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DOCTORAL DISSERTATION

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Sveučilište u Zagrebu

PRIRODOSLOVNO–MATEMATIČKI FAKULTET

MATEMATIČKI ODSJEK

Marin Bužančić

Izvod modela homogenizirane elastično-plastične ploče

DOKTORSKI RAD

Mentor:

prof. dr. sc. Igor Velčić

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Summary

In this thesis we consider a lower dimensional homogenized thin plate model within the framework of linearized elasto-plasticity. Starting from the energetic formulation of the quasistatic evolution, we analyse the behavior of the elastic energies and dissipation potentials, as well as the displacements and strain tensors, when the period of oscillation of the heterogeneous material and the thickness of the thin body simultaneously tend to zero. In order to derive convergence results for energy functionals and the associated energy minimizers, we base our approach on Γ-convergence techniques and the two-scale convergence method adapted to dimension reduction.

Keywords: quasistatic evolution, perfect elasto-plasticity, thin plates, dimension reduction, periodic homogenization, two-scale convergence

SAŽETAK

U ovoj disertaciji promatramo niżedimenzionalni homogenizirani model tanke ploče u okviru linearizirane elasto-plastičnosti. Polazeći od energetske formulacije kvazistatične evolucije, analiziramo ponašanje energetskih funkcionala i disipacijskih potencijala te elastičnog i plastičnog tenzora deformacije kada period oscilacije heterogenog materijala i debljina tankog tijela simultano teže prema nuli. Kako bismo dobili rezultate konvergencije za energetske funkcionale i pridružene minimizatore energije, naš pristup temeljimo na tehnici Γ-konvergencije i metodi dvoskalne konvergencije prilagođenoj redukciji dimenzije.

Ključne riječi: kvazistatična evolucija, idealna elasto-plastičnost, tanke ploče, redukcija dimenzije, periodična homogenizacija, dvoskalna konvergencija

Contents

INTRODUCTION

The aim of the dissertation is to derive model equations for a heterogeneous elasto-plastic plate for composite materials with a periodic microstructure. We analyze the asymptotic behavior of the quasistatic evolutions in small-strain elasto-plasticity as the periodicity scale and the thickness of the plate both converge to zero. Different effective models will be obtained with simultaneous homogenization and dimension reduction depending on the ratio of the parameters - the oscillation rate of the microstructure and the thickness of the plate - and that the obtained models depend both on the macroscopic and microscopic variables, since it is known that the two-scale structure of the effective model cannot be eliminated when applying homogenization in elasto-plasticity. In particular, we will obtain a new compactness result by means of two-scale convergence for the sequences of symmetrically scaled gradients in the spaces of functions with bounded deformation.

In this work, convergence results for energy functionals and the associated energy minimizers will be obtained by simultaneous homogenization and reduction of dimensions within the framework of linearized elasto-plasticity. This will provide a rigorous mathematical justification for effective models that are more suitable for mathematical analysis and numerical solving, and contribute to a proper understanding of the interaction of the microscopic and macroscopic properties of materials.

LITERATURE OVERVIEW

The rigorous derivation of lower-dimensional models for thin structures - such as plates, membranes, rods, and strings - has proved to be important in engineering and material science. One of the approaches is based on ansatzes that describe the lower-dimensional models as a three-dimensional body subjected to additional constitutive restrictions. Other approaches derive lower-dimensional models starting from the three-dimensional thin bodies and proving convergence when one or two dimensions of the body tend to zero. Dimension reduction problems in the context of elasticity by applying asymptotic methods (and proving convergence in the linear case) have been performed in books [14] for plates and [15] for shells. Models of curved rods are derived in the paper [32]. The variational approach to dimension reduction based on the Γ-convergence method proved to be suitable for nonlinear problems. The first results on dimension reduction problems using Γ-convergence, in nonlinear elasticity, were given in seminal papers [1] and [35]. Different higher-order models in nonlinear elasticity, depending on various elastic energy scales, by Γ-convergence, were derived in the seminal papers [27] and [26] for thin plates and in the papers [38] and [44] for rod models.

Within linearized elasto-plasticity, reduced plate models were derived by methods of evolutionary Γ-convergence in the paper [36] in the case of a linearly elastic-hardening plastic material and in the paper [18] in the case of a linearly elastic-perfectly plastic material. The functional analysis is much simpler in the case of hardening material than in perfect plasticity, where the formulation lies in the spaces of functions with bounded deformation and bounded Radon measures. The main existence result for the three-dimensional quasistatic evolution, for linearly elastic-perfectly plastic material, in such a variational framework was proved in the seminal paper [17].

Another area of research in materials science is the derivation of effective or homogenized models that simplify calculations and provide a good approximation of the description of the (average) behavior of heterogeneous materials when the materials are mixed at small scales. Non-triviality stems from the fact that such materials, obtained by mixing two or more materials at fine scales, have different properties than the averaged properties of the materials that make them. Mathematically, the derivation of the effective properties of such mixtures is obtained by analyzing the behavior of differential equations (of the energy functional in the variational approach) with fast oscillating coefficients when the oscillation parameter tends to zero. In that respect, different methods within homogenization theory have been developed, including the two-scale convergence method (suitable for periodic homogenization) in the seminal papers [41] and [2], as well as the mentioned Γ-convergence method. For our purposes, the most important are the existence results and periodic homogenization for the quasistatic evolution in perfect plasticity from the papers [24] and [25].

Analysis of mathematical problems in which there are many very small parameters, such as highly heterogeneous thin structures, where homogenization and dimension reduction are performed simultaneously, is challenging. It has been shown that, apart from depending on elastic energy scales, different effective models are obtained depending on the assumptions about the relationship between the oscillation of the microstructure and the thickness of the body. Complete asymptotics for heterogeneous rods or plates is performed in the book [42] under the assumption that the oscillations of the microstructure and the thickness of the body are equal. We also mention the earlier paper [33] where a linearized rod model with a composite microstructure along the cross-section was derived, and a more recent paper [29] where a heterogeneous rod model was derived using the so-called ”unfolding operator” (for homogenization problems) and Griso’s decomposition (for dimension reduction problems). Problems of simultaneous homogenization and dimension reduction in the context of nonlinear elasticity were tackled in [9] and [7] for the membrane case using the Γ-convergence methods. Higher-order models, such as von Kármán’s regime and bending regimes for plates, rods, and shells, have been studied in a series of works (see [39], [40], [30], [31], [11]). In the paper [12] a new approach for simultaneous dimension reduction and non-periodic homogenization is introduced.

Chapter overview

This thesis is divided into five chapters. In Chapter 1 we give definitions and basic results from the analysis of partial differential equations, geometric measure theory and convex analysis, which are used thought the rest of the thesis. In Chapter 2 we describe the framework of a periodic multi-phase elasto-plastic plate, and we state the basic assumptions in each of the three regimes ($\gamma \in (0,+\infty)$, $\gamma = 0$ and $\gamma = +\infty$) on the interfaces and admissible stresses needed to obtain our results. We describe the formulation of the rescaled three-dimensional problem and detail the properties of the reduced problem. Finally, we discuss the quasistatic evolution of the h-problem.

In Chapter 3 we present the first contributions of the thesis. We consider a general framework with which we analyze the properties spaces of bounded measures whose appropriate derivatives are also bounded measures. We then give some auxiliary results, which we use to characterize the two-scale limit of scaled symmetrized gradients. This structure theorem represents the fundamental compactness result. Further, we introduce the notion of the unfolding measure adapted to dimension reduction and prove results regarding the unfolding of scaled symmetrized gradients of *BD* functions. We apply these results in the following chapter to establish a lower semicontinuity result for the dissipation potentials in the regime $\gamma \in (0, +\infty)$.

In Chapter 4 we give meaning to pairings between stress fields (which belong to some Lebesgue space) and plastic strains (which are bounded measure) defined on an appropriate cell. In order to apply this for configurations defined in both variables x and y, we proceed to state disintegration results for kinematic fields and approximation results for stresses. Applying all of these results, we prove the principle of maximum plastic work.

In Chapter 5 we are finally able to state and prove the main result of the thesis, namely the quasistatic evolution for two-scale homogenized limits.

1. PRELIMINARIES

1.1. NOTATION

We will write any point $x \in \mathbb{R}^3$ as a pair (x', x_3), with $x' \in \mathbb{R}^2$ and $x_3 \in \mathbb{R}$. and we will use the notation $\nabla_{x'}$ to denote the gradient with respect to x'. We denote by $y \in \mathcal{Y}$ the points on a flat 2-dimensional torus. In what follows we will also adopt the following notation for scaled gradients and symmetrized scaled gradients:

$$\nabla_h v := \left[\nabla_{x'} v \,\middle|\, \tfrac{1}{h}\partial_{x_3} v \right], \quad E_h v := \operatorname{sym} \nabla_h v,$$
$$\widetilde{\nabla}_\gamma v := \left[\nabla_y v \,\middle|\, \tfrac{1}{\gamma}\partial_{x_3} v \right], \quad \widetilde{E}_\gamma v := \operatorname{sym} \widetilde{\nabla}_\gamma v.$$

The scaled divergence operators div_h and $\widetilde{\operatorname{div}}_\gamma$ are defined as (formal) adjoints of the respective scaled gradients.

If $a, b \in \mathbb{R}^N$, we write $a \cdot b$ for the Euclidean scalar product, and we denote by $|a| := \sqrt{a \cdot a}$ the Euclidean norm. We write $\mathbb{M}^{N\times N}$ for the set of real $N \times N$ matrices. If $A, B \in \mathbb{M}^{N\times N}$, we use the Frobenius scalar product $A : B := \sum_{i,j} A_{ij} B_{ij}$ and the associated norm $|A| := \sqrt{A : A}$. We denote by $\mathbb{M}^{N\times N}_{\text{sym}}$ the space of real symmetric $N \times N$ matrices, and by $\mathbb{M}^{N\times N}_{\text{dev}}$ the set of real deviatoric matrices, respectively, i.e. the subset of $\mathbb{M}^{N\times N}_{\text{sym}}$ given by matrices having null trace. For every matrix $A \in \mathbb{M}^{N\times N}$ we denote its trace by $\operatorname{tr} A$, and its deviatoric part by A_{dev} will be given by

$$A_{\text{dev}} = A - \frac{1}{N} \operatorname{tr} A.$$

The *symmetrized tensor product* $a \odot b$ of two vector $a, b \in \mathbb{R}^N$ is the symmetric matrix with entries $(a \odot b)_{ij} := \frac{a_i b_j + a_j b_i}{2}$. Note that $\operatorname{tr}(a \odot b) = a \cdot b$, and that $|a \odot b|^2 = \frac{1}{2}|a|^2|b|^2 + \frac{1}{2}(a \cdot b)^2$, so that

$$\frac{1}{\sqrt{2}}|a||b| \le |a \odot b| \le |a||b|.$$

Given a vector $v \in \mathbb{R}^3$, we will use the notation v' to denote the vector

$$v' := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Analogously, given a matrix $A \in \mathbb{M}^{3\times 3}$, we will denote by A'' the minor

$$A'' := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

The Lebesgue measure in $\mathbb{R}^N$ and the $(N-1)$-dimensional Hausdorff measure are denoted by $\mathscr{L}^N$ and $\mathscr{H}^{N-1}$, respectively. Given an open subset $U \subset \mathbb{R}^N$ and a finite dimensional Euclidean space E, we use standard notations for Lebesgue spaces $L^p(U;E)$ and Sobolev spaces $H^1(U;E)$ or $W^{1,p}(U;E)$.

We will write $C^k(U;E)$ for the space of all k-times continuously differentiable functions $\varphi : U \to E$ and $C^\infty(U;E) := \bigcap_{k=0}^{\infty} C^k(U;E)$ for the space of infinitely differentiable function. We will distinguish between the spaces $C^k_c(U;E)$ (C^k functions with compact support contained in U) and $C^k_0(U;E)$ (C^k functions ”vanishing on ∂U”). We will write $C(\mathscr{Y};E)$ to denote the space of all continuous functions $\varphi : \mathbb{R}^2 \to E$ which are $[0,1])^2$-periodic, and set $C^k(\mathscr{Y};E) := C^k(\mathbb{R}^2;E) \cap C(\mathscr{Y};E)$. We will identify $C^k(\mathscr{Y};E)$ with the space of all C^k functions on the 2-dimensional torus.

We will frequently make use of the *standard mollfier* $\rho \in C^\infty(\mathbb{R}^N)$, defined by

$$\rho(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where the constant $C > 0$ is selected so that $\int_{\mathbb{R}^N} \rho(x)\,dx = 1$, and the associated family $\{\rho_\varepsilon\}_{\varepsilon>0} \subset C^\infty(\mathbb{R}^N)$ with

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^N} \rho\left(\frac{x}{\varepsilon}\right).$$

Throughout the text, the letter C stands for generic constants which may vary from line to line.

1.2. TWO-SCALE CONVERGENCE

Two-scale convergence adapted to dimension reduction

Let $\Omega = \omega \times I$, where $\omega \subset \mathbb{R}^2$ is bounded and has Lipschitz boundary and $I = (-1/2, 1/2)$, and let $\varepsilon_h > 0$ be a sequence such that $\varepsilon_h \to 0$ as $h \to 0$ so that

$$\lim_{h\to 0} \frac{h}{\varepsilon_h} = \gamma \in [0, \infty].$$

Definition 1.2.1. We say a bounded sequence $\{u^h\}_{h>0}$ in $L^2(\Omega)$ two-scale converges to $u \in L^2(\Omega \times Y)$ and we write $u^h \xrightharpoonup{2} u$, if

$$\lim_{h\to 0} \int_\Omega u^h(x)\psi\left(x, \frac{x'}{\varepsilon_h}\right) dx = \int_{\Omega\times\mathscr{Y}} u(x,y)\,\psi(x,y)\,dxdy$$

for all $\psi \in C_0^\infty(\Omega; C(\mathscr{Y}))$. When $||u^h||_{L^2(\Omega)} \to ||u||_{L^2(\Omega\times\mathscr{Y})}$ in addition, we say that u^h strongly two-scale converges to u and write $u^h \xrightarrow{2} u$. For vector-valued functions, two-scale convergence is defined componentwise.

If we identify functions on ω with their trivial extension to Ω, the definition above contains the standard notion of two-scale convergence on $\omega \times \mathscr{Y}$ as a special case. Indeed, when $\{u^h\}_{h>0}$ is a sequence in $L^2(\omega)$, then $u^h \xrightarrow{2} u$ is equivalent to

$$\lim_{h\to 0} \int_\omega u^h(x')\psi\left(x', \frac{x'}{\varepsilon_h}\right) dx = \int_{\omega\times\mathscr{Y}} u(x',y)\,\psi(x',y)\,dy\,dx'$$

for all $\psi \in C_0^\infty(\omega; C(\mathscr{Y}))$.

Let us recall some well-known properties of two-scale convergence. We refer to [2, 37, 48] for proofs.

Lemma 1.2.2. (i) Any sequence that is bounded in $L^2(\Omega)$ admits a two-scale convergent subsequence.

(ii) Let $\tilde{u} \in L^2(\Omega \times \mathscr{Y})$ and let $u^h \in L^2(\Omega)$ be such that $u^h \xrightharpoonup{2} \tilde{u}$. Then $u^h \rightharpoonup \int_{\mathscr{Y}} \tilde{u}(\cdot, y)\,dy$ weakly in $L^2(\Omega)$.

(iii) Let u^0 and $u^h \in L^2(\Omega)$ be such that $u^h \rightharpoonup u^0$ weakly in $L^2(\Omega)$. Then (after passing to subsequences) we have $u^h \xrightharpoonup{2} u^0(x) + \tilde{u}$ for some $\tilde{u} \in L^2(\Omega \times \mathscr{Y})$ with $\int_{\mathscr{Y}} \tilde{u}(\cdot, y)\,dy = 0$ almost everywhere in Ω.

(iv) Let u^0 and $u^h \in H^1(\Omega)$ be such that $u^h \to u^0$ strongly in $L^2(\Omega)$. Then $u^h \xrightarrow{2} u^0$, where we extend u^0 trivially to $\Omega \times \mathcal{Y}$.

The following theorem is given in [39]. We will not use it directly, but it a useful result which captures the structure of the limit of scaled gradients, which are natural objects to consider when dealing with dimension reduction.

Theorem 1.2.3. Let $(u^{\varepsilon_h})_{h>0}$ be a weakly convergent sequence in $H^1(\Omega;\mathbb{R}^3)$ with limit u and suppose that

$$\limsup_{h\to 0} ||\nabla_h u^{\varepsilon_h}||_{L^2(\Omega;\mathbb{R}^3)} < \infty. \tag{1.1}$$

1. (a) If $\gamma \in (0,\infty)$ then there exists a function $w \in L^2(\omega; H^1(I \times \mathcal{Y};\mathbb{R}^3))$ and a subsequence (not relabeled) such that

$$\nabla_h u^{\varepsilon_h}(x) \xrightharpoonup{2} \big(\nabla_{\hat{x}} u(\hat{x})\,|\,0\big) + \widetilde{\nabla}_\gamma w(x,y).$$

   (b) If $\gamma \in (0,\infty)$ and in addition to (1.1) we assume that

$$\limsup_{h\to 0} h^{-1} \left\|u^{\varepsilon_h}\right\|_{L^2(\Omega;\mathbb{R}^3)} < \infty,$$

   then there exists a function $w \in L^2(\omega; H^1(I \times \mathcal{Y};\mathbb{R}^3))$ and a subsequence (not relabeled) such that

$$h^{-1} u^{\varepsilon_h}(x) \xrightharpoonup{2} w(x,y), \quad \nabla_h u^{\varepsilon_h}(x) \xrightharpoonup{2} \widetilde{\nabla}_\gamma w(x,y).$$

2. If $\gamma = 0$ then there exits $w \in L^2(\omega; H^1(\mathcal{Y};\mathbb{R}^3))$ and $g \in L^2(\Omega \times Y;\mathbb{R}^3)$ such that

$$\nabla_h u^{\varepsilon_h}(x) \xrightharpoonup{2} \big(\nabla_{\hat{x}} u(\hat{x})\,|\,0\big) + \big(\nabla_y w\,|\,g\big).$$

3. If $\gamma = \infty$ then there exists $w \in L^2(\Omega; H^1(\mathcal{Y};\mathbb{R}^3))$, $g \in L^2(\Omega;\mathbb{R}^3)$ such that

$$\nabla_h u^{\varepsilon_h}(x) \xrightharpoonup{2} \big(\nabla_{\hat{x}} u(\hat{x})\,|\,0\big) + \big(\nabla_y w\,|\,g\big).$$

1.3. MEASURES

We first recall some basic notions from measure theory that we will use throughout the thesis (see, e.g. [23]).

Given a Borel set $U \subset \mathbb{R}^N$ and a finite dimensional Hilbert space X, we denote by $\mathscr{M}_b(U;X)$ the space of bounded Borel measures on U taking values in X, and endowed with the norm $\|\mu\|_{\mathscr{M}_b(U;X)} := |\mu|(U)$, where $|\mu| \in \mathscr{M}_b(U)$ is the total variation of the measure μ. For every $\mu \in \mathscr{M}_b(U;X)$ we consider the Lebesgue decomposition $\mu = \mu^a + \mu^s$, where μ^a is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^N$ and μ^s is singular with respect to $\mathscr{L}^N$. If $\mu^s = 0$, we always identify μ with its density with respect to $\mathscr{L}^N$, which is a function in $L^1(U;X)$.

If the relative topology of U is locally compact, by Riesz representation theorem the space $\mathscr{M}_b(U;X)$ can be identified with the dual of $C_0(U;X)$, which is the space of all continuous functions $\varphi : U \to X$ such that the set $\{|\varphi| \geq \delta\}$ is compact for every $\delta > 0$. The weak* topology on $\mathscr{M}_b(U;X)$ is defined using this duality.

The *restriction* of $\mu \in \mathscr{M}_b(U;X)$ to a subset $E \in U$ is the measure $\mu\lfloor E \in \mathscr{M}_b(E;X)$ defined by

$$\mu\lfloor E(B) := \mu(E \cap B), \qquad \text{for every Borel set } B \subset U.$$

Given two real-valued measures μ_1, $\mu_2 \in \mathscr{M}_b(U)$ we write $\mu_1 \geq \mu_2$ if $\mu_1(B) \geq \mu_2(B)$ for every Borel set $B \subset U$.

Convex functions of measures

Let U be an open set of $\mathbb{R}^N$. For every $\mu \in \mathscr{M}_b(U;X)$ let $\frac{d\mu}{d|\mu|}$ be the Radon-Nikodym derivative of μ with respect to its variation $|\mu|$. Let $H : X \to [0,+\infty)$ be a convex and positively one-homogeneous function such that

$$r|\xi| \leq H(\xi) \leq R|\xi| \quad \text{for every } \xi \in X, \tag{1.2}$$

where r and R are two constants, with $0 < r \leq R$.

Using the theory of convex functions of measures, developed in [28] and [21], we introduce the nonnegative Radon measure $H(\mu) \in \mathscr{M}_b^+(U)$ defined by

$$H(\mu)(A) := \int_A H\left(\frac{d\mu}{d|\mu|}\right) d|\mu|,$$

for every Borel set $A \subset U$. We also consider the functional $\mathscr{H} : \mathscr{M}_b(U;X) \to [0,+\infty)$ defined by

$$\mathscr{H}(\mu) := H(\mu)(U) = \int_U H\left(\frac{d\mu}{d|\mu|}\right) d|\mu|.$$

One can prove that $\mathscr{H}$ is lower semicontinuous on $\mathscr{M}_b(U;X)$ with respect to weak* convergence (see, e.g., [5, Theorem 2.38]).

Let $a, b \in [0,T]$ with $a \le b$. The *total variation* of a function $\mu : [0,T] \to \mathscr{M}_b(U;X)$ on $[a,b]$ is defined by

$$\mathscr{V}(\mu;a,b) := \sup\left\{\sum_{i=1}^{n} \|\mu(t_{i+1}) - \mu(t_i)\|_{\mathscr{M}_b(U;X)} : a = t_1 < t_2 < \ldots < t_n = b,\ n \in \mathbb{N}\right\}.$$

Analogously, we define the $\mathscr{H}$*-variation* of a function $\mu : [0,T] \to \mathscr{M}_b(U;X)$ on $[a,b]$ as

$$\mathscr{D}_{\mathscr{H}}(\mu;a,b) := \sup\left\{\sum_{i=1}^{n} \mathscr{H}\left(\mu(t_{i+1}) - \mu(t_i)\right) : a = t_1 < t_2 < \ldots < t_n = b,\ n \in \mathbb{N}\right\}.$$

From (1.2) it follows that

$$r\mathscr{V}(\mu;a,b) \le \mathscr{D}_{\mathscr{H}}(\mu;a,b) \le R\mathscr{V}(\mu;a,b). \tag{1.3}$$

Disintegration of a measure

Let S and T be measurable spaces and let μ be a measure on S. Given a measurable function $f : S \to T$, we denote by $f_\#\mu$ the *push-forward* of μ under the map f, defined by

$$f_\#\mu(B) := \mu\left(f^{-1}(B)\right), \qquad \text{for every measurable set } B \subset T.$$

In particular, for any measurable function $g : T \to \overline{\mathbb{R}}$ we have

$$\int_S g \circ f \, d\mu = \int_T g \, d(f_\#\mu).$$

Note that in the previous formula $S = f^{-1}(T)$.

Let $S_1 \subset \mathbb{R}^{n_1}$, $S_2 \subset \mathbb{R}^{n_2}$ be open sets, and let $\eta \in \mathscr{M}_b^+(S_1)$. We say that a function $x_1 \in S_1 \to \mu_{x_1} \in \mathscr{M}_b(S_2;\mathbb{R}^N)$ is η-measurable if $x_1 \in S_1 \to \mu_{x_1}(B)$ is η-measurable for every Borel set $B \subseteq S_2$.

Given a η-measurable function $x_1 \to \mu_{x_1}$, the *generalized product* $\eta \overset{\text{gen.}}{\otimes} \mu_{x_1} \in \mathscr{M}_b(S_1 \times S_2;\mathbb{R}^N)$ is a well defined measure such that

$$\langle \eta \overset{\text{gen.}}{\otimes} \mu_{x_1}, \varphi\rangle := \int_{S_1}\left(\int_{S_2} \varphi(x_1,x_2)\, d\mu_{x_1}(x_2)\right) d\eta(x_1),$$

for every bounded Borel function $\varphi : S_1 \times S_2 \to \mathbb{R}$ such that $\mathrm{supp}(\varphi) \subset K \times S_2$, where $K \subset S_1$ is any compact set.

Moreover, the following disintegration result holds (c.f. [5, Theorem 2.28 and Corrolary 2.29]):

Theorem 1.3.1. Let $\mu \in \mathscr{M}_b^+(S_1 \times S_2; \mathbb{R}^N)$ and let $proj : S_1 \times S_2 \to S_1$ be the projection on the first factor. Assume that the push-forward measure $\eta := proj_\#|\mu| \in \mathscr{M}_b^+(S_1)$ is a Radon measure, i.e. $|\mu|(K \times S_2) < \infty$ for any compact set $K \subset S_1$. Then there exists a unique family of bounded Radon measures $\{\mu_{x_1}\}_{x_1 \in S_1} \subset \mathscr{M}_b(S_2; \mathbb{R}^N)$ such that $x_1 \to \mu_{x_1}$ is η-measurable, and

$$\mu = \eta \overset{\text{gen.}}{\otimes} \mu_{x_1}.$$

For every $\varphi \in L^1(S_1 \times S_2, d|\mu|)$ we have

$$\varphi(x_1, \cdot) \in L^1(S_2, d|\mu_{x_1}|) \quad \text{for } \eta\text{-a.e. } x_1 \in S_1,$$
$$x_1 \to \int_{S_2} \varphi(x_1, x_2)\, d\mu_{x_1}(x_2) \ \in L^1(S_1, d\eta),$$
$$\int_{S_1 \times S_2} \varphi(x_1, x_2)\, d\mu(x_1, x_2) = \int_{S_1} \left(\int_{S_2} \varphi(x_1, x_2)\, d\mu_{x_1}(x_2) \right) d\eta(x_1).$$

Furthermore,

$$|\mu| = \eta \overset{\text{gen.}}{\otimes} |\mu_{x_1}|.$$

Arguing as in [25, Remark 5.5], we have the following:

Proposition 1.3.2. With the same notation as in Theorem 1.3.1, for η-a.e. $x_1 \in S_1$

$$\frac{d\mu}{d|\mu|}(x_1, \cdot) = \frac{d\mu_{x_1}}{d|\mu_{x_1}|} \quad |\mu_{x_1}|\text{-a.e. on } S_2.$$

Proof. Since $\frac{d\mu}{d|\mu|} \in L^1(S_1 \times S_2, d|\mu|)$, from Theorem 1.3.1 we have $\frac{d\mu}{d|\mu|}(x_1, \cdot) \in L^1(S_2, d|\mu_{x_1}|)$ for η-a.e. $x_1 \in S_1$. Thus,

$$\eta \overset{\text{gen.}}{\otimes} \frac{d\mu_{x_1}}{d|\mu_{x_1}|} |\mu_{x_1}| = \eta \overset{\text{gen.}}{\otimes} \mu_{x_1} = \mu = \frac{d\mu}{d|\mu|} |\mu| = \eta \overset{\text{gen.}}{\otimes} \frac{d\mu}{d|\mu|}(x_1, \cdot) |\mu_{x_1}|,$$

from which we have the claim. ■

1.4. Spaces of functions with measures as derivatives

Functions with bounded variation

Let U be an open set of $\mathbb{R}^N$. The space $BV(U)$ of functions with *bounded variation* is the space of all functions $u \in L^1(U)$ whose gradient Du (in the sense of distributions) satisfies $Du \in \mathscr{M}_b(U;\mathbb{R}^N)$. The measure Du can be decomposed as

$$Du = \nabla u \mathscr{L}^N + (u^+ - u^-) \otimes \nu_u \mathscr{H}^{N-1} \lfloor S_u + D^c u,$$

where ∇u is the Radon-Nikodym derivative of Du with respect to the Lebesgue measure $\mathscr{L}^N$, which coincides with the approximate gradient of u. The jump set S_u is a countably $\mathscr{H}^{N-1}$-rectifiable Borel set (see [5, Definition 2.57]), ν_u is an approximate unit normal to S_u, and $u^\pm$ are the one-sided Lebesgue limits of u on S_u. The measure $D^c u$ is the Cantor part of Du which has the property of vanishing on any finite set with respect to the $(N-1)$-dimensional Hausdorff measure $\mathscr{H}^{N-1}$. The general properties of the space $BV(U)$ can be found in [5, 8].

Functions with bounded deformation

Let U be an open set of $\mathbb{R}^N$. The space $BD(U)$ of functions with *bounded deformation* is the space of all functions $u \in L^1(U;\mathbb{R}^N)$ whose symmetric gradient $Eu := \operatorname{sym} Du$ (in the sense of distributions) satisfies $Eu \in \mathscr{M}_b(U;\mathbb{M}^{N\times N}_{sym})$. It is a Banach space endowed with the norm

$$\|u\|_{BD(U)} = \|u\|_{L^1(U;\mathbb{R}^N)} + |Eu|(U).$$

It was proved in [47, Proposition 2.5] that $BD(U)$ can be identified with the dual of a Banach space, and therefore it can be endowed with a natural weak* topology. We say that a sequence $\{u_k\}_k$ converges to u weakly* in $BD(U)$ if and only if

$$\begin{cases} u_k \to u & \text{strongly in } L^1(U;\mathbb{R}^N), \\ Eu_k \overset{*}{\rightharpoonup} Eu & \text{weakly* in } \mathscr{M}_b(U;\mathbb{M}^{N\times N}_{sym}). \end{cases}$$

Every bounded sequence in $BD(U)$ has a weakly* converging subsequence.

An intermediate notion of convergence between weak* and strong convergences is the so-called *strict convergence*: a sequence $\{u_k\}_k \subset BD(U)$ converges strictly to some $u \in BD(U)$ if and only if $u_k \overset{*}{\rightharpoonup} u$ weakly* in $BD(U)$ and $|Eu_k|(U) \to |Eu|(U)$.

If U is bounded and has Lipschitz boundary, $BD(U)$ can be continuously embedded into $L^{N/(N-1)}(U;\mathbb{R}^N)$. Furthemore, the injection of $BD(U)$ into $L^p(U;\mathbb{R}^N)$ is compact for all $1 \leq p < N/(N-1)$. If Γ is a nonempty open subset of ∂U, there exists a constant $C > 0$, depending only on U and Γ, such that

$$\|u\|_{L^1(U;\mathbb{R}^N)} \leq C\left(\|u\|_{L^1(\Gamma;\mathbb{R}^N)} + |Eu|(U)\right) \tag{1.4}$$

(see [45, Chapter II, Proposition 2.4 and Remark 2.5]).

Let M be a C^1-hypersurface contained in U. It is well known (see [45, Chapter II]) that the one sided Lebesgue limits $u^\pm(x)$ on both sides of M exist for $\mathscr{H}^{N-1}$ almost every $x \in M$ and satisfy

$$Eu\lfloor M = (u^+ - u^-) \odot \nu\, \mathscr{H}^{N-1}\lfloor M,$$

where ν is a unit normal to M. As shown in [4], the measure Eu can be decomposed as

$$Eu = \mathscr{E}(u)\mathscr{L}^N + (u^+ - u^-) \odot \nu_u \mathscr{H}^{N-1}\lfloor J_u + E^c u,$$

where $\mathscr{E}(u) = \frac{\nabla u + \nabla u^T}{2}$, ∇u is the approximate gradient of u. The jump set J_u is $\mathscr{H}^{N-1}$-rectifiable, ν_u is an approximate unit normal to J_u, and $u^\pm$ are the one-sided Lebesgue limits of u on J_u. The measure $E^c u$ is the Cantor part of Eu, defined as the restriction $E^c u := E^s u\lfloor(U \setminus J_u)$.

Let $\mathscr{R}$ be the class of *rigid motions* in $\mathbb{R}^N$, i.e., affine maps of the form $Ax+b$ such that A is a skew-symmetric $N \times N$ matrix and $b \in \mathbb{R}^N$. The following Poincaré type inequality for *BD* functions follows from [45, Proposition 2.2 and Remark 1.1 of Chapter II].

Theorem 1.4.1. Let U be a bounded connected open set with Lipschitz boundary and let $\Pi : BD(U) \to \mathscr{R}$ be a continuous linear map which leaves the elements of $\mathscr{R}$ fixed. Then there exists a constant C, depending only on U and Π, such that

$$\int_U |u - \Pi(u)|\, dx \leq C|Eu|(U),$$

for all $u \in BD(U)$.

Let $U \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. From [6, Theorem 3.2] we have that there exists a unique linear continuous trace operator from $T : BD(U) \to L^1(\partial U; \mathbb{R}^N)$ such that the following integration by parts formula holds: for every $u \in BD(U)$ and $\varphi \in C^1(\mathbb{R}^N)$

$$\int_U u \odot \nabla\varphi \, dx + \int_U \varphi \, dEu = \int_{\partial U} T(u) \odot \nu\varphi \, d\mathscr{H}^{N-1},$$

where ν is the outer unit normal to ∂U. In addition,

$$T(u) = u|_{\partial U} \text{ for all } u \in C(\overline{U}; \mathbb{R}^N) \cap BD(U).$$

Furthermore, if $u \in BD(U)$ and $\{u_k\}_k \subset C^\infty(\overline{U}; \mathbb{R}^N)$ is such that $u_k \to u$ strictly in $BD(U)$, then $T(u_k) \to T(u)$ strongly in $L^1(\partial U; \mathbb{R}^N)$.

Functions with bounded Hessian

Let U be an open set of $\mathbb{R}^N$. The space $BH(U)$ of functions with *bounded Hessian* is the space of all functions $u \in W^{1,1}(U)$ whose Hessian D^2u (in the sense of distributions) satisfies $D^2u \in \mathscr{M}_b(U; \mathbb{M}^{N\times N}_{sym})$. It is a Banach space endowed with the norm

$$\|u\|_{BH(U)} = \|u\|_{W^{1,1}(U)} + |D^2u|(U).$$

If U has the cone property, then $BH(U)$ coincides with the space of functions in $L^1(U)$ whose Hessian belongs to $\mathscr{M}_b(U; \mathbb{M}^{N\times N}_{sym})$. If U is bounded and has Lipschitz boundary, then $BH(U)$ can be embedded into $W^{1,N/(N-1)}(U)$. If U is bounded and has C^2 boundary, then for every function $u \in BH(U)$ one can define the traces of u and of ∇u (still denoted by u and ∇u) which satisfy $u \in W^{1,1}(\partial U)$, $\nabla u \in L^1(\partial U; \mathbb{R}^N)$, and $\frac{\partial u}{\partial \tau} = \nabla u \cdot \tau$ in $L^1(\partial U)$, where τ is any tangent vector to ∂U. If, in addition, $N = 2$, then $BH(U)$ embeds into $C(\overline{U})$. The general properties of the space $BH(U)$ can be found in [20].

1.5. STRESS-STRAIN DUALITY

Traces of stresses

We suppose here that U is an open bounded set of class C^2. If $\sigma \in L^2(U;\mathbb{M}^{N\times N}_{\rm sym})$ and $\mathrm{div}\sigma \in L^2(U;\mathbb{R}^N)$, then we can define a distribution $[\sigma\nu]$ on ∂U by

$$[\sigma\nu](\psi) := \int_U \psi\cdot \mathrm{div}\sigma\, dx + \int_U \sigma : E\psi\, dx, \tag{1.5}$$

for every $\psi \in H^1(U;\mathbb{R}^N)$. It turns out that $[\sigma\nu] \in H^{-1/2}(\partial U;\mathbb{R}^N)$ (see, e.g., [46, Chapter 1, Theorem 1.2]). If, in addition, $\sigma \in L^\infty(U;\mathbb{M}^{N\times N}_{\rm sym})$ and $\mathrm{div}\sigma \in L^N(U;\mathbb{R}^N)$, then (1.5) holds for $\psi \in W^{1,1}(U;\mathbb{R}^N)$. By Gagliardo's extension theorem, in this case we have $[\sigma\nu] \in L^\infty(\partial U;\mathbb{R}^N)$ and that

$$[\sigma_k\nu] \overset{*}{\rightharpoonup} [\sigma\nu] \quad \text{weakly* in } L^\infty(\partial U;\mathbb{R}^N),$$

whenever $\sigma_k \overset{*}{\rightharpoonup} \sigma$ weakly* in $L^\infty(U;\mathbb{M}^{N\times N}_{\rm sym})$ and $\mathrm{div}\sigma_k \rightharpoonup \mathrm{div}\sigma$ weakly in $L^N(U;\mathbb{R}^N)$.

We will consider the normal and tangential parts of $[\sigma\nu]$, defined by

$$[\sigma\nu]_\nu := ([\sigma\nu]\cdot\nu)\nu, \quad [\sigma\nu]_\nu^\perp := [\sigma\nu] - ([\sigma\nu]\cdot\nu)\nu.$$

Since $\nu \in C^1(\partial U;\mathbb{R}^N)$, we have that $[\sigma\nu]_\nu, [\sigma\nu]_\nu^\perp \in H^{-1/2}(\partial U;\mathbb{R}^N)$. If, in addition, $\sigma_{\rm dev} \in L^\infty(U;\mathbb{M}^{N\times N}_{\rm dev})$, then it was proved in [34, Lemma 2.4] that $[\sigma\nu]_\nu^\perp \in L^\infty(\partial U;\mathbb{R}^N)$ and

$$\|[\sigma\nu]_\nu^\perp\|_{L^\infty(\partial U;\mathbb{R}^N)} \le \frac{1}{\sqrt{2}}\|\sigma_{\rm dev}\|_{L^\infty(U;\mathbb{M}^{N\times N}_{\rm dev})}.$$

More generally, if U has Lipschitz boundary and such that there exists a compact set $S \subset \partial U$ with $\mathscr{H}^{N-1}(S) = 0$ such that $\partial U \setminus S$ is C^2-hypersurface, then arguing as in [24, Section 1.2] we can uniquely determine $[\sigma\nu]_\nu^\perp$ as an element of $L^\infty(\partial U;\mathbb{R}^N)$ through any approximating sequence $\{\sigma_n\} \subset C^\infty(\overline{U};\mathbb{M}^{N\times N}_{\rm sym})$ such that

$$\begin{aligned}
&\sigma_n \to \sigma \quad \text{strongly in } L^2(U;\mathbb{M}^{N\times N}_{\rm sym}),\\
&\mathrm{div}\sigma_n \to \mathrm{div}\sigma \quad \text{strongly in } L^2(U;\mathbb{R}^N),\\
&\|(\sigma_n)_{\rm dev}\|_{L^\infty(U;\mathbb{M}^{N\times N}_{\rm dev})} \le \|\sigma_{\rm dev}\|_{L^\infty(U;\mathbb{M}^{N\times N}_{\rm dev})}.
\end{aligned}$$

The duality theorems

In the following, let $U \subset \mathbb{R}^N$ be an open, bounded set of with C^2 boundary. Let us recall certain results obtained in [34].

Proposition 1.5.1. For any $u \in BV(U)$ and $\sigma \in L^\infty(U;\mathbb{R}^N)$ with $\operatorname{div}\sigma \in L^N(U)$, let $[\sigma \cdot Du]$ denote the distribution on U defined for $\varphi \in C_c^\infty(U)$ by:

$$[\sigma \cdot Du](\varphi) := -\int_U u \operatorname{div}\sigma\,\varphi\,dx - \int_U u\,\sigma \cdot \nabla\varphi\,dx$$

Then $[\sigma \cdot Du]$ may be extended as a bounded measure on U which is absolutely continuous with respect to $|Du|$, whose variation satisfies

$$|[\sigma \cdot Du]| \leq \|\sigma\|_{L^\infty(U;\mathbb{R}^N)}|Du| \quad \text{ in } \mathscr{M}_b(U).$$

Moreover, the following integration by parts formula holds

$$\int_{\partial U} \varphi\,[\sigma\nu]\,u\,d\mathscr{H}^{N-1} = \int_U \varphi\,d[\sigma \cdot Du] + \int_U u \operatorname{div}\sigma\,\varphi\,dx + \int_U u\,\sigma \cdot \nabla\varphi\,dx$$

for every $\varphi \in C^1(\overline{U})$.

Proposition 1.5.2. The set

$$\mathscr{S}(U) := \left\{\sigma \in L^2(U;\mathbb{M}^{N\times N}_{\text{sym}}) : \operatorname{div}\sigma \in L^N(U;\mathbb{R}^N),\ \sigma_{\text{dev}} \in L^\infty(U;\mathbb{M}^{N\times N}_{\text{dev}})\right\},$$

is a subset of $L^p(U;\mathbb{M}^{N\times N}_{\text{sym}})$ for every $1 \leq p < \infty$, and

$$\|\sigma\|_{L^p(U;\mathbb{M}^{N\times N}_{\text{sym}})} \leq C_p\left(\|\sigma\|_{L^2(U;\mathbb{M}^{N\times N}_{\text{sym}})} + \|\operatorname{div}\sigma\|_{L^N(U;\mathbb{R}^N)} + \|\sigma_{\text{dev}}\|_{L^\infty(U;\mathbb{M}^{N\times N}_{dev})}\right).$$

Proposition 1.5.3. Given $u \in BD(U)$ with $\operatorname{div} u \in L^2(U)$, and $\sigma \in L^2(U;\mathbb{M}^{N\times N}_{\text{sym}})$ with $\operatorname{div}\sigma \in L^N(U;\mathbb{R}^N)$, $\sigma_{\text{dev}} \in L^\infty(U;\mathbb{M}^{N\times N}_{\text{dev}})$, let $[\sigma_{\text{dev}} : E_{\text{dev}}u]$ denote the distribution on U defined for $\varphi \in C_c^\infty(U)$ by:

$$[\sigma_{\text{dev}} : E_{\text{dev}}u](\varphi) := -\int_U \varphi \operatorname{div}\sigma \cdot u\,dx - \int_U \sigma : (u \odot \nabla\varphi)\,dx - \frac{1}{N}\int_U \varphi \operatorname{tr}\sigma \operatorname{div} u\,dx$$

Then $[\sigma_{\text{dev}} : E_{\text{dev}}u]$ may be extended as a bounded measure on U which is absolutely continuous with respect to $|E_{\text{dev}}u|$, whose variation satisfies

$$|[\sigma_{\text{dev}} : E_{\text{dev}}u]| \leq \|\sigma_{\text{dev}}\|_{L^\infty(U;\mathbb{M}^{N\times N}_{\text{dev}})}|E_{\text{dev}}u| \quad \text{ in } \mathscr{M}_b(U).$$

Moreover, the following integration by parts formula holds

$$\int_{\partial U} \varphi\,[\sigma\nu]\cdot u\,d\mathscr{H}^{N-1} = \int_U \varphi\,d[\sigma_{\rm dev} : E_{\rm dev}u] + \frac{1}{N}\int_U \varphi\,\mathrm{tr}\,\sigma\,\mathrm{div}u\,dx + \int_U \varphi\,\mathrm{div}\,\sigma\cdot u\,dx + \int_U \sigma : (u\odot\nabla\varphi)\,dx$$

for every $\varphi \in C^1(\overline{U})$.

A similar result was given in [16, Section 3].

Proposition 1.5.4. For any $u \in BD(U)$ and $\sigma \in L^\infty(U;\mathbb{M}^{N\times N}_{\rm sym})$ with $\mathrm{div}\,\sigma \in L^N(U;\mathbb{R}^N)$, let $[\sigma : Eu]$ denote the distribution on U defined for $\varphi \in C^\infty_c(U)$ by:

$$[\sigma : Eu](\varphi) := -\int_U \varphi\,\mathrm{div}\,\sigma\cdot u\,dx - \int_U \sigma : (u\odot\nabla\varphi)\,dx$$

Then $[\sigma : Eu]$ may be extended as a bounded measure on U which is absolutely continuous with respect to $|Eu|$, whose variation satisfies

$$|[\sigma : Eu]| \leq \|\sigma\|_{L^\infty(U;\mathbb{M}^{N\times N}_{\rm sym})}|Eu| \quad \text{ in } \mathscr{M}_b(U).$$

Moreover, the following integration by parts formula holds

$$\int_{\partial U} \varphi\,[\sigma\nu]\cdot u\,d\mathscr{H}^{N-1} = \int_U \varphi\,d[\sigma : Eu] + \int_U \varphi\,\mathrm{div}\,\sigma\cdot u\,dx + \int_U \sigma : (u\odot\nabla\varphi)\,dx$$

for every $\varphi \in C^1(\overline{U})$.

We also recall the following construction from [19]:

Proposition 1.5.5. For any $u \in BH(U)$ and $\sigma \in L^\infty(U;\mathbb{M}^{N\times N}_{\rm sym})$ with $\mathrm{div}\,\mathrm{div}\,\sigma \in L^2(U)$, let $[\sigma : D^2u]$ denote the distribution on U defined for $\varphi \in C^\infty_c(U)$ by:

$$[\sigma : D^2u](\varphi) := \int_U u\,\mathrm{div}\,\mathrm{div}\,\sigma\,\varphi\,dx - 2\int_U \sigma : \big(\nabla u\odot\nabla\varphi\big)\,dx - \int_U u\,\sigma : \nabla^2\varphi\,dx$$

Then $[\sigma : D^2u]$ may be extended as a bounded measure on U which is absolutely continuous with respect to $|D^2u|$, whose variation satisfies

$$|[\sigma : D^2u]| \leq \|\sigma\|_{L^\infty(U;\mathbb{M}^{N\times N}_{\rm sym})}|D^2u| \quad \text{ in } \mathscr{M}_b(U).$$

1.6. Basics of convex analysis

We recall several definitions and basic facts from convex analysis (see, e.g. [22] and [43, Sections 13 and 23]).

Let X be a normed vector space, X^* its topological dual space and $\langle \cdot,\cdot \rangle$ the duality pairing on $X^* \times X$.

Definition 1.6.1. Consider $f : X \to \overline{\mathbb{R}}$. We say that

(a) f is a proper function if $f(x) > -\infty$ for every $x \in X$, and it is not identically equal to $+\infty$.

(b) f is a convex function if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \text{ for every } x,y \in X \text{ and } \lambda \in [0,1].$$

(c) f is a lower semicontinuous (or closed) function if

$$\liminf_{y \to x} f(y) \geq f(x), \text{ for every } x \in X.$$

(d) f is a positively 1-homogeneous function if

$$f(\lambda x) = \lambda f(x), \text{ for every } x \in X \text{ and } \lambda \geq 0.$$

Definition 1.6.2. The *convex subdifferential* of f at $x \in X$ is the set

$$\partial f(x) := \{x^* \in X^* : f(y) \geq f(x) + \langle x^*, y - x \rangle \ \ \forall y \in X\},$$

for $f(x) \in \mathbb{R}$. Otherwise, $\partial f(x) := \emptyset$.

Definition 1.6.3. The *conjugate function* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}.$$

We collect below some elementary properties of the subdifferential and conjugate function.

Proposition 1.6.4. For a given $f : X \to \overline{\mathbb{R}}$ we have:

(a) The set $\partial f(x)$ is closed and convex.

(b) Fermat's rule: $x \in \operatorname{Arg\,min}_X f \iff 0 \in \partial f(x)$.

Proposition 1.6.5. For a given $f : X \to \overline{\mathbb{R}}$ we have:

(a) The function f^* is convex and weak* lower semicontinuous.

(b) Young-Fenchel inequality: $f(x) + f^*(x^*) \geq \langle x^*, x\rangle$.

The subdifferential and conjugate of a convex function are dual notions. This can be seen from the following property.

Theorem 1.6.6. Let f be a proper convex function. Then the following conditions are all equivalent:

(i) $x^* \in \partial f(x)$;

(ii) $x \in \operatorname{Arg\,max}_{y \in X} \{\langle x^*, y\rangle - f(y)\}$;

(iii) $f(x) + f^*(x^*) = \langle x^*, x\rangle$.

If f is lower semicontinuous, then all of the above conditions are equivalent to:

(iv) $x \in \partial f(x^*)$;

(v) $x^* \in \operatorname{Arg\,max}_{y^* \in X^*} \{\langle y^*, x\rangle - f^*(y^*)\}$.

Definition 1.6.7. The *biconjugate function* of f is the function $f^{**} : X \to \overline{\mathbb{R}}$ defined by

$$f^{**}(x) := \sup_{x^* \in X^*} \{\langle x^*, x\rangle - f^*(x^*)\}.$$

When X^* is endowed with the weak* topology, then $f^{**} = (f^*)^*$.

Proposition 1.6.8. Let $f : X \to \overline{\mathbb{R}}$ be convex. If $\partial f(x) \neq \emptyset$, then $f(x) = f^{**}(x)$.

Definition 1.6.9. If $f_1, f_2 : X \to \overline{\mathbb{R}}$ are proper functions, then the *infimal convolution* of f_1 and f_2 is defined as

$$(f_1 \Box f_2)(x) := \inf_{x' \in X} \left\{f_1(x - x') + f_2(x')\right\}.$$

Note that if both f_1 and f_2 are convex, then so it is $f_1 \Box f_2$. It can be shown that

$$f_1 \Box f_2 = (f_1^* + f_2^*)^*.$$

Subdifferential of 1-homogeneous functions

We stress the following special structure of the subdifferential of a positively 1-homogeneous function.

Lemma 1.6.10. Let $h : X \to \overline{\mathbb{R}}$ be a proper, positively 1-homogeneous function. For $x \in X$ we have

$$\partial h(x) = \{x^* \in X^* : h(x) = \langle x^*, x\rangle \text{ and } h(y) \geq \langle x^*, y\rangle \ \ \forall y \in X\}.$$

Proof. Consider the set $S = \{x^* \in X^* : h(x) = \langle x^*, x\rangle \text{ and } h(y) \geq \langle x^*, y\rangle \ \ \forall y \in X\}$. Then, by subtracting the defining conditions of S, for $x^* \in S$ we have

$$h(y) - h(x) \geq \langle x^*, y - x\rangle, \quad \forall y \in X,$$

from which the inclusion $S \subseteq \partial h(x)$ directly follows.

Conversely, for $x^* \in \partial h(x)$ the above inequality holds for all $y \in X$. In particular, we can first choose $y = 2x$ and then $y = \frac{1}{2}x$, and use the 1-homogeneity to conclude

$$\begin{aligned} h(x) &\geq \langle x^*, x\rangle, \\ \frac{1}{2}h(x) &\leq \frac{1}{2}\langle x^*, x\rangle. \end{aligned}$$

Hence, we have $h(x) = \langle x^*, x\rangle$. The remaining inequality now follows from the definition of the subdifferential. ∎

The above lemma has the following consequence.

Proposition 1.6.11. Let $h : X \to \overline{\mathbb{R}}$ be a proper, positively 1-homogeneous function. Then, the following holds:

(a) $\partial h(x) \subseteq \partial h(0)$ for all $x \in X$.

(b) $\partial h(x) = \{x^* \in \partial h(0) : h(x) = \langle x^*, x\rangle\}$.

Proof. Since 1-homogeneity of h implies that $h(0) = 0$, from Lemma 1.6.10 we get

$$\partial h(0) = \{x^* \in X^* : h(y) \geq \langle x^*, y\rangle \ \ \forall y \in X\}. \tag{1.6}$$

The results then follows by substituting the appropriate property in the structure given by Lemma 1.6.10. ∎

Remark 1.6.12. Let $f : X \to \overline{\mathbb{R}}$ be a proper, convex, lower semicontinuous function. Then f has no values other than 0 and $+\infty$ if and only if its conjugate f^* is positively 1-homogeneous.

Indicator and support functions

Here we will assume that X is a reflexive normed space, i.e. $X^{**} = X$.

Definition 1.6.13. The *indicator function* of a set $A \subseteq X$ is the function $\iota_A : X \to \overline{\mathbb{R}}$ given by

$$\iota_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

Using characterizations based on the notion of the epigraph of a function, we can easily conclude the following:

(a) The function ι_A is proper if and only if A is non-empty.

(b) The function ι_A is convex if and only if A is a convex set in X.

(c) The function ι_A is lower semicontinuous if and only if A is a closed set in X.

Definition 1.6.14. The *normal cone* to $A \subseteq X$ is the set $N_A(x)$ defined by

$$N_A(x) := \begin{cases} \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0 \ \ \forall y \in A\} & \text{if } x \in A, \\ \emptyset & \text{otherwise.} \end{cases}$$

The convex subdifferential of the indicator function ι_A of a set $A \subseteq X$ is the normal cone of A, i.e. $N_A(x) = \partial \iota_A(x)$ for every $x \in X$. Indeed, for $x \in A$ we have

$$\begin{aligned} x^* \in N_A(x) &\iff \langle x^*, y - x \rangle \leq 0 \ \ \forall y \in A \\ &\iff \iota_A(y) \geq \iota_A(x) + \langle x^*, y - x \rangle \ \ \forall y \in X \\ &\iff x^* \in \partial \iota_A(x). \end{aligned}$$

Definition 1.6.15. The *support function* of a set $A \subseteq X$ is the function $h_A : X^* \to \overline{\mathbb{R}}$ given by

$$h_A(x^*) := \sup_{x \in A} \langle x^*, x \rangle.$$

The indicator function ι_C and the support function h_C of a closed convex set $C \subseteq X$ are conjugate to each other, i.e. $\iota_C^* = h_C$ and $h_C^* = \iota_C$.

Proposition 1.6.16. Let $C \subseteq X$ be a non-empty closed convex set. Then, for each $x^* \in X^*$, the set $\partial h_C(x^*)$ consists of all $x \in X$ such that

$$\langle x^*, x \rangle = \sup_{y \in C} \langle x^*, y \rangle.$$

Proposition 1.6.17. Let $C \subseteq X$ be a non-empty closed convex set. Then $\partial h_C(0) = C$. In particular, the following relations are equivalent:

(i) $x \in C$;

(ii) $\langle x^*, x \rangle \leq h_C(x^*)$ for all $x^* \in X^*$.

Proof. Using the equivalences in Proposition 1.6.6 and Proposition 1.6.4, we have

$$\begin{aligned} x \in \partial h_C(0) &\iff 0 \in \partial h_C^*(x) = \partial \iota_C(x) \\ &\iff x \in \operatorname*{Arg\,min}_X \iota_C \\ &\iff \iota_C(x) = 0 \\ &\iff x \in C, \end{aligned}$$

which proves the first claim. In view of (1.6), the second claim directly follows from the equivalence shown above. ■

Remark 1.6.18. More generally, for any positively 1-homogeneous, convex function $h : X^* \to \overline{\mathbb{R}}$, the conjugate function $h^* : X \to \overline{\mathbb{R}}$ is the indicator function ι_C of the set $C = \{x \in X : \langle x^*, x \rangle \leq h(x^*) \;\; \forall x^* \in X^*\} = \partial h(0)$.

1.7. Star-shaped domains

Definition 1.7.1. We say that an open set $U \subseteq \mathbb{R}^N$ is star-shaped with respect to one of its points x_0 if the segment joining it to any other $x \in U$ is contained in U.

A set U which is star-shaped with respect to the origin can be equivalently characterized by the relation

$$\alpha U \subseteq U, \text{ for all } \alpha \in [0,1], \tag{1.7}$$

or

$$U \subseteq \alpha U, \text{ for all } \alpha \geq 1. \tag{1.8}$$

Definition 1.7.2. Let $U \subseteq \mathbb{R}^N$ be an open set, and $x_0 \in U$. We say that U is strongly star-shaped with respect to x_0 if it is star-shaped with respect to x_0, and if for every $x \in \overline{U}$ the half open line segment joining x_0 and x, and not containing x, is contained in U.

We say that an open set $U \subseteq \mathbb{R}^N$ is strongly star-shaped if there exists $x_0 \in U$ such that U is strongly star-shaped with respect to x_0

Proposition 1.7.3. Let $U \subseteq \mathbb{R}^N$ be an open set, $x_0 \in U$ be such that U is strongly star-shaped with respect to x_0. Then

$$x_0 + \alpha(U - x_0) \text{ is strongly star-shaped with respect to } x_0 \text{ for every } \alpha \in (0, +\infty),$$

$$\overline{x_0 + \alpha(U - x_0)} \subseteq U, \text{ for every } \alpha \in [0,1),$$

$$\overline{U} \subseteq x_0 + \alpha(U - x_0), \text{ for every } \alpha > 1.$$

In particular, any set U which is strongly star-shaped with respect to the origin satisfies

$$\overline{\alpha U} \subseteq U, \text{ for every } \alpha \in [0,1), \tag{1.9}$$

and

$$\overline{U} \subseteq \alpha U, \text{ for every } \alpha > 1. \tag{1.10}$$

The following covering result is proved in [13, Proposition 2.5.4]

Proposition 1.7.4. Let $U \subseteq \mathbb{R}^N$ be a bounded, open set with Lipschitz boundary. Then there exists a finite open covering $\{U_i\}$ of $\overline{U}$ such that $U \cap U_i$ is strongly star-shaped with Lipschitz boundary.

Remark 1.7.5. Examining the proof of Proposition 1.7.4 given in [13] shows that sets U_i satisfying $U_i \subset U$ can be replaced by open balls, whereas sets U_i intersecting ∂U can be chosen of the form (upon relabeling and and reorienting the coordinate axis)

$$U_i = B \times (-\varepsilon, \varepsilon),$$

where B is an open ball in $\mathbb{R}^{N-1}$ centered in the origin, and $\varepsilon > 0$ is small enough.

In the case when $N = 2$ this implies that we can select a covering such that sets which intersect the boundary are open rectangles.

2. SETTING OF THE PROBLEM

Let $\omega \subset \mathbb{R}^2$ be a bounded, connected, and open set with a C^2 boundary, and consider the open interval $I = \left(-\frac{1}{2}, \frac{1}{2}\right)$. Given a small positive number $h > 0$, we define a three-dimensional thin plate

$$\Omega^h := \omega \times (hI),$$

with the boundary partitioned into the lateral surface $\partial\omega \times (hI)$ and the transverse boundary $\omega \times \partial(hI)$. We assume a non-zero Dirichlet boundary condition set on the whole lateral surface, i.e. the Dirichlet boundary of Ω^k is given by $\Gamma_D^h := \gamma_D \times (hI)$, where $\gamma_D = \partial\omega$.

Throughout this paper, we assume that the body is only submitted to a hard device on Γ_D^h and that there are no applied loads, i.e. what drives the evolution is the boundary condition that depends on time. It is also possible to consider more general boundary conditions, together with volume and surfaces forces (see [17, 18, 24]).

2.1. PHASE DECOMPOSITION

We recall here some basic notation and assumptions from [25].

Let $\mathcal{Y} = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus, let $Y := [0,1)^2$ be its associated periodicity cell, and denote by $\mathcal{I} : \mathcal{Y} \to Y$ their canonical identification. We denote by $\mathcal{C}$ the set

$$\mathcal{C} := \mathcal{I}^{-1}(\partial Y).$$

For any $\mathcal{Z} \subset \mathcal{Y}$, we denote

$$\mathcal{Z}_\varepsilon := \left\{ x \in \mathbb{R}^2 : \frac{x}{\varepsilon} \in \mathbb{Z}^2 + \mathcal{I}(\mathcal{Z}) \right\}, \tag{2.1}$$

and for any function $F : \mathscr{Y} \to X$ we associate the ε-periodic function $F_\varepsilon : \mathbb{R}^2 \to X$, given by

$$F_\varepsilon(x) := F(y_\varepsilon), \text{ for } \frac{x}{\varepsilon} - \left\lfloor \frac{x}{\varepsilon} \right\rfloor = \mathscr{I}(y_\varepsilon) \in Y.$$

With a slight abuse of notation we will also write $F_\varepsilon(x) = F\left(\frac{x}{\varepsilon}\right)$.

The torus $\mathscr{Y}$ is assumed to be made up of finitely many phases $\mathscr{Y}_i$ together with their interfaces. We assume that those phases are pairwise disjoint open sets with Lipschitz boundary. Then we have $\mathscr{Y} = \bigcup_i \overline{\mathscr{Y}}_i$ and we denote the interfaces by

$$\Gamma := \bigcup_{i,j} \partial \mathscr{Y}_i \cap \partial \mathscr{Y}_j.$$

Furthermore, the interfaces are assumed to have a negligible intersection with the set $\mathscr{C}$, i.e. for every i

$$\mathscr{H}^1(\partial \mathscr{Y}_i \cap \mathscr{C}) = 0. \tag{2.2}$$

We will write

$$\Gamma := \bigcup_{i \neq j} \Gamma_{ij},$$

where Γ_{ij} stands for the interface between $\mathscr{Y}_i$ and $\mathscr{Y}_j$.

We assume that ω is composed of the finitely many phases $(\mathscr{Y}_i)_\varepsilon$, and that $\Omega^h \cup \Gamma_D^h$ is a geometrically admissible multi-phase domain in the sense of [24, Subsection 1.2]. Additionally, we assume that Ω^h is a specimen of an elasto-perfectly plastic material having periodic elasticity tensor and dissipation potential.

We are interested in the situation when the period ε is a function of the thickness h, i.e. $\varepsilon = \varepsilon_h$, and we assume that the limit

$$\gamma := \lim_{h \to 0} \frac{h}{\varepsilon_h}.$$

exists in $[0, +\infty]$. Depending on the limit, we additionally have assumptions on Γ as follows:

(i) For $\gamma \in (0, +\infty]$, we assume that there exists a compact set $S \subset \Gamma$ with $\mathscr{H}^1(S) = 0$ such that $\Gamma \setminus S$ is a C^2-hypersurface.

(ii) For $\gamma = 0$, we assume that each $\mathscr{Y}_i$ has C^2 boundary.

We say that a multi-phase torus $\mathscr{Y}$ is *geometrically admissible* if it satisfies the above assumptions.

Remark 2.1.1. In the case $\gamma \in (0, +\infty]$ we assume greater regularity than that in [25], where the interface $\Gamma \setminus S$ was allowed to be a C^1-hypersurface. There the tangential part of the trace of an admissible stress $[\sigma \nu]_\nu^\perp$ at a point x on $\Gamma \setminus S$ is not defined independently of the the considered approximation sequence, while we will avoid dealing with this situation.

We also remark that in the case $\gamma = 0$, because of the assumed strict regularity on the whole interface Γ, we can not have three or more phases intersecting at any point on the interface.

The set of admissible stresses.

We assume there exist convex compact sets $K_i \in \mathbb{M}^{3\times 3}_{\text{dev}}$ for each phase $\mathscr{Y}_i$. We further assume there exist two constants r_K and R_K, with $0 < r_K \leq R_K$, such that for every i

$$\{\xi \in \mathbb{M}^{3\times 3}_{\text{sym}} : |\xi| \leq r_K\} \subseteq K_i \subseteq \{\xi \in \mathbb{M}^{3\times 3}_{\text{sym}} : |\xi| \leq R_K\}.$$

Finally, we define

$$K(y) := K_i, \quad \text{for } y \in \mathscr{Y}_i.$$

In case $\gamma = 0$ or $\gamma = +\infty$, ordering between the phases is assumed on the interface. Suppose $K_1 \subseteq K_2 \subseteq \ldots \subseteq K_N$, then:

(i) For $\gamma = 0$,

$$K(y) := K_{min\{i,j\}}, \quad \text{if } y \in \partial\mathscr{Y}_i \cap \partial\mathscr{Y}_j. \tag{2.3}$$

(ii) For $\gamma = +\infty$,

$$K(y) := K_{min\{i,j\}}, \quad \text{if } y \in \left(\partial\mathscr{Y}_i \cap \partial\mathscr{Y}_j\right) \setminus S. \tag{2.4}$$

Remark 2.1.2. In case $\gamma \in (0, +\infty)$, we will define the dissipation potential through inf-convolution, as in [24, 25]. This requires us to prove the lower semicontinuity result for the dissipation functional.

On the other hand, the restrictive assumption $K_1 \subseteq K_2 \subseteq \ldots \subseteq K_N$ together with (2.3) and (2.4) will allow us to use Reshetnyak's lower semicontinuity theorem to obtain the lower semicontinuity result for the dissipation functional in case $\gamma = 0$ or $\gamma = +\infty$.

The elasticity tensor.

Let $\mathbb{C}$ be the *elasticity tensor*, considered as a map from $\mathscr{Y}$ taking values in the set of symmetric positive definite linear operators, $\mathbb{C} : \mathscr{Y} \times \mathbb{M}^{3\times3}_{\text{sym}} \to \mathbb{M}^{3\times3}_{\text{sym}}$, defined as

$$\mathbb{C}(y)\xi := \mathbb{C}_{\text{dev}}(y)\,\xi_{\text{dev}} + (k(y)\,\text{tr}\xi)\,I_{3\times3} \quad \text{for every } y \in \mathscr{Y} \text{ and } \xi \in \mathbb{M}^{3\times3},$$

where $\mathbb{C}_{\text{dev}}(y) = (\mathbb{C}_{\text{dev}})_i$ and $k(y) = k_i$ for every $y \in \mathscr{Y}_i$, and exist two constants r_c and R_c, with $0 < r_c \leq R_c$, such that

$$\begin{aligned} & r_c|\xi|^2 \leq (\mathbb{C}_{\text{dev}})_i\xi : \xi \leq R_c|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{\text{dev}}, \\ & r_c \leq k_i \leq R_c. \end{aligned}$$

Let $Q : \mathscr{Y} \times \mathbb{M}^{3\times3}_{\text{sym}} \to [0, +\infty)$ be the quadratic form associated with $\mathbb{C}$, given by

$$Q(y,\ \xi) := \frac{1}{2}\mathbb{C}(y)\xi : \xi \quad \text{for every } y \in \mathscr{Y} \text{ and } \xi \in \mathbb{M}^{3\times3}_{\text{sym}}.$$

It follows that Q satisfies

$$r_c|\xi|^2 \leq Q(\xi) \leq R_c|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{\text{sym}}. \tag{2.5}$$

The dissipation potential.

For each i, let $H_i : \mathscr{Y} \times \mathbb{M}^{3\times3}_{\text{dev}} \to [0, +\infty)$ be the support function of the set K_i, i.e

$$H_i(\xi) = \sup_{\tau \in K_i} \tau : \xi.$$

It follows that H_i is convex, positively 1-homogeneous, and satisfies

$$r_k|\xi| \leq H_i(\xi) \leq R_k|\xi| \quad \text{for every}\, \xi \in \mathbb{M}^{3\times3}_{\text{dev}}. \tag{2.6}$$

Then we define the dissipation potential $H : \mathscr{Y} \times \mathbb{M}^{3\times3}_{\text{dev}} \to [0, +\infty]$ as follows:

(i) For every $y \in \mathscr{Y}_i$, we take

$$H(y,\ \xi) := H_i(\xi).$$

(ii) For a point $y \in \Gamma \setminus S$ on the interface between $\mathscr{Y}_i$ and $\mathscr{Y}_j$, such that the associated normal $\nu(y)$ points from $\mathscr{Y}_j$ to $\mathscr{Y}_i$, we set

$$H(y,\ \xi) := \begin{cases} H_{ij}(a,\ \nu(y)) & \text{if } \xi = a \odot \nu(y) \in \mathbb{M}^{3\times3}_{\text{dev}}, \\ +\infty & \text{otherwise on } \mathbb{M}^{3\times3}_{\text{dev}}, \end{cases}$$

where for $a \in \mathbb{R}^3$ and $\nu \perp a \in \mathbb{S}^2$,

$$H_{ij}(a,\nu) := \inf\Big\{H_i(a_i \odot \nu) + H_j(-a_j \odot \nu) : \\ a = a_i - a_j,\ a_i \perp \nu,\ a_j \perp \nu\Big\}.$$

(iii) For $y \in S$, we define H arbitrarily (e.g. $H(y,\ \xi) := r_k\,|\xi|$).

Remark 2.1.3. We point out that H is a Borel function on $\mathscr{Y} \times \mathbb{M}^{3\times 3}_{\text{dev}}$. Furthermore, for each $y \in \mathscr{Y}$, the function $\xi \mapsto H(y,\ \xi)$ is positively 1-homogeneous and convex. However, the function $(y,\ \xi) \mapsto H(y,\ \xi)$ is not necessarily lower semicontinous. This will only be satisfied in case $\gamma = 0$ or $\gamma = +\infty$ where we assumed an ordering between phases, since then the above definition amount to $H(y,\ \xi) = H_{min\{i,j\}}(\xi)$ on the interface Γ_{ij}.

Admissible triples and energy.

On Γ^h_D we prescribe a boundary datum being the trace of a map $w^h \in H^1(\Omega^h;\mathbb{R}^3)$ of the following form:

$$w^h(z) := \left(\bar{w}_1(z') - \frac{z_3}{h}\partial_1\bar{w}_3(z'),\ \bar{w}_2(z') - \frac{z_3}{h}\partial_2\bar{w}_3(z'),\ \frac{1}{h}\bar{w}_3(z')\right) \text{ for a.e. } z = (z', z_3) \in \Omega^h, \tag{2.7}$$

where $\bar{w}_\alpha \in H^1(\omega)$, $\alpha = 1,2$, and $\bar{w}_3 \in H^2(\omega)$. The *set of admissible displacements and strains* for the boundary datum w^h is denoted by $\mathscr{A}(\Omega^h, w^h)$ and is defined as the class of all triples $(v, f, q) \in BD(\Omega^h) \times L^2(\Omega^h;\mathbb{M}^{3\times 3}_{\text{sym}}) \times \mathscr{M}_b(\Omega^h;\mathbb{M}^{3\times 3}_{\text{dev}})$ satisfying

$$Ev = f + q \quad \text{in } \Omega^h,$$
$$q = (w^h - v) \odot \nu_{\partial\Omega^h}\mathscr{H}^2 \quad \text{on } \Gamma^h_D.$$

The function v represents the *displacement* of the plate, while f and q are called the *elastic* and *plastic strain*, respectively.

For every admissible triple $(v, f, q) \in \mathscr{A}(\Omega^h, w^h)$ we define the associated *energy* as

$$\mathscr{E}_h(v, f, q) := \int_{\Omega^h} Q\left(\frac{z'}{\varepsilon_h}, f(z)\right)dz + \int_{\Omega^h \cup \Gamma^h_D} H\left(\frac{z'}{\varepsilon_h}, \frac{dq}{d|q|}\right)d|q|.$$

The first term represents the elastic energy, while the second term accounts for plastic dissipation.

2.2. The rescaled problem

As usual in dimension reduction problems, it is convenient to perform a change of variables in such a way to rewrite the system on a fixed domain independent of h. To this purpose, we consider the open interval $I = \left(-\frac{1}{2}, \frac{1}{2}\right)$ and set

$$\Omega := \omega \times I, \qquad \Gamma_D := \partial\omega \times I.$$

We consider the change of variables $\psi_h : \overline{\Omega} \to \overline{\Omega^h}$, defined as

$$\psi_h(x', x_3) := (x', hx_3) \quad \text{for every } (x', x_3) \in \overline{\Omega}, \tag{2.8}$$

and the linear operator $\Lambda_h : \mathbb{M}^{3\times3}_{\text{sym}} \to \mathbb{M}^{3\times3}_{\text{sym}}$ given by

$$\Lambda_h \xi := \begin{pmatrix} \xi_{11} & \xi_{12} & \frac{1}{h}\xi_{13} \\ \xi_{21} & \xi_{22} & \frac{1}{h}\xi_{23} \\ \frac{1}{h}\xi_{31} & \frac{1}{h}\xi_{32} & \frac{1}{h^2}\xi_{33} \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{\text{sym}}. \tag{2.9}$$

To any triple $(v, f, q) \in \mathscr{A}(\Omega^h, w^h)$ we associate a triple $(u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\text{sym}}) \times \mathscr{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}_{\text{sym}})$ defined as follows:

$$u := (v_1, v_2, hv_3) \circ \psi_h, \qquad e := \Lambda_h^{-1} f \circ \psi_h, \qquad p := \tfrac{1}{h}\Lambda_h^{-1} \psi_h^{\#}(q).$$

Here the measure $\psi_h^{\#}(q) \in \mathscr{M}_b(\Omega; \mathbb{M}^{3\times3})$ is the pull-back measure of q, satisfying

$$\int_{\Omega \cup \Gamma_D} \varphi : d\psi_h^{\#}(q) = \int_{\Omega^h \cup \Gamma_D^h} (\varphi \circ \psi_h^{-1}) : dq \quad \text{for every } \varphi \in C_0(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}).$$

According to this change of variable we have

$$\mathscr{E}_h(v, f, q) = h\mathscr{Q}_h(\Lambda_h e) + h\mathscr{H}_h(\Lambda_h p),$$

where

$$\mathscr{Q}_h(\Lambda_h e) = \int_\Omega Q\left(\frac{x'}{\varepsilon_h}, \Lambda_h e\right) dx \tag{2.10}$$

and

$$\mathscr{H}_h(\Lambda_h p) = \int_{\Omega \cup \Gamma_D} H\left(\frac{x'}{\varepsilon_h}, \frac{d\Lambda_h p}{d|\Lambda_h p|}\right) d|\Lambda_h p|. \tag{2.11}$$

We also introduce the scaled Dirichlet boundary datum $w \in H^1(\Omega; \mathbb{R}^3)$, given by

$$w(x) := (\bar{w}_1(x') - x_3\partial_1 w_3(x'), \bar{w}_2(x') - x_3\partial_2 w_3(x'), w_3(x')) \quad \text{for a.e. } x \in \Omega.$$

By the definition of the class $\mathscr{A}(\Omega^h, w^h)$ it follows that the scaled triple (u, e, p) satisfies the equalities

$$Eu = e + p \quad \text{in } \Omega, \tag{2.12}$$

$$p = (w - u) \odot \nu_{\partial\Omega} \mathscr{H}^2 \quad \text{on } \Gamma_D, \tag{2.13}$$

$$p_{11} + p_{22} + \tfrac{1}{h^2} p_{33} = 0 \quad \text{in } \Omega \cup \Gamma_D. \tag{2.14}$$

We are thus led to introduce the class $\mathscr{A}_h(w)$ of all triples $(u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\text{sym}}) \times \mathscr{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}_{\text{sym}})$ satisfying (2.12)–(2.14), and to define the functional

$$\mathscr{J}_h(u, e, p) := \mathscr{Q}_h(\Lambda_h e) + \mathscr{H}_h(\Lambda_h p) \tag{2.15}$$

for every $(u, e, p) \in \mathscr{A}_h(w)$. In the following we will study the asymptotic behaviour of the minimizers of $\mathscr{J}_h$ and of the quasistatic evolution associated with $\mathscr{J}_h$, as $h \to 0$ and $\varepsilon \to 0$.

Kirchhoff-Love admissible triples and limit energy.

We consider the set of *Kirchhoff-Love displacements*, defined as

$$KL(\Omega) := \big\{ u \in BD(\Omega) : \ (Eu)_{i3} = 0 \quad \text{for } i = 1, 2, 3 \big\}.$$

We note that $u \in KL(\Omega)$ if and only if $u_3 \in BH(\omega)$ and there exists $\bar{u} \in BD(\omega)$ such that

$$u_\alpha = \bar{u}_\alpha - x_3 \partial_{x_\alpha} u_3, \quad \alpha = 1, 2. \tag{2.16}$$

In particular, if $u \in KL(\Omega)$, then

$$Eu = \begin{pmatrix} \begin{matrix} E\bar{u} - x_3 D^2 u_3 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}. \tag{2.17}$$

If, in addition, $u \in W^{1,p}(\Omega; \mathbb{R}^3)$ for some $1 \le p \le \infty$, then $\bar{u} \in W^{1,p}(\omega; \mathbb{R}^2)$ and $u_3 \in W^{2,p}(\omega)$. We call $\bar{u}, u_3$ the *Kirchhoff-Love components* of u.

For every $w \in H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ we define the class $\mathscr{A}_{KL}(w)$ of *Kirchhoff-Love admissible triples* for the boundary datum w as the set of all triples $(u, e, p) \in KL(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\text{sym}}) \times \mathscr{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}_{\text{sym}})$ satisfying

$$Eu = e + p \quad \text{in } \Omega, \qquad p = (w - u) \odot \nu_{\partial\Omega} \mathscr{H}^2 \quad \text{on } \Gamma_D, \tag{2.18}$$

$$e_{i3} = 0 \quad \text{in } \Omega, \quad p_{i3} = 0 \quad \text{in } \Omega \cup \Gamma_D, \quad i = 1, 2, 3. \tag{2.19}$$

Note that the space

$$\{\xi \in \mathbb{M}^{3\times 3}_{\text{sym}} : \ \xi_{i3} = 0 \text{ for } i = 1,2,3\}$$

is canonically isomorphic to $\mathbb{M}^{2\times 2}_{\text{sym}}$. Therefore, in the following, given a triple $(u,e,p) \in \mathscr{A}_{KL}(w)$ we will usually identify e with a function in $L^2(\Omega;\mathbb{M}^{2\times 2}_{\text{sym}})$ and p with a measure in $\mathscr{M}_b(\Omega \cup \Gamma_D;\mathbb{M}^{2\times 2}_{\text{sym}})$. Note also that the class $\mathscr{A}_{KL}(w)$ is always nonempty as it contains the triple $(w, Ew, 0)$.

To provide a useful characterisation of admissible triplets in $\mathscr{A}_{KL}(w)$, let us first recall the definition of zeroth and first order moments of functions.

Definition 2.2.1. For $f \in L^2(\Omega;\mathbb{M}^{2\times 2}_{\text{sym}})$ we denote by $\bar{f}$, $\hat{f} \in L^2(\omega;\mathbb{M}^{2\times 2}_{\text{sym}})$ and $f^\perp \in L^2(\Omega;\mathbb{M}^{2\times 2}_{\text{sym}})$ the following orthogonal components (with respect to the scalar product of $L^2(\Omega;\mathbb{M}^{2\times 2}_{\text{sym}})$) of f:

$$\bar{f}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x',x_3)\,dx_3, \qquad \hat{f}(x') := 12\int_{-\frac{1}{2}}^{\frac{1}{2}} x_3 f(x',x_3)\,dx_3$$

for a.e. $x' \in \omega$, and

$$f^\perp(x) := f(x) - \bar{f}(x') - x_3\hat{f}(x')$$

for a.e. $x \in \Omega$. We name $\bar{f}$ the *zero-th order moment* of f and $\hat{f}$ the *first order moment* of f.

The coefficient in the definition of $\hat{f}$ is chosen from the computation $\int_I x_3^2\,dx_3 = \frac{1}{12}$. It ensures that if f is of the form $f(x) = x_3 g(x')$, for some $g \in L^2(\omega;\mathbb{M}^{2\times 2}_{\text{sym}})$, then $\hat{f} = g$.

Analogously, we have the following definition of zeroth and first order moments of measures.

Definition 2.2.2. For $\mu \in M_b(\Omega \cup \Gamma_D;\mathbb{M}^{2\times 2}_{\text{sym}})$ we define $\bar{\mu}$, $\hat{\mu} \in M_b(\omega \cup \gamma_D;\mathbb{M}^{2\times 2}_{\text{sym}})$ and $\mu^\perp \in M_b(\Omega \cup \Gamma_D;\mathbb{M}^{2\times 2}_{\text{sym}})$ as follows:

$$\int_{\omega\cup\gamma_D} \varphi : d\bar{\mu} := \int_{\Omega\cup\Gamma_D} \varphi : d\mu, \qquad \int_{\omega\cup\gamma_D} \varphi : d\hat{\mu} := 12\int_{\Omega\cup\Gamma_D} x_3\varphi : d\mu$$

for every $\varphi \in C_0(\omega \cup \gamma_D;\mathbb{M}^{2\times 2}_{\text{sym}})$, and

$$\mu^\perp := \mu - \bar{\mu} \otimes \mathscr{L}^1_{x_3} - \hat{\mu} \otimes x_3\mathscr{L}^1_{x_3},$$

where $\otimes$ is the usual product of measures, and $\mathscr{L}^1_{x_3}$ is the Lebesgue measure restricted to the third component of $\mathbb{R}^3$. We name $\bar{\mu}$ the *zero-th order moment* of μ and $\hat{\mu}$ the *first order moment* of μ.

Remark 2.2.3. More generally, for any function f which is integrable over I, we will use the short-hand notation

$$\bar{f} := \int_I f(\cdot, x_3)\, dx_3, \qquad \hat{f} := 12 \int_I x_3 f(\cdot, x_3)\, dx_3.$$

We are now ready to recall the following characterisation of $\mathscr{A}_{KL}(w)$, given in [18, Proposition 4.3].

Proposition 2.2.4. Let $w \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ and let $(u,e,p) \in KL(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_{\text{sym}}) \times \mathscr{M}_b(\Omega \cup \Gamma_D;\mathbb{M}^{3\times3}_{\text{dev}})$. Then $(u,e,p) \in \mathscr{A}_{KL}(w)$ if and only if the following three conditions are satisfied:

(i) $E\bar{u} = \bar{e} + \bar{p}$ in ω and $\bar{p} = (\bar{w} - \bar{u}) \odot \nu_{\partial\omega}\mathscr{H}^1$ on γ_D;

(ii) $D^2 u_3 = -(\hat{e} + \hat{p})$ in ω, $u_3 = w_3$ on γ_D, and $\hat{p} = (\nabla u_3 - \nabla w_3) \odot \nu_{\partial\omega}\mathscr{H}^1$ on γ_D;

(iii) $p^\perp = -e^\perp$ in Ω and $p^\perp = 0$ on Γ_D.

Proof. The statement easily follows from the preceding definitions and (2.17). ■

2.3. THE REDUCED PROBLEM

The reduced elasticity tensor.

For a fixed $y \in \mathscr{Y}$, let $\mathbb{A}_y : \mathbb{M}^{2\times 2}_{\text{sym}} \to \mathbb{M}^{3\times 3}_{\text{sym}}$ be the operator given by

$$\mathbb{A}_y \xi := \begin{pmatrix} & \xi & \lambda_1^y(\xi) \\ & & \lambda_2^y(\xi) \\ \lambda_1^y(\xi) & \lambda_2^y(\xi) & \lambda_3^y(\xi) \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{\text{sym}},$$

where for every $\xi \in \mathbb{M}^{2\times 2}_{\text{sym}}$ the triple $(\lambda_1^y(\xi), \lambda_2^y(\xi), \lambda_3^y(\xi))$ is the unique solution to the minimum problem

$$\min_{\lambda_i^y \in \mathbb{R}} Q\left(y, \begin{pmatrix} & \xi & \lambda_1^y \\ & & \lambda_2^y \\ \lambda_1^y & \lambda_2^y & \lambda_3^y \end{pmatrix} \right). \tag{2.20}$$

We observe that for every $\xi \in \mathbb{M}^{2\times 2}_{\text{sym}}$, the matrix $\mathbb{A}_y\xi$ is given by the unique solution of the linear system

$$\mathbb{C}(y)\mathbb{A}_y\xi : \begin{pmatrix} 0 & 0 & \lambda_1^y \\ 0 & 0 & \lambda_2^y \\ \lambda_1^y & \lambda_2^y & \lambda_3^y \end{pmatrix} = 0 \quad \text{for every } \lambda_1^y, \lambda_2^y, \lambda_3^y \in \mathbb{R}.$$

This implies, in particular, for every $y \in \mathscr{Y}$ that $\mathbb{A}_y$ is a linear map.

Let $Q_r : \mathscr{Y} \times \mathbb{M}^{2\times 2}_{\text{sym}} \to [0, +\infty)$ be the quadratic form defined as

$$Q_r(y,\ \xi) := Q(y,\ \mathbb{A}_y\xi) \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{\text{sym}}.$$

By properties of Q, we have that $Q_r(y, \cdot)$ is positive definite on symmetric matrices.

We also define the tensor $\mathbb{C}_r : \mathscr{Y} \times \mathbb{M}^{2\times 2}_{\text{sym}} \to \mathbb{M}^{3\times 3}_{\text{sym}}$, given by

$$\mathbb{C}_r(y)\xi := \mathbb{C}(y)\mathbb{A}_y\xi \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{\text{sym}}.$$

We remark that by (2.20) there holds

$$\mathbb{C}_r(y)\xi : \zeta = \mathbb{C}(y)\mathbb{A}_y\xi : \begin{pmatrix} \zeta'' & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{\text{sym}},\ \zeta \in \mathbb{M}^{3\times 3}_{\text{sym}},$$

and

$$Q_r(y,\,\xi) = \frac{1}{2}\mathbb{C}_r(y)\xi : \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{\text{sym}}.$$

The reduced dissipation potential.

The set $K_r(y) \subset \mathbb{M}^{2\times 2}_{\text{sym}}$ represents the set of admissible stresses in the reduced problem and can be characterised as follows (see [18, Section 3.2]):

$$\xi \in K_r(y) \quad \Longleftrightarrow \quad \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}(\text{tr}\xi)I_{3\times 3} \in K(y), \tag{2.21}$$

where $I_{3\times 3}$ is the identity matrix in $\mathbb{M}^{3\times 3}$.

The *plastic dissipation potential* $H_r : \mathscr{Y} \times \mathbb{M}^{2\times 2}_{\text{sym}} \to [0, +\infty)$ is given by the support function of $K_r(y)$, i.e

$$H_r(y,\,\xi) := \sup_{\sigma \in K_r(y)} \sigma : \xi \quad \text{for every}\, \xi \in \mathbb{M}^{2\times 2}_{\text{sym}}.$$

It follows that $H_r(y, \cdot)$ is convex and positively 1-homogeneous, and there are two constants $0 < r_H \le R_H$ such that

$$r_H|\xi| \le H_r(y,\,\xi) \le R_H|\xi| \quad \text{for every}\, \xi \in \mathbb{M}^{2\times 2}_{\text{sym}}.$$

Therefore $H_r(y, \cdot)$ satisfies the triangle inequality

$$H_r(y,\,\xi_1 + \xi_2) \le H_r(y,\,\xi_1) + H_r(y,\,\xi_2) \quad \text{for every}\, \xi_1,\, \xi_2 \in \mathbb{M}^{2\times 2}_{\text{sym}}.$$

Finally, for a fixed $y \in \mathscr{Y}$, we can deduce the property

$$K_r(y) = \partial H_r(y,\,0).$$

2.4. QUASISTATIC EVOLUTIONS

Recalling Section 1.3, the $\mathscr{H}_h$-variation of a function $p^h : [0,T] \to \mathscr{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times 3}_{\text{dev}})$ on $[a,b]$ is defined as

$$\mathscr{D}_{\mathscr{H}_h}(P;a,b) := \sup\left\{\sum_{i=1}^{n} \mathscr{H}\left(P(t_{i+1}) - P(t_i)\right) : a = t_1 < t_2 < \ldots < t_n = b,\ n \in \mathbb{N}\right\}.$$

For every $t \in [0,T]$ we prescribe a boundary datum $w(t) \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ and we assume the map $t \mapsto w(t)$ to be absolutely continuous from $[0,T]$ into $H^1(\Omega;\mathbb{R}^3)$.

Definition 2.4.1. Let $h > 0$. An *h-quasistatic evolution* for the boundary datum $w(t)$ is a function $t \mapsto (u^h(t), e^h(t), p^h(t))$ from $[0,T]$ into $BD(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{\text{sym}}) \times \mathscr{M}_b(\Omega \cup \Gamma_D;\mathbb{M}^{3\times 3}_{\text{dev}})$ that satisfies the following conditions:

$(\text{qs1})_h$ for every $t \in [0,T]$ we have $(u^h(t), e^h(t), p^h(t)) \in \mathscr{A}_h(w(t))$ and

$$\mathscr{Q}_h(\Lambda_h e^h(t)) \leq \mathscr{Q}_h(\Lambda_h \eta) + \mathscr{H}_h(\Lambda_h \pi - \Lambda_h p^h(t)),$$

for every $(\upsilon, \eta, \pi) \in \mathscr{A}_h(w(t))$.

$(\text{qs2})_h$ the function $t \mapsto p^h(t)$ from $[0,T]$ into $\mathscr{M}_b(\Omega \cup \Gamma_D;\mathbb{M}^{3\times 3}_{\text{dev}})$ has bounded variation and for every $t \in [0,T]$

$$\mathscr{Q}_h(\Lambda_h e^h(t)) + \mathscr{D}_{\mathscr{H}_h}(\Lambda_h p^h; 0, t) = \mathscr{Q}_h(\Lambda_h e^h(0)) + \int_0^t \int_\Omega \mathbb{C}\left(\tfrac{x'}{\varepsilon_h}\right) \Lambda_h e^h(s) : E\dot{w}(s)\, dx ds.$$

The following existence result of a quasi-static evolution for a general multi-phase material is given by [24, Theorem 2.7].

Theorem 2.4.2. Let $h > 0$ and let $(u_0^h, e_0^h, p_0^h) \in \mathscr{A}_h(w(0))$ satisfy the global stability condition $(\text{qs1})_h$. Then, there exists a two-scale quasistatic evolution $t \mapsto (u^h(t), e^h(t), p^h(t))$ for the boundary datum $w(t)$ such that $u^h(0) = u_0$, $e^h(0) = e_0^h$, and $p^h(0) = p_0^h$.

3. COMPACTNESS RESULTS

In this section, we provide a characterization of two-scale limits of symmetrized scaled gradients. We will consider sequences of deformations $\{v^h\}$ such that $v^h \in BD(\Omega^h)$ for every $h > 0$, their L^1-norms are uniformly bounded, and their symmetrized gradients Ev^h form a sequence of uniformly bounded Radon measures. We associate to the sequence $\{v^h\}$ above a rescaled sequence of maps $\{u^h\} \subset BD(\Omega)$, defined as

$$u^h := (v_1^h, v_2^h, hv_3^h) \circ \psi_h,$$

where ψ_h is defined in (2.8). The symmetric gradients of the maps $\{v^h\}$ and $\{u^h\}$ are related as follows

$$Ev^h = \psi_h^\# \begin{pmatrix} E_{x'}(u^h)' & \frac{1}{2h}\left(D_{x'}u_3^h + \partial_{x_3}(u^h)'\right) \\ \frac{1}{2h}\left(D_{x'}u_3^h + \partial_{x_3}(u^h)'\right)^T & \frac{1}{h^2}\,\partial_{x_3}u_3^h \end{pmatrix} \tag{3.1}$$

In the following, we use the notation $E_{ij}(u^h)$ to denote the measure $E_{ij}(u^h) := \frac{\partial_i u_j^h + \partial_j u_i^h}{2}$. We first recall a compactness result for sequences of non-oscillating fields with uniformly bounded symmetric gradients.

Proposition 3.0.1. Let $\{v^h\}_{h>0}$ be such that $v^h \in BD(\Omega^h)$ for every h, and there exists a constant C for which $\|v^h\|_{BD(\Omega^h)} \le C$. Denote by u^h the map $u^h := (v_1^h, v_2^h, hv_3^h) \circ \psi_h$. Then, there exist functions $\bar{u} = (\bar{u}_1, \bar{u}_2) \in BD(\omega)$ and $u_3 \in BH(\omega)$ such that, up to subsequences, there holds

$$E_{\alpha\beta}(u^h) \xrightharpoonup{*} \frac{1}{2}(\partial_\alpha \bar{u}_\beta + \partial_\beta \bar{u}_\alpha) - x_3 \partial_{\alpha\beta} u_3 \quad \text{weakly* in } \mathcal{M}_b(\Omega). \tag{3.2}$$

The proposition above has been proved in [18]. We briefly sketch the main arguments below for convenience of the reader.

Proof of Proposition 3.0.1. From the boundedness of the sequence $\{v^h\}$, we conclude that the sequence $\{u^h\}$ is bounded in $BD(\Omega)$, and that the right-hand side of (3.1) is bounded in $\mathscr{M}_b(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$, and thus there exist $u \in BD(\Omega)$, and $\lambda_{i3} \in \mathscr{M}_b(\Omega)$, $i = 1,2,3$, such that, up to the extraction of a (not relabeled) subsequence,

$$u^h \xrightarrow{*} u \quad \text{weakly* in } BD(\Omega),$$

$$\frac{1}{h}E_{\alpha3}(u^h) \xrightarrow{*} \lambda_{\alpha3} \quad \text{weakly* in } \mathscr{M}_b(\Omega), \quad \alpha = 1,2, \tag{3.3}$$

$$\frac{1}{h^2}E_{33}(u^h) \xrightarrow{*} \lambda_{33} \quad \text{weakly* in } \mathscr{M}_b(\Omega). \tag{3.4}$$

In particular, in view of (3.3) and (3.4) we have

$$E_{i3}(u^h) \to 0 \quad \text{strongly in } \mathscr{M}_b(\Omega). \tag{3.5}$$

By (3.5) we deduce that $\frac{1}{2}(\partial_i u_3 + \partial_3 u_i) = 0$, for $i = 1,2,3$, and that $u_3 \in L^1(\omega)$. This implies that $u_3 \in W^{1,1}(\omega)$ and that $u_\alpha = \bar{u}_\alpha - x_3\partial_\alpha u_3$, for $\alpha = 1,2$, where $\bar{u} \in L^1(\omega;\mathbb{R}^2)$. Finally we conclude that $(\bar{u}_1,\bar{u}_2) \in BD(\omega)$ and $u_3 \in BH(\omega)$. ■

Now we turn to identifying the two-scale limits of the sequence $\Lambda_h E u^h$. We will adapt some results and definitions from [25].

Definition 3.0.2. Let $\Omega \subset \mathbb{R}^3$ be an open set. Let $\{\mu^h\}_{h>0}$ be a family in $\mathscr{M}_b(\Omega)$ and consider $\mu \in \mathscr{M}_b(\Omega \times \mathscr{Y})$. We say that

$$\mu_h \xrightarrow{2-*} \mu \quad \text{two-scale weakly* in } \mathscr{M}_b(\Omega \times \mathscr{Y}),$$

if for every $\chi \in C_0(\Omega \times \mathscr{Y})$

$$\lim_{h\to0}\int_\Omega \chi\left(x, \frac{x'}{\varepsilon_h}\right) d\mu^h(x) = \int_{\Omega\times\mathscr{Y}} \chi(x,y)\, d\mu(x,y).$$

The convergence above is called *two-scale weak* convergence*.

Proposition 3.0.3. (i) Any sequence that is bounded in $\mathscr{M}_b(\Omega)$ admits a two-scale weakly* convergent subsequence.

(ii) Let $\mathscr{D} \subset \mathscr{Y}$ and assume that $\mathrm{supp}(\mu_h) \subset \Omega \cap (\mathscr{D}_{\varepsilon_h} \times I)$. If $\mu_h \xrightarrow{2-*} \mu$ two-scale weakly* in $\mathscr{M}_b(\Omega \times \mathscr{Y})$, then $\mathrm{supp}(\mu) \subset \Omega \times \overline{\mathscr{D}}$.

3.1. Corrector properties and duality results

In order to define and analyze the space of measures which arise as two-scale limits of scaled symmetrized gradients of *BD* functions, we will consider the following general framework.

Let V and W be finite-dimensional Euclidean spaces of dimensions N and M, respectively. We will consider k^{th} order linear homogeneous partial differential operators with constant coefficients $\mathscr{A} : C_c^\infty(\mathbb{R}^n;V) \to C_c^\infty(\mathbb{R}^n;W)$. More precisely, the operator $\mathscr{A}$ acts on functions $u : \mathbb{R}^n \to V$ as

$$\mathscr{A} u := \sum_{|\alpha|=k} A_\alpha \partial^\alpha u.$$

where the coefficients $A_\alpha \in W \otimes V^* \cong \text{Lin}(V;W)$ are constant tensors, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index and $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ denotes the distributional partial derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

We define the space

$$BV^{\mathscr{A}}(U) = \left\{ u \in L^1(U;V) : \mathscr{A} u \in \mathscr{M}_b(U;W) \right\}$$

of *functions with bounded $\mathscr{A}$-variations* on an open subset U of $\mathbb{R}^n$. This is a Banach space endowed with the norm

$$\|u\|_{BV^{\mathscr{A}}(U)} := \|u\|_{L^1(U;V)} + |\mathscr{A} u|(U).$$

Here, the distributional $\mathscr{A}$-gradient is defined and extended to distributions via the duality

$$\int_U \varphi \cdot d\mathscr{A} u := \int_U \mathscr{A}^* \varphi \cdot u \, dx, \quad \varphi \in C_c^\infty(U;W^*),$$

where $\mathscr{A}^* : C_c^\infty(\mathbb{R}^n;W^*) \to C_c^\infty(\mathbb{R}^n;V^*)$ is the formal L^2-adjoint operator of $\mathscr{A}$

$$\mathscr{A}^* := (-1)^k \sum_{|\alpha|=k} A_\alpha^* \partial^\alpha.$$

The *total $\mathscr{A}$-variation* of $u \in L^1_{loc}(U;V)$ is defined as

$$|\mathscr{A} u|(U) := \sup \left\{ \int_U \mathscr{A}^* \varphi \cdot u \, dx : \varphi \in C_c^k(U;W^*),\ |\varphi| \le 1 \right\}.$$

Let $\{u_n\} \subset BV^{\mathscr{A}}(U)$ and $u \in BV^{\mathscr{A}}(U)$. We say that $\{u_n\}$ converges weakly* to u if $u_n \to u$ in $L^1(U;V)$ and $\mathscr{A} u_n \overset{*}{\rightharpoonup} \mathscr{A} u$ in $\mathscr{M}_b(U;W)$.

In order to characterize the two-scale weak* limit of the scaled symmetrized gradients, we will generally consider two domains $\Omega_1 \subset \mathbb{R}^{n_1}$, $\Omega_2 \subset \mathbb{R}^{n_2}$ and assume that the operator $\mathscr{A}_{x_2}$ is defined through partial derivatives only with respect to the entries of the n_2-tuple x_2. In the spirit of [25, Section 4.2], we will define the space

$$\mathscr{X}^{\mathscr{A}_{x_2}}(\Omega_1) := \Big\{\mu \in \mathscr{M}_b(\Omega_1 \times \Omega_2;V) : \mathscr{A}_{x_2}\mu \in \mathscr{M}_b(\Omega_1 \times \Omega_2;W),$$
$$\mu(F \times \Omega_2) = 0 \text{ for every Borel set } S \subseteq \Omega_1\Big\}.$$

We will assume that $BV^{\mathscr{A}_{x_2}}(\Omega_2)$ satisfies the following weak* compactness property:

Assumption 1. If $\{u_n\} \subset BV^{\mathscr{A}_{x_2}}(\Omega_2)$ is uniformly bounded in the $BV^{\mathscr{A}_{x_2}}$-norm, then there exists a subsequence $\{u_m\} \subseteq \{u_n\}$ and a function $u \in BV^{\mathscr{A}_{x_2}}(\Omega_2)$ such that $\{u_m\}$ converges weakly* to u in $BV^{\mathscr{A}_{x_2}}(\Omega_2)$, i.e.

$$u_m \to u \text{ in } L^1(\Omega_2;V) \text{ and } \mathscr{A}_{x_2}u_m \overset{*}{\rightharpoonup} \mathscr{A}_{x_2}u \text{ in } \mathscr{M}_b(\Omega_2;W).$$

Furthermore, there exists a countable collection $\{U^k\}$ of open subsets of $\mathbb{R}^{n_2}$ that increases to Ω_2 (i.e. $\overline{U^k} \subset U^{k+1}$ for all k, and $\Omega_2 = \bigcup_k U^k$) such that $BV^{\mathscr{A}_{x_2}}(U^k)$ also satisfies the weak* compactness property.

The following theorem is the main disintegration result for measures in $\mathscr{X}^{\mathscr{A}_{x_2}}(\Omega_1)$, which will allow us to define the duality result for admissible two-scale configurations. The proof is an adaptation of the arguments in [25, Proposition 4.7].

Proposition 3.1.1. Let $\mu \in \mathscr{X}^{\mathscr{A}_{x_2}}(\Omega_1)$. Then there exist $\eta \in \mathscr{M}_b^+(\Omega_1)$ and a Borel map $(x_1,x_2) \in \Omega_1 \times \Omega_2 \mapsto \mu_{x_1}(x_2) \in V$ such that, for η-a.e. $x_1 \in \Omega_1$,

$$\mu_{x_1} \in BV^{\mathscr{A}_{x_2}}(\Omega_2), \qquad \int_{\Omega_2} \mu_{x_1}(x_2)\,dx_2 = 0, \qquad |\mathscr{A}_{x_2}\mu_{x_1}|(\Omega_2) \neq 0, \tag{3.6}$$

and

$$\mu = \mu_{x_1}(x_2)\,\eta \otimes \mathscr{L}^{n_2}_{x_2}. \tag{3.7}$$

Moreover, the map $x_1 \mapsto \mathscr{A}_{x_2}\mu_{x_1} \in \mathscr{M}_b(\Omega_2;W)$ is η-measurable and

$$\mathscr{A}_{x_2}\mu = \eta \overset{\text{gen.}}{\otimes} \mathscr{A}_{x_2}\mu_{x_1}.$$

Proof. By assumption, we have $\mu \in \mathscr{M}_b(\Omega_1 \times \Omega_2; V)$ and $\lambda := \mathscr{A}_{x_2}\mu \in \mathscr{M}_b(\Omega_1 \times \Omega_2; W)$. Setting

$$\eta := proj_\#|\mu| + proj_\#|\lambda| \in \mathscr{M}_b^+(\Omega_1),$$

where $proj_\#$ is the push-forward by the projection of $\Omega_1 \times \Omega_2$ on Ω_1, we obtain the disintegrations

$$\mu = \eta \overset{\text{gen.}}{\otimes} \mu_{x_1} \text{ and } \lambda = \eta \overset{\text{gen.}}{\otimes} \lambda_{x_1}, \tag{3.8}$$

with $\mu_{x_1} \in \mathscr{M}_b(\Omega_2; V)$ and $\lambda_{x_1} \in \mathscr{M}_b(\Omega_2; W)$. Further, if we set $S := \{x_1 \in \Omega_1 : |\lambda_{x_1}|(\Omega_2) \neq 0\}$, then $\lambda = \eta \lfloor S \overset{\text{gen.}}{\otimes} \lambda_{x_1}$.

For every $\varphi^{(1)} \in C_c^\infty(\Omega_1)$ and $\varphi^{(2)} \in C_c^\infty(\Omega_2; W^*)$ we have

$$\begin{aligned}
\int_{\Omega_1} \varphi^{(1)}(x_1) \left\langle \mu_{x_1}, \mathscr{A}_{x_2}^* \varphi^{(2)} \right\rangle \cdot d\eta(x_1) &= \int_{\Omega_1} \left(\int_{\Omega_2} \varphi^{(1)}(x_1) \mathscr{A}_{x_2}^* \varphi^{(2)}(x_2) \cdot d\mu_{x_1}(x_2) \right) \cdot d\eta(x_1) \\
&= \left\langle \eta \overset{\text{gen.}}{\otimes} \mu_{x_1}, \varphi^{(1)} \mathscr{A}_{x_2}^* \varphi^{(2)} \right\rangle = \left\langle \mu, \mathscr{A}_{x_2}^* \left(\varphi^{(1)} \varphi^{(2)} \right) \right\rangle \\
&= \left\langle \mathscr{A}_{x_2}\mu, \varphi^{(1)} \varphi^{(2)} \right\rangle = \left\langle \eta \lfloor S \overset{\text{gen.}}{\otimes} \lambda_{x_1}, \varphi^{(1)} \varphi^{(2)} \right\rangle \\
&= \int_{\Omega_1} \left(\int_{\Omega_2} \varphi^{(1)}(x_1) \varphi^{(2)}(x_2) \cdot d\lambda_{x_1}(x_2) \right) \mathbb{1}_S(x_1) \cdot d\eta(x_1) \\
&= \int_{\Omega_1} \varphi^{(1)}(x_1) \left\langle \mathbb{1}_S(x_1) \lambda_{x_1}, \varphi^{(2)} \right\rangle \cdot d\eta(x_1).
\end{aligned}$$

From this we infer that for η-a.e. $x_1 \in \Omega_1$ and for every $\varphi \in C_c^\infty(\Omega_2; W^*)$

$$\left\langle \mu_{x_1}, \mathscr{A}_{x_2}^* \varphi \right\rangle = \left\langle \mathbb{1}_S(x_1) \lambda_{x_1}, \varphi \right\rangle. \tag{3.9}$$

We can consider μ_{x_1} and λ_{x_1} as measures on $\mathbb{R}^{n_2}$ if we extend the measure μ by zero on the complement of Ω_1. Then, using the standard mollifiers $\{\rho_\varepsilon\}_{\varepsilon>0}$ on $\mathbb{R}^{n_2}$, we define the functions $\mu_{x_1}^\varepsilon := \mu_{x_1} * \rho_\varepsilon$ and $\lambda_{x_1}^\varepsilon := \lambda_{x_1} * \rho_\varepsilon$, which are smooth functions and uniformly bounded in $L^1(\Omega_2; V)$ and $L^1(\Omega_2; W)$, respectively. For every $\varphi \in C_c^k(\Omega_2; W^*)$, $\text{supp}(\varphi) \subset U^k$ for k large enough. Furthermore, the support of $\varphi * \rho_\varepsilon$ is contained in Ω_2

provided ε is sufficiently small, and thus from (3.9) we have

$$\begin{aligned}\langle \mu_{x_1}^{\varepsilon}, \mathscr{A}_{x_2}^{*}\varphi\rangle &= \int_{\mathbb{R}^{n_2}} (\mu_{x_1} * \rho_\varepsilon)\cdot \mathscr{A}_{x_2}^{*}\varphi\, dx_2 = \int_{\mathbb{R}^{n_2}} \left(\mathscr{A}_{x_2}^{*}\varphi * \rho_\varepsilon\right)\cdot d\mu_{x_1} \\ &= \int_{\mathbb{R}^{n_2}} \mathscr{A}_{x_2}^{*}(\varphi * \rho_\varepsilon)\cdot d\mu_{x_1} = \langle \mu_{x_1}, \mathscr{A}_{x_2}^{*}(\varphi * \rho_\varepsilon)\rangle \\ &= \langle \mathbb{1}_S(x_1)\lambda_{x_1}, \varphi * \rho_\varepsilon\rangle = \int_{\mathbb{R}^{n_2}} (\varphi * \rho_\varepsilon)\cdot \mathbb{1}_S(x_1)\, d\lambda_{x_1} \\ &= \int_{\mathbb{R}^{n_2}} \mathbb{1}_S(x_1)\,(\lambda_{x_1} * \rho_\varepsilon)\cdot \varphi\, dx_2 \\ &= \langle \mathbb{1}_S(x_1)\lambda_{x_1}^{\varepsilon}, \varphi\rangle.\end{aligned}$$

Hence, for η-a.e. $x_1 \in \Omega_1$ the sequence $\{\mu_{x_1}^{\varepsilon}\}$ is eventually bounded in $BV^{\mathscr{A}_{x_2}}(U^k)$. By Assumption 1, this implies strong convergence in $L^1(U^k;V)$ up to a subsequence. As $\varepsilon \to 0$, we have both $\varphi * \rho_\varepsilon \to \varphi$ and $\mathscr{A}_{x_2}^{*}\varphi * \rho_\varepsilon \to \mathscr{A}_{x_2}^{*}\varphi$ uniformly, so by the Lebesgue's dominated convergence theorem we obtain, for η-a.e. $x_1 \in \Omega_1$,

$$\langle \mu_{x_1}^{\varepsilon}, \mathscr{A}_{x_2}^{*}\varphi\rangle \to \langle \mu_{x_1}, \mathscr{A}_{x_2}^{*}\varphi\rangle \text{ and } \langle \mathbb{1}_S(x_1)\lambda_{x_1}^{\varepsilon}, \varphi\rangle \to \langle \mathbb{1}_S(x_1)\lambda_{x_1}, \varphi\rangle.$$

From the convergence above, we conclude for η-a.e. $x_1 \in \Omega_1$ that $\mu_{x_1}^{\varepsilon} \to \mu_{x_1}$ strongly in $L^1(U^k;V)$. Since μ_{x_1} has bounded total variation, we have that $\mu_{x_1} \in L^1(\Omega_2;V)$ for η-a.e. $x_1 \in \Omega_1$. This, together with (3.9), implies

$$\mu_{x_1} \in BV^{\mathscr{A}_{x_2}}(\Omega_2) \text{ and } \mathscr{A}_{x_2}\mu_{x_1} = \mathbb{1}_S(x_1)\lambda_{x_1}.$$

Furthermore, from (3.8) we now have that μ is absolutely continuous with respect to $\eta \otimes \mathscr{L}_{x_2}^{n_2}$. Consequently, for η-a.e. $x_1 \in \Omega_1$ there exists a Borel measurable function which is equal to μ_{x_1} for $\mathscr{L}_{x_2}^{n_2}$-a.e. $x_2 \in \Omega_2$, so that (3.7) immediately follows.

Finally, since $\mu(F \times \Omega_2) = 0$ for every Borel set $F \subseteq \Omega_1$, we have

$$\int_{\Omega_1} f(x_1)\left(\int_{\Omega_2} \mu_{x_1}(x_2)\, dx_2\right) d\eta(x_1) = \int_{\Omega_1\times\Omega_2} f(x_1)\, d\mu(x_1,x_2) = 0$$

for every $f \in C_c(\Omega_1)$, from which we obtain the second claim in (3.6). This concludes the proof. ■

Lastly, we give a necessary and sufficient condition with which we can characterize the $\mathscr{A}_{x_2}$-gradient of a measure, under the following two assumptions.

Assumption 2. For every $\chi \in C_0(\Omega_1 \times \Omega_2;W)$ with $\mathscr{A}_{x_2}^{*}\chi = 0$ (in the sense of distributions), there exists a sequence of smooth functions $\{\chi_n\} \subset C_c^{\infty}(\Omega_1 \times \Omega_2;W)$ such that $\mathscr{A}_{x_2}^{*}\chi_n = 0$ for every n, and $\chi_n \to \chi$ in $L^{\infty}(\Omega_1 \times \Omega_2;W)$.

Assumption 3. The following Poincaré-Korn type inequality holds in $BV^{\mathscr{A}_{x_2}}(\Omega_2)$:

$$\left\| u - \int_{\Omega_2} u \, dx_2 \right\|_{L^1(\Omega_2;V)} \leq C|\mathscr{A}_{x_2} u|(\Omega_2), \quad \forall u \in BV^{\mathscr{A}_{x_2}}(\Omega_2).$$

Proposition 3.1.2. Let $\lambda \in \mathscr{M}_b(\Omega_1 \times \Omega_2; W)$. Then, the following items are equivalent:

(i) For every $\chi \in C_0(\Omega_1 \times \Omega_2; W)$ with $\mathscr{A}^*_{x_2}\chi = 0$ (in the sense of distributions) we have

$$\int_{\Omega_1 \times \Omega_2} \chi(x_1, x_2) \cdot d\lambda(x_1, x_2) = 0.$$

(ii) There exists $\mu \in \mathscr{X}^{\mathscr{A}_{x_2}}(\Omega_1)$ such that $\lambda = \mathscr{A}_{x_2}\mu$.

Proof. Let $\chi \in C_0(\Omega_1 \times \Omega_2; W)$ with $\mathscr{A}^*_{x_2}\chi = 0$ (in the sense of distributions) and let $\{\chi_n\}$ be an approximating sequence of χ as in Assumption 2. Assume that (ii) holds, then we have

$$\begin{aligned}\int_{\Omega_1 \times \Omega_2} \chi(x_1, x_2) \cdot d\lambda(x_1, x_2) &= \int_{\Omega_1 \times \Omega_2} \chi(x_1, x_2) \cdot d\mathscr{A}_{x_2}\mu(x_1, x_2) \\ &= \lim_n \int_{\Omega_1 \times \Omega_2} \chi_n(x_1, x_2) \cdot d\mathscr{A}_{x_2}\mu(x_1, x_2).\end{aligned}$$

Then, by integrating by parts, (i) follows.

Let us prove that the space

$$\mathscr{E}^{\mathscr{A}_{x_2}} = \left\{ \mathscr{A}_{x_2}\mu : \mu \in \mathscr{X}^{\mathscr{A}_{x_2}}(\Omega_1) \right\}$$

is weakly* closed in $\mathscr{M}_b(\Omega_1 \times \Omega_2; W)$. By the Krein-Šmulian theorem it is enough to show that the intersection of $\mathscr{E}^{\mathscr{A}_{x_2}}$ with every closed ball in $\mathscr{M}_b(\Omega_1 \times \Omega_2; W)$ is weakly* closed. This implies, since the weak* topology is metrizable on any closed ball of $\mathscr{M}_b(\Omega_1 \times \Omega_2; W)$, that it is enough to prove that $\mathscr{E}^{\mathscr{A}_{x_2}}$ is sequentially weakly* closed.

Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathscr{E}^{\mathscr{A}_{x_2}}$ and $\lambda \in \mathscr{M}_b(\Omega_1 \times \Omega_2; W)$ be such that

$$\lambda_n \overset{*}{\rightharpoonup} \lambda \ \text{ in } \mathscr{M}_b(\Omega_1 \times \Omega_2; W).$$

By the definition of the space $\mathscr{E}^{\mathscr{A}_{x_2}}$, there exist measures $\mu_n \in \mathscr{M}_b(\Omega_1 \times \Omega_2; V)$ such that $\lambda_n = \mathscr{A}_{x_2}\mu_n$. By Proposition 3.1.1, for every $n \in \mathbb{N}$ we have that there exist $\eta_n \in \mathscr{M}^+_b(\Omega_1)$ and $\mu^n_{x_1} \in BV^{\mathscr{A}_{x_2}}(\Omega_2)$ such that, for η_n-a.e. $x_1 \in \Omega_1$,

$$\mu_n = \mu^n_{x_1}(x_2)\, \eta_n \otimes \mathscr{L}^{n_2}_{x_2}, \qquad \mathscr{A}_{x_2}\mu_n = \eta_n \overset{\text{gen.}}{\otimes} \mathscr{A}_{x_2}\mu^n_{x_1}.$$

Furthermore, $\mu_{x_1}^n$ satisfy $\int_{\Omega_2} \mu_{x_1}^n(x_2)\,dx_2 = 0$. Then, by Assumption 3, there is a constant C independent of n such that

$$\begin{aligned}|\mu_n|(\Omega_1 \times \Omega_2) &= \int_{\Omega_1 \times \Omega_2} |\mu_n(x_1,x_2)|\,dx_1 dx_2 = \int_{\Omega_1} \left(\int_{\Omega_2} |\mu_{x_1}^n(x_2)|\,dx_2 \right) d\eta_n(x_1) \\ &\leq C \int_{\Omega_1} |\mathscr{A}_{x_2}\mu_{x_1}^n|(\Omega_2)\,d\eta_n(x_1) = C \int_{\Omega_1} \left(\int_{\Omega_2} d|\mathscr{A}_{x_2}\mu_{x_1}^n|(x_2) \right) d\eta_n(x_1) \\ &= C \int_{\Omega_1 \times \Omega_2} d\left(\eta_n \overset{\text{gen.}}{\otimes} |\mathscr{A}_{x_2}\mu_{x_1}^n| \right) = C|\mathscr{A}_{x_2}\mu_n|(\Omega_1 \times \Omega_2) \leq C.\end{aligned}$$

Hence there exists a subsequence of $\{\mu_n\}$, not relabeled, and an element $\mu \in \mathscr{M}_b(\Omega_1 \times \Omega_2; V)$ such that

$$\mu_n \xrightharpoonup{*} \mu \ \text{ in } \mathscr{M}_b(\Omega_1 \times \Omega_2; V).$$

Then, for every $\varphi \in C_c^\infty(\Omega_1 \times \Omega_2; W^*)$ we have

$$\begin{aligned}\langle \lambda, \varphi \rangle &= \lim_n \langle \lambda_n, \varphi \rangle = \lim_n \langle \mathscr{A}_{x_2}\mu_n, \varphi \rangle \\ &= \lim_n \langle \mu_n, \mathscr{A}_{x_2}^*\varphi \rangle = \langle \mu, \mathscr{A}_{x_2}^*\varphi \rangle.\end{aligned}$$

From the convergence we deduce that $\lambda = \mathscr{A}_{x_2}\mu \in \mathscr{E}^{\mathscr{A}_{x_2}}$. This implies that $\mathscr{E}^{\mathscr{A}_{x_2}}$ is weakly* closed in $\mathscr{M}_b(\Omega_1 \times \Omega_2; W) = (C_0(\Omega_1 \times \Omega_2; W^*))'$.

Assume now that (i) holds. If $\lambda \notin \mathscr{E}^{\mathscr{A}_{x_2}}$, by Hahn-Banach's theorem, there exists $\chi \in C_0(\Omega_1 \times \Omega_2; W^*)$ such that

$$\int_{\Omega_1 \times \Omega_2} \chi \cdot d\lambda = 1, \tag{3.10}$$

and, for every $u \in BV^{\mathscr{A}_{x_2}}(\Omega_1 \times \Omega_2)$,

$$\int_{\Omega_1 \times \Omega_2} \chi \cdot d\mathscr{A}_{x_2} u = 0. \tag{3.11}$$

In particular, choosing u to be a smooth function, (3.11) implies that $\mathscr{A}_{x_2}^*\chi = 0$ (in the sense of distributions). As a consequence, (3.10) contradicts (i). Thus, $\lambda \in \mathscr{E}^{\mathscr{A}_{x_2}}$. ■

3.1.1. Case $\gamma \in (0, +\infty)$

If we consider $\mathscr{A}_{x_2} = \widetilde{E}_\gamma$, $\mathscr{A}_{x_2}^* = \widetilde{\operatorname{div}}_\gamma$, $\Omega_1 = \omega$ with points $x_1 = x'$, and $\Omega_2 = I \times \mathscr{Y}$ with points $x_2 = (x_3, y)$, then we denote the associated spaces from the previous section by:

$$BD_\gamma(I \times \mathscr{Y}) := \left\{ u \in L^1(I \times \mathscr{Y}; \mathbb{R}^3) : \widetilde{E}_\gamma u \in \mathscr{M}_b(I \times \mathscr{Y}; \mathbb{M}^{3\times 3}_{\text{sym}}) \right\},$$

$$\mathscr{X}_\gamma(\omega) := \Big\{\mu \in \mathscr{M}_b(\Omega\times\mathscr{Y};\mathbb{R}^3) : \widetilde{E}_\gamma\mu \in \mathscr{M}_b(\Omega\times\mathscr{Y};\mathbb{M}^{3\times3}_{\rm sym}),$$
$$\mu(F\times I\times\mathscr{Y}) = 0 \text{ for every Borel set } F\subseteq\omega\Big\}.$$

Remark 3.1.3. For each $u\in BD_\gamma(I\times\mathscr{Y})$, we can associate a function $v := \left(\frac{1}{\gamma}u_1, \frac{1}{\gamma}u_2, u_3\right)$. Then

$$E_{y,x_3}v = \begin{pmatrix} \frac{1}{\gamma}E_y u' & \frac{1}{2}\Big(D_y u_3 + \frac{1}{\gamma}\partial_{x_3}u'\Big) \\ \frac{1}{2}\Big(D_y u_3 + \frac{1}{\gamma}\partial_{x_3}u'\Big)^T & \partial_{x_3}u_3 \end{pmatrix},$$

from which we can see that $v\in BD(I\times\mathscr{Y})$.

Alternatively, we can define the change of variables $\psi : (\gamma I)\times\mathscr{Y}\to I\times\mathscr{Y}$ given by $\psi(x_3,y) := \left(\frac{1}{\gamma}x_3, y\right)$ and consider the function $w := u\circ\psi$. Then $w\in BD((\gamma I)\times\mathscr{Y})$ and we have

$$\widetilde{E}_\gamma u = \psi_\#(E_{y,x_3}w).$$

Using any one of these scalings, we can prove that $BD_\gamma(I\times\mathscr{Y})$ satisfies the weak* compactness property Assumption 1.

Remark 3.1.4. For any $\chi\in C_0(\Omega\times\mathscr{Y};\mathbb{M}^{3\times3}_{\rm sym})$ with $\widetilde{\operatorname{div}}_\gamma\chi(x,y) = 0$ (in the sense of distributions), we construct an approximating sequence which satisfies Assumption 2. To see this we take $\chi\in C_0(\Omega\times\mathscr{Y};\mathbb{M}^{3\times3}_{\rm sym})$, extend it by zero outside Ω and define

$$\tilde{\chi}^\varepsilon(x,y) := \Lambda_{1+\varepsilon}\chi\left(\varphi^\varepsilon(x')x', (1+\varepsilon)x_3, y\right),$$

where $\Lambda_{1+\varepsilon}$ is the linear operator described in (2.9), and $\varphi^\varepsilon : \omega\to[0,1]$ is a continuous function that is zero in a neighbourhood of $\partial\omega$ and equal to 1 for $x'\in\omega$ such that $\operatorname{dist}(x',\partial\omega)\geq\varepsilon$. Notice that $\tilde{\chi}^\varepsilon\in C_c(\Omega\times\mathscr{Y};\mathbb{M}^{3\times3}_{\rm sym})$, $\tilde{\chi}^\varepsilon\to\chi$ as $\varepsilon\to0$ in L^∞ and $\widetilde{\operatorname{div}}_\gamma\tilde{\chi}^\varepsilon = 0$ (in the sense of distributions). The final argument goes by convoluting $\tilde{\chi}^\varepsilon$.

Remark 3.1.5. In view of Remark 3.1.3, to show that $BD_\gamma(I\times\mathscr{Y})$ satisfies Assumption 3 it is enough to Poincaré-Korn type inequality holds in the case $\gamma = 1$. We detail this below.

Theorem 3.1.6. There exists a constant $C>0$ such that

$$\int_{I\times\mathscr{Y}} |u|\,dx_3dy \leq C|E_{y,x_3}u|(I\times\mathscr{Y})$$

for each function $u\in BD(I\times\mathscr{Y})$ with $\int_{I\times\mathscr{Y}} u\,dx_3dy = 0$.

Proof. Assume otherwise, then there exists a sequence $\{u_n\}_n \subset BD(I\times\mathscr{Y})$ such that

$$\int_{I\times\mathscr{Y}} |u_n|\,dx_3dy > n|E_{y,x_3}u_n|(I\times\mathscr{Y}),\ \text{ with } \int_{I\times\mathscr{Y}} u_n\,dx_3dy = 0.$$

We can normalize the sequence such that

$$\int_{I\times\mathscr{Y}} |u_n|\,dx_3dy = 1,\ \text{ and } |E_{y,x_3}u_n|(I\times\mathscr{Y}) < \frac{1}{n}.$$

In particular the sequence $\{u_n\}$ is bounded in $BD(I\times\mathscr{Y})$.

Now by the weak* compactness property, there exists a subsequence $\{u_m\}\subseteq\{u_n\}$ and a function $u\in BD(I\times\mathscr{Y})$ such that $\{u_m\}$ converges weakly* to u in $BD(I\times\mathscr{Y})$, i.e.

$$u_m \to u\ \text{ in } L^1(I\times\mathscr{Y};\mathbb{R}^3),\ \text{ and } E_{y,x_3}u_m \xrightharpoonup{*} E_{y,x_3}u\ \text{ in } \mathscr{M}_b(I\times\mathscr{Y};\mathbb{M}^{3\times3}_{\text{sym}}).$$

It's clear that the limit satisfies

$$\int_{I\times\mathscr{Y}} |u|\,dx_3dy = 1,\ \text{ with } \int_{I\times\mathscr{Y}} u\,dx_3dy = 0. \tag{3.12}$$

Also, by the weak* lower semicontinuity of the total variation of measures, we have

$$|E_{y,x_3}u|(I\times\mathscr{Y}) = 0, \tag{3.13}$$

which implies $E_{y,x_3}u = 0$. As a result, the limit u is a rigid deformation, i.e. is of the form

$$u(x_3,y) = A\begin{pmatrix} y_1\\ y_2\\ x_3\end{pmatrix} + b,\ \text{ where } A\in\mathbb{M}^{3\times3}_{\text{skew}}, b\in\mathbb{R}^3.$$

Further, (3.13) implies that u has no jumps along C^1 hypersurfaces contained in $I\times\mathscr{Y}$. Hence, due to the form of skew-symmetric matrices, u must be a constant vector. However, this contradicts with (3.12). ■

The following two propositions are now a consequence of Proposition 3.1.1 and Proposition 3.1.2, respectively.

Proposition 3.1.7. Let $\mu\in\mathscr{X}_\gamma(\omega)$. Then there exist $\eta\in\mathscr{M}_b^+(\omega)$ and a Borel map $(x',x_3,y)\in\Omega\times\mathscr{Y}\mapsto\mu_{x'}(x_3,y)\in\mathbb{R}^3$ such that, for η-a.e. $x'\in\omega$,

$$\mu_{x'}\in BD_\gamma(I\times\mathscr{Y}),\qquad \int_{I\times\mathscr{Y}}\mu_{x'}(x_3,y)\,dx_3dy = 0,\qquad |\widetilde{E}_\gamma\mu_{x'}|(I\times\mathscr{Y})\neq 0, \tag{3.14}$$

and

$$\mu = \mu_{x'}(x_3, y)\,\eta \otimes \mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y. \tag{3.15}$$

Moreover, the map $x' \mapsto \widetilde{E}_\gamma \mu_{x'} \in \mathcal{M}_b(I \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{sym}})$ is η-measurable and

$$\widetilde{E}_\gamma \mu = \eta \overset{\text{gen.}}{\otimes} \widetilde{E}_\gamma \mu_{x'}.$$

Proposition 3.1.8. Let $\lambda \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{sym}})$. The following items are equivalent:

(i) For every $\chi \in C_0(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{sym}})$ with $\widetilde{\operatorname{div}}_\gamma \chi(x, y) = 0$ (in the sense of distributions) we have

$$\int_{\Omega \times \mathcal{Y}} \chi(x, y) : d\lambda(x, y) = 0.$$

(ii) There exists $\mu \in \mathcal{X}_\gamma(\omega)$ such that $\lambda = \widetilde{E}_\gamma \mu$.

Additionally, we state the following property, which will be used in the proof of Lemma 3.4.4. The proof is analogous to [25, Proposition 4.7. item (b)].

Proposition 3.1.9. Let $\mu \in \mathcal{X}_\gamma(\omega)$. For any C^1-hypersurface $\mathcal{D} \subseteq \mathcal{Y}$, if ν denotes a continuous unit normal vector field to $\mathcal{D}$, then

$$\widetilde{E}_\gamma \mu \lfloor \Omega \times \mathcal{D} = a(x, y) \odot \nu(y)\,\eta \otimes (\mathcal{H}^2_{x_3, y} \lfloor I \times \mathcal{D}),$$

where $a : \Omega \times \mathcal{D} \mapsto \mathbb{R}^3$ is a Borel function.

3.1.2. Case $\gamma = 0$

If we consider $\mathcal{A}_{x_2} = E_y$, $\mathcal{A}^*_{x_2} = \operatorname{div}_y$, $\Omega_1 = \omega$ with points $x_1 = x'$, and $\Omega_2 = \mathcal{Y}$ with points $x_2 = y$, then $BV^{\mathcal{A}_{x_2}}(\Omega_2) = BD(\mathcal{Y})$ and we denote the associated corrector space by

$$\mathcal{X}_0(\omega) := \Big\{ \mu \in \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{R}^2) : E_y \mu \in \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\text{sym}}),$$
$$\mu(F \times \mathcal{Y}) = 0 \text{ for every Borel set } F \subseteq \omega \Big\}.$$

Remark 3.1.10. We note that $\mathcal{X}_0(\omega)$ is the 2-dimensional variant of the set introduced in [25, Section 4.2], where they proved its main properties.

Further, if we consider $\mathscr{A}_{x_2} = D^2_y$, $\mathscr{A}^*_{x_2} = \mathrm{div}_y\mathrm{div}_y$, $\Omega_1 = \omega$ with points $x_1 = x'$, and $\Omega_2 = \mathscr{Y}$ with points $x_2 = y$, then $BV^{\mathscr{A}_{x_2}}(\Omega_2) = BH(\mathscr{Y})$ and we denote the associated corrector space by

$$\Upsilon_0(\omega) := \Big\{ \kappa \in \mathscr{M}_b(\omega \times \mathscr{Y}) : D^2_y \kappa \in \mathscr{M}_b(\omega \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}), \\ \kappa(F \times \mathscr{Y}) = 0 \text{ for every Borel set } F \subseteq \omega \Big\}.$$

Remark 3.1.11. It is known that that Assumption 1 and Assumption 2 are satisfied in $BH(\mathscr{Y})$, so we only need to justify Assumption 3.

Owing to [20, Remarque 1.3], there exists a constant $C > 0$ such that

$$\|u - p(u)\|_{BH(\mathscr{Y})} \leq C|D^2_y u|(\mathscr{Y}),$$

where $p(u)$ is given by

$$p(u) = \int_{\mathscr{Y}} \nabla_y u \, dy \cdot y + \int_{\mathscr{Y}} u \, dy - \int_{\mathscr{Y}} \nabla_y u \, dy \cdot \int_{\mathscr{Y}} y \, dy.$$

However, since integrating first derivatives of periodic functions over the period is zero, we precisely obtain the desired Poincaré-Korn type inequality.

As a consequence of Proposition 3.1.1 and Proposition 3.1.2, we have the following results.

Proposition 3.1.12. Let $\mu \in \mathscr{X}_0(\omega)$ and $\kappa \in \Upsilon_0(\omega)$. Then there exist $\eta \in \mathscr{M}^+_b(\omega)$ and Borel maps $(x', y) \in \omega \times \mathscr{Y} \mapsto \mu_{x'}(y) \in \mathbb{R}^2$ and $(x', y) \in \omega \times \mathscr{Y} \mapsto \kappa_{x'}(y) \in \mathbb{R}$ such that, for η-a.e. $x' \in \omega$,

$$\mu_{x'} \in BD(\mathscr{Y}), \qquad \int_{\mathscr{Y}} \mu_{x'}(y) \, dy = 0, \qquad |E_y \mu_{x'}|(\mathscr{Y}) \neq 0,$$
$$\kappa_{x'} \in BH(\mathscr{Y}), \qquad \int_{\mathscr{Y}} \kappa_{x'}(y) \, dy = 0, \qquad |D^2_y \kappa_{x'}|(\mathscr{Y}) \neq 0,$$

and

$$\mu = \mu_{x'}(y) \, \eta \otimes \mathscr{L}^2_y, \qquad \kappa = \kappa_{x'}(y) \, \eta \otimes \mathscr{L}^2_y.$$

Moreover, the maps $x' \mapsto E_y \mu_{x'} \in \mathscr{M}_b(\mathscr{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ and $x' \mapsto D^2_y \kappa_{x'} \in \mathscr{M}_b(\mathscr{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ are η-measurable and

$$E_y \mu = \eta \overset{\text{gen.}}{\otimes} E_y \mu_{x'}, \qquad D^2_y \kappa = \eta \overset{\text{gen.}}{\otimes} D^2_y \kappa_{x'}.$$

Proposition 3.1.13. Let $\lambda \in \mathscr{M}_b(\omega \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\text{sym}})$. The following items are equivalent:

(i) For every $\chi \in C_0(\omega \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\text{sym}})$ with $\text{div}_y \chi(x', y) = 0$ (in the sense of distributions) we have

$$\int_{\omega \times \mathscr{Y}} \chi(x', y) : d\lambda(x', y) = 0.$$

(ii) There exists $\mu \in \mathscr{X}_0(\omega)$ such that $\lambda = E_y \mu$.

Proposition 3.1.14. Let $\lambda \in \mathscr{M}_b(\omega \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\text{sym}})$. The following items are equivalent:

(i) For every $\chi \in C_0(\omega \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\text{sym}})$ with $\text{div}_y \text{div}_y \chi(x', y) = 0$ (in the sense of distributions) we have

$$\int_{\omega \times \mathscr{Y}} \chi(x', y) : d\lambda(x', y) = 0.$$

(ii) There exists $\kappa \in \Upsilon_0(\omega)$ such that $\lambda = D^2_y \kappa$.

3.1.3. Case $\gamma = +\infty$

If we consider $\mathscr{A}_{x_2} = E_y$, $\mathscr{A}^*_{x_2} = \text{div}_y$, $\Omega_1 = \Omega$ with points $x_1 = x$, and $\Omega_2 = \mathscr{Y}$ with points $x_2 = y$, then $BV^{\mathscr{A}_{x_2}}(\Omega_2) = BD(\mathscr{Y})$ and we denote the associated corrector space by

$$\mathscr{X}_\infty(\Omega) := \Big\{ \mu \in \mathscr{M}_b(\Omega \times \mathscr{Y}; \mathbb{R}^2) : E_y \mu \in \mathscr{M}_b(\Omega \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\text{sym}}), \\ \mu(F \times \mathscr{Y}) = 0 \text{ for every Borel set } F \subseteq \Omega \Big\},$$

Further, if we consider $\mathscr{A}_{x_2} = D_y$, $\mathscr{A}^*_{x_2} = \text{div}_y$, $\Omega_1 = \Omega$ with points $x_1 = x$, and $\Omega_2 = \mathscr{Y}$ with points $x_2 = y$, then $BV^{\mathscr{A}_{x_2}}(\Omega_2) = BV(\mathscr{Y})$ and we denote the associated corrector space by

$$\Upsilon_\infty(\Omega) := \Big\{ \kappa \in \mathscr{M}_b(\Omega \times \mathscr{Y}) : D_y \kappa \in \mathscr{M}_b(\Omega \times \mathscr{Y}; \mathbb{R}^2), \\ \kappa(F \times \mathscr{Y}) = 0 \text{ for every Borel set } F \subseteq \Omega \Big\}.$$

Clearly Assumption 1, Assumption 2 and Assumption 3 are satisfied in $BD(\mathscr{Y})$ and $BV(\mathscr{Y})$. Thus, we can state the following propositions as consequences of Proposition 3.1.1 and Proposition 3.1.2.

Proposition 3.1.15. Let $\mu \in \mathscr{X}_\infty(\Omega)$ and $\kappa \in \Upsilon_\infty(\Omega)$. Then there exist $\eta \in \mathscr{M}_b^+(\Omega)$ and Borel maps $(x,y) \in \Omega \times \mathscr{Y} \mapsto \mu_x(y) \in \mathbb{R}^2$ and $(x,y) \in \Omega \times \mathscr{Y} \mapsto \kappa_x(y) \in \mathbb{R}^2$ such that, for η-a.e. $x \in \Omega$,

$$\mu_x \in BD(\mathscr{Y}), \qquad \int_{\mathscr{Y}} \mu_x(y)\,dy = 0, \qquad |E_y\mu_x|(\mathscr{Y}) \neq 0,$$
$$\kappa_x \in BV(\mathscr{Y}), \qquad \int_{\mathscr{Y}} \kappa_x(y)\,dy = 0, \qquad |D_y\kappa_x|(\mathscr{Y}) \neq 0,$$

and

$$\mu = \mu_x(y)\,\eta \otimes \mathscr{L}^2_y, \qquad \kappa = \kappa_x(y)\,\eta \otimes \mathscr{L}^2_y.$$

Moreover, the maps $x \mapsto E_y\mu_x \in \mathscr{M}_b(\mathscr{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ and $x \mapsto D_y\kappa_x \in \mathscr{M}_b(\mathscr{Y};\mathbb{R}^2)$ are η-measurable and

$$E_y\mu = \eta \overset{\text{gen.}}{\otimes} E_y\mu_x, \qquad D_y\kappa = \eta \overset{\text{gen.}}{\otimes} D_y\kappa_x.$$

Proposition 3.1.16. Let $\lambda \in \mathscr{M}_b(\Omega \times \mathscr{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}})$. The following items are equivalent:

(i) For every $\chi \in C_0(\Omega \times \mathscr{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ with $\mathrm{div}_y\chi(y) = 0$ (in the sense of distributions) we have
$$\int_{\mathscr{Y}} \chi(x,y) : d\lambda(x,y) = 0.$$

(ii) There exists $\mu \in \mathscr{X}_\infty(\Omega)$ such that $\lambda = E_y\mu$.

Proposition 3.1.17. Let $\lambda \in \mathscr{M}_b(\Omega \times \mathscr{Y};\mathbb{R}^2)$. The following items are equivalent:

(i) For every $\chi \in C_0(\Omega \times \mathscr{Y};\mathbb{R}^2)$ with $\mathrm{div}_y\chi(y) = 0$ (in the sense of distributions) we have
$$\int_{\mathscr{Y}} \chi(x,y) : d\lambda(x,y) = 0.$$

(ii) There exists $\kappa \in \Upsilon_\infty(\Omega)$ such that $\lambda = D_y\kappa$.

3.2. AUXILIARY RESULTS

3.2.1. Case $\gamma \in (0,+\infty)$

We will need the following result, which is connected with the compactly supported De Rham cohomology.

Proposition 3.2.1. (a) Let $\mathscr{Y}^{(3)}$ be a flat torus in $\mathbb{R}^3$ and let $\chi \in C^\infty(\mathscr{Y}^{(3)};\mathbb{R}^3)$ be such that $\operatorname{div}\chi = 0$ and $\int_{\mathscr{Y}^{(3)}} \chi = 0$. Then there exists $F \in C^\infty(\mathscr{Y}^{(3)};\mathbb{R}^3)$ such that $\operatorname{rot} F = \chi$.

(b) Let $\mathscr{Y}$ be a flat torus in $\mathbb{R}^2$ and let $\chi \in C_c^\infty(\mathscr{Y} \times I;\mathbb{R}^3)$ be such that $\operatorname{div}_{y,x_3}\chi = 0$ and $\int_{\mathscr{Y}\times I} \chi = 0$. Then there exists $F \in C_c^\infty(\mathscr{Y} \times I;\mathbb{R}^3)$ such that $\operatorname{rot}_{y,x_3} F = \chi$.

Proof. The first claim is standard and can be easily proved by, e.g, Fourier transforms. For the second claim, extending χ by periodicity to $\mathscr{Y}^{(3)}$, by the first part of the statement we obtain $\tilde{F} \in C^\infty(\mathscr{Y}^{(3)};\mathbb{R}^3)$ such that $\operatorname{rot}\tilde{F} = \chi$ on $\mathscr{Y}^{(3)}$. Since χ has compact support in $\mathscr{Y} \times I$, there exists $0 < \delta < \frac{1}{2}$ such that $\operatorname{rot}_{y,x_3}\tilde{F} = 0$ on $\mathscr{Y} \times \tilde{I}_\delta$, where $\tilde{I}_\delta = \{(\frac{1}{2}-\delta,\frac{1}{2}) \cup (-\frac{1}{2},-\frac{1}{2}+\delta)\}$. Let now $\tilde{\varphi} \in C^\infty(S_\delta)$, where $S_\delta = (0,1)^2 \times \tilde{I}_\delta$, be such that $\tilde{F} = \nabla_{y,x_3}\tilde{\varphi}$ on S_δ. For $\alpha \in \{1,2\}$, let

$$\sum_{k\in\mathbb{Z}} a_k^\alpha(y_2,x_3)e^{2\pi i k y_1}$$

be the exponential Fourier series of $\tilde{F}_\alpha = \partial_{y_\alpha}\tilde{\varphi}$ with respect to the variable y_1. Note that the coefficients $\{a_k^\alpha(y_2,x_3)\}_{k\in\mathbb{Z}}$ are smooth functions and periodic with respect to the variable y_2 and x_3. Furthermore, the Fourier series of smooth functions converges uniformly, and the result of differentiating or integrating the series term by term will converge to the derivative or integral of the original series. Hence, we can infer that

$$\tilde{\varphi}(y,x_3) = a_0^1(y_2,x_3)y_1 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{a_k^1(y_2,x_3)}{2\pi i k} e^{2\pi i k y_1} + b^1(y_2,x_3) \text{ on } S_\delta, \tag{3.16}$$

for a suitable smooth function $b^1(y_2,x_3)$. Then, differentiating with respect to y_1 and y_2, we have that

$$\partial_{y_1 y_2}\tilde{\varphi}(y,x_3) = \partial_{y_2}a_0^1(y_2,x_3) + \sum_{k\in\mathbb{Z}\setminus\{0\}} \partial_{y_2}a_k^1(y_2,x_3)e^{2\pi i k y_1} \text{ on } S_\delta.$$

However, since

$$\partial_{y_1 y_2}\tilde{\varphi}(y,x_3) = \partial_{y_1}\tilde{F}_2(y,x_3) = \sum_{k\in\mathbb{Z}\setminus\{0\}} 2\pi i k a_k^2(y_2,x_3) e^{2\pi i k y_1} \text{ on } S_\delta,$$

by the uniqueness of the Fourier expansion we have that $\partial_{y_2} a_0^1(y_2,x_3) = 0$, i.e.

$$a_0^1(y_2,x_3) = c_1(x_3), \tag{3.17}$$

for some $c_1 \in C^\infty(\tilde{I}_\delta)$. Further, differentiating (3.16) with respect to y_2, we have that

$$\partial_{y_2}\tilde{\varphi}(y,x_3) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\partial_{y_2} a_k^1(y_2,x_3)}{2\pi i k} e^{2\pi i k y_1} + \partial_{y_2} b^1(y_2,x_3) \text{ on } S_\delta.$$

Since $\partial_{y_2}\tilde{\varphi} = \tilde{F}_2$ is periodic, we can conclude that $\partial_{y_2} b^1$ is also periodic with respect to the variable y_2 and we can consider its Fourier series. Let $c_2 \in C^\infty(\tilde{I}_\delta)$ be the corresponding zero-th term. Then the antiderivative of $\partial_{y_2} b^1 - c_2$ with respect to y_2 is a periodic function. Combining this fact with (3.16) and (3.17), we deduce that there exists a smooth function $\hat{\varphi} \in C^\infty(\tilde{I}_\delta; C^\infty(\mathscr{Y}))$ such that $\tilde{\varphi}$ can be rewritten as

$$\tilde{\varphi}(y,x_3) = \hat{\varphi}(y,x_3) + c_1(x_3)\, y_1 + c_2(x_3)\, y_2 \text{ on } \mathscr{Y} \times \tilde{I}_\delta.$$

From this, differentiating with respect to x_3, we have that

$$\tilde{F}_3(y,x_3) = \partial_{x_3}\hat{\varphi}(y,x_3) + c_1'(x_3)\, y_1 + c_2'(x_3)\, y_2 \text{ on } \mathscr{Y} \times \tilde{I}_\delta.$$

As a consequence of the periodicity of $\tilde{F}_3$ and $\partial_{x_3}\hat{\varphi}$ in the variables y_1 and y_2, we conclude that $c_1' = 0$ and $c_2' = 0$. Since $\mathscr{Y} \times \tilde{I}_\delta$ is a union of two disjoint open sets, we have that c_1, c_2 are constant on each connected component. Using the fact that, for $\alpha \in \{1,2\}$,

$$\partial_{y_\alpha}\tilde{\varphi}(y,x_3) = \partial_{y_\alpha}\hat{\varphi}(y,x_3) + c_\alpha(x_3) \text{ on } \mathscr{Y} \times \tilde{I}_\delta, \tag{3.18}$$

the periodicity of $\tilde{F}_\alpha = \partial_{y_\alpha}\tilde{\varphi}$ implies that c_1, c_2 are in fact constant. This can be seen by integrating the equation (3.18) over the plane $x_3 = -\frac{1}{2}$ and $x_3 = \frac{1}{2}$. Thus we conclude that

$$\tilde{F}(y,x_3) = \nabla_{y,x_3}\hat{\varphi}(y,x_3) + \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix} \text{ on } \mathscr{Y} \times \tilde{I}_\delta. \tag{3.19}$$

Consider now the exponential Fourier series of $\tilde{F}_3$ with respect to the x_3 variable, such that

$$\tilde{F}_3(y,x_3) = \sum_{k\in\mathbb{Z}} a_k^3(y)e^{2\pi ikx_3} \text{ on } \mathscr{Y}\times\tilde{I}_\delta.$$

Integrating the third component in (3.19) with respect to x_3, we have that there exists a smooth function $b^3(y,x_3)$, which has values $b_+^3(y)$ and $b_-^3(y)$ on each of the two parts of $\mathscr{Y}\times\tilde{I}_\delta$, such that

$$\hat{\varphi}(y,x_3) = a_0^3(y)x_3 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{a_k^3(y)}{2\pi ik}e^{2\pi ikx_3} + b^3(y,x_3) \text{ on } \mathscr{Y}\times\tilde{I}_\delta.$$

From this and (3.18) we have, for $\alpha\in\{1,2\}$,

$$\tilde{F}_\alpha(y,x_3) - c_\alpha = \partial_{y_\alpha}a_0^3(y)x_3 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\partial_{y_\alpha}a_k^3(y)}{2\pi ik}e^{2\pi ikx_3} + \partial_{y_\alpha}b^3(y,x_3) \text{ on } \mathscr{Y}\times\tilde{I}_\delta.$$

Considering the continuity and periodicity in x_3 of the above terms, evaluating in $x_3=-\frac{1}{2}$ and $x_3=\frac{1}{2}$ gives $\partial_{y_\alpha}a_0^3(y) = \partial_{y_\alpha}b_-^3(y) - \partial_{y_\alpha}b_+^3(y)$. From this we have that there exists a constant c_3 and $\varphi\in C^\infty(\mathscr{Y}\times\tilde{I}_\delta)$ such that it and all its derivatives are periodic in the x_3 variable, for which

$$\hat{\varphi}(y,x_3) = \varphi(y,x_3) + c_3x_3 \text{ on } \mathscr{Y}\times\tilde{I}_\delta.$$

From this and (3.19) we conclude

$$\tilde{F}(y,x_3) = \nabla_{y,x_3}\varphi(y,x_3) + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \text{ on } \mathscr{Y}\times\tilde{I}_\delta.$$

Finally, we consider a smooth function $k: I\to\mathbb{R}$ that is zero on the set $\left[-\frac{1}{2}+\delta, \frac{1}{2}-\delta\right]$ and one in a neighbourhood of $x_3=-\frac{1}{2}$, $x_3=\frac{1}{2}$. By taking

$$F := \tilde{F} - \nabla_{y,x_3}(k\varphi) - \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \text{ on } \mathscr{Y}\times I.$$

we have the claim. ■

Remark 3.2.2. By considering functions scaled by γ in the third component and by $\frac{1}{\gamma}$ in the direction x_3, one can apply the proof item (b) in Proposition 3.2.1 so that the statement is valid for maps in the space $C_c^\infty(\mathscr{Y}\times(\gamma I);\mathbb{R}^3)$.

Consequently, for $\chi \in C_c^\infty(\mathscr{Y} \times I;\mathbb{R}^3)$ such that $\widetilde{\operatorname{div}}_\gamma \chi = 0$ and $\int_{\mathscr{Y}\times I} \chi = 0$ there exists $F \in C_c^\infty(\mathscr{Y} \times I;\mathbb{R}^3)$ such that $\widetilde{\operatorname{rot}}_\gamma F = \chi$, which can be easily seen by rescaling in the direction x_3.

Remark 3.2.3. If $\chi \in C_c^\infty(\Omega \times \mathscr{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$ is such that $\widetilde{\operatorname{div}}_\gamma \chi = 0$, then for a.e. $x' \in \omega$

$$\int_{I\times\mathscr{Y}} \chi_{3i}(x,y)\, dx_3 dy = 0, \quad i = 1,2,3.$$

Indeed, by putting

$$\varphi(x) = \begin{pmatrix} 2\gamma x_3\, c_1(x') \\ 2\gamma x_3\, c_2(x') \\ \gamma x_3\, c_3(x') \end{pmatrix},$$

for $c \in C_c^\infty(\omega;\mathbb{R}^3)$, it is easy to see that

$$\widetilde{E}_\gamma \varphi = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & c_2 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

and the conclusion results from testing χ with $\widetilde{E}_\gamma\varphi$ on $I \times \mathscr{Y}$, and by the arbitrariness of the maps $c_i,\ i = 1,2,3$.

3.2.2. Case $\gamma = 0$

In order to simplify the proof of the structure result for the two-scale limits of symmetrized scaled gradients, we will use the following lemma.

Lemma 3.2.4. Let $\{\mu^h\}_{h>0}$ be a bounded family in $\mathscr{M}_b(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ such that

$$\mu^h \xrightarrow{2-*} \mu \quad \text{two-scale weakly* in } \mathscr{M}_b(\Omega\times\mathscr{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}}).$$

for some $\mu \in \mathscr{M}_b(\Omega\times\mathscr{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ as $h \to 0$. Assume that

(i) $\bar{\mu}^h \xrightarrow{2-*} \lambda_1$ two-scale weakly* in $\mathscr{M}_b(\omega\times\mathscr{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}})$, for some $\lambda_1 \in \mathscr{M}_b(\omega\times\mathscr{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}})$;

(ii) For every $\chi \in C_c^\infty(\omega\times\mathscr{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ such that $\operatorname{div}_y\operatorname{div}_y \chi(x',y) = 0$ we have

$$\lim_{h\to 0}\int_\omega \chi\Big(x',\tfrac{x'}{\varepsilon_h}\Big) : d\hat{\mu}^h(x')\, dx' = \int_{\omega\times\mathscr{Y}} \chi(x',y) : d\lambda_2(x',y),$$

for some $\lambda_2 \in \mathscr{M}_b(\omega\times\mathscr{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}})$;

(iii) There exists an open set $\widetilde{I} \supset I$ which compactly contains I such $(\mu^h)^\perp \xrightarrow{2-*} 0$ two-scale weakly* in $\mathcal{M}_b(\omega \times \widetilde{I} \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$.

Then there exists $\kappa \in \Upsilon_0(\omega)$ such that

$$\mu = \lambda_1 \otimes \mathcal{L}^1_{x_3} + \left(\lambda_2 + D^2_y \kappa\right) \otimes x_3 \mathcal{L}^1_{x_3}.$$

Proof. Every μ^h determines a measure ν^h on $\omega \times \widetilde{I} \times \mathcal{Y}$ with the relations

$$\nu^h(B) := \mu^h(B \cap (\Omega \times \mathcal{Y}))$$

for every Borel set $B \subseteq \omega \times \widetilde{I} \times \mathcal{Y}$. With a slight abuse of notation, we will still write μ^h instead of ν^h.

Let ν be the measure such that

$$\mu^h \xrightarrow{2-*} \nu \quad \text{two-scale weakly* in } \mathcal{M}_b(\omega \times \widetilde{I} \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}).$$

We first observe that, from the assumption (i) and (iii), it follows that $\bar{\nu} = \lambda_1$ and $\nu^\perp = 0$. Furthermore, $\mu^h \xrightarrow{2-*} \nu$ two-scale weakly* in $\mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$.

Let $\chi \in C^\infty_c(\Omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$. If we consider the following orthogonal decomposition

$$\chi(x,y) = \bar{\chi}(x',y) + x_3 \hat{\chi}(x',y) + \chi^\perp(x,y),$$

then we have that

$$\begin{aligned}
&\int_{\Omega \times \mathcal{Y}} \chi(x,y) : d\nu(x,y) = \lim_{h \to 0} \int_\Omega \chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big) : d\mu^h(x') \\
&= \lim_{h \to 0} \int_\omega \bar{\chi}\Big(x', \tfrac{x'}{\varepsilon_h}\Big) : d\bar{\mu}^h(x') + \frac{1}{12} \lim_{h \to 0} \int_\omega \hat{\chi}\Big(x', \tfrac{x'}{\varepsilon_h}\Big) : d\hat{\mu}^h(x') + \lim_{h \to 0} \int_\Omega \chi^\perp\Big(x, \tfrac{x'}{\varepsilon_h}\Big) : d(\mu^h)^\perp(x) \\
&= \int_{\omega \times \mathcal{Y}} \bar{\chi}\,(x',y) : d\lambda_1(x',y) + \frac{1}{12} \lim_{h \to 0} \int_\omega \hat{\chi}\Big(x', \tfrac{x'}{\varepsilon_h}\Big) : d\hat{\mu}^h(x').
\end{aligned}$$

Suppose now that $\chi(x,y) = x_3 \hat{\chi}(x',y)$ with $\mathrm{div}_y \mathrm{div}_y \hat{\chi}(x',y) = 0$. Then the above equality yields

$$\int_{\omega \times \mathcal{Y}} \hat{\chi}(x',y) : d\hat{\nu}(x',y) = \lim_{h \to 0} \int_\omega \hat{\chi}\Big(x', \tfrac{x'}{\varepsilon_h}\Big) : d\hat{\mu}^h(x') = \int_{\omega \times \mathcal{Y}} \hat{\chi}(x',y) : d\lambda_2(x',y).$$

By a density argument, we infer that

$$\int_{\omega \times \mathcal{Y}} \hat{\chi}(x',y) : d\left(\hat{\nu}(x',y) - \lambda_2(x',y)\right) = 0,$$

for every $\hat{\chi} \in C_0(\omega \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\text{sym}})$ with $\operatorname{div}_y \operatorname{div}_y \hat{\chi}(x', y) = 0$ (in the sense of distributions). From this and Proposition 3.1.14 we conclude that there exists $\kappa \in \Upsilon_0(\omega)$ such that

$$\hat{v} - \lambda_2 = D^2_y \kappa.$$

Since $\mu = \nu$ on $\Omega \times \mathscr{Y}$, we have the claim. ■

3.2.3. Case $\gamma = +\infty$

The following result will be in the proof of the structure result for the two-scale limits of symmetrized scaled gradients. We note, however, that this result is independent of the limit value γ.

Proposition 3.2.5. Let $\{v^h\}_{h>0}$ be a bounded family in $BD(\Omega)$ such that

$$v^h \xrightarrow{*} v \quad \text{weakly* in } BD(\Omega),$$

for some $v \in BD(\Omega)$. Then there exists $\mu \in \mathscr{X}_\infty(\Omega)$ such that

$$\left(Ev^h\right)'' \xrightarrow{2-*} E_{x'}v' \otimes \mathscr{L}^2_y + E_y\mu \quad \text{two-scale weakly* in } \mathscr{M}_b(\Omega \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\text{sym}}).$$

Proof. The proof follows closely that of [25, Proposition 4.10].

By compactness, the exists $\lambda \in \mathscr{M}_b(\Omega \times \mathscr{Y}; \mathbb{M}^{3\times 3}_{\text{sym}})$ such that (up to a subsequence)

$$Ev^h \xrightarrow{2-*} \lambda \quad \text{two-scale weakly* in } \mathscr{M}_b(\Omega \times \mathscr{Y}; \mathbb{M}^{3\times 3}_{\text{sym}}).$$

Since $v^h \to v$ strongly in $L^1(\Omega;\mathbb{R}^3)$, we have compontentwise

$$v^h_i \xrightarrow{2-*} v_i(x)\mathscr{L}^3_x \otimes \mathscr{L}^2_y \quad \text{two-scale weakly* in } \mathscr{M}_b(\Omega \times \mathscr{Y}), \quad i = 1,2,3.$$

Consider $\chi \in C^\infty_c(\Omega \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\text{sym}})$ such that $\operatorname{div}_y \chi(x,y) = 0$. Then

$$\begin{aligned}
&\lim_{h\to 0} \int_\Omega \chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big) : d\left(Ev^h\right)''(x) = \lim_{h\to 0} \int_\Omega \chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big) : dE_{x'}(v^h)'(x) \\
&= -\lim_{h\to 0} \int_\Omega (v^h)'(x) \cdot \operatorname{div}_{x'}\left(\chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\right) dx \\
&= -\lim_{h\to 0} \left(\int_\Omega (v^h)'(x) \cdot \operatorname{div}_{x'}\chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx + \frac{1}{\varepsilon_h} \int_\Omega (v^h)'(x) \cdot \operatorname{div}_y \chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx \right) \\
&= -\lim_{h\to 0} \int_\Omega (v^h)'(x) \cdot \operatorname{div}_x \chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx \\
&= -\int_{\Omega\times\mathscr{Y}} v'(x) \cdot \operatorname{div}_{x'}\chi(x,y)\, dxdy \\
&= \int_{\Omega\times\mathscr{Y}} \chi(x,y) : d\left(E_{x'}v' \otimes \mathscr{L}^2_y\right).
\end{aligned}$$

By a density argument, we infer that

$$\int_{\Omega\times\mathscr{Y}} \chi(x,y) : d\left(\lambda(x,y) - E_{x'}v' \otimes \mathscr{L}_y^2\right) = 0,$$

for every $\chi \in C_0(\Omega \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ with $\mathrm{div}_y \chi(x,y) = 0$ (in the sense of distributions). From this and Proposition 3.1.16 we conclude that there exists $\mu \in \mathscr{X}_\infty(\Omega)$ such that

$$\lambda - E_{x'}v' \otimes \mathscr{L}_y^2 = E_y \mu.$$

From this we have the claim. ■

3.3. TWO-SCALE LIMITS OF SCALED SYMMETRIZED GRADIENTS

We are now ready to prove the first main result of this section.

Theorem 3.3.1. Let $\{v^h\}_{h>0}$ be such that $v^h \in BD(\Omega^h)$ for every $h > 0$, and there exists a constant C for which

$$\|v^h\|_{BD(\Omega^h)} \leq C, \quad \text{for every } h > 0.$$

Denote by u^h the map $u^h := (v_1^h, v_2^h, hv_3^h) \circ \psi_h$. Then there exist

$$\bar{u} = (\bar{u}_1, \bar{u}_2) \in BD(\omega), \quad u_3 \in BH(\omega), \quad \widetilde{E} \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{sym}}),$$

and a (not relabeled) subsequence of $\{u^h\}_{h>0}$ which satisfy

$$\Lambda_h E u^h \xrightarrow{2-*} \begin{pmatrix} E\bar{u} - x_3 D^2 u_3 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathcal{L}^2_y + \widetilde{E} \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{sym}}).$$

(a) If $\gamma \in (0, +\infty)$, then there exists $\mu \in \mathcal{X}_\gamma(\omega)$ such that

$$\widetilde{E} = \widetilde{E}_\gamma \mu(x, y).$$

(b) If $\gamma = 0$, then there exist $\mu \in \mathcal{X}_0(\omega)$, $\kappa \in \Upsilon_0(\omega)$ and $\zeta \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{R}^3)$ such that

$$\widetilde{E} = \begin{pmatrix} E_y \mu(x', y) - x_3 D^2_y \kappa(x', y) & \zeta'(x, y) \\ (\zeta'(x, y))^T & \zeta_3(x, y) \end{pmatrix}.$$

(c) If $\gamma = +\infty$, then there exist $\mu \in \mathcal{X}_\infty(\Omega)$, $\kappa \in \Upsilon_\infty(\Omega)$ and $\zeta \in \mathcal{M}_b(\Omega; \mathbb{R}^3)$ such that

$$\widetilde{E} = \begin{pmatrix} E_y \mu(x, y) & \zeta'(x) + D_y \kappa(x, y) \\ (\zeta'(x) + D_y \kappa(x, y))^T & \zeta_3(x) \end{pmatrix}.$$

Proof. Owing to [45, Chapter II, Remark 3.3], we can assume without loss of generality that u^h are smooth functions for every $h > 0$. Further, the uniform boundedness of the sequence $\{Ev^h\}$ implies that

$$\int_\Omega |\partial_{x_\alpha} u_3^h + \partial_{x_3} u_\alpha^h| \, dx \leq Ch, \quad \text{for } \alpha = 1, 2, \tag{3.20}$$

$$\int_\Omega |\partial_{x_3} u_3^h| \, dx \leq Ch^2. \tag{3.21}$$

In the following, we will consider $\lambda \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$ such that

$$\Lambda_h E u^h \xrightarrow{2-*} \lambda \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}}).$$

Step 1. We prove the statement in the case $\frac{h}{\varepsilon_h} \to \gamma \in (0, +\infty)$.

By using Proposition 3.0.1 we have that there exist $(\bar{u}_1, \bar{u}_2) \in BD(\omega)$, $u_3 \in BH(\omega)$ such that

$$(Eu^h)''(x) \xrightarrow{*} E\bar{u}(x') - x_3 D^2 u_3(x') \quad \text{weakly* in } \mathcal{M}_b(\Omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}}).$$

Let $\chi \in C_c^\infty(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$ be such that $\widetilde{\operatorname{div}}_\gamma \chi = 0$. We have

$$\begin{aligned}
&\int_{\Omega\times\mathcal{Y}} \chi(x,y) : d\lambda(x,y) \\
&= \lim_{h\to 0} \int_\Omega \chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big) : d\left(\Lambda_h E u^h(x)\right) = -\lim_{h\to 0} \int_\Omega u^h(x) \cdot \operatorname{div}\left(\Lambda_h \chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\right) dx \\
&= -\lim_{h\to 0} \sum_{\alpha=1,2} \int_\Omega u^h_\alpha(x) \left(\partial_{x_1}\chi_{\alpha 1} + \partial_{x_2}\chi_{\alpha 2}\right)\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx - \lim_{h\to 0} \frac{1}{h} \int_\Omega u^h_3(x) \left(\partial_{x_1}\chi_{31} + \partial_{x_2}\chi_{32}\right)\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx \\
&\quad - \lim_{h\to 0} \sum_{\alpha=1,2} \frac{1}{\varepsilon_h} \int_\Omega u^h_\alpha(x) \left(\partial_{y_1}\chi_{\alpha 1} + \partial_{y_2}\chi_{\alpha 2}\right)\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx - \lim_{h\to 0} \frac{1}{h\varepsilon_h} \int_\Omega u^h_3(x) \left(\partial_{y_1}\chi_{31} + \partial_{y_2}\chi_{32}\right)\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx \\
&\quad - \lim_{h\to 0} \sum_{\alpha=1,2} \frac{1}{h} \int_\Omega u^h_\alpha(x)\, \partial_{x_3}\chi_{\alpha 3}\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx - \lim_{h\to 0} \frac{1}{h^2} \int_\Omega u^h_3(x)\, \partial_{x_3}\chi_{33}\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx \\
&= -\lim_{h\to 0} \sum_{\alpha=1,2} \int_\Omega u^h_\alpha \cdot \left(\partial_{x_1}\chi_{\alpha 1} + \partial_{x_2}\chi_{\alpha 2}\right)\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx - \lim_{h\to 0} \frac{1}{h} \int_\Omega u^h_3 \cdot \left(\partial_{x_1}\chi_{31} + \partial_{x_2}\chi_{32}\right)\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx \\
&\quad + \lim_{h\to 0} \left(\frac{h}{\varepsilon_h \gamma} - 1\right) \left(\sum_{\alpha=1,2} \frac{1}{h} \int_\Omega u^h_\alpha \cdot \partial_{x_3}\chi_{\alpha 3}\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx + \frac{1}{h^2} \int_\Omega u^h_3 \cdot \partial_{x_3}\chi_{33}\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx\right),
\end{aligned} \tag{3.22}$$

where in the last equality we used that $\frac{1}{\varepsilon_h}\partial_{y_1}\chi_{i1} + \frac{1}{\varepsilon_h}\partial_{y_2}\chi_{i2} + \frac{1}{h}\partial_{y_3}\chi_{i3} = \left(\frac{1}{h} - \frac{1}{\varepsilon_h\gamma}\right)\partial_{y_3}\chi_{i3}$.

From the proof of Proposition 3.0.1 we know that we have the following convergences:

$$\begin{aligned}
&u^h_\alpha \to \bar{u}_\alpha - x_3 \partial_{x_\alpha} u_3, \quad \text{strongly in } L^1(\Omega), \quad \alpha = 1,2, \\
&u^h_3 \to u_3, \quad \text{strongly in } L^1(\Omega).
\end{aligned}$$

Notice that

$$\begin{aligned}
&\lim_{h\to 0}\sum_{\alpha=1,2}\int_\Omega u^h_\alpha(x)\,(\partial_{x_1}\chi_{\alpha 1}+\partial_{x_2}\chi_{\alpha 2})\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\\
&=\sum_{\alpha=1,2}\int_\Omega(\bar u_\alpha - x_3\partial_{x_\alpha}u_3)\left(\partial_{x_1}\int_{\mathscr{Y}}\chi_{\alpha 1}(x,y)\,dy+\partial_{x_2}\int_{\mathscr{Y}}\chi_{\alpha 2}(x,y)\,dy\right)dx\\
&=-\int_{\Omega\times\mathscr{Y}}\chi(x,y):d\left(\begin{pmatrix} E\bar u(x')-x_3D^2u_3(x') & 0\\ 0 & 0\end{pmatrix}\otimes\mathscr{L}^2_y\right). \qquad (3.23)
\end{aligned}$$

Next, in view of Remark 3.2.3, we can use item (b) in Proposition 3.2.1 to conclude that there exists $F\in C^\infty_c(\Omega\times\mathscr{Y};\mathbb{R}^3)$ such that $\widetilde{\operatorname{rot}}_\gamma F=(\chi_{3i})_{i=1,2,3}$. Thus we have

$$\chi_{31} = \partial_{y_2}F_3-\frac{1}{\gamma}\partial_{x_3}F_2, \qquad (3.24)$$

$$\chi_{32} = \frac{1}{\gamma}\partial_{x_3}F_1-\partial_{y_1}F_3. \qquad (3.25)$$

Next we compute

$$\begin{aligned}
\lim_{h\to 0}\frac{1}{\varepsilon_h}\int_\Omega u^h_3(x)\,\partial_{x_1y_2}F_3\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx &= \lim_{h\to 0}\int_\Omega u^h_3(x)\,\partial_{x_2}\Big(\partial_{x_1}F_3\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\Big)\,dx\\
&\quad -\lim_{h\to 0}\int_\Omega u^h_3(x)\,\partial_{x_1x_2}F_3\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx. \qquad (3.26)
\end{aligned}$$

Notice that

$$\lim_{h\to 0}\int_\Omega u^h_3(x)\,\partial_{x_1x_2}F_3\Big(x,\tfrac{x'}{\varepsilon_h}\Big)=\int_{\Omega\times\mathscr{Y}}u_3\,\partial_{x_1x_2}F_3(x,y)dxdy=\int_\Omega\partial_{x_1x_2}u_3\int_{\mathscr{Y}}F_3(x,y)dydx. \qquad (3.27)$$

Recalling (3.20), we compute

$$\begin{aligned}
\lim_{h\to 0}\int_\Omega u^h_3(x)\,\partial_{x_2}\Big(\partial_{x_1}F_3\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\Big)\,dx &= -\lim_{h\to 0}\int_\Omega\partial_{x_2}u^h_3(x)\,\partial_{x_1}F_3\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\\
&= \lim_{h\to 0}\int_\Omega\partial_{x_3}u^h_2\,\partial_{x_1}F_3\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\\
&= -\lim_{h\to 0}\int_\Omega u^h_2\,\partial_{x_1x_3}F_3\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\\
&= -\int_{\Omega\times\mathscr{Y}}(\bar u_2-x_3\partial_{x_2}u_3)\,\partial_{x_1x_3}F_3(x,y)dxdy\\
&= \int_\Omega\partial_{x_1x_2}u_3\int_{\mathscr{Y}}F_3(x,y)dydx. \qquad (3.28)
\end{aligned}$$

From (3.26), (3.27), (3.28) we conclude

$$\begin{aligned}
\lim_{h\to 0}\frac{1}{h}\int_\Omega u^h_3(x)\,\partial_{x_1y_2}F_3\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx &= \lim_{h\to 0}\frac{1}{\varepsilon_h\gamma}\int_\Omega u^h_3(x)\,\partial_{x_1y_2}F_3\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\\
&= 0. \qquad (3.29)
\end{aligned}$$

In the similar way for u_3^h (recalling (3.21)), we conclude

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}\int_\Omega u_3^h(x)\,\partial_{x_1x_3}F_2\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx &= -\lim_{h\to 0}\frac{1}{h}\int_\Omega \partial_{x_3}u_3^h(x)\,\partial_{x_1}F_2\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\\ &= 0. \end{aligned}\tag{3.30}$$

From (3.24), (3.29), (3.30) we conclude

$$\lim_{h\to 0}\frac{1}{h}\int_\Omega u_3^h(x)\,\partial_{x_1}\chi_{31}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx = 0. \tag{3.31}$$

In a similar way we obtain

$$\lim_{h\to 0}\frac{1}{h}\int_\Omega u_3^h(x)\,\partial_{x_2}\chi_{32}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx = 0. \tag{3.32}$$

Lastly, using similar arguments as above, we compute

$$\begin{aligned}
&\lim_{h\to 0}\Big(\frac{h}{\varepsilon_h\gamma}-1\Big)\Big(\sum_{\alpha=1,2}\frac{1}{h}\int_\Omega u_\alpha^h(x)\,\partial_{x_3}\chi_{\alpha 3}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx+\frac{1}{h^2}\int_\Omega u_3^h(x)\,\partial_{x_3}\chi_{33}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\Big)\\
&=\lim_{h\to 0}\Big(\frac{h}{\varepsilon_h\gamma}-1\Big)\Big(-\sum_{\alpha=1,2}\frac{1}{h}\int_\Omega \partial_{x_3}u_\alpha^h(x)\,\chi_{\alpha 3}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx+\frac{1}{h^2}\int_\Omega u_3^h(x)\,\partial_{x_3}\chi_{33}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\Big)\\
&=\lim_{h\to 0}\Big(\frac{h}{\varepsilon_h\gamma}-1\Big)\Big(\sum_{\alpha=1,2}\frac{1}{h}\int_\Omega \partial_{x_\alpha}u_3^h(x)\,\chi_{\alpha 3}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx+\frac{1}{h^2}\int_\Omega u_3^h(x)\,\partial_{x_3}\chi_{33}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\Big)\\
&=\lim_{h\to 0}\Big(\frac{h}{\varepsilon_h\gamma}-1\Big)\Big(-\frac{1}{h}\int_\Omega u_3^h(x)\,(\partial_{x_1}\chi_{31}+\partial_{x_2}\chi_{32})\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx+\Big(\frac{h}{\varepsilon_h\gamma}+1\Big)\frac{1}{h^2}\int_\Omega u_3^h(x)\,\partial_{x_3}\chi_{33}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\Big)\\
&=0.
\end{aligned}\tag{3.33}$$

From (3.22), (3.23), (3.31), (3.32), (3.33) we have that

$$\int_{\Omega\times\mathscr{Y}}\chi(x,y):d\left(\lambda(x,y)-\begin{pmatrix}E\bar{u}(x')-x_3D^2u_3(x') & 0\\ 0 & 0\end{pmatrix}\otimes\mathscr{L}_y^2\right)=0.$$

From this and Proposition 3.1.8 we conclude that there exists $\mu\in\mathscr{X}_\gamma(\omega)$ such that

$$\lambda-\begin{pmatrix}E\bar{u}-x_3D^2u_3 & 0\\ 0 & 0\end{pmatrix}\otimes\mathscr{L}_y^2=\widetilde{E}_\gamma\mu.$$

From this we have the claim.

Step 2. We consider the case $\gamma=0$, i.e. $\frac{h}{\varepsilon_h}\to 0$.

By the Poincaré inequality in $L^1(I)$, there is a constant C independent of h such that

$$\int_I |u_3^h-\overline{u}_3^h|\,dx_3\le C\int_I|\partial_{x_3}u_3^h|\,dx_3,$$

for a.e. $x' \in \omega$. Integrating over ω we obtain that

$$\int_\Omega |u_3^h - \overline{u}_3^h| \, dx \leq C \int_\Omega |\partial_{x_3} u_3^h| \, dx \leq Ch^2. \tag{3.34}$$

Then we can define a sequence $\{\vartheta_3^h\}_{h>0}$ by

$$\vartheta_3^h(x) := \frac{u_3^h(x) - \overline{u}_3^h(x')}{h^2},$$

which is uniformly bounded in $L^1(\Omega)$. We can construct a sequence of antiderivatives $\{\theta_3^h\}_{h>0}$ by

$$\theta_3^h(x) := \int_{-\frac{1}{2}}^{x_3} \vartheta_3^h(x', z_3) \, dz_3 - C_{\vartheta_3^h},$$

where we choose $C_{\vartheta_3^h}$ such that $\overline{\theta}_3^h = 0$. Note that the constructed sequence is also uniformly bounded in $L^1(\Omega)$. Next, for $\alpha \in \{1,2\}$, we construct sequences $\{\theta_\alpha^h\}_{h>0}$ by

$$\theta_\alpha^h(x) := \frac{u_\alpha^h(x) - \overline{u}_\alpha^h(x') + x_3 \partial_{x_\alpha} \overline{u}_3^h(x')}{h} + h \partial_{x_\alpha} \theta_3^h(x).$$

Then $\overline{\theta}_\alpha^h = 0$ and

$$\partial_{x_3} \theta_\alpha^h = \frac{\partial_{x_3} u_\alpha^h + \partial_{x_\alpha} \overline{u}_3^h}{h} + h \partial_{x_\alpha} \vartheta_3^h = \frac{\partial_{x_3} u_\alpha^h + \partial_{x_\alpha} u_3^h}{h},$$

since $\partial_{x_3} \theta_3^h = \vartheta_3^h$. Thus, by the Poincaré inequality in $L^1(I)$ and integrating over ω, we obtain that

$$\int_\Omega |\theta_\alpha^h| \, dx \leq C \int_\Omega |\partial_{x_3} \theta_\alpha^h| \, dx \leq C. \tag{3.35}$$

From the above constructions, we have that

$$u_\alpha^h(x) = \overline{u}_\alpha^h(x') - x_3 \partial_{x_\alpha} \overline{u}_3^h(x') + h^2 \partial_{x_\alpha} \theta_3^h(x) + h \theta_\alpha^h(x), \quad \alpha = 1,2. \tag{3.36}$$

For the 2×2 minors of the scaled symmetrized gradients, a simple calculation then shows

$$\begin{aligned} &\int_{\Omega \times \mathscr{Y}} \chi(x,y) : d\lambda''(x,y) \\ &= \lim_{h \to 0} \int_\Omega \chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big) : \Big(E(\overline{u}^h)'(x') - x_3 D^2 \overline{u}_3^h(x') + h^2 D_{x'}^2 \theta_3^h(x) + h E_{x'}(\theta^h)'(x)\Big) \, dx, \end{aligned} \tag{3.37}$$

for every $\chi \in C_c^\infty(\omega; C^\infty(I \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}))$. Notice that the last two terms in (3.37) are negligible in the limit. Indeed, we have

$$\begin{aligned}
&\lim_{h\to 0} \int_\Omega \chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big) : h^2 D^2_{x'} \theta_3^h(x)\, dx \\
&= \lim_{h\to 0} h^2 \int_\Omega \theta_3^h(x)\, \mathrm{div}_{x'} \mathrm{div}_{x'} \left(\chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\right) dx \\
&= \lim_{h\to 0} h^2 \sum_{\alpha,\beta=1,2} \int_\Omega \theta_3^h(x)\, \partial_{x_\alpha} \left(\partial_{x_\beta} \chi_{\alpha\beta}\Big(x, \tfrac{x'}{\varepsilon_h}\Big) + \frac{1}{\varepsilon_h} \partial_{y_\beta} \chi_{\alpha\beta}\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\right) dx \\
&= \lim_{h\to 0} \sum_{\alpha,\beta=1,2} \int_\Omega \theta_3^h(x) \left(h^2 \partial_{x_\alpha x_\beta} \chi_{\alpha\beta}\Big(x, \tfrac{x'}{\varepsilon_h}\Big) + \frac{h^2}{\varepsilon_h} \partial_{y_\alpha x_\beta} \chi_{\alpha\beta}\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \right. \\
&\qquad\qquad \left. + \frac{h^2}{\varepsilon_h} \partial_{x_\alpha y_\beta} \chi_{\alpha\beta}\Big(x, \tfrac{x'}{\varepsilon_h}\Big) + \frac{h^2}{{\varepsilon_h}^2} \partial_{y_\alpha y_\beta} \chi_{\alpha\beta}\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \right) dx \\
&= 0. \qquad (3.38)
\end{aligned}$$

Similarly we compute

$$\begin{aligned}
&\lim_{h\to 0} \int_\Omega \chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big) : h E_{x'} (\theta^h)'(x)\, dx \\
&= -\lim_{h\to 0} h \int_\Omega (\theta^h)'(x) \cdot \mathrm{div}_{x'} \left(\chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\right) dx \\
&= -\lim_{h\to 0} \sum_{\alpha,\beta=1,2} \int_\Omega \theta_\alpha^h(x) \left(h \partial_{x_\beta} \chi_{\alpha\beta}\Big(x, \tfrac{x'}{\varepsilon_h}\Big) + \frac{h}{\varepsilon_h} \partial_{y_\beta} \chi_{\alpha\beta}\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \right) dx \\
&= 0. \qquad (3.39)
\end{aligned}$$

Thus, if we consider an open set $\widetilde{I} \supset I$ which compactly contains I, we can infer that

$$\Big(E_{\alpha\beta}(u^h)\Big)^\perp \xrightarrow{2-*} 0 \quad \text{two-scale weakly* in } \mathscr{M}_b(\omega \times \widetilde{I} \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}). \qquad (3.40)$$

Since $\{(\overline{u}^h)'\}$ is bounded in $BD(\omega)$ with $(\overline{u}^h)' \xrightarrow{*} \bar{u}$ weakly* in $BD(\omega)$, by [25, Proposition 4.10] there exists $\mu \in \mathscr{X}_0(\omega)$ such that

$$E(\overline{u}^h)' \xrightarrow{2-*} E\bar{u} \otimes \mathscr{L}^2_y + E_y \mu \quad \text{two-scale weakly* in } \mathscr{M}_b(\omega \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}). \qquad (3.41)$$

From the proof of Proposition 3.0.1 we know that we have

$$\begin{aligned}
&u_\alpha^h \to \bar{u}_\alpha - x_3 \partial_{x_\alpha} u_3, \quad \text{strongly in } L^1(\Omega), \quad \alpha = 1, 2, \\
&u_3^h \to u_3, \quad \text{strongly in } L^1(\Omega).
\end{aligned}$$

thus we infer that

$$\overline{u}_3^h \xrightarrow{2-*} u_3(x') \mathscr{L}^2_{x'} \otimes \mathscr{L}^2_y \quad \text{two-scale weakly* in } \mathscr{M}_b(\omega \times \mathscr{Y}) \qquad (3.42)$$

Further, multiplying (3.36) with x_3 and integrating over ω, we obtain

$$\partial_{x_\alpha}\overline{u}_3^h(x') = -\widehat{u}_\alpha^h(x') + h^2\partial_{x_\alpha}\widehat{\theta}_3^h(x') + h\widehat{\theta}_\alpha^h(x'), \quad \alpha = 1,2.$$

Using similar calculations as in (3.38) and (3.39), we can show that only the first term is not negligible in the limit, from which we conclude that, for any $\varphi \in C_c^\infty(\omega \times \mathscr{Y})$

$$\lim_{h\to 0}\int_\omega \partial_{x_\alpha}\overline{u}_3^h(x')\,\varphi\big(x', \tfrac{x'}{\varepsilon_h}\big)\,dx' = \int_{\omega\times\mathscr{Y}} \partial_{x_\alpha}u_3(x')\,\varphi\big(x',y\big)\,dx'dy, \quad \alpha = 1,2. \tag{3.43}$$

Consider $\chi \in C_c^\infty(\omega\times\mathscr{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ such that $\mathrm{div}_y\mathrm{div}_y\chi(x',y) = 0$. Then

$$\begin{aligned}
&\lim_{h\to 0}\int_\omega \chi\big(x', \tfrac{x'}{\varepsilon_h}\big) : D^2\overline{u}_3^h(x')\,dx' \\
&= \lim_{h\to 0}\int_\omega \overline{u}_3^h(x')\,\mathrm{div}_{x'}\mathrm{div}_{x'}\Big(\chi\big(x', \tfrac{x'}{\varepsilon_h}\big)\Big)\,dx' \\
&= \lim_{h\to 0}\sum_{\alpha,\beta=1,2}\int_\omega \overline{u}_3^h(x')\Big(\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\big(x', \tfrac{x'}{\varepsilon_h}\big) + \frac{1}{\varepsilon_h}\partial_{y_\alpha x_\beta}\chi_{\alpha\beta}\big(x', \tfrac{x'}{\varepsilon_h}\big) \\
&\qquad\qquad + \frac{1}{\varepsilon_h}\partial_{x_\alpha y_\beta}\chi_{\alpha\beta}\big(x', \tfrac{x'}{\varepsilon_h}\big) + \frac{1}{{\varepsilon_h}^2}\partial_{y_\alpha y_\beta}\chi_{\alpha\beta}\big(x', \tfrac{x'}{\varepsilon_h}\big)\Big)\,dx' \\
&= \lim_{h\to 0}\sum_{\alpha,\beta=1,2}\int_\omega \overline{u}_3^h(x')\Big(\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\big(x', \tfrac{x'}{\varepsilon_h}\big) + \frac{2}{\varepsilon_h}\partial_{y_\alpha x_\beta}\chi_{\alpha\beta}\big(x', \tfrac{x'}{\varepsilon_h}\big)\Big)\,dx' \\
&= \lim_{h\to 0}\sum_{\alpha,\beta=1,2}\int_\omega \overline{u}_3^h(x')\,\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\big(x', \tfrac{x'}{\varepsilon_h}\big)\,dx' + 2\int_\omega\Big(\partial_{x_\alpha}\Big(\overline{u}_3^h(x')\,\partial_{x_\beta}\chi_{\alpha\beta}\big(x', \tfrac{x'}{\varepsilon_h}\big)\Big) \\
&\qquad\qquad - \partial_{x_\alpha}\overline{u}_3^h(x')\,\partial_{x_\beta}\chi_{\alpha\beta}\big(x', \tfrac{x'}{\varepsilon_h}\big) - \overline{u}_3^h(x')\,\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\big(x', \tfrac{x'}{\varepsilon_h}\big)\Big)\,dx' \\
&= \lim_{h\to 0}\sum_{\alpha,\beta=1,2}\Big(-\int_\omega \overline{u}_3^h(x')\,\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\big(x', \tfrac{x'}{\varepsilon_h}\big)\,dx' - 2\int_\omega \partial_{x_\alpha}\overline{u}_3^h(x')\,\partial_{x_\beta}\chi_{\alpha\beta}\big(x', \tfrac{x'}{\varepsilon_h}\big)\,dx'\Big),
\end{aligned}$$

where in the last equality we used Green's theorem. Passing to the limit and using (3.42) and (3.43), we have

$$\begin{aligned}
&\lim_{h\to 0}\int_\omega \chi\big(x', \tfrac{x'}{\varepsilon_h}\big) : D^2\overline{u}_3^h(x')\,dx' \\
&= \sum_{\alpha,\beta=1,2}\Big(-\int_{\omega\times\mathscr{Y}} u_3(x')\,\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\,(x',y)\,dx'dy - 2\int_{\omega\times\mathscr{Y}} \partial_{x_\alpha}u_3(x')\,\partial_{x_\beta}\chi_{\alpha\beta}\,(x',y)\,dx'dy\Big) \\
&= \sum_{\alpha,\beta=1,2}\Big(-\int_{\omega\times\mathscr{Y}} u_3(x')\,\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\,(x',y)\,dx'dy \\
&\qquad - 2\int_{\omega\times\mathscr{Y}}\Big(\partial_{x_\alpha}\Big(u_3(x')\,\partial_{x_\beta}\chi_{\alpha\beta}\,(x',y)\Big) - u_3(x')\,\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\,(x',y)\Big)\,dx'dy\Big) \\
&= \sum_{\alpha,\beta=1,2}\int_{\omega\times\mathscr{Y}} u_3(x')\,\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\,(x',y)\,dx'dy \\
&= \int_{\omega\times\mathscr{Y}} \chi(x',y) : d\left(D^2u_3\otimes\mathscr{L}_y^2\right).
\end{aligned} \tag{3.44}$$

From (3.41), (3.44), (3.40) and Lemma 3.2.4, we have that

$$\lambda'' = E\bar{u} \otimes \mathscr{L}^2_y + E_y\mu - x_3 D^2 u_3 \otimes \mathscr{L}^2_y - x_3 D^2_y \kappa.$$

Finally, we consider the vector $\zeta^h(x)$ given by the third column of $\Lambda_h E u^h$, for every $h > 0$. The boundedness of the sequence of functions $v^h \in BD(\Omega^h)$ implies that $\{\zeta^h\}_{h>0}$ is a uniformly bounded sequence in $L^1(\Omega;\mathbb{R}^3)$. Consequently, we can extract a subsequence which two-scale weakly* converges in $\mathscr{M}_b(\Omega \times \mathscr{Y};\mathbb{R}^3)$ such that

$$\begin{aligned} &\frac{1}{h} E_{\alpha 3}(u^h) \xrightarrow{2-*} \zeta_\alpha \quad \text{two-scale weakly* in } \mathscr{M}_b(\Omega \times \mathscr{Y}), \quad \alpha = 1,2, \\ &\frac{1}{h^2} E_{33}(u^h) \xrightarrow{2-*} \zeta_3 \quad \text{two-scale weakly* in } \mathscr{M}_b(\Omega \times \mathscr{Y}), \end{aligned}$$

for a suitable $\zeta \in \mathscr{M}_b(\Omega \times \mathscr{Y};\mathbb{R}^3)$, which concludes the proof.

Step 3. Finally, we consider the case $\gamma = +\infty$, i.e. $\frac{\varepsilon_h}{h} \to 0$.

For the 2×2 minors of two-scale limit, by Proposition 3.2.5 and the proof Proposition 3.0.1, we have that there exists $\mu \in \mathscr{X}_\infty(\Omega)$ such that

$$\lambda'' = \left(E\bar{u} - x_3 D^2 u_3\right) \otimes \mathscr{L}^2_y + E_y \mu.$$

Let $\chi^{(1)} \in C^\infty_c(\Omega)$ and $\chi^{(2)} \in C^\infty(\mathscr{Y};\mathbb{M}^{2\times 2}_{\text{sym}})$ such that $\int_{\mathscr{Y}} \chi^{(2)}\, dy = 0$. We consider a test function $\chi(x,y) = \chi^{(1)}(x)\chi^{(2)}\big(\frac{x'}{\varepsilon_h}\big)$, such that

$$\int_{\Omega\times\mathscr{Y}} \chi(x,y) : d\lambda(x,y) = \lim_{h\to 0} \int_\Omega \chi^{(1)}(x)\chi^{(2)}\Big(\frac{x'}{\varepsilon_h}\Big) : d\left(\Lambda_h E u^h(x)\right).$$

For each $i = 1,2,3$, let G_i denote the unique solution in $C^\infty(\mathscr{Y})$ to the Poisson's equation

$$-\triangle_y G_i = \chi^{(2)}_{3i}, \qquad \int_{\mathscr{Y}} G_i\, dy = 0.$$

Then, if we consider the limit of

$$\int_{\Omega\times\mathscr{Y}} \chi_{33}(x,y) : d\lambda_{33}(x,y) = \lim_{h\to 0} \frac{1}{h^2} \int_\Omega \partial_{x_3} u^h_3(x)\, \chi^{(1)}(x)\chi^{(2)}_{33}\Big(\frac{x'}{\varepsilon_h}\Big)\, dx,$$

we have

$$\begin{aligned}
&\int_{\Omega\times\mathscr{Y}} \chi_{33}(x,y) : d\lambda_{33}(x,y) \\
&= -\lim_{h\to 0}\frac{1}{h^2}\sum_{\alpha=1,2}\int_\Omega \partial_{x_3}u_3^h(x)\,\chi^{(1)}(x)\partial_{y_\alpha y_\alpha}G_3\big(\tfrac{x'}{\varepsilon_h}\big)\,dx \\
&= \lim_{h\to 0}\frac{1}{h^2}\sum_{\alpha=1,2}\int_\Omega u_3^h(x)\,\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha y_\alpha}G_3\big(\tfrac{x'}{\varepsilon_h}\big)\,dx \\
&= \lim_{h\to 0}\frac{\varepsilon_h}{h^2}\sum_{\alpha=1,2}\Big(\int_\Omega u_3^h(x)\,\partial_{x_\alpha}\Big(\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_3\big(\tfrac{x'}{\varepsilon_h}\big)\Big)\,dx - \int_\Omega u_3^h(x)\,\partial_{x_\alpha x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_3\big(\tfrac{x'}{\varepsilon_h}\big)\,dx\Big) \\
&= \lim_{h\to 0}\frac{\varepsilon_h}{h^2}\sum_{\alpha=1,2}\Big(-\int_\Omega \partial_{x_\alpha}u_3^h(x)\,\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_3\big(\tfrac{x'}{\varepsilon_h}\big)\,dx + \int_\Omega \partial_{x_3}u_3^h(x)\,\partial_{x_\alpha}\chi^{(1)}(x)\partial_{y_\alpha}G_3\big(\tfrac{x'}{\varepsilon_h}\big)\,dx\Big).
\end{aligned}$$

Recalling (3.20) and (3.21), we have

$$\begin{aligned}
&\int_{\Omega\times\mathscr{Y}} \chi_{33}(x,y) : d\lambda_{33}(x,y) \\
&= \lim_{h\to 0}\frac{\varepsilon_h}{h^2}\sum_{\alpha=1,2}\int_\Omega \partial_{x_3}u_\alpha^h(x)\,\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_3\big(\tfrac{x'}{\varepsilon_h}\big)\,dx \\
&= -\lim_{h\to 0}\frac{\varepsilon_h}{h^2}\sum_{\alpha=1,2}\int_\Omega u_\alpha^h(x)\,\partial_{x_3x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_3\big(\tfrac{x'}{\varepsilon_h}\big)\,dx \\
&= -\lim_{h\to 0}\frac{{\varepsilon_h}^2}{h^2}\sum_{\alpha=1,2}\Big(\int_\Omega u_\alpha^h(x)\,\partial_{x_\alpha}\Big(\partial_{x_3x_3}\chi^{(1)}(x)G_3\big(\tfrac{x'}{\varepsilon_h}\big)\Big)\,dx - \int_\Omega u_3^h(x)\,\partial_{x_\alpha x_3x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_3\big(\tfrac{x'}{\varepsilon_h}\big)\,dx\Big) \\
&= \lim_{h\to 0}\frac{{\varepsilon_h}^2}{h^2}\sum_{\alpha=1,2}\int_\Omega \partial_{x_\alpha}u_\alpha^h(x)\,\partial_{x_3x_3}\chi^{(1)}(x)G_3\big(\tfrac{x'}{\varepsilon_h}\big)\,dx \\
&= 0. \qquad (3.45)
\end{aligned}$$

Thus, recalling that $\int_{\mathscr{Y}}\chi_{33}^{(2)}\,dy = 0$, and since for arbitrary test function we can subtract its mean value over $\mathscr{Y}$ to obtain a function with mean value zero, we infer that there exists $\zeta_3 \in \mathscr{M}_b(\Omega)$ such that

$$\lambda_{33} = \zeta_3 \otimes \mathscr{L}_y^2.$$

Similarly, if we consider the limit of

$$\begin{aligned}
&\int_{\Omega\times\mathscr{Y}} \chi_{13}(x,y) : d\lambda_{13}(x,y) + \int_{\Omega\times\mathscr{Y}} \chi_{23}(x,y) : d\lambda_{23}(x,y) \\
&= \lim_{h\to 0}\frac{1}{2h}\sum_{\alpha=1,2}\int_\Omega \Big(\partial_{x_\alpha}u_3^h(x) + \partial_{x_3}u_\alpha^h(x)\Big)\,\chi^{(1)}(x)\chi_{3\alpha}^{(2)}\big(\tfrac{x'}{\varepsilon_h}\big)\,dx,
\end{aligned}$$

we can write

$$\int_{\Omega\times\mathscr{Y}} \chi_{13}(x,y) : d\lambda_{13}(x,y) + \int_{\Omega\times\mathscr{Y}} \chi_{23}(x,y) : d\lambda_{23}(x,y)$$
$$= \lim_{h\to 0} \frac{1}{2h} \sum_{\alpha,\beta=1,2} \left(\int_\Omega \partial_{x_\alpha} u_3^h(x)\, \chi^{(1)}(x) \partial_{y_\beta y_\beta} G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\, dx + \int_\Omega \partial_{x_3} u_\alpha^h(x)\, \chi^{(1)}(x) \partial_{y_\beta y_\beta} G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\, dx \right). \tag{3.46}$$

Suppose now that $\mathrm{div}_y \chi_{3\alpha}^{(2)} = 0$, i.e. $\sum_{\alpha,\beta=1,2} \partial_{y_\alpha y_\beta y_\beta} G_\alpha = 0$. Then we have

$$\lim_{h\to 0} \frac{1}{2h} \sum_{\alpha,\beta=1,2} \int_\Omega \partial_{x_\alpha} u_3^h(x)\, \chi^{(1)}(x) \partial_{y_\beta y_\beta} G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\, dx$$
$$= \lim_{h\to 0} \frac{1}{2h} \sum_{\alpha,\beta=1,2} \left(-\int_\Omega u_3^h(x)\, \partial_{x_\alpha} \chi^{(1)}(x) \partial_{y_\beta y_\beta} G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\, dx - \frac{1}{\varepsilon_h} \int_\Omega u_3^h(x)\, \chi^{(1)}(x) \partial_{y_\alpha y_\beta y_\beta} G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\, dx \right)$$
$$= -\lim_{h\to 0} \frac{1}{2h} \sum_{\alpha,\beta=1,2} \int_\Omega u_3^h(x)\, \partial_{x_\alpha} \chi^{(1)}(x) \partial_{y_\beta y_\beta} G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\, dx$$
$$= \lim_{h\to 0} \frac{\varepsilon_h}{2h} \sum_{\alpha,\beta=1,2} \left(\int_\Omega \partial_{x_\beta} u_3^h(x)\, \partial_{x_\alpha} \chi^{(1)}(x) \partial_{y_\beta} G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\, dx + \int_\Omega u_3^h(x)\, \partial_{x_\alpha x_\beta} \chi^{(1)}(x) \partial_{y_\beta} G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\, dx \right)$$
$$= \lim_{h\to 0} \frac{\varepsilon_h}{2h} \sum_{\alpha,\beta=1,2} \int_\Omega \partial_{x_\beta} u_3^h(x)\, \partial_{x_\alpha} \chi^{(1)}(x) \partial_{y_\beta} G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\, dx$$
$$= -\lim_{h\to 0} \frac{\varepsilon_h}{2h} \sum_{\alpha,\beta=1,2} \int_\Omega \partial_{x_3} u_\beta^h(x)\, \partial_{x_\alpha} \chi^{(1)}(x) \partial_{y_\beta} G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\, dx$$
$$= \lim_{h\to 0} \frac{\varepsilon_h}{2h} \sum_{\alpha,\beta=1,2} \int_\Omega u_\beta^h(x)\, \partial_{x_\alpha x_3} \chi^{(1)}(x) \partial_{y_\beta} G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\, dx$$
$$= 0. \tag{3.47}$$

Furthermore,

$$
\begin{aligned}
&\lim_{h\to 0}\frac{1}{2h}\sum_{\alpha,\beta=1,2}\int_\Omega \partial_{x_3}u^h_\alpha(x)\,\chi^{(1)}(x)\partial_{y_\beta y_\beta}G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\,dx\\
&=-\lim_{h\to 0}\frac{1}{2h}\sum_{\alpha,\beta=1,2}\int_\Omega u^h_\alpha(x)\,\partial_{x_3}\chi^{(1)}(x)\partial_{y_\beta y_\beta}G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\,dx\\
&=\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\Big(\int_\Omega \partial_{x_\beta}u^h_\alpha(x)\,\partial_{x_3}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\,dx+\int_\Omega u^h_\alpha(x)\,\partial_{x_\beta x_3}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\,dx\Big)\\
&=\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\int_\Omega \partial_{x_\beta}u^h_\alpha(x)\,\partial_{x_3}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\,dx\\
&=-\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\int_\Omega \partial_{x_\alpha}u^h_\beta(x)\,\partial_{x_3}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\,dx\\
&=\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\Big(\int_\Omega u^h_\beta(x)\,\partial_{x_\alpha x_3}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)+\frac{1}{\varepsilon_h}\int_\Omega u^h_\beta(x)\,\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha y_\beta}G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\,dx\Big)\\
&=\lim_{h\to 0}\frac{1}{2h}\sum_{\alpha,\beta=1,2}\int_\Omega u^h_\beta(x)\,\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha y_\beta}G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\,dx\\
&=-\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\Big(\int_\Omega \partial_{x_\beta}u^h_\beta(x)\,\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\,dx+\int_\Omega u^h_\beta(x)\,\partial_{x_\beta x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_\alpha\Big(\tfrac{x'}{\varepsilon_h}\Big)\,dx\Big)\\
&=0. \qquad (3.48)
\end{aligned}
$$

From (3.46), (3.47) and (3.48), and Proposition 3.1.17, and recalling that $\int_{\mathscr{Y}}\chi^{(2)}_{13}\,dy=0$ and $\int_{\mathscr{Y}}\chi^{(2)}_{23}\,dy=0$, we can conclude that there exist $\kappa\in\Upsilon_\infty(\Omega)$ and $\zeta'\in\mathscr{M}_b(\Omega;\mathbb{R}^2)$ such that

$$
\begin{pmatrix}\lambda_{13}\\ \lambda_{23}\end{pmatrix}=\zeta'\otimes\mathscr{L}^2_y+D_y\kappa.
$$

which concludes the proof. ■

3.4. Unfolding adapted to dimension reduction

Here we follow [25, Section 4.3].

For every $\varepsilon > 0$ and $i \in \mathbb{Z}^2$, let

$$Q^i_\varepsilon := \left\{ x \in \mathbb{R}^2 : \frac{x - \varepsilon i}{\varepsilon} \in Y \right\}.$$

Given an open set $\omega \subseteq \mathbb{R}^2$, we will set

$$I_\varepsilon(\omega) := \left\{ i \in \mathbb{Z}^2 : Q^i_\varepsilon \subset \omega \right\}.$$

Given $\mu_\varepsilon \in \mathcal{M}_b(\omega \times I)$ and $Q^i_\varepsilon \subset \omega$, we define $\mu^i_\varepsilon \in \mathcal{M}_b(\mathcal{Y} \times I)$ such that

$$\int_{\mathcal{Y} \times I} \psi(y, x_3)\, d\mu^i_\varepsilon(y, x_3) = \frac{1}{\varepsilon^2} \int_{Q^i_\varepsilon \times I} \psi\left(\frac{x'}{\varepsilon}, x_3\right) d\mu_\varepsilon(x), \quad \psi \in C(\mathcal{Y} \times I).$$

Definition 3.4.1. Given $\varepsilon > 0$, *the unfolding measure* associated with μ_ε is the measure $\tilde{\lambda}_\varepsilon \in \mathcal{M}_b(\omega \times \mathcal{Y} \times I)$ defined by

$$\tilde{\lambda}_\varepsilon := \sum_{i \in I_\varepsilon(\omega)} \left(\mathcal{L}^2_{x'} \lfloor Q^i_\varepsilon \right) \otimes \mu^i_\varepsilon.$$

The following proposition gives the relationship between the two-scale weak* convergence and unfolding measures. The proof is analogous to [25, Proposition 4.11.].

Proposition 3.4.2. Let $\omega \subseteq \mathbb{R}^2$ be an open set and let $\{\mu_\varepsilon\} \subset \mathcal{M}_b(\omega \times I)$ be a bounded family such that

$$\mu_\varepsilon \xrightarrow{2-*} \mu_0 \quad \text{two-scale weakly* in } \mathcal{M}_b(\omega \times \mathcal{Y} \times I).$$

Let $\{\tilde{\lambda}_\varepsilon\} \subset \mathcal{M}_b(\omega \times \mathcal{Y} \times I)$ be the associated family of unfolding measure with $\{\mu_\varepsilon\}$. Then

$$\tilde{\lambda}_\varepsilon \xrightarrow{*} \mu_0 \quad \text{weakly* in } \mathcal{M}_b(\omega \times \mathcal{Y} \times I).$$

To analyze the sequences of symmetrized scaled gradients of *BD* function in the context of unfolding, we will need to consider auxiliary spaces

$$BD_{\frac{h}{\varepsilon}}(\mathcal{Y} \times I) := \left\{ u \in L^1(\mathcal{Y} \times I; \mathbb{R}^3) : \widetilde{E}_{\frac{h}{\varepsilon}} u \in \mathcal{M}_b(\mathcal{Y} \times I; \mathbb{M}^{3\times3}_{\text{sym}}) \right\},$$

$$BD_{\frac{h}{\varepsilon}}\left((0,1)^2 \times I\right) := \left\{ u \in L^1\left((0,1)^2 \times I; \mathbb{R}^3\right) : E_{\frac{h}{\varepsilon}} u \in \mathcal{M}_b\left((0,1)^2 \times I; \mathbb{M}^{3\times3}_{\text{sym}}\right) \right\},$$

where $\widetilde{E}_{\frac{h}{\varepsilon}}$ and $E_{\frac{h}{\varepsilon}}$ denote the distributional symmetrized scaled gradients of the form

$$\widetilde{E}_{\frac{h}{\varepsilon}}u(y,x_3) := \operatorname{sym}\left[\, D_y u(y,x_3) \,\middle|\, \tfrac{\varepsilon}{h}\partial_{x_3}u(y,x_3)\,\right],$$
$$E_{\frac{h}{\varepsilon}}u(x',x_3) := \operatorname{sym}\left[\, D_{x'} u(x',x_3) \,\middle|\, \tfrac{\varepsilon}{h}\partial_{x_3}u(x',x_3)\,\right].$$

Similarly as in Remark 3.1.3, scaling in the the first two components shows that these auxiliary spaces are equivalent to the usual BD space on the appropriate domain.

Proposition 3.4.3. Let $\omega \subseteq \mathbb{R}^2$ be an open set and let $\mathscr{B} \subseteq \mathscr{Y}$ be an open set with Lipschitz boundary. Let $\gamma_0 \in (0,1]$ and let h, $\varepsilon > 0$ be such that

$$\gamma_0 \leq \frac{h}{\varepsilon} \leq \frac{1}{\gamma_0}.$$

If $u_\varepsilon \in BD(\omega \times I)$, the unfolding measure associated with $\Lambda_h E u_\varepsilon \lfloor (\mathscr{B}_\varepsilon \setminus \mathscr{C}_\varepsilon) \times I$ is given by

$$\sum_{i \in I_\varepsilon(\omega)} \left(\mathscr{L}^2_{x'} \lfloor Q^i_\varepsilon\right) \otimes \widetilde{E}_{\frac{h}{\varepsilon}} \hat{u}^i_{h,\varepsilon} \lfloor (\mathscr{B} \setminus \mathscr{C}) \times I, \tag{3.49}$$

where $\hat{u}^i_{h,\varepsilon} \in BD_{\frac{h}{\varepsilon}}(\mathscr{Y} \times I)$ is such that

$$\int_{\partial\mathscr{B} \times I} |\hat{u}^i_{h,\varepsilon}|\, d\mathscr{H}^2 + |\widetilde{E}_{\frac{h}{\varepsilon}} \hat{u}^i_{h,\varepsilon}|\left((\mathscr{B} \cap \mathscr{C}) \times I\right) \leq \frac{C}{\varepsilon^2} |\Lambda_h E u_\varepsilon| \left(int(Q^i_\varepsilon) \times I\right), \tag{3.50}$$

for some constant C independent of i, h and ε.

Proof. Since $\mathscr{B}_\varepsilon$ has Lipschitz boundary, $u_\varepsilon \mathbb{1}_{\mathscr{B}_\varepsilon \times I} \in BD_{loc}(\omega \times I)$ with

$$E u_\varepsilon \lfloor \mathscr{B}_\varepsilon \times I = E\left(u_\varepsilon \mathbb{1}_{\mathscr{B}_\varepsilon \times I}\right) + [u_\varepsilon \lfloor \partial\mathscr{B}_\varepsilon \times I \odot \nu]\, \mathscr{H}^2 \lfloor \partial\mathscr{B}_\varepsilon \times I,$$

where $u_\varepsilon \lfloor \partial\mathscr{B}_\varepsilon \times I$ denotes the trace of $u_\varepsilon \mathbb{1}_{\mathscr{B}_\varepsilon \times I}$ on $\partial\mathscr{B}_\varepsilon \times I$, while ν is the exterior normal to $\partial\mathscr{B}_\varepsilon \times I$. We note that ν has the third component zero.

Remark that $\mathscr{C}_\varepsilon = \left(\cup_i \partial Q^i_\varepsilon\right) \cap \omega$. Accordingly, for $i \in I_\varepsilon(\omega)$ and $\psi \in C^1(\mathscr{Y} \times I; \mathbb{M}^{3\times3}_{\mathrm{sym}})$,

$$\int_{Q^i_\varepsilon \times I} \psi\left(\frac{x'}{\varepsilon}, x_3\right) : d\left(\Lambda_h E u_\varepsilon \lfloor (\mathscr{B}_\varepsilon \setminus \mathscr{C}_\varepsilon) \times I\right)(x) = \int_{int(Q^i_\varepsilon) \times I} \psi\left(\frac{x'}{\varepsilon}, x_3\right) : d\left(\Lambda_h E u_\varepsilon \lfloor \mathscr{B}_\varepsilon \times I\right)(x)$$
$$= \int_{int(Q^i_\varepsilon) \times I} \psi\left(\frac{x'}{\varepsilon}, x_3\right) : d\Lambda_h E\left(u_\varepsilon \mathbb{1}_{\mathscr{B}_\varepsilon \times I}\right)(x)$$
$$+ \int_{int(Q^i_\varepsilon) \times I} \psi\left(\frac{x'}{\varepsilon}, x_3\right) : \Lambda_h [u_\varepsilon \lfloor \partial\mathscr{B}_\varepsilon \times I \odot \nu]\, d\mathscr{H}^2 \lfloor \partial\mathscr{B}_\varepsilon \times I(x).$$

We set $v^i_{h,\varepsilon}(x) := \operatorname{diag}\left(1,1,\frac{1}{h}\right) u_\varepsilon(\varepsilon i+\varepsilon x', x_3)$ for $x \in (0,1)^2 \times I$. Then $v^i_{h,\varepsilon} \in BD_{\frac{h}{\varepsilon}}\left((0,1)^2 \times I\right)$, and it is easy to check that $E_{\frac{h}{\varepsilon}} v^i_{h,\varepsilon}(x) = \varepsilon \Lambda_h E u_\varepsilon(\varepsilon i + \varepsilon x', x_3)$. With change of variables we have

$$\begin{aligned}
&\int_{Q^i_\varepsilon \times I} \psi\left(\frac{x'}{\varepsilon}, x_3\right) : d\left(\Lambda_h E u_\varepsilon \lfloor (\mathscr{B}_\varepsilon \setminus \mathscr{C}_\varepsilon) \times I\right)(x) \\
&= \varepsilon \int_{(0,1)^2 \times I} \psi(x) : dE_{\frac{h}{\varepsilon}}\left(v^i_{h,\varepsilon} \mathbb{1}_{\mathscr{I}(\mathscr{B}) \times I}\right)(x) \\
&\quad + \varepsilon \int_{(0,1)^2 \times I} \psi(x) : \Lambda_h \left[\operatorname{diag}(1,1,h)\, v^i_{h,\varepsilon} \lfloor \mathscr{I}(\partial \mathscr{B}) \times I \odot \nu\right] d\mathscr{H}^2(x) \\
&= \varepsilon \int_{(0,1)^2 \times I} \psi(x) : dE_{\frac{h}{\varepsilon}}\left(v^i_{h,\varepsilon} \mathbb{1}_{\mathscr{I}(\mathscr{B}) \times I}\right)(x) + \varepsilon \int_{(0,1)^2 \times I} \psi(x) : \left[v^i_{h,\varepsilon} \lfloor \mathscr{I}(\partial \mathscr{B}) \times I \odot \nu\right] d\mathscr{H}^2(x).
\end{aligned}$$

Notice that we can assume that

$$\int_{\partial(0,1)^2 \times I} |v^i_{h,\varepsilon} \lfloor \partial(0,1)^2 \times I| \, d\mathscr{H}^2 \leq C |E_{\frac{h}{\varepsilon}} v^i_{h,\varepsilon}|\left((0,1)^2 \times I\right) = \frac{C}{\varepsilon} |\Lambda_h E u_\varepsilon|\left(int(Q^i_\varepsilon) \times I\right),$$

for some constant C independent of i, h and ε. Note that subtracting a rigid deformation to u_ε on $Q^i_\varepsilon \times I$ corresponds to subtracting an element of the kernel of $E_{\frac{h}{\varepsilon}}$ and E to $v^i_{h,\varepsilon}$ and w, respectively, which does not modify the calculations done thus far. Hence, we can use the trace theorem and Poincaré-Korn's inequality in $BD\left((0,1)^2 \times I\right)$ to get the desired inequality.

We define $\hat{u}^i_{h,\varepsilon}(y, x_3) := \frac{1}{\varepsilon} v^i_{h,\varepsilon}\left(\mathscr{I}(y), x_3\right)$. Then we obtain

$$\begin{aligned}
|\widetilde{E}_{\frac{h}{\varepsilon}} \hat{u}^i_{h,\varepsilon}|(\mathscr{Y} \times I) &\leq \int_{\mathscr{C} \times I} |\hat{u}^i_{h,\varepsilon} \lfloor \mathscr{C} \times I| \, d\mathscr{H}^2 + |\widetilde{E}_{\frac{h}{\varepsilon}} \hat{u}^i_{h,\varepsilon}|\left((\mathscr{Y} \setminus \mathscr{C}) \times I\right) \\
&= \frac{1}{\varepsilon} \int_{\partial(0,1)^2 \times I} |v^i_{h,\varepsilon} \lfloor \partial(0,1)^2 \times I| \, d\mathscr{H}^2 + \frac{1}{\varepsilon} |E_{\frac{h}{\varepsilon}} v^i_{h,\varepsilon}|\left((0,1)^2 \times I\right) \\
&\leq \frac{C+1}{\varepsilon} |E_{\frac{h}{\varepsilon}} v^i_{h,\varepsilon}|\left((0,1)^2 \times I\right) = \frac{C+1}{\varepsilon^2} |\Lambda_h E u_\varepsilon|\left(int(Q^i_\varepsilon) \times I\right).
\end{aligned}$$

Furthermore,

$$\varepsilon \int_{(0,1)^2 \times I} \psi : dE_{\frac{h}{\varepsilon}}\left(v^i_{h,\varepsilon} \mathbb{1}_{\mathscr{I}(\mathscr{B}) \times I}\right) = \varepsilon^2 \int_{(\mathscr{Y} \setminus \mathscr{C}) \times I} \psi : d\widetilde{E}_{\frac{h}{\varepsilon}}\left(\hat{u}^i_{h,\varepsilon} \mathbb{1}_{\mathscr{B} \times I}\right)$$

and

$$\varepsilon \int_{(0,1)^2 \times I} \psi : \left[v^i_{h,\varepsilon} \lfloor \mathscr{I}(\partial \mathscr{B}) \times I \odot \nu\right] d\mathscr{H}^2 = \varepsilon^2 \int_{(\mathscr{Y} \setminus \mathscr{C}) \times I} \psi : \left[\hat{u}^i_{h,\varepsilon} \lfloor (\partial \mathscr{B} \setminus \mathscr{C}) \times I \odot \nu\right] d\mathscr{H}^2.$$

So we get

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_{Q_\varepsilon^i\times I}\psi\left(\frac{x'}{\varepsilon},x_3\right):d\left(\Lambda_h E u_\varepsilon\lfloor(\mathscr{B}_\varepsilon\setminus\mathscr{C}_\varepsilon)\times I\right)(x)\\
&=\int_{(\mathscr{Y}\setminus\mathscr{C})\times I}\psi(y,x_3):d\widetilde{E}_{\frac{h}{\varepsilon}}\left(\hat{u}^i_{h,\varepsilon}\mathbb{1}_{\mathscr{B}\times I}\right)(y,x_3)\\
&\quad+\int_{(\mathscr{Y}\setminus\mathscr{C})\times I}\psi(y,x_3):\left[\hat{u}^i_{h,\varepsilon}\lfloor(\partial\mathscr{B}\setminus\mathscr{C})\times I\odot\nu\right]d\mathscr{H}^2(y,x_3)\\
&=\int_{\mathscr{Y}\times I}\psi(y,x_3):d\widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}\lfloor(\mathscr{B}\setminus\mathscr{C})\times I(y,x_3),
\end{aligned}$$

from which (3.49) follows. It remains to prove (3.50). Again, up to adding a rigid body motion to $\hat{u}^i_{h,\varepsilon}$ on $\mathscr{B}\times I$, we can assume

$$\begin{aligned}
&\int_{\partial\mathscr{B}\times I}|\hat{u}^i_{h,\varepsilon}|\,d\mathscr{H}^2+|\widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}|\left((\mathscr{B}\cap\mathscr{C})\times I\right)\\
&\leq C|\widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}|\left(\mathscr{B}\times I\right)+|\widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}|\left((\mathscr{B}\cap\mathscr{C})\times I\right)\leq C|\widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}|\left(\mathscr{Y}\times I\right)\\
&\leq\frac{C}{\varepsilon^2}|\Lambda_h E u_\varepsilon|\left(int(Q_\varepsilon^i)\times I\right).
\end{aligned}$$

This concludes the proof of the theorem. ■

The prior result can be used to prove the following lemma, which in turn will be used in the proof of the lower semicontinuity of $\mathscr{H}^{hom}$ in Section 4.5.

Lemma 3.4.4. Let $\mathscr{B}\subseteq\mathscr{Y}$ be an open set with Lipschitz boundary, such that $\partial\mathscr{B}\setminus\mathscr{T}$ is a C^1-hypersurface, for some compact set $\mathscr{T}$ with $\mathscr{H}^1(\mathscr{T})=0$. Additionally we assume that $\partial\mathscr{B}\cap\mathscr{C}\subseteq\mathscr{T}$. Let $v^h\in BD(\Omega)$ be such that

$$v^h\xrightarrow{*}v\quad\text{weakly* in } BD(\Omega)$$

and

$$\Lambda_h E v^h\lfloor\Omega\cap(\mathscr{B}_{\varepsilon_h}\times I)\xrightarrow{2-*}\pi\quad\text{two-scale weakly* in } \mathscr{M}_b(\Omega\times\mathscr{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}}).$$

Then π is supported in $\Omega\times\bar{\mathscr{B}}$ and

$$\pi\lfloor\Omega\times(\partial\mathscr{B}\setminus\mathscr{T})=a(x,y)\odot\nu(y)\,\zeta,\tag{3.51}$$

where $\zeta\in\mathscr{M}_b^+(\Omega\times(\partial\mathscr{B}\setminus\mathscr{T}))$, $a:\Omega\times(\partial\mathscr{B}\setminus\mathscr{T})\to\mathbb{R}^3$ is a Borel map, and ν is the exterior normal to $\partial\mathscr{B}$.

Proof. Denote by $\tilde{\pi} \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\text{sym}})$ the two-scale weak* limit (up to a subsequence) of

$$\Lambda_h E v^h \lfloor \Omega \cap ((\mathcal{B}_{\varepsilon_h} \setminus \mathcal{C}_{\varepsilon_h}) \times I) \in \mathcal{M}_b(\Omega; \mathbb{M}^{3\times3}_{\text{sym}}).$$

Then it is enough to prove the analogue of (3.51) for $\tilde{\pi}$. Indeed, the two-scale weak* limit (up to a subsequence) of

$$\Lambda_h E v^h \lfloor \Omega \cap ((\mathcal{B}_{\varepsilon_h} \cap \mathcal{C}_{\varepsilon_h}) \times I) \in \mathcal{M}_b(\Omega; \mathbb{M}^{3\times3}_{\text{sym}})$$

is supported on $\Omega \times \overline{\mathcal{B} \cap \mathcal{C}}$. Since by assumption $\partial\mathcal{B} \cap \mathcal{C} \subseteq \mathcal{T}$, we have that $\partial\mathcal{B} \setminus \mathcal{T}$ and $\overline{\mathcal{B} \cap \mathcal{C}}$ are disjoint sets, which implies

$$\pi \lfloor \Omega \times (\partial\mathcal{B} \setminus \mathcal{T}) = \tilde{\pi} \lfloor \Omega \times (\partial\mathcal{B} \setminus \mathcal{T}).$$

By Theorem 3.4.3 we have that the unfolding measure associated with $\Lambda_h E v^h \lfloor (\mathcal{B}_{\varepsilon_h} \setminus \mathcal{C}_{\varepsilon_h}) \times I$ is given by

$$\sum_{i \in I_{\varepsilon_h}(\omega)} \left(\mathcal{L}^2_{x'} \lfloor Q^i_{\varepsilon_h} \right) \otimes \widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h} \lfloor (\mathcal{B} \setminus \mathcal{C}) \times I, \tag{3.52}$$

where $\hat{v}^i_{\varepsilon_h} \in BD(\mathcal{Y} \times I)$ are such that

$$\int_{\partial\mathcal{B} \times I} |\hat{v}^i_{\varepsilon_h}| \, d\mathcal{H}^2 + |\widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h}| \left((\mathcal{B} \cap \mathcal{C}) \times I \right) \leq \frac{C}{\varepsilon_h{}^2} |\Lambda_h E v^h| \left(int(Q^i_{\varepsilon_h}) \times I \right). \tag{3.53}$$

Further, by Theorem 3.4.2, the family of associated measures in (3.52) converge weakly* to $\tilde{\pi}$ in $\mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\text{sym}})$. Then, for every $\chi \in C^\infty_c(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\text{sym}})$ with $\widetilde{\operatorname{div}}_\gamma \chi(x,y) = 0$, we get

$$\begin{aligned}
&\int_{\Omega \times \mathcal{Y}} \chi(x,y) : d\tilde{\pi}(x,y) \\
&= \lim_h \int_{\Omega \times \mathcal{Y}} \chi(x,y) : d \left(\sum_{i \in I_{\varepsilon_h}(\omega)} \left(\mathcal{L}^2_{x'} \lfloor Q^i_{\varepsilon_h} \right) \otimes \widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h} \lfloor (\mathcal{B} \setminus \mathcal{C}) \times I \right) \\
&= \lim_h \sum_{i \in I_{\varepsilon_h}(\omega)} \int_{Q^i_{\varepsilon_h}} \left(\int_{(\mathcal{B} \setminus \mathcal{C}) \times I} \chi(x,y) : d\widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h} \right) dx' \\
&= \lim_h \sum_{i \in I_{\varepsilon_h}(\omega)} \int_{Q^i_{\varepsilon_h}} \left(\int_{\mathcal{B} \times I} \chi(x,y) : d\widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h} - \int_{(\mathcal{B} \cap \mathcal{C}) \times I} \chi(x,y) : d\widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h} \right) dx'.
\end{aligned}$$

If we denote by $\widetilde{\operatorname{div}}_{\frac{h}{\varepsilon_h}}$ the scaled divergence operator associated with $\widetilde{E}_{\frac{h}{\varepsilon_h}}$, then by the integration by parts formula for BD functions over $\mathscr{B}\times I$ we have

$$\begin{aligned}
&\int_{\Omega\times\mathscr{Y}}\chi(x,y):d\tilde{\pi}(x,y)\\
&=\lim_h\sum_{i\in I_{\varepsilon_h}(\omega)}\int_{Q^i_{\varepsilon_h}}\Bigg(-\int_{\mathscr{B}\times I}\widetilde{\operatorname{div}}_{\frac{h}{\varepsilon_h}}\chi(x,y)\cdot\hat{v}^i_{\varepsilon_h}(y,x_3)\,dydx_3+\int_{\partial\mathscr{B}\times I}\chi(x,y):\left[\hat{v}^i_{\varepsilon_h}(y,x_3)\odot\nu\right]d\mathscr{H}^2(y,x_3)\\
&\qquad-\int_{(\mathscr{B}\cap\mathscr{C})\times I}\chi(x,y):d\widetilde{E}_{\frac{h}{\varepsilon_h}}\hat{v}^i_{\varepsilon_h}\Bigg)dx'\\
&=\lim_h\sum_{i\in I_{\varepsilon_h}(\omega)}\int_{Q^i_{\varepsilon_h}}\Bigg(-\left(\frac{\varepsilon_h}{h}-\frac{1}{\gamma}\right)\int_{\mathscr{B}\times I}\partial_{x_3}\chi(x,y)\cdot\hat{v}^i_{\varepsilon_h}(y,x_3)\,dydx_3\\
&\qquad+\int_{\partial\mathscr{B}\times I}\chi(x,y):\left[\hat{v}^i_{\varepsilon_h}(y,x_3)\odot\nu\right]d\mathscr{H}^2(y,x_3)-\int_{(\mathscr{B}\cap\mathscr{C})\times I}\chi(x,y):d\widetilde{E}_{\frac{h}{\varepsilon_h}}\hat{v}^i_{\varepsilon_h}\Bigg)dx'.
\end{aligned}$$

Owing to Poincaré-Korn's inequality on $BD(\mathscr{Y}\times I)$ and (3.53), we conclude that the integrals $\int_{\mathscr{B}\times I}\partial_{x_3}\chi(x,y)\cdot\hat{v}^i_{\varepsilon_h}(y,x_3)\,dydx_3$ are bounded. Further, in view of (3.53) we can rewrite the above limit as

$$\begin{aligned}
&\int_{\Omega\times\mathscr{Y}}\chi(x,y):d\tilde{\pi}(x,y)\\
&=\lim_h\left(\int_{\Omega\times\mathscr{Y}}\chi(x,y):d\lambda_1^h(x,y)+\int_{\Omega\times\mathscr{Y}}\chi(x,y):d\lambda_2^h(x,y)\right),
\end{aligned}\tag{3.54}$$

with $\lambda_1^h,\lambda_2^h\in\mathscr{M}_b(\Omega\times\mathscr{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})$, such that (up to a subsequence)

$$\lambda_1^h\overset{*}{\rightharpoonup}\lambda_1\ \text{ and }\ \lambda_2^h\overset{*}{\rightharpoonup}\lambda_2\quad\text{weakly* in }\mathscr{M}_b(\Omega\times\mathscr{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})$$

for suitable $\lambda_1,\lambda_2\in\mathscr{M}_b(\Omega\times\mathscr{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})$. Then, we have $\operatorname{supp}(\lambda_1)\subseteq\Omega\times\partial\mathscr{B}$ and $\operatorname{supp}(\lambda_2)\subseteq\Omega\times(\overline{\mathscr{B}\cap\mathscr{C}})$.

By a density argument given in Remark 3.1.4, we can conclude that (3.54) holds for every $\chi\in C_0(\Omega\times\mathscr{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})$ with $\widetilde{\operatorname{div}}_\gamma\chi=0$. The definition of λ_1 and λ_2 then yields

$$\int_{\Omega\times\mathscr{Y}}\chi(x,y):d\left(\tilde{\pi}-\lambda_1-\lambda_2\right)(x,y)=0.$$

From this and Proposition 3.1.8 we conclude that there exists $\mu\in\mathscr{X}_\gamma(\omega)$ such that

$$\tilde{\pi}-\lambda_1-\lambda_2=\widetilde{E}_\gamma\mu.$$

Recalling the assumption that $\partial\mathscr{B}\cap\mathscr{C}\subseteq\mathscr{T}$ and using the same argument as above, we can see that

$$\tilde{\pi}\lfloor\Omega\times(\partial\mathscr{B}\setminus\mathscr{T})=\lambda_1\lfloor\Omega\times(\partial\mathscr{B}\setminus\mathscr{T})+\widetilde{E}_\gamma\mu\lfloor\Omega\times(\partial\mathscr{B}\setminus\mathscr{T})$$

In view of Proposition 3.1.9 and recalling the assumption that $\partial\mathscr{B}\setminus\mathscr{T}$ is a C^1-hypersurface, we are left to prove the analogue of (3.51) for λ_1.

We consider

$$\hat{v}^h(x,y) = \sum_{i\in I_{\varepsilon_h}(\omega)} \mathbb{1}_{Q^i_{\varepsilon_h}}(x')\, \hat{v}^i_{\varepsilon_h}(y,x_3),$$

so that $\lambda_1^h(x,y) = \left[\hat{v}^h(y,x_3)\odot\nu\right]\mathscr{L}^2_{x'}\otimes(\mathscr{H}^2_{x_3,y}\lfloor I\times\partial\mathscr{B})$. Then $\{\hat{v}^h\}$ is bounded in $L^1(\Omega\times\partial\mathscr{B};\mathbb{R}^3)$ by (3.53). Up to a subsequence,

$$\hat{v}^h\,\mathscr{L}^2_{x'}\otimes(\mathscr{H}^2_{x_3,y}\lfloor I\times\partial\mathscr{B}) \stackrel{*}{\rightharpoonup} \eta \quad \text{weakly* in } \mathscr{M}_b(\Omega\times\partial\mathscr{B};\mathbb{R}^3)$$

for a suitable $\eta\in\mathscr{M}_b(\Omega\times\partial\mathscr{B};\mathbb{R}^3)$. Since ν is continuous on $\partial\mathscr{B}\setminus\mathscr{T}$, we conclude

$$\lambda_1\lfloor\Omega\times(\partial\mathscr{B}\setminus\mathscr{T}) = \frac{\eta}{|\eta|}(x,y)\odot\nu(y)\,|\eta|\lfloor\Omega\times(\partial\mathscr{B}\setminus\mathscr{T}),$$

which concludes the proof. ■

4. TWO-SCALE STATICS AND DUALITY

4.1. STRESS-PLASTIC STRAIN DUALITY ON THE CELL

4.1.1. Case $\gamma \in (0,+\infty)$

Definition 4.1.1. Let $\gamma \in (0,+\infty)$. The set $\mathcal{K}_\gamma$ of admissible stresses is defined as the set of all elements $\Sigma \in L^2(I \times \mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$ satisfying:

(i) $\widetilde{\operatorname{div}}_\gamma \Sigma = 0$ in $I \times \mathcal{Y}$,

(ii) $\Sigma \vec{e}_3 = 0$ on $\partial I \times \mathcal{Y}$,

(iii) $\Sigma_{\mathrm{dev}}(x_3,y) \in K(y)$ for $\mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y$-a.e. $(x_3,y) \in I \times \mathcal{Y}$.

Since condition (iii) implies that $\Sigma_{\mathrm{dev}} \in L^\infty(I \times \mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$, for every $\Sigma \in \mathcal{K}_\gamma$ we can deduce from Proposition 1.5.2 that $\Sigma \in L^p(I \times \mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$ for every $1 \leq p < \infty$.

Definition 4.1.2. Let $\gamma \in (0,+\infty)$. The family $\mathcal{A}_\gamma$ of admissible configurations is given by the set of triplets

$$u \in BD_\gamma(I \times \mathcal{Y}), \qquad E \in L^2(I \times \mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}}), \qquad P \in \mathcal{M}_b(I \times \mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{dev}}),$$

such that

$$\widetilde{E}_\gamma u = E\, \mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y + P \quad \text{in } I \times \mathcal{Y}.$$

Definition 4.1.3. Let $\Sigma \in \mathcal{K}_\gamma$ and let $(u,E,P) \in \mathcal{A}_\gamma$. We define the distribution $[\Sigma_{\mathrm{dev}} : P]$ on $\mathbb{R} \times \mathcal{Y}$ by

$$[\Sigma_{\mathrm{dev}} : P](\varphi) := -\int_{I\times\mathcal{Y}} \varphi \Sigma : E \, dx_3 dy - \int_{I\times\mathcal{Y}} \Sigma : \left(u \odot \widetilde{\nabla}_\gamma \varphi\right) dx_3 dy, \tag{4.1}$$

for every $\varphi \in C_c^\infty(\mathbb{R} \times \mathcal{Y})$.

Remark 4.1.4. Note that the second integral in (4.1) is well defined since $BD(I \times \mathcal{Y})$ is embedded into $L^{3/2}(I \times \mathcal{Y};\mathbb{R}^3)$. Moreover, it is easy to check that the definition of $[\Sigma_{\text{dev}} : P]$ is independent of the choice of (u,E), so (4.1) defines a meaningful distribution on $\mathbb{R} \times \mathcal{Y}$.

The following results can be established from the proofs of [24, Theorem 6.2] and [24, Proposition 3.9] respectively, by treating the relative boundary of the "Dirichlet" part as empty, the "Neumann" part as $\partial I \times \mathcal{Y}$, and considering approximating sequences which must be periodic in $\mathcal{Y}$.

Proposition 4.1.5. Let $\Sigma \in \mathcal{K}_\gamma$ and $(u,E,P) \in \mathcal{A}_\gamma$. Then $[\Sigma_{\text{dev}} : P]$ can be extended to a bounded Radon measure on $\mathbb{R} \times \mathcal{Y}$, whose variation satisfies

$$|[\Sigma_{\text{dev}} : P]| \leq \|\Sigma_{\text{dev}}\|_{L^\infty(I\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}})}|P| \quad \text{in } \mathcal{M}_b(\mathbb{R} \times \mathcal{Y}).$$

Proposition 4.1.6. Let $\Sigma \in \mathcal{K}_\gamma$ and $(u,E,P) \in \mathcal{A}_\gamma$. If $\mathcal{Y}$ is a geometrically admissible multi-phase torus, then

$$H\left(y, \frac{dP}{d|P|}\right)|P| \geq [\Sigma_{\text{dev}} : P] \quad \text{in } \mathcal{M}_b(I \times \mathcal{Y}).$$

4.1.2. Case $\gamma = 0$

Definition 4.1.7. The set $\mathcal{K}_0$ of admissible stresses is defined as the set of all elements $\Sigma \in L^2(I \times \mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}})$ satisfying:

(i) $\Sigma_{i3}(x_3,y) = 0$ for $i = 1,2,3$,

(ii) $\Sigma_{\text{dev}}(x_3,y) \in K(y)$ for $\mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y$-a.e. $(x_3,y) \in I \times \mathcal{Y}$,

(iii) $\text{div}_y\bar{\Sigma} = 0$ in $\mathcal{Y}$,

(iv) $\text{div}_y\text{div}_y\hat{\Sigma} = 0$ in $\mathcal{Y}$,

where $\bar{\Sigma}$, $\hat{\Sigma} \in L^2(\mathcal{Y};\mathbb{M}^{2\times2}_{\text{sym}})$ are the zeroth and first order moments of the 2×2 minor of Σ.

Recalling (2.21), by conditions (i) and (ii) we may identify $\Sigma \in \mathscr{K}_0$ with an element of $L^\infty(I \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\rm sym})$ such that $\Sigma(x_3, y) \in K_r(y)$ for $\mathscr{L}^1_{x_3} \otimes \mathscr{L}^2_y$-a.e. $(x_3, y) \in I \times \mathscr{Y}$. Thus, in this regime it will be natural to define the family of admissible configurations defined with a relation in $\mathbb{M}^{2\times 2}_{\rm sym}$.

Definition 4.1.8. The family $\mathscr{A}_0$ of admissible configurations is given by the set of quadruplets

$$\bar{u} \in BD(\mathscr{Y}), \qquad u_3 \in BH(\mathscr{Y}), \qquad E \in L^2(I \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\rm sym}), \qquad P \in \mathscr{M}_b(I \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\rm sym}),$$

such that

$$E_y \bar{u} - x_3 D^2_y u_3 = E\, \mathscr{L}^1_{x_3} \otimes \mathscr{L}^2_y + P \quad \textit{in } I \times \mathscr{Y}.$$

Recalling the definitions of zeroth and first order moments of functions and measures (see Definition 2.2.1 and Definition 2.2.2), we introduce the following analogue of the duality between moments of stresses and plastic strains.

Definition 4.1.9. Let $\Sigma \in \mathscr{K}_0$ and let $(\bar{u}, u_3, E, P) \in \mathscr{A}_0$. We define distributions $[\bar{\Sigma} : \bar{P}]$ and $[\hat{\Sigma} : \hat{P}]$ on $\mathscr{Y}$ by

$$[\bar{\Sigma} : \bar{P}](\varphi) := -\int_{\mathscr{Y}} \varphi\, \bar{\Sigma} : \bar{E}\, dy - \int_{\mathscr{Y}} \bar{\Sigma} : \left(\bar{u} \odot \nabla_y \varphi\right) dy, \tag{4.2}$$

$$[\hat{\Sigma} : \hat{P}](\varphi) := -\int_{\mathscr{Y}} \varphi\, \hat{\Sigma} : \hat{E}\, dy + 2\int_{\mathscr{Y}} \hat{\Sigma} : \left(\nabla_y u_3 \odot \nabla_y \varphi\right) dy + \int_{\mathscr{Y}} u_3\, \hat{\Sigma} : \nabla^2_y \varphi\, dy, \tag{4.3}$$

for every $\varphi \in C^\infty(\mathscr{Y})$.

Remark 4.1.10. Note that the second integral in (4.2) is well defined since $BD(\mathscr{Y})$ is embedded into $L^2(\mathscr{Y}; \mathbb{R}^2)$. Similarly, the second and third integrals in (4.3) are well defined since $BH(\mathscr{Y})$ is embedded into $H^1(\mathscr{Y})$. Moreover, the definitions are independent of the choice of (u, E), so (4.2) and (4.3) define a meaningful distributions on $\mathscr{Y}$.

Arguing as in [18, Section 7], one can prove that $[\bar{\Sigma} : \bar{P}]$ and $[\hat{\Sigma} : \hat{P}]$ are bounded Radon measures on $\mathscr{Y}$. We are now in a position to introduce a duality pairing between admissible stresses and plastic strains.

Definition 4.1.11. Let $\Sigma \in \mathscr{K}_0$ and let $(\bar{u}, u_3, E, P) \in \mathscr{A}_0$. Then we can define a bounded Radon measure $[\Sigma : P]$ on $I \times \mathscr{Y}$ by setting

$$[\Sigma : P] := [\bar{\Sigma} : \bar{P}] \otimes \mathscr{L}^1_{x_3} + \frac{1}{12}[\hat{\Sigma} : \hat{P}] \otimes \mathscr{L}^1_{x_3} - \Sigma^\perp : E^\perp,$$

so that

$$\int_{I\times\mathscr{Y}} \varphi\, d[\Sigma : P] = -\int_{I\times\mathscr{Y}} \varphi \Sigma : E\, dx_3 dy - \int_{\mathscr{Y}} \bar{\Sigma} : \left(\bar{u} \odot \nabla_y \varphi\right) dy \\ + \frac{1}{6}\int_{\mathscr{Y}} \hat{\Sigma} : \left(\nabla_y u_3 \odot \nabla_y \varphi\right) dy + \frac{1}{12}\int_{\mathscr{Y}} u_3 \hat{\Sigma} : \nabla_y^2 \varphi\, dy, \tag{4.4}$$

for every $\varphi \in C^2(\mathscr{Y})$.

Proposition 4.1.12. Let $\Sigma \in \mathscr{K}_0$ and $(\bar{u}, u_3, E, P) \in \mathscr{A}_0$. If $\mathscr{Y}$ is a geometrically admissible multi-phase torus and $K(y)$ satisfies the ordering assumption (2.3), then

$$\int_{I\times\mathscr{Y}} \varphi(y) H_r\left(y, \frac{dP}{d|P|}\right) d|P| \geq \int_{I\times\mathscr{Y}} \varphi(y)\, d[\Sigma : P],$$

for every $\varphi \in C(\mathscr{Y})$ such that $\varphi \geq 0$.

Proof. The proof is divided into two steps.

Step 1. We first consider the case of a two-phase material and require that the yield surface of one phase be included in that of the other phase, i.e. $\mathscr{Y}$ is made of two phases $\mathscr{Y}_1$ and $\mathscr{Y}_2$ such that

$$K_1 \subseteq K_2, \tag{4.5}$$

$$K(y) = K_1, \text{ if } y \in \partial\mathscr{Y}_1 \cap \partial\mathscr{Y}_2. \tag{4.6}$$

Further, we assume that $\mathscr{Y}_1$ is star-shaped with respect to one of its points.

Let us consider a covering $\{\mathscr{Z}_1, \mathscr{Z}_2\}$ of $\mathscr{Y}$ made of open C^2-subdomains, such that $\overline{\mathscr{Y}}_1 \subset \mathscr{Z}_1$ and $\mathscr{Y}_1 \cap \mathscr{Z}_2 = \emptyset$. Let $\{\psi_1, \psi_2\}$ be an associated partition of unity of $\mathscr{Y}$. We can establish the stated inequality by considering the behavior of the measures on $\mathscr{Z}_1$ and $\mathscr{Z}_2$ respectively.

First, consider the inequality on $\mathscr{Z}_1$. Let ρ be the standard mollifier on $\mathbb{R}^2$ which is $[0,1]^2$-periodic, and let us define the planar dilation $d_n(x_3, y) = \left(x_3, \frac{n}{n+1} y\right)$, for every $n \in \mathbb{N}$. We then set

$$\Sigma_n^{(1)}(x_3, y) := \left((\Sigma \circ d_n)(x_3, \cdot) * \rho_{\frac{1}{n+1}}\right)(y). \tag{4.7}$$

We obtain a sequence $\Sigma_n^{(1)} \in C^\infty(\mathscr{Z}_1; L^2(I; \mathbb{M}^{3\times 3}_{\text{sym}}))$ such that

$$\Sigma_n^{(1)} \to \Sigma \quad \text{strongly in } L^2(I \times \mathscr{Z}_1; \mathbb{M}^{3\times 3}_{\text{sym}}),$$

$$\text{div}_y \bar{\Sigma}_n^{(1)} = 0 \text{ in } \mathscr{Z}_1,$$

$$\text{div}_y \text{div}_y \hat{\Sigma}_n^{(1)} = 0 \text{ in } \mathscr{Z}_1.$$

Furthermore, $(\Sigma_n^{(1)}(x_3,y))_{\text{dev}} \in K(y)$ for a.e. $x_3 \in I$ and every $y \in \mathscr{Z}_1$. Consider the the orthogonal decomposition

$$P = \bar{P} \otimes \mathscr{L}^1_{x_3} + \hat{P} \otimes x_3 \mathscr{L}^1_{x_3} + P^\perp,$$

where $\bar{P}, \hat{P} \in \mathscr{M}_b(\mathscr{Y}; \mathbb{M}^{2\times 2}_{\text{sym}})$ and $P^\perp \in L^2(I \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\text{sym}})$. We can infer that $|P|$ is absolutely continuous with respect to the measure

$$\Pi := |\bar{P}| \otimes \mathscr{L}^1_{x_3} + |\hat{P}| \otimes \mathscr{L}^1_{x_3} + \mathscr{L}^3_{x_3,y},$$

As a consequence, for $|\Pi|$-a.e. $(x_3,y) \in I \times \mathscr{Z}_1$ we have

$$H_r\left(y, \frac{dP}{d|\Pi|}\right) \geq \Sigma_n^{(1)} : \frac{dP}{d|\Pi|}.$$

Thus for every $\varphi^{(1)} \in C_c(\mathscr{Z}_1)$, such that $\varphi^{(1)} \geq 0$, we obtain

$$\begin{aligned}
\int_{I\times\mathscr{Z}_1} \varphi^{(1)}(y)\, H_r\left(y, \frac{dP}{d|P|}\right) d|P| &= \int_{I\times\mathscr{Z}_1} \varphi^{(1)}(y)\, H_r\left(y, \frac{dP}{d|\Pi|}\right) d|\Pi| \\
&\geq \int_{I\times\mathscr{Z}_1} \varphi^{(1)}(y)\, \Sigma_n^{(1)} : \frac{dP}{d|\Pi|}\, d|\Pi| \\
&= \int_{I\times\mathscr{Z}_1} \varphi^{(1)}(y)\, \Sigma_n^{(1)} : \frac{dP}{d|P|}\, d|P| \\
&= \int_{I\times\mathscr{Z}_1} \varphi^{(1)}(y)\, d[\Sigma_n^{(1)} : P].
\end{aligned}$$

Since $\bar{\Sigma}_n^{(1)}$, $\hat{\Sigma}_n^{(1)}$ and $(\Sigma_n^{(1)})^\perp$ are smooth with respect to the variable y, we can conclude that

$$\begin{gathered}
[\bar{\Sigma}_n^{(1)} : \bar{P}] \xrightharpoonup{*} [\bar{\Sigma}^{(1)} : \bar{P}] \quad \text{weakly* in } \mathscr{M}_b(\mathscr{Z}_1), \\
[\hat{\Sigma}_n^{(1)} : \hat{P}] \xrightharpoonup{*} [\hat{\Sigma}^{(1)} : \hat{P}] \quad \text{weakly* in } \mathscr{M}_b(\mathscr{Z}_1), \\
\int_{I\times\mathscr{Z}_1} \varphi^{(1)}(y)(\Sigma_n^{(1)})^\perp : P^\perp\, dx_3 dy \to \int_{I\times\mathscr{Z}_1} \varphi^{(1)}(y)(\Sigma^{(1)})^\perp : P^\perp\, dx_3 dy.
\end{gathered}$$

Passing to the limit we have

$$\int_{I\times\mathscr{Z}_1} \varphi^{(1)}(y)\, H_r\left(y, \frac{dP}{d|P|}\right) d|P| \geq \int_{I\times\mathscr{Z}_1} \varphi^{(1)}(y)\, d[\Sigma : P],$$

Next, consider the inequality on $\mathscr{Z}_2$. If we regularize Σ just by convolution with respect to y, we obtain a sequence $\Sigma_n^{(2)} \in C^\infty(\mathscr{Z}_2; L^2(I; \mathbb{M}^{2\times 2}_{\text{sym}}))$ such that

$$\begin{gathered}
\Sigma_n^{(2)} \to \Sigma \quad \text{strongly in } L^2(I \times \mathscr{Z}_2; \mathbb{M}^{3\times 3}_{\text{sym}}), \\
\operatorname{div}_y \bar{\Sigma}_n^{(2)} = 0 \text{ in } \mathscr{Z}_2, \\
\operatorname{div}_y \operatorname{div}_y \hat{\Sigma}_n^{(2)} = 0 \text{ in } \mathscr{Z}_2,
\end{gathered}$$

such that $(\Sigma_n^{(2)}(x_3,y))_{\mathrm{dev}} \in K_2$ for a.e. $x_3 \in I$ and every $y \in \mathscr{Z}_2$. Using the same argument as above, we can conclude that for every $\varphi^{(2)} \in C_c(\mathscr{Z}_2)$ with $\varphi^{(2)} \geq 0$

$$\int_{I\times\mathscr{Z}_2} \varphi^{(2)}(y)\, H_r\left(y, \frac{dP}{d|P|}\right) d|P| \geq \int_{I\times\mathscr{Z}_2} \varphi^{(2)}(y)\, d[\Sigma : P].$$

Finally, let $\varphi \in C(\mathscr{Y})$ such that $\varphi \geq 0$. We have

$$\begin{aligned}
&\int_{I\times\mathscr{Y}} \varphi(y)\, H_r\left(y, \frac{dP}{d|P|}\right) d|P| \\
&= \int_{I\times\mathscr{Z}_1} \psi_1(y)\varphi(y)\, H_r\left(y, \frac{dP}{d|P|}\right) d|P| + \int_{I\times\mathscr{Z}_2} \psi_2(y)\varphi(y)\, H_r\left(y, \frac{dP}{d|P|}\right) d|P| \\
&\geq \int_{I\times\mathscr{Z}_1} \psi_1(y)\varphi(y)\, d[\Sigma : P] + \int_{I\times\mathscr{Z}_2} \psi_2(y)\varphi(y)\, d[\Sigma : P] \\
&= \int_{I\times\mathscr{Y}} \varphi(y)\, d[\Sigma : P].
\end{aligned}$$

Step 2. We will now consider the case of a multiphase torus.

Since, for each i, $\mathscr{Y}_i$ is a bounded open set with piecewise C^2 boundary (in particular, with Lipschitz boundary) by Proposition 1.7.4 there exists a finite open covering $\{\mathscr{U}_k^{(i)}\}$ of $\overline{\mathscr{Y}}_i$ such that $\mathscr{Y}_i \cap \mathscr{U}_k^{(i)}$ is (strongly) star-shaped with Lipschitz boundary.

For each i, let $\{\psi_k^{(i)}\}$ be a smooth partition of unity subordinate to the covering $\{\mathscr{U}_k^{(i)}\}$, i.e. $\psi_k^{(i)} \in C^\infty(\overline{\mathscr{Y}}_i)$, with $0 \leq \psi_k^{(i)} \leq 1$, such that $\mathrm{supp}(\psi_k^{(i)}) \subset \mathscr{U}_k^{(i)}$ and $\sum_k \psi_k^{(i)} = 1$ on $\overline{\mathscr{Y}}_i$. We can then modify sets $\mathscr{U}_k^{(i)}$ so that $\mathscr{Y}_i \cap \mathscr{U}_k^{(i)}$ is (strongly) star-shaped with C^2 boundary, while $\mathrm{supp}(\psi_k^{(i)}) \subset \mathscr{U}_k^{(i)}$ still holds.

The result now follows from Step 1, by taking a finite covering $\{\mathscr{U}_k\}$ of $\mathscr{Y}$ made of open C^2-subdomains, such that for each $y \in \partial\mathscr{Y}_i \cap \partial\mathscr{Y}_j$ there exists a covering element $\mathscr{U}_k$ such that $\mathscr{Y}_{min\{i,j\}} \cap \mathscr{U}_k$ is star-shaped with respect to one of its points and considering an associated partition of unity of $\mathscr{Y}$. ■

4.1.3. Case $\gamma = +\infty$

Definition 4.1.13. The set $\mathscr{K}_\infty$ of admissible stresses is defined as the set of all elements $\Sigma \in L^2(\mathscr{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$ satisfying:

(i) $\mathrm{div}_y \Sigma = 0$ in $\mathscr{Y}$,

(ii) $\Sigma_{\mathrm{dev}}(y) \in K(y)$ for $\mathscr{L}^2_y$-a.e. $y \in \mathscr{Y}$.

Definition 4.1.14. The family $\mathscr{A}_\infty$ of admissible configurations is given by the set of quintuplets

$$\bar{u}\in BD(\mathscr{Y}),\qquad u_3\in BV(\mathscr{Y}),\qquad v\in\mathbb{R}^3,\qquad E\in L^2(\mathscr{Y};\mathbb{M}^{3\times3}_{\rm sym}),\qquad P\in\mathscr{M}_b(\mathscr{Y};\mathbb{M}^{3\times3}_{\rm dev}),$$

such that

$$\begin{pmatrix} E_y\bar{u} & v'+D_yu_3 \\ (v'+D_yu_3)^T & v_3 \end{pmatrix} = E\,\mathscr{L}^2_y + P \quad \textit{in } \mathscr{Y}.$$

Definition 4.1.15. Let $\Sigma\in\mathscr{K}_\infty$ and let $(\bar{u},u_3,v,E,P)\in\mathscr{A}_\infty$. We define the distribution $[\Sigma_{\rm dev}:P]$ on $\mathscr{Y}$ by

$$\begin{aligned}[\Sigma_{\rm dev}:P](\varphi) := -\int_{\mathscr{Y}}\varphi\,\Sigma:E\,dy - \int_{\mathscr{Y}}\Sigma'':\big(\bar{u}\odot\nabla_y\varphi\big)\,dy \\ -2\int_{\mathscr{Y}}u_3\begin{pmatrix}\Sigma_{13}\\ \Sigma_{23}\end{pmatrix}\cdot\nabla_y\varphi\,dy \\ +2v'\cdot\int_{\mathscr{Y}}\varphi\begin{pmatrix}\Sigma_{13}\\ \Sigma_{23}\end{pmatrix}dy + v_3\int_{\mathscr{Y}}\varphi\,\Sigma_{33}\,dy,\end{aligned} \tag{4.8}$$

for every $\varphi\in C^\infty(\mathscr{Y})$.

Remark 4.1.16. Note that integral are well defined since $BD(\mathscr{Y})$ and $BV(\mathscr{Y})$ are both embedded into $L^2(\mathscr{Y};\mathbb{R}^2)$. Moreover, the definition is independent of the choice of $(\bar{u},u_3,v,E)$, so (4.8) defines a meaningful distributions on $\mathscr{Y}$.

Proposition 4.1.17. Let $\Sigma\in\mathscr{K}_\infty$ and $(\bar{u},u_3,v,E,P)\in\mathscr{A}_\infty$. Then $[\Sigma_{\rm dev}:P]$ can be extended to a bounded Radon measure on $\mathscr{Y}$, whose variation satisfies

$$|[\Sigma_{\rm dev}:P]| \le \|\Sigma_{\rm dev}\|_{L^\infty(\mathscr{Y};\mathbb{M}^{3\times3}_{\rm sym})}|P| \quad \text{in } \mathscr{M}_b(\mathscr{Y}).$$

Proof. Using a convolution argument we can find a sequence $\{\Sigma_n\}\subset C^\infty(\mathscr{Y};\mathbb{M}^{3\times3}_{\rm sym})$ such that

$$\begin{aligned}&\Sigma_n\to\Sigma \quad \text{strongly in } L^2(\mathscr{Y};\mathbb{M}^{3\times3}_{\rm sym}),\\ &{\rm div}_y\Sigma_n = 0 \text{ in } \mathscr{Y},\\ &\|(\Sigma_n)_{\rm dev}\|_{L^\infty(\mathscr{Y};\mathbb{M}^{3\times3}_{\rm dev})} \le \|\Sigma_{\rm dev}\|_{L^\infty(\mathscr{Y};\mathbb{M}^{3\times3}_{\rm dev})}.\end{aligned}$$

According to the integration by parts formulas for $BD(\mathscr{Y})$ and $BV(\mathscr{Y})$, we have for every $\varphi \in C^1(\mathscr{Y})$

$$\int_{\mathscr{Y}} \varphi \operatorname{div}_y(\Sigma_n)'' \cdot \bar{u}\, dy + \int_{\mathscr{Y}} \varphi\, (\Sigma_n)'' : dE_y\bar{u} + \int_{\mathscr{Y}} (\Sigma_n)'' : \big(\bar{u} \odot \nabla_y \varphi\big)\, dy = 0,$$

$$\int_{\mathscr{Y}} \varphi\, u_3 \operatorname{div}_y \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} dy + \int_{\mathscr{Y}} \varphi \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot dD_y u_3 + \int_{\mathscr{Y}} u_3 \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot \nabla_y \varphi\, dy = 0.$$

From these two equalities, together with the above convergence and the expression in Equation (4.8), we compute

$$\begin{aligned}
&[\Sigma_{\mathrm{dev}} : P](\varphi) \\
&= \lim_n \Big[-\int_{\mathscr{Y}} \varphi\, \Sigma_n : E\, dy - \int_{\mathscr{Y}} (\Sigma_n)'' : \big(\bar{u} \odot \nabla_y \varphi\big)\, dy \\
&\qquad - 2\int_{\mathscr{Y}} u_3 \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot \nabla_y \varphi\, dy + 2v' \cdot \int_{\mathscr{Y}} \varphi \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} dy + v_3 \int_{\mathscr{Y}} \varphi\, (\Sigma_n)_{33}\, dy \Big] \\
&= \lim_n \Big[-\int_{\mathscr{Y}} \varphi\, \Sigma_n : E\, dy + \int_{\mathscr{Y}} \varphi \operatorname{div}_y(\Sigma_n)'' \cdot \bar{u}\, dy + \int_{\mathscr{Y}} \varphi\, (\Sigma_n)'' : dE_y\bar{u} \\
&\qquad + 2\int_{\mathscr{Y}} \varphi\, u_3 \operatorname{div}_y \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} dy + 2\int_{\mathscr{Y}} \varphi \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot dD_y u_3 \\
&\qquad + 2v' \cdot \int_{\mathscr{Y}} \varphi \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} dy + v_3 \int_{\mathscr{Y}} \varphi\, (\Sigma_n)_{33}\, dy \Big] \\
&= \lim_n \Big[\int_{\mathscr{Y}} \varphi \operatorname{div}_y(\Sigma_n) \cdot \begin{pmatrix} \bar{u} \\ u_3 \end{pmatrix} dy + \int_{\mathscr{Y}} \varphi\, \Sigma_n : dP \Big] \\
&= \lim_n \int_{\mathscr{Y}} \varphi\, (\Sigma_n)_{\mathrm{dev}} : dP.
\end{aligned}$$

In view of the L^∞-bound on $\{(\Sigma_n)_{\mathrm{dev}}\}$, taking the limit yields

$$|[\Sigma_{\mathrm{dev}} : P]|(\varphi) \le \|\Sigma_{\mathrm{dev}}\|_{L^\infty(\mathscr{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})} \int_{\mathscr{Y}} |\varphi|\, d|P|,$$

from which the claims follow. ■

Proposition 4.1.18. Let $\Sigma \in \mathscr{K}_\infty$. Then, for $\mathscr{H}^1$-a.e. $y \in \Gamma$,

$$[\Sigma'' \nu]_\nu^\perp(y) \in (K''_{min\{i,j\}} \nu)_\nu^\perp. \tag{4.9}$$

Furthermore, if $(\bar{u}, u_3, v, E, P) \in \mathscr{A}_\infty$, then for every $i \neq j$,

$$[\Sigma_{\mathrm{dev}} : P] \lfloor \Gamma_{ij} = \left([\Sigma'' \nu]_\nu^\perp \cdot (\bar{u}^i - \bar{u}^j) + 2\left(\begin{pmatrix} \Sigma_{13} \\ \Sigma_{23} \end{pmatrix} \cdot \nu \right) (u_3^i - u_3^j) \right) \mathscr{H}^1 \lfloor \Gamma_{ij}, \tag{4.10}$$

where $\bar{u}^i, u_3^i$ and $\bar{u}^j, u_3^j$ are the traces on Γ_{ij} of the restrictions of $\bar{u}, u_3$ to $\mathscr{Y}_i$ and $\mathscr{Y}_j$ respectively, assuming that ν points from $\mathscr{Y}_j$ to $\mathscr{Y}_i$.

Proof. To prove (4.10), let $\varphi \in C^1(\mathscr{Y})$ be such that its support is contained $\mathscr{Y}_i \cup \mathscr{Y}_j \cup \Gamma_{ij}$. Let $\mathscr{U} \subset\subset \mathscr{Y}$ be a compact set containing $\mathrm{supp}(\varphi)$, and consider any smooth approximating sequence $\{\Sigma_n\} \subset C^\infty(\mathscr{U};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$ such that

$$\begin{aligned}
&\Sigma_n \to \Sigma \quad \text{strongly in } L^2(\mathscr{U};\mathbb{M}^{3\times 3}_{\mathrm{sym}}),\\
&\mathrm{div}_y \Sigma_n = 0 \text{ in } \mathscr{U},\\
&\|(\Sigma_n)_{\mathrm{dev}}\|_{L^\infty(\mathscr{U};\mathbb{M}^{3\times 3}_{\mathrm{dev}})} \le \|\Sigma_{\mathrm{dev}}\|_{L^\infty(\mathscr{U};\mathbb{M}^{3\times 3}_{\mathrm{dev}})}.
\end{aligned}$$

Note that $((\Sigma_n)''\nu)^\perp_\nu = \big((\Sigma_n)''_{\mathrm{dev}}\nu\big)^\perp_\nu$ and

$$\big((\Sigma_n)''_{\mathrm{dev}}\nu\big)^\perp_\nu \xrightharpoonup{*} [\Sigma''_{\mathrm{dev}}\nu]^\perp_\nu \quad \text{weakly* in } L^\infty(\Gamma_{ij};\mathbb{R}^2).$$

Since $\varphi\,\bar{u} \in BD(\mathscr{Y})$ and $\varphi\, u_3 \in BD(\mathscr{Y})$, with

$$\begin{aligned}
E_y(\varphi\,\bar{u}) &= \varphi\, E_y\bar{u} + \bar{u}\odot\nabla_y\varphi,\\
D_y(\varphi\, u_3) &= \varphi\, D_y u_3 + u_3\,\nabla_y\varphi,
\end{aligned}$$

we compute

$$\begin{aligned}
&[\Sigma_{\mathrm{dev}} : P](\varphi)\\
&= \lim_n \Big[-\int_{\mathscr{Y}_i\cup\mathscr{Y}_j} \varphi\,\Sigma_n : E\,dy - \int_{\mathscr{Y}_i\cup\mathscr{Y}_j} (\Sigma_n)'' : \big(\bar{u}\odot\nabla_y\varphi\big)\,dy\\
&\qquad -2\int_{\mathscr{Y}_i\cup\mathscr{Y}_j} u_3 \begin{pmatrix}(\Sigma_n)_{13}\\(\Sigma_n)_{23}\end{pmatrix}\cdot\nabla_y\varphi\,dy + 2\nu'\cdot\int_{\mathscr{Y}_i\cup\mathscr{Y}_j}\varphi\begin{pmatrix}(\Sigma_n)_{13}\\(\Sigma_n)_{23}\end{pmatrix}dy + \nu_3\int_{\mathscr{Y}_i\cup\mathscr{Y}_j}\varphi\,(\Sigma_n)_{33}\,dy\Big]\\
&= \lim_n \Big[-\int_{\mathscr{Y}_i\cup\mathscr{Y}_j} \varphi\,\Sigma_n : E\,dy - \int_{\mathscr{Y}_i\cup\mathscr{Y}_j} (\Sigma_n)'' : dE_y(\varphi\,\bar{u}) + \int_{\mathscr{Y}_i\cup\mathscr{Y}_j}\varphi\,(\Sigma_n)'' : E_y\bar{u}\\
&\qquad -2\int_{\mathscr{Y}_i\cup\mathscr{Y}_j}\begin{pmatrix}(\Sigma_n)_{13}\\(\Sigma_n)_{23}\end{pmatrix}\cdot dD_y(\varphi\,u_3) + 2\int_{\mathscr{Y}_i\cup\mathscr{Y}_j}\varphi\begin{pmatrix}(\Sigma_n)_{13}\\(\Sigma_n)_{23}\end{pmatrix}\cdot dD_y u_3\\
&\qquad +2\nu'\cdot\int_{\mathscr{Y}_i\cup\mathscr{Y}_j}\varphi\begin{pmatrix}(\Sigma_n)_{13}\\(\Sigma_n)_{23}\end{pmatrix}dy + \nu_3\int_{\mathscr{Y}_i\cup\mathscr{Y}_j}\varphi\,(\Sigma_n)_{33}\,dy\Big]\\
&= \lim_n \Big[-\int_{\mathscr{Y}_i\cup\mathscr{Y}_j}(\Sigma_n)'' : dE_y(\varphi\,\bar{u}) - 2\int_{\mathscr{Y}_i\cup\mathscr{Y}_j}\begin{pmatrix}(\Sigma_n)_{13}\\(\Sigma_n)_{23}\end{pmatrix}\cdot dD_y(\varphi\,u_3) + \int_{\mathscr{Y}_i\cup\mathscr{Y}_j}\varphi\,\Sigma_n : dP\Big].
\end{aligned}$$

Owing to the assumption on $\mathrm{supp}(\varphi)$, we have that the only relevant part of the boundary of $\mathscr{Y}_i \cup \mathscr{Y}_j$ is Γ_{ij}, i.e. by integration by parts we have

$$\begin{aligned}
&[\Sigma_{\mathrm{dev}} : P](\varphi) \\
&= \lim_n \Big[\int_{\Gamma_{ij}} \varphi \left((\Sigma_n)'' \nu \right) \cdot (\bar{u}^i - \bar{u}^j)\, d\mathscr{H}^1 - 2 \int_{\Gamma_{ij}} \varphi \left(\begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot \nu \right) (u_3^i - u_3^j)\, d\mathscr{H}^1 \\
&\qquad + \int_{\mathscr{Y}_i \cup \mathscr{Y}_j} \varphi\, (\Sigma_n)_{\mathrm{dev}} : dP \Big].
\end{aligned}$$

Now

$$P \lfloor \Gamma_{ij} = \begin{pmatrix} E_y \bar{u} & D_y u_3 \\ (D_y u_3)^T & 0 \end{pmatrix} \lfloor \Gamma_{ij} = \begin{pmatrix} (\bar{u}^j - \bar{u}^i) \odot \nu & (u_3^j - u_3^i)\, \nu \\ (u_3^j - u_3^i)\, \nu^T & 0 \end{pmatrix} \mathscr{H}^1$$

and $\mathrm{tr}\, P = 0$ imply that $\bar{u}^i(y) - \bar{u}^j(y) \perp \nu(y)$ for $\mathscr{H}^1$-a.e. $y \in \Gamma_{ij}$. The above computation then yields

$$\begin{aligned}
[\Sigma_{\mathrm{dev}} : P](\varphi) = \int_{\Gamma_{ij}} \varphi\, [\Sigma'' \nu]_\nu^\perp \cdot (\bar{u}^i - \bar{u}^j)\, d\mathscr{H}^1 - 2 \int_{\Gamma_{ij}} \varphi \left(\begin{pmatrix} \Sigma_{13} \\ \Sigma_{23} \end{pmatrix} \cdot \nu \right) (u_3^i - u_3^j)\, d\mathscr{H}^1 \\
+ \lim_n \int_{\mathscr{Y}_i \cup \mathscr{Y}_j} \varphi\, (\Sigma_n)_{\mathrm{dev}} : dP.
\end{aligned} \tag{4.11}$$

If we define $\lambda_n \in \mathscr{M}_b(\mathscr{Y}_i \cup \mathscr{Y}_j \cup \Gamma_{ij})$ as

$$\lambda_n(\varphi) := \int_{\mathscr{Y}_i \cup \mathscr{Y}_j} \varphi\, (\Sigma_n)_{\mathrm{dev}} : dP,$$

then the L^∞-bound on $\{(\Sigma_n)_{\mathrm{dev}}\}$ ensures that it satisfies

$$|\lambda_n| \leq C\, |P| \lfloor (\mathscr{Y}_i \cup \mathscr{Y}_j),$$

and we infer from (4.11) that

$$\lambda_n \xrightharpoonup{*} \lambda \quad \text{weakly* in } \mathscr{M}_b(\mathscr{Y}_i \cup \mathscr{Y}_j \cup \Gamma_{ij})$$

for a suitable $\lambda \in \mathscr{M}_b(\mathscr{Y}_i \cup \mathscr{Y}_j \cup \Gamma_{ij})$ with

$$|\lambda| \leq C\, |P| \lfloor (\mathscr{Y}_i \cup \mathscr{Y}_j), \tag{4.12}$$

and

$$[\Sigma_{\text{dev}} : P](\varphi) = \int_{\Gamma_{ij}} \varphi\, [\Sigma'' \nu]^{\perp}_{\nu} \cdot (\bar{u}^i - \bar{u}^j)\, d\mathscr{H}^1 - 2\int_{\Gamma_{ij}} \varphi \left(\begin{pmatrix} \Sigma_{13} \\ \Sigma_{23} \end{pmatrix} \cdot \nu \right) (u_3^i - u_3^j)\, d\mathscr{H}^1$$
$$+ \lambda(\varphi).$$

Since (4.12) implies $\lambda \lfloor \Gamma_{ij} = 0$, the result directly follows. ■

Proposition 4.1.19. Let $\Sigma \in \mathscr{K}_\infty$ and $(\bar{u}, u_3, v, E, P) \in \mathscr{A}_\infty$. If $\mathscr{Y}$ is a geometrically admissible multi-phase torus and $K(y)$ satisfies the ordering assumption (2.4), then

$$H\left(y, \frac{dP}{d|P|}\right) |P| \geq [\Sigma_{\text{dev}} : P] \quad \text{in } \mathscr{M}_b(\mathscr{Y}).$$

Proof. We can establish the stated inequality by considering the behavior of the measures on each phase $\mathscr{Y}_i$ and inteface Γ_{ij} respectively.

First, consider an opet set $\mathscr{U}$ such that $\overline{\mathscr{U}} \subset \mathscr{Y}_i$ for some i. Regularizing by convolution, we obtain a sequence $\Sigma_n \in C^\infty(\mathscr{U}; \mathbb{M}^{3\times 3}_{\text{sym}})$ such that

$$\Sigma_n \to \Sigma \quad \text{strongly in } L^2(\mathscr{U}; \mathbb{M}^{3\times 3}_{\text{sym}}),$$
$$\text{div}_y \Sigma_n = 0 \text{ in } \mathscr{U}.$$

Furthermore, $(\Sigma_n(y))_{\text{dev}} \in K_i$ for every $y \in \mathscr{U}$. As a consequence, for $|P|$-a.e. $y \in \mathscr{U}$ we have

$$H\left(y, \frac{dP}{d|P|}\right) = H_i\left(\frac{dP}{d|P|}\right) \geq \Sigma_n : \frac{dP}{d|P|}.$$

Thus for every $\varphi \in C(\mathscr{U})$, such that $\varphi \geq 0$, we obtain

$$\int_{\mathscr{U}} \varphi H\left(y, \frac{dP}{d|P|}\right) d|P| \geq \int_{\mathscr{U}} \varphi \Sigma_n : \frac{dP}{d|P|}\, d|P| = \int_{\mathscr{U}} \varphi\, d[\Sigma_n : P].$$

Since Σ_n is smooth, we can conclude that

$$[\Sigma_n : \bar{P}] \xrightharpoonup{*} [\Sigma : \bar{P}] \quad \text{weakly* in } \mathscr{M}_b(\mathscr{U}).$$

Passing to the limit we have

$$\int_{\mathscr{U}} \varphi H\left(y, \frac{dP}{d|P|}\right) d|P| \geq \int_{\mathscr{U}} \varphi\, d[\Sigma : P].$$

The inequality on the phase $\mathscr{Y}_i$ now follows by considering a collection of open subsets that increases to $\mathscr{Y}_i$.

Next, for every $i \neq j$,

$$H\left(y, \frac{dP}{d|P|}\right)|P|\lfloor\Gamma_{ij} = H_{min\{i,j\}}\left(\begin{pmatrix} (\bar{u}^j - \bar{u}^i) \odot \nu & (u_3^j - u_3^i)\,\nu \\ (u_3^j - u_3^i)\,\nu^T & 0 \end{pmatrix}\right)\mathscr{H}^1\lfloor\Gamma_{ij}.$$

where $\bar{u}^i, u_3^i$ and $\bar{u}^j, u_3^j$ are the traces on Γ_{ij} of the restrictions of $\bar{u}, u_3$ to $\mathscr{Y}_i$ and $\mathscr{Y}_j$ respectively, assuming that ν points from $\mathscr{Y}_j$ to $\mathscr{Y}_i$. The claim then directly follows in view of Proposition 4.1.18. ■

4.2. DISINTEGRATION OF ADMISSIBLE CONFIGURATIONS

From now onward, we consider the open and bounded $\widetilde{\omega} \subseteq \mathbb{R}^2$ such that $\omega \subset \widetilde{\omega}$ and $\widetilde{\omega} \cap \partial\omega = \gamma_D$. We also denote by $\widetilde{\Omega} = \widetilde{\omega} \times I$ the associated reference domain.

In order to make sense of the duality between the two-scale limits of stresses and plastic strains, we will need to disintegrate the two-scale limits of the kinematically admissible fields in such a way to obtain elements of $\mathscr{A}_\gamma$, for $\gamma \in [0, +\infty]$.

4.2.1. Case $\gamma \in (0, +\infty)$

Definition 4.2.1. Let $w \in H^1(\widetilde{\Omega};\mathbb{R}^3) \cap KL(\widetilde{\Omega})$. We define the class $\mathscr{A}_\gamma^{hom}(w)$ of admissible two-scale configurations relative to the boundary datum w as the set of triplets (u, E, P) with

$$u \in KL(\widetilde{\Omega}), \qquad E \in L^2(\widetilde{\Omega} \times \mathscr{Y};\mathbb{M}^{3\times3}_{\text{sym}}), \qquad P \in \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y};\mathbb{M}^{3\times3}_{\text{dev}}),$$

such that

$$u = w, \qquad E = Ew, \qquad P = 0 \qquad \text{on } (\widetilde{\Omega} \setminus \overline{\Omega}) \times \mathscr{Y},$$

and also such that there exists $\mu \in \mathscr{X}_\gamma(\widetilde{\omega})$ with

$$Eu \otimes \mathscr{L}^2_y + \widetilde{E}_\gamma \mu = E\,\mathscr{L}^3_x \otimes \mathscr{L}^2_y + P \qquad \text{in } \widetilde{\Omega} \times \mathscr{Y}. \tag{4.13}$$

Lemma 4.2.2. Let $(u, E, P) \in \mathscr{A}_\gamma^{hom}(w)$ with the associated $\mu \in \mathscr{X}_\gamma(\widetilde{\omega})$, and let $\bar{u} \in BD(\widetilde{\omega})$ and $u_3 \in BH(\widetilde{\omega})$ be the Kirchhoff-Love components of u. Set

$$\eta := \mathscr{L}^2_{x'} + (proj_\#|P|)^s \in \mathscr{M}_b^+(\widetilde{\omega}).$$

Then the following disintegrations hold true:

$$Eu \otimes \mathscr{L}^2_y = \begin{pmatrix} A_1(x') + x_3 A_2(x') & 0 \\ 0 & 0 \end{pmatrix} \eta \otimes \mathscr{L}^1_{x_3} \otimes \mathscr{L}^2_y, \tag{4.14}$$

$$E\,\mathscr{L}^3_x \otimes \mathscr{L}^2_y = C(x')E(x,y)\,\eta \otimes \mathscr{L}^1_{x_3} \otimes \mathscr{L}^2_y \tag{4.15}$$

$$P = \eta \overset{\text{gen.}}{\otimes} P_{x'}. \tag{4.16}$$

Above, $A_1, A_2 : \widetilde{\omega} \to \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ and $C : \widetilde{\omega} \to [0,+\infty]$ are respective Radon-Nikodym derivatives of $E\bar{u}$, $-D^2u_3$ and $\mathscr{L}^2_{x'}$ with respect to η, $E(x,y)$ is a Borel representative of E, and $P_{x'} \in \mathscr{M}_b(I\times\mathscr{Y};\mathbb{M}^{3\times 3}_{\mathrm{dev}})$ for η-a.e. $x' \in \widetilde{\omega}$.

Furthermore, we can choose a Borel map $(x',x_3,y) \in \widetilde{\Omega}\times\mathscr{Y} \mapsto \mu_{x'}(x_3,y) \in \mathbb{R}^3$ such that, for η-a.e. $x' \in \widetilde{\omega}$,

$$\mu = \mu_{x'}(x_3,y)\,\eta\otimes\mathscr{L}^1_{x_3}\otimes\mathscr{L}^2_y, \quad \widetilde{E}_\gamma\mu = \eta \overset{\text{gen.}}{\otimes} \widetilde{E}_\gamma\mu_{x'}, \tag{4.17}$$

where $\mu_{x'} \in BD_\gamma(I\times\mathscr{Y})$, $\int_{I\times\mathscr{Y}} \mu_{x'}(x_3,y)\,dx_3dy = 0$.

Proof. The proof is analogous to [25, Lemma 5.4]. The only difference is the statement and argument for the disintregration of $Eu\otimes\mathscr{L}^2_y$, that we detail below.

First we note that $proj_\#\left(\widetilde{E}_\gamma\mu\right)_{\alpha\beta} = proj_\#\left(E_y\mu\right)_{\alpha\beta} = 0$ for $\alpha,\beta = 1,2$. Then, from (4.13) we get

$$\begin{aligned}(E\bar{u})_{\alpha\beta} = proj_\#\left(Eu\otimes\mathscr{L}^2_y\right)_{\alpha\beta} &= \left(\int_{I\times\mathscr{Y}} E_{\alpha\beta}(x,y)\,dx_3dy\right)\mathscr{L}^2_{x'} + proj_\#(P)_{\alpha\beta}\\ &\le e^{(1)}_{\alpha\beta}(x')\,\mathscr{L}^2_{x'} + (proj_\#|P|)^s_{\alpha\beta},\end{aligned}$$

where we set $e^{(1)}(x') := \int_{I\times\mathscr{Y}} E(x,y)\,dx_3dy + (proj_\#|P|)^a \in L^2(\widetilde{\omega};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$. Similarly, after multipliying equation (4.13) by x_3, we have that

$$\begin{aligned}\left(-D^2u_3\right)_{\alpha\beta} = \frac{1}{12}proj_\#\left(x_3Eu\otimes\mathscr{L}^2_y\right)_{\alpha\beta} &= \frac{1}{12}\left(\int_{I\times\mathscr{Y}} x_3E_{\alpha\beta}(x,y)\,dx_3dy\right)\mathscr{L}^2_{x'} + \frac{1}{12}proj_\#(x_3P)\\ &\le e^{(2)}_{\alpha\beta}(x')\,\mathscr{L}^2_{x'} + \frac{1}{12}(proj_\#|x_3P|)^s_{\alpha\beta},\end{aligned}$$

where we set $e^{(2)}(x') := \frac{1}{12}\int_{I\times\mathscr{Y}} x_3E(x,y)\,dx_3dy + \frac{1}{12}(proj_\#|x_3P|)^a \in L^2(\widetilde{\omega};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$. Consequently, measures $E\bar{u}$ and $-D^2u_3$ are absolutely continuous with respect to η, so we can write.

$$\begin{aligned}E\bar{u}\otimes\mathscr{L}^2_y &= A_1(x')\,\eta\otimes\mathscr{L}^1_{x_3}\otimes\mathscr{L}^2_y,\\ -D^2u_3\otimes\mathscr{L}^2_y &= A_2(x')\,\eta\otimes\mathscr{L}^1_{x_3}\otimes\mathscr{L}^2_y,\end{aligned}$$

for suitable $A_1, A_2 : \widetilde{\omega} \to \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ such that (4.14) hold true. ■

Remark 4.2.3. From the above disintegration, we have that, for η-a.e. $x' \in \widetilde{\omega}$,

$$\widetilde{E}_\gamma\mu_{x'} = \left[C(x')E(x,y) - \begin{pmatrix} A_1(x') + x_3A_2(x') & 0\\ 0 & 0\end{pmatrix}\right]\mathscr{L}^1_{x_3}\otimes\mathscr{L}^2_y + P_{x'} \quad in\ I\times\mathscr{Y}.$$

Thus, the triple

$$\left(\mu_{x'}, \left[C(x')E(x,y) - \begin{pmatrix} A_1(x') + x_3 A_2(x') & 0 \\ 0 & 0 \end{pmatrix}\right], P_{x'}\right)$$

is an element of $\mathscr{A}_\gamma$.

4.2.2. Case $\gamma = 0$

Definition 4.2.4. Let $w \in H^1(\widetilde{\Omega};\mathbb{R}^3) \cap KL(\widetilde{\Omega})$. We define the class $\mathscr{A}_0^{hom}(w)$ of admissible two-scale configurations relative to the boundary datum w as the set of triplets (u,E,P) with

$$u \in KL(\widetilde{\Omega}), \qquad E \in L^2(\widetilde{\Omega} \times \mathscr{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}}), \qquad P \in \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}}),$$

such that

$$u = w, \qquad E = Ew, \qquad P = 0 \qquad \text{on } (\widetilde{\Omega} \setminus \overline{\Omega}) \times \mathscr{Y},$$

and also such that there exist $\mu \in \mathscr{X}_0(\widetilde{\omega})$, $\kappa \in \Upsilon_0(\widetilde{\omega})$ with

$$Eu \otimes \mathscr{L}^2_y + E_y\mu - x_3 D^2_y \kappa = E\,\mathscr{L}^3_x \otimes \mathscr{L}^2_y + P \qquad \text{in } \widetilde{\Omega} \times \mathscr{Y}. \tag{4.18}$$

Lemma 4.2.5. Let $(u,E,P) \in \mathscr{A}_0^{hom}(w)$ with the associated $\mu \in \mathscr{X}_0(\widetilde{\omega})$, $\kappa \in \Upsilon_0(\widetilde{\omega})$, and let $\bar{u} \in BD(\widetilde{\omega})$ and $u_3 \in BH(\widetilde{\omega})$ be the Kirchhoff-Love components of u. Set

$$\eta := \mathscr{L}^2_{x'} + (proj_{\#}|P|)^s \in \mathscr{M}^+_b(\widetilde{\omega}).$$

Then the following disintegrations hold true:

$$Eu \otimes \mathscr{L}^2_y = \big(A_1(x') + x_3 A_2(x')\big)\,\eta \otimes \mathscr{L}^1_{x_3} \otimes \mathscr{L}^2_y, \tag{4.19}$$

$$E\,\mathscr{L}^3_x \otimes \mathscr{L}^2_y = C(x')E(x,y)\,\eta \otimes \mathscr{L}^1_{x_3} \otimes \mathscr{L}^2_y \tag{4.20}$$

$$P = \eta \overset{\text{gen.}}{\otimes} P_{x'}. \tag{4.21}$$

Above, $A_1, A_2 : \widetilde{\omega} \to \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ and $C : \widetilde{\omega} \to [0,+\infty]$ are respective Radon-Nikodym derivatives of $E\bar{u}$, $-D^2 u_3$ and $\mathscr{L}^2_{x'}$ with respect to η, $E(x,y)$ is a Borel representative of E, and $P_{x'} \in \mathscr{M}_b(I \times \mathscr{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ for η-a.e. $x' \in \widetilde{\omega}$.

Furthermore, we can choose Borel maps $(x',y) \in \widetilde{\omega} \times \mathscr{Y} \mapsto \mu_{x'}(y) \in \mathbb{R}^2$ and $(x',y) \in \widetilde{\omega} \times \mathscr{Y} \mapsto \kappa_{x'}(y) \in \mathbb{R}$ such that, for η-a.e. $x' \in \widetilde{\omega}$,

$$\mu = \mu_{x'}(y)\,\eta \otimes \mathscr{L}^2_y, \quad E_y\mu = \eta \overset{\text{gen.}}{\otimes} E_y\mu_{x'}, \tag{4.22}$$

$$\kappa = \kappa_{x'}(y)\,\eta \otimes \mathcal{L}^2_y, \quad D^2_y\kappa = \eta \overset{\text{gen.}}{\otimes} D^2_y\kappa_{x'}, \tag{4.23}$$

where $\mu_{x'} \in BD(\mathcal{Y})$, $\int_{\mathcal{Y}} \mu_{x'}(y)\,dy = 0$ and $\kappa_{x'} \in BH(\mathcal{Y})$, $\int_{\mathcal{Y}} \kappa_{x'}(y)\,dy = 0$.

Remark 4.2.6. From the above disintegration, we have that, for η-a.e. $x' \in \widetilde{\omega}$,

$$E_y\mu_{x'} - x_3 D^2_y\kappa_{x'} = \left[C(x')E(x,y) - \left(A_1(x') + x_3A_2(x')\right)\right]\mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y + P_{x'} \quad \text{in } I \times \mathcal{Y}.$$

Thus, the quadruplet

$$\left(\mu_{x'}, \kappa_{x'}, \left[C(x')E(x,y) - \left(A_1(x') + x_3A_2(x')\right)\right], P_{x'}\right)$$

is an element of $\mathcal{A}_0$.

4.2.3. Case $\gamma = +\infty$

Definition 4.2.7. Let $w \in H^1(\widetilde{\Omega};\mathbb{R}^3) \cap KL(\widetilde{\Omega})$. We define the class $\mathcal{A}^{hom}_\infty(w)$ of admissible two-scale configurations relative to the boundary datum w as the set of triplets (u,E,P) with

$$u \in KL(\widetilde{\Omega}), \qquad E \in L^2(\widetilde{\Omega} \times \mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}}), \qquad P \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y};\mathbb{M}^{3\times3}_{\text{dev}}),$$

such that

$$u = w, \qquad E = Ew, \qquad P = 0 \qquad \text{on } (\widetilde{\Omega} \setminus \overline{\Omega}) \times \mathcal{Y},$$

and also such that there exist $\mu \in \mathcal{X}_\infty(\widetilde{\Omega})$, $\kappa \in \mathcal{X}_\infty(\widetilde{\Omega})$, $\zeta \in \mathcal{M}_b(\Omega;\mathbb{R}^3)$ with

$$Eu \otimes \mathcal{L}^2_y + \begin{pmatrix} E_y\mu & \zeta' + D_y\kappa \\ (\zeta' + D_y\kappa)^T & \zeta_3 \end{pmatrix} = E\,\mathcal{L}^3_x \otimes \mathcal{L}^2_y + P \qquad \text{in } \widetilde{\Omega} \times \mathcal{Y}. \tag{4.24}$$

Lemma 4.2.8. Let $(u,E,P) \in \mathcal{A}^{hom}_\infty(w)$ with the associated $\mu \in \mathcal{X}_\infty(\widetilde{\Omega})$, $\kappa \in \mathcal{X}_\infty(\widetilde{\Omega})$, $\zeta \in \mathcal{M}_b(\Omega;\mathbb{R}^3)$ and let $\bar{u} \in BD(\widetilde{\omega})$ and $u_3 \in BH(\widetilde{\omega})$ be the Kirchhoff-Love components of u. Set

$$\eta := \mathcal{L}^3_x + (proj_\#|P|)^s \in \mathcal{M}^+_b(\widetilde{\Omega}).$$

Then the following disintegrations hold true:

$$Eu \otimes \mathcal{L}^2_y = \left(A_1(x') + x_3A_2(x')\right)\eta \otimes \mathcal{L}^2_y, \tag{4.25}$$

$$\zeta \otimes \mathcal{L}^2_y = z(x)\,\eta \otimes \mathcal{L}^2_y, \tag{4.26}$$

$$E\,\mathcal{L}^3_x \otimes \mathcal{L}^2_y = C(x)E(x,y)\,\eta \otimes \mathcal{L}^2_y \tag{4.27}$$

$$P = \eta \overset{\text{gen.}}{\otimes} P_x. \tag{4.28}$$

Above, $A_1, A_2 : \widetilde{\omega} \to \mathbb{M}^{2\times 2}_{\text{sym}}$, $z : \widetilde{\omega} \to \mathbb{R}^3$ and $C : \widetilde{\Omega} \to [0, +\infty]$ are respective Radon-Nikodym derivatives of $E\bar{u}$, $-D^2 u_3$, ζ and $\mathscr{L}^3_x$ with respect to η, $E(x,y)$ is a Borel representative of E, and $P_x \in \mathscr{M}_b(\mathscr{Y}; \mathbb{M}^{3\times 3}_{\text{dev}})$ for η-a.e. $x \in \widetilde{\Omega}$.

Furthermore, we can choose Borel maps $(x,y) \in \widetilde{\Omega} \times \mathscr{Y} \mapsto \mu_x(y) \in \mathbb{R}^2$ and $(x,y) \in \widetilde{\Omega} \times \mathscr{Y} \mapsto \kappa_x(y) \in \mathbb{R}$ such that, for η-a.e. $x \in \widetilde{\Omega}$,

$$\mu = \mu_x(y)\, \eta \otimes \mathscr{L}^2_y, \quad E_y \mu = \eta \overset{\text{gen.}}{\otimes} E_y \mu_x, \tag{4.29}$$

$$\kappa = \kappa_x(y)\, \eta \otimes \mathscr{L}^2_y, \quad D^2_y \kappa = \eta \overset{\text{gen.}}{\otimes} D^2_y \kappa_x, \tag{4.30}$$

where $\mu_x \in BD(\mathscr{Y})$, $\int_{\mathscr{Y}} \mu_x(y)\, dy = 0$ and $\kappa_x \in BV(\mathscr{Y})$, $\int_{\mathscr{Y}} \kappa_x(y)\, dy = 0$.

Remark 4.2.9. From the above disintegration, we have that, for η-a.e. $x \in \widetilde{\Omega}$,

$$\begin{pmatrix} E_y \mu_x & z' + D_y \kappa_x \\ (z' + D_y \kappa_x)^T & z_3 \end{pmatrix} = \left[C(x) E(x,y) - \begin{pmatrix} A_1(x') + x_3 A_2(x') & 0 \\ 0 & 0 \end{pmatrix} \right] \mathscr{L}^2_y + P_x \quad \textit{in } \mathscr{Y}.$$

Thus, the quintuplet

$$\left(\mu_x, \kappa_x, z, \left[C(x) E(x,y) - \begin{pmatrix} A_1(x') + x_3 A_2(x') & 0 \\ 0 & 0 \end{pmatrix} \right], P_x \right)$$

is an element of $\mathscr{A}_\infty$.

4.3. Admissible stress configurations and approximations

For every $e^h \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$ we denote $\sigma^h(x) := \mathbb{C}\left(\frac{x'}{\varepsilon_h}\right)\Lambda_h e^h(x)$. Then, in view of [24, Theorem 3.6], we introduce

$$\mathscr{K}_h = \Big\{\sigma^h \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}) : \mathrm{div}_h\sigma^h = 0 \text{ in } \Omega,\ \sigma^h\nu = 0 \text{ in } \partial\Omega\setminus\overline{\Gamma}_D,$$
$$\sigma^h_{\mathrm{dev}}(x',x_3) \in K\left(\frac{x'}{\varepsilon_h}\right) \text{ for a.e. } x'\in\omega,\, x_3\in I\Big\}.$$

If we consider the weak limit $\sigma \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$ of the sequence $\sigma^h \in \mathscr{K}_h$ as $h\to 0$, then $\sigma_{i3} = 0$ for $i = 1,2,3$. To see this, let $v \in C^\infty_c(\Omega;\mathbb{R}^3)$ and $V \in C^\infty(\overline{\Omega};\mathbb{R}^3)$ be defined by

$$V(x',x_3) := \int_{-\frac{1}{2}}^{x_3} v(x',\zeta)\,d\zeta.$$

From the condition $\mathrm{div}_h\sigma^h = 0$ in Ω, for every $\varphi \in H^1(\Omega;\mathbb{R}^3)$ with $\varphi = 0$ on Γ_D we have

$$\int_\Omega \sigma^h(x) : E_h\varphi(x)\,dx = 0. \tag{4.31}$$

By putting

$$\varphi(x) = \begin{pmatrix} 2hV_1(x) \\ 2hV_2(x) \\ hV_3(x) \end{pmatrix},$$

and passing to the limit, it is easy to see that

$$\int_\Omega \sigma(x) : \begin{pmatrix} 0 & 0 & v_1(x) \\ 0 & 0 & v_2(x) \\ v_1(x) & v_2(x) & v_3(x) \end{pmatrix} dx = \int_\Omega \sigma(x) : \begin{pmatrix} 0 & 0 & \partial_{x_3}V_1(x) \\ 0 & 0 & \partial_{x_3}V_2(x) \\ \partial_{x_3}V_1(x) & \partial_{x_3}V_2(x) & \partial_{x_3}V_3(x) \end{pmatrix} dx = 0,$$

and consequently $\sigma_{i3} = 0$ since v was arbitrary.

Furthermore, since the uniform boundedness of sets $K(y)$ implies that the deviatoric part of the weak limit, i.e. $\sigma_{\mathrm{dev}} = \sigma - \frac{1}{3}\mathrm{tr}\,\sigma I_{3\times3}$, is bounded in $L^\infty(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$, we have that

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} \sigma_{11}+\sigma_{22} & 0 & 0 \\ 0 & \sigma_{11}+\sigma_{22} & 0 \\ 0 & 0 & \sigma_{11}+\sigma_{22} \end{pmatrix} \text{ is bounded in } L^\infty(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}).$$

Hence, we can concluded that the components $\sigma_{\alpha\beta}$ are all bounded in $L^\infty(\Omega)$.

Lastly, let $\bar{\varphi} \in C_c^\infty(\omega;\mathbb{R}^3)$. If we choose the function

$$\varphi(x) = \begin{pmatrix} \bar{\varphi}_1(x') - x_3\, \partial_{x_1} \bar{\varphi}_3(x') \\ \bar{\varphi}_2(x') - x_3\, \partial_{x_2} \bar{\varphi}_3(x') \\ \frac{1}{h}\, \bar{\varphi}_3(x') \end{pmatrix},$$

we deduce from (4.31) that

$$\int_\Omega \sigma^h(x) : \begin{pmatrix} E\,\bar{\varphi}(x') - x_3 D^2 \bar{\varphi}_3(x') & 0 \\ 0 & 0 \end{pmatrix} dx = 0.$$

Passing to the limit, we immediately get that

$$\operatorname{div}_{x'} \bar{\sigma} = 0 \text{ in } \omega, \text{ and } \operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma} = 0 \text{ in } \omega.$$

4.3.1. Case $\gamma \in (0,+\infty)$

Definition 4.3.1. The set $\mathscr{K}_\gamma^{hom}$ is the set of all elements $\Sigma \in L^2(\Omega \times \mathscr{Y};\mathbb{M}^{3\times 3}_{\text{sym}})$ satisfying:

(i) $\widetilde{\operatorname{div}}_\gamma \Sigma(x',\cdot) = 0$ in $I \times \mathscr{Y}$ for a.e. $x' \in \omega$,

(ii) $\Sigma(x',\cdot)\,\vec{e}_3 = 0$ on $\partial I \times \mathscr{Y}$ for a.e. $x' \in \omega$,

(iii) $\Sigma_{\text{dev}}(x,y) \in K(y)$ for $\mathscr{L}^3_x \otimes \mathscr{L}^2_y$-a.e. $(x,y) \in \Omega \times \mathscr{Y}$,

(iv) $\sigma_{i3}(x) = 0$ for $i = 1,2,3$,

(v) $\operatorname{div}_{x'} \bar{\sigma} = 0$ in ω,

(vi) $\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma} = 0$ in ω,

where $\sigma := \int_{\mathscr{Y}} \Sigma(\cdot,y)\,dy$, and $\bar{\sigma}$, $\hat{\sigma} \in L^2(\omega;\mathbb{M}^{2\times 2}_{\text{sym}})$ are the zero-th and first order moments of the 2×2 minor of σ.

Proposition 4.3.2. Let $\{\sigma^h\}$ be a bounded family in $L^2(\Omega;\mathbb{M}^{3\times 3}_{\text{sym}})$ such that $\sigma^h \in \mathscr{K}_h$ and

$$\sigma^h \xrightharpoonup{2} \Sigma \quad \text{two-scale weakly in } L^2(\Omega \times \mathscr{Y};\mathbb{M}^{3\times 3}_{\text{sym}}).$$

Then $\Sigma \in \mathscr{K}_\gamma^{hom}$.

Proof. We consider the test function $\varepsilon_h \phi\left(x, \frac{x'}{\varepsilon_h}\right)$, for $\phi \in C_c^\infty(\omega; C^\infty(\bar{I} \times \mathscr{Y}; \mathbb{R}^3))$. We can see that

$$\nabla_h \left(\varepsilon_h \phi\left(x, \frac{x'}{\varepsilon_h}\right)\right) = \left[\, \varepsilon_h \nabla_{x'} \phi\left(x, \frac{x'}{\varepsilon_h}\right) + \nabla_y \phi\left(x, \frac{x'}{\varepsilon_h}\right) \,\middle|\, \frac{\varepsilon_h}{h} \partial_{x_3} \phi\left(x, \frac{x'}{\varepsilon_h}\right) \right]$$

converges strongly in $L^2(\Omega \times \mathscr{Y}; \mathbb{M}^{3\times 3})$. Hence, taking such a test function in (4.31) and passing to the limit, we get

$$\int_{\Omega\times\mathscr{Y}} \Sigma(x,y) : \widetilde{E}_\gamma \phi\,(x,y)\, dxdy = 0.$$

Suppose now that $\phi(x,y) = \psi^{(1)}(x')\, \psi^{(2)}(x_3,y)$ for $\psi^{(1)} \in C_c^\infty(\omega)$ and $\psi^{(2)} \in C^\infty(\bar{I} \times \mathscr{Y}; \mathbb{R}^3)$. Then

$$\int_\omega \psi^{(1)}(x') \left(\int_{I\times\mathscr{Y}} \Sigma(x,y) : \widetilde{E}_\gamma \psi^{(2)}(x_3,y)\, dx_3 dy \right) dx' = 0,$$

from which we can deduce that, for a.e. $x' \in \omega$,

$$\begin{aligned} 0 &= \int_{I\times\mathscr{Y}} \Sigma(x,y) : \widetilde{E}_\gamma \psi^{(2)}(x_3,y)\, dx_3 dy \\ &= -\int_{I\times\mathscr{Y}} \widetilde{\operatorname{div}}_\gamma \Sigma(x,y) \cdot \psi^{(2)}(x_3,y)\, dx_3 dy + \int_{\partial(I\times\mathscr{Y})} \Sigma(x,y)\, \nu \cdot \psi^{(2)}(x_3,y)\, d\mathscr{H}^2(x_3,y) \\ &= -\int_{I\times\mathscr{Y}} \widetilde{\operatorname{div}}_\gamma \Sigma(x,y) \cdot \psi^{(2)}(x_3,y)\, dx_3 dy + \int_{\partial I\times\mathscr{Y}} \Sigma(x,y)\, \vec{e}_3 \cdot \psi^{(2)}(x_3,y)\, d\mathscr{H}^2(x_3,y), \end{aligned}$$

from which we can conclude $\widetilde{\operatorname{div}}_\gamma \Sigma(x',\cdot) = 0$ in $I \times \mathscr{Y}$ and $\Sigma(x',\cdot)\, \vec{e}_3 = 0$ on $\partial I \times \mathscr{Y}$.

Finally, we define

$$\Sigma^h(x,y) = \sum_{i \in I_{\varepsilon_h}(\widetilde{\omega})} \mathbb{1}_{Q^i_{\varepsilon_h}}(x')\, \sigma^h(\varepsilon_h i + \varepsilon_h \mathscr{I}(y), x_3), \tag{4.32}$$

and consider the set

$$S = \{\Xi \in L^2(\Omega \times \mathscr{Y}; \mathbb{M}^{3\times 3}_{\text{sym}}) : \Xi_{\text{dev}}(x,y) \in K(y) \ \text{ for } \mathscr{L}^3_x \otimes \mathscr{L}^2_y\text{-a.e. } (x,y) \in \Omega \times \mathscr{Y}\}.$$

The construction of Σ^h from $\sigma^h \in \mathscr{K}_h$ ensures that $\Sigma^h \in S$ and that $\Sigma^h \rightharpoonup \Sigma$ weakly in $L^2(\Omega \times \mathscr{Y}; \mathbb{M}^{3\times 3}_{\text{sym}})$. Since compactness of $K(y)$ implies that S is is convex and weakly closed in $L^2(\Omega \times \mathscr{Y}; \mathbb{M}^{3\times 3}_{\text{sym}})$, we have that $\Sigma \in S$, which concludes the proof. ∎

Lemma 4.3.3. Let $\omega \subset \mathbb{R}^2$ be an open bounded set that is star-shaped with respect to one of its points and let $\Sigma \in \mathscr{K}_\gamma^{hom}$. Then, there exists a sequence $\Sigma_n \in L^2(\mathbb{R}^2 \times I \times \mathscr{Y}; \mathbb{M}^{3\times 3}_{\text{sym}})$ such that the following holds:

(a) $\Sigma_n \in C^\infty(\mathbb{R}^2; L^2(I \times \mathscr{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}))$ and $\Sigma_n \to \Sigma$ strongly in $L^2(\omega \times I \times \mathscr{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$,

(b) $\widetilde{\mathrm{div}}_\gamma \Sigma_n(x', \cdot) = 0$ on $I \times \mathscr{Y}$ for every $x' \in \mathbb{R}^2$,

(c) $\Sigma_n(x', \cdot)\vec{e}_3 = 0$ on $\partial I \times \mathscr{Y}$ for every $x' \in \mathbb{R}^2$,

(d) $(\Sigma_n(x,y))_{\mathrm{dev}} \in K(y)$ for every $x' \in \mathbb{R}^2$ and $\mathscr{L}^1_{x_3} \otimes \mathscr{L}^2_y$-a.e. $(x_3, y) \in I \times \mathscr{Y}$.

Further, if we set $\sigma_n(x) := \int_{\mathscr{Y}} \Sigma_n(x,y)\,dy$, and $\bar{\sigma}_n$, $\hat{\sigma}_n \in L^2(\omega; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of σ_n, then:

(e) $\sigma_n \in C^\infty(\mathbb{R}^2 \times I; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ and $\sigma_n \to \sigma$ strongly in $L^2(\omega \times I; \mathbb{M}^{3\times3}_{\mathrm{sym}})$,

(f) $\mathrm{div}_{x'}\bar{\sigma}_n = 0$ in ω,

(g) $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}_n = 0$ in ω.

Proof. After a translation we may assume that ω is star-shaped with respect to the origin.

We can extend Σ to $\mathbb{R}^2 \times I \times \mathscr{Y}$ by setting $\Sigma = 0$ outside $\Omega \times \mathscr{Y}$. Let ρ be the standard mollifier on $\mathbb{R}^2$ and let us define the planar dilation $d_n(x') = \left(\frac{n}{n+1}x'\right)$, for every $n \in \mathbb{N}$. Owing to (1.8), we can find a vanishing sequence $\varepsilon_n > 0$ such that for every map $\varphi \in C^\infty_c(\omega; \mathbb{R}^2)$

$$\mathrm{supp}(\rho_{\varepsilon_n} * \varphi) \subset\subset \tfrac{n+1}{n}\omega = d_n^{-1}(\omega) \implies \mathrm{supp}\left((\rho_{\varepsilon_n} * \varphi) \circ d_n^{-1}\right) \subset\subset \omega. \tag{4.33}$$

We then set

$$\Sigma_n(x', x_3, y) := \left((\Sigma \circ d_n)(\cdot, x_3, y) * \rho_{\varepsilon_n}\right)(x'). \tag{4.34}$$

With a slight abuse of notation, it is immediate to see that

$$\begin{aligned}
\sigma_n(x', x_3) &= \left((\sigma \circ d_n)(\cdot, x_3) * \rho_{\varepsilon_n}\right)(x'),\\
\bar{\sigma}_n(x') &= \left((\bar{\sigma} \circ d_n) * \rho_{\varepsilon_n}\right)(x'),\\
\hat{\sigma}_n(x') &= \left((\hat{\sigma} \circ d_n) * \rho_{\varepsilon_n}\right)(x').
\end{aligned}$$

From the above construction items (a) and (e) immediately follow, while item (d) follows from Jensen's inequality since $K(y)$ is convex. Next, we can see that for $x' \in \mathbb{R}^2$

$$\widetilde{\mathrm{div}}_\gamma \Sigma_n(x', \cdot) = \widetilde{\mathrm{div}}_\gamma (\Sigma \circ d_n) * \rho_{\varepsilon_n} = 0 \text{ in } I \times \mathscr{Y},$$

which proves item (b).

Item (f) follows from the computation that, for every map $\varphi \in C_c^\infty(\omega;\mathbb{R}^2)$,

$$\begin{aligned}
\langle \operatorname{div}_{x'} \bar{\sigma}_n, \varphi \rangle &= -\int_{\mathbb{R}^2} \bar{\sigma}_n : \nabla_{x'} \varphi \, dx' \\
&= -\int_{\mathbb{R}^2} (\bar{\sigma} \circ d_n) : (\rho_{\varepsilon_n} * \nabla_{x'} \varphi) \, dx' \\
&= -\int_{\mathbb{R}^2} (\bar{\sigma} \circ d_n) : \nabla_{x'} (\rho_{\varepsilon_n} * \varphi) \, dx' \\
&= -(\tfrac{n+1}{n})^2 \int_{\mathbb{R}^2} \bar{\sigma} : [\nabla_{x'} (\rho_{\varepsilon_n} * \varphi) \circ d_n^{-1}] \, dx' \\
&= -(\tfrac{n+1}{n}) \int_{\mathbb{R}^2} \bar{\sigma} : \nabla_{x'} [(\rho_{\varepsilon_n} * \varphi) \circ d_n^{-1}] \, dx' \\
&= (\tfrac{n+1}{n}) \langle \operatorname{div}_{x'} \bar{\sigma}, (\rho_{\varepsilon_n} * \varphi) \circ d_n^{-1} \rangle = 0,
\end{aligned}$$

where in last equation we used that $\operatorname{div}_{x'} \bar{\sigma} = 0$ in ω and (4.33).

Similarly for item (g), for every map $\varphi \in C_c^\infty(\omega)$ we have

$$\begin{aligned}
\langle \operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma}_n, \varphi \rangle &= \int_{\mathbb{R}^2} \bar{\sigma}_n : \nabla_{x'}^2 \varphi \, dx' \\
&= \int_{\mathbb{R}^2} (\hat{\sigma} \circ d_n) : (\rho_{\varepsilon_n} * \nabla_{x'}^2 \varphi) \, dx' \\
&= \int_{\mathbb{R}^2} (\hat{\sigma} \circ d_n) : \nabla_{x'}^2 (\rho_{\varepsilon_n} * \varphi) \, dx' \\
&= (\tfrac{n+1}{n})^2 \int_{\mathbb{R}^2} \hat{\sigma} : [\nabla_{x'}^2 (\rho_{\varepsilon_n} * \varphi) \circ d_n^{-1}] \, dx' \\
&= \int_{\mathbb{R}^2} \hat{\sigma} : \nabla_{x'}^2 [(\rho_{\varepsilon_n} * \varphi) \circ d_n^{-1}] \, dx' \\
&= \langle \operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma}, (\rho_{\varepsilon_n} * \varphi) \circ d_n^{-1} \rangle = 0,
\end{aligned}$$

where in last equation we used that $\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma} = 0$ in ω and (4.33). ■

4.3.2. Case $\gamma = 0$

Definition 4.3.4. The set $\mathscr{K}_0^{hom}$ is the set of all elements $\Sigma \in L^\infty(\Omega \times \mathscr{Y}; \mathbb{M}_{\text{sym}}^{3\times 3})$ satisfying:

(i) $\Sigma_{i3}(x,y) = 0$ for $i = 1,2,3$,

(ii) $\Sigma_{\text{dev}}(x,y) \in K(y)$ for $\mathscr{L}_x^3 \otimes \mathscr{L}_y^2$-a.e. $(x,y) \in \Omega \times \mathscr{Y}$,

(iii) $\operatorname{div}_y \bar{\Sigma}(x', \cdot) = 0$ in $\mathscr{Y}$ for a.e. $x' \in \omega$,

(iv) $\operatorname{div}_y \operatorname{div}_y \hat{\Sigma}(x', \cdot) = 0$ in $\mathscr{Y}$ for a.e. $x' \in \omega$,

(v) $\operatorname{div}_{x'} \bar{\sigma} = 0$ in ω,

(vi) $\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma} = 0$ in ω,

where $\bar{\Sigma}, \hat{\Sigma} \in L^\infty(\omega \times \mathscr{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of Σ, $\sigma := \int_{\mathscr{Y}} \Sigma(\cdot, y)\, dy$, and $\bar{\sigma}, \hat{\sigma} \in L^\infty(\omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of σ.

Proposition 4.3.5. Let $\{\sigma^h\}$ be a bounded family in $L^2(\Omega; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$ such that $\sigma^h \in \mathscr{K}_h$ and

$$\sigma^h \xrightarrow{2} \Sigma \quad \text{two-scale weakly in } L^2(\Omega \times \mathscr{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}}).$$

Then $\Sigma \in \mathscr{K}_0^{hom}$.

Proof. First, let $\phi \in C_c^\infty(\omega; C^\infty(\mathscr{Y}; \mathbb{R}^3))$ and consider the test function

$$\varphi(x) = \varepsilon_h \begin{pmatrix} \phi_1(x', \frac{x'}{\varepsilon_h}) \\ \phi_2(x', \frac{x'}{\varepsilon_h}) \\ 0 \end{pmatrix} + \varepsilon_h{}^2 \begin{pmatrix} -x_3\, \partial_{x_1} \phi_3(x', \frac{x'}{\varepsilon_h}) - \frac{x_3}{\varepsilon_h} \partial_{y_1} \phi_3(x', \frac{x'}{\varepsilon_h}) \\ -x_3\, \partial_{x_2} \phi_3(x', \frac{x'}{\varepsilon_h}) - \frac{x_3}{\varepsilon_h} \partial_{y_2} \phi_3(x', \frac{x'}{\varepsilon_h}) \\ \frac{1}{h} \phi_3(x', \frac{x'}{\varepsilon_h}) \end{pmatrix}.$$

By direct computation we can see that

$$E_h \varphi(x) \to \begin{pmatrix} E_y \phi'(x', y) - x_3 D_y^2 \phi_3(x', y) & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & 0 \end{pmatrix} \quad \text{strongly in } L^2(\Omega \times \mathscr{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}}).$$

Hence, taking such a test function in (4.31) and passing to the limit, we get

$$\int_{\Omega \times \mathscr{Y}} \Sigma(x, y) : \begin{pmatrix} E_y \phi' - x_3 D_y^2 \phi_3 & 0 \\ 0 & 0 \end{pmatrix} dx dy = 0.$$

Suppose now that $\phi\,(x', y) = \psi^{(1)}(x')\, \psi^{(2)}(y)$ for $\psi^{(1)} \in C_c^\infty(\omega)$ and $\psi^{(2)} \in C^\infty(\mathscr{Y}; \mathbb{R}^3)$. Then

$$\int_\omega \psi^{(1)}(x') \left(\int_{I \times \mathscr{Y}} \Sigma(x, y) : \begin{pmatrix} E_y (\psi^{(2)})'(y) - x_3 D_y^2 \psi_3^{(2)}(y) & 0 \\ 0 & 0 \end{pmatrix} dx_3 dy \right) dx' = 0,$$

from which we can deduce that, for a.e. $x' \in \omega$,

$$\begin{aligned}0 &= \int_{I\times\mathscr{Y}} \Sigma(x,y) : \begin{pmatrix} E_y(\psi^{(2)})'(y) - x_3 D_y^2 \psi_3^{(2)}(y) & 0 \\ 0 & 0 \end{pmatrix} dx_3 dy \\ &= \int_{\mathscr{Y}} \bar{\Sigma}(x',y) : E_y(\psi^{(2)})'(y)\, dy - \frac{1}{12}\int_{\mathscr{Y}} \hat{\Sigma}(x',y) : D_y^2 \psi_3^{(2)}(y)\, dy \\ &= -\int_{\mathscr{Y}} \mathrm{div}_y \bar{\Sigma}(x',y)\cdot(\psi^{(2)})'(y)\, dy - \frac{1}{12}\int_{\mathscr{Y}} \mathrm{div}_y \mathrm{div}_y \hat{\Sigma}(x',y)\cdot \psi_3^{(2)}(y)\, dy,\end{aligned}$$

from which we can conclude $\mathrm{div}_y \bar{\Sigma}(x',\cdot) = 0$ in $\mathscr{Y}$ and $\mathrm{div}_y \mathrm{div}_y \hat{\Sigma}(x',\cdot) = 0$ in $\mathscr{Y}$.

Next, let $\psi \in C_c^\infty(\omega; C^\infty(\bar{I}\times\mathscr{Y};\mathbb{R}^3))$ and consider the test function $h\psi\left(x, \frac{x'}{\varepsilon_h}\right)$. We can see that

$$\nabla_h\left(h\psi\left(x,\frac{x'}{\varepsilon_h}\right)\right) = \left[\, h\nabla_{x'}\psi\left(x,\frac{x'}{\varepsilon_h}\right) + \frac{h}{\varepsilon_h}\nabla_y\psi\left(x,\frac{x'}{\varepsilon_h}\right) \,\middle|\, \partial_{x_3}\psi\left(x,\frac{x'}{\varepsilon_h}\right)\right]$$

converges strongly in $L^2(\Omega\times\mathscr{Y};\mathbb{M}^{3\times 3})$. Hence, taking such a test function in (4.31) and passing to the limit, we get

$$\int_{\Omega\times\mathscr{Y}} \Sigma(x,y) : \begin{pmatrix} 0 & 0 & \partial_{x_3}\psi_1(x,y) \\ 0 & 0 & \partial_{x_3}\psi_2(x,y) \\ \partial_{x_3}\psi_1(x,y) & \partial_{x_3}\psi_2(x,y) & \partial_{x_3}\psi_3(x,y) \end{pmatrix} dxdy = 0,$$

which is sufficient to conclude that $\Sigma_{i3}(x,y) = 0$ for $i = 1,2,3$.

Finally, if we choose the approximating sequence (4.32), the same argument as in the proof of Proposition 4.3.2 implies the stress constraint $\Sigma_{\mathrm{dev}}(x,y) \in K(y)$ for $\mathscr{L}_x^3 \otimes \mathscr{L}_y^2$-a.e. $(x,y) \in \Omega\times\mathscr{Y}$, which concludes the proof. ■

Lemma 4.3.6. Let $\omega \subset \mathbb{R}^2$ be an open bounded set that is star-shaped with respect to one of its points and let $\Sigma \in \mathscr{K}_0^{hom}$. Then, there exists a sequence $\Sigma_n \in L^\infty(\mathbb{R}^2\times I\times\mathscr{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$ such that the following holds:

(a) $\Sigma_n \in C^\infty(\mathbb{R}^2; L^\infty(I\times\mathscr{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}}))$ and $\Sigma_n \to \Sigma$ strongly in $L^p(\omega\times I\times\mathscr{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$, for $1 \le p < +\infty$.

(b) $(\Sigma_n)_{i3}(x,y) = 0$ for $i = 1,2,3$,

(c) $(\Sigma_n(x,y))_{\mathrm{dev}} \in K(y)$ for every $x' \in \mathbb{R}^2$ and $\mathscr{L}_{x_3}^1 \otimes \mathscr{L}_y^2$-a.e. $(x_3,y) \in I\times\mathscr{Y}$,

(d) $\mathrm{div}_y \bar{\Sigma}_n(x',\cdot) = 0$ in $\mathscr{Y}$ for every $x' \in \omega$,

(e) $\mathrm{div}_y\mathrm{div}_y\hat{\Sigma}_n(x',\cdot)=0$ in $\mathscr{Y}$ for every $x'\in\omega$,

where $\bar{\Sigma}_n, \hat{\Sigma}_n \in L^\infty(\omega\times\mathscr{Y};\mathbb{M}^{2\times2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of Σ_n. Further, if we set $\sigma_n(x):=\int_{\mathscr{Y}}\Sigma_n(x,y)\,dy$, and $\bar{\sigma}_n$, $\hat{\sigma}_n\in L^\infty(\omega;\mathbb{M}^{2\times2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of σ_n, then:

(f) $\sigma_n\in C^\infty(\mathbb{R}^2\times I;\mathbb{M}^{3\times3}_{\mathrm{sym}})$ and $\sigma_n\to\sigma$ strongly in $L^p(\omega\times I;\mathbb{M}^{3\times3}_{\mathrm{sym}})$, for $1\le p<+\infty$.

(g) $\mathrm{div}_{x'}\bar{\sigma}_n=0$ in ω,

(h) $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}_n=0$ in ω.

Proof. The proof is analogous to that of Lemma 4.3.3. ■

4.3.3. Case $\gamma=+\infty$

Definition 4.3.7. The set $\mathscr{K}^{hom}_\infty$ is the set of all elements $\Sigma\in L^2(\Omega\times\mathscr{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})$ satisfying:

(i) $\mathrm{div}_y\Sigma(x,\cdot)=0$ in $\mathscr{Y}$ for a.e. $x\in\Omega$,

(ii) $\Sigma_{\mathrm{dev}}(x,y)\in K(y)$ for $\mathscr{L}^3_x\otimes\mathscr{L}^2_y$-a.e. $(x,y)\in\Omega\times\mathscr{Y}$,

(iii) $\sigma_{i3}(x)=0$ for $i=1,2,3$,

(iv) $\mathrm{div}_{x'}\bar{\sigma}=0$ in ω,

(v) $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}=0$ in ω,

where $\sigma:=\int_{\mathscr{Y}}\Sigma(\cdot,y)\,dy$, and $\bar{\sigma}$, $\hat{\sigma}\in L^2(\omega;\mathbb{M}^{2\times2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of σ.

Proposition 4.3.8. Let $\{\sigma^h\}$ be a bounded family in $L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$ such that $\sigma^h\in\mathscr{K}_h$ and

$$\sigma^h \xrightharpoonup{2} \Sigma \quad \text{two-scale weakly in } L^2(\Omega\times\mathscr{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}}).$$

Then $\Sigma\in\mathscr{K}^{hom}_\infty$.

Proof. We consider the test function $\varepsilon_h \phi\left(x, \frac{x'}{\varepsilon_h}\right)$, for $\phi \in C_c^\infty(\omega; C^\infty(\bar{I} \times \mathscr{Y}; \mathbb{R}^3))$. We can see that

$$\nabla_h \left(\varepsilon_h \phi\left(x, \frac{x'}{\varepsilon_h}\right) \right) = \left[\varepsilon_h \nabla_{x'} \phi\left(x, \frac{x'}{\varepsilon_h}\right) + \nabla_y \phi\left(x, \frac{x'}{\varepsilon_h}\right) \middle| \frac{\varepsilon_h}{h} \partial_{x_3} \phi\left(x, \frac{x'}{\varepsilon_h}\right) \right]$$

converges strongly in $L^2(\Omega \times \mathscr{Y}; \mathbb{M}^{3\times 3})$. Hence, taking such a test function in (4.31) and passing to the limit, we get

$$\int_{\Omega \times \mathscr{Y}} \Sigma(x,y) : E_y \phi(x,y)\, dxdy = 0.$$

Suppose now that $\phi(x,y) = \psi^{(1)}(x)\, \psi^{(2)}(y)$ for $\psi^{(1)} \in C_c^\infty(\omega; C^\infty(\bar{I}))$ and $\psi^{(2)} \in C^\infty(\mathscr{Y}; \mathbb{R}^3)$. Then

$$\int_\Omega \psi^{(1)}(x) \left(\int_{\mathscr{Y}} \Sigma(x,y) : E_y \psi^{(2)}(y)\, dy \right) dx = 0,$$

from which we can deduce that $\mathrm{div}_y \Sigma(x,\cdot) = 0$ in $\mathscr{Y}$ for a.e. $x \in \Omega$.

To conclude the proof, it remains to show the stress constraint $\Sigma_{\mathrm{dev}}(x,y) \in K(y)$ for $\mathscr{L}^3_x \otimes \mathscr{L}^2_y$-a.e. $(x,y) \in \Omega \times \mathscr{Y}$. To do this we can define the approximating sequence (4.32) and argue as in the proof of Proposition 4.3.2. ■

Lemma 4.3.9. Let $\omega \subset \mathbb{R}^2$ be an open bounded set that is star-shaped with respect to one of its points and let $\Sigma \in \mathscr{K}^{hom}_\infty$. Then, there exists a sequence $\Sigma_n \in L^2(\mathbb{R}^2 \times I \times \mathscr{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$ such that the following holds:

(a) $\Sigma_n \in C^\infty(\mathbb{R}^3; L^2(\mathscr{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}}))$ and $\Sigma_n \to \Sigma$ strongly in $L^2(\omega \times I \times \mathscr{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$,

(b) $\mathrm{div}_y \Sigma_n(x,\cdot) = 0$ on $\mathscr{Y}$ for every $x \in \mathbb{R}^3$,

(c) $(\Sigma_n(x,y))_{\mathrm{dev}} \in K(y)$ for every $x \in \mathbb{R}^3$ and $\mathscr{L}^2_y$-a.e. $y \in \mathscr{Y}$.

Further, if we set $\sigma_n(x) := \int_{\mathscr{Y}} \Sigma_n(x,y)\, dy$, and $\bar{\sigma}_n, \hat{\sigma}_n \in L^2(\omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of σ_n, then:

(d) $\sigma_n \in C^\infty(\mathbb{R}^2 \times I; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$ and $\sigma_n \to \sigma$ strongly in $L^2(\omega \times I; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$,

(e) $\mathrm{div}_{x'} \bar{\sigma}_n = 0$ in ω,

(f) $\mathrm{div}_{x'} \mathrm{div}_{x'} \hat{\sigma}_n = 0$ in ω.

Proof. The proof is analogous to that of Lemma 4.3.3. The only difference is that the convolution and dilation used to define Σ_n in Step 1 are taken in $\mathbb{R}^3$ instead of $\mathbb{R}^2$. ■

4.4. THE PRINCIPLE OF MAXIMUM PLASTIC WORK

The aim of this section is to prove the following inequality between two-scale dissipation and plastic work, which will be used to prove the global stability condition of the two-scale quasistatic evolution.

Proposition 4.4.1. Let $\gamma \in [0,+\infty]$. Then

$$\mathcal{H}^{hom}(P) \geq -\int_{\Omega\times\mathcal{Y}} \Sigma : E \,dxdy + \int_{\omega} \bar{\sigma} : E\bar{w}\,dx' - \frac{1}{12}\int_{\omega} \hat{\sigma} : D^2 w_3\,dx',$$

for every $\Sigma \in \mathcal{K}_{\gamma}^{hom}$ and $(u,E,P) \in \mathcal{A}_{\gamma}^{hom}(w)$.

The proof of the above inequality is an immediate consequence of the results given below (see Remark 4.4.4, Remark 4.4.7 and Remark 4.4.10).

4.4.1. Case $\gamma \in (0,+\infty)$

Proposition 4.4.2. Let $\Sigma \in \mathcal{K}_{\gamma}^{hom}$ and $(u,E,P) \in \mathcal{A}_{\gamma}^{hom}(w)$ with the associated $\mu \in \mathcal{X}_{\gamma}(\widetilde{\omega})$. There exists an element $\lambda \in \mathcal{M}_b(\widetilde{\Omega}\times\mathcal{Y})$ such that for every $\varphi \in C_c^2(\widetilde{\omega})$

$$\begin{aligned}\langle\lambda,\varphi\rangle = &-\int_{\Omega\times\mathcal{Y}} \varphi(x')\,\Sigma : E\,dxdy + \int_{\omega} \varphi\,\bar{\sigma} : E\bar{w}\,dx' - \frac{1}{12}\int_{\omega} \varphi\,\hat{\sigma} : D^2 w_3\,dx' \\ &-\int_{\omega} \bar{\sigma} : ((\bar{u}-\bar{w})\odot\nabla\varphi)\,dx' - \frac{1}{6}\int_{\omega} \hat{\sigma} : \big(\nabla(u_3-w_3)\odot\nabla\varphi\big)\,dx' \\ &-\frac{1}{12}\int_{\omega}(u_3-w_3)\,\hat{\sigma} : \nabla^2\varphi\,dx'.\end{aligned}$$

Furthermore, the mass of λ is given by

$$\lambda(\widetilde{\Omega}\times\mathcal{Y}) = -\int_{\Omega\times\mathcal{Y}} \Sigma : E\,dxdy + \int_{\omega} \bar{\sigma} : E\bar{w}\,dx' - \frac{1}{12}\int_{\omega} \hat{\sigma} : D^2 w_3\,dx'. \tag{4.35}$$

Proof. The proof is divided into two steps.

Step 1. Suppose that ω is star-shaped with respect to one of its points.

Let $\{\Sigma_n\} \subset C^\infty(\mathbb{R}^2; L^2(I\times\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}}))$ be sequence given by Lemma 4.3.3. We define the sequence

$$\lambda_n := \eta \overset{\text{gen.}}{\otimes} [(\Sigma_n)_{\mathrm{dev}}(x',\cdot) : P_{x'}] \in \mathcal{M}_b(\widetilde{\Omega}\times\mathcal{Y}),$$

where the duality $[(\Sigma_n)_{\text{dev}}(x',\cdot):P_{x'}]$ is a well defined bounded measure on $I\times\mathscr{Y}$ for η-a.e. $x'\in\widetilde{\omega}$. Further, in view of Remark 4.2.3, the expression from (4.1) gives

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathscr{Y}}\psi\,d[(\Sigma_n)_{\text{dev}}(x',\cdot):P_{x'}]\\
&=-\int_{I\times\mathscr{Y}}\psi(x_3,y)\,\Sigma_n(x,y):\left[C(x')E(x,y)-\begin{pmatrix}A_1(x')+x_3A_2(x') & 0\\ 0 & 0\end{pmatrix}\right]dx_3dy\\
&\quad-\int_{I\times\mathscr{Y}}\Sigma_n(x,y):\big(\mu_{x'}(x_3,y)\odot\widetilde{\nabla}_\gamma\psi(x_3,y)\big)\,dx_3dy,
\end{aligned}$$

for every $\psi\in C^1(\mathbb{R}\times\mathscr{Y})$, and

$$|[(\Sigma_n)_{\text{dev}}(x',\cdot):P_{x'}]|\le\|(\Sigma_n)_{\text{dev}}(x',\cdot)\|_{L^\infty(I\times\mathscr{Y};\mathbb{M}^{3\times3}_{\text{sym}})}|P_{x'}|\le C|P_{x'}|,$$

where the last inequality stems from item (d) in Lemma 4.3.3. This in turn implies that

$$|\lambda_n|=\eta\overset{\text{gen.}}{\otimes}|[(\Sigma_n)_{\text{dev}}(x',\cdot):P_{x'}]|\le C\,\eta\overset{\text{gen.}}{\otimes}|P_{x'}|=C|P|,$$

from which we conclude that is $\{\lambda_n\}$ is a bounded sequence.

Let now $\widetilde{I}\supset I$ be an open set which compactly contains I. Let ξ be a smooth cut-off function with $\xi\equiv1$ on I, with support contained in $\widetilde{I}$. Finally, we consider a test function $\phi(x,y):=\varphi(x')\xi(x_3)$, for $\varphi\in C_c^\infty(\widetilde{\omega})$. Then, since $\widetilde{\nabla}_\gamma\phi(x,y)=0$, we have

$$\begin{aligned}
\langle\lambda_n,\phi\rangle&=\int_{\widetilde{\omega}}\left(\int_{I\times\mathscr{Y}}\phi(x,y)\,d[(\Sigma_n)_{\text{dev}}(x',\cdot):P_{x'}]\right)d\eta(x')\\
&=-\int_{\widetilde{\Omega}\times\mathscr{Y}}\varphi(x')\,\Sigma_n(x,y):\left[C(x')E(x,y)-\begin{pmatrix}A_1(x')+x_3A_2(x') & 0\\ 0 & 0\end{pmatrix}\right]d\left(\eta\otimes\mathscr{L}^1_{x_3}\otimes\mathscr{L}^2_y\right)\\
&=-\int_{\widetilde{\Omega}\times\mathscr{Y}}\varphi(x')\,\Sigma_n(x,y):E(x,y)\,dxdy+\int_{\widetilde{\Omega}}\varphi(x')\,\sigma_n(x):\begin{pmatrix}A_1(x')+x_3A_2(x') & 0\\ 0 & 0\end{pmatrix}d\left(\eta\otimes\mathscr{L}^1_{x_3}\right)\\
&=-\int_{\widetilde{\Omega}\times\mathscr{Y}}\varphi(x')\,\Sigma_n(x,y):E(x,y)\,dxdy+\int_{\widetilde{\Omega}}\varphi(x')\,\sigma_n(x):dEu(x)
\end{aligned}$$

Since $u\in KL(\widetilde{\Omega})$, we have

$$\int_{\widetilde{\Omega}}\varphi(x')\,\sigma_n(x):dEu(x)=\int_{\widetilde{\omega}}\varphi(x')\,\bar{\sigma}_n(x'):dE\bar{u}(x')-\frac{1}{12}\int_{\widetilde{\omega}}\varphi(x')\,\hat{\sigma}_n(x'):dD^2u_3(x'),$$

where $\bar{u}\in BD(\widetilde{\omega})$ and $u_3\in BH(\widetilde{\omega})$ are the Kirchhoff-Love components of u. From the characterization given in Proposition 2.2.4, we can thus conclude that

$$\begin{aligned}\int_{\widetilde{\Omega}} \varphi(x')\,\sigma_n(x) : dEu(x) &= \int_{\widetilde{\omega}} \varphi(x')\,\bar{\sigma}_n(x') : \bar{e}(x')\,dx' + \int_{\widetilde{\omega}} \varphi(x')\,\bar{\sigma}_n(x') : d\bar{p}(x') \\ &\quad + \frac{1}{12}\int_{\widetilde{\omega}} \varphi(x')\,\hat{\sigma}_n(x') : \hat{e}(x')\,dx' + \frac{1}{12}\int_{\widetilde{\omega}} \varphi(x')\,\hat{\sigma}_n(x') : d\hat{p}(x') \\ &= \int_{\widetilde{\omega}} \varphi(x')\,\bar{\sigma}_n(x') : \bar{e}(x')\,dx' + \int_{\widetilde{\omega}} \varphi(x')\,d[\bar{\sigma}_n : \bar{p}](x') \\ &\quad + \frac{1}{12}\int_{\widetilde{\omega}} \varphi(x')\,\hat{\sigma}_n(x') : \hat{e}(x')\,dx' + \frac{1}{12}\int_{\widetilde{\omega}} \varphi(x')\,d[\hat{\sigma}_n : \hat{p}](x'),\end{aligned}$$

where in the last equality we used that $\bar{\sigma}_n$ and $\hat{\sigma}_n$ are smooth functions. Notice that, since $\bar{p} \equiv 0$ and $\hat{p} \equiv 0$ outside of $\omega \cup \gamma_D$, we have

$$\int_{\widetilde{\omega}} \varphi\,d[\bar{\sigma}_n : \bar{p}] = \int_{\omega\cup\gamma_D} \varphi\,d[\bar{\sigma}_n : \bar{p}], \quad \int_{\widetilde{\omega}} \varphi\,d[\hat{\sigma}_n : \hat{p}] = \int_{\omega\cup\gamma_D} \varphi\,d[\hat{\sigma}_n : \hat{p}].$$

Furthermore, since $e = E = E\bar{w} - x_3 D^2 w_3$ on $\widetilde{\Omega} \setminus \Omega$, we can conclude that

$$\begin{aligned}\langle \lambda_n, \phi \rangle &= -\int_{\widetilde{\Omega}\times\mathscr{Y}} \varphi(x')\,\Sigma_n : E\,dxdy + \int_{\widetilde{\omega}} \varphi\,\bar{\sigma}_n : \bar{e}\,dx' + \frac{1}{12}\int_{\widetilde{\omega}} \varphi\,\hat{\sigma}_n : \hat{e}\,dx' \\ &\quad + \int_{\omega\cup\gamma_D} \varphi\,d[\bar{\sigma}_n : \bar{p}] + \frac{1}{12}\int_{\omega\cup\gamma_D} \varphi\,d[\hat{\sigma}_n : \hat{p}] \\ &= -\int_{\Omega\times\mathscr{Y}} \varphi(x')\,\Sigma_n : E\,dxdy + \int_{\omega} \varphi\,\bar{\sigma}_n : \bar{e}\,dx' + \frac{1}{12}\int_{\omega} \varphi\,\hat{\sigma}_n : \hat{e}\,dx' \\ &\quad + \int_{\omega\cup\gamma_D} \varphi\,d[\bar{\sigma}_n : \bar{p}] + \frac{1}{12}\int_{\omega\cup\gamma_D} \varphi\,d[\hat{\sigma}_n : \hat{p}].\end{aligned}$$

Considering $\mathrm{div}_{x'}\bar{\sigma}_n = 0$ in ω, from [18, Proposition 7.2] we have for every $\varphi \in C^1(\overline{\omega})$

$$\int_{\omega\cup\gamma_D} \varphi\,d[\bar{\sigma}_n : \bar{p}] + \int_{\omega} \varphi\,\bar{\sigma}_n : (\bar{e} - E\bar{w})\,dx' + \int_{\omega} \bar{\sigma}_n : ((\bar{u} - \bar{w}) \odot \nabla\varphi)\,dx' = 0.$$

Likewise considering $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}_n = 0$ in ω and $u_3 = w_3$ on γ_D, from [18, Proposition 7.6] we have for every $\varphi \in C^2(\overline{\omega})$

$$\begin{aligned}&\int_{\omega\cup\gamma_D} \varphi\,d[\hat{\sigma}_n : \hat{p}] + \int_{\omega} \varphi\,\hat{\sigma}_n : (\hat{e} + D^2 w_3)\,dx' \\ &+ 2\int_{\omega} \hat{\sigma}_n : \big(\nabla(u_3 - w_3) \odot \nabla\varphi\big)\,dx' + \int_{\omega} (u_3 - w_3)\,\hat{\sigma}_n : \nabla^2\varphi\,dx' = 0.\end{aligned}$$

Let now $\lambda \in \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y})$ be such that (up to a subsequence)

$$\lambda_n \xrightharpoonup{*} \lambda \quad \text{weakly* in } \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y}).$$

By items (a) and (e) in Lemma 4.3.3, we have in the limit

$$
\begin{aligned}
\langle \lambda, \phi \rangle &= \lim_n \langle \lambda_n, \phi \rangle \\
&= \lim_n \Big[- \int_{\Omega \times \mathscr{Y}} \varphi(x') \Sigma_n : E \, dx dy + \int_\omega \varphi \, \bar{\sigma}_n : E\bar{w} \, dx' - \frac{1}{12} \int_\omega \varphi \, \hat{\sigma}_n : D^2 w_3 \, dx' \\
&\qquad - \int_\omega \bar{\sigma}_n : ((\bar{u} - \bar{w}) \odot \nabla \varphi) \, dx' - \frac{1}{6} \int_\omega \hat{\sigma}_n : \big(\nabla (u_3 - w_3) \odot \nabla \varphi\big) \, dx' \\
&\qquad - \frac{1}{12} \int_\omega (u_3 - w_3) \, \hat{\sigma}_n : \nabla^2 \varphi \, dx' \Big] \\
&= - \int_{\Omega \times \mathscr{Y}} \varphi(x') \Sigma : E \, dx dy + \int_\omega \varphi \, \bar{\sigma} : E\bar{w} \, dx' - \frac{1}{12} \int_\omega \varphi \, \hat{\sigma} : D^2 w_3 \, dx' \\
&\quad - \int_\omega \bar{\sigma} : ((\bar{u} - \bar{w}) \odot \nabla \varphi) \, dx' - \frac{1}{6} \int_\omega \hat{\sigma} : \big(\nabla (u_3 - w_3) \odot \nabla \varphi\big) \, dx' \\
&\quad - \frac{1}{12} \int_\omega (u_3 - w_3) \, \hat{\sigma} : \nabla^2 \varphi \, dx'.
\end{aligned}
$$

Taking $\varphi \nearrow \mathbb{1}_{\widetilde{\omega}}$, we deduce (4.35).

Step 2. If ω is not star-shaped, then since ω is a bounded C^2 domain (in particular, with Lipschitz boundary) by Proposition 1.7.4 there exists a finite open covering $\{U_i\}$ of $\overline{\omega}$ such that $\omega \cap U_i$ is (strongly) star-shaped with Lipschitz boundary.

Let $\{\psi_i\}$ be a smooth partition of unity subordinate to the covering $\{U_i\}$, i.e. $\psi_i \in C^\infty(\overline{\omega})$, with $0 \leq \psi_i \leq 1$, such that $\operatorname{supp}(\psi_i) \subset U_i$ and $\sum_i \psi_i = 1$ on $\overline{\omega}$.

For each i, let

$$
\Sigma^i(x,y) := \begin{cases} \Sigma(x,y) & \text{if } x' \in \omega \cap U_i, \\ 0 & \text{otherwise.} \end{cases}
$$

Since $\Sigma^i \in \mathscr{K}_\gamma^{hom}$, the construction in Step 1 yields that there exist sequences $\{\Sigma_n^i\} \subset C^\infty(\mathbb{R}^2; L^2(I \times \mathscr{Y}; \mathbb{M}^{3\times 3}_{\text{sym}}))$ and

$$
\lambda_n^i := \eta \overset{\text{gen.}}{\otimes} [(\Sigma_n^i)_{\text{dev}}(x', \cdot) : P_{x'}] \in \mathscr{M}_b((\omega \cap U_i) \times I \times \mathscr{Y}),
$$

such that

$$
\lambda_n^i \xrightharpoonup{*} \lambda^i \quad \text{weakly* in } \mathscr{M}_b((\omega \cap U_i) \times I \times \mathscr{Y}),
$$

with

$$
\begin{aligned}
\langle \lambda^i, \varphi \rangle &= - \int_{(\omega \cap U_i) \times I \times \mathscr{Y}} \varphi(x') \Sigma : E \, dx dy + \int_{\omega \cap U_i} \varphi \, \bar{\sigma} : E\bar{w} \, dx' - \frac{1}{12} \int_{\omega \cap U_i} \varphi \, \hat{\sigma} : D^2 w_3 \, dx' \\
&\quad - \int_{\omega \cap U_i} \bar{\sigma} : ((\bar{u} - \bar{w}) \odot \nabla \varphi) \, dx' - \frac{1}{6} \int_{\omega \cap U_i} \hat{\sigma} : \big(\nabla (u_3 - w_3) \odot \nabla \varphi\big) \, dx' \\
&\quad - \frac{1}{12} \int_{\omega \cap U_i} (u_3 - w_3) \, \hat{\sigma} : \nabla^2 \varphi \, dx'.
\end{aligned}
$$

for every $\varphi \in C^2_c(\overline{\omega} \cap U_i)$. This allows us to define measures on $\widetilde{\Omega} \times \mathscr{Y}$ by letting, for every $\phi \in C_0(\widetilde{\Omega} \times \mathscr{Y})$,

$$\langle \lambda_n, \phi \rangle := \sum_i \langle \lambda_n^i, \psi_i(x')\, \phi \rangle,$$

and

$$\langle \lambda, \phi \rangle := \sum_i \langle \lambda^i, \psi_i(x')\, \phi \rangle.$$

Then we can see that $\lambda_n \xrightharpoonup{*} \lambda$ weakly* in $\mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y})$, and λ satisfies all the required properties. ■

Theorem 4.4.3. Let $\Sigma \in \mathscr{K}_\gamma^{hom}$ and $(u,E,P) \in \mathscr{A}_\gamma^{hom}(w)$ with the associated $\mu \in \mathscr{X}_\gamma(\widetilde{\omega})$. Then

$$H\left(y, \frac{dP}{d|P|}\right) |P| \geq \lambda,$$

where $\lambda \in \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y})$ is given by Proposition 4.4.2.

Proof. Let $\{\Sigma_n^i\}$, $\{\lambda_n^i\}$ and λ^i be defined as in Step 2 of the proof of Proposition 4.4.2. Item (d) in Lemma 4.3.3 implies that

$$(\Sigma_n^i)_{\mathrm{dev}}(x,y) \in K(y) \text{ for every } x' \in \omega \text{ and } \mathscr{L}^1_{x_3} \otimes \mathscr{L}^2_y\text{-a.e. } (x_3,y) \in I \times \mathscr{Y}.$$

By Proposition 4.1.6, we have for η-a.e. $x' \in \widetilde{\omega}$

$$H\left(y, \frac{dP_{x'}}{d|P_{x'}|}\right) |P_{x'}| \geq [(\Sigma_n^i)_{\mathrm{dev}}(x',\cdot) : P_{x'}] \text{ as measures on } I \times \mathscr{Y}.$$

Since $\frac{dP}{d|P|}(x,y) = \frac{dP_{x'}}{d|P_{x'}|}(x_3,y)$ for $|P_{x'}|$-a.e. $(x_3,y) \in I \times \mathscr{Y}$ by Proposition 1.3.2, we can conclude that

$$\begin{aligned} H\left(y, \frac{dP}{d|P|}\right) |P| &= \eta \overset{\text{gen.}}{\otimes} H\left(y, \frac{dP}{d|P|}\right) |P_{x'}| = \eta \overset{\text{gen.}}{\otimes} H\left(y, \frac{dP_{x'}}{d|P_{x'}|}\right) |P_{x'}| \\ &= \sum_i \psi_i \eta \overset{\text{gen.}}{\otimes} H\left(y, \frac{dP_{x'}}{d|P_{x'}|}\right) |P_{x'}| \\ &\geq \sum_i \psi_i \eta \overset{\text{gen.}}{\otimes} [(\Sigma_n^i)_{\mathrm{dev}}(x',\cdot) : P_{x'}] \\ &= \sum_i \psi_i \lambda_n^i = \lambda_n. \end{aligned}$$

By passing to the limit, we have the desired inequality. ■

Remark 4.4.4. As a consequence of the above theorem and (4.35), we have the proof of Proposition 4.4.1 for $\gamma \in (0,+\infty)$.

4.4.2. Case $\gamma = 0$

Proposition 4.4.5. Let $\Sigma \in \mathscr{K}_0^{hom}$ and $(u, E, P) \in \mathscr{A}_0^{hom}(w)$ with the associated $\mu \in \mathscr{X}_0(\widetilde{\omega})$, $\kappa \in \Upsilon_0(\widetilde{\omega})$. There exists an element $\lambda \in \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y})$ such that for every $\varphi \in C_c^2(\widetilde{\omega})$

$$\begin{aligned}\langle \lambda, \varphi \rangle = &- \int_{\Omega \times \mathscr{Y}} \varphi(x')\, \Sigma : E\, dx dy + \int_\omega \varphi\, \bar{\sigma} : E\bar{w}\, dx' - \frac{1}{12} \int_\omega \varphi\, \hat{\sigma} : D^2 w_3\, dx' \\ &- \int_\omega \bar{\sigma} : ((\bar{u} - \bar{w}) \odot \nabla \varphi)\, dx' - \frac{1}{6} \int_\omega \hat{\sigma} : \big(\nabla(u_3 - w_3) \odot \nabla \varphi\big)\, dx' \\ &- \frac{1}{12} \int_\omega (u_3 - w_3)\, \hat{\sigma} : \nabla^2 \varphi\, dx'.\end{aligned}$$

Furthermore, the mass of λ is given by

$$\lambda(\widetilde{\Omega} \times \mathscr{Y}) = - \int_{\Omega \times \mathscr{Y}} \Sigma : E\, dx dy + \int_\omega \bar{\sigma} : E\bar{w}\, dx' - \frac{1}{12} \int_\omega \hat{\sigma} : D^2 w_3\, dx'. \qquad (4.36)$$

Proof. Suppose that ω is star-shaped with respect to one of its points.

Let $\{\Sigma_n\} \subset C^\infty(\mathbb{R}^2; L^2(I \times \mathscr{Y}; \mathbb{M}^{3\times3}_{\text{sym}}))$ be sequence given by Lemma 4.3.6. We define the sequence

$$\lambda_n := \eta \overset{\text{gen.}}{\otimes} [\Sigma_n(x', \cdot) : P_{x'}] \in \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y}),$$

where the duality $[\Sigma_n(x', \cdot) : P_{x'}]$ is a well defined bounded measure on $I \times \mathscr{Y}$ for η-a.e. $x' \in \widetilde{\omega}$. Further, in view of Remark 4.2.6, the expressions from (4.4) gives

$$\begin{aligned}&\int_{I \times \mathscr{Y}} \psi\, d[\Sigma_n(x', \cdot) : P_{x'}] \\ &= - \int_{I \times \mathscr{Y}} \psi(y)\, \Sigma_n(x, y) : \big[C(x')E(x, y) - \big(A_1(x') + x_3 A_2(x')\big)\big]\, dx_3 dy \\ &\quad - \int_{\mathscr{Y}} \bar{\Sigma}_n(x', y) : \big(\mu_{x'}(y) \odot \nabla_y \psi(y)\big)\, dy \\ &\quad + \frac{1}{6} \int_{\mathscr{Y}} \hat{\Sigma}_n(x', y) : \big(\nabla_y \kappa_{x'}(y) \odot \nabla_y \psi(y)\big)\, dy + \frac{1}{12} \int_{\mathscr{Y}} \kappa_{x'}(y)\, \hat{\Sigma}_n(x', y) : \nabla_y^2 \psi(y)\, dy,\end{aligned}$$

for every $\psi \in C^2(\mathscr{Y})$, and

$$|[\Sigma_n(x', \cdot) : P_{x'}]| \leq \|\Sigma_n(x', \cdot)\|_{L^\infty(I \times \mathscr{Y}; \mathbb{M}^{2\times2}_{\text{sym}})} |P_{x'}| \leq C\, |P_{x'}|,$$

where the last inequality stems from item (c) in Lemma 4.3.6. This in turn implies that

$$|\lambda_n| = \eta \overset{\text{gen.}}{\otimes} |[\Sigma_n(x', \cdot) : P_{x'}]| \leq C\, \eta \overset{\text{gen.}}{\otimes} |P_{x'}| = C\, |P|,$$

from which we conclude that is $\{\lambda_n\}$ is a bounded sequence.

Let now $\widetilde{I} \supset I$ be an open set which compactly contains I. Let ξ be a smooth cut-off function with $\xi \equiv 1$ on I, with support contained in $\widetilde{I}$. Finally, we consider a test function $\phi(x,y) := \varphi(x')\xi(x_3)$, for $\varphi \in C_c^\infty(\widetilde{\omega})$. Then, since $\nabla_y \phi(x,y) = 0$ and $\nabla_y^2 \phi(x,y) = 0$, we have

$$\begin{aligned}
\langle \lambda_n, \phi \rangle &= \int_{\widetilde{\omega}} \left(\int_{I \times \mathscr{Y}} \phi(x,y) \, d[\Sigma_n(x',\cdot) : P_{x'}] \right) d\eta(x') \\
&= - \int_{\widetilde{\Omega} \times \mathscr{Y}} \varphi(x') \Sigma_n(x,y) : \left[C(x')E(x,y) - \left(A_1(x') + x_3 A_2(x') \right) \right] d\left(\eta \otimes \mathscr{L}^1_{x_3} \otimes \mathscr{L}^2_y \right) \\
&= - \int_{\widetilde{\Omega} \times \mathscr{Y}} \varphi(x') \Sigma_n(x,y) : E(x,y) \, dx dy + \int_{\widetilde{\Omega}} \varphi(x') \, \sigma_n(x) : \left(A_1(x') + x_3 A_2(x') \right) d\left(\eta \otimes \mathscr{L}^1_{x_3} \right) \\
&= - \int_{\widetilde{\Omega} \times \mathscr{Y}} \varphi(x') \Sigma_n(x,y) : E(x,y) \, dx dy + \int_{\widetilde{\Omega}} \varphi(x') \, \sigma_n(x) : dEu(x)
\end{aligned}$$

From this point on, the proof is exactly the same as the proof of Proposition 4.4.2. ■

Theorem 4.4.6. Let $\Sigma \in \mathscr{K}_0^{hom}$ and $(u,E,P) \in \mathscr{A}_0^{hom}(w)$ with the associated $\mu \in \mathscr{X}_0(\widetilde{\omega})$, $\kappa \in \Upsilon_0(\widetilde{\omega})$. Then

$$\int_{\widetilde{\Omega} \times \mathscr{Y}} \varphi(y) H_r\left(y, \frac{dP}{d|P|}\right) |P| \geq \int_{\widetilde{\Omega} \times \mathscr{Y}} \varphi(y) \, d\lambda, \quad \text{for every } \varphi \in C(\mathscr{Y}), \varphi \geq 0,$$

where $\lambda \in \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y})$ is given by Proposition 4.4.5.

Proof. Let $\{\Sigma_n^i\}$, $\{\lambda_n^i\}$ and λ^i be defined as in Step 2 of the proof of Proposition 4.4.5. Item (c) in Lemma 4.3.6 implies that

$$(\Sigma_n^i)_{\text{dev}}(x,y) \in K(y) \ \text{ for every } x' \in \omega \text{ and } \mathscr{L}^1_{x_3} \otimes \mathscr{L}^2_y\text{-a.e. } (x_3,y) \in I \times \mathscr{Y}.$$

By Proposition 4.1.12, we have for η-a.e. $x' \in \widetilde{\omega}$

$$\int_{I \times \mathscr{Y}} \varphi(y) H_r\left(y, \frac{dP_{x'}}{d|P_{x'}|}\right) d|P_{x'}| \geq \int_{I \times \mathscr{Y}} \varphi(y) \, d[\Sigma_n^i : P_{x'}], \quad \text{for every } \varphi \in C(\mathscr{Y}), \varphi \geq 0.$$

Since $\frac{dP}{d|P|}(x,y) = \frac{dP_{x'}}{d|P_{x'}|}(x_3,y)$ for $|P_{x'}|$-a.e. $(x_3,y) \in I \times \mathscr{Y}$ by Proposition 1.3.2, we can conclude that

$$\begin{aligned}
H_r\left(y, \frac{dP}{d|P|}\right) |P| &= \eta \overset{\text{gen.}}{\otimes} H_r\left(y, \frac{dP}{d|P|}\right) |P_{x'}| = \eta \overset{\text{gen.}}{\otimes} H_r\left(y, \frac{dP_{x'}}{d|P_{x'}|}\right) |P_{x'}| \\
&= \sum_i \psi_i \eta \overset{\text{gen.}}{\otimes} H_r\left(y, \frac{dP_{x'}}{d|P_{x'}|}\right) |P_{x'}|.
\end{aligned}$$

Consequently,

$$\int_{\widetilde{\Omega}\times\mathscr{Y}} \varphi(y) H_r\left(y, \frac{dP}{d|P|}\right) d|P| = \sum_i \int_{\widetilde{\omega}} \psi_i(x') \left(\int_{I\times\mathscr{Y}} \varphi(y) H_r\left(y, \frac{dP_{x'}}{d|P_{x'}|}\right) |P_{x'}| \right) d\eta(x')$$
$$\geq \sum_i \int_{\widetilde{\omega}} \psi_i(x') \left(\int_{I\times\mathscr{Y}} \varphi(y)\, d[\Sigma_n^i : P_{x'}] \right) d\eta(x')$$
$$= \sum_i \int_{\widetilde{\Omega}\times\mathscr{Y}} \psi_i(x')\varphi(y)\, d\lambda_n^i(x,y) = \int_{\widetilde{\Omega}\times\mathscr{Y}} \varphi(y)\, d\lambda_n.$$

By passing to the limit, we have the desired inequality. ■

Remark 4.4.7. As a consequence of the above theorem and (4.36), we have the proof of Proposition 4.4.1 for $\gamma = 0$.

4.4.3. Case $\gamma = +\infty$

Proposition 4.4.8. Let $\Sigma \in \mathscr{K}_\infty^{hom}$ and $(u,E,P) \in \mathscr{A}_\infty^{hom}(w)$ with the associated $\mu \in \mathscr{X}_\infty(\widetilde{\Omega})$, $\kappa \in \mathscr{X}_\infty(\widetilde{\Omega})$, $\zeta \in \mathscr{M}_b(\Omega;\mathbb{R}^3)$. There exists an element $\lambda \in \mathscr{M}_b(\widetilde{\Omega}\times\mathscr{Y})$ such that for every $\varphi \in C_c^2(\widetilde{\omega})$

$$\langle \lambda, \varphi \rangle = -\int_{\Omega\times\mathscr{Y}} \varphi(x')\Sigma : E\, dxdy + \int_\omega \varphi\, \bar{\sigma} : E\bar{w}\, dx' - \frac{1}{12}\int_\omega \varphi\, \hat{\sigma} : D^2 w_3\, dx'$$
$$- \int_\omega \bar{\sigma} : ((\bar{u} - \bar{w}) \odot \nabla\varphi)\, dx' - \frac{1}{6}\int_\omega \hat{\sigma} : \big(\nabla(u_3 - w_3) \odot \nabla\varphi\big)\, dx'$$
$$- \frac{1}{12}\int_\omega (u_3 - w_3)\, \hat{\sigma} : \nabla^2\varphi\, dx'.$$

Furthermore, the mass of λ is given by

$$\lambda(\widetilde{\Omega}\times\mathscr{Y}) = -\int_{\Omega\times\mathscr{Y}} \Sigma : E\, dxdy + \int_\omega \bar{\sigma} : E\bar{w}\, dx' - \frac{1}{12}\int_\omega \hat{\sigma} : D^2 w_3\, dx'. \tag{4.37}$$

Proof. Suppose that ω is star-shaped with respect to one of its points.

Let $\{\Sigma_n\} \subset C^\infty(\mathbb{R}^3; L^2(\mathscr{Y}; \mathbb{M}^{3\times3}_{\text{sym}}))$ be sequence given by Lemma 4.3.9. We define the sequence

$$\lambda_n := \eta \overset{\text{gen.}}{\otimes} [(\Sigma_n)_{\text{dev}}(x,\cdot) : P_x] \in \mathscr{M}_b(\widetilde{\Omega}\times\mathscr{Y}),$$

where the duality $[(\Sigma_n)_{\text{dev}}(x,\cdot) : P_x]$ is a well defined bounded measure on $\mathscr{Y}$ for η-a.e.

$x \in \widetilde{\Omega}$. Further, in view of Remark 4.2.9, the expressions from (4.8) gives

$$\begin{aligned}
&\int_{\mathscr{Y}} \psi \, d[(\Sigma_n)_{\mathrm{dev}}(x,\cdot) : P_x] \\
&= -\int_{\mathscr{Y}} \psi \Sigma_n : \left[C(x)E(x,y) - \begin{pmatrix} A_1(x') + x_3 A_2(x') & 0 \\ 0 & 0 \end{pmatrix} \right] dy \\
&\quad - \int_{\mathscr{Y}} (\Sigma_n)''(x,y) : \big(\mu_x(y) \odot \nabla_y \psi(y)\big) \, dy \\
&\quad - \sum_{\alpha=1,2} \int_{\mathscr{Y}} \kappa_x(y) \, (\Sigma_n)_{\alpha 3}(x,y) \, \partial_{y_\alpha} \psi(y) \, dy + \sum_{i=1,2,3} z_i \int_{\mathscr{Y}} \psi(y) \, (\Sigma_n)_{i3}(x,y) \, dy,
\end{aligned}$$

for every $\psi \in C^1(\mathscr{Y})$, and

$$|[(\Sigma_n)_{\mathrm{dev}}(x,\cdot) : P_x]| \le \|(\Sigma_n)_{\mathrm{dev}}(x,\cdot)\|_{L^\infty(\mathscr{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})} |P_x| \le C\,|P_x|,$$

where the last inequality stems from item (c) in Lemma 4.3.9. This in turn implies that

$$|\lambda_n| = \eta \overset{\text{gen.}}{\otimes} |[(\Sigma_n)_{\mathrm{dev}}(x,\cdot) : P_x]| \le C\,\eta \overset{\text{gen.}}{\otimes} |P_x| = C\,|P|,$$

from which we conclude that is $\{\lambda_n\}$ is a bounded sequence.

Let now $\widetilde{I} \supset I$ be an open set which compactly contains I. Let ξ be a smooth cut-off function with $\xi \equiv 1$ on I, with support contained in $\widetilde{I}$. Finally, we consider a test function $\phi(x,y) := \varphi(x')\xi(x_3)$, for $\varphi \in C_c^\infty(\widetilde{\omega})$. Then, since $\nabla_y \phi(x,y) = 0$, $\partial_{y_\alpha} \phi(x,y) = 0$ and $\int_{\mathscr{Y}} (\Sigma_n)_{i3}(x,y) \, dy = 0$, we have

$$\begin{aligned}
\langle \lambda_n, \phi \rangle &= \int_{\widetilde{\Omega}} \left(\int_{\mathscr{Y}} \phi(x,y) \, d[(\Sigma_n)_{\mathrm{dev}}(x,\cdot) : P_x] \right) d\eta(x) \\
&= -\int_{\widetilde{\Omega}\times\mathscr{Y}} \varphi(x') \Sigma_n(x,y) : \left[C(x)E(x,y) - \begin{pmatrix} A_1(x') + x_3 A_2(x') & 0 \\ 0 & 0 \end{pmatrix} \right] d\left(\eta \otimes \mathscr{L}^2_y\right) \\
&= -\int_{\widetilde{\Omega}\times\mathscr{Y}} \varphi(x') \Sigma_n(x,y) : E(x,y) \, dx dy + \int_{\widetilde{\Omega}} \varphi(x') \, \sigma_n(x) : \big(A_1(x') + x_3 A_2(x')\big) \, d\eta \\
&= -\int_{\widetilde{\Omega}\times\mathscr{Y}} \varphi(x') \Sigma_n(x,y) : E(x,y) \, dx dy + \int_{\widetilde{\Omega}} \varphi(x') \, \sigma_n(x) : dEu(x)
\end{aligned}$$

From this point on, the proof is exactly the same as the proof of Proposition 4.4.2. ■

Theorem 4.4.9. Let $\Sigma \in \mathscr{K}^{hom}_\infty$ and $(u,E,P) \in \mathscr{A}^{hom}_\infty(w)$ with the associated $\mu \in \mathscr{X}_\infty(\widetilde{\Omega})$, $\kappa \in \mathscr{X}_\infty(\widetilde{\Omega})$, $\zeta \in \mathscr{M}_b(\Omega;\mathbb{R}^3)$. Then

$$H\left(y, \frac{dP}{d|P|}\right) |P| \ge \lambda,$$

where $\lambda \in \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y})$ is given by Proposition 4.4.8.

Proof. Let $\{\Sigma_n^i\}$, $\{\lambda_n^i\}$ and λ^i be defined as in Step 2 of the proof of Proposition 4.4.8. Item (c) in Lemma 4.3.9 implies that

$$(\Sigma_n^i)_{\mathrm{dev}}(x,y) \in K(y) \text{ for every } x \in \Omega \text{ and } \mathcal{L}_y^2\text{-a.e. } y \in \mathcal{Y}.$$

By Proposition 4.1.19, we have for η-a.e. $x \in \widetilde{\Omega}$

$$H\left(y, \frac{dP_x}{d|P_x|}\right)|P_x| \geq [(\Sigma_n^i)_{\mathrm{dev}}(x,\cdot) : P_x] \text{ as measures on } \mathcal{Y}.$$

Since $\frac{dP}{d|P|}(x,y) = \frac{dP_x}{d|P_x|}(y)$ for $|P_x|$-a.e. $y \in \mathcal{Y}$ by Proposition 1.3.2, we can conclude that

$$\begin{aligned} H\left(y, \frac{dP}{d|P|}\right)|P| &= \eta \overset{\text{gen.}}{\otimes} H\left(y, \frac{dP}{d|P|}\right)|P_x| = \eta \overset{\text{gen.}}{\otimes} H\left(y, \frac{dP_x}{d|P_x|}\right)|P_x| \\ &= \sum_i \psi_i(x')\eta \overset{\text{gen.}}{\otimes} H\left(y, \frac{dP_x}{d|P_x|}\right)|P_x| \\ &\geq \sum_i \psi_i(x')\eta \overset{\text{gen.}}{\otimes} [(\Sigma_n^i)_{\mathrm{dev}}(x,\cdot) : P_x] \\ &= \sum_i \psi_i(x')\lambda_n^i = \lambda_n. \end{aligned}$$

By passing to the limit, we have the desired inequality. ∎

Remark 4.4.10. As a consequence of the above theorem and (4.37), we have the proof of Proposition 4.4.1 for $\gamma = +\infty$.

4.5. Lower semicontinuity of energy functionals

For $(u,e,p) \in \mathscr{A}_h(w)$, we recall the definition of energy functionals $\mathscr{Q}_h$ and $\mathscr{H}^{hom}$ given in (2.10) and (2.11). For $(u,E,P) \in \mathscr{A}_\gamma^{hom}(w)$ we now define

$$\mathscr{Q}^{hom}(E) := \int_{\Omega\times\mathscr{Y}} Q(y,E)\,dxdy \tag{4.38}$$

and

$$\mathscr{H}^{hom}(P) := \int_{\overline{\Omega}\times\mathscr{Y}} H\left(y, \frac{dP}{d|P|}\right) d|P|. \tag{4.39}$$

Theorem 4.5.1. Let $\gamma \in [0,+\infty]$. Let $(u^h,e^h,p^h) \in \mathscr{A}_h(w)$ be such that

$$u^h \xrightharpoonup{*} u \quad \text{weakly* in } BD(\widetilde{\Omega}), \tag{4.40}$$

$$\Lambda_h e^h \xrightharpoonup{2} E \quad \text{two-scale weakly in } L^2(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\text{sym}}), \tag{4.41}$$

$$\Lambda_h p^h \xrightharpoonup{2-*} P \quad \text{two-scale weakly* in } \mathscr{M}_b(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\text{dev}}), \tag{4.42}$$

with $(u,E,P) \in \mathscr{A}_\gamma^{hom}(w)$. Then we get

$$\mathscr{Q}^{hom}(E) \le \liminf_h \mathscr{Q}_h(\Lambda_h e^h) \tag{4.43}$$

and

$$\mathscr{H}^{hom}(P) \le \liminf_h \mathscr{H}_h(\Lambda_h p^h). \tag{4.44}$$

Proof. Let $\varphi \in C_c^\infty(\Omega\times\mathscr{Y};\mathbb{M}^{3\times3}_{\text{sym}})$. From the coercivity condition of the quadratic form $\mathscr{Q}_h$ we obtain the inequality

$$0 \le \frac{1}{2}\int_\Omega \mathbb{C}\left(\frac{x'}{\varepsilon_h}\right)\left(\Lambda_h e^h(x) - \varphi\left(x,\frac{x'}{\varepsilon_h}\right)\right) : \left(\Lambda_h e^h(x) - \varphi\left(x,\frac{x'}{\varepsilon_h}\right)\right) dx.$$

Since $\mathbb{C}\left(\frac{x'}{\varepsilon_h}\right)\Lambda_h e^h(x) \xrightharpoonup{2} \mathbb{C}(y)E(x,y)$ two-scale weakly $L^2(\Omega\times\mathscr{Y};\mathbb{M}^{3\times3}_{\text{sym}})$, we can apply $\liminf$ to the above inequality to obtain

$$\int_{\Omega\times\mathscr{Y}} \mathbb{C}(y)E(x,y) : \varphi(x,y)\,dxdy - \frac{1}{2}\int_{\Omega\times\mathscr{Y}} \mathbb{C}(y)\varphi(x,y) : \varphi(x,y)\,dx \le \liminf_h \mathscr{Q}_h(\Lambda_h e).$$

Choosing φ such that $\varphi \to E$ strongly in $L^2(\Omega\times\mathscr{Y};\mathbb{M}^{3\times3}_{\text{sym}})$ proves (4.43).

To prove (4.44), let us first note that in the case $\gamma = 0$ and $\gamma = +\infty$, as previously noted in Remark 2.1.3, the dissipation potential H is a lower semicontinuous function, positively 1-homogeneous and convex in the second variable. Thus, the desired lower semicontinuity property follows directly the version of Reshetnyak's lower semicontinuity theorem adapted for two-scale convergence (see [25, Lemma 4.6]).

Let now $\gamma \in (0,+\infty)$. We can assume without loss of generality that

$$\liminf_h \mathscr{H}_h(\Lambda_h p^h) < \infty. \tag{4.45}$$

We can write

$$p^h = \sum_i p_i^h + \sum_{i \neq j} p_{ij}^h \tag{4.46}$$

where $p_i^h := p^h \lfloor \Omega \cap ((\mathscr{Y}_i)_{\varepsilon_h} \times I)$ and $p_{ij}^h := p^h \lfloor \widetilde{\Omega} \cap ((\Gamma_{ij} \setminus S)_{\varepsilon_h} \times I)$. Up to a subsequence,

$$\Lambda_h p_i^h \xrightarrow{2-*} P_i \quad \text{two-scale weakly* in } \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y}; \mathbb{M}^{3\times3}_{\mathrm{dev}}),$$
$$\Lambda_h p_{ij}^h \xrightarrow{2-*} P_{ij} \quad \text{two-scale weakly* in } \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y}; \mathbb{M}^{3\times3}_{\mathrm{dev}}).$$

Clearly,

$$P = \sum_i P_i + \sum_{i \neq j} P_{ij},$$

with $\mathrm{supp}(P_i) \subseteq \widetilde{\Omega} \times \overline{\mathscr{Y}}_i$ and $\mathrm{supp}(P_{ij}) \subseteq \widetilde{\Omega} \times \Gamma_{ij}$. Furthermore, considering (4.41), we can concluded that

$$\Lambda_h E u^h \lfloor \widetilde{\Omega} \cap ((\mathscr{Y}_i)_{\varepsilon_h} \times I) \xrightarrow{2-*} E\, \mathbb{1}_{\widetilde{\Omega} \times \mathscr{Y}_i} \mathscr{L}^3_x \otimes \mathscr{L}^2_y + P_i \quad \text{two-scale weakly* in } \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$$

Recalling (2.2), we can additionally assume that $\Gamma_{ij} \cap \mathscr{C} \subseteq S$. Then, with a normal ν on Γ_{ij} that points from $\mathscr{Y}_j$ to $\mathscr{Y}_i$ for every $j \neq i$, Lemma 3.4.4 implies that

$$P_i \lfloor \widetilde{\Omega} \times (\Gamma_{ij} \setminus S) = -a_{ij}(x,y) \odot \nu(y)\, \eta_{ij} \tag{4.47}$$

for suitable $\eta_{ij} \in \mathscr{M}_b^+(\widetilde{\Omega} \times (\Gamma_{ij} \setminus S))$ and a Borel map $a_{ij} : \widetilde{\Omega} \times (\Gamma_{ij} \setminus S) \to \mathbb{R}^3$ such that $a_{ij} \perp \nu$ for η_{ij}-a.e. $(x,y) \in \widetilde{\Omega} \times (\Gamma_{ij} \setminus S)$.

Using a version of Reshetnyak's lower semicontinuity theorem adapted for two-scale convergence (see [25, Lemma 4.6]), we get

$$
\begin{aligned}
&\liminf_h \int_{\Omega\cup\Gamma_D} H\left(\frac{x'}{\varepsilon_h}, \frac{d\Lambda_h p_i^h}{d|\Lambda_h p_i^h|}\right) d|\Lambda_h p_i^h| \\
&= \liminf_h \int_{\widetilde{\Omega}} H_i\left(\frac{d\Lambda_h p_i^h}{d|\Lambda_h p_i^h|}\right) d|\Lambda_h p_i^h| \geq \int_{\widetilde{\Omega}\times\mathscr{Y}} H_i\left(\frac{dP_i}{d|P_i|}\right) d|P_i| \\
&= \int_{\widetilde{\Omega}\times\mathscr{Y}_i} H_i\left(\frac{dP_i}{d|P_i|}\right) d|P_i| + \int_{\widetilde{\Omega}\times\Gamma} H_i\left(\frac{dP_i}{d|P_i|}\right) d|P_i| \\
&\geq \int_{\widetilde{\Omega}\times\mathscr{Y}_i} H\left(y, \frac{dP_i}{d|P_i|}\right) d|P_i| + \sum_{j\neq i} \int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H_i\left(\frac{dP_i}{d|P_i|}\right) d|P_i| \\
&\geq \int_{\widetilde{\Omega}\times\mathscr{Y}_i} H\left(y, \frac{dP_i}{d|P_i|}\right) d|P_i| + \sum_{j\neq i} \int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H_i\left(-a_{ij}(x,y)\odot\nu(y)\right) d\eta_{ij}.
\end{aligned} \tag{4.48}
$$

Next, we have

$$
\begin{aligned}
\Lambda_h p_{ij}^h &= \Lambda_h \left[(u_i^h - u_j^h)\odot\nu\left(\frac{x'}{\varepsilon_h}\right)\right] \mathscr{H}^2 \lfloor \widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I) \\
&= \left[\operatorname{diag}\left(1,1,\frac{1}{h}\right)(u_i^h - u_j^h)\odot\nu\left(\frac{x'}{\varepsilon_h}\right)\right] \mathscr{H}^2 \lfloor \widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I),
\end{aligned}
$$

where u_i^h and u_j^h are the traces on $\widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I)$ of the restrictions of u^h to $(\mathscr{Y}_i)_{\varepsilon_h}\times I$ and $(\mathscr{Y}_j)_{\varepsilon_h}\times I$ respectively, such that $u_i^h - u_j^h$ is perpendicular to ν. Then, since the infimum in the inf-convolution definition of H on $\Gamma\setminus S$ is actually a minimum, we get

$$
\begin{aligned}
&\int_{\Omega\cup\Gamma_D} H\left(\frac{x'}{\varepsilon_h}, \frac{d\Lambda_h p_{ij}^h}{d|\Lambda_h p_{ij}^h|}\right) d|\Lambda_h p_{ij}^h| \\
&= \int_{\widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I)} H\left(\frac{x'}{\varepsilon_h}, \frac{d\Lambda_h p_{ij}^h}{d|\Lambda_h p_{ij}^h|}\right) d|\Lambda_h p_{ij}^h| \\
&= \int_{\widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I)} H\left(\frac{x'}{\varepsilon_h}, \left[\operatorname{diag}\left(1,1,\frac{1}{h}\right)(u_i^h - u_j^h)\odot\nu\left(\frac{x'}{\varepsilon_h}\right)\right]\right) d\mathscr{H}^2(x) \\
&= \int_{\widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I)} H_{ij}\left(\operatorname{diag}\left(1,1,\frac{1}{h}\right)(u_i^h - u_j^h), \nu\left(\frac{x'}{\varepsilon_h}\right)\right) d\mathscr{H}^2(x) \\
&= \int_{\widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I)} \left[H_i\left(b_i^{h,ij}(x)\odot\nu\left(\frac{x'}{\varepsilon_h}\right)\right) + H_j\left(-b_j^{h,ij}(x)\odot\nu\left(\frac{x'}{\varepsilon_h}\right)\right)\right] d\mathscr{H}^2(x)
\end{aligned} \tag{4.49}
$$

for suitable Borel functions $b_i^{h,ij}, b_j^{h,ij} : \widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I) \to \mathbb{R}^3$ which are perpendicular to ν for $\mathscr{H}^2$-a.e. $x\in(\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I$ and such that

$$
b_i^{h,ij} - b_j^{h,ij} = \operatorname{diag}\left(1,1,\frac{1}{h}\right)(u_i^h - u_j^h) \qquad \text{for } \mathscr{H}^2\text{-a.e. } x\in(\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I.
$$

The fact that $b_i^{h,ij}$, $b_j^{h,ij}$ are Borel functions can be argued by approximating $u_i^h - u_j^h$ along $(\Gamma_{ij} \setminus S)_{\varepsilon_h} \times I$ by simple functions, and recalling ν is continuous. From the coercivity condition of the dissipation potential H and (4.45), we obtain

$$\int_{\widetilde{\Omega} \cap ((\Gamma_{ij} \setminus S)_{\varepsilon_h} \times I)} \left[\left| b_i^{h,ij}(x) \odot \nu\left(\frac{x'}{\varepsilon_h}\right) \right| + \left| b_j^{h,ij}(x) \odot \nu\left(\frac{x'}{\varepsilon_h}\right) \right| \right] d\mathscr{H}^2(x) \leq C,$$

for some constant $C > 0$. This bound implies that the measures

$$\eta_i^{h,ij} = b_i^{h,ij} \mathscr{H}^2 \lfloor \widetilde{\Omega} \cap ((\Gamma_{ij} \setminus S)_{\varepsilon_h} \times I) \text{ and } \eta_j^{h,ij} = b_j^{h,ij} \mathscr{H}^2 \lfloor \widetilde{\Omega} \cap ((\Gamma_{ij} \setminus S)_{\varepsilon_h} \times I)$$

are bounded in h. Thus, by two-scale compactness, we can assume that up to a subsequence

$$\eta_i^{h,ij} \xrightarrow{2-*} \eta_i^{ij} \quad \text{two-scale weakly* in } \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y}; \mathbb{R}^3),$$
$$\eta_j^{h,ij} \xrightarrow{2-*} \eta_j^{ij} \quad \text{two-scale weakly* in } \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y}; \mathbb{R}^3).$$

We denote by b_i^{ij} and b_j^{ij} the Radon-Nikodym derivatives $\frac{d\eta_i^{ij}}{d|\eta_i^{ij}|}$ and $\frac{d\eta_j^{ij}}{d|\eta_j^{ij}|}$, respectively. Then, since the normal vector field ν is continuous on $(\Gamma_{ij} \setminus S)_{\varepsilon_h} \times I$,

$$b_i^{h,ij} \odot \nu\left(\frac{x'}{\varepsilon_h}\right) \mathscr{H}^2 \lfloor \widetilde{\Omega} \cap ((\Gamma_{ij} \setminus S)_{\varepsilon_h} \times I) \xrightarrow{2-*} b_i^{ij} \odot \nu(y) \, |\eta_i^{ij}| \quad \text{two-scale weakly* in } \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y}; \mathbb{R}^3),$$
$$b_j^{h,ij} \odot \nu\left(\frac{x'}{\varepsilon_h}\right) \mathscr{H}^2 \lfloor \widetilde{\Omega} \cap ((\Gamma_{ij} \setminus S)_{\varepsilon_h} \times I) \xrightarrow{2-*} b_j^{ij} \odot \nu(y) \, |\eta_j^{ij}| \quad \text{two-scale weakly* in } \mathscr{M}_b(\widetilde{\Omega} \times \mathscr{Y}; \mathbb{R}^3).$$

In view of the Reshetnyak's lower semicontinuity theorem adapted for two-scale convergence, (4.49) yields

$$\begin{aligned}
&\liminf_h \int_{\Omega \cup \Gamma_D} H\left(\frac{x'}{\varepsilon_h}, \frac{d\Lambda_h p_{ij}^h}{d|\Lambda_h p_{ij}^h|}\right) d|\Lambda_h p_{ij}^h| \\
&= \liminf_h \int_{\widetilde{\Omega} \cap ((\Gamma_{ij} \setminus S)_{\varepsilon_h} \times I)} \left[H_i\left(b_i^{h,ij}(x) \odot \nu\left(\frac{x'}{\varepsilon_h}\right)\right) + H_j\left(-b_j^{h,ij}(x) \odot \nu\left(\frac{x'}{\varepsilon_h}\right)\right) \right] d\mathscr{H}^2(x) \\
&\geq \liminf_h \int_{\widetilde{\Omega} \cap ((\Gamma_{ij} \setminus S)_{\varepsilon_h} \times I)} H_i\left(b_i^{h,ij}(x) \odot \nu\left(\frac{x'}{\varepsilon_h}\right)\right) d\mathscr{H}^2(x) \\
&\quad + \liminf_h \int_{\widetilde{\Omega} \cap ((\Gamma_{ij} \setminus S)_{\varepsilon_h} \times I)} H_j\left(-b_j^{h,ij}(x) \odot \nu\left(\frac{x'}{\varepsilon_h}\right)\right) d\mathscr{H}^2(x) \\
&\geq \int_{\widetilde{\Omega} \times (\Gamma_{ij} \setminus S)} H_i\left(b_i^{ij}(x) \odot \nu(y)\right) d|\eta_i^{ij}| + \int_{\widetilde{\Omega} \times (\Gamma_{ij} \setminus S)} H_j\left(-b_j^{ij}(x) \odot \nu(y)\right) d|\eta_j^{ij}| \qquad (4.50)
\end{aligned}$$

Recaling (4.47), we have

$$\begin{aligned}
P \lfloor \widetilde{\Omega} \times (\Gamma_{ij} \setminus S) &= -a_{ij}(x,y) \odot \nu(y) \, \eta_{ij} + a_{ji}(x,y) \odot \nu(y) \, \eta_{ji} + b_i^{ij} \odot \nu(y) \, |\eta_i^{ij}| - b_j^{ij} \odot \nu(y) \, |\eta_j^{ij}| \\
&= \left(c^i(x,y) - c^j(x,y)\right) \odot \nu(y) \, \zeta_{ij},
\end{aligned}$$

for $\zeta_{ij} = \eta_{ij} + \eta_{ji} + |\eta_i^{ij}| + |\eta_j^{ij}| \in \mathscr{M}_b^+(\widetilde{\Omega} \times (\Gamma_{ij} \setminus S))$, and suitable Borel functions c^i, c^j : $\widetilde{\Omega} \times (\Gamma_{ij} \setminus S) \to \mathbb{R}^3$ which are is perpendicular to ν for ζ_{ij}-a.e. $(x,y) \in \widetilde{\Omega} \times (\Gamma_{ij} \setminus S)$ and such that

$$\begin{aligned} c^i(x,y) \odot \nu(y)\, \zeta_{ij} &= -a_{ij}(x,y) \odot \nu(y)\, \eta_{ij} + b_i^{ij} \odot \nu(y)\, |\eta_i^{ij}|, \\ c^j(x,y) \odot \nu(y)\, \zeta_{ij} &= -a_{ji}(x,y) \odot \nu(y)\, \eta_{ji} + b_j^{ij} \odot \nu(y)\, |\eta_j^{ij}|. \end{aligned}$$

Now, in view of (4.46), from (4.48) and (4.50) we get

$$\begin{aligned}
&\liminf_h \mathscr{H}_h(\Lambda_h p^h) \\
&\geq \sum_i \liminf_h \int_{\Omega\cup\Gamma_D} H\left(\frac{x'}{\varepsilon_h}, \frac{d\Lambda_h p_i^h}{d|\Lambda_h p_i^h|}\right) d|\Lambda_h p_i^h| + \sum_{i\neq j} \liminf_h \int_{\Omega\cup\Gamma_D} H\left(\frac{x'}{\varepsilon_h}, \frac{d\Lambda_h p_{ij}^h}{d|\Lambda_h p_{ij}^h|}\right) d|\Lambda_h p_{ij}^h| \\
&\geq \sum_i \left(\int_{\widetilde{\Omega}\times\mathscr{Y}_i} H\left(y, \frac{dP_i}{d|P_i|}\right) d|P_i| + \sum_{j\neq i} \int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H_i\left(-a_{ij}(x,y)\odot\nu(y)\right) d\eta_{ij} \right) \\
&\quad + \sum_{i\neq j} \left(\int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H_i\left(b_i^{ij}(x)\odot\nu(y)\right) d|\eta_i^{ij}| + \int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H_j\left(-b_j^{ij}(x)\odot\nu(y)\right) d|\eta_j^{ij}| \right) \\
&= \int_{\widetilde{\Omega}\times(\cup_i\mathscr{Y}_i)} H\left(y, \frac{dP}{d|P|}\right) d|P| \\
&\quad + \sum_{i\neq j} \left(\int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H_i\left(-a_{ij}(x,y)\odot\nu(y)\right) d\eta_{ij} + \int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H_j\left(a_{ji}(x,y)\odot\nu(y)\right) d\eta_{ji} \right. \\
&\quad \left. + \int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H_i\left(b_i^{ij}(x)\odot\nu(y)\right) d|\eta_i^{ij}| + \int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H_j\left(-b_j^{ij}(x)\odot\nu(y)\right) d|\eta_j^{ij}| \right) \\
&= \int_{\widetilde{\Omega}\times(\cup_i\mathscr{Y}_i)} H\left(y, \frac{dP}{d|P|}\right) d|P| \\
&\quad + \sum_{i\neq j} \int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} \left[H_i\left(c^i(x,y)\odot\nu(y)\right) d\zeta_{ij} + \int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H_j\left(-c^j(x,y)\odot\nu(y)\right) \right] d\zeta_{ij} \\
&= \int_{\widetilde{\Omega}\times(\cup_i\mathscr{Y}_i)} H\left(y, \frac{dP}{d|P|}\right) d|P| + \sum_{i\neq j} \int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H\left(y, \left(c^i(x,y)-c^j(x,y)\right)\odot\nu(y)\right) d\zeta_{ij} \\
&= \int_{\widetilde{\Omega}\times(\cup_i\mathscr{Y}_i)} H\left(y, \frac{dP}{d|P|}\right) d|P| + \sum_{i\neq j} \int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H\left(y, \frac{dP}{d|P|}\right) d|P| \\
&= \mathscr{H}^{hom}(P),
\end{aligned}$$

which concludes the proof. ■

5. TWO-SCALE QUASI-STATIC EVOLUTIONS

We recall the definition of energy functionals $\mathcal{Q}^{hom}$ and $\mathcal{H}^{hom}$ given in (4.38) and (4.39). The associated $\mathcal{H}^{hom}$-variation of a function $P : [0,T] \to \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{dev}})$ on $[a,b]$ is then defined as

$$\mathcal{D}_{\mathcal{H}^{hom}}(P;a,b) := \sup\left\{\sum_{i=1}^{n} \mathcal{H}^{hom}(P(t_{i+1}) - P(t_i)) : a = t_1 < t_2 < \ldots < t_n = b,\ n \in \mathbb{N}\right\}.$$

We now give the notion of the limiting quasistatic elasto-plastic evolution.

Definition 5.0.1. A *two-scale quasistatic evolution* for the boundary datum $w(t)$ is a function $t \mapsto (u(t), E(t), P(t))$ from $[0,T]$ into $KL(\widetilde{\Omega}) \times L^2(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{sym}}) \times \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{dev}})$ which satisfies the following conditions:

$(\text{qs1})^{hom}_{\gamma}$ for every $t \in [0,T]$ we have $(u(t), E(t), P(t)) \in \mathcal{A}^{hom}_{\gamma}(w(t))$ and

$$\mathcal{Q}^{hom}(E(t)) \leq \mathcal{Q}^{hom}(H) + \mathcal{H}^{hom}(\Pi - P(t)),$$

for every $(\upsilon, H, \Pi) \in \mathcal{A}^{hom}_{\gamma}(w(t))$.

$(\text{qs2})^{hom}_{\gamma}$ the function $t \mapsto P(t)$ from $[0,T]$ into $\mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{dev}})$ has bounded variation and for every $t \in [0,T]$

$$\mathcal{Q}^{hom}(E(t)) + \mathcal{D}_{\mathcal{H}^{hom}}(P;0,t) = \mathcal{Q}^{hom}(E(0)) + \int_0^t \int_{\Omega \times \mathcal{Y}} \mathbb{C}(y)E(s) : E\dot{w}(s)\, dxdyds.$$

Recalling the definition of a h-quasistatic evolution for the boundary datum $w(t)$ given in Definition 2.4.1, we are in a position to formulate the main result of the thesis.

Theorem 5.0.2. Let $t \mapsto w(t)$ be absolutely continuous from $[0,T]$ into $H^1(\widetilde{\Omega};\mathbb{R}^3) \cap KL(\widetilde{\Omega})$. Assume that there exists a sequence of triples $(u_0^h, e_0^h, p_0^h) \in \mathscr{A}_h(w(0))$ such that

$$u_0^h \xrightharpoonup{*} u_0 \quad \text{weakly* in } BD(\widetilde{\Omega}), \tag{5.1}$$

$$\Lambda_h e_0^h \xrightarrow{2} E_0 \quad \text{two-scale strongly in } L^2(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\text{sym}}), \tag{5.2}$$

$$\Lambda_h p_0^h \xrightharpoonup{2-*} P_0 \quad \text{two-scale weakly* in } \mathscr{M}_b(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\text{dev}}), \tag{5.3}$$

for $(u_0,E_0,P_0) \in \mathscr{A}_\gamma^{hom}(w(0))$ if $\gamma \in (0,+\infty]$ and $(u_0,E_0'',P_0'') \in \mathscr{A}_0^{hom}(w(0))$ if $\gamma = 0$. For every $h > 0$, let

$$t \mapsto (u^h(t), e^h(t), p^h(t))$$

be a h-quasistatic evolution for the boundary datum $w(t)$ such that $u^h(0) = u_0^h$, $e^h(0) = e_0^h$, and $p^h(0) = p_0^h$. Then, there exists a two-scale quasistatic evolution

$$t \mapsto (u(t), E(t), P(t))$$

for the boundary datum $w(t)$ such that $u(0) = u_0$, $E(0) = E_0$, and $P(0) = P_0$, and such that (up to subsequences) for every $t \in [0,T]$

$$u^h(t) \xrightharpoonup{*} u(t) \quad \text{weakly* in } BD(\widetilde{\Omega}), \tag{5.4}$$

$$\Lambda_h e^h(t) \xrightharpoonup{2} E(t) \quad \text{two-scale weakly in } L^2(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\text{sym}}), \tag{5.5}$$

$$\Lambda_h p^h(t) \xrightharpoonup{2-*} P(t) \quad \text{two-scale weakly* in } \mathscr{M}_b(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\text{dev}}), \tag{5.6}$$

in case $\gamma \in (0,+\infty]$, and

$$u^h(t) \xrightharpoonup{*} u(t) \quad \text{weakly* in } BD(\widetilde{\Omega}), \tag{5.7}$$

$$e^h(t) \xrightharpoonup{2} \mathbb{A}_y E(t) \quad \text{two-scale weakly in } L^2(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\text{sym}}), \tag{5.8}$$

$$p^h(t) \xrightharpoonup{2-*} \begin{pmatrix} P(t) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{two-scale weakly* in } \mathscr{M}_b(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\text{sym}}), \tag{5.9}$$

in case $\gamma = 0$.

Proof. The proof is divided into steps, in the spirit of evolutionary Γ-convergence. We present the proof in the case $\gamma \in (0,+\infty)$, while the arguments for cases $\gamma = 0$ and $\gamma = +\infty$ are identical upon replacing the appropriate structures in the statement of Theorem 3.3.1 and definition of $\mathscr{A}_\gamma^{hom}(w)$.

Step 1: ***Compactness.***

Firstly, we can prove that that there exists a constant C, depending only on the initial and boundary data, such that

$$\sup_{t\in[0,T]} \left\|\Lambda_h e^h(t)\right\|_{L^2(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})} \leq C \text{ and } \mathscr{D}_{\mathscr{H}_h}(\Lambda_h p^h;0,T) \leq C, \tag{5.10}$$

for every $h > 0$. Indeed, the energy balance of the h-quasistatic evolution $(\mathrm{qs2})_h$ and (2.5) imply

$$\begin{aligned}
&r_c \left\|\Lambda_h e^h(t)\right\|_{L^2(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})} + \mathscr{D}_{\mathscr{H}_h}(\Lambda_h p^h;0,t) \\
&\leq R_c \left\|\Lambda_h e^h(0)\right\|_{L^2(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})} + 2R_c \sup_{t\in[0,T]} \left\|\Lambda_h e^h(t)\right\|_{L^2(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})} \int_0^T \|E\dot{w}(s)\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\mathrm{sym}})}\, ds,
\end{aligned}$$

where the last integral is well defined as $t \mapsto E\dot{w}(t)$ belongs to $L^1([0,T];L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\mathrm{sym}}))$. In view of the boundedness of $\Lambda_h e^h_0$ that is implied by (5.2), property (5.10) now follows by the Cauchy-Schwarz inequality.

Secondly, from the latter inequality in (5.10) and (2.6), we infer that

$$r_k \left\|\Lambda_h p^h(t) - \Lambda_h p^h_0\right\|_{\mathscr{M}_b(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\mathrm{dev}})} \leq \mathscr{H}_h\left(\Lambda_h p^h(t) - \Lambda_h p^h_0\right) \leq \mathscr{D}_{\mathscr{H}_h}(\Lambda_h p^h;0,t) \leq C,$$

for every $t \in [0,T]$, which together with (5.3) implies

$$\sup_{t\in[0,T]} \left\|\Lambda_h p^h(t)\right\|_{\mathscr{M}_b(\widetilde{\Omega}\times\mathscr{Y};\mathbb{M}^{3\times3}_{\mathrm{dev}})} \leq C. \tag{5.11}$$

Next, we note that $\|\cdot\|_{L^1(\widetilde{\Omega}\setminus\overline{\Omega};\mathbb{M}^{3\times3}_{\mathrm{sym}})}$ is a continuous seminorm on $BD(\widetilde{\Omega})$ which is also a norm on the set of rigid motions. Then, using a variant of Poincaré-Korn's inequality (see [45, Chapter II, Proposition 2.4]) and the fact $(u^h(t),e^h(t),p^h(t)) \in \mathscr{A}_h(w(t))$, we can conclude that, for every $h > 0$ and $t \in [0,T]$,

$$\begin{aligned}
\left\|u^h(t)\right\|_{BD(\widetilde{\Omega})} &\leq C\left(\left\|u^h(t)\right\|_{L^1(\widetilde{\Omega}\setminus\overline{\Omega};\mathbb{R}^3)} + \left\|Eu^h(t)\right\|_{\mathscr{M}_b(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\mathrm{sym}})}\right) \\
&\leq C\left(\|w(t)\|_{L^1(\widetilde{\Omega}\setminus\overline{\Omega};\mathbb{R}^3)} + \left\|e^h(t)\right\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\mathrm{sym}})} + \left\|p^h(t)\right\|_{\mathscr{M}_b(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\mathrm{dev}})}\right) \\
&\leq C\left(\|w(t)\|_{L^2(\widetilde{\Omega};\mathbb{R}^3)} + \left\|\Lambda_h e^h(t)\right\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\mathrm{sym}})} + \left\|\Lambda_h p^h(t)\right\|_{\mathscr{M}_b(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\mathrm{dev}})}\right).
\end{aligned}$$

In view of the assumption $w \in H^1(\widetilde{\Omega};\mathbb{R}^3)$, from (5.11) and the former inequality in (5.10) it follows that the sequences $\{u^h(t)\}$ are bounded in $BD(\widetilde{\Omega})$ uniformly with respect to t.

Owing to (1.3), we can conclude that $\mathcal{D}_{\mathcal{H}_h}$ and $\mathcal{V}$ are equivalent norms, which immediately implies

$$\mathcal{V}(\Lambda_h p^h;0,T) \le C, \tag{5.12}$$

for every $h > 0$. Hence, by a generalized version of Helly's selection theorem (see [17, Lemma 7.2]), there exists a (not relabeled) subsequence, independent of t, and $P \in BV(0,T;\mathcal{M}_b(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{dev}}))$ such that

$$\Lambda_h p^h(t) \xrightarrow{2-*} P(t) \quad \text{two-scale weakly* in } \mathcal{M}_b(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{dev}}),$$

for every $t \in [0,T]$, and $\mathcal{V}(P;0,T) \le C$. We extract a further subsequence (possibly depending on t),

$$u^{h_t}(t) \xrightarrow{*} u(t) \quad \text{weakly* in } BD(\widetilde{\Omega}),$$
$$\Lambda_{h_t} e^{h_t}(t) \xrightarrow{2} E(t) \quad \text{two-scale weakly in } L^2(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}}),$$

for every $t \in [0,T]$. From the proof of Proposition 3.0.1, we can conclude for every $t \in [0,T]$ that $u(t) \in KL(\widetilde{\Omega})$. Furthermore, according to Theorem 3.3.1, one can choose the above subsequence in a way such that there exists $\mu(t) \in \mathcal{X}_\gamma(\widetilde{\omega})$ for which

$$\Lambda_h E u^{h_t}(t) \xrightarrow{2-*} Eu(t) \otimes \mathcal{L}^2_y + \widetilde{E}_\gamma \mu(t).$$

Since, $\Lambda_{h_t} E u^{h_t}(t) = \Lambda_{h_t} e^{h_t}(t) + \Lambda_{h_t} p^{h_t}(t)$ in $\widetilde{\Omega}$ for every $h > 0$ and $t \in [0,T]$, we deduce that $(u(t), E(t), P(t)) \in \mathcal{A}^{hom}_\gamma(w(t))$.

Lastly, we consider for every $t \in [0,T]$

$$\sigma^{h_t}(t) := \mathbb{C}\left(\tfrac{x'}{\varepsilon_{h_t}}\right) \Lambda_{h_t} e^{h_t}(t).$$

Then we can choose a (not relabeled) subsequence, such that

$$\sigma^{h_t}(t) \xrightarrow{2} \Sigma(t) \quad \text{two-scale weakly in } L^2(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}}), \tag{5.13}$$

where $\Sigma(t) := \mathbb{C}(y)E(t)$. Since $\sigma^{h_t}(t) \in \mathcal{K}_{h_t}$ for every $t \in [0,T]$, by Proposition 4.3.2 we can conclude $\Sigma(t) \in \mathcal{K}^{hom}_\gamma$.

Step 2: ***Global stability.***

Since from Step 1 we have $(u(t), E(t), P(t)) \in \mathcal{A}^{hom}_\gamma(w(t))$ with the associated $\mu(t) \in \mathcal{X}_\gamma(\widetilde{\omega})$, then for every $(\upsilon, H, \Pi) \in \mathcal{A}^{hom}_\gamma(w(t))$ with the associated $\nu \in \mathcal{X}_\gamma(\widetilde{\omega})$ we have

$$(\upsilon - u(t), H - E(t), \Pi - P(t)) \in \mathcal{A}^{hom}_\gamma(0).$$

Furthermore, since from the first step of the proof $\mathbb{C}(y)E(t) \in \mathscr{K}_\gamma^{hom}$, by Proposition 4.4.1 we have

$$\mathscr{H}^{hom}(\Pi - P(t)) \geq -\int_{\omega\times I\times\mathscr{Y}} \mathbb{C}(y)E(t) : (H - E(t))\, dxdy$$
$$= \mathscr{Q}^{hom}(E(t)) + \mathscr{Q}^{hom}(H - E(t)) - \mathscr{Q}^{hom}(H),$$

where the last equality is a straightforward computation. From the above, we immediately deduce

$$\mathscr{H}^{hom}(\Pi - P(t)) + \mathscr{Q}^{hom}(H) \geq \mathscr{Q}^{hom}(E(t)) + \mathscr{Q}^{hom}(H - E(t)) \geq \mathscr{Q}^{hom}(E(t)),$$

hence the global stability of the two-scale quasistatic evolution $(\text{qs1})_\gamma^{hom}$.

Now we can prove that limit functions $u(t)$ and $E(t)$ do not depend on the subsequence. Assume $(\upsilon(t), H(t), P(t)) \in \mathscr{A}_\gamma^{hom}(w(t))$ with the associated $\nu(t) \in \mathscr{X}_\gamma(\widetilde{\Omega})$ also satisfy the global stability of the two-scale quasistatic evolution. By the strict convexity of $\mathscr{Q}^{hom}$, we immediately obtain that

$$H(t) = E(t).$$

Then, using (4.13), we have that

$$E\upsilon(t) \otimes \mathscr{L}_y^2 + \widetilde{E}_\gamma \nu(t) = H(t)\, \mathscr{L}_x^3 \otimes \mathscr{L}_y^2 + P(t)$$
$$= E(t)\, \mathscr{L}_x^3 \otimes \mathscr{L}_y^2 + P(t)$$
$$= Eu(t) \otimes \mathscr{L}_y^2 + \widetilde{E}_\gamma \mu(t).$$

Identifing $Eu(t), E\upsilon(t)$ with elements of $\mathscr{M}_b(\widetilde{\Omega}; \mathbb{M}^{2\times 2}_{\text{sym}})$ and integrating over $\mathscr{Y}$, we obtain

$$E\upsilon(t) = Eu(t).$$

Using the variant of Poincaré-Korn's inequality as in Step 1, we can infer that $\upsilon(t) = u(t)$ on $\widetilde{\Omega}$.

This implies that the whole sequences converge without depening on t, i.e.

$$u^h(t) \xrightharpoonup{*} u(t) \quad \text{weakly* in } BD(\widetilde{\Omega}),$$
$$\Lambda_h e^h(t) \xrightharpoonup{2} E(t) \quad \text{two-scale weakly in } L^2(\widetilde{\Omega} \times \mathscr{Y}; \mathbb{M}^{3\times 3}_{\text{sym}}).$$

Step 3: ***Energy balance.***

In order to prove energy balance of the two-scale quasistatic evolution $(qs2)^{hom}_{\gamma}$, it is enough (by arguing as in, e.g. [17, Theorem 4.7] and [24, Theorem 2.7]) to prove the energy inequality

$$\begin{aligned}&\mathcal{Q}^{hom}(E(t)) + \mathcal{D}_{\mathcal{H}^{hom}}(P;0,t)\\&\leq \mathcal{Q}^{hom}(E(0)) + \int_0^t \int_{\Omega\times\mathcal{Y}} \mathbb{C}(y)E(s) : E\dot{w}(s)\,dxdyds.\end{aligned} \tag{5.14}$$

For a fixed $t \in [0,T]$, let us consider a subdivision $0 = t_1 < t_2 < \ldots < t_n = t$ of $[0,t]$. In view of the lower semicontinuity of $\mathcal{Q}^{hom}$ and $\mathcal{H}^{hom}$ (see (4.43) and (4.44)), from $(qs2)_h$ we have

$$\begin{aligned}&\mathcal{Q}^{hom}(E(t)) + \sum_{i=1}^{n} \mathcal{H}^{hom}\left(P(t_{i+1}) - P(t_i)\right)\\&\leq \liminf_h \left(\mathcal{Q}_h(\Lambda_h e^h(t)) + \sum_{i=1}^{n} \mathcal{H}_h\left(\Lambda_h p^h(t_{i+1}) - \Lambda_h p^h(t_i)\right)\right)\\&\leq \liminf_h \left(\mathcal{Q}_h(\Lambda_h e^h(t)) + \mathcal{D}_{\mathcal{H}_h}(\Lambda_h p^h;0,t)\right)\\&= \liminf_h \left(\mathcal{Q}_h(\Lambda_h e^h(0)) + \int_0^t \int_{\Omega} \mathbb{C}\left(\tfrac{x'}{\varepsilon_h}\right) \Lambda_h e^h(s) : E\dot{w}(s)\,dxds\right).\end{aligned}$$

In view of strong convergence assumed in (5.2) and (5.13), by the Lebesgue's dominated convergence theorem we get

$$\begin{aligned}&\lim_h \left(\mathcal{Q}_h(\Lambda_h e^h(0)) + \int_0^t \int_{\Omega} \mathbb{C}\left(\tfrac{x'}{\varepsilon_h}\right) \Lambda_h e^h(s) : E\dot{w}(s)\,dxds\right)\\&= \mathcal{Q}^{hom}(E(0)) + \int_0^t \int_{\Omega\times\mathcal{Y}} \mathbb{C}(y)\Lambda_h E(s) : E\dot{w}(s)\,dxdyds.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\mathcal{Q}^{hom}(E(t)) + \sum_{i=1}^{n} \mathcal{H}^{hom}\left(P(t_{i+1}) - P(t_i)\right)\\&\leq \mathcal{Q}^{hom}(E(0)) + \int_0^t \int_{\Omega\times\mathcal{Y}} \mathbb{C}(y)\Lambda_h E(s) : E\dot{w}(s)\,dxdyds\end{aligned}$$

Taking the supremum over all partitions of $[0,t]$ yields (5.14), which concludes the proof.

■

CONCLUSION

In this thesis we rigorously derived the convergence of quasistatic evolutions models for perfectly plastic plates, in terms of periodic homogenization. Our analysis covered different regimes, which depended on different orders of magnitudes between the oscillation of the microstructure and the thickness of the body. We obtained a compactness results for a sequence of scaled symmetrized gradients of *BD* function in terms of two-scale convergence of measures and described the general framework in which one can analyze measures which result from the kinematics of elasto-plasticity.

We also established new notions of stress-plastic strain duality, which we then used to prove different inequalities between dissipation and plastic work, under various conditions on the regularity of the interfaces. The problem of attaining these results in a general situation for the regimes $\gamma = 0$ and $\gamma = +\infty$ seems to be a nontrivial issue, as additional compactness results on the interfaces are needed. As a simple problem, one can analyze the homogenization of the 2D plate equation, which also requires new compactness results. We leave these problems for a future work.

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Curriculum Vitae

Marin Bužančić was born on October 21, 1991 in Zagreb, Croatia, where he finished primary and secondary school. In 2010 he enrolled in the Undergraduate University Programme in Mathematics at the Department of Mathematics, University of Zagreb, Faculty of Science, and in 2013 in the Graduate University Programme in Theoretical Mathematics at the same faculty. He obtained his master's degree in 2015 with the master thesis *Bellman Function Technique in Mathematical Analysis* under supervision of Prof. Dr. Vjekoslav Kovač. In December 2015, he enrolled in the PhD Programme in Mathematics at the Faculty of Science, University of Zagreb, and in April 2016 started working as a teaching assistant at the Department of Mathematics at the Faculty of Chemical Engineering and Technology, University of Zagreb.

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