Semilinear equations for non-local operators

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Semilinearne jednadžbe za nelokalne operatore

DOKTORSKI RAD

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SUMMARY

This doctoral thesis deals with semilinear equations for non-local operators in bounded domains in higher dimensions: For a bounded domain $D \subset \mathbb{R}^d$, $d \geq 2$, a non-local operator L, and a function $f : D \times \mathbb{R} \to \mathbb{R}$, the following problem is being solved

$$Lu(x) = f(x, u(x)), \quad x \in D, \tag{1}$$

where we also impose boundary conditions in D^c and/or on ∂D, depending on the type of the non-local operator L.

The first type of non-local operators that are studied are infinitesimal generators of transient subordinate Brownian motions. Here the domain D can be any bounded domain in $\mathbb{R}^d$, $d \geq 2$, and we impose boundary conditions both in D^c and on ∂D. A Martin representation formula for non-negative generalized harmonic functions is proved and a new type of boundary trace operator for this non-local setting is developed. A solution to the problem (1) is found for a large class of functions f and conditions on f are given such that there is no solution to (1).

The second type of non-local operators that are observed are infinitesimal generators of subordinate killed Brownian motions. Here a smoothness assumption on the boundary of the domain D is imposed as well as the boundary condition on ∂D in addition to (1). An integral representation formula for non-negative harmonic functions is given and also an equivalence between non-negative harmonic functions and non-negative functions that satisfy a certain mean-value property with respect to the underlying subordinate killed Brownian motion is established. A solution to the problem (1) is found for a large class of functions f and conditions on f are given such that there is no solution to (1).

Keywords: semilinear differential equations, non-local operators, killed subordinate Brownian motion, subordinate killed Brownian motion, harmonic functions

PROŠIRENI SAŽETAK

U ovoj disertaciji proučavaju se semilinearne jednadžbe za nelokalne operatore u omeđenim domenama u višim dimenzijama, tj. u omeđenoj domeni $D \subset \mathbb{R}^d$, $d \geq 2$, i za nelokalni operator L, i funkciju $f : D \times \mathbb{R} \to \mathbb{R}$, rješava se sljedeći problem

$$Lu(x) = f(x, u(x)), \quad x \in D, \tag{2}$$

gdje još dodatno, u ovisnosti o tipu nelokalnog operatora L, postavljamo i rubne uvjete na D^c i/ili na ∂D.

Prvi tip nelokalnog operatora koji se promatra je infinitezimalni generator prolaznog subordiniranog Brownovog gibanja pri čemu je Laplaceov eksponent subordinatora potpuna Bernsteinova funkcija koja zadovoljava slabo skaliranje u beskonačnosti. Problem (2) se promatra u proizvoljnoj omeđenoj domeni D, a rubni uvjeti su dani i na D^c i na ∂D. Kako bi se problem (2) uspješno riješio, prvo se razvijaju pomoćne tehnike i objekti. Proučavaju se generalizirane harmonijske funkcije u odnosu na dani proces subordiniranog Brownovog gibanja te se pokazuje da su takve generalizirane harmonijske funkcije glatke te da ih pripadni operator L poništava u slabom smislu. Također, proučava se i relativna oscilacija kvocijenta generaliziranih harmonijskih funkcija te se uz pomoć toga pokazuje Martinova integralna reprezentacija nenegativnih generaliziranih harmonijskih funkcija. Kao dio dokaza te reprezentacije, definira se i novi tip rubnog operatora, tj. operatora traga, koji je pogodan za analizu semilinearnog problema i za generalnije nelokalne operatore, a definicija mu ne zahtjeva glatkoću ruba domene D. Nakon tih pripremnih rezultata, razvija se metoda sub- i superrješenja za (2) koja se potom primjenjuje na nelinearnost f koja zadovoljava ocjenu

$$|f(x,t)| \leq \rho(x)\Lambda(|t|), \quad x \in D, t \in \mathbb{R}, \tag{3}$$

uz određene uvjete integrabilnosti funkcija ρ i Λ. Glavni oslonac u rješavanju problema (2) je pristup s gledišta teorije potencijala pa je tako rješenje problema (2) prikazano kao suma Greenovog, Poissonovog i Martinovog potencijala. U slučaju glatkog ruba domene D, daju se istančaniji uvjeti na funkciju f uz koje problem (2) ima rješenje. Također, daje se i uvjet na f uz koji problem nema rješenja. Kako bi bilo moguće dati ljepše uvjete na f, dobivene su i oštre ograde za Greenov, Poissonov i Martinov integral. Svi rezultati uspoređuju se s poznatim rezultatima u slučaju frakcionalnog Laplaceovog operatora.

Drugi tip nelokalnog operatora koji se promatra dolazi kao infinitezimalni generator subordiniranog ubijenog Brownovog gibanja te je i ovdje Laplaceov eksponent subordinatora potpuna Bernsteinova funkcija koja zadovoljava slabo skaliranje u beskonačnosti. Ovaj operator spektralnog je tipa te se u disertaciji pokazuje da može biti definiran spektralno, točkovno, ali i slabo (distribucijski), uz ekvivalentnost definicije kada operator djeluje na dovoljno glatkim funkcijama. Iz Greenove funkcije za dani nelokalni operator pomoću derivacije na rubu u smjeru normale definira se i Poissonova jezgra. Poissonova jezgra intenzivno se koristi kako bi se pokazala integralna reprezentacija nenegativnih harmonijskih funkcija s obzirom na dani nelokalni operator. Također, pokazuje se i 1-1 korespondencija između nenegativnih klasičnih harmonijskih funkcija i nenegativnih harmonijskih funkcija s obzirom na promatrano subordinirano ubijeno Brownovo gibanje. Prije nego se napadne semilinearan problem, pokazuje se nekoliko rezultata regularnosti Poissonovog i Greenovog integrala. Semilinearni problem (2) rješava se uz pomoć Katove nejednakosti dokazane u disertaciji za ovaj tip nelokalnog operatora. Razvija se metoda sub- i superrješenja za (2). Metoda se primjenjuje na razne tipove funkcija f gdje je ponovno uvedena pretpostavka da funkcija f zadovoljava ocjenu (3) uz određene uvjete integrabilnosti funkcija ρ i Λ. Osim metode sub- i superrješenja, daje se primjer upotrebe metode monotonih iteracija. Na kraju se promatra specijalan slučaj spektralnog frakcionalnog Laplaceovog operatora te nelinarnost polinomnog tipa $f(x,t) = \pm t^p$ za koje se može dobiti oštra ograda na parametar $p \in \mathbb{R}$ za koju jednadžba (2) ima rješenje, tj. možemo iskazati u kojim slučajevima jednadžba (2) nema rješenje.

Ključne riječi: semilinearne diferencijalne jednadžbe, nelokalni operatori, ubijeno subordinirano Brownovo gibanje, subordinirano ubijeno Brownovo gibanje, harmonijske funkcije

Contents

Introduction

Let $D \subset \mathbb{R}^d$, $d \geq 2$, be an open bounded set, $f : D \times \mathbb{R} \to \mathbb{R}$ a function, and L a non-local operator. In this thesis we deal with the following semilinear problem

$$Lu(x) = f(x, u(x)), \quad x \in D, \tag{4}$$

where we also impose boundary conditions in D^c and/or on ∂D, depending on the type of the non-local operator L.

We are interested in two types of operators. The first type of the operator L is an integro-differential operator which can be written in the form $L = \phi(-\Delta)$, where $\phi : (0,\infty) \to (0,\infty)$ is a complete Bernstein function satisfying a certain weak scaling condition at infinity. More precisely,

$$\phi(-\Delta)u(x) = \text{P.V.} \int_{\mathbb{R}^d} \big(u(x) - u(y)\big) j(|x-y|) dy, \tag{5}$$

if u is a sufficiently smooth function and where the singular kernel j is completely determined by the function ϕ. The operator $-\phi(-\Delta)$ can be viewed as an infinitesimal generator of a subordinate Brownian motion where the subordinator has ϕ as its Laplace exponent. Our cornerstone for obtaining a solution to the problem (4) for the operator $\phi(-\Delta)$ is the potential theory for the underlying subordinate Brownian motion developed in recent years, which we heavily exploit. The best known example of this operator is the fractional Laplacian $(-\Delta)^{\alpha/2}$, where $\phi(\lambda) = \lambda^{\alpha/2}$. Here the underlying process is called the rotationally invariant (or isotropic) α-stable process. In this setting, since the underlying subordinate Brownian motion can exit the set D with a jump, we impose boundary conditions both in D^c and on ∂D.

The second type of the operator L we are interested in is also an elliptic type of non-local operator which can be written in the form $L = \phi(-\Delta|_D)$ where, again, $\phi : (0,\infty) \to$

$(0,\infty)$ is a complete Bernstein function satisfying a certain weak scaling condition at infinity. The operator $\phi(-\Delta|_D)$ can be written as a principal value integral in the following way

$$\phi(-\Delta|_D)u(x) = \text{P.V.}\int_D \big(u(x)-u(y)\big)J_D(x,y)dy + \kappa(x)u(x), \tag{6}$$

if u is a sufficiently smooth function and where the singular kernel J_D and the function κ are completely determined by the function ϕ. Here the operator $-\phi(-\Delta|_D)$ can be viewed as an infinitesimal generator of a subordinate killed Brownian motion where the subordinator has ϕ as its Laplace exponent as we show in the thesis. In this setting, the subordinate killed Brownian motion is a strong Markov process which has a strong potential-theoretic connection to the underlying Brownian motion which had been developed in recent years. Again, we use this connection and the potential theory as a cornerstone for obtaining a solution to (4) for the operator $\phi(-\Delta|_D)$. In addition to (4), we will impose a boundary condition on ∂D since our underlying subordinate killed Brownian motion is killed inside the set D. The best known example of this operator is the spectral fractional Laplacian $(-\Delta_{|D})^{\alpha/2}$, where $\phi(\lambda)=\lambda^{\alpha/2}$.

Before we move into the details, let us say a word about the work that has been done before this thesis in the theory of semilinear equations. In the classical (local) setting of the Laplacian, semilinear problems have been studied for a long time now. In the monograph [64] it is said that this study is at least 50 years old now. However, the study of semilinear problems for non-local operators is quite recent and mostly oriented to the problems driven by the fractional Laplacian, see e.g. [1, 2, 6–8, 20, 27, 38, 39]. To the best of our knowledge, for more general operators than the fractional Laplacian which fall into our setting (or which generalize our setting) only linear problems were discussed, see e.g. [43, 47]. A direct contribution of this thesis to the theory of semilinear equations for operators more general than the fractional Laplacian is already published in [10, 11], and these results are given in every detail in Chapters 1 and 2. We note that this breakthrough was possible by the recent development of such operators in potential-theoretical and in analytical sense, see [12, 13, 47–49].

The spectral fractional Laplacian is also a fairly known operator and most of work has been done in comparing the spectral to the regular fractional Laplacian, see [22,25,41,72].

However, there are just a few articles discussing the semilinear Dirichlet problem for the spectral fractional Laplacian, see [3, 36]. To the best of our knowledge, the work done in this thesis, which will be presented in Chapter 3, is the first one to study semilinear problems for spectral-type operators more general then the spectral fractional Laplacian. We note that these results were possible by the recent potential-theoretic and analytic developments in the theory of subordinated killed Brownian motions, see [58, 60, 74].

Before we continue, let us note that a typical difference between local and non-local setting is that in the non-local setting even solutions of the linear Dirichlet problem can explode at the boundary whereas in the local setting this does not happen. To be more precise, there exist harmonic functions with respect to $\phi(-\Delta)$ and $\phi(-\Delta|_D)$ which explode at the boundary. In this thesis we will restrict ourselves to the so-called moderate blow-up solutions, that is those bounded by harmonic functions with respect to the underlying operator.

A brief overview of chapters

Let us now give a brief description of the content of each chapter in the thesis. A more detailed one can be found at the beginning of each chapter.

Our main assumption on the operators $\phi(-\Delta)$ and $\phi(-\Delta|_D)$ is on the function ϕ:

(WSC). The function ϕ is a complete Bernstein function which satisfies the weak scaling condition at infinity: There exist constants δ_1, $\delta_2 \in (0,1)$ and $a_1, a_2 > 0$ such that

$$a_1\lambda^{\delta_1}\phi(r) \le \phi(\lambda r) \le a_2\lambda^{\delta_2}\phi(r), \quad \lambda \ge 1, r \ge 1. \tag{WSC}$$

In Chapter 1 we prove a representation formula for non-negative generalizred harmonic functions with respect to a class of subordinate Brownian motions in a general open set $D \subset \mathbb{R}^d$, $d \ge 2$, where the Laplace exponent of the corresponding subordinator satisfies **(WSC)**. We look at pairs (f, λ) such that f is a function on D and λ is a measure on D^c that we call, following [21], functions with outer charge. We prove the following result: if f is a non-negative harmonic function in D with a non-negative outer charge λ, then there is a unique finite measure μ on ∂D such that

$$f = P_D\lambda + M_D\mu, \quad \text{in } D.$$

Here $P_D\lambda$ denotes the Poisson integral of the measure λ and $M_D\mu$ the Martin integral of the measure μ, with respect to the subordinate Brownian motion. In the chapter we also study the boundary trace operator W_D, see Definition 1.4.5. The operator W_D plays a significant role in the semilinear Dirichlet problem for the fractional Laplacian. We generalize the operator for the case of the subordinate Brownian motion and use it as a tool to obtain the finite measure for the Martin integral in the representation of generalized harmonic functions. Motivated by the article [21] where harmonic functions with outer charge were introduced for the case of the isotropic α-stable process, we use the same concept to define $\phi(-\Delta)$-harmonic functions with outer charge. We prove that a function is $\phi(-\Delta)$-harmonic if and only if the operator $\phi(-\Delta)$ annihilates it in the weak sense and that every $\phi(-\Delta)$-harmonic function is infinitely differentiable.

In Chapter 2 we study the semilinear problem for $\phi(-\Delta)$. Let $D \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open set, $f : D \times \mathbb{R} \to \mathbb{R}$ a function, λ a signed measure on $D^c = \mathbb{R}^d \setminus D$ and μ a signed measure on ∂D. We study the problem

$$\begin{aligned} \phi(-\Delta)u(x) &= f(x,u(x)) && \text{in } D \\ u &= \lambda && \text{in } D^c \\ W_D u &= \mu && \text{on } \partial D. \end{aligned} \tag{7}$$

Recall that the operator $\phi(-\Delta)$ can be written as a principal value integral in the form (5), and in the fractional case, i.e. when $\phi(t) = t^{\alpha/2}$, $\alpha \in (0,2)$, $\phi(-\Delta)$ is the fractional Laplacian $(-\Delta)^{\alpha/2}$ and the kernel $j(|x-y|)$ is proportional to $|x-y|^{-d-\alpha}$. We consider solutions of (7) in the weak dual sense, see Definition 2.3.1, and show that for bounded $C^{1,1}$ open sets this is equivalent to the notion of weak L^1 solution as in [1, Definition 1.3]. For the nonlinearity f throughout the chapter we assume the condition

(F). $f : D \times \mathbb{R} \to \mathbb{R}$ is continuous in the second variable and there exist a function $\rho : D \to [0,\infty)$ and a continuous function $\Lambda : [0,\infty) \to [0,\infty)$ such that $|f(x,t)| \leq \rho(x)\Lambda(|t|)$.

The goal of this chapter is to generalize results from [1, 20] and at the same time to provide a unified approach. The first main contribution of this part of the thesis is that we replace the fractional Laplacian with a more general non-local operator. The second main contribution is that we obtain some of the results from [1] (which deals with $C^{1,1}$ open sets) for regular open subsets of $\mathbb{R}^d$.

In Chapter 3 we deal with the problem

$$\begin{aligned} \phi(-\Delta|_D)u(x) &= f(x,u(x)) && \text{in } D, \\ \frac{u}{P_D^\phi\sigma} &= \zeta && \text{on } \partial D, \end{aligned} \tag{8}$$

where D is a bounded $C^{1,1}$ domain. The operator $\phi(-\Delta)$ in its principal value form is given in (6). The notion of the boundary condition is a bit abstract, but it can be interpreted as a limit at the boundary of $u/P_D^\phi\sigma$ in the pointwise sense, or in the weak sense of (3.77), depending on the smoothness of the boundary datum, where $P_D^\phi\sigma$ is a reference function defined as the Poisson potential (with respect to the subordinate killed Brownian motion) of the $d-1$ dimensional Hausdorff measure on ∂D. We consider solutions of (8) in the weak dual sense, see Definition 3.4.1. The main goal of this chapter is to generalize results from [3] where the semilinear problem was studied for the spectral fractional Laplacian, and to generalize results from Chapter 2 to a slightly different type of a non-local operator in the special case of $C^{1,1}$ bounded domain. For the nonlinearity f in (8) in our results we again assume (**F**).

The thesis also contains Appendix, which consists of five parts. The first part deals with an approximation of excessive functions - a known technique applied to our setting of subordinate (killed) Brownian motions. In the second part we provide technical proofs of sharp bounds of Green and Poisson potentials, where the proofs are modelled by the proofs of known results in the case of the fractional Laplacian. In the third part we provide a proof of the sharp bound of the Green function of the subordinated killed Brownian motion - a slightly strengthened result already proved for a large class of subordinate killed Brownian motions. In the forth part of the Appendix we prove a uniform integrability property of a class of functions in which we find a solution to the semilinear problem in Chapter 3. In the final part of the Appendix we prove that the transition density of the killed Brownian motion is regular up to the boundary in the case of bounded $C^{1,\alpha}$ domains - a result which appears to be known but for which we could not find an exact reference.

Notation

For an open set $D\subset\mathbb{R}^d$: $C(D)$ denotes the family of all continuous functions on D, $C_b(D)$ the family of all bounded $C(D)$ functions, $C_0(D)$ the family of all continuous functions vanishing at infinity (i.e. $f \in C_0(D)$ if for every $\varepsilon > 0$ there exists a compact subset $K \subset D$ such that $|f(x)| < \varepsilon$ for all $x \in D \setminus K$), $C^k(D)$, $k \in \mathbb{N}$, k times continuously differentiable functions on D, $C^\infty(D)$ infinitely differentiable functions on D, and $C_c^\infty(D)$ infinitely differentiable functions with compact support on D. For $\alpha \in (0,1]$ and $k \in \mathbb{N}$: $C^{k,\alpha}$ denotes the space of k times differentiable functions whose all k-th partial derivatives are Hölder continuous on D with exponent α. Also, by e.g. $C^k(\overline{D})$ we denote the family of functions in $C^k(D)$ whose all derivatives of order less than k have continuous extension to $\overline{D}$, by e.g. $C^{1,\alpha}(\overline{D})$ we denote functions in $C^1(\overline{D})$ whose first partial derivatives are uniformly Hölder continuous in D with exponent α, etc. All these function spaces are equipped with their standard supremum norms: e.g.

$$\|f\|_{C^{k,\alpha}(D)} = \sum_{|\beta| \le k} \sup_{x \in D} \{|D^\beta f(x)|\} + \sum_{|\beta| = k} \sup_{x,y \in D} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha},$$

where β in the sums above denotes a multi-index, and $D^\beta f(x)$ denotes the partial derivate of f at x of order β.

For a set $D \subset \mathbb{R}^d$ and a measure μ on D: $L^p(D,\mu)$, $1 \le p \le \infty$, is the space of all p-integrable functions on D with respect to μ, and $L^p_{loc}(D,\mu)$ the space of all locally p-integrable functions on D with respect to μ. If μ is the Lebesgue measure restricted on D, we simply write $L^p(D)$ and $L^p_{loc}(D)$. Also, if in addition $D = \mathbb{R}^d$, we write L^p and L^p_{loc}. The Sobolev space $W^{1,2}(D)$ denotes the space of $L^2(D)$ functions whose weak partial derivatives belong to $L^2(D)$, equipped with the standard Sobolev norm. The space $H_0^1(D)$ denotes the closure of $C_c^\infty(D)$ with respect to the Sobolev norm in the Sobolev space $W^{1,2}(D)$.

When we say ν is a measure, we mean that ν is a non-negative measure on $\mathbb{R}^d$. By $|\nu|$ we denote the total variation of a signed measure ν. When we say ν is a signed measure on $D \subset \mathbb{R}^d$, we mean that ν is a signed measure on $\mathbb{R}^d$ and $|\nu|(D^c) = 0$. The Dirac measure of a point $x \in \mathbb{R}^d$ is denoted by δ_x. The set $\mathscr{B}(\mathbb{R}^d)$ denotes Borel measurable sets in $\mathbb{R}^d$ as well as the set of Borel measurable functions in $\mathbb{R}^d$ and its usage will be clarified each time. We suppose that all functions in the article are Borel functions and all signed measures are Borel signed measures. For $D \subset \mathbb{R}^d$, $\mathscr{M}(D)$ denotes σ-finite signed measures on D.

The boundary of the set D is denoted by ∂D. Notation $U \subset\subset D$ means that U is a nonempty bounded open set such that $U \subset \overline{U} \subset D$ where $\overline{U}$ denotes the closure of U. By $|x|$ we denote the Euclidean norm of $x \in \mathbb{R}^d$ and $B(x,r)$ denotes the ball around $x \in \mathbb{R}^d$ with radius $r > 0$. We abbreviate $B_r := B(0,r)$. For $A, B \subset \mathbb{R}^d$ let $\delta_A(x) = \inf\{|x-y| : y \in A^c\}$ and $\operatorname{dist}(A,B) = \inf\{|x-y| : x \in A, y \in B\}$. Unimportant constants in the article will be denoted by small letters c, c_1, c_2, $\ldots$, and their labeling starts anew in each new statement. By a big letter C we denote some more important constants, where e.g. $C(a,b)$ means that the constant C depends only on parameters a and b. However, the dependence on the dimension d will not be mentioned explicitly. All constants are positive finite numbers. For two positive functions f and g we write $f \asymp g$ ($f \lesssim g$, $f \gtrsim g$) if there exist a finite positive constant c such that $c^{-1} f \le g \le c f$ ($f \le c g$, $f \ge c g$). Finally, $a \wedge b = \min\{a,b\}$, $a \vee b = \max\{a,b\}$.

1. Generalized harmonic functions

In this chapter we prove a representation formula for non-negative harmonic functions with respect to a class of subordinate Brownian motions in a general open set $D \subset \mathbb{R}^d$, $d \geq 2$, where the Laplace exponent of the corresponding subordinator is a complete Bernstein function satisfying the weak scaling condition at infinity- **(WSC)**. In this setting, the novelty is that we look at pairs (f, λ) such that f is a function on D and λ is a measure on D^c that we call, following [21], functions with outer charge. We prove the following result: if f is a non-negative harmonic function in D with a non-negative outer charge λ, then there is a unique finite measure μ on ∂D such that

$$f = P_D\lambda + M_D\mu, \quad \text{in } D. \tag{1.1}$$

Here $P_D\lambda$ denotes the Poisson integral of the measure λ and $M_D\mu$ the Martin integral of the measure μ, with respect to the subordinate Brownian motion, see Theorem 1.5.13. Such representation was proved for the case of the isotropic α-stable process in [21] more than 10 years ago. A similar representation for functions (in the classical sense) was proved recently for more general Markov processes in bounded open sets in [62], and in *nice* and general open sets in [55]. Analogous result for non-negative classical harmonic functions on the ball $B(x,r)$, i.e. harmonic functions with respect to the Brownian motion, is better known as Riesz-Herglotz theorem, cf. [5]. In the chapter the case $d = 1$ is excluded since it would require somewhat different potential-theoretic methods.

On the way to obtaining the representation, motivated by results in [21], we study the relative oscillation of the quotient of Poisson integrals. The novelty of these results is that we prove that the oscillation can be uniformly tamed. To be more precise, for a positive function f on a set D we define the relative oscillation of the function f by

$$\mathrm{RO}_D f := \frac{\sup_D f}{\inf_D f}.$$

We prove that for every $\eta > 0$ there is $\delta > 0$ such that for every $D \subset B(0,R)$ and measures λ_1 and λ_2 on $B(0,R)^c$ we have

$$\mathrm{RO}_{D\cap B(0,\delta)} \frac{P_D\lambda_1}{P_D\lambda_2} \le 1+\eta,$$

see Lemma 1.5.5. Uniformity lies in the fact that δ is independent of the set D and the measures λ_1 and λ_2. Similar claims on the relative oscillation of harmonic functions were recently proved for more general processes in [55, Proposition 2.5 & Proposition 2.11] and [54, Theorem 2.4 & Theorem 2.8] but the claims lack the aforementioned uniformity.

In the chapter we also study the boundary trace operator W_D, see Definition 1.4.5. The operator W_D was introduced in [20] building on results in [21]. In [20] it plays a significant role in the semilinear Dirichlet problem for the fractional Laplacian. We generalize the operator for the case of the subordinate Brownian motion and use it as a tool to obtain the finite measure for the Martin integral in the representation.

Motivated by the article [21] where harmonic functions with outer charge were introduced for the case of the isotropic α-stable process, we use the same concept to define $\phi(-\Delta)$-harmonic functions with outer charge, see Definition 1.3.7. Here $\phi(-\Delta)$ stands for the integrodifferential operator which generates the subordinate Brownian motion, see (1.14). In Theorem 1.3.16 we prove that $\phi(-\Delta)$ annihilates all $\phi(-\Delta)$-harmonic functions in the weak sense. In Theorem 1.3.18 we prove a converse claim, i.e. if $\phi(-\Delta)$ annihilates a (generalized) function in the weak sense, then the function is $\phi(-\Delta)$ harmonic. Also, the novelty of the study of $\phi(-\Delta)$-harmonic functions is that we prove that all such functions are continuous, see Proposition 1.3.9, whereas in [21] the continuity condition was used as a part of the definition. Moreover, motivated by results in [44], in Theorem 1.3.12 we prove even an stronger result which says that every $\phi(-\Delta)$-harmonic function is infinitely differentiable.

The chapter is organized as follows. Section 1.1 is a lengthy preliminary section where we define the process of interest - subordinate Brownian motion and its killed version, introduce the Green and the Poisson kernels, and state some well-known results on the process that will be needed in this and the following chapter. A short Section 1.2 deals with elementary properties of Green potentials. In Section 1.3 we prove basic results on the Poisson kernel, define $\phi(-\Delta)$-harmonic functions and study their basic properties as well, as some more sophisticated properties such as smoothness. In Section 1.4 we recall already known facts on the theory of the Martin kernel and connect them to $\phi(-\Delta)$-harmonic functions. Section 1.5 begins with results on the boundary trace operator W_D. After we prove results on the relative oscillations of the Poisson integrals, we finish the chapter by proving the representation formula for non-negative $\phi(-\Delta)$-harmonic functions.

1.1. PRELIMINARIES

Let $S=(S_t)_{t\geq 0}$ be a subordinator with the Laplace exponent ϕ, i.e. S is an increasing Lévy process with $S_0=0$ and

$$\mathbb{E}[e^{-\lambda S_t}]=e^{-t\phi(\lambda)}, \quad \lambda, t\geq 0.$$

It is well known that ϕ is a Bernstein function of the form

$$\phi(\lambda)=b\lambda+\int_0^\infty (1-e^{-\lambda t})\mu(dt), \quad \lambda>0, \tag{1.2}$$

where $b\geq 0$ and μ is a measure on $(0,\infty)$ satisfying $\int_0^\infty (1\wedge t)\mu(dt)<\infty$, see [71, Chapter 5]. The measure μ is called the Lévy measure and b the drift of the subordinator.

Throughout the chapter we impose two following assumption on ϕ.

(WSC). The function ϕ is a complete Bernstein function, i.e. the Lévy measure $\mu(dt)$ has a completely monotone density $\mu(t)$, and ϕ satisfies the following weak scaling condition at infinity: There exist $a_1, a_2>0$ and $\delta_1,\delta_2\in(0,1)$ satisfying

$$a_1\lambda^{\delta_1}\phi(t)\leq \phi(\lambda t)\leq a_2\lambda^{\delta_2}\phi(t), \quad t\geq 1, \lambda\geq 1. \tag{1.3}$$

This assumption yields that $b=0$. The condition $r\geq 1$ in (1.3) is important in the sense that the scaling holds true away from zero. By using the continuity of ϕ, it is easy to show that if $R_0>0$, then (1.3) is also valid for $r\geq R_0$ but with different constants

a_1 and a_2 (δ_1 and δ_2 remain the same). Further, the scaling condition (1.3) implies the well-known bound

$$\phi'(\lambda) \asymp \frac{\phi(\lambda)}{\lambda}, \quad \lambda \geq 1, \tag{1.4}$$

where, in fact, the upper bound holds for every Bernstein function and every $\lambda > 0$, and the lower bound follows from (1.3).

The second assumption on ϕ will be important for the transience of the subordinate Brownian motion defined a few paragraphs below.

(T). If $d = 2$, we assume that

$$\int_0^1 \frac{d\lambda}{\phi(\lambda)} < \infty.$$

The best-known subordinator with the properties **(WSC)** and **(T)** is the α-stable subordinator where $\phi(\lambda) = \lambda^{\alpha/2}$, for some $\alpha \in (0,2)$, which satisfies exact (and even global) scaling condition (1.3). However, there are many other interesting subordinators which fall into our setting. For a short list of these, see e.g. [57, p. 3].

Let $W = (W_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^d$, $d \geq 2$, independent of S with the characteristic exponent $\xi \mapsto |\xi|^2$, $\xi \in \mathbb{R}^d$. This means that the transition densities $p(t,x,y)$ of W are given by

$$p(t,x,y) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}, \quad t > 0,\, x,\, y \in \mathbb{R}^d. \tag{1.5}$$

The process $X = ((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d})$ defined as $X_t = W_{S_t}$ is called a subordinate Brownian motion in $\mathbb{R}^d$. Here $\mathbb{P}_x$ denotes the probability under which the process X starts from $x \in \mathbb{R}^d$, and by $\mathbb{E}_x$ we denote the corresponding expectation. For an open set $D \subset \mathbb{R}^d$, let $\tau_D := \inf\{t > 0 : X_t \notin D\}$ be the first exit time of the process X from D. We define the killed process X^D by

$$X_t^D := \begin{cases} X_t, & t < \tau_D, \\ \partial, & t \geq \tau_D, \end{cases}$$

where ∂ is an adjoint point to $\mathbb{R}^d$ called the cemetery. The process X^D is called the killed subordinate Brownian motion. We deal with semilinear problems driven by this process in Chapter 2.

Since we assume that ϕ satisfies the weak scaling condition (1.3), thus $b = 0$, $X = (X_t)_{t\geq 0} = (W_{S_t})_{t\geq 0}$ is a pure-jump rotationally symmetric Lévy process with the characteristic exponent $\xi \mapsto \Psi(\xi) = \phi(|\xi|^2)$. The exponent has the following form

$$\Psi(\xi) = \phi(|\xi|^2) = \int_{\mathbb{R}^d} (1-\cos(\xi\cdot x))\,J(dx), \quad \xi \in \mathbb{R}^d,$$

where the measure J satisfies $\int_{\mathbb{R}^d}(1\wedge|x|^2)J(dx) < \infty$ and it is called the Lévy measure of the process X. Also, J has a density given by $J(x) = j(|x|)$, $x\in\mathbb{R}^d$, where

$$j(r) := \int_0^\infty p(t,x,y)\mu(t)dt = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)}\mu(t)dt, \quad r>0,$$

Obviously, the density j is positive, continuous, decreasing and satisfies $\lim\limits_{r\to\infty} j(r) = 0$. For the details on the subordination see e.g. [68, Section 30 & 31] and [71, Chapter 13].

It is well known that under the assumption **(WSC)** the kernel j enjoys sharp two-sided estimates for small $r>0$: For every $R>0$ there exists $C = C(R)\geq 1$ such that

$$C^{-1}\phi(r^{-2})r^{-d} \leq j(r) \leq C\phi(r^{-2})r^{-d}, \qquad 0<r<R, \tag{1.6}$$

see for example [19, Eq. (15) & Corollary 22]. Moreover, since ϕ is a complete Bernstein function, the following properties of j hold. There exists $C = C(\phi) > 0$ such that

$$j(r) \leq Cj(r+1), \qquad r >\geq 1. \tag{1.7}$$

For every $M>0$ there exists $C = C(M,\phi) > 0$ such that

$$j(r) \leq Cj(2r), \quad r\in(0,M), \tag{1.8}$$

cf. [51, (2.11), (2.12)]. Further, for every $n\in\mathbb{N}$ there exists $C_n = C_n(\phi) > 0$ such that

$$\left|\left(\frac{d}{dr}\right)^n j(r)\right| \leq C_n j(r), \qquad r\geq 1, \tag{1.9}$$

cf. [18, Proposition 7.2]. Finally, by [54, Lemma 4.3], for every $r_0\in(0,1)$,

$$\lim_{\delta\to 0}\sup_{r>r_0}\frac{j(r)}{j(r+\delta)} = 1\,. \tag{1.10}$$

Properties (1.7)–(1.10) are used in some of the results that we quote later. Similarly as before, since j is continuous, inequalities (1.7) and (1.9) also hold for $r\geq R_0$ but with different constants $C = C(\phi,R_0) > 0$.

By using (1.10) we easily prove the following technical lemma.

Lemma 1.1.1. Let $R > 0$, $\varepsilon > 0$, and $0 < q \le 1$. There exists $p = p(q, \varepsilon, R) < q$ such that for all $z \in B_{pR}$ and $y \in B^c_{qR}$

$$\frac{1}{1+\varepsilon} j(|y|) \le j(|y-z|) \le (1+\varepsilon) j(|y|).$$

Recall that throughout the chapter we assume (**T**). This means that X is transient, i.e. $\mathbb{P}_x(\lim_{t\to\infty} |X_t| = \infty) = 1$, $x \in \mathbb{R}^d$. Indeed, the Chung-Fuchs condition implies that

$$\begin{aligned} X \text{ is transient} \iff & \int\limits_{B(0,R)} \frac{d\xi}{\Psi(\xi)} < \infty, \quad \text{for some } R > 0 \\ \iff & \int\limits_0^R \frac{\lambda^{d-1}}{\phi(\lambda^2)} d\lambda < \infty, \quad \text{for some } R > 0 \\ \iff & \int\limits_0^1 \frac{\lambda^{d/2-1}}{\phi(\lambda)} d\lambda < \infty, \end{aligned} \tag{1.11}$$

see [68, Corrollary 37.6]. Since it holds

$$1 \wedge \lambda \le \frac{\phi(\lambda r)}{\phi(r)} \le 1 \vee \lambda, \quad \lambda, r > 0, \tag{1.12}$$

which easily follows from (1.2), we have that X is always transient if $d \ge 3$ and if $d = 2$, the transience is achieved if and only if $\int_0^1 \phi(\lambda)^{-1} d\lambda < \infty$, which is exactly our assumption (**T**).

1.1.1. Additional assumptions

In some results dealing with unbounded sets we will occasionally make additional assumptions on the density j and the exponent ϕ. The first assumption strengthens the assumption (**WSC**).

(GWSC). (Global weak scaling condition) There exist constants δ_1, $\delta_2 \in (0,1)$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{\delta_1} \phi(r) \le \phi(\lambda r) \le a_2 \lambda^{\delta_2} \phi(r), \quad \lambda \ge 1, r > 0. \tag{GWSC}$$

Note that if (**GWSC**) holds, then by (1.11) we immediately have that X is transient even if $d = 2$.

The second assumption comes as an addition to Lemma 1.1.1.

(E). For every $R \geq 1$, $\varepsilon > 0$, and $q \in (1,\infty)$, there exists $p = p(q,\varepsilon,R) > q$ such that for all $z \in B_{pR}^c$ and $y \in B_{qR}$

$$\frac{1}{1+\varepsilon} j(|z|) \leq j(|y-z|) \leq (1+\varepsilon) j(|z|).$$

To the best of our knowledge it is not clear if the assumption (E) is true for every density j generated by a complete Bernstein function. However, it is known that if for some $\alpha \in (0,2)$ we have $\lim_{\lambda\to 0} \frac{\phi(\lambda^2)}{\lambda^\alpha l(\lambda)} = 1$, where l is a slowly varying function at 0, then the condition (E) is satisfied, see [54, Lemma 4.6(a)].

Note that the isotropic α-stable process, $\alpha \in (0,2)$, satisfies all mentioned assumptions, since in that case we have $\phi(\lambda) = \lambda^{\alpha/2}$ and $j(r) = c(d,\alpha)\frac{1}{r^{d+\alpha}}$.

1.1.2. The semigroup, the operator and the potential kernel

For a bounded or non-negative function $u \in \mathscr{B}(\mathbb{R}^d)$ and $t \geq 0$, define $R_t u(x) := \mathbb{E}_x[u(X_t)]$. Then $(R_t)_{t\geq 0}$ is the semigroup corresponding to X. It is well known that this semigroup has the Feller property, i.e., $R_t : C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$.

Under our assumptions, the process X is also strongly Feller, i.e., $R_t : L^\infty(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$. Indeed, by using (WSC), for $t > 0$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} \left|\mathbb{E}_0\left[e^{i\xi\cdot X_t}\right]\right| |\xi|^n \, d\xi &= \int_{\mathbb{R}^d} e^{-t\Psi(\xi)}|\xi|^n d\xi = \int_{\mathbb{R}^d} e^{-t\phi(|\xi|^2)}|\xi|^n d\xi \\ &\leq c(d)\left(\int_0^1 r^{d-1+n} dr + \int_1^\infty e^{-ta_1\phi(1)r^{\delta_1}} r^{d-1+n}\right) < \infty,\end{aligned}$$

which implies that X_t has a density $r(t,x,y) = r(t,y-x)$ given by the inverse Fourier transform

$$r(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \cos(x\cdot\xi) e^{-t\Psi(\xi)} \, d\xi, \quad t > 0, x \in \mathbb{R}^d.$$

The density $r(t,x)$ is smooth in x and decays to 0 when $|x| \to 0$, see e.g. [68, Proposition 28.1]. This immediately implies the strong Feller property. Also, $r(t,x,y)$ are transition densities of X (or the heat kernel) in the sense that $R_t f(x) = \int_{\mathbb{R}^d} r(t,x,y) f(y)\, dy$. Furthermore, since X is obtained by the subordination of the Brownian motion, the density $r(t,x,y)$ can be also written as

$$r(t,x,y) = \int_0^\infty p(s,x,y)\mathbb{P}(S_t \in ds) = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{-\frac{|x-y|^2}{4s}} \mathbb{P}(S_t \in ds). \qquad (1.13)$$

From the representation (1.13) it is obvious that $x \mapsto r(t,x)$ is radially decreasing.

Since X is a Lévy process, the space $C_c^\infty(\mathbb{R}^d)$ is contained in the domain of the infinitesimal generator of the semigroup $(R_t)_t$, which we denote as $-\phi(-\Delta)$, and for $u \in C_c^\infty(\mathbb{R}^d)$ it holds that

$$\begin{aligned}\phi(-\Delta)u(x) &= \int_{\mathbb{R}^d} \left(u(x)-u(y)+\nabla u(x)\cdot(y-x)\mathbf{1}_{\{|y-x|\le 1\}}\right) j(|y-x|)\,dy \\ &= \lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon} (u(x)-u(y))\, j(|y-x|)\,dy, \end{aligned} \tag{1.14}$$

see [68, Section 31]. In the familiar case of the isotropic stable process the operator $\phi(-\Delta)$ is the fractional Laplacian.

We extend the definition of $\phi(-\Delta)$ by (1.14) for every $x \in \mathbb{R}^d$ and $u : \mathbb{R}^d \to \mathbb{R}$ such that the limit above exists. We note that if $\varphi \in C_c^2(\mathbb{R}^d)$, i.e. φ is a twice continuously differentiable function with compact support, then $\phi(-\Delta)\varphi(x)$ exists for every $x \in \mathbb{R}^d$. Indeed, since j is radial, for $\varphi \in C_c^2(\mathbb{R}^d)$ we have

$$\begin{aligned}\phi(-\Delta)\varphi(x) &= \lim_{\varepsilon\to 0^+}\int_{|y-x|>\varepsilon} (\varphi(x)-\varphi(y))j(|y-x|)dy \\ &= \lim_{\varepsilon\to 0^+}\int_{|y-x|>\varepsilon} \left(\varphi(x)-\varphi(y)+\nabla\varphi(x)\cdot(y-x)\mathbf{1}_{\{|x-y|\le 1\}}\right)dy \\ &= \int_{\mathbb{R}^d} \left(\varphi(x)-\varphi(y)+\nabla\varphi(x)\cdot(y-x)\mathbf{1}_{\{|x-y|\le 1\}}\right)j(|y-x|)dy.\end{aligned}$$

Therefore, $\phi(-\Delta)\varphi(x)$ exists for every $x \in \mathbb{R}^d$ since by Taylor's theorem

$$|\varphi(x)-\varphi(y)+\nabla\varphi(x)\cdot(y-x)\mathbf{1}_{\{|x-y|\le 1\}}| \le c(1\wedge |x-y|^2), \quad y \in \mathbb{R}^d,$$

and since the density j satisfies $\int_{\mathbb{R}^d}(1\wedge|x|^2)j(|x|) < \infty$. In fact, it is easy to see that there is a constant $C = C(K,\phi) > 0$, where $\operatorname{supp}\varphi \subset K \subset\subset \mathbb{R}^d$, such that

$$|\phi(-\Delta)\varphi(x)| \le C||\varphi||_{C^2(\mathbb{R}^d)}(1\wedge j(|x|)), \quad x \in \mathbb{R}^d. \tag{1.15}$$

Here $||\cdot||_{C^2(\mathbb{R}^d)}$ denotes the standard norm for twice differentiable functions.

For functions $u \in \mathscr{L}^1 := L^1(\mathbb{R}^d, (1\wedge j(|x|))dx)$ we define the distribution $\widetilde{\phi(-\Delta)}u$ as

$$\langle \widetilde{\phi(-\Delta)}u, \varphi\rangle := \langle u, \phi(-\Delta)\varphi\rangle := \int_{\mathbb{R}^d} u(x)\phi(-\Delta)\varphi(x)dx, \quad \varphi \in C_c^\infty(\mathbb{R}^d).$$

The condition $u \in \mathscr{L}^1$ is needed to ensure that the integral above is well defined, see (1.15). Also, note that since j is positive, we have $\mathscr{L}^1 \subset L^1_{loc}$. By following exactly the same calculations as in [15, Section 3], it is easy to show that if $u \in C^2(D) \cap \mathscr{L}^1$, then $\phi(-\Delta)u(x)$ exists for every $x \in D$ and $\widetilde{\phi(-\Delta)}u = \phi(-\Delta)u$ as distributions on D, i.e.

$$\langle \widetilde{\phi(-\Delta)}u, \varphi \rangle = \langle \phi(-\Delta)u, \varphi \rangle, \quad \varphi \in C_c^\infty(D).$$

Furthermore, we extend the definition of $\widetilde{\phi(-\Delta)}$ to measures in the following way

$$\langle \widetilde{\phi(-\Delta)}\lambda, \varphi \rangle := \langle \lambda, \phi(-\Delta)\varphi \rangle := \int_{\mathbb{R}^d} \phi(-\Delta)\varphi(x)\lambda(dx), \quad \varphi \in C_c^\infty(\mathbb{R}^d), \tag{1.16}$$

for all signed measures λ such that $\int_{\mathbb{R}^d}(1 \wedge j(|x|))|\lambda|(dx) < \infty$. This last condition ensures that the integral in (1.16) is well defined.

Since the process X is transient, we can define the potential kernel of X, i.e. the Green function of X, by

$$G(x) := \int_0^\infty r(t,x)dt, \quad x \in \mathbb{R}^d. \tag{1.17}$$

The kernel G is the density of the mean occupation time for X, i.e. for $f \geq 0$ we have

$$\int_{\mathbb{R}^d} G(x-y)f(y)dy = \mathbb{E}_x\left[\int_0^\infty f(X_t)dt\right], \quad x \in \mathbb{R}^d.$$

From [53, Lemma 3.2(b)] it follows that for every $M > 0$ there is a constant $C = C(\phi, M) > 0$ such that

$$C^{-1}\frac{1}{|x|^d\phi(|x|^{-2})} \leq G(x) \leq C\frac{1}{|x|^d\phi(|x|^{-2})}, \quad |x| \leq M. \tag{1.18}$$

In particular, G is finite for $x \neq 0$. Further, (1.17) implies that G is rotationally symmetric and radially decreasing so we will slightly abuse notation by denoting $G(x,y) = G(x-y) = g(|x-y|)$.

1.1.3. The potential kernel for the killed process and the Poisson kernel

The killed subordinate Brownian motion X^D has the semigroup $(R_t^D)_{t\geq 0}$ defined by

$$R_t^D f(x) = \mathbb{E}_x[f(X_t^D)] = \mathbb{E}_x[f(X_t), t < \tau_D], \quad f \in L^\infty(D).$$

By applying the strong Markov property, we get that X^D has transition densities which are for $t > 0$ and $x, y \in \mathbb{R}^d$ given by

$$r^D(t,x,y) = r(t,x,y) - \mathbb{E}_x[r(t-\tau_D, X_{\tau_D}, y)\mathbf{1}_{\{\tau_D < t\}}], \tag{1.19}$$

i.e. $R_t^D f(x) = \int_D r^D(t,x,y) f(y) dy$.

It follows that $0 \le r^D \le r$ and by repeating the proof of [33, Theorem 2.4] we get that r^D is symmetric. Also, [60, Proposition 2.3] implies that r_D is jointly continuous in $D \times D$, hence $(R_t^D)_t$ is strongly Feller. Since the process X has right continuous paths, it follows that $r^D(t,x,y) = 0$ if $x \in \overline{D}^c$ or $y \in \overline{D}^c$.

The Green function of the killed process X^D is defined by

$$G_D(x,y) := \int_0^\infty r^D(t,x,y)dt, \quad x,y \in \mathbb{R}^d,$$

which is the density of the mean occupation time for X^D, i.e. for $f \ge 0$ we have

$$\int_D G_D(x,y) f(y) dy = \mathbb{E}_x\left[\int_0^{\tau_D} f(X_t)dt\right], \quad x \in \mathbb{R}^d,$$

since we extended every Borel function f on D to $D \cup \{\partial\}$ by letting $f(\partial) = 0$. Also, note that $G = G_{\mathbb{R}^d}$.

For $x \in \mathbb{R}^d$ the $\mathbb{P}_x$ distribution of X_{τ_D} is denoted by ω_D^x, i.e.

$$\mathbb{P}_x(X_{\tau_D} \in A) = \omega_D^x(A), \quad A \in \mathscr{B}(\mathbb{R}^d).$$

The measure ω_D^x is concentrated on D^c and since we are in the transient case, (1.19) implies the following formula for $x,y \in \mathbb{R}^d$

$$G_D(x,y) = G(x,y) - \mathbb{E}_x[G(X_{\tau_D},y)] = G(x,y) - \int_{D^c} G(w,y)\omega_D^x(dw). \tag{1.20}$$

It follows from (1.19) that G_D is non-negative and symmetric. On $(D \times D) \setminus \{(x,x) : x \in D\}$ the kernel G_D is jointly continuous which we get by using the joint-continuity of the densities r_D, the bound $0 \le r_D \le r$, see also (1.13), and by the dominated convergence theorem.

Further, by using the strong Markov property and (1.20) it follows that for all open $U \subset D$ and $x,y \in \mathbb{R}^d$ we have

$$G_D(x,y) = G_U(x,y) + \int_{U^c} G_D(w,y)\omega_U^x(dw). \tag{1.21}$$

Equation (1.19) also yields that $G_D(x,y) = 0$ if $x \in \overline{D}^c$ or $y \in \overline{D}^c$. Furthermore, if $y \in \partial D$, then $G_D(x,y) = 0$ for all $x \in D$ if and only if y is a *regular* point for D. A point $x \in \partial D$ is regular for D if $\mathbb{P}_x(\tau_D = 0) = 1$, i.e. if $\omega_D^x = \delta_x$. A point at ∂D which is not regular is called *irregular* and it is well known that the set of irregular points is polar, i.e. the process X never enters the set of irregular points for D almost surely. This property will be used many times throughout the chapter.

Equation (1.21) yields that for every $x,y \in \mathbb{R}^d$ and open $U \subset D$ we have $G_U(x,y) \le G_D(x,y)$. In fact, if we have open sets $D_1 \subset D_2 \subset \cdots \subset D$ and $\cup_n D_n = D$, then $G_{D_n}(x,y) \uparrow G_D(x,y)$, for every $x,y \in \mathbb{R}^d$ except if x or y are irregular for D. This follows from (1.20), the continuity of G off the diagonal and the quasi-left-continuity of X.

For an open set $D \subset \mathbb{R}^d$, we define P_D, the Poisson kernel of D with respect to X, by

$$P_D(x,y) := \int_D G_D(x,w)j(|w-y|)dw, \quad (x,y) \in \mathbb{R}^d \times D^c. \tag{1.22}$$

If $x \in D$ the measure ω_D^x is absolutely continuous with respect to the Lebesgue measure in the interior of D^c and its Radon-Nikodym derivative is $P_D(x,\cdot)$, see [54, Eq. (1.1)]. Further, if the boundary of D possesses enough regularity, e.g. if D is a Lipschitz set, then

$$\omega_D^x(dy) = P_D(x,y)dy, \quad \text{on the whole } D^c, \tag{1.23}$$

see [65, Proposition 4.1].

By integrating (1.21) with respect to $j(|y-z|)dy$ on $\mathbb{R}^d$, with $z \in D^c$, and by using Fubini's theorem we get

$$P_D(x,z) = P_U(x,z) + \int_{D\setminus U} P_D(w,z)\omega_U^x(dw), \quad (x,z) \in U \times D^c, \tag{1.24}$$

where we used that the sets of irregular points at ∂D and ∂U are polar.

Suppose for a moment that the set of irregular points of D is empty. E.g. this holds if D is a Lipschitz set. Let $\nu \in \mathscr{M}(D)$ and set $u(y) = G_D\nu(y) := \int_D G_D(y,v)\nu(dv)$. Then for $z \in D^c$ we have $u(z) = 0$, hence

$$\begin{aligned}
-\phi(-\Delta)u(z) &= \lim_{\varepsilon\to 0}\int_{|y-z|>\varepsilon}(u(y)-u(z))\,j(|y-z|)\,dy = \int_{\mathbb{R}^d} G_D\nu(y)\,j(|y-z|)\,dy \\
&= \int_D\left(\int_D G_D(y,v)\nu(dv)\right) j(|y-z|)\,dy \\
&= \int_D\left(\int_D G_D(y,v)\,j(|y-z|)\,dy\right)\nu(dv) \\
&= \int_D P_D(v,z)\,\nu(dv),
\end{aligned}$$

if the last integral absolutely converges. In particular, if $\nu = \delta_x$ for $x \in D$, where δ_x is the Dirac measure at x, then $u(y) = G_D(x,y)$ and

$$-\phi(-\Delta)G_D(x,\cdot)(z) = P_D(x,z),$$

which gives an alternative expression for the Poisson kernel. Further, let $\psi : D \to \mathbb{R}$ be bounded, $u = G_D\psi$ and $\lambda \in \mathscr{M}(D^c)$. Then

$$\begin{aligned} -\int_{D^c} \phi(-\Delta)u(z)\,\lambda(dz) &= \int_{D^c}\left(\int_D P_D(y,z)\psi(y)\,dy\right)\lambda(dz) \\ &= \int_D \psi(y)\left(\int_{D^c} P_D(y,z)\,\lambda(dz)\right)dy = \int_D \psi(y)P_D\lambda(y)\,dy. \end{aligned} \tag{1.25}$$

This alternative expression of the Poisson kernel will be further discussed in Chapter 2.

1.1.4. Harmonic functions and Harnack inequality

A function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be harmonic with respect to the process X in an open set $D \subset \mathbb{R}^d$ if for every open $U \subset\subset D$ and all $x \in U$ it holds that $\mathbb{E}_x[|u(X_{\tau_U})|] < \infty$ and

$$u(x) = \mathbb{E}_x[u(X_{\tau_U})]. \tag{1.26}$$

We say that u is regular harmonic in D if (1.26) holds with $U = D$. If u is harmonic in D and $u = 0$ in $\overline{D}^c$, then u is said to be singular harmonic. From (1.21) we see that for $y \in D$ the function $x \mapsto G_D(x,y)$ is harmonic in $D \setminus \{y\}$ and regular harmonic in $D \setminus \overline{B(y,\varepsilon)}$ for every $\varepsilon > 0$.

We say that the scale invariant Harnack inequality is valid if there exists $r_0 > 0$ and a constant $c = c(r_0) > 0$ such that for every $x_0 \in \mathbb{R}^d$, every $r \in (0, r_0)$ and every function $u : \mathbb{R}^d \to [0,\infty)$ which is harmonic in the ball $B(x_0,r)$ it holds that

$$u(x) \le cu(y), \qquad x,y \in B(x_0, r/2).$$

It is well known that the scale invariant Harnack inequality is valid under the weak scaling condition **(WSC)**, cf. [42, Theorem 1, Theorem 7]. Moreover, it was proved in [44, Theorem 1.7] that if u is harmonic in an open set D, then $u \in C^\infty(D)$.

1.2. GREEN POTENTIALS

Definition 1.2.1. Let $D \subset \mathbb{R}^d$ be an open set and $f : D \to [-\infty, \infty]$. The Green potential of f is defined by

$$G_D f(x) := \int_D G_D(x,y) f(y) dy, \tag{1.27}$$

for all $x \in \mathbb{R}^d$ such that the integral above converges absolutely.

Lemma 1.2.2. Let $f \geq 0$. If the integral $\int_D G_D(x_0,y) f(y) dy$ converges at one point $x_0 \in D$, then $G_D f < \infty$ a.e., $G_D f \in \mathscr{L}^1$ and $f \in L^1_{loc}(D)$. In particular, if D is bounded, then $G_D f \in L^1(D)$.

Proof. Let $0 < s < \delta_D(x_0)$, and denote just for this proof $B = B(x_0,s)$. By using the strong Markov property we have

$$\begin{aligned} \infty > G_D f(x_0) &\geq \mathbb{E}_{x_0}\left[\int_{\tau_B}^{\tau_D} f(X_t) dt\right] = \mathbb{E}_{x_0}\left[\mathbb{E}_{X_{\tau_B}}\left[\int_0^{\tau_D} f(X_t) dt\right]\right] \\ &= \mathbb{E}_{x_0}[G_D f(X_{\tau_B})] = \int_{B^c} G_D f(y) P_B(x_0,y) dy. \end{aligned} \tag{1.28}$$

From [44, Lemma 2.2] we have that $P_B(x_0,y) \geq c_1 j(|x_0 - y|)$, $y \in \overline{B}^c$. Further, since j decreases and vanishes at infinity, pick $r_0 \in (1,\infty)$ such that $j(|y|) \leq 1$, for $|y| \geq r_0$. Inequality (1.7) implies that there is a constant $c_2 > 0$ such that $j(|y|) \leq c_2 j(|x_0 - y|)$, for all $|y| \geq r_0$. Let $m := \inf\{j(|x_0 - y|) : y \in B^c, |y| \leq r_0\} > 0$. Thus, for $y \in \overline{B}^c$ we have

$$1 \wedge j(|y|) \leq \max\{c_2, 1/m\} j(|x_0 - y|).$$

Therefore, there is $c_3 > 0$ such that $P_B(x_0,y) \geq c_3(1 \wedge j(|y|)) > 0$, $y \in \overline{B}^c$. This yields

$$\int_{B^c} G_D f(y)(1 \wedge j(|y|)) dy < \infty,$$

hence $G_D f < \infty$ a.e. on B^c. Starting the calculations again from the point $\tilde{x} \in D \setminus \overline{B}$ such that $G_D f(\tilde{x}) < \infty$, we also get $\int_B G_D f(y)(1 \wedge j(|y|)) dy < \infty$. Hence, $G_D f < \infty$ a.e. and $G_D f \in \mathscr{L}^1$. Since j is non-negative, this also means $G_D f \in L^1(D)$ if D is bounded.

To prove that $f \in L^1_{loc}(D)$ take $U \subset\subset D$ and $x \in D \setminus \overline{U}$ such that $G_D f(x) < \infty$. Recall that the function $y \to G_D(x,y)$ is harmonic in $D \setminus \{x\}$, hence bounded from below and above on U by the Harnack inequality, see Subsection 1.1.4 or [42, Theorem 7], so

$$\infty > G_D f(x) \geq \int_U G_D(x,y) f(y) \geq c \int_U f(y) dy.$$

■

The following proposition is an extension of [16, Lemma 5.3] to more general non-local operators.

Proposition 1.2.3. Let D be an open set. If $f : D \to [-\infty, \infty]$ satisfies $G_D|f|(x) < \infty$ for some $x \in D$, then $\widetilde{\phi(-\Delta)}(G_D f) = f$ in D.

Proof. In [43, Lemma 3.5] the claim was proved for bounded D and for $f \in L^1(D)$. Recall that Lemma 1.2.2 yields that $G_D f$ is well defined almost everywhere and $f \in L^1_{loc}(D)$. Without loss of generality we can assume that $f \geq 0$.

Suppose that D is unbounded and $f \in L^1_{loc}(D)$. There is an increasing sequence of open precompact sets $(D_n)_n$ in D such that $\cup_n D_n = D$. Define $f_n := f \mathbf{1}_{D_n} \in L^1(D_n)$ and note that D_n is bounded. Further, $G_{D_n} f = G_{D_n} f_n$ so by the monotone convergence theorem we get $G_{D_n} f_n \uparrow G_D f$ a.e. in D and also in $\mathscr{L}^1$ by Lemma 1.2.2. Hence, due to (1.15) for all $\varphi \in C_c^\infty(D)$ we get

$$\begin{aligned}\langle \widetilde{\phi(-\Delta)} G_D f, \varphi \rangle &= \langle G_D f, \phi(-\Delta)\varphi \rangle = \lim_{n\to\infty} \langle G_{D_n} f_n, \phi(-\Delta)\varphi \rangle \\ &= \lim_{n\to\infty} \langle f_n, \varphi \rangle = \langle f, \varphi \rangle,\end{aligned}$$

where in the third equality we used [43, Lemma 3.5] since for all large enough $n \in \mathbb{N}$ we have $\operatorname{supp}\varphi \subset D_n$. ■

1.3. POISSON KERNEL AND $\phi(-\Delta)$-HARMONIC FUNCTIONS

Proposition 1.3.1. Let D be an open set. Then $P_D : D \times \overline{D}^c \to (0,\infty)$ is jointly continuous.

Proof. We imitate the proof of the similar claim for the isotropic α-stable process, see [76, Theorem 5.7]. Let $(x_n)_n \subset D$ and $(z_n)_n \subset \overline{D}^c$ such that $x_n \to x \in D$, and $z_n \to z \in \overline{D}^c$. Let $0 < \varepsilon, \delta < 1$ such that $\delta_D(x) > 2\delta$ and $\delta_{D^c}(z) > 2\varepsilon$. Then for all large enough $n \in \mathbb{N}$ we have $\delta_D(x_n) > \delta$ and $\delta_{D^c}(z_n) > \varepsilon$. By (1.22) we have

$$
\begin{aligned}
|P_D(x_n,z_n) - P_D(x,z)| &= \left| \int_D G_D(x_n,y)j(|y-z_n|)dy - \int_D G_D(x,y)j(|y-z|)dy \right| \\
&\le \left| \int_{D\cap B(x,2\delta)^c} G_D(x_n,y)j(|y-z_n|)dy - \int_{D\cap B(x,2\delta)^c} G_D(x,y)j(|y-z|)dy \right| \\
&\quad + \int_{B(x,2\delta)} G_D(x_n,y)j(|y-z_n|)dy + \int_{B(x,2\delta)} G_D(x,y)j(|y-z|)dy.
\end{aligned}
$$

Recall that j is continuous and that G_D is continuous off the diagonal. Thus, for the first term we have by the dominated convergence theorem

$$
\lim_{n\to\infty} \int_{D\cap B(x,2\delta)^c} G_D(x_n,y)j(|y-z_n|)dy = \int_{D\cap B(x,2\delta)^c} G_D(x,y)j(|y-z|)dy.
$$

Indeed, we can apply the dominated convergence theorem since $G_{\mathbb{R}^d}$ is radially decreasing so there is $c_1 > 0$ such that $G_D(w,y) \le G_{\mathbb{R}^d}(w,y) \le c_1$ for all $w \in B(x,\delta)$ and $y \in B(x,2\delta)^c$. Also, by using (1.7) there is $c_2 > 0$ such that $j(|y-q|) \le c_2 j(|y-z|)$ for $q \in B(z,\varepsilon)$ and $y \in D \cap B(x,2\delta)^c$.

For the other two integrals we use the estimate (1.18), i.e. we use

$$
G_D(x,y) \le G_{\mathbb{R}^d}(x,y) \le c_3 \frac{1}{|x-y|^d \phi(|x-y|^{-2})}, \quad |x-y| < 3,
$$

where $c_3 = c_3(\phi) > 0$. Now for all $w \in B(x,\delta)$, and $q \in B(z,\varepsilon)$ we have

$$\begin{aligned}\int\limits_{B(x,2\delta)} G_D(w,y) j(|y-q|)dy &\leq j(\varepsilon) \int\limits_{B(x,2\delta)} G_D(w,y)dy \\ &\leq j(\varepsilon)\left(\int\limits_{B(x,2\delta)\cap B(w,\delta)} G_D(w,y)dy + \int\limits_{B(x,2\delta)\cap B(w,\delta)^c} G_D(w,y)dy \right) \\ &\leq j(\varepsilon)c_3 \left(\int\limits_0^{\delta} \frac{dr}{r\phi(r^{-2})} + \int_{\delta}^{3\delta} \frac{dr}{r\phi(r^{-2})} \right) \leq j(\varepsilon)c_3 \left(\int\limits_0^{3\delta} \frac{dr}{r\phi(r^{-2})} \right) \overset{\delta\to 0}{\longrightarrow} 0,\end{aligned}$$

where for the convergence of the integral part we use for example (1.3). ■

Definition 1.3.2. Let $D \subset \mathbb{R}^d$ be an open set and let λ be a σ-finite signed measure on D^c such that for all $x \in D$

$$\int_{D^c} P_D(x,y)|\lambda|(dy) < \infty. \tag{1.29}$$

The Poisson integral of λ is defined by

$$P_D\lambda(x) := \int_{D^c} P_D(x,y)\lambda(dy), \quad x \in D.$$

We extend the definition of the Poisson integral for non-negative σ-finite measures by the same formula, i.e. for σ-finite measure λ we define

$$P_D\lambda(x) := \int_{D^c} P_D(x,y)\lambda(dy) \in [0,\infty], \quad x \in D.$$

Although this seems as an extension of the definition, it will follow from Theorem 1.3.5 that either $P_D|\lambda| \equiv \infty$ or $P_D|\lambda| < \infty$ in D, see Remark 1.3.6.

It will be of considerable interest to extend $P_D\lambda$ to the whole $\mathbb{R}^d$ in the following sense. We define the (signed) measure $P_D^*\lambda$ by

$$P_D^*\lambda(dy) = P_D\lambda(y)\mathbf{1}_D(y)dy + \mathbf{1}_{D^c}(y)\lambda(dy), \tag{1.30}$$

i.e. $P_D^*\lambda$ is on D the (signed) measure with the density function $P_D\lambda$ and on D^c it is the (signed) measure λ. This extension was introduced in [21, Eq. (25)] for the case of the isotropic α-stable process.

Remark 1.3.3. Suppose that $P_D|\lambda|(x) < \infty$ for all $x \in D$. Then λ is finite on compact subsets of $\overline{D}^c$. Indeed, let K be a compact subset of $\overline{D}^c$ and let $s \in (0,1)$ such that $\overline{B(x,s)} \subset$

D. For $y \in \overline{D}^c$ by [51, Proposition 4.7] we have $P_D(x,y) \geq P_{B(x,s)}(x,y) \geq c_1\, j(|x-y|)$, where $c_1 > 0$. Thus, since j is continuous and strictly positive, we have

$$\infty > \int_{D^c} P_D(x,y)|\lambda|(dy) \geq c_2\,|\lambda|(K),$$

where $c_2 > 0$. Furthermore, in Remark 1.4.2 we will see that λ can have some mass on ∂D but only on the specific part of the boundary at so-called inaccessible points.

Lemma 1.3.4.

(a) Let $R \in (0,1)$. There is a constant $C = C(\phi) > 0$ such that if λ is a σ-finite measure supported on B_R^c, and $D \subset B_R$, then for all $x \in D \cap B_{R/2}$ it holds

$$C^{-1}\,\mathbb{E}_x\tau_D \int\limits_{B_{R/2}^c} j(|y|)P_D^*\lambda(dy) \leq P_D\lambda(x) \leq C\,\mathbb{E}_x\tau_D \int\limits_{B_{R/2}^c} j(|y|)P_D^*\lambda(dy). \tag{1.31}$$

(b) Suppose **(GWSC)** and let $R \geq 1$. There is a constant $C = C(\phi) > 0$ such that if λ is a σ-finite measure supported on $\overline{B}_R$, and $D \subset \overline{B}_R^c$, then for all $x \in D \cap \overline{B}_{2R}^c$ it holds

$$C^{-1}P_D(x,0) \int\limits_{\overline{B}_{2R}} P_D^*\lambda(dy) \leq P_D\lambda(x) \leq C\,P_D(x,0) \int\limits_{\overline{B}_{2R}} P_D^*\lambda(dy). \tag{1.32}$$

Proof. For the part (a) we will use [51, Lemma 5.4]. The inequality from the statement of [51, Lemma 5.4] applied in our case when $z_0 = 0$, $U = D$, and $r = R$ gives us

$$\begin{aligned} C^{-1}\,\mathbb{E}_x\tau_D &\left(\int_{D\setminus B_{R/2}} j(|w|)P_D(w,y)dw + j(|y|)\right) \\ &\leq P_D(x,y) \leq C\,\mathbb{E}_x\tau_D\left(\int_{D\setminus B_{R/2}} j(|w|)P_D(w,y)dw + j(|y|)\right), \end{aligned} \tag{1.33}$$

for all $(x,y) \in (D \cap B_{R/2}) \times B_R^c$. In [51, Lemma 5.4] the second point y had to be in $B_R^c \cap \overline{D}^c$ but the claim is also true for the points $y \in \partial D$ which can be seen by inspecting the proof of the lemma since [51, Eq. (5.1)] can be extended to (1.24). Hence, to finish the proof we just need to integrate the inequality (1.33) with respect to the measure $\lambda(dy)$.

For the part (b) we will use [52, Lemma 3.4]. Similarly as above, the inequality from the statement of [52, Lemma 3.4] applied in our case with $a = 2$, $U = D$ and $r = R$ gives us

$$\begin{aligned} C^{-1}\,P_D(x,0) &\left(\int_{D\cap B_{2R}} P_D(w,y)dw + 1\right) \\ &\leq P_D(x,y) \leq C\,P_D(x,0)\left(\int_{D\cap B_{2R}} P_D(w,y)dw + 1\right), \end{aligned} \tag{1.34}$$

which is valid for any $(x,y) \in (D \cap \overline{B_{2R}}^c) \times \overline{B_R}$ and not only for $(x,y) \in (D \cap \overline{B_{2R}}^c) \times B_R$ which can be checked by inspecting the proof. Again, the only difference is in the fact that [52, Eq. (3.10)] can be extended to (1.24). To finish the proof we need to integrate the inequality (1.34) with respect to the measure $\lambda(dy)$. ■

Lemma 1.3.4 yields the following version of a uniform boundary Harnack principle.

Theorem 1.3.5.

(a) There is a constant $C = C(\phi) > 1$ such that for every $R \in (0,1)$, for all open $D \subset \mathbb{R}^d$, $x_1, x_2 \in D \cap B_{R/2}$, $y_1, y_2 \in D^c \cap B_R^c$, and for all σ-finite measures ρ, λ on B_R^c we have

$$P_D(x_1,y_1)P_D(x_2,y_2) \le C\,P_D(x_1,y_2)P_D(x_2,y_1) \tag{1.35}$$

and

$$P_D\rho(x_1)P_D\lambda(x_2) \le C\,P_D\rho(x_2)P_D\lambda(x_1). \tag{1.36}$$

(b) Suppose **(GWSC)**. There is a constant $C = C(\phi) > 1$ such that for every $R \ge 1$, for all open $D \subset \mathbb{R}^d$, $x_1, x_2 \in D \cap \overline{B}_{2R}^c$, $y_1, y_2 \in D^c \cap \overline{B}_R$, and for all σ-finite measures ρ, λ on $\overline{B}_R$ we have

$$P_D(x_1,y_1)P_D(x_2,y_2) \le C\,P_D(x_1,y_2)P_D(x_2,y_1) \tag{1.37}$$

and

$$P_D\rho(x_1)P_D\lambda(x_2) \le C\,P_D\rho(x_2)P_D\lambda(x_1). \tag{1.38}$$

The first part of this theorem is an extension of [51, Theorem 1.1(ii)] with $\overline{D}^c$ being replaced by D^c, i.e. the difference is that points y_1 and y_2 in (1.35) can be at ∂D. This subtle difference comes as a consequence of Lemma 1.3.4 and will play a very important role in proving the results on the relative oscillation of Poisson integrals, e.g. Lemma 1.5.5.

Proof of Theorem 1.3.5. We give the proof of the first claim. The second claim follows similarly.

Let $D_R = D \cap B_R$. From (1.24) follows that for $x_i \in D \cap B_{R/2}$, $i \in \{1,2\}$, and $y_j \in D^c \cap B_R^c$, $j \in \{1,2\}$, we have

$$\begin{aligned} P_D(x_i, y_j) &= P_{D_R}(x_i, y_j) + \int_{D_R^c} P_D(w, y_j) \omega_{D_R}^{x_i}(dw) \\ &= P_{D_R}(x_i, y_j) + \int_{D_R^c} P_D(w, y_j) P_{D_R}(x_i, w) dw. \end{aligned}$$

Indeed, the last equality holds true since the boundary of D_R has a part $D \cap \partial B_R$ which is smooth, and for the other part of the boundary of D_R we use the fact that the irregular points for D at ∂D are polar. Thus, we proved that $P_D(x_i, y_j) = P_{D_R}\lambda_j(x_i)$ for some measure λ_j supported on B_R^c. Now (1.35) follows from Lemma 1.3.4. By integrating (1.35) with respect to the measures $\rho(dy_1)$ and $\lambda(dy_2)$ we get (1.36). ■

Remark 1.3.6. Note that for the σ-finite measures ρ and λ appearing in Theorem 1.3.5 we do not assume (1.29). However, by fixing $\rho = \delta_{y_2}$, where $y_2 \in \overline{D}^c$, it follows from (1.36) that if for a σ-finite signed measure λ on D^c we have $P_D|\lambda|(x) < \infty$ for some $x \in D$, then we have $P_D|\lambda|(x) < \infty$ for all $x \in D$. This means that either $P_D|\lambda| \equiv \infty$ or $P_D|\lambda| < \infty$ in D.

Now we bring a generalization of harmonic functions with respect to X, see Subsection 1.1.4.

Definition 1.3.7. Let $D \subset \mathbb{R}^d$ be an open set. We say that $f : D \to \mathbb{R}$ is $\phi(-\Delta)$-harmonic in D with outer charge λ if λ is a σ-finite (signed) measure on D^c and if for every $U \subset\subset D$ and $x \in U$ we have

$$f(x) = \int_{D^c} P_U(x,y)\lambda(dy) + \int_{D \setminus U} f(y) \omega_U^x(dy), \tag{1.39}$$

where the integrals converge absolutely.

The definition above was first used in [21] for the isotropic α-stable process with an additional assumption of the continuity of the function f. We prove in Proposition 1.3.9 that this additional assumption can be dropped. Moreover, in Theorem 1.3.12 we prove that $f \in C^\infty(D)$. Furthermore, note that a function $u : \mathbb{R}^d \to \mathbb{R}$ which is harmonic in D is $\phi(-\Delta)$-harmonic in D with outer charge $\lambda(dy) = u(y)dy$. Indeed, take $U \subset\subset D$ and

$x \in U$. Equation (1.26) implies

$$
\begin{aligned}
u(x) = \mathbb{E}_x[u(X_{\tau_U})] &= \int_{U^c} u(y)\omega_U^x(dy) \\
&= \int_{D^c} P_U(x,y)u(y)dy + \int_{D\setminus U} u(y)\omega_U^x(dy),
\end{aligned}
\tag{1.40}
$$

where we used that $P_U(x,\cdot)$ is the density of ω_U^x in the interior of U^c. Hence, every harmonic function is $\phi(-\Delta)$-harmonic. Furthermore, if u is $\phi(-\Delta)$-harmonic in D with outer charge λ such that λ is absolutely continuous with respect to the Lebesgue measure on D^c, then u is harmonic in D. In particular, if u has zero outer charge, i.e. $\lambda \equiv 0$, then u is a singular harmonic function.

If f is $\phi(-\Delta)$-harmonic in D with outer charge λ we sometimes abbreviate notation by saying (f,λ) is $\phi(-\Delta)$-harmonic in D. Property (1.39) is often referred to as the mean-value property because of the connection with (1.40). Similarly as in (1.30), integrating with respect to (f,λ) means that we integrate with respect to the measure $f(y)\mathbf{1}_D(y)dy + \mathbf{1}_{D^c}(y)\lambda(dy)$.

We continue with a few basic properties of $\phi(-\Delta)$-harmonic functions.

Lemma 1.3.8. Let D be an open set. If (f,λ) is $\phi(-\Delta)$-harmonic in D, then

$$
\int_{\mathbb{R}^d} (1\wedge j(|y|))(|f|,|\lambda|)(dy) < \infty. \tag{1.41}
$$

In particular, $f \in L^1_{loc}(D)$ and if D is bounded, we have $f \in L^1(D)$.

Proof. Let $B(x,s) \subset\subset D$. With the same calculations as in Lemma 1.2.2 we get that $P_{B(x,s)}(x,y) \geq c(1\wedge j(|y|)) > 0$, $y \in B(x,s)^c$. Hence, from the absolute finiteness of (1.39) for $U = B(x,s)$ we obtain

$$
\infty > \int_{D^c} (1\wedge j(|y|))|\lambda|(dy) + \int_{D\setminus B(x,s)} |f(y)|(1\wedge j(|y|))(dy). \tag{1.42}
$$

By considering $\tilde{x} \in D\setminus \overline{B}(x,s)$ and repeating the argument which lead to (1.42) we also get that

$$
\int_{B(x,s)} |f(y)|(1\wedge j(|y|))(dy) < \infty. \tag{1.43}
$$

Thus, $f \in L^1_{loc}(D)$ and (1.41) holds. Obviously, if D is bounded, then $y \mapsto (1\wedge j(|y|))$ is bounded from below and above, so we have $f \in L^1(D)$. ■

Proposition 1.3.9. Let D be an open set. If (f,λ) is $\phi(-\Delta)$-harmonic in D, then $f \in C(D)$.

Proof. Let $x \in D$ and $(x_n)_n \subset D$ such that $x_n \to x$. Let $0 < \varepsilon < 1$ be such that $\delta_D(x) > \varepsilon$. Without loss of generality suppose that for all $n \in \mathbb{N}$ we have $x_n \in B(x,\varepsilon/2)$. By using (1.39) with $U = B(x,\varepsilon)$ and applying (1.23) we have

$$f(x) = \int_{D^c} P_{B(x,\varepsilon)}(x,y)\lambda(dy) + \int_{D\setminus B(x,\varepsilon)} f(y)P_{B(x,\varepsilon)}(x,y)dy,$$
$$f(x_n) = \int_{D^c} P_{B(x,\varepsilon)}(x_n,y)\lambda(dy) + \int_{D\setminus B(x,\varepsilon)} f(y)P_{B(x,\varepsilon)}(x_n,y)dy.$$

Note that Proposition 1.3.1 yields $P_{B(x,\varepsilon)}(x_n,y) \to P_{B(x,\varepsilon)}(x,y)$. Also, inequality (1.35) for $D = B(x,\varepsilon)$ implies that there is a constant $c > 0$ such that $P_{B(x,\varepsilon)}(x_n,y) \le c\,P_{B(x,\varepsilon)}(x,y)$, for all $n \in \mathbb{N}$ and all $y \in B(x,s)^c$. Now by the dominated convergence theorem we have $f(x_n) \to f(x)$. ■

In Theorem 1.3.12 we strengthen the previous proposition by proving that $f \in C^\infty(D)$. This is achieved using the same technique as in [44, Proposition 3.2 & Theorem 1.7]. First we invoke [44, Proposition 3.2] and its consequences.

Lemma 1.3.10. Let $0 \le q < r < \infty$. There is a radial kernel function $\overline{P}_{q,r} : \mathbb{R}^d \to \mathbb{R}$, a constant $C = C(\phi,q,r) > 0$, and a probability measure $\mu_{q,r}$ on $[q,r]$ with the following properties:

(a) $0 \le \overline{P}_{q,r} \le C$ in $\mathbb{R}^d$, $\overline{P}_{q,r} = 0$ in B_q, $\overline{P}_{q,r} = C$ in $B_r \setminus B_q$, $\overline{P}_{q,r}$ is radially decreasing, and $\overline{P}_{q,r}(z) \le P_{B_r}(0,z)$, for $|z| > r$;

(b) for any $A \in \mathscr{B}(\mathbb{R}^d)$ it holds

$$\int_A \overline{P}_{q,r}(z)dz = \int_{[q,r]} \int_A P_{B_s}(0,y)dy\,\mu_{q,r}(ds). \tag{1.44}$$

Equality (1.44) implies the following claim.

Lemma 1.3.11. Let $0 \le q < r < \infty$ and $\varepsilon > 0$. If (f,λ) is $\phi(-\Delta)$-harmonic in $B_{r+\varepsilon}$, then

$$f(0) = \int_{\mathbb{R}^d\setminus B_q} \overline{P}_{q,r}(z)(f,\lambda)(dz).$$

In particular, if (f,λ) is $\phi(-\Delta)$-harmonic in $B_{2r+\varepsilon}$, then for $x \in B_r$ it holds

$$f(x) = \int_{\mathbb{R}^d \setminus B(x,q)} \overline{P}_{q,r}(x-z)(f,\lambda)(dz),$$

i.e. $f = (f,\lambda) * \overline{P}_{q,r}$ in B_r.

The following theorem is a generalization of [44, Theorem 1.7] to $\phi(-\Delta)$-harmonic functions.

Theorem 1.3.12. Let D be an open set. If f is $\phi(-\Delta)$-harmonic in D with outer charge λ, then $f \in C^\infty(D)$.

Proof. The claim can be proved in the same way as in [44]. However, in [44] it was assumed that f is bounded so we will repeat and slightly extend the first part of the proof to justify the calculations that follow.

Due to the translation invariance of the process X, we can assume that $0 \in D$. Let $r \in (0,1)$ and $k \in \mathbb{N}$ be such that $B_{2(k+1)r} \subset\subset D$. Set $q = 0$ and let C_r denote $C(\phi,0,r) > 0$ of Lemma 1.3.10. Further, let κ be a non-negative smooth radial function which takes values in $[0,1]$, which is equal to 1 in $B_{3r/2}$, and which is equal to 0 in B^c_{2r}. Define $\pi_r(z) = \overline{P}_{0,r}(z)\kappa(z)$, and $\Pi_r(z) = \overline{P}_{0,r}(z)(1-\kappa(z))$.

Note that Proposition 1.3.9 yields that f is bounded on $B_{2(k+1)r}$ so set $m := \sup_{B_{2(k+1)r}} |f| < \infty$. From Lemma 1.3.10$(a)$ for $x \in B_{2kr}$ we have

$$\begin{aligned}(|f|,|\lambda|) * \overline{P}_{0,r}(x) &= \int\limits_{B(x,2r)} |f(z)|\overline{P}_{0,r}(x-z)dz + \int\limits_{B(x,2r)^c} \overline{P}_{0,r}(x-z)(|f|,|\lambda|)(dz) \\ &\leq c_1 m C_r + \int\limits_{B(x,2r)^c} P_{B(x,r)}(x,z)(|f|,|\lambda|)(dz), \qquad (1.45)\end{aligned}$$

where $c_1 = c_1(r) > 0$ is the volume of a ball with radius r. From [51, Proposition 4.7] we have $P_{B(x,r)}(x,z) \leq c_2 j(|x-z|-r)$, where $c_2 = c_2(\phi,r) > 0$. Thus, by using (1.7), we get that there is $c_3 = c_3(\phi,k,r) > 0$ such that for all $x \in B_{2kr}$ and $z \in B^c(x,2r)$ it holds

$$P_{B(x,r)}(x,z) \leq c_3(1 \wedge j(|z|)).$$

Applying this inequality in (1.45) and recalling Lemma 1.3.8 we get that there is $M = M(\phi,k,r,f,\lambda) < \infty$ such that

$$(|f|,|\lambda|) * \overline{P}_{0,r} \leq M, \quad \text{in } B_{2kr}. \qquad (1.46)$$

Obviously, since $\overline{P}_{0,r} = \pi_r + \Pi_r$, we have $|f| * \pi_r \le M$ and $(|f|, |\lambda|) * \Pi_r \le M$ in B_{2kr}. Also, since $f = (f,\lambda) * \overline{P}_{0,r}$ in B_{2kr}, we have

$$f = f * \pi_r + (f,\lambda) * \Pi_r, \qquad \text{in } B_{2kr}. \tag{1.47}$$

Finally, inequality (1.46) implies that the convolution property (1.47) can be used iteratively to get that for $x \in B_r$ it holds

$$f = (\delta_0 + \pi_r + \pi_r^{*2} + \dots, \pi_r^{*(k-1)}) * \Pi_r * (f,\lambda) + \pi_r^{*k} * f.$$

Since all derivatives of the jumping kernel j exist and are absolutely integrable in B_ε^c, for every $\varepsilon > 0$, see [18, Proposition 7.2], we may proceed with the proof in the same way as in [44, Theorem 1.7]. ■

Corollary 1.3.13. Let D be an open set. If λ is a σ-finite signed measure on D^c satisfying (1.29), then for every $x \in U \subset D$

$$P_D\lambda(x) = \int_{D^c} P_U(x,y)\lambda(dy) + \int_{D\setminus U} P_D\lambda(y)\omega_U^x(dy). \tag{1.48}$$

In particular, $P_D\lambda$ is $\phi(-\Delta)$-harmonic in D with outer charge λ and $P_D\lambda \in C^\infty(D) \cap L^1_{loc}(D)$. Also, if D is bounded $P_D\lambda \in L^1(D)$.

Proof. Take $U \subset D$ and $x \in U$. By integrating (1.24) with respect to $\lambda(dz)$ we get (1.48). In particular, $P_D\lambda$ is $\phi(-\Delta)$-harmonic in D with outer charge λ. Hence by Theorem 1.3.12 and Lemma 1.3.8 we have $P_D\lambda \in C^\infty(D) \cap L^1_{loc}(D)$ and if D is bounded, then $P_D\lambda \in L^1(D)$. ■

Remark 1.3.14. Note that (1.48) holds for every $U \subset D$ which is a lot stronger than needed in (1.39). This property will be heavily used in proving results on the relative oscillation of Poisson integrals.

We finish this section by proving two theorems about the connection between harmonic functions and the operator $\phi(-\Delta)$. First we prove an auxiliary result.

Lemma 1.3.15. Let D be an open set and λ be a σ-finite signed measure on D^c such that (1.29) is satisfied. Then $\widetilde{\phi(-\Delta)}(P_D^*\lambda) = 0$ in D.

Proof. First recall that for $\varphi \in C_c^\infty(D)$ we have

$$\phi(-\Delta)\varphi(x) = \text{P.V.}\int_{\mathbb{R}^d} (\varphi(x)-\varphi(y))\, j(|x-y|)dy,$$

and

$$\begin{aligned}\langle \widetilde{\phi(-\Delta)}(P_D^*\lambda), \varphi\rangle = \langle P_D^*\lambda, \phi(-\Delta)\varphi\rangle &= \int_D P_D\lambda(x)\phi(-\Delta)\varphi(x)dx + \int_{D^c}\phi(-\Delta)\varphi(x)\lambda(dx)\\ &=: I_1 + I_2.\end{aligned}$$

Note that $P_D\lambda(x) = G_D f(x)$, $x\in D$, where $f(z) = \int_{D^c} j(|z-y|)\lambda(dy)$. Hence, by Proposition 1.2.3 we have

$$I_1 = \int_D P_D\lambda(x)\phi(-\Delta)\varphi(x)dx = \int_D \left(\int_{D^c} j(|x-y|)\lambda(dy)\right)\varphi(x)dx. \tag{1.49}$$

For the integral I_2 recall that $\text{supp}\varphi \subset D$ and $\varphi = 0$ on D^c. Hence

$$\begin{aligned} I_2 = \int_{D^c}\phi(-\Delta)\varphi(x)\lambda(dx) &= \int_{D^c}\left(\int_D -\varphi(y)j(|x-y|)dy\right)\lambda(dx)\\ &= -\int_D \varphi(y)\left(\int_{D^c} j(|x-y|)\lambda(dx)\right)dy, \end{aligned}\tag{1.50}$$

where we can change the order of integration by Fubini's theorem since $f\in L^1_{loc}(D)$. Thus, from (1.49) and (1.50) we obtain $\langle \widetilde{\phi(-\Delta)}(P_D^*\lambda), \varphi\rangle = 0$ for all $\varphi\in C_c^\infty(D)$. ■

Theorem 1.3.16. Let D be an open set and u $\phi(-\Delta)$-harmonic in D with outer charge λ. Then $\widetilde{\phi(-\Delta)}(u,\lambda) = 0$ in D.

Proof. Let $\varphi\in C_c^\infty(D)$. There is $U \subset\subset D$ with Lipschitz boundary such that $\text{supp}\varphi\subset U$, i.e. $\varphi\in C_c^\infty(U)$. From the mean-value property (1.39) for u and U, we have $u = P_U\tilde{\lambda}$ in U, where $\tilde{\lambda}(dy) = u(y)\mathbf{1}_{D\setminus U}(y)dy + \mathbf{1}_{D^c}(y)\lambda(dy)$. This means that u is the Poisson integral on U so Lemma 1.3.15 implies

$$\begin{aligned}&\int_D \phi(-\Delta)\varphi(x)u(x)dx + \int_{D^c}\phi(-\Delta)\varphi(x)\lambda(dx)\\ &\qquad = \int_U \phi(-\Delta)\varphi(x)P_U\tilde{\lambda}(x)dx + \int_{U^c}\phi(-\Delta)\varphi(x)\tilde{\lambda}(dx) = 0.\end{aligned}$$

Since φ was arbitrary, we have the claim. ■

Remark 1.3.17. The proof of the previous theorem is valid in a much greater generality. Indeed, the only non-trivial part of the proof was the property $\widetilde{\phi(-\Delta)}(G_D f) = f$ in D proved in Proposition 1.2.3. One can check that Proposition 1.2.3 is true with the same proof for the isotropic unimodal Lévy process with the condition (1.7) on the jumping kernel since the auxiliary results [43, Lemma 3.5] and Lemma 1.2.2 also hold in this setting.

In the next theorem we prove a converse of Theorem 1.3.16, which implies that $\phi(-\Delta)$-harmonic functions and generalized functions which are annihilated by $\phi(-\Delta)$ are essentially the same. This equivalency is known for classical functions in slightly more general non-local setting, see [43, Theorem 3.4].

Theorem 1.3.18. Let D be an open set and (u,λ) such that (1.41) holds. If $\widetilde{\phi(-\Delta)}(u,\lambda) = 0$ in D, then u has a modification $\tilde{u}$ in D such that $(\tilde{u},\lambda)$ is $\phi(-\Delta)$-harmonic in D.

Proof. The proof relies on the proofs of [43, Lemma 3.2 & Lemma 3.3].

First we prove the claim for $u \in C^2(D)$ by following the proof of [43, Lemma 3.2]. Let $D_1 \subset\subset D$ be a Lipschitz set and define $\widetilde{u} = P^*_{D_1}\big[(u,\lambda)\big]$. By Corollary 1.3.13 $\widetilde{u}$ is $\phi(-\Delta)$-harmonic in D_1 and $\widetilde{u} \in C^\infty(D_1)$, thus $\phi(-\Delta)\widetilde{u}(x) = 0$, $x \in D_1$, by Theorem 1.3.16. Now we prove that $\widetilde{u}$ is also continuous up to the boundary of D_1. Indeed, take D_2 such that $D_1 \subset\subset D_2 \subset\subset D$ and note that

$$\widetilde{u}(x) = \int_{D_2 \cap D_1^c} P_{D_1}(x,y)u(y)dy + \int_{D_2^c} P_{D_1}(x,y)(u,\lambda)(dy). \tag{1.51}$$

Note that $u \in L^\infty(D_2)$ since $u \in C_2(D)$, and that $P_{D_1}(x,z) \le c(1 \wedge j(|z|))$ for $x \in D_1$ and $z \in D_2$. Since for (u,λ) the integrability condition (1.41) holds, we see that $\lim\limits_{D_1 \ni x \to x_0 \in \partial D_1} \widetilde{u}(x) = u(x_0)$, i.e. $\widetilde{u}$ in $C(\overline{D}_1) \cap L^\infty(D_1)$. Define $h = \widetilde{u} - u$ on D_1 and extend the function h by 0 outside the set D_1. From what we have already proved, it follows that $h \in C^2(D_1) \cap L^\infty(D_1) \cap C(\overline{D}_1)$. Hence, there is $x_0 \in D_1$ such that $x_0 = \arg\max_{x \in D_1} h(x)$. Notice that $\phi(-\Delta)h(x) = 0$, $x \in D_1$. Hence, for x_0 it holds that

$$0 = \phi(-\Delta)h(x_0) = \text{P.V.} \int_{R^d} \big(h(x_0) - h(y)\big) j(|x_0 - y|)dy,$$

i.e. $h \le 0$ since j is strictly positive. Similarly we get $g \ge 0$. This means that $\widetilde{u} = u$ in D_1, i.e. (u,λ) satisfies the mean-value property (1.39) for every Lipschitz set $D_1 \subset\subset D$.

Now we prove that u satisfies (1.39) for every $U \subset\subset D$. In essence, this follows by using the strong Markov property. Take $x \in U \subset\subset D$ and find a Lipschitz set V such that $U \subset\subset V \subset\subset D$. Since V is a Lipschitz set, it holds that

$$u(x) = \int_{D\setminus V} u(y)P_V(x,y)dy + \int_{D^c} P_V(x,y)\lambda(dy), \quad x \in V. \tag{1.52}$$

Now implement (1.24) for sets U and V in (1.52) to get

$$\begin{aligned} u(x) &= \int_{D\setminus V} u(y)P_U(x,y)dy + \int_{V\setminus U}\int_{D\setminus V} u(y)P_V(w,y)dy\omega_U^x(dw) \\ &\qquad + \int_{D^c} P_U(x,y)\lambda(dy) + \int_{V\setminus U}\int_{D^c} P_V(w,y)\lambda(dy)\omega_U^x(dw) \\ &= \int_{D\setminus V} u(y)P_U(x,y)dy + \int_{V\setminus U} u(w)\omega_U^x(dw) + \int_{D^c} P_U(x,y)\lambda(dy) \\ &= \int_{D\setminus U} u(y)\omega_U^x(dy) + \int_{D^c} P_U(x,y)\lambda(dy), \end{aligned} \tag{1.53}$$

i.e. (u,λ) satisfies the mean-value property (1.39).

Take now general (u,λ) such that $\widetilde{\phi(-\Delta)}(u,\lambda) = 0$. We follow the proof of [43, Lemma 3.3]. Take a Lipschitz set $\Omega \subset\subset D$, define $\rho = (1 \wedge \mathrm{dist}(\Omega, D^c))/2$ and let $V = \Omega + B_\rho := \{x \in \mathbb{R}^d : x = v + b, v \in V, b \in B_\rho\}$. For $0 < \varepsilon < \rho/4$ consider a standard mollifier ϕ_ε. The translation invariance of $\phi(-\Delta)$ implies $\widetilde{\phi(-\Delta)}\big(\phi_\varepsilon * (u,\lambda)\big) = \phi_\varepsilon * \widetilde{\phi(-\Delta)}\big(u,\lambda\big)$ in $V_\varepsilon := \{x \in D : \mathrm{dist}(x, V^c) > \varepsilon\}$. By the first part of the proof, since $\phi_\varepsilon * u$ is smooth, we have

$$\phi_\varepsilon * u(x) = P_\Omega\left(\phi_\varepsilon * (u,\lambda)\right)(x), \quad x \in \Omega. \tag{1.54}$$

Also, since $u \in L^1_{loc}(D)$ we have, up to the subsequence,

$$\lim_{\varepsilon\to 0} \phi_\varepsilon * u(x) = u(x), \quad \text{a.e. in } D. \tag{1.55}$$

We proceed with the proof in the same way as in [43] by showing that

$$\lim_{\varepsilon\to 0} P_\Omega\left(\phi_\varepsilon * (u,\lambda)\cdot \mathbf{1}_{V_{\rho/2}}\right)(x) = P_\Omega\left((u,\lambda)\cdot \mathbf{1}_{V_{\rho/2}}\right)(x). \tag{1.56}$$

It is worth noting that in $V_{\rho/2}$ we have $\phi_\varepsilon * (u,\lambda) = \phi_\varepsilon * u$.

To prove the relation

$$\lim_{\varepsilon\to 0} P_\Omega\left(\phi_\varepsilon * (u,\lambda)\cdot \mathbf{1}_{V^c_{\rho/2}}\right)(x) = P_\Omega\left((u,\lambda)\cdot \mathbf{1}_{V^c_{\rho/2}}\right)(x), \tag{1.57}$$

we first split the integral $P_\Omega\left(\phi_\varepsilon*(u,\lambda);V^c_{\rho/2}\right)$ by the part which uses u and by the part which uses λ as the integrator, and then change the order of the integration:

$$\begin{aligned}P_\Omega\left(\phi_\varepsilon*(u,\lambda);V^c_{\rho/2}\right)(x) &= \int_{V^c_{\rho/2}}\int_{D\cap B(y,\varepsilon)}\phi_\varepsilon(y-z)u(z)dzP_\Omega(x,y)dy\\ &\qquad+\int_{V^c_{\rho/2}}\int_{D^c\cap B(y,\varepsilon)}\phi_\varepsilon(y-z)\lambda(dz)P_\Omega(x,y)dy\\ &= \int_{V^c_{\rho/2}}\int_{D\cap B(y,\varepsilon)}\phi_\varepsilon(y-z)u(z)dzP_\Omega(x,y)dy\\ &\qquad+\int_{D^c}\int_{B(y,\varepsilon)}\phi_\varepsilon(y-z)P_\Omega(x,y)dy\,\lambda(dz).\end{aligned}$$

By using (1.55) for the first integral where the convergence also holds in $\mathscr{L}^1$, see [43, Lemma 2.9], and by using $\phi_\varepsilon*P_\Omega\to P_\Omega$ in D^c and in $\mathscr{L}^1$ for the second integral, as well as the integrability condition (1.41) for (u,λ), we get (1.57).

Thus, we have obtained that for every Lipschitz set $\Omega\subset\subset D$ and a.e. $x\in\Omega$ we have

$$u(x)=\lim_{\varepsilon\to0}\phi_\varepsilon*u(x)=\lim_{\varepsilon\to0}P_\Omega\left(\phi_\varepsilon*(u,\lambda)\right)(x)=P_\Omega\left((u,\lambda)\right)(x). \tag{1.58}$$

Now we define $\widetilde{u}$ which will be a modification of u such that $(\widetilde{u},\lambda)$ is $\phi(-\Delta)$-harmonic in D. For $x\in D$ choose some Lipschitz $U\subset\subset D$ such that $x\in U$ and define $\tilde{u}(x)=P_U\big((u,\lambda)\big)(x)$. Let us show that $\tilde{u}$ is well defined. Suppose that we have Lipschitz sets $U_1\subset\subset D$ and $U_2\subset\subset D$ such that $x\in U_1\cap U_2$ and $P_{U_1}\big((u,\lambda)\big)(x)>P_{U_2}\big((u,\lambda)\big)(x)$. By Corollary 1.3.13 $P_{U_j}\big((u,\lambda)\big)$ is continuous in U_j, $j\in\{1,2\}$, so there is $\varepsilon>0$ such that for every $y\in B(x,\varepsilon)\subset U_1\cap U_2$ we have $P_{U_1}\big((u,\lambda)\big)(y)>P_{U_2}\big((u,\lambda)\big)(y)+\varepsilon$. But $u=P_{U_1}\big((u,\lambda)\big)=P_{U_2}\big((u,\lambda)\big)$ a.e. in $U_1\cap U_2$ by (1.58) so we have a contradiction. Hence, $\tilde{u}$ is well defined.

Recall that since D is an open set, it is a countable union of balls. Also, every ball is a Lipschitz set so it is obvious from the construction of $\tilde{u}$ that $u=\tilde{u}$ a.e. in D.

Now we prove that $\tilde{u}$ is harmonic in D. Note that since $u=\tilde{u}$ a.e. in D, we have for all Lipschitz sets $V\subset\subset D$ and all $x\in V$

$$\begin{aligned}\tilde{u}(x)&=\int_{D\setminus V}u(y)P_V(x,y)dy+\int_{D^c}P_V(x,y)\lambda(dy)\\ &=\int_{D\setminus V}\widetilde{u}(y)P_V(x,y)dy+\int_{D^c}P_V(x,y)\lambda(dy)=P_V\big((\widetilde{u},\lambda)\big)(x).\end{aligned}$$

In other words, $(\widetilde{u},\lambda)$ satisfies the mean-value property (1.39) for every Lipschitz set $V \subset\subset D$. By repeating the calculation of (1.53) we have that $(\widetilde{u},\lambda)$ is $\phi(-\Delta)$-harmonic in D. ■

1.4. ACCESSIBLE POINTS AND MARTIN KERNEL

In this section we give a summary of results concerning the Martin boundary. All of the results are already known but some are not plainly stated. Our goal is to state and prove results that are important in this chapter for the reader's convenience.

In the case where only **(WSC)** holds many results concerning the Martin kernel can be proved only for bounded sets so the additional assumptions **(GWSC)** and **(E)** will be occasionally assumed to get results for unbounded sets.

For $D \subset \mathbb{R}^d$ let us denote

$$D^* := \begin{cases} \overline{D}, & \text{if } D \text{ is bounded,} \\ \overline{D} \cup \{\infty\}, & \text{if } D \text{ is unbounded,} \end{cases} \qquad \partial^* D := \begin{cases} \partial D, & \text{if } D \text{ is bounded,} \\ \partial D \cup \{\infty\}, & \text{if } D \text{ is unbounded,} \end{cases}$$

where ∞ is an additional point in Alexandroff compactification and it is called *the point at infinity*.

Definition 1.4.1. Let D be an open set. A point $y \in \partial D$ is called accessible from D if

$$P_D(x_0,y) = \int_D G_D(x_0,z)j(|z-y|)dz = \infty, \qquad \text{for some } x_0 \in D.$$

The point at infinity is accessible from D if

$$\mathbb{E}_{x_0}\tau_D = \int_D G_D(x_0,y)dy = \infty, \qquad \text{for some } x_0 \in D.$$

If $y \in \partial^* D$ is not accessible it is called inaccessible. The set of all accessible points is denoted by $\partial_M D$.

Remark 1.4.2. In [55, Proposition 4.1 & Remark 4.2] the following claims were proved.

(a) Let $y \in \partial D$. If $P_D(x_0,y) < \infty$ for some $x_0 \in D$, then $P_D(x,y) < \infty$ for all $x \in D$.

(b) Assume **(GWSC)**. If $\mathbb{E}_{x_0}\tau_D < \infty$ for some $x_0 \in D$, then $\mathbb{E}_x\tau_D < \infty$ for all $x \in D$.

Note that we could also get the claim (a) directly from Theorem 1.3.5(a). Also, from the definition of accessible points it is clear that if λ is a signed measure on D^c such that $P_D|\lambda| < \infty$, then λ is concentrated on $\mathbb{R}^d \setminus (D \cup \partial_M D)$, i.e. λ can have no mass on the set of accessible points.

For an open $D \subset \mathbb{R}^d$ we fix an arbitrary point $x_0 \in D$ and define the Martin kernel on D by

$$\begin{aligned} M_D(x,y) &:= \frac{G_D(x,y)}{G_D(x_0,y)}, \quad x,y \in D,\, y \neq x_0, \\ M_D(x,z_0) &:= \lim_{D \ni v \to z_0} \frac{G_D(x,v)}{G_D(x_0,v)}, \quad x \in D,\, z_0 \in \partial^* D. \end{aligned} \tag{1.59}$$

In [54] and [55] many important and useful results about the Martin kernel of more general processes than the subordinate Brownian motion were proved. E.g. it was proved that $M_D(x,z_0)$ exists, is finite and strictly positive for every $z_0 \in \partial^* D$ (with the additional assumptions **(GWSC)** and **(E)** if z_0 is the point at infinity). We summarize some of those results in the following theorem.

Theorem 1.4.3. Let D be an open set, and $z_0 \in \partial^* D$.

(a) Let $z_0 \in \partial_M D$ and if $z_0 = \infty$, assume **(GWSC)**. The function $x \mapsto M_D(x,z_0)$ is $\phi(-\Delta)$-harmonic in D with zero outer charge and for every open $U \subset\subset D$ it holds

$$M_D(x,z_0) = \int_{D \setminus U} M_D(y,z_0)\, \omega_U^x(dy), \quad x \in U.$$

(b) Let $z_0 \notin \partial_M D$ (for $z_0 = \infty$ assume **(GWSC)** and **(E)**). The function $x \mapsto M_D(x,z_0)$ is not $\phi(-\Delta)$-harmonic in D with zero outer charge and for every open $U \subset\subset D$ it holds

$$M_D(x,z_0) > \int_{D \setminus U} M_D(y,z_0)\, \omega_U^x(dy), \quad x \in U.$$

Proof. First notice that by adding the assumptions **(GWSC)** and **(E)** where needed all assumptions of claims from [54] and [55] are satisfied, see [54, Section 4.1] and [55, Section 4.1]. Also, recall that Lemma 1.1.1 is exactly the assumption **E1** of [54].

Suppose that $z_0 \notin \partial_M D$. From [54, Theorem 3.1] we have that

$$M_D(x,z_0) = \begin{cases} \frac{P_D(x,z_0)}{P_D(x_0,z_0)}, & \text{if } z_0 \in \partial D, \\ \frac{\mathbb{E}_x \tau_D}{\mathbb{E}_{x_0} \tau_D}, & \text{if } z_0 = \infty. \end{cases}$$

Hence, for finite $z_0 \notin \partial_M D$, $x \mapsto M_D(x,z_0)$ is $\phi(-\Delta)$-harmonic with outer charge $\delta_{z_0}/P_D(x_0,z_0)$ but it is not $\phi(-\Delta)$-harmonic with zero outer charge, see Corollary 1.3.13. Also, for every

$x \in U \subset\subset D$ we have by the mean-value property of $\phi(-\Delta)$-harmonic functions

$$\begin{aligned}M_D(x,z_0) &= \int_{D\setminus U} M_D(y,z_0)\omega_U^x(dy) + \frac{P_U(x,z_0)}{P_D(x_0,z_0)}\\ &> \int_{D\setminus U} M_D(y,z_0)\omega_U^x(dy), \quad x \in U.\end{aligned}$$

If $z_0 = \infty$, then $M_D(x,\infty)$ is not $\phi(-\Delta)$-harmonic with zero outer charge because for $x \in U \subset\subset D$ we have

$$\begin{aligned}\int_{D\setminus U} M_D(y,\infty)\omega_U^x(dy) &= \frac{1}{\mathbb{E}_{x_0}\tau_D}\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_U}}\tau_D\right] = \frac{1}{\mathbb{E}_{x_0}\tau_D}\mathbb{E}_x\left[\int_{\tau_U}^{\tau_D}\mathbf{1}dt\right]\\ &< \frac{\mathbb{E}_x\tau_D}{\mathbb{E}_{x_0}\tau_D} = M_D(x,\infty),\end{aligned}$$

where the strict inequality comes from the fact that for $x \in U$ there is $\varepsilon > 0$ such that $B(x,\varepsilon) \subset U$ and $\mathbb{E}_x\tau_U \geq \mathbb{E}_x\tau_{B(x,\varepsilon)} > 0$ by [51, Lemma 4.3].

Suppose now that $z_0 \in \partial_M D$. Then we have that $x \mapsto M_D(x,z_0)$ is $\phi(-\Delta)$-harmonic with zero outer charge. For the finite point z_0 this follows from [54, Theorem 1.2(b)] (see the proof), or [55, Theorem 1.1], and for the point at infinity we apply [54, Theorem 1.4(b)], or [55, Theorem 1.3]. In either case by the mean-value property of $\phi(-\Delta)$-harmonic functions we get for every $U \subset\subset D$ and all $x \in U$

$$M_D(x,z_0) = \int_{D\setminus U} M_D(y,z_0)\omega_U^x(dy).$$

■

Remark 1.4.4. It will be very useful to note that in [55] two specific mean-value formulae were proved. If $z_0 \in \partial_M D \setminus \{\infty\}$, then for every $r < \frac{1}{4}|z_0 - x_0|$ and $U_r := D \setminus \overline{B(z_0,r)}$ it holds that

$$M_D(x,z_0) = \int_{U_r^c} M_D(y,z_0)\omega_{U_r}^x(dy), \quad x \in U_r, \tag{1.60}$$

see [55, (3.14)].

Also, if $z_0 = \infty \in \partial_M D$ and if we additionally assume **(GWSC)**, then for every $R > 4|x_0|$ and $U_R := D \cap B(0,R)$ it holds that

$$M_D(x,\infty) = \int_{U_R^c} M_D(y,\infty)\omega_{U_R}^x(dy), \quad x \in U_R, \tag{1.61}$$

see [55, (3.4)].

In fact, from (1.60) it follows by using the strong Markov property that (1.60) is true for every $U \subset D$ open such that $z_0 \notin \overline{U}$. By similar reasoning (1.61) holds for every $U \subset D$ open and bounded such that $U_R \subset U$ for some $R > 4|x_0|$.

Definition 1.4.5. Let $D \subset \mathbb{R}^d$ be an open set and μ a finite signed measure on $\partial^* D$ concentrated on $\partial_M D$. The Martin integral of μ is defined by

$$M_D\mu(x) := \int_{\partial_M D} M_D(x,y)\mu(dy), \quad x \in \mathbb{R}^d.$$

Remark 1.4.6. Let μ be a finite measure concentrated on $\partial_M D$. From $M_D(x_0, z) = 1$, $z \in \partial^* D$, we see that $M_D\mu(x_0) = \mu(\partial_M D)$. It will follow from Corollary 1.5.8 that $M_D\mu$ is finite at some point (or all points) if and only if μ is finite. Also, due to harmonicity of $x \mapsto M_D(x, z_0)$ for $z_0 \in \partial_M D$, it is easy to check that $M_D\mu$ is $\phi(-\Delta)$-harmonic in D with outer charge zero. That is the reason why we look, regarding the Martin integral, at finite measures concentrated on $\partial_M D$ in what follows.

1.5. Representation of $\phi(-\Delta)$-harmonic functions

Let D be an open set, $u : D \to [-\infty, \infty]$, and let $U \subset\subset D$ be a set with Lipschitz boundary such that $x_0 \in U$, where x_0 is the fixed point from the definition of the Martin kernel. We define the signed measure $\eta_U u$ by

$$\eta_U u(A) = \int_A G_U(x_0, z) \left(\int_{D \setminus U} j(|z-y|)u(y)dy \right) dz, \quad A \in \mathscr{B}(\mathbb{R}^d).$$

Definition 1.5.1. If $(\eta_U |u|(D))_U$ is bounded as $U \uparrow D$ and $(\eta_U u)_U$ weakly converges to a signed measure μ as $U \uparrow D$, then we denote $W_D u = \mu$, i.e. $W_D u := \lim\limits_{U \uparrow D} \eta_U u$.

The boundary trace operator W_D was used in [20] as the boundary condition in the Dirichlet problem for the fractional Laplacian and it was used as a tool to get the representation of non-negative α-harmonic functions in [21]. As one can see, the definition of W_D is rather delicate. Also, for a bounded function f we have $W_D f = 0$ since

$$\eta_U |f|(D) = \int_{U^c} P_U(x_0, z)|f|(z)dz \lesssim \int_{U^c} P_U(x_0, z)dz \downarrow 0, \quad U \uparrow D.$$

However, W_D can be applied to many more functions, e.g. we will show that $W_D(M_D\mu) = \mu$ and $W_D(G_D f) = W_D(P_D\lambda) = 0$, see also Proposition 2.4.8. In what follows, we prove that some important properties of W_D are also true in the case of subordinate Brownian motions and at the end of the chapter we will use the operator to get the representation of non-negative $\phi(-\Delta)$-harmonic functions.

Also, the operator W_D will appear as the boundary condition in the Dirichlet problem for the non-local operator $\phi(-\Delta)$ in Chapter 2, and more on the boundary trace operators in the setting of the operator $\phi(-\Delta)$ will be said therein.

Lemma 1.5.2. *$W_D u$ is concentrated on $\partial^* D$.*

Proof. Let $A \subset\subset D$. Then there is a Lipschitz set $U_A \subset\subset D$ such that $x_0 \in U_A$ and $A \subset\subset U_A$. Now we will show that $G_U(x_0, y) \asymp G_D(x_0, y)$, for all $y \in A$ and for all Lipschitz U such that $U_A \subset\subset U \subset\subset D$.

Let $\varepsilon > 0$ be such that $\overline{B(x_0, 2\varepsilon)} \subset U_A$. For $y \in B(x_0, \varepsilon)$ and all Lipschitz U such that $U_A \subset\subset U \subset\subset D$ we have

$$G_{B(x_0,2\varepsilon)}(x_0, y) \le G_U(x_0, y) \le G_{\mathbb{R}^d}(x_0, y) \le C\, G_{B(x_0,2\varepsilon)}(x_0, y), \tag{1.62}$$

where $C > 1$ is independent of U. Indeed, by (1.18) and [44, Theorem 1.3] we have for $y \in B(x_0, \varepsilon)$

$$\begin{aligned} G_{\mathbb{R}^d}(x_0, y) &\le c_1 \frac{1}{|x_0 - y|^d \phi(|x_0 - y|^{-2})}, \\ G_{B(x_0,2\varepsilon)}(x_0, y) &\ge \frac{1}{c_2} \frac{j(|x_0 - y|)}{(K(|x_0 - y|) + L(|x_0 - y|))^2}. \end{aligned} \tag{1.63}$$

where $K(r) = \int_{B(0,r)} \frac{|z|^2}{r^2} j(|z|)dz$ and $L(r) = \int_{B(0,r)^c} j(|z|)dz$. Define $h(r) = K(r) + L(r) = \int_{\mathbb{R}^d} \left(1 \wedge \frac{|z|^2}{r^2}\right) j(|z|)dz$. By [19, Eq. (6) and Lemma 1] we have that $h(r) \asymp \phi(\frac{1}{r^2})$ so by using [51, Theorem 2.3] for all small enough $q > 0$ we have that

$$K(q) \le K(q) + L(q) = h(q) \le c_3 \phi(\frac{1}{q^2}) \le c_4 j(q) q^d.$$

By using this inequality with the inequalities (1.63) we get (1.62).

For $y \in B(x_0, \varepsilon)^c \cap A$ notice that $0 < c_5 \le G_{U_A}(x_0, y) \le G_U(x_0, y) \le G_D(x_0, y) \le c_6 < \infty$ because Green functions are continuous and strictly positive on $B(x_0, r)^c \cap A$ since $A \subset\subset U_A \subset\subset D$. Thus, $G_U(x_0, y) \asymp G_D(x_0, y)$, for all $y \in A$ and for all Lipschitz U such that $U_A \subset\subset U \subset\subset D$.

Hence, for all such U we have

$$\eta_U |u|(A) \asymp \int_A G_D(x_0, y) \underbrace{\int_{D \setminus U} j(|z - y|)|u(z)|dz dy}_{\downarrow 0 \text{ as } U \uparrow D} \xrightarrow{U \uparrow D} 0$$

by the dominated convergence theorem. In other words, we proved that $W_D u$ does not have any mass in D. ■

Remark 1.5.3.

(a) If we take a closer look at the proof of the previous lemma, we have actually proved that if $(\eta_U |u|(D))_U$ is bounded as $U \uparrow D$, then for every $A \subset\subset D$ we have

$$\lim_{U \uparrow D} \eta_U |u|(A) = 0.$$

(b) The measures $(\eta_U u)_U$ depend on $x_0 \in D$ but we can prove quite simply that for any other $x \in D$, the measures

$$\eta_U^x |u|(dy) := G_U(x,y) \left(\int_{D\setminus U} j(|z-y|)|u(z)|dz \right) dy$$

are also bounded as $U \uparrow D$ if $(\eta_U |u|)_U$ are. Indeed, let $M := \limsup_{U\uparrow D} \eta_U |u|(D)$. Notice that by Fubini's theorem

$$\begin{aligned}\eta_U |u|(D) &= \int_D G_U(x_0,z) \left(\int_{D\setminus U} j(|z-y|)|u(y)|dy \right) dz \\ &= \int_{D\setminus U} P_U(x_0,y)|u(y)|dy.\end{aligned}$$

Find $R \in (0,1)$ such that $\delta_D(x_0) > 2R$ and let $(U_n)_n$ be some increasing sequence of Lipschitz sets such that $x_0 \in U_1$, $\delta_{U_1}(x_0) > R$, and such that for all $n \in \mathbb{N}$ it holds $U_n \subset\subset D$ and $\cup_n U_n = D$. Also, fix some $\tilde{y} \in \overline{D}^c$. Theorem 1.3.5 yields that there is $C > 0$ such that for all $n \in \mathbb{N}$, all $x \in B(x_0, R/2)$, and all $y \in U_n^c$

$$P_{U_n}(x,y) \le C \frac{P_{U_n}(x,\tilde{y})}{P_{U_n}(x_0,\tilde{y})} P_{U_n}(x_0,y).$$

Notice that

$$\frac{P_{U_n}(x,\tilde{y})}{P_{U_n}(x_0,\tilde{y})} \le \frac{P_D(x,\tilde{y})}{P_{U_1}(x_0,\tilde{y})} \le \frac{\max_{z\in B(x_0,R/2)} P_D(z,\tilde{y})}{P_{B(x_0,R/2)}(x_0,\tilde{y})} \le c_1 < \infty,$$

where $c_1 > 0$ depends on x_0, R and $\tilde{y}$ but it is independent of $n \in \mathbb{N}$ and $x \in B(x_0, R/2)$. Finiteness of c_1 is due to the continuity of the Poisson kernel. Thus, there is $c_2 > 0$ such that for all $n \in \mathbb{N}$, all $x \in B(x_0, R/2)$ and all $y \in U_n^c$ we have $P_{U_n}(x,y) \le c_2 P_{U_n}(x_0,y)$. Hence

$$\begin{aligned}\eta_{U_n}^x |u|(D) &= \int_{D\setminus U_n} P_{U_n}(x,y)|u(y)|dy \\ &\le c_2 \int_{D\setminus U_n} P_{U_n}(x_0,y)|u(y)|dy \le c_2 \cdot M,\end{aligned}$$

i.e. $(\eta_U^x |u|(D))_U$ is bounded as $U \uparrow D$, for all $x \in D$.

Proposition 1.5.4. Let D be an open set, $f : D \to [-\infty, \infty]$ such that $G_D|f|(x) < \infty$ for some $x \in D$, and λ a σ-finite signed measure on D^c such that (1.29) holds. Then

$$W_D(G_D f) = W_D(P_D \lambda) = 0.$$

Proof. The proof is exactly the same as in the isotropic α-stable case, see [20, Lemma 1.17]. ■

We now focus on proving the mentioned property $W_D(M_D\mu) = \mu$. We use an adaptation of the technique used in [21] where the property was shown for the isotropic α-stable process. In the next few results we have twofold statements - for sets near the origin, and for sets away of the origin. In the isotropic α-stable case the Kelvin transform allowed the authors to deal only with sets near the origin but in our setting this is not the case.

Let us recall the definition of the relative oscillation of a positive function f on a nonempty set D

$$\mathrm{RO}_D f := \frac{\sup_{x\in D} f(x)}{\inf_{x\in D} f(x)}.$$

If $D = \emptyset$ we put $\mathrm{RO}_D f = 1$.

The first lemma is the one that generalizes [21, Lemma 8].

Lemma 1.5.5.

(a) For every $R \in (0,1)$ and $\eta > 0$ there exists $\delta > 0$ such that for all open $D \subset B_R$ and all σ-finite measures λ_1, λ_2 on B_R^c satisfying (1.29) we have

$$\mathrm{RO}_{D\cap B_\delta} \frac{P_D\lambda_1}{P_D\lambda_2} \le 1+\eta. \tag{1.64}$$

(b) Assume **(GWSC)** and **(E)**. For every $R \ge 1$ and $\eta > 0$ there exists $\delta > 0$ such that for all open $D \subset \overline{B}_R^c$ and all σ-finite measures λ_1, λ_2 on $\overline{B}_R$ satisfying (1.29) we have

$$\mathrm{RO}_{D\cap \overline{B}^c_{1/\delta}} \frac{P_D\lambda_1}{P_D\lambda_2} \le 1+\eta. \tag{1.65}$$

Before we bring the proof let us emphasize the results of the previous lemma. In both parts of the lemma δ is chosen independently of the set D, and the measures λ_1 and λ_2. In similar results on the relative oscillation of harmonic functions, e.g. [55, Proposition 2.5, Proposition 2.11], δ is dependent on the set D, see also the proofs of [54, Theorem 2.4, Theorem 2.8]. This subtle but big difference will be used as a crucial and indispensable step in proving $W_D(M_D\mu) = \mu$, see (1.77).

Moreover, the previous lemma yields that the Martin kernel $M_D(x,z)$ is well defined and strictly positive for $x \in D$ and $z \in \partial^* D$. Indeed, recall that $M_D(x,z) = \lim_{y\to z} \frac{G_D(x,y)}{G_D(x_0,y)}$, where if $z = \infty$ we look at the limit as $|y| \to \infty$. Since the process X is translation invariant, we can assume that for the finite point z it holds $z = 0$. Further, notice that from (1.21) we have for $\rho > 0$

$$G_D(\tilde{x}, y) = P_{D\cap B_\rho}[G_D(\tilde{x}, v)dv](y), \quad \tilde{x} \in D \setminus \overline{B}_\rho, y \in D \cap B_\rho,$$

and

$$G_D(\tilde{x}, y) = P_{D\cap \overline{B}_\rho^c}[G_D(\tilde{x}, v)dv](y), \quad \tilde{x} \in D \cap B_\rho, y \in B \setminus \overline{B}_\rho.$$

Now the claim follows from (1.64) and (1.65). However, for this result the uniformity of δ was not important.

Proof of Lemma 1.5.5. We prove only part (b). The proof of part (a) is almost identical to the proof of the [21, Lemma 8]. The only difference is that instead of the unit ball B we look at the ball B_R and instead of [21, Eq. (48)] we use Lemma 1.1.1. The proof of part (b) follows the same idea and we present the proof to emphasize the differences. To establish a connection between our proof and the proof of [21, Lemma 8] we will keep a similar notation.

For an open set D and $R, p, q > 0$ denote by

$$\begin{aligned} D_p &= D \cap \overline{B}_p^c, \\ D_p^R &= (D \setminus D_p) \cup \overline{B}_R, \\ D_{p,q} &= D_q \setminus D_p. \end{aligned}$$

For a measure μ let

$$\begin{aligned} \Lambda_{0,p}(\mu) &= \int_{\overline{B}_p} \mu(dy), \\ \Lambda_{0,p,q}(\mu) &= \int_{D_{p,q}} \mu(dy). \end{aligned}$$

Fix $R \geq 1$, $D \subset \overline{B}_R^c$, and σ-finite measures λ_1 and λ_2 on $\overline{B}_R$ satisfying (1.29). We will see at the end of the proof that δ will not depend on D, λ_1 or λ_2, so this is not a loss of generality. Let c denote $C(\phi) > 1$ of Lemma 1.3.4(b) and notice that Theorem 1.3.5(b) holds with the constant $C = c^4$. Thus, (1.65) holds for $\delta = \frac{1}{2}$ with $1 + \eta$ replaced by c^4.

We denote

$$f_i = P_D\lambda_i, \qquad f_i^{pR,qR} = P_{D_{pR}}[\mathbf{1}_{D_{pR,qR}}P_D^*\lambda_i],\ \widetilde{f}_i^{pR,qR} = P_{D_{pR}}[\mathbf{1}_{D_{qR}^R}P_D^*\lambda_i],$$
$$f_i^* = P_D^*\lambda_i, \qquad f_i^{pR,qR*} = P_{D_{pR}}^*[\mathbf{1}_{D_{pR,qR}}P_D^*\lambda_i],\ \widetilde{f}_i^{pR,qR*} = P_{D_{pR}}^*[\mathbf{1}_{D_{qR}^R}P_D^*\lambda_i].$$

Recall that $P_D\lambda$ satisfies the mean-value formula for every $U \subset D$ by Corollary 1.3.13. Hence, by using (1.23) we have $f_i = f_i^{pR,qR} + \widetilde{f}_i^{pR,qR}$ and $f_i^* = f_i^{pR,qR*} + \widetilde{f}_i^{pR,qR*}$, for $i = 1,2$. For $\delta \in (0, \frac{1}{2}]$ we denote $m_{R/\delta} = \inf_{D_{R/\delta}}(f_1/f_2)$ and $M_{R/\delta} = \sup_{D_{R/\delta}}(f_1/f_2)$. As we have already noted we have $M_{R/\delta} \le c^4 m_{R/\delta}$.

Let $\varepsilon > 0$ such that $1+\varepsilon < c$ and let $q \ge 2$. Assumption **(E)** yields that there is $p = p(q,\varepsilon,R) > 2q$ such that for $z \in D_{pR/2}$ and $y \in \overline{B}_{qR}$ we have

$$\frac{1}{1+\varepsilon} j(|z|) \le j(|z-y|) \le (1+\varepsilon) j(|z|). \tag{1.66}$$

Thus, for $x \in D_{pR/2}$ we have

$$\begin{aligned}\widetilde{f}_i^{pR/2,qR}(x) &= \int_{D_{qR}^R}\int_{D_{pR/2}} G_{D_{pR/2}}(x,z) j(|z-y|) dz f_i^*(dy)\\ &\le (1+\varepsilon)\Lambda_{0,qR}(f_i^*) P_{D_{pR/2}}(x,0),\end{aligned}$$

and similarly

$$\widetilde{f}_i^{pR/2,qR}(x) \ge (1+\varepsilon)^{-1}\Lambda_{0,qR}(f_i^*) P_{D_{pR/2}}(x,0).$$

Let us examine consequences of the following assumption:

$$\Lambda_{0,pR,qR}(f_i^*) \le \varepsilon \Lambda_{0,qR}(f_i^*), \quad i = 1,2. \tag{1.67}$$

If (1.67) is true, then by using Lemma 1.3.4(b) we have for $x \in D_{pR}$

$$\begin{aligned}f_i^{pR/2,qR}(x) &\le c P_{D_{pR/2}}(x,0)\Lambda_{0,pR}(f_i^{pR/2,qR*}) \le c P_{D_{pR/2}}(x,0)\Lambda_{0,pR,qR}(f_i^*)\\ &\le c\varepsilon P_{D_{pR/2}}(x,0)\Lambda_{0,qR}(f_i^*).\end{aligned}$$

Recall that $f_i = f_i^{pR/2,qR} + \widetilde{f}_i^{pR/2,qR}$ so if (1.67) holds, we have for $x \in D_{pR}$

$$\frac{(1+\varepsilon)^{-1}\Lambda_{0,qR}(f_1^*)}{(c\varepsilon+1+\varepsilon)\Lambda_{0,qR}(f_2^*)} \le \frac{f_1(x)}{f_2(x)} \le \frac{(c\varepsilon+1+\varepsilon)\Lambda_{0,qR}(f_1^*)}{(1+\varepsilon)^{-1}\Lambda_{0,qR}(f_2^*)} \tag{1.68}$$

and finally

$$\mathrm{RO}_{D_{pR}}\frac{f_1}{f_2} \le (c\varepsilon+1+\varepsilon)^2(1+\varepsilon)^2. \tag{1.69}$$

We are satisfied with (1.69) for now.

Let $2 \le \bar{q} < \bar{p}/4 < \infty$, $g = f_1^{\bar{p}R/2,\bar{q}R} - m_{\bar{q}R} f_2^{\bar{p}R/2,\bar{q}R}$, and $h = M_{\bar{q}R} f_2^{\bar{p}R/2,\bar{q}R} - f_1^{\bar{p}R/2,\bar{q}R}$. Note that on $D_{\bar{p}R/2}$ the functions g and h are the Poisson integrals of non-negative measures. If $D_{\bar{p}R} \neq \emptyset$, then by (1.38)

$$\begin{aligned}
\sup_{D_{\bar{p}R}} \frac{f_1^{\bar{p}R/2,\bar{q}R}}{f_2^{\bar{p}R/2,\bar{q}R}} - m_{\bar{q}R} = \sup_{D_{\bar{p}R}} \frac{g}{f_2^{\bar{p}R/2,\bar{q}R}} &\le c^4 \inf_{D_{\bar{p}R}} \frac{g}{f_2^{\bar{p}R/2,\bar{q}R}} \\
&= c^4 \left(\inf_{D_{\bar{p}R}} \frac{f_1^{\bar{p}R/2,\bar{q}R}}{f_2^{\bar{p}R/2,\bar{q}R}} - m_{\bar{q}R} \right),
\end{aligned}$$

and similarly

$$M_{\bar{q}R} - \inf_{D_{\bar{p}R}} \frac{f_1^{\bar{p}R/2,\bar{q}R}}{f_2^{\bar{p}R/2,\bar{q}R}} \le c^4 \left(M_{\bar{q}R} - \sup_{D_{\bar{p}R}} \frac{f_1^{\bar{p}R/2,\bar{q}R}}{f_2^{\bar{p}R/2,\bar{q}R}} \right).$$

By adding these two inequalities we obtain

$$(c^4+1)\left(\sup_{D_{\bar{p}R}} \frac{f_1^{\bar{p}R/2,\bar{q}R}}{f_2^{\bar{p}R/2,\bar{q}R}} - \inf_{D_{\bar{p}R}} \frac{f_1^{\bar{p}R/2,\bar{q}R}}{f_2^{\bar{p}R/2,\bar{q}R}} \right) \le (c^4-1)(M_{\bar{q}R} - m_{\bar{q}R}). \tag{1.70}$$

Let us examine consequences of the following assumption:

$$\Lambda_{0,\bar{q}R}(f_i^*) \le \varepsilon \Lambda_{0,\bar{p}R/2,\bar{q}R}(f_i^*), \tag{1.71}$$

for $\bar{p}$ big enough such that $j(|z-y|) \le c\, j(|z|)$ for all $z \in D_{\bar{p}R/2}$ and $y \in \overline{B}_{\bar{q}R}$ (see (1.66)). We have for all $x \in D_{\bar{p}R/2}$ and $y \in \overline{B}_{\bar{q}R}$

$$P_{D_{\bar{p}R/2}}(x,y) = \int_{D_{\bar{p}R/2}} G_{D_{\bar{p}R/2}}(x,z) j(|z-y|)dz \le c P_{D_{\bar{p}R/2}}(x,0),$$

hence

$$\widetilde{f}_i^{\bar{p}R/2,\bar{q}R}(x) = \int_{D_{\bar{q}R}^R} P_{D_{\bar{p}R/2}}(x,y) f_i^*(dy) \le c P_{D_{\bar{p}R/2}}(x,0) \Lambda_{0,\bar{q}R}(f_i^*).$$

From the previous inequality, by using the assumption (1.71) and Lemma 1.3.4(b), we have for $x \in D_{\bar{p}R}$

$$\begin{aligned}
\widetilde{f}_i^{\bar{p}R/2,\bar{q}R}(x) &\le c\,\varepsilon P_{D_{\bar{p}R/2}}(x,0) \Lambda_{0,\bar{p}R/2,\bar{q}R}(f_i^*) \\
&\le c\,\varepsilon P_{D_{\bar{p}R/2}}(x,0) \Lambda_{0,\bar{p}R}(f_i^{\bar{p}R/2,\bar{q}R*}) \le c^2 \varepsilon f_i^{\bar{p}R/2,\bar{q}R}(x).
\end{aligned}$$

Recall $f_i = f_i^{\bar{p}R/2,\bar{q}R} + \widetilde{f}_i^{\bar{p}R/2,\bar{q}R}$ on $D_{\bar{p}R/2}$ so the previous inequality and (1.70) yield

$$(c^4+1)\left(M_{\bar{p}R}/(1+c^2\varepsilon) - m_{\bar{p}R}(1+c^2\varepsilon)\right) \le (c^4-1)(M_{\bar{q}R} - m_{\bar{q}R}).$$

Since $m_{\bar{p}R} \ge m_{\bar{q}R}$, dividing by $m_{\bar{q}R}$ we finally get

$$\mathrm{RO}_{D_{\bar{p}R}} \frac{f_1}{f_2} \le (1+c^2\varepsilon)^2 + (1+c^2\varepsilon)\frac{c^4-1}{c^4+1}\left(\mathrm{RO}_{D_{\bar{q}R}} \frac{f_1}{f_2} - 1\right). \tag{1.72}$$

We now come to the conclusion of our considerations. Let $\eta > 0$. If ε is small enough, then the right hand side of (1.69) is smaller than $1+\eta$ and the right hand side of (1.72) does not exceed $\varphi(\mathrm{RO}_{D_{\bar{q}R}}(f_1/f_2))$, where

$$\varphi(t) = 1 + \frac{\eta}{2} + \frac{c^4}{c^4+1}(t-1), \quad t \ge 1.$$

Let $\varphi^1 = \varphi$, $\varphi^{l+1} = \varphi \circ \varphi^l$, $l \in \mathbb{N}$. Observe that φ is an increasing linear contraction with a fixed point $t = 1 + \eta(c^4+1)/2$. Thus the l-fold compositions $\varphi^l(c^4)$ converge to $1+\eta(c^4+1)/2$ as $l \to \infty$. In what follows let l be such that

$$\varphi^l(c^4) < 1 + \eta(c^4+1).$$

Let k be the smallest integer such that $k-1 > c^2/\varepsilon^2$. We denote $n = lk$. Note that n depends only on η and ϕ. Let $q_0 = 2$, $q_{j+1} = p(q_j, \varepsilon, R)$ for $j = 0, 1, \ldots, n-1$, from (1.66), and $\delta = \frac{1}{q_n}$. Note that δ depends only on η, R and ϕ. If for any $j < n$ the relation (1.67) holds with $q = q_j$ and $p = p(q) = q_{j+1}$, then

$$\mathrm{RO}_{D_{R/\delta}} \frac{f_1}{f_2} \le \mathrm{RO}_{D_{q_{j+1}R}} \frac{f_1}{f_2} \le 1 + \eta,$$

by the definition of ε and (1.69). Otherwise for $j = 0, \ldots, n-1$, we have $\Lambda_{0,q_{j+1}R,q_jR}(f_i^*) > \varepsilon \Lambda_{0,q_jR}(f_i^*)$ for $i = 1$ or $i = 2$. Note that by Lemma 1.3.4(*b*)

$$c^{-1}\frac{f_i(x)}{\Lambda_{0,q_jR}(f_i^*)} \le P_{D_{q_jR/2}}(x,0) \le c\frac{f_{3-i}(x)}{\Lambda_{0,q_jR}(f_{3-i}^*)}, \quad x \in D_{q_{j+1}R,q_jR}.$$

Hence $\Lambda_{0,q_{j+1}R,q_jR}(f_i^*)/\Lambda_{0,q_jR}(f_i^*) \le c^2 \Lambda_{0,q_{j+1}R,q_jR}(f_{3-i}^*)/\Lambda_{0,q_jR}(f_{3-i}^*)$ and so $\Lambda_{0,q_{j+1}R,q_jR}(f_i^*) \ge c^{-2}\varepsilon\Lambda_{0,q_jR}(f_i^*)$ for both $i = 1$ and $i = 2$ (and all $j = 0, \ldots, n-1$). If $0 \le j < l$ and $\bar{p} = q_{(j+1)k}$, $\bar{q} = q_{jk}$, then

$$\Lambda_{0,\bar{p}R/2,\bar{q}R}(f_i^*) \ge \Lambda_{0,q_{(j+1)k-1}R,q_{jk}R}(f_i^*) \ge (k-1)\frac{\varepsilon}{c^2}\Lambda_{0,\bar{q}R}(f_i^*) \ge \varepsilon^{-1}\Lambda_{0,\bar{q}R}(f_i^*),$$

so that (1.71) is satisfied. We conclude that (1.72) holds. Recall that $q_0 = 2$ and $\mathrm{RO}_{D_{2R}}(f_1/f_2) \le c^4$. By the definition of l and the monotonicity of φ

$$\mathrm{RO}_{D_{q_{lk}R}} \frac{f_1}{f_2} \le \varphi\left(\mathrm{RO}_{D_{q_{(l-1)k}R}} \frac{f_1}{f_2}\right) \le \cdots \le \varphi^l\left(\mathrm{RO}_{D_{q_0 R}} \frac{f_1}{f_2}\right) \le 1 + \eta(c^4+1),$$

i.e. $\mathrm{RO}_{D_{R\delta}} \frac{f_1}{f_2} \le 1 + \eta(c^4+1)$. Since $\eta > 0$ was arbitrary and δ is dependant only on η, R and ϕ, the proof is complete. ■

Corollary 1.5.6. Let D be an open set, D_{reg} the set of all regular points for D, $z \in \partial D$, and $0 < r < 1 \le R$.

(a) Let f_1 and f_2 be non-negative functions which are regular harmonic in $D \cap B(z,r)$ and $f_i = 0$ on $(\overline{D}^c \cup D_{reg}) \cap B(z,r)$, $i = 1,2$. Then

$$\lim_{D \ni x \to z} \frac{f_1(x)}{f_2(x)}$$

exists and is finite.

(b) Assume **(GWSC)** and **(E)**. If f_1 and f_2 are non-negative functions which are regular harmonic in $D \cap \overline{B}_R^c$ and $f_i = 0$ on $(\overline{D}^c \cup D_{reg}) \cap \overline{B}_R^c$, $i = 1,2$, then

$$\lim_{D \ni x \to \infty} \frac{f_1(x)}{f_2(x)}$$

exists and is finite.

Moreover, the speed of convergence in the limits above does not depend on the set D.

The previous corollary is an immediate consequence of Lemma 1.5.5, cf. [54, Theorem 2.4, Theorem 2.8] and [55, Corollary 2.6, Corollary 2.12] where the speed of convergence depends on the set D.

Proof of Corollary 1.5.6. For the part (a) it is enough to notice that from the assumptions of the corollary we have for $x \in D \cap B(z,r)$ and both $i = 1,2$

$$f_i(x) = \int_{D^c \cup B(z,r)^c} f_i(y)\,\omega^x_{D \cap B(z,r)}(dy) = \int_{B(z,r)^c} P_{D \cap B(z,r)}(x,y) f_i(y)\,dy. \tag{1.73}$$

Indeed, the boundary part $D \cap \partial B(z,r)$ of $D \cap B(z,r)$ is smooth, $f_i = 0$ on $D_{reg} \cap \overline{B}_R^c$, and the irregular points for D at ∂D are polar, so we can replace $\omega^x_{D \cap B(z,r)}(dy)$ by $P_{D \cap B(z,r)}(x,y)dy$ in (1.73).

The claim now follows from Lemma 1.5.5(a). Part (b) follows similarly. ■

The following results generalize [21, Lemma 12].

Lemma 1.5.7. For every $0<\rho<1$ and $\eta>0$ there is $r>0$ such that for all open D it holds

$$\mathrm{RO}_{y\in\overline{D}\cap B_r} M_D(x,y)\le 1+\eta, \quad \text{if } x,x_0\in D\setminus\overline{B}_\rho, \tag{1.74}$$

and with the additional assumptions **(GWSC)** and **(E)** it holds

$$\mathrm{RO}_{y\in D^*\setminus\overline{B}_{1/r}} M_D(x,y)\le 1+\eta, \quad \text{if } x,x_0\in D\cap B_{1/\rho}. \tag{1.75}$$

Proof. Let $1>\rho>r>0$. Note that

$$\sup_{y\in\overline{D}\cap B_r} M_D(x,y)=\sup_{y\in D\cap B_r}\frac{G_D(x,y)}{G_D(x_0,y)},\quad \inf_{y\in\overline{D}\cap B_r} M_D(x,y)=\inf_{y\in D\cap B_r}\frac{G_D(x,y)}{G_D(x_0,y)}.$$

Since $G_D(\tilde{x},y)=P_{D\cap B_\rho}[G_D(\tilde{x},v)dv](y)$ for $\tilde{x}\in D\setminus\overline{B}_\rho$, the claim (a) follows from Lemma 1.5.5(a). For the part (b) we apply Lemma 1.5.5(b) in a similar way. ∎

Corollary 1.5.8. Let D be an open set. If D is unbounded suppose **(GWSC)** and **(E)**.

(a) For every fixed $x\in D$, the function $z\mapsto M_D(x,z)$ is continuous on ∂^*D.

(b) Let μ be a measure on $\partial_M D$. Then $M_D\mu(x)=\infty$ for some $x\in D$ if and only if $M_D\mu\equiv\infty$ in D, and in that case μ is an infinite measure.

Proof. Lemma 1.5.7 directly yields that $\partial^*D\ni z\mapsto M_D(x,z)$ is continuous for every $x\in D$, which proves the part (a). Moreover, Lemma 1.5.7 also yields that $z\mapsto M_D(x,z)$ is bounded from below and above from which the the part (b) easily follows. ∎

Now we state two lemmas that appeared in [21] for the case of the isotropic α-stable process. The lemmas will be useful for proving uniqueness of the representation of non-negative $\phi(-\Delta)$-harmonic functions with zero outer charge.

Lemma 1.5.9. Let D be an open set. Suppose that $0\le g\le f$ on D, and that f, g are $\phi(-\Delta)$-harmonic in D with zero outer charge. If $U\subset D$ and $f(x)=\int_{U^c} f(y)\omega_U^x(dy)$, $x\in U$, then $g(x)=\int_{U^c} g(y)\omega_U^x(dy)$, $x\in U$.

Proof. The proof is exactly the same as the proof of [21, Lemma 9]. ∎

Lemma 1.5.10. Let D_1 and D_2 be open sets such that

$$\mathrm{dist}(D_1 \setminus D_2, D_2 \setminus D_1) > 0.$$

Set $D = D_1 \cup D_2$ and assume that $\omega_D^x(D^c) > 0$ for one (and therefore for all) $x \in D$. Let $f \geq 0$ be a function on $\mathbb{R}^d$ such that $f = 0$ on D^c, and for $i = 1,2$ and all $x \in D_i$ we have

$$f(x) = \int f(y)\omega_{D_i}^x(dy).$$

Let D_1 be bounded and if D_2 is unbounded assume **(GWSC)**. Then $f = 0$ on the whole of D.

Proof. The proof is the same as the proof [21, Lemma 10] where for inequalities (70) and (71) of [21] we use Harnack inequality for the subordinate Brownian motion [42, Theorem 7]. ∎

Now we have a generalization of [21, Lemma 14].

Proposition 1.5.11 (Martin representation)**.** Let D be an open set. If D is unbounded we additionally assume **(GWSC)** and **(E)**. Suppose $f \geq 0$ is $\phi(-\Delta)$-harmonic on D with zero outer charge. Then there is a unique finite measure $\mu \geq 0$ on $\partial_M D$ such that

$$f(x) = \int_{\partial_M D} M_D(x,y)\mu(dy), \quad x \in D, \tag{1.76}$$

and we have $W_D f = \mu$. Conversely, if μ is a finite measure on $\partial_M D$ and if we define $f(x) := \int_{\partial_M D} M_D(x,y)\mu(dy)$, $x \in D$, then f is $\phi(-\Delta)$-harmonic in D with zero outer charge.

Before we prove the proposition we connect the result with the Martin boundary of D with respect to X^D in the sense of Kunita-Watanabe, see [61]. From [54,55] it follows that in our setting the (abstract) Martin boundary of the set D can be identified with $\partial^* D$. Also, the *minimal* Martin boundary can be identified with $\partial_M D$. However, in [55, Corollary 1.2 & Corollary 1.4] the Martin representation of harmonic functions with respect to X^D was proved only for the case $\partial_M D = \partial^* D$, cf. [61, Theorem 4]. Hence, the proposition above extends [55, Corollary 1.2 & Corollary 1.4] on more general sets but on less general processes.

Proof of Proposition 1.5.11. The second claim is almost trivial. Since μ is a finite measure on $\partial_M D$, we have that $f := M_D\mu$ is well-defined by Corollary 1.5.8. It is also $\phi(-\Delta)$-harmonic in D with zero outer charge since for every $z \in \partial_M D$ the Martin kernel $M_D(\cdot, z)$ is $\phi(-\Delta)$-harmonic in D with zero outer charge, see Theorem 1.4.3.

The first claim is proved similarly as in [21, Lemma 14] but because of some differences at the end of the proof we give the full proof for the reader's convenience. Let $(D_n)_n$ denote an increasing sequence of open sets with Lipschitz boundary such that for all $n \in \mathbb{N}$ we have $D_n \subset\subset D$ and $D = \bigcup_{n=1}^{\infty} D_n$. By the mean-value property we have for $x \in D_n$

$$\begin{aligned} f(x) &= \int_{D\setminus D_n} P_{D_n}(x,y) f(y) dy \\ &= \int_{D_n} M_{D_n}(x,v) \left(G_{D_n}(x_0,v) \int_{D\setminus D_n} j(|v-y|) f(y) dy \right) dv \\ &= \int_{D_n} M_{D_n}(x,v) \eta_{D_n} f(dv), \end{aligned}$$

where $\eta_{D_n} f$ is the measure from Definition 1.5.1. For brevity's sake, we write η_n for $\eta_{D_n} f$. Since $\eta_n(D) = f(x_0) < \infty$, by considering a subsequence we may assume that the sequence $(\eta_n)_n$ weakly converges on D^* to a finite non-negative measure μ^*. It follows from Lemma 1.5.2, more precisely Remark 1.5.3(a), that μ^* is supported on $\partial^* D$.

Let $\varepsilon > 0$ and $x \in D$. By Lemma 1.5.7 for every $y \in \partial^* D$ there exists a neighbourhood V_y of y such that

$$\mathrm{RO}_{V_y \cap U^*} M_U(x,\cdot) \le 1 + \varepsilon, \tag{1.77}$$

for all $U \in \{D, D_1, D_2, \dots\}$. From $\{V_y : y \in \partial^* D\}$, we select a finite family $\{V_j : j = 1, \dots, m\}$ such that $V := V_1 \cup \dots \cup V_m \supset \partial^* D$. For $j \in \{1, \dots, m\}$ let $z_j \in D \cap V_j$. Let k be so large that for $n > k$ we have $z_j \in D_n$ and

$$(1+\varepsilon)^{-1} \le \frac{M_D(x,z_j)}{M_{D_n}(x,z_j)} \le (1+\varepsilon), \quad j = 1, \dots, m.$$

The last inequality can be achieved because $G_{D_n} \uparrow G_D$ pointwise in D as $n \to \infty$. If $v \in D_n \cap V_j$, then by (1.77) and the last inequality we get

$$(1+\varepsilon)^{-3} \le \frac{M_D(x,v)}{M_D(x,z_j)} \cdot \frac{M_D(x,z_j)}{M_{D_n}(x,z_j)} \cdot \frac{M_{D_n}(x,z_j)}{M_{D_n}(x,v)} \le (1+\varepsilon)^3.$$

Therefore

$$(1+\varepsilon)^{-3} \leq \frac{\int_{D\cap V} M_D(x,y)\eta_n(dy)}{\int_{D\cap V} M_{D_n}(x,y)\eta_n(dy)} \leq (1+\varepsilon)^3, \quad n>k. \tag{1.78}$$

Notice that $(\eta_n)_n$ also weakly converges to μ^* on $D^*\cap V$ and that $x,x_0 \notin D\cap V$. Since $M_D(x,\cdot)$ is continuous and bounded on $D^*\cap V$ by Lemma 1.5.7, we have

$$\int_{D\cap V} M_D(x,y)\eta_n(dy) \to \int_{D^*\cap V} M_D(x,y)\mu^*(dy) = \int_{\partial^* D} M_D(x,y)\mu^*(dy).$$

Further, note that $f(x) = \int_{D\cap V} M_{D_n}(x,y)\eta_n(dy) + \int_{D\cap V^c} M_{D_n}(x,y)\eta_n(dy)$ and that there is k so large such that $D\cap V^c \subset D_k$. Hence for $n>k$ we have

$$\begin{aligned}
\int_{D\cap V^c} M_{D_n}(x,y)\eta_n(dy) &\leq \int_{D_k} M_{D_n}(x,y)\eta_n(dy) \\
&= \int\limits_{D_k} G_{D_n}(x,v) \int\limits_{D\setminus D_n} j(|v-y|)f(y)dydv \\
&\leq c_k \left(\int\limits_{D_k} G_D(x,v)dv\right)\left(\int\limits_{D\setminus D_n} f(y)(1\wedge j(|y|))dy\right) \xrightarrow{n\to\infty} 0,
\end{aligned}$$

since $f\in\mathscr{L}^1$ by Lemma 1.3.8 and since $G_D(x,\cdot)\in L^1_{loc}$ which we get from $G_D(x,\cdot)\leq G_{\mathbb{R}^d}(x,\cdot)$ and (1.18). By letting $n\to\infty$ in (1.78) we obtain

$$(1+\varepsilon)^{-3} \leq \frac{\int_{\partial^* D} M_D(x,y)\mu^*(dy)}{f(x)} \leq (1+\varepsilon)^3.$$

i.e. $f(x) = \int_{\partial^* D} M_D(x,y)\mu^*(dy)$.

We now prove that the measure μ^* is concentrated on $\partial_M D$. Let $x\in U \subset\subset D$. If $y\in\partial^* D$, then by Theorem 1.4.3 $M_D(x,y) \geq \int_{D\setminus U} M_D(z,y)\omega_U^x(dz)$ and equality holds if and only if $y\in\partial_M D$. By Fubini's theorem we have

$$0 = f(x) - \int\limits_{D\setminus U} f(z)\omega_U^x(dz) = \int\limits_{\partial^* D}\left(M_D(x,y) - \int_{D\setminus U} M_D(z,y)\omega_U^x(dz)\right)\mu^*(dy),$$

hence $\mu^*(\partial^* D\setminus\partial_M D) = 0$.

Now we prove uniqueness. Consider first the case $f(\,\cdot\,) = M_D(\,\cdot\,,z_0) = M_D\delta_{z_0}(\,\cdot\,)$ and suppose that there is another measure μ on $\partial_M D$ such that $f = M_D\mu$. If z_0 is finite, then the uniqueness is proved in the same way as in [21]. Therefore, we deal with the case $z_0=\infty$. For $s>0$ define $D_s = D\cap B_s$ and take $R>0$ such that (1.61) is true, i.e.

$M_D(x,\infty) = \mathbb{E}_x[M_D(X_{\tau_{D_R}},\infty)]$, $x \in D_R$. Define the function $g : \mathbb{R}^d \to [0,\infty)$ as $g(x) = \int_{|y|<R} M_D(x,y)\mu(dy)$. For $x \in D \setminus D_{2R}$, by Fubini's theorem and the comment about (1.60) in Remark 1.4.4, we have that

$$\begin{aligned}\int_{D_{2R}} g(z)\omega^x_{D\setminus D_{2R}}(dz) &= \int_{|y|<R} \left(\int_{D_{2R}} M_D(z,y)\omega^x_{D\setminus D_{2R}}(dz)\right)\mu(dy)\\ &= \int_{|y|<R} M_D(x,y)\mu(dy) = g(x).\end{aligned}$$

Further, for $x \in D_{3R}$ it holds that $g(x) = \int_{D\setminus D_{3R}} g(z)\omega^x_{D_{3R}}(dz)$. Indeed, $g \le f$ and $f(x) = \int_{D\setminus D_{3R}} f(z)\omega^x_{D_{3R}}(dz)$ because of (1.61) so Lemma 1.5.9 yields the claim. Lemma 1.5.10 yields $g = 0$ on whole D, in particular $g(x_0) = \mu(\{|y| < R\}) = 0$. Since this is true for all big $R > 0$, we see that μ is concentrated at the point at infinity. Thus, we have uniqueness for the function $f(\cdot) = M_D(\cdot,\infty)$.

Consider now $f = M_D\mu$ for a finite measure μ on $\partial_M D$ and let $(\eta_{D_n} f)_n$ be the corresponding sequence of measures for f from the beginning of the proof. We want to show that $\mu^* = \mu$. Since $(\eta_{D_n} f)_n$ converges weakly to μ^*, by the uniqueness of the weak limit it is enough to show that for every relatively open set $A \subset \overline{D}$ we have $\liminf_n \eta_{D_n} f(A) \ge \mu(A)$. To this end, by using Fubini's theorem, Fatou's lemma, and what was already proven for the case of the Dirac measures we have

$$\begin{aligned}&\liminf_{n\to\infty} \eta_{D_n} f(A) = \liminf_{n\to\infty} \int\limits_A G_{D_n}(x_0,v)\left(\int\limits_{D\setminus D_n} j(v,y) M_D\mu(y)dy\right)dv\\ &= \liminf_{n\to\infty} \int\limits_{\partial_M D}\left(\int\limits_A G_{D_n}(x_0,v)\left(\int\limits_{D\setminus D_n} j(|v-y|)M_D(y,z)dy\right)dv\right)\mu(dz)\\ &\ge \int\limits_{\partial_M D} \liminf_{n\to\infty}\left(\int\limits_A G_{D_n}(x_0,v)\left(\int\limits_{D\setminus D_n} j(|v-y|)M_D(y,z)dy\right)dv\right)\mu(dz)\\ &= \int\limits_{\partial_M D} \liminf_{n\to\infty} \eta_{D_n}\big(M_D(\cdot,z)\big)(A)\mu(dz) \ge \int\limits_{\partial_M D} \delta_z(A)\mu(dz) = \mu(A).\end{aligned}$$

Thus, we have proved uniqueness.

Notice that due to uniqueness of the measure μ, any choice of the sequence $(D_n)_n$ from the beginning of the proof gives μ as the limit of $\eta_{D_n} f$ so we have proved that $W_D f$ is well defined and that $W_D f = \mu$. ∎

Remark 1.5.12. Since for a finite measure μ on $\partial_M D$ we have that $M_D\mu$ is $\phi(-\Delta)$-harmonic with zero outer charge, we have that $M_D\mu \in C^\infty(D)\cap\mathscr{L}^1$ and if D is bounded we have $M_D\mu \in L^1(D)$, see Lemma 1.3.8, and Theorem 1.3.12.

Combining Propositions 1.5.4 and 1.5.11, we get that (under the additional assumptions **(GWSC)** and **(E)** if D is unbounded)

$$W_D\big(G_D f + P_D\lambda + M_D\mu\big) = \mu. \tag{1.79}$$

Theorem 1.5.13 (Representation of non-negative $\phi(-\Delta)$-harmonic functions)**.** Let D be an open set. If D is unbounded additionally assume **(GWSC)** and **(E)**. If f is a non-negative function, $\phi(-\Delta)$-harmonic in D with a non-negative outer charge λ, then there is a unique finite measure μ_f on $\partial_M D$ such that $f = P_D\lambda + M_D\mu_f$ on D.

Proof. The proof is exactly the same as the proof of [21, Lemma 13]. ■

The following corollary is an extension of [21, Proposition 1] to more general non-local setting.

Corollary 1.5.14. Let D be an open set and $x\in D$. If D is unbounded assume (GWSC) and (E). The harmonic measure ω_D^x is absolutely continuous in $D^c\setminus\partial_M D$ with respect to the Lebesgue measure with the density $P_D(x,\cdot)$.

Proof. The proof is almost identical to the proof of [21, Proposition 1] and we give it for the reader's convenience.

Let K be a compact set in $D^c\setminus\partial_M D$. Note that the function $\mathbb{R}^d \ni x\mapsto \omega_D^x(K)$ is by the strong Markov property harmonic in D, hence $\phi(-\Delta)$-harmonic with outer charge $\mathbf{1}_K$, and also by the strong Markov property the mean-value property (1.39) holds for $\omega_D^x(K)$ for every $U\subset D$. Let $f:\mathbb{R}^d\to[0,\infty)$ such that $f(x) = \omega_D^x(K) - P_D\big(\mathbf{1}_K\big)(x)$, $x\in\mathbb{R}^d$. Obviously, f is $\phi(-\Delta)$-harmonic with zero outer charge. Note that by Corollary 1.3.13 the function $P_D\big(\mathbf{1}_K\big)$ satisfies the mean-value property (1.39) for every $U\subset D$, hence f satisfies it, too. Also, Theorem implies that there exists a finite measure μ on $\partial_M D$ such that $f = M_D\mu$.

We now show that $\mu\equiv 0$ which implies $\omega_D^x(K) = P_D\big(\mathbf{1}_K\big)(x)$ for every $x\in D$. Let $L\subset\partial_M D$ be a compact set and define $g = M_D\big(\mu\mathbf{1}_L\big)$. Obviously, $0\le g\le f$ and g is

$\phi(-\Delta)$-harmonic with zero outer charge. Since for fixed $x \in D$ we have $z \mapsto M_D(x,z) \asymp 1$, to get $\mu \equiv 0$ it is enough to show $g \equiv 0$. To this end define $\varepsilon = (\mathrm{diam} D \wedge 1)/2$ and

$$D_1 = \{x \in D : \mathrm{dist}(x,L) < 2\varepsilon\},$$
$$D_2 = \{x \in D : \mathrm{dist}(x,L) > \varepsilon\}.$$

Since $g \le f$ and f satisfies the mean-value property (1.39) for $U = D_1$, by Lemma 1.5.9 so does g. Also, by Remark 1.4.4, see the comment about (1.60), it follows that g satisfies the mean-value property (1.39) for $U = D_2$. Therefore, since D_1 is bounded, by Lemma 1.5.10 it follows that $g \equiv 0$. ■

2. SEMILINEAR PROBLEM FOR $\phi(-\Delta)$

Let $D \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open set, $f : D \times \mathbb{R} \to \mathbb{R}$ a function, λ a signed measure on $D^c = \mathbb{R}^d \setminus D$ and μ a signed measure on ∂D. In this chapter we study the semilinear problem

$$\begin{aligned} \phi(-\Delta)u(x) &= f(x,u(x)) && \text{in } D \\ u &= \lambda && \text{in } D^c \\ W_D u &= \mu && \text{on } \partial D. \end{aligned} \tag{2.1}$$

The operator $\phi(-\Delta)$ is an integro-differential operator where $\phi : (0,\infty) \to (0,\infty)$ is a complete Bernstein function without drift satisfying the weak scaling condition at infinity - the condition **(WSC)**. Recall that by (1.14), $\phi(-\Delta)$ can be written as a principal value integral

$$\phi(-\Delta)u(x) = \text{P.V.} \int_{\mathbb{R}^d} (u(x) - u(y))\, j(|y-x|)\, dy,$$

where the singular kernel j is completely determined by the function ϕ. Recall also that if $\phi(t) = t^{\alpha/2}$, $\alpha \in (0,2)$, $\phi(-\Delta)$ is the fractional Laplacian $(-\Delta)^{\alpha/2}$ and the kernel $j(|y-x|)$ is proportional to $|y-x|^{-d-\alpha}$.

The operator W_D is a boundary trace operator first introduced in [20] in the case of the fractional Laplacian, and extended to more general non-local operators in Chapter 1 – see the beginning of Section 1.5 for the precise definition.

Motivated by the recent preprint [4] we consider solutions of (2.1) in the weak dual sense, cf. Definition 2.3.1, and show that for bounded $C^{1,1}$ open sets this is equivalent to the notion of weak L^1 solution as in [1, Definition 1.3].

For the nonlinearity f throughout the chapter we assume the condition

(F) $f : D \times \mathbb{R} \to \mathbb{R}$ is continuous in the second variable and there exist a function $\rho : D \to [0,\infty)$ and a continuous function $\Lambda : [0,\infty) \to [0,\infty)$ such that $|f(x,t)| \leq \rho(x)\Lambda(|t|)$.

The study of semilinear problems for this type of non-local operators is quite recent and is mostly focused on the fractional Laplacian, see [1, 2, 6–8, 20, 27, 38, 39]. One of the important differences between the local and non-local equations is that in the non-local case the boundary blow-up solutions are possible even for linear equations. To be more precise, there exist non-negative harmonic functions for the operator $\phi(-\Delta)$ that blow up at the boundary. In this chapter we will restrict ourselves to the so called moderate blow-up solutions, that is those bounded by harmonic functions with respect to the operator $\phi(-\Delta)$. This restriction is a consequence of the problem (2.1) itself, namely of the boundary trace requirement on the solution. In this respect we follow [1, 20] where the boundary behaviour of solutions was also imposed. Note that in [1] the theory was developed for the fractional Laplacian in a bounded $C^{1,1}$ open set D, while [20] extends part of the theory to regular open sets. This extension was possible mainly due to potential-theoretic results from [21].

The goal of this chapter is to generalize results from [1, 20] and at the same time to provide a unified approach. The first main contribution of this part of the thesis is that we replace the fractional Laplacian with a more general non-local operator. This is possible due to potential-theoretic and analytic properties of such operators developed in the last ten years, see [12, 13, 47–49]. The second main contribution is that we obtain some of the results from [1] (which deals with $C^{1,1}$ open sets) for regular open subsets of $\mathbb{R}^d$. To achieve this goal we combine methods from [1] with those of [20].

Let us now describe the content of the chapter in more detail. In Section 2.1 we briefly recall the preliminary notions and results from Chapter 1 since in both chapters we work in the same setting. In Section 2.2 we invoke Kato class of functions and there we show some auxiliary results on Green potentials.

Section 2.3 is central to the chapter and contains two main results on the existence of a solution to the semilinear problem (2.1) in arbitrary bounded open sets. The first result, Theorem 2.3.6, can be thought of as a generalization of [1, Theorem 1.5]. It assumes the existence of a subsolution and a supersolution to the problem (2.1) and gives several sufficient conditions for the existence of a solution. As in almost all existence proofs of semilinear problems, the solution is obtained by using Schauder's fixed point theorem. As a corollary of the third part of that theorem, in Corollary 2.3.8 we obtain a generalization of the main result of [20]. Theorem 2.3.10 deals with non-positive function f and is a generalization of [1, Theorem 1.7]. The main novelty of our approach is contained in using Lemma 2.3.9 to approximate a non-negative harmonic function by an increasing sequence of Green potentials. This replaces the approximation used in [1] which works only in smooth open sets.

In the last two sections we look at the semilinear problem for $\phi(-\Delta)$ in bounded $C^{1,1}$ open sets and at some related questions. In Section 2.4 we first recall the notion of the renewal function whose importance comes form the fact that it gives exact decay rate of harmonic functions at the boundary. We then state known sharp two-sided estimates for the Green function, Poisson kernel, Martin kernel and the killing function in terms of the renewal function. Subsection 2.4.3 may be of independent interest - there we give the boundary behaviour of the Green potential and the Poisson potential of a function of the distance to the boundary. We next provide a sufficient integral condition (in terms of the renewal function) for a function of the distance to the boundary to be in the Kato class. In Subsection 2.4.6 we invoke a powerful result from [47] to show the existence of generalized normal derivative at the boundary which is used in the equivalent formulation of the weak dual solution. We end the section with a discussion on the relationship of the boundary trace operator W_D with the boundary operator used in [1,2].

The last section revisits Theorem 2.3.10 and Corollary 2.3.8 in bounded $C^{1,1}$ sets. In case when $f(x,t) = W(\delta_D(x))\Lambda(t)$ for some function W, we give a sufficient and necessary integral condition for (a version of) Theorem 2.3.10 to hold in terms of W, Λ and the renewal function. Building on Lemma 2.4.5 we next give a sufficient condition for Corollary 2.3.8 to hold in a bounded $C^{1,1}$ set. Finally, we end by establishing Theorem 2.5.3 that extends Corollary 2.3.8 for non-negative nonlinearities f. This result generalizes [1, Theorem 1.9].

To the chapter is also connected a part of Appendix. In the first part of Appendix we provide a proof of Lemma 2.3.9 in a more general context. In the second part, we give quite technical proofs of Propositions 2.4.1 and 2.4.2. The proof of Proposition 2.4.1 is modelled after the proof of [4, Theorem 3.4], while the proof of Proposition 2.4.2 is somewhat simpler.

2.1. PRELIMINARIES

The setting in which we work in this chapter is the same as the one in Chapter 1. The only addition or restriction is that in this chapter we always assume that D is bounded. Here we will just briefly recall the assumptions and the objects from the preliminary section of Chapter 1.

Let $S = (S_t)_{t\geq 0}$ be a subordinator with the Laplace exponent $\phi(\lambda) = \int_0^\infty (1-e^{-\lambda t})\mu(t)dt$ which satisfies the assumptions **(WSC)** and **(T)**. Let $W = (W_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^d$, $d \geq 2$, independent of S with the characteristic exponent $\xi \mapsto |\xi|^2$, $\xi \in \mathbb{R}^d$. The underlying process of this chapter will be subordinate the Brownian motion $X = ((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d})$ defined as $X_t = W_{S_t}$ and its killed version X^D - the process X killed upon exiting the set D.

Recall from Subsection 1.1.2 that the operator $\phi(-\Delta)$ is defined as

$$\phi(-\Delta)u(x) = \lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon} (u(y)-u(x))\, j(|y-x|)\, dy,$$

for every u such that the previous relation is well defined. This is true if e.g. $u \in C^2(D) \cap L^1(\mathbb{R}^d, (1\wedge j(|x|))dx)$, where

$$j(r) = \int_0^\infty p(t,x,y)\mu(t)dt = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)}\mu(t)dt.$$

With G_D we continue to denote the Green function of X in D, and with P_D the Poisson function of X in D, see Subsection 1.1.3. Finally, please recall that by M_D we denoted the Martin kernel, and that $\partial_M D$ denoted the set of all accessible points from D, see Section 1.4.

2.2. Kato class and Green potentials

Recall that D is a bounded open subset of $\mathbb{R}^d$. We say that a function $q : D \to [-\infty, \infty]$ is in the Kato class $\mathscr{J}$ with respect to X if the family of functions $\{G_D(x,y)|q|(y) : x \in D\}$ is uniformly integrable (with respect to the Lebesgue measure on D). Obviously, if $|v| \le |q|$ and $q \in \mathscr{J}$ then $v \in \mathscr{J}$.

Next, we show that a function $q : D \to [-\infty, \infty]$ satisfying

$$\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<\varepsilon} |q(y)| \phi(|x-y|^{-2})^{-1} |x-y|^{-d} \, dy = 0 \tag{2.2}$$

is in the Kato class $\mathscr{J}$. Extend the function q to all of $\mathbb{R}^d$ by setting $q(y) = 0$ for $y \in D^c$. Since $G_D(x,y) \le G(x,y)$, to show that $q \in \mathscr{J}$ it suffices to show that the family of functions $\{G(x,y)|q(y)| : x \in \mathbb{R}^d\}$ is uniformly integrable. By using (1.18), one can check that [79, (24) & Lemma 5] holds true. Hence, we can apply [79, Theorem 1] with $A(t) := \int_0^t |q(X_s)| ds$, which together with (1.18) implies that (2.2) is equivalent to

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[\int_0^t |q(X_s)| \, ds \right] = 0, \tag{2.3}$$

i.e. q is in the classical Kato class $K(X)$ from [31] and [29]. By (2.2), $q \in L^1(D)$ and therefore by repeating the proof of [29, Theorem 2.1(ii)] in our setting, we get that $q \in K_\infty(X)$, i.e.

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall B \in \mathscr{B}(\mathbb{R}^d) \text{ such that } \lambda(B) < \delta \ \Rightarrow \ \sup_{x \in \mathbb{R}^d} \int_B |q(y)| G(x,y) dy < \varepsilon. \tag{2.4}$$

cf. [31, Definition 2.1(ii)]. Furthermore, by [31, Proposition 2.1], $q \in K_\infty(X)$ implies that q is Green bounded. Together with boundedness of D, [69, Theorem 16.8(iii)] gives that the family $\{G(x,y)|q(y)| : x \in \mathbb{R}^d\}$ is uniformly integrable, and therefore $q \in \mathscr{J}$.

Note that under **(WSC)**, the condition (2.2) is satisfied for $q \in \mathscr{B}_b(\mathbb{R}^d)$, so every bounded function q is in the Kato class $\mathscr{J}$.

Recall that the boundary point $z \in \partial D$ is said to be regular (for D^c) if $\mathbb{P}_z(\tau_D = 0) = 1$. The set D is regular if every boundary point is regular. The same proof as in [20, Proposition 1.31] shows that if D is regular, then $q \in \mathscr{J}$ if and only if $G_D|q| \in C_0(D)$, and then $G_D q \in C_0(D)$.

Let $z \in \partial D$ be regular. Then for all $x \in D$,

$$\lim_{y \to z, y \in D} G_D(x,y) = 0.$$

A proof of this well-known result can be found in [59, Proposition 6.2]. The next result is also known – we include the proof for the sake of completeness.

Lemma 2.2.1. Let D be a bounded open subset of $\mathbb{R}^d$. Then $G_D\mathbf{1} \in C(D)$ and $\lim_{x\to z} G_D\mathbf{1}(x) = 0$ for every regular boundary point $z \in \partial D$.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be any sequence of points in D. Since the constant function $\mathbf{1}$ is in $\mathscr{J}$, the family $\{G_D(x_n,\cdot) : n \in \mathbb{N}\}$ is uniformly integrable. If $x_n \to x \in D$, then $\lim_{n\to\infty} G_D(x_n,y) = G_D(x,y)$ for a.e. $y \in D$, hence by Vitali's theorem, see [69, Theorem 16.6 $(ii) \iff (iii)$], it follows that

$$\lim_{n\to\infty} \int_D G_D(x_n,y)dy = \int_D G_D(x,y)dy,$$

proving that $G_D\mathbf{1} \in C(D)$. If $x_n \to z \in \partial D$ with z regular, then $\lim_{n\to\infty} G_D(x_n,y) = 0$ for all $y \in D$. Again by Vitali's theorem we get that $\lim_{n\to\infty} \int_D G_D(x_n,y)\,dy = 0$. ∎

Denote by D^{reg} the set of all regular boundary points of D. For $\delta > 0$, let $D_\delta := \{x \in D : \text{dist}(x,\partial D) > \delta\}$.

Lemma 2.2.2. Let $v : D \to [0,\infty)$ be a locally bounded function and $\rho : D \to [0,\infty)$ such that $G_D\rho \in C(D)$ and $\rho v G_D\mathbf{1} \in L^1(D)$. Then, for every $x \in D$ it follows that

$$\lim_{w\to x} \int_D |G_D(x,y) - G_D(w,y)|\rho(y)v(y)dy = 0.$$

Proof. Let $r > 0$ such that $\overline{B(x,r)} \subset D$ and take a sequence $(x_n)_n \subset B(x,r/2)$ such that $x_n \to x$. Since v is locally bounded in D, there exists a constant $c_1 > 0$ such that $v(y) \le c_1$ for all $y \in B(x,r)$. Therefore,

$$\begin{aligned}\int_D |G_D(x_n,y) - G_D(x,y)|\rho(y)v(y)dy &\le c_1 \int_D |G_D(x_n,y) - G_D(x,y)|\rho(y)dy \\ &\quad + \int_{D\cap B(x,r)^c} |G_D(x_n,y) - G_D(x,y)|\rho(y)v(y)dy\end{aligned}$$

Since $G_D(x_n,y)\rho(y) \to G_D(x,y)\rho(y)$ as $n \to \infty$, for a.e. $y \in D$, by Vitali's convergence theorem, [69, Theorem 16.6 $(i) \iff (iii)$], it is enough to show that

$$\begin{aligned}&\lim_{n\to\infty} \int_D G_D(x_n,y)\rho(y)dy = \int_D G_D(x,y)\rho(y)dy \quad \text{and} \\ &\lim_{n\to\infty} \int_{D\cap B(x,r)^c} G_D(x_n,y)\rho(y)v(y)dy = \int_{D\cap B(x,r)^c} G_D(x,y)\rho(y)v(y)dy.\end{aligned}$$

The first limit follows directly from the assumption $G_D\rho \in C(D)$. For the second integral, we will show that there exists a constant $c_2 > 0$ such that

$$G_D(w,y) \le c_2 G_D\mathbf{1}(y)\,, \quad w \in B(x,r/2), \;\; y \in D \cap B(x,r)^c. \tag{2.5}$$

Therefore, since $\rho v G_D\mathbf{1} \in L^1(D)$ and $x_n \in B(x,r/2)$, we can apply the dominated convergence theorem to obtain

$$\lim_{n\to\infty} \int_{D\cap B(x,r)^c} G_D(x_n,y)\rho(y)v(y)\,dy = \int_{D\cap B(x,r)^c} G_D(x,y)\rho(y)v(y)\,dy.$$

It remains to show (2.5). First note that $G_D(\cdot,y)$ are harmonic functions in $B(x,r)$ for all $y \in D \cap B(x,r)^c$. By the Harnack principle, there exists $c_3 > 0$ such that

$$G_D(w,y) \le c_3 G_D(x,y)\,, \quad \text{for all } w \in B(x,r/2) \text{ and all } y \in D \cap B(x,r)^c. \tag{2.6}$$

Let $\psi : D \to [0,1]$ be a function with support in $B(x,r/2)$. Then both $G_D\psi$ and $G_D(x,\cdot)$ are regular harmonic in $D \cap B(x,r)^c$ and vanish in the sense of the limit on D^{reg} and by definition on $\overline{D}^c$.

Let $z \in \partial D$. By [54, Theorem 1.1], there exists a finite limit

$$a(z) := \lim_{y\to z, y\in D} \frac{G_D(x,y)}{G_D\psi(y)}.$$

Therefore, there exists a $0 < \varepsilon(z) < \text{dist}(B(x,r),\partial D)/2$ such that

$$\frac{G_D(x,y)}{G_D\psi(y)} \le a(z)+1\,, \quad \text{for all } y \in D \cap B(z,\varepsilon(z)).$$

By compactness of ∂D, there are finitely many points $z_1, z_2, \dots, z_n \in \partial D$ and $\delta > 0$ such that $\partial D \subset \overline{D} \setminus D_\delta \subset \cup_{j=1}^n B(z_j,\varepsilon(z_j))$. Thus for any $y \in D \setminus D_\delta$ it holds that

$$\frac{G_D(x,y)}{G_D\psi(y)} \le \max_{j=1,\dots,n}(a(z_j)+1) =: c_4. \tag{2.7}$$

Further, since both $G_D\psi$ and $G_D(x,\cdot)$ are continuous (and strictly positive) on the compact set $\overline{D_\delta} \cap B(x,r)^c$, we get that

$$\frac{G_D(x,y)}{G_D\psi(y)} \le c_5\,, \quad y \in \overline{D_\delta} \cap B(x,r)^c. \tag{2.8}$$

Combining (2.6)–(2.8) together with $G_D\psi \le G_D\mathbf{1}$, we get (2.5). ■

Lemma 2.2.3. Let $|g| \le f$ such that $G_Df \in C_0(D)$. Then $G_Dg \in C_0(D)$.

Proof. Let $(x_n)_n \subset D$ be a sequence that converges to $x \in D$. We have

$$\begin{aligned} |G_D g(x_n) - G_D g(x)| &\leq \int_D |G_D(x_n, y) - G_D(x, y)||g(y)|dy \\ &\leq \int_D |G_D(x_n, y) - G_D(x, y)|f(y)dy. \end{aligned} \tag{2.9}$$

Since $G_D(x_n, y)f(y) \to G_D(x, y)f(y)$ as $n \to \infty$ and $G_D f \in C_0(D)$ by Vitali's theorem [69, Theorem 16.6 $(i) \iff (iii)$] we have that the right-hand side of (2.9) tends to 0. Hence, $G_D g \in C(D)$.

To see that $G_D g \in C_0(D)$ it is enough to notice that $0 \leq |G_D g(x)| \leq G_D f(x)$ in D so when $x \to z \in \partial D$ we have $G_D g(x) \to 0$. ■

2.3. THE SEMILINEAR PROBLEM IN BOUNDED OPEN SET

Let us now turn to the semilinear problem. For functions $f : D \times \mathbb{R} \to \mathbb{R}$ and $u : D \to \mathbb{R}$ let $f_u : D \to \mathbb{R}$ be a function defined by

$$f_u(x) = f(x, u(x)).$$

Definition 2.3.1. Let $f : D \times \mathbb{R} \to \mathbb{R}$ be a function, $\lambda \in \mathscr{M}(D^c)$ and $\mu \in \mathscr{M}(\partial D)$ a measure concentrated on $\partial_M D$, such that $P_D|\lambda| + M_D|\mu| < \infty$ on D. A function $u \in L^1(D)$ is called a weak dual solution to the semilinear problem

$$\begin{aligned} \phi(-\Delta)u(x) &= f(x,u(x)) && \text{in } D \\ u &= \lambda && \text{in } D^c \\ W_D u &= \mu && \text{on } \partial D \end{aligned} \tag{2.10}$$

if u satisfies the equality

$$\begin{aligned} \int_D u(x)\psi(x)dx = &\int_D f(x,u(x)) G_D\psi(x)dx \\ &+ \int_{D^c}\int_D P_D(x,z)\psi(x)\,dx\,\lambda(dz) \\ &+ \int_{\partial_M D}\int_D M_D(x,z)\psi(x)\,dx\,\mu(dz), \end{aligned} \tag{2.11}$$

for every $\psi \in C_c^\infty(D)$. If in the equation above we have $\geq$ ($\leq$) instead of the equality and the inequality holds for every non-negative $\psi \in C_c^\infty(D)$, we say that u is a supersolution (subsolution) to (2.10).

Remark 2.3.2. Let us give short comments on the previous definition.

(i) Recall from Remark 1.3.6 and Corollary 1.5.8(b) that if $P_D|\lambda|(x) + M_D|\mu|(x) < \infty$ for some $x \in D$, then $P_D|\lambda|(x) + M_D|\mu|(x) < \infty$ for all $x \in D$. Also, since $P_D|\lambda| < \infty$, λ is a measure on $\mathbb{R}^d \setminus (D \cup \partial_M D)$, see Subsection 1.4, so conditions in (2.10) in D^c and on ∂D are indeed complementary.

(ii) Note that by Fubini's theorem and symmetry of G_D, the above definition implies that the weak dual solution u of (2.10) satisfies

$$u(x) = G_D f_u(x) + P_D\lambda(x) + M_D\mu(x),$$

for almost every $x \in D$. Moreover, if we set $g = P_D\lambda + M_D\mu$, then (3.2) is equivalent to

$$\int_D u(x)\psi(x)dx = \int_D f(x,u(x))G_D\psi(x)dx + \int_D g(x)\psi(x)dx. \tag{2.12}$$

Also, suppose that $u \in L^1_{loc}(D)$ satisfies (2.10). This also implies that $u = G_D f_u + P_D\lambda + M_D\mu$ a.e. in D. Since $G_D f_u$, $P_D\lambda$, $M_D\mu \in L^1(D)$, see Lemma 1.2.2, Corollary 1.3.13 and Remark 1.5.12, we have $u \in L^1(D)$, i.e. every function that satisfies (2.10) must be in $L^1(D)$.

Before we show an existence and uniqueness theorem for a wide class of problems we show an auxiliary result. For a Borel set $A \subset D$ and $x \in A$, let $\omega_A^x(dz) := \mathbb{P}_x(X_{\tau_A} \in dz)$ denote the harmonic measure. If $u : \mathbb{R}^d \to [-\infty,\infty]$, let $\tilde{P}_A u(x) := \mathbb{E}_x[u(X_{\tau_A})] = \int_{\mathbb{R}^d} u(y)\omega_A^x(dy)$ whenever the integral makes sense. Note that the functions $\tilde{P}_A u$ and $P_A u$ are not the same, but they are same if A is e.g. a Lipschitz set. We also recall that $G_A(x,y) = 0$ if $y \in \overline{A}^c \cup A_{reg}$ and that the set of irregular points for A is polar. Finally, if the function u is defined only on D, we extend it to all of $\mathbb{R}^d$ by setting $u(x) = 0$ for $x \notin D$, and denote the extended function by $u\mathbf{1}_D$.

Lemma 2.3.3. Let D be an open bounded set in $\mathbb{R}^d$, $f : D \to [-\infty,\infty]$ a function on D and $\lambda \in \mathscr{M}(D^c)$ such that

$$G_D|f|(x_0), P_D|\lambda|(x_0) < \infty \text{ for some } x_0 \in D.$$

Let u be a function on D satisfying

$$u(x) = G_D f(x) + P_D\lambda(x) \text{ for a.e. } x \in D$$

and $A \subset D$ an open set. Then for a.e. $x \in A$,

$$u(x) = G_A f(x) + \tilde{P}_A(u\mathbf{1}_D)(x) + \int_{D^c} P_A(x,y)\lambda(dy). \tag{2.13}$$

Proof. First recall that if $G_D|f|(x_0), P_D|\lambda|(x_0) < \infty$ for some $x_0 \in D$ then $G_D|f|(x), P_D|\lambda|(x) < \infty$ for almost every $x \in D$, see Lemma 1.2.2 and Remark 1.3.6. By the strong Markov property we have that

$$G_D(x,y) = G_A(x,y) + \int_{D\setminus A} G_D(z,y)\omega_A^x(dz), \quad x \in A, y \in D,$$

and then (1.22) implies that

$$P_D(x,y) = P_A(x,y) + \int_{D\setminus A} P_D(z,y)\omega_A^x(dz), \quad x \in A, y \in D^c.$$

Therefore, for a.e. $x \in A$ we have

$$\begin{aligned}
u(x) &= \int_A G_D(x,y)f(y)dy + \int_{D\setminus A} G_D(x,y)f(y)dy + \int_{D^c} P_D(x,y)\lambda(dy) \\
&= \int_A G_A(x,y)f(y)dy + \int_A \int_{D\setminus A} G_D(z,y)\omega_A^x(dz)f(y)dy \\
&\quad + \int_{D\setminus A}\int_{D\setminus A} G_D(z,y)\omega_A^x(dz)f(y)dy + \int_{D^c} P_D(x,y)\lambda(dy) \\
&= \int_A G_A(x,y)f(y)dy + \int_{D\setminus A}\left(\int_D G_D(z,y)f(y)dy\right)\omega_A^x(dz) \\
&\quad + \int_{D^c} P_D(x,y)\lambda(dy) \\
&= \int_A G_A(x,y)f(y)dy + \int_{D\setminus A} u(z)\omega_A^x(dz) \\
&\quad - \int_{D\setminus A}\left(\int_{D^c} P_D(z,y)\lambda(dy)\right)\omega_A^x(dz) + \int_{D^c} P_D(x,y)\lambda(dy) \\
&= \int_A G_A(x,y)f(y)dy + \int_{D\setminus A} u(z)\omega_A^x(dz) + \int_{D^c} P_A(x,y)\lambda(dy).
\end{aligned}$$

■

Remark 2.3.4. Let $u = G_D f + P_D\lambda$ as above and set $u = \lambda$ on D^c. For an open set $A \subset D$ with a Lipschitz boundary consider the linear problem $-Lu_A = f$ in A, $u_A = u$ in A^c, and $W_A u_A = 0$ on ∂A. Then Lemma 2.3.3 says that $u_A = u$ in A.

Proposition 2.3.5. Let $D \subset \mathbb{R}^d$ be a bounded open set and let $f : D\times\mathbb{R} \to \mathbb{R}$ be a function which is non-increasing in the second variable. Then the continuous weak dual solution to (2.10) is unique.

Proof. Let u_1 and u_2 be two continuous solutions to (2.10). Remark 2.3.2(ii) yields that $u_i = G_D f_{u_i} + P_D\lambda + M_D\mu$ a.e. on D, $i = 1,2$, hence $u_1 - u_2 = G_D f_{u_1} - G_D f_{u_2}$ a.e. on D. Note that $A := \{x \in D : u_1(x) > u_2(x)\}$ is open and that $f(x,u_1(x)) \le f(x,u_2(x))$, $x \in A$, since f is non-increasing. Using Lemma 2.3.3 we get for a.e. $x \in A$

$$0 < u_1(x) - u_2(x) = G_A(f_{u_1} - f_{u_2})(x) + \tilde{P}_A\big((u_1-u_2)\mathbf{1}_D\big)(x) \le 0$$

hence $A = \emptyset$. Similarly we get $\{x \in D : u_2(x) > u_1(x)\} = \emptyset$. ■

Let us recall the condition **(F)** on the function f:

(F). $f : D \times \mathbb{R} \to \mathbb{R}$ is continuous in the second variable and there exist a function $\rho : D \to [0,\infty)$ and a continuous function $\Lambda : [0,\infty) \to [0,\infty)$ such that $|f(x,t)| \le \rho(x)\Lambda(|t|)$.

Theorem 2.3.6. Let $D \subset \mathbb{R}^d$ be a bounded open set and let $f : D \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the condition (F). Let $\lambda \in \mathcal{M}(D^c)$ such that $P_D|\lambda| < \infty$ and $\mu \in \mathcal{M}(\partial D)$ be a finite measure concentrated on $\partial_M D$. Assume that the nonlinear problem (2.10) admits a weak dual subsolution $\underline{u} \in L^1(D) \cap C(D)$ and a weak dual supersolution $\overline{u} \in L^1(D) \cap C(D)$ such that $\underline{u} \le \overline{u}$. Set $g := P_D\lambda + M_D\mu$ and $h := |\overline{u}| \vee |\underline{u}|$. If one of the following conditions holds

(i) $\mu \equiv 0$, $G_D\rho \in C_0(D)$ and $\underline{u}, \overline{u} \in L^\infty(D)$ such that for every open subset $A \subset D$ and a.e. $x \in A$

$$\underline{u}(x) \le G_A f_{\underline{u}}(x) + \tilde{P}_A(\underline{u}\mathbf{1}_D)(x) + P_A\lambda(x), \tag{2.14}$$

$$\overline{u}(x) \ge G_A f_{\overline{u}}(x) + \tilde{P}_A(\overline{u}\mathbf{1}_D)(x) + P_A\lambda(x); \tag{2.15}$$

(ii) $\mu \equiv 0$, Λ is non-decreasing, $G_D(\rho\Lambda(h)) \in C_0(D)$ and $\underline{u}$ and $\overline{u}$ satisfy (2.14) and (2.15), respectively;

(iii) Λ is non-decreasing, $G_D(\rho\Lambda(h)) \in C_0(D)$ and there exists a constant $C > 0$ such that, on D, $G_D(\rho\Lambda(h)) \le C$ and $\underline{u} - g \le -C < C \le \overline{u} - g$;

then (2.10) has a weak dual solution $u \in L^1(D) \cap C(D)$ satisfying

$$\underline{u} \le u \le \overline{u}. \tag{2.16}$$

If, in addition, f is non-increasing in the second variable, then u is a unique continuous weak dual solution to (2.10).

Remark 2.3.7. Note that by Lemma 2.3.3 a supersolution $\overline{u}$ to the nonlinear problem (2.10) satisfies the condition (2.15) if, for example, $\overline{u}$ is a solution to the nonlinear problem

$$\begin{aligned} \phi(-\Delta)u(x) &= f(x,u(x)) && \text{in } D \\ u &= \widetilde{\lambda} && \text{in } D^c \\ W_D u &= 0 && \text{on } \partial D \end{aligned}$$

for some $\widetilde{\lambda} \in \mathcal{M}(D^c)$ such that $P_D|\widetilde{\lambda}| < \infty$ on D and $\lambda \le \widetilde{\lambda}$ (for details see the proof of Theorem 2.3.10 and the functions $u_{n,k}$).

Proof of Theorem 2.3.6. First note that by using (2.12) and (1.79), a function $u \in L^1(D)$ is the solution to (2.10) if and only if $u - g$ is the solution to the homogeneous problem

$$\begin{aligned} \phi(-\Delta)w(x) &= f(x, w+g) && \text{in } D \\ w &= 0 && \text{in } D^c \\ W_D w &= 0 && \text{on } \partial D. \end{aligned} \tag{2.17}$$

Thus, we solve (2.17). For general $v \in C_0(D)$, the function f_v does not need to satisfy the Kato condition $G_D|f_v| \in C_0(D)$, so we define a modification of f in the following way:

$$F(x,t) = \begin{cases} f(x, \overline{u}(x)), & t > \overline{u}(x) - g(x) \\ f(x, t + g(x)), & \underline{u}(x) - g(x) \le t \le \overline{u}(x) - g(x) \\ f(x, \underline{u}(x)), & t < \underline{u}(x) - g(x). \end{cases} \tag{2.18}$$

Note that F is continuous in the second variable. Furthermore,

$$\text{if } v \in C_0(D), \text{ then } G_D|F_v| \in C_0(D), \tag{2.19}$$

since

- under (i), $G_D\rho \in C_0(D)$ and

  $$|F(x, v(x))| \le \rho(x) \max_{y \in [0,M]} \Lambda(y), \tag{2.20}$$

  where $M := \max\{\|\underline{u}\|_\infty, \|\overline{u}\|_\infty\}$ and $c_1 := \max_{y \in [0,M]} \Lambda(y) < \infty$ so the claim now follows from Lemma 2.2.3;

- under (ii) and (iii), $G_D(\rho\Lambda(h)) \in C_0(D)$ and

  $$|F(x, v(x))| \le \rho(x)\Lambda(|\underline{u}(x)| \vee |\overline{u}(x)|) = \rho(x)\Lambda(h(x)), \tag{2.21}$$

  and the claim again follows from Lemma 2.2.3.

Next we consider an auxiliary problem

$$\begin{aligned} \phi(-\Delta)u(x) &= F(x, u) && \text{in } D \\ u &= 0 && \text{in } D^c \\ W_D u &= 0 && \text{on } \partial D, \end{aligned} \tag{2.22}$$

whose solution will be given by the Schauder fixed point theorem. To this end,

- under (i), set $C := \|G_D\rho\|_{L^\infty(D)}\|\Lambda\|_{L^\infty([0,M])}$;
- under (ii), set $C := \|G_D(\rho\Lambda(h))\|_{L^\infty(D)}$;
- under (iii), let C be the constant from the assumption (iii);

and let $K = \{v \in C_0(D) : \|v\|_\infty \le C\}$. Define the operator T by

$$Tv(x) = \int_D F(y, v(y))G_D(x,y)dy, \ v \in C_0(D). \tag{2.23}$$

From (2.19) we have $Tv \in C_0(D)$. We now prove the continuity of T. Suppose the opposite, i.e. suppose that there are $\varepsilon > 0$, $(x_n)_n \subset D$, $(v_n)_n \subset C_0(D)$ and $v \in C_0(D)$ such that $||v_n - v||_\infty \to 0$ and $|Tv_n(x_n) - Tv(x_n)| \ge \varepsilon$, for all $n \in \mathbb{N}$. Since $\overline{D}$ is compact there is $x \in \overline{D}$ and a subsequence of $(x_n)_n$ denoted again by $(x_n)_n$ such that $x_n \to x$. We have

$$\varepsilon \le |Tv_n(x_n) - Tv(x_n)| \le |Tv_n(x) - Tv(x)| + |Tv_n(x_n) - Tv_n(x)| + |Tv(x) - Tv(x_n)|. \tag{2.24}$$

Note that if $x \in \partial D$, then $Tv_n(x) = Tv(x) = 0$ by (2.23). Since F is continuous in the second variable using the dominated convergence theorem with bounds from (2.20) and (2.21) for the first term, for $x \in D$ we have $|Tv_n(x) - Tv(x)| \to 0$ as $n \to \infty$. For the second and the third term let us also look first at the case $x \in \partial D$. Note that from (2.20) and (2.21) we have

- under (i)

  $$|Tw(x_n)| \le c_1 \int_D G_D(x_n, y)\rho(y)dy = c_1 G_D\rho(x_n) \to 0, \quad \text{as } x_n \to x, \quad w \in \{v, v_n\},$$

  since $G_D\rho \in C_0(D)$;

- under (ii) and (iii)

  $$|Tw(x_n)| \le \int_D G_D(x_n, y)\rho(y)\Lambda(h(y))dy = G_D(\rho\Lambda(h))(x_n) \to 0, \quad \text{as } x_n \to x, \quad w \in \{v, v_n\},$$

  since $G_D(\rho\Lambda(h)) \in C_0(D)$.

If $x \in D$ then $G_D(x_n, y) \to G_D(x, y)$ so using [69, Theorem 16.6 $(i) \iff (iii)$]

- under (i)

  $$|Tw(x_n) - Tw(x)| \le c_1 \int_D |G_D(x_n, y) - G_D(x, y)|\rho(y)dy \to 0, \quad \text{as } x_n \to x, \quad w \in \{v, v_n\},$$

  since $G_D\rho \in C_0(D)$;

- under (ii) and (iii)

  $$|Tw(x_n) - Tw(x)| \le \int_D |G_D(x_n, y) - G_D(x, y)|\rho(y)\Lambda(h(y))dy \to 0, \quad \text{as } x_n \to x, \quad w \in \{v, v_n\},$$

  since $G_D(\rho\Lambda(h)) \in C_0(D)$.

Thus, we have a contradiction with (2.24), i.e. T is continuous.

Also, from (2.20), (2.21) and the choice of constant C we get $T(K) \subset K$.

We are left to prove that $T(K)$ is a precompact subset of K. By Arzelà-Ascoli theorem it suffices to note that the functions $\{Tv : v \in K\}$ are equicontinuous by the same calculations as above.

Hence by the Schauder fixed point theorem there is a function $u \in K$ such that

$$u(x) = \int_D F(y, u(y))G_D(x, y)dy,$$

i.e. u is a weak dual solution to (2.22). It follows immediately from (2.18) that, if $\underline{u} - g \le u \le \overline{u} - g$, then u is also a weak dual solution to (2.17). Finally, we show that the obtained solution u to (2.22) is between $\underline{u} - g$ and $\overline{u} - g$. In case of assumption (iii), this is obvious. Under (i) or (ii), set $A = \{x \in D : u(x) > \overline{u}(x) - g(x)\}$. Note that $F_u(y) = f_{\overline{u}}(y)$ for all $y \in A$ and that A is an open subset of D, since both u and $\overline{u} - g$ are continuous on D. Then, for every $x \in A$, by (2.13) we have

$$\begin{aligned} u(x) + g(x) &= G_A F_u(x) + \tilde{P}_A((u+g)\mathbf{1}_D)(x) + P_A\lambda(x) \\ &\le G_A f_{\overline{u}}(x) + \tilde{P}_A(\overline{u}\mathbf{1}_D)(x) + P_A\lambda(x) \\ &\le \overline{u}(x), \end{aligned}$$

where the first inequality comes only from the middle term and the second one is (2.15). This implies that $A = \emptyset$. By using (2.14), one can analogously show that $\{x \in D : u(x) \le \underline{u}(x) - g(x)\} = \emptyset$.

Uniqueness follows from Proposition 2.3.5. ∎

In the following corollary we extend the main result from [20] to our setting of more general non-local operators.

Corollary 2.3.8. Let $D \subset \mathbb{R}^d$ be a bounded open set and let $f : D \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the condition (F) with Λ non-decreasing. Let $\lambda \in \mathscr{M}(D^c)$ such that $P_D|\lambda| < \infty$ and $\mu \in \mathscr{M}(\partial D)$ a finite measure concentrated on $\partial_M D$. Set $g := P_D\lambda + M_D\mu$ and $\overline{g} := P_D|\lambda| + M_D|\mu|$. Assume that $G_D\rho \in C_0(D)$, $G_D(\rho\Lambda(2\overline{g})) \in C_0(D)$, and that either (a) Λ is sublinearly increasing, $\lim_{t\to\infty}\Lambda(t)/t = 0$, or (b) m is sufficiently small. Then the semilinear problem

$$\begin{aligned} \phi(-\Delta)u(x) &= mf(x,u(x)) && \text{in } D \\ u &= \lambda && \text{in } D^c \\ W_D u &= \mu && \text{on } \partial D \end{aligned} \tag{2.25}$$

has a weak dual solution $u \in L^1(D) \cap C(D)$ such that $|u| \le \overline{g} + C$, for some $C > 0$.

If, in addition, f is non-increasing in the second variable, u is a unique continuous weak dual solution to (2.25).

Proof. We use Theorem 2.3.6(iii) with mf instead of f and first choose the constant $C > 0$. Set $r_1 := \sup_{x\in D} G_D\rho(x)$ and $r_2 := \sup_{x\in D} G_D(\rho\Lambda(2\overline{g}))(x)$. By the assumption, we have that $r_1 < \infty$ and $r_2 < \infty$. If (b) holds, given any $C > 0$ we can find m small enough such that $m(\Lambda(2C)r_1 + r_2) \le C$. If (a) holds, then since Λ is sublinearly increasing, we can find $C > 0$ large enough so that again $m(\Lambda(2C)r_1 + r_2) \le C$.

Let $\overline{u} := C + \overline{g}$, $\underline{u} := -\overline{u}$ and $h := |\overline{u}| \vee |\underline{u}| = C + \overline{g}$. Clearly, $\overline{u}$ and $\underline{u}$ belong to $L^1(D) \cap C(D)$ and satisfy $\underline{u} - g \le -C < C \le \overline{u} - g$. We check that $\overline{u}$ is a supersolution of (2.25). Indeed,

$$\begin{aligned} &|G_D(mf_{\overline{u}}) + g| \le mG_D|f_{C+\overline{g}}| + \overline{g} \le mG_D(\rho\Lambda(C+\overline{g})) + \overline{g} \\ &\le \; mG_D\big(\rho\big(\Lambda(2C) + \Lambda(2\overline{g})\big)\big) + \overline{g} \le m\big(\Lambda(2C)r_1 + r_2\big) + \overline{g} \le C + \overline{g} = \overline{u}. \end{aligned}$$

In the same way we see that $\underline{u}$ is a subsolution. It remains to check that $G_D(m\rho\Lambda(h)) \in C_0(D)$ and $G_D(m\rho\Lambda(h)) \le C$. By the same computations as above we have

$$\begin{aligned} G_D(m\rho\Lambda(h)) &\le m\Lambda(2C)G_D\rho + mG_D\rho\Lambda(2\overline{g})) \\ &\le m(\Lambda(2C)r_1 + r_2) \le C. \end{aligned} \tag{2.26}$$

Since $G_D\rho \in C_0(D)$ and $G_D(\rho\Lambda(2\overline{g})) \in C_0(D)$, by (2.26) and Lemma 2.2.3 we also have $G_D(m\rho\Lambda(h)) \in C_0(D)$.

Uniqueness follows from Proposition 2.3.5. ■

Our next goal is to extend Corollary 2.3.8 to a wider class of non-positive functions f. First we show an additional auxiliary result. This result provides an approximation of a non-negative harmonic function on D by an increasing sequence of potentials. It is a consequence of a rather well-known fact that we prove in the appendix, see Proposition 4.1.3. We can use this result because the semigroup $(P_t^D)_{t\geq 0}$ is strongly Feller, the process X^D is transient, non-negative harmonic functions are excessive, and the potential $G_D\mathbf{1}$ is continuous and satisfies $0 < G_D\mathbf{1} < \infty$ on D.

Lemma 2.3.9. Let $h : D \to [0,\infty)$ be a harmonic function with respect to the process X^D. There exists a sequence $(\widetilde{f}_k)_{k\geq 1}$ of non-negative, bounded and continuous functions such that $G_D\widetilde{f}_k \uparrow h$.

Theorem 2.3.10. Let $D \subset \mathbb{R}^d$ be a bounded open set. Let $f : D\times\mathbb{R} \to (-\infty,0]$ be a function that satisfies (F) with $G_D\rho \in C_0(D)$. Assume, additionally, that $f(x,0) = 0$. Let $\lambda \in \mathscr{M}(\mathbb{R}^d \setminus \overline{D})$ be a non-negative measure such that $P_D\lambda < \infty$ and $\mu \in \mathscr{M}(\partial D)$ be a finite non-negative measure concentrated on $\partial_M D$. Let $g := P_D\lambda + M_D\mu$. If the semilinear problem (2.10) satisfies one of the following conditions:

(i) $\mu \equiv 0$;

(ii) $\mu \not\equiv 0$, the function Λ is non-decreasing and $\rho\Lambda(g)G_D\mathbf{1} \in L^1(D)$;

then the problem (2.10) has a non-negative weak dual solution $u \in L^1(D)\cap C(D)$. If, in addition, f is non-increasing in the second variable, then u is a unique continuous solution to (2.10).

Proof. Let $(\widetilde{f}_k)_k$ be a sequence of non-negative, bounded and continuous functions on D from Lemma 2.3.9 such that $G_D\widetilde{f}_k \uparrow M_D\mu$. Let $(K_n)_n$ be an increasing sequence of compact sets such that $K_n \uparrow \overline{D}^c$. Then, for $n \in \mathbb{N}$ the measure $\lambda_n(\cdot) = \lambda(\cdot \cap K_n)$ is a finite non-negative measure on $\overline{D}^c$. Consider the following semilinear problem

$$\begin{aligned} \phi(-\Delta)u(x) &= f(x,u(x))+\widetilde{f}_k(x) && \text{in } D \\ u &= \lambda_n && \text{in } D^c \\ W_D u &= 0 && \text{on } \partial D. \end{aligned} \tag{2.27}$$

Since $f(x,0)=0$ and $\widetilde{f}_k \geq 0$, $\underline{u}\equiv 0$ is a subsolution to (2.27). Furthermore, since f is non-positive, as a supersolution to (2.27) we take the solution $u_k^{(n)} = G_D\widetilde{f}_k + P_D\lambda_n$ of the linear problem

$$\begin{aligned} \phi(-\Delta)u(x) &= \widetilde{f}_k(x) && \text{in } D \\ u &= \lambda_n && \text{in } D^c \\ W_D u &= 0 && \text{on } \partial D. \end{aligned}$$

Fix $k\in\mathbb{N}$. Notice that $u_k^{(n)} \in C(D)$ and that, by Lemma 2.3.3, $u_k^{(n)}$ satisfies (2.13). Moreover, since λ_n is finite and

$$\sup_{x\in D, z\in K_n} P_D(x,z) \leq j(\mathrm{dist}(D,K_n)) \sup_{x\in D} G_D\mathbf{1}(x) < \infty,$$

$u_k^{(n)}$ is bounded. This means that we can apply Theorem 2.3.6(i) so that for $n=1$ the problem (2.27) has a solution $u_{1,k}\in C(D)\cap L^\infty(D)$ such that $0\leq u_{1,k}\leq u_k^{(1)}$. Note that since $\lambda_1\leq\lambda_2$, $u_{1,k}$ is also a subsolution to the problem (2.27) for $n=2$ such that (2.14) holds for every open subset $A\subset D$, that is for a.e. $x\in A$

$$\begin{aligned} u_{1,k}(x) &= G_A f_{u_{1,k}}(x) + G_A\widetilde{f}_k(x) + \tilde{P}_A(u_{1,k}\mathbf{1}_D)(x) + P_A\lambda_1(x) \\ &\leq G_A f_{u_{1,k}}(x) + G_A\widetilde{f}_k(x) + \tilde{P}_A(u_{1,k}\mathbf{1}_D)(x) + P_A\lambda_2(x). \end{aligned}$$

Since $u_{1,k}\leq u_k^{(1)}\leq u_k^{(2)}$, again by Theorem 2.3.6(i), there exists a solution $u_{2,k}\in C(D)\cap L^\infty(D)$ to the problem (2.27) with λ_2 on D^c, such that $u_{1,k}\leq u_{2,k}\leq u_k^{(2)}$. By iterating this procedure, we obtain an increasing sequence $(u_{n,k})_{n\in\mathbb{N}}$ of solutions to problems (2.27) for different $n\in\mathbb{N}$. Moreover, the sequence $(u_{n,k})_{n\in\mathbb{N}}$ is dominated by the function u_k^0 associated with the linear problem

$$\begin{aligned} \phi(-\Delta)u_k^0(x) &= \widetilde{f}_k(x) && \text{in } D \\ u_k^0 &= \lambda && \text{in } D^c \\ W_D u_k^0 &= 0 && \text{on } \partial D. \end{aligned}$$

Hence, the pointwise limit $\lim_{n\to\infty} u_{n,k} = u_k$ is well defined in D. We will now show that u_k is a weak dual solution to the problem

$$\begin{aligned}
\phi(-\Delta)u(x) &= f(x,u(x)) + \widetilde{f}_k(x) && \text{in } D\\
u &= \lambda && \text{in } D^c \\
W_D u &= 0 && \text{on } \partial D.
\end{aligned} \tag{2.28}$$

Take any $\psi \in C_c^\infty(D)$, $\psi \geq 0$. Then by Fatou's lemma and the continuity of the function f in the second variable, we get that

$$\begin{aligned}
-\int_D f(x,u_k(x))G_D\psi(x)dx &\leq -\limsup_{n\to\infty}\int_D f(x,u_{n,k}(x))G_D\psi(x)dx\\
&= -\limsup_{n\to\infty}\int_D u_{n,k}(x)\psi(x)dx + \int_D \widetilde{f}_k(x)G_D\psi(x)dx\\
&\quad + \int_D P_D\lambda(x)\psi(x)dx\\
&= -\int_D u_k(x)\psi(x)dx + \int_D \widetilde{f}_k(x)G_D\psi(x)dx + \int_D P_D\lambda(x)\psi(x)dx,
\end{aligned}$$

where we used the monotone convergence theorem in the last line. The inequality above implies that u_k is a weak dual subsolution to (2.28). To show that u_k is also a supersolution to the same problem, set $D' = \operatorname{supp}\psi \subset\subset D$ and build a sequence $(D_l)_{l\in\mathbb{N}}$ of sets with Lipschitz boundaries such that $D' \subset\subset D_l \subset\subset D$ and $D_l \uparrow D$. Obviously, $\psi \in C_c^\infty(D_l)$, and both $G_{D_l}\psi \uparrow G_D\psi$ and $P_{D_l}\lambda \uparrow P_D\lambda$ pointwise in D. Also, notice that $u_k^0 = G_D\widetilde{f}_k + P_D\lambda$ is continuous, hence locally bounded. Furthermore, in D_l we have

$$|f(x,u_{n,k}(x))|G_{D_l}\psi(x) \leq C\rho(x)G_{D_l}\psi(x),$$

where $C := \max_{y\in D_l}\Lambda(u_k^0(y)) < \infty$, and $\rho G_{D_l}\psi \in L^1(D)$ since $\int_D \rho G_{D_l}\psi = \int_D \psi G_{D_l}\rho \leq \int_D \psi G_D\rho < \infty$. By using the dominated convergence theorem in the first equality and

Lemma 2.3.3 in the second, we have

$$\begin{aligned}
\int_{D_l}[f(x,u_k(x))+\widetilde{f}_k(x)]G_{D_l}\psi(x)dx &= \lim_{n\to\infty}\int_{D_l}[f(x,u_{n,k}(x))+\widetilde{f}_k(x)]G_{D_l}\psi(x)dx \\
&= \lim_{n\to\infty}\left(\int_{D_l}u_{n,k}(x)\psi(x)dx-\int_{D_l}P_{D_l}u_{n,k}(x)\psi(x)dx-\int_{D_l}P_{D_l}\lambda_n(x)\psi(x)dx\right) \\
&\le \lim_{n\to\infty}\left(\int_{D_l}u_{n,k}(x)\psi(x)dx-\int_{D_l}P_{D_l}\lambda_n(x)\psi(x)dx\right) \\
&= \int_{D_l}u_k(x)\psi(x)dx+\int_{D_l}P_{D_l}\lambda(x)\psi(x)dx.
\end{aligned}$$

Letting $l\to\infty$ we obtain

$$\int_D[f(x,u_k(x))+\widetilde{f}_k(x)]G_D\psi(x)dx \le \int_D u_k(x)\psi(x)dx+\int_D P_D\lambda(x)\psi(x)dx,$$

which proves that u_k is a supersolution, and therefore the solution to (2.28). Notice that for $\mu\equiv 0$ we have $\widetilde{f}_k\equiv 0$ so we have found a solution to the problem (2.10) under the assumption (i).

Suppose that we have a function Λ with properties as in the assumption (ii) of this theorem. With the Arzelà-Ascoli theorem we will now find a suitable subsequence of $(u_k)_k$ that converges to a function u that is a solution to the problem (2.10). To this end first notice that u_k is given by

$$\begin{aligned}
u_k(x) &= \int_D G_D(x,y)[f(y,u_k(y))+\widetilde{f}_k(y)]dy+\int_{D^c}P_D(x,y)\lambda(dy) \\
&= \int_D G_D(x,y)f(y,u_k(y))dy+G_D\widetilde{f}_k(x)+P_D\lambda(x). \qquad (2.29)
\end{aligned}$$

Since f is non-positive, $u_k\le g=P_D\lambda+M_D\mu$ so we have the pointwise boundedness of the family $(u_k)_k$. Since $G_D\widetilde{f}_k$ increases to the continuous function $M_D\mu$, by Dini's theorem the convergence is locally uniform so the usual 3ε-argument gives equicontinuity of the family $(G_D\widetilde{f}_k)_k$ at every point $x\in D$. Also, $P_D\lambda$ is continuous in D so it remains to analyse the first term. We have

$$\begin{aligned}
&\left|\int_D G_D(x,y)f(y,u_k(y))dy-\int_D G_D(z,y)f(y,u_k(y))dy\right| \\
&\le \int_D|G_D(x,y)-G_D(z,y)|\rho(y)\Lambda(u_k(y))dy \\
&\le \int_D|G_D(x,y)-G_D(z,y)|\rho(y)\Lambda(g(y))dy.
\end{aligned}$$

Equicontinuity of the first term in (2.29) now follows from Lemma 2.2.2. Now by Arzelà-Ascoli theorem we extract a subsequence $(u_{k_l})_l$ which converges pointwise to a continuous function u. Without loss of generality, assume that $u_k \to u$. It remains to prove that u is a weak solution to (2.10), i.e., for every $\psi \in C_c^\infty(D)$

$$\int_D u(x)\psi(x)dx = \int_D f(x,u(x))G_D\psi(x)dx + \int_D P_D\lambda(x)\psi(x)dx + \int_D M_D\mu(x)\psi(x)dx. \tag{2.30}$$

We know that u_k satisfies

$$\int_D u_k(x)\psi(x)dx = \int_D f(x,u_k(x))G_D\psi(x)dx + \int_D P_D\lambda(x)\psi(x)dx + \int_D G_D\widetilde{f}_k(x)\psi(x)dx. \tag{2.31}$$

Since $u_k \to u$ pointwise and $u_k \le g$, by the dominated convergence theorem the left-hand side of (2.31) converges to the left-hand side of (2.30). Furthermore, by the monotone convergence theorem the last term of (2.31) converges to the last term of (2.30). To show the convergence of the first term on the right-hand side, note that

$$|f(x,u_k(x))G_D\psi(x)| \le c_1\rho(x)\Lambda(g(x))G_D\mathbf{1}(x).$$

Now the assumption (ii) implies boundedness in $L^1(D)$, so the convergence follows from the dominated convergence theorem. Hence, u is a solution to the problem (2.10). Uniqueness follows from Proposition 2.3.5. ■

Remark 2.3.11. (i) Note that the condition $\rho\Lambda(g)G_D\mathbf{1} \in L^1(D)$ from Theorem 2.3.10 is weaker than the condition $G_D(\rho\Lambda(2\overline{g})) \in C_0(D)$ from Corollary 2.3.8.

(ii)Recall that if D is regular then $q \in \mathscr{J}$ if and only if $G_D|q| \in C_0(D)$. Hence, if we assume that D is regular in Theorem 2.3.6 then we can equivalently assume $\rho \in \mathscr{J}$ and $\rho\Lambda(h) \in \mathscr{J}$ instead of $G_D\rho \in C_0(D)$ and $G_D(\rho\Lambda(h)) \in C_0(D)$, respectively. Obviously, a similar argument applies to Corollary 2.3.8 and Theorem 2.3.10.

2.4. Auxiliary results in bounded $C^{1,1}$ open sets

2.4.1. The renewal function

We start this section by introducing a function which plays a prominent role in studying the boundary behaviour in $C^{1,1}$ open sets.

Let $Z = (Z_t)_{t\geq 0}$ be a one-dimensional subordinate Brownian motion with the characteristic exponent $\phi(\theta^2)$, $\theta \in \mathbb{R}$. We can think of Z as one of the components of the process X. Let $M_t := \sup_{0\leq s\leq t} Z_s$ be the supremum process of Z and let $L = (L_t)_{t\geq 0}$ be the local time of $M_t - Z_t$ at zero. We refer the readers to [9, Chapter VI] for details. The inverse local time $L_t^{-1} := \inf\{s > 0 : L_s > t\}$ is called the ascending ladder time process of Z. Define the ascending ladder height process $H = (H_t)_{t\geq 0}$ of Z by $H_t := M_{L_t^{-1}} = Z_{L_t^{-1}}$ if $L_t^{-1} < \infty$ and $H_t = \infty$ otherwise. The renewal function of the process H is defined as

$$V(t) := \int_0^\infty \mathbb{P}(H_s \leq t)\,ds, \quad t \in \mathbb{R}.$$

Then $V(t) = 0$ for $t < 0$, $V(0) = 0$, $V(\infty) = \infty$, and V is strictly increasing. The importance of the renewal function V lies in the fact that $V_{|(0,\infty)}$ is harmonic with respect to the killed process $Z^{(0,\infty)}$. This fact was for the first time used in [56] in order to obtain the precise rate of decay of harmonic functions of d-dimensional subordinate Brownian motion.

In the case of the isotropic α-stable process, it holds that $V(t) = t^{\alpha/2}$. In general, the function V is not known explicitly, but under the weak scaling condition **(WSC)** it is known, see e.g. [56], that there is a constant $C = C(R_0) \geq 1$ such that

$$C^{-1}\phi(t^{-2})^{-1/2} \leq V(t) \leq C\phi(t^{-2})^{-1/2}, \quad 0 < t < R_0. \tag{2.32}$$

For more general results, covering also the case $R_0 = \infty$, see [63, Theorem 4.4 and Remark 4.7].

Note that (2.32) and weak scaling (1.3) of ϕ imply that for all $R_1 \geq 1$ there are constants $0 < \widetilde{a}_1 \leq \widetilde{a}_2$ depending on R_1 such that

$$\widetilde{a}_1 \left(\frac{t}{s}\right)^{\delta_1} \leq \frac{V(t)}{V(s)} \leq \widetilde{a}_2 \left(\frac{t}{s}\right)^{\delta_2}, \quad 0 < s \leq t \leq R_1. \tag{2.33}$$

2.4.2. Estimates in $C^{1,1}$ open set

Recall that an open set D in $\mathbb{R}^d$ $(d \geq 2)$ is said to be a $C^{1,1}$ open set if there exist a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$ function $\psi = \psi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\psi(0) = 0$, $\nabla\psi(0) = (0,\dots,0)$, $\|\nabla\psi\|_\infty \leq \Lambda$, $|\nabla\psi(x) - \nabla\psi(z)| \leq \Lambda|x-z|$, and an orthonormal coordinate system CS_z: $y = (y_1,\cdots,y_{d-1},y_d) := (\widetilde{y}, y_d)$ with origin at z such that

$$B(z,R)\cap D = \{y = (\widetilde{y}, y_d) \in B(0,R) \text{ in } CS_z : y_d > \psi(\widetilde{y})\}.$$

The pair (R,Λ) is called the characteristics of the $C^{1,1}$ open set D. We remark that in some literature, the $C^{1,1}$ open set defined above is called a uniform $C^{1,1}$ open set since (R,Λ) is universal for all $z \in \partial D$.

From now until the end of this section let D be a bounded open $C^{1,1}$ set. It is well known that all boundary points of a $C^{1,1}$ open set are regular and accessible. Thus, $\partial_M D = \partial D$. Recall that $\delta_D(x)$ denotes the distance of the point $x \in D$ to the boundary ∂D, while $\delta_{D^c}(z)$ denotes the distance of $z \in \overline{D}^c$ to ∂D.

Under the weak scaling condition **(WSC)** the following sharp two-sided estimates of the Green function, Martin kernel and the Poisson kernel are known. The comparability constant depends on the constants in (1.3) and the diameter of D. We give the estimates in terms of the renewal function V:

$$G_D(x,y) \asymp \left(1\wedge\frac{V(\delta_D(x))}{V(|x-y|)}\right)\left(1\wedge\frac{V(\delta_D(y))}{V(|x-y|)}\right)\frac{V(|x-y|)^2}{|x-y|^d}, \quad x,y\in D, \tag{2.34}$$

$$M_D(x,z) \asymp \frac{V(\delta_D(x))}{|x-z|^d}, \quad x\in D, z\in\partial D, \tag{2.35}$$

$$P_D(x,z) \asymp \frac{V(\delta_D(x))}{V(\delta_{D^c}(z))(1+V(\delta_{D^c}(z)))}\frac{1}{|x-z|^d}, \quad x\in D, z\in\overline{D}^c. \tag{2.36}$$

For (2.34) see [30, Theorem 7.3(iv)], (2.35) follows immediately from (1.59) and (2.34), while (2.36) is proved in [46, Theorem 1.3]. We will also need sharp two-sided estimates of the killing function

$$\kappa_D(x) := \int_{D^c} j(|y-x|)\,dy, \quad x\in D. \tag{2.37}$$

It holds that

$$\kappa_D(x) \asymp V(\delta_D(x))^{-2}, \qquad x\in D. \tag{2.38}$$

The upper bound is straightforward and valid in any open set D, while the lower bound holds in open sets satisfying the outer cone condition, see e.g. [58, proof of Lemma 5.7].

2.4.3. Green and Poisson potentials

In this subsection we state two results which should be of independent interest. The first one gives sharp two-sided estimates of the Green potential of the function $x \mapsto U(\delta_D(x))$ for a function $U : (0,\infty) \to [0,\infty)$ satisfying certain assumptions. The estimates are given in terms of the function U and the renewal function V. A similar result was shown in [4, Theorem 3.4]. Since our proof is modelled after and is very similar to the one in [4], we defer the proof to Appendix. The second result is a sort of a counterpart of the first one and gives sharp two sided estimates of the Poisson potential of the function $z \mapsto \widetilde{U}(\delta_{D^c}(z))$ for a function $\widetilde{U} : (0,\infty) \to [0,\infty)$. The proof of this second result is simpler and will be also given in Appendix.

To be more precise, let $U : (0,\infty) \to [0,\infty)$ be a function satisfying the following conditions:

(U1) Integrability condition: It holds that

$$\int_0^1 U(t)V(t)\,dt < \infty; \tag{2.39}$$

(U2) Almost non-increasing condition: There exists $C > 0$ such that

$$U(t) \leq CU(s), \quad 0 < s \leq t \leq 1; \tag{2.40}$$

(U3) Reverse doubling condition: There exists $C > 0$ such that

$$U(t) \leq CU(2t), \quad t \in (0,1); \tag{2.41}$$

(U4) Boundedness away from zero: U is bounded from above on $[c,\infty)$ for each $c > 0$.

Note that if $U(t) = t^{-\beta}$, $\beta \in \mathbb{R}$, satisfies (2.39), then it satisfies **(U1)**-**(U4)**. In particular, if the process X is isotropic α-stable, then (2.39) (hence **(U1)**-**(U4)**) is equivalent to $-\beta + \alpha/2 > -1$.

Proposition 2.4.1. Assume that a function $U:(0,\infty)\to[0,\infty)$ satisfies conditions (**U1**)-(**U4**). Then

$$G_D(U(\delta_D))(x) \asymp \frac{V(\delta_D(x))}{\delta_D(x)}\int_0^{\delta_D(x)} U(t)V(t)\,dt + V(\delta_D(x))\int_{\delta_D(x)}^{\mathrm{diam}(D)} \frac{U(t)V(t)}{t}\,dt\,. \tag{2.42}$$

Morover, if U is positive and bounded on every bounded subset of $(0,\infty)$, then

$$G_D(U(\delta_D))(x) \asymp V(\delta_D(x)).$$

The asymptotic behaviour of $G_D(U(\delta_D))$ is given by the largest term that appears in (2.42). In this generality, this is not easy to determine (but see [4, Theorem 3.4]). It will follow from the proof that $G_D(U(\delta_D))<\infty$ if and only if (2.39) holds true. Clearly, if $f:D\to[0,\infty)$ is such that $f(x)\asymp U(\delta_D(x))$, then $G_Df(x)$ is asymptotically equal to the right-hand side of (2.42).

Proposition 2.4.2. Let $g:\overline{D}^c\to[0,\infty)$ be such that

$$g(y)\asymp \widetilde{U}(\delta_{D^c}(y)),\ y\in\overline{D}^c, \tag{2.43}$$

holds for some function $\widetilde{U}:(0,\infty)\to[0,\infty)$. Assume that $\widetilde{U}$ is bounded on every compact subset of $(0,\infty)$ and satisfies

$$\int_0^1 \frac{\widetilde{U}(t)}{V(t)}dt + \int_1^\infty \frac{\widetilde{U}(t)}{V(t)^2 t}dt < \infty. \tag{2.44}$$

Then

$$P_Dg(x)\asymp V(\delta_D(x))\int_0^{\mathrm{diam}(D)} \frac{\widetilde{U}(t)}{V(t)(\delta_D(x)+t)}dt,\ x\in D, \tag{2.45}$$

and

$$P_Dg(x)\lesssim \frac{V(\delta_D(x))}{\delta_D(x)},\ x\in D. \tag{2.46}$$

Remark 2.4.3. In the case of the fractional Laplacian and the power function $\widetilde{U}(t)=t^{-\beta}$, condition (2.44) becomes $-\alpha<\beta<1-\alpha/2$. Further, it is easy to see that for $-\beta<\alpha/2$, the integral in (2.45) is comparable to $\delta_D(x)^{-\beta-\alpha/2}$, in the case $\beta=-\alpha/2$ it is comparable to $\log(1/\delta_D(x))$, while for $-\beta>\alpha/2$ it is comparable to a constant. We conclude that for $g(y)=\delta_{D^c}(y)^{-\beta}$

$$P_Dg(x)\asymp \begin{cases} \delta_D(x)^{-\beta}, & -\alpha<\beta<-\alpha/2,\\ \delta_D(x)^{\alpha/2}\log(1/\delta_D(x)), & \beta=-\alpha/2,\\ \delta_D(x)^{\alpha/2}, & -\alpha/2<\beta<1-\alpha/2. \end{cases}$$

2.4.4. Boundary estimates of harmonic functions

Let σ denote the $(d-1)$-dimensional Hausdorff measure on ∂D. It follows immediately from (2.35) and the estimate

$$\int_{\partial D} \frac{\sigma(dz)}{|x-z|^d} \asymp \frac{1}{\delta_D(x)}, \quad x \in D,$$

that

$$M_D\sigma(x) = \int_{\partial D} M_D(x,z)\sigma(dz) \asymp \frac{V(\delta_D(x))}{\delta_D(x)}, \quad x \in D. \tag{2.47}$$

The following result appears as [17, Theorem 4.2] for the fractional Laplacian.

Proposition 2.4.4. Let $h \in L^1(\partial D, \sigma)$ and let $\mu(d\zeta) = h(\zeta)\sigma(d\zeta)$. If h is continuous at $z \in \partial D$, then

$$\lim_{x\to z, x\in D} \frac{M_D\mu(x)}{M_D\sigma(x)} = h(z). \tag{2.48}$$

Since the proof is essentially the same as the proof of [17, Theorem 4.2], we omit it. Proposition 2.4.4 has the following two consequences. Assume that h is non-negative, continuous in D, not identically equal to zero, and set $\mu(d\zeta) = h(\zeta)\sigma(d\zeta)$. Then since both $M_D\mu$ and $M_D\sigma$ are continuous and D is bounded, we first conclude that there exists $C = C(h) > 0$ such that

$$M_D\mu(x) \le CM_D\sigma(x), \quad x \in D.$$

Secondly, there exist $z \in \partial D$, $\varepsilon > 0$, and $C = C(h) > 0$ such that

$$M_D\mu(x) \ge CM_D\sigma(x), \quad x \in D \cap B(z,\varepsilon).$$

Together with (2.47), these last two estimates imply that there is a constant $C = C(h) > 1$ such that

$$M_D\mu(x) \le C\frac{V(\delta_D(x))}{\delta_D(x)}, \quad x \in D, \tag{2.49}$$

$$M_D\mu(x) \ge C^{-1}\frac{V(\delta_D(x))}{\delta_D(x)}, \quad x \in D \cap B(z,\varepsilon). \tag{2.50}$$

2.4.5. Kato class revisited

In this subsection we give a sufficient condition for a function of the distance to the boundary to be in the Kato class $\mathscr{J}$. First, note that by (1.22) and (2.37), we have that

$$\sup_{x\in D} G_D\kappa_D(x) \le 1. \tag{2.51}$$

Recall from (2.38) that $\kappa_D(x) \asymp V(\delta_D(x))^{-2}$. The first part of the following result is an analogue of [20, Lemma 1.26].

Lemma 2.4.5. Let $f:(0,\infty)\to[0,\infty)$ be bounded on $(0,M]$ for every $M>0$, and $\lim_{t\to\infty} f(t)/t = 0$.

(a) Let D be a bounded open set, $h>0$ a locally bounded function on D such that $h\to\infty$ at ∂D and

$$\sup_{x\in D}\int_D G_D(x,y)h(y)\,dy < \infty. \tag{2.52}$$

Then $f\circ h \in \mathscr{J}$.

(b) Let D be a bounded $C^{1,1}$ open set. Then $x\mapsto f(V(\delta_D(x))^{-2})$ is in the Kato class $\mathscr{J}$.

(c) Let D be a bounded $C^{1,1}$ open set and let $U:(0,\infty)\to[0,\infty)$ satisfy condition (U4). If

$$\lim_{s\to 0} U(s)V(s)^2 = 0, \tag{2.53}$$

then $x\mapsto U(\delta_D(x))$ is in the Kato class $\mathscr{J}$.

Proof. (a) We will take advantage of the equivalence of (i) and (ii) of [69, Theorem 16.8]. Denote $c := \sup_{x\in D}\int_D G_D(x,y)h(y)\,dy$ and let $\eta>0$. There is $t_0>0$ such that $f(t)/t < \frac{\eta}{c}$ for every $t\ge t_0$. Also, since $h\to\infty$ at ∂D there is $F\subset\subset D$ such that $h>t_0$ on $D\setminus F$ and since h is locally bounded we have $M:=\sup_F h<\infty$. Hence

$$\begin{aligned}\sup_{x\in D}\int_D G_D(x,y)f(h(y))dy &\le \sup_{x\in D}\int_F G_D(x,y)f(h(y))dy + \sup_{x\in D}\int_{D\setminus F} G_D(x,y)f(h(y))dy\\ &\le (\sup_{(0,M]} f)\,\sup_{x\in D}\mathbb{E}_x[\tau_D] + \eta < \infty,\end{aligned}$$

i.e. we have property (a) of (ii) in [69, Theorem 16.8]. Note that $\mathbf{1}\in\mathscr{J}$ since D is bounded so there is $w_\eta\in L^1_+(D)$ and $\delta>0$ such that for all $B\subset D$ with $\int_B w_\eta<\delta$ we have $\sup_{x\in D}\int_B G_D(x,y)dy < \frac{\eta}{\sup_{(0,M]} f}$. Hence, for all such B it holds that

$$\begin{aligned}\sup_{x\in D}\int_B G_D(x,y)f(h(y))dy &\le \sup_{x\in D}\int_{B\cap F} G_D(x,y)f(h(y))dy + \sup_{x\in D}\int_{B\setminus F} G_D(x,y)f(h(y))dy\\ &\le (\sup_{(0,M]} f)\left(\sup_{x\in D}\int_B G_D(x,y)dy\right) + \eta \le 2\eta.\end{aligned}$$

Since η was arbitrary we have (b) of (ii) in [69, Theorem 16.8.], i.e. $f\circ h\in\mathscr{J}$.

(b) This follows immediately from (a) by using (2.51) and (2.38).

(c) Define $f(t) := U(V^{-1}(t^{-1/2}))$ so that $f(V(t)^{-2}) = U(t)$. By the assumption on U, the function f is locally bounded. Moreover, by using the substitution $t = V(s)^{-2}$ and the assumption (2.53), we get

$$\lim_{t\to\infty} \frac{f(t)}{t} = \lim_{s\to 0} \frac{f(V(s)^{-2})}{V(s)^{-2}} = \lim_{s\to 0} U(s)V(s)^2 = 0.$$

The claim now follows from (b). ■

2.4.6. Generalized normal derivative, modified Martin kernel and equivalent formulation of the weak dual solution

We now invoke the powerful recent result from [47] on boundary regularity of the solution to the equation

$$\begin{aligned} \phi(-\Delta)u(x) &= \psi(x) && \text{in } D \\ u &= 0 && \text{in } D^c \end{aligned}$$

where ψ is a bounded continuous function on D. It is proved in [47, Theorem 1.2] (see also [47, Theorem 3.10]), that $u = G_D\psi$ is the (viscosity) solution to the above equation, $u/V(\delta_D) \in C^\gamma(D)$, and

$$\left\| \frac{u}{V(\delta_D)} \right\|_{C^\gamma(D)} \le C\|\psi\|_\infty,$$

for some constants $\gamma > 0$ and $C > 0$ depending only on d, D and ϕ. Here $C^\gamma(D)$ is the space of γ-Hölder continuous functions on D with the corresponding Hölder norm. It follows that $u/V(\delta_D)$ can be continuously extended to $\overline{D}$. In particular, for any bounded and continuous function $\psi : D \to \mathbb{R}$ and for every $z \in \partial D$, there exists a finite limit

$$\frac{d}{dV}(G_D\psi)(z) := \lim_{y\to z, y\in D} \frac{G_D\psi(y)}{V(\delta_D(y))}. \tag{2.54}$$

We can think of $d(G_D\psi)/dV$ as the generalized normal derivative of the function $G_D\psi$ – instead of the distance function δ_D we use $V(\delta_D)$.

If ψ is non-negative and has compact support, then $G_D\psi$ is regular harmonic in $D \setminus \mathrm{supp}(\psi)$. By [54, Theorem 1.1], for any $x \in D$, there exists a finite limit

$$\lim_{y\to z, y\in D} \frac{G_D\psi(y)}{G_D(x,y)}.$$

Combining with (2.54), we see that for every $x \in D$ and every $z \in \partial D$, there exists

$$K_D(x,z) := \lim_{y\to z, y\in D} \frac{G_D(x,y)}{V(\delta_D(y))}. \tag{2.55}$$

We call $K_D(x,z)$ the modified Martin kernel, because given $x_0 \in D$, we have that

$$\frac{K_D(x,z)}{K_D(x_0,z)} = \lim_{y\to z, y\in D} \frac{\frac{G_D(x,y)}{V(\delta_D(y))}}{\frac{G_D(x_0,y)}{V(\delta_D(y))}} = \lim_{y\to z} \frac{G_D(x,y)}{G_D(x_0,y)} = M_D(x,z). \tag{2.56}$$

Lemma 2.4.6. Let D be a bounded open set and let $\psi : D \to \mathbb{R}$ be a bounded function with compact support and set $u = G_D\psi$. Then

$$\frac{d}{dV}u(z) = \int_D K_D(y,z)\psi(y)\,dy.$$

Proof. Let $2\varepsilon = \mathrm{dist}(\mathrm{supp}(\psi), \partial D)$, $z \in \partial D$, and $x \in D$ such that $|x-z| < \varepsilon$. By using (2.34), we get that for $y \in \mathrm{supp}(\psi)$,

$$\frac{G_D(x,y)}{V(\delta_D(x))} \le c\frac{V(|x-y|)}{|x-y|^d} \le c\frac{V(\mathrm{diam}(D))}{\varepsilon^d}.$$

Thus we can use the bounded convergence theorem to conclude from (2.55) that

$$\frac{d}{dV}u(z) = \lim_{x\to z, x\in D} \frac{G_D\psi(x)}{V(\delta_D(x))} = \lim_{x\to z, x\in D} \int_D \frac{G_D(x,y)}{V(\delta_D(x))}\psi(y)\,dy = \int_D K_D(y,z)\psi(y)\,dy.$$

■

Recall the weak dual formulation (2.11) of the semilinear problem (2.10). We will now rewrite the last two integrals in (2.11) in the case when D is a $C^{1,1}$ bounded domain. Let $\psi \in C_c^\infty(D)$ and set $\varphi = G_D\psi$. First, by using (1.25) we see that

$$\int_{D^c}\int_D P_D(x,z)\psi(x)\,dx\,\lambda(dz) = -\int_{D^c} \phi(-\Delta)\varphi(z)\,\lambda(dz).$$

Further, for $\mu \in \mathscr{M}(\partial D)$, let $\widetilde{\mu}(dz) := K_D(x_0,z)^{-1}\mu(dz)$. By Lemma 2.4.6 and (2.56)

$$\int_{\partial D}\int_D M_D(x,z)\psi(x)\,dx\,\mu(dz) = \int_{\partial D}\int_D K_D(x,z)\psi(x)\,dx\,\widetilde{\mu}(dz) = \int_{\partial D}\frac{d}{dV}\varphi(z)\widetilde{\mu}(dz).$$

Since $\psi = \phi(-\Delta)\varphi$, we see that the function u is a weak dual solution to the problem (2.10) if and only if

$$\int_D u(x)\phi(-\Delta)\varphi(x)\,dx = \int_D f(x,u(x))\varphi(x)\,dx - \int_{\overline{D}^c} \phi(-\Delta)\varphi(z)\,\lambda(dz) + \int_{\partial D}\frac{d}{dV}\varphi(z)\widetilde{\mu}(dz).$$

This formulation of a solution to the problem (2.10) (in bounded $C^{1,1}$ open sets) can be found in [1] in the case of the fractional Laplacian.

2.4.7. Another boundary operator

Following [1, Subsection 1.2] (see also [26, Eq. (2.2) and Appendix B]) we now introduce another boundary operator. For a measure $\mu \in \mathscr{M}(\partial D)$ set $K_D\mu(x) := \int_{\partial D} K_D(x,z)\mu(dz)$, $x \in D$. Note that by Remark 2.4.7(i), $K_D(x_0,\cdot)$ is continuous on ∂D. In the context of the Proposition 2.4.4, let $\mu(d\zeta) := f(\zeta)\sigma(d\zeta)$, $\widetilde{\mu}(d\zeta) := K_D(x_0,\zeta)\mu(d\zeta)$ and $\nu(d\zeta) := K_D(x_0,\zeta)\sigma(d\zeta)$. Then

$$\lim_{D\ni x\to z} \frac{K_D\mu(x)}{K_D\sigma(x)} = \lim_{D\ni x\to z} \frac{M_D\widetilde{\mu}(x)}{M_D\nu(x)} = \frac{K_D(x_0,z)f(z)}{K_D(x_0,z)} = f(z). \tag{2.57}$$

For $u: D \to \mathbb{R}$ and $z \in \partial D$, let

$$E_D u(z) := \lim_{D\ni x\to z} \frac{u(x)}{K_D\sigma(x)},$$

whenever the limit exists and is finite.

Remark 2.4.7. We will need the following elementary calculations several times below.

(i) Let $u: D \to \mathbb{R}$ be a function and assume that for every $z \in \partial D$ there exists a finite limit

$$\widetilde{u}(z) := \lim_{D\ni x\to z} u(x). \tag{2.58}$$

Then, by applying the usual 2ε-argument, it follows that $\widetilde{u}: \partial D \to \mathbb{R}$ is continuous.

(ii) Assume further that D is bounded and $\widetilde{u}(z) = 0$ for all $z \in \partial D$. Then convergence in (2.58) is uniform in the sense that for every $\varepsilon > 0$ there exists a compact set $F \subset D$ such that $|u(x)| < \varepsilon$ for all $x \in D \setminus F$. Indeed, due to compactness of ∂D we easily find a finite cover $V := \cup_{i=1}^n B(z_i, r_i)$, $z_i \in \partial D$, of ∂D such that $|u| \le \varepsilon$ on $D \cap V$.

Proposition 2.4.8. Let $u: D \to \mathbb{R}$. If $E_D u(z)$ exists for every $z \in \partial D$, then $W_D u$ exists and

$$W_D u(dz) = E_D u(z) K_D(x_0,z)\sigma(dz).$$

Proof. Assume that $E_D u(z)$ exists for every $z \in \partial D$. By Remark 2.4.7(i), $E_D u$ is continuous on ∂D. Let $\nu(dz) = K_D(x_0,z)\sigma(dz)$, $\mu(dz) = E_D u(z)\nu(dz)$ and

$$v(x) := M_D\mu(x) = \int_{\partial D} M_D(x,z) E_D u(z)\nu(dz) = \int_{\partial D} K_D(x,z) E_D u(z)\sigma(dz).$$

By (2.57), for every $z \in \partial D$,

$$\lim_{x \to z, x \in D} \frac{v(x)}{K_D\sigma(x)} = E_D u(z),$$

hence $E_D v = E_D u$, so that $\lim_{x\to z, x\in D}(u(x) - v(x))/K_D\sigma(x) = 0$ for every $z \in \partial D$. By Remark 2.4.7(ii), this implies that for every $\varepsilon > 0$ there exists a compact set $F \subset D$, such that

$$\frac{|u(x) - v(x)|}{K_D\sigma(x)} < \varepsilon, \quad \text{for all } x \in D \setminus F.$$

Since $K_D\sigma$ is a non-negative harmonic function, the same proof as [20, Lemma 1.16] gives that $W_D(u - v) = 0$. Notice that the set of functions on D for which W_D is defined is a vector space and W_D is linear on that space. We conclude that $W_D u$ exists and $W_D u = W_D v + W_D(u - v) = W_D v = W_D(M_D\mu) = \mu$ by (1.79). ■

2.5. THE SEMILINEAR PROBLEM IN BOUNDED $C^{1,1}$ OPEN SET

2.5.1. Corollary 2.3.8 revisited

Recall that in Corollary 2.3.8 we assumed that the function $f : D \times \mathbb{R} \to \mathbb{R}$ satisfies **(F)** with Λ non-decreasing and that $G_D\rho \in C_0(D)$ and $G_D(\rho\Lambda(2\overline{g})) \in C_0(D)$, where $\overline{g} = P_D|\lambda| + M_D|\mu|$. We give sufficient conditions for these assumptions in case of a bounded $C^{1,1}$ open set. We will additionally assume that $\rho(x) = W(\delta_D(x))$ for a function $W : (0,\infty) \to [0,\infty)$ and that Λ satisfies the following doubling condition: There exists $C \geq 1$ such that

$$\Lambda(2t) \leq C\Lambda(t), \quad t > 0. \tag{2.59}$$

This implies that for all $c_1 > 1$ there exists $c_2 = c_2(C, c_1)$ such that

$$\Lambda(c_1 t) \leq c_2\Lambda(t), \qquad t > 0,$$

which can be rewritten as follows: For every $\widetilde{c}_1 \in (0,1)$, there exists $\widetilde{c}_2 > 0$ such that

$$\Lambda(\widetilde{c}_1 t) \geq \widetilde{c}_2\Lambda(t), \quad t > 0. \tag{2.60}$$

Secondly, assume that

$$\overline{g}(x) \lesssim \frac{V(\delta_D(x))}{\delta_D(x)}, \quad x \in D.$$

By (2.46) and (2.49), this will be the case provided $\mu(dz) = h(z)\sigma(dz)$ for a continuous function $h : \partial D \to \mathbb{R}$, and $\lambda(dy) = g(y)dy$ with $|g(y)| \lesssim \widetilde{U}(\delta_{D^c}(y))$ where $\widetilde{U}$ is non-negative, bounded on compact subsets of $(0,\infty)$ and satisfies (2.44). Then we have

$$\rho(x)\Lambda(2\overline{g})(x) \leq cW(\delta_D(x))\Lambda\left(\frac{V(\delta_D(x))}{\delta_D(x)}\right), \quad x \in D,$$

for some $c > 0$. By using Lemma 2.4.5(c), we see that $G_D(\rho\Lambda(2\overline{g})) \in C_0(D)$ if

$$\lim_{t\to 0} W(t)\Lambda\left(\frac{V(t)}{t}\right)V(t)^2 = 0,$$

while $G_D\rho \in C_0(D)$ if $\lim_{t\to 0} W(t)V(t)^2 = 0$.

In the case of the fractional Laplacian, $W(t) = t^{-\beta}$ and $\Lambda(t) = t^p$, these two conditions become $\beta + p(1-\alpha/2) < \alpha$.

2.5.2. Theorem 2.3.10 in bounded $C^{1,1}$ open set

In this subsection we revisit Theorem 2.3.10(ii) in case of a bounded $C^{1,1}$ open set D. Recall that the assumptions of that theorem were that $f : D\times\mathbb{R}\to(-\infty,0]$ satisfies **(F)** with $G_D\rho\in C_0(D)$, $f(x,0)=0$ and the function Λ is non-decreasing. As in the previous subsection, we will additionally assume that $\rho(x)=W(\delta_D(x))$ for a function $W:(0,\infty)\to[0,\infty)$ and that Λ satisfies the doubling condition (2.59).

Proposition 2.5.1. Let $D\subset\mathbb{R}^d$ be a bounded $C^{1,1}$ open set. Let $f:D\times\mathbb{R}\to(-\infty,0]$ be a function that satisfies **(F)** with $\rho(x)=W(\delta_D(x))$, where $W:(0,\infty)\to[0,\infty)$ is bounded away from zero, and such that Λ is a non-decreasing function satisfying the doubling condition (2.59). Assume that

$$\lim_{t\to 0} W(t)V(t)^2=0. \tag{2.61}$$

Let $\lambda(dy)=\widetilde{U}(\delta_{D^c}(y))dy$ where $\widetilde{U}:(0,\infty)\to[0,\infty)$ is bounded on every compact subset of $(0,\infty)$ and satisfies (2.44), and let $\mu(dz)=h(z)\sigma(dz)$ where $h:\partial D\to[0,\infty)$ is continuous and not identically equal to zero. If for some $\eta>0$

$$\int_0^{\eta} W(t)V(t)\Lambda\left(\frac{V(t)}{t}\right)dt<\infty, \tag{2.62}$$

then the semilinear problem (2.10) has a non-negative weak dual solution $u\in L^1(D)\cap C(D)$.

Proof. We first note that the assumption (2.61) implies by Lemma 2.4.5(c) that $\rho=W(\delta_D)\in\mathscr{J}$, and thus by Subsection 2.2, $\rho\in C_0(D)$. Hence, in order to see that the semilinear problem (2.10) has a non-negative solution it suffices to check that $\rho\Lambda(g)G_D\mathbf{1}\in L^1(D)$ where $g=P_D\lambda+M_D\mu$. By (2.46) and (2.49) there exists a constant $c_1>0$ such that

$$g(x)\le c_1\frac{V(\delta_D(x))}{\delta_D(x)},\quad x\in D.$$

Together with (2.59) this implies that

$$\Lambda(g(x))\le\Lambda\left(c_1\frac{V(\delta_D(x))}{\delta_D(x)}\right)\le c_2\Lambda\left(\frac{V(\delta_D(x))}{\delta_D(x)}\right),\quad x\in D,$$

for some $c_2>0$. Therefore, there is $c_3>0$ such that

$$\rho(x)\Lambda(g(x))G_D\mathbf{1}(x)\le c_3W(\delta_D(x))\Lambda\left(\frac{V(\delta_D(x))}{\delta_D(x)}\right)V(\delta_D(x)),\quad x\in D. \tag{2.63}$$

By using boundedness of $W(\delta_D)\Lambda\left(\frac{V(\delta_D)}{\delta_D}\right)V(\delta_D)$ inside D and the co-area formula near the boundary of D with the assumption (2.62) we see that

$$\int_D W(\delta_D(x))\Lambda\left(\frac{V(\delta_D(x))}{\delta_D(x)}\right)V(\delta_D(x))dx < \infty.$$

Now it follows from (2.63) that $\rho\Lambda(g)G_D\mathbf{1} \in L^1(D)$. ■

Remark 2.5.2. (a) Proposition 2.5.1 allows a partial converse. Assume that $f(x,t) = -W(\delta_D(x))\Lambda(|t|)$ where $\Lambda : (0,\infty) \to (0,\infty)$ is a non-decreasing and unbounded function satisfying (2.59) Assume further that there exists $\eta_0 > 0$ such that for all $\eta \in (0,\eta_0]1$

$$\int_0^\eta W(t)V(t)\Lambda\left(\frac{V(t)}{t}\right)dt = +\infty. \tag{2.64}$$

Let $\mu(d\zeta) = h(\zeta)\sigma(d\zeta)$ with non-negative continuous h, $h \neq 0$. Then the semilinear problem (2.10) does not have a non-negative weak dual solution $u \in L^1(D)$ such that $E_D u$ is well defined. To show this, suppose that there exists a non-negative u that solves (2.10). Then $u(x) = G_D f_u(x) + P_D\lambda(x) + M_D\mu(x)$ a.e. Since by assumption, $E_D u$ exists, by Proposition 4.8 $W_D u$ also exists and $W_D u(d\zeta) = E_D u(\zeta)K_D(x_0,\zeta)\sigma(d\zeta)$. On the other hand, since $u = G_D f_u + P_D\lambda + M_D\mu$, we have by (2.15) that $W_D u = W_D(M_D\mu) = \mu$. Since $\mu(d\zeta) = h(\zeta)\sigma(d\zeta)$, we get

$$E_D u(\zeta) = \frac{h(\zeta)}{K_D(x_0,\zeta)} \quad \sigma(d\zeta) - \text{a.e.}$$

Choose $z \in \partial D$ such that $E_D u(z) = h(z)/K_D(x_0,z) > 0$. Since

$$E_D u(z) = \lim_{x\to z, x\in D} \frac{u(x)}{K_D\sigma(x)},$$

there exists $\varepsilon > 0$ such that

$$u(x) \geq \frac{1}{2}E_D u(z)K_D\sigma(x) = \frac{1}{2}\frac{h(z)}{K_D(x_0,z)}K_D\sigma(x) = c_1 K_D\sigma(x), \qquad \text{for all } x \in D\cap B(z,\varepsilon),$$

where $c_1 = c_1(z,h) > 0$. By using (2.34) and (2.55) to get the same estimate of $K_D(x,z)$ as the one of $M_D(x,z)$ in (2.35), we see in the same way as for (2.47) that

$$K_D\sigma(x) \asymp \frac{V(\delta_D(x))}{\delta_D(x)}, \quad x \in D.$$

This implies that there exists $c_2 = c_2(z,h) > 0$ such that

$$u(x) \geq c_2\frac{V(\delta_D(x))}{\delta_D(x)}, \qquad \text{for all } x \in D\cap B(z,\varepsilon).$$

Therefore, by using (2.60) this implies that for some $c_3 > 0$

$$\Lambda(u(y)) \geq \Lambda\left(c_2 \frac{V(\delta_D(y))}{\delta_D(y)}\right) \geq c_3 \Lambda\left(\frac{V(\delta_D(y))}{\delta_D(y)}\right), \quad \text{for all } y \in D \cap B(z,\varepsilon).$$

Choose $x \in D$ so that $\delta_D(x) \asymp |x-y| \asymp 1$ whenever $y \in D \cap B(z,\varepsilon)$. By (2.34), there exists $c_4 > 0$ such that $G_D(x,y) \geq c_4 V(\delta_D(y))$. Hence,

$$G_D f_u(x) = \int_D G_D(x,y) f(y,u(y))dy \leq -c_3 c_4 \int_{D\cap B(z,\varepsilon)} V(\delta_D(y)) W(\delta_D(y)) \Lambda\left(\frac{V(\delta_D(y))}{\delta_D(y)}\right) dy.$$

By use of the co-area formula it follows that the last integral is equal to some constant multiplied by

$$\int_0^\varepsilon V(t) W(t) \Lambda\left(\frac{V(t)}{t}\right) dt.$$

By (2.64) it follows that $G_D f_u(x) = -\infty$ for points x in some open subset of D. This is a contradiction with $G_D f_u > -\infty$ a.e. which follows from $u \geq 0$, $P_D\lambda < \infty$ and $M_D\mu < \infty$.

(b) Note that the power function $\Lambda(t) = t^p$ is increasing and satisfies the doubling condition (2.59). Assume that $W(t) = t^{-\beta}$ and the underlying process is an isotropic α-stable process (so that $V(t) = t^{\alpha/2}$). Then (2.61) reads $\beta < \alpha$, while the integral criterion (2.62) is equivalent to $\beta + p(1-\alpha/2) < 1+\alpha/2$. In case $f(x,t) = -t^p$, we see that the problem (2.10) has a non-negative solution u if $p < (2+\alpha)/(2-\alpha)$, while in case $p \geq (2+\alpha)/(2-\alpha)$ a non-negative solution u such that $E_D u$ is well defined does not exist.

2.5.3. Extending Corollary 2.3.8 to a wider class of non-negative nonlinearities

Our next goal is to extend the results of Corollary 2.3.8 for non-negative nonlinearities f. Unlike Theorem 2.3.10, this approach relies heavily on the estimates of Green and Poisson potentials in bounded $C^{1,1}$ domains.

Theorem 2.5.3. Let $f : D \times \mathbb{R} \to [0,\infty)$ be a function, non-decreasing in the second variable, satisfying (F), with $\rho = W(\delta_D)$ for some function $W : (0,\infty) \to [0,\infty)$, Λ non-decreasing and satisfying the doubling condition (2.59). Let $\lambda \in \mathscr{M}(D^c)$ be a non-negative measure which is absolutely continuous with respect to the Lebesgue measure with density $\widetilde{U}(\delta_{D^c})$, where $\widetilde{U}:(0,\infty)\to[0,\infty)$ is a function bounded on compact subsets of $(0,\infty)$ satisfying (2.44). Let $h:\partial D\to[0,\infty)$ be a continuous function and let $\mu(d\zeta)=h(\zeta)\sigma(d\zeta)$ be a measure on ∂D. Suppose that one of the following conditions hold:

(i) the function $t\mapsto W(t)\Lambda\left(\frac{V(t)}{t}\right)$, $t>0$, satisfies the conditions **(U1)-(U4)**;

(ii) $h\equiv 0$ and the function $W\Lambda(\widetilde{U})$ satisfies the conditions **(U1)-(U4)**. Morover assume that

$$\int_0^{\mathrm{diam}(D)}\frac{\widetilde{U}(t)}{V(t)(s+t)}\,dt\lesssim\frac{\widetilde{U}(s)}{V(s)}, \tag{2.65}$$

$$\begin{aligned}&\int_0^s W(t)V(t)\Lambda(\widetilde{U}(t))dt\lesssim\frac{s\widetilde{U}(s)}{V(s)},\\ &\int_s^{\mathrm{diam}(D)}\frac{W(t)V(t)\Lambda(\widetilde{U}(t))}{t}\,dt\lesssim\frac{\widetilde{U}(s)}{V(s)},\end{aligned} \tag{2.66}$$

where the constants do not depend on $0<s\le\frac{\mathrm{diam}(D)}{2}$.

Then there exists a constant $m_1>0$ such that for every $m\in[0,m_1]$ the semilinear problem

$$\begin{aligned}-Lu(x)&=mf(x,u(x)) && \text{in } D\\ u&=\lambda && \text{in } \overline{D}^c\\ W_Du&=\mu && \text{on } \partial D\end{aligned} \tag{2.67}$$

has a non-negative weak dual solution $u\in L^1(D)$.

Proof. First we prove the theorem under assumption (i). Since f is non-negative, the function $u_0=P_D\lambda+M_D\mu$ is a subsolution to (2.67). Recall from (2.46) and (2.49) that there exists a constant $c_1>0$ such that

$$u_0(x)\le c_1\frac{V(\delta_D(x))}{\delta_D(x)},\ x\in D.$$

Next we construct a supersolution $\overline{u}$ for (2.67) of the form

$$\overline{u}(x)=c_2\frac{V(\delta_D(x))}{\delta_D(x)},$$

i.e. find a constant $c_2>c_1$ such that

$$\overline{u}(x)\ge mG_Df_{\overline{u}}(x)+u_0(x),\ x\in D, \tag{2.68}$$

for m small enough. To be exact, we show that for every $c_2 > c_1$ there exists $m_1 > 0$ such that (2.68) holds for every $m \in [0, m_1]$. Fix $c_2 > c_1$. First note that by **(F)** and the doubling property (2.59) for Λ we have

$$f\left(x, c_2 \frac{V(\delta_D(x))}{\delta_D(x)}\right) \le W(\delta_D(x))\Lambda\left(c_2 \frac{V(\delta_D(x))}{\delta_D(x)}\right) \le c_3 W(\delta_D(x))\Lambda\left(\frac{V(\delta_D(x))}{\delta_D(x)}\right)$$

for some constant $c_3 > 0$. Now by Proposition 2.4.1 there exists $c_4 > 0$ such that

$$G_D f_{\overline{u}}(x) \le c_3 G_D \left[W(\delta_D)\Lambda\left(\frac{V(\delta_D)}{\delta_D}\right)\right](x) \le c_4 \frac{V(\delta_D(x))}{\delta_D(x)}.$$

By choosing $m_1 = \frac{c_2 - c_1}{c_4}$ we get that for every $m \le m_1$

$$mG_D f_{\overline{u}}(x) + u_0(x) \le (mc_4 + c_1)\frac{V(\delta_D(x))}{\delta_D(x)} = \frac{mc_4 + c_1}{c_2}\overline{u}(x) \le \overline{u}(x).$$

Now we can apply the classical iteration scheme in the following way: For $k \in \mathbb{N}$ let u_k be the weak L^1 solution to the linear problem

$$\begin{aligned} \phi(-\Delta)u_k(x) &= mf(x, u_{k-1}(x)) && \text{in } D \\ u_k &= \lambda && \text{in } \overline{D}^c \\ W_D u_k &= \mu && \text{on } \partial D. \end{aligned}$$

The constructed sequence $(u_k)_k$ is non-decreasing and dominated by $\overline{u}$. To see this, take $x \in D$. Since f is non-negative, we have that

$$u_1(x) - u_0(x) = mG_D f_{u_0}(x) \ge 0.$$

Furthermore, since f is non-decreasing in the second variable and $u_0 \le \overline{u}$, we have that

$$u_1(x) = mG_D f_{u_0}(x) + u_0(x) \le mG_D f_{\overline{u}}(x) + u_0(x) \le \overline{u}(x).$$

Assume now that $u_{k-1}(x) \le u_k(x) \le \overline{u}(x)$ for some $k \in \mathbb{N}$. This implies that $f_{u_{k-1}}(x) \le f_{u_k}(x) \le f_{\overline{u}}(x)$, so

$$u_{k+1}(x) - u_k(x) = mG_D f_{u_k}(x) - mG_D f_{u_{k-1}}(x) \ge 0$$

and

$$u_{k+1}(x) = mG_D f_{u_k}(x) + u_0(x) \le mG_D f_{\overline{u}}(x) + u_0(x) \le \overline{u}(x).$$

The claim now follows by induction.

Therefore, we can define a pointwise limit $u := \lim_{k\to\infty} u_k$ which, by the monotone convergence theorem and the continuity of f in the second variable, satisfies

$$\begin{aligned} u(x) &= \lim_{k\to\infty} \int_D f(y,u_k(y))G_D(x,y)dy + u_0(x) \\ &= \int_D \lim_{k\to\infty} f(y,u_k(y))G_D(x,y)dy + u_0(x) \\ &= \int_D f(y,u(y))G_D(x,y)dy + u_0(x), \end{aligned}$$

i.e. u is a weak L^1 solution to (2.67).

Next, consider the proof of the theorem under the assumptions (ii). Note that we only need to find a supersolution $\overline{u} \ge u_0 = P_D\lambda$ satisfying (2.68). The rest of the proof then follows from the proof of (i). Note first that (2.45) and (2.65) imply that there exists a constant $c_5 > 0$ such that

$$u_0(x) \le c_5 \widetilde{U}(\delta_D(x)),\ x \in D.$$

Therefore, in this case we fix a constant $c_6 > c_5$ and show that the function $\overline{u}$ of the form

$$\overline{u}(x) = c_6 \widetilde{U}(\delta_D(x)),$$

is indeed a supersolution to (2.67) for m small enough. As in the previous case, by **(F)** and the doubling property for Λ

$$f\left(x, c_6\widetilde{U}(\delta_D(x))\right) \le W(\delta_D(x))\Lambda\left(c_6\widetilde{U}(\delta_D(x))\right) \le c_7 W(\delta_D(x))\Lambda\left(\widetilde{U}(\delta_D(x))\right)$$

for some constant $c_7 > 0$. Now by Proposition 2.4.1 and (2.66) it follows that

$$G_D f_{\overline{u}}(x) \le c_7 G_D\left[W(\delta_D)\Lambda\left(\widetilde{U}(\delta_D(x))\right)\right](x) \le c_8 \widetilde{U}(\delta_D(x)).$$

By choosing $m_1 = \frac{c_6 - c_5}{c_8}$ we get that for every $m \le m_1$

$$mG_D f_{\overline{u}}(x) + u_0(x) \le (mc_8 + c_5)\widetilde{U}(\delta_D(x)) = \frac{mc_8 + c_5}{c_6}\overline{u}(x) \le \overline{u}(x).$$

■

Assume that functions W and Λ satisfy (2.62), W satisfies conditions **(U2)**-**(U4)**, and Λ is non-decreasing and satisfies the doubling condition (2.59). Then the function $U(t) =$

$W(t)\Lambda\left(\frac{V(t)}{t}\right)$ satisfies conditions **(U1)**-**(U4)**. Indeed, since W is almost non-increasing and Λ is non-decreasing it follows that

$$\frac{W(t)\Lambda\left(\frac{V(t)}{t}\right)}{W(s)\Lambda\left(\frac{V(s)}{s}\right)} \lesssim \frac{\Lambda\left(\frac{V(t)}{t}\right)}{\Lambda\left(\frac{V(s)}{s}\right)} \overset{(2.33)}{\lesssim} \frac{\Lambda\left(\widetilde{a}_2\frac{V(s)}{s}\right)}{\Lambda\left(\frac{V(s)}{s}\right)} \overset{(2.59)}{\lesssim} 1, \quad s<t\leq 1.$$

Furthermore, since W satisfies the reverse doubling condition (2.41) and Λ is non-decreasing, we have that

$$\frac{W(t)\Lambda\left(\frac{V(t)}{t}\right)}{W(2t)\Lambda\left(\frac{V(2t)}{2t}\right)} \lesssim \frac{\Lambda\left(\frac{V(t)}{t}\right)}{\Lambda\left(\frac{V(2t)}{2t}\right)} \overset{(2.33)}{\lesssim} \frac{\Lambda\left(\frac{V(t)}{t}\right)}{\Lambda\left(\widetilde{a}_1 2^{\delta_1-1}\frac{V(t)}{t}\right)} \overset{(2.59)}{\lesssim} 1, \quad t\in(0,1).$$

Finally, note that U is bounded away from zero, since both W and $t\mapsto\frac{V(t)}{t}$ satisfy **(U4)** and Λ is non-decreasing. Similarly, note that the function $U=W\Lambda(\widetilde{U})$ satisfies conditions **(U2)**-**(U4)** if we additionally assume that $\widetilde{U}$ satisfies **(U2)**-**(U4)**.

Remark 2.5.4. (i) Consider the isotropic α-stable case and take $\Lambda(t)=t^p$ and $W(t)=t^{-\beta_1}$ for some $p>0$ and $\beta_1\geq 0$, as in Remark 2.5.2. The function $U(t)=W(t)\Lambda\left(\frac{V(t)}{t}\right)$ satisfies conditions **(U1)**-**(U4)** if and only if $\beta_1+p(1-\alpha/2)<1+\alpha/2$. Hence, if $f(x,t)=t^p$, then Theorem 2.5.3 holds for $p<\frac{2+\alpha}{2-\alpha}$.

(ii) When $\widetilde{U}(t)=t^{-\beta_2}$, the function $W\Lambda(\widetilde{U})$ satisfies conditions **(U1)**-**(U4)** if and only if $\beta_1+p\beta_2<1+\alpha/2$. The condition (2.65) is satisfied for $\beta_2<1-\alpha/2$. When $\beta_1=0$ the conditions in (2.66) are satisfied when $\beta_2(p-1)\leq\alpha$. Since $\beta_2<1-\alpha/2$ we have that $\frac{\alpha}{\beta_2}+1<\frac{1+\alpha/2}{\beta_2}$, so Theorem 2.5.3 states that the solution exists for $p<\frac{\alpha}{\beta_2}+1$.

3. SEMILINEAR PROBLEM FOR $\phi(-\Delta|_D)$

Let $D \subset \mathbb{R}^d$, $d \geq 2$, be a bounded $C^{1,1}$ domain, $f : D \times \mathbb{R} \to \mathbb{R}$ a function, and ζ a signed measure on ∂D. In this chapter we study the semilinear problem

$$\begin{aligned} \phi(-\Delta|_D)u(x) &= f(x,u(x)) && \text{in } D \\ \frac{u}{P_D^\phi \sigma} &= \zeta && \text{on } \partial D, \end{aligned} \tag{3.1}$$

where $\phi : (0,\infty) \to (0,\infty)$ is a complete Bernstein function without drift satisfying the weak scaling condition at infinity - **(WSC)**. The boundary condition will be described below whereas the operator $\phi(-\Delta|_D)$ can be written in its spectral form as well as a principal value integral:

$$\phi(-\Delta|_D)u(x) = \sum_{j=1}^{\infty} \phi(\lambda_j)\widehat{u}_j\varphi_j = \text{P.V.}\int_D (u(x)-u(y))J_D(x,y)\,dy + \kappa(x)u(x), \quad x \in D.$$

Here $(\lambda_j, \varphi_j)_{j\in\mathbb{N}}$ are eigenpairs of the Dirichlet Laplacian in D, and the singular kernel J_D as well as the function κ are completely determined by the function ϕ. This said, $\phi(-\Delta|_D)$ is a non-local operator of elliptic type which in case $\phi(\lambda) = \lambda^{\alpha/2}$, $\alpha \in (0,2)$, is the spectral fractional Laplacian $(-\Delta_{|D})^{\alpha/2}$. The operator $-\phi(-\Delta|_D)$ can be also viewed as the infinitesimal generator of the subordinate killed Brownian motion, where the subordinator has ϕ as its Laplace exponent.

The notion of the boundary condition is a bit abstract but, at this point, let us say that it can be understood as a limit at the boundary of $u/P_D^\phi\sigma$ in the pointwise sense, or in the weak sense of (3.77), depending on the smoothness of the boundary datum, where $P_D^\phi\sigma$ is a reference function defined as the Poisson potential of the $d-1$ dimensional Hausdorff measure on ∂D.

Motivated by the recent article [3], see also the preprint [4], we consider solutions of (3.1) in the weak dual sense, see Definition 3.4.1, and we prove that the solutions have a special form of a sum of the Green and the Poisson potential, see Theorem 3.3.3.

In the past there have been just a few articles discussing the semilinear Dirichlet problem for the spectral fractional Laplacian, see [3, 36]. To the best of our knowledge the work in this thesis is the first one to study semilinear problems for spectral-type operators more general then the spectral fractional Laplacian.

In this setting a typical difference between local and non-local setting is experienced, i.e. even solutions of the linear Dirichlet problem can explode at the boundary whereas in the local setting this does not happen. To be more precise, there exists a harmonic function with respect to $\phi(-\Delta|_D)$ which explodes at the boundary, e.g. the reference function $P_D^\phi\sigma$ is such one for which we prove the explosion rate, see (3.61).

The main goal of this chapter is to generalize results from [3] where the semilinear problem was studied for the spectral fractional Laplacian, and to generalize results from Chapter 2 to a slightly different type of a non-local operator in the special case of $C^{1,1}$ bounded domain. To achieve this goal, we intensively use the potential-theoretic and analytic properties of the killed Brownian motion subordinated by a subordinator with the Laplace exponent ϕ, the process that gives $-\phi(-\Delta|_D)$ as its infinitesimal generator. Some of these properties are well known for a long time and belong to the general potential theory. However, some properties are pretty recently proved such as the sharp bounds for the potential kernel and the jumping kernel, the (boundary) Harnack principle, etc., see [58, 60].

Let us now describe the central results of this chapter which are given in Section 3.4. For the nonlinearity f in (3.1) in our results we assume that

(F). $f : D\times\mathbb{R}\to\mathbb{R}$ is continuous in the second variable, and there exist a locally bounded function $\rho : D\to[0,\infty]$ and a non-decreasing function $\Lambda : [0,\infty)\to[0,\infty)$ such that $|f(x,t)|\le \rho(x)\Lambda(|t|)$, $x\in D$, $t\in\mathbb{R}$.

First in Proposition 3.4.4 we prove Kato's inequality for $\phi(-\Delta|_D)$ using which we develop a method of sub- and supersolution for $\phi(-\Delta|_D)$ in Theorem 3.4.9. This theorem directly generalizes [3, Theorem 32] to our setting of more general non-local operators and also extends Theorem 2.3.6 to slightly different non-local operators. In Theorem 3.4.10 we prove existence of a solution when the nonlinearity f is non-positive and when the boundary measure ζ is non-negative. This theorem comes as a generalization of [3, Theorem 8] to our setting of more general non-local operators. Moreover, we consider a more general boundary condition which can also be a measure, whereas in [3, Theorem 8] only continuous functions where considered. The nonlinearity in our theorem is also slightly more general then the one in [3, Theorem 8]. A similar result in a different non-local setting can be found in Theorem 2.3.10. By the method of monotone iterations, in Theorem 3.4.14 we find a solution to the semilinear problem when both f and ζ are non-negative. Finally, for a signed f and a signed ζ, in Theorem 3.4.16 we find a solution by the technique used in [20, Theorem 2.4]. After each theorem, we give a comment on the existence (and non-existence) of a solution in the spectral fractional Laplacian case for the power-like nonlinearity f, see Remarks 3.4.13, 3.4.15 and 3.4.17.

Let us now give a short summary of the rest of the chapter. In Section 3.1 we introduce assumptions on ϕ and recall the known results on the Green kernel. We connect the operator $\phi(-\Delta|_D)$ to the subordinate killed Brownian motion as its infinitesimal generator, give a pointwise characterization of $\phi(-\Delta|_D)$, and study regularity of the Green potentials. The last part of the section deals with Poisson potentials and harmonic functions. In Proposition 3.1.19 we prove existence of the Poisson kernel as a normal derivative of the Green function and in Theorem 3.1.22 we prove an integral representation formula for non-negative harmonic functions for $\phi(-\Delta|_D)$. We finish the section by proving that harmonic functions are continuous, and by Theorem 3.1.26 in which we show that non-negative harmonic functions are those which satisfy the mean-value property with respect to the subordinate killed Brownian motion.

Section 3.2 deals with a boundary behaviour of potential integrals. Here we emphasize Theorem 3.2.6 which gives the boundary behaviour of the Green potentials. This theorem generalizes [36, Proposition 7] to our setting of more general non-local operators and more general functions. Furthermore, this theorem with Proposition 3.2.4 shows that in some cases the boundary condition (3.1) can be understood as a limit at the boundary in the pointwise sense. Finally, the section also contains Proposition 3.2.5 and Proposition 3.2.7 which show that the boundary condition in (3.1) can be viewed as a limit at the boundary in the weak sense.

Section 3.3 contains the basic properties of the linear Dirichlet problem where we prove that every weak solution to the Dirichlet problem is a sum of the Green and the Poisson potential, see Theorem 3.3.3.

Section 3.4 contains already described main results.

Lastly, in connection to this chapter, we refer to the following results in the Appendix. Firstly, we provide a proof of a Green function sharp estimate in our setting, see Lemma 4.3.1, modelled after [58, Theorem 3.1]. We also give a technical proof of Theorem 3.2.6 modelled after the proof of [4, Theorem 3.4], as well as prove Lemma 4.2.4, which is an additional and a bit lengthy calculation providing an interpretation of the boundary condition. In the final part of the Appendix we prove that the heat kernel of the killed Brownian motion upon exiting a $C^{1,\alpha}$ domain is differentiable up to the boundary - a fact that appears to be known but for which we could not find an exact reference.

3.1. Preliminaries

The setting of this chapter is slightly different from the one from Chapters 1 and 2. Here instead of the killed subordinate process we look at the subordinate killed process, i.e. we change the order of the subordination and the killing. These two processes are different which will be easily seen in this preliminary section.

Recall that by $(W_t)_{t\geq 0}$ we denoted the Brownian motion in $\mathbb{R}^d$, $d\geq 2$, with the characteristic exponent $\xi\mapsto|\xi|^2$, $\xi\in\mathbb{R}^d$. Let D be a non-empty open set, and $\tau_D:=\inf\{t>0: W_t\notin D\}$ the first exit time from the set D. We define the killed process W^D upon exiting the set D by

$$W_t^D := \begin{cases} W_t, & t<\tau_D,\\ \partial, & t\geq\tau_D,\end{cases}$$

where ∂ is an additional point added to $\mathbb{R}^d$ called the cemetery.

Recall that S was a subordinator independent of W with the Laplace exponent

$$\lambda\mapsto\phi(\lambda)=b\lambda+\int_0^\infty(1-e^{-\lambda t})\mu(dt). \tag{3.2}$$

The process $Y^D=((Y_t^D)_{t\geq 0},(\mathbb{P}_x)_{x\in D})$ defined by $Y_t^D:=(W^D)_{S_t}$ is called the subordinate killed Brownian motion. Here $\mathbb{P}_x$ denotes the probability under which the process Y starts from $x\in D$, and by $\mathbb{E}_x$ we denote the corresponding expectation.

3.1.1. Assumptions

The first assumption that we impose throughout the chapter concerns the set D. Although some results will be valid for general open sets, we always assume that D is a bounded $C^{1,1}$ domain.

Again, same as in Chapters 1 and 2 we assume:

(WSC). The function ϕ is a complete Bernstein function, i.e. the Lévy measure $\mu(dt)$ has a completely monotone density $\mu(t)$, and ϕ satisfies the following weak scaling condition at infinity: There exist $a_1, a_2 > 0$ and $\delta_1, \delta_2 \in (0,1)$ satisfying

$$a_1\lambda^{\delta_1}\phi(t) \le \phi(\lambda t) \le a_2\lambda^{\delta_2}\phi(t), \quad t \ge 1, \lambda \ge 1. \tag{3.3}$$

However, we do not assume that **(T)** holds.

Allow us to give some comments on the assumptions above. Since we assume **(WSC)**, the function $\phi^*(\lambda) := \frac{\lambda}{\phi(\lambda)}$ is a complete Bernstein function, too, see [71, Proposition 7.1], and ϕ^* is called the conjugate Bernstein function of ϕ. We easily see that (3.3) also holds for ϕ^* but with different constants a_1, a_2, and δ_1 and δ_2, thus **(WSC)** also holds for ϕ^*. By $\nu(dt) = \nu(t)dt$ we denote the Lévy measure of ϕ^*.

In what follows we discuss properties of ϕ and the same will hold for ϕ^* or, to be more precise, for the counterparts of the function ϕ^*. Recall that for ϕ, by **(WSC)**, it holds that

$$\phi'(\lambda) \asymp \frac{\phi(\lambda)}{\lambda}, \quad \lambda \ge 1, \tag{3.4}$$

see the comments below (1.4). We will use (3.4) many times through the chapter. The Lévy measure $\mu(dt)$ is infinite, see [71, p. 160], and the density $\mu(t)$ cannot decrease too fast, i.e. there is $c = c(\phi) > 1$ such that

$$\mu(t) \le c\mu(t+1), \quad t \ge 1,$$

see [56, Lemma 2.1]. Moreover, it holds that

$$\mu(t) \le (1-2e^{-1})^{-1}\frac{\phi'(t^{-1})}{t^2}, \; t > 0, \quad \text{and} \quad \mu(t) \ge c\,\frac{\phi(t^{-1})}{t^2}, \; 0 < t \le M, \tag{3.5}$$

for $M > 0$ and $c = c(\phi, M) > 0$, see [58, Eq. (2.13)] and [50, Proposition 3.3].

The potential measure U of the subordinator S, defined by $U(A) := \int_0^\infty \mathbb{P}(S_t \in A)dt$, $A \in \mathscr{B}(\mathbb{R})$, has a decreasing density $\mathfrak{u}$ which satisfies $\int_0^1 \mathfrak{u}(t)dt < \infty$, see [71, Theorem 11.3]. In addition, it holds that

$$\mathfrak{u}(t) \leq (1-2e^{-1})^{-1}\frac{\phi'(t^{-1})}{t^2\phi(t^{-1})^2}, \ t>0, \quad \text{and} \quad \mathfrak{u}(t) \geq c\frac{\phi'(t^{-1})}{t^2\phi(t^{-1})^2}, \ 0<t\leq M, \tag{3.6}$$

for $M>0$ and $c=c(\phi,M)>0$, see [58, Eq. (2.11)] and [50, Proposition 3.4]. The potential density of ϕ^* will be denoted by $\mathfrak{v}$.

Recall also that for a general Bernstein function from (3.2) a version of a global scaling condition holds

$$1\wedge\lambda \leq \frac{\phi(\lambda t)}{\phi(t)} \leq 1\vee\lambda, \quad \lambda>0, t>0. \tag{3.7}$$

In [58, 60] important aspects of the potential theory of the process Y^D were developed such as scale invariant Harnack principle and boundary Harnack principle. Our assumption **(WSC)** implies (A1)-(A4) but not (A5) (that is, our assumption **(T)**, which we do not assume in this chapter) from [58, 60], so each time we use a result from [58, 60] we will explain how the assumption (A5) (or in our notation **(T)**) can be avoided.

Again, in the chapter the case $d=1$ is excluded since it would require somewhat different potential theoretic methods.

3.1.2. Green function

With $p(t,x,y)$ we denoted the transition density of the Brownian motion W, i.e.

$$p(t,x,y) = (4\pi t)^{-d/2}e^{-\frac{|x-y|^2}{4t}}, \quad x,y\in\mathbb{R}^d, t>0. \tag{3.8}$$

Then the transition density of the killed Brownian motion W^D is given by

$$p_D(t,x,y) = p(t,x,y) - \mathbb{E}_x[p(t-\tau_D, W_{\tau_D}, y)\mathbf{1}_{\{\tau_D<t\}}], \quad x,y\in\mathbb{R}^d. \tag{3.9}$$

It is well known that $p_D(t,\cdot,\cdot)$ is symmetric and it seems that it is known that $p_D(\cdot,\cdot,\cdot) \in C^1((0,\infty)\times\overline{D}\times\overline{D})$ since D is a $C^{1,1}$ open domain. However, as we were unable to find an exact reference for the regularity up to the boundary of the transition density, we prove it in Appendix in Lemma 4.5.1. Furthermore, the following heat kernel estimate holds: There exist constants $T_0, c_1, c_2 > 0$ (dependent on D) such that for all $x, y \in D$ and $t \in (0, T_0]$ it holds that

$$\left[\frac{\delta_D(x)\delta_D(y)}{t} \wedge 1\right] \frac{1}{c_1 t^{d/2}} e^{-\frac{|x-y|^2}{c_2 t}} \le p_D(t,x,y) \le \left[\frac{\delta_D(x)\delta_D(y)}{t} \wedge 1\right] \frac{c_1}{t^{d/2}} e^{-\frac{c_2|x-y|^2}{t}}. \quad (3.10)$$

Moreover, the right hand side inequality in (3.10) holds for every $t > 0$. For the proofs see [78, Theorem 1.1] and [73, Theorem 3.1 & Theorem 3.8].

The semigroup $(P_t^D)_{t\ge 0}$ of the process W^D is given by

$$P_t^D f(x) = \int_D p_D(t,x,y) f(y) dy = \mathbb{E}_x[f(W_t); t < \tau_D] = \mathbb{E}_x[f(W_t^D)], \quad f \in L^\infty(D), \quad (3.11)$$

where $f(\partial) = 0$ for all Borel functions on D by convention. It is well known that the semigroup $(P_t^D)_{t\ge 0}$ is strongly Feller since D is $C^{1,1}$, i.e. $P_t^D(L^\infty) \subset C_b(D)$, and can be uniquely extended to a $L^2(D)$ semigroup. For details see e.g. [33, Chapter 2].

The potential kernel of W^D (or the Green function of W^D) is defined as

$$G_D^1(x,y) = \int_0^\infty p_D(t,x,y) dt, \quad x, y \in \mathbb{R}^d.$$

The kernel G_D^1 is symmetric, non-negative, finite off the diagonal and jointly continuous in the extended sense, see [33, Theorem 2.6], and it is the density of the mean occupation time for W^D, i.e. for $f \ge 0$ we have

$$\int_D G_D^1(x,y) f(y) dy = \mathbb{E}_x\left[\int_0^\infty f(W_t^D) dt\right], \quad x \in D.$$

The superscript 1 in G_D^1 will obtain its meaning in Lemma 3.1.1 and it also allows us to differentiate between G_D - the Green function in D of the subordinate Brownian motion, and the Green function of W^D.

By $(Q_t^D)_t$ we denote the $L^2(D)$ transition semigroup of Y^D. It is well known that for every $t > 0$ we have

$$Q_t^D f = \int_0^\infty P_s^D f \, \mathbb{P}(S_t \in ds), \quad f \in L^2(D),$$

see [71, Proposition 13.1], hence Q_t^D admits the density

$$q_D(t,x,y) = \int_0^\infty p_D(s,x,y) \, \mathbb{P}(S_t \in ds).$$

The semigroup $(Q_t^D)_t$ is also strongly Feller since $(P_t^D)_t$ is, see [14, Proposition V.3.3]. The process Y^D has the potential kernel (i.e. the Green function of Y^D) which is given by

$$G_D^{\phi}(x,y) = \int_0^\infty q_D(t,x,y)dt = \int_0^\infty p_D(t,x,y)\mathfrak{u}(t)dt, \quad x,y \in \mathbb{R}^d. \tag{3.12}$$

The kernel G_D^{ϕ} is symmetric, non-negative, and by the bound (3.10) finite off the diagonal. Moreover, G_D^{ϕ} is the density of the mean occupation time for Y^D, i.e. for $f \geq 0$ we have

$$\int_D G_D^{\phi}(x,y)f(y)dy = \mathbb{E}_x\left[\int_0^\infty f(Y_t^D)dt\right], \quad x \in D.$$

The closed form of G_D^{ϕ} is not known, but in [58, Theorem 3.1] the sharp estimate was obtained, i.e. we have

$$G_D^{\phi}(x,y) \asymp \left(\frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \wedge 1\right)\frac{1}{|x-y|^d\phi(|x-y|^{-2})}, \quad x,y \in D, \tag{3.13}$$

where the constant of comparability depends only on d, D and ϕ. We note that the usage of the transience assumption (A5) from [58] in [58, Theorem 3.1] can be avoided, see Lemma 4.3.1 in the Appendix for the details. Further, by using the upper bound of (3.10) and the bounds (3.13), we can repeat the proof of [58, Proposition 3.3] to get that G_D^{ϕ} is infinite on the diagonal and jointly continuous in extended sense in $D \times D$.

By the characterization of Bernstein functions the conjugate Bernstein function ϕ^* generates a subordinator $(T_t)_{t\geq 0}$, see [71, Chapter 5]. From the previous subsection it follows that $(T_t)_{t\geq 0}$ has a potential measure which also has the decreasing density which we denote by $V(dt) = \mathfrak{v}(t)dt$, see [71, Theorem 11.3 & Corollary 11.8]. We define the potential kernel generated by ϕ^* with

$$G_D^{\phi^*}(x,y) = \int_0^\infty p_D(t,x,y)\mathfrak{v}(t)dt, \quad x,y \in \mathbb{R}^d. \tag{3.14}$$

Since ϕ^* satisfies **(WSC)**, $G_D^{\phi^*}$ is also symmetric, finite off the diagonal, jointly continuous in extended sense $D \times D$ and satisfies the sharp bound (3.13) where ϕ is replaced by ϕ^*. Of course, the kernel $G_D^{\phi^*}$ can be viewed as the potential kernel of the subordinate killed Brownian motion $\big((W^D)_{T_t}\big)_{t\geq 0}$.

The kernels G_D^1, G_D^{ϕ} and $G_D^{\phi^*}$ are also connected by the following well-known factorization.

Lemma 3.1.1. For $x,y \in D$ it holds that

$$\int_D G_D^{\phi}(x,\xi)G_D^{\phi^*}(\xi,y)d\xi = G_D^1(x,y). \tag{3.15}$$

Proof. The claim follows from [71, Proposition 14.2(ii)] where we set $\gamma = \delta_y$. ■

3.1.3. Operator $\phi(-\Delta|_D)$

Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a Hilbert basis of $L^2(D)$ consisting of eigenfunctions of the Dirichlet Laplacian $-\Delta|_D$, associated to the eigenvalues λ_j, $j \in \mathbb{N}$, i.e. $\varphi_j \in H_0^1(D)\cap C^\infty(D)\cap C^{1,1}(\overline{D})$ and

$$-\Delta|_D\,\varphi_j = \lambda_j\varphi_j, \quad \text{in } D, \tag{3.16}$$

see [23, Theorem 9.31] and [40, Section 8.11]. Here (3.16) can be viewed in various equivalent ways, e.g. as a distributional or a pointwise relation. Also, $\Delta|_D$ in (3.16) can be viewed as the $L^2(D)$-infinitesimal generator of the semigroup $(P_t^D)_t$, i.e.

$$\Delta|_D u = \lim_{t\to 0}\frac{P_t^D u - u}{t}, \quad u \in \mathscr{D}(\Delta|_D),$$

where $\mathscr{D}(\Delta|_D) = H_0^1(D)$ is the domain of the generator $\Delta|_D$ and the limit is taken with respect to $L^2(D)$ norm. For the details, see [33, Chapter 2] and [23, Chapter 9]. Note that since φ_j is an eigenfunction, we have

$$P_t^D\varphi_j = e^{-\lambda_j t}\varphi_j, \tag{3.17}$$

see [70, Lemma 7.10]. Further, since we assume that D is $C^{1,1}$, it is well known that $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$, and by the Weyl's asymptotic law we have

$$\lambda_j \asymp j^{2/d}, \quad j \in \mathbb{N}. \tag{3.18}$$

Also, we choose the basis $\{\varphi_j\}_{j\in\mathbb{N}}$ such that $\varphi_1 > 0$ in D, see [23, Chapter 9]. Hence, another very important sharp estimate for φ_1 holds:

$$\varphi_1(x) \asymp \delta_D(x), \quad x \in D. \tag{3.19}$$

The interior estimate is trivial since φ_1 is smooth and positive. The boundary bound follows from Hopf's lemma, see e.g. [37, Hopf's lemma in Section 6.4.2].

Consider the Hilbert space

$$H_D(\phi) := \left\{ v = \sum_{j=1}^{\infty} \widehat{v}_j \varphi_j \in L^2(D) : \|v\|_{H_D(\phi)}^2 := \sum_{j=0}^{\infty} \phi(\lambda_j)^2 |\widehat{v}_j|^2 < \infty \right\}.$$

The spectral operator $\phi(-\Delta|_D) : H_D(\phi) \to L^2(D)$ is defined as

$$\phi(-\Delta|_D)u = \sum_{j=1}^{\infty} \phi(\lambda_j)\widehat{u}_j \varphi_j, \quad u \in H_D(\phi). \tag{3.20}$$

Note that $H_D(\phi) \hookrightarrow L^2(D)$ and we will show in the next proposition that $C_c^\infty(D) \subset H_D(\phi)$, see (3.27). Now it is obvious that $\phi(-\Delta|_D)$ is unbounded operator, densely defined in $L^2(D)$ and has the bounded inverse $[\phi(-\Delta|_D)]^{-1} : L^2(D) \to H_D(\phi)$ given by

$$[\phi(-\Delta|_D)]^{-1} u = \sum_{j=1}^{\infty} \frac{1}{\phi(\lambda_j)} \widehat{u}_j \varphi_j, \quad u \in L^2(D). \tag{3.21}$$

In the next proposition we prove that a potential relative to G_D^ϕ is the inverse of $\phi(-\Delta|_D)$. The proof is similar to [3, Lemma 9] but we give the complete proof for the reader's convenience since some elements of the proof will be important in what follows.

Proposition 3.1.2. Let $f \in L^2(D)$. For a.e. $x \in D$ it holds that $G_D^\phi(x,\cdot)f(\cdot) \in L^1(D)$ and

$$[\phi(-\Delta|_D)]^{-1} f(x) = \int_D G_D^\phi(x,y) f(y) dy. \tag{3.22}$$

Proof. First we prove (3.22) for $f = \varphi_1 \geq 0$. Fubini's theorem yields

$$\begin{aligned} \int_D G_D^\phi(x,y)\varphi_1(y)dy &= \int_0^\infty \mathfrak{u}(t) \int_D p_D(t,x,y)\varphi_1(y)dydt = \int_0^\infty e^{-\lambda_1 t}\varphi_1(x)\mathfrak{u}(t)dt \\ &= \frac{1}{\phi(\lambda_1)}\varphi_1(x) = [\phi(-\Delta|_D)]^{-1}\varphi_1(x), \quad \text{for a.e. } x \in D, \end{aligned} \tag{3.23}$$

where in the third equality we used (3.17) and in the last equality (3.21). By the elliptic regularity there exist constants $C = C(d,D)$ and $k = k(d)$ such that $\|\nabla\varphi_j\|_{L^\infty(D)} \leq (C\lambda_j)^k \|\varphi_j\|_{L^2(D)} = (C\lambda_j)^k$, see (4.43). Recall that $\varphi_j \in C^{1,1}(\overline{D})$ and that φ_j vanishes on the boundary so the mean value theorem implies

$$\left\| \frac{\varphi_j}{\delta_D} \right\|_{L^\infty(D)} \leq (C\lambda_j)^k. \tag{3.24}$$

Since $\varphi_1 \asymp \delta_D$, by the previous inequality, Fubini's theorem, and the same calculations as in (3.23), we have that (3.22) holds for every φ_j, $j \in \mathbb{N}$. By linearity the same is true for the linear span of $\{\varphi_j : j \in \mathbb{N}\}$.

Let

$$\mathbb{G}f(x) := \int_D G_D^\phi(x,y)f(y)dy, \tag{3.25}$$

for $f \in L^2(D)$ and $x \in D$ such that the integral exists. In what was proved, $\mathbb{G}f(x)$ is well defined for every $f \in \operatorname{span}\{\varphi_j : j \in \mathbb{N}\}$ and a.e. $x \in D$. Moreover, from $\mathbb{G}\varphi_j = \frac{1}{\phi(\lambda_j)}\varphi_j = [\phi(-\Delta|_D)]^{-1}\varphi_j$ it follows that for $f \in \operatorname{span}\{\varphi_j : j \in \mathbb{N}\}$ we have

$$\|\mathbb{G}f\|^2_{H_D(\phi)} = \|f\|^2_{L^2(D)}. \tag{3.26}$$

Hence, the map $f \mapsto \mathbb{G}f$ uniquely extends to a linear isometry from $L^2(D)$ to $H_D(\phi)$ which coincides with $[\phi(-\Delta|_D)]^{-1}$. Further, a consequence of (3.23) is that $G_D^\phi(x,\cdot) \in L^1(D)$ for a.e. $x \in D$ since by Fubini's theorem

$$\int_D \left(\int_D G_D^\phi(x,y)dy\right)\varphi_1(x)dx = \frac{1}{\phi(\lambda_1)}\int_D \varphi_1(y)dy < \infty.$$

Next we prove that (3.22) holds a.e. in D for $f = \psi = \sum_{j=1}^\infty \widehat{\psi}_j\varphi_j \in C_c^\infty(D)$. Take the approximating sequence $f_n = \sum_{j=1}^n \widehat{\psi}_j\varphi_j$, $n \in \mathbb{N}$, and note that $\mathbb{G}f_n = [\phi(-\Delta|_D)]^{-1} f_n \to [\phi(-\Delta|_D)]^{-1} f = \mathbb{G}f$ in $L^2(D)$ since $f_n \to f$ in $L^2(D)$. Moreover, by integrating by parts $m \in \mathbb{N}$ times we get

$$\widehat{\psi}_j = \int_D \psi(x)\varphi_j(x)dx = \frac{(-1)^m}{\lambda_j^m}\int_D \Delta^m\psi(x)\varphi_j(x)dx,$$

which implies

$$|\widehat{\psi}_j| \le \frac{\|\Delta^m\psi\|_{L^2(D)}}{\lambda_j^m} =: C(m,\psi)\frac{1}{\lambda_j^m}. \tag{3.27}$$

Hence, by using (3.18), (3.24), and (3.27) for large enough $m \in \mathbb{N}$, it follows that f_n converges uniformly in D to $f = \psi$. This implies that $\mathbb{G}f_n = \int_D G_D^\phi(\cdot,y)f_n(y)dy \to \int_D G_D^\phi(\cdot,y)f(y)dy$ a.e. in D since $G_D^\phi(x,\cdot) \in L^1(D)$ for a.e. $x \in D$. Thus, by uniqueness of the limit $\mathbb{G}f = [\phi(-\Delta|_D)]^{-1} f = \int_D G_D^\phi(\cdot,y)f(y)dy$ a.e. in D.

Take now $f \in L^2(D)$, and let $(f_n)_n \subset C_c^\infty(D)$ which converges to f in $L^2(D)$. Hence, $\mathbb{G}f_n = [\phi(-\Delta|_D)]^{-1} f_n \to [\phi(-\Delta|_D)]^{-1} f = \mathbb{G}f$ in $L^2(D)$. On the other hand,

$$\begin{aligned}\int_D \left|\int_D G_D^\phi(x,y)(f_n(y)-f(y))dy\right|\varphi_1(x)dx &\le \frac{1}{\phi(\lambda_1)}\int_D \varphi_1(y)|f_n(y)-f(y)|dy \\ &\le \frac{1}{\phi(\lambda_1)}||f_n - f||_{L^2(D)} \to 0,\end{aligned}$$

which shows that $G_D^\phi(\cdot,y)f(y) \in L^1(D)$ a.e. in D and by taking the subsequence we get

$$\mathbb{G}f_n = \int_D G_D^\phi(\cdot,y)f_n(y)dy \to \int_D G_D^\phi(\cdot,y)f(y)dy \quad \text{a.e. in } D,$$

thus $[\phi(-\Delta|_D)]^{-1} f = \int_D G_D^\phi(\cdot,y)f(y)dy$ a.e. in D. ■

In what follows, for the operator $\mathbb{G}$ from the proof of the previous lemma we will write

$$G_D^\phi f(x) := \int_D G_D^\phi(x,y)f(y)dy = \mathbb{G}f(x), \quad x \in D. \tag{3.28}$$

Remark 3.1.3. Proposition 3.1.2 implies that $G_D^\phi\big(L^2(D)\big) = H_D(\phi)$ and that

$$\phi(-\Delta|_D)(G_D^\phi f) = f, \quad f \in L^2(D).$$

By the general theory of semigroups, this means that $-\phi(-\Delta|_D)$ defined by (3.20) is the infinitesimal generator of the semigroup $(Q_t^D)_t$ and that $H_\phi(D)$ is the domain of $-\phi(-\Delta|_D)$, see e.g. [67,77]. In particular, $H_0^1(D) = \mathscr{D}(\Delta|_D) \subset \mathscr{D}\big(-\phi(-\Delta|_D)\big) = H_\phi(D)$ by [71, Theorem 13.6].

For sufficiently regular functions $\phi(-\Delta|_D)u$ can be expressed pointwisely. At this point we only consider $u \in C^{1,1}(D) \cap H_0^1(D)$ but later on in Proposition 3.1.15 we will prove the pointwise representation of $\phi(-\Delta|_D)$ for $u \in C^{1,1}(D) \cap H_\phi(D)$.

Lemma 3.1.4. Let $u \in C^{1,1}(D) \cap H_0^1(D)$. Then for a.e. $x \in D$

$$\phi(-\Delta|_D)u(x) = \text{P.V.}\int\limits_D [u(x)-u(y)]J_D(x,y)dy + \kappa(x)u(x), \tag{3.29}$$

where

$$J_D(x,y) := \int_0^\infty p_D(t,x,y)\mu(t)dt, \quad \kappa(x) := \int_0^\infty \left(1 - \int_D p_D(t,x,y)dy\right)\mu(t)dt.$$

In particular, (3.29) holds for $u \in C_c^\infty(D)$.

Remark 3.1.5. The function J_D is called the jumping density and the function κ is called the killing function of the process Y^D. Obviously, J_D is non-negative and symmetric. It is also finite off the diagonal and satisfies $\int_D \big(1 \wedge |x-y|^2\big)J_D(x,y)dy < \infty$ since the following estimate holds

$$J_D(x,y) \asymp \left(\frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \wedge 1\right) \frac{\phi(|x-y|^{-2})}{|x-y|^d}, \quad x,y \in D. \tag{3.30}$$

Here the constant of comparability depends only on d, D and ϕ and the proof of (3.30) is essentially the same as the proof of (3.13). By applying comments given for the proof of [58, Proposition 3.5] and using similar manipulations as in the proof of Lemma 4.3.1 to avoid using (A5) from [58], we easily obtain (3.30), so we skip the proof.

The killing function κ is continuous and $\kappa \in L^1(D, \delta_D(x)dx)$. Indeed, since the semigroup P_t^D is strongly Feller, $1 - P_t^D\mathbf{1}(x) = \mathbb{P}_x(\tau_D \le t)$ is continuous in x. Further, for $\varepsilon > 0$ such that $\varepsilon < 2\delta_D(x)$ it holds that $\mathbb{P}_x(\tau_D \le t) \le \mathbb{P}_x(\tau_{B(x,\varepsilon)} \le t) = \mathbb{P}_0(\tau_{B(0,1)} \le \frac{t}{\varepsilon^2}) \le c_1(\varepsilon)(1 \wedge t)$, where the last inequality follows by e.g. [45, Theorem 1]. Now the dominated convergence theorem yields the continuity of κ. Finally, $\int_D \kappa(x)\varphi_1(x)dx = \phi(\lambda_1)\int_D \varphi_1(x)dx$ by (3.17), so (3.19) yields $\kappa \in L^1(D, \delta_D(x)dx)$.

Proof of Lemma 3.1.4. It is known that for all $u \in H_0^1(D)$ it holds that

$$\phi(-\Delta|_D)u = \int_0^\infty \left(u - P_t^D u\right)\mu(t)dt \tag{3.31}$$

see [71, Theorem 13.6], since $H_0^1(D) \subset H_\phi(D)$ by Remark 3.1.3. The rest of the proof is dedicated to showing that the right hand sides of (3.31) and (3.29) are equal.

Let $u \in C^{1,1}(D) \cap H_0^1(D)$ and $x \in D$. First we show that the principal value integral in (3.29) is well defined. Indeed, fix $\delta > 0$ such that $\delta < (1 \wedge \delta_D(x)/4)$ and let $\varepsilon > 0$ such that $\varepsilon < \delta$. We have

$$\begin{aligned}
&\int_{D\setminus B(x,\varepsilon)} \big(u(x) - u(y)\big)J_D(x,y)dy \\
&\qquad = \int_{D\setminus B(x,\varepsilon)} \big(u(x) - u(y) + \nabla u(x)\cdot(y-x)\mathbf{1}_{B(x,\delta)}(y)\big)J_D(x,y)dy \\
&\qquad\qquad - \int_{B(x,\delta)\setminus B(x,\varepsilon)} \nabla u(x)\cdot(y-x)J_D(x,y)dy \\
&\qquad = I_1 - I_2.
\end{aligned}$$

By a $C^{1,1}$ version of Taylor's theorem we have

$$|u(x) - u(y) + \nabla u(x)\cdot(y-x)\mathbf{1}_{B(x,\delta)}(y)| \le c_1\left(1 \wedge |x-y|^2\right), \tag{3.32}$$

where $c_1 > 0$ depends on δ and $\|u\|_{C^{1,1}(B(x,\delta_D(x)/2))}$. Hence, the integral I_1 is finite and converges as $\varepsilon \to 0$ by the dominated convergence theorem.

For the second integral, by Fubini's theorem and (3.9), we have

$$
\begin{aligned}
I_2 = &\int_0^\infty \int_{B(x,\delta)\setminus B(x,\varepsilon)} \nabla u(x)\cdot (y-x) p(t,x,y)dy\mu(t)dt \\
&\qquad - \int_0^\infty \int_{B(x,\delta)\setminus B(x,\varepsilon)} \nabla u(x)\cdot (y-x)\mathbb{E}_x[p(t-\tau_D, W_{\tau_D}, y)\mathbf{1}_{\{\tau_D<t\}}]dy\,\mu(t)dt \\
=:& J_1 - J_2.
\end{aligned}
$$

The integral J_1 is zero for all $\varepsilon<\delta$ since the kernel $p(t,x,y)$ is symmetric in y around x, and since the region of integration is symmetric around x. For the integral J_2 note that $|\nabla u(x)\cdot(y-x)|\le c_2\delta,\ y\in B(x,\delta)$, where $c_2=c_2(u)=\max_{B(x,\delta_D(x)/2)}|\nabla u(x)|$, i.e. c_2 depends on local properties of u around x. Also,

$$
p(t-\tau_D, W_{\tau_D}, y)\mathbf{1}_{\{\tau_D<t\}} \le \frac{(4\pi)^{-d/2}}{(t-\tau_D)^{d/2}} e^{-\frac{\delta_D(x)^2}{16(t-\tau_D)}}\mathbf{1}_{\{\tau_D<t\}} \le c_3(1\wedge t), \quad y\in B(x,\delta), \tag{3.33}
$$

where $c_3=c_3(d,\delta_D(x))>0$. Thus,

$$
\int_{B(x,\delta)\setminus B(x,\varepsilon)} |\nabla u(x)\cdot(y-x)|\,\mathbb{E}_x[p(t-\tau_D,W_{\tau_D},y)\mathbf{1}_{\{\tau_D<t\}}]dy \le c_4\delta^{d+1}(1\wedge t), \quad t>0, \tag{3.34}
$$

where $c_4=c_4(d,D,u,\delta_D(x))>0$. In other words, we showed that

$$
|I_2|\le c_6\delta^{d+1},
$$

where $c_6=c_6(d,D,u,\delta_D(x),\mu)>0$. Moreover, the bounds (3.33) and (3.34) imply that the integral J_2 converges as $\varepsilon\to 0$ by the dominated convergence theorem. Hence, I_2 converges as $\varepsilon\to 0$. Finally, this means that the principal value integral in (3.29) is well defined.

Now we prove (3.29). For the fixed $\delta>0$ from above, by using (3.31) we have

$$
\begin{aligned}
\phi(-\Delta|_D)u(x) &= \int_0^\infty \big(u(x)-u(x)P_t^D\mathbf{1}(x)+u(x)P_t^D\mathbf{1}(x)-P_t^Du(x)\big)\mu(t)dt \\
&= \int_0^\infty \Big(\int_D \big(u(x)-u(y)\big)p_D(t,x,y)dy\Big)\mu(t)dt + \kappa(x)u(x) \\
&= \int_0^\infty \Big(\lim_{\varepsilon\searrow 0}\int_{D\setminus B(x,\varepsilon)} \big(u(x)-u(y)+\nabla u(x)\cdot(y-x)\mathbf{1}_{B(x,\delta)}(y)\big)p_D(t,x,y)dy\Big)\mu(t)dt \\
&\qquad - \int_0^\infty \Big(\lim_{\varepsilon\searrow 0}\int_{B(x,\delta)\setminus B(x,\varepsilon)} \big(\nabla u(x)\cdot(y-x)\big)p_D(t,x,y)dy\Big)\mu(t)dt + \kappa(x)u(x) \\
&= \lim_{\varepsilon\searrow 0}\int_{D\setminus B(x,\varepsilon)} \big(u(x)-u(y)\big)J_D(x,y) + \kappa(x)u(x),
\end{aligned}
$$

where the change of the order of integration, as well as taking the limit outside the integral, was justified by (3.32), (3.33) and (3.34). ■

Remark 3.1.6. Lemma 3.1.4 suggest the pointwise definition of the operator $\phi(-\Delta|_D)$, i.e. we define

$$\phi_p(-\Delta|_D)u(x) = \text{P.V.}\int_D [u(x)-u(y)]J_D(x,y)dy + \kappa(x)u(x), \tag{3.35}$$

for every function u and $x \in D$ for which (3.35) is well defined. E.g. this is true for every $x \in D$ if $u \in C^{1,1}(D) \cap L^1(D, \delta_D(x)dx)$ by the proof of Lemma 3.1.4 and the bound (3.30).

To conclude the subsection, we bring the well-known factorization of the Dirichlet Laplacian $-\Delta_{|D}$ which is closely related to Lemma 3.1.1. Since ϕ^* satisfies **(WSC)**, the operator $\phi^*(-\Delta|_D)$ can be defined in the same way as $\phi(-\Delta|_D)$, and the same properties hold for $\phi^*(-\Delta|_D)$. In what follows, such comments on the objects defined relative to ϕ and relative to ϕ^* will be skipped.

Lemma 3.1.7. For $\psi \in C_c^\infty(D)$, it holds that

$$\phi(-\Delta|_D)\circ\phi^*(-\Delta|_D)\psi = \phi^*(-\Delta|_D)\circ\phi(-\Delta|_D)\psi = (-\Delta_{|D})\psi, \quad \text{a.e. in } D.$$

Further, $(-\Delta_{|D})\psi = -\Delta\psi$.

Proof. Recall that the operator $\Delta_{|D}$ is the infinitesimal generator of the semigroup $(P_t^D)_t$ which on $C_c^\infty(D)$ functions acts like the standard Laplacian Δ. Hence, the claim follows from [71, Corollary 13.25] since $C_c^\infty(D) \subset H_0^1(D) = \mathscr{D}(\Delta_{|D})$. ■

3.1.4. Green potentials

In this subsection we prove some useful identities related to the Green potentials, develop some integrability conditions and prove two regularity properties for $G_D^\phi f$.

The next lemma says that the definition of the Green potential $G_D^\phi f$ in (3.28) makes sense for $f \in L^1(D, \delta_D(x)dx)$, too, and that the operator $f \mapsto G_D^\phi f$ is bounded from $L^1(D, \delta_D(x)dx)$ to itself.

Lemma 3.1.8. It holds that

$$G_D^\phi \delta_D(x) \asymp \delta_D(x), \quad x \in D, \tag{3.36}$$

where the constant of comparability depends only on d, D and ϕ. Further, if $\lambda \in \mathscr{M}(D)$ such that $\int_D \delta_D(x)|\lambda|(dx) < \infty$ then

$$x \mapsto G_D^\phi \lambda(x) := \int_D G_D^\phi(x,y)\lambda(dy) \in L^1(D, \delta_D(x)dx), \tag{3.37}$$

and there is $C = C(d,D,\phi) \geq 1$ such that $\|G_D^\phi \lambda\|_{L^1(D,\delta_D(x)dx)} \leq C \int_D \delta_D(x)|\lambda|(dx)$.

Proof. Recall that $\varphi_1(x) \asymp \delta_D(x)$, $x \in D$, by (3.19), thus by (3.23)

$$G_D^\phi \delta_D(x) \asymp G_D^\phi \varphi_1(x) = \frac{1}{\phi(\lambda_1)}\varphi_1(x) \asymp \delta_D(x), \quad x \in D.$$

The second and the third claim follow from Fubini's theorem and (3.36). ■

Corollary 3.1.9. There is $C = C(d,D,\phi) > 0$ such that for every $f \in L^1(D, \delta_D(x)dx)$ it holds that $\|G_D^\phi f\|_{L^1(D,\delta_D(x)dx)} \leq C\|f\|_{L^1(D,\delta_D(x)dx)}$.

Remark 3.1.10. Let us note that by using (3.13) it easily follows that $G_D^\phi f \in L^\infty(D)$ for $f \in L^\infty(D)$.

Operator $\phi(-\Delta|_D)$ revisited

In the next lemma we prove the boundary estimate of $\phi(-\Delta|_D)\psi$ for $\psi \in C_c^\infty(D)$ which will allow us to the define the operator $\phi(-\Delta|_D)$ in the distributional sense.

Lemma 3.1.11. For $\psi \in C_c^\infty(D)$ there is $C_1 = C_1(d,D,\phi,\psi) > 0$ such that

$$|\phi(-\Delta|_D)\psi(x)| \leq C_1 \delta_D(x), \quad x \in D. \tag{3.38}$$

In addition, if $\psi \geq 0$, $\psi \not\equiv 0$, then there is $C_2 = C_2(d,D,\phi,\psi) > 0$ such that

$$\phi(-\Delta|_D)\psi(x) \leq -C_2\delta_D(x), \quad x \in D \setminus \operatorname{supp}\psi. \tag{3.39}$$

Proof. Let $\psi \in C_c^\infty(D)$ and note that $\phi(\lambda) \leq (1 \wedge \lambda)$ by (3.7). Thus, from (3.18), (3.24), and (3.27) for large enough $m \in \mathbb{N}$, we have

$$\frac{|\phi(-\Delta|_D)\psi(x)|}{\delta_D(x)} \leq \sum_{j=1}^\infty |\widehat{\psi}_j|\phi(\lambda_j)\left\|\frac{\varphi_j}{\delta_D}\right\|_{L^\infty(D)} \leq C_1(d,D,\phi,\psi).$$

For the other bound let $x^* = \arg\max_{x \in D}\psi(x)$, and let $r > 0$ such that $B(x^*,2r) \subset \operatorname{supp}\psi$ and $\psi \geq c > 0$ on $B(x^*,2r)$. For $x \in D \setminus \operatorname{supp}\psi$, by using the representation (3.29) and the bound (3.30), we have

$$\phi(-\Delta|_D)\psi(x) = -\int_{\operatorname{supp}\psi}\psi(y)J_D(x,y)dy \leq -\int_{B(x^*,r)} c_1\delta_D(x)dy \leq -C_2\,\delta_D(x), \tag{3.40}$$

where $C_2 = C_2(d,D,\psi,\phi) > 0$. ■

Definition 3.1.12. For $f \in L^1(D, \delta_D(x)dx)$ we define the distribution $\widetilde{\phi}(-\Delta|_D)f$ in D by

$$\langle \widetilde{\phi}(-\Delta|_D)f, \psi\rangle := \langle f, \phi(-\Delta|_D)\psi\rangle := \int_D f(x)\phi(-\Delta|_D)\psi(x)dx, \quad \psi \in C_c^\infty(D).$$

Remark 3.1.13. Sometimes for $\widetilde{\phi}(-\Delta|_D)f$ we say $\phi(-\Delta|_D)f$ in the distributional sense. Notice that Lemma 3.1.11 implies that the integral defining $\widetilde{\phi}(-\Delta|_D)f$ is well defined.

By following the calculations from [15, Section 3], we get that for $f \in C^{1,1}(D) \cap L^1(D, \delta_D(x)dx)$ we have $\widetilde{\phi}(-\Delta|_D)f = \phi_p(-\Delta|_D)f$.

The next proposition says that the relation from Remark 3.1.3 can be also extended to $\widetilde{\phi}(-\Delta|_D)$.

Proposition 3.1.14. Let $\mu \in \mathscr{M}(D)$ such that $\int_D \delta_D(x)|\mu|(dx) < \infty$. Then $\widetilde{\phi}(-\Delta|_D)G_D^\phi\mu = \mu$.

Proof. Let $\psi \in C_c^\infty(D)$ and recall that $\phi(-\Delta|_D)\psi \in L^2(D)$ which follows by taking $m \in \mathbb{N}$ large enough in (3.27). Hence, by Proposition 3.1.2 we have a.e. in D

$$\psi = [\phi(-\Delta|_D)]^{-1}(\phi(-\Delta|_D)\psi) = G_D^\phi(\phi(-\Delta|_D)\psi).$$

Thus, by using Lemma 3.1.8 and Lemma 3.1.11, Fubini's theorem gives us

$$\begin{aligned}
\langle \widetilde{\phi}(-\Delta|_D)G_D^\phi\mu, \psi\rangle &= \langle G_D^\phi\mu, \phi(-\Delta|_D)\psi\rangle \\
&= \int_D \left(\int_D G_D^\phi(x,y)\mu(dy)\right)\phi(-\Delta|_D)\psi(x)dx \\
&= \int_D \left(\int_D G_D^\phi(x,y)\phi(-\Delta|_D)\psi(x)dx\right)\mu(dy) = \int_D \psi(y)\mu(dy).
\end{aligned}$$

■

The following proposition connects the spectral, the distributional and the pointwise definition of $\phi(-\Delta|_D)$ for nice enough functions.

Proposition 3.1.15. If $u \in C^{1,1}(D) \cap H_\phi(D)$, then

$$\phi(-\Delta|_D)u = \widetilde{\phi}(-\Delta|_D)u = \phi_p(-\Delta|_D)u$$

holds a.e. in D.

Proof. Let $u \in C^{1,1}(D) \cap H_\phi(D)$. Recall that $H_\phi(D) = G_D^\phi\big(L^2(D)\big) \subset L^2(D) \subset L^1(D, \delta_D(x)dx)$, so $u = G_D^\phi h$ for some $h \in L^2(D)$, and $\phi(-\Delta|_D)u = h$. However, $u \in C^{1,1}(D)$ so $\widetilde{\phi}(-\Delta|_D)u = \phi_p(-\Delta|_D)u$ by Remark 3.1.13, and $\widetilde{\phi}(-\Delta|_D)u = h$ by Proposition 3.1.14. ■

Regularity of Green potentials

In two following claims we deal with regularity properties of $G_D^\phi f$. The first claim says that Green potentials are continuous and this fact is rather simple to see and prove. We also prove that the Green potential of a $C_c^\infty(D)$ function is a $C^{1,1}(\overline{D})$ function, i.e. we prove a smoothness result for a specific class of functions.

Proposition 3.1.16. If $f \in L^1(D, \delta_D(x)dx)$, then $G_D^\phi f \in C(D)$.

Proof. First note that $L^1(D, \delta_D(x)dx) \subset L^\infty_{loc}(D)$ since $\delta_D \asymp 1$ away from ∂D.

Let $x \in D$, $\eta \in (0, \delta_D(x)/2)$ and $(x_n)_n \subset D$ such that $x_n \to x$ and $|x_n - x| < \eta/2$, $n \in \mathbb{N}$. We have

$$\begin{aligned} |G_D^\phi f(x_n) - G_D^\phi f(x)| &\le \int_D |G_D^\phi(x_n,y) - G_D^\phi(x,y)||f(y)|dy \\ &\le \int_{D\cap B(x,\eta)^c} |G_D^\phi(x_n,y) - G_D^\phi(x,y)||f(y)|dy \qquad (3.41) \\ &\quad + \int_{B(x,\eta)} G_D^\phi(x_n,y)|f(y)|dy \qquad (3.42) \\ &\quad + \int_{B(x,\eta)} G_D^\phi(x,y)|f(y)|dy. \qquad (3.43) \end{aligned}$$

The first integral (3.41) goes to 0 as $n \to \infty$ by the dominated convergence theorem since G_D^ϕ is continuous, $f \in L^1(D, \delta_D(x)dx)$, and since the bound (3.13) holds.

For the integrals (3.42) and (3.43) note that $M := \sup_{y \in B(x, \delta_D(x)/2)} |f(y)| < \infty$ since $f \in L^1(D, \delta_D(x)dx) \subset L^\infty_{loc}(D)$. Further, by (3.13) for all $w \in B(x, \eta/2)$ we have

$$\begin{aligned} \int_{B(x,\eta)} G_D^\phi(w,y)|f(y)|dy &\le c_1 M \int_{B(w,\frac{3}{2}\eta)} \frac{1}{|w-y|^d \phi(|w-y|^{-2})} dy \\ &\le c_2 M \int_0^{\frac{3}{2}\eta} \frac{dr}{r\phi(r^{-2})} \le c_3 M \int_0^{\frac{3}{2}\eta} \frac{\phi'(r^{-2})}{r^3 \phi(r^{-2})^2} = \frac{c_3 M}{\phi(\frac{4}{9\eta^2})}, \qquad (3.44) \end{aligned}$$

where in the last equality we used the substitution $t = \phi(r^{-2})$ and $c_3 = c_3(d, D, \phi) > 0$. Thus, the second and the third integral can be made arbitrarily small. ∎

Remark 3.1.17. From Proposition 3.1.16 it follows that

$$\lim_{\xi \to x} \int_D |G_D^\phi(\xi,y) - G_D^\phi(x,y)||f(y)|dy = 0, \qquad (3.45)$$

uniformly on compact subsets of D.

Indeed, fix a compact set $K \subset D$ and $\varepsilon > 0$. First choose $\eta > 0$ from Proposition 3.1.16 such that $\mathrm{dist}(K, \partial D) > 2\eta$ and $(c_3 M)/\phi(\frac{4}{9\eta^2}) < \varepsilon/3$, where $M = \sup_{y \in K+B(0,\eta)} |f(y)|$, see (3.44). Thus, we tamed the integrals (3.42) and (3.43). For the integral (3.41) notice that the convergence $\lim_{\xi \to x} G_D^\phi(\xi, y) = G_D^\phi(x, y)$ is uniform in $x \in K$ and $y \in D \cap B(x, \eta)^c$ since G_D^ϕ is jointly continuous and since G_D^ϕ continuously vanishes at the boundary by (3.13). Hence, (3.45) holds uniformly on compact sets.

Proposition 3.1.18. If $f \in C_c^\infty(D)$, then $G_D^\phi f \in C^{1,1}(\overline{D})$.

Proof. By Proposition 3.1.2 we have $G_D^\phi f = \sum_{j=1}^\infty \frac{1}{\phi(\lambda_j)} \widehat{f}_j \varphi_j$ a.e. in D. However, $G_D^\phi f \in C(D)$ by Proposition 3.1.16. Also, recall that there is $c_1 = c_1(m, f) > 0$ such that $|\widehat{f}_j| \le c_1 \lambda_j^{-m}$, $j \in \mathbb{N}$, by (3.27), hence in the light of (4.41) and (4.42), for large enough $m \in \mathbb{N}$ we have

$$\left\| \sum_{j=1}^\infty \frac{1}{\phi(\lambda_j)} \widehat{f}_j \varphi_j \right\|_{C^{1,1}(\overline{D})} \le \sum_{j=1}^\infty \frac{c_2}{\phi(\lambda_j) \lambda_j^m} (1 + \lambda_j)^{d/4+1} < \infty$$

by (3.18) and by (3.7), where $c_2 = c_2(d, D, m, f) > 0$.

In other words, $G_D^\phi f = \sum_{j=1}^\infty \frac{1}{\phi(\lambda_j)} \widehat{f}_j \varphi_j$ everywhere in D and $G_D^\phi f \in C^{1,1}(\overline{D})$. ■

3.1.5. Poisson kernel and harmonic functions

Recall that the Poisson kernel of the Brownian motion (i.e. of the Dirichlet Laplacian) can be defined as

$$P_D(x, z) = -\frac{\partial}{\partial \mathbf{n}} G_D^1(x, z), \quad x \in D, z \in \partial D, \tag{3.46}$$

see [37, Section 2.2.4], where $\frac{\partial}{\partial \mathbf{n}}$ denotes the derivate in the direction of the inner normal. In this subsection we study the Poisson kernel of the process Y^D which we define as the normal derivative of the Green kernel of the process Y^D and we study harmonic functions relative to $\phi(-\Delta|_D)$, or, as we show at the end of the subsection, relative to Y^D.

Proposition 3.1.19. The function

$$P_D^\phi(x, z) := -\frac{\partial}{\partial \mathbf{n}} G_D^\phi(x, z), \quad x \in D, z \in \partial D, \tag{3.47}$$

is well defined and $(x, z) \mapsto P_D^\phi(x, z) \in C(D \times \partial D)$. Moreover,

$$P_D^\phi(x, z) \asymp \frac{\delta_D(x)}{|x - z|^{d+2} \phi(|x - z|^{-2})}, \quad x \in D, z \in \partial D, \tag{3.48}$$

where the constant of comparability depends only on d, D and ϕ. Finally, it holds that

$$\int_D G_D^{\phi^*}(x,\xi)P_D^{\phi}(\xi,z)d\xi = P_D(x,z), \quad x\in D, z\in\partial D. \tag{3.49}$$

Proof. Let $x\in D$ and $z\in\partial D$. For $y\in D$ we have

$$\frac{G_D^{\phi}(x,y)}{\delta_D(y)} = \int_0^{\infty}\frac{1}{\delta_D(y)}p_D(t,x,y)\mathfrak{u}(t)dt.$$

In what follows, we always consider $y\in D$ which is in the direction of the normal derivative in z, close enough to z so that $\delta_D(x)\le 2|x-y|$.

Recall that $p_D\in C^1((0,\infty)\times\overline{D}\times\overline{D})$ since D is $C^{1,1}$, see Lemma 4.5.1, hence $-\frac{\partial}{\partial\mathbf{n}}p_D(t,x,z) = \lim\limits_{y\to z}\frac{p_D(t,x,y)}{\delta_D(y)}$ exists. Further, from (3.10) it follows that

$$\frac{p_D(t,x,y)\mathfrak{u}(t)}{\delta_D(y)} \le c_1\frac{\delta_D(x)}{t^{d/2+1}}e^{-\frac{c_2|x-y|^2}{t}}\mathfrak{u}(t) \le c_1\frac{\delta_D(x)}{t^{d/2+1}}e^{-\frac{c_2\delta_D(x)^2}{4t}}\mathfrak{u}(t). \tag{3.50}$$

Recall that $\mathfrak{u}$ is decreasing and that $\int_0^1\mathfrak{u}(t)dt<\infty$, hence the right hand side of (3.50) is in $L^1\big((0,\infty),dt\big)$. By using the dominated convergence theorem we conclude that $P_D^{\phi}(x,z)$ is well defined and

$$P_D^{\phi}(x,z) = \lim_{y\to z}\frac{G_D^{\phi}(x,y)}{\delta_D(y)} = -\int_0^{\infty}\frac{\partial}{\partial\mathbf{n}}p_D(t,x,z)\mathfrak{u}(t)dt.$$

Moreover, (3.48) immediately follows from the definition of P_D^{ϕ} and from (3.13).

Now we show that P_D^{ϕ} is jointly continuous on $D\times\partial D$. Let $(x_n)_n\subset D$ such that $x_n\to x\in D$ and such that $\delta_D(x_n)\ge\delta_D(x)/2$. Also, take $(z_n)_n\subset\partial D$ such that $z_n\to z\in\partial D$. By taking the limit $y\to z$ in the first inequality in (3.50) without the term $\mathfrak{u}(t)$, we obtain for all $n\in\mathbb{N}$ and all $t\in(0,\infty)$

$$0\le -\frac{\partial}{\partial\mathbf{n}}p_D(t,x_n,z_n)\le c_1\frac{\delta_D(x_n)}{t^{d/2+1}}e^{-\frac{c_2|x_n-z_n|^2}{t}} \le c_1\frac{\delta_D(x_n)}{t^{d/2+1}}e^{-\frac{c_2\delta_D(x_n)^2}{t}}, \tag{3.51}$$

which also holds for z instead of z_n. Since $\frac{\partial}{\partial\mathbf{n}}p_D(t,x,z)\in C((0,\infty)\times\overline{D}\times\partial D)$, see Lemma 4.5.1, by using the dominated convergence theorem with the bound derived from (3.51) we get

$$|P_D^{\phi}(x,z)-P_D^{\phi}(x_n,z_n)|\le\int_0^{\infty}\left|\frac{\partial}{\partial\mathbf{n}}p_D(t,x_n,z_n)-\frac{\partial}{\partial\mathbf{n}}p_D(t,x,z)\right|\mathfrak{u}(t)dt\to 0, \quad \text{as } n\to\infty.$$

We are left to prove (3.49). Obviously, Lemma 3.1.1 implies

$$-\frac{\partial}{\partial\mathbf{n}}\left(\int_D G_D^{\phi^*}(x,\xi)G_D^{\phi}(\xi,\cdot)d\xi\right)(z) = P_D(x,z), \quad x\in D, z\in\partial D.$$

We need to justify that the normal derivative can go inside the integral. To this end, let $x\in D$, $z\in\partial D$, and $\varepsilon>0$ such that $\delta_D(x)>3\varepsilon$. Again, we only consider $y\in D$ which is in the direction of the normal derivative. For $|z-y|\le\varepsilon/2$ we have

$$\begin{aligned}\int_D G_D^{\phi^*}(x,\xi)\frac{G_D^{\phi}(\xi,y)}{\delta_D(y)}d\xi &= \int_{D\cap B(z,\varepsilon)^c} G_D^{\phi^*}(x,\xi)\frac{G_D^{\phi}(\xi,y)}{\delta_D(y)}d\xi+\int_{D\cap B(z,\varepsilon)} G_D^{\phi^*}(x,\xi)\frac{G_D^{\phi}(\xi,y)}{\delta_D(y)}d\xi\\ &=: I_1+I_2.\end{aligned}$$

For the integral I_1 by the sharp bounds (3.13) we have

$$\frac{G_D^{\phi}(\xi,y)}{\delta_D(y)}\lesssim\frac{\delta_D(\xi)}{|\xi-y|^{d+2}\phi(|\xi-y|^{-2})}. \tag{3.52}$$

Thus, if $\xi\in D\cap B(z,\varepsilon)^c$, we have $\frac{G_D^{\phi}(\xi,y)}{\delta_D(y)}\le c_3\delta_D(\xi)$, where $c_3=c_3(\phi,D,d,\varepsilon)>0$. Further, $G_D^{\phi^*}\delta_D\asymp\delta_D$ by Lemma 3.1.8, hence the integral I_1 converges to

$$\int_{D\cap B(y,\varepsilon)^c} G_D^{\phi^*}(x,\xi)P_D^{\phi}(\xi,z)d\xi,$$

as $y\to z$.

The integral I_2 we break in two additional integrals

$$\begin{aligned}I_2&=\int_{D\cap B(z,\varepsilon)} G_D^{\phi^*}(x,\xi)\frac{G_D^{\phi}(\xi,y)}{\delta_D(y)}d\xi\\ &\le\int_{B\left(y,\frac{\delta_D(y)}{2}\right)} G_D^{\phi^*}(x,\xi)\frac{G_D^{\phi}(\xi,y)}{\delta_D(y)}d\xi+\int_{D\cap B\left(y,\frac{\delta_D(y)}{2}\right)^c\cap B(y,2\varepsilon)} G_D^{\phi^*}(x,\xi)\frac{G_D^{\phi}(\xi,y)}{\delta_D(y)}d\xi\\ &=: J_1+J_2.\end{aligned}$$

Recall that $3\varepsilon\le\delta_D(x)$ so $\frac{1}{6}|x-z|\le|x-\xi|\le2|x-z|$ for all $\xi\in B(y,2\varepsilon)$. Hence, (3.13) applied on $G_D^{\phi^*}$ implies

$$G_D^{\phi^*}(x,\xi)\le c_4\delta_D(\xi),\quad \xi\in B(y,2\varepsilon)\cap D, \tag{3.53}$$

where $c_4=c_4(d,D,\phi^*,|x-z|)>0$ and is independent of ε in the sense if $\varepsilon\to0$, the constant c_4 remains the same.

For J_1 note that $\delta_D(\xi)\le\frac{3}{2}\delta_D(y)$ for $\xi\in B(y,\delta_D(y)/2)$ so by using the bounds (3.13) and (3.53) we have

$$\begin{aligned}J_1&\le c_5\int_{B\left(y,\frac{\delta_D(y)}{2}\right)}\frac{\delta_D(\xi)}{\delta_D(y)}\frac{1}{|\xi-y|^d\phi(|\xi-y|^{-2})}\le c_6\int_{B\left(y,\frac{\delta_D(y)}{2}\right)}\frac{1}{|\xi-y|^d\phi(|\xi-y|^{-2})}\\ &\le c_7\int_0^{\delta_D(y)/2}\frac{\phi'(r^{-2})}{r^3\phi(r^{-2})^2}dr\le c_8\frac{1}{\phi(4/\delta_D(y)^2)},\end{aligned}$$

where c_8 is independent of y and ε. In the second to last inequality we used (3.4) and for the last one we used the substitution $t = \phi(r^{-2})$.

For J_2 note that $\delta_D(\xi) \le \delta_D(y) + |y - \xi| \le 3|\xi - y|$, for $\xi \in B(y, \delta_D(y)/2)^c$, hence by the sharp bounds (3.13) we have

$$J_2 \le c_9 \int_{B\left(y,\frac{\delta_D(y)}{2}\right)^c \cap B(y,2\varepsilon)} \frac{1}{|\xi - y|^d \phi(|\xi - y|^{-2})} \le c_{10} \int_{\delta_D(y)/2}^{2\varepsilon} \frac{\phi'(r^{-2})}{r^3 \phi(r^{-2})^2} dr \le c_{11} \frac{1}{\phi(\frac{1}{4\varepsilon^2})},$$

where c_{11} is independent of y and ε. Hence, for sufficiently small ε the integral I_2 can be made sufficiently small. Thus, (3.49) holds. ■

Now we deal with harmonic functions with respect to the operator $\phi(-\Delta|_D)$. Our first goal is to show the integral representation of positive harmonic functions which we show in Theorem 3.1.22. After that, in Theorem 3.1.24 we show the continuity of harmonic functions and at the end of the subsection we connect harmonic functions with functions that satisfy a certain mean-value property with respect to Y^D, see Theorem 3.1.26.

Definition 3.1.20. A function $h \in L^1(D, \delta_D(x)dx)$ is called harmonic in D if $\widetilde{\phi}(-\Delta|_D)h = 0$ in D.

First we present a connection between harmonic functions and classical harmonic functions.

Proposition 3.1.21. A function $h \in L^1(D, \delta_D(x)dx)$ is harmonic in D if and only if $G_D^{\phi^*} h$ is a classical harmonic function in D. In particular, for every $z \in \partial D$, the function $x \mapsto P_D^{\phi}(x,z)$ is harmonic in D.

Proof. The first part of the claim follows by the following calculation. Take $\psi \in C_c^\infty(D)$. Then by using Lemma 3.1.7, Proposition 3.1.2, and Fubini's theorem we have

$$\begin{aligned}\int_D h(x)\phi(-\Delta|_D)\psi(x)dx &= \int_D h(x) \left[\phi^*(-\Delta|_D)^{-1} \circ (-\Delta)\psi(x)\right] dx \\ &= \int_D h(x) G_D^{\phi^*}((-\Delta)\psi)(x)dx \\ &= -\int_D G_D^{\phi^*} h(x)\, \Delta\psi(x)dx,\end{aligned}$$

i.e. h is harmonic if and only if $G_D^{\phi^*} h$ is a classical harmonic function in D (since $G_D^{\phi^*} h \in C(D)$).

If $z \in \partial D$, then $P_D^\phi(\cdot, z) \in L^1(D, \delta_D(x)dx)$ by the bound (3.48), see also the beginning of the proof of Theorem 3.1.22 with $\zeta = \delta_z$. The second claim now follows from (3.49) and the fact that the kernel $P_D^1(\cdot, z)$ is classical harmonic function. ■

Theorem 3.1.22. If a non-negative function $h \in L^1(D, \delta_D(x)dx)$ is harmonic in D, then there exists a finite non-negative measure $\zeta \in \mathscr{M}(\partial D)$ such that

$$h(x) = \int_{\partial D} P_D^\phi(x,z)\zeta(dz), \quad \text{for a.e. } x \in D. \tag{3.54}$$

Moreover, there is $C = C(d, D, \phi) > 0$ such that

$$\|h\|_{L^1(D,\delta_D(x)dx)} \leq C\|\zeta\|_{\mathscr{M}(\partial D)}. \tag{3.55}$$

Conversely, every function of the form (3.54) is harmonic in D.

Proof. Let h be represented as (3.54). Since $P_D^\phi(x, \cdot) \in C(\partial D)$ for fixed $x \in D$ by Proposition 3.1.19, hence bounded, the function h is well defined. Further, since $\delta_D(x) \leq |x - z|$, $z \in \partial D$, from (3.48) and Fubini's theorem we get

$$\begin{aligned}
\int_D h(x)\delta_D(x)dx &\leq c_1 \int_{\partial D}\int_D \frac{\delta_D(x)^2}{|x-z|^{d+2}} \frac{1}{\phi(|x-z|^{-2})} dx\, \zeta(dz) \\
&\leq c_1 \int_{\partial D}\int_{B(z,\mathrm{diam}D)} \frac{1}{|x-z|^d \phi(|x-z|^{-2})} dx\, \zeta(dz) \\
&\leq c_2 \int_{\partial D} \frac{\zeta(dz)}{\phi(\mathrm{diam}D^{-2})} < \infty,
\end{aligned}$$

where $c_2 = c_2(d, D, \phi) > 0$, i.e. $h \in L^1(D, \delta_D(x)dx)$ and $\|h\|_{L^1(D,\delta_D(x)dx)} \leq C\|\zeta\|_{\mathscr{M}(\partial D)}$. Take now $\psi \in C_c^\infty(D)$. Fubini's theorem and Proposition 3.1.21 yield

$$\int_D P_D^\phi \zeta(x) \phi(-\Delta|_D)\psi(x)dx = \int_{\partial D} \left(\int_D P_D^\phi(x,z)\phi(-\Delta|_D)\psi(x)dx \right) \zeta(dz) = 0,$$

i.e. h is harmonic in D.

Conversely, let h be a non-negative harmonic function in D. Then $G_D^{\phi^*}h$ is a classical non-negative harmonic function in D by Proposition 3.1.21. Note that $G_D^{\phi^*}h \in C(D)$ by Proposition 3.1.16 so by the representation of non-negative classical harmonic functions there is a non-negative finite measure $\zeta \in \mathscr{M}(\partial D)$ such that

$$G_D^{\phi^*}h(x) = \int_{\partial D} P_D(x,z)\zeta(dz), \quad x \in D. \tag{3.56}$$

Applying (3.49) to the right hand side of (3.56) we get

$$\int_D G_D^{\phi^*}(x,\xi)h(\xi)d\xi = \int_D G_D^{\phi^*}(x,\xi)\left[\int_{\partial D} P_D^{\phi}(\xi,z)\zeta(dz)\right]d\xi, \quad x \in D. \tag{3.57}$$

By using Proposition 3.1.14 in (3.57) we obtain

$$h(\xi) = \int_{\partial D} P_D^{\phi}(\xi,z)\zeta(dz), \quad \text{for a.e. } \xi \text{ in } D.$$

■

Motivated by the previous theorem, we introduce the definition of the Poisson integral.

Definition 3.1.23. For a finite signed measure $\zeta \in \mathcal{M}(\partial D)$ we define the Poisson integral of ζ by

$$P_D^{\phi}\zeta(x) := \int_{\partial D} P_D^{\phi}(x,z)\zeta(dz), \quad x \in D.$$

Note that finiteness of the (signed) measure ζ in the previous definition is a necessary and sufficient condition for the integral defining $P_D^{\phi}\zeta$ to be finite, see (3.48). If $\zeta \in L^1(\partial D)$, we slightly abuse the notation in Definition 3.1.23 where we set $P_D^{\phi}\zeta(x) = \int_{\partial D} P_D^{\phi}(x,z)\zeta(z)\sigma(dz)$, where σ is the $d-1$ dimensional Hausdorff measure on ∂D. Since the set D is $C^{1,1}$, the measure σ is finite so we can define the Poisson integral of σ

$$P_D^{\phi}\sigma(x) = \int_{\partial D} P_D^{\phi}(x,z)\sigma(dz), \quad x \in D, \tag{3.58}$$

which will be of great importance for the boundary condition of the semilinear problem.

We finish the subsection with two properties of harmonic functions of the form $P_D^{\phi}\zeta$.

Theorem 3.1.24. A non-negative harmonic function in D is continuous in D (after a modification on the Lebesgue null set). Furthermore, for every finite (signed) measure $\zeta \in \mathcal{M}(\partial D)$, we have $P_D^{\phi}\zeta \in C(D)$.

Proof. Let $h \in L^1(D, \delta_D(x)dx)$ be a non-negative harmonic function in D. By Theorem 3.1.22 there exists a finite non-negative measure $\zeta \in \mathcal{M}(\partial D)$ such that $h = P_D^{\phi}\zeta$ a.e. in D. In Proposition 3.1.19 it was proved that the function $P_D^{\phi}(\cdot,\cdot)$ is continuous in the first variable and that the sharp bounds (3.48) hold, so we can use the dominated convergence theorem to get $P_D^{\phi}\zeta \in C(D)$. ■

In the theory of Markov processes, harmonicity of a function is considered relative to the process itself, i.e. it is said that a function $f : D \to [-\infty, \infty]$ is harmonic in D with respect to Y^D if for every $U \subset\subset D$ and $x \in U$

$$h(x) = \mathbb{E}_x[h(Y^D_{\tau_U^{Y^D}})] \tag{3.59}$$

holds, where $\tau_U^{Y^D} = \inf\{t > 0 : Y_t^D \notin U\}$ and where we implicitly assume $\mathbb{E}_x[|h(Y^D_{\tau_U^{Y^D}})|] < \infty$ for every $x \in U \subset\subset D$. The relation (3.59) is often referred to as the mean-value property of the function f with respect to Y^D. In order not to confuse, if f is harmonic in D with respect to Y^D, we will say that f satisfies the mean-value property with respect to Y^D. We note that $\mathbb{E}_x[|h(Y^D_{\tau_U^{Y^D}})|] < \infty$ for every $x \in U \subset\subset D$ implies that $f \in L^1(D, \delta_D(x)dx)$, see the proof of [58, Lemma 3.6] where instead of the inequality $U^{D,B}(x,y) \le G_X(x,y)$ use $U^{D,B}(x,y) \le G_D^\phi(x,y)$.

The connection between non-negative functions that satisfy the mean-value property with respect to Y^D and non-negative functions that satisfy the mean-value property with respect to W^D is known due to [74, Theorem 3.6] which we cite in the next claim.

Theorem 3.1.25. If a non-negative function h satisfies the mean-value property in D with respect to Y^D, then $s := G_D^{\phi^*} h$ satisfies the mean-value property in D with respect to W^D. Conversely, if a non-negative function s satisfies the mean-value property in D with respect to W^D, then

$$h(x) := \int_0^\infty \left(s(x) - P_t^D s(x)\right) \nu(t)dt = \phi_p^*(-\Delta|_D)s(x), \quad x \in D, \tag{3.60}$$

satisfies the mean-value property in D with respect to Y^D, h is continuous and $G_D^{\phi^*} h = s$.

Proof. Everything follows from [74, Theorem 3.6] except the second equality in (3.60). To finish the proof, it follows from the proof of [74, Lemma 3.4] that

$$|s(x) - P_t^D s(x)| \le c(1 \wedge t), \quad x \in K,$$

where K is any compact subset of D and $c = c(d, D, s_{|K}) > 0$. Also, $s \in C^\infty(D)$ since it is a classical harmonic function so by the same calculations as in Lemma 3.1.4 we get that

$$\int_0^\infty \left(s(x) - P_t^D s(x)\right) \nu(t)dt = \phi_p^*(-\Delta|_D)s.$$

■

The following theorem says that non-negative harmonic functions and non-negative functions with the mean-value property with respect to Y^D are essentially the same.

Theorem 3.1.26. If a non-negative function $h \in L^1(D, \delta_D(x)dx)$ is harmonic in D, then (after a modification on the Lebesgue null set) h satisfies the mean-value property with respect to Y^D. Conversely, if $h \geq 0$ satisfies the mean-value property with respect to Y^D, then h is harmonic in D.

Proof. Let $h \geq 0$ be harmonic in D. Theorem 3.1.22 implies that we can modify h such that $h = P_D^\phi \zeta$ in the whole D for some non-negative and finite $\zeta \in \mathscr{M}(\partial D)$. This also means that $h \in C(D)$ by Theorem 3.1.24. Since $G_D^{\phi^*} h = P_D^1 \zeta$ in D by (3.49), the claim follows from Theorem 3.1.25 because $P_D \zeta$ is a (smooth) classical harmonic function, hence it satisfies the mean-value property with respect to W^D.

Conversely, if $h \geq 0$ satisfies the mean-value property with respect to Y^D, then $G_D^{\phi^*} h$ satisfies the mean-value property with respect to W^D by Theorem 3.1.25. By the classical theory of harmonic functions, $G_D^{\phi^*} h$ is a classical harmonic function in D. Proposition 3.1.21 now implies that h is harmonic in D. ■

3.2. Boundary behaviour of potential integrals

In this section we study the boundary behaviour of Poisson and Green integrals which will serve as a foundation for understanding of the boundary condition of the (semi)linear problem and for understanding of the connection between weak and distributional solutions in the next section. However, these problems are also interesting in itself. First we give a sharp bound for $P_D^\phi \sigma$.

Lemma 3.2.1. It holds that

$$P_D^\phi \sigma(x) \asymp \frac{1}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})}, \quad x \in D, \tag{3.61}$$

where the constant of comparability depends only on d, D and ϕ.

Proof. In Proposition 3.1.19 we have proved that

$$P_D^\phi(x,z) \asymp \frac{\delta_D(x)}{|x-z|^{d+2}\phi(|x-z|^{-2})}, \quad x \in D, z \in \partial D,$$

where the constant of comparability depends only on d, D and ϕ. Also, in the following calculations, it is easy to check that every comparability constant remains to depend only on d, D and ϕ.

For the upper bound, note that $\delta_D(x) \le |x-z|$, $z \in \partial D$ so by using (3.7) we have $\delta_D(x)^2 \phi(\delta_D(x)^{-2}) \le |x-z|^2 \phi(|x-z|^{-2})$, thus

$$P_D^\phi \sigma(x) \asymp \int_{\partial D} \frac{\delta_D(x)}{|x-z|^{d+2}\phi(|x-z|^{-2})} \sigma(dz) \le \frac{1}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})},$$

since $\int_{\partial D} |x-z|^{-d} \asymp \delta_D(x)^{-1}$, $x \in D$.

For the lower bound fix $x \in D$ and choose $\Gamma = \{z \in \partial D : |x-z| \le 2\delta_D(x)\}$. Recall that ϕ is increasing so

$$\begin{aligned} P_D^\phi \sigma(x) &\asymp \int_{\partial D} \frac{\delta_D(x)}{|x-z|^{d+2}\phi(|x-z|^{-2})} \sigma(dz) \ge \frac{1}{4\delta_D(x)^2 \phi(\delta_D(x)^{-2})} \delta_D(x) \int_\Gamma \frac{\sigma(dz)}{|x-z|^d} \\ &\asymp \frac{1}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})}, \end{aligned}$$

since $\int_\Gamma |x-z|^{-d} \asymp \delta_D(x)^{-1}$, $x \in D$, by reducing to the flat case, see Lemma 4.2.1. ∎

Remark 3.2.2. For the classical Poisson kernel P_D^1, defined in (3.46), it is well known that $P_D^1(x,z) \asymp \frac{\delta_D(x)}{|x-z|^d}$, for $x \in D$ and $z \in \partial D$. Moreover, since P_D^1 is the density of W_{τ_D}, we have $P_D^1\sigma(x) = \mathbb{E}_x[\mathbf{1}(W_{\tau_D})] = 1$. In particular, by the sharp bound (3.61) and by the scaling condition (3.3), $P_D^\phi\sigma$ explodes when approaching the boundary of D whereas $P_D^1\sigma$ obviously does not.

Remark 3.2.3. In what follows we will need the following inequality

$$\frac{P_D^\phi(x,z)}{P_D^\phi\sigma(x)} \lesssim \frac{\delta_D(x)}{|x-z|^d}, \quad x \in D, \tag{3.62}$$

which holds by the sharp bounds (3.48) and (3.61), and since by (3.7) it holds that $\delta_D(x)^2\phi(\delta_D(x)^{-2}) \le |x-z|^2\phi(|x-z|^{-2})$, for $x \in D$ and $z \in \partial D$.

Two following propositions deal with the boundary behaviour of Poisson integrals. They generalize [3, Proposition 25 & Theorem 26] to our more general non-local setting.

Proposition 3.2.4. Let $\zeta \in C(\partial D)$. It holds

$$\lim_{D\ni x\to z\in\partial D} \frac{P_D^\phi\zeta(x)}{P_D^\phi\sigma(x)} = \zeta(z)$$

uniformly on ∂D.

Proof. Note that ζ is uniformly continuous since D is bounded and let $M = 2\sup_{z\in\partial D}|\zeta(z)|$. For $\varepsilon > 0$ choose $\eta > 0$ such that if $y,z \in \partial D$ and $|y-z| < \eta$, then $|\zeta(y)-\zeta(z)| \le \varepsilon$. For $z \in \partial D$ let $\Gamma_z = \{y \in \partial D : |y-z| < \eta\}$. Now if $|x-z| \le \frac{\eta}{2}$, then by using (3.62) we have

$$\begin{aligned}
\left|\frac{P_D^\phi\zeta(x)}{P_D^\phi\sigma(x)} - \zeta(z)\right| &\le \frac{1}{P_D^\phi\sigma(x)}\int_{\partial D} P_D^\phi(x,y)\,|\zeta(y)-\zeta(z)|\,\sigma(dy) \\
&\le c_1\delta_D(x)\int_{\Gamma_z}\frac{|\zeta(y)-\zeta(z)|}{|x-y|^d}\sigma(dy) + c_1\delta_D(x)\int_{\partial D\setminus\Gamma_z}\frac{|\zeta(y)-\zeta(z)|}{|x-y|^d}\sigma(dy) \\
&\le c_2\varepsilon + c_1\delta_D(x)M\sigma(\partial D)\left(\frac{\eta}{2}\right)^{-d},
\end{aligned}$$

where in the last inequality for the first term we used $\delta_D(x) \asymp \int_{\partial D}|x-y|^{-d}\sigma(dy)$, hence $c_2 = c_2(d,D,\phi) > 0$. Now the claim follows by taking x close enough to z. ■

Proposition 3.2.5. For any $\zeta \in L^1(\partial D)$ and any $\varphi \in C(\overline{\Omega})$ it holds that

$$\frac{1}{t}\int_{\{\delta_D(x)\le t\}}\frac{P_D^\phi\zeta(x)}{P_D^\phi\sigma(x)}\varphi(x)dx \xrightarrow{t\downarrow 0} \int_{\partial D}\varphi(y)\zeta(y)d\sigma(y).$$

Proof. We can repeat the proof of [3, Theorem 26] almost to the letter. Indeed, take $\varphi \in C(\overline{D})$ and note the h_1 of [3] is our $P_D^\phi \sigma$, and ϕ of [3] is our φ. We repeat the proof up to the definition of

$$\Phi(t,y) := \frac{1}{t}\int_{\{\delta_D(x)<t\}} \frac{P_D^\phi(x,y)}{P_D^\phi \sigma(x)}\,\varphi(x)\,dx.$$

Now we use Remark 3.2.3 and the boundedness of φ to obtain

$$|\Phi(t,y)| \le c_1 \frac{\|\varphi\|_{L^\infty(D)}}{t}\int_{\{\delta_D(x)<t\}} \frac{\delta_D(x)}{|x-y|^d}dx \le c_2,$$

by the reduction to the flat boundary, see [3, Lemma 40], where $c_2 = c_2(\phi, D, d, \varphi) > 0$. The rest of the proof is now the same as in [3]. ■

Now we turn to the boundary behaviour of Green integrals. Here the pointwise limits are harder to get and we must assume some kind of uniformity of the integrating function.

Theorem 3.2.6. Let $U : (0,\infty) \to [0,\infty)$ such that

($\widetilde{\mathbf{U}}\mathbf{1}$) integrability condition holds

$$\int_0^1 U(t)t\,dt < \infty; \tag{3.63}$$

($\widetilde{\mathbf{U}}\mathbf{2}$) almost non-increasing condition holds, i.e. there exists $C > 0$ such that

$$U(t) \le CU(s), \quad 0 < s \le t \le 1; \tag{3.64}$$

($\widetilde{\mathbf{U}}\mathbf{3}$) reverse doubling condition holds, i.e. there exists $C > 0$ such that

$$U(t) \le CU(2t), \quad t \in (0,1); \tag{3.65}$$

($\widetilde{\mathbf{U}}\mathbf{4}$) boundedness away from zero holds, i.e. U is bounded from above on $[c,\infty)$ for each $c > 0$.

Then $U(\delta_D) \in L^1(D, \delta_D(x)dx)$ and

$$G_D^\phi\big(U(\delta_D)\big)(x) \asymp \frac{1}{\delta_D(x)^2\phi(\delta_D(x)^{-2})}\int_0^{\delta_D(x)} U(t)t\,dt + \delta_D(x) + \delta_D(x)\int_{\delta_D(x)}^{\mathrm{diam}D} \frac{U(t)}{t^2\phi(t^{-2})}\,dt\,. \tag{3.66}$$

In particular,

$$\lim_{D\ni x\to z\in\partial D} \frac{G_D^\phi[U(\delta_D)](x)}{P_D^\phi\sigma(x)} = 0. \tag{3.67}$$

This theorem generalizes [36, Proposition 7] to more general non-local operators and more general functions since in [36] this result was proved in the case of the spectral fractional Laplacian and for functions of the form $U(t) = t^\beta$.

Proof of Theorem 3.2.6. The proof of this claim is very technical and follows the proof of [4, Theorem 3.4], hence we moved it to Appendix, see Section 4.2.

■

The following proposition appears as [3, Theorem 27] for the case of the spectral fractional Laplacian but in our more general setting the proof gets a little more complicated, cf. [3, Eq. (46)] and (3.70).

Proposition 3.2.7. Let $\lambda \in \mathscr{M}(D)$ such that $\int_D \delta_D(x)|\lambda|(dx) < \infty$. Then

$$\frac{1}{t}\int_{\{\delta_D(x)\le t\}} \frac{G_D^\phi \lambda(x)}{P_D^\phi \sigma(x)} \varphi(x)dx \xrightarrow{t\downarrow 0} 0, \quad \varphi \in C(\overline{D}). \tag{3.68}$$

Proof. Without loss of generality we may assume that λ is a non-negative measure. It is enough to prove that (3.68) holds for $\varphi \equiv 1$. By using Fubini's theorem it follows that

$$\frac{1}{t}\int_{\{\delta_D(x)\le t\}} \frac{G_D^\phi \lambda(x)}{P_D^\phi \sigma(x)} dx = \int_D \left(\frac{1}{t}\int_{\{\delta_D(x)\le t\}} \frac{G_D^\phi(x,y)}{P_D^\phi \sigma(x)} dx \right) \lambda(dy). \tag{3.69}$$

Lemma 4.2.3(b) for $U \equiv 1 \gtrsim 1/P_D^\phi \sigma$ and Lemma 4.2.4 imply that there is $C = C(d,D,\phi) > 0$ such that

$$\int_{\{\delta_D(x)\le t\}} \frac{G_D^\phi(x,y)}{P_D^\phi \sigma(x)} dx \le \begin{cases} Ct\delta_D(y), & \delta_D(y) < \frac{t}{2}, \\ C\widetilde{f}(y,t), & \delta_D(y) \ge \frac{t}{2}, \end{cases} \tag{3.70}$$

where $0 \le \widetilde{f}(y,t) \le t\,\delta_D(y)$ in $\{\delta_D(y) \ge \frac{t}{2}\}$ and $f(y,t)/t \to 0$ as $t \to 0$ for every $y \in D$. Hence, (3.69) and (3.70) imply

$$\frac{1}{t}\int_{\{\delta_D(x)\le t\}} \frac{G_D^\phi \lambda(x)}{P_D^\phi \sigma(x)} dx \le C\int_{\{\delta_D(y)<\frac{t}{2}\}} \delta_D(y)\lambda(dy) + C\int_{\{\delta_D(y)\ge\frac{t}{2}\}} \frac{\widetilde{f}(y,t)}{t}\lambda(dy)$$

from which the claim of the lemma follows by using the dominated convergence theorem.

■

3.3. Linear Dirichlet problem

In this section we deal with a linear Dirichlet problem for $\phi(-\Delta|_D)$ and develop some basic properties of a weak solution to the problem. At the end of the section, we connect the weak formulation of the problem with the distributional.

Definition 3.3.1. Let $\lambda \in \mathscr{M}(D)$ and $\zeta \in \mathscr{M}(\partial D)$ such that

$$\int_D \delta_D(x)|\lambda|(dx) + |\zeta|(\partial D) < \infty. \tag{3.71}$$

We say that $u \in L^1_{loc}(D)$ is a weak solution to the problem

$$\begin{cases} \phi(-\Delta|_D)u = \lambda, & \text{in } D, \\ \frac{u}{P_D^\phi \sigma} = \zeta, & \text{on } \partial D, \end{cases} \tag{3.72}$$

if for every $\psi \in C_c^\infty(D)$ it holds that

$$\int_D u(x)\psi(x)dx = \int_D G_D^\phi \psi(x)\lambda(dx) - \int_{\partial D} \frac{\partial}{\partial \mathbf{n}} G_D^\phi \psi(z)\zeta(dz). \tag{3.73}$$

If in (3.73) we have $\leq$ ($\geq$) instead of the equality and the inequality holds for every non-negative $\psi \in C_c^\infty(D)$, then we say u is a weak subsolution (supersolution) to the problem (3.72).

Remark 3.3.2. (a) Let $\psi \in C_c^\infty(D)$. From the calculations in the proof of Proposition 3.1.19, see also (3.47) and (3.48), it follows that $\frac{\partial}{\partial \mathbf{n}} G_D^\phi \psi(z)$ is well defined and

$$-\frac{\partial}{\partial \mathbf{n}} G_D^\phi \psi(z) = \int_D P_D^\phi(y,z)\psi(y)dy, \quad z \in \partial D,$$

holds, hence $\frac{\partial}{\partial \mathbf{n}} G_D^\phi \psi \in L^\infty(\partial D)$. Moreover, Lemma 3.1.8 implies that $|G_D^\phi \psi(x)| \lesssim \delta_D(x)$, thus the condition (3.71) ensures that the integrals in (3.73) are well defined.

(b) If u is a solution to the linear problem (3.72), then by using Fubini's theorem in (3.73) we get that

$$u = G_D^\phi \lambda + P_D^\phi \zeta, \quad \text{a.e. in } D. \tag{3.74}$$

This implies that $u \in L^1(D, \delta_D(x)dx)$. Indeed, $G_D^\phi \lambda \in L^1(D, \delta_D(x)dx)$ by Lemma 3.1.8, and $P_D^\phi \zeta \in L^1(D, \delta_D(x)dx)$ by (3.55).

Conversely, the function defined in (3.74) is the solution to linear problem (3.72) which we also get by using Fubini's theorem in (3.73).

The following theorem summarizes the previous remark.

Theorem 3.3.3. Let $\lambda \in \mathscr{M}(D)$ and $\zeta \in \mathscr{M}(\partial D)$ such that (3.71) holds. Then the linear problem (3.72) has a unique weak solution u for which it holds that $u \in L^1(D, \delta_D(x)dx)$ and

$$u(x) = G_D^\phi \lambda(x) + P_D^\phi \zeta(x), \quad \text{for a.e. } x \in D.$$

Furthermore, there is $C = C(d, D, \phi) > 0$ such that

$$\|u\|_{L^1(D,\delta_D(x)dx)} \leq C\left(\int_D \delta_D(x)|\lambda|(dx) + |\zeta|(\partial D)\right). \tag{3.75}$$

In the next corollary we bring a version of a maximum principle for the weak solution.

Corollary 3.3.4. Let $\lambda \in \mathscr{M}(D)$ and $\zeta \in \mathscr{M}(\partial D)$ such that (3.71) holds. If $\lambda \geq 0$ and $\zeta \geq 0$, then the unique solution u of the linear problem (3.72) satisfies $u \geq 0$ a.e. in D.

Now we connect the weak and the distributional formulation of the Dirichlet problem. First we give the definition of the distributional solution.

Definition 3.3.5. We say that $u \in L^1(D, \delta_D(x)dx)$ is a distributional solution to (3.72) if for every $\psi \in C_c^\infty(D)$ it holds that

$$\int_D u(x)\phi(-\Delta|_D)\psi(x)dx = \int_D \psi(x)\lambda(dx), \tag{3.76}$$

and if for every $\varphi \in C(\overline{D})$ it holds that

$$\lim_{t\downarrow 0} \frac{1}{t}\int_{\{\delta_D(x)\leq t\}} \frac{u(x)}{P_D^\phi \sigma(x)}\varphi(x)dx = \int_{\partial D} \varphi(z)\zeta(dz). \tag{3.77}$$

Proposition 3.3.6. Let $\lambda \in \mathscr{M}(D)$ and $\zeta \in L^1(\partial D)$ such that (3.71) holds. Then the weak solution to (3.72) is also a distributional solution to (3.72).

Proof. The weak solution is given by $u = G_D^\phi \lambda + P_D^\phi \zeta$ so the relation (3.76) follows from Proposition 3.1.14 and Theorem 3.1.22. The boundary condition (3.77) follows from Proposition 3.2.5 and Proposition 3.2.7. ∎

3.4. SEMILINEAR DIRICHLET PROBLEM

In this section we study the following semilinear problem.

Definition 3.4.1. Let $f : D \times \mathbb{R} \to \mathbb{R}$ and $\zeta \in \mathscr{M}(\partial D)$ such that $|\zeta|(\partial D) < \infty$. We say that $u \in L^1_{loc}(D)$ is a weak solution to the problem

$$\begin{cases} \phi(-\Delta|_D)u(x) = f(x,u(x)), & \text{in } D, \\ \frac{u}{P_D^\phi \sigma} = \zeta, & \text{on } \partial D, \end{cases} \tag{3.78}$$

if

$$\int_D u(x)\psi(x) = \int_D G_D^\phi \psi(x) f(x,u(x))dx - \int_{\partial D} \frac{\partial}{\partial \mathbf{n}} G_D^\phi \psi(z)\zeta(dz), \quad \psi \in C_c^\infty(D). \tag{3.79}$$

If in the equation above we have $\leq$ ($\geq$) instead of the equality and the inequality holds for every non-negative $\psi \in C_c^\infty(D)$, then we say u is a weak subsolution (supersolution) to (3.78).

Note that if u is a solution to the semilinear problem (3.78), then it is implicitly assumed that $x \mapsto f(x,u(x)) \in L^1(D, \delta_D(x)dx)$ since only then the first integral in (3.79) is well defined. For the sake of brevity, we will frequently use the notation $f_u(x) := f(x,u(x))$, $x \in D$, which is also known as Nemytskii operator. Further, in the same way as in the linear case we can see that if u is a weak solution to (3.78), then by Fubini's theorem used in (3.79) we get

$$u = G_D^\phi f_u + P_D^\phi \zeta. \tag{3.80}$$

Conversely, if u satisfies (3.80), then u is a weak solution to (3.78).

In the following subsection we prove Kato's inequality in our setting. This will help us to obtain existence and uniqueness results for various different nonlinearities f in the semilinear problem, which we do in the final subsection of the chapter.

3.4.1. Kato's inequality

The proof of Kato's inequality in our setting, i.e. Proposition 3.4.4, is motivated by the proofs of Kato's inequality found in [3,28] for the case of the spectral fractional Laplacian and the fractional Laplacian, respectively. First we need a lemma.

Lemma 3.4.2. Let w be the weak solution to the linear problem

$$\begin{cases} \phi(-\Delta|_D)u = h, & \text{in } D, \\ \frac{u}{P_D^\phi \sigma} = 0, & \text{on } \partial D, \end{cases}$$

for $h \in L^1(D, \delta_D(x)dx)$. Let $\Lambda \in C^2(\mathbb{R})$ be a convex function such that $\Lambda(0) = 0$, and such that $|\Lambda'| \le C$ for some $C > 0$. Then

$$\int_D \Lambda(w(x))\phi(-\Delta|_D)\psi(x)dx \le \int_D \Lambda'(w(x))h(x)\psi(x)dx, \quad \psi \in C_c^\infty(D), \tag{3.81}$$

and

$$\Lambda(w) \le G_D^\phi\left[\Lambda'(w)h\right] \quad \text{a.e. in } D. \tag{3.82}$$

Proof. Recall that $w = G_D^\phi h \in L^1(D, \delta_D(x)dx)$.

Let $h \in C_c^\infty(D)$. Then by Proposition 3.1.18 we have $w = G_D^\phi h \in C^{1,1}(\overline{D})$ from which we can calculate $\phi(-\Delta|_D)w$ and $\phi(-\Delta|_D)\Lambda(w)$ pointwisely, see Proposition 3.1.15. We have

$$\begin{aligned} \phi(-\Delta|_D)[\Lambda \circ w](x) &= \mathrm{P.V.} \int_D [\Lambda(w(x)) - \Lambda(w(y))]\, J_D(x,y)\, dy + \kappa(x)\, \Lambda(w(x)) \\ &= \Lambda'(w(x))\, \mathrm{P.V.} \int_D [w(x) - w(y)]\, J_D(x,y)\, dy + \kappa(x)\, \Lambda(w(x)) \\ &\quad - \mathrm{P.V.} \int_D \left([w(x) - w(y)]^2 J_D(x,y) \int_0^1 \Lambda''(w(x) + t[w(y) - w(x)])(1-t)\, dt \right) dy \\ &\le \Lambda'(w(x))\, \phi(-\Delta|_D)w(x), \end{aligned}$$

where we have used that $\Lambda'' \ge 0$ in $\mathbb{R}$ and that $\Lambda(t) \le t\Lambda'(t)$, which follows from $\Lambda(0) = 0$ and the fact that Λ' is non-decreasing. Integrating the previous inequality with respect to $\psi(x)dx$, where $0 \le \psi \in C_c^\infty(D)$, we get (3.81). Furthermore, since $w \in L^\infty(D)$, both sides of the previous inequality are in $L^\infty(D)$ so we can apply Proposition 3.1.2 to get

$$\Lambda(w) = G_D^\phi[\phi(-\Delta|_D)\Lambda(w)] \le G_D^\phi\left[\Lambda'(w)h\right] \quad \text{a.e. in } D,$$

i.e. (3.82) holds.

Let $h \in L^1(D, \delta_D(x)dx)$ and $(h_n)_n \subset C_c^\infty(D)$ such that $h_n \to h$ in $L^1(D, \delta_D(x)dx)$ and a.e. in D. By Corollary 3.1.9 we have $w_n := G_D^\phi h_n \to w$ in $L^1(D, \delta_D(x)dx)$ so by considering a subsequence we may assume that $w_n \to w$ a.e., too. From the first part of the proof we know

$$\int_D \Lambda(w_n(x))\phi(-\Delta|_D)\psi(x)dx \le \int_D \Lambda'(w_n(x))h_n(x)\psi(x)dx \tag{3.83}$$

$$\text{and} \qquad \Lambda(w_n) \le G_D^\phi\left[\Lambda'(w_n)h_n\right] \quad \text{a.e. in } D, \tag{3.84}$$

for all $n \in \mathbb{N}$ and all $0 \le \psi \in C_c^\infty(D)$.

Now we will take n in (3.83) and (3.84) to infinity. Recall that $|\phi(-\Delta|_D)\psi| \le C_1\delta_D$ by Lemma 3.1.11. Also, since $|\Lambda'| \le C$, we have $|\Lambda(t) - \Lambda(s)| \le C|t-s|$. By using these two facts and the fact that both $w_n \to w$ and $h_n \to h$ in $L^1(D, \delta_D(x)dx)$, both sides of (3.83) converge. Hence, by taking the limit in (3.83) we obtain

$$\int_D \Lambda(w(x))\phi(-\Delta|_D)\psi(x)dx \le \int_D \Lambda'(w(x))h(x)\psi(x)dx.$$

Before we take the limit in equality (3.84), note that $\Lambda \in C^2(\mathbb{R})$ so $\Lambda(w_n) \to \Lambda(w)$ and $\Lambda'(w_n) \to \Lambda'(w)$ a.e. in D. Further, again by $|\Lambda'| \le C$ and the fact that $h_n \to h$ in $L^1(D, \delta_D(x)dx)$ we have

$$\begin{aligned}\left|G_D^\phi\left[\Lambda'(w_n)h_n\right] - G_D^\phi\left[\Lambda'(w)h\right]\right| &\le G_D^\phi\left[|\Lambda'(w) - \Lambda'(w_n)||h|\right] \\ &\quad + G_D^\phi\left[|\Lambda'(w_n)||h-h_n|\right] \to 0, \quad n \to \infty,\end{aligned}$$

where the first term goes to zero by the dominated convergence theorem, and the second by the continuity of G_D^ϕ acting on $L^1(D, \delta_D(x)dx)$, i.e. by Lemma 3.1.8. This calculation justifies taking the limit in (3.84) to get

$$\Lambda(w) \le G_D^\phi\left[\Lambda'(w)h\right] \quad \text{a.e. in } D.$$

■

Remark 3.4.3. For $h \in L^\infty(D)$ the inequalities (3.81) and (3.82) hold for every convex function $\Lambda \in C^2(\mathbb{R})$ such that $\Lambda(0) = 0$ since the assumption $|\Lambda'| \le C$ was used only as a technical tool to justify the usage of the dominated convergence theorem for general $h \in L^1(D, \delta_D(x)dx)$.

In the next proposition we prove Kato's inequality which says that we can take $\Lambda(t) = t^+ = t \vee 0$ in Lemma 3.4.2.

Proposition 3.4.4 (Kato's inequality)**.** Let w be the weak solution to the linear problem

$$\begin{cases} \phi(-\Delta|_D)u = h, & \text{in } D, \\ \frac{u}{P_D^\phi \sigma} = 0, & \text{on } \partial D, \end{cases}$$

for $h \in L^1(D, \delta_D(x)dx)$. Then for every $\psi \in C_c^\infty(D)$, $\psi \geq 0$, it holds that

$$\int_D w(x)^+ \phi(-\Delta|_D)\psi(x)dx \leq \int_{\{w>0\}} h(x)\psi(x)dx. \tag{3.85}$$

Moreover, it holds that

$$w^+ \leq G_D^\phi \left[\mathbf{1}_{\{w>0\}}h\right], \quad \text{a.e. in } D. \tag{3.86}$$

Proof. First, let us prove (3.85). Set $\Lambda(t) = t \vee 0$ and $w = G_D^\phi h$ where $h \in L^1(D, \delta_D(x)dx)$. Also, for every $n \in \mathbb{N}$ let $\Lambda_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$\Lambda_n(t) = \begin{cases} 0, & t \leq 0 \\ \frac{n^2t^3}{6}, & t \in (0, \frac{1}{n}] \\ \frac{1}{3n} - t + nt^2 - \frac{n^2t^3}{6}, & t \in (\frac{1}{n}, \frac{2}{n}] \\ t - \frac{1}{n}, & t > \frac{2}{n}. \end{cases} \tag{3.87}$$

We have that $\Lambda_n \in C^2(\mathbb{R})$, $0 \leq \Lambda_n \leq \Lambda$, and $0 \leq \Lambda_n' \leq 1$ in $\mathbb{R}$. Also, $\Lambda_n \to \Lambda$ and $\Lambda_n' \to \mathbf{1}_{(0,\infty)}$ in $\mathbb{R}$ as $n \to \infty$. Thus, Lemma 3.4.2 yields

$$\int_D \Lambda_n(w(x))\phi(-\Delta|_D)\varphi(x)dx \leq \int_D \Lambda_n'(w(x))h(x)\varphi(x)dx \tag{3.88}$$

and the relation (3.85) follows from (3.88) by using the dominated convergence theorem.

Let us now turn to (3.86). Consider again the sequence Λ_n defined above. Lemma 3.4.2 yields

$$\Lambda_n(w) \leq G_D^\phi \left[\Lambda_n'(w)h\right], \quad \text{a.e. in } D \text{ and for all } n \in \mathbb{N}. \tag{3.89}$$

Again, by taking $n \to \infty$ and by using the dominated convergence theorem we get

$$w^+ \leq G_D^\phi \left[\mathbf{1}_{\{w>0\}}h\right], \quad \text{a.e. in } D.$$

∎

Remark 3.4.5. By modifying the proof of the previous proposition we also get

$$\int_D w(x)^+ \phi(-\Delta|_D)\psi(x)dx \le \int_{\{w\ge 0\}} h(x)\psi(x)dx, \tag{3.90}$$

and

$$w^+ \le G_D^\phi\left[\mathbf{1}_{\{w\ge 0\}}h\right], \quad \text{a.e. in } D. \tag{3.91}$$

Indeed, in the proof we only need to change Λ_n to $\tilde{\Lambda}_n \in C^2(\mathbb{R})$ such that $\tilde{\Lambda}_n(t) = \Lambda_n(t + \frac{2}{n}) - \frac{1}{n}$. For $\tilde{\Lambda}_n$ it holds that

$$-\frac{1}{n} \le \tilde{\Lambda}_n \le \Lambda, \quad 0 \le \tilde{\Lambda}_n' \le 1, \quad \lim_n \tilde{\Lambda}_n = \Lambda, \text{ and } \lim_n \tilde{\Lambda}_n' = \mathbf{1}_{[0,\infty)}$$

in $\mathbb{R}$. By repeating the procedure in the proof of the previous proposition we get the claim.

Remark 3.4.6. Note that Kato's inequality was proved only for weak solutions of linear problems with a zero boundary condition whereas the classical Kato's inequality holds for subsolutions even if the considered linearity is a measure, see [24]. To the best of our knowledge it is not clear whether the inequality (3.85) holds for subsolutions since the non-local nature of the operator $\phi(-\Delta|_D)$ causes problems in the calculations in Proposition 3.4.4. Even in simpler non-local cases as in [3] and [28] Kato's inequality was proved only for solutions, see [3, Lemma 31] and [28, Proposition 2.4].

In the next corollary we bring a simple consequence of Kato's inequality which is the fact interesting in itself.

Corollary 3.4.7. Let u and v be weak solutions of (3.78). Then $\max\{u,v\}$ is a subsolution to (3.78).

Proof. Applying Proposition 3.4.4 to the $w := u - v$ and $h(x) := f(x,u(x)) - f(x,v(x))$ we get

$$\int_D w^+(x)\phi(-\Delta|_D)\psi(x)dx \le \int_{u>v} \left[f(x,u(x)) - f(x,v(x))\right]\psi(x)dx, \quad \psi \in C_c^\infty(D),\ \psi \ge 0.$$

Since $\max\{u,v\} = v + (u-v)^+ = v + w^+$ we have for all non-negative $\psi \in C_c^\infty(D)$

$$\begin{aligned}
\int_D \max\{u,v\}(x)\phi(-\Delta|_D)\psi(x)dx \\
&\le \int_D f(x,v(x))\psi(x)dx - \int_{\partial D} \frac{\partial}{\partial \mathbf{n}} G_D^\phi \psi(z)\zeta(dz) \\
&\qquad + \int_{u>v} \left[f(x,u(x)) - f(x,v(x))\right]\psi(x)dx \\
&= \int_{u\le v} f(x,v(x))\psi(x)dx - \int_{\partial D} \frac{\partial}{\partial \mathbf{n}} G_D^\phi \psi(z)\zeta(dz) \\
&\qquad + \int_{u>v} f(x,u(x))\psi(x)dx \\
&= \int_D f(x,\max\{u,v\}(x))\psi(x)dx - \int_{\partial D} \frac{\partial}{\partial \mathbf{n}} G_D^\phi \psi(z)\zeta(dz).
\end{aligned}$$

■

3.4.2. Semilinear problem

In this subsection we prove existence and uniqueness results for the semilinear problem (3.78). As such, the subsection is central for the chapter.

For the nonlinearity f in the following problems we will almost always assume that the following condition holds true.

(F). $f : D\times\mathbb{R} \to \mathbb{R}$ is continuous in the second variable, and there exist a locally bounded function $\rho : D \to [0,\infty]$ and a non-decreasing function $\Lambda : [0,\infty) \to [0,\infty)$ such that $|f(x,t)| \le \rho(x)\Lambda(|t|)$, $x\in D$, $t\in\mathbb{R}$.

From now on, the function f will be solely used as a nonlinearity in the semilinear problem and the functions ρ and Λ are solely used as the functions in the condition **(F)** for f.

Our first result is the uniqueness theorem for general nonlinearity f which is non-increasing in the second variable.

Proposition 3.4.8. If the nonlinearity f in (3.78) is non-increasing in the second variable, then the weak solution to (3.78), if it exists, is unique (up to the modification on the Lebesgue null set).

Proof. Let u and v be two solutions of (3.78). Then $w := u - v$ solves the linear problem

$$\begin{cases} \phi(-\Delta|_D)w(x) = f(x,u(x)) - f(x,v(x)), & \text{in } D, \\ \frac{w}{P_D^\phi \sigma} = 0, & \text{on } \partial D. \end{cases}$$

By Kato's inequality (3.86), since f is non-increasing in the second variable, we have

$$w^+ \leq G_D^\phi \left[\mathbf{1}_{\{u>v\}} \cdot \left(f_u - f_v \right) \right] \leq 0. \tag{3.92}$$

Thus, $u \leq v$ a.e. in D. Reversing the roles of u and v we get $u \geq v$ a.e. in D, hence $u = v$ a.e. in D. ∎

The next theorem, Theorem 3.4.9, deals with a semilinear problem with a zero boundary condition and it is a generalization of [3, Theorem 32] to our setting of more general non-local operators. Theorem 3.4.9 will be of great importance for a general semilinear problem (with a non-zero boundary condition), and it is, in fact, the cornerstone of the proof of Theorem 3.4.10. A somewhat similar role for the semilinear problem in a slightly different non-local setting is played by Theorem 2.3.6.

Theorem 3.4.9. Let f satisfy (F). Assume that there exist a supersolution $\overline{u}$ and a subsolution $\underline{u}$ to the semilinear problem

$$\begin{cases} \phi(-\Delta|_D)u(x) = f(x,u(x)), & \text{in } D, \\ \frac{u}{P_D^\phi \sigma} = 0, & \text{on } \partial D, \end{cases} \tag{3.93}$$

of the form $\underline{u} = G_D^\phi \underline{h}$ and $\overline{u} = G_D^\phi \overline{h}$ such that $\underline{u} \leq \overline{u}$, $\underline{h}(x) \leq f(x,\underline{u}(x))$ and $f(x,\overline{u}(x)) \leq \overline{h}(x)$ a.e. in D, and such that $\overline{u}, \underline{u} \in L^1(D, \delta_D(x)dx)$. Further, assume that $\rho\Lambda(|\underline{u}| \vee |\overline{u}|) \in L^1(D, \delta_D(x)dx)$.

Then there exist weak solutions $u_1, u_2 \in L^1(D, \delta_D(x)dx)$ of (3.93) such that every solution to (3.93) with property $\underline{u} \leq u \leq \overline{u}$ satisfies

$$\underline{u} \leq u_1 \leq u \leq u_2 \leq \overline{u}.$$

Further, every weak solution u of (3.93) with property $\underline{u} \leq u \leq \overline{u}$ is continuous after the modification on a Lebesgue null set.

Additionally, if the nonlinearity f is non-increasing in the second variable, the weak solution to (3.93) is unique.

Proof. Step 1: existence of a solution to (3.93). Define the function $F : D\times\mathbb{R}\to\mathbb{R}$ by

$$F(x,t)=\begin{cases} f(x,\underline{u}(x)), & t<\underline{u}(x),\\ f(x,t), & \underline{u}\le t\le\overline{u},\\ f(x,\overline{u}(x)), & \overline{u}(x)<t,\end{cases}$$

and denote by $F_v(x):=F(x,v(x))$. Note that since f is continuous in the second variable, so is F. Further, $|F_v|\le\rho\Lambda(|\underline{u}|\vee|\overline{u}|)$, hence $F_v\in L^1(D,\delta_D(x)dx)$, for all $v\in L^1(D,\delta_D(x)dx)$.

Also, the mapping $v\mapsto F_v$ is continuous from $L^1(D,\delta_D(x)dx)$ to $L^1(D,\delta_D(x)dx)$. Indeed, take $v_n\to v$ in $L^1(D,\delta_D(x)dx)$ and let $(v_{n_k})_k$ be a subsequence of $(v_n)_n$ which converges to v a.e. By Lemma 4.4.1 the family $\{F_{v_{n_k}}:k\in\mathbb{N}\}$ is uniformly integrable with respect to the measure $\delta_D(x)dx$, hence by Vitali's theorem [69, Theorem 16.6], we get $F_{v_{n_k}}\to F_v$ in $L^1(D,\delta_D(x)dx)$ because F is continuous in the second variable. However, the limit does not depend on the subsequence $(v_{n_k})_k$ so $v\mapsto F_v$ is continuous.

Next we prove that the operator $\mathscr{K}:L^1(D,\delta_D(x)dx)\to L^1(D,\delta_D(x)dx)$ defined by

$$\mathscr{K}v(x)=\int_D G_D^\phi(x,y)F(y,v(y))dy,\quad x\in D,$$

is compact. Since $v\mapsto F_v$ is continuous in $L^1(D,\delta_D(x)dx)$, Corollary 3.1.9 implies that $\mathscr{K}$ is continuous $L^1(D,\delta_D(x)dx)$, too. To have compactness, we are left to prove that $\mathscr{K}$ maps bounded sets to relatively compact sets. To this end, take a bounded sequence $(v_n)_n\subset L^1(D,\delta_D(x)dx)$. Recall $|F_{v_n}|\le\rho\Lambda(|\underline{u}|\vee|\overline{u}|)$ so $(\mathscr{K}v_n)_n$ are pointwisely bounded by Proposition 3.1.16 and equicontinuous by Remark 3.1.17. By Arzelà-Ascoli theorem, there is a subsequence $(\mathscr{K}v_{n_k})_k$ of $(\mathscr{K}v_n)_n$ which converges pointwisely to some $u\in C(D)\cap L^1(D,\delta_D(x)dx)$. Since $\mathscr{K}v_n=G_D^\phi F_{v_n}$, Lemma 3.1.8 implies that that $\{\mathscr{K}v_{n_k}:k\in\mathbb{N}\}$ is uniformly integrable with respect to the measure $\delta_D(x)dx$ since $\{F_{v_{n_k}}:k\in\mathbb{N}\}$ is. However, $\mathscr{K}v_{n_k}\to u$ pointwisely so by Vitali's theorem [69, Theorem 16.6] we have $\mathscr{K}v_{n_k}\to u$ in $L^1(D,\delta_D(x)dx)$.

This means that $\mathscr{K}$ is compact so by Schauder's fixed point theorem there is $u\in L^1(D,\delta_D(x)dx)$ such that $\mathscr{K}u=u$ in D, i.e. u solves

$$\begin{cases}\phi(-\Delta|_D)u(x)=F(x,u(x)), & \text{in } D,\\ \frac{u}{P_D^\phi\sigma}=0, & \text{on } \partial D.\end{cases}$$

We need to prove that $\underline{u} \le u \le \overline{u}$ in D which would mean that u also solves (3.93). For this step we will use Kato's inequality. More precisely, applying Proposition 3.4.4 to $w = u - \overline{u} = G_D^\phi(F_u - \overline{h})$ we get

$$(u-\overline{u})^+ \le G_D^\phi\left[\mathbf{1}_{\{u>\overline{u}\}} \cdot \left(F_u - \overline{h}\right)\right] \le G_D^\phi\left[\mathbf{1}_{\{u>\overline{u}\}} \cdot \left(f_{\overline{u}} - f_{\overline{u}}\right)\right] = 0, \tag{3.94}$$

where the second inequality holds since $F(x,u(x)) = f(x,\overline{u}(x))$ on $\{u \ge \overline{u}\}$ and since we assume $f(x,\overline{u}) \le \overline{h}$ a.e. in D. This means $u \le \overline{u}$ a.e. in D. Similarly we get that $\underline{u} \le u$ a.e. in D. Hence, we found a solution to the problem (3.93).

Step 2: finding the maximal and the minimal solution. We adapt a method from [34, Theorem 1.3] which uses Zorn's lemma.

Let $\mathscr{P} := \{u \in L^1(D,\delta_D(x)dx) : \underline{u} \le u \le \overline{u}$ and u solves (3.93)$\}$. Let $\{u_i\}_{i\in\mathscr{I}}$ be a totally ordered subset of $\mathscr{P}$. Since $u_i \in \mathscr{P}$ and since we assume $\rho\Lambda(|\underline{u}| \vee |\overline{u}|) \in L^1(D,\delta_D(x)dx)$, it follows that $\{u_i\}_{i\in\mathscr{I}}$ is equicontinuous in D. In fact, by Remark 3.1.17 the set $\{u_i\}_{i\in\mathscr{I}}$ is equicontinuous on every compact subset of D. Hence, the function $u := \sup_{i\in\mathscr{I}} u_i$ is continuous and u can be approximated by $\{u_i\}_{i\in\mathscr{I}}$ uniformly on compact subsets of D. Moreover, D is σ-compact so we can choose an increasing sequence $(u_n)_n \subset \{u_i\}_{i\in\mathscr{I}}$ such that $\lim_n u_n(x) = u(x)$ for all $x \in D$.

By the dominated convergence theorem, since $|f_{u_n}| \le \rho\Lambda(|\underline{u}| \vee |\overline{u}|)$, it easily follows by the continuity of f in the second variable that $u = \lim_n u_n = \lim_n G_D^\phi(f_{u_n}) = G_D^\phi(f_u)$, i.e. $u \in \mathscr{P}$. Now Zorn's lemma implies that there exists the maximal solution u_2 of (3.93). We find the minimal solution u_1 in the same way.

Step 3: continuity of solutions. We prove that every solution to (3.93) with property $\underline{u} \le u \le \overline{u}$ is continuous up to the modification. Indeed, every solution satisfies $u = G_D^\phi f_u$ a.e. in D. Furthermore, since $\underline{u} \le u \le \overline{u}$ and $\rho\Lambda(|\underline{u}| \vee |\overline{u}|) \in L^1(D,\delta_D(x)dx)$, we have $G_D^\phi f_u \in C(D)$ by Proposition 3.1.16. Finally, $\tilde{u} := G_D^\phi f_u$ is a continuous modification of u, hence $f_u = f_{\tilde{u}}$ a.e. in D, hence $\tilde{u} = G_D^\phi f_{\tilde{u}}$ in D.

Step 4: uniqueness of solution. In the case when f is non-increasing in the second variable, uniqueness follows from Proposition 3.4.8. ■

By using the previous theorem, a method of sub- and super-solutions and the approximation of harmonic functions, we solve a semilinear problem which deals with a non-positive nonlinearity f and a non-negative boundary condition ζ. Theorem 3.4.10

generalizes [3, Theorem 8] to our setting of more general non-local operators. Moreover, we consider a more general boundary condition which can also be a measure, whereas in [3, Theorem 8] only continuous functions where considered. The nonlinearity in our theorem is also slightly more general then the one in [3, Theorem 8]. A similar result in a slightly different non-local setting can be found in Theorem 2.3.10.

Theorem 3.4.10. Let $f : D \times \mathbb{R} \to (-\infty, 0]$ such that $f(x,0) = 0$, $x \in D$, and such that f satisfies (**F**). Further, let $\zeta \in \mathscr{M}(\partial D)$ be a finite non-negative measure such that

$$\rho\Lambda(P_D^\phi \zeta) \in L^1(D, \delta_D(x)dx).$$

Then the problem

$$\begin{cases} \phi(-\Delta|_D)u(x) = f(x, u(x)), & \text{in } D, \\ \frac{u}{P_D^\phi \sigma} = \zeta, & \text{on } \partial D, \end{cases} \tag{3.95}$$

has a weak solution $u \in C(D) \cap L^1(D, \delta_D(x)dx)$.

Additionally, if f is non-increasing in the second variable, the continuous weak solution to (3.95) is unique.

Proof. Let $(\widetilde{f}_k)_k$ be a non-negative sequence of bounded functions such that $G_D^\phi \widetilde{f}_k \uparrow P_D^\phi \zeta$ in D. This sequence exists by Subsection 4.1 in Appendix since the semigroup $(Q_t^D)_t$ is strongly Feller, $G_D^\phi \delta_D \asymp \delta_D$ by Lemma 3.1.8, and since $P_D^\phi \zeta$ is a continuous function with the mean-value property with respect to Y^D, see Theorem 3.1.24 and Theorem 3.1.26.

We build a sequence of solutions to the following semilinear problems

$$\begin{cases} \phi(-\Delta|_D)u(x) = f(x, u(x)) + \widetilde{f}_k, & \text{in } D, \\ \frac{u}{P_D^\phi \sigma} = 0, & \text{on } \partial D. \end{cases} \tag{3.96}$$

For every $k \in \mathbb{N}$, a subsolution to (3.96) is $\underline{u} = 0$ since $f(x,0) = 0$ and since $\widetilde{f}_k \geq 0$. A supersolution to (3.96) is $\overline{u} = G_D^\phi \widetilde{f}_k$ because f is non-positive. Note that both $\underline{u}$ and $\overline{u}$ are bounded functions, so it is trivial to check that the assumptions of Theorem 3.4.9 are satisfied. Hence, for every $k \in \mathbb{N}$ there is a solution $u_k \geq 0$ to (3.96) which is also continuous in D and satisfies

$$u_k = G_D^\phi f_{u_k} + G_D^\phi \widetilde{f}_k, \quad \text{in } D. \tag{3.97}$$

Now we find an appropriate subsequence of $(u_k)_k$ which converges to a solution to (3.95). Since $G_D^\phi \widetilde{f}_k$ is continuous and increases to the continuous function $P_D^\phi \zeta$, by Dini's theorem the convergence is locally uniform so the usual 3ε-argument gives equicontinuity of the family $(G_D^\phi \widetilde{f}_k)_k$. Also, since $|f_{u_k}| \le \rho\Lambda(P_D^\phi \zeta)$, equicontinuity of $(G_D^\phi(f_{u_k}))_k$ follows by Proposition 3.1.16 and Remark 3.1.17. Hence, Arzelà-Ascoli theorem gives us a subsequence, denoted again by $(u_k)_k$, which converges to a continuous function u.

Now we show that u is a solution to (3.95). Obviously, since $u = \lim_{k\to\infty} u_k$ and $0 \le u_k \le G_D^\phi \widetilde{f}_k \le P_D^\phi \zeta < \infty$, u is non-negative and finite. Further, $G_D^\phi \widetilde{f}_k \uparrow P_D^\phi \zeta$, so we are left to prove that $G_D^\phi f_{u_k} \to G_D^\phi f_u$. However, this is easy since $|f_{u_k}| \le \rho\Lambda(P_D^\phi \zeta)$, so continuity of f in the second variable and the dominated convergence imply $G_D^\phi f_{u_k} \to G_D^\phi f_u$.

Uniqueness, if f is non-increasing in the second variable, follows from Proposition 3.4.8. ■

Remark 3.4.11. Applying the Zorn's lemma argument from the proof of Theorem 3.4.9 we get that for the problem (3.95) there exists a minimal solution u_1 and a maximal solution u_2 such that for every solution u of (3.95) we have

$$0 \le u_1 \le u \le u_2 \le P_D^\phi \zeta, \quad \text{in } D.$$

We say that $\Lambda : [0,\infty) \to [0,\infty)$ satisfies the doubling condition if there exists $C > 0$ such that

$$\Lambda(2t) \le C\Lambda(t), \quad t \ge 1. \tag{3.98}$$

If Λ is non-decreasing, the condition (3.98) implies that for every $c_1 > 1$ there is $c_2 = c_2(C, c_1) > 0$ such that

$$\Lambda(c_1 t) \le c_2 \Lambda(t), \quad t \ge 1. \tag{3.99}$$

Corollary 3.4.12. Let $f : D \times \mathbb{R} \to (-\infty, 0]$ such that $f(x,0) = 0$, $x \in D$. Let f also satisfy **(F)** such that Λ satisfies the doubling condition (3.98).

If $\rho\Lambda\left(\frac{1}{\delta_D^2 \phi(\delta_D^{-2})}\right) \in L^1(D, \delta_D(x)dx)$, then the problem

$$\begin{cases} \phi(-\Delta|_D)u(x) = f(x,u(x)), & \text{in } D, \\ \frac{u}{P_D^\phi \sigma} = \zeta, & \text{on } \partial D, \end{cases} \tag{3.100}$$

has a continuous weak solution for every non-negative function $\zeta \in C(D)$. Additionally, if f is non-increasing in the second variable, the continuous weak solution is unique.

In particular, if $f(x,t) = -|t|^p$, then the equation (3.100) has a unique continuous weak solution for $p < \frac{1}{1-\delta_1}$, where δ_1 comes from (3.3).

Proof. Note that for $\zeta \in C(D)$ we have $P_D^\phi \zeta \le c_1 \frac{1}{\delta_D^2 \phi(\delta_D^{-2})}$ by Lemma 3.2.1 since ζ is bounded on ∂D. Thus, from the doubling condition we have

$$\rho\Lambda(P_D^\phi \zeta) \le c_2 \rho\Lambda\left(\frac{1}{\delta_D^2 \phi(\delta_D^{-2})}\right) \in L^1(D, \delta_D(x)dx)$$

so we can apply Theorem 3.4.10 to get the claim.

In the special case $f(x,t) = -|t|^p$ we have $\rho \equiv 1$ and $\Lambda(t) = t^p$ so (3.3) and the reduction to the flat case give us

$$\int_D \rho\Lambda\left(\frac{1}{\delta_D^2 \phi(\delta_D^{-2})}\right)\delta_D dx \asymp \int_D \frac{\delta_D}{\delta_D^{2p}\phi(\delta_D^{-2})^{2p}} dx \lesssim \int_0^1 t^{1-2p+2p\delta_1} dt$$

which is finite if $p < \frac{1}{1-\delta_1}$. ■

Remark 3.4.13. Assume that we are in the spectral fractional Laplacian case in the previous corollary, i.e. if $\phi(\lambda) = \lambda^s$, for some $s \in (0,1)$. Then we can find a solution to (3.100) for $f(x,t) = -|t|^p$ and for every non-negative $\zeta \in C(\partial D)$ if $p < \frac{1}{1-s}$ since $\delta_1 = s$ in this case.

Conversely, if $f(x,t) = -|t|^p$ for $p \ge \frac{1}{1-s}$, and we additionally demand that the boundary condition holds pointwisely for a non-negative $\zeta \in C(\partial D)$ such that $\zeta \not\equiv 0$, then the problem (3.100) does not have a solution. Indeed, assume that u is a solution to (3.100) and that the boundary condition holds pointwisely. Then $u \gtrsim \delta_D^{2s-2}$ near $z \in \partial D$ such that $\zeta(z) > 0$ since $P_D^\phi \zeta \asymp \delta_D^{2s-2}$ near such z, see Proposition 3.2.4. Thus, $|u|^p \notin L^1(D, \delta_D(x)dx)$ since $p \ge \frac{1}{1-s}$, i.e. $G_D^\phi f_u = \infty$ in D by Lemma 3.1.8, which is a contradiction.

One of the weaknesses of Theorem 3.4.9 is that one has to have a supersolution and a subsolution which are strictly Green potentials, i.e. a supersolution and a subsolution cannot consist of Poisson integrals which are annulled by $\phi(-\Delta|_D)$, since only then we may use Kato's inequality (3.86). However, in some cases we can exploit some other methods for obtaining a solution to a semilinear problem. For example, in the next theorem we deal with a non-negative nonlinearity f and a non-negative boundary condition ζ and we use a method of monotone iterations to obtain a solution.

Theorem 3.4.14. Let $f : D\times\mathbb{R}\to[0,\infty)$ satisfy (F), and let f be a non-decreasing function in the second variable. Let ζ be a non-negative finite measure on ∂D such that

$$G_D^\phi\big(\rho\Lambda(2P_D^\phi\zeta)\big)\le P_D^\phi\zeta,\quad \text{in } D. \tag{3.101}$$

There there is a continuous non-negative solution to

$$\begin{cases}\phi(-\Delta|_D)u(x)=f(x,u(x)), & \text{in } D,\\ \frac{u}{P_D^\phi\sigma}=\zeta, & \text{on } \partial D.\end{cases} \tag{3.102}$$

Proof. We use a method of monotone iterations. Let $u_0=0$, and define for $n\ge 1$

$$u_n=G_D^\phi\big(f_{u_{n-1}}\big)+P_D^\phi\zeta.$$

Since f is non-negative and non-decreasing in the second variable, it follows that $(u_n)_n$ is non-negative and non-decreasing, too. However, by induction it is easy to see that $0\le u_n\le 2P_D^\phi\zeta$. Indeed, for u_0 this fact is trivial, and for $n\ge1$ by (3.101) we have

$$u_n=G_D^\phi\big(f_{u_{n-1}}\big)+P_D^\phi\zeta\le G_D^\phi\big(\rho\Lambda(2P_D^\phi\zeta)\big)+P_D^\phi\zeta\le 2P_D^\phi\zeta.$$

This means that $u=\uparrow\lim_{n\to\infty}u_n$ is well defined. Since f is continuous in the second variable by (F) and since the integrability condition (3.101) holds, by the dominated convergence theorem we get

$$u=G_D^\phi f_u+P_D^\phi\zeta,$$

i.e. we found a solution to (3.102).

For the continuity of u, note that since $u\le 2P_D^\phi\zeta$, the condition (3.101) implies that $f_u\in L^1(D,\delta_D(x)dx)$ in the following way

$$\int_D f_u(x)\delta_D(x)dx\asymp\int_D f_u(x)G_D^\phi\delta_D(x)dx=\int_D G_D^\phi(f_u)(x)\delta_D(x)\le\int_D P_D^\phi\zeta(x)\delta_D(x)dx<\infty.$$

Now Proposition 3.1.16 and Theorem 3.1.24 give $u\in C(D)$. ∎

Remark 3.4.15. If we are in the spectral fractional Laplacian case in the previous theorem, i.e. if $\phi(\lambda)=\lambda^s$, for some $s\in(0,1)$, then there exists a solution to (3.102) for any non-negative $\zeta \in C(\partial D)$ and for the nonlinearity $f(x,t) = m|t|^p$, where $m > 0$ is sufficiently small and $p < \frac{1}{1-s}$. Indeed, in this case $P_D^\phi \zeta \lesssim \delta_D(x)^{2-2s}$, and $(P_D^\phi \zeta)^p \in L^1(D, \delta_D(x)dx)$ if $p < \frac{1}{1-s}$. Obviously, we chose the parameter $m > 0$ so small so that (3.101) holds.

Conversely, if $p \geq \frac{1}{1-s}$, then the problem (3.102) does not have a solution for $f(x,t) = m|t|^p$ for any $m > 0$ and for any non-negative $\zeta \in C(\partial D)$ such that $\zeta \not\equiv 0$. Indeed, assume that u solves (3.102). Then $u \geq P_D^\phi \zeta$ since $f \geq 0$ and $P_D^\phi \zeta \gtrsim \delta_D^{2-2s}$, near $z \in \partial D$ such that $\zeta(z) > 0$, see Proposition 3.2.4. Hence for $p \geq \frac{1}{1-s}$ the function $(P_D^\phi \zeta)^p \notin L^1(D, \delta_D(x)dx)$ which implies $u = G_D^\phi f_u + P_D^\phi \gtrsim G_D^\phi\left((P_D^\phi \zeta)^p\right) = \infty$ in D, by Lemma 3.1.8.

To obtain a solution to a semilinear problem with an unsigned nonlinearity f and an unsigned boundary condition ζ we need some stronger assumptions on the nonlinearity f. The following theorem is in spirit same as [20, Theorem 2.4] and Corollary 2.3.8 which were proved in a different non-local setting.

Theorem 3.4.16. Let $f : D \times \mathbb{R} \to \mathbb{R}$ satisfy (**F**) and let ζ be a finite measure on ∂D. Assume that $G_D^\phi \rho \in C_0(D)$ and $G_D^\phi\big(\rho\Lambda(2P_D^\phi|\zeta|)\big) \in C_0(D)$. Assume additionally that: (a) Λ is sublinearly increasing, i.e. $\lim_{t\to\infty} \Lambda(t)/t = 0$, or (b) $m > 0$ is sufficiently small. Then the semilinear problem

$$\begin{cases} \phi(-\Delta|_D)u(x) = m f(x,u(x)), & \text{in } D, \\ \frac{u}{P_D^\phi \sigma} = \zeta, & \text{on } \partial D. \end{cases} \tag{3.103}$$

has a weak continuous solution u such that $|u| \leq C + P_D^\phi|\zeta|$, for some constant $C \geq 0$.

If, in addition, f is non-increasing in the second variable, u is a unique weak solution to (3.103).

Proof. The proof follows the proof of [20, Theorem 2.4] and we repeat the main steps for the reader's convenience.

Define the operator T on $C_0(D)$ by

$$Tv(x) = \int_D G_D^\phi(x,y) m f(y, v(y) + P_D^\phi \zeta) dy, \quad v \in C_0(D),\ x \in D.$$

Our goal is to get a fixed point of the operator T from which we will extract a solution to (3.103).

Let $r_\rho = \sup_{x\in D} G_D^\phi \rho(x) < \infty$ and $r_\zeta = \sup_{x\in D} G_D^\phi\big(\rho\Lambda(2P_D^\phi|\zeta|)\big)(x) < \infty$. Let $C \geq 0$ and define $K := \{v \in C_0(D) : \|v\|_\infty \leq C\}$. It is easy to show that for $a,b > 0$ we have $\Lambda(a+b) \leq \Lambda(2a) + \Lambda(2b)$. Hence,

$$|f(y,v(y)+P_D^\phi\zeta(y))| \leq \rho(y)\Lambda(|v(y)|+P_D^\phi|\zeta|(y)) \leq \rho(y)\Lambda(2C)+\rho\Lambda(2P_D^\phi|\zeta|(y)), \quad v \in K,$$

so $Tv \in C_0(D)$ by the upper bound and the same calculations as in Proposition 3.1.16. Moreover,

$$\begin{aligned}\|Tv\|_\infty &= \sup_{x\in D}\Big|\int_D G_D^\phi(x,y)mf(y,v(y)+P_D^\phi\zeta(y))dy\Big| \\ &\leq \sup_{x\in D}\int_D G_D^\phi(x,y)m\big(\rho(y)\Lambda(2C)+\rho\Lambda(2P_D^\phi|\zeta|(y))\big)dy \leq m\big(r_\rho\Lambda(2C)+r_\zeta\big).\end{aligned}$$

If m is sufficiently small or Λ sublinearly increases, there is $C>0$ such that $m\big(r_\rho\Lambda(2C)+r_\zeta\big) \leq C$. Fix this C. We will now use Schauder's fixed point theorem on T. By the choice of C, we have $T[K] \subset K$. Also, T is a continuous operator on K. This is proved by assuming the opposite as in the proof of Theorem 2.3.6 (iii) for the operator defined in (2.23), see also (2.24). Further, the family $\{Tv : v \in K\}$ is equicontinuous in D by the inequality

$$|Tv(x)-Tv(\xi)| \leq \int_D |G_D^\phi(x,y)-G_D^\phi(\xi,y)|m\big(\rho(y)\Lambda(2C)+\rho\Lambda(2P_D^\phi|\zeta|(y))\big)dy, \quad v \in K,$$

and by the Remark 3.1.17. Arzelà-Ascoli theorem implies that $T[K]$ is precompact in K, thus, by Schauder's fixed point theorem there exist $u_0 \in K$ such that $Tu_0 = u_0$. To finish the proof, notice that the function

$$u(x) := u_0(x)+P_D^\phi\zeta(x) = \int_D G_D^\phi(x,y)mf(y,u(y))dy + P_D^\phi\zeta(x)$$

solves (3.103), and it holds that $u \in C(D)$ and $|u| \leq C + P_D^\phi|\zeta|$. ∎

Remark 3.4.17. In the spectral fractional case where $\phi(\lambda) = \lambda^s$, for some $s \in (0,1)$, when $\zeta \in C(\partial D)$, we have a solution to (3.103) for the nonlinearity f which satisfies $|f(x,t)| \lesssim |t|^p$ if $p < \frac{s}{1-s}$. Indeed, in that case $P_D^\phi|\zeta| \lesssim \delta_D^{2s-2}$, hence $(P_D^\phi|\zeta|)^p \in L^1(D,\delta_D(x)dx)$ and $G_D^\phi\big((P_D^\phi|\zeta|)^p\big) \in C_0(D)$ by Theorem 3.2.6, or see [36, Proposition 7]. Note that the range $p < \frac{s}{1-s}$ is worse then the one for Corollary 3.4.12 and Theorem 3.4.14, see Remarks 3.4.13 and 3.4.15.

4. APPENDIX

4.1. APPROXIMATION OF EXCESSIVE FUNCTIONS

Let $(X_t, \mathbb{P}_x)$ be a Hunt process on a locally compact space D and let $(P_t)_{t\geq 0}$ denote its semigroup. Let U be the potential operator of X, that is

$$Uf(x) = \mathbb{E}_x \int_0^\infty f(X_t)\,dt = \mathbb{E}_x \int_0^\zeta f(X_t)dt = \int_0^\infty P_t f(x)dt.$$

Here ζ denotes the lifetime of the process. We assume that X is transient in the sense that there exists a non-negative measurable function h such that $0 < Uh < \infty$, see [32, p. 86], and also that (P_t) is strongly Feller. What follows essentially comes from [32, Section 3.2]. Recall that a measurable function $f : E \to [0, \infty]$ is said to be excessive relative to $(P_t)_{t\geq 0}$ if $f \geq P_t f$ for all $t \geq 0$ and $f = \lim_{t\to 0} P_t f$ (see for example [32, Section 2.1]).

Lemma 4.1.1. Suppose that f is excessive, $P_t f < \infty$ for all $t \geq 0$ and $\lim_{t\to\infty} P_t f = 0$. Then there exists a sequence $(g_n)_{n\geq 1}$ of non-negative measurable functions such that $f = \uparrow \lim U g_n$. Moreover, if f is continuous and bounded, then one can choose g_n to be continuous.

Proof. This is proved as [32, Theorem 6, p. 82]. The function g_n is given by

$$g_n = n(f - P_{1/n} f).$$

If f is bounded, then $P_{1/n} f$ is continuous (by the strong Feller property). If f is also continuous, then $f - P_{1/n} f$ is continuous. ■

Remark 4.1.2. Transience is not needed in this result. The assumption $P_t f < \infty$ is satisfied if $f < \infty$ since $P_t f \leq f$. The assumption $\lim_{t\to\infty} P_t f = 0$ is *not* satisfied for harmonic functions (since they are invariant).

Proposition 4.1.3. Let f be excessive. If (P_t) is transient, there exists a sequence $(g_n)_{n\geq 1}$ of bounded measurable functions such that $f =\uparrow \lim_{n\to\infty} Ug_n$. Moreover, assume that there exists $h > 0$ such that $0 < Uh < \infty$ and Uh is continuous. If f is continuous and (P_t) is strongly Feller, then one can choose g_n to be continuous.

Proof. Let $h_n = nh$ with $0 < Uh < \infty$. and put

$$f_n = f \wedge Uh_n \wedge n.$$

By [32, Theorem 8, p. 104], f_n is excessive (minimum of excessive function is excessive). Note that under additional assumptions, f_n is continuous (and clearly bounded). By Lemma 4.1.1, there exists a sequence $(g_{nk})_{k\geq 1}$ such that $f_n =\uparrow \lim_{k\to\infty} Ug_{nk}$. In fact,

$$g_{nk} = k(f_n - P_{1/k} f_n) \leq kn\,.$$

Under additional assumptions, g_{nk} are continuous. From the proof of Lemma 4.1.1, cf. [32, Theorem 6, p. 82],

$$Ug_{nk} = k\int_0^{1/k} P_s f_n\, ds \leq n\,.$$

For each n, Ug_{nk} increases with k (this is part of Lemma 4.1.1); for each k, Ug_{nk} increases with n (this follows from $f_n \leq f_{n+1}$). Now, by [32, Lemma 1, p. 80],

$$\uparrow \lim_{n\to\infty} f_n =\uparrow \lim_{n\to\infty} \uparrow \lim_{k\to\infty} Ug_{nk} =\uparrow \lim_{n\to\infty} Ug_{nn}.$$

On the other hand, by the same [32, Lemma 1, p.80] and monotone convergence

$$\uparrow \lim_{n\to\infty} f_n =\uparrow \lim_{n\to\infty} \uparrow \lim_{t\downarrow 0} P_t f_n =\uparrow \lim_{t\downarrow 0} \uparrow \lim_{n\to\infty} P_t f_n \uparrow \lim_{t\downarrow 0} P_t f = f.$$

Therefore, by setting $g_n = g_{nn}$,

$$f =\uparrow \lim_{n\to\infty} Ug_n.$$

∎

4.2. BOUNDARY BEHAVIOUR OF POTENTIAL INTEGRALS - AUXILIARY RESULTS

Lemma 4.2.1. For $\Gamma = \{y \in \partial D : |x-y| \le 2\delta_D(x)\}$ it holds that

$$\int_\Gamma |x-y|^{-d} \asymp \delta_D(x)^{-1}, \quad x \in D.$$

Proof. Since D is a $C^{1,1}$ set, for small enough $\delta_D(x)$ the boundary part Γ can be described as $\Gamma = \{q \in \mathbb{R}^{d-1} : |\delta_D(x) - f(q)|^2 + |q|^2 \le 4\delta_D(x)^2\}$, for some $C^{1,1}$ function f on $\mathbb{R}^{d-1}$ such that $f(0) = 0$ and $\nabla f(0) = 0$, whereas x can be viewed as $x = (0, \dots, 0, \delta_D(x))$. Hence

$$\begin{aligned}\int_\Gamma \frac{\delta_D(x)}{|x-y|^d} d\sigma(y) &\asymp \delta_D(x) \int\limits_{\{q\in\mathbb{R}^{d-1}:|\delta_D(x)-f(q)|^2+|q|^2\le 4\delta_D(x)^2\}} \frac{\sqrt{1+|\nabla f(q)|^2}}{(|\delta_D(x)-f(q)|^2+|q|^2)^{d/2}} dq \\ &\asymp \int\limits_{\{z\in\mathbb{R}^{d-1}:|1-f(\delta_D(x)z)/\delta_D(x)|^2+|z|^2\le 4\}} \frac{1}{\left(\left|1-\frac{f(\delta_D(x)z)}{\delta_D(x)}\right|^2+|z|^2\right)^{d/2}} dz,\end{aligned}$$

where we first used that $|\nabla f|$ is bounded by the Lipschitz property and then the substitution $q = \delta_D(x)z$. Since $f \in C^{1,1}(\mathbb{R}^{d-1})$ such that $f(0) = 0$ and $\nabla f(0) = 0$, by the dominated convergence theorem the last integral converges to

$$\int\limits_{\{z\in\mathbb{R}^{d-1}:1+|z|^2\le 4\}} \frac{1}{(1+|z|^2)^{d/2}} dz < \infty$$

as $\delta_D(x) \to 0$.

■

Let $\varepsilon = \varepsilon(D) > 0$ be such that the map $\Phi : \partial D \times (-\varepsilon, \varepsilon) \to \mathbb{R}^d$ defined by $\Phi(y, \delta) = y + \delta\mathbf{n}(y)$ defines a diffeomorphism to its image, cf. [4, Remark 3.1]. Here $\mathbf{n}$ denotes the unit interior normal. Without loss of generality assume that $\varepsilon < \mathrm{diam}(D)/20$.

Lemma 4.2.2. Let $\eta < \varepsilon$.

(a) Assume that conditions **(U1)-(U4)** of Proposition 2.4.1 hold true. Then for any $x \in D$ such that $\delta_D(x) < \eta/2$,

$$\begin{aligned} G_D\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) \;\asymp\; & \frac{V(\delta_D(x))}{\delta_D(x)} \int_0^{\delta_D(x)} U(t)V(t)\,dt + V(\delta_D(x)) \int_0^\eta U(t)V(t)\,dt \\ & + V(\delta_D(x)) \int_{3\delta_D(x)/2}^\eta \frac{U(t)V(t)}{t}\,dt\,. \end{aligned} \tag{4.1}$$

In particular, $G_D\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) < \infty$ if and only if the integrability condition (**U1**) holds true.

(b) Assume that conditions ($\widetilde{\mathbf{U}}$**1**)-($\widetilde{\mathbf{U}}$**4**) of Theorem 3.2.6 hold true. Then for any $x \in D$ such that $\delta_D(x) < \eta/2$,

$$\begin{aligned} G_D^\phi\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) \asymp \frac{1}{\delta_D(x)^2\phi(\delta_D(x)^{-2})}\int_0^{\delta_D(x)} U(t)t\,dt + \delta_D(x)\int_0^\eta U(t)t\,dt \\ + \delta_D(x)\int_{3\delta_D(x)/2}^\eta \frac{U(t)}{t^2\phi(t^{-2})}\,dt\,. \end{aligned} \tag{4.2}$$

In particular, $G_D^\phi\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) < \infty$ if and only if the integrability condition (3.63) holds true.

Proof. We will prove both claims simultaneously. Note that (**U2**)-(**U4**) of Proposition 2.4.1 are the same conditions as ($\widetilde{\mathbf{U}}$**2**)-($\widetilde{\mathbf{U}}$**4**) of Theorem 3.2.6. Fix some $r_0 < \varepsilon$ and $x \in D$ as in the statement. Define

$$\begin{aligned} D_1 &= B(x, \delta_D(x)/2) \\ D_2 &= \{y : \delta_D(y) < \eta\} \setminus B(x, r_0) \\ D_3 &= \{y : \delta_D(y) < \delta_D(x)/2\} \cap B(x, r_0) \\ D_4 &= \{y : 3\delta_D(x)/2 < \delta_D(y) < \eta\} \cap B(x, r_0) \\ D_5 &= \{y : \delta_D(x)/2 < \delta_D(y) < 3\delta_D(x)/2\} \cap (B(x, r_0) \setminus B(x, \delta_D(x)/2)). \end{aligned}$$

Thus we have that

$$\begin{aligned} G_D\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) = \sum_{j=1}^5 \int_{D_j} G_D(x,y)U(\delta_D(y))\,dy =: \sum_{j=1}^5 I_j, \\ G_D^\phi\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) = \sum_{j=1}^5 \int_{D_j} G_D^\phi(x,y)U(\delta_D(y))\,dy =: \sum_{j=1}^5 \widetilde{I}_j. \end{aligned}$$

Estimate of I_1 and $\widetilde{I}_1$: We show that

$$\begin{aligned} I_1 &\lesssim U(\delta_D(x))V(\delta_D(x))^2 \lesssim \frac{V(\delta_D(x))}{\delta_D(x)}\int_0^{\delta_D(x)} U(t)V(t)\,dt\,, \\ \widetilde{I}_1 &\asymp \frac{U(\delta_D(x))}{\phi(\delta_D(x)^{-2})} \lesssim \frac{1}{\delta_D(x)^2\phi(\delta_D(x)^{-2})}\int_0^{\delta_D(x)} U(t)\,dt. \end{aligned}$$

Indeed, let $y \in D_1$. Then $\delta_D(y) > \delta_D(x)/2 > |y-x|$ implying that

$$G_D(x,y) \asymp G_D^{\phi}(x,y) \asymp \frac{1}{|x-y|^d \phi(|x-y|^{-2})},$$

since $V(t) \asymp \phi(t^{-2})^{-1/2}$ by (2.32). Further, by using first **(U2)** and then **(U3)** we have that

$$U(\delta_D(y)) \le c_1 U(\delta_D(x)/2) \le c_2 U(\delta_D(x)). \tag{4.3}$$

Therefore, by using weak scaling of ϕ in the penultimate asymptotic equality,

$$\begin{aligned} I_1 \asymp \widetilde{I}_1 &\asymp \int_{D_1} U(\delta_D(y)) \frac{1}{|x-y|^d \phi(|x-y|^{-2})}\, dy \\ &\lesssim U(\delta_D(x)) \int_{|y-x|<\delta_D(x)/2} \frac{1}{|x-y|^d \phi(|x-y|^{-2})}\, dy \lesssim U(\delta_D(x)) \int_0^{\delta_D(x)} \frac{1}{r\phi(r^{-2})} dr \\ &\asymp U(\delta_D(x)) \frac{1}{\phi(\delta_D(x)^{-2})} \asymp U(\delta_D(x)) V(\delta_D(x))^2. \end{aligned}$$

Finally, by **(U2)** and the upper weak scaling (2.33) of V,

$$\begin{aligned} \frac{1}{\delta_D(x)} \int_0^{\delta_D(x)} U(t)V(t)\, dt &\gtrsim \frac{U(\delta_D(x))V(\delta_D(x))}{\delta_D(x)} \int_0^{\delta_D(x)} \frac{V(t)}{V(\delta_D(x))}\, dt \\ &\gtrsim \frac{U(\delta_D(x))V(\delta_D(x))}{\delta_D(x)} \int_0^{\delta_D(x)} \left(\frac{t}{\delta_D(x)}\right)^{\delta_2} dt \\ &\asymp U(\delta_D(x))V(\delta_D(x)). \end{aligned}$$

Similarly,

$$\frac{1}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})} \int_0^{\delta_D(x)} U(t)t\, dt \gtrsim \frac{U(\delta_D(x))}{\phi(\delta_D(x)^{-2})}.$$

Estimate of I_2 and $\widetilde{I}_2$: Next, we show that

$$I_2 \asymp V(\delta_D(x)) \int_0^{\eta} U(t)V(t)\, dt\,, \tag{4.4}$$

$$\widetilde{I}_2 \asymp \delta_D(x) \int_0^{\eta} U(t)t\, dt\,. \tag{4.5}$$

Let $y \in D_2$. Then $r_0 < |y-x| < \mathrm{diam}(D)$ so that $|y-x| \asymp 1$. This implies that $G_D(x,y) \asymp V(\delta_D(x))V(\delta_D(y))$ and $G_D^{\phi}(x,y) \asymp \delta_D(x)\delta_D(y)$. Therefore

$$I_2 \asymp V(\delta_D(x)) \int_{D_2} U(\delta_D(y))V(\delta_D(y))\, dy \asymp V(\delta_D(x)) \int_{\delta_D(y)<\eta} U(\delta_D(y))V(\delta_D(y))\, dy.$$

Finally, (4.4) follows by the co-area formula. Similarly

$$\widetilde{I}_2 \asymp \delta_D(x) \int_{\delta_D(y)<\eta} U(\delta_D(y))\delta_D(y) \asymp \delta_D(x) \int_0^{\eta} U(t)t\, dt.$$

In estimates for I_3, I_4 and I_5 (and for $\widetilde{I}_3$, $\widetilde{I}_4$ and $\widetilde{I}_5$) we will use the change of variables formula based on a diffeomorphism $\Phi : B(x,r_0) \to B(0,r_0)$ satisfying

$$\Phi(D \cap B(x,r_0)) = B(0,r_0) \cap \{z \in \mathbb{R}^d : z \cdot e_d > 0\},$$
$$\Phi(y) \cdot e_d = \delta_D(y) \text{ for any } y \in B(x,r_0), \quad \Phi(x) = \delta_D(x) e_d,$$

see [4, p. 38]. For the point $z \in \mathbb{R}^d_+ = \{z \in \mathbb{R}^d : z \cdot e_d > 0\}$ we will write $z = (\widetilde{z}, z_d)$. Several times we also use the following integrals:

$$\int_0^a \frac{s^{d-2}}{(1+s)^d}\, ds = \frac{(1+1/a)^{1-d}}{(d-1)}, \quad a > 0, \tag{4.6}$$
$$\int_0^a \frac{s^{d-2}}{(1+s)^{d+2}}\, ds = \frac{a^{d-1}}{(1+a)^{d+1}} \left(2a(1+a) + d + 2ad + d^2\right), \quad a > 0. \tag{4.7}$$

Estimate of I_3 and $\widetilde{I}_3$: It holds that

$$I_3 \asymp \frac{V(\delta_D(x))}{\delta_D(x)} \int_0^{\delta_D(x)} U(t)V(t)\, dt, \tag{4.8}$$
$$\widetilde{I}_3 \asymp \frac{1}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})} \int_0^{\delta_D(x)} U(t) t\, dt. \tag{4.9}$$

To see this, take $y \in D_3$. Then $\delta_D(y) \le \delta_D(x)/2$ implying $|x-y| \ge \delta_D(x)/2$, and thus

$$G_D(x,y) \asymp \frac{V(\delta_D(x))}{V(|x-y|)} \frac{V(\delta_D(y))}{V(|x-y|)} \frac{V(|x-y|)^2}{|x-y|^d} = \frac{V(\delta_D(x))V(\delta_D(y))}{|x-y|^d}, \tag{4.10}$$
$$G_D^{\phi}(x,y) \asymp \frac{\delta_D(x)\delta_D(y)}{|x-y|^{d+2}\phi(|x-y|^{-2})}. \tag{4.11}$$

Therefore

$$\begin{aligned}
I_3 &\asymp V(\delta_D(x))\int_{D_3}\frac{U(\delta_D(y))V(\delta_D(y))}{|x-y|^d}\,dy\\
&\asymp V(\delta_D(x))\int_{\{0<z_d<\delta_D(x)/2\}\cap B(0,r_0)}\frac{U(z_d)V(z_d)}{(|\delta_D(x)-z_d|+|\widetilde{z}|)^d}\,dz\\
&\asymp V(\delta_D(x))\int_{|\widetilde{z}|<r_0}\int_0^{\delta_D(x)/2}\frac{U(z_d)V(z_d)}{(|\delta_D(x)-z_d|+|\widetilde{z}|)^d}\,dz_d\,d\widetilde{z}\\
&\asymp V(\delta_D(x))\int_0^{r_0}t^{d-2}\int_0^{\delta_D(x)/2}\frac{U(z_d)V(z_d)}{(|\delta_D(x)-z_d|+t)^d}\,dz_d\,dt\\
&= V(\delta_D(x))\int_0^{r_0/\delta_D(x)}s^{d-2}\int_0^{1/2}\frac{U(\delta_D(x)h)V(\delta_D(x)h)}{\big((1-h)+s\big)^d}\,dh\,ds\\
&\asymp V(\delta_D(x))\int_0^{r_0/\delta_D(x)}\frac{s^{d-2}}{(1+s)^d}\,ds\int_0^{1/2}U(\delta_D(x)h)V(\delta_D(x)h)\,dh\\
&= V(\delta_D(x))\frac{(1+\delta_D(x)/r_0)^{1-d}}{d-1}\int_0^{1/2}U(\delta_D(x)h)V(\delta_D(x)h)\,dh\\
&\asymp V(\delta_D(x))\int_0^{1/2}U(\delta_D(x)h)V(\delta_D(x)h)\,dh\\
&= \frac{V(\delta_D(x))}{\delta_D(x)}\int_0^{\delta_D(x)/2}U(t)V(t)\,dt.
\end{aligned}$$

This proves the upper bound in (4.8). For the lower bound, note that by the upper weak scaling (2.33) of V and the almost non-increasing condition **(U2)**, we have

$$\begin{aligned}
\int_0^{\delta_D(x)/2}U(t)V(t)\,dt &= 2\int_0^{\delta_D(x)}U(t/2)V(t/2)\,dt \geq 2\int_0^{\delta_D(x)}c_3U(t)\widetilde{a}_1^{-1}2^{-\delta_1}V(t)\,dt\\
&= c_4\int_0^{\delta_D(x)}U(t)V(t)\,dt\,.
\end{aligned}$$

By repeating the first five lines of the calculations for the integral I_3 with the bound (4.11) for G_D^ϕ, we get

$$\begin{aligned}
\widetilde{I}_3 &\asymp \int_0^{r_0/\delta_D(x)}s^{d-2}\int_0^{1/2}\frac{U(\delta_D(x)h)h}{\big((1-h)+s\big)^{d+2}\phi\big(\delta_D(x)^{-2}((1-h)+s)^{-2}\big)}dh\,ds\\
&\asymp \int_0^{r_0/\delta_D(x)}s^{d-2}\int_0^{1/2}\frac{U(\delta_D(x)h)h}{(1+s)^{d+2}\phi\big(\delta_D(x)^{-2}(1+s)^{-2}\big)}dh\,ds,
\end{aligned}\tag{4.12}$$

where the last line comes from $\frac{1}{2}\leq h\leq 1$. Further, for ϕ it holds that

$$(1+s)^{-2}\phi(\delta_D(x)^{-2})\leq\phi\big(\delta_D(x)^{-2}(1+s)^{-2}\big)\leq\phi(\delta_D(x)^{-2}),\tag{4.13}$$

see (1.12). Since we have

$$\int_0^{r_0/\delta_D(x)}\frac{s^{d-2}}{(1+s)^d}ds\asymp 1,\quad \int_0^{r_0/\delta_D(x)}\frac{s^{d-2}}{(1+s)^{d+2}}ds\asymp 1,\tag{4.14}$$

by (4.6) and (4.7), by applying the inequalities (4.13) to (4.12) and by using (4.14) we obtain

$$\begin{aligned}\widetilde{I}_3 &\asymp \frac{1}{\phi(\delta_D(x)^{-2})}\int_0^{1/2} U(\delta_D(x)h)h\,dh\\ &\asymp \frac{1}{\delta_D(x)^2\phi(\delta_D(x)^{-2})}\int_0^{\delta_D(x)/2} U(t)t\,dt.\end{aligned}$$

This proves (4.9) since the almost non-increasing condition ($\widetilde{\mathbf{U2}}$) implies

$$\int_0^{\delta_D(x)/2} U(t)t\,dt = \int_0^{\delta_D(x)} U(t/2)t\,dt \gtrsim \int_0^{\delta_D(x)} U(t)t\,dt.$$

Estimate of I_4 and $\widetilde{I}_4$: By applying the same change of variables as in the previous step, we show that

$$I_4 \asymp V(\delta_D(x))\int_{3\delta_D(x)/2}^{\eta} \frac{U(t)V(t)}{t}\,dt\,, \tag{4.15}$$

$$\widetilde{I}_4 \asymp \delta_D(x)\int_{3\delta_D(x)/2}^{\eta} \frac{U(t)}{t^2\phi(t^{-2})}\,dt\,. \tag{4.16}$$

Let $y \in D_4$. Then $|x-y| \geq \delta_D(x)/2$ and $|x-y| \geq \delta_D(y)/3$, hence $G_D(x,y)$ is of the form (4.10) and $G_D^{\phi}(x,y)$ is of the form (4.11). By following the first five lines in the computation of I_3, we arrive at

$$\begin{aligned} I_4 &\asymp V(\delta_D(x))\int_0^{r_0/\delta_D(x)} s^{d-2}\int_{3/2}^{\eta/\delta_D(x)} \frac{U(\delta_D(x)h)V(\delta_D(x)h)}{\big((h-1)+s\big)^d}\,dh\,ds\\ &\asymp V(\delta_D(x))\int_{3/2}^{\eta/\delta_D(x)} \frac{U(\delta_D(x)h)V(\delta_D(x)h)}{h-1}\int_0^{\frac{r_0}{(h-1)\delta_D(x)}} \frac{r^{d-2}}{(1+r)^d}\,dr\,dh\\ &= V(\delta_D(x))\int_{3/2}^{\eta/\delta_D(x)} \frac{U(\delta_D(x)h)V(\delta_D(x)h)}{h-1}\frac{(1+(h-1)\delta_D(x)/r_0)^{1-d}}{d-1}\,dh\\ &\asymp V(\delta_D(x))\int_{3/2}^{\eta/\delta_D(x)} \frac{U(\delta_D(x)h)V(\delta_D(x)h)}{h-1}\,dh\\ &\asymp V(\delta_D(x))\int_{3/2}^{\eta/\delta_D(x)} \frac{U(\delta_D(x)h)V(\delta_D(x)h)}{h}\,dh\\ &= V(\delta_D(x))\int_{3\delta_D(x)/2}^{\eta} \frac{U(t)V(t)}{t}\,dt\,.\end{aligned}$$

By following the computation for the integral I_4 we have

$$\begin{aligned}
\widetilde{I}_4 &\asymp \int_0^{r_0/\delta_D(x)} s^{d-2} \int_{3/2}^{\eta/\delta_D(x)} \frac{U(\delta_D(x)h)h}{\left((h-1)+s\right)^{d+2}\phi\left(\delta_D(x)^{-2}((h-1)+s)^{-2}\right)} dh\,ds \\
&\asymp \int_0^{r_0/\delta_D(x)} s^{d-2} \int_{3/2}^{\eta/\delta_D(x)} \frac{U(\delta_D(x)h)h}{\left(h+s\right)^{d+2}\phi\left(\delta_D(x)^{-2}(h+s)^{-2}\right)} dh\,ds \\
&\asymp \int_0^{r_0/(\delta_D(x)h)} s^{d-2} \int_{3/2}^{\eta/\delta_D(x)} \frac{U(\delta_D(x)h)}{h^2\left(1+s\right)^{d+2}\phi\left((\delta_D(x)h)^{-2}(1+s)^{-2}\right)} dh\,ds, \qquad (4.17)
\end{aligned}$$

where the second line comes from $\frac{1}{3}h \le h-1 \le h$. By applying (4.13) in (4.17), since the relations (4.14) also hold for $r_0/(\delta_D(x)h)$ instead of r_0/δ_D, we get

$$\widetilde{I}_4 \asymp \int_{3/2}^{\eta/\delta_D(x)} \frac{U(\delta_D(x)h)}{h^2\phi\left((\delta_D(x)h)^{-2}\right)} dh \asymp \delta_D(x) \int_{3\delta_D(x)/2}^{\eta} \frac{U(t)}{t^2\phi\left(t^{-2}\right)} dt.$$

Estimate of I_5 and $\widetilde{I}_5$: Under the almost non-increasing condition **(U2)** and the doubling condition **(U3)** it holds that

$$I_5 \lesssim U(\delta_D(x))V(\delta_D(x))^2 \lesssim \frac{V(\delta_D(x))}{\delta_D(x)} \int_0^{\delta_D(x)} U(t)V(t)\,dt\,, \qquad (4.18)$$

$$\widetilde{I}_5 \lesssim \frac{U(\delta_D(x))}{\phi(\delta_D(x)^{-2})} \lesssim \frac{1}{\delta_D(x)^2\phi(\delta_D(x)^{-2})} \int_0^{\delta_D(x)} U(t)t\,dt\,. \qquad (4.19)$$

Indeed, let $y \in D_5$. Then $|x-y| > \delta_D(x)/2 > \delta_D(y)/3$, hence $G_D(x,y)$ is of the form (4.10) and $G_D^{\phi}(x,y)$ of the form (4.11). Also, the estimate (4.3) and the analogous one with V hold true. Therefore

$$\begin{aligned}
I_5 &\asymp V(\delta_D(x)) \int_{D_5} \frac{U(\delta_D(y))V(\delta_D(y))}{|x-y|^d}\,dy \\
&\lesssim U(\delta_D(x))V(\delta_D(x))^2 \int_{D_5} \frac{1}{|x-y|^d}\,dy\,.
\end{aligned}$$

Similarly,

$$\widetilde{I}_5 \asymp \delta_D(x) \int_{D_5} \frac{U(\delta_D(y))\delta_D(y)}{|x-y|^{d+2}\phi(|x-y|^{-2})}\,dy \lesssim \frac{U(\delta_D(x))}{\phi(\delta_D(x)^{-2})} \int_{D_5} \frac{1}{|x-y|^d}\,dy.$$

It is shown in [4, page 42] that the integral $\int_{D_5} \frac{1}{|x-y|^d}\,dy$ is comparable to 1. This proves the first approximate inequalities in (4.18) and (4.19), while the second were already proved in the estimate of I_1 and $\widetilde{I}_1$.

The proof is finished by noting that $I_1 + I_5 \lesssim I_3$ and $\widetilde{I}_1 + \widetilde{I}_5 \lesssim \widetilde{I}_3$. ∎

Lemma 4.2.3. Let $\eta < \varepsilon$.

(a) Assume that conditions (**U1**)-(**U4**) from Proposition 2.4.1 hold true. There exists $c(\eta) > 0$ such that for any $x \in D$ satisfying $\delta_D(x) \geq \eta/2$,

$$G_D\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) \leq c(\eta)\,. \tag{4.20}$$

(b) Assume that conditions ($\widetilde{\mathbf{U}}$**1**)-($\widetilde{\mathbf{U}}$**4**) from Theorem 3.2.6 hold true. There exists $c(\eta) > 0$ such that for any $x \in D$ satisfying $\delta_D(x) \geq \eta/2$,

$$G_D^\phi\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) \leq c(\eta)\,. \tag{4.21}$$

Proof. Again, we will prove both claims simultaneously. Fix $x \in D$ as in the statement and define

$$\begin{aligned} D_1 &= \{y : \delta_D(y) < \eta/4\}, \\ D_2 &= \{y : \eta/4 \leq \delta_D(y) < \eta\}. \end{aligned}$$

Then

$$G_D\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) = \sum_{j=1}^{2} \int_{D_j} G_D(x,y)U(\delta_D(y))\,dy =: \sum_{j=1}^{2} J_j, \tag{4.22}$$

$$G_D^\phi\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) = \sum_{j=1}^{2} \int_{D_j} G_D^\phi(x,y)U(\delta_D(y))\,dy =: \sum_{j=1}^{2} \widetilde{J}_j. \tag{4.23}$$

Estimate of J_1 and $\widetilde{J}_1$: We show that

$$J_1 \lesssim \frac{1}{\eta}\int_0^\eta U(t)V(t)\,dt, \tag{4.24}$$

$$\widetilde{J}_1 \lesssim \frac{1}{\eta^2\phi(\eta^{-2})}\int_0^\eta U(t)t\,dt. \tag{4.25}$$

Let $y \in D_1$. Then $\delta_D(y) < \eta/4 \leq \delta_D(x)/2$, hence by using $|x-y| \geq \delta_D(x) - \delta_D(y)$ we have that $|x-y| > \delta_D(y)$ and $|x-y| > \delta_D(x)/2$. This implies that $G_D(x,y)$ satisfies (4.10) and $G_D^\phi(x,y)$ satisfies (4.11). Therefore,

$$J_1 \asymp V(\delta_D(x))\int_{D_1} \frac{U(\delta_D(y))V(\delta_D(y))}{|x-y|^d}\,dy \lesssim \int_{D_1} \frac{U(\delta_D(y))V(\delta_D(y))}{|x-y|^d}\,dy$$

and

$$\widetilde{J}_1 \asymp \delta_D(x)\int_{D_1} \frac{U(\delta_D(y))\delta_D(y)}{|x-y|^{d+2}\phi(|x-y|^{-2})}\,dy \lesssim \frac{\delta_D(x)}{\eta^2\phi(\eta^{-2})}\int_{D_1} \frac{U(\delta_D(y))\delta_D(y)}{|x-y|^d}\,dy,$$

since on D_1 we have $|x-y| \geq \eta/4$, hence $|x-y|^2\phi(|x-y|^{-2}) \gtrsim \eta^2\phi(\eta^{-2})$ by (1.12).

By using the co-area formula we get (below dy denotes the Hausdorff measure on $\{\delta_D(y)=t\}$)

$$\begin{aligned} J_1 &\asymp \int_0^{\eta/4} U(t)V(t)\Big(\int_{\delta_D(y)=t}\frac{1}{|x-y|^d}\,dy\Big)\,dt, \quad \text{and} \\ \widetilde{J}_1 &\lesssim \frac{\delta_D(x)}{\eta^2\phi(\eta^{-2})}\int_0^{\eta/4} U(t)\,t\Big(\int_{\delta_D(y)=t}\frac{1}{|x-y|^d}\,dy\Big)\,dt. \end{aligned} \tag{4.26}$$

The inner integral is estimated as follows: For $\delta_D(y)=t$ it holds that $|x-y| \geq \delta_D(x)-t$, hence $|x-y|^{-d} \leq (\delta_D(x)-t)^{-d}$. The Hausdorff measure of $\{\delta_D(y)=t\}$ is larger than or equal to the Hausdorff measure of the sphere around x of radius $\delta_D(x)-t$ which is comparable to $(\delta_D(x)-t)^{d-1}$. This implies that the inner integral is estimated from above by a constant times $(\delta_D(x)-t)^{-1}$. Thus

$$\begin{aligned} J_1 &\lesssim \int_0^{\eta/4} U(t)V(t)(\delta_D(x)-t)^{-1}\,dt, \quad \text{and} \\ \widetilde{J}_1 &\lesssim \frac{\delta_D}{\eta^2\phi(\eta^{-2})}\int_0^{\eta/4} U(t)\,t\,(\delta_D(x)-t)^{-1}. \end{aligned}$$

If $t<\eta/4$, then $t<\delta_D(x)/2$, implying $\delta_D(x)/2 < \delta_D(x)-t < \delta_D(x)$. Therefore,

$$\begin{aligned} J_1 &\lesssim \frac{1}{\delta_D(x)}\int_0^{\eta/4} U(t)V(t)\,dt \lesssim \frac{1}{\eta}\int_0^{\eta} U(t)V(t)\,dt\,, \quad \text{and} \\ \widetilde{J}_1 &\lesssim \frac{1}{\eta^2\phi(\eta^{-2})}\int_0^{\eta} U(t)t\,dt. \end{aligned}$$

Estimate of J_2 and $\widetilde{J}_2$: It holds that

$$J_2 \lesssim U(\eta/4) \quad \text{and} \quad \widetilde{J}_2 \lesssim U(\eta/4). \tag{4.27}$$

Let $y \in D_2$. By the almost non-increasing condition (U2) we have $U(\delta_D(y)) \leq c_1 U(\eta/4)$, hence

$$\begin{aligned} J_2 &\lesssim \int_{\eta/4<\delta_D(y)<\eta} U(\delta_D(y))\frac{V(|x-y|)^2}{|x-y|^d}\,dy \lesssim U(\eta/4)\int_{\eta/4<\delta_D(y)<\eta}\frac{V(|x-y|)^2}{|x-y|^d}\,dy \\ &\leq U(\eta/4)\int_{B(x,2\mathrm{diam}(D))}\frac{V(|x-y|)^2}{|x-y|^d}\,dy \lesssim U(\eta/4). \end{aligned}$$

The last estimate uses the fact that the integral is not singular. Identically,

$$\widetilde{J}_2 \lesssim U(\eta/4).$$

By putting together estimates for J_1 and J_2 (and $\widetilde{J_1}$ and $\widetilde{J_2}$), we see that there exists $c_2 > 0$ such that

$$G_D\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) \le c_2\left(\frac{1}{\eta}\int_0^\eta U(t)V(t)\,dt + U(\eta/4)\right) =: c(\eta),$$
$$G_D^\phi\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) \le c_2\left(\frac{1}{\eta^2\phi(\eta^{-2})}\int_0^\eta U(t)t\,dt + U(\eta/4)\right) =: c(\eta).$$

■

Proof of Proposition 2.4.1 and Theorem 3.2.6. We prove the proposition and the theorem simultaneously. Recall that the assumptions (**U2**)-(**U4**) for the proposition are the same as the assumptions ($\widetilde{\mathbf{U}}$**2**)-($\widetilde{\mathbf{U}}$**4**) for the theorem.

Fix some $\eta < \varepsilon$ and treat it as a constant. Note that on $\{\delta_D(y) \ge \eta\}$ it holds that U is bounded (by the assumption ($\widetilde{\mathbf{U}}$**4**)). Therefore

$$G_D^\phi\big(U(\delta_D)\mathbf{1}_{(\delta_D\ge\eta)}\big)(x) \lesssim G_D^\phi\delta_D(x) \asymp \delta_D(x), \quad x \in D, \tag{4.28}$$

by Lemma 3.1.8. For the lower bound of this term note that on $\{\delta_D(x) \ge \eta/2\}$ we have

$$G_D^\phi\big(U(\delta_D)\mathbf{1}_{(\delta_D\ge\eta)}\big)(x) \gtrsim \int_{B(x,\eta/4)} \frac{1}{|x-y|^d\phi(|x-y|^{-2})}dy \asymp \frac{1}{\phi(16/\eta^2)} \gtrsim 1, \tag{4.29}$$

and on $\{\delta_D(x) \le \eta/2\}$ we have

$$G_D^\phi\big(U(\delta_D)\mathbf{1}_{(\delta_D\ge\eta)}\big)(x) \gtrsim \delta_D(x)\int_{\delta_D(y)\ge\eta} \frac{\delta_D(y)}{|x-y|^{d+2}\phi(|x-y|^{-2})}dy \gtrsim \delta_D(x). \tag{4.30}$$

Since $\delta_D(x) \asymp 1$ on $\{\delta_D(x) \ge \eta/2\}$, we have just obtained $G_D^\phi\big(U(\delta_D)\mathbf{1}_{(\delta_D\ge\eta)}\big)(x) \asymp \delta_D(x)$ in D. Further, by Lemma 4.2.3, if $\delta_D(x) \ge \eta/2$, then $G_D^\phi\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) \le c(\eta)$. Hence,

$$G_D^\phi(U(\delta_D))(x) \asymp 1, \quad \delta_D(x) \ge \eta/2.$$

Similarly

$$G_D(U(\delta_D)\mathbf{1}_{(\delta_D\ge\eta)})(x) \asymp G_D\mathbf{1}(x) \asymp V(\delta_D(x)), \tag{4.31}$$

and if $\delta_D(x) \ge \eta/2$, then $G_D\big(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)}\big)(x) \le c(\eta)$ by Lemma 4.2.3. Hence,

$$G_D(U(\delta_D))(x) \asymp 1, \quad \delta_D(x) \ge \eta/2.$$

Since for $\delta_D(x) \ge \eta/2$ the right-hand sides of (2.42) and (3.66) are also comparable to 1, this proves the claims for this case.

Assume now that $\delta_D(x) < \eta/2$. By Lemma 4.2.2 and (4.31) we have that

$$\begin{aligned}
G_D(U(\delta_D))(x) &= G_D(U(\delta_D)\mathbf{1}_{(\delta_D<\eta)})(x) + G_D(U(\delta_D)\mathbf{1}_{(\delta_D\geq\eta)})(x)\\
&\asymp \frac{V(\delta_D(x))}{\delta_D(x)}\int_0^{\delta_D(x)} U(t)V(t)\,dt + V(\delta_D(x))\int_0^{\eta} U(t)V(t)\,dt\\
&+ V(\delta_D(x))\int_{3\delta_D(x)/2}^{\eta} \frac{U(t)V(t)}{t}\,dt \ + V(\delta_D(x))\\
&\asymp \frac{V(\delta_D(x))}{\delta_D(x)}\int_0^{\delta_D(x)} U(t)V(t)\,dt + V(\delta_D(x))\int_{\delta_D(x)}^{\eta} \frac{U(t)V(t)}{t}\,dt\,.
\end{aligned}$$

Clearly, in the last integral we can replace η by $\mathrm{diam}(D)$.

Similarly, by following the previous calculations, and by using (4.28) and (4.30), we obtain that for $\delta_D(x) < \eta/2$ the relation (3.66) holds.

Assume that the function U from Proposition 2.4.1 is bounded on every bounded subset of $(0,\infty)$. Obviously, by (4.31),

$$G_D(U(\delta_D))(x) \lesssim G_D\mathbf{1}(x) \asymp V(\delta_D(x)).$$

On the other hand, analogously as in (4.4),

$$G_D(U(\delta_D))(x) \geq \int_{D_2} U(\delta_D(y))G_D(x,y)\,dy \asymp V(\delta_D(x))\int_0^{\eta} U(t)V(t)\,dt\,,$$

hence $G_D\big(U(\delta_D)\big) \asymp V(\delta_D)$ in D. This finishes the proof of Proposition 2.4.1.

To finish the proof of Theorem 3.2.6, we prove that $G_D^{\phi}(U(\delta_D))(x)/P_D^{\phi}\sigma(x) \to 0$ as $x \to \partial D$. It is obvious that $P_D^{\phi}\sigma$ annihilates the first and the second term of (3.66). For the third term, note that on $\{t \geq \delta_D(x)\}$ we have $t^2\phi(t^{-2}) \gtrsim \delta_D(x)^2\phi(\delta_D(x)^{-2})$ and $U(t)\delta_D(x) \leq U(t)t$. By applying the dominate convergence theorem we obtain

$$\frac{\delta_D(x)\int_{\delta_D(x)}^{\mathrm{diam}D}\frac{U(t)}{t^2\phi(t^{-2})}\,dt}{P_D^{\phi}\sigma(x)} \lesssim \int_{\delta_D(x)}^{\mathrm{diam}D} U(t)\delta_D(x)dt \to 0,$$

as $\delta_D(x) \to 0$.

∎

Lemma 4.2.4. Let $t < r_0$. There exists $C = C(d,D,\phi) > 0$ such that for $\delta_D(x) \geq \frac{t}{2}$ it holds that

$$\int_{\delta_D(y)\leq t} \frac{G_D^{\phi}(x,y)}{P_D^{\phi}\sigma(y)}dy \leq C\widetilde{f}(x,t),$$

where $0 \leq \widetilde{f}(x,t) \leq t\,\delta_D(x)$ on $\{\delta_D(x) \geq t/2\}$ and $\widetilde{f}(x,t)/t \to 0$ as $t \to 0$ for every fixed $x \in D$.

Proof. We need a little adaptation of Lemma 4.2.3. We break the set D_2 in three pieces. Fix $x \in D$ as in the statement and define

$$D_1 = \{y : \delta_D(y) < t/4\},$$
$$D_2 = \{y : t/4 \le \delta_D(y) < t\} \cap B(x,t/4),$$
$$D_3 = \{y : t/4 \le \delta_D(y) < t\} \cap B(x,t/4)^c \cap B(x,r_0),$$
$$D_4 = \{y : t/4 \le \delta_D(y) < t\} \cap B(x,r_0)^c.$$

Then

$$\int_{\delta_D(y)\le t} \frac{G_D^\phi(x,y)}{P_D^\phi \sigma(y)} dy = \sum_{i=1}^4 \int_{D_i} \frac{G_D^\phi(x,y)}{P_D^\phi \sigma(y)} dy = \sum_{i=1}^4 J_i.$$

Estimate of J_1: We prove

$$J_1 \lesssim t^2. \tag{4.32}$$

Let $y \in D_1$. Then $\delta_D(y) < t/4 \le \delta_D(x)/2$, hence $|x-y| > \delta_D(x)/2 > \delta_D(y)$. This implies that $G_D^\phi(x,y)$ satisfies (4.11), and $G_D^\phi(x,y)/P_D^\phi\sigma(y) \lesssim \delta_D(x)\delta_D(y)/|x-y|^d$. Therefore, by using the co-area formula in the second comparison, we get

$$J_1 \lesssim \delta_D(x) \int_{D_1} \frac{\delta_D(y)}{|x-y|^d} \asymp \delta_D(x) \int_0^{t/4} h \left(\int_{\delta_D(y)=h} \frac{\sigma(dy)}{|x-y|^d} \right) dh.$$

The inner integral is estimated as before, see the paragraph under (4.26), i.e. the inner integral is bounded from above by a constant times $(\delta_D(x)-h)^{-1}$. Thus

$$J_1 \lesssim \delta_D(x) \int_0^{t/4} \frac{h}{\delta_D(x)-h} dh.$$

However, when $h < t/4$ we have $\frac{1}{2}\delta_D(x) \le \delta_D(x) - h \le \delta_D(x)$, therefore

$$J_1 \lesssim \int_0^{t/4} h\, dh \lesssim t^2.$$

In the following integral estimates we have $y \in D$ such that $t/4 \le \delta_D(y) \le t$ so $P_D^\phi \sigma(y) \asymp \frac{1}{t^2\phi(t^{-2})}$.

Estimate of J_2: We prove

$$J_2 \lesssim t^2. \tag{4.33}$$

On D_2 we obviously have $G_D^\phi(x,y) \asymp \frac{1}{|x-y|^d \phi(|x-y|^{-2})}$, hence

$$J_2 \lesssim t^2\phi(t^{-2}) \int_{B(x,t/4)} \frac{1}{|x-y|^d \phi(|x-y|^{-2})} dy \asymp t^2\phi(t^{-2}) \frac{1}{\phi(t^{-2})} \asymp t^2.$$

Estimate of J_3: We prove that $J_3 \lesssim f(x,t)$ for a function f which satisfies $0 \le f(x,t)/t \lesssim \delta_D(x)$ and $f(x,t)/t \to 0$ as $t \to 0$ for every fixed $x \in D$.

To this end, since $y \in D_3$, hence $|x-y| \ge t/4$, it holds that $G_D^\phi(x,y) \asymp \frac{\delta_D(x)t}{|x-y|^{d+2}\phi(|x-y|^{-2})}$. Hence,

$$J_3 \asymp t^3\phi(t^{-2})\delta_D(x)\int_{D_3}\frac{1}{|x-y|^{d+2}\phi(|x-y|^{-2})}dy =: f(x,t). \tag{4.34}$$

Since $|x-y| \ge t/4$, we have $|x-y|^2\phi(|x-y|^{-2}) \gtrsim t^2\phi(t^{-2})$ by (1.12), hence

$$f(x,t)/t \lesssim \delta_D(x)\int_{D_3}\frac{1}{|x-y|^d}dy. \tag{4.35}$$

Also, by reducing to the flat case we have

$$\begin{aligned}\int_{D_3}\frac{1}{|x-y|^d}dy &\asymp \int_{t/4}^{r_0}\int_{t/4}^{t}\frac{r^{d-2}}{(|\delta_D(x)-h|+r)^d}dh\,dr\\ &\asymp \int_{t/4}^{r_0}\int_{(t/4-\delta_D(x))/r}^{(t-\delta_D(x))/r}\frac{1}{r(|\rho|+1)^d}d\rho\,dr.\end{aligned}$$

Since $\rho \mapsto 1/(|\rho|+1)$ is bell-shaped, and the inner interval $[(t/4-\delta_D(x))/r,(t-\delta_D(x))/r]$ has fixed length, the inner integral is maximal when the inner interval is symmetric (which is when $\delta_D(x) = \frac{5}{8}t$), thus, we get

$$\begin{aligned}\int_{D_3}\frac{1}{|x-y|^d} &\lesssim \int_{t/4}^{r_0}\int_{-3t/(8r)}^{3t/(8r)}\frac{1}{r(|\rho|+1)^d}d\rho\,dr\\ &= 2\int_{t/4}^{r_0}\int_{0}^{3t/(8r)}\frac{1}{r(\rho+1)^d}d\rho\,dr.\end{aligned}$$

Further, $1 \le \rho+1 \le 3t/(8r)+1 \le 3$ so we get

$$\int_{D_3}\frac{1}{|x-y|^d} \lesssim \int_{t/4}^{r_0}\frac{t}{r^2}dr \lesssim 1. \tag{4.36}$$

Inserting the bound (4.36) into (4.35), we get that $0 \le f(x,t)/t \lesssim \delta_D(x)$ where the constant of comparability depends only on d, D and ϕ.

Further, if we fix x and let $t \to 0$, then it is clear that $\mathbf{1}_{D_3} \to 0$, and that $|x-y|^{-d-2}\phi(|x-y|^{-2})^{-1} \le c$ for every $y \in D_3$ for all small enough $t > 0$. Hence, $f(x,t)/t \to 0$ as $t \to 0$.

Estimate of J_4: We prove

$$J_4 \lesssim t^3\phi(t^{-2})\,\delta_D(x). \tag{4.37}$$

For $y \in D_4$ we have $G_D^\phi(x,y) \lesssim \frac{\delta_D(x)\delta_D(y)}{|x-y|^{d+2}\phi(|x-y|^{-2})} \lesssim \delta_D(x)t$ since $r_0 \le |x-y| \le \mathrm{diam}D$. Hence, $J_4 \lesssim t^3\phi(t^{-2})\delta_D(x)$.

To finish the proof, note that we can take $\widetilde{f}(x,t) = c\big(t^2 + f(x,t) + t^3\phi(t^{-2})\delta_D(x)\big)$ for some constant $c = c(d,D,\phi) > 0$. ■

Proof of Proposition 2.4.2. Fix $\eta < \varepsilon$. Let $x \in D$ and $r_0 > \delta_D(x) + \eta$. We split D^c into three parts,

$$D_1 = \{z \in D^c : \delta_{D^c}(z) \geq \eta\}$$
$$D_2 = \{z \in D^c \cap B(x,r_0) : \delta_{D^c}(z) < \eta\}$$
$$D_3 = \{z \in D^c \setminus B(x,r_0) : \delta_{D^c}(z) < \eta\}$$

and apply (2.43) to get that

$$\begin{aligned} P_D g(x) &\asymp \int_{D_1} \widetilde{U}(\delta_{D^c}(z)) P_D(x,z)dz + \int_{D_2} \widetilde{U}(\delta_{D^c}(z)) P_D(x,z)dz + \int_{D_3} \widetilde{U}(\delta_{D^c}(z)) P_D(x,z)dz \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Estimate of I_1: For $z \in D^c$ such that $\delta_{D^c}(z) \geq \eta$, the estimate (2.36) is equivalent to

$$P_D(x,z) \asymp \frac{V(\delta_D(x))}{V(\delta_{D^c}(z))^2 \delta_{D^c}(z)^d}.$$

By applying this estimate and the co-area formula to I_1, we arrive to

$$\begin{aligned} I_1 &\asymp V(\delta_D(x)) \int_{D_1} \frac{\widetilde{U}(\delta_{D^c}(z))}{V(\delta_{D^c}(z))^2 \delta_{D^c}(z)^d} dz \\ &\asymp V(\delta_D(x)) \int_\eta^\infty \frac{\widetilde{U}(t)}{V(t)^2 t^d} \int_{D^c} \mathbf{1}_{\delta_D(w)=t}\, dw dt \\ &\asymp V(\delta_D(x)) \int_\eta^\infty \frac{\widetilde{U}(t)}{V(t)^2 t} dt. \end{aligned}$$

As before, dw in the first two lines denotes the Hausdorff measure on $\delta_D(w) = t$ and we used that

$$|\{w \in D^c : \delta_{D^c}(w) = t\}| \asymp t^{d-1},\ t \geq \eta.$$

Estimate of I_2: First note that for $z \in D^c \cap B(x,r_0)$ estimate (2.36) implies that

$$P_D(x,z) \asymp \frac{V(\delta_D(x))}{V(\delta_{D^c}(z))|x-z|^d}.$$

Next, as in the proof of Lemma 4.2.2 we will use the change of variables formula based on a diffeophormism $\Phi : B(x,r_0) \to B(0,r_0)$ satisfying

$$\Phi(\overline{D}^c \cap B(x,r_0)) = B(0,r_0) \cap \{w \in \mathbb{R}^d : w \cdot e_d < 0\},$$
$$|\Phi(z) \cdot e_d| = \delta_D(z) \text{ for any } z \in \overline{D}^c \cap B(x,r_0), \quad \Phi(x) = \delta_D(x) e_d\,.$$

Similarly as before, for the point $w \in \mathbb{R}^d_- = \{w \in \mathbb{R}^d : w \cdot e_d < 0\}$ we will write $w = (\widetilde{w}, w_d)$. Therefore, by the change of variables given by the diffeomorphism Φ it follows that

$$
\begin{aligned}
I_2 &\asymp V(\delta_D(x)) \int_{D_2} \frac{\widetilde{U}(\delta_{D^c}(z))}{V(\delta_{D^c}(z))|x-z|^d} dz \\
&\asymp V(\delta_D(x)) \int_{\{w \in B(0,r_0): -\eta < w_d < 0\}} \frac{\widetilde{U}(-w_d)}{V(-w_d)(|\delta_D(x) - w_d| + |\widetilde{w}|)^d} dw.
\end{aligned}
$$

Next, we apply the substitution $w_d = -t$ and switch to polar coordinates for $\widetilde{w}$ to obtain that

$$
\begin{aligned}
I_2 &\asymp V(\delta_D(x)) \int_0^\eta \frac{\widetilde{U}(t)}{V(t)} \int_0^{r_0} \frac{s^{d-2}}{(\delta_D(x) + t + s)^d} ds\, dt \\
&\overset{(4.6)}{\asymp} V(\delta_D(x)) \int_0^\eta \frac{\widetilde{U}(t)}{V(t)(\delta_D(x) + t)} dt \\
&\le \frac{V(\delta_D(x))}{\delta_D(x)} \int_0^\eta \frac{\widetilde{U}(t)}{V(t)} dt.
\end{aligned}
$$

Estimate of I_3: Lastly, note that for $z \in D^c \setminus B(x, r_0)$ such that $\delta_D(z) < \eta$, estimate (2.36) is equivalent to

$$
P_D(x,z) \asymp \frac{V(\delta_D(x))}{V(\delta_{D^c}(z))}.
$$

Therefore, similarly as in the estimate of I_1 we have

$$
\begin{aligned}
I_3 &\asymp V(\delta_D(x)) \int_{D_3} \frac{\widetilde{U}(\delta_{D^c}(z))}{V(\delta_{D^c}(z))} dz \\
&\asymp V(\delta_D(x)) \int_0^\eta \frac{\widetilde{U}(t)}{V(t)} \int_{D^c \setminus B(x,r_0)} \mathbf{1}_{\delta_D(w)=t}\, dw dt \\
&\asymp V(\delta_D(x)) \int_0^\eta \frac{\widetilde{U}(t)}{V(t)} dt.
\end{aligned}
$$

Since for $t < \eta$ we have that $\delta_D(x) + t < \operatorname{diam}(D) + \eta$, it follows that $I_3 \lesssim I_2$.

This proves that

$$
P_D g(x) \asymp V(\delta_D(x)) \left(\int_0^\eta \frac{\widetilde{U}(t)}{V(t)(\delta_D(x) + t)} dt + \int_\eta^\infty \frac{\widetilde{U}(t)}{V(t)^2 t} dt \right), \; x \in D.
$$

By fixing η and noting that

$$
\int_\eta^{\operatorname{diam}(D)} \frac{\widetilde{U}(t)}{V(t)(\delta_D(x) + t)} dt + \int_\eta^\infty \frac{\widetilde{U}(t)}{V(t)^2 t} dt \asymp 1
$$

we obtain (2.45). Inequality (2.46) follows immediately. ■

4.3. Estimate of the Green function G_D^ϕ

Lemma 4.3.1. Under assumption (WSC) it holds that

$$G_D^\phi(x,y) \asymp \left(\frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \wedge 1\right) \frac{1}{|x-y|^d \phi(|x-y|^{-2})}, \quad x,y \in D, \tag{4.38}$$

where the constant of comparability depends only on d, D and ϕ.

Proof. We slightly modify the proof of [58, Theorem 3.1] where the claim was proved under assumptions (A1)-(A5) from [58]. Since (WSC) implies (A1)-(A4) from [58], we show that assumption (A5), which assumes that $\int_0^1 \phi(\lambda)^{-1} d\lambda < \infty$, can be dropped in our setting. To shorten the proof, we note that every constant of comparability in the proof will depend at most on d, D and ϕ.

The lower bound proved in [58] does not use (A5) so we need to modify just the calculations for the upper bound.

Similarly as in [58], let us define

$$\begin{aligned}
I_1(r) &:= \int_0^{r^2} \left(\frac{\delta_D(x)\delta_D(y)}{t} \wedge 1\right) t^{-d/2} e^{-\frac{cr^2}{t}} \mathfrak{u}(t)dt,\\
I_2(r) &:= \int_{r^2}^{(2\mathrm{diam}D)^2} \left(\frac{\delta_D(x)\delta_D(y)}{t} \wedge 1\right) t^{-d/2} e^{-\frac{cr^2}{t}} \mathfrak{u}(t)dt,\\
L &:= \int_{(2\mathrm{diam}D)^2}^{\infty} e^{-\lambda_1 t} \delta_D(x)\delta_D(y)\mathfrak{u}(t)dt,
\end{aligned}$$

where λ_1 is the first eigenvalue of $-\Delta_{|D}$, see Subsection 3.1.3, and the constant c is the constant c_2 from (3.10). In addition to the bounds (3.10), there is another one for all big enough $t > 0$:

$$p_D(t,x,y) \asymp e^{-\lambda_1 t} \varphi_1(x)\varphi_1(y) \overset{(3.19)}{\asymp} e^{-\lambda_1 t} \delta_D(x)\delta_D(y), \quad x,y \in D, t \geq \mathrm{diam}D,$$

see [35, Theorem 4.2.5] and [73, Remark 3.3]. Hence,

$$\begin{aligned}
G_D^\phi(x,y) &= \int_0^\infty p_D(t,x,y)\mathfrak{u}(t)dt = \left(\int_0^{|x-y|^2} + \int_{|x-y|^2}^{(2\mathrm{diam}D)^2} + \int_{(2\mathrm{diam}D)^2}^{\infty}\right) p_D(t,x,y)\mathfrak{u}(t)dt\\
&\lesssim I_1(|x-y|) + I_2(|x-y|) + L.
\end{aligned} \tag{4.39}$$

Obviously,

$$L \leq \delta_D(x)\delta_D(y) \int_0^\infty e^{-\lambda_1 t} u(t) = \frac{\delta_D(x)\delta_D(y)}{\phi(\lambda_1)} \lesssim \left(\frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \wedge 1\right) \frac{1}{|x-y|^d \phi(|x-y|^{-2})},$$

since $|x-y|^{-d}\phi(|x-y|^{-2})^{-1}$ explodes at $x=y$ by (**WSC**).

For I_1 we imitate the calculations for [58, Eq. (3.7)]. Since $\mathfrak{u}(t)\lesssim \frac{\phi'(t^{-1})}{t^2\phi(t^{-1})^2}$ by (3.5) for all $t>0$, and since $t\mapsto \phi'(t^{-1})/\phi(t^{-1})^2$ increases, by the change of variables $cr^2/t=s$ we have

$$\begin{aligned}
I_1(r) &\lesssim \int_0^{r^2}\left(\frac{\delta_D(x)\delta_D(y)}{t}\wedge 1\right)t^{-d/2}e^{-\frac{cr^2}{t}}\frac{\phi'(t^{-1})}{t^2\phi(t^{-1})^2}dt\\
&\le \frac{\phi'(r^{-2})}{\phi(r^{-2})^2}\int_0^{r^2}\left(\frac{\delta_D(x)\delta_D(y)}{t}\wedge 1\right)t^{-d/2-2}e^{-\frac{cr^2}{t}}dt\\
&\lesssim \left(\frac{\delta_D(x)\delta_D(y)}{r^2}\wedge 1\right)\frac{\phi'(r^{-2})}{r^{d+2}\phi(r^{-2})^2}\int_c^{\infty}s^{d/2+1}e^{-s}ds\lesssim \left(\frac{\delta_D(x)\delta_D(y)}{r^2}\wedge 1\right)\frac{1}{r^d\phi(r^{-2})},
\end{aligned}$$

where the last inequality follows from (1.4).

The calculation for I_2 is slightly different then the one for [58, Eq. (3.8)]. Note that $\mathfrak{u}(t)\lesssim \frac{\phi'(t^{-1})}{t^2\phi(t^{-1})^2}\lesssim \frac{1}{t\phi(t^{-1})}\lesssim \frac{t^{\delta_2-1}}{r^{2\delta_2}\phi(r^{-2})}$, for $r^2\le t\le (2\mathrm{diam}D)^2$, where in the last approximate inequality we used 1.3. Hence

$$\begin{aligned}
I_2(r) &\lesssim \frac{1}{r^{2\delta_2}\phi(r^{-2})}\int_{r^2}^{(2\mathrm{diam}D)^2}\left(\frac{\delta_D(x)\delta_D(y)}{t}\wedge 1\right)t^{\delta_2-1-d/2}e^{-\frac{cr^2}{t}}dt\\
&\le \left(\frac{\delta_D(x)\delta_D(y)}{r^2}\wedge 1\right)\frac{1}{r^{2\delta_2}\phi(r^{-2})}\int_{r^2}^{\infty}t^{\delta_2-d/2-1}dt\lesssim \left(\frac{\delta_D(x)\delta_D(y)}{r^2}\wedge 1\right)\frac{1}{r^d\phi(r^{-2})}.
\end{aligned}$$

The claim now follows from (4.39).

■

4.4. Uniform integrability of some classes of functions

Lemma 4.4.1. Let $f : D \times \mathbb{R} \to \mathbb{R}$ be continuous in the second variable and $u_1, u_2 \in L^1(D, \delta_D(x)dx)$ such that $u_1 \leq u_2$. Assume that for every $u \in L^1(D, \delta_D(x)dx)$ such that $u_1 \leq u \leq u_2$ a.e. in D it holds that $x \mapsto f(x, u(x)) \in L^1(D, \delta_D(x)dx)$. Then the family

$$\mathscr{F} := \{ f(\cdot, u(\cdot)) \in L^1(D, \delta_D(x)dx) : u_1 \leq u \leq u_2 \text{ a.e. in } D \}$$

is uniformly integrable in D with respect to the measure $\delta_D(x)dx$, hence bounded in $L^1(D, \delta_D(x)dx)$.

Proof. Before we start the proof, we refer the reader to [69, Chapter 16] for details on the uniform integrability. Also, the proof is motivated by the proof of the similar claim which can be found in [66, Section 2].

Suppose that the family $\mathscr{F}$ is not uniformly integrable. Then there is $\varepsilon > 0$, a sequence $(v_n)_n \subset L^1(D, \delta_D(x)dx)$ such that $u_1 \leq v_n \leq u_2$ a.e. in D, and a sequence $(E_n)_n$ consisting of measurable subsets of D with property

$$\int_{E_n} |f(x, v_n(x))| \delta_D(x)dx \geq \varepsilon, \quad n \in \mathbb{N}.$$

Now use [66, Lemma 2.1] with $w_n(\cdot) = |f(\cdot, v_n(\cdot))| \delta_D(\cdot)/\varepsilon \in L^1(D)$ to extract a subsequence $(v_{n_k})_k$ of $(v_n)_n$ and disjoint sets $F_k \subset E_{n_k}$ such that

$$\int_{F_k} |f(x, v_{n_k}(x))| \delta_D(x)dx \geq \frac{\varepsilon}{2}, \quad k \in \mathbb{N}.$$

To finish the proof, define

$$v(x) = \begin{cases} v_{n_k}(x), & x \in F_k, \\ u_1(x), & x \in \cap_{k=1}^{\infty} F_k^c. \end{cases}$$

We have $u_1 \leq v \leq u_2$ in D, hence $v \in L^1(D, \delta_D(x)dx)$. Further,

$$\int_D |f(x, v(x))| \delta_D(x)dx \geq \sum_{k=1}^{\infty} \int_{F_k} |f(x, v_{n_k}(x))| \delta_D(x)dx = \infty,$$

which is a contradiction. ■

4.5. Regularity of transition densities

The following result on the regularity up to the boundary of the transition kernel of the killed Brownian motion appears to be well known, but we were unable to find an exact reference. In Chapter 3 we assumed that D is a $C^{1,1}$ bounded domain, but this result we give for a slightly more general open set since the claim is important in itself.

Lemma 4.5.1. Let D be an open bounded $C^{1,\alpha}$ domain for some $\alpha \in (0,1]$. For the transition density $p_D(\cdot,\cdot,\cdot)$ of the killed Brownian motion upon exiting the set D it holds that $p_D \in C^1((0,\infty)\times\overline{D}\times\overline{D})$.

Remark 4.5.2. Moreover, we will see in the proof of the previous lemma that p_D is somehow independently regular, variable by variable. E.g. we can differentiate $p_D(t,x,y)$ in x up to the boundary, then differentiate the obtained function in y up to the boundary, and then differentiate in t as many times as we want. This can be done up to $C^{1,\alpha}(\overline{D})$ regularity in the second and the third variable and up to $C^\infty(0,\infty)$ regularity in the first variable.

Proof of Lemma 4.5.1. Note that $p_D(t,x,y) \le p(t,x,y)$ everywhere by (3.9) so for fixed $t>0$ and $x\in D$ we have that the mapping $y \mapsto p_D(t,x,y)$ is in $L^\infty(D)\subset L^2(D)$. Hence, by the spectral representation of $L^2(D)$ functions we have

$$p_D(t,x,y) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\varphi_j(x)\varphi_j(y), \tag{4.40}$$

where we have used (3.17).

Now we show that the sum in (4.40) converges uniformly and is bounded in a certain strong sense. First note that $\varphi_j \in C^{1,\alpha}(\overline{D})$ by [40, Theorem 8.34]. Furthermore, by [40, Theorem 8.33] the following estimate holds

$$\|\varphi_j\|_{C^{1,\alpha}(D)} \le c_1(1+\lambda_j)\|\varphi_j\|_{L^\infty(D)}, \tag{4.41}$$

where $\|\cdot\|_{C^{1,\alpha}(D)}$ is the standard $C^{1,\alpha}(D)$ Hölder norm and $c_1 = c_1(d,D)>0$. Also, the eigenvalues satisfy the well known estimate

$$\|\varphi_j\|_{L^\infty(D)} \le c_2\lambda_j^{d/4}\|\varphi_j\|_{L^2(D)} = c_2\lambda_j^{d/4}, \tag{4.42}$$

see e.g. [75, Theorem 1.6], where $c_2 = c_2(d) > 0$. In particular, this inequality and the inequality in (4.41) imply

$$\|\nabla\varphi_j\|_{L^\infty(D)} \le C_1(1+\lambda_j)\|\varphi_j\|_{L^\infty(D)} \le c_3(1+\lambda_j)^{d/4+1}, \tag{4.43}$$

for $c_3 = c_3(d,D) > 0$. Also note that since φ_j vanishes on the boundary, by the mean-value theorem for every $x \in D$ there is some $\widetilde{x}$ between x and the closest boundary point to x such that

$$\left|\frac{\varphi_j(x)}{\delta_D(x)}\right| = \left|\nabla\varphi_j(\widetilde{x})\right| \le \|\nabla\varphi_j\|_{L^\infty(D)}.$$

Hence, for the sum in (4.40) the following uniform bound holds

$$\begin{aligned}\sum_{j=1}^{\infty} e^{-\lambda_j t}|\varphi_j(x)||\varphi_j(y)| &\le \sum_{j=1}^{\infty} e^{-\lambda_j t}\|\varphi_j\|_{L^\infty(D)}\|\varphi_j\|_{L^\infty(D)} \\ &\le c_2^2\sum_{j=1}^{\infty} e^{-\lambda_j t}\lambda_j^{d/2} < \infty,\end{aligned} \tag{4.44}$$

$$\begin{aligned}\sum_{j=1}^{\infty} e^{-\lambda_j t}|\varphi_j(x)|\left|\frac{\varphi_j(y)}{\delta_D(y)}\right| &\le \sum_{j=1}^{\infty} e^{-\lambda_j t}\|\varphi_j\|_{L^\infty(D)}\|\nabla\varphi_j\|_{L^\infty(D)} \\ &\le c_4\sum_{j=1}^{\infty} e^{-\lambda_j t}(1+\lambda_j)^{d/2+1} < \infty,\end{aligned} \tag{4.45}$$

where $c_4 = c_4(d,D) > 0$ and the sums converge by Weyl's law, see (3.18). Similar bounds hold if we take the derivate by the variable t or by the variable y.

Since $\varphi_j \in C^{1,\alpha}(\overline{D})$ and since the bounds (4.44) and (4.45) hold, we can pass the needed limits inside the sum (4.40) to get $p_D \in C^1((0,\infty)\times\overline{D}\times\overline{D})$.

In addition, since the bounds (4.41)-(4.43) hold, we can pass the limits inside the sum in the representation (4.40) to get that the density p_D is regular, variable by variable up to $C^{1,\alpha}(\overline{D})$ regularity in the second and the third variable and up to $C^\infty(0,\infty)$ regularity in the first variable, see Remark 4.5.2. ■

CONCLUSION

The aim of this thesis was to solve the following semilinear problem in a bounded domain $D \subset \mathbb{R}^d$, $d \geq 2$, for a non-local operator L:

$$Lu(x) = f(x, u(x)), \quad x \in D. \tag{4.46}$$

This was done for two types of non-local operators. The first type of the operator L was the operator $\phi(-\Delta)$, where $-\phi(-\Delta|_D)$ is the infinitesimal generator of the subordinate Brownian motion where the subordinator has ϕ as its Laplace exponent. On the exponent ϕ, the following was imposed:

(WSC). The function ϕ is a complete Bernstein function which satisfies the weak scaling condition at infinity: There exist constants δ_1, $\delta_2 \in (0,1)$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{\delta_1} \phi(r) \leq \phi(\lambda r) \leq a_2 \lambda^{\delta_2} \phi(r), \quad \lambda \geq 1, r \geq 1. \tag{WSC}$$

In addition to the equation (4.46) we imposed boundary conditions in D^c and on ∂D. For the boundary condition on ∂D a new type of a boundary trace operator was constructed - the operator W_D, motivated by the recent developments of such an operator for the case of the fractional Laplacian. In the thesis for the operator $\phi(-\Delta)$ it was assumed that the nonlinearity f from (4.46) satisfies

(F). $f : D \times \mathbb{R} \to \mathbb{R}$ is continuous in the second variable and there exist a function $\rho : D \to [0,\infty)$ and a continuous function $\Lambda : [0,\infty) \to [0,\infty)$ such that $|f(x,t)| \leq \rho(x)\Lambda(|t|)$.

There were given a number of sufficient conditions on f (i.e. on ρ and Λ from (**F**)) such that the problem (4.46) has a so-called weak dual solution. Such solutions are continuous as it is proved in the thesis. The set D in the problem (4.46) was an arbitrary bounded domain. In the special case when the set D is a bounded $C^{1,1}$ domain, the sufficient and necessary conditions on f such that (4.46) has a solution were given. In order to obtain a solution to (4.46), the theory of generalized harmonic functions was developed. The Martin integral representation of such generalized harmonic functions was shown, as well as the fact that such harmonic functions are smooth inside D.

The second type of the studied non-local operator was the operator $\phi(-\Delta|_D)$, where $-\phi(-\Delta|_D)$ is the infinitesimal generator of the subordinate killed Brownian motion, with the Laplace exponent ϕ of the subordinator. Again, **(WSC)** was imposed on ϕ. Further, in this setting only bounded $C^{1,1}$ domains D were considered, and the boundary condition was given only on ∂D. In the thesis the existence of Poisson kernel relative to $\phi(-\Delta|_D)$ was proved, and the kernel was used to obtain an integral representation of non-negative harmonic functions relative to $\phi(-\Delta|_D)$. Furthermore, an equivalence between non-negative harmonic functions relative to $\phi(-\Delta|_D)$ and non-negative functions that satisfy a certain mean-value property with respect to the underlying subordinate killed Brownian motion was obtained. Under assumption **(F)**, many sufficient conditions on f were given such that the problem (4.46) for $\phi(-\Delta|_D)$ has a so-called weak solution. Furthermore, it was proved that these solutions are continuous. Finally, in the special case when $\phi(\lambda) = \lambda^s$, $s \in (0,1)$, and $f(x,t) = \pm t^p$, there were given sharp bounds on the parameter $p \in \mathbb{R}$ such that the problem (4.46) has a solution.

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CURRICULUM VITAE

Ivan Biočić was born on October 9th, 1993, in Zagreb. He spent his youth in Brckovljani, attended elementary school in nearby Božjakovina, and graduated from Gymnasium in Sesvete. He finished elementary music school Zlatko Grgošević in Sesvete and secondary music school Elly Bašić in Zagreb in the class of Prof. Ante Čagalj. In 2015 he got his Bachelor's degree in mathematics, and in 2017 he got his Master's degree in mathematical statistics, both at the Faculty of Science in Zagreb. The topic of his Master's thesis was *Random walks with cookies in random and deterministic environments*, done under the supervision of Prof. Zoran Vondraček. During this period he received scholarships from the Rotary Club Sesvete, the Republic of Croatia, and the Adris Foundation.

In 2017 he enrolled in the Postgraduate program in Mathematics at the University of Zagreb and in 2018 he became a research and teaching assistant at the University of Zagreb, Faculty of Science, Department of Mathematics. During his doctoral studies, he visited many summer schools, workshops, and conferences. Among others, he gave a talk at the workshop Deterministic and stochastic fractional differential equations and jump processes at Isaac Newton Institute, Cambridge, UK.

In 2014 he attended the Modern Mathematics Summer School for Students in Lyon where he shook hands with John H. Conway, thus becoming five handshakes away from Gauss (Gauss-Dedekind-Cantor-Russell-Conway).

The work on his thesis resulted in two published articles thus far:

- I. Biočić. Representation of harmonic functions with respect to subordinate Brownian motion. *Journal of Mathematical Analysis and Applications*, **506**(1):125554, 2022.
- I. Biočić, Z. Vondraček, and V. Wagner. Semilinear equations for non-local operators: Beyond the fractional Laplacian. *Nonlinear Analysis*, **207**:112303, 2021.