Sigma models, generalized geometry and applications in field and string theories

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Sigma modeli, generalizirana geometrija i primjene u teorijama polja i struna

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Abstract

In this thesis we study the interplay between topological sigma models and generalized geometry. Firstly, they are related through generalized gauging procedure by the notion of Lie algebroids replacing the more conventional situation of Lie algebras. Furthermore, the gauging of the 2-dimensional string models in the target space with the background metric g and a closed 3-form H is closely related to Dirac structures of exact Courant algebroids. Here we expand on this, firstly by showing that the coupling to the metric can only be minimal, and secondly by allowing the 3-form to be non-closed, a situation that arises in the context of heterotic strings. It is shown that this is again related to Dirac structures, but this time of transitive Courant algebroid.

The gauged action obtained by gauging the 2-dimensional string model is related to Dirac sigma models, which as the special case contains twisted Poisson sigma model (TPSM). Thus, Dirac sigma models can be considered as a generalization of the TPSM. Another such generalization is introduced here in the form of Jacobi sigma model, a new 2-dimensional sigma model. Geometrically, this corresponds to Jacobi structures which can be considered as a generalization of Poisson structures to which the Poisson sigma model corresponds.

The final generalization of TPSM of relevance here is that of (twisted) R-Poisson sigma model which, unlike the previously mentioned Dirac and Jacobi sigma models, is a higher dimensional one. This sigma model corresponds to R-Poisson structures, which adds an additional multibracket to the Poisson bracket of the Poisson manifolds.

Finally, the classical BV action has been constructed for the above sigma models as a first step towards their quantization. The significance of this result is that, despite being topological field theories, their BV action cannot obtained through the AKSZ construction due to the absence of the QP-structure on the target space, thus requiring different methods for the constructions of the BV action. Thus, the methods used here lead to the classical BV action for two significant theories (Dirac and R-Poisson sigma models), but also directly show different strategies for the construction of the BV action when more convenient methods are not available.

Keywords: Sigma models, Gauging procedure, Lie algebroids, Courant algebroids, Dirac sigma model, Jacobi sigma model, Batalin-Vilkovisky formalism

Prošireni sažetak

Teorija struna jedan je od glavnih kandidata moguće teorije kvantne gravitacije koja bi ujedno i davala ujedinjenu teoriju svih temeljnih interakcija u prirodi. Opća teorija relativnosti, koja vrlo uspješno opisuje gravitacijsku interakciju na klasičnoj razini, sadržana je u teoriji struna gdje se pojavljuje kao niskoenergijski limes. S tim na umu, u ovom radu detaljnije opisujemo određene aspekte topoloških sigma modela, koji mogu proizaći iz određenih modela unutar teorije struna.

Kao početnu točku uzimamo model koji opisuje propagaciju strune u ciljnom prostoru M koji kao pozadinska polja sadrži metriku g i zatvorenu 3-formu H. Pripadna akcija koja opisuje ovu propagaciju dana je s:

$$S[X] = -\int_{\Sigma_2} \frac{1}{2} g_{ij}(X) \mathrm{d}X^i \wedge * \mathrm{d}X^j - \int_{\Sigma_3} \frac{1}{3!} H_{ijk}(X) \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \,,$$

gdje Σ_2 označava 2-dimensionalnu svjetsku opnu strune, dok je Σ_3 3-dimenzionalna mnogostrukost čiji je rub jednak Σ_2. Kao stupnjevi slobode u ovoj teoriji nam služe polja $X : \Sigma_2 \to M$ koja opisuju položaj strune unutar ciljnog prostora M i njihova dinamika tada opisuje propagaciju strune. Iz ovog modela tada tražimo pripadnu baždarnu teoriju. Međutim, za razliku od više tradicionalnog pristupa u kojem bismo prvo identificirali simetrije početne teorije i zatim ih baždarili u obliku minimalnog vezanja, ovdje koristimo drugačiju metodu. Generalno, baždarna teorija inducira folijaciju na ciljnoj mnogostrukosti. Stoga, umjesto da krećemo od simetrija teorije, ovdje ćemo krenuti od neke zadane folijacije na M i postaviti pitanje je li moguće pronaći pripadnu baždarnu teoriju. Rezultat ovakvog pristupa se pokazao puno općenitijim i ne zahtijeva postojanje ikakve simetrije od samog početka. Unatoč tome, ovakav pristup je pokazao da postoji neka baždarna teorija.

Pod pretpostavkom minimalnog vezanja na metriku, ovakav pristup daje baždarnu teoriju opisanu akcijom:

$$S[X,A] = -\int_{\Sigma_2} \left(\frac{1}{2} g_{ij}(X) F^i \wedge * F^j + A^a \wedge \theta_a(X) + \frac{1}{2} \gamma_{ab}(X) A^a \wedge A^b \right) - \int_{\Sigma_3} H(X) \,,$$

gdje su A^a baždarna polja koja su 1-forme na svjetskoj opni te poprimaju vrijednosti u nekom Liejevom algebroidu $(L, \rho, [\cdot,\cdot])$ nad M. Ovdje su γ_{ab} i θ_a funkcija i 1-forma nad M, respektivno. Pripadne baždarne transformaciju za ovu baždarnu teoriju glase:

$$\begin{aligned}
\delta X^i &= \rho^i_a \varepsilon^a , \\
\delta A^a &= \mathrm{d}\varepsilon^a + C^a{}_{bc} A^b \varepsilon^c + \omega^a_{bi} \varepsilon^b F^i + \phi^a_{bi} \varepsilon^b * F^i ,
\end{aligned}$$

gdje su $C^a{}_{bc}$ strukturne funkcije Liejeve zagrade Liejevog algebroida. Može se pokazati da je ovakvo baždarenje moguće pod uvjetom da $\rho + \theta$ predstavljaju prereze Diracove strukture generaliziranog tangentnog svežnja $TM \oplus T^*M$ promatranog kao Courantov algebroid. Nadalje, $\omega \pm \phi$ iz baždarne transformacije od A nisu jedinstveno određeni, nego samo moraju predstavljati komponente koneksije na L pri čemu je njihova torzija dobro definirana pozadinskim poljima.

U ovom radu želimo generalizirati opisani baždarni proces. Prvo, promatramo mogućnost neminimalnog vezanja na metriku te pokazujemo da je u tim slučajevima moguće redefinirati baždarno polje tako da se vratimo na oblik minimalno vezane akcije, na taj način pokazujući da se baždarenje ne može generalizirati u tom obliku. Nadalje, iduću generalizaciju koju provodimo je proširivanje pozadinske strukture, u smislu da dopuštamo 3-formi H da bude nezatvorena. U ovom slučaju generalizacija dovodi do novih struktura tako da su baždarni podaci i dalje ograničeni na Diracovu strukturu, ali općenitijeg Courantovog algebroida. Umjesto generaliziranog tangentnog svežnja, u ovom slučaju se pojave tranzitivni Courantovi algebroidi, koji uključuju dodatni svežanj kvadratnih Liejevih algebri i generalno imaju oblik $\mathcal{F} \oplus \mathcal{G} \oplus \mathcal{F}^*$, gdje je $\mathcal{F}$ podsvežanj tangentnog svežnja jednak slici funkcije usidrenja ρ Liejevog algebroida, a $\mathcal{G}$ navedeni svežanj kvadratnih Liejevih algebri.

Nadalje, u kontekstu dvodimenzionalnih sigma modela, uvodimo novi model nazvan Jacobijev sigma model. On predstavlja generalizaciju Poissonovog sigma modela, slično kao što Jacobijeva mnogostrukost predstavlja generalizaciju Poissonove mnogostrukosti. Ona je povezana s prijašnjim postupkom baždarenja na dva načina. Prvo, kada kao $\mathcal{G}$ uzmemo Liejevu algebru Abelove grupe, dobiveni Courantov algebroid je kontaktna mnogostrukost koja je uvijek opremljena s pripadnom Jacobijevom strukturom. Drugo, oba sigma modela moguće je prikazati kroz dimenzionalnu redukciju ranije poznatih sigma modela. U slučaju baždarne teorije s nezatvorenom 3-formom H, riječ je o dimenzionalnoj redukciji Diracovog sigma modela, dok je u Jacobijevom slučaju riječ o dimenzionalnoj redukciji Poissonovog sigma modela, koji je specijalan slučaj Diracovog sigma modela.

Specifično, Jacobijev sigma model opisan je akcijom:

$$S_{\mathrm{JSM}}[X,\Phi,A,A_0] = \int_{\Sigma_2} A_i \wedge \mathrm{d}X^i + A_0 \wedge \mathrm{d}\Phi + \tfrac{1}{2} e^{-\Phi}\Pi^{ij}(X) A_i \wedge A_j + e^{-\Phi} V^i(X) A_0 \wedge A_i \,,$$

s pripadnim baždarnim transformacijama:

$$\begin{aligned} \delta X^i &= e^{-\Phi}\left(\Pi^{ji}\varepsilon_j + V^i \varepsilon_0\right) \,, \\ \delta\Phi &= -e^{-\Phi} V^i \varepsilon_i \,, \\ \delta A_i &= \mathrm{d}\varepsilon_i + e^{-\Phi}\partial_i \Pi^{jk} A_j \varepsilon_k - e^{-\Phi}\partial_i V^j \left(A_j \varepsilon_0 - A_0 \varepsilon_j\right) \,, \\ \delta A_0 &= \mathrm{d}\varepsilon_0 - e^{-\Phi}\Pi^{jk} A_j \varepsilon_k + e^{-\Phi} V^j \left(A_j \varepsilon_0 - A_0 \varepsilon_j\right) \,. \end{aligned}$$

Ovdje Π i V definiraju Jacobijevu strukturu na ciljnoj mnogostrukosti tako da vrijedi:

$$\begin{aligned} [\Pi,\Pi] &= -2V \wedge \Pi \,, \\ [\Pi,V] &= 0 \,, \end{aligned}$$

gdje je $[\cdot,\cdot]$ Schouten-Nijenhuisova zagrada. Za razliku od Poissonovog sigma modela, Jacobijev sigma model uključuje dva dodatna polja Φ i A_0, prvo od kojih je moguće dobiti kroz dimenzionalnu redukciju Poissonovog sigma modela u kojem slučaju predstavlja dodatnu koordinatu.

Konačno, u posljednjem poglavlju promatramo proces kvantizacije određenih sigma modela, specifično Diracovog sigma modela i R-Poissonovog sigma modela. U oba slučaja, riječ je o topološkim baždarnim teorijama polja s otvorenim algebrama, odnosno algebrama koje su zatvorene samo do na jednadžbe gibanja. Činjenica da su im baždarne algebre otvorene zahtjeva upotrebu Batalin-Vilkovisky (BV) formalizma. Specifično, ovdje se koncentriramo na pronalaženje klasične BV akcije za ova dva modela. Općenito, budući da su ovo topološke teorije polja, obično bi najbolji pristup ovom problemu uključivao AKSZ konstrukciju, koja ima geometrijski princip u konstrukciji klasične BV akcije. Međutim, AKSZ konstrukcija se oslanja na QP-strukturu na ciljnoj mnogostrukosti koja je narušena za dva navedena modela zbog prisutnosti Wess-Zumino člana. Stoga AKSZ konstrukciju nije moguće upotrijebiti i bez ikakve druge poznate metode, ovdje smo morali pronaći način kako dobiti traženi rezultat. U slučaju Diracovog sigma modela koristili smo razvoj klasične BV akcije po broju antipolja te zatim rješavali klasičnu glavnu jednadžbu red po red. Ovaj pristup je u ovom slučaju moguće iskoristiti jer je Diracov sigma model dvodimenzionalan. To znači da razvoj BV akcije u broju antipolja završava s drugim članom. Viši članovi ne postoje jer ne postoje skalarna antipolja te stoga nije moguće konstruirati član s 3 antipolja koji je ujedno i 2-forma na svjetskoj opni. S druge strane, nulti član jednostavno predstavlja klasičnu akciju, dok je prvi član određen baždarnim simetrijama klasične akcije, odnosno specifičnije, pripadnim BRST transformacijama. Stoga je jedini nepoznati član u razvoju drugi član koji sadrži dva antipolja.

S druge strane, kod R-Poissonovog sigma modela razvoj po broju antipolja se pokazao nepraktičnim jer je R-Poissonov sigma model $(p+1)$-dimenzionalan. Stoga će razvoj po antipoljima uključivati članove do $(p+1)$-vog reda. Stoga, ovdje koristimo drugačiju taktiku. Umjesto direktne konstrukcije klasične BV akcije S_{BV}, konstruiramo BV operator $s=(S_{BV},\cdot)$, gdje je $(\cdot,\cdot)$ antizagrada na prostoru polja i antipolja. Osnovna svojstva ovog operatora su da je nilpotentan, tj $s^2=0$ te da u limesu kada su sva antipolja jednaka nuli, mora reproducirati poznati BRST operator. Konstrukcijom takvog operatora ustanovljeno je da nije jedinstveno određen te da postoji više različitih operatora koji zadovoljavaju navedena svojstva. Za određivanje pravog BV operatora, trebalo je iskoristiti njegovu osnovnu definiciju. Iako postoji više mogućih operatora koji su nilpotentni i daju prvi limes kada antipolja iščezavaju, pokazalo da samo jedan od njih je moguće prikazati u obliku antizagrade s nekom funkcijom. Stoga je to jedina mogućnost za BV operator dok je pripadna funkcija u antizagradi upravo tražena BV akcija.

Ključne riječi: Sigma modeli, Baždarenje, Liejev algebroid, Courantov algebroid, Diracov sigma model, Jacobijev sigma model, Batalin-Vilkovisky formalizam

Contents

Chapter 1

Introduction

String theory [1, 2] is one of the main frameworks that tries to unify gravity with the rest of the fundamental interactions. It postulates the existence of 1-dimensional objects (strings) instead of point particles. As a result of this first postulate, gravity emerges quite naturally from the quantization of the closed strings, without the need to explicitly add it to the theory. What is more, the general relativity, a classical theory of gravity, is perfectly reproduced by string theory in the low energy limit, thus giving the necessary credibility.

The extended degrees of freedom in string theory give rise to extra symmetries not seen previously, such as T-duality [3, 4]. It relates two different target spaces with the full equivalence of the string spectra upon exchange of the momentum and winding modes. As a result, it is possible for strings to propagate even in non-geometric backgrounds [5], in the context of flux compactification which cannot be described with standard geometric methods. T-dualizing certain backgrounds can create situations in which the fields cannot be patched using the usual geometrical symmetries (diffeomorphisms and gauge symmetries). In order to give a proper geometrical interpretation here, one needs to consider a bit more general structures, specifically complex and generalized geometry [6, 7]. For example, it was shown that the topological T-duality can be interpreted as an isomorphism between Courant algebroids [8].

One of the emerging models in string theory is the 2-dimensional (non-linear) sigma model which serves in this context as an effective theory for string interactions, though sigma models were first introduced in [9] in a different setting, serving as an attempt to describe charged pion decay rates. The model describes the spontaneous symmetry breaking, with the scalar field in the sigma model being the corresponding Goldstone boson. In a bit more general situation, sigma models can be used to describe scalar Goldstone bosons for different situations of spontaneously broken symmetries. However, in this thesis we shall be concerned mostly with its relations to strings.

Of special importance here is the class of topological sigma models, many of which can be related to string theory. A better known example of such instances are A/B topological string models [10, 11]. Another example of topological sigma model is the Chern-Simons theory [12], which is a 3-dimensional sigma model that can be used to reformulate 3-dimensional gravity as a gauge theory. Finally, another important example of special interest here is that of Poisson sigma model [13, 14] which is a 2-dimensional sigma model. Originally, Poisson sigma model arose as an attempt to unify different 2-dimensional gravity theories, e.g. dilaton gravity and R^2 gravity. It was later shown in [15, 16] that its corresponding 3-point function gives the Kontsevich's formula for the star product [17] in the context of deformation quantization of Poisson manifolds. It can also be related to topological sigma models like the A/B models mentioned above. On the more geometrical side, the Poisson sigma model gives realisation of the Poisson manifold structure in the form of gauge symmetry of the model.

In this thesis we shall consider three possible generalizations (of different type) of the (twisted) Poisson sigma model. First is that of Dirac sigma models [18], which are 2-dimensional sigma models relating geometrically to the Dirac manifolds [19] in a similar way the (twisted) Poisson sigma models relate to (twisted) Poisson manifolds. On the other hand, Dirac sigma models also serve as a generalization of G/G Wess-Zumino-Witten (WZW) model [20] which is the way they were first introduced. However, Dirac sigma models can emerge independently of the (twisted) Poisson sigma model through the gauging of the string models with the metric g and a closed 3-form H as the background [21, 22]. It turns out that the Dirac structures of an exact Courant algebroid are related to the gauging consistency [23, 24, 25, 26, 27]. As a further generalizations, one can consider situations when the 3-form H is not closed, as was assumed in all of the mentioned work. However, if one considers the heterotic string theory, the 3-form can obtain α'-corrections and thus making it non-closed. Such generalization has been studied in [28], being one of the papers upon which this thesis is formulated.

The gauging procedure and Dirac sigma models give rise to the structure of Courant algebroid [29], where such structure arises naturally. It has been shown that the specific class of Courant algebroid one obtains through the gauging procedure is directly related to the gauge symmetries of the model whether one studies closed or non-closed 3-form backgrounds. In the case of closed 3-form, one always obtains what is known as exact Courant algebroids, while non-closed 3-form produces a much wider and more general class of non-exact Courant algebroids, which can in turn produce a much richer gauge theory. Specifically, the non-exact Courant algebroid include additional group structure that can be implemented in the theory [30], a fact that is especially important for heterotic strings which require such additional gauge symmetries.

The second generalization of the Poisson sigma model is that of the Jacobi sigma model [28, 31]. Jacobi manifolds are a natural generalization of the Poisson manifolds [32] in several different ways outlined in the first chapter. What is more, the Jacobi manifolds arise as a generalization of the traditional Hamiltonian mechanics. Specifically, in the point-particle classical mechanics, one conventionally studies only non-dissipative systems, at least in the regards with Hamilotnian formulation. It has been shown in [33, 34] that this can be extended to include dissipative systems as well, in which case the Jacobi geometry replaces the more traditional Poisson one, thus showing the interplay between the two in a physical setting. Since the Poisson manifolds can be related to Poisson sigma models, it is explored here whether the same kind of correspondence exists for Jacobi manifolds, thus giving rise to Jacobi sigma model. This is considered a generalization of the Poisson since every Poisson sigma model is also a Jacobi sigma model in the same way that every Poisson manifold is a special case of the Jacobi manifold. However, this is true if one considers only untwisted Poisson/Jacobi structures. While it turns out to be possible to twist a Jacobi sigma model, one cannot recover the twisted Poisson sigma model from it.

The final generalization of the Poisson sigma model considered here is that of the R-Poisson sigma model [35]. Unlike the previous generalizations in which the resulting sigma model was also a 2-dimensional one, here one attempts to generalize Poisson sigma model to $p+1$ dimensions. The standard Poisson manifold is equipped with a Poisson bivector Π satisfying:

$$[\Pi,\Pi] = 0\,, \tag{1.0.1}$$

where the bracket is the Schouten-Nijenhuis bracket. In order to generalize this to higher dimensions, but still retaining some Poisson structure, one generalizes only one of the Πs in the above expression to a completely antisymmetric $(p+1)$-multivector R, thus giving the so called R-Poisson structure:

$$[\Pi,R] = 0\,. \tag{1.0.2}$$

Notice that in this case, one cannot recover the original Poisson structure from the R-Poisson one just by going to appropriate dimension. This is because in attempting to generalize Poisson manifold to higher dimensions, a new background field had to be introduced. Just by going to 2 dimensions does not force this multivector R to equal Poisson. Instead, one obtains a 2-dimensional structure with 2 bivectors whose Schouten-Nijenhuis bracket vanish. In order to recover the Poisson, one needs to add extra condition $R=\Pi$ by hand. Similar things happen when considering the twisted versions of these manifolds. Nevertheless, just as one can construct the Poisson sigma model from the Poisson manifold, it is also possible to construct the corresponding sigma model, called the R-Poisson sigma model, from the R-Poisson manifold.

Sigma models discussed up to now are all gauge theories with open gauge algebras, meaning that the algebra closes only on-shell. Thus, if one wishes to discuss quantum versions of these models, it would be necessary to employ the Batalin-Vilkovisky (BV) quantization [36, 37, 38, 39]. It is a general method for quantization of the theory with gauge redundancies, and in the case of theories with open gauge algebras or those with reducible gauge structure, it is the only viable option. In [40] was shown that the solutions to the classical master equation correspond to graded supermanifolds with a QP-structure. These are graded supermanifolds equipped with a cohomological vector field Q and a Q-invariant graded symplectic structure P. This structure is used to build a whole array of topological field theories in different dimensions. As an example, the Poisson sigma model shows up as the first model in this array [13, 14], while the second one corresponds to the so called Courant sigma model [41, 42, 43, 44]. Despite the success of AKSZ construction, it relies on the assumption of the QP-structure. Thus, it cannot be used for all topological field theories. It turns out that even though Poisson and Courant sigma models posses the required QP-structure, their twisted versions do not. The addition of the Wess-Zumino term to the action breaks the QP-structure, thus rendering the AKSZ construction unusable. Up to present date, no formal method exists that could replace the AKSZ in such instances. The absence of such method might be the reason that the classical BV action of the twisted Poisson sigma model was only found recently in [45], despite the untwisted version already being known in [40]. As a result, here we consider the construction of the BV action for two theories: Dirac sigma model [46] and R-Poisson sigma model [47]. First of those has a twisted Poisson sigma model as its special case, so the said construction follows similarly as the one for the twisted Poisson version. For the R-Poisson, we have employed a different option. Instead of constructing the BV action directly, one tries to construct the corresponding BV operator. As a result, we managed to find a closed form for the BV operator in the untwisted case, and the full BV operator (and the corresponding BV action) for the 3-dimensional twisted case, which also a special case of the Courant sigma model.

In chapter 2 we introduce the main mathematical concepts that shall be used throughout the rest of this thesis. We start by considering Poisson and Jacobi geometry, along with some motivation for those structures in physics. After that the concepts of Lie algebroid and Courant algebroid are introduced, both of which become important when considering gauging procedures. Finally, as a preparation for the BV formalism, we discuss the basics of the graded geometry.

In chapter 3 we start with a brief overview of string theory, which serves as a starting point for the construction of the Dirac sigma models through the gauging of string models. The gauging procedure here is done in three different steps, each of which included extra relaxation of the assumption. Thus we managed to provide an improvement on the generalized gauging theorem presented in [26]. The obtained gauged action is then related to Dirac sigma models. Being the special case of the latter, the Poisson sigma model is presented as well. After that, we discuss other generalizations of the Poisson sigma model: Jacobi sigma model and R-Poisson sigma model.

In the fourth chapter, we start with a brief overview of the BV formalism. After that we discuss the construction of the BV action for the Dirac sigma models through antifield expansion. Finally, we finish with the construction of the BV action for the R-Poisson sigma model by constructing the BV operator. Here, the BV action was constructed fully only for the 3-dimensional case, while in higher dimensions, only the untwisted version was considered, in which case the closed form of the BV operator is presented.

We finish this thesis with a conclusion and an outlook.

Chapter 2

Generalized Geometry

In this chapter we give a summary of various mathematical concepts that shall be used in later chapters. First, a short overview of Poisson geometry is given, along with a physical motivation for the study of such geometry. Next, Jacobi geometry is introduced as a generalization of the Poisson geometry, including both mathematical generalization and that of physical motivation. Both of these serve as a starting point for the introduction of certain sigma models, specifically, Poisson, Jacobi and R-Poisson sigma models. After that, the concepts of Lie and Courant algebroids are discussed, since they play an important role in the study of generalized gauging procedure and gauging conditions for the two-dimensional sigma models that will be discussed in the next chapter. Finally, a short introduction into graded geometry is presented, which will be used in the final chapter in the context of Batalin-Vilkovisky formalism.

2.1 Poisson geometry

Poisson geometry emerges very naturally in the study of Hamiltonian systems and as such, has a very strong physical motivation. Here we shall mostly follow the work presented in [48, 49]. Consider a non-dissipative, Hamiltonian system. The set of states of such a system is its phase space M, which has a geometric structure of a symplectic manifold. In general, symplectic manifold is an even-dimensional manifold endowed with a symplectic form ω, which is a non-degenerate, closed 2-form. Non-degenerate here means that ω, when viewed as a map from $\Gamma(TM)$ to $\Gamma(T^*M)$, is an isomorphism. Furthermore, this non-degeneracy also implies that:

$$\omega^n \neq 0\,, \tag{2.1.1}$$

where $2n$ is a dimension of M, and the power of a symplectic form is taken with regards to the exterior product $\wedge$.

To give a more direct connection with physical interpretation, one can make use of the Darboux theorem, which states that on a symplectic manifold there exist locally a coordinate system $q^1,\dots,q^n,p_1,\dots,p_n$ in which the symplectic form takes the form:

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i\,. \tag{2.1.2}$$

In terms of classical mechanics, these coordinates can be interpreted as generalized coordinates and generalized momenta for the system. Furthermore, since ω is closed, locally there exist a 1-form α such that $\omega = \mathrm{d}\alpha$. In Darboux coordinates one can take α to be:

$$\alpha = p_i \mathrm{d}q^i\,. \tag{2.1.3}$$

While this choice isn't unique, for the present purposes, the ambiguity in the choice of α is irrelevant.

An important notion when considering Hamiltonian systems is that of a Hamiltonian function $H \in C^\infty(M)$, which serves as a generator for the possible physical trajectories of the system. First, associated to any Hamiltonian function is a Hamiltonian vector field $X_H \in \Gamma(TM)$ defined through:

$$\iota_{X_H}\omega = -\mathrm{d}H\,. \tag{2.1.4}$$

In Darboux coordinates, the Hamiltonian vector field takes the form:

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}\,. \tag{2.1.5}$$

The trajectories of the system are then just the integral curves of the Hamiltonian vector field. This gives the very well known Hamilton equations of motion in Darboux coordinates:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}\,, \tag{2.1.6}$$

where the dot denotes a derivative with respect to the curve parameter.

Other than the Hamiltonian vector field, the Hamilton equations of motion can be obtained through the use of the Poisson bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ as well. Its form in

Darboux coordinates is very well known:

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\,, \tag{2.1.7}$$

for all functions $f,g \in C^\infty(M)$. Thus, a derivative of any function on M along a trajectory is then given as the Poisson bracket of that function with the Hamilton function. Specially, this is the case for the coordinates themselves, thus producing Hamilton equations.

The Poisson bracket forms an algebra on the space of functions, known as Poisson algebra. It is a skew-symmetric bracket that satisfies the Jacobi identity, so it can be classified as a Lie algebra as well. However, the form of the bracket given by (2.1.7) has a major drawback in the fact that is given in terms of coordinates. However, this can be easily remedied. First, one can introduce a bivector $\Pi \in \Gamma(\wedge^2 TM)$ such that:

$$\{f,g\} = \Pi(f,g)\,, \tag{2.1.8}$$

which then gives its form in Darboux coordinates:

$$\Pi = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}\,. \tag{2.1.9}$$

This bivector is antisymmetric since the Poisson bracket is skew-symmetric, while the Jacobi identity of the Poisson bracket translates to:

$$\Pi^{d[c}\partial_d \Pi^{ab]} = 0\,, \tag{2.1.10}$$

where the indices a,b,c,d can take values for all the Darboux coordinates. But one can recognize this as Schouten-Nijenhuis bracket of Π with itself:

$$[\Pi,\Pi]_{\mathrm{SN}} = -\Pi^{dc}\partial_d \Pi^{ab}\partial_a \wedge \partial_b \wedge \partial_c\,, \tag{2.1.11}$$

which is of the same form in any coordinates, not necessarily just Darboux. Thus, Π can be defined as an antisymmetric bivector for which the Schouten-Nijenhuis bracket with itself vanishes, i.e. $[\Pi,\Pi]_{\mathrm{SN}} = 0$. This kind of bivector is known as the Poisson bivector, and in general, any manifold endowed with a Poisson bivector is called a Poisson manifold.

While the Poisson bivector is now successfully defined in coordinate free fashion, its definition does not relate it to the symplectic form. Actually, one can define different Poisson bivectors on any symplectic manifold and each one would produce their own Poisson bracket.

However, for the correct physical interpretation, one needs a specific Poisson bracket, and thus, a specific Poisson bivector. By a straightforward calculation in Darboux coordinates, one can check that the necessary Poisson bivector and the symplectic form satisfy the following identity:

$$\iota_{\Pi(\xi)}\omega = -\xi\,, \qquad \forall\xi \in \Gamma(T^*M)\,. \tag{2.1.12}$$

Furthermore, it turns out that this condition is enough to recover the Darboux form of Π from the Darboux form of ω, thus giving the required association. In addition, this also proves that any symplectic manifold is also a Poisson manifold.

2.2 Jacobi geometry

While the previous section gives a nice geometric interpretation of classical mechanics, it is valid only for non-dissipative, time-independent systems. Time-dependency can be easily introduced and has been extensively studied in [49, 50, 51]. However, all those works focus only on non-dissipative systems. This has been remedied in [33, 34], results of which are presented here.

Instead of symplectic manifold, one defines a contact manifold M when dealing with dissipative systems. It is a $(2n+1)$-dimensional manifold endowed with a contact form η, which is a 1-form such that:

$$\eta \wedge \mathrm{d}\eta^n \neq 0\,. \tag{2.2.1}$$

Here, η serves as a generalization of α in symplectic geometry. As such, this condition is a generalization of the non-degeneracy of the symplectic form.

Like for symplectic manifolds, there is version of Darboux theorem for contact manifolds. It states that locally there exists a system of coordinates $\{q^1,\dots,q^n,p_1,\dots p_n,S\}$ in which the contact form is expressed as:

$$\eta = \mathrm{d}S - p_i\mathrm{d}q^i\,. \tag{2.2.2}$$

Notice that:

$$\mathrm{d}\eta = -\mathrm{d}p_i \wedge \mathrm{d}q^i := -\omega\,, \tag{2.2.3}$$

so projecting out the S coordinate locally recovers the structure of the symplectic manifold.

Contact form defines an isomorphism $g : \Gamma(TM) \to \Gamma(T^*M)$ such that:

$$g(X) = \iota_X\mathrm{d}\eta + (\iota_X\eta)\eta\,, \qquad \forall X \in \Gamma(TM)\,. \tag{2.2.4}$$

This isomorphism can also be viewed as tensor in $\Gamma(\otimes^2 TM)$ in which case it takes the form:

$$g = \frac{1}{2}\eta \vee \eta - \omega. \tag{2.2.5}$$

Since g is an isomorphism, it has a well defined inverse. One then defines the so called Reeb vector field $V \in \Gamma(TM)$ as:

$$V = g^{-1}(\eta)\,. \tag{2.2.6}$$

This is equivalent to the statement that:

$$\iota_V \eta = 1\,, \qquad \iota_V \mathrm{d}\eta = 0\,. \tag{2.2.7}$$

In Darboux coordinates, the Reeb vector takes the form:

$$V = \frac{\partial}{\partial S}\,. \tag{2.2.8}$$

Thus, it can be interpreted as giving direction to the *extra* dimension.

As was the case for the symplectic manifolds, one can introduce a Hamiltonian function $\mathcal{H} \in C^\infty(M)$, also sometimes referred to as contact Hamiltonian function in this setting, and the corresponding Hamiltonian vector field $X_\mathcal{H} \in \Gamma(TM)$ through:

$$\mathcal{H} = -\iota_{X_\mathcal{H}}\eta\,, \qquad \mathcal{L}_{X_\mathcal{H}}\eta = f\eta\,, \tag{2.2.9}$$

for some function $f \in C^\infty(M)$. This function appears arbitrary, but is in fact fully defined by the above expression:

$$f = -\iota_V \mathrm{d}\mathcal{H}\,. \tag{2.2.10}$$

In Darboux coordinates, the function f and the Hamiltonian vector field take to form:

$$f = -\frac{\partial \mathcal{H}}{\partial S}\,, \tag{2.2.11}$$

$$X_\mathcal{H} = \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial \mathcal{H}}{\partial q^i} + p_i\frac{\partial \mathcal{H}}{\partial S}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}\right)\frac{\partial}{\partial S}\,. \tag{2.2.12}$$

Provided that the trajectories of the system are defined as integral curves of the Hamiltonian vector field (which is the whole reason why $\mathcal{H}$ is called a Hamiltonian function), the appropriate equations of motion are then:

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i} - p_i \frac{\partial \mathcal{H}}{\partial S}\,, \qquad \dot{S} = p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}\,. \tag{2.2.13}$$

The derivative of an arbitrary function $F \in C^\infty(M)$ is then given by (in Darboux coordinates):

$$\dot{F} = \left(\frac{\partial F}{\partial q^i} + p_i \frac{\partial F}{\partial S}\right)\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial F}{\partial p_i}\left(\frac{\partial \mathcal{H}}{\partial q^i} + p_i \frac{\partial \mathcal{H}}{\partial S}\right) - \mathcal{H}\frac{\partial F}{\partial S}\,. \tag{2.2.14}$$

In the correspondence to the Poisson case, the time derivative of an arbitrary function along the trajectory prompts the definition of the so called Jacobi bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$:

$$\{F,G\} = \left(\frac{\partial F}{\partial q^i} + p_i \frac{\partial F}{\partial S}\right)\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\left(\frac{\partial G}{\partial q^i} + p_i \frac{\partial G}{\partial S}\right)\,, \tag{2.2.15}$$

and, as in the Poisson case, it is governed by the appropriate bivector $\Pi \in \Gamma(\wedge^2 TM)$ which in Darboux coordinates takes the form:

$$\Pi = \left(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial S}\right) \wedge \frac{\partial}{\partial p_i}\,. \tag{2.2.16}$$

As the Poisson bivector, the Jacobi bivector is antisymmetric because the Jacobi bracket is skew-symmetric. However, unlike the Poisson bracket, the Jacobi bracket does not satisfy the Jacobi identity, but its breaking is controlled by the Reeb vector field:

$$\{\{F,G\},H\} + c.p. = -\frac{1}{2}\{G,H\}\iota_V \mathrm{d}F + c.p.\,, \tag{2.2.17}$$

where c.p. denotes cyclic permutations in F, G and H. Furthermore, the Reeb vector does not only control the breaking of the Jacobi identity, it is also a derivative for the Jacobi bracket:

$$\iota_V \mathrm{d}\{F,G\} = \{\iota_V \mathrm{d}F, G\} + \{F, \iota_V \mathrm{d}G\}\,. \tag{2.2.18}$$

These two properties of the Jacobi bracket translate to certain properties of the bivector Π, which can again be written through the use of Schouten-Nijenhuis bracket:

$$[\Pi,\Pi]_{\mathrm{SN}} = -2V \wedge \Pi\,, \tag{2.2.19}$$

$$[\Pi,V]_{\mathrm{SN}} = 0\,. \tag{2.2.20}$$

For the sake of completeness, we also note the expression for the Schouten-Nijenhuis bracket of Π and V in arbitrary local coordinates:

$$[\Pi, V]_{\mathrm{SN}} = -\left(\frac{1}{2}V^c\partial_c\Pi^{ab} + \Pi^{ca}\partial_c V^b\right)\partial_a \wedge \partial_b\,. \tag{2.2.21}$$

In general, any bivector Π that satisfies (2.2.19) and (2.2.20) is known as the Jacobi bivector, and any manifold endowed with the Jacobi bivector and the appropriate Reeb vector field V is known as the Jacobi manifold. The properties of the Jacobi manifolds presented here follow what is shown in [32, 52, 53].

Finally, it remains to find a systematic, coordinate-free way to assign appropriate Jacobi bivector to any contact form η, as was done for the Poisson case. This can be done nicely through the use of the isomorphism g, so that the sufficient and necessary condition turns out to be:

$$g(\Pi(\xi)) = \xi - (\iota_V\xi)\eta\,, \qquad \forall\xi \in \Gamma(T^*M)\,. \tag{2.2.22}$$

This is equivalent to:

$$\iota_{\Pi(\xi)}\mathrm{d}\eta = \xi - (\iota_V\xi)\eta \qquad \text{and} \qquad \iota_{\Pi(\xi)}\eta = 0\,. \tag{2.2.23}$$

In addition, the Jacobi bivector can be used to express the inverse of isomorphism g in a closed form:

$$g^{-1} = \Pi + \frac{1}{2}V \vee V\,. \tag{2.2.24}$$

2.2.1 Relation between Poisson and Jacobi manifolds

As have been seen so far, Poisson and Jacobi structures are similar to each other, one relating to the symplectic structure and the other to the contact structure. One may even be tempted to say that the Poisson manifolds are even-dimensional, while the Jacobi manifolds are odd-dimensional. That is not the case however as can be shown very easily. Suppose M is a Poisson manifold with a Poisson bivector $\mathcal{P}$ and let $f \in C^\infty(M)$ be an arbitrary function. A straightforward calculation then gives:

$$[f\mathcal{P}, f\mathcal{P}]_{\mathrm{SN}} = 2f\mathcal{P} \wedge \mathcal{P}(f)\,. \tag{2.2.25}$$

Identifying $\Pi = f\mathcal{P}$ and $V = -\mathcal{P}(f)$, this expression becomes:

$$[\Pi, \Pi]_{\mathrm{SN}} = -2V \wedge \Pi\,. \tag{2.2.26}$$

In addition, it is easy to check that:

$$[\Pi, V]_{\mathrm{SN}} = 0\,. \tag{2.2.27}$$

This means that it is always possible to construct Jacobi bivector from the Poisson bivector, and additionally, that all Poisson manifolds are also Jacobi manifolds, and not only in the trivial sense (when the Jacobi bivector is just Poisson and the Reeb vector field vanishes).

The above procedure can also be somewhat reversed. Given a Jacobi structure (M, Π, V), if there exist a function $f \in C^\infty(M)$ such that $\Pi(f) = 0$, then:

$$[f\Pi, f\Pi]_{\mathrm{SN}} = 0\,, \tag{2.2.28}$$

meaning that $f\Pi$ is a Poisson bivector. Of course, in this case one cannot say that every Jacobi manifold is also a Poisson manifold. That depends on the existence of the function f.

Despite the fact that Jacobi structure on a manifold M does not necessarily lead to a Poisson structure, it is possible to construct a new manifold that is necessarily Poisson. Let $\widetilde{M} = M \times \mathbb{R}$ and Φ a coordinate on $\mathbb{R}$. By defining a bivector:

$$\widetilde{\mathcal{P}} = e^{-\Phi}\left(\Pi + V \wedge \frac{\partial}{\partial \Phi}\right)\,, \tag{2.2.29}$$

one can easily check that the Schouten-Nijenhuis bracket of this bivector with itself vanishes, thus making it a Poisson bivector. This process is called a Poissonization of the Jacobi structure and it corresponds to dimensional oxidation, i.e. the procedure of obtaining a higher-dimensional theory from the lower-dimensional one. In the special case when the Jacobi bivector was obtained from the Poisson bivector $\mathcal{P}$, such that $\Pi = f\mathcal{P}$ and $V = -\mathcal{P}(f)$, we obtain the dimensional oxidation of the Poisson structure:

$$\widetilde{\mathcal{P}} = e^{-\Phi}\left(f\mathcal{P} - \mathcal{P}(f) \wedge \frac{\partial}{\partial \Phi}\right)\,, \tag{2.2.30}$$

which shows that the Poisson structure can exist in any dimension, not just even ones.

2.3 Lie algebroids

The concept of Lie algebroids was first introduced in [54] and researched in more detail in [55, 7]. It can be defined as a vector bundle L over a manifold M, equipped with a Lie bracket

$[\cdot,\cdot]_L : \Gamma(L)\times\Gamma(L)\to\Gamma(L)$ and an anchor map $\rho : L \to TM$ satisfying the Leibniz rule:

$$[X, fY]_L = f[X,Y]_L + (\iota_{\rho(X)}\mathrm{d}f)Y\,, \quad \forall X,Y\in\Gamma(L), \forall f\in C^\infty(M)\,. \tag{2.3.1}$$

As an immediate consequence, it follows that the anchor map is a homomorphism from the Lie algebra on the algebroid to the Lie algebra on the tangent bundle with the standard commutator as the algebra operation:

$$\rho([X,Y]_L) = [\rho(X),\rho(Y)]\,, \quad \forall X,Y\in\Gamma(L)\,. \tag{2.3.2}$$

One example of the Lie algebroid is the tangent bundle itself, with algebra operation being the commutator of the vector fields and anchor a simple identity map. As a result, one can think of the Lie algebroid as a certain generalization of the tangent bundle.

As in the case of standard tangent bundle, one can use the Lie algebroid to define an exterior derivative d_L on the exterior algebra $\Gamma(\wedge^\bullet L^*)$ in a similar fashion:

$$\begin{aligned} d_L\omega(X_1,\ldots,X_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\rho(X_i)\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1}) + \\ &\quad + \sum_{i=1}^{k}\sum_{j=i+1}^{k+1}(-1)^{i+j}\omega([X_i,X_j]_L,X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{k+1}), \end{aligned} \tag{2.3.3}$$

for all $X_1,\ldots,X_{k+1}\in\Gamma(L)$ and $\omega\in\Gamma(\wedge^k L^*)$, and where the hat in the brackets denotes the exclusion of an element in an array. By a straightforward calculation it is easy to show that this operator is nilpotent, i.e. $d_L^2 = 0$. So, not only does the concept of the Lie algebroid generalizes the tangent bundle, but it also gives a generalization of the standard exterior algebra $\Gamma(\wedge^\bullet T^*M)$.

Furthermore, the Lie algebra structure on the Lie algebroid sections can be naturally extended to include multivector fields as well. In the standard case of a tangent bundle this is achieved through the use of Schouten-Nijenhuis bracket which can then be generalized to arbitrary Lie algebroids in a straightforward way:

$$[X_1\wedge\ldots\wedge X_k, Y_1\wedge\ldots\wedge Y_l]_L = \sum_{i=1}^{k}\sum_{j=1}^{l}(-1)^{i+j}[X_i,Y_j]_L\wedge X_1\wedge\ldots\hat{X}_i\ldots\wedge X_k\wedge Y_1\wedge\ldots\hat{Y}_j\ldots\wedge Y_l\,, \tag{2.3.4}$$

for all $X_1,\ldots,X_k,Y_1,\ldots,Y_l\in\Gamma(L)$. With this, the standard algebra on tangent and cotangent bundles is completely generalized to arbitrary Lie algebroids.

The introduction of the Lie algebroid derivative d_L and Schouten-Nijenhuis bracket allows the construction of a Lie bialgebroid. Given some Lie algebroid L with its dual bundle L^*

having the Lie algebroid structure as well, their sum $L \oplus L^*$ is called a Lie bialgebroid if the Lie algebroid derivative d_L of L is a derivative operator on the algebra of $\Gamma(\wedge^\bullet L^*)$ with respect to the Schouten-Nijenhuis $[\cdot,\cdot]_{L^*}$ bracket:

$$d_L[\omega_1, \omega_2]_{L^*} = [d_L\omega_1, \omega_2]_{L^*} + [\omega_1, d_L\omega_2]_{L^*} \,. \tag{2.3.5}$$

In addition, it is also true that d_{L^*} is a derivative operator on the algebra of $\Gamma(\wedge^\bullet L)$ with respect to its Schouten-Nijenhuis bracket $[\cdot,\cdot]_L$ [56]. The simplest nontrivial example of Lie bialgebroid is $TM \oplus T^*M$ with TM having the standard Lie algebroid structure and T^*M having the trivial structure, i.e. zero bracket and zero anchor.

Returning to the general case of Lie algebroids, it is often useful to introduce a basis of local sections $e_a \in \Gamma(L)$, $a = 1, \ldots, r$, with r being the rank of L. This gives rise to structure functions $C^c{}_{ab} \in C^\infty(M)$ defined as:

$$[e_a, e_b]_L = C^c{}_{ab} e_c \,. \tag{2.3.6}$$

The important thing to note here is that these are not structure constants one usually sees when dealing with Lie algebras, but functions on the manifold M. As in the case of the standard Lie algebras, the structure functions are antisymmetric in their lower indices since the Lie bracket on the algebroid is antisymmetric. On the other hand, the Jacobi identity of the Lie bracket gives the condition:

$$C^a{}_{e[b} C^e{}_{cd]} - \iota_{\rho_{[b}} \mathrm{d} C^a{}_{cd]} = 0 \,, \tag{2.3.7}$$

which is modified with respect to the pure Lie algebra case because C is a function instead of a constant. That said, Lie algebras are a special case of Lie algebroids when for the base manifold M one takes a single point. From the local basis standpoint, the structure functions become constants in such a situation, recovering the usual form of the Jacobi identity.

Using the anchor map, the local sections e_a define vector fields $\rho_a = \rho(e_a) \in \Gamma(TM)$. Since the anchor map is an algebra homomorphism, it follows that the structure functions for these vector fields are the same as those for the local sections in $\Gamma(L)$:

$$[\rho_a, \rho_b] = C^c{}_{ab} \rho_c \,, \tag{2.3.8}$$

which is expected since the anchor map just translates the algebraic structure from the Lie algebroid L to the tangent bundle.

Finally, it is important to understand the geometrical implications the Lie algebroid has on the base manifold M. As described in [19], the image of the anchor map is a foliation on M. As such, the vector fields ρ_a span the leaves of this foliation.

2.3.1 Poisson manifold as Lie algebroid

An example of special importance is that of a Poisson manifold. It admits a Lie algebroid structure that is a bit more interesting than the standard tangent bundle example. In this case, a cotangent bundle T^*M is taken as the Lie algebroid bundle. The anchor is given by the Poisson bivector:

$$\rho = -\Pi\,, \tag{2.3.9}$$

and the appropriate Lie bracket is Koszul-Schouten bracket [57]:

$$[\xi,\eta]_{\mathrm{KS}} = \iota_{\Pi(\eta)}\mathrm{d}\xi - \mathcal{L}_{\Pi(\xi)}\eta\,. \tag{2.3.10}$$

The Lie algebroid derivative d_L then turns out to be the Schouten-Nijenhuis bracket with the Poisson bivector, i.e. $d_L = [\Pi,\cdot]_{\mathrm{SN}}$. Furthermore, the Koszul-Schouten bracket is related to the Poisson bracket by de Rham differential which serves as homomorphism between the two algebras:

$$\mathrm{d}\{f,g\} = [\mathrm{d}f,\mathrm{d}g]_{\mathrm{KS}}\,. \tag{2.3.11}$$

This Lie algebroid structure on Poisson manifolds can be even more generalized to include twisted Poisson manifolds as well. Those are manifolds equipped with a closed 3-form H in addition to Π. However, Π is not a Poisson bivector anymore, but its deviation from the Poisson case is controlled by the 3-form H. Specifically:

$$[\Pi,\Pi]_{\mathrm{SN}} = \frac{1}{3}\Pi^{li}\Pi^{mj}\Pi^{nk}H_{lmn}\partial_i\wedge\partial_j\wedge\partial_k\,. \tag{2.3.12}$$

As in the untwisted case, the anchor is given by the twisted Poisson bivector, i.e. $\rho = -\Pi$, while the Lie bracket has to be modified:

$$[\xi,\eta]_{\mathrm{KS}} = \iota_{\Pi(\eta)}\mathrm{d}\xi - \mathcal{L}_{\Pi(\xi)}\eta - \iota_{\Pi(\eta)}\iota_{\Pi(\xi)}H\,. \tag{2.3.13}$$

Besides admitting a Lie algebroid structure on its cotangent bundle, Poisson manifolds are closely connected with Lie bialgebroids. Let $L\oplus L^*$ be a Lie bialgebroid over some arbitrary base manifold M. The bialgebroid structure gives rise to an algebra on $C^\infty(M)$ with the algebra bracket given by:

$$\{f,g\} = \langle d_L f, d_{L^*}g\rangle\,. \tag{2.3.14}$$

It can be proven [56] that this bracket is a Poisson bracket, and thus, defines a Poisson bivector in $\Gamma(T^*M)$. This means that any manifold that admits the Lie bialgebroid structure is necessarily a Poisson manifold.

2.3.2 Connections induced by Lie algebroids

The structure of the Lie algebroid can be used to induce connections on other vector bundles. Let $(L, \rho, [\cdot,\cdot]_L)$ be a Lie algebroid over a base manifold M and $\mathcal{F}$ be an arbitrary vector bundle over the same base manifold (specially, $\mathcal{F}$ can be L itself). The standard connection on the vector bundle $\mathcal{F}$ would be a function from $\Gamma(TM) \times \Gamma(\mathcal{F})$ to $\Gamma(\mathcal{F})$ giving notion of the covariant derivative in the direction of some vector from the vector bundle. As was already mentioned, Lie algebroids can be viewed as a generalization of the vector bundle. Thus, the notion of covariant derivative can be generalized as well, but with sections of the tangent bundle replaced with sections of an arbitrary Lie algebroid. In this fashion, an L-connection, as discussed in [58], is defined as a bilinear map ${}^L\nabla : \Gamma(L) \times \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F})$, $(e, X) \mapsto {}^L\nabla_e X$ with the following Leibniz rule:

$$
\begin{aligned}
{}^L\nabla_{fe}X &= f\,{}^L\nabla_e X\,, && (2.3.15)\\
{}^L\nabla_e(fX) &= f\,{}^L\nabla_e X + (\iota_{\rho(e)}\mathrm{d}f)X\,. && (2.3.16)
\end{aligned}
$$

As with the standard connection, one may use L-connection to define the corresponding curvature ${}^LR \in \Gamma(\mathcal{F} \otimes \mathcal{F}^* \otimes \wedge^2 L^*)$, called an L-curvature:

$$
{}^LR(e,e') = {}^L\nabla_e {}^L\nabla_{e'} - {}^L\nabla_{e'} {}^L\nabla_e - {}^L\nabla_{[e,e']_L}\,. \tag{2.3.17}
$$

In the special case when $\mathcal{F} = L$, one can also define an L-torsion ${}^LT \in \Gamma(L \otimes \wedge^2 L^*)$ of this L-connection:

$$
{}^LT(e,e') = {}^L\nabla_e e' - {}^L\nabla_{e'} e - [e,e']_L\,. \tag{2.3.18}
$$

If the Lie algebroid L is also equipped with a (standard) connection $\nabla : \Gamma(TM) \times \Gamma(L) \to \Gamma(L)$, then this induces two additional L-connections, called basic connections, one on the Lie algebroid L and one on the tangent bundle TM:

$$
\begin{aligned}
{}^L\nabla^{bas}_e e' &= \nabla_{\rho(e')} e + [e,e']_L\,, && (2.3.19)\\
{}^L\nabla^{bas}_e X &= \rho(\nabla_X e) + [\rho(e), X]\,. && (2.3.20)
\end{aligned}
$$

These two connections are related through the anchor map:

$$
{}^L\nabla^{bas} \circ \rho = \rho \circ {}^L\nabla^{bas}\,, \tag{2.3.21}
$$

where the the first ${}^L\nabla^{bas}$ denotes the L-connection on TM, while the second one denotes the L-connection on L. These two connection can now be used to define a basic curvature $S \in \Gamma(L \otimes TM \otimes \wedge^2 L^*)$ of the connection ∇:

$$S(e,e')(X) = \nabla_X[e,e']_L - [\nabla_X e, e']_L - [e, \nabla_X e']_L - \nabla_{{}^L\nabla^{bas}_{e'}X} e + \nabla_{{}^L\nabla^{bas}_{e}X} e' \,. \tag{2.3.22}$$

The basic curvature can also be expressed in terms of the L-torsion and L-curvature of the basic connection on L:

$$S = \nabla({}^L T) + 2\mathrm{Alt}(\rho({}^L R)) \,. \tag{2.3.23}$$

2.4 R-Poisson manifolds

Having introduced twisted Poisson manifolds, we can now try and generalize the notion. Let us start with a twisted Poisson manifold structure M with a closed 3-form H and H-twisted Poisson bivector Π:

$$\frac{1}{2}[\Pi,\Pi]_{\mathrm{SN}} = -\left\langle \otimes^3 \Pi, H \right\rangle , \tag{2.4.1}$$

where the contraction with H is carried out with even indices of $\otimes^3\Pi$, i.e.

$$\left\langle \otimes^3 \Pi, H \right\rangle = \Pi^{il}\Pi^{jm}\Pi^{kn} H_{lmn} \partial_i \otimes \partial_j \otimes \partial_k \,. \tag{2.4.2}$$

For the sake of simplification of some of the formulas, we introduce the notation of raising indices on H with Π, with the contraction carried out on the second index of Π. For example:

$$H_i{}^{jk} = \Pi^{jl}\Pi^{km} H_{ilm} \,, \tag{2.4.3}$$

and analogously for other combinations. With this notation, the condition for the twisted Poisson structure can be written as:

$$[\Pi,\Pi]_{SN} = -\frac{1}{3} H^{ijk} \partial_i \wedge \partial_j \wedge \partial_k \,. \tag{2.4.4}$$

The (twisted) Poisson manifold structure can now be generalized by replacing one of the Πs in the above condition by another bivector (also antisymmetric), denoted here by R, thus giving:

$$[\Pi,R]_{\mathrm{SN}} = -\left\langle \otimes^3 \Pi, H \right\rangle , \tag{2.4.5}$$

where the factor of $1/2$ was interpreted as the combinatorical factor and as such was lost in the generalized version. However, this condition by itself is hardly related to the Poisson structure which is supposed to be a generalization of. For that reason, we add an extra condition:

$$[\Pi,\Pi]_{\mathrm{SN}} = 0\,, \tag{2.4.6}$$

meaning that Π is a Poisson bivector. Notice that if one puts the same kind of condition on R, a double Poisson structure would be obtained.

The structure we just obtained can be easily generalized even more. Instead of R being a bivector, we allow it to be any kind of multivector, let us say $(p+1)$-multivector. The Π still remains the Poisson bivector while condition between Π and R becomes:

$$[\Pi,R]_{\mathrm{SN}} = -\left\langle \otimes^{p+2}\Pi, H\right\rangle\,. \tag{2.4.7}$$

This kind of structure is called an R-Poisson structure in [35]. If one adds an extra condition that the Schouten-Nijenhuis of R with itself vanishes, than R becomes a certain generalization of Poisson structure and would define the corresponding $(p+1)$-bracket that would be a generalization of the Poisson bracket.

2.5 Courant algebroids

The Courant algebroid, first introduced in [29] and more extensively explored in [59, 30, 60], is a vector bundle E over a manifold M equipped with a nondegenerate, symmetric, bilinear form $\langle\cdot,\cdot\rangle : \Gamma(E)\times\Gamma(E)\to C^\infty(M)$, a skew-symmetric bracket $[\cdot,\cdot]_E : \Gamma(E)\times\Gamma(E)\to\Gamma(E)$ called a Courant bracket and an anchor map $\rho : E\to TM$ such that the following hold, for all $A,B,C\in\Gamma(E)$ and $f\in C^\infty(M)$:

1. The anchor map is an algebra homomorphism:

$$\rho([A,B]_E) = [\rho(A),\rho(B)]\,. \tag{2.5.1}$$

2. The Courant bracket satisfies the modified Jacobi identity:

$$\mathrm{Jac}(A,B,C) = \mathcal{D}\mathrm{Nij}(A,B,C)\,. \tag{2.5.2}$$

where $\mathcal{D} : C^\infty(M)\to\Gamma(E)$ is an induced differential operator defined through:

$$\langle A,\mathcal{D}f\rangle = \frac{1}{2}\iota_{\rho(A)}\mathrm{d}f\,, \forall A\in\Gamma(E)\,, \forall f\in C^\infty(M)\,, \tag{2.5.3}$$

and Jac and Nij denote the Jacobiator and Nijenhuis operators defined as:

$$\begin{aligned} \mathrm{Jac}(A,B,C) &= [[A,B]_E,C]_E + [[B,C]_E,A]_E + [[C,A]_E,B]_E\,, &(2.5.4)\\ \mathrm{Nij}(A,B,C) &= \frac{1}{3}\left(\langle [A,B]_E,C\rangle + \langle [B,C]_E,A\rangle + \langle [C,A]_E,B\rangle\right). &(2.5.5)\end{aligned}$$

3. The Courant bracket satisfies the Leibniz rule:

$$[A,fB]_E = f[A,B]_E + (\iota_{\rho(A)}\mathrm{d}f)B - \langle A,B\rangle \mathcal{D}f\,. \tag{2.5.6}$$

4. The image of the induced derivative $\mathcal{D}$ is in the kernel of an anchor map, i.e. $\rho\circ\mathcal{D}=0$. This is equivalent to saying that $\langle \mathcal{D}f,\mathcal{D}g\rangle = 0$ for all function $f,g\in C^\infty(M)$.

5. There is a compatibility condition between the bilinear form and the Courant bracket:

$$\iota_{\rho(A)}\langle B,C\rangle = \langle [A,B]_E + \mathcal{D}\langle A,B\rangle, C\rangle + \langle B,[A,C]_E + \mathcal{D}\langle A,C\rangle\rangle\,. \tag{2.5.7}$$

However, not all of these properties are independent. For the minimal set of axioms only three of these five properties are enough, provided the two of those three are the modified Jacobi identity and the compatibility condition [61].

Instead of Courant bracket, the Courant algebroids are often defined using the so called Dorfman bracket which will be denoted here by $\circ$. It can be defined in terms of the Courant bracket as:

$$A\circ B = [A,B]_E + \mathcal{D}\langle A,B\rangle\,. \tag{2.5.8}$$

Note that unlike the Courant bracket, the Dorfman bracket is not skew-symmetric. In fact, one can recover the Courant bracket from the Dorfman bracket by antisymmetrization:

$$[A,B]_E = \frac{1}{2}(A\circ B - B\circ A)\,,\forall A,B\in\Gamma(E)\,. \tag{2.5.9}$$

Of course, some of the axioms for the Courant algebroid take different form when expressed in terms of the Dorfman bracket. Specifically, the fourth one doesn't change since it relates $\mathcal{D}$ with ρ and does not deal with the bracket. Because of this, ρ is also a homomorphism for the Dorfman bracket as well since the difference between the two brackets lies in the kernel of ρ. Finally, the Jacobi identity, the Leibniz rule and the compatibility condition change their form:

1. Homomorphism:

$$\rho(A\circ B) = [\rho(A),\rho(B)]\,. \tag{2.5.10}$$

2. Jacobi identity:
$$A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C)\,. \tag{2.5.11}$$

3. Leibniz rule:
$$A \circ (fB) = f(A \circ B) + (\iota_{\rho(A)}\mathrm{d}f)B\,. \tag{2.5.12}$$

5. The compatibility condition:
$$\iota_{\rho(A)}\mathrm{d}\langle B,C\rangle = \langle A \circ B, C\rangle + \langle B, A \circ C\rangle\,. \tag{2.5.13}$$

One can immediately notice that expressed in terms of the Dorfman bracket, the axioms look a bit simpler. Specifically, the Jacobi identity and the Leibniz rule contain less terms in their expressions, while the compatibility condition is expressed much more naturally. However, even if the Dorfman bracket is definitely more natural for the compatibility condition, one needs to take care with regards with the other two. This is because the Dorfman bracket is not skew-symmetric so the Leibniz rule and the Jacobi identity take their simplest form exactly as they have been written above. For example, the other Leibniz rule, $(fX) \circ Y$ will be more complicated than the Leibniz rule stated above, but also than the Leibniz rule for the Courant bracket. In the end, since the usage of the Courant or Dorfman bracket is equivalent with regards to the Courant algebroid structure, the usage of one or the other is usually a thing of specific situation and personal preference.

2.5.1 Generalized tangent bundle as the Courant algebroid

A standard example of the Courant algebroid, first introduced in [19], is that of $TM \oplus T^*M$, for some manifold M. This bundle is often called a generalized tangent bundle. The anchor map here is just a projection to the tangent bundle TM, while the Courant bracket is defined as a generalization of a Lie bracket on the tangent bundle:

$$[\mathbb{X}_1, \mathbb{X}_2]_E = [X_1, X_2] + \mathcal{L}_{X_1}\eta_2 - \mathcal{L}_{X_2}\eta_1 - \frac{1}{2}\mathrm{d}(\iota_{X_1}\eta_2 - \iota_{X_1}\eta_2)\,, \tag{2.5.14}$$

where $\mathbb{X} = X + \eta \in TM \oplus T^*M$. Here, $[X_1, X_2]$ denotes the standard commutator of the vector fields. Finally, the bilinear form is defined as:

$$\langle \mathbb{X}_1, \mathbb{X}_2\rangle = \frac{1}{2}(\iota_{X_1}\eta_2 + \iota_{X_2}\eta_1)\,. \tag{2.5.15}$$

With this data, the derivative operator $\mathcal{D}$ turns out to be the standard de Rham differential d. Of course, one can now find the corresponding Dorfman bracket as well:

$$\mathbb{X}_1 \circ \mathbb{X}_2 = [X_1, X_2] + \mathcal{L}_{X_1}\eta_2 - \iota_{X_2}\mathrm{d}\eta_1 \,. \tag{2.5.16}$$

This example is also what is known as an exact Courant algebroid, which is in general defined as a Courant algebroid with a short exact sequence:

$$0 \to T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \to 0 \,. \tag{2.5.17}$$

As described in [59], exact Courant algebroids are classified by the Ševera class, i.e. a degree-3 class in the de Rham cohomology.

The Lie bracket of the vector fields is invariant under diffeomorphisms and those are its only symmetries. As a generalization of this Lie bracket, one can ask the question of what are the symmetries of the Courant bracket. Just as the Lie bracket, the Courant bracket is invariant under diffeomorphisms as well, but those are not its only symmetries. There is another transformation, called in [7] a B-field transformation, that preserves the Courant bracket. Let B be a 2-form on M which can be viewed as a function from TM to T^*M. One can then define a map $\mathcal{B} : \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$ such that:

$$\mathcal{B}(\mathbb{X}) = \mathbb{X} + \iota_X B \,. \tag{2.5.18}$$

With this definition, the Courant bracket transforms as:

$$[\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2)]_E = \mathcal{B}([\mathbb{X}_1, \mathbb{X}_2]_E) - \iota_{X_1}\iota_{X_2}\mathrm{d}B \,. \tag{2.5.19}$$

From this follow that this transformation is indeed a symmetry of the Courant bracket, but only if the 2-form B is closed, i.e. $\mathrm{d}B = 0$.

While the B-field transformation represents a symmetry of the Courant bracket if and only if B is closed, it does give a generalization of the Courant bracket in all the other cases. For any 3-form H, one can define the so called twisted Courant bracket on the generalized tangent bundle as:

$$[\mathbb{X}_1, \mathbb{X}_2]_E = [X_1, X_2] + \mathcal{L}_{X_1}\eta_2 - \mathcal{L}_{X_2}\eta_1 - \frac{1}{2}\mathrm{d}(\iota_{X_1}\eta_2 - \iota_{X_1}\eta_2) - \iota_{X_1}\iota_{X_2}H \,, \tag{2.5.20}$$

with its corresponding twisted Dorfman bracket:

$$\mathbb{X}_1 \circ \mathbb{X}_2 = [X_1, X_2] + \mathcal{L}_{X_1}\eta_2 - \iota_{X_2}\mathrm{d}\eta_1 - \iota_{X_1}\iota_{X_2}H\,. \tag{2.5.21}$$

This bracket is still Courant bracket with the same anchor and bilinear form as the untwisted version. The B-field transformation of this bracket has the same form as the untwisted case (2.5.19). Thus, the B-field transformation transforms the H-twisted Courant bracket to $(H + \mathrm{d}B)$-twisted Courant bracket. It is still a symmetry of the twisted version only when B is closed, but in other cases it transforms one twist to the other. However, since $\mathrm{d}(H + \mathrm{d}B) = \mathrm{d}H$, the B-field transformation does not change the cohomology class of H, giving rise to the classification of all these twisted Courant brackets into cohomology classes of H.

2.5.2 Dissection of Courant algebroids

Just as in the case of Lie algebroids, the anchor map defines a foliation on M which we shall denote as $\mathcal{F} = \rho(E)$. In general, a foliation is called regular if its leaves are all of the same dimension, while otherwise, it is called singular. In correspondence to this, a Courant algebroid is called regular if its anchor map defines a regular foliation on M and singular otherwise.

The anchor map of any Courant algebroid has a nontrivial kernel $\ker\rho$, which is easily seen from the Courant algebroid axioms which state that the image of the derivative operator must be contained in this kernel. In addition, it is possible to define its complement $(\ker\rho)^{\perp}$ with respect to the bilinear form of the Courant algebroid. It turns out that this complement is contained in the kernel itself which can be seen quite easily. Assume that $X \in (\ker\rho)^{\perp}$. That means that $\langle X, Y\rangle = 0$ for all $Y \in \ker\rho$. Specifically, $\langle X, \mathcal{D}f\rangle = 0$ for all $f \in C^{\infty}(M)$ since $\mathrm{Im}\,\mathcal{D} \subset \ker\rho$. But by definition of the derivative operator, it follows then that $\iota_{\rho(X)}\mathrm{d}f = 0$. Since this has to be valid for all functions f, it must be that $\rho(X) = 0$, or in other words $X \in \ker\rho$. Given that this X was chosen from $(\ker\rho)^{\perp}$ arbitrarily, it results in the conclusion that $(\ker\rho)^{\perp} \subset \ker\rho$.

For the case of regular Courant algebroids, the kernel of the anchor map and its complement are smooth subbundles of E and the quotients of E by them are Lie algebroids. In addition, there also exists a canonical isomorphism between $E/\ker\rho$ and $\mathcal{F}$. Furthermore, the quotient $\mathcal{G} = \ker\rho/(\ker\rho)^{\perp}$ is also defined. This $\mathcal{G}$ is bundle with its fiber being a quadratic Lie algebra, meaning it is an algebra with a nondegenerate, ad-invariant inner product.

There now exist an isomorphism between vector bundles $\Psi : \mathcal{F} \oplus \mathcal{G} \oplus \mathcal{F}^* \to E$, called in [30] a dissection of E, such that:

$$\langle \Psi(X_1 + s_1 + \xi_1), \Psi(X_2 + s_2 + \xi_2)\rangle = \frac{1}{2}(\iota_{X_1}\xi_2 + \iota_{X_2}\xi_1) + \langle s_1, s_2\rangle^{\mathcal{G}}\,, \tag{2.5.22}$$

for all $X_1, X_2 \in \Gamma(\mathcal{F})$, $s_1, s_2 \in \Gamma(\mathcal{G})$, $\xi_1, \xi_2 \in \Gamma(\mathcal{F}^*)$, and where $\langle\cdot,\cdot\rangle^{\mathcal{G}}$ is an inner product on $\mathcal{G}$. In the case of exact Courant algebroids, the kernel of the anchor map and its complement coincide, so $\mathcal{G}$ is trivial. Thus, the isomorphism Ψ gives a nice distinction between exact and non-exact Courant algebroids. In the latter case, it gives an additional definition of the structure that is often better fitted to the current needs. Instead of the usual data $(E, \rho, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_E)$, one can consider the quintuple $(\mathcal{F}, \mathcal{G}, \nabla, F, \mathcal{H})$ for the definition of the structure. It consist of a distribution[1] $\mathcal{F}$, a bundle of quadratic Lie algebras $\mathcal{G}$, a covariant derivative $\nabla : \Gamma(\mathcal{F}) \times \Gamma(\mathcal{G}) \to \Gamma(\mathcal{G})$ on $\mathcal{G}$, a C^∞ bilinear map $F : \Gamma(\wedge^2 \mathcal{F}) \to \Gamma(\mathcal{G})$ and a 3-form $\mathcal{H} \in \Gamma(\wedge^3 \mathcal{F}^*)$. As is usual, the covariant derivative ∇ follows the following linearity property and the Leibniz rule:

$$
\begin{aligned}
\nabla_{fX} s &= f \nabla_X s, \forall X \in \Gamma(\mathcal{F}), \forall f \in C^\infty(M), \forall s \in \Gamma(\mathcal{G}), && (2.5.23)\\
\nabla_X (fs) &= f \nabla_X s + (X(f))s, \forall X \in \Gamma(\mathcal{F}), \forall f \in C^\infty(M), \forall s \in \Gamma(\mathcal{G}). && (2.5.24)
\end{aligned}
$$

This data then defines the Courant algebroid if the following identities are satisfied:

$$
\begin{aligned}
\mathrm{d}\mathcal{H} &= \langle F \wedge F \rangle, && (2.5.25a)\\
\mathcal{L}_X \langle s, s' \rangle^{\mathcal{G}} &= \langle \nabla_X s, s' \rangle^{\mathcal{G}} + \langle s, \nabla_X s' \rangle^{\mathcal{G}}, && (2.5.25b)\\
\mathcal{L}_X [s, s']^{\mathcal{G}} &= [\nabla_X s, s']^{\mathcal{G}} + [s, \nabla_X s']^{\mathcal{G}}, && (2.5.25c)\\
\nabla_{X_1} F(X_2, X_3) - F([X_1, X_2], X_3) + \text{c.p.} &= 0, && (2.5.25d)\\
(\nabla_{X_1} \nabla_{X_2} - \nabla_{X_2} \nabla_{X_1}) s - \nabla_{[X_1, X_2]} s &= [F(X_1, X_2), s]^{\mathcal{G}}, && (2.5.25e)
\end{aligned}
$$

for all $X, X_1, X_2, X_3 \in \Gamma(\mathcal{F})$, $s, s' \in \Gamma(\mathcal{G})$, and the 4-form $\langle F \wedge F \rangle$ is defined as:

$$
\langle F, F \rangle (X_1, X_2, X_3, X_4) = \frac{1}{4} \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) \langle F(X_{\sigma(1)}, X_{\sigma(2)}), F(X_{\sigma(3)}, F_{\sigma(4)}) \rangle^{\mathcal{G}}, \tag{2.5.26}
$$

with S_n denoting the permutation group of n elements. The corresponding Courant bracket defined by this data then turns out to be:

$$
\begin{aligned}
[\mathbb{X}_1, \mathbb{X}_2] = \Big(& [X_1, X_2], \nabla_{X_1} s_2 - \nabla_{X_2} s_1 + [s_1, s_2]^{\mathcal{G}} + \iota_{X_1} \iota_{X_2} F,\\
& \mathcal{L}_{X_1} \eta_2 - \mathcal{L}_{X_2} \eta_1 - \frac{1}{2} \mathrm{d}(\iota_{X_1} \eta_2 - \iota_{X_2} \eta_1) - \iota_{X_1} \iota_{X_2} \mathcal{H} +\\
& + \langle \nabla s_1, s_2 \rangle^{\mathcal{G}} - \langle \nabla s_2, s_1 \rangle^{\mathcal{G}} + 2 \langle s_1, \iota_{X_2} F \rangle^{\mathcal{G}} - 2 \langle s_2, \iota_{X_1} F \rangle^{\mathcal{G}} \Big), \qquad (2.5.27)
\end{aligned}
$$

[1]A distribution $\mathcal{F}$ over a manifold M is a set $\{V_x \subset T_x M\}_{\{x \in M\}}$, with V_x being a subspace of $T_x M$ for every $x \in M$, such that for every $x \in M$ there exists a neighbourhood O_x and a set of vector fields $X_1, \ldots, X_k$ with a property that $X_1(y), \ldots, X_k(y)$ span V_y for every $y \in O_x$.

where the notation $\mathbb{X} = (X, s, \eta) \in \Gamma(\mathcal{F} \oplus \mathcal{G} \oplus \mathcal{F}^*)$ have been used. In the special case of exact Courant algebroids, the expression (2.5.27) has the same form as the twisted Courant bracket on the generalized tangent bundle meaning that any regular exact Courant algebroid is isomorphic to some Courant algebroid of the generalized tangent bundle. The similar thing holds true for non-exact Courant algebroids as well, but with the addition of the bundle of quadratic Lie algebras $\mathcal{G}$.

2.5.3 Dirac structures

As already mentioned when discussing the generalized tangent bundle, the Courant bracket is a certain generalization of the Lie bracket. One can also see this by looking at the Courant algebroid axioms, specifically at the Jacobi identity and the Leibniz rule which are modified in comparison to the Lie algebroid case by a few extra terms. However, all those extra terms include a bilinear form. Since those two axioms are the only ones important for the Lie algebroid, and since the bilinear form is irrelevant for the Lie algebroid structure, one can define a Lie algebroid as a subbundle L of the Courant algebroid E which is involutive:

$$[\Gamma(L), \Gamma(L)]_E \subset \Gamma(L)\,, \tag{2.5.28}$$

and isotropic:

$$\langle \Gamma(L), \Gamma(L) \rangle = 0\,. \tag{2.5.29}$$

This Lie algebroid, first introduced in [62], is called a Dirac structure and its Lie bracket is given as the restriction of the Courant bracket of E to L. Also note that by (2.5.8), the Dorfman bracket coincides to the Courant bracket on Dirac structures since their difference is controlled by the bilinear form. Thus, both Courant and Dorfman bracket become Lie brackets when restricted to Dirac structures.

In the case of the generalized tangent bundle, it is easily seen from the bilinear form that isotropic subbundles cannot have rank higher than the dimension of the base manifold M. While this kind of statement is not valid for the general Courant algebroids, it remains true that the rank of the Dirac structure is strictly lesser than the rank of the full Courant algebroid. In correspondence to this, Dirac structures of maximal rank are called full Dirac structures, while those that are not are called small Dirac structures [63].

Dirac structures relate the concept of Courant algebroids to that of Lie bialgebroids. Suppose L and L' are two transverse Dirac structures of the Courant algebroid E, i.e. $E = L \oplus L'$. Using the bilinear form of the Courant algebroid, it is possible to identify L' with L^*. Thus, both L and L^* are Lie algebroids and it turns out that $L\oplus L^*$ really does have the Lie bialgebroid structure [29].

The converse of the previous statement is also true. Given a Lie bialgebroid $L\oplus L^*$, one can construct a Courant algebroid from it. The appropriate bilinear form is simply:

$$\langle \mathbb{X}_1, \mathbb{X}_2\rangle = \frac{1}{2}\left(\iota_{X_1}\eta_2 + \iota_{X_2}\eta_1\right) . \tag{2.5.30}$$

Then the Courant bracket can be found using the general expression (2.5.27), but instead of the standard commutator of the vector fields one needs to use the Lie bracket of the Lie algebroid. Also, since there are two copies of Lie algebroid in Lie bialgebroid, the general formula is also used as a double copy finally giving:

$$\begin{aligned}[\mathbb{X}_1, \mathbb{X}_2]_E \;\; = \;\; & [X_1, X_2]_L + \mathcal{L}^{(L^*)}_{\eta_1} X_2 - \mathcal{L}^{(L^*)}_{\eta_2} X_1 - \frac{1}{2}\mathrm{d}_{L^*}\left(\iota_{X_1}\eta_2 - \iota_{X_2}\eta_1\right) + \\ & + [\eta_1, \eta_2]_{L^*} + \mathcal{L}^{(L)}_{X_1}\eta_2 - \mathcal{L}^{(L)}_{X_2}\eta_1 - \frac{1}{2}\mathrm{d}_L\left(\iota_{X_1}\eta_2 - \iota_{X_2}\eta_1\right) ,\end{aligned} \tag{2.5.31}$$

where $\mathcal{L}^{(L)}_X$ denotes a Lie derivative on L given by the Cartan formula with the corresponding differential:

$$\mathcal{L}^{(L)}_X = \iota_X \mathrm{d}_L + \mathrm{d}_L \iota_X , \tag{2.5.32}$$

and similarly for the $\mathcal{L}^{(L^*)}_X$. In addition to the Courant bracket and the bilinear form, an anchor map is still needed to fully define the Courant algebroid structure. If ρ_L and ρ_{L^*} are anchor maps for the Lie algebroids L and L^*, respectively, then the anchor map for the Courant brackets is given by their sum:

$$\rho_E = \rho_L + \rho_{L^*} . \tag{2.5.33}$$

The derivative operator $\mathcal{D}$ of this Courant algebroid is then just the sum of the derivative operators of the two Lie algebroids:

$$\mathcal{D} = \mathrm{d}_L + \mathrm{d}_{L^*} . \tag{2.5.34}$$

In the special case when the Lie bialgebroid $L\oplus L^*$ arises from some Courant algebroid E, the above procedure then recovers the original Courant algebroid.

2.6 Graded geometry

In standard differential geometry, one encounters the exterior algebra of differential forms. If M is some manifold, the algebra in question is $\Gamma(\wedge^\bullet T^*M)$, with respect to the exterior product

$\wedge$. Given two differential forms $\omega_p \in \Gamma(\wedge^p T^*M)$ and $\omega_q \in \Gamma(\wedge^q T^*M)$, their product satisfies the property:

$$\omega_p \wedge \omega_q = (-1)^{pq} \omega_q \wedge \omega_p \,. \tag{2.6.1}$$

In other words, two differential forms can either commute or anticommute with regards to the exterior product, depending on the order of the forms. Thus, the exterior product is neither commutative nor anticommutative in general. Instead, this commutation property will be called graded commutative.

The exterior algebra of differential forms is a standard example of such graded commutative property, but this notion can be easily generalized to include much more structures resulting in what is known as graded geometry. In this section we aim to give a short introduction to graded geometry, following the work presented in [64, 65, 66, 67].

2.6.1 Graded algebras

The first step is to generalize the notion of the differential form order, or as is usually called in this setting, its grading. This is done by the notion of a graded vector space which is a vector space V that can be written as a sum of different *smaller* vector spaces V_n:

$$V = \bigoplus_{n \in \mathbb{Z}} V_n \,. \tag{2.6.2}$$

To each vector space V_n is associated an integer n known as its grading or degree. Furthermore, for each vector $v \in V_n$, one associates the grading or degree as well, denoted as $|v|$, as being equal to the grading of the vector space it is contained in. Notice that the grading is not a function on V though. For example, two vectors $v \in V_0$ and $w \in V_1$ have well defined gradings, specifically being 0 and 1, respectively, but their sum $v + w \in V$ does not. Furthermore, we also denote with $V[n]$ a vector space V with every degree shifted by n. Specifically if $v \in V$ with a degree $|v|_V$ with respect to V, then its degree with respect to $V[n]$ is $|v|_{V[n]} = |v|_V + n$.

Having the notion of the graded vector space, one can proceed to define the graded algebra structure in a similar fashion one would define an algebra on an ordinary vector space. It is an ordered pair $(V, \cdot)$ of a graded vector space V over a field $\mathbb{F}$ and a multiplication operation $\cdot : V \times V \to V$ which is distributive with respect to addition in V and compatible with the scalar multiplication:

$$
\begin{aligned}
x\cdot(y+z) &= x\cdot y + x\cdot z\,, && (2.6.3)\\
(x+y)\cdot z &= x\cdot z + y\cdot z\,, && (2.6.4)\\
(\alpha x)\cdot(\beta y) &= (\alpha\beta)(x\cdot y)\,, && (2.6.5)
\end{aligned}
$$

for all vectors $x,y,z\in V$ and scalars $\alpha,\beta\in\mathbb{F}$. Additionally, the graded algebra is called an associative graded algebra if the multiplication operation is also associative:

$$x\cdot(y\cdot z) = (x\cdot y)\cdot z\,. \tag{2.6.6}$$

From now on we shall assume that every graded algebra is an associative graded algebra without explicitly stating. In addition, we shall assume that it is graded commutative, i.e. that the it satisfies the graded commutative property:

$$x\cdot y = (-1)^{|x||y|}y\cdot x\,. \tag{2.6.7}$$

When dealing with regular algebras, they could be further classified depending on whether the multiplication satisfies certain additional properties. One such example is a Lie algebra defined as an anticommutative algebra whose product satisfies the Jacobi identity. The same kind of principle can be applied to graded algebras as well. One such interesting class of graded algebras is a graded Lie algebra, whose product is a graded Lie bracket $[\cdot,\cdot]$ of degree $-n$, defined as graded anticommutative bracket, but with an additional n shift:

$$[x,y] = -(-1)^{(|x|+n)(|y|+n)}[y,x]\,, \tag{2.6.8}$$

and satisfies graded Jacobi identity:

$$[x,[y,z]] = [[x,y],z] + (-1)^{(|x|+n)(|y|+n)}[y,[x,z]]\,. \tag{2.6.9}$$

Batalin-Vilkovisky algebras

Having introduced the concept of graded algebras, one can go one step further and define *double* graded algebras. Specifically, we define what is known as graded Poisson algebra of degree n, or simply n-Poisson algebra, on a graded vector space with two multiplication operations $\cdot : V\times V\to V$ and $[\cdot,\cdot]: V\times V\to V[-n]$. First of these is a graded commutative product of degree 0, while the second is graded Lie bracket of degree $-n$. In addition, these two operations are required to satisfy the compatibility condition:

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{|y|(|x|+n)} y \cdot [x, z], \tag{2.6.10}$$

or in other words, the bracket has to be a derivation of the product. The special case of 0-graded Poisson algebra is simply the standard Poisson algebra, while 1-graded Poisson algebra is also known as a Gerstenhaber algebra.

In n-Poisson algebra for odd n, a linear map $\Delta : V \to V[-n]$ of degree $-n$ is known as a graded Poisson bracket generator if it satisfies:

$$\Delta(x \cdot y) = \Delta(x) \cdot y + (-1)^{n|x|} x \cdot \Delta(y) + (-1)^{n|x|} [x, y]. \tag{2.6.11}$$

Furthermore, this generator is called exact if it is 2-nilpotent as well, i.e. if $\Delta^2 = 0$. In this special case it is also a derivation of the graded Poisson bracket:

$$\Delta([x, y]) = [\Delta(x), y] - (-1)^{n|x|} [x, \Delta(y)]. \tag{2.6.12}$$

Such an n-Poisson algebra, endowed with an exact generator, is known as an n-Batalin-Vilkovisky (BV) algebra. Specially, in what follows we shall refer to 1-BV algebras simply as BV-algebras.

2.6.2 Graded manifolds

An important notion in graded geometry is that of a graded manifold. To define this, start with a manifold M and endow it with a sheaf of local rings $\mathcal{R}$. This means that for every open set $U \subset M$ there exists a ring $R_U \in \mathcal{R}$. Furthermore, for every pair open sets $U, V \subset M$ related by inclusion, let us say $V \subset U$, there exist a restriction function $\phi_{V,U} : R_U \to R_V$ satisfying some properties:

1. For every open set $U \subset M$, a restriction function $\phi_{U,U}$ is an identity on R_U.
2. For all open sets $U, V, W \subset M$ such that $W \subset V \subset U$, the composition of restriction functions $\phi_{W,V}$ and $\phi_{V,U}$ has to be equal to the restriction function $\phi_{W,U}$:

$$\phi_{W,V} \circ \phi_{V,U} = \phi_{W,U}. \tag{2.6.13}$$

3. For every open set $U \subset M$ and open cover $\mathcal{U} = \{U_n\}_{n \in J}$ of U, if $\phi_{U_n,U}(x) = \phi_{U_n,U}(y)$ for all $x, y \in R_U$ and $n \in J$, then $x = y$.

4. For every open set $U \subset M$, open cover $\mathcal{U} = \{U_n\}_{n\in J}$ of U and every set $\{x_n \in R_{U_n}\}_{n\in J}$, if $\phi_{U_n\cap U_m,U}(x_n) = \phi_{U_n\cap U_m,U}(x_m)$ for all $n,m \in J$, then there exists $x \in R_U$ such that $\phi_{U_n,U}(x) = x_n$ for all $n \in J$.

This structure $(M,\mathcal{R})$ is known as a locally ringed space.

To get a graded manifold, one extra condition is necessary, specifically, that the sheaf of local rings $(M,\mathcal{R})$ is locally isomorphic to $(U, C^\infty(U)\otimes S(W^*))$, where U is some open subset of $\mathbb{R}^n$, W some graded vector space and $S(W^*)$ a symmetric algebra over W^*, and with all isomorphism being degree-preserving. Here, M is called a body of the graded manifold.

Graded vector bundle E over manifold M can now be defined as a graded manifold with M as its body and $\Gamma(S(E^*))$ as the functions in the sheaf. In general, E can be decomposed as:

$$E = \bigoplus_{n\in\mathbb{Z}} E_n\,, \tag{2.6.14}$$

with each E_n being a graded vector bundle of fixed degree n. A standard example of a graded vector bundle is $T[1]M$ with M as its body and $\Gamma(S(T^*[-1]M))$ as its functions. Notice that this is equivalent to the bundle of differential forms, meaning that a differential form over M can be considered as a function on $T[1]M$.

Along with functions, one can also consider vector fields on graded manifolds. In ordinary geometry, vector fields are defined as maps on the space of functions. This translates nicely to graded situation. A graded vector field X of degree n on a graded manifold $\mathcal{M}$ can be defined as a linear map from $C^\infty(\mathcal{M})$ to $C^\infty(\mathcal{M})[n]$ that satisfies the (graded) Leibniz rule:

$$X(fg) = X(f)g + (-1)^{n|f|} fX(g)\,. \tag{2.6.15}$$

Of special importance with regards to the graded vector fields is a graded commutator $[\cdot,\cdot]$ defined as:

$$[X,Y] = XY - (-1)^{|X||Y|} YX\,, \tag{2.6.16}$$

for any two vector fields X and Y. This graded commutator now turns the space of graded vector fields into a graded Lie algebra. On the other hand the graded commutator is important since it allows to define a special class of graded vector fields that are of special interest. Those are degree-1 graded vector fields that commute with themselves, and are known as cohomological vector fields.

A graded manifold $\mathcal{M}$ endowed with the cohomological vector field Q is called a differential graded manifold or a Q-manifold. The term differential here comes from the fact that Q is a differential on the algebra of functions in the sense that:

$$Q^2 = 0\,. \tag{2.6.17}$$

This follows directly from the definition of cohomological vector fields, since for degree-1 vector fields, $[Q,Q]$ and Q^2 are proportional to each other (or more generally, for any odd degree vector field).

In a similar fashion as the vector fields, one can define differential forms as well, but instead of mapping functions to functions, differential forms map vector fields to functions. To be more precise, graded differential 1-forms are smooth maps on $T[1]\mathcal{M}$. The extra shift of 1 in the tangent bundle means that in local coordinates $\{x^i\}$, the corresponding differential forms have their degree increased by 1, i.e. $|\mathrm{d}x^i| = |x^i| + 1$.

2.6.3 Graded symplectic geometry

A symplectic graded manifold of degree n is a graded manifold endowed with a graded symplectic form ω of degree n, which is a non-degenerate 2-form, closed with respect to the de Rham differential. Since ω is non-degenerate it automatically induces an isomorphism of vector bundles:

$$\omega : T\mathcal{M} \to T^*[n]\mathcal{M}\,. \tag{2.6.18}$$

Symplectic graded manifolds are also often referred to as P-manifolds. It is interesting to note that graded symplectic forms of non-vanishing degree are also exact and not just closed [64].

In non-graded geometry, symplectic manifolds are used to describe Hamiltonian systems and it is possible to extend such notions to the graded setting as well. A vector field X is called symplectic if the Lie derivative of the (graded) symplectic form along X vanishes, i.e.

$$\mathcal{L}_X \omega = 0\,, \tag{2.6.19}$$

and it called a Hamiltonian vector field if its contraction with the (graded) symplectic form is an exact form, i.e. if there exists a function H such that:

$$\iota_X \omega = -\mathrm{d}H\,. \tag{2.6.20}$$

Here, H is known as a Hamiltonian function. It can be shown that a symplectic vector field X is also Hamiltonian as long as $|X| + |\omega| \neq 0$.

Just as in ungraded case, the graded symplectic form induces a bracket $(\cdot,\cdot)$ through:

$$(f,g) = (-1)^{|f|+1} X_f(g)\,, \tag{2.6.21}$$

where X_f is a graded vector field defined through:

$$\iota_{X_f}\omega = -\mathrm{d}f\,. \tag{2.6.22}$$

Suppose the symplectic, graded manifold is also equipped with a symplectic, cohomological vector field Q. Such a symplectic, graded manifold, endowed with a symplectic, cohomological vector field is known as a symplectic differential graded manifold, or QP-manifold for short. Since the degree of Q is equal to 1, it follows that it is also a Hamiltonian vector field unless a degree of ω is -1.[2] In those cases, it is possible to write Q by the means of an induced bracket as:

$$Q = (S,\cdot)\,, \tag{2.6.23}$$

where S is an appropriate Hamiltonian function. Since

$$[Q,Q]f = ((S,S),f)\,, \tag{2.6.24}$$

for an arbitrary function f, it follows that the condition $[Q,Q]=0$ is equivalent to (S,S) being a constant. By simple degree counting, for $|\omega| \neq -2$, it turns out that this constant has to vanish, i.e.

$$(S,S) = 0\,. \tag{2.6.25}$$

This equation is known as the classical master equation.

[2]Note that even when $|\omega| = -1$, symplectic, cohomological vector field can still be Hamiltonian. It is just that in other cases every symplectic, cohomological vector field must be Hamiltonian as well.

Chapter 3

Non-linear Sigma Models

3.1 String theory

In its simplest form, string theory [1, 2, 68] describes propagation of 1-dimensional objects, called strings, in some target space M. Mathematically, the target space often has an n-manifold structure, as we shall assume here, though this is not strictly necessary, allowing for the propagation of the strings in non-geometric backgrounds [5]. Physically, this usually denotes some spacetime through which the string is propagating. In this spacetime, propagating strings form a (curved) surface, or more precisely, a 2-dimensional manifold Σ_2, known as the world-sheet of the string, that is embedded in M.

Let σ^0, σ^1 be local coordinates on the world-sheet and let:

$$\gamma = \frac{1}{2}\gamma_{\mu\nu}\mathrm{d}\sigma^{\mu} \vee \mathrm{d}\sigma^{\nu} \tag{3.1.1}$$

be a metric on Σ_2. Then the corresponding volume form on the world-sheet is given by:

$$\varepsilon = \sqrt{|\gamma|}\,\mathrm{d}\sigma^0 \wedge \mathrm{d}\sigma^1\,, \tag{3.1.2}$$

where $|\gamma|$ denotes an absolute value of the determinant of the metric. In order to describe the propagation of the strings in the target space M, the embedding function $X : \Sigma_2 \to M$ is used. By introducing local coordinates x^i, $i = 1, \ldots, n$ on M, this embedding function can be separated into components $X^i = X^* x^i$. With all of this, it is possible write down the action functional that describes the actual dynamics of the strings, known as the Polyakov action:

$$S[X,\gamma] = -\frac{T}{2}\int_{\Sigma_2} g_{ij}(X)\gamma^{\mu\nu}\partial_{\mu}X^i\partial_{\nu}X^j\sqrt{|\gamma|}\mathrm{d}^2\sigma\,. \tag{3.1.3}$$

Here, g_{ij} are components of the metric on M:

$$g = \frac{1}{2} g_{ij} \mathrm{d}x^i \vee \mathrm{d}x^j \,, \tag{3.1.4}$$

$g_{ij}(X)$ is the pull-back of this metric with respect to the map X, i.e. $g_{ij}(X) = X^* g_{ij}$, and the T is constant of dimension mass/length, known as the string tension. The string tension is often written in terms of the so called Regge slope α':

$$T = \frac{1}{2\pi\alpha'} \,. \tag{3.1.5}$$

The above action is often written in a different form, taking advantage of the differential forms:

$$S[X,\gamma] = -\frac{T}{2} \int_{\Sigma_2} \frac{1}{2} g_{ij}(X) \mathrm{d}X^i \wedge * \mathrm{d}X^j \,, \tag{3.1.6}$$

where d denotes the exterior derivative on the $\Gamma(\wedge^\bullet T^* \Sigma_2)$. Notice that the dependence of the action on the world-sheet metric γ is not explicit anymore. Instead, that dependence is hidden inside the Hodge dual operator $*$.

In the first step, the metric g on M is usually taken to be a flat metric. In other words, one usually starts by looking at the propagation of a single string in a Minkowski spacetime, though not necessarily 4-dimensional. While this is certainly possible, it is not strictly necessary for the present purposes, so the metric on the target space is left arbitrary.

An important thing to note about the Polyakov action are its (world-sheet) symmetries. First, it is invariant to reparametrization of the world-sheet. This is evident in the ability to write it in the coordinate independent form (3.1.6). Locally, the infinitesimal transformation of the reparametrization invariance can be written as:

$$\delta X^i = -\xi^\mu \partial_\mu X^i \,, \tag{3.1.7}$$

$$\delta\gamma = -\mathcal{L}_\xi \gamma, \tag{3.1.8}$$

where $\xi \in \Gamma(T\Sigma_2)$. In addition to reparametrization invariance, the Polyakov action is scale invariant as well, i.e. it is invariant to Weyl transformations:

$$\delta X^i = 0 \,, \tag{3.1.9}$$

$$\delta\gamma_{\mu\nu} = 2\Lambda\gamma_{\mu\nu} \,, \tag{3.1.10}$$

with Λ being a scalar parameter. These two transformations can be combined to produce very interesting results. In total, they are governed by three scalar parameters. In contrast, since the world-sheet is 2-dimensional, its metric γ is determined by three scalar parameters as well. First, it is possible to use reparametrization invariance of the Polyakov action to choose coordinates on the world-sheet in which the world-sheet metric becomes conformally flat, i.e. it is proportional to the flat metric η:

$$\gamma = \Omega^2 \eta \,. \tag{3.1.11}$$

After that, the Weyl transformation may be used to transform the metric into pure flat metric. Thus, locally, the symmetries of the Polyakov action allow for the choice of specific gauge, known as the conformal gauge, in which the world-sheet metric is the flat metric. Of course, this does not say anything about the global properties of the world-sheet, nor does it say that the world-sheet is a flat manifold, but it represents a convenient tool that can be used in practical calculation that do not involve global properties of the world-sheet.

Having considered the symmetries[1], the next thing to look at are the equations of motion for the Polyakov action. A simple calculation of the functional derivatives of the action (3.1.3) leads to two field equations:

$$\begin{aligned}
\Box X^i &= \frac{1}{\sqrt{|\gamma|}}\partial_\mu\left(\sqrt{|\gamma|}\gamma^{\mu\nu}\partial_\nu X^i\right) + \gamma^{\mu\nu}\Gamma^i_{jk}\partial_\mu X^j \partial_\nu X^k = 0\,, && (3.1.12)\\
T_{\mu\nu} &= \frac{4\pi}{\sqrt{|\gamma|}}\frac{\delta S}{\delta\gamma^{\mu\nu}} = -\frac{1}{\alpha'}\left(g_{ij}\partial_\mu X^i \partial_\nu X^j - \frac{1}{2} g_{ij}\gamma_{\mu\nu}\gamma^{\sigma\rho}\partial_\sigma X^i \partial_\rho X^j\right) = 0\,, && (3.1.13)
\end{aligned}$$

where $\Box$ denotes a Laplacian, $T_{\mu\nu}$ is the energy-momentum tensor and Γ^i_{jk} are the components of the Levi-Civita connection on M. Equation (3.1.13) was found before the gauge fixing for γ has been done. This is important because the gauge-fixed action cannot be varied with respect to γ after gauge-fixing has been done since γ becomes a constant in that setting. Thus, one field equation would have been lost and one look at the energy-momentum tensor is enough to see that it does not vanish identically when γ becomes flat. Thus, even when gauge-fixing condition is implemented, and the action loses a degree of freedom in metric, there exists an additional constraint, known as the Virasoro constraint, for the energy-momentum tensor defined above to vanish.

The above field equations specify the dynamics of the field propagation both in classical and quantum setting (with the appropriate quantization procedure). While no major problem arises on a classical level, on the quantum level things are a bit different. Specifically, the Virasoro constraints lead to an infinite number of conserved charges, which on a quantum level satisfy

[1] Actually, the Polyakov action may have additional symmetries, depending on the target space metric, but those aren't relevant at the moment and will be considered later in this chapter.

the Virasoro algebra and produce an anomaly, at least in general. It turns that this anomaly can be removed in specific number of dimension, specifically for bosonic strings it is $n = 26$. While the anomaly here can be taken care of, the spectrum still remains problematic. It turns out that bosonic strings have a scalar tachyion as the ground state. This means that the theory needs to be modified further. Leaving that aside for a moment, we note that the first excited state of closed strings consists of a massless spin-2 particle, an antisymmetric tensor and a massless scalar. These are, respectively, called the graviton, the Kalb-Ramond field and the dilaton.

Finally, one need to deal with the tachyon that shows up in the spectrum of bosonic strings, leading to superstring theory. Here, a supersymmetric extension of the bosonic strings is constructed, thus giving a supersymmetric theory with anticommuting degrees of freedom. With them it is possible to remove the tachyon from the string spectrum. We also note that critical dimension for the superstrings is 10 instead of 26.

The quantization procedure for the superstrings gives rise to five supersymmetric string theories: Type I for open strings, Type IIa and Type IIb for closed strings and two heterotic string theories. When considering closed bosonic strings in light-cone coordinates, the field equation for X^i can be separated into two distinct and (mostly) independent sectors, each being dependent on only one of the light-cone coordinates. The corresponding solutions are known as left-movers and right-movers. Classically, these are completely independent, while on a quantum level there is an extra condition, known as the level-matching condition, that needs to be satisfied. The introduction of fermionic degrees of freedom in the superstrings still leave left and right-movers independent. Thus, one can form heterotic string theories in which the right-movers are supersymmetric, while the left-movers are left purely bosonic. As a result left and right-movers have different critical dimensions. Thus, to make sense of such a theory, the extra 16 bosons for the left-movers are considered as internal bosonic degrees of freedom and they have to be compactified on a torus. The compactification lattice is not completely general however. In fact, in order to have a consistent theory, there are only two options for the lattice, representing two gauge groups for the heterotic strings: $E_8 \times E_8$ and $SO(32)$.

As mentioned previously, when looking at the propagation of the single string, the target space is usually taken to be a flat Minkowski spacetime. This is a reasonable assumption in such circumstances. However, we are not interested in having just a single string in an empty flat spacetime, but also in the situation when there are many strings that can interact. While it is possible to look at these interactions directly, it is possible to construct an effective field theory that describes this interaction. This kind of theory would describe a single string propagating in the target space filled with other strings that affect the dynamics of this one specific string. The effects of the background sea of strings can then be described by introducing the effective metric

g, the Kalb-Ramond field B and the dilaton Φ produced by the other strings. The appropriate action is then given by:

$$S[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_{\Sigma_2}\left(\frac{1}{2}g_{ij}(X)\mathrm{d}X^i\wedge *\mathrm{d}X^j + \frac{1}{2}B_{ij}(X)\mathrm{d}X^i\wedge \mathrm{d}X^j + \frac{1}{2}\alpha' R\Phi(X)\varepsilon\right), \quad (3.1.14)$$

where R is the world-sheet Ricci scalar and, as is the case for the metric, $B_i j(X)$ and $\Phi(X)$ represent pull-backs of the Kalb-Ramond field and the dilaton with respect to the embedding function X, respectively. This kind of theory is known as 2-dimensional non-linear sigma model.

In what follows, we shall discuss only the bosonic sector of closed strings. The results can of course be implemented in the superstring theory as well, but the fermionic degrees of freedom shall not be discussed here. Furthermore, we shall look at strings only at leading order terms meaning that, in the spectrum, the only relevant contributions are the metric and the Kalb-Ramond field, while the dilaton contributes only at 1-loop level. Finally, the units in which the theory shall be considered are the ones in which $\alpha' = 1/4\pi$ in order to simplify the notion a little bit.

Finally, there is another way one can write the 2-dimensional sigma model. It turns out that the only contribution of importance the B-field has is through its curvature:

$$H = \mathrm{d}B. \quad (3.1.15)$$

Thus, one may replace the B-field by this H-field in the action. Since H is a 3-form, and Σ_2 is 2-dimensional, one needs to extend the world-sheet to some larger manifold. Let Σ_3 be a 3-dimensional manifold with Σ_2 as its boundary. Then the 2-dimensional sigma model (3.1.14) is equivalent to (with the exclusion of the dilaton):

$$S[X] = -\int_{\Sigma_2}\frac{1}{2}g_{ij}(X)\mathrm{d}X^i\wedge *\mathrm{d}X^j - \int_{\Sigma_3}\frac{1}{3!}H_{ijk}(X)\mathrm{d}X^i\wedge \mathrm{d}X^j\wedge \mathrm{d}X^k, \quad (3.1.16)$$

where the function X has been extended to be a function from Σ_3. This H term in the action is often called a Wess-Zumino (WZ) term. Since H is closed, the Stokes theorem guarantees the equivalence of the two actions. However, it is possible to extend the theory to include not only exact 3-forms, but general closed 3-forms. In this case, the WZ term cannot be simply projected to Σ_2. However, the field equation for this action is:

$$\left(\frac{1}{2}\partial_i g_{jk} - \partial_k g_{ij}\right)\mathrm{d}X^j\wedge *\mathrm{d}X^k - g_{ij}\mathrm{d}*\mathrm{d}X^j + \frac{1}{2}H_{ijk}\mathrm{d}X^j\wedge \mathrm{d}X^k = 0. \quad (3.1.17)$$

Since H is closed, locally it can always be written as $\mathrm{d}b$ for some 2-form b, thus giving the local equivalence of the two theories on a classical level. The quantum version of this equivalence does exist, and furthermore, the action with the general WZ term can give additional effects (cf. e.g. [69]), but we shall not concern ourselves with that here since it is not relevant in this thesis.

3.1.1 Symmetries of 2-dimensional sigma model

Consider a 2-dimensional sigma model with a WZ-term, determined by an action functional (3.1.16). As was mentioned before, this sigma model possesses world-sheet symmetries (reparametrization invariance and possible Weyl invariance for the flat target), but it might also have additional symmetries from the target space. Specifically, consider a transformation of the form:

$$\delta X^i = \rho^i \varepsilon \,, \tag{3.1.18}$$

where $\rho \in \Gamma(TM)$ and ε is a real parameter. The corresponding transformation of the action is then:

$$\delta S = -\int_{\Sigma_2} \varepsilon \left(\frac{1}{2} \left(\mathcal{L}_\rho g\right)_{ij} \mathrm{d}X^i \wedge *\mathrm{d}X^j + \frac{1}{2} \left(\iota_\rho H\right)_{ij} \mathrm{d}X^i \wedge \mathrm{d}X^j \right) . \tag{3.1.19}$$

Thus the action is invariant under this transformation if g and H satisfy the following conditions:

$$\begin{aligned} \mathcal{L}_\rho g &= 0\,, & (3.1.20)\\ \iota_\rho H &= \mathrm{d}\beta\,, & (3.1.21) \end{aligned}$$

for some 2-form β. The second condition, when combined with the fact that the 3-form H is closed, is equivalent to the vanishing of the Lie derivative of H:

$$\mathcal{L}_\rho H = 0\,. \tag{3.1.22}$$

It is possible that the vector ρ does not exist for given g and H, but let us assume it does. Furthermore, it is also possible that more than one vector ρ satisfies the above conditions. Let ρ_a, $a = 1, \ldots, r$ denote all such possible vector fields, each having its own parameter ε^a. The full transformation of the fields X^i is then given by:

$$\delta X^i = \rho^i_a \varepsilon^a \,. \tag{3.1.23}$$

Since all possible vector fields have been taken into account, they need to form an algebra. This can be seen by considering two transformations δ_ε and $\delta_{\varepsilon'}$ for two sets of parameters ε and ε', or more specifically their commutator $[\delta_\varepsilon, \delta_{\varepsilon'}]$. Since both transformations leave the action invariant, the same is true of their commutator. Thus $[\delta_\varepsilon, \delta_{\varepsilon'}]X^i$ needs to be of the form (3.1.23) for some parameter ε'' that can depend on ε and ε'. A straightforward calculation gives:

$$[\delta_\varepsilon, \delta_{\varepsilon'}]X^i = [\rho_a, \rho_b]^i \varepsilon^a \varepsilon'^b \,, \tag{3.1.24}$$

thus requiring that vector fields ρ_a form an algebra with respect to the commutator of vector fields. Let $C^c{}_{ab}$ be corresponding structure functions, such that:

$$[\rho_a, \rho_b] = C^c{}_{ab} \rho_c \,. \tag{3.1.25}$$

Then the commutator of the transformation is:

$$[\delta_\varepsilon, \delta_{\varepsilon'}]X^i = \delta_{\varepsilon''} X^i \,, \tag{3.1.26}$$

with ε'' being given by:

$$\varepsilon''^c = C^c{}_{ab} \varepsilon^a \varepsilon'^b \,. \tag{3.1.27}$$

Since the operation in this algebra is just a commutator of vector fields, this algebra is a Lie algebra.

Symmetries described above can be viewed in a bit of a different way. Let L denote a subbundle of the tangent bundle TM spanned by the vector fields ρ_a. Then, if one defines a map $\rho : L \to TM$ as just a simple inclusion map[2], L becomes a Lie algebroid. Indeed, it is not difficult to see that L really satisfies all of the Lie algebroid axioms. The Lie bracket on it is just commutator restricted to include only appropriate vector fields and thus, it also satisfies the Leibniz rule by default. This viewpoint of symmetry transformations allows us to extend the possible symmetries of the 2-dimensional sigma model. Instead of starting with the symmetries of the action, it is possible to start with any Lie algebroid $(L, \rho, [\cdot,\cdot])$ over the target space M and induce the transformations in the sigma model. For every section $\varepsilon \in \Gamma(L)$, one can define a corresponding transformation:

$$\delta X^i = \rho^i(\varepsilon) \,. \tag{3.1.28}$$

To put in different language, if one chooses a local basis e_a on $\Gamma(L)$, and defines the corresponding vector fields $\rho_a = \rho(e_a)$, then the transformation X^i takes the form:

$$\delta X^i = \rho^i_a \varepsilon^a \,, \tag{3.1.29}$$

[2] If A and B are two sets such that $A \subset B$, then inclusion map $f : A \to B$ is defined as $f(x) = x$ for all $x \in A$.

where ε^a denote components of the section ε in the local basis e_a, i.e.

$$\varepsilon = \varepsilon^a e_a\,. \tag{3.1.30}$$

Whether this transformation will be a symmetry for the 2-dimensional sigma model depends on whether the vector fields obtained by projection satisfy the necessary symmetry conditions (3.1.20) and (3.1.21). In any case, one can conclude that the data of a Lie algebroid over the target space M induces the transformations of the 2-dimensional sigma model, with the sections of this Lie algebroid serving as parameters for the transformation.

An additional advantage of the Lie algebroid picture is a possibility to easily define an action of an arbitrary Lie group G on the sigma model, as long as this group allows for representations of the same dimension as the dimension of the target space. Let $\mathfrak{g}$ be a Lie algebra of this group G. Then one can define a Lie algebroid L over M to be simply $M \times \mathfrak{g}$. At any point p in M, one can define a representation of G in the tangent space T_pM, thus giving the anchor map by simply projecting from L to its appropriate representation elements. This again gives the same kind of transformation as before, but giving a bit of a different context. The same kind of principle could be applied to different field theories as well, not just sigma models. The most famous example of such an implementation is probably the Yang-Mills theory.

3.2 Gauging the 2-dimensional sigma model

Consider some field theory on a manifold Σ with the target space M and an action functional S. Furthermore, assume that this field theory possesses a symmetry determined by the group G (which is implemented through the Lie algebroid construction on the target space). Then the physical information of the theory is fully contained in the quotient space M/G [38]. Note that this space does not have to be a manifold, though it certainly could be. From a more geometrical viewpoint, this quotient space can be described as the foliation the corresponding Lie algebroid induces. In this sense, whether the quotient space is a manifold or not depends on the regularity of this foliation.

The standard way to deal with (local) symmetries is to gauge them. This is done by introducing a gauge field A that is a 1-form on Σ and takes values in the space of section of the Lie algebroid $M \times \mathfrak{g}$, with $\mathfrak{g}$ being a Lie algebra of the group G. Then those gauge fields are minimally coupled to the other fields of the theory. Take the 2-dimensional sigma model for example. The first thing one does is to define covariant derivatives of the fields X^i:

$$F^i = \mathrm{d}X^i - \rho^i(A) \tag{3.2.1}$$

Then the gauged extension of the original action is simply obtained by replacing de Rham differential by the corresponding covariant ones:

$$S_{\text{gauged}}[X,A] = -\int_{\Sigma_2} \left(\frac{1}{2} g_{ij}(X) F^i \wedge F^j + \frac{1}{2} B_{ij}(X) F^i \wedge F^j \right) \tag{3.2.2}$$

The full gauge transformations of this gauged theory must then include the transformation of the gauge fields:

$$\delta A = \mathrm{d}\varepsilon + [A, \varepsilon], \tag{3.2.3}$$

for an arbitrary section $\varepsilon \in \Gamma(L)$, while the transformations of X is given as usual:

$$\delta X = \rho(\varepsilon). \tag{3.2.4}$$

Instead of having some gauge group producing the foliation on the target space and extending the theory to its gauged version through minimal coupling, we can do the reverse. Forgetting about the gauge group, imagine we are given a foliation on M and we want to construct the corresponding gauge theory. This turns out to be much more general procedure and we will closely follow the work done in [25, 26].

In this context, the question we want to answer is what conditions does the geometric data of the target space M needs to satisfy in order for a given foliation $\mathcal{F}$ to properly define a gauge theory $S[X,A]$. But before one can tackle this question, it is necessary to define what a gauge theory is in this context. It is a field theory with the action $S[X,A]$ that reduces to that of the 2-dimensional sigma model $S_0[X]$ when all the gauge fields are set to zero, i.e. $S[X,A=0] = S_0[X]$. Furthermore, S has a gauge symmetry which on fields X^i has the form of the transformation along the leaves of the foliation.

In order to deal with this kind of setting, define a bundle L over M that is anchored, i.e. there exist a map $\rho : L \to M$. Furthermore, let e_a be a local basis of sections in $\Gamma(L)$ and assume that their projections $\rho_a = \rho(e_a)$ generate the foliation $\mathcal{F}$. Note that the vector fields ρ_a can be linearly dependent and that L does not need to have any special structure. The existence of L is then guaranteed by the existence of the vector fields that can generate the foliation. It is simply a matter of choosing L and ρ as to project to the right part of the tangent space. In addition, since the vector fields ρ_a need to generate the foliation, they have to form an algebra as well, thus defining the appropriate structure functions:

$$[\rho_a, \rho_b] = C^c{}_{ab}\rho_c\,. \tag{3.2.5}$$

The gauged action depends on the map $X : \Sigma_2 \to M$ and the gauge field A. As before, this gauge field is a 1-form on the world-sheet, but unlike before, it is not evident where does it take values from. When there was a gauge group, the gauge fields took values in the corresponding Lie algebroid, but in this case the Lie algebroid does not exist (in general). That is why the bundle L is necessary. While it is not Lie algebroid, it can be used to define the gauge field A in the sense that A takes values in $\Gamma(L)$. Thus, A can be considered as a map from $\Gamma(T\Sigma_2)$ to $\Gamma(L)$. Combining X and A gives a map $a : T\Sigma_2 \to L$. In accordance with this, the gauged action is a functional of a.

Having defined the basic structure of this gauge theory, it is time to look at appropriate gauge transformation, with those of X^i being already defined by the definition of the gauged action:

$$\delta X^i = \rho^i_a \varepsilon^a\,, \tag{3.2.6}$$

where $\varepsilon = \varepsilon^a e_a \in \Gamma(X^*L)$. This can also be written in terms of the anchor map ρ instead of vector fields ρ_a:

$$\delta X = \rho(\varepsilon)\,. \tag{3.2.7}$$

Though this transformation was immediately defined, the gauge transformation of the gauge fields still needs to be determined. There is another way to look at this. Instead of posing the question of what are the gauge transformation of the gauge field A, it is possible to pose an equivalent question of how to lift the transformation of X to that of a. In this sense, the gauging procedure becomes a lift from the field theory that has Σ_2 as the source and M as a target to field theory that has $T\Sigma_2$ as the source and L as the target, with the appropriate projection giving back the original theory. The problem in question here then becomes finding this lift in order for the given foliation to represent the gauge symmetry of this lifted theory.

At this point it becomes inconvenient that L does not have any additional algebraic structure. This can be easily remedied however, by defining a skew-symmetric bracket $[\cdot,\cdot]$ on the space of sections $\Gamma(L)$ such that ρ is an algebra homomorphism from this algebra to the algebra of vector fields ρ_a. In local basis in $\Gamma(L)$ this means that:

$$\rho([e_a, e_b]) = [\rho_a, \rho_b]\,. \tag{3.2.8}$$

Furthermore, this bracket is required to satisfy the Leibniz rule of the Lie algebroid:

$$[e, fe'] = f[e,e'] + (\iota_{\rho(e)}\mathrm{d}f)e', \tag{3.2.9}$$

for all sections $e, e' \in \Gamma(L)$ and functions $f \in C^\infty(M)$. This kind of bracket can always be constructed in the following way. First one defines the bracket between the basis sections using the structure functions from the algebra of vector fields ρ_a:

$$[e_a, e_b] = C^c{}_{ab} e_c, \tag{3.2.10}$$

There is no obstruction to doing this since the bracket has no prior structure other than being skew-symmetric and since the structure functions are antisymmetric in their down indices, this property is automatically satisfied. Next, one defines $[e_a, fe_b]$ for any function $f \in C^\infty(M)$ such that the Leibniz rule is satisfied:

$$[e_a, fe_b] = fC^c{}_{ab} e_c + (\iota_{\rho_a}\mathrm{d}f)e_b, \tag{3.2.11}$$

and $[fe_a, e_b]$ in order for the bracket to be skew-symmetric:

$$[fe_a, e_b] = -[e_b, fe_a]. \tag{3.2.12}$$

Finally, the bracket of two arbitrary sections is defined by requiring the bracket to be bilinear. This bracket now satisfies all of the required properties, but it does not necessarily give L a structure of a Lie algebroid. The bracket does not have to satisfy Jacobi identity. Only the projection of the Jacobiator via the anchor ρ has to vanish, not the Jacobiator itself. We shall refer to such a structure as an almost Lie algebroid.

To proceed further, one needs to make an Ansatz for the form of gauge transformation of A and the gauged action. For the gauge transformation, we shall take the form given by the minimal coupling with the additional terms:

$$\delta A^a = \mathrm{d}\varepsilon^a + C^a{}_{bc} A^b \varepsilon^c + \omega^a_{bi}\varepsilon^b F^i + \phi^a_{bi}\varepsilon^b * F^i + \Delta A^a, \tag{3.2.13}$$

where F^i is the covariant derivative of X^i of the same form as was in the case of minimal coupling:

$$F^i = \mathrm{d}X^i - \rho^i_a A^a, \tag{3.2.14}$$

and ω^a_{bi} and ϕ^a_{bi} are yet undetermined coefficients. The contribution ΔA^a leaves the possibility of extra terms, not taken into account explicitly. Notice that the term containing the Hodge dual of the covariant derivative F is only possible here because the world-sheet is 2-dimensional, for otherwise $*F$ and A would have different form degrees. As for the gauged action, it is possible to consider only minimal coupling, but here we shall consider a more general version, though leaving the minimal coupling at the metric sector for the moment. Thus, the general Ansatz for the gauged action has the form:

$$S[X,A] = -\int_{\Sigma_2}\left(\frac{1}{2}g_{ij}(X)F^i\wedge *F^j + A^a\wedge\theta_a(X) + \frac{1}{2}\gamma_{ab}(X)A^a\wedge A^b\right) - \int_{\Sigma_3}H(X)\,, \quad (3.2.15)$$

Here $\gamma_{ab}(X)$ and $\theta_a(X) = \theta_{ai}(X)\mathrm{d}X^i$ are a function and a 1-form on M, respectively, both pulled-back by X.

With the above Ansatz, it is possible to find the transformation of the gauged action S and the conditions under which that transformation vanishes. First thing to note is that ΔA cannot depend on ε. However, it can depend on some other gauge parameters, as was explored in [26]. Here, we shall simply assume that this contribution does not exist and focus on the rest. Then, by requiring that $\delta S = 0$, one is led to two conditions for the fields g and H:

$$\begin{aligned} \mathcal{L}_{\rho_a}g &= \omega_a^b\vee\iota_{\rho_b}g + \phi_a^b\vee\theta_b\,, && (3.2.16)\\ \iota_{\rho_a}H &= \mathrm{d}\theta_a - \omega_a^b\wedge\theta_b - \phi_a^b\wedge\iota_{\rho_b}g\,. && (3.2.17)\end{aligned}$$

In addition to these conditions, there are two extra constraints on the gauging data:

$$\begin{aligned} \gamma_{ab} &= \iota_{\rho_a}\theta_b\,, && (3.2.18)\\ \mathcal{L}_{\rho_a}\theta_b &= C^c{}_{ab}\theta_c + \iota_{\rho_b}\mathrm{d}\theta_a + \iota_{\rho_a}\iota_{\rho_b}H\,. && (3.2.19)\end{aligned}$$

The first of these constraints simply expresses γ in terms of ρ and θ. However, since γ is antisymmetric, this poses a constraint of ρ and θ:

$$\iota_{\rho_a}\theta_b + \iota_{\rho_b}\theta_a = 0\,. \quad (3.2.20)$$

Remembering that ρ is a section of the tangent bundle and θ is a section of the cotangent bundle, those two can be combined into section $\rho_a + \theta_a$ of the generalized tangent bundle $TM\oplus T^*M$. Since the generalized tangent bundle is automatically equipped with a bilinear form:

$$\langle\rho+\theta,\rho'+\theta'\rangle = \frac{1}{2}\left(\iota_\rho\theta' + \iota_{\rho'}\theta\right)\,, \quad (3.2.21)$$

the last constraint can be written as:

$$\langle \rho_a + \theta_a, \rho_b + \theta_b \rangle = 0\,. \tag{3.2.22}$$

In other words, sections $\rho_a + \theta_a$ are constrained to an isotropic subbundle of the generalized tangent bundle.

The second constraint also has a simple geometric interpretation. It forces $C^c{}_{ab}\theta_c$ to lie in the form part of the twisted Dorfman bracket (2.5.21) of the generalized tangent bundle. Remembering that the vector of the Dorfman bracket is just commutator of the vector fields, this constraint can finally be written in the form:

$$(\rho_a + \theta_a) \circ (\rho_b + \theta_b) = C^c{}_{ab}(\rho_c + \theta_c)\,. \tag{3.2.23}$$

Since sections $\rho_a + \theta_a$ must lie in the isotropic subbundle, in which Dorfman and Courant bracket coincide, it is possible to replace this Dorfman bracket with the Courant bracket, thus giving the equivalent condition:

$$[\rho_a + \theta_a, \rho_b + \theta_b] = C^c{}_{ab}(\rho_c + \theta_c)\,. \tag{3.2.24}$$

This means that the isotropic subbundle containing these sections must be involutive as well, in the sense that it is closed under the Courant/Dorfman bracket, thus making it a Dirac structure. This has an immediate consequence since Dirac structures are Lie algebroids. In this case, the important thing is that the image of the anchor of this Dirac structure (which is just the projection to the tangent bundle) is the same as $\rho(L)$, meaning that the two define the same foliation on M. As such, one can take L to be this Dirac structure giving it the geometrical structure of the Lie algebroid (not just almost Lie algebroid). In what follows, we shall assume this has been done.

Finally, one can also find a geometrical interpretation for the coefficients ω^a_{bi} and ϕ^a_{bi}. In order to do so, consider a change of basis in the vector bundle L:

$$e_a \to \Lambda^b_a(X) e_b\,. \tag{3.2.25}$$

Since the gauge field $A = A^a e_a$ is independent of the choice of this basis, its components then need to transform as:

$$A^a \to (\Lambda^{-1}(X))^a_b A^b\,. \tag{3.2.26}$$

Next, the algebra of basis sections gives the transformation of the structure functions:

$$C^c{}_{ab} \to (\Lambda^{-1})^c_d \Lambda^e_a \Lambda^f_b C^d{}_{ef} + 2(\Lambda^{-1})^c_d \Lambda^e_{[a} \iota_{\rho_{|e|}} \mathrm{d}\Lambda^d_{b]} \,. \tag{3.2.27}$$

As expected, these do not transform as tensors. Finally, requiring that the gauge transformations after the change of basis have the same form, we obtain the transformations of ω and ϕ:

$$\omega^a_{bi} \quad \to \quad (\Lambda^{-1})^a_c \omega^c_{di} \Lambda^d_b - \Lambda^c_b \partial_i (\Lambda^{-1})^a_c \,, \tag{3.2.28}$$

$$\phi^a_{bi} \quad \to \quad (\Lambda^{-1})^a_c \phi^c_{di} \Lambda^d_b \,. \tag{3.2.29}$$

It is immediately evident that ϕ^a_b transforms as a tensor, while ω^a_b does not. Instead, it transforms as a connection, as was already mentioned in [70]. Thus ω^a_b can be interpreted as the coefficients of the connection ∇ on L:

$$\nabla e_a = \omega^b_a \otimes e_b \,. \tag{3.2.30}$$

On the other hand, ϕ is a 1-form on M that is valued in the space of endomorphisms of L, i.e. $\phi \in \Gamma(T^*M \otimes L \otimes L^*)$. Remembering that the sum of a connection and an endomorphism is again a connection, it is possible to define:

$$\Omega^{\pm a}{}_b = \omega^a_b \pm \phi^a_b \,, \tag{3.2.31}$$

which are the coefficients of the connections:

$$\nabla^\pm = \nabla \pm \phi \,, \tag{3.2.32}$$

such that:

$$\nabla^\pm e_a = \Omega^{\pm b}{}_a \otimes e_b \,. \tag{3.2.33}$$

In the context of gauging, this means that the geometrical data on the target space, specifically g and H need to be related to two connections $\nabla^\pm$ through the conditions (3.2.16) and (3.2.17). Rewriting those conditions in terms of $\Omega^\pm$ produces:

$$\mathcal{L}_{\rho_a} g \quad = \quad \frac{1}{2}\Omega^{+b}{}_a \vee (\iota_{\rho_b} g + \theta_b) + \frac{1}{2}\Omega^{-b}{}_a \vee (\iota_{\rho_b} g - \theta_b) \,, \tag{3.2.34}$$

$$\iota_{\rho_a} H \quad = \quad \mathrm{d}\theta_a - \frac{1}{2}\Omega^{+b}{}_a \wedge (\theta_b + \iota_{\rho_b} g) + \frac{1}{2}\Omega^{-b}{}_a \wedge (\iota_{\rho_b} g - \theta_b) \,. \tag{3.2.35}$$

This prompts the definition of sections $\mathcal{G}_{\pm} \in \Gamma(T^*M \otimes L^*)$ as:

$$\mathcal{G}_{\pm} = \theta \pm \rho^*, \tag{3.2.36}$$

where $\rho^* = g(\rho)$. Therefore, the above expressions can now be more neatly written:

$$\begin{aligned} \mathcal{L}_{\rho_a} g &= \frac{1}{2}\left(\Omega^{+}{}_{a}^{b} \vee \mathcal{G}_{+b} - \Omega^{-}{}_{a}^{b} \vee \mathcal{G}_{-b}\right), && (3.2.37)\\ \iota_{\rho_a} H &= \mathrm{d}\theta_a - \frac{1}{2}\left(\Omega^{+}{}_{a}^{b} \wedge \mathcal{G}_{+b} + \Omega^{-}{}_{a}^{b} \wedge \mathcal{G}_{-b}\right). && (3.2.38) \end{aligned}$$

To proceed further, one needs a theorem, proven in [25], that states that $\mathcal{G}_{\pm}$, viewed as maps from the Dirac structure to the cotangent bundle T^*M, are invertible, as long as the Dirac structure is of the full rank. Here, we shall assume that is the case and only consider full Dirac structures without explicitly stating it further. This has several important consequences. The first one is upon field equations of the gauged theory. Variation of the gauged action (3.2.15) with respect to the gauge field gives the following field equation:

$$(\theta_{ai} - (\iota_{\rho_a} g)_i *) F^i = 0. \tag{3.2.39}$$

But the operator in parenthesis is $\mathcal{G}_{-}$, up to the Hodge dual which does not spoil the invertibility [18]. Thus, the field equation for the gauged theory simplifies to:

$$F^i = 0. \tag{3.2.40}$$

The other field equation, obtained from the variation of the action with respect to the fields X^i, is:

$$G_i = \mathrm{d}(\theta_{ai} A^a) + \frac{1}{2}\left(\rho_b^j \partial_i \theta_{aj} - \theta_{aj} \partial_i \rho_b^j + \rho_a^j \rho_b^k H_{ijk}\right) A^a \wedge A^b = 0. \tag{3.2.41}$$

Apart from field equations, the invertibility of $\mathcal{G}_{\pm}$ makes a significant impact on the gauging possibilities. It was shown in [25] that the gauging conditions (3.2.37) and (3.2.38) can be inverted as well, giving the connection components $\Omega^{\pm}$ in terms of the geometrical data on the target space (g and H):

$$\Omega^{\pm}{}_{bi}^{a} = (\mathcal{G}_{\pm}^{-1})^{aj}\left(\partial_i \mathcal{G}_{\pm bj} - \mathring{\Gamma}^k_{ji} \mathcal{G}_{\pm bk} - \frac{1}{2}\rho_b^k H_{ijk}\right), \tag{3.2.42}$$

where $\mathring{\Gamma}$ are components of any torsion-free connection M.[3] This means that the gauging of

[3] In this sense, the invertibility of gauging conditions is not exact, but it would more precise to say that the

the 2-dimensional sigma model is always possible without any extra conditions on geometrical data of M.

3.2.1 Non-minimal coupling to the metric sector

Up to now, we have considered only the gauging with a minimal coupling to the metric sector. However, one can look for more general gauged actions, which we explore in this section.

As a more general Ansatz here, we take for the gauged action:

$$S[X,A] = S_0[X] - \int_{\Sigma_2} \left(A^a \wedge \theta_a(X) + A^a \wedge *\widetilde{\theta}_a(X) + \frac{1}{2}\gamma_{ab}(X) A^a \wedge A_b + \frac{1}{2}\widetilde{\gamma}_{ab}(X) A^a \wedge *A^b \right) , \tag{3.2.43}$$

where $\theta_a(X) = \theta_{ai}(X)\mathrm{d}X^i$ and $\widetilde{\theta}_a = \widetilde{\theta}_{ai}(X)\mathrm{d}X^i$ are 1-forms, γ_{ab} and $\widetilde{\gamma}_{ab}$ functions on M, all pulled back to Σ_2 by X, and $S_0[X]$ is just the ungauged action of the 2-dimensional sigma model. It is now necessary to find conditions on the background fields g and H, as well as the constraints on the gauging data θ, $\widetilde{\theta}$, γ, $\widetilde{\gamma}$ and ρ for this more general gauging to be possible. To do that, the gauge transformations of the fields are required. The transformation of X is known, while for the gauge fields we make a more general Ansatz than before

$$\begin{aligned} \delta X^i &= \rho^i_a(X)\varepsilon_a , && (3.2.44)\\ \delta A^a &= r^a{}_b(X)\mathrm{d}\varepsilon^b + s^a{}_b(X) * \mathrm{d}\varepsilon^b + \omega^a{}_{bi}(X)\varepsilon^b F^i + \phi^a{}_{bi}(X)\varepsilon^b * F^i + \\ &\quad + \chi^a{}_{bc}(X) A^b \varepsilon^c + \psi^a{}_{bc}(X) * A^b \varepsilon^c , && (3.2.45) \end{aligned}$$

where $\varepsilon^a \in \Gamma(X^*L)$ is the scalar gauge parameter, $r^a{}_b(X)$, $s^a{}_b(X)$, $\chi^a{}_{bc}(X)$ and $\psi^a{}_{bc}(X)$ are functions and $\omega^a{}_b(X)$ and $\phi^a{}_b$ 1-forms on M, pulled back to Σ_2.

Given these gauge transformations of X and A, it is easy to find the gauge transformation of the action (3.2.43). Requirement that such a transformation vanish gives conditions for the fields g and H:

$$\begin{aligned} \mathcal{L}_{\rho_a} g &= -\omega^b{}_a \vee \widetilde{\theta}_b + \phi^b{}_a \vee \theta_b , && (3.2.46)\\ \iota_{\rho_a} H &= \mathrm{d}\left(r^b{}_a \theta_b - s^b{}_a \widetilde{\theta}_b \right) - \omega^b{}_a \wedge \theta_b + \phi^b{}_a \wedge \widetilde{\theta}_b , && (3.2.47)\\ \iota_{\rho_a} g &= s^b{}_a \theta_b - r^b{}_a \widetilde{\theta}_b , && (3.2.48) \end{aligned}$$

gauging condition equations can be solved for the connection components $\Omega^{\pm}$. There are infinitely many solutions here since there are infinitely many possibilities to define torsion-free connection on M.

in addition to the following constraints:

$$\begin{aligned}
\iota_{\rho_a}\theta_b &= r^c{}_a\gamma_{cb} - s^c{}_a\widetilde{\gamma}_{cb}\,, &(3.2.49)\\
\iota_{\rho_a}\widetilde{\theta}_b &= s^c{}_a\gamma_{cb} - r^c{}_a\widetilde{\gamma}_{cb}\,, &(3.2.50)\\
\mathcal{L}_{\rho_a}\theta_b &= -\chi^c{}_{ba}\theta_c + \psi^c{}_{ba}\widetilde{\theta}_c + \iota_{\rho_b}\mathrm{d}\left(r^c{}_a\theta_c - s^c{}_a\widetilde{\theta}_c\right) + \iota_{\rho_a}\iota_{\rho_b}H + \\
&\quad + \left(\gamma_{cb} + \iota_{\rho_b}\theta_c\right)\omega^c{}_a - \left(\widetilde{\gamma}_{cb} + \iota_{\rho_b}\widetilde{\theta}_c\right)\phi^c{}_a\,, &(3.2.51)\\
\mathcal{L}_{\rho_a}\widetilde{\theta}_b &= -\chi^c{}_{ba}\widetilde{\theta}_c + \psi^c{}_{ba}\theta_c - \iota_{\rho_b}\mathcal{L}_{\rho_a}g + \\
&\quad + \left(\gamma_{cb} + \iota_{\rho_b}\theta_c\right)\phi^c{}_a - \left(\widetilde{\gamma}_{cb} + \iota_{\rho_b}\widetilde{\theta}_c\right)\omega^c{}_a\,, &(3.2.52)\\
\frac{1}{2}\mathcal{L}_{\rho_a}\gamma_{cb} &= \gamma_{d[c}\chi^d{}_{b]a} - \widetilde{\gamma}_{d[c}\psi^d{}_{b]a} - \gamma_{d[c}\iota_{\rho_{b]}}\omega^d{}_a + \widetilde{\gamma}_{d[c}\iota_{\rho_{b]}}\phi^d{}_a\,, &(3.2.53)\\
\frac{1}{2}\mathcal{L}_{\rho_a}\widetilde{\gamma}_{cb} &= -\widetilde{\gamma}_{d[c}\chi^d{}_{b]a} + \gamma_{d[c}\psi^d{}_{b]a} + \widetilde{\gamma}_{d[c}\iota_{\rho_{b]}}\omega^d{}_a - \gamma_{d[c}\iota_{\rho_{b]}}\phi^d{}_a\,. &(3.2.54)
\end{aligned}$$

At this point it is convenient to consider redefined quantities:

$$\begin{aligned}
\theta^\pm_a &= \theta_a \pm \widetilde{\theta}_a\,, &(3.2.55)\\
\gamma^\pm_{ab} &= \gamma_{ab} \pm \widetilde{\gamma}_{ab}\,, &(3.2.56)\\
r^{\pm a}{}_b &= \frac{1}{2}\left(r^a{}_b \pm s^a{}_b\right), &(3.2.57)\\
\Omega^{\pm a}{}_b &= \omega^a{}_b \pm \phi^a{}_b\,, &(3.2.58)\\
\mathcal{C}^{\pm c}{}_{ab} &= -\chi^c{}_{ab} \mp \psi^c{}_{ab} + \frac{1}{2}\iota_{\rho_a}\Omega^{\pm c}{}_b\,. &(3.2.59)
\end{aligned}$$

In the terms of these new quantities, the above conditions become:

$$\begin{aligned}
\mathcal{L}_{\rho_a}g &= \frac{1}{2}\left(\Omega^{+b}{}_a \vee \theta^-_b - \Omega^{-b}{}_a \vee \theta^+_b\right), &(3.2.60)\\
\iota_{\rho_a}H &= \mathrm{d}\left(r^{+b}{}_a\theta^-_b + r^{-b}{}_a\theta^+_b\right) - \frac{1}{2}\left(\Omega^{+b}{}_a \wedge \theta^-_b + \Omega^{-b}{}_a \wedge \theta^+_b\right), &(3.2.61)\\
\iota_{\rho_a}g &= r^{+b}{}_a\theta^-_b - r^{-b}{}_a\theta^+_b\,, &(3.2.62)
\end{aligned}$$

and the constraints simplify to:

$$\begin{aligned} \frac{1}{2}\iota_{\rho_a}\theta^{\pm}_b &= r^{\pm c}{}_a\gamma^{\mp}_{cb}\,, & (3.2.63)\\ \mathcal{L}_{\rho_a}\theta^{\pm}_b &= C^{\mp c}{}_{ba}\,\theta^{\pm}_c+\frac{1}{2}\Omega^{\pm c}{}_a\gamma^{\mp}_{cb}\,, & (3.2.64)\\ \frac{1}{2}\mathcal{L}_{\rho_a}\gamma^{\pm}_{cb} &= \gamma^{\mp}_{d[b}C^{\pm d}{}_{c]a}\,. & (3.2.65)\end{aligned}$$

Under the assumption that $r^{\pm}$ are invertible, it is possible to redefine the gauge field as:

$$\widetilde{A}^a=\frac{1}{4}\left((r^+)^{-1}\right)^a{}_b(A^b+*A^b)+\frac{1}{4}\left((r^-)^{-1}\right)^a{}_b(A^b-*A^b)\,. \qquad (3.2.66)$$

With this new field, using (3.2.62) and (3.2.63), the gauged action becomes:

$$S[X,\widetilde{A}]=-\int_{\Sigma_2}\left(\frac{1}{2}g_{ij}(X)\widetilde{F}^i\wedge*\widetilde{F}^j+\widetilde{A}^a\wedge\theta'_a(X)+\frac{1}{2}\gamma'_{ab}(X)\widetilde{A}^a\wedge\widetilde{A}^b\right)-\int_{\Sigma_3}H(X)\,, \quad (3.2.67)$$

where $\widetilde{F}$, θ'_a and γ'_{ab} are defined as:

$$\begin{aligned} \widetilde{F}^i &= \mathrm{d}X^i-\rho^i_a\widetilde{A}^a\,, & (3.2.68)\\ \theta'_a &= r^{-b}{}_a\theta^+_b+r^{+b}{}_a\theta^-_b\,, & (3.2.69)\\ \gamma'_{ab} &= 2r^{-c}{}_a r^{+d}{}_b\gamma^+_{cd}+2r^{+c}{}_a r^{-d}{}_b\gamma^-_{cd}\,. & (3.2.70)\end{aligned}$$

But this is just the minimal coupling to the metric sector. So the only nonstandard gauging is in the topological sector. This also forces the simplification of gauge transformations to:

$$\delta\widetilde{A}^a=\mathrm{d}\varepsilon^a+C^a{}_{bc}(X)\widetilde{A}^b\varepsilon^c+\omega'^a{}_{bi}(X)\varepsilon^b\widetilde{F}^i+\phi'^a{}_{bi}(X)\varepsilon^b*\widetilde{F}^i\,, \qquad (3.2.71)$$

with $\omega'^a{}_b$ and $\phi'^a{}_b$ defined as:

$$\begin{aligned} \omega'^a{}_b &= \frac{1}{4}\left((r^+)^{-1}\right)^a{}_c\Omega^{+c}{}_b+\frac{1}{4}\left((r^-)^{-1}\right)^a{}_c\Omega^{-c}{}_b\,, & (3.2.72)\\ \phi'^a{}_b &= \frac{1}{4}\left((r^+)^{-1}\right)^a{}_c\Omega^{+c}{}_b-\frac{1}{4}\left((r^-)^{-1}\right)^a{}_c\Omega^{-c}{}_b\,. & (3.2.73)\end{aligned}$$

The above simplification is valid only when $r^{\pm}$ are invertible. For all the other cases, it is possible to prove that the gauge theory would not be diffeomorphism invariant. Specifically, it is impossible to construct covariant field strength for the fields X^i. First notice that the field strength F^i in gauge transformation (3.2.45) of the gauge fields is the covariant field strength for the case when the gauge fields are minimally coupled to the metric, i.e. when s vanishes and r is an identity. In the present situation it is necessary to construct a new covariant field strength $\widetilde{F}^i$. Since it represents a covariant derivative of the fields X^i, it has to be equal to $\mathrm{d}X^i$ plus some extra terms depending on the gauge fields. Furthermore, the transformation of $\widetilde{F}$ cannot contain $\mathrm{d}\varepsilon$ for it to be a covariant field strength. Because transformation of $\mathrm{d}X^i$ contains $\rho^i_a \mathrm{d}\varepsilon^a$, the transformation of extra terms needs to contain $-\rho^i_a \mathrm{d}\varepsilon^a$. Given that $\mathrm{d}\varepsilon$ in transformation A does not include ρ, the extra terms in $\widetilde{F}^i$ need to be equal to:

$$-\rho^i_a(\alpha^a_b A^b + \widetilde{\alpha}^a_b * A^b)\,, \tag{3.2.74}$$

where $\alpha^a_b(X)$ and $\widetilde{\alpha}^a_b(X)$ are some functions that have to be determined. If $r^\pm$ are invertible, a direct calculation yields:

$$\alpha^a_b = ((r^+)^{-1})^a_b + ((r^-)^{-1})^a_b\,, \tag{3.2.75}$$

$$\widetilde{\alpha}^a_b = ((r^+)^{-1})^a_b - ((r^-)^{-1})^a_b\,. \tag{3.2.76}$$

In order to deal with other cases, it is more convenient to define $\alpha^\pm = \alpha \pm \widetilde{\alpha}$. Since the transformation of $\mathrm{d}X^i$ includes only $\mathrm{d}\varepsilon^a$ and not $*\mathrm{d}\varepsilon^a$, the same must be true for (3.2.74). This produces a condition on $\alpha^\pm$:

$$\alpha^+ r^+ = \alpha^- r^-\,, \tag{3.2.77}$$

making the only relevant term in the transformation of (3.2.74) equal to:

$$-2\rho^i_a {\alpha^+}^a_b {r^+}^b_c \mathrm{d}\varepsilon^c\,. \tag{3.2.78}$$

This means that $\alpha^+ r^+$ must be proportional to identity, but it would also be true of $\alpha^- r^-$ because of (3.2.77). But this can only hold if both r^+ and r^- are invertible which is in contradiction with the original assumption. We conclude that the covariant field strengths $\widetilde{F}^i$ do not exist, meaning that the resulting gauge theory could not be diffeomorphism invariant and so is of no interest here. Thus, the gauging in the metric sector can always be put in the form of minimal coupling in all of relevant situations (meaning that we require diffeomorphism invariance).

3.3 Dirac sigma model

Dirac sigma models, first introduced in [18], are 2-dimensional sigma models that are of significant interest here. Let M be a target space for the model, equipped with a metric g and closed 3-form H, and D some Dirac structure of the generalized tangent bundle, viewed as an H-twisted Courant algebroid. Then the action functional for the model is given by:

$$S_{\mathrm{DSM}}[X, v+\eta] = -\int_{\Sigma_2}\left(\frac{1}{2}g_{ij}(X)F^i \wedge *F^j + \eta_i \wedge \mathrm{d}X^i - \frac{1}{2}\eta_i \wedge v^i\right) - \int_{\Sigma_3} H\,, \tag{3.3.1}$$

where $v+\eta$ is a 1-form on the world-sheet Σ_2 and taking values in the pull-back (with respect to X) of D, while F^i is a covariant derivative of X^i defined as:

$$F^i = \mathrm{d}X^i - v^i\,. \tag{3.3.2}$$

This model is topological[4] if D is taken to be a full Dirac structure [18], while for small Dirac structures one obtains a non-topological sigma model [63]. Here, we shall only consider topological ones, i.e. we shall assume D is a full Dirac structure.

The Dirac sigma model is closely related to the gauging of 2-dimensional sigma models. By defining:

$$v = \rho(A)\,, \tag{3.3.3}$$
$$\eta = \theta(A)\,, \tag{3.3.4}$$

in the gauged sigma model action (3.2.15) and its corresponding gauge symmetries, the action takes the form of the Dirac sigma model. Furthermore, because of the invertibility of $\mathcal{G}_\pm$ introduced in the same section, this process is always possible to do in reverse as well [25], i.e. the topological Dirac sigma model can always be written as the gauged action (3.2.15). As a result, when considering the Dirac sigma models in what follows, we shall immediately assume the gauged action (3.2.15), since it is much more convenient for the present purposes.

For future convenience, we now construct some additional geometric structures and define some notation. First, notice that the connections $\nabla^\pm$ on L induce connections $\nabla^{*\pm}$ on T^*M:

$$\nabla^{*\pm} = \mathcal{G}_\pm \circ \nabla^\pm \circ \mathcal{G}_\pm^{-1}\,. \tag{3.3.5}$$

[4] A sigma model is considered topological if physical results do not depend on the metric of the world-sheet.

Their connection components in the local basis are then equal to:

$$\Gamma^{*\pm k}_{\ \ ij} = -\mathring{\Gamma}^{k}_{ij} + \frac{1}{2}\left(\mathcal{G}_{\pm}^{-1}\right)^{ka}(\iota_{\rho_a} H)_{ij} \,. \tag{3.3.6}$$

As is immediately seen from the second term on the right-hand side, these connections are not torsion-free, but their torsion tensors (or more precisely, the corresponding components) are equal to:

$$\Theta^{\pm k}_{\ \ ij} = 2\Gamma^{*\pm k}_{\ \ [ij]} = \left(\mathcal{G}_{\pm}^{-1}\right)^{ka}(\iota_{\rho_a} H)_{ij} \,. \tag{3.3.7}$$

These torsions play a role if we consider the curvatures of the original connections $\nabla^{\pm}$:

$$R^{\pm a}_{\ \ b} = \mathrm{d}\Omega^{\pm a}_{\ \ b} + \Omega^{\pm a}_{\ \ c} \wedge \Omega^{\pm c}_{\ \ b} \,, \tag{3.3.8}$$

or more precisely, the Bianchi identity they satisfy:

$$\nabla^{\pm}_{[i} R^{\pm a}_{\ \ bjk]} + \Theta^{\pm l}_{\ \ [ij} R^{\pm a}_{\ \ bk]l} = 0 \,, \tag{3.3.9}$$

where the antisymmetrization includes only indices of M.

Furthermore, using the structures introduced in Section 2.3.2, the connections $\nabla^{\pm}$ on L can be used to define two L-connections on L as well:

$$^{L}\nabla^{\pm}_{e} e' = \nabla^{\pm}_{\rho(e)} e' \,, \tag{3.3.10}$$

with their L-torsion tensors being:

$$T^{\pm c}_{\ \ ab} = -C^{c}_{\ ab} + 2\iota_{\rho_{[a}}\Omega^{\pm c}_{\ \ b]} \,. \tag{3.3.11}$$

The corresponding basic curvatures are then given by:

$$S^{\pm a}_{\ \ bc} = \nabla^{\pm} T^{\pm a}_{\ \ bc} + 2\iota_{\rho_{[b}} R^{\pm a}_{\ \ c]} \,. \tag{3.3.12}$$

These torsions and curvatures satisfy a handful of useful identities that shall become significant in the next chapter:

$$\begin{aligned}
T^{\pm c}{}_{ab}\,\rho^i_c &= -2\rho^j_{[a}\nabla^\pm_j\rho^i_{b]} + \rho^j_a\rho^k_b\Theta^{\pm i}{}_{jk}\,, && (3.3.13)\\
\iota_{\rho_{[b}}\iota_{\rho_a}R^{\pm d}{}_{c]} &= \rho^i_{[a}\nabla^\pm_i T^{\pm d}{}_{bc} + T^{\pm d}{}_{e[a}\,T^{\pm e}{}_{bc]}\,, && (3.3.14)\\
\left[\nabla^\pm_i,\nabla^\pm_j\right]T^{\pm a}{}_{bc} &= -\Theta^{\pm k}{}_{ij}\nabla^\pm_k T^{\pm a}{}_{bc} + T^{\pm d}{}_{bc}\,R^{\pm a}{}_{dij} - T^{\pm a}{}_{dc}\,R^{\pm d}{}_{bij} - T^{\pm a}{}_{bd}\,R^{\pm d}{}_{cij}\,, && (3.3.15)\\
T^{\pm c}{}_{ab} &= (\mathcal{G}^{-1}_\pm)^{ci}\left(\rho^j_{[a}\nabla^\pm_i\theta_{b]j} - \theta_{[bj}\nabla^\pm_i\rho^j_{a]} - \rho^j_{[a}\theta_{b]k}\Theta^{\pm k}{}_{ji}\right), && (3.3.16)
\end{aligned}$$

with $\nabla^\pm$ acting both as the original $\nabla^\pm$ and as $\nabla^{*\pm}$ (or its dual), the first one on indices from the Dirac structure and latter on indices from the target space.

3.3.1 Target space covariance

Up to now, the action, the field equations and the gauge transformations of the Dirac sigma model has been presented with manifest spacetime covariance, but not target space covariance. Here we show how the connections $\nabla^\pm$ guarantee this covariance.

Let us first consider gauge transformations. The one for X was already written in the basis-independent way:

$$\delta X = \rho(\varepsilon)\,. \tag{3.3.17}$$

For the other gauge transformation, one should first note that the 1-form gauge field is $A = A^a \otimes e_a$, meaning that the full transformation of A should include, besides gauge transformation of A^a, the transformation coming from the frame change due to the change of base points. Any of the connections $\nabla^\pm$ can be used to take into account this change of frame so $\delta e_a = \Omega^{\pm b}{}_{ai}\delta X^i e_b$, which then gives the transformation of A:

$$\delta A = (\delta A^a + \iota_{\rho_c}\Omega^{\pm a}{}_b A^b\varepsilon_c)\otimes e_a\,. \tag{3.3.18}$$

Also, δA^a has to be rewritten in terms of connection which then gives:

$$\delta A^a = \mathrm{D}^\pm\varepsilon^a - \left(T^{\pm a}{}_{bc} + \iota_{\rho_c}\Omega^{\pm a}{}_b\right)A^b\varepsilon^c + \frac{1}{2}\left(\Omega^{+a}{}_{bi}(1+*) + \Omega^{-a}{}_{bi}(1-*) - 2\Omega^{\pm a}{}_{bi}\right)\varepsilon^b F^i\,. \tag{3.3.19}$$

This then expresses δA in terms of Ω^+ or Ω^-. By adding those two options together, the final form of the gauge transformation is obtained:

$$\delta A = \frac{\mathrm{D}^+ + \mathrm{D}^-}{2}\varepsilon - \frac{T^+ + T^-}{2}(A,\varepsilon) + \left\langle\frac{\Omega^+ - \Omega^-}{2}, *F\right\rangle(\varepsilon)\,, \tag{3.3.20}$$

where $\mathrm{D}^{\pm}$ denote exterior derivatives corresponding to the connections $\nabla^{\pm}$. Notice that $(\Omega^{+}-\Omega^{-})/2$ is equal to the tensor ϕ, so this is indeed the tensorial form of the gauge transformation.

Having written the gauge transformation in the target space covariant form, one can do the same with the field equations. The field equations for A is already target space covariant:

$$F=\mathrm{d}X-\rho(A)\,. \tag{3.3.21}$$

To covariantize the other field equations, it is useful to define $\mathfrak{a}=\theta(A)$. Then the field equation becomes:

$$G=\mathrm{D}^{\pm}\mathfrak{a}-\frac{1}{2}T^{\pm}_{\mathcal{G}_{\pm}}(A,A)\mp\frac{1}{2}\Theta^{\pm}_{*}(\rho(A),\rho(A),\cdot)\,, \tag{3.3.22}$$

where $T^{\pm}_{\mathcal{G}_{\pm}}=\mathcal{G}_{\pm a}\otimes T^{\pm a}\in\Gamma(T^{*}M\otimes\Lambda^{2}E^{*})$ and $\Theta^{\pm}_{*}$ is the contraction of the torsion tensor Θ with the metric, such that its components are $\Theta^{\pm}_{*ijk}=g_{il}\Theta^{\pm l}{}_{jk}$.

Finally, the action itself has to be rewritten in the manifestly target space covariant form. This is easy to do using the maps $\rho:E\to TM$ and $\theta:E\to T^{*}M$:

$$S_0=-\int_{\Sigma_2}\left(||F||^{2}+\left\langle(X^{*}\theta)(A),\mathrm{d}X+\frac{1}{2}(X^{*}\rho)(A)\right\rangle\right)-\int_{\Sigma_3}X^{*}H\,, \tag{3.3.23}$$

where $||F||^{2}=(X^{*}g)(F\overset{\wedge}{,}*F)$.

3.3.2 Twisted Poisson sigma model

Let us consider a specific example of the Dirac sigma model. Suppose the target space M is a twisted Poisson manifold with a twisted Poisson bivector Π and the corresponding 3-form H. In the context of gauging, it is possible to choose L to be a Lie algebroid induced by this structure, as described in section 2.3.1. Here, L is just a cotangent bundle so for the basis section e_a od L, one can simply take the section obtained from the local coordinates on M, i.e. $\mathrm{d}x^{a}$.[5] Projecting these to the tangent bundle using the anchor gives:

$$\rho^{a}=\rho(\mathrm{d}x^{a})=\Pi^{ai}\partial_i\,. \tag{3.3.24}$$

[5]In this section we retain the notation containing different types indices $a,b,c,\dots$ and $i,j,k,\dots$ in order to make it clearer how this special case is related to the general one. The only difference shall be index-positioning, meaning that, e.g. elements of local basis were denoted with lower indices, while here they are denoted by upper indices. Note that in this case, these two types of indices take the same values and there is no fundamental difference between the sets.

It is now a simple matter to compute their commutator:

$$[\rho^a, \rho^b] = (\partial_c \Pi^{ab} + \Pi^{ad}\Pi^{be}H_{cde})\Pi^{ci}\partial_i\,, \tag{3.3.25}$$

thus giving the structure functions:

$$C_c{}^{ab} = \partial_c \Pi^{ab} + \Pi^{ad}\Pi^{be}H_{cde}\,. \tag{3.3.26}$$

Alternatively, these can be computed using the Koszul-Schouten bracket directly on the sections on the cotangent bundle.

Up to now, all of the gauging data has been completely specified by the choice of the Lie algebroid. However, the rest do not have to be uniquely determined, only satisfy the necessary constraints. Specifically, θ have to be chosen in order for $\rho + \theta$ to lie on a Dirac structure. In order to make the appropriate choice here, we shall make use of the property of the Poisson bivector that the maps $\mathrm{id} \pm g \circ \Pi$ are invertible. Identifying $g \circ \Pi$ as ρ^* suggests the choice of $\theta = \mathrm{id}$ to be a possible one. Indeed, with this choice of θ_a corresponding to $\mathrm{d}X^a$, the gauged action becomes:

$$S[X,A] = -\int_{\Sigma_2}\left(\frac{1}{2}g_{ij}(X)F^i \wedge *F^j + A_a \wedge \mathrm{d}X^a + \frac{1}{2}\Pi^{ab}A_a \wedge A_b\right) - \int_{\Sigma_3} H(X)\,. \tag{3.3.27}$$

For the gauge transformations, one needs to determine ω and ϕ, which is easily done using the general formulas, thus obtaining [25]:

$$\begin{aligned} \Phi^b_{ai} &= -\left((1 - g\Pi g\Pi)^{-1}\right)^c_a g_{cd}\left(\mathring{\nabla}_i \Pi^{db} + \frac{1}{2}\Pi^{de}\Pi^{jf}H_{efi}\right), & (3.3.28)\\ \omega^b_{ai} &= \Gamma^b_{ai} + g_{ac}\Pi^{cd}\Phi^b_{di} + \frac{1}{2}\Pi^{bc}H_{cai}\,, & (3.3.29) \end{aligned}$$

where $\mathring{\nabla}$ denotes the Levi-Civita connection here and Γ its components. It is worth noting that it is possible to put g to zero here, in which case both Φ and Γ vanish, while ω becomes:

$$\omega^b_{ai} = \frac{1}{2}\Pi^{bc}H_{cai} \tag{3.3.30}$$

Thus obtained model is known as the (twisted) Poisson sigma model, originally obtained by other means [14, 13, 71, 72].

3.4 Gauging sigma model with non-closed 3-form

Up to now, we have been studying the 2-dimensional sigma models with a closed 3-form in the Wess-Zumino term. However, this is not always the case. For example, in heterotic string theory, the 3-form acquires α'-corrections, which make it non-closed. Another example where such a thing can arise is in the dimensional reduction of the target space, like when it has the structure of the circle fibration [73, 74], or more generally in the case of reduction of Courant algebroids [59, 60, 30, 75, 76].

Let us start with string model action in the target space that has the structure of a circle fibration with d-dimensional base M and S^1 fibers and it is equipped with a metric[6] $\widetilde{G}$ and a closed 3-form $\widetilde{H}$:

$$\widetilde{S}_0 = -\int_{\Sigma_2} \frac{1}{2}\widetilde{G}_{\mu\nu}(\widetilde{X})\mathrm{d}\widetilde{X}^\mu \wedge *\mathrm{d}\widetilde{X}^\nu - \int_{\Sigma_3} \frac{1}{3!}\widetilde{H}_{\mu\nu\sigma}(\widetilde{X})\mathrm{d}\widetilde{X}^\mu \wedge \mathrm{d}\widetilde{X}^\nu \wedge \mathrm{d}\widetilde{X}^\sigma\,, \tag{3.4.1}$$

where $\widetilde{X}$ is the sigma model map to the $(d+1)$-dimensional target space. To proceed with dimensional reduction, we make the further assumption that the background fields are invariant under the isometry generated by the Killing vector field on S^1. In adapted coordinates $\widetilde{x}^\mu = (x^i, \Phi)$ where the Killing vector is simply $\partial/\partial\Phi$, this assumption simply translates to the metric and 3-form being independent of Φ. This corresponds to a Kaluza-Klein reduction[7], where the Ansatz for the metric $\widetilde{G}$ takes the characteristic form (in terms of the line element)

$$\widetilde{G} = \frac{1}{2}G_{ij}(x)\mathrm{d}x^i \vee \mathrm{d}x^j + G_{\Phi\Phi}(x)(\mathrm{d}\Phi + \mathfrak{a})^2\,, \tag{3.4.2}$$

with $\mathfrak{a} = \mathfrak{a}_i \mathrm{d}x^i$ being the Kaluza-Klein vector with field strength $R = \mathrm{d}\mathfrak{a}$. In the usual Kaluza-Klein parametrization, $G_{\Phi\Phi}$ is the exponential of a scalar field, however we keep the discussion general here. Furthermore, the 3-form is decomposed accordingly as

$$\widetilde{H} = H + \Omega \wedge (\mathrm{d}\Phi + \mathfrak{a})\,, \tag{3.4.3}$$

where the 3-form H and the 2-form Ω are differential forms on lower-dimensional manifold M, i.e. they are independent of Φ. Then, given that $\mathrm{d}\widetilde{H} = 0$, one finds that the lower-dimensional

[6]In this section we denote the metric with a upper-case letter G in order to reserve the lower-case version g for another object.

[7]Note that in more general cases one could in principle consider Scherk-Schwarz reductions, where the higher-dimensional fields depend on the internal coordinates but the dependence drops out in the lower-dimensional theory due to the symmetry structure of the higher-dimensional one. Such cases shall not be considered here.

3-form is not closed but instead satisfies

$$\mathrm{d}H = -\Omega \wedge R\,, \tag{3.4.4}$$

whereas Ω is itself a closed 2-form, $\mathrm{d}\Omega = 0$.

The obtained model is a simple example of how non-closed 3-forms may arise, but here we study more general situations, with the example serving only as a motivation for the set up [28]. Let the target space be equipped with a metric G, a non-closed 3-form H, but also with two additional 2-forms Ω and R such that:

$$\mathrm{d}H = -\langle R \wedge \Omega \rangle\,, \tag{3.4.5}$$

where the angle brackets denote the fact that the 2-forms may be also valued in additional bundles; for example they can have a gauge index, as will be the case here, in which case a trace should be taken in the right-hand side. In this general setting, we therefore consider $R, \Omega \in \Gamma(\mathcal{G} \otimes \wedge^2 T^*M)$, where $\mathcal{G}$ is a bundle of quadratic Lie algebras over M. The corresponding action functional

$$S_0 = S_{0,\mathrm{kin}} + S_{0,\mathrm{WZ}} \tag{3.4.6}$$

is split into a kinetic sector and one that contains Wess-Zumino terms, corresponding precisely to the Kaluza-Klein reduction Ansatz (3.4.2) and (3.4.3) respectively:

$$\begin{aligned} S_{0,\mathrm{kin}} &= -\int_{\Sigma_2} \left(\frac{1}{2} G_{ij}(X) \mathrm{d}X^i \wedge * \mathrm{d}X^j + \frac{1}{2} G_{\Phi\Phi}(X) \Lambda \wedge * \Lambda \right), & (3.4.7) \\ S_{0,\mathrm{WZ}} &= -\int_{\Sigma_3} \left(\frac{1}{3!} H_{ijk}(X) \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k + \frac{1}{2!} \Omega_{ij}(X) \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \Lambda \right). & (3.4.8) \end{aligned}$$

Here, Λ is the (pull-back of the) 1-form $\Lambda = \mathrm{d}\Phi + \mathfrak{a}$, and $\Phi(\sigma)$ is the additional scalar corresponding to the reduction picture explained above, which is a function of the local coordinates on the world-sheet. Note that none of the components of the background fields depend on this additional field.

The action (3.4.1) can have target space symmetries generated by a set of vector fields $\widetilde{\rho}_a = \rho_a^\mu(\widetilde{X})\partial_\mu$ that satisfy a non-Abelian Lie algebra

$$[\widetilde{\rho}_a, \widetilde{\rho}_b] = C^c{}_{ab} \widetilde{\rho}_c\,, \tag{3.4.9}$$

with $C^c{}_{ab}$ being the corresponding structure functions. However, in the present section, their dependence on X will be somewhat neglected; from now on, one may safely assume that $C^c{}_{ab}$ are constants, albeit keeping in mind that this can be generalized in a straightforward way. In the reduction spirit employed here, these vector fields may be decomposed as

$$\widetilde{\rho}_a = \rho^i_a(X)\partial_i + \widetilde{f}_a(X)\partial_\Phi\,, \tag{3.4.10}$$

where $\widetilde{f}_a$ are functions on M (see also Section 5.3.6 of [73].) This leads to a set of vector fields ρ_a satisfying the algebra

$$[\rho_a,\rho_b] = C^c{}_{ab}\rho_c\,, \tag{3.4.11}$$

with the same structure constants as before. Moreover, it immediately follows that

$$2\iota_{\rho_{[a}}\mathrm{d}\widetilde{f}_{b]} = C^c{}_{ab}\widetilde{f}_c\,. \tag{3.4.12}$$

In the following, we consider another parametrization for these functions, specifically

$$\widetilde{f}_a = f_a - \iota_{\rho_a}\mathfrak{a}\,, \tag{3.4.13}$$

in which case (3.4.12) becomes an equation for the functions f_a and reads

$$2\iota_{\rho_{[a}}\mathrm{d}f_{b]} = C^c{}_{ab}f_c - \iota_{\rho_a}\iota_{\rho_b}R\,, \tag{3.4.14}$$

where $R = \mathrm{d}\mathfrak{a}$ is the Abelian field strength of the 1-form $\mathfrak{a}$. The symmetries generated by ρ_a for the action functional (3.4.6) manifest themselves upon considering the following transformations for the scalar fields X^i and Φ:

$$\delta X^i = \rho^i_a\varepsilon^a\,, \tag{3.4.15}$$

$$\delta\Phi = \widetilde{f}_a\varepsilon^a\,, \tag{3.4.16}$$

where ε^a are rigid symmetry parameters. Moreover, for future reference, it is useful to calculate the transformation of the 1-form Λ, which turns out to be

$$\delta\Lambda = \mathrm{d}(f_a\varepsilon^a) + (\iota_{\rho_a}R)\varepsilon^a\,, \tag{3.4.17}$$

We observe that in reference to Λ, the combination $f_a\varepsilon^a$ behaves as a single transformation parameter, and moreover the second 2-form R appears in the transformation rule. Note that in the Abelian case, both R and Ω are closed, namely $\mathrm{d}R = 0 = \mathrm{d}\Omega$.

One can now compute the transformation of the action S_0 under (3.4.15) and (3.4.16) for constant transformation parameters ε^a and examine under which conditions the action is invariant. Taking into account that $\mathrm{d}H = -R\wedge\Omega$, one can show that this is true if and only if

$$\mathcal{L}_{\rho_a} G = 0\,, \qquad \mathcal{L}_{\rho_a} G_{\Phi\Phi} = 0\,, \qquad \iota_{\rho_a} R = -\mathrm{d}f_a\,, \tag{3.4.18}$$

$$\iota_{\rho_a} H + f_a\Omega - g_a R = \mathrm{d}\theta_a\,, \qquad \iota_{\rho_a}\Omega = \mathrm{d}g_a\,, \tag{3.4.19}$$

where θ_a and g_a are arbitrary 1-form and function respectively. Note that ρ_a are Killing vector fields for the metric $G = G_{ij}(x)\mathrm{d}x^i\otimes\mathrm{d}x^j$ of the Kaluza-Klein Ansatz (3.4.2). However, one should keep in mind that this is not the full nonlinear coupling in the kinetic term of the scalar fields X. As can be seen from (3.4.7), the components of the full metric should be identified with $G_{ij} + G_{\Phi\Phi}a_i a_j$, i.e. the usual Kaluza-Klein metric in the lower-dimensional space. The vectors ρ_a are not isometries of this metric though. This is expected in view of the higher-dimensional origin of these conditions; indeed, the higher-dimensional vector $\widetilde{\rho}_a$ is a Killing vector for $\widetilde{G}$.

A similar discussion follows for the non-Abelian case, where Ω^α and R^α are $\mathcal{G}$-valued and there is a worth of $\dim\mathcal{G}$ 1-forms $\Lambda^\alpha = \mathrm{d}\Phi^\alpha + a^\alpha$, where α is a gauge index. The transformation rule of the scalars X^i remains the same as in (3.4.15), whereas the one of Λ^α reads as

$$\delta\Lambda^\alpha = \mathrm{d}(f_a^\alpha\varepsilon^a) + (\iota_{\rho_a}\mathrm{d}a^\alpha)\varepsilon^a\,, \tag{3.4.20}$$

Note that in our conventions, the non-Abelian field strength is given as

$$R^\alpha = \mathrm{d}a^\alpha - \frac{1}{2}K^\alpha{}_{\beta\gamma}a^\beta\wedge a^\gamma\,, \tag{3.4.21}$$

where $K^\alpha{}_{\beta\gamma}$ are the structure constants of $\mathcal{G}$; however, it should be noted that only $\mathrm{d}a^\alpha$ appears in the transformation of Λ^α. The ungauged action functional has the form

$$\begin{aligned} S_0 \;&=\; S_{0,\mathrm{kin}} - \int_{\Sigma_3}\left(\frac{1}{3!}H_{ijk}(X)\mathrm{d}X^i\wedge\mathrm{d}X^j\wedge\mathrm{d}X^k + \frac{1}{2!}\Omega_{\alpha ij}(X)\mathrm{d}X^i\wedge\mathrm{d}X^j\wedge\Lambda^\alpha\right) \\ &\quad - \int_{\Sigma_3}\left(-\frac{1}{2}K_{\alpha\beta\gamma}\Lambda^\alpha\wedge\Lambda^\beta\wedge a^\gamma + \frac{1}{3!}K_{\alpha\beta\gamma}\Lambda^\alpha\wedge\Lambda^\beta\wedge\Lambda^\gamma\right)\,, \end{aligned} \tag{3.4.22}$$

where we use the same symbol S_0 as in the Abelian case, since the latter is simply obtained from the non-Abelian one in an obvious way. Note that the gauge indices are raised and lowered with the $\mathcal{G}$-metric (essentially the Killing form), which in a basis of Lie algebra generators $\{T^\alpha\}$ we denote as $k^{\alpha\beta}$. Then the conditions (3.4.19) generalize in the non-Abelian case to

$$\iota_{\rho_a}H + f_a^{\alpha}\Omega_{\alpha} - g_{a\alpha}\mathrm{d}a^{\alpha} - \frac{1}{2}K_{\alpha\beta\gamma}\widetilde{f}_a^{\gamma}a^{\alpha}\wedge a^{\beta} = \mathrm{d}\theta_a\,, \tag{3.4.23a}$$

$$\iota_{\rho_a}\left(\Omega_{\alpha} - \tfrac{1}{2}K_{\alpha\beta\gamma}a^{\beta}\wedge a^{\gamma}\right) = \mathrm{d}g_{a\alpha}\,, \tag{3.4.23b}$$

$$\iota_{\rho_a}\left(R^{\alpha} + \frac{1}{2}K^{\alpha}{}_{\beta\gamma}a^{\beta}\wedge a^{\gamma}\right) = -\mathrm{d}f_a^{\alpha}\,, \tag{3.4.23c}$$

provided that

$$\mathrm{d}\Omega_{\alpha} + K_{\alpha\beta\gamma}a^{\beta}\wedge \mathrm{d}a^{\gamma} = 0 \tag{3.4.24}$$

and that $K_{\alpha\beta\gamma}\widetilde{f}_a^{\gamma}$ is constant. These conditions and their interpretation will be revisited in more detail in the gauged version of the theory.

The global symmetries generated by the vector fields $\rho_a + \widetilde{f}_a\partial_{\phi}$ can be promoted to local symmetries of the action upon considering transformation parameters $\varepsilon^a = \varepsilon^a(\sigma)$ that depend on the world-sheet coordinates. This allows us to consider gaugings of the action (3.4.6) along the foliation generated by the corresponding vector fields.

According to the above, we consider additional world-sheet 1-forms A^a that we wish to couple to the theory. These gauge fields take values in some gauge bundle $\mathcal{E}$, in which we may consider a local basis of sections e_a such that $A = A^a e_a$. $\mathcal{E}$ is taken to be a Lie algebroid, with a bundle map homomorphism ρ to the tangent bundle of M such that $\rho(e_a) = \rho_a$. As such, if the bracket operation on $\mathcal{E}$ satisfies

$$[e_a, e_b]_{\mathcal{E}} = C_{ab}{}^{c}e_c\,, \tag{3.4.25}$$

then (3.4.9) follows. For the present purposes, we consider that $\mathcal{E}$ is a direct sum bundle $\mathcal{E} = L\oplus L'$ over M, for some bundles L and L'. For instance, this allows us to consider Lie algebroids such as $TM\oplus\mathbb{R}$ and $T^*M\oplus\mathbb{R}$ in the Abelian case; the fact that these are indeed Lie algebroids is discussed for example in [77] (see also [78]).

As typical for sigma model actions with Wess-Zumino term, gauging proceeds by a non-minimal coupling of the gauge fields A^a to the topological sector of the theory, while they are coupled minimally to the kinetic sector. This is facilitated by the following candidate gauged action functional

$$\begin{aligned} S \;=\; & S_{\text{kin}} - \int_{\Sigma_2}\left(A^a\wedge\theta_a + g_{a\alpha}A^a\wedge\Lambda^{\alpha} + \frac{1}{2}\gamma_{ab}A^a\wedge A^b\right) \\ & - \int_{\Sigma_3}\left(\frac{1}{3!}H_{ijk}(X)\mathrm{d}X^i\wedge\mathrm{d}X^j\wedge\mathrm{d}X^k + \frac{1}{2!}\Omega_{\alpha ij}(X)\mathrm{d}X^i\wedge\mathrm{d}X^j\wedge\Lambda^{\alpha}\right) \\ & - \int_{\Sigma_3}\left(-\frac{1}{2}K_{\alpha\beta\gamma}\Lambda^{\alpha}\wedge\Lambda^{\beta}\wedge a^{\gamma} + \frac{1}{3!}K_{\alpha\beta\gamma}\Lambda^{\alpha}\wedge\Lambda^{\beta}\wedge\Lambda^{\gamma}\right)\,, \end{aligned} \tag{3.4.26}$$

where $\theta_a = \theta_{ai}\mathrm{d}X^i$, $g_{a\alpha}$, γ_{ab} are arbitrary 1-form and functions of X respectively. The gauge fields A^a transform as is typical for a nonlinear gauge theory[8]

$$\delta A^a = \mathrm{d}\varepsilon^a + C^a{}_{bc}A^b\varepsilon^c\,. \tag{3.4.27}$$

Moreover, the gauged kinetic sector takes the form

$$S_{\text{kin}} = -\int_{\Sigma_2} \frac{1}{2}G_{ij}F^i \wedge *F^j + \frac{1}{2}G_{\alpha\beta}\hat{\Lambda}^\alpha \wedge *\hat{\Lambda}^\beta\,, \tag{3.4.28}$$

where F and $\hat{\Lambda}$ are 1-forms defined as

$$F^i \;=\; \mathrm{d}X^i - \rho^i_a A^a\,, \tag{3.4.29}$$

$$\hat{\Lambda}^\alpha \;=\; D\Phi^\alpha + \mathfrak{a}^\alpha_i F^i\,, \tag{3.4.30}$$

with $D\Phi^\alpha = \mathrm{d}\Phi^\alpha - \widetilde{f}^\alpha_a A^a$. They are both covariant as world-sheet 1-forms, since they transform as

$$\delta F^i = \partial_j\rho^i_a\varepsilon^a F^i \quad \text{and} \quad \delta\hat{\Lambda} = (\partial_i f_b + (\iota_{\rho_b}\mathrm{d}\mathfrak{a})_i)\varepsilon^b F^i\,. \tag{3.4.31}$$

Note that in principle one can also consider additional equation of motion symmetries in the transformation of A^a, also called trivial gauge transformations [38], which are important when the gauge algebra of the model is open. In the present section we consider only gauge algebras that close off shell, thus omitting such trivial gauge transformations.

We now examine under which conditions the action S is invariant under the above gauge transformations. As for the kinetic sector, this is separately gauge invariant provided that (3.4.18), or the corresponding non-Abelian extension of them, hold. On the other hand, the topological sector imposes additional constraints. In order to cancel all terms supported on Σ_3,

[8]As have been shown before, this is certainly not the most general transformation for the gauge field in this context, but it is the one considered in the original papers [21, 22]. It corresponds to the standard case where the gauge symmetry originates from a corresponding global symmetry, which we also assume here. More generally, one can examine whether a gauge theory exists even without an underlying global symmetry. We do not examine this apparently more general situation here. However, this does not affect the main result which is presented below, since in both cases one obtains conditions for Dirac structures of a Courant algebroid. The only difference regards the background fields, which in the present case are assumed invariant, while in the general case this assumption is relaxed.

we take into account the following identities

$$\mathrm{d}\mathcal{H} = -R^{\alpha} \wedge \Omega_{\alpha}\,, \tag{3.4.32a}$$

$$\mathrm{d}R^{\alpha} = -K^{\alpha}{}_{\beta\gamma}\mathrm{d}\mathfrak{a}^{\beta} \wedge \mathfrak{a}^{\gamma}\,, \tag{3.4.32b}$$

$$\mathrm{d}\Omega^{\alpha} = K^{\alpha}{}_{\beta\gamma}\mathrm{d}\mathfrak{a}^{\beta} \wedge \mathfrak{a}^{\gamma}\,, \tag{3.4.32c}$$

where we have defined the improved 3-form

$$\mathcal{H} = H - \frac{1}{3!}K_{\alpha\beta\gamma}\mathfrak{a}^{\alpha} \wedge \mathfrak{a}^{\beta} \wedge \mathfrak{a}^{\gamma}\,. \tag{3.4.33}$$

Note that although R and Ω seem related in view of (3.4.32), we have treated them separately because in the Abelian case they can be independent. Then it is straightforward to show that the topological sector of the action S transforms as

$$\begin{aligned}
\delta S_{\mathrm{top}} \;=\; & -\int_{\Sigma_2}\left(\gamma_{ab} - \rho_a^i\theta_{bi} - f_a^{\alpha}g_{b\alpha}\right)\mathrm{d}\varepsilon^a \wedge A^b \\
& - \int_{\Sigma_2}\varepsilon^a\Bigg[\left(\frac{1}{2}\Omega_{\alpha ij}f_a^{\alpha} + \frac{1}{2}\rho_a^k H_{ijk} - g_{a\alpha}\partial_i\mathfrak{a}_j^{\alpha} - \partial_i\theta_{aj}\right)\mathrm{d}X^i \wedge \mathrm{d}X^j \\
& + \left(-\partial_i g_{a\alpha} + \rho_a^j\Omega_{\alpha ji} - K_{\alpha\beta\gamma}f_a^{\beta}\mathfrak{a}_i^{\gamma}\right)\mathrm{d}X^i \wedge \Lambda^{\alpha} + \frac{1}{2}K_{\alpha\beta\gamma}\left(f_a^{\gamma} - \rho_a^i\mathfrak{a}_i^{\gamma}\right)\Lambda^{\alpha} \wedge \Lambda^{\beta} \\
& + \left(C_{ab}^c\theta_{ci} - \rho_a^j\partial_j\theta_{bi} - \theta_{bj}\partial_i\rho_a^j - g_{b\alpha}\partial_i f_a^{\alpha} - g_{b\alpha}(\iota_{\rho_a}\mathrm{d}\mathfrak{a}^{\alpha})_i\right)\mathrm{d}X^i \wedge A^b \\
& + \left(C_{ab}^c g_{c\alpha} - \rho_a^i\partial_i g_{b\alpha}\right)\Lambda^{\alpha} \wedge A^b + \left(\frac{1}{2}\rho_a^i\partial_i\gamma_{bc} - C_{ab}^d\gamma_{dc}\right)A^b \wedge A^c\Bigg]\,,
\end{aligned} \tag{3.4.34}$$

where we have also used the Jacobi identity for the structure constants $K_{\alpha\beta\gamma}$. Requiring invariance of the action thus leads to the following conditions on the background fields,

$$\iota_{\rho_a}H + f_a^{\alpha}\Omega_{\alpha} - g_{a\alpha}\mathrm{d}\mathfrak{a}^{\alpha} = \mathrm{d}\theta_a\,, \tag{3.4.35a}$$

$$\iota_{\rho_a}\Omega_{\alpha} - K_{\alpha\beta\gamma}f_a^{\beta}\mathfrak{a}^{\gamma} = \mathrm{d}g_{a\alpha} \tag{3.4.35b}$$

$$K_{\alpha\beta\gamma}\widetilde{f}_a^{\gamma} = 0\,. \tag{3.4.35c}$$

Note that in comparison to the conditions found in the rigid case, $K_{\alpha\beta\gamma}\widetilde{f}_a^{\gamma}$ is zero rather than constant because ε is not a rigid parameter. This affects only the non-Abelian case, since in the Abelian case $K_{\alpha\beta\gamma} = 0$ anyway. In the non-Abelian case it implies a relation between the functions f_a^{α} and the contraction of the fundamental gauge field $\mathfrak{a}^{\alpha}$ with the vector fields ρ_a. We comment further on this below. In addition, there are three constraints that must be satisfied in order to obtain a consistent gauge theory. First, we note from the first term in the variation of the action that

$$\gamma_{ab} = \iota_{\rho_a}\theta_b + k_{\alpha\beta} f_a^\alpha g_b^\beta \,. \tag{3.4.36}$$

Then the three constraints read as

$$\gamma_{(ab)} = 0 \tag{3.4.37a}$$

$$\mathcal{L}_{\rho_a}\theta_b = C_{ab}^c\theta_c - g_b^\alpha\left(\mathrm{d}f_{a\alpha} + \iota_{\rho_a}R_\alpha + K_{\alpha\beta\gamma}f_a^\beta \mathfrak{a}^\gamma\right) , \tag{3.4.37b}$$

$$\iota_{\rho_a}\mathrm{d}g_b^\alpha = C_{ab}^c g_c^\alpha \,. \tag{3.4.37c}$$

Finally, there is a further requirement due to the very last, quadratic in As, term in (3.4.34),

$$\mathcal{L}_{\rho_a}\gamma_{bc} = C^d_{a[c}\gamma_{b]d} \,, \tag{3.4.38}$$

which, however, is identically satisfied once the previous conditions are taken into account.

Summarizing the findings of this section, the gauged action functional (3.4.26) is invariant under the infinitesimal transformations (3.4.15), (3.4.20) and (3.4.27) if and only if the conditions (3.4.35) hold and the constraints (3.4.37) are satisfied, and at the same time the vector fields ρ_a generate isometries for the metric G, respectively the higher-dimensional vector fields $\rho_a + \widetilde{f}_a^\alpha \partial_{\phi^\alpha}$ generate isometries for the metric $\widetilde{G}$. Our next goal is to understand the geometric meaning of these constraints.

3.4.1 The Abelian case and contact Courant algebroids

Let us first investigate in more detail the Abelian case where the bundle of quadratic Lie algebras is $\mathcal{G} = M \times \mathbb{R} \oplus \mathbb{R}$, in which case $K_{\alpha\beta\gamma} = 0$. The gauged action functional (3.4.26) then contains only the terms appearing in the first two lines. Moreover, from (3.4.32) we learn that both R and Ω are closed 2-forms, while the exterior derivative of the 3-form H is the opposite of their wedge product. Note that nothing necessitates any further relation between the two 2-forms in the present case. Moreover, the third of conditions (3.4.35) is identically satisfied and does not play any role, in particular it does not relate f_a with $\iota_{\rho_a}\mathfrak{a}$.

We proceed in analysing the three constraints (3.4.37), taking into account the first two conditions (3.4.35). First, the constraint (3.4.37c) may be rewritten as

$$C_{ab}^c g_c = \rho_a^i \partial_i g_b = 2\rho^i_{[a}\partial_{|i|}g_{b]} + \rho_b^i\partial_i g_a \overset{(3.4.35b)}{=} 2\rho^i_{[a}\partial_{|i|}g_{b]} + \rho_b^i\rho_a^j\Omega_{ji} \,, \tag{3.4.39}$$

or

$$C^c_{ab} g_c = 2\iota_{\rho_{[a}} \mathrm{d} g_{b]} - \iota_{\rho_a}\iota_{\rho_b}\Omega\,, \tag{3.4.40}$$

where (anti)symmetrization is taken with weight 1 and vertical bars denote exclusion from it. This equation looks like closure of a bracket operation and indeed we are going to interpret it as such. Turning to the constraint (3.4.37b), a similar rewriting is possible as follows:

$$\begin{aligned}
C^c_{ab}\theta_c &= \mathcal{L}_{\rho_a}\theta_b + g_b \mathrm{d} f_a + g_b \iota_{\rho_a} R \\
&= 2\mathcal{L}_{\rho_{[a}}\theta_{b]} - \mathrm{d}\iota_{\rho_{[a}}\theta_{b]} + \tfrac{1}{2}\mathrm{d}\left(\iota_{\rho_a}\theta_b + \iota_{\rho_b}\theta_a\right) + \iota_{\rho_b}\mathrm{d}\theta_a + g_b\mathrm{d} f_a + g_b\iota_{\rho_a} R \\
&\overset{(3.4.35a)}{=} 2\mathcal{L}_{\rho_{[a}}\theta_{b]} - \mathrm{d}\iota_{\rho_{[a}}\theta_{b]} + \mathrm{d}\iota_{\rho_{(a}}\theta_{b)} - \iota_{\rho_a}\iota_{\rho_b} H + 2g_{[b}\iota_{\rho_{a]}} R + g_b \mathrm{d} f_a + f_a \iota_{\rho_b}\Omega \\
&\overset{(3.4.35b)}{=} 2\mathcal{L}_{\rho_{[a}}\theta_{b]} - \mathrm{d}\iota_{\rho_{[a}}\theta_{b]} - \iota_{\rho_a}\iota_{\rho_b} H + 2g_{[b}\iota_{\rho_{a]}} R - 2f_{[b}\iota_{\rho_{a]}}\Omega + g_{[b}\mathrm{d} f_{a]} + f_{[b}\mathrm{d} g_{a]}\,,
\end{aligned} \tag{3.4.41}$$

where in the second line we used the Cartan relation

$$\mathcal{L} = \iota\circ\mathrm{d} + \mathrm{d}\circ\iota\,, \tag{3.4.42}$$

and in the last line we also used the constraint (3.4.37a), which may be written explicitly as

$$\frac{1}{2}\left(\iota_{\rho_a}\theta_b + \iota_{\rho_b}\theta_a + f_a g_b + f_b g_a\right) = 0\,. \tag{3.4.43}$$

In order to provide a geometric interpretation of these results, let us recall the definitions of twisted contact Courant algebroids and their Dirac structures. Contact Courant algebroids twisted by a 3-form H and two 2-forms R and Ω are defined in Ref. [73][9] as follows. Consider the vector bundle $E = TM\oplus\mathbb{R}\oplus T^*M\oplus\mathbb{R}$, whose sections are $\mathbb{X} = (X, f, \eta, g)$, where X is a vector field, η an 1-form and f, g are functions. Then the twisted contact Courant algebroid is given by the data $(E, [\cdot,\cdot]_E, \langle\cdot,\cdot\rangle, a: E\to TM)$ of the above vector bundle, a skew-symmetric

[9]The version in terms of non-skew-symmetric bracket is found in [76].

bracket, a nondegenerate symmetric bilinear form and an anchor map, given as

$$\begin{aligned}
[\mathbb{X}_1,\mathbb{X}_2]_E &= ([X_1,X_2], X_1(f_2)-X_2(f_1)+\iota_{X_1}\iota_{X_2}R, \\
&\quad \mathcal{L}_{X_1}\eta_2-\mathcal{L}_{X_2}\eta_1+g_2\iota_{X_1}R-g_1\iota_{X_2}R-\tfrac{1}{2}\mathrm{d}(\iota_{X_1}\eta_2-\iota_{X_2}\eta_1) \\
&\quad +\tfrac{1}{2}(g_2\mathrm{d}f_1-g_1\mathrm{d}f_2-f_1\mathrm{d}g_2+f_2\mathrm{d}g_1)-\iota_{X_1}\iota_{X_2}H-f_2\iota_{X_1}\Omega+f_1\iota_{X_2}\Omega, \\
&\quad X_1(g_2)-X_2(g_1)-\iota_{X_1}\iota_{X_2}\Omega)\,, && (3.4.44) \\
\langle\mathbb{X}_1,\mathbb{X}_2\rangle &= \frac{1}{2}(\iota_{X_1}\eta_2+\iota_{X_2}\eta_1+f_1g_2+f_2g_1)\,, && (3.4.45) \\
a(\mathbb{X}) &= X\,, && (3.4.46)
\end{aligned}$$

with $\mathrm{d}R = 0 = \mathrm{d}\Omega$ and $\mathrm{d}H = -R\wedge\Omega$. (Note that in comparison to [73] we have a different sign convention for H and Ω.) It is now clear what the constraints we derived above mean in this geometric setting. First, the 1-forms θ_a and the functions g_a can be associated to the following maps

$$\begin{aligned}
\theta : \mathcal{E} &\to T^*M & g : \mathcal{E} &\to \mathbb{R} \\
e_a &\mapsto \theta(e_a) := \theta_a & e_a &\mapsto g(e_a) := g_a\,. \qquad (3.4.47)
\end{aligned}$$

Together with ρ and f, one then obtains a map

$$\begin{aligned}
\hat{\rho} := \rho\oplus f\oplus\theta\oplus g : \mathcal{E} &\to E \\
e_a &\mapsto \hat{\rho}(e_a) := \rho_a + f_a + \theta_a + g_a := \hat{\rho}_a\,. \qquad (3.4.48)
\end{aligned}$$

Then, combining (3.4.9) and (3.4.14) with the two constraints written in the form (3.4.40) and (3.4.41), we directly obtain

$$[\hat{\rho}_a,\hat{\rho}_b]_E = C^c_{ab}\hat{\rho}_c\,, \qquad (3.4.49)$$

in other words the generalised sections $\hat{\rho}$ as defined above are closed under the bracket of the twisted contact Courant algebroid. In addition, the remaining constraint (3.4.43) states that

$$\langle\hat{\rho}_a,\hat{\rho}_b\rangle = 0\,. \qquad (3.4.50)$$

This means that the generalized sections $\hat{\rho}$ are constrained in a subbundle of E, which is isotropic and involutive with respect to the twisted Courant bracket (3.4.44). This is analogous to the case with a closed 3-form. Here we encounter the contact Courant algebroid on $TM\oplus\mathbb{R}\oplus T^*M\oplus\mathbb{R}$, with the respective contact Dirac structures as constrained subbundles.

From an alternative point of view, the gauging of the original sigma model can be interpreted as a constraining of a different action functional, which is essentially the dimensional reduction of the universal action functional. The starting point for this interpretation is the introduction of two auxiliary 1-forms V^i and W_i taking values in T^*M and TM respectively, and two additional auxiliary 1-forms v and w taking values in $C^\infty(M)$. Then we can write the action functional

$$\begin{aligned}\mathcal{S} &= -\int_{\Sigma_2} \frac{1}{2} G_{ij} F^i \wedge *F^j + \frac{1}{2} G_{\Phi\Phi} \lambda \wedge *\lambda - \int_{\Sigma_3} H + \Omega \wedge \Lambda \\ &\quad - \int_{\Sigma_2} W_i \wedge \left(\mathrm{d}X^i - \frac{1}{2} V^i \right) - \int_{\Sigma_2} w \wedge \left[\mathrm{d}\Phi - \frac{1}{2} v + \mathfrak{a}_i \left(\mathrm{d}X^i - \frac{1}{2} V^i \right) \right] , \end{aligned} \tag{3.4.51}$$

where $F^i = \mathrm{d}X^i - V^i$ and $\lambda = \mathrm{d}\Phi - v + \mathfrak{a}_i(\mathrm{d}X^i - V^i)$. This action functional has the property that when the auxiliary fields are unconstrained it is equivalent to the ungauged action functional S_0, whereas when the auxiliary fields are appropriately constrained it yields the gauged versions of the model corresponding to the action functional S. Regarding the first part of this statement, the field equations for the four auxiliary fields read

$$\begin{aligned} &V^i = 2\mathrm{d}X^i , \quad v + \mathfrak{a}_i V^i = 2(\mathrm{d}\Phi + \mathfrak{a}_i \mathrm{d}X^i) , \quad w = 2G_{\Phi\Phi} * \lambda , \\ &W_i + \mathfrak{a}_i w = 2G_{ij} * F^j + \mathfrak{a}_i G_{\Phi\Phi} * \lambda . \end{aligned} \tag{3.4.52}$$

Substituting them in $\mathcal{S}$, one directly obtains the ungauged action S_0. On the other hand, if the auxiliary fields are constrained to live on a Dirac structure of the contact Courant algebroid on $TM \oplus \mathbb{R} \oplus \mathbb{R} \oplus T^*M$, then $\mathcal{S}$ yields the gauged action S upon the identifications

$$V^i = \rho^i_a A^a , \quad v = \widetilde{f}_a A^a , \quad W_i = \theta_{ai} A^a , \quad w = g_a A^a . \tag{3.4.53}$$

This identification only works if $\gamma_{ab} = \iota_{\rho_a} \theta_b + f_a g_b$, which is precisely the relation obtained from the consistency of the gauging procedure. Thus we see that one may think of the gauging not as an extension of the given action S_0 by additional gauge fields but as a restriction of the equivalent action $\mathcal{S}$ to a constrained set of fields.

3.4.2 The non-Abelian case and non-exact Courant algebroids

Let us now examine the meaning of the constraints arising from the gauged action functional in the non-Abelian case. First of all, condition (3.4.35c) implies that

$$f^\alpha_a = \iota_{\rho_a} \mathfrak{a}^\alpha . \tag{3.4.54}$$

Secondly, the fact that $\mathrm{d}R^\alpha = -\mathrm{d}\Omega^\alpha \neq 0$ prompts us to identify[10]

$$\Omega^\alpha = -R^\alpha\,, \tag{3.4.55}$$

therefore the first of (3.4.32) becomes

$$\mathrm{d}\mathcal{H} = k_{\alpha\beta} R^\alpha \wedge R^\beta\,. \tag{3.4.56}$$

Then the first constraint is written explicitly as

$$\frac{1}{2}\left(\iota_{\rho_a}\theta_b + \iota_{\rho_b}\theta_a + k_{\alpha\beta} f_a^\alpha g_b^\beta + k_{\alpha\beta} f_b^\alpha g_a^\beta\right) = 0\,. \tag{3.4.57}$$

Furthermore, similar manipulations to the Abelian case lead to the following form of constraints (3.4.37b) and (3.4.37c) respectively,

$$2\iota_{\rho_{[a}}\mathrm{d}g_{b]}^\alpha - \iota_{\rho_a}\iota_{\rho_b}\Omega^\alpha - K^\alpha{}_{\beta\gamma} f_a^\beta f_b^\gamma = C_{ab}^c g_c^\alpha \tag{3.4.58}$$

and

$$C_{ab}^c\theta_c \;=\; 2\mathcal{L}_{\rho_{[a}}\theta_{b]} - \mathrm{d}\iota_{\rho_{[a}}\theta_{b]} - \iota_{\rho_a}\iota_{\rho_b}\mathcal{H} + 2\left(g_{[b}^\alpha + f_{[b}^\alpha\right)\iota_{\rho_{a]}}R^\alpha + g_{[b}^\alpha \nabla f_{a]\alpha} + f_{[b}^\alpha \nabla g_{a]\alpha}\,, \tag{3.4.59}$$

where we have defined

$$(\nabla g_a)^\alpha := \mathrm{d}g_a^\alpha + K^\alpha{}_{\beta\gamma} g_a^\beta \mathfrak{a}^\gamma\,, \tag{3.4.60}$$

and similarly for f_a^α. In addition to these constraints, condition (3.4.54) implies that:

$$\begin{aligned}
C_{ab}^c f_c^\alpha \;&=\; C_{ab}^c \iota_{\rho_c} a^\alpha \overset{(3.4.9)}{=} \iota_{[\rho_a,\rho_b]} a^\alpha = \mathcal{L}_{\rho_a}\iota_{\rho_b} a^\alpha - \iota_{\rho_b}\mathcal{L}_{\rho_a} a^\alpha \\
&=\; \iota_{\rho_a}\mathrm{d}\iota_{\rho_b} a^\alpha - \iota_{\rho_a}\mathrm{d}\iota_{\rho_b} a^\alpha - \iota_{\rho_b}\iota_{\rho_a}\mathrm{d}a^\alpha \\
&=\; 2\iota_{\rho_{[a}}\mathrm{d}f_{b]}^\alpha - \iota_{\rho_b}\iota_{\rho_a}R^\alpha + K^\alpha{}_{\beta\gamma} f_a^\beta f_b^\gamma \\
&=\; 2\iota_{\rho_{[a}}\left(\nabla f_{b]}\right)^\alpha - \iota_{\rho_b}\iota_{\rho_a}R^\alpha - K^\alpha{}_{\beta\gamma} f_a^\beta f_b^\gamma\,,
\end{aligned} \tag{3.4.61}$$

which is a non-Abelian version of the assumption (3.4.14) that we have made in the beginning. This means that, unlike the Abelian case, in the non-Abelian case this follows directly from the conditions on the background fields.

[10] More precisely, at this stage the two 2-forms differ by an exact 2-form. The properties of Courant algebroids with nonclosed 3-form [30] provide an a posteriori justification for the proportionality of the 2-forms Ω and R.

In a similar spirit to the Abelian case, our goal now is to identify the resulting constraints in terms of a bracket and bilinear in an appropriate Courant algebroid. Note that exactly because of the additional structure $\mathcal{G}$, this cannot be an exact Courant algebroid. Instead, we can make use of Courant algebroid dissections presented in section 2.5.2.

We are now ready to provide an interpretation of the gauging constraints. To this end we consider $\mathcal{G} = \mathcal{G}_L \oplus \mathcal{G}_R$ for the bundle of quadratic Lie algebras, and decompose its sections accordingly as $s = (f, g)$ with $f \in \mathcal{G}_L$ and $g \in \mathcal{G}_R$. For the inner product in $\mathcal{G}$ we identify

$$\langle s_a, s_b \rangle^{\mathcal{G}} = \frac{1}{2} k_{\alpha\beta} (f_a^\alpha g_b^\beta + g_a^\alpha f_b^\beta) \,. \tag{3.4.62}$$

Then the first constraint (3.4.57) takes the general form

$$\langle \Psi(\hat{\rho}), \Psi(\hat{\rho}') \rangle = 0 \tag{3.4.63}$$

in terms of the dissection Ψ, where $\hat{\rho}, \hat{\rho}' \in \Gamma(E)$. Furthermore, comparing the bracket (2.5.27) and the gauging constraints (3.4.58) and (3.4.59), the above identification directly leads to the closure of the bracket, namely

$$[\hat{\rho}_a, \hat{\rho}_b] = C^c_{ab} \hat{\rho}_c \,, \tag{3.4.64}$$

with the Lie bracket in $\mathcal{G}$ being

$$[(f_a, g_a), (f_b, g_b)]^\alpha = K^\alpha{}_{\beta\gamma} (f_a^\beta f_b^\gamma, f_a^\beta f_b^\gamma + f_a^\beta g_b^\gamma - f_b^\beta g_a^\gamma) \,. \tag{3.4.65}$$

All properties (2.5.25) are satisfied with this identification, as they correspond to the conditions (3.4.35) on the background fields and the choice of connection (3.4.60). This directly generalizes the corresponding result of the Abelian case. The fields of the gauged theory are constrained on Dirac structures of the nonexact Courant algebroid over $TM \oplus \mathcal{G} \oplus T^*M$.

Before we close this section, it is worth comparing in more general terms the Abelian and non-Abelian cases. It should be clear that the Abelian case is not fully contained in the non-Abelian one. One difference, as already mentioned, is the absence of condition (3.4.35c) in the Abelian case. As a result, condition (3.4.61) is absent in the Abelian case. Furthermore, the fields R and Ω are independent in the Abelian case, while in the non-Abelian they are related through (3.4.55). Finally, it would seem that the Courant bracket obtained in the Abelian version is different from the non-Abelian one. However, the way that the Courant bracket is defined here does include the Abelian case as well. It turns out that there is more freedom in satisfying properties (2.5.25) if R is closed since in that case it is possible for R to act independently on

$\mathcal{G}_L$ and $\mathcal{G}_R$. Specifically, we take

$$R((f_a,g_a),(f_b,g_b)) = (R(f_a,f_b), -\Omega(g_a,g_b)) \,. \tag{3.4.66}$$

In the non-Abelian case this is not possible since the bracket would not satisfy all of the necessary properties.

3.4.3 Examples

Let us now discuss a couple of simple examples of gauging along vector fields ρ_a in the presence of a nonclosed 3-form H. Both examples refer to the Abelian case of subsection 3.4.1 and they differ in that ρ_a form an Abelian algebra in the first and a non-Abelian one in the second.

In both examples, we take as M a real, Euclidean 4-manifold with coordinates $x^i = (x,y,z,w)$—and we denote the corresponding pull-backs X^i with the same lower case letters. The additional direction (of the circle bundle) is denoted by Φ as before. Furthermore, we consider the following 3-form and 2-forms

$$H = x\,\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}w\,, \quad \Omega = \mathrm{d}y\wedge\mathrm{d}z\,, \quad R = -\mathrm{d}x\wedge\mathrm{d}w\,. \tag{3.4.67}$$

Evidently, both Ω and R are closed 2-forms, whereas the 3-form H is not closed and it satisfies $\mathrm{d}H = -R\wedge\Omega$. We should also specify the manifold, in particular its metric G, which we do separately in each case.

First, we consider a single vector field ρ_x, specifically

$$\rho_x = y\partial_z - z\partial_y\,. \tag{3.4.68}$$

This vector field generates rotations and any manifold M with metric G which is invariant under this rotation, namely for which ρ_x is a Killing vector field, is admissible. Here we consider the flat metric for simplicity,

$$G = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2\,. \tag{3.4.69}$$

Our goal now is to specify f_x, g_x and θ_x such that all consistency conditions (3.4.35) and constraints (3.4.37) for gauging along the vector field ρ_x are satisfied—in the present Abelian case, for $K_{\alpha\beta\gamma} = 0$ and α taking only one value. First, from (3.4.35b), we can determine the function g_x, at least up to exact terms. We find

$$g_x = -\frac{y^2+z^2}{2}\,. \tag{3.4.70}$$

It may be directly checked that the constraint (3.4.37c) is then identically satisfied. Next, condition (3.4.35a) is consistent only if $\mathrm{d}f_x = 0$, as can be checked by acting on it with the exterior derivative d. Thus, f_x should be a real constant. Then, one can determine the 1-form θ_x up to exact terms,

$$\theta_x = -\frac{x}{2}\left(y^2+z^2\right)\mathrm{d}w + \frac{f}{2}\left(y\mathrm{d}z - z\mathrm{d}y\right) . \tag{3.4.71}$$

This makes sure that the constraints (3.4.37a) and (3.4.37b) are identically satisfied, therefore all the necessary and sufficient conditions for consistent gauging hold. The gauged action functional, containing the single gauge field A_x, reads explicitly as

$$\begin{aligned} S \;=\; & -\int_{\Sigma_2} ||\mathrm{d}x||^2 + ||\mathrm{d}y + zA_x)||^2 + ||\mathrm{d}z - yA_x||^2 + ||\mathrm{d}w||^2 \\ & -\int_{\Sigma_2} \frac{f}{2}A_x\wedge(y\mathrm{d}z - z\mathrm{d}y) + \frac{x}{2}(y^2+z^2)A_x\wedge\mathrm{d}w - \frac{1}{2}(y^2+z^2)A_x\wedge\Lambda \\ & -\int_{\Sigma_3} x\,\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}w + \mathrm{d}y\wedge\mathrm{d}z\wedge\Lambda\,, \end{aligned} \tag{3.4.72}$$

where we denote the inner product $||\omega||^2 = \frac{1}{2}\omega\wedge *\omega$ and the fiber 1-form is $\Lambda = \mathrm{d}\Phi - x\mathrm{d}w$. The infinitesimal gauge transformations are

$$\delta y = -z\varepsilon\,, \quad \delta z = y\varepsilon\,, \quad \delta\Phi = f\varepsilon\,, \quad \delta A_x = \mathrm{d}\varepsilon\,, \tag{3.4.73}$$

where ε is the single gauge parameter of the model. It is worth emphasizing that from the point of view of the higher-dimensional geometry of the target space $M\times S^1$, the 3-form supported on Σ_3 is closed; indeed, it can be written as $\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}\Phi$, which is then the Wess-Zumino term for the action S. However, from a dimensional reduction perspective, lower dimensional fields are not identified using $\mathrm{d}\Phi$ as an expansion 1-form but using instead the 1-form $\Lambda = \mathrm{d}\Phi + \mathfrak{a}$, where presently $\mathfrak{a} = -x\mathrm{d}w$ is the "Kaluza-Klein" gauge field. This is precisely what we did above by identifying H and Ω as in (3.4.67). Indeed, then the full 3-form in the higher-dimensional theory is closed, whereas the 3-form of the dimensionally reduced theory is not and it is given as above. It is precisely this Kaluza-Klein perspective that prompted us to write the Wess-Zumino term in (3.4.72) as such.

As a second example, we would like to consider a set of non-Abelian vector fields ρ_a. To this end, we consider the same background fields in the topological sector as in the first example, and the vector fields

$$\rho_x = \partial_x\,, \qquad \rho_y = \partial_y + z\partial_x\,, \qquad \rho_z = \partial_z\,. \tag{3.4.74}$$

We observe that there is one nonvanishing commutator, namely

$$[\rho_y, \rho_z] = -\rho_x \,, \tag{3.4.75}$$

therefore one nonvanishing structure constant $C^x_{zy} = 1$. This is nothing but a three-dimensional nilpotent Lie algebra. The background metric should then be invariant under this symmetry, and this is indeed the case for the following metric,

$$G = (\mathrm{d}x - z\mathrm{d}y)^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2 \,. \tag{3.4.76}$$

Given the above background data, one may readily check that all conditions (3.4.35) and constraints (3.4.37) are satisfied for the following

$$f_x = 0 \,, \qquad f_y = -1 = f_z \,, \tag{3.4.77}$$
$$g_x = 1 \,, \qquad g_y = z \,, \qquad g_z = -y \,, \tag{3.4.78}$$
$$\theta_x = x\mathrm{d}w \,, \quad \theta_y = \mathrm{d}x - y\mathrm{d}z + xz\mathrm{d}w \,, \quad \theta_z = \mathrm{d}x - y\mathrm{d}z - xy\mathrm{d}w \,. \tag{3.4.79}$$

This leads to a gauged action functional of the form,

$$\begin{aligned} S \;=\; & -\int_{\Sigma_2} ||\mathrm{d}x - z\mathrm{d}y - A_x||^2 + ||\mathrm{d}y - A_y||^2 + ||\mathrm{d}z - A_z||^2 + ||\mathrm{d}w||^2 \\ & -\int_{\Sigma_2} xA_x \wedge \mathrm{d}w + A_y \wedge (\mathrm{d}x - y\mathrm{d}z + xz\mathrm{d}w) + A_z \wedge (\mathrm{d}x - y\mathrm{d}z - xy\mathrm{d}w) \\ & -\int_{\Sigma_2} (A_x + zA_y - yA_z) \wedge \Lambda + A_x \wedge (A_y + A_z) - (y+z) A_y \wedge A_z \\ & -\int_{\Sigma_3} x\,\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}w + \mathrm{d}y \wedge \mathrm{d}z \wedge \Lambda \,, \end{aligned} \tag{3.4.80}$$

where similar comments to the ones below (3.4.72) apply here.

3.5 Jacobi sigma models and twists

Contact structures, such as the ones encountered in section 3.4.1, may be viewed as special cases of Jacobi structures for odd dimensional manifolds, as was shown in section 2.2. Our purpose in this section is to investigate which sigma models are associated to Jacobi structures and whether such theories can be twisted in a sense to be explained below [28, 31]. In essence, the analysis and results of the present section are independent from section 3.4. However, they can be viewed as a natural generalisation in the direction of allowing the background fields to depend on the additional spacetime direction. This is in contrast to trivial dimensional reduction and we comment on this in the following.

3.5.1 Jacobi Sigma Models

Our main goal here is to investigate the construction and symmetries of nonlinear sigma models when the target space is associated to a Jacobi structure. In other words, we wish to show that there is a correspondence between Jacobi manifolds and a certain class of sigma models, henceforth called Jacobi Sigma Models.

In order to proceed, one may start with the Poisson sigma model described in section 3.3.2, whose action functional is simply

$$S_{\mathrm{PSM}}[X,A] = \int_{\Sigma_2} A_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\mathcal{P}^{ij}(X) A_i \wedge A_j\,, \tag{3.5.1}$$

and the corresponding infinitesimal gauge transformations are given by:

$$\delta X^i = \mathcal{P}^{ji}\varepsilon_j\,, \tag{3.5.2}$$

$$\delta A_i = \mathrm{d}\varepsilon_i + \partial_i \mathcal{P}^{jk} A_j \varepsilon_k\,. \tag{3.5.3}$$

Motivated by the discussion in Section 2.2.1, we consider multiplying the Poisson structure with a function f, which we henceforth parametrize as $f = e^{-\Phi}$ for later convenience. This will directly spoil the symmetries of the Poisson sigma model, and in order to restore gauge invariance one must add a BF-type term involving the function f and a topological term based on the (pull-back of) the components of a vector field V. In order to be explicit, we should note that eventually the function f, or equivalently Φ, will depend on the local coordinates σ^α of the two-dimensional world-sheet. Here, in a similar spirit to section 3.4, we consider only an explicit such dependence, thus $\Phi = \Phi(\sigma)$ being a map from Σ_2 to $\mathbb{R}$. Since we are still considering the sigma model map $X : \Sigma_2 \to M$ from the two-dimensional source manifold Σ_2 to the d-dimensional target manifold M, one may also think in terms of the combination of the $d+1$ scalar fields to a map $\hat{X} = (X^i, \Phi) : \Sigma_2 \to M \times \mathbb{R}$; this points to a higher-dimensional viewpoint where the target space is $(d+1)$-dimensional, as was the case in the concept of Poissonization discussed in section 2.2.1.

The action functional we consider for a Jacobi sigma model is

$$S_{\mathrm{JSM}}[X,\Phi,A,A_0] = \int_{\Sigma_2} A_i \wedge \mathrm{d}X^i + A_0 \wedge \mathrm{d}\Phi + \tfrac{1}{2} e^{-\Phi}\Pi^{ij}(X) A_i \wedge A_j + e^{-\Phi} V^i(X) A_0 \wedge A_i\,. \tag{3.5.4}$$

One may view the combination of the world-sheet 1-forms A_i and A_0 as taking values in $T^*M \oplus \mathbb{R}$. The background fields now are $\Pi^{ij}(X)$ and $V^i(X)$. In order to interpret the action functional (3.5.4) as a nonlinear gauge theory, we now consider the following infinitesimal gauge transformations,

$$\begin{aligned}
\delta X^i &= e^{-\Phi}\left(\Pi^{ji}\varepsilon_j + V^i\varepsilon_0\right) , && (3.5.5)\\
\delta\Phi &= -e^{-\Phi}V^i\varepsilon_i , && (3.5.6)\\
\delta A_i &= \mathrm{d}\varepsilon_i + e^{-\Phi}\partial_i\Pi^{jk}A_j\varepsilon_k - e^{-\Phi}\partial_i V^j\left(A_j\varepsilon_0 - A_0\varepsilon_j\right) , && (3.5.7)\\
\delta A_0 &= \mathrm{d}\varepsilon_0 - e^{-\Phi}\Pi^{jk}A_j\varepsilon_k + e^{-\Phi}V^j\left(A_j\varepsilon_0 - A_0\varepsilon_j\right) , && (3.5.8)
\end{aligned}$$

where ε_i and ε_0 are two Σ_2-dependent scalar gauge parameters and $\delta = \delta_{(\varepsilon_i,\varepsilon_0)}$. Our aim is to find under which conditions these gauge transformations leave the action functional (3.5.4) invariant. By straightforward calculations, the transformation of (3.5.4) is found to be

$$\begin{aligned}
\delta S_{\mathrm{JSM}} = \int_{\Sigma_2} e^{-2\Phi}\Big(&\varepsilon_i A_j\wedge A_k\left(\tfrac{1}{2}\Pi^{il}\partial_l\Pi^{jk} + \Pi^{kl}\partial_l\Pi^{ij} + \tfrac{1}{2}V^i\Pi^{jk} + V^k\Pi^{ij}\right) + \\
&+\varepsilon_0 A_i\wedge A_j\left(\tfrac{1}{2}V^k\partial_k\Pi^{ij} + \Pi^{jk}\partial_k V^i\right) + \\
&+\varepsilon_i A_0\wedge A_j\left(\Pi^{ik}\partial_k V^j - \Pi^{jk}\partial_k V^i - V^k\partial_k\Pi^{ij}\right)\Big) . \qquad (3.5.9)
\end{aligned}$$

This immediately shows that the action functional is gauge invariant if and only if the bivector Π and the vector V constitute a Jacobi structure, i.e.

$$\delta S_{\mathrm{JSM}} = 0 \quad \Leftrightarrow \quad [\Pi,V]_{\mathrm{S}} = 0 \quad \text{and} \quad [\Pi,\Pi]_{\mathrm{S}} = -2V\wedge\Pi , \qquad (3.5.10)$$

which justifies the name of the model under consideration. Note that the target space in this case is $M\times\mathbb{R}$, with M being a Jacobi manifold.

For completeness, one may proceed to find the equations of motion of the model and the structure of its gauge algebra. The field equations for the action functional (3.5.4) are

$$\frac{\delta S_{\rm JSM}}{\delta A_i} \equiv DX^i := \mathrm{d}X^i + e^{-\Phi}\Pi^{ij}(X)A_j - e^{-\Phi}V^i(X)A_0 = 0\,, \tag{3.5.11}$$

$$\frac{\delta S_{\rm JSM}}{\delta X^i} \equiv DA_i := \mathrm{d}A_i + \tfrac{1}{2}e^{-\Phi}\partial_i\Pi^{jk}(X)A_j\wedge A_k + e^{-\Phi}\partial_iV^j(X)A_0\wedge A_j = 0\,, \tag{3.5.12}$$

$$\frac{\delta S_{\rm JSM}}{\delta A_0} \equiv D\Phi := \mathrm{d}\Phi + e^{-\Phi}V^i(X)A_i = 0\,, \tag{3.5.13}$$

$$\frac{\delta S_{\rm JSM}}{\delta \Phi} \equiv DA_0 := \mathrm{d}A_0 - \tfrac{1}{2}e^{-\Phi}\Pi^{ij}(X)A_i\wedge A_j - e^{-\Phi}V^i(X)A_0\wedge A_i = 0\,. \tag{3.5.14}$$

One can check that the equations of motion indeed transform covariantly, in other words D acting on the fields is a covariant world-sheet derivative. For example, using the Jacobi structure conditions, we find

$$\begin{aligned} \delta DX^i &= e^{-\Phi}\left(\varepsilon_k\partial_j\Pi^{ki} + \varepsilon_0\partial_jV^i\right)DX^j + e^{-\Phi}\left(\Pi^{ij}\varepsilon_j - V^i\varepsilon_0\right)D\Phi\,,\\ \delta D\Phi &= e^{-\Phi}\varepsilon_iV^iD\Phi - e^{-\Phi}\varepsilon_j\partial_iV^jDX^i\,. \end{aligned}$$

Much like the Poisson sigma model, the gauge algebra in the Jacobi case is open, in other words it only closes on-shell. The commutator algebra is found to be

$$[\delta(\varepsilon),\delta(\varepsilon')]X^i = \delta(\varepsilon'')X^i\,, \tag{3.5.15}$$

$$[\delta(\varepsilon),\delta(\varepsilon')]\Phi = \delta(\varepsilon'')\Phi\,, \tag{3.5.16}$$

$$[\delta(\varepsilon),\delta(\varepsilon')]A_i = \delta(\varepsilon'')A_i + \varepsilon''_iD\Phi + e^{-\Phi}\left(\partial_iQ_j{}^{kl}\varepsilon_k\varepsilon'_l - \partial_i\widetilde{Q}_j{}^k(\varepsilon_k\varepsilon'_0 - \varepsilon_0\varepsilon'_k)\right)DX^j\,, \tag{3.5.17}$$

$$[\delta(\varepsilon),\delta(\varepsilon')]A_0 = \delta(\varepsilon'')A_0 - \varepsilon''_0D\Phi - \varepsilon''_iDX^i\,, \tag{3.5.18}$$

with the gauge parameters on the right hand side being

$$\varepsilon''_i := e^{-\Phi}\left(Q_i{}^{jk}\varepsilon_j\varepsilon'_k - \widetilde{Q}_i{}^j(\varepsilon_j\varepsilon'_0 - \varepsilon_0\varepsilon'_j)\right)\,, \tag{3.5.19}$$

$$\varepsilon''_0 := -e^{-\Phi}\left(\Pi^{ij}\varepsilon_i\varepsilon'_j - V^i(\varepsilon_i\varepsilon'_0 - \varepsilon_0\varepsilon'_i)\right)\,, \tag{3.5.20}$$

where we have defined the structure functions

$$Q_i{}^{jk} := \partial_i\Pi^{jk} \quad \text{and} \quad \widetilde{Q}_i{}^j := \partial_iV^j\,. \tag{3.5.21}$$

Moreover, as is typical for gauge theories, there are also trivial gauge symmetries (with gauge transformations being proportional to the field equations, which is always possible).

Finally, one can now see that Poissonization applies directly to the two-dimensional sigma model presented here. Recall that we defined the map $\hat{X} = (\hat{X}^I) = (X^i, \Phi) : \Sigma_2 \to M \times \mathbb{R}$, with $I = 1, \ldots, \dim M + 1$. Similarly, the world-sheet 1-forms are packaged accordingly as $A_I = (A_i, A_0)$. Then the action functional of the Jacobi sigma model may be rewritten as

$$S_{\text{JSM}}[\hat{X}, A] = \int_{\Sigma_2} A_I \wedge \mathrm{d}\hat{X}^I + \tfrac{1}{2} \widetilde{\mathcal{P}}^{ij}(\hat{X}) A_I \wedge A_J \,, \tag{3.5.22}$$

which is a Poisson sigma model with target space $M \times \mathbb{R}$, distinct from the one defined by (3.5.1).

3.5.2 Examples

Conformal symplectic manifolds

A list of examples of Jacobi structures may be found for instance in Ref. [79]. One of the basic classes is locally conformal symplectic manifolds. Assume that M is an even-dimensional manifold. It is a locally conformal symplectic manifold if it is endowed with a pair (Ω, ω) of a nondegenerate 2-form and a globally defined 1-form (called the Lee form) respectively, such that

$$\mathrm{d}\omega = 0 \quad \text{and} \quad \mathrm{d}\Omega + \omega \wedge \Omega = 0 \,. \tag{3.5.23}$$

This structure can be obtained from a Jacobi structure as follows. One defines the vector field V and the bivector field Π to satisfy

$$i_V \Omega = -\omega \quad \text{and} \quad i_{\Pi(\xi)} \Omega = -\xi \tag{3.5.24}$$

for every $\xi \in T^*M$. As such, Ω and Π satisfy

$$\Omega \circ \Pi = \mathrm{Id} \quad \Rightarrow \quad \Omega_{ij} \Pi^{jk} = \delta_i^k \,, \tag{3.5.25}$$

where we presented the component expression for clarity and because it will be useful in the following. Note that in the index-free equation (3.5.25) we made an abuse of notation, in that Ω and Π denote also the corresponding isomorphisms from TM to T^*M and vice versa.

Let us now examine the corresponding Jacobi sigma model. We consider the equations of motion for A_i and A_0, given by Eqs. (3.5.11) and (3.5.13) respectively. Since A_i appears only algebraically, if we are able to solve for it we could substitute it back into the action. Since we have assumed nondegeneracy, this is presently possible. First, using (3.5.25), we may rewrite

(3.5.11) as

$$A_i = -e^{\Phi}\Omega_{ij}\mathrm{d}X^j + \omega A_0\,. \tag{3.5.26}$$

where in the last term on the right-hand side we also used the first defining equation (3.5.24). Turning then to the second field equation (3.5.13), substituting (3.5.26) and using once more the defining conditions for the Jacobi structure, we readily obtain

$$\omega = -\,\mathrm{d}\Phi\,. \tag{3.5.27}$$

Plugging these equations back in the original action functional of the Jacobi sigma model, one obtains as a result a sigma model with locally conformal symplectic target given by

$$S_{\mathrm{LCSSM}} = \int_{\Sigma_2} \tfrac{1}{2}e^{\Phi}\Omega_{ij}(X)\mathrm{d}X^i \wedge \mathrm{d}X^j + 2\,\omega(X)\wedge A_0\,. \tag{3.5.28}$$

Moreover, one may observe that although Ω is not a symplectic form, $\widetilde{\Omega} = e^{\Phi}\Omega$ is. On the other hand, setting ω to zero, one obtains the starting point of the topological A-model [10].

Contact manifolds

Another example of Jacobi structure is given by a contact manifold. Let M be a manifold of dimension $2n+1$ equipped with a (contact) 1-form ω. One may then construct the Jacobi bivector Π and the Reeb vector V as was described in section 2.2. The conditions on Π and V can be written in component form:

$$\begin{aligned} \Pi^{ij}\omega_j &= 0\,, & (3.5.29)\\ 2\Pi^{jk}\partial_{[k}\omega_{i]} &= -\delta_i^j + V^j\omega_i\,. & (3.5.30) \end{aligned}$$

We can now examine the corresponding Jacobi sigma model. Multiplying the equation of motion for A_i, given by Eq. (3.5.11), with ω_i, we get an expression for A_0:

$$A_0 = e^{\Phi}\omega_i \mathrm{d}X^i\,. \tag{3.5.31}$$

On the other hand, if we multiply the equation of motion for A_i with $(\mathrm{d}\omega)_{ik}$ and use the equation of motion for A_0, given by Eq. (3.5.13), we get an expression for A_i:

$$A_i = e^{\Phi}(\mathrm{d}\omega)_{ij}\mathrm{d}X^j - e^{\Phi}\omega_i\mathrm{d}\Phi\,. \tag{3.5.32}$$

Plugging these equations back in the action of the Jacobi sigma model, and assuming a nontrivial boundary for Σ_2, we obtain a 1-dimensional sigma model:

$$S_\omega = \int_{\Sigma_2} -\mathrm{d}\left(e^\Phi \omega_i \mathrm{d}X^i\right) = -\int_{\partial\Sigma_2} e^\Phi \omega_i \mathrm{d}X^i \,. \tag{3.5.33}$$

Thinking in terms of a nontrivial path $\gamma = \partial\Sigma_2$, and neglecting the additional scalar field Φ, the action functional

$$\begin{aligned} S_\omega : C^\infty(S^1, M) &\to \mathbb{R} \\ \gamma &\mapsto \int_\gamma \omega \,, \end{aligned} \tag{3.5.34}$$

determines the closed trajectories of the Reeb vector field, and therefore the Reeb dynamics [80].

3.5.3 Twisting the Jacobi Sigma Model

In the case of Poisson sigma model, we considered a generalization of the Poisson sigma model in the spirit of WZW models by assuming that the target space is equipped with a closed 3-form H, $dH = 0$, thus relating the resulting twisted Poisson sigma model with the twisted Poisson structure. Our aim in this section is to study the analog of this twisted Poisson structure in the case of the Jacobi sigma model. Although one might think that this is completely straightforward due to the Poissonization trick, we will see that a proper twisted Jacobi structure requires a cautious definition.

Given that apart from X^i, we now also have the additional scalar field Φ, there are two membrane terms one may add to the model. They are the pull-backs by the sigma model map $\hat{X} = (X, \Phi)$ of a 3-form $\hat{H} \in \Gamma(\bigwedge^3(T^*M \oplus \mathbb{R}))$ and a 2-form $\hat{\Omega} \in \Gamma(\bigwedge^2(T^*M \oplus \mathbb{R}))$ on the target space. Note that in principle one may also consider a second 2-form and a vector, and then add their corresponding terms on Σ_2; however, we will argue that these terms belong to the same cohomology classes of $\hat{H}$ and $\hat{\Omega}$ respectively. Finally, we mention that unlike the Poisson sigma model, here the 3-form $\hat{H}$ is not assumed to be closed from the beginning; we examine its properties below. According to the above, we consider the action functional

$$S_{\mathrm{WZJSM}} = S_{\mathrm{JSM}} + \int_{\Sigma_3} \frac{1}{6} \hat{H}_{ijk}(\hat{X}) \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k + \int_{\Sigma_3} \frac{1}{2} \hat{\Omega}_{ij}(\hat{X}) \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}\Phi \,. \tag{3.5.35}$$

For the time being, we do not specify further the dependence of the components $\hat{H}_{ijk}$ and $\hat{\Omega}_{ij}$ on the scalar field Φ. Furthermore, we suggest the following extended infinitesimal gauge transformations for the 1-form fields, denoted as $\hat{\delta}$, now including contributions from the components of the Wess-Zumino terms,

$$
\begin{aligned}
\hat{\delta}A_i &= \delta A_i - \tfrac{1}{2}e^{-\Phi}\left(\hat{H}_{ijk}(\Pi^{kl}\varepsilon_l + V^k\varepsilon_0) + \hat{\Omega}_{ij}V^l\varepsilon_l\right)(\mathrm{d}X^j - e^{-\Phi}\Pi^{jm}A_m + e^{-\Phi}V^jA_0) + \\
&\quad + \tfrac{1}{2}e^{-\Phi}\hat{\Omega}_{ij}(\Pi^{jl}\varepsilon_l + V^j\varepsilon_0)(\mathrm{d}\Phi - e^{-\Phi}V^mA_m)\,, \qquad (3.5.36)\\
\hat{\delta}A_0 &= \delta A_0 - \tfrac{1}{2}e^{-\Phi}\hat{\Omega}_{jk}(\Pi^{kl}\varepsilon_l + V^k\varepsilon_0)(\mathrm{d}X^j - e^{-\Phi}\Pi^{jm}A_m + e^{-\Phi}V^jA_0)\,, \qquad (3.5.37)
\end{aligned}
$$

up to trivial gauge transformations. On the other hand, the transformations for the scalar fields X^i and Φ remain unchanged. Note that the rightmost parentheses of each new term contain combinations starting with $\mathrm{d}X^i$ and $\mathrm{d}\Phi$; clearly these are not the derivatives DX^i and $D\Phi$ and they do not transform covariantly themselves.

Next we examine the gauge invariance of the action functional (3.5.35). First, a necessary condition is that

$$\mathrm{d}\hat{H} + \mathrm{d}\hat{\Omega}\wedge\mathrm{d}\Phi = 0\,. \qquad (3.5.38)$$

Thus we observe that at face value the 3-form $\hat{H}$ is not closed. Then the transformation of (3.5.35) becomes:

$$
\begin{aligned}
\hat{\delta}S_{\mathrm{WZJSM}} &= \int_{\Sigma_2} e^{-2\Phi}\Bigg(\frac{1}{2}\varepsilon_0 A_i\wedge A_j\left(-[\Pi,V]_{\mathrm{S}} + e^{-\Phi}\left(\hat{H}(\Pi,\Pi,V) - \hat{\Omega}(\Pi,V)\wedge V\right)\right)^{ij} + \\
&\quad + \frac{1}{2}\varepsilon_k A_i\wedge A_j\left(V\wedge\Pi + \frac{1}{2}[\Pi,\Pi]_{\mathrm{S}} - e^{-\Phi}\left(\hat{H}(\Pi,\Pi,\Pi) + \hat{\Omega}(\Pi,\Pi)\wedge V\right)\right)^{ijk} + \\
&\quad + \varepsilon_j A_0\wedge A_i\left(-[\Pi,V]_{\mathrm{S}} + e^{-\Phi}\left(\hat{H}(\Pi,\Pi,V) - \hat{\Omega}(\Pi,V)\wedge V\right)\right)^{ij}\Bigg)\,. \qquad (3.5.39)
\end{aligned}
$$

Therefore, gauge invariance is achieved if and only if the following conditions hold:

$$
\begin{aligned}
[\Pi,\Pi]_{\mathrm{S}} &= -2\left(V\wedge\Pi - e^{-\Phi}\hat{H}(\Pi,\Pi,\Pi) - e^{-\Phi}V\wedge\hat{\Omega}(\Pi,\Pi)\right)\,, \qquad (3.5.40)\\
[\Pi,V]_{\mathrm{S}} &= -e^{-\Phi}V\wedge\hat{\Omega}(\Pi,V) + e^{-\Phi}\hat{H}(\Pi,\Pi,V)\,. \qquad (3.5.41)
\end{aligned}
$$

Moreover, the result does not change essentially if we include from the beginning additional terms on the world-sheet Σ_2. Such terms could be the pull-back of a 2-form B and a 1-form C of the type

$$\int_{\Sigma_2}\frac{1}{2}B_{ij}(\hat{X})\mathrm{d}X^i\wedge\mathrm{d}X^j + C_i(\hat{X})\mathrm{d}X^i\wedge\mathrm{d}\Phi\,. \qquad (3.5.42)$$

The only difference then would be that the above conditions hold using instead the modified by exact terms 3-form and 2-form

$$\hat{H}\to\hat{H}+\mathrm{d}B \quad \text{and} \quad \hat{\Omega}\to\hat{\Omega}+\mathrm{d}C\,, \qquad (3.5.43)$$

which belong to the same cohomology class as $\hat{H}$ and $\hat{\Omega}$.

As a simple check at this stage, we observe that setting Φ and V to zero the twisted Poisson condition (2.3.12) is immediately recovered. Then one could be tempted to call (3.5.40) and (3.5.41) the twisted Jacobi structure. However, a puzzle arises by noticing that the left-hand side of these conditions is independent of Φ, while the right-hand side contains Φ explicitly. To resolve this puzzle, we are led to specify the dependence of $\hat{H}$ and $\hat{\Omega}$ on Φ by refining their definitions as follows:

$$\hat{H}(X,\Phi) = e^{\Phi}H(X) \quad \text{and} \quad \hat{\Omega}(X,\Phi) = e^{\Phi}\Omega(X)\,. \tag{3.5.44}$$

Taking into account Eq. (3.5.38), it turns out that H is exact (and therefore closed), and in particular

$$H = \mathrm{d}\Omega\,. \tag{3.5.45}$$

We conclude that the twisted Jacobi structure is appropriately defined as a triple (Π, V, Ω) of a bivector field, a vector field and a nonclosed 2-form such that the following two conditions hold,

$$\frac{1}{2}[\Pi,\Pi]_{\mathrm{S}} = -V\wedge\Pi + \mathrm{d}\Omega(\Pi,\Pi,\Pi) + V\wedge\Omega(\Pi,\Pi)\,, \tag{3.5.46}$$
$$[\Pi,V]_{\mathrm{S}} = -V\wedge\Omega(\Pi,V) + \mathrm{d}\Omega(\Pi,\Pi,V)\,. \tag{3.5.47}$$

This is in agreement with the corresponding definition of Ref. [81], which we obtained here using a field-theoretic approach and gauge invariance. Moreover, these observations show that the action functional of the twisted Jacobi sigma model ends up being

$$S_{\mathrm{WZJSM}} = S_{\mathrm{JSM}} + \int_{\Sigma_3}\frac{1}{2}e^{\Phi}\partial_i\Omega_{jk}(X)\mathrm{d}X^i\wedge\mathrm{d}X^j\wedge\mathrm{d}X^k + \int_{\Sigma_3}\frac{1}{2}e^{\Phi}\Omega_{ij}(X)\mathrm{d}X^i\wedge\mathrm{d}X^j\wedge\mathrm{d}\Phi\,,$$

and noting that $\mathrm{d}e^{\Phi} = e^{\Phi}\mathrm{d}\Phi$, the Wess-Zumino term is a total derivative and thus drops to the boundary. Then the resulting theory is simply

$$\begin{aligned} S_{\mathrm{WZJSM}} = & \int_{\Sigma_2} A_i\wedge\mathrm{d}X^i + A_0\wedge\mathrm{d}\Phi + \\ & + \tfrac{1}{2}e^{-\Phi}\Pi^{ij}(X)A_i\wedge A_j + e^{-\Phi}V^i(X)A_0\wedge A_i + e^{\Phi}\Omega_{ij}(X)\mathrm{d}X^i\wedge\mathrm{d}X^j\,. \end{aligned} \tag{3.5.48}$$

We remark that although the Poisson structure is a special case of Jacobi, this is no longer strictly true for the corresponding twisted structures. Setting $V = 0$ to (3.5.46) and (3.5.47), one obtains a twisted Poisson structure, albeit for an exact 3-form only. In general though the

3-form does not have to be exact but only closed. This shows that if one wishes to recover the general twisted Poisson structure, the case should be distinguished already at the level of conditions (3.5.40) and (3.5.41). Indeed, our previous argumentation for defining the twisted Jacobi structure assumed that Φ is nonvanishing. However, when Φ is zero, then (3.5.38) yields $\hat{H}$ closed and with V also vanishing the aforementioned conditions result in the twisted Poisson structure as already noted before.

3.6 R-Poisson sigma model

Jacobi sigma model arose from the generalization of the Poisson sigma model motivated by Jacobi structures as generalization of Poisson structures. But there is another generalization of Poisson structures, that of R-Poisson structures, introduced in Section 2.4. Here we wish to explore the corresponding generalization of the Poisson sigma model, resulting in R-Poisson sigma model [35].

Given a twisted R-Poisson structure of order $p+1$, there exists a topological field theory in $p+1$ dimensions with target space the corresponding twisted R-Poisson manifold M [35]. The fields of the theory are of four different types, specifically (a) a set of scalar fields $X^i, i = 1,\dots \dim M$, which are identified with the components of a sigma model map $X : \Sigma_{p+1} \to M$, where Σ_{p+1} is the $(p+1)$-dimensional spacetime where the theory is defined (the world volume; in the few instances when we consider a local coordinate system on it, we refer to its coordinates as σ^μ with $\mu = 0,\dots p$), (b) world volume 1-forms $A_i = A_{i\mu}(\sigma)\mathrm{d}\sigma^\mu$ taking values in the pull-back bundle X^*T^*M, (c) world volume $(p-1)$-forms Y^i taking values in the pull-back bundle TM and (d) world volume p-forms Z_i taking values in the pull-back bundle T^*M. Summarizing, the field content of the theory is

$$(X^i, A_i, Y^i, Z_i) \quad \text{of form degrees} \quad (0,1,p-1,p)\,. \tag{3.6.1}$$

With the above field content, one can write down a general action functional in $p+1$ dimensions with $p \geq 1$, which has the form of a topological sigma model, specifically

$$\begin{aligned} S^{(p+1)} \;=\; & \int_{\Sigma_{p+1}} \Big(Z_i \wedge \mathrm{d}X^i - A_i \wedge \mathrm{d}Y^i + \Pi^{ij}(X) Z_i \wedge A_j - \frac{1}{2}\partial_k \Pi^{ij}(X) Y^k \wedge A_i \wedge A_j + \\ & + \frac{1}{(p+1)!} R^{i_1\dots i_{p+1}}(X) A_{i_1} \wedge \dots \wedge A_{i_{p+1}} \Big) + \int_{\Sigma_{p+2}} X^*H\,, \end{aligned} \tag{3.6.2}$$

where the last term is a Wess-Zumino one, obtained as the pull-back of the $(p+2)$-form H on

M,

$$X^*H = \frac{1}{(p+2)!} H_{i_1\dots i_{p+2}}(X)\,\mathrm{d}X^{i_1}\wedge\ldots\wedge\mathrm{d}X^{i_{p+2}} \tag{3.6.3}$$

and supported on an open $(p+2)$-brane Σ_{p+2} whose boundary is Σ_{p+1}. The $(p+2)$-form H is further assumed to be closed, $\mathrm{d}H = 0$, so that its variation drops to the boundary and its contribution to the field equations is only through the map X and not its extension that is necessary to define the higher-dimensional term in (3.6.2). As usual, the quantum theory is well-defined provided that the homology class $[X(\Sigma_{p+1})] \in H_{p+1}(M)$ vanishes and that H defines an integer cohomology class [82].

Although the action functional (3.6.2) is written on a local patch of the target space M, it can be naturally defined globally once the relevant target space connections are introduced. Then the apparently non-tensorial coefficient $\partial_k\Pi^{ij}$ is completed to a tensor and the full theory can be written without using a local coordinate system on the target. Since this is not necessary for the present purposes, we refer to [35] where a complete discussion of the covariant formulation appears.

Provided that (M,Π,R,H) is a twisted R-Poisson manifold of order $p+1$, it was shown in [35] that the theory given by (3.6.2) is invariant under the following set of gauge transformations:

$$\begin{aligned}
\delta X^i &= \Pi^{ji}\varepsilon_j\,, && (3.6.4)\\
\delta A_i &= \mathrm{d}\varepsilon_i + \partial_i\Pi^{jk}A_j\varepsilon_k\,, && (3.6.5)\\
\delta Y^i &= (-1)^{p-1}\mathrm{d}\chi^i + \Pi^{ji}\psi_j - \partial_j\Pi^{ik}\left(\chi^j A_k + Y^j\varepsilon_k\right) + \frac{1}{(p-1)!}R^{iji_1\dots i_{p-1}}A_{i_1}\ldots A_{i_{p-1}}\varepsilon_j\,, && (3.6.6)\\
\delta Z_i &= (-1)^p\mathrm{d}\psi_i + \partial_i\Pi^{jk}\left(Z_j\varepsilon_k + \psi_j A_k\right) - \partial_i\partial_j\Pi^{kl}\left(Y^jA_k\varepsilon_l - \frac{1}{2}A_kA_l\chi^j\right) + \\
&\quad + \frac{(-1)^p}{p!}\partial_i R^{ji_1\dots i_p}A_{i_1}\ldots A_{i_p}\varepsilon_j - \frac{1}{(p+1)!}\Pi^{kj}H_{ijl_1\dots l_p}\Omega^{l_1\dots l_p}\varepsilon_k\,, && (3.6.7)
\end{aligned}$$

where wedge products between differential forms are implicit. It is observed that there are three gauge parameters $(\varepsilon_i,\chi^i,\psi_i)$ of form degrees $(0,p-2,p-1)$ respectively. The gauge transformation of the scalar fields is controlled by the Poisson structure Π. Notably, the only appearance of the components of the $(p+2)$-form H is in the ultimate term of the highest differential form field Z_i. They are combined with the world volume p-form $\Omega^{l_1\dots l_p}$ defined as

$$\Omega^{l_1\dots l_p} = \sum_{r=1}^{p+1}(-1)^r\prod_{s=1}^{r-1}\mathrm{d}X^{l_s}\prod_{t=r}^{p}\Pi^{l_tm_t}A_{m_t}\,, \tag{3.6.8}$$

which contains all possible combinations of $\mathrm{d}X$ and $\Pi(A)$ that yield a p-form. This is essentially tailor-made to cancel the contribution of the Wess-Zumino term to the gauge variation of $S^{(p+1)}$.

The field equations obtained from the action for the twisted R-Poisson sigma model read

$$F^i := \mathrm{d}X^i + \Pi^{ij}A_j = 0\,, \tag{3.6.9}$$

$$G_i := \mathrm{d}A_i + \frac{1}{2}\partial_i\Pi^{jk}A_j\wedge A_k = 0\,, \tag{3.6.10}$$

$$\mathcal{F}^i := \mathrm{d}Y^i + (-1)^p\Pi^{ij}Z_j + \partial_k\Pi^{ij}A_j\wedge Y^k - \frac{1}{p!}R^{ij_1\dots j_p}A_{j_1}\wedge\dots\wedge A_{j_p} = 0\,, \tag{3.6.11}$$

$$\begin{aligned}\mathcal{G}_i := {} & (-1)^{p+1}\mathrm{d}Z_i + \partial_i\Pi^{jk}Z_j\wedge A_k - \frac{1}{2}\partial_i\partial_j\Pi^{kl}Y^j\wedge A_k\wedge A_l + \\ & + \frac{1}{(p+1)!}\partial_i R^{j_1\dots j_{p+1}}A_{j_1}\wedge\dots\wedge A_{j_{p+1}} + \frac{1}{(p+1)!}H_{ij_1\dots j_{p+1}}\mathrm{d}X^{j_1}\wedge\dots\wedge\mathrm{d}X^{j_{p+1}} = 0\,.\end{aligned} \tag{3.6.12}$$

Using the first field equation, i.e. the one of the highest form Z_i, the gauge transformation rule of Z_i can be rewritten in an equivalent and more useful form as

$$\begin{aligned}\delta Z_i = {} & (-1)^p\mathrm{d}\psi_i + \partial_i\Pi^{jk}\left(Z_j\epsilon_k + \psi_j A_k\right) - \partial_i\partial_j\Pi^{kl}\left(Y^jA_k\epsilon_l - \frac{1}{2}A_kA_l\chi^j\right) + \\ & + \frac{1}{p!}f_i^{\,i_1\dots i_p j}A_{i_1}\dots A_{i_p}\epsilon_j - \\ & - \frac{1}{(p+1)!}\Pi^{kj}H_{ijl_1\dots l_p}\sum_{r=1}^{p}(-1)^{r+1}\binom{p+1}{r+1}\prod_{s=1}^{r}F^{l_s}\prod_{t=r+1}^{p}\Pi^{l_tm_t}A_{m_t}\epsilon_k\,,\end{aligned} \tag{3.6.13}$$

where we have defined

$$f_i^{\,i_1\dots i_{p+1}} := \partial_i R^{i_1\dots i_{p+1}} + H_i{}^{i_1\dots i_{p+1}}\,. \tag{3.6.14}$$

We can now readily observe that the transformation of Z_i contains the field strength of the scalar fields X^i, which is the field equation of Z_i. Remarkably, although for the 2D model ($p = 1$) this transformation only contains F^i linearly and without it appearing together with the field A_i, this ceases to be true in every other dimension higher than two. In contrast, for example in 3D one finds that[11]

$$\delta Z_i \supset \frac{1}{2}H_{il}{}^{mk}F^lA_m\epsilon_k + \frac{1}{3!}H_{ilm}{}^kF^lF^m\epsilon_k\,. \tag{3.6.15}$$

Thus both a product of F^i with A_i appears as well as a quadratic term in the field equation.

[11] In this section and section 4.3, we use the subset symbol $\supset$ to mean that the right-hand side appears in the full expression of the left-hand side along with other terms that are not shown. It will mostly be used to provide heuristic explanations that clarify the often complicated structure of the quantities we compute.

Clearly the situation becomes even more non-linear in higher dimensions. This general feature of this class of theories is unusual and it reproduces itself in the closure of the gauge algebra and the square of the BRST operator that we will encounter in section 4.3. Although in gauge theories we are used to having gauge algebras that only close on-shell or BRST operators that are nilpotent only on-shell, we are not aware of particular examples where products of field equations appear. This should not be discouraging however, since the general statements of on-shell closure or on-shell nilpotency are still valid. Therefore one expects that these features can still be treated within the BV/BRST formalism and we show in the section 4.3 that this is indeed the case.

Chapter 4

Classical BV Action

In this section, we start with the brief review of the quantization procedure for gauge theories, and in particular, the Batalin-Vilkovisky (BV) formalism [83, 64, 84, 85]. The rest of the chapter is then dedicated to the construction of the classical BV action for the Dirac sigma model and R-Poisson sigma model.

4.1 BV formalism

Start with some classical field theory, which is specified by the following data: an underlying space M on which the theory is defined, a space of fields $\mathcal{F}$ and an action functional $S : \mathcal{F} \to \mathbb{R}$. The space M is mathematically a manifold, but the physical interpretation of it can vary greatly. For example, in the Yang-Mills theory, this is a 4-dimensional Minkowski spacetime, while in the string theory, it is a world-sheet of the string. The space of fields is usually some kind of sheaf or bundle over M, but the specifics depend on the details of the theory. Finally, the action functional is usually required to satisfy the locality condition, meaning that it can be written as an integral of some function over M:

$$S[\Phi] = \int_M L, \tag{4.1.1}$$

where Φ denote all the possible fields in the theory. Here, L is called a Lagrangian and it is a map from some finite order jet space[1] of $\mathcal{F}$ and taking values in $\Gamma(\wedge^{\dim M} T^*M)$. As is usual, the equations of motion are determined by the variational principle thus giving the possible classical solutions of the theory.

[1]Effectively, this means that L depends on the coordinates on the manifold M, the fields Φ and derivatives of the fields with respect to the coordinates on M.

Here, the focus is on quantum field theory and the process of quantization itself, which immediately brings up 2 questions: What is quantum field theory? What does it mean to quantize a classical field theory? The first of these questions cannot be easily answered in its full generality, but for the present purposes we shall consider a simplified definition, which has been used in [85]. A quantum field theory is primarily a quantum theory meaning that it can be defined by specifying a Hilbert space of states and the set of physical observables. The field theory part of it is specified by this set of observables. It is very similar to the classical situation for it is a space $\mathcal{F}$, considered as a sheaf or a bundle over an underlying space M, but the fields in $\mathcal{F}$ take values in the space of linear operators of the above mentioned Hilbert space. The final ingredient would be the analogue of the action functional, something that describes the dynamics of the theory. For the sake of simplicity, here we take that to be equations of motion, both for the fields and for the states.

Given the definitions of the classical and quantum field theories, it becomes possible to specify what is meant by the notion of quantization. It is a procedure that assigns a quantum field theory to a classical one, such that the classical theory is reproduced from the quantum in the classical limit.[2] While this sounds simple enough, in practice it can become quite problematic. First thing to note here is that the definitions of the classical and quantum field theories only give their mathematical structure, but do not contain any physical information. Thus, two (mathematically) different classical/quantum field theories can be physically equivalent if they produce same physical, i.e. observable results. On the other hand, it is also possible that a single field theory is mathematically redundant from the physics point of view, in the sense that different mathematical configurations of fields and/or states represent the same physical state. Here, we shall focus only on the latter situation.

The mathematical redundancies exist in any quantum theory, whether it is a field theory or not. An example is the phase transformation in the Hilbert space of states. These are easily handled and we shall not refer to them any further, but focus on a more troublesome redundancies, the gauge symmetries. At the classical level, gauge theories include situations where the same physical state is described by multiple field configurations. The gauge group $\mathcal{G}$ of the theory consists of all transformations that map a single configuration to all of its physically equivalent configurations. Thus, the space of physical states can be obtained as quotient space of the configuration space over $\mathcal{G}$ and this procedure is known as gauge-fixing.

[2]The definition of quantization only assigns the name to a specific procedure if the conditions are met. There is no guarantee that such a thing necessarily even exists for a given classical field theory, and if it does, it does not have to be unique. For the present purposes, it is always assumed that such a thing really does exist, while the uniqueness part is not relevant. Additionally, any difficulties regarding the proper definitions of the classical limit for the quantum field theory shall not be explored here.

The real trouble begins when one tries to quantize classical gauge theory. One option here would be to quantize the space of physical configurations directly. This can be problematic since the quotient space, unlike the original configuration space, need not be a manifold, thus leading to various difficulties. Instead, it is much easier to quantize the full theory and then do the gauge-fixing at the quantum level. As a result the space of physical states does not equal the corresponding Hilbert space of the theory, not even up to phase transformations, but is instead a proper subspace of it. It is then possible to define a hermitian operator s on the Hilbert space, whose kernel is equal to the subspace of physical states [85].

As a consequence of its definition, it follows that a scalar product of two vectors from the $\ker s$ remains unchanged when an s-exact term is added to one of them. Since the definition of s is still not unique, one can then add a requirement that the image of s lies in its kernel. This means that the two states are physically equivalent if they differ by an s-exact term. Furthermore, it also forces s to be nilpotent of order 2, i.e. $s^2 = 0$. As a result, s defines cohomology on the Hilbert space, and the corresponding space of cohomology classes is isomorphic to the space of physical states. This kind of operator is known as the BRST operator. Thus, the task of defining a space of physical states is equivalent to the one of defining the BRST operator.

It is obvious that defining the BRST operator is just as easy/hard as defining the space of physical states, at least when one tries to define it directly on the Hilbert space. Instead, it is possible to shift the action of s to the fields. The way to do this, as described in [83], is to construct a differential graded manifold $\mathcal{F}_{BRST}$ whose 0-th order is the space of fields $\mathcal{F}$. The grading on this manifold is usually referred to as ghost number, while the additional fields are known as ghost fields. The BRST operator can then be interpreted as a cohomological vector field on this graded manifold.

Having introduced the ghost-graded manifold $\mathcal{F}_{BRST}$ and the action of BRST operator on the fields, the BV formalism requires us to go one step further and define a space $\mathcal{F}_{BV} = T^*[-1]\mathcal{F}_{BRST}$. Thus, for every field $\Phi \in \mathcal{F}_{BRST}$ there exists the so called antifield $\Phi^* \in \mathcal{F}_{BV}$, such that the sum of ghost degrees of the field and its corresponding antifield is -1. The idea here is to replace the original field theory by one on this extended space that contains the original in some subset.

The BV manifold $\mathcal{F}_{BV}$ is naturally equipped with a graded symplectic form:

$$\omega = \sum_{\Phi} \delta\Phi \wedge \delta\Phi^* , \tag{4.1.2}$$

which then induces a Gerstenhaber bracket $(\cdot,\cdot)$ on $\mathcal{F}_{BV}$, known as the antibracket, in the same fashion a standard symplectic form induces a Poisson bracket on symplectic manifolds. Furthermore, the BV manifold is also equipped with a measure:

$$\mu = \prod_{\Phi} \mathcal{D}\Phi\mathcal{D}\Phi^* \tag{4.1.3}$$

that is going to be used in path integrals. More details about thin can be found in e.g. [86]. The above measure then defines a divergence operator ∇ on vector fields X of $\mathcal{F}_{BV}$ such that:

$$\int_{\mathcal{F}_{BV}} X(f)\,\mu = \int_{\mathcal{F}_{BV}} (\nabla X) f\,\mu\,, \tag{4.1.4}$$

for all functions f on $\mathcal{F}_{BV}$. This in turn defines a Laplacian operator Δ:

$$\Delta f = \frac{(-1)^{|f|}}{2} \nabla((f,\cdot))\,. \tag{4.1.5}$$

This definition of Laplacian operator is the same as in [67], though it is not the only possible one. Additional ways to construct the Laplacian operator can be found in [64, 84], where some of its properties are described as well.

Since a BV manifold $\mathcal{F}_{BV}$ is a (graded) symplectic manifold, it is possible to define its Lagrangian submanifolds as those for which the corresponding projection of the (graded) symplectic form ω vanishes, and, in addition, whose dimension is equal to the half of the dimension of $\mathcal{F}_{BV}$. One example of a Lagrangian submanifold is exactly the original space $\mathcal{F}_{BRST}$. However, it is possible to find a much wider class of Lagrangian submanifolds by relaxing the condition that the antifields vanish. For that purpose, one introduces the gauge-fixing fermion Ψ, which is a function on $\mathcal{F}_{BRST}$. Then, instead of requiring that Φ^* equals 0, one can require it to be equal to the derivative of Ψ with respect to Φ. The point of this is that the choice of the Lagrangian submanifold corresponds to the specific choice of gauge. Specifically, in the path integral formalism, instead of integrating over the whole space of fields, which as we know is problematic because of gauge symmetry, one only needs to integrate over a single Lagrangian submanifold, with the corresponding induced measure on it. However, instead of original action S_{cl}, one needs to use some corresponding BV action S_{BV} in those path integrals. It has been shown in [87] that the necessary and sufficient condition to obtain the correct expectation values using the BV action is:

$$\Delta \mathrm{e}^{iS_{BV}/\hbar} = 0\,, \tag{4.1.6}$$

where we have written the Planck constant $\hbar$ explicitly for the moment. This expression is equivalent to:

$$(S_{BV}, S_{BV}) + i\hbar\Delta S_{BV} = 0\,, \tag{4.1.7}$$

which is known as the quantum master equation. By expanding S_{BV} in terms of powers of $\hbar$:

$$S_{BV} = S_0 + \hbar S_1 + \hbar^2 S_2 + \ldots, , \tag{4.1.8}$$

and keeping only the lowest order terms in the quantum master equation, one obtains the classical master equation:

$$(S_0, S_0) = 0 . \tag{4.1.9}$$

As already mentioned previously, there is a convenient geometrical way to construct BV action for some of the topological string theories, known as the AKSZ construction [40]. Consider a sigma model with Σ as the source manifold, M as the target space and X as the sigma model maps. It is assumed that the target space can be extended to a QP-manifold $\mathcal{M}$ with M as its 0-th degree component. One can then extend the original maps X to $\phi : T[1]\Sigma \to \mathcal{M}$ and formulate the BV action as the sigma model from $T[1]\Sigma$. For the cohomological vector field Q on $\mathcal{M}$ there exists a Hamiltonian Θ such that $Q = \{\Theta, \cdot\}$. Then the BV action is simply given by the pull-back of Θ:

$$S_{BV}[\phi] = \int \left(\frac{1}{2} \omega_{ab} \mathrm{d}\phi^a \wedge \mathrm{d}\phi^b + \phi^* \Theta \right) , \tag{4.1.10}$$

where ω is the P-structure (graded symplectic form) on $\mathcal{M}$. This procedure gives the BV action in terms of geometrical data, but it does have an assumption of QP-structure on $\mathcal{M}$, which is not the case for the Dirac sigma model and twisted R-Poisson sigma model. In what follows we consider alternative methods for the construction of the BV action for the two mentioned models.

For the rest of this chapter, we shall focus only on the classical master equation. First we construct the classical BV action S_0 for the Dirac sigma model and then for the 3-dimensional R-Poisson sigma model. At this point we specify the conventions we are using throughout the rest of the chapter. First, the antibracket of two functions F and G equals:

$$(F, G) = \int \mathrm{d}^n\sigma \mathrm{d}^n\sigma' \sum_{\Phi} \left(\frac{\delta_R F}{\delta\Phi(\sigma)} \frac{\delta_L G}{\delta\Phi^*(\sigma')} - \frac{\delta_R F}{\delta\Phi^*(\sigma)} \frac{\delta_L G}{\delta\Phi(\sigma')} \right) \delta(\sigma - \sigma') , \tag{4.1.11}$$

where L and R as indices in the derivatives denote left and right derivative, which are defined through:

$$\delta S = \int \sum_{\Phi} \delta\Phi \frac{\delta_L S}{\delta\Phi} = \int \sum_{\Phi} \frac{\delta_R S}{\delta\Phi} \delta\Phi . \tag{4.1.12}$$

Instead of Φ^* here, we shall use its Hodge dual:

$$\Phi^+ = *\Phi^*\,, \tag{4.1.13}$$

which is more useful than Φ^* because all of the fields we shall be considering are differential forms on the world-sheet (world-volume). With this convention, one can find two useful identities for a theory with $(p+1)$-dimensional world-volume:

$$\begin{aligned}\left(\int a \wedge \Phi^+, \Phi\right) &= (-1)^{p \cdot f(\Phi)} a\,, &(4.1.14)\\ \left(\int b \wedge \Phi, \Phi^+\right) &= -b\,, &(4.1.15)\end{aligned}$$

where $f(\Phi)$ is the form degree of Φ and a and b have form and ghost degrees such that $a \wedge \Phi^+$ and $b \wedge \Phi$ are $(p+1)$-forms of vanishing ghost degree. Furthermore, a and b here are independent of Φ^+ and Φ, respectively. Such additional cases follow from this specific ones by adding appropriate combinatorical factors.

4.2 BV action of the Dirac sigma model

In this section we present a procedure for the construction of the classical BV action for the Dirac sigma models. First we define the BRST operator, then introduce the corresponding antifields, and then finally construct the classical BV action by solving the classical master equation. The procedure here is analogous to the one presented in [45] for the H-twisted Poisson sigma model, which is a special case of the Dirac sigma model.

4.2.1 BRST operator and field-antifield content

The gauge transformations for the Dirac sigma model contain a single scalar gauge parameter. Thus, it is enough to introduce a single scalar ghost field c^a of ghost degree 1. The BRST operator s, which is of ghost degree 1, by its action on the fields:

$$\begin{aligned} sX^i &= \rho^i_a c^a\,, &(4.2.1)\\ sA^a &= \mathrm{d}c^a + C^a{}_{bc} A^b c^c + \omega^a{}_{bi} c^b F^i + \phi^a{}_{bi} c^b * F^i\,, &(4.2.2)\\ sc^a &= -\frac{1}{2} C^a{}_{bc} c^b c^c\,. &(4.2.3)\end{aligned}$$

(Anti)Field	X^i	A^a	c^a	X_i^+	A_a^+	c_a^+
Ghost degree	0	0	1	-1	-1	-2
Form degree	0	1	0	2	1	2

Table 4.1: The classical basis with ghost and form degrees for Dirac sigma models.

The basic property of the BRST operator is that it should be nilpotent, at least on shell. Indeed it is straightforward to confirm that $s^2X^i = 0$ due to the involutivity of the vector fields ρ_a and that $s^2c^a = 0$ due to the Jacobi identity of the Lie bracket. Finally, calculating the action of s^2 on the 1-form gauge fields A^a, one finds that it is not identically zero but rather proportional to terms that vanish on the classical equations of motion of the model. Specifically the result is

$$s^2A^a = \frac{1}{2}S^a{}_{bci}c^bc^cF^i + \frac{1}{2}\widetilde{S}^a{}_{bci}c^bc^c * F^i\,, \tag{4.2.4}$$

with the two curvature tensors S and $\widetilde{S}$ given by

$$S^a{}_{bc} = \frac{1}{2}\left(S^{+a}{}_{bc} + S^{-a}{}_{bc}\right)\,, \tag{4.2.5}$$

$$\widetilde{S}^a{}_{bc} = \frac{1}{2}\left(S^{+a}{}_{bc} - S^{-a}{}_{bc}\right)\,, \tag{4.2.6}$$

with $S^\pm$ defined in (3.3.12). The fact that s^2 does not vanish on all fields reflects the openness of the gauge algebra, namely that it closes only on shell. For this reason the BRST formalism is not sufficient to construct the extended action of the classical model and one should reside to the more general BV formalism.

The next step in implementing the BV strategy is a further extension of the field content by the introduction of antifields for each field and ghost of the theory. In the present case we have three antifields X_i^+,A_a^+ and c_a^+. We collect all fields and antifields with their ghost and form degree in Table 4.1.

The stage is now set to move on to the last part of the classical BV contruction and determine the BV action for the Dirac sigma model. We introduce a symplectic form on the space of fields and antifields:

$$\omega_{\mathrm{BV}} = \int_\Sigma \left(\delta X^i \wedge \delta X_i^+ + \delta A^a \wedge \delta A_a^+ + \delta c^a \wedge \delta c_a^+\right) , \tag{4.2.7}$$

which induces the antibracket shown in (4.1.11).

4.2.2 BV action and the classical master equation

The BV action $\mathcal{S}_{\mathrm{BV}}$ needs to satisfy classical master equation:

$$(\mathcal{S}_{\mathrm{BV}}, \mathcal{S}_{\mathrm{BV}})_{\mathrm{BV}} = 0 . \tag{4.2.8}$$

In general, the BV action can be expanded in the number of the antifields:

$$\mathcal{S}_{\mathrm{BV}} = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2 , \tag{4.2.9}$$

where the subscripts on the right-hand side denote the number of antifields in each sector and $\mathcal{S}_0$ is the classical action (3.2.15) that contains no antifields. While it is possible in principal for this expansion to contain higher terms as well, that is not the case here. The next term S_3 would need to contain at least 3 antifields. Since there are no scalar antifields, S_3 would have to contain 3 and higher order forms, which is impossible for the 2-dimensional world-sheet. Moreover, the sector with one antifield is essentially fixed by the BV formalism since it should reflect the gauge invariance of the classical action. In particular, it is given as

$$\mathcal{S}_1 = \int_\Sigma \left(X_i^+ sX^i - A_a^+ \wedge sA^a - c_a^+ sc^a\right) . \tag{4.2.10}$$

On the other hand, the sector with two antifields is not fixed a priori and it should be such that the classical master equation is satisfied. Therefore, anticipating the final result, we make the following Ansatz for this sector of the BV action,

$$\mathcal{S}_2 = \int_\Sigma \frac{1}{4} \left(Y^{ab}{}_{cd}(X) A_a^+ \wedge A_b^+ + Z^{ab}{}_{cd}(X) A_a^+ \wedge *A_b^+\right) c^c c^d , \tag{4.2.11}$$

with Y and Z being X-dependent coefficients to be determined through the classical master equation. At this stage they are arbitrary, but share the property that they are antisymmetric in their two lower indices, whereas Y is symmetric and Z antisymmetric in their two upper indices.

The next step is to impose the classical master equation (4.2.8). This comprises several terms with different structure that should be separately set to zero. The simplest of them is $(\mathcal{S}_0, \mathcal{S}_0)_{\mathrm{BV}}$, which vanishes identically because the classical action does not contain any antifields and there is no nonvanishing component of the BV bracket that does not include antifields. Moreover, it is evident that $(\mathcal{S}_2,\mathcal{S}_2)_{\mathrm{BV}}$ vanishes, even without knowing the form of Y and Z. This is due to the fact that it contains only A^+ antifields but no A fields and the only nonvanishing BV brackets of A^+ include necessarily A. Furthermore, a straightforward calculation establishes that $(\mathcal{S}_0,\mathcal{S}_1)_{\mathrm{BV}}$ also vanishes provided the conditions (3.2.37), (3.2.38), (3.2.19) and $\gamma_{ab} = \iota_{\rho_a}\theta_b$ hold. This is nothing but the invariance of the classical action $\mathcal{S}_0$ under the gauge transformations (3.2.6) and (3.2.13) implemented in the classical master equation, as expected. The next condition to be satisfied is:

$$(\mathcal{S}_1,\mathcal{S}_1)_{\mathrm{BV}} + 2\,(\mathcal{S}_0,\mathcal{S}_2)_{\mathrm{BV}} \overset{!}{=} 0\,. \tag{4.2.12}$$

Calculating each of the two terms results in

$$\begin{aligned}
(\mathcal{S}_1,\mathcal{S}_1)_{\mathrm{BV}} &= \int \left(\widetilde{S}^a{}_{cdi}F^i\wedge *A_a^+ - S^a{}_{cdi}F^i\wedge A_a^+\right)c^c c^d\,, && (4.2.13)\\
(\mathcal{S}_0,\mathcal{S}_2)_{\mathrm{BV}} &= \int \frac{1}{2}\Big[\left(Y^{(ab)}{}_{cd}\,(\iota_{\rho_b}g)_i - Z^{[ab]}{}_{cd}\,\theta_{bi}\right)F^i\wedge *A_a^+ + \\
&\qquad + \left(Y^{(ab)}{}_{cd}\,\theta_{bi} - Z^{[ab]}{}_{cd}\,(\iota_{\rho_b}g)_i\right)F^i\wedge A_a^+\Big]\,c^c c^d\,, && (4.2.14)
\end{aligned}$$

where S and $\widetilde{S}$ are given in (4.2.5) and (4.2.6) respectively. This means that imposing (4.2.12) fixes the two so far unknown quantities Y and Z in terms of the known quantities S and $\widetilde{S}$. Specifically, one directly finds the inverse relations, namely $S(Y,Z)$ and $\widetilde{S}(Y,Z)$:

$$\begin{aligned}
S^a{}_{cd} &= Y^{(ab)}{}_{cd}\theta_b - Z^{[ab]}{}_{cd}\iota_{\rho_b}g\,, && (4.2.15)\\
\widetilde{S}^a{}_{cd} &= -Y^{(ab)}{}_{cd}\iota_{\rho_b}g + Z^{[ab]}{}_{cd}\theta_b\,. && (4.2.16)
\end{aligned}$$

This prompts us to define the sum and difference of the quantities Y and Z,

$$Y^{\pm ab}{}_{cd} = (Y\pm Z)^{ab}{}_{cd}\,, \tag{4.2.17}$$

which allows us to write:

$$S^{\pm a}{}_{cd} = Y^{\pm ab}{}_{cd}\;(\mathcal{G}_{\mp})_b\,. \tag{4.2.18}$$

Due to the invertibility of the operators $\mathcal{G}_{\pm}$, we can invert this to express $Y^{\pm}$:

$$Y^{ab}_{\pm}{}_{cd} = \langle(\mathcal{G}_{\mp}^{-1})^b, S^a_{\pm cd}\rangle\,. \tag{4.2.19}$$

Then $\mathcal{S}_2$ is fully specified and the condition (4.2.12) holds.

In order to confirm that the obtained action is actually the BV action, it is necessary to determine the antibracket of $\mathcal{S}_1$ and $\mathcal{S}_2$. A straightforward calculation leads to:

$$(\mathcal{S}_1,\mathcal{S}_2)_{\mathrm{BV}} = \int_\Sigma \frac{1}{4}\left(I^{ab}{}_{cde}A^+_a\wedge A^+_b + J^{ab}{}_{cde}A^+_a\wedge *A^+_b\right)c^c c^d c^e\,, \tag{4.2.20}$$

where we defined the shorthand quantities

$$I^{ab}{}_{cde} = \rho^i_{[e}\partial_i Y^{ab}{}_{cd]} - 2(C^{(a}{}_{p[e} - \rho^i_p\omega^{(a}_{[ei})Y^{b)p}{}_{cd]} - 2\rho^i_p\phi^{(a}_{[ei}Z^{b)p}{}_{cd]} - Y^{ab}{}_{p[e}C^p{}_{cd]}\,, \tag{4.2.21}$$

$$J^{ab}{}_{cde} = \rho^i_{[e}\partial_i Z^{ab}{}_{cd]} + 2(C^{[a}{}_{p[e} - \rho^i_p\omega^{[a}_{[ei})Z^{b]p}{}_{cd]} + 2\rho^i_p\phi^{[a}_{[ei}Y^{b]p}{}_{cd]} - Z^{ab}{}_{p[e}C^p{}_{cd]}\,. \tag{4.2.22}$$

One can recognize that these expressions both correspond to Bianchi identities. Indeed, since Y and Z are fully determined, one can first decouple these two equations by adding and subtracting them, thus expressing them in terms of $Y_\pm$. The latter can be substituted for via (4.2.19), leading to Bianchi identities for the curvature combinations $S_\pm$. The latter are

$$\begin{aligned}\rho^\mu_{[e}\nabla^\pm_\mu\langle(\mathcal{G}^{-1}_\mp)^b, S^{\pm a}{}_{cd]}\rangle - T^{\pm f}{}_{[cd}\langle(\mathcal{G}^{-1}_\mp)^b, S^{\pm a}{}_{e]f}\rangle + T^{\pm a}{}_{f[e}\langle(\mathcal{G}^{-1}_\mp)^b, S^{\pm f}{}_{cd]}\rangle +&\\ + T^{\mp b}{}_{f[e}\langle(\mathcal{G}^{-1}_\mp)^f, S^{\pm a}{}_{cd]}\rangle = 0\,,&\end{aligned} \tag{4.2.23}$$

and they can be proven by direct calculation using the identities (3.3.9) and (3.3.13)-(3.3.16). This proves that the antibracket of $\mathcal{S}_1$ and $\mathcal{S}_2$ vanishes. Therefore the BV action for Dirac sigma model is fully determined and equal to:

$$\begin{aligned}\mathcal{S}_{\mathrm{BV}} = &-\int_\Sigma\left(\frac{1}{2}g_{ij}(X)F^i\wedge *F^j + A^a\wedge\theta_a(X) + \frac{1}{2}\gamma_{ab}(X)A^a\wedge A^b\right) - \int_{\widehat{\Sigma}} X^*H\\ &+\int_\Sigma\left(\rho^i_a(X)X^+_i c^a + \frac{1}{2}C^a_{bc}(X)c^+_a c^b c^c\right)\\ &-\int_\Sigma A^+_a\wedge\left(\mathrm{d}c^a + C^a{}_{bc}(X)A^b c^c + \omega^a{}_{bi}(X)c^b F^i + \phi^a{}_{bi}(X)c^b * F^i\right)\\ &+\int_\Sigma\frac{1}{8}\left(\langle(\mathcal{G}^{-1}_-)^b, S^{+a}{}_{cd}\rangle(X) + \langle(\mathcal{G}^{-1}_+)^b, S^{-a}{}_{cd}\rangle(X)\right)A^+_a\wedge A^+_b c^c c^d\\ &+\int_\Sigma\frac{1}{8}\left(\langle(\mathcal{G}^{-1}_-)^b, S^{+a}{}_{cd}\rangle(X) - \langle(\mathcal{G}^{-1}_+)^b, S^{-a}{}_{cd}\rangle(X)\right)A^+_a\wedge *A^+_b c^c c^d\,.\end{aligned} \tag{4.2.24}$$

4.2.3 Manifestly target space covariant form of the BV action

It is expected that the BV action (4.2.24) is covariant with the respect to change of coordinates on the target space M. However, this covariance is not manifest due to the terms involving the antifield X^+. Here we aim to make this covariance manifest in the BV action.

First notice that the gauge field A, the ghost c and their antifields A^+ and c^+ are covariant objects. Thus, if we change coordinates on M and let $M^i_j(x)$ be the corresponding Jacobian matrix, with $M^a_b(x)$ the induced Jacobian matrix on E, the components of the field A transform as

$$\widetilde{A}^a = M^a_b A^b\,, \tag{4.2.25}$$

and similarly for the other fields.

The symplectic form (4.2.7) needs to be invariant under the change of coordinates, or in other words, it needs to behave as a scalar. Knowing the transformations of the fields X, A and c, as well as the antifields A^+ and c^+, this leads to the transformation of the antifield X^+,

$$\widetilde{X}^+_i = (M^{-1})^j_i X^+_j - (M^{-1})^c_b \partial_i M^a_c (A^+_a \wedge A^b + c^+_a c^b)\,. \tag{4.2.26}$$

This transformation can now be used to check the covariance of the BV action. In order to make the covariance manifest, we have to covariantize the antifield X^+ as:

$$X^{+\nabla}_i = X^+_i - \omega^a_{bi}\left(A^+_a \wedge A^b + c^+_a c^b\right)\,. \tag{4.2.27}$$

which now transforms tensorially. In terms of this field, the BV action takes the form:

$$\begin{aligned}
\mathcal{S}_{BV} \;=\; & -\int_\Sigma \left(||F||^2 + \left\langle \theta(A), \mathrm{d}X + \frac{1}{2}\rho(A) \right\rangle\right) - \int_{\widehat{\Sigma}} X^* H \\
& -\int_\Sigma \left(\left\langle X^{+\nabla}, \rho(c)\right\rangle + \frac{1}{4}(T^+ + T^-)(c^+, c, c) + \langle \phi, *F\rangle (A^+, c)\right) \\
& -\frac{1}{2}\int_\Sigma \left(\left((D^+ + D^-)c\right)(A) - (T^+ + T^-)(A^+, A, c)\right) \\
& +\frac{1}{8}\int_\Sigma \left(\left\langle S^+(A^+, c, c), \mathcal{G}_-^{-1}(A)\right\rangle + \left\langle S^-(A^+, c, c), \mathcal{G}_+^{-1}(A)\right\rangle\right) \\
& +\frac{1}{8}\int_\Sigma \left(\left\langle S^+(A^+, c, c), \mathcal{G}_-^{-1}(*A)\right\rangle - \left\langle S^-(A^+, c, c), \mathcal{G}_+^{-1}(*A)\right\rangle\right)\,,
\end{aligned} \tag{4.2.28}$$

where pull-backs via the map X are understood. This is then the final covariant form of the BV action for topological Dirac sigma models.

4.3 BV action of the R-Poisson sigma model

In this section, we are interested in determining the classical BV action for the R-Poisson sigma model, which will be the solution to the classical master equation. The BV extension is necessary here since the gauge algebra of the R-Poisson sigma models is open, i.e. it closes only on-shell.

4.3.1 Ghosts and the BRST operator

The first step towards quantization is to construct the classical basis of fields and ghosts. The ghosts correspond to the gauge parameters of the theory, promoted to fields of ghost number 1. To avoid introducing too much new notation, we denote the ghosts with the same letters as the gauge parameters. Thus the degree-1 ghosts are $(\varepsilon_i, \chi^i, \psi_i)$. However, the theory contains differential forms of form degree greater than 1 and therefore there will necessarily exist gauge transformations that are not independent. This means that the theory is highly reducible as a constrained Hamiltonian system and we must introduce additional ghosts for ghosts that take care of this redundancy. Indeed, the ghosts for the higher differential forms Y^i and Z_i are χ^i and ψ_i of form degree $p-2$ and $p-1$ respectively. Being differential forms themselves means that we must include in the theory fields of ghost degree 2, say $\chi^i_{(1)}$ and $\psi_i^{(1)}$ of differential form degree $p-3$ and $p-2$. This process continues until we reach the top ghosts for ghost for each of the χ and ψ series, which will be spacetime scalars. Thus we find that the classical basis contains the fields

$$(X^i, A_i, Y^i, Z_i, \varepsilon_i, \chi^i_{(r)}, \psi_i^{(r)})\,, \tag{4.3.1}$$

where the counter r takes values from 0 to $p-2$ for the χ-series $\chi^i_{(r)}$ and from 0 to $p-1$ for the ψ-series $\psi_i^{(r)}$.[3] Thus the classical basis contains a total of $2p+4$ fields of diverse ghost and form degree. At this stage, this is in accordance with the AKSZ construction; for example in the 3D case where $p=2$ we find 8 fields (4 ordinary fields, 3 ghosts and 1 ghost for ghost) as expected for the Courant sigma model [41, 42, 43, 44]. All the fields for the general case, along with their ghost and differential form degrees, are collected in Table 4.2.

Next we define the BRST operator on the fields and ghosts, denoted as s_0 and raising the ghost degree by 1. Its action on the fields is simply the gauge transformation rule appearing in (3.6.4)-(3.6.7) with the gauge parameters replaced by the corresponding ghosts. Since we use the same notation for ghosts, we do not repeat these expressions here. The BRST operator

[3] This discrepancy in the range is irrelevant; one could just state that the upper value is $p-1$ and the ghost $\chi^i_{(p-1)}$ does not exist since otherwise it would have negative form degree.

Field/Ghost	X^i	A_i	Y^i	Z_i	ε_i	$\chi^i_{(r)}$	$\psi_i^{(r)}$
Ghost degree	0	0	0	0	1	$r+1$	$r+1$
Form degree	0	1	$p-1$	p	0	$p-2-r$	$p-1-r$

Table 4.2: The fields and ghosts of the twisted R-Poisson sigma model in $p+1$ dimensions. The range of r is $r = 0,\dots,p-1$ and we make the identifications $\chi^i_{(0)} \equiv \chi^i$, $\psi_i^{(0)} \equiv \psi_i$, so that we use a collective notation for the $p-1$ ghosts χ and the p ghosts ψ. Obviously, $\chi^i_{(p-1)} = 0$, since this ghost does not exist.

should be nilpotent on-shell,

$$s_0^2(\cdot) \stackrel{!}{\approx} 0\,, \tag{4.3.2}$$

where $(\cdot)$ is a placeholder for any field or ghost and $\approx$ denotes that the field equations of the theory have been taken into account, or in other words that the square of the BRST operator is proportional to equations of motion. This requirement fixes the BRST transformation of the ghost fields. In particular, observing that

$$s_0^2 X^i = \Pi^{lk}\partial_k\Pi^{ji}\varepsilon_l\varepsilon_j + \Pi^{ji}s_0\varepsilon_i\,, \tag{4.3.3}$$

and since X^i and ε_i are scalars, their BRST transformations cannot contain field equations. The BRST transformation of ε_i is then completely fixed due to Π being a Poisson bivector:

$$s_0\varepsilon_i = -\frac{1}{2}\partial_i\Pi^{jk}\varepsilon_j\varepsilon_k\,. \tag{4.3.4}$$

One may then check that $s_0^2\varepsilon_i = 0$, as it should. Knowing the BRST transformation of ε_i allows us to compute the square of the BRST operator on A_i and find

$$s_0^2 A_i = -\frac{1}{2}\partial_i\partial_l\Pi^{jk}F^l\varepsilon_j\varepsilon_k\,. \tag{4.3.5}$$

We observe that it is proportional to the field equation for Z_i and thus it vanishes only on-shell. This already dictates that the BV formalism must be used. Following this logic for the rest of the fields, leads to the BRST transformations of the ghosts in the χ and ψ series. They all follow the same pattern and therefore they can be presented collectively as

$$\begin{aligned} s_0\chi^i_{(r)} &= \mathrm{d}\chi^i_{(r+1)} + \partial_k\Pi^{ij}\left(A_j\chi^k_{(r+1)} - \varepsilon_j\chi^k_{(r)}\right) - (-1)^{p+r}\Pi^{ij}\psi_j^{(r+1)} + \\ &\quad - \frac{\beta_{(r)}}{(r+2)!(p-r-2)!}R^{ij_1\dots j_{r+2}k_1\dots k_{p-r-2}}\varepsilon_{j_1}\dots\varepsilon_{j_{r+2}}A_{k_1}\dots A_{k_{p-r-2}}\,, \end{aligned} \tag{4.3.6}$$

$$\begin{aligned} s_0\psi_i^{(r)} &= \mathrm{d}\psi_i^{(r+1)} + \partial_i\Pi^{jk}\left(A_j\psi_k^{(r+1)} - \varepsilon_j\psi_k^{(r)}\right) + \\ &\quad +(-1)^{p+r}\partial_i\partial_l\Pi^{jk}\left(\frac{1}{2}\varepsilon_j\varepsilon_k\chi^l_{(r-1)} - \varepsilon_jA_k\chi^l_{(r)} - \frac{1}{2}A_jA_k\chi^l_{(r+1)}\right) - \\ &\quad +\frac{(-1)^p\beta_{(r)}}{(r+2)!(p-r-1)!}f_i^{j_1\dots j_{r+2}k_1\dots k_{p-r-1}}\varepsilon_{j_1}\dots\varepsilon_{j_{r+2}}A_{k_1}\dots A_{k_{p-r-1}} + \\ &\quad +\sum_{s=1}^{p-r-1}\frac{(-1)^{p(s+1)}\beta_{(r)}}{(s+1)!(r+2)!(p-r-s-1)!}H_{il_1\dots l_s}{}^{j_1\dots j_{r+2}k_1\dots k_{p-r-s-1}}\times \\ &\qquad \times\varepsilon_{j_1}\dots\varepsilon_{j_{r+2}}A_{k_1}\dots A_{k_{p-r-s-1}}F^{l_1}\dots F^{l_s}\,, \end{aligned} \tag{4.3.7}$$

where

$$\beta_{(r)} = (-1)^{p+r(r+1)/2}\,. \tag{4.3.8}$$

A useful remark is that the fields Y^i and Z_i may be seen as the "-1" elements in the χ and ψ series, as is confirmed by inspection of the degrees in Table 4.2. Indeed, if we identify

$$\chi^i_{(-1)} := (-1)^{p+1}Y^i \qquad \text{and} \qquad \psi_i^{(-1)} := (-1)^pZ_i\,, \tag{4.3.9}$$

then the general formulas (4.3.6) and (4.3.7) are identical to the BRST transformations of Y^i and Z_i for $r=-1$, given in (3.6.6) and (3.6.7) with the gauge parameters replaced by the corresponding ghosts. This includes the term in s_0Z_i containing field equations explicitly. Note that none of the ghosts in the χ-series contains explicit equation of motion terms in their BRST transformation, whereas all ghosts in the ψ-series do, save the top one which is anyway a scalar. The advantage of this identification is that once we compute the square of the BRST operator on the ghosts, the one for the fields Y^i and Z_i simply follows. A straightforward calculation leads to the results

$$
\begin{aligned}
s_0^2\chi^i_{(r)} \;=\;& -\frac{\beta_{(r+1)}}{(r+3)!(p-r-4)!}R^{ilj_1\dots j_{r+3}k_1\dots k_{p-r-4}}\varepsilon_{j_1}\dots\varepsilon_{j_{r+3}}A_{k_1}\dots A_{k_{p-r-4}}G_l -\\
&-\frac{(-1)^p\beta_{(r+1)}}{(r+3)!(p-r-3)!}\partial_l R^{ij_1\dots j_{r+3}k_1\dots k_{p-r-3}}\varepsilon_{j_1}\dots\varepsilon_{j_{r+3}}A_{k_1}\dots A_{k_{p-r-3}}F^l +\\
&+\sum_{s=1}^{p-r-2}\frac{(-1)^{(p+1)s}\beta_{(r)}}{(s+1)!(r+3)!(p-r-s-2)!}H_{l_1\dots l_s}{}^{ij_1\dots j_{r+3}k_1\dots k_{p-r-s-2}}\times\\
&\qquad\times\varepsilon_{j_1}\dots\varepsilon_{j_{r+3}}A_{k_1}\dots A_{k_{p-r-s-2}}F^{l_1}\dots F^{l_s}+\\
&+\partial_k\partial_l\Pi^{ij}F^k\left(A_j\chi^l_{(r+2)}-\varepsilon_j\chi^l_{(r+1)}\right)+\partial_k\Pi^{ij}\left(G_j\chi^k_{(r+2)}-\psi_j^{(r+2)}F^k\right),
\end{aligned}
\tag{4.3.10}
$$

for the χ-series of ghosts, and

$$
\begin{aligned}
s_0^2\psi_i^{(r)} \;=\;& \frac{(-1)^p\beta_{(r)}}{(r+3)!(p-r-3)!}\partial_i R^{lj_1\dots j_{r+3}k_1\dots k_{p-r-3}}\varepsilon_{j_1}\dots\varepsilon_{j_{r+3}}A_{k_1}\dots A_{k_{p-r-3}}G_l -\\
&-\frac{\beta_{(r)}}{(r+3)!(p-r-2)!}\partial_l\partial_i R^{j_1\dots j_{r+3}k_1\dots k_{p-r-2}}\varepsilon_{j_1}\dots\varepsilon_{j_{r+3}}A_{k_1}\dots A_{k_{p-r-2}}F^l +\\
&+\sum_{s=0}^{p-r-3}\frac{(-1)^{p(s+1)}\beta_{(r)}}{(s+2)!(r+3)!(p-r-s-3)!}H_{il_1\dots l_s}{}^{mj_1\dots j_{r+3}k_1\dots k_{p-r-s-3}}\times\\
&\qquad\times G_m\varepsilon_{j_1}\dots\varepsilon_{j_{r+3}}A_{k_1}\dots A_{k_{p-r-s-3}}F^{l_1}\dots F^{l_s}+\\
&+2\sum_{s=1}^{p-r-1}\frac{(-1)^{p(s+1)+s}\beta_{(r)}}{(s+1)!(r+3)!(p-r-s-1)!}\partial_{(i}H_{l_1)l_2\dots l_s}{}^{j_1\dots j_{r+3}k_1\dots k_{p-r-s-1}}\times\\
&\qquad\times\varepsilon_{j_1}\dots\varepsilon_{j_{r+3}}A_{k_1}\dots A_{k_{p-r-s-1}}F^{l_1}\dots F^{l_s}+\\
&+\partial_i\Pi^{jk}G_j\psi_k^{(r+2)}+\partial_i\partial_j\Pi^{kl}F^j\left(A_k\wedge\psi_l^{(r+2)}-\varepsilon_k\psi_l^{(r+1)}\right)+\\
&+(-1)^{p+r}\partial_i\partial_l\Pi^{jk}G_j\left(A_k\chi^l_{(r+2)}-\varepsilon_k\chi^l_{(r+1)}\right)+\\
&+(-1)^{p+r}\partial_i\partial_j\partial_m\Pi^{kl}F^j\left(\frac{1}{2}A_kA_l\chi^m_{(r+2)}+\varepsilon_kA_l\chi^m_{(r+1)}-\frac{1}{2}\varepsilon_k\varepsilon_l\chi^m_{(r)}\right),
\end{aligned}
\tag{4.3.11}
$$

for the ψ-series of ghosts. We observe that in both cases the field equations of Y^i and Z_i appear in all terms on the right-hand side. Moreover, according to the discussion above, the square of the BRST operator on Y^i is found to be

$$\begin{aligned} s_0^2 Y^i &= \frac{1}{2(p-3)!} R^{ilj_1j_2k_1\dots k_{p-3}} \varepsilon_{j_1}\varepsilon_{j_2} A_{k_1}\dots A_{k_{p-3}} G_l + (-1)^p \partial_k \Pi^{ij}\left(\psi_j^{(1)} F^k - G_j \chi^k_{(1)}\right) + \\ &+ \frac{(-1)^p}{2(p-2)!}\partial_l R^{ij_1j_2k_1\dots k_{p-2}} \varepsilon_{j_1}\varepsilon_{j_2} A_{k_1}\dots A_{k_{p-2}} F^l + (-1)^p \partial_k\partial_l \Pi^{ij} F^k \left(\varepsilon_j \chi^l - A_k \chi^l_{(1)}\right) - \\ &- \sum_{s=1}^{p-1} \frac{(-1)^{(p+1)s}}{2(s+1)!(p-s-1)!} H_{l_1\dots l_s}{}^{ij_1j_2k_1\dots k_{p-s-1}} \varepsilon_{j_1}\varepsilon_{j_2} A_{k_1}\dots A_{k_{p-s-1}} F^{l_1}\dots F^{l_s}\,. \end{aligned} \tag{4.3.12}$$

For the corresponding expression of Z_i, which could alternatively be calculated directly from (3.6.7), we find

$$\begin{aligned} s_0^2 Z_i &= \frac{(-1)^p}{2(p-2)!}\partial_i R^{lj_1j_2k_1\dots k_{p-2}} \varepsilon_{j_1}\varepsilon_{j_2} A_{k_1}\dots A_{k_{p-2}} G_l + (-1)^p \partial_i \Pi^{jk} G_j \psi_k^{(1)} - \\ &- \frac{1}{2(p-1)!}\partial_l\partial_i R^{j_1j_2k_1\dots k_{p-1}} \varepsilon_{j_1}\varepsilon_{j_2} A_{k_1}\dots A_{k_{p-1}} F^l + (-1)^p \partial_j\partial_i \Pi^{kl} F^j \left(A_k \psi_l^{(1)} - \varepsilon_k \psi_l\right) + \\ &+ \partial_i\partial_l \Pi^{jk} G_j \left(\varepsilon_k \chi^l - A_k \chi^l_{(1)}\right) + \frac{(-1)^p}{2}\partial_i\partial_l \Pi^{jk}\varepsilon_j\varepsilon_k \mathcal{F}^l - \\ &- \partial_i\partial_j\partial_m \Pi^{kl} F^j \left(\frac{1}{2} A_k A_l \chi^m_{(1)} + \varepsilon_k A_l \chi^m + \frac{(-1)^p}{2}\varepsilon_k\varepsilon_l Y^m\right) + \\ &+ \sum_{s=0}^{p-2} \frac{(-1)^{p(s+1)}}{2(s+2)!(p-s-2)!} H_{il_1\dots l_s}{}^{mj_1j_2k_1\dots k_{p-s-2}} G_m \varepsilon_{j_1}\varepsilon_{j_2} A_{k_1}\dots A_{k_{p-s-2}} F^{l_1}\dots F^{l_s} + \\ &+ \sum_{s=1}^{p} \frac{(-1)^{p(s+1)+s}}{(s+1)!(p-s)!}\partial_{(i} H_{l_1)l_2\dots l_s}{}^{j_1j_2k_1\dots k_{p-s}} \varepsilon_{j_1}\varepsilon_{j_2} A_{k_1}\dots A_{k_{p-s}} F^{l_1}\dots F^{l_s}\,. \end{aligned} \tag{4.3.13}$$

This completes the calculation of the square of the BRST operator on all fields. Since in most cases it does not vanish off-shell, the BV formalism is necessary to solve the classical master equation.

Nevertheless, before proceeding with the BV formalism, it is worth listing the fields and ghosts whose BRST transformation is already nilpotent off-shell. First, we saw that this is the case for X^i and ε_i. There exist, however, two more ghosts with this property. These are the top ghosts in each of the χ and ψ series, namely $\chi^i_{(p-2)}$ and $\psi_i^{(p-1)}$, both being spacetime scalars.

The general formulas yield

$$s_0\chi^i_{(p-2)} = -\partial_k\Pi^{ij}\varepsilon_j\chi^k_{(p-2)} - \Pi^{ij}\psi_j^{(p-1)} - \frac{\beta_{(p-2)}}{p!}R^{ij_1\dots j_p}\varepsilon_{j_1}\dots\varepsilon_{j_p}\,, \tag{4.3.14}$$

$$s_0\psi_i^{(p-1)} = -\partial_i\Pi^{jk}\varepsilon_j\psi_k^{(p-1)} - \frac{1}{2}\partial_i\partial_l\Pi^{jk}\varepsilon_j\varepsilon_k\chi^l_{(p-2)} + \frac{(-1)^p\beta_{(p-1)}}{(p+1)!}f_i^{j_1\dots j_{p+1}}\varepsilon_{j_1}\dots\varepsilon_{j_{p+1}}\,. \tag{4.3.15}$$

Either by direct computation or simply by inspection of the results (4.3.10) and (4.3.11), we find

$$s_0^2\chi^i_{(p-2)} = 0 = s_0^2\psi_i^{(p-1)}\,. \tag{4.3.16}$$

We conclude that only 4 of the $2p+4$ fields in (4.3.1), naturally being the four scalars, have nilpotent BRST operator acting on them. Therefore, for these fields there is no need to modify this operator, or in other words the BRST and the BV operator are identical for them. Thus we denote

$$sX^i := s_0X^i\,, \quad s\varepsilon_i := s_0\varepsilon_i\,, \quad s\chi^i_{(p-2)} = s_0\chi^i_{(p-2)}\,, \quad s\psi_i^{(p-1)} := s_0\psi_i^{(p-1)}\,, \tag{4.3.17}$$

and s^2 vanishes on these fields.

4.3.2 Antifields and the untwisted BV operator

To pave the way towards determining the solution of the classical master equation of a twisted R-Poisson sigma model, we start as usual, by enlarging the space of fields and ghosts by inclusion of the corresponding antifields and antighosts. For any field φ we denote them as φ_+ (or φ^+, depending on the index position). The full set of $2p+4$ antifields and antighosts with their degrees appears in Table 4.3. In total the fields and antifields are $4(p+2)$ in number, a multiple of 4. This is to be expected, since without the Wess-Zumino term one could have used the AKSZ contruction with source space the graded manifold $T[1]\Sigma$ and would have constructed 4 superfields containing the sum of all fields and antifields of total degree (the sum of ghost and form degrees) $0,1,p-1,p$ respectively. In particular, the superfield $\mathbf{X}^i$ of total degree 0 would contain $(X^i, Z^i_+, \psi^i_{+(r)})$, the total degree-1 superfield $\mathbf{A}_i$ would contain $(A_i, \varepsilon_i, Y_i^+, \chi_i^{+(r)})$, the total degree-$(p-1)$ superfield $\mathbf{Y}^i$ would contain $(Y^i, \chi^i_{(r)}, A^i_+, \varepsilon^i_+)$ and the total degree-p superfield $\mathbf{Z}_i$ would contain $(Z_i, \psi_i^{(r)}, X_i^+)$. The BV action would then be of the same form as the classical action but with superfields instead of fields. The Wess-Zumino term given by the pull-back of the 4-form H is the sole reason that this would not be sufficient to determine the correct BV action.

Antifield	X_i^+	A_+^i	Y_i^+	Z_+^i	ε_+^i	$\chi_i^{+(r)}$	$\psi_{+(r)}^i$
Ghost degree	-1	-1	-1	-1	-2	$-r-2$	$-r-2$
Form degree	$p+1$	p	2	1	$p+1$	$r+3$	$r+2$

Table 4.3: The antifields and antighosts of the twisted R-Poisson sigma model in $p+1$ dimensions. The range of r is the same as for the corresponding fields.

Next, one could use the BRST transformations found in the previous section to write down the extension of the classical action $S^{(p+1)}$ of (3.6.2) by all terms that contain one antifield and subsequently extend this action with all allowed terms with two and more antifields such that the classical master equation is satisfied, the way it was done for the Dirac sigma model. Alternatively, one could directly determine the BV operator s, i.e. the extension of the BRST operator s_0 that is nilpotent off-shell, using the fact that its action on the antifields produces equations of motion. Here we will work in this latter approach.

Let us describe our approach in a heuristic way before presenting the details of the procedure. We have already found in Section 4.3.1 that in all cases when the square of the BRST operator on the fields does not vanish, it is proportional to the field equations F^i, G_i and $\mathcal{F}^i$, the latter appearing only in $s_0^2 Z_i$. Therefore, we will certainly need the antifields from Table 4.3 whose transformation gives these field equations. These are Z_+^i, Y_i^+ and A_+^i, whose BV transformation will contain

$$sZ_+^i \supset (-1)^{p+1} F^i\,, \tag{4.3.18}$$

$$sY_i^+ \supset G_i\,, \tag{4.3.19}$$

$$sA_+^i \supset (-1)^p \mathcal{F}^i\,, \tag{4.3.20}$$

among other terms that we will determine. The goal then is to extend the BRST transformations by terms proportional to these antifields such that the square of the resulting operator vanishes. However, one should be careful with two more issues. The first issue is that once Z_+^i- and Y_i^+-dependent terms are included in the transformation of some field, the lower field which transforms as the derivative of the previous field will contain terms proportional to $\mathrm{d}Z_+^i$ and

$\mathrm{d}Y_i^+$. This issue is ameliorated by noting that

$$s\psi_+^i \supset (-1)^p \,\mathrm{d}Z_+^i\,, \tag{4.3.21}$$

$$s\chi_i^+ \supset (-1)^p \,\mathrm{d}Y_i^+\,, \tag{4.3.22}$$

and so on for the χ and ψ series, since in general

$$s\psi_{+(r)}^i \supset \mathrm{d}\psi_{+(r-1)}^i\,, \tag{4.3.23}$$

$$s\chi_i^{+(r)} \supset -\,\mathrm{d}\chi_i^{+(r-1)}\,. \tag{4.3.24}$$

The second issue regards the appearance of explicit field equations in s_0Z_i and in fact in all ghosts of the ψ-series. One may then ask whether any of the antifields will also contain explicit field equations in their BV transformation. The answer is necessarily yes and it will turn out to be very important in determining the correct BV action. Crucially, we will find that sA_+^i contains Z_i dependence and this will lead to a modification of its BV transformation by explicit F^i-dependent terms. Higher antifields will also get corrected accordingly, but it will become obvious that this will not be crucial for finding the BV action and can be determined a posteriori. This feature of higher ghosts and antifields having BV operator that contains field equations is one that does not exist in ordinary AKSZ constructions.

In summary, this heuristic discussion establishes the strategy for determining the BV operator on the fields. First we recall that the BV operator, denoted in general as s_{BV} should satisfy the following three properties:

I) When antifields are set to zero, it reduces to the BRST operator s_0.

II) It is strictly nilpotent, $s_{\text{BV}}^2 = 0$, without using the field equations.

III) It is obtained from a BV action as $s_{\text{BV}}\cdot = (S_{\text{BV}},\cdot)$, with respect to the BV antibracket $(\cdot,\cdot)$.

Note that there can (and will) exist operators s other than s_{BV} that satisfy the first two properties. It is the third property that establishes the right $s = s_{\text{BV}}$ that corresponds to a solution of the classical master equation. Our strategy then goes as follows. Consider the square of the BRST operator and add terms linear in the antifields to s_0, say s_1 such that the field equations cancel. Then compute the square of the modified BRST operator s_0+s_1, which will also be proportional to the field equations in general. Modify the operator s_0+s_1 by some antifield-dependent s_2 and repeat the procedure until the point that the modified operator is nilpotent off-shell. Then properties I and II are addressed. Property III is nearly automatic in the untwisted case but harder to satisfy in the twisted case. The strategy would be to add all possible additional H-dependent terms and solve a complicated set of consistency conditions. In the following, we will apply the above strategy in the untwisted case in arbitrary dimensions and we will also solve explicitly the twisted case in three dimensions, addressing the complicated property III.

Let us now apply this procedure, starting with the simplest case of A_i, for which the square of the BRST operator is given in (4.3.5). Using (4.3.18), we refine it to

$$(s_0+s_1)A_i = \mathrm{d}\varepsilon_i + \partial_i\Pi^{jk}A_j\varepsilon_k - \frac{(-1)^p}{2}\partial_i\partial_l\Pi^{jk}Z^l_+\varepsilon_j\varepsilon_k\,. \tag{4.3.25}$$

The square of the modified operator can be easily calculated; requiring that it vanishes fixes completely the transformation of Z^i_+ too. Specifically, for

$$sZ^i_+ = (-1)^{p+1}F^i + \partial_j\Pi^{ik}Z^j_+\varepsilon_k\,, \tag{4.3.26}$$

we find that $(s_0+s_1)^2A_i = 0$ identically. Therefore the BV operator on A_i is

$$sA_i = \mathrm{d}\varepsilon_i + \partial_i\Pi^{jk}A_j\varepsilon_k - \frac{(-1)^p}{2}\partial_i\partial_l\Pi^{jk}Z^l_+\varepsilon_j\varepsilon_k\,. \tag{4.3.27}$$

However, one should now cross check that $s^2Z^i_+ = 0$ too. This is a non-trivial consistency check, whose validity is easily established via an easy calculation, using also the modified transformation of F^i which is directly computed to be

$$sF^i = \partial_k\Pi^{ji}F^k\varepsilon_j - \frac{(-1)^p}{2}\Pi^{ij}\partial_j\partial_k\Pi^{lm}Z^k_+\varepsilon_l\varepsilon_m\,. \tag{4.3.28}$$

In this way we have determined the BV operator on A_i and Z^i_+. The fact that the procedure stopped quickly is only a feature of the low degree differential form A_i. For the rest of the fields reducibility kicks in and the procedure must be repeated multiple times. Fortunately, the common pattern of their BRST transformation allows us, at least in the untwisted case, to perform this task once for each of the χ and ψ series, before turning to the fields Y^i and Z_i.

For all ghosts $\chi^i_{(r)}$, we find the BV transformation

$$
\begin{aligned}
s\chi^i_{(r)} &= \mathrm{d}\chi^i_{(r+1)} + \sum_{s=0}^{p-r-2} \frac{(-1)^p}{s!}\partial_k\partial_{l_1}\dots\partial_{l_s}\Pi^{ij}\sum_{s'=0}^{p-r-s-2}(-1)^{s'}O^{l_1\dots l_s}(s,s')\mathfrak{X}^k_j(s,s') + \\
&+ \sum_{s=0}^{p-r-2}\frac{1}{s!}(-1)^{p+r+s-1}\partial_{l_1}\dots\partial_{l_s}\Pi^{ij}\sum_{s'=0}^{p-r-s-2}(-1)^{s'}O^{l_1\dots l_s}(s,s')\psi_j^{(r+s+s'+1)} + \\
&- \sum_{t=0}^{\lfloor\frac{p-r-2}{2}\rfloor}\sum_{s=0}^{p-r-2t-2}\sum_{s'=0}^{p-r-s-2t-2}\sum_{t'=0}^{p-r-s-s'-2t-2}\frac{(-1)^{(t+1)p+s'}}{s!t!}\frac{\beta_{(t-1)}\beta_{(a-2)}}{a!(p-a-t)!}\times \\
&\times\ \partial_{l_1}\dots\partial_{l_s}R^{ij_1\dots j_ak_1\dots k_{p-a}}O^{l_1\dots l_s}(s,s')\widetilde{O}_{k_1\dots k_t}(t,t')\varepsilon_{j_1}\dots\varepsilon_{j_a}A_{k_{t+1}}\dots A_{k_{p-a}}\,, \qquad (4.3.29)
\end{aligned}
$$

where we denote $a := r+s+s'+t+t'+2$ and we define the following operators,

$$
\begin{aligned}
O^{l_1\dots l_s}(s,s') &= \sum_{\substack{m_i=-1\\1\le i\le s-1}}^{s'-1}\left(\prod_{u=1}^{s-1}\psi^{l_u}_{+(m_u)}\right)\psi^{l_s}_{+(s'-s-\sum_{i=1}^{s-1}m_i)}\,,\\
\widetilde{O}_{k_1\dots k_t}(t,t') &= \sum_{\substack{m_i=-1\\1\le i\le t-1}}^{t'-1}(-1)^{\sum_{q=0}^{\lfloor t/2\rfloor-1}(1+m_{t-1-2q})}\left(\prod_{u=1}^{t-1}\chi^{+\,(m_u)}_{k_u}\right)\chi^{+}_{k_t\,(t'-t-\sum_{i=1}^{t-1}m_i)}\,,\\
\mathfrak{X}^k_j(s,s') &= \sum_{u=0}^{p-r-s-s'-2}\widetilde{O}_j(1,u-2)\chi^k_{(r+s+s'+u)}\,, \qquad (4.3.30)
\end{aligned}
$$

with starting values

$$
O(0,s') = \delta_{0,s'}\,, \qquad (4.3.31)
$$

$$
\widetilde{O}(0,t') = \delta_{0,t'}\,. \qquad (4.3.32)
$$

We observe that the operators O and $\widetilde{O}$ contain all products of antighosts of the ψ_+ and χ^+ series and their fusion appears in the last term of the BV operator for the ghosts $\chi^i_{(r)}$. A few further remarks are in order. In these formulas, the antighosts $\chi_i^{+(r)}$ have been extended to include the values $r=-1,-2,-3$, which by inspection of Tables 4.2 and 4.3 are identified with

$$
\chi_i^{+(-1)} \equiv (-1)^{p-1}Y_i^+\,, \quad \chi_i^{+(-2)} \equiv (-1)^pA_i\,, \quad \chi_i^{+(-3)} \equiv (-1)^{p-1}\varepsilon_i\,. \qquad (4.3.33)
$$

There is nothing deep about these identifications, it is just one that uniformizes the presentation of the diverse expressions. In particular, it does not mean that A_i and ε_i are antighosts, but only that they can be alternatively included in the antighost series for presentation purposes.

Similarly, for all ghosts $\psi_i^{(r)}$ we find the BV transformation

$$
\begin{aligned}
s\psi_i^{(r)} &= \mathrm{d}\psi_i^{(r+1)} + \sum_{s=0}^{p-r-1} \frac{(-1)^p}{s!}\partial_i\partial_{l_1}\dots\partial_{l_s}\Pi^{jk}\sum_{s'=0}^{p-r-s-1}(-1)^{s'}O^{l_1\dots l_s}(s,s')\widetilde{\mathfrak{X}}_{jk}(s,s') + \\
&+ \frac{(-1)^{p+r}}{2}\sum_{s=0}^{p-r-1}\frac{(-1)^s}{s!}\partial_i\partial_l\partial_{l_1}\dots\partial_{l_s}\Pi^{jk}\sum_{s'=0}^{p-r-s-1}(-1)^{s+s'}O^{l_1\dots l_s}(s,s')\times \\
&\times \sum_{t=0}^{p-r-s-s'-1}\sum_{t'=0}^{t}(-1)^{t'}\widetilde{O}_j(1,t'-2)\widetilde{O}_k(1,t-t'-2)\chi^l_{(t+r+s+s'-1)} + \\
&+ \sum_{t=0}^{\lfloor\frac{p-r-1}{2}\rfloor}\sum_{s=0}^{p-r-2t-1}\sum_{s'=0}^{p-r-s-2t-1}\sum_{t'=0}^{p-r-s-s'-2t-1}\frac{(-1)^{pt+s'}}{s!t!}\frac{\beta_{(t-1)}\beta_{(a-2)}}{a!(p-a-t+1)!}\times \\
&\times \partial_{l_1}\dots\partial_{l_s}\partial_i R^{j_1\dots j_a k_1\dots k_{p-a+1}}O^{l_1\dots l_s}(s,s')\widetilde{O}_{k_1\dots k_t}(t,t')\varepsilon_{j_1}\dots\varepsilon_{j_a}A_{k_{t+1}}\dots A_{k_{p-a+1}}\,, \quad (4.3.34)
\end{aligned}
$$

where the only new operator that appears is defined as

$$
\widetilde{\mathfrak{X}}_{jk} = \sum_{u=0}^{p-r-s-s'-1}\widetilde{O}_j(1,u-2)\psi_k^{(r+s+s'+u)}\,. \quad (4.3.35)
$$

Once again, the fusion of the operators O and $\widetilde{O}$ appears in the last term. What remains is to determine the BV operator acting on the fields Y^i and Z_i. These are however just special values of the above general formulas by means of the identifications in (4.3.9). The above universal formulas give the desired result of the operator that satisfies properties I, II and III. This can be alternatively found via the AKSZ construction since we have set $H=0$ to find these expressions and thus the QP structure is restored. Nevertheless, it is worth emphasizing that the BV operator found via AKSZ would at face value look much more complicated than (4.3.29) and (4.3.34). These formulas organise the different terms in a neat and simple way and they are valid in any dimension.

Turning on H, the task of finding a closed expression for the BV operator with all H-dependent terms included becomes complicated. Nevertheless, the strategy we employed can still be applied, at least in a case by case fashion. In general, the requirement I and the fact that we have determined the form of the BRST operator for all fields including the H-dependence already indicates that $s\psi_i^{(r)}$ is modified to

$$
s\psi_i^{(r)}|_{H=0} \mapsto s\psi_i^{(r)}|_{H=0} + \Delta s\,\psi_i^{(r)}\,, \quad (4.3.36)
$$

where the additional H- and F-dependent term $\Delta s\,\psi_i^{(r)}$, which vanishes in absence of H, is given as

$$
\begin{aligned}
\Delta s\,\psi_i^{(r)} &= \sum_{s=1}^{p-r-1} \frac{(-1)^{p(s+1)}\beta_{(r)}}{(s+1)!(r+2)!(p-r-s-1)!} H_{il_1\dots l_s}{}^{j_1\dots j_{r+2}k_1\dots k_{p-r-s-1}} \times \\
&\qquad \times \varepsilon_{j_1}\dots\varepsilon_{j_{r+2}} A_{k_1}\dots A_{k_{p-r-s-1}} F^{l_1}\dots F^{l_s} + \\
&+ \sum_{t=0}^{\lfloor\frac{p-r-1}{2}\rfloor} \sum_{s=0}^{p-r-2t-1} \sum_{s'=0}^{p-r-s-2t-1} \sum_{t'=0}^{p-r-s-s'-2t-1} \frac{(-1)^{pt+s'}}{s!t!} \frac{\beta_{(t-1)}\beta_{(a-2)}}{a!(p-a-t+1)!} \times \\
&\times\ \partial_{l_1}\dots\partial_{l_s} H_i{}^{j_1\dots j_a k_1\dots k_{p-a+1}} O^{l_1\dots l_s}(s,s')\widetilde{O}_{k_1\dots k_t}(t,t')\varepsilon_{j_1}\dots\varepsilon_{j_a} A_{k_{t+1}}\dots A_{k_{p-a+1}} \\
&+\ \Delta_i(H)\,,
\end{aligned}
$$

where the explicit contributions guarantee that property I is satisfied and $\Delta_i(H)$ with $\Delta_i \overset{H\to 0}{\longrightarrow} 0$ has to be determined such that properties II and III are satisfied too. In addition, $s\chi^i_{(r)}$ is also modified with a corresponding term that should be determined. One should then apply the same algorithmic procedure of taking the square of the modified operator and refining it with suitable antifields as many times as necessary such that eventually its square vanishes. Once this is achieved, one must determine the relative weight of each of the unknown terms in the two series of χ's and ψ's such that the nilpotent operator is indeed one obtained from a BV action through the antibracket. In the next section we apply this approach to the twisted R-Poisson sigma model in 3D.

4.4 Twisted R-Poisson-Courant sigma models in 3D

In this section we apply the general formalism developed above to a specific example, essentially the simplest non-trivial one that can be fully solved including the twist. This is a 3D Courant sigma model with a 4-form Wess-Zumino term. Such H-twisted Courant sigma models were considered from the viewpoint of first class constrained systems and 4-form-twisted Courant algebroids in [88]. Here we study one such topological field theory that has the structure of a twisted R-Poisson sigma model. Apart from determining for the first time the BV action for twisted Courant sigma models, this task will be helpful in exemplifying (and of course extending to the twisted case) the rather complicated closed formulas derived in the previous section.

We consider the action functional (3.6.2) in three dimensions ($p=2$),

$$
\begin{aligned}
S^{(3)} &= \int_{\Sigma_3}\Big(Z_i\wedge \mathrm{d}X^i - A_i\wedge \mathrm{d}Y^i + \Pi^{ij}(X)Z_i\wedge A_j - \frac{1}{2}\partial_k\Pi^{ij}(X)Y^k\wedge A_i\wedge A_j + \\
&\quad + \frac{1}{3!}R^{ijk}(X)A_i\wedge A_j\wedge A_k\Big) + \int_{\Sigma_4} X^*H\,, \qquad (4.4.1)
\end{aligned}
$$

with the Wess-Zumino term being the pull-back of a 4-form on the target space M, which is equipped with a twisted R-Poisson structure, consisting of a Poisson bivector Π and an anti-symmetric trivector R that satisfy

$$
[\Pi,R] = \langle \otimes^4\Pi, H\rangle\,. \qquad (4.4.2)
$$

In absence of H, this is a Bianchi identity for the derivation

$$
\mathrm{d}_\Pi(\cdot) := [\Pi,(\cdot)]\,, \qquad (4.4.3)
$$

which is nilpotent due to the Poisson condition $[\Pi,\Pi]=0$. In this case one notices that Y^i is a spacetime 1-form and it may be combined with A_i to a 1-form $V^I=(A_i,Y^i)$ taking values in the pull-back of the generalized tangent bundle $TM\oplus T^*M$, where the index I takes its $2\dim M$ values. This observation is helpful in identifying the action (4.4.1) with the general form of a Courant sigma model with Wess-Zumino term, which reads in our conventions as

$$
\begin{aligned}
S^{(\text{WZ-CSM})} &= \int_{\Sigma_3}\Big(Z_i\wedge \mathrm{d}X^i - \frac{1}{2}\eta_{IJ}V^I\wedge \mathrm{d}V^J + \rho^i_I(X)Z_i\wedge V^I + \\
&\quad + \frac{1}{3!}T_{IJK}(X)V^I\wedge V^J\wedge V^K\Big) + \int_{\Sigma_4} X^*H\,, \qquad (4.4.4)
\end{aligned}
$$

with η_{IJ} the $O(\dim M,\dim M)$ covariant metric

$$
\eta = (\eta_{IJ}) = \begin{pmatrix} 0 & \mathbf{1}_{\dim M} \\ \mathbf{1}_{\dim M} & 0 \end{pmatrix}\,, \qquad (4.4.5)
$$

ρ^i_I the components of the anchor map $\rho : E = TM\oplus T^*M \to TM$ of a Courant algebroid with vector bundle E and T_{IJK} the structure functions of the Courant bracket in a local basis. The example we use has anchor map components given by the Poisson bivector Π and Courant bracket the twisted Koszul one. For $H=0$, it is called a Poisson Courant algebroid or a contravariant Courant algebroid on a Poisson manifold [89, 90]. In presence of the Wess-Zumino term there is a departure from this Courant algebroid structure to a twisted one in the sense of [88], or a pre-Courant algebroid in the sense of [91], which in our example becomes the twisted R-Poisson structure. More details on this relation are found in [35].

Our goal now is to determine the corresponding BV action of the classical action (4.4.1). According to our discussion in previous section, there exist 16 fields and antifields, specifically the four fields X^i, A_i, Y^i, Z_i, their four antifields, three ghosts $\epsilon_i, \chi^i, \psi_i$ and their three antighosts and one ghost for ghost $\widetilde{\psi}_i \equiv \psi_i^{(1)}$ and its antighost. First we briefly recall that when $H = 0$ the BV action can be found using the AKSZ construction, see [44]. In short, the above 16 fields are collected in four superfields of degrees $0,1,1,2$,

$$\begin{aligned}
\boldsymbol{X}^i &= X^i + Z^i_+ + \psi^i_+ + \widetilde{\psi}^i_+ \,, && (4.4.6)\\
\boldsymbol{A}_i &= \epsilon_i + A_i + Y^+_i + \chi^+_i \,, && (4.4.7)\\
\boldsymbol{Y}^i &= \chi^i + Y^i + A^i_+ + \epsilon^i_+ \,, && (4.4.8)\\
\boldsymbol{Z}_i &= \widetilde{\psi}_i + \psi_i + Z_i + X^+_i \,, && (4.4.9)
\end{aligned}$$

defined on the graded Q-manifold $T[1]\Sigma_3$ and taking values on the QP manifold $T^*[2]T^*[1]M$, which is isomorphic to $T^*[2]T[1]M$, which is typically associated to Courant sigma models. Then the BV action is simply [44]

$$S^{(3)}_{\text{AKSZ}} = \int_{T[1]\Sigma_3} \left(\boldsymbol{Z}_i \mathrm{d}\boldsymbol{X}^i - \frac{1}{2}\eta_{IJ}\boldsymbol{V}^I \mathrm{d}\boldsymbol{V}^J + \rho^i_I(\boldsymbol{X})\boldsymbol{Z}_i\boldsymbol{V}^I + \frac{1}{3!}T_{IJK}(\boldsymbol{X})\boldsymbol{V}^I\boldsymbol{V}^J\boldsymbol{V}^K \right) \,, \quad (4.4.10)$$

with $\boldsymbol{V}^I$ the superfield that combines $\boldsymbol{A}_i$ and $\boldsymbol{Y}^i$ and d the cohomological vector field on $T[1]\Sigma_3$.

Once the twist H is turned on though, this simple sequence of steps does not work, as already argued and as proven in [45] for the 2D AKSZ sigma model after twisting it by a 3-form. For the sake of completeness and for examplifying the general formulas of the previous section, we now present the BV operator on the eight fields as obtained by applying the general formulas in this case. First of all, due to (4.3.17) and (4.3.27), we already have the BV operator on five of them,

$$
\begin{aligned}
sX^i &= \Pi^{ji}\epsilon_j\,, && (4.4.11)\\
s\epsilon_i &= -\frac{1}{2}\partial_i\Pi^{jk}\epsilon_j\epsilon_k\,, && (4.4.12)\\
s\chi^i &= -\partial_k\Pi^{ij}\epsilon_j\chi^k - \Pi^{ij}\widetilde{\psi}_j - \frac{1}{2}R^{ijk}\epsilon_j\epsilon_k\,, && (4.4.13)\\
s\widetilde{\psi}_i &= -\partial_i\Pi^{jk}\epsilon_j\widetilde{\psi}_k - \frac{1}{2}\partial_i\partial_l\Pi^{jk}\epsilon_j\epsilon_k\chi^l - \frac{1}{3!}f_i^{jkl}\epsilon_j\epsilon_k\epsilon_l\,. && (4.4.14)\\
sA_i &= \mathrm{d}\epsilon_i + \partial_i\Pi^{jk}A_j\epsilon_k - \frac{1}{2}\partial_i\partial_l\Pi^{jk}Z_+^l\epsilon_j\epsilon_k\,, && (4.4.15)
\end{aligned}
$$

where we recall that

$$
f_i^{jkl} = \partial_i R^{jkl} + H_i^{\ jkl}\,. \qquad (4.4.16)
$$

We observe that in the above BV transformations only the one of the ghost for ghost $\widetilde{\psi}_i$ receives a correction due to the twist H, whereas the rest are identical to the AKSZ result. For Y^i, partially guided by the formula (4.3.29) for $p=2$ and $r=-1$ (recalling that $Y^i = -\chi^i_{(-1)}$), and adding a suitable H-dependent correction, we obtain

$$
\begin{aligned}
sY^i &= -\mathrm{d}\chi^i - \partial_k\Pi^{ij}(\epsilon_jY^k + A_j\chi^k) + \partial_k\partial_l\Pi^{ij}Z_+^l\epsilon_j\chi^k + \\
&\quad + \Pi^{ji}\psi_j + \partial_k\Pi^{ij}Z_+^k\widetilde{\psi}_j + \\
&\quad + R^{ijk}\epsilon_jA_k + \frac{1}{2}\left(\partial_lR^{ijk} + \frac{1}{2}H_l^{\ ijk}\right)Z_+^l\epsilon_j\epsilon_k\,, \qquad (4.4.17)
\end{aligned}
$$

where we wrote the terms exactly in order of appearance in (4.3.29). Note that the form of the H-correction in the final term is necessary so as to satisfy all three required properties of the BV operator eventually. Similarly, for ψ_i we apply the formula (4.3.34) for $p=2$ and $r=0$ keeping the order of appearance and add suitable H-dependent terms to obtain

$$
\begin{aligned}
s\psi_i &= \mathrm{d}\widetilde{\psi}_i + \partial_i\Pi^{jk}(-\epsilon_j\psi_k + A_j\widetilde{\psi}_k) - \partial_i\partial_j\Pi^{jk}Z_+^l\epsilon_j\widetilde{\psi}_k - \\
&\quad - \partial_i\partial_l\Pi^{jk}(\epsilon_jA_k\chi^l + \frac{1}{2}\epsilon_j\epsilon_kY^l) - \frac{1}{2}\partial_i\partial_l\partial_m\Pi^{jk}Z_+^m\epsilon_j\epsilon_k\chi^l - \\
&\quad + \frac{1}{4}H_{il}^{\ \ jk}\epsilon_j\epsilon_kF^l + \frac{1}{2}f_i^{jkl}\epsilon_j\epsilon_kA_l - \frac{1}{3!}\partial_{(m}f_{i)}^{jkl}Z_+^m\epsilon_j\epsilon_k\epsilon_l\,. \qquad (4.4.18)
\end{aligned}
$$

Finally, collecting together terms of the same type, for the field Z_i we find that

$$\begin{aligned}
sZ_i &= \mathrm{d}\psi_i + \partial_i\Pi^{jk}(-\varepsilon_j Z_k + A_j\psi_k - Y_j^+\widetilde{\psi}_k) + \\
&+ \partial_i\partial_l\Pi^{jk}\left(\frac{1}{2}\varepsilon_j\varepsilon_k A_+^l - \varepsilon_j A_k Y^l + \frac{1}{2}A_jA_k\chi^l + \varepsilon_j\psi_k Z_+^l - A_j\widetilde{\psi}_k Z_+^l - \varepsilon_j Y_k^+\chi^l + \varepsilon_k\widetilde{\psi}_k\psi_+^l\right) + \\
&+ \partial_i\partial_l\partial_m\Pi^{jk}\left(\frac{1}{2}\varepsilon_j\varepsilon_k Y^l Z_+^m + \varepsilon_j A_k\chi^l Z_+^m - \frac{1}{2}\varepsilon_j\widetilde{\psi}_k Z_+^l Z_+^m + \frac{1}{2}\varepsilon_j\varepsilon_k\chi^l\psi_+^m\right) - \\
&- \frac{1}{4}\partial_i\partial_l\partial_m\partial_n\Pi^{jk}Z_+^m Z_+^n\varepsilon_j\varepsilon_k\chi^l + \frac{1}{2}f_i^{jkl}\varepsilon_j A_k A_l + \frac{1}{6}H_{ikl}{}^j\varepsilon_j F^k F^l + \frac{1}{2}H_{il}{}^{jk}\varepsilon_j A_k F^l + \\
&+ \partial_{(i}f_{m)}^{jkl}\left(\frac{1}{6}\varepsilon_j\varepsilon_k\varepsilon_l\psi_+^m - \frac{1}{2}\varepsilon_j\varepsilon_k A_l Z_+^m\right) - \frac{1}{2}\left(\partial_i R^{jkl} + \frac{1}{2}H_i{}^{jkl}\right)\varepsilon_j\varepsilon_k Y_l^+ - \\
&- \frac{1}{6}\partial_{(i}H_{m)l}{}^{jk}\varepsilon_j\varepsilon_k F^l Z_+^m - \left(\frac{1}{12}\partial_{(m}\partial_n f_{i)}^{jkl} + \frac{1}{8}\partial_{(m}\partial_n\Pi^{jp}H_{i)p}{}^{kl}\right)\varepsilon_j\varepsilon_k\varepsilon_l Z_+^m Z_+^n
\end{aligned} \tag{4.4.19}$$

To verify that all the BV operators shown above are nilpotent off-shell, the complete ones for the antifields Z_+^i, Y_i^+, A_+^i and ψ_+^i are needed too. They are found to be

$$sZ_+^i = -F^i - \partial_k\Pi^{ij}\varepsilon_j Z_+^k\,, \tag{4.4.20}$$

$$sY_i^+ = G_i - \partial_i\Pi^{jk}\varepsilon_j Y_k^+ + \partial_i\partial_l\Pi^{jk}\left(\frac{1}{2}\varepsilon_j\varepsilon_k\psi_+^l - \varepsilon_j A_k Z_+^l\right) - \frac{1}{4}\partial_i\partial_l\partial_m\Pi^{jk}\varepsilon_j\varepsilon_k Z_+^l Z_+^m\,, \tag{4.4.21}$$

$$\begin{aligned}
sA_+^i &= \mathcal{F}^i - \partial_k\Pi^{ij}(\varepsilon_j A_+^k - Y_j^+\chi^k + \psi_j Z_+^k + \widetilde{\psi}_j\psi_+^k) - \\
&\quad - \partial_k\partial_l\Pi^{ij}(A_j\chi^k Z_+^l + \varepsilon_j Y^k Z_+^l - \frac{1}{2}\widetilde{\psi}_j Z_+^k Z_+^l + \varepsilon_j\chi^k\psi_+^l) + \frac{1}{2}\partial_k\partial_l\partial_m\Pi^{ij}\varepsilon_j\chi^k Z_+^l Z_+^m + \\
&\quad + R^{ijk}\varepsilon_j Y_k^+ + \left(\partial_l R^{ijk} + \frac{1}{2}H_l{}^{ijk}\right)\left(\varepsilon_j A_k Z_+^l - \frac{1}{2}\varepsilon_j\varepsilon_k\psi_+^l\right) - \\
&\quad - \frac{1}{6}H_{kl}{}^{ij}\varepsilon_j F^k Z_+^l + \left(\frac{1}{4}\partial_l f_m^{ijk} - \frac{1}{12}\Pi^{in}\partial_l H_{mn}{}^{jk}\right)\varepsilon_j\varepsilon_k Z_+^l Z_+^m
\end{aligned} \tag{4.4.22}$$

$$s\psi_+^i = \mathrm{d}Z_+^i + \Pi^{ij}Y_j^+ + \partial_k\Pi^{ij}(A_j Z_+^k - \varepsilon_j\psi_+^k) + \frac{1}{2}\partial_k\partial_l\Pi^{ij}\varepsilon_j Z_+^k Z_+^l\,. \tag{4.4.23}$$

Apart from confirming that the BV operator on the fields is nilpotent, a long yet straightforward calculation leads to the result that its action on these four antifields is also nilpotent, as desired.

With the above data, we can now write the candidate BV action for the 4-form-twisted R-Poisson sigma model in three dimensions. To present it in a compact way, let $\varphi^\alpha, \alpha = 1,\dots,8$ be a collective notation for the eight distinct fields and ghosts of the theory, whose BV operator is given above. The BV action is simply given as

$$S_{\text{BV}}^{(3)}=S^{(3)}-\sum_{\alpha}\int(-1)^{\text{gh}(\varphi)}\varphi_{\alpha}^{+}\,s_0\varphi^{\alpha}+\int\Big(L_k Z_+^k+M_{kl}Z_+^kZ_+^l+N_{klm}Z_+^kZ_+^lZ_+^m\Big)\,, \tag{4.4.24}$$

with $S^{(3)}$ as in (4.4.1) and

$$\begin{aligned}
L_k &= -\partial_k\Pi^{ij}\widetilde{\psi}_jY_i^+ +\partial_k\partial_l\Pi^{ij}(\frac{1}{2}\varepsilon_i\varepsilon_jA_+^l-\varepsilon_j\chi^kY_i^+ +\varepsilon_i\widetilde{\psi}_j\psi_+^k)+\\
&\quad +\frac{1}{2}\partial_k\partial_l\partial_m\Pi^{ij}\varepsilon_i\varepsilon_j\chi^l\psi_+^m-\frac{1}{2}(\partial_kR^{ijl}+\frac{1}{2}H_k{}^{ijl})\varepsilon_j\varepsilon_lY_i^+ +\frac{1}{6}\partial_{(k}f_{m)}{}^{ijl}\varepsilon_i\varepsilon_j\varepsilon_l\psi_+^m\,, &(4.4.25)\\
M_{kl} &= \frac{1}{2}\partial_k\partial_l\Pi^{ij}(\varepsilon_i\psi_j-A_i\widetilde{\psi}_j)+\frac{1}{2}\partial_k\partial_l\partial_m\Pi^{ij}(\varepsilon_iA_j\chi^m+\frac{1}{2}\varepsilon_i\varepsilon_jY^m)-\\
&\quad -\frac{1}{4}\partial_{(k}f_{l)}{}^{ijm}\varepsilon_i\varepsilon_jA_m-\frac{1}{12}\partial_{(k}H_{l)m}{}^{ij}\varepsilon_i\varepsilon_jF^m\,, &(4.4.26)\\
N_{klm} &= -\frac{1}{6}\partial_k\partial_l\partial_m\Pi^{ij}\varepsilon_i\widetilde{\psi}_j-\frac{1}{12}\partial_k\partial_l\partial_m\partial_n\Pi^{ij}\varepsilon_i\varepsilon_j\chi^n-\\
&\quad -\left(\frac{1}{36}\partial_{(k}\partial_lf_{m)}{}^{ijn}+\frac{1}{24}\partial_{(k}\partial_l\Pi^{ip}H_{m)p}{}^{jn}\right)\varepsilon_i\varepsilon_j\varepsilon_n\,. &(4.4.27)
\end{aligned}$$

That this is indeed the BV action, or in other words that it is the solution to the classical master equation $(S_{\text{BV}},S_{\text{BV}})$ with respect to the BV antibracket $(\cdot,\cdot)$ can be seen as follows. The BV operator on the fields should satisfy the three properties I, II and III mentioned in section 4.3.2. To confirm that S_{BV} as given in (4.4.24) satisfies the classical master equation, it suffices to show that all the nilpotent operators s derived above are indeed the unique BV operator stemming from S_{BV} and moreover that the remaining four ones on the antifields of $X^i,\varepsilon_i,\chi^i,\widetilde{\psi}_i$ are also strictly nilpotent. Then the classical master equation follows due to the graded Jacobi identity for the antibracket. This is not trivial because the operator s can have additional off-shell ambiguities, terms that are proportional to the classical equations of motion of the theory. In particular, there are more than one ways to satisfy properties I and II, and the point is to show that property III completely fixes s to be s_{BV} without further ambiguities.

To show this, first of all notice that only terms proportional to the field equation F^i constitute possible ambiguities. This is proven as follows. An ambiguity proportional to the field equation $\mathcal{G}_i$ can only potentially appear in the 3-form antifields $X_i^+,\varepsilon_+^i,\chi_i^+,\widetilde{\psi}_+^i$, since $\mathcal{G}_i$ is a 3-form; such ambiguity terms are a product of a scalar and $\mathcal{G}_i$. Since there are no scalar antifields, the ghost degree of a scalar that multiplies $\mathcal{G}_i$ has to be nonnegative which means that this type of correction can exist only for the antifields of ghost degree -1. The only such antifield is X_i^+ in which case the scalar multiplying $\mathcal{G}_i$ would have vanishing ghost degree, meaning that it is a function of X. But such terms in sX_i^+ are completely determined by the classical part of the BV action and cannot be modified which then eliminates all the ambiguities proportional to $\mathcal{G}_i$.

The ambiguities proportional to 2-form field equations G_i and $\mathcal{F}^i$ are possible only in 3-form antifields X_i^+, ε_+^i, χ_i^+, $\widetilde{\psi}_+^i$, 2-form antifields A_+^i, Y_i^+, ψ_+^i and a 2-form field Z_i. Here 2-form antifields cannot receive such corrections for the same reason why 3-form antifields could not receive corrections proportional to $\mathcal{G}_i$. In sZ_i such corrections would not contain any antifields (because there are no scalar antifields), but all such terms are determined by the BRST operator (property I). Finally, correction terms in 3-form antifields would have 1-form multiplying the field equation. This 1-form would need to contain an antifield, for otherwise, such a term would be determined by the classical part of the BV action. Since there are no scalar antifields and the only 1-form antifield is Z_+^i, the term would contain only Z_+^i and no other antifields. However, this would produce terms in sZ_i that contain no antifields and all such terms are determined by the BRST operator.

Finally, the only possibility are the ambiguities proportional to the field equation F^i. With this, 1-form fields A_i and Y^i cannot receive any corrections since those kind of terms cannot contain any antifields and as such are determined by the BRST operator. Similarly, Z_+^i cannot receive those corrections as well because that would require sZ_i to receive corrections that contain no antifields and that part is again determined by the BRST operator. On the other hand, there are no obstructions for $s\psi_i$ and sZ_i to receive corrections proportional to F^i (with the correction in sZ_i containing at least one antifield). The remaining antifields would then receive corrections proportional to F^i as well, but all those would be determined by the corrections of $s\psi_i$ and sZ_i. So, all the possible independent ambiguities are those proportional to the field equation F^i in $s\psi_i$ and sZ_i. In addition, property I is now completely satisfied. However, properties II and III still have to be taken into account. Taking into account all possible corrections, a straightforward calculation finally removes any remaining ambiguities.

A final cross-check is to confirm that s_{BV}^2 vanishes on $X_i^+, \varepsilon_+^i, \chi_i^+$ and $\widetilde{\psi}_+^i$. First of all, using property III we determine

$$\begin{aligned}
s\widetilde{\psi}_+^i &= \mathrm{d}\psi_+^i + \Pi^{ji}\chi_+^j - \partial_k\Pi^{ij}(\varepsilon_j\widetilde{\psi}_+^k - Z_+^k Y_j^+ - A_j\psi_+^k) + \\
&\quad + \partial_k\partial_l\Pi^{ij}(\varepsilon_j Z_+^l\psi_+^k - \frac{1}{2}A_j Z_+^l Z_+^k) - \frac{1}{3!}\partial_k\partial_l\partial_m\Pi^{ij}\varepsilon_j Z_+^k Z_+^l Z_+^m, \qquad (4.4.28)\\
s\chi_i^+ &= \mathrm{d}Y_i^+ + \partial_i\Pi^{jk}(-\varepsilon_j\chi_k^+ + A_j Y_k^+) + \\
&\quad + \partial_i\partial_l\Pi^{jk}(\varepsilon_j Z_+^l Y_k^+ + \varepsilon_j A_k\psi_+^l - \frac{1}{2}\varepsilon_j\varepsilon_k\widetilde{\psi}_+^l + \frac{1}{2}A_j A_k Z_+^l) + \\
&\quad + \frac{1}{2}\partial_i\partial_l\partial_m\Pi^{jk}\varepsilon_j(\varepsilon_k Z_+^m\psi_+^l - A_k Z_+^l Z_+^m) - \frac{1}{2}\partial_i\partial_l\partial_m\partial_n\Pi^{jk}\varepsilon_j\varepsilon_k Z_+^l Z_+^m Z_+^n \,, \qquad (4.4.29)
\end{aligned}$$

for the 3-form antighosts of the scalar ghosts of the theory, and moreover

$$
\begin{aligned}
s\varepsilon_i^+ &= -\mathrm{d}A_+^i + \Pi^{ij}X_j^+ - \partial_k\Pi^{ij}(\varepsilon_j\varepsilon_+^k - \chi^k\chi_j^+ + \widetilde{\psi}_j\widetilde{\psi}_+^k + A_jA_+^k - Y^kY_j^+ - \psi_j\psi_+^k + Z_jZ_+^k) \\
&\quad - \partial_k\partial_l\Pi^{ij}(\varepsilon_j\chi^l\widetilde{\psi}_+^k + \varepsilon_jZ_+^lA_+^k + \chi^kZ_+^lY_j^+ + \varepsilon_jY^l\psi_+^k + A_j\chi^l\psi_+^k - \widetilde{\psi}_jZ_+^l\psi_+^k + \\
&\qquad + A_jY^lZ_+^k - \frac{1}{2}\psi_jZ_+^kZ_+^l) + \\
&\quad + \partial_k\partial_l\partial_m\Pi^{ij}(\varepsilon_j\chi^lZ_+^m\psi_+^k + \frac{1}{2}\varepsilon_jY^lZ_+^kZ_+^m + \frac{1}{2}A_j\chi^lZ_+^kZ_+^m - \frac{1}{6}\widetilde{\psi}_jZ_+^kZ_+^lZ_+^m) - \\
&\quad - \frac{1}{6}\partial_k\partial_l\partial_m\partial_m\Pi^{ij}\varepsilon_j\chi^lZ_+^kZ_+^mZ_+^n - R^{ijk}(\varepsilon_j\chi_k^+ + A_kY_j^+) - \\
&\quad - \partial_lR^{ijk}(\frac{1}{2}\varepsilon_j\varepsilon_k\widetilde{\psi}_+^l + \varepsilon_kZ_+^lY_j^+ - \varepsilon_jA_k\psi_+^l - \frac{1}{2}A_jA_kZ_+^l) + \qquad (4.4.30) \\
&\quad + \frac{1}{2}\partial_l\partial_mR^{ijk}(\varepsilon_j\varepsilon_kZ_+^l\psi_+^m - \varepsilon_jA_kZ_+^lZ_+^m) - \frac{1}{12}\partial_l\partial_m\partial_nR^{ijk}\varepsilon_j\varepsilon_kZ_+^lZ_+^mZ_+^n + \Delta s\,\varepsilon_i^+\,, \\
sX_i^+ &= (S_{\mathrm{BV}}, X_i^+)\,, \qquad (4.4.31)
\end{aligned}
$$

where we refrain from presenting the full result for X_i^+ since it contains all possible partial derivatives with respect to X on every term of the BV action and is hence a very long expression. The H-dependent part of the transformation on ε_i^+ is hidden in Δs, which is given as

$$
\begin{aligned}
\Delta s\,\varepsilon_+^i &= -\frac{1}{2}H_l{}^{ijk}(\varepsilon_j\varepsilon_k\widetilde{\psi}_+^l + \varepsilon_kZ_+^lY_j^+ - 2\varepsilon_jA_k\psi_+^l - A_jA_kZ_+^l) + \\
&\quad + \frac{1}{2}H_{kl}{}^{ij}F^l(\varepsilon_j\psi_+^k - A_jZ_+^k) - \frac{1}{6}\partial_{(l}H_{m)k}{}^{ij}\varepsilon_jF^kZ_+^lZ_+^m + \frac{1}{6}H_{jkl}{}^iF^kF^lZ_+^j + \\
&\quad + \frac{1}{2}\partial_{(m}H_{l)}{}^{ijk}(\varepsilon_j\varepsilon_kZ_+^m\psi_+^l - \varepsilon_jA_kZ_+^lZ_+^m) - \\
&\quad - \left(\frac{1}{12}\partial_{(m}\partial_nH_{l)}{}^{ijk} + \frac{1}{6}\partial_{(m}\partial_n\Pi^{ip}H_{l)p}{}^{jk}\right)\varepsilon_j\varepsilon_kZ_+^lZ_+^mZ_+^n\,. \qquad (4.4.32)
\end{aligned}
$$

With a straightforward calculation it is possible to confirm that s^2 on these fields vanish.

To facilitate the comparison with the BV operators and the BV action found through the AKSZ theory in the $H=0$ case, hence called s_{AKSZ}, we may rewrite the above expressions as

$$
s\varphi^\alpha = s_{\text{AKSZ}}\varphi^\alpha + \Delta s\,\varphi^\alpha\,, \qquad (4.4.33)
$$

where φ^α are the eight distinct fields and ghosts. Then Δs vanishes for four of them, namely for

$X^i, \varepsilon_i, \chi^i$ and A_i, whereas for the remaining four we have found

$$\Delta s Y^i = \frac{1}{4} H_l{}^{ijk} Z_+^l \varepsilon_j \varepsilon_k \,, \tag{4.4.34}$$

$$\Delta s \widetilde{\psi}_i = -\frac{1}{3!} H_i{}^{jkl} \varepsilon_j \varepsilon_k \varepsilon_l \,, \tag{4.4.35}$$

$$\Delta s \psi_i = \left(\frac{1}{4} H_{il}{}^{jk} F^l + \frac{1}{2} H_i{}^{jkl} A_l - \frac{1}{3!} \partial_{(m} H_{i)}{}^{jkl} Z_+^m \varepsilon_l \right) \varepsilon_j \varepsilon_k \tag{4.4.36}$$

$$\begin{aligned} \Delta s Z_i = & \left\{ \frac{1}{3!} H_{ikl}{}^j F^k F^l + \frac{1}{2} H_{il}{}^{jk} A_k F^l + \frac{1}{4} H_i{}^{jkl} \left(A_k A_l + \varepsilon_k Y_l^+ \right) + \frac{1}{3!} \partial_{(i} H_{m)l}{}^{jk} F^l Z_+^m \varepsilon_k \right. \\ & \left. + \frac{1}{3!} \partial_{(m} H_{i)}{}^{jkl} \left(\varepsilon_l \psi_+^m + 3 A_l Z_+^m \right) \varepsilon_k - \frac{1}{2 \cdot 3!} \partial_{(m} \partial_n H_{i)}{}^{jkl} \varepsilon_k \varepsilon_l Z_+^m Z_+^n \right\} \varepsilon_j \,. \end{aligned} \tag{4.4.37}$$

This leads us to an alternative presentation of the BV action for the 4-form-twisted R-Poisson-Courant sigma model, which reads[4]

$$S_{\mathrm{BV}}^{(3)} = S_{\mathrm{AKSZ}}^{(3)} + \Delta S^{(3)} \,, \tag{4.4.38}$$

where $S_{\mathrm{AKSZ}}^{(3)}$ is the AKSZ action for the untwisted R-Poisson-Courant sigma model given in (4.4.10), and $\Delta S^{(3)}$ is the H-dependent correction to it, given by

$$\begin{aligned} \Delta S^{(3)} = & \int_{\Sigma_3} \left(-\frac{1}{6} H_l{}^{ijk} \varepsilon_i \varepsilon_j \varepsilon_k \widetilde{\psi}_+^l - \frac{1}{4} H_{kl}{}^{ij} \varepsilon_i \varepsilon_j F^k \psi_+^l + \frac{1}{2} H_l{}^{ijk} \varepsilon_i \varepsilon_j A_k \psi_+^l + \frac{1}{6} H_{jkl}{}^i \varepsilon_i F^j F^k Z_+^l \right. \\ & + \frac{1}{6} \partial_{(m} H_{l)}{}^{ijk} \varepsilon_i \varepsilon_j \varepsilon_k Z_+^l \psi_+^m - \frac{1}{4} H_l{}^{ijk} \varepsilon_i \varepsilon_j Y_k^+ Z_+^l + \frac{1}{2} H_l{}^{ijk} \varepsilon_i A_j A_k Z_+^l - \\ & - \frac{1}{2} H_{kl}{}^{ij} \varepsilon_i A_j F^k Z_+^l - \frac{1}{4} \partial_m H_l{}^{ijk} \varepsilon_i \varepsilon_j A_k Z_+^l Z_+^m + \frac{1}{12} \partial_m H_{kl}{}^{ij} \varepsilon_i \varepsilon_j F^k Z_+^l Z_+^m - \\ & \left. - \left(\frac{1}{36} \partial_m \partial_n H_l{}^{ijk} + \frac{1}{24} \partial_m \partial_n \Pi^{kp} H_{lp}{}^{ij} \right) \varepsilon_i \varepsilon_j \varepsilon_k Z_+^l Z_+^m Z_+^n \right) + \int_{\Sigma_4} X^* H \,. \end{aligned} \tag{4.4.39}$$

Obviously, when $H = 0$ then $\Delta S^{(3)} = 0$ and the correct solution of the classical master equation is given by the AKSZ action.

[4]To avoid confusion, note that it is only $S_{\mathrm{BV}}^{(3)}$ that satisfies the classical master equation. In the present context $S_{\mathrm{AKSZ}}^{(3)}$ does not satisfy the classical master equation in general, but only when $H = 0$.

Chapter 5

Conclusion and outlook

In this thesis we studied different aspects of the topological sigma models. First, we improved on work done in [25, 26] about the generalized gauging procedure of the 2-dimensional string model in the background with the metric g and the 3-form H. Originally, only backgrounds with closed 3-forms and gauging with minimal coupling to the metric were considered. Here, we first improved the latter, allowing for the more general coupling to the metric sector, successfully proving that non-minimal coupling is equivalent to the minimal.

After non-minimal coupling, the general gauging theorem was further improved by allowing the background 3-form to be non-closed. This was done through the process of Kaluza-Klein style dimensional reduction on the target space. Here we obtained that gauging data is constrained to lie in the Dirac structure of some Courant algebroid. This is a direct generalization of the original version which stated that gauging data is constrained on the Dirac structure of an exact Courant algebroid. However, the gauging was not done in its full generality, lacking the terms in the gauge transformations proportional to the field equations. This kind of generalization is direct and it would probably produce similar result as the exact version, that of the gauge transformation being parametrized by 2 connections.

Furthermore, we introduced a new kind of 2-dimensional sigma model, the Jacobi sigma model. Its significance can be motivated in several ways. First, it is a generalization of the Poisson sigma model in a way similar to the twisted Poisson sigma model. In the latter, breaking of the Poisson structure is controlled by an additional 3-form, while in the Jacobi case, the breaking of the Poisson structure was controlled by an additional vector field. Furthermore, as was shown in 2.2.1, the Poisson manifolds are all Jacobi manifolds, both in trivial and non-trivial way. Specifically, the Poisson condition $[\Pi,\Pi]=0$ is automatically the Jacobi condition with the additional vector field being 0. However, multiplying the Poisson bivector by some function f makes $f\Pi$ into a pure Jacobi bivector that is no longer Poisson. In that sense, the possible Poisson structure on certain manifolds might be hidden as an emergent Jacobi structure. On the other hand, the Jacobi structure is rigid in this regard, meaning that product of the Jacobi bivector and some function is still a Jacobi bivector, only for the different Reeb vector field. Thus Jacobi sigma models take not only much wider variety of target spaces into account, but also provide a different viewpoint for the Poisson manifolds as well. What is more, they can always give rise to new Poisson sigma models in one dimension higher through the Poissonization procedure. Finally, the Jacobi structure is related to gauging the string models with non-closed 3-form in the Abelian case, since it was shown one can obtain contact manifolds in such situations, all of which carry a Jacobi structure as well.

These kind of results are interesting in the context of heterotic string theory, where the Bianchi identity of the 3-form is modified with respect to the TypeII superstrings by additional α' corrections, having the general form studied here. It would be very nice if it turns out to be possible to construct heterotic membrane sigma models in analogy to the Courant sigma models [42] in which the heterotic Bianchi identity could be obtained through classical master equation of the corresponding BV formulation. A further goal would be to find a geometrical way to understand α' corrections. The corrections to the bracket and bilinear form of the Courant sigma models that would account for the α' corrections has already been suggested in [92, 93, 94] while their relation to transitive Courant algebroids was explored in [74]. In addition, a possible method for obtaining higher α' corrections was given in [95] which would be interesting to reformulate (if possible) in the context of Courant algebroids.

Finally, we studied the mechanics involved in constructing the (classical) BV action for the topological field theories without the *QP*-structure on the target space. For such theories, the famous AKSZ construction of the BV action cannot be used, thus requiring some other methods. Here we explored the possibilities of what other kind of methods could be used and in the process constructed the BV action for the Dirac sigma model and 3-dimensional R-Poisson sigma model. A notable property the latter possesses are the products of field equations in the gauge transformations of the theory. Specifically, the 3-dimensional case has a term quadratic in the field equations that shows up in the gauge transformations, which is property not encountered before. In addition to finding the full BV action for the 3-dimensional case, we presented the results for arbitrary (in any dimension) non-twisted R-Poisson sigma model. While this is technically not a new result, since this can be obtained through the AKSZ construction, the actual expressions one would get from such a method would be expanded version of the formulas derived here. Thus the main result here is the way this result is written in a neatly compact form. A significant improvement would be to expand the result to include the twisted version in any dimension, this turns out extremely difficult to do because of the higher products of field equations in the gauge transformation and a significant amount of work needs to be done before such a thing could be possible. On the other hand one may extend the work done here, by going from the classical BV action to the quantum one which should not be overly difficult, but would be a significant result on its own, giving the fully quantized version of the theories presented here. Finally, and what might be interesting most of all, is the possibility to generalize the *refinement* method used to construct the BV operator in the R-Poisson case. While the actual procedure can be used for any field theory, the question of whether one could always produce a BV operator from it is still open. Specifically, in the case of R-Poisson sigma model we relied on the uniqueness of the BV operator in the sense that there was only one operator that satisfied all the properties the BV operator needs to satisfy, thus making it the BV operator. However, if one would lose this uniqueness, the described procedure would produce more than one potential BV operators. So, the open question remaining here is in what cases is the constructed operator unique, and if it is not is there a way to determine the correct BV operator from the set of all possibilities.

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Curriculum vitae

Grgur Šimunić was born in 1993 in Zagreb, Croatia. In 2012 they enrolled into the Physics programme at the Department of Physics, Faculty of Science, University of Zagreb, from which they graduated in 2017 on the subject "Computing Black Hole Entropy via Near-Horizon Symmetries" under the supervision of dr. sc. Maro Cvitan, thus obtaining the title of Master of Science in Physics. A few months later they enrolled in doctoral programme at the Department of Physics, Faculty of Science, University of Zagreb working as a research assistant in Quantum Gravity and Mathematical Physics Group led by dr. sc. Larisa Jonke at the Division of Theoretical Physics, Rudjer Boskovic Institute, while their doctoral research was supervised by dr. sc. Athanasios Chatzistavrakidis. During this time they gained some teaching experience which included auditory and practical exercises as well as administering written exams to undergraduate courses: Quantum Physics, Differential Geometry in Physics and General Relativity.

List of publications

- A. Chatzistavrakidis, G. Šimunić. Gauged sigma-models with nonclosed 3-form and twisted Jacobi structures. *Journal of High Energy Physics*, 2020(11):1-33, 2020
- A. Chatzistavrakidis, N. Ikeda, G. Šimunić. The BV action of 3D twisted R-Poisson sigma models. *Journal of High Energy Physics*, 2022(10), 2022
- A. Chatzistavrakidis, L. Jonke, T. Strobl, G. Šimunić. Topological Dirac sigma models and the classical master equation. *arXiv preprint arXiv: 2206.14258*, 2022
- G. Šimunić. Dirac sigma models from gauging the nonlinear sigma models and its BV action. *arXiv preprint arXiv:2208.01530*, 2022