Time and causal ordering in quantum mechanics and quantum field theory

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VREMENSKO I KAUZALNO UREĐENJE U KVANTNOJ MEHANICI I KVANTNOJ TEORIJI POLJA

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I would like to thank my family for all the love and support. To all the friends and extraordinary people who influenced me in this span of time. Especially thanks to my dear colleagues, Nicolás, Fatemeh and Sebastijan for being there in the most crucial times.

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In order to facilitate the comprehension of this inconceivable thesis, a heresiarch of the eleventh century devised the sophism of the nine copper coins, whose scandalous renown is in Tlön equivalent to that of the Eleatic paradoxes. There are many versions of this "specious argument," with varying number of coins and discoveries; the following is the most common:

"*On Tuesday, X crosses a deserted road and loses nine copper coins. On Thursday, Y finds in the road four coins, somewhat rusted by Wednesday's rain. On Friday, Z discovers three coins in the road. On Friday morning, X finds two coins on the veranda of his house.*"

The heresiarch would deduce from this story the reality – i.e., the continuity in time - of those nine recovered coins. "It is absurd" he said "to imagine that four of the coins did not exist between Tuesday and Thursday, three between Tuesday and Friday afternoon, two between Tuesday and Friday morning. It is logical to think that they in fact *did* exist – albeit in some secret way, hidden from the comprehension of men – at every moment of those three periods of time."

Jorge Luis Borges

1

[1] As a reminder of one passionate discussion on "*Do events exist (operationally) if there are no (quantum) clocks to measure them?*"

Vremensko i kauzalno uređenje u kvantnoj mehanici i kvantnoj teoriji polja

Sažetak

Jedna od glavnih poteškoća koja se javlja u pokušajima pomirenja kvantne teorije i opće teorije relativnosti proizlazi iz dubokih razlika u načinima na koje su pojmovi prostora, vremena, referentnog sustava i kauzalnosti implementirani u navedene teorije. Često se tvrdi da bi obje teorije trebale istrpiti kompromis u konceptualnim razumijevanjima spomenutog, kako bi njihovo spajanje (u teoriju kvantne gravitacije) bilo moguće.

Potaknuti tom mišlju, u ovom ćemo se radu usredotočiti na pretpostavke iznesene unutar novouspostavljenog okvira *kvantne kauzalnosti* i formulacije vremena preko uvjetnog probabilističkog pristupa (tzv. *Page-Wottersov formalizam*). Ovim proširenjima konceptualne razlike već na razini usporedbe kvantne mehanike s obzirom na kvantnu teoriju polja postaju snažne, a njihovo razumijevanje predstavlja preliminarni korak prema idejama kvantne gravitacije. Potaknuti time, cilj ove disertacije bio je preispitati pojmove vremenskog i kauzalnog uređenja u kontekstu modernih pristupa kvantnoj teoriji u odnosu na pristupe unutar perturbativne teoriji polja.

Kao zanimljivu okosnica za ispitivanje našeg razumijevanja uzeli smo operator vremenskog uređenja i istražili ga u nekoliko različitih, iako povezanih konteksta: (1) kako se pojavljuje u Schrödingerovom rješenju standardne kvantne mehanike s vremenski ovisnim Hamiltonijanom, (2) u Page-Wottersovom formalizam, gdje smo razmatrali jednadžbe ograničenja s nekoliko kvantnih satova i (3) u kontekstu kvantne teorije polja i Feynmanovog propagatora. Cilj istraživanja bio je razumjeti može li se promatrati djelovanje operatora vremenskog uređenja kao omogućavanje superpozicije različitih vremenski uređenih konfiguracija, uvodeći vremensku neodređenost u gore navedene postavke. Što se tiče kvantnih polja, doveli smo u pitanje mogućnost operativnog tumačenja procesa virtualne izmjene čestica (teorija raspršenja), koji se često shvaća u terminima dva vremenski uređena procesa koji se odvijaju putem izmjene čestica/antičestica.

Naš je pristup bio promatrati vremenski uređene eksponencijale do drugog reda i pokušati izolirati jednu od 'grana' reda, projicirajući superpoziciju na određeno

stanje. Otkrili smo da se superpozicija vremenskog uređenja eksponencijalne ekspanzije ne može proicirati na definitno stanje jedne od 'grana' superpoziciju, u kontekstu standardne kvantne mehanike, spajanjem s potencijalom pomoćnog sustava; međutim, ova vrsta projekcije može se izvesti u kontekstu Page-Wotters formalisma s dva ili više kvantnih satova. Zatim smo pristupili ovim razmatranjima u kontekstu kvantne teorije polja, koristeći Schrödingerovu funkcionalnu reprezentaciju, nastojeći implementirati Page-Wotters pristup i prethodno razvijeni 'toy model'.

Ključne riječi: superpozicija vremenskih uređenja, kvantna kauzalnost, perturbativna kvantna teorija polja, Feynmanov propagator, Page-Wottersov formalizam (bezvremenski formalizam), Schrödingerova funkcionalna reprezentacija

Time and causal ordering in quantum mechanics and quantum field theory

Abstract

One of the grates difficulties arising within the attempts to reconcile the quantum theory and general relativity stems from the profound differences in the ways the notions of space, time, reference frame, and causation enter the two formulations. It is often argued that both theories should compromise on their respective understandings for this merging (in the theory of quantum gravity) to be possible.

Motivated by this thought, in the thesis we will focus on the assumptions brought up within the newly established framework of *quantum causality* and the formulation of time within the conditional probabilistic approach (so-called *Page-Wotters formalism*). With these extensions, there are already strong conceptual differences at the level of comparison between quantum mechanics and quantum field theory, the resolution of which would represent a preliminary step towards the ideas of quantum gravity. Stirred by this friction, the goal of the dissertation was to reexamine the notions of temporal and causal ordering in the context of modern approaches to quantum theory in relation to approaches within perturbative field theory.

As an interesting playground for challenging our understanding we took time ordering operator and investigated it in several different, albeit related contexts: (1) as it appears in the Schrödinger solution of standard quantum mechanics with time-dependent Hamiltonian, (2) in Page-Wotters formalism, considering a constraint equation with several quantum clocks and (3) in the context of quantum field theory and Feynman propagator. The research objective was to understand whether one can view the action of the time ordering operator as enabling a superposition of different time-ordered configurations, introducing time indefiniteness in the aforementioned settings. Regarding quantum fields, we questioned the possibility of the operational interpretation of the (scattering theory) virtual particle exchange process, often understood in terms of two time-ordered processes happening via particle/antiparticle exchange.

Our approach was to look at the time-ordered exponentials up to the second order and attempt to isolate one of the ordering 'branches,' projecting a superposition to

a definite state. We found that one cannot break the time ordering superposition of the exponential expansion in the context of standard quantum mechanics coupling it with the ancilla potential; however, one can perform this kind of projection in the context of timeless quantum mechanics with two or more quantum clocks. We then approached these considerations in the context of quantum fields, using the Schrödinger functional representation, seeking to implement the Page-Wotters approach and the previously developed toy model.

Keywords: superposition of temporal orders, quantum causality, perturbative quantum field theory, Feynman propagator, Page-Wotters formalism, Schrödinger functional representation

Contents

1 Introduction and motivation

All established physical paradigms presuppose in a certain way a notion of space and time, or spacetime, in which matter content evolves. Spacetime is a set of events [2] subjected to the causal and time ordering. Thus, time is essentially different from space in the way that it comes with a preferred orientation, known as *the arrow of time*, imposing an ordering of the events in a fixed manner. Particular choice of orientation implements the fact that our experiences are time asymmetric. Such asymmetry seems to be lacking in the fundamental dynamical laws, nevertheless it is often ascribed to the second law of thermodynamics and low entropy initial states. A question sometimes posed is whether the arrow of time can be understood as a feature of how we as the agents probe the world and not intrinsic to nature itself? [74]

Accepting the asymmetry of ordering in time, in the standard sense we relate it to asymmetry of causation; causes precede the effects in time. However, the two orderings are essentially different. Whereas the causal ordering implies the temporal one, the temporal ordering is, for the causal one, a necessary but not sufficient condition. The differences are most prominently manifested when going beyond non-relativistic physics. There, the ordering in time establishes a relation between every pair of events, which we call a *total* ordering over the set, but bears the observer dependence. On the other hand, the causal ordering will be *partial*, due to the spacelike region where events lack causal relation, and *absolute* in sense of observer independence. Even so, *'. . . if one arranges phenomena in a series such that every term contains the reason for all those which come after it in the series, the causal order of the phenomena so defined will coincide with their temporal order of succession.'*[61].

To elucidate these points one should notice that although all physical paradigms work with the notions of 'time', 'event', and the 'observer', the way these enter the

[2] Here we will distinguish abstract pointlike events in a sense of a set which constitutes a spacetime manifold, from the events defined in the operational sense via some transformation on the system i.e. some physically meaningful occurrence. Theories such as general relativity and quantum field theory will postulate the events as points, as elements of the set existing a priori where the rest of the physical reality is represented by mathematical structures over the sets of these points. On the other hand, theories set up operationally will assume that physical transformations representing an event will occur in some finite extend of space and time. Moreover, in probabilistic theories such as quantum mechanics, the events in a spacetime are defined with a spread of probability distribution assigned to that event and not (in general sense) sharply.

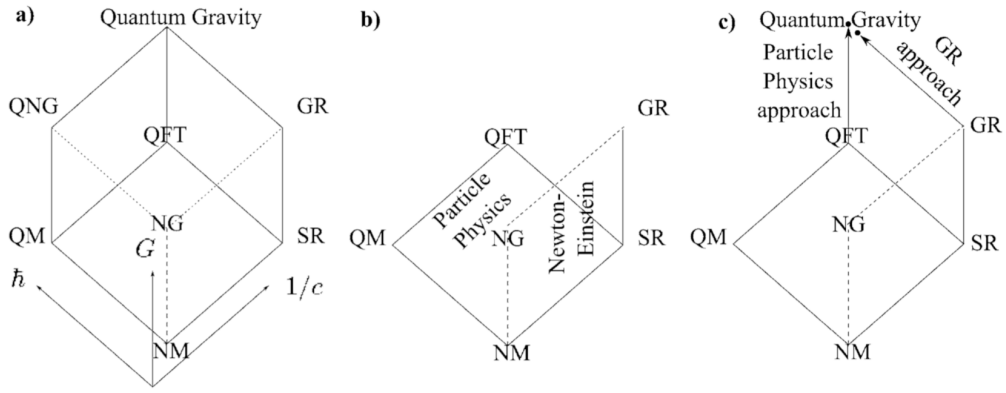


Figure 1.1: (a) Three fundamental constant represent three major paradigm shifts. Going along the edge of a cube represents making a specific constant significant relative to the of the rest of physical structure. Going along the edge representing Newton's gravitational constant G, thus means considering the realms where gravitational force is non-negligible in comparison to the rest of the forces. Similarly, going along the edge of $1/c$ we are approaching theories where the speeds are comparable to the speed of light. Going along the edge with $\hbar$, represents considering scales comparable to minimal length scale. (b) Shows the regimes we often deal with separately. Namely, 'Newton-Einstein' classical plane and 'Particle physics' non-gravitational plane. (c) Here we see the way of approaching 'Quantum gravity' vertex without extending the Planckian cube. If one aims to extend the notions of regular quantum mechanics to combine with general relativity, this extension will need to be implemented into quantum field theory contexts first. [2]

formalism often differ significantly across those, representing a conceptual problem in the regimes where two theories should combine. Namely, if one considers the architecture of modern theoretical physics, it seems to be spanned by the three known fundamental constants of Nature: Newton's gravitational constant G, Planck's constant $\hbar$ and the reciprocal of the speed of light c, as seen in Figure (1.1). One should notice that out of all vertices of the Planckian cube, for all except one we have a formalism with working predictive power. The vertex which, after almost a century of research effort [3] remains unestablished, is the one of quantum gravity. The main difficulty can be understood as coming from the disparate nature of its ingredient theories, as it attempts to reconcile general relativity with quantum mechanics which belongs to a different class with respect to how the notions of space, time, and causality are implemented within

This is often recognized in a division on *background dependent* and *background independent* theories. Quantum mechanics then, along with Newtonian physics, spe-

[3]The first speculations came from Einstein's 1916. paper on gravitational waves, culminating in the first attempt by Rosenfeld in 1930.

cial relativity, and quantum field theory, is background-dependent, since it is set in the fixed, rigid structure of space and time, completely external to its dynamical law. On the other hand, in general relativity, both matter field and spacetime are possibly evolving, treating both structures as subjected to the dynamical law present within. This arises from the relational nature of general relativity, background independence coming from the diffeomorphism invariance of the theory. Contrasting with the standard quantum mechanics, not only that quantum theory lack such symmetry, but it fully leaves the notions of time outside its regime: time is not observable in the theory, nor is it subjected to quantum indefiniteness, rather it is treated as an external classical parameter. Both time and causal ordering are thus, within standard treatments of quantum mechanics and quantum field theory, unaffected by the quantum regime. Naturally, this represents a difficulty when considering possible quantum gravity contexts where spacetime, and thus time itself, should be quantized. There, one might expect that the notion of causality should succumb to the principle of superposition, introducing a 'fuzziness' in the spacetime structure, making the background both dynamical and probabilistic [15].

Nevertheless, in the recent efforts, the idea of causal relations being subjected to quantum indefiniteness was posed [12][58][45][47]. Interestingly, the framework of these considerations lies within the realm of information theory and computation [46], where the configuration of quantum circuit is contrasted with the unraveling of the events in the structure of spacetime. It has been proposed [22] that the geometry of the wires between the gates could be controlled by the quantum state of a controlled qubit, introducing coherent superposition in the ordering of the processes. Such a realization is known as *the quantum switch* and represents an example of a causally non-separable process. These processes, incompatible with definite ordering of operations but, in certain subsets still embeddable in definite spacetime background, were proven to exist by employing so-called "causal witnesses" [3][75]. Even beyond this, the framework of process matrices was established [63], where resources incompatible with the definite causal order of quantum operations were introduced. The main mathematical tool for implementing the ordering indefiniteness are quantum supermaps [19]. Within this framework, recent considerations addressed also the possibility of backwards-facing agents probing quantum processes

with an inverted time arrow or even with an arrow in a superposition of orientations[21][4]. However, as seen from the (1.1), to make these considerations significant for the quantum gravity vertex these features should first be lifted to the quantum field theory context.

It is then legitimate to ask, can we find an example of ordering indefiniteness, e.g. as the one appearing on the level of quantum switch, within the framework of standard physics?[5] One of the possible places to look is Feynman's interpretation of antiparticles [35], and the spacetime interpretation of interaction processes in terms of Feynman diagrams [34]. Namely, by labeling the events representing transformations as, $A =$' particle created' and $B =$' particle annihilated', we would have a similar ordering indefiniteness, on the level of superposition of orderings in the Feynman propagator. Within formalism, this indefiniteness of operations is accounted for with the time ordering operator, having role to chronologically reorder the series of operators.

In this thesis we aimed, therefore, to investigate the ordering coming from the time ordering operator. In particular, we considered the time ordering operator as having two 'branches', corresponding to the two definite ordering realizations. Contrasting with the superposition of orderings as appearing in the quantum switch, we aimed to "isolate" one of these branches by making suitable choices of the preparations and the measurements. Taking these questions as a motivation, the goal of the thesis was to investigate common points and points of departure of state-of-the-art understanding of time and causality in quantum mechanics, with respect to the one in quantum field theory contexts. In these considerations we will embed our original results, regarding the superposition of orders within time ordering operator. This thesis will not investigate algebraic formulations of quantum field theory, rather it will focus on the perturbative approach and corresponding interpretations. We will also suppose that the concept of a point-like particle and point evaluated field operators are viable working idealisation. The thesis is organized as follows:

- In Preliminaries (2) we will introduce theoretical context of our considerations;
  - In subsection (2.1) we will unravel the notions of time and symmetry in

[4]Superposition of thermodynamical arrow of time was already mentioned prior in [39]

[5]even without encountering metric indefiniteness suspected in the quantum gravity realm

quantum mechanics and quantum field theory. We will touch upon the extension of the covariance principle in quantum mechanics by considering in section (2.1.3) quantum reference frame transformations.

  - In subsection (2.2) we will state the implementations of causality in quantum mechanics, surpassing the classical ways of treating causality in the quantum regime. Here we will introduce the notion of the quantum switch and quantum time flip.
  - Subsection (2.3) will portray how causality is implemented in quantum field theory, by considering Klein-Gordon and Dirac field. We will consider the interpretation regarding the antiparticles as negative energy solutions propagating backward in time. We will then proceed to analyze the structure of the time ordering operator as appearing in the Dyson perturbative expansion and consider it in the context of the transition amplitude of Feynman diagrams. Lastly, we will contrast the ordering appearing on the level of virtual particle exchange with the old-fashioned perturbation where diagrams come with a fixed order.

- In section (3) we will show an attempt to couple the ancillary system to a particular ordering configuration, attempting to use it as a control enabling us to project on a definite ordering state.

- In section (4) we introduce the Page-Wotters formalism which surpasses the classical ways of treating time and extends the Schrödinger equation if several quantum clocks are used. Within the context of Page-Wotters formalism with multiple clocks, we were able to make the entangling preparation and measurement and to isolate one branch of time-ordered exponential, as appearing this time in the context of history state.

- In (5) we then attempted to put these considerations back into the context of scattering amplitude and Feynman diagrams by proposing a way of implementing the history state in the Schrödinger functional representation of quantum field theory.

- In (6) we conclude.

2 Preliminaries

2.1 Time, clock and symmetries in quantum mechanics and quantum field theory

2.1.1 Time and clock in quantum mechanics

1927. Heisenberg in his paper [49] stated the uncertainty relations, with particular interpretational difficulty rising around the statement of time-energy uncertainty:

$$\Delta E \Delta t \geq \frac{\hbar}{2} \,. \tag{2.1}$$

Namely, since the uncertainty relations come from the non-commutativity of quantum mechanical observables [73]:

$$(\Delta_\psi A) \cdot (\Delta_\psi B) \geq \frac{1}{2} \left| \left\langle \psi \left| \left[\hat{A}, \hat{B} \right] \right| \psi \right\rangle \right| \,, \tag{2.2}$$

equation (2.1) would presuppose relation of the form:

$$[\hat{T}, \hat{H}] = i\hbar \,. \tag{2.3}$$

Even though Heisenberg didn't formulate uncertainty relation in a general way of equation (2.2), his reasoning also relied on the arguments of incompatibility of the observables, stating the commutator relation with time and energy as operators.
As one of the responses to Heisenberg's time-energy uncertain relation came Pauli's criticism [66] regarding the impossibility of the time operator appearing in equation (2.3). Namely, if $\hat{T}$ would be a self-adjoint operator satisfying such a commutation relation, then $\hat{H}$ as its conjugate operator, would have an unbounded spectrum. This is a general result; we say that any two operators satisfying such a commutator relation are unitarily equivalent to the position and momentum operators on the real line, as a consequence of the Stone-von-Neumann theorem[6]. We can see this also by noting that $\hat{T}$ would now be a generator of energy shifts:

[6]The statement regarding the unboundedness imposed by the commutator is usually known under the uniqueness of unitary representation of the Heisenberg group on finitely many generators (the Stone-von-Neumann theorem). Namely, the theorem ensures us that the representation in terms of unbounded operators on a Hilbert space of square-integrable functions, as in the case of position and momentum $[\hat{x}, \hat{p}] = i\hbar$, is unique up to an isomorphism. The theorem does not hold in the case of infinitely many generators (i.e. in a QFT context).

$$e^{\hat{T}\Delta E}|E\rangle = |E + \Delta E\rangle \tag{2.4}$$

This means that, by considering negative energy displacements $-\Delta E$, we could generate displacements going below the minimal energy E_{min} bound on the Hamiltonian operator, implying that the spectrum has no lower bound. Such a situation is not physical since no system should be able to radiate energy infinitely.

In Pauli's conclusion:

... from the C.R. written above, it follows that $\hat{H}$ possesses continuously all eigenvalues from $-\infty$ to $+\infty$. [...] *We, therefore, conclude that the introduction of an operator t is basically forbidden and the time t must necessarily be considered as an ordinary number ('c-number').* [66]

Nonetheless, one can overcome the criticism of Pauli, if one relates time observables to POV measures, characterized by covariance under time translations. In other words, any time observable $\hat{T}$ should possess covariance with respect to a group of time translations

$$\alpha_g : \mathbf{R} \to \mathbf{R}, \quad t \mapsto t + b \tag{2.5}$$

inducing a group homomorphism from the additive group $G = (R, +)$ to a group of unitary operators $e^{-i\hat{H}t}$. The covariance condition is thus, [52]

$$e^{i\hat{H}t'} E(t) e^{-i\hat{H}t'} = E(t + t') \quad t \in \mathbf{R} \tag{2.6}$$

Then time operator can be constructed via non-orthogonal resolution

$$\hat{T} = \int_{\mathbf{R}} t d\hat{E}(t) \tag{2.7}$$

where the properties of POV measure are (as introduced in Section (A)),

$$\begin{aligned} E(\tau) &\geq 0 \\ E\left(\cup_i \tau_i\right) &= \sum_i E\left(\tau_i\right) \text{ for all disjoint sequences } (\tau_i) \subset \mathbf{R} \\ \int_{\mathbf{R}} dt E(t) &= \mathbb{1} \\ P\left[T = \tau\right] &= \mathrm{tr}[\rho E(\tau)] \end{aligned} \tag{2.8}$$

Let us now take POV measures to be 1-dim projector on a clock states $E(t) = |t\rangle\langle t|$, such that the clock states corresponding to different t are not necessarily orthogonal. Then, the covariance of $\hat{T}$ with respect to the group G generated by $\hat{H}$, is equivalent to the following relation between clocks states

$$|t + t'\rangle = e^{-i\hat{H}t'}|t\rangle \tag{2.9}$$

which we get by integrating relation (2.6),

$$\int e^{i\hat{H}t'}|t\rangle\langle t|e^{-i\hat{H}t'}dt = \int |t + t'\rangle\langle t + t'|dt \tag{2.10}$$

On the other hand, if the states associated with the projectors $E(t) = |t\rangle\langle t|$ are clock eigenstates, we get back to the situation of self-adjoint $\hat{T}$ and unbounded $\hat{H}$. Nevertheless, in this case the clock behaves perfectly in the sense that the clock states are perfectly distinguishable, $\langle t|t'\rangle = \delta(t - t')$ [7]. Even though the perfect clock is not physically realisable, in Chapter (4) we will take it as a working assumption, keeping in mind the possibility of generalization.

Much of the discussion following the Heisenberg uncertainty relation (2.1) cleared a careful distinction between:

- External time: background classical time, inherited from Newtonian physics. It is measured by an external, classical clock and is not affected by the laws of quantum mechanics.
- Observable time: some dynamical variable of the system. It is based on nonstationary observables and defined as the time determined by the scale through which $\langle\hat{O}\rangle$ changes significantly.

As discussed, time operator $\hat{T}$ did not enter to be a part of standard quantum mechanics, precluding Δt to have a clear meaning in terms of its standard deviation. This is why the character of time in time-energy uncertainty is still disputable, and the relation is often misused to justify various desired interpretations. Nevertheless, one typical way of understanding Δt is as the minimum time in which the mean

[7] To a perfect clock we will usually additionally associate a property that it doesn't run backwards, i.e. $\langle t'|e^{-i\hat{H}\tilde{t}}|t\rangle = 0$ for all $\tilde{t} > 0$ if $t > t'$.

value of the observable changes by more than one standard deviation. To show this, let us choose $A = H$ and $B = Q$, in the generalized uncertainty principle (2.2),

$$\sigma_H^2 \sigma_Q^2 \geq \left(\frac{1}{2i} \langle [\hat{H}, \hat{Q}] \rangle \right)^2 \tag{2.11}$$

where we introduced the notation $\sigma_A \equiv \Delta_\psi A$. We now use the equation for the rate of change of the expectation value

$$\frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle \tag{2.12}$$

assuming that Q does not depend explicitly on t, $\frac{\partial \hat{Q}}{\partial t} = 0$,

$$\sigma_H^2 \sigma_Q^2 \geq \left(\frac{1}{2i} \frac{\hbar}{i} \frac{d\langle Q \rangle}{dt} \right)^2 = \left(\frac{\hbar}{2} \right)^2 \left(\frac{d\langle Q \rangle}{dt} \right)^2 \tag{2.13}$$

Or simply,

$$\Rightarrow \sigma_H \sigma_Q \geq \frac{\hbar}{2} \left| \frac{d\langle Q \rangle}{dt} \right| \tag{2.14}$$

We define

$$\begin{aligned} \Delta E &\equiv \sigma_H \\ \Delta t &\equiv \frac{\sigma_Q}{|d\langle Q \rangle / dt|} \end{aligned} \tag{2.15}$$

from which we get $\Delta E \Delta t \geq \frac{\hbar}{2}$. We see now that Δt is the time it takes for the expectation value to change by one standard deviation. The uncertainty then simply means that, if ΔE is very small, the rate of change of any observable Q must be gradual. Or vice versa, if the rate of change of all the observables is abrupt (for example, in the case of particles living for a very short time), then the uncertainty in energy is very large. We will refer to this in Chapter (2.4.3)

2.1.2 Symmetries of quantum theory

Let us recall the symmetries of quantum mechanics. The state space of a quantum system is a Hilbert space $\mathcal{H}$. This means that set of all *linear* maps that preserve

Hilbert space structure $\hat{U} : \mathcal{H} \to \mathcal{H}$, must be *unitary*

$$\left\langle \hat{U}\psi \mid \hat{U}\phi \right\rangle = \langle \psi \mid \phi \rangle \tag{2.16}$$

for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$, since from

$$\begin{aligned} \left\langle \hat{U}\psi \mid \hat{U}\phi \right\rangle = \left\langle \psi \left| U^\dagger \hat{U} \right| \phi \right\rangle \\ \Rightarrow \hat{U}^\dagger \hat{U} = \mathbb{1} \end{aligned} \tag{2.17}$$

Nevertheless, what we actually measure are probabilities, $|\langle \psi \mid \phi \rangle|^2$. This allows for another type of transformation,

$$\left\langle \tilde{\hat{U}}\psi \mid \tilde{\hat{U}}\phi \right\rangle = \langle \phi \mid \psi \rangle \tag{2.18}$$

These are *antilinear* [8] and *antiunitary*. Antiunitary operators can correspond only to discrete transformations (for example time reversals) whereas unitary transformations are continuously connected to unity and thus can correspond to both continuous and discrete transformations.

Theorem 1. [52] Any automorphism of the set of quantum states $\hat{\rho}(\mathcal{H})$ has the form

$$\hat{\rho} \to \hat{V}\hat{\rho}\hat{V}^\dagger \tag{2.19}$$

where $\hat{V}$ is a unitary or anti-unitary operator on the Hilbert space $\mathcal{H}$.

Demanding unitary representation in the context of quantum field theory will have an important consequence, as we shall see below. Note that the unitarity condition implies that the adjoint $U^\dagger$ is a compositional inverse to U, hence making every unitary operator invertible. Therefore, unitary operators on the Hilbert space $\mathcal{H}$ form a group, which we will denote $U(\mathcal{H})$.

We will call transformation a symmetry, if it leaves the functional form of the Hamiltonian (and consequently a dynamical law) invariant. This means that we

[8] $\tilde{\hat{U}}(|\psi\rangle + |\varphi\rangle) = \tilde{\hat{U}}|\psi\rangle + \tilde{\hat{U}}|\varphi\rangle$, $\tilde{\hat{U}}c|\psi\rangle = c^*\tilde{\hat{U}}|\psi\rangle$

impose further that,

$$\hat{H}'(x_i) = \hat{H}(x_i) = \hat{U}\hat{H}\hat{U}^\dagger + i\hbar\frac{d\hat{U}}{dt}\hat{U}^\dagger \tag{2.20}$$

where in the case of time-independent transformation, $\hat{U}(g,t) \equiv \hat{U}(g)$ descends to the condition that the transformation commutes with the Hamiltonian,

$$[\hat{U}, \hat{H}] = 0 \tag{2.21}$$

Non-relativistic physics admits a symmetry group of Galileian transformations $G = SO(3) \rtimes R^3$, representing isometries of Euclidean $3D$ space. As discussed, in the context of quantum mechanics, these transformations will be represented by the unitary operators. Each such transformation can be represented in the form of exponential $\hat{U}(\epsilon) = e^{i\epsilon\hat{G}}$, where $\hat{G}$ is hermitian operator called the *generator*, as a consequence of Stone's theorem. Thus, for the dynamical symmetries of non-relativistic quantum mechanics we have:

- translations in time $\hat{T}_a t = t + a \rightarrow \hat{U}_t = e^{-i\hat{H}t}$
- translations in space $\hat{T}_{\vec{a}}\vec{r} = \vec{r} + \vec{a} \rightarrow \hat{U}_t = e^{-i\hat{\vec{p}}\vec{r}}$
- rotations in space $\hat{R}_\phi \vec{v} = \vec{v}' \rightarrow \hat{U}_\phi = e^{-i\hat{\vec{J}}\phi}$
- Galileaian boosts (change from one to another inertial, non-relativistic reference frame)
  $\hat{T}_{\vec{V}}\vec{r} = \vec{r} + \vec{V}t, \hat{T}_{\vec{V}}\vec{p} = \vec{p} - \vec{V}m \rightarrow \hat{U}_{\vec{V}} = e^{-i\vec{V}(t\vec{p} - m\vec{r})}$, yielding $\hat{U}_{\vec{V}}|\vec{r}\rangle = e^{-im\vec{V}\vec{r}}|\vec{r} + \vec{V}t\rangle$ and $\hat{U}_{\vec{V}}|\vec{p}\rangle = e^{-it\vec{V}\vec{p}}|\vec{p} + \vec{V}m\rangle$

2.1.3 Quantum reference frames

In classical mechanics, reference frames consist of a spatial Cartesian coordinate system and a clock. Each time we write a state, we assume a choice of some classical reference frame. For example, writing $|\psi\rangle = \int dx \psi(x)|x\rangle$, assumes some measuring rod labeling with $'x'$ a position of a coordinate basis. Now, since reference frames are ultimately realized as physical systems, it can be assumed they are subjected also to a quantum regime [31]. Such quantum transformations are now taking into account

that one reference frame, from the perspective of another, might appear in a superposition or even become entangled with the system. In other words, the group of symmetries established via classical reference frames doesn't encompass the symmetries of a full set of fundamental transformations.

Let us illustrate the idea of a quantum reference frame transformation via an example. We describe the situation in which C is the initial reference frame, A is the reference frame to which perspective we want to move, and B is, in general, a composite system. To change from the C reference frame to A, we need to change relative position coordinates as seen from C to the relative position coordinates as seen from A,

$$\begin{aligned} \hat{x}_A &\mapsto -\hat{q}_C, \qquad \hat{p}_A \mapsto -\left(\hat{\pi}_B + \hat{\pi}_C\right), \\ \hat{x}_B &\mapsto \hat{q}_B - \hat{q}_C, \qquad \hat{p}_B \mapsto \hat{\pi}_B, \end{aligned} \tag{2.22}$$

generalizing the classical transformations in a sense that coordinates and momenta are now quantum and satisfying canonical commutation relations,

$$\begin{aligned} (\hat{x}_A, \hat{p}_A) &\mapsto (\hat{q}_C, \hat{\pi}_C), \quad [\hat{x}_A, \hat{p}_A] = [\hat{q}_C, \hat{\pi}_C] = i\hbar, \\ (\hat{x}_B, \hat{p}_B) &\mapsto (\hat{q}_B, \hat{\pi}_B), \quad [\hat{x}_B, \hat{p}_B] = [\hat{q}_B, \hat{\pi}_B] = i\hbar. \end{aligned} \tag{2.23}$$

Then the quantum reference frame transformation will be defined as a quantum map $\hat{S}$,

$$\hat{S} : \mathcal{H}_A^{(C)} \otimes \mathcal{H}_B^{(C)} \to \mathcal{H}_B^{(A)} \otimes \mathcal{H}_C^{(A)} \tag{2.24}$$

such that it preserves the structure of canonical commutation relations. For the transformation from the perspective of C to the perspective of A, this map will take the form

$$\hat{S} = \hat{\mathcal{P}}_{AC} e^{i\hat{x}_A \hat{p}_B}. \tag{2.25}$$

It contains two pieces which we will elaborate on more carefully. One is the *translation operator* $e^{\frac{i}{\hbar}\hat{x}_A\hat{p}_B}$ and another is *parity swap operator* $\hat{\mathcal{P}}_{AC}$. The translation operator $e^{\frac{i}{\hbar}\hat{x}_A\hat{p}_B}$ can be understood by looking first at the bottom case of Figure (2.1). We are considering a system B, described from the perspective of system C. Now, if we want to translate to a reference frame A, we simply have to apply a unitary generating spatial translation to move the system B by the amount of parameter x_A, $e^{ix_A\hat{p}_B}|\phi\rangle_B$. What if now we have a situation as depicted in the upper part of

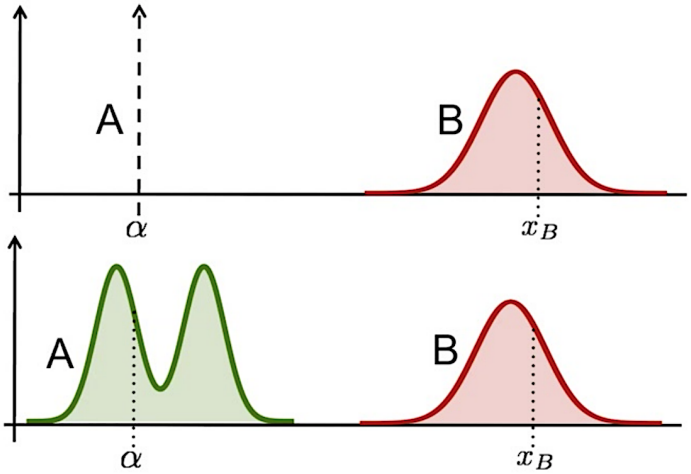


Figure 2.1: Example of relative states in a reference frame of particle C. The upper figure assumes A to be sharply localized at the position $x_A = \alpha$, whereas the upper figure assumes a delocalized quantum state, serving as a new reference frame to which we want to jump.[13]

Figure(2.1), where A is a delocalized quantum system? Since the position of A, in this case, is not sharp, we again consider translations of B but this time for each x'_A constituting the positions covered by the system A. Thus, the translation of the system B will be controlled by the position of the system A, in a sense that total translation will take the form:

$$\int dx'_A \psi(x'_A) e^{ix'_A \hat{p}_B} |\phi\rangle_B |x'_A\rangle \tag{2.26}$$

where we summed over all x'_A's, weighted by the probability amplitude. This is now

$$\int dx'_A e^{i\hat{x}_A \hat{p}_B} |\phi\rangle_B |x'_A\rangle \langle x'_A|\psi\rangle_A = e^{i\hat{x}_A \hat{p}_B} |\psi\rangle_A |\phi\rangle_B \tag{2.27}$$

The crucial point then is that now x_A *parameter* of standard translation transformation, became an *operator which acts on the Hilbert space serving as a reference frame*.

To account fully for the change of perspective, we additionally need *parity swap operator*. Namely, as seen from C, Hilbert spaces assigned to A and B are $\mathcal{H}_A^{(C)}$ and $\mathcal{H}_B^{(C)}$. In the new reference frame A, we assign $\mathcal{H}_B^{(A)}$ to B and $\mathcal{H}_C^{(A)}$ to C. We can thus

define a *parity swap operator* $\hat{\mathcal{P}}_{AC}$,

$$\begin{aligned} \hat{\mathcal{P}}_{AC} : \mathcal{H}_A^{(C)} &\to \mathcal{H}_C^{(A)} \\ \psi_A(x) &\mapsto \psi_C(-x) \\ \Rightarrow \hat{\mathcal{P}}_{AC}\hat{x}_A\hat{\mathcal{P}}_{AC}^\dagger &= -\hat{q}_C \\ \hat{\mathcal{P}}_{AC}\hat{p}_A\hat{\mathcal{P}}_{AC}^\dagger &= -\hat{\pi}_C \end{aligned} \tag{2.28}$$

Given (2.25) we conclude that the state of B and C relative to A is given by,

$$\rho_{BC}^{(A)} = \hat{S}\rho_{BA}^{(C)}\hat{S}^\dagger \tag{2.29}$$

Let us derive now the Schrödinger equation as seen from a quantum reference frame. Consider the Schrödinger equation in the reference frame of particle C

$$i\hbar\frac{d\hat{\rho}_{AB}^{(C)}}{dt} = \left[\hat{H}_{AB}^{(C)}, \hat{\rho}_{AB}^{(C)}\right] \tag{2.30}$$

Schrödinger equation in A's reference frame is then

$$\begin{aligned} i\hbar\frac{d\hat{\rho}_{BC}^{(A)}}{dt} &= \left[\hat{H}_{BC}^{(A)}, \hat{\rho}_{BC}^{(A)}\right] \\ \hat{H}_{BC}^{(A)} &= \hat{S}\hat{H}_{AB}^{(C)}\hat{S}^\dagger + i\hbar\frac{d\hat{S}}{dt}\hat{S}^\dagger \end{aligned} \tag{2.31}$$

This implies that provided a transformation via $\hat{S}$ operator exists, starting from a classical reference frame, the evolution can be described unitarily also from any quantum reference frame. Then, we can also define a symmetry transformation as:

$$\hat{S}\hat{H}\left(\{m_i, \hat{x}_i, \hat{p}_i\}_{i=A,B}\right)\hat{S}^\dagger + i\hbar\frac{d\hat{S}}{dt}\hat{S}^\dagger = \hat{H}\left(\{m_i, \hat{x}_i, \hat{p}_i\}_{i=B,C}\right) \tag{2.32}$$

a transformation leaving the functional form of the Hamiltonian invariant.

The important consequence of this is that the notions of entanglement and superposition may now be reference frame dependent, as depicted in Figure(2.2)

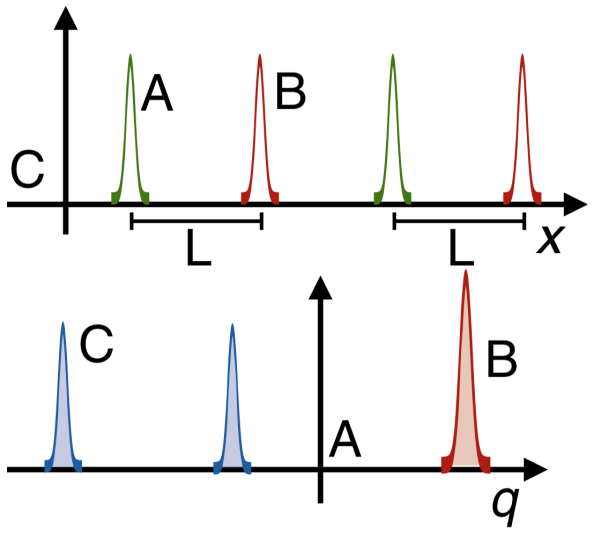


Figure 2.2: In a reference frame of C, A and B are entangled and perfectly correlated such that the relative distance between them is always L. In the reference frame of particle A, B is in a well-defined position and C is in a superposition of positions.[31]

2.1.4 Time and clock in quantum field theory

Time in quantum field theory is the one of special relativity and Minkowski spacetime structure. In non-relativistic quantum mechanics, the notion of time is inherited from Newtonian physics [9]. We will thus here briefly contrast the notions of time in Newtonian physics compared to relativistic physics.

We define a process of *datation*, assigning a real number $t \in \mathbf{R}$ to any event of a spacetime $\mathcal{M}$. Formally, we define a spacetime to be an ordered pair $(\mathcal{M}, g_{\mu\nu})$. Here $g_{\mu\nu}$ is *metric tensor*: smooth, symmetric and nondegenerate $(0,2)$-tensor field defined on a manifold $\mathcal{M}$. This defines *chronological (time) ordering* via a function $f : \mathcal{M} \to \mathbf{R}$ called *timefunction*. Time ordering defines a *total ordering* over the set of spacetime points; with respect to any given event, we can define a subset of events constituting its chronological past, present, and future.

The causal structure, imprinted in the metric, encodes a well-defined causal ordering in spacetime. The crucial difference between Newtonian physics and relativistic physics lies in the fact that the metric in relativity is of semi-Riemannian type

[9] Although the parameter serving as time in quantum theory can still be compatible with the context of relativistic quantum mechanics, one should keep in mind that the formalism of quantum theory was built assuming the Newtonian notion of time. Since the formalism of quantum theory has subtle conceptual and interpretational caveats, the nature of time assumed in its foundations should be kept in mind while coupling this parameter to the symmetries of spacetime (i.e. adding Lorentz transformations).

(more concretely Lorentzian, having $(1,3)$ signature). Namely, space and time in Newtonian physics $\mathbf{R} \times \mathbf{R}^3$ admit separate metrics of Riemannian (Euclidian) type. Physical events may thus be described by the independent quantities $t \in \mathbf{R}$ and $\mathbf{x} \in \mathbf{R}^3$. The action of a metric tensor on two vectors gives us an inner product $g_{\mu\nu} : T_pM \times T_pM \to \mathbf{R}$ which induces a notion of *length*. Namely, we define a line element as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.33}$$

For the spatial components of Newtonian spacetime $g_{\mu\nu} = (1,1,1)$, we get back familiar Euclidian distance formula $ds^2 = dx^2 + dy^2 + dz^2$. On the other hand, for special relativity we have $\eta_{\mu\nu} \equiv (1,-1,-1,-1)$[10], yielding a line element of the type,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2 \tag{2.34}$$

This is called a *spacetime interval*. A consequence of such a metric is the division of spacetime directions into three families:

- timelike $x_\mu x^\mu < 0$,
- spacelike $x_\mu x^\mu > 0$,
- lightlike $x_\mu x^\mu = 0$.

Consequently, spacetime trajectories are classified into timelike, spacelike and lightlike, depending on the type of tangent vector field on it. Massive particles will move on timelike trajectories, massless particles on lightlike, and particles of imaginary mass (sometimes referred to as tachyons) on spacelike trajectories. Standardly, we assume no signals or particles propagating through the spacelike region.

Such a structure of Minkowski spacetime also implies a difference in the notion of *duration*. Namely, the duration of a process in the Newtonian sense is merely a difference between the times of its initial and final events, whereas in relativistic physics it depends on the trajectory through the spacetime, i.e. it is given by the spacetime arc length.

Newtonian physics has well defined, *unique* timefunction - presupposing an absolute notion of *simultaneity*. Its time ordering coincides with the causal one, which is again

[10]The important thing is a flip of the sign in the first slot, i.e. that the metric is of Lorentzian type. Equivalent choice of the metric can be $\eta_{\mu\nu} \equiv (-1,1,1,1)$.

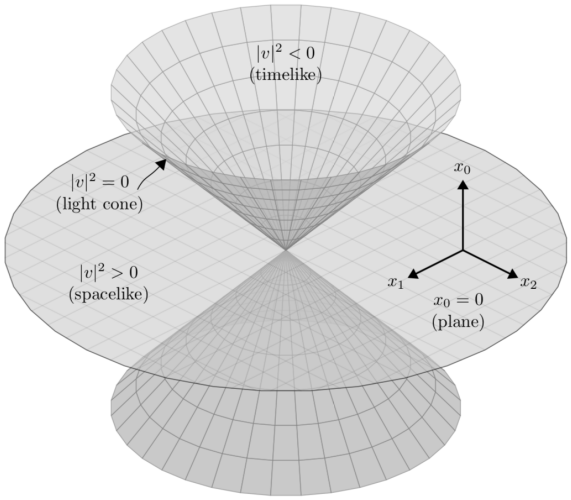


Figure 2.3: Lorentzian metric splits up spacetime into three regions. Given a point $p \in \mathcal{M}$, we define set of tangent vectors $v \in T_p\mathcal{M}$ to be timelike if $g(v,v) > 0$, lightlike if $g(v,v) = 0$ and spacelike if $g(v,v) < 0$.

then absolute. The existence of a timefunction in a spacetime induces a decomposition of spacetime into a product of space and time, $\mathcal{M} = \Sigma \times \mathbf{R}$. We call this *folliation* of spacetime in terms of Σ non-intersecting *hypersurfaces*. If spacetime admits such splitting, we call it *globally hyperbolic* [11]. This enables us to have well-defined time evolution from initial data defined on each hypersurface (Cauchy surface). Of such type are both Newtonian (Euclidian space + time) and Minkowski spacetime of special relativity. Nevertheless, the difference between the two spacetimes is that the unique timefunction of Newtonian induces anisotropy, due to the preferential status this puts on the time direction (we can rotate in space but not in space + time). Isotropy in Newtonian physics exists only in the spatial part, implying the existence of $SO(3)$ symmetry of spatial rotations. [12]

On the other hand, in globally hyperbolic spacetimes one has infinite choices of timefunctions, implying isotropy in time directions. Then, only upon choosing a time function, does this symmetry gets broken. In this thesis we will restrict our considerations to Minkowski spacetime, which has even further symmetries; it is both

[11] not all solutions of Einstein equations are of this type

[12] This is to be contrasted with pre-Newtonian physics which distinguished horizontal, $2-$dimensional isotropic plane, and $1-$dimensional vertical direction exterior to it [59][57]. Namely, due to the effects of a gravitational field, the isotropy in all three spatial directions was not obvious. We will now have a similar shift in the perspective, where isotropy lifted to include also temporal directions will imply the existence of symmetry rotations which will mix temporal and spatial components, in a way $3D$ rotations mix vertical and horizontal components.

homogeneous and isotropic in all directions. As such it is called *maximally symmetric spacetime*, admitting a maximal group of isometries. Here we see the departure between time and causal ordering in relativistic physics, whereas there are infinitely many possible time orderings, depending on the choice of an observer, the causal ordering will be invariant.

Different observers will thus in relativistic considerations be associated with different timefunctions, slicing up spacetime with a different set of hypersurfaces. These hypersurfaces define each of his/her notions of simultaneity. They will nevertheless agree on the causal ordering between the events. What defines how the observer splits spacetime, is the notion of velocity at which one is traveling. Each observer is tracing a path in spacetime called a *worldline*. A worldline is a path γ through spacetime, parameterized by λ, having associated tangent vector $u^\mu = dx^\mu/d\lambda$. For simplicity, we take this parameter to correspond to the *proper time* $\lambda \to \tau$ [13]. Proper time is a measure of a distance traveled through a spacetime, up to the minus sign (reflecting the fact that the shortest possible path between the two events will maximize the elapsed time [14]). Now the tangent to the worldline becomes a 4-velocity, $u^\mu = dx^\mu/d\tau$. Thus, the direction in which future points depend on the velocity one is traveling through spacetime. The perspective of any point observer traveling along a worldine[15], spacetime decomposes as $\mathbf{R}u^\mu \otimes \Sigma(u^\mu)_\perp$.

2.1.5 Symmetries of special relativity

Every physical theory admits a certain set of equally valid reference frames with respect to which it forms a covariance principle. In Newtonian, those are inertial reference frames at low velocities, where physical laws admit symmetry under *Galilean group* of transformation $G = SO(3) \rtimes R^3$. For special relativity, those are *all* inertial reference frames, and changes of reference frames that leave the physics of special relativity invariant are translations, rotations, and spacetime rotations, called *boosts*. All these transformations define *Poincaré group*, $\mathcal{P} = SO(1,3) \rtimes \mathbf{R}^{1,3}$, often also called

[13] The theory is reparametrization invariant. Choosing a parameter corresponding to proper time, reduces some redundancies and simplifies the equations of motion

[14] This is why a twin who stays at Earth gets older!

[15] In local rest frame, 4-velocity is $u^\mu = (1, \mathbf{0})$

inhomogeneous Lorentz group, since it is build out of Lorentz group and spacetime translations. With respect to the Galilean group thus, we now consider translations and boosts in spacetime that mix up spatial and temporal directions. Let us thus first define the Lorentz group, which describes spacetime boosts and rotations. We define *Lorentz transformation* Λ as

$$x^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} x^{\beta} \tag{2.35}$$

a transformation that leaves Minkowski metric invariant,

$$\begin{aligned} (\Delta s)^2 &= \eta_{\alpha'\gamma'} dx^{\alpha'} dx^{\gamma'} \\ &= \eta_{\alpha'\gamma'} \Lambda^{\alpha'}{}_{\beta} \Lambda^{\gamma'}{}_{\delta} dx^{\beta} dx^{\delta} \\ &\equiv \eta_{\beta\delta} dx^{\beta} dx^{\delta} \end{aligned} \tag{2.36}$$

$$\begin{aligned} \Rightarrow \eta_{\beta\delta} &= \eta_{\alpha'\gamma'} \Lambda^{\alpha'}{}_{\beta} \Lambda^{\gamma'}{}_{\delta} \\ \eta &= \Lambda^{\mathrm{T}} \eta \Lambda \end{aligned} \tag{2.37}$$

The set of Lorentz transformations defines a *Lorentz group*,

Definition 1 (Lorentz group)**.** [88] The Lorentz group $O(1,3)$ is the group of linear transformations preserving the Minkowski space inner product on $\mathbf{R}^{1,3}$.

Lorentz group has four disconnected components, as seen in Figure (2.4), representing solutions to the condition (2.37). From the defying relation, we get the condition $\det \Lambda = \pm 1$ by taking the determinant of both sides. We often consider a subgroup called *proper* Lorentz group, defined via the extra restriction $\det \Lambda = 1$. The components corresponding to $\det \Lambda = -1$ are called *improper*. Let us consider timelike components $\mu\nu = 00$ of (2.37),

$$\begin{aligned} 1 &= \left(\Lambda^0_0\right)^2 - \sum_i \left(\Lambda^i_0\right)^2 \\ \Rightarrow \left(\Lambda^0_0\right)^2 &\geq 1 \end{aligned} \tag{2.38}$$

Thus, we have two components $\Lambda^0_0 \geq 1$ called *orthochronous* and $\Lambda^0_0 \leq -1$ *non-orthochronous*. With respect to $\det \Lambda = \pm 1$ and $\Lambda^0_0 \geq \pm 1$, we will have 4 disconnected

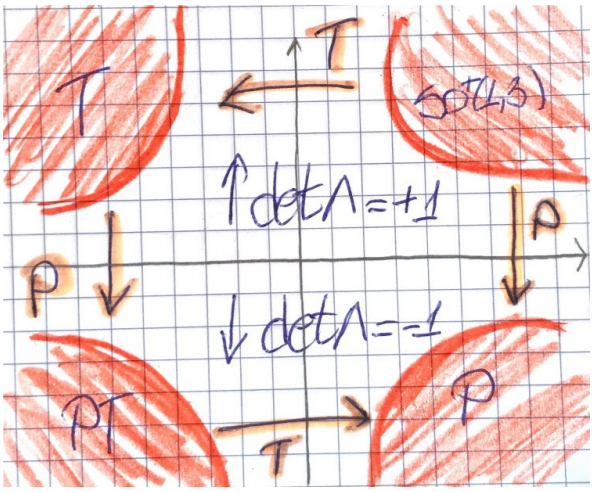


Figure 2.4: The Lorentz group consists of four separate components. The proper orthochronous Lorentz subgroup $SO(1,3)^+$ is the component continuously connected to the identity. The other components can be reached using time reversal, parity swap, or a combination of both.

components of Lorentz group,

$$\begin{aligned}
\{(\Lambda, a)\} \equiv L &= L_+^\uparrow \cup \left(L_+^\uparrow \otimes P\right) \cup \left(L_+^\uparrow \otimes T\right) \cup \left(L_+^\uparrow \otimes PT\right) \\
L_+^\uparrow &\equiv \left\{(\Lambda, a), \det \Lambda = 1, \Lambda_0^0 \geq 1\right\} \\
P &= \begin{pmatrix} 1 & \\ & -\mathbf{1}_{3\times 3} \end{pmatrix} \\
T &= \begin{pmatrix} -1 & \\ & \mathbf{1}_{3\times 3} \end{pmatrix}
\end{aligned} \tag{2.39}$$

as shown in Figure (2.4). Our focus will be *restricted* Lorentz group, representing proper, orthochronous component; the part of the Lorentz group continuously connected to the unity $L_+^\uparrow = SO(1,3)^+$. The other three components are obtained by multiplying with discrete symmetries, as denoted in Figure (2.4).

Lorentz group has six parameters, three corresponding to boost velocities $\beta\boldsymbol{n}$ and the three to rotation angles $\boldsymbol{\alpha}$. Each $\Lambda \in SO(3,1)^+$ can be reconstructed from a Lorentz boost with velocity $\beta = \frac{v}{c}$ in direction $\boldsymbol{n}$ and spatial rotation $R(\boldsymbol{\alpha}) \in SO(3)$. Whereas rotations form a subgroup of the Lorentz group, boosts do not. There are two important consequences the Lorentz group will inherit from these transformations; namely, since it contains boosts, it will be *non-compact* and as such it will admit only infinite-dimensional irreducible unitary representation. On the other hand,

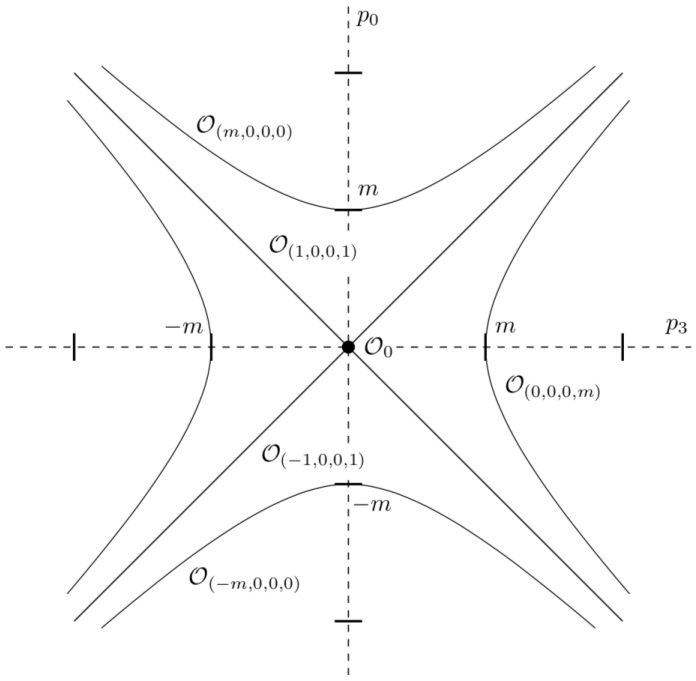


Figure 2.5: Orbits of the Lorentz group [88]

since it contains rotations it will not be simply connected, implying the projective representation. As in the case of $SO(3)$, one can deal with this by looking at the covering group (in the case of $SO(3)$ it is $SU(2)$) which will have the same algebra. Thus, when considering $SO(1,3)^+$, we will consider covering group $SL(2,\mathbb{C})$. These considerations are important in spinor representations.

Let us consider orbits of restricted Lorentz group $SO(1,3)$ on $\mathbf{R}^{1,3}$ by considering its action on 4-momentum $p_\mu \in \mathbf{R}^{1,3} : p_\mu p^\mu = m^2$. Setting different values of mass will yield different types of orbits. Namely,

- *positive energy* particles $p = (m,0,0,0)$ will, upon boosting, trace upper sheet of the two sheeted hyperboloid

$$H_m^+ = \left\{ p_\mu \in \mathbf{R}^{1,3} : p_\mu p^\mu = m^2, \quad p^0 > 0 \right\} \tag{2.40}$$

  This will correspond to particles with mass m moving *forward in time*

- *negative energy* particles $p = (-m,0,0,0)$ will trace bottom sheet of the two sheeted hyperboloid

$$H_m^- = \left\{ p_\mu \in \mathbf{R}^{1,3} : p_\mu p^\mu = m^2, \quad p^0 < 0 \right\} \tag{2.41}$$

passing *backward in time*. This corresponds to *antiparticle solutions*

- massless $m = 0$ particles travelling forward through time

$$H_0^+ = \left\{ p_\mu \in \mathbf{R}^{1,3} : p_\mu p^\mu = 0, \quad p^0 > 0 \right\} \tag{2.42}$$

  associated to forward lightcone

- massless $m = 0$ particles travelling backwards in time

$$H_0^- = \left\{ p_\mu \in \mathbf{R}^{1,3} : p_\mu p^\mu = 0, \quad p^0 < 0 \right\} \tag{2.43}$$

  associated with backwards lightcone

- particle with imaginary mass im

$$H_{im} = \left\{ p_\mu \in \mathbf{R}^{1,3} : p_\mu p^\mu = -m^2 \right\} \tag{2.44}$$

  passing through spacelike regions are associated with one sheeted hyperboloid.

The particle is specified with the mass and the spin. Here mass will determine the size of the hyperboloid whereas the spin will be determined by the representation of the stabilizer group acting on the orbit. The part with the spin comes from considering a representation of the Poincaré group.

Let us now go back to what is a full symmetry group of special relativity, that is the Poincaré group. Poincaré transformations are defined as

$$x' = T(\lambda, a)x = \Lambda x + a \tag{2.45}$$

[16]resulting in 10-parameter group which now contains, along with rotations and boosts, also spacetime translations. In analogy to the above, the component which contains the identity $T(1,0)$ is called $ISO(3,1)^+$, where I stands for inhomogeneous.

Symmetries of quantum field theory are characterized by the *unitary* representation of Poincare group $\mathcal{P} = SO(1,3) \rtimes \mathbf{R}^{1,3}$. This implements both symmetries

[16]where we used matrix notation $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu \to x' = \Lambda x$

of quantum mechanics (in a sense of preserving Hilbert's space structure) and special relativity. As we said, the Lorentz group is non-compact and admits infinite-dimensional irreducible unitary representation. This is the reason why field theory admits infinite dimensional Fock space structure on which field operators act. Generators that define a unitary representation of the Poincaré algebra on the state space are $\hat{M}^{\mu\nu}$, which contains six generators of the Lorentz group, and $\hat{P}^{\mu}$ which represents generators of spacetime translations. We can thus represent the Poincaré group as an operator

$$U(\Lambda, a) = e^{\frac{i}{2}\varepsilon_{\mu\nu}\hat{M}^{\mu\nu}} e^{ia_{\mu}\hat{P}^{\mu}} = 1 + \frac{i}{2}\varepsilon_{\mu\nu}\hat{M}^{\mu\nu} + ia_{\mu}\hat{P}^{\mu} + \dots \tag{2.46}$$

acting on a Hilbert space (Fock space of field theory). The algebra these operators satisfy is given by the following commutation relations,

$$\begin{aligned} i\left[M^{\mu\nu}, M^{\rho\sigma}\right] &= g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho} \\ i\left[P^{\mu}, M^{\rho\sigma}\right] &= g^{\mu\rho}P^{\sigma} - g^{\mu\sigma}P^{\rho} \\ \left[P^{\mu}, P^{\nu}\right] &= 0 \end{aligned} \tag{2.47}$$

which defines the algebra of the Poincaré group. We can cast the Poincaré algebra relations in a less compact form, defining the generator of $SO(3)$ rotations $\boldsymbol{J}$ (the angular momentum) and the generator of boosts $\boldsymbol{K}$ via

$$M^{ij} = -\varepsilon_{ijk}J^{k} \quad \Leftrightarrow \quad J^{i} = -\frac{1}{2}\varepsilon_{ijk}M^{jk}, \quad M^{0i} = K^{i} \tag{2.48}$$

then the commutator relations take the form

$$\begin{aligned} \left[J^{i}, J^{j}\right] &= i\varepsilon_{ijk}J^{k}, \quad \left[J^{i}, P^{j}\right] = i\varepsilon_{ijk}P^{k}, \quad \left[P^{i}, P^{j}\right] = 0 \\ \left[J^{i}, K^{j}\right] &= i\varepsilon_{ijk}K^{k}, \quad \left[K^{i}, P^{j}\right] = i\delta_{ij}P_{0}, \quad \left[J^{i}, P_{0}\right] = 0 \\ \left[K^{i}, K^{j}\right] &= -i\varepsilon_{ijk}J^{k}, \quad \left[K^{i}, P_{0}\right] = iP^{i}, \quad \left[P^{i}, P_{0}\right] = 0 \end{aligned} \tag{2.49}$$

We see that whereas the angular momentum is hermitian, $\boldsymbol{K}$ is antihermitian for all finite-dimensional representations, preventing them from being unitary. Regarding commutation relations, we see that boosts and rotations will generally not commute, except if their axes coincide. Moreover, P_0 commutes with rotations and spatial translations but not with boosts. Therefore the eigenvalues of boost operator

$\boldsymbol{K}$ cannot be used for labelling physical states.

The Poincaré group can be classified by recalling that one can associate irreducible representations of semi-direct products $M \rtimes N$ to M-orbit O_α for $\alpha \in M$, and an irreducible representation of the corresponding little group N_α (see for example [88], Chapter 20). Thus, referring to the discussion above, we need to further specify the representation of the stabilizer group N_p on the eigenspace of the momentum operators with eigenvalue p. Irreducible representations of this group are classified by the spin. For spin 0, points on the hyperboloid can be identified with positive energy solutions of the Klein-Gordon equation, and functions on the hyperboloid both correspond to the space of all solutions of this equation and carry an irreducible representation of the Poincaré group. The case of spin 1 will correspond to the solutions of the Dirac equation, where one must use the double cover $SU(2)$ of $SO(3)$. The Poincaré group representation will be on functions on the orbit that take values in two copies of the spinor representation of $SU(2)$. For choices of higher spin representations of the stabilizer group, one can again find appropriate wave equations and construct Poincaré group representations on their space of solutions. For an alternative classification via Casimir operators, see [88], for more detailed discussion refer to Wigner's paper [87].

The important point we want to make here is that unitary irreps of the Poincaré group specify the mass and the spin, which completely determines the particle. We can state the definition of a particle a la Wigner,

Definition 2. [32] A quantum mechanical particle is a projective, irreducible unitary representation of the Poincaré group.

2.2 Quantum causality

Quantum theory has, in many ways, challenged our understanding of causality. Early discussions questioned the causality in quantum realms due to the probabilistic nature of the theory [17]. In the words *'...an event is causally determined if it can be predicted with certainty.'* [68] the attitude of what is now known as causal determinism

[17] Given identical initial state preparation, the outcomes of the experiments follow probabilistic distributions.

is reflected. Coming as naturally inherited from Newtonian physics, this view was by that time, a general way of understanding causation. Nevertheless, the advent of quantum theory disentangled the notions of causality and determinism, putting forward an understanding of causal relations as genuinely probabilistic. [18]

The other challenging point was the notion of quantum non-locality. Namely, in 1964. Bell [7] proved that the predictions of quantum theory are incompatible with the notion of, what is known as the *local realism*. The assumptions made by the local realism hypothesis were that the results of the observations on the individual systems are predetermined by the elements of reality (realism) and independent of whatever measurements might be performed distantly (locality). Here one should distinguish this notion of locality from the one defined in the context of special relativity. Namely, while relativity imposes locality in terms of a finite speed of information propagation, the notion of locality referred to in local realism assumes the impossibility of a correlation between separated systems without dynamic and causal relations. Here by causal relations, one understands either direct or indirect relations, later ones described by Reichenbach's common cause principle [19] For clarity, in the following, we will refer to this notion of the locality as *Bell's locality*. As showed by Bell, quantum mechanics exhibits correlations between spacelike separated measurements without the set of common classical past influences, amusingly by Einstein attributed with the phrase "spooky action at a distance" [28].

Aiming to disprove the completeness of quantum mechanics, the aforementioned assumptions which came to be known as the local realism, ware made in the famous Einstein Podolsky Rosen (EPR) argument [29]. Since these considerations will serve as a good analogy to understanding the structure of causal correlations and since

[18]Plank believed that the probabilistic understanding of causality is not inevitable due to causal relations being subjected to intrinsic uncertainty, but rather something to be attributed to the subject's imperfection, "*We have to stick to our starting point, the statement, that an event is causally determined if it can be predicted with certainty. Otherwise, we should lose our only basis. On the other hand, we remain bound to the other statement that it is in no case possible to predict an event with certainty. Consequently, we have to modify the first statement in order to maintain the principle of causality in nature. [But] an ideal spirit (idealer Geist), who knows perfectly all the physical events of today, would be capable of predicting the weather of tomorrow in all details with absolute certainty; and that applies to any other prediction of physical events.*"[54]

[19]Reichenbach's principle. If two quantities are correlated, then one quantity directly influences the other, or both have a common cause

Bell's experiment represents one of the most important results in quantum mechanics, let us briefly sketch the structure of the EPR argument. The argument starts by presupposing that theory can establish isomorphism with objective reality, only if it satisfies the condition of *correctness* and *completeness*. Regarding correctness, it is defined with respect to the degree of agreement between the conclusions of the theory and human experience [20]. Completeness can then be stated with the following definition,

Definition 3 (Completeness)**.** [29] A theory is complete if every element of physical reality has a counterpart in it.

Related to the criterion of correctness, they state the *sufficient criterion of physical reality*

Definition 4 (Sufficient criterion of physical reality)**.** [29] If without in any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an element of the physical reality corresponding to this physical quantity.

EPR argument focused on the aspect of completeness, using, stated in their paper less explicitly, the notion of what we will call *Bell's locality*,

Definition 5 (Bell locality)**.** Measurements performed on dynamically and causally independent systems, must be independent of each other. More concretely, if outcomes of measurements performed on spacelike separated, dynamically independent systems are *not independent*, we should be able to identify *a set of past factors*, described by some variables λ, having a joint causal influence on both outcomes, which fully account for the dependence between the outcomes.

The structure of the EPR argument then went as follows,

$$[(\text{ Suff. Cond. Reality }) \wedge (\text{ Bell's locality})] \underset{\text{for QM}}{\Longrightarrow} \neg \text{ Completeness}, \tag{2.50}$$

where the assumptions on the left-hand side would be, as they assumed, the assumptions of any physically valid theory. We will call these the assumptions of local

[20] *"This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement. It is the second question that we wish to consider here, as applied to quantum mechanics."*[29]

realism. What invalidated completeness in the case of quantum mechanics is the uncertainty principle, as they conclude: *'More generally, it is shown in quantum mechanics that, if the operators corresponding to two physical quantities, say A and B, do not commute, that is, if $AB \neq BA$, then the precise knowledge of one of them precludes such a knowledge of the other. Furthermore, any attempt to determine the latter experimentally will alter the state of the system in such a way as to destroy the knowledge of the first. From this follows that either (1) the quantum mechanical description of reality given by the wave function is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality.'*[29]. In other words, given the stated assumptions (of local realism) and the non-commutativity of quantum mechanics, EPR concludes the incompleteness of quantum theory.

Nevertheless, as a response to the EPR argument came Bell's experiment showing that the assumptions of the local realistic theory are not consistent with quantum mechanics. To state Bell's argument, consider Alice and Bob and a source S distributing to each a physical system, as shown in Figure (2.6).

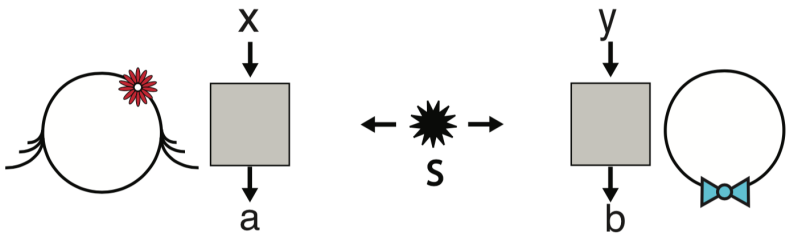


Figure 2.6: Bell experiment. Two distant parties, Alice and Bob, are receiving a system coming from the source S. Variables x and y label measurement choices available to Alice and Bob respectively, whereas a and b denote obtained measurement outcomes. [14]

Each agent performs an experiment on the corresponding subsystem [21]. We will denote the measurement settings of Alice and Bob by x and y respectively. Chosen variables are said to be *free*, in the sense "that the values of such variables have implications only in their future light cones" [8] [22]. We will denote measurement outcomes by a and b respectively. Measurement results are assumed to be non-deterministic in

[21] subsystems being entangled due to the past interaction.

[22] One interpretation of free variables is that they can be set directly by an experimenter. A simple example of a free variable would be a light switch by which we control a light bulb which will have the switch in its past lightcone. [6]

general, i.e. for a fixed set-up, we will have some probability distribution of possible outcomes. Thus, the experiment will be described with the probability distribution $p(ab \mid xy)$. What can be shown is that when such an experiment is performed, outcomes on both sides are not statistically independent from each other:

$$p(ab \mid xy) \neq p(a|x)p(b|y). \tag{2.51}$$

To explain this within the assumptions of local realism, we should be able to account fully for the dependence between outcomes a and b by identifying a set of past variables, having a joint causal influence on both outcomes. These random variables representing a common past influence on the statistics of both parties are called *hidden variables* λ, sometimes referred to also as *shared randomness* (ref. Figure(2.7)).

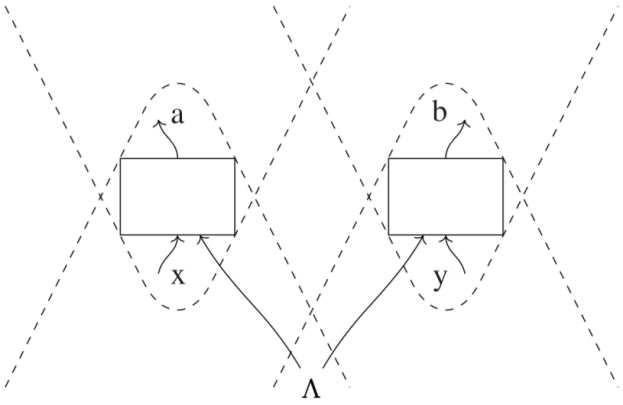


Figure 2.7: Two boxes represent local operations of Alice, on the left, and Bob, on the right. Alice chooses some measurement setting x and obtains the measurement result a, whereas Bob does similarly, adjusting y and obtaining b. The dashed lines represent light cones, for measurement setting, their future causal region, and for measurement outcomes, their past causal region, assuming time to flow from bottom to top. Λ represents assumed shared causal influence on the measurement settings. [6]

Bell's principle of locality then requires a factorization of probability distribution as follows:

$$p(ab \mid xy) = \int_{\Lambda} d\lambda q(\lambda) p(a|x,\lambda) p(b|y,\lambda) \tag{2.52}$$

where $p(\lambda)$ represents probability distribution for λ [23].

Bell's argument was based on stating so-called *Bell's inequality*. In the following, we will portray the idea based on a slightly modified version, stated by Clauser,

[23] here we implicitly assume that the measurements x and y can be chosen independently of λ, i.e., that $q(\lambda|x,y) = q(\lambda)$

Horne, Shimony, and Holt in 1969. [23] in terms of *CHSH inequality* [24]. Let us assume only two measurement choices per observer $x, y \in \{0, 1\}$ and possible outcomes taking only two values $a, b \in \{-1, +1\}$. We define expectation values of the products ab, for certain measurement choices (x, y) as,

$$\langle a_x b_y \rangle = \sum_{a,b} abp(ab|xy) \tag{2.53}$$

We now define S, a function of the probabilities $p(ab|xy)$,

$$S = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle \tag{2.54}$$

Under the assumption that these probabilities satisfy the locality decomposition (2.52), we arrive at the CHSH inequality:

$$S = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle \leq 2 \tag{2.55}$$

Namely, the assumption of local hidden variable theory, enables us to write expectation values of a product ab as a product of local expectations $\langle a_x \rangle_\lambda = \sum_a ap(a|x, \lambda)$ and $\langle b_y \rangle_\lambda = \sum_b bp(b|y, \lambda)$ taking values in $[-1, 1]$,

$$\langle a_x b_y \rangle = \int \mathrm{d}\lambda q(\lambda) \langle a_x \rangle_\lambda \langle b_y \rangle_\lambda \tag{2.56}$$

We then derive (2.55) via, [7]

$$S = \int \mathrm{d}\lambda q(\lambda) S_\lambda \tag{2.57}$$

with

$$\begin{aligned} S_\lambda &= \langle a_0 \rangle_\lambda \langle b_0 \rangle_\lambda + \langle a_0 \rangle_\lambda \langle b_1 \rangle_\lambda + \langle a_1 \rangle_\lambda \langle b_0 \rangle_\lambda - \langle a_1 \rangle_\lambda \langle b_1 \rangle_\lambda \\ &= \langle a_0 \rangle_\lambda (\langle b_0 \rangle_\lambda + \langle b_1 \rangle_\lambda) + (\langle b_0 \rangle_\lambda - \langle b_1 \rangle_\lambda) \langle a_1 \rangle_\lambda \end{aligned} \tag{2.58}$$

Since each $\langle a_0 \rangle_\lambda, \langle a_1 \rangle_\lambda, \langle b_0 \rangle_\lambda, \langle b_1 \rangle_\lambda \in [-1, 1]$. If we assume that λ is deterministic conditioning, we have either $(\langle b_0 \rangle_\lambda + \langle b_1 \rangle_\lambda)$ or $(\langle b_0 \rangle_\lambda - \langle b_1 \rangle_\lambda)$ is 0 while the other is ± 2. However, if the variable λ is not constant for all runs of the experiment, then we can marginalize the absolute value $S_\lambda \leq |\langle b_0 \rangle_\lambda + \langle b_1 \rangle_\lambda| + |\langle b_0 \rangle_\lambda - \langle b_1 \rangle_\lambda|$. We can

[24] Bell's inequality is a special case of CHSH that holds when one has perfect anticorrelation of outcomes in one measurement run.

assume now assume, without loss of generality, that $\langle b_0 \rangle_\lambda \geq \langle b_1 \rangle_\lambda \geq 0$ which yields $S_\lambda = 2 \langle b_0 \rangle_\lambda \leq 2$, and thus $S = \int \mathrm{d}\lambda q(\lambda) S_\lambda \leq 2$.

In conclusion, if quantum mechanics is consistent with local hidden variable theory, inequality (2.55) always holds. We will now show with a counterexample that this does not hold.

Consider Alice and Bob sharing a singlet state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Suppose that each has access to measurement choices x and y respectively. Let each measurement choice correspond to a vector $\vec{x}$ or $\vec{y}$, such that corresponding to measurements on fist or second qubit are associated to $\vec{x} \cdot \vec{\sigma}$ and $\vec{y} \cdot \vec{\sigma}$ respectively. We then have the expectations

$$\langle a_x b_y \rangle = -\vec{x} \cdot \vec{y} \tag{2.59}$$

Let the two settings $x \in \{0, 1\}$ correspond to measurements in the orthogonal directions $\hat{e}_1$ and $\hat{e}_2$, whereas $y \in \{0, 1\}$ to measurements in the directions $-(\hat{e}_1 + \hat{e}_2)/\sqrt{2}$ and $(-\hat{e}_1 + \hat{e}_2)/\sqrt{2}$. By plugging in the direction choices, we get $\langle a_0 b_0 \rangle = \langle a_0 b_1 \rangle = \langle a_1 b_0 \rangle = 1/\sqrt{2}$ and $\langle a_1 b_1 \rangle = -1/\sqrt{2}$. Therefore, in contradiction with (2.55) we get,

$$S = 2\sqrt{2} > 2 \tag{2.60}$$

which proves the non-local character of quantum theory.

In conclusion, the result of Bell's experiment was to show that the predictions of quantum theory, in general, do not admit a decomposition of the form (2.52). This implies that *the established correlations are stronger than the ones which can be attributed to a classical common cause*, which might seem at odds with relativistic causality. Nevertheless, the no-signalling principle is retained by the fact that probability distributions satisfy the following constraints:

- Alice and Bob spacelike separated

$$\begin{aligned} p(a \mid x) &= \sum_b p(ab \mid xy) = \sum_b p\left(ab \mid xy'\right), \\ p(b \mid y) &= \sum_a p(ab \mid xy) = \sum_a p\left(ab \mid x'y\right), \end{aligned} \tag{2.61}$$

Namely, the local marginal probabilities of Alice $p(a|x)$, representing her local

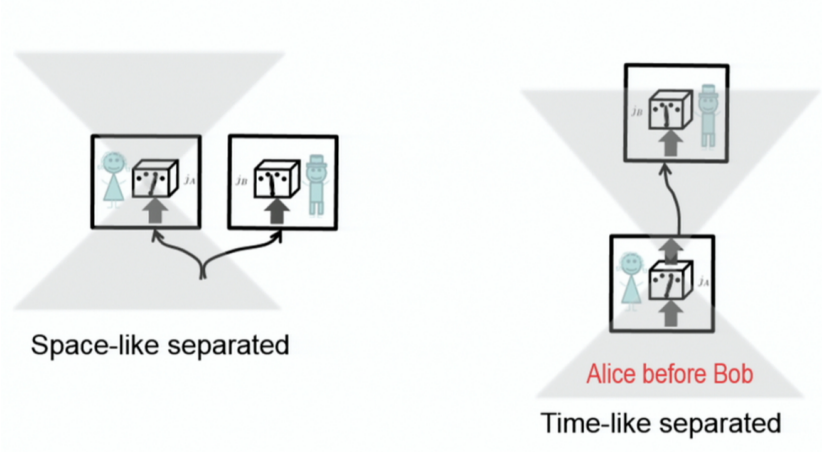


Figure 2.8: Local laboratories embedded in spacetime structure. Causal relations coming from the spacetime structure will induce different constraints on the joint probability distributions. [11]

statistics, are independent of Bob's measurement setting y, and so Bob cannot signal any message to Alice (and converse). Quantum mechanical distributions satisfying (2.61) are called *no-signalling correlations*.

On the other hand, if we now assume that Alice is at a timelike separation from Bob, we can define one-directional signalling in terms of relations:

- Alice and Bob timelike separated

$$\begin{aligned} p(a \mid x) &= \sum_b p(a, b \mid x, y) \\ p(b \mid y, x) &= \sum_a p(a, b \mid x, y) \end{aligned} \tag{2.62}$$

where we have assumed that Alice signals to Bob, as in the right of Figure(2.8). This defines the arrow of causation from Alice to Bob. In quantum mechanics, space-like correlations generally arise from local measurements on joint quantum states, while time-like correlations are established via quantum channels.

The models we will consider are set up within operational framework. Namely, physical situations will be modeled by parties interacting with physical systems, establishing correlations. Physical quantities will be understand as random variables. We can either define constraints on correlations out of knowing their spacetime relations,

Definition 6 (Space-time past, space-time future)**.** [6] A random variable A is in the

space-time past of the random variable B, iff a signal at the speed of light or slower could travel from A to B. We denote this relation with $A \preceq_{\mathrm{ST}} B$.

or we can obtain causal order and spacetime relations from the correlations defined on the set of physical variables via causal inference (see [67]). For example, we will say that A has a causal influence on B if conditional probability $p(B|A)$ for B observably changes under the free variation of A [12]. As in ([6]), one can even take the definition of *freeness* of a variable to be more fundamental then the causal relations, enabling us to derive causal relations from correlations and postulated freeness,

Definition 7 (Causal past, causal future, cause, and effect.)**.** [6] A random variable A is said to be in the causal past of the random variable B, iff A and B are correlated and A is free. This relation is expressed by $A \preceq B$, where $\preceq$ denotes partial ordering [25] The random variable A is called a cause and B is called an effect.

Let us also note that one should distinguish entanglement as a resource from nonlocality [38]. Namely, not all entangled states are such that violate Bell's inequality [86]. Let us we first define the notion of *separability*. Suppose we have parties $A, B, C, \ldots$. A state, shared by these parties, is said to be separable if it can be written in the form

$$\rho_{ABC\ldots} = \sum_n p_n \rho_A^n \otimes \rho_B^n \otimes \rho_C^i \otimes \ldots \tag{2.63}$$

where p_n corresponds to the probability of the n-th outcome within given probability distribution, $p_n > 0$ and $\sum_n p_n = 1$. Such decomposition reflects the fact that a separable state can be prepared by local quantum operations (LO) and classical communication (CC) between parties [62]. Importantly, converse also holds and thus a *quantum state may be generated perfectly using LOCC iff it is separable*. On the other hand, the state is called entangled if it cannot be written in the form (2.63). Entanglement is then operationally defined as a resource that allows parties to overcome the LOCC constraint solving certain multipartite tasks (*games*).

We can choose a different set of restrictions on possible operations, namely the local operations and shared randomness (*LOSR*). In *LOSR* parties are forbidden communication during the game but are allowed to share in advance common classical

[25] A relation on a nonempty set is called a partial ordering or a partial-order relation if it is reflexive, antisymmetric, and transitive.

random variable. As in *LOCC*, using *LOSR* only separable states can be created 'ex nihilo' but now the resource to overcome *LOSR* is nonlocality.

The states violating CHSH inequality are (Bell) nonlocal (those that can overcome LOSR constraint). We will understand this as a crucial feature of quantum correlations. CHSH inequality defines thus the boundary between the region of *local correlations* $\mathcal{L}$ and *quantum correlations* $\mathcal{Q}$, both satisfying no-signalling constraints (2.61). As it turns out [70], there exists even bigger set of correlations satisfying no-signaling constraints, as is shown in Figure(2.10). We will refer to these as simply *no-signalling correlations* $\mathcal{NS}$, where it holds $\mathcal{L} \subset \mathcal{Q} \subset \mathcal{NS}$. All those sets are closed, bounded, and convex [26]. Local $\mathcal{L}$ and no-signaling $\mathcal{NS}$ correlations form convex polytope[27] whose facets correspond to the (Bell's) inequalities.

Namley, let us first introduce terminology (based on [85]) where we will refer to the set $\mathbf{p} = \{p(ab|xy)\}$ of all probabilities as a *behavior*. The behaviour can be understood as as the correlations characterizing black box shared by Alice and Bob, i.e. as a point in the probability space $\mathbf{p} \in \mathcal{P} \subset \mathbf{R}^N$. Here the dimension N will be defined by the dimension of possible inputs and outputs on both sides. I.e if $x, y = \{1, ..., n\}$ and $a, b = \{1, ..., l\}$, then $N = l^2n^2$. Due to the normalization constraints $\mathcal{P}$ is a subspace of dimension $\dim \mathcal{P} = (l^2 - 1)n^2$.

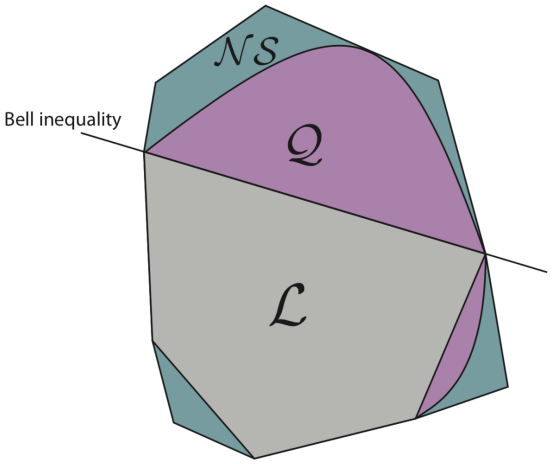


Figure 2.9: Sketch of the no-signaling $\mathcal{NS}$, quantum $\mathcal{Q}$ and local $\mathcal{L}$ sets. $\mathcal{NS}$ and $\mathcal{L}$ are can be polytopes while the set $\mathcal{Q}$ is convex, but not a polytope. The hyperplane representing a boundary between $\mathcal{L}$ and $\mathcal{Q}$ set correspond to Bell inequalities.[14]

Now, for each $\mathbf{p} \in \mathbf{R}^t$ that does not belong to one of the sets, there exists a hyperplane that separates this $\mathbf{p}$ from the corresponding set [28]. For example, if $\mathbf{p} \notin \mathcal{K}$,

[26] The sets of correlations are defined to be closed, bounded, and convex, such that if $\mathbf{p}_1$ and $\mathbf{p}_2$ belong to one of these sets, then the mixture $\mu\mathbf{p}_1 + (1 - \mu)\mathbf{p}_2$ with $0 \leq \mu \leq 1$ also belongs.

[27] Convex polytope has the structure of a simplex; an n-simplex being defined as a convex polytope generated by $n + 1$ vertices that are not confined to any $(n - 1)$-dimensional subspace.

[28] this can be shown by the hyperplane separation theorem

where $\mathcal{K} = \mathcal{L}, \mathcal{Q}$ or $\mathcal{NS}$, there exists an inequality of the form

$$\mathbf{s} \cdot \mathbf{p} = \sum_{abxy} s_{xy}^{ab} p(ab \mid xy) \leq S_k \tag{2.64}$$

satisfied by all $\mathbf{p} \in \mathcal{K}$ but violated by all $\mathbf{p} \notin \mathcal{K} : \mathbf{s} \cdot \mathbf{p} > S_k$, where s is arbitrary $\mathbf{p} \in \mathbf{R}^N$.

In the following, we will see that somewhat similar considerations hold for the case of (non)causal and causally (non)separable correlations, as for the considered Bell (non)local and (non)separable (entangled) states.

2.2.1 Quantum correlations with no causal order

As mentioned in the introduction, standard quantum mechanics is a background-dependent theory. As such it a priori assumes some fixed causal configuration of laboratories, forcing an ordering with respect to some global notion of time. Nonetheless, even in a fixed background structure, causal order itself can be a random variable: there might be a situation where Alice exists before Bob with a probability of $0 \leq q \leq 1$ and Bob exists before Alice with a probability of $1-q$. This will then be represented as probabilistic mixture of two possible orderings $p(a, b \mid x, y) = \lambda p^{\mathrm{A} \preceq \mathrm{B}}(a, b \mid x, y) + (1 - \lambda) p^{\mathrm{B} \preceq \mathrm{A}}(a, b \mid x, y)$. One can also formulate a the notion of causality which doesn't pertain to a "fixed background" [63], [47], by letting the order of events depend on the operations performed at the locations of these events. Namely, dependent on the choice of its measurement setting, an operation at A could influence the order in which B and C, laying in the future of A, occur. This makes causal order a random function of random events rather than the ordering of underlying spacetime locations in which events happen.

Such considerations assume a set of agents, embedded in the *environment*[29], performing a local experiments in respective local laboratories. A local argent in a local laboratory is what is called a *party*, whereas corresponding experiment is what defines the *event*. Every party is isolated from all the other parties and interacts with the environment only once. In each run of the experiment, agent receives a physical

[29]The environment is simply a channel that takes the outputs of the parties and produces the inputs to the parties.

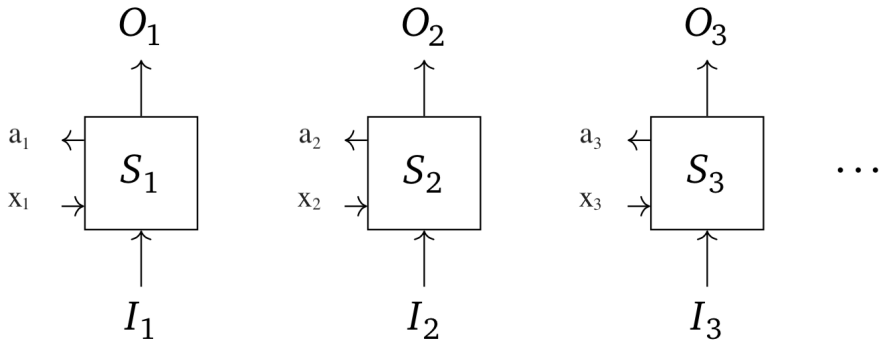


Figure 2.10: A party S_j has a free variable x_j and a channel C_j that transforms the inputs (received from the environment) I_j, to the outputs (sent back to the environment) a_j and O_j. Parties are assumed to be isolated such that every party can perform operations only on their random variables.[6]

system from the environment, chooses a measurement setting, performs a measurement, obtains an outcome and lets the transformed system out. As one may notice, this already presupposed a notion of time on the scale of each party. Nevertheless, globally, there is no notion of time or fix causal ordering over the set of parties. Causal relations, now treated as random and dynamical, are then implemented in the following way:

Definition 8 (Party and causal relation between parties.)**.** [6] A party $S_j = (x_j, a_j, C_j)$ is a triple that consists of a free random variable x_j[30], a non-free random variable a_j, and a local operation C_j. The variable x_j is called input of S_j, and the variable a_j is called output of S_j. For two parties S_j and S_k, we say that S_j is in the causal past of S_k iff $x_j \preceq a_k$. This relation is expressed by $S_j \preceq S_k$, where $\preceq$ is defined in Definition (7).

In other words, causality between parties is defined such that the choice of a setting in the local experiment cannot affect events occurring in the past or simultaneously, nor the causal configuration of these events.

Causal correlations are those that can be embedded in the fixed background spacetime. As already mentioned, in a definite causal structure it may still be the case that the causal relations between events are not known with certainty. Such correlations, for the case of two parties, will be defined as:

[30]A (free) random variable can also be composed of multiple (free) random variables, e.g., $x_j = (x'_j, x''_j, x'''_j)$.

Definition 9 (Biparty causal correlations)**.** [6] Let S_1 and S_2 be two parties. The probability distribution $p(a_1, a_2 \mid x_1, x_2)$ is called causal if and only if it can be written as

$$\begin{aligned} p(a_1, a_2 \mid x_1, x_2) &= qp^{S_1 \preceq S_2}(a_1, a_2 \mid x_1, x_2) + (1-q)p^{S_2 \preceq S_1}(a_1, a_2 \mid x_1, x_2) \\ &= qp(a_1 \mid x_1)p(a_2 \mid a_1, x_1, x_2) + (1-q)p(a_1 \mid a_2, x_1, x_2)p(a_2 \mid x_2) \end{aligned} \tag{2.65}$$

where q is a probability.

Now the question was posed: do more general causal relations exist, such that they would extend to the new set of correlations similarly to the way quantum correlations, violating Bell's inequalities, extend beyond the set of a local polytope? And indeed, as was shown in [63], the possibility of causal correlations not compatible with definite underlying causal ordering is exists. Then, in similar manner to nonlocality considerations, for a fixed number of settings and outcomes, causal correlations form a causal polytope whose facets define *causal inequalities* [64] [10] [3]. If probability distribution violates causal inequalities, it cannot be decomposed in terms of relation (2.65), i.e. $p(a, b \mid x, y) \neq \lambda p^{\mathrm{A} \preceq \mathrm{B}}(a, b \mid x, y) + (1-\lambda)p^{\mathrm{B} \preceq \mathrm{A}}(a, b \mid x, y)$ and it lies outside the causal polytope.

Let us consider a causal inequality in the example of a communication task between two parties. Consider Alice S_1 and Bob S_2, each receiving a system in their respective laboratory. After a given party receives the system, by some means they generate random variables x (referring to Alice's laboratory) and y (referring to Bob's). Bob will additionally generate another random bit y', that determines whether he should guess the bit of Alice, or she should guess his bit. We will label produced guess for Alice and Bob denoted by a and b respectively. The task of their game is to maximize the probability of success. Let us denote Alice $S_1 = \{x_1 = x, a_1 = a\}$ and Bob $S_2 = \{x_2 = (y, y'), a_2 = b\}$. We consider a situation in which Bob is in the past of Alice, $B \preceq A$. Bob is given additional bit y', such that if $y' = 0$, Bob will send his bit y to Alice, enabling her to guess now his bit *perfectly* $p(a = y \mid y' = 0) = 1$, while if $y' = 1$, he will have to guess the bit x *at random* $p(b = x \mid y' = 1) = 1/2$, yielding

$$p^{\mathrm{succ}} := \frac{1}{2}\left(p(a = y \mid y' = 0) + p(b = x \mid y' = 1)\right) \leq \frac{3}{4} \tag{2.66}$$

As it turns out, any ordering in $S_1 \preceq_{\mathrm{ST}} S_2$, or $S_1 \succeq_{\mathrm{ST}} S_2$, or probabilistic mixture of these, does not outperform a success probability of $p^{\mathrm{succ}} = 3/4$. For causal distri-

butions, thus the highest value for success probability is $3/4$, reflecting the fact that at best a party can send the value of one's free variable to the other party, enabling the other to guess perfectly, while oneself making a random guess.

It is important to note that the causal inequalities are *theory-independent* constraints: they are formulated independently of the physics assumed to be valid in each local laboratory. To get the classification of *quantum* correlations that allows indefinite causal structures, we will now open 'black boxes' defining parties, and assume the validity of quantum mechanics on the level of each local laboratory, leading us to the framework of *process matrices*. The process matrix framework represents a general framework for studying correlations between local experiments without the assumption of a predefined causal order. Similarly to the theory-independent framework, we can now ask whether all situations are encompassed with the decomposition of the form (2.65), where now we are assuming that the maps performed on systems inside laboratories are quantum. The decomposition can now be stated in terms of *processes*, separability of which will define a class of so-called *causally separable processes*,

Definition 10 (Biparty causally separable process.)**.** Let us define a *process* $\mathcal{W}^{AB}$ for a biparty set of local experiments $S = \{A, B\}$ as the collection of probabilities $\mathcal{W}^{AB} = p(\mathcal{M}_1, \mathcal{M}_2)$ obtained through a quantum operations $\mathcal{M}_1, \mathcal{M}_2$ performed within each local laboratory. Causally separable processes are those which can be written as convex mixtures of ordered processes:

$$\mathcal{W}^{AB} = q\mathcal{W}^{A\preceq B} + (1-q)\mathcal{W}^{B\preceq A} \tag{2.67}$$

where q is a probability.

Correlations stemming from the processes that are *causally non-separable*, are referred to as *quantum correlations with indefinite causal order*, yet they may still be realized in a global causal structure. Namely, the difference between causally separable and causal processes is the similar as between a separable (non-entangled) quantum state and Bell local (one that does not violate Bell's inequalities) state: a separable quantum state is Bell local, but the reverse is known not always to be true (as we discussed above, there exist entangled states which don't violate Bell's in-

equalities). Physically realized causally non-separable processes are processes that have some system (quantum or classical) as control over the order, making the ordering of the events indefinite. This can still be realized in the background causal structure, as was experimentally realized and proved via so-called causal witness [75] [40]. On the other hand, a process that would violate causal inequality cannot be realized in definite background ordering, so it may come as no surprise that the physical realization of such processes is not yet known. [31].

2.2.2 Quantum switch

One of the examples of causally non-separable processes is so called *quantum switch*, first discussed in the context of quantum computation [22]. Causally non-separable processes can be understood as overcoming the uncertainty in the order of events that could be attributed to classical randomness, introducing the idea of causal relations being subjected to quantum coherences. As we already mentioned, one way of implementing such quantum coherence is by introducing an extra variable, serving as a quantum control over the circuit wires, and thus over the order between gates. We can thus have a linear superposition of orders $A \preceq B$ and $B \preceq A$, as seen in the Figure(2.11). This may come as natural if one thinks, via *Choi-Jamiołkowski (CJ) isomorphism*, of quantum operations as quantum systems, subjected to quantum superposition. The maps between these systems (representing quantum operations) will be 'higher order' maps, so-called *supermaps*. The framework of quantum supermaps enables us to implement superpositions of channels and encompass in a precise way the indefinite causal order.

Let us consider a party S_j, having free (classical) random variable x_j and non-free (classical) random variable a_j, receiving a quantum system from the environment I_j, performing a local operation C_j, and returning quantum transformed quantum system O_j to the environment. This local operation C_j can be considered as a *quantum operation* from I_j to O_j. We often describe quantum operations in the diagrammatic language (the language of quantum circuits), as shown in Figure (2.12).

[31] but might be expected in the context of quantum gravity

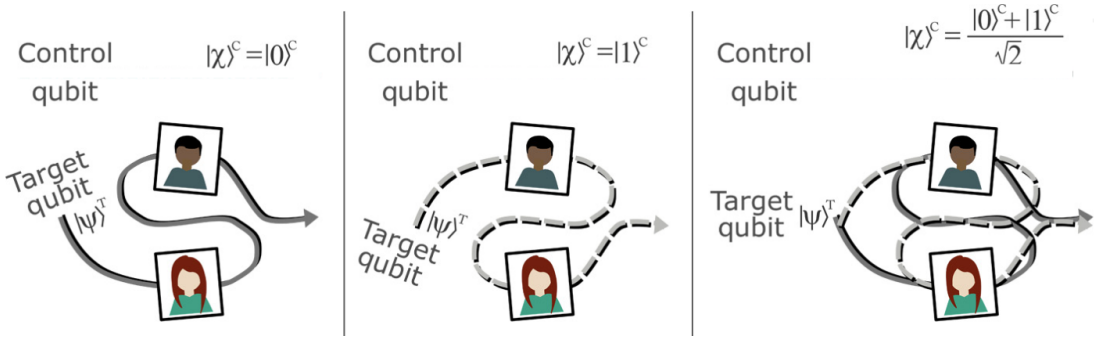


Figure 2.11: The scheme of a quantum switch. The state of the control qubit determines the order in which two parties act on a target qubit depends on the state of a control qubit. Preparing the control qubit in superposition induces controlled superposition of configurations, Alice $\prec$ Bob and Bob $\prec$ Alice.[75]

Transformations on a quantum systems are represented with *gates* (or boxes), out of which *wires*, representing systems, are coming in and out. The quantum state of the system then evolves through a sequence of quantum gates, ordered in time, from left to right. Gates representing *quantum channels* are transformations of single systems. It may be a unitary gate or in general, some noisy quantum channel. A box with multiple wires describes an interaction between the corresponding systems.

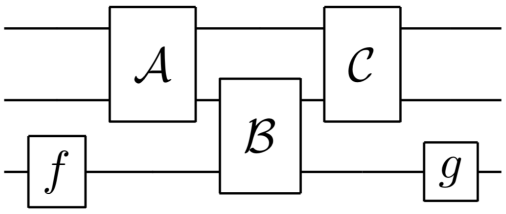


Figure 2.12: Systems in a quantum circuit are represented as wires and operations as boxes. The state of the system evolves through a sequence of quantum gates. The time is assumed to flow from left to right. [22]

Mathematically, we describe quantum channels via superoperators. We will refer to superoperators and quantum channels as the most general evolution of the density matrix preserving the normalization (i.e. trace) [71][32]:

$$\mathcal{C} : \hat{\rho} \mapsto \hat{\rho}' = \mathcal{C}(\hat{\rho}) \tag{2.68}$$

under the restriction that the map:

- *linear*:

$$\forall \hat{\rho}_A, \hat{\rho}_A \in \mathcal{B}(\mathcal{H}), \alpha, \beta \in \mathbf{C} : \quad \mathcal{C}(\alpha\hat{\rho}_A + \beta\hat{\rho}_B) = \alpha\mathcal{C}(\hat{\rho}_A) + \beta\mathcal{C}(\hat{\rho}_B) \tag{2.69}$$

[32]whereas quantum operation and the quantum process are more generally restricted to be trace non-increasing

where $\mathcal{B}(\mathcal{H})$ denotes bounded linear operators on Hilbert space

- $\mathcal{C}$ preserves hermiticity: $\hat{\rho}'$ hermitian if $\hat{\rho}$ is
- $\mathcal{C}$ is complete positive:
  - *positive*: $\hat{\rho}' \geq 0 \Rightarrow \hat{\rho} \geq 0$ [33]
  - *complete positive*: $\mathcal{C}_A$ is completely positive on $\mathcal{H}_A$ if for all possible extensions $\mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_B$, map $\mathcal{C}_A \otimes I_B$ is positive
- $\mathcal{C}$ is trace-preserving: $Tr\hat{\rho}' = \mathbf{1}$ if $Tr\hat{\rho} = \mathbf{1}$.

We will use a representation of this map in terms of Kraus operators:

$$\mathcal{C}(\hat{\rho}) = \sum_i M_i \hat{\rho} M_i^\dagger \quad \text{with} \quad \sum_i M_i^\dagger M_i = I \tag{2.70}$$

The simplest example is then unitary channel, for which we have: $\mathcal{U}(\rho) = U\rho U^\dagger$ with $U^\dagger U = UU^\dagger = I$.

Such a completely positive, trace preserving map ($CPTP$ map) can be decomposed into a collection of completely-positive trace non-increasing (CP) maps, which define a most general operation on a system. A CP map can be understood as a generalization of a measurement, where the probability that a particular map is applied equals the trace of the resulting state. In most general terms, a quantum map is defined as a map that satisfies two axioms:

Axiom 1. A Quantum map must map from the set of quantum states back to itself.

i.e. it maps quantum states into quantum states, and

Axiom 2. All quantum maps must be convex-linear on the set of quantum states.

Namely, if we have some $\hat{\rho} = \sum_{i \in I} p_i \hat{\rho}_i$ representing the ensemble of mixed states $\{(\hat{\rho}_i, p_i) \mid i \in I\}$, for a given transformation $\hat{\rho} \mapsto \mathcal{C}(\hat{\rho})$ then we expect to have $\{(\mathcal{C}(\hat{\rho}_i), p_i) \mid i \in I\}$, representing new ensemble where all the components of the initial state evolved according to the given transformation.

[33] where $\hat{\rho}' \geq 0$ means $\langle\psi|\hat{\rho}|\psi\rangle \geq 0$ for $\forall\psi$

To introduce the concept of supermap, first, we need to introduce the idea of Choi-Jamiołkowski isomorphism. Establishing the isomorphism between CP maps and linear operators on the tensor-product space, Choi-Jamiołkowski isomorphism enables us to view quantum maps in terms of quantum states representations, [20]

Definition 11 (Choi-Jamiołkowski isomorphism)**.** Given some CP map, from a set of linear maps $\mathcal{L}$ over some Hilbert space,

$$\mathcal{M} : \mathcal{L}\left(\mathcal{H}_A\right) \to \mathcal{L}\left(\mathcal{H}_B\right), \tag{2.71}$$

there exists a map $\mathcal{C}$, establishing one to one correspondence between linear maps $\mathcal{M} \in \mathcal{L}(\mathcal{L}\left(\mathcal{H}_A\right), \mathcal{L}\left(\mathcal{H}_B\right))$ and linear operators on $\mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ as follows,

$$M = \mathcal{C}(\mathcal{M}) = \left(\mathcal{I}_{\mathcal{H}_A} \otimes \mathcal{M}\right) |\Phi\rangle \langle\Phi| \in \mathcal{L}\left(\mathcal{H}_A \otimes \mathcal{H}_B\right) \tag{2.72}$$

Here $\mathcal{I}_{\mathcal{H}_A}$ denotes the identity map on $\mathcal{H}_A$ and $|\Phi\rangle$ is defined as the unnormalized maximally entangled state:

$$|\Phi\rangle = \sum_i |i\rangle_A |i\rangle_A \in \mathcal{H}_A \otimes \mathcal{H}_A \tag{2.73}$$

Such map $\mathcal{C}$

$$\mathcal{C} : \mathcal{M} \in \mathcal{L}(\mathcal{L}\left(\mathcal{H}_A\right), \mathcal{L}\left(\mathcal{H}_B\right)) \to M \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \tag{2.74}$$

is called *Choi-Jamiołkowski isomorphism*.

Given some Choi map $M \in \mathcal{L}\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)$, the corresponding CP map is

$$\begin{aligned} \mathcal{M} : \mathcal{L}\left(\mathcal{H}_A\right) &\to \mathcal{L}\left(\mathcal{H}_B\right) \\ \rho_A &\mapsto \mathrm{Tr}_A\left(\left(\rho_A^T \otimes \mathbf{1}_B\right) M\right) \end{aligned} \tag{2.75}$$

where the superscript T denotes the transposition. If the Choi map represents a $CPTP$ map, we have

$$\mathrm{Tr}_B\, M = \mathbf{1}_A \tag{2.76}$$

if M is a CP map, it holds

$$\mathrm{Tr}_B\, M < \mathbf{1} \tag{2.77}$$

The probability that some CP map M_k is applied to an input state ρ_A is

$$p_k = \mathrm{Tr}\left(\left(\rho_A^T \otimes \mathbf{1}_B\right) M_k\right) \tag{2.78}$$

If we denote a collection of CP maps by $\{M_k\}_{\mathcal{K}}$, then from the normalisation condition we can see that a $CPTP$ map M can be written as the sum of CP maps: $\{M_k\}_{\mathcal{K}}$:

$$1 = \sum_k p_k = \sum_k \mathrm{Tr}\left(\left(\rho_A^T \otimes \mathbf{1}_B\right) M_k\right) = \mathrm{Tr}\left(\left(\rho_A^T \otimes \mathbf{1}_B\right) M\right) \tag{2.79}$$

Quantum operations (CP) thus can be viewed as positive linear matrices on a tensor product state via CJ isomorphism.

A quantum switch is defined as a special type of map between two quantum operations, called a *quantum switch supermap*. Namely, since a quantum operation can be understood as a generalized notion of a quantum state, we can define a so-called 'higher order map' called a *supermap* defined as a transformation of quantum maps, such as channels, namely

$$\mathcal{S} : \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}_{\text{in}}\right), \mathcal{L}\left(\mathcal{H}_{\text{out}}\right)\right) \to \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}_{\text{in}'}\right), \mathcal{L}\left(\mathcal{H}_{\text{out}'}\right)\right) \tag{2.80}$$

for some Hilbert spaces $\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}, \mathcal{H}_{\text{in}'}$ and $\mathcal{H}_{\text{out}'}$, where $\mathcal{L}(\mathcal{H})$ is space of linear operators on the Hilbert space.

Since quantum supermaps must represent physical transformations of quantum maps, we impose:

Axiom 3. All quantum supermaps map quantum maps to qunatum maps.

Axiom 4. All quantum supermaps must be convex-linear on the set of quantum maps.

As in the case of quantum maps, if the input quantum channel $\mathcal{C}$ describes a statistical ensemble of quantum channels $\{(\mathcal{C}_i, p_i) \mid i \in I\}$, then the output quantum channel $\mathcal{S}(\mathcal{C})$ must describe the ensemble $\{(\mathcal{S}\left(\mathcal{C}_i\right), p_i) \mid i \in I\}$. Thus, the axioms are in complete analogy with axioms for quantum maps.

We then define a quantum switch via the supermap $\mathcal{S}$ mapping

$$\mathcal{S}(\mathcal{A}\otimes\mathcal{B})(\rho_T) := \mathcal{B}\mathcal{A}\,\langle 0|_Q\,\rho_T|0\rangle_Q + \mathcal{A}\mathcal{B}\langle 1|_Q\rho_T|1\rangle_Q \tag{2.81}$$

or in Kraus representation

$$\mathcal{S}\left(\mathcal{A},\mathcal{B}\right)\left(\rho_T\right) = \sum_{i,j} S_{ij}\rho_T S_{ij}^{\dagger} \tag{2.82}$$

where

$$S_{ij} := A_iB_j\otimes|0\rangle\langle 0|_Q + B_jA_i\otimes|1\rangle\langle 1|_Q \tag{2.83}$$

where $\mathcal{A}(\rho_T) = \sum_i A_i\rho_T A_i^{\dagger}$ and $\mathcal{B}(\rho_T) = \sum_j B_j\rho_T B_j^{\dagger}$

In other words, a quantum switch consists of two quantum systems: target ρ_T and control, ρ_C and two CP maps $\mathcal{A},\mathcal{B}$ acting on a target. The control system serves as a control over the order of maps on a target, i.e. if $|\psi\rangle_C = |0\rangle$, the resulting operation is $\mathcal{A}\circ\mathcal{B}$, and for $|\psi\rangle_C = |1\rangle$ we have $\mathcal{B}\circ\mathcal{A}$. Thus the coherence in the control system translates to coherence in the order of operations. For example, a control qubit in superposition, $|\psi\rangle_C = \left(|0\rangle_C + |1\rangle_C\right)/\sqrt{2}$ results in superposition of two different orders.

We can also have a quantum n-switch, for the n-th permutation of operations [72]

$$S_n|x\rangle_c|\Psi\rangle_t = |x\rangle_c\Pi_x|\Psi\rangle_t \tag{2.84}$$

here $|\Psi\rangle_t$ denotes the target, which can be arbitrary dimensional; $|x\rangle_c$ denotes the control, which now needs to be $n!$ dimensional; $\Pi_x := U_{\sigma_x(N-1)}\cdots U_{\sigma_x(1)}U_{\sigma_x(0)}$, denotes the product of the n unitary gates from a some given set $U := \{U_A, U_B, \ldots\}$.

2.2.3 Quantum time flip

The main motivation of our work was to compare the indefiniteness appearing on the level of Feynman propagator, as sketched in (2.13), with the indefiniteness appearing on the level of quantum switch.

As will be shown in Chapter (2.4), Feynman propagator is defined as the sum of the processes depicted on the left and right spacetime sketch, where the comparison with quantum switch is depicted on the rightmost picture. This kind of situation

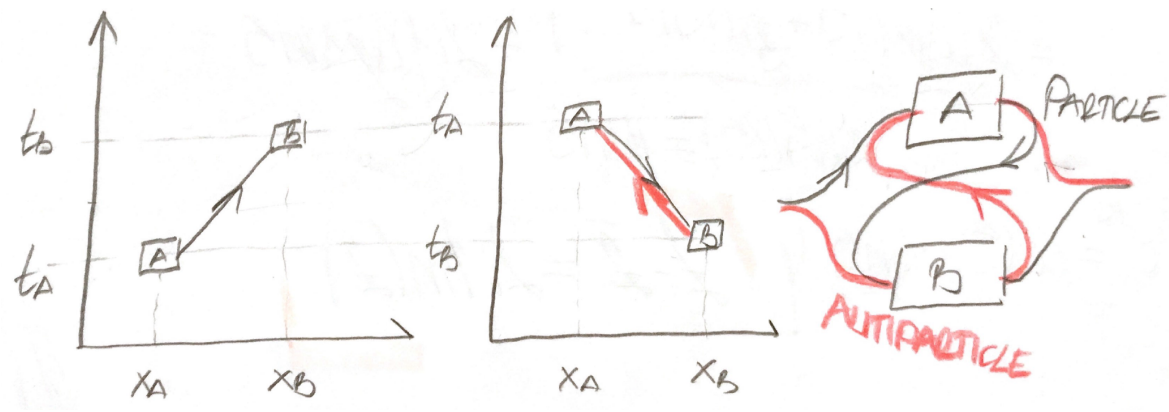


Figure 2.13: The Feynman propagator as a superposition of diagrams corresponding to particle and antiparticle degrees of freedom, compared with the structure of a quantum switch.

comes about due to the second order partial derivative in a timelike component in the differential equation describing the field of interest, splitting the energies into a positive and negative spectrum. The grey line in the figure represents the same kind of entity, where left diagram corresponds to positive part of the spectrum and right diagram to negative part. As it will be explained bellow, one can either view the situation within the framework admitting only forward in time propagation, with the necessity of implementing C transformation as denoted on the sketch with red line. This now changes the description of only particle degrees of freedom to particles and antiparticles, differing in charge. Alternatively, we can view the situation within the scope of only particle degrees of freedom, in the manner usually depicted on Feynman diagrams. This then admits also backwards in time motion for the negative energy solutions. But quantum switch description would not be compatible with such backwards in time description! Namely, from Figure (2.13) we see that the time arrow is always forward in the quantum switch, even though the ordering is indefinite. Nevertheless, recently processes with indefinite time arrow were also discussed [21] [39], precise implementation stated via so called *quantum time flip*.

Within the framework of supermaps, we can put quantum operations into superposition. In the context of superposition of time arrows of operations, we define a channel $\mathcal{C}$ as a superposition of channels $\mathcal{A}$ and $\mathcal{B}$, with Kraus representation $(A_i)_{i=1}^k$ and $(B_i)_{i=1}^k$;

$$C_i = A_i \otimes |0\rangle\langle 0| + B_i \otimes |1\rangle\langle 1| \tag{2.85}$$

following closely discussion in [21]. Imagine now putting in a superposition two channels probing the box representing a quantum process from two different directions. Namely, let us denote a forward process $\mathcal{C}$ with Kraus representation $(C_i)_{i=1}^k$ and the channel oriented in backward direction with $\theta\left(C_i\right)$. The coherently-controlled time reversal is then defined as the process,

$$\mathcal{F} : \rho \mapsto \mathcal{F} = \sum_i F_i \rho F_i^\dagger \tag{2.86}$$

with Kraus representation;

$$F_i := C_i \otimes |0\rangle \langle 0| + \theta\left(C_i\right) \otimes |1\rangle \langle 1| \tag{2.87}$$

Not all processes can admit such a superposition of directionality. Namely, for the process $\mathcal{F}_{\mathrm{C}}$ to be a valid quantum channel we need $\mathcal{C}$ to be *bistochastic*. In general, we will call the processes that can be probed both in forward and backward direction *bidirectional processes*. Here we are assuming some hypothetical backward facing agent whose description will be defined via $\theta : \mathcal{C} \to \theta(\mathcal{C})$, as denoted in the Figure (2.14). We call this *change of description* from the forward agent, to the hypothetical backward agent via the map θ a *time reversal*. This change of description is assumed to satisfy the following axioms:[21]

Axiom 5. Time reversal inverts the order of processes.

$$\Theta\left(\mathcal{C}_1 \circ \mathcal{C}_2\right) = \Theta\left(\mathcal{C}_2\right) \circ \Theta\left(\mathcal{C}_1\right) \tag{2.88}$$

Axiom 6. All reversible processes admit a time reversal, and their time reversal is a reversible process.

Axiom 7. The time reversal of a mixture of channels is the mixture of their time reversals.

$$\Theta\left(p\mathcal{C}_1 + (1-p)\mathcal{C}_2\right) = p\Theta\left(\mathcal{C}_1\right) + (1-p)\Theta\left(\mathcal{C}_2\right) \tag{2.89}$$

Axiom 8. Time reversal maps distinct processes into distinct processes

$$\mathcal{C}_1 \neq \mathcal{C}_2 \Longrightarrow \Theta\left(\mathcal{C}_1\right) \neq \Theta\left(\mathcal{C}_2\right) \tag{2.90}$$

In general case, the set of such processes will be bistochasic,

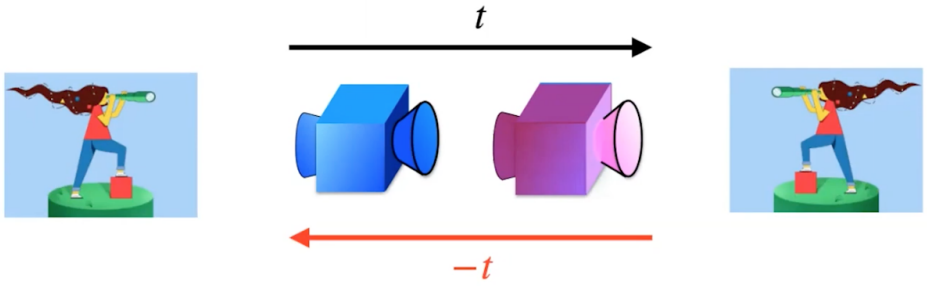


Figure 2.14: Forward and backwards-facing agents probing the boxes in the opposite order

Theorem 2. The largest set of quantum channels admitting a time reversal satisfying aforementioned axioms is the set of bistochastic channels, that is, maps of the form

$$\mathcal{C} : \rho \mapsto \sum_i C_i \rho C_i^\dagger \tag{2.91}$$

with $\sum_i C_i^\dagger C_i = \sum_i C_i C_i^\dagger = I$

while in the case of unitary channels we will have two possibilities,

Theorem 3. Up to unitary equivalence, there are only two possible time reversals satisfying aforementioned axioms: the adjoint $U \mapsto U^\dagger$ and the transpose $U \mapsto U^T$

In other words, if we would take the interpretation of backwards in time motion literally, and if we would lift the notion of a quantum channel to quantum field theory context, it seems, within the formalism worked out so far, that the total channel showing one of the sketches in Figure (2.13) should be bistochastic. In our analogy, the Feynman propagator would be understood as a quantum supermap, mapping two quantum processes into superposition. Nevertheless, in this thesis, we didn't formalize what would the notion of a quantum channel, quantum supermap, or Choi-Jamiołkowski isomorphism translate to in the QFT context, thus technical parallel is not yet clear and modifications to this constraints, in the QFT context, would be expectable.

2.3 Causality in quantum field theory and reason for antiparticles

Quantum field theory is, roughly speaking, quantum mechanics applied to dynamical systems of fields (i.e. systems having infinite degrees of freedom). The main reason

for quantum field theory is to couple symmetries of spacetime, induced by the action of the Poincaré group, with the symmetries of quantum mechanics, represented via unitarity transformations. Therefore, quantum field theory will be defined on a vector space which is a unitary representation of the Poincaré group. To demonstrate the need for theory beyond standard fix-particle quantum mechanics, we consider the notion of causality which forces us to invoke quantum fields and, more concretely, antiparticle solutions.

Let us consider probability amplitude for a particle to propagate from $\mathbf{x}_0$ to $\mathbf{x}$,

$$\mathcal{A} = \left\langle \mathbf{x} \left| e^{-iHt} \right| \mathbf{x}_0 \right\rangle \tag{2.92}$$

in the context of non-relativistic quantum mechanics, for a free particle we have:

$$\begin{aligned} \mathcal{A} &= \langle \mathbf{x} | e^{-i\frac{\mathbf{p}^2 t}{2m}} | \mathbf{x}_0 \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \langle \mathbf{x} | e^{-i\frac{\mathbf{p}^2 t}{2m}} | \mathbf{p} \rangle \langle \mathbf{p} \mid \mathbf{x}_0 \rangle \\ &= \frac{1}{(2\pi)^3} \int d^3p e^{-i\frac{\mathbf{p}^2 t}{2m}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} \\ &= \left(\frac{m}{2\pi i t}\right)^{3/2} e^{im(\mathbf{x}-\mathbf{x}_0)^2/2t} \end{aligned} \tag{2.93}$$

The expression is nonzero for all $\mathbf{x}$ and t. This conflicts with a theory of relativity since this would imply that a particle can propagate between any two points of spacetime, violating causality. The problem cannot be fixed just by invoking the relativistic expression for the energy $E = \sqrt{p^2 + m^2}$.

$$\begin{aligned} \mathcal{A} &= \langle \mathbf{x} | e^{-it\sqrt{\mathbf{p}^2+m^2}} | \mathbf{x}_0 \rangle \\ &= \frac{1}{(2\pi)^3} \int d^3p e^{-it\sqrt{\mathbf{p}^2+m^2}} \cdot e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} \\ &= \frac{1}{2\pi^2 |\mathbf{x}-\mathbf{x}_0|} \int_0^\infty dp p \sin\left(p |\mathbf{x}-\mathbf{x}_0|\right) e^{-it\sqrt{p^2+m^2}} \end{aligned} \tag{2.94}$$

One can evaluate this integral explicitly in terms of Bessel functions. Nevertheless, for the purposes of our discussion we will consider only its asymptotic behaviour $x^2 \gg t^2$ (well outside the light cone). Within the approximation, the integral can now be evaluated using the method of stationary phase (for more details consider [76][53]). It yields:

$$\mathcal{A} \sim e^{-m\sqrt{x^2-t^2}} \tag{2.95}$$

The propagation amplitude decays outside the light cone but is still non-vanishing and thus causality is again violated. Such an attempt to merely consider the expression for energy in its relativistic form was also made while trying to establish a relativistic analogue of the Schrödinger equation:

$$E = \frac{\hat{p}^2}{2m} \quad \rightarrow \quad E^2 = \hat{\mathbf{p}}^2 + m^2 \tag{2.96}$$

where $E = i\partial_t$ and $\hat{\mathbf{p}} = -i\nabla$ gives:

$$i\frac{\partial\psi(t)}{\partial t} = \hat{H}\psi(t) \quad \Rightarrow \quad (\partial^\mu\partial_\mu + m^2)\psi(x) = 0 \tag{2.97}$$

[34] This is known as the Klein-Gordon equation and in the case of single particle relativistic mechanics gives rise to negative probability densities. For this reason, such a single particle extension to relativistic realms is not possible. Nonetheless, the Klein-Gordon equation is an important equation in the QFT context where it describes multi-particle excitations of a scalar quantum field.

Causality in QFT is established by invoking antiparticle solutions, cancelling the 'leaking' of probability amplitude outside the timelike region.

2.3.1 Causality of Klein-Gordon field

The main purpose of quantum fields is to implement the notion of locality [35]. Whereas in nonrelativistic physics, we assume that an observer has access to all possible measurements of a system, within the context of special relativity, an observer can access only its local measurements, reflecting the fact that information can travel with the maximal limit being the speed of light. The locality is then implemented by defining operator-valued fields on which so-called *microcausality condition* will be imposed. Namely, we demand of these field operators (1) not to allow transmission of information faster than the speed of light and (2) be Hermitian operators such that they correspond to quantum mechanical observables. The first condition is implemented

[34] Here $x^\mu = (t, \mathbf{x})$ denotes a 4-vector. If 4-vector is contracted with its dual, or if it appears as the argument in a function, then superscript μ will be dropped. Bold symbols will denote 3-vectors through the text. The metric convention we will be using in the following chapters will be $\eta_{\mu\nu} = (1, -1, -1, -1)$. Four derivative is given by $\partial_\mu = (\partial_0, \nabla)$.

[35] Where here we are talking of locality, which refers to limit on information propagation and not locality in the sense of Bell.

via the microcausality condition,

$$[\hat{O}(x), \hat{O}(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0 \tag{2.98}$$

[36]

implementing a fact that there is no difference in the ordering of measurements $\hat{O}(x)\hat{O}(y)$ or $\hat{O}(y)\hat{O}(x)$ in the spacelike region and the results of spacelike separated measurements cannot affect each other.

Let us now consider an equation (2.97):

$$(\partial^\mu \partial_\mu + m^2)\phi(x) = 0 \tag{2.99}$$

Here we changed the notation from $\psi(x)$ of the equation (2.97), where it denotes quantum mechanical wave function, to distinguish it from the field $\phi(x)$, which will promote to a quantum field. [37]

If additionally, we impose that the equation

$$\Rightarrow p^2 = p_\mu p^\mu = m^2 \tag{2.100}$$

is satisfied, we can write *a* solution to (2.99) as a plane wave,

$$\phi(x) = e^{-\mathrm{i}p\cdot x} = e^{-\mathrm{i}Et + \mathrm{i}\mathbf{p}\cdot\mathrm{x}} \tag{2.101}$$

This condition is often referred to as being on *mass-shell*. Particle is said to be on mass-shell if it satisfies $E^2 = p^2 + m^2$. We can write further

$$\begin{aligned} \Rightarrow \left(\frac{\partial^2}{\partial t^2} - \mathbf{p}^2 + m^2\right) e^{-\mathrm{i}Et + \mathrm{i}\mathbf{p}\cdot\mathrm{x}} &= 0 \\ \ddot{\phi}(x) + \omega_p^2 \phi(x) &= 0 \end{aligned} \tag{2.102}$$

[36] where the microcausality condition for fermions will be given in terms of anticommutator instead

[37] A field is formally just a function assigning a quantity to each point of space(time). Thus, technically, the standard wave function can also be thought of as a scalar (classical) field. Nevertheless, physically quantum fields are related to the construction of a Fock space which structure allows superposition of configurations having different number of particle excitations $\mathcal{F}_\nu(\mathcal{H}) = \bigoplus_{n=0}^{\infty} S_\nu \mathcal{H}^{\otimes n} = \mathbf{C} \oplus \mathcal{H} \oplus (S_\nu(\mathcal{H}\otimes\mathcal{H})) \oplus (S_\nu(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H})) \oplus \cdots$. Excitations of the quantum field are indistinguishable particles, which can be bosonic (spin number integer-valued) or fermionic (spin number half integer-valued) which can be shown to impose (anti)symmetrization $S_\nu = \pm 1$ where $+1$ is symmetrization for bosons and -1 is antisymmetrization for fermions. For more details see [9].

where

$$\omega_p = \pm E = \pm\sqrt{\mathbf{p}^2 + m^2} \tag{2.103}$$

from which we see that for each momentum, we get the equation of harmonic oscillator. Since the relativistic equation for energy (2.103) has a square of energy, we now have both positive and negative solutions in the spectrum. While in quantum mechanics this seemed worrisome inconsistency [38], in quantum field theory, negative energy solutions will play an important role, enabling us to satisfy the condition of microcausality (2.98). What represented a much worse interpretational inconsistency for the Klein-Gordon equation was the presence of negative probabilities: $\rho \sim \pm|E|$. In the context of field theory, however, ρ now represents charge density for a field and as such can be both positive and negative.

For each momentum thus we have two independent solutions: $\mathrm{e}^{-\mathrm{i}\omega_p t+\mathrm{i}\mathbf{p}\cdot\mathbf{x}}$ and $\mathrm{e}^{+\mathrm{i}\omega_p t+\mathrm{i}\mathbf{p}\cdot\mathbf{x}}$. The general solution of (2.99) is then

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3\mathbf{p}}{\sqrt{2\omega_p}} \left(a(\mathbf{p})\mathrm{e}^{-\mathrm{i}(\omega_p t-\mathbf{p}\cdot\mathbf{x})} + b^\dagger(-\mathbf{p})\mathrm{e}^{\mathrm{i}(\omega_p t+\mathbf{p}\cdot\mathbf{x})}\right) \tag{2.104}$$

where $a(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ are complex coefficients. Upon making a change $\mathbf{p} \to -\mathbf{p}$ in the second term, we get

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3\mathbf{p}}{\sqrt{2\omega_p}} \left(a(\mathbf{p})\mathrm{e}^{-\mathrm{i}p\cdot x} + b^\dagger(\mathbf{p})\mathrm{e}^{\mathrm{i}p\cdot x}\right) \tag{2.105}$$

Notice also that if $\phi(x)$ is a real field we have $a(\mathbf{k}) = b(\mathbf{k})$[39].

As a part of well known quantization recipe, in order to quantize field ϕ we promote coefficients in the expansion (2.194) into ladder operators $a(\mathbf{p}), b(\mathbf{p}) \to \hat{a}(\mathbf{p}), \hat{b}(\mathbf{p})$ such that it holds:

$$\begin{aligned} \left[a(\mathbf{k}), a^\dagger(\mathbf{q})\right] &= \delta^{(3)}(\mathbf{k}-\mathbf{q}) \\ [a(\mathbf{k}), a(\mathbf{q})] &= \left[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{q})\right] = 0 \end{aligned} \tag{2.106}$$

[38] In the end, negative energy solutions managed to find their meaning even in the context of relativistic quantum mechanics, in the context of Dirac equation. Even though the Dirac equation ran into the same problem of negative energies, it didn't encounter the problem of negative probability densities.

[39] This case where a particle is its own antiparticle is called Majorana particle.

and similarly for $\hat{b}(\mathbf{p})$. From (2.107) we can also show:

$$\begin{aligned}[\hat{\phi}(\mathbf{x},t),\dot{\hat{\phi}}(\mathbf{y},t)] &= \mathrm{i}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\ [\hat{\phi}(\mathbf{x},t),\hat{\phi}(\mathbf{y},t)] &= [\dot{\hat{\phi}}(\mathbf{x},t),\dot{\hat{\phi}}(\mathbf{y},t)] = 0,\end{aligned} \tag{2.107}$$

Now, the quantum Klein-Gordon field (in the Heisenberg picture)[40] is represented as

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(\hat{a}(\mathbf{p})e^{-ip\cdot x} + \hat{b}^\dagger(\mathbf{p})e^{ip\cdot x}\right) \tag{2.108}$$

and

$$\hat{\phi}^\dagger(y) = \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\left(\hat{a}^\dagger(\mathbf{q})e^{iq\cdot y} + \hat{b}(\mathbf{q})e^{-iq\cdot y}\right) \tag{2.109}$$

We will call the negative part of the spectrum *antiparticle solutions*, for which a detailed account will be given in a later section.

Here $\hat{\phi}^\dagger(x)$ represents the operator which creates a *particle, and annihilates the antiparticle*:

$$\hat{\phi}^\dagger(x)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^{\frac{3}{2}}\left(2E_{\boldsymbol{p}}\right)^{\frac{1}{2}}}|\boldsymbol{p}\rangle \mathrm{e}^{\mathrm{i}p\cdot x} \tag{2.110}$$

We interpret this as particle being created at the point $(x^0, \mathbf{x})$. One can see this as plausible by comparison with the x representation of the nonrelativistic one-particle momentum state. Consider the overlap with one particle momentum state, (in Schrödinger picture)

$$(2\pi)^{\frac{3}{2}}\left(2E_{\boldsymbol{p}}\right)^{\frac{1}{2}}\langle \boldsymbol{q}|\hat{\phi}^\dagger(\boldsymbol{x})|0\rangle = \int \mathrm{d}^3p \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\langle \boldsymbol{q} \mid \boldsymbol{p}\rangle = \int \mathrm{d}^3p \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\delta^{(3)}(\boldsymbol{q}-\boldsymbol{p}) = e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} \tag{2.111}$$

we see that this yields a similar overlap as

$$\langle \boldsymbol{q} \mid \boldsymbol{x}\rangle \sim e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} \tag{2.112}$$

and thus we interpret $\hat{\phi}^\dagger(\boldsymbol{x})|0\rangle \sim| \boldsymbol{x}\rangle$. The expression (2.111) $= \mathrm{e}^{\mathrm{i}q\cdot x}$ tells us how much probability amplitude there is in the q-th momentum mode if we create a scalar particle at spacetime point x.

Now, similarly, we have for $\hat{\phi}(y)$. This operator *creates antiparticles and annihilates particles*,

[40] One can show that $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ are time independent operators

$$\hat{\phi}(y)|0\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^{\frac{3}{2}}\,(2E_{\boldsymbol{p}})^{\frac{1}{2}}} |\boldsymbol{p}\rangle \mathrm{e}^{\mathrm{i}p\cdot y} \tag{2.113}$$

Going back to causality considerations, let us define now the amplitude of propagation. We define it as the overlap between (2.112) and (2.113):

$$\begin{aligned} G(x-y) &\equiv \langle 0|\hat{\phi}(x)\hat{\phi}^{\dagger}(y)|0\rangle \\ &= \left\langle 0 \left| \begin{pmatrix} \text{Particle annihilated} \\ \text{at } (x^0, \boldsymbol{x}) \end{pmatrix} \begin{pmatrix} \text{Particle created} \\ \text{at } (y^0, \boldsymbol{y}) \end{pmatrix} \right| 0 \right\rangle \\ &= \left\langle 0 \left| e^{\mathrm{i}\hat{H}x^0} \hat{\phi}(\boldsymbol{x}) e^{-\mathrm{i}\hat{H}\left(x^0 - y^0\right)} \hat{\phi}^{\dagger}(\boldsymbol{y}) e^{-\mathrm{i}\hat{H}y^0} \right| 0 \right\rangle \end{aligned} \tag{2.114}$$

containing the following steps:

(1) state $|0\rangle$ is evolved to a time y^0 via $e^{-\mathrm{i}\hat{H}y^0}$

(2) then $\hat{\phi}^{\dagger}(\boldsymbol{y})$ adds to the $e^{-\mathrm{i}\hat{H}y^0}|0\rangle$ state a particle at time y^0 at a position $\boldsymbol{y}$.

(3)$e^{-i\hat{H}\left(x^0-y^0\right)}\hat{\phi}^{\dagger}(\boldsymbol{y})e^{-\mathrm{i}\hat{H}y^0}|0\rangle$ is that state evolved to a time x^0.

(4) second disturbance, given by the operator $e^{\mathrm{i}\hat{H}x^0}\hat{\phi}(\boldsymbol{x})$ removes the particle at time x^0

(5) by overlapping with the vacuum state, we finally ask for the probability amplitude that this second disturbance restored the original state $|0\rangle$

For a given field, the propagation amplitude equals to

$$G(x-y) \equiv \langle 0|\hat{\phi}(x)\hat{\phi}^{\dagger}(y)|0\rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip(x-y)} \tag{2.115}$$

Let us now analyze this expression with respect to spacetime intervals. Let us consider first the case of the timelike interval $(x-y)^2 > 0$:

- To make calculations simpler, we will choose a system where $\vec{x} - \vec{y} = 0$ and define $x^0 - y^0 = t$. Transitioning to polar coordinates, we have

$$\begin{aligned} G(x-y) &= \frac{4\pi^3}{2\pi} \int_0^{\infty} dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}t} \\ &= \frac{1}{4\pi^2} \int_m^{\infty} dE \sqrt{E^2 - m^2} e^{-iEt} \\ \Rightarrow &=\sim e^{-imt} = e^{-im(x^0-y^0)} \end{aligned} \tag{2.116}$$

Importantly, notice that $G(x-y)$ is *not symmetric* on the exchange $x \leftrightarrow y$ i.e. $x^0 \leftrightarrow y^0$.

For the spacelike interval $(x-y)^2 < 0$:

- Now let us choose a system where $x^0 - y^0 = 0$ and let us define $\vec{r} = \vec{x} - \vec{y}$. We get:

$$\begin{aligned} G(x-y) &= \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^{\infty} dp \frac{p e^{ipr}}{\sqrt{p^2+m^2}} \\ &= \frac{1}{4\pi^2 r}\int_{m}^{\infty} d\rho \frac{\rho e^{-\rho\tau}}{\sqrt{\rho^2-m^2}} \\ \Rightarrow &= \sim e^{-mr} \end{aligned} \tag{2.117}$$

  $G(x-y)$ is now dependent on $|\vec{r}|$ and thus symmetric on the exchange $x \leftrightarrow y$ i.e. $x^0 \leftrightarrow y^0$.

What is alarming here is that the probability amplitude for the spacelike transition is exponentially decaying but non-vanishing. Nevertheless, our condition of locality referred to the locality of the measurements and the finite speed of information propagation. Thus, as Peskin and Schroeder nicely put it:
'To really discuss causality however we should ask not whether particles can propagate over spacelike intervals but whether a measurement performed at one point can affect a measurement at another point whose separation from the first is spacelike'[76] Let us thus focus on the causality condition of the field $\hat{\phi}(x)$ measurements. Commutator of the field in two different points is given by: For the complex Klein-Gordon field we have three kinds of commutators: $\left[\hat{\phi}(x), \hat{\phi}(y)\right]$, $\left[\hat{\phi}^\dagger(x), \hat{\phi}^\dagger(y)\right]$, $\left[\hat{\phi}^\dagger(x), \hat{\phi}(y)\right]$, where the only commutator which is not by itself vanishing is:

$$\begin{aligned} \left[\hat{\phi}^\dagger(x), \hat{\phi}(y)\right] &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}} \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{q}}}} \\ &\quad \times \left[\left(a^\dagger_{\vec{p}} e^{ip\cdot x} + b_{\vec{p}} e^{-ip\cdot x}\right), \left(a_{\vec{q}} e^{-iq\cdot y} + b^\dagger_{\vec{q}} e^{iq\cdot y}\right)\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) \\ &= G(x-y) - G(y-x). \end{aligned} \tag{2.118}$$

The first contribution is due to only antiparticle degrees of freedom and represents

the propagation of antiparticles from point y to point x. The other contribution is due to only particle degrees of freedom and represents particle propagating from x to y. These two contributions cancel each other out in the spacelike region if to every particle solution there exists an antiparticle solution with the same mass but opposite quantum numbers, propagating in the exactly opposite manner. Thus, we conclude the following:[35]

(1) *The existence of the antiparticle solutions is crucial for preserving causality in QFT.*

(2) *Antiparticle behavior is completely determined by particle behavior.*

2.3.2 Dirac field equation and antiparticles

All the similar considerations we can do for the case of the Dirac field. Here we will also pause to understand how negative energy solutions arise and how they can be reinterpreted as antiparticles. Dirac equation is,

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0 \tag{2.119}$$

[41] Representing via momentum operator $p_{\mu} = i\partial_{\mu}$,

$$\left(\gamma_0 \hat{p}^0 - \gamma \cdot \hat{\boldsymbol{p}} - m\right)\psi(x) = 0 \tag{2.120}$$

where now due to the structure of γ matrices we have

$$\left[\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} - \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} \\ -\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} & 0 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\right]\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0 \tag{2.121}$$

which can be rewritten as

$$\begin{aligned} (E - \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})\,\psi_{\mathrm{L}} &= m\psi_{\mathrm{R}} \\ (E + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})\,\psi_{\mathrm{R}} &= m\psi_{\mathrm{L}} \end{aligned} \tag{2.122}$$

[41] where $\gamma^{\mu} = (\gamma^0, \gamma)$ such that

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$$

In other words, four-component eigenstate $\psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix}$ splits into two two-component states, ψ_{L} and ψ_{R} called *Weyl spinors*. Spinor solutions can be found by taking the two simplest orthogonal choices for ψ_{L} and ψ_{R},

$$\psi_{\mathrm{L,R}}^{(s=1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_{\mathrm{L,R}}^{(s=2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.123}$$

where we introduced s index to refer to the spin degree of freedom. The corresponding components are coupled due to the presence of mass,

$$\psi_{\mathrm{R}} = \frac{\sigma \cdot \mathbf{p}}{E+m}\psi_{\mathrm{L}} = \frac{1}{E+m}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\psi_{\mathrm{L}}, \tag{2.124}$$

$$\psi_{\mathrm{L}} = \frac{\sigma \cdot \mathbf{p}}{E-m}\psi_{\mathrm{R}} = \frac{1}{E-m}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\psi_{\mathrm{R}}, \tag{2.125}$$

In total, we get four solution of the form:

$$\psi_i = u_i(E, \mathbf{p})e^{i(-Et+\mathbf{p}\cdot\mathbf{x})} = u_i(E, \mathbf{p})e^{-ip\cdot x} \tag{2.126}$$

where

$$u_1 = N_1\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, u_2 = N_2\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{\frac{E+m}{Ep_3}} \\ \frac{-p+m}{E+i} \end{pmatrix}, u_3 = N_3\begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \text{ and } u_4 = N_4\begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix} \tag{2.127}$$

We call these four component objects *Dirac spinors* [42]. Upon substituting back into

[42] a bispinor that transforms spinorial under the action of the Lorentz group

the equation we get the equation:

$$(\gamma^\mu p_\mu - m)u(E, \mathbf{p}) = 0 \tag{2.128}$$

To see that two solutions correspond to negative energies, consider a rest frame of a particle $\mathbf{p} = \mathbf{0}$ where

$$\psi = u(E, 0)e^{-iEt}, \tag{2.129}$$

and thus equation reduces to

$$(E\gamma^0 - m)u = 0 \tag{2.130}$$

This can be expressed as an eigenvalue equation for the components of the spinor

$$E \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = m \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}. \tag{2.131}$$

We see that we have two positive energy solutions and two negative energy solutions:

- for $E = +m$ we have

$$u_1(E, 0) = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2(E, 0) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{2.132}$$

- for $E = -m$ we have

$$u_3(E, 0) = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_4(E, 0) = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.133}$$

In conclusion, the spinors u_1 and u_2 are $E > 0$ spinors and u_3 and u_4 are $E < 0$

spinors where E is again given by the relativistic energy relation:

$$E = \pm \left| \sqrt{\mathrm{p}^2 + m^2} \right| \tag{2.134}$$

2.3.3 Physical interpretation of negative energy solutions (Stückelberg and Feynman interpretation)

["Wheeler saying:] Feynman, I know why all electrons have the same charge and the same mass" "Why?" "Because, they are all the same electron!" And, then he explained on the telephone, "suppose that the world lines which we were ordinarily considering before in time and space – instead of only going up in time were a tremendous knot, and then, when we cut through the knot, by the plane corresponding to a fixed time, we would see many, many world lines and that would represent many electrons, except for one thing. If in one section this is an ordinary electron world line, in the section in which it reversed itself and is coming back from the future we have the wrong sign to the proper time – to the proper four velocities – and that's equivalent to changing the sign of the charge, and, therefore, that part of a path would act like a positron." "But, Professor", I said, "there aren't as many positrons as electrons." "Well, maybe they are hidden in the protons or something", he said. I did not take the idea that all the electrons were the same one from him as seriously as I took the observation that positrons could simply be represented as electrons going from the future to the past in a back section of their world lines. That, I stole! - Richard P. Feynman, Nobel Lecture, December 11, 1965 [34]

For the sake of clarity, let us introduce a new kind of notation for the energy of a physical entity: $E_{real} = |E|$. Negative energy solutions, corresponding to $E = -E_{real}$, are interpreted as *negative energy particles* propagating *backwards in time*. By associating negative energy solutions to the backward in time motion, we are yielding mathematically the same evolution as for the dynamics of positive energy propagating forwards in time. Namely,

$$\hat{H}\psi_{3,4} = i\frac{\partial \psi_{3,4}}{\partial t} = -E_{real}\psi_{3,4} \quad \Rightarrow i\frac{\partial \psi_{3,4}}{\partial(-t)} = E_{real}\psi_{3,4} \tag{2.135}$$

where we interpreted t to label values of 'backward' process, thus $-t$ labels values of 'forward' process [43] i.e *time arrow of the process with negative energy is flipped with*

[43] it can be also stated other way around, what is important is the flip reflected in the minus sign

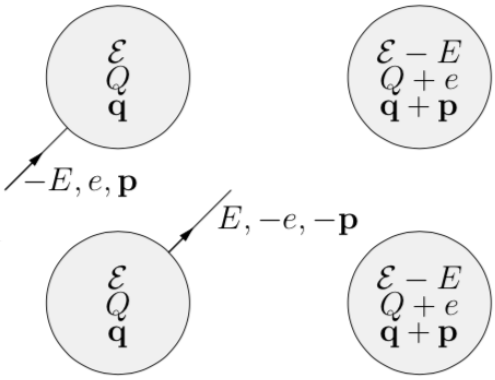


Figure 2.15: Absorption of negative energy particle E, with positive charge e and momentum p is equivalent to emission of an antiparticle in a positive energy state.[9]

respect to time arrow of the process with positive energy. Let us show this by employing an example. Suppose negative energy, with say negative charge, is flowing from point $(t_i, \mathbf{x_i} = \mathbf{B})$ to a point $(t_f, \mathbf{x_f} = \mathbf{A})$. This means that point B is gaining energy (and getting positive in charge), whereas point A is vice versa. We now claim that this situation describing the negative energy solution is backward in the time motion of some original situation moving forward in time. Then, forward in time motion would see entity at the initial point $(t'_i, \mathbf{x}'_i = A)$ and at final point $(t'_f, \mathbf{x}'_f = B)$. For physical consequences to stay the same (for B to gain positive energy and charge), the particle needs to be positive in energy and charge. [44] Thus, we will conclude:

$$(\textit{absorption of } E<0 \textit{ particle of } -\vec{S}, -\vec{p}) \quad \leftrightarrow \quad (\textit{emission of } E>0 \textit{ antiparticle of }, \vec{S}, \vec{p}) \tag{2.136}$$

as shown in Figure (2.15). Thus, negative energy is not a problem if we flip also the sign of 3-momentum. Going back to equation (2.126), we re-express everything in terms of physical quantities and redefine negative energy particle spinors by introducing *antiparticle* spinors v_i:

[44]Notice that this is different than just running footage of a process backwards. Here we demand that in both forward and backward time, physical consequences are indistinguishable (at the end of both processes B becomes more positive), trading off with the identity of a propagating entity, whereas if we just ran the movie backward, this would represent a physically nonequivalent process. Namely, say we filming point B while the negative charge is flowing from B to A. By the end of the movie, B will become more positive in charge. If we now run the movie backward, we would have still a negative charge going from A to B. Now in the opening scene, B starts as positive and becomes negative by the last scene. In contrast to that, what we wanted in the main text are two movies showing B starting negative and becoming positive in time.

$$
\begin{aligned}
&u_{4,3}(E_{4,3} = -E_{real}, \mathbf{p}_{4,3} = -\mathbf{p}_{real}) e^{i(-(-E_{real})t + (-\mathbf{p}_{real})\cdot\mathbf{x})} \\
&= \tilde{u}_{4,3}(\tilde{E}_{4,3} = E_{real}, \tilde{\mathbf{p}}_{4,3} = \mathbf{p}_{real}) e^{i(-(+E_{real})\cdot(-t) + \mathbf{p}_{real}\cdot(-\mathbf{x}))} \\
&\equiv v_{1,2}(E_{real}, \mathbf{p}_{real}) \underbrace{e^{i(E_{real}t - \mathbf{p}_{real}\mathbf{x})}}_{e^{i\mathbf{p}_{real}\cdot\mathbf{x}}}
\end{aligned}
\tag{2.137}
$$

such that:

$$
\begin{aligned}
u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} &\Rightarrow v_2 = N_3 \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{+E+m} \\ 1 \\ 0 \end{pmatrix} \\
u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix} &\Rightarrow v_1 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
\tag{2.138}
$$

we swapped the signs in momenta and energy. In conclusion, whereas u_4, u_3 were *particle states* solutions which had unphysical *negative energy*, propagating *backwards in time*, with some momentum $\mathbf{p}_{4,3}$ (positive or negative), v_1, v_2 are *antiparticle states conjugated in charge*, propagating *forward in time*, expressed in terms of *positive valued physical energy* and *opposite signed momentum*. One can also get antiparticle spinors directly, by taking the ansatz $\psi_i = v_i e^{i\mathbf{p}_{real}\cdot\mathbf{x}}$. This would again yield four spinor solutions, two with positive and two with negative energy. We would then have eight solutions in total, but only four independent: $\{u_1, u_2, v_1, v_2\}$, all corresponding to positive energies:[45]

$$
\psi(x) = \begin{bmatrix} \text{Incoming positive} \\ \text{energy particle} \\ \propto \mathrm{e}^{-\mathrm{i}(Et - \boldsymbol{p}\cdot\boldsymbol{x})} \end{bmatrix} + \begin{bmatrix} \text{Outgoing positive} \\ \text{energy antiparticle} \\ \propto \mathrm{e}^{+\mathrm{i}(Et - \boldsymbol{p}\cdot\boldsymbol{x})} \end{bmatrix}
\tag{2.139}
$$

where $\psi(x)$ denotes a solution to Dirac equation (2.119). Note also that the same structure has also the solution of Klein-Gordon equation (2.156).

[45] From now on we will denote with E strictly positive values of energy $E = \left|\sqrt{\mathrm{p}^2 + m^2}\right|$.

2.3.4 Operators and the antiparticle spinors

Now that we have antiparticle spinors written in terms of the physical energy and momenta:

$$\psi = v(E, \mathbf{p})e^{i(Et - \mathbf{p}\cdot\mathbf{x})} \tag{2.140}$$

we need to redefine also corresponding Hamiltonian and momentum operators. Namely, in the world where negative energy solutions are actually $E > 0$, one is actually using the opposite sign of coordinatisation from the one appearing in particle solution part $(t', \mathbf{x}') = (-t, -\mathbf{x})$. This naturally represents a problem, since we want the meaning of t and x to be in correspondence with what our rods and clocks measure, regardless if we measure particle or antiparticle, i.e. to be consistent on the level of all expressions. Nevertheless, since $(t, \mathbf{x})$ is just representing the coordiantisation of spacetime, they are not directly observable. Thus, we can fix this by implementing a minus sign in the antiparticle observables.
Namely, as written in (2.113), we need to fix the operators which give the physical energy and momenta of the antiparticle spinors are, therefore:

$$\hat{H}^{(v)}\psi \equiv -i\frac{\partial\psi}{\partial t} = (E > 0)\psi \tag{2.141}$$

and also momentum operator would give now opposite signed momentum, thus we redefine:

$$\hat{\mathbf{p}}\psi = -i\nabla\psi = -\mathbf{p} \quad \Rightarrow \quad \hat{\mathbf{p}}^{(v)}\psi = +i\nabla\psi \tag{2.142}$$

The orbital angular momentum of a particle goes to:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}_{real} \rightarrow \mathbf{r} \times (-\mathbf{p}_{real}) = -\mathbf{L} \tag{2.143}$$

For the commutator, $\left[\hat{H}, \hat{\mathbf{L}} + \hat{\mathbf{S}}\right]$ to remain zero for the antiparticle spinors, the operator giving the physical spin states of the v spinors must have

$$\hat{\mathbf{S}}^{(v)} = -\hat{\mathbf{S}} \tag{2.144}$$

Let us now consider charge conjugation. To get the Dirac equation for the electron with charge $q = -e$ in the presence of an electromagnetic field, we make a minimal

substitution

$$i\partial_\mu \to i\partial_\mu - qA_\mu \tag{2.145}$$

which gives

$$\gamma^\mu \left(\partial_\mu - ieA_\mu\right)\psi + im\psi = 0 \tag{2.146}$$

Now, we want to introduce an operator doing transformation

$$\psi' = \hat{C}\psi = i\gamma^2\psi^* \tag{2.147}$$

such that for ψ' we get the same equation as (2.146) just with the opposite charge. By first taking the complex conjugate and then pre-multiplying by $-i\gamma^2$, we get:

$$\begin{aligned} \gamma^\mu \left(\partial_\mu + ieA_\mu\right) i\gamma^2\psi^* + imi\gamma^2\psi^* = 0 \\ \gamma^\mu \left(\partial_\mu + ieA_\mu\right)\psi' + im\psi' = 0 \end{aligned} \tag{2.148}$$

which we notice precisely as the equation for ψ', using (2.147)

$$\gamma^\mu \left(\partial_\mu + ieA_\mu\right)\psi' + im\psi' = 0 \tag{2.149}$$

The $\hat{C}$ operator as given above would then be *charge conjugation operator*. [46]

As discussed above, now we can show concretely that the charge conjugation operator $\hat{C}$, transforms a particle wavefunction into the corresponding antiparticle wavefunction. Namely, consider particle Dirac spinor:

$$\psi = u_1 e^{i(\mathbf{p}\cdot\mathbf{x} - Et)} \tag{2.150}$$

which we charge conjugate:

$$\psi' = i\gamma^2 u_1^* e^{-i(\mathbf{p}\cdot\mathbf{x} - Et)}. \tag{2.151}$$

[46] In the Dirac-Pauli representation of the γ-matrices, $\left(\gamma^0\right)^* = \gamma^0, \left(\gamma^1\right)^* = \gamma^1, \left(\gamma^2\right)^* = -\gamma^2$ and $\left(\gamma^3\right)^* = \gamma^3$. Using these relations and $\gamma^2\gamma^\mu = -\gamma^\mu\gamma^2$ for $\mu \neq 2$

where

$$i\gamma^2 u_1^* = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}^* = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = v_1, \tag{2.152}$$

Thus, we conclude

$$\psi = u_1 e^{i(\mathbf{p}\cdot\mathbf{x}-Et)} \xrightarrow{\hat{C}} \psi' = v_1 e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)} \tag{2.153}$$

and likewise (up to a unobservable overall complex phase) the effect on u_2 is

$$\psi = u_2 e^{i(\mathbf{p}\cdot\mathbf{x}-Et)} \xrightarrow{\hat{C}} \psi' = v_2 e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)}. \tag{2.154}$$

2.3.5 Causality of Dirac field

Putting all together, the plain wave solutions of Dirac equation have the form:

$$\begin{aligned} &\text{Particles:} \quad \psi(x) = u^s(p)\mathrm{e}^{-\mathrm{i}p\cdot x} = \begin{pmatrix} u^s_{\mathrm{L}}(p) \\ u^s_{\mathrm{R}}(p) \end{pmatrix}\mathrm{e}^{-\mathrm{i}p\cdot x} \\ &\text{Antiparticles:} \quad \psi(x) = v^s(p)\mathrm{e}^{\mathrm{i}p\cdot x} = \begin{pmatrix} v^s_{\mathrm{L}}(p) \\ v^s_{\mathrm{R}}(p) \end{pmatrix}\mathrm{e}^{\mathrm{i}p\cdot x} \end{aligned} \tag{2.155}$$

where s denotes a spin degree of freedom, as defined in the equation (2.123). Following the same procedure as in the case Klein-Gordon field, from the pain wave solutions, we can construct now the Dirac field

$$\begin{aligned} \hat{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \left(\hat{a}^s_{\mathbf{p}} u^s(p)e^{-ip\cdot x} + \hat{b}^s_{\mathbf{p}} v^s(p)e^{ip\cdot x}\right) \\ \hat{\psi}^\dagger(y) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \left(\hat{a}^{s\dagger}_{\mathbf{p}} \bar{u}^s(p)e^{ip\cdot y} + \hat{b}^{s\dagger}_{\mathbf{p}} \bar{v}^s(p)e^{-ip\cdot y}\right) \end{aligned} \tag{2.156}$$

To check how this field implements microcausality (i.e locality) condition, we

consider a commutator:

$$\begin{aligned}\left[\hat{\psi}_a(x), \hat{\psi}_b^\dagger(y)\right] &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s \left(u_a^s(p)\bar{u}_b^s(p)e^{-ip\cdot(x-y)} + v_a^s(p)\bar{v}_b^s(p)e^{ip\cdot(x-y)}\right) \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left((\not{p}+m)_{ab}e^{-ip\cdot(x-y)} + (\not{p}-m)_{ab}e^{ip\cdot(x-y)}\right) \\ &= (i\ \not{\partial}_x + m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) \\ &= (i\ \not{\partial}_x + m)_{ab}\left[\hat{\phi}(x), \hat{\phi}(y)\right]\end{aligned} \tag{2.157}$$

Since the commutator of a real Klein-Gordon field $[\hat{\phi}(x), \hat{\phi}(y)]$ vanishes outside the light-cone, this quantity does also.

2.3.6 Propagators

Green's function specifies how a system responds to a localized pointlike disturbance. For the Schrödinger equation $\hat{H}\phi(x,t) = \mathrm{i}\frac{\partial\phi(x,t)}{\partial t}$

$$\phi\left(x, t_x\right) = \int \mathrm{d}y G^+\left(x, t_x, y, t_y\right)\phi\left(y, t_y\right) \tag{2.158}$$

Green's function is evolving wave function from the spacetime point (y, t_y) to (x, t_x). Plus-sign superscript means that we constrain G^+ to have non-vanishing values only for $t_x > t_y$, propagating particles only forward in time. From equation (2.158), we can get:

$$G^+ = \theta(t_x - t_y)\langle x, t_x | y, t_y\rangle \tag{2.159}$$

This is called *retarded propagator*. Similarly, we can also define *advanced propagator* as:

$$G^- = \theta(t_y - t_x)\langle y, t_y | x, t_x\rangle \tag{2.160}$$

Nevertheless, the propagator of the second type doesn't exist for Schrödinger's equation. Namely, advanced and retarded propagators are Green functions for the hyperbolic differential operators, such as the Klein-Gordon equation, on manifolds with causal structure. The corresponding integral kernels say how a point excitation propagates into the future or past, respectively, via the given differential equation.

- **Propagators for Klein-Gordon field**

In Section (2.3.1) we noted that transition amplitude of a particle to go from y to x is given by:

$$G(x-y) \equiv \langle \mathbb{0}|\phi(x)\phi^{\dagger}(y)|\mathbb{0}\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip(x-y)} \tag{2.161}$$

Nevertheless, this correlator propagates particle degrees of freedom in both directions of time, i.e. both in the future and past, depending on the possible relation between x^0 and y^0. To implement the discussion regarding antiparticles, Feynman introduced a propagator which propagates particle degrees of freedom towards the future and antiparticle towards the past:

$$G(x,y) = \theta\left(x^0-y^0\right)\left\langle \mathbb{0}\left|\hat{\phi}(x)\hat{\phi}^{\dagger}(y)\right|\mathbb{0}\right\rangle + \theta\left(y^0-x^0\right)\left\langle \mathbb{0}\left|\hat{\phi}(y)^{\dagger}\hat{\phi}(x)\right|\mathbb{0}\right\rangle$$

Correlator in equation (2.161) is a satisfies free Klein-Gordon equation,

$$\left(\partial_\mu\partial^\mu + m^2\right)G_X(x-y) = 0 \tag{2.162}$$

whereas (2.3.6) is the Green function of the Klein-Gordon equation, i.e. it satisfies the equation with a delta source term:

$$\left(\partial_\mu\partial^\mu + m^2\right)G_X(x-y) = -i\delta(x-y), \quad X = R, A, F, D \tag{2.163}$$

Where we denote with R = retarded, A = advanced, F = Feynman, D = Dyson propagator. Four different Green functions come from the four different ways of circling the pole. Namely, let us write the Green function as $4D$ Fourier integral:

$$G(x-y) = \int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\tilde{G}_R(p) \tag{2.164}$$

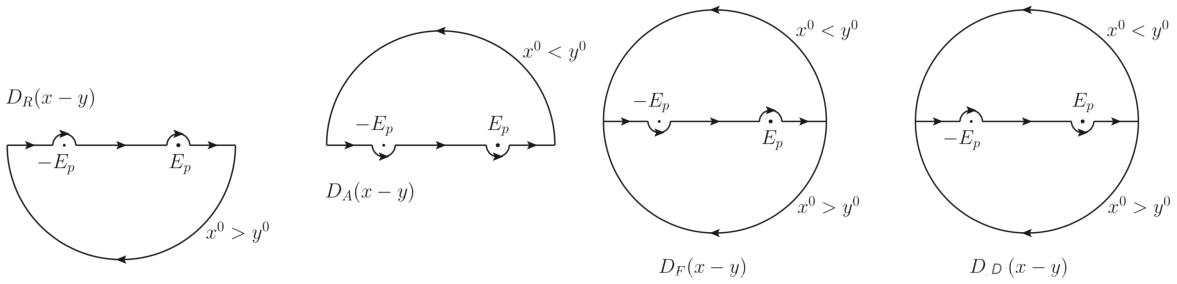
Upon plugging in Klein-Gordon equation [47]

$$\left(-p^2+m^2\right)\tilde{G}(p) = -i. \tag{2.165}$$

we get

$$G(x-y) = \int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{i}{p^2-m^2} \tag{2.166}$$

[47] while using $\delta(x-y) = \frac{1}{(2\pi)^4}\int e^{ip\cdot(x-y)}\mathrm{d}^dp$



(a) Retarded and advanced contour

(b) Feynman and Dyson contour

Figure 2.16: Four ways of circling around the poles [53]

When the particle is on mass-shell, the function diverges due to the presence of poles at $p^2 = m^2 \Rightarrow p^0 = \pm\sqrt{m^2 + \mathbf{p}} = \pm E_p$. As a standard procedure of complex integration, to yield non-divergent contribution we need to integrate along the contour which will circle around the pole. Thus, we add infinitesimal contribution $\pm i\eta << 0$ which will displace us from the real axis where poles are.

From Figure (2.16) we see there are four different ways to circle around the pole, yielding four different results for the Green function. We will denote them with $\mathrm{R} : s_A = s_B = 1,\ \mathrm{A} : s_A = s_B = -1,\ \mathrm{F} : s_A = -s_B = 1$, $\mathrm{D} : -s_A = s_B = 1$ such that we have:

$$\begin{aligned}
G_X(x-y) &= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{\left(p^0 - E_p + is_A\eta\right)\left(p^0 + E_p + is_B\eta\right)} \\
&= \int \frac{d^3p}{(2\pi)^3} \left[s_A\theta\left(s_A\left(x^0-y^0\right)\right) \frac{\exp\left[-iE_p\left(x^0-y^0\right) + i\vec{p}(\vec{x}-\vec{y})\right]}{2E_p} \right. \\
&\left. -s_B\theta\left(s_B\left(x^0-y^0\right)\right) \frac{\exp\left[iE_p\left(x^0-y^0\right) - i\vec{p}(\vec{x}-\vec{y})\right]}{2E_p} \right] \\
&= s_A\theta\left(s_A\left(x^0-y^0\right)\right) \left\langle \mathbb{0} \left| \phi(x)\phi^\dagger(y) \right| \mathbb{0} \right\rangle - s_B\theta\left(s_B\left(x^0-y^0\right)\right) \left\langle \mathbb{0} \left| \phi^\dagger(y)\phi(x) \right| \mathbb{0} \right\rangle \\
&\overset{R}{=} \left\langle \mathbb{0} \left| \theta\left(x^0-y^0\right)\left(\phi(x)\phi^\dagger(y) - \phi^\dagger(y)\phi(x)\right) \right| \mathbb{0} \right\rangle = G_R(x-y) \\
&\overset{A}{=} \left\langle \mathbb{0} \left| -\theta\left(y^0-x^0\right)\left(\phi(x)\phi^\dagger(y) - \phi^\dagger(y)\phi(x)\right) \right| \mathbb{0} \right\rangle = G_A(x-y) \\
&\overset{F}{=} \langle \mathbb{0} | \theta\left(x^0-y^0\right)\phi(x)\phi^\dagger(y) + \theta\left(y^0-x^0\right)\phi^\dagger(y)\phi(x) \Big) | \mathbb{0} \rangle = G_F(x-y) \\
&\overset{D}{=} \langle \mathbb{0} | - \theta\left(y^0-x^0\right)\phi(x)\phi^\dagger(y) - \theta\left(x^0-y^0\right)\phi^\dagger(y)\phi(x) \Big) | \mathbb{0} \rangle = G_D(x-y)
\end{aligned} \tag{2.167}$$

Given the structure in terms of a commutator, we see that retarded and advanced propagators are strictly causal (in the sense that they are vanishing outside the light-cone):

$$
\begin{aligned}
G_R(x-y) &= \theta\left(x^0-y^0\right)\left\langle \mathbb{0}\,\middle|\,\left[\phi(x),\phi^\dagger(y)\right]\,\middle|\,\mathbb{0}\right\rangle \\
G_A(x-y) &= \theta\left(y^0-x^0\right)\left\langle \mathbb{0}\,\middle|\,\left[\phi^\dagger(y),\phi(x)\right]\,\middle|\,\mathbb{0}\right\rangle
\end{aligned}
\tag{2.168}
$$

whereas Dyson's and Feynman's propagators are not strictly causal, in the sense that they decay outside the lightcone. Namely, since they are built out of contributions of the form (2.161) for which we have at a spacelike distance (2.117) (due to Heaviside functions in different domains of spacetime so that contributions do not cancel out). Thus, at spacelike region Feynman propagator decays with $\sim e^{-m|x-y|}$. Nevertheless, notice that advanced and retarded propagators propagate both particle and antiparticle degrees of freedom in either future or past lightcone. On the other hand, Feynman propagator is designed to propagate particles and antiparticles to future lightcone, as shown in Figure (2.16) [48]. Feynman propagator is thus causal in the sense that it always first produces and then annihilates (anti)particles. In that sense, the Dyson propagator is called anticausal, since it first annihilates and then produces (anti)particle.

In a similar manner we can construct propagators for the Dirac equation. See for example [76].

2.4 Time ordering and Feynman diagrams

2.4.1 Time ordered product

The time ordering operator is a mathematical object appearing in the context of perturbation expansions in non-relativistic quantum mechanics and field theory settings. It represents an operation of re-ordering a string of operators according to their causal ordering. Namely, for the case of field observables associated to the distinct events $x, y \in \mathcal{M}$, where $\mathcal{M}$ denotes spacetime, the time-ordered product is defined by

$$
T\left(\boldsymbol{\psi}(x)\boldsymbol{\psi}(y)\right) := \begin{cases} \boldsymbol{\psi}(x)\boldsymbol{\psi}(y) & x \text{ not in the past of } y \\ \pm\boldsymbol{\psi}(y)\boldsymbol{\psi}(x) & \text{otherwise} \end{cases}
\tag{2.169}
$$

'$\pm$' refers to the fact that for the bosonic fields we use commutation relations to re-order field operators chronologically, corresponding to the '$+$' sing, whereas for

[48] Recall that in Feynman diagrams we sketch always in terms of particle u_i solutions having positive or negative energy propagating forward vs. backwards in time.

the fermionic fields we use anticommutation relations, corresponding to the '−'.
Let us consider the first context of non-relativistic quantum mechanics. The dynamics of the states are given by deterministic evolution, defined by the Schrödinger equation:

$$i\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}(t)|\psi(t)\rangle \tag{2.170}$$

with the solution

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle \tag{2.171}$$

To find the form of the $U(t,t_0)$, let us consider infinitesimal expansion around some point t:

$$\begin{aligned} |\psi(t+\delta t)\rangle &= [1 - i\delta t \cdot \hat{H}(t)]\psi(t) \\ &= e^{-i\delta t\cdot\hat{H}(t)}|\psi(t)\rangle \end{aligned} \tag{2.172}$$

Accordingly the value of ψ at any instant t_f can be expressed in terms of its value at some initial instant $t_i\,(<t_f)$ as the product of deterministic evolutions over all the infinitesimal intervals δt_α between t_i and t_f:

$$\psi\left(t_f\right) = \left(\prod_i^f e^{-i\delta t_a \hat{H}(t_a)}\right)\psi\left(t_i\right) \tag{2.173}$$

Naively, in the $\lim(\delta\to 0)$ this would give the solution of the form:

$$\hat{U} \sim \exp\left(-i\int_{t_0}^{t}\hat{H}(t)dt\right) \tag{2.174}$$

which would upon expansion look like:

$$\hat{U}\left(t,t_0\right) = 1 - i\int_{t_0}^{t} dt_1\hat{H}(t_1) + \frac{(-i)^2}{2}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\hat{H}(t_1)\hat{H}\left(t_2\right) + \ldots \tag{2.175}$$

Notice however that this expansion assumes that Hamiltonians pertaining to different instants of time commute. Nevertheless, generally $[\hat{H}(t),\hat{H}(t')] \neq 0$ and more care must be taken. Namely, let us consider the second order, as this is the lowest order in which the problem occurs. Double integral going from t_0 to t represents integration over the entire square. Nonetheless, operator actions should be ordered such that the first operators defined at earlier times act, then the later ones. From the Figure (2.17) it may be noticed that the integral should be split into two parts,

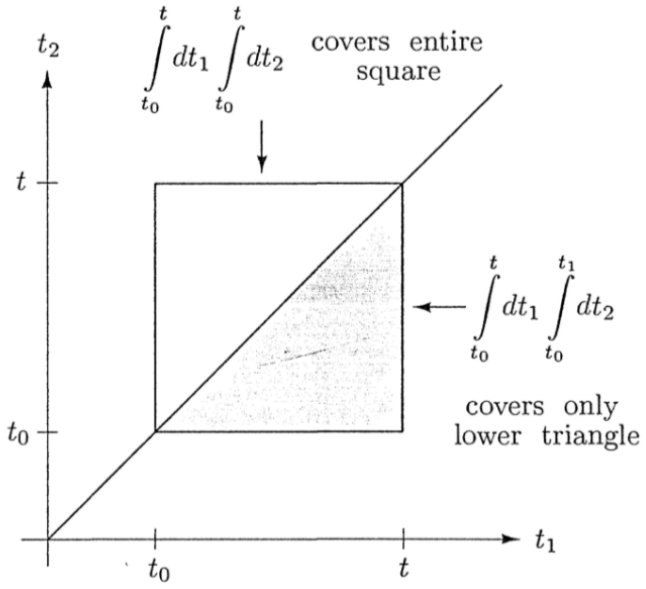


Figure 2.17: Geometrical interpretation of time ordering in the second order term. [76]

each pertaining to one triangle where operators should be appropriately ordered: ordering associated with the lower triangle should be $\hat{H}(t_1)\hat{H}(t_2)$, taking into account that in that region of integration $t_1 > t_2$, whereas ordering associated to the upper triangle should be $\hat{H}(t_2)\hat{H}(t_1)$, taking into account that in that region of integration $t_2 > t_1$. This can be done by defining the time ordering operator; for the second order we have:

$$T\left(\hat{H}_I\left(t_1\right)\hat{H}_I\left(t_2\right)\right) = \hat{H}_I\left(t_1\right)\hat{H}_I\left(t_2\right)\theta(t_1 - t_2) + \hat{H}_I\left(t_2\right)\hat{H}_I\left(t_1\right)\theta(t_2 - t_1) \tag{2.176}$$

The time ordering operator splits the integration into integration through upper and lower triangle, giving appropriate ordering of the operators. The expansion can now be written as:

$$\hat{U}\left(t, t_0\right) = 1 - i\int_{t_0}^{t} dt_1 \hat{H}(t_1) + \frac{(-i)^2}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T\left\{\hat{H}_I\left(t_1\right)\hat{H}_I\left(t_2\right)\right\} + \ldots \tag{2.177}$$

The solution for the case when $[\hat{H}(t), \hat{H}(t')] \neq 0$ is thus:

$$\hat{U}\left(t, t_0\right) \equiv T\exp\left[-i\int_{t_0}^{t} dt' \hat{H}(t')\right] \tag{2.178}$$

where T is the time ordering operator, as defined above. We can see the time

ordering arising also by considering defining the equation for $\hat{U}(t,t_0)$:

$$\frac{\partial}{\partial t}\hat{U}\left(t,t_0\right)=-i\hat{H}(t)\hat{U}\left(t,t_0\right) \tag{2.179}$$

and solving the equation iteratively. Starting from:

$$\hat{U}\left(t,t_0\right)=1-i\int_{t_0}^{t}dt_1\hat{H}(t_1)\hat{U}\left(t_1,t_0\right) \tag{2.180}$$

substitute $U(t_1,t_0)$ back to the equation:

$$\begin{aligned}\hat{U}\left(t,t_0\right)&=1-i\int_{t_0}^{t}dt_1\hat{H}(t_1)\left[1-i\int_{t_0}^{t_1}dt_2\hat{H}\left(t_2\right)\hat{U}\left(t_2,t_0\right)\right]\\&=1+(-i)\int_{t_0}^{t}dt_1\hat{H}(t_1)+(-i)^2\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\hat{H}(t_1)\hat{H}\left(t_2\right)\hat{U}\left(t_2,t_0\right)\end{aligned} \tag{2.181}$$

Now notice the implied time order of variables of integration: $t_0<t_2<t_1<t$. The second order can be viewed as integration over the lower triangle represented in Figure 2.17. This can be rewritten in terms of integration through the whole square as:

$$\begin{aligned}&\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\hat{H}_I\left(t_1\right)\hat{H}_I\left(t_2\right)=\frac{1}{2}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2T\left\{\hat{H}_I\left(t_1\right)\hat{H}_I\left(t_2\right)\right\}\\&=\frac{1}{2}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\hat{H}_I\left(t_1\right)\hat{H}_I\left(t_2\right)\theta(t_1-t_2)+\hat{H}_I\left(t_2\right)\hat{H}_I\left(t_1\right)\theta(t_2-t_1)\end{aligned} \tag{2.182}$$

where we introduced the time order operator which, as stated before, splits the integration into integration through the upper and lower triangle. We have:

$$\hat{U}\left(t,t_0\right)=1-i\int_{t_0}^{t}dt_1\hat{H}(t_1)+\frac{(-i)^2}{2}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2T\left\{\hat{H}_I\left(t_1\right)\hat{H}_I\left(t_2\right)\right\}\hat{U}\left(t_2,t_0\right) \tag{2.183}$$

Proceeding iteratively we get:

$$\hat{U}\left(t,t_0\right)\equiv T\exp\left[-i\int_{t_0}^{t}dt'\hat{H}(t')\right] \tag{2.184}$$

the expansion of this operator is usually called Dyson series.

Each order in the Dyson series can be visualized as going through different paths, di-

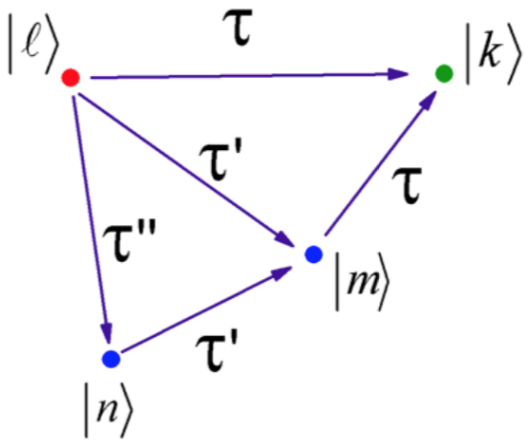


Figure 2.18: Starting from the state $|\psi(t_0)\rangle = |l\rangle$ and aiming to the target state $|\psi(t)\rangle = |k\rangle$ Dyson operator describes all possible paths between initial and final state. [83]

vided into more and more intermediate states as order increases (see Figure (2.18).

In the context of field theory, the Dyson series is considered in the interaction picture and features in the central object of the theory namely, the scattering matrix:

$$\mathcal{S}\left(\hat{V}_I\right) := \lim_{t\to\infty} T\left(\exp\left(-i\int_{-t}^{t}\hat{V}_I(t)dt\right)\right) = \lim_{t\to\infty}\hat{U}(t,-t) \tag{2.185}$$

which sends asymptotic in states $|\psi(-\infty)\rangle_I$ to the asymptotic out states $|\psi(+\infty)\rangle_I$.

At first, one might expect that time ordering depends on the specific choice of time function with respect to which the ordering is done. Nevertheless, the claim that time ordering comes from causal axioms imposed on the S-matrix implies invariance of the time ordering operation. To illuminate this fact, let us consider first time-like separated points, x_1 and x_2, $(x_1 - x_2)^2 > 0$. In this case we have causal ordering between x_1 and x_2 and thus the sign of $t_2 - t_1$, where t_1 and t_2 represent respective timelike components, is independent of the frame of reference. However, in the case of space-like separated events, $(x_1 - x_2)^2 < 0$, the sign of $t_2 - t_1$ is not independent of the frame of reference and thus Lorentz transformation may reverse the sequence of time instants. Invariance of time order is here provided by invoking

the microcausality condition

$$[O(x_i), O(x_j)] = 0 \quad \text{for} \quad (x_i - x_j)^2 < 0 \tag{2.186}$$

Namely, recall that non-commutativity of operators physically means that corresponding quantities cannot be measured simultaneously. However, no matter what are the operators related to the space-like separated points, since there is no causal connection between them simultaneous measurements can always be done. Given the commutation, the factors in the time ordering product can always be restored according to their chronological order. Thus, in both scenarios, time ordering product stays invariant. Notice that here we used the following implication:

$$\text{microcausality} \quad \Rightarrow \quad \text{invariance of time ordering product} \tag{2.187}$$

2.4.2 From S matrix to Feynman diagrams

Rigorous models of quantum field theory are applicable so far only for a small number of toy models, non-interacting theories, or axiomatizations in lower dimensional spacetimes. For this reason, perturbative quantum field theory plays a crucial role in phenomenological considerations. The central mathematical object of these considerations is the scattering matrix (S-matrix), expanded in a Dyson series, as power series in the coupling constant. S-matrix enables us to calculate probability amplitude $\mathcal{A}$ and further scattering cross sections σ, which is a quantity directly measured in scattering experiments. Scattering experiments consist of incoming particles, *'in' states*, starting at a large enough distance so that at the beginning of the experiment, $t = -\infty$ they can be considered as *free*. Then the beams of the particles are made to collide, a collision lasting for a very short time. In the end, we measure the outgoing distribution of particles, again at large enough distances so that particles can be considered again as free *'out' states*. In perturbative quantum field theory we work in an interaction picture Hamiltonian is split in free and interacting part $\hat{H} = \hat{H}_0 + \hat{V}$ where interaction is considered weak perturbation of the non-interacting case. S-matrix represents unitary transformation which evolves 'in' states to 'out' states:

$$\hat{S} = \hat{U}_I(-\infty, \infty) = T \exp\left(-\mathrm{i} \int_{-\infty}^{\infty} \hat{V}_I(t) \mathrm{d}t\right) \tag{2.188}$$

Then, the probability amplitude for a process with two 'in' and several 'out' states is simply defined as:

$$\mathcal{A} = {}^{\text{out}}\langle q_1 q_2 ... \mid p_2 p_1 \rangle^{\text{in}} = {}_0\left\langle q_1 q_2 ... \left| \hat{S} \right| p_2 p_1 \right\rangle_0 \tag{2.189}$$

The scattering process contains the creation and annihilation of particles. The interactions between 'in' and 'out' particles are carried via mediating bosonic particles. This can be represented via famously known *Feynman diagrams*, which make calculations of scattering amplitude easier.

We will consider the procedure of computing Feynman diagrams by employing an example. Let us consider one of the simplest theories, ϕ^4 theory through which scalar field interacts with itself. The Lagrangian density contains a free scalar field and ϕ^4 potential,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi(x)\right)^2 - \frac{1}{2}m^2\phi(x)^2 - \frac{\lambda}{4!}\phi(x)^4 \tag{2.190}$$

[49]

Free Hamiltonian density

$$\hat{\mathcal{H}}_0 = \frac{1}{2}\left(\left(\partial_t \hat{\phi}\right)^2 + (\boldsymbol{\nabla}\hat{\phi})^2 + m^2\hat{\phi}^2\right) \tag{2.191}$$

evolves in time the field operators $\hat{\phi}(x) = \mathrm{e}^{\mathrm{i}\hat{H}_0 t}\hat{\phi}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\hat{H}_0 t}$, whereas states evolve via the interaction Hamiltonian:

$$\hat{\mathcal{H}}_I = -\frac{\lambda}{4!}\phi(x)^4 \tag{2.192}$$

such that $\mathrm{i}\frac{\partial}{\partial t}|\psi(t)\rangle_{\mathrm{I}} = \hat{V}_{\mathrm{I}}(t)|\psi(t)\rangle_{\mathrm{I}}$ where $\hat{V}_{\mathrm{I}}(t) = \int \hat{\mathcal{H}}_I(x)\mathrm{d}^3x$.

For simplicity, we will consider a case of one 'in' particle and one 'out' particle,

$$\begin{aligned}\mathcal{A} &= {}^{\text{out}}\langle q|p\rangle^{\text{in}} = {}_0\langle q|\hat{S}|p\rangle_0 \\ &= (2\pi)^3\left(2E_q\right)^{\frac{1}{2}}\left(2E_p\right)^{\frac{1}{2}}\left\langle 0\left|\hat{a}_q\hat{S}\hat{a}_p^\dagger\right|0\right\rangle\end{aligned} \tag{2.193}$$

where we used momentum state with relativistic normalisation $|p\rangle = (2\pi)^{\frac{3}{2}}\left(2E_p\right)^{\frac{1}{2}}\hat{a}_p^\dagger|0\rangle$.

[49] There is also a quick trick giving pieces of Feynman diagrams directly from Lagrangian density. Namely, first, we notice how many different fields there are (when we here discuss fields, it doesn't matter if there is a direct field appearing or a field derivative). These pieces give legs to the diagram. Terms quadratic in fields (and field derivatives) will give rise to a propagator. Other terms, having in total more than two field terms, will give a vertex. Then the terms are pieced in all possible ways.

Upon expanding:

$$\begin{aligned}\mathcal{A} =&\langle q|\hat{S}|p\rangle \\ =&\langle q|T\left(-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t\hat{V}_{\mathrm{I}}(t)\right)|p\rangle \\ =&\left\langle q\left|T\left(1-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t_1\hat{V}_{\mathrm{I}}(t_1)+\frac{(-\mathrm{i})^2}{2}\int_{-\infty}^{\infty}\mathrm{d}t_1\,\mathrm{d}t_2\hat{V}_{\mathrm{I}}(t_1)\hat{V}_{\mathrm{I}}(t_2)+\ldots\right)\right|p\right\rangle\end{aligned} \tag{2.194}$$

and inserting particular form of interaction,

$$\begin{aligned}\mathcal{A} =&\left\langle q\left|T\left(1-\frac{\mathrm{i}\lambda}{4!}\int\mathrm{d}^4z\hat{\phi}(z)^4+\frac{(-\mathrm{i})^2}{2!}\left(\frac{\lambda}{4!}\right)^2\int\mathrm{d}^4y\,\mathrm{d}^4w\hat{\phi}(y)^4\hat{\phi}(w)^4+\ldots\right)\right|p\right\rangle \\ =&(2\pi)^3\left(2E_{\boldsymbol{q}}\right)^{\frac{1}{2}}\left(2E_{\boldsymbol{p}}\right)^{\frac{1}{2}}\hat{T}\left[\left\langle 0\left|\hat{a}_{\boldsymbol{q}}\hat{a}^{\dagger}_{\boldsymbol{p}}\right|0\right\rangle+\left(\frac{-\mathrm{i}\lambda}{4}\right)\int\mathrm{d}^4z\left\langle 0\left|\hat{a}_{\boldsymbol{q}}\hat{\phi}(z)^4\hat{a}^{\dagger}_{\boldsymbol{p}}\right|0\right\rangle\right. \\ &\left.+\frac{(-\mathrm{i})^2}{2!}\left(\frac{\lambda}{4!}\right)^2\int\mathrm{d}^4y\,\mathrm{d}^4w\left\langle 0\left|\hat{a}_q\hat{\phi}(y)^4\hat{\phi}(w)^4\hat{a}^{\dagger}\right|0\right\rangle+\ldots\right]\end{aligned} \tag{2.195}$$

where we will refer to $\mathcal{A}=\mathcal{A}^{(0)}+\mathcal{A}^{(1)}+\mathcal{A}^{(2)}...$ as zeroth order, first order and second order term. The crucial tool in the following will be Wick's theorem, enabling us to simplify time-ordered strings of operators. Namely, the Wick theorem (Appendix B) gives an expression,

$$T[\hat{A}\hat{B}\hat{C}\ldots\hat{Z}]=N\left[\hat{A}\hat{B}\hat{C}\ldots\hat{Z}+\begin{array}{c}\text{all possible contractions of}\\ \hat{A}\hat{B}\hat{C}\ldots\hat{Z}\end{array}\right] \tag{2.196}$$

where N represents normal ordering. The purpose of introducing a normal ordering is to make vacuum expectation values vanish, by placing annihilation operators on the right and creation operators on the left. We will be left with terms involving just contractions, enabling us to break up the strings into smaller pieces.

Let us try to compute $\mathcal{A}^{(1)}$. We have to unravel a term of the form:

$$\left\langle 0\left|T(\hat{a}_q\hat{\phi}(x)^4\hat{a}^{\dagger}_p)\right|0\right\rangle\equiv\left\langle 0\left|T(\hat{a}_q\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z)\hat{a}^{\dagger}_p)\right|0\right\rangle\equiv(*) \tag{2.197}$$

Using Wick theorem following contractions:

$$(*) = \left\langle 0 \left| \overbrace{\hat{a}_q, \hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z)\hat{a}^\dagger}^{} \right| 0 \right\rangle = \left\langle 0 \left| \hat{a}_q \hat{a}_p^\dagger \right| 0 \right\rangle \langle 0|T\hat{\phi}(z)\hat{\phi}(z)|0\rangle \langle 0|T\hat{\phi}(z)\hat{\phi}(z)|0\rangle \tag{2.198}$$

where we first considered the contraction of ladder operators with ladder operators and fields with fields. This term will come in the amplitude multiplied with a symmetry factor $S = 3$ since there are three different ways to do contraction of this type. Namely, one can contract a ladder operator with a ladder operator, for which there is only one option, and field operators with field operators, for which there are three ways, all giving indistinguishable contributions. Notice that contractions involving ladder operators aren't time ordered since they are defined only at asymptotic regions, $t = -\infty$ and $t = \infty$. Now we can also consider the contraction of ladder operators with field operators,

$$(*) = \left\langle 0 \left| \overbrace{\hat{a}_q, \hat{\phi}(z)}\hat{\phi}(z)\overbrace{\hat{\phi}(z)}\hat{\phi}(z)\hat{a}^\dagger \right| 0 \right\rangle = \left\langle 0 \left| \hat{a}_{\boldsymbol{q}} \hat{\phi}(z) \right| 0 \right\rangle \langle 0|T\hat{\phi}(z)\hat{\phi}(z)|0\rangle \left\langle 0 \left| \hat{\phi}(z)\hat{a}_{\boldsymbol{p}}^\dagger \right| 0 \right\rangle \tag{2.199}$$

we can contract first the ladder operator by choosing one out of four field operators, giving a symmetry factor of 4. Then we can contract other ladder operator with one of the rest three field operators, giving a symmetry factor of 3. In total, symmetry factor of this term is $S = 3 \cdot 4 = 12$.

The resulting amplitude in the first-order term is

$$\begin{aligned}\mathcal{A}^{(1)} = (2\pi)^3 (2E_{\boldsymbol{q}})^{\frac{1}{2}} (2E_{\boldsymbol{p}})^{\frac{1}{2}} \frac{(-i)\lambda}{4!} \int \mathrm{d}^4 z \Big(& 3 \left\langle 0 \left| \hat{a}_{\boldsymbol{q}} \hat{a}_{\boldsymbol{p}}^\dagger \right| 0 \right\rangle \langle 0|\hat{\phi}(z)\hat{\phi}(z)|0\rangle \langle 0|\hat{\phi}(z)\hat{\phi}(z)|0\rangle \\ & + 12 \left\langle 0 \left| \hat{a}_{\boldsymbol{q}} \hat{\phi}(z) \right| 0 \right\rangle \langle 0|T\hat{\phi}(z)\hat{\phi}(z)|0\rangle \left\langle 0 \left| \hat{\phi}(z)\hat{a}_{\boldsymbol{p}}^\dagger \right| 0 \right\rangle \Big)\end{aligned} \tag{2.200}$$

We can now compute the correlators we got from the contractions,

- From a contractions between two fields we get a free propagator

$$\overbrace{\hat{\phi}(y)\hat{\phi}(z)} = \langle 0|T(\hat{\phi}(y)\hat{\phi}(z))|0\rangle = \Delta(y - z) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{ie^{-\mathrm{i}q\cdot(y-z)}}{q^2 - m^2 + \mathrm{i}\epsilon} \tag{2.201}$$

Here we calculated the Feynman propagator, along with the discussion represented in (2.3.6). For the expression to be well defined, particles propagated should be off mass-shell, i.e. $q^2 \neq m^2$. In conclusion, the Feynman propagator introduces a new kind of propagating particle, called *virtual particle*. In the discussion of the

section below, we will see that either we could have kept on mass-shell particles, paying the price of energy temporarily not being conserved in the interacting vertex, and amplitude expression not being invariant; or we introduce virtual particles so that the amplitude is invariant and 4-momentum is conserved in the vertex, but with a price of introducing the virtual particles which do not satisfy relativistic energy-momentum relation.

- From the contractions between field and the creation operator we get a term

$$\begin{aligned}\overbrace{\hat{\phi}(z)\hat{a}_p}^{\dagger} = \left\langle 0 \left| \hat{\phi}(z)\hat{a}_p^\dagger \right| 0 \right\rangle &= \int \frac{\mathrm{d}^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\boldsymbol{q}})^{\frac{1}{2}}} \left\langle 0 \left| \left(\hat{a}_{\boldsymbol{q}} \mathrm{e}^{-\mathrm{i}q\cdot z} + \hat{a}_{\boldsymbol{q}}^\dagger \mathrm{e}^{\mathrm{i}q\cdot z}\right) \hat{a}_{\boldsymbol{p}}^\dagger \right| 0 \right\rangle \\ &= \int \frac{\mathrm{d}^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\boldsymbol{q}})^{\frac{1}{2}}} \left\langle 0 \left| \left(\hat{a}_{\boldsymbol{q}} \mathrm{e}^{-\mathrm{i}q\cdot z} + \hat{a}_{\boldsymbol{q}}^\dagger \mathrm{e}^{\mathrm{i}q\cdot z}\right) \right| \boldsymbol{p} \right\rangle \\ &= \int \frac{\mathrm{d}^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\boldsymbol{q}})^{\frac{1}{2}}} \mathrm{e}^{-\mathrm{i}q\cdot z} \delta^{(3)}(\boldsymbol{q}-\boldsymbol{p}) \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\boldsymbol{p}})^{\frac{1}{2}}} \mathrm{e}^{-\mathrm{i}p\cdot z}\end{aligned} \tag{2.202}$$

Factors $(2\pi)^{\frac{-3}{2}} (2E_{\boldsymbol{p}})^{\frac{-1}{2}}$ will exactly cancel with the factors in the amplitude (2.200) coming from the relativistic normalization $|p\rangle = (2\pi)^{\frac{3}{2}} (2E_{\boldsymbol{p}})^{\frac{1}{2}}$. In total, we will have a resulting contraction

$$\overbrace{\hat{\phi}(x)|p\rangle} = \mathrm{e}^{-\mathrm{i}p\cdot x} \tag{2.203}$$

This will represent an *incoming* particle.

In a similar manner, the contraction between a field operator and a final state annihilation operator, corresponding to *outgoing* particle.

$$\begin{aligned}\overbrace{\hat{a}_p^\dagger \hat{\phi}(x)} &= \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_q)^{\frac{1}{2}}} e^{\mathrm{i}q\cdot z} \\ \Rightarrow \overbrace{\langle q|\hat{\phi}(x)} &= e^{\mathrm{i}q\cdot z}\end{aligned} \tag{2.204}$$

- Finally, contractions between initial and final particles will simply yield a delta function

$$\overbrace{\hat{a}_q \hat{a}_p}^{\dagger} = \left\langle 0 \left| \hat{a}_q \hat{a}_p^\dagger \right| 0 \right\rangle = \delta^{(3)}(\boldsymbol{q}-\boldsymbol{p}) \tag{2.205}$$

Here let us notice that time order does not affect the outer legs, rather only the propagator part. This means that breaking time order would affect only what is hap-

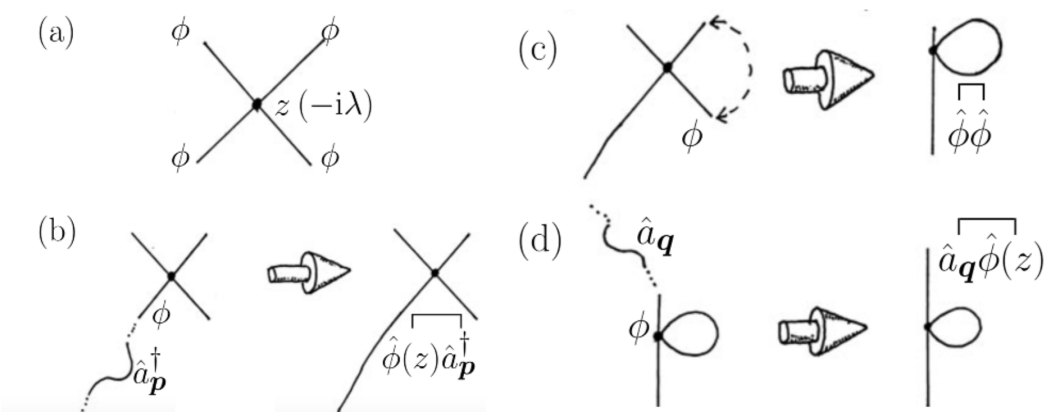


Figure 2.19: Steps in drawing the Feynman diagram. [9]

pening on the level of the propagator.

We can now draw Feynman diagrams, corresponding to our amplitude expansion. Order of amplitude expansion will determine number of interaction vertices; zeroth order has no interactions, first has one (corresponding to $\hat{V}_I(z_1)$), second has two (corresponding to $\hat{V}_I(z_1)\hat{V}_I(z_2)$) etc. Incoming (2.203) and outgoing particles (2.204) are emerging from the vertices. They are called the legs of the diagram. The Wick contractions of field parts (2.201) represent propagators, joining the legs of the diagram. For example, let us consider the term

$$-i\frac{12\lambda}{4!}\int \mathrm{d}^4 z \left\langle 0 \left| \hat{a}_{\boldsymbol{p}}\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z)\hat{a}^\dagger_p \right| 0 \right\rangle \tag{2.206}$$

corresponding to one part of first-order amplitude expansion. This means we draw one vertex in some spacetime point z. Following the structure given in (2.19), we draw an incoming particle going into one of the vertex legs. Propagator contraction then connects two of the other field lines. Lastly, the outgoing particle connects with the remaining free leg.

Now that we have an idea of how things work, let us briefly comment on how to get a scattering second order diagram, which is our main focus. Now we have two incoming and outgoing particles,

$$\mathcal{A} = \left\langle q_1 q_2 | \hat{S} | p_2 p_1 \right\rangle = \left\langle 0 \left| \hat{a}_{\boldsymbol{q}_1}\hat{a}_{\boldsymbol{q}_2}\hat{S}\hat{a}^\dagger_{\boldsymbol{p}_2}\hat{a}^\dagger_{\boldsymbol{p}_1} \right| 0 \right\rangle \tag{2.207}$$

For example, take scattering in Yukawa's $\psi^\dagger\psi\phi$ theory, which describes a complex scalar field ψ and a real scalar field ϕ interacting. Lagrangian density is the following,

$$\mathcal{L} = \partial^\mu\psi^\dagger\partial_\mu\psi - m^2\psi^\dagger\psi + \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}\mu^2\phi^2 - g\psi^\dagger\psi\phi \tag{2.208}$$

Interaction part is $\mathcal{H}_{\mathrm{I}} = -g\psi^\dagger\psi\phi$. We start by writing down the mode expansions of each of the free fields: [50]

We expand the $\hat{S}$-operator using Dyson's expansion

$$\begin{aligned}\left\langle 0\left|\hat{a}_{\boldsymbol{q}_1}\hat{a}_{\boldsymbol{q}_2}\hat{S}\hat{a}^\dagger_{\boldsymbol{p}_2}\hat{a}^\dagger_{\boldsymbol{p}_1}\right|0\right\rangle =& \left\langle 0\middle|\hat{a}_{\boldsymbol{q}_1}\hat{a}_{\boldsymbol{q}_2}\Big(1 - \mathrm{i}g\int \mathrm{d}^4z\hat{\psi}^\dagger(z)\hat{\psi}(z)\hat{\phi}(z)\right.\\ &\left. + \frac{(-\mathrm{i}g)^2}{2!}\int \mathrm{d}^4y\,\mathrm{d}^4w\left[\hat{\psi}^\dagger(y)\hat{\psi}(y)\hat{\phi}(y)\right]\left[\hat{\psi}^\dagger(w)\hat{\psi}(w)\hat{\phi}(w)\right] + \dots\Big)\hat{a}^\dagger_{\boldsymbol{p}_2}\hat{a}^\dagger_{\boldsymbol{p}_1}\middle|0\right\rangle\end{aligned} \tag{2.209}$$

Let us consider the second-order term[51]

$$\mathcal{A}^{(2)} = \frac{(-\mathrm{i}g)^2}{2!}(2\pi)^3\left(2E_{\boldsymbol{p}}\right)^{\frac{1}{2}}\left(2E_{\boldsymbol{q}}\right)^{\frac{1}{2}}\left\langle 0\left|\hat{a}_{\boldsymbol{q}_1}\hat{a}_{\boldsymbol{q}_2}\hat{\psi}^\dagger(y)\hat{\psi}(y)\hat{\phi}(y)\hat{\psi}^\dagger(w)\hat{\psi}(w)\hat{\phi}(w)\hat{a}^\dagger_{\boldsymbol{p}_2}\hat{a}^\dagger_{\boldsymbol{p}_1}\right|0\right\rangle \tag{2.210}$$

One possible way of contracting this is the following,

$$\left\langle 0\left|\hat{a}_{q_1}\hat{a}_{q_2}\hat{\psi}^\dagger(y)\hat{\psi}(y)\hat{\phi}(y)\hat{\psi}^\dagger(w)\hat{\psi}(w)\hat{\phi}(w)\hat{a}^\dagger_{p_2}\hat{a}^\dagger_{p_1}\right|0\right\rangle \tag{2.211}$$

this represent t- process. We have also u-channel, given by:

$$\left\langle 0\left|\hat{a}_{q_1}\hat{a}_{q_2}\hat{\psi}^\dagger(y)\hat{\psi}(y)\hat{\phi}(y)\hat{\psi}^\dagger(w)\hat{\psi}(w)\hat{\phi}(w)\hat{a}^\dagger_{p_2}\hat{a}^\dagger_{p_1}\right|0\right\rangle \tag{2.212}$$

as shown in Figure(2.20). Here initial and final particles have changed places

[50]
$$\begin{aligned}\hat{\psi}(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^{\frac{3}{2}}}\frac{1}{\left(2E_{\boldsymbol{p}}\right)^{\frac{1}{2}}}\left(\hat{a}_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}p\cdot x} + \hat{b}^\dagger_{\boldsymbol{p}}\mathrm{e}^{\mathrm{i}p\cdot x}\right)\\ \hat{\psi}^\dagger(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^{\frac{3}{2}}}\frac{1}{\left(2E_{\boldsymbol{p}}\right)^{\frac{1}{2}}}\left(\hat{a}^\dagger_{\boldsymbol{p}}\mathrm{e}^{\mathrm{i}p\cdot x} + \hat{b}_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}p\cdot x}\right)\\ \hat{\phi}(x) &= \int \frac{\mathrm{d}^3q}{(2\pi)^{\frac{3}{2}}}\frac{1}{\left(2\varepsilon_{\boldsymbol{q}}\right)^{\frac{1}{2}}}\left(\hat{c}_{\boldsymbol{q}}\mathrm{e}^{-\mathrm{i}q\cdot x} + \hat{c}^\dagger_{\boldsymbol{q}}\mathrm{e}^{\mathrm{i}q\cdot x}\right)\end{aligned}$$

where $E_{\boldsymbol{p}} = \left(\boldsymbol{p}^2 + m^2\right)^{\frac{1}{2}}$ and $\varepsilon_{\boldsymbol{q}} = \left(\boldsymbol{q}^2 + \mu^2\right)^{\frac{1}{2}}$ Here the $\hat{a}$-operators describe the creation and annihilation of psions, and the $\hat{b}$-operators describe the creation and annihilation of antipsions and the $\hat{c}$-operators

[51] First order will end up vanishing.

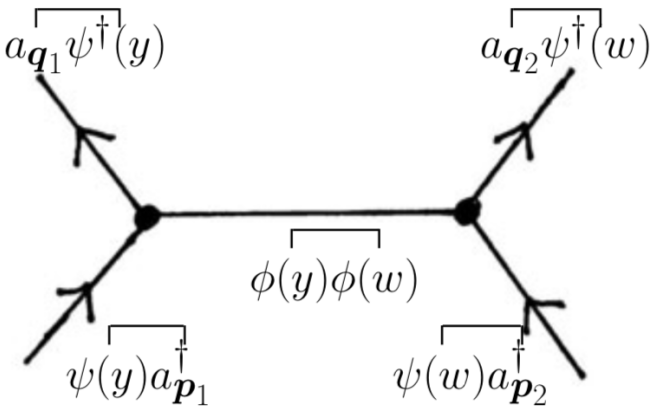


Figure 2.20: Representation of Feynman diagram with the considered Wick contractions [9]

after the interaction and thus both diagrams contribute to the process if the particles are indistinguishable.

As one can notice from the zeroth term in the expansion (2.194), even in interacting theory, there will exist a possibility that particles do not interact. Thus, $S-$matrix is often split into a trivial part and the rest, to isolate that contribution. We call the rest *transition matrix* T,

$$S_{fi} = \delta_{fi} + iT_{fi} \tag{2.213}$$

where

$$T_{fi} = (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum q_f \right) \mathcal{M} \left(p_1, p_2 \to \{q_f\} \right) \tag{2.214}$$

where p_1, p_2 and q_f are on mass-shell $4-$momenta. Term $\delta^{(4)} \left(p_1 + p_2 - \sum q_f \right)$ actually comes from diagram calculations when expressed in momentum space, making $S-$matrix conserve $4-$momenta. This is here factored out to make amplitude $\mathcal{M}$ Lorentz invariant. We're now ready to reveal a wonderful simplification. [52]. Thus,

$$T_{fi} = \sum \left(\begin{array}{c} \text{All connected, amputated Feynman} \\ \text{diagrams with incoming momentum } p_{\text{i}} \\ \text{and outgoing momentum } p_{\text{f}} \end{array} \right) \tag{2.215}$$

[52]Single Feynman diagram can be constituted of one or several connected pieces. If it contains only one connected piece, we call it a connected diagram. Otherwise, a diagram is disconnected. Disconnected diagrams correspond to physical processes which do not interact with each other, thus, only fully connected diagrams contribute to the T-matrix. There are also diagrams with loops attached to external legs. Such diagrams represent self-energy contributions and yield non-trivial infinities. Those are also excluded from the $T-$ matrix, and in fact, excluded from the whole $S-$ matrix through the procedure called amputation.

Invariant amplitude $\mathcal{M}$ is what relates directly to differential cross section - quantity measured in scattering experiments such as ones at LHC, through the formula,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\mathrm{CM}}^2} \tag{2.216}$$

2.4.3 Old fashioned perturbation theory

As one may notice, the time ordering operation was a crucial part of simplifying string operator correlators and calculating the propagators. Time ordering dropped out on the level of external legs, since there was no time ordering ambiguity, but stayed of crucial relevance in the propagators of Feynman diagrams. An alternative way of doing perturbation in QFT is to do it in a way of standard quantum mechanics, as it was done in calculations somewhere till $the1960$s. In fact, Julian Schwinger, the biggest rival of Feynman, never embraced Feynman diagrams and continued to use the approach which we will call here *old fashioned perturbation*. As it was eventually shown in the works of Dyson [26] and others, the old-fashioned perturbation yield equivalent results.
Here, we will look at independent diagrams with definite time ordering, each giving non-invariant amplitude $\mathcal{M}$. Summing over all possible orderings will give a corresponding Feynman propagator. This is consistent with interpreting Feynman propagator as superposition of two processes with a definite time arrow, into a total process where the ordering remains indefinite.

In quantum mechanics, we had a picture based on scattering in a potential where particles act as sources of fields that give rise to a potential in which other particles scatter. The transition rate Γ_{fi} between initial state i and final state f is given by Fermi's golden rule $\Gamma_{fi} = 2\pi \left|T_{fi}\right|^2 \rho\left(E_f\right)$, where T_{fi} is the transition matrix element, given by the perturbation expansion:

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} + \cdots \tag{2.217}$$

Here j denotes some intermediate state through which scattering occurs. With the description in terms of the potential, we have some issues. For example, when a particle scatters in potential there is a transfer of momentum from one particle to

another without any apparent mediating body. The other problem is that if a distant particle were moved suddenly, the potential due to that particle would change instantaneously at all points in space, in violation of the special theory of relativity. This is why in QFT we have interactions between particles modeled as being mediated by the exchange of other particles.

To analyze the interaction, split the Hamiltonian into a free and interacting parts,

$$H = H_0 + V \tag{2.218}$$

Here H_0 is the free part with eigenstates that are known exactly, and interaction part V gives corrections to those eigenstates. We will also denote two sets of states, $\{|\phi\rangle\}$ for free part and $\{|\psi\rangle\}$ for the full Hamiltonian, where we have $\lim_{V\to 0}|\psi\rangle = |\phi\rangle$. We will also assume that in asymptotic future and past of scattering event interaction is negligible $\lim_{t\to\pm\infty}|\psi\rangle = |\phi\rangle$. Suppose that the incident state has fixed energy E at some initial time,

$$H_0|\phi\rangle = E|\phi\rangle \tag{2.219}$$

In QFT the energies E are continuous, thus we should be able to find an eigenstate $|\psi\rangle$ of the full Hamiltonian with the same eigenvalue

$$H|\psi\rangle = E|\psi\rangle \tag{2.220}$$

From this we get,

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0}V|\psi\rangle \tag{2.221}$$

which is known as the Lippmann-Schwinger equation. The potential acts at intermediate times, inducing transitions among the free states $|\phi\rangle$ via Green function $G = \frac{1}{E-H_0}$. We would like to express $|\psi\rangle$ entirely in terms of $|\phi\rangle$. We define an operator T by

$$V|\psi\rangle = T|\phi\rangle \tag{2.222}$$

T is known as the transfer matrix. Inserting this back in (2.221) we get:

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0} T|\phi\rangle \tag{2.223}$$

which gives $|\psi\rangle$ in terms of $|\phi\rangle$. Multiplying by V and some state $\langle\phi_j|$,

$$\langle\phi_j|V|\psi\rangle \overset{(2.222)}{=} \langle\phi_j|T|\phi\rangle = \langle\phi_j|V|\phi\rangle + \left\langle \phi_j \left| V \frac{1}{E - H_0} T \right| \phi \right\rangle \tag{2.224}$$

Since it hold for any $|\phi_j\rangle$ and $|\phi_j\rangle$, an operator equation for T must hold

$$\Rightarrow T = V + V \frac{1}{E - H_0} T \tag{2.225}$$

We can then solve perturbatively in V to get

$$T = V + V \frac{1}{E - H_0} V + V \frac{1}{E - H_0} V \frac{1}{E - H_0} V + \cdots \tag{2.226}$$

If we now insert complete set $\sum_j |\phi_j\rangle \langle\phi_j|$ of eigenstates $|\phi_j\rangle$ of H_0, and write $T_{fi} = \langle\phi_f|T|\phi_i\rangle$ and $V_{ij} = \langle\phi_i|V|\phi_j\rangle$ we get:

$$T_{fi} = V_{fi} + \sum_n V_{fn} \frac{1}{E_i - E_n} V_{ni} + \cdots \tag{2.227}$$

where E_n is the energy of the intermediate state. Notice that we recreated the expression (2.217).

Let us now consider the particle interaction for some process $a + b \to c + d$, as depicted in Figure(2.21). The process can occur via an intermediate state corresponding to the exchange of a particle X but there is also another possible spacetime picture with the same external legs, the one via intermediate state corresponding to the exchange antiparticle $\tilde{X}$. Let us consider the first-order perturbation in the first spacetime picture. We have the initial state:

$$|i\rangle \equiv |a, b\rangle \tag{2.228}$$

the intermediate state

$$|j\rangle \equiv |c, b, X\rangle \tag{2.229}$$

and the final state

$$|f\rangle \equiv |c,d\rangle \tag{2.230}$$

Particle a can be thought of as emitting the exchanged particle X, and then at a later time X is absorbed by particle b. Let us now consider energies and momenta. For both time-ordered diagrams, the exchange particle will be on mass-shell; meaning that the energy of the exchanged particle will be related to its momentum by the usual energy-momentum relation, $E_X^2 = \mathbf{p}_X^2 + m_X^2$. From momentum conservation we have:

$$\mathbf{p}_X = (\mathbf{p}_a - \mathbf{p}_c) \tag{2.231}$$

for the first-time order process. In the case of the second time-ordered process we have:

$$\mathbf{p}_{\tilde{X}} = (\mathbf{p}_b - \mathbf{p}_d) = -(\mathbf{p}_a - \mathbf{p}_c) \tag{2.232}$$

Consequently, for both time-ordered diagrams, the energy of the exchanged particle can be written as

$$E_X^2 = \mathbf{p}_X^2 + m_X^2 = (\mathbf{p}_a - \mathbf{p}_c)^2 + m_X^2 \tag{2.233}$$

The corresponding term in the perturbation expansion is

$$\begin{aligned} T_{fi}^{ab} = \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} &= \frac{\langle c,d|V|c,b,X\rangle\langle c,b,X|V|a,b\rangle}{(E_a+E_b)-(E_c+E_X+E_b)} \\ &= \frac{\langle d|V|b,X\rangle\langle c,X|V|a\rangle}{(E_a+E_b)-(E_c+E_X+E_b)} \end{aligned} \tag{2.234}$$

Let us now notice that the energy is *not conserved* in the vertex since this would produce the divergences in the denominator. The standard interpretation of this is that the violation of conservation of energy for a short period of time can be justified via energy-time uncertainty relation $\Delta E \Delta t \geq \hbar/2$. [53] On the other hand, Feynman diagrams introduced the idea of off mass-shell particle $p^2 \neq m^2$, referred to as *virtual particle*, trading off 'mass-shellness' for the conservation of 4-momentum conservation in the vertex.

[53] Nevertheless, as we mentioned in the chapter (2.1.1), the character of time in energy-time uncertainty relation is disputable, since it does not refer to the standard deviation of some time operator. In this context time is taken to mean internal time, stating that the since the lifetime of an exchange particle is very short, the energy of a particle has great uncertainty.

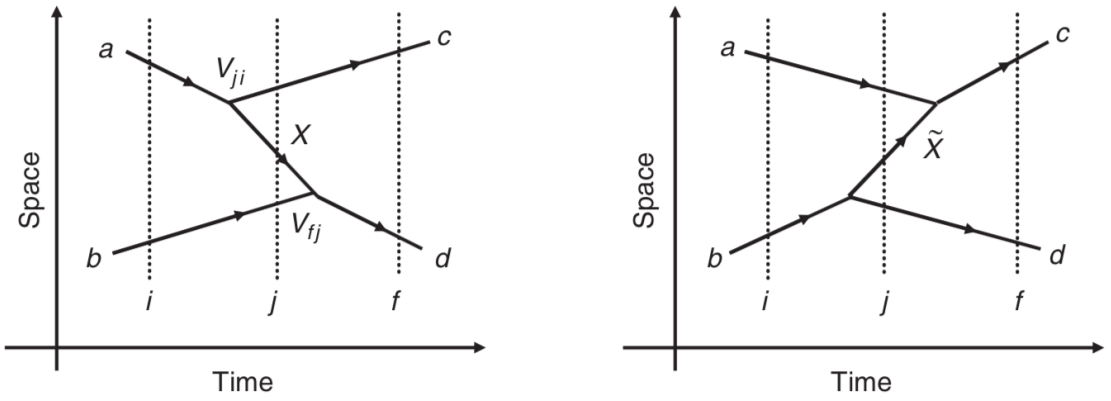


Figure 2.21: Two possible time-orderings for the process a + b → c + d [82]

The interactions at the two vertices are defined by the non-invariant matrix elements

$$V_{ji} = \langle c, X|V|a\rangle \quad \text{and} \quad V_{fj} = \langle d|V|X, b\rangle \tag{2.235}$$

The non-invariant matrix element V_{ji} is related to the Lorentz-invariant matrix element $\mathcal{M}_{ji}$ by

$$V_{ji} = \mathcal{M}_{ji} \prod_k (2E_k)^{-1/2} \tag{2.236}$$

where the index k runs over the energies of the particles involved. We have:

$$V_{ji} = \langle c, X|V|a\rangle = \frac{\mathcal{M}_{a\to c+X}}{(2E_a 2E_c 2E_X)^{1/2}} \tag{2.237}$$

where $\mathcal{M}_{a\to c+X}$ is the Lorentz invariant matrix element for the fundamental interaction $a \to c + X$. The matrix element $\mathcal{M}_{a\to c+X}$ will be given by the scalar, denoting strength of interaction $\mathcal{M}_{a\to c+X} \equiv g_a$, and thus

$$V_{ji} = \langle c, X|V|a\rangle = \frac{g_a}{(2E_a 2E_c 2E_X)^{1/2}} \tag{2.238}$$

g_a is constant denoting the magnitude of the coupling and it represents a measure of the strength of the scalar interaction. Similarly

$$V_{fj} = \langle d|V|X, b\rangle = \frac{g_b}{(2E_b 2E_d 2E_X)^{1/2}} \tag{2.239}$$

where g_b is the coupling strength at the $b + X \to d$ interaction vertex. Therefore,

the first-order term in the perturbation series gives

$$\begin{aligned} T_{fi}^{ab} &= \frac{\langle d|V|X,b\rangle\langle c,X|V|a\rangle}{(E_a+E_b)-(E_c+E_X+E_b)} \\ &= \frac{1}{2E_X}\cdot\frac{1}{(2E_a2E_b2E_c2E_d)^{1/2}}\cdot\frac{g_ag_b}{(E_a-E_c-E_X)} \end{aligned} \tag{2.240}$$

The Lorentz invariant matrix element for the process $a+b\to c+d$ is related to the corresponding transition matrix element by,

$$\mathcal{M}_{fi}^{ab} = (2E_a2E_b2E_c2E_d)^{1/2}\,T_{fi}^{ab} \tag{2.241}$$

Thus we get:

$$\mathcal{M}_{fi}^{ab} = \frac{1}{2E_X}\cdot\frac{g_ag_b}{(E_a-E_c-E_X)} \tag{2.242}$$

For the second possible time-ordering for the process $a+b\to c+d$ we have that the state b emitting $\tilde{X}$ which is subsequently absorbed by a. The exchanged particle is in this case $\tilde{X}$. It is assumed that $\tilde{X}$ has the same mass as X but has the opposite charge. This must be the case if the charge is to be conserved at each vertex. For the second time-ordering process we get:

$$\mathcal{M}_{fi}^{ba} = \frac{1}{2E_X}\cdot\frac{g_ag_b}{(E_b-E_d-E_X)} \tag{2.243}$$

Since both processes are possible, we need to sum over both to get the total amplitude. The total amplitude (in the lowest order) is given by:

$$\begin{aligned} \mathcal{M}_{fi} &= \mathcal{M}_{fi}^{ab}+\mathcal{M}_{fi}^{ba} \\ &= \frac{g_ag_b}{2E_X}\cdot\left(\frac{1}{E_a-E_c-E_X}+\frac{1}{E_b-E_d-E_X}\right) \end{aligned} \tag{2.244}$$

which, using energy conservation $E_b-E_d=E_c-E_a$, can be written

$$\begin{aligned} \mathcal{M}_{fi} &= \frac{g_ag_b}{2E_X}\cdot\left(\frac{1}{E_a-E_c-E_X}-\frac{1}{E_a-E_c+E_X}\right) \\ &= \frac{g_ag_b}{(E_a-E_c)^2-E_X^2} \end{aligned} \tag{2.245}$$

Substituting this expression for E_X^2 (2.233) leads to

$$\begin{aligned}\mathcal{M}_{fi} &= \frac{g_a g_b}{(E_a - E_c)^2 - (\mathbf{p}_a - \mathbf{p}_c)^2 - m_X^2} \\ &= \frac{g_a g_b}{(p_a - p_c)^2 - m_X^2}\end{aligned} \tag{2.246}$$

where p_a and p_c are the respective four-momenta of particles a and c. Finally writing the four-momentum of the exchanged virtual particle X as

$$q = p_a - p_c \tag{2.247}$$

gives

$$\mathcal{M}_{fi} = \frac{g_a g_b}{q^2 - m_X^2} \tag{2.248}$$

We see that the sum over the two possible time-ordered diagrams in second-order perturbation theory has produced an expression for the interaction matrix element that depends on the four-vector scalar product q^2 and is, therefore, *manifestly Lorentz invariant*. The term

$$\frac{1}{q^2 - m_X^2} \tag{2.249}$$

is referred to as the *propagator*, and is associated with the exchanged particle. This propagator corresponds to the Feynman propagator. In the case of particular time-ordering, we have propagation via an advanced or retarded propagator (which diagram is advanced or retarded depends on what we call the source), which after summing up yields the Feynman propagator. Modern QFT perturbation theory is thus a theory with respect to Feynman diagrams, working in terms of virtual particles (off-shell particles) and yielding manifestly Lorentz invariant results. On the other hand, old-fashioned perturbation theory considers diagrams with respect to some fixed time ordering, where all states are *physical*, in a sense that they are on mass-shell at all times. This means that the 3-momentum is conserved at each vertex, but the energy is not (which is justified via time-energy uncertainty).

3 Ancilla approach

In the following, we will present the original results of the thesis where we will analyze a time ordering operator as it appears in the time-ordered exponential formula. We will focus on its expansion in the interaction picture, up to the second-order term. As explained in detail in Section (2.4), the time ordering operator has a sum along all possible ordering configurations, where each term in the sum will, in the following notes, we call a 'branch' of the time ordering operator. Motivated by the discussion in Section (2.2), we will think of the system as being in a superposition of going through different possible branches of time configurations. The main question of the work is now whether one can isolate a branch of time ordering by means of coupling a system to an ancilla and performing an indirect measurement on a system, as introduced in (A). The idea is that the ancilla interaction could couple to a particular time ordering configuration, such that the measurement on the ancilla would collapse the superposition of configurations, isolating a particular ordering.

In the interaction picture, time ordered exponential reads,

$$U(t,t_0) = Te^{-i\int_{t_0}^{t} \mathrm{d}t' H_I(t')} \tag{3.1}$$

where T denotes time ordering operator. Upon expanding:

$$U\left(t,t_0\right) = 1 + (-i)\int_{t_0}^{t} dt_1 H_I\left(t_1\right) + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 dt_2 T\left(H_I\left(t_1\right) H_I\left(t_2\right)\right) + \cdots \tag{3.2}$$

We will focus on the second order term, where time ordering begins to be relevant;

$$\frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \left(H_I\left(t_1\right) H_I\left(t_2\right)\theta\left(t_1 - t_2\right) + H_I\left(t_2\right) H_I\left(t_1\right)\theta\left(t_2 - t_1\right)\right) \tag{3.3}$$

Notice that the integral contributions from the two different time orderings are the same, making them indistinguishable.

Let us now try to project on $t_1 > t_2$ or $t_1 < t_2$ branch of evolution; to do so, let us consider a system $|\psi(t)\rangle$ coupled with the continuous spectrum ancillary system

$|a(t)\rangle$:

$$|\Psi(t)\rangle = |\psi(t)\rangle|a(t)\rangle \tag{3.4}$$

We now have $H_I(t) \to V(t) \otimes A(t) : \mathcal{H}_S(t) \otimes \mathcal{H}_A(t) \longrightarrow \mathcal{H}_S(t) \otimes \mathcal{H}_A(t)$. We define the ancillary system to initially be in the state:

$$|a(t_0)\rangle = |a_0\rangle \tag{3.5}$$

and define the eigenstates of the ancilla potential operator $\hat{A}(t)$ to satisfy:

$$\langle c\,(t') \mid c(t)\rangle \equiv \langle c' \mid c\rangle = \delta\,(c' - c) \tag{3.6}$$

where we switched to notation $\{c(t_i)\} \equiv \{c_i\}$. Let us take $\{|c\rangle : c \in X\}$ and take ancilla $\hat{A}(t)$ to be general ancilla interaction. Expanded via spectral theorem,

$$\hat{A}(t) = \int_X c|c\rangle\langle c|dc \tag{3.7}$$

Let us consider now the evolution of the total state

$$|\Psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle|a_0\rangle \tag{3.8}$$

Upon inserting the expansion (3.2),

$$|\Psi(t)\rangle = \left(1 - i\int_{t_0}^{t} dt_1 V(t_1) \otimes A(t_1) + \frac{(-i)^2}{2!}\int_\gamma dt_1 dt_2 T(V(t_1) \otimes A(t_1)V(t_2) \otimes A(t_2) + \cdots\right)|\psi(t_0)\rangle|a_0\rangle \tag{3.9}$$

Inserting the interaction yields

$$\begin{aligned}|\Psi(t)\rangle = \Big(1 - i\int_{t_0}^{t} dt_1 \int_{X_1} dc_1 V(t_1) \otimes c_1|c_1\rangle_A\langle c_1| \\ + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_X dc_1 \int_X dc_2 T(V(t_1) \otimes c_1|c_1\rangle_A\langle c_1|V(t_2) \otimes c_2|c_2\rangle_A\langle c_2|)) + \cdots\Big)|\psi(t_0)\rangle|a_0\rangle\end{aligned} \tag{3.10}$$

Let us now project on the ancillary system by measurement over some finite in-

terval. As mentioned above, we will focus on the second-order term:

$$
\begin{aligned}
&\int_c |c\rangle\langle c|\Psi(z)\rangle dc = \\
&= \Big(\cdots + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_c dc \int_{c_1} dc_1 \int_{c_2} dc_2 \theta(t_1 - t_2)(V(t_1) \otimes c_1\langle c|c_1\rangle\langle c_1|V(t_2) \otimes c_2|c_2\rangle\langle c_2|) \\
&+ \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_c dc \int_{c_1} dc_1 \int_{c_2} dc_2 \theta(t_2 - t_1)(V(t_2) \otimes c_2\langle c|c_2\rangle\langle c_2|V(t_1) \otimes c_1|c_1\rangle\langle c_1|)\Big)|\psi(t_0)\rangle|a_0\rangle|c\rangle \\
&= \Big(\cdots + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_c dc \int_{c_1} dc_1 \int_{c_2} dc_2 \theta(t_1 - t_2)(V(t_1) \otimes c_1\langle c|c_1\rangle\langle c_1|V(t_2) \otimes c_2|c_2\rangle\langle c_2|a_0\rangle \\
&+ \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_c dc \int_{c_1} dc_1 \int_{c_2} dc_2 \theta(t_2 - t_1)(V(t_2) \otimes c_2\langle c|c_2\rangle\langle c_2|V(t_1) \otimes c_1|c_1\rangle\langle c_1|a_0\rangle\Big)|\psi(t_0)\rangle|c\rangle \\
&= \Big(\cdots + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_c dc \int_{c_1} dc_1 \int_{c_2} dc_2 \theta(t_1 - t_2)V(t_1)V(t_2)c_1c_2\delta(c - c_1)\delta(c_1 - c_2)\delta(c_2 - a_0) \\
&+ \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_c dc \int_{c_1} dc_1 \int_{c_2} dc_2 \theta(t_2 - t_1)V(t_2)V(t_1)c_1c_2\delta(c - c_2)\delta(c_2 - c_1)\delta(c_1 - a_0)\Big)|\psi(t_0)\rangle|c\rangle
\end{aligned}
\tag{3.11}
$$

To yield a non-vanishing result, delta functions obligate us to satisfy the condition:

$$c = c_1 = c_2 = a_0 \tag{3.12}$$

making at the same time both contributions non-vanishing. If c_1 and c_2 appearing in one branch would not be exactly c_1 and c_2 in the other branch (as will be discussed bellow) we would be able to distinguish by demanding appropriate conditions. Let

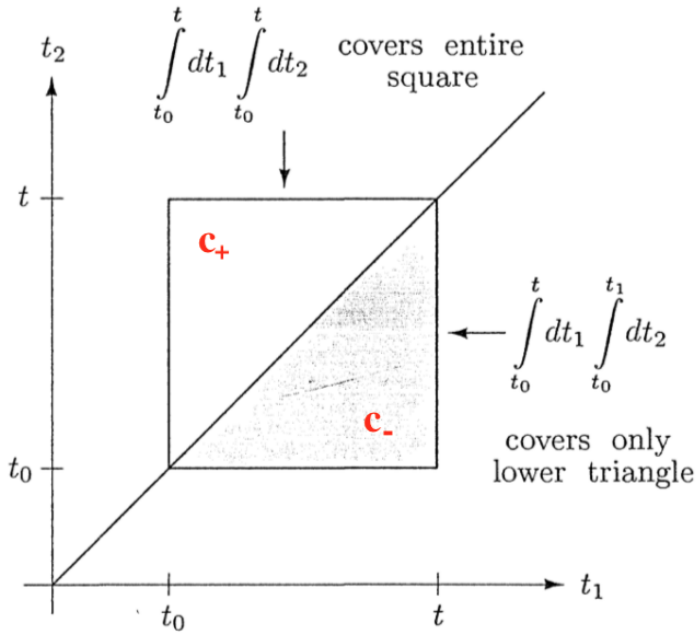


Figure 3.1: Geometrical representation of 2nd order Dyson expansion term. In order to couple to a particular time ordering, ancilla potential needs to depend on the function in $2D$ parametric space.

us, therefore, suppose that coefficients c could acquire different values if associated with a different time ordering configuration. As an example, consider Figure(3.1) where we see how the values of c coupled to a particular time ordering configuration, making the ancilla potential couple differently to two-time orderings. Namely, we would now have

$$\begin{array}{ccc} t & \longrightarrow & t \otimes v \\ \downarrow & & \downarrow \\ V(t) & \longrightarrow & V(t) \otimes A(v) \end{array}$$

where $v \coloneqq v(t_1, t_2)$ would be some hypotetical function in $2D$ parameter space of 2-nd order term. Here one should not that this is already problematic and cannot make sense on the level of (3.1). Nevertheless, for the sake of example take:

$$v(t_1, t_2) = t_1 - t_2 \tag{3.13}$$

and

$$\begin{aligned} \hat{A}(t_1, t_2) &= \hat{A}(v > 0)\theta(t_1 - t_2) + \hat{A}(v < 0)(t_2 - t_1)\theta(t_2 - t_1) \\ &= \int_{X_+} c_+|c_+\rangle\langle c_+|dc_+\theta(v > 0) + \int_{X_-} c_-|c_-\rangle\langle c_-|dc_-\theta(v < 0) \end{aligned} \tag{3.14}$$

We would then get:

$$\begin{aligned} &\int_c |c\rangle\langle c|\Psi(z)\rangle dc \\ &= \int_c |c\rangle\langle c|dc\big(... + \frac{(-i)^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 \hat{V}(t_1)\hat{V}(t_2) \otimes \hat{A}\left(v(t_1, t_2)\right)\hat{A}\left(v(t_1, t_2)\right)\theta\left(t_1 - t_2\right) \\ &\quad + \frac{(-i)^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 \hat{V}(t_2)\hat{V}(t_1) \otimes \hat{A}\left(v(t_1, t_2)\right)A\left(v(t_1, t_2)\right)\theta\left(t_2 - t_1\right) + \cdots\big)|\psi(t_0)\rangle|a_0\rangle \\ &= \int_c |c\rangle\langle c|dc\big(... + \frac{(-i)^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 \hat{V}(t_1)\hat{V}(t_2) \otimes \hat{A}(v > 0)\hat{A}(v > 0)\theta(v) \\ &\quad + \frac{(-i)^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 \hat{V}(t_2)\hat{V}(t_1) \otimes \hat{A}(v < 0)\hat{A}(v < 0)\theta(-v) + \cdots\big)|\psi(t_0)\rangle|a_0\rangle \\ &\quad = \Big(... + \frac{(-i)^2}{2!}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 \int_c dc \int_{c_1^-} dc_1^- \int_{c_2^-} dc_2^- dz_1 dz_2 \theta(t_1 - t_2)\cdot \\ &\qquad\qquad \cdot (V(t_1)V(t_2))c_1^- c_2^- \delta(c - c_1^-)\delta(c_1^- - c_2^-)\delta(c_2^- - a_0) \\ &\quad + \frac{(-i)^2}{2!}\int_{\gamma_+} dz_1 dz_2 \int_c dc \int_{c_1^+} dc_1^+ \int_{c_2^+} dc_2^+ \theta(t_2 - t_1)\cdot \\ &\qquad\qquad \cdot (V(t_1)V(t_2))c_1^+ c_2^+ \delta(c - c_1^+)\delta(c_1^+ - c_2^+)\delta(c_2^+ - a_0)\ldots\Big)|\psi(t_0)\rangle|v\rangle \end{aligned} \tag{3.15}$$

Now we could impose some initial and final conditions, such that:

$$c = c_1^+ = c_2^+ = a_0 \tag{3.16}$$

to project on an upper or

$$c = c_1^- = c_2^- = a_0 \tag{3.17}$$

to project on the lower branch. However, as already mentioned, such construction wouldn't make sense since it would not be well defined in all orders, for $v(t_1, t_2)$ is a function defined in the parametric space of 2-nd order term. In conclusion, this kind of dependence cannot exist on the level of equation (3.1) and we are left with the correct construction (3.11) which cannot distinguish two contributions. One may attribute this to the fact that there is no real indefiniteness in the evolution since external time always flows in the same direction from t_0 to t. As explained in the section (2.4), time-ordered exponential is just a product of infinitesimal deterministic steps, as given by the formula:

$$\hat{U}(t_0, t) = \prod_i^f e^{-i\delta t_a \hat{H}(t_a)} \tag{3.18}$$

In order to introduce indefiniteness in the dynamic, one would expect to introduce indefiniteness on the level of each infinitesimal time step, as discussed in [33].

4 Page-Wotters approach: expansion of time ordered exponentials using timeless state formalism

One of the attempts to reconcile the differences between how gravitational versus quantum physics addresses time, is to overcome the notion of time as an external parameter in quantum mechanics. Serving as a parameter of time translations, time as an external parameter appears in Schrödinger's equation

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = \hat{H}|\psi(t)\rangle \tag{4.1}$$

Operationally one can interpret it as the 'time measured by the classical, laboratory clock'. As such, it is not observable in the quantum mechanical sense; it stands outside the realm of the theory. The external character of time is at odds with how time is implemented in the theory of general relativity. Namely, as mentioned in the Introduction (1), diffeomorphism symmetry Diff(M) of general relativity induces background independence (see e.g. [69]), succumbing the spacetime to the dynamical law presented within. In such a theory, space and time in their unified fabric of spacetime are treated as internal variables. Coming as an unease in the attempts to reconcile quantum theory into a unified framework with the theory of general relativity, the disparate nature of time in the two represents a seed for the cluster of problems in literature known as 'The problem of time'.

In order to make time internal to the quantum theory, we will promote a clock to a quantum degree of freedom. Namely, we will consider some quantum system, assigned with the Hilbert space $\mathcal{H}_T$ on which time operator $\hat{T}$ acts, serving as a temporal quantum reference frame[54] i.e. physical (quantum) clock. Considering now the system coupled to the physical clock will enable us to describe the change in time via conditional probabilities, replacing 'being at a time' by timeless correlations. That is to say, the temporal behavior we observe, depends on relations between some internal clock and the system, instead of on an external coordinate time parameter. Let us now look more thoroughly this formalism to which we will refer as *Page-Wootters formalism* (sometimes known also as *Timeless approach to quantum mechanics*).

[54] As we saw in Chapter(2.1.3), using quantum degree of freedom as a reference frame will promote parameter of translation into an operator.

4.1 Page-Wootters formalizam

In the timeless formulation, one considers a global, timeless state $|\Psi\rangle\!\rangle$ called *history state*, which can be viewed as the total system composed of a clock and the system of interest. Since any isolated system is in an eigenstate of energy and thus stationary with respect to the coordinate time, the timeless state is usually considered to be the state of the whole Universe (as this is the smallest truly isolated system). This state is famously present in Wheeler–DeWitt equation,

$$\hat{H}|\Psi\rangle\!\rangle = 0 \tag{4.2}$$

appearing within Dirac quantization approach to quantum gravity (canonical quantum gravity). As with any constraint equation, (4.2) reduces the kinematical space of states (all possible states) to physical state space (states satisfying the constraint). If we now consider the dynamics of the system given by Schrödinger equation (4.1), Wheeler–DeWitt equation (4.2) implies

$$\Rightarrow \quad i\hbar\frac{d|\Psi\rangle\!\rangle}{dt} = 0 \tag{4.3}$$

That is to say, our state $|\Psi\rangle\!\rangle$ is deprived of time evolution. This kind of situation appears generally within time relationalism and is known as the *Frozen formalism problem*.

The question we now face is, how do we yield everyday dynamics from this frozen global state $|\Psi\rangle\!\rangle$? One of the observations we can make is that the notion of time is not well defined at the level of such a global state. The nature of time itself is not fully clear to us, yet time is an undeniable fact of our everyday experience and these experiences always concern *subsystems* rather than the Universe as a whole [55]. That is to say, *No observation is possible without an observer.* [25][56], hence shifting a

[55] This is at the heart of temporal relationalism for which it is characteristic to be manifestly reparametrization invariant (as it is the case with general relativity) and to get constraint equations forcing the frozen formalism

[56] In fact, Crane even goes further and states: *'Hence there is no Hilbert space associated with a closed universe.'* but we shall sustain from going into his formulation.

perspective to what one observes, implies shifting a perspective to one of a subsystem. Along these lines, in 1982. Page and Wootters formulated conditional probabilities approach to timeless quantum mechanics. Namely, they defined a history state $|\Psi\rangle\!\rangle$ to be the correlated state of the system and the clock:

$$|\Psi\rangle\!\rangle = \int dt |t\rangle_C \otimes |\psi(t)\rangle_S \tag{4.4}$$

such that it satisfies the equation (4.2). It encodes the full dynamics of the system represented in terms of some internal observer measuring time evolution $|\psi(t)\rangle$, with respect to some subsystem serving as a clock. We will define *a perfect clock* as a subsystem associated with an infinite-dimensional Hilbert space $\mathcal{H}_C$, isomorphic to the Hilbert space of a particle on a line, having canonical coordinates $\hat{T}_C$ and $\hat{H}_C$, satisfying the Heisenberg algebra $[\hat{T}_C, \hat{H}_C] = i$. We define a dynamical state of the internal observer to be the state one gets upon conditioning the static, global state by projecting on the eigenvector of the clock:

$$|\psi(t)\rangle = {}_C\langle t|\Psi\rangle \ . \tag{4.5}$$

This state is called 'reduced state' and corresponds to the state of the system S when clock C shows time t. Comparing with Einstein's definition of time: *"The "time" of an event is that which is given simultaneously with the event by a stationary clock located at the place of the event'*[30], we further assume that the clock is stationary with respect to the observer.

Here we introduce $|t\rangle_C$ eigenstates of the time operator $\hat{T}_C$,

$$\hat{T}_C|t\rangle_C = t|t\rangle_C \tag{4.6}$$

and interpret the eigenvalue t as the outcome of a measurement on the clock showing time t. As mentioned in Chapter (2.1.1), we should keep in mind that this kind of perfect clock is not physical since the commutator

$$[\hat{T}, \hat{H}] = i \tag{4.7}$$

cannot hold for a physically realistic Hamiltonian, as discussed in 2.1.1. Nevertheless, such a model can serve as a working idealization.

As mentioned above, history state contains all the information about the correlations between the system and the clock and it belongs to the *physical* Hilbert space. Generally, it satisfies, some constraint equation:

$$\hat{C}|\Psi\rangle\rangle = 0 \tag{4.8}$$

Starting from an element of kinematical Hilbert space $|\Phi\rangle$, we can restrict the solutions to the physical Hilbert space using the equation (see [60]):

$$|\Psi\rangle\rangle = \int d\alpha e^{-i\alpha\hat{C}}|\Phi\rangle \tag{4.9}$$

In jargon, we sometimes say that $|\Phi\rangle$ is 'projected' onto the physical Hilbert space. Despite this, one should note that kinematical $\mathcal{H}_{kin}$ and physical Hilbert space $\mathcal{H}_{phy}$ do not have the same inner product, and thus $\mathcal{H}_{phy} \not\subset \mathcal{H}_{kin}$. (4.10) is therefore not formally a projection. For our model, we have the following constraint equation:

$$\hat{C} = \hat{H}_C \otimes \mathbb{1}_S + \mathbb{1}_C \otimes \hat{H}_S \tag{4.10}$$

such that the reduced state satisfies the usual Scrödinger equation upon conditioning equation (5.18) on the state of the clock in time t:

$$\begin{aligned} {}_C\langle t|\hat{C}|\Psi\rangle\rangle = {}_C\langle t|(\hat{H}_C \otimes \mathbb{1}_S + \mathbb{1}_C \otimes \hat{H}_S)|\Psi\rangle\rangle = 0 \\ \Rightarrow \left(i\frac{\partial}{\partial t} - \hat{H}_S\right)|\psi(t)\rangle = 0 \end{aligned} \tag{4.11}$$

4.1.1 Multiple time measurements

Performing sequential measurements on the system was not adequately defined in the context of Page-Wotter's original proposal. Following the lines of Kuchař's criticism [56], we can see that such formulation leads to the wrong expression for propagators. To see this, we ask for the probability of the state $|\psi(t)\rangle$ to be in the state $|q^*\rangle$ at time $t = t^*$, we would look for the conditional probability $P\left(q^*|t^*\right)$. Let

$$\hat{\Pi}_{t^*} := |t^*\rangle\langle t^*| \tag{4.12}$$

on $\mathcal{H}_T$ and the projection operator

$$\hat{\Pi}_{q^*} := |q^*\rangle \langle q^*| \tag{4.13}$$

on $\mathcal{H}_S$. The conditional probability is then:

$$P(q^*|t^*) = P(q^* \text{ when } t^*) = \frac{\langle\langle\Psi|(|t^*\rangle\langle t^*| \otimes |q^*\rangle\langle q^*|)|\Psi\rangle\rangle}{\langle\langle\Psi|(|t^*\rangle\langle t^*| \otimes \mathbb{1})|\Psi\rangle\rangle} = |\langle q^*|\psi_S(t^*)\rangle|^2 \tag{4.14}$$

So far, (4.14) is consistent with quantum theory. However, if one now asks for the probability of finding a particle in state q_1 at time t_1, given that the particle was in the state q_2 at time t_2, we have:

$$\begin{aligned} P(q_2 \text{ when } t_2 \mid q_1 \text{ when } t_1) &= \frac{\langle\langle\Psi\,|\Pi(t_1,q_1)\,\Pi(t_2,q_2)\,\Pi(t_1,q_1)|\,\Psi\rangle\rangle}{\langle\langle\Psi\,|\Pi(t_1,q_1)|\,\Psi\rangle\rangle} \\ &= |\delta(t_2-t_1)|^2\,|\langle q_1 \mid q_2\rangle|^2 \neq |\langle q_2\,|U_S(t_2,t_1)|\,q_1\rangle|^2 \end{aligned} \tag{4.15}$$

Equation (4.15) seems to be radically wrong, as it gives non-vanishing probability only if $t_1 = t_2$, implying the frozen formalism.

To resolve this criticism, one can argue that it is operationally more meaningful to compute the probability of finding a particle at q_2, given that the time is t_2 and there is a *record* of a particle being at q_1 at time t_1. In other words, we will introduce an ancillary system serving as a memory. This was worked out in [37] but the idea that the resolution lies along these lines, was already argued by Page himself, *'Here I simply wish to argue that all of the testable predictions of ordinary quantum mechanics appear to arise from one-time conditional probabilities. [...] We can never directly test what happened yesterday, but we can check the consequences that a hypothetical scenario for yesterday has on the situation today.'* [65]

In other words, after dividing the space into $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_R$, where C denotes clock and R denotes 'rest', we further divide subsystem R into the system of interest S, and a set of ancillary systems a_i serving as a measurement device that records the information about the i-th measurement. By doing so we are purifying the measurement: modeling the measurement on S by explicitly including an interaction Hamiltonian between S and the ancillary systems a_i in the constraint $\hat{C}$. For example, in the case

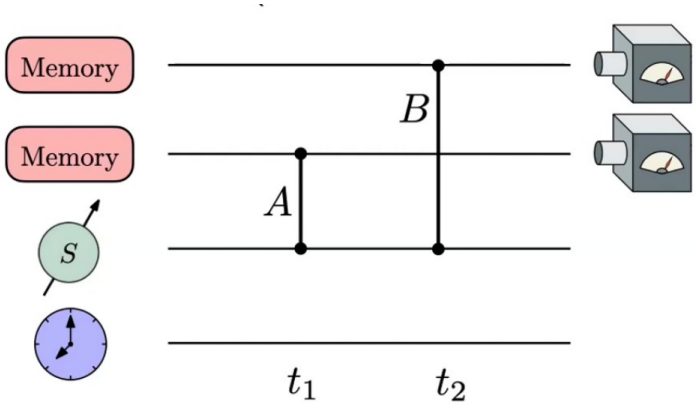


Figure 4.1: For each measurement, we need to introduce one ancillary system serving as a memory [50]

of two time measurements we would have:

$$\left(\hat{H}_S + \delta\left(\hat{T}_{\mathrm{A}} - t_1\right)\hat{H}_{Sa_1} + \delta\left(\hat{T}_{\mathrm{A}} - t_2\right)\hat{H}_{Sa_2}\right)|\Psi\rangle\!\rangle = 0 \tag{4.16}$$

where we introduced $\hat{H}_{Sa_i}$, for $i = 1, 2$, an operator acting on $\mathcal{H}_S \otimes \mathcal{H}_{a_i}$, $\mathcal{H}_{a_i}$ being the Hilbert space associated to the i-th ancilla. We used Dirac delta distribution to model interactions as sharply localized. Now the history state, besides encoding correlations with the clock states, describes all the measurement interactions between the system S and the ancillas a_i.

To see how one now obtains the history state of the form, consider von Neumann's formulation of a measurement apparatus describing the process of measurement in terms of pointer observable $\hat{O}_{\mathcal{A}}$, corresponding to the apparatus $\mathcal{A}$ and observable $\hat{O}_S$, corresponding of a system of interest S. The general state of the pointer is $|\mathcal{A}\rangle = \sum_a c_a |a\rangle_M$, where we labeled $\{|a\rangle_M\}$. Initially, pointer will be in some 'ready' state $|\mathcal{A}(0)\rangle = |r\rangle_M$. Since we assume no free dynamics for the pointer (or memory), the change in its state will be only due to the measurement. The measurement is described as an instantaneous transformation which at time t^* induces entangling unitary mapping:

$$|\psi(t^*)\rangle_S \otimes |r\rangle_M \xrightarrow{\hat{V}_{SM}} \sum_a \hat{K}_a |\psi(t^*)\rangle_S \otimes |a\rangle_M \tag{4.17}$$

where $\left\{\hat{K}_a\right\}$ are Kraus operators satisfying the normalization condition $\sum_a \hat{K}_a^\dagger \hat{K}_a = \hat{\mathbb{1}}$. In the case of 1-dimensional $\hat{K}_a = |a\rangle\langle a|$ corresponding to the eigenspace of $\hat{O}_{\mathcal{A}}$.

In this case,

$$|\psi\left(t^*\right)\rangle_S \otimes |r\rangle_M \rightarrow \sum_a \psi_a\left(t^*\right)|a\rangle_S \otimes |a\rangle_M, \tag{4.18}$$

were $\psi_a\left(t^*\right) := \langle a \mid \psi\left(t^*\right)\rangle$. Accordingly, the probability of getting the outcome a is given by

$$P\left(a \mid t^*\right) := |\hat{K}_a\,|\psi\left(t^*\right)\rangle_S\,|^2 \tag{4.19}$$

and the update rule,

$$|\phi_a\rangle_Q := \frac{\hat{K}_a\,|\psi\left(t^*\right)\rangle_Q}{\sqrt{P\left(a \mid t^*\right)}} \tag{4.20}$$

describing the state of the system immediately after memory has recorded an event. For the one time measurement, we have the form,

$$\hat{H}_{\tilde{S}}(t) = \hat{H}_S(t) + \delta\left(t - t^*\right)\hat{H}_{SM} \tag{4.21}$$

where the total evolution would take the form,

$$\hat{U}_{\tilde{S}}\left(t, t_0\right) = \begin{cases} \hat{U}_S\left(t, t_0\right) & \forall t < t^* \\ \hat{U}_S\left(t, t^*\right)\hat{V}_{SM}\hat{U}_S\left(t^*, t_0\right) & \forall t > t^* \end{cases} \tag{4.22}$$

and the history state would accordingly become

$$\begin{aligned} |\Psi\rangle\rangle = &\int_{-\infty}^{t^*} dt|t\rangle_T \otimes |\psi(t)\rangle_S \otimes |r\rangle_M \\ &+ \int_{t^*}^{\infty} dt|t\rangle_T \otimes \sum_a \hat{U}_S\left(t, t^*\right)\hat{K}_a\,|\psi\left(t^*\right)\rangle_S \otimes |a\rangle_M, \end{aligned} \tag{4.23}$$

(For alternative resolution of the criticism, see [51].) In case of two time measurement, we would then similarly have,

$$\hat{U}_{\tilde{S}}\left(t, t_0\right) = \begin{cases} \hat{U}_S\left(t, t_0\right) & \forall t < t_1 \\ \hat{U}_S\left(t, t_1\right)\hat{V}_{SM}\hat{U}_S\left(t_1, t_0\right) & \forall t : t_2 > t > t_1 \\ \hat{U}_S\left(t, t_2\right)\hat{V}_{SM'}\hat{U}_S\left(t_2, t_1\right)\hat{V}_{SM}\hat{U}_S\left(t_1, t_0\right) & \forall t > t_2 \end{cases} \tag{4.24}$$

where the history state is now of the form

$$|\Psi\rangle\rangle = \int_{-\infty}^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^{\infty} dt|t\rangle_T \\ \otimes \sum_{a_1 a_2} \hat{U}_S(t,t_2)\,\hat{K}_{a_2}\hat{U}_S(t_2,t_1)\,\hat{K}_{a_1}\,|\psi(t_1)\rangle_S \otimes |a_1\rangle_{M1} \otimes |a_2\rangle_{M_2}, \tag{4.25}$$

Suppose that we are now looking for the probability of obtaining a result a_1, corresponding to the measurement performed at t_1, and a result a_2, corresponding to the measurement performed at t_2, given that clock shows time $t > t_2$. The probabilities for the results of measurements are obtained by projecting the history state only once:

$$P(a_1, a_2|t > t_2) = \langle t| \otimes \langle a_1| \otimes \langle a_2| \cdot |\Psi\rangle\rangle|^2 \tag{4.26}$$

From this we would get:

$$P((a_2|t_2)|(a_1|t_1)) = \frac{|\langle t_2| \otimes \langle a_1| \otimes \langle a_2| \cdot |\Psi\rangle\rangle|^2}{|\langle t_1| \otimes \langle a_1| \cdot |\Psi\rangle\rangle|^2} = |\langle a_2|U(t_2,t_1)|a_1\rangle|^2 \tag{4.27}$$

Which gives the correct two time probabilities.

4.1.2 Extension of PaW to several clocks

Let us consider an extension of Page-Wotters formalism extended to include several clocks [17], as this will be of importance when considering our results. We will assume that clocks are non-interacting. We will define as an event any operation on S and use clocks to label the temporal localization of the events. The constraint equation (4.11) now reads:

$$\hat{\mathrm{C}} = \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{B}} + \hat{f}_{\mathrm{A}}\left(\hat{T}_{\mathrm{A}}\right) + \hat{f}_{\mathrm{B}}\left(\hat{T}_{\mathrm{B}}\right) \tag{4.28}$$

while the history state gets promoted to,

$$|\Psi\rangle = \int \mathrm{d}\alpha e^{-i\alpha\left(\hat{H}_{\mathrm{A}}+\hat{H}_{\mathrm{B}}\right)} \mathrm{T} e^{-i\int_0^\alpha \mathrm{d}s\left(\hat{f}_{\mathrm{A}}\left(s+\hat{T}_{\mathrm{A}}\right)+\hat{f}_{\mathrm{B}}\left(s+\hat{T}_{\mathrm{B}}\right)\right)}|\varphi\rangle \tag{4.29}$$

where the procedure for obtaining this will be explained in more details in the section (4.2). T appearing in the (4.29) is time ordering operator, introduced in Chapter (2.4). By casting the history state with respect to different clock reference frames,

we can show that each clock will describe the state of all the other subsystems as evolving unitarily:

$$|\Psi\rangle\!\rangle = \int dt |t\rangle_A \otimes \hat{U}_{\bar{A}}(t_A)|\psi_A(0)\rangle_{\bar{A}} \tag{4.30}$$

where clock is marked with A and we denoted with bar all the subsystems other than clock. To show this, let us write the state of kinematical Hilbert space $|\varphi\rangle$ as

$$|\varphi\rangle = \int \mathrm{d}t'_{\mathrm{A}}\mathrm{d}t'_{\mathrm{B}}\varphi\left(t'_{\mathrm{A}}, t'_{\mathrm{B}}\right)|t'_{\mathrm{A}}, t'_{\mathrm{B}}\rangle_{\mathrm{AB}} \otimes |\chi\rangle_{\mathrm{R}} \tag{4.31}$$

and plug it back into equation (4.29):

$$|\Psi\rangle\!\rangle = \int \mathrm{d}t'_{\mathrm{A}}\mathrm{d}t'_{\mathrm{B}}\mathrm{d}\alpha\varphi\left(t'_{\mathrm{A}}, t'_{\mathrm{B}}\right)e^{-i\alpha\left(\hat{H}_{\mathrm{A}}+\hat{H}_{\mathrm{B}}\right)}\mathrm{T}e^{-i\int_0^\alpha \mathrm{d}s\left(\hat{f}_{\mathrm{A}}\left(s+\hat{T}_{\mathrm{A}}\right)+\hat{f}_{\mathrm{B}}\left(s+\hat{T}_{\mathrm{B}}\right)\right)}|t'_{\mathrm{A}}, t'_{\mathrm{B}}\rangle_{\mathrm{AB}} \otimes |\chi\rangle_{\mathrm{R}} \tag{4.32}$$

Next, we evaluate the operators acting on the state:

$$\begin{aligned}|\Psi\rangle\!\rangle &= \int \mathrm{d}t'_{\mathrm{A}}\mathrm{d}t'_{\mathrm{B}}\mathrm{d}\alpha\varphi\left(t'_{\mathrm{A}}, t'_{\mathrm{B}}\right)e^{-i\alpha\left(\hat{H}_{\mathrm{A}}+\hat{H}_{\mathrm{B}}\right)}\mathrm{T}e^{-i\int_0^\alpha \mathrm{d}s\left(\hat{f}_{\mathrm{A}}\left(s+t'_{\mathrm{A}}\right)+\hat{f}_{\mathrm{B}}\left(s+t'_{\mathrm{B}}\right)\right)}|t'_{\mathrm{A}}, t'_{\mathrm{B}}\rangle_{\mathrm{AB}} \otimes |\chi\rangle_{\mathrm{R}} \\ &= \int \mathrm{d}t'_{\mathrm{A}}\mathrm{d}t'_{\mathrm{B}}\mathrm{d}\alpha\varphi\left(t'_{\mathrm{A}}, t'_{\mathrm{B}}\right)\mathrm{T}e^{-i\int_0^\alpha \mathrm{d}s\left(\hat{f}_{\mathrm{A}}\left(s+t'_{\mathrm{A}}\right)+\hat{f}_{\mathrm{B}}\left(s+t'_{\mathrm{B}}\right)\right)}|t'_{\mathrm{A}}+\alpha\rangle_{\mathrm{A}} \otimes |t'_{\mathrm{B}}+\alpha\rangle_{\mathrm{AB}} \otimes |\chi\rangle_{\mathrm{R}}\end{aligned} \tag{4.33}$$

We want to eliminate α and write the integral in terms of time measured by the clock A. We define the variable $t_{\mathrm{A}}(\alpha) = t'_{\mathrm{A}} + \alpha$, and rewrite the integral:

$$|\Psi\rangle\!\rangle = \int \mathrm{d}t_{\mathrm{A}}\mathrm{d}t'_{\mathrm{A}}\mathrm{d}t'_{\mathrm{B}}\varphi\left(t'_{\mathrm{A}}, t'_{\mathrm{B}}\right)\mathrm{T}e^{-i\int_0^{t_{\mathrm{A}}-t'_{\mathrm{A}}} \mathrm{d}s\left(\hat{f}_{\mathrm{A}}\left(s+t'_{\mathrm{A}}\right)+\hat{f}_{\mathrm{B}}\left(s+t'_{\mathrm{B}}\right)\right)}|t_{\mathrm{A}}\rangle_{\mathrm{A}} \otimes |t_{\mathrm{A}}+t'_{\mathrm{B}}-t'_{\mathrm{A}}\rangle_{\mathrm{B}} \otimes |\chi\rangle_{\mathrm{R}} \tag{4.34}$$

Furthermore, in the integral in s, we make the change of variable $s \longrightarrow s - t'_{\mathrm{A}}$, so that:

$$e^{-i\int_0^{t_{\mathrm{A}}-t'_{\mathrm{A}}} \mathrm{d}s\left(\hat{f}_{\mathrm{A}}\left(s+t'_{\mathrm{A}}\right)+\hat{f}_{\mathrm{B}}\left(s+t'_{\mathrm{B}}\right)\right)} = e^{-i\int_{t'_{\mathrm{A}}}^{t_{\mathrm{A}}} \mathrm{d}s\left(\hat{f}_{\mathrm{A}}(s)+\hat{f}_{\mathrm{B}}\left(s+t'_{\mathrm{B}}\right)\right)} \tag{4.35}$$

Lastly, divide the integral in the exponent in one from 0 to t_{A} and another one from t'_{A} to 0 and rearrange the terms:

$$|\Psi\rangle = \int \mathrm{d}t_{\mathrm{A}}\,|t_{\mathrm{A}}\rangle_{\mathrm{A}} \otimes e^{-it_{\mathrm{A}}\hat{H}_{\mathrm{B}}}\mathrm{T}e^{-i\int_0^{t_{\mathrm{A}}} \mathrm{d}s\left(\hat{f}_{\mathrm{A}}(s)+\hat{f}_{\mathrm{B}}\left(s+\hat{T}_{\mathrm{B}}\right)\right)}|\psi_{\mathrm{A}}(0)\rangle_{\overline{\mathrm{A}}} \tag{4.36}$$

Where we put part not depending on t_A into the state:

$$|\psi_{\mathrm{A}}(0)\rangle_{\overline{\mathrm{A}}} = \int \mathrm{d}t'_{\mathrm{A}}\mathrm{d}t'_{\mathrm{B}}\mathrm{T}e^{i\int_0^{t'_{\mathrm{A}}}\mathrm{d}s\left(\hat{f}_{\mathrm{A}}(s)+\hat{f}_{\mathrm{B}}\left(s+\hat{T}_{\mathrm{B}}\right)\right)}\varphi\left(t'_{\mathrm{A}},t'_{\mathrm{B}}\right)|t'_{\mathrm{B}}-t'_{\mathrm{A}}\rangle_{\mathrm{B}}\otimes|\chi\rangle_{\mathrm{R}}. \tag{4.37}$$

We can write this state also as just, after integrating over t'_{A}:

$$|\psi_{\mathrm{A}}(0)\rangle_{\overline{\mathrm{A}}} = \int \mathrm{d}t'_{\mathrm{B}}\varphi\left(t'_{\mathrm{B}}\right)|t'_{\mathrm{B}}\rangle_{\mathrm{B}}\otimes|\chi\rangle_{\mathrm{R}}. \tag{4.38}$$

Notice that we have the same form as equation (4.30), where unitary is given by:

$$\hat{U}_{\bar{A}}(t_A) = e^{-it_{\mathrm{A}}\hat{H}_{\mathrm{B}}}\mathrm{T}e^{-i\int_0^{t_{\mathrm{A}}}\mathrm{d}s\left(\hat{f}_{\mathrm{A}}(s)+\hat{f}_{\mathrm{B}}\left(s+\hat{T}_{\mathrm{B}}\right)\right)} \tag{4.39}$$

Thus, the important conclusion is that even if the clock, from the perspective of another, might be entangled with some third clock, it will nevertheless always in its reference frame see just unitary evolution of the rest with respect to it.

In the following chapters, we will focus on the constraints having multiple systems serving as clocks.

4.2 Time ordered exponential in the solution with two quantum clocks

In this section, we will present the original results of the thesis where we used Page-Wotters formalism with several clocks to investigate time ordering appearing in the time ordering operator. Let us recall that in Section (2.4.2) we showed how the time ordering operation appearing on the level of Feynman propagator is the one coming precisely from the time ordered exponential, analyzed in Chapter (3) (where the ordering ambiguity dropped off on the level of external legs). The main motivation of our work was to compare two orderings appearing on the level of the Feynman propagator, with the controlled superposition of orders appearing in the quantum switch. As seen in Chapter (2.4), the Feynman propagator is defined as the sum of

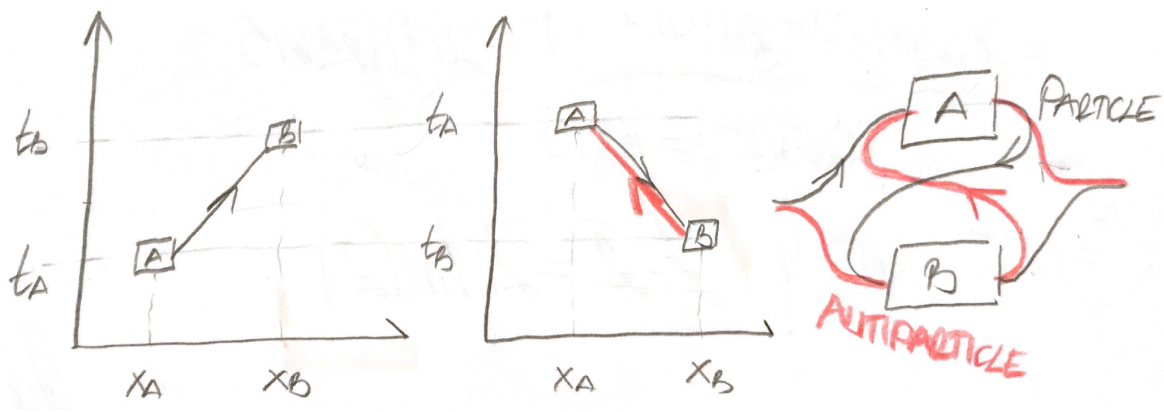


Figure 4.2: The Feynman propagator as a superposition of diagrams corresponding to particle and antiparticle degrees of freedom, compared with the structure of a quantum switch. For the operational setup of the calculation, imagine that the vertices of spacetime diagram A and B represent local laboratories in which Alice and Bob have their entangled clocks. Clocks are in a superposition in an entangled basis with respect to some third observer who will perceive a situation as the superposition of two depicted diagrams. The shared system is in this case a field and the kicks correspond to vertices.

the processes

$$\left\langle \mathbb{0} \left| T\left(\hat{\phi}(x)\hat{\phi}^{\dagger}(y)\right) \right| \mathbb{0} \right\rangle = \underbrace{\left\langle \mathbb{0} \left| \hat{\phi}(x)\hat{\phi}^{\dagger}(y) \right| \mathbb{0} \right\rangle}_{\text{particle DOFs}} \theta(x^0 - y^0) + \underbrace{\left\langle \mathbb{0} \left| \hat{\phi}^{\dagger}(y)\hat{\phi}(x) \right| \mathbb{0} \right\rangle}_{\text{antiparticle DOFs}} \theta(y^0 - x^0)$$

$$= \left\langle \mathbb{0} \left| \begin{pmatrix} \text{Particle annihilated} \\ \text{at } (x^0, \boldsymbol{x}) \end{pmatrix} \begin{pmatrix} \text{Particle created} \\ \text{at } (y^0, \boldsymbol{y}) \end{pmatrix} \right| \mathbb{0} \right\rangle \theta(x^0 - y^0)$$

$$+ \left\langle \mathbb{0} \left| \begin{pmatrix} \text{Antiparticle annihilated} \\ \text{at } (y^0, \boldsymbol{y}) \end{pmatrix} \begin{pmatrix} \text{Antiparticle created} \\ \text{at } (x^0, \boldsymbol{x}) \end{pmatrix} \right| \mathbb{0} \right\rangle \theta(y^0 - x^0) \tag{4.40}$$

as depicted on the left and right spacetime sketch of Figure (4.2), where the comparison with the quantum switch is shown on the rightmost picture. We can thus make an (unrigorous) parallel between the structure of a quantum switch, as discussed in Chapter (2.11), and the structure of Feynman propagator by making the following associations:

- *Target system* is now a vacuum state of the field $|\mathbb{0}\rangle$ on which the agents will act.

- *Agents*[57], labelled by A and B, are associated with the vertices of the prop-

[57] In this context, 'laboratories' and 'agents' should be taken figuratively. The important thing is only to find the local input-output structure and the notion of a map between the maps.

agator where they are performing local CP transformations on some localized region representing a subspace of the total Hilbert space corresponding to the target system. These transformations will concretely be particle creation and annihilation, which we will label as: $\hat{\phi}^{\dagger}(y) = \mathcal{A} \equiv$'particle created' and $\hat{\phi}(x) = \mathcal{B} \equiv$'particle annihilated'. Since the state space should be preserved with the action of field operators, considered maps comply with the desired description. The lines on the diagrams represent produced field excitations.

- *'Free variables'*[58] in this case are the spacetime positions on which agents choose to create or annihilate the particle. In other words, agents are maps which assign the algebras of the observables to desired spacetime region.

- *Control* is now Heaviside θ function, determining the time relation between time components of the events.

We can thus view Feynman propagator as analogous to a *supermap*, in the sense that it is mapping two CP maps $\mathcal{A} = \hat{\phi}^{\dagger}(x) : |\mathbb{0}\rangle \rightarrow \hat{\phi}^{\dagger}(x)|\mathbb{0}\rangle$ and $\mathcal{B} = \hat{\phi}(y) : |\mathbb{0}\rangle \rightarrow \hat{\phi}(y)|\mathbb{0}\rangle$ into a superposition (4.40). This is a bit different structure from the structure of supermap appearing in (2.80) due to both unconventional notion of the control appearing and the fact that we are working in the Heisenberg representation.
One should also note that in this case upon projecting on the particular state of the control, we would get a single amplitude which violates the causality, as described in Chapter (2.2), and contributes to the propagation in the spacetime region. [59]

Time directionality serving as a control seems problematic because it makes the control inaccessible within the context of standard considerations. What we presented in Chapter (3) can be understood precisely as the attempt to access this control degree of freedom; or in other sense, to implement the accessible control. However, we saw that the only possibility was to propose a hypothetical ancillary potential, in the form of relation (3.14), which would depend not on time but on *time directionality*, inconsistently with the starting expression for a time-ordered exponential.

[58] In some natural QFT scenario, they wouldn't really be free in a sense of having only future light-cone.

[59] Thus, if such a controlled superposition could be projected, it might be only for the short scales, possibly within the quantum gravity regimes.

In the following, we will show that a suitable extension on of our observations where implementation of such a control is possible is the framework of Page-Wotters formalism. Similarly to the approach presented in Chapter (3), in the following we will attempt to isolate one 'branch' of the time ordering operator, or in other words, to implement the control over the branches of superposition, but this time in the context of the solution to the constraint equation, which in the case of several clocks represents a generalization to the Schrödinger equation and therefore extends beyond standard quantum mechanics. The reason for considering constraint equation with more then one system serving as a clock is because, as explained in this Chapter (4), Page-Wotters formalism with one clock gives back the ordinary quantum dynamics and reduces to already tried out attempt.

Let us consider system S coupled with two clocks A and B, such that we have a composite system $|\Phi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$. Dynamics is specified by the following constraint:

$$\hat{C} = \hat{H}_A + \hat{H}_B + \hat{H}_S + \hat{V}_S(\hat{T}_A) + \hat{V}_S(\hat{T}_B) \tag{4.41}$$

defining our physical states (history states) via constraint equation:

$$\hat{C}|\Psi\rangle\!\rangle = 0 \tag{4.42}$$

Using group averaging[60](see [60]) we solve for the history states:

$$|\Psi\rangle\!\rangle = \int d\alpha e^{-i\alpha\hat{C}}|\Phi\rangle_{A,B,S} \tag{4.43}$$

where

$$|\Phi\rangle_{A,B,S} = |t_A, t_B\rangle \otimes |\phi\rangle_S \tag{4.44}$$

Computing the action of the exponential in equation (4.43), as shown in Supplements (C). The equation for the history state takes the form:

$$|\Psi\rangle\!\rangle = \int d\alpha e^{-i\alpha(\hat{H}_A+\hat{H}_B)} T e^{-i\int_0^\alpha d\lambda\left(\hat{V}_S(\hat{T}_A+\lambda)+\hat{V}_S(\hat{T}_B+\lambda)\right)} |t_A, t_B, \phi\rangle \tag{4.45}$$

where T represents time ordering operator (introduced in Chapter (2.4) with

[60]Averaging over all possible elements of the group, such that the average element is in the kernel of the constraint operator.

respect to scale λ:

$$T(\hat{f}(\lambda_1)\hat{g}(\lambda_2)) = \theta(\lambda_1 - \lambda_2)\hat{f}(\lambda_1)\hat{g}(\lambda_2) + \theta(\lambda_2 - \lambda_1)\hat{g}(\lambda_2)\hat{f}(\lambda_1) \tag{4.46}$$

Now, let us consider the expansion of time ordering exponential, up to the second order,

$$\begin{aligned} &Te^{-i\int_0^\alpha d\lambda \hat{V}_S(\hat{T}_A+\lambda)+\hat{V}_S(\hat{T}_B+\lambda)} \approx \mathbb{1} - i\int_0^\alpha d\lambda(\hat{V}_S(\hat{T}_A+\lambda)+\hat{V}_S(\hat{T}_B+\lambda)) \\ &+ \frac{(-i)^2}{2!}\underbrace{T\int_0^\alpha d\lambda_1 d\lambda_2(\hat{V}_S(\hat{T}_A+\lambda_1)+\hat{V}_S(\hat{T}_B+\lambda_1))(\hat{V}_S(\hat{T}_A+\lambda_2)+\hat{V}_S(\hat{T}_B+\lambda_2))}_{\equiv *} \end{aligned} \tag{4.47}$$

Focusing only on the second order term:

$$\begin{aligned} * &= T\int_0^\alpha d\lambda_1 d\lambda_2(\hat{V}_S(\hat{T}_A+\lambda_1)\hat{V}_S(\hat{T}_A+\lambda_2) + \hat{V}_S(\hat{T}_A+\lambda_1)\hat{V}_S(\hat{T}_B+\lambda_2) \\ &+ \hat{V}_S(\hat{T}_B+\lambda_1)\hat{V}_S(\hat{T}_A+\lambda_2) + \hat{V}_S(\hat{T}_B+\lambda_1)\hat{V}_S(\hat{T}_B+\lambda_2)) \\ &= T\int_0^\alpha d\lambda_1 d\lambda_2 \sum_{I,J=A,B} \hat{V}_S(\hat{T}_I+\lambda_1)\hat{V}_S(\hat{T}_J+\lambda_2) \\ &= \int_0^\alpha d\lambda_1 d\lambda_2 \Big[\textstyle\sum_{I,J=A,B} \underbrace{\hat{V}_S(\hat{T}_I+\lambda_1)\hat{V}_S(\hat{T}_J+\lambda_2)\theta(\lambda_1-\lambda_2)}_{(1)} + \underbrace{\hat{V}_S(\hat{T}_J+\lambda_2)\hat{V}_S(\hat{T}_I+\lambda_1)\theta(\lambda_2-\lambda_1)}_{(2)}\Big] \end{aligned} \tag{4.48}$$

Now, we will make a choice of the potential. We will take the potential with respect to each clock to be such that it 'kicks' once in some predefined instances t_1^* and t_2^* (local times with respect to each clock):

$$\begin{aligned} \hat{V}_S(\hat{T}_A+\lambda_{1,2}) &= \hat{V}_S(t_1^*)\delta(\hat{T}_A+\lambda_{1,2}-t_1^*) \\ \hat{V}_S(\hat{T}_B+\lambda_{1,2}) &= \hat{V}_S(t_2^*)\delta(\hat{T}_B+\lambda_{1,2}-t_2^*) \end{aligned} \tag{4.49}$$

Going back into equation (4.48) using equation (4.71), we get:

$$\begin{aligned} (1) &= \int_0^\alpha d\lambda_1 d\lambda_2 \Big(\hat{V}_S(\hat{T}_A+\lambda_1)\hat{V}_S(\hat{T}_A+\lambda_2) + \hat{V}_S(\hat{T}_A+\lambda_1)\hat{V}_S(\hat{T}_B+\lambda_2) \\ &+ \hat{V}_S(\hat{T}_B+\lambda_2)\hat{V}_S(\hat{T}_A+\lambda_2) + \hat{V}_S(\hat{T}_B+\lambda_1)\hat{V}_S(\hat{T}_B+\lambda_2)\Big)\theta(\lambda_1-\lambda_1) \\ &= \hat{V}_S(t_1^*)\hat{V}_S(t_1^*)\theta(t_1^*-\hat{T}_A-t_1^*+\hat{T}_A) + \hat{V}_S(t_1^*)\hat{V}_S(t_2^*)\theta(t_1^*-\hat{T}_A-t_2^*+\hat{T}_B) \\ &+ \hat{V}_S(t_2^*)\hat{V}_S(t_1^*)\theta(t_2^*-\hat{T}_B-t_1^*+\hat{T}_A) + \hat{V}_S(t_2^*)\hat{V}_S(t_2^*)\theta(t_2^*-\hat{T}_B-t_2^*+\hat{T}_B) \end{aligned} \tag{4.50}$$

$$
\begin{aligned}
(2) = \int_0^{\alpha} d\lambda_1 d\lambda_2 \Big(& \hat{V}_S(\hat{T}_B + \lambda_2)\hat{V}_S(\hat{T}_B + \lambda_1) + \hat{V}_S(\hat{T}_B + \lambda_2)\hat{V}_S(\hat{T}_A + \lambda_1) \\
& + \hat{V}_S(\hat{T}_A + \lambda_2)\hat{V}_S(\hat{T}_B + \lambda_1) + \hat{V}_S(\hat{T}_A + \lambda_2)\hat{V}_S(\hat{T}_A + \lambda_1)\Big)\theta(\lambda_2 - \lambda_1) \\
= & \hat{V}_S(t_2^*)\hat{V}_S(t_2^*)\theta(t_2^* - \hat{T}_B - t_2^* + \hat{T}_B) + \hat{V}_S(t_2^*)\hat{V}_S(t_1^*)\theta(t_2^* - \hat{T}_B - t_1^* + \hat{T}_A) \\
& + \hat{V}_S(t_1^*)\hat{V}_S(t_2^*)\theta(t_1^* - \hat{T}_A - t_2^* + \hat{T}_B) + \hat{V}_S(t_1^*)\hat{V}_S(t_1^*)\theta(t_1^* - \hat{T}_A - t_1^* + \hat{T}_A)
\end{aligned}
\tag{4.51}
$$

$$
\begin{aligned}
\Rightarrow \sum_{m,n=1,2} \sum_{I,J=A,B} \Big\{ & \theta(t_n^* - \hat{T}_I - t_m^* + \hat{T}_J)\hat{V}_S(t_n^*)\hat{V}_S(t_m^*) \\
& + \theta(t_n^* - \hat{T}_J - t_m^* + \hat{T}_I)\hat{V}_S(t_n^*)\hat{V}_S(t_m^*)\Big\}
\end{aligned}
\tag{4.52}
$$

From equation (4.52) we get:

$$
\begin{aligned}
\Rightarrow 2 \sum_{m,n=1,2} \Big\{ & 2\theta(t_n^* - t_m^*)\hat{V}_S(t_n^*)\hat{V}_S(t_m^*) \\
& + \theta(t_n^* - \hat{T}_A - t_m^* + \hat{T}_B)\hat{V}_S(t_n^*)\hat{V}_S(t_m^*) + \theta(t_n^* - \hat{T}_B - t_m^* + \hat{T}_A)\hat{V}_S(t_n^*)\hat{V}_S(t_m^*)\Big\} \\
= 2\Big\{ & 2\theta(t_1^* - t_2^*)\hat{V}_S(t_1^*)\hat{V}_S(t_2^*) + 2\theta(t_2^* - t_1^*)\hat{V}_S(t_2^*)\hat{V}_S(t_1^*) \\
& + 2\theta(t_1^* - t_1^*)\hat{V}_S(t_1^*)\hat{V}_S(t_1^*) + 2\theta(t_2^* - t_2^*)\hat{V}_S(t_2^*)\hat{V}_S(t_2^*) \\
& + \theta(t_1^* - \hat{T}_A - t_1^* + \hat{T}_B)\hat{V}_S(t_1^*)\hat{V}_S(t_1^*) + \theta(t_1^* - \hat{T}_A - t_2^* + \hat{T}_B)\hat{V}_S(t_1^*)\hat{V}_S(t_2^*) \\
& + \theta(t_1^* - \hat{T}_B - t_1^* + \hat{T}_A)\hat{V}_S(t_1^*)\hat{V}_S(t_1^*) + \theta(t_1^* - \hat{T}_B - t_2^* + \hat{T}_A)\hat{V}_S(t_1^*)\hat{V}_S(t_2^*) \\
& + \theta(t_2^* - \hat{T}_A - t_1^* + \hat{T}_B)\hat{V}_S(t_2^*)\hat{V}_S(t_1^*) + \theta(t_2^* - \hat{T}_A - t_2^* + \hat{T}_B)\hat{V}_S(t_2^*)\hat{V}_S(t_2^*) \\
& + \theta(t_2^* - \hat{T}_B - t_1^* + \hat{T}_A)\hat{V}_S(t_2^*)\hat{V}_S(t_1^*) + \theta(t_2^* - \hat{T}_B - t_2^* + \hat{T}_A)\hat{V}_S(t_2^*)\hat{V}_S(t_2^*)\Big\}
\end{aligned}
\tag{4.53}
$$

Introducing notation:

$$
\begin{aligned}
\delta t^* &\equiv t_2^* - t_1^* \\
\delta\hat{T} &\equiv \hat{T}_B - \hat{T}_A \\
\hat{W} &\equiv \hat{V}_S(t_2^*)\hat{V}_S(t_1^*) \\
\hat{Z} &\equiv \hat{V}_S(t_1^*)\hat{V}_S(t_2^*)
\end{aligned}
\tag{4.54}
$$

In the following, we will ignore terms of the form $\hat{V}_S(t_1^*)\hat{V}_S(t_1^*)$, $\hat{V}_S(t_2^*)\hat{V}_S(t_2^*)$ as they are irrelevant for the time ordering. We will ignore also all the other terms, up

to the second order, which do not show different time ordering. We get,

$$Te^{-i\int_0^\alpha d\lambda \hat{V}_S(t_A+\lambda)+\hat{V}_S(t_B+\lambda)} =$$
$$= \cdots + 2\Big\{2\theta(-\delta t^*)\hat{Z} + 2\theta(\delta t^*)\hat{W} + \theta(-\delta t^* + \delta\hat{T})\hat{Z} + \theta(-\delta t^* - \delta\hat{T})\hat{Z} + \theta(\delta t^* + \delta\hat{T})\hat{W} + \theta(\delta t^* - \delta\hat{T})\hat{W}\Big\} \tag{4.55}$$

Since times of the potential kicks are predefined, δt^* will have definite value and sign. Without loss of generality, let us take:

$$\delta t^* > 0 \tag{4.56}$$

Notice also that the expression (4.55) has two ordering of $\hat{V}_S$ with respect to t^* values. We will look for the preparations and the measurements such that only one time ordering of $\hat{V}_S$ operators remains. Let us partition the space $\mathcal{H}_A \otimes \mathcal{H}_B$ into centre of mass $\mathcal{H}_{CM}$ and relative coordinates $\mathcal{H}_{rel}$, such that $|t_A, t_B\rangle = |t_{CM}\rangle \otimes |t_{rel}\rangle$, where $|t_{rel}\rangle = |\delta t\rangle$ is eigenstate of $\delta\hat{T}|\delta t\rangle = \delta t|\delta t\rangle$. We have $|t_A, t_B, \phi\rangle \to |t_{CM}\rangle \otimes |t_{rel}\rangle \otimes |\phi\rangle$. Let us consider now preparation of the clock states such that:

$$|t_{rel}\rangle_{AB} = \eta\,|\delta t \leq \delta t^*\rangle + \xi\,|\delta t > \delta t^*\rangle \tag{4.57}$$

Here one should note that such preparation states might arise quite naturally. For example, when clocks at different positions interact with each other and with other systems, as in the case of gravitational quantum switch [89]. Or if the clocks are in different states of relative motion, causing them to succumb time dilatation [78]. Going back to equation (4.45)

$$|\Psi\rangle\!\rangle = \int d\alpha e^{-i\alpha(\hat{H}_A+\hat{H}_B)}\cdot \underbrace{Te^{-i\int_0^\alpha d\lambda\left(\hat{V}_S(\hat{T}_A+\lambda)+\hat{V}_S(\hat{T}_B+\lambda)\right)}\Big\{\eta\,|\delta t \leq \delta t^*\rangle + \xi\,|\delta t > \delta t^*\rangle\Big\} \otimes |t_{CM}\rangle \otimes |\phi\rangle_{\mathrm{S}}}_{**} \tag{4.58}$$

$$\Rightarrow ** = Te^{-i\int_0^\alpha \mathrm{d}\lambda\left(\hat{V}_S(\hat{T}_A+\lambda)+\hat{V}_S(\hat{T}_B+\lambda)\right)}\Big\{\eta\,|\delta t \leq \delta t^*\rangle + \xi\,|\delta t > \delta t^*\rangle\Big\} \otimes |t_{CM}\rangle \otimes |\phi\rangle_{\mathrm{S}} \tag{4.59}$$

Using expression (4.55), which is up to the second order:

$$\begin{aligned}
** &= 2\Big\{ \cdots + 2\hat{W} + \theta(-\delta t^* + \delta t)\hat{Z} + \theta(-\delta t^* - \delta t)\hat{Z} + \theta(\delta t^* + \delta t)\hat{W} + \theta(\delta t^* - \delta t)\hat{W} \Big\} \\
&\cdot \Big\{ \eta \left|\delta t \leq \delta t^*\right\rangle + \xi \left|\delta t > \delta t^*\right\rangle \Big\} \otimes |t_{CM}\rangle \otimes |\phi\rangle_{\mathrm{S}} \\
&= \Big\{ \cdots + 8\eta\hat{W} \left|\delta t \leq \delta t^*\right\rangle + 2\xi(\hat{Z} + 3\hat{W}) \left|\delta t > \delta t^*\right\rangle \Big\} \otimes |\phi\rangle_{\mathrm{S}}
\end{aligned} \tag{4.60}$$

where '...' represent all the other terms in the expansion. Now, for the measurement, consider a state:

$$|\zeta\rangle_{\mathrm{AB}} = \gamma \left|\delta t \leq \delta t^*\right\rangle + \beta \left|\delta t > \delta t^*\right\rangle \tag{4.61}$$

Which gives (leaving out parts not relevant for time ordering):

$$\begin{aligned}
{}_{\mathrm{AB}}\langle\zeta| \cdot |\Psi\rangle &\approx \int d\alpha e^{-i\alpha(\hat{H}_S + \hat{H}_A + \hat{H}_B)} {}_{\mathrm{AB}}\langle\zeta|(**) \\
&\approx \int d\alpha e^{-i\alpha(\hat{H}_S + \hat{H}_A + \hat{H}_B)} \cdot \Big(\gamma \left\langle \delta t \leq \delta t^* \right| + \beta \left\langle \delta t > \delta t^* \right| \Big) \\
&\cdot \Big\{ 8\eta\hat{W} \left|\delta t \leq \delta t^*\right\rangle + 2\xi(\hat{Z} + 3\hat{W}) \left|\delta t > \delta t^*\right\rangle \Big\} \otimes |t_{CM}\rangle \otimes |\phi\rangle_{\mathrm{S}} \\
&\approx \int d\alpha e^{-i\alpha(\hat{H}_S + \hat{H}_A + \hat{H}_B)} \Big(8\hat{W}\gamma\eta + 2(\hat{Z} + 3\hat{W})\beta\xi \Big) \otimes |t_{CM}\rangle \otimes |\phi\rangle_{\mathrm{S}}
\end{aligned} \tag{4.62}$$

Under the conditions:

$$\begin{aligned}
8\hat{W}\gamma\eta + 6\hat{W}\beta\xi = 0 \\
\beta\xi = -\frac{4}{3}\gamma\eta
\end{aligned} \tag{4.63}$$

we isolate one ordering configuration of the second order term of time ordered expansion of (4.45),

$${}_{\mathrm{AB}}\langle\zeta| \cdot |\Psi\rangle\!\rangle \approx \int d\alpha e^{-i\alpha(\hat{H}_A + \hat{H}_B)} 2\hat{V}_S(t_1^*)\hat{V}_S(t_2^*)\beta\xi \otimes |\phi\rangle_{\mathrm{S}} \tag{4.64}$$

where ordering was considered with respect to some parameter λ.

In conclusion, by considering constraint equation with two quantum clocks, we managed to construct a history state which can be projected to the subspace pertaining to a particular time ordering branch of time ordering operator T (ordering with respect to some classical scale λ). In other words, within the given framework we

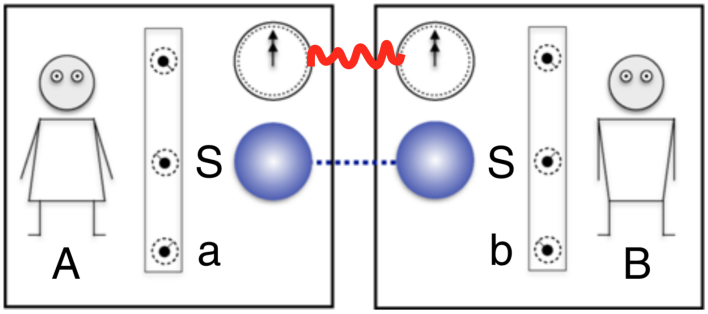


Figure 4.3: At the beginning of the experiment Alice and Bob prepare their clocks in the entangled state $|t_{rel}\rangle_{AB}$. They proceed the experiment by measuring their local clock which triggers a 'kick' on a subsystem accessible from their local laboratory. Red wavy line denotes the entanglement. [17]

managed to obtain quantum controlled superposition of time orderings of considered time ordered exponential, where now relative degrees of freedom of clocks served as a control. Therefore, we can understand the history state as having a superposition of time orderings, with respect to a scale λ. To make a parallel with the superposition of orderings considered in the context of the Feynman propagator, we could attribute this scale to the classical clock of some third observer who will see the superposition of processes corresponding to different orderings within time ordered operator, where internal degrees of freedom of scattering particles could now represent considered quantum clocks.[61]

Going back to our toy model, let us suppose for the operational setup that we have Alice and Bob making measurements on their clocks in their local laboratories and sending their statistics to the observer with a classical clock, ticking according to the scale λ. Measurement of the clocks will trigger a 'kick' on the system and set up an event in a physically meaningful way. Two clocks are in the entangled state of relative degrees of freedom. The state is in the superposition in the entangled basis, where eigenspaces are split with respect to fixing the temporal distance between two kicks δt^*. Each clock will then set up an event performing a transformation on a subsystem of a system S when their local time shows some predetermined values t_1^* or t_2^*. We can imagine that the clocks are programmed in advance to kick in these times. Within the analogy with the Feynman propagator, we can assume that the system

[61] We should emphasize that generally, it is not necessary to associate this classical parameter with some classical clock. Classical clocks can be completely removed within the framework of Page-Wotters formalism.

is a field and that Alice and Bob are performing transformations on some localized subsystem. We will label $V(t_1^*) = A$ and $V(t_2^*) = B$ to make a connection with Figure 2.13. Particular preselections and the postselections on the systems A and B will result in scenarios where the observer C, will be able to see either $A \prec B$ or $B \prec A$, but will generally see the superposition of orders $(A \prec B)$ and $(B \prec A)$. Namely, let us note that the considered situation reduces to the usual case under the preparations which set up $\delta t = 0$, i.e. when the clock tick synchronized with each other.

Now, one may argue that the discussion could be lifted to the quantum field theory context, by promoting the history state of the wave function of finite particle state $|\psi(t)\rangle$ to the history state of the wave functional, as it appears in the Schrödinger functional formulation, which we will consider in Chapter (5). In light of that, we will consider the following a generalization of our two-clock toy model to the case of N clocks as a toy model for the general functional case.

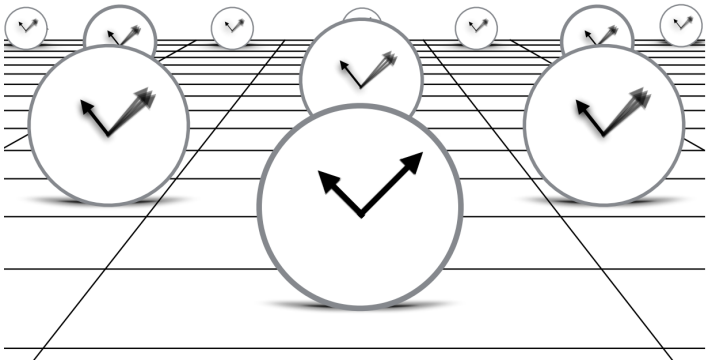


Figure 4.4: The grid of clocks imagined on a hypersurface. [16]

4.3 Time order exponential with N quantum clocks and continuous limit

Let us now extend the discussion to the case of system S coupled to N clocks. Following the previous discussion, one can imagine a classical distant observer, splitting up spacetime and associating a grid of clocks to a particular hypersurface, as sketched in Figure (4.4). The motivation behind such consideration can be understod as the attempt to definine a spacetime operationally (instead of defining a spacetime with

respect to abstract points, we will imagine it as defined with respect to physical rods and clocks).

The constraint with N clocks now reads:

$$\hat{C} = \sum_{x=1}^{N} \hat{H}_C^x + \sum_{x=1}^{N} \hat{V}_S(\hat{T}_C^x) + \hat{H}_S \tag{4.65}$$

here x labels each clock. History state is again defined via

$$|\Psi\rangle\!\rangle = \int d\alpha e^{-i\alpha\hat{C}} |\Phi\rangle_{1\ldots N,S} \tag{4.66}$$

where

$$|\Phi\rangle_{1\ldots N,S} = |t_1 \ldots t_N\rangle \otimes |\phi\rangle_S \tag{4.67}$$

Generalizing equation (4.45):

$$|\Psi\rangle\!\rangle = \int d\alpha e^{-i\alpha\left(\hat{H}_S + \sum_{x=1}^{N} \hat{H}_C^x\right)} \underbrace{T e^{-i\sum_{x=1}^{N} \int_0^\alpha d\lambda \hat{V}_S(\hat{T}_x + \lambda)}}_{(*)} |t_1 \ldots t_N, \phi\rangle \tag{4.68}$$

Expanding (*):

$$\begin{aligned} & T e^{-i\sum_{x=1}^{N} \int_0^\alpha d\lambda \hat{V}_S(\hat{T}_x+\lambda)} \\ & \approx \mathbb{1} - i\int_0^\alpha d\lambda \left(\sum_{x=1}^{N} \hat{V}_S(\hat{T}_x + \lambda)\right) + \frac{(-i)^2}{2!} \underbrace{T\int_0^\alpha d\lambda_1 d\lambda_2 \left(\sum_{x_1=1}^{N} \hat{V}_S(\hat{T}_{x_1} + \lambda_1)\right)\left(\sum_{x_2=1}^{N} \hat{V}_S(\hat{T}_{x_2} + \lambda_2)\right)}_{(**)} \end{aligned} \tag{4.69}$$

Focusing on the second order:

$$\begin{aligned} & T\int_0^\alpha d\lambda_1 d\lambda_2 \left(\sum_{x_1=1}^{N} \hat{V}_S(\hat{T}_{x_1} + \lambda_1)\right)\left(\sum_{x_2=1}^{N} \hat{V}_S(\hat{T}_{x_2} + \lambda_2)\right) \\ & = \int_0^\alpha d\lambda_1 d\lambda_2 \Bigg[\left(\sum_{x_1=1}^{N} \hat{V}_S(\hat{T}_{x_1} + \lambda_1)\right)\left(\sum_{x_2=1}^{N} \hat{V}_S(\hat{T}_{x_2} + \lambda_2)\right)\theta(\lambda_1 - \lambda_2) \\ & \qquad + \left(\sum_{x_2=1}^{N} \hat{V}_S(\hat{T}_{x_2} + \lambda_2)\right)\left(\sum_{x_1=1}^{N} \hat{V}_S(\hat{T}_{x_1} + \lambda_1)\right)\theta(\lambda_2 - \lambda_1)\Bigg] \end{aligned} \tag{4.70}$$

Assuming again form of potential:

$$\hat{V}_S(\hat{T}_{x_i} + \lambda_{1,2}) = \hat{V}_S(t^*_{x_i})\delta(\hat{T}_{x_i} + \lambda_{1,2} - t^*_{x_i}) \tag{4.71}$$

upon which we have,

$$\Rightarrow = \int_0^{\alpha} d\lambda_1 d\lambda_2 \Bigg[\sum_{x_{1,2}=1}^{N} \left(\hat{V}_S(t^*_{x_1}) \hat{V}_S(t^*_{x_2}) \delta(\hat{T}_{x_1} + \lambda_1 - t^*_{x_1}) \delta(\hat{T}_{x_2} + \lambda_2 - t^*_{x_2}) \right) \theta(\lambda_1 - \lambda_2) + \sum_{x_{1,2}=1}^{N} \left(\hat{V}_S(t^*_{x_2}) \hat{V}_S(t^*_{x_1}) \delta(\hat{T}_{x_2} + \lambda_2 - t^*_{x_2}) \delta(\hat{T}_{x_1} + \lambda_1 - t^*_{x_1}) \right) \theta(\lambda_2 - \lambda_1) \Bigg] \tag{4.72}$$

Keeping only parts relevant for the time ordering will imply taking only $x_1 \neq x_2$ pairs,

$$\Rightarrow \approx \sum_{x_{1,2}=1; x_1 \neq x_2}^{N} \left\{ \hat{V}_S(t^*_{x_1}) \hat{V}_S(t^*_{x_2}) \theta(t^*_{x_1} - \hat{T}_{x_1} - t^*_{x_2} + \hat{T}_{x_2}) + \hat{V}_S(t^*_{x_2}) \hat{V}_S(t^*_{x_1}) \theta(t^*_{x_2} - \hat{T}_{x_2} - t^*_{x_1} + \hat{T}_{x_1}) \right\} \tag{4.73}$$

Introduce notation:

$$\begin{aligned} \delta t^*_{x_1 x_2} &\equiv t^*_{x_2} - t^*_{x_1} \\ \delta \hat{T}_{x_1 x_2} &\equiv \hat{T}_{x_2} - \hat{T}_{x_1} \\ \hat{W}_{x_1 x_2} &\equiv \hat{V}_S(t^*_{x_2}) \hat{V}_S(t^*_{x_1}) \\ \hat{Z}_{x_1 x_2} &\equiv \hat{V}_S(t^*_{x_1}) \hat{V}_S(t^*_{x_2}) \end{aligned} \tag{4.74}$$

We see now that we need to pick two clocks x_1 and x_2 to compare the kicks and readings between them. Upon picking these two clocks, calculation reduces to the one stated in the previous section and we get,

$${}_{\mathrm{x_1,x_2}}\langle \zeta | \cdot | \Psi \rangle\!\rangle = \int d\alpha e^{-i\alpha(\hat{H}_S + \sum_{x=1}^{N} \hat{H}_x)} 2 \hat{V}_S(t^*_1) \hat{V}_S(t^*_2) \beta_{x_{1,2}} \xi_{x_{1,2}} \otimes |\phi\rangle_{\mathrm{S}} \tag{4.75}$$

where we used preparations,

$$|t_{rel}\rangle_{1,\ldots,N} = \eta_{x_1,x_2} \left| \delta t_{x_1,x_2} \leq \delta t^*_{x_1 x_2} \right\rangle + \xi_{x_1,x_2} \left| \delta t_{x_1,x_2} > \delta t^*_{x_1 x_2} \right\rangle \tag{4.76}$$

and a measurement,

$$|\zeta\rangle_{\mathrm{x_1,x_2}} = \left\{ \gamma_{x_1,x_2} \left| \delta t_{x_1,x_2} \leq \delta t^* \right\rangle + \beta_{x_1,x_2} \left| \delta t_{x_1,x_2} > \delta t^* \right\rangle \right\} \tag{4.77}$$

We again considered time ordering relative to two clocks picked from the set of N clocks.

Expression generalises to continuous case by replacing sums with the integral over the space and considering time ordering with respect to any two spatial points. Namely, we would then have:

$$\hat{C} = \int dx \left(\hat{H}_C(x) + \hat{V}_S(\hat{T}_C(x)) \right) + \hat{H}_S \tag{4.78}$$

here x labels each clock. This generalizes (4.65). History state is again defined via

$$|\Psi\rangle\!\rangle = \int d\alpha e^{-i\alpha\hat{C}} |\Phi\rangle_{1\ldots N,S} \tag{4.79}$$

where

$$|\Phi\rangle_{1\ldots N,S} = \Pi_x |t(x)\rangle \otimes |\phi\rangle_S \tag{4.80}$$

would now be infinite dimensional tensor product state. Generalizing equation (4.45):

$$|\Psi\rangle\!\rangle = \int d\alpha e^{-i\alpha\left(\hat{H}_S + \int_x dx \hat{H}_C(x)\right)} \underbrace{T e^{-i\int_x \int_0^\alpha d\lambda \hat{V}_S(\hat{T}(x)+\lambda)}}_{(*)} \Pi_x \, |t(x), \phi\rangle \tag{4.81}$$

By doing preparations and measurements, we would get

$${}_{(x_1,x_2)}\langle\zeta| \cdot |\Psi\rangle\!\rangle = \int d\alpha e^{-i\alpha(\hat{H}_S + \int_x dx \hat{H}_C(x)} 2\hat{V}_S(t_1^*)\hat{V}_S(t_2^*)\beta(x_1,x_2)\xi(x_1,x_2) \otimes |\phi\rangle_S \tag{4.82}$$

One should here note that taking an infinite tensor product has some caveats. For example, the index set should be countable, and in our case it is dense and uncountable [5]. The prime problem comes from the topology of the index set. This is why the construction which takes into account Hilbert space over each spacetime point is set into the context of fibre bundles, where one can implement nontrivial topology via glueing trivial bundles defined over finite spaces [4].
The main point of the continuous limit was to construct a field of clocks. In Chapter (5) we will approach this consideration from a slightly different perspective, starting already with a field.

Let us also note that the comparison we were making, regarding quantum switch and Feynman propagator, is yet not fully refined and should be taken figuratively. There is yet no clear formalism which would translate the notions of quantum channels and quantum supermaps in a precise way to the quantum field theory context.

Our considerations may be understood as a step in this direction. [62] As we already mentioned, the notion of supermaps is a mathematical way to put quantum channels in superposition. Thus, it might be relevant, from the viewpoint of quantum gravity, to introduce this concept in the field theory. Recall also that Page-Wotters formalism was tailored to capture the nature of time one may expect in such a regime. Therefore, it might as well be that the context where our considerations would realise themselves naturally, would be the quantum gravity regime.

In the following chapter, we will implement a continuous limit of our N-clock toy model, constructing a quantum field serving as a clock within the Schrödinger functional formulation.

[62] The way we could approach these notions more rigorously may be in the context of categorical formulation of quantum theory (the framework of process theories [1]) with respect to categorical formulations of field theory (functorial quantum field theory [84]) as this gives clear common underlying mathematical structure.

5 Schrödinger functional representation and Page-Wotters formalism

Quantum field theory is essentially quantum mechanics of infinite degrees of freedom, meaning that it too succumbs to Schrödinger's formulation. Nonetheless, the particularity of quantum field theory with respect to quantum mechanics is the implementation of spacetime symmetry. Precisely the lack of manifestly Lorentz invariance when cast in the form of Schrödinger's representation, along with the fact that its renormalizability was proven relatively late [80], was the reason why Schrödinger's representation of quantum field theory never attained much popularity. Nonetheless, lately, it gained attention in some cosmological [36][42][42] and conceptual considerations (as for our purposes, where it is suitable for implementing history state formulation of quantum mechanics).

We can work with this formalism in any curved globally hyperbolic spacetime since there one can introduce a time function t with a globally valid time direction. This allows us to make a foliation of spacetime $\mathbf{R} \times \Sigma_t$ globally. We can also assume the construction on a finite interval I encompassing encompassing relevant events, allowing for the folliation $(\Sigma_t)_{t \in I \subset \mathbf{R}}$ locally. In our considerations, we will restrict to Minkowski spacetime.

Roughly speaking, we can use most of the basic intuition from quantum theory while substituting the word wave *function* with wave *functional*, making coordinate representations of *state vector functional*. In other words, in this representation states $|\Psi\rangle$ are functionals of a time-independent field $\hat{\phi}(\mathbf{x})$. We will restrict our attention to scalar field configurations, defined as a map $\phi : \mathcal{M} \to \mathbf{C}$. The space in which $|\Psi\rangle$ lives is the space of square-integrable *functionals* $L^2\left(\mathcal{C}\left(\Sigma_t\right)\right), \mathcal{C}\left(\Sigma_t\right)$ denoting a space of all possible instantaneous field configurations. Eigenvectors of the operators, defined via

$$\hat{\Phi}(\vec{x})|\phi\rangle = \phi(\vec{x})|\phi\rangle \tag{5.1}$$

will define 'coordinate representation' in functional space, where equation (5.1) mimics the role of $\hat{X}|x\rangle = x|x\rangle$ of one-particle QM. Here $\phi(\vec{x})$ denoted just an ordinary scalar function. The coordinate representation of the state $|\Psi\rangle$ is now time

dependent wave functional $\Psi[\phi(\vec{x}),t] = \langle\phi(x)|\Psi(t)\rangle$, for which it holds:

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle \tag{5.2}$$

$\Psi[\phi(\vec{x}),t] = \langle\phi|\Psi(t)\rangle$ represents probability amplitude of observing the field configuration on a spacelike hypersurface of constant time t.

The space of states is also equipped with a functional norm $\mathcal{D}\phi$, such that we have a normalization:

$$\|\Psi_t\|^2 := \int_{\mathcal{C}(\Sigma_t)} \mathcal{D}\phi\, |\Psi_t[\phi]|^2 \tag{5.3}$$

Here we will take what is standard procedure in constructing a space of square integrable functions [79]; namely, we denote with L^2 space quotiented out with the set of configurations which are almost everywhere the same $\Psi_1 \sim \Psi_2$ (i.e. which differ on the set of measure zero)

$$L^2\left(\mathcal{C}\left(\Sigma_t\right), \mathcal{D}\phi\right) := \mathcal{L}^2\left(\mathcal{C}\left(\Sigma_t\right)\right) / \sim \tag{5.4}$$

such that we can say $\|\Psi_t\| = 0 \Rightarrow \Psi_t \equiv 0$. To define a probability density, we use indicator functional $\lambda_{\mathcal{A}}$ on a measurable subset of $\mathcal{A} \subset \mathcal{C}\left(\Sigma_t\right)$ such that $\left|\lambda_{\mathcal{A}}\Psi_t[\phi]\right|^2$ can be understood as the probability for the field configuration to be given by some $\phi \in \mathcal{A}$. On this space, we can define now an inner product as,

$$\langle\Psi_1 \mid \Psi_2\rangle = \int \mathcal{D}\phi \Psi_1^*[\phi]\Psi_2[\phi] \tag{5.5}$$

To quantize the theory, we must impose the canonical commutation relations. This fixes the form of the canonical momentum field operator to be of the form of a functional derivative,

$$\Pi(\mathbf{x}) = -i\frac{\delta}{\delta\phi(\mathbf{x})} \tag{5.6}$$

such that commutation relations hold:[63]

$$[\Phi(\mathbf{y}), \Pi(\mathbf{x})] = i\delta(\mathbf{x} - \mathbf{y}) \tag{5.8}$$

Operators of the theory are represented as time independent functional kernels

$$\hat{O}|\psi\rangle = \int \mathcal{D}\phi \langle\phi'|\hat{O}|\phi\rangle\langle\phi|\psi\rangle = \int \mathcal{D}\phi \mathcal{O}(\phi, \phi')\Psi[\phi] \tag{5.9}$$

For the canonical variables we have,

$$\begin{aligned} \Phi(\mathbf{x}) &\to \langle\phi'|\Phi(\mathbf{x})|\phi\rangle = \phi(\mathbf{x})\delta[\phi - \phi'] \\ \Pi(\mathbf{x}) &\to \langle\phi'|\Pi(\mathbf{x})|\phi\rangle = -i\frac{\delta}{\delta\phi(\mathbf{x})}\delta\left[\phi - \phi'\right] \end{aligned} \tag{5.10}$$

where we made use of a functional delta function and functional differentiation. The equation (5.10) defines the action of the field operator $\Phi(\mathbf{x})$ as the multiplication with $\phi(\mathbf{x})$,

$$\Phi(\mathbf{x})|\Psi\rangle \to \phi(\mathbf{x})\Psi[\phi] \tag{5.11}$$

The action of any operator is then defined as,

$$\mathcal{O}(\Pi, \Phi)|\Psi\rangle \longrightarrow \mathcal{O}\left(\frac{1}{i}\frac{\partial}{\partial\phi}, \phi\right)\Psi[\phi] \tag{5.12}$$

In the case of the Klein-Gordon field, we have the Hamiltonian:

$$H = \frac{1}{2}\int d^3x \left(\Pi^2 + |\nabla\phi|^2 + m^2\phi^2\right) \tag{5.13}$$

which, upon inserting (5.6) turns into a functional differential operator,

$$H = \frac{1}{2}\int d^3x \left(-\frac{\delta^2}{\delta\phi^2(\mathbf{x})} + |\nabla\phi|^2 + m^2\phi^2\right) \tag{5.14}$$

[63] since for the functional derivative holds:

$$\frac{\delta}{\delta\phi(\mathbf{x})}\phi(\mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \tag{5.7}$$

and Schrödinger equation into a functional differential equation,

$$i\frac{\partial}{\partial t}\Psi[\phi(\mathbf{x}),t]=\frac{1}{2}\int d^3x\left(-\frac{\delta^2}{\delta\phi^2(\mathbf{x})}+|\nabla\phi|^2+m^2\phi^2\right)\Psi[\phi(\mathbf{x}),t]. \tag{5.15}$$

Since the Hamiltonian does not explicitly depend on time, we may separate out the time dependence of the wave functional $\Psi[\phi,t]$ and write

$$\Psi[\phi(\mathbf{x}),t]=e^{-i\hat{H}_S t}\Psi[\phi(\mathbf{x})]. \tag{5.16}$$

where $\Psi[\phi(\mathbf{x})]$ satisfies the time-independent Schrödinger functional equation,

$$\frac{1}{2}\int d^3x\left(-\frac{\delta^2\Psi[\phi(\mathbf{x})]}{\delta\phi^2(\mathbf{x})}+\left(|\nabla\phi|^2+m^2\phi^2\right)\Psi[\phi(\mathbf{x})]\right)=E\Psi[\phi(\mathbf{x})] \tag{5.17}$$

For more details, refer to [48], [27].

5.1 Schrödinger functional history state

We can also think now of promoting constraint equation

$$\hat{C}=\hat{H}_C\otimes\mathbb{1}_S+\mathbb{1}_C\otimes\hat{H}_S \tag{5.18}$$

into a functional differential equation such that the reduced state satisfies the functional differential Scrödinger equation upon conditioning on the state of the clock. Notice that in the equation (5.15) we still have standard partial derivative with respect to time. Interpreting via Page-Wotters formalism, this implies that (at this stage) time operator still corresponds to the standard coordinate operator, for which we have eigenvalue equation $\hat{T}|t\rangle=t|t\rangle$. We can thus write the history state in terms of *clock being one particle system*:

$$|\Psi\rangle\!\rangle=\int dt|t\rangle_C\otimes|\Psi(t)\rangle_S \tag{5.19}$$

which takes the form of standard Page-Wotters history state, reflecting a mere fact that quantum field theory satisfies standard Schrödinger equation (5.2). Nevertheless, one should keep in mind that $|\Psi(t)\rangle$ now represents a general state in the space

$L^2\left(\mathcal{C}\left(\Sigma_t\right)\right)$, such that in 'coordinate' representation,

$$|\Psi[\phi]\rangle = \int dt |t\rangle_C \otimes \Psi[t,\phi]_S \tag{5.20}$$

Equation (5.20) corresponds to the situation of taking a perspective of one particle clock degree of freedom and looking at the history state of field state configuration with respect to its preferred slicing.
Nevertheless, as we stated in Chapter (2.1.5), in order to implement symmetries of spacetime, we need to look for unitary representation of Poincaré group which is infinite dimensional. Thus, we cannot couple to the 1-dimensional system without breaking relativistic covariance. To account for this, we can take into considerations continuous limit of clocks, such that they correspond to the field τ

$$\hat{\tau}(\vec{x})|\tau\rangle = \tau(\vec{x})|\tau\rangle \tag{5.21}$$

where $\hat{\tau}(\vec{x})$ represents operator corresponding to the field of clocks and $\tau(\vec{x})$ is a scalar function representing field configuration. This now means that every degree of freedom of a field can be associated to some time function with corresponding foliation. For the clock momentum we would now have,

$$\hat{\omega}(\vec{x}) = -i\frac{\delta}{\delta\tau(\vec{x})} \tag{5.22}$$

and the constraint equation gets promoted into functional differential equation with respect to two functions:

$$_C\langle\tau|\hat{C}|\Psi\rangle = {}_C\langle\tau|(\hat{H}_C \otimes \mathbb{1}_S + \mathbb{1}_C \otimes \hat{H}_S)|\Psi\rangle\rangle = 0 \tag{5.23}$$

$$\left(-i\frac{\delta}{\delta\tau(\vec{x})} + H_S[\phi(\vec{x})]\right)|\Psi[\tau(\vec{x})]\rangle = 0 \tag{5.24}$$

where for the clock Hamiltonian we assumed the perfect clock form

$$\hat{H}_C = \hat{\omega} = -i\frac{\delta}{\delta\tau(\vec{x})} \tag{5.25}$$

In 'coordinate' representation we have the equation:

$$i\frac{\delta}{\delta\tau(\vec{x})}\Psi[\phi(\vec{x}),\tau(\vec{x})] = H_S[\phi(\vec{x})]\Psi[\phi(\vec{x}),\tau(\vec{x})] \tag{5.26}$$

This can be recognized as the form of *Schwinger-Tomonaga equation*:

$$\left(H(x) - i\frac{\delta}{\delta\sigma(x)}\right)\Psi[\sigma] = 0 \tag{5.27}$$

where the formal solution of this equation will be given by the state functional[81]

$$\Psi[\sigma] = \mathcal{U}\left[\sigma,\sigma_0\right]\Psi\left[\sigma_0\right] \tag{5.28}$$

Here σ_0 denotes the initial condition on the hyper-surface and $\Psi\left[\sigma_0\right]$ denotes the state functional corresponding to this initial condition. Thus, Schwinger-Tomonaga equation describes the evolution of the the state of a quantum field from one initial Cauchy surface, to another and was introduced by Schwinger and Tomonaga [77] attempting to make invariant generalization of Schrödinger equation. It might seem interesting how we arrived to the same equation, bearing similar conceptual input, out of somewhat different context. Nevertheless, it was argued in [18] that the transformation functional will not be unitarily implemented on the Fock space representation of the quantum field for the case of interacting fields, for the dimension of spacetime grater then two, even in the flat spacetime. Particular problem seems to be that the evolution from an initial surface to a final surface is dependent of the choice of foliation, as was discussed by Kuchař in [56]. However, it is not so obvious that this problems cannot be addressed in any way and that criticism cannot be surpassed. Keeping that in mind, we will for the sake of discussion proceed, recalling that the motivation of using this approach was to lift the discussion of Section (4.2) to the field theory context.

The history state would now be written as,

$$|\Psi\rangle\!\rangle = \int \mathcal{D}\tau|\tau\rangle_C \otimes |\Psi[\tau(\vec{x})]\rangle_S \tag{5.29}$$

where $\mathcal{D}\tau$ is the measure of functional integration. This would represent evolution of the field state $|\Psi\rangle$, spanned by the basis elements of $\{|\phi\rangle\}$, evolving relative

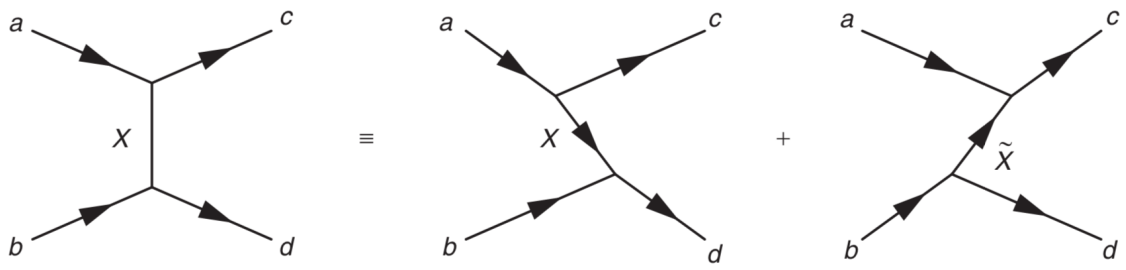


Figure 5.1: Two time ordering of Feynman propagator. [82]

to the field configurations of a field $\tau(\vec{x})$, given by (5.21).

5.2 Scattering amplitude

Let us recall that the idea of our considerations was to view the situation of two time orderings appearing in the Feynman propagator (see Figure(**??**)), in the sense of Hilbert state superposition of orders $A \prec B$ and $B \prec A$, analogous to [89]. Recall that Feynman diagrams are directly related to the scattering amplitude, which for two particle 'in' and 'out' states takes the form:

$$\mathcal{A} = {}^{\text{out}}\left\langle q_1 q_2 \mid p_2 p_1 \right\rangle^{\text{in}} = {}_0\left\langle q_1 q_2 \left| \hat{S} \right| p_2 p_1 \right\rangle_0 \tag{5.30}$$

where $|\psi\rangle = |p_2 p_1\rangle_0$ represent asymptotic free states existing at the beginning of the experiment $t \to -\infty$, $|\phi\rangle = |q_2 q_1\rangle_0$ asymptotic state at the end of the experiment $t \to \infty$. The definition of the S-matrix is given by (as discussed in (2.4.2)):

$$S \equiv U(\infty, -\infty) = T \exp\left(-\mathrm{i} \int_{-\infty}^{\infty} \mathcal{H}_{\text{int}}^{\text{I}}(t) \mathrm{d}t\right) \tag{5.31}$$

where the Dyson operator comes as a solution of Schrödinger's equation which one gets from the constraint equation (5.18). Let us now implement Page-Wotters formalism into the expression of scattering amplitude.

In standard quantum mechanics, we define the propagator between some initial $|I\rangle_S$ state at time t_I and some final state $|F\rangle_S$ at time t_F:

$$G\left(F, t_F; I, t_I\right) := {}_S\left\langle F \left| \hat{U}_S\left(t_F, t_I\right) \right| I \right\rangle_S \tag{5.32}$$

To cast the propagator in terms of Page-Wotters formalism we simply fix the initial

condition of history state $|\Psi\rangle\rangle$, i.e. we identify t_0 with the time t_I and $|\psi\left(t_0\right]\rangle_S$ with $|I\rangle_S$ in the equation:

$$|\Psi[\phi]\rangle = \int dt|t\rangle_C \otimes \hat{U}(t,t_0)_S \psi[t_0;\phi]_S \tag{5.33}$$

The propagator then simply takes the form (see [37] eqv. (22)):

$$G\left(F,t_F;I,t_I\right) = {}_C\left\langle t_F\right|\otimes_S\langle F \mid \Psi\rangle\rangle \tag{5.34}$$

Now we can extend this to QFT scenarios via Schrödinger functional representaiton. The propagator will now go from some initial hypersurface field configuration Σ_x at t_x to final Σ_y at t_y:

$$G\left(F,t_F;I,t_I\right) = {}_C\left\langle t_y\right|\otimes_S\langle\Psi(t_y)|\Psi\rangle\rangle_{(t_x,\Sigma_x)} \tag{5.35}$$

which enables us immediately to cast the scattering amplitude in terms of history state. Namely, equation (5.30) would now have the form:

$$\mathcal{A} = \lim_{t_F\to\infty}\langle t_F|\otimes {}_0\left\langle q_1q_2 \mid \Psi\right\rangle\rangle \tag{5.36}$$

where

$$|\Psi\rangle\rangle = \int dt|t\rangle_C \otimes \hat{U}(t,-\infty)_S|p_2p_1\rangle_{0,S} \tag{5.37}$$

Here $\hat{U}(t,-\infty)_S$ is a solution to equation (5.2). In the coordinate representation,

$$|\Psi[\phi]\rangle = \int dt|t\rangle_C \otimes \hat{U}(t,-\infty)_S \mathbf{a}^\dagger\left(\vec{p}_2\right)\mathbf{a}^\dagger\left(\vec{p}_1\right)\Psi_0^{(0)}[\phi]_S \tag{5.38}$$

where we used $\langle\phi \mid p_1p_2\rangle = \Psi^{(0)}_{p_1p_2}[\phi] = \mathbf{a}^\dagger\left(\vec{p}_2\right)\mathbf{a}^\dagger\left(\vec{p}_1\right)\Psi_0^{(0)}[\phi]$ [48] and thus,

$$\mathcal{A} = \int \mathcal{D}\phi \lim_{t_F\to\infty}\langle t_F|\otimes\int dt|t\rangle_C \otimes \Psi_0^{(0)\dagger}[\phi]_S \mathbf{a}\left(\vec{q}_2\right)\mathbf{a}\left(\vec{q}_1\right)\hat{U}(t,-\infty)_S\mathbf{a}^\dagger\left(\vec{p}_2\right)\mathbf{a}^\dagger\left(\vec{p}_1\right)\Psi_0^{(0)}[\phi]_S \tag{5.39}$$

Here $\Psi_0^{(0)}$ denotes noninteracting ground state. This equation assumes that the clock is some external, non-interacting system and *does not* refer to Schwinger-Tomonaga formulation. In this case, we can implement considerations stated in (4.2) since we would again expect some time ordered exponential but this time acting on

the kinematical state functional, constraining functional Fock space.

If we now want to implement the equation with respect to (5.29),

$$\mathcal{A} = \lim_{t_F \to \infty} \langle t_F| \otimes {}_0\langle q_1 q_2 \mid \Big(\int \mathcal{D}\tau |\tau\rangle_C \otimes \tilde{\mathcal{U}}(\tau, \tau_0) |p_2 p_1\rangle_{0,S} \Big) \tag{5.40}$$

where $\tilde{\mathcal{U}}$ now refers to the solution of (5.26). Or, we can write it as:

$$\mathcal{A} = \int \mathcal{D}\phi \lim_{t_F \to \infty} \langle t_F| \otimes_0 \Psi_0^{(0)\dagger}[\tau, \phi]_S \mathbf{a}\left(\vec{q}_2\right) \mathbf{a}\left(\vec{q}_1\right) \int \mathcal{D}\tau |\tau\rangle_C \otimes \tilde{\mathcal{U}}(\tau, \tau_0) \mathbf{a}^\dagger\left(\vec{p}_2\right) \mathbf{a}^\dagger\left(\vec{p}_1\right) \Psi_0^{(0)}[\tau_0; \phi_0]_S \tag{5.41}$$

where, if we are choosing internal degrees of freedom of particles as clocks, we can define initial clock configuration τ_0 to correspond to the $|\tau_0\rangle_C = \bigotimes_{n=2} |t_n\rangle$, i.e. an initial state of the experiment with respect to the initial particles.

In this way we can argue the connection between the toy model results of Chapter (4.2) and the motivation given in the context of time ordering and Feynman propagators. Namely, one can argue that given this construction, one can make specific initial preparations and measurements such that time ordering, which is expected to appear in the context of the solution (5.26) [81], would correspond with isolating one or the other branch of the process, as depicted in Figure (5.1).

6 Summary and outlook

In this thesis, we explored the notions of time and causal ordering as they appear in quantum mechanics and perturbative quantum field theory. Special attention was dedicated to the structure of time order exponentials, expanded into the Dyson series, where we attempted to understand their time ordering in a sense of ordering appearing in a quantum switch. Namely, the main motivation of our contemplation was to compare the orderings appearing on the level of the Feynman propagator, with the superposition of a quantum switch. As it was explained in the Chapter (2.4) the ordering operator plays an important role in the Feynman propagator and the interaction process viewed as an exchange over the virtual particle, can be understood as a superposition of two processes exchanging on mass-shell particle or antiparticle. It seems plausible then to say that isolating one branch of the ordering configuration in Dyson series expansion, would mean isolating one of the mass-shell exchange processes.

Having in mind this motivation, we first considered time ordered exponential in the context of standard quantum mechanics, as it appears in the solution to time dependent Schrödinger equation with $[\hat{H}(t), \hat{H}(t')] \neq 0$. This setting was analyzed in Chapter (3) where we attempted to couple the ancillary system with an interaction that would couple differently to one ordering branch or the other, enabling us to use it as the control over the orders. Namely, if such coupling were to be possible, one could use the ancillary system to serve as control over the ordering branches, enabling us to isolate one of the branches via certain PVM measurements on the ancillary state. Nevertheless, we convinced ourselves that it was not possible to introduce a control degree of freedom, within the scope of the presented methods. The puzzling aspect of these considerations is that according to the no-go theorem stated in [24] one cannot have a superposition of orders without a control system. We will thus conclude that there might be some inaccessible background degree of freedom serving as a control, enabling a superposition of ordering configurations, or that the superposition of orders is merely an artifact of this mathematical formalism and should not be taken in the exact sense of a quantum switch. We emphasize that these considerations rely on interpretations attached to a perturbative understanding

of the theory and reasoning should be compared with algebraic description and that it is yet not fully clear how to understand the notions of supermaps in QFT context. Since the notion of supermap is mathematical tool for encoding superpositions of quantum evolutions, efforts towards the implementation of supermaps into QFT context bares potential significant in the quantum gravity context. Our work can be understood as the initiative in that direction.

In Chapter (4) we proceeded with the attempt to isolate one ordering branch in time ordered exponential, this time in the context of history state with several clocks, which now goes beyond the context of standard quantum mechanics (understood in terms of Schrödinger's equation). Here we managed to isolate contributions of time ordering by suitable choice of preparations and the measurement of states in the Hilbert space of relative degrees of freedom. We offered an interpretation in terms of parameters appearing in the history state to have the meaning of classical parameter corresponding to a time of some classical external clock. This means that we will understand our results as indefiniteness appearing with respect to a classical clock observing operations done by two entangled quantum clocks attached to two different time functions, in the context of fixed Minkowski background. The fact that the attempt succeeded might be understood as the fact that the history state encodes all the correlated configurations between the system and the clocks, where now relative degrees of freedom of the clocks may be understood as control over the order enabling a superposition. As a preliminary step towards the final construction, we extended our model to N-clocks.

In the last section, we described Schrödinger's functional representation of quantum field theory, emphasizing its importance in the contexts of these discussions. We argued that the extension to the Page-Wotters formalism is in this setting simple if one takes one degree of freedom serving as a clock with respect to which hypersurfaces of field configurations evolve. Nevertheless, since the symmetry of field theory space has a unitary representation of the Poincaré group, which is infinite-dimensional as discussed in Chapter (2.4), coupling one degree of freedom would not be compatible with it and such a history state would not be Lorentz invariant. We thus extended considerations by assuming a field whose space of eigenstates will serve as a clock for

the evolution of another field with respect to it. With this simple argumentation inspired by the Page-Wotters formalism, we arrived at an already known equation, the Schwinger-Tomonaga equation. This construction gave *the evolution of one field with respect to field configurations of the clock field.* Nevertheless, as it was discussed in [18][56], the Schwinger-Tomonaga equation suffers its difficulties, the biggest being the dependence of the evolution from one Cauchy surface to another on the choice of foliation and nonunitary for the spacetimes $dim > 2$. Given our considerations, if the Schwinger-Tomonaga equation cannot be implemented in the quantum gravity context, the consequence might be also that the program of the Page-Wotters approach itself cannot be covariantly implemented. To lift Page-Wotter's approach to field theory setting, the construction of Schrödinger functional space should be understood more rigorously and the problems of the Schwinger-Tomonaga equation need to be circumvented.

Appendices

Appendix A Probabilistic approach to quantum mechanics

Any experiment can be understood in terms of *preparation*, when a particular experimental set-up determines some initial conditions and the *measurement* in which the prepared system is coupled to a measuring device, resulting in some output data. Let us suppose a state $\tilde{\rho}$ is prepared. Then a measurement will lead to some outcome a_i. In the context of quantum mechanics, the final results of the experiment are described by probability distributions. Namely, deterministic dependence on the initial preparation is replaced by statistical: *'...the individual results in a sequence of identical, independent realizations of an experiment may vary, but the occurrence of one or another result in a long enough sequence of realizations can be characterized by a definite stable frequency.'*

For a given observable A, associated with the outcomes $\{a_i\}$ measurement can be described via mapping:

$$(\tilde{\rho}, a_i)^A \mapsto p^A(a_i|\tilde{\rho}) := \mu_\rho^A(a_i) \tag{A.1}$$

where $\tilde{\rho}$ denotes input state. One should note that we can have different input states, yielding the same measuring statistics for a same measuring procedure; thus, we will define states instead as an equivalence class $\rho = [\tilde{\rho}] \in \mathcal{S}$ corresponding to the indis-

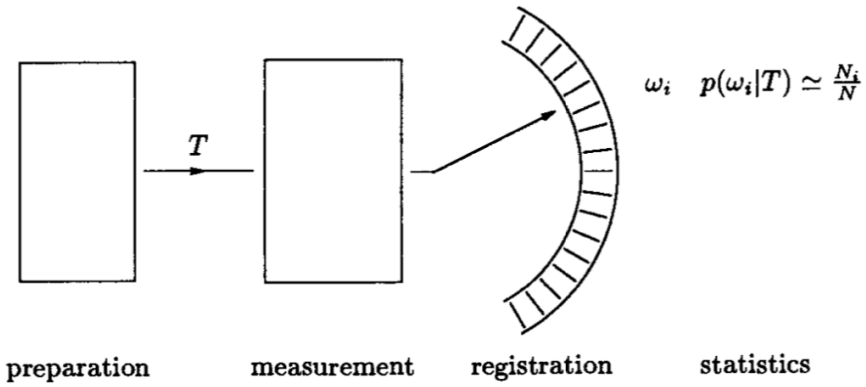


Figure A.1: [55]

tinguishable statistics, where $\mathcal{S}$ will denote set of states.

The structure of possible output data can be either discrete or continuous. For example, we can have a discrete set of pointer readings or we can have a measurement resulting in a picture of a whole trajectory. To encompass both cases, we assume that the outcomes of measurement form a *measurable space*, defined on a set U with σ-algebra Σ given on a collection of open sets $\mathcal{A}(U)$. [64] A set U is topological space, thus σ-algebra is *Borel σ-algebra*. The result of the measurement a will lie in a measurable subset $B \subset U$ and will represents an event. On a measurable space, we can define a measure μ as a map $\mu : \Sigma \to [0, \infty]$ on a σ-algebra Σ such that
(1) $\mu(\emptyset) = 0$
(2) $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu\left(A_j\right)$ if $A_j \cap A_k = \emptyset$ for all $j \neq k$ (σ-additivity)
As we said, algebra will be Borel σ-algebra $\mathcal{B}(U)$ and the corresponding measure will be Borel measure.
The measure is called a probability measure, if further conditions are satisfied:
1) positivity $\left(\mu_\rho^A\left(a_i\right) \geq 0\right)$ and
2) normalisation condition $\left(\sum_i \mu_\rho^A\left(a_i\right) = 1\right)$
Here we stated definition in terms of discrete measurement, assuming a finite set of pointer readings $\{a_1, \ldots, a_n\}$.

We define a statistical model as a pair $(\mathcal{S}, \mathcal{M})$ where $\mathcal{S}$ is a set of states and $\mathcal{M}$ represents a class of affine maps such that they map $\mathcal{S}$ into the collections of probability distributions on some measurable spaces U. These maps represent measurements. Namely, we define *measurement* as a map which transforms a state into a probability distribution over possible outcomes,

$$\rho \mapsto \mu_\rho^A \tag{A.2}$$

In quantum mechanics, a set of states $\mathcal{S}$ will correspond to a set of complex Hermitian matrices ρ, satisfying

$$\rho \geq 0, \quad \operatorname{Tr} \rho = 1 \tag{A.3}$$

called density matrices. The set of all such matrices is a convex set $\mathcal{S}$, extreme

[64] Taking that an algebra is closed under countable unions, defines that it is σ-algebra.

points being the one-dimensional projections $\rho_\psi = \mid \psi\rangle\langle\psi \mid$. We will impose that measurement maps any mixture of states into the corresponding convex mixture of probabilities i.e. to preserve the convex structure of the state space [65]. Meaning, for a sequence of independent experiments, such that in each the object is prepared in some of the states ρ_α, with corresponding probabilities p_α [66]. The total state will be called a mixed state and will be denoted by

$$\rho = \rho\left(\{\rho_\alpha\}, \{p_\alpha\}\right) = \sum_a p_\alpha \rho_\alpha \tag{A.4}$$

we have for any measurement $\rho \to \mu_\rho$

$$\mu_\rho(\ \mathrm{d}a) = \sum_a p_\alpha \mu_{\rho_\alpha}(\mathrm{d}a) \tag{A.5}$$

Furthermore, referring to map $(A.1)$, we see that any outcome a_i induces a state functional E_i:

$$E_i : \rho \to E_i(\rho) := p(a_i, \rho) \tag{A.6}$$

called an *effect*. We can define now a *measured observable* as a map associating to each outcome a_i corresponding effect.

$$E : a_i \to E_i \tag{A.7}$$

Effects will define either orthogonal or nonorthogonal measurement, depending on whether they satisfy condition of projectivity.

Namely, a mapping $E : \mathcal{B}(\mathbf{R}) \to \mathcal{L}(\mathcal{H})$ is a projection valued (PV) measure, corresponding to orthogonal measurement, if

$$\begin{aligned} E(X) &= E(X)^\dagger = E(X)^2 \text{ for all } X \in \mathcal{B}(\mathbf{R}) \\ E(\mathbf{R}) &= I \\ E\left(\cup X_i\right) &= \sum E\left(X_i\right) \text{ for all disjoint sequences } (X_i) \subset \mathcal{B}(\mathbf{R}) \end{aligned} \tag{A.8}$$

where the series converges in the weak operator topology of $\mathcal{L}(\mathcal{H})$. Here $\mathcal{L}(\mathcal{H})$

[65] This is a property we will impose on a general quantum map, of which measurement is an example.

[66] Preparing such mixture can be understood as a consequence of some fluctuations in the measurement apparatus.

denote the set of bounded linear operators on $\mathcal{H}$. This is a Banach space with respect to the operator norm, $\|A\| = \sup\{\|A\varphi\| \mid \varphi \in \mathcal{H}, \|\varphi\| = 1\}$. If the measurable space $(U, \mathcal{B})$ underlying a PV measure E is the real Borel space $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$, then E determines a unique self-adjoint operator A, and conversely, any real PV measure is determined by a unique selfadjoint operator. [67]

These properties of a PV measure E, along with continuity and the additivity properties of the inner product guarantee that for any unit vector φ the mapping

$$\begin{aligned} &\mu_\varphi : \mathcal{B}(\mathbf{R}) \to [0, \infty] \\ &X \in \mathcal{B}(\mathbf{R}) \mapsto \mu_\varphi(X) := \langle \varphi | E(X) | \varphi \rangle \end{aligned} \tag{A.9}$$

is a probability measure. If $\varphi \in \mathcal{H}$ is an arbitrary vector, then $X \mapsto \langle \varphi | E(X) | \varphi \rangle$ is a real measure, with normalisation $\langle \varphi \mid E(\mathbf{R})\varphi \rangle = \langle \varphi \mid \varphi \rangle$, $\mu_\phi(\mathbf{R}) = \|\phi\|^2 < \infty$. This enables us to define integration with respect to our projection-valued measure, and define self-adjoint operators with the continuous spectre. Namely, $A := \int x dE(x)$ with its domain of definition $D(A)$ consisting of those vectors $\varphi \in \mathcal{H}$ for which the integral $\int x^2 d\mu_\varphi^E(x)$ is convergent. Namely, let A be a self-adjoint operator with the domain $D(A) \subset \mathcal{H}$. There is a unique PV measure $E : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ such that

$$D(A) = \left\{ \varphi \in \mathcal{H} \mid \int_{\mathbf{R}} x^2 d\langle \varphi | E(x) | \varphi \rangle < \infty \right\} \tag{A.10}$$

and for any $\varphi \in D(A)$

$$\langle \varphi | A | \varphi \rangle = \int_{\mathbf{R}} x d\langle \varphi | E(x) | \varphi \rangle \tag{A.11}$$

Let U be a nonempty set and $\mathcal{B}$ Borel σ-algebra of subsets of U so that $(U, \mathcal{B})$ is a measurable space. A normalised positive [68] operator valued (POV) measure

[67] An operator $A \in \mathcal{L}(\mathcal{H})$ is self-adjoint if A equals its adjoint A^*, and it is a projection operator if $A = A^* = A^2$. The notion of adjoint operator can be extended also to unbounded operators defined on a dense domain $D(A)$. Then an operator is self-adjoint whenever $D(A) = D(A^*)$ and $A = A^*$.

[68] An operator $A \in \mathcal{L}(\mathcal{H})$ is positive, $A \geq O$, if $\langle \varphi \mid A\varphi \rangle \geq 0$ for all vectors $\varphi \in \mathcal{H}$. A positive operator is always self-adjoint. The relation $A \geq B$ (or $B \leq A$), defined as $A - B \geq O$, is an ordering on the set of self-adjoint bounded operators.

$E : \mathcal{B}(\mathbf{R}) \to \mathcal{L}(\mathcal{H})$ on $(U, \mathcal{B})$ is defined through the properties:[41]

$$\begin{aligned} E(X) &\geq O \text{ for all } X \in \mathcal{B} \\ E(\Omega) &= I \\ E\left(\cup X_i\right) &= \sum E\left(X_i\right) \text{ for all disjoint sequences } (X_i) \subset \mathcal{B} \end{aligned} \tag{A.12}$$

where the series converges in the weak operator topology of $\mathcal{L}(\mathcal{H})$. For any POV measure $E : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ the following two conditions are equivalent:

$$\begin{aligned} E(X)^2 &= E(X) \text{ for all } X \in \mathcal{F} \\ E(X \cap Y) &= E(X)E(Y) \text{ for all } X, Y \in \mathcal{F} \end{aligned} \tag{A.13}$$

Thus a positive operator valued measure is a projection valued measure exactly when it is multiplicative.

Now, one can show [44] that any affine functional $\mu(\rho)$ has the form $\mu(S) = \operatorname{Tr} SE$, where E is a Hermitean matrix. Then, the relation

$$\mu_\rho(u) = \operatorname{Tr} SE_u, \quad u \in U, \tag{A.14}$$

establishes the one-to-one correspondence between affine maps $\rho \to \mu_\rho$ of the set of density matrices into the set of probability distributions on U and the resolutions of identity $\{E_u; u \in U\}$, i.e., the collections of Hermitean matrices $\{E_u\}$, satisfying

$$E_u \geq 0, \quad \sum_{u \in U} E_u = I. \tag{A.15}$$

According to this, any pair (E, ρ) of an observable E and a state ρ induces a probability measure μ_ρ^E on the value space $(U, \mathcal{B})$ of E. The number $\mu_\rho^E(X)$ is the probability that a measurement of the observable E performed on the system ρ leads to a result in the set U. The notion of an observable provides a representation of the possible events occurring as outcomes of a measurement. In this sense an observable is defined as and identified with a POV measure $E : \mathcal{B} \to \mathcal{L}(\mathcal{H}), U \mapsto E(U)$ on a measurable space $(U, \mathcal{B})$, describing the possible measurement outcomes.

According to spectral theorem, we can then write a general observable as:

$$X = \int \lambda E(\,\mathrm{d}\lambda) \tag{A.16}$$

If the measurements correspond to a discrete set, we will have,

$$E(\,\mathrm{d}\lambda) = \left[\sum_k \delta\left(\lambda - \lambda_k\right) E_k\right] \mathrm{d}\lambda \tag{A.17}$$

where $\lambda_k \in \mathcal{B}(\mathbf{R})$. This then simply yields,

$$X = \sum_k \lambda_k E_k \tag{A.18}$$

Appendix B Wick's theorem

To derive Wick's theorem, we split the operators into positive (which contains only $\hat{a}$ operators and positive e^{-iEt} phases) and negative part of the spectrum (which contains only $a^\dagger$ operators and negative part of the spectrum e^{+iEt}),

$$\begin{aligned}
\phi_I^+(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} a_{\vec{p}} e^{-ip\cdot x} \\
\phi_I^-(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} a_{\vec{p}}^\dagger e^{ip\cdot x}
\end{aligned} \tag{B.1}$$

We then introduce T product, as defined in Chapter (2.4),

$$\begin{aligned}
T\left(\phi_I(x)\phi_I(y)\right) &= \theta\left(x^0 - y^0\right)\phi_I(x)\phi_I(y) + \theta\left(y^0 - x^0\right)\phi_I(y)\phi_I(x) \\
&= \theta\left(x^0 - y^0\right)\left(\phi_I^+(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y)\right) \\
&+ \theta\left(y^0 - x^0\right)\left(\phi_I^+(y)\phi_I^+(x) + \phi_I^+(y)\phi_I^-(x) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(y)\phi_I^-(x)\right) \\
&= \theta\left(x^0 - y^0\right)\left(\phi_I^+(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + \left[\phi_I^+(x), \phi_I^-(y)\right]\right) \\
&+ \theta\left(y^0 - x^0\right)\left(\phi_I^+(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + \left[\phi_I^+(y), \phi_I^-(x)\right]\right) \\
&= N\left(\phi_I(x)\phi_I(y)\right) + \phi(x)\phi(y))
\end{aligned} \tag{B.2}$$

where normal ordering, denoted by N, is defined to put annihilation operators

on the right and creation operators on the left, so that for every operator we have:

$$N(\hat{O}) = \hat{O} - \langle 0|\hat{O}|0\rangle \tag{B.3}$$

and where the contractions give Feynman propagator,

$$\begin{aligned}
\overbrace{\hat{\phi}(x)\hat{\phi}}(y) &\equiv \theta\left(x^0 - y^0\right)\left[\phi_I^+(x), \phi_I^-(y)\right] + \theta\left(y^0 - x^0\right)\left[\phi_I^+(y), \phi_I^-(x)\right] \\
&= \theta\left(x^0 - y^0\right)\langle 0\,|\phi_I(x)\phi_I(y)|\,0\rangle + \theta\left(y^0 - x^0\right)\langle 0\,|\phi_I(y)\phi_I(x)|\,0\rangle \\
&= \theta\left(x^0 - y^0\right) D(x-y) + \theta\left(y^0 - x^0\right) D(y-x) \\
&= D_F(x-y)
\end{aligned} \tag{B.4}$$

This construction then generally gives,

$$T[\hat{A}\hat{B}\hat{C}\ldots\hat{Z}] = N\left[\hat{A}\hat{B}\hat{C}\ldots\hat{Z} + \begin{array}{c}\text{all possible contractions of} \\ \hat{A}\hat{B}\hat{C}\ldots\hat{Z}\end{array}\right] \tag{B.5}$$

For more details, see [76].

Appendix C Supplements

In order to compute the action of the exponential in equation (4.43), we rewrite the integrand using the Trotter product formula [69]:

$$e^{-i\alpha\hat{C}}\,|t_A, t_B, \phi\rangle_{A,B,S} = \lim_{N\to\infty}\left(\underbrace{e^{-i\frac{\alpha}{N}(\hat{H}_A+\hat{H}_B+\hat{H}_S)}e^{-i\frac{\alpha}{N}(\hat{V}_S(\hat{T}_A)+\hat{V}_S(\hat{T}_B))}}_{\equiv\hat{L}}\right)^N |t_A, t_B, \phi\rangle_{A,B,S} \tag{C.1}$$

Let us take $\hat{H}_S = 0$ to switch to the interaction picture. Consider the action on

[69] Trotter product formula

$$e^{t(A+B)} = \lim_{n\to\infty}\left(e^{tA/n}e^{tB/n}\right)^n$$

first few terms:

$$
\begin{aligned}
& n = 1 \\
\hat{L}\,|t_A, t_B, \phi\rangle &= e^{-i\frac{\alpha}{N}(\hat{H}_A+\hat{H}_B)} e^{-i\frac{\alpha}{N}(\hat{V}_S(\hat{T}_A)+\hat{V}_S(\hat{T}_B))}\,|t_A, t_B, \phi\rangle \\
&= e^{-i\frac{\alpha}{N}(\hat{V}_S(t_A)+\hat{V}_S(t_B))} e^{-i\frac{\alpha}{N}(\hat{H}_A+\hat{H}_B)}\,|t_A, t_B, \phi\rangle \\
&= e^{-i\frac{\alpha}{N}(\hat{V}_S(t_A)+\hat{V}_S(t_B))} \big|t_A + \tfrac{\alpha}{N}, t_B + \tfrac{\alpha}{N}, \phi\big\rangle
\end{aligned}
\tag{C.2}
$$

Repeat iteratively:

$$
\begin{aligned}
& n = 2 \\
& \hat{L}^2\,|t_A, t_B, \phi\rangle = e^{-i\frac{\alpha}{N}(\hat{H}_A+\hat{H}_B)} e^{-i\frac{\alpha}{N}(\hat{V}_S(\hat{T}_A)+\hat{V}_S(\hat{T}_B))} e^{-i\frac{\alpha}{N}(\hat{V}_S(t_A)+\hat{V}_S(t_B))} \big|t_A + \tfrac{\alpha}{N}, t_B + \tfrac{\alpha}{N}, \phi\big\rangle \\
& = e^{-i\frac{\alpha}{N}(\hat{H}_A+\hat{H}_B)} e^{-i\frac{\alpha}{N}(\hat{V}_S(\hat{T}_A)+\hat{V}_S(\hat{T}_B))} \big|t_A + \tfrac{\alpha}{N}, t_B + \tfrac{\alpha}{N}\big\rangle e^{-i\frac{\alpha}{N}\hat{H}_S} e^{-i\frac{\alpha}{N}(\hat{V}_S(t_A)+\hat{V}_S(t_B))}\,|\phi\rangle \\
& = e^{-i\frac{\alpha}{N}(\hat{H}_A+\hat{H}_B)} \big|t_A + \tfrac{\alpha}{N}, t_B + \tfrac{\alpha}{N}\big\rangle e^{-i\frac{\alpha}{N}(\hat{V}_S(t_A+\frac{\alpha}{N})+\hat{V}_S(t_B+\frac{\alpha}{N}))} e^{-i\frac{\alpha}{N}\hat{H}_S} e^{-i\frac{\alpha}{N}(\hat{V}_S(t_A)+\hat{V}_S(t_B))}\,|\phi\rangle \\
& = e^{-i\frac{\alpha}{N}(\hat{V}_S(t_A+\frac{\alpha}{N})+\hat{V}_S(t_B+\frac{\alpha}{N}))} e^{-i\frac{\alpha}{N}\hat{H}_S} e^{-i\frac{\alpha}{N}(\hat{V}_S(t_A)+\hat{V}_S(t_B))} \big|t_A + \tfrac{2\alpha}{N}, t_B + \tfrac{2\alpha}{N}\big\rangle \\
& \vdots \\
& n = N \\
& \hat{L}^N\,|t_A, t_B, \phi\rangle = \\
& = e^{-i\frac{\alpha}{N}(\hat{V}_S(t_A+\frac{\alpha(n-1)}{N})+\hat{V}_S(t_B+\frac{\alpha(n-1)}{N}))} \dots e^{-i\frac{\alpha}{N}\hat{H}_S} e^{-i\frac{\alpha}{N}(\hat{V}_S(t_A)+\hat{V}_S(t_B))}\,|t_A + \alpha, t_B + \alpha, \phi\rangle \\
& = \overleftarrow{\prod}_{n=1}^{N} e^{-i\frac{\alpha}{N}(\hat{V}_S(t_A+\frac{\alpha(n-1)}{N})+\hat{V}_S(t_B+\frac{\alpha(n-1)}{N}))}\,|t_A + \alpha, t_B + \alpha, \phi\rangle
\end{aligned}
\tag{C.3}
$$

where we used $\overleftarrow{\prod}$ to denote that the product is ordered with respect to the arguments of potentials appearing in the exponentials (lowest arguments appearing rightmost). Under the ordering condition, effectively $[\hat{V}(t), \hat{V}(t')] = 0$ since regardless of the permutations of the string of operators, the resulting sequence will simply be re-ordered with respect to the chronological sequence of a chosen parameter. Thus, using the Baker–Campbell–Hausdorff formula[70] formula:

$$
\lim_{N\to\infty} \hat{L}^N\,|t_A, t_B, \phi\rangle = e^{-i\alpha(\hat{H}_S+\hat{H}_A+\hat{H}_B)} e^{-i\overleftarrow{\sum}_{n=1}^{N}\hat{V}_S(\hat{T}_A+\lambda)+\hat{V}_S(\hat{T}_B+\lambda)}\,|t_A, t_B, \phi\rangle \tag{C.4}
$$

To consider continuous limit $N \to \infty$, define $\lambda \equiv \frac{\alpha(n-1)}{N}$ such that $n = 1 \Rightarrow \lambda = 0$

[70] $$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]+\cdots}$$

and $n = N \Rightarrow \lambda = \alpha$. It follows:

$$\lim_{N\to\infty} \hat{L}^N \left|t_A, t_B, \phi\right\rangle = e^{-i\alpha(\hat{H}_A+\hat{H}_B)} T e^{-i\int_0^\alpha d\lambda \hat{V}_S(\hat{T}_A+\lambda)+\hat{V}_S(\hat{T}_B+\lambda)} \left|t_A, t_B, \phi\right\rangle \tag{C.5}$$

where we introduced time ordered product with respect to λ:

$$T(\hat{f}(\lambda_1)\hat{g}(\lambda_2)) = \theta(\lambda_1-\lambda_2)\hat{f}(\lambda_1)\hat{g}(\lambda_2) + \theta(\lambda_2-\lambda_1)\hat{g}(\lambda_2)\hat{f}(\lambda_1) \tag{C.6}$$

7 Prošireni sažetak

7.1 Uvod

Iako sve paradigme fizike pretpostavljaju pojam 'vremena', 'događaja' i 'promatrača', način na koji su ti pojmovi implementirani u formalizam često se značajno razlikuje među njima, što predstavlja konceptualni problem u režimima u kojima bi se dvije teorije trebale kombinirati. Jedna od prepoznatih podjela s obzirom na navedene pojmove je podjela na teorije ovisne i neovisne o pozadini prostorvremena. Kvantna mehanika, uz Newtonovu fiziku, specijalnu teoriju relativnosti i kvantum teorija polja, ovisi o pozadini, budući da je postavljena u fiksnu, nedinamičnu strukturu prostora i vremena, potpuno netaknutu pretpostavljenim dinamičkim zakonom. S druge strane, u općoj teoriji relativnosti, i polje materije i prostorvrijeme su podvrgnute dinamičkom zakonu koji teorija pretpostavlja. Ovo proizlazi iz relacijske strukture opće teorije relativnosti, gdje se neovisnost o pozadini prostorvremena može pokazati kao posljedicu simetrije s obzirom na proizvoljne infinitezimalne difeomorfizme. Suprotno tome, ne samo da kvantna teorija nema takvu simetriju, već ona u potpunosti prepušta pojam vremena izvan svog režima. Naime, vrijeme nije observabla teorije, niti je podvrgnuto kvantnoj neodređenosti, već se tretira kao vanjski klasični parametar. I vremenski i kauzalni poredak zaparavo su, u kontekstu standardne kvantne mehanike i kvantne teorije polja, potpuno netaknuti kvantnim režimom. Naravno, ovo predstavlja poteškoću kada se razmatraju mogući konteksti kvantne gravitacije u kojima prostorvrijeme, a time i samo vrijeme, treba kvantizirati. Tu bi se moglo očekivati da bi pojam kauzalnosti trebao podleći načelu superpozicije, uvodeći kvantnu neodređenost u strukturu prostorvremena, čineći pozadinu i dinamičnom i probabilističnom [15].

Imajući to u vidu, u nedavnim istraživanjima javila se ideja o kauzalnim odnosima podvrgnutim kvantnoj neodređenosti [12][58][45][47]. Iznenađujuće je možda to što je kontekst ovih razmatranja smješten u kontekst teorije informacija i kvantnog računarstva, gdje je konfiguracija kvantnog kruga (eng. *quantum circuit*) komparirana s razlaganjem događaja u strukturi prostorvremena. Predloženo je [22] da se geometrija žica između vrata (eng. *gate*) može kontrolirati kvantnim stanjem kontroliranog dvorazinskog sustava (eng. *qubita*), uvodeći koherentnu neodređenost u

poredak procesa. Takva realizacija poznata je kao *kvantna sklopka* (eng. *quantum switch*) i predstavlja primjer kauzalno neseparabilnog procesa. Ovi procesi, nekompatibilni s određenim vremenskim redoslijedom, ali u određenom podskupu još uvijek kompatibilni sa fiksnom kauzalnom strukturom, eksperimentalno su dokazani korištenjem takozvanih "kauzalnih svjedoka" [3][75]. Čak i iznad toga, uspostavljen je okvir procesnih matrica (eng. *process matrices*) [63], gdje su uvedeni resursi nekompatibilni s definitnim kauzalnim redoslijedom kvantnih operacija. Glavni matematički alat za implementaciju neodređenosti uređenja jesu kvantne supermape [22]. S obzirom na motivaciju u kontekstu kvantne gravitacije, spomenuta je razmatranja potrebno najprije implementirati u kontekst kvantne teorije polja, gdje naše istraživanje predstavlja korak u tom smjeru.
Stoga smo se pitali može li pronaći primjer neodređenosti uređenja, poput one koja se pojavljuje na razini kvantne sklopke, unutar okvira standardne fizike?[71] Jedan od mogućih mjesta za traženje je Feynmanova interpretacija antičestica [35] i prostorno-vremenska interpretacija procesa interakcije u smislu Feynmanovih dijagrama [34]. Naime, označavanjem događaja koji predstavljaju transformacije kao, $A =$' čestica stvorena' i $B =$' čestica anihilirana', imali bismo sličnu neodređenost uređenja, na razini superpozicije poredaka u Feynmanovu propagatoru. Unutar formalizma, ova neodređenost operacija se manifestira operatorom vremenskog uređenja, koji ima ulogu kronološkog poretka nizova operatora.

Iz tog je razloga glavni cilj istraživanja bio razmotriti poredak koji dolazi od operatora vremenskog uređenja. Namjeravali smo promatrati operator vremenskog uređenja u smislu dvije 'grane', koje odgovaraju dvjema definitivnim realizacijama uređenja. U suprotnosti sa superpozicijom poredaka koji se pojavljuju u kvantnom prekidaču, nastojali smo izolirati jednu od tih grana odgovarajućim odabirom priprema i mjerenja. Uzimajući za motivaciju ova pitanja, cilj diplomskog rada bio je istražiti dodirne točke i točke razilaženja suvremenog razumijevanja vremena i kauzalnosti u kvantnoj mehanici, s obzirom na ono u kontekstu kvantne teorije polja. U ova razmatranja ćemo ugraditi naše originalne rezultate, u vezi sa superpozicijom unutar operatora vremenskog uređenja. Ovaj diplomski rad neće istraživati algebarsku formulacije kvantne teorije polja, već će se usredotočiti na perturbativni pristup i

[71] čak i bez susreta s neodređenom metrikom koju bismo očekivali u kontekstu kvantne gravitacije

odgovarajuće interpretacije. Također ćemo pretpostaviti kao radnu idealizaciju koncept idealizirane čestice definirane u točki, kao i operatore polja definirani u točki.

7.2 Kvantne korelacije bez uzročnog reda

Kao što je spomenuto u uvodu, standardna kvantna mehanika je teorija koja ovisi o fiksnoj pozadini prostorvremena. Kao takva, a priori pretpostavlja neku fiksnu kauzalnu konfiguraciju laboratorija, namećući poredak s obzirom na neki globalni pojam vremena. Unatoč tome, čak i u fiksnoj pozadinskoj strukturi, uzročni poredak može biti slučajna varijabla: može postojati situacija u kojoj Alice postoji prije Boba s vjerojatnošću od $0 \leq q \leq 1$ i Bob postoji prije Alice s vjerojatnošću od $1 - q$. To će tada biti predstavljeno kao probabilistička mješavina dvaju mogućih poredaka $p(a, b \mid x, y) = \lambda p^{\mathrm{A} \preceq \mathrm{B}}(a, b \mid x, y) + (1 - \lambda) p^{\mathrm{B} \preceq \mathrm{A}}(a, b \mid x, y)$. Također se može formulirati pojam uzročnosti koji se ne odnosi na "fiksnu pozadinu" [63], [47], dopuštajući da redoslijed događaja ovisi o operacijama koje se izvode na lokacijama tih događanja. Naime, ovisno o izboru mjerenja, operacija na A može utjecati na redoslijed u kojem se pojavljuju B i C, koji leže u budućnosti A. Ovo čini uzročni poredak slučajnom funkcijom slučajnih događaja, a ne redoslijedom temeljnih prostorno-vremenskih lokacija u kojima se događaji događaju.

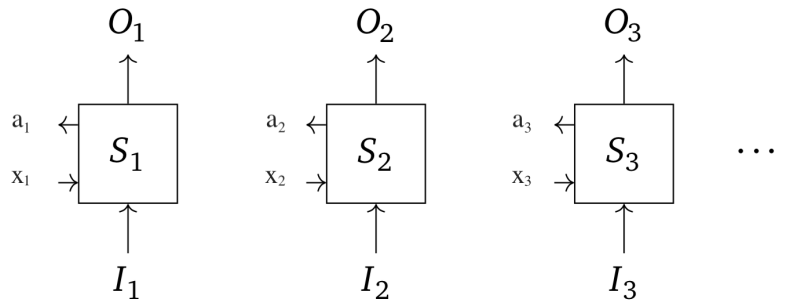


Figure 7.1: Stranka S_j ima slobodnu varijablu x_j i kanal C_j koji transformira ulaze (primljene iz okoline) I_j u izlaze (poslano natrag u okolinu) a_j i O_j. Pretpostavlja se da su stranke izolirane tako da svaka stranka može izvoditi operacije samo na svojim slučajnim varijablama.[6]

Takva razmatranja pretpostavljaju skup agenata (eng. *agents*), ugrađenih u *okolinu*[72], izvodeći lokalne pokuse u odgovarajućim lokalnim laboratorijima. Lokalni agent u

[72]Okolina je kanal koji preuzima izlaze agenata i proizvodi inpute agentima.

lokalnom laboratoriju je ono što se naziva *stranka*, dok je odgovarajući eksperiment ono što definira *događaj*. Svaka stranka je izolirana od svih ostalih stranaka i komunicira s okolinom samo jednom. U svakom izvođenju eksperimenta, agent prima fizički sustav iz okoline, odabire postavku mjerenja, izvodi mjerenje, dobiva ishod i pušta transformirani sustav van. Kao što se može primijetiti, to je već pretpostavljalo pojam vremena na lokalnoj skali svake stranke. Usprkos tome, globalno gledano, ne postoji pojam vremena ili fiksnog uzročnog reda nad skupom stranaka. Uzročni odnosi, koji se sada tretiraju kao slučajni i dinamički, tada se implementiraju na sljedeći način:

Definition 12 (Stranka i uzročni odnos između stranaka.)**.** [6] Stranka $S_j = (x_j, a_j, C_j)$ je trojka koja se sastoji od slobodne slučajne varijable x_j[73], neslobodna slučajna varijabla a_j i lokalna operacija C_j. Varijabla x_j naziva se ulaz od S_j, a varijabla a_j se naziva izlaz od S_j. Za dvije stranke S_j i S_k, kažemo da je S_j u uzročnoj prošlosti S_k akko $x_j \preceq a_k$. Ova relacija je izražena sa $S_j \preceq S_k$.

Drugim riječima, uzročnost među strankama definirana je tako da izbor postavki u lokalnom eksperimentu ne može utjecati na događaje koji su se dogodili u prošlosti ili istodobno, niti na uzročnu konfiguraciju tih događaja.

Uzročne korelacije su one koje se mogu ugraditi u fiksnu pozadinu prostorvremena. Kao što je već spomenuto, u određenoj kauzalnoj strukturi još uvijek može biti slučaj da uzročne veze između događaja nisu sa sigurnošću poznate. Takve korelacije, za slučaj dviju strana, bit će definirane kao:

Definition 13 (Kauzalne korelacije između dvije stranke)**.** [6] Neka su S_1 i S_2 dvije stranke. Distribucija vjerojatnosti $p(a_1, a_2 \mid x_1, x_2)$ naziva se uzročnom ako i samo ako se može napisati kao

$$\begin{aligned} p(a_1, a_2 \mid x_1, x_2) &= qp^{S_1 \preceq S_2}(a_1, a_2 \mid x_1, x_2) + (1-q)p^{S_2 \preceq S_1}(a_1, a_2 \mid x_1, x_2) \\ &= qp(a_1 \mid x_1)p(a_2 \mid a_1, x_1, x_2) + (1-q)p(a_1 \mid a_2, x_1, x_2)p(a_2 \mid x_2) \end{aligned} \tag{7.7}$$

gdje je q vjerojatnost.

Sada se postavlja pitanje: postoje li općenitiji uzročni odnosi, takvi da bi se proširili na novi skup korelacija slično kao što se kvantne korelacije, kršeći Bellove nejednakosti, protežu izvan skupa lokalnog politopa?[14] I doista, kao što je pokazano

[73]Gdje (slobodna) slučajna varijabla također može biti sastavljena od više (slobodnih) slučajnih varijabli, npr. $x_j = \left(x'_j, x''_j, x'''_j\right)$.

u [63], postoji mogućnost kauzalnih korelacija koje nisu kompatibilne s definitivnim temeljnim kauzalnim poretkom. Zatim, na sličan način kao i razmatranja nelokalnosti, za fiksni broj postavki i ishoda, uzročne korelacije tvore uzročni politop čiji aspekti definiraju *uzročne nejednakosti* [64] [10] [3]. Ako distribucija vjerojatnosti narušava uzročne nejednakosti, ne može se rastaviti u smislu $p(a,b \mid x,y) \neq \lambda p^{\mathrm{A}\preceq\mathrm{B}}(a,bx,y) + (1-\lambda)p^{\mathrm{B}\preceq\mathrm{A}}(a,bx,y)$ a nalazi se izvan kauzalnog politopa.

Razmotrimo uzročnu nejednakost na primjeru komunikacijskog zadatka između dviju stranke. Razmotrimo Alice S_1 i Boba S_2, koji svaki primaju sustav u svom laboratoriju. Nakon što određena strana primi sustav, na neki način generiraju slučajne varijable x (odnosi se na Alicein laboratorij) i y (odnosi se na Bobov). Bob će dodatno generirati još jedan nasumični bit y', koji određuje treba li on pogoditi bit od Alice ili ona treba pogoditi njegov bit. Proizvedeno pogađanje za Alice i Boba označit ćemo s a odnosno b. Zadatak njihove igre je maksimizirati vjerojatnost uspjeha. Označimo Alice $S_1 = \{x_1 = x, a_1 = a\}$ i Boba $S_2 = \{x_2 = (y,y'), a_2 = b\}$. Razmatramo situaciju u kojoj je Bob u prošlosti Alice, $B \preceq A$. Bobu je dan dodatni bit y', tako da ako je $y' = 0$, Bob će poslati svoj bit y Alice, omogućujući joj da sada pogodi njegov bit *savršeno* $p\left(a = y \mid y' = 0\right) = 1$, a ako je $y' = 1$, morat će pogoditi bit x *nasumično* $p(b = x \mid y' = 1) = 1/2$, daje

$$p^{\mathrm{succ}} := \frac{1}{2}\left(p\left(a = y \mid y' = 0\right) + p\left(b = x \mid y' = 1\right)\right) \leq \frac{3}{4} \tag{7.8}$$

Kako se ispostavilo, svako uređenje u $S_1 \preceq_{\mathrm{ST}} S_2$, ili $S_1 \succeq_{\mathrm{ST}} S_2$, ili probabilistička mješavina ovih, ne nadmašuje vjerojatnost uspjeha od $p^{\mathrm{succ}} = 3/4$. Za kauzalne distribucije, najveća vrijednost za vjerojatnost uspjeha je $3/4$, odražavajući činjenicu da u najboljem slučaju stranka može poslati vrijednost svoje slobodne varijable drugoj stranki, omogućujući drugoj da savršeno pogodi, dok sama tada mora pogoditi nasumično..

Važno je primijetiti da su uzročne nejednakosti ograničenja *neovisna o teoriji*: formulirana su neovisno o fizici za koju se pretpostavlja da je važeća u svakom lokalnom laboratoriju. Da bismo dobili klasifikaciju *kvantnih* korelacija koja dopušta neodređene kauzalne strukture, sada ćemo otvoriti 'crne kutije' koje definiraju stranke i pretpostaviti valjanost kvantne mehanike na razini svakog lokalnog labora-

torija, što nas vodi do formalizma *matrica procesa* (eng. *process matrix formalism*). Formalizam matrica procesa predstavlja opći fromalizam za proučavanje korelacija između lokalnih eksperimenata bez pretpostavke unaprijed definiranog uzročnog poretka. Slično teorijski neovisnom formalizmu, sada se možemo zapitati jesu li sve situacije obuhvaćene dekompozicijom oblika (7.7), gdje sada pretpostavljamo da su mape izvedene na sustavima unutar laboratorija kvantne. Dekompozicija se sada može izraziti u terminima *procesa*, čija će separabilnost definirati klasu takozvanih *kauzalno separabilnih procesa*,

Definition 14 (Kauzalno separabilni procesi između dvije strane)**.** Definirajmo *proces* $\mathcal{W}^{AB}$ za dvostrani skup lokalnih eksperimenata $S = \{A, B\}$ kao skup vjerojatnosti $\mathcal{W}^{AB} = p(\mathcal{M}_1, \mathcal{M}_2)$ dobiveno kroz kvantne operacije $\mathcal{M}_1, \mathcal{M}_2$ izvedeno unutar svakog lokalnog laboratorija. Kauzalno separabilni procesi su oni koji se mogu napisati kao konveksne mješavine uređenih procesa:

$$\mathcal{W}^{AB} = q\mathcal{W}^{A \preceq B} + (1-q)\mathcal{W}^{B \preceq A} \tag{7.9}$$

gdje je q vjerojatnost.

Korelacije koje proizlaze iz procesa koji su *kauzalno neseparabilni* nazivaju se *kvantnim korelacijama s neodređenim uzročnim poretkom*, ali se ipak mogu ostvariti u globalnoj kauzalnoj strukturi. Naime, razlika između kauzalno separabilnih i kauzalnih procesa slična je kao između separabilnih (neseparabilnih) kvantnog stanja i Bellovog lokalnog (onog koje ne narušava Bellove nejednakosti) stanja: separabilno kvantno stanje je Bellovo lokalno, ali obrnuto je poznato je da nije uvijek istinito (postoje neseparabilna stanja koja ne krše Bellove nejednakosti). Fizikalno realizirani kauzalno seseparabilni procesi su procesi koji imaju neki sustav (kvantni ili klasični) kao kontrolu nad poretkom, čime je poredak događaja neodređen. To se još uvijek može spoznati u pozadinskoj kauzalnoj strukturi, kao što je eksperimentalno spoznato i dokazano putem takozvanog kauzalnog svjedoka [75] [40]. S druge strane, proces koji bi narušio kauzalnu nejednakost ne može se realizirati u određenom pozadinskom poretku, stoga ne čudi da fizička realizacija takvih procesa još nije poznata. [74].

[74] ali se može očekivati u kontekstu kvantne gravitacije

7.2.1 Kvantna sklopka

Jedan od primjera uzročno neodvojivih procesa je takozvana *kvantna sklopka* (eng. *quantum switch*), o kojem se prvi put govori u kontekstu kvantnog računanja (eng. *quantum computing*) [22]. Kauzalno neseparabilni procesi mogu se shvatiti kao prevladavanje neodređenosti u redoslijedu događaja koji bi se mogli pripisati klasičnoj slučajnosti, uvodeći ideju o uzročno-posljedičnim odnosima koji su podvrgnuti kvantnim koherencijama. Kao što smo već spomenuli, jedan od načina implementacije takve kvantne koherencije je uvođenje dodatne varijable, koja služi kao kvantna kontrola nad žicama kruga, a time i nad redoslijedom između vrata. Stoga možemo imati linearnu superpoziciju redoslijeda $A \preceq B$ i $B \preceq A$), kao što se vidi na slici (7.2). Ovo se može prirodno shvatiti ako se razmišlja, u terminima *Choi-Jamiołkowskog (CJ) izomorfizma*, o kvantnim operacijama kao o kvantnim sustavima, podvrgnutim kvantnoj superpoziciji. Preslikavanja između ovih sustava (koje predstavljaju kvantne operacije) bit će preslikavanja 'višeg reda', takozvane *supermape*. Okvir kvantnih supermapa omogućuje nam implementaciju superpozicija kanala i na precizan način obuhvatimo neodređeni kauzalni poredak.

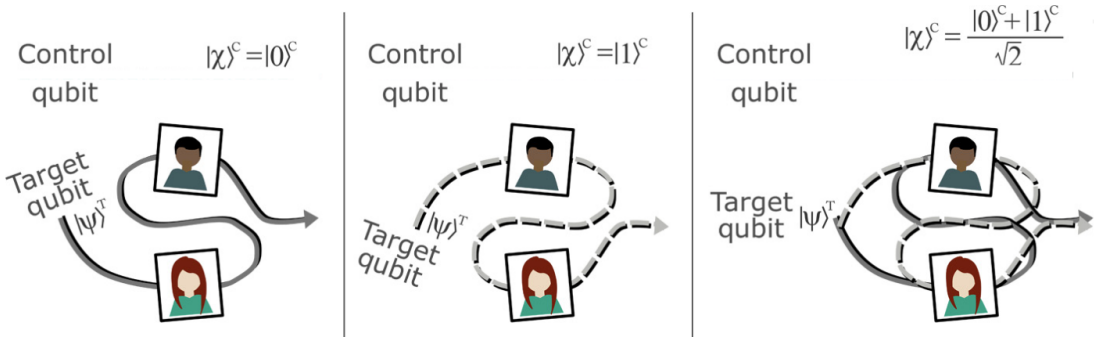


Figure 7.2: Shema kvantne sklopke. Stanje kontrolnog kubita određuje redoslijed kojim dvije strane djeluju na ciljni kubit, a ovisi o stanju kontrolnog kubita. Priprema kontrolnog kubita u superpoziciji inducira kontroliranu superpoziciju konfiguracija, Alice $\prec$ Bob i Bob $\prec$ Alice.[75]

Uzmimo u obzir stranku S_j koja ima slobodnu (klasičnu) slučajnu varijablu x_j i neslobodnu (klasičnu) slučajnu varijablu a_j koja prima kvantni sustav iz okoline I_j, izvodi lokalnu operaciju C_j i vraća transformirani kvantni sustav O_j u okolinu. Ova lokalna operacija C_j može se smatrati *kvantnom operacijom* od I_j do O_j. Često opisujemo kvantne operacije u dijagramskom jeziku (jezik kvantnih sklopova), kao

što je prikazano na slici (2.12).

Transformacije na kvantnim sustavima predstavljene su *vratima* (eng. *gates*) (ili kutijama), iz kojih ulaze i izlaze *žice*, koje predstavljaju sustave. Kvantno stanje sustava zatim se razvija kroz slijed kvantnih vrata, poredanih u vremenu, s lijeva na desno. Vrata koja predstavljaju *kvantne kanale* su transformacije pojedinačnih sustava. To mogu biti unitarna vrata ili općenito, kvantni kanal sa šumom (eng. *noisy channel*). Kutija s više žica opisuje interakciju između odgovarajućih sustava.

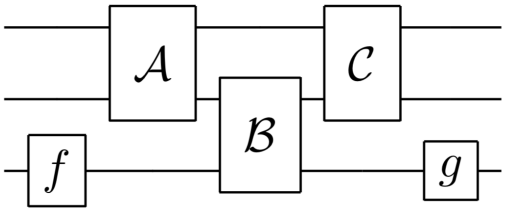


Figure 7.3: Sustavi u kvantnom krugu predstavljeni su kao žice, a operacije kao kutije. Stanje sustava razvija se kroz niz kvantnih vrata. Pretpostavlja se da vrijeme teče s lijeva na desno. [22]

Matematički, kvantne kanale opisujemo preko superoperatora. Pozivat ćemo se na superoperatore i kvantne kanale kao na najopćenitiju evoluciju matrice gustoće koja čuva normalizaciju (tj. trag) [71][75] :

$$\mathcal{C} : \hat{\rho} \mapsto \hat{\rho}' = \mathcal{C}(\hat{\rho}) \tag{7.10}$$

uz ograničenje da je preslikavanje:

- *linearno*:

$$\forall \hat{\rho}_A, \hat{\rho}_A \in \mathcal{B}(\mathcal{H}), \alpha, \beta \in \mathbf{C} : \quad \mathcal{C}(\alpha\hat{\rho}_A + \beta\hat{\rho}_B) = \alpha\mathcal{C}(\hat{\rho}_A) + \beta\mathcal{C}(\hat{\rho}_B) \tag{7.11}$$

  gdje $\mathcal{B}(\mathcal{H})$ označava omeđene linearne operatore na Hilbertovom prostoru

- $\mathcal{C}$ čuva hermitičnost: $\hat{\rho}'$ hermitičan ako $\hat{\rho}$ hermitičan

- $\mathcal{C}$ je potpuno pozitivan

  - *pozitivan*: $\hat{\rho}' \geq 0 \Rightarrow \hat{\rho} \geq 0$ [76]

  - *potpuna pozitivnost*: $\mathcal{C}_A$ je potpuno pozitivan na $\mathcal{H}_A$ ako za sve moguće ekstenzije $\mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_B$, preslikavanje $\mathcal{C}_A \otimes I_B$ je pozitivno

[75] dok su kvantne operacije i kvantni proces općenito ograničeni na preslikavanja koja ne povećavaju trag

[76] gdje $\hat{\rho}' \geq 0$ znači $\langle\psi|\hat{\rho}|\psi\rangle \geq 0$ za $\forall\psi$

- $\mathcal{C}$ čuva trag: $Tr\hat{\rho}' = \mathbf{1}$ if $Tr\hat{\rho} = \mathbf{1}$.

Koristit ćemo Krausovu reprezentaciju ovog preslikavanja:

$$\mathcal{C}(\hat{\rho}) = \sum_i M_i \hat{\rho} M_i^\dagger \quad \text{with} \quad \sum_i M_i^\dagger M_i = I \tag{7.12}$$

Najjednostavniji primjer je unitarni kanal za koji vrijedi $\mathcal{U}(\rho) = U\rho U^\dagger$ gdje $U^\dagger U = UU^\dagger = I$.

Takva potpuno pozitivna preslikavanja koja čuva trag ($CPTP$ preslikavanja) mogu se rastaviti na skup potpuno pozitivnih preslikavanja bez povećanja traga (CP), koji definiraju najopćenitiju operaciju na sustavu. CP preslikavanje može se shvatiti kao generalizacija mjerenja, gdje je vjerojatnost da je određeno preslikavanje primijenjeno jednaka tragu rezultirajućeg stanja. U najopćenitijim terminima, kvantna mapa se definira kao mapa koja zadovoljava dva aksioma:

Axiom 9. Kvantna mapa mora preslikavati skupa kvantnih stanja u skup kvantnih stanja.

Axiom 10. Sve kvantne mape moraju biti konveksno-linearne na skupu kvantnih stanja.

Naime, ako imamo neki $\hat{\rho} = \sum_{i\in I} p_i \hat{\rho}_i$ koji predstavlja ansambl mješovitih stanja $\{(\hat{\rho}_i, p_i) \mid i \in I\}$, za danu transformaciju $\hat{\rho} \mapsto \mathcal{C}(\hat{\rho})$ tada očekujemo da ćemo imati $\{(\mathcal{C}(\hat{\rho}_i), p_i) \mid i \in I\}$, predstavljajući novi ansambl u kojem su sve komponente početnog stanja evoluirale prema danoj transformaciji.

Da bismo predstavili koncept supermape, prvo moramo uvesti ideju izomorfizma Choi-Jamiołkowskog. Uspostavljajući izomorfizam između CP mapa i linearnih operatora na prostoru tenzorskih produkata, izomorfizam Choi-Jamiołkowski nam omogućuje da kvantne mape promatramo u smislu prikaza kvantnih stanja, [20]

Definition 15 (Choi-Jamiołkowski izomorfizam)**.** Neka je dato neko CP preslikavanje, iz skupa linearnih preslikavanja $\mathcal{L}$ preko nekog Hilbertovog prostora,

$$\mathcal{M} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B), \tag{7.13}$$

tada postoji preslikavanje $\mathcal{C}$, uspostavljajući korespondenciju jedan prema jedan između linearnih mapa $\mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_A), \mathcal{L}(\mathcal{H}_B))$ i linearni operatori na $\mathcal{L}(\mathcal{H}_A \otimes mathcalH_B)$ kako slijedi,

$$M = \mathcal{C}(\mathcal{M}) = (\mathcal{I}_{\mathcal{H}_A} \otimes \mathcal{M}) |\Phi\rangle \langle\Phi| \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \tag{7.14}$$

Ovdje $\mathcal{I}_{\mathcal{H}_A}$ označava preslikavanje identiteta na $\mathcal{H}_A$, a $|\Phi\rangle$ je definiran kao nenormalizirano maksimalno prepleteno stanje:

$$|\Phi\rangle = \sum_i |i\rangle_A |i\rangle_A \in \mathcal{H}_A \otimes \mathcal{H}_A \tag{7.15}$$

Takvo preslikavanje $\mathcal{C}$

$$\mathcal{C} : \mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_A), \mathcal{L}(\mathcal{H}_B)) \to M \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \tag{7.16}$$

zove se *Choi-Jamiołkowski izomorfizam*.

Neka je dato neko Choijevo preslikavanje $M \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, odgovarajuće CP preslikavanje je

$$\begin{aligned} \mathcal{M} : \mathcal{L}(\mathcal{H}_A) &\to \mathcal{L}(\mathcal{H}_B) \\ \rho_A &\mapsto \mathrm{Tr}_A \left(\left(\rho_A^T \otimes \mathbb{1}_B \right) M \right) \end{aligned} \tag{7.17}$$

gdje superskript T označava transpoziciju. Ako Choi mapa predstavlja $CPTP$ preslikavanje, imamo

$$\mathrm{Tr}_B M = \mathbb{1}_A \tag{7.18}$$

ako M je CP preslikavanje, tada vrijedi

$$\mathrm{Tr}_B M < \mathbb{1} \tag{7.19}$$

Vjerojatnost da se neko CP preslikavanje M_k primijeni na ulazno stanje ρ_A je

$$p_k = \mathrm{Tr} \left(\left(\rho_A^T \otimes \mathbb{1}_B \right) M_k \right) \tag{7.20}$$

Ako skup CP mapa označimo s $\{M_k\}_{\mathcal{K}}$, tada iz uvjeta normalizacije možemo vidjeti da $CPTP$ preslikavanje M se može napisati kao zbroj CP preslikavanja: $\{M_k\}_{\mathcal{K}}$

:

$$1 = \sum_k p_k = \sum_k \mathrm{Tr}\left(\left(\rho_A^T \otimes \mathbb{1}_B\right) M_k\right) = \mathrm{Tr}\left(\left(\rho_A^T \otimes \mathbb{1}_B\right) M\right) \tag{7.21}$$

Kvantne operacije (CP) stoga se mogu promatrati kao pozitivne linearne matrice na stanju tenzorskog umnoška preko CJ izomorfizma.

Kvantna sklopka definirana je kao posebna vrsta preslikavanja između dviju kvantnih operacija, nazvana *supermapa kvantne sklopke*. Naime, budući da se kvantna operacija može shvatiti kao generalizirani pojam kvantnog stanja, možemo definirati takozvano 'preslikavanje višeg reda' nazvano *supermapa*, definirano kao transformacija kvantnih mapa,

$$\mathcal{S} : \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}_{\text{in}}\right), \mathcal{L}\left(\mathcal{H}_{\text{out}}\right)\right) \to \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}_{\text{in}'}\right), \mathcal{L}\left(\mathcal{H}_{\text{out}'}\right)\right) \tag{7.22}$$

za neke Hilbertove prostore $\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}, \mathcal{H}_{\text{in}'}$ i $\mathcal{H}_{\text{out}'}$, gdje je $\mathcal{L}(\mathcal{H})$ prostor linearnih operatora na Hilbertovom prostoru.

Budući da kvantne supermape moraju predstavljati fizičke transformacije kvantnih preslikavanja, namećemo:

Axiom 11. Sve kvantne supermape preslikavaju kvantna preslikavanja u kvantna preslikavanja.

Axiom 12. Sve kvantne supermape moraju biti konveksno-linearne na skupu kvantnih preslikavanja.

Kao u slučaju kvantnih preslikavanja, ako ulazni kvantni kanal $\mathcal{C}$ opisuje statistički skup kvantnih kanala $\{(\mathcal{C}_i, p_i) \mid i \in I\}$, tada izlazni kvantni kanal $\mathcal{S}(\mathcal{C})$ mora opisivati ansambl $\{(\mathcal{S}\left(\mathcal{C}_i\right), p_i) \mid i \in I\}$. Dakle, aksiomi su u potpunoj analogiji s aksiomima za kvantne mape.

Zatim definiramo kvantnu sklopku preko supermapa $\mathcal{S}$,

$$\mathcal{S}(\mathcal{A} \otimes \mathcal{B})(\rho_T) := \mathcal{B}\mathcal{A}\,\langle 0|_Q\, \rho_T |0\rangle_Q + \mathcal{A}\mathcal{B}\langle 1|_Q \rho_T |1\rangle_Q \tag{7.23}$$

tj. u Krausovoj reprezentaciji

$$\mathcal{S}\left(\mathcal{A},\mathcal{B}\right)\left(\rho_T\right)=\sum_{i,j}S_{ij}\rho_T S_{ij}^{\dagger} \tag{7.24}$$

gdje

$$S_{ij}:=A_iB_j\otimes|0\rangle\langle 0|_Q+B_jA_i\otimes|1\rangle\langle 1|_Q \tag{7.25}$$

gdje $\mathcal{A}(\rho_T)=\sum_i A_i\rho_T A_i^{\dagger}$ i $\mathcal{B}(\rho_T)=\sum_j B_j\rho_T B_j^{\dagger}$

Drugim riječima, kvantna sklopka sastoji se od dva kvantna sustava: ciljni sustav ρ_T i kontrolni sustav, ρ_C te dva CP preslikavanja $\mathcal{A},\mathcal{B}$ koje djeluju na ciljni sustav. Kontrolni sustav služi kao kontrola nad redoslijedom preslikavanja na ciljnom sustavu, tj. ako je $|\psi\rangle_C=|0\rangle$, rezultirajuća operacija je $\mathcal{A}\circ\mathcal{B}$, a za $|\psi\rangle_C=|1\rangle$ imamo $\mathcal{B}\circ\mathcal{A}$. Stoga se koherentnost kntorolnog sustava prevodi se u koherentnost u redoslijedu operacija. Na primjer, kontrolni kubit u superpoziciji, $|\psi\rangle_c=\left(|0\rangle_c+|1\rangle_c\right)/\sqrt{2}$ rezultira superpozicijom dva različita reda operacija.

Također možemo imati kvantnu n-sklopku, za n-tu permutaciju operacija [72]

$$S_n|x\rangle_c|\Psi\rangle_t=|x\rangle_c\Pi_x|\Psi\rangle_t \tag{7.26}$$

ovdje $|\Psi\rangle_t$ označava ciljni sustav, koji može biti proizvoljne dimenzije; $|x\rangle_c$ označava kontrolu, koja sada mora biti $n!$ dimenzionalna; $\Pi_x:=U_{\sigma_x(N-1)}\cdots U_{\sigma_x(1)}U_{\sigma_x(0)}$, označava produkt n unitarnih vrata iz nekog zadanog skupa $U:=\{U_A,U_B,\ldots\}$.

7.3 Vremensko uređenje

Operator vremenskog uređenja matematički je objekt koji se pojavljuje u kontekstu perturbativnog razvoja u nerelativističkoj kvantnoj mehanici i teoriji polja. Predstavlja operaciju prslagivanja niza operatora prema njihovom uzročnom redoslijedu. Naime, za slučaj observabli polja povezanih s različitim događajima $x, y \in \mathcal{M}$, gdje $\mathcal{M}$ označava prostorvrijeme, vremenski uređen proizvod definiran je s

$$T\left(\boldsymbol{\psi}(x)\boldsymbol{\psi}(y)\right) := \begin{cases} \boldsymbol{\psi}(x)\boldsymbol{\psi}(y) & x \text{ not in the past of } y \\ \pm\boldsymbol{\psi}(y)\boldsymbol{\psi}(x) & \text{otherwise} \end{cases} \tag{7.27}$$

'$\pm$' se odnosi na činjenicu da za bozonska polja koristimo komutacijske relacije za kronološki redoslijed operatora polja, što odgovara znaku '$+$', dok za fermionska polja koristimo antikomutacijske relacije, koje odgovaraju '$-$'.
Razmotrimo prvi kontekst nerelativističke kvantne mehanike. Dinamika stanja određena je determinističkom evolucijom, definiranom Schrödingerovom jednadžbom:

$$i\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}(t)|\psi(t)\rangle \tag{7.28}$$

with the solution

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle \tag{7.29}$$

Da bismo pronašli oblik $U(t, t_0)$, razmotrimo infinitezimalni razvoj oko neke točke t:

$$\begin{aligned} |\psi(t+\delta t)\rangle &= [1 - i\delta t \cdot \hat{H}(t)]\psi(t) \\ &= e^{-i\delta t\cdot\hat{H}(t)}|\psi(t)\rangle \end{aligned} \tag{7.30}$$

Prema tome, vrijednost ψ u bilo kojem trenutku t_f može se izraziti u smislu njegove vrijednosti u nekom početnom trenutku $t_i\,(< t_f)$ kao produkt determinističke evolucije u svim infinitezimalnim intervalima δt_α između t_i i t_f:

$$\psi\left(t_f\right) = \left(\prod_i^f e^{-i\delta t_a \hat{H}(t_a)}\right)\psi\left(t_i\right) \tag{7.31}$$

Naivno, u $\lim(\delta \to 0)$ to bi dalo rješenje oblika:

$$\hat{U} \sim \exp\left(-i\int_{t_0}^{t}\hat{H}(t)dt\right) \tag{7.32}$$

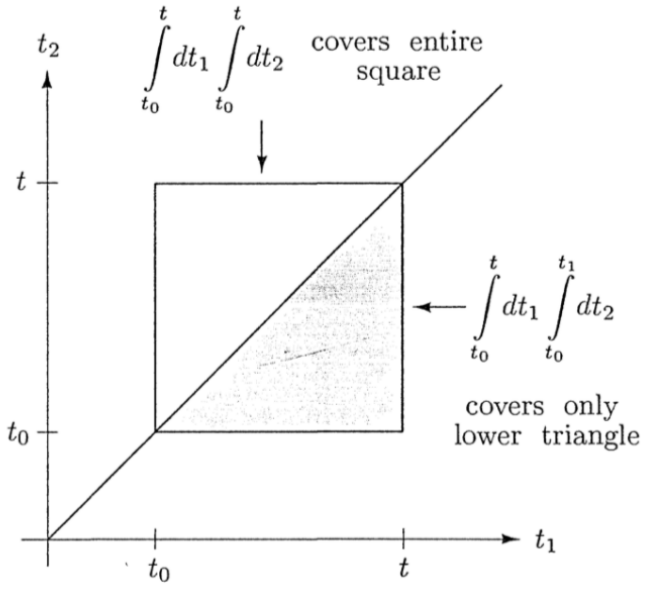


Figure 7.4: Geometrijska interpretacija vremenskog uređenja u članu drugog reda razvoja. [76]

koji bi nakon razvoja izgledao:

$$\hat{U}\left(t,t_0\right)=1-i\int_{t_0}^{t}dt_1\hat{H}(t_1)+\frac{\left(-i\right)^2}{2}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\hat{H}(t_1)\hat{H}\left(t_2\right)+\ldots \tag{7.33}$$

Međutim, primijetite da ovaj razvoj pretpostavlja da Hamiltonijani koji se odnose na različite trenutke vremena komutiraju. Unatoč tome, općenito se mora voditi računa o $[\hat{H}(t),\hat{H}(t')]\neq 0$. Naime, razmotrimo drugi red, jer je to najniži red u kojem se javlja problem. Dvostruki integral koji ide od t_0 do t predstavlja integraciju po cijelom kvadratu. Unatoč tome, radnje operatora trebaju biti poredane tako da djeluju prvi operatori definirani ranije, a zatim oni kasniji. Na slici (7.4) može se primijetiti da bi integral trebao biti podijeljen na dva dijela, od kojih svaki pripada jednom trokutu gdje bi operatori trebali biti odgovarajuće poredani: poredak povezan s donjim trokutom trebao bi biti $\hat{H}(t_1)\hat{H}(t_2)$, uzimajući u obzir da u tom području integracije $t_1>t_2$, dok bi poredak povezan s gornjim trokutom trebao biti $\hat{H}(t_2)\hat{H}(t_1)$, uzimajući u obzir da je u tom području integracije $t_2>t_1$. To se može učiniti definiranjem operatora vremenskog uređenja; za drugi red imamo:

$$T\left(\hat{H}_I\left(t_1\right)\hat{H}_I\left(t_2\right)\right)=\hat{H}_I\left(t_1\right)\hat{H}_I\left(t_2\right)\theta(t_1-t_2)+\hat{H}_I\left(t_2\right)\hat{H}_I\left(t_1\right)\theta(t_2-t_1) \tag{7.34}$$

Operator za vremensko uređenja dijeli integraciju na integraciju kroz gornji i donji trokut, dajući odgovarajući poredak operatora. Razvoj se sada može napisati kao:

$$\hat{U}\left(t,t_0\right)=1-i\int_{t_0}^{t}dt_1\hat{H}(t_1)+\frac{(-i)^2}{2}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2T\left\{\hat{H}_I\left(t_1\right)\hat{H}_I\left(t_2\right)\right\}+\ldots \tag{7.35}$$

Riješenje za slučaj $[\hat{H}(t),\hat{H}(t')]\neq 0$ stoga glasi:

$$\hat{U}\left(t,t_0\right)\equiv T\exp\left[-i\int_{t_0}^{t}dt'\hat{H}(t')\right] \tag{7.36}$$

gdje je T operator za vremensko uređenja, kako je gore definirano. Možemo vidjeti da vremenski poredak nastaje i razmatranjem definiranja jednadžbe za $\hat{U}(t,t_0)$:

$$\frac{\partial}{\partial t}\hat{U}\left(t,t_0\right)=-i\hat{H}(t)\hat{U}\left(t,t_0\right) \tag{7.37}$$

i iterativno rješavanje jednadžbe. Počevši od:

$$\hat{U}\left(t,t_0\right)=1-i\int_{t_0}^{t}dt_1\hat{H}(t_1)\hat{U}\left(t_1,t_0\right) \tag{7.38}$$

nakon uvrštavanja $U(t_1,t_0)$ natrag u jednadžbu:

$$\begin{aligned}\hat{U}\left(t,t_0\right)&=1-i\int_{t_0}^{t}dt_1\hat{H}(t_1)\left[1-i\int_{t_0}^{t_1}dt_2\hat{H}\left(t_2\right)\hat{U}\left(t_2,t_0\right)\right]\\&=1+(-i)\int_{t_0}^{t}dt_1\hat{H}(t_1)+(-i)^2\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\hat{H}(t_1)\hat{H}\left(t_2\right)\hat{U}\left(t_2,t_0\right)\end{aligned} \tag{7.39}$$

Sada primijetite implicirani vremenski poredak varijabli integracije: $t_0<t_2<t_1<t$. Drugi red se može promatrati kao integracija preko donjeg trokuta predstavljenog na slici 7.4. Ovo se može prepisati u smislu integracije kroz cijeli kvadrat kao:

$$\begin{aligned}\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\hat{H}_I\left(t_1\right)\hat{H}_I\left(t_2\right)&=\frac{1}{2}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2T\left\{\hat{H}_I\left(t_1\right)\hat{H}_I\left(t_2\right)\right\}\\&=\frac{1}{2}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\hat{H}_I\left(t_1\right)\hat{H}_I\left(t_2\right)\theta(t_1-t_2)+\hat{H}_I\left(t_2\right)\hat{H}_I\left(t_1\right)\theta(t_2-t_1)\end{aligned} \tag{7.40}$$

gdje smo uveli operator vremenskog uređivanja koji, kao što je prije rečeno, dijeli

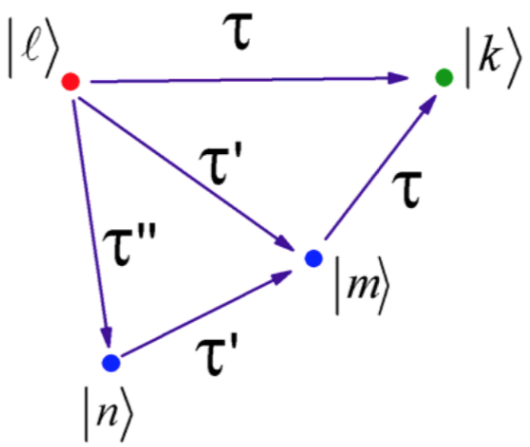


Figure 7.5: Polazeći od stanja $|\psi(t_0)\rangle = |l\rangle$ do nekog konačnog stanja $|\psi(t)\rangle = |k\rangle$ Dysonov operator opisuje sve moguće putove između početno i konačno stanje. [83]

integraciju na integraciju kroz gornji i donji trokut. Imamo:

$$\hat{U}(t,t_0) = 1 - i\int_{t_0}^{t} dt_1 \hat{H}(t_1) + \frac{(-i)^2}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T\left\{\hat{H}_I(t_1)\,\hat{H}_I(t_2)\right\}\hat{U}(t_2,t_0) \tag{7.41}$$

Nastavimo li iterativno:

$$\hat{U}(t,t_0) \equiv T\exp\left[-i\int_{t_0}^{t} dt' \hat{H}(t')\right] \tag{7.42}$$

ovaj se operator obično naziva Dysonov operator.
Svaki red u Dysonovom operatoru može se vizualizirati kako prolazi kroz različite puteve, podijeljen na sve više međustanja kako se red povećava (vidi sliku 7.5).

U kontekstu teorije polja, Dysonov operator razmatra se u slici interakcije i značajkama u središnjem objektu teorije, naime, matrici raspršenja:

$$\mathcal{S}\left(\hat{V}_I\right) := \lim_{t\to\infty} T\left(\exp\left(-i\int_{-t}^{t}\hat{V}_I(t)dt\right)\right) = \lim_{t\to\infty}\hat{U}(t,-t) \tag{7.43}$$

koji šalje asimptotička ulazna stanja $|\psi(-\infty)\rangle_I$ u asimptotička izlazna stanja $|\psi(+\infty)\rangle_I$.

Isprva bi se moglo očekivati da vremenski poredak ovisi o specifičnom izboru vremenske funkcije s obzirom na koji se vrši poredak. Unatoč tome, tvrdnja da vremensko uređenje dolazi iz kauzalnih aksioma nametnutih S-matrici (vidi [43]) implicira

nepromjenjivost operacije vremenskog uređenja. Da bismo rasvijetlili ovu činjenicu, razmotrimo prvo vremenoliko odvojene točke, x_1 i x_2, $(x_1 - x_2)^2 > 0$. U ovom slučaju imamo uzročni poredak između x_1 i x_2, a time i predznak $t_2 - t_1$, gdje su t_1 i t_2 predstavljaju odgovarajuće vremenolike komponente, neovisno je o referentnom sustavu. Međutim, u slučaju prostorno odvojenih događaja, $(x_1 - x_2)^2 < 0$, predznak $t_2 - t_1$ nije neovisan o referentni okvir i stoga Lorentzova transformacija mogu obrnuti slijed vremenskih trenutaka. Invarijantnost vremenskog poretka ovdje je osigurana pozivanjem na uvjet mikrokauzalnosti

$$[O(x_i), O(x_j)] = 0 \quad \text{for} \quad (x_i - x_j)^2 < 0 \tag{7.44}$$

Naime, podsjetimo da nekomutativnost operatora fizikalno znači da se odgovarajuće veličine ne mogu mjeriti istovremeno. Međutim, bez obzira koji su operatori vezani uz prostorno odvojene točke, budući da između njih ne postoji kauzalna veza, uvijek se mogu vršiti simultana mjerenja. Uzimajući u obzir komutaciju, faktori u produktu vremenskog slijeda uvijek se mogu poredati prema njihovom kronološkom redoslijedu. Stoga, u oba scenarija, operator vremenskog uređenja ostaje nepromjenjiv. Primijetite da smo ovdje koristili sljedeću implikaciju:

$$\text{mikrokauzalnost} \quad \Rightarrow \quad \text{invarijantnost vremenskog uređenja} \tag{7.45}$$

8 Pristup s ancillom

U nastavku ćemo predstaviti orginalne rezultate disertacije u kojoj ćemo analizirati operator vremenskog uređenja u kontekstu u vremenski uređenih eksponencijala, rješenja vremenski ovisne Schrödingerove jednadžbe. Usredotočit ćemo se na ekspanziju vremensko uređenog eksponencijala u interakcijskoj slici, do člana drugog reda. Kao što je detaljno objašnjeno u Odjeljku (7.3), operator vremenskog uređenja sadrži superpoziciju duž svih mogućih konfiguracija uređenja, gdje ćemo svaki izraz koji predstavlja zasebnu konfiguraciju, u ovom tekstu, nazvati 'granom' uređenja vremenskog operatora. Potaknuti raspravom u odjeljku (7.2), razmišljat ćemo o sustavu kao o superpoziciji prolaska kroz različite moguće grane vremenskih konfiguracija. Glavno pitanje rada sada je može li se izolirati grana vremenskog uređenja pomoću vezanja sustava na pomoćni sustav, koji ćemo nazivati *ancillom*, te izvođenja neizravnog mjerenja na sustavu. Ideja je da bi se ancilla interakcija mogla vezati na određenu konfiguraciju vremenskog poretka, tako da bi mjerenje na ancilli urušilo superpoziciju konfiguracija, izolirajući određeni poredak.

Na slici interakcije, vremenski uređen eksponencijal glasi,

$$U(t,t_0) = Te^{-i\int_{t_0}^{t} \mathrm{d}t' H_I(t')} \tag{8.1}$$

gdje T označava operator vremenskog uređenja. Nakon razvoja imamo:

$$U\left(t,t_0\right) = 1 + (-i)\int_{t_0}^{t} dt_1 H_I\left(t_1\right) + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 dt_2 T\left(H_I\left(t_1\right) H_I\left(t_2\right)\right) + \cdots \tag{8.2}$$

Usredotočit ćemo se na član razvoja do drugog reda, gdje vremensko uređenje počinje biti relevantno;

$$\frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \left(H_I\left(t_1\right) H_I\left(t_2\right)\theta\left(t_1 - t_2\right) + H_I\left(t_2\right) H_I\left(t_1\right)\theta\left(t_2 - t_1\right)\right) \tag{8.3}$$

Primijetite da su integralni doprinosi dvaju različitih vremenskih redoslijeda isti, što ih čini nerazlučivim.

Pokušajmo sada projicirati na $t_1 > t_2$ ili $t_1 < t_2$ granu evolucije; da bismo to učinili,

razmotrimo sustav $|\psi(t)\rangle$ vezan s pomoćnim sustavom kontinuiranog spektra $|a(t)\rangle$:

$$|\Psi(t)\rangle = |\psi(t)\rangle|a(t)\rangle \tag{8.4}$$

Sada imamo $H_I(t) \to V(t) \otimes A(t) : \mathcal{H}_S(t) \otimes \mathcal{H}_A(t) \longrightarrow \mathcal{H}_S(t) \otimes \mathcal{H}_A(t)$. Definiramo da pomoćni sustav inicijalno bude u stanju:

$$|a(t_0)\rangle = |a_0\rangle \tag{8.5}$$

i definirajmo da vlastita stanja operatora ancilla potencijala $\hat{A}(t)$ zadovoljavaju:

$$\langle c\,(t') \mid c(t)\rangle \equiv \langle c' \mid c\rangle = \delta\,(c' - c) \tag{8.6}$$

gdje smo prešli na notaciju $\{c(t_i)\} \equiv \{c_i\}$. Uzmimo $\{|c\rangle : c \in X\}$ i $\hat{A}(t)$ kao općenit potencijal ancille. Prošireno preko spektralnog teorema,

$$\hat{A}(t) = \int_X c|c\rangle\langle c|dc \tag{8.7}$$

Razmotrimo sada evoluciju ukupnog stanja

$$|\Psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle|a_0\rangle \tag{8.8}$$

Nakon umetanja ekspanzije (8.2),

$$|\Psi(t)\rangle = \left(1 - i\int_{t_0}^{t} dt_1 V(t_1) \otimes A(t_1) + \frac{(-i)^2}{2!}\int_\gamma dt_1 dt_2 T(V(t_1) \otimes A(t_1)V(t_2) \otimes A(t_2) + \cdots\right)|\psi(t_0)\rangle|a_0\rangle \tag{8.9}$$

Umetanje interakcije daje

$$\begin{aligned}|\Psi(t)\rangle = \Big(1 - i\int_{t_0}^{t} dt_1 \int_{X_1} dc_1 V(t_1) \otimes c_1|c_1\rangle_A\langle c_1| \\ + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_X dc_1 \int_X dc_2 T(V(t_1) \otimes c_1|c_1\rangle_A\langle c_1|V(t_2) \otimes c_2|c_2\rangle_A\langle c_2|)) + \cdots\Big)|\psi(t_0)\rangle|a_0\rangle\end{aligned} \tag{8.10}$$

Projicirajmo sada na pomoćni sustav mjerenjem preko nekog konačnog intervala.

Kao što je gore spomenuto, usredotočit ćemo se na pojam drugog reda:

$$
\begin{aligned}
&\int_c |c\rangle\langle c|\Psi(z)\rangle dc = \\
&= \Big(\cdots + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_c dc \int_{c_1} dc_1 \int_{c_2} dc_2 \theta(t_1-t_2)(V(t_1)\otimes c_1\langle c|c_1\rangle\langle c_1|V(t_2)\otimes c_2|c_2\rangle\langle c_2|) \\
&+ \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_c dc \int_{c_1} dc_1 \int_{c_2} dc_2 \theta(t_2-t_1)(V(t_2)\otimes c_2\langle c|c_2\rangle\langle c_2|V(t_1)\otimes c_1|c_1\rangle\langle c_1|)\Big)|\psi(t_0)\rangle|a_0\rangle|c\rangle \\
&= \Big(\cdots + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_c dc \int_{c_1} dc_1 \int_{c_2} dc_2 \theta(t_1-t_2)(V(t_1)\otimes c_1\langle c|c_1\rangle\langle c_1|V(t_2)\otimes c_2|c_2\rangle\langle c_2|a_0\rangle \\
&+ \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_c dc \int_{c_1} dc_1 \int_{c_2} dc_2 \theta(t_2-t_1)(V(t_2)\otimes c_2\langle c|c_2\rangle\langle c_2|V(t_1)\otimes c_1|c_1\rangle\langle c_1|a_0\rangle\Big)|\psi(t_0)\rangle|c\rangle \\
&= \Big(\cdots + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_c dc \int_{c_1} dc_1 \int_{c_2} dc_2 \theta(t_1-t_2)V(t_1)V(t_2)c_1c_2\delta(c-c_1)\delta(c_1-c_2)\delta(c_2-a_0) \\
&+ \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_c dc \int_{c_1} dc_1 \int_{c_2} dc_2 \theta(t_2-t_1)V(t_2)V(t_1)c_1c_2\delta(c-c_2)\delta(c_2-c_1)\delta(c_1-a_0)\Big)|\psi(t_0)\rangle|c\rangle
\end{aligned}
\tag{8.11}
$$

Da bismo dobili rezultat koji ne iščezava, delta funkcija nas obvezuju da zadovoljimo uvjet:

$$c = c_1 = c_2 = a_0 \tag{8.12}$$

činećи u isto vrijeme oba doprinosa neiščezavajućima. Ako c_1 i c_2 koji se pojavljuju u jednoj grani ne bi bili točno c_1 i c_2 u drugoj grani (kao što će biti objašnjeno u nastavku), mogli bismo razlikovati grane zahtijevajući odgovarajuće uvjete.

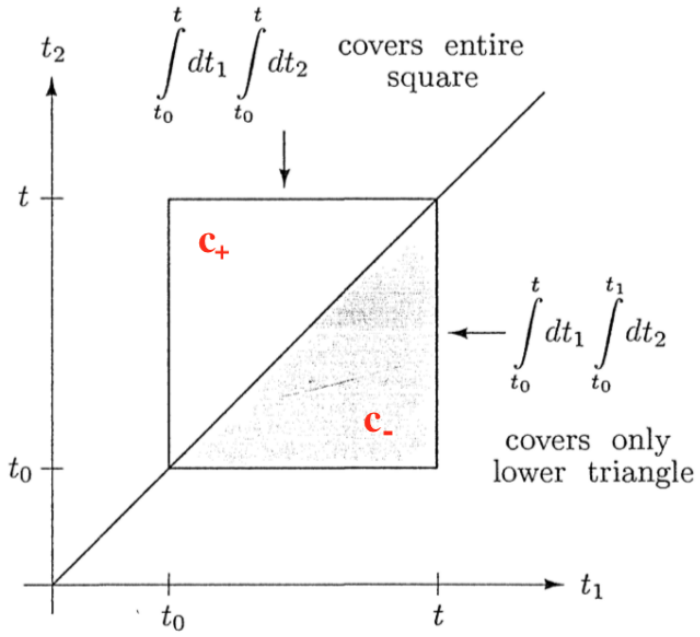


Figure 8.1: Geometrijski prikaz člana drugog reda Dysonovog razvoja. Kako bi se povezao s određenim vremenskim poretkom, pomoćni potencijal treba ovisiti o funkciji u $2D$ parametarskom prostoru.

Pretpostavimo, dakle, da bi koeficijenti c mogli poprimiti različite vrijednosti ako su povezani s različitom konfiguracijom vremenskog poretka. Kao primjer, razmotrite sliku (8.1) na kojoj vidimo kako su vrijednosti c vezane s određenom konfiguracijom vremenskog poretka. Naime, tada bismo imali

$$\begin{array}{ccc} t & \longrightarrow & t \otimes v \\ \downarrow & & \downarrow \\ V(t) & \longrightarrow & V(t) \otimes A(v) \end{array}$$

gdje bi $v \coloneqq v(t_1, t_2)$ bila neka hipotetska funkcija u $2D$ prostoru parametara člana drugog reda. Čitatelj bi trebao uočiti da je ovo problematično i nema zapravo smisla na razini relacije (8.1). Ipak, nastavit ćemo da pokažemo poantu, da upravo takvim potencijalom bismo postigli željeno. Uzmimo:

$$v(t_1, t_2) = t_1 - t_2 \tag{8.13}$$

i

$$\begin{aligned} \hat{A}(t_1, t_2) &= \hat{A}(v > 0)\theta(t_1 - t_2) + \hat{A}(v < 0)(t_2 - t_1)\theta(t_2 - t_1) \\ &= \int_{X_+} c_+|c_+\rangle\langle c_+|dc_+\theta(v > 0) + \int_{X_-} c_-|c_-\rangle\langle c_-|dc_-\theta(v < 0) \end{aligned} \tag{8.14}$$

Tada bismo dobili:

$$\begin{aligned} &\int_c |c\rangle\langle c|\Psi(z)\rangle dc \\ &= \int_c |c\rangle\langle c|dc\big(... + \frac{(-i)^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 \hat{V}(t_1)\hat{V}(t_2) \otimes \hat{A}\left(v(t_1,t_2)\right)\hat{A}\left(v(t_1,t_2)\right)\theta\left(t_1 - t_2\right) \\ &\quad + \frac{(-i)^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 \hat{V}(t_2)\hat{V}(t_1) \otimes \hat{A}\left(v(t_1,t_2)\right)A\left(v(t_1,t_2)\right)\theta\left(t_2 - t_1\right) + \cdots\big)|\psi(t_0)\rangle|a_0\rangle \\ &= \int_c |c\rangle\langle c|dc\big(... + \frac{(-i)^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 \hat{V}(t_1)\hat{V}(t_2) \otimes \hat{A}(v > 0)\hat{A}(v > 0)\theta(v) \\ &\quad + \frac{(-i)^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 \hat{V}(t_2)\hat{V}(t_1) \otimes \hat{A}(v < 0)\hat{A}(v < 0)\theta(-v) + \cdots\big)|\psi(t_0)\rangle|a_0\rangle \\ &\quad = \Big(... + \frac{(-i)^2}{2!}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 \int_c dc \int_{c_1^-} dc_1^- \int_{c_2^-} dc_2^- dz_1 dz_2 \theta(t_1 - t_2)\cdot \\ &\qquad\qquad \cdot (V(t_1)V(t_2))c_1^- c_2^- \delta(c - c_1^-)\delta(c_1^- - c_2^-)\delta(c_2^- - a_0) \\ &\quad + \frac{(-i)^2}{2!}\int_{\gamma_+} dz_1 dz_2 \int_c dc \int_{c_1^+} dc_1^+ \int_{c_2^+} dc_2^+ \theta(t_2 - t_1)\cdot \\ &\qquad\qquad \cdot (V(t_1)V(t_2))c_1^+ c_2^+ \delta(c - c_1^+)\delta(c_1^+ - c_2^+)\delta(c_2^+ - a_0)\ldots\Big)|\psi(t_0)\rangle|v\rangle \end{aligned} \tag{8.15}$$

Sada bismo mogli nametnuti neke početne i konačne uvjete, kao što su:

$$c = c_1^+ = c_2^+ = a_0 \tag{8.16}$$

projicirati na gornji ili

$$c = c_1^- = c_2^- = a_0 \tag{8.17}$$

projicirati na donju granu. Međutim, kao što je već spomenuto, takva konstrukcija ne bi imala smisla jer ne bi bila dobro definirana u svim redovima. Naime, $v(t_1, t_2)$ je funkcija definirana u parametarskom prostoru drugog reda. Ova vrsta ovisnosti ne može postojati na razini jednadžbe (8.1) i ostaje nam ispravna konstrukcija (8.11) koja ne može razlikovati dva doprinosa. To se može pripisati činjenici da u evoluciji nema stvarne neodređenosti jer vanjsko vrijeme uvijek teče u istom smjeru od t_0 do t. Kao što je objašnjeno u odjeljku (7.3), vremenski uređen eksponencijal samo je proizvod infinitezimalnih deterministčkih koraka, kao što je dano formulom:

$$\hat{U}(t_0, t) = \prod_i^f e^{-i\delta t_a \hat{H}(t_a)} \tag{8.18}$$

Kako bi se uvela neodređenost u dinamiku, očekivalo bi se uvođenje neodređenosti na razini svakog infinitezimalnog vremenskog koraka, kao što je objašnjeno u [33].

9 Page-Wottersov pristup: razvoj vremenski uređenih eksponencijala korištenjem formalizma bezvremenskog stanja

Jedan od pokušaja da se pomire razlike u načinu na koji gravitacijska naspram kvantne fizike tretira vrijeme jest prevladavanje pojma vremena kao vanjskog parametra u kvantnoj mehanici. Služeći kao parametar vremenskih translacija, vrijeme kao vanjski parametar pojavljuje se u Schrödingerovoj jednadžbi

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = \hat{H}|\psi(t)\rangle \tag{9.1}$$

Operativno ga se može protumačiti kao 'vrijeme mjereno klasičnim, laboratorijskim satom'. Kao takvo, ono nije kvantnomehanička opservable te stoji izvan režima teorije. Takav vanjski karakter vremena je u suprotnosti s načinom na koji je vrijeme implementirano u teoriji opće relativnosti. Naime, kao što je spomenuto u Uvodu (7.1), simetrija difeomorfizma Diff(M) opće relativnosti inducira neovisnost o pozadini (vidi npr. [69]), podvrgavajući prostorvrijeme dinamičkom zakonu predstavljen unutar. U takvoj teoriji prostor i vrijeme u njihovoj jedinstvenoj strukturi prostorvremena tretiraju se kao unutarnje varijable. Ovo predstavlja napetost prilikom pokušaja da se kvantna teorija pomiri u jedinstveni okvir s teorijom opće relativnosti, različita priroda vremena u te dvije teorije predstavlja sjeme za skup problema u literaturi poznatih kao 'Problem vremena' [2].

Kako bismo vrijeme učinili internim za kvantnu teoriju, promovirat ćemo sat u kvantni stupanj slobode. Naime, razmotrit ćemo neki kvantni sustav, kojem je pridružen Hilbertov prostor $\mathcal{H}_T$ na koji djeluje vremenski operator $\hat{T}$, koji služi kao vremenski kvantni referentni sustav[77] tj. fizikalni (kvantni) sat. Uzimajući u obzir sada, sustav povezan s fizikalnim satom omogućit će nam da opišemo promjenu vremena putem uvjetnih vjerojatnosti, zamjenjujući 'biti u trenutku' bezvremenskim korelacijama. Odnosno, vremensko ponašanje koje promatramo ovisi o odnosima između nekog unutarnjeg sata i sustava, umjesto o vanjskom koordinatnom parametru

[77] Kao što vidi u poglavlju (2.1.3), korištenje kvantnog stupnja slobode kao referentnog sustav promovirat će parametar translacije u operator.

vremena. Pogledajmo sada detaljnije ovaj formalizam koji ćemo nazivati *Page-Wottersov formalizam* (ponekad poznat i kao *Bezvremenski pristup kvantnoj mehanici*).

9.1 Page-Wootters formalizam

U bezvremenoj formulaciji, razmatra se globalno, bezvremeno stanje $|\Psi\rangle\rangle$ zvano *stanje povijesti*, koje se može promatrati kao ukupni sustav sastavljen od sata i sustava od interesa. Budući da je svaki izolirani sustav u vlastitom stanju energije i stoga stacionaran u odnosu na koordinatno vrijeme, bezvremeno stanje se obično smatra stanjem cijelog Svemira (budući da je ovo najmanji istinski izolirani sustav). Ovo stanje je poznato u Wheeler-DeWittovoj jednadžbi,

$$\hat{H}|\Psi\rangle\rangle = 0 \tag{9.2}$$

koji se pojavljuje unutar Diracovog pristupa kvantizaciji gravitaciji (kanonska kvantna gravitacija). Kao i kod svake jednadžbe ograničenja, (9.2) reducira kinematički prostor stanja (sva moguća stanja) na fizički prostor stanja (stanja koja zadovoljavaju ograničenje). Ako sada razmotrimo dinamiku sustava koju daje Schrödingerova jednadžba (9.1), Wheeler–DeWittova jednadžba (9.2) implicira

$$\Rightarrow \quad i\hbar\frac{d|\Psi\rangle\rangle}{dt} = 0 \tag{9.3}$$

Odnosno, naše stanje $|\Psi\rangle\rangle$ je lišeno vremenske evolucije. Ova vrsta situacije općenito se pojavljuje unutar vremenskog relacionizma i poznata je kao *problem zamrznutog formalizma*.

Pitanje s kojim se sada suočavamo je, kako iz ovog zamrznutog globalnog stanja $|\Psi\rangle\rangle$ izvući svakodnevnu dinamiku? Jedno od zapažanja koje možemo iznijeti jest da pojam vremena nije dobro definiran na razini takvog globalnog stanja. Priroda vremena sama po sebi nije nam potpuno jasna, no vrijeme je neporeciva činjenica našeg svakodnevnog iskustva a ta se iskustva uvijek tiču *podsustava*, a ne Svemira kao cjeline [78]. Odnosno, *Nijedno promatranje nije moguće bez promatrača.* [25],

[78] Ovo je srž temporalnog relacionizma za koje je karakteristično da su očito reparametrizacijski invarijantni (kao što je slučaj s općom relativnošću) i da dobivaju jednadžbe ograničenja koje forsiraju zamrznuti formalizam

dakle pomicanje perspektive na ono što se promatra, implicira pomicanje perspektive na jedan od podsustava. U skladu s tim, 1982. Page i Wootters formulirali su pristup uvjetnih vjerojatnosti bezvremenskoj kvantnoj mehanici. Naime, definirali su bezvremensko stanje $|\Psi\rangle\rangle$ kao korelirano stanje sustava i sata:

$$|\Psi\rangle\rangle = \int dt |t\rangle_C \otimes |\psi(t)\rangle_S \tag{9.4}$$

tako da zadovoljava jednadžbu (9.2). Ono sadrži punu dinamiku sustava nekog unutarnjeg promatrača koji mjeri evoluciju vremena $|\psi(t)\rangle$, s obzirom na neki podsustav koji služi kao sat. Definirat ćemo *savršeni sat* kao podsustav povezan s beskonačnodimenzionalnim Hilbertovim prostorom $\mathcal{H}_C$, izomorfnim Hilbertovom prostoru čestice na liniji, s kanonskim koordinatama $\hat{T}_C$ i $\hat{H}_C$, zadovoljavajući Heisenbergovu algebru $[\hat{T}_C, \hat{H}_C] = i$. Dinamičko stanje unutarnjeg promatrača definiramo kao stanje koje se dobiva uvjetovanjem statičkog, globalnog stanja projiciranjem na svojstveni vektor sata:

$$|\psi(t)\rangle = {}_C\langle t|\Psi\rangle \ . \tag{9.5}$$

Ovo stanje se naziva 'reducirano stanje' i odgovara stanju sustava S kada sat C pokazuje vrijeme t. Uspoređujući s Einsteinovom definicijom vremena: *'Vrijeme' nekog događaja je ono koje je zadano istovremeno s događajem pomoću stacionarnog sata koji se nalazi na mjestu događaja'*[30], nadalje pretpostavljamo da sat miruje u odnosu na promatrača.

Ovdje uvodimo $|t\rangle_C$ svojstvena stanja vremenskog operatora $\hat{T}_C$,

$$\hat{T}_C|t\rangle_C = t|t\rangle_C \tag{9.6}$$

i interpretirajte svojstvenu vrijednost t kao rezultat mjerenja na satu koji pokazuje vrijeme t. Trebali bismo imati na umu da ova vrsta savršenog sata nije fizikalna budući da komutator

$$[\hat{T}, \hat{H}] = i \tag{9.7}$$

implicira neomeđeni spektar oba operatora po Stone-von-Neumannovom teoremu. Ipak, takav model može poslužiti kao radna idealizacija.

Bezvremensko stanje sadrži sve informacije o korelacijama između sustava i sata i

pripada fizičkom Hilbertovom prostoru, budući da zadovoljava jednadžbu ograničenja:

$$\hat{C}|\Psi\rangle\rangle = 0 \tag{9.8}$$

Polazeći od općeg elementa kinematičkog Hilbertovog prostora $|\Phi\rangle$, možemo ograničiti rješenja na Hilbertov prostor fizikalnih stanja pomoću jednadžbe (vidi [60]):

$$|\Psi\rangle\rangle = \int d\alpha e^{-i\alpha\hat{C}}|\Phi\rangle \tag{9.9}$$

U žargonu ponekad kažemo da je $|\Phi\rangle$ 'projiciran' na Hilbertov prostor fizikalnih stanja. Unatoč tome, treba primijetiti da kinematički $\mathcal{H}_{kin}$ i fizikalni Hilbertov prostor $\mathcal{H}_{phy}$ nemaju isti skalarni produkt, pa stoga $\mathcal{H}_{phy} \neq\subset \mathcal{H}_{kin}$. (9.9) stoga formalno nije projekcija. Za naš model imamo sljedeću jednadžbu ograničenja:

$$\hat{C} = \hat{H}_C \otimes \mathbb{1}_S + \mathbb{1}_C \otimes \hat{H}_S \tag{9.10}$$

tako da reducirano stanje zadovoljava uobičajenu Scrödingerovu jednadžbu nakon uvjetovanja (10.19) na stanje sata u vremenu t:

$$\begin{aligned} {}_C\langle t|\hat{C}|\Psi\rangle = {}_C\langle t|(\hat{H}_C \otimes \mathbb{1}_S + \mathbb{1}_C \otimes \hat{H}_S)|\Psi\rangle\rangle = 0 \\ \Rightarrow \left(i\frac{\partial}{\partial t} - \hat{H}_S\right)|\psi(t)\rangle = 0 \end{aligned} \tag{9.11}$$

9.2 Eksponencijalno uređeno vrijeme u rješenju s dva kvantna sata

U ovom ćemo odjeljku predstaviti izvorne rezultate disertacije u kojoj smo koristili Page-Wottersov formalizam s nekoliko satova kako bismo istražili konfiguracije vremenskih uređenja koje se javljaju u operatoru vremenskog uređenja. Operacija vremenskog uređenja koja se pojavljuje na razini Feynmanovog propagatora je ona koja dolazi upravo iz vremenski uređene eksponencijalne, analizirane u Poglavlju (8). Glavna motivacija našeg rada bila je usporedba dvije konfiguracije koje se javljaju na razini Feynmanova propagatora, s kontroliranom superpozicijom poredaka koji se pojavljuju u kvantnoj sklopki. Kao što se vidi u poglavlju (7.3), Feynmanov

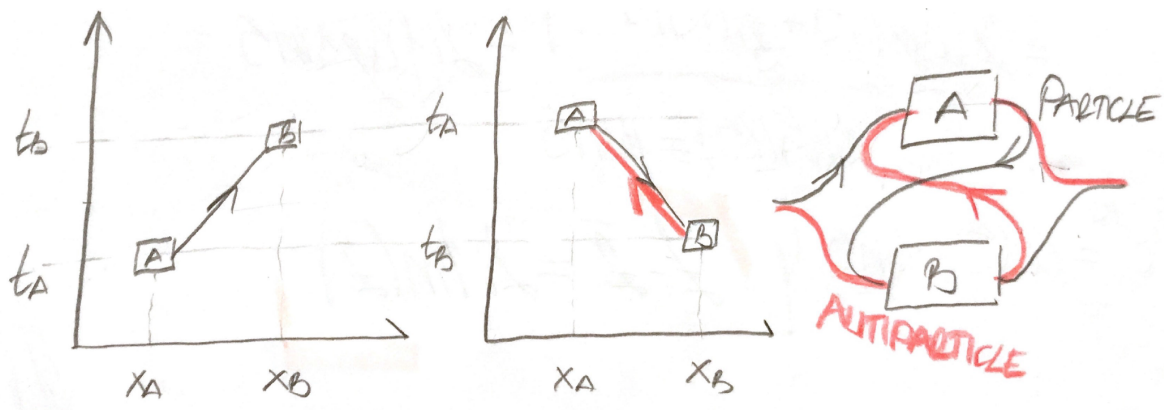


Figure 9.1: Feynmanov propagator kao superpozicija dijagrama koji odgovaraju čestičnim i antičestičnim stupnjevima slobode, u usporedbi sa strukturom kvantne sklopke. Za operativno razumijevanje, zamislite da vrhovi dijagrama prostorvremena A i B predstavljaju lokalne laboratorije u kojima Alice i Bob imaju svoje prepletene satove. Satovi su u superpoziciji u prepletenoj bazi u odnosu na nekog trećeg promatrača koji će situaciju vidjeti kao superpoziciju dvaju prikazanih dijagrama. Zajednički sustav je u ovom slučaju polje na koji agenti vrše operacije.

propagator je definiran kao zbroj procesa

$$\left\langle \mathbb{0} \left| T\left(\hat{\phi}(x)\hat{\phi}^\dagger(y)\right) \right| \mathbb{0} \right\rangle = \underbrace{\left\langle \mathbb{0} \left| \hat{\phi}(x)\hat{\phi}^\dagger(y) \right| \mathbb{0} \right\rangle}_{\text{čestični stupnjevi slobode}} \theta(x^0 - y^0) + \underbrace{\left\langle \mathbb{0} \left| \hat{\phi}^\dagger(y)\hat{\phi}(x) \right| \mathbb{0} \right\rangle}_{\text{antičestični stupnjevi slobode}} \theta(y^0 - x^0)$$

$$= \left\langle \mathbb{0} \left| \begin{pmatrix} \text{Poništavanje čestice} \\ \text{na } (x^0, \boldsymbol{x}) \end{pmatrix} \begin{pmatrix} \text{Stvaranje čestice} \\ \text{na } (y^0, \boldsymbol{y}) \end{pmatrix} \right| \mathbb{0} \right\rangle \theta(x^0 - y^0)$$

$$+ \left\langle \mathbb{0} \left| \begin{pmatrix} \text{Poništavanje antičestice} \\ \text{at } (y^0, \boldsymbol{y}) \end{pmatrix} \begin{pmatrix} \text{Stvaranje antičestice} \\ \text{na } (x^0, \boldsymbol{x}) \end{pmatrix} \right| \mathbb{0} \right\rangle \theta(y^0 - x^0) \tag{9.12}$$

kao što je prikazano na lijevoj i desnoj prostornovremenskoj skici slike (9.1), gdje je usporedba s kvantnom sklopkom prikazana na krajnjoj desnoj slici. Stoga možemo napraviti (nerigoroznu) paralelu između strukture kvantne sklopke, kao što je objašnjeno u poglavlju (7.2.1), i strukture Feynmanova propagatora praveći sljedeće asocijacije:

- *Ciljni sustav* sada je stanje vakuuma polja $|\mathbb{0}\rangle$ na koje će agenti djelovati.
- *Agenti*[79], označene s A i B, pridružene su vrhovima propagatora gdje izvode lokalni CP transformacije na nekom lokaliziranom području koje predstavlja potprostor ukupnog Hilbertovog prostora koji odgovara ciljnom sustavu. Ove

[79]U ovom kontekstu, 'laboratoriji' i 'agenti' trebaju se shvatiti figurativno. Važno je samo pronaći lokalnu ulazno-izlaznu strukturu i pojam preslikavanja između mapa.

transformacije će konkretno biti stvaranje i poništavanje čestica, koje ćemo označiti kao: $\hat{\phi}^{\dagger}(y) = \mathcal{A} \equiv$'čestica stvorena' i $\hat{phi}(x) = \mathcal{B} \equiv$'anihilirana čestica'. Budući da prostor stanja treba očuvati djelovanjem operatora polja, razmatrana preslikavanja odgovaraju željenom opisu. Crte na dijagramima predstavljaju proizvedene pobude polja.

- *'Slobodne varijable'*[80] u ovom slučaju su prostorno-vremenski položaju na kojima agenti odlučuju stvoriti ili uništiti česticu. Drugim riječima, agenti su preslikavanja koja dodjeljuju algebre observabli željenom prostorno-vremenskom području.

- *Kontrolni sustav* sada je Heavisideova θ funkcija, koja određuje vremenski odnos između vremenskih komponenti događaja.

Stoga Feynmanov propagator možemo promatrati kao analogon *supermapi*, u smislu da preslikava dvije CP mape $\mathcal{A} = \hat{\phi}^{\dagger}(x) : |\mathbb{0}\rangle \rightarrow \hat{\phi}(x)|\mathbb{0}\rangle$ i $\mathcal{B} = \hat{\phi}(y) : |\mathbb{0}\rangle \rightarrow \hat{\phi}(y)|\mathbb{0}\rangle$ u superpoziciju (4.40). Ovo je malo drugačija struktura od strukture supermape koja se pojavljuje u (7.22) zbog nekonvencionalnog pojma pojavljivanja kontrole i činjenice da radimo u Heisenbergovoj reprezentaciji.
Treba također primijetiti da bismo u ovom slučaju nakon projekcije na određeno stanje kontrole dobili jednu amplitudu koja narušava kauzalnost, i pridonosi širenju u području prostor-vremena . [81]

Usmjerenost vremena koja služi kao kontrola čini se problematičnom jer čini kontrolu nedostupnom u kontekstu standardnih razmatranja. Ono što smo predstavili u poglavlju (8) može se shvatiti upravo kao pokušaj pristupa tom kontrolnom stupnju slobode; ili u drugom smislu, implementirati pristupačnu kontrolu. Međutim, vidjeli smo da je jedina mogućnost bila predložiti hipotetski pomoćni potencijal, u obliku relacije (8.14), koji ne bi ovisio o vremenu, već o *time directionality*, što nije u skladu s početnim izrazom za vremenski uređen eksponencijal.

U nastavku ćemo pokazati da je prikladno proširenje naših opažanja gdje je moguća

[80] Ovo nisu slobodne varijable u doslovnom smislu. Naime, slobodne varijable su definirane kao varijable koje imaju samo budući svjetlosni stožac, reflektirajući slobodu izbora eksperimentatora.
[81] Dakle, ako bi se takva kontrolirana superpozicija mogla projicirati, to bi moglo biti samo za kratka mjerila, vjerojatno unutar režima kvantne gravitacije.

implementacija takve kontrole okvir Page-Wottersovog formalizma. Slično pristupu predstavljenom u poglavlju (8), u nastavku ćemo pokušati izolirati jednu 'granu' operatora vremenskog uređenja, ili drugim riječima, implementirati kontrolu nad granama superpozicije, ali ovo vremena u kontekstu rješenja jednadžbe ograničenja, koja u slučaju nekoliko satova predstavlja generalizaciju Schrödingerove jednadžbe i stoga se proteže izvan standardne kvantne mehanike. Razlog za razmatranje jednadžbe ograničenja s više od jednog sustava koji služi kao sat je zato što, kao što je objašnjeno u ovom poglavlju (9), Page-Wottersov formalizam s jednim satom vraća uobičajenu kvantnu dinamiku i svodi se na već isprobanu pokušaj.

Razmotrimo sustav S spojen s dva sata A i B, tako da imamo kompozitni sustav $|\Phi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$. Dinamika je određena sljedećim ograničenjem:

$$\hat{C} = \hat{H}_A + \hat{H}_B + \hat{H}_S + \hat{V}_S(\hat{T}_A) + \hat{V}_S(\hat{T}_B) \tag{9.13}$$

definiranje naših fizičkih stanja putem jednadžbe ograničenja:

$$\hat{C}|\Psi\rangle\rangle = 0 \tag{9.14}$$

Koristeći grupno usrednjavanje[60], rješavamo povijesna stanja:

$$|\Psi\rangle\rangle = \int d\alpha e^{-i\alpha\hat{C}}|\Phi\rangle_{A,B,S} \tag{9.15}$$

gdje

$$|\Phi\rangle_{A,B,S} = |t_A, t_B\rangle \otimes |\phi\rangle_S \tag{9.16}$$

Izračunavanje djelovanja eksponencijala u jednadžbi (9.15), kao što je prikazano u Dodacima (C). Jednadžba za povijesno stanje ima oblik:

$$|\Psi\rangle = \int d\alpha e^{-i\alpha(\hat{H}_A+\hat{H}_B)} T e^{-i\int_0^\alpha d\lambda\left(\hat{V}_S(\hat{T}_A+\lambda)+\hat{V}_S(\hat{T}_B+\lambda)\right)} |t_A, t_B, \phi\rangle \tag{9.17}$$

gdje T predstavlja operator vremenskog reda (uveden u poglavlju (7.3) s obzirom na mjerilo λ:

$$T(\hat{f}(\lambda_1)\hat{g}(\lambda_2)) = \theta(\lambda_1 - \lambda_2)\hat{f}(\lambda_1)\hat{g}(\lambda_2) + \theta(\lambda_2 - \lambda_1)\hat{g}(\lambda_2)\hat{f}(\lambda_1) \tag{9.18}$$

Sada, razmotrimo eksponencijalno širenje vremenskog reda, do drugog reda,

$$\begin{aligned}
&Te^{-i\int_0^\alpha d\lambda \hat{V}_S(\hat{T}_A+\lambda)+\hat{V}_S(\hat{T}_B+\lambda)} \approx \mathbb{1} - i\int_0^\alpha d\lambda(\hat{V}_S(\hat{T}_A+\lambda)+\hat{V}_S(\hat{T}_B+\lambda)) \\
&+\frac{(-i)^2}{2!}\underbrace{T\int_0^\alpha d\lambda_1 d\lambda_2(\hat{V}_S(\hat{T}_A+\lambda_1)+\hat{V}_S(\hat{T}_B+\lambda_1))(\hat{V}_S(\hat{T}_A+\lambda_2)+\hat{V}_S(\hat{T}_B+\lambda_2))}_{\equiv *}
\end{aligned} \tag{9.19}$$

Usredotočujući se samo na član drugog reda:

$$\begin{aligned}
* &= T\int_0^\alpha d\lambda_1 d\lambda_2(\hat{V}_S(\hat{T}_A+\lambda_1)\hat{V}_S(\hat{T}_A+\lambda_2)+\hat{V}_S(\hat{T}_A+\lambda_1)\hat{V}_S(\hat{T}_B+\lambda_2) \\
&+\hat{V}_S(\hat{T}_B+\lambda_1)\hat{V}_S(\hat{T}_A+\lambda_2)+\hat{V}_S(\hat{T}_B+\lambda_1)\hat{V}_S(\hat{T}_B+\lambda_2)) \\
&= T\int_0^\alpha d\lambda_1 d\lambda_2 \sum_{I,J=A,B}\hat{V}_S(\hat{T}_I+\lambda_1)\hat{V}_S(\hat{T}_J+\lambda_2) \\
&= \int_0^\alpha d\lambda_1 d\lambda_2\Big[\sum_{I,J=A,B}\underbrace{\hat{V}_S(\hat{T}_I+\lambda_1)\hat{V}_S(\hat{T}_J+\lambda_2)\theta(\lambda_1-\lambda_2)}_{(1)}+\underbrace{\hat{V}_S(\hat{T}_J+\lambda_2)\hat{V}_S(\hat{T}_I+\lambda_1)\theta(\lambda_2-\lambda_1)}_{(2)}\Big]
\end{aligned} \tag{9.20}$$

Sada ćemo napraviti izbor potencijala. Potencijal u odnosu na svaki sat uzet ćemo kao takav da 'okrene' jednom u nekim unaprijed definiranim slučajevima t_1^* i t_2^* (lokalna vremena svakog sata):

$$\begin{aligned}
\hat{V}_S(\hat{T}_A+\lambda_{1,2}) &= \hat{V}_S(t_1^*)\delta(\hat{T}_A+\lambda_{1,2}-t_1^*) \\
\hat{V}_S(\hat{T}_B+\lambda_{1,2}) &= \hat{V}_S(t_2^*)\delta(\hat{T}_B+\lambda_{1,2}-t_2^*)
\end{aligned} \tag{9.21}$$

Vraćajući se na jednadžbu (9.20) koristeći jednadžbu (9.21), dobivamo:

$$\begin{aligned}
(1) &= \int_0^\alpha d\lambda_1 d\lambda_2\Big(\hat{V}_S(\hat{T}_A+\lambda_1)\hat{V}_S(\hat{T}_A+\lambda_2)+\hat{V}_S(\hat{T}_A+\lambda_1)\hat{V}_S(\hat{T}_B+\lambda_2) \\
&+\hat{V}_S(\hat{T}_B+\lambda_2)\hat{V}_S(\hat{T}_A+\lambda_2)+\hat{V}_S(\hat{T}_B+\lambda_1)\hat{V}_S(\hat{T}_B+\lambda_2)\Big)\theta(\lambda_1-\lambda_1) \\
&= \hat{V}_S(t_1^*)\hat{V}_S(t_1^*)\theta(t_1^*-\hat{T}_A-t_1^*+\hat{T}_A)+\hat{V}_S(t_1^*)\hat{V}_S(t_2^*)\theta(t_1^*-\hat{T}_A-t_2^*+\hat{T}_B) \\
&+\hat{V}_S(t_2^*)\hat{V}_S(t_1^*)\theta(t_2^*-\hat{T}_B-t_1^*+\hat{T}_A)+\hat{V}_S(t_2^*)\hat{V}_S(t_2^*)\theta(t_2^*-\hat{T}_B-t_2^*+\hat{T}_B)
\end{aligned} \tag{9.22}$$

$$
\begin{aligned}
(2) = \int_0^\alpha d\lambda_1 d\lambda_2 \Big(& \hat{V}_S(\hat{T}_B + \lambda_2)\hat{V}_S(\hat{T}_B + \lambda_1) + \hat{V}_S(\hat{T}_B + \lambda_2)\hat{V}_S(\hat{T}_A + \lambda_1) \\
& + \hat{V}_S(\hat{T}_A + \lambda_2)\hat{V}_S(\hat{T}_B + \lambda_1) + \hat{V}_S(\hat{T}_A + \lambda_2)\hat{V}_S(\hat{T}_A + \lambda_1)\Big)\theta(\lambda_2 - \lambda_1) \\
= {} & \hat{V}_S(t_2^*)\hat{V}_S(t_2^*)\theta(t_2^* - \hat{T}_B - t_2^* + \hat{T}_B) + \hat{V}_S(t_2^*)\hat{V}_S(t_1^*)\theta(t_2^* - \hat{T}_B - t_1^* + \hat{T}_A) \\
& + \hat{V}_S(t_1^*)\hat{V}_S(t_2^*)\theta(t_1^* - \hat{T}_A - t_2^* + \hat{T}_B) + \hat{V}_S(t_1^*)\hat{V}_S(t_1^*)\theta(t_1^* - \hat{T}_A - t_1^* + \hat{T}_A)
\end{aligned}
\tag{9.23}
$$

$$
\begin{aligned}
\Rightarrow \sum_{m,n=1,2} \sum_{I,J=A,B} \Big\{ & \theta(t_n^* - \hat{T}_I - t_m^* + \hat{T}_J)\hat{V}_S(t_n^*)\hat{V}_S(t_m^*) \\
& + \theta(t_n^* - \hat{T}_J - t_m^* + \hat{T}_I)\hat{V}_S(t_n^*)\hat{V}_S(t_m^*)\Big\}
\end{aligned}
\tag{9.24}
$$

Uvodeći notaciju

$$
\begin{aligned}
\delta t^* &\equiv t_2^* - t_1^* \\
\delta\hat{T} &\equiv \hat{T}_B - \hat{T}_A \\
\hat{W} &\equiv \hat{V}_S(t_2^*)\hat{V}_S(t_1^*) \\
\hat{Z} &\equiv \hat{V}_S(t_1^*)\hat{V}_S(t_2^*)
\end{aligned}
\tag{9.25}
$$

U nastavku ćemo zanemariti članove oblika $\hat{V}_S(t_1^*)\hat{V}_S(t_1^*)$, $\hat{V}_S(t_2^*)\hat{V}_S(t_2^*)$ jer su nevažni za naručivanje vremena. Također ćemo zanemariti sve ostale termine, do drugog reda, koji ne pokazuju drugačiji vremenski poredak. dobivamo,

$$
\begin{aligned}
& Te^{-i\int_0^\alpha d\lambda \hat{V}_S(t_A+\lambda)+\hat{V}_S(t_B+\lambda)} = \\
& = \cdots + 2\Big\{2\theta(-\delta t^*)\hat{Z} + 2\theta(\delta t^*)\hat{W} + \theta(-\delta t^* + \delta\hat{T})\hat{Z} + \theta(-\delta t^* - \delta\hat{T})\hat{Z} + \theta(\delta t^* + \delta\hat{T})\hat{W} + \theta(\delta t^* - \delta\hat{T})\hat{W}\Big\}
\end{aligned}
\tag{9.26}
$$

Budući da su vremena potencijalnih udaraca unaprijed definirana, δt^* će imati određenu vrijednost i predznak. Bez gubitka općenitosti, uzmimo:

$$
\delta t^* > 0 \tag{9.27}
$$

Također primijetite da izraz (9.26) ima dva reda $\hat{V}_S$ s obzirom na t^* vrijednosti. Tražit ćemo pripreme i mjerenja tako da ostane samo jedno vremensko naređivanje $\hat{V}_S$ operatora. Podijelimo prostor $\mathcal{H}_A \otimes \mathcal{H}_B$ na centar mase $\mathcal{H}_{CM}$ i relativne koordinate $\mathcal{H}_{rel}$, tako da $|t_A, t_B\rangle = |t_{CM}\rangle \otimes |t_{rel}\rangle$, gdje je $|t_{rel}\rangle = |\delta t\rangle$ svojstveno stanje $\delta\hat{T}|\delta t\rangle = \delta t|\delta t\ rang$. Imamo $|t_A, t_B, \phi\rangle \to |t_{CM}\rangle \otimes |t_{rel}\rangle \otimes |\phi\rangle$. Razmotrimo sada pripremu stanja

sata tako da:

$$|t_{rel}\rangle_{AB} = \eta\,|\delta t \leq \delta t^*\rangle + \xi\,|\delta t > \delta t^*\rangle \tag{9.28}$$

Ovdje treba napomenuti da takva stanja pripreme mogu nastati sasvim prirodno. Na primjer, kada satovi na različitim položajima djeluju međusobno i s drugim sustavima, kao u slučaju gravitacijske kvantne sklopke [89]. Ili ako su satovi u različitim stanjima relativnog gibanja, uzrokujući da podlegnu vremenskoj dilataciji [78]. Vratimo se jednadžbi (9.17)

$$\begin{aligned}|\Psi\rangle = \int d\alpha e^{-i\alpha(\hat{H}_A+\hat{H}_B)}\cdot \\ \underbrace{\cdot\, Te^{-i\int_0^\alpha d\lambda\left(\hat{V}_S(\hat{T}_A+\lambda)+\hat{V}_S(\hat{T}_B+\lambda)\right)}\Big\{\eta\,|\delta t \leq \delta t^*\rangle + \xi\,|\delta t > \delta t^*\rangle\Big\}\otimes|t_{CM}\rangle\otimes|\phi\rangle_{\mathrm{S}}}_{**}\end{aligned} \tag{9.29}$$

$$\Rightarrow ** = Te^{-i\int_0^\alpha \mathrm{d}\lambda\left(\hat{V}_S(\hat{T}_A+\lambda)+\hat{V}_S(\hat{T}_B+\lambda)\right)}\Big\{\eta\,|\delta t \leq \delta t^*\rangle + \xi\,|\delta t > \delta t^*\rangle\Big\}\otimes|t_{CM}\rangle\otimes|\phi\rangle_{\mathrm{S}} \tag{9.30}$$

Koristeći izraz (9.26), koji je do drugog reda:

$$\begin{aligned}** &= 2\Big\{\cdots + 2\hat{W} + \theta(-\delta t^*+\delta t)\hat{Z} + \theta(-\delta t^*-\delta t)\hat{Z} + \theta(\delta t^*+\delta t)\hat{W} + \theta(\delta t^*-\delta t)\hat{W}\Big\} \\ &\cdot\Big\{\eta\,|\delta t \leq \delta t^*\rangle + \xi\,|\delta t > \delta t^*\rangle\Big\}\otimes|t_{CM}\rangle\otimes|\phi\rangle_{\mathrm{S}} \\ &= \Big\{\cdots + 8\eta\hat{W}\,|\delta t \leq \delta t^*\rangle + 2\xi(\hat{Z}+3\hat{W})\,|\delta t > \delta t^*\rangle\Big\}\otimes|\phi\rangle_{\mathrm{S}}\end{aligned} \tag{9.31}$$

gdje '...' predstavljaju sve ostale pojmove u ekspanziji. Sada, za mjerenje, razmotrite stanje:

$$|\zeta\rangle_{\mathrm{AB}} = \gamma\,|\delta t \leq \delta t^*\rangle + \beta\,|\delta t > \delta t^*\rangle \tag{9.32}$$

Što daje (izostavljajući dijelove koji nisu relevantni za vremensko određivanje):

$$\begin{aligned}{}_{\mathrm{AB}}\langle\zeta|\cdot|\Psi\rangle &\approx \int d\alpha e^{-i\alpha(\hat{H}_S+\hat{H}_A+\hat{H}_B)}{}_{\mathrm{AB}}\langle\zeta|(**) \\ &\approx \int d\alpha e^{-i\alpha(\hat{H}_S+\hat{H}_A+\hat{H}_B)}\cdot\Big(\gamma\,\langle\delta t \leq \delta t^*| + \beta\,\langle\delta t > \delta t^*|\Big) \\ &\cdot\Big\{8\eta\hat{W}\,|\delta t \leq \delta t^*\rangle + 2\xi(\hat{Z}+3\hat{W})\,|\delta t > \delta t^*\rangle\Big\}\otimes|t_{CM}\rangle\otimes|\phi\rangle_{\mathrm{S}} \\ &\approx \int d\alpha e^{-i\alpha(\hat{H}_S+\hat{H}_A+\hat{H}_B)}\Big(8\hat{W}\gamma\eta + 2(\hat{Z}+3\hat{W})\beta\xi\Big)\otimes|t_{CM}\rangle\otimes|\phi\rangle_{\mathrm{S}}\end{aligned} \tag{9.33}$$

Pod uvjetima:

$$\begin{aligned} 8\hat{W}\gamma\eta + 6\hat{W}\beta\xi = 0 \\ \beta\xi = -\frac{4}{3}\gamma\eta \end{aligned} \tag{9.34}$$

izoliramo jednu konfiguraciju uređenja člana drugog reda vremenski uređenog širenja (9.17),

$${}_{\mathrm{AB}}\langle\zeta| \cdot |\Psi\rangle \approx \int d\alpha e^{-i\alpha(\hat{H}_A+\hat{H}_B)} 2\hat{V}_S(t_1^*)\hat{V}_S(t_2^*)\beta\xi \otimes |\phi\rangle_{\mathrm{S}} \tag{9.35}$$

gdje se poredak razmatrao s obzirom na neki parametar λ.

Zaključno, razmatrajući jednadžbu ograničenja s dva kvantna sata, uspjeli smo konstruirati povijesno stanje koje se može projicirati na potprostor koji se odnosi na određenu granu vremenskog uređenja operatora vremenskog uređenja T (poredak s obzirom na neku klasičnu skalu $lambda$). Drugim riječima, unutar zadanog okvira uspjeli smo dobiti kvantno kontroliranu superpoziciju vremenskih poredaka razmatrane vremenski uređene eksponencijalne, gdje su sada relativni stupnjevi slobode satova služili kao kontrola. Stoga, povijesno stanje možemo shvatiti kao superpoziciju vremenskih poredaka, s obzirom na skalu λ. Da napravimo paralelu sa superpozicijom poredaka razmatranih u kontekstu Feynmanova propagatora, mogli bismo pripisati ovu ljestvicu klasičnom satu nekog trećeg promatrača koji će vidjeti superpoziciju procesa koji odgovaraju različitim redoslijedima unutar vremenski uređenog operatora, gdje unutarnji stupnjevi Sloboda raspršenih čestica sada bi mogla predstavljati kvantne satove.[82]

Vraćajući se našem toy modelu, pretpostavimo za radnu postavku da imamo Alice i Boba koji vrše mjerenja na svojim satovima u svojim lokalnim laboratorijima i šalju svoje statistike promatraču s klasičnim satom koji otkucava prema skali λ . Mjerenje satova pokrenut će "udarac" na sustavu i postaviti događaj na fizički smislen način. Dva sata su u isprepletenom stanju relativnih stupnjeva slobode. Stanje je u superpoziciji u zapletenoj bazi, gdje su svojstveni prostori podijeljeni s obzirom na fiksiranje vremenske udaljenosti između dva udarca δt^*. Svaki će sat zatim postaviti

[82]Trebamo naglasiti da općenito nije potrebno ovaj klasični parametar povezivati s nekim klasičnim satom. Klasični satovi mogu se potpuno ukloniti unutar okvira formalizma Page-Wottersa.

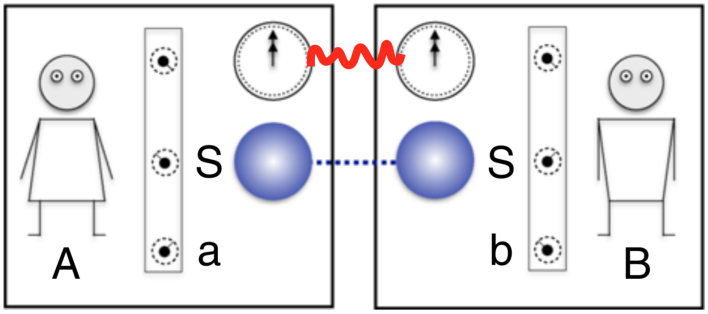


Figure 9.2: Na početku eksperimenta Alice i Bob pripremaju svoje satove u zapetljano stanje $|t_{rel}\rangle_{AB}$. Oni nastavljaju s eksperimentom mjerenjem svog lokalnog sata koji pokreće 'udarac' na podsustavu dostupnom iz njihovog lokalnog laboratorija. Crvena valovita linija označava prepletenost. [17]

događaj koji izvodi transformaciju na podsustavu sustava S kada njihovo lokalno vrijeme pokaže neke unaprijed određene vrijednosti t_1^* ili t_2^*. Možemo zamisliti da su satovi unaprijed programirani da rade u ovim vremenima. Unutar analogije s Feynmanovim propagatorom, možemo pretpostaviti da je sustav polje i da Alice i Bob izvode transformacije na nekom lokaliziranom podsustavu. Označit ćemo $V(t_1^*) = A$ i $V(t_2^*) = B$ kako bismo uspostavili vezu sa slikom 8.1. Određeni prethodni odabiri i naknadni odabiri na sustavima A i B rezultirat će scenarijima u kojima će promatrač C moći vidjeti ili $A \prec B$ ili $B \prec A$, ali će općenito vidjeti superpozicija naloga $(A \prec B) + (B \prec A)$. Naime, napomenimo da se razmatrana situacija svodi na uobičajeni slučaj pri pripremama koje postavljaju $\delta t = 0$, odnosno kada su otkucaji sata međusobno sinkronizirani.

Sada se može tvrditi da bi se rasprava mogla podići u kontekst kvantne teorije polja, promicanjem povijesnog stanja valne funkcije stanja konačnih čestica $|\psi(t)\rangle$ u povijesno stanje valnog funkcionala, kako se pojavljuje u Schrödingerovoj funkcionalnoj formulaciji, koju ćemo razmotriti u poglavlju (10). U svjetlu toga, razmotrit ćemo sljedeću generalizaciju našeg modela igračke s dva sata na slučaj N satova kao modela igračke za opći funkcionalni slučaj.

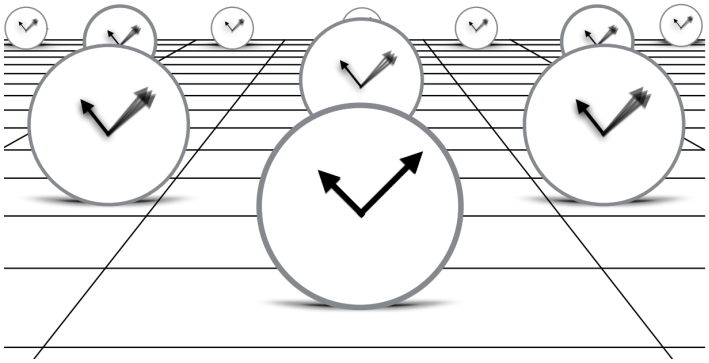


Figure 9.3: Mreža satova zamišljena na hiperpovršini. [16]

10 Schrödingerova funkcionalna reprezentacija i Page-Wottersov formalizam

Kvantna teorija polja je u biti kvantna mehanika beskonačnih stupnjeva slobode, što znači da također podliježe Schrödingerovoj formulaciji. Ipak, posebnost kvantne teorije polja u odnosu na kvantnu mehaniku je implementacija prostorno-vremenske simetrije. Upravo nedostatak očigledne Lorentzove invarijantnosti kada se izrazi u obliku Schrödingerove reprezentacije, zajedno s činjenicom da je njena renormalizabilnost dokazana relativno kasno [80], bio je razlog zašto Schrödingerova reprezentacija kvantne teorije polja nikada nije postigla veliku popularnost. Unatoč tome, u posljednje je vrijeme privukao pozornost u nekim kozmološkim [36][42][42] i konceptualnim razmatranjima (za naše svrhe, gdje je prikladan za implementaciju bezvremenskog formalizma).
S ovim formalizmom možemo raditi u bilo kojem zakrivljenom globalno hiperboličnom prostorvremenu budući da se tamo može uvesti vremenska funkcija t s globalno valjanim vremenskim smjerom. To nam omogućuje da napravimo globalnu folijaciju prostorvremena $\mathbf{R}\times\Sigma_t$. Također možemo pretpostaviti konstrukciju na konačnom intervalu I koji obuhvaća relevantne događaje, dopuštajući folijaciju $(\Sigma_t)_{t\in I\subset\mathbf{R}}$ lokalno. U našim razmatranjima ograničit ćemo se na prostorvrijeme Minkowskog.

Grubo govoreći, možemo koristiti većinu osnovne intuicije iz kvantne teorije dok zamjenjujemo riječ valna *funkcija* s valnim *funkcionalom*, stvarajući koordinatne prikaze *funkcionalnog vektora stanja*. Drugim riječima, u ovoj reprezentaciji stanja $|\Psi\rangle$ su

funkcionali vremenski neovisnog polja $\hat{\phi}(\mathbf{x})$. Ograničit ćemo našu pozornost na skalarne konfiguracije polja, definirane kao preslikavanje $\phi : \mathcal{M} \to \mathbf{C}$. Prostor kojem pripada $|\Psi\rangle$ je prostor kvadratno integrabilnih *funkcionala* $L^2(\mathcal{C}(\Sigma_t))$, $\mathcal{C}(\Sigma_t)$ označavajući prostor svih mogućih trenutnih konfiguracija polja. Vlastiti vektori operatora definirani putem

$$\hat{\Phi}(\vec{x})|\phi\rangle = \phi(\vec{x})|\phi\rangle \tag{10.1}$$

će definirati 'koordinatnu reprezentaciju' u funkcionalnom prostoru, gdje jednadžba (5.1) oponaša ulogu $\hat{X}|x\rangle = x|x\rangle$ jednočestičnog QM. Ovdje $\phi(\vec{x})$ označava samo običnu skalarnu funkciju. Koordinatni prikaz stanja $|\Psi\rangle$ sada je valni funkcional ovisan o vremenu $\Psi[\phi(\vec{x}), t] = \langle\phi(x)|\Psi(t)\rangle$, za koji vrijedi:

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle \tag{10.2}$$

$\Psi[\phi(\vec{x}), t] = \langle\phi|\Psi(t)\rangle$ predstavlja amplitudu vjerojatnosti promatranja konfiguracije polja na hiperpovršini konstantnog vremena t.

Prostor stanja također je opremljen funkcionalnom normom $\mathcal{D}\phi$, tako da imamo normalizaciju:

$$\|\Psi_t\|^2 := \int_{\mathcal{C}(\Sigma_t)} \mathcal{D}\phi \, |\Psi_t[\phi]|^2 \tag{10.3}$$

Ovdje ćemo uzeti ono što je standardni postupak u konstruiranju prostora kvadratno integrabilnih funkcija [79]; naime, s L^2 označavamo prostor kvocijentiran sa skupom konfiguracija koje su gotovo posvuda iste $\Psi_1 \sim \Psi_2$ (tj. koje se razlikuju na skupu mjere nula)

$$L^2(\mathcal{C}(\Sigma_t), \mathcal{D}\phi) := \mathcal{L}^2(\mathcal{C}(\Sigma_t)) / \sim \tag{10.4}$$

tako da možemo reći $\|\Psi_t\| = 0 \Rightarrow \Psi_t \equiv 0$. Za definiranje gustoće vjerojatnosti koristimo indikatorski funkcional $\lambda_{\mathcal{A}}$ na mjerljivom podskupu $\mathcal{A} \subset \mathcal{C}(\Sigma_t)$ tako da se $|\lambda_{\mathcal{A}}\Psi_t[\phi]|^2$ može shvatiti kao vjerojatnost da konfiguracija polja bude dana nekim $\phi \in$

$$\langle\Psi_1 \mid \Psi_2\rangle = \int \mathcal{D}\phi \Psi_1^*[\phi]\Psi_2[\phi] \tag{10.5}$$

Da bismo kvantizirali teoriju, moramo nametnuti kanonske komutacijske relacije. To implicira da oblik kanonskog operatora polja momenta bude u obliku funkcionalne

derivacije,

$$\Pi(\mathbf{x}) = -i\frac{\delta}{\delta\phi(\mathbf{x})} \tag{10.6}$$

tako da vrijede komutacijske relacije:[83]

$$[\Phi(\mathbf{y}), \Pi(\mathbf{x})] = i\delta(\mathbf{x}-\mathbf{y}) \tag{10.8}$$

Operatori teorije predstavljeni su kao vremenski neovisne funkcionalne jezgre

$$\hat{O}|\psi\rangle = \int \mathcal{D}\phi \left\langle \phi' | \hat{O} | \phi \right\rangle \langle \phi | \psi \rangle = \int \mathcal{D}\phi \mathcal{O}(\phi, \phi')\Psi[\phi] \tag{10.9}$$

Za kanonske varijable imamo,

$$\Phi(\mathbf{x}) \to \langle \phi' | \Phi(\mathbf{x}) | \phi \rangle = \phi(\mathbf{x})\delta[\phi - \phi'] \tag{10.10}$$

$$\Pi(\mathbf{x}) \to \langle \phi' | \Pi(\mathbf{x}) | \phi \rangle = -i\frac{\delta}{\delta\phi(\vec{x})}\delta\left[\phi - \phi'\right] \tag{10.11}$$

gdje smo koristili funkcionalnu delta funkciju i funkcionalno diferenciranje. Jednadžba (10.10) definira djelovanje operatora polja $\Phi(\mathbf{x})$ kao množenje s $\phi(\mathbf{x})$,

$$\Phi(\mathbf{x})|\Psi\rangle \to \phi(\mathbf{x})\Psi[\phi] \tag{10.12}$$

Akcija bilo kojeg operatora se tada definira kao,

$$\mathcal{O}(\Pi, \Phi)|\Psi\rangle \longrightarrow \mathcal{O}\left(\frac{1}{i}\frac{\partial}{\partial\phi}, \phi\right)\Psi[\phi] \tag{10.13}$$

U slučaju Klein-Gordonovog polja, imamo Hamiltonian:

$$H = \frac{1}{2}\int d^3x \left(\Pi^2 + |\nabla\phi|^2 + m^2\phi^2\right) \tag{10.14}$$

[83] jer za funkcionalnu derivaciju vrijedi:

$$\frac{\delta}{\delta\phi(\mathbf{x})}\phi(\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) \tag{10.7}$$

koji se nakon umetanja (10.6) pretvara u funkcionalni diferencijalni operator,

$$H = \frac{1}{2}\int d^3x \left(-\frac{\delta^2}{\delta\phi^2(\mathbf{x})} + |\nabla\phi|^2 + m^2\phi^2 \right) \tag{10.15}$$

i Schrödingerovu jednadžbu u funkcionalnu diferencijalnu jednadžbu,

$$i\frac{\partial}{\partial t}\Psi[\phi(\mathbf{x}),t] = \frac{1}{2}\int d^3x \left(-\frac{\delta^2}{\delta\phi^2(\mathbf{x})} + |\nabla\phi|^2 + m^2\phi^2 \right) \Psi[\phi(\mathbf{x}),t]. \tag{10.16}$$

Budući da Hamiltonijan eksplicitno ne ovisi o vremenu, možemo odvojiti vremensku ovisnost valnog funkcionala $\Psi[\phi,t]$ i napisati

$$\Psi[\phi(\mathbf{x}),t] = e^{-i\hat{H}_S t}\Psi[\phi(\mathbf{x})]. \tag{10.17}$$

gdje $\Psi[\phi(\mathbf{x})]$ zadovoljava Schrödingerovu funkcionalnu jednadžbu neovisnu o vremenu,

$$\frac{1}{2}\int d^3x \left(-\frac{\delta^2\Psi[\phi(\mathbf{x})]}{\delta\phi^2(\mathbf{x})} + \left(|\nabla\phi|^2 + m^2\phi^2\right)\Psi[\phi(\mathbf{x})] \right) = E\Psi[\phi(\mathbf{x})] \tag{10.18}$$

Za više detalja, vidi [48], [27].

10.1 Schrödingerovo funkcionalno bezvremensko stanje

Sada također možemo razmišljati o promicanju jednadžbe ograničenja

$$\hat{C} = \hat{H}_C \otimes \mathbb{1}_S + \mathbb{1}_C \otimes \hat{H}_S \tag{10.19}$$

u funkcionalnu diferencijalnu jednadžbu tako da smanjeno stanje zadovoljava funkcionalnu diferencijalnu Scrödingerovu jednadžbu nakon uvjetovanja stanjem sata. Primijetite da u jednadžbi (5.15) još uvijek imamo standardnu parcijalnu derivaciju u odnosu na vrijeme. Tumačenjem putem Page-Wottersovog formalizma, to implicira da (u ovoj fazi) vremenski operator još uvijek odgovara standardnom koordinatnom operatoru, za koji imamo jednadžbu svojstvene vrijednosti $\hat{T}|t\rangle = t|t\rangle$. Stoga možemo napisati povijesno stanje u terminima *sata kao sustava jedne čestice*:

$$|\Psi\rangle\!\rangle = \int dt |t\rangle_C \otimes \Psi(t)\rangle_S \tag{10.20}$$

koji ima oblik standardnog Page-Wottersovog bezvremenskog stanja, odražavajući puku činjenicu da kvantna teorija polja zadovoljava standardnu Schrödingerovu jednadžbu (10.2). Ipak, treba imati na umu da $|\Psi(t)\rangle$ sada predstavlja opće stanje u prostoru $L^2\left(\mathcal{C}\left(\Sigma_t\right)\right)$, u 'koordinatnom' prikazu,

$$|\Psi[\phi]\rangle = \int dt|t\rangle_C \otimes \Psi(t,\phi]_S \tag{10.21}$$

(oznaka $\Psi(t,\phi]$ naglašava da je Ψ funkcija od t, ali funkcional od ϕ [84]). Jednadžba (10.21) odgovara situaciji uzimanja perspektive jednog stupnja slobode kvantnog sata i promatranja stanja povijesti konfiguracije stanja polja s obzirom na preferiranu foliaciju.
Ipak, kao što smo naveli u poglavlju (2.1.5), da bismo implementirali simetrije prostorvremena, trebamo tražiti unitarnu reprezentaciju Poincaréove grupe koja je beskonačno dimenzionalna. Stoga polje ne možemo vezati na 1-dimenzionalni sustav bez da narušimo relativističku kovarijantnost. Da bismo to uzeli u obzir, možemo razmatrati kontinuirano sustav satova, tako da oni odgovaraju polju τ

$$\hat{\tau}(\vec{x})|\tau\rangle = \tau(\vec{x})|\tau\rangle \tag{10.22}$$

gdje $\hat{\tau}(\vec{x})$ predstavlja operator koji odgovara polju satova, a $\tau(\vec{x})$ je skalarna funkcija koja predstavlja konfiguraciju polja. To sada znači da se svaki stupanj slobode polja može pridružiti nekoj vremenskoj funkciji s odgovarajućom folijacijom. Za impuls sata koji bismo sada imali,

$$\hat{\omega}(\vec{x}) = -i\frac{\delta}{\delta\tau(\vec{x})} \tag{10.23}$$

a jednadžba ograničenja se promiče u funkcionalnu diferencijalnu jednadžbu s obzirom na dvije funkcije:

$${}_C\langle\tau|\hat{C}|\Psi\rangle = {}_C\langle\tau|(\hat{H}_C \otimes \mathbb{1}_S + \mathbb{1}_C \otimes \hat{H}_S)|\Psi\rangle\rangle = 0 \tag{10.24}$$

$$\left(-i\frac{\delta}{\delta\tau(\vec{x})} + H_S[\phi(\vec{x})]\right)|\Psi[\tau(\vec{x})]\rangle = 0 \tag{10.25}$$

[84] Ovo je preuzeto iz [56]

gdje smo za sat Hamiltonian pretpostavili savršen Hamiltonijan sata

$$\hat{H}_C = \hat{\omega} = -i\frac{\delta}{\delta\tau(\vec{x})} \tag{10.26}$$

U 'koordinatnom' prikazu imamo jednadžbu:

$$i\frac{\delta}{\delta\tau(\vec{x})}\Psi[\phi(\vec{x}),\tau(\vec{x})] = H_S[\phi(\vec{x})]\Psi[\phi(\vec{x}),\tau(\vec{x})] \tag{10.27}$$

Ovo se može prepoznati kao oblik *Schwinger-Tomonagine jednadžbe*:

$$\left\{H(x) - i\frac{\delta}{\delta\sigma(x)}\right\}\Psi[\sigma] = 0 \tag{10.28}$$

gdje će formalno rješenje ove jednadžbe biti dano funkcionalom stanja[81]

$$\Psi[\sigma] = \mathcal{U}\left[\sigma,\sigma_0\right]\Psi\left[\sigma_0\right] \tag{10.29}$$

Ovdje σ_0 označava početni uvjet na hiper-plohi, a $\Psi\left[\sigma_0\right]$ označava funkcional stanja koji odgovara ovom početnom uvjetu. Dakle, Schwinger-Tomonagina jednadžba opisuje evoluciju stanja kvantnog polja od jedne početne Cauchyjeve površine do druge, a uveli su je Schwinger i Tomonaga [77] pokušavajući napraviti invarijantnu generalizaciju Schrödingerove jednadžbe. Moglo bi se činiti zanimljivim kako smo došli do iste jednadžbe, sa sličnim konceptualnim inputom, iz ponešto drugačijeg konteksta. Usprkos tome, u [18] se tvrdilo da transformacijski funkcional neće biti jedinstveno implementiran na Fockovu prostornu reprezentaciju kvantnog polja za slučaj međudjelovanja polja, za dimenziju prostor-vremena veću od dvije, čak ni u ravnom prostor-vremenu . Čini se da je poseban problem to što evolucija od početne površine do konačne površine ovisi o izbor folijacije, kao što je raspravljao Kuchař u [56]. No, nije toliko očito da se tim problemima ne može nikako pristupiti i da se kritike ne mogu nadmašiti. Imajući to na umu, radi rasprave ćemo nastaviti, podsjećajući da je motivacija korištenja ovog pristupa bila podizanje rasprave o odjeljku 9.2 u kontekst teorije polja.

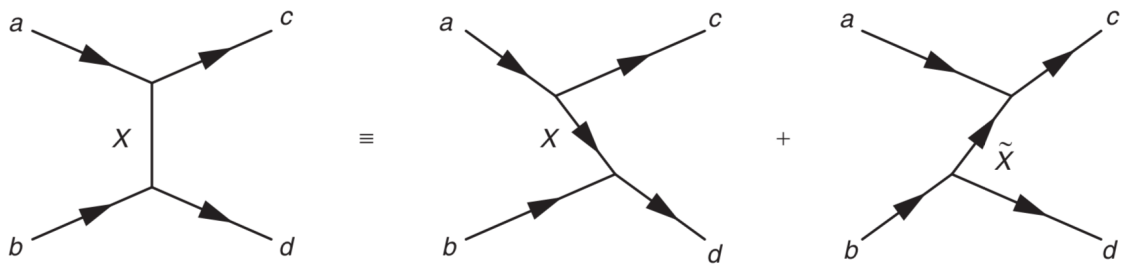


Figure 10.1: Vremenska uređenja Feynmanovog propagatora. [82]

Stanje povijesti sada bi bilo zapisano kao,

$$|\Psi\rangle\!\rangle = \int \mathcal{D}\tau |\tau\rangle_C \otimes |\Psi[\tau(\vec{x})]\rangle_S \tag{10.30}$$

gdje je $\mathcal{D}\tau$ mjera funkcionalne integracije. Ovo bi predstavljalo evoluciju stanja polja $|\Psi\rangle$, razapeto baznim elementima $\{|\phi\rangle\}$, koje se razvija relativno u odnosu na konfiguracije polja $\tau(\vec{x})$.

10.2 Amplituda raspršenja u bezvremenskom formalizmu

Prisjetimo se da je ideja naših razmatranja bila sagledati situaciju dvaju vremenskih poredaka koji se pojavljuju u Feynmanovu propagatoru (vidi sliku(10.1)), u smislu Hilbertove superpozicije stanja uređenja $A \prec B$ i $B \prec A$, analogno [89]. Podsjetimo se da su Feynmanovi dijagrami izravno povezani s amplitudom raspršenja, koja za dva stanja čestice 'in' i 'out' ima oblik:

$$\mathcal{A} = {}^{\text{out}}\langle q_1 q_2 \mid p_2 p_1 \rangle^{\text{in}} = {}_0\left\langle q_1 q_2 \left| \hat{S} \right| p_2 p_1 \right\rangle_0 \tag{10.31}$$

gdje $|\psi\rangle = |p_2 p_1\rangle_0$ predstavljaju asimptotska slobodna stanja koja postoje na početku eksperimenta $t \to -\infty$, $|\phi\rangle = |q_2 q_1\rangle_0$ asimptotsko stanje na kraju eksperimenta $t \to \infty$. Definicija S-matrice dana je (kako je objašnjeno u (2.4.2)):

$$S \equiv U(\infty, -\infty) = T \exp\left(-\mathrm{i}\int_{-\infty}^{\infty} \mathcal{H}^{\mathrm{I}}_{\mathrm{int}}(t)\mathrm{d}t\right) \tag{10.32}$$

gdje Dysonov operator dolazi kao rješenje Schrödingerove jednadžbe koja se dobiva iz jednadžbe ograničenja (5.18). Implementirajmo sada Page-Wottersov formalizam u izraz amplitude raspršenja.

U standardnoj kvantnoj mehanici, definiramo propagator između nekog početnog stanja $|I\rangle_S$ u trenutku t_I i nekog konačnog stanja $|F\rangle_S$ u trenutku t_F:

$$G\left(F,t_F;I,t_I\right) := {}_S\left\langle F\left|\hat{U}_S\left(t_F,t_I\right)\right|I\right\rangle_S \tag{10.33}$$

Da bismo uspostavili propagator u smislu Page-Wottersovog formalizma, jednostavno fiksiramo početni uvjet povijesnog stanja $|\Psi\rangle\rangle$, tj. identificiramo t_0 s vremenom t_I i $|\psi\left(t_0\right)\rangle_S$ s $|I\rangle_S$ u jednadžbi:

$$|\Psi[\phi]\rangle = \int dt|t\rangle_C \otimes \hat{U}(t,t_0)_S\psi(t_0;\phi]_S \tag{10.34}$$

Propagator tada jednostavno poprima oblik (vidi [37] jedn. (22)):

$$G\left(F,t_F;I,t_I\right) = {}_C\left\langle t_F\right|\otimes_S\langle F\mid\Psi\rangle\rangle \tag{10.35}$$

Sada to možemo proširiti na QFT scenarij pomoću Schrödingerove funkcionalne reprezentacije. Propagator će sada ići od početne konfiguracije polja hiperpovršine Σ_x na t_x do konačne Σ_y na t_y:

$$G\left(F,t_F;I,t_I\right) = {}_C\left\langle t_y\right|\otimes_S\langle\Psi(t_y)|\Psi\rangle\rangle_{(t_x,\Sigma_x)} \tag{10.36}$$

što nam omogućuje da odmah odredimo amplitudu raspršenja u smislu bezvremenskog stanja. Naime, jednadžba 10.31 sada bi imala oblik:

$$\mathcal{A} = \lim_{t_F\to\infty}\langle t_F|\otimes{}_0\left\langle q_1q_2\mid\Psi\right\rangle\rangle \tag{10.37}$$

gdje

$$|\Psi\rangle\rangle = \int dt|t\rangle_C \otimes \hat{U}(t,-\infty)_S|p_2p_1\rangle_{0,S} \tag{10.38}$$

U koordinatnoj reprezentaciji

$$|\Psi[\phi]\rangle = \int dt|t\rangle_C \otimes \hat{U}(t,-\infty)_S\mathbf{a}^\dagger\left(\vec{p}_2\right)\mathbf{a}^\dagger\left(\vec{p}_1\right)\Psi_0[\phi]_S \tag{10.39}$$

gdje smo koristili $\langle \phi \mid p_1 p_2 \rangle = \Psi_{p_1 p_2}[\phi] = \mathbf{a}^\dagger(\vec{p}_2)\,\mathbf{a}^\dagger(\vec{p}_1)\,\Psi_0[\phi]$ [48] te stoga,

$$\mathcal{A} = \int \mathcal{D}\phi \lim_{t_F \to \infty} \langle t_F | \otimes \int dt |t\rangle_C \otimes \Psi_0^\dagger(t;\phi]_S \mathbf{a}(\vec{q}_2)\,\mathbf{a}(\vec{q}_1)\,\hat{U}(t,-\infty)_S \mathbf{a}^\dagger(\vec{p}_2)\,\mathbf{a}^\dagger(\vec{p}_1)\,\Psi_0(t;\phi]_S \tag{10.40}$$

Ova jednadžba pretpostavlja da je sat neki vanjski sustav bez interakcije i ne odnosi se na Schwinger-Tomonaginu formulaciju. U ovom slučaju, možemo primijeniti razmatranja navedena u (4.2) jer bismo opet očekivali neki vremenski uređen eksponencijalni, ali ovaj put koji djeluje na funkcional kinematičkog stanja, ograničavajući funkcionalni Fockov prostor.

Ako sada želimo implementirati jednadžbu s obzirom na (10.30),

$$\mathcal{A} = \int \mathcal{D}\phi \lim_{t_F \to \infty} \langle t_F | \otimes_0 \Psi_0^{(0)\dagger}[\tau,\phi]_S \mathbf{a}(\vec{q}_2)\,\mathbf{a}(\vec{q}_1) \int \mathcal{D}\tau |\tau\rangle_C \otimes \tilde{\mathcal{U}}(\tau,\tau_0) \mathbf{a}^\dagger(\vec{p}_2)\,\mathbf{a}^\dagger(\vec{p}_1)\,\Psi_0^{(0)}[\tau_0;\phi_0]_S \tag{10.41}$$

gdje, ako biramo unutarnje stupnjeve slobode čestica kao satove, možemo definirati početnu konfiguraciju sata τ_0 da odgovara $|\tau_0\rangle_C = \bigotimes_{n=2} |t_n\rangle$, tj. početno stanje eksperimenta u odnosu na početne čestice.

Na ovaj način možemo argumentirati vezu između rezultata toy modela iz poglavlja (9.2) i motivacije dane u kontekstu vremenskog uređenja i Feynmanovih propagatora. Naime, s obzirom na ovu konstrukciju, analogno diskusiji u poglavlju (8), bi se određenim početnim stanjima i odabirom mjerenja, eventualno izolirao vremenski poredak, za koji se očekuje da će se pojaviti u kontekstu rješenja (10.27) [81].

11 Zaključak

U ovom smo radu istražili pojmove vremena i kauzalnog uređenja u kvantnoj mehanici i perturbativnoj kvantnoj teoriji polja. Posebna pažnja posvećena je strukturi vremensko uređenih eksponencija, proširenoj u Dysonov razvoj, gdje smo pokušali razumjeti njihovo vremensko uređenje u smislu superpozicije poredaka koji se pojavljuju u kvantnoj sklopki. Naime, glavni motiv našeg promišljanja bio je usporediti superpoziciju vremenskih uređenja koja se pojavljuje na razini Feynmanova propagatora s neodređenošću vremenskih poredaka u kvantnoj sklopki. Kao što je objašnjeno u poglavlju (7.3) operator uređenja igra važnu ulogu u Feynmanovu propagatoru, a proces interakcije promatran kao razmjena preko virtualne čestice može se shvatiti kao superpozicija dvaju procesa koji se izmjenjuju na ljusci mase čestice ili antičestice. Tada se čini uvjerljivim reći da bi izdvajanje jedne grane konfiguracije uređenja u Dysonovom širenju niza značilo izdvajanje jednog od procesa izmjene na ljusci mase.

Imajući na umu ovu motivaciju, prvo smo razmotrili vremenski uređenu eksponencijalnu funkciju u kontekstu standardne kvantne mehanike, kako se pojavljuje u rješenju vremenski ovisne Schrödingerove jednadžbe s $[\hat{H}(t), \hat{H}(t')] \neq 0$. Ova je postavka analizirana u poglavlju (8) gdje smo pokušali vezanjem na pomoćni sustav interakcijom koja bi razlučivala jednu ili drugu konfiguraciju vremenskog uređenja. Naime, kada bi takvo vezanje bilo moguće, moglo bi se koristiti pomoćni sustav da služi kao kontrola nad granama reda, omogućujući nam da izoliramo jednu od grana putem mjerenja na pomoćnom sustavu. Ipak, uvjerili smo se da nije moguće uvesti kontrolni stupanj slobode, u okviru prikazanih metoda. Zagonetan aspekt ovih razmatranja je taj da se prema teoremu o zabranjivanju navedenom u [24] ne može imati superpozicija naloga bez kontrolnog sustava. Stoga ćemo zaključiti da bi mogao postojati neki nedostupni pozadinski stupanj slobode koji služi kao kontrola, omogućujući superpoziciju konfiguracija uređenja, ili da je superpozicija poredaka samo artefakt ovog matematičkog formalizma i ne treba je shvatiti u točnom smislu kvantni prekidač. Naglašavamo da se ova razmatranja oslanjaju na tumačenja povezana s perturbativnim razumijevanjem teorije i razmišljanje treba usporediti s aksiomatskim opisom.

U poglavlju (9) nastavili smo s pokušajem izolacije jedne od grana vremenskog uređenja u eksponencijalnom razvoju, ovaj put u kontekstu bezvremenskog stanja s nekoliko satova, što sada nadilazi kontekst standardne kvantne mehanike (shvaćene u terminima Schrödingerove jednadžbe). Ovdje smo uspjeli izolirati doprinose uređenja odgovarajućim izborom priremljenog i mjerenog stanja u Hilbertovom prostoru relativnih stupnjeva slobode. Ponudili smo tumačenje klasičnog parametara koji se pojavljuju u bezvremenskom stanju da imaju značenje klasičnog parametra koji odgovara vremenu nekog klasičnog vanjskog sata. To znači da ćemo naše rezultate shvatiti kao neodređenost koja se pojavljuje u odnosu na klasične operacije promatranja sata koje obavljaju dva isprepletena kvantna sata povezana s dvije različite vremenske funkcije, u kontekstu fiksne Minkowskijeve pozadine. Činjenica da je pokušaj uspio može se shvatiti kao činjenica da stanje povijesti sadrži sve korelirane konfiguracije između sustava i satova, gdje se sada relativni stupnjevi slobode satova mogu shvatiti kao kontrola nad redoslijedom koji omogućuje superpoziciju.

U posljednjem odjeljku opisali smo Schrödingerov funkcionalni prikaz kvantne teorije polja, naglašavajući njezinu važnost u kontekstu ovih rasprava. Tvrdili smo da je proširenje Page-Wottersovog formalizma u ovoj postavci jednostavno ako se uzme jedan stupanj slobode koji služi kao sat u odnosu na koji se radi folijacija prostorvremena. Unatoč tome, budući da simetrija prostorvremena ima ireducibilnu unitarnu reprezentaciju Poincaréove grupe, koja je beskonačnodimenzionalna, spajanje jednog stupnja slobode ne bi bilo kompatibilno s tom simetrijom i takvo bezvremensko stanje ne bi bilo Lorentz invarijantno. Stoga smo proširili razmatranja pretpostavkom polja čiji će prostor vlastitih stanja služiti kao sat za evoluciju drugog polja u odnosu na njega. Tj. svojstvena stanja pripisana konfiguraciji jednog polja služe kao referentni sustav za konfiguracije drugog. Ovom jednostavnom argumentacijom inspiriranom Page-Wottersovim formalizmom, došli smo do već poznate jednadžbe, Schwinger-Tomonagine jednadžbe. Unatoč tome, kao što je raspravljeno u [18][56], Schwinger-Tomonagina jednadžba ima poteškoća, a najveća je ovisnost evolucije od jedne Cauchyjeve površine do druge o izboru folijacije i neunitarnosti za prostorvremena $dim > 2$. Obzirom na naša razmatranja, ako se Schwinger-Tomonagina jednadžba ne može implementirati u kontekstu kvantne gravitacije, posljedica bi također

mogla biti da se sam program Page-Wottersovog pristupa ne može kovarijantno implementirati. Da bi se Page-Wotterov pristup implementirao u teoriju polja, konstrukciju Schrödingerovog funkcionalnog prostora treba shvatiti rigoroznije i probleme Schwinger-Tomonagine jednadžbe treba zaobići.

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