{L^2(\mathbb{R};\mathbb{R}^{2\times2})}\le C\|\nabla^2\boldsymbol{b}\|_{L^2(\mathbb{R};\mathbb{R}^{2\times2\times2})}\|\boldsymbol{b}\|_{L^2(\mathbb{R};\mathbb{R}^2)},$$

we clearly have

$$\|\varepsilon\nabla\boldsymbol{b}\|_{L^2(\omega;\mathbb{R}^{2\times2})}\le C\varepsilon^{\frac{\gamma+2}{4}}\|\boldsymbol{g}\|_{L^2(\omega;\mathbb{R}^2)}.$$

In other words, by replacing $\mathcal{M}_\varepsilon^{\text{memb}}$ with $\mathcal{M}_0^{\text{bend}}$ in (3.35), the error of approximation remains of order $\varepsilon^{\frac{\gamma+2}{4}}$ at worst. ■

Remark 3.4.7. Here we make a remark regarding the spectrum of the limit resolvents. First, notice that the force momentum operator $\mathcal{M}_0^{\text{bend}}$ is clearly a partial isometry since

$$\mathcal{M}_0^{\text{bend}} \left(\mathcal{M}_0^{\text{bend}}\right)^* = I, \quad \text{on } H^2(\mathbb{R};\mathbb{R}^2).$$

Due to this and the fact that $\sigma\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}}\right) = [0,\infty\rangle$ (a simple argument via Fourier transform), it is evident that

$$\sigma\left((\mathcal{M}_0^{\text{bend}})^* \left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}} + I\right)^{-1} \mathcal{M}_0^{\text{bend}}\right) = \sigma\left(\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}} + I\right)^{-1}\right) = [0,1].$$

On the other hand, by defining a inner product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\text{stretch}} := \int_{\mathbb{R}} C^{\text{stretch}} \boldsymbol{u}\cdot\boldsymbol{v}, \quad \boldsymbol{u},\boldsymbol{v} \in L^2(\mathbb{R};\mathbb{R}^2),$$

one clearly has

$$\begin{aligned}
&(\mathcal{M}^{\text{stretch}})^* \left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{stretch}} + C^{\text{stretch}}\right)^{-1} \mathcal{M}^{\text{stretch}} \\
&= (\mathcal{M}^{\text{stretch}})^* \left(\frac{1}{\varepsilon^{\gamma-2}}\left(C^{\text{stretch}}\right)^{-1}\mathcal{A}^{\text{stretch}} + I\right)^{-1} \left(C^{\text{stretch}}\right)^{-1} \mathcal{M}^{\text{stretch}},
\end{aligned}$$

and can easily verify that $(\mathcal{M}^{\text{stretch}})^*$ is the adjoint of $\left(C^{\text{stretch}}\right)^{-1}\mathcal{M}^{\text{stretch}}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\text{stretch}}$ on $L^2(\mathbb{R};\mathbb{R}^2)$ (in pair with the usual inner product on $L^2(\mathbb{R};\mathbb{R}^3)$). Notice that

$$\left(C^{\text{stretch}}\right)^{-1}\mathcal{M}^{\text{stretch}}(\mathcal{M}^{\text{stretch}})^* = I.$$

The operator $\left(C^{\text{stretch}}\right)^{-1}\mathcal{A}^{\text{stretch}}$ is clearly symmetric (with respect to the inner product $\langle\cdot,\cdot\rangle_{\text{stretch}}$), and again we have

$$\sigma\left((\mathcal{M}^{\text{stretch}})^* \left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{stretch}} + C^{\text{stretch}}\right)^{-1} \mathcal{M}^{\text{stretch}}\right) = [0,1].$$

The norm-resolvent estimates allow us to easily estimate the gaps in the spectrum of the operator $\mathcal{A}_\varepsilon$, for every $\varepsilon > 0$.

Corollary 3.4.8. *Suppose that the assumptions on the material symmetries 3.1.1 hold. Let $\gamma > -2$ be the parameter of spectral scaling. Let $M > 0$. Then*

$$\sup_{\substack{[a,b]\subset[0,M\varepsilon^\gamma] \\ [a,b]\cap\sigma(\mathcal{A}_\varepsilon|_{L^2_{\text{bend}}})=\emptyset}} |[a,b]| \le C(M+1)^2\varepsilon^{\frac{5\gamma+2}{4}}, \qquad \sup_{\substack{[a,b]\subset[0,M\varepsilon^\gamma] \\ [a,b]\cap\sigma(\mathcal{A}_\varepsilon|_{L^2_{\text{stretch}}})=\emptyset}} |[a,b]| \le C(M+1)^2\varepsilon^{\frac{3\gamma+2}{4}}.$$

Proof. It is known that, for a bounded perturbation A of a self-adjoint operator H in a Hilbert space $\mathcal{H}$, one has the following spectral inclusion:

$$\sigma(H+A) \subset \{\lambda : \operatorname{dist}(\lambda, \sigma(\mathrm{H})) \le \|\mathrm{A}\|\}.$$

Thus, by plugging

$$H = \left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon + I\right)^{-1}|_{L^2_{\mathrm{bend}}},$$
$$A = \left(\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon + I\right)^{-1}|_{L^2_{\mathrm{bend}}} - (\mathcal{M}_0^{\mathrm{bend}})^* \left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\mathrm{bend}} + I\right)^{-1} \mathcal{M}_0^{\mathrm{bend}}\right),$$

and employing the norm-resolvent estimate (3.36), we obtain:

$$[0,1] \subset \left\{\lambda \in [0,1], \quad \operatorname{dist}\left(\lambda, \sigma\left(\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon + \mathrm{I}\right)^{-1}|_{\mathrm{L}^2_{\mathrm{bend}}}\right)\right) < \mathrm{C}\varepsilon^{\frac{\gamma+2}{4}}\right\}.$$

We have that in each interval of size $C\varepsilon^{\frac{\gamma+2}{4}}$ in $[0,1]$ there must exist a member of the spectrum of the operator $\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon + I\right)^{-1}|_{L^2_{\mathrm{bend}}}$. In other words, the maximal gap between the members of the spectrum of $\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon + I\right)^{-1}|_{L^2_{\mathrm{bend}}}$ is $C\varepsilon^{\frac{\gamma+2}{4}}$.

Let $\lambda_1, \lambda_2 \in \sigma(\mathcal{A}_\varepsilon|_{L^2_{\mathrm{bend}}})$ be such that $\frac{1}{\varepsilon^\gamma}\lambda_1, \frac{1}{\varepsilon^\gamma}\lambda_2 \le M$, $\langle\lambda_1, \lambda_2\rangle \cap \sigma(\mathcal{A}_\varepsilon) = \emptyset$. Due to the fact that

$$\left|\frac{\frac{1}{\varepsilon^\gamma}\lambda_1 - \frac{1}{\varepsilon^\gamma}\lambda_2}{\left(\frac{1}{\varepsilon^\gamma}\lambda_1 + 1\right)\left(\frac{1}{\varepsilon^\gamma}\lambda_2 + 1\right)}\right| = \left|\frac{1}{\frac{1}{\varepsilon^\gamma}\lambda_1 + 1} - \frac{1}{\frac{1}{\varepsilon^\gamma}\lambda_2 + 1}\right| \le C\varepsilon^{\frac{\gamma+2}{4}},$$

we have:

$$\left|\frac{1}{\varepsilon^\gamma}\lambda_1 - \frac{1}{\varepsilon^\gamma}\lambda_2\right| \le C\varepsilon^{\frac{\gamma+2}{4}}(M+1)^2,$$

hence

$$|\lambda_1 - \lambda_2| \le C\varepsilon^{\frac{5\gamma+2}{4}}(M+1)^2.$$

The stretching case goes analogously.

■

3.4.2. $L^2 \to H^1$ norm resolvent estimates

In order to state the results we define the following operators which take the zero order terms to the associated first order corrector terms:

$$\mathcal{B}^{\chi,\mathrm{corr}}_{1,\mathrm{stretch}} : \mathbb{C}^2 \to H^{\mathrm{stretch}}, \quad \mathcal{B}^{\chi,\mathrm{corr}}_{1,\mathrm{bend}} : \mathbb{C}^2 \to H^{\mathrm{bend}},$$

with the following identities:

$$\mathcal{B}^{\chi,\mathrm{corr}}_{1,\mathrm{stretch}}\begin{bmatrix} m_3 \\ m_4 \end{bmatrix} := \boldsymbol{u}_1, \quad \operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_1 = -\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda^{\mathrm{stretch}}_{\chi,m_3,m_4}(x), \quad \boldsymbol{u}_1 \in H^{\mathrm{stretch}},$$

$$\mathcal{B}^{\chi,\mathrm{corr}}_{1,\mathrm{bend}}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} := \boldsymbol{u}_1, \quad \operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_1 = -\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda^{\mathrm{bend}}_{\chi,m_1,m_2}(x), \quad \boldsymbol{u}_1 \in H^{\mathrm{bend}}.$$

Thus, the first order corrector operators depending on the spectral parameter $z \in \mathbb{C}$ can be defined for each fibre $\chi \in [-\pi,\pi]\setminus\{0\}$ with the following formulae:

$$\mathcal{A}^{\mathrm{stretch}}_{\chi,\mathrm{corr}}(z) := \mathcal{B}^{\chi,\mathrm{corr}}_{1,\mathrm{stretch}}\left(zC^{\mathrm{stretch}} - \frac{1}{|\chi|^2}\mathbb{C}^{\mathrm{stretch}}_{\chi}\right)^{-1}\mathcal{M}^{\mathrm{stretch}}_{\chi},$$
$$\mathcal{A}^{\mathrm{bend}}_{\chi,\mathrm{corr}}(z) := \mathcal{B}^{\chi,\mathrm{corr}}_{1,\mathrm{bend}}\left(zI - \frac{1}{|\chi|^4}\mathbb{C}^{\mathrm{bend}}_{\chi}\right)^{-1}\mathcal{M}^{\mathrm{bend}}_{\chi}.$$

Next we define the rescaled versions:

$$\mathcal{A}^{\mathrm{stretch}}_{\chi,\varepsilon,\mathrm{corr}} = \oint_{\Gamma^{\mathrm{stretch}}} g_{\varepsilon,\chi}(z)\mathcal{A}^{\mathrm{stretch}}_{\chi,\mathrm{corr}}(z)dz, \quad \mathcal{A}^{\mathrm{bend}}_{\chi,\varepsilon,\mathrm{corr}} = \oint_{\Gamma^{\mathrm{bend}}} f_{\varepsilon,\chi}(z)\mathcal{A}^{\mathrm{bend}}_{\chi,\mathrm{corr}}(z)dz, \quad \varepsilon > 0,$$

where Γ^{bend}, $\Gamma^{\mathrm{stretch}}$ are contours which uniformly enclose the scaled eigenvalues of $\mathbb{C}^{\mathrm{bend}}_{\chi}$, $\mathbb{C}^{\mathrm{stretch}}_{\chi}$, respectively. Notice here that we have:

$$\mathcal{A}^{\mathrm{stretch}}_{\chi,\varepsilon,\mathrm{corr}} := \mathcal{B}^{\chi,\mathrm{corr}}_{1,\mathrm{stretch}}\left(\frac{1}{\varepsilon^2}\mathbb{C}^{\mathrm{stretch}}_{\chi} + C^{\mathrm{stretch}}\right)^{-1}\mathcal{M}^{\mathrm{stretch}}_{\chi}, \quad \mathcal{A}^{\mathrm{bend}}_{\chi,\varepsilon,\mathrm{corr}} := \mathcal{B}^{\chi,\mathrm{corr}}_{1,\mathrm{bend}}\left(\frac{1}{\varepsilon^4}\mathbb{C}^{\mathrm{bend}}_{\chi} + I\right)^{-1}\mathcal{M}^{\mathrm{bend}}_{\chi}.$$

Finally we are able to define the following corrector operators:

$$\mathcal{A}^{\mathrm{corr}}_{\mathrm{stretch}}(\varepsilon) = \mathcal{G}_{\varepsilon}^{-1}\mathcal{A}^{\mathrm{stretch}}_{\chi,\varepsilon,\mathrm{corr}}\mathcal{G}_{\varepsilon}, \quad \mathcal{A}^{\mathrm{corr}}_{\mathrm{bend}}(\varepsilon) = \mathcal{G}_{\varepsilon}^{-1}\mathcal{A}^{\mathrm{bend}}_{\chi,\varepsilon,\mathrm{corr}}\mathcal{G}_{\varepsilon}, \quad \varepsilon > 0. \tag{3.37}$$

Let us start with the stretching case. Our goal is to prove the following theorem:

Theorem 3.4.9. *Suppose that the assumptions on the material symmetries 3.1.1 hold and that the forces $\boldsymbol{f}$ belong to L^2_{stretch}. Let $\gamma > -2$ be the parameter of spectral scaling. Let*

$\delta \geq 0$ be the parameter of force scaling. Then we have the following estimates:

$$\left\| P_1 \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} \boldsymbol{f} - x_2 \pi_1 \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}^{\text{stretch}} + C^{\text{stretch}} \right)^{-1} \mathcal{M}^{\text{stretch}} \Xi_\varepsilon \boldsymbol{f} - P_1 \mathcal{A}^{\text{corr}}_{\text{stretch}}(\varepsilon) \boldsymbol{f} \right\|_{H^1(\omega \times \mathbb{R})} \leq C \max \left\{ \varepsilon^{\gamma+1}, \varepsilon^{\frac{\gamma+2}{2}} \right\} \|\boldsymbol{f}\|_{L^2(\omega \times \mathbb{R}; \mathbb{R}^3)},$$

$$\left\| P_2 \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} \boldsymbol{f} + x_1 \pi_1 \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}^{\text{stretch}} + C^{\text{stretch}} \right)^{-1} \mathcal{M}^{\text{stretch}} \Xi_\varepsilon \boldsymbol{f} - P_2 \mathcal{A}^{\text{corr}}_{\text{stretch}}(\varepsilon) \boldsymbol{f} \right\|_{H^1(\omega \times \mathbb{R})} \leq C \max \left\{ \varepsilon^{\gamma+1}, \varepsilon^{\frac{\gamma+2}{2}} \right\} \|\boldsymbol{f}\|_{L^2(\omega \times \mathbb{R}; \mathbb{R}^3)},$$

$$\left\| P_3 \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} \boldsymbol{f} - \pi_2 \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}^{\text{stretch}} + C^{\text{stretch}} \right)^{-1} \mathcal{M}^{\text{stretch}} \Xi_\varepsilon \boldsymbol{f} - P_3 \mathcal{A}^{\text{corr}}_{\text{stretch}}(\varepsilon) \boldsymbol{f} \right\|_{H^1(\omega \times \mathbb{R})} \leq C \max \left\{ \varepsilon^{\gamma+1}, \varepsilon^{\frac{\gamma+2}{2}} \right\} \|\boldsymbol{f}\|_{L^2(\omega \times \mathbb{R}; \mathbb{R}^3)}.$$

Proof. In order to prove the $H^1(\omega \times \mathbb{R}^3)$ estimate from the statement of the above Theorem, the first resolvent estimate in (3.19) does not suffice, the reason being that, the Gelfand pullback would ruin the order of the estimate in the third variable. On the other hand, we do not need the whole expression in the second estimate from (3.19), either. This is because we can neglect the element $\boldsymbol{u}_0^{(1)}$ in the $H^1(\omega \times Y; \mathbb{C}^3)$ norm. Indeed, from (3.17) we note that:

$$\|\boldsymbol{u}_0^{(1)}\|_{L^2(\omega \times Y; \mathbb{C}^3)} \leq C|\chi| \|\boldsymbol{f}\|_{L^2(\omega \times Y; \mathbb{C}^3)}, \quad \|\partial_{x_\alpha} \boldsymbol{u}_0^{(1)}\|_{L^2(\omega \times Y; \mathbb{C}^3)} \leq C|\chi| \|\boldsymbol{f}\|_{L^2(\omega \times Y; \mathbb{C}^3)}, \quad \alpha = 1, 2,$$
$$\|\partial_y \boldsymbol{u}_0^{(1)}\|_{L^2(\omega \times Y; \mathbb{C}^3)} = 0,$$

Hence the useful estimates which we obtain from (3.19) are

$$\left\| \partial_{x_\alpha} \left(\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \end{bmatrix} - \begin{bmatrix} m_3 x_2 \\ -m_3 x_1 \\ m_4 \end{bmatrix} - \begin{bmatrix} (\boldsymbol{u}_1)_1 \\ (\boldsymbol{u}_1)_2 \\ (\boldsymbol{u}_1)_3 \end{bmatrix} \right) \right\|_{L^2(\omega \times Y, \mathbb{C}^3)} \leq C|\chi| \|\boldsymbol{f}\|_{L^2(\omega \times Y; \mathbb{C}^3)}.$$

$$\left\| \partial_y \left(\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \end{bmatrix} - \begin{bmatrix} m_3 x_2 \\ -m_3 x_1 \\ m_4 \end{bmatrix} - \begin{bmatrix} (\boldsymbol{u}_1)_1 \\ (\boldsymbol{u}_1)_2 \\ (\boldsymbol{u}_1)_3 \end{bmatrix} \right) \right\|_{L^2(\omega \times Y, \mathbb{C}^3)} \leq C|\chi|^2 \|\boldsymbol{f}\|_{L^2(\omega \times Y; \mathbb{C}^3)}. \tag{3.38}$$

We will focus only on the norm of derivative with respect to y. For this derivative, we need a higher order estimate in $|\chi|$ in order to obtain the desired estimate by applying the inverse of scaled Gelfand transform. By rewriting the estimate (3.38) we are able to deduce:

$$\left\|\partial_y\left(P_1\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi+I\right)^{-1}-x_2\pi_1\left(\frac{1}{|\chi|^2}\mathbb{C}_\chi^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}_\chi^{\text{stretch}}-P_1\mathcal{A}_{\chi,\text{corr}}^{\text{stretch}}\right)\boldsymbol{f}\right\|_{L^2(\omega\times Y)}$$
$$\leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},$$
$$\left\|\partial_y\left(P_2\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi+I\right)^{-1}+x_1\pi_1\left(\frac{1}{|\chi|^2}\mathbb{C}_\chi^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}_\chi^{\text{stretch}}-P_2\mathcal{A}_{\chi,\text{corr}}^{\text{stretch}}\right)\boldsymbol{f}\right\|_{L^2(\omega\times Y)}$$
$$\leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},$$
$$\left\|\partial_y\left(P_3\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi+I\right)^{-1}-\pi_2\left(\frac{1}{|\chi|^2}\mathbb{C}_\chi^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}_\chi^{\text{stretch}}-P_3\mathcal{A}_{\chi,\text{corr}}^{\text{stretch}}\right)\boldsymbol{f}\right\|_{L^2(\omega\times Y)}$$
$$\leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

We use the method introduced in the estimates for $L^2\to L^2$ operator norm. By defining the function $g_{\varepsilon,\chi}(z)$ as in (3.28), choose $\Gamma\subset\mathbb{C}$ such that:

$$\left\|\partial_y\left(P_1\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi+I\right)^{-1}-x_2\pi_1\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}_\chi^{\text{stretch}}-P_1\mathcal{A}_{\chi,\varepsilon,\text{corr}}^{\text{stretch}}\right)\boldsymbol{f}\right\|_{L^2(\omega\times Y)}$$
$$\leq\frac{1}{2\pi}\oint_\Gamma|g_{\varepsilon,\chi}(z)|\left\|\partial_y\left(P_1\left(zI-\frac{1}{|\chi|^2}\mathcal{A}_\chi\right)^{-1}-x_2\pi_1\left(zC^{\text{stretch}}-\frac{1}{|\chi|^2}\mathbb{C}_\chi^{\text{stretch}}\right)^{-1}\mathcal{M}_\chi^{\text{stretch}}\right.\right.$$
$$\left.\left.-P_1\mathcal{B}_{1,\text{stretch}}^{\chi,\text{corr}}\left(zC^{\text{stretch}}-\frac{1}{|\chi|^2}\mathbb{C}_\chi^{\text{stretch}}\right)^{-1}\mathcal{M}_\chi^{\text{stretch}}\right)\boldsymbol{f}\right\|_{L^2(\omega\times Y)}dz$$
$$\leq C|\chi|^2\max\left\{\frac{|\chi|^2}{\varepsilon^{\gamma+2}},1\right\}^{-1}\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}$$
$$\leq C\varepsilon^{\gamma+2}\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

By applying the Gelfand transform we obtain:

$$\left\|\partial_{x_3}\left(P_1\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon+I\right)^{-1}\boldsymbol{f}-x_2\pi_1\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}^{\text{stretch}}\boldsymbol{f}-P_1\mathcal{A}_{\text{stretch}}^{\text{corr}}(\varepsilon)\boldsymbol{f}\right)\right\|_{L^2(\omega\times\mathbb{R})}$$
$$\leq C\varepsilon^{\gamma+1}\|\boldsymbol{f}\|_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)}$$

For the second and the third component we have analogous results:

$$\left\|\partial_{x_3}\left(P_2\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon+I\right)^{-1}\boldsymbol{f}+x_1\pi_1\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}^{\text{stretch}}\boldsymbol{f}-P_2\mathcal{A}_{\text{stretch}}^{\text{corr}}(\varepsilon)\boldsymbol{f}\right)\right\|_{L^2(\omega\times\mathbb{R})}$$

$$\leq C\varepsilon^{\gamma+1}\|\boldsymbol{f}\|_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)},$$

$$\left\|\partial_{x_3}\left(P_3\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon+I\right)^{-1}\boldsymbol{f}-\pi_2\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}}+C^{\text{stretch}}\right)^{-1}\mathcal{M}^{\text{stretch}}\boldsymbol{f}-P_3\mathcal{A}^{\text{corr}}_{\text{stretch}}(\varepsilon)\boldsymbol{f}\right)\right\|_{L^2(\omega\times\mathbb{R})}$$

$$\leq C\varepsilon^{\gamma+1}\|\boldsymbol{f}\|_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)},$$

due to (3.29), (3.37) and from the fact that $\varepsilon\frac{\partial}{\partial x_3}=\mathcal{G}_\varepsilon^{-1}\frac{\partial}{\partial y}\mathcal{G}_\varepsilon+O(\chi\varepsilon)$. For the remaining derivatives we obtain the estimates similarly as in Theorem 3.4.1. ■

Next is the analogous result for the bending case.

Theorem 3.4.10. *Suppose that the assumptions on the material symmetries 3.1.1 hold. Let the forces $\boldsymbol{f}$ belong to L^2_{bend}. Let $\gamma>-2$ be the parameter of spectral scaling. Let $\delta\geq 0$ be the parameter of force scaling. Then we have the following estimates:*

$$\left\|P_i\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon+I\right)^{-1}S_{\varepsilon^\delta}\boldsymbol{f}-\pi_i\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}}+I\right)^{-1}\mathcal{M}^{\text{bend}}_\varepsilon\Xi_\varepsilon S_{\varepsilon^\delta}\boldsymbol{f}-P_i\mathcal{A}^{\text{corr}}_{\text{bend}}(\varepsilon)S_{\varepsilon^\delta}\boldsymbol{f}\right\|_{H^1(\omega\times\mathbb{R})}$$

$$\leq C\max\left\{\varepsilon^{\frac{\gamma+2}{4}},\varepsilon^{\frac{\gamma}{2}}\right\}\max\left\{\varepsilon^{\frac{\gamma+2}{4}-\delta},1\right\}\|\boldsymbol{f}\|_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)},\quad i=1,2,$$

$$\left\|P_3\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon+I\right)^{-1}S_{\varepsilon^\delta}\boldsymbol{f}+\varepsilon\begin{bmatrix}x_1\\x_2\end{bmatrix}^T\cdot\frac{d}{dx_3}\left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}}+I\right)^{-1}\mathcal{M}^{\text{bend}}_\varepsilon\Xi_\varepsilon S_{\varepsilon^\delta}\boldsymbol{f}-P_3\mathcal{A}^{\text{corr}}_{\text{bend}}(\varepsilon)S_{\varepsilon^\delta}\boldsymbol{f}\right\|_{H^1(\omega\times\mathbb{R})}$$

$$\leq C\max\left\{\varepsilon^{\frac{\gamma+2}{2}},\varepsilon^{\frac{3\gamma+2}{4}}\right\}\max\left\{\varepsilon^{\frac{\gamma+2}{4}-\delta},1\right\}\|\boldsymbol{f}\|_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)}.$$

Proof. We start with the second couple of estimates in (3.24) and we neglect the corrector term $\boldsymbol{u}_0^{(1)}$ in the $H^1(\omega\times Y;\mathbb{C}^3)$ norm. The argument for this is again that from (3.23) we have for $\alpha=1,2$:

$$\|(\boldsymbol{u}_0^{(1)})_i\|_{L^2(\omega\times Y)}\leq C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},\qquad \|(\boldsymbol{u}_0^{(1)})_3\|_{L^2(\omega\times Y)}\leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},$$
$$\|\partial_{x_\alpha}(\boldsymbol{u}_0^{(1)})_i\|_{L^2(\omega\times Y)}\leq C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},\quad \|\partial_{x_i}(\boldsymbol{u}_0^{(1)})_3\|_{L^2(\omega\times Y)}\leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},$$
$$\|\partial_y(\boldsymbol{u}_0^{(1)})_i\|_{L^2(\omega\times Y)}=0,\quad \|\partial_y(\boldsymbol{u}_0^{(1)})_3\|_{L^2(\omega\times Y)}=0.$$

Again we focus only on the norm of the derivative with respect to y. Thus we consider the estimates

$$\left\|\partial_y\left(\begin{bmatrix}\boldsymbol{u}_1\\\boldsymbol{u}_2\end{bmatrix}-\begin{bmatrix}m_1\\m_2\end{bmatrix}-\begin{bmatrix}(\boldsymbol{u}_1)_1\\(\boldsymbol{u}_1)_2\end{bmatrix}\right)\right\|_{L^2(\omega\times Y,\mathbb{C}^2)}\leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},$$

$$\left\|\partial_y\left(\boldsymbol{u}_3+i\chi(m_1x_1+m_2x_2)-(\boldsymbol{u}_1)_3\right)\right\|_{L^2(\omega\times Y)}\leq C|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

In other words, we have that the following estimates hold for $i=1,2$:

$$\left\|\partial_y\left(P_i\left(\frac{1}{|\chi|^4}\mathcal{A}_\chi+I\right)^{-1}-\pi_i\left(\frac{1}{|\chi|^4}\mathbb{C}_\chi^{\text{bend}}+I\right)^{-1}\mathcal{M}_\chi^{\text{bend}}-P_i\mathcal{A}_{\chi,\text{corr}}^{\text{bend}}\right)S_{|\chi|}\boldsymbol{f}\right\|_{L^2(\omega\times Y)}$$
$$\leq|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},$$

$$\left\|\partial_y\left(P_3\left(\frac{1}{|\chi|^4}\mathcal{A}_\chi+I\right)^{-1}+i\chi\begin{bmatrix}x_1\\x_2\end{bmatrix}^T\cdot\left(\frac{1}{|\chi|^4}\mathbb{C}_\chi^{\text{bend}}+I\right)^{-1}\mathcal{M}_\chi^{\text{bend}}-P_3\mathcal{A}_{\chi,\text{corr}}^{\text{bend}}\right)S_{|\chi|}\boldsymbol{f}\right\|_{L^2(\omega\times Y)}$$
$$\leq|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

By defining the function $f_{\varepsilon,\chi}(z)$ as in (3.31), we can provide norm resolvent estimates for the operators

$$\partial_y\left(P_i\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi+I\right)^{-1}-\pi_i\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\text{bend}}+I\right)^{-1}\mathcal{M}_\chi^{\text{bend}}-P_i\mathcal{A}_{\chi,\varepsilon,\text{corr}}^{\text{bend}}\right),\quad i=1,2,$$

$$\partial_y\left(P_3\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi+I\right)^{-1}+i\chi\begin{bmatrix}x_1\\x_2\end{bmatrix}^T\cdot\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\text{bend}}+I\right)^{-1}\mathcal{M}_\chi^{\text{bend}}-P_3\mathcal{A}_{\chi,\varepsilon,\text{corr}}^{\text{bend}}\right).$$

Indeed we have that for $i=1,2$:

$$\left\|\partial_y\left(P_i\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi+I\right)^{-1}-\pi_i\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\text{bend}}+I\right)^{-1}\mathcal{M}_\chi^{\text{bend}}-P_i\mathcal{A}_{\chi,\varepsilon,\text{corr}}^{\text{bend}}\right)S_{\varepsilon^\delta}\boldsymbol{f}\right\|_{L^2(\omega\times Y)}$$
$$\leq\frac{1}{2\pi}\oint_\Gamma|f_{\varepsilon,\chi}(z)|\left\|\partial_y\left(P_i\left(zI-\frac{1}{|\chi|^4}\mathcal{A}_\chi\right)^{-1}-\pi_i\left(zI-\frac{1}{|\chi|^4}\mathbb{C}_\chi^{\text{bend}}\right)^{-1}\mathcal{M}_\chi^{\text{bend}}\right.\right.$$
$$\left.\left.-P_i\mathcal{B}_{1,\text{bend}}^{\chi,\text{corr}}\left(zI-\frac{1}{|\chi|^4}\mathbb{C}_\chi^{\text{bend}}\right)^{-1}\mathcal{M}_\chi^{\text{bend}}\right)S_{|\chi|}S_{\varepsilon^\delta/|\chi|}\boldsymbol{f}\right\|_{L^2(\omega\times Y)}dz$$
$$\leq C|\chi|^2\left(\max\left\{\frac{|\chi|^4}{\varepsilon^{\gamma+2}},1\right\}\right)^{-1}\max\left\{|\chi|\varepsilon^{-\delta},1\right\}\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}$$
$$\leq C\varepsilon^{\frac{\gamma+2}{2}}\max\left\{\varepsilon^{\frac{\gamma+2}{4}-\delta},1\right\}\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

Analogously for the third component we have

$$\left\| \partial_y \left(P_3 \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} - i\chi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \cdot \left(\frac{1}{\varepsilon^{\gamma+2}} \mathbb{C}_\chi^{\text{bend}} + I \right)^{-1} \mathcal{M}_\chi^{\text{bend}} - P_3 \mathcal{A}_{\chi,\varepsilon,\text{corr}}^{\text{bend}} \right) S_\varepsilon \boldsymbol{f} \right\|_{L^2(\omega\times Y)}$$

$$\leq \frac{1}{2\pi} \oint_\Gamma |f_{\varepsilon,\chi}(z)| \left\| \partial_y \Big(P_3 \left(zI - \frac{1}{|\chi|^4} \mathcal{A}_\chi \right)^{-1} - i\chi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \cdot \left(zI - \frac{1}{|\chi|^4} \mathbb{C}_\chi^{\text{bend}} \right)^{-1} \mathcal{M}_\chi^{\text{bend}} \right.$$

$$\left. - P_3 \mathcal{B}_{1,\text{bend}}^{\chi,\text{corr}} \left(zI - \frac{1}{|\chi|^4} \mathbb{C}_\chi^{\text{bend}} \right)^{-1} \mathcal{M}_\chi^{\text{bend}} \Big) S_{|\chi|} S_{\varepsilon^\delta/|\chi|} \boldsymbol{f} \right\|_{L^2(\omega\times Y)} dz$$

$$\leq C|\chi|^3 \left(\max\left\{ \frac{|\chi|^4}{\varepsilon^{\gamma+2}}, 1 \right\} \right)^{-1} \max\left\{ |\chi|\varepsilon^{-\delta}, 1 \right\} \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}$$

$$\leq C\varepsilon^{\frac{3(\gamma+2)}{4}} \max\left\{ \varepsilon^{\frac{\gamma+2}{4}-\delta}, 1 \right\} \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)} \quad (\text{when } |\chi|^4 \approx \varepsilon^{\gamma+2}).$$

Now by passing back to the real domain we obtain:

$$\left\| \partial_{x_3} \left(P_i \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} - \pi_i \left(\frac{1}{\varepsilon^{\gamma-2}} \mathcal{A}^{\text{bend}} + I \right)^{-1} \mathcal{M}_\varepsilon^{\text{bend}} \Xi_\varepsilon - P_i \mathcal{A}_{\text{corr}}^{\text{bend}}(\varepsilon) \right) S_{\varepsilon^\delta} \boldsymbol{f} \right\|_{L^2(\omega\times\mathbb{R})}$$

$$\leq C\varepsilon^{\frac{\gamma}{2}} \max\left\{ \varepsilon^{\frac{\gamma+2}{4}-\delta}, 1 \right\} \|\boldsymbol{f}\|_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)}.$$

Analogously:

$$\left\| \partial_{x_3} \left(P_3 \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \cdot \varepsilon \frac{d}{dx_3} \left(\frac{1}{\varepsilon^{\gamma-2}} \mathcal{A}^{\text{bend}} + I \right)^{-1} \mathcal{M}_\varepsilon^{\text{bend}} \Xi_\varepsilon - P_3 \mathcal{A}_{\text{corr}}^{\text{bend}}(\varepsilon) \right) S_{\varepsilon^\delta} \boldsymbol{f} \right\|_{L^2(\omega\times\mathbb{R})}$$

$$\leq C\varepsilon^{\frac{3\gamma+2}{4}} \max\left\{ \varepsilon^{\frac{\gamma+2}{4}-\delta}, 1 \right\} \|\boldsymbol{f}\|_{L^2(\omega\times\mathbb{R};\mathbb{R}^3)}.$$

The original statement follows using (3.32), (3.33) and the definition of the corrector operator. ■

3.4.3. Higher order $L^2 \to L^2$ norm resolvent estimates

We define the leading order term corrector operators as follows:

$$\widetilde{\mathcal{A}}_{\chi,\text{corr}}^{\text{stretch}} : L^2(\omega\times Y;\mathbb{C}^3) \to L^2(\omega\times Y;\mathbb{C}^3), \quad \widetilde{\mathcal{A}}_{\chi,\text{corr}}^{\text{bend}} : L^2(\omega\times Y;\mathbb{C}^3) \to L^2(\omega\times Y;\mathbb{C}^3)$$

such that according to the asymptotic procedure in the last section (equations (3.17) and (3.23)) we have:

$$\widetilde{\mathcal{A}}^{\mathrm{stretch}}_{\chi,\mathrm{corr}}\boldsymbol{f} := (\boldsymbol{u}_0^{(1)})_{\mathrm{stretch}}, \quad \widetilde{\mathcal{A}}^{\mathrm{bend}}_{\chi,\mathrm{corr}}\boldsymbol{f}_\chi := (\boldsymbol{u}_0^{(1)})_{\mathrm{bend}}.$$

We reflect on the asymptotic procedure from the last section and consider now the resolvent problems depending on the spectral parameter $z \in \mathbb{C}$. Our aim is to vaguely express the operators $\widetilde{\mathcal{A}}^{\mathrm{stretch}}_{\chi,\mathrm{corr}}(z)$ and $\widetilde{\mathcal{A}}^{\mathrm{bend}}_{\chi,\mathrm{corr}}(z)$ in a closed form, where $z \in \mathbb{C}$ is a spectral parameter. To this end, we focus first on the stretching case.

Notice that, due to the structure and linearity of the equation (3.16), one can express the corrector term $\boldsymbol{u}_2(z)$ as

$$\boldsymbol{u}_2(z) = \widehat{\mathcal{B}}_\chi\left(zC^{\mathrm{stretch}} - \frac{1}{|\chi|^2}\mathbb{C}^{\mathrm{stretch}}_\chi\right)^{-1}\mathcal{M}^{\mathrm{stretch}}_\chi\boldsymbol{f} + \mathcal{B}^{\chi,\mathrm{corr}}_{1,\mathrm{stretch}}P_{H^\perp_{\mathrm{stretch}}}\boldsymbol{f},$$

where $\hat{\mathcal{B}}_\chi$ is a bounded linear operator which can be defined through (3.16). Furthermore, due to (3.17), we have:

$$\begin{aligned}\boldsymbol{u}_0^{(1)}(z) = &\left(\mathcal{M}^{\mathrm{stretch}}_\chi\right)^*\left(zC^{\mathrm{stretch}} - \frac{1}{|\chi|^2}\mathbb{C}^{\mathrm{stretch}}_\chi\right)^{-1}\breve{\mathcal{B}}_\chi\left(zC^{\mathrm{stretch}} - \frac{1}{|\chi|^2}\mathbb{C}^{\mathrm{stretch}}_\chi\right)^{-1}\mathcal{M}^{\mathrm{stretch}}_\chi\boldsymbol{f}\\ &+\left(\mathcal{M}^{\mathrm{stretch}}_\chi\right)^*\left(zC^{\mathrm{stretch}} - \frac{1}{|\chi|^2}\mathbb{C}^{\mathrm{stretch}}_\chi\right)^{-1}\widetilde{\mathcal{B}}_\chi\boldsymbol{f},\end{aligned}$$

where the bounded operators $\widetilde{\mathcal{B}}_\chi$, $\breve{\mathcal{B}}_\chi$ are introduced via (3.17). Equivalently:

$$\begin{aligned}\widetilde{\mathcal{A}}^{\mathrm{stretch}}_{\chi,\mathrm{corr}}(z) = &\left(\mathcal{M}^{\mathrm{stretch}}_\chi\right)^*\left(zC^{\mathrm{stretch}} - \frac{1}{|\chi|^2}\mathbb{C}^{\mathrm{stretch}}_\chi\right)^{-1}\breve{\mathcal{B}}_\chi\left(zC^{\mathrm{stretch}} - \frac{1}{|\chi|^2}\mathbb{C}^{\mathrm{stretch}}_\chi\right)^{-1}\mathcal{M}^{\mathrm{stretch}}_\chi\\ &+\left(\mathcal{M}^{\mathrm{stretch}}_\chi\right)^*\left(zC^{\mathrm{stretch}} - \frac{1}{|\chi|^2}\mathbb{C}^{\mathrm{stretch}}_\chi\right)^{-1}\widetilde{\mathcal{B}}_\chi.\end{aligned}$$

The same structure is valid for the operator $\widetilde{\mathcal{A}}^{\mathrm{bend}}_{\chi,\mathrm{corr}}(z)$ as well. Next, since we are dealing with a finite dimensional spaces, it is clear that we have the following matrix structure:

$$\left(zC^{\mathrm{stretch}} - \frac{1}{|\chi|^2}\mathbb{C}^{\mathrm{stretch}}_\chi\right)^{-1}\breve{\mathcal{B}}_\chi\left(zC^{\mathrm{stretch}} - \frac{1}{|\chi|^2}\mathbb{C}^{\mathrm{stretch}}_\chi\right)^{-1} = \mathbb{B}_\chi\begin{bmatrix}\mathcal{B}^\chi_{1,1}(z) & \mathcal{B}^\chi_{1,2}(z)\\ \mathcal{B}^\chi_{2,1}(z) & \mathcal{B}^\chi_{2,2}(z)\end{bmatrix}\mathbb{D}_\chi,$$

where the coordinate functions depend on the spectral parameter $z \in \mathbb{C}$ in the following way:

$$\mathcal{B}^\chi_{i,j}(z) = \frac{a^\chi_{i,j}}{(z - \frac{1}{|\chi|^2}\lambda^\chi_1)^2} + \frac{b^\chi_{i,j}}{(z - \frac{1}{|\chi|^2}\lambda^\chi_1)(z - \frac{1}{|\chi|^2}\lambda^\chi_2)} + \frac{c^\chi_{i,j}}{(z - \frac{1}{|\chi|^2}\lambda^\chi_2)^2},$$

where λ_1^χ and λ_2^χ are the eigenvalues of the matrix $\mathbb{C}_\chi^{\rm stretch}$. Recall that those two eigenvalues, when scaled with $1/|\chi|^2$ are positioned in a fixed interval uniformly in χ.

Lemma 3.4.11. *Let Γ be a closed contour enclosing both eigenvalues λ_1^χ, λ_2^χ of the matrix $\mathbb{C}_\chi^{\rm stretch}$ uniformly in χ. Then we have the following:*

$$\frac{1}{2\pi i}\oint_\Gamma \frac{g_{\varepsilon,\chi}(z)}{(z-\frac{1}{|\chi|^2}\lambda_i^\chi)^2}dz = -\frac{|\chi|^2}{\varepsilon^{\gamma+2}}\frac{1}{(\frac{1}{\varepsilon^{\gamma+2}}\lambda_i^\chi+1)^2}, \quad i=1,2,$$
$$\frac{1}{2\pi i}\oint_\Gamma \frac{g_{\varepsilon,\chi}(z)}{(z-\frac{1}{|\chi|^2}\lambda_1^\chi)(z-\frac{1}{|\chi|^2}\lambda_2^\chi)}dz = -\frac{|\chi|^2}{\varepsilon^{\gamma+2}}\frac{1}{(\frac{1}{\varepsilon^{\gamma+2}}\lambda_1^\chi+1)(\frac{1}{\varepsilon^{\gamma+2}}\lambda_2^\chi+1)},$$
$$\frac{1}{2\pi i}\oint_\Gamma \frac{f_{\varepsilon,\chi}(z)}{(z-\frac{1}{|\chi|^4}\lambda_i^\chi)^4}dz = -\frac{|\chi|^4}{\varepsilon^{\gamma+2}}\frac{1}{(\frac{1}{\varepsilon^{\gamma+2}}\lambda_i^\chi+1)^2}, \quad i=1,2,$$
$$\frac{1}{2\pi i}\oint_\Gamma \frac{f_{\varepsilon,\chi}(z)}{(z-\frac{1}{|\chi|^4}\lambda_1^\chi)(z-\frac{1}{|\chi|^4}\lambda_2^\chi)}dz = -\frac{|\chi|^4}{\varepsilon^{\gamma+2}}\frac{1}{(\frac{1}{\varepsilon^{\gamma+2}}\lambda_1^\chi+1)(\frac{1}{\varepsilon^{\gamma+2}}\lambda_2^\chi+1)},$$

Similar is true for the matrix $\mathbb{C}_\chi^{\rm bend}$ if we replace $g_{\varepsilon,\chi}$ with $f_{\varepsilon,\chi}$.

The previous Lemma allows us to conclude the following structure:

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_\Gamma g_{\varepsilon,\chi}(z)\widetilde{\mathcal{A}}^{\rm stretch}_{\chi,\rm corr}(z)dz \\
&= -\frac{|\chi|^2}{\varepsilon^{\gamma+2}}\left(\mathcal{M}_\chi^{\rm stretch}\right)^*\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\rm stretch}+C^{\rm stretch}\right)^{-1}\breve{\mathcal{B}}_\chi\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\rm stretch}+C^{\rm stretch}\right)^{-1}\mathcal{M}_\chi^{\rm stretch} \\
&+\left(\mathcal{M}_\chi^{\rm stretch}\right)^*\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\rm stretch}+C^{\rm stretch}\right)^{-1}\widetilde{\mathcal{B}}_\chi =: \widetilde{\mathcal{A}}^{\rm stretch}_{\chi,\varepsilon,\rm corr}, \\
&\frac{1}{2\pi i}\oint_\Gamma f_{\varepsilon,\chi}(z)\widetilde{\mathcal{A}}^{\rm bend}_{\chi,\rm corr}(z)dz \\
&= \frac{|\chi|^4}{\varepsilon^{\gamma+2}}\left(\mathcal{M}_\chi^{\rm bend}\right)^*\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\rm bend}+I\right)^{-1}\breve{\mathcal{B}}_\chi\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\rm bend}+I\right)^{-1}\mathcal{M}_\chi^{\rm bend} \\
&+\left(\mathcal{M}_\chi^{\rm bend}\right)^*\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\rm bend}+I\right)^{-1}\widetilde{\mathcal{B}}_\chi =: \widetilde{\mathcal{A}}^{\rm bend}_{\chi,\varepsilon,\rm corr},
\end{aligned}$$

We denote the Gelfand pullback of these operators:

$$\widetilde{\mathcal{A}}^{\rm corr}_{\rm bend}(\varepsilon) = \mathcal{G}_\varepsilon^{-1}\widetilde{\mathcal{A}}^{\rm bend}_{\chi,\varepsilon,\rm corr}\mathcal{G}_\varepsilon, \quad \widetilde{\mathcal{A}}^{\rm corr}_{\rm stretch}(\varepsilon) = \mathcal{G}_\varepsilon^{-1}\widetilde{\mathcal{A}}^{\rm stretch}_{\chi,\varepsilon,\rm corr}\mathcal{G}_\varepsilon. \tag{3.39}$$

Remark 3.4.12. Using the above expressions as well as (3.16), (3.17) and (3.23) we can conclude that there exist operators $K_1, K_2 : L^2(\mathbb{R};\mathbb{R}^n) \to L^2(\mathbb{R};\mathbb{R}^n)$ such that

$$
\begin{aligned}
&\widetilde{\mathcal{A}}^{\text{corr}}_{\text{stretch}}(\varepsilon)\boldsymbol{f} \\
&= \varepsilon^{1-\gamma}(\mathcal{M}^{\text{stretch}})^* \left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}} + C^{\text{stretch}}\right)^{-1} \mathbf{A}_1 \frac{d^3}{dx_3^3} \left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}} + C^{\text{stretch}}\right)^{-1} \mathcal{M}^{\text{stretch}} \Xi_\varepsilon \boldsymbol{f} \\
&\qquad + \varepsilon^{1-\gamma}(\mathcal{M}^{\text{stretch}})^* \left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}^{\text{stretch}} + C^{\text{stretch}}\right)^{-1} \mathbf{A}_2 \frac{d^3}{dx_3^3} \Xi_\varepsilon K_1 \boldsymbol{f}, \\
&\widetilde{\mathcal{A}}^{\text{corr}}_{\text{bend}}(\varepsilon)\boldsymbol{f} = \varepsilon^{3-\gamma}(\mathcal{M}^{\text{bend}}_\varepsilon)^* \left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}} + I\right)^{-1} \mathbf{A}_3 \frac{d^5}{dx_3^5} \left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}} + I\right)^{-1} \mathcal{M}^{\text{bend}}_\varepsilon \Xi_\varepsilon \boldsymbol{f} \\
&\qquad + \varepsilon^{3-\gamma}(\mathcal{M}^{\text{bend}})^* \left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}} + I\right)^{-1} \mathbf{A}_4 \frac{d^5}{dx_3^5} \Xi_\varepsilon K_2 \boldsymbol{f},
\end{aligned}
$$

where $\mathbf{A}_i \in \mathbb{R}^{2\times 2}$, for $i = 1,\dots,4$ are diagonal matrices.

Much in the similar fashion as before we are able to prove the following result:

Theorem 3.4.13 (Higher order $L^2 \to L^2$ norm-resolvent estimate)**.** *Suppose that the assumptions on the material symmetries 3.1.1 hold. Let $\gamma > -2$ be the parameter of spectral scaling. Let $\delta \geq 0$ be the parameter of force scaling. Then there exists $C > 0$ such that for every $\varepsilon > 0$ we have:*

$$
\left\| \left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon + I\right)^{-1} |_{L^2_{\text{stretch}}} - (\mathcal{M}^{\text{stretch}})^* \left(\frac{1}{\varepsilon^\gamma}\mathcal{A}^{\text{stretch}} + C^{\text{stretch}}\right)^{-1} \mathcal{M}^{\text{stretch}} \Xi_\varepsilon - \mathcal{A}^{\text{corr}}_{\text{stretch}}(\varepsilon) - \widetilde{\mathcal{A}}^{\text{corr}}_{\text{stretch}}(\varepsilon) \right\|_{L^2\to L^2} \leq C\varepsilon^{\gamma+2},
$$

$$
\begin{aligned}
&\left\| P_i \left(\left(\frac{1}{\varepsilon^\gamma}\mathcal{A}_\varepsilon + I\right)^{-1} |_{L^2_{\text{bend}}} - (\mathcal{M}^{\text{bend}}_\varepsilon)^* \left(\frac{1}{\varepsilon^{\gamma-2}}\mathcal{A}^{\text{bend}} + I\right)^{-1} \mathcal{M}^{\text{bend}}_\varepsilon \Xi_\varepsilon - \mathcal{A}^{\text{corr}}_{\text{bend}}(\varepsilon) - \widetilde{\mathcal{A}}^{\text{corr}}_{\text{bend}}(\varepsilon) \right) S_{\varepsilon^\delta} \right\|_{L^2\to L^2} \\
&\qquad \leq \begin{cases} C\varepsilon^{\frac{\gamma+2}{2}} \max\left\{\varepsilon^{\frac{\gamma+2}{4}-\delta}, 1\right\}, & i = 1,2; \\ C\varepsilon^{\frac{3(\gamma+2)}{4}} \max\left\{\varepsilon^{\frac{\gamma+2}{4}-\delta}, 1\right\}, & i = 3. \end{cases}
\end{aligned}
$$

3.5. The analysis of the general elasticity tensor

In this section, we drop the assumptions on the material symmetries 3.1.1. Separately, we investigate and develop asymptotics for the solution of two resolvent problems with different scalings, one for each of the orders of magnitudes of the operator eigenspaces.

3.5.1. The asymptotics of $|\chi|^2$ resolvent problem

We begin with the asymptotics for the following resolvent problem:
Find $\boldsymbol{u} \in H^1_\#(Y; H^1(\omega;\mathbb{C}^3))$ such that

$$\frac{1}{|\chi|^2}\int_{\omega\times Y}\mathbb{C}(\operatorname{sym}\nabla\boldsymbol{u}+iX_\chi\boldsymbol{u}):\overline{(\operatorname{sym}\nabla\boldsymbol{v}+iX_\chi\boldsymbol{v})}+\int_{\omega\times Y}\boldsymbol{u}\overline{\boldsymbol{v}}=\int_{\omega\times Y}\boldsymbol{f}\overline{\boldsymbol{v}},\quad \forall\boldsymbol{v}\in H^1_\#(Y;H^1(\omega;\mathbb{C}^3)),\tag{3.40}$$

or, equivalently:

$$\frac{1}{|\chi|^2}\left(\operatorname{sym}\nabla^*+\left(iX_\chi\right)^*\right)\mathbb{C}(y)\left(\operatorname{sym}\nabla+iX_\chi\right)\boldsymbol{u}+\boldsymbol{u}=\boldsymbol{f}.$$

1) The first approximation cycle

Consider the solution $m\in\mathbb{C}^4$ to the following equation:

$$\left(\frac{1}{|\chi|^2}\mathbb{C}^{\mathrm{rod}}_\chi+C^{\mathrm{rod}}_\chi(\omega)\right)m=\mathcal{M}^{\mathrm{rod}}_\chi\boldsymbol{f}.\tag{3.41}$$

The solution satisfies the estimate:

$$|m|\le C\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

The following estimate is crucial for the continuation of our procedure since it allows us to disregard this term from the definition of the corrector term $\boldsymbol{u}_2$, thus obtaining a well posed problem. This term will, however, need to be canceled at some point, and it indeed will, with the definition of the corrector term $\boldsymbol{u}_2^{(1)}$.

Lemma 3.5.1. *The solution $m\in\mathbb{C}^4$ of (3.41) satisfies:*

$$\left|\begin{bmatrix}m_1\\ m_2\\ 0\end{bmatrix}-\begin{bmatrix}\int_{\omega\times Y}\boldsymbol{f}_1\\ \int_{\omega\times Y}\boldsymbol{f}_2\\ 0\end{bmatrix}\right|\le C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.\tag{3.42}$$

Proof. The estimate is proven by testing the equation (3.41) with a constant vector $d := \left(m_1 - \int_{\omega\times Y} \boldsymbol{f}_1, m_2 - \int_{\omega\times Y} \boldsymbol{f}_2, 0\right)^T$, to obtain:

$$\left(m_1 - \int_{\omega\times Y} \boldsymbol{f}_1\right)^2 + \left(m_1 - \int_{\omega\times Y} \boldsymbol{f}_2\right)^2 = -\chi^2 \left(c_1(\omega) m_1 \left(m_1 - \int_{\omega\times Y} \boldsymbol{f}_1\right) + c_2(\omega) m_2 \left(m_2 - \int_{\omega\times Y} \boldsymbol{f}_2\right)\right)$$
$$+ \left(i\chi \int_{\omega\times Y} \boldsymbol{f}_3 x_1\right)\left(m_1 - \int_{\omega\times Y} \boldsymbol{f}_1\right) + \left(i\chi \int_{\omega\times Y} \boldsymbol{f}_3 x_2\right)\left(m_2 - \int_{\omega\times Y} \boldsymbol{f}_2\right) - \mathbb{C}_\chi^{\mathrm{rod}} m \cdot \overline{d},$$

and noticing that

$$\mathbb{C}_\chi^{\mathrm{rod}} m \cdot \overline{d} \le |\chi|^3 |m| \sqrt{\left(m_1 - \int_{\omega\times Y} \boldsymbol{f}_1\right)^2 + \left(m_1 - \int_{\omega\times Y} \boldsymbol{f}_2\right)^2}.$$

■

The leading order term

$$\boldsymbol{u}_0 := \mathcal{I}^{\mathrm{rod}}(m) \in H^1_\#(Y; H^1(\omega; \mathbb{C}^3))$$

clearly satisfies:

$$\|\boldsymbol{u}_0\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

The correctors are defined with the following set of equations:

$$\operatorname{sym}\nabla^* \mathbb{C}(y) \operatorname{sym}\nabla \boldsymbol{u}_1 = -\operatorname{sym}\nabla^* \mathbb{C}(y) \Lambda^{\mathrm{rod}}_{\chi,m}, \quad \boldsymbol{u}_1 \in H. \tag{3.43}$$

$$\frac{1}{|\chi|^2} \operatorname{sym}\nabla^* \mathbb{C}(y) \operatorname{sym}\nabla \boldsymbol{u}_2 = -\frac{1}{|\chi|^2}\left(\left(iX_\chi\right)^* \mathbb{C}(y) \operatorname{sym}\nabla \boldsymbol{u}_1 + \operatorname{sym}\nabla^* \mathbb{C}(y) iX_\chi \boldsymbol{u}_1 + \left(iX_\chi\right)^* \mathbb{C}(y) \Lambda^{\mathrm{rod}}_{\chi,m}(x)\right)$$
$$- \begin{bmatrix} m_3 x_2 \\ -m_3 x_1 \\ m_4 - i\chi(m_1 x_1 + m_2 x_2) \end{bmatrix} + \begin{bmatrix} \boldsymbol{f}_1 - \int_{\omega\times Y} \boldsymbol{f}_1 dxdy \\ \boldsymbol{f}_2 - \int_{\omega\times Y} \boldsymbol{f}_2 dxdy \\ \boldsymbol{f}_3 \end{bmatrix}, \quad \boldsymbol{u}_2 \in H. \tag{3.44}$$

It is easy to check that the right hand side of (3.43) is orthogonal to Y-periodic rigid motions, making this a well posed problem. The orthogonality with respect to $(C_1, C_2, 0)^T$ is due to (3.43), while the orthogonality with respect to $(C_3 x_2, -C_3 x_1, C_4)^T$ is due to (3.43) and (3.41). We have the following estimates:

$$\|\boldsymbol{u}_1\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, \quad \|\boldsymbol{u}_2\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},.$$

2) The second approximation cycle

Next, we update the leading order term with $m^{(1)} \in \mathbb{C}^4$ satisfying:

$$\left(\frac{1}{|\chi|^2}\mathbb{C}_\chi^{\mathrm{rod}} + C_\chi^{\mathrm{rod}}(\omega)\right) m^{(1)} \cdot \overline{d^T} = -\frac{1}{|\chi|^2}\int_{\omega\times Y} \mathbb{C}(y)\left(\operatorname{sym}\nabla \boldsymbol{u}_2 + iX_\chi \boldsymbol{u}_1\right) : \overline{\Lambda_{\chi,d}^{\mathrm{rod}}(x)} dxdy, \quad \forall d \in \mathbb{C}^4.$$

By testing with $m^{(1)}$ and by using the ellipticity estimates of $\mathbb{C}_\chi^{\mathrm{rod}}$, as well as estimates on the right hand side, we see that:

$$|m^{(1)}| \le C|\chi| \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

Additionally, by setting the stretching components d_3, d_4 to 0, we obtain a sharper estimate:

$$|m_1^{(1)}, m_2^{(1)}| \le C|\chi|^2 \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

Nevertheless, for $\boldsymbol{u}_0^{(1)} := \mathcal{I}^{\mathrm{rod}} m^{(1)}$ we have:

$$\left\|\boldsymbol{u}_0^{(1)}\right\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C|\chi| \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

Naturally, the next corrector is defined with:

$$\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla \boldsymbol{u}_1^{(1)} = -\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda_{\chi,m^{(1)}}^{\mathrm{rod}}(x), \quad \boldsymbol{u}_1^{(1)} \in H. \tag{3.45}$$

It satisfies

$$\|\boldsymbol{u}_1^{(1)}\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C|\chi|^2 \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}$$

The next corrector is defined as to eliminate the remaining terms of order $|\chi|$.

$$\begin{aligned}
\frac{1}{|\chi|^2}\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla \boldsymbol{u}_2^{(1)} = -\frac{1}{|\chi|^2}&\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla(\boldsymbol{u}_2 + \boldsymbol{u}_1^{(1)}) + \operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi(\boldsymbol{u}_2 + \boldsymbol{u}_1^{(1)})\right) \\
&-\frac{1}{|\chi|^2}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\Lambda_{\chi,m^{(1)}}^{\mathrm{rod}}(x) + \left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi \boldsymbol{u}_1\right) \\
-\begin{bmatrix} m_3^{(1)}x_2 \\ -m_3^{(1)}x_1 \\ m_4^{(1)} - i\chi(m_1^{(1)}x_1 + m_2^{(1)}x_2) \end{bmatrix} - \begin{bmatrix} m_1 \\ m_2 \\ 0 \end{bmatrix} &+ \begin{bmatrix} \int_{\omega\times Y} \boldsymbol{f}_1 dxdy \\ \int_{\omega\times Y} \boldsymbol{f}_2 dxdy \\ 0 \end{bmatrix} - \boldsymbol{u}_1, \quad \boldsymbol{u}_2^{(1)} \in H.
\end{aligned} \tag{3.46}$$

Obviously, due to (3.45), the right hand side of (3.46) is orthogonal to stretching rigid motions. In order to verify that it is orthogonal to $(C_1, C_2, 0)^T$, $C_1, C_2 \in \mathbb{C}$, we make the following calculation:

$$
\begin{aligned}
&-\frac{1}{|\chi|^2}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla(\boldsymbol{u}_2+\boldsymbol{u}_1^{(1)})+\Lambda^{\mathrm{rod}}_{\chi,m^{(1)}}(x)+iX_\chi\boldsymbol{u}_1\right):iX_\chi\overline{\begin{bmatrix}C_1\\C_2\\0\end{bmatrix}}\\
=&\frac{1}{|\chi|^2}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla(\boldsymbol{u}_2+\boldsymbol{u}_1^{(1)})+\Lambda^{\mathrm{rod}}_{\chi,m^{(1)}}(x)+iX_\chi\boldsymbol{u}_1\right):\overline{\operatorname{sym}\nabla\begin{bmatrix}0\\0\\-i\chi(C_1x_1+C_2x_2)\end{bmatrix}}\\
=&\frac{1}{|\chi|^2}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_2+iX_\chi\boldsymbol{u}_1\right):\overline{\operatorname{sym}\nabla\begin{bmatrix}0\\0\\-i\chi(C_1x_1+C_2x_2)\end{bmatrix}}\qquad\text{(due to (3.45))}\\
=&-\frac{1}{|\chi|^2}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_1+\Lambda^{\mathrm{rod}}_{\chi,m}(x)\right):\overline{iX_\chi\begin{bmatrix}0\\0\\-i\chi(C_1x_1+C_2x_2)\end{bmatrix}}\\
&-\int_{\omega\times Y}\begin{bmatrix}0\\0\\-i\chi(m_1x_1+m_2x_2)\end{bmatrix}\overline{\begin{bmatrix}0\\0\\-i\chi(C_1x_1+C_2x_2)\end{bmatrix}}+\int_{\omega\times Y}\begin{bmatrix}0\\0\\\boldsymbol{f}_3\end{bmatrix}\overline{\begin{bmatrix}0\\0\\-i\chi(C_1x_1+C_2x_2)\end{bmatrix}}\\
&\text{(due to (3.44))}\\
=&-\frac{1}{|\chi|^2}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_1+\Lambda^{\mathrm{rod}}_{\chi,m}(x)\right):\overline{\Lambda^{\mathrm{bend}}_{\chi,C_1,C_2}(x)}\\
&-\int_{\omega\times Y}\begin{bmatrix}0\\0\\-i\chi(m_1x_1+m_2x_2)\end{bmatrix}\overline{\begin{bmatrix}0\\0\\-i\chi(C_1x_1+C_2x_2)\end{bmatrix}}+\int_{\omega\times Y}\begin{bmatrix}0\\0\\\boldsymbol{f}_3\end{bmatrix}\overline{\begin{bmatrix}0\\0\\-i\chi(C_1x_1+C_2x_2)\end{bmatrix}}\\
=&\int_{\omega\times Y}\begin{bmatrix}m_1\\m_2\\-i\chi(m_1x_1+m_2x_2)\end{bmatrix}\overline{\begin{bmatrix}C_1\\C_2\\-i\chi(C_1x_1+C_2x_2)\end{bmatrix}}-\int_{\omega\times Y}\begin{bmatrix}\boldsymbol{f}_1\\\boldsymbol{f}_2\\\boldsymbol{f}_3\end{bmatrix}\overline{\begin{bmatrix}C_1\\C_2\\-i\chi(C_1x_1+C_2x_2)\end{bmatrix}}\\
&-\int_{\omega\times Y}\begin{bmatrix}0\\0\\-i\chi(m_1x_1+m_2x_2)\end{bmatrix}\overline{\begin{bmatrix}0\\0\\-i\chi(C_1x_1+C_2x_2)\end{bmatrix}}+\int_{\omega\times Y}\begin{bmatrix}0\\0\\\boldsymbol{f}_3\end{bmatrix}\overline{\begin{bmatrix}0\\0\\-i\chi(C_1x_1+C_2x_2)\end{bmatrix}}\\
&\text{(due to (3.41), (3.43))}\\
=&\begin{bmatrix}m_1\\m_2\\0\end{bmatrix}\overline{\begin{bmatrix}C_1\\C_2\\0\end{bmatrix}}-\begin{bmatrix}\int_{\omega\times Y}\boldsymbol{f}_1\\\int_{\omega\times Y}\boldsymbol{f}_2\\0\end{bmatrix}\overline{\begin{bmatrix}C_1\\C_2\\0\end{bmatrix}}=\left(\begin{bmatrix}m_1\\m_2\\0\end{bmatrix}-\begin{bmatrix}\int_{\omega\times Y}\boldsymbol{f}_1\\\int_{\omega\times Y}\boldsymbol{f}_2\\0\end{bmatrix}\right)\overline{\begin{bmatrix}C_1\\C_2\\0\end{bmatrix}}.
\end{aligned}
$$

Thus, the problem is well posed and the solution satisfies:

$$\|\boldsymbol{u}_2^{(1)}\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^3 \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},$$

where we have used (3.42) in order to obtain this estimate.

4) The final approximation

The final approximation

$$\tilde{\boldsymbol{u}}_{\text{approx}} := \boldsymbol{u}_0 + \boldsymbol{u}_0^{(1)} + \boldsymbol{u}_1 + \boldsymbol{u}_1^{(1)} + \boldsymbol{u}_2 + \boldsymbol{u}_2^{(1)}$$

satisfies the following equation:

$$\frac{1}{|\chi|^2}\left(\operatorname{sym}\nabla^* + \left(iX_\chi\right)^*\right)\mathbb{C}(y)\left(\operatorname{sym}\nabla + iX_\chi\right)\tilde{\boldsymbol{u}}_{\text{approx}} + \tilde{\boldsymbol{u}}_{\text{approx}} - \boldsymbol{f} = \tilde{R}_\chi,$$

with the residual $\tilde{R}_\chi$ given with:

$$\begin{aligned}\tilde{R}_\chi = \frac{1}{|\chi|^2}&\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_2^{(1)} + \operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2^{(1)} + \left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2^{(1)} + \left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2\right)\\ &+\frac{1}{|\chi|^2}\left(\left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2 + \left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_1^{(1)}\right) + (m_1^{(1)}, m_2^{(1)}, 0)^T + \boldsymbol{u}_1^{(1)} + \boldsymbol{u}_2 + \boldsymbol{u}_2^{(1)}.\end{aligned}$$

We have the following estimate on the residual:

$$\|\tilde{R}_\chi\|_{H^{-1}_{\#}(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^2 \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},$$

for which the error of the approximation:

$$\boldsymbol{u}_{\text{error}} := \boldsymbol{u} - \tilde{\boldsymbol{u}}_{\text{approx}}$$

can be calculated from:

$$\frac{1}{|\chi|^2}\left(\operatorname{sym}\nabla^* + \left(iX_\chi\right)^*\right)\mathbb{C}(y)\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u}_{\text{error}} + \boldsymbol{u}_{\text{error}} = -\tilde{R}_\chi.$$

By employing the estimates from (3.18), we can deduce the estimate on the error:

$$\|\boldsymbol{u}_{\text{error}}\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^2 \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

By leaving out higher order terms, we can estimate the error in the approximation by lower order terms:

Proposition 3.5.2. *Let $\boldsymbol{u} \in H^1_\chi(Y; H^1(\omega;\mathbb{C}^3))$ be the quasiperiodic solution of problem* (3.40). *Then, the following estimates are valid:*

$$
\begin{aligned}
\left\| \boldsymbol{u} - e^{i\chi} \mathcal{I}^{\mathrm{rod}}_\chi m \right\|_{H^1(\omega\times Y,\mathbb{C}^3)} &\leq C|\chi| \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, \\
\left\| \boldsymbol{u} - \mathcal{I}^{\mathrm{rod}}_\chi m - \mathcal{I}^{\mathrm{rod}}_\chi m^{(1)} - e^{i\chi y}\boldsymbol{u}_1 \right\|_{H^1(\omega\times Y,\mathbb{C}^3)} &\leq C|\chi|^2 \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)},
\end{aligned}
\tag{3.47}
$$

where $m, m^{(1)}, \boldsymbol{u}_1$ are defined with the approximation procedure above.

3.5.2. The asymptotics of $|\chi|^4$ resolvent problem

Here we focus on deriving the asymptotics for the following resolvent problem:
Find $\boldsymbol{u} \in H^1_\#(Y; H^1(\omega;\mathbb{C}^3))$ such that

$$
\frac{1}{|\chi|^4}\int_{\omega\times Y} \mathbb{C}(\mathrm{sym}\,\nabla \boldsymbol{u} + iX_\chi \boldsymbol{u}) : \overline{(\mathrm{sym}\,\nabla \boldsymbol{v} + iX_\chi \boldsymbol{v})} + \int_{\omega\times Y} \boldsymbol{u}\overline{\boldsymbol{v}} = \int_{\omega\times Y} S_{|\chi|}\boldsymbol{f}\overline{\boldsymbol{v}}, \quad \forall \boldsymbol{v} \in H^1_\#(Y; H^1(\omega;\mathbb{C}^3)),
\tag{3.48}
$$

or, equivalently:

$$
\frac{1}{|\chi|^4}\left(\mathrm{sym}\,\nabla^* + \left(iX_\chi\right)^*\right)\mathbb{C}(y)\left(\mathrm{sym}\,\nabla + iX_\chi\right)\boldsymbol{u} + \boldsymbol{u} = S_{|\chi|}\boldsymbol{f}.
$$

1) The first approximation cycle

The leading order term in the asymptotic expansion is defined with the solution to the following homogenized equation:

$$
\left(\frac{1}{|\chi|^4}\mathbb{C}^{\mathrm{rod}}_\chi + C^{\mathrm{rod}}_\chi(\omega)\right) m = \mathcal{M}^{\mathrm{rod}}_\chi S_{|\chi|}\boldsymbol{f}.
$$

By using the apriori estimates on $\mathbb{C}^{\mathrm{rod}}_\chi$ we derive the following estimates by testing with $m \in \mathbb{C}^4$:

$$
\begin{aligned}
\frac{1}{|\chi|^4}\left(|\chi|^4|(m_1,m_2)^T|^2 + |\chi|^2|(m_3,m_4)^T|^2\right) &\leq \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}\left(|(m_1,m_2)^T| + |m_3, \frac{1}{|\chi|}m_4|\right) \\
&\leq \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}\left(|(m_1,m_2)^T| + \frac{1}{|\chi|}|(m_3,m_4)^T|\right),
\end{aligned}
$$

from where we read:

$$
|(m_1,m_2)^T| \leq \|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, \quad |(m_3,m_4)^T| \leq |\chi|\,\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.
$$

The leading order term, defined with $\boldsymbol{u}_0 := \mathcal{I}^{\mathrm{rod}}_\chi m$ satisfies the following estimates:

$$
\|(\boldsymbol{u}_0)_\alpha\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, \quad \|(\boldsymbol{u}_0)_3\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}
$$

Next two corrector terms are defined as the solutions to the following set of well posed problems:

$$\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla \boldsymbol{u}_1 = -\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda^{\mathrm{rod}}_{\chi,m}(x), \quad \boldsymbol{u}_1 \in H,$$

$$\frac{1}{|\chi|^4}\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla \boldsymbol{u}_2 = -\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla \boldsymbol{u}_1 + \operatorname{sym}\nabla^*\mathbb{C}(y) iX_\chi \boldsymbol{u}_1 + \left(iX_\chi\right)^*\mathbb{C}(y)\Lambda^{\mathrm{rod}}_{\chi,m}(x)\right)$$
$$-\begin{bmatrix} m_3 x_2 \\ -m_3 x_1 \\ m_4 - i\chi(m_1 x_1 + m_2 x_2) \end{bmatrix} + \begin{bmatrix} \boldsymbol{f}_1 - \int_{\omega\times Y}\boldsymbol{f}_1 dxdy \\ \boldsymbol{f}_2 - \int_{\omega\times Y}\boldsymbol{f}_2 dxdy \\ \frac{1}{|\chi|}\boldsymbol{f}_3 \end{bmatrix},$$

These correctors satisfy:

$$\|\boldsymbol{u}_1\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, \quad \|\boldsymbol{u}_2\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

2) The second approximation cycle

We proceed with the leading order term update: Define $\boldsymbol{u}_0^{(1)} = \mathcal{I}^{\mathrm{rod}}_\chi m^{(1)}$, where $m^{(1)}$ is the solution of the following problem:

$$\left(\frac{1}{|\chi|^4}\mathbb{C}^{\mathrm{rod}}_\chi + C^{\mathrm{rod}}_\chi(\omega)\right) m^{(1)}\cdot\overline{d} = -\frac{1}{|\chi|^4}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla \boldsymbol{u}_2 + iX_\chi \boldsymbol{u}_1\right) : \overline{\Lambda^{\mathrm{rod}}_{\chi,d}(x)}dxdy, \quad \forall d\in\mathbb{C}^4.$$

We have

$$|m_1^{(1)}, m_2^{(1)}| \leq C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)}, \quad |m_3^{(1)}, m_4^{(1)}| \leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}$$
$$\|(\boldsymbol{u}_0^{(1)})_\alpha\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, \quad \|(\boldsymbol{u}_0^{(1)})_3\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}$$

We define the next corrector $\boldsymbol{u}_1^{(1)}$ with the relation:

$$\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla \boldsymbol{u}_1^{(1)} = -\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda^{\mathrm{rod}}_{\chi,m^{(1)}}(x), \quad \boldsymbol{u}_1^{(1)} \in H.$$

This yields the estimate

$$\|\boldsymbol{u}_1^{(1)}\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

The next corrector is defined with:

$$\frac{1}{|\chi|^4}\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_2^{(1)} = -\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla(\boldsymbol{u}_2+\boldsymbol{u}_1^{(1)}) + \operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi(\boldsymbol{u}_2+\boldsymbol{u}_1^{(1)})\right)$$

$$-\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\Lambda^{\mathrm{rod}}_{\chi,m^{(1)}}(x) + \left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_1\right)$$

$$-\begin{bmatrix} m_3^{(1)}x_2 \\ -m_3^{(1)}x_1 \\ m_4^{(1)} - i\chi(m_1^{(1)}x_1 + m_2^{(1)}x_2)\end{bmatrix} + \begin{bmatrix}\int_{\omega\times Y}\boldsymbol{f}_1 dxdy \\ \int_{\omega\times Y}\boldsymbol{f}_2 dxdy \\ 0\end{bmatrix} - \begin{bmatrix} m_1 \\ m_2 \\ 0\end{bmatrix}, \quad \boldsymbol{u}_2^{(1)} \in H.$$

The solution satisfies the following:

$$\|\boldsymbol{u}_2^{(1)}\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C|\chi|^4\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

3) The third approximation cycle

The correctors $\boldsymbol{u}_0^{(2)} = \mathcal{I}^{\mathrm{rod}}_\chi m^{(2)}, \boldsymbol{u}_1^{(2)}, \boldsymbol{u}_2^{(2)} \in H$ further decrease the error of approximation. They are gradually built with the following relations:

$$\left(\frac{1}{|\chi|^4}\mathbb{C}^{\mathrm{rod}}_\chi + C^{\mathrm{rod}}_\chi(\omega)\right)m^{(2)}\cdot\overline{d} = -\frac{1}{|\chi|^4}\int_{\omega\times Y}\mathbb{C}(y)\left(\operatorname{sym}\nabla\boldsymbol{u}_2^{(1)} + iX_\chi\boldsymbol{u}_1^{(1)} + iX_\chi\boldsymbol{u}_2\right) : \overline{\Lambda^{\mathrm{rod}}_{\chi,d}(x)}dxdy,$$

$$\forall d\in\mathbb{C}^4.$$

$$\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_1^{(2)} = -\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda^{\mathrm{rod}}_{\chi,m^{(2)}}(x), \quad \boldsymbol{u}_1^{(2)} \in H.$$

$$\frac{1}{|\chi|^4}\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla\boldsymbol{u}_2^{(2)} = -\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla(\boldsymbol{u}_1^{(2)}+\boldsymbol{u}_2^{(1)}) + \operatorname{sym}\nabla^*\mathbb{C}(y)iX_\chi(\boldsymbol{u}_1^{(2)}+\boldsymbol{u}_2^{(1)})\right)$$

$$-\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\Lambda^{\mathrm{rod}}_{\chi,m^{(2)}}(x) + \left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi(\boldsymbol{u}_2+\boldsymbol{u}_1^{(1)})\right)$$

$$-\begin{bmatrix} m_1^{(1)} \\ m_2^{(1)} \\ 0\end{bmatrix} - \begin{bmatrix} m_3^{(2)}x_2 \\ -m_3^{(2)}x_1 \\ m_4^{(2)} - i\chi(m_1^{(2)}x_1 + m_2^{(2)}x_2)\end{bmatrix}, \quad \boldsymbol{u}_2^{(2)} \in H.$$

All of these problems are well posed which can be seen by reviewing the relations throughout the process, thus concluding that the right hand sides vanish when tested against functions in H. These approximations satisfy the following estimates:

$$\|(\boldsymbol{u}_0^{(2)})_\alpha\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, \quad \|(\boldsymbol{u}_0^{(2)})_3\|_{H^1(\omega\times Y;\mathbb{C}^3)} \le C|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}$$

$$||\boldsymbol{u}_1^{(2)}||_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^4||\boldsymbol{f}||_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

$$||\boldsymbol{u}_2^{(2)}||_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^5||\boldsymbol{f}||_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

4) The fourth approximation cycle

The final approximation cycle consists of defining the corrector terms $\boldsymbol{u}_0^{(3)} = \mathcal{I}_\chi^{\mathrm{rod}} m^{(3)}, \boldsymbol{u}_1^{(3)}, \boldsymbol{u}_2^{(3)} \in H$ with the following relations:

$$\left(\frac{1}{|\chi|^4}\mathbb{C}_\chi^{\mathrm{rod}} + C_\chi^{\mathrm{rod}}(\omega)\right) m^{(3)} \cdot \overline{d} = -\frac{1}{|\chi|^2}\int_{\omega\times Y} \mathbb{C}(y)\left(\operatorname{sym}\nabla \boldsymbol{u}_2^{(2)} + iX_\chi \boldsymbol{u}_1^{(2)} + iX_\chi \boldsymbol{u}_1^{(1)}\right) : \overline{\Lambda_{\chi,d}^{\mathrm{rod}}(x)} dxdy,$$
$$+\left(-i\chi\int_{\omega\times Y} x_1\boldsymbol{u}_1, -i\chi\int_{\omega\times Y} x_2\boldsymbol{u}_2\right)^T \cdot \overline{(d_1,d_2)^T}.$$

$$\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla \boldsymbol{u}_1^{(3)} = -\operatorname{sym}\nabla^*\mathbb{C}(y)\Lambda_{\chi,m^{(3)}}^{\mathrm{rod}}(x), \quad \boldsymbol{u}_1^{(3)} \in H.$$

$$\frac{1}{|\chi|^4}\operatorname{sym}\nabla^*\mathbb{C}(y)\operatorname{sym}\nabla \boldsymbol{u}_2^{(3)} = -\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\operatorname{sym}\nabla(\boldsymbol{u}_1^{(3)} + \boldsymbol{u}_2^{(2)}) + \operatorname{sym}\nabla^*\mathbb{C}(y) iX_\chi(\boldsymbol{u}_1^{(3)} + \boldsymbol{u}_2^{(2)})\right)$$
$$-\frac{1}{|\chi|^4}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\Lambda_{\chi,m^{(3)}}^{\mathrm{rod}}(x) + \left(iX_\chi\right)^*\mathbb{C}(y) iX_\chi(\boldsymbol{u}_2^{(1)} + \boldsymbol{u}_1^{(2)})\right)$$
$$-\begin{bmatrix} m_1^{(2)} \\ m_2^{(2)} \\ 0 \end{bmatrix} - \begin{bmatrix} m_3^{(3)} x_2 \\ -m_3^{(3)} x_1 \\ m_4^{(3)} - i\chi(m_1^{(3)}x_1 + m_2^{(3)}x_2) \end{bmatrix} - \boldsymbol{u}_1, \quad \boldsymbol{u}_2^{(2)} \in H.$$

All of these problems define unique correctors which satisfy the following estimates:

$$||(\boldsymbol{u}_0^{(3)})_\alpha||_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^3||\boldsymbol{f}||_{L^2(\omega\times Y;\mathbb{C}^3)}, \quad ||(\boldsymbol{u}_0^{(3)})_3||_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^4||\boldsymbol{f}||_{L^2(\omega\times Y;\mathbb{C}^3)}$$

$$||\boldsymbol{u}_1^{(3)}||_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^5||\boldsymbol{f}||_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

$$||\boldsymbol{u}_2^{(3)}||_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^6||\boldsymbol{f}||_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

5) The final approximation

The function $\tilde{\boldsymbol{u}}_{\mathrm{approx}}$, defined with:

$$\tilde{\boldsymbol{u}}_{\mathrm{approx}} := \boldsymbol{u}_0 + \boldsymbol{u}_0^{(1)} + \boldsymbol{u}_0^{(2)} + \boldsymbol{u}_0^{(3)} + \boldsymbol{u}_1 + \boldsymbol{u}_1^{(1)} + \boldsymbol{u}_1^{(2)} + \boldsymbol{u}_1^{(3)} + \boldsymbol{u}_2 + \boldsymbol{u}_2^{(1)} + \boldsymbol{u}_2^{(2)} + \boldsymbol{u}_2^{(3)},$$

is the solution to the following problem:

$$\frac{1}{|\chi|^4}\left(\operatorname{sym}\nabla^* + \left(iX_\chi\right)^*\right)\mathbb{C}(y)\left(\operatorname{sym}\nabla + iX_\chi\right)\tilde{\boldsymbol{u}}_{\mathrm{approx}} + \tilde{\boldsymbol{u}}_{\mathrm{approx}} - S_{|\chi|}\boldsymbol{f} = \tilde{R}_\chi,$$

where the residual $\tilde{R}_\chi$ is given with:

$$\tilde{R}_\chi = \frac{1}{|\chi|^2}\left(\left(iX_\chi\right)^*\mathbb{C}(y)\,\mathrm{sym}\,\nabla\boldsymbol{u}_2^{(3)} + \mathrm{sym}\,\nabla^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2^{(3)} + \left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2^{(3)} + \left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2^{(2)}\right)$$

$$+\frac{1}{|\chi|^2}\left(\left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_2^{(2)} + \left(iX_\chi\right)^*\mathbb{C}(y)iX_\chi\boldsymbol{u}_1^{(3)}\right) + \begin{bmatrix} m_1^{(3)} \\ m_2^{(3)} \\ 0 \end{bmatrix} + \boldsymbol{u}_2 + \boldsymbol{u}_1^{(1)} + \boldsymbol{u}_2^{(1)} + \boldsymbol{u}_1^{(2)} + \boldsymbol{u}_2^{(2)} + \boldsymbol{u}_1^{(3)} + \boldsymbol{u}_2^{(3)}.$$

The final estimate on the residual is:

$$\|\tilde{R}_\chi\|_{H_\#^{-1}(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y,\mathbb{C}^3)}.$$

The error of the approximation

$$\boldsymbol{u}_{\mathrm{error}} := \boldsymbol{u} - \tilde{\boldsymbol{u}}_{\mathrm{approx}}.$$

satisfies

$$\frac{1}{|\chi|^4}\left(\mathrm{sym}\,\nabla^* + \left(iX_\chi\right)^*\right)\mathbb{C}(y)\left(\mathrm{sym}\,\nabla + iX_\chi\right)\boldsymbol{u}_{\mathrm{error}} + \boldsymbol{u}_{\mathrm{error}} = -\tilde{R}_\chi.$$

Finally, by using (3.18), we have:

$$\|\boldsymbol{u}_{\mathrm{error}}\|_{H^1(\omega\times Y;\mathbb{C}^3)} \leq C|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}.$$

Easily, we obtain the error in the approximation by lower order terms:

Proposition 3.5.3. *Let $\boldsymbol{u} \in H^1_\chi(Y;H^1(\omega;\mathbb{C}^3))$ be the quasiperiodic solution of problem* (3.48). *Then, the following estimates are valid:*

$$\left\|P_i\left(\boldsymbol{u} - e^{i\chi y}\mathcal{I}_\chi^{\mathrm{rod}}m\right)\right\|_{H^1(\omega\times Y,\mathbb{C}^2)} \leq \begin{cases} C|\chi|\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, & i = 1,2; \\ C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, & i = 3. \end{cases}$$

$$\left\|P_i\left(\boldsymbol{u} - e^{i\chi y}\mathcal{I}_\chi^{\mathrm{rod}}m - e^{i\chi y}\mathcal{I}_\chi^{\mathrm{rod}}m^{(1)} - e^{i\chi y}\boldsymbol{u}_1\right)\right\|_{H^1(\omega\times Y,\mathbb{C}^2)} \leq \begin{cases} C|\chi|^2\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, & i = 1,2; \\ C|\chi|^3\|\boldsymbol{f}\|_{L^2(\omega\times Y;\mathbb{C}^3)}, & i = 3. \end{cases} \tag{3.49}$$

where $m, m^{(1)}, \boldsymbol{u}_1$ are defined with the approximation procedure above.

3.5.3. Norm resolvent estimates for the general elasticity tensor

Theorem 3.5.4 ($L^2 \to L^2$ norm-resolvent estimate)**.** *Let $\gamma > -2$ be the parameter of spectral scaling. There exists $C > 0$ such that for every $\varepsilon > 0$ we have:*

$$\left\| P_i \left(\left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} - (\mathcal{M}_\varepsilon^{\mathrm{rod}})^* \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon^{\mathrm{rod}} + C^{\mathrm{rod}}(\omega) \right)^{-1} \mathcal{M}_\varepsilon^{\mathrm{rod}} \Xi_\varepsilon \right) \right\|_{L^2 \to L^2} \leq \begin{cases} C\varepsilon^{\frac{\gamma+2}{4}}, & i = 1,2; \\ C\varepsilon^{\frac{\gamma+2}{2}}, & i = 3. \end{cases} \tag{3.50}$$

Proof. As the proof of this result goes analogously as the proofs of the Theorem 3.1.3, we will focus only on the differences. First, we substitute C^{rod} with I much in the same manner as before. Here in the case of general elasticity tensor, the operator $\mathcal{A}_\chi$ has a spectrum consisting of two eigenvalues of order $|\chi|^4$, two eigenvalues of order $|\chi|^2$ and the rest of order one. Thus, by providing estimates of the scaled resolvent problems (3.40) and (3.48), we have actually estimated the resolvents in these two eigenspaces with eigenvalues of different orders. In order to combine the two, we make the following argument:
By using the uniform estimates on the eigenvalues, obtained by the Proposition 3.2.8, we can obtain a closed contour $\Gamma_{|\chi|^2}$ surrounding the two eigenvalues of order one of the operator $\frac{1}{|\chi|^2}\mathcal{A}_\chi$, and a closed contour $\Gamma_{|\chi|^4}$ surrounding the two eigenvalues of order one of the operator $\frac{1}{|\chi|^4}\mathcal{A}_\chi$. Even the notation suggests that these contours depend on $|\chi|$, this is not the case. The notation here is purely aesthetic. Next, we use the scaling functions $g_{\varepsilon,\chi}$, $f_{\varepsilon,\chi}$ which are analytic on the neighbourhoods of $\Gamma_{|\chi|^4}$, $\Gamma_{|\chi|^2}$, given with:

$$f_{\varepsilon,\chi}(z) := \left(\frac{|\chi|^{\gamma+2}}{\varepsilon^4} z + 1 \right)^{-1}, \quad g_{\varepsilon,\chi}(z) := \left(\frac{|\chi|^2}{\varepsilon^{\gamma+2}} z + 1 \right)^{-1}, \ \Re(z) > 0,$$

It is clear that

$$\begin{aligned} &\left(P_{\Gamma_{|\chi|^4}} + P_{\Gamma_{|\chi|^2}} \right) \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} \left(P_{\Gamma_{|\chi|^4}} + P_{\Gamma_{|\chi|^2}} \right) \\ &= P_{\Gamma_{|\chi|^2}} \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} P_{\Gamma_{|\chi|^2}} + P_{\Gamma_{|\chi|^4}} \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}_\chi + I \right)^{-1} P_{\Gamma_{|\chi|^4}} \\ &= \frac{1}{2\pi i} \oint_{\Gamma_{|\chi|^2}} g_{\chi,\varepsilon}(z) \left(zI - \frac{1}{|\chi|^2} \mathcal{A}_\chi \right)^{-1} dz + \frac{1}{2\pi i} \oint_{\Gamma_{|\chi|^4}} f_{\chi,\varepsilon}(z) \left(zI - \frac{1}{|\chi|^4} \mathcal{A}_\chi \right)^{-1} dz. \end{aligned} \tag{3.51}$$

Also:

$$\left\| \left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi + I\right)^{-1} - \left(P_{\Gamma_{|\chi|^4}} + P_{\Gamma_{|\chi|^2}}\right)\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi + I\right)^{-1}\left(P_{\Gamma_{|\chi|^4}} + P_{\Gamma_{|\chi|^2}}\right)\right\|_{L^2\to H^1}$$
$$= \left\| (I - P_{\Gamma_{|\chi|^4}} - P_{\Gamma_{|\chi|^2}})\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi + I\right)^{-1}(I - P_{\Gamma_{|\chi|^4}} - P_{\Gamma_{|\chi|^2}})\right\|_{L^2\to H^1} \leq C\varepsilon^{\gamma+2},$$

so it is enough to estimate only the projections on these two eigenspaces. The optimal estimate is obtained by separately estimating the two terms in (3.51). For this, we employ the first row of fiberwise norm-resolvent estimates in (3.49), (3.47). By using the estimates of functions functions $g_{\varepsilon,\chi}$, $f_{\varepsilon,\chi}$, we are able to get the following:

$$\left\| P_i\left(\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi + I\right)^{-1} - (\mathcal{M}_\chi^{\mathrm{rod}})^*\left(\frac{1}{\varepsilon^{\gamma+2}}\mathbb{C}_\chi^{\mathrm{hom}} + C^{\mathrm{rod}}(\omega)\right)^{-1}\mathcal{M}_\chi^{\mathrm{rod}}\right)\right\|_{L^2\to L^2}$$
$$\leq \begin{cases} C|\chi|\left(\max\left\{\frac{|\chi|^2}{\varepsilon^{\gamma+2}},1\right\}\right)^{-1} + C|\chi|\left(\max\left\{\frac{|\chi|^4}{\varepsilon^{\gamma+2}},1\right\}\right)^{-1}, & i = 1,2; \\ C|\chi|\left(\max\left\{\frac{|\chi|^2}{\varepsilon^{\gamma+2}},1\right\}\right)^{-1} + C|\chi|^2\left(\max\left\{\frac{|\chi|^4}{\varepsilon^{\gamma+2}},1\right\}\right)^{-1}, & i = 3; \end{cases}$$
$$\leq \begin{cases} C\varepsilon^{\frac{\gamma+2}{4}}, & i = 1,2; \\ C\varepsilon^{\frac{\gamma+2}{2}}, & i = 3. \end{cases}$$

The proof is finished after applying the inverse Gelfand transform. ■

Remark 3.5.5. The results for the $L^2 \to H^1$ norm and higher accuracy in $L^2 \to L^2$ norm are done analogously as in the case of invariant subspaces, and combining with the argumentation provided in the Theorem 3.5.4.

Remark 3.5.6. A similar argument as in the Corollary 3.4.5 can be used to demonstrate that one can drop the smoothing operator from the $L^2 \to L^2$ norm-resolvent estimates (3.50) while keeping the same order of accuracy. Namely, there exists $C > 0$ such that for every $\varepsilon > 0$ we have:

$$\left\| P_i\left(\left(\frac{1}{\varepsilon^{\gamma}}\mathcal{A}_\varepsilon + I\right)^{-1} - (\mathcal{M}_\varepsilon^{\mathrm{rod}})^*\left(\frac{1}{\varepsilon^{\gamma}}\mathcal{A}_\varepsilon^{\mathrm{rod}} + C^{\mathrm{rod}}(\omega)\right)^{-1}\mathcal{M}_\varepsilon^{\mathrm{rod}}S_\infty\right)\right\|_{L^2\to L^2} \leq \begin{cases} C\varepsilon^{\frac{\gamma+2}{4}}, & i = 1,2; \\ C\varepsilon^{\frac{\gamma+2}{2}}, & i = 3. \end{cases}$$

Remark 3.5.7. Much in the same fashion as in Corollary 3.4.8, one can prove the following bound on the spectral gaps in the general case: Let $\gamma > -2$ be the parameter of spectral scaling. Let $M > 0$. Then

$$\sup_{\substack{[a,b]\subset[0,M\varepsilon^\gamma] \\ [a,b]\cap\sigma(\mathcal{A}_\varepsilon)=\emptyset}} |[a,b]| \leq C(M+1)^2\varepsilon^{\frac{5\gamma+2}{4}}.$$

4. Appendix

4.1. Operators, forms and resolvent formalism

Let a be a densely defined non-negative bilinear form on a subspace $\mathcal{D}(a) \leq H$, $a : \mathcal{D}(a) \times \mathcal{D}(a) \to \mathbb{R}$. We define the subspace $\mathcal{D}(A) \leq H$ with the following expression:

$$\mathcal{D}(\mathcal{A}) := \{\boldsymbol{u} \in \mathcal{D}(a); \quad \exists \boldsymbol{v} \in H, \text{ such that:} \quad a(\boldsymbol{u}, \boldsymbol{w}) = \langle \boldsymbol{v}, \boldsymbol{w} \rangle_H, \quad \forall \boldsymbol{w} \in \mathcal{D}(a)\}.$$

Since the set $\mathcal{D}(a)$ is dense in H, the vector $\boldsymbol{v} \in H$, representing the form a, is unique. Therefore, we can define a map $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to H$ with: $\mathcal{A}\boldsymbol{u} := \boldsymbol{v}$. We have:

$$a(\boldsymbol{u}, \boldsymbol{v}) = \langle \mathcal{A}\boldsymbol{u}, \boldsymbol{v} \rangle, \quad \forall \boldsymbol{u} \in \mathcal{D}(\mathcal{A}), \forall \boldsymbol{v} \in \mathcal{D}(a).$$

We refer to this linear operator as the operator associated with the bilinear form a.

Theorem 4.1.1. *Let a be densely defined on H, continuous on $\mathcal{D}(a)$ with respect to $\|\cdot\|_a := \sqrt{\|\cdot\|_H + a(\cdot, \cdot)}$, closed bilinear form such that $a(\boldsymbol{u}, \boldsymbol{u}) \geq 0$, $\forall \boldsymbol{u} \in \mathcal{D}(a)$. Then the associated operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to H$ is closed, densely defined and*

$$\{\lambda \in \mathbb{R}, \quad \lambda < 0\} \subset \rho(\mathcal{A}), \quad \|(\mathcal{A} + \alpha I)^{-1}\| \leq \frac{1}{\alpha}, \quad \forall \alpha > 0.$$

For a closed operator $\mathcal{A}$ on a Banach space X, with domain $\mathcal{D}(\mathcal{A})$, we associate its resolvent set

$$\rho(\mathcal{A}) := \left\{ z \in \mathbb{C}; (zI - \mathcal{A}) : \mathcal{D}(\mathcal{A}) \to X \text{ is bijective } \right\}.$$

For every $z \in \rho(\mathcal{A})$, the resolvent of A is given with:

$$R(z, \mathcal{A}) := (zI - \mathcal{A})^{-1} : X \to X.$$

It is known that we have the following two identities:

$$\text{First resolvent identity:} \quad R(z,\mathcal{A}) - R(w,\mathcal{A}) = (w-z)R(z,\mathcal{A})R(w,\mathcal{A}), \quad \forall z,w \in \rho(\mathcal{A}),$$

$$\text{Second resolvent identity:} \quad R(z,\mathcal{A}) - R(z,\mathcal{B}) = R(z,\mathcal{A})(B-A)R(z,\mathcal{B}), \quad \forall z \in \rho(\mathcal{A}) \cap \rho(\mathcal{B}).$$

To establish a norm-resolvent estimate is to provide the estimate for the difference of two resolvents in a strong operator norm topology. We make a remark here that estimates of resolvents which depend on the spectral parameter $z \in \mathbb{C}$ can all be reduced to a single resolvent estimate where the dependance on the spectral parameter is hidden in the right hand side. Namely, we have the following Lemma:

Lemma 4.1.2. *Let $w,z \in \rho(\mathcal{A}) \cap \rho(\mathcal{B})$, where $\mathcal{A}$, $\mathcal{B}$ are closed operators on X. Then we have:*

$$\|R(z,\mathcal{A}) - R(z,\mathcal{B})\|_X \leq C(z,w)\,\|R(w,\mathcal{A}) - R(w,\mathcal{B})\|_X\,,$$

where

$$C(z,w) := \max\left\{1, \frac{|z-w|}{\operatorname{dist}(z,\sigma(\mathcal{A}))}\right\} \max\left\{1, \frac{|z-w|}{\operatorname{dist}(z,\sigma(\mathcal{B}))}\right\}.$$

Proof. We have the following identity:

$$\begin{aligned} R(z,\mathcal{A}) - R(z,\mathcal{B}) =& R(z,\mathcal{A}) - R(w,\mathcal{A}) + R(w,\mathcal{A}) - R(w,\mathcal{B}) + R(w,\mathcal{B}) - R(z,\mathcal{B}) \\ &= (w-z)R(z,\mathcal{A})R(w,\mathcal{A}) + R(w,\mathcal{A}) - R(w,\mathcal{B}) + (z-w)R(w,\mathcal{B})R(z,\mathcal{B}) \\ &= (w-z)\left[R(z,\mathcal{A})R(w,\mathcal{A}) - R(w,\mathcal{B})R(z,\mathcal{B})\right] + R(w,\mathcal{A}) - R(w,\mathcal{B}). \end{aligned}$$

A clear consequence of the first resolvent identity is the following:

$$R(z,\mathcal{B})R(w,\mathcal{B}) = R(w,\mathcal{B})R(z,\mathcal{B}).$$

We have the following:

$$R(z,\mathcal{A})R(w,\mathcal{A}) - R(z,\mathcal{B})R(w,\mathcal{B}) = R(z,\mathcal{A})\left[R(w,\mathcal{A}) - R(w,\mathcal{B})\right] + \left[R(z,\mathcal{A}) - R(z,\mathcal{B})\right]R(w,\mathcal{B})$$

By combining this, we obtain:

$$\left(R(z,\mathcal{A}) - R(z,\mathcal{B})\right)\left[I - (w-z)R(w,\mathcal{B})\right] = \left[I + (w-z)R(z,\mathcal{A})\right]\left(R(w,\mathcal{A}) - R(w,\mathcal{B})\right).$$

From this we have:

$$R(z,\mathcal{A}) - R(z,\mathcal{B}) = \left[I + (w-z)R(z,\mathcal{A})\right]\left(R(w,\mathcal{A}) - R(w,\mathcal{B})\right)\left[I - (w-z)R(w,\mathcal{B})\right]^{-1}.$$

For the norm we have:

$$\|R(z,\mathcal{A})-R(z,\mathcal{B})\|_X \leq \|I+(w-z)R(z,\mathcal{A})\|_X \, \|R(w,\mathcal{A})-R(w,\mathcal{B})\|_X \left\|(I-(w-z)R(w,\mathcal{B}))^{-1}\right\|_X .$$

We define the following complex function:

$$f_{w,z}(\lambda) := 1+\frac{w-z}{z-\lambda} = \frac{w-\lambda}{z-\lambda} = \left(1-\frac{w-z}{w-\lambda}\right)^{-1}, \quad \lambda \in \mathbb{C} \setminus (\rho(\mathcal{A}) \cap \rho(\mathcal{B})) .$$

It is clear that:

$$f_{w,z}(\mathcal{A}) = I+(w-z)R(z,\mathcal{A}), \quad f_{w,z}(\mathcal{B}) = (I-(w-z)R(w,\mathcal{B}))^{-1} .$$

From this, we conclude that:

$$\begin{aligned}
\|I+(w-z)R(z,\mathcal{A})\|_X &\leq \max\left\{1, \frac{|z-w|}{\operatorname{dist}(z,\sigma(\mathcal{A}))}\right\}, \\
\left\|(I-(w-z)R(w,\mathcal{B}))^{-1}\right\|_X &\leq \max\left\{1, \frac{|z-w|}{\operatorname{dist}(z,\sigma(\mathcal{B}))}\right\}.
\end{aligned}$$

■

For an extensive overview of operator theory and spectral theory we refer to books [8] and [29].

4.2. Griso's decomposition

Theorem 4.2.1 (Korn inequalities)**.** *[34] Let $p > 1$, $\Omega \subset \mathbb{R}^n$ and suppose that $\Gamma \subset \partial\Omega$ has a positive measure. There exist positive constants C_{K}^1, C_{K}^2 and C_{K}^{Γ}, which depend on p, Ω, and Γ only, such that then the following inequalities hold for all $\boldsymbol{\psi} \in W^{1,p}(\Omega;\mathbb{R}^n)$:*

$$\|\boldsymbol{\psi}\|_{W^{1,p}}^p \le C_{\mathrm{K}}^1 \left(\|\boldsymbol{\psi}\|_{L^p}^p + \|\operatorname{sym}\nabla\boldsymbol{\psi}\|_{L^p}^p\right),$$

$$\inf_{\boldsymbol{A}\in\mathbb{R}^{n\times n}_{\mathrm{skew}},\boldsymbol{b}\in\mathbb{R}^n} \|\boldsymbol{\psi} - \boldsymbol{A}x - \boldsymbol{b}\|_{W^{1,p}}^p \le C_{\mathrm{K}}^2 \|\operatorname{sym}\nabla\boldsymbol{\psi}\|_{L^p}^p, \tag{4.1}$$

$$\|\boldsymbol{\psi}\|_{W^{1,p}(\Omega;\mathbb{R}^n)}^p \le C_{\mathrm{K}}^{\Gamma} \left(\|\boldsymbol{\psi}\|_{L^p(\Gamma)}^p + \|\operatorname{sym}\nabla\boldsymbol{\psi}\|_{L^p}^p\right). \tag{4.2}$$

Within this appendix, we will also use the following version of Korn's inequality.

Proposition 4.2.2. *Suppose that $\omega \subset \mathbb{R}^2$ has Lipschitz boundary. Then for every $\boldsymbol{\psi} \in H^1(\omega;\mathbb{R}^2)$ one has*

$$\left\|\boldsymbol{\psi} - \fint_\omega \boldsymbol{\psi}\right\|_{L^2} \le C\big(\|\operatorname{sym}\nabla\boldsymbol{\psi}\|_{L^2} + \operatorname{dist}(\psi,\mathcal{G})\big),$$

where $C > 0$ depends only on ω,

$$\mathcal{G} := \big\{\nabla\phi : \phi \in H^1(\omega)\big\},$$

and the distance is understood in the sense of the L^2 metric.

Proof. The proof follows a standard contradiction argument. Suppose the claim is false, i.e. there exists a sequence $(\boldsymbol{\psi}^n)_{n\in\mathbb{N}} \subset H^1(\omega;\mathbb{R}^2)$ such that

$$\int_\omega \boldsymbol{\psi}^n = 0,$$

$$\|\boldsymbol{\psi}^n\|_{L^2} \ge n\big(\|\operatorname{sym}\nabla\boldsymbol{\psi}^n\|_{L^2} + \operatorname{dist}(\boldsymbol{\psi}^n,\mathcal{G})\big) \qquad \forall n \in \mathbb{N}. \tag{4.3}$$

Without loss of generality,

$$\|\boldsymbol{\psi}^n\|_{L^2} = 1.$$

and (4.3) can be written as

$$\|\operatorname{sym}\nabla\boldsymbol{\psi}^n\|_{L^2} + \operatorname{dist}(\boldsymbol{\psi}^n,\mathcal{G}) \le n^{-1} \qquad \forall n \in \mathbb{N}.$$

By Korn's inequality, it follows that $\boldsymbol{\psi}^n \rightharpoonup \boldsymbol{\psi}$, weakly in $H^1(\omega;\mathbb{R}^2)$. Combining this with (4.3), we infer that $\operatorname{sym}\nabla\boldsymbol{\psi} = 0$ and $\boldsymbol{\psi} \in \mathcal{G}$. From $\operatorname{sym}\nabla\boldsymbol{\psi} = 0$ we obtain

$$\boldsymbol{\psi} = (-ax_2, ax_1)^\top + \boldsymbol{b}, \quad a \in \mathbb{R}, \boldsymbol{b} \in \mathbb{R}^2.$$

Together with (4.2) and $\boldsymbol{\psi} \in \mathcal{G}$ this implies $\boldsymbol{\psi} = 0$, which contradicts (4.2). ■

Theorem 4.2.3 (Griso's decomposition, [31]). *Let $\omega \subset \mathbb{R}^2$ with Lipschitz boundary and $\boldsymbol{\psi} \in H^1(\Omega;\mathbb{R}^3)$. Then one has*

$$\boldsymbol{\psi} = \hat{\boldsymbol{\psi}}(x') + \boldsymbol{r}(x') \wedge x_3\boldsymbol{e}_3 + \bar{\boldsymbol{\psi}}(x) = \begin{cases} \hat{\psi}_1(x') + \ r_2(x')x_3 + \bar{\psi}_1(x), \\ \hat{\psi}_2(x') - r_1(x')x_3 + \bar{\psi}_2(x), \\ \hat{\psi}_3(x') + \bar{\psi}_3(x), \end{cases} \tag{4.4}$$

where

$$\hat{\boldsymbol{\psi}}(x') = \int_I \boldsymbol{\psi}(x', x_3)\mathrm{d}x_3, \quad \boldsymbol{r}(x') = \frac{3}{2}\int_I x_3\boldsymbol{e}_3 \wedge \boldsymbol{\psi}(x', x_3)\mathrm{d}x_3,$$

the following inequality holds for arbitrary $h > 0$, with a constant $C > 0$ that depends on ω only:

$$\left\|\operatorname{sym}\nabla_h(\hat{\boldsymbol{\psi}} + \boldsymbol{r} \wedge x_3\boldsymbol{e}_3)\right\|_{L^2}^2 + \left\|\nabla_h\bar{\boldsymbol{\psi}}\right\|_{L^2(\Omega;\mathbb{R}^{3\times3})}^2 + h^{-2}\|\bar{\boldsymbol{\psi}}\|_{L^2}^2 \le C\left\|\operatorname{sym}\nabla_h\boldsymbol{\psi}\right\|_{L^2}^2. \tag{4.5}$$

Remark 4.2.4. Notice that

$$\begin{aligned} \left\|\operatorname{sym}\nabla_h(\hat{\boldsymbol{\psi}} + \boldsymbol{r} \wedge x_3e_3)\right\|_{L^2}^2 &= \left\|\operatorname{sym}\nabla_{\hat{x}}(\hat{\psi}_1, \hat{\psi}_2)^\top\right\|_{L^2}^2 + \frac{1}{12}\left\|\operatorname{sym}\nabla_{\hat{x}}(r_2, -r_1)^\top\right\|_{L^2}^2 \\ &\quad + h^{-2}\left\|\partial_1(h\hat{\psi}_3) + r_2\right\|_{L^2}^2 + h^{-2}\left\|\partial_2(h\hat{\psi}_3) - r_1\right\|_{L^2}^2. \end{aligned} \tag{4.6}$$

Thus from Korn's inequality it follows

$$\begin{aligned} h^2\left\|\pi_{1/h}\hat{\boldsymbol{\psi}}\right\|_{H^1}^2 + \left\|(r_1, r_2)^\top\right\|_{H^1}^2 + h^{-2}\left\|\partial_1(h\hat{\psi}_3) + r_2\right\|_{L^2}^2 + h^{-2}\left\|\partial_2(h\hat{\psi}_3) - r_1\right\|_{L^2}^2 \\ \le C\left(\left\|\operatorname{sym}\nabla_h(\hat{\boldsymbol{\psi}} + \boldsymbol{r} \wedge x_3\boldsymbol{e}_3)\right\|_{L^2}^2 + \|\boldsymbol{r}\|_{L^2}^2 + h^2\|\pi_{1/h}\hat{\boldsymbol{\psi}}\|_{L^2}^2\right) \\ \le C\left(\left\|\operatorname{sym}\nabla_h(\hat{\boldsymbol{\psi}} + \boldsymbol{r} \wedge x_3\boldsymbol{e}_3)\right\|_{L^2}^2 + h^2\|\pi_{1/h}\boldsymbol{\psi}\|_{L^2}^2\right). \end{aligned} \tag{4.7}$$

The following corollary is the direct consequence of (4.5), (4.7), and Korn inequalities.

Corollary 4.2.5 (Korn's inequality for thin domains)**.** *Suppose that $\omega \subset \mathbb{R}^2$ is such that $\gamma \subset \partial\omega$ has positive measure. Then there exist constants $C_{\mathrm{T}}, C_{\mathrm{T}}^{\gamma} > 0$ that depend on ω and γ only, such that the following inequalities hold for all $\boldsymbol{\psi} \in H^1(\Omega;\mathbb{R}^3)$:*

$$\|\pi_{1/h}\boldsymbol{\psi}\|_{H^1}^2 \leq C_{\mathrm{T}} \left(\|\pi_{1/h}\boldsymbol{\psi}\|_{L^2}^2 + h^{-2}\|\operatorname{sym}\nabla_h \boldsymbol{\psi}\|_{L^2}^2 \right),$$
$$\|\pi_{1/h}\boldsymbol{\psi}\|_{H^1}^2 \leq C_{\mathrm{T}}^{\gamma} \left(\|\pi_{1/h}\boldsymbol{\psi}\|_{L^2(\Gamma;\mathbb{R}^3)}^2 + h^{-2}\left\|\operatorname{sym}\nabla_h \boldsymbol{\psi}\right\|_{L^2}^2 \right).$$

Remark 4.2.6. If it is known that the components ψ_α, $\alpha = 1,2$, are even in x_3 and ψ_3 is odd in x_3, then additionally $\boldsymbol{r} = 0$, $\hat{\psi}_3 = 0$.

Remark 4.2.7. If a sequence of deformations $(\boldsymbol{\psi}^h)_{h>0}$ is such that $(\boldsymbol{\psi}^h)_{h>0}$ and $(\operatorname{sym}\nabla_h \boldsymbol{\psi}^h)_{h>0}$ are bounded in L^2 and if $\boldsymbol{r}^h$, $\hat{\boldsymbol{\psi}}^h$ and $\bar{\boldsymbol{\psi}}^h$ are the terms in the decomposition (4.4), then the relations (4.5)–(4.7) imply that $\boldsymbol{r}^h \overset{H^1}{\rightharpoonup} 0$ and $h\hat{\boldsymbol{\psi}}^h \overset{H^1}{\to} 0$.

The following lemma provides additional information on the weak limit of sequences with bounded symmetrised scaled gradients and is proved in [10, Lemma A.4] as a direct consequence of Griso's decomposition.

Lemma 4.2.8. *Consider a bounded set $\omega \subset \mathbb{R}^2$ with Lipschitz boundary. Suppose that a sequence $(\boldsymbol{\psi}^h)_{h>0} \subset H^1_{\gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)$ is such that*

$$\limsup_{n\to\infty} \left\|\operatorname{sym}\nabla_h \boldsymbol{\psi}^h\right\|_{L^2} < \infty.$$

Then there exists a subsequence (still labelled by $h > 0$) for which

$$\boldsymbol{\psi}^h = \begin{pmatrix} \mathfrak{a}_1 - x_3\partial_1\mathfrak{b} \\ \mathfrak{a}_2 - x_3\partial_2\mathfrak{b} \\ h^{-1}\mathfrak{b} \end{pmatrix} + \tilde{\boldsymbol{\psi}}^h,$$

in particular

$$\operatorname{sym}\nabla_h \boldsymbol{\psi}^h = \iota\left(-x_3\nabla_{\hat{x}}^2\mathfrak{b} + \operatorname{sym}\nabla_{\hat{x}}\mathfrak{a}\right) + \operatorname{sym}\nabla_h \tilde{\boldsymbol{\psi}}^h,$$

for some $\mathfrak{b} \in H^2_{\gamma_{\mathrm{D}}}(\omega)$, $\mathfrak{a} \in H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)$, and the sequence $(\tilde{\boldsymbol{\psi}}^h)_{h>0} \subset H^1_{\gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)$ satisfies $h\pi_{1/h}\tilde{\boldsymbol{\psi}}^h \overset{L^2}{\to} 0$.

Remark 4.2.9. It can be easily seen that $\mathfrak{a}$ is the weak limit of $(\hat{\psi}_1^h, \hat{\psi}_2^h)^\top$ in $H^1(\omega;\mathbb{R}^2)$ and $\mathfrak{b}$ is the weak limit on $h\hat{\psi}_3^h$ in $H^1(\omega)$. (More precisely, in this case $(-r_2^h, r_1^h)^\top \overset{H^1}{\rightharpoonup} x_3\nabla\mathfrak{b}$, where $\boldsymbol{r}^h$ and $\hat{\boldsymbol{\psi}}^h$ come from the decomposition (4.4) of $\boldsymbol{\psi}^h$.)

We will now establish a decomposition that can be viewed as a consequence of Griso's decomposition. Note that another proof of the first part of Lemma 4.2.10 is given in [13]. The proof provided here uses the strategy of [69].

Lemma 4.2.10. *Suppose that $\omega \subset \mathbb{R}^2$ is a connected set with $C^{1,1}$ boundary and $\boldsymbol{\psi} \in H^1(\Omega;\mathbb{R}^3)$.*

1. *There exist $\boldsymbol{a} \in \mathbb{R}^3$, $\boldsymbol{B} \in \mathbb{R}^{3\times 3}_{\text{skew}}$, $v \in H^2(\omega)$, $\tilde{\boldsymbol{\psi}} \in H^1(\Omega;\mathbb{R}^3)$ such that*

$$\boldsymbol{\psi} = \boldsymbol{a} + \boldsymbol{B}\begin{pmatrix} x_1 \\ x_2 \\ hx_3 \end{pmatrix} + \begin{pmatrix} -x_3\partial_1 v \\ -x_3\partial_2 v \\ h^{-1}v \end{pmatrix} + \tilde{\boldsymbol{\psi}}, \tag{4.8}$$

   and the estimate

$$\|v\|^2_{H^2} + \|\tilde{\boldsymbol{\psi}}\|^2_{L^2} + \|\nabla_h \tilde{\boldsymbol{\psi}}\|^2_{L^2} \le C(\omega)\|\operatorname{sym}\nabla_h \boldsymbol{\psi}\|^2_{L^2}$$

   holds for some $C(\omega) > 0$.

2. *If $\boldsymbol{\psi} \in H^1(\Omega,\mathbb{R}^3)$, $\boldsymbol{\psi} = 0$ on $\partial\omega \times I$, then in (4.8) one can take $\boldsymbol{a} = \boldsymbol{B} = 0$. In addition, v, $\tilde{\boldsymbol{\psi}}$ can be chosen so that $v = \nabla v = 0$ on $\partial\omega$ and $\tilde{\boldsymbol{\psi}} = 0$ on $\partial\omega \times I$.*

3. *If a sequence $(\boldsymbol{\psi}^h)_{h>0} \subset H^1(\Omega;\mathbb{R}^3)$ is such that*

$$h\pi_{1/h}\boldsymbol{\psi}^h \xrightarrow{L^2} 0, \qquad \limsup_{n\to\infty}\left\|\operatorname{sym}\nabla_h \boldsymbol{\psi}^h\right\|_{L^2} < \infty,$$

   then there exist sequences $(\varphi^h)_{h>0} \subset H^2(\omega)$, $(\tilde{\boldsymbol{\psi}}^h)_{h>0} \subset H^1(\Omega;\mathbb{R}^3)$ such that

$$\operatorname{sym}\nabla_h \boldsymbol{\psi}^h = -x_3\iota(\nabla^2_{\hat{x}}\varphi^h) + \operatorname{sym}\nabla_h \tilde{\boldsymbol{\psi}}^h + o^h,$$

   where $(o^h)_{h>0} \subset L^2(\Omega;\mathbb{R}^{3\times 3})$ is such that $o^h \xrightarrow{L^2} 0$, and the following properties hold:

$$\lim_{h\to 0}\left(\|\varphi^h\|_{H^1} + \|\tilde{\boldsymbol{\psi}}^h\|_{L^2}\right) = 0, \qquad \limsup_{h\to 0}\left(\|\varphi^h\|_{H^2} + \|\nabla_h\tilde{\boldsymbol{\psi}}^h\|_{L^2}\right) \le C\limsup_{n\to\infty}\left\|\operatorname{sym}\nabla_h\boldsymbol{\psi}^h\right\|_{L^2},$$

   where $C > 0$ depends on ω only. Moreover, one has

$$\psi^h_3 = h^{-1}\varphi^h + w^h + \tilde{\psi}^h_3,$$

   where $w^h \in H^1(\omega)$ with

$$\|w^h\|_{H^1} \le C\left(\left\|\operatorname{sym}\nabla_h\boldsymbol{\psi}^h\right\|_{L^2} + \left\|h\pi_{1/h}\boldsymbol{\psi}^h\right\|_{L^2}\right),$$

   for some $C > 0$ that depends on ω only.

Proof. We decompose $\boldsymbol{\psi}$ as in (4.4). From (4.1), (4.5) and Proposition 4.2.2 we conclude that there exist $c \in \mathbb{R}$, $\boldsymbol{d} \in \mathbb{R}^2$ such that

$$\left\|\boldsymbol{r} - \fint_\omega \boldsymbol{r}\right\|_{H^1}^2 + \left\|(\hat{\psi}_1, \hat{\psi}_2)^\top - (-cx_2, cx_1)^\top - \boldsymbol{d}\right\|_{H^1} + h^{-2}\left\|\bar{\boldsymbol{\psi}}^h\right\|_{L^2}^2 + \left\|\nabla_h \bar{\boldsymbol{\psi}}\right\|_{L^2}^2$$
$$+ h^{-2}\left\|\partial_1(h\hat{\psi}_3) + r_2\right\|_{L^2}^2 + h^{-2}\left\|\partial_2(h\hat{\psi}_3) - r_1\right\|_{L^2}^2 \le C\left\|\operatorname{sym}\nabla_h \boldsymbol{\psi}\right\|_{L^2}^2.$$

We do the regularization of $\boldsymbol{r}$, i.e., we look for the solution of the problem

$$\min_{\varphi \in H^1(\omega),\ \int_\omega \varphi = h\int_\omega \hat{\psi}_3} \int_\omega \left|\nabla_{\hat{x}}\varphi + (r_2, -r_1)^\top\right|^2 dx'. \tag{4.9}$$

The Euler-Lagrange equation and the associated boundary conditions for the problem (4.9) read

$$-\Delta'\varphi = \nabla_{\hat{x}} \cdot (r_2, -r_1)^\top \text{ in } \omega, \qquad \partial_\nu \varphi = -(r_2, -r_1)^\top \cdot \nu \text{ on } \partial\omega.$$

Since $\nabla_{\hat{x}} \cdot (r_2, -r_1) \in L^2$, by standard regularity estimates we obtain the inclusion $\varphi \in H^2(\omega)$ and the estimate

$$\left\|\varphi - h\fint_\Omega \hat{\psi}_3\right\|_{H^2(\omega)} \le C(\omega)\|\boldsymbol{r}\|_{H^1(\omega;\mathbb{R}^2)},$$

for which we require the $C^{1,1}$ regularity of $\partial\omega$. In particular, one has

$$\left\|\varphi + \fint_\omega r_2 x_1 - \fint_\omega r_1 x_2 - h\fint_\omega \hat{\psi}_3\right\|_{H^2} \le C\left\|\boldsymbol{r} - \fint_\omega \boldsymbol{r}\right\|_{H^1}.$$

Furthermore, from (4.9) we have the following inequalities:

$$\|\partial_1\varphi + r_2\|_{L^2}^2 + \|\partial_2\varphi - r_1\|_{L^2(\omega)}^2 \le \|\partial_1(h\hat{\psi}_3) + r_2\|_{L^2}^2 + \|\partial_2(h\hat{\psi}_3) - r_1\|_{L^2}^2$$

$$\begin{aligned}\left\|\nabla_{\hat{x}}\left(\hat{\psi}_3 - h^{-1}\varphi\right)\right\|_{L^2}^2 &\le h^{-2}\|\partial_1(h\hat{\psi}_3) + r_2\|_{L^2}^2 + h^{-2}\|\partial_1\varphi + r_2\|_{L^2}^2 \\ &\quad + h^{-2}\|\partial_2(h\hat{\psi}_3) - r_1\|_{L^2}^2 + h^{-2}\|\partial_2\varphi - r_1\|_{L^2}^2 \\ &\le 2(\|\partial_1(h\hat{\psi}_3) + r_2\|_{L^2}^2 + h^{-2}\|\partial_2(h\hat{\psi}_3) - r_1\|_{L^2}^2).\end{aligned}$$

The claim follows by taking

$$\boldsymbol{a} = \left(d_1, d_2, \fint_\omega \hat{\psi}_3\right)^\top, \quad \boldsymbol{B} = \begin{pmatrix} 0 & -c & h^{-1}\fint r_2 \\ c & 0 & -h^{-1}\fint r_1 \\ -h^{-1}\fint r_2 & h^{-1}\fint r_1 & 0 \end{pmatrix},$$

$$v = \varphi + \fint_\omega r_2 x_1 - \fint_\omega r_1 x_2 - h \fint_\omega \hat{\boldsymbol{\psi}}_3,$$
$$\tilde{\boldsymbol{\psi}} = \bar{\psi} + \left(\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3 - h^{-1}\varphi\right)^\top + (x_3(r_2 + \partial_1\varphi), x_3(-r_1 + \partial_2\varphi), 0)^\top - (-cx_2, cx_1, 0)^\top - (\boldsymbol{d}, 0)^\top.$$

This finishes the proof of part 1 of the lemma. To prove part 2, we only need to note that if $\boldsymbol{\psi} = 0$ on $\partial\omega$, then $\boldsymbol{r} = 0$, $\hat{\boldsymbol{\psi}} = 0$ on $\partial\omega$. Combining this (4.6), we infer

$$\|\boldsymbol{r}\|^2_{H^1} + \|(\hat{\psi}_1, \hat{\psi}_2)^\top\|_{H^1} + h^{-2}\|\bar{\boldsymbol{\psi}}^h\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla_h \bar{\boldsymbol{\psi}}\|^2_{L^2}$$
$$+ h^{-2}\|\partial_1(h\hat{\psi}_3) + r_2\|^2_{L^2} + h^{-2}\|\partial_2(h\hat{\psi}_3) - r_1\|^2_{L^2} \leq C\|\operatorname{sym}\nabla_h \boldsymbol{\psi}\|^2_{L^2}.$$

Furthermore, due to the condition $\boldsymbol{\psi} = 0$ on $\partial\omega$, in the "regularisation" of $(r_2, -r_1)^\top$ provided by the variational problem (4.9) we can minimise over $\varphi \in H^1_0(\omega)$ and we immediately obtain that

$$\|\varphi\|_{H^2} \leq C(\omega)\|\boldsymbol{r}\|_{H^1},$$

which replaces (4.2).

Part 3 is proved in [69]; alternatively, one can follow the argument used for part 1, taking into account (4.7) and noting that $h\pi_{1/h}\boldsymbol{\psi}^h \to 0$ in $L^2(\Omega;\mathbb{R}^3)$ implies the convergence $\boldsymbol{r}^h \rightharpoonup 0$ in $H^1(\omega;\mathbb{R}^2)$, $h\pi_{1/h}\hat{\boldsymbol{\psi}}^h \rightharpoonup 0$ in $H^1(\omega;\mathbb{R}^3)$, where r^h and $\hat{\boldsymbol{\psi}}^h$ are from the decomposition (4.4) applied to $\boldsymbol{\psi}^h$. (Note that within the described argument here one can set to zero the vectors $\boldsymbol{a}^h$, $\boldsymbol{B}^h$ in the decomposition (4.8) for $\boldsymbol{\psi}^h$.)

■

Remark 4.2.11. Following Remark 4.2.6, we note that if ψ_α, $\alpha = 1, 2$, are even in the variable and ψ_3 is odd in x_3 variable, then on has $v = 0$, $a_3 = 0$, $B_{13} = B_{23} = 0$ in (4.8). Moreover, the estimate

$$\|\psi_3\|_{L^2} = \|\tilde{\psi}_3\|_{L^2} = \left\|\tilde{\psi}_3 - \fint_I \tilde{\psi}_3\right\|_{L^2} \leq C\|\partial_3\tilde{\psi}_3\|_{L^2} \leq Ch\|\operatorname{sym}\nabla_h \boldsymbol{\psi}\|_{L^2}.$$

holds with $C > 0$ that depends on ω only.

4.3. Two-scale convergence

In this chapter we assume that $\Omega \subset \mathbb{R}^n$, if not otherwise stated, is a bounded open set with Lipschitz boundary. As before, we use the notation $I = (-1/2, 1/2)$. For $x \in \mathbb{R}^n$, we denote by $\hat{x}$ the first $n-1$ coordinates, thus $x = (\hat{x}, x_n)$. Depending on the context, the unit cell is $Y = [0,1)^n$ or $Y = [0,1)^{n-1}$, while $\mathcal{Y}$ denotes the unit flat torus in $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$, respectively.

Definition 4.3.1. (dimension-reduction two-scale convergence). Let $(u^\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $L^2(\Omega)$. We say that u^ε weakly two-scale converges to $u \in L^2(\Omega \times Y)$ with respect to Y if (in settings where $Y = [0,1)^{n-1}$)

$$\int_\Omega u^\varepsilon(x)\phi\left(x, \frac{\hat{x}}{\varepsilon}\right) dx \to \int_\Omega \int_Y u(x,y)\phi(x,y)\, dydx \qquad \forall \phi \in C_c^\infty\big(\Omega; C(\mathcal{Y})\big),$$

or (in settings where $Y = [0,1)^n$)

$$\int_\Omega u^\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Y u(x,y)\phi(x,y)\, dydx \qquad \forall \phi \in C_c^\infty\big(\Omega; C(\mathcal{Y})\big).$$

We write

$$u^\varepsilon \xrightharpoonup{dr-2} u(x,y).$$

Furthermore, we say that $(u^\varepsilon)_{\varepsilon>0}$ strongly two-scale converges to $u \in L^2(\Omega \times Y)$ if

$$\int_\Omega u^\varepsilon(x)\phi^\varepsilon(x)\, dx \to \int_\Omega \int_Y u(x,y)\phi(x,y)\, dydx,$$

for every weakly two-scale convergent sequence $\phi^\varepsilon(x) \xrightharpoonup{dr-2} \phi(x,y)$. We write

$$u^\varepsilon \xrightarrow{dr-2} u(x,y).$$

The following theorem is given in [48, Theorem 6.3.3].

Theorem 4.3.1. *Let $\Omega = \omega \times I$, where $\omega \subset \mathbb{R}^2$ is bounded and has Lipschitz boundary, and let $\varepsilon_h > 0$ be a sequence such that $\varepsilon_h \to 0$ as $h \to 0$ so that $\lim_{h\to 0} h/\varepsilon_h = \delta \in [0,\infty]$. Let $(\boldsymbol{u}^{\varepsilon_h})_{h>0}$ be a weakly convergent sequence in $H^1(\Omega;\mathbb{R}^3)$ with limit $\boldsymbol{u}$ and suppose that*

$$\limsup_{h\to 0} \|\nabla_h \boldsymbol{u}^{\varepsilon_h}\|_{L^2(\Omega;\mathbb{R}^3)} < \infty. \tag{4.10}$$

1. (a) *If $\delta \in (0,\infty)$ then there exists a function $\boldsymbol{w} \in L^2(\omega; H^1(I \times \mathcal{Y}; \mathbb{R}^3))$ and a subsequence (not relabelled) such that*

$$\nabla_h \boldsymbol{u}^{\varepsilon_h}(x) \xrightarrow{dr-2} \big(\nabla_{\hat{x}} \boldsymbol{u}(\hat{x}) | 0\big) + \widetilde{\nabla}_\delta \boldsymbol{w}(x,y).$$

   (b) *If $\delta \in (0,\infty)$ and in addition (4.10) we assume that*

$$\limsup_{h \to 0} h^{-1} \big\| \boldsymbol{u}^{\varepsilon_h} \big\|_{L^2(\Omega;\mathbb{R}^3)} < \infty,$$

   then there exists a function $\boldsymbol{w} \in L^2(\omega; H^1(I \times \mathcal{Y}; \mathbb{R}^3))$ and a subsequence (not relabeled) such that

$$h^{-1} \boldsymbol{u}^{\varepsilon_h}(x) \xrightarrow{dr-2} \boldsymbol{w}(x,y), \quad \nabla_h \boldsymbol{u}^{\varepsilon_h}(x) \xrightarrow{dr-2} \widetilde{\nabla}_\delta \boldsymbol{w}(x,y).$$

2. *If $\delta = 0$ then there exits $\boldsymbol{w} \in L^2(\omega; H^1(\mathcal{Y}; \mathbb{R}^3))$ and $\boldsymbol{g} \in L^2(\Omega \times Y; \mathbb{R}^3)$ such that*

$$\nabla_h \boldsymbol{u}^{\varepsilon_h}(x) \xrightarrow{dr-2} \big(\nabla_{\hat{x}} \boldsymbol{u}(\hat{x}) | 0\big) + \big(\nabla_y \boldsymbol{w} | \boldsymbol{g}\big).$$

3. *If $\delta = \infty$ then there exists $\boldsymbol{w} \in L^2(\Omega; H^1(\mathcal{Y}; \mathbb{R}^3))$, $\boldsymbol{g} \in L^2(\Omega; \mathbb{R}^3)$ such that*

$$\nabla_h \boldsymbol{u}^{\varepsilon_h}(x) \xrightarrow{dr-2} \big(\nabla_{\hat{x}} \boldsymbol{u}(\hat{x}) | 0\big) + \big(\nabla_y \boldsymbol{w} | \boldsymbol{g}\big).$$

We will need the following helpful lemma.

Lemma 4.3.2. *1. Suppose that $(\varphi^\varepsilon)_{\varepsilon>0} \subset H^1(\Omega)$ be a bounded sequence in $L^2(\Omega)$ such that $\varphi^\varepsilon \xrightarrow{dr-2} \varphi(x,y) \in L^2(\Omega \times \mathcal{Y})$. Suppose additionally that $\varepsilon \varphi^\varepsilon \to 0$ strongly in $H^1(\Omega)$. Then $\varphi(x,y)$ depends on x only.*

2. Suppose $(\varphi^\varepsilon)_{\varepsilon>0} \subset H^2(\Omega)$ be a bounded sequence in $L^2(\Omega)$ such that $\varphi^\varepsilon \xrightarrow{dr-2} \varphi(x,y) \in L^2(\Omega \times \mathcal{Y})$. Suppose additionally that $\varepsilon^2 \varphi^\varepsilon \to 0$ strongly in $H^2(\Omega)$. Then $\varphi(x,y)$ depends on x only.

Proof. We write

$$\varphi(x,y) = \sum_{k \in \mathbb{Z}^n} a_k(x) \exp\big(2\pi \mathrm{i}(k,y)\big), \qquad a_k \in L^2(\Omega, \mathbb{C}^n), \quad \sum_{k \in \mathbb{Z}^n} \int |a_k(x)|^2 < \infty.$$

We want to show that for $k \neq 0$ we have that $a_k = 0$. We take an arbitrary $b \in C_0^\infty(\Omega)$ and $i \in \{1, \ldots, n\}$ such that $k_i \neq 0$ and calculate

$$\int_\Omega a_k(x) b(x) dx = \int_{\Omega \times Y} \varphi(x,y) b(x) \exp\big(2\pi \mathrm{i}(k,y)\big) dx dy$$

$$= \lim_{\varepsilon\to 0}\int_\Omega \varphi^\varepsilon(x)b(x)\exp\Big(2\pi\mathrm{i}\Big(k,\frac{x}{\varepsilon}\Big)\Big)dx$$

$$= \lim_{\varepsilon\to 0}\int_\Omega \frac{\varepsilon}{2\pi\mathrm{i}k_i}\varphi^\varepsilon(x)b(x)\partial_{x_i}\Big(\exp\Big(2\pi\mathrm{i}\Big(k,\frac{x}{\varepsilon}\Big)\Big)\Big)dx$$

$$= -\lim_{\varepsilon\to 0}\Big\{\int_\Omega \frac{\varepsilon}{2\pi\mathrm{i}k_i}\partial_{x_i}\varphi^\varepsilon(x)b(x)\exp\Big(2\pi\mathrm{i}\Big(k,\frac{x}{\varepsilon}\Big)\Big)$$

$$+\int_\Omega \frac{\varepsilon}{2\pi\mathrm{i}k_i}\varphi^\varepsilon\partial_{x_i}b(x)\exp\Big(2\pi\mathrm{i}\Big(k,\frac{x}{\varepsilon}\Big)\Big)\Big\} = 0.$$

From this we infer that for $k \neq 0$, $a_k = 0$, and the claim follows. The proof of the second claim is similar. ∎

The following claim can be proved directly by integration by parts.

Lemma 4.3.3. *1. Let $(\varphi^\varepsilon)_{\varepsilon>0} \subset H^2(\Omega)$ be a bounded sequence. Suppose that $\varphi^\varepsilon \to \varphi_0$ strongly in $L^2(\Omega)$ and $\nabla^2\varphi^\varepsilon \xrightarrow{dr-2} \psi$, where $\psi \in L^2(\Omega\times Y;\mathbb{R}^{n\times n})$. Then there exists $\varphi_1 \in L^2(\Omega;H^2(\mathcal{Y}))$ such that*

$$\nabla^2\varphi^\varepsilon \xrightarrow{dr-2} \nabla^2\varphi_0(x) + \nabla_y^2\varphi_1(x,y).$$

2. *Suppose that $(\varphi^{\varepsilon_h})_{h>0} \subset H^2(\Omega)$ is a bounded sequence such that $h^{-1}\varphi^{\varepsilon_h} \xrightarrow{dr-2} \varphi(x,y)$ and $\lim_{h\to 0}{\varepsilon_h}^{-2}h = \kappa \in [0,\infty)$. Then $\nabla^2\varphi^{\varepsilon_h} \xrightarrow{dr-2} \kappa\nabla_y^2\varphi(x,y)$.*

3. (a) *Let $(\varphi^\varepsilon)_{\varepsilon>0} \subset H^2(\omega)$ be such that the sequences $(\varphi^\varepsilon)_{\varepsilon>0}$, $(\varepsilon\nabla\varphi^\varepsilon)_{\varepsilon>0}$ are bounded in the corresponding L^2 spaces. Suppose that $\varphi^\varepsilon \xrightarrow{dr-2} \varphi(x,y) \in L^2(\Omega\times Y)$. Then $\varphi \in L^2(\Omega;H^1(\mathcal{Y}))$ and $\varepsilon\nabla\varphi^\varepsilon \xrightarrow{dr-2} \nabla_y\varphi(x,y)$.*

   (b) *Let $(\varphi^\varepsilon)_{\varepsilon>0} \subset H^2(\omega)$ be such that the sequences $(\varphi^\varepsilon)_{\varepsilon>0}$, $(\varepsilon\nabla\varphi^\varepsilon)_{\varepsilon>0}$, $(\varepsilon^2\nabla^2\varphi^\varepsilon)_{\varepsilon>0}$ are bounded in L^2. Suppose that $\varphi^\varepsilon \xrightarrow{dr-2} \varphi(x,y) \in L^2(\Omega\times Y)$. Then $\varphi \in L^2(\Omega;H^2(\mathcal{Y}))$ and $\varepsilon\nabla\varphi^\varepsilon \xrightarrow{dr-2} \nabla_y\varphi(x,y)$, $\varepsilon^2\nabla^2\varphi^\varepsilon \xrightarrow{dr-2} \nabla_y^2\varphi(x,y)$.*

We will prove the following lemma.

Lemma 4.3.4. *Let $\Omega = \omega\times I$, where $\omega \subset \mathbb{R}^2$ a bounded set with Lipschitz boundary and let $(\psi^{\varepsilon_h})_{h>0} \subset H^1(\Omega)$ be such that there exists $C > 0$ such that*

$$\|\psi^{\varepsilon_h}\|^2_{L^2(\Omega)} + \varepsilon_h^2\|\nabla_h\psi^{\varepsilon_h}\|^2_{L^2(\Omega;\mathbb{R}^3)} \leq C. \tag{4.11}$$

1. *If $h \ll \varepsilon_h$ then there exist $\psi_1 \in H^1(\omega \times \mathcal{Y})$, $\psi_2 \in L^2(\Omega \times \mathcal{Y})$ such that (on a subsequence)*

$$\psi^{\varepsilon_h} \xrightarrow{dr-2} \psi_1, \quad \varepsilon_h \nabla_h \psi^{\varepsilon_h} \xrightarrow{dr-2} (\partial_{y_1}\psi_1, \partial_{y_2}\psi_1, \psi_2). \tag{4.12}$$

The opposite claim is also valid, i.e., for every $\psi_1 \in H^1(\omega \times \mathcal{Y})$, $\psi_2 \in H^1(\Omega \times \mathcal{Y})$ we have that there exists $(\psi^{\varepsilon_h})_{h>0} \subset H^1(\Omega)$ such that (4.11) and (4.12) are satisfied.

2. *If $\varepsilon_h \ll h$ then there exists $\psi \in L^2(\Omega, H^1(\mathcal{Y}))$ such that*

$$\psi^{\varepsilon_h} \xrightarrow{dr-2} \psi, \quad \varepsilon_h \nabla_h \psi^{\varepsilon_h} \xrightarrow{dr-2} (\partial_{y_1}\psi, \partial_{y_2}\psi, 0). \tag{4.13}$$

The opposite claim is also valid, i.e., for every $\psi \in L^2(\Omega; H^1(\mathcal{Y}; \mathbb{R}^2))$ there exists $(\psi^{\varepsilon_h})_{h>0} \subset H^1(\Omega)$ such that (4.11) and (4.13) hold.

Proof. To prove the first part of the lemma, we take $\psi_1 \in L^2(\Omega \times Y; \mathbb{R}^3)$ such that $\psi^{\varepsilon_h} \xrightarrow{dr-2} \psi_1$ on a subsequence. Since, by assumption,

$$\|\partial_{x_3}\psi^{\varepsilon_h}\|_{L^2(\Omega)} \leq C\frac{h}{\varepsilon_h},$$

we immediately conclude that ψ_1 does not depend on x_3. Denote the two-scale limit of $h^{-1}\varepsilon_h \partial_{x_3}\psi^{\varepsilon_h}$ by ψ_2. Invoking integration by parts in a standard fashion, it is easy to check that

$$\varepsilon_h(\partial_1\psi^{\varepsilon_h}, \partial_2\psi^{\varepsilon_h}) \xrightarrow{dr-2} (\partial_{y_1}\psi_1, \partial_{y_2}\psi_1).$$

In order to prove the second claim of part 1, it suffices to consider the case $\psi_1 \in C^1(\omega; C^1(\mathcal{Y}))$, $\psi_2 \in C^1(\Omega; C^1(\mathcal{Y}))$. We can then take

$$\psi^{\varepsilon_h} := \psi_1\left(x_1, x_2, \frac{x_1}{\varepsilon_h}, \frac{x_2}{\varepsilon_h}\right) + \frac{h}{\varepsilon_h}\int_{-1/2}^{x_3} \psi_2\left(x_1, x_2, s, \frac{x_1}{\varepsilon_h}, \frac{x_2}{\varepsilon_h}\right) ds.$$

This completes the proof of part 1.

To prove part 2, we take $\psi \in L^2(\Omega \times Y; \mathbb{R}^3)$ such that $\psi^{\varepsilon_h} \xrightarrow{dr-2} \psi$ on a subsequence. Again, using integration by parts, we obtain

$$\varepsilon_h(\partial_1\psi^{\varepsilon_h}, \partial_2\psi^{\varepsilon_h}) \xrightarrow{dr-2} (\partial_{y_1}\psi, \partial_{y_2}\psi).$$

Next, for $b \in C_0^1(\Omega), v \in C^1(\mathcal{Y})$ we have

$$\int_\Omega \frac{\varepsilon_h}{h}\partial_{x_3}\psi^{\varepsilon_h} b(x) v\left(\frac{\hat{x}}{\varepsilon_h}\right) dx = -\int_\Omega \frac{\varepsilon_h}{h}\psi^{\varepsilon_h}\partial_{x_3} b(x) v\left(\frac{\hat{x}}{\varepsilon_h}\right) dx \to 0.$$

It follows that $\partial_{x_3}\psi^{\varepsilon_h} \xrightarrow{dr-2} 0$. To prove the last claim, we set

$$\psi^{\varepsilon_h} := \psi\left(x, \frac{x_1}{\varepsilon_h}, \frac{x_2}{\varepsilon_h}\right)$$

for $\psi \in C^1(\Omega, C^1(\mathcal{Y}))$ and pass to the limit as $h \to 0$. ■

The definition of two-scale convergence (Definition 4.3.1) naturally extends to time dependent spaces.

Definition 4.3.2. Let $(u^\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $L^2([0,T];L^2(\Omega))$. We say that $(u^\varepsilon)_{\varepsilon>0}$ weakly two-scale converges to $u \in L^2([0,T];L^2(\Omega\times Y))$, and write

$$u^\varepsilon \xrightarrow{\text{t,dr}-2} u(t,x,y),$$

if

$$\int_0^T\int_\Omega u^\varepsilon(t,x)\phi\left(x,\frac{\hat{x}}{\varepsilon}\right)\varphi(t)\,dxdt \to \int_0^T\int_\Omega\int_Y u(t,x,y)\phi(x,y)\varphi(t)\,dydxdt,$$

i.e.,

$$\int_0^T\int_\Omega u^\varepsilon(t,x)\phi\left(x,\frac{x}{\varepsilon}\right)\varphi(t)\,dxdt \to \int_0^T\int_\Omega\int_Y u(t,x,y)\phi(x,y)\varphi(t)\,dydxdt,$$

for every $\phi \in C_c^\infty(\Omega;C(\mathcal{Y}))$, $\varphi \in C(0,T)$. If in addition one has

$$u^\varepsilon(t,x) \xrightarrow{dr-2} u(t,x,y) \quad \text{a.e. } t \in [0,T]$$

and

$$\int_0^T \left\|u^\varepsilon(t,\cdot)\right\|^2_{L^2(\Omega)}dt \to \int_0^T \left\|u(t,\cdot)\right\|^2_{L^2(\Omega)}dt,$$

then we will say that $(u^\varepsilon)_{\varepsilon>0}$ strongly two-scale converges to u and write

$$u^\varepsilon \xrightarrow{\text{t,dr}-2} u(t,x,y).$$

Similarly, we define the notions of weak two-scale convergence and strong two-scale convergence of sequences in $L^p([0,T];L^2(\Omega))$, for any $1 \le p < \infty$, denoted by $\xrightarrow{t,p,\text{dr}-2}$ and $\xrightarrow{t,p,\text{dr}-2}$, respectively. The convergence $\xrightarrow{t,\infty,\text{dr}-2}$ is understood in the weak* sense with respect to the time variable t, while $\xrightarrow{t,\infty,\text{dr}-2}$ is understood in the sense of simultaneous pointwise convergence (4.3.2) and boundedness of $\|u^\varepsilon(t,\cdot)\|_{L^2(\Omega)}$, $t \in [0,T]$, in the space $L^\infty(0,T)$. The following lemma is standard (see e.g. the proof of [53, Lemma 4.7].

Lemma 4.3.5. *If $(u^\varepsilon)_{\varepsilon>0}$ is a bounded sequence in $L^p([0,T];L^2(\Omega))$, $p > 1$, then it has a subsequence that converges weakly two-scale in the sense of Definition 4.3.2.*

4.4. Extension theorems

4.4.0.1 Asymptotic regime $h \sim \varepsilon_h$

We use the extension theory in order to decompose the sequence of displacements into two functions for which we can get enough compactness for passing to the limit as $h \to 0$ in the original equations for displacements. We use the following result, which can be found in [51]:

Theorem 4.4.1. *For every $h > 0$ there exists a linear extension operator $\tilde{\cdot} : H^1(\Omega_1^{\varepsilon_h}; \mathbb{R}^3) \to H^1(\Omega; \mathbb{R}^3)$ such that $\tilde{\boldsymbol{u}} = \boldsymbol{u}$ on $\Omega_1^{\varepsilon_h}$ and*

$$\left\| \operatorname{sym} \nabla_h \tilde{\boldsymbol{u}} \right\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \le C \left\| \operatorname{sym} \nabla_h \boldsymbol{u} \right\|_{L^2(\Omega_1^{\varepsilon_h};\mathbb{R}^{3\times 3})}.$$

Proof. The way in which we introduce the extensions here is to look at every single cell inside the thin domain Ω^h. The extension of the function $\boldsymbol{u}$ (defined on Ω^h) is constructed as follows. First we "inflate" the cell (with the scaling factors ε_h^{-1} and h^{-1} in the in-plane and out-of plane directions, respectively) and translate it to the reference cell $Y \times I$. On this reference cell we apply a linear extension operator $E : H^1(Y_1 \times I) \to H^1(Y \times I)$ (see [51, Lemma 4.1] to extend $\boldsymbol{u}$ to the function $E\boldsymbol{u} =: \tilde{\boldsymbol{u}}$. By passing to the original coordinates and concatenating the extensions we construct functions $\tilde{\boldsymbol{u}} \in H^1(\Omega^h; \mathbb{R}^3)$ which is the extension of $\boldsymbol{u}$ from $\Omega_1^{h,\varepsilon_h}$ to Ω^h and satisfy the estimate

$$\left\| \operatorname{sym} \nabla \tilde{\boldsymbol{u}} \right\|_{L^2(\Omega^h;\mathbb{R}^{3\times 3})} \le C \left\| \operatorname{sym} \nabla \boldsymbol{u} \right\|_{L^2(\Omega_1^{h,\varepsilon_h};\mathbb{R}^{3\times 3})},$$

where the constant C does not depend on the thickness h. We finish the proof by rescaling the estimates back to Ω. ■

Remark 4.4.2. It is not difficult to see that if we have a sequence $\{(h, \varepsilon_h)\}$ such that $0 < \alpha < h/\varepsilon_h < \beta < \infty$, where α, β do not depend of h, then the constant in Theorem 4.4.1 depends on α, β only.

Remark 4.4.3. By inspecting the construction of the extension operator in [51, Lemma 4.1], it can be easily seen that if $\boldsymbol{u} \in L^{2,\mathrm{bend}}(\Omega; \mathbb{R}^3)$ or $\boldsymbol{u} \in L^{2,\mathrm{memb}}(\Omega; \mathbb{R}^3)$, then the same inclusion holds for $\tilde{\boldsymbol{u}}$.

Using the result above we conclude the following lemma.

Lemma 4.4.4. *For all $\mathring{\boldsymbol{u}} \in H^1(\Omega;\mathbb{R}^3)$ such that $\mathring{\boldsymbol{u}}|_{\Omega_1^{\varepsilon_h}} = 0$, the following Poincaré and Korn inequalities hold:*

$$\|\mathring{\boldsymbol{u}}\|_{L^2} \leq C\varepsilon_h \|\nabla_h \mathring{\boldsymbol{u}}\|_{L^2}, \tag{4.14}$$

$$\|\nabla_h \mathring{\boldsymbol{u}}\|_{L^2} \leq C \left\| \operatorname{sym} \nabla_h \mathring{\boldsymbol{u}} \right\|_{L^2}, \tag{4.15}$$

where the constant C is independent of h.

Proof. We again look the problem on the physical domain Ω^h. The function $\boldsymbol{u}$ (scaled properly), when restricted to a single cell within the domain and then rescaled and translated to $Y \times I$, satisfies the Poincare inequality, as well as Korn's inequality (4.2) with a constant determined by the domain $Y \times I$. Scaling back to the physical domain Ω^h and summing up the norms over all cells, we obtain a version of the estimates (4.14) and (4.15) for Ω^h. Finally, rescaling to Ω, we obtain (4.14) and (4.15). ■

Remark 4.4.5. Using continuity of embeddings into spaces on the boundary, it follows immediately that

$$\|\mathring{\boldsymbol{u}}\|_{L^2(\Gamma\cap\overline{\Omega}_0)} \leq C\varepsilon_h \left\| \operatorname{sym} \nabla_h \mathring{\boldsymbol{u}} \right\|_{L^2(\Omega;\mathbb{R}^{3\times 3})},$$

where $\Gamma := \omega \times \{-1/2, 1/2\}$.

4.4.0.2 Asymptotic regime $h \ll \varepsilon_h$

In this section we assume that $h \ll \varepsilon_h$. We will assume that $Y_0 \subset Y$ does not touch the boundary of Y and is of class $C^{1,1}$. First, we provide an extension property, in the spirit of Theorem 4.4.1. We denote by $\Omega_\alpha^{\varepsilon_h}$, $\alpha = 1, 2$, the same sets as in Section 2.1. We have the following theorem.

Theorem 4.4.6. *There exists a linear extension $E^{\varepsilon_h} : H^1(\Omega_1^{\varepsilon_h};\mathbb{R}^3) \to H^1(\Omega;\mathbb{R}^3)$ such that for every $\boldsymbol{u} \in H^1(\Omega,\mathbb{R}^3)$, $E^{\varepsilon_h}\boldsymbol{u} = \boldsymbol{u}$ on $\Omega_1^{\varepsilon_h}$ and*

$$\left\| \operatorname{sym} \nabla_h E^{\varepsilon_h} \boldsymbol{u} \right\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C \left\| \operatorname{sym} \nabla_h \boldsymbol{u} \right\|_{L^2(\Omega_1^{\varepsilon_h},\mathbb{R}^{3\times 3})}. \tag{4.16}$$

for some $C > 0$ independent of h. Moreover, there exist $\mathring{v} \in H^2(\omega)$ and $\mathring{\boldsymbol{\psi}} \in H^1(\Omega;\mathbb{R}^3)$ such that $\mathring{v} = \mathring{\boldsymbol{\psi}} = 0$ on $\Omega_1^{\varepsilon_h}$ and

$$\mathring{\boldsymbol{u}} := \boldsymbol{u} - E^{\varepsilon_h}\boldsymbol{u} = \begin{pmatrix} -\varepsilon_h x_3 \partial_1 \mathring{v} \\ -\varepsilon_h x_3 \partial_2 \mathring{v} \\ h^{-1}\varepsilon_h \mathring{v} \end{pmatrix} + \mathring{\boldsymbol{\psi}}, \tag{4.17}$$

with the estimate

$$\varepsilon_h{}^{-2}\|\mathring{v}\|_{L^2}^2 + \|\nabla \mathring{v}\|_{L^2}^2 + \varepsilon_h^2 \left\|\nabla^2 \mathring{v}\right\|_{L^2}^2 + \varepsilon_h{}^{-2}\left\|\mathring{\boldsymbol{\psi}}\right\|_{L^2}^2 + \left\|\nabla_h \mathring{\boldsymbol{\psi}}\right\|_{L^2}^2 \le C\left\|\operatorname{sym}\nabla_h \mathring{\boldsymbol{u}}\right\|_{L^2}^2, \tag{4.18}$$

where C depends on Y_0 only.

Proof. We consider a domain $\tilde{Y}_0 \times (h/\varepsilon_h)I$, such that $\tilde{Y}_0$ has $C^{1,1}$ boundary, $\overline{Y}_0 \subset \tilde{Y}_0$, and $\tilde{Y}_0 \backslash \overline{Y}_0$ is connected. For the extension $\boldsymbol{\psi} \in H^1((\tilde{Y}_0\backslash\overline{Y}_0)\times I;\mathbb{R}^3)$, we apply the decomposition of Part 1 of Lemma 4.2.10 to obtain

$$\boldsymbol{\psi} = \boldsymbol{a} + \boldsymbol{B}(x_1, x_2, \varepsilon_h^{-1} h x_3)^\top + \begin{pmatrix} -x_3\partial_1 v \\ -x_3 \partial_2 v \\ h^{-1}\varepsilon_h v \end{pmatrix} + \tilde{\boldsymbol{\psi}}, \tag{4.19}$$

where $\boldsymbol{a} \in \mathbb{R}^3$, $\boldsymbol{B} \in \mathbb{R}^{3\times 3}_{\text{skew}}$, $v \in H^2(\tilde{Y}_0\backslash\overline{Y}_0)$, $\tilde{\boldsymbol{\psi}} \in H^1((\tilde{Y}_0\backslash\overline{Y}_0)\times I;\mathbb{R}^3)$, and the following estimate holds:

$$\|v\|^2_{H^2(\tilde{Y}_0\backslash\overline{Y}_0)} + \|\tilde{\boldsymbol{\psi}}\|_{L^2((\tilde{Y}_0\backslash\overline{Y}_0)\times I;\mathbb{R}^3)} + \|\nabla_{h/\varepsilon_h}\tilde{\boldsymbol{\psi}}\|^2_{L^2((\tilde{Y}_0\backslash\overline{Y}_0)\times I;\mathbb{R}^{3\times3})} \le C\|\operatorname{sym}\nabla_{h/\varepsilon_h}\boldsymbol{\psi}\|^2_{L^2((\tilde{Y}_0\backslash\overline{Y}_0)\times I;\mathbb{R}^{3\times3})}, \tag{4.20}$$

where C depends on Y_0 only. It is not difficult to construct extension operators

$$E_1 : H^2(\tilde{Y}_0\backslash\overline{Y}_0) \to H^2(\tilde{Y}_0), \qquad E_2 : H^1\big((\tilde{Y}_0\backslash\overline{Y}_0)\times I\big) \to H^1(\tilde{Y}_0 \times I)$$

such that $E_1\varphi = \varphi$ on $\tilde{Y}_0\backslash Y_0$ and $E_2 w = w$ on $(\tilde{Y}_0\backslash Y_0)\times I$ and

$$\|E_1\varphi\|_{L^2(\tilde{Y}_0)} \le C\|\varphi\|_{L^2(\tilde{Y}_0\backslash\overline{Y}_0)}, \quad \|E_1\varphi\|_{H^2(\tilde{Y}_0)} \le C\|\varphi\|_{H^2(\tilde{Y}_0\backslash\overline{Y}_0)} \qquad \forall\varphi \in H^2(\tilde{Y}_0\backslash\overline{Y}_0),$$

$$\|E_2 w\|_{L^2(\tilde{Y}_0)\times I)} \le C\|w\|_{L^2((\tilde{Y}_0\backslash\overline{Y}_0))\times I)},$$

$$\|\nabla_{h/\varepsilon_h} E_2 w\|_{L^2(\tilde{Y}_0)\times I;\mathbb{R}^3)} \le C\|\nabla_{h/\varepsilon_h} w\|_{L^2((\tilde{Y}_0\backslash\overline{Y}_0))\times I;\mathbb{R}^3)} \qquad \forall w \in H^1(\tilde{Y}_0\backslash\overline{Y}_0 \times I), \tag{4.21}$$

for some $C>0$. Indeed, E_1 is constructed by using the standard reflection principle. Also using the reflection principle, we first construct $\tilde{E}_2 : H^1(\tilde{Y}_0\backslash\overline{Y}_0) \to H^1(\tilde{Y}_0)$ such that

$$\|\tilde{E}_2\varphi\|_{L^2(\tilde{Y}_0)} \le C\|\varphi\|_{L^2(\tilde{Y}_0\backslash\overline{Y}_0)}, \quad \|\tilde{E}_2\varphi\|_{H^1(\tilde{Y}_0)} \le C\|\varphi\|_{H^1(\tilde{Y}_0\backslash\overline{Y}_0)} \qquad \forall\varphi\in H^1(\tilde{Y}_0\backslash\overline{Y}_0),$$

for some $C>0$. On the basis of $\tilde{E}_2$, we construct E_2 as follows. For $w\in C^2(\overline{\tilde{Y}_0\backslash\overline{Y}_0\times I})$ we set $E_2w(\cdot,x_3)=\tilde{E}_2w(\cdot,x_3)$ for all $x_3\in I$. It is easy to check that for $w\in C^2(\overline{\tilde{Y}_0\backslash\overline{Y}_0\times I})$ one has $\partial_{x_3}E_2w=E_2(\partial_{x_3}w)$, from which we infer the property (4.21) for $w\in C^2(\overline{\tilde{Y}_0\backslash\overline{Y}_0\times I})$. We then extend E_2 to the whole of $H^1(\tilde{Y}_0\backslash\overline{Y}_0\times I)$ by density, which concludes the construction.

For $\boldsymbol{\psi}\in H^1(\tilde{Y}_0\backslash\overline{Y}_0\times I;\mathbb{R}^3)$, using the expression (4.19), we define $\tilde{E}^{\varepsilon_h}\boldsymbol{\psi}\in H^1(\tilde{Y}_0\times I;\mathbb{R}^3)$ as follows:

$$\tilde{E}^{\varepsilon_h}\boldsymbol{\psi}=\boldsymbol{a}+\boldsymbol{B}(x_1,x_2,\varepsilon_h^{-1}hx_3)^\top+\begin{pmatrix}-x_3\partial_1E_1v\\ -x_3\partial_2E_1v\\ \varepsilon_h h^{-1}E_1v\end{pmatrix}+E_2\tilde{\boldsymbol{\psi}}. \tag{4.22}$$

Recalling (4.20), we obtain the estimate

$$\|E_1v\|^2_{H^2(\tilde{Y}_0)}+\|E_2\tilde{\boldsymbol{\psi}}\|^2_{L^2(\tilde{Y}_0;\mathbb{R}^3)}+\|\nabla_{h/\varepsilon_h}E_2\tilde{\boldsymbol{\psi}}\|^2_{L^2(\tilde{Y}_0;\mathbb{R}^{3\times3})}\le C\|\operatorname{sym}\nabla_{h/\varepsilon_h}\boldsymbol{\psi}\|^2_{L^2((\tilde{Y}_0\backslash\overline{Y}_0)\times I;\mathbb{R}^{3\times3})}.$$

We construct $E^{\varepsilon_h}\boldsymbol{u}$ by considering $z\in\mathbb{Z}^2$ such that $\varepsilon_h(Y+z)\subset\omega$ and applying the extension $\tilde{E}^{\varepsilon_h}$ to the function $x\mapsto\boldsymbol{u}(\varepsilon_h\hat{x}+\varepsilon_h z,x_3)$. In this way we obtain

$$E^{\varepsilon_h}\boldsymbol{u}\big|_{\varepsilon_h(\tilde{Y}_0+z)\times I}=\boldsymbol{a}_z^{\varepsilon_h}+\boldsymbol{B}_z^{\varepsilon_h}(x_1,x_2,hx_3)^\top+\begin{pmatrix}-x_3\partial_1v_z\\ -x_3\partial_2v_z\\ h^{-1}v_z\end{pmatrix}+\boldsymbol{\psi}_z, \tag{4.23}$$

with the estimate

$$\begin{aligned}\varepsilon_h^{-4}\|v_z\|^2_{L^2(\varepsilon_h(\tilde{Y}_0+z))}&+\varepsilon_h^{-2}\|\nabla v_z\|^2_{L^2(\varepsilon_h(\tilde{Y}_0+z);\mathbb{R}^2)}+\left\|\nabla^2v_z\right\|^2_{L^2(\varepsilon_h(\tilde{Y}_0+z);\mathbb{R}^{2\times2})}\\ &+\varepsilon_h^{-2}\|\boldsymbol{\psi}_z\|^2_{L^2(\varepsilon_h(\tilde{Y}_0+z)\times I;\mathbb{R}^3)}+\|\nabla_h\boldsymbol{\psi}_z\|^2_{L^2(\varepsilon_h(\tilde{Y}_0+z)\times I;\mathbb{R}^{3\times3})}\\ &\le C\|\operatorname{sym}\nabla_h\boldsymbol{u}\|^2_{L^2(\varepsilon_h(\tilde{Y}_0\backslash\bar{Y}_0+z)\times I;\mathbb{R}^{3\times3})}.\end{aligned}$$

This concludes the proof of (4.16). To prove (4.17), for each $z\in\mathbb{Z}^2$ consider the deformation

$$\boldsymbol{u}-E^{\varepsilon_h}\boldsymbol{u}|_{\varepsilon(\tilde{Y}_0+z)\times I}$$

and apply the above rescaling as well as the second claim of Lemma 4.2.10. The linearity of E^{ε_h} follows from the decompositions in Theorem 4.2.3 and Lemma 4.2.10. ■

Remark 4.4.7. Notice that as a consequence of Corollary 4.2.5 and the estimate (4.16), if $\boldsymbol{u} = 0$ on $\gamma_{\rm D} \times I$, where $\gamma_{\rm D} \subset \partial\omega$ is the set of positive measure, then

$$\left\|\pi_{1/h}(E^{\varepsilon_h}\boldsymbol{u})\right\|^2_{H^1(\Omega;\mathbb{R}^3)} \leq C\Big(\|\pi_{1/h}\boldsymbol{u})\|^2_{L^2(\Gamma_D;\mathbb{R}^3)} + h^{-2}\|\operatorname{sym}\nabla_h\boldsymbol{u}\|^2_{L^2(\Omega_1^{\varepsilon_h},\mathbb{R}^{3\times3})}\Big),$$

where C is obtained by combining $C_{\rm T}^{\gamma}$ in (4.2.5) and the constant in the extension inequality (4.16).

Remark 4.4.8. We can assume, without loss of generality, that in the above proof E_1 maps affine expressions $a_1x_1 + a_2x_2 + a_3$, for $a_1, a_2, a_3 \in \mathbb{R}$, to themselves. (Indeed, as in the proof of Proposition 2.2.4, on the orthogonal complement of affine maps in L^2 the extension is constructed by reflection.) From (4.23), recalling (4.19) and (4.22), we also have the estimate

$$\|E^{\varepsilon_h}\boldsymbol{u}\|_{L^2(\Omega;\mathbb{R}^3)} \leq C\left(\|\boldsymbol{u}\|_{L^2(\Omega_1^{\varepsilon_h};\mathbb{R}^3)} + \|\operatorname{sym}\nabla_h\boldsymbol{u}\|_{L^2(\Omega_1^{\varepsilon_h};\mathbb{R}^{3\times3})}\right),$$

for some $C > 0$. From (4.17), (4.22) we then additionally obtain that

$$\varepsilon_h h^{-1}\|\mathring{v}\|_{L^2(\omega)} \leq C\left(\|\boldsymbol{u}\|_{L^2(\Omega;\mathbb{R}^3)} + \|\operatorname{sym}\nabla_h\boldsymbol{u}\|_{L^2(\Omega_1^{\varepsilon_h};\mathbb{R}^{3\times3})} + \varepsilon_h\|\operatorname{sym}\nabla_h\boldsymbol{u}\|_{L^2(\Omega_0^{\varepsilon_h};\mathbb{R}^{3\times3})}\right).$$

Remark 4.4.9. Following Remark 4.2.6 and Remark 4.2.11 we infer that if u_α is even in the variable x_3 for $\alpha = 1,2$ and u_3 is odd in the variable x_3, the extension $E^{\varepsilon_h}\boldsymbol{u}$ has the same properties. Noting that by Lemma 4.2.10 and Theorem 4.2.3 one has $a^{\varepsilon_h}_{z,3} = B^{\varepsilon_h}_{z,13} = B^{\varepsilon_h}_{z,23} = v^{\varepsilon_h}_z = \mathring{v}^{\varepsilon_h} = 0$ in (4.23), we also infer that

$$\left\|(E^{\varepsilon_h}\boldsymbol{u})_3\right\|_{L^2(\Omega)} \leq Ch\|\operatorname{sym}\nabla_h\boldsymbol{u}\|_{L^2(\Omega_1^{\varepsilon_h};\mathbb{R}^{3\times3})},$$

$$\|\boldsymbol{u}_3\|_{L^2(\Omega)} \leq \|\mathring{\boldsymbol{\psi}}_3\|_{L^2(\Omega;\mathbb{R}^3)} \leq C\varepsilon_h{}^{-1}h\varepsilon_h\|\operatorname{sym}\nabla_h\boldsymbol{u}\|_{L^2(\Omega;\mathbb{R}^{3\times3})} = Ch\|\operatorname{sym}\nabla_h\boldsymbol{u}\|_{L^2(\Omega;\mathbb{R}^{3\times3})},$$

for some $C > 0$.

4.4.0.3 Asymptotic regime $\varepsilon_h \ll h$

In this regime the extension theorem is analogous to Theorem 4.4.1.

Theorem 4.4.10. *For every $\varepsilon_h > 0$ there exists a linear extension operator $u \mapsto \tilde{u} : H^1(\Omega_1^{\varepsilon_h};\mathbb{R}^3) \to H^1(\Omega;\mathbb{R}^3)$ such that $\tilde{\boldsymbol{u}} = \boldsymbol{u}$ on $\Omega_1^{\varepsilon_h}$ and*

$$\left\| \operatorname{sym} \nabla_h \tilde{\boldsymbol{u}} \right\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C \left\| \operatorname{sym} \nabla_h \boldsymbol{u} \right\|_{L^2(\Omega_1^{\varepsilon_h};\mathbb{R}^{3\times 3})}.$$

Proof. Since $\varepsilon_h \ll h$, the way in which we introduce the extensions here is to look at every single cell inside the thin domain Ω^h. The cells are ε_h-cubes $\varepsilon_h(Y+z)\times[k\varepsilon_h,(k+1)\varepsilon_h]$, where $z \in \mathbb{Z}^2$, $k \in \mathbb{Z}$, are chosen so that each cube is entirely inside Ω. We use the extension operator on the cube $Y \times I$ (see [51, Lemma 4.1]), followed by a scaling argument. We label the resulting extension by E_1. The problem is that we can have a mismatch at the lines $x_3 = k\varepsilon_h$, where $k \in \mathbb{Z}$, and there are possible "boundary layers" at the sides $x_3 = \pm h/2$ where the extension is not defined (due to the fact that h/ε_h is not necessarily an integer). We deal with this by introducing another series of extensions to cubes

$$\varepsilon_h(Y+z)\times\left[\left(k+\frac{1}{2}\right)\varepsilon_h, \left(k+\frac{3}{2}\right)\varepsilon_h\right], \qquad z \in \mathbb{Z}^2, \quad k \in \mathbb{Z},$$

from the complements of the corresponding "perforations"

$$\varepsilon_h(Y_0+z)\times\left[\left(k+\frac{3}{4}\right)\varepsilon_h, \left(k+\frac{5}{4}\right)\varepsilon_h\right].$$

(On the parts

$$\varepsilon_h(Y_0+z)\times\left[\left(k+\frac{1}{2}\right)\varepsilon_h, \left(k+\frac{3}{4}\right)\varepsilon_h\right], \quad \varepsilon_h(Y_0+z)\times\left[\left(k+\frac{5}{4}\right)\varepsilon_h, \left(k+\frac{3}{2}\right)\varepsilon_h\right]$$

we continue using the extension E_1.) In this way we correct the first extension and eliminate the mismatch. We denote the resulting extension by E_2. We deal with the upper layers at $x_3 = \pm h/2$ in a different way, namely, we first consider the extension on the cubes

$$\varepsilon_h(Y+z)\times\left(\left[\frac{h}{2}-\varepsilon_h, \frac{h}{2}\right] \cup \left[-\frac{h}{2} - \frac{h}{2}+\varepsilon_h\right]\right)$$

(referring to this as E_3), and then we correct the possible mismatch between E_2 and E_3 by performing extensions to the cubes

$$\varepsilon_h(Y+z)\times\left(\left[\frac{h}{2}-\frac{3}{2}\varepsilon_h, \frac{h}{2}-\frac{1}{2}\varepsilon_h\right] \cup \left[-\frac{h}{2}+\frac{1}{2}\varepsilon_h, -\frac{h}{2}+\frac{3}{2}\varepsilon_h\right]\right)$$

from the complements of the corresponding perforations

$$\varepsilon_h(Y_0+z)\times\left(\left[\frac{h}{2}-\frac{5}{4}\varepsilon_h, \frac{h}{2}-\frac{3}{4}\varepsilon_h\right] \cup \left[-\frac{h}{2}+\frac{3}{4}\varepsilon_h, -\frac{h}{2}+\frac{5}{4}\varepsilon_h\right]\right)$$

(On the part

$$\varepsilon(Y_0+z)\times\left(\left[\frac{h}{2}-\frac{3}{2}\varepsilon_h,\frac{h}{2}-\frac{5}{4}\varepsilon_h\right]\cup\left[-\frac{h}{2}+\frac{5}{4}\varepsilon_h,-\frac{h}{2}+\frac{3}{2}\varepsilon_h\right]\right)$$

we take the above extension E_2, while on the part

$$\varepsilon_h(Y_0+z)\times\left(\left[\frac{h}{2}-\frac{3}{4}\varepsilon_h,\frac{h}{2}-\frac{1}{2}\varepsilon_h\right]\cup\left[-\frac{h}{2}+\frac{1}{2}\varepsilon_h,-\frac{h}{2}+\frac{3}{4}\varepsilon_h\right]\right)$$

we take the extension E_3.) We refer to this extension on the cube

$$\varepsilon_h(Y+z)\times\left(\left[\frac{h}{2}-\frac{3}{2}\varepsilon_h,\frac{h}{2}\right]\cup\left[-\frac{h}{2},-\frac{h}{2}+\frac{3}{2}\varepsilon_h\right]\right)$$

as E_4. The final extension is given by E_4 on the layer

$$\left\{(x_1,x_2,x_3)\in\Omega^h : x_3\in\left[\frac{h}{2}-\frac{3}{2}\varepsilon_h,\frac{h}{2}\right]\cup\left[-\frac{h}{2},-\frac{h}{2}+\frac{3}{2}\varepsilon_h\right]\right\}$$

and by E_2 on the remaining part of Ω^h. The required extension on Ω is now then by scaling in x_3. (A procedure analogous to this has been described in [51, Chapter 4] for some specific domains.) ■

Remark 4.4.11. It is easy to see that if $\boldsymbol{u}\in L^{2,\mathrm{bend}}(\Omega;\mathbb{R}^3)$ or $\boldsymbol{u}\in L^{2,\mathrm{memb}}(\Omega;\mathbb{R}^3)$ then the same is valid for $\tilde{\boldsymbol{u}}$ (see the extension operator in [51, Lemma 4.1]).

The following statement is proved analogously to Theorem 4.4.10, see also the proof of Lemma 4.4.4.

Lemma 4.4.12. *For $\mathring{\boldsymbol{u}}\in V(\Omega)$ be such that $\mathring{\boldsymbol{u}}^{\varepsilon_h}|_{\Omega_1^{\varepsilon_h}}=0$, the following Poincaré and Korn inequalities hold:*

$$\|\mathring{\boldsymbol{u}}\|_{L^2}\le\varepsilon_h C\|\nabla_h\mathring{\boldsymbol{u}}\|_{L^2}, \qquad \|\nabla_h\mathring{\boldsymbol{u}}\|_{L^2}\le C\left\|\mathrm{sym}\,\nabla_h\mathring{\boldsymbol{u}}\right\|_{L^2},$$

where the constant C does not depend on h.

Proof. As in the case of Theorem 4.4.10, we work on the physical domain Ω^h. The restrictions of the inequalities (4.4.12) to those cylinders $\varepsilon_h(Y_0+z)\times[-h/2+k\varepsilon_h,-h/2+(k+1)\varepsilon_h]$, $z\in\mathbb{Z}^2$, $k\in\mathbb{N}_0$, that are contained in Ω are obtained by combining a scaling argument, the Korn inequality, and the Poincare inequality on the unit cube. We similarly obtain the restrictions of the inequalities (4.4.12) to the cylinders $\varepsilon_h(Y_0+z)\times[h/2-\varepsilon_h,h/2]$, $z\in\mathbb{Z}^2$. The argument for the physical domain Ω^h is now completed by summing up all above inequalities. To obtain the inequality on the canonical domain Ω (where gradients are replaced by scaled gradients), we simply perform the corresponding rescaling. ■

4.5. Hyperbolic evolution problems of second order

Let $\mathcal{A}$ be a non-negative, self-adjoint operator with domain $\mathcal{D}(\mathcal{A})$ in a separable Hilbert space H, and let $\mathcal{A}^{1/2}$ its unique non-negative self-adjoint root in H. We define the following norm on $V := \mathcal{D}(\mathcal{A}^{1/2})$:

$$\|\boldsymbol{u}\|_V^2 := \left((\mathcal{A}+\mathcal{I})^{1/2}\boldsymbol{u}, (\mathcal{A}+\mathcal{I})^{1/2}\boldsymbol{u}\right)_H, \quad \boldsymbol{u} \in \mathcal{D}(\mathcal{A}^{1/2}).$$

V is a Hilbert space and the inclusion $V \geq \mathcal{D}(\mathcal{A})$ is dense. Let V^* be the dual of V making (V, H, V^*) the Gelfand triple [4, 28]. Due to the density argument, the operators $\mathcal{A}$, $\mathcal{A}^{1/2}$ can be uniquely extended to bounded linear operators:

$$\mathcal{A} : V \to V^*, \quad \mathcal{A}^{1/2} : H \to V^*.$$

Moreover, $\mathcal{A}+\mathcal{I} : V \to V^*$ as well as $(\mathcal{A}+\mathcal{I})^{1/2}$, $\mathcal{A}^{1/2}+\mathcal{I}$, viewed both as operators from V to H and form H to V^*, are isomorphisms. Consider also the following evolution problem:

$$\begin{aligned} &\partial_{tt}\boldsymbol{u}(t) + \mathcal{A}\boldsymbol{u}(t) = \boldsymbol{f}(t), \\ &\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \partial_t\boldsymbol{u}(0) = \boldsymbol{u}_1, \\ &\boldsymbol{u}_0 \in V, \quad \boldsymbol{u}_1 \in H, \quad \boldsymbol{f} \in L^2([0,T]; V^*). \end{aligned} \tag{4.24}$$

Definition 4.5.1. We say that $\boldsymbol{u} \in L^2([0,T]; V)$ is a weak solution of the problem (4.24) if it satisfies:

$$\begin{aligned} &\boldsymbol{u} \in C([0,T]; V), \quad \partial_t\boldsymbol{u} \in C([0,T]; H), \\ &\partial_t(\partial_t\boldsymbol{u}(t), \boldsymbol{v})_H + a(\boldsymbol{u}(t), \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_{V^*,V} \quad \forall \boldsymbol{v} \in V \text{ in the sense of distributions on } (0,T), \\ &\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \partial_t\boldsymbol{u}(0) = \boldsymbol{u}_1. \end{aligned} \tag{4.25}$$

The problem (4.24) can be restated as a first-order form, as follows. Consider the product space $E = V \times H$ endowed with the inner product

$$(\vec{v}, \vec{w})_E = \left((\boldsymbol{v}_1, \boldsymbol{v}_2)^\top, (\boldsymbol{w}_1, \boldsymbol{w}_2)^\top\right)_E := \left((\mathcal{A}+\mathcal{I})^{1/2}\boldsymbol{v}_1, (\mathcal{A}+\mathcal{I})^{1/2}\boldsymbol{w}_1\right)_H + (\boldsymbol{v}_2, \boldsymbol{w}_2)_H,$$

and set

$$\mathbb{A} := \begin{bmatrix} 0 & -\mathcal{I} \\ \mathcal{A} & 0 \end{bmatrix}, \quad \mathcal{D}(\mathbb{A}) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{1/2}), \tag{4.26}$$

with the embedding $\mathcal{D}(\mathbb{A}) \hookrightarrow E$ being dense. It is easily seen that

$$((\mathbb{A}+\mathbb{I})\vec{\boldsymbol{v}},\vec{\boldsymbol{v}})_E = (\mathcal{A}\boldsymbol{v}_1,\boldsymbol{v}_1)+(\boldsymbol{v}_1,\boldsymbol{v}_1)+(\boldsymbol{v}_2,\boldsymbol{v}_2)-(\boldsymbol{v}_1,\boldsymbol{v}_2) \geq ((\boldsymbol{v}_1,\boldsymbol{v}_1)+(\boldsymbol{v}_2,\boldsymbol{v}_2))/2, \tag{4.27}$$

where by $\mathbb{I}$ we have denoted the identity operator on E. Moreover for $\lambda \in \mathbb{R}\backslash\{0\}$ one has

$$(\mathbb{A}+\lambda\mathbb{I})\vec{\boldsymbol{v}} = \vec{\boldsymbol{f}} \quad \Longleftrightarrow \quad \boldsymbol{v}_1 = (\mathcal{A}+\lambda^2\mathcal{I})^{-1}(\lambda\boldsymbol{f}_1+\boldsymbol{f}_2), \quad \boldsymbol{v}_2 = \lambda\boldsymbol{v}_1-\boldsymbol{f}_1, \tag{4.28}$$

which implies

$$(\mathbb{A}+\lambda\mathbb{I})^{-1} = \begin{bmatrix} \lambda(\mathcal{A}+\lambda^2\mathcal{I})^{-1} & (\mathcal{A}+\lambda^2\mathcal{I})^{-1} \\ \lambda^2(\mathcal{A}+\lambda^2\mathcal{I})^{-1}-\mathcal{I} & \lambda(\mathcal{A}+\lambda^2\mathcal{I})^{-1} \end{bmatrix}, \qquad \lambda \neq 0. \tag{4.29}$$

As a consequence of (4.27) we can conclude that $-(\mathbb{A}+\mathbb{I})$ is a dissipative operator in the sense that

$$\left\|(\lambda\mathbb{I}+\mathbb{I}+\mathbb{A})\vec{\boldsymbol{v}}\right\|_E \geq \sqrt{\lambda^2+\lambda}\,\|\vec{\boldsymbol{v}}\|_E \geq \lambda\|\vec{\boldsymbol{v}}\|_E, \qquad \forall\lambda > 0. \tag{4.30}$$

The property (4.28) implies that $\mathbb{A}+\mathbb{I}$ is a closed operator. (Note that a dissipative operator $\mathcal{S}$ is closed if there exists $\lambda > 0$ such that the range of $\lambda\mathcal{I}-\mathcal{S}$ is closed.) From (4.28) and (4.30) we conclude that $(\lambda+1)\mathbb{I}+\mathbb{A}$ is a bijection

$$\left\|((\lambda+1)\mathbb{I}+\mathbb{A})^{-1}\right\| \leq \lambda^{-1}, \quad \forall\lambda > 0.$$

It follows that $-(\mathbb{A}+\mathbb{I})$ generates a contraction semigroup (by Hille-Yosida Theorem) and

$$e^{-t\mathbb{A}} = e^t e^{-t(\mathbb{A}+\mathbb{I})}, \quad \left\|e^{-t(\mathbb{A}+\mathbb{I})}\right\| \leq 1, \quad \left\|e^{-t\mathbb{A}}\right\| \leq e^t. \tag{4.31}$$

The problem (4.24) can be formally written in the form

$$\partial_t\vec{\boldsymbol{u}}(t)+\mathbb{A}\vec{\boldsymbol{u}}(t) = \vec{\boldsymbol{f}}(t), \qquad \vec{\boldsymbol{u}}(0) = \vec{\boldsymbol{u}}_0, \tag{4.32}$$

with $\vec{\boldsymbol{u}} = (\boldsymbol{u},\partial_t\boldsymbol{u})^\top$, $\vec{\boldsymbol{u}}_0 := (\boldsymbol{u}_0,\boldsymbol{u}_1)^\top$, $\vec{\boldsymbol{f}} = (0,\boldsymbol{f})^\top \in E$. The following two theorems establish sufficient conditions for the problem (4.24) to be well posed.

Theorem 4.5.1. *Under the additional assumption $\boldsymbol{f} \in L^2([0,T];H)$, there exists a unique weak solution of the problem* (4.24)*, understood in the sense of Definition 4.5.1.*

Proof. The existence can be obtained by the variation of constants formula

$$\vec{\boldsymbol{u}}(t) = e^{-t\mathbb{A}}\vec{\boldsymbol{u}}_0 + \int_0^T e^{-(t-s)\mathbb{A}}\vec{\boldsymbol{f}}(s)ds. \tag{4.33}$$

The uniqueness is given by parabolic regularisation and can be found in [28, Theorem 3, p. 572] in a more general setting, where also a proof of existence is obtained by the same method while working directly with the problem (4.24). ∎

Remark 4.5.2. It can be easily seen from (4.25) that in Theorem 4.5.1 one additionally has $\partial_{tt}\boldsymbol{u} \in L^2([0,T];V^*)$. Furthermore, if $\boldsymbol{f} \in L^\infty([0,T];H)$ then $\partial_{tt}\boldsymbol{u} \in L^\infty([0,T];V^*)$.

Remark 4.5.3. It follows from (4.31) and (4.33) that there exists $C > 0$ such that

$$\|\vec{\boldsymbol{u}}\|_{L^\infty([0,T];E)} \leq Ce^T\left(\|\vec{\boldsymbol{u}}_0\| + \|\vec{\boldsymbol{f}}\|_{L^1([0,T];E)}\right),$$

from which one directly concludes that

$$\|\boldsymbol{u}\|_{L^\infty([0,T];V)} + \|\partial_t\boldsymbol{u}\|_{L^\infty([0,T];H)} \leq Ce^T\left(\|\boldsymbol{u}_0\|_V + \|\boldsymbol{u}_1\|_H + \|\boldsymbol{f}\|_{L^1([0,T];H)}\right). \tag{4.34}$$

Note that as a consequence of (4.28) the operator $\mathbb{A} + \lambda\mathbb{I}$ has bounded inverse for every $\lambda \neq 0$.

Theorem 4.5.4. *Assume that $\boldsymbol{f}, \partial_t\boldsymbol{f} \in L^2([0,T];V^*)$ Then, there exists a unique weak solution in the sense of Definition 4.5.1 of the problem* (4.24).

Proof. Notice that we actually have $\boldsymbol{f} \in C([0,T];V^*)$. The existence of solution follows from the formula

$$\vec{\boldsymbol{u}}(t) = e^{-t\mathbb{A}}\vec{\boldsymbol{u}}_0 - \int_0^T e^{(s-t)\mathbb{A}}(\mathbb{A}+\mathbb{I})^{-1}\left(\partial_s\vec{\boldsymbol{f}}(s) - \vec{\boldsymbol{f}}(s)\right)ds + (\mathbb{A}+\mathbb{I})^{-1}\vec{\boldsymbol{f}}(t) - e^{-t\mathbb{A}}(\mathbb{A}+\mathbb{I})^{-1}\vec{\boldsymbol{f}}(0), \tag{4.35}$$

which can be obtained formally from (4.33) by using integration by parts. Here we also use the fact that (see (4.29)):

$$(\mathbb{A}+\mathbb{I})^{-1}(0,\boldsymbol{f})^\top = \left((\mathcal{A}+\mathcal{I})^{-1}\boldsymbol{f}, (\mathcal{A}+\mathcal{I})^{-1}\boldsymbol{f}\right), \tag{4.36}$$

from which it follows that if $f, \partial_t\boldsymbol{f} \in L^2([0,T];V^*)$ then $(\mathbb{A}+\mathbb{I})^{-1}(0,\boldsymbol{f})^\top$, i.e., $(\mathbb{A}+\mathbb{I})^{-1}(0,\partial_t\boldsymbol{f})^\top$ is an element of $L^2([0,T];V)$. As in the proof of Theorem 4.5.1, the uniqueness follows from [28, Theorem 3, p. 572]. (Notice that considering the difference of the solutions associated with two different load densities gives us a solution to the problem with zero load density, which is then necessarily zero by the cited theorem.) ∎

Remark 4.5.5. It can be easily seen that in Theorem 4.5.4 one additionally has $\partial_{tt}\boldsymbol{u} \in L^\infty([0,T];V^*)$. This follows from (4.25) and the fact that $\boldsymbol{f} \in L^\infty([0,T];V^*)$.

Remark 4.5.6. It follows from (4.31) and (4.35) that there exists $C > 0$ such that

$$\|\vec{\boldsymbol{u}}\|_{L^\infty([0,T];E)} \le Ce^T \left(\|\vec{\boldsymbol{u}}_0\| + \|\vec{\boldsymbol{f}}(0)\|_{V^*} + \|\partial_t \vec{\boldsymbol{f}}\|_{L^1([0,T];V^*)}\right),$$

from which one directly concludes that

$$\|\boldsymbol{u}\|_{L^\infty([0,T];V)} + \|\partial_t \boldsymbol{u}\|_{L^\infty([0,T];H)} \le Ce^T \left(\|\boldsymbol{u}_0\|_V + \|\boldsymbol{u}_1\|_H + \|\boldsymbol{f}(0)\|_{V^*} + \|\partial_t \boldsymbol{f}\|_{L^1([0,T];V^*)}\right). \tag{4.37}$$

We will now give an overview of the results of [53], which we will then extend with the concept of solution discussed in Theorem 4.5.4. While this extension is not considered in [53], its validity follows from the formula (4.35).

Suppose that we are given a sequence of Hilbert spaces $(H_\varepsilon)_{\varepsilon>0}$ endowed with norms $\|\cdot\|_{H_\varepsilon}$ and some type of weak convergence $\boldsymbol{u}^\varepsilon \xrightharpoonup{H_\varepsilon} \boldsymbol{u} \in H$ of sequences $(\boldsymbol{u}^\varepsilon) \subset H_\varepsilon$. Our assumption on this type of weak convergence is that every weakly convergent sequence $(\boldsymbol{u}^\varepsilon)_{\varepsilon>0}$ is bounded, i.e. $\limsup_{\varepsilon\to 0} \|\boldsymbol{u}^\varepsilon\|_{H_\varepsilon} < \infty$. We additionally assume that the "limit space" H is separable.

Definition 4.5.2. We say that a sequence $(\boldsymbol{u}^\varepsilon)_{\varepsilon>0} \subset H_\varepsilon$ strongly converges to $\boldsymbol{u} \in H$ and write $\boldsymbol{u}^\varepsilon \xrightarrow{H_\varepsilon} \boldsymbol{u}$ if $\boldsymbol{u}^\varepsilon \xrightharpoonup{H_\varepsilon} \boldsymbol{u}$ and

$$\lim_{\varepsilon\to 0} (\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)_{H_\varepsilon} = (\boldsymbol{u}, \boldsymbol{v})_H, \tag{4.38}$$

for every weakly convergent sequence $\boldsymbol{v}^\varepsilon \xrightharpoonup{H_\varepsilon} \boldsymbol{v} \in H$, $\boldsymbol{v}^\varepsilon \in H_\varepsilon$.

Additionally, we assume the following properties of this abstract weak convergence:

(compactness principle)	Every bounded sequence contains a weakly convergent subsequence.
(approximation principle)	For every $\boldsymbol{u} \in H$, there exists a sequence $(\boldsymbol{u}^\varepsilon)_{\varepsilon>0} \subset H_\varepsilon$, such that $\boldsymbol{u}^\varepsilon \xrightarrow{H_\varepsilon} \boldsymbol{u}$.
(norm convergence)	If $\boldsymbol{u}^\varepsilon \xrightarrow{H_\varepsilon} \boldsymbol{u}$, then $\lim_{\varepsilon\to 0} \|\boldsymbol{u}^\varepsilon\|_{H_\varepsilon} = \|\boldsymbol{u}\|_H$.

As a consequence of these properties, to guarantee strong convergence $\boldsymbol{u}^\varepsilon \xrightarrow{H_\varepsilon} \boldsymbol{u}$ it suffices to have the property (4.38) or, alternatively, to have $\boldsymbol{u}^\varepsilon \xrightharpoonup{H_\varepsilon} \boldsymbol{u}$ and norm convergence (see the proof of Lemma [53, Lemma 6.3]). The following kind of operator convergence is convenient in the analysis of parameter-dependent problems.

Definition 4.5.3. Let $(\mathcal{A}_\varepsilon)_{\varepsilon>0}$ be a sequence of non-negative self-adjoint operators acting on the respective spaces H_ε. Suppose that $\mathcal{A}$ is a non-negative self-adjoint operator on some closed subspace H_0 of H, and consider the orthogonal projection $P : H \to H_0$. We say that $\mathcal{A}_\varepsilon$ converge to $\mathcal{A}$ in the weak resolvent sense if

$$\forall \lambda > 0, \quad (\mathcal{A}_\varepsilon + \lambda \mathcal{I})^{-1} \boldsymbol{f}^\varepsilon \xrightharpoonup{H_\varepsilon} (\mathcal{A} + \lambda \mathcal{I})^{-1} P\boldsymbol{f}, \quad \forall (\boldsymbol{f}^\varepsilon)_{\varepsilon>0}, \quad \boldsymbol{f}^\varepsilon \in H_\varepsilon, \quad \boldsymbol{f}^\varepsilon \xrightharpoonup{H_\varepsilon} \boldsymbol{f} \in H. \tag{4.39}$$

Similarly, we say that $\mathcal{A}_\varepsilon$ converge to $\mathcal{A}$ in the strong resolvent sense if

$$\forall \lambda > 0, \quad (\mathcal{A}_\varepsilon + \lambda \mathcal{I})^{-1} \boldsymbol{f}^\varepsilon \xrightarrow{H_\varepsilon} (\mathcal{A} + \lambda \mathcal{I})^{-1} P\boldsymbol{f}, \quad \forall (\boldsymbol{f}^\varepsilon)_{\varepsilon>0}, \quad \boldsymbol{f}^\varepsilon \in H_\varepsilon, \quad \boldsymbol{f}^\varepsilon \xrightarrow{H_\varepsilon} \boldsymbol{f} \in H. \tag{4.40}$$

Lemma 4.5.7. *The convergence* (4.39) *is equivalent to the convergence* (4.40).

Proof. The proof is based on a duality argument. Take $\lambda > 0$ and consider $(\boldsymbol{f}^\varepsilon)_{\varepsilon>0}$ such that $\boldsymbol{f}^\varepsilon \xrightharpoonup{H_\varepsilon} \boldsymbol{f}$ and $(\boldsymbol{g}^\varepsilon)_{\varepsilon>0}$ such that $\boldsymbol{g}^\varepsilon \xrightarrow{H_\varepsilon} \boldsymbol{g}$. Then one has

$$\begin{aligned}\lim_{\varepsilon\to 0} \left(\boldsymbol{f}^\varepsilon, (\mathcal{A}_\varepsilon + \lambda \mathcal{I})^{-1} \boldsymbol{g}^\varepsilon\right)_{H_\varepsilon} &= \lim_{\varepsilon\to 0} \left((\mathcal{A}_\varepsilon + \lambda \mathcal{I})^{-1} \boldsymbol{f}^\varepsilon, \boldsymbol{g}^\varepsilon\right)_{H_\varepsilon} \\ &= \left((\mathcal{A} + \lambda I)^{-1} P\boldsymbol{f}, \boldsymbol{g}\right)_H = \left(\boldsymbol{f}, (\mathcal{A} + \lambda \mathcal{I})^{-1} P\boldsymbol{g}\right)_H.\end{aligned}$$

These equalities show that (4.39) implies (4.40). In a similar fashion, one shows that (4.40) implies (4.39). ∎

Henceforth we work within the framework of Definition 4.5.3. For the sequence $(\mathcal{A}_\varepsilon)_{\varepsilon>0}$ we construct the associated Hilbert spaces V_ε endowed with norms $\|\cdot\|_{V_\varepsilon}$, defined as follows:

$$\|\boldsymbol{u}^\varepsilon\|^2_{V_\varepsilon} := ((\mathcal{A}_\varepsilon + \mathcal{I})^{1/2} \boldsymbol{u}^\varepsilon, (\mathcal{A}_\varepsilon + \mathcal{I})^{1/2} \boldsymbol{u}^\varepsilon)_{H_\varepsilon}.$$

Similarly, we define $\|\boldsymbol{u}\|_V$, where $V = \mathcal{D}(\mathcal{A}^{1/2})$. It is easily seen that, since H is separable, the spaces H_0 and V are also separable. (Indeed, H_0 is a subspace of H, and if $\{h_n\}_{n\in\mathbb{N}}$ is a dense subset of H, then $\{(\mathcal{A} + I)^{-1/2} h_n\}_{n\in\mathbb{N}}$ is a dense subset of V.)

Definition 4.5.4. Suppose that $(\boldsymbol{u}^\varepsilon)_{\varepsilon>0} \subset V_\varepsilon$, $\boldsymbol{u} \in V$. We say that $\boldsymbol{u}^\varepsilon$ converge weakly to $\boldsymbol{u}$ in V_ε, and write $\boldsymbol{u}^\varepsilon \xrightharpoonup{V_\varepsilon} \boldsymbol{u}$, if

$$\boldsymbol{u}^\varepsilon \xrightharpoonup{H_\varepsilon} \boldsymbol{u} \quad \text{and} \quad \limsup_{\varepsilon>0} \|\boldsymbol{u}^\varepsilon\|_{V_\varepsilon} < \infty.$$

Additionally, we say that $(\boldsymbol{u}^\varepsilon)_{\varepsilon>0}$ converges strongly to $\boldsymbol{u}$ in V_ε, and write $\boldsymbol{u}^\varepsilon \xrightarrow{V_\varepsilon} \boldsymbol{u}$, if

$$(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)_{V_\varepsilon} \to (\boldsymbol{u}, \boldsymbol{v})_V \quad \text{for all} \quad \boldsymbol{v}^\varepsilon \xrightharpoonup{V_\varepsilon} \boldsymbol{v}.$$

The following statement is [53, Lemma 6.2, Lemma 6.3].

Lemma 4.5.8. *If $\boldsymbol{u}^\varepsilon \xrightarrow{V_\varepsilon} \boldsymbol{u}$, then $\boldsymbol{u}^\varepsilon \xrightarrow{H_\varepsilon} \boldsymbol{u}$. Moreover, one has*

$$\boldsymbol{u}^\varepsilon \xrightarrow{V_\varepsilon} \boldsymbol{u} \iff \boldsymbol{u}^\varepsilon \xrightharpoonup{V_\varepsilon} \boldsymbol{u} \ \textit{ and } \ \|\boldsymbol{u}^\varepsilon\|_{V_\varepsilon} \to \|\boldsymbol{u}\|_V.$$

It can be shown that there exists a dense subset $S \subset V$, such that for every $\boldsymbol{z} \in S$ there exists a subsequence $(\boldsymbol{z}^\varepsilon)_{\varepsilon>0}$ such that $\boldsymbol{z}^\varepsilon \xrightarrow{V_\varepsilon} \boldsymbol{z}$ (see [53, Lemma 6.5]).

We also introduce convergence notions convenient for the analysis of time-dependent problems.

Definition 4.5.5. Suppose that a sequence $(\boldsymbol{u}^\varepsilon)_{\varepsilon>0} \subset L^2([0,T];H_\varepsilon)$ is bounded. We say that $(\boldsymbol{u}^\varepsilon)_{\varepsilon>0}$ weakly converges to $\boldsymbol{u} \in L^2([0,T];H)$, and write $\boldsymbol{u}^\varepsilon \xrightharpoonup{t,H_\varepsilon} \boldsymbol{u}$, if

$$\int_0^T (\boldsymbol{u}^\varepsilon(t), \boldsymbol{v}^\varepsilon)_{H_\varepsilon} \varphi(t) dt \to \int_0^T (\boldsymbol{u}(t), \boldsymbol{v}) \varphi(t) dt,$$

for all $\boldsymbol{v}^\varepsilon \xrightarrow{H_\varepsilon} \boldsymbol{v}$ and $\varphi \in L^2(0,T)$.

Definition 4.5.6. Suppose that $(\boldsymbol{u}^\varepsilon)_{\varepsilon>0} \subset L^2([0,T];V_\varepsilon)$. We say that $(\boldsymbol{u}^\varepsilon)_{\varepsilon>0}$ weakly converges to $\boldsymbol{u} \in L^2([0,T];V)$, and write $\boldsymbol{u}^\varepsilon \xrightharpoonup{t,V_\varepsilon} \boldsymbol{u}$, if

$$\boldsymbol{u}^\varepsilon \xrightharpoonup{t,H_\varepsilon} \boldsymbol{u} \quad \text{and} \quad \limsup_{\varepsilon\to 0} \int_0^T (\mathcal{A}_\varepsilon^{1/2} \boldsymbol{u}^\varepsilon, \mathcal{A}_\varepsilon^{1/2} \boldsymbol{u}^\varepsilon)_{H_\varepsilon} dt < \infty.$$

In the same way we can define the weak convergence in $L^p([0,T];H_\varepsilon)$ and $L^p([0,T];V_\varepsilon)$, $1 \le p \le \infty$ (which denote by $\xrightharpoonup{t,p,H_\varepsilon}$ and $\xrightharpoonup{t,p,V_\varepsilon}$, respectively.) The following lemma is stated in [53, Lemma 4.7].

Lemma 4.5.9. *The spaces $L^2([0,T];H_\varepsilon)$ and $L^2([0,T];V_\varepsilon)$ satisfy the weak compactness principle.*

The next lemma can also be easily established, see [53, Lemma 2.3, Lemma 4.5].

Lemma 4.5.10. *If $\boldsymbol{u}^\varepsilon \xrightarrow{H_\varepsilon} \boldsymbol{u}$, then $\liminf_{\varepsilon\to 0} \|\boldsymbol{u}^\varepsilon\|_{H_\varepsilon} \geq \|\boldsymbol{u}\|_H$. The same is valid for the weak convergence in V_ε. If $\boldsymbol{u}^\varepsilon \xrightarrow{t,H_\varepsilon} \boldsymbol{u}$, then $\liminf_{\varepsilon\to 0} \|\boldsymbol{u}^\varepsilon\|_{L^2([0,T];H_\varepsilon)} \geq \|\boldsymbol{u}\|_{L^2([0,T];H)}$. The same is valid for the weak convergence in $L^2([0,T];V_\varepsilon)$.*

In the natural way, by components, we define the weak and strong convergence in $E_\varepsilon = V_\varepsilon \times H_\varepsilon$ as well as the weak convergence in $L^2([0,T];E_\varepsilon)$. Also the space $E_0 = V \times H_0$ and the projection P onto E_0 are defined in a natural way, the latter being given by $P\vec{v} = (v_1, Pv_2)^\top$. In an obvious way we also define operators $\mathbb{A}_\varepsilon$ and $\mathbb{A}$. The following theorem is a basic tool for proving weak or strong convergence of solutions to ε-parametrised evolution problems, understood in the sense of Definition 4.5.1. The theorem can be found in [53, Theorem 5.2, Theorem 7.1]. The first part is easily proved by combining the Laplace transform and a compactness result as in Theorem 4.5.13.

Theorem 4.5.11. *Let $(\mathcal{A}_\varepsilon)_{\varepsilon>0}$ be a sequence of non-negative self-adjoint operators in H_ε that converge to a non-negative self-adjoint operator $\mathcal{A}$ in some subspace $H_0 \leq H$ in the sense of weak resolvent convergence.*

1. *If $\vec{\boldsymbol{f}}^\varepsilon \xrightarrow{E_\varepsilon} \vec{\boldsymbol{f}}$, then*

$$(\mathbb{A}_\varepsilon + \lambda\mathbb{I})^{-1}\vec{\boldsymbol{f}}^\varepsilon \xrightarrow{E_\varepsilon} (\mathbb{A} + \lambda\mathbb{I})^{-1}\vec{\boldsymbol{f}}, \quad \forall \lambda > 1.$$

2. *If $\vec{\boldsymbol{f}}^\varepsilon \xrightarrow{E_\varepsilon} \vec{\boldsymbol{f}}$, then $e^{-t\mathbb{A}_\varepsilon}\vec{\boldsymbol{f}}^\varepsilon \xrightarrow{t,E_\varepsilon} e^{-t\mathbb{A}}P\boldsymbol{f}$ for every $T > 0$. If $\vec{\boldsymbol{f}}^\varepsilon \xrightarrow{E_\varepsilon} \vec{\boldsymbol{f}} \in E_0$, then $e^{-t\mathbb{A}_\varepsilon}\vec{\boldsymbol{f}}^\varepsilon \to e^{-t\mathbb{A}}\vec{\boldsymbol{f}}$ for every $t \geq 0$.*

Remark 4.5.12. The pointwise convergence $e^{-t\mathbb{A}_\varepsilon}\vec{\boldsymbol{f}}^\varepsilon \xrightarrow{E_\varepsilon} e^{-t\mathbb{A}}P\boldsymbol{f}$ in Theorem 4.5.11 does not necessarily hold if we only assume that $\vec{\boldsymbol{f}}^\varepsilon \xrightarrow{E_\varepsilon} \vec{\boldsymbol{f}} \in E$, see [53, p. 2267].

A version of the following theorem can be found in [53, Theorem 5.2, Theorem 5.3].

Theorem 4.5.13. *Let $(\mathcal{A}_\varepsilon)_{\varepsilon>0}$ be a sequence of non-negative self-adjoint operators in H_ε that converge to a non-negative self-adjoint operator $\mathcal{A}$ in some subspace $H_0 \leq H$ in the sense of weak resolvent convergence. Let $T > 0$ and $(\boldsymbol{u}^\varepsilon)_{\varepsilon>0}$ be a sequence of weak solutions of the evolution problems (4.24) where $\mathcal{A}$ is replaced by $\mathcal{A}_\varepsilon$, with initial data $\boldsymbol{u}_0^\varepsilon \in V_\varepsilon$, $\boldsymbol{u}_1^\varepsilon \in H_\varepsilon$ and right-hand sides $\boldsymbol{f}^\varepsilon \in L^2([0,T];H_\varepsilon)$ such that*

$$\boldsymbol{u}_0^\varepsilon \xrightarrow{V_\varepsilon} \boldsymbol{u}_0 \in V, \quad \boldsymbol{u}_1^\varepsilon \xrightarrow{H_\varepsilon} \boldsymbol{u}_1 \in H, \quad \boldsymbol{f}^\varepsilon \xrightarrow{t,H_\varepsilon} \boldsymbol{f} \in L^2([0,T];H). \tag{4.41}$$

Then one has

$$\boldsymbol{u}^\varepsilon \xrightarrow{t,V_\varepsilon} \boldsymbol{u} \in L^2([0,T];V), \quad \partial_t \boldsymbol{u}^\varepsilon \xrightarrow{t,H_\varepsilon} \partial_t \boldsymbol{u} \in L^2([0,T];H_0), \tag{4.42}$$

where $\boldsymbol{u}$ *is the weak solution of the evolution problem* (4.24) *for the limit operator* $\mathcal{A}$*, with the initial data* $\boldsymbol{u}_0 \in V$*,* $P\boldsymbol{u}_1 \in H_0$ *and the right-hand side* $P\boldsymbol{f} \in L^2([0,T];H_0)$*.*

Proof. We write the problem (4.24) in the form (4.32). As a consequence of (4.41), (4.34), and Lemma 4.5.9, there exist $\vec{\boldsymbol{u}}_l = (\boldsymbol{u}_l, \partial_t \boldsymbol{u}_l) \in L^2([0,T];E_0)$ such that the convergence (4.42) holds. Due to above mentioned bounds and weak convergence, we have

$$\mathcal{L}(\vec{\boldsymbol{u}}^\varepsilon)(\lambda) \xrightarrow{E_\varepsilon} \mathcal{L}(\vec{\boldsymbol{u}}_l)(\lambda), \qquad \lambda > 1, \tag{4.43}$$

where $\mathcal{L}$ denotes the Laplace transform (where extend $\boldsymbol{f}^\varepsilon$ and $\boldsymbol{f}$ by zero on (T,∞)). On the one hand, the Laplace transform $\mathcal{L}$ of the solution of the ε-parametrised equation is then given by

$$\mathcal{L}(\vec{\boldsymbol{u}}^\varepsilon)(\lambda) = (\mathbb{A}_\varepsilon + \lambda\mathbb{I})^{-1}\mathcal{L}(\vec{\boldsymbol{f}}^\varepsilon)(\lambda) + (\mathbb{A}_\varepsilon + \lambda\mathbb{I})^{-1}\vec{\boldsymbol{u}}_0^\varepsilon, \qquad \lambda > 1. \tag{4.44}$$

On the other hand, the Laplace transform of the solution of the limit problem is given by

$$\mathcal{L}(\vec{\boldsymbol{u}})(\lambda) = (\mathbb{A} + \lambda\mathbb{I})^{-1}\mathcal{L}(P\vec{\boldsymbol{f}})(\lambda) + (\mathbb{A} + \lambda\mathbb{I})^{-1}P\vec{\boldsymbol{u}}_0, \qquad \lambda > 1. \tag{4.45}$$

Using (4.29), we infer that for every sequence $(\vec{\boldsymbol{f}}^\varepsilon)_{\varepsilon>0} \subset E_\varepsilon$ such that $\vec{\boldsymbol{f}}^\varepsilon \xrightarrow{E_\varepsilon} \vec{\boldsymbol{f}} \in E$, and every $\lambda \neq 0$ we have (see (4.39))

$$(\mathbb{A} + \lambda\mathbb{I})^{-1}\vec{\boldsymbol{f}}^\varepsilon \xrightarrow{E_\varepsilon} (\mathbb{A} + \lambda\mathbb{I})^{-1}P\vec{\boldsymbol{f}}.$$

Using (4.43), (4.44), (4.45), and the fact that for $\lambda > 1$ one has $\mathcal{L}(\vec{\boldsymbol{f}}^\varepsilon)(\lambda) \xrightarrow{E_\varepsilon} \mathcal{L}(\vec{\boldsymbol{f}})(\lambda)$ and $\mathcal{L}(P\vec{\boldsymbol{f}})(\lambda) = P\mathcal{L}(\vec{\boldsymbol{f}})(\lambda)$, we infer that for every $\lambda > 1$ one has $\mathcal{L}(\vec{\boldsymbol{u}}_l) = \mathcal{L}(\vec{\boldsymbol{u}})$, and the claim follows. ■

We proceed to the strong convergence analogue of Theorem 4.5.13, see [53, Theorem 7.2].

Theorem 4.5.14. *Let* $(\mathcal{A}_\varepsilon)$ *be a sequence of non-negative self-adjoint operators in* H_ε *that converges to a non-negative self-adjoint operator* $\mathcal{A}$ *in some subspace* $H_0 \leq H$ *in the sense of strong resolvent convergence. Let* $T > 0$ *and* $(\boldsymbol{u}^\varepsilon)_{\varepsilon>0}$ *be a sequence of weak* *solutions of the evolution problems* (4.24) *where* $\mathcal{A}$ *is replaced by* $\mathcal{A}_\varepsilon$, *with initial data* $\boldsymbol{u}_0^\varepsilon \in V_\varepsilon$, $\boldsymbol{u}_1^\varepsilon \in H_\varepsilon$ *and right-hand sides* $\boldsymbol{f}^\varepsilon \in L^2([0,T];H_\varepsilon)$ *such that*

$$\begin{aligned} &\boldsymbol{u}_0^\varepsilon \xrightarrow{V_\varepsilon} \boldsymbol{u}_0 \in V, \quad \boldsymbol{u}_1^\varepsilon \xrightarrow{H_\varepsilon} \boldsymbol{u}_1 \in H_0, \\ &\boldsymbol{f}_\varepsilon(t) \xrightarrow{H_\varepsilon} \boldsymbol{f}(t) \in H_0 \quad a.e.\ t \in [0,T], \quad \int_0^T \left\|\boldsymbol{f}^\varepsilon(s)\right\|_{H_\varepsilon}^2 ds \to \int_0^T \left\|\boldsymbol{f}(s)\right\|_H^2 ds. \end{aligned} \tag{4.46}$$

Then for every $t \in [0,T]$ *one has*

$$\boldsymbol{u}_\varepsilon(t) \xrightarrow{V_\varepsilon} \boldsymbol{u}(t) \in V, \quad \partial_t \boldsymbol{u}_\varepsilon(t) \xrightarrow{H_\varepsilon} \partial_t \boldsymbol{u}(t) \in H_0, \tag{4.47}$$

where $\boldsymbol{u}$ *is the weak solution of the evolution problem of* (4.24) *for the operator* $\mathcal{A}$, *with the initial data* $\boldsymbol{u}_0 \in V$, $\boldsymbol{u}_1 \in H_0$ *and the right-hand side* $\boldsymbol{f} \in L^2([0,T];H_0)$.

Proof. Again we use (4.32), formula (4.33) and Theorem 4.5.11. The proof follows by using Lebesgue theorem on Dominated convergence and the fact that as a consequence of Lemma 4.5.10 we have $\int_0^t \|\boldsymbol{f}^\varepsilon(s)\|_{H_\varepsilon}^2 ds \to \int_0^t \|\boldsymbol{f}(s)\|_H^2 ds$, for every $t \le T$. ■

Theorem 4.5.14 can be generalized as follows.

Theorem 4.5.15. *Let* $(\mathcal{A}_\varepsilon)_{\varepsilon>0}$ *be a sequence of non-negative self-adjoint operators in* H_ε *that converge to a non-negative self-adjoint operator* $\mathcal{A}$ *in some subspace* $H_0 \le H$ *in the sense of strong resolvent convergence. Let* $T > 0$ *and* $(\boldsymbol{u}^\varepsilon)_{\varepsilon>0}$ *be a sequence of weak solutions of the evolution problems* (4.24) *where* $\mathcal{A}$ *is replaced by* $\mathcal{A}_\varepsilon$, *with initial data* $\boldsymbol{u}_0^\varepsilon \in V_\varepsilon$, $\boldsymbol{u}_1^\varepsilon \in H_\varepsilon$ *and right-hand sides* $\boldsymbol{f}^\varepsilon \in L^2([0,T];V_\varepsilon^*)$, $\partial_t \boldsymbol{f}^\varepsilon \in L^2([0,T];V_\varepsilon^*)$ *such that the sequences* $(\boldsymbol{f}^\varepsilon)_{\varepsilon>0}$, $(\partial_t \boldsymbol{f}^\varepsilon)_{\varepsilon>0}$ *are bounded in* $L^2([0,T];V_\varepsilon^*)$ *and*

$$\begin{gathered} \boldsymbol{u}_0^\varepsilon \xrightarrow{V_\varepsilon} \boldsymbol{u}_0 \in V, \quad \boldsymbol{u}_1^\varepsilon \xrightarrow{H_\varepsilon} \boldsymbol{u}_1 \in H, \\ (\mathcal{A}_\varepsilon + \mathcal{I})^{-1} \boldsymbol{f}^\varepsilon \xrightarrow{t,V_\varepsilon} (\mathcal{A} + \mathcal{I})^{-1} \boldsymbol{f} \in L^2([0,T];V), \end{gathered}$$

where $\boldsymbol{f}, \partial_t \boldsymbol{f} \in L^2(O,T;V^*)$. *Then one has*

$$\boldsymbol{u}^\varepsilon \xrightarrow{t,V_\varepsilon} \boldsymbol{u} \in L^2([0,T];V), \quad \partial_t \boldsymbol{u}^\varepsilon \xrightarrow{t,H_\varepsilon} \partial_t \boldsymbol{u} \in L^2([0,T];H_0),$$

where $\boldsymbol{u}$ *is the weak solution of the evolution problem* (4.24) *for the operator* $\mathcal{A}$, *with the initial data* $\boldsymbol{u}_0 \in V$, $P\boldsymbol{u}_1 \in H_0$ *and the right-hand side* $\boldsymbol{f} \in L^2([0,T];V^*)$. *Furthermore, if* *we assume that*

$$\begin{aligned}
&\boldsymbol{u}_0^\varepsilon \xrightarrow{V_\varepsilon} \boldsymbol{u}_0 \in V, \quad \boldsymbol{u}_1^\varepsilon \xrightarrow{H_\varepsilon} \boldsymbol{u}_1 \in H_0,\\
&(\mathcal{A}_\varepsilon + \mathcal{I})^{-1} \boldsymbol{f}^\varepsilon(0) \xrightarrow{V_\varepsilon} (\mathcal{A} + \mathcal{I})^{-1} \boldsymbol{f}(0) \in V,\\
&(\mathcal{A}_\varepsilon + \mathcal{I})^{-1} \partial_t \boldsymbol{f}^\varepsilon(t) \xrightarrow{V_\varepsilon} (\mathcal{A} + \mathcal{I})^{-1} \partial_t \boldsymbol{f}(t) \in V, \quad \textit{for a.e. } t \in [0,T],\\
&\int_0^T \left\|(\mathcal{A}_\varepsilon + \mathcal{I})^{-1} \partial_s \boldsymbol{f}^\varepsilon(s)\right\|_{V_\varepsilon}^2 ds \to \int_0^T \left\|(\mathcal{A} + \mathcal{I})^{-1} \partial_s \boldsymbol{f}(s)\right\|_V^2 ds,
\end{aligned} \tag{4.48}$$

where $\boldsymbol{f}$, $\partial_t \boldsymbol{f} \in L^2([0,T];V^*)$, *then we have*

$$\boldsymbol{u}_\varepsilon(t) \xrightarrow{V_\varepsilon} \boldsymbol{u}(t) \in V, \quad \partial_t \boldsymbol{u}_\varepsilon(t) \xrightarrow{H_\varepsilon} \partial_t \boldsymbol{u}(t) \in H_0, \quad \forall t \in [0,T],$$

where $\boldsymbol{u}$ *is the weak solution of the evolution problem* (4.24) *for the operator* $\mathcal{A}$, *with the initial data* $\boldsymbol{u}_0 \in V$, $\boldsymbol{u}_1 \in H_0$ *and the right hand side* $\boldsymbol{f} \in L^2([0,T];V^*)$.

Proof. The argument follows the proofs of Theorem 4.5.13 and Theorem 4.5.14, by using the formula (4.35) instead of (4.33) (see also (4.36)).

To prove the first part, notice that the Laplace transform of the solution $\vec{\boldsymbol{u}}^\varepsilon$, respectively $\vec{\boldsymbol{u}}$, is given by formula (4.44), respectively (4.45). (Note that in (4.45) we replace $P\vec{\boldsymbol{f}}$ with $\vec{\boldsymbol{f}}$.) This is established by a density argument, using the fact that $L^2([0,T];H_\varepsilon)$ is dense in $L^2([0,T];V_\varepsilon^*)$, respectively that $L^2([0,T];H)$ is dense in $L^2([0,T];V^*)$.

To prove the second part, use the second part of Theorem 4.5.11 and notice that (4.48) implies that for all $t \in [0,T]$ one has

$$\begin{aligned}
&(\mathcal{A}_\varepsilon + \mathcal{I})^{-1} \boldsymbol{f}^\varepsilon(t) \xrightarrow{V_\varepsilon} (\mathcal{A}_\varepsilon + \mathcal{I})^{-1} \boldsymbol{f}(t),\\
&\int_0^t e^{(s-t)\mathbb{A}_\varepsilon} (\mathbb{A}_\varepsilon + \mathbb{I})^{-1} \left(0, \boldsymbol{f}^\varepsilon(s)\right)^\top ds \xrightarrow{E_\varepsilon} \int_0^t e^{(s-t)\mathbb{A}} (\mathbb{A} + \mathbb{I})^{-1} \left(0, \boldsymbol{f}(s)\right)^\top ds,\\
&\int_0^t e^{(s-t)\mathbb{A}_\varepsilon} (\mathbb{A}_\varepsilon + \mathbb{I})^{-1} \left(0, \partial_s \boldsymbol{f}^\varepsilon(s)\right)^\top ds \xrightarrow{E_\varepsilon} \int_0^t e^{(s-t)\mathbb{A}} (\mathbb{A} + \mathbb{I})^{-1} \left(0, \partial_s \boldsymbol{f}(s)\right)^\top ds,
\end{aligned}$$

as a consequence of the dominated convergence theorem and Lemma 4.5.10. ■

Remark 4.5.16. It is easy to see that in Theorem 4.5.13, Theorem 4.5.14, and Theorem 4.5.15 the sequences $(\|\boldsymbol{u}^\varepsilon\|_{V_\varepsilon})_{\varepsilon>0}$ and $(\|\partial_t \boldsymbol{u}^\varepsilon\|_{H_\varepsilon})_{\varepsilon>0}$ are bounded in $L^\infty(0,T)$.

Remark 4.5.17. The claims of Theorem 4.5.13, Theorem 4.5.14 and Theorem 4.5.15 can be strengthened slightly. Namely, it suffices to require that $\boldsymbol{f}^\varepsilon \xrightarrow{t,1,H_\varepsilon} \boldsymbol{f} \in L^1\big([0,T];H\big)$ in

(4.41) to obtain the convergence

$$\boldsymbol{u}^{\varepsilon} \xrightarrow{t,\infty,V_{\varepsilon}} \boldsymbol{u} \in L^{\infty}\big([0,T];V\big), \qquad \partial_t \boldsymbol{u}^{\varepsilon} \xrightarrow{t,\infty,H_{\varepsilon}} \partial_t \boldsymbol{u} \in L^{\infty}\big([0,T];H_0\big).$$

Similarly, it suffices to require $\int_0^T \|\boldsymbol{f}^{\varepsilon}\|_{H_{\varepsilon}}\, ds \to \int_0^T \|\boldsymbol{f}(s)\|\, ds$ in (4.46) to obtain (4.47). The statement of Theorem 4.5.15 can also be strengthened accordingly.

4.6. Additional claims

In the membrane space, under no additional assumptions on the symmetries of the set Y_0, the function $\tilde{\beta}^{\text{memb}}(\lambda)$ (see (2.2.2.1) as well as the parts A of Sections 2.2.2.2, 2.2.2.3) is a symmetric matrix that is not necessarily diagonal. We will first prove two lemmata and then a proposition concerning the problem (2.35), which involves the function $\tilde{\beta}_\delta^{\text{memb}}(\lambda)$ defined in Section 2.2.2.

Lemma 4.6.1. *There exists $C > 0$ such that for every $\lambda > 0$ we have:*

$$\left(\left(\tilde{\beta}_\delta^{\text{memb}}\right)'(\lambda)\xi, \xi \right) > C|\xi|^2, \quad \forall \xi \in \mathbb{R}^2, \xi \neq 0.$$

Proof. We will give the proof for $\delta \in (0,\infty)$, and the other cases can be treated analogously. Notice that for the function $\lambda \mapsto (\tilde{\eta}_n - \lambda)^{-1}\lambda^2$ one has

$$\left(\frac{\lambda^2}{\tilde{\eta}_n - \lambda} \right)' = -1 + \frac{\tilde{\eta}_n^2}{(\tilde{\eta}_n - \lambda)^2}.$$

It follows that

$$\begin{aligned}
\left(\tilde{\beta}_\delta^{\text{memb}}\right)'(\lambda) &= \boldsymbol{I}_{2\times 2}\langle \rho \rangle + \sum_{n\in\mathbb{N}} \frac{\tilde{\eta}_n^2}{(\tilde{\eta}_n - \lambda)^2} \left\langle \rho_0 \overline{(\tilde{\boldsymbol{\varphi}}_n)_*} \right\rangle \cdot \left\langle \rho_0 \overline{(\tilde{\boldsymbol{\varphi}}_n)_*} \right\rangle^\top - \sum_{n\in\mathbb{N}} \left\langle \rho_0 \overline{(\tilde{\boldsymbol{\varphi}}_n)_*} \right\rangle \cdot \left\langle \rho_0 \overline{(\tilde{\boldsymbol{\varphi}}_n)_*} \right\rangle^\top \\
&= \boldsymbol{I}_{2\times 2}\langle \rho_1 \rangle + \sum_{n\in\mathbb{N}} \frac{\tilde{\eta}_n^2}{(\tilde{\eta}_n - \lambda)^2} \left\langle \rho_0 \overline{(\tilde{\boldsymbol{\varphi}}_n)_*} \right\rangle \cdot \left\langle \rho_0 \overline{(\tilde{\boldsymbol{\varphi}}_n)_*} \right\rangle^\top.
\end{aligned}$$

Here $\tilde{\boldsymbol{\varphi}}_n$, $n \in \mathbb{N}$, are those eigenfunctions of the operator $\tilde{\mathcal{A}}_{00,\delta}$ associated with the eigenvalues $\tilde{\eta}_n$, $n \in \mathbb{N}$, that satisfy

$$\left\langle \rho_0 \overline{(\tilde{\boldsymbol{\varphi}}_n)_*} \right\rangle \neq 0,$$

and we have used the identity

$$\boldsymbol{I}_{2\times 2}\langle \rho_0 \rangle = \sum_{n\in\mathbb{N}} \left\langle \rho_0 \overline{(\tilde{\boldsymbol{\varphi}}_n)_*} \right\rangle \cdot \left\langle \rho_0 \overline{(\tilde{\boldsymbol{\varphi}}_n)_*} \right\rangle^\top, \quad \alpha, \beta = 1, 2.$$

The proof of the required estimate is concluded by noting that for $\xi \neq 0$ one has

$$\left(\left(\tilde{\beta}_\delta^{\text{memb}}\right)'(\lambda)\xi, \xi \right) = \langle \rho_1 \rangle |\xi|^2 + \sum_{n\in\mathbb{N}} \frac{\tilde{\eta}_n^2}{(\tilde{\eta}_n - \lambda)^2} \left(\left\langle \rho_0 \overline{(\tilde{\boldsymbol{\varphi}}_n)_*} \right\rangle, \xi \right)^2 > \langle \rho_1 \rangle |\xi|^2.$$

∎

Lemma 4.6.2. *Let $\lambda_0 > 0$ be such that there exists a nontrivial solution $\mathfrak{a} \in H^1_{\gamma_D}(\omega;\mathbb{R}^2)$ of the problem* (2.35)*. Then there exists $\eta > 0$ such that for each $\lambda \in (\lambda_0, \lambda_0 + \eta)$ the problem* (2.35) *has only the trivial solution.*

Proof. The problem (2.35) can be reformulated as follows:

$$\left((\mathcal{A}_\delta^{\mathrm{memb}})^{1/2}\mathfrak{a}, (\mathcal{A}_\delta^{\mathrm{memb}})^{1/2}\boldsymbol{\varphi}\right) = \left(\tilde{\beta}_\delta^{\mathrm{memb}}(\lambda)\mathfrak{a}, \boldsymbol{\varphi}\right), \quad \forall \boldsymbol{\varphi} \in H^1_{\gamma_D}(\omega;\mathbb{R}^2),$$

where $(\mathcal{A}_\delta^{\mathrm{memb}})^{1/2}$ is a self-adjoint positive square root of the operator $\mathcal{A}_\delta^{\mathrm{memb}}$, which has compact inverse. Thus, the above problem can be rewritten as

$$\left((\mathcal{A}_\delta^{\mathrm{memb}})^{1/2}\mathfrak{a}, (\mathcal{A}_\delta^{\mathrm{memb}})^{1/2}\boldsymbol{\varphi}\right) = \left((\mathcal{A}_\delta^{\mathrm{memb}})^{-1/2}\tilde{\beta}_\delta^{\mathrm{memb}}(\lambda)\mathfrak{a}, (\mathcal{A}_\delta^{\mathrm{memb}})^{1/2}\boldsymbol{\varphi}\right), \quad \forall \boldsymbol{\varphi} \in H^1_{\gamma_D}(\omega;\mathbb{R}^2).$$

By substituting $\boldsymbol{v} = (\mathcal{A}_\delta^{\mathrm{memb}})^{1/2}\mathfrak{a}$, we have reduced the problem (2.35) to the following equivalent problem: find $\boldsymbol{v} \in L^2(\omega;\mathbb{R}^2)$ that is an eigenfunction for $(\mathcal{A}_\delta^{\mathrm{memb}})^{-1/2}\tilde{\beta}_\delta^{\mathrm{memb}}(\lambda)(\mathcal{A}_\delta^{\mathrm{memb}})^{-1/2}$ with eigenvalue $\mu^\lambda = 1$, i.e.

$$(\mathcal{A}_\delta^{\mathrm{memb}})^{-1/2}\tilde{\beta}_\delta^{\mathrm{memb}}(\lambda)(\mathcal{A}_\delta^{\mathrm{memb}})^{-1/2}\boldsymbol{v} = \boldsymbol{v}.$$

The operator $(\mathcal{A}_\delta^{\mathrm{memb}})^{-1/2}\tilde{\beta}_\delta^{\mathrm{memb}}(\lambda)(\mathcal{A}_\delta^{\mathrm{memb}})^{-1/2}$ is compact and its positive eigenvalues, in decreasing order, are characterised by the variational principle

$$\mu_k^\lambda = \max_{V < L^2(\omega;\mathbb{R}^2),\ \dim V = k} \ \min_{x \in V,\ \|x\|=1} \left(\mathcal{A}_{\mathrm{memb}}^{-1/2}\tilde{\beta}^{\mathrm{memb}}(\lambda)\mathcal{A}_{\mathrm{memb}}^{-1/2}x, x\right), \qquad k = 1, 2, \dots.$$

Denote by k_1 the index of the eigenvalue $1 = \mu_{k_1}^{\lambda_0}$, which can clearly be done due to the assumption on λ_0. Next, denote by k_2 the index of the next smaller eigenvalue $\mu_{k_2}^{\lambda_0} < 1$. Furthermore, notice that for $\lambda > \lambda_0$ one has

$$\begin{aligned}(\mathcal{A}_\delta^{\mathrm{memb}})^{-1/2}\tilde{\beta}_\delta^{\mathrm{memb}}(\lambda)(\mathcal{A}_\delta^{\mathrm{memb}})^{-1/2} &= (\mathcal{A}_\delta^{\mathrm{memb}})^{-1/2}\tilde{\beta}_\delta^{\mathrm{memb}}(\lambda_0)(\mathcal{A}_\delta^{\mathrm{memb}})^{-1/2} \\ &\quad + (\lambda - \lambda_0)(\mathcal{A}_\delta^{\mathrm{memb}})^{-1/2}(\tilde{\beta}_\delta^{\mathrm{memb}})'(\lambda_0)(\mathcal{A}_\delta^{\mathrm{memb}})^{-1/2} + O(|\lambda - \lambda_0|^2),\end{aligned}$$

where $\|O(|\lambda - \lambda_0|^2\| \le C|\lambda - \lambda_0|^2$ for some $C > 0$. For this reason, by virtue of Lemma 4.6.1, one has $\mu_{k_1}^\lambda > 1, \dots, \mu_{k_2-1}^\lambda > 1$, $\mu_{k_2}^\lambda > \mu_{k_2}^{\lambda_0}$ whenever $\lambda \in (\lambda_0, \lambda_0 + \eta)$, for some $\eta > 0$. Due to the continuity of $\mu_{k_2}^\lambda$ with respect to λ, we can redefine $\eta > 0$ so that for every $\lambda \in (\lambda_0, \lambda_0 + \eta)$ we have $\mu_{k_2}^{\lambda_0} < \mu_{k_2}^\lambda < 1$. Thus, for $\lambda \in (\lambda_0, \lambda_0 + \eta)$, unity is not the eigenvalue of $\mathcal{A}_{\mathrm{memb}}^{-1/2}\tilde{\beta}^{\mathrm{memb}}(\lambda)\mathcal{A}_{\mathrm{memb}}^{-1/2}$. ∎

Proposition 4.6.3. *The set of all $\lambda > 0$ for which the problem (2.35) has a nontrivial solution $\mathfrak{a} \in H^1_{\gamma_D}(\omega;\mathbb{R}^2)$ is at most countable.*

Proof. Using the preceding lemma, we can define the following family of disjoint intervals:

$$\mathcal{F} = \{[\lambda, \lambda+\eta_\lambda) \subset \mathbb{R}, \quad \lambda > 0 \text{ is such that the problem (2.35) has a nontrivial solution}\},$$

where η_λ is provided by Lemma 4.6.2. Such a family can be at most countable, from which the claim follows. ∎

Finally, we prove a lemma whose variants we used on several occasions within Section 2.3.3. We begin by introducing some notation. We define the following norm and seminorm on $H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)$:

$$\|\boldsymbol{u}\|_{h,\varepsilon_h} = \|\boldsymbol{u}\|_{L^2(\mathbb{R}^3)} + \|\operatorname{sym}\nabla_h \tilde{\boldsymbol{u}}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \varepsilon_h \|\operatorname{sym}\nabla_h \mathring{\boldsymbol{u}}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})},$$

$$\|\boldsymbol{u}\|_{s,h,\varepsilon} = \|\operatorname{sym}\nabla_h \tilde{\boldsymbol{u}}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \varepsilon_h \|\operatorname{sym}\nabla_h \mathring{\boldsymbol{u}}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})},$$

where we have for every $\boldsymbol{u} \in H^1_{\Gamma_\mathrm{D}}$ a decomposition $\boldsymbol{u} = \tilde{\boldsymbol{u}} + \mathring{\boldsymbol{u}}$ is employed, with both $\tilde{\boldsymbol{u}}$ and $\mathring{\boldsymbol{u}}$ depending on $\boldsymbol{u}$ in a linear manner. We also assume that $\mathcal{A}$ is a non-negative self-adjoint operator whose domain is a subset of $L^2(\Omega;\mathbb{R}^3)$ such that there exist $c_1, c_2 > 0$ such that

$$c_1 \|\boldsymbol{u}\|^2_{s,h,\varepsilon_h} \le (\mathcal{A}\boldsymbol{u}, \boldsymbol{u}) \le c_2 \|\boldsymbol{u}\|^2_{s,h,\varepsilon_h} \quad \forall \boldsymbol{u} \in \mathcal{D}(\mathcal{A}). \tag{4.49}$$

Lemma 4.6.4. *Suppose that $\mathcal{A}$ is as above and let $\lambda \notin \sigma(\mathcal{A})$. Assume that $\boldsymbol{u} = \tilde{\boldsymbol{u}} + \mathring{\boldsymbol{u}} \in H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)$ satisfies*

$$\left(\mathcal{A}^{1/2}\boldsymbol{u}, \mathcal{A}^{1/2}\boldsymbol{\xi}\right) - \lambda(\boldsymbol{u}, \boldsymbol{\xi}) = \int_\Omega \left(\boldsymbol{f}_1 : \operatorname{sym}\nabla_h \tilde{\boldsymbol{\xi}} + \varepsilon_h \boldsymbol{f}_2 : \operatorname{sym}\nabla_h \mathring{\boldsymbol{\xi}} + \boldsymbol{f}_3 \cdot \boldsymbol{\xi}\right) dx \qquad \forall \boldsymbol{\xi} = \tilde{\boldsymbol{\xi}} + \mathring{\boldsymbol{\xi}} \in H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3).$$

where $\boldsymbol{f}_1 \in L^2(\Omega,\mathbb{R}^{3\times 3})$, $\boldsymbol{f}_2 \in L^2(\Omega,\mathbb{R}^{3\times 3})$ and $\boldsymbol{f}_3 \in L^2(\Omega;\mathbb{R}^3)$. Then one has

$$\|\boldsymbol{u}\|_{h,\varepsilon_h} \le \frac{C(\lambda)}{\operatorname{dist}\big(\lambda, \sigma(\mathcal{A})\big)} \left(\|\boldsymbol{f}_1\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \|\boldsymbol{f}_2\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \|\boldsymbol{f}_3\|_{L^2(\Omega;\mathbb{R}^3)}\right),$$

for some $C(\lambda)$ that is bounded on bounded intervals.

Proof. By virtue of the Riesz representation theorem and (4.49), we know that there exists $\boldsymbol{f} \in L^2(\mathbb{R}^3)$ and $C > 0$, which depends on c_1, c_2 only, such that

$$\|\boldsymbol{f}\|_{L^2(\Omega;\mathbb{R}^3)} \le C \left(\|\boldsymbol{f}_1\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \|\boldsymbol{f}_2\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \|\boldsymbol{f}_3\|_{L^2(\Omega;\mathbb{R}^3)}\right),$$

$$(\mathcal{A}\boldsymbol{u},\boldsymbol{\xi}) - \lambda(\boldsymbol{u},\boldsymbol{\xi}) = \int_\Omega \boldsymbol{f}\cdot\left(\mathcal{A}^{1/2}\boldsymbol{\xi}+\boldsymbol{\xi}\right) \qquad \forall\boldsymbol{\xi}\in H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3). \tag{4.50}$$

We can now use the spectral theorem (see e.g. [55]): there exists a measurable space (M,μ) with a finite measure μ and a unitary operator

$$\mathcal{U} : L^2(\Omega;\mathbb{R}^3) \to L^2(M)$$

and a non-negative real-valued function a, which is an element of $L^p(M)$, $p\in[1,\infty)$, such that

- $\boldsymbol{\psi}\in\mathcal{D}(\mathcal{A})$ if and only if $a(\cdot)\mathcal{U}\boldsymbol{\psi}(\cdot)\in L^2(M)$;
- $\boldsymbol{\psi}\in\mathcal{U}\mathcal{D}(\mathcal{A})$, $\mathcal{U}\mathcal{A}\mathcal{U}^{-1}\boldsymbol{\psi}(\cdot) = a(\cdot)\boldsymbol{\psi}(\cdot)$.

Notice that the second claim implies

- $\mathcal{U}\mathcal{A}^{1/2}\mathcal{U}^{-1}\boldsymbol{\psi}(\cdot) = \sqrt{a(\cdot)}\boldsymbol{\psi}(\cdot)$.
- $\sigma(\mathcal{A}) = \operatorname{EssRan} a := \left\{r : \forall\varepsilon>0,\ \mu\{m\in M : r-\varepsilon\le a(m)\le r+\varepsilon\}>0\right\}$.

Furthermore, (4.50) implies

$$\big(a(\cdot)-\lambda\big)\mathcal{U}\boldsymbol{u}(\cdot) = \mathcal{U}\boldsymbol{f}(\cdot)\big(\sqrt{a(\cdot)}+1\big),$$

from which, by virtue of $\lambda\notin\operatorname{EssRan} a$, one has

$$\big(\sqrt{a(\cdot)}+1\big)\mathcal{U}\boldsymbol{u}(\cdot) = \mathcal{U}\boldsymbol{f}(\cdot)\frac{\big(\sqrt{a(\cdot)}+1\big)^2}{\big(a(\cdot)-\lambda\big)}.$$

Therefore, there exists $C(\lambda)>0$, which is uniformly bounded on compact intervals of λ, such that

$$\left\|(\mathcal{A}^{1/2}+\mathcal{I})\boldsymbol{u}\right\|_{L_2} \le \frac{C(\lambda)}{\operatorname{dist}\big(\lambda,\sigma(\mathcal{A})\big)}\|\boldsymbol{f}\|_{L^2},$$

from which the claim follows immediately. ■

Remark 4.6.5. Throughout the paper, we also use some variants of the above lemma, see the discussions around (2.3.3), (2.3.3), (2.3.3), (2.3.3). They generically apply to setups that can be put the form (4.50), and they result in estimates of the type (4.6). The key ingredient for their validity is the fact that the right-hand side of the equation is in the dual of $\mathcal{D}(\mathcal{A}^{1/2}+\mathcal{I})$ with respect to the graph norm.

Conclusion

The thesis consists of two parts. In the first part we have established the approximation properties of the resolvents associated with the operators describing the heterogeneous elastic plates in the linear theory of elasticity, in terms of two-scale convergence. In the analysis we have covered several interesting regimes, depending on the mutual relations of the parameters describing the thickness of the plate, the period of material oscillations and the scaling of time/density. The analysed composite materials are assumed to be composed of the soft inclusions embedded into the stiff matrix with material coefficients being in high contrast. This property yielded various interesting phenomena in the effective model which can all be described as "metamaterial" phenomena. These phenomena include: memory effects, band-gap structure of the spectrum, the occurrence of evanescent waves, etc. In addition to these qualitative results, what remains to be done is to establish quantitative results in terms of the operator norm-resolvent estimates which would then yield the complete picture on the approximation properties for such materials by effective lower dimensional models.

However, in the second part of this thesis, sharp operator norm-resolvent estimates are obtained in the case of thin heterogeneous elastic rods in moderate contrast. Here, by the means of asymptotic expansion of resolvent operators, we have established estimates on the $L^2 \to L^2$ distance of the resolvent operators to their associated effective resolvents. These estimates have a lot of consequences such as: estimates on the band gaps, estimates on the associated semigroups, etc. What remains to be done here is to employ these newly derived operator norm-resolvent estimates to answer pending questions on the evolution of heterogeneous elastic rods.

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Curriculum Vitae

Josip Žubrinić was born on 10th of January 1993, in Zagreb, Croatia. He finished primary and secondary school in Zagreb, and in September 2011 he started his studies at Department of Mathematics, Faculty of Science, University of Zagreb. He obtained his bachelor's degree in July 2014, and his master's degree in July 2016 with the thesis title *Porous medium equation* under the supervision of prof. Mladen Jurak.

He started working as a teaching assistant at the Faculty of Electrical Engineering and Computing and enrolled his PhD studies at the Department of Mathematics, Faculty of Science in October 2016, under supervision of prof. Igor Velčić. During his PhD studies he actively participated in numerous international summer/winter schools, workshops and conferences where he held presentations about his work.

He was a member of a Croatian Science Foundation project *Mathematical Analysis of Multi-Physics Problems Involving Thin and Composite Structures and Fluids*, and is currently a member of a Croatian Science Foundation project *Homogenisation, dimension reduction and structural optimisation in continuum mechanics*.

His scientific interest includes partial differential equations arising from the theory of thin heterogeneous structures in elasticity. He is an active member of the *Seminar for differential equations and numerical analysis* at Department of Mathematics, Faculty of Science, University of Zagreb.