The geometry of the Standard Model

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THE GEOMETRY OF THE STANDARD MODEL

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Master Thesis

The geometry of the Standard Model

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I would like to thank my mentor dr. Tajron Jurić for giving me so many opportunities to work and learn. I would also like to thank my dear cousin Iva for her tremendous help in writing this thesis in English which is not my native language. Finally, I must express my gratitude to my parents.

Geometrija Standardnog Modela

Sažetak

Ovaj diplomski rad je gotovo samodostatno rigorozno izlaganje Standardmog modela (*SM*) kao baždarne teorije. Izgradili smo pojmove iz diferencijalne geometrije, diferencijalne topologije i spinske strukture nužne za matematički opis baždarnih teorija i usput ih demonstrirali primjerima na Općoj teoriji relativnosti (*OTR*). Posebna pozornost je dana glavnim i pridruženih svežnjevima koji su centralni matematički objekti za definiranje baždarnih teorija. Zatim je ukratko izložen SM u stilu diplomskih kolegija Fizika elementarnih čestica, fokusirajući se najviše na njegove simetrije i strukturu njegovog lagranžijana. Na kraju, primijenjene su razvijene matematičke strukture na konkretnu baždarnu teoriju Standardnog modela rigoroznim definiranjem svakog člana njegovog lagranžijana na geometrijski način.
Ključne riječi: Standardni model, diferencijalna geometrija, topologija, simetrije, glavni svežanj, pridruženi svežanj, spinor

The geometry of the Standard Model

Abstract

This Master's Thesis is an almost completely self-contained outline of the Standard Model (*SM*) in the framework of gauge theory. The tools from differential geometry, differential topology and spin structure necessary for mathematical description of gauge theories were built and their examples in General theory of relativity (*GR*) were shown. Special attention was paid to principal and associated bundles which are central mathematical objects used to define gauge theories. Then the SM was shortly outlined in the style of undergraduate courses Elementary particle physics, focusing mostly on its symmetries and the structure of its Lagrangian. In the end, the developed mathematical structures were applied on the concrete gauge theory of the Standard Model by rigorously defining every term of its Lagrangian in a geometrical way.

Keywords: Standard Model, differential geometry, topology, symmetries, principal bundle, associated bundle, spinor

Contents

1 Mathematical introduction to gauge theories

1.1 Introduction

Gauge theories are theories that represent distinct physical states as equivalence classes of field configurations. The goal of this chapter is to rigorously define the geometric and spin building blocks for describing gauge gauge theories on manifolds (for the definition of a manifold, see Appendix A.1.1). We will start the first Chapter with a section that focuses on examples of a classical and quantum gauge theory and later we will build the theory of bundles which is a natural formalism for describing gauge theories.

1.1.1 Classical Electrodynamics

Classical Electrodynamics (*CE*) is the first gauge theory physicists encounter during their studies, specifically, CE is a $U(1)$ gauge theory. That is, Maxwell equations

$$\begin{aligned}
&\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \\
&\nabla \times \vec{E} = \frac{-1}{c}\frac{\partial \vec{B}}{\partial t} \\
&\vec{\nabla} \cdot \vec{B} = 0 \\
&\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}
\end{aligned} \tag{1.1}$$

can be simplified into one equation

$$\Box A^{\mu} - \partial^{\mu}\left(\partial_{\alpha} A^{\alpha}\right) = J^{\mu} \;, \tag{1.2}$$

by introducing the four-vector electromagnetic potential $A^{\mu} = (\frac{\phi}{c}, \vec{A})$ and the four-current $J^{\mu} = (\rho, \vec{J})$, where A^{μ} is defined in a way such that the following two relations are true

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \tag{1.3}$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A} \;. \tag{1.4}$$

It can be deduced from the definitive properties of A^μ that the CE is invariant to transformations

$$A^\mu \to A^\mu + \frac{1}{q}\partial^\mu \chi \tag{1.5}$$

and this is where the reduction from 4 equations to only one equation is hidden. That is, Maxwell equations have a unique solution, while the electromagnetic potential determined by equation (1.2) and some boundary conditions is not the only field which yields the same physically observable quantities $\vec{E}$ and $\vec{B}$. The choice of A^μ which we are going to do calculations with is called *gauge fixing*[1]. The previous relation (1.5) can be written in a different way:

$$A^\mu \to e^{-i\chi} A^\mu e^{i\chi} - \frac{i}{q} e^{-i\chi} \partial^\mu e^{i\chi} \ , \tag{1.6}$$

and this is a familiar expression from which it can be seen that the CE is a $U(1)$ gauge theory, which, in other words, means that CE is invariant under local (the ones which depend pointwise) $U(1)$ transformations. Sometimes undergraduate courses might not mention the expression (1.6) and that the theory is exactly a $U(1)$ gauge theory, but it is certainly true that the CE is an example of a gauge theory. Also, this whole theory can be reformulated using the Lagrangian of CE

$$\mathcal{L}_{\text{CE}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \tag{1.7}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the Faraday tensor. It can be shown that the Euler-Lagrange equations generated with this Lagrangian are Maxwell equations and that this Lagrangian is invariant under transformations (1.6) if the equation of continuity holds ($\partial_\mu J^\mu = 0$), i.e., for conserved currents. Later, we are going to see that transformation properties (1.6) are not unique to CE, but that they are the way in which the connection pullback to the base manifold transforms under gauge transformations.

1.1.2 Local phase invariance of quantum electrodynamics

The next example we are going to study is how we can enrich relativistic quantum mechanics with an electromagnetic interaction if we insist on invariance of the theory under local phase transformations of the wave function/spinor. Let us consider the

[1] In quantum field theory it is impossible to define the photon propagator without gauge fixing.

Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi \tag{1.8}$$

and apply a local gauge transformation

$$\psi \to e^{-i\chi(x)}\psi \ . \tag{1.9}$$

Then the transformed Lagrangian is

$$\mathcal{L}' = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + \bar{\psi}\chi\!\!\!/\psi \ . \tag{1.10}$$

We can see that the Lagrangian (1.10) does not differ from the starting one just by a surface term, therefore, we conclude that we would generate different Euler-Lagrange equations from the starting ones which, in turn, means that the relativistic quantum mechanics is not gauge invariant. It is very important to notice this fact, because we know that the physical probability density $|\psi|^2$ does not depend on a point-wise phase factor (nor the global phase, obviously). The Dirac Lagrangian, which describes the quantum theory of fermions, becomes gauge invariant if we modify the Lagrangian with a function A^μ, which transforms under gauge transformations like (1.6)

$$\mathcal{L}_{\text{gauge invariant}} = \bar{\psi}\left(i\partial\!\!\!/ - qA\!\!\!/ - m\right)\psi \ . \tag{1.11}$$

We have acquired the fermion sector of the Lagrangian of quantum electrodynamics whose total Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{QED} &= \bar{\psi}\left(i\partial\!\!\!/ - qA\!\!\!/ - m\right)\psi + \mathcal{L}_{CE} \\ &= \bar{\psi}\left(iD\!\!\!/ - m\right)\psi + \mathcal{L}_{CE} \end{aligned} \tag{1.12}$$

where we have used $D_\mu = \partial_\mu + iqA_\mu$, which is referred to in literature as *(gauge) covariant derivative*. Covariant derivatives are naturally described with the mathematics of bundles which we are going to develop throughout this chapter. Also, more on this introduction (and almost all definitions which we are going to use in this chapter) can be found in e.g. [1] or in Schuller's lectures on Geometrical anatomy of theoretical physics [2].

1.2 Introduction to differential geometry

For physicists, one of the most important concepts from differential geometry are *tensors on manifolds* because they allow us to define tensor quantities on spacetime (e.g., Faraday tensor, stress energy tensor, vector fields, etc.). We are going to construct tensors by taking tensor products of tangent vectors and 1-forms.

Definition 1.1. Tangent vector (definition as in [3]) at the point p on the manifold M, $X|_p$ (sometimes denoted as X_p or, if it is completely clear, just X for short), is a function $X|_p : C^\infty_p(M) \to \mathbb{R}$ which for all $f, g \in C^\infty_p$ and $\lambda, \mu \in \mathbb{R}$ satisfies:

1. Linearity
$$X|_p(\lambda f + \mu g) = \lambda X|_p(f) + \mu X|_p(g)$$

2. Leibniz' rule
$$X|_p(fg) = f(p)X|_p(g) + g(p)X|_p(f)$$

The set of all tangent vectors at a point p is denoted as T_pM and we call it *the tangent space at* p.

Theorem 1.1. *It can be shown that the dimension of the space T_pM is equal to the dimension of the manifold M. Also, it can be shown that for every chart $(O, \psi) = (O, (x^1, ..., x^{dimM}))$ around the point p, the base of the space T_pM is the set $\left\{\frac{\partial}{\partial x^\mu}\left(\equiv \frac{\partial}{\partial x^\mu}|_p\right) = \partial_\mu\right\}$. In the end, if $(O, (x^\mu))$ and $(V, (y^{\mu'}))$ are two charts around the point p and if the components of the vector $X|_p$ in the bases defined by coordinate systems (x^μ) i $(y^{\mu'})$ are:*

$$\begin{aligned} X|_p &= X|_p^\mu \frac{\partial}{\partial x^\mu} \\ X|_p &= X|_p^{\mu'} \frac{\partial}{\partial y^{\mu'}} \end{aligned} \tag{1.13}$$

Then the connection between components $X|_p^\mu$ and $X|_p^{\mu'}$ is given by

$$X|_p^{\mu'} = \frac{\partial y^{\mu'}}{\partial x^\alpha} X|_p^\alpha \, . \tag{1.14}$$

Definition 1.2. 1-form at the point $p \in M$, $\omega|_p$ (or ω_p for short) is an element of the vector space dual to the tangent space T_pM. Duals of vectors (which come from the chart $(O,(x^\mu)))$ ∂_μ are denoted by dx^μ. T_p^*M denotes the set of all 1-forms at the point p.

Theorem 1.2. *Let $(O,(x^\mu))$ and $(V,(y^{\mu\prime}))$ be two coordinate charts around the point p. If the components of the 1-form $\omega|_p$ in bases defined by coordinate systems (x^μ) and $(y^{\mu\prime})$ are*

$$\begin{aligned} \omega|_p &= \omega|_{p_\mu}\, dx^\mu \\ \omega|_p &= \omega|_{p_{\mu\prime}}\, dy^{\mu\prime} \end{aligned} \tag{1.15}$$

Then the following connection between components $\omega|_{p_\mu}$ and $\omega|_{p_{\mu\prime}}$ is true

$$\omega|_{p_{\mu\prime}} = \frac{\partial x^\alpha}{\partial y^{\mu\prime}} \omega|_{p_\alpha} \tag{1.16}$$

We are now able to define tensors at the point p of arbitrary rank using linear combinations for appropriate tensor products of basis vectors ∂_μ and basis covectors dx^μ. Also, a *vector field* (or more generally, a *tensor field*) is a function which maps points $p \in M$ to tangent vectors (tangent tensors) $X_p \in T_pM$. A problem arises when we think about smooth tensor fields of any rank. That is, our previous definitions do not contain enough information to classify the map $p \mapsto X|_p$ as smooth. We use bundle theory to work with smooth vector (tensor) fields and their generalizations. Also, it is worth mentioning that *differential 1-forms* are smooth (where smoothness is defined with respect to a natural topology which we will discuss later) functions that map $p \mapsto \omega_p \in T_p^*M$.

1.3 Introduction to bundle theory

In this section we are going to define the topological objects called *bundles*.

1.3.1 Fiber bundles

Definition 1.3. Bundle is a triple (E,π,M) consisting of smooth manifolds (see Appendix A.1) E and M and of a continuous surjection $\pi : E \to M$. We call E the *total space* and we call M the *base space*, while the surjection π is often called the *projection map*. Also, sometimes we denote bundle (E,π,M) as $E \xrightarrow{\pi} M$. For every $p \in M$, we call the set $E_p := \pi^{-1}(p)$ *the fiber at the point p*.

The first additional structure that we define over bundles is the *standard fiber*.

Definition 1.4. Fiber bundle with standard fiber F is a quadruple (E, π, M, F), such that (E, π, M) is a bundle with a *local trivialization*, i.e., for every point $p \in M$ there exists a neighborhood O_p and a homeomorphism (local trivialization) ψ_p such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(O_p) \arrow[r, "\psi_p"] \arrow[d, "\pi"'] & O_p \times F \arrow[dl, "\pi_1"] \\ O_p & \end{tikzcd}$$

where π_1 is the projection to the first variable, $\pi_1(p, f) = p \ \forall p \in M, \forall f \in F$. If π is a smooth map, we call the fiber bundle (E, π, M, F) a *smooth fiber bundle (with fiber F)*. If F is a vector space, then we say that the fiber bundle (E, π, M, F) is a **vector bundle**.

We are also going to define *sections* of bundles, functions that we will recognize as tensor fields in a special case of the total space.

Definition 1.5. Bundle section is any function $\sigma : M \to E$ which satisfies $\sigma \circ \pi = id_M$. The set of all sections of the bundle (E, π, M) is denoted as $\Gamma(E)$.

Notice that the condition $\sigma \circ \pi = id_m$ means that we are only regarding functions which map to every $p \in M$ a point in its fiber E_p.

1.3.1.1 Example of a fiber bundle

It is worth commenting on stated definitions. Fiber bundles are bundles such that the total space locally ”looks like” the product space $M \times F$. One often mentioned example of a fiber bundle is the Möbius strip, depicted on the Figure 1. As it can be seen, the standard fiber of a Möbius strip (with respect to the projection like on the Figure 1) is a segment. Bundles are useful for describing geometrical objects which are hard to define using methods like implicit defining, but are also applicable to physicists, e.g., in General Theory of Relativity (GR) where spacetime can be chosen for the base manifold and the total manifold can be chosen to be the *tangent bundle*, with standard fiber isomorphic to tangent vector spaces T_pM [4].

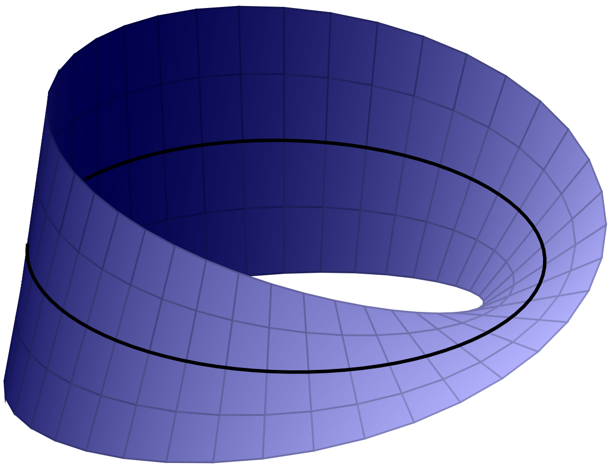

Figure 1.1: Möbius strip as the total space to the circle base space - thin, perpendicular gray lines are projected onto the black curve.

1.3.2 Tangent bundle

As we have noticed before, at first it is not clear how to define continuous or smooth vector fields. The problem arises in the fact that even though all the spaces T_pM are isomorphic as vector spaces, still for $p \neq q$, the spaces T_pM and T_qM are different vector spaces so we can not actually compare tangent vectors from different tangent spaces. We got away with it in special relativity because we worked with a flat spacetime which has the structure of a vector space, but in GR the same idea will not work because manifolds, in general, are not vector spaces. The solution is to define a topology over the (disjoint) union of tangent spaces.

Theorem 1.3. *The set*

$$TM := \bigsqcup_{p \in M} T_pM \tag{1.17}$$

with a natural topology (details on that topology are in references [3, 5]) is a smooth manifold and we call it the ***tangent bundle****. If M and N are diffeomorphic (definition in Appendix A.1) manifolds, then so are TM and TN.*

With the topology [3, 5] we have a smooth vector bundle $TM \xrightarrow{\pi} M$ with the standard fiber $\mathbb{R}^{\dim M}$ and now we have made all the necessary definitions in order to define smooth vector fields as smooth sections of the tangent bundle. Of course, for general tensor fields of rank (l, k) we can use an analogous bundle, except that in the disjoint union (1.17) we do not use tangent spaces but instead we use tangent tensor spaces of the rank (l, k). Also, for every tensor bundle T^l_kM, there exist generalizations of pointwise relations (1.14) and (1.16), which can be derived using linear algebra on each tangent space.

1.3.3 Bundle morphisms and bundle restrictions

Let us continue with the definitions of subbundles and bundle restrictions.

Definition 1.6. **Subbundle** of the bundle $E \xrightarrow{\pi} M$ is a bundle $E' \xrightarrow{\pi'} M'$ such that $E' \subseteq E$, $M' \subseteq M$ and $\pi' = \pi|_{E'}$.

Definition 1.7. **Bundle restriction** $E \xrightarrow{\pi} M$ to the manifold $N \subseteq M$ is the bundle $E \xrightarrow{\pi'} N$ with projection

$$\pi' = \pi|_{\pi^{-1}(N)} \ .$$

Now let us define morphisms of the types of bundles we have defined so far.

Definition 1.8. **Bundle morphism** of bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ is a pair of functions $u : E \to E'$ and $v : M \to M'$ for which the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{v} & M' \end{array}$$

If u and v are diffeomorphisms, we specially say that the bundle morphism is a **bundle isomorphim**. Additionally, if a bundle is isomorphic to the *product bundle* $(M \times F, \pi_1, M)$, we say that it is **trivial**.

Definition 1.9. We say that $E \xrightarrow{\pi} M$ is **locally isomorphic** to the bundle $E' \xrightarrow{\pi'} M'$ if for every point $p \in M$, there exists a neighborhood U_p such that the restriction of $E \xrightarrow{\pi} M$ to U_p is isomorphic to $E' \xrightarrow{\pi'} M'$. If a bundle is locally isomorphic to the product bundle, we say that it is **locally trivial**.

The definition of *bundle pullbacks* will be the last one in this section.

Definition 1.10. **Bundle pullback** $E \xrightarrow{\pi} M$ induced by a function $f : M' \to M$ is the bundle $E' \xrightarrow{\pi'} M'$ where

$$E' \left(\equiv M' \times_M E\right) = \{(m', e) \in M' \times E : f(m') := \pi(e)\}$$

and $\pi' = \pi_1$, i.e., $\pi'(m', e) = m'$.

1.4 *Principal and associated bundles*

In order to study gauge theories, it is necessary to have a Lie group action on a manifold. Roughly speaking, a principal bundle is a fiber bundle with the Lie group G for its standard fiber, while an associated bundle is a bundle upon whose fiber the group G acts. In this section, we are going to study principal and associated bundles and give physical examples.

1.4.1 Lie group actions on manifolds

Definition 1.11. Let $(G, \cdot)$ be a Lie Group and M a smooth manifold. A smooth function

$$\begin{aligned} \triangleright : G \times M &\to M \\ (g, p) &\mapsto g \triangleright p \end{aligned} \tag{1.18}$$

which satisfies

1. $\forall p \in M : e \triangleright p = p$
2. $\forall g_1, g_2 \in G, \forall p \in M : (g_1 \cdot g_2) \triangleright p = g_1 \triangleright (g_2 \triangleright p)$

is called a **left (Lie) group action** or a **left G-action** on M. We call the manifold M, on which we have defined $\triangleright$, a *left G-manifold*

We similarly define right group action.

Definition 1.12. Right (Lie) group action or **right G-action** of the group $(G, \cdot)$ on the manifold M is a smooth function

$$\begin{aligned} \triangleleft : M \times G &\to M \\ (p, g) &\mapsto p \triangleleft g \end{aligned} \tag{1.19}$$

which satisfies

1. $\forall p \in M : p \triangleleft e = p$
2. $\forall g_1, g_2 \in G, \forall p \in M : p \triangleleft (g_1 \cdot g_2) = (p \triangleleft g_1) \triangleleft g_2$

Definition 1.13. Let $\triangleright\colon G \times M \to M$ be a left G-action and M a smooth manifold. For every point $p \in M$ we define the **orbit** of the point p as the set

$$G_p \coloneqq \{q \in M : \exists g \in G \text{ such that } q = g \triangleright p\} \ . \tag{1.20}$$

Definition 1.14. Let $\triangleright\colon G \times M \to M$ be a left G-action on the manifold M. The **stabilizer** of the point p, S_p, is the set

$$S_p = \{g \in G : g \triangleright p = p\} \ . \tag{1.21}$$

Note that S_p is always a subgroup of G.

Definition 1.15. Let us define an equivalence relation $\sim$ on the manifold M

$$p \sim q :\Leftrightarrow \exists g \in G \text{ such that } q = g \triangleright p \ . \tag{1.22}$$

We define the **orbit space** M/G as the quotient space of the relation $\sim$

$$M/G \coloneqq G/_{\sim} = \{G_p : p \in M\} \ . \tag{1.23}$$

Definition 1.16. Left (analogously right) G-action $\triangleright$ on the manifold M is defined as

1. **Free** if $\forall p \in M : S_p = \{e\}$
2. **Transitive** if $\forall p, q \in M : \exists g \in G$ such that $p = g \triangleright q$.

A useful property of free left (analogously for right) G-actions is

$$g_1 \triangleright p = g_2 \triangleright p \iff g_1 = g_2 \ . \tag{1.24}$$

1.4.1.1 Example of a free action

An example of a non-free left action would be the action of the $SO(2)$ rotations on the plane $\mathbb{R}^2$. For every non-zero element $p \in \mathbb{R}^2 : S_p = \{e\}$, but, since the stabilizer of 0 is $SO(2)$, the group action can not be free. On the other hand, $SO(2)$ is free on $\mathbb{R}^2 \setminus (0,0)$. This claim is illustrated on Figure 1.2, along with orbits of the points p and 0.

Another example of a free action would be the free left G-action on G (regarded as a smooth manifold) simply defined as multiplication. Since G is a group, $g_1 \cdot x = g_2 \cdot x$ implies $g_1 = g_2$ because we can always multiply with x^{-1} from the right. This canonical example is also called *the (left) regular representation of* G.

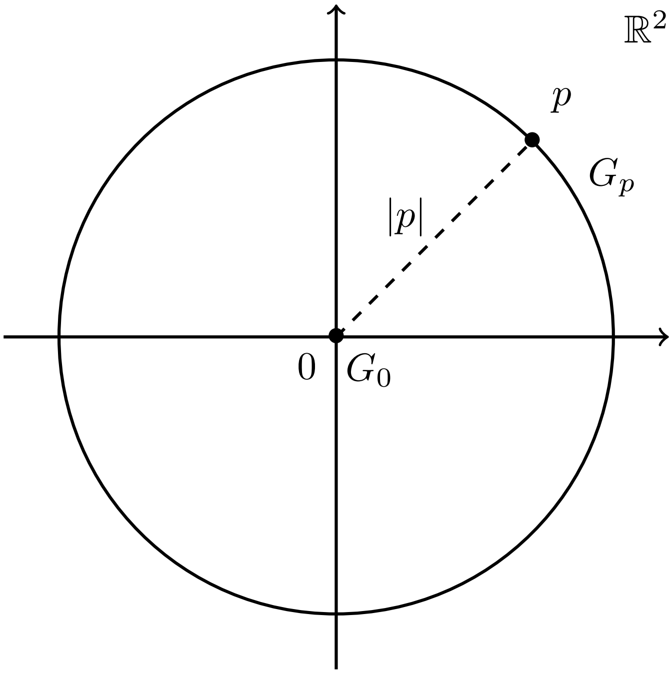

Figure 1.2: Orbits of two points from $\mathbb{R}^2$ under the action of $SO(2)$.

1.4.2 Principal bundles

Definition 1.17. Let G be a Lie group. We call a smooth bundle $E \xrightarrow{\pi} M$ a **principal G-bundle** if there is a free right G-action defined over E and if the smooth bundle $E \xrightarrow{\pi} M$ is isomorphic to the bundle $E \xrightarrow{\rho} E/G$, where ρ is a projection which maps every point to its equivalence class, i.e.,

$$p \overset{\rho}{\mapsto} [p] = G_p \ . \tag{1.25}$$

Remark, because the action is free (so the useful property (1.24) holds), we have that the fiber of any arbitrary point $G_p \in E/G$

$$\rho^{-1}(G_p \in E/G) = G_p \in E$$

is equal to that same orbit G_p (now regarded as the set of points in E and not as an equivalence class) and it is diffeomorphic (i.e., topologically isomorphic) to the group G by the useful property (1.24).

1.4.2.1 Geometry of GR - frame bundle LM as a principal bundle

In physics (especially in GR) the *frame bundle* is of exceptional importance. It is defined using a smooth manifold[2] M of dimension d and a special right action from which it becomes a principal $GL(d,\mathbb{R})$-bundle (for the definition of $GL(d,\mathbb{R})$ see Appendix A.1). Define the space

$$L_pM := \{(e_1, ..., e_d) : \{e_1, ..., e_d\} \text{ is a basis for } T_pM\} \ . \tag{1.26}$$

One can see that L_pM is the set of all ordered bases of the tangent space of $p \in M$. It is clear from the definition (considering the d-tuple of vectors as a $d \times d$ matrix) that $L_pM \cong M_d(\mathbb{R})$ (they are isomorphic as real vector spaces because they have the same finite dimension).

Definition 1.18. The **frame bundle** LM is defined as the disjoint union

$$LM := \bigsqcup_{p \in M} L_pM \tag{1.27}$$

And, using a similar idea (see, e.g., [4]) as when we defined the tangent bundle, we endow it with a differentiable structure. Projection is defined in a quite obvious way, we simply map $(e_1, ..., e_d) \in L_pM$ to p, the point whose tangent space the vectors $(e_1, ..., e_d) \in L_pM$ are the basis of.

In order to arrive at the structure of a principal $GL(d,\mathbb{R})$-bundle, we also need a right action of the general linear group. We define it in the following way:

$$(e_1, ..., e_d) \triangleleft g := (g^a_1 e_a, ..., g^a_d e_a) \tag{1.28}$$

where g^i_j are the components of the group element g with respect to the standard basis of $\mathbb{R}^d$. It is easy to show that such left action is well defined. It is also evident that this action is free because the only $GL(d,\mathbb{R})$ element that does not change any of the basis vectors is the identity map.

[2]When talking about frame bundles in physics, the base manifold M is almost always the spacetime manifold.

It remains to show that $LM \xrightarrow{\pi} M$ is isomorphic to the bundle $LM \xrightarrow{\rho} LM/GL(d,\mathbb{R})$. In other words, we are looking for diffeomorphisms $u : LM \to LM$ and $f : M \to LM/GL(d,\mathbb{R})$ for which the following diagram commutes:

$$\begin{tikzcd} LM \arrow[r, shift left, "u"] \arrow[d, "\pi"'] & LM \arrow[l, shift left, "u^{-1}"] \arrow[d, "\rho"] \\ M \arrow[r, shift left, "f"] & LM/GL(d,\mathbb{R}) \arrow[l, shift left, "f^{-1}"] \end{tikzcd}$$

The choice $u = id_{LM}$ is quite obvious, while for f we can choose the function which maps $p \in M$ to the entire set $GL(d,\mathbb{R})$ in any basis from L_pM. Function f defined so is surely an injection, because for different $p \neq q$ the bases from L_pM and L_qM will be from different vector spaces so it follows that $f(p) \neq f(q)$, but it is also clear that it is surjective because every orbit from $LM/GL(d,\mathbb{R})$ is surely an orbit of some basis from L_pM for some $p \in M$. It is now clear that the previous diagram commutes and that we have arrived at an example of a principal $GL(d,\mathbb{R})$-bundle. Later, in other "Geometry of GR" comments (1.4.4.2, 1.5.3.2, 1.5.4.1, 1.5.6.1), we are going to see that the frame bundle is useful for defining many geometrical tools that we use often in GR.

1.4.3 Principal bundle morphisms

Next we are going to define principal G-bundle morphisms and morphisms between principal G-bundles and principal H-bundles.

Definition 1.19. Let (P,π,M) and (Q,π',N) be principal G-bundles. **Principal G-bundle morphism** is a pair of smooth functions (u,f) such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "u"] & Q \\ P \arrow[u, "\triangleleft G"] \arrow[r, "u"] \arrow[d, "\pi"'] & Q \arrow[u, "\blacktriangleleft G"'] \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

i.e., $\forall p \in M$ and $\forall g \in G$ the following is true:

$$\begin{aligned} (f \circ \pi)(p) &= (\pi' \circ u)(p) \\ u(p \lhd g) &= u(p) \blacktriangleleft g \ . \end{aligned} \tag{1.29}$$

We say that the morphism of principal G-bundles is a **principal G-bundle isomorphism** if it is also a bundle isomorphism. In the end, a the principal G-bundle is **trivial** if it is isomorphic as a principal G-bundle to the trivial bundle $(M \times G, \pi_1, M)$.

After this definition we arrive at an important theorem.

Theorem 1.4. *Principal G-bundle (P, π, M) is trivial if and only if there exists a (global) smooth section $\sigma : M \to P$ (such that $\sigma \circ \pi = id_M$).*

The proof of Theorem 1.4 can be found in [1], section 9.2. It is also worth mentioning a more general version of the Definition 1.19, which generalizes to the case of morphisms between principal bundles of different Lie groups. Firstly, it will be necessary to define the notion of *equivariant* functions.

Definition 1.20. Let G and H be Lie groups, $\rho : G \to H$ a Lie group homomorphism and let

$$\begin{aligned} \lhd &: M \times G \to M \\ \blacktriangleleft &: N \times H \to N \end{aligned} \tag{1.30}$$

be left G- and H-actions on some smooth manifolds M and N. We say that a smooth function $f : M \to N$ is ρ **equivariant** if the following diagram commutes:

$$\begin{tikzcd} M \times G \arrow[r, "f \times \rho"] \arrow[d, "\lhd"'] & N \times H \arrow[d, "\blacktriangleleft"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

i.e., if $\forall m \in M, \forall g \in G : f(m \lhd g) = f(m) \blacktriangleleft \rho(g)$. The definition for right actions is completely analogous so we will not specify equivariant functions as left or right equivariant whenever it is clear from context.

Definition 1.21. Let (P, π, M) be a principal G-bundle, let (Q, π', N) be a principal H-bundle and let $\rho : G \to H$ be a homomorphisom of Lie groups G and H. **Morphism of**

principal bundles from (P, π, M) to (Q, π', N) is an ordered pair of smooth functions (u, f) such that the following diagram commutes (i.e., such that u is a ρ-equivariant function and $f \circ \pi = \pi' \circ u$):

$$\begin{tikzcd}
P \arrow[r, "u"] & Q \\
P \times G \arrow[u, "\triangleleft"] \arrow[r, "u \times \rho"] & Q \times H \arrow[u, "\blacktriangleleft"'] \\
P \arrow[u, "i_1"] \arrow[r, "u"] \arrow[d, "\pi"'] & Q \arrow[u, "i_1"'] \arrow[d, "\pi'"] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

where i_1 is any funcion for which $\pi_1 \circ i_1 = id$ (we use it to draw a commutative diagram from which one recognizes ρ-equivariance). The morphism between a principal G-bundle and a principal H-bundle is an **isomorphism** if ρ is a Lie group isomorphism and if (u, f) defines a bundle isormophism.

The Definition 1.21 is the most general morphism between principal bundles and now we are ready to study associated bundles.

1.4.4 Associated bundles

We are going to define associated bundles using principal bundles in such a way that they are capable of reproducing known transformation rules of objects we use in physics. But, associated bundles with familiar transformation rules are only special cases and we can define associated bundles with more general transformation rules.

Definition 1.22. Let (P, π, M) be a principal G-bundle and let F be a smooth manifold equipped with a left G-action $\triangleright$. We define the smooth manifold

$$P_F := (P \times F)/\!\sim_G$$

where $\sim_G$ is defined as follows

$$(p, f) \sim_G (p', f') :\Leftrightarrow \exists g \in G \text{ such that } p' = p \triangleleft g, f' = g^{-1} \triangleright f \ .$$

We are going to label points from P_F as $[p, f]$ because P_F is a quotient space. Additionally, we define the projection π_F as follows:

$$\begin{aligned} \pi_F : P_F &\to M \\ [p, f] &\mapsto \pi(p) \ . \end{aligned}$$

Notice that such π_F is well defined because any other element of the class $[p, f]$ is of the form $[p \triangleleft g, g^{-1} \triangleleft f]$ and we know that $p \triangleleft g$ for all $g \in G$ belong to the same fiber. With all that said, the **associated bundle** (to the bundle (P, π, M) with F and $\triangleright$) is the bundle (P_F, π_F, M).

1.4.4.1 Associated vector bundles

In the case when F is a vector space and the left G-action $\triangleright$ respects its vector space structure, i.e. $g \triangleright$ is a linear operator[3] on F for every $g \in G$, we will call (P_F, π_F, M) an **associated vector bundle** and denote it for short $P_F =: P \times_\triangleright F$.

1.4.4.2 Geometry of GR - tangent bundle as an associated bundle

Remember (from 1.4.2.1) that (LM, π, M) is a principal $GL(d, \mathbb{R})$-bundle. If for F we take $F = \mathbb{R}^d$ and define a left action

$$\begin{aligned} \triangleright : GL(d, \mathbb{R}) \times \mathbb{R}^d &\to \mathbb{R}^d \\ (g, x) &\mapsto g \triangleright x \quad ((g \triangleright x)^a := g^a{}_b x^b) \end{aligned} \tag{1.31}$$

we get an associated bundle $(LM_{\mathbb{R}^d}, \pi_{\mathbb{R}^d}, M)$ isomorphic (as a bundle) to the bundle (TM, π, M)

$$\begin{tikzcd} LM_{\mathbb{R}^d} \arrow[r, "u"] \arrow[d, "\pi_{\mathbb{R}^d}"'] & TM \arrow[d, "\pi"] \\ M \arrow[r, "id_M"] & M \end{tikzcd}$$

[3]From the property 2. in Definition 1.11, we can see that $\triangleright$ is a representation of G on F.

using a function u defined as follows:

$$\begin{aligned} u : LM_{\mathbb{R}^d} &\to TM \\ [(e_1, ..., e_d), x] &\mapsto x^a e_a \ . \end{aligned} \tag{1.32}$$

The function u is surely a bijection because every $X \in TM$ can, at every point $p \in M$, be expanded in any basis $(e_1, ..., e_d) \in L_pM$. Pick any such basis and map

$$X \xmapsto{u^{-1}} [(e_1, ..., e_d), X_{\mathbb{R}^d}] \ , \tag{1.33}$$

where $X_{\mathbb{R}^d}$ is a vector from $\mathbb{R}^d$ with components X^a of the vector X w.r.t. the basis $(e_1, ..., e_d)$, however, (1.33) is well defined (in the sense that it does not depend on the choice of basis from L_pM) because it maps to equivalence classes. It is worth mentioning that the isomorphism of bundles $(LM_{\mathbb{R}^d}, \pi_{\mathbb{R}^d}, M)$ and (TM, π, M) is largely due to the appropriate choice of the left action (1.31), so we can conclude that the formalism of associated bundles can be used to construct many general theories (by choosing different left Lie actions), but it still covers the one we already know. Additionally, one can also find an isomorphism between the tensor bundle (of any rank) and some associated bundle (associated to the frame bundle). The isomorphism for tensors of rank (l, k) is constructed with the fiber $F := \underbrace{\mathbb{R}^d \times ... \times \mathbb{R}^d}_{l \text{ times}} \times \underbrace{\mathbb{R}^{d*} \times ... \times \mathbb{R}^{d*}}_{k \text{ times}}$.

1.4.5 Associated bundle morphisms

We are concluding the introduction to the theory of associated bundles with the definition of the morphism.

Definition 1.23. Let (P_F, π_F, M) and (Q_F, π'_F, N) be associated bundles (with the same fiber F) of principal G-bundles (P, π, M) and (Q, π', N). A **morphism of associated bundles** is a morphism of bundles $(\tilde{u}, v)$ such that, for some u, (u, v) is a principal G-bundle morphism (of the underlying principal bundles) and the following relation between $\tilde{u}$ and u holds:

$$\tilde{u}\left([p, f]\right) = [u(p), f] \ , \tag{1.34}$$

or equivalently, if the following two diagrams commute (and the relation (1.34) is satisfied):

$$
\begin{tikzcd}
P_F \arrow[r, "\tilde{u}"] \arrow[d, "\pi_F"'] & Q_F \arrow[d, "\pi'_F"] \\
M \arrow[r, "v"] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "u"] & Q \\
P \arrow[u, "\triangleleft G"] \arrow[r, "u"] \arrow[d, "\pi"'] & Q \arrow[u, "\blacktriangleleft G"'] \arrow[d, "\pi'"] \\
M \arrow[r, "v"] & N
\end{tikzcd}
$$

We say that $(\tilde{u}, v)$ is a **principal bundle isomorphism** if $\tilde{u}$ and v are bijections and $(\tilde{u}^{-1}, v^{-1})$ also defines an associated bundle morphism. In the end, the associated bundle (P_F, π_F, M) is said to be **trivial** if its underlying principal G-bundle (P, π, M) is trivial as a G-bundle.

It is worth mentioning after the Definition 1.23 that an associated bundle can be trivial as a fiber bundle without having its underlying principal bundle being trivial (and as such not being a trivial associated bundle). On the other hand, a trivial associated bundle is necessarily trivial as a fiber bundle.

1.5 Geometry on bundles

In this section, we are going study geometry on bundles. We are going to define connections on bundles and connection 1-forms. Then, we are going to define the connection curvature, and in the end, the covariant derivative. The objects defined in this section are the most basic geometric structures commonly used in physics and mathematics and we are going to show some of their most important properties as well as examples when they are applied to GR.

1.5.1 Connections

The connection is an additional structure on principal bundles which, in a special way, associates to each point $p \in M$ one special vector space compatible with the right action of the principal bundle. It can be shown that the choice of these vector spaces is equivalent to the choice of a Lie algebra valued differential form.

Definition 1.24. The **vertical subspace** at the point $p \in P$ of the bundle (P, π, M) is

$$V_pP := \ker(\pi_*) = \{X|_p \in T_pP : \pi_*(X|_p) = 0\} \tag{1.35}$$

Where π_* is the push-forward (see Appendix A.1.5 for the definition) of tangent vectors using the projection map.

Additionally, one often encounters in literature the function $i_p : T_eG \to T_pP$, which maps an element $A \in T_eG$ of the Lie algebra to the tangent vector X_p^A in the following way:

$$\begin{aligned} &X_p^A : C_p^\infty(P) \to \mathbb{R} \\ &f \mapsto [f(p \lhd \exp(tA))]'(0) \; . \end{aligned} \tag{1.36}$$

It can be shown that i_p is a Lie algebra isomorphism. Also, it is easy to see using the definition that $\forall A \in T_eG$ and $\forall p \in P : i_p(A) = X_p^A \in V_pP$.

Definition 1.25. The **horizontal subspace** at the point p of the bundle (P, π, M) is a subspace $H_pP \leq T_pP$ which is complementary to the vertical subspace, i.e.:

$$T_pP = H_pP \oplus V_pP \; . \tag{1.37}$$

It is clear that the choice of H_pP is not unique at any point p, but, once we have made the choice, we get a unique decomposition to the vertical and horizontal part of vectors:

$$X_p(= X|_p) = \text{hor}(X_p) + \text{ver}(X_p) \; . \tag{1.38}$$

We are ready to define the *connection* on principal bundles.

Definition 1.26. The **connection** on a principal G-bundle (P, π, M) is the choice of horizontal subspaces H_pP for every point $p \in P$ such that:

1. for every $g \in G$ we have

$$(\lhd g)_* H_p = H_{p \lhd g}P \; , \tag{1.39}$$

2. for every smooth vector field $X \in \Gamma(TP)$, the summands in the unique decomposition

$$X_p = \text{hor}(X_p) + \text{ver}(X_p) \tag{1.40}$$

generate smooth vector fields $\text{hor}(X)$ and $\text{ver}(X)$.

The Definition 1.26 completely formalizes "smoothness" of the choice of horizontal spaces both inside the fiber (1.39) and between fibers (1.40). Also, even though it may not be clear at first, but in general both the horizontal and the vertical part in the decomposition (1.40) depend on the choice of H_pP (unless the vector is completely contained in the vertical subspace).

1.5.2 Connection 1-forms

Next we are going to define the *connection 1-form*, the choice of which is equivalent to the choice of connection. Firstly, a Lie algebra valued 1-form ω is a map which maps every point $p \in P$ to a linear operator $\omega_p : T_pP \longrightarrow T_eG$.

Definition 1.27. Let ω_p be a function such that:

$$\begin{aligned} &\omega_p : T_pP \to T_eG \\ &X_p \mapsto \omega_p\left(X_p\right) := i_p^{-1}(\text{ver}\left(X_p\right)) \ . \end{aligned} \tag{1.41}$$

We call the function ω, which maps every point $p \in P$ to ω_p the **connection 1-form with respect to the connection**. We sometimes denote ω as an element of the set $\Omega^1(T_pP, T_eG)$.

We justify the previous definition as follows. Namely, if someone gives us a function ω and claims that it is the connection 1-form with respect to the connection, then we are able to reconstruct the horizontal subspace H_pP as

$$H_pP = \ker\left(\omega_p\right) \ .$$

Of course, not every Lie algebra valued 1-form will generate a connection in an acceptable way (in the sense of Definition 1.26), but the following theorem states the necessary and sufficient conditions a form has to obey in order to generate a connection.

Theorem 1.5. *ω is a connection 1-form if and only if it satisfies:*

1. *$\forall p \in P$ we have $\omega_p(X_p^A) = A$, i.e., $\omega_p \circ i_p = id_{T_eG}$. Diagramatically shown*

$$
\begin{tikzcd}
T_eG \arrow[r, "i_p"] \arrow[dr, "id_{T_eG}"'] & V_pP \arrow[d, "\omega_p|_{V_pP}"] \\
& T_eG
\end{tikzcd}
$$

2. *$\forall p \in P$ and $\forall X_p \in T_pP$ the following is true*

$$\left(\left(\lhd\, g\right)^* \omega\right)_p \left(X_p\right) = \left(Ad_{g^{-1}}\right)_* \left(\omega_p \left(X_p\right)\right) \tag{1.42}$$

Where $Ad_{g^{-1}}$ is the adjoint map, i.e., $Ad_g(h \in G) = g \cdot h \cdot g^{-1}$. Equivalently, for all $p \in P$ the following diagram commutes:

$$
\begin{tikzcd}
T_pP \arrow[r, "\omega_p"] \arrow[dr, "((\lhd g)^*\omega)_p"'] & T_eG \arrow[d, "\left(\mathrm{Ad}_{g^{-1}}\right)_*"] \\
& T_eG
\end{tikzcd}
$$

3. *ω is a smooth function.*

The proof of this theorem and the map from connections to connection 1-forms can be found in [7], Theorem 5.2.2.

1.5.3 Local representations of connection forms on the base manifold

Our next goal is to express the connection 1-form on the base manifold. That is useful to us because in physics, spacetime is always regarded as the base manifold.

Definition 1.28. Let $\sigma : U \subseteq M \to P$ be a **local section**[4] of the principal G-bundle (P, π, M), i.e., $\pi|_U \circ \sigma = id_U$. The given local section induces:

- A **Yang-Mills field** $\omega^U : \Gamma(TU) \to \Gamma\left(T_eG\right)$ given as

$$\omega^U := \sigma^* \omega \ .$$

[4]Local sections are, in the case of principal bundles, also called local gauges.

- A **Local trivialisation** h of the principal bundle P

$$\begin{aligned} h &: U \times G \to P \\ (m,g) &\mapsto \sigma(m) \lhd g \end{aligned}$$

which is a G-equivariant diffeomorphism, meaning that h^{-1} is a local trivialization in the sense of Definition 1.4.

- A **local representation** of ω on U:

$$h^*\omega : \Gamma\left(T\left(U \times G\right)\right) \to \Gamma\left(T_e G\right)\ .$$

When discussing local representations of ω on U, it is worth mentioning that the tangent space of the point (m,g), $T_{(m,g)}$, is isomorphic as a Lie algebra to the algebra $T_m U \oplus T_g G$

$$T_{(m,g)}\left(U \times G\right) \cong T_m U \oplus T_g G\ . \tag{1.43}$$

1.5.3.1 Maurer-Cartan form and gauge function Ω

The following connection between a Yang-Mills field and a local representation (generated by the same local section) holds:

Theorem 1.6. *The following is true for all $v \in T_m U$ and $\gamma \in T_g G$:*

$$\left(h^*\omega\right)_{(m,g)}\left(v,\gamma\right) = \left(Ad_{g^{-1}}\right)_*\left(\omega^U\left(v\right)\right) + \Xi_g\left(\gamma\right) \tag{1.44}$$

Where $\Xi_g : T_g G \to T_e G$ is the ***Maurer-Cartan form*** *and it is defined as the inverse of the push-forward by left translation*[5] $(l_g)_*$

$$\begin{aligned} (l_g)_* &: T_e G \to T_g G \\ A &\mapsto X_g^A\ . \end{aligned} \tag{1.45}$$

Sometimes, in physics, we need a global picture of Yang-Mills fields, but only have access to Yang-Mills fields induced by local sections. This means we need a way

[5] The left translation $l_g : G \to G$, $l_g(h) = g \cdot h$ is a Lie group automorphism and as such can be used to push forward Lie group's tangent vectors. The right translation is defined analogously, $r_g(h) = h \cdot g$.

to identify Yang-Mills fields at intersections of domains of local sections. Moreover, Theorem 1.4 tells us that on principal bundles it is not even possible to define smooth global sections (a smooth section whose domain is the entire base manifold) except for the case when the principal bundle is trivial (product). The gauge function, which we will define next, is used to identify local Yang-Mills fields on the intersections of their domains.

Definition 1.29. Let $U_1, U_2 \subseteq M$ be open sets on M and consider Yang-Mills fields associated to two local sections σ_1 and σ_2 like on the following diagram:

$$\begin{tikzcd}
 & P \arrow[dl, "\pi"] \arrow[dr, "\pi"'] & \\
U_1 \subseteq M \arrow[ur, hook, bend left, "\sigma_1"] & & U_2 \subseteq M \arrow[ul, hook, bend right, "\sigma_2"']
\end{tikzcd}$$

The **gauge function** Ω is a function $\Omega : U_1 \cap U_2 \to G$ such that

$$\sigma_2(m) = \sigma_1(m) \lhd \Omega(m) \ . \tag{1.46}$$

Because the Lie action on principal bundles is (by definition) free, the relation (1.46) uniquely defines the gauge function Ω.

Theorem 1.7. *Under the assumptions of Definition 1.29, the connection between ω^{U_2} and ω^{U_1} is given as:*

$$(\omega^{U_2})_m = \left(Ad_{\Omega^{-1}(m)}\right)_* \left(\omega^{U_1}\right) + \left(\Omega^* \Xi_g\right)_m \ . \tag{1.47}$$

Theorem 1.7 supplies us with the connection between two Yang-Mills fields on the intersections of their domains.

1.5.3.2 Geometry of GR - Christoffel symbols as Yang-Mills fields

We are now going to show that in the case when $P = LM$ and $G = GL(d, \mathbb{R})$, Christoffel symbols can be considered as Yang-Mills fields and their transformations between reference frames follow from the Theorem 1.7. It can be directly shown (see, e.g. [4]) that for $GL(d, \mathbb{R})$, the Maurer-Cartan form has the form:

$$\left(\Xi_g\right)^i{}_j = \left(g^{-1}\right)^i{}_k \left(dx^k{}_j\right)_g , \quad (dx^k{}_j)_g \in T^*_g G \otimes T_e G \ . \tag{1.48}$$

The second summand in Theorem 1.7. expands in the basis $\left(\frac{\partial}{\partial x^\mu}\right)_p$ as follows:

$$\left(\Omega^*\Xi\right)_{p\,j}^{\;i}\left(\left(\frac{\partial}{\partial x^\mu}\right)_p\right)=\Xi_{\Omega(p)j}^{\;i}\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}= \tag{1.49}$$

using the relation (1.48)

$$=\left(\Omega(p)^{-1}\right)^i{}_k\left(dx^k{}_j\right)_{\Omega(p)}\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}= \tag{1.50}$$

by definition of differentials acting on vectors

$$=\left(\Omega(p)^{-1}\right)^i{}_k\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\left(x^k{}_j\right)= \tag{1.51}$$

definition of vector pushforward

$$=\left(\Omega(p)^{-1}\right)^i{}_k\left(\frac{\partial}{\partial x^\mu}\right)_p\left(x^k{}_j\circ\Omega\right)(p)=\left(\Omega(p)^{-1}\right)^i{}_k\left(\frac{\partial}{\partial x^\mu}\right)_p\Omega(p)^k_j\;. \tag{1.52}$$

To calculate the first summand one needs to remark that $\forall A\in T_eGL(d,\mathbb{R})$

$$\left(\left(Ad_g\right)_*A\right)^i{}_j=g^i_kA^k_l\left(g^{-1}\right)^l{}_j\;, \tag{1.53}$$

and then, because ω^{U_1} is a Lie algebra valued 1-form, we can directly apply (1.53)

$$\left(\left(Ad_{\Omega^{-1}(m)}\right)_*\omega^{U_1}\right)^i{}_j=\left(\Omega^{-1}(m)\right)^i{}_k\left(\omega^{U_1}\right)^k{}_l\left(\Omega(m)\right)^l{}_j\;. \tag{1.54}$$

All together, the connection between Yang-Mills fields on the intersection of open sets U_1 and U_2 is

$$\left(\omega^{U_2}\right)^i{}_{j\mu}=\left(\Omega^{-1}\right)^i{}_k\left(\omega^{U_1}\right)^k{}_{l\mu}\Omega^l_j+\left(\Omega^{-1}\right)^i{}_k\partial_\mu\Omega^k_j\;. \tag{1.55}$$

It is important to note that in the expression (1.55) we still have not explicitly chosen the local sections of bundles, $\sigma_1:U_1\to P$ and $\sigma_2:U_2\to P$, so the result we arrived at is completely general and independent of the choice of sections. If we choose the natural sections induced by coordinate charts (U_1, x) and (U_2, y),

$$
\begin{aligned}
\sigma_1 &: m \mapsto x^\mu \left(\frac{\partial}{\partial x^\mu}\right)_m \\
\sigma_2 &: m \mapsto y^\mu \left(\frac{\partial}{\partial y^\mu}\right)_m
\end{aligned}
\tag{1.56}
$$

we conclude that obviously $\Omega^i_j = \frac{\partial y^i}{\partial x^j}$ (as the consequence of (1.14)). With all of this said, ω^{U_2} expressed using coordinates y and the form ω^{U_1} is:

$$
\left(\omega^{U_2}\right)^i{}_{j\nu} = \frac{\partial y^\mu}{\partial x^\nu}\left(\frac{\partial x^i}{\partial y^k}\left(\omega^{U_1}\right)^k{}_{l\mu}\frac{\partial y^l}{\partial x^j} + \frac{\partial x^i}{\partial y^k}\frac{\partial^2 y^k}{\partial x^\mu \partial x^j}\right) \tag{1.57}
$$

and this is exactly the transformation rule of Christoffel symbols. We arrived at this rule using the most natural choices of sections and maps, exactly the ones we use in GR.

1.5.4 Curvature on principal bundles

It is often stated in literature that curvature is a property of covariant derivative. We will show in this subsection that in order to define curvature on principal bundles one only needs a connection. Firstly, in order to define curvature we need the concept of *exterior covariant derivative*.

Definition 1.30. Let (P, π, M) be a principal G-bundle with the connection 1-form ω and let ϕ be a module V (see Appendix A.1.7 for the definition) valued k-form. We define the **exterior covariant derivative** of ϕ, $D\phi$, as the $(k+1)$-form

$$
\begin{aligned}
&D\phi : \Gamma\left(T_0^{k+1}\right) \longrightarrow V \\
&(X_1, ..., X_{k+1}) \mapsto d\phi\left(\text{hor}\left(X_1\right), ..., \text{hor}\left(X_{k+1}\right)\right) \ .
\end{aligned}
\tag{1.58}
$$

If the horizontal subspaces are generated from a connection 1-form ω, we also denote the exterior covariant derivative as d^ω.

We are now ready to define *curvature*.

Definition 1.31. Let ω be the connection 1-form on a principal G-bundle (P, π, M). **Curvature** of the connection 1-form is the Lie algebra (T_eG) valued 2-form $\mathcal{F}^\omega$ on P defined as

$$
\mathcal{F}^\omega := D\omega = d^\omega\omega \ . \tag{1.59}
$$

The following theorem is useful for calculations.

Theorem 1.8. *Let ω be the connection 1-form and $\mathcal{F}^\omega$ its curvature. Then the following equality holds:*

$$\mathcal{F}^\omega = d\omega + \omega \wedge \omega \ , \tag{1.60}$$

where the exterior product of Lie algebra valued 1-forms is naturally defined using the Lie algebra commutator

$$\omega \wedge \omega \left(X, Y\right) := [\![\omega\left(X\right), \omega\left(Y\right)]\!] \ . \tag{1.61}$$

Theorem 1.9. ***First Bianchi identity**. Let $\mathcal{F}^\omega$ be the connection curvature. Then the following equality holds:*

$$D\mathcal{F}^\omega \ \left(= D^2\omega\right) = 0 \ . \tag{1.62}$$

In contrast to the regular differential, the operator $D^2 \neq 0$ in general, but we can see (using Theorem 1.9) that it has a nontrivial kernel after all.
As we shall see next, *Yang-Mils field strength* is an object closely tied to the connection curvature.

Definition 1.32. Let (P, π, M) be a principal G-bundle and let F^ω be the curvature of its connection 1-form ω. Let $\sigma : U \subseteq M \to P$ be a local section. Then we call the 2-form

$$\sigma^*\mathcal{F}^\omega \in \Omega^2(U) \otimes T_eG$$

the **Yang-Mills field strength**. Sometimes we will denote $\sigma^*\mathcal{F}^\omega$ as F, F^ω, W, B, G or Riem, depending on the context.

1.5.4.1 Geometry of GR - Riemann tensor as Yang-Mills field strength

Let us again study the case when $P = LM$ and $G = GL(d, \mathbb{R})$. In the comment 1.5.3.2 we have shown that ω^U carries matrix indices and have shown its transformation properties in the relation (1.57). If we plug in such a Yang-Mills field into the relation (1.60) of the Theorem 1.8, we arrive at (using commutativity of differentials and pullbacks) the Yang-Mills field strength which we will now denote by R,

$$\begin{aligned} R^i_{j\mu\nu} &= \left(d\omega^U\right)^i{}_{j\mu\nu} + \left(\omega^U\right)^i_{k\mu} \wedge \left(\omega^U\right)^k_{j\nu} = \\ &= \partial_\nu \left(\omega^U\right)^i{}_{j\mu} - \partial_\mu \left(\omega^U\right)^i{}_{j\nu} + \\ &+ \left(\omega^U\right)^i{}_{k\mu} \left(\omega^U\right)^k{}_{j\nu} - \left(\omega^U\right)^i{}_{k\nu} \left(\omega^U\right)^k{}_{j\mu} \,. \end{aligned} \tag{1.63}$$

This Yang-Mills strength, R, using the explained correspondence from 1.5.3.2,

$$\left(\omega^U\right)^i{}_{k\mu} \equiv \Gamma^i_{k\mu} \ ,$$

exactly matches the definition of the Riemann tensor.

1.5.5 Forms with values in Ad(P)

We will now show how the difference of two connection 1-forms on P (where P is the total space of a principal bundle) can be understood as a form on the base manifold M with values in a special vector bundle, Ad$(P) := P \times_{\text{Ad}} \mathfrak{g}$, where Ad is the adjoint representation defined in the Appendix A.3.6. We are studying this correspondence because in physics, what we usually consider excitations of gauge fields, i.e., gauge bosons, are actually differences of two gauge fields, one representing the excited state and one representing the vacuum state. We will later see that gauge fields are connection 1-forms, which means (as a consequence of Theorem 1.5) that the vacuum state can not be simply a constant 1-form with values equal to 0 (because the 1-form $\omega = 0$ does not generate a connection).
We start with the definition of *horizontal* forms and forms *of the type* Ad.

Definition 1.33. Let $\omega \in \Omega^k(P, \mathfrak{g})$ be a k-form on P with values in the Lie algebra $\mathfrak{g} = T_e G$. We say that ω is:

1. **Horizontal** if for all $p \in P$

$$\omega_p(X_1, ..., X_k) = 0 \tag{1.64}$$

   whenever at least one of the vectors X_i is vertical (equals its vertical part).

2. **Of type** Ad if

$$\left(g \rhd\right)^* \omega = \text{Ad}_{g^{-1}} \circ \omega, \quad \forall g \in G \,. \tag{1.65}$$

We denote the subset of $\Omega^k(P,\mathfrak{g})$, consisting of horizontal k-forms of type Ad on P with values of $\mathfrak{g}$, as $\Omega^k_{\text{hor}}(P,\mathfrak{g})^{\text{Ad}}$.

The following theorem explains some of our previous constructions in terms of horizontal k-forms of type Ad.

Theorem 1.10. *Let $\pi : P \to M$ be a principal G-bundle . The following statements are true:*

1. *Let A and A' be connection 1-forms on P with values in $\mathfrak{g}$. Then*

$$A - A' \in \Omega^1_{hor}(P,\mathfrak{g})^{Ad} \ . \tag{1.66}$$

   Additionally, for any $\omega \in \Omega^1_{hor}(P,\mathfrak{g})^{Ad}$, $A+\omega$ is a connection 1-for on P with values in $\mathfrak{g}$.

2. *The curvature 2-form of A on P is an element of $\Omega^2_{hor}(P,\mathfrak{g})^{Ad}$.*

3. *The vector space $\Omega^k_{hor}(P,\mathfrak{g})^{Ad}$ is isomorphic to $\Omega^k(P,Ad(P))$ with the isomorphism given with*

$$\begin{aligned}&\Lambda : \Omega^k_{hor}(P,\mathfrak{g})^{Ad} \to \Omega^k(P,Ad(P))\\ &\omega \mapsto \Lambda(\omega)_x(X_1,...,X_k) = [p,\omega_p(X_1,...,X_k)] \in Ad(P)_x \quad \textit{where } \pi(p) = x \ .\end{aligned} \tag{1.67}$$

1.5.6 Covariant derivative

What remains is to define the covariant derivative. We are going to supply an abstract definition of the covariant derivative on associated bundles. There exists also a geometrically intuitive way to define covariant derivative which relies on parallel transport, but because of the sheer number of necessary definitions and the fact that the end result (although completely equivalent to the abstract approach) is very computationally inefficient, it is deemed outside of the scope of this Thesis. The geometrical idea of covariant derivatives is to consider associated vector bundles (their fiber F is a vector space) and linear (in the second argument) left Lie actions $\triangleright : G \times F \to F$. Because F is a vector space, we can subtract vectors, which means we can compare vectors from some point's fiber with parallelly transported vectors from some nearby

point's fiber. Depending on the curve over which we have transported, we will obtain the covariant derivative in the direction of the curve's tangent vector field.
Our goal is to construct an operator ∇ which maps local sections $\sigma : U \to P_F$ along with a vector field $X \in TU$ into a local section $\nabla_X \sigma : U \to P_F$, while at the same time obeying standard properties that we expect from the covariant derivative:

1. $\forall f, g \in C_p^\infty(M)$, $\forall X, Y \in TU$:
$$\nabla_{fX+gY}\sigma = f\nabla_X\sigma + g\nabla_Y\sigma \tag{1.68}$$

2. $\forall X \in TU$:
$$\nabla_X (\sigma + \tau) = \nabla_X\sigma + \nabla_X\tau \tag{1.69}$$

3. $\forall f \in C_p^\infty(M)$:
$$\nabla_X f\sigma = X(f)\,\sigma + f\nabla_X\sigma \tag{1.70}$$

Let us begin with the theorem which uniquely connects G-equivariant (for the definition see Appendix A.1.4) functions and local sections of the associated bundle.

Theorem 1.11. *Let (P, π, M) be a principal G-bundle and (P_F, π_F, M) its associated bundle. The set of local sections $U \subseteq M \to P_F$ is in a 1-to-1 correspondence with the set of G-equivariant functions $\phi : \pi^{-1}(U) \subseteq P \longrightarrow F$.*

The previous theorem allows us to define the covariant derivative on the codomain F and then to equivalently (using the theorem's correspondence) move it to the associated bundle.

Theorem 1.12. *Let $\phi : P \to F$ be a G-equivariant function, let $s : U \subseteq M \to P$ be a local section and $X \in TP$. Then the following identity holds:*

$$(s^*D\phi)(X) = (ds^*\phi)(X) + \omega^U(X) \triangleright (s^*\phi)\ . \tag{1.71}$$

With identifications

$$\begin{aligned} s^*\phi &\longleftrightarrow \sigma : U \to P_F \\ (s^*D\phi)(X) &\longleftrightarrow \nabla_X\sigma \end{aligned} \tag{1.72}$$

one arrives at the construction of the ***directional covariant derivative*** *with properties (*1.68*), (*1.69*) and (*1.70*) from the previous discussion. It is important to mention that σ and $\nabla_X\sigma$ are obtained from the bijective pairs of ϕ and $D\phi$ (the pairs, also lazily denoted ϕ and $D\phi$ in the identification (*1.72*), are in the sense of Theorem 1.11) because otherwise the codomains of σ and $\nabla_X\sigma$ would be F instead of P_F.*

Using the Theorem 1.12., we have constructed the (directional) covariant derivative operator and found that it takes the form (now with the included identifications from the theorem)

$$\nabla_X\sigma = d\sigma\left(X\right) + \omega^U\left(X\right) \triangleright \sigma \ . \tag{1.73}$$

The second summand in (1.73) is actually two bijective identifications in the sense of Theorem 1.11. First we choose a local section σ and take its G-equivariant pair. We act on the pair with $\omega^U\left(X\right) \triangleright$ and then the new G-equivariant function produces (again using the bijective pairing) the final local section (which we denote $\omega^U\left(X\right) \triangleright \sigma$).

It is worth mentioning that the properties of the operator depend on, generally, two independent choices: on the choice of the connection ω and on the choice of the left *linear* action $\triangleright$ on F.

1.5.6.1 Geometry of GR - covariant derivative

Theorem 1.12. finishes our sequence of comments about geometry of GR in which we apply results to the frame bundle (LM, π, M) and reproduce mathematical tools used in GR. Namely, if for the left linear action on $F(\simeq \mathbb{R}^d)$ we pick

$$\left(g \triangleright f\right)^i := g^i_j f^j \ , \tag{1.74}$$

where g^i_j are the components of $g \in T_eGL(d,\mathbb{R})$ in the standard basis of $\mathbb{R}$, and if ω^U is like in the previous two comments (1.5.3.2 and 1.5.4.1), we obtain the covariant derivative used in GR:

$$\forall X \in TM : \left(\nabla_X\sigma\right)^i = X^\nu\partial_\nu\sigma^i + X^\nu\left(\omega^U\right)^i{}_{j\nu}\sigma^j \ . \tag{1.75}$$

1.6 *Classical Electrodynamics as a $U(1)$ gauge theory*

Throughout the first part of the chapter we have shown that GR can be rightfully considered a $GL(d,\mathbb{R})$ theory (with $d = 4$ in our universe, of course). Next, we are going to construct a principal $U(1)$ bundle from which it follows that Classical Electrodynamics is a $U(1)$ gauge theory.

1.6.1 Principal bundle for electrodynamics

Let $(M \times U(1), \pi_1, M)$ be a trivial bundle where M is Minkowski spacetime with its metric g. On this bundle we define the right Lie action as follows:

$$(x^\mu, g) \triangleleft g' := (x^\mu, gg') \tag{1.76}$$

which makes this bundle a principal $U(1)$-bundle because the action $\triangleleft$ in (1.76) is obviously free, as discussed in 1.4.1.1. If we define a connection on $M \times U(1)$, we can pull it back to M with a global section (which exists according to Theorem 1.4. because $(M \times U(1), \pi_1, M)$ is a trivial bundle) and obtain a global Yang-Mills field. This is not surprising because in CE we often do have a global potential A_μ which we have already planned to identify with a Yang-Mills field (it will turn to correspond to a Yang-Mills field up to a factor).

1.6.2 Potential A_μ and connection

Now we are going to derive the transformation rule for the 4-potential under gauge transformations. Let $\mathcal{A}^1_\mu$ be some Yang-Mills field on M obtained by pulling back the connection with a global section

$$\sigma_1 : m \in M \mapsto \left(x^\mu, e^{i\alpha(m)}\right) \ . \tag{1.77}$$

Let

$$\sigma_2 : m \in M \mapsto \left(x^\mu, e^{i\beta(m)}\right) \tag{1.78}$$

be a second global section for the same functions x^μ which pulls back the connection, defining a Yang-Mills fields $\mathcal{A}^2_\mu$. Then the gauge function Ω is given as

$$\Omega(m) = e^{i\chi(m)} = e^{i(\beta(m)-\alpha(m))} \ . \tag{1.79}$$

Repeating the same procedure (but now without the $GL(d,\mathbb{R})$) matrix indices) like in the comment 1.5.3.2, we see that the second summand in Theorem 1.7. is

$$(\Omega^*\Xi)_m \left(\left(\frac{\partial}{\partial x^\mu} \right)_m \right) = \Omega(m)^{-1}\partial_\mu \Omega(m) \ , \tag{1.80}$$

while the first summand in Theorem 1.7. equals

$$\left(Ad_{\Omega^{-1}(m)}\right)_* \left(\mathcal{A}^1_\mu\right) = \Omega^{-1}(p)\mathcal{A}^1_\mu \Omega(p) \ . \tag{1.81}$$

All together, the connection between two gauge fixed Yang-Mills fields is:

$$\mathcal{A}^2_\mu = \Omega^{-1}\mathcal{A}^1_\mu \Omega + \Omega^{-1}\partial_\mu \Omega \ . \tag{1.82}$$

If we suppose that every Yang-Mills field $\mathcal{A}_\mu$ defines one 4-vector potential A_μ as $\mathcal{A}_\mu = iqA_\mu$, we get, by inserting the expression for Ω, the connection between electromagnetic potentials under gauge transformations:

$$A^2_\mu = e^{-i\chi}A^1_\mu e^{i\chi} - \frac{i}{q}e^{-i\chi}\partial_\mu e^{i\chi} \tag{1.83}$$

which is identical to the expression (1.6) from this chapter's introduction. We conclude that electromagnetic potential is simply (up to a factor) a Yang-Mills field. If we can not define a global section over the total manifold (e.g., in the case of Dirac monopoles) we are forced to use local sections so the best we can do is obtain local potentials and compare them on the intersections of their domains using Theorem 1.7.

1.6.3 Faraday tensor and connection curvature

Our next goal is to represent the Faraday tensor as the Yang-Mills field strength. If $\mathcal{A}_\mu$ is some $U(1)$ Yang-Mills field, then the Yang-Mills field strength $\mathcal{F}_{\mu\nu}$ is by definition equal to

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \ . \tag{1.84}$$

One has to take into accoung that the summand $\mathcal{A} \wedge \mathcal{A}$ vanishes because $U(1)$ is a commutative group. All together, if we define the Faraday tensor as $F_{\mu\nu} := \frac{1}{iq}\mathcal{F}_{\mu\nu}$, we obtain the famous expression

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{1.85}$$

1.6.4 Covariant derivative

Given the principal $U(1)$-bundle (P, π_1, M) (where $P = M \times U(1)$), we can define an associated vector bundle $(P_\mathbb{C}, \pi_\mathbb{C}, M)$. On its fiber, $\mathbb{C}$, we define left Lie action as the multiplication in complex numbers

$$g \rhd z \coloneqq g \cdot z \ . \tag{1.86}$$

By Theorem 1.12., the directional covariant derivative has the form

$$\nabla_X = X^\mu \partial_\mu + X^\mu \mathcal{A}_\mu = X^\mu \partial_\mu + iq \cdot X^\mu A_\mu = X^\mu D_\mu \ . \tag{1.87}$$

We can see that the directional (gauge) covariant derivative is in this case a contraction of vectors and the covariant derivative which we have defined in the introduction 1.1.2.

It is important to mention that this gauge covariant derivative can differentiate between functions which are gauge scalars and functions which are gauge vectors. For example, the classical neutron field n is a scalar for the gauge covariant derivative so the gauge covariant derivative reduces to partial derivatives, as it should.

$$\begin{aligned} \nabla_X n &= X^\mu \partial_\mu n + iq \cdot X^\mu A_\mu n = \\ &= [q = 0 \text{ for neutrons}] = \\ &= X^\mu \partial_\mu n \ . \end{aligned} \tag{1.88}$$

On the other hand, the classical electron field e is a gauge vector so the gauge covariant derivative does not simply reduce to partial derivatives.

1.7 Spin Structure

In the previous section we have defined the objects from CE using the tools from principal and associated bundle theory. We have done a similar thing, throughout the whole chapter, with GR. With that said, we have explained the geometry of two

classical theories and defined them as gauge theories. In order to geometrically define gauge boson fields, the mathematical tools up to this point are sufficient because gauge bosons are of tensorial structure. Matter (fermion) fields, on the other hand, are spinorial objects and require additional definitions. In this section we are going to define spinor fields as sections of the *spinor bundle*, which will be defined as an associated vector bundle to the principal $\text{Spin}^+(p,q)$-bundle, which itself is closely related to the *orthogonal frame bundle* OLM.
Finally, we are going to define *Dirac forms*; which will be used to define Dirac mass terms, the *spin covariant derivatives* (twisted and twisted chiral) and the *Dirac operator*; which will be used to define kinetic and interaction terms.

1.7.1 Spinor bundle as an associated vector bundle

Just like we have used the frame bundle to define tensorial objects as sections of associated vector bundles, we will use the spinor frame bundle to define spinorial objects as sections of associated *spinor bundles*. Spinor frame bundle is defined using the *ortohognal frame bundle*.

Definition 1.34. Let M be a smooth manifold with the metric g_{ab} (the definition is analogous for metrics of Lorentz and Riemann type). Consider the subset of the frame bundle LM, OLM, which is defined using sets OL_pM

$$OL_pM := \{(e_1, ..., e_{dimM}) : (e_1, ..., e_{dimM}) \text{ is an orthonormal basis with respect to } g\} \tag{1.89}$$

which are taken into a disjoint union, just like in (1.27)

$$OLM := \bigsqcup_{p\in M} OL_pM \; . \tag{1.90}$$

In the end, the space OLM should be endowed with a topology (see reference [4]) in order to finish the construction of the **orthogonal frame bundle**.

1.7.1.1 OLM as an $SO^+(p,q)$-principal bundle

For a manifold with the metric of signature (p,q), (OLM,π_{OLM}, M) is an $SO^+(p,q)$-principal bundle[6]. The definition is completely analogous to 1.4.2.1. We define the

[6] $SO^+(p,q)$ is the orientation preserving subgroup of $SO(p,q)$.

projection

$$\begin{aligned}\pi &: OLM \to M \\ (e_1, ..., e_{\dim M}) &\in OL_pM \mapsto p\end{aligned} \tag{1.91}$$

and the right action $\blacktriangleleft$ identically as in 1.4.2.1, but change the group to $SO^+(p,q)$. Sometimes (OLM, π_{OLM}, M) is denoted as $(SO^+(M), \pi_{SO}, M)$. We call a local section $e = (e_1, ..., e_{\dim M})$ of $SO^+(M)$ a **local vielbein** or a **local tetrad**.
Next we are defining the *spinor frame bundle* as a $\text{Spin}^+(p,q)$-principal bundle with a special kind of double covering.

Definition 1.35. Spinor frame bundle (sometimes called **the spin structure**) is a $\text{Spin}^+(p,q)$-principal bundle ($\text{Spin}^+(M)$,π_{Spin}, M) with a λ-equivariant function

$$\Lambda : \text{Spin}^+(M) \to OLM \tag{1.92}$$

such that the following diagram commutes:

$$\begin{tikzcd}
\text{Spin}^+(M) \times \text{Spin}^+(p,q) \arrow[r, "\pi_1"] \arrow[d, "\Lambda\times\lambda"'] & \text{Spin}^+(M) \arrow[d, "\Lambda"'] \arrow[dr, "\pi_{\text{Spin}}"] & \\
OLM \times SO^+(M) \arrow[r, "\pi_1"'] & OLM \arrow[r, "\pi_{OLM}"'] & M
\end{tikzcd}$$

where λ is a restriction of π, defined in Appendix A.2.8

$$\lambda = \pi|_{\text{Spin}^+(p,q)} : \text{Spin}^+(p,q) \to SO^+(p,q) \tag{1.93}$$

and it is a double covering of $SO^+(p,q)$.

It can be proven (see, e.g. [7], Theorem 6.9.7) that the spin structure, if it exists[7], is unique up to a principal bundle isomorphism. Also, we call any manifold that can have a spin structure defined on it a **spin manifold**. Associated bundles to spinor frame bundles are called spinor bundles and are our next point of interest. Very

[7]The spin structure over M exists if and only if M is orientable (i.e. the first Stiefel-Whitney class of M, $w_1(M)$, vanishes) and the second Stiefel-Whitney class of M, $w_2(M)$, vanishes, see [7].

importantly, it can also be proven that all (Lorentzian) tensor bundles on M can be recovered as associated bundles to the spinor frame bundle (see, e.g., [7]).

Definition 1.36. Let $\Delta \equiv \Delta_n = \mathbb{C}^N$ be the **vector space of Dirac spinors** for some n and let[8]

$$\kappa : \mathrm{Spin}^+(p,q) \to GL(\Delta) \tag{1.94}$$

be a spinor representation. The **(Dirac) spinor bundle** is the associated bundle $S = \mathrm{Spin}^+(M) \times_\kappa \Delta$. Sections of S are called **spinor fields** or **spinors**. Also, if κ is a nontrivial representation, S is called a **charged spinor bundle**.

1.7.2 Dirac form

In order to define Dirac forms, we need to define a product between $\mathbb{R}^{p,q}$ vectors and Dirac vectors from Δ_n. Dirac forms will be used in our intrinsic Lagrangian formulation to replicate the spinor contractions from the fermionic Lagrangian sectors defined in the Chapter 2.

Definition 1.37. The (mathematical[9]) **Clifford multiplication** is the bilinear map

$$\mathbb{R}^{p,q} \times \Delta_n \longrightarrow \Delta_n \tag{1.95}$$

given as

$$(X, \psi) \mapsto X \cdot \psi := \rho(\gamma(X))\psi \tag{1.96}$$

where ρ is the representation whose representation space is Δ_n and γ is defined in the Appendix A.2.4.

Definition 1.38. Let $\Delta = \Delta_n$ be the complex spinor representation of $\mathrm{Cl}(p,q) = \mathrm{Cl}(\mathbb{R}^{p,q})$ and fix a constant $\delta = \pm 1$. We call a non-degenerate bilinear form a **Dirac form**

$$\langle \cdot, \cdot \rangle : \Delta \times \Delta \longrightarrow \mathbb{C} \tag{1.97}$$

if it satisfies:

[8]The connection between N and n is the result of the structure theorem for complex Clifford algebras, see [7] Theorem 6.3.21. In this Thesis, n in Δ_n will always equal the spacetime dimension $\dim M$.

[9]The physical multiplication is given as the mathematical times $-i$.

1. $\forall X \in \mathbb{R}^{p,q}, \forall \psi, \phi \in \Delta$:
$$\langle X \cdot \psi, \phi \rangle = \delta \langle \psi, X \cdot \phi \rangle$$

2. $\forall \psi, \phi \in \Delta$:
$$\langle \psi, \phi \rangle = \langle \phi, \psi \rangle^*$$

3. $\forall \psi, \phi \in \Delta, \forall c \in \mathbb{C}$:
$$\langle \psi, c\phi \rangle = c \langle \psi, \phi \rangle \ .$$

We will sometimes denote $\langle \psi, \phi \rangle$ as $\bar{\psi}\phi$.
For every Dirac form (1.97) there exists a matrix A such that

1. $\forall \psi, \phi \in \Delta$
$$\langle \phi, \psi \rangle = \psi^\dagger A \phi$$

2. $\gamma_a^\dagger = \delta A \gamma_a A^{-1}$

3. $A = A^\dagger$

where γ_a is the a-th mathematical gamma matrix, defined in Appendix A.2.4. Also, the Dirac form is Spin$^+(p,q)$-invariant so it, by Appendix A.3.2, generates an associated bundle metric on S, $\langle \cdot, \cdot \rangle_S$.

Now follows a theorem about spinor bundles that guarantees the well definedness of the Clifford multiplication in spinor bundles and also defines Weyl spinor bundles and their Clifford multiplication properties.

Theorem 1.13. *Let $S = Spin^+(M) \times_\kappa \Delta_n$ be the spinor bundle associated to the spin structure on M. The following statements are true:*

1. *There exists a well defined Clifford multiplication on the level of bundles*
$$\begin{aligned} \cdot : TM \times S &\to S \\ (X, \psi) &\mapsto X \cdot \psi \end{aligned} \tag{1.98}$$
which is obtained by Clifford multiplying at every point, i.e., $(X \cdot \psi)_p = X_p \cdot \psi_p$.

2. *If the dimension n of M is even, then S splits into a direct sum of* ***Weyl spinor bundles*** $S = S_+ \oplus S_-$ *defined as*

$$S_\pm = \mathit{Spin}^+(M) \times_\kappa \Delta_n^\pm \tag{1.99}$$

Where $\Delta_n^\pm$ are defined in the Appendix A.2.6. In this case, Clifford multiplication maps $S_\pm$ to $S_\mp$.

1.7.3 Spin covariant derivative

If given a Levi-Civita covariant derivative (often incorrectly called Levi-Civita connection) and a spin structure, one can uniquely determine the spin covariant derivative which acts on sections of the spinor bundle $S = \mathrm{Spin}^+(p,q) \times_\kappa M$. In this subsection, we are going to define spin covariant derivatives which generalize into twisted spinor covariant derivatives and twisted chiral spinor covariant derivatives, former of which will be used to define the gauge covariant derivative for QCD and latter of which will be used to define the gauge covariant derivative for the electroweak sector.

1.7.3.1 Spin covariant derivative and Clifford connection

Given a local vielbein (a local section of the OLM bundle) $e = (e_1, ..., e_{\dim M})$, one can act on it using the Levi-Civita covariant derivative, since vectors e_i are elements of TM, and expand the derivative in the basis e:

$$\nabla_X e_a = \omega_{ab}(X)\eta^{bc} e_c \ . \tag{1.100}$$

The same procedure can be applied to any basis e of TM, but in the case when the basis e is orthonormal with respect to the manifold metric, we call ω the **Clifford connection**[10] with respect to the (orthonormal) basis e.
We can convince ourselves that the Clifford connection is uniquely determined from the Levi-Civita covariant derivative. Firstly, we can express ω in terms of **anholonomy coefficients** $\Omega_{ab}{}^c$ as

$$\omega_{cab} = \omega_{ab}(e_c) = \frac{1}{2}\left(\Omega_{cab} - \Omega_{abc} + \Omega_{bca}\right) \tag{1.101}$$

[10]We call it a connection similarly to how people call ∇ the Levi-Civita connection.

where $\Omega_{abc} = \Omega_{ab}{}^{d}\eta_{dc}$ and

$$[e_a, e_b] = \Omega_{ab}{}^{c} e_c \ . \tag{1.102}$$

We can see that (1.102) completely determines the Clifford connection, but (1.102) contains Levi-Civita covariant derivatives because it is known that for every two vector fields X, Y, their commutator is given as

$$[X, Y] = \nabla_X Y - \nabla_Y X \ . \tag{1.103}$$

Now we will state the theorem which expresses the spin covariant derivative in local coordinates on M.

Theorem 1.14. *Let e be a local vielbein, $p \in M$ be an arbitrary point and x^μ be a local coordinate chart on an open set $U \ni p$.*
For any spinor $\Psi = [\epsilon, \psi] \in \Gamma(S)$, where ϵ is a local gauge of the spinor frame bundle such that[11] $\Lambda \circ \epsilon = e$, and $\psi : U \to \Delta$, and any local vector field $X \in \Gamma(TU)$, we can express the ***spin covariant derivative*** *$\nabla_X \Psi$ as*

$$\nabla_X \Psi = [\epsilon, \nabla_X \psi] \ ,$$

where $\nabla_X \psi$ equals

$$\nabla_X \psi = d\psi(X) - \frac{1}{4}\omega_{ab}(X)\Gamma^{ab}\psi = X^\mu \partial_\mu \psi - X^c \frac{1}{4}\omega_{cab}\Gamma^{ab}\psi. \tag{1.104}$$

In (1.104), X^c are the components of X in the local vielbein e and X^μ the components of X in the induced coordinate vector fields ∂_μ.
Also, the spin covariant derivative is compatible with the Levi-Civita covariant derivative in the sense that the following relation holds for every $X, Y \in \Gamma(TU)$ and $\Psi \in \Gamma(S)$:

$$\nabla_X (Y \cdot \Psi) = (\nabla_X Y) \cdot \Psi + Y \cdot (\nabla_X \Psi) \ , \tag{1.105}$$

where $\cdot$ is the mathematical Clifford multiplication defined in Definition 1.37. The proofs of all the claims in this theorem can be found in [7], section 6.10.2.

[11]For every local vielbein e, there always exist exactly two local gauges $\epsilon_\pm$ such that $\Lambda \circ \epsilon_\pm = e$, where Λ is the λ equivariant function from the Definition 1.35. The proof of this claim can be found in [7], Lemma 6.9.11.

1.7.3.2 Twisted spinor bundle and twisted spin covariant derivative

In this paragraph we are introducing twisted spinor bundles and twisted spin covariant derivatives. They are a way to connect spin structure with principal G-bundles, introducing interaction with gauge fields to spinors.
Let (P, π, M) be a principal G-bundle and $\rho : G \to GL(V)$ a complex representation defining an associated vector bundle $E = P \times_\rho V$. Let $S = \text{Spin}^+(M) \times_\kappa \Delta$ be the spinor bundle associated to the spin structure on M.

Definition 1.39. We call the associated vector bundle $S \otimes E$ the **twisted spinor bundle** or **gauge multiplet bundle**. For the definition of the fibre product, see Appendix A.1.8.

The sections of the twisted spinor bundle are defined as follows. Let $s : U \to P$ be a local gauge, we can then, using a map $v : U \to V$, define a local section τ of E as

$$\tau(p) = [s(p), v(p)]. \tag{1.106}$$

If we choose a basis $v_1, ..., v_r$ for V, we can define a basis of local sections τ_i for E (since fibers of E are vector spaces isomorphic to V), this is analogous to how we can define a basis for TM.
Similarly, if we take a local gauge $\epsilon : U \to \text{Spin}^+(M)$, then all local sections of S are given as

$$[\epsilon(p), f(p)] \tag{1.107}$$

for functions[12] $f : M \to \Delta_n$.
Finally, from the definition of the tensor product bundle, any section $\Psi \in \Gamma(S \otimes E)$ can be written as the tensor product of a section of S and a section of E. If we expand the section of E in the basis τ_i, we obtain an expression for the section

$$\Psi = \sum_{i=1}^{r} \Psi_i \otimes \tau_i \tag{1.108}$$

with $\Psi_i \in \Gamma(S)$. Equivalently, the section Ψ can be expressed locally as

$$\Psi = [\epsilon \times s, \psi] \tag{1.109}$$

[12]Local sections of associated bundles are in a 1 to 1 correspondence with functions from the base manifold to the associated bundle's fibre. To see this construction, see Appendix A.1.9.

where ψ is a **multiplet** of the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_r \end{pmatrix} : U \longrightarrow \Delta_n \otimes \mathbb{C}^r \tag{1.110}$$

and ψ_i are all maps to Δ_n. It is worth noting that in (1.110) we have identified V with $\mathbb{C}^r$ for the sake of conveniently writing ψ as a column vector.

Definition 1.40. Let A be a connection 1-form on the principal G-bundle P and let s and ϵ be local gauges for P and Spin$^+(M)$ from $U \subseteq M$.
We define the directional **twisted spin covariant derivative** ∇^A_X locally on $\Psi = [s \times \epsilon, \psi]$ as

$$\nabla^A_X \Psi \coloneqq [s \times \epsilon, \nabla^A_X \psi] \tag{1.111}$$

where $\nabla^A_X \psi$ is defined as follows

$$\nabla^A_X \psi = d\psi(X) - \frac{1}{4}\omega_{bc}(X)\Gamma^{bc}\psi + \left(\rho_* A^U(X)\right)\psi \, . \tag{1.112}$$

In the previous equation, $\rho_* A^U$ is the push-forward of the Yang-Mills field A^U.

1.7.3.3 Twisted chiral spin bundle and twisted chiral spinor covariant derivative

We are now defining twisted chiral bundles and twisted chiral covariant derivatives which couple chiral spinors to gauge fields.
Suppose the spacetime M dimension is even. Then, in accordance with Theorem 1.13, the spinor bundle $S = \text{Spin}^+(M) \times_\kappa \Delta_n$ splits into a direct sum of Weyl spinor bundles $S_\pm$. Let $P \to M$ be a principal G-bundle and

$$\rho_\pm : G \longrightarrow GL(V_\pm) \tag{1.113}$$

be two (possibly distinct) complex representations of G on vector spaces $V_\pm$ and let $E_\pm = P \times_{\rho_\pm} V_\pm$ be the associated vector bundles.

Definition 1.41. We call

$$(S \otimes E)_+ := (S_+ \otimes E_+) \oplus (S_- \otimes E_-) \tag{1.114}$$

the **twisted chiral spinor bundle**. Sometimes, the bundle $(S \otimes E)_-$ is also studied and it is defined as follows:

$$(S \otimes E)_- := (S_- \otimes E_+) \oplus (S_+ \otimes E_-)\ . \tag{1.115}$$

Sections of the twisted chiral spinor bundle are given as

$$\Psi = \Psi_+ + \Psi_- \tag{1.116}$$

where $\Psi_\pm$ are sections of $S_\pm \otimes E_\pm$.

Definition 1.42. Let A be a connection 1-form on the principal G-bundle P and let s and ϵ be local gauges for P and Spin$^+(M)$ from $U \subseteq M$.
The directional **twisted chiral spin covariant derivative** ∇^A_X on the twisted chiral spinor bundle is defined (analogously to the Definition 1.40) as

$$\nabla^A_X \Psi = [s \times \epsilon, \nabla^A_X \psi] \tag{1.117}$$

where $\nabla^A_X \psi$ is defined as follows

$$\nabla^A_X \psi = d\psi(X) - \frac{1}{4}\omega_{bc}\Gamma^{bc}\psi + \left(\rho_{+*}A^U\right)\psi + \left(\rho_{-*}A^U\right)\psi\ . \tag{1.118}$$

1.7.4 Dirac operator

We are now defining Dirac operators which correspond to the $\not{D} = \gamma^\mu D_\mu$ operator from QFT. We will define three types of Dirac operators, one acting on sections of spinor bundles, one acting on sections of twisted spinor bundles and one acting on sections of twisted chiral spinor bundles.

1.7.4.1 Spinor bundle case

Definition 1.43. The **Dirac operator** $D : \Gamma(S) \to \Gamma(S)$ is defined (using a local gauge $\epsilon : M \to \text{Spin}^+(M)$ and a local vielbein e) as

$$D\Psi = [\epsilon, D\psi] \tag{1.119}$$

where $D\psi$ equals

$$D\psi = \gamma^a \nabla_{e_a} \psi = i\Gamma^a \left(d\psi(e_a) - \frac{1}{4}\omega_{abc}\Gamma^{bc}\psi \right) . \tag{1.120}$$

It is worth noting that the Dirac operator (all three versions) does not depend on the choice of ϵ and e. The proof of this claim can be found in [7], section 6.10.3.

1.7.4.2 Twisted spinor bundle case

Definition 1.44. The **Dirac operator** on a twisted spinor bundle with a connection 1-form A (on the G-bundle P), with local gauges $s : U \to P, \epsilon : U \to \text{Spin}^+(M)$ and a local vielbein e, is expressed locally as

$$\begin{aligned} &D_A : \Gamma(S \otimes E) \longrightarrow \Gamma(S \otimes E) \\ &D_A \Psi = [s \times \epsilon, D_A \psi] \end{aligned} \tag{1.121}$$

where $D_A\psi$ equals

$$D_A\psi = i\Gamma^a \left(d\psi(e_a) - \frac{1}{4}\omega_{abc}\Gamma^{bc}\psi + \left(\rho_* A^U\right)\psi \right) . \tag{1.122}$$

1.7.4.3 Twisted chiral spinor bundle case

Definition 1.45. The **Dirac operator** on twisted chiral spinor bundles is given locally (in terms of a local vielbein e, inducing a local gauge ϵ, and a local gauge s) as

$$\begin{aligned} &D_A\Psi : \Gamma\left((S \otimes E)_+\right) \longrightarrow \left((S \otimes E)_-\right) \\ &D_A \Psi = [s \times \epsilon, \psi] \, , \end{aligned} \tag{1.123}$$

where $D_A\psi$ equals

$$D_A\psi = i\Gamma^a \left(d\psi(e_a) - \frac{1}{4}\omega_{abc}\Gamma^{ab}\psi + \left(\rho_{+*} A^U \psi_+\right) + \left(\rho_{-*} A^U \psi_-\right) \right) . \tag{1.124}$$

One can also naturally decompose the Dirac operator into $D_{A\pm} : \Gamma\left(S_{\pm} \otimes E_{\pm}\right) \longrightarrow \Gamma\left(S_{\mp} \otimes E_{\pm}\right)$ defined by

$$D_{A_{\pm}} \psi_{\pm} = i\Gamma^{a} \left(d\psi_{\pm}(e_a) - \frac{1}{4}\omega_{abc}\psi_{\pm} + \left(\rho_{\pm *} A^{U}\right) \psi_{\pm} \right) . \qquad (1.125)$$

This concludes our mathematical introduction to gauge theories. Using the tools developed in this chapter and in the Appendices, we proceed to outline Standard Model's Lagrangian geometrically. In the next chapter we are going to somewhat heuristically define SM's Lagrangian and discuss all of its symmetries and historical development.

2 The Standard Model

2.1 Introduction

The Standard Model (*SM*) is the best known and experimentally tested theory describing the fundamental interactions of elementary particles. It is also the most rigorously tested physical theory, correctly predicting many physical quantities, the most successful of which is the fine structure constant up to 12 significant digits [8]. However, it is agreed upon that the SM can not be a Theory Of Everything (*TOE*) because, firstly, it does not describe gravitational interaction, and, secondly, there are some open questions in physics, which aren't known to be related to quantizing gravity, that the SM apparently does not describe, e.g., neutrino masses or dark matter. More precisely, the SM is a quantum field theory describing the electroweak, strong and Higgs interactions of the three generations of quarks and leptons mediated by the bosonic force carriers. The particle content of the SM can be seen on the Figure 2.1. In this chapter we are going to arrive at the Lagrangian of the SM and describe its symmetries. But before doing so, we are going to define quantum field theory.

2.2 Quantum field theory

A quantum field theory is the application of quantum mechanical tools to systems of *quantum fields*. Most commonly one arrives at QFTs by quantizing classical field theories. The most common quantization methods are the *canonical quantization* and *path integral quantization*. The canonical quantization is the most optimal for the first encounters with quantum field theory while the path integral quantization is much harder to rigorously define[1] but is more suitable for our geometric approach that we will employ in the third chapter. Since the mathematics developed in the first chapter is only suitable for classical field theories, we will use it on the SM's Lagrangian to rigorously define the classical SM which can then be quantized using the path integral approach. Historically, path integral quantization was discovered after canonical quantization approach. The development of quantum field theory has started with Paul Dirac's discovery of the relativistic quantum mechanics and Dirac equation. In the following three subsections we are going to derive the Dirac equation

[1] In fact, path integral quantization to this day still remains a heuristic approach in many cases, even though it has been rigorously defined for some simple Lagrangians.

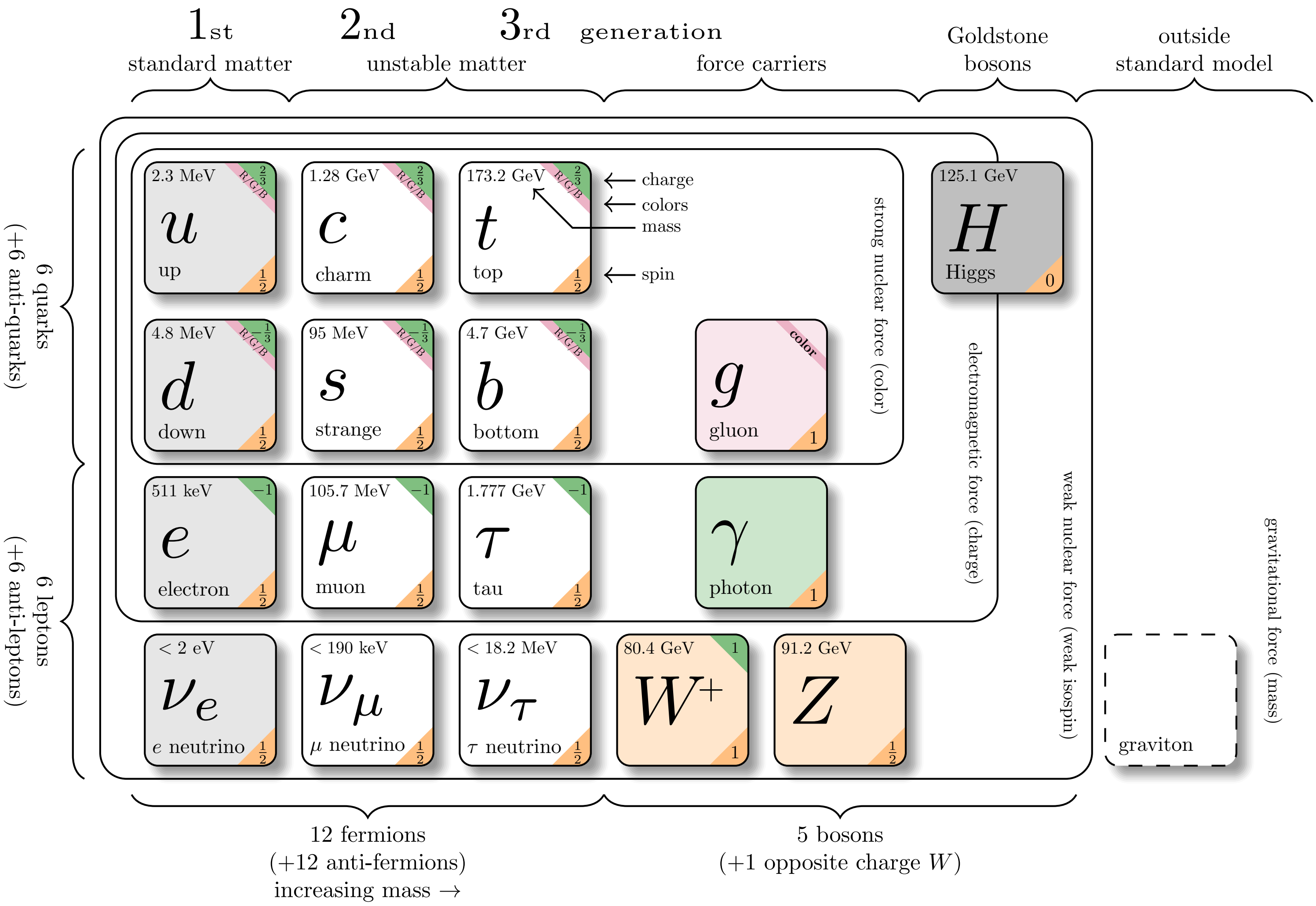

Figure 2.1: The entire particle content in the SM including the masses, spins and charges of particles. The diagram was made at the CERN Webfest [9].

and the Dirac Lagrangian and then we are going to explain how one quantizes the theory defined with the Dirac's Lagrangian using canonical quantization approach.

2.2.1 Dirac equation

In this subsection we are going to follow Dirac's derivation of Dirac's equation using the *canonical quantization* approach, i.e., by promoting observables into hermitian operators and then, after finding the Dirac equation, we will guess a Lagrangian that has the Dirac equation as its equation of motion. The Dirac equation is a relativistic quantum equation governing the dynamics of a free (massive spin $\frac{1}{2}$) fermion.

2.2.2 Derivation of Dirac equation

In order for a theory to be a valid relativistic quantum theory, the momentum and energy should satisfy the Einstein dispersion relation

$$E^2 = \vec{p}^{\,2} + m^2 \; . \tag{2.1}$$

The simplest possible quantized model is arrived at by promoting the equation (2.1) into an operator equation

$$\hat{E}^2\psi = \hat{\vec{p}}^{\,2}\psi + m^2\psi \tag{2.2}$$

where $\hat{E}$ and $\hat{p}$ are the hermitian operators corresponding to the energy and momentum observables. But, following the usual correspondence $\hat{E} = i\frac{\partial}{\partial t}$ and $\hat{\vec{p}} = -i\vec{\nabla}$ one arrives at the Klein-Gordon equation

$$\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi + m^2\psi \tag{2.3}$$

whose plane wave solutions have negative probability densities which are, of course, not physical and mathematically undefined.
To circumvent the problem of negative energy and probability densities, Dirac tried a more complicated equation of motion than (2.2) which will still reproduce the energy momentum relation (2.1), namely

$$\hat{E}\psi\left(= i\frac{\partial}{\partial t}\psi\right) = (\vec{\alpha}\cdot\vec{p} + \beta m)\,\psi \;= \left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi \; , \tag{2.4}$$

where $\vec{\alpha}$ and β are, for now, unknown mathematical objects which are yet to be determined from physical conditions the theory should satisfy. Firstly, by squaring the operator equation underlying (2.4), one should arrive back at the Klein-Gordon equation. This condition is equivalent to the following set of conditions:

$$\begin{aligned} &\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1 \\ &\alpha_i\beta + \beta\alpha_i = 0 \\ &\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \quad (i \neq j) \; . \end{aligned} \tag{2.5}$$

The simplest structure that can satisfy anticommutative relations (2.5) is the set of matrices. From the cyclic properties of trace and the fact that $\alpha_i^2 = \beta^2 = 1$, we can deduce that all matrices α_i and β are traceless

$$\text{Tr}(\alpha_i) = \text{Tr}(\alpha_i \beta\beta) = \text{Tr}(\beta\alpha_i\beta) = -\text{Tr}(\alpha_i\beta\beta) = -\text{Tr}(\alpha_i) = 0 \; . \tag{2.6}$$

It can also be shown (again using the property $\alpha_i^2 = \beta^2 = 1$) that all eigenvalues of α_i and β are ± 1, meaning that they are even-dimensional matrices since they are traceless.
Secondly, from the condition that the Dirac Hamiltonian, $H_D = -\vec{\alpha} \cdot \vec{\nabla} + \beta m$, is hermitian we can deduce that α_i and β are hermitian.
The last observation one needs to make is that two-dimensional square matrices do not contain four (linearly independent) anticommuting traceless hermitian matrices since there are only three linearly independent matrices, Pauli matrices σ_i, satisfying all the desired properties. This means the next try should be in the set of 4×4 matrices, and it turns out, it contains enough matrices with all the desired properties. In order to make the Dirac equation (2.4) covariant, one can define the *gamma matrices* γ^μ as

$$\gamma^0 := \beta, \; \gamma^i := \beta\alpha_i \tag{2.7}$$

which convert the Dirac equation into its most famous form

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0 \tag{2.8}$$

and satisy the following relations

$$\begin{aligned} &\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \\ &\left(\gamma^0\right)^2 = -\left(\gamma^i\right)^2 = 1 \\ &\gamma^{0\dagger} = \gamma^0 \; \gamma^{i\dagger} = -\gamma^i \; . \end{aligned} \tag{2.9}$$

It is also worth noting that the (four-component) object ψ is a spinor and that the probability current j^μ is given by

$$j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi \equiv \bar{\psi}\gamma^\mu\psi \; . \tag{2.10}$$

2.2.3 Dirac Lagrangian

The starting point of all Lagrangians describing the interactions of fermions is the Dirac Lagrangian, which equals

$$\mathcal{L} = \bar{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi\ . \tag{2.11}$$

The equations of motion for $\bar{\psi}$ and ψ are the Dirac equation and the conjugate Dirac equation respectively. The Lagrangian (2.11) is the minimal Lagrangian reproducing the Dirac equation as the Euler-Lagrange equation of motion.

2.2.4 Quantizing the Dirac Lagrangian

We are now going to describe the process of applying canonical quantization to the Dirac Lagrangian. As per usual procedure, we promote the fields and canonically conjugate fields in the Lagrangian into operators on the Hilbert space of quantum states. This is done by imposing suitable equal-time (anti)commutation relations to the fields and conjugate momenta which then give rise to suitable[2] (anti)commutation relations of the Fourier coefficients in the general expansion of the classical solution[3]. This is not very different to Dirac's approach to deriving Dirac's equation, with the main difference being that the procedure we have just described can be used to quantize general classical field theories, and not just theories for point-like particles. Imposing the following anticommutation relations to the Dirac fields (the conjugate momentum is proportional to $\psi^\dagger$, $\pi_\psi = i\psi^\dagger$, so we do not need to list any other anticommutation relations)

$$\begin{aligned}\{\psi_a(x), \psi_b^\dagger(y)\}_{x^0=y^0} &= \delta^3(\vec{x}-\vec{y})\delta_{ab}\\ \{\psi_a(x), \psi_b(y)\}_{x^0=y^0} &= \{\psi_a(x)^\dagger, \psi_b(y)^\dagger\}_{x^0=y^0} = 0\ .\end{aligned} \tag{2.12}$$

[2]Depending on the statistics we want the quantum field to obey, we need to choose commutation or anticommutation relations. For example, fermionic fields need to obey Pauli exclusion principle which forces us to use anticommutation relations or the canonical Hamiltian will not be positive definite.

[3]The word classical is used because the solution we are referring to is of the classical field theory. For the Dirac Lagrangian, the classical solution to the Dirac equation is a plane wave solution. See, e.g., [10] 4.6. for the exact form of the solution

and using the general expansion of the Dirac field

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=-1/2}^{1/2} \left(a_{\vec{p}}^s u^s(p) e^{-ip\cdot x} + b_{\vec{p}}^{s\dagger} v^s(p) e^{ip\cdot x} \right) \tag{2.13}$$

we obtain the following anticommutation relations of the Fourier coefficients (with all other combinations equaling 0)

$$\left\{ a_{\vec{p}}^r, a_{\vec{q}}^{s\dagger} \right\} = \left\{ b_{\vec{p}}^r, b_{\vec{q}}^{s\dagger} \right\} = (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \delta^{rs} \ , \tag{2.14}$$

which concludes the quantization of the Dirac Lagrangian.

2.3 Quantum electrodynamics

Quantum electrodynamics (QED) was the first fundamental interaction fully described in terms of quantum field theory. Its Lagrangian is invariant to $U(1)$ transformations and it is the result of the minimal modification to the Dirac's Lagrangian that can reproduce the invariance.

2.3.1 QED Lagrangian

The Lagrangian for the QED, as already mentioned in the Mathematical introduction to gauge theories section 1.1.2, equals (without external currents J^μ_{ext}):

$$\mathcal{L}_{\text{QED}} = \bar{\psi} \left(i\gamma^\mu D_\mu - m \right) \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{2.15}$$

where $D_\mu = \partial_\mu + iqA_\mu$ is the gauge covariant derivative and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the Faraday tensor. The Euler-Lagrange equations for this theory are the Dirac equation and the Maxwell equations

$$\begin{aligned} &\frac{\delta \mathcal{L}}{\delta \bar{\psi}} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \bar{\psi}} = 0 \iff \left(i\gamma^\mu D_\mu - m \right) \psi = 0 \ , \\ &\frac{\delta \mathcal{L}}{\delta A_\nu} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} = 0 \iff \partial_\mu F^{\nu\mu} = e\bar{\psi}\gamma^\mu \psi = ej^\nu = J^\nu \ . \end{aligned} \tag{2.16}$$

The first equation is the Dirac equation in the presence of an electromagnetic potential while the second Euler-Lagrange equation is the covariant form of Gauss-Ampere law with the electric current being the probability current multiplied with the elementary charge. The second two Maxwell equations in the covariant form are just the Bianchi identity for $F_{\mu\nu}$

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \ . \tag{2.17}$$

.

2.3.2 $U(1)$ symmetry

$U(1)$ is a compact abelian Lie group generated by $1 \in \mathbb{R}$ whose Lie algebra is $\mathfrak{u}(1) = T_1 U(1) = \mathbb{R}$. We say that a Lagrangian is invariant to $U(1)$ gauge transformations if, after applying a gauge transformation, the transformed Lagrangian differs the starting Lagrangian up to a surface term[4].
It is easy to see that, the $U(1)$ transformations

$$\begin{aligned} \psi &\mapsto U(x)\psi = e^{i\alpha(x)}\psi \\ \bar{\psi} &\mapsto \bar{\psi}U^\dagger(x) = e^{-i\alpha(x)}\bar{\psi} \\ A_\mu &\mapsto A_\mu - \partial_\mu \alpha(x) \ , \end{aligned} \tag{2.18}$$

leave the QED Lagrangian invariant. With all that in mind (and as explained in 1.1.2), we can see that QED is, in some sense, a minimal $U(1)$ theory originating from Dirac Lagrangian. The condition for a theory to satisfy $U(1)$ invariance is physical because it corresponds to local phase invariance which does not change the probability density.
In the following sections, we are going to construct minimal gauge invariant theories for other Lie groups, namely $SU(3)$ and $SU(2)$ for which there was also physical and experimental motivation. As it turns out, the minimal Lagrangians corresponding to the mentioned Lie groups will be the main components for constructing the entire SM Lagrangian.

2.4 Quantum chromodynamics

Quantum chromodynamics (QCD) is the interaction also known as the strong nuclear force and it is responsible for the interaction between quarks and gluons. Histori-

[4]A surface term is any function that can be represented as the 4-divergence of some vector field.

cally, the development of the description of QCD heavily relied on symmetries. For example, it was conjectured that the strong interaction does not distinguish between neutrons and protons and that every difference between the two particles arises from their difference in electric charge.

Trying to describe a force that did not distinguish protons and neutrons, Heisenberg introduced the concept of *isospin*, a quality of a theoretical particle *nucleon* that is mathematically analogous to spin 1/2 but its projections were interpreted as the *proton state* $|p\rangle$ and *neutron state* $|n\rangle$. The strong interaction then, not depending on the projections of isospin, was conjectured as a force invariant under $SU(2)$ transformations which would rotate a nucleon state around the $|p\rangle - |n\rangle$ plane. Its mediators were conjectured to be the pion particles π^+, π^- and π^0.

After the discovery of constituent particles of the neutron and proton, *up* and *down* (u and d) quarks, the idea of isospin was translated to the quarks. It turns out that isospin is a very good approximation to low energy limit of QCD in which only up and down quarks exist. But, after the discovery of the *strange* quark, the idea of $SU(2)$ symmetry was upgraded to the $SU(3)$ flavor symmetry which was not as good of an approximation for QCD at the strange quark energy scales, but it was a model none the less. The idea of isospin and its generalisation slowly died out, but the Lie group $SU(3)$ returned as the proposed symmetry group of the three color charges for the strong force, namely, r≡red, g≡green and b≡blue. Of course, only particles that carry color charge couple to the force carriers of QCD, *gluons*. Nowadays, QCD is regarded as the $SU(3)$ invariant gauge theory of interactions between 6 flavors of quarks and 8 gluons, in fact, a quark of any flavor has 3 colors and the strong interaction does not differ between flavors - in that sense strong interactions are always between 3 colored quarks and 8 gluons and not dependent on flavor.

2.4.1 QCD Lagrangian

The QCD Lagrangian is very similar to the QED Lagrangian with a few modifications which arise from the $SU(3)$'s non-abelian nature and the multitude of bosons and quark flavors:

$$\mathcal{L}_{\text{QCD}} = \sum_q \bar{\psi}_{qi} \left(i\gamma^\mu (D_\mu^{\text{QCD}})_{ij} - m\delta_{ij} \right) \psi_{qj} - \frac{1}{4} G^{a\mu\nu} G^a_{\mu\nu} \, . \tag{2.19}$$

There are some clarifications to be made:

- $(D_\mu^{\text{QCD}})_{ij} = \partial_\mu + ig_S G_\mu^a (T^a)_{ij}$ is the gauge covariant derivative for $SU(3)$, with g_S being the *strong coupling constant*.
- Functions $G_\mu^a(x)$, with $1 \leq a \leq 8$, are the classical[5] *gluon fields*. The index a ranges over 8 values because the Lie group $SU(3)$ has 8 generators.
- The matrices $T^a = \frac{1}{2}\Lambda^a$ are the 3×3 defining matrix representation of $SU(3)$ generators. For example, Gell-Mann representation Λ^a is an often used representation in QCD. The structure constants f^{abc} of the representations T and Λ are defined as follows (for exact values, see, e.g. [11]):

$$\left[T^a, T^b\right] = i f^{abc} T^c \tag{2.20}$$

- $G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - g_S f^{abc} G_\mu^b G_\nu^c$ is the Yang-Mills field strength[6] for the field G_μ^a, or the generalised Faraday tensor.
- ψ_{qi} are the Dirac 4-component spinors, index q ranging over each of the six quark flavors, while i is the color index (ranging between r,g and b).

2.4.2 $SU(3)$ symmetry

Every $SU(3)$ transformation can be represented as an exponential of an element of the Lie algebra $\mathfrak{su}(3)$. A general element of the Lie algebra is given as the linear combination of generators, meaning that for every β^a (1$\leq a \leq 8$) we have

$$\text{SU(3)} \ni U = e^{ig_S \beta^a T^a} = \left(e^{ig_S \beta^a T^a}\right)_{ij} \ . \tag{2.21}$$

This is easily promoted into a local transformation $U(x)$ by allowing coefficients β^a to depend on spacetime coordinates x

$$U(x) = e^{ig_S \beta^a(x) T^a} \ . \tag{2.22}$$

[5]It is important to remember that Lagrangians always contain classical fields. Only after establishing a Lagrangian one can go to quantize it, preserving symmetries and other properties.

[6]Notice the similarity between this expression and expression (1.60)

The SU(3) infinitesimal transformations on classical fields are then given by :

$$\begin{aligned}
\psi_{qi} &\mapsto U(x)_{ij}\psi_{qi} \\
\bar{\psi}_{qi} &\mapsto \bar{\psi}_{qi}U^{\dagger}(x)_{ij} \\
G^a_\mu(x) &\mapsto G^a_\mu(x) - \partial_\mu\beta^a(x) - g_s f^{abc}\beta^b G^c_\mu(x)
\end{aligned} \tag{2.23}$$

and it can be shown by direct calculation that the QCD Lagrangian (2.19) is invariant to the transformation (2.23). Also, since in the SM, $SU(3)$ is the symmetry group acting on the color space, we will denote it as $SU(3)_C$.

2.5 *Electroweak interaction*

Electroweak interaction is the unified interaction of the weak force and QED. In the following subsections we are going to discuss the weak force and explain the mechanism underlying the electroweak unification. We will start off with discrete symmetries in field theory and explain the violation of parity symmetry in the weak force (which was first experimentally confirmed in [12]).

2.5.1 Discrete symmetries of the Standard model

This subsection will be devoted to discrete symmetries of the SM.
As opposed to gauge symmetries, the groups underlying discrete symmetries are discrete groups and thus they do not generate conserved charges. The discrete symmetries of classical physics (Newtonian gravity and classical electromagnetism) are the charge conjugation $\mathcal{C}$, parity $\mathcal{P}$ and time inversion $\mathcal{T}$ transformations which generate the group of discrete symmetries.

2.5.1.1 Parity transformation $\mathcal{P}$

Parity transformation is defined on the phase space as

$$\begin{aligned}
x^\mu &\mapsto \mathcal{P}x^\mu = \mathcal{P}^\mu{}_\nu x^\nu = (x^0, -\vec{x}) \\
p^\mu &\mapsto \mathcal{P}p^\mu = \mathcal{P}^\mu{}_\nu p^\nu = (p^0, -\vec{p})
\end{aligned} \tag{2.24}$$

and on Dirac spinors as

$$\mathcal{P}\psi(x)\mathcal{P}^{-1} = \gamma^0\psi(\mathcal{P}x) \; . \tag{2.25}$$

QCD and QED are invariant under this transformation. On the other hand, experimentally measured violation of parity symmetry in the processes involving the weak force was one of the main constraints for the theoretical description of the weak interaction.

It is worth noting that $\mathcal{P}$ is a linear[7] and unitary operator.

2.5.1.2 Time inversion transformation $\mathcal{T}$

Time inversion is another discrete transformation that is often studied in QFT. Its action on the phase space is as follows

$$
\begin{aligned}
x^\mu &\mapsto \mathcal{T}x^\mu = \mathcal{T}^\mu{}_\nu x^\nu = (-x^0, \vec{x}) \\
p^\mu &\mapsto \mathcal{T}p^\mu = \mathcal{T}^\mu{}_\nu p^\nu = (-p^0, \vec{p}) \ ,
\end{aligned}
\tag{2.26}
$$

while its action on Dirac spinors is given as (in 3+1-dimensional spacetime[8])

$$
\mathcal{T}\psi(x)\mathcal{T}^{-1} = -i\gamma_5\gamma^2\gamma^0\psi(\mathcal{T}x) \ . \tag{2.27}
$$

Time inversion is an antilinear and antiunitary transformation.

2.5.1.3 Charge conjugation transformation $\mathcal{C}$

Charge conjugation operator $\mathcal{C}$ is defined on the Dirac spinors as follows:

$$
\mathcal{C}\psi = i\gamma^2\psi^* \ . \tag{2.28}
$$

It does not act at all on the phase space.

Even though some QFT-s violate $\mathcal{C}, \mathcal{P}$ or $\mathcal{T}$ individually, every known local theory is symmetric to their composition[9], $\mathcal{CPT}$.

---

[7]According to Wigner's theorem, the only two options are linear unitary operator and antilinear antiunitary operator.

[8]The exact from of the equation (2.27) depends on the spacetime dimension.

[9]This is a consequence of the CPT theorem which states that every quantum field theory with Poincare symmetry necessarily remains invariant under $\mathcal{CPT}$. See the proof in [13].

2.5.2 Weak force

Historically, QCD and QED were unable to describe β decay process. A model was devised whose interaction term (part of the covariant derivative) was as follows[10]:

$$\mathcal{L}_{\text{int.}} = \frac{-g_W}{\sqrt{2}} \bar{u}\gamma^\mu W_\mu \frac{1}{2}(1-\gamma_5)d \tag{2.29}$$

and it was capable of describing the β decay process.
In relativistic quantum mechanics, chirality projection operators are defined as

$$\begin{aligned} P_L &= \frac{1}{2}(1-\gamma_5) \\ P_R &= \frac{1}{2}(1+\gamma_5) \end{aligned} \tag{2.30}$$

which are used to define left and right chirality spinors (and bar spinors):

$$\begin{aligned} \psi_L &= P_L\psi \quad \psi_R = P_R\psi \\ \bar{\psi}_L &= \bar{\psi}P_R \quad \bar{\psi}_R = \bar{\psi}P_L \ . \end{aligned} \tag{2.31}$$

With all that said, the interaction (2.29) can be written as follows

$$\mathcal{L}_{\text{int.}} = \frac{-g_W}{\sqrt{2}} \bar{u}_L\gamma^\mu d_L W_\mu \tag{2.32}$$

and it can be seen that the W particle changes the flavor of quarks while only mediating the interaction of left-chirality particles, violating parity symmetry. It turns out that such an interaction can be arrived to from the *vector-axial vector interaction*[11] and the gauge invariance principle for the Lie group $SU(2)$.

2.5.3 $SU(2)$ symmetry

Supported by interaction term (2.29) and our isospin discussion during the outline of QCD, we are searching for an $SU(2)$ invariant gauge theory because we can see that

[10] From here on in this chapter, we are going to be suppressing color indices whereever there is no $SU(3)$ generators mixing them. The interaction term (2.29) was devised by Fermi [10].

[11] Feynman and Gell-Mann first proposed this type of interaction as a candidate to describe Fermi interaction (2.29) [14].

β decay changes isospin states[12]. In analogy to QED and QCD, there will be three[13] gauge bosons W^1, W^2, W^3. The following Lagrangian, with $W^i_{\mu\nu}$ being gauge bosons' Yang-Mills field strength tensors,

$$\mathcal{L} = \sum_D \bar{D}_L \gamma^\mu \Big\{ i\partial_\mu - \frac{g}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \left(W^1_\mu - iW^2_\mu\right) - \frac{g}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \left(W^1_\mu + iW^2_\mu\right) - \frac{g}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} W^3_\mu \Big\} D_L + \sum_{\text{fermions}} \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R + \frac{1}{4} W^{i\mu\nu} W^i_{\mu\nu} \,, \tag{2.33}$$

where D are *weak isospin doublets* and they range over *quark doublets* Q and *lepton doublets* L

$$\begin{aligned} Q &= \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix} \\ L &= \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix} \end{aligned} \tag{2.34}$$

and where the components of doublets are Dirac spinors for the particles in the doublets, is invariant to the following infinitesimal $SU(2)$ transformations :

$$\begin{aligned} &Q_L \mapsto e^{ig\alpha_i(x)\tau_i} Q_L \\ &\psi_R \mapsto \psi_R \\ &W^i_\mu(x) \mapsto W^i_\mu(x) - \partial_\mu \alpha_i(x) - g\epsilon_{ijk}\alpha_j(x) W^k_\mu(x) \,. \end{aligned} \tag{2.35}$$

It is important to mention that, even though left neutrinos constitute left lepton doublets, we do not incorporate right neutrino spinors in any way whatsoever into the theory.

There are some things physically wrong with the Lagrangian (2.33).

Firstly, even though the fields $W^1_\mu \mp iW^2_\mu$ physically correctly correspond to $W^\pm$ boson fields (up to a normalisation factor $\frac{1}{\sqrt{2}}$), the W^3 boson field can not be the measured Z^0 boson, because Z^0 interacts with right-handed particles too.

[12]Today it is understood that u and d quarks form a *weak isospin doublet pair*, but there are 5 more weak isospin pairs (see (2.34)).

[13]Corresponding to three basis vectors of $\mathfrak{su}(2)$, $\tau_i = \frac{1}{2}\sigma_i$ with the respective commutation relation $[\tau_i, \tau_j] = i\epsilon_{ijk}\tau_k$

Secondly, the Lagrangian does not contain any mass terms. That is because fermion mass terms of the form

$$\bar{\psi}m\psi = \bar{\psi}_L m\psi_R + \bar{\psi}_R m\psi_L \tag{2.36}$$

are, by themselves, not invariant to $SU(2)$ transformations (2.35).

Thirdly, weak force's gauge bosons are known to be massive, and yet there are no mass terms of the form

$$\frac{1}{2}m^2 W^\mu W_\mu \tag{2.37}$$

in the Lagrangian (2.33). That is because such mass terms alone are not invariant to gauge transformations of the form (2.35), (2.23) or (2.18).

The first issue will be adressed in the next subsection with the introduction of the electroweak unification of the weak force and QED. The second and third issue are resolved later by the means of Higgs mechanism.

2.5.4 Unification with the QED

Electroweak unification is a gauge theory by Glaslow, Shalam and Weinberg [15]. It is a gauge theory whose Lie group is $SU(2)\times U(1)$, more commonly written as $SU(2)_L \times U(1)_Y$, where $SU(2)_L$ reminds us that $SU(2)$ acts nontrivially only on left doublets and $U(1)_Y$ stands for the $U(1)$ Lie group whose charge is the *weak hypercharge*[14] *Y*, i.e, it is generated by a single generator

$$Y = 2(Q - \tau_3)\ , \tag{2.38}$$

where Q is the electric charge, and τ_3 the weak isospin's third component.

The gauge boson fields for $SU(2)_L \times U(1)_Y$ are $W_\mu^{1,2,3}(x)$ and $B_\mu(x)$. As was previously said, we will recognise

$$W_\mu^\pm(x) = W_\mu^1(x) \mp iW_\mu^2(x) \tag{2.39}$$

which leaves us with W_μ^3 and B_μ which we need to combine into the Z^0 boson and photon.

[14]Unlike the QED's electric charge, Y is the hypercharge given as the difference of electric charge and τ_3 isospin projection, τ_3 projection is 0 for $SU(2)_L$ singlets, i.e., for the right chirality particles. $\tau_3 = \frac{1}{2}$ for the upper particles in the doublet (2.34) and $\tau_3 = -\frac{1}{2}$ for the lower particles in the doublet (2.34).

The neutral current sector of the electroweak Lagrangian, $\mathcal{L}_{\text{nc}}$ equals

$$\mathcal{L}_{\text{nc}} = \sum_D \bar{D}_L \gamma^\mu (-ig\tau_3 W^3_\mu(x) - g'\frac{Y}{2}B^\mu) D_L + \sum_{\text{fermions}} \bar{\psi}_R \gamma^\mu (-g'\frac{Y}{2}B^\mu)\psi_R \ . \tag{2.40}$$

While the charged-current sector equals

$$\begin{aligned}\mathcal{L}_{\text{cc}} = &\sum_D \bar{D}_L \gamma^\mu \Big\{ i\partial_\mu - \frac{g}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} W^+_\mu - \frac{g}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} W^-_\mu \Big\} D_L + \ . \\ &\sum_{\text{fermions}} \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R \ .\end{aligned} \tag{2.41}$$

From the condition for the theory to reproduce QED interaction sector, we can find the coefficients for the general expansion of A_μ and Z^0_μ in the basis W^3_μ and B_μ

$$\begin{aligned} A_\mu(x) &= \cos(\theta_W) B_\mu(x) + \sin(\theta_W) W^3_\mu(x) \\ Z^0_\mu(x) &= -\sin(\theta_W) B_\mu(x) + \cos(\theta_W) W^3_\mu(x) \ . \end{aligned} \tag{2.42}$$

The coefficients are then given as the solutions[15] to:

$$\begin{aligned} eQ &= g'\frac{Y}{2}\cos(\theta_W) \\ eQ &= g\tau_3 \sin(\theta_W) + g'\frac{Y}{2}\cos(\theta_W) \ . \end{aligned} \tag{2.43}$$

All in all, the electroweak Lagrangian is given as

$$\mathcal{L}_{\text{EW}} = \mathcal{L}_{\text{cc}} + \mathcal{L}_{\text{nc}} + \frac{1}{4} W^{i\mu\nu} W^i_{\mu\nu} + B^{\mu\nu} B_{\mu\nu} \tag{2.44}$$

and it reproduces QED interaction terms while correctly describing interaction terms for the weak force. What remains to be done to describe electroweak interaction fully is finding a mechanism to include particle masses into the Lagrangian without violating $SU(2)_L$ symmetry.

[15]Note that this is an operator equation, Q is an operator which assigns electric charge to left and right spinors and τ_3 is an operator which assigns weak isospin to left and right spinors, i.e., left and right chirality spinors are eingevectors for Q and τ_3. The parameter θ_W is the numerical coefficient which satisfies (2.42) and is called the *Weinberg angle*. θ_W is a free parameter of the theory and is measured experimentally, [16].

2.6 *Higgs interaction*

The solution to including massive $SU(2)$ gauge boson masses into the theory is to add the following terms to the Lagrangian (2.44)

$$\mathcal{L}_{\text{Higgs}} = \left(D^{\text{EW}\,\mu}\phi\right)^\dagger \left(D^{\text{EW}}_\mu \phi\right) - \left(\underbrace{\mu^2\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2}_{V(\phi)}\right) \quad (\mu^2 < 0) \tag{2.45}$$

where ϕ is an $SU(2)$ doublet of a charged complex field ϕ^+ and a chargeless complex field ϕ^0

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad \phi^\dagger = \left((\phi^+)^* \equiv \phi^- \quad (\phi^0)^* \, (\neq \phi^0)\right) \tag{2.46}$$

and where D^{EW}_μ is the gauge $SU(2)$ covariant derivative, $D^{\text{EW}}_\mu = \partial_\mu + igW^i_\mu \tau_i + ig' B_\mu$. By direct expansion of the first term in (2.45) in the unitary $SU(2)$ gauge (where $\frac{v}{\sqrt{2}}$ is the minimum of the function $V(x)$)

$$\phi' = U\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{2.47}$$

one obtains a summand

$$\frac{1}{2} M_{ab} W^a_\mu W^{b\mu} \; (a, b = 1, 2, 3, 4 \text{ and } W^4_\mu \equiv B_\mu)\,. \tag{2.48}$$

Eigenvalues of the matrix M_{ab} are

$$\begin{aligned} m_{W^+} &= m_{W^-} = \frac{1}{2} g v \\ m_{Z^0} &= \frac{1}{2} v \sqrt{g^2 + g'^2} \\ m_\gamma &= 0 \end{aligned} \tag{2.49}$$

and its eigenvectors for Z^0 and γ bosons are consistent with (2.43).

2.6.0.1 Spontaneous symmetry breaking

We will now explain the seeming loss of gauge $SU(2)$ invariance in the theory. The potential $V(\phi)$ (depicted in the Figure 2.2) regarded as a real function has a continuous local minimum at the radius $\frac{v}{\sqrt{2}} = \frac{-\mu^2}{2\lambda}$.

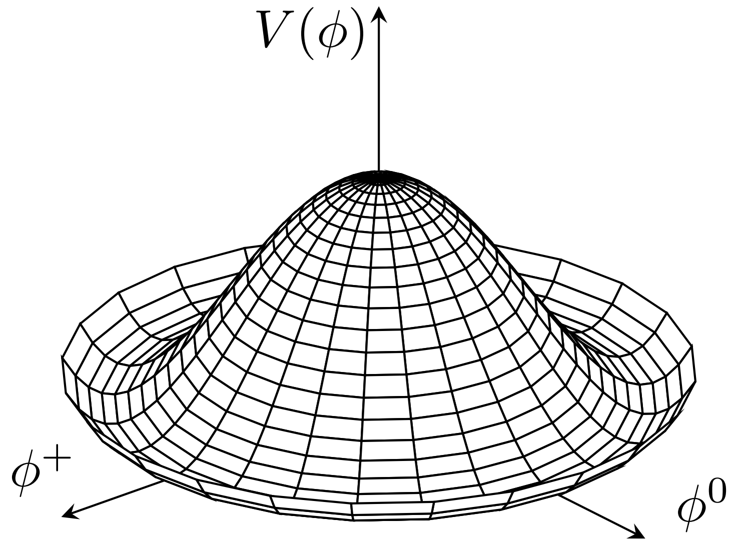

Figure 2.2: The potential $V(\phi)$ regarded as a real function of variable $\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$. This figure is a modification of the open source diagram [17].

Perturbations of the lowest energy state correspond to perturbations of the field

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{2.50}$$

or of any field arrived to by $SU(2)$ transformations of (2.50) in the $\phi^+ - \phi^0$ plane. In nature, since there is a continuous degeneracy of the vacuum energy, a vacuum state which is perturbed by $h(x)$ is chosen at random. In the unitary gauge (2.47), three degrees of freedom of the massless scalar ϕ are transformed into the third polarisation degree of freedom of (now massive) bosons $W^{\pm}$ and Z^0 so the total number of degrees of freedom is conserved[16]. The Lagrangian is still invariant to $SU(2)$ gauge transformations, unitary gauge is just the gauge that is the most naturally interpreted in terms of measurements, but some other gauges are more appropriate for calculations, e.g. 't Hooft-Feynman gauge is often used in one-loop calculations involving electroweak force.

[16]This property is a special case of the Goldstone theorem which explains spontaneous symmetry breaking in the unitary gauge for general gauge theories [18].

2.6.1 Fermion masses

Now we are applying the Higgs mechanism to introduce fermion mass terms into the Lagrangian.
As already explained in 2.5.3, mass terms for fermions equaling

$$\bar{\psi} m \psi = \bar{\psi}_L m \psi_R + \text{h.c.} \,, \tag{2.51}$$

are not $SU(2)_L$ invariant. That is because left spinors transform like vectors to $SU(2)_L$ while right handed spinors transform like scalars. To define an $SU(2)$ invariant quantity, we need an analogy to the scalar product $\vec{a} \cdot \vec{b}$ and not something of the form $a\vec{b}$.
If we consider the following combination

$$-g_d \bar{Q}_L \cdot \phi \cdot d_R + \text{h.c.} \tag{2.52}$$

we will obtain a mass term for the down quark (and analogously for all the generations of electrons and down quarks) and an interaction term that couples two down quarks (electrons or higher generations) and a Higgs boson

$$-\frac{g_d v}{\sqrt{2}} \bar{d}_L d_R + \frac{g_d}{\sqrt{2}} h \bar{d}_L d_R \,. \tag{2.53}$$

The mass of the down quark then equals

$$m_d = \frac{g_d v}{\sqrt{2}} \tag{2.54}$$

where g_d is the *Yukawa coupling for the down quark*.
In order to obtain mass terms for the $\tau_3 = +\frac{1}{2}$ fermions, the following term will be invariant to $SU(2)_L$ while generating appropriate mass term

$$-g_u \bar{Q}_L \cdot \phi_c \cdot u_R + h.c., \quad \phi_c = i\sigma_2 \phi^* \,. \tag{2.55}$$

In (2.55), g_u is the *Yukawa coupling for the up quark*. The mass term is analogously introduced for all other generations for the up quark. The neutrino mass[17] can be

[17]The mass of neutrinos is thought to be very small, but non zero, because (experimentally confirmed) neutrino flavor oscillations can only be exibited by massive neutrinos. The neutrino oscil-

generated in the same way, but this way of introducing neutrino mass gives no insight to why the Yukawa coupling for neutrinos would be so small, i.e., why the neutrino mass is so small.

All in all, since (2.52) and (2.55) are given as products of two $SU(2)_L$ doublets and an $SU(2)_L$ singlet, the terms we introduced are invariant to $SU(2)_L$ gauge transformations. They generate appropriate mass terms but at the expense of introducing mass terms by allowing a new type of interaction in the theory, an interaction between two fermions and a Higgs boson.

2.6.2 CKM matrix

One final phenomenon in the SM that remains to be described is the fermions' transitioning between flavors. For example, there exist nonzero amplitudes for the transitions like

$$u \leftrightarrow b, \quad d \leftrightarrow b, \quad d \leftrightarrow t, \text{ etc.}. \tag{2.56}$$

More specifically, quarks were observed to have nonzero transition amplitudes both inside their doublet pairs and between generations. The most general Yukawa interaction term which mixes quark flavors is

$$\mathcal{L}_{\text{Yuk}} = -g_{ij}^{(d)} \bar{Q}_{iL} \phi d_{jR} - g_{ij}^{(u)} \bar{Q}_L \phi_c u_{jR} + \text{h.c.} \,, \tag{2.57}$$

where indices i, j range over quark generations. $g_{ij}^{(d)}$ and $g_{ij}^{(u)}$ are *Yukawa coupling matrices*[18] and we have suppressed color indices of quarks. Equation (2.57) translates to mixed mass terms in the unitary gauge

$$\mathcal{L}_{\text{Yuk}} \supset -g_{ij}^{(d)} \frac{v}{\sqrt{2}} \bar{d'}_{iL} d'_{jR} - g_{ij}^{(u)} \frac{v}{\sqrt{2}} \bar{u'}_{iL} u'_{jR} + \text{h.c.} \,, \tag{2.58}$$

where we have, for later aesthetic purposes, added primes to the spinor labels. It is a known fact from linear algebra, that any arbitrary complex matrix A can be diagonalized with two unitary matrices $S(A)$ and $T(A)$ such that $D = S(A)^\dagger \cdot A \cdot T(A)$

lations are encapsulated in the Langrangian if one introduces right chirality neutrino spinors and neutrino Yukawa mass term. Then one can do analogous derivation as we will do for the CKM matrix and obtain a formalism which can allow neutrino oscillations and the PMNS matrix.

[18] They are non-diagonal matrix generalizations of g_d and g_u which form two 3×3 diagonal matrices, one for top quark generations and one for down quark generations. Yukawa coupling matrices are introduced because we do not have a good reason to not include such a trivial generalization from diagonal $g^{(d)}$ and $g^{(u)}$ matrices to non-diagonal ones.

where D is a diagonal matrix. Applying this to the matrices in (2.58), we can define matrices

$$m^{(d)} = \frac{v}{\sqrt{2}} V_L^{(d)} g^{(d)} V_R^{(d)\dagger}, \quad m^{(u)} = \frac{v}{\sqrt{2}} V_L^{(u)} g^{(u)} V_R^{(u)\dagger} \tag{2.59}$$

and by renaming

$$\begin{aligned} &V_{L\,ij}^{(d)\dagger} d'_{jL} \to d_{iL} \\ &V_{R\,ij}^{(d)\dagger} d'_{jR} \to d_{iR} \\ &(+\text{ analogously for } u \text{ quark}) \end{aligned} \tag{2.60}$$

we obtain a diagonalized mass part of the Lagrangian

$$\mathcal{L}_{\text{Yuk}} \supset \sum_i d_{iL} m_{ii} d_{iR} + \sum_i u_{iL} m_{ii}^{(u)} u_{iR} + \text{h.c.}\ . \tag{2.61}$$

Since (2.60) is a unitary transformation, the Lagrangian mostly keeps the same form:

- Every term that only has γ^μ matrices in between spinors is invariant to (2.60) because

  $$\bar{\psi}\gamma^\mu\psi \sim \bar{\psi}_R\gamma^\mu\psi_R + \bar{\psi}_L\gamma^\mu\psi_L$$

  and barred spinors transform with complex conjugate of (2.60). This leaves entire $\mathcal{L}_{\text{QCD}}$ intact. It also leaves the kinetic terms invariant for all fermions (because kinetic terms come with $\gamma^\mu\partial_\mu$).

- The only part of the Lagrangian that transforms under this unitary redefinition is $\mathcal{L}_{\text{cc}}$[19]. The transformed part (in terms of new, redefined d,u and lepton spinors) is given by

  $$\begin{aligned} \mathcal{L}_{cc} =& \bar{u'}_{iL}\left(-\frac{g}{\sqrt{2}}\gamma^\mu W_\mu^+\right) d'_{iL} + \text{h.c.} + \text{lepton part} \\ =& \bar{u}_{iL} V_{L\,ij}^{(u)} \left(-\frac{g}{\sqrt{2}}\gamma^\mu W_\mu^+\right) V_{R\,jk}^{(d)\dagger} d_{kL} + \text{h.c.} + \text{lepton part}\ . \end{aligned} \tag{2.62}$$

  If we define the matrix $V^{\text{CKM}} = V_L^{(u)} V_R^{(d)\dagger}$, we obtain the charged current sector in the unitary transformed mass basis:

  $$\mathcal{L}_{\text{cc}} = \bar{u}_{iL}\left(-\frac{g}{\sqrt{2}}\gamma^\mu W_\mu^+\right) V_{ij}^{\text{CKM}} d_{jL} + \text{h.c.} + \text{lepton part}\ . \tag{2.63}$$

[19] Z^0 and photon interaction terms are also unaffected

The matrix V^{CKM} is called the *CKM matrix*. Its non-diagonal elements give rise to non-zero transition amplitudes between generations of quarks, unlike what we first modeled in 2.5.4 where we only had transitions inside generations mediated by $W^{\pm}$ bosons.

2.7 *The Standard Model*

So far in Chapter 2, we have outlined the building blocks for the SM. We have developed the QED and QCD sectors of the Lagrangian and then we have extended QED into the electroweak interaction with its gauge group $SU(2)_L \times U(1)_Y$. We have successfully included massive gauge boson masses and reintroduced fermion masses into the Lagrangian, without breaking gauge symmetry, by the means of Higgs mechanism. In the end, we have generalized Yukawa couplings to non-diagonal matrix elements and then, via a unitary transformation, diagonalized the Yukawa coupling matrix at the expense of losing diagonality in the $\mathcal{L}_{\text{cc}}$ sector.

2.7.1 The SM Lagrangian

The full SM Lagrangian is given by

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{EW}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{Yuk}} \; . \tag{2.64}$$

Because we have reintroduced fermion mass terms in $\mathcal{L}_{\text{Yuk}}$, we are implicity excluding them from $\mathcal{L}_{\text{QCD}}$ and $\mathcal{L}_{\text{EW}}$.
Similarly, we group all fermionic kinetic terms in the $\mathcal{L}_{\text{kin}}$ Lagrangian sector. All in all, the expanded Lagrangian sectors are given by

$$\begin{aligned}
\mathcal{L}_{\text{QCD}} &= \frac{1}{4} G^{a\mu\nu} G^a_{\mu\nu} \\
\mathcal{L}_{\text{EW}} &= \frac{1}{4} W^{i\mu\nu} W^i_{\mu\nu} + \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \\
\mathcal{L}_{\text{Yuk}} &= \sum_{\text{generations } i} -1 \left(g^{(d)}_{ij} \bar{Q}_{iL} \phi d_{jR} + g^{(u)}_{ij} \bar{Q}_{iL} \phi_c u_{jR} \right) + \text{h.c.} + \text{lepton part} \\
\mathcal{L}_{\text{Higgs}} &= \left(D^{\text{EW}\,\mu} \phi \right)^\dagger \left(D^{\text{EW}}_\mu \phi \right) - \left(\mu^2 \phi^\dagger \phi + \lambda \left(\phi^\dagger \phi \right)^2 \right) \quad (\mu^2 < 0) \\
\mathcal{L}_{\text{kin}} &= \sum_{\text{quarks } q} \bar{q}_i \cdot i\gamma^\mu \left(\left(D^{\text{QCD}}_\mu \right)_{ij} + D^{\text{EW}}_\mu \delta_{ij} - \partial_\mu \delta_{ij} \right) q_j + \sum_{\text{leptons } l} \bar{l} \left(i\gamma^\mu D^{\text{EW}}_\mu \right) l
\end{aligned} \tag{2.65}$$

where D_μ^{EW} and D_μ^{QCD} are the electroweak ($SU(2)_L$) and the QCD ($SU(3)_C$) gauge covariant derivatives. In the unitary gauge and the mass basis one can recover the form of the Lagrangian which is physically interpreted in terms of known particles and gauge bosons.

2.7.2 $U(1)_Y \times SU(2)_L \times SU(3)_C$ symmetry

The SM Lagrangian is, by construction, invariant to $U(1)_Y \times SU(2)_L \times SU(3)_C$, which in the infinitesimal case reduce to

$$\begin{aligned}
\text{quarks } q: \quad & q_R \mapsto e^{ig_S\beta^a(x)T^a} q_R \\
& q_L \mapsto e^{ig_S\beta^a(x)T^a + ig\alpha_i(x)\tau_i + ig'\gamma(x)Y} q_L \\
\text{leptons } l: \quad & l_R \mapsto e^{ig\alpha_3(x)\tau_3 + ig'\gamma(x)Y} l_R \\
& l_L \mapsto e^{ig\alpha_i(x)\tau_i + ig'\gamma(x)Y} l_L \\
\text{gluons } G_\mu^a(x): \quad & G_\mu^a(x) \mapsto G_\mu^a(x) - \partial_\mu\beta^a(x) - g_s f^{abc}\beta^b(x)G_\mu^c(x) \\
SU(2)_L \text{ bosons } W_\mu^i(x): \quad & W_\mu^i(x) \mapsto W_\mu^i(x) - \partial_\mu\alpha_i(x) - g\epsilon_{ijk}\alpha_j(x)W_\mu^k(x) \\
U(1)_Y \text{ boson } B_\mu(x): \quad & B_\mu(x) \mapsto B_\mu(x) - \partial_\mu\gamma(x) \ ,
\end{aligned} \tag{2.66}$$

and this concludes our outline of SM Lagrangian's symmetries and properties.

3 Geometry of The Standard Model

3.1 Introduction

In this chapter we are applying the geometric objects developed in the Chapter 1 in order to describe the Lagrangian from the Chapter 2. We are starting by defining the gauge bosons as appropriate Yang-Mills fields. Then we are going to define spinor bundles for all matter fields, we will define charge conjugation and antimatter bundles followed by a Higgs bundle. After this we will use the defined fields, the Dirac operator and Dirac forms in order to again define the SM's Lagrangian which will be invariant to $G = U(1)_Y \times SU(2)_L \times SU(3)_C$.

3.2 Gauge boson fields

We are now defining principal G-bundles whose Yang-Mills fields will correspond to gauge boson fields, which we will couple to fermion fields through Dirac operators defined in 1.7.4.

3.2.1 $SU(3)_C$ gauge fields - gluons

As defined in the Chapter 1, connection 1-forms are Lie algebra valued forms on a principal G-bundle P. In the case of gluons and QCD, the Lie group G is, as explained in Chapter 2, $SU(3)_C$.

3.2.1.1 Principal $SU(3)$-bundle

We can choose the trivial bundle $P := M \times SU(3)$, where M is spacetime with its metric g, for our principal $SU(3)$-bundle's total space. In order to make this trivial bundle a principal $SU(3)$-bundle, we are defining the right $SU(3)$-action as follows:

$$(x^\mu, g) \lhd g' := (x^\mu, gg') \quad \forall g, g' \in \mathrm{SU}(3) \tag{3.1}$$

while the projection π is defined obviously as

$$\pi(x^\mu, g) \mapsto x^\mu. \tag{3.2}$$

3.2.1.2 $SU(3)$-bundle connection and gauge function

The principal $SU(3)$-bundle's connection 1-form, expanded in the basis $ig_S T^a$ of the Lie algebra $\mathfrak{su}(3)$ (i.e., the generators of $SU(3)$), is given as

$$\mathcal{G}_\mu(p) = \sum_{a=1}^{8} \mathcal{G}_\mu^a(p) ig_S T^a \quad p \in P\ , \tag{3.3}$$

where we identify each pullback $\sigma^* \left(\mathcal{G}_\mu^a(p)\right) \equiv G_\mu^a(x)$ with a gluon field. To confirm the validity of our definition, we want to verify the transformation rule (2.66) for gluons and the formula for the gluon field strength tensor $G_{\mu\nu}^a$ from 2.4.1.

Choose two global sections (gauges) (analogously to how we defined the gauge field A_μ for classical electrodynamics in 1.6.2) defined as

$$\begin{aligned} \sigma_1 &: m \in M \mapsto \left(x^\mu, e^{ig_S \beta_1^a(x) T^a}\right) \\ \sigma_2 &: m \in M \mapsto \left(x^\mu, e^{ig_S \beta_2^a(x) T^a}\right)\ , \end{aligned} \tag{3.4}$$

then the gauge function Ω (defined such that $\sigma_2 = \sigma_1 \lhd \Omega$) equals

$$\Omega\left(x\left(m \in M\right)\right) = e^{-ig_S \beta_1^a(x) T^a} e^{ig_S \beta_2^a(x) T^a}. \tag{3.5}$$

Following the derivation from 1.6.2, we obtain the following expression for the connection between two Yang-Mills fields $G^{(1,2)}$ generated by sections $\sigma_{1,2}$:

$$G_\mu^{(2)} = \Omega^{-1} G_\mu^{(1)} \Omega + \Omega^{-1}\left(\partial_\mu \Omega\right) \tag{3.6}$$

which in the infinitesimal case reduces (component $ig_s T^a$ wise) to the expression (2.66) with $\beta^a(x) = \beta_1^a(x) - \beta_2^a(x)$. Also, after finding the commutator relation for the basis $ig_S T^a$ for $\mathfrak{su}(3)$, $\left[ig_S T^a, ig_S T^b\right] = -g_S f^{abc} ig_S T^c$, we also obtain the correct expression for the gluon field strength tensor $G_{\mu\nu}^a$ as it was defined in 2.4.1.

3.2.2 $SU(2)_L$ gauge fields - W bosons

We are now defining $SU(2)_L$ bosons as Yang-Mills fields associated to a trivial $SU(2)$-bundle.

3.2.2.1 Principal $SU(2)_L$-bundle

We define the principal $SU(2)$-bundle as the trivial $SU(2)$-bundle over M. The projection and right Lie actions are defined analogously as in the pricipal $SU(3)$-bundle case 3.2.1.1.

3.2.2.2 $SU(2)$-bundle connection and gauge function

Again, the derivation analogously follows the QCD case. Expanded in the $\mathfrak{su}(2)$ basis $ig\tau_i$, a general $\mathfrak{su}(2)$-algebra valued 1-form on the pricipal $SU(2)$-bundle equals

$$\mathcal{W}_\mu(x) = \sum_{i=1}^{3} \mathcal{W}_\mu^i ig\tau_i \tag{3.7}$$

and we define a W boson field as $W_\mu^i = \sigma^* \mathcal{W}_\mu^i$, where σ is a local section

$$\sigma(m) = (x^\mu(m), e^{ig\alpha_i(x)\tau_i}) \tag{3.8}$$

on the $SU(2)$-bundle. Using the analogous gauge function to (3.5), we can reproduce the infinitesimal transformation rule from 2.66 by renaming $\alpha_i(x) = \alpha_i^1(x) - \alpha_i^2(x)$. The proof of this is a straight forward expansion of the equality

$$W_\mu^{(2)} = \Omega^{-1} W_\mu^{(1)} \Omega + \Omega^{-1} \left(\partial_\mu \Omega\right) \tag{3.9}$$

and it relies on the commutation relation $[ig\tau_i, ig\tau_j] = -g\epsilon_{ijk} ig\tau_k$. Also, this definition of W boson fields produces the same expression for the W boson field strength tensor $W_{\mu\nu}$ as in (2.35).

3.2.3 $U(1)_Y$ gauge field - B boson

Analogously to the definitions of W bosons and gluons, we define the B boson field as the Yang-Mills field $B_\mu(x) = \sigma^* \mathcal{B}_\mu$ where $\mathcal{B}$ is an $\mathfrak{u}(1)$ valued connection 1-form of the principal $U(1)$-bundle. For the basis of $\mathfrak{u}(1)$ we choose $ig' \cdot 1$.

3.2.4 Principal $U(1) \times SU(2) \times SU(3)$-bundle

Instead of defining seperate $SU(3)$, $SU(2)$ and $U(1)$ bundles, we could have instead defined an $SU(3) \times SU(2) \times U(1)$ bundle. We can choose the trivial bundle with the right action defined as $(x^\mu, g) \lhd g' = (x^\mu, gg')$, and the projection being the projection to the first variable, π_1. Since the Lie group of G is $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, we can construct a basis for $\mathfrak{g}$ using the discussed bases for $\mathfrak{su}(3)$, $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$. We can choose two local sections $\sigma_{1,2}$ defined using the exponential of general Lie group elements

$$\begin{aligned}\sigma_1(m) &= \left(x^\mu(m), e^{ig_S\alpha_1^a(x)T^a + ig\beta_i^1(x)\tau_i + ig'\gamma_1(x)1}\right)\\ \sigma_2(m) &= \left(x^\mu(m), e^{ig_S\alpha_2^a(x)T^a + ig\beta_i^2(x)\tau_i + ig'\gamma_2(x)1}\right)\ ,\end{aligned} \tag{3.10}$$

which will induce infinitesimal transformation rules that reproduce (2.66). Also, because generators of different Lie groups commute, there will be no mixing of bosons from different gauge groups in the transformation rules, which is as expected. It is also worth noting that for **global** transformations, gauge bosons transform in the adjoint representation of G (see Appendix A.3.6 for the definition of the adjoint representation) because the gauge function Ω then does not depend on x and as such has vanishing derivative $\partial_\mu\Omega$.

3.3 Matter fields

As explained before, matter fields are spinor fields on the spacetime. In the next subsection we are going to define the twisted chiral spinor bundles for each particle and then find the associated vector bundle of the total fermionic content obtained as the direct sum of associated vector bundles for each particle.

3.3.1 Matter field bundles

Let $G = U(1)_Y \times SU(2)_L \times SU(3)_C$. Matter fields are sections of the twisted chiral bundle

$$(S \otimes E)_+ = (S_- \otimes E_-) \oplus (S_+ \otimes E_+) \equiv (S_L \otimes F_L) \oplus (S_R \otimes F_R)\ , \tag{3.11}$$

where S_L and S_R are left-handed and right-handed spinor bundles over the 4-dimensional flat Minkowski spacetime, while F_L and F_R are associated vector bundles defined by complex unitary representations (with their respective fibres/representation spaces V_L and V_R) of the gauge group G.

Since the transformation rules under the Lie group G are determined by its representations on vector spaces, we are now stating the representations for all fermions. We will, for each particle, use tensor products of three complex representations, one for each constituent Lie group in the product for G, and then take direct sums for all particles.

We will assume (both $SU(3)$ and $SU(2)$) singlet representations from the representation space $\mathbb{C}$. For $SU(2)$ and $SU(3)$ fundamental representations, we will be exclusively using the representation spaces $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively, while for the group U$(1)_Y$ we will be using representations ρ_Y defined in Appendix A.3.7 with their representation spaces $\mathbb{C}_Y$.

3.3.1.1 Left chirality quarks

We know that left quark doublets transform under the fundamental representation for $SU(3)_C$ and $SU(2)_L$. Their weak hypercharge Y equals 1/3, meaning that, for each generation $i = 1, 2, 3$, their representation space is given as

$$Q_L^i = \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}_{1/3} \ . \tag{3.12}$$

This representation has the dimension equal to 6, for each generation, meaning that left chirality quarks contribute 18 dimensions to the total SM's fermionic representation. We can define the total left quark representation as the direct sum of left quark representations and its representation space is given as

$$Q_L \equiv Q_L^1 \oplus Q_L^2 \oplus Q_L^3 \ . \tag{3.13}$$

3.3.1.2 Right chirality quarks

Right quark singlets transform under the fundamental representation for $SU(3)$. Their right chirality, by definition, means that they transform under the trivial representation for $SU(2)$. Up quark generations have weak isospin $Y = 4/3$ while down quark generations have $Y = -2/3$ which means that (u_i, d_i) transforms under $U(1)_Y$ as $\mathbb{C}_{4/3} \oplus \mathbb{C}_{-2/3}$. All together, right chirality quarks transform under the representation

$$Q_R^i = \left(\mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{4/3}\right) \oplus \left(\mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{-2/3}\right) \tag{3.14}$$

This representation has the dimension equal to 6, meaning that right chirality quarks contribute in total 18 dimensions to the total SM's fermionic representation. The total right quark representation space is defined as the direct sum

$$Q_R = Q_R^1 \oplus Q_R^2 \oplus Q_R^3 \ . \tag{3.15}$$

3.3.1.3 Left chirality leptons

Leptons, unlike quarks, do not interact via the strong force, meaning that their representation for $SU(3)$ is trivial. On the other hand, left lepton doublets transform under the nontrival fundamental $SU(2)$ representation, while their weak hypercharge equals $Y = -1$ (both in the case of electron and electron neutrino), meaning that their $U(1)_Y$ representation is $\mathbb{C}_{-1}$. All together, for each generation i, the left leptons transform under the total representation

$$L_L^i = \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}_{-1} \ . \tag{3.16}$$

This representation contributes 2 dimensions per generation to the total SM's representation space, 6 in total. The total left lepton representation space is the direct sum over generations

$$L_L = L_L^1 \oplus L_L^2 \oplus L_L^3 \ . \tag{3.17}$$

3.3.1.4 Right chirality electrons

Unlike left chirality neutrinos, right chirality neutrinos are not included in the SM's Lagrangian. Electrons are singlets both for $SU(3)_C$ and $SU(2)_L$, while their hypercharge is $Y = -2$. Translating these facts into the representation for G, we get for each generation i

$$L_R^i = \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}_{-2} \ , \tag{3.18}$$

where we have kept the convention L_R^i instead od using e_R^i, this is done purely for aesthetic reasons. Electron and its higher generations each contribute 1 dimension to the SM's representation space, i.e, 3 in total. The total right electron representation space is given as

$$L_R = L_R^1 \oplus L_R^2 \oplus L_R^3 \ . \tag{3.19}$$

3.3.2 Charge conjugation

We know that every fermion (a consequence of the Dirac Lagrangian) has an antiparticle. Antiparticles are sections of the complex conjugate bundle (defined in Appendix A.1.10)

$$\overline{(S \otimes E)_+} = \overline{(S_L \otimes E_L)} \oplus \overline{(S_R \otimes E_R)} \; . \tag{3.20}$$

There are isomorphisms

$$\begin{aligned} \Delta_L &\cong \overline{\Delta_R} \\ \Delta_R &\cong \overline{\Delta_L} \end{aligned} \tag{3.21}$$

and if we set

$$\begin{aligned} V_L^C &\equiv \overline{V_R} \\ V_R^C &\equiv \overline{V_L} \end{aligned} \tag{3.22}$$

and extend this notation to representations too, we can express (3.20) as

$$\overline{(S \otimes E)_+} = \left(S_L \otimes F_L^C\right) \oplus \left(S_R \otimes F_R^C\right) \; . \tag{3.23}$$

3.3.2.1 Antimatter represetations

The representations for left (right) antiparticles are arrived to from the right (left) particles' representations by changing fundamental representations to antifudnamental and by changing $\mathbb{C}_Y$ to $\mathbb{C}_{-Y}$. This means that, in total, antiparticles contribute the same number of dimensions to the total representation space of the SM.

3.3.3 Higgs bundle

Higgs bundle is defined as the vector bundle

$$E = \underline{\mathbb{C}} \otimes E \; , \tag{3.24}$$

where $\underline{\mathbb{C}}$ is the trivial line-bundle coming from the trivial representation of the spin group, while E is the associated vector bundle generated by the representation $\mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}$ (which reflects the fact the Higgs field is an $SU(2)$ doublet composed of complex scalar fields).

3.3.3.1 Total matter content bundle

The total particle content associated bundle is obtained by taking the direct sums of all bundles defined from 3.3.1.1 up until 3.3.1.4 and the Higgs bundle. Every matter field is then obtained as a section that has all components equal to 0 except for the selected matter particle's component. We are including the Higgs field in this bundle because the Higgs field is not a gauge boson so it is natural to include it (also because of its definition as a section of an associated bundle) in the matter content bundle.

3.4 Intrinsic formulation of the Standard Model's Lagrangian

In this section we will again define the SM Lagrangian using the fields defined in 3.2 and 3.3.

We will individually study the main components of the SM's Lagrangian through subsections devoted to each of the components.

3.4.1 Yang-Mills-Dirac Lagrangian

We fix the following data:

- a connection 1-form A on a (trivial) principal G bundle P. Let $\mathcal{F}^A$ be the curvature 2-form associated to A which, according to Theorem 1.10, corresponds to a form $\mathcal{F}^A_M \in \Omega^2(P, \text{Ad}(P))$ with values in $\text{Ad}(P)$ and let F^A be the Yang-Mills field strength generated with a local section $s : M \to P$
- a section of a twisted spinor bundle $S \otimes E$ with $E = P \times_\rho V$.
- a Dirac form $\langle \cdot, \cdot \rangle$ on Δ_n which generates a spinor bundle metric $\langle \cdot, \cdot \rangle_S$
- An Ad-invariant scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ which generates a bundle metric $\langle \cdot, \cdot \rangle_{\text{Ad}(P)}$ on $\text{Ad}(P)$.

3.4.1.1 Yang-Mills Lagrangian

The **Yang-Mills** Lagrangian which is invariant to G gauge transformation is defined as follows

$$\mathcal{L}_{YM}[A] = -\frac{1}{2} \left\langle \mathcal{F}^A_M, \mathcal{F}^A_M \right\rangle_{\text{Ad}(P)} = -\frac{1}{4} \left\langle F^A_{\mu\nu}, F^{A\mu\nu} \right\rangle_{\mathfrak{g}} , \tag{3.25}$$

where the second equality holds locally. This Lagrangian is the generalization of terms of the form $G^a_{\mu\nu} G^{a\mu\nu}$ from the Chapter 2.

It is worth noting that since F^A is a Lie algebra valued form, it already contains all of the field strength tensors $G^a_{\mu\nu}$ for $a = 1, ..., \dim G$ since F^A is given as a linear combination of forms G^a multiplying basis vectors of $\mathfrak{g}$.

3.4.1.2 Dirac Lagrangian

The Dirac Lagrangian, which is invariant to Spin$^+(p, q)$ is defined as follows

$$\mathcal{L}_D[\Psi] = \text{Re}\,\langle \Psi, D\Psi \rangle_S - m\,\langle \Psi, \Psi \rangle_S \equiv \text{Re}\left(\bar{\Psi} D \Psi\right) - m\bar{\Psi}\Psi \tag{3.26}$$

where we have denoted $\langle \Psi, D\Psi \rangle_S$ as $\bar{\Psi} D \Psi$ and D is the Dirac operator for spinor fields. Also, taking the real part is necessary in order to ensure that the Lagrangian is real, even though it can be shown that the term $\bar{\Psi} D \Psi$ is real up to a surface term which doesn't contribute to equations of motion.

3.4.1.3 Yang-Mills-Dirac Lagrangian

The Lagrangian that is important for the SM is the **Yang-Mills-Dirac** Lagrangian which couples a gauge field to a fermion. It is defined as follows

$$\mathcal{L}_{\text{YMD}} = \text{Re}\,\langle \Psi, D_A \Psi \rangle_S - m\bar{\Psi}\Psi - \frac{1}{2}\left\langle \mathcal{F}^A_M, \mathcal{F}^A_M \right\rangle_{\text{Ad}(P)}\,, \tag{3.27}$$

where D_A is the Dirac operator for twisted spinor bundles. The first term in (3.27) is called the **kinetic term** for fermions while the third term in (3.27) is called the **kinetic term** for gauge bosons. The second term is called the **fermionic mass term** and it will not be invariant to the $SU(2)_L$ subgroup of G so we will include the masses of fermions and $SU(2)$ bosons via the Higgs mechanism.

3.4.2 Higgs Lagrangian

We fix the following data:

- a principal G-bundle P and a connection 1-form A on it
- a complex representation $\rho : G \to GL(W)$ and an associated vector bundle $E = P \times_\rho W$, where ρ and W reflect the interaction of the Higgs with gauge bosons (in the case of SM, $W = \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}$ as discussed in 3.3.3)

- a G-invariant scalar product $\langle\cdot,\cdot\rangle_W$ on W which induces an associated bundle metric $\langle\cdot,\cdot\rangle_E$ on E.

The spinor part of the twisted spinor Higgs bundle is an associated line bundle generated by the trivial representation of the spin group, as discussed in 3.3.3.

3.4.2.1 Higgs Lagrangian

The **Higgs Lagrangian** for a potential $V:\mathbb{R}\to\mathbb{R}$ is given as

$$\mathcal{L}_{\mathrm{H}}[\Phi, A] = \langle d_A\Phi, d_A\Phi\rangle_E - V\left(\langle\Phi,\Phi\rangle_E\right) \ . \tag{3.28}$$

The Lagrangian (3.28) is invariant to G gauge transformations. The first term in (3.28) is called the **scalar field kinetic term**. It is worth reminding ourselves that $d_A\Phi$ is the exterior covariant derivative of Φ with respect to the connection 1-form A. Locally, $\langle d_A\Phi, d_A\Phi\rangle_E$ can be expressed using a local gauge of ϵ Spin$^+(M)$ and a function $\phi: M\to W$ (with $\Phi=[\epsilon,\phi]$) as

$$\langle d_A\Phi, d_A\Phi\rangle_E = \left\langle \nabla^{A\mu}\Phi, \nabla^A_\mu\Phi\right\rangle_E = \left\langle \nabla^{A\mu}\phi, \nabla^A_\mu\phi\right\rangle_W \ . \tag{3.29}$$

3.4.3 Yukawa coupling

For representation spaces V_L, V_R and W of unitary representations we define the **Yukawa form** as a map

$$\tau: V_L\times W\times V_R \longrightarrow \mathbb{C} \tag{3.30}$$

that is invariant under G actions on $V_L\times W\times V_R$, complex antilinear in V_L, complex linear in W and complex linear in V_R. For every real constant g_Y, Yukawa form defines a **Yukawa coupling** as the G-invariant scalar

$$\begin{aligned} &Y:(\Delta_L\otimes V_L)\times W\times(\Delta_R\otimes V_R)\longrightarrow\mathbb{R} \\ &Y\left(\lambda_L\otimes v_L, \phi, \lambda_R\otimes v_R\right) := -2g_Y \mathrm{Re}\left(\langle\lambda_L,\lambda_R\rangle\cdot\tau\left(v_L,\phi,v_R\right)\right) \ , \end{aligned} \tag{3.31}$$

where $\langle\lambda_L,\lambda_R\rangle$ is the Dirac form of Dirac vectors $\lambda_L\in\Delta_L\subseteq\Delta_n$ and $\lambda_R\in\Delta_R\subseteq\Delta_n$. Since fermion fields and the Higgs field are sections of twisted spinor bundles which are generated from the vector spaces $\Delta_L\otimes V_L$, $\Delta_R\otimes V_R$ and $\mathbb{C}\otimes W$, the equation (3.31) uniquely determines a Yukawa coupling on sections of twisted spinor bundles.

The map (3.31) generalized to spinor sections defines the **Yukawa Lagrangian** and it is written for short as follows

$$\begin{aligned}\mathcal{L}_{\text{Yuk}}[\Psi_L, \Phi, \Psi_R] =& Y(\Psi_L, \Phi, \Psi_R) = -2g_Y \text{Re}\left(\overline{\Psi_L}\Phi\Psi_R\right) = \\ =& - g_Y \left(\overline{\Psi_L}\Phi\Psi_R\right) - g_Y \left(\overline{\Psi_L}\Phi\Psi_R\right)^* \ .\end{aligned} \tag{3.32}$$

3.4.3.1 Yukawa coupling of quarks

In the SM, the (G-invariant) Yukawa form for quarks is given as

$$\tau_Q(q_L, \phi, q_R) = g_{ij}^{(d)} q_{Li}^\dagger \phi \cdot d_{Rj} + g_{ij}^{(u)} q_{Li}^\dagger \phi_c \cdot u_{Rj} \ , \tag{3.33}$$

where $q_L \in Q_L$, $\phi \in W$, $q_R \in Q_R$ and $\phi_c = i\sigma_2\phi^*$. Also, the scalar product over color space $\mathbb{C}^3$ is implicit.

Matrices $g^{(u)}$ and $g^{(d)}$ are non-diagonal, but we can, as described in 2.6.2, diagonalize them using pairs of unitary matrices $V_L^{(u)}, V_R^{(u)}$ and $V_L^{(d)}, V_R^{(d)}$ in order to obtain mass term interpretation of Yukawa coupling. The matrices $V_{L/R}^{(u/d)}$ are defined such that

$$\begin{aligned} V_L^{(u)} g^{(u)} V_R^{(u)\dagger} &= \text{diag}(g_u, g_c, g_t) \\ V_L^{(d)} g^{(d)} V_R^{(d)\dagger} &= \text{diag}(g_d, g_s, g_b) \ . \end{aligned} \tag{3.34}$$

Diagonalizing matrices $g^{(u)}$ and $g^{(d)}$ corresponds to finding quark masses in the unitary gauge

$$m_f = \frac{1}{\sqrt{2}} g_f v \quad f = u, d, c, s, t, b \ , \tag{3.35}$$

and we recognize the mass eigenvectors as the physical quarks.

3.4.3.2 CKM matrix

The unitary transformations $U_{L/R}^{(u/d)}$ change the quark basis from one that is diagonal in the electroweak sector to the one that is diagonal in the Yukawa sector. The latter basis is easier to physically explain - we find the masses of quarks at the expense of dealing with electroweak vertices that mix left quarks both inside and between generations. The CKM matrix is defined as

$$V^{\text{CKM}} = V_L^{(u)} V_L^{(d)\dagger} \tag{3.36}$$

and it visibly affects the quark electroweak sector by changing the Lagrangian in electroweak basis from

$$\mathcal{L}_{\text{cc}} = \bar{u}_{iL} \left(-\frac{g}{\sqrt{2}} \Gamma^{\mu} W^{+}_{\mu} \right) d_{iL} + \text{h.c.} \tag{3.37}$$

to the Lagrangian that mixes mass basis generations

$$\mathcal{L}_{\text{cc}} = \bar{u}_{iL} \left(-\frac{g}{\sqrt{2}} \Gamma^{\mu} W^{+}_{\mu} \right) V^{\text{CKM}}_{ij} d_{jL} + \text{h.c.} \ , \tag{3.38}$$

both written locally with coordinate charts x^{μ}. All the other Lagrangian sectors have the same form as in the electroweak basis because only the $SU(2)$ sector mixes up and down quarks which transform with different unitary transformations.

3.4.3.3 Yukawa coupling for electrons

In the SM, we use the following Yukawa form in order to reproduce experimental data:

$$\begin{aligned} &\tau^{i}_{L} : L^{i}_{L} \times W \times L^{i}_{R} \\ &(l_L \phi, l_R) \mapsto g_i l^{\dagger}_{L} \phi l_R \ , \end{aligned} \tag{3.39}$$

and it, in the unitary gauge, generates mass terms for generations of electrons

$$m_e = \frac{1}{\sqrt{2}} g_e v \ , \quad m_{\mu} = \frac{1}{\sqrt{2}} g_{\mu} v \ , \quad m_{\tau} = \frac{1}{\sqrt{2}} g_{\tau} v \ . \tag{3.40}$$

3.4.3.4 Yukawa Lagrangian for the Standard Model

The total Yukawa Lagrangian for the SM is defined using τ^{i}_{L} and τ_Q and it equals, using the shorter notation,

$$\mathcal{L}_{\text{Yuk}} = \sum_{i,j=1}^{3} -\left(g^{(d)}_{ij} \bar{Q}_{Li} \cdot \phi \, d_{Rj} + g^{(u)}_{ij} \bar{Q}_{Li} \cdot \phi_c \, u_{Rj} \right) + \sum_{i=1}^{3} -g_i \left(\bar{L}_{Li} \cdot \phi \, e_{Ri} \right) + \text{h.c.} \ , \tag{3.41}$$

where Q_{Li} is the i-th generational component of the left quark doublet spinor bundle and d_{Rj} and u_{Rj} are the j-th d and u generational components of the right quark spinor bundle. ϕ is the section of the Higgs (spinor[1]) bundle.

[1]Higgs bundle is a spinor bundle but, as we have already mentioned, its spin group representation is trivial.

3.4.4 Lagrangian of the Standard Model

In this subsection we are stating the SM's Lagrangian in terms of fields that are arrived to from the bundle formalism. Since the Lagrangian sectors we defined in the previous subsections are, by definition, coordinate free (and invariant to G), we will have defined a Lagrangian that is defined purely using geometrical objects without any use of coordinatization. We fix the following data:

- The trivial principal G-bundle (P, π, M), where $G = SU(3)_C \times SU(2)_L \times U(1)_Y$ and M is the Minkowski spacetime with its Lorentizan metric of the signature (1,3).

- A connection 1-form $A = \mathcal{G} + \mathcal{W} + \mathcal{B}$ with values in the Lie algebra $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ and its curvature 2-form $\mathcal{F}^A$ associated to A (which generates a form $\mathcal{F}^A_M$ with values in Ad(P)).

- An Ad-invariant scalar product on $\mathfrak{g}$ which generates a bundle metric $\langle \cdot, \cdot \rangle_{\text{Ad}(P)}$ on Ad(P).

- All the matter bundles from 3.3 with their appropriate G and Spin$^+(1,3)$ representations.

- A real function $V : \mathbb{R} \to \mathbb{R}$ defined as

$$V(x) = \mu^2 x + \lambda x^2 \tag{3.42}$$

  where $\mu^2 < 0$.

3.4.4.1 Intrinsic Lagrangian for the Standard Model

The Lagrangian for the SM, defined using geometric objects and which is equivalent in terms of phenomenology to the Lagrangian (2.65) is as follows

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{YMD}} + \mathcal{L}_{\text{Yuk}} + \mathcal{L}_{\text{H}} , \tag{3.43}$$

where $\mathcal{L}_{\text{YMD}}$ is the Yang-Mills-Dirac Lagrangian for massless fermions coupled to G gauge bosons, $\mathcal{L}_{\text{Yuk}}$ is the Yukawa coupling sector and $\mathcal{L}_{\text{H}}$ is the Higgs Lagrangian defined with the potential $V(x)$ that we have fixed in (3.42):

$$\mathcal{L}_{\text{YMD}} = \sum_{\text{fermions } f} \text{Re} \left\langle \Psi_f, D_A \Psi_f \right\rangle_{S_f} - \frac{1}{2} \left\langle \mathcal{F}_M^A, \mathcal{F}_M^A \right\rangle_{\text{Ad}(P)}$$
$$\mathcal{L}_{\text{Yuk}} = \sum_{i,j=1}^{3} - \left(g_{ij}^{(d)} \bar{Q}_{Li} \cdot \phi \, d_{Rj} + g_{ij}^{(u)} \bar{Q}_{Li} \cdot \phi_c \, u_{Rj} \right) + \sum_{i=1}^{3} -g_i \left(\bar{L}_{Li} \cdot \phi \, e_{Ri} \right) + \text{h.c.}$$
$$\mathcal{L}_{\text{H}} = \left\langle d_A \Phi, d_A \Phi \right\rangle_{E_{\text{H}}} - V \left(\left\langle \Phi, \Phi \right\rangle_{E_{\text{H}}} \right) \ . \tag{3.44}$$

3.5 *Remarks*

In this chapter we have applied the formalism from the Chapter 1 to the physics of the SM summarized in Chapter 2. By doing so we have completely geometrically described the SM's Lagrangian, without invoking any coordinate frames or gauges. Though, in order to completely understand the SM, it is necessary to fix an $SU(2)_L$ gauge, but the Lagrangian itself is invariant under gauge transformations of the full symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$.

The SM is described by fixing 18 parameters (fundamental constants) that have to be determined in experiments, namely:

- the 3 coupling constants g_s, g and g'
- the 2 parameters λ and μ of the Higgs potential, or equivelently, the mass of the Higgs boson m_H and the absolute value of v
- 3 Yukawa couplings for leptons and 6 Yukawa couplings for quarks, or equivalently, the masses of 3 leptons and 6 quarks
- the 3 quark mixing angles and the KM phase δ characterizing the CKM matrix.

4 Concluding remarks and perspectives

4.1 Summary

In the first Chapter, we have made a brief introduction to gauge theories and differential geometry. Then we have started developing fiber bundle formalism and illustrated it on the GR on multiple occasions (in comments 1.4.2.1, 1.4.4.2, 1.5.3.2, 1.5.4.1, 1.5.6.1). We have finished the development of the theory of principal and associated bundles by studying CE as a $U(1)$ gauge theory. The first Chapter was finished with definitions of spinor bundles and constructions over them (Dirac forms, etc.) which we needed to do in order to describe fermion fields as sections of spinor bundles. It is also important to note that in the Chapter 1, our approach is valid for a general smooth Manifold, only in the Chapter 3 have we considered the special case of Minkowski spacetime. The approach presented here is applicable to curved spacetime of arbitrary metric signature, as long as the spacetime manifold admits a spin structure.

In the Chapter 2, we have studied the Standard Model with special focus on its symmetries. We have defined all the parts of Standard Model's Lagrangian, described the Higgs Mechanism and Yukawa coupling which leads to the transitions between generations of quarks. Note that one can analogously describe neutrino masses and oscillations by introducing the PMNS matrix.

In the Chapter 3 we have defined gauge boson fields as Yang Mills fields for specific principal bundles and fermion fields as sections of specific twisted chiral spinor bundles. We have also defined the Higgs field as a section of the spinor bundle with the trivial spin group representation. After this, we have used the tools from the Chapter 1 and fields from Chapter 2 in order to define all constituent parts of the Standard Model's Lagrangian. Summing the constituent parts, we have arrived at the Standard Model's Lagrangian in a coordinate free, geometric manner.

Our approach to studying gauge theories offers a very rigorous understanding of every term in the Lagrangian. Just the rigor itself allowed us to discover similarities between gauge theory nature of General Relativity and Standard Model. However, there are some aspects of the Standard Model and geometry that are outside of the scope of this Thesis and thus have not been mentioned at all so far. Namely, the topological aspects of the Standard Model and the noncommutative approach to geometry. We will now discuss them shortly.

4.2 *Topological aspects of gauge theories*

Gauge theories that have degenerate vacuum states have a chance to admit a **topological defect** which, depending on the type of the defect (which we will classify later), are solutions to equations of motion which exhibit particle-like properties[1] and exist purely because of topological properties of the gauge group. It can be shown, e.g., in [20] and [21], that the existence of topological defects can be characterized by the homotopy groups of G/H where G is the full symmetry group of the gauge theory and H is the stabilizer subgroup of G arrived to by the spontaneous symmetry breaking, i.e., it is the subgroup of G whose action preserves the vacuum state ϕ_0 after symmetry breaking.
More precisely, if the n-th homotopy group (defined in Appendix A.1.11) π_n is nontrivial, and we define $d \coloneqq s - 1 - n$, then the theory admits the following topological defect cases :

- $d = s - 1$ *domain wall*
- $d = s - 2$ *vortex/cosmic string*
- $d = s - 3$ *monopole*
- $d = s - 4$ *texture*

where s is the number of spacial dimensions on the spacetime manifold, and we call d the dimension of the defect. Topological defects are also called **topological solitons** and they are important not only in gauge theory but also in solid state physics [21]. Some examples of solitons are skyrmions, instantons, magnetic monopoles etc.

4.2.1 Magnetic monopole

In [22], it was shown that in the case of a $3+1$-dimensional $SO(3)$ gauge theory with 3-component Higgs fields, there exists a solution to the equations of motion which, asymptotically, exhibits the properties of a magnetic monopole. This is reflected in the fact that, after symmetry breaking, the gauge group reduces to $U(1)$

[1] i.e., they are solutions whose non-vacuum excitations are stable and localised in (at least) some space and/or time dimensions.

and $\pi_2(SO(3)/U(1))$ is non trivial. From this we can read $n = 2$, $s = 3$ (since the theory is $3 + 1$-dimensional) and we find that $d = 3 - 1 - 2 = 3 - 3$, i.e., the theory admits a monopole solution. The monopole solution demonstrated in the book is called the **'t Hooft-Polyakov monopole**.

This property is interesting because the theory is formulated only using electric charges, but still admits solutions which behave like magnetic charges.

Dirac showed an important fact that any electromagnetic theory that has non-zero magnetic charge solutions necessarily implies that both magnetic and electric charges are quantized, which means that finding a magnetic monopole in our universe would explain the charge quantization. This can also be found in [22].

4.2.2 Topological defects in the Standard Model

In the Standard Model, Higgs mechanism breaks down the group $G = SU(3)_C \times SU(2)_L \times U(1)_Y$ into $SU(3)_C \times U(1)_\mathrm{Q}$, where $U(1)_Q$ is the $U(1)$ group for electromagnetism. Finding the homotopy groups for this case proves that the Standard Model does not admit any[2] topological defects.

It is important to note that some *Grand Unified Theory*[3] (GUT)[4] models that spontaneously break down to the Standard Model do admit monopole defects, so finding a magnetic monopole would provide a significant hint towards the validity of GUT models whose characteristic energy scales are many orders of magnitude larger than what is producable in particle collider experiments on Earth[5].

4.3 *Noncommutative geometry*

Apart from the approach to geometry we have taken in this Thesis, which has its roots in mathematical analysis, there is also an important algebraic generalization of geometry and it rests upon the **Gelfand duality**, which is a mathematical theorem that gives rise to a duality between topological spaces and algebras. The noncommutative geometry formalism was first developed by A. Connes and he wrote an extensive

[2]Apart from, possibly, texture defects.

[3]A Grand Unified Theory is any theory that merges the electroweak and strong force, similarly to how electroweak unification is the theory of unified weak and electromagnetic interaction.

[4]Namely, SU(5) or SO(10) gauge theories.

[5]12 to be exact, highest energies achieved in CERN are of the or magnitude $\Lambda_\mathrm{CERN} \approx 10^4$ GeV, while the GUT scale is of the order of magnitute $\Lambda_\mathrm{GUT} \approx 10^{16}$ GeV, see [23]. This large gap in energy scales where no new unknown interactions appear, is called the *desert*

book [24] on its formalism with an application to the Standard Model at the end. Noncommutative geometry presents itself as a promising approach to the physics beyond Standard Model because it predicts relations between coupling constants in the case of Standard Model (just like GUTs do) and it automatically couples gauge theories to Einstein's gravity. It also naturally implements Higgs mechanism to theories with spontaneous symmetry breaking. See references [25] and [26] for more information on these claims.

In the following subsections we will display core ideas of noncommutative approach to geometry and its application to the Standard Model.

4.3.1 Gelfand duality

Gelfand duality is a theorem which accomplishes a 1-to-1 correspondence between topological spaces and **spectral triples** $(A, \mathcal{H}, D)$ where:

- A is a commutative $*$-algebra for finite topological spaces or a commutative C^*-algebra for compact Hausdorff topological spaces,
- $\mathcal{H}$ is a Hilbert space (finite-dimensional for finite topological spaces and infinite-dimensional for infinite compact Hausdorff spaces) that is a representation space of an irreducible representation $\pi : A \to L(\mathcal{H})$,
- D is a symmetric operator on $\mathcal{H}$ called the **Dirac operator**.

If the starting topological space is finite, or if we study a $*$-algebra with finite-dimensional Hilbert space, we call the spectral triple a *finite spectral triple*.

It can be shown (e.g., [27]) that for every spin manifold M there exists a canonical spectral triple consisting of:

- $A = C^\infty(M)$, the pointwise algebra (inherited from $\mathbb{C}$) of smooth functions on M,
- $\mathcal{H} = L^2(S)$, the Hilbert space of square-integrable sections of a spinor bundle $S \to M$,
- D the Dirac operator on S associated to the Levi-Civita connection.

Since every spectral triple $(A, \mathcal{H}, D)$, with A a commutative C^* algebra, corresponds to a compact spin manifold (via a unitary equivalence with some canonical spectral triple), one can generalize the notion of spin manifolds to noncommutative spin manifolds by studying spectral triples with noncommutative C^* algebras. In the book [27], the space of differential 1-forms (and much more) was developed for canonical spectral triples and generalized to the noncommutative case. Also, the notion of *almost-commutative* manifolds was defined as the spectral triple obtained by taking the spectral triple product of a canonical spectral triple and a *noncommutative finite spectral triple*, which is a generalization of finite spectral triples to noncommutative $*$-algebras.

4.3.2 The Standard Model in noncommutative geometry

In [27] similar work has been done to the work from our Chapter 3 but with noncommutative geometry formalism. A suitable almost commutative manifold was constructed which contained[6] $G = SU(3)_C \times SU(2)_L \times U(1)_Y$ as its gauge group and then the Standard Model Lagrangian was constructed using objects in noncommutative geometry.

[6]It was shown in [27] that the gauge group of an almost commutative manifold depends on the choice of the finite spectral triple. Choosing an adequate finite spectral triple generates G for the gauge group.

Appendices

Appendix A Supplementary definitions

A.1 Topology and linear algebra

A.1.1 Topological manifold

Topological manifold of the dimension m, or **topological *m* manifold** for short, is (see [3]) a Hausdorff topological space of dimension $m \in \mathbb{N}_0$ that is locally Euclidean with a countable basis. A topological manifold is said to be a **smooth manifold** if it is endowed with a globally defined differential structure (see [3]).

A.1.2 Diffeomorphism

A **diffeomorphism** (definition as in [3]) between smooth manifolds M (with an atlas $\mathcal{A}_M$) and N (with an atlas $\mathcal{A}_N$) is a smooth homeomorphism $f : M \to N$ whose inverse $f^{-1} : N \to M$ is also smooth.

A.1.3 General linear group

The **General linear group over the vector space *V*** is the set of all automorphisms $A : V \to V$ and we denote it as $GL(V)$. If $V = \mathbb{R}^d$ (or $\mathbb{C}^d$) it is customary to write $GL(d, \mathbb{R})$ (or $GL(d, \mathbb{C})$).

A.1.4 G-equivariant function

Let (P, π, M) be a principal G-bundle, (P_F, π_F, M) an associated bundle and $U \subseteq M$ an open neighborhood. We say that $\phi : \pi^{-1}(U) \subseteq P \to F$ is a ***G*-equivariant** (see, e.g., [19]) function if

$$\forall g \in G : \forall p \in \pi^{-1}(U) : \phi(p \triangleleft g) = g^{-1} \triangleright \phi(p) \tag{A.1}$$

where $\pi^{-1}(U)$ is the preimage of the open neighborhood U.

A.1.5 Push-forward of vectors

Let M and N be smooth manifolds and $F : M \to N$ a smooth map. For every point $p \in M$ we define the map

$$\begin{aligned} &F_{*p} : T_pM \to T_F(p)M \\ &\left(F_{*p}(X_p)\right)(f) \coloneqq X_p(f \circ F) \quad \forall f \in c_p^\infty(M). \end{aligned} \tag{A.2}$$

$(F_{*p}(X_p))$ is called the **push-forward** of the tangent vector X_p. This definition follows [3].

A.1.6 Pullback of differential 1-forms

Let M and N be smooth manifolds and $F : M \to N$ a smooth map. For every point $p \in M$ we define the map

$$\begin{aligned} &F_p^* : T_F^*(p) \to T_p^*M \\ &\left(F_p^*\left(\omega_p\right)\right)(X_p) \coloneqq \omega_{F(p)}(F_* X_p) \quad \forall X_p \in T_pM. \end{aligned} \tag{A.3}$$

$\left(F_p^*\left(\omega_p\right)\right)$ is called the **pullback** of the 1-form ω_p. This definition follows [3].

A.1.7 Module

A module is a quadruple $(V, R, \cdot, +)$ where $(V, +)$ is an abelian group and R is a ring such that the following axioms hold (the definition as in [6]):

1. $\forall r \in R, x, y \in V$:
$$r \cdot (x + y) = r \cdot x + r \cdot y \tag{A.4}$$

2. $\forall r, s \in R, x \in V$:
$$(r + s) \cdot x = r \cdot x + s \cdot x \tag{A.5}$$

3. $\forall r, s \in R, x \in V$:
$$(rs) \cdot x = r \cdot (s \cdot x) \tag{A.6}$$

4. $\forall x \in V$:
$$1 \cdot x = x \tag{A.7}$$

We can see that the the only difference between vector spaces and modules is in the fact that vector spaces are defined over fields, while modules are defined over the weaker algebraic structure of rings.

A.1.8 Fiber product of associated vector bundles

The fiber product [19], sometimes called the spliced product, of a principal G-bundle (P, π_P, M) and a principal H-bundle (Q, π_Q, M) is the principal $G \times H$-bundle $P \times_M Q$ defined as the set

$$P \times_M Q := \{(p, q) \in P \times Q : \pi_P(p) = \pi_Q(q)\} \ , \tag{A.8}$$

whose projection is canonically defined as $\pi_1 \circ (\pi_P \times \pi_Q)$. Specifically, the tensor product of associated vector bundles $P \times_\rho V$ and $Q \times_\sigma W$ is an associated vector bundle to the principal bundle $P \times_M Q$ with the standard fiber $V \otimes W$ and the left Lie-action defined as the product representation $\rho \otimes \sigma$. One can also analogously define the direct sum of associated bundles.

A.1.9 The bijection between local sections of associated bundles and maps to the fiber

Let (P, π, M) be a principal G-bundle, $E = P \times_\rho V$ an associated vector bundle and let $s : U \to P$ be a local gauge. Then there is a 1-to-1 relation between smooth sections $\tau : U \to E$ and smooth maps $f : U \to V$ given as

$$\tau(x) = [s(x), f(x)] \ . \tag{A.9}$$

A.1.10 Complex conjugate bundle

Let (P, π, M) be a principal G-bundle and $E = P \times_\rho V$ its associated vector bundle. The **complex conjugate bundle** of the associated bundle E is defined as

$$\overline{E} = P \times_{\bar{\rho}} \overline{V} \ , \tag{A.10}$$

where $\bar{\rho}$ is the complex conjugate representation with respect to ρ and $\overline{V}$ is the complex conjugate vector space with respect to V.

A.1.11 Homotopy groups π_n

This definition follows [21].Let X be a smooth manifold. We say that an **n-loop** at $x_0 \in X$ is a continuous map $\alpha : I^n \to X$ such that

$$\alpha\left(\partial I^n\right) = \{x_0\}\ , \tag{A.11}$$

where I^n is the unit n-cube $[0,1]^n$. Two n-loops α and β are **path-homotopic** if there exists a continuous $F : I^n \times I \to X$ such that

$$\begin{aligned} &F(s,0) = \alpha(s) \\ &F(s,1) = \beta(s) \\ &F(s,t) = x_0 \quad \forall s \in \partial I^n, t \in I\ . \end{aligned} \tag{A.12}$$

Path homotopy of n-loops is an equivalence relation. We also define the **product of n-loops** as follows

$$\left(\alpha * \beta\right)\left(s_1, ..., s_n\right) = \begin{cases} \alpha(2s_1, s_2, ..., s_n), & s_1 \in \left[0, \frac{1}{2}\right] \\ \beta(2s_1 - 1, s_2, ..., s_n), & s_1 \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{A.13}$$

and the **inverse** of an n-loop as follows

$$\alpha^{-1}(s_1, ..., s_n) = \alpha(1 - s_1, s_2, ..., s_n)\ . \tag{A.14}$$

Finally, the **n-th homotopy group** of X at x_0, $\pi_n(X, x_0)$ is the set of n-loop equivalence classes (with respect to path homotopy relation) with the product $*$ and the inverse $^{-1}$. If X is path connected, then $\pi_n(X, x_0) \cong \pi(X, x_1)$ for all $x_0, x_1 \in X$ so we just write $\pi_n(X)$.

A.2 *Clifford algebras and spin structure*

A.2.1 Double covering

Let G and H be Lie groups. We say that G is a **double covering** of the Lie group H is there exists a Lie group homomorphism $\rho : G \to H$ for which $\mathrm{Ker}(\rho) \simeq \mathbb{Z}_2$.

A.2.2 Clifford algebra

Let V be a finite dimensional vector space (with $\mathrm{dim}V = n$) over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let $B : V \times V \to \mathbb{F}$ be a non-degenerate (not necessarily a positive definite) bilinear form. The Clifford algebra with respect to V and B, $\mathrm{Cl}(V, B)$, is defined as the quotient

$$\mathrm{Cl}(V, B) = \mathcal{T}(V)/\mathcal{I}(V, B) \tag{A.15}$$

where $\mathcal{T}(V)$ is the **tensor algebra of V**

$$\mathcal{T}(V) = \mathbb{F} \oplus \bigoplus_{n=1}^{\infty} V^{\otimes n} \tag{A.16}$$

and $\mathcal{I}(V, B)$ is the ideal[7] in $\mathcal{T}(V)$ generated by the set

$$\{v \otimes w + w \otimes v - 2B(v, w)\} \ . \tag{A.17}$$

The multiplication in the Clifford algebra $\mathrm{Cl}(V, B)$ is given as

$$[a] \cdot [b] = [a \otimes b] \quad \forall a, b \in \mathcal{T}(V) \tag{A.18}$$

If a Clifford algebra is over the field of complex numbers, we will denote it as $\mathbb{C}\mathrm{l}(V, B)$.

A.2.3 Even and odd parts of the Clifford algebra

Given a vector space V with a bilinear form B, one can generate the Clifford algebra $\mathrm{Cl}(V, B)$ as in A.2.2. Since $\mathrm{Cl}(V, B)$ is generated by products of up to n (with $n = \mathrm{dim}V$) basis vectors e_i (the same basis as in A.2.7), we can decompose $\mathrm{Cl}(V, B)$ into a direct sum

$$\mathrm{Cl}(V, B) = \mathrm{Cl}^0(V, B) \oplus \mathrm{Cl}^1(V, B) \ , \tag{A.19}$$

where $\mathrm{Cl}^0(V, B)$ and $\mathrm{Cl}^1(V, B)$ are generated by products of even number and odd number of basis vectors respectively. We call $\mathrm{Cl}^0(V, B)$ the **even part** of the Clifford Algebra, and $\mathrm{Cl}^1(V, B)$ the **odd part** of the Clifford algebra.

[7]Ideals in rings are generalized naturally to ideals in algebras, so since $\mathcal{T}(V)$ is an algebra, it is possible to define its ideals.

Clifford algebra Cl(V, B) has the structure of a $\mathbb{Z}_2$-**graded Lie algebra**, or in other words, a **graded Lie superalgebra** because it satisfies

$$\mathrm{Cl}^i(V, B) \cdot \mathrm{Cl}^j(V, B) \subset \mathrm{Cl}^{i+j \bmod 2}(V, B) . \tag{A.20}$$

A.2.4 Gamma matrices in Clifford algebras

We define the map $\gamma : V \to \mathrm{Cl}(V, B)$ as the composition

$$\gamma = \pi \circ i , \tag{A.21}$$

where i is the **canonical embedding**

$$\begin{aligned} &i : V \to \mathcal{T}(V) \\ &i(v) = v \quad \text{(regarded as an element of } \mathcal{T}\text{(V))} , \end{aligned} \tag{A.22}$$

and π is the **canonical projection**

$$\begin{aligned} &\pi : \mathcal{T}(V) \to \mathrm{Cl}(V, B) \\ &\pi(v) = [v] . \end{aligned} \tag{A.23}$$

Now let $(V, B) = \mathbb{R}^{p,q} \equiv (\mathbb{R}^{p+q}, \eta)$ with standard basis $(e_1, ..., e_{p+q})$ and suppose

$$\rho : \mathrm{Cl}(V, B) \to \mathrm{End}(\Sigma) \tag{A.24}$$

is a representation of Cl(V, B) on an $\mathbb{F}$-vector space $\Sigma = \mathbb{F}^N$. We define for $a = 1, ..., N$ the **mathematical gamma matrices**

$$\gamma_a = \rho \circ \gamma(e_a) , \tag{A.25}$$

and **physical gamma matrices**

$$\Gamma_a = (-i)\gamma_a . \tag{A.26}$$

The anticommutators of gamma matrices are given as

$$
\begin{aligned}
\{\gamma_a, \gamma_b\} &= -2\eta_{ab} I_N \\
\{\Gamma_a, \Gamma_b\} &= 2\eta_{ab} I_N
\end{aligned}
\tag{A.27}
$$

where I_N is the N×N identity matrix, we can denote the commutators as

$$
\begin{aligned}
\gamma_{ab} &= \frac{1}{2}\left[\gamma_a, \gamma_b\right] \\
\Gamma_{ab} &= \frac{1}{2}\left[\Gamma_a, \Gamma_b\right] \ .
\end{aligned}
\tag{A.28}
$$

Also, we define gamma matrices with raised indices naturally

$$
\begin{aligned}
\gamma^a &= \eta^{ab}\gamma_b \\
\Gamma^a &= \eta^{ab}\Gamma_b \ .
\end{aligned}
\tag{A.29}
$$

A.2.5 Chirality element

For spaces $(V, B) = \mathbb{R}^{p,q}$ where $p+q$ is even, we define the **mathematical chirality operator**

$$
\gamma_{p+q+1} = -i^{p+q}\gamma_1 ... \gamma_{p+q}
\tag{A.30}
$$

and the **physical chirality operator**

$$
\Gamma_{p+q+1} = -i^{p+q}\Gamma_1 ... \Gamma_{p+q} \ .
\tag{A.31}
$$

The physical chirality operator is an arbitrary (even) dimensional analog to the γ_5 matrix in the $(p, q) = (1, 3)$ case that we have studied in Chapter 2.

A.2.6 Weyl spinors and chirality in even dimensions

Let $\mathbb{C}\mathrm{l}(\mathbb{R}^n, \eta)$ be a complex Clifford algebra and let $\rho : \mathbb{C}\mathrm{l}(\mathbb{R}^n, \eta) \to \mathrm{End}(\Delta_n)$ be a representation to the vector space od Dirac spinors (defined in Definition 1.36). If n is even, the representation ρ splits into two irreducible represenations $\rho_\pm$ such that

$$
\begin{aligned}
&\rho = \rho_+ \oplus \rho_- \\
&\rho_+ : \mathbb{C}\mathrm{l}(\mathbb{N}^n, \eta) \to \mathrm{End}(\Delta_n^+) \\
&\rho_- : \mathbb{C}\mathrm{l}(\mathbb{N}^n, \eta) \to \mathrm{End}(\Delta_n^-)
\end{aligned}
\tag{A.32}
$$

Where $\Delta_n^{\pm}$ are the (± 1) eigenspaces of the physical chirality operator Γ_{n+1}. The proof of this statement can be found in [7], Proposition 6.4.5.
We call the spaces $\Delta_n^{\pm}$ the vector spaces of **right/left chirality Weyl spinors**.

A.2.7 Pin group Pin(V)

If we are given an orthonormal basis (with respect to B), $\{e_1, ..., e_n\}$ for V, then

$$\begin{aligned} &[e_i] \cdot [e_j] = -[e_j] \cdot [e_i] \quad \text{for } i \neq j \\ &[e_i]^2 = 1 \end{aligned} \tag{A.33}$$

in Cl(V, B). This implies that Cl(V, B) is a 2^n dimensional vector space because it is spanned by products of basis vectors e_i, but because of (A.33), any product containing more than n basis vectors e_i necessarily reduces to a product of at most n basis vectors. Relations (A.33) also imply that for every unit vector v, $v^2 = 1$, because for $i \neq j$

$$(\alpha e_i + \beta e_j)^2 = \alpha^2 e_i^2 + \beta^2 e_j^2 + \alpha \cdot \beta \{e_i, e_j\} = \alpha^2 e_i^2 + \beta^2 e_j^2 = \alpha^2 + \beta^2 \ , \tag{A.34}$$

so we can see that the set of unit vectors in V defines a group in Cl(V, B) where every unit vector is its own inverse. This group is denoted as **Pin(V)**.

A.2.8 Spin group Spin(p,q)

It can be shown (see e.g., [7]) that for every $v \in$ Pin(V) and $w \in V$, vwv^{-1} is the reflection of w across the orthogonal (with respect to B) complement of v in V. Therefore, $\pi(w) \mapsto vwv^{-1}$ is an (surjective) orthogonal map and π induces a group homomorphism

$$\pi : \text{Pin}(V) \to O(V, B) \supset SO(V, B) \ . \tag{A.35}$$

It is a known fact that for every group endomorphism $\rho : G \to H$, if $K \leq H$, then the preimage $\rho^{-1}(K) \leq G$.
Applying this to π, we can define the group Spin(V, B) as the preimage $\pi^{-1}(SO(V, B))$.
If $V = \mathbb{R}^{p,q}$, then we denote Spin(V,B) as **Spin(*p*,*q*)**. It is also possible to prove that every element of Spin(V) is a product of an even number of unit vectors in Pin(V).

A.2.9 Ortochronous spin group Spin$^+$(p,q)

If η is the symmetric bilinear form for $\mathbb{R}^{p,q}$, then we define the following subsets of $\mathbb{R}^{p,q}$

$$\begin{aligned} S_-^{p,q} &= \{v \in \mathbb{R}^{p,q} : \eta(v,v) = -1\} \\ S_+^{p,q} &= \{v \in \mathbb{R}^{p,q} : \eta(v,v) = +1\} \\ S_\pm^{p,q} &= S_-^{p,q} \cup S_+^{p,q} \ . \end{aligned} \tag{A.36}$$

The spin group Spin(p,q) can alternatively be defined as the set

$$\text{Spin}(p,q) = \{v_1, ..., v_{2r} : v_i \in S_\pm^{p,q}, r \geq 0\} \ , \tag{A.37}$$

while the **ortochronous spin group** Spin$^+(p,q)$ is defined as the set

$$\{v_1...v_{2n} w_1...w_{2m} : v_i \in S_+^{p,q}, w_j \in S_-^{p,q}, m \geq 0, n \geq 0\} \ . \tag{A.38}$$

A.3 Representation theory of Lie groups

A.3.1 G-invariant scalar product

One very important property of Lie groups is that for every representation $\rho : G \to$ GL(V), there exists a G-**invariant positive definite scalar product**[8] given as

$$\langle v, w \rangle = \int_G \tau_{v,w} \sigma \ , \tag{A.39}$$

where $\tau_{v,w} = \langle\langle v, w \rangle\rangle$ and $\langle\langle \cdot, \cdot \rangle\rangle$ is any scalar product on V (not necessarily G-invariant), while σ is a dim$G$-form on G

$$\sigma = \omega^1 \wedge ... \wedge \omega^n \ , \tag{A.40}$$

with ω^i being a right-invariant basis of $\Omega(G, T^*G)$. A right-invariant basis ω^i is defined as the dual basis to the basis of right-invariant tangent vectors $X_i \in T_eG$, i.e., they satisfy $r_{g*}X_i = X_i$ for all i and right translation r_g is defined as $r_g(h) = h \cdot g$.

[8] This is really a scalar product and not just a contraction with the metric on M.

A.3.2 Associated bundle metric

Let (P, π, M) be a principal G-bundle, $\rho : G \to GL(V)$ a representation of G and $E = P \times_\rho V$ an associated vector bundle.

For every G-invariant scalar product on V, $\langle \cdot, \cdot \rangle_V$, there exists a **bundle metric** $\langle \cdot, \cdot \rangle_E$ on E given by

$$\langle [p, v], [p, w] \rangle_{E_x} := \langle v, w \rangle_V \tag{A.41}$$

and it is well defined, i.e., it does not depend on the choice of $p \in P_x$ for every x.

A.3.3 Ad-invariant scalar product

Let G be a compact Lie group. There exists a scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on the Lie group $\mathfrak{g}$ which is Ad-invariant and it is, up to a factor[9], unique. The Ad-invariant scalar product on $\mathfrak{g}$ determines a bundle metric $\langle \cdot, \cdot \rangle_{\mathrm{Ad}(P)}$ on $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g}$.

A.3.4 Fundamental representation of $SU(N)$

Lie groups $GL(n, \mathbb{C})$ (analogously for real Lie groups) have canonical representations on $\mathbb{C}^n$ given as the matrix multiplication on column vectors in the standard basis of $\mathbb{C}^n$. This type of representation is called the **fundamental**, **defining** or **standard** representation and it is also valid for all subgroups of $GL(n, \mathbb{C})$. Fundamental representation of $GL(n, \mathbb{C})$ (or any of its subgroups) is sometimes denoted as $\mathbf{N}$ or simply $\mathbb{C}^n$.

A.3.5 Antifundamental representation of $SU(N)$

The **antifundamental representation** of $GL(n, \mathbb{C})$ is the complex conjugate of the fundamental representation. Given a representation $\rho : G \to GL(V)$, the complex conjugate representation is a representation $\bar{\rho} : G \to GL(n, \mathbb{C})$ whose representation space is $\overline{V}$, the complex conjugate vector space to V, and $\bar{\rho}(g) = \rho(g)^*$ for all $g \in G$. Antifundamental representation of $GL(n, \mathbb{C})$ (or any of its subgroups) is sometimes denoted as $\bar{\mathbf{N}}$ or simply $\bar{\mathbb{C}}^n$.

[9]The factor connecting the chosen scalar product and the canonical scalar product on $\mathfrak{g}$ is called the **coupling constant**.

A.3.6 Adjoint representation

Let $c_g = l_g \circ r_{g^{-1}}$, i.e., $c_g(h) = g \cdot h \cdot g^{-1}$ be the group element conjugation.
The **adjoint representation** or **adjoint action of the Lie group** is defined as

$$\begin{aligned} \mathrm{Ad} : G &\to GL\left(\mathfrak{g}\right) \\ g &\mapsto \mathrm{Ad}(g) = \mathrm{Ad}_g = \left(c_g\right)_* \end{aligned} \tag{A.42}$$

and is sometimes denoted Ad_G instead of Ad.

A.3.7 $U(1)_Y$ representations $\mathbb{C}_Y$

The $U(1)_Y$ representations used in the description of the Standard Model are representations of the form

$$\begin{aligned} \rho_Y : U(1) &\longrightarrow \mathbb{C}_Y \\ z &\mapsto e^{3Y} z \ , \end{aligned} \tag{A.43}$$

where $\mathbb{C}_Y$ is just the field of complex numbers, with the added index Y to remind us what representation $\mathbb{C}_Y$ is the representation space of.

5 Prošireni sažetak

5.1 Matematički uvod u baždarne teorije

Ovo poglavlje se fokusiralo na matematičke alate iz teorije diferencijalne geometrije, diferencijalne topologije i spinske strukture.

Prvo je u potpoglavlju 1.1 napravljena motivacija za matematičkim opisom baždarnih teorija gdje su dani primjeri dviju baždarnih teorija. Klasična elektrodinamika i kvantna elektrodinamika kao $U(1)$ baždarne teorije.

Zatim je u potpoglavlju 1.2 napravljen uvod osnovnih pojmova iz diferencijalne geometrije - tangentni vektori, 1-forme i tenzori.

U potpoglavlju 1.3 su definirani vlaknasti svežnjevi te je tangentni svežanj prikazan kao primjer vlaknastog svežnja. Potpoglavlje je završeno definicijom morfizma vlaknastih svežnjeva i definicijom restrikcije svežnja.

Potpoglavlje 1.4 je uvelo Liejeva djelovanja koja se koriste za definiciju desnog i lijevog djelovanja u glavnim i pridruženim svežnjevima. Definirani su glavni svežnjevi i morfizmi glavnih svežnjeva te je svežanj tetrada demonstriran kao primjer glavnog svežnja koji se koristi u fizici. Definirani su i pridruženi svežnjevi te je tangenti svežanj prikazan kao pridruženi svežanj svežnju tetrada.

Duž potpoglavlja 1.5 su konstruirane najvažnije geometrijske strukture na svežnjevima. Definiran je pojam koneksije i koneksijske 1-fore te je iskazan teorem koji povezuje ta dva objekta. Zatim je definirana lokalna reprezentacija koneksijse forme na baznoj mnogostrukosti koja se ponekada zove i Yang-Mills polje. Christoffelovi simboli su pokazani kao Yang-Mills polje, a Riemmanov tenzor kao lokalna reprezentacija zakrivljenosti koneksije. Definirane su forme s vrijednostima u Ad(P) te kovarijantna derivacija.

U potpoglavlju 1.6 se klasična elektrodinamika prikazuje kao $U(1)$ baždarna teorija u formalizmu koji je razvijen kroz prijašnja potpoglavlja. Prvo se konstruira glavni $U(1)$-svežanj za elektrodinamiku. Zatim se pokaže da je A_μ Yang-Mills polje, a Faradayev tenzor lokalni prikaz zakrivljenosti koneksije. Potpoglavlje završava definicijom baždarne kovarijante derivacije za elektrodinamiku.

Prvo poglavlje završava potpoglavljem 1.7 koje uvodi spinsku strukturu. Poglavlja do sada su dovoljna da se polja baždarnih bozona definiraju na geometrijski način, ali budući da lagranžijan Standardnog modela sadrži i spinorna polja, potrebno je i njih definirati na geometrijski način. Potpoglavlje započinje nizom definicija koja vode do definicije spinornog svežnja kao pridruženog vektorskog svežnja. Potom se uvode Diracove forme, spinska kovarijantna derivacija i Diracov operator.

5.2 Standardni model

Drugo poglavlje diplomskog rada izlaže Standardni model na način poput na kolegijima Fizika elementarnih čestica, odnosno bez korištenja diferencijalne geometrije i svežnjeva.

Poglavlje započinje uvodnim potpoglavljem 2.1 o Standardnom modelu.

Potpoglavlje 2.1 je o kvantnoj teoriji polja. Izvede se Diracova jednadžba na način kao što ju je povijesno izveo Dirac. Zatim se iskaže Diracov lagranžijan i kvantizira ga se kanonskom kvantizacijom.

U potpoglavlju 2.3 se izlaže kvantna elektrodinamika (QED). Iskazuje se njezin lagranžijan i njegove jednadžbe gibanja. Potpoglavlje završava diskusijom o $U(1)$ baždarnoj simetriji lagranžijana za kvantnu elektrodinamiku.

Nastavlja se potpoglavljem 2.4 o kvantnoj kromodinamici (QCD). Radi se povijesni uvod u kojem se spominje $SU(2)$ i $SU(3)$ približna simetrija izospina. Zatim se iskazuje lagranžijan kvantne kromodinamike i iskazuju se $SU(3)$ baždarne transformacije spinornih i gluonskih polja na koje je lagranžijan QCD-a invarijantan.

Potpoglavlje 2.5 je o elektroslaboj interakciji. Potpoglavlje započinje diskusijom o diskretnim simetrijama Standardnog modela te se spominje $\mathcal{CPT}$ teorem. Radi se uvod u slabu silu, njezin povijesni razvoj te se iskazuje $SU(2)$ baždarna simetrija slabe sile. Potpoglavlje završava diskusijom o elektroslabom ujedinjenju, uvodi se Weinbergov kut θ_W i iskazuje se lagranžijan elektroslabe slike koji je invarijantan na $SU(2)$ baždarne transformacije. Navodi se da je potreban mehanizam koji uvodi masu u lagranžijan elektroslabe sile bez narušenja $SU(2)_L$ simetrije.

Zatim slijedi potpoglavlje 2.6 o Higgsovoj interakciji. Uvodi se masa baždarnih bozona slabe sile te se uvodi masa fermiona bez narušavanja $SU(2)_L$ baždarne simetrije. Potpoglavlje završava izlaganjem CKM matrice.

Drugo poglavlje završava potpoglavljem 2.7 koje iskazuje cijeli lagranžijan Standardnog modela te zapisuje ukupnu baždarnu $U(1)_Y \times SU(2)_L \times SU(3)_C$ transformaciju na koju je lagranžijan Standardnog modela invarijantan.

5.3 Geometrija Standardnog modela

Treće poglavlje diplomskog rada je o geometriji Standardnog modela. Alatima iz poglavlja 1 se reproducira lagranžijan s kraja poglavlja 2.

Potpoglavlje 3.1 je uvod u poglavlje u kojem se najavljuje da će se rigorozno definirati lagranžijan s kraja poglavlja 2.

Potpoglavlje 3.2 definira polja baždarnih bozona kao Yang-Mills polja posebnih svežnjeva. Konstruira se glavni $SU(3)_C$ svežanj čija Yang-Mills polja odgovaraju gluonima, navodi se i da je izbor konvencija takav da potpuno reproducira sva tranformacijska svojstva i predznake iz poglavlja 2. Ponovi se postupak i za $SU(2)_L$ i $U(1)_Y$-svežnjeve i time se geometrijski definiraju polja W i B bozona. Potpoglavlje završava konstrukcijom glavnog $U(1) \times SU(2) \times SU(3)$-svežnja čija Yang-Mills polja su svi baždarni bozoni iz Standardnog modela.

Nastavlja se potpoglavljem 3.3 u kojem se definiraju materijska polja fermiona. Definiraju se prikladne reprezentacije iz kojih se grade spinorni svežnjevi s transformacijsim svojstvima koja reproduciraju pojedine fermione (npr, lijevi elektron, desni kvarkovi itd.). Nastavlja se diskusijom o konjugaciji naboja koja se ostvaruje pomoću kompleksno konjugiranog svežnja, a potpoglavlje završava svežnjem čiji prerezi odgovaraju Higgsovom polju.

Potpoglavlje 3.4 geometrijski definira tri vrste lagranžijana koje u posebnim slučajevima čine osnovne komponente lagranžijana Standardnog modela. Započinje se definicijom Yang-Mills-Dirac lagranžijana koji predstavlja kinetički član za fermione i kinetički član za bozonska polja. Zatim se uvodi Higgsov lagranžijan te Yukawin lagranžijan koji uvodi mase fermiona u lagranžijan Standardnog modela. Potpoglavlje završava izlaganjem ukupnog lagranžijana Standarnog modela koji je definiran svežnjevima iz potpoglavalja 3.2 i 3.3.

Poglavlje završava potpoglavljem o opaskama u kojem se navode fundamentalne konstante koje fiksiraju Standardni model u onaj koji se opazuje u našem svemiru.

5.4 Završne opaske i perspektive

Posljednje poglavlje ovog diplomskog rada je o završnim opaskama i perspektivama.

Potpoglavlje 4.1 sumira što se napravilo u diplomskom radu.

U potpoglavlju 4.2 se govori o topološkim aspektima baždarnih teorija. Započinje se općenitom diskusijom o topološkim defektima u baždarnim teorijama koja se primjenjuje na magnetske monopole. Potpoglavlje završava diskusijom da Standardni model nema topoloških defekata, ali da neke GUT teorije dozvoljavaju njihovo postojanje.

Završno potpoglavlje 4.3 je o drugom pristupu geometriji, algebarskom. Uvodi se Gelfandova dualnost i čitatelja se upućuje prema literaturi o nekomutativnoj geometriji i Gelfandovoj dualnosti. Potpoglavlje završava kratkom diskusijom o nekomutativnoj geometriji Standardnog modela u kojoj se čitatelja upućuje prema udžbeniku koji formalizam nekomutativne geometrije koristi za konstrukciji lagranžijana Standardnog modela.

5.5 HR nazivi slika i tablica

Prijevodi opisa slika :

Slika 1.1: Möbiusova vrpca kao totalni prostor baznog prostora kružnice - tanke, okomite sive linije su projicirane na crnu krivulju.
Slika 1.2: Orbite dviju točki iz $\mathbb{R}^2$ pod djelovanjem grupe $SO(2)$.
Slika 2.1: Čestični sadržaj Standardnog modela koji uključuje mase, spinove i naboje čestica. Ovaj dijagram je napravljem na CERN Webfest-u [9].
Slika 2.2: Potencijal $V(\phi)$ smatran realnom funkcijom varijable $\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$. Ovaj graf je modifikacija open source dijagrama [17].

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