Classification of boundary conditions for Freidrichs systems

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To my parents

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SUMMARY

This work is concerned with the classification of boundary conditions for the classical Friedrichs systems by developing new results on abstract Friedrichs operators. The concept of the classical Friedrichs system was introduced by Friedrichs in 1958 as a symmetric positive system of first-order linear partial differential equations. The main motivation was to treat the equations that change their type, like the Tricomi equation (which appears in the transonic flow). Moreover, it allows a unified treatment of a wide variety of elliptic, parabolic, hyperbolic and mixed-type equations. On the other hand, the theory of abstract Friedrichs operators was introduced in 2007, which is formulated in terms of Hilbert space theory. Further development of this theory on a purely operator-theoretic approach allows us to work beyond the realm of partial differential operators.

We derive a von Neumann decomposition-type formula for the graph space of abstract Friedrichs operators. This decomposition ensures that the classification of all boundary conditions depends (only) on the kernels of the adjoint operators. We recognise the potential connection between the theory of abstract Friedrichs operators and symmetric operators. By representing an abstract Friedrichs operator as the sum of a skew-symmetric and a bounded self-adjoint operator with a strictly positive bottom, we introduce a von-Neumann extension theory for abstract Friedrichs operators, enabling a comprehensive classification of boundary conditions—a distinct approach from the general Grubb extension theory.

Furthermore, we present a complete classification of boundary conditions for classical Friedrichs operators in the one-dimensional case. This classification involves an explicit formulation of the boundary operator, depending on the coefficient matrix. We argue using total projections and prove a result that relates the dimensions of the kernels to the rank of the coefficient matrix at the endpoints of the interval. We illustrate the theory on a second order ordinary differential equation.

Non-stationary theory for Friedrichs operators using semigroup theory is studied at the end. It turns out that a wide class of boundary conditions for a given pair of abstract Friedrichs operators gives rise to the generators of contractive C_0-semigroups. A subclass of these boundary conditions is related to the skew-selfadjoint extensions of the skew-symmetric operators. The boundary quadruple approach is used to give another classification of these special types of boundary conditions.

Keywords: symmetric positive first-order systems of partial differential equations, non-selfadjoint operators, extension theory of closed operators, dual pairs, indefinite inner product space, perturbation of matrices, C_0-semigroup.

SAŽETAK

Ovaj rad proučava klasifikaciju rubnih uvjeta za klasične Friedrichsove sustave razvijajući nove rezultate o apstraktnim Friedrichsovim operatorima. Pojam klasičnog Friedrichsovog sustava uveo je Friedrichs 1958. godine kao simetrični pozitivan sustav linearnih parcijalnih diferencijalnih jednadžbi prvog reda. Glavna motivacija bila je tretirati jednadžbe koje mijenjaju svoj tip, poput Tricomijeve jednadžbe (koja se pojavljuje pri opisu toka kod prijelaza brzine zvuka). To omogućuje objedinjeni pristup širokom rasponu eliptičnih, paraboličnih, hiperboličnih i jednadžbi mješovitog tipa. Teorija apstraktnih Friedrichsovih operatora uvedena je 2007. godine, a formulirana je u terminima teorije Hilbertovih prostora. Daljnji razvoj ove teorije čisto operatorskim pristupom omogućuje nam istraživanje izvan područja parcijalnih diferencijalnih operatora.

U radu je prepoznata povezanost teorije apstraktnih Friedrichsovih operatora i simetričnih operatora. Naime, reprezentiranjem apstraktnih Friedrichsovih operatora kao zbroja antisimetričnog i ograničenog pozitivnog hermitskog operatora, dobivamo formulu za rastav prostor grafa apstraktnih Friedrichsovih operatora von-Neumannovog tipa. Ovaj rastav osigurava da klasifikacija svih rubnih uvjeta ovisi (samo) o jezgrama adjunigiranih operatora. Uvođenjem teorije von-Neumannovog proširenja za apstraktne Friedrichsove operatore, omogućujuje se sveobuhvatna klasifikacija rubnih uvjeta – pristup koji se razlikuje od opće Grubbine teorije proširenja.

Nadalje, prezentiramo potpunu klasifikaciju rubnih uvjeta za klasične Friedrichsove operatore u jednodimenzionalnom slučaju. Ova klasifikacija uključuje eksplicitnu formulaciju rubnog operatora, ovisno o matričnom koeficijentu. Dokaz se temelji na korištenju ukupne projekcije (u terminu projekcije na svojstvene potprostore) i dokazujemo rezultat koji povezuje dimenzije jezgara s rangom matričnog koeficijenta na rubovima intervala. Tu teoriju ilustriramo na običnim diferencijalnim jednadžbama drugog reda.

Na kraju se proučava nestacionarna teorija za Friedrichsove operatore koristeći teoriju polugrupa. Pokazuje se da širok spektar rubnih uvjeta za dani par apstraktnih Friedrichsovih operatora dovodi do generatora kontrakcijske C_0-polugrupe. Štoviše, pokazana je povezanost tih rubnih uvjeta s antihermitskim proširenjima antisimetirčnih operatora. Recentni pristup s rubnim četvorkama koristi se za alternativnu klasifikacije ovog posebnog tipa rubnih uvjeta.

Rad je organiziran kako slijedi.

Čitatelja se upoznaje u Uvodu s teorijom Friedrichsovih sustava, gdje je dan i kratki pregled rezultata koji su obrađeni u ostatku Rada.

Poglavlje 1 sadrži osnove teorije klasičnih Friedrichsovih operatora. Sažeti su osnovni rezultati teorije dobre postavljenosti zajedno s primjerima klasičnih Friedrichsovih operatora. Nadalje, raspravljaju se različiti načini zadavanja rubnih uvjeta.

Poglavlje 2 uvodi apstraktne Friedrichsove operatore zajedno s njihovom formulacijom u terminima teorije Hilbertovih prostora. Naglasak je stavljen na teoriju dobre postavljenosti koristeći tzv. *konusni formalizam* koji je uveden u [39]. Također se diskutiraju rezultati o višestrukosti i klasifikaciji koristeći teoriju Kreĭnovih prostora (vidi [3, 9]). Na kraju ovog poglavlja proučavaju se odabrani primjeri od interesa.

Poglavlje 3 uvodi novu karakterizaciju apstraktnih Friedrichsovih operatora u terminima zbroja antisimetričnog i pozitivnog hermitskog operatora. Izvodi se dekompozicija prostora grafa apstraktnih Friedrichsovih operatora von Neumannovog tipa. Poglavlje se zaključuje proučavanjem klasifikacije rubnih uvjeta u duhu teorije von Neumanna o proširenjima, te pripadnom poveznicom s teorijom za simetrične operatore.

U poglavlje 4 se primjenjuju dobiveni rezultati prethodnog poglavlja na Friedrichsove sustave na intervalu. Pretpostavke u vezi s koeficijentima prilično su općenite i obuhvaćaju situacije koje uključuju singularne jednadžbe (ili sustave). Pruža se potpuna klasifikacija rubnih uvjeta za skalarni slučaj. Analizira se i vektorski slučaj, gdje se dokazuje rezultat koji povezuje jezgre pripadnih maksimalnih operatora sa svojstvenim vrijednostima matričnih koeficijenata na rubovima intervala.

Poglavlje 5 se fokusira na nestacionarnu teoriju apstraktnih Friedrichsovih operatora. Preciznije, dokazuje se da bijektivne realizacije apstraktnih Friedrichsovih operatora s rubnim preslikavanjem s predznakom, kao i odgovarajuće realizacije antisimetričnih di-

jelova, daju generatore kontrakcijskih C_0-polugrupa, te da su to jedine realizacije s takvim svojstvom. Rezultati su povezani s novouvedenom teorijom rubnih četvorki za antisimetrične opratore [10].

Ključne riječi: simetrični pozitivni sustav parcijalnih diferencijalnih jednadžbi prvog reda, nehermitski operator, teorija proširenja zatvorenih operatora, dualni par, prostor indefinitnog skalarnog produkta, perturbacija matrice, C_0-polugrupa.

Contents

INTRODUCTION

Friedrichs introduced the concept of the *positive symmetric system* [41] (following his research on symmetric hyperbolic systems [40]), which is today customarily referred to as the *Friedrichs system*. It was a historical effort to provide a unified framework that incorporates the study of a wide range of differential equations. Although it represents a class of (initial-)boundary value problems consisting of first-order linear partial differential equations, the casting of second-order elliptic, parabolic, and hyperbolic equations into Friedrichs systems is well-studied. Also, many other equations, such as diffusion equations, advection-diffusion-reaction equations, div-grad problems, linear elasticity problems, the Klein-Gordon equation, Maxwell's equations, and magnetohydrodynamics equations, can be analysed within this framework. Friedrichs' primary motivation was to study partial differential equations (appearing in a number of physical phenomena) that change their type, such as the *Tricomi equation* appearing in transonic fluid flow:

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \,. \tag{1}$$

Regions of subsonic flow correspond to a local model of elliptic type $y > 0$, while regions of supersonic flow are represented by a hyperbolic equation $y < 0$.

When it comes to providing a unified treatment for such a diverse class of equations, it is inevitable that the solutions to Friedrichs systems incorporate the mathematical characteristics of solutions of the individual differential equations. To elaborate: considering the characteristics of hyperbolic equations, the solution to a Friedrichs system may be discontinuous. On the other hand, a solution may exhibit poles at corners of the domain, which is not unusual considering the nature of solutions of elliptic equations. Referring to many other familiar features such as holomorphy, boundary and interior layers, the list could be further extended. Clearly, trying to capture all of these characteristics simultaneously is a major challenge, and it is still of great interest to study such problems. One of the main difficulties is the correct implementation of the boundary conditions. Friedrichs introduced a very clever technique to characterise admissible boundary conditions. This technique involves a non-uniquely defined, positive matrix boundary field having peculiar algebraic properties. One difficulty of the theory developed by Friedrichs is that it is not intrinsic, due to the nonuniqueness of the boundary field used to enforce boundary conditions. Moreover, the theory involves boundary values of the solution to the PDE whose meaning is not clear i.e. due to lack of regularity, it is subtle how to define and integrate these values. Finally, he was only able to prove the existence of weak solutions and uniqueness of strong ones, leaving the general question open on the joint existence and uniqueness.

In the following years, a number of improvements were made to this theory (particularly to clarify the meaning of traces in Friedrichs systems), mostly by Friedrichs' collaborators and former students. In [42], another equivalent way to impose boundary conditions was introduced by Friedrichs and Lax, and a third equivalent way to impose boundary conditions was introduced by Phillips and Sarason in [52]. Each of these methods of imposing boundary conditions provides a different perspective to the Friedrichs theory and, along with an improvement, also governs the strength of the theory in a given direction. Although there was some progress in very specific points, the topic appeared to be less active from the mid-1960s to the late-1990s.

New interest in Friedrichs systems arose from numerical analysis, thanks to their feature of providing a convenient unified framework for numerical solvers to partial differential equations of different types, together with the fact that the structure of first-order equations is beneficial for developing numerical schemes (see e.g. [35, 46]). A comprehensive overview of the theory from this perspective can be found in [47]. This interest derived from numerical analysis motivated the introduction of abstract theory at the beginning of 2000s. While we continue our focus on abstract theory from now on, here we mention the following references for the development of different numerical schemes [19, 24, 25, 26, 37, 38].

Ern, Guermond, Caplain introduced the theory of abstract Friedrichs operators [39, 2007] for real Hilbert spaces (in [5], the theory is interpreted for complex Hilbert spaces).

They reformulated the theory in terms of operators acting on Hilbert spaces and were able to avoid invoking traces at the boundary. To impose the boundary conditions, they introduced a set of geometric conditions, the so-called *cone formalism*. In this framework, the main problem of well-posedness can be understood as follows: To find a subspace $\mathscr{V}$ of a Hilbert space $\mathscr{H}$, for any abstract Friedrichs operator T, such that the problem:

$$\text{For a given } f \in \mathscr{H}\text{, find } u \in \mathscr{V} \text{ such that } Tu = f\,, \tag{2}$$

is well-posed i.e., $T|_{\mathscr{V}}$ is bijective (this operator-theoretic reformulation of partial differential equations could be traced back to works of Višik [54, 55]). The way of imposing boundary conditions could be seen as the quest for such subspaces $\mathscr{V}$. The first well-posedness result was proved in the aforementioned reference under the assumption of the existence of subspaces satisfying the condition given by the cone formalism. The abstract theory not only encompasses the classical theory; it goes beyond the realm of PDEs. Analogously to the three ways of imposing boundary conditions in the classical theory, we have three ways in the abstract theory as well. In fact, the cone-formalism is analogous to the one introduced by Phillips and Sarason in [52], and the *boundary operator formalism* is analogous to Friedrichs' condition. Equivalence among these three conditions has been investigated in the following references [3, 39]. This new development attracted the community for further theoretical and numerical investigations. For example, studies of different representations of boundary conditions and the relation with the classical theory [3, 4, 5, 6, 8, 9, 12], applications to various (initial-)boundary value problems of elliptic, hyperbolic, and parabolic type [7, 12, 23, 29, 31, 36, 49], and the development of different numerical schemes [18, 19, 24, 36, 37, 38].

A characterisation of the cone formalism in terms of an indefinite inner product is studied in [3], which, in a quotient by its isotropic part, gives a *Kreĭn space*. In the same reference, the authors proved the equivalence among the three boundary conditions in the abstract setting in full generality, where the existence of the required subspaces is evident by the Kreĭn space theory. The use of the indefinite inner product structure plays a key role in answering the following generalised well-posedness problem:

$$\text{For a given Friedrichs system } Tu = f\,, \tag{3}$$

(1) **Existence:** the existence of boundary conditions for which the problem is well-

posed.

(2) **Multiplicity:** the possibility that there exist infinitely many different boundary conditions of well-posedness.

(3) **Classification:** an efficient classification of such boundary conditions for which the problem is well-posed.

In 2017, Antonić, Erceg, and Michelangeli presented a purely operator-theoretic description of abstract Friedrichs systems, and using the Kreĭn space theory, they proved the existence of boundary conditions for which the problem (3) is well-posed (note that in [39], the well-posedness result was obtained under the assumption that such boundary conditions exist). Moreover, they provided the multiplicity result, i.e., necessary and sufficient conditions for the existence of infinitely many such boundary conditions. The operator-theoretic approach allowed an application of the universal extension theory (see e.g., [43] and [44, Chapter 13]), and this answers the problem of classification, i.e. we get a complete classification of all such boundary conditions.

Let us elaborate in more details on the organisation and context of the dissertation, while at the same time emphasising the main novelty of this work.

In Chapter 1 we recall the calssical theory of Friedrichs systems, presenting the classical theory along with the three ways to impose the boundary conditions in Section 1.1. We elaborate on the casting of some well-known examples (stationary diffusion equation and Maxwell's equation in the elliptic regime) as classical Friedrichs operators in Section 1.2, and we conclude Chapter 1 with a summary of well-posedness results of the classical theory.

In Section 2.1, we introduce the theory of abstract Friedrichs operators and provide a complete description in terms of an operator-theoretic approach. The boundary operator and the connection to the indefinite inner product space and the Kreĭn space theory are covered in Section 2.2. A discussion about classical Friedrichs operators being encompassed by the abstract Friedrichs operators is presented as well. A brief review of the Kreĭn space theory in the context of the abstract theory of Friedrichs systems is provided in Appendix. A rigorous discussion on the cone-formalism and its operator-theoretic formulation is provided in Section 2.3, while proof of well-posedness result along with the multiplicity result is covered in 2.4. In Section 2.5, we discuss the equivalence among the three ways to impose boundary conditions in the abstract setting. We briefly recall the general extension theory in Section 3.1.

From Section 3.2 onwards, most of the material is a novel contribution, so we will be more detailed in describing the context. It turns out that the classification of boundary conditions using the general extension theory is not the only way to approach this theory. The well-known von-Neumann extension theory for symmetric operators is also applicable to the abstract Friedrichs operators. However, we present a different approach for the proofs which allows us to go in the direction of symmetric operator theory via abstract Friedrichs operators. We start by proving a decomposition of the graph space in terms of the so-called *minimal space* and kernels of the adjoint operators in Section 3.2. As a consequence, a pair of boundary conditions is explicitly obtained. To establish the decomposition formula of the graph space, we obtain an alternate decomposition in terms of the *reference operator* of the abstract Friedrichs operator and then achieve the equality of both decompositions using the Hilbert space theory. This decomposition reveals that the room for choosing any boundary condition for the abstract Friedrichs operators is completely dependent on the study of kernels of the adjoint operators. In Section 3.3, we present a characterisation of the abstract Friedrichs operators in terms of a skew-symmetric operator and a bounded self-adjoint operator with strictly positive bottom. Let us emphasise the fact that the entire theory is equivalently applicable to the formal adjoint operator of a Friedrichs operator. In fact, we call them together a pair of abstract Friedrichs operators. In the same section, we discuss the *deficiency indices (defect numbers)* and prove that the kernels of the adjoints being *isomorphic* is equivalent to the existence of the same admissible boundary condition (the cone formalism) for the Friedrichs operator and its formal adjoint. In Section 3.4, we develop the von-Neumann extension theory for abstract Friedrichs operators and provide a complete classification of boundary conditions (even the closed ones) for abstract Friedrichs operators. When compared to the universal Grubb's extension theory (3.1), this approach turns out to be more suitable when studying an important class of bijective realisations with signed boundary maps. In Section 3.5, we provide an application of the developed theory in the case of symmetric operators.

One of our main concerns is the application of abstract theory to classical Friedrichs systems, including the classification of the boundary conditions of interest. In Section 3.6, we provide a more straightforward proof of the equivalence between the cone formalism and the boundary operator formalism for abstract Friedrichs operators. This allows us to construct a boundary operator for each boundary condition given by the cone formalism more directly. Then, we turn our attention to the classical Friedrichs systems on an interval (one-dimensional case). For the scalar case, dealing with some difficulty related to the singularity of the coefficients, we provide a complete analysis and classification of boundary conditions in Section 4.1. In the vectorial case, the problem becomes particularly challenging because of the non-smoothness of eigenvectors of the coefficient matrix. We develop some preliminary results using the concept of total projections, which enables us to define the boundary operator and the minimal space explicitly. The main result of this part is in connecting the dimensions of the kernels to the coefficient matrix evaluated at the end-points of the interval. We elaborate the results of this part in two examples, one is a 2×2 system of ordinary differential equations and another is a second order ordinary differential equation. This approach can be more helpful in dealing with the singular coefficients.

Finally, we turn our attention to the non stationary theory for Friedrichs systems. More precisely, we prove that the bijective realisations of abstract Friedrichs operators with signed boundary maps as well as the corresponding realisations of the skew-symmetric parts give rise to the generators of contractive C_0-semigroup. Moreover, some special bijective realisations, which are related to the skew-selfadjoint realisations of the skew-symmetric parts generate C_0-group and the skew-selfadjoint parts generate unitary C_0-group. For skew-symmetric operators the theory has been developed ([10]). However, our approach gives an alternate approach for the same theory and extends the theory to non skew-symmetric operators, which is again a demonstration of the strength of the von Neumann classification theory developed in Chapter 3. We illustrate the theory on some some examples.

Notation. Most of our notations are standard, let us only emphasise the following. For the sake of generality, in this dissertation we work on complex vector spaces. Thus, by $\mathscr{H}$ we denote a complex Hilbert space with scalar product $\langle\cdot\mid\cdot\rangle$, which we take to be linear in the first and anti-linear in the second entry. The corresponding norm is given by

$\|\cdot\| := \sqrt{\langle\cdot\mid\cdot\rangle}$. For $\mathscr{H} = \mathbb{C}^r$ we shall often use an alternative notation: $\langle \mathsf{x} \mid \mathsf{y}\rangle = \mathsf{x}\cdot\mathsf{y}$, $\mathsf{x},\mathsf{y} \in \mathbb{C}^r$. The topological (anti)dual $\mathscr{H}'$ will be identified with $\mathscr{H}$ by means of the usual duality (the Riesz representation theorem). For any Banach space $\mathscr{X}$ by ${}_{\mathscr{X}'}\langle\cdot,\cdot\rangle_{\mathscr{X}}$ we denote the corresponding dual product between $\mathscr{X}$ and its (anti)dual $\mathscr{X}'$. The annihilator of $S \subseteq \mathscr{X}$, denoted by S^0, is a closed subspace of $\mathscr{X}'$ given by $S^0 = \{f \in \mathscr{X}' : (\forall u \in S)\ {}_{\mathscr{X}'}\langle f,u\rangle_{\mathscr{X}} = 0\}$. For a subspace $\mathscr{Y} \subseteq \mathscr{X}$ we denote by $\mathrm{cl}_{\mathscr{X}}\,\mathscr{Y}$ its closure within $\mathscr{X}$.

For a densely defined linear operator $A : \mathscr{H} \to \mathscr{H}$ we denote by $\operatorname{dom}A$, $\ker A$, $\operatorname{ran}A$, $\overline{A}$, A^* its *domain*, *kernel*, *range* (or *image*), *closure* (if it exists), and *adjoint*, respectively. For $S \subseteq \mathscr{H}$, the *restriction* of A to S is denoted by $A|_S$. For two linear operators A, B in $\mathscr{H}$ by $A \subseteq B$ we mean that $\operatorname{dom}A \subseteq \operatorname{dom}B$ and $B|_{\operatorname{dom}A} = A$. By $\langle\cdot\mid\cdot\rangle_A := \langle\cdot\mid\cdot\rangle + \langle A\cdot\mid A\cdot\rangle$ we denote the *graph scalar product*, while the corresponding norm $\|\cdot\|_A := \sqrt{\langle\cdot\mid\cdot\rangle_A}$ is called the *graph norm*. If $A = A^*$, then A is said to be *self-adjoint*, while the infimum of its spectrum is called the *bottom*. The *identity* operator is denoted by $\mathbb{I}$. For a *direct* sum between two vector spaces we use the symbol $\dot{+}$. We write $\ominus$ for the *orthogonal difference* in order to express in which Hilbert space the orthogonal complement is taken.

For any complex number $z \in \mathbb{C}$ we denote by $\Re z$ and $\Im z$ the real and the imaginary part of z, respectively.

1. Classical Friedrichs systems

This chapter contains a brief overview of the theory of classical Friedrichs systems. Most of the content is from the paper by Friedrichs [41]. Generalisation of some results and more details on the theory can be found also in [42] and [52].

1.1. Definition and boundary conditions

Definition 1.1.1 (Classical Friedrichs systems)**.** For an open and bounded set $\Omega \subseteq \mathbb{R}^d$ with Lipschitz boundary Γ, let the quadratic matrix functions $\mathbf{A}_k \in W^{1,\infty}(\Omega; \mathrm{M}_r(\mathbb{C}))$, $k = 1, 2, \ldots, d$, and $\mathbf{B} \in L^\infty(\Omega; \mathrm{M}_r(\mathbb{C}))$ satisfy

$$\mathbf{A}_k = \mathbf{A}_k^* \qquad \text{on } \Omega \tag{F1}$$

and

$$(\exists \mu_0 > 0) \quad \mathbf{B} + \mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geq 2\mu_0 \mathbf{I} \qquad \text{a.e. on } \Omega\,. \tag{F2}$$

Then the first-order differential operator $L : L^2(\Omega)^r \longrightarrow \mathscr{D}'(\Omega)^r$ defined by

$$L\mathsf{u} \;:=\; \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{B}\mathsf{u} \tag{CFO}$$

(here derivatives are taken in the distributional sense) is called *the (classical) Friedrichs operator* or *the symmetric positive operator*, while (for a given $\mathsf{f} \in L^2(\Omega)^r$) the first-order system of partial differential equations $L\mathsf{u} = \mathsf{f}$ is called *the (classical) Friedrichs system* or *the symmetric positive system*.

Condition (F1) is the *symmetry condition* and (F2) is the *positivity condition*. The formal adjoint $\widetilde{L} : L^2(\Omega)^r \longrightarrow \mathscr{D}'(\Omega)^r$ of operator L is given by

$$\widetilde{L}\mathsf{u} \;:=\; -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k^* \mathsf{u}) + \Big(\mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k\Big)\mathsf{u}\,.$$

Definition 1.1.2 (Matrix-valued boundary conditions)**.** Under the assumptions in the definition of classical Friedrichs system, let $\boldsymbol{\nu} = (\nu_1, ..., \nu_d)$ be the outward unit normal on the boundary Γ and $\mathbf{M} : \Gamma \to \mathbf{M}_r$ be given *matrix-valued boundary field*, then the boundary condition is given by

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}|_{\Gamma} = 0\,,$$

where,

$$\mathbf{A}_{\boldsymbol{\nu}} := \sum_{k=1}^{d} \nu_k \mathbf{A}_k \in L^{\infty}(\Gamma; \mathbf{M}_r)\ .$$

Not all matrix-valued boundary fields $\mathbf{M}$ would lead to the well-posedness problem. Friedrichs proposed a concrete definition of the *admissible boundary condition* with respect to the matrix-valued boundary field.

Definition 1.1.3 (Admissible boundary condition (FM))**.** Let $\mathbf{M}$ be a matrix-valued boundary field. For a.e. $\mathbf{x} \in \mathbb{C}^r$ we introduce two conditions:

(FM1) Positivity condition:

$$(\forall \boldsymbol{\xi} \in \mathbb{C}^r) \quad (\mathbf{M}(\mathbf{x}) + \mathbf{M}^*(\mathbf{x}))\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq 0\,,$$

(FM2) Maximality condition:

$$\mathbb{C}^r = \ker\big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\big) + \ker\big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\big)\ .$$

If both conditions are satisfied, then the boundary condition

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}|_{\Gamma} = 0\,,$$

is called an *admissible boundary condition*.

Remark 1.1.4. (FM1) condition implies

$$(\forall \boldsymbol{\xi} \in \mathbb{C}^r) \quad \Re(\mathbf{M}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi}) \geq 0\,,$$
$$\text{and} \quad (\forall \boldsymbol{\xi} \in \mathbb{C}^r) \quad \Re(\mathbf{M}^*(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi}) \geq 0\,.$$

Remark 1.1.5. If $\boldsymbol{\xi} \in \ker\big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\big) \cap \ker\big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\big)$, then

$$\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} = \mathbf{M}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} = -\mathbf{M}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} = 0\,,$$

thus we have

$$\ker\big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\big) \cap \ker\big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\big) = \{\boldsymbol{\xi} \in \mathbb{C}^r :\ \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} = \mathbf{M}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} = 0\}\ .$$

The sole condition (FM1) is the criterion for *semi-admissibility*. If both conditions (FM1) and (FM2) hold for $\mathbf{M}$, then they also hold for $\mathbf{M}^*$. As a consequence, we have the following:

Remark 1.1.6. The boundary condition $(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}|_\Gamma = 0$ is (semi-)admissible if and only if the boundary condition $(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}^*)\mathsf{u}|_\Gamma = 0$ is (semi-)admissible. Where (by (F1)) we have

$$\mathbf{A}_{\boldsymbol{\nu}}^* \;=\; \mathbf{A}_{\boldsymbol{\nu}}\,.$$

Another, but equivalent concept for imposing boundary condition was introduced by Friedrichs and Lax in [42].

Definition 1.1.7. (Admissible boundary condition (FX)) Let $N = \{N(\mathbf{x}) : \mathbf{x} \in \Gamma\}$ be a family of subspaces of $\mathbb{C}^r$ of constant dimensions depending on the boundary Γ. For a.e. $\mathbf{x} \in \Gamma$, we define two conditions:

(FX1) (Positivity condition): $N(\mathbf{x})$ is non-negative with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, i.e.

$$(\forall \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq 0\,,$$

(FX2) (Maximality condition): There is no subspace of $\mathbb{C}^r$ which contains $N(\mathbf{x})$ properly, and non-negative with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$.

If both conditions are satisfied, then the admissible boundary condition is given by

$$\mathsf{u}(\mathbf{x}) \in N(\mathbf{x}), \quad \text{for a.e. } \mathbf{x} \in \Gamma\,.$$

Finally, the third set of boundary conditions (still equivalent to the previous ones) was introduced by Phillips and Sarason in [52]. In this setup, a dual subspace of $N(\mathbf{x})$ is introduced as $\widetilde{N}(\mathbf{x}) := (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^\perp$.

Definition 1.1.8. (Admissible boundary condition (FV)) For a.e. $\mathbf{x} \in \Gamma$, we define two conditions:

(FV1) (Sign condition): The subspace $N(\mathbf{x})$ is non-negative and the dual subspace $\widetilde{N}(\mathbf{x})$ non-positive with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, i.e.

$$(\forall \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq 0\,,$$

$$(\forall \boldsymbol{\xi} \in \widetilde{N}(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leq 0\,,$$

(FV2) (Maximality condition): Maximality condition is equivalent to the following condition:

$$\widetilde{N}(\mathbf{x}) = \big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x})\big)^{\perp} \quad \text{and} \quad N(\mathbf{x}) = \big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\widetilde{N}(\mathbf{x})\big)^{\perp} .$$

If both conditions are satisfied, then the admissible boundary condition is given by

$$\mathsf{u}(\mathbf{x}) \in N(\mathbf{x}), \quad \text{for a.e. } \mathbf{x} \in \Gamma .$$

The equivalence between (FX) and (FV) boundary conditions is straightforward. The sets $N(\mathbf{x})$ and $\widetilde{N}(\mathbf{x})$ are non-negative and non-positive with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, respectively. The condition of being mutually orthogonal with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$ is equivalent to both sets being maximal non-negative and maximal non-positive with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, respectively. If we set $N(\mathbf{x}) := \ker(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x}))$, then (FM) and (FX) conditions are also equivalent. The part that (FX) implies (FM) requires existence of non-unique projectors to construct the operator $\mathbf{M}(\mathbf{x})$, which was proved by Friedrichs himself in [41]. We refer to aforementioned paper for the details. However, a proof of equivalence of the corresponding boundary conditions in the case of abstract Friedrichs operators is discussed in chapters 2 and 3.

1.2. EXAMPLES OF CLASSICAL FRIEDRICHS SYSTEM

The concept of Friedrichs systems is a historical effort to provide one general model for various differential equations. An analysis for the Tricomi equation appearing in the transonic fluid flow can be found in the paper by Friedrichs [41]. Here we mention some well known examples, which can be cast into this framework. To elaborate the strength of the theory we picked one mixed type example (the Tricomi equation), one elliptic equation and one hyperbolic equation. For an example of a parabolic equation we refer to [7].

1.2.1. Tricomi's equation

The Tricomi equation (on a bounded domain $\Omega \subseteq \mathbb{R}^2$) is given by

$$y\partial_x^2 u + \partial_y^2 u = 0\,. \tag{1.1}$$

It is a second order linear partial differential equation of mixed type. For $y > 0$ the equation is elliptic, and for $y < 0$ it becomes hyperbolic. Interchanging the roles of positive and negative $y-$axes, we can rewrite the equation as

$$y\partial_x^2 u - \partial_y^2 u = 0\,. \tag{1.2}$$

Let us represent this equation as a classical Friedrichs system. Define, $v_1 := e^{-\lambda x}\partial_x u$ and $v_2 := e^{-\lambda x}\partial_y u$, $\mathsf{v} := [v_1, v_2]^\top$, the equation (1.2) can be written as the following system.

$$\left(\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}\partial_x + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\partial_y + \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}\lambda\right)\mathsf{v} = 0\,. \tag{1.3}$$

The system is symmetric, but not positive. Let us multiply it from the left by the matrix

$$\begin{bmatrix} 1 & y \\ 1 & 1 \end{bmatrix},$$

(which is a regular matrix for $y \neq 1$) to get

$$\left(\begin{bmatrix} y & y \\ y & 1 \end{bmatrix}\partial_x + \begin{bmatrix} -y & -1 \\ -1 & -1 \end{bmatrix}\partial_y + \begin{bmatrix} y & y \\ y & 1 \end{bmatrix}\lambda\right)\mathsf{v} = 0\,. \tag{1.4}$$

For

$$\mathbf{A}_1 = \begin{bmatrix} y & y \\ y & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} -y & -1 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1+\lambda y & \lambda y \\ \lambda y & \lambda \end{bmatrix},$$

the operator

$$L\mathsf{v} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{v}) + \mathbf{B}\mathsf{v}$$

is a classical Friedrichs operator. Indeed, (F1) condition is satisfied due to the symmetry of A_1 and A_2, and (F2) is satisfied because,

$$\mathbf{B} + \mathbf{B}^* + \partial_x \mathbf{A}_1 + \partial_y \mathbf{A}_2 = \begin{bmatrix} 1+2\lambda y & 2\lambda y \\ 2\lambda y & 2\lambda \end{bmatrix},$$

is positive definite for sufficiently small $\lambda > 0$.

1.2.2. Stationary diffusion equation

Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set with Lipschitz boundary Γ. Consider the following equation

$$-\operatorname{div}(\mathbf{A}\nabla u) + cu = f\,, \tag{1.5}$$

where, $\mathbf{A} \in W^{1,\infty}(\Omega)$ is a bounded and uniformly positive symmetric matrix, $f \in L^2(\Omega)$ and $c \in L^\infty(\Omega)$ is bounded and uniformly positive. Equation 1.5 can be written as a system of first order PDEs in the form

$$\begin{cases} \mathbf{p} = -\mathbf{A}\nabla u \\ \operatorname{div} \mathbf{p} + cu = f\,. \end{cases} \tag{1.6}$$

Let us define the coefficient matrices $\mathbf{A}_k = \mathsf{e}_k \otimes \mathsf{e}_{d+1} + \mathsf{e}_{d+1} \otimes \mathsf{e}_k \in \mathbf{M}_{d+1}(\mathbb{R})$ for $k = 1, \ldots, d$, where $(\mathsf{e}_1, \ldots, \mathsf{e}_{d+1})$ is the standard basis for $\mathbb{R}^{d+1}$, and the block-diagonal matrix-valued function

$$\mathbf{B} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ 0 & c \end{bmatrix}.$$

Here, $\mathbf{B}$ becomes uniformly positive. Let us define the operator

$$L\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{B}\mathsf{u}\,,$$

where, $\mathsf{u} = \begin{bmatrix} \mathbf{p} \\ u \end{bmatrix}$. Note that for $\mathsf{u} := \begin{bmatrix} -\mathbf{A}\nabla u \\ u \end{bmatrix}$ and $\mathsf{f} = \begin{bmatrix} f \\ 0 \end{bmatrix}$, (1.6) reads as

$$L\mathsf{u} = \mathsf{f}\ .$$

Let us check that L is a classical Friedrichs operator. Since, each A_k $(k = 1, ..., d)$ is symmetric, (F1) condition holds, while

$$\mathbf{B} + \mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k = 2\mathbf{B} \geq 2\alpha\mathbb{I}\,,$$

implying that (F2) holds as well.

1.2.3. Maxwell's equation in the elliptic regime

Let, $\Omega \subseteq \mathbb{R}^3$ be open, bounded with Lipschitz boundary Γ. $\sigma, \mu \in L^\infty(\Omega)$ are two positive functions uniformly bounded away from zero. For $f_1, f_2 \in L^2(\Omega;\mathbb{R}^3))$ and two unknown functions $H, E : \Omega \to \mathbb{R}^3$, consider the following system of PDE's

$$\begin{cases} \mu H + \nabla \times E = f_1\,, \\ \sigma E - \nabla \times H = f_2\ . \end{cases} \tag{1.7}$$

For $k = 1, 2, 3$ let us define the coefficient matrices $\mathbf{A}_k \in \mathrm{M}_6(\mathbb{R})$ by

$$\mathbf{A}_k = \left(\begin{array}{c|c} \mathbf{0} & R_k \\ \hline R_k^\top & \mathbf{0} \end{array} \right),$$

where, $R_k = [\varepsilon_{ikj}], 1 \leq i, j, k \leq 3$, ε_{ikj} being Levi-Civita permutations, making each $\mathbf{A}_k$ a symmetric matrix. The matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{bmatrix} \mu & 0 \\ 0 & \sigma \end{bmatrix}.$$

For

$$\mathsf{u} = [H, E]^\top, \quad \mathsf{f} = [f_1, f_2]^\top$$

the operator $L : L^2(\Omega;\mathbb{R}^6) \to L^2(\Omega;\mathbb{R}^6)$ defined by

$$L\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{B}\mathsf{u}$$

is a classical Friedrichs operator. Indeed, (F1) condition holds due to symmetry of each $\mathbf{A}_k$ and since

$$\mathbf{B}+\mathbf{B}^*+\sum_{k=1}^{d}\partial_k\mathbf{A}_k \;=\; 2\mathbf{B} \;=\; 2\begin{bmatrix}\mu & 0\\ 0 & \sigma\end{bmatrix},$$

where both μ,σ are uniformly bounded away from zero, condition (F2) also holds.

1.3. WELL-POSEDNESS

Under a semi-admissible boundary condition, the following uniqueness result was obtained for the strong solutions in [41], and later the results were slightly strengthened in [42] and [52]. Here we present the most general form of the corresponding results, while we still refer to the original results given in [41] (cf. [47]).

Theorem 1.3.1. [41, Theorem 3.1] If the classical Friedrichs system $L\mathsf{u} = \mathsf{f}$ admits a strong solution $\mathsf{u} \in C^1(\Omega)^r$ under the semi-admissible boundary condition $(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}|_\Gamma = 0$, then the solution is unique.

On the other hand, existence results are available for weak solutions only. Suppose $f \in L^2(\Omega)^r$, then $\mathsf{u} \in L^2(\Omega)^r$ is a *weak solution* of $L\mathsf{u} = f$ with the boundary condition $\mathbf{M}\mathsf{u} = 0$ if

$$\langle \mathsf{v} \mid f \rangle = \langle L^* \mathsf{v} \mid \mathsf{u} \rangle ,$$

for all $\mathsf{v} \in C^1(\Omega)^r$ satisfying $\mathbf{M}^* \mathsf{v} = 0$ at Γ.

Theorem 1.3.2. [41, Theorem 4.1] If $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$ is of constant rank and the boundary Γ is of class C^2, then for any $f \in L^2(\Omega)^r$ the classical Friedrichs system $L\mathsf{u} = f$ equipped with semi-admissible boundary condition $(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}|_\Gamma = 0$ admits a weak solution.

Finally, we have the following result regarding existence of (semi-)admissible boundary conditions.

Theorem 1.3.3. [41, Section 5] If $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$ is of constant rank near boundary Γ (Γ being of class C^2) and $L : L^2(\Omega)^r \longrightarrow \mathscr{D}'(\Omega)^r$ is a classical Friedrichs operator, then there exists an admissible boundary condition associated with L.

Remark 1.3.4. It was not clear that any given boundary condition associated with the classical Friedrichs operator can be realised as an admissible boundary condition in the given sense of admissibility.

To elaborate the admissibility criteria, let us consider the stationary diffusion equation in the form of a system given in (1.6) (for a detailed analysis on the same, we refer to

[20]). The classical Friedrichs operator $L : L^2(\Omega;\mathbb{R}^{d+1}) \to \mathscr{D}'(\Omega;\mathbb{R}^{d+1})$ is defined as

$$L\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{B}\mathsf{u},$$

while

$$\mathbf{A}_{\boldsymbol{\nu}} = \sum_{k=1}^{d} \boldsymbol{\nu}_k \mathbf{A}_k = \begin{bmatrix} 0 & \dots & 0 & \nu_1 \\ 0 & \dots & 0 & \nu_2 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & \nu_d \\ \nu_1 & \dots & \nu_d & 0 \end{bmatrix}.$$

The choice of matrices $\mathbf{M}$ gives different boundary conditions. For example, Dirichlet boundary condition can be imposed by choosing

$$\mathbf{M} = \begin{bmatrix} 0 & \dots & 0 & -\nu_1 \\ 0 & \dots & 0 & -\nu_2 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & -\nu_d \\ \nu_1 & \dots & \nu_d & 0 \end{bmatrix}.$$

The boundary condition corresponding to the above choice is given by

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}) \begin{bmatrix} \mathbf{p} \\ u \end{bmatrix} \Bigg|_{\Gamma} = 0, \tag{1.8}$$

which holds if and only if

$$\forall k = 1, \dots, d \qquad \nu_k u|_{\Gamma} = 0,$$

which is equivalent to $u|_{\Gamma} = 0$. With this information let us check the criteria for admissibility. For such $\mathbf{M}$, it is straightforward that we have

$$\forall \boldsymbol{\xi} \in \mathbb{R}^d \times \{0\} \qquad \mathbf{M}\boldsymbol{\xi} \cdot \boldsymbol{\xi} = 0,$$

hence (FM1) condition is satisfied. Moreover,

$$\ker(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}) = \{(\boldsymbol{\xi}_d, \xi_{d+1})^\top \in \mathbb{R}^{d+1} : \xi_{d+1} = 0\},$$
$$\ker(\mathbf{A}_{\boldsymbol{\nu}} + \mathbf{M}) = \{(\boldsymbol{\xi}_d, \xi_{d+1})^\top \in \mathbb{R}^{d+1} : \boldsymbol{\nu} \cdot \boldsymbol{\xi}_d = 0\}.$$

Clearly, $\mathbb{R}^{d+1} = \ker(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}) + \ker(\mathbf{A}_{\boldsymbol{\nu}} + \mathbf{M})$, which is the condition (FM2). Hence, the boundary condition

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}|_{\Gamma} = 0\,,$$

is admissible and for each $\mathsf{f} \in L^2(\Omega;\mathbb{R}^{d+1})$, the problem

$$\begin{cases} L\mathsf{u} = \mathsf{f}\,, \\ (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}|_{\Gamma} = 0\,. \end{cases}$$

is well-posed.

Remark 1.3.5. The choice of matrix $\mathbf{M}$ is generally not unique. Indeed, if we take

$$\mathbf{M} = \begin{bmatrix} 0 & \dots & 0 & -\nu_1 \\ 0 & \dots & 0 & -\nu_2 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & -\nu_d \\ \nu_1 & \dots & \nu_d & \alpha \end{bmatrix},$$

where $\alpha > 0$ is a constant, then the condition

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}) \begin{bmatrix} \mathbf{p} \\ u \end{bmatrix} \Bigg|_{\Gamma} = 0\,, \tag{1.9}$$

is fulfilled if and only if

$$\forall k = 1,...,d \qquad \nu_k u|_{\Gamma} = 0 \quad \text{and} \quad \alpha u|_{\Gamma} = 0\,,$$

which in turn is equivalent to $u|_{\Gamma} = 0$. Hence, this choice of $\mathbf{M}$ again corresponds to the Dirichlet boundary condition.

Obviously $-\mathbf{M}$ is admissible as well. In particular, it is easy to see that this choice leads to the homogeneous Neumann boundary condition $(\boldsymbol{\nu}\cdot\mathbf{p})|_{\Gamma} = 0$.

The Robin boundary condition can be imposed by choosing

$$\mathbf{M} = \begin{bmatrix} 0 & \dots & 0 & \nu_1 \\ 0 & \dots & 0 & \nu_2 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & \nu_d \\ -\nu_1 & \dots & -\nu_d & 2\alpha \end{bmatrix},$$

where $\alpha > 0$ is a constant. The boundary condition

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}) \begin{bmatrix} \mathbf{p} \\ u \end{bmatrix} \Big|_{\Gamma} \;=\; 0\,, \tag{1.10}$$

is satisfied if and only if $(\boldsymbol{\nu} \cdot \mathbf{p} - \alpha u)|_{\Gamma} = 0$. In terms of the original equation (1.5), it corresponds to the Robin boundary condition

$$(\boldsymbol{\nu} \cdot \nabla u + \alpha u)|_{\Gamma} = 0\,. \tag{1.11}$$

In this case

$$\begin{aligned}
\ker(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}) &= \{(\boldsymbol{\xi}_d, \xi_{d+1})^{\top} \in \mathbb{R}^{d+1} : \boldsymbol{\nu} \cdot \boldsymbol{\xi}_d - \alpha \xi_{d+1} = 0\}\,, \\
\ker(\mathbf{A}_{\boldsymbol{\nu}} + \mathbf{M}) &= \{(\boldsymbol{\xi}_d, \xi_{d+1})^{\top} \in \mathbb{R}^{d+1} : \xi_{d+1} = 0\}\,.
\end{aligned}$$

2. Abstract Friedrichs operatos

In the theory of classical Friedrichs operators the matrix-valued boundary field that enforces the boundary conditions is not uniquely defined, hence the theory is not intrinsic. Moreover, the boundary values (traces) of the solutions are involved whose meaning are not clear. In 2007, Ern, Guermond and Caplain revisited the theory of Friedrichs operators to avoid invoking the traces at the boundary. They introduced the theory of abstract Friedrichs operators as acting on Hilbert spaces in [39] for real vector spaces, while it has been studied over complex vector spaces in [5]. We elaborate the theory in the complex setting and provide proofs of the results to supplement some technical things and make the material self contained.

2.1. Definition and Hilbert space formulation

Let us start with the definition of abstract Friedrichs operators (relation to the classical Framework is disscused in Subsection 2.6.1).

Definition 2.1.1. A (densely defined) linear operator T on a complex Hilbert space $\mathscr{H}$ is called an *abstract Friedrichs operator* if it admits another (densely defined) linear operator $\widetilde{T}$ on $\mathscr{H}$ with the following properties:

(T1) T and $\widetilde{T}$ have a common domain $\mathscr{D}$, which is dense in $\mathscr{H}$, satisfying

$$\langle T\varphi \mid \psi\rangle = \langle \varphi \mid \widetilde{T}\psi\rangle , \qquad \varphi,\psi \in \mathscr{D};$$

(T2) there is a constant $c > 0$ for which

$$\|(T+\widetilde{T})\varphi\| \leqslant c\|\varphi\| , \qquad \varphi \in \mathscr{D};$$

(T3) there exists a constant $\mu_0 > 0$ such that

$$\langle (T+\widetilde{T})\varphi \mid \varphi \rangle \;\geqslant\; 2\mu_0 \|\varphi\|^2 \,, \qquad \varphi \in \mathscr{D} \,.$$

The pair $(T,\widetilde{T})$ is referred to as a *joint pair of abstract Friedrichs operators* (the definition is indeed symmetric in T and $\widetilde{T}$).

By (T1), $T \subseteq \widetilde{T}^*$ and $\widetilde{T} \subseteq T^*$. Which means both T^* and $\widetilde{T}^*$ are densely defined and thus T and $\widetilde{T}$ are closable. $(\overline{T}, \mathscr{D}(\overline{T}))$ and $(\overline{\widetilde{T}}, \mathscr{D}(\overline{\widetilde{T}}))$ are their closures respectively in $(\mathscr{H}, \langle \cdot \mid \cdot \rangle)$. The completion of $\mathscr{D}$ with respect to the *graph norm* $\| \cdot \|_T$ defined by the *graph inner product* $\langle \cdot \mid \cdot \rangle_T := \langle \cdot \mid \cdot \rangle + \langle T\cdot \mid T\cdot \rangle$ is denoted by $\mathscr{W}_0$, by (T2), the graph norms $\| \cdot \|_T$ and $\| \cdot \|_{\widetilde{T}}$ are equivalent and thus the completion of $\mathscr{D}$ with respect to the graph norm $\| \cdot \|_{\widetilde{T}}$ is again $\mathscr{W}_0$. We also call the space $(\mathscr{W}_0, \langle \cdot \mid \cdot \rangle_T)$ (or equivalently, $(\mathscr{W}_0, \langle \cdot \mid \cdot \rangle_{\widetilde{T}})$) as the *minimal domain* of the abstract Friedrichs operators T and $\widetilde{T}$. Before we discuss the adjoint operators, let us observe that the minimal space $\mathscr{W}_0$ is continuously embedded in $\mathscr{H}$ (since T is closable) and the image (of the embedding) is precisely $\mathscr{D}(\overline{T}) = \mathscr{D}(\overline{\widetilde{T}})$. Let us consider the following construction for a given pair of abstract Friedrichs operators $(T,\widetilde{T})$, introduced in [39]:

- The operators T and $\widetilde{T}$ extend uniquely to bounded linear operators from $\mathscr{W}_0$ to $\mathscr{H}$, say T_0 and $\widetilde{T}_0$ respectively. In fact, these extensions coincide with $\overline{T}$ and $\overline{\widetilde{T}}$ respectively.

- We have the Gel'fand triplet

  $$\mathscr{W}_0 \hookrightarrow \mathscr{H} \equiv \mathscr{H}' \hookrightarrow \mathscr{W}_0' \,,$$

  where $\mathscr{H}'$ and $\mathscr{W}_0'$ are the (anti-)duals of $\mathscr{H}$ and $\mathscr{W}_0$ respectively. Due to the Reisz representation theorem we identify $\mathscr{H} \equiv \mathscr{H}'$. The operators $T_0, \widetilde{T}_0 : \mathscr{W}_0 \to \mathscr{H}$ are continuous linear operators, where $\mathscr{W}_0$ is equipped with the graph-norm topology and $\mathscr{H}$ is with its usual topology. Let

  $$T_0',\, \widetilde{T}_0{}' :\; \mathscr{H} \to \mathscr{W}_0' \,,$$

  be the Banach adjoints of T_0 and $\widetilde{T}_0$, respectively, i.e.

  $$\begin{aligned} (\forall u \in \mathscr{H})(\forall \varphi \in \mathscr{W}_0) \quad & {}_{\mathscr{W}_0'}\langle T_0' u, \varphi \rangle_{\mathscr{W}_0} := \langle u \mid T_0 \varphi \rangle \,, \\ \text{and} \quad & {}_{\mathscr{W}_0'}\langle \widetilde{T}_0' u, \varphi \rangle_{\mathscr{W}_0} := \langle u \mid \widetilde{T}_0 \varphi \rangle \,, \end{aligned} \tag{2.1}$$

where ${}_{\mathscr{W}_0'}\langle\cdot,\cdot\rangle_{\mathscr{W}_0}$ represents the pairing between $\mathscr{W}_0$ and its dual $\mathscr{W}_0'$. We clearly have $T_0 = \widetilde{T}_0'|_{\mathscr{W}_0}$. Therefore, $T_0 : \mathscr{W}_0 \to \mathscr{H} \hookrightarrow \mathscr{W}_0'$ is a continuous linear operator from $(\mathscr{W}_0, \|\cdot\|_{\mathscr{H}})$ to $\mathscr{W}_0'$, whose unique extension to the whole $\mathscr{H}$ is the operator $\widetilde{T}_0'$ (the same holds for $\widetilde{T}_0$ and T_0').

- **Graph Space**: Since, $\mathscr{H} \subseteq \mathscr{W}_0'$, it makes sense to define the following:

  The space

$$\mathscr{W} := \{u \in \mathscr{H} :\ \widetilde{T_0}' u \in \mathscr{H}\} = \{u \in \mathscr{H} :\ T_0' u \in \mathscr{H}\} \subseteq \mathscr{H}\,, \tag{2.2}$$

  (the equality is due to condition (T2)) equipped with the graph inner product is a Hilbert space and called the *graph space*. Let us note that $T_1 := \widetilde{T}_0'|_{\mathscr{W}} = \widetilde{T}_0^* = \widetilde{T}^*$ and $\widetilde{T}_1 := T_0'|_{\mathscr{W}} = T_0^* = T^*$ by Lemma 2.1.3 below. Since $\mathscr{W}_0 \subseteq \mathscr{W}$, by $T_0 = \widetilde{T}_0'|_{\mathscr{W}_0}$ we have $T_0 \subseteq T_1$ and $\widetilde{T}_0 \subseteq \widetilde{T}_1$. Since, $\mathscr{W} = \operatorname{dom} T_0^* = \operatorname{dom} \widetilde{T}_0^*$, it is clear that it is a Hilbert space when equipped with the graph inner product. The graph space $(\mathscr{W}, \langle\cdot\mid\cdot\rangle_{T_1})$ is also called the *maximal domain* related to the abstract Friedrichs operators and it contains $\mathscr{W}_0$.

- In particular, the restriction of Banach adjoints to $\mathscr{D}$ are $\widetilde{T}_0'|_{\mathscr{D}} = T$ and $T_0'|_{\mathscr{D}} = \widetilde{T}$, and hence $(\widetilde{T}_0' + T_0')|_{\mathscr{D}} = T + \widetilde{T}$. The operator $\overline{T+\widetilde{T}}$ is everywhere defined on $\mathscr{H}$ and from the condition $(T2)$ it is bounded, while on the dense subsapce $\mathscr{D}$, it coincides with $T + \widetilde{T}$ and so with $\widetilde{T}_0' + T_0'$. Due to continuous embedding of $\mathscr{H}$ into $\mathscr{W}_0'$, both maps are continuous. Thus on $\mathscr{H}$, $\widetilde{T}_0' + T_0' = \overline{T+\widetilde{T}}$. The continuity of the operator also gives the equivalence of the norms $\|\cdot\|_{\widetilde{T}_0'}$ and $\|\cdot\|_{T_0'}$. Moreover, due to density of $\mathscr{W} = \operatorname{dom}(T^* + \widetilde{T}^*)$ on $\mathscr{H} = \operatorname{dom}(\overline{T+\widetilde{T}})$, we obtain $T^* + \widetilde{T}^* \subseteq (T+\widetilde{T})^*$. Since, $T + \widetilde{T} \subseteq T_1 + \widetilde{T}_1 = T^* + \widetilde{T}^*$, we conclude the symmetry of $\overline{T+\widetilde{T}}$. Therefore, $\overline{T+\widetilde{T}}$ is an everywhere defined, bounded and symmetric operator and hence it is selfadjoint. By part (ii), $\overline{T+\widetilde{T}}|_{\mathscr{W}} = (\widetilde{T}_0' + T_0')|_{\mathscr{W}} = T^* + \widetilde{T}^*$ and the strictly positive bottom part is due to boundedness and condition (T3).

For an illustration, for $\mathscr{H} = L^2(\Omega)$ and a certain choice of operators we can achieve that $\mathscr{W}$ and $\mathscr{W}_0$ are Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ respectively.

The preceding construction is rephrased as *Hilbert space construction of Friedrichs operators*. Here we summarise some properties related to the construction. The arguments are already given above, while for a more detailed proof we refer to [9, Theorem 7].

Theorem 2.1.2. Let $(T,\widetilde{T})$ be a joint pair of abstract Friedrichs operators on Hilbert space $\mathscr{H}$ and $(T_0,\widetilde{T}_0)$ and $(T_1,\widetilde{T}_1)$ are as in the preceding construction.

(i) $T_0 = \overline{T}$ and $\widetilde{T}_0 = \overline{\widetilde{T}}$. The pair $(\overline{T},\overline{\widetilde{T}})$ satisfies condition (T1)–(T3) on $\mathscr{W}_0$ and the corresponding graph norms $\|\cdot\|_{\overline{T}}$ and $\|\cdot\|_{\overline{\widetilde{T}}}$ are equivalent.

(ii) $\widetilde{T}_0'|_{\mathscr{W}_0} = \overline{T}$, $T_0'|_{\mathscr{W}_0} = \overline{\widetilde{T}}$ and $\widetilde{T}_0' + T_0' = \overline{T+\widetilde{T}}$ is a (everywhere defined) bounded operator in $\mathscr{H}$. The graph norms $\|\cdot\|_{\widetilde{T}_0'}$ and $\|\cdot\|_{T_0'}$ are equivalent in $\mathscr{W}$.

(iii) $T_1 = \widetilde{T}^*$ and $\widetilde{T}_1 = T^*$. $\overline{T+\widetilde{T}}$ is a bounded selfadjoint operator in $\mathscr{H}$ with strictly positive bottom and $\overline{T+\widetilde{T}}|_{\mathscr{W}} = T^* + \widetilde{T}^*$.

Lemma 2.1.3. Let $A : \mathrm{dom} A \subseteq \mathscr{H} \to \mathscr{H}$ be a closed densely defined operators on $\mathscr{H}$. If we denote by $A' : \mathscr{H} \to (\mathrm{dom} A)'$, the Banach adjoint of A, then the domain of the (Hilbert) adjoint of A is given by $\mathrm{dom} A^* = \{u \in \mathscr{H} : A'u \in \mathscr{H}\}$.

Proof. Let $A_1 = A'|_{\{u\in\mathscr{H}:A'u\in\mathscr{H}\}}$. For any $u \in \mathscr{H}$ such that $A'u \in \mathscr{H}$ and $v \in \mathrm{dom} A$, we have

$$\langle A_1 u \mid v \rangle = {}_{\mathscr{H}'}\langle A'u, v\rangle_{\mathscr{H}} = \langle u \mid Av \rangle ,$$

which means $A_1 \subseteq A^*$. Conversely, for any $u \in \mathrm{dom} A^*$ and $v \in \mathrm{dom} A$, we have

$${}_{(\mathrm{dom} A)'}\langle A'u, v\rangle_{\mathrm{dom} A} = \langle u \mid Av \rangle = \langle A^* u \mid v \rangle$$

thus $A'u = A^*u \in \mathscr{H}$, implying $A^* \subseteq A_1$. Hence, $\mathrm{dom} A^* = \{u \in \mathscr{H} : A'u \in \mathscr{H}\}$.

■

It is worth noting that in all the previous discussions the condition (T3) is used only to obtain the strictly positive bottom result for the operator $\overline{T+\widetilde{T}}$. We establish most of the results with only conditions (T1)–(T2) and so going forward we shall be specific with the use of condition (T3).

By previous analysis of the construction, the concept of abstract Friedrichs operators can be entirely formulated in the language of Hilbert spaces. Indeed, (T1) can be seen as a symmetry condition and together with (T2), we get an everywhere defined selfadjoint bounded operator. The following characterisation is given in [9, Theorem 8].

Theorem 2.1.4. A pair of operators $(T,\widetilde{T})$ on a Hilbert space $\mathscr{H}$ is a joint pair of abstract Friedrichs operators if and only if $T \subseteq \widetilde{T}^*$, $\widetilde{T} \subseteq T^*$ and $\overline{T+\widetilde{T}}$ is an everywhere defined, bounded, selfadjoint operator with strictly positive bottom.

Remark 2.1.5. Note that we can formulate the definition of abstract Friedrichs operators only in terms of a single operator. Indeed, for a densely defined operator T on $\mathscr{H}$, let us define $\widetilde{T} = T^*|_{\operatorname{dom} T}$. Then we can say that T is an abstract Friedrichs operator if and only if the pair $(T,\widetilde{T})$ is a joint pair of abstract Friedrichs operators. A more explicit description is given in Theorem 3.3.1 below.

Operators A, B on a Hilbert space $\mathscr{H}$ with the property that $A \subseteq B^*$ and $B \subseteq A^*$ are called *dual pairs* or *symmertic pairs*. Thus, by the previous theorem the operators forming a pair of abstract Friedrichs operators are dual pairs (in fact it is equivalent to condition (T1)). Since, a given pair of abstract Friedrichs operators $(T,\widetilde{T})$ on $\mathscr{H}$ are closable, we can start with the assumption that the operators $(T,\widetilde{T})$ are closed, which will often be the case in the rest of the dissertation.

The boundary value problem in this abstract setting can be interpreted as follows. For any abstract Friedrichs operator T in a Hilbert space $\mathscr{H}$ find a domain $\mathscr{V}$ (i.e. boundary conditions), such that the abstract problem:

$$\textit{for a given } f \in \mathscr{H} \textit{ find } u \in \mathscr{V} \text{ such that } Tu = f\,,$$

is well-posed. After the reformulation in the sense of Hilbert space theory, we can be more precise. The well-posedness problem can be formulated in the following way:

to find restrictions of T_1 to a suitable subspace $\mathscr{V}$, with $\mathscr{W}_0 \subseteq \mathscr{V} \subseteq \mathscr{W}$, such that $T_1|_{\mathscr{V}} : \mathscr{V} \to \mathscr{H}$ is an isomorphism, namely a continuous bijection, when equipped with the graph-norm topology.

In the above we used that the restriction of T_1 to any closed subspace of $\mathscr{W}$ is continuous when the domain $\mathscr{V}$ is equipped with the graph norm. Let us emphasise that $T_1|_{\mathscr{V}}$ is called the *realisation* or *extension* of T (on $\mathscr{V}$). So, the main goal is to obtain the bijectivity condition. It turns out that the above question for T_1 is immediately related to the same question for $\widetilde{T}_1$.

In the next section we discuss about the boundary operators, which in the case of classical Friedrichs operators carry the information about the boundary conditions.

2.2. THE BOUNDARY OPERATOR

In this section and the rest of the chapter, we shall use the revised notation $T = T_0$ and $\widetilde{T} = \widetilde{T}_0$.

Definition 2.2.1. Let $(\mathscr{W}, \langle \cdot \mid \cdot \rangle_{T_1})$ be the graph space related to a joint pair of abstract Friedrichs operators $(T, \widetilde{T})$ in a Hilbert space $\mathscr{H}$. The operator $D : (\mathscr{W}, \langle \cdot \mid \cdot \rangle_{T_1}) \to (\mathscr{W}, \langle \cdot \mid \cdot \rangle_{T_1})'$ defined by

$$(\forall u, v \in \mathscr{W}) \qquad {}_{\mathscr{W}'}\langle Du, v \rangle_{\mathscr{W}} := \langle T_1 u \mid v \rangle - \langle u \mid \widetilde{T}_1 v \rangle \,,$$

is called the *boundary operator* associated with the pair $(T_1, \widetilde{T}_1)$.

Both T_1 and $\widetilde{T}_1$ are bounded linear operators in $\mathscr{L}(\mathscr{W}; \mathscr{H})$ and so the operator D is well-defined and $D \in \mathscr{L}(\mathscr{W}; \mathscr{W}')$.

Remark 2.2.2. The boundary operator in the case of differential operators can be compared to the integral over boundary in the Green's identity.

Here we summarise some properties of the boundary operator D (see [5, Lemma 1] and [39, Lemma 2.4].

Lemma 2.2.3. Let $(T, \widetilde{T})$ satisfies conditions (T1)–(T2), then the boundary operator D satisfies the following:

(i) $(\forall u, v \in \mathscr{W}) \quad {}_{\mathscr{W}'}\langle Du, v \rangle_{\mathscr{W}} = \overline{{}_{\mathscr{W}'}\langle Dv, u \rangle_{\mathscr{W}}}\,,$

(ii) $\ker D = \mathscr{W}_0\,,$

(iii) $\operatorname{ran} D = \mathscr{W}_0^0\,,$

where ^{0} stands for the annihilator.

Proof. (i) Let $u, v \in \mathscr{W}$, then

$$\begin{aligned} {}_{\mathscr{W}'}\langle Du, v \rangle_{\mathscr{W}} - \overline{{}_{\mathscr{W}'}\langle Dv, u \rangle_{\mathscr{W}}} &= (\langle T_1 u \mid v \rangle - \langle u \mid \widetilde{T}_1 v \rangle) - (\langle \widetilde{T}_1 u \mid v \rangle - \langle v \mid T_1 u \rangle) \\ &= \langle (T_1 + \widetilde{T}_1) u \mid v \rangle - \langle u \mid (T_1 + \widetilde{T}_1) v \rangle = 0 \,, \end{aligned}$$

in the last equality we used part (iii) of Theorem 2.1.2.

(ii) Let $\varphi \in \mathscr{W}_0$, then for any $v \in \mathscr{W}$, we have

$$\begin{aligned} {}_{\mathscr{W}'}\langle D\varphi, v\rangle_{\mathscr{W}} &= \langle T_1\varphi \mid v\rangle - \langle \varphi \mid \widetilde{T}_1 v\rangle = \langle T\varphi \mid v\rangle - \langle \varphi \mid \widetilde{T}_1 v\rangle \\ &= \langle \varphi \mid \widetilde{T}_1 v\rangle - \langle \varphi \mid \widetilde{T}_1 v\rangle = 0\,, \end{aligned}$$

where in the second equality we used $T_1|_{\mathscr{W}_0} = T$ and the third equality is due to $T^* = \widetilde{T}_1$. Thus, $\mathscr{W}_0 \subseteq \ker D$. Conversely, due to the relation $\ker D = (\operatorname{ran} D)^0$, which holds by part (i), it is enough to prove that $\mathscr{W}_0^0 \subseteq \operatorname{ran} D$. Let $u \in \mathscr{W}_0^0$, by the Riesz representation theorem, there exists some $x \in \mathscr{W}$ such that for any $v \in \mathscr{W}$

$${}_{\mathscr{W}'}\langle u, v\rangle_{\mathscr{W}} = \langle x \mid v\rangle + \langle T_1 x \mid T_1 v\rangle\ .$$

For any $\varphi \in \mathscr{W}_0$ we have

$${}_{\mathscr{W}_0'}\langle T'T_1 x, \varphi\rangle_{\mathscr{W}_0} = \langle T_1 x \mid T\varphi\rangle = \langle T_1 x \mid T_1\varphi\rangle = {}_{\mathscr{W}'}\langle u, \varphi\rangle_{\mathscr{W}} - \langle x \mid \varphi\rangle = -\langle x \mid \varphi\rangle\,,$$

where $T' : \mathscr{H} \to \mathscr{W}_0'$ is the Banach adjoint of T and in the penultimate inequality we used the identity above. Therefore, $T'T_1 x = -x \in \mathscr{H}$ implying $w := T_1 x \in \mathscr{W}$. Hence, we have $\widetilde{T}_1 w = T' w = -x$. Moreover, for any $z \in \mathscr{W}$ we get

$$\begin{aligned} \overline{{}_{\mathscr{W}'}\langle Dw, z\rangle_{\mathscr{W}}} &= {}_{\mathscr{W}'}\langle Dz, w\rangle_{\mathscr{W}} = \langle T_1 z \mid w\rangle - \langle z \mid \widetilde{T}_1 w\rangle \\ &= \langle T_1 z \mid T_1 x\rangle + \langle z \mid x\rangle = \overline{{}_{\mathscr{W}'}\langle u, z\rangle_{\mathscr{W}}}\ . \end{aligned}$$

This gives us $u = Dw \in \operatorname{ran} D$ and thus $\mathscr{W}_0^0 \subseteq \operatorname{ran} D$.

(iii) From the proof in part (ii), we have

$$\mathscr{W}_0^0 \subseteq \operatorname{ran} D \subseteq (\ker D)^0 \subseteq \mathscr{W}_0^0.$$

Hence, $\operatorname{ran} D = \mathscr{W}_0^0$.

■

Remark 2.2.4. For any $u \in \mathscr{W}$, ${}_{\mathscr{W}'}\langle Du, u\rangle_{\mathscr{W}}$ and $\langle (T + \widetilde{T})u \mid u\rangle$ are real numbers.

In the context of this theory, the boundary operator plays a vital role. The structure given by the boundary operator, namely

$$[\,\cdot \mid \cdot\,] := {}_{\mathscr{W}'}\langle D(\cdot), \cdot\rangle_{\mathscr{W}} \tag{2.3}$$

defines an indefinite inner product on $\mathscr{W}$. Indeed, the linearity follows from the linearity of the map D, while conjugate symmetry is due to Lemma 2.2.3(i). Moreover, by Lemma 2.2.3(ii) we have that all vectors from $\mathscr{W}_0$ are neutral, i.e. for any $u \in \mathscr{W}_0$ we have $[u \mid u] = 0$. Therefore, $(\mathscr{W}, [\cdot \mid \cdot])$ is indeed an indefinite inner product space and also it is degenerate (see Definition 5.4.7).

Lemma 2.2.5. Let $\mathscr{W}$ be the graph space related to a joint pair of abstract Friedrichs operators $(T, \widetilde{T})$ in a Hilbert space $\mathscr{H}$ and $[\cdot \mid \cdot] = {}_{\mathscr{W}'}\langle D(\cdot), \cdot \rangle_{\mathscr{W}}$ is the indefinite inner product defined by the boundary operator. Then,

(i) $(\mathscr{W}, [\cdot \mid \cdot])$ is an *indefinite inner product space*.

(ii) Orthogonal complement of a subset S of $\mathscr{W}$ with respect to $[\cdot \mid \cdot]$ is defined by

$$S^{[\perp]} := \{u \in \mathscr{W} : \ (\forall v \in S) \ [u \mid v] = 0\}.$$

Moreover, $S^{[\perp]}$ is a closed subspace of $\mathscr{W}$ with respect to the graph norm.

(iii) For $L \subseteq S \subseteq \mathscr{W}$, $S^{[\perp]} \subseteq L^{[\perp]}$, $\mathscr{W}^{[\perp]} = \mathscr{W}_0$ and $\mathscr{W}_0^{[\perp]} = \mathscr{W}$.

Proof. The proof of part (i) is already argued before the lemma, so let us discuss the remaining two parts.

(ii) Let (u_n) be a sequence in $S^{[\perp]}$ and u be the limit in $\mathscr{W}$ (with respect to the graph norm). This means $(u_n), (Tu_n)$ converge to u and Tu in $\mathscr{H}$, respectively (as T is closed). We have for any $v \in S$ and $n \in \mathbb{N}$, $[u_n \mid v] = 0$. Hence,

$$0 = \lim_{n\to\infty} [u_n \mid v] = \lim_{n\to\infty} (\langle Tu_n \mid v \rangle - \langle u_n \mid \widetilde{T}v \rangle) = \langle Tu \mid v \rangle - \langle u \mid \widetilde{T}v \rangle = [u \mid v],$$

which means $u \in S^{[\perp]}$ and thus $S^{[\perp]}$ is closed with respect the graph norm.

(iii) Let $u \in S^{[\perp]}$, which means for any $v \in S$ we have $[u \mid v] = 0$. In particular, for any $v \in L$ we have $[u \mid v] = 0$. So, $u \in L^{[\perp]}$ and hence $S^{[\perp]} \subseteq L^{[\perp]}$. From parts (i) and (ii) of Lemma 2.2.3 we have $\mathscr{W}^{[\perp]} = \mathscr{W}_0$ and $\mathscr{W}_0^{[\perp]} = \mathscr{W}$.

∎

As a consequence of 2.2.5(iii), the quotient space $\mathscr{W}/\mathscr{W}_0$ equipped with the indefinite inner product $[\,\hat{u}\mid\hat{v}\,]_{\widehat{\mathscr{W}}} := [\,u\mid v\,]$, becomes non-degenerate. In particular, it was recognised in [3] that this quotient space is a *Kreĭn space* (not all non-degenerate spaces are Kreĭn spaces). While we refer to Appendix for a brief review of the *Kreĭn space* theory, here we present a proof of the above statement.

Proposition 2.2.6. [3, Lemma 8] The quotient space $\widehat{\mathscr{W}} = \mathscr{W}/\mathscr{W}_0$ is a *Kreĭn space* and the corresponding inner product on $\widehat{\mathscr{W}}$ is defined as

$$[\,\hat{u}\mid\hat{v}\,]_{\widehat{\mathscr{W}}} := [\,u\mid v\,],$$

where $\hat{u} = u + \mathscr{W}_0,\ \hat{v} = v + \mathscr{W}_0$ and $u, v \in \mathscr{W}$ are representatives of $\hat{u}, \hat{v}$, respectively.

Proof. In order to prove that $\widehat{\mathscr{W}}$ is a Kreĭn space, we show that it admits a Gramm operator G (see Definition 5.4.13) with closed range and $\ker G = \mathscr{W}_0$.

Since $\mathscr{W}$ equipped with the graph inner product is a Hilbert space, by the Riesz representation theorem, we have the isomorphism $J : \mathscr{W}' \to \mathscr{W}$ such that

$$(\forall f \in \mathscr{W}')(\forall u \in \mathscr{W}) \quad {}_{\mathscr{W}'}\langle f, u\rangle_{\mathscr{W}} = \langle J(f) \mid u\rangle_{\mathscr{W}}\,.$$

Let us define $G := J \circ D$. G is obviously continuous on $\mathscr{W}$ (equipped with the graph norm) and $\ker G = \ker D = \mathscr{W}_0$. For any $u, v \in \mathscr{W}$, we have

$$\langle Gu \mid v\rangle_{\mathscr{W}} = \langle J(D(u)) \mid v\rangle_{\mathscr{W}} = {}_{\mathscr{W}'}\langle Du, v\rangle_{\mathscr{W}} = [\,u\mid v\,]\,.$$

Thus, G is a Gramm operator on $\mathscr{W}$ (it is selfadjoint, since D is symmetric). Since $\operatorname{ran} G = J(\operatorname{ran} D) = J(\mathscr{W}_0^0)$ and J is an isomorphism, $\operatorname{ran} G$ is closed. Hence, by Theorem 5.4.14, $\widehat{\mathscr{W}} = \mathscr{W}/\mathscr{W}_0$ is a Kreĭn space. ■

Remark 2.2.7. There exists a *canonical (fundamental)* decomposition of the Kreĭn space $(\widehat{\mathscr{W}}, [\,\cdot\mid\cdot\,]_{\widehat{\mathscr{W}}})$. That is there exist subspaces $X_+, X_- \subseteq \widehat{\mathscr{W}}$ with $X_+ \cap X_- = \{0\}$ such that

$$\widehat{\mathscr{W}} = X_+ [\dot{+}] X_-\,,$$

where $(X_+, [\,\cdot\mid\cdot\,]_{\widehat{\mathscr{W}}})$ and $(X_-, -[\,\cdot\mid\cdot\,]_{\widehat{\mathscr{W}}})$ are Hilbert spaces and $[\dot{+}]$ denotes $[\,\cdot\mid\cdot\,]_{\widehat{\mathscr{W}}}$-orthogonal direct sum.

Using the operator-theoretic approach in Hilbert spaces, our goal is to obtain well-posedness and classification results for abstract Friedrichs operators and apply them to classical Friedrichs operators.

2.3. THE CONE FORMALISM

Any vector $u \in \mathscr{W}$ can be characterised as positive, negative or neutral with respect to the indefinite inner product $[\cdot \mid \cdot]$ depending on the sign of $[u \mid u]$. We can decompose the indefinite inner product space $(\mathscr{W}, [\cdot \mid \cdot])$ in the following two sets (cones)

$$\mathscr{W}^+ := \{u \in \mathscr{W} : \ [u \mid u] \geq 0\} \qquad \text{and} \qquad \mathscr{W}^- := \{u \in \mathscr{W} : \ [u \mid u] \leq 0\}\,. \tag{2.4}$$

Let us introduce two assumptions on which the cone formalism is based.

Definition 2.3.1 ((V)-boundary conditions)**.** Let $(T, \widetilde{T})$ be a joint pair of closed abstract Friedrichs operators on a Hilbert space $\mathscr{H}$. A pair of subspaces $(\mathscr{V}, \widetilde{\mathscr{V}})$ of the graph space $\mathscr{W}$ is said to allow (V)-*boundary conditions* related to $(T, \widetilde{T})$ if the following conditions are satisfied:

(V1) The boundary operator has *opposite* signs on these spaces. More precisely, $\mathscr{V} \subseteq \mathscr{W}^+$ and $\widetilde{\mathscr{V}} \subseteq \mathscr{W}^-$, i.e.

$$\begin{aligned} (\forall u \in \mathscr{V}) \qquad [u \mid u] \ &\geq \ 0\,, \\ (\forall v \in \widetilde{\mathscr{V}}) \qquad [v \mid v] \ &\leq \ 0\,. \end{aligned}$$

(V2) The subspaces $\mathscr{V}, \widetilde{\mathscr{V}}$ are $[\,\cdot\,]$-orthogonal complement to each other, i.e.

$$\mathscr{V} \ = \ \widetilde{\mathscr{V}}^{[\perp]} \quad \text{and} \quad \widetilde{\mathscr{V}} \ = \ \mathscr{V}^{[\perp]}\,.$$

Remark 2.3.2. Condition (V2) has two immediate consequences. Taking complement $[\perp]$ produces closed subspaces (see Lemma 2.2.5(ii)), which means $\mathscr{V}$ and $\widetilde{\mathscr{V}}$ are closed subspaces of $\mathscr{W}$ in the graph norm. Another consequence is that (recall that $\mathscr{W}^{[\perp]} = \mathscr{W}_0$)

$$\ker D \ = \ \mathscr{W}_0 \ \subseteq \ \mathscr{V} \cap \widetilde{\mathscr{V}}\,,$$

which is interesting in the sense that $\mathscr{V}$ and $\widetilde{\mathscr{V}}$ are not presumed to contain $\mathscr{W}_0$.

Despite that $\mathscr{V}$ and $\widetilde{\mathscr{V}}$ are closed subspaces of $\mathscr{W}$ the closedness of $\mathscr{V} + \widetilde{\mathscr{V}}$ is not guaranteed (see [3, Theorem 5]). In particular, we can not claim that $\mathscr{V} + \widetilde{\mathscr{V}} = \mathscr{W}$. But if we assume that $\mathscr{V} + \widetilde{\mathscr{V}} = \mathscr{W}$ holds, then we have a refinement on the intersection of the spaces $\mathscr{V}$ and $\widetilde{\mathscr{V}}$.

Lemma 2.3.3. [3, Lemma 13] If $(\mathscr{V},\widetilde{\mathscr{V}})$ satisfies (V2)-condition and $\mathscr{V}+\widetilde{\mathscr{V}}=\mathscr{W}$, then

$$\mathscr{W}_0 \;=\; \ker D \;=\; \mathscr{V}\cap\widetilde{\mathscr{V}} .$$

Proof. The part $\mathscr{W}_0\subseteq\mathscr{V}\cap\widetilde{\mathscr{V}}$ follows from Remark 2.3.2. To prove the other inclusion, let $u\in\mathscr{V}\cap\widetilde{\mathscr{V}}=\mathscr{W}$ and $v\in\mathscr{W}$. Since $\mathscr{V}+\widetilde{\mathscr{V}}$ we can write $v=\nu+\tilde{\nu}$, where $\nu\in\mathscr{V}$ and $\tilde{\nu}\in\widetilde{\mathscr{V}}$, then

$$[\,u\mid v\,]=[\,u\mid \nu\,]+[\,u\mid \tilde{\nu}\,] .$$

Using $\mathscr{V}=\widetilde{\mathscr{V}}^{[\perp]}$ and that $u\in\mathscr{V},\ \tilde{\nu}\in\widetilde{\mathscr{V}}$, we get $[\,u\mid\tilde{\nu}\,]=0$. Similarly, from other part of (V2) condition we have $[\,u\mid\nu\,]=0$. Therefore, by the arbitrariness of v, $[\,u\mid v\,]=0$ implies $\mathscr{V}\cap\widetilde{\mathscr{V}}\subseteq\mathscr{W}^{[\perp]}=\mathscr{W}_0$. Which completes the proof. ■

We can characterise (V)-conditions (particularly (V2)) in the *purely Hilbert space language*.

Theorem 2.3.4. [9, Theorem 9] Let $(T,\widetilde{T})$ be a pair of closed operators on $\mathscr{H}$ satisfying conditions (T1)–(T2), and let $(\mathscr{W},[\cdot\mid\cdot])$ be the indefinite inner product space as in Lemma 2.2.5. Let $(T_1|_{\mathscr{V}},\widetilde{T}_1|_{\widetilde{\mathscr{V}}})$ be a pair of realisations related to $(T,\widetilde{T})$, i.e. $\mathscr{V}$ and $\widetilde{\mathscr{V}}$ are subspaces of $\mathscr{W}$ that contain $\mathscr{W}_0$. Then

$$(T_1|_{\mathscr{V}})^* \;=\; \widetilde{T}_1|_{\widetilde{\mathscr{V}}} \iff \widetilde{\mathscr{V}} \;=\; \mathscr{V}^{[\perp]}$$

and

$$(\widetilde{T}_1|_{\widetilde{\mathscr{V}}})^* \;=\; T_1|_{\mathscr{V}} \iff \mathscr{V} \;=\; \widetilde{\mathscr{V}}^{[\perp]} .$$

In particular, if $\mathscr{V}$ is closed in $\mathscr{W}$, then condition $\widetilde{\mathscr{V}}=\mathscr{V}^{[\perp]}$ is sufficient to have that the operators $T_1|_{\mathscr{V}}$ and $\widetilde{T}_1|_{\widetilde{\mathscr{V}}}$ are mutually adjoint.

Remark 2.3.5. (V2)-condition can also be referred to as *mutually adjoint-condition*.

Proof. Let us first prove that $(T_1|_{\mathscr{V}})^*=\widetilde{T}_1|_{\mathscr{V}^{[\perp]}}$ and $(\widetilde{T}_1|_{\widetilde{\mathscr{V}}})^*=T_1|_{\mathscr{V}^{[\perp]}}$. We prove the first claim only since the second claim is analogous to it. Since $\mathscr{W}_0\subseteq\mathscr{V}$ we have $T\subseteq T_1|_{\mathscr{V}}$, which implies $(T_1|_{\mathscr{V}})^*\subseteq T^*=\widetilde{T}_1$. Thus, it is sufficient to prove that $\operatorname{dom}(T_1|_{\mathscr{V}})^*=\mathscr{V}^{[\perp]}$. Let $v\in\mathscr{V}^{[\perp]}$, then for any $u\in\mathscr{V}$, we have

$$\langle\,(T_1|_{\mathscr{V}})u\mid v\,\rangle=\langle\,T_1u\mid v\,\rangle=[\,u\mid v\,]+\langle\,u\mid\widetilde{T}_1v\,\rangle=\langle\,u\mid(T_1|_{\mathscr{V}})^*v\,\rangle ,$$

implying $v \in \mathrm{dom}(T_1|_{\mathscr{V}})^*$ and thus $\mathscr{V}^{[\perp]} \subseteq \mathrm{dom}(T_1|_{\mathscr{V}})^*$. To prove the other inclusion, let $v \in \mathrm{dom}(T_1|_{\mathscr{V}})^*$. Then for any $u \in \mathscr{V}$, we have

$$\langle (T_1|_{\mathscr{V}})u \mid v \rangle = \langle u \mid (T_1|_{\mathscr{V}})^* v \rangle = \langle u \mid \widetilde{T}_1 v \rangle = \langle (T_1|_{\mathscr{V}})u \mid v \rangle - [\, u \mid v \,] \,,$$

implying $[\, u \mid v \,] = 0$. Since $u \in \mathscr{V}$ is arbitrary, we have $\mathrm{dom}(T_1|_{\mathscr{V}})^* \subseteq \mathscr{V}^{[\perp]}$. which proves the claim.

To conclude, if $\mathscr{V}$ is a closed subspace of $\mathscr{W}$ containing $\mathscr{W}_0$ and $\widetilde{\mathscr{V}} = \mathscr{V}^{[\perp]}$, by Lemma 5.4.20 (see also Proposition 2.2.6) we have $\widetilde{\mathscr{V}}^{[\perp]} = \mathscr{V}^{[\perp][\perp]} = \mathscr{V}$, which enables us to use the identities we obtained above, completing the proof.

■

As already mentioned, our goal is to find and classify all such closed subspaces $\mathscr{V}$ of $\mathscr{W}$ such that for any $f \in \mathscr{H}$ the abstract problem $(T_1|_{\mathscr{V}})u = f$ is well-posed. This means that $T_1|_{\mathscr{V}}$ is a closed, densely defined, bijective, linear operator on $\mathscr{H}$. Here we list some properties of the corresponding adjoint problem and the inverses.

Remark 2.3.6. Let $(T, \widetilde{T})$ be a joint pair of closed abstract Friedrichs operators on $\mathscr{H}$ and let $\mathscr{V}$ be a closed subspace of $\mathscr{W}$ containing $\mathscr{W}_0$. If for any $f \in \mathscr{H}$ the abstract problem $T_1|_{\mathscr{V}} = f$ has a unique solution, i.e. $T_1|_{\mathscr{V}}$ is bijective, then

(i) Since $T_1|_{\mathscr{V}} : (\mathscr{V}, \|\cdot\|_{T_1}) \to \mathscr{H}$ is bounded (note that in the domain now we consider the graph norm) and bijective, the inverse operator $(T_1|_{\mathscr{V}})^{-1} : \mathscr{H} \to (\mathscr{V}, \|\cdot\|_{T_1})$ is everywhere defined and bounded. Of course, the latter holds if we consider the weaker norm of the Hilbert space $\mathscr{H}$ in the codomain of the inverse operator. Hence, $(T_1|_{\mathscr{V}})^{-1}$ is bounded on $\mathscr{H}$. The adjoint of the inverse operator $\big((T_1|_{\mathscr{V}})^{-1}\big)^*$ is then also bounded on $\mathscr{H}$ and injective (cf. [44, Theorem 12.7]).

(ii) If we define $\widetilde{\mathscr{V}} := \mathrm{dom}(T_1|_{\mathscr{V}})^*$, then $\widetilde{\mathscr{V}}$ is closed in $\mathscr{W}$, contains $\mathscr{W}_0$ and $(T_1|_{\mathscr{V}})^* = \widetilde{T}_1|_{\widetilde{\mathscr{V}}}$ (see Theorem 2.3.4). Moreover, $\widetilde{T}_1|_{\widetilde{\mathscr{V}}} : \widetilde{\mathscr{V}} \to \mathscr{H}$ is bijective as well. Indeed, by [44, Theorem 12.7] we know that $\widetilde{T}_1|_{\widetilde{\mathscr{V}}}$ is injective and

$$(\widetilde{T}_1|_{\widetilde{\mathscr{V}}})^{-1} \;=\; \big((T_1|_{\mathscr{V}})^*\big)^{-1} \;=\; \big((T_1|_{\mathscr{V}})^{-1}\big)^* \,.$$

Hence, by part (i), $\mathrm{dom}(\widetilde{T}_1|_{\widetilde{\mathscr{V}}})^{-1} = \mathscr{H}$. Since $\mathrm{ran}\,\widetilde{T}_1|_{\widetilde{\mathscr{V}}} = \mathrm{dom}(\widetilde{T}_1|_{\widetilde{\mathscr{V}}})^{-1}$, the claim holds.

As a consequence of Theorem 2.3.4 and Remark 2.3.6, we have the following: *if* $T_1|_{\mathscr{V}}$ *is a closed bijective realisation of* T_0*, then* $(T_1|_{\mathscr{V}})^* = \widetilde{T}_1|_{\widetilde{\mathscr{V}}}$ *is a closed bijective realisation of* $\widetilde{T}_0$. Therefore, without any loss of generality we can simultaneously study both the original problem $T_1 u = f$ and the associated adjoint problem $\widetilde{T}_1 v = g$. This means that our main goal can be reformulated: we seek for pairs $(T_1|_{\mathscr{V}}, \widetilde{T}_1|_{\widetilde{\mathscr{V}}})$ of mutually adjoint *bijective* realisations relative to $(T_0, \widetilde{T}_0)$.

Before we proceed to the well-posedness results, let us address the following question:

Proposition 2.3.7. [9, Proposition 14] For any joint pair of closed abstract Friedrichs operators $(T, \widetilde{T})$ in a Hilbert space $\mathscr{H}$, there exists a pair of closed subspaces $(\mathscr{V}, \widetilde{\mathscr{V}})$ of the graph space $\mathscr{W}$ satisfying (V)-boundary conditions.

Proof. From Remark 2.2.7, we have the following canonical (fundamental) decomposition of the Kreĭn space $\widehat{\mathscr{W}}$,

$$\widehat{\mathscr{W}} = X_+ [\dot{+}] X_- \,. \tag{2.5}$$

We claim that the pair of subspaces $(\mathscr{V}, \widetilde{\mathscr{V}})$ defined by

$$\mathscr{V} := \{u \in \mathscr{W} : \hat{u} \in X_+\}, \quad \widetilde{\mathscr{V}} := \{v \in \mathscr{W} : \hat{v} \in X_-\},$$

is a pair of closed subspaces of the graph space $\mathscr{W}$ satisfying (V)-boundary conditions. *Proof of the claim*:

- (V1)-condition: For any $u \in \mathscr{V}$, we have $\hat{u} \in X_+$. From the canonical decomposition

  $$[\, u \mid u \,] = [\, \hat{u} \mid \hat{u} \,]_{\widehat{\mathscr{W}}} \geq 0 \,.$$

  Similarly, for any $v \in \widetilde{\mathscr{V}}$, we have that $[\, v \mid v \,] \leq 0$. Thus (V1)-condition is satisfied.

- (V2)-condition: Let $v \in \widetilde{\mathscr{V}}$, then $\hat{v} \in X_-$. For any $u \in \mathscr{V}$ we have $\hat{u} \in X_+$ and

  $$[\, u \mid v \,] = [\, \hat{u} \mid \hat{v} \,]_{\widehat{\mathscr{W}}} = 0 \,,$$

  second equality is due to the decomposition (2.5). Therefore, $v \in \mathscr{V}^{[\perp]}$ which means $\widetilde{\mathscr{V}} \subseteq \mathscr{V}^{[\perp]}$. Conversely, if $v \in \mathscr{V}^{[\perp]} \subseteq \mathscr{W}$, i.e. $\hat{v} \in \widehat{\mathscr{W}}$, then there exist $\hat{v}_+ \in X_+$ and $\hat{v}_- \in X_-$ such that $\hat{v} = \hat{v}_+ + \hat{v}_-$. For any $u \in \mathscr{V}$, i.e. $\hat{u} \in X_-$, we have $[\, \hat{u} \mid \hat{v}_- \,]_{\widehat{\mathscr{W}}} = 0$,

  $$0 = [\, u \mid v \,] = [\, \hat{u} \mid \hat{v} \,]_{\widehat{\mathscr{W}}} = [\, \hat{u} \mid \hat{v}_+ \,]_{\widehat{\mathscr{W}}} \,.$$

In particular, for $\hat{u} = \hat{v}_+$, we have $[\,\hat{v}_+ \mid \hat{v}_+\,]_{\widehat{\mathscr{W}}} = 0$. Since, $(X_+, [\,\cdot \mid \cdot\,]_{\widehat{\mathscr{W}}})$ is a Hilbert space, we conclude that $\hat{v}_+ = \hat{0}$. Hence, $\hat{v} = \hat{v}_- \in X_-$, implying that $v \in \widetilde{\mathscr{V}}$. Thus, $\mathscr{V}^{[\perp]} \subseteq \widetilde{\mathscr{V}}$.

From Remark 2.3.2 both $\mathscr{V}$ and $\widetilde{\mathscr{V}}$ are closed subspace of the graph space $\mathscr{W}$ containing $\mathscr{W}_0$. ■

Remark 2.3.8. In the previous proposition, the subspaces $\mathscr{V}$ and $\widetilde{\mathscr{V}}$ satisfy $\mathscr{W} = \mathscr{V} + \widetilde{\mathscr{V}}$, which is a direct consequence of the canonical decomposition $\widehat{\mathscr{W}} = X_+ [\dot{+}] X_-$. A concrete pair satisfying that the sum is the whole graph space will be given in Corollary 3.2.6. The property that the sum of two domains is closed is beneficial in studying the relation of different abstract concepts of imposing boundary conditions (see Section 2.5).

2.4. WELL-POSEDNESS RESULT

Let us start this section with our focus on the role of (T3)-condition. That is, there exists a constant $\mu_0 > 0$ such that

$$\langle (T+\widetilde{T})\varphi \mid \varphi \rangle \geqslant 2\mu_0 \|\varphi\|^2 , \qquad \varphi \in \mathscr{D} .$$

Since $\overline{T+\widetilde{T}}$ is bounded on $\mathscr{H}$ (see theorems 2.1.2 and 2.5.11), the above can be extended to $\mathscr{H}$.

Lemma 2.4.1. [39, Lemma 3.2] and [5, Lemma 2] Let $(T,\widetilde{T})$ be a joint pair of closed abstract Friedrichs operators in a Hilbert space $\mathscr{H}$ and $(\mathscr{V},\widetilde{\mathscr{V}})$ be a pair of subspaces of the graph space $\mathscr{W}$ satisfying (V1)-condition. Then, $T_1|_{\mathscr{V}}$ and $\widetilde{T}_1|_{\widetilde{\mathscr{V}}}$ are $\mathscr{H}$-coercive, i.e.

$$\begin{aligned} (\forall u \in \mathscr{V}) \qquad & |\langle T_1 u \mid u \rangle| \geq \mu_0 \|u\|^2 , \\ (\forall v \in \widetilde{\mathscr{V}}) \qquad & |\langle \widetilde{T}_1 v \mid v \rangle| \geq \mu_0 \|v\|^2 . \end{aligned}$$

Proof. For any $u \in \mathscr{W}$

$$\begin{aligned} [u \mid u] = \langle T_1 u \mid u \rangle - \langle u \mid \widetilde{T}_1 u \rangle &= (\langle T_1 u \mid u \rangle + \langle u \mid T_1 u \rangle) - \langle u \mid (T_1 + \widetilde{T}_1) u \rangle \\ &= 2\Re \langle T_1 u \mid u \rangle - \langle (T_1 + \widetilde{T}_1) u \mid u \rangle . \end{aligned}$$

Here we used that $\langle u \mid (T_1 + \widetilde{T}_1) u \rangle$ is real (note that $\overline{T+\widetilde{T}}$ is selfadjoint). Thus we have for any $u \in \mathscr{W}$

$$\Re \langle T_1 u \mid u \rangle - \frac{1}{2}[u \mid u] = \frac{1}{2} \langle (T_1 + \widetilde{T}_1) u \mid u \rangle .$$

Due to (V1) and (T3) conditions the above implies

$$\Re \langle T_1 u \mid u \rangle \geq \Re \langle T_1 u \mid u \rangle - \frac{1}{2}[u \mid u] \geq \mu_0 \|u\|^2 .$$

Since $|\langle T_1 u \mid u \rangle| \geq \Re \langle T_1 u \mid u \rangle$, we conclude

$$(\forall u \in \mathscr{V}) \qquad |\langle T_1 u \mid u \rangle| \geq \mu_0 \|u\|^2 .$$

A similar calculation leads to the other identity. ■

Corollary 2.4.2. If a pair of closed subspace of $(\mathscr{V},\widetilde{\mathscr{V}})$ of $\mathscr{W}$ containing $\mathscr{W}_0$ satisfies (V1) condition, the we have the following:

(i) $T_1|_{\mathscr{V}}$ is injective;

(ii) $\operatorname{ran} T_1|_{\mathscr{V}}$ is closed;

(iii) The following decomposition holds

$$\mathscr{H} \;=\; \operatorname{ran} T_1|_{\mathscr{V}} \,\oplus\, \ker(T_1|_{\mathscr{V}})^* \,.$$

The same statements hold for $\widetilde{T}_1|_{\widetilde{\mathscr{V}}}$.

Proof. Due to coercivity in Lemma 2.4.1, if for some $v \in \mathscr{V}$ we have $T_1 v = 0$, then

$$0 = |\langle T_1 v \mid v \rangle| \geq \mu_0 \|v\|^2 ,$$

implying $v = 0$. Hence, $T_1|_{\mathscr{V}}$ is injective. For any $v \in \mathscr{V}$, we also have

$$\|T_1 v\| \geq \mu_0 \|v\| \,.$$

Since $T_1|_{\mathscr{V}}$ is injective, inserting $Tv = w \in \operatorname{ran} T_1|_{\mathscr{V}} = \operatorname{dom}(T_1|_{\mathscr{V}})^{-1}$ we get

$$\|(T_1|_{\mathscr{V}})^{-1} w\| = \|v\| \leq \mu_0^{-1} \|w\| \,.$$

The operator $T_1|_{\mathscr{V}}$ is closed, implying that T^{-1} is a closed and bounded operator. Hence, $\operatorname{dom}(T_1|_{\mathscr{V}})^{-1} = \operatorname{ran} T_1|_{\mathscr{V}}$ is closed. Finally, from Part (ii) and the standard result for linear operators, we get

$$(\ker(T_1|_{\mathscr{V}})^*)^{\perp} = \overline{\operatorname{ran} T_1|_{\mathscr{V}}} = \operatorname{ran} T_1|_{\mathscr{V}} ,$$

which leads to part (iii). ∎

Remark 2.4.3. A trivial pair satisfying condition (V1) is $(\mathscr{W}_0, \mathscr{W}_0)$ since $\ker D = \mathscr{W}_0$. This implies that closed operators $T_0 = T_1|_{\mathscr{W}_0}$ and $\widetilde{T}_0 = \widetilde{T}_1|_{\mathscr{W}_0}$ are $\mathscr{H}-$coercive, hence injective. In particular, their ranges $\operatorname{ran} T_0$ and $\operatorname{ran} \widetilde{T}_0$ are closed in $\mathscr{H}$. Therefore, the following orthogonal decompositions of $\mathscr{H}$ hold:

$$\begin{aligned} \mathscr{H} \;&=\; \operatorname{ran} T_0 \oplus \ker \widetilde{T}_1 \\ &=\; \operatorname{ran} \widetilde{T}_0 \oplus \ker T_1 \,. \end{aligned} \tag{2.6}$$

Theorem 2.4.4 (Banach-Načas-Babuška (BNB))**.** Let $\mathscr{K}$ and $\mathscr{X}$ be complex Banach spaces. The following statements are equivalent:

(i) $T \in \mathscr{L}(\mathscr{K};\mathscr{X})$ is bijective.

(ii) There exists a constant $\alpha > 0$ such that

$$(\forall u \in \mathscr{K}) \qquad \sup_{v\in\mathscr{X}'\setminus\{0\}} \frac{|_{\mathscr{X}'}\langle Tv, v\rangle_{\mathscr{X}}|}{\|v\|_{\mathscr{X}'}} \geq \alpha\|u\|_{\mathscr{K}}\,, \tag{2.7}$$

and

$$(\forall v \in \mathscr{X}') \qquad \big((\forall u \in \mathscr{K}) \quad {}_{\mathscr{X}'}\langle v, Tu\rangle_{\mathscr{X}} = 0 \implies v = 0\big)\,. \tag{2.8}$$

The well-posedness result was proved in [39, Theorem 3.1] under the condition that there exists a pair of subspaces $(\mathscr{V}, \widetilde{\mathscr{V}})$ satisfying (V)-condition.

Theorem 2.4.5. [39, Theorem 3.1] Let $(T,\widetilde{T})$ be a joint pair of closed abstract Friedrichs operators in a Hilbert space $\mathscr{H}$ and the $(\mathscr{V}, \widetilde{\mathscr{V}})$ satisfies (V)-condition. Then, $(T_1|_{\mathscr{V}}, \widetilde{T}_1|_{\widetilde{\mathscr{V}}})$ is a pair of mutually adjoint bijective realisations related to $(T,\widetilde{T})$

Remark 2.4.6. Proposition 2.3.7 ensures the existence of a pair of subspaces $(\mathscr{V}, \widetilde{\mathscr{V}})$ satisfying (V)-condition is ensured in . Moreover, $(T_1|_{\mathscr{V}}, \widetilde{T}_1|_{\widetilde{\mathscr{V}}})$ is called a pair of mutually adjoint bijective realisations with *signed boundary map* related to $(T,\widetilde{T})$ if $(\mathscr{V}, \widetilde{\mathscr{V}})$ satisfy (V)-condition. There are other bijective realisations *without* signed boundary maps (see Example 3.3.4).

Proof. Consider the pair of subspaces $(\mathscr{V}, \widetilde{\mathscr{V}})$ satisfying (V)-condition as in Proposition 2.3.7. We only prove the bijectivity here, as mutual adjointness is ensured from Theorem 2.3.4. It is enough to prove that $T_1|_{\mathscr{V}} : \mathscr{V} \to \mathscr{H}$ is bijective ($\mathscr{V}$ is equipped with the graph norm), since the other part is analogous to it. We shall use Theorem 2.4.4. Let us check that the operator satisfies the requirements.

(a) For $u \in \mathscr{V}$ with $u \neq 0$,

$$\frac{|\langle T_1 u \mid u\rangle|}{\|u\|} \leq \sup_{v\in\mathscr{H}\setminus\{0\}} \frac{|\langle T_1 u \mid v\rangle|}{\|v\|}\,.$$

Using Lemma 2.4.1, the left-hand side is greater or equal to $\mu_0\|u\|$. Thus, we get

$$\begin{aligned}\|u\| + \|T_1 u\| &\leq \frac{1}{\mu_0}\sup_{v\in\mathscr{H}\setminus\{0\}} \frac{|\langle T_1 u \mid v\rangle|}{\|v\|} + \sup_{v\in\mathscr{H}\setminus\{0\}} \frac{|\langle T_1 u \mid v\rangle|}{\|v\|} \\ &= \big(1+\frac{1}{\mu_0}\big)\sup_{v\in\mathscr{H}\setminus\{0\}} \frac{|\langle T_1 u \mid v\rangle|}{\|v\|}\,,\end{aligned}$$

which satisfies equation (2.7).

(b) Let $v \in \mathscr{H}$ such that for any $u \in \mathscr{V}$ we have $\langle T_1|_{\mathscr{V}} u \mid v \rangle = 0$. By Corollary 2.4.2(iii), $v \in \ker(T_1|_{\mathscr{V}})^*$. By Theorem 2.3.4, $v \in \ker \widetilde{T}_{1|_{\widetilde{\mathscr{V}}}}$, implying $\widetilde{T}_1|_{\widetilde{\mathscr{V}}} v = 0$. In particular, we get $v \in \mathscr{W}$. For any $u \in \mathscr{V}$, we have

$$[\, u \mid v \,] = \langle T_1 u \mid v \rangle - \langle u \mid \widetilde{T}_1 v \rangle = 0 - 0 = 0 \,,$$

implying $v \in \mathscr{V}^{[\perp]} = \widetilde{\mathscr{V}}$ and due to coercivity in Lemma 2.4.1, we have

$$0 = |\langle \widetilde{T}_1 v \mid v \rangle| \geq \mu_0 \|v\|^2 \,.$$

Which implies $v = 0$, thus (2.8) is satisfied. Hence by Theorem 2.4.4, $T_1|_{\mathscr{V}} : \mathscr{V} \to \mathscr{H}$ is bijective, completing the proof.

■

Remark 2.4.7. By Theorem 2.4.5 and Proposition 2.3.7 we know that there exists a closed subspace $\mathscr{V} \subseteq \mathscr{W}$ containing $\mathscr{W}_0$ such that $T_1|_{\mathscr{V}}$ is bijective. This in particular implies that T_1 is surjective. The same holds for $\widetilde{T}_1$.

By Proposition 2.3.7 and Theorem 2.4.5 we know that there is always at least one bijective realisation of each T and $\widetilde{T}$, which are in fact *with signed boundary map*. In the following we study the question of the number of such bijective realisations. Before coming to that point, let us establish the following results on $\ker T_1$ and $\ker \widetilde{T}_1$.

Lemma 2.4.8. $(\widehat{\ker T_1}, -[\cdot \mid \cdot]_{\widehat{\mathscr{W}}})$ and $(\widehat{\ker \widetilde{T}_1}, [\cdot \mid \cdot]_{\widehat{\mathscr{W}}})$ are Hilbert spaces.

Proof. Let $v \in \ker T_1$, then

$$-[\, v \mid v \,] = -(\langle T_1 v \mid v \rangle - \langle v \mid \widetilde{T}_1 v \rangle) = \langle v \mid (T_1 + \widetilde{T}_1) v \rangle \geq 2\mu_0 \|v\|^2 \,.$$

Which implies that $-[\cdot \mid \cdot]$ is a definite inner product on $\ker T_1$. Since $\ker T_1$ is a closed subspace of the graph space, $\widehat{\ker T_1}$ is a closed subspace of $\widehat{\mathscr{W}}$. Hence, $(\widehat{\ker T_1}, -[\cdot \mid \cdot]_{\widehat{\mathscr{W}}})$ is a Hilbert space. The proof of the other part is similar. ■

Remark 2.4.9. $(\mathscr{W}_0 + \ker \widetilde{T}_1, \mathscr{W}_0 + \ker T_1)$ satisfies (V1)-condition.

Remark 2.4.10. The subspaces $\mathscr{W}_0$, $\ker T_1$ and $\ker \widetilde{T}_1$ are pairwise $[\perp]$-orthogonal to each other. Indeed, let $u_0 \in \mathscr{W}_0$, $v_1 \in \ker T_1$ and $\tilde{v}_1 \in \ker \widetilde{T}_1$ be arbitrary. Since $\mathscr{W}_0 = \ker D$, i.e. $\mathscr{W}^{[\perp]} = \mathscr{W}_0$, we have

$$[\, u_0 \mid v_1 \,] = [\, u_0 \mid \tilde{v}_1 \,] = 0 \,.$$

Moreover,

$$[\, v_1 \mid \tilde{v}_1 \,] = \langle\, T_1 v_1 \mid \tilde{v}_1 \,\rangle - \langle\, v_1 \mid \widetilde{T}_1 \tilde{v}_1 \,\rangle = 0 - 0 = 0\,.$$

Hence, $\mathscr{W}_0$, $\ker T_1$ and $\ker \widetilde{T}_1$ are mutually $[\perp]$-orthogonal.

Remark 2.4.11. From [14, Theorems V.3.4 and V.3.5] it is evident that $\widehat{\mathscr{W}}$ allows a canonical decomposition $\widehat{\mathscr{W}} = X_+ \widehat{[+]} X_-$ such that $\widehat{\ker T_1} \in X_-$ and $\widehat{\ker \widetilde{T}_1} \in X_+$. Which means there exists a pair of subspaces $(\mathscr{V}, \widetilde{\mathscr{V}})$ of the graph space $\mathscr{W}$ satisfying (V)-boundary conditions such that

$$\mathscr{W}_0 + \ker T_1 \subseteq \widetilde{\mathscr{V}} \qquad \text{and} \qquad \mathscr{W}_0 + \ker \widetilde{T}_1 \subseteq \mathscr{V}\,.$$

The multiplicity result can be characterised in terms of the kernels.

Theorem 2.4.12. [9, Theorem 13] Let $(T, \widetilde{T})$ be a joint pair of closed abstract Friedrichs operators in a Hilbert space $\mathscr{H}$ and $(T_1, \widetilde{T}_1)$ be the pair of corresponding adjoint operators.

(i) If $\ker T_1 \neq \{0\}$ and $\ker \widetilde{T}_1 \neq \{0\}$, then there exists uncoultably many mutually adjoint pair of bijective realisations related to the pair $(T, \widetilde{T})$.

(ii) If $\ker T_1 = \{0\}$ or $\ker \widetilde{T}_1 = \{0\}$, then there is exactly one mutually adjoint pair of bijective realisations related to $(T, \widetilde{T})$. More precisely, if $\ker T_1 = \{0\}$, then $(T_1, \widetilde{T})$ and if $\ker \widetilde{T}_1 = \{0\}$, then $(T, \widetilde{T}_1)$ are the mutually adjoint pair of bijective realisations. When $\ker T_1 = \ker \widetilde{T}_1 = \{0\}$, then $(T, \widetilde{T})$ is the desired pair.

Proof. From Remark 2.4.11, there exists a canonical decomposition of $(\widehat{\mathscr{W}}, \widehat{[\cdot \mid \cdot]})$, say $\widehat{\mathscr{W}} = X_+ \widehat{[+]} X_-$ such that $\widehat{\ker \widetilde{T}_1} \subseteq X_+$ and $\widehat{\ker T_1} \subseteq X_-$.

(i) Since both kernels are non-trivial, so are X^+ and X^- and thus by Lemma 5.4.12, there are uncountably many canonical decompositions of $\widehat{\mathscr{W}}$. Each canonical decomposition produces a pair of subspaces $(\mathscr{V}, \widetilde{\mathscr{V}})$ of $\mathscr{W}$ each containing $\mathscr{W}_0$, which satisfies (V)-conditions (by Proposition 2.3.7) and thus by Theorem 2.4.5, there are uncoultably many mutually adjoint pair of bijective realisations related to the pair $(T, \widetilde{T})$.

(ii) Let $\ker T_1 = \{0\}$ and $(T_\mathrm{r}, T_\mathrm{r}^*)$ be a pair of mutually adjoint bijective realisations with signed boundary map related to $(T, \widetilde{T})$. We have $T \subseteq T_\mathrm{r} \subseteq T_1$. Suppose, $T_\mathrm{r} \neq T_1$, then there exits $u \in \mathscr{W} \setminus \operatorname{dom} T_\mathrm{r}$, $u \neq 0$. Due to bijectivity of T_r, there exists unique $v \in \operatorname{dom} T_\mathrm{r}$, such that $T_\mathrm{r} v = T_1 u$. We have $T_1(u - v) = T_1 u - T_1 v = T_\mathrm{r} u - T_1 v = 0$, implying $u - v \in \ker T_1 = \{0\}$. Thus, $u = v \in \operatorname{dom} T_\mathrm{r}$, which is a contradiction to the choice of u. Therefore, $T_\mathrm{r} = T_1$ and $T_\mathrm{r}^* = T_1^* = \widetilde{T}$. Hence, $(T_1, \widetilde{T})$ is the only pair of mutually adjoint bijective realisations related to $(T, \widetilde{T})$. The case $\ker \widetilde{T}_1 = \{0\}$ is completely analogous.

■

2.5. Boundary conditions

In this section we discuss about different ways to pose boundary conditions for abstract Friedrichs operators, analogous to the set of boundary conditions in the case of classical Friedrichs operators (see Section 1.1). It turns out that these conditions are also equivalent. (V)-boundary conditions (the cone formalism) are analogous to (FV)-boundary conditions for classical Friedrichs operators. We extend the ideas to analogous boundary conditions (X) and (M) that are related to (FX) and (FM) conditions respectively in the classical theory. First, let us discuss about the maximality of the subspaces $\mathscr{V}$ and $\widetilde{\mathscr{V}}$ introduced in (V)-boundary conditions. These subspaces can not be extended further in the cones in which they are contained. For a discussion on different ways to pose boundary conditions in the abstract setting, we refer [3, 39].

We introduce the set of boundary conditions corresponding to (FX)-boundary conditions in the classical setting.

Definition 2.5.1 ((X)-boundary conditions)**.** A subspace $\mathscr{V}$ of $(\mathscr{W}, [\cdot \mid \cdot])$ is called maximal non-negative if it satisfies the following conditions:

(X1) $\mathscr{V}$ is non-negative with respect to the indefinite inner product $[\cdot \mid \cdot]$, i.e. $\mathscr{V} \subseteq \mathscr{W}^+$.

(X2) There is no non-negative subspace of $(\mathscr{W}, [\cdot \mid \cdot])$ containing $\mathscr{V}$ properly.

Theorem 2.5.2. [39, Theorem 3.3] Let $(\mathscr{V}, \widetilde{\mathscr{V}})$ satisfies (V)-boundary conditions. Then, $\mathscr{V}$ is maximal in $\mathscr{W}^+$ and $\widetilde{\mathscr{V}}$ is maximal in $\mathscr{W}^-$.

Proof. From (V1) condition $\mathscr{V}$ is non-negative in $\mathscr{W}$ and then $\widehat{\mathscr{V}}$ is non-negative in $\widehat{\mathscr{W}}$. Similarly, $\widehat{\widetilde{\mathscr{V}}}$ is non-positive in $\widehat{\mathscr{W}}$. From (V2) condition we have $\widetilde{\mathscr{V}} = \mathscr{V}^{[\perp]}$ and by Lemma 5.4.21 we get

$$\widehat{\widetilde{\mathscr{V}}} = \widehat{\mathscr{V}^{[\perp]}} = (\widehat{\mathscr{V}})^{\widehat{[\perp]}} .$$

From Theorem 5.4.15, the closure of $\widehat{\mathscr{V}}$ is maximal non-negative in $\widehat{\mathscr{W}}$. Since $\mathscr{V}$ is closed subspace of $\mathscr{W}$ containing $\mathscr{W}_0$, from Lemma 5.4.19 $\widetilde{\mathscr{V}}$ is closed in $\widehat{\mathscr{W}}$. Therefore, $\widehat{\mathscr{V}}$ is maximal non-negative in $\widehat{\mathscr{W}}$ and thus by Lemma 5.4.22(ii), $\mathscr{V}$ is maximal non-negative subspace of $\mathscr{W}$. The proof of the other part is similar. ■

The equivalence between (V) and (X) conditions is fairly straightforward. Theorem 2.5.2 gives (V) $\implies$ (X), as the subspace $\mathscr{V}$ is maximal non-negative. The following theorem covers the part (X) $\implies$ (V) and the proof follows form Lemma 5.4.10.

Theorem 2.5.3. [3, Theorem 2] Let $\mathscr{V}$ be a maximal non-negative subspace in $(\mathscr{W},[\cdot\mid\cdot])$, then $\widetilde{\mathscr{V}} := \mathscr{V}^{[\perp]}$ is maximal non-positive and $(\mathscr{V},\widetilde{\mathscr{V}})$ satisfies (V)-boundary conditions.

Proof. From Lemma 5.4.10, $[\cdot\mid\cdot]-$orthogonal complement of maximal non-negative subspace is non-positive, which proves (V1). From Lemma 5.4.22(i), $\widehat{\mathscr{V}}$ is maximal non-negative in $\widehat{\mathscr{W}}$ and by theorems 5.4.15(ii) and 5.4.16 it is closed and equal to $(\widehat{\mathscr{V}}^{\widehat{[\perp]}})^{\widehat{[\perp]}}$. From Lemma 5.4.21 we get

$$\widehat{\mathscr{V}} = (\widehat{\mathscr{V}}^{\widehat{[\perp]}})^{\widehat{[\perp]}} = \widehat{(\mathscr{V}^{[\perp]})}^{\widehat{[\perp]}} = \widehat{(V^{[\perp][\perp]})}\,.$$

Since $\mathscr{V}$ is maximal non-negative, by Lemma 5.4.9 it contains $\mathscr{W}_0$, while we trivially have $\mathscr{W}_0 \subseteq \mathscr{V}^{[\perp][\perp]}$. Hence, $\mathscr{V} = \mathscr{V}^{[\perp][\perp]} = \widetilde{\mathscr{V}}^{[\perp]}$. Which proves (V2). ■

Corollary 2.5.4. Let $(T,\widetilde{T})$ be a joint pair of closed abstract Friedrichs operators in a Hilbert space $\mathscr{H}$ and let $\mathscr{V}$ be a maximal non-negative subspace of $(\mathscr{W},[\cdot\mid\cdot])$. Define $\widetilde{\mathscr{V}} := \mathscr{V}^{[\perp]}$, then $(T|_{\mathscr{V}},\widetilde{T}|_{\widetilde{\mathscr{V}}})$ is a pair of mutually adjoint bijective realisations related to $(T,\widetilde{T})$.

Proof. By Theorem 2.5.3, $(\mathscr{V},\widetilde{\mathscr{V}})$ satisfies (V)-conditions and by Theorem 2.4.5, pair $(T|_{\mathscr{V}},\widetilde{T}|_{\widetilde{\mathscr{V}}})$ is a pair of mutually adjoint bijective realisations related to $(T,\widetilde{T})$. ■

Remark 2.5.5. For a given pair of abstract Friedrichs operators $(T,\widetilde{T})$ on a Hilbert space $\mathscr{H}$, the existence of a pair of subspaces $(\mathscr{V},\widetilde{\mathscr{V}})$ satisfying (V)-conditions can be guaranteed using Corollary 2.5.4, which provides an alternate proof of Proposition 2.3.7. Indeed, from Remark 2.4.9, we know that $\mathscr{W}_0 \dotplus \ker\widetilde{T}_1$ is a non-negative subspace. Consider the maximal non-negative subspace $\mathscr{V}$ such that $\mathscr{W}_0 \dotplus \ker\widetilde{T}_1 \subseteq \mathscr{V}$ (it exists by Zorn's lemma; cf. [14, Section I.6]). Hence, for $\widetilde{\mathscr{V}} = \mathscr{V}^{[\perp]}$, the pair $(\mathscr{V},\widetilde{\mathscr{V}})$ satisfies (V)-boundary conditions.

Here we introduce a set of boundary conditions corresponding to the original idea of Friedrichs in the classical sense, i.e. (FM)-boundary conditions.

Definition 2.5.6 ((M)-boundary conditions)**.** Let $M \in \mathscr{L}(\mathscr{W};\mathscr{W}')$. (M)-boundary conditions are given by:

(M1) Non-negativity:

$$(\forall u \in \mathscr{W}) \qquad \Re_{\mathscr{W}'}\langle Mu, u\rangle_{\mathscr{W}} \geq 0\,,$$

(M2) D is the boundary operator. Then

$$\mathscr{W} = \ker(D-M) + \ker(D+M)\,.$$

The operator $M^* \in \mathscr{L}(\mathscr{W};\mathscr{W}')$ is the adjoint operator of M defined as follows:

$$(\forall u, v \in \mathscr{W}) \qquad {}_{\mathscr{W}'}\langle M^* u, v\rangle_{\mathscr{W}} := \overline{{}_{\mathscr{W}'}\langle Mv, u\rangle_{\mathscr{W}}}\,.$$

The idea is to define a pair of subspaces $\mathscr{V} := \ker(D-M)$ and $\widetilde{\mathscr{V}} := \ker(D+M^*)$ satisfying (V)-boundary conditions. In fact the part (M) $\Longrightarrow$ (V) is given by this construction. The challenging part is the converse, i.e. the following question: *For a given pair of subspaces* $(\mathscr{V}, \widetilde{\mathscr{V}})$ *satisfying* (V)*-boundary conditions, does there exists an operator* $M \in \mathscr{L}(\mathscr{W};\mathscr{W}')$ *satisfying* (M)*-boundary conditions?* Which is similar to the concern pointed out in Remark 1.3.4 in the classical sense. In some cases, this question boils down to closedness of the subspace $\mathscr{V} + \widetilde{\mathscr{V}}$ in the graph space $\mathscr{W}$ (see Remark 2.3.8). Before we address this question, let us see some properties of the operator M, when (M)-conditions are satisfied.

Lemma 2.5.7. [39, Lemma 4.1] Let $M \in \mathscr{L}(\mathscr{W}';\mathscr{W})$ satisfies (M)-boundary conditions. Then,

$$\ker D \;=\; \ker M \;=\; \ker M^* \quad \text{and} \quad \operatorname{ran} D \;=\; \operatorname{ran} M \;=\; \operatorname{ran} M^*\,.$$

Proof. First we prove that $\ker M = \ker M^*$ and second $\operatorname{ran} D = \operatorname{ran} M$. Using selfadjointness of D we have $(\operatorname{ran} D)^0 = \ker D^* = \ker D$ and the second claim implies $\ker D = \ker M^*$. Moreover, $\operatorname{ran} D = (\ker D)^0$ as in Lemma 2.2.3(iii), and the first claim imply $\operatorname{ran} D = \operatorname{ran} M^*$. Let us prove both claims.

(i) Let $u \in \ker M$, then for any $v \in \mathscr{W}$ and for any $\lambda \in \mathbb{C}$, we have

$$0 \leq \Re_{\mathscr{W}'}\langle M(v+\lambda u), (v+\lambda u)\rangle_{\mathscr{W}} = \Re_{\mathscr{W}'}\langle Mv, v\rangle_{\mathscr{W}} + \Re(\bar{\lambda}\,{}_{\mathscr{W}'}\langle Mv, u\rangle_{\mathscr{W}})\,.$$

Since $\lambda \in \mathbb{C}$ is arbitrary, we have ${}_{\mathscr{W}'}\langle Mv, u\rangle_{\mathscr{W}} 0$, implying ${}_{\mathscr{W}'}\langle M^*u, v\rangle_{\mathscr{W}} = 0$. Which means $u \in \ker M^*$ and thus $\ker M \subseteq \ker M^*$. Proof of $\ker M^* \subseteq \ker M$ is similar.

(ii) Let $u \in \operatorname{ran} D$, which means there exists $v \in \mathscr{W}$ such that $Dv = u$. Using (M2), we can write $v = v_1 + v_2$ where $v_1 \in \ker(D - M)$ and $v_1 \in \ker(D + M)$, then

$$0 = (D-M)(v_1) + (D+M)v_2 = D(v_1+v_2) - M(v_1 - v_2) = u - M(v_1 - v_2),$$

thus $u = M(v_1 - v_2) \in \operatorname{ran} M$. Hence, $\operatorname{ran} D \subseteq \operatorname{ran} M$. Proof of the other inclusion is similar.

■

Remark 2.5.8. Since $\mathscr{W}_0 = \ker D = \ker M$, the operator M is also called boundary operator.

Theorem 2.5.9. [39, Theorem 4.2] Let $M \in \mathscr{L}(\mathscr{W}; \mathscr{W}')$ be an operator satisfying (M)-conditions and define the subspace $\mathscr{V} := \ker(D - M)$ and $\widetilde{\mathscr{V}} := \ker(D + M^*)$. Then, $(\mathscr{V}, \widetilde{\mathscr{V}})$ satisfies (V)-conditions.

Proof. (V1) Let $u \in \mathscr{V} = \ker(D - M)$, then $Du = Mu$ and using (M1) condition, we have $\Re_{\mathscr{W}'}\langle Mu, u\rangle_{\mathscr{W}} \geq 0$ and thus $[u \mid u] \geq 0$. If $v \in \widetilde{\mathscr{V}} = \ker(D + M^*)$, then we have $M^*v = -Dv$. Since

$$0 \leq \Re_{\mathscr{W}'}\langle Mv, v\rangle_{\mathscr{W}} = \Re_{\mathscr{W}'}\langle M^*v, v\rangle_{\mathscr{W}}$$

implying $[v \mid v] \leq 0$. Hence $(\mathscr{V}, \widetilde{\mathscr{V}})$ satisfies (V1) condition.

(V2) Let $v \in \widetilde{\mathscr{V}}$, for any $u \in \mathscr{V}$, we have

$$\begin{aligned} {}_{\mathscr{W}'}\langle Du, v\rangle_{\mathscr{W}} &= {}_{\mathscr{W}'}\langle (D-M)u, v\rangle_{\mathscr{W}} + {}_{\mathscr{W}'}\langle (D+M)u, v\rangle_{\mathscr{W}} \\ &= {}_{\mathscr{W}'}\langle (D+M^*)v, u\rangle_{\mathscr{W}} = 0, \end{aligned}$$

in the second equality we used that D is symmetric. Thus, $v \in \mathscr{V}^{[\perp]}$. Proof of the other inclusion is similar.

■

Corollary 2.5.10. Let $(T,\widetilde{T})$ be a pair of closed abstract Friedrichs operators. If $M \in \mathcal{L}(\mathcal{W},\mathcal{W}')$ satisfies (M)-conditions, then $(T|_{\mathcal{V}},\widetilde{T}|_{\widetilde{\mathcal{V}}})$ is a mutually adjoint pair of bijective realisations related to $(T,\widetilde{T})$, where $\mathcal{V} := \ker(D-M)$ and $\widetilde{\mathcal{V}} := \ker(D+M^*)$.

Proof. By Theorem 2.5.9, $(\mathcal{V},\widetilde{\mathcal{V}})$ satisfies (V)-boundary condition and due to Theorem 2.4.5, $(T|_{\mathcal{V}},\widetilde{T}|_{\widetilde{\mathcal{V}}})$ is a mutually adjoint pair of bijective realisations related to $(T,\widetilde{T})$. ■

Let us return to the question of (V) $\Longrightarrow$ (M). The construction of operator M requires some additional assumptions on the subspaces $\mathcal{V}$ and $\widetilde{\mathcal{V}}$. In [39, Lemma 4.4], the authors proved the construction under the assumption that $\mathcal{V}+\widetilde{\mathcal{V}}$ is closed. Later, using the theory of indefinite inner products in [3, Theorem 8] another construction is presented which always holds. More precisely, the existence of operator M is equivalent to the existence of a non-positive subspace that together with $\mathcal{V}$ spans the whole graph space, which is known to exist [3, Theorem 9].

Theorem 2.5.11. [3, Theorem 8] Let $\mathcal{W}$ and D be the graph space and the boundary operator, respectively.

(i) Let $(\mathcal{V},\widetilde{\mathcal{V}})$ satisfies (V)-conditions.
Existence: If there exists a closed subspace $\mathcal{W}_2 \subseteq \mathcal{W}^-$ of $\mathcal{W}$ such that $\mathcal{V} \dot{+} \mathcal{W}_2 = \mathcal{W}$, then there exists an operator $M \in \mathcal{L}(\mathcal{W};\mathcal{W}')$ satisfying (M)-conditions and $\mathcal{V} = \ker(D-M)$.
Construction: Let $\mathcal{W}_1$ is the orthogonal complement of $\mathcal{W}_0$ in $\mathcal{V}$, so that $\mathcal{W} = \mathcal{W}_0 \dot{+} \mathcal{W}_1 \dot{+} \mathcal{W}_2$ and p_0, p_1, p_2 are corresponding non-orthogonal projectors, then one such operator M is given by $M = D(p_1 - p_2)$.

(ii) Let $M \in \mathcal{L}(\mathcal{W};\mathcal{W}')$ is an operator satisfying (M)-conditions and $\mathcal{V} = \ker(D-M)$. Then the subspace $\mathcal{W}_2$ is given by the orthogonal complement of $\mathcal{W}_0$ in $\ker(D+M)$. Moreover, $\mathcal{W}_2 \subseteq \mathcal{W}^-$ is closed subspace of the graph space $\mathcal{W}$ and $\mathcal{V} \dot{+} \mathcal{W}_2 = \mathcal{W}$.

While in [3], the existence of such $\mathcal{W}_2$ is confirmed using Kreĭn space theory, in the next chapter, we present that $\mathcal{W}_2$ can be taken as $\ker T_1$ regardless of the choice of $(\mathcal{V},\widetilde{\mathcal{V}})$ satisfying (V)-conditions under (T3) condition, i.e. if $(T,\widetilde{T})$ is a pair of closed abstract Friedrichs operators.

Remark 2.5.12. For a given subspace $\mathscr{V}$ the operator M is unique in the sense that it depends on the decomposition led by $\mathscr{V}$. However, the room for non-uniqueness is determined by possible choices of subspaces of $\mathscr{W}_2$.

2.6. EXAMPLES

2.6.1. Classical is abstract

Example 2.6.1 (Classical Friedrichs operators)**.** Let $d, r \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^d$ be an open and bounded set with Lipschitz boundary Γ. Here we present how the theory of abstract Friedrichs operators can encompass classical Friedrichs differential operators, while for details we refer to [39, Subsection 5.1].

We consider the restriction of operator L (CFO) to $C_c^\infty(\Omega;\mathbb{C}^r)$ and denote it by T, i.e.

$$T\mathsf{u} = \sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{B}\mathsf{u}\,, \quad \mathsf{u} \in C_c^\infty(\Omega;\mathbb{C}^r)$$

(here the derivatives can be understood in the classical sense as derivatives of smooth functions are equal to their distributional derivatives). Since $\mathbf{B} \in L^\infty(\Omega;\mathrm{M}_r(\mathbb{C}))$ and $\mathbf{A}_k \in W^{1,\infty}(\Omega;\mathrm{M}_r(\mathbb{C}))$ (for any k), it is obvious that $T : C_c^\infty(\Omega;\mathbb{C}^r) \to L^2(\Omega;\mathbb{C}^r)$.

For the second operator we take $\widetilde{T} : C_c^\infty(\Omega;\mathbb{C}^r) \to L^2(\Omega;\mathbb{C}^r)$ given by

$$\widetilde{T}\mathsf{u} = -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) + \Big(\mathbf{B}^* + \sum_{k=1}^{d} \partial_k\mathbf{A}_k\Big)\mathsf{u}\,, \quad \mathsf{u} \in C_c^\infty(\Omega;\mathbb{C}^r)\,.$$

Then one can easily see that $(T, \widetilde{T})$ is a joint pair of abstract Friedrichs operators, where $\mathscr{H} = L^2(\Omega;\mathbb{C}^r)$ and $\mathscr{D} = C_c^\infty(\Omega;\mathbb{C}^r)$. Indeed, (T1) is obtained by integration by parts and using (F1), the boundedness of coefficients implies (T2), while (T3) follows from (F2) (a more general case where $\mathscr{H}$ is taken to be a closed subspace of $L^2(\Omega;\mathbb{C}^r)$ can be found in [5, Example 2]).

The domain of adjoint operators $T_1 = \widetilde{T}^*$ and $\widetilde{T} = T^*$ (the graph space) reads

$$\begin{aligned}
\mathscr{W} &= \Big\{\mathsf{u} \in L^2(\Omega;\mathbb{C}^r) : \sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{B}\mathsf{u} \in L^2(\Omega;\mathbb{C}^r)\Big\} \\
&= \Big\{\mathsf{u} \in L^2(\Omega;\mathbb{C}^r) : \sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) \in L^2(\Omega;\mathbb{C}^r)\Big\}\,.
\end{aligned}$$

The action of T_1 and $\widetilde{T}_1$ is (formally) the same as the action of T and $\widetilde{T}$, respectively (we have just that the classical derivatives are replaced by the distributional ones). It is known that $C_c^\infty(\mathbb{R}^d;\mathbb{C}^r)$ is dense in $\mathscr{W}$ [1, Theorem 4] (cf. [47, Chapter 1]) and that the boundary operator, for $\mathsf{u},\mathsf{v} \in C_c^\infty(\mathbb{R}^d;\mathbb{C}^r)$, is given by

$$ {}_{\mathscr{W}'}\langle D\mathsf{u},\mathsf{v}\rangle_{\mathscr{W}} = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\mathsf{u}|_\Gamma(\mathbf{x})\cdot\mathsf{v}|_\Gamma(\mathbf{x})\,dS(\mathbf{x})\,, \tag{2.9}$$

where $\mathbf{A}_{\boldsymbol{\nu}} := \sum_{k=1}^d \nu_k\mathbf{A}_k$ and $\boldsymbol{\nu} = (\nu_1,\nu_2,\ldots,\nu_d) \in L^\infty(\Gamma;\mathbb{R}^d)$ is the unit outward normal on Γ. In the one-dimensional case ($d=1$) for $\Omega = (a,b)$, $a<b$, the above formula simplifies to

$$ {}_{\mathscr{W}'}\langle D\mathsf{u},\mathsf{v}\rangle_{\mathscr{W}} = \mathbf{A}(b)\mathsf{u}(b)\cdot\mathsf{v}(b) - \mathbf{A}(a)\mathsf{u}(a)\cdot\mathsf{v}(a)\,. \tag{2.10}$$

By the definition, we have that the domain of closures $T_0 = \overline{T}$ and $\widetilde{T}_0 = \overline{\widetilde{T}}$ is given by $\mathscr{W}_0 = \mathrm{cl}_{\mathscr{W}}\, C_c^\infty(\Omega;\mathbb{C}^r)$, while by Lemma 2.2.3(ii) and the identity above we have

$$\begin{aligned}\mathscr{W}_0 \cap C_c^\infty(\mathbb{R}^d;\mathbb{C}^r) = \Big\{\mathsf{u}\in C_c^\infty(\mathbb{R}^d;\mathbb{C}^r) : (\forall \mathsf{v}\in C_c^\infty(\mathbb{R}^d;\mathbb{C}^r))& \\ \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\mathsf{u}|_\Gamma(\mathbf{x})\cdot\mathsf{v}|_\Gamma(\mathbf{x})\,dS(\mathbf{x}) = 0&\Big\}\,.\end{aligned}$$

A more specific characterisation involving the trace operator on the graph space can be found in [1, 47].

Let us illustrate the theory of abstract Friedrichs operators on some classical Friedrichs operators discussed in Section 1.3.

2.6.2. Stationary diffusion equation

Consider the classical Friedrichs system given in Subsection 1.2.2. This fits into the abstract framework as explained in Subsection 2.6.1. The graph space and the minimal space are identified as

$$\begin{aligned} \mathscr{W} &:= L^2_{\mathrm{div}}(\Omega;\mathbb{C}^d)\times H^1(\Omega;\mathbb{C}) \\ \text{and}\quad \mathscr{W}_0 &:= L^2_{\mathrm{div},0}(\Omega;\mathbb{C}^d)\times H^1_0(\Omega;\mathbb{C})\,,\end{aligned}$$

respectively. We also have

$$\mathbf{A}_{\boldsymbol{\nu}} = \begin{bmatrix} 0 & \ldots & 0 & \nu_1 \\ 0 & \ldots & 0 & \nu_2 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & \nu_d \\ \nu_1 & \ldots & \nu_d & 0 \end{bmatrix}.$$

For any $\mathsf{u} \in C_c^\infty(\mathbb{R}^d;\mathbb{C}^{d+1})$, we have

$$\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\mathsf{u}(\mathbf{x})|_\Gamma = \begin{bmatrix} \nu_1 u(\mathbf{x}) \\ \vdots \\ \nu_d u(\mathbf{x}) \\ \boldsymbol{\nu}\cdot\mathbf{p}(\mathbf{x}) \end{bmatrix}\Bigg|_\Gamma .$$

Since $H^1(\Omega;\mathbb{C})$ have traces in $H^{1/2}(\Gamma)$ and the vector fields in $L^2_{\mathrm{div}}(\Omega;\mathbb{C}^d)$ have traces in $H^{-1/2}(\Gamma)$, the boundary operator (2.9) can be characterised more explicitly. Let $\mathbf{T}_0$ and $\mathbf{T}_{\boldsymbol{\nu}}$ be the corresponding trace maps, then the boundary can be characterised as,

$$\left(\forall \begin{bmatrix}\mathbf{p}\\ u\end{bmatrix}, \begin{bmatrix}\mathbf{q}\\ v\end{bmatrix} \in \mathscr{W}\right) \qquad {}_{\mathscr{W}'}\langle D\mathsf{u},\mathsf{v}\rangle_{\mathscr{W}} = {}_{-\frac{1}{2}}\langle \mathbf{T}_{\boldsymbol{\nu}}\mathbf{p}, \mathbf{T}_0 v\rangle_{\frac{1}{2}} + {}_{-\frac{1}{2}}\langle \mathbf{T}_{\boldsymbol{\nu}}\mathbf{q}, \mathbf{T}_0 u\rangle_{\frac{1}{2}}, \tag{2.11}$$

where, ${}_{-\frac{1}{2}}\langle\cdot,\cdot\rangle_{\frac{1}{2}}$ denotes the duality pairing between the space $H^{1/2}(\Gamma)$ and its dual $H^{-1/2}(\Gamma)$.

Homogeneous Dirichlet boundary condition can be imposed by the choice $\mathscr{V} = \widetilde{\mathscr{V}} = L^2_{\mathrm{div}}(\Omega;\mathbb{C}^d)\times H^1_0(\Omega;\mathbb{C}) = \{(\mathbf{p},u)\in\mathscr{W} : \mathbf{T}_0 u = 0\}$.

Lemma 2.6.2. The pair $(\mathscr{V},\widetilde{\mathscr{V}}) = (L^2_{\mathrm{div}}(\Omega;\mathbb{C}^d)\times H^1_0(\Omega;\mathbb{C}), L^2_{\mathrm{div}}(\Omega;\mathbb{C}^d)\times H^1_0(\Omega;\mathbb{C}))$ satisfies (V)-boundary condition.

Proof. Let $\mathsf{u},\mathsf{v}\in\mathscr{V}$, then $\mathbf{T}_0 u = \mathbf{T}_0 v = 0$, which implies

$${}_{\mathscr{W}'}\langle D\mathsf{u},\mathsf{v}\rangle_{\mathscr{W}} = 0 .$$

Thus, (V1) condition is satisfied and $\mathscr{V}\subseteq\mathscr{V}^{[\perp]}$. For the second inclusion, let $\mathsf{u} = \begin{bmatrix}\mathbf{p}\\ u\end{bmatrix} \in \mathscr{V}^{[\perp]}$, then for any $\mathsf{v} = \begin{bmatrix}\mathbf{q}\\ 0\end{bmatrix} \in \mathscr{V}$, we have

$${}_{\mathscr{W}'}\langle D\mathsf{u},\mathsf{v}\rangle_{\mathscr{W}} = {}_{-\frac{1}{2}}\langle \mathbf{T}_{\boldsymbol{\nu}}\mathbf{p}, v\rangle_{\frac{1}{2}} + {}_{-\frac{1}{2}}\langle \mathbf{T}_{\boldsymbol{\nu}}\mathbf{q}, u\rangle_{\frac{1}{2}} = {}_{-\frac{1}{2}}\langle \mathbf{T}_{\boldsymbol{\nu}}\mathbf{q}, u\rangle_{\frac{1}{2}} = 0 .$$

For any $\theta\in H^{-1/2}(\Gamma)$, there exists $\mathbf{q}\in L^2_{\mathrm{div}}(\Omega;\mathbb{C}^d)$ such that $\mathbf{T}_{\boldsymbol{\nu}}\mathbf{q} = \theta$. Which means arbitrariness of $\begin{bmatrix}\mathbf{q}\\ 0\end{bmatrix}$ is enough is conclude that $u = 0$, implying $\mathsf{u}\in\mathscr{V}$. Hence, $\mathscr{V}^{[\perp]}\subseteq\mathscr{V}$. ■

By Theorem 2.4.5, the operator $(T_1|_{\mathscr{V}}, \widetilde{T}_1|_{\mathscr{V}})$ is a pair of mutually adjoint bijective realisations related to the Friedrichs operator $(T, \widetilde{T})$, where $T = L$ and L is defined as in Subsection (1.2.2).

Similarly, homogeneous Neumann boundary condition can be imposed by the choice of

$$\mathscr{V} = \widetilde{\mathscr{V}} := \left\{ \begin{bmatrix} \mathbf{p} \\ u \end{bmatrix} \in \mathscr{W} : \mathbf{T}_{\boldsymbol{\nu}}\mathbf{p} = 0 \right\},$$

and Robin boundary conditions are imposed by

$$\mathscr{V} := \left\{ \begin{bmatrix} \mathbf{p} \\ u \end{bmatrix} \in \mathscr{W} : \mathbf{T}_{\boldsymbol{\nu}}\mathbf{p} = \alpha \mathbf{T}_0 u \right\}$$

$$\text{and} \quad \widetilde{\mathscr{V}} := \left\{ \begin{bmatrix} \mathbf{p} \\ u \end{bmatrix} \in \mathscr{W} : \mathbf{T}_{\boldsymbol{\nu}}\mathbf{p} = -\alpha \mathbf{T}_0 u \right\}$$

(compare with (1.11)).

2.6.3. Maxwell's equation in the elliptic regime

Consider the example from Subsection 1.2.3, in particular the system 1.7, which is a classical Friedrichs system. The graph space $\mathscr{W}$ consists of the vectors $(H, E) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3)$ such that $(\nabla \times H, \nabla \times E) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3)$. That is the graph space is given by

$$\mathscr{W} := L^2_{\mathrm{rot}}(\Omega; \mathbb{R}^3) \times L^2_{\mathrm{rot}}(\Omega; \mathbb{R}^3),$$

and the minimal space is given by

$$\mathscr{W}_0 := L^2_{\mathrm{rot},0}(\Omega; \mathbb{R}^3) \times L^2_{\mathrm{rot},0}(\Omega; \mathbb{R}^3),$$

The domain Ω has Lipschitz boundary Γ, so $H^1(\Omega; \mathbb{R}^3)$ is dense in $L^2_{\mathrm{rot}}(\Omega; \mathbb{R}^3)$ and the tangential traces of vector fields in $L^2_{\mathrm{rot}}(\Omega; \mathbb{R}^3)$ are characterised in $H^{-1/2}(\Gamma)$ (see [17]). For outward normal vector $\boldsymbol{\nu}$ the following formula holds for all $h \in H^1(\Omega; \mathbb{R}^3)$ and $e \in L^2_{\mathrm{rot}}(\Omega; \mathbb{R}^3)$,

$$\langle \nabla \times e \mid h \rangle - \langle e \mid \nabla \times h \rangle = {}_{-\frac{1}{2}}\langle \boldsymbol{\nu} \times e, h \rangle_{\frac{1}{2}} . \tag{2.12}$$

By density argument (2.12) can be extended to $L^2_{\mathrm{rot}}(\Omega; \mathbb{R}^3)$ instead of $H^1(\Omega; \mathbb{R}^3)$.

The boundary operator can be characterised as

$$(\forall (H,E),(h,e)\in \mathscr{W}) \qquad {}_{\mathscr{W}'}\langle D(H,E),(h,e)\rangle_{\mathscr{W}} = \langle \nabla\times E \mid h\rangle - \langle E\mid \nabla\times h\rangle + \langle H\mid \nabla\times e\rangle - \langle \nabla\times H\mid e\rangle\,. \tag{2.13}$$

When H and E are smooth functions, the right side of the above equation can be interpreted as the boundary integral

$$\int_\Gamma (\boldsymbol{\nu}\times E)\cdot h + (\boldsymbol{\nu}\times e)\cdot H$$

One acceptable boundary condition can be imposed by the choice of

$$\mathscr{V} = \widetilde{\mathscr{V}} = L^2_{\mathrm{rot}}(\Omega;\mathbb{R}^3)\times L^2_{\mathrm{rot},0}(\Omega;\mathbb{R}^3) = \{(H,E)\in\mathscr{W} : (E\times\boldsymbol{\nu})|_\Gamma = 0\}\,, \tag{2.14}$$

where the formula appearing in the last description should be understood in terms of the tangential normal trace.

Lemma 2.6.3. Let V and $\widetilde{\mathscr{V}}$ be defined by (2.14). Then, (V)-condition holds.

Proof. Let $(H,E)\in\mathscr{V}$. The extension of (2.12) to $L^2_{\mathrm{rot}}(\Omega)$ implies

$${}_{\mathscr{W}'}\langle D(H,E),(H,E)\rangle_{\mathscr{W}} = 0\,,$$

which means (V1) holds and $\mathscr{V}\subseteq\mathscr{V}^{[\perp]}$. For the second inclusion, let $(h,e)\in\mathscr{V}^{[\perp]}$. Let $(H,E)\in H^1(\Omega)\times L^2_{\mathrm{rot},0}(\Omega)$, then

$$\begin{aligned}0 = {}_{\mathscr{W}'}\langle D(H,E),(H,E)\rangle_{\mathscr{W}} &= \langle H\mid\nabla\times e\rangle - \langle\nabla\times H\mid e\rangle\\ &= {}_{-\frac{1}{2}}\langle e\times\boldsymbol{\nu}, H\rangle_{\frac{1}{2}}\,.\end{aligned}$$

Since, $H\in H^1(\Omega)$ is arbitrary and traces of vector fields in $H^1(\Omega)$ span $H^{1/2}(\Gamma)$, we conclude that $e\times\boldsymbol{\nu}=0$, that is, $(h,e)\in\mathscr{V}$. Hence, $\mathscr{V}^{[\perp]}\subseteq\mathscr{V}$.

■

3. The von Neumann extension theory

The theme of this chapter is to develop results for the classification of all boundary conditions for a given pair of abstract Friedrichs operators. To elaborate on the term *classification of boundary conditions*: Let $(T,\widetilde{T})$ be a pair of abstract Friedrichs operators on a Hilbert space $\mathscr{H}$, with $\mathscr{W}_0,\mathscr{W}$ being the minimal space and the maximal/graph space respectively. We wish to find all pairs of closed subspaces $(\mathscr{V},\widetilde{\mathscr{V}})$ of the graph space $\mathscr{W}$ with $\mathscr{W}_0 \subseteq \mathscr{V}, \mathscr{W}_0 \subseteq \widetilde{\mathscr{V}}$, such that $(T_1|_{\mathscr{V}},\widetilde{T}|_{\widetilde{\mathscr{V}}})$ is a pair of bijective realisations/extensions of $(T,\widetilde{T})$. Moreover, we are also interested in the classification of all bijective realisations with signed boundary map.

The operator theoretic approach of the abstract Friedrichs operators is useful in the application of the *general extension theory* (see [43] and [44, Chapter 13]). An adaption of this theory for abstract Friedrichs operators is studied in [9], which we recall in Section 3.1. In the rest of the chapter we focus on developing the von Neumann extension theory for abstract Friedrichs operators. The theory is well studied for symmetric operators and it is useful in the classification of all selfadjoint (even closed) extensions of symmetric operators.

3.1. General extension theory

Hilbert space formulation of the theory of abstract Friedrichs operators allows an application of the *general extension theory* ([43] and [44, Chapter 13.1]) of (closed and) densely defined operators on Hilbert spaces. While we refer to [44, Chapter 13.1] for details and proofs, here we briefly recall the theory.

Let $(A_0, \widetilde{A}_0)$ be a pair densely defined closed operators in a Hilbert space $\mathscr{H}$ satisfying (T1)-condition. That is for $(\widetilde{A}_0)^* =: A_1$, $(A_0)^* =: \widetilde{A}_1$ the following is satisfied

$$A_0 \subseteq A_1 \quad \text{and} \quad \widetilde{A}_0 \subseteq \widetilde{A}_1\ . \tag{3.1}$$

Let $(A_\mathrm{r}, A_\mathrm{r}^*)$ be a pair of reference operators that are closed, satisfy $A_0 \subseteq A_\mathrm{r} \subseteq A_1$, equivalently $\widetilde{A}_0 \subseteq A_\mathrm{r}^* \subseteq \widetilde{A}_1$, and are invertible with everywhere defined bounded inverses A_r^{-1} and $(A_\mathrm{r}^*)^{-1}$ with $(A_\mathrm{r}^{-1})^* = (A_\mathrm{r}^*)^{-1}$. The domains of the reference operators $\mathrm{dom} A_\mathrm{r}$ and $\mathrm{dom} A_\mathrm{r}^*$ are closed with respect to the graph norms $\|\cdot\|_{A_1}$ and $\|\cdot\|_{\widetilde{A}_1}$. With regard to the reference operators the following decompositions of $\mathrm{dom}(A_1)$ and $\mathrm{dom}(\widetilde{A}_1)$ hold.

Lemma 3.1.1. There are decompositions

$$\mathrm{dom} A_1 \;=\; \mathrm{dom} A_\mathrm{r} \dotplus \ker A_1 \qquad \text{and} \qquad \mathrm{dom} \widetilde{A}_1 \;=\; \mathrm{dom} A_\mathrm{r}^* \dotplus \ker \widetilde{A}_1\,,$$

the corresponding (non-orthogonal) projections

$$\begin{aligned} p_\mathrm{r} &: \mathrm{dom} A_1 \to \mathrm{dom} A_\mathrm{r}\,, \qquad & p_{\tilde{\mathrm{r}}} &: \mathrm{dom} \widetilde{A}_1 \to \mathrm{dom} A_\mathrm{r}^*\,,\\ p_\mathrm{k} &: \mathrm{dom} A_1 \to \ker A_1\,, \qquad & p_{\tilde{\mathrm{k}}} &: \mathrm{dom} \widetilde{A}_1 \to \ker \widetilde{A}_1\,, \end{aligned}$$

satisfy

$$\begin{aligned} p_\mathrm{r} &= A_\mathrm{r}^{-1} A_1\,, \qquad & p_{\tilde{\mathrm{r}}} &= (A_\mathrm{r}^*)^{-1} \widetilde{A}_1\,,\\ p_\mathrm{k} &= \mathbb{1} - p_\mathrm{r}\,, \qquad & p_{\tilde{\mathrm{k}}} &= \mathbb{1} - p_{\tilde{\mathrm{r}}}\,, \end{aligned}$$

and are continuous with respect to the graph norms.

Let $P_\mathscr{V}$ denotes the orthogonal projector from $\mathscr{H}$ to $\mathscr{V}$. Using the previous decompositions we have the following.

Lemma 3.1.2. For $u \in \mathrm{dom} A_1$ and $v \in \mathrm{dom} \widetilde{A}_1$, we have

$$\begin{aligned} \langle A_1 u \mid v \rangle - \langle u \mid \widetilde{A}_1 v \rangle &= \langle A_1 u \mid p_{\tilde{\mathrm{k}}} v \rangle - \langle p_\mathrm{k} u \mid \widetilde{A}_1 v \rangle\\ &= \langle P_{\ker \widetilde{A}_1} A_1 u \mid p_{\tilde{\mathrm{k}}} v \rangle - \langle p_\mathrm{k} u \mid P_{\ker A_1} \widetilde{A}_1 v \rangle \end{aligned}$$

Let (A, A^*) be a pair of mutually adjoint operators such that $A_0 \subseteq A \subseteq A_1$ and equivalently $\widetilde{A}_0 \subseteq A^* \subseteq \widetilde{A}_1$. Any pair of such operators can be parameterised by a pair of densely defined mutually adjoint operators through the closed subspaces of $\ker A_1$ and $\ker \widetilde{A}_1$. More precisely we have the following (cf. [9, Theorem 17]).

Theorem 3.1.3. There is a one-to-one correspondence between all pairs of mutually adjoint operators (A, A^*) with $A_0 \subseteq A \subseteq A_1$, equivalently $\widetilde{A}_0 \subseteq A^* \subseteq \widetilde{A}_1$, and all pairs of densely defined mutually adjoint operators $B : \mathscr{Z} \to \widetilde{\mathscr{Z}}$ and $B^* : \widetilde{\mathscr{Z}} \to \mathscr{Z}$, with domains $\operatorname{dom} B \subseteq \mathscr{Z}$ and $\operatorname{dom} B^* \subseteq \widetilde{\mathscr{Z}}$, where $\mathscr{Z}$ and $\widetilde{\mathscr{Z}}$ run through all closed subspaces of $\ker A_1$ and $\ker \widetilde{A}_1$. The correspondence is given by

$$\begin{aligned}\operatorname{dom} A &= \left\{ u \in \operatorname{dom} A_1 \,:\, p_{\mathrm{k}} u \in \operatorname{dom} B\,,\ P_{\widetilde{\mathscr{Z}}}(A_1 u) = B(p_{\mathrm{k}} u) \right\},\\ \operatorname{dom} A^* &= \left\{ v \in \operatorname{dom} \widetilde{A}_1 \,:\, p_{\tilde{\mathrm{k}}} v \in \operatorname{dom} B^*\,,\ P_{\mathscr{Z}}(\widetilde{A}_1 v) = B^*(p_{\tilde{\mathrm{k}}} v) \right\},\end{aligned} \tag{3.2}$$

and conversely, by

$$\begin{aligned}\operatorname{dom} B &= p_{\mathrm{k}} \operatorname{dom} A\,, & \mathscr{Z} &= \overline{\operatorname{dom} B}\,, & B(p_{\mathrm{k}} u) &= P_{\widetilde{\mathscr{Z}}}(A_1 u)\,,\\ \operatorname{dom} B^* &= p_{\tilde{\mathrm{k}}} \operatorname{dom} A^*\,, & \widetilde{\mathscr{Z}} &= \overline{\operatorname{dom} B^*}\,, & B^*(p_{\tilde{\mathrm{k}}} v) &= P_{\mathscr{Z}}(\widetilde{A}_1 v)\,,\end{aligned}$$

where $P_{\mathscr{Z}}$ and $P_{\widetilde{\mathscr{Z}}}$ are the *orthogonal* projections from $\mathscr{H}$ onto $\mathscr{Z}$ and $\widetilde{\mathscr{Z}}$.

Moreover, in the correspondence above, A is injective, resp. surjective, resp. bijective, if and only if so is B.

Remark 3.1.4. The formulation is completely symmetric in A and A^*, and in B and B^*. The pair (A, A^*) is completely determined by the closed operator A. So it is sufficient to mention only the operator A (and similarly B).

If A_B and A_B^* correspond to the operators B and B^* respectively, then the following is a useful description of the domains.

Theorem 3.1.5. When A_B corresponds to B as in (3.1.3), then

$$\operatorname{dom} A_B = \left\{ w_0 + (A_{\mathrm{r}})^{-1}(B\nu + \tilde{\nu}) + \nu \;\middle|\; \begin{array}{c} w_0 \in \operatorname{dom} A_0 \\ \nu \in \operatorname{dom} B \\ \tilde{\nu} \in \ker \widetilde{A}_1 \ominus \widetilde{\mathscr{Z}} \end{array} \right\},$$

$$A_B\big(w_0 + (A_{\mathrm{r}})^{-1}(B\nu + \tilde{\nu}) + \nu\big) = A_0 w_0 + B\nu + \tilde{\nu}$$

and

$$\operatorname{dom}(A_B)^* = \left\{ \tilde{w}_0 + (A_{\mathrm{r}}^*)^{-1}(B^*\tilde{\mu} + \mu) + \tilde{\mu} \;\middle|\; \begin{array}{c} \tilde{w}_0 \in \operatorname{dom} \widetilde{A}_0 \\ \tilde{\mu} \in \operatorname{dom} B^* \\ \mu \in \ker A_1 \ominus \mathscr{Z} \end{array} \right\},$$

$$(A_B)^*\big(\tilde{w}_0 + (A_{\mathrm{r}}^*)^{-1}(B^*\tilde{\mu} + \mu) + \tilde{\mu}\big) = \widetilde{A}_0 \tilde{w}_0 + B^*\tilde{\mu} + \mu\,,$$

and, moreover

$$(A_B)^* = A_{B^*}.$$

For the trivial choice $\mathscr{Z} = \widetilde{\mathscr{Z}} = \{0\}$ one has $A_B = A_{\mathrm{r}}$.

Here $\ker A_1 \ominus \mathscr{Z}$ denotes the orthogonal complement of $\mathscr{Z}$ in $\ker A_1$.

For a pair of closed abstract Friedrichs operator $(T_0, \widetilde{T}_0)$ on a Hilbert space $\mathscr{H}$, the existence of a pair of reference operators $(T_{\mathrm{r}}, \widetilde{T}_{\mathrm{r}})$ is guaranteed by Proposition 2.3.7 and Theorem 2.4.5 (see also Remark 2.3.6). Since (T1) condition is satisfied by $(T_0, \widetilde{T}_0)$, this general extension theory is applicable in the case of abstract Friedrichs operators (cf. Theorem 2.1.4).

3.2. DECOMPOSITION OF THE GRAPH SPACE

The graph space $\mathscr{W}$ (maximal domain) can be written as a direct sum of the minimal domain $\mathscr{W}_0$ and the kernels of the adjoint operators i.e. $\ker T_1$ and $\ker \widetilde{T}_1$. Before we prove this result, let us establish some preliminary results. In the rest of the section we proceed with the assumption that $(T, \widetilde{T})$ is a joint pair of closed abstract Friedrichs operators.

Theorem 3.2.1. Let $(T, \widetilde{T})$ be a joint pair of closed abstract Friedrichs operators in a Hilbert space $\mathscr{H}$. For any bijective realisation T_{r} of T, we have

$$\operatorname{dom} T_{\mathrm{r}} \;=\; \mathscr{W}_0 \dotplus T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \quad \text{and} \quad \operatorname{dom} T_{\mathrm{r}}^* \;:=\; \mathscr{W}_0 \dotplus (T_{\mathrm{r}}^*)^{-1}(\ker T_1) \tag{3.3}$$

Proof. We prove only the first identity in (3.3), as the second one in completely analogous to it.

- Let us first prove that the sum is direct. Let $u \in \mathscr{W}_0 \cap T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1)$, then for some $\tilde{v} \in \ker \widetilde{T}_1$,

  $$u = T_{\mathrm{r}}^{-1}\tilde{v} \implies T_{\mathrm{r}} u = \tilde{v} \;.$$

  Since, $T_{\mathrm{r}}|_{\mathscr{W}_0} = T$ and $\operatorname{ran} T \cap \ker \widetilde{T}_1 = \{0\}$, we get that $Tu = \tilde{v} = 0$. Injectivity of T gives $u = 0$ (injectivity of T holds since $T \subseteq T_{\mathrm{r}}$ and T_{r} is injective).

- Since $T \subseteq T_{\mathrm{r}}$, we have $\mathscr{W}_0 \subseteq \operatorname{dom} T_{\mathrm{r}}$, while inclusion $T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \subseteq \operatorname{dom} T_{\mathrm{r}}$ is trivial. Hence, $\mathscr{W}_0 + T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \subseteq \operatorname{dom} T_{\mathrm{r}}$. Now let u be an arbitrary element in $\operatorname{dom} T_{\mathrm{r}}$. Since $T_{\mathrm{r}} u \in \mathscr{H} = \operatorname{ran} T \oplus \ker \widetilde{T}_1$ (see Remark 2.4.3), there exist $u_0 \in \mathscr{W}_0$ and $\tilde{v} \in \ker \widetilde{T}_1$ such that $T_{\mathrm{r}} u = T u_0 + \tilde{v} = T_{\mathrm{r}} u_0 + \tilde{v}$. Thus, using that $T_1|_{\operatorname{dom} T_{\mathrm{r}}} = T_{\mathrm{r}}$ is a bijection, we have

  $$u = T_{\mathrm{r}}^{-1} T_{\mathrm{r}} u = T_r^{-1}(T_{\mathrm{r}} u_0 + \tilde{v}) = u_0 + T_{\mathrm{r}}^{-1}(\tilde{v}) \;,$$

  implying $u \in \mathscr{W}_0 + T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1)$, which completes the proof.

■

Corollary 3.2.2. Let T_{r} be a bijective realisation of T, then

$$\mathscr{W} \;=\; \mathscr{W}_0 \dotplus T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \dotplus \ker T_1 \tag{3.4}$$

$$\;=\; \mathscr{W}_0 \dotplus (T_{\mathrm{r}}^*)^{-1}(\ker T_1) \dotplus \ker \widetilde{T}_1 \;. \tag{3.5}$$

Proof. We prove the first identity, as the second one is completely analogous to it. From Grubb's decomposition given by Lemma 3.1.1, we have

$$\mathscr{W} \;=\; \operatorname{dom} T_{\mathrm{r}} \dotplus \ker T_1 \;.$$

Using Theorem 3.2.1 it holds

$$\operatorname{dom} T_{\mathrm{r}} \;=\; \mathscr{W}_0 \dotplus T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \;.$$

Hence, $\mathscr{W} \;=\; \mathscr{W}_0 \dotplus T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \dotplus \ker T_1$. ■

Let us prove that $\mathscr{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1$ is a closed subspace of $\mathscr{W}$.

Lemma 3.2.3. The sum $\mathscr{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1$ is direct and closed in $\mathscr{W}$.

Proof. Let us prove this result in two steps. First we prove that the sum is direct.

- Lemma 2.4.8 implies that $(\mathscr{W}_0 + \ker \widetilde{T}_1, \mathscr{W}_0 + \ker T_1)$ satisfies (V1)-condition and by Lemma 2.4.1, $T_1|_{\mathscr{W}_0 + \ker \widetilde{T}_1}$ and $\widetilde{T}_1|_{\mathscr{W}_0 + \ker T_1}$ are $\mathscr{H}$-coercive. More precisely,

$$\begin{aligned}
&(\forall u \in \mathscr{W}_0 + \ker \widetilde{T}_1) \qquad |\langle\, T_1 u \mid u \,\rangle| \geq \mu_0 \|u\|^2 \,,\\
&(\forall v \in \mathscr{W}_0 + \ker T_1) \qquad |\langle\, \widetilde{T}_1 v \mid v \,\rangle| \geq \mu_0 \|v\|^2 \,.
\end{aligned}$$

  In particular, $T_1|_{\mathscr{W}_0 + \ker \widetilde{T}_1}$ and $\widetilde{T}_1|_{\mathscr{W}_0 + \ker T_1}$ are injective operators. Which means, $\mathscr{W}_0 \cap \ker T_1 = \{0\}$ and $\mathscr{W}_0 \cap \ker \widetilde{T}_1 = \{0\}$. Let $u \in \ker T_1 \cap \ker \widetilde{T}_1$, then $u \in \mathscr{W}_0 + \ker \widetilde{T}_1$ and $T_1 u = 0$. Thus, injectivity implies $u = 0$ and hence $\mathscr{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1$ is a direct sum.

- To prove that $\mathscr{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1$ is a closed subspace of the graph space $\mathscr{W}$, let the sequence $u_n = u_n^0 + \nu_n + \tilde{\nu}_n \in \mathscr{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1$ ($u_n^0 \in \mathscr{W}_0$, $\nu_n \in \ker T_1$, $\tilde{\nu}_n \in \ker \widetilde{T}_1$) converges to $u \in \mathscr{W}$ with respect to the graph norm. $T_1(u_n^0 + \nu_n + \tilde{\nu}_n) = T_1(u_n^0 + \tilde{\nu}_n)$ is a Cauchy sequence in $\mathscr{H}$ and due to coercivity of $T_1|_{\mathscr{W}_0 + \ker \widetilde{T}_1}$, $(u_n^0 + \tilde{\nu}_n)$ is also a Cauchy sequence in $\mathscr{H}$. Let us define $w := \lim_n (u_n^0 + \tilde{\nu}_n) \in \mathscr{H}$ and $\nu := u - w \in \mathscr{H}$. We have,

$$\begin{aligned}
\|\nu_n - \nu\| &= \|(u_n^0 + \nu_n + \tilde{\nu}_n) - u - (u_n^0 + \tilde{\nu}_n - w)\|\\
&\leq \|u_n^0 + \nu_n + \tilde{\nu}_n - u\| + \|u_n^0 + \tilde{\nu}_n - w\| \,,
\end{aligned}$$

implying (v_n) converges to v in $\mathscr{H}$. Since, $\ker T_1$ is closed in $\mathscr{H}$, we get $v \in \ker T_1$ and that (v_n) converges to v in the graph norm. Therefore, $u_n^0 + \tilde{v}_n$ converges to $u - v$ in the graph norm. Which means, $\widetilde{T}_1(u_n^0 + \tilde{v}_n) = \widetilde{T}_1(u_n^0) = \widetilde{T}(u_n^0)$ is a Cauchy sequence in $\mathscr{H}$ and due to the coercivity as above, u_n^0 is also a Cauchy sequence in $\mathscr{H}$. Closedness of $\widetilde{T}$ implies that in fact (u_n^0) is convergent in $\mathscr{W}_0$ (in the graph norm) and let us denote its limit by $u_0 \in \mathscr{W}_0$. Finally, let us define $\tilde{v} := u - u_0 - v$. Analogously as for (v_n), we get that $\tilde{v}_n \xrightarrow{\mathscr{W}} \tilde{v} \in \ker \widetilde{T}_1$. Thus, $u_n^0 + v_n + \tilde{v}_n \xrightarrow{\mathscr{W}} u_0 + v + \tilde{v}$. Uniqueness of the limit implies that $u = u_0 + v + \tilde{v} \in \mathscr{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1$.

■

Theorem 3.2.4. [32, Theorem 3.1] Let $(T, \widetilde{T})$ be a joint pair of closed abstract Friedrichs operators in a Hilbert space $\mathscr{H}$. Then the following decomposition holds:

$$\mathscr{W} \; = \; \mathscr{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1 \; . \tag{3.6}$$

Proof. Lemma 5.4.20 (see also Proposition 2.2.6) together with Lemma (3.2.3) implies that

$$(\mathscr{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1)^{[\perp][\perp]} \; = \; \mathscr{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1 \; . \tag{3.7}$$

By Lemma 2.2.5, $\mathscr{W} = \mathscr{W}_0^{[\perp]}$. Hence, it is sufficient to prove that

$$(\mathscr{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1)^{[\perp]} \; = \; \mathscr{W}_0 \; . \tag{3.8}$$

Since $\mathscr{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1 \subseteq \mathscr{W}$ and $\mathscr{W}^{[\perp]} = \mathscr{W}_0$, we have $\mathscr{W}_0 \subseteq (\mathscr{W}_0 \dotplus \ker T_1 \dotplus \widetilde{T}_1)^{[\perp]}$. To prove the other inclusion, let $u \in (\mathscr{W}_0 \dotplus \ker T_1 \dotplus \widetilde{T}_1)^{[\perp]}$ and let T_{r} be a bijective realisation of T (existence is ensured due to Theorem 2.4.5 and Proposition 2.3.7). Since $u \in \mathscr{W}$, by Corollary 3.2.2 there exist unique $u_0 \in \mathscr{W}_0$, $v \in \ker T_1$ and $\tilde{v} \in \ker \widetilde{T}_1$ such that $u = u_0 + T_{\mathrm{r}}^{-1}(\tilde{v}) + v$. For arbitrary $v_0 \in \mathscr{W}_0$, $v_1 \in \ker T_1$ and $\tilde{v}_1 \in \ker \widetilde{T}_1$ we have

$$\begin{aligned} 0 = [\, u \mid v_0 + v_1 + \tilde{v}_1 \,] &= [\, u_0 + T_{\mathrm{r}}^{-1}(\tilde{v}) + v \mid v_0 + v_1 + \tilde{v}_1 \,] \\ &= [\, T_{\mathrm{r}}^{-1}(\tilde{v}) + v \mid v_1 + \tilde{v}_1 \,] \\ &= [\, T_{\mathrm{r}}^{-1}(\tilde{v}) \mid v_1 \,] + [\, T_{\mathrm{r}}^{-1}(\tilde{v}) \mid \tilde{v}_1 \,] + [\, v \mid v_1 \,] + [\, v \mid \tilde{v}_1 \,] \\ &= [\, T_{\mathrm{r}}^{-1}(\tilde{v}) \mid v_1 \,] + [\, T_{\mathrm{r}}^{-1}(\tilde{v}) \mid \tilde{v}_1 \,] + [\, v \mid v_1 \,] \, . \end{aligned}$$

In the third equality we used $\ker D = \mathscr{W}_0$ (i.e. $\mathscr{W}_0 = \mathscr{W}^{[\perp]}$) and in the last equality we used $\ker T_1 [\perp] \ker \widetilde{T}_1$ (see Remark 2.4.10). If we choose $v_1 = 0$ and $\tilde{v}_1 = \tilde{v}$, then we get

$$0 = [\, T_{\mathrm{r}}^{-1}(\tilde{v}) \mid \tilde{v}\,] = \langle\, T_1 T_{\mathrm{r}}^{-1}(\tilde{v}) \mid \tilde{v}\,\rangle = \langle\, \tilde{v} \mid \tilde{v}\,\rangle = \|\tilde{v}\|^2 ,$$

where in the second equality we used the definition of $[\,\cdot \mid \cdot\,]$ (see (2.3)) and $\tilde{v} \in \ker \widetilde{T}_1$, while in the third equality we used the fact that $T_{\mathrm{r}} \subseteq T_1$. Thus, $\tilde{v} = 0$ and we have $0 = [\, v \mid v_1\,]$. But, we can choose $v_1 = v$ and apply the $\mathscr{H}$-coercivity of $\widetilde{T}_1|_{\mathscr{W}_0 \dot{+} \ker T_1}$, to get

$$0 = |[\, v \mid v\,]| = |\langle \widetilde{T}_1 v \mid v\,\rangle| \geq \mu_0 \|v\|^2 .$$

Therefore, $v = 0$ as well. Thus $u = u_0 \in \mathscr{W}_0$ and hence $(\mathscr{W}_0 \dot{+} \ker T_1 \dot{+} \widetilde{T}_1)^{[\perp]} \subseteq \mathscr{W}_0$. This completes the proof. ■

Remark 3.2.5. The above decomposition (3.6) of the graph space implies that $\mathscr{W}/\mathscr{W}_0 \cong \ker T_1 \dot{+} \ker \widetilde{T}_1$, which reveals that the *room* for choosing different boundary conditions for abstract problem $T_1 u = f$ is given by $\ker T_1 \dot{+} \ker \widetilde{T}_1$. Thus, the knowledge of adjoint operators completely describes the problem.

Another consequence of the decomposition (3.6) is that $(T_1|_{\mathscr{W}_0 \dot{+} \ker \widetilde{T}_1}, \widetilde{T}_1|_{\mathscr{W}_0 \dot{+} \ker T_1})$ is a pair of mutually adjoint bijective realisations relative to $(T_0, \widetilde{T}_0)$.

Corollary 3.2.6. The pair of subspaces $(\mathscr{W}_0 \dot{+} \ker \widetilde{T}_1, \mathscr{W}_0 \dot{+} \ker T_1)$ satisfies (V)-boundary conditions.

Proof. By Lemma 2.4.8, the pair satisfies (V1)-condition. It is sufficient to prove (V2)-condition. More precisely, we need to prove

$$\mathscr{W}_0 \dot{+} \ker T_1 = (\mathscr{W}_0 \dot{+} \ker \widetilde{T}_1)^{[\perp]} \qquad \text{and} \qquad \mathscr{W}_0 \dot{+} \ker \widetilde{T}_1 = (\mathscr{W}_0 \dot{+} \ker T_1)^{[\perp]} .$$

Here, we present the proof of the first equality only, as the proof of the second one is completely analogous to it. Let $u_0 + v_1 \in \mathscr{W}_0 \dot{+} \ker T_1$, then for any $v_0 + \tilde{v}_1 \in \mathscr{W}_0 \dot{+} \ker \widetilde{T}_1$, we have

$$[\, u_0 + v_1 \mid v_0 + \tilde{v}_1\,] = [\, v_1 \mid \tilde{v}_1\,] = 0 .$$

We used that $\ker D = \mathscr{W}_0$ and $\ker T_1 [\perp] \ker \widetilde{T}_1$ (see Remark 2.4.10). This means, $u_0 + v_1 \in (\mathscr{W}_0 \dot{+} \ker \widetilde{T}_1)^{[\perp]}$. To prove the other inclusion, let $u \in (\mathscr{W}_0 \dot{+} \ker \widetilde{T}_1)^{[\perp]}$. Then by the decomposition (3.6) there exist $u_0 \in \mathscr{W}_0$, $\nu \in \ker T_1$ and $\tilde{\nu} \in \ker \widetilde{T}_1$ such that $u = u_0 + \nu + \tilde{\nu}$. For any $v_0 \in \mathscr{W}_0$ and $\tilde{v}_1 \in \ker \widetilde{T}_1$ we have

$$\begin{aligned} 0 = [\, v_0 + \tilde{v}_1 \mid u \,] &= [\, v_0 + \tilde{v}_1 \mid u_0 + \nu + \tilde{\nu} \,] \\ &= [\, \tilde{v}_1 \mid \nu \,] + [\, \tilde{v}_1 \mid \tilde{\nu} \,] \\ &= [\, \tilde{v}_1 \mid \tilde{\nu} \,], \end{aligned}$$

here we again used $\ker D = \mathscr{W}_0$ and $\ker T_1 [\perp] \ker \widetilde{T}_1$. By choosing, $\tilde{v}_1 = \tilde{\nu}$, we get $[\, \tilde{\nu} \mid \tilde{\nu} \,] = 0$. Further,

$$[\, \tilde{\nu} \mid \tilde{\nu} \,] = \langle T_1 \tilde{\nu} \mid \tilde{\nu} \rangle = \langle (T_1 + \widetilde{T}_1)\tilde{\nu} \mid \tilde{\nu} \rangle \geq 2\mu_0 \|\tilde{\nu}\|^2 ,$$

implies $\tilde{\nu} = 0$. In the last inequality we used (T3)-condition. Therefore, $u = u_0 + \nu \in \mathscr{W}_0 \dot{+} \ker T_1$, completing the proof. ■

Remark 3.2.7. We have a concrete pair $(T_1|_{\mathscr{W}_0 \dot{+} \ker \widetilde{T}_1}, \widetilde{T}_1|_{\mathscr{W}_0 \dot{+} \ker T_1})$ of mutually adjoint bijective realisations for a given pair of Friedrichs operators. Moreover, for this pair we have that the sum of the domains is equal to $\mathscr{W}$. In particular, the sum is closed.

Remark 3.2.8. In the case of finite dimensional kernels, i.e. when $\dim \ker T_1 < \infty$ and $\dim \ker \widetilde{T}_1 < \infty$, the statement of Theorem 3.2.4 is a direct consequence of Corollary 3.2.2. Here, we have that $\mathscr{W}/\mathscr{W}_0 \cong T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \dot{+} \ker T_1$ and since, $T_r : \operatorname{dom} T_r \to \mathscr{H}$ is a bijection, we get

$$\begin{aligned} \dim(\mathscr{W}/\mathscr{W}_0) &= \dim\big(T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \dot{+} \ker T_1\big) \\ &= \dim(\ker \widetilde{T}_1) + \dim(\ker T_1) < \infty , \end{aligned}$$

hence the codimension is finite.

On the other hand, obviously $(\mathscr{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1)/\mathscr{W}_0 \subseteq \mathscr{W}/\mathscr{W}_0$ and (since the sum is direct)

$$\begin{aligned} \dim\big((\mathscr{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1)/\mathscr{W}_0\big) &= \dim(\ker T_1) + \dim(\ker \widetilde{T}_1) \\ &= \dim(\mathscr{W}/\mathscr{W}_0) . \end{aligned}$$

Therefore, we get $(\mathscr{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1)/\mathscr{W}_0 = \mathscr{W}/\mathscr{W}_0$, and hence $\mathscr{W} = \mathscr{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1$.

Lemma 3.2.9. The non-orthogonal projections

$$p_{\mathrm{k}} : \mathscr{W} \to \ker T_1 \quad \text{and} \quad p_{\tilde{\mathrm{k}}} : \mathscr{W} \to \ker \widetilde{T}_1 \,,$$

corresponding to the decomposition $\mathscr{W} = \mathscr{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1$, are continuous with respect to the graph norm. Moreover,

$$(\forall u, v \in \mathscr{W}) \qquad [\, u \mid v \,] = [\, p_{\mathrm{k}} u \mid p_{\mathrm{k}} v \,] + [\, p_{\tilde{\mathrm{k}}} u \mid p_{\tilde{\mathrm{k}}} v \,] \,.$$

Proof. On the kernels the graph norm and the standard norm of $\mathscr{H}$ are the same, so we can assume kernels are equipped with the $\mathscr{H}$ norm. Let $u_n := u_{0,n} + \nu_{1,n} + \tilde{\nu}_{1,n} \in \mathscr{W}$ be a sequence that converges to $u := u_0 + \nu_1 + \tilde{\nu}_1 \in \mathscr{W}$. Following the argument in the proof of Lemma 3.2.3, we get that $u_{0,n}$, $\nu_{1,n}$ and $\tilde{\nu}_{1,n}$ converge to u_0, ν_1 and $\tilde{\nu}_1$ respectively. This means $p_{\mathrm{k}} u_n$ and $p_{\tilde{\mathrm{k}}} u_n$ converge to $p_{\mathrm{k}} u$ and $p_{\tilde{\mathrm{k}}} u$ respectively, therefore p_{k} and $p_{\tilde{\mathrm{k}}}$ are continuous. Finally, for any $u, v \in \mathscr{W}$, we have

$$[\, u \mid v \,] = [\, p_{\mathrm{k}} u + p_{\tilde{\mathrm{k}}} u \mid p_{\mathrm{k}} v + p_{\tilde{\mathrm{k}}} v \,] = [\, p_{\mathrm{k}} u \mid p_{\mathrm{k}} v \,] + [\, p_{\tilde{\mathrm{k}}} u \mid p_{\tilde{\mathrm{k}}} v \,] \,.$$

In the first equality we used $\mathscr{W}_0 = \ker D$ and the second equality is due to the fact that $\ker T_1 [\perp] \ker \widetilde{T}_1$ (see Remark 2.4.10). ■

Theorem 3.2.10. Let $\mathscr{V}$ be a closed subspace of $\mathscr{W}$ such that $\mathscr{W}_0 \subseteq \mathscr{V}$. Then $T_1|_{\mathscr{V}}$ is bijective if and only if $\mathscr{V} \dotplus \ker T_1 = \mathscr{W}$.

Proof. The first implication is followed from the decomposition given in Lemma 3.1.1.

For the converse, $\mathscr{V} \cap \ker T_1 = \{0\}$ implies that $T_1|_{\mathscr{V}}$ is injective. Now let $f \in \mathscr{H}$. Since $T_1 : \mathscr{W} \to \mathscr{H}$ is surjective (see Remark 2.4.7), there exists $u \in \mathscr{W}$ such that $T_1 u = f$. We also have for some $v \in \mathscr{V}$ and $\nu \in \ker T_1$, $u = v + \nu$. Thus,

$$f = T_1 u = T_1(v + \nu) = T_1 v = (T_1|_{\mathscr{V}})v$$

implies that $T_1|_{\mathscr{V}}$ is surjective. Hence, $T_1|_{\mathscr{V}}$ is a bijective realisation. ■

Remark 3.2.11. For any closed subspace $\mathscr{V}$ of $\mathscr{W}$ such that $\mathscr{W}_0 \subseteq \mathscr{V}$ and $T_1|_{\mathscr{V}}$ is bijective we have $\mathscr{V} / \mathscr{W}_0 \cong \ker \widetilde{T}_1$.

Remark 3.2.12. Not all bijective realisations, characterised in Theorem 3.2.10, are bijective realisations with signed boundary map (e.g. see Example 3.3.4 below), i.e. $\mathscr{V} \subseteq$

$\mathscr{W}^+$ and $\mathscr{V}^{[\perp]} \subseteq \mathscr{W}^-$ is only a sufficient condition. However, we have the following equivalence: $T_1|_{\mathscr{V}}$ is bijective with $\mathscr{V} \subseteq \mathscr{W}^+$ if and only if $\widetilde{T}_1|_{\mathscr{V}^{[\perp]}}$ is bijective with $\mathscr{V}^{[\perp]} \subseteq \mathscr{W}^-$. Thus, there is no need in considering pairs of bijective realisations with signed boundary map, but we can denote (in this case) each of $T_1|_{\mathscr{V}}$ and $\widetilde{T}_1|_{\mathscr{V}^{[\perp]}}$ as a bijective realisation with signed boundary map of T_1 and $\widetilde{T}_1$, respectively. Let us comment on this.

By Remark 2.3.6 we have equivalence on the level of bijectivity. Let us assume that $\mathscr{V} \subseteq \mathscr{W}^+$ and let us denote by $\mathscr{V}_1$ a maximal subspace of $\mathscr{W}$ such that $\mathscr{V} \subseteq \mathscr{V}_1 \subseteq \mathscr{W}^+$ (it exists by Zorn's lemma; cf. [14, Section I.6]). Then $\mathscr{V}_1^{[\perp]} \subseteq \mathscr{W}^-$ (cf. [14, Lemma 6.3] or [3, Theorem 2(b)] for a perspective in the context of abstract Friedrichs operators), hence $T_1|_{\mathscr{V}_1}$ is bijective as well. Since $\mathscr{V} \subseteq \mathscr{V}_1$, it must be $\mathscr{V} = \mathscr{V}_1$. Hence, $\mathscr{V}^{[\perp]} = \mathscr{V}_1^{[\perp]} \subseteq \mathscr{W}^-$. The opposite implication can be proved analogously.

The previous argument actually shows that a subspace $\mathscr{V} \subseteq \mathscr{W}^+$ with the property of $T_1|_{\mathscr{V}}$ being bijective is maximal nonnegative subspace, i.e. if for a subspace $\mathscr{V}_1 \subseteq \mathscr{W}$ we have $\mathscr{V} \subseteq \mathscr{V}_1 \subseteq \mathscr{W}^+$, then it must be $\mathscr{V} = \mathscr{V}_1$. Then it is known that such $\mathscr{V}$ defines a pair of bijective realisations with signed boundary map (see [3, Theorem 2]). An argument on the existence of such subspaces using (X)-condition is made in Remark 2.5.5.

3.3. CHARACTERISATION OF ABSTRACT FRIEDRICHS OPERATORS

The von Neumann theory is well studied for the symmetric operators (classification of all self-adjoint extensions). We derive a simple and explicit characterisation of abstract Friedrichs operators (see Definition 2.1.1 and the previous characterisation given by Theorem 2.1.2) that allows us to connect the theory of abstract Friedrichs operators with the well-established theory of symmetric operators. We also present the von Neumann theory for abstract Friedrichs operators. The results of this and the following section are available in [33].

3.3.1. Characterisation

Theorem 3.3.1. [33, Theorem 3.1] A pair of densely defined operators $(T_0,\widetilde{T}_0)$ on $\mathscr{H}$ satisfies (T1) and (T2) if and only if there exist a densely defined skew-symmetric operator L_0 and a bounded self-adjoint operator S, both on $\mathscr{H}$, such that

$$T_0 = L_0 + S \qquad \text{and} \qquad \widetilde{T}_0 = -L_0 + S\,. \tag{3.9}$$

For a given pair, the decomposition (3.9) is unique.

If in the above we include condition (T3), then the same holds with S being strictly positive, i.e.

$$\langle Su \mid u\rangle \geq \mu_0\|u\|^2\,, \quad u\in\mathscr{H}\,,$$

where $\mu_0 > 0$ is the constant appearing in (T3).

Proof. Let $(T_0,\widetilde{T}_0)$ satisfies (T1) and (T2). Then we define $S := \frac{1}{2}\overline{T_0+\widetilde{T}_0}$, which is a bounded and self-adjoint operator by Theorem 2.1.2(iii). Moreover, if condition (T3) is satisfied as well, it has a positive lower bound. Therefore, by

$$T_0 = \frac{T_0-\widetilde{T}_0}{2} + \frac{T_0+\widetilde{T}_0}{2} \qquad \text{and} \qquad \widetilde{T}_0 = -\frac{T_0-\widetilde{T}_0}{2} + \frac{T_0+\widetilde{T}_0}{2}$$

it is left to prove that $L_0 := \frac{T_0-\widetilde{T}_0}{2}$ is skew-symmetric. Since $\operatorname{dom} L_0 = \operatorname{dom} T_0 \cap \operatorname{dom}\widetilde{T}_0 = \mathscr{D}$, L_0 is densely defined. Furthermore, using $L_0 = S-\widetilde{T}_0$ and the boundedness of S we get

$$L_0^* = S^* - \widetilde{T}_0^* = S - T_1 = S|_{\mathscr{W}} - T_1 = \frac{T_1 + \widetilde{T}_1}{2} - T_1 = -\frac{T_1 - \widetilde{T}_1}{2} \supseteq -L_0 \,,$$

where in addition we used the second part of Theorem 2.1.2(iii). The uniqueness of operators L_0 and S follows by a standard argument.

The converse follows easily by direct inspection. ■

Remark 3.3.2. From the proof of the previous lemma it is easy to see that for general mutually adjoint closed realisations $(T_1|_{\mathscr{V}}, \widetilde{T}_1|_{\mathscr{V}^{[\perp]}})$ (see Theorem 2.3.4) we have

$$\big((T_1 - \widetilde{T}_1)|_{\mathscr{V}}\big)^* = -(T_1 - \widetilde{T}_1)|_{\mathscr{V}^{[\perp]}} \,.$$

Note that for $\mathscr{V} = \mathscr{W}_0$ we have the identity obtained in the proof of the previous theorem.

By Theorem 3.3.1, the study of (pairs of) abstract Friedrichs operators is reduced to the study of operators of the form (3.9), which, in our opinion, makes the situation much more explicit (cf. [25, Remark 4.3]). Let us illustrate some straightforward conclusions. For a pair of abstract Friedrichs operators $(T_0, \widetilde{T}_0)$, let L_0 and S be operators given in Theorem 3.3.1. If we denote $L_1 := -L_0^* \supseteq L_0$, then we have

$$\begin{aligned} T_0 &= L_0 + S \,, & \widetilde{T}_0 &= -L_0 + S \,, \\ T_1 &= L_1 + S \,, & \widetilde{T}_1 &= -L_1 + S \,. \end{aligned}$$

In particular, $\mathscr{W}_0 = \operatorname{dom} \overline{L}_0$ and $\mathscr{W} = \operatorname{dom} L_1$, i.e. spaces $\mathscr{W}_0$ and $\mathscr{W}$ are independent of S. This is also clear by noting that the graph norms $\|\cdot\|_{T_1}$ and $\|\cdot\|_{L_1}$ are equivalent, due to the boundedness of S. The same holds for the sesquilinear map $[\,\cdot \mid \cdot\,] := {}_{\mathscr{W}'}\langle D(\cdot), \cdot \rangle_{\mathscr{W}}$ (see (2.3) and Lemma 2.2.5(i)) since

$$[\, u \mid v \,] = \langle L_1 u \mid v \rangle + \langle u \mid L_1 v \rangle \,, \quad u, v \in \mathscr{W} \,. \tag{3.10}$$

Thus, all conditions on subspaces $\mathscr{V} \subseteq \mathscr{W}$ given in Theorem 2.4.5 depend only on L_1 (i.e. L_0). In particular, we can formulate the following corollary.

Corollary 3.3.3. Let $(T_0, \widetilde{T}_0)$ be a joint pair of abstract Friedrichs operators on $\mathscr{H}$ and let $\mathscr{V} \subseteq \mathscr{W}$ be a closed subspace containing $\mathscr{W}_0$ such that $\mathscr{V} \subseteq \mathscr{W}^+$ and $\mathscr{V}^{[\perp]} \subseteq \mathscr{W}^-$ (with respect to $(T_0, \widetilde{T}_0)$). For any joint pair of abstract Friedrichs operators $(A_0, \widetilde{A}_0)$ on $\mathscr{H}$ such that

$$(A_0 - \widetilde{A}_0)^* = (T_0 - \widetilde{T}_0)^*$$

we have that $\big((\widetilde{A}_0)^*|_{\mathscr{V}},(A_0)^*|_{\mathscr{V}^{[\perp]}}\big)$ is a pair of bijective realisations with signed boundary map.

Not all domains of bijective realisations have the feature described in the previous corollary, i.e. there are subspaces $\mathscr{V}\subseteq\mathscr{W}$ such that realisations $T=L_1|_{\mathscr{V}}+S$ are bijective for some admissible S, but not all. Furthermore, if a subspace $\mathscr{V}\subseteq\mathscr{W}$ defines bijective realisations for any admissible S, that does not imply that $\mathscr{V}\subseteq\mathscr{W}^+$ and $\mathscr{V}^{[\perp]}\subseteq\mathscr{W}^-$. All this can be illustrated by the following simple example.

Example 3.3.4. Let $a<b$, $\mathscr{H}=L^2((a,b);\mathbb{R})$ (for simplicity we consider only real functions) and $\mathscr{D}=C_c^\infty(a,b)$. For $\mu>0$ and $\beta\in L^\infty(a,b)$ such that $\beta\geq\mu$ a.e. on (a,b), we consider operators $T_0,\widetilde{T}_0:\mathscr{D}\to\mathscr{H}$ given by

$$T_0u=u'+\beta u\,,\quad \widetilde{T}_0u=-u'+\beta u\,.$$

Then it is easy to see that $(T_0,\widetilde{T}_0)$ is a joint pair of abstract Friedrichs operators, while $\mathscr{W}=H^1(a,b)$ (which is embedded into $C([a,b])$; see [16, Theorem 8.2] or [44, Theorem 4.13]) and $\mathscr{W}_0=H_0^1(a,b)$. Of course, in the notation of Theorem 3.3.1, here we have $L_0u=u'$ and $Su=\beta u$. Then $L_1u:=-L_0^*u=u'$ (here the derivative is in the weak sense), $T_1=L_1+S$, $\widetilde{T}_1=-L_1+S$, and

$$[\,u\mid v\,]=u(b)v(b)-u(a)v(a)\,,\quad u,v\in\mathscr{W}\,.$$

Let us comment on all bijective realisations of operators T_0 and $\widetilde{T}_0$, i.e. all bijective restrictions of T_1 and $\widetilde{T}_1$.

We define closed subspaces $\mathscr{V}_\alpha\subseteq\mathscr{W}$, $\alpha\in\mathbb{R}\cup\{\infty\}$ (here we identify $-\infty$ and $+\infty$), by

$$\mathscr{V}_\alpha:=\big\{u\in\mathscr{W}:u(b)=\alpha u(a)\big\}\,,\quad \alpha\in\mathbb{R}\,,$$

and $\mathscr{V}_\infty:=\big\{u\in\mathscr{W}:u(a)=0\big\}$. Since $(L_1|_{\mathscr{V}_\alpha})^*=-L_1|_{\mathscr{V}_{\frac{1}{\alpha}}}$ (this can be verified by direct calculations), we have $(T_1|_{\mathscr{V}_\alpha})^*=\widetilde{T}_1|_{\mathscr{V}_{\frac{1}{\alpha}}}$, where we use the convention: $\frac{1}{\infty}=0$ and $\frac{1}{0}=\infty$. Thus, we want to see for which values of α,

$$\big(T_1|_{\mathscr{V}_\alpha},\widetilde{T}_1|_{\mathscr{V}_{\frac{1}{\alpha}}}\big) \tag{3.11}$$

is a pair of mutually adjoint bijective realisations (with signed boundary map).

By Theorem 3.2.10 (see also Remark 4.2.3 below) we have that *all* mutually adjoint bijective realisations are given for $\alpha \in \mathbb{R} \cup \{\infty\} \setminus \{\alpha_\beta\}$, where $\alpha_\beta := e^{-\int_a^b \beta(y)dy}$. Indeed, since $\ker T_1 = \operatorname{span}\{e^{-\int_a^x \beta(y)dy}\}$ we have $\ker T_1 \subseteq \mathscr{V}_{\alpha_\beta}$. Note that for $\alpha \in \{-1, 1\}$ we have $\mathscr{V}_\alpha = \mathscr{V}_{\frac{1}{\alpha}}$, hence $L_1|_{\mathscr{V}_\alpha}$ is skew-selfadjoint.

By direct inspection we get that only for $\alpha \in (-1, 1)$ bijective realisations are not with signed boundary map (one can also consider $\beta \equiv \mu$ by Corollary 3.3.3 and read the result from [2, Example 1]). This is in the correspondence with the above result since $\alpha_\beta \in (-1, 1)$. More precisely, we have $\alpha_\beta \in (0, 1)$ and by varying μ and β one can get any number in that interval for α_β.

Therefore, for $\alpha \in \mathbb{R} \cup \{\infty\} \setminus (-1, 1)$ the corresponding domains, i.e. boundary conditions, give rise to bijective realisations independent of the choice of admissible β (see Corollary 3.3.3). The same holds for $\alpha \in (-1, 0]$ although these bijective realisations are not with signed boundary map. Here the reason lies in the fact that $\mathscr{V}_\alpha \cap \ker T_1 = \{0\}$ (Theorem 3.2.10) for any $\alpha \in (-1, 0]$ and any choice of admissible β. Finally, there is no α in $(0, 1)$ with this property. However, for fixed β, all $\alpha \in (0, 1)$ but one ($\alpha = \alpha_\beta$) correspond to mutually adjoint bijective realisations (3.11).

We will return to this example to consider general symmetric parts.

Remark 3.3.5. If we consider $T_0 = L_0 + C$ where L_0 is skew-symmetric and C bounded, then it is easy to see that the discussion preceding Corollary 3.3.3 still holds. More precisely, spaces $\mathscr{W}_0$, $\mathscr{W}$ and indefinite inner product $[\cdot \mid \cdot]$ are independent of C ((3.10) holds precisely as it is; recall that $L_1 = -L_0^*$) and the graph norm is equivalent with $\|\cdot\|_{L_1}$ (cf. [22, Subsection 2.2]). Thus, these objects depend only on unbounded part of the skew-symmetric part of T_0.

3.3.2. Deficiency indices

The previous example illustrates that in order to get all bijective realisations (not only with signed boundary map) it is not enough to consider L_0 alone, we must also bring the symmetric part S into play. In particular, information on kernels $\ker T_1$ and $\ker \widetilde{T}_1$ is essential. By Theorem 3.2.4 we have

$$\mathscr{W} = \mathscr{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1 .$$

Hence, the sum of dimensions of the kernels is constant and equals the codimension of $\mathscr{W}_0$ in $\mathscr{W}$. However, from here we cannot conclude that (cardinal) numbers $\dim\ker T_1$ and $\dim\ker\widetilde{T}_1$ are independent of S, where $T_1 = L_1 + S$. If so, this would be beneficial in the analysis (see Example 3.3.8 below). Let us motivate why one should expect such a result. Since L_0 is skew-symmetric, we have that $-iL_0$ is symmetric. Thus, for any positive constant $\beta > 0$ we have

$$\dim\ker(L_1 + \beta\mathbb{1}) = \dim\ker(iL_0^* - i\beta\mathbb{1}) = \dim\ker((-iL_0)^* - i\beta\mathbb{1}) = d_+(-iL_0)\,,$$

where $\mathbb{1}$ denotes the identity operator and on the right we have the deficiency index (or the defect number) of $-iL_0$ (note that $d_+ := \dim\ker(A - i\mathbb{1})$ and $d_+ := \dim\ker(A + i\mathbb{1})$ are called the deficiency indices of the operator A), which is known to be independent of $\beta > 0$ (see [53, Section 3.1]). Analogously, $\dim\ker(L_1 - \beta\mathbb{1}) = d_-(-iL_0)$. Therefore, all that we need is to show that instead of $\beta\mathbb{1}$ we can put an arbitrary bounded self-adjoint strictly positive operator. Below is a slightly more general statement.

Lemma 3.3.6. Let L_0 be a densely defined skew-symmetric operator and let us denote $L_1 := -L_0^*$. For a bounded linear operator C with strictly positive symmetric part $\frac{1}{2}(C + C^*)$, we define

$$d_+^C(L_0) := \dim\ker(L_1 + C) \quad \text{and} \quad d_-^C(L_0) := \dim\ker(L_1 - C)\,.$$

Then $d_+^C(L_0)$ and $d_-^C(L_0)$ are independent of C, i.e. $d_\pm^C(L_0) = d_\pm(-iL_0)$.

Proof. Since L_0 is closable and $d_\pm^C(L_0) = d_\pm^C(\overline{L_0})$, we can assume that L_0 is closed. We shall prove the claim for $d_+^C(L_0)$, while the same argument applies on $d_-^C(L_0)$.

Let us take arbitrary bounded operators C and C' with strictly positive symmetric parts, and let us denote by μ and μ' the greatest lower bounds of their symmetric parts, respectively. We shall first argue in a specific situation when $\|C - C'\| < \min\{\mu, \mu'\}$, where here $\|\cdot\|$ denotes the operator norm. Before we start, let us note that according to Theorem 3.3.1, both operators $L_0 + C$ and $L_0 + C'$ define a (pair of) abstract Friedrichs operators (the skew-symmetric part is equal to the sum of L_0 (unbounded part) and the skew-symmetric part of C (bounded part)). Hence, all results related to abstract Friedrichs operators are applicable.

If $d_+^C(L_0) > d_+^{C'}(L_0)$, then there exists $0 \neq v \in \ker(L_1 + C) \cap \ker(L_1 + C')^\perp$ ([53, Lemma 2.3]). Since L_0 is closed, we have $\ker(L_1 + C')^\perp = \operatorname{ran}(-L_0 + C')$. Thus, we also have $v \in \operatorname{ran}(-L_0 + C)^\perp \cap \operatorname{ran}(-L_0 + C')$. Let $0 \neq u \in \operatorname{dom} L_0$ be such that $v = (-L_0 + C')u$. Then it holds

$$\langle (-L_0 + C')u \mid (-L_0 + C)u \rangle = 0 . \tag{3.12}$$

Since the identity (3.12) is symmetric with respect to C and C', the same holds even if we start with the assumption $d_+^C(L_0) < d_+^{C'}(L_0)$. However, then we have $(-L_0 + C)u \neq 0$. Repeating the last calculations with C' and C swapping places, we come to the analogous conclusion: $\mu \leq \|C - C'\| < \mu$. Therefore, it must be $d_+^C(L_0) = d_+^{C'}(L_0)$.

Finally, it is left to prove the statement without the additional assumption $\|C - C'\| < \min\{\mu, \mu'\}$. This easily follows by noting that the set of all bounded operators on $\mathscr{H}$ with strictly positive symmetric part is convex. More precisely, for each $\lambda \in [0,1]$ we have that $C_\lambda := \lambda C + (1-\lambda)C'$ is bounded and the greatest lower bound of its symmetric part is $\lambda\mu + (1-\lambda)\mu' \geq \min\{\mu, \mu'\}$. Moreover, $\|C_{\lambda_1} - C_{\lambda_2}\| = |\lambda_1 - \lambda_2| \|C - C'\|$. Thus, we can pick finitely many values $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_m = 1$ such that $\|C_{\lambda_j} - C_{\lambda_{j+1}}\| < \min\{\mu, \mu'\}$, $j = 1, 2, \ldots, , m-1$. Therefore, by applying the previously obtained result, we get

$$d_+^{C'}(L_0) = d_+^{C_{\lambda_1}}(L_0) = d_+^{C_{\lambda_2}}(L_0) = \cdots = d_+^{C_{\lambda_m}}(L_0) = d_+^C(L_0) ,$$

concluding the proof. ■

Remark 3.3.7. For a densely defined skew-symmetric operator L_0 we will refer to the cardinal numbers $d_\pm(-iL_0)$ as the deficiency indices (or the defect numbers) of L_0, and we introduce the notation $d_\pm(L_0) := d_\pm(-iL_0)$. The definition is not ambiguous because depending on whether the operator is symmetric or skew-symmetric the corresponding definition applies.

Let us return to the analysis of Example 3.3.4.

Example 3.3.8. In Example 3.3.4 we studied specific (multiplicative) symmetric parts. Let us now consider a general situation where

$$T_0 u = u' + Cu , \quad \widetilde{T}_0 u = -u' + Cu ,$$

for an arbitrary bounded linear operator C with strictly positive symmetric part $\frac{1}{2}(C+C^*)$.

First note that by Example 3.3.4 (cf. [9, Subsection 6.1]) and Lemma 3.3.6 we have $\dim\ker T_1 = \dim \widetilde{T}_1 = 1$ (for any admissible C).

The conclusion of Example 3.3.4 for the range $\mathbb{R}\cup\{\infty\}\setminus(-1,1)$ remains the same (cf. Corollary 3.3.3), i.e. for these values of α we get for any admissible C bijective realisations (even with signed boundary map).

Let us take $\alpha\in(-1,0)$. Since the codimension of $\mathscr{V}_\alpha$ in $\mathscr{W}$ equals 1, and $\dim\ker T_1 = 1$, by Theorem 3.2.10 it is sufficient to prove that $\mathscr{V}_\alpha$ and $\ker T_1 = \operatorname{span}\{\varphi_C\}$ have a trivial intersection to get that the corresponding realisations are bijective. Let us assume on the contrary that $\varphi_C\in\mathscr{V}_\alpha$. Since $\alpha<0$, we have $\varphi_C(a)\varphi_C(b)<0$. Thus, recalling that $\mathscr{W}\hookrightarrow C([a,b])$, there exists $c\in(a,b)$ such that $\varphi_C(c)=0$. Moreover, $\varphi_C\in\ker T_1$ implies that $\varphi_C'+C\varphi_C=0$ in (c,b) as well. This together with $\varphi_C(c)=0$ implies that $\varphi_C\equiv 0$ in (c,b). Indeed, just recall that $\mathscr{V}_\infty$ defines a bijective realisation. In particular, we have $\varphi_C(b)=0$, implying $\alpha=0$, which is a contradiction. Therefore, for any $\alpha\in(-1,0)$ we get bijective realisations independently of the choice of C. Note that here we were not able to capture the value $\alpha=0$.

The argument given in Example 3.3.4 is sufficient to conclude that in general there is no α in $(0,1)$ with the property that the pair of domains $(\mathscr{V}_\alpha,\mathscr{V}_{\frac{1}{\alpha}})$ gives rise to (mutually adjoint) bijective realisations (3.11) for any choice of admissible C. On the other hand, since $\dim\ker T_1=1$, for fixed C all $\alpha\in[0,1)$ but one corresponds to mutually adjoint bijective realisations.

A particularly interesting case of bijective realisations with signed boundary map is when $\mathscr{V}=\mathscr{V}^{[\perp]}$ (see Introduction). By Remark 3.3.2 this occurs if and only if the associated realisation of the skew-symmetric part L_0 is skew-selfadjoint. Thus, such subspace $\mathscr{V}$ exists if and only if $d_+(L_0)=d_-(L_0)$ (see [53, Theorem 13.10]). Applying Lemma 3.3.6 we can formulate the following corollary.

Corollary 3.3.9. Let $(T_0,\widetilde{T}_0)$ be a joint pair of abstract Friedrichs operators on $\mathscr{H}$. There exists a closed subspace $\mathscr{V}$ of $\mathscr{W}$ with $\mathscr{W}_0\subseteq\mathscr{V}$ and such that $(T_1|_{\mathscr{V}},\widetilde{T}_1|_{\mathscr{V}})$ is a pair of mutually adjoint bijective realisations related to $(T_0,\widetilde{T}_0)$ if and only if $\ker T_1$ and $\ker\widetilde{T}_1$ are isomorphic.

Remark 3.3.10. The notion of isomorphism of Hilbert spaces used in the previous corollary is the standard (and natural) one (cf. [28, I.5.1. Definition]): two Hilbert spaces are *isomorphic* if there exists a linear surjective isometry (*isomorphism* or *unitary transformation*) between them.

One can find several characterisations, e.g. two Hilbert spaces are isomorphic if and only if

i) they have the same dimension.

ii) there exists a linear bounded bijection between them.

For the first claim we refer to [28, I.5.4. Theorem], while in the latter one needs to discuss only the converse. This can be done in a straigtforward constructive way. Indeed, if we denote by A a linear bounded bijection between two given Hilbert spaces, then $U := A(A^*A)^{-\frac{1}{2}}$ is an isomorphism (in the above sense).

Remark 3.3.11. For $\tilde{v} \in \ker \widetilde{T}_1$ we have

$$
\begin{aligned}
[\,\tilde{v} \mid \tilde{v}\,] &= \langle\, T_1 \tilde{v} \mid \tilde{v}\,\rangle - \langle\, v \mid \widetilde{T}_1 \tilde{v}\,\rangle \\
&= \langle\, T_1 \tilde{v} \mid \tilde{v}\,\rangle \\
&= \langle\, (T_1 + \widetilde{T}_1)\tilde{v} \mid \tilde{v}\,\rangle \;.
\end{aligned}
$$

Thus, by the boundedness of $T_1 + \widetilde{T}_1$ on $\mathscr{H}$ the norms $\sqrt{[\cdot \mid \cdot]}$ and $\|\cdot\|$ are equivalent on $\ker \widetilde{T}_1$, implying that $\ker \widetilde{T}_1$ is a Hilbert space when both equipped with $[\cdot \mid \cdot]$ and $\langle \cdot \mid \cdot \rangle$. Moreover, the identity map $i : (\ker \widetilde{T}_1, \langle \cdot \mid \cdot \rangle) \to (\ker \widetilde{T}_1, [\cdot \mid \cdot])$ is continuous (due to the boundedness of $T_1 + \widetilde{T}_1$ on $\mathscr{H}$), it is irrelevant which Hilbert space structure we consider on $\ker \widetilde{T}_1$ in Corollary 3.3.9. The same applies on $\ker T_1$ as well, with the only difference that $[\cdot \mid \cdot]$ should be replaced by $-[\cdot \mid \cdot]$.

3.4. Classification of the von Neumann type

Applying von Neumann's extension theory of symmetric operators (cf. [53, Theorem 13.9]) on $-iL_0$, we can classify all skew-selfadjoint (even closed skew-symmetric) realisations of L_0 in terms of unitary transformations between (closed subspaces of) $\ker(L_1+\mathbb{1})$ and $\ker(L_1-\mathbb{1})$. Of course, by Lemma 3.3.6 (see also Remark 3.3.10) this can also be done when $\ker(L_1+\mathbb{1})$ and $\ker(L_1-\mathbb{1})$ are replaced by $\ker T_1$ and $\ker\widetilde{T}_1$. We are about to focus on this situation since, as it was demonstrated in examples 3.3.4 and 3.3.8, often it is desirable to keep abstract Friedrichs operators T_0 and $\widetilde{T}_0$ as a whole (e.g. not all bijective realisations of $T_0 = L_0+S$ correspond to skew-symmetric realisations of L_0). Also, in terms of partial differential operators (especially with variable coefficients), it is sometimes easier to work with the operator L_0+C, for some C, than with skew-symmetric operator L_0 itself. However, we will not make use of [53, Theorem 13.9], but develop an independent constructive proof. This will allow for an explicit classification and at the same time provide an alternative proof in the symmetric setting (for even more general situations).

3.4.1. Preliminaries.

Lemma 3.4.1. Let $(T_0,\widetilde{T}_0)$ be a joint pair of abstract Friedrichs operators on $\mathscr{H}$. Let $\mathscr{V}\subseteq\mathscr{W}$ be a closed subspace containing $\mathscr{W}_0$ and let us define $\mathscr{G} := p_{\mathrm{k}}(\mathscr{V})$ and $\widetilde{\mathscr{G}} := p_{\tilde{\mathrm{k}}}(\mathscr{V})$, where p_{k} and $p_{\tilde{\mathrm{k}}}$ are given by Lemma 3.2.9. Then, we have the following.

i) $T_1|_{\mathscr{V}}$ is a bijective realisation of T_0 if and only if $\mathscr{V}\cap\ker T_1=\{0\}$ and $\widetilde{\mathscr{G}}=\ker\widetilde{T}_1$.

ii) Let $\mathscr{V}\cap\ker T_1=\{0\}$. Then $U:\widetilde{\mathscr{G}}\to\mathscr{G}$ defined by

$$U(p_{\tilde{\mathrm{k}}}(u)) = p_{\mathrm{k}}(u), \quad u\in\mathscr{V}, \tag{3.13}$$

is a well-defined closed linear map. Moreover, U is bounded if and only if $\widetilde{\mathscr{G}}$ is closed in $\ker\widetilde{T}_1$ (cf. Remark 3.3.11).

iii) If $\mathscr{V} \subseteq \mathscr{W}^+$ (see (2.4)), then $\widetilde{\mathscr{G}}$ is closed and $U : (\widetilde{\mathscr{G}}, [\cdot \mid \cdot]) \to (\ker T_1, -[\cdot \mid \cdot])$ has norm (with respect to the indicated norms) less than or equal to 1.

iv) If $\mathscr{V} \subseteq \mathscr{W}^+ \cap \mathscr{W}^-$ (see (2.4)), then both $\widetilde{\mathscr{G}}$ and $\mathscr{G}$ are closed and $U : (\widetilde{\mathscr{G}}, [\cdot \mid \cdot]) \to (\mathscr{G}, -[\cdot \mid \cdot])$ is a unitary transformation (cf. Remark 3.3.10).

v) Let $\mathscr{V} \cap \ker T_1 = \{0\}$. Then $\mathscr{V}$ coincides with $\mathscr{V}_U$ given by

$$\mathscr{V}_U := \left\{u_0 + U\tilde{v} + \tilde{v} : u_0 \in \mathscr{W}_0,\, \tilde{v} \in \widetilde{\mathscr{G}}\right\}, \tag{3.14}$$

where U is defined by (3.13), and $T_1|_{\mathscr{V}}(u_0 + U\tilde{v} + \tilde{v}) = \overline{T}_0 u_0 + (\overline{T_0 + \widetilde{T}_0})\tilde{v}$.

Moreover, such U is unique, i.e. if for a subspace $\widetilde{\mathscr{G}} \subseteq \mathscr{W}$ and a closed operator $U : \widetilde{\mathscr{G}} \to \ker T_1$ we have $\mathscr{V} = \mathscr{V}_U$, where $\mathscr{V}_U$ is defined by the formula above, then U is given by (3.13).

Remark 3.4.2. The assumption of part iv) is equivalent to $\mathscr{V} \subseteq \mathscr{V}^{[\perp]}$. The part $\mathscr{V} \subseteq \mathscr{W}^+ \cap \mathscr{W}^-$ implies $\mathscr{V} \subseteq \mathscr{V}^{[\perp]}$ is trivial, while the converse is a consequence of the polarisation formula [14, (2.3)].

Note that $\mathscr{V} \subseteq \mathscr{W}^+$ implies that $\mathscr{V} \cap \ker T_1 = \{0\}$. indeed, if that were not the case then for $v \in \mathscr{V} \cap \ker T_1$ we have

$$[\, v \mid v\,] = -\langle\, v \mid (T_1 + \widetilde{T}_1)v < 0)\,\rangle\,,$$

contradicting $\mathscr{V} \subseteq \mathscr{W}^+$.

Remark 3.4.3. It is clear that for any given $U : \widetilde{\mathscr{G}} \to \ker T_1$, $\mathscr{V}_U$ defined by (3.14) is a subspace of $\mathscr{W}$ which contains $\mathscr{W}_0$. Moreover, if U is closed, then $\mathscr{V}_U$ is closed as well. Indeed, from $u_0^n + U\tilde{v}_n + \tilde{v}_n \to u_0 + v + \tilde{v} \in \mathscr{W}$ it follows (see the proof of Lemma 3.2.3) that $U\tilde{v}_n \to v$ and $\tilde{v}_n \to \tilde{v}$. Hence, for closed U we have $\tilde{v} \in \operatorname{dom} U$ and $v = U\tilde{v}$, implying that $u_0 + v + \tilde{v} = u_0 + U\tilde{v} + \tilde{v} \in \mathscr{V}_U$.

Proof of Lemma 3.4.1. i) Let us assume that $T_1|_{\mathscr{V}}$ is a bijective realisation. Injectivity implies that $\mathscr{V} \cap \ker T_1 = \{0\}$, while $p_{\tilde{k}}(\mathscr{V}) \subseteq \ker \widetilde{T}_1$ is trivial. Let us prove the opposite inclusion. By Theorem 3.2.10, we have $\mathscr{W} = \mathscr{V} \dot{+} \ker T_1$. Thus, for any $\tilde{v} \in \ker \widetilde{T}_1$ (recall that $\ker \widetilde{T}_1 \subseteq \mathscr{W}$) there exist unique $u \in \mathscr{V}$ and $v \in \ker T_1$, such that $\tilde{v} = u + v$. This implies $u = -v + \tilde{v}$, so $\tilde{v} = p_{\tilde{k}}(u) \in p_{\tilde{k}}(\mathscr{V})$.

For the converse, we shall make use of Theorem 3.2.10 again. Let us take an arbitrary $w \in \mathscr{W}$. By Theorem 3.2.4 there exist unique $w_0 \in \mathscr{W}_0$, $\nu \in \ker T_1$ and $\tilde{\nu} \in \ker \widetilde{T}_1$ such that $w = w_0 + \nu + \tilde{\nu}$. The assumption ensures existence of $u \in \mathscr{V}$ such that $u = u_0 + \mu + \tilde{\nu}$, for some $u_0 \in \mathscr{W}_0$ and $\mu \in \ker T_1$. By subtracting $\tilde{\nu}$ from the second equation and inserting it into the first, we get

$$w = (w_0 - u_0 + u) + (\nu - \mu) .$$

Since the first term on the right hand side belongs to $\mathscr{V}$ (note that $\mathscr{W}_0 \subseteq \mathscr{V}$) and the second one to $\ker T_1$, Theorem 3.2.10 is applicable. In conclusion, $T_1|_{\mathscr{V}}$ is bijective.

ii) Let us start by showing that U is a well-defined function. Let $u, v \in \mathscr{V}$ be such that $p_{\tilde{\mathrm{k}}}(u) = p_{\tilde{\mathrm{k}}}(v)$. By the decomposition given in Theorem 3.2.4, there exist $u_0, v_0 \in \mathscr{W}_0$, such that

$$u = u_0 + p_{\mathrm{k}}(u) + p_{\tilde{\mathrm{k}}}(u) , \quad v = v_0 + p_{\mathrm{k}}(v) + p_{\tilde{\mathrm{k}}}(v) .$$

Thus,

$$u - v = (u_0 - v_0) + \big(p_{\mathrm{k}}(u) - p_{\mathrm{k}}(v)\big) .$$

Since $(u - v) - (u_0 - v_0) \in \mathscr{V}$, we get $p_{\mathrm{k}}(u) - p_{\mathrm{k}}(v) \in \mathscr{V} \cap \ker T_1 = \{0\}$. Hence, U is well-defined.

Linearity follows from the linearity of projections p_{k} and $p_{\tilde{\mathrm{k}}}$. Let us show that U is closed. Take (u_n) in $\mathscr{V}$ such that $p_{\tilde{\mathrm{k}}}(u_n) \to \tilde{\nu}$ in $\ker \widetilde{T}_1$ and $U p_{\tilde{\mathrm{k}}}(u_n) = p_{\mathrm{k}}(u_n) \to \nu$ in $\ker T_1$ (as n tends to infinity). This implies that $\big(p_{\tilde{\mathrm{k}}}(u_n) + p_{\mathrm{k}}(u_n)\big)$ converges to $\tilde{\nu} + \nu$ in $\mathscr{W}$. Since $\mathscr{W}_0 \subseteq \mathscr{V}$, for each $n \in \mathbb{N}$ we have $p_{\tilde{\mathrm{k}}}(u_n) + p_{\mathrm{k}}(u_n) \in \mathscr{V}$. Thus, by the closedness of $\mathscr{V}$, we obtain $u := \tilde{\nu} + \nu \in \mathscr{V}$, implying $\tilde{\nu} = p_{\tilde{\mathrm{k}}}(u) \in p_{\tilde{\mathrm{k}}}(\mathscr{V})$ and $U\tilde{\nu} = U p_{\tilde{\mathrm{k}}}(u) = p_{\mathrm{k}}(u) = \nu$.

Similarly as with the closedness of U, in the last part the goal is to exploit the fact that $\mathscr{V}$ is closed. Indeed, let us assume that U is bounded, i.e. there exists $c > 0$ such that $\|p_{\mathrm{k}}(u)\| \leq c\|p_{\tilde{\mathrm{k}}}(u)\|$, $u \in \mathscr{V}$. Let us take (u_n) in $\mathscr{V}$ such that $p_{\tilde{\mathrm{k}}}(u_n) \to \tilde{\nu}$ in $\ker \widetilde{T}_1$, as n tends to infinity. Using the boundedness of U we get that the sequence $(p_{\mathrm{k}}(u_n))$ is a Cauchy sequence in $\ker T_1$, hence convergent. Now we get $\tilde{\nu} \in p_{\tilde{\mathrm{k}}}(\mathscr{V})$

following the previous reasoning. On the other hand, if $p_{\tilde{k}}(\mathscr{V})$ is closed, then U is a closed linear map between two Hilbert spaces $p_{\tilde{k}}(\mathscr{V})$ and $\ker T_1$. Thus, U is bounded by the closed graph theorem.

iii) Let $u \in \mathscr{V} \subseteq \mathscr{W}$. Using $u = u_0 + p_k(u) + p_{\tilde{k}}(u)$, $\mathscr{W}^{[\perp]} = \mathscr{W}_0$ (see Theorem 2.2.5(iii)) and $\mathscr{V} \subseteq \mathscr{W}^+$, we get

$$0 \leq [u \mid u] = [p_k(u) \mid p_k(u)] + [p_{\tilde{k}}(u) \mid p_{\tilde{k}}(u)]\,, \tag{3.15}$$

which in terms of the operator U reads

$$-[U p_{\tilde{k}}(u) \mid U p_{\tilde{k}}(u)] \leq [p_{\tilde{k}}(u) \mid p_{\tilde{k}}(u)]\,.$$

Thus, $U : \widetilde{\mathscr{G}} \to \ker T_1$ is bounded (hence $\widetilde{\mathscr{G}}$ is closed by part ii)) and $\|U\| \leq 1$ (with respect to the norms $\sqrt{[\cdot \mid \cdot]}$ and $\sqrt{-[\cdot \mid \cdot]}$, respectively).

iv) When $\mathscr{V} \subseteq \mathscr{W}^+ \cap \mathscr{W}^-$, then in (3.15) we have equality. This allows us to follow the last part of the proof of part ii) to conclude that $\mathscr{G}$ is closed as well. Furthermore, U is obviously a unitary transformation between Hilbert spaces $(\widetilde{\mathscr{G}}, [\cdot \mid \cdot])$ and $(\mathscr{G}, -[\cdot \mid \cdot])$.

v) Since for any $u \in \mathscr{V}$ there exists $u_0 \in \mathscr{W}_0$ such that $u = u_0 + p_k(u) + p_{\tilde{k}}(u) = u_0 + U p_{\tilde{k}}(u) + p_{\tilde{k}}(u)$, it is clear that $\mathscr{V} = \mathscr{V}_U$. For arbitrary $u_0 \in \mathscr{W}_0$ and $\tilde{v} \in \widetilde{\mathscr{G}}$ we have

$$T_1|_{\mathscr{V}}(u_0 + U\tilde{v} + \tilde{v}) = T_1(u_0 + U\tilde{v} + \tilde{v}) = T_1 u_0 + T_1\tilde{v} = \overline{T}_0 u_0 + (T_1 + \widetilde{T}_1)\tilde{v}\,,$$

where we have used $\mathscr{V} \subseteq \mathscr{W}$, $T_1(U\tilde{v}) = 0$, $T_1|_{\mathscr{W}_0} = \overline{T}_0$ and $\widetilde{T}_1\tilde{v} = 0$.

Let us take two subspaces $\widetilde{\mathscr{G}_i} \subseteq \ker \widetilde{T}_1$, $i \in \{1,2\}$, and two closed operators $U_i : \widetilde{\mathscr{G}_i} \to \ker T_1$, $i \in \{1,2\}$, such that $\mathscr{V}_{U_1} = \mathscr{V}_{U_2}$. This means that for an arbitrary $\tilde{v}_1 \in \widetilde{\mathscr{G}_1}$ there exist $u_0 \in \mathscr{W}_0$ and $\tilde{v}_2 \in \widetilde{\mathscr{G}_2}$ such that

$$U_1\tilde{v}_1 + \tilde{v}_1 = u_0 + U_2\tilde{v}_2 + \tilde{v}_2\,.$$

Applying Theorem 3.2.4 (the fact that it is a direct sum) we get $u_0 = 0$, $\tilde{v}_1 = \tilde{v}_2$ and $U_1\tilde{v}_1 = U_2\tilde{v}_2$. Hence, $U_1 \subseteq U_2$. By the symmetry we can conclude that in fact we have $U_1 = U_2$. This proves that such U is unique, and by the first part we have that U is necessarily given by (3.13).

■

Remark 3.4.4. For $\mathscr{V}_U$ given by (3.14) we can explicitly write $\mathscr{V}_U^{[\perp]}$ (i.e. the domain of $(T_1|_{\mathscr{V}_U})^*$; see Theorem 2.3.4) in terms of the adjoint operator U^*. For simplicity, let us elaborate on this only in the case of bounded $U : \widetilde{\mathscr{G}} \to \ker T_1$, i.e. $\widetilde{\mathscr{G}} = p_{\tilde{\mathrm{k}}}(\mathscr{V})$ is closed (see Lemma 3.4.1(ii)). Then $U^* : \ker T_1 \to \widetilde{\mathscr{G}}$ is also bounded and $p_{\mathrm{k}}(\mathscr{V}^{[\perp]}) = \ker T_1$. Moreover,

$$U^* p_{\mathrm{k}}(v) = U^* P_{\bar{\mathscr{G}}} p_{\mathrm{k}}(v) = \widetilde{P}_{\widetilde{\mathscr{G}}} p_{\tilde{\mathrm{k}}}(v)\,, \quad v \in \mathscr{V}^{[\perp]}\,,$$

where $P_{\bar{\mathscr{G}}}$ and $\widetilde{P}_{\widetilde{\mathscr{G}}}$ denote the orthogonal projections on $\bar{\mathscr{G}}$ and $\widetilde{\mathscr{G}}$ within spaces $(\ker T_1, -[\cdot\,|\,\cdot])$ and $(\ker \widetilde{T}_1, [\cdot\,|\,\cdot])$, respectively ($\bar{\mathscr{G}}$ is the closure of $\mathscr{G} = p_{\mathrm{k}}(\mathscr{V})$ in $\ker T_1$). Furthermore, $\mathscr{V}_U^{[\perp]}$ is then given by

$$\mathscr{V}_U^{[\perp]} := \big\{ v_0 + \mu_1 + U^*\mu_1 + \mu_2 + \tilde{\mu}_2 : v_0 \in \mathscr{W}_0,\, \mu_1 \in \bar{\mathscr{G}},$$
$$\mu_2 \in \mathscr{G}^{[\perp]} \cap \ker T_1,\, \tilde{\mu}_2 \in \widetilde{\mathscr{G}}^{[\perp]} \cap \ker \widetilde{T}_1 \big\}\,.$$

Note that $\mathscr{G}^{[\perp]} \cap \ker T_1$ is the orthogonal complement of $\mathscr{G}$ within the space $(\ker T_1, -[\cdot\,|\,\cdot])$, and analogously for $\widetilde{\mathscr{G}}^{[\perp]} \cap \ker \widetilde{T}_1$.

As an example, let us consider $\mathscr{V} = \mathscr{W}_0 + \ker \widetilde{T}_1$. Clearly, we have $p_{\tilde{\mathrm{k}}}(\mathscr{V}) = \ker \widetilde{T}_1$ and $\mathscr{V} \subseteq \mathscr{W}^+$. It is also easy to check that $\mathscr{V}^{[\perp]} = \mathscr{W}_0 + \ker T_1$ (see Corollary 3.2.6). For this $\mathscr{V}$ we get $\widetilde{\mathscr{G}} = \ker \widetilde{T}_1$ and $\mathscr{G} = \{0\}$, hence both U and U^* are zero operators. Since $\mathscr{G}^{[\perp]} \cap \ker T_1 = \ker T_1$ and $\widetilde{\mathscr{G}}^{[\perp]} \cap \ker \widetilde{T}_1 = \{0\}$, it is easy to read that the above expression for $\mathscr{V}_U^{[\perp]}$ gives the right space $\mathscr{W}_0 + \ker T_1$.

Remark 3.4.5. Of course, in part iv) it is implicitly required that $\widetilde{\mathscr{G}}$ and $\mathscr{G}$ are isomorphic.

A trivial situation when the assumption $\mathscr{V} \subseteq \mathscr{W}^+ \cap \mathscr{W}^-$ is satisfied occurs for $\mathscr{V} = \mathscr{W}_0$. Then $\mathscr{G} = \widetilde{\mathscr{G}} = \{0\}$, hence they are obviously isomorphic.

Remark 3.4.6. By Remark 3.3.10, in the regime $\mathscr{V} \subseteq \mathscr{W}^+ \cap \mathscr{W}^-$ of part iv) of the previous lemma from the mapping U, given by (3.13), we can construct an isomorphism between spaces $\widetilde{\mathscr{G}}$ and $\mathscr{G}$ when both are equipped with the standard inner product $\langle \cdot\,|\,\cdot \rangle$. Indeed, if we denote by $\tilde{\iota} : \widetilde{\mathscr{G}} \hookrightarrow \mathscr{H}$ and $\iota : \mathscr{G} \hookrightarrow \mathscr{H}$ canonical embeddings (note that then $\tilde{\iota}^*$ and ι^* are orthogonal projections in $\mathscr{H}$ onto $\widetilde{\mathscr{G}}$ and $\mathscr{G}$, respectively), then a unitary transformation is given by

$$U\Big(\tilde{\iota}^*\, \overline{T_0 + \widetilde{T}_0}\, U^* \iota^* (\overline{T_0 + \widetilde{T}_0})^{-1} U\Big)^{-\frac{1}{2}}\,.$$

A trivial situation is when $\overline{T_0+\widetilde{T}_0}=\alpha\mathbb{1}$, for some $\alpha\in\mathbb{C}$, since then the expression above equals U.

3.4.2. Classification.

In this subsection we formulate and prove the result concerning a classification of realisations of abstract Friedrichs operators in the spirit of von Neumann's theory.

Theorem 3.4.7. Let $(T_0,\widetilde{T}_0)$ be a joint pair of abstract Friedrichs operators on $\mathscr{H}$ and let T be a closed realisation of T_0, i.e. $T_0\subseteq T\subseteq T_1$. In what follows we use $\mathscr{V}_U$ to denote the space (3.14) for a given U.

i) T is bijective if and only if there exists a bounded operator $U:\ker\widetilde{T}_1\to\ker T_1$ such that $\operatorname{dom}T=\mathscr{V}_U$.

ii) $\operatorname{dom}T\subseteq\mathscr{W}^+$ if and only if there exist a closed subspace $\widetilde{\mathscr{G}}\subseteq\ker\widetilde{T}_1$ and a continuous linear operator $U:(\widetilde{\mathscr{G}},[\cdot\mid\cdot])\to(\ker T_1,-[\cdot\mid\cdot])$ with the norm (with respect to the indicated norms) less than or equal to 1 (i.e. U is a non-expansive map) such that $\operatorname{dom}T=\mathscr{V}_U$.

iii) T is a bijective realisation with signed boundary map if and only if there exists a continuous linear operator $U:(\ker\widetilde{T}_1,[\cdot\mid\cdot])\to(\ker T_1,-[\cdot\mid\cdot])$ with the norm (with respect to the indicated norms) less than or equal to 1 such that $\operatorname{dom}T=\mathscr{V}_U$.

iv) $\operatorname{dom}T\subseteq\operatorname{dom}T^*$ if and only if there exist closed subspaces $\widetilde{\mathscr{G}}\subseteq\ker\widetilde{T}_1$ and $\mathscr{G}\subseteq\ker T_1$ and a unitary transformation $U:(\widetilde{\mathscr{G}},[\cdot\mid\cdot])\to(\mathscr{G},-[\cdot\mid\cdot])$ (i.e. the norm of U is equal to 1) such that $\operatorname{dom}T=\mathscr{V}_U$.

v) $\operatorname{dom}T=\operatorname{dom}T^*$ if and only if there exists a unitary transformation $U:(\ker\widetilde{T}_1,[\cdot\mid\cdot])\to(\ker T_1,-[\cdot\mid\cdot])$ such that $\operatorname{dom}T=\mathscr{V}_U$.

vi) In each of the above cases there is one-to-one correspondence between realisations T, i.e. $\operatorname{dom}T$, and classifying operators U. A correspondence is given by $U\mapsto T_1|_{\mathscr{V}_U}$.

Proof. Existence of such U in all parts is a direct consequence of Lemma 3.4.1. Thus, it remains to comment only the converse of each claim.

Since in all parts U is bounded, by Remark 3.4.3 we have that $\mathscr{V}_U$ is a closed subspace of $\mathscr{W}$ containing $\mathscr{W}_0$. This means that $T_1|_{\mathscr{V}_U}$ is indeed a closed realisation of T_0.

In part i) it is evident that $p_{\tilde{k}}(\mathscr{V}_U) = \ker \widetilde{T}_1$, hence we just apply Lemma 3.4.1(i) to conclude.

For parts ii) and iii) we need to show that $\mathscr{V}_U \subseteq \mathscr{W}^+$ (bijectivity in part iii) is again consequence of Lemma 3.4.1(i); see also Remark 3.2.12). For an arbitrary $u = u_0 + U\tilde{\nu} + \tilde{\nu} \in \mathscr{V}_U$ we have

$$[\, u \mid u \,] = [\, U\tilde{\nu} \mid U\tilde{\nu} \,] + [\, \tilde{\nu} \mid \tilde{\nu} \,] \geq -[\, \tilde{\nu} \mid \tilde{\nu} \,] + [\, \tilde{\nu} \mid \tilde{\nu} \,] = 0\,,$$

where we have used that the norm of U is less than or equal to 1.

Let us recall that by Theorem 2.3.4 we have $\operatorname{dom}(T_1|_{\mathscr{V}_U})^* = \mathscr{V}_U^{[\perp]}$. Thus, for parts iv) and v) we need to show that $\mathscr{V}_U \subseteq \mathscr{V}_U^{[\perp]}$ and (only for part v)) $\mathscr{V}_U^{[\perp]} \subseteq \mathscr{V}_U$. This can be done using Remark 3.4.4, but let us present here a direct proof. For arbitrary $u = u_0 + U\tilde{\nu} + \tilde{\nu}$ and $v = v_0 + U\tilde{\mu} + \tilde{\mu}$ from $\mathscr{V}_U$, similarly as in the previous calculations, we have

$$[\, u \mid v \,] = [\, U\tilde{\nu} \mid U\tilde{\mu} \,] + [\, \tilde{\nu} \mid \tilde{\mu} \,] = -[\, \tilde{\nu} \mid \tilde{\mu} \,] + [\, \tilde{\nu} \mid \tilde{\mu} \,] = 0\,,$$

where we have used that U is an isometry. Thus, $\mathscr{V}_U \subseteq \mathscr{V}_U^{[\perp]}$.

Let us prove now the opposite inclusion for U given in part v). Let $v \in \mathscr{V}_U^{[\perp]} \subseteq \mathscr{W}$. By Theorem 3.2.4, there exist $v_0 \in \mathscr{W}_0$, $\mu \in \ker T_1$, $\tilde{\mu} \in \ker \widetilde{T}_1$, such that $v = v_0 + \mu + \tilde{\mu}$. For any $u = u_0 + U\tilde{\nu} + \tilde{\nu} \in \mathscr{V}_U$, we have

$$\begin{aligned} 0 = [\, u \mid v \,] &= [\, U\tilde{\nu} \mid \mu \,] + [\, \tilde{\nu} \mid \tilde{\mu} \,] \\ &= [\, U\tilde{\nu} \mid \mu \,] - [\, U\tilde{\nu} \mid U\tilde{\mu} \,] = [\, U\tilde{\nu} \mid \mu - U\tilde{\mu} \,]\,, \end{aligned}$$

where we have used that U is a unitary transformation. The identity above holds for any $\tilde{\nu} \in \ker \widetilde{T}_1$. Since U is surjective and $(\ker T_1, -[\,\cdot \mid \cdot\,])$ is a Hilbert space, we get $\mu = U\tilde{\mu}$. Thus, $v \in \mathscr{V}_U$ and hence, $V_U^{[\perp]} \subseteq \mathscr{V}_U$.

Surjectivity of the map $U \mapsto T$ follows from parts i)-v), while injectivity holds by Lemma 3.4.1(v). ■

Remark 3.4.8. In [9, Section 4] Grubb's classification was applied on abstract Friedrichs operators, which differs significantly from the method of the previous theorem. For instance, in the result just developed a realisation is bijective if and only if the classifying operator is defined on the whole kernel (Theorem 3.4.7(i)), while in the theory of [44, Chapter 13] (see also [9, Theorem 17]) the same holds for bijective classifying operators. Another difference is that Grubb's classification is developed around the reference operator, while such distinguished operator is not needed here. One can notice the same also in the symmetric case (see e.g. [25]) when comparing von Neumann's (*absolute*) theory (cf. [53, Section 13.2]) and the (*relative*) theory developed by Kreĭn, Višik and Birman (cf. [44, Section 13.2]).

If we focus on bijective realisations with signed boundary map, then the result of part iii) of the previous theorem (see also part vi)) offers a full and explicit characterisation contrast to [9, Theorem 18] where the result is optimal only when kernels are isomorphic (cf. Corollary 3.3.9).

By Theorem 3.4.7 we know that the number of certain type of realisations agrees with the number of corresponding classifying operators U. For instance, it is easy to deduce the number of isomorphisms between Hilbert spaces. Hence, having this point of view at our hands, we can formulate the following straightforward quantitative generalisation of Corollary 3.3.9.

Corollary 3.4.9. Let $(T_0, \widetilde{T}_0)$ be a joint pair of abstract Friedrichs operators on $\mathscr{H}$ and let us denote by m the cardinality of the set of all subspaces $\mathscr{V}$ of $\mathscr{W}$ such that $\mathscr{V} = \mathscr{V}^{[\perp]}$, i.e. such that $(T_1|_{\mathscr{V}}, \widetilde{T}_1|_{\mathscr{V}})$ is a pair of mutually adjoint bijective realisations related to $(T_0, \widetilde{T}_0)$.

i) If $\dim\ker T_1 \neq \dim\ker \widetilde{T}_1$, then $m = 0$.

ii) If $\dim\ker T_1 = \dim\ker \widetilde{T}_1 = 0$, then $m = 1$.

iii) If $\dim\ker T_1 = \dim\ker \widetilde{T}_1 = 1$, then $m = 2$ in the real case, and $m = \infty$ in the complex case.

iv) If $\dim\ker T_1 = \dim\ker \widetilde{T}_1 \geq 2$, then $m = \infty$.

Example 3.4.10. a) In Example 3.3.8, it is commented that $\dim\ker T_1 = \dim\ker \widetilde{T}_1 = 1$. Thus, since the problem was addressed in the real setting, by Corollary 3.4.9 there are two closed subspaces with the property that $\mathscr{V} = \mathscr{V}^{[\perp]}$. They are precisely $\mathscr{V}_\alpha$, $\alpha \in \{-1, 1\}$ (see Example 3.3.4).

b) Let us consider operators from examples 3.3.4 and 3.3.8 on $(0,\infty)$, instead of the bounded interval (a,b), i.e. $L_0 u = u'$ and $\mathscr{H} = L^2((0,\infty);\mathbb{R})$. Then (see [53, Example 3.2]) we have $d_+(L_0) = d_+(-iL_0) = 1$ and $d_-(L_0) = d_-(-iL_0) = 0$. Thus, there is no closed subspace $\mathscr{V} \subseteq \mathscr{W}$ such that $\mathscr{V} = \mathscr{V}^{[\perp]}$, or in other words there is no skew-selfadjoint realisation of the operator $-iL_0$.

c) The previous example is very specific since there is also only one bijective realisation. This can be justified by Theorem 3.4.7(i) (note that the zero operator is the only bounded operator between $\ker \widetilde{T}_1$ and $\ker T_1$), but we also refer to [9, Theorem 13].

   Let us now present an example where still there is no closed subspace $\mathscr{V} \subseteq \mathscr{W}$ such that $\mathscr{V} = \mathscr{V}^{[\perp]}$, but for which there are infinitely many bijective realisations. More precisely, we need $\min\{\dim\ker T_1, \dim\ker \widetilde{T}_1\} \geq 1$ (see Theorem 2.4.12) and $\dim\ker T_1 \neq \dim\ker \widetilde{T}_1$ (see Corollary 3.3.9).

   Let $\mathscr{H} = L^2((0,1);\mathbb{C}^2)$ (all conclusions are also valid for the real case) and $\mathscr{D} = C_c^\infty((0,1);\mathbb{C}^2)$. For $\mathsf{u} \in \mathscr{D}$ and

$$\mathbf{A}(x) := \begin{bmatrix} 1 & 0 \\ 0 & 1-x \end{bmatrix}$$

   we define $T_0\mathsf{u} := (\mathbf{A}\mathsf{u})' + \mathsf{u}$ and $\widetilde{T}_0\mathsf{u} := -(\mathbf{A}\mathsf{u})' + \mathbf{A}'\mathsf{u} + \mathsf{u}$. It is easy to see that $(T_0, \widetilde{T}_0)$ is a joint pair of abstract Friedrichs operators (just apply Theorem 3.3.1 or notice that T_0 satisfies Definition 1.1.1, i.e. it is a classical Friedrichs operator). As usual, we put $T_1 := \widetilde{T}_0^*$ and $\widetilde{T}_1 := T_0^*$. Since both T_1 and $\widetilde{T}_1$ are of a block structure, calculations of the kernels can be done by studying each component separately. More precisely, $\mathsf{u} = (u_1, u_2) \in \ker T_1$ if and only if

$$u_1' + u_1 = 0 \quad \text{and} \quad (a_2 u_2)' + u_2 = 0\,,$$

   where $a_2(x) := 1 - x$. Thus, we can apply the available results for scalar ordinary differential equations (see e.g. the second example of Subsection 4.2.6).

   Informally speaking, the equation above for the first component u_1 contributes with 1 for both $\dim\ker T_1$ and $\dim\ker \widetilde{T}_1$. On the other hand, the second equation contributes with 1 for $\dim\ker T_1$ and 0 for $\dim\ker \widetilde{T}_1$. The overall result then reads

$$\dim\ker T_1 = 2 \quad \text{and} \quad \dim\ker \widetilde{T}_1 = 1 \ ,$$

which corresponds to what we wanted to get.

3.5. The symmetric case

In this section we focus on symmetric operators and present several results that can be directly extracted from the theory developed in the previous section.

Corollary 3.5.1. Let A be a densely defined symmetric operator on $\mathscr{H}$ and let S_1, S_2 be bounded self-adjoint linear operators such that S_2 is in addition strictly positive. Define an indefinite inner product on $\operatorname{dom} A^*$ by

$$[u \mid v]_A := i\Big(\langle A^*u \mid v\rangle - \langle u \mid A^*v\rangle\Big)\,, \quad u,v \in \operatorname{dom} A^*\,. \tag{3.16}$$

Then we have the following.

i) It holds

$$\dim\ker(A^* - S_1 - iS_2) = d_+(A) \quad \text{and} \quad \dim\ker(A^* - S_1 + iS_2) = d_-(A)\,,$$

where $d_\pm(A)$ denote deficiency indices of A (cf. [53, Section 3.1]).

ii) $\operatorname{dom} A^* = \operatorname{dom}\overline{A} \dotplus \ker(A^* - S_1 - iS_2) \dotplus \ker(A^* - S_1 + iS_2)$, where the sums are direct and all spaces on the right-hand side are pairwise $[\cdot \mid \cdot]_A$-orthogonal.

iii) There is one-to-one correspondence between all closed symmetric realisations of A and all unitary transformations U between any closed isomorphic subspaces of $(\ker(A^* - S_1 + iS_2), [\cdot \mid \cdot]_A)$ and $(\ker(A^* - S_1 - iS_2), -[\cdot \mid \cdot]_A)$, respectively.

iv) There is one-to-one correspondence between all self-adjoint realisations of A and all unitary transformations $U : (\ker(A^* - S_1 + iS_2), [\cdot \mid \cdot]_A) \to (\ker(A^* - S_1 - iS_2), -[\cdot \mid \cdot]_A)$.

v) Correspondences of parts iii) and iv) can be expressed by $U \mapsto A_U = A^*|_{\operatorname{dom} A_U}$, where

$$\operatorname{dom} A_U := \left\{u_0 + U\tilde{v} + \tilde{v} : u_0 \in \operatorname{dom}\overline{A},\, \tilde{v} \in \operatorname{dom} U\right\},$$

and $A_U(u_0 + U\tilde{v} + \tilde{v}) = \overline{A}u_0 + (S_1 + iS_2)\tilde{v} + (S_1 - iS_2)U\tilde{v}$.

Proof. If we define $T_0 := iA - iS_1 + S_2$ and $\widetilde{T}_0 := -iA + iS_1 + S_2$, then the pair $(T_0, \widetilde{T}_0)$ is a joint pair of abstract Friedirchs operators by Theorem 3.3.1. Moreover, corresponding indefinite inner product (see (2.3) and (3.10)) agrees with $[\cdot \mid \cdot]_A$ (see Remark 3.3.5).

Therefore, the statements of the corollary follow from Lemma 3.3.6, Theorem 3.2.4 (see also Remark 2.4.10) and Theorem 3.4.7 (note that $\operatorname{dom} A_U$ agrees with (3.14) for the above choice of $(T_0, \widetilde{T}_0)$). ■

If $S_i = \alpha_i \mathbb{1}$, $i = 1, 2$, where $\alpha_1 \in \mathbb{R}$ and $\alpha_2 > 0$, then the statement of the previous theorem is well-known and can be found in many textbooks on unbounded linear operators. For instance, in [53] part i) is present in Section 3.1, part ii) in Proposition 3.7 (von Neumann's formula) and parts iii)-v) are studied in Section 13.2 as part of the von Neumann extension theory (see also [28, Chapter X]). Moreover, the correspondence given in part v) completely agrees with the one of [53, Theorem 13.9] since for this choice of bounded operators S_i, $i = 1, 2$, the same U represents a unitary transformation when the standard inner product of the Hilbert space $\mathscr{H}$ is considered (see Remark 3.4.6).

Let us just remark that the geometrical point of view provided in part ii), i.e. orthogonality with respect to $[\cdot \mid \cdot]_A$, is something that is not commonly present, although $[\cdot \mid \cdot]_A$ is (up to a multiplicative constant). More precisely, in [53, Definition 3.4] (see also Lemma 3.5 there) the indefinite inner product $-i[\cdot \mid \cdot]_A$ is referred to as the *boundary form* and it is an important part of the extension theory of boundary triplets ([53, Chapter 14]; see also [44, Section 13.4]). A more advanced study of boundary forms for Hilbert complexes can be found in a recent work [45].

Of course, in the standard theory of symmetric operators it is usually satisfactory to observe only the case $S_1 = 0$ and $S_1 = \mathbb{1}$. Thus, the preceding corollary may seem like an excessive technical complication. Here we want to stress one more time that our primary focus was in developing a classification result for abstract Friedrichs operators where such approach can be justified, e.g., by perceiving that not all bijective realisations of $T_0 = L_0 + S$ correspond to skew-symmetric realisations of L_0 (see Section 3.4). Therefore, our intention is to see the last corollary principally as a way to connect two theories, while an additional abstraction can sometimes offer a better sense of the underlying structure.

3.6. On (M)-boundary conditions

Let us conclude this chapter with some recent insights on (M)-boundary conditions. The equivalence between the abstract formulation of Friedrichs' boundary conditions, i.e. (M)-boundary conditions, and the boundary conditions given via the cone-formalism, i.e. (V) boundary conditions, has been studied under different conditions. In [3] authors proved the equivalence in the most general sense, which requires no further assumption (see Theorem 2.5.11). In this section, we use (T3)-condition of the abstract Friedrichs operators more effectively. We expect to reproduce more direct proof of the equivalence for a pair of abstract Friedrichs operators. Moreover, we present a concrete operator satisfying (M)-conditions and hence admissible boundary conditions.

Theorem 3.6.1. Let $(T_0,\widetilde{T_0})$ be a joint pair of closed abstract Friedrichs operators in a Hilbert space $\mathscr{H}$ and $(\mathscr{V},\widetilde{\mathscr{V}})$ be a pair of subspaces satisfying (V)-conditions. With respect to the decomposition $\mathscr{W}=\mathscr{V}\dotplus\ker T_1$, let p_1,p_2 are corresponding non-orthogonal projectors, then the operator $M:=D(1-2p_2)$ satisfies (M)-conditions.

Proof. (M1)-*condition*: Let $u\in\mathscr{W}$ then,

$$ {}_{\mathscr{W}'}\langle Mu,u\rangle_{\mathscr{W}} = [\,(1-2p_2)u\mid u\,] = [\,(p_1-p_2)u\mid(p_1+p_2)u\,]\;. $$

Since, $[\,p_1u\mid p_2u\,]-[\,p_2u\mid p_1u\,]$ has no real part and $\mathscr{V}\subseteq\mathscr{W}^+,\ker T_1\subseteq\mathscr{W}^-$, we conclude that $\Re_{\mathscr{W}'}\langle Mu,u\rangle_{\mathscr{W}}\geq 0$. Hence, M satisfies (M1) condition.

(M2)-*condition*: Let us first prove that $\mathscr{V}=\ker(D-M)$. Let $u\in\mathscr{V}$, then

$$ (D-M)u=(D-D(p_1-p_2))u=Du-Dp_1u=Du-Du=0\;. $$

Which implies $u\in\ker(D-M)$. Hence, $\mathscr{V}\subseteq\ker(D-M)$. Conversely, let $u\in\ker(D-M)$. That is $(D-M)=0$, which further implies $Dp_2u=0$ and so, $p_2u\in\mathscr{W}_0$ (since, $\mathscr{W}_0=\ker D$). Moreover, $p_2u\in\ker T_1$. Since $\ker T_1\cap\mathscr{W}_0=\{0\}$, we conclude that $p_2u=0$, implying $u=p_1u\in\mathscr{V}$. Hence, $\ker(D-M)\subseteq\mathscr{V}$.

To obtain $\mathscr{W}=\ker(D-M)+\ker(D+M)$ it is sufficient (due to the decomposition given in Corollary 3.2.2; see also Theorem 3.2.10) to prove that $\ker T_1\subseteq\ker(D+M)$. Let $u\in\ker T_1$, then $p_1u=0$. For any $v\in\mathscr{W}$, we have

$$ (D+M)u=2Dp_1u=0\;, $$

thus, $u \in \ker(D+M)$. Which completes the proof. ■

Remark 3.6.2. i) In the proof of the previous theorem we showed that $\ker(D-M) = \mathscr{V}$ and $\ker(D+M) \supseteq \ker T_1$. The latter can be improved to $\ker(D+M) = \mathscr{W}_0 + \ker T_1$. In fact, the analogous identity holds for a more general construction given in Theorem 2.5.11(i), i.e. $\ker(D+M) = \mathscr{W}_0 + \mathscr{W}_2$.

Indeed, since $\ker(D+M)$ is a subspace, from $\mathscr{W}_0 \subseteq \ker(D+M)$ (which is a straightforward consequence of lemmata 2.2.3 and 2.5.7) and $\mathscr{W}_2 \subseteq \ker(D+M)$ (by the definition and already proved in Theorem 2.5.11), we have $\mathscr{W}_0 + \mathscr{W}_2 \subseteq \ker(D+M)$. Let us take an arbitrary $u \in \ker(D+M)$. Since $D+M = 2Dp_1$, $(D+M)u = 0$ implies $p_1 u \in \mathscr{W}_0$. Hence, $u = p_1 u + p_2 u \subseteq \mathscr{W}_0 + \mathscr{W}_2$.

ii) From the identity $\ker(D+M) = \mathscr{W}_0 + \mathscr{W}_2$ it is clear that the mapping $\mathscr{W}_2 \mapsto M$, where M is given by Theorem 2.5.11(i), is injective, while surjecitvity follows from the second part of the same theorem.

Let us elaborate on injectivity in a more detail. Let us take two closed subspaces $\mathscr{W}_2, \mathscr{W}_2' \subseteq \mathscr{W}$, $\mathscr{W}_2 \neq \mathscr{W}_2'$, both satisfying the assumption of Theorem 2.5.11(i) and let M and M' be the corresponding operators. Take $u \in \mathscr{W}_2$ such that $u \notin \mathscr{W}_2'$. Then by the previous part of this remark $u \in \ker(D+M)$, but $u \notin \ker(D+M')$. Hence, $Mu = -Du \neq M'u$, implying $M \neq M'$.

iii) For the fixed $(\mathscr{V}, \widetilde{\mathscr{V}})$ satisfying (V)-conditions, by the previous part we have that we have as many operators M satisfying (M)-conditions and corresponding to the pair $(\mathscr{V}, \widetilde{\mathscr{V}})$ as we have closed subspaces $\mathscr{W}_2 \subseteq \mathscr{W}$ such that $\mathscr{V} \dotplus \mathscr{W}_2 = \mathscr{W}$. Furthermore, the number of subspaces $\mathscr{W}_2$ one can be explicitly obtain using e.g. the representation of the subspace $\mathscr{V}$ given by (3.14) (see Theorem 3.4.7).

On the other hand, Theorem 3.6.1 guarantees that for each pair $(\mathscr{V}, \widetilde{\mathscr{V}})$ we can find a suitable M by the choice $\mathscr{W}_2 = \ker T_1$. In other words, the mapping $(\mathscr{V}, \widetilde{\mathscr{V}}) \mapsto M$ given by Theorem 3.6.1 (i.e. the construction of Theorem 2.5.11(i) for $\mathscr{W}_2 = \ker T_1$) is well defined and injective. Of course, this mapping is not surjective, i.e. there are operators M satisfying (M)-conditions that cannot be obtained with this construction.

iv) In principle with this remark we closed the question on the multiplicity of M's. Now the interesting question is to see how this reflects to the classical theory. For example, do different operators M necessarily correspond to different classical representations $\mathbf{M}$? We leave this question as the subject of future investigation.

4. Friedrichs systems on an interval

The classification results discussed in the previous chapter are now the subject of application on the classical Friedrichs operators treated as abstract Friedrichs operators. In this chapter we study the classical Friedrichs systems on an interval (one dimensional case) in full generality. We start with the scalar case (Section 4.1) following [32] and then consider systems of ordinary differential equations in the rest of the chapter, which is studied in [34].

4.1. 1-d scalar case

Our goal is to present a classification of boundary conditions of classical Friedrichs operators in one dimensional ($d=1$) scalar case ($r=1$) in full generality. In this section we develop some preliminary results in this direction. The domain $\Omega=(a,b)$ is an open interval $a<b$ and the spaces are $\mathscr{D}=C_c^\infty(a,b)$ and $\mathscr{H}=L^2(a,b)$. We adjust the notation of $T,\widetilde{T}:\mathscr{D}\to\mathscr{H}$ given in Example 2.6.1 in the following way:

$$T\varphi := (\alpha\varphi)' + \beta\varphi \qquad \text{and} \qquad \widetilde{T}\varphi := -(\alpha\varphi)' + (\overline{\beta}+\alpha')\varphi\,, \tag{4.1}$$

where $\alpha\in W^{1,\infty}((a,b);\mathbb{C})$, $\beta\in L^\infty((a,b);\mathbb{C})$, $\alpha=\overline{\alpha}$ and for some $\mu_0>0$ we have $2\Re\beta+\alpha'\geq 2\mu_0>0$ ($\Re z$ denotes the real part of complex number z and $'$ the derivative).

It is already realised in Example 2.6.1 that the pair $(T,\widetilde{T})$ is a joint pair of abstract Friedrichs operators. In addition, without loss of generality we take these operators to be closed.

Graph Space: Let $u \in \mathscr{H}$ and $\beta \in L^\infty(a,b)$, then $\|\beta u\| \le \|\beta\|_\infty \|u\| < \infty$. Which means $Tu \in \mathscr{H}$ if and only if $(\alpha u)' \in \mathscr{H}$. Therefore, the graph space $\mathscr{W}$ is given by

$$\mathscr{W} := \{u \in \mathscr{H} : (\alpha u)' \in \mathscr{H}\}.$$

and the graph norm is (equivalent to)

$$\|u\|_{\mathscr{W}} = \|u\| + \|(\alpha u)'\| .$$

We also denote the graph norm as $\|\cdot\|_T$ or $\|\cdot\|_{\widetilde{T}}$.

Here, $u \in \mathscr{W}$ if and only if $\alpha u \in H^1(a,b)$. By the Sobolev embedding theorem (see e.g. [16, Theorem 8.2] or [44, Theorem 4.13]) for any $u \in \mathscr{W}$ we have $\alpha u \in C([a,b])$. This in particular implies that for any $u \in \mathscr{W}$ and $x \in [a,b]$, the evaluation map $(\alpha u)(x)$ is well-defined. However, in general $u \notin C([a,b])$ implying the evaluation $u(x)$ and thus $\alpha(x)u(x)$ is not defined. This evaluation is important in the description of the boundary operator. The following result gives a better description of the graph space.

Lemma 4.1.1. Let $I := [a,b] \setminus \alpha^{-1}(\{0\})$. Then $\mathscr{W} \subseteq H^1_{\rm loc}(I)$, i.e. for any $u \in \mathscr{W}$ and $[c,d] \subseteq I$, $c < d$, we have $u|_{[c,d]} \in H^1(c,d)$.

Proof. Here $\alpha \in W^{1,\infty}((a,b);\mathbb{R})$ and due to Sobolev embedding $\alpha \in C([a,b];\mathbb{R})$. Which means $I = [a,b] \setminus \alpha^{-1}(\{0\})$ is relatively open set in $[a,b]$. Let $[c,d] \subseteq I$, $(c < d)$ (if such subinterval does not exist, then $\alpha \equiv 0$ and $I = \emptyset$). The subinterval $[c,d]$ is compact, due to continuity of α we define $\alpha_0 = \min_{x\in[c,d]} |\alpha(x)|$. Obviously $\alpha_0 > 0$.

Let $u \in C_c^\infty(\mathbb{R})$, then

$$\begin{aligned}
\|u'\|_{L^2(c,d)} = \|\frac{1}{\alpha}\alpha u'\|_{L^2(c,d)} &\le \frac{1}{\alpha_0}\|\alpha u'\|_{L^2(c,d)} \\
&\le \frac{1}{\alpha_0}\Big(\|(\alpha u)'\|_{L^2(c,d)} + \|\alpha' u\|_{L^2(c,d)}\Big) \\
&\le \frac{1}{\alpha_0}\Big(\|(\alpha u)'\| + \|\alpha'\|_{L^\infty(a,b)}\|u\|\Big) \\
&\le \frac{1 + \|\alpha\|_{W^{1,\infty}(a,b)}}{\alpha_0}\|u\|_{\mathscr{W}} .
\end{aligned}$$

The space $C_c^\infty(\mathbb{R})$ is dense in $\mathscr{W}$. We conclude that

$$(\forall u \in \mathscr{W}) \qquad \|u\|_{H^1(c,d)}\| \le C\|u\|_{\mathscr{W}} .$$

Here, $C = 1 + \frac{1+\|\alpha\|_{W^{1,\infty}(a,b)}}{\alpha_0}$, thus $u|_{[c,d]} \in H^1(c,d)$. ■

Remark 4.1.2. The set $\alpha^{-1}(\{0\})$ can have some peculiar behavior, in any case, the set I is relatively open in $[a,b]$. Hence, the meaning of $H^1_{\mathrm{loc}}(I)$, as elaborated in the statement of the previous lemma, makes sense. To illustrate one such case, we consider the function

$$\alpha(x) = \begin{cases} x^2\sin(1/x) & , x \in (0,1]\,, \\ 0 & , x = 0\,, \end{cases}$$

which is a Lipschitz continuous function on $[0,1]$. The set $\{x \in [0,1] : \alpha(x) = 0\}$ is not well-behaved near the end-point $x = 0$, i.e. 0 is an accumulation point. However, the set is still closed and hence I is relatively open in $[0,1]$.

Remark 4.1.3. Here we list some immediate consequences of the previous lemma.

i) If $x \in I$, where I is defined in the statement of the previous lemma, then there exists a subinterval $[c,d] \subseteq I$ such that $x \in [c,d]$ and $u|_{[c,d]} \in H^1(c,d)$. Again by the Sobolev embedding we get $u|_{[c,d]} \in C([c,d])$ and since α is already in $C([c,d])$, we have $(\alpha u)(x) = \alpha(x)u(x)$. Then it is natural to expect that $(\alpha u)(x) = 0$ if $x \notin I$, i.e. $(\alpha u)(x) = 0$ whenever $\alpha(x) = 0$. Let us elaborate more on this: Let $u \in \mathscr{W}$ and (u_n) be a sequence in $C_c^\infty(\mathbb{R})$ such that $u_n \xrightarrow{\mathscr{W}} u$ (cf. [1, Theorem 4]). This implies that $u_n \to u$ and $(\alpha u_n)' \to (\alpha u)'$ in L^2. Since $\alpha \in W^{1,\infty}(a,b)$, we also have $\alpha u_n \to \alpha u$ in L^2. Thus, $\alpha u_n \xrightarrow{H^1} \alpha u$. Now, using the Sobolev embedding, for any $x \in [a,b]$ we have $\alpha(x)u_n(x) = (\alpha u_n)(x) \to (\alpha u)(x)$. Hence, for $x \notin I$, i.e. $\alpha(x) \neq 0$, we have $(\alpha u)(x) = 0$.

ii) From the proof of the previous lemma we can deduce that $\mathscr{W}$ is continuously embedded in $H^1_{\mathrm{loc}}(I)$, which we write as $\mathscr{W} \hookrightarrow H^1_{\mathrm{loc}}(I)$. The embedding is strict, an argument being that $\mathscr{W}$ a Hilbert space, in particular it is a Fréchet space, while $H^1_{\mathrm{loc}}(I)$ is not.

   On the other hand, it follows easily from Lemma 4.1.1 that $H^1(a,b) \hookrightarrow \mathscr{W}$ and $H^1(a,b) = \mathscr{W}$ if and only if α has no zeros on $[a,b]$.

The boundary operator in the case of classical Friedrichs operators understood as abstract Friedrichs operators is defined explicitly on the dense subspace $C_c^\infty(\mathbb{R})$ as mentioned in (2.10). Using previous lemma we can extend it uniquely on $\mathscr{W}$ by density argument, as follows,

$$[\, u \mid v \,] = {}_{\mathscr{W}'}\langle Du, v\rangle_{\mathscr{W}} = \big(\alpha u\bar{v}\big)(b) - \big(\alpha u\bar{v}\big)(a)\,, \quad u,v \in \mathscr{W}\,, \tag{4.2}$$

where we have (see Remark 4.1.3(i))

$$\big(\alpha u\bar{v}\big)(x) := \begin{cases} 0 & , \quad \alpha(x) = 0 \\ \alpha(x)u(x)\overline{v(x)} & , \quad \alpha(x) \neq 0 \end{cases} \quad , \quad x \in [a,b]\,. \tag{4.3}$$

The minimal space i.e. the closures of T and $\widetilde{T}$ is $\mathscr{W}_0 = \mathrm{cl}_{\mathscr{W}} C_c^\infty(\mathbb{R})$. Using (4.2) $\ker D = \mathscr{W}_0$ can be characterised more explicitly.

Lemma 4.1.4. The space $\mathscr{W}_0$ can be characterised as

$$\mathscr{W}_0 = \Big\{ u \in \mathscr{W} : (\alpha u)(a) = (\alpha u)(b) = 0 \Big\}\,,$$

where $(\alpha u)(x)$ is to be understood as in (4.3).

Proof. Let us define the set $S := \Big\{ u \in \mathscr{W} : (\alpha u)(a) = (\alpha u)(b) = 0 \Big\}$. Since, $\ker D = \mathscr{W}_0$, we prove that $S = \ker D$. Let $u \in S$, then $(\alpha u)(a) = (\alpha u)(b) = 0$. For any $v \in \mathscr{W}$,

$$[\, u \mid v \,] = \big(\alpha u\bar{v}\big)(b) - \big(\alpha u\bar{v}\big)(a) = 0 - 0 = 0\,,$$

according to (4.3) (of course we deal with this in cases depending on the value of α at boundary points, but in each case the result is 0). Hence, $S \subseteq \ker D$. To prove the other inclusion, let $u \in \ker D$. Then, for any $v \in C_c^\infty(\mathbb{R}) \subseteq \mathscr{W}$, we have

$$0 = [\, u \mid v \,] = (\alpha u\bar{v})(b) - (\alpha u\bar{v})(a) = (\alpha u)(b)\overline{v(b)} - (\alpha u)(a)\overline{v(a)}\,.$$

If we choose $v(x) = x - a$, then$(\alpha u)(b) = 0$, while for $v(x) = x - b$ we get $(\alpha u)(a) = 0$, implying $u \in S$, thus $\ker D \subseteq S$. Which completes the proof. ∎

From the decomposition of the graph space (3.6) we see that $\ker T_1 + \ker \widetilde{T}_1$, or equivalently $\mathscr{W}/\mathscr{W}_0$, plays an important role in studying boundary conditions associated to T (or $\widetilde{T}$). In fact, in the case of finite dimensional kernels, it is enough to know their dimensions or codimension of the space $\mathscr{W}/\mathscr{W}_0$. We realise that in this situation the codimension completely depends on sign of α at the end-points of the interval (a,b).

Lemma 4.1.5.

$$\dim(\mathscr{W}/\mathscr{W}_0) \;=\; \begin{cases} 2 & , \quad \alpha(a)\alpha(b) \neq 0\,, \\ 1 & , \quad \big(\alpha(a) = 0 \wedge \alpha(b) \neq 0\big) \vee \big(\alpha(a) \neq 0 \wedge \alpha(b) = 0\big)\,, \\ 0 & , \quad \alpha(a) = \alpha(b) = 0\,. \end{cases}$$

Proof. Since $C_c^\infty(\mathbb{R}) \subseteq \mathscr{W}$, we choose $\varphi, \psi \in C_c^\infty(\mathbb{R})$ such that $\varphi(a) = 1$, $\varphi(b) = 0$ and $\psi(a) = 0$, $\psi(b) = 1$. Define $\hat{\varphi} := \varphi + \mathscr{W}_0$ and $\hat{\psi} := \psi + \mathscr{W}_0$.

- Consider the case $\alpha(a)\alpha(b) \neq 0$. and let $B := \{\hat{\varphi}, \hat{\psi}\}$. It is enough to prove that the set B forms a basis of $\mathscr{W}/\mathscr{W}_0$ and hence it has dimension 2. Contrarily, let us first check that B is linearly independent. Let r be a scalar, such that $\hat{\psi} = r\hat{\varphi}$. Which means, $\hat{\psi} - r\hat{\varphi} = \hat{0}$ and then $\psi - r\varphi \in \mathscr{W}_0$. From the characterisation of $\mathscr{W}_0$ in Lemma 4.1.4, we have that

$$\big(\alpha(\psi - r\varphi)\big)(a) = \big(\alpha(\psi - r\varphi)\big)(b) = 0.$$

However,

$$\big(\alpha(\psi - r\varphi)\big)(a) = \alpha(a)\psi(a) - r\alpha(a)\varphi(a) = -r\alpha(a) \neq 0\,,$$

here we used that $\alpha(a) \neq 0$. This contradiction proves that B is indeed linearly independent. Also any $\hat{u} \in \mathscr{W}/\mathscr{W}_0$ can be written as $\hat{u} = u(a)\hat{\varphi} + u(b)\hat{\psi}$, because

$$u - u(a)\varphi - u(b)\psi \in \mathscr{W}_0\ ,$$

(of course $u(a)$ and $u(b)$ here make sense due to the condition $\alpha(a)\alpha(b) \neq 0$) which implies that B spans $\mathscr{W}/\mathscr{W}_0$. Hence, $\dim(\mathscr{W}/\mathscr{W}_0) = 2$.

- If $\alpha(a) = 0$ and $\alpha(b) \neq 0$, then we have

$$(\alpha\varphi)(a) = \alpha(a)\varphi(a) = 0 \quad \text{and} \quad (\alpha\varphi)(b) = \alpha(b)\varphi(b) = 0\,,$$

implying $\varphi \in \mathscr{W}_0$, so $\mathscr{W}/\mathscr{W}_0 = \operatorname{span}\{\hat{\psi}\}$ and $\dim(\mathscr{W}/\mathscr{W}_0) = 1$.
Similarly, if $\alpha(a) \neq 0$ and $\alpha(b) = 0$, we also have $\dim(\mathscr{W}/\mathscr{W}_0) = 1$.

- If $\alpha(a) = \alpha(b) = 0$, then the boundary operator $D = 0$, which means $\mathscr{W} = \ker(D) = \mathscr{W}_0$. Hence, $\dim(\mathscr{W}/\mathscr{W}_0) = 0$.

■

This result is interesting in the sense that the nature of $\alpha \in W^{1,\infty}((a,b),\mathbb{R})$ in the interior of the interval $[a,b]$ does not affect the codimension.

Remark 4.1.6. i) If $\min_{x\in[a,b]}|\alpha(x)| > \alpha_0 > 0$ i.e. α is uniformly away from 0 in the interval (a,b), then the statement of the previous lemma reveals a well known fact that $\dim\big(H^1(a,b)/H^1_0(a,b)\big) = 2$.

ii) By the decomposition (3.6) we have

$$\dim(\ker T_1) + \dim(\ker \widetilde{T}_1) = \dim \mathscr{W}/\mathscr{W}_0\,.$$

Thus, by the previous lemma and Theorem 2.4.12 (about multiplicity) we can immediately conclude that in the case $\alpha(a)\alpha(b) = 0$ there is only one bijective realisation of T_0. Moreover, in the opposite case $\alpha(a)\alpha(b) \neq 0$ there are infinitely many bijective realisations if and only if $\dim(\ker T_1) = \dim(\ker \widetilde{T}_1)$.

4.2. 1-D SCALAR CASE-CLASSIFICATION

For abstract Friedrichs operators, the classification schemes mentioned in Chapter 2 using operator-theoretic approach can be applied to classical Friedrichs operators. In this section we achieve a classification of one dimensional ($d = 1$) scalar ($r = 1$) classical Friedrichs operators in full generality. The classification is divided into three cases depending on the sign$(\alpha(a)\alpha(b))$. Somehow all the cases are trivial except $\alpha(a)\alpha(b) > 0$, where we achieve infinitely many pairs of mutually adjoint bijective realisations and we provide an explicit classification for all. Let us recall that we want to find $\mathcal{V} \subseteq \mathcal{W}$ containing $\mathcal{W}_0$ such that $(T_1|_{\mathcal{V}}, \widetilde{T}_1|_{\widetilde{\mathcal{V}}})$ is a pair of bijective realisations, of course $\widetilde{\mathcal{V}} = \mathcal{V}^{[\perp]}$.

4.2.1. Case 1: $\alpha(a)\alpha(b) = 0$.

Since the boundary map is given by the values at end-points of the interval (a,b), let us investigate the effect of this case on the boundary map.

(i) If $\alpha(a) = \alpha(b) = 0$, then the boundary map is trivial $D = 0$ and we have $\ker D = \mathcal{W}_0 = \mathcal{W}$. There is only one pair of mutually adjoint bijective realisations given by $(\mathcal{V}, \widetilde{\mathcal{V}}) = (\mathcal{W}, \mathcal{W})$.

(ii) If $\alpha(a) = 0$ and $\alpha(b) > 0$, then the boundary map is given by

$$(\forall u, v \in \mathcal{W}) \quad {}_{\mathcal{W}'}\langle Du, v\rangle_{\mathcal{W}} = \alpha(b)u(b)\overline{v(b)}\,, \quad u, v \in \mathcal{W}\,,$$

and by Lemma 4.1.4, the space $\mathcal{W}_0$ is

$$\mathcal{W}_0 = \{u \in \mathcal{W} : (\alpha u)(b) = 0\}\,.$$

For any $u \in \mathcal{W}$, we have $[\,u \mid u\,] = \alpha(b)|u(b)|^2 \geq 0$. Which means $(\mathcal{W}, \mathcal{W}_0)$ satisfies (V1)-condition. By Lemma 2.2.5 $\mathcal{W}_0$ and $\mathcal{W}$ are $[\perp]-$orthogonal to each other and hence by the well-posedness result 2.4.5, $(T_1|_{\mathcal{W}}, \widetilde{T}_1|_{\mathcal{W}_0}) = (T_1, \widetilde{T}_0)$ is a pair of mutually adjoint bijective realisations. Since this implies that $\ker T_1 = \{0\}$, by Theorem 2.4.12(ii), $(T_1, \widetilde{T}_0)$ is the only pair of mutually adjoint bijective realisations relative to $(T, \widetilde{T})$.

(iii) The analysis for other subcases is completely analogous to the previous one. Here we list the pair of mutually adjoint bijective realisations depending on the other three subcases.

α at end-points	$(\mathscr{V},\widetilde{\mathscr{V}})$
$\alpha(a)=0,\alpha(b)<0$	$(\mathscr{W}_0,\mathscr{W})$
$\alpha(a)>0,\alpha(b)=0$	$(\mathscr{W}_0,\mathscr{W})$
$\alpha(a)<0,\alpha(b)=0$	$(\mathscr{W},\mathscr{W}_0)$

Overall conclusion for this case can be given as follows,

Lemma 4.2.1. For $\alpha(a)\alpha(b)=0$, all pairs of mutually adjoint bijective realisations can be classified as

$$(\mathscr{V},\widetilde{\mathscr{V}}) = \begin{cases} (\mathscr{W},\mathscr{W}_0) \quad , \quad \big(\alpha(a)=0\wedge\alpha(b)\geq 0\big)\vee\big(\alpha(a)\leq 0\wedge\alpha(b)=0\big) \\ (\mathscr{W}_0,\mathscr{W}) \quad , \quad \big(\alpha(a)=0\wedge\alpha(b)\leq 0\big)\vee\big(\alpha(a)\geq 0\wedge\alpha(b)=0\big) \end{cases} , \tag{4.4}$$

i.e we always have only one pair of bijective realisations. In the above we also included the first case $\alpha(a)=\alpha(b)=0$ as then $(\mathscr{W},\mathscr{W}_0)=(\mathscr{W}_0,\mathscr{W})=(\mathscr{W},\mathscr{W})$.

This agrees with Remark 4.1.6(ii) (note that Lemma 4.1.5 is not used). Although we have fully characterised bijective realisations, let us say a little more about kernels of T_1 and $\widetilde{T}_1$.

In the case $\alpha(a)=\alpha(b)=0$ it is clear that $\ker T_1=\ker\widetilde{T}_1=\{0\}$. This means that both equations

$$(\alpha\varphi)'+\beta\varphi=0 \quad \text{and} \quad -(\alpha\varphi)'+(\overline{\beta}+\alpha')\varphi=0$$

do not have any non-trivial solution in $\mathscr{W}$.

If exactly one of numbers $\alpha(a)$ and $\alpha(b)$ is equal to zero, from Remark 4.1.6(ii) we have $\dim(\ker T_1)+\dim(\ker\widetilde{T}_1)=1$, while the analysis above implies that one of dimensions equals 0 (the one associated to the operator for which we took the whole graph space $\mathscr{W}$ as the domain of the bijective realisation – see (4.4)). To be more specific, let us stick to the case $\alpha(a)=0$ and $\alpha(b)>0$. Then, $\dim(\ker T_1)=0$, hence $\dim(\ker\widetilde{T}_1)=1$. Let us denote by $\tilde{\varphi}\in\mathscr{W}$ a function that forms a basis of $\ker\widetilde{T}_1$.

If α does not have any zeros in the open interval (a,b), then $\tilde{\varphi}$ is just a non-trivial solution of

$$-(\alpha\tilde{\varphi})' + (\overline{\beta} + \alpha')\tilde{\varphi} = 0$$

in (a,b).

On the other hand, if $\alpha^{-1}(\{0\}) \cap (a,b) \neq \emptyset$, let us define

$$x^0_{\min} := \min\Big(\alpha^{-1}(\{0\}) \cap (a,b)\Big) \,, \quad x^0_{\max} := \max\Big(\alpha^{-1}(\{0\}) \cap (a,b)\Big) \qquad (4.5)$$

(one should note that in this case $x^0_{\min}$ might not exist, however in the following analysis only $x^0_{\max}$ is required). Since in particular $\tilde{\varphi}$ should satisfy the differential equation above in $(a, x^0_{\max})$, where we have $\alpha(a) = \alpha(x^0_{\max}) = 0$, by the conclusion of the first subcase ($\alpha(a) = \alpha(b) = 0$) we have that a.e. $\tilde{\varphi}|_{(a,x^0_{\max})} = 0$. Thus, $\operatorname{supp}\tilde{\varphi} \subseteq [x^0_{\max}, b]$ (see Figure 4.1).

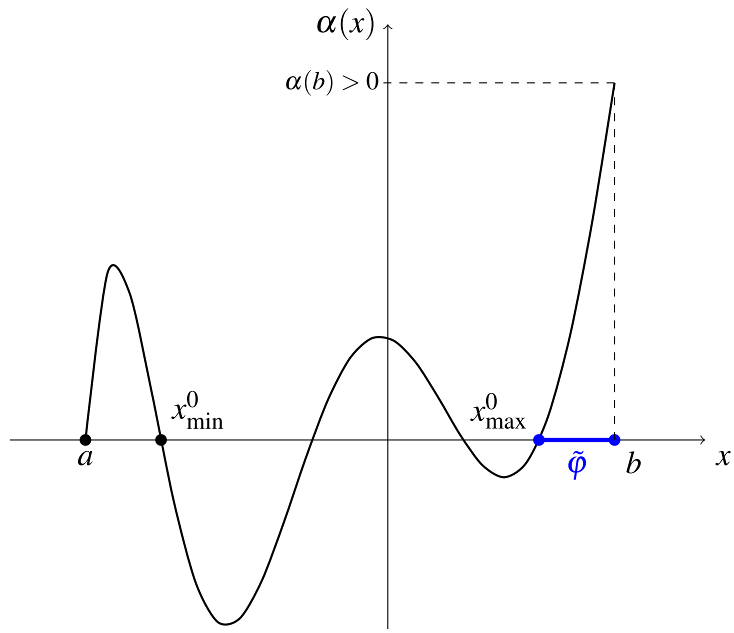

Figure 4.1: For α satisfying $\alpha(a) = 0$ and $\alpha(b) > 0$ we denoted on the graph points $x^0_{\min}$ and $x^0_{\max}$. The bold blue line segment contains the support of $\tilde{\varphi}$.

In other cases the only differences are whether $\dim(\ker T_1) = 1$ or $\dim(\ker \widetilde{T}_1) = 1$, and whether a function forming a basis is supported in $[a, x^0_{\min}]$ or $[x^0_{\max}, b]$. Here we summarise the subcases with dimensions.

$\alpha(a)\alpha(b)=0$	$(\mathscr{V},\widetilde{\mathscr{V}})$	$(\dim\ker T_1, \dim\ker\widetilde{T}_1)$
$\alpha(a)=0, \alpha(b)=0$	$(\mathscr{W}_0,\mathscr{W}_0)$	$(0,0)$
$\alpha(a)>0$ or $\alpha(b)<0$	$(\mathscr{W}_0,\mathscr{W})$	$(1,0)$
$\alpha(a)<0$ or $\alpha(b)>0$	$(\mathscr{W},\mathscr{W}_0)$	$(0,1)$

4.2.2. Case 2: $\alpha(a)\alpha(b)<0$.

This case is divided further in two subcases. However, the boundary map and the minimal subspace remain the same i.e. $\mathscr{W}_0 = \{u \in \mathscr{W} : (\alpha u)(a) = (\alpha u)(b) = 0\}$.

(i) When $\alpha(a)>0, \alpha(b)<0$, we have

$$(\forall u \in \mathscr{W}) \quad [\, u \mid u\,] = \alpha(b)|u(b)|^2 - \alpha(a)|u(a)|^2 \leq 0\,,$$

here again the evaluations $u(a)$ and $u(b)$ are well-defined due to Lemma 4.1.1. Hence, by Theorem 2.4.5 and with similar reasoning as in the previous case we get that $(T_0,\widetilde{T}_1) = (T_1|_{\mathscr{V}},\widetilde{T}_1|_{\widetilde{\mathscr{V}}})$ is the only pair of mutually adjoint bijective realisations relative to $(T,\widetilde{T})$.

(ii) When $\alpha(a)<0, \alpha(b)>0$, this subcase is completely analogous to the previous subcase. Here, $(T_1,\widetilde{T}_0)$ is the only pair of mutually adjoint bijective realisations relative to $(T,\widetilde{T})$.

Although (the codimension) Lemma 4.1.5 suggests that $\dim\ker T_1 + \ker\widetilde{T}_1 = 2$, we have only one pair of bijective realisations. Hence, this analysis together with Theorem 2.4.12 suggests the following:

α at end-points	$(\mathscr{V},\widetilde{\mathscr{V}})$	$(\dim\ker T_1, \dim\ker\widetilde{T}_1)$
$\alpha(a)>0, \alpha(b)<0$	$(\mathscr{W}_0,\mathscr{W})$	$(2,0)$
$\alpha(a)<0, \alpha(b)>0$	$(\mathscr{W},\mathscr{W}_0)$	$(0,2)$

Let us focus on the case $\alpha(a)<0$ and let us study $\ker\widetilde{T}_1$. We define $x^0_{\min}$ and $x^0_{\max}$ as in (4.5). They are well-defined since $\alpha(a)\alpha(b)<0$ and α is continuous, hence $\alpha^{-1}(\{0\})$ is not empty. With the same argument as in the previous case we can conclude that for any $\tilde{\varphi} \in \ker \widetilde{T}_1$ we have that a.e. $\tilde{\varphi}|_{[x^0_{\min}, x^0_{\max}]} = 0$. Moreover, on both subintervals $(a, x^0_{\min})$ and $(x^0_{\max}, b)$ we are in the same case regarding (4.4), and this is precisely the reason why we have that one kernel is trivial, while the other being two-dimensional.

Thus, if we take $\tilde{\varphi}_1, \tilde{\varphi}_2 \in \mathscr{W}$ such that $\tilde{\varphi}_2 = 0$ on $[a, x^0_{\max}]$ and in $(x^0_{\max}, b)$ to be a non-trivial solution to the corresponding differential equation, while $\tilde{\varphi}_1 = 0$ on $[x^0_{\min}, b]$ and in $(a, x^0_{\min})$ to be a non-trivial solution to the corresponding differential equation (see Figure 4.2), then $\{\tilde{\varphi}_1, \tilde{\varphi}_2\}$ is a basis for $\ker \widetilde{T}_1$.

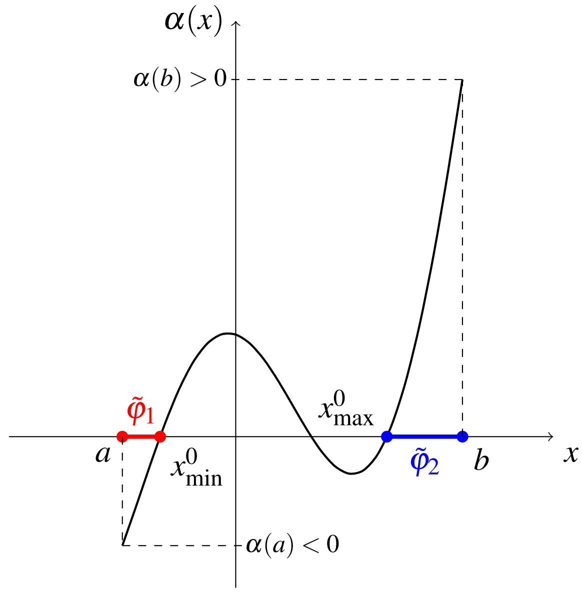

Figure 4.2: For α satisfying $\alpha(a) < 0$ and $\alpha(b) > 0$ we denoted on the graph points $x^0_{\min}$ and $x^0_{\max}$. The bold red and blue line segments contain supports of $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$, respectively.

4.2.3. Case 3: $\alpha(a)\alpha(b) > 0$.

Similar to the previous case, here we have $\mathscr{W}_0 = \{u \in \mathscr{W} : u(a) = u(b) = 0\}$, and the boundary operator reads (see (4.2)):

$$[\, u \mid v \,] = \alpha(b)u(b)\overline{v(b)} - \alpha(a)u(a)\overline{v(a)}\,, \quad u, v \in \mathscr{W}\,.$$

Let us define a subspace $\mathscr{V}$ of $\mathscr{W}$ as

$$\mathscr{V} := \left\{ u \in \mathscr{W} : u(b) = \sqrt{\frac{\alpha(a)}{\alpha(b)}}\, u(a) \right\}. \tag{4.6}$$

Here again the evaluations $u(a), u(b)$ are well defined due to Lemma 4.1.1. We prove that the pair $(\mathscr{V}, \widetilde{\mathscr{V}})$ satisfies (V)-conditions, then by Theorem 2.4.5 we have that operators T_{r} and T_{r}^*, where $T_{\mathrm{r}} := T_1|_{\mathscr{V}}$, form a mutually adjoint pair of bijective realisations relative to $(T, \widetilde{T})$.

Lemma 4.2.2. Let $\mathscr{V}$ be defined as above in (4.6), then $(\mathscr{V}, \mathscr{V})$ satisfies (V)-conditions.

Proof. For any $u \in \mathscr{V}$ and $v \in \mathscr{W}$ we have

$$
\begin{aligned}
[u \mid v] &= \alpha(b)u(b)\overline{v(b)} - \alpha(a)u(a)\overline{v(a)} \\
&= \alpha(b)\left(u(b)\overline{v(b)} - \sqrt{\frac{\alpha(a)}{\alpha(b)}}u(a)\sqrt{\frac{\alpha(a)}{\alpha(b)}}\overline{v(a)}\right) \\
&= \alpha(b)u(b)\overline{\left(v(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}v(a)\right)}\,.
\end{aligned}
$$

In particular, for any $u \in \mathscr{V}$ we have $[u \mid u] = 0$. Which means $(\mathscr{V}, \mathscr{V})$ satisfies (V1)-condition and $\mathscr{V} \subseteq \mathscr{V}^{[\perp]}$. To prove (V2)-condition, it is left to show $\mathscr{V}^{[\perp]} \subseteq \mathscr{V}$. Let $v \in \mathscr{V}^{[\perp]}$, then following the previous calculation, for any $u \in \mathscr{V}$ we have

$$
\alpha(b)u(b)\overline{\left(v(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}v(a)\right)} = 0\,.
$$

Since $\alpha(b) \neq 0$ and there exists $u \in \mathscr{V}$ such that $u(b) \neq 0$ (e.g. just consider the linear function $u(x) = \left(\sqrt{\frac{\alpha(a)}{\alpha(b)}} - 1\right)\left(\frac{x-a}{b-a}\right) + 1$), this implies $v(b) = \sqrt{\frac{\alpha(a)}{\alpha(b)}}v(a)$, i.e. $v \in \mathscr{V}$. Which completes the proof. ∎

Therefore, $(T_{\mathrm{r}}, T_{\mathrm{r}}^*)$ is indeed a mutually adjoint pair of bijective realisations relative to $(T, \widetilde{T})$. It is evident that $\mathscr{W}_0 \subsetneqq \mathscr{V} \subsetneqq \mathscr{W}$, hence by Theorem 2.4.12(ii) there are infinitely many bijective realisations. In particular, using the same theorem, we can conclude that both $\dim(\ker T_1)$ and $\dim(\ker \widetilde{T}_1)$ are greater or equal to 1. Now Remark 4.1.6(ii) implies that in fact we have $\dim(\ker T_1) = \dim(\ker \widetilde{T}_1) = 1$. Let us emphasise that this conclusion is in accordance with Corollary 3.3.9. Let φ and $\tilde{\varphi}$ span $\ker T_1$ and $\ker \widetilde{T}_1$ respectively. We shall discuss more about the explicit form of these vectors later. First, let us determine all bijective realisations in this case (in terms of φ and $\tilde{\varphi}$) using classification schemes from Chapter 2.

We are looking for *bijective* realisations, and thus the classifying operator B, densely defined over a closed non-trivial subspace $\mathscr{Z}$ of $\ker T_1$ an mapping to closed subspace $\widetilde{\mathscr{Z}}$

of $\ker \widetilde{T}_1$, in Theorem 3.1.3 should be bijective as well. Both kernels of T_1 and $\widetilde{T}_1$ are one-dimensional, hence the only (non-trivial) choice is $\operatorname{dom} B = \mathscr{Z} = \ker T_1$ and $\widetilde{\mathscr{Z}} = \operatorname{dom} \widetilde{T}_1$ (then also $\operatorname{dom} B^* = \ker \widetilde{T}_1$). Then there exists $(c+id) \in \mathbb{C}$ such that $B\varphi = (c+id)\tilde{\varphi}$. Therefore, all bijective realisations are indexed by $c+id \in \mathbb{C} \setminus \{0\}$ (for these values B is an isomorphism). The operator corresponding to B we denote by $T_{c,d} = T_B$. Recall that $T_0 \subseteq T_{c,d} \subseteq T_1$. From (3.2) we have, $u \in \mathscr{W}$ belongs to $\operatorname{dom} T_{c,d}$ if and only if

$$P_{\ker \widetilde{T}_1}(T_1 u) = B(p_{\mathrm{k}} u)\,, \tag{4.7}$$

where $P_{\ker \widetilde{T}_1}$ is the orthogonal projection from $\mathscr{H}$ onto $\ker \widetilde{T}_1$ and p_{k} is the non-orthogonal projector corresponding to the decomposition $\mathscr{W} = \mathscr{V} \dotplus \ker T_1$ which is due to Theorem 3.2.10. For any $u \in \mathscr{W}$ there exist unique $u_{\mathrm{r}} \in \mathscr{V}$ and $u_{\mathrm{k}} \in \ker T_1$ such that $u = u_{\mathrm{r}} + u_{\mathrm{k}}$. Moreover, u_{k} is just a scalar multiple of φ, i.e of the form $C_u \varphi$, so we get

$$\begin{aligned} u(a) &= u_{\mathrm{r}}(a) + C_u \varphi(a)\,, \\ u(b) &= u_{\mathrm{r}}(b) + C_u \varphi(b)\,. \end{aligned}$$

Since, $u_{\mathrm{r}} \in \mathscr{V}$, we have $u_{\mathrm{r}}(b) = \sqrt{\frac{\alpha(a)}{\alpha(b)}} u_{\mathrm{r}}(a)$, hence we get

$$C_u = \frac{u(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}} u(a)}{\varphi(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}} \varphi(a)} \tag{4.8}$$

here $\varphi \in \ker T_1$ and since $\ker T_1 \cap \mathscr{V} = \{0\}$, we have $\varphi \notin \mathscr{V}$ implying that the denominator is non-zero and C_u is well defined. Thus, the corresponding non-orthogonal projection $p_{\mathrm{k}} : \mathscr{W} \to \ker T_1$ is given by $p_{\mathrm{k}}(u) = C_u \varphi$. Similarly, $p_{\tilde{\mathrm{k}}} : \mathscr{W} \to \ker \widetilde{T}_1$ is given by $p_{\tilde{\mathrm{k}}}(u) = \tilde{C}_u \tilde{\varphi}$, where

$$\tilde{C}_u = \frac{u(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}} u(a)}{\tilde{\varphi}(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}} \tilde{\varphi}(a)}\,.$$

Now, let $u \in \mathscr{W}$, since the orthogonal projection of $T_1 u$ onto $\ker \widetilde{T}_1$ is same as the orthogonal projection onto $\tilde{\varphi}$, we have

$$\begin{aligned} P_{\ker \widetilde{T}_1}(T_1 u) &= \frac{1}{\|\tilde{\varphi}\|^2} \langle T_1 u \mid \tilde{\varphi} \rangle \tilde{\varphi} \\ &= \frac{1}{\|\tilde{\varphi}\|^2} [T_1 u \mid \tilde{\varphi}] \tilde{\varphi} \\ &= \frac{1}{\|\tilde{\varphi}\|^2} \Big(\alpha(b) u(b) \overline{\tilde{\varphi}(b)} - \alpha(a) u(a) \overline{\tilde{\varphi}(a)} \Big) \tilde{\varphi}\,, \end{aligned}$$

in the second equality we used $\widetilde{T}_1\tilde{\varphi}=0$. Substituting the values in the equation (4.7), we get

$$\frac{1}{\|\tilde{\varphi}\|^2}\Big(\alpha(b)u(b)\overline{\tilde{\varphi}(b)}-\alpha(a)u(a)\overline{\tilde{\varphi}(a)}\Big)\tilde{\varphi} \;=\; (c+id)\left(\frac{u(b)-\sqrt{\frac{\alpha(a)}{\alpha(b)}}u(a)}{\varphi(b)-\sqrt{\frac{\alpha(a)}{\alpha(b)}}\varphi(a)}\right)\tilde{\varphi}\;.$$

We reorganise this equation to get in more desirable form

$$\begin{aligned}&\left(\frac{\alpha(b)\overline{\tilde{\varphi}(b)}}{\|\tilde{\varphi}\|^2}-\frac{(c+id)}{\varphi(b)-\sqrt{\frac{\alpha(a)}{\alpha(b)}}\varphi(a)}\right)u(b)\\ &\quad=\left(\frac{\alpha(a)\overline{\tilde{\varphi}(a)}}{\|\tilde{\varphi}\|^2}-\frac{(c+id)\sqrt{\frac{\alpha(a)}{\alpha(b)}}}{\varphi(b)-\sqrt{\frac{\alpha(a)}{\alpha(b)}}\varphi(a)}\right)u(a)\;.\end{aligned}\tag{4.9}$$

Hence, $u\in\mathscr{W}$ is in $\operatorname{dom}T_B$ if and only if it satisfies the above condition (4.9). The adjoint operator $B^*:\ker\widetilde{T}_1\to\ker T_1$ reads

$$B^*(\tilde{\varphi}) \;=\; \frac{\|\tilde{\varphi}\|^2}{\|\varphi\|^2}(c-id)\varphi\;.$$

Indeed, we have

$$\langle B\varphi\mid\tilde{\varphi}\rangle=\langle\,(c+id)\tilde{\varphi}\mid\tilde{\varphi}\,\rangle=(c+id)\|\tilde{\varphi}\|^2\,,$$

and

$$\langle\,\varphi\mid B^*\tilde{\varphi}\,\rangle=\langle\,\varphi\mid\frac{\|\tilde{\varphi}\|^2}{\|\varphi\|^2}(c-id)\varphi\,\rangle=\frac{\|\tilde{\varphi}\|^2}{\|\varphi\|^2}(c+id)\langle\,\varphi\mid\varphi\,\rangle=(c+id)\|\tilde{\varphi}\|^2\,,$$

satisfying $\langle B\varphi\mid\tilde{\varphi}\,\rangle=\langle\,\varphi\mid B^*\tilde{\varphi}\,\rangle$. Following a similar calculation as above we obtain that $u\in\mathscr{W}$ is in $\operatorname{dom}T^*_{c,d}$ if and only if

$$\begin{aligned}&\left(\alpha(b)\overline{\varphi(b)}-\frac{\|\tilde{\varphi}\|^2(c-id)}{\tilde{\varphi}(b)-\sqrt{\frac{\alpha(a)}{\alpha(b)}}\tilde{\varphi}(a)}\right)u(b)\\ &\quad=\left(\alpha(a)\overline{\varphi(a)}-\frac{\|\tilde{\varphi}\|^2(c-id)\sqrt{\frac{\alpha(a)}{\alpha(b)}}}{\tilde{\varphi}(b)-\sqrt{\frac{\alpha(a)}{\alpha(b)}}\tilde{\varphi}(a)}\right)u(a)\;.\end{aligned}\tag{4.10}$$

Therefore, the set of all pairs of mutually adjoint bijective realisations relative to $(T,\widetilde{T})$ in this case is given by

$$\Big\{(T_{c,d},T^*_{c,d}):c,d\in\mathbb{R}^2\setminus\{(0,0)\}\Big\}\bigcup\big\{(T_{\mathrm{r}},T^*_{\mathrm{r}})\big\}\;.\tag{4.11}$$

All bijective realisations are parameterised by *one* complex parameter $(c+id)$, which is in parallel to the fact that the dimension of both kernels $\ker T_1$ and $\ker \widetilde{T}_1$ is *one*.

Note that $\operatorname{dom} T_{c,d} = \mathscr{W}_0 + \ker \widetilde{T}_1$ (see Corollary 3.2.6) if and only if $\tilde{\varphi} \in \operatorname{dom} T_{c,d}$. Indeed, then $\mathscr{W}_0 + \ker \widetilde{T}_1 \subseteq \operatorname{dom} T_{c,d}$ and the inclusion cannot be strict as in that case it would be impossible that both operators $T_{c,d}$ and $T_1|_{\mathscr{W}_0 + \ker \widetilde{T}_1}$ are bijective. From the above it can be easily seen that $\tilde{\varphi} \in \operatorname{dom} T_{c,d}$ is achieved if and only if

$$c + id = \frac{[\,\tilde{\varphi} \mid \tilde{\varphi}\,]}{\|\tilde{\varphi}\|^2 C_{\tilde{\varphi}}} \,.$$

Let us go back to kernels of T_1 and $\widetilde{T}_1$, so that we can derive some properties of functions φ and $\tilde{\varphi}$.

If $\min_{x\in[a,b]} |\alpha(x)| > 0$, then we get φ and $\tilde{\varphi}$ simply by taking non-trivial solutions of

$$(\alpha\varphi)' + \beta\varphi = 0 \qquad \text{and} \qquad -(\alpha\tilde{\varphi})' + (\overline{\beta} + \alpha')\tilde{\varphi} = 0 \tag{4.12}$$

on (a,b). Thus, a possible choice is ($x \in [a,b]$):

$$\varphi(x) = \frac{1}{\alpha(x)} \exp\Big(-\int \frac{\beta(x)}{\alpha(x)}\,dx\Big) \qquad \text{and} \qquad \tilde{\varphi}(x) = \exp\Big(\int \frac{\overline{\beta(x)}}{\alpha(x)}\,dx\Big)\,. \tag{4.13}$$

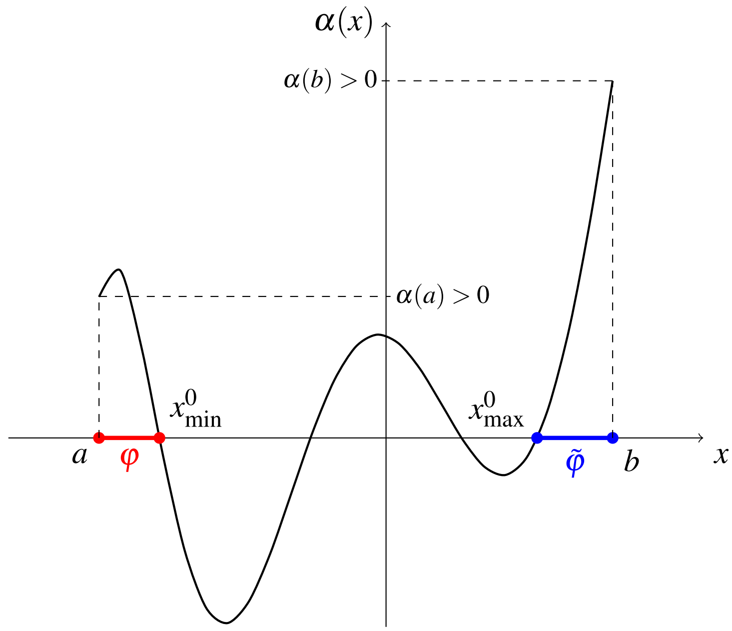

Figure 4.3: For α satisfying $\alpha(a) > 0$ and $\alpha(b) > 0$ we denoted on the graph points $x^0_{\min}$ and $x^0_{\max}$. The bold red and blue line segments contain supports of φ and $\tilde{\varphi}$, respectively.

If $\alpha^{-1}(\{0\}) \cap (a,b)$ is not empty, we define $x^0_{\min}$ and $x^0_{\max}$ as in (4.5). Here we can apply the same inference as in Case 1 to conclude that functions φ and $\tilde{\varphi}$ are supported on $[a, x^0_{\min}]$ or $[x^0_{\max}, b]$, while on the supports we just use (4.13) (one needs to be aware that now integrals are improper, but for sure convergent as we know that such non-trivial φ and $\tilde{\varphi}$ should exist in $\mathscr{W}$). To be more specific, let us assume that $\alpha(a) > 0$ and $\alpha(b) > 0$. Then any solution in $\mathscr{W}$ of the first equation in (4.12) must satisfy $\varphi|_{[x^0_{\min}, b]} = 0$, while for the second equation we have $\tilde{\varphi}|_{[a, x^0_{\max}]} = 0$ (see Figure 4.3). In particular, under this assumption we have $\varphi(b) = \tilde{\varphi}(a) = 0$, which could be used to simplify (4.9) and (4.10). Moreover, $\mathscr{W}_0 + \ker \widetilde{T}_1 = \{u \in \mathscr{W} : u(a) = 0\}$. This should hold since $r\tilde{\varphi}(b) = \delta\tilde{\varphi}(a)$ implies $\tilde{\varphi}(b) = 0$ or $r = 0$. The first option is not possible since $\tilde{\varphi} \equiv 0$ (solution of ODE on $(x^0_{\max}, b)$ with 0 for both boundary conditions), implying $r = 0$ thus the condition reads $u(a) = 0$ ($\delta \neq 0$ since in contrary we would have $\mathscr{W}_0 \dotplus \ker \widetilde{T}_1 = \mathscr{W}$, which is not possible since $\ker T_1 \neq \{0\}$). Also, $\mathscr{W}_0 + \ker T_1 = \{u \in \mathscr{W} : u(b) = 0\}$ holds following a similar argument as above.

Remark 4.2.3. Equation (4.10) covers all (linear) boundary conditions which are of the form $\gamma u(b) = \delta u(a)$, where $(\gamma, \delta) \in \mathbb{C}^2 \setminus \{(0,0)\}$, except the one that is satisfied by all functions from $\ker T_1$ (and then also φ). To justify this claim let us just study the case $c = d = 0$ (the only case which does not lead to a bijective realisation). We get

$$\alpha(b)\overline{\tilde{\varphi}(b)}u(b) = \alpha(a)\overline{\tilde{\varphi}(a)}u(a) \; ,$$

implying $[\, u \mid \tilde{\varphi} \,] = 0$, which concludes to $u \in \mathscr{W}_0 \dotplus \ker T_1$ using (3.6). Thus, the above boundary condition is satisfied by functions from $\ker T_1$. It is straightforward to justify the remaining part, that is the fact that all other boundary conditions are attained.

The approach using the universal classification theory has some additional advantages when studying, e.g. the spectrum and the resolvent of realisations. Moreover, once the classification is established, choosing the desired properties for realisations comes down to choosing the same properties for operator B, which is often easier to control.

4.2.4. The von Neumann approach

Using Theorem 3.2.10, all bijective realisations can be characterised in a more concise way. Indeed, all possible domains of the bijective realisations are given by

$$\mathscr{V} = \mathscr{W}_0 \dotplus \operatorname{span}\{\tilde{\varphi} + \lambda\varphi\}\,, \quad \lambda \in \mathbb{C}\,. \tag{4.14}$$

This can serve as another evidence to the discussion of Remark 4.2.3. In fact, von Neumann's approach leads to the classification of the boundary conditions of the type (4.14). In addition, we are able to distinguish the boundary conditions with signed boundary map. Let us elaborate on this in more details.

- If $\alpha(a)\alpha(b) = 0$, then $\dim\ker T_1 = \dim\ker\widetilde{T}_1 = 0$. In this case $\mathscr{W} = \mathscr{W}_0$, and we have only one realisation $\mathscr{V}\tilde{=}\mathscr{V} = \mathscr{W}_0$.

- If $\alpha(a)\alpha(b) < 0$, then either $(\mathscr{W}, \mathscr{W}_0)$ or $(\mathscr{W}_0, \mathscr{W})$ serve as $(\mathscr{V}, \widetilde{\mathscr{V}})$.

- If $\alpha(a)\alpha(b) > 0$, then $\dim\ker T_1 = \dim\ker\widetilde{T}_1 = 1$. Let $\ker T_1 = \operatorname{span}\{\varphi\}$ and $\ker\widetilde{T}_1 = \operatorname{span}\{\tilde{\varphi}\}$. We can parameterise all bijective extensions/realisations in this case using Theorem 3.4.7. Let $U : (\ker\widetilde{T}_1, [\cdot \mid \cdot]) \to (\ker T_1, -[\cdot \mid \cdot])$ be a contraction, i.e. $\|U\| \le 1$. The structure of U is completely given by the multiplication $U(\tilde{\varphi}) = C_U\varphi$, where $C_U \in \mathbb{C}$. The operator U is contractive if and only if

$$-|C_U|^2 \le \frac{[\tilde{\varphi} \mid \tilde{\varphi}]}{[\varphi \mid \varphi]} = \frac{\alpha(b)|\tilde{\varphi}(b)|^2 - \alpha(a)|\tilde{\varphi}(a)|^2}{\alpha(b)|\varphi(b)|^2 - \alpha(a)|\varphi(a)|^2}\,. \tag{4.15}$$

  Moreover, the boundary conditions with $\mathscr{V} = \widetilde{\mathscr{V}}$ are given by

$$\mathscr{V} = \mathscr{W}_0 \dotplus \left\{ C_U\varphi + \tilde{\varphi} : |C_U|^2 = -\frac{\alpha(b)|\tilde{\varphi}(b)|^2 - \alpha(a)|\tilde{\varphi}(a)|^2}{\alpha(b)|\varphi(b)|^2 - \alpha(b)|\varphi(a)|^2},\ C_U \in \mathbb{C} \right\}\,.$$

  All the boundary conditions with signed boundary map are described in (4.15). On the other hand, the bijective extensions without signed boundary map correspond to $\|U\| > 1$, which leads to

$$-|C_U|^2 > \frac{[\tilde{\varphi} \mid \tilde{\varphi}]}{[\varphi \mid \varphi]} = \frac{\alpha(b)|\tilde{\varphi}(b)|^2 - \alpha(a)|\tilde{\varphi}(a)|^2}{\alpha(b)|\varphi(b)|^2 - \alpha(a)|\varphi(a)|^2}\,. \tag{4.16}$$

The von Neumann extension theory is useful in the classification of boundary conditions with signed boundary map, which are important from the perspective of semigroup theory for abstract Friedrichs operators (see Chapter 5). In contrast to the general extension theory, this theory makes a bridge between the theories of (skew-)symmetric operators and abstract Friedrichs operators.

4.2.5. Summary

Depending on the values of α at end-points, the pairs of subspaces $(\mathscr{V},\widetilde{\mathscr{V}})$ for which we obtain bijective realisations, i.e. such that $(T_1|_{\mathscr{V}},\widetilde{T}_1|_{\widetilde{\mathscr{V}}})$ is a pair of mutually adjoint bijective realisations relative to $(T,\widetilde{T})$, where T and $\widetilde{T}$ are given by (4.1), are:

<table>
<tr><th>α at end-points</th><th>No. of bij. realisations</th><th colspan="2">$(\mathscr{V},\widetilde{\mathscr{V}})$</th></tr>
<tr><td rowspan="2">$\alpha(a)\alpha(b)\leq 0$</td><td rowspan="2">1</td><td>$\alpha(a)\geq 0\wedge\alpha(b)\leq 0$</td><td>$(\mathscr{W}_0,\mathscr{W})$</td></tr>
<tr><td>$\alpha(a)\leq 0\wedge\alpha(b)\geq 0$</td><td>$(\mathscr{W},\mathscr{W}_0)$</td></tr>
<tr><td>$\alpha(a)\alpha(b)>0$</td><td>∞</td><td colspan="2">(4.11) (see (4.9)–(4.10))</td></tr>
</table>

Thus, a classification of bijective realisations is needed only in the case when α has the same sign at both end-points.

4.2.6. Examples

Let us see some examples related to the cases discussed previously and investigate the information about kernels e.g dimensions and supports.

1. We take the interval $(0,2)$ and coefficients $\alpha(x)=1-x$ and $\beta=1$. Then, the corresponding pair of operators $(T,\widetilde{T})$ are given by

$$T\varphi=((1-x)\varphi)'+\varphi$$

   and

$$\widetilde{T}\varphi=-((1-x)\varphi)'\,.$$

   Both α and β are smooth functions and that $2\Re\beta+\alpha'=2-1=1>0$ on $(0,2)$, which meaning that $(T,\widetilde{T})$ is a pair of classical and thus abstract Friedrichs operators in the interval $(0,2)$. Here, $\alpha(0)=1$ and $\alpha(2)=-1$, giving $\alpha(0)\alpha(2)<0$, which means this example belongs to Case 2. Furthermore, $\alpha(0)>0,\alpha(2)<0$

means $(\mathscr{W}_0, \mathscr{W})$ gives the domains of only pair of mutually adjoint bijective realisations and that $\dim\ker T_1 = 2, \dim\ker \widetilde{T}_1 = 0$. Let us calculate the kernels. Using (4.13) on $(0,1)$ and $(1,2)$ separately we get

$$\varphi(x) = \frac{1}{\alpha(x)} \exp\Big(-\int \frac{\beta(x)}{\alpha(x)}\,dx\Big) = \frac{1}{1-x} \exp\Big(-\int \frac{dx}{1-x}\Big) = c\,.$$

Here, c is an arbitrary constant. Which implies, for $\varphi \in \ker T_1$ necessarily

$$\varphi = \begin{cases} c_1 & , \quad \text{in } (0,1) \\ c_2 & , \quad \text{in } (1,2)\,, \end{cases}$$

for some constants $c_1, c_2 \in \mathbb{C}$. We have $\varphi \in \mathscr{W}$. Indeed, it is evident that $\varphi \in L^2(0,2)$, while for $\psi \in C_c^\infty(0,2)$ we have

$$\begin{aligned} \int_0^2 (1-x)\varphi(x)\psi'(x)\,dx &= c_1 \int_0^1 (1-x)\psi'(x)\,dx + c_2 \int_1^2 (1-x)\psi'(x)\,dx \\ &= c_1 \int_0^1 \psi(x)\,dx + c_2 \int_1^2 \psi(x)\,dx \\ &= \int_0^2 \varphi(x)\psi(x)\,dx\,. \end{aligned}$$

This means $((1-x)\varphi)' = -\varphi \in L^2(0,2)$, thus $\varphi \in \mathscr{W}$. Therefore, $\dim\ker T_1 = 2$ (since we have two parameters in the definition of φ).

On the other hand, $\tilde{\varphi} \in \ker \widetilde{T}_1$ implies

$$\tilde{\varphi}(x) = \begin{cases} \frac{d_1}{1-x} & , \quad x \in (0,1) \\ \frac{d_2}{1-x} & , \quad x \in (1,2)\,, \end{cases}$$

for some constants $d_1, d_2 \in \mathbb{C}$. But, the integrals

$$\int_0^1 \frac{dx}{(1-x)^2} \quad \text{and} \quad \int_1^2 \frac{dx}{(1-x)^2}$$

are unbounded, implying $\tilde{\varphi} \in L^2(0,2)$ if and only if $d_1 = d_2 = 0$. Hence, $\ker \widetilde{T}_1 = \{0\}$ and $\dim\ker \widetilde{T}_1 = 0$, justifying the results obtained in Case 2.

It is interesting to note that for $c_1 \neq c_2$ we have $\varphi' \notin L^2(0,2)$, because $\varphi' = (c_2 - c_1)\delta_1$ (here δ_1 is the Dirac measure at 1) and so $\varphi \notin H^1(0,2)$. Thus, $H^1(0,2) \subsetneqq \mathscr{W}$.

Moreover, it is evident that $\tilde{\varphi} \in H^1_{\text{loc}}([0,2] \setminus \{1\})$ for any choice of parameters d_1, d_2. Indeed, for any subinterval $[c,d] \subseteq [0,2] \setminus \{1\}$ we have $\tilde{\varphi}|_{(c,d)} \in \mathrm{H}^1(c,d)$.

Since $\tilde{\varphi} \notin \mathscr{W}$, this shows that the graph space $\mathscr{W}$ is indeed a proper subspace of $H^1_{\mathrm{loc}}([0,2] \setminus \{1\})$, i.e. $\mathscr{W} \subsetneqq H^1_{\mathrm{loc}}([0,2] \setminus \{1\})$.

2. Take the same example as above, but now on the interval $(0,1)$. Here, $\alpha(0) = 1 > 0$ and $\alpha(1) = 0$. Using (4.13) again we get that $\varphi \in \ker T_1$ implies that $\varphi = c$, for some constant $c \in \mathbb{C}$. Since $\varphi \in H^1(0,1)$, it is contained in the graph space $\mathscr{W}$. Hence $\ker T_1 = \mathrm{span}\{1\}$ and $\dim \ker T_1 = 1$.

   Furthermore, for $\tilde{\varphi} \in \ker \widetilde{T}_1$ necessarily
   $$\tilde{\varphi}(x) = \frac{d}{(x-1)} , \quad x \in (0,1) ,$$
   for some constant $d \in \mathbb{C}$. But $\tilde{\varphi} \in L^2(0,1)$ if and only if $d = 0$. Hence, $\ker \widetilde{T}_1 = \{0\}$ and $\dim \ker \widetilde{T}_1 = 0$.

   This coincides with the results obtained in Case 1.

3. Let us consider another example that fits into the setting of Case 1. Take $\alpha(x) = x(x-1)$ and $\beta = 1$ on the interval $(0,1)$. Here $\alpha'(x) = 2x-1$, so we have $2\Re\beta + \alpha' = 2x+1 \geq 1 > 0$ in $(0,1)$. By (4.13), $\varphi \in \ker T_1$ and $\tilde{\varphi} \in \ker \widetilde{T}_1$ imply
   $$\varphi(x) = \frac{c}{(x-1)^2} , \qquad \tilde{\varphi}(x) = d\left(\frac{x-1}{x}\right) ,$$
   for some constants $c, d \in \mathbb{C}$. But $\varphi, \tilde{\varphi} \in L^2(0,1)$ if and only if $c = d = 0$. Hence, $\ker T_1 = \ker \widetilde{T}_1 = \{0\}$.

4. Take $\alpha(x) = (x-1)(x-2)$ and $\beta = 2$ on the interval $(0,3)$. Then α has two zeroes on the interval $(0,3)$. Here $\alpha'(x) = 2x-3$, hence we have $2\Re\beta + \alpha' \geq 1 > 0$ in $(0,3)$. Again using (4.13) on subintervals $(0,1)$, $(1,2)$ and $(2,3)$ separately we get that $\varphi \in \ker T_1$ implies
   $$\varphi(x) = \begin{cases} c_1 \left(\frac{x-1}{x-2}\right)^2 & , \quad x \in (0,1) \\ c_2 \left(\frac{x-1}{x-2}\right)^2 & , \quad x \in (1,2) \\ c_2 \left(\frac{x-1}{x-2}\right)^2 & , \quad x \in (2,3) , \end{cases}$$

for some constants $c_1, c_2, c_3 \in \mathbb{C}$. But $\varphi \in L^2(0,3)$ if and only if $c_2 = c_3 = 0$. Moreover, for $c_2 = c_3 = 0$ we have $\varphi \in \mathscr{W}$, implying $\dim\ker T_1 = 1$ and that φ has support in $(0,1)$.

On the other hand, $\tilde{\varphi} \in \ker \widetilde{T}_1$ implies

$$\tilde{\varphi}(x) = \begin{cases} d_1 \left(\frac{x-2}{x-1}\right)^2 & , \quad x \in (0,1) \\ d_2 \left(\frac{x-2}{x-1}\right)^2 & , \quad x \in (1,2) \\ d_3 \left(\frac{x-2}{x-1}\right)^2 & , \quad x \in (2,3) \, , \end{cases}$$

for some constants $d_1, d_2, d_3 \in \mathbb{C}$. But $\tilde{\varphi} \in L^2(0,3)$ if and only if $d_1 = d_2 = 0$, and for $d_1 = d_2 = 0$ we have $\tilde{\varphi} \in \mathscr{W}$. So, $\dim\ker \widetilde{T}_1 = 1$ and $\tilde{\varphi}$ has support in $(0,3)$, which is in accordance with Case 3.

4.3. 1-D VECTORIAL CASE

The previous section covers one dimensional ($d = 1$) scalar ($r = 1$) Friedrichs systems in full generality. Non-smoothness of functions from the graph space posts a challenge in defining boundary operator explicitly. Moreover, the coefficient matrix $\mathbf{A}(x)$ has eigenvectors of peculiar properties, which makes it difficult than the case of $(r = 1)$ (see Remark 4.3.2 below). Let us see this situation more explicitly.

For the domain we take an open interval $\Omega = (a,b)$, $a < b$. Then $\mathscr{D} = C_c^\infty((a,b),\mathbb{C}^r)$ and $\mathscr{H} = L^2((a,b),\mathbb{C}^r)$. We adjust the notation of $T,\widetilde{T} : \mathscr{D} \to \mathscr{H}$ given in Subsection 2.6.1 in the following way:

$$T\mathsf{u} := (\mathbf{A}\mathsf{u})' + \mathbf{B}\mathsf{u} \qquad \text{and} \qquad \widetilde{T}\mathsf{u} := -(\mathbf{A}\mathsf{u})' + (\mathbf{B}^* + \mathbf{A}')\mathsf{u}\,, \tag{4.17}$$

where $\mathbf{A} = \mathbf{A}^* \in W^{1,\infty}((a,b);\mathrm{M}_r(\mathbb{C}))$, $\mathbf{B} \in L^\infty((a,b);\mathrm{M}_r(\mathbb{C}))$ and for some $\mu_0 > 0$ we have $\mathbf{B}^* + \mathbf{B} + \mathbf{A}' \geq 2\mu_0\mathbb{1} > 0$ ($\mathbb{1}$ is the identity matrix and $'$ the derivative). It is commented in Example 2.6.1 that $(T,\widetilde{T})$ is a joint pair of abstract Friedrichs operators. Let us recall that in the one-dimensional case ($d = 1$) for $\Omega = (a,b)$, $a < b$, the graph space simplifies to

$$\mathscr{W} \;=\; \left\{\mathsf{u} \in \mathscr{H} : (\mathbf{A}\mathsf{u})' \in \mathscr{H}\right\}, \tag{4.18}$$

while the graph norm is (equivalent to)

$$\|\cdot\|_{T_1} \;=\; \|\cdot\| + \|(\mathbf{A}\,\cdot)'\|\,. \tag{4.19}$$

The boundary operator D is given by,

$$[\,\mathsf{u} \mid \mathsf{v}\,] = \big(\mathbf{A}\mathsf{u}\cdot\mathsf{v}\big)(b) - \big(\mathbf{A}\mathsf{u}\cdot\mathsf{v}\big)(a)\,, \quad \mathsf{u},\mathsf{v} \in C_c^\infty(\mathbb{R};\mathbb{C}^r)\,, \tag{4.20}$$

and the minimal domain is described as

$$\mathscr{W}_0 \;=\; \{\mathsf{u} \in \mathscr{W} : (\mathbf{A}\mathsf{u})(a) = (\mathbf{A}\mathsf{u})(b) = 0\}\,, \tag{4.21}$$

(here $\|\cdot\|$ stands, as usual, for the norm on $\mathscr{H}$ induced by the standard inner product, i.e. the L^2 norm on (a,b)). In fact, $\mathsf{u} \in \mathscr{H}$ belongs to $\mathscr{W}$ if and only if $\mathbf{A}\mathsf{u} \in H^1((a,b);\mathbb{C}^r)$. Thus, by the standard Sobolev embedding theorem (see e.g. [16, Theorem 8.2]) for any $\mathsf{u} \in \mathscr{W}$ we have $\mathbf{A}\mathsf{u} \in C([a,b];\mathbb{C}^r)$. This in particular implies that for any $\mathsf{u} \in \mathscr{W}$ and

$x \in [a,b]$ evaluation $(\mathbf{A}\mathsf{u})(x)$ is well defined. Here, $\mathbf{A}(x)\mathsf{u}(x)$ is not necessarily meaningful as u itself is not necessarily continuous. We dealt with this situation in scalar case by developing a smoothness result in Lemma 4.1.1. Natural generalisation of this result requires a different strategy. In this section we proceed in the direction of total projections to get a generalised smoothness result. Further, we can connect dimensions of the kernels with the ranks of coefficient matrices at end-points of the interval.

We divide this section into several parts. First we recall some properties related to the eigenvalues and some results on codimensions. Then, we prove a smoothness result involving total projections. Next we discuss about explicit forms of the boundary operator and the minimal space and in the last part we obtain the result of the dimensions of the kernels.

4.3.1. Preliminaries

In our setting, the eigenvalues and eigenprojections of a Lipschitz continuous matrix function $\mathbf{A}(x)$ in the definition of Friedrichs operators (e.g. see Example 2.6.1) are required to be Lipschitz continuous functions in some neighborhood of the end-points a and b of the interval $[a,b]$. The following theorem gives us the smoothness of the eigenvalues.

Theorem 4.3.1 (Hoffman-Wielandt Inequality [13])**.** Let $\mathbf{A}$ and $\mathbf{B}$ be Herimitian matrices of order n and $\lambda_1(\mathbf{A}) \geq ... \geq \lambda_n(\mathbf{A})$ and $\lambda_1(\mathbf{B}) \geq ... \geq \lambda_n(\mathbf{B})$ be their eigenvalues, respectively. Then we have

$$\sum_{i=1}^{n} |\lambda_i(\mathbf{A}) - \lambda_i(\mathbf{B})|^p \leq |\mathbf{A}-\mathbf{B}|_p^p \,.$$

Where, $p \geq 1$ and $|\cdot|_p$ is $p-$norm.

Using Theorem 4.3.1 we get that all eigenvalues of $\mathbf{A}(x), x \in [a,b]$, are Lipschitz continuous functions of $\mathbf{A}(x)$ and $\mathbf{A}$ is Lipschitz continuous in the interval $[a,b]$. Thus, all the eigenvalues are Lipschitz continuous on $[a,b]$. A similar result on eigenvectors is not expected. Here we present an argument on eigenprojections which resolves our specific situation.

We adapt our definitions and notations from [48].

(i) For the definition and details of $\lambda-$*group*, we refer to [48, Chapter II, Section 1.2]. Let $\lambda = \lambda(x_0)$ be an eigenvalue of $\mathbf{A}(x_0)$. In general there are several cycles with the same center λ and all the eigenvalues of $\mathbf{A}(x)$ belonging to the cycles with the center λ are said to depart from the unperturbed eigenvalue λ by *splitting* at $x = x_0$. The set of these eigenvalues is called the $\lambda-$*group*, since they cluster around λ for small $|x-x_0|$.

Let $\lambda_1(x) \geq \lambda_2(x) \geq, \ldots, \geq \lambda_r(x)$ be the eigenvalues of $\mathbf{A}(x)$ (which are Lipschitz continuous). Note that we repeat eigenvalues with higher multiplicity (so-called repeated eigenvalues). Also, let $I(x)$ denotes the set of indices such that for any x (close to x_0) $\{\lambda_i(x) : i \in I(x)\}$ is the set of all eigenvalues of $\mathbf{A}(x)$ forming the $\lambda-$group, where we are not repeating eigenvalues of higher multiplicity. Thus, $I(x)$ really depends on x.

The *total projection* corresponding to the $\lambda-$group is given by

$$\mathbf{P}_\lambda(x) \ = \ \sum_{i \in I(x)} \mathbf{P}_i(x)\,, \tag{4.22}$$

where $\mathbf{P}_i(x)$ denotes the eigenprojection corresponding to $\lambda_i(x)$. For any $x \in [a,b]$ it holds $\mathbf{P}_i^2(x) = \mathbf{P}_i(x) = \mathbf{P}_i^*(x)$, while since $\mathbf{A}(x)$ is Hermitian we have in addition

$$\mathbf{P}_i(x)\mathbf{P}_j(x) = \delta_{i,j}\mathbf{P}_j(x),\ i,j \in \{1,2,\ldots,r\}$$

and

$$\sum_{\lambda \in \sigma(\mathbf{A}(x))} \mathbf{P}_\lambda(x) = \mathbb{1}\,.$$

Let us consider the following example to elaborate on the definitions that we just introduced. Consider the following function on $[0,1]$

$$f(x) \ = \ \begin{cases} x^2 \sin(1/x) & x > 0\,, \\ 0 & x = 0\,. \end{cases}$$

Let $\mathbf{A}(x) = \begin{bmatrix} x & 0 \\ 0 & f(x) \end{bmatrix}$, which is a continuous function on $[0,1]$. The eigenvalues for $x \neq 0$ are $\lambda_1(x) = x$, $\lambda_2(x) = x^2\sin(1/x)$. At $x = 0$, we have $\lambda_1(0) = \lambda_2(0) = 0$

(the multiplicity is 2), while for $\varepsilon > 0$ (sufficiently small) and any $x \in (0,\varepsilon)$ we have

$$I(x) \;=\; \begin{cases} \{1\}, & (\exists k \in \mathbb{N})\; x = \frac{2}{k\pi}\,, \\ \{1,2\}, & \text{otherwise}\,. \end{cases}$$

(ii) The resolvent definition of the *total projection* can be found in [48, Chapter II, Section 1.4]. Let $x_0 \in [a,b]$ and λ be an eigenvalue of $\mathbf{A} = \mathbf{A}(x_0)$ with multiplicity m. Let Γ be a positively oriented curve, say a circle, in the resolvent set $\rho(\mathbf{A})$ of $\mathbf{A}$ enclosing λ but no other eigenvalues of $\mathbf{A}$. The second Neumann series is then convergent for x sufficiently close to x_0, uniformly for $\xi \in \Gamma$. The existence of the resolvent $\mathbf{R}(\xi,x) := (\mathbf{A}(x) - \mathbb{1}\xi)^{-1}$ of $\mathbf{A}(x)$ for $\xi \in \Gamma$ implies that there are no eigenvalues of $\mathbf{A}(x)$ on Γ. The operator

$$\mathbf{P}_\lambda(x) \;=\; -\frac{1}{2\pi i}\int_\Gamma \mathbf{R}(\xi,x)d\xi\,, \tag{4.23}$$

is a projection and is equal to the sum of the eigenprojections of all the eigenvalues of $\mathbf{A}(x)$ lying inside Γ. The eigenvalues of $\mathbf{A}(x)$ lying inside Γ form exactly the $\lambda-$group and $\mathbf{P}(x)$ defined in (4.23) is called the *total projection* corresponding to the $\lambda-$group. Here $\mathbf{P}(x_0)$ is precisely the eigenprojection corresponding to the eigenvalue λ of $\mathbf{A}$ and the following holds

$$\operatorname{rank}\mathbf{P}_\lambda(x) = \operatorname{rank}\mathbf{P}_\lambda(x_0) = m\,,$$

for x sufficiently close to x_0.

Let x,y be sufficiently close to x_0 as previously. More precisely, let $\varepsilon > 0$ be small and $x,y \in (x_0-\varepsilon, x_0+\varepsilon)$. For a fixed $\xi \in \Gamma$, using the resolvent identity

$$\mathbf{R}(\xi,y) - \mathbf{R}(\xi,x) = \mathbf{R}(\xi,y)(\mathbf{A}(x) - \mathbf{A}(y))\mathbf{R}(\xi,x) \tag{4.24}$$

we get

$$\begin{aligned} |\mathbf{R}(\xi,y) - \mathbf{R}(\xi,x)\| &\leq |\mathbf{R}(\xi,y)||\mathbf{R}(\xi,x)||\mathbf{A}(y) - \mathbf{A}(x)| \\ &\leq \Big(\max_{\xi\in\Gamma, z\in(x_0-\varepsilon,x_0+\varepsilon)} |\mathbf{R}(\xi,z)|^2\Big)|\mathbf{A}(x) - \mathbf{A}(y)|\,, \end{aligned} \tag{4.25}$$

where, $(\xi,x) \mapsto \mathbf{R}(\xi,x)$ is continuous [48, Sec. II.5.1]. Hence, from (4.25), we conclude that $\mathbf{R}(\xi,x)$ is Lipschitz continuous in x variable on $(x_0-\varepsilon,x_0+\varepsilon)$. Therefore, from the resolvent definition of total projection (4.23), we get that $\mathbf{P}_\lambda(x)$ is Lipschitz continuous on $(x_0-\varepsilon, x_0+\varepsilon)$.

As mentioned in [48, Chapter II, Section 5.3], in general, the eigenprojections do not have these continuity results, thus total projections can not be replaced by eigenprojections. Let us consider the following example to elaborate on this.

$$\mathbf{A}(x) = e^{-1/x^2}\begin{bmatrix}\cos(2/x) & \sin(2/x)\\ \sin(2/x) & -\cos(2/x)\end{bmatrix}, \quad \mathbf{A}(0)=0\,.$$

The matrix $\mathbf{A}(x)$ is infinitely differentiable on $\mathbb{R}$. It has eigenvalues

$$\lambda(x) \;=\; \begin{cases} \pm e^{-1/x^2} & x\neq 0\,,\\ 0 & x=0\,,\end{cases}$$

which are also infinitely differentiable. The corresponding eigenprojection for $x\neq 0$ are

$$\begin{bmatrix}\cos^2(1/x) & \cos(1/x)\sin(1/x)\\ \cos(1/x)\sin(1/x) & \sin^2(1/x)\end{bmatrix}, \begin{bmatrix}\sin^2(1/x) & -\cos(1/x)\sin(1/x)\\ -\cos(1/x)\sin(1/x) & \cos^2(1/x)\end{bmatrix}.$$

These matrix functions are infinitely differentiable on any interval which does not contain $x=0$, but they can not be continued to $x=0$ as continuous functions. Furthermore, the eigenvectors are

$$\begin{bmatrix}\cos(1/x)\\ \sin(1/x)\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}\sin(1/x)\\ -\cos(1/x)\end{bmatrix}, \qquad (x\neq 0)$$

and at $x=0$, the eigenvectors are

$$\begin{bmatrix}1\\0\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}0\\1\end{bmatrix}.$$

Hence, there does not exist any eigenvector of $\mathbf{A}(x)$ that is continuous in the neighborhood of $x=0$.

Remark 4.3.2. In our case, the matrix $\mathbf{A}$ is Lipschitz continuous Hermitian matrix. One can think of diagonalising the matrix $\mathbf{A}=\mathbf{QDQ}^{-1}$ and transform the given Friedrichs system accordingly, so that we work with the diagonal matrix $\mathbf{D}$ instead of $\mathbf{A}$. However, the matrix $\mathbf{Q}$ consists of eigenvectors of matrix $\mathbf{A}$, and, as evident in the previous example, the eigenvectors may not be continuous (even measurability is not expected in some cases). In conclusion, the transformed system, using the diagonalisation of $\mathbf{A}$, can have some peculiar properties, very different from the original Friedrichs system. The role of total projections is vital in dealing with the smoothness results (see Lemma 4.3.5 below), which can not be obtained using eigenprojections only.

4.3.2. Codimension

In order to obtain the result of Lemma 4.3.8 below, we make use of the following elementary identities.

Lemma 4.3.3. [15, Sec. II.7, Proposition 6] If M and N are two subspaces of a vector space V of finite codimensions, then both $M+N$ and $M\cap N$ have finite codimensions and

$$\operatorname{codim}(M+N)+\operatorname{codim}(M\cap N)=\operatorname{codim}(M)+\operatorname{codim}(N)\ . \tag{4.26}$$

Lemma 4.3.3 can be generalised as follows:

Lemma 4.3.4. If $\{M_k\}_{k=1,2,\ldots,n}$ are subspaces of a vector space V of finite codimensions, then

$$\operatorname{codim}\bigcap_{k=1}^{n} M_k=\sum_{k=1}^{n}\operatorname{codim}M_k+\sum_{j=2}^{n}(-1)^{n+j-1}\operatorname{codim}\Big(\bigcap_{k=1}^{j-1} M_k+M_j\Big)\ . \tag{4.27}$$

Proof. For $n=2$ the result follows from Lemma 4.3.3. Suppose for some $n\geq 2$ and $n\in\mathbb{N}$ that (4.27) is true. Let us define $M:=\bigcap_{k=1}^{n} M_k$. By Lemma 4.3.3, we have

$$\operatorname{codim}(M\cap M_{n+1})=\operatorname{codim}(M)+\operatorname{codim}(M_{n+1})-\operatorname{codim}(M+M_{n+1})\ . \tag{4.28}$$

Using the hypothesis, we get

$$\begin{aligned}\operatorname{codim}(M\cap M_{n+1})&=\sum_{k=1}^{n}\operatorname{codim}M_k+\sum_{j=2}^{n}(-1)^{n+j-1}\operatorname{codim}\Big(\bigcap_{k=1}^{j-1} M_k+M_j\Big)\\&\quad+\operatorname{codim}(M_{n+1})-\operatorname{codim}(M+M_{n+1})\,,\\&=\sum_{k=1}^{n+1}\operatorname{codim}M_k+\sum_{j=2}^{n+1}(-1)^{n+j}\operatorname{codim}\Big(\bigcap_{k=1}^{j-1} M_k+M_j\Big)\ .\end{aligned}$$

The result follows by the induction. ∎

4.3.3. Total projections.

Let us return to the study of (4.17). Let $\mathbf{P}_\lambda(x_0)$ denotes the total projection corresponding to the eigenvalue $\lambda = \lambda(x_0)$ of $\mathbf{A}(x_0)$. As mentioned in Subsection 4.3.1, for any $x_0 \in [a,b]$ there exists $\varepsilon =: \varepsilon(x_0) > 0$, such that the operator $\mathbf{P}_\lambda(x_0)$ is Lipschitz continuous in the interval $[a,b] \cap [x_0 - \varepsilon, x_0 + \varepsilon] =: I_{\lambda,x_0}$. Which means $\mathbf{P}_\lambda \in W^{1,\infty}(I_{\lambda,x_0}; \mathbb{C}^{r\times r})$.

Also, eigenvalues $\lambda(x)$ of the matrix $\mathbf{A}(x)$ are in $W^{1,\infty}((a,b),\mathbb{R})$.

Lemma 4.3.5. Let $\lambda(x)$ be an eigenvalue of $\mathbf{A}(x)$ and $x_0 \in [a,b]$, such that $\lambda := \lambda(x_0) \neq 0$. Then there exists $\varepsilon > 0$ such that for any $\mathsf{u} \in \mathscr{W}$,

$$(\mathbf{P}_\lambda \mathsf{u})|_{I_{\lambda,x_0}} \in H^1(I_{\lambda,x_0}; \mathbb{C}^r)\,,$$

where $I_{\lambda,x_0} = [a,b] \cap [x_0 - \varepsilon, x_0 + \varepsilon]$.

Proof. Let $\varepsilon > 0$ such that $\mathbf{P}_\lambda \in W^{1,\infty}(I_{\lambda,x_0}; \mathbb{C}^{r\times r})$ and for any $x \in I_{\lambda,x_0}$, eigenvalues of $\mathbf{A}(x)$ forming the $\lambda-$group (see Appendix) are $\lambda/2$ close to λ (for simplicity, we assumed $\lambda > 0$). Hence, for any $x \in I_{\lambda,x_0}$ and $i \in I(x)$, it holds

$$|\lambda_i(x) - \lambda| \leq \lambda/2 \iff \lambda_i(x) \in [\lambda/2, 3\lambda/2]\,,$$

where by $I(x)$, $x \in I_{\lambda,x_0}$, we denote the index set of eigenvalues forming the $\lambda-$group. Let $x \in I_{\lambda,x_0}$. For an arbitrary $\nu \in \mathbb{C}^r$, we have

$$\begin{aligned}
|\mathbf{P}_\lambda(x)\nu|^2 &= (\mathbf{P}_\lambda(x)\nu)\cdot(\mathbf{P}_\lambda(x)\nu) \\
&= \Big(\sum_{i\in I(x)} \mathbf{P}_i(x)\nu\Big)\cdot\Big(\sum_{j\in I(x)} \mathbf{P}_j(x)\nu\Big) \\
&= \sum_{i,j\in I(x)} \big(\mathbf{P}_j^*(x)\mathbf{P}_i(x)\nu\big)\cdot\nu\,.
\end{aligned}$$

Here, $\mathbf{P}_j(x)$ is the eigenprojection corresponding to the eigenvalue $\lambda_j(x)$. Applying, $\mathbf{P}_j^*(x) = \mathbf{P}_j(x)$ and $\mathbf{P}_i(x)\mathbf{P}_j(x) = \delta_{i,j}\mathbf{P}_i(x)$ (the latter holds since $\mathbf{A}(x)$ is Hermitian), we get

$$|\mathbf{P}_\lambda(x)\nu|^2 = \sum_{i\in I(x)} |\mathbf{P}_i(x)\nu|^2\,. \tag{4.29}$$

Since, $\lambda/2 \leq \lambda_i(x)$, and $\mathbf{P}_i(x)$ and $\mathbf{A}(x)$ commute (note that $\mathbf{A}(x)$ is Hermitian), $i \in I(x)$, we have

$$|\mathbf{P}_i(x)\nu|^2 \leq \frac{4}{\lambda^2}|\lambda_i(x)\mathbf{P}_i(x)\nu|^2 = \frac{4}{\lambda^2}|\mathbf{A}(x)\mathbf{P}_i(x)\nu|^2 = \frac{4}{\lambda^2}|\mathbf{P}_i(x)\mathbf{A}(x)\nu|^2\,.$$

Replacing ν by $\mathbf{A}(x)\nu$ in the previous calculation, we obtain

$$\sum_{i\in I(x)} |\mathbf{P}_i(x)\mathbf{A}(x)\nu|^2 = |\mathbf{P}_\lambda(x)\mathbf{A}(x)\nu|^2 \leq |\mathbf{A}(x)\mathsf{u}(x)|^2\,,$$

where we used that $\mathbf{P}_\lambda(x)$ is an orthogonal projection. Thus, for any $x \in I_{\lambda,x_0}$ we have

$$|\mathbf{P}_\lambda(x)\nu|^2 \leq \frac{4}{\lambda^2}|\mathbf{A}(x)\nu|^2\,.$$

Now, let us take $\mathsf{u} \in C_c^\infty(\mathbb{R};\mathbb{C}^r)$. Then, the above inequality holds for $\mathsf{u}(x)$ as well as $\mathsf{u}'(x)$ in place of ν. Integrating over I_{λ,x_0} and taking the square root, we get

$$\|\mathbf{P}_\lambda \mathsf{u}'\|_{L^2(I_{\lambda,x_0})} \leq \frac{2}{\lambda}\|\mathbf{A}\mathsf{u}'\|_{L^2(I_{\lambda,x_0})} \leq \frac{2(1+\|A'\|_{L^\infty(I_{\lambda,x_0})})}{\lambda}\|\mathsf{u}\|_{\mathscr{W}}\,.$$

Hence,

$$\|(\mathbf{P}_\lambda \mathsf{u})'\|_{L^2(I_{\lambda,x_0})} \leq C\|\mathsf{u}\|_{\mathscr{W}}\,.$$

Where, $C = 2\max\{\|\mathbf{P}_\lambda'\|_{L^\infty(I_{\lambda,x_0})}, 2\lambda^{-1}(1+\|A'\|_{L^\infty(I_{\lambda,x_0})})\}$. Since, $C_c^\infty(\mathbb{R};\mathbb{C}^r)$ is dense in $\mathscr{W}$, we conclude that for any $\mathsf{u} \in \mathscr{W}$, we have

$$(\mathbf{P}_\lambda \mathsf{u})|_{I_{\lambda,x_0}} \in H^1(I_{\lambda,x_0};\mathbb{C}^r)\,.$$

■

Remark 4.3.6. We have the following immediate consequences of the previous result:

(i) Using Sobolev embedding, we get $(\mathbf{P}_\lambda \mathsf{u})|_{I_{\lambda,x_0}} \in C(I_{\lambda,x_0};\mathbb{C}^r)$ and so pointwise evaluation is well-defined. Of course, if for $x \in I_{\lambda,x_0}$ the evaluation $\mathsf{u}(x)$ is well-defined, then

$$(\mathbf{P}_\lambda \mathsf{u})(x) = \mathbf{P}_\lambda(x)\mathsf{u}(x)\,,$$

where we used that $\mathbf{P}_\lambda$ is continuous.

(ii) For $\lambda(x_0) = 0$ it is natural to take $(\mathbf{P}_\lambda \mathsf{u})(x_0) = 0$.

With the result of Lemma 4.3.5, we can write the boundary map and the space $\mathscr{W}_0$ explicitly.

Lemma 4.3.7. Let, $\sigma(\mathbf{A}(x))$ denotes the spectrum of matrix $\mathbf{A}(x)$.

1. For any $\mathsf{u},\mathsf{v}\in\mathscr{W}$, the boundary operator can be characterised as

$$[\,\mathsf{u}\mid\mathsf{v}\,]=\sum_{\substack{\lambda\in\sigma(\mathbf{A}(b))\\ \lambda\neq0}}\lambda(\mathbf{P}_\lambda\mathsf{u})(b)\cdot(\mathbf{P}_\lambda\mathsf{v})(b)-\sum_{\substack{\lambda\in\sigma(\mathbf{A}(a))\\ \lambda\neq0}}\lambda(\mathbf{P}_\lambda\mathsf{u})(a)\cdot(\mathbf{P}_\lambda\mathsf{v})(a)\,.$$

2. The minimal domain $\mathscr{W}_0$ is characterised as

$$\mathscr{W}_0=\big\{\mathsf{u}\in\mathscr{W}:(\forall x\in\{a,b\})\;(\forall\lambda\in\sigma(\mathbf{A}(x))\setminus\{0\})\;(\mathbf{P}_\lambda\mathsf{u})(x)=0\big\}\,.$$

Proof. (i) Let $\mathsf{u},\mathsf{v}\in C_c^\infty(\mathbb{R},\mathbb{C}^r)$, then (see (4.20))

$$\begin{aligned}[\,\mathsf{u}\mid\mathsf{v}\,]&=\mathbf{A}(b)\mathsf{u}(b)\cdot\mathsf{v}(b)-\mathbf{A}(a)\mathsf{u}(a)\cdot\mathsf{v}(a)\\&=\sum_{\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}}\lambda(\mathbf{P}_\lambda\mathsf{u})(b)\cdot\mathsf{v}(b)-\sum_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}}\lambda(\mathbf{P}_\lambda\mathsf{u})(a)\cdot\mathsf{v}(a)\,.\end{aligned}$$

Using $\mathbf{P}_\lambda(x)=\mathbf{P}_\lambda(x)^2$ and $\mathbf{P}_\lambda^*(x)=\mathbf{P}_\lambda(x)$ (see also Remark 4.3.6(i)), we get

$$[\,\mathsf{u}\mid\mathsf{v}\,]=\sum_{\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}}\lambda(\mathbf{P}_\lambda\mathsf{u})(b)\cdot(\mathbf{P}_\lambda\mathsf{v})(b)-\sum_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}}\lambda(\mathbf{P}_\lambda\mathsf{u})(a)\cdot(\mathbf{P}_\lambda\mathsf{v})(a)\,.$$

Now, let $\mathsf{u},\mathsf{v}\in\mathscr{W}$, then there exist sequences $\mathsf{u}_n,\mathsf{v}_n\in C_c^\infty(\mathbb{R},\mathbb{C}^r)$ such that $\mathsf{u}_n,\mathsf{v}_n$ converge in $\mathscr{W}$ to u,v, respectively. By Remark 4.1.3, $(\mathbf{P}_\lambda\mathsf{u}_n)(a)\to(\mathbf{P}_\lambda\mathsf{u})(a)$ and $(\mathbf{P}_\lambda\mathsf{u}_n)(b)\to(\mathbf{P}_\lambda\mathsf{u})(b)$. Hence, for any $\mathsf{u},\mathsf{v}\in\mathscr{W}$,

$$[\,\mathsf{u}\mid\mathsf{v}\,]=\sum_{\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}}\lambda(\mathbf{P}_\lambda\mathsf{u})(b)\cdot(\mathbf{P}_\lambda\mathsf{v})(b)-\sum_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}}\lambda(\mathbf{P}_\lambda\mathsf{u})(a)\cdot(\mathbf{P}_\lambda\mathsf{v})(a)\,.$$

(ii) The form of $\mathscr{W}_0$ is described in (4.21). Let $\mathsf{u}\in C_c^\infty((a,b);\mathbb{C}^r)$, then

$$\mathbf{A}(a)\mathsf{u}(a)=\sum_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}}\lambda\mathbf{P}_\lambda(a)\mathsf{u}(a)=0\,.$$

Since $\mathbf{A}(a)$ is Hermitian, the eigenprojections are orthogonal to each other. Which implies that $\{\mathbf{P}_\lambda(a)\mathsf{u}(a):\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}\}$ is an orthogonal set. Thus, we have

$$\mathbf{A}(a)\mathsf{u}(a)=0\iff(\forall\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\})\quad(\mathbf{P}_\lambda\mathsf{u})(a)=\mathbf{P}_\lambda(a)\mathsf{u}(a)=0\,.$$

Now, let $\mathsf{u}\in\mathscr{W}_0$ and take the sequence $\mathsf{u}_n\in C_c^\infty((a,b),\mathbb{C}^r)$ such that u_n converges to u in $\mathscr{W}$. By Remark 4.3.6, $(\mathbf{P}_\lambda\mathsf{u}_n)(a)\to(\mathbf{P}_\lambda\mathsf{u})(a)$. Hence, we have

$$(\forall\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\})\quad(\mathbf{P}_\lambda\mathsf{u})(a)=0\,.$$

Similarly,

$$(\forall \lambda \in \sigma(\mathbf{A}(b)) \setminus \{0\}) \quad (\mathbf{P}_\lambda \mathsf{u})(b) = 0\,,$$

which completes the first inclusion. For the converse, assume that for some $\mathsf{u} \in \mathscr{W}$,

$$(\forall x \in \{a,b\}, \forall \lambda \in \sigma(\mathbf{A}(x)) \setminus \{0\}) \quad (\mathbf{P}_\lambda \mathsf{u})(x) = 0\,.$$

Here, the evaluations are well defined by Remark 4.3.6(i). Due to the density of $C_c^\infty(\mathbb{R},\mathbb{C}^r)$ in $\mathscr{W}$ ([1, Theorem 4]), there exist $\mathsf{u}_n \in C_c^\infty(\mathbb{R},\mathbb{C}^r)$ such that $\mathsf{u}_n \to \mathsf{u}$ in $\mathscr{W}$. For $x \in \{a,b\}$ we have

$$(\mathbf{A}\mathsf{u}_n)(x) = \mathbf{A}(x)\mathsf{u}_n(x) = \sum_{\lambda \in \sigma(\mathbf{A}(x))\setminus\{0\}} \lambda \mathbf{P}_\lambda(x)\mathsf{u}_n(x) = \sum_{\lambda \in \sigma(\mathbf{A}(x))\setminus\{0\}} \lambda (\mathbf{P}_\lambda \mathsf{u}_n)(x)\,.$$

By the Sobolev embedding theorem (see Lemma 4.3.5 and Remark 4.3.6(i)), by letting $n \to \infty$ we get

$$(\mathbf{A}\mathsf{u})(x) = \sum_{\lambda \in \sigma(\mathbf{A}(x))\setminus\{0\}} \lambda (\mathbf{P}_\lambda \mathsf{u})(x) = 0\,.$$

Hence,

$$(\mathbf{A}\mathsf{u})(a) = (\mathbf{A}\mathsf{u})(b) = 0\,,$$

implying $\mathsf{u} \in \mathscr{W}_0$ by (4.21).

■

4.3.4. Dimension of kernels

In this section we shall see how to construct a pair of subspaces $(\mathscr{V}, \widetilde{\mathscr{V}})$ of $\mathscr{W}$, i.e. how to impose suitable boundary conditions, in order to get bijective realisations of (4.17). The strategy is to reduce the problem to the one-dimensional setting and use the approach of Section 4.2. Of course, in the diagonal case it is easy to proceed with this plan (Subection 4.3.5 below), while in the general case we shall make use of the results on total projections just developed (as the system obtained after diagonalisation can have some peculiar behavior).

Depending on the sign of the eigenvalues at the end-points, we can construct a pair of subspaces $(\mathscr{V}, \widetilde{\mathscr{V}})$ of $\mathscr{W}$, forming a pair of mutually adjoint bijective realisations of (4.1).

We first define the subspaces $\{\mathscr{V}_{\lambda,a}, \widetilde{\mathscr{V}}_{\lambda,a}\}_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}}$ and $\{\mathscr{V}_{\lambda,b}, \widetilde{\mathscr{V}}_{\lambda,b}\}_{\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}}$ of $\mathscr{W}$ as follows:

For $\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}$

Sign of λ	$\mathscr{V}_{\lambda,a}$	$\widetilde{\mathscr{V}}_{\lambda,a}$
$\lambda=0$	$\mathscr{W}$	$\mathscr{W}$
$\lambda>0$	$\{\mathsf{u}\in\mathscr{W} : (\mathbf{P}_\lambda\mathsf{u})(a)=0\}$	$\mathscr{W}$
$\lambda<0$	$\mathscr{W}$	$\{\mathsf{u}\in\mathscr{W} : (\mathbf{P}_\lambda\mathsf{u})(a)=0\}$

and for $\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}$

Sign of λ	$\mathscr{V}_{\lambda,b}$	$\widetilde{\mathscr{V}}_{\lambda,b}$
$\lambda=0$	$\mathscr{W}$	$\mathscr{W}$
$\lambda>0$	$\mathscr{W}$	$\{\mathsf{u}\in\mathscr{W} : (\mathbf{P}_\lambda\mathsf{u})(b)=0\}$
$\lambda<0$	$\{\mathsf{u}\in\mathscr{W} : (\mathbf{P}_\lambda\mathsf{u})(b)=0\}$	$\mathscr{W}$

Now we shall see that the subspaces

$$\mathscr{V}=\mathscr{V}_a\cap\mathscr{V}_b \quad\text{and}\quad \widetilde{\mathscr{V}}=\widetilde{\mathscr{V}}_a\cap\widetilde{\mathscr{V}}_b\,, \tag{4.30}$$

satisfy the condition (V), where

$$\begin{aligned}\mathscr{V}_a&:=\bigcap_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}}\mathscr{V}_{\lambda,a}\,, &\qquad \mathscr{V}_b&:=\bigcap_{\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}}\mathscr{V}_{\lambda,b}\,,\\ \widetilde{\mathscr{V}}_a&:=\bigcap_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}}\widetilde{\mathscr{V}}_{\lambda,a}\,, &\qquad \widetilde{\mathscr{V}}_b&:=\bigcap_{\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}}\widetilde{\mathscr{V}}_{\lambda,b}\,.\end{aligned} \tag{4.31}$$

Lemma 4.3.8. The pair of subspaces $(\mathscr{V},\widetilde{\mathscr{V}})$ of $\mathscr{W}$, defined as above, satisfies (V)-conditions.

Proof. By construction (4.30),

$$(\forall\mathsf{u}\in\mathscr{V})\qquad \sum_{\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}}\lambda|(\mathbf{P}_\lambda\mathsf{u})(b)|^2-\sum_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}}\lambda|(\mathbf{P}_\lambda\mathsf{u})(a)|^2\geq 0\,.$$

Similarly,

$$(\forall\mathsf{u}\in\widetilde{\mathscr{V}})\qquad \sum_{\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}}\lambda|(\mathbf{P}_\lambda\mathsf{u})(b)|^2-\sum_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}}\lambda|(\mathbf{P}_\lambda\mathsf{u})(a)|^2\leq 0\,.$$

Hence, by Lemma 4.3.7(i), (V1) condition is satisfied.

Let $\mathsf{v} \in \widetilde{\mathscr{V}}$, then for any $\mathsf{u} \in \mathscr{V}$, we have

$$[\,\mathsf{u} \mid \mathsf{v}\,] = \sum_{\lambda \in \sigma(\mathbf{A}(b))\setminus\{0\}} \lambda(\mathbf{P}_\lambda \mathsf{u})(b) \cdot (\mathbf{P}_\lambda \mathsf{v})(b) - \sum_{\lambda \in \sigma(\mathbf{A}(a))\setminus\{0\}} \lambda(\mathbf{P}_\lambda \mathsf{u})(a) \cdot (\mathbf{P}_\lambda \mathsf{v})(a)\,.$$

For $\lambda \in \sigma(\mathbf{A}(a)) \setminus \{0\}$,

$$\begin{cases} (\mathbf{P}_\lambda \mathsf{u})(a) = 0, & \text{if } \lambda > 0 \\ (\mathbf{P}_\lambda \mathsf{v})(a) = 0, & \text{if } \lambda < 0\,. \end{cases}$$

Similarly, for $\lambda \in \sigma(\mathbf{A}(b)) \setminus \{0\}$,

$$\begin{cases} (\mathbf{P}_\lambda \mathsf{v})(b) = 0, & \text{if } \lambda > 0 \\ (\mathbf{P}_\lambda \mathsf{u})(b) = 0, & \text{if } \lambda < 0\,. \end{cases}$$

Which gives $[\,\mathsf{u} \mid \mathsf{v}\,] = 0$. Thus, $\widetilde{\mathscr{V}} \subseteq \mathscr{V}^{[\perp]}$.

For the converse, let $\mathsf{v} \in \mathscr{V}^{[\perp]}$, then for any $\mathsf{u} \in \mathscr{V}$,

$$[\,\mathsf{u} \mid \mathsf{v}\,] = 0\,.$$

If all the eigenvalues of $\mathbf{A}(b)$ are nonpositive and all the eigenvalues of $\mathbf{A}(a)$ are nonnegative i.e. $\mathscr{V} = \mathscr{W}_0$, then $\widetilde{\mathscr{V}} = \mathscr{W}$. Hence, the inclusion $\mathscr{V}^{[\perp]} \subseteq \widetilde{\mathscr{V}}$ is trivial. Let us assume that there exists a strictly positive eigenvalue of $\mathbf{A}(b)$ (the left end-point is treated in an analogous manner). Let us denote by $\lambda > 0$, an arbitrary such eigenvalue. We choose $\mathsf{u} \in H^1((a,b);\mathbb{C}^r)$ such that $\mathsf{u}(a) = 0, (\mathbf{P}_\lambda \mathsf{u})(b) \neq 0$, and for any $\lambda' \in \sigma(\mathbf{A}(b)) \setminus \{\lambda\})$ we have $(\mathbf{P}_{\lambda'}\mathsf{u})(b) = 0$ (e.g. for $\mathsf{e} \in \mathbb{C}^r$ such that $\mathbf{P}_\lambda(b)\mathsf{e} \neq 0$, we can take $\mathsf{u}(x) = \big(\frac{x-a}{x-b}\big)\mathbf{P}_\lambda(b)\mathsf{e}$). It is evident that $\mathsf{u} \in \mathscr{V}$. By inserting this u in the identity above, we get

$$\lambda(\mathbf{P}_\lambda \mathsf{u})(b)(\mathbf{P}_\lambda \mathsf{v})(b) = 0 \implies \mathsf{v} \in \widetilde{\mathscr{V}}_{\lambda,b}\,.$$

Since, $\lambda \in \sigma(\mathbf{A}(b)) \setminus \{0\}$, $\lambda > 0$, was arbitrary, we get $\mathsf{v} \in \bigcap_{\lambda \in \sigma(\mathbf{A}(b))\setminus\{0\}} \widetilde{\mathscr{V}}_{\lambda,b}$. Similarly, $\mathsf{v} \in \bigcap_{\lambda \in \sigma(\mathbf{A}(a))\setminus\{0\}} \widetilde{\mathscr{V}}_{\lambda,a}$. Hence, $\mathsf{v} \in \widetilde{\mathscr{V}}$, concluding the proof.

∎

An immediate consequence of the previous lemma (see Theorem 2.4.5) is that the pair $(T_1|_{\mathscr{V}}, \widetilde{T}_1|_{\widetilde{\mathscr{V}}})$ is a pair of mutually adjoint bijective realisations. Having this information available, we can by means of Theorem 3.2.10 get some information on the kernels, which is fundamental in describing all bijective realisations. Before we state and prove the main result of this Section, let us prove the following useful result.

Lemma 4.3.9. Let $\mathscr{V}$ and $\widetilde{\mathscr{V}}$ be given by (4.30). Then,

$$\dim\big(\mathscr{W}/\mathscr{V}\big) = \sum_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}} \dim\big(\mathscr{W}/\mathscr{V}_{\lambda,a}\big) + \sum_{\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}} \dim\big(\mathscr{W}/\mathscr{V}_{\lambda,b}\big)\,,$$

$$\text{and}\quad \dim\big(\mathscr{W}/\widetilde{\mathscr{V}}\big) = \sum_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}} \dim\big(\mathscr{W}/\widetilde{\mathscr{V}}_{\lambda,a}\big) + \sum_{\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}} \dim\big(\mathscr{W}/\widetilde{\mathscr{V}}_{\lambda,b}\big)\,.$$

Proof. We shall prove the first equality only, as the second one is completely analogous to it. Let us first show that $\mathscr{W} = \mathscr{V}_a + \mathscr{V}_b$, where $\mathscr{V}_a$ and $\mathscr{V}_b$ are given by (4.31). The subspaces $\mathscr{V}_a$ and $\mathscr{V}_b$ can be rewritten as,

$$\mathscr{V}_a = \{\mathsf{u}\in\mathscr{W} : (\forall\lambda\in\sigma(\mathbf{A}(a)), \lambda>0)\ (\mathbf{P}_\lambda\mathsf{u})(a)=0\}$$
$$\text{and}\quad \mathscr{V}_b = \{\mathsf{u}\in\mathscr{W} : (\forall\lambda\in\sigma(\mathbf{A}(b)), \lambda>0)\ (\mathbf{P}_\lambda\mathsf{u})(b)=0\}\,.$$

Let φ be a smooth function on $[a,b]$ such that $\varphi(a)=1$ and $\varphi(b)=0$. For $\mathsf{u}\in\mathscr{W}$, we have $\mathsf{u}=(1-\varphi)\mathsf{u}+\varphi\mathsf{u}$. It is clear that $(1-\varphi)\mathsf{u}, \varphi\mathsf{u}\in\mathscr{W}$ and (see Remark 4.3.6(i))

$$(\forall\lambda\in\sigma(\mathbf{A}(b))\wedge\lambda<0)\quad (\mathbf{P}_\lambda(\varphi\mathsf{u}))(b) = \varphi(b)(\mathbf{P}_\lambda\mathsf{u})(b)=0\,,$$
$$(\forall\lambda\in\sigma(\mathbf{A}(a))\wedge\lambda>0)\quad (\mathbf{P}_\lambda(1-\varphi)\mathsf{u})(a) = (1-\varphi(a))(\mathbf{P}_\lambda\mathsf{u})(a)=0$$

(in fact the above holds regardless of the sign of λ). This implies that $\varphi\mathsf{u}\in\mathscr{V}_b$ and $(1-\varphi)\mathsf{u}\in\mathscr{V}_a$. Since $\mathsf{u}\in\mathscr{W}$ was arbitrary, $\mathscr{W}\subseteq\mathscr{V}_a+\mathscr{V}_b$. From the construction, $\mathscr{V}_a,\mathscr{V}_b\subseteq\mathscr{W}$, hence $\mathscr{W}=\mathscr{V}_a+\mathscr{V}_b$. Thus, $\dim(\mathscr{W}/(\mathscr{V}_a+\mathscr{V}_b))=0$. Using Lemma 4.3.3, we get

$$\dim(\mathscr{W}/\mathscr{V}) \;=\; \dim(\mathscr{W}/\mathscr{V}_a) \;+\; \dim(\mathscr{W}/\mathscr{V}_b)\,.$$

It is enough to prove that

$$\dim(\mathscr{W}/\mathscr{V}_a) \;=\; \sum_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}} \dim\big(\mathscr{W}/\mathscr{V}_{\lambda,a}\big)\,.$$

Let $\lambda_{i_1},\lambda_{i_2},\ldots,\lambda_{i_n}$ $(n\le r)$ be distinct positive eigenvalues of $\mathbf{A}(a)$ and $\mathscr{V}_k := \mathscr{V}_{\lambda_{i_k},a}$, $\mathbf{P}_k := \mathbf{P}_{\lambda_{i_k}}$ for $k\in\{1,2,\ldots,n\}$. We claim that for each $k\in\{2,3,\ldots,n\}$,

$$\bigcap_{i=1}^{k-1}\mathscr{V}_i + \mathscr{V}_k = \mathscr{W}\,. \tag{4.32}$$

Let $\varepsilon>0$ be such that all $\mathbf{P}_k$, $k=1,2,\ldots,n$, are well-defined and Lipschitz continuous on $[a,a+2\varepsilon]$ (see Subsection 4.3.1). Let ϑ be a smooth function on $[a,b]$ such that

$$\vartheta(x) \;=\; \begin{cases} 1\,, & x\in[a,a+\varepsilon] \\ 0\,, & x\in[a+2\varepsilon,b] \end{cases}\,.$$

For an arbitrary $\mathsf{v} \in \mathscr{W}$ we have $(1-\vartheta)\mathsf{v} \in \mathscr{V}_k$. Thus, it is sufficient to study $\mathsf{u} := \vartheta\mathsf{v}$. Since $\mathbf{P}_k$ is well-defined on the support of ϑ, we can write

$$\mathsf{u} = \mathbf{P}_k\mathsf{u} + (\mathbb{1} - \mathbf{P}_k)\mathsf{u} \,,$$

where both functions on the right-hand side belongs to the graph space $\mathscr{W}$. From the following simple series of equalities

$$(\mathbf{P}_k(\mathbb{1}-\mathbf{P}_k)\mathsf{u})(a) = (\mathbf{P}_k\mathsf{u})(a) - (\mathbf{P}_k\mathbf{P}_k\mathsf{u})(a) = (\mathbf{P}_k\mathsf{u})(a) - (\mathbf{P}_k\mathsf{u})(a) = 0\,,$$

we get $(\mathbb{1}-\mathbf{P}_k)\mathsf{u} \in \mathscr{V}_k$. Also, $(\mathbf{P}_k\mathbf{P}_k\mathsf{u})(a) = (\mathbf{P}_k\mathsf{u})(a)$. If $(\mathbf{P}_k\mathsf{u})(a) = 0$, then $\mathsf{u} \in \mathscr{V}_k$. Let us assume that $(\mathbf{P}_k\mathsf{u})(a) \neq 0$. Then, for any $1 \leq i < k$,

$$(\mathbf{P}_i\mathbf{P}_k\mathsf{u})(a) = 0\,,$$

implying $\mathbf{P}_k\mathsf{u} \in \bigcap_{i=1}^{k-1}\mathscr{V}_i$. Since, $\mathsf{v} \in \mathscr{W}$ was arbitrary, $\mathscr{W} \subseteq \bigcap_{i=1}^{k-1}\mathscr{V}_i + \mathscr{V}_k$, which proves the claim (4.32). Hence,

$$(\forall k \in \{2,3,\ldots,n\}) \quad \dim\Big(\mathscr{W} \setminus \bigcap_{i=1}^{k-1}\mathscr{V}_i + \mathscr{V}_k\Big) = 0\,.$$

Using Lemma 4.3.4, finally we obtain

$$\dim(\mathscr{W}/\mathscr{V}_a) \;=\; \sum_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}} \dim\big(\mathscr{W}/\mathscr{V}_{\lambda,a}\big)\,,$$

which should have been shown. ∎

Here we present the main result of this section.

Theorem 4.3.10. Let n_x^+, n_x^- denote the number of positive and negative eigenvalues of the matrix $\mathbf{A}(x)$ respectively. Then,

$$\dim\ker T_1 = n_a^+ + n_b^- \quad \text{and} \quad \dim\ker \widetilde{T}_1 = n_a^- + n_b^+\,.$$

Proof. For any closed subsapce $\mathscr{V} \subseteq \mathscr{W}$, $\mathscr{W}_0 \subseteq \mathscr{V}$, such that $T_1|_{\mathscr{V}}$ is bijective, we have by Theorem 3.2.10

$$\dim\ker T_1 = \dim(\mathscr{W}/\mathscr{V})\,.$$

By Lemma 4.3.8 one such $\mathscr{V}$ is given by (4.30). Applying Lemma 4.3.9, we have

$$\dim\ker T_1 \;=\; \sum_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}} \dim\big(\mathscr{W}/\mathscr{V}_{\lambda,a}\big) + \sum_{\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}} \dim\big(\mathscr{W}/\mathscr{V}_{\lambda,b}\big)\,.$$

Hence, it is left to determine $\dim\big(\mathscr{W}/\mathscr{V}_{\lambda,a}\big)$ and $\dim\big(\mathscr{W}/\mathscr{V}_{\lambda,b}\big)$ for any λ.

From the construction, for any $\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}$,

$$\dim(\mathscr{W}/\mathscr{V}_{\lambda,a}) \;=\; \begin{cases} \operatorname{rank}\mathbf{P}_\lambda(a), & \lambda>0\,,\\ 0, & \lambda<0\,.\end{cases}$$

Similarly, for any $\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}$,

$$\dim(\mathscr{W}/\mathscr{V}_{\lambda,b}) \;=\; \begin{cases} \operatorname{rank}\mathbf{P}_\lambda(b), & \lambda<0\,,\\ 0, & \lambda>0\,.\end{cases}$$

This implies

$$\sum_{\lambda\in\sigma(\mathbf{A}(a))\setminus\{0\}} \dim\big(\mathscr{W}/\mathscr{V}_{\lambda,a}\big) = n_a^+ \text{ and } \sum_{\lambda\in\sigma(\mathbf{A}(b))\setminus\{0\}} \dim\big(\mathscr{W}/\mathscr{V}_{\lambda,b}\big) = n_b^-\,,$$

Therefore,

$$\dim\ker T_1 = n_a^+ + n_b^-\,.$$

In a completely analogous way, one obtains

$$\dim\ker \widetilde{T}_1 = n_a^- + n_b^+\,.$$

∎

Having the result of Theorem 4.3.10 at our hands, we can formulte the following corollary.

Corollary 4.3.11. Codimension of the graph space $\mathscr{W}$ over minimal space $\mathscr{W}_0$ is given by,

$$\dim\big(\mathscr{W}/\mathscr{W}_0\big) \;=\; \operatorname{rank}\mathbf{A}(a) \;+\; \operatorname{rank}\mathbf{A}(b)\,.$$

Proof. By the decomposition (3.6) we have

$$\dim(\ker T_1) + \dim(\ker\widetilde{T}_1) = \dim\big(\mathscr{W}/\mathscr{W}_0\big)\,.$$

So, by the previous theorem, we have

$$\dim\big(\mathscr{W}/\mathscr{W}_0\big) = n_a^+ + n_b^- + n_a^- + n_b^+ = \operatorname{rank}\mathbf{A}(a) + \operatorname{rank}\mathbf{A}(b)$$

■

Remark 4.3.12. The triplet (n_x^+, n_x^0, n_x^-) is called the inertia of the Hermitian matrix $\mathbf{A}(x)$ which is relevant in Sylvester's law of inertia, where n_x^0 denotes the multiplicity of zero eigenvalues of $\mathbf{A}(x)$.

Remark 4.3.13. When $n_a^+ + n_b^- = n_a^- + n_b^+$, then $\ker T_1 \cong \ker \widetilde{T}_1$. Thus, by Corollary 3.3.9 there exists a subspace $\mathscr{V}$ of $\mathscr{W}$ with $\mathscr{W}_0 \subseteq \mathscr{V}$, such that $(T_1|_{\mathscr{V}}, \widetilde{T}_1|_{\mathscr{V}})$ is a pair of mutually adjoint bijective realisations related to $(T, \widetilde{T})$. Another point of view is to say that in this case the skew-symmetric operator $T - \widetilde{T}$ (see (4.1)) admits skew-selfadjoint extensions (see Section 3.4).

Remark 4.3.14. If all the eigenvalues of $\mathbf{A}(x)$ are strictly positive or strictly negative in $[a,b]$, then the graph space is $\mathscr{W} = H^1((a,b);\mathbb{C}^r)$ and the minimal space is $\mathscr{W}_0 = H_0^1((a,b);\mathbb{C}^r)$. Using Theorem 4.3.10, $\dim\ker T_1 + \dim\ker \widetilde{T}_1 = r + r = 2r$, which reveals a well-known fact that $H^1((a,b);\mathbb{C}^r)/H_0^1((a,b);\mathbb{C}^r) = 2r$.

Remark 4.3.15. The result of Theorem 4.3.10 is valid even if $\mathbf{A}(x)$ is singular in some (or even all) points of the interval $[a,b]$. Thus we cover what might be called *singular systems* of differential equations of the first order. These problems have a long history where often the problem of well-posedness was studied by obtaining the explicit formula in terms of (formal) series [27, Chapter 4]

4.3.5. Example 1

Let us consider one-dimensional $(d = 1)$ vectorial $(r = 2)$ Friedrichs operator defined as follows

$$\mathbf{A}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1-x \end{bmatrix}, \ \mathbf{B}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ x \in (0,1)\ .$$

Here, $\mathbf{B}(x) + \mathbf{B}^*(x) + \mathbf{A}'(x) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, hence conditions (F1) and (F2) are satisfied (see Example 2.6.1).

Due to the diagonal structure of this system, we can use the information obtained in $(r=1)$ case (see Section 4.1), to evaluate the dimensions of the kernels. Indeed, it is easy to see that $(\varphi_1, \varphi_2)^\top \in \ker T_1$ if and only if for any $x \in (0,1)$ we have

$$\varphi_1'(x) + \varphi_1(x) = 0 \quad \text{and} \quad ((1-x)\varphi_2)'(x) + \varphi_2(x) = 0\,.$$

Both equations contribute by 1 in the dimension of $\ker T_1$, i.e.

$$\dim\ker T_1 = 2\,. \tag{4.33}$$

Analogously, for $\ker \widetilde{T}_1$ we need to study ($x \in (0,1)$)

$$-\varphi_1'(x) + \varphi_1(x) = 0 \quad \text{and} \quad -((1-x)\varphi_2)'(x) + \varphi_2(x) = 0\,,$$

which leads to

$$\dim\ker \widetilde{T}_1 = 1\,. \tag{4.34}$$

Now we shall test the result of Theorem 4.3.10 by investigating the ranks of matrices

$$\mathbf{A}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\,.$$

Obviously, $n_0^+ = 2, n_0^- = 0, n_1^+ = 1, n_1^- = 0$. By Theorem 4.3.10, we have

$$\dim\ker T_1 = 2 \quad \text{and} \quad \dim\ker \widetilde{T}_1 = 1\,,$$

which is in accordance with the information obtained in (4.33) and (4.34).

Remark 4.3.16. The choice of $\mathbf{B}$ in this example is irrelevant. More precisely, by Theorem 4.3.10, for any bounded $\mathbf{B}$ such that the condition (F2) is satisfied (i.e. the corresponding operators are Friedrichs operators) we have the same conclusion on the kernels. Moreover, the same holds for general operators (see Section 3.3).

4.3.6. Second order linear ODE

Let $I = (a,b)$ be an open interval. For $f \in L^2(I;\mathbb{C})$, $p \in W^{1,\infty}(I;\mathbb{R})$, $q \in L^\infty(I;\mathbb{C})$, such that $p \geq \mu_0 > 0$, $\Re q \geq \mu_0 > 0$, consider the following ordinary differential equation on I,

$$-(p(x)u'(x))' + q(x)u(x) = f(x)\,. \tag{4.35}$$

Let $\mathsf{u} = (u_1, u_2)$ and consider the following system, for $\mathsf{f} \in L^2(I;\mathbb{C}^2)$,

$$T\mathsf{u} := (\mathbf{A}\mathsf{u})' + \mathbf{B}\mathsf{u} = \mathsf{f}\,,$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -p \\ -p & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} q & 0 \\ p' & p \end{bmatrix}.$$

Here

$$\mathbf{A}' + \mathbf{B} + \mathbf{B}^* = \begin{bmatrix} 2\Re q & 0 \\ 0 & 2p \end{bmatrix} \geq 2\mu_0 \mathbf{I}\,.$$

For $\widetilde{T}\mathsf{u} := -(\mathbf{A}\mathsf{u})' + (\mathbf{A}' + \mathbf{B}^*)\mathsf{u}$, the pair $(T, \widetilde{T})$ forms a joint pair of abstract Friedrichs operators on $\mathscr{H} = L^2(I;\mathbb{C}^2)$ with domain $\mathscr{D} := C_c^\infty(I;\mathbb{C}^2)$. For $u_1 = u$, $u_2 = u'$ and $\mathsf{f} = (f, 0)^\top$, the above system represents (4.35). Therefore, one can easily transfer all well-posedness results for the system developed above to the original second-order equation (4.35).

The graph space is $\mathscr{W} = H^1(I;\mathbb{C}^2)$, which by the Sobolev embedding theorem means that for any $\mathsf{u} \in \mathscr{W}$ and $x \in [a,b]$, the evaluation $\mathsf{u}(x)$ is well-defined. The boundary operator is given by

$$\begin{aligned}(\forall \mathsf{u}, \mathsf{v} \in \mathscr{W}) \quad [\,\mathsf{u} \mid \mathsf{v}\,] = &- p(b)u_2(b)\overline{v_1(b)} - p(b)u_1(b)\overline{v_2(b)} \\ &+ p(a)u_2(a)\overline{v_1(a)} + p(a)u_1(a)\overline{v_2(a)}\,,\end{aligned}$$

and the minimal space $\mathscr{W}_0$ is given by

$$\mathscr{W}_0 = \{\mathsf{u} \in \mathscr{W} : u_1(a) = u_1(b) = u_2(a) = u_2(b) = 0\} = H_0^1(I, \mathbb{C}^2)\,.$$

Here the number of positive and negative eigenvalues are equal to 1 for both $\mathbf{A}(b)$ and $\mathbf{A}(a)$, i.e. $n_a^+ = n_a^- = n_b^+ = n_b^- = 1$ (hence we are in the regime discussed in Remark 4.3.14). Using Theorem 4.3.10, the dimensions of the kernels are equal to

$$\dim\ker T_1 = 2 \quad \text{and} \quad \dim\ker \widetilde{T}_1 = 2\,.$$

Since both kernels are non-trivial, Theorem 2.4.12 guarantees the existence of infinitely many bijective realisations related to $(T, \widetilde{T})$. The kernels are also isomorphic, thus by

Corollary 3.3.9, there exists a subspace $\mathscr{V}$ with $\mathscr{W}_0 \subseteq \mathscr{V} \subseteq \mathscr{W}$ such that $(T_1|_{\mathscr{V}}, \widetilde{T}_1|_{\mathscr{V}})$ is a pair of bijective realisations related to $(T, \widetilde{T})$. In fact one such subspace is given by

$$\mathscr{V}_1 = \{\mathsf{u} \in \mathscr{W} : u_1(a) = u_1(b) = 0\} = H_0^1(I;\mathbb{C}) \times H^1(I;\mathbb{C}) \; .$$

Due to Theorem 2.4.5 it is enough to check that $(\mathscr{V}_1, \mathscr{V}_1)$ satisfies (V)-boundary conditions. For any $\mathsf{u}, \mathsf{v} \in \mathscr{V}_1$ we clearly have $[\,\mathsf{u} \mid \mathsf{v}\,] = 0$, which in particular gives (V1)-boundary condition and $\mathscr{V}_1 \subseteq \mathscr{V}_1^{[\perp]}$. Now for $\mathsf{v} \in \mathscr{V}_1^{[\perp]}$ and for any $\mathsf{u} \in \mathscr{V}_1$ the expression $[\,\mathsf{u} \mid \mathsf{v}\,] = 0$ reads

$$-p(b)u_2(b)\overline{v_1(b)} + p(a)u_2(a)\overline{v_1(a)} = 0 \; .$$

Choice of $\mathsf{u} \in \mathscr{V}_1$, such that $u_2(a) \neq 0$, $u_2(b) = 0$, gives $v_1(a) = 0$, and analogously for $v_1(b)$. Thus, $\mathsf{v} \in \mathscr{V}_1$, implying $\mathscr{V}_1^{[\perp]} \subseteq \mathscr{V}_1$ and hence (V2)-boundary condition also holds.

Note that the previous choice gives the homogeneous Dirichlet boundary condition for (4.35). On the other hand the choice

$$\mathscr{V}_2 = \widetilde{\mathscr{V}_2} = \{\mathsf{u} \in \mathscr{W} : u_2(a) = u_2(b) = 0\} = H^1(I;\mathbb{C}) \times H_0^1(I;\mathbb{C}) \, ,$$

gives the homogeneous Neumann boundary condition.

The restriction on the coefficient q seems to be very strict, but thanks to the non-uniqueness of the representation of the equations as Friedrichs systems, we can ease this condition. For the choice $\mathsf{u} = (u_1, u_2)^\top = (e^{-\beta x}u', e^{-\beta x}u)^\top$, for $\beta \in \mathbb{R}$, the equation (4.35) can be rewritten as

$$(\mathbf{A}\mathsf{u})' + \mathbf{B}\mathsf{u} = \mathsf{f} \, ,$$

with

$$\mathbf{A} = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \, , \quad \mathbf{B} = \begin{bmatrix} \beta p & -q \\ -p & \beta p - p' \end{bmatrix} \, , \quad \mathsf{f} = \begin{bmatrix} e^{-\beta x} f \\ 0 \end{bmatrix} \; .$$

Here

$$\mathbf{A}' + \mathbf{B} + \mathbf{B}^* = \begin{bmatrix} 2\beta p + p' & -(p+q) \\ -(p+\overline{q}) & 2\beta p - p' \end{bmatrix} \, ,$$

is positive for $p \geq \mu_0 > 0$, for any bounded q and sufficiently large $\beta \in \mathbb{R}$.

The graph space and the minimal space of the corresponding Friedrichs operator are again given by $\mathscr{W} = H^1(I;\mathbb{C}^2)$ and $\mathscr{W}_0 = H^1_0(I;\mathbb{C}^2)$, but the boundary operator differs: $\mathsf{u},\mathsf{v} \in \mathscr{W}$,

$$[\,\mathsf{u} \mid \mathsf{v}\,] = p(b)u_1(b)\overline{v_1(b)} + p(b)u_2(b)\overline{v_2(b)} - p(a)u_1(a)\overline{v_1(a)} - p(b)u_2(a)\overline{v_2(a)}\,.$$

$\mathbf{A}(a)$ and $\mathbf{A}(b)$ both have two positive eigenvalues and no negative eigenvalues. Hence, by Theorem 4.3.10, $\dim\ker T_1 = 2$ and $\dim\ker \widetilde{T}_1 = 2$.

In this representation both the homogeneous Dirichlet and the homogeneous Neumann boundary conditions are still admissible (the realisations are bijective). However, here we cannot take the same boundary conditions for both operators T and $\widetilde{T}$, as it was the case in the previous representation. Namely, both pairs $(\mathscr{V}_1,\mathscr{V}_2)$ and $(\mathscr{V}_2,\mathscr{V}_1)$ define bijective realisations, i.e. if we take the homogeneous Dirichelt boundary condition for T, then the homogeneous Neumann boundary condition should be imposed for $\widetilde{T}$.

5. SEMIGROUP THEORY

In this chapter we shall consider an (initial-)boundary value problem for a non-stationary Friedrichs system. To be specific we shall consider the abstract Cauchy problem:

$$\begin{cases} \frac{d}{dt}u(t) + Au(t) = f\,, \\ u(0) = u_0\,, \end{cases} \tag{5.1}$$

where $u : [0,T) \to \mathscr{H}$, $T > 0$, is the unknown function, while $f : [0,T) \to \mathscr{H}$, the initial data $u_0 \in \mathscr{H}$ and the abstract (Friedrichs) operator A are given. Our goal is to show that all bijective realisations of a given pair of Friedrichs operators with signed boundary map give rise to the operators which generate a contractive C_0-semigroup. Moreover, this family of realisations, i.e. bijective realisations with signed boundary map (see Remark 2.4.6) contains all the realisations with such property of generating a contractive C_0-semigroup. The study of the same has been done in [21], however we adapt a slightly different approach and extend the theory. More precisely, in [21], it was proved that all bijective realisations with signed boundary map related to a joint pair of abstract Friedrichs operator $(T_0, \widetilde{T}_0)$ on $\mathscr{H}$ give rise to the operators which generate contraction C_0-semigroups. Here we prove that the converse is also true, i.e. the bijective realisations which give rise to the generators of contractive C_0-semigroups, are precisely the ones with signed boundary maps. In parallel, we show a nice entanglement between the theory of abstract Friedrichs operators and skew symmetric operators in the same sense. This illustrates the strength of the characterisation from Section 3.3.

5.1. PRELIMINARIES

In this section we recall the theory of strongly continuous semigroups which will be applicable in our case. Most of the material from this section can be found in [30, Chapter II]

Definition 5.1.1. A family $(T(t))_{t\geq 0}$ of mappings on a vector space $\mathscr{X}$ satisfying

$$\begin{cases} T(t+s) = T(t)T(s) \quad \forall t,s \geq 0\,, \\ T(0) = \mathbb{1}\ . \end{cases} \tag{5.2}$$

is called a (one-parameter) *operator semigroup*.

strongly continuous semigroup (or C_0-*semigroup*) on a Banach space $\mathscr{X}$ is a family $(T(t))_{t\geq 0}$ of bounded linear operators that satisfies (5.2) and for every $x \in \mathscr{X}$ the mappings $t \mapsto T(t)x$ are continuous from $\mathbb{R}_+$ into $\mathscr{X}$.

Finally, if these properties hold for $t \in \mathbb{R}$, then $(T(t))_{t\geq 0}$ is called C_0-*group* of bounded operators.

The following equivalent characterisation of strongly continuous semigroup can be found in [30, Chapter I, Proposition 1.3].

Proposition 5.1.2. For a semigroup $(T(t))_{t\geq 0}$ on a Banach space $\mathscr{X}$, the following assertions are equivalent.

(a) $(T(t))_{t\geq 0}$ is strongly continuous;

(b) $\lim_{t\downarrow 0} T(t)x = x$ for all $x \in \mathscr{X}$;

(c) $\exists \delta > 0, M \geq 1$, and a dense subset $\mathscr{D} \subset \mathscr{X}$ such that

  (i) $\|T(t)\| \leq M$ for all $t \in [0,\delta]$,

  (ii) $\lim_{t\downarrow 0} T(t)x = x$ for all $x \in \mathscr{D}$.

Proposition 5.1.3. [30, Chapter I, Proposition 1.4] For every C_0-semigroup $(T(t))_{t\geq 0}$, there exist $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{\omega t}\,,$$

for all $t \geq 0$.

We shall call such semigroup of type (M,ω). The semigroup is called *bounded* if we can take $\omega = 0$ and *contractive* if we can take $M = 1, \omega = 0$.

Another equivalent characterisation of strongly continuous semigroups which is more useful in the Hilbert space setting is the following.

Theorem 5.1.4. [30, Chapter I, Theorem 1.6] A semigroup $(T(t))_{t\geq 0}$ on a Banach space $\mathscr{X}$ is strongly continuous if and only if it is weakly continuous, i.e. if the mappings

$$t \mapsto {}_{\mathscr{X}'}\langle x', T(t)x\rangle_{\mathscr{X}}$$

from $\mathbb{R}_+$ to $\mathbb{C}$ are continuous for each $x \in \mathscr{X}, x' \in \mathscr{X}'$.

In general, the *adjoint semigroup* $(T(t)')_{t\geq 0}$ consisting of all Banach adjoint operators $T(t)'$ on the dual space $\mathscr{X}'$ is *not* strongly continuous (even though $T(t)$ is a strongly continuous semigroup). This is due to the difference between *weak* and *weak** topology. But, in the case of reflexive spaces (in particular on Hilbert spaces) these two topologies coincide, and hence by Theorem 5.1.4 the adjoint semigroup is also strongly continuous.

Definition 5.1.5. The *generator* $A : \operatorname{dom} A \subseteq \mathscr{X} \to \mathscr{X}$ of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $\mathscr{X}$ is the operator

$$Ax := \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x)$$

defined for every $x \in \operatorname{dom} A := \{x \in X : \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x) \text{ exists}\}$.

The generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.

Lemma 5.1.6. [30, Chapter II, Lemma 1.3] For the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ in a Banach space $\mathscr{X}$, the following properties hold.

(i) $A : \operatorname{dom} A \subseteq \mathscr{X} \to \mathscr{X}$ is a linear operator.

(ii) If $x \in \operatorname{dom} A$, then $T(t)x \in \operatorname{dom} A$ and

$$\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x \qquad \forall t \geq 0\,. \tag{5.3}$$

(iii) For every $t \geq 0$ and $x \in \mathscr{X}$, one has

$$\int_0^t T(s)x\,ds \in \operatorname{dom} A\,.$$

(iv) For every $t \geq 0$, one has

$$\begin{aligned} T(t)x - x &= A\int_0^t T(s)x\,ds \quad \text{if x} \in \mathscr{X}\,, \\ &= \int_0^t T(s)Ax\,ds \quad \text{if x} \in \operatorname{dom} \mathrm{A}\,. \end{aligned}$$

Theorem 5.1.7. [30, Chapter II, Theorem 1.10] Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space $\mathscr{X}$. Let $\omega \in \mathbb{R}$, $M \geq 1$ be the constants given by Proposition 5.1.3. Then the following hold.

(i) If $\lambda \in \mathbb{C}$ such that $R(\lambda)x := \int_0^\infty e^{-\lambda s}T(s)x\,ds$, then $\lambda \in \rho(A)$ and $R(\lambda, A) = R(\lambda)$.

(ii) If $\Re\lambda > \omega$, then $\lambda \in \rho(A)$ and the resolvent is given by the previous integral.

(iii) For any $\Re\lambda > \omega$,

$$\|R(\lambda, A)\| \leq \frac{M}{\Re\lambda - \omega}\,.$$

Definition 5.1.8. A linear operator A on a Banach space $\mathscr{X}$ is called *dissipative* if

$$\|(\lambda\mathbb{1} - A)x\| \geq \lambda\|x\|\,, \tag{5.4}$$

for all $\lambda > 0$ and $x \in \operatorname{dom} A$. If $\mathscr{X}$ is a Hilbert space, then an equivalent condition for A being dissipative is

$$\Re\langle Ax \mid x\rangle \leq 0\,, \tag{5.5}$$

for all $x \in \mathscr{X}$. An operator A is called *maximal dissipative* if $A \subseteq B$ and both A and B are dissipative, then $A = B$. Furthermore, an operator A is called *$m-$dissipative* if A is dissipative and $\mathbb{1} - A$ is surjective. The operator A is called accretive, $m-$accretive, maximal accretive if and only if $-A$ is dissipative, $m-$dissipative and maximal dissipative, respectively.

Theorem 5.1.9. [51, Corollary of Theorem 1.1.1][Phillips] Let $\mathscr{X}$ be a Banach space and A an operator on $\mathscr{H}$. Then A is $m-$dissipative if and only if A is maximal dissipative and $\operatorname{dom} A$ is dense.

Theorem 5.1.10. [30, Chapter II, Theorem 3.15][Lumer Phillips] An operator A on a Banach space $\mathscr{X}$ generates a contractive C_0-semigroup if and only if A is $m-$dissipative.

Theorem 5.1.11. [30, Chapter III, Theorem 1.3] If A generates a semigroup of type (M,ω) in a Banach space $\mathscr{X}$ and $B \in \mathscr{L}(\mathscr{X})$, then $A+B$ with $\operatorname{dom}(A+B) = \operatorname{dom} A$ generates a semigroup of type $(M,\omega+\|B\|)$.

5.2. Semigroup theory for abstract Friedrichs operators

Let us recall the characterisation from Theorem 3.3.1. Let $(T_0, \widetilde{T}_0)$ be a joint pair of abstract Friedrichs operators on $\mathscr{H}$. The characterisation in terms of a skew-symmetric and a bounded selfadjoint operator with strictly positive bottom, is

$$T_0 = L_0 + S \quad \text{and} \quad \widetilde{T}_0 = -L_0 + S \, .$$

If we denote $L_1 := -L_0^* \supseteq L_0$, then we have

$$T_1 := (\widetilde{T}_0)^* = L_1 + S \qquad \text{and} \qquad \widetilde{T}_1 := T_0^* = -L_1 + S \, .$$

For an extension T such that $T_0 \subseteq T \subseteq T_1$ we denote the corresponding extension of L_0 by L and thus we have $L_0 \subseteq L \subseteq L_1$.

Lemma 5.2.1. Let T be an extension of T_0 such that $T_0 \subseteq T \subseteq T_1$. T is an accretive realisation of T_0 if and only if L is an accretive realisation of L_0.

Proof. Let $u \in \operatorname{dom} T$, then (see the proof of Lemma 2.4.1 and (3.10))

$$\Re\langle Tu \mid u\rangle = \frac{1}{2}\big(\langle (T_1 + \widetilde{T}_1)u \mid u\rangle + [\,u \mid u\,]\big) \, , \tag{5.6}$$

and

$$\Re\langle Lu \mid u\rangle = [\,u \mid u\,] \, .$$

Since $\langle (T_1 + \widetilde{T}_1)u \mid u\rangle \geq 2\mu_0 \|u\|^2$, it is easy to see that if L is accretive then so is T.

For the converse, suppose T is accretive. Take an arbitrary $u \in \operatorname{dom} T$. By Theorem 3.2.4 there exist $u_0 \in \mathscr{W}_0$, $\nu \in \ker T_1$ and $\tilde{\nu} \in \ker \widetilde{T}_1$ such that $u = u_0 + \nu + \tilde{\nu}$. Since $\operatorname{dom} T_0 \subseteq \operatorname{dom} T$, for any $v_0 \in \mathscr{W}_0$ we also have that $v = v_0 + \nu + \tilde{\nu} \in \operatorname{dom} T$. Since $[\,u \mid u\,] = [\,\nu \mid \nu\,] + [\,\tilde{\nu} \mid \tilde{\nu}\,]$ (see 2.4.10), we have (using (T2) condition)

$$\Re\langle Tu \mid u\rangle \geq 0 \implies [\,u \mid u\,] = [\,v \mid v\,] \geq -c\|v_0 + \nu + \tilde{\nu}\|^2 \, .$$

Since $\operatorname{dom} T_0$ is dense in $\mathscr{H}$, we can take v_0 arbitrary close to $-\nu - \tilde{\nu}$ (with respect to norm $\|\cdot\|$). Hence,

$$\Re\langle Lu \mid u\rangle = [\, u \mid u\,] \geq 0\,,$$

implying L is accretive. ■

Theorem 5.2.2. Let $(T_0, \widetilde{T}_0)$ be a joint pair of abstract Friedrichs operators on $\mathscr{H}$ and let T be a closed realisation of T_0, i. e. $T_0 \subseteq T \subseteq T_1$. The following assertions are equivalent:

(i) T is $m-$accretive;

(ii) L is $m-$accretive;

(iii) T is a bijective realisation with signed boundary map.

Proof. Let us first prove the equivalence between (i) and (iii). Let T be an accretive extension of T_0. Since for any $u \in \operatorname{dom} T$,

$$0 \leq 2\Re\langle Tu \mid u\rangle = \langle\, (T_1 + \widetilde{T}_1)u \mid u\rangle + [\, u \mid u\,]\,,$$

and $\langle\, (T_1 + \widetilde{T}_1)u \mid u\rangle \geq 0$, one has $[\, u \mid u\,] \geq -c\|u\|^2$. Now like in the proof of Lemma 5.2.1 one can get $[\, u \mid u\,] \geq 0$. Since $u \in \operatorname{dom} T$ is arbitrary, we get that $\operatorname{dom} T \subseteq \mathscr{W}^+$. Assume that $\operatorname{dom} T$ is not maximal non-negative and let $\operatorname{dom} T \subsetneq \mathscr{V}\,(\subseteq \mathscr{W}^+)$. But then $T_1|_{\mathscr{V}}$ is again accretive (see the proof of Lemma 2.4.1 and (5.6)), which contradicts that T is $m-$accretive. Hence, $\operatorname{dom} T$ is maximal non-negative and by Corollary 2.5.4, T is a bijective realisation with signed boundary map.

For the converse, if T is a bijective realisation with signed boundary map then for any $u \in \operatorname{dom} T$, $[\, u \mid u\,] \geq 0$, which by (5.6) and (T3) condition implies T is accretive. Following the approach discussed in Section 3.3, we get that $\mathbb{1} + T$ is also an abstract Friedrichs operator and from Corollary 3.3.3, $\mathbb{1} + T$ is a bijective realisation with signed boundary map. In particular $\mathbb{1} + T$ is surjective. Since $\operatorname{dom} T$ is dense in $\mathscr{H}$, we conclude that T is $m-$accretive.

For the equivalence between (i) and (ii), from Lemma 5.2.1 we have that T is accretive if and only if L is accretive. Since $\operatorname{dom} T = \operatorname{dom} L$ and it is easy to to see that if one is maximal accretive then so is the other, the equivalence follows. ■

Remark 5.2.3. Recall that (by Theorem 3.4.7) (iii) is equivalent to the fact that there exists a linear contraction $U : (\ker \widetilde{T}_1, [\cdot \mid \cdot]) \to (\ker T_1, -[\cdot \mid \cdot])$ i. e. $\|U\| \leq 1$ such that $\operatorname{dom} T = \mathscr{W}_0 \dot{+} \{\tilde{v} + U\tilde{v} : \tilde{v} \in \ker \widetilde{T}_1\}$. Hence, all $m-$accertive realisations of T can be parameterised by such contractive mappings U between the kernels.

Now we state the main result of this section which is now a direct consequence of the Lumer Phillips Theorem 5.1.10.

Corollary 5.2.4. The following are equivalent:

(i) T is a bijective realisation with signed boundary map;

(ii) $-T$ is a generator of a contractive C_0-semigroup;

(iii) $-L$ is a generator of a contractive C_0-semigroup.

In [21, Theorem 2], (i) implies (ii) was proved. Here in Corollary 5.2.4, we proved that the converse is also true, i.e. the bijective realisations which give rise to the generators of contractive C_0-semigroups, are precisely the ones with signed boundary maps.

There are bijective realisations without signed boundary maps. The question about generation of semigroups related to these bijective realisations remains open. However, we give an example (see Remark 5.3.8) where a bijective realisation without signed boundary maps does not give rise to a generator of strongly continuous semigroup.

Remark 5.2.5. Let $(T_0, \widetilde{T}_0)$ be a joint pair of abstract Friedrichs operators on $\mathscr{H}$ and let T be a closed realisation (extension) of T_0, i.e. $T_0 \subseteq T \subseteq T_1$. If (T, T^*) satisfies the (V)-conditions, then both T and T^* are $m-$accretive. In addition, $-T$, $-T^*$, $L_1|_{\operatorname{dom} T}$ and $-L_1|_{\operatorname{dom} \widetilde{T}}$ are generators of C_0-semigroup.

5.3. Dissipative extensions of skew-symmetric operators

In the previous section we proved, using the theory developed on Chapter 3, that all bijective realisations with signed boundary map related to a joint pair of abstract Friedrichs operator $(T_0, \widetilde{T}_0)$ on $\mathscr{H}$ give rise to the operators which generate contractive C_0-semigroups, and vice-versa. It turns out that the characterisation from Section 3.3 allows us to use the well-developed theory of boundary quadruple for skew-symmetric operators on abstract Friedrichs operators. As a consequence, we have another way to classify all the boundary conditions which give rise to the generators of contractive C_0-semigroups. Thus, we obtain an alternate approach to classify all the bijective realisations with signed boundary maps.

The classification of all $m-$dissipative realisations of skew-symmetric operators has been studied in [10]. Here we recall the theory about the skew-symmetric operators, while for the details and proofs we refer to the paper [10].

Definition 5.3.1. Let A_0 be a densely defined skew-symmetric operator on $\mathscr{H}$ and $A_1 = -A_0^*$. A *boundary quadruple* $(\mathscr{K}_-, \mathscr{K}_+, G_-, G_+)$ consists of pre-Hilbert spaces $\mathscr{K}_-, \mathscr{K}_+$ and surjective linear mappings $G_- : \mathrm{dom} A_1 \to \mathscr{K}_-$ and $G_+ : \mathrm{dom} A_1 \to \mathscr{K}_+$ such that

$$\langle A_1 u \mid v \rangle + \langle u \mid A_1 v \rangle \;=\; \langle G_+ u \mid G_+ v \rangle_{\mathscr{K}_+} - \langle G_- u \mid G_- v \rangle_{\mathscr{K}_-}\,, \tag{5.7}$$

for all $u, v \in \mathrm{dom} A$, with the additional condition

$$\ker G_- + \ker G_+ \;=\; \mathrm{dom} A_1\,. \tag{5.8}$$

We shall see (in Proposition 5.3.2) that the assumptions (5.7) and (5.8) are enough to guarantee that the mappings G_- and G_+ are continuous and the images $\mathscr{K}_-$ and $\mathscr{K}_+$ are Hilbert spaces.

Proposition 5.3.2. [10, Section 3] Let L_0 be a densely defined skew-symmetric operator on $\mathscr{H}$ and $A_1 = -A_0^*$. Then,

(i) There exists a boundary quadruple $(\mathscr{K}_-, \mathscr{K}_+, G_-, G_+)$;

(ii) The operators $G_- : \mathrm{dom} A_1 \to \mathscr{K}_-$, $G_+ : \mathrm{dom} A_1 \to \mathscr{K}+$ are continuous;

(iii) The spaces $\mathscr{K}_-$ and $\mathscr{K}_+$ are Hilbert spaces;

(iv) One has $\mathrm{dom}\overline{A}_0 = \ker G_+ \cap \ker G_-$ and $\overline{A}_0 u \subseteq A_1$ for all $u \in \mathrm{dom}\overline{A}_0$.

Lemma 5.3.3. [10, Lemma 3.2] Let $x_- \in \mathscr{K}_-$ and $x_+ \in \mathscr{K}_+$. Then there exists $u \in \mathrm{dom} A_1$ such that $G_- u = x_-$ and $G_+ u = x_+$. In fact, this interpolation property is equivalent to

$$\ker G_- + \ker G_+ \; = \; \mathrm{dom} A_1 \, . \tag{5.9}$$

Now we can state the main result of this section. For $\Phi \in \mathscr{L}(\mathscr{K}_-, \mathscr{K}_+)$ i.e. a bounded linear operator for $\mathscr{K}_-$ to $\mathscr{K}_+$, we define the operator A_Φ on $\mathscr{H}$ by

$$\mathrm{dom} A_\Phi \; := \; \{u \in \mathrm{dom} A_1 : \Phi G_- u \; = \; G_+ u\} \quad \text{and} \quad A_\Phi u \; := \; A_1 u \, , \tag{5.10}$$

for all $u \in \mathrm{dom} A_\Phi$. Clearly, $A_0 \subseteq A_\Phi \subseteq A_1$ (see Proposition 5.3.2(iv)).

Theorem 5.3.4. [10, Theorem 3.10] Let A be an extension of the operator A_0 on $\mathscr{H}$, then the following assertions are equivalent:

(i) A is $m-$dissipative;

(ii) There exists a linear contraction $\Phi : \mathscr{K}_- \to \mathscr{K}_+$ such that $A = A_\Phi$.

Theorem 5.3.5. [10, Theorem 4.2] Let A be an extension of the operator A_0 on $\mathscr{H}$, then the following assertions are equivalent:

(i) A is skew-selfadjoint;

(ii) There exists a unitary operator $\Phi : \mathscr{K}_- \to \mathscr{K}_+$ such that $A = A_\Phi$.

Remark 5.3.6. The previous theorem holds even without the assumption that $A \subseteq A_1$.

Let us elaborate, how this theory relates to our situation. Consider the characterisation of abstract Friedrichs operators presented in Section 3.3. That is, a joint pair of densely defined operators $(T_0, \widetilde{T}_0)$ on $\mathscr{H}$ is a pair of abstract Friedrichs operators if and only if there exist a densely defined skew-symmetric operator L_0 and a bounded self-adjoint operator S with strictly positive bottom, both on $\mathscr{H}$, such that

$$T_0 = L_0 + S \qquad \text{and} \qquad \widetilde{T}_0 = -L_0 + S \, . \tag{5.11}$$

If we denote $L_1 := -L_0^* \supseteq L_0$, then we have

$$T_1 := L_1 + S \qquad \text{and} \qquad \widetilde{T}_1 := -L_1 + S \,.$$

If we set $A_0 = -L_0$, then $A_1 = -L_1$ and by Theorem 5.2.2, the search of all $m-$dissipative realisations of A_0 is equivalent to the quest of all bijective realisations of $(T_0, \widetilde{T}_0)$ with signed boundary map.

Let us illustrate the theory on the following example (see [10, Example 6.3]).

Example 5.3.7. Consider the first derivative operator $A_0 u = u'$ on $L^2((0,1);\mathbb{R})$ with $\operatorname{dom} A_0 = C_c^\infty((0,1);\mathbb{R})$. The minimal space is $\mathscr{W}_0 = H_0^1((0,1);\mathbb{R})$ and the graph space is $\mathscr{W} = H^1((0,1);\mathbb{R})$. A boundary quadruple $(\mathscr{K}_-, \mathscr{K}_+, G_-, G_+)$ related to the A_0 is given by $\mathscr{K}_- = G_- H^1((a,b);\mathbb{R})$ and $\mathscr{K}_+ = G_+ H^1((0,1);\mathbb{R})$, where:

$$G_+ u = u(1) \quad \text{and} \quad G_- u = u(0) \,.$$

By Theorem 5.3.4, a realisation A such that $A_0 \subseteq A \subseteq A_1$ is $m-$dissipative if and only if there exists a linear contraction $\Phi : \mathscr{K}_- \to \mathscr{K}_+$ such that

$$\operatorname{dom} A = \{u \in H^1((0,1),\mathbb{R}) : \Phi u(0) = u(1)\} \,.$$

Note that $\mathscr{K}_-, \mathscr{K}_+ = \mathbb{R}$ and so, Φ can be characterised as a multiplicative operator (by constant).

Our results are related to those bijective realisations of the abstract Friedrichs operators (analogously, skew-symmetric operators) which correspond to the signed boundary maps (see Theorem 5.2.2 and Corollary 5.2.4). There are bijective realisations without signed boundary maps, which are also classified in Theorem 3.4.7. For sure, if these realisations produce a generator of a C_0-semigroup, then it will not be contractive (see theorem 5.1.10 and 5.2.2), still it will be interesting to investigate the semigroup properties of these realisations in future. Here we explain by an example that it might happen that at least some of these realisations do not give rise to generators of a C_0-semigroups.

Remark 5.3.8. Consider the first derivative operator $L_0 u = u'$ on $L^2((0,1);\mathbb{R})$ with $\operatorname{dom} L_0 = C_c^\infty((0,1);\mathbb{R})$. The minimal space is $\mathscr{W}_0 = H_0^1((0,1);\mathbb{R})$ and the graph space is $\mathscr{W} = H^1((0,1);\mathbb{R})$, while the boundary map is given by (see also Remark 4.1.3)

$$(\forall u, v \in \mathscr{W}) \qquad [\, u \mid v\,] = u(1)v(1) - u(0)v(0) \,.$$

The operator $T_0 = L_0 + \mathbb{1}$ is an abstract Friedrichs operator (see Example 3.3.4) and the realisations T with $T_0 \subseteq T \subseteq T_1$ such that

$$\operatorname{dom} T = \left\{ u \in \mathscr{W} : u(1) = \alpha u(0) \right\}, \quad \alpha \in \mathbb{R},$$

are bijective realisations (except for one $\alpha \in (-1,1)$, see Example 3.3.4). By Theorem 5.2.2, the search of $m-$accretive realisations of T_0 and L_0 are equivalent. The bijective realisations corresponding to $\alpha \notin (-1,1)$ are with signed boundary maps and hence, they are $m-$accretive (see Theorem 5.2.2).

Since it is equivalent to study the operator $\widetilde{T}_0 = -L_0 + \mathbb{1}$, instead of T_0 and it is also compatible with the analysis of Example 5.3.7, for $A_0 = -L_0$, we focus on the realisations $\widetilde{T}$ with $\widetilde{T}_0 \subseteq \widetilde{T} \subseteq \widetilde{T}_1$ such that

$$\operatorname{dom} \widetilde{T} = \left\{ u \in \mathscr{W} : \beta u(1) = u(0) \right\}, \quad \alpha \in \mathbb{R}.$$

A direct inspection on the resolvent operator corresposding to the operator L with $\operatorname{dom} L = \left\{ u \in \mathscr{W} : u(0) = 0 \right\}$ (i.e. $\beta = 0$), confirms that L does not generate a C_0-semigroup, although $\widetilde{T}$ for $\beta = 0$ is a bijective realisation (see Example 3.3.4). We shall use the (general) Hille-Yosida Theorem for the same (see [50, Chapter I, Theorem 5.2]).

The resolvent operator can be characterised as follows. For any $f \in L^2((0,1);\mathbb{R})$,

$$\mathbf{R}(\lambda, L) f(x) = -\int_0^x e^{\lambda(x-y)} f(y) dy \, .$$

For constant function $f \equiv 1$, the resolvent becomes

$$\mathbf{R}(\lambda, L) f(x) = -\lambda^{-1} (e^{\lambda x} - 1) \, .$$

Thus, we have

$$\|\mathbf{R}(\lambda, L)\|^2 \geq \|\mathbf{R}(\lambda, L) f\|^2 = \lambda^{-2} (e^{2\lambda} - 4e^{\lambda} - 5) \, ,$$

implying that it is not $O(\lambda^{-1})$ (here, O is the big-O notation). Hence, by the Hille-Yosida Theorem, L and $-\widetilde{T}$ (see Theorem 5.2.2) with $\operatorname{dom} \widetilde{T} = \{ u \in H^1((0,1);\mathbb{R}) : u(0) = 0 \}$ do not generate a C_0-semigroup.

5.4. Stationary diffusion equation

Let us consider T_0 to be the abstract Firedrichs operator given by the stationary diffusion equation defined in sections 1.2.2 and 2.6.2. Here we are working in the real setting, but one can easily generalised it to complex setting. The closed skew-symmetric part is given by

$$L_0 \begin{bmatrix} \mathbf{p} \\ u \end{bmatrix} = \begin{bmatrix} \nabla u \\ \operatorname{div} \mathbf{p} \end{bmatrix},$$

for all $\begin{bmatrix} \mathbf{p} \\ u \end{bmatrix} \in L^2_{\mathrm{div},0}(\Omega) \times H^1_0(\Omega)$. Then $A := (-A_0)^*$ is given by

$$\left(\forall \begin{bmatrix} \mathbf{p} \\ u \end{bmatrix} \in \mathscr{W} = L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d) \times H^1(\Omega;\mathbb{R})\right) \qquad A \begin{bmatrix} \mathbf{p} \\ u \end{bmatrix} = - \begin{bmatrix} \nabla u \\ \operatorname{div} \mathbf{p} \end{bmatrix}, \tag{5.12}$$

The boundary map is given by (see (2.11))

$$\left(\forall \begin{bmatrix} \mathbf{p} \\ u \end{bmatrix}, \begin{bmatrix} \mathbf{q} \\ v \end{bmatrix} \in \mathscr{W}\right) \qquad {}_{\mathscr{W}'}\langle D\mathsf{u}, \mathsf{v} \rangle_{\mathscr{W}} = {}_{-\frac{1}{2}}\langle \mathbf{T}_{\boldsymbol{\nu}} \mathbf{p}, \mathbf{T}_0 v \rangle_{\frac{1}{2}} + {}_{-\frac{1}{2}}\langle \mathbf{T}_{\boldsymbol{\nu}} \mathbf{q}, \mathbf{T}_0 u \rangle_{\frac{1}{2}}, \tag{5.13}$$

which in terms of A can be written as

$$(\forall \mathsf{u}, \mathsf{v} \in \mathscr{W}) \qquad [\, \mathsf{u} \mid \mathsf{v} \,] = -(\langle A\mathsf{u} \mid \mathsf{v} \rangle + \langle \mathsf{u} \mid A\mathsf{v} \rangle) .$$

Let $\mathbf{p} \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)$. Then

$$\int_\Omega \operatorname{div} \mathbf{p} v + \int_\Omega \mathbf{p} \cdot \nabla v = 0,$$

for all $v \in H^1_0(\Omega;\mathbb{R})$. Thus there exists a unique functional $\mathbf{T}_{\boldsymbol{\nu}} \mathbf{p} \in H^{-1/2}(\Gamma) = H^{1/2}(\Gamma)'$ defined by

$${}_{-\frac{1}{2}}\langle \mathbf{T}_{\boldsymbol{\nu}} \mathbf{p}, \mathbf{T} v \rangle_{\frac{1}{2}} = \int_\Omega \operatorname{div} \mathbf{p} v + \int_\Omega \mathbf{p} \cdot \nabla v, \qquad v \in H^1(\Omega;\mathbb{R}) .$$

If $u \in H^1(\Omega : \mathbb{R})$ such that $\Delta u \in L^2(\Omega;\mathbb{R})$, then $\nabla u \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)$ and we let $\partial_{\boldsymbol{\nu}} u := \mathbf{T}_{\boldsymbol{\nu}} \nabla u \in H^{-1/2}(\Gamma)$. Thus

$${}_{-\frac{1}{2}}\langle \partial_{\boldsymbol{\nu}} u, \mathbf{T}_0 v \rangle_{\frac{1}{2}} = \int_\Omega \Delta u v + \int_\Omega \nabla u \nabla v$$

for all $v \in H^1(\Omega;\mathbb{R})$.

Denote by $R : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ the Riesz isomorphism defined by

$$_{-\frac{1}{2}}\langle \varphi, \mathbf{T}_0 v\rangle_{\frac{1}{2}} = \langle R\varphi \mid \mathbf{T}_0 v\rangle_{1/2}\,,$$

for all $\varphi \in H^{-1/2}(\Gamma)$, $v \in H^1(\Gamma)$. It is well known that the linear mapping

$$u \in L^2_{\mathrm{div}}(\Omega) \mapsto R(\mathbf{T}_{\boldsymbol{\nu}} u) \in H^{1/2}(\Gamma) \tag{5.14}$$

is continuous and surjective.

Now let us construct a boundary quadruple.

Theorem 5.4.1. Let $\mathscr{K}_- = \mathscr{K}_+ := H^{1/2}(\Gamma)$ and define

$$G_-, G_+ : \mathscr{W} = L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d) \times H^1(\Omega;\mathbb{R}) \to H^{1/2}(\Gamma)$$

by

$$G_+\mathsf{u} \;=\; -\frac{1}{2}\mathbf{T}_0 u + R(\mathbf{T}_{\boldsymbol{\nu}}\mathbf{p}) \quad \text{and} \quad G_-\mathsf{u} \;=\; -\frac{1}{2}\mathbf{T}_0 u - R(\mathbf{T}_{\boldsymbol{\nu}}\mathbf{p})\,,$$

where $\mathsf{u} = \begin{bmatrix}\mathbf{p}\\ u\end{bmatrix}$. Then $(\mathscr{K}_-, \mathscr{K}_+, G_-, G_+)$ is a boundary quadruple for A_0.

Proof. For $\mathsf{u},\mathsf{v} \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d) \times H^1(\Omega;\mathbb{R})$, we have

$$\begin{aligned}
\langle A\mathsf{u} \mid \mathsf{v}\rangle + \langle \mathsf{u} \mid A\mathsf{v}\rangle &= -[\mathsf{u} \mid \mathsf{v}] \\
&= -\,_{-\frac{1}{2}}\langle \mathbf{T}_{\boldsymbol{\nu}}\mathbf{p}, \mathbf{T}_0 v\rangle_{\frac{1}{2}} - \,_{-\frac{1}{2}}\langle \mathbf{T}_{\boldsymbol{\nu}}\mathbf{q}, \mathbf{T}_0 u\rangle_{\frac{1}{2}} \\
&= -\langle R(\mathbf{T}_{\boldsymbol{\nu}}\mathbf{p}) \mid \mathbf{T}_0 v\rangle_{1/2} - \langle R(\mathbf{T}_{\boldsymbol{\nu}}\mathbf{q}) \mid \mathbf{T}_0 u\rangle_{1/2} \\
&= \langle G_+ u \mid G_+ v\rangle_{1/2} - \langle G_- u \mid G_- v\rangle_{1/2}\,.
\end{aligned}$$

It remains to show the surjectivity. Let $h_-, h_+ \in H^{1/2}(\Gamma)$. Using (5.14) we find $u \in H^1(\Omega : \mathbb{R})$ such that $\mathbf{T}_0 u = -(h_- + h_+)$, and $\mathbf{p} \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)$ such that $R(\mathbf{T}_{\boldsymbol{\nu}}\mathbf{p}) = \frac{1}{2}(h_+ - h_-)$. Thus

$$G_+(\mathbf{p}, u) = -\frac{1}{2}\mathbf{T}_0 u + R(\mathbf{T}_{\boldsymbol{\nu}}\mathbf{p}) = \frac{1}{2}(h_+ + h_-) + \frac{1}{2}(h_+ - h_-) = h_+$$

and

$$G_-(\mathbf{p}, u) = -\frac{1}{2}\mathbf{T}_0 u - R(\mathbf{T}_{\boldsymbol{\nu}}\mathbf{p}) = \frac{1}{2}(h_+ + h_-) - \frac{1}{2}(h_+ - h_-) = h_-\,.$$

The surjectivity of $(G_+, G_-) : \mathscr{W} \times \mathscr{W} \to H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$ follows from Lemma 5.3.3. Hence, $(\mathscr{K}_-, \mathscr{K}_+, G_-, G_+)$ is a boundary quadruple. ■

We apply Theorem 5.3.4 on the boundary quadruple constructed in Theorem 5.4.1 to classify all $m-$dissipative realisations of A_0.

Now that we have the classification of all $m-$dissipative realisations of the skew-symmetric operator $A_0 = -L_0$, we can formulate the main result of this section.

Corollary 5.4.2. The following assertions are equivalent:

(i) A is $m-$dissipative;

(ii) There exists a contraction $\Phi \in \mathscr{L}(H^{1/2}(\Gamma))$ such that $A = A_\Phi$;

(iii) $T_\Phi = T_1|_{\mathrm{dom}A}$ is a bijective realisation with signed boundary map, where

$$\mathrm{dom}A \ := \left\{ \begin{bmatrix} \mathbf{p} \\ u \end{bmatrix} \in \mathrm{dom}A_1 : \ \Phi(-\frac{1}{2}\mathbf{T}_0 u - R(\mathbf{T}_{\boldsymbol{\nu}}\mathbf{p})) \ = \ -\frac{1}{2}\mathbf{T}_0 u + R(\mathbf{T}_{\boldsymbol{\nu}}\mathbf{p}) \right\}.$$

CONCLUSION

In this dissertation we developed the von-Neumann's extension theory for a joint pair of abstract Friedrichs operators which provides an alternate way to classify all boundary conditions of interest, for a pair of abstract Friedrichs operators. This method differs significantly from the universal extension theory. Both methods have their own advantages. The development of von-Neumann's theory for abstract Friedrichs operators relies on the fact that the abstract Friedrichs operators can be characterised as a sum of a skew-symmetric operator and a bounded selfadjoint operator with strictly positive bottom. This characterisation works both ways, thus allowing us to connect the theory of abstract Friedrichs operators to the theory of skew-symmetric and selfadjoint operators and vice-versa. We demonstrate this fact by applying our theory on symmetric operators.

The key was to obtain a decomposition of the graph space (maximal domain) of an abstract Friedrichs operator in terms of the minimal domain and the kernels of the adjoint operators. This is similar to the von-Neumann type decomposition for symmetric operators. The decomposition reveals that the room for choosing the boundary conditions depends on the study of kernels of the adjoint operators. We were able to achieve the classification of all the realisations of interest in terms of bounded operators acting between the kernels of the adjoint operators. We made further classifications depending on the norms of these operators. For example, the boundary conditions corresponding to the contractive (non-expansive) maps satisfy the (V)-conditions of the cone-formalism. Moreover, the unitary maps are important from the perspective of skew-selfadjoint realisations of the skew-symmetric part of the abstract Friedrichs operators. These special types of boundary conditions are useful for the non-stationary theory of abstract Friedrichs operators. In fact, we proved that the latter boundary conditions correspond to the m-accretive realisations of the abstract Friedrichs operators, thus the negative operator generates a contractive

C_0-semigroup of contractions. Moreover, the corresponding skew-symmetric parts of the corresponding realisations (see the characterisation in Section 3.3) also give rise to the generators of contractive C_0-semigroups. The bijective realisations with signed boundary maps related to a pair of abstract Friedrichs operators are the only realisations which give rise to the generators of contractive C_0-semigroups, i.e. we proved a necessary and sufficient condition for this part. The operators satisfying (V)-boundary conditions seem to have the properties that lead to the construction of the boundary triplets for dual pairs [44, Chapter 13.4]. The abstract Friedrichs operators are bounded perturbations of skew-symmetric operators, so the theory of skew-symmetric operators can be adapted in our case to some extent. The semigroup theory studied in [10] using the boundary quadruple approach is discussed and elaborated on some examples in Chapter 5. It will be interesting to investigate the theory further using boundary quadruple/triplet approach.

We also connected (M)-boundary conditions to (V)-boundary conditions with an explicit construction of M-operators. More precisely, the M-operators can be constructed using the boundary operator and the non-orthogonal projectors corresponding to the spaces satisfying (V)-conditions. The construction of these operators is non-unique; we also argued about the multiplicity of such constructions.

We studied the Friedrichs systems on an interval, both in the scalar and vectorial case in full generality. In the scalar case, we used the universal extension theory approach to write all the boundary conditions that give the bijective realisations of interest. There is detailed analysis on kernels depending on the coefficient function evaluated at the end-points. The challenging part was the singularity of the coefficient matrices, which became even more challenging in the vectorial case. We proved a smoothness result in the scalar case, but in the vectorial case, the argument is completed using the theory of total projections. These smoothness results allowed us to define the boundary operator and the minimal space explicitly. Our main result in this direction is connecting the dimensions of kernels to the eigenvalues of the coefficient matrix at the end-points of the interval.

We investigated the semigroup theory of abstract Friedrichs operators and proved that all bijective realisations with signed boundary maps are the only realisations which give rise to the generators of C_0-semigroups and vice-versa. The result is also connected to the corresponding realisations of the skew-symmetric operators. We illustrate the results on stationary diffusion equation.

APPENDIX

In the Hilbert operator theoretic framework of abstract Friedrichs operators, the theories of indefinite inner products and the Kreĭn spaces play an important role. The boundary operator (2.3) gives an indefinite inner product on the graph space (see Lemma 2.2.5), while the existence of subspaces in the cone formalism is a consequence of the canonical decomposition of a Kreĭn space (see Remark 2.2.7 and Proposition 2.3.7). In this section, we briefly recall this theory that we use directly in chapters 2 and 3. For more details on this topic we refer to [11] and [14], and also [3] for a systematic overview of the material.

Kreĭn spaces

Definition 5.4.3. Let $\mathscr{W}$ is a complex vector space equipped with a sesquilinear functional $[\cdot \mid \cdot] : \mathscr{W} \times \mathscr{W} \to \mathbb{C}$ i.e. it satisfies the following.

$$(\forall \lambda_1, \lambda_1 \in \mathbb{C})(\forall u_1, u_2, v \in \mathscr{W}) \quad [\lambda_1 u_1 + \lambda_2 u_2 \mid v] = \lambda_1 [u_1 \mid v] + \lambda_2 [u_2 \mid v]$$
$$(\forall u, v \in \mathscr{W}) \quad [u \mid v] = \overline{[v \mid u]}\,.$$

Such a functional is called *indefinite inner product* on $\mathscr{W}$.

From the second condition for any $u \in \mathscr{W}$, $[u \mid u]$ is a real number and that any element u of $\mathscr{W}$ has three possibilities with respect to the indefinite inner product, $[u \mid u] > 0$, $[u \mid u] < 0$ or $[u \mid u] = 0$. Accordingly u is called a positive, negative or neutral vector in $\mathscr{W}$ with respect to $[\cdot \mid \cdot]$. Every indefinite inner product space is also an indefinite inner product space, we shall call them trivial, so by a non-trivial indefinite inner product we refer to the ones which contain at least one positive and one negative vectors.

Lemma 5.4.4. [14, Chapter 1, Corollary 2.7] If $\mathscr{W}$ is a non-trivial indefinite inner product space, then none of the positive, negative, non-positive or non-negative sets is a vector subspace of $\mathscr{W}$.

Lemma 5.4.5. [14, Chapter 1, Lemma 2.1] Every non-trivial indefinite inner product contains non-zero neutral vectors.

Definition 5.4.6 (Orthogonality)**.** Vectors $u, v \in \mathscr{W}$ are $[\cdot \mid \cdot]-$orthogonal if $[u \mid v] = 0$ and we denote it as $u[\perp]v$. Two subsets $V_1, V_2 \subseteq \mathscr{W}$ are $[\cdot \mid \cdot]-$orthogonal if for any $v_1 \in V_1$ and $v_2 \in V_2$, $v_1[\perp]v_2$. $[\cdot \mid \cdot]-$orthogonal complement of a subset $V \subseteq \mathscr{W}$ is defined as

$$V^{[\perp]} = \{u \in \mathscr{W} : (\forall v \in V)\, [u \mid v] = 0\} .$$

Using the first condition in (5.4.3), $V^{[\perp]}$ is vector subspace of $\mathscr{W}$. Moreover, if $V_1 \subseteq V_2 \subseteq \mathscr{W}$ then $V_2^{[\perp]} \subseteq V_1^{[\perp]}$.

Definition 5.4.7. [Isotropic vectors] Let $\mathscr{V}$ be a subspace of $\mathscr{W}$. A vector $v \in \mathscr{V}$ is called *isotropic* in $\mathscr{V}$ if $v \in \mathscr{V}^{[\perp]}$. In particular, an isotropic vector is a neutral vector. The collection of all isotropic vectors of $\mathscr{V}$ is $\mathscr{V} \cap \mathscr{V}^{[\perp]}$.

If $\mathscr{V} \cap \mathscr{V}^{[\perp]} = \{0\}$, then V is called *non-degenerate*, otherwise it is called *degenerate*.

Lemma 5.4.8. The quotient space $\widehat{\mathscr{W}} = \mathscr{W} / (\mathscr{V} \cap \mathscr{V}^{[\perp]})$ is an indefinite inner product space and the corresponding inner product on $\widehat{\mathscr{W}}$ is defined as

$$[\hat{u} \mid \hat{v}]_{\widehat{\mathscr{W}}} := [u \mid v],$$

where $\hat{u} = u + (\mathscr{V} \cap \mathscr{V}^{[\perp]})$, $\hat{v} = v + (\mathscr{V} \cap \mathscr{V}^{[\perp]})$ and $u, v \in \mathscr{W}$ are the representatives of u, v in $\mathscr{W}$ correspondingly. Moreover, $\widehat{\mathscr{W}}$ with corresponding indefinite inner product is a non-degenerate space.

A subspaces $\mathscr{V} \subseteq \mathscr{W}$ is maximal positive if $\mathscr{V}$ is positive and there is no other positive subspace $\mathscr{V}_1$ such that $\mathscr{V} \neq \mathscr{V}_1$ and $\mathscr{V} \subseteq \mathscr{V}_1$. Similarly, we define maximal negative, maximal neutral, maximal non-positive and maximal non-negative subspaces. Moreover, we call maximal positive or maximal negative subspaces as maximal definite and maximal non-negative or maximal non-positive subspaces as maximal semi-definite.

Lemma 5.4.9. [11, p. 7] Each maximal semi-definite subspace of $\mathscr{W}$ contains all the isotropic vectors.

Lemma 5.4.10. [14, p. 13] The orthogonal complement of each maximal non-negative subspace with respect to $[\cdot \mid \cdot]$ is a maximal non-positive subspace and vice-versa.

If $\mathscr{V}_1, \mathscr{V}_2$ are two subspaces of $\mathscr{W}$ which are mutually $[\cdot \mid \cdot]-$orthogonal and such that $\mathscr{V}_1 \cap \mathscr{V}_2 = \{0\}$, then we denote their direct sum as $\mathscr{V}_1[\dot{+}]\mathscr{V}_2$. The decomposition $\mathscr{W} = \mathscr{W}^+[\dot{+}]\mathscr{W}^-$ is called a *canonical decomposition* of $\mathscr{W}$ if $\mathscr{W}^+$ is a positive and $\mathscr{W}^-$ a negative subspace. Moreover, if $\mathscr{W}$ is a direct sum of a positive and a negative subspace then it is non-degenerate.

Definition 5.4.11 (Keriň space)**.** An indefinite inner product space $(\mathscr{W}, [\cdot \mid \cdot])$ that admits a canonical decomposition $\mathscr{W} = \mathscr{W}^+[\dot{+}]\mathscr{W}^-$ such that $(\mathscr{W}^+, [\cdot \mid \cdot])$ and $(\mathscr{W}^-, -[\cdot \mid \cdot])$ are Hilbert spaces, is called a *Kreĭn space*.

Lemma 5.4.12. [9, Lemma 15] Let $\mathscr{W} = \mathscr{W}^+[\dot{+}]\mathscr{W}^-$ be a canonical decomposition of a given Kreĭn space $\mathscr{W}$. If both $\mathscr{W}^+, \mathscr{W}^- \neq \{0\}$, then there are uncountably many distinct canonical decompositions, whereas if $\mathscr{W}^+ = \{0\}$ or $\mathscr{W}^- = \{0\}$, then there is only one canonical decomposition.

Definition 5.4.13 (Gramm operator)**.** Let $(\mathscr{W}, \langle \cdot \mid \cdot \rangle)$ be a Hilbert space. A linear operator $G : \mathscr{W} \to \mathscr{W}'$ is called a *Gramm operator* if it is bounded and symmetric (i.e. $G = G^*$), and the corresponding indefinite inner product on $\mathscr{W}$ is defined by:

$$(\forall u, v \in \mathscr{W}) \quad [u \mid v] := {}_{\mathscr{W}'}\langle Gu, v \rangle_{\mathscr{W}} \, .$$

The isotropic part $\mathscr{W} \cap \mathscr{W}^{[\perp]}$ of $\mathscr{W}$ (with respect to $[\cdot \mid \cdot]$) is equal to $\ker G$.

It is a well known fact that the indefinite inner product on any Kreĭn space can be expressed by a Gramm operator in some Hilbert scalar product on that space. As this topology does not depend on the choice of the Gramm operator and the Hilbert scalar product, it is usually considered as the standard topology on the Kreĭn space.

Theorem 5.4.14. [11, p. 40] Let G be a Gramm operator on a Hilbert space $(\mathscr{W}, \langle \cdot \mid \cdot \rangle)$ and $(\mathscr{W}, [\cdot \mid \cdot])$ is the corresponding indefinite inner product space. The quotient space $\widehat{W} = \mathscr{W} / \ker G$ is a Kreĭn space if and only if $\operatorname{ran} G$ is closed.

Theorem 5.4.15. [14, p. 106] Let $\overline{\mathscr{V}}$ denotes the closure of $\mathscr{V}$ in $\mathscr{W}$.

(i) If $\mathscr{V}$ is a non-negative (non-positive) subspace of a Kreĭn space such that $\mathscr{V}^{[\perp]}$ is non-positive (non-negative), then $\overline{\mathscr{V}}$ is maximal non-negative (non-positive).

(ii) Each maximal semi-definite subspace of a Kreĭn space is closed.

Theorem 5.4.16. [14, p. 69, 101–102] A subspace $\mathscr{V}$ of a Kreĭn space is closed if and only if $\mathscr{V} = \mathscr{V}^{[\perp][\perp]}$.

Theorem 5.4.17. [11, p. 44] A subspace $\mathscr{V}$ of a Kreĭn space $\mathscr{W}$ is non-degenerate (i.e. $V \cap V^{[\perp]} = \{0\}$) if and only if $\overline{\mathscr{V} + \mathscr{V}^{[\perp]}} = \mathscr{W}$.

Theorem 5.4.18. [14, p. 112] For each maximal non-negative (non-positive) $\mathscr{V}_1$ of a Kreĭn space $\mathscr{W}$ there is a non-positive (non-negative) subspace $\mathscr{V}_2$ such that $\mathscr{V}_1 \dot{+} \mathscr{V}_2 = \mathscr{W}$. One of the possible choices of $\mathscr{V}_2$ is $\mathscr{W}^-(\mathscr{W}^+)$ from the canonical decomposition of $\mathscr{W}$.

For the rest of the section, let $(\mathscr{W}, \langle \cdot \mid \cdot \rangle_{\mathscr{W}})$ be a Hilbert space and let $(\mathscr{W}, [\cdot \mid \cdot])$ be an indefinite inner product space given by a Gramm operator such that $\widehat{\mathscr{W}} = \mathscr{W}/\mathscr{W}_0$ is a Kreĭn space, where $\mathscr{W}_0$ is the kernel the corresponding Gramm operator, i.e. the isotropic part of $\mathscr{W}$.

Lemma 5.4.19. [3, Lemma 7] A subspace $\mathscr{V}$ of $\mathscr{W}$ containing $\mathscr{W}_0$ is closed in $\mathscr{W}$ if and only if $\widehat{\mathscr{V}}$ is closed in the quotient space $\widehat{\mathscr{W}}$.

All the closed subspaces of $\mathscr{W}$ in the graph norm can be characterised as follows.

Lemma 5.4.20. Let $\mathscr{V}$ be a subspace of $\mathscr{W}$ such that $\mathscr{W}_0 \subseteq \mathscr{V} \subseteq \mathscr{W}$. Then $\mathscr{V}$ is closed in $\mathscr{W}$ (with respect to graph norm) if and only if $\mathscr{V} = \mathscr{V}^{[\perp][\perp]}$.

Properties of orthogonality and maximality transfer from $\mathscr{W}$ to $\widehat{\mathscr{W}}$ and vice-versa.

Lemma 5.4.21. [3, Lemma 9] For any subspace $\mathscr{V}$ of $\mathscr{W}$ we have

$$(\widehat{\mathscr{V}})^{\widehat{[\perp]}} = \widehat{\mathscr{V}^{[\perp]}}\,.$$

Lemma 5.4.22. [3, Lemma 10] For any subspace $\mathscr{V}$ of $\mathscr{W}$:

(i) If $\mathscr{V}$ is maximal non-negative (non-positive) in $\mathscr{W}$, then $\widehat{\mathscr{V}}$ is maximal non-negative (non-positive) in $\widehat{\mathscr{W}}$.

(ii) If $\mathscr{W}_0 \subseteq \mathscr{V}$ and $\widehat{\mathscr{V}}$ is maximal non-negative (non-positive) is $\widehat{\mathscr{W}}$, then $\mathscr{V}$ is maximal non-negative (non-positive) in $\mathscr{W}$.

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Curriculum Vitae

Sandeep Kumar Soni was born on January 4, 1995, in Hazaribag, India. After completing his Bachelor's degree in Mathematics from St. Columba's College, Hazaribag, he enrolled in the Master's program in Mathematics at TIFR CAM, Bangalore, in 2017. In December 2019, he commenced his Doctoral program in Mathematics at the Department of Mathematics, University of Zagreb under the supervision of Prof. Nenad Antonić and Assoc. Prof. Marko Erceg.

During his career, he participated in many professional development programs; he attended 12 conferences, summer schools, and workshops, and delivered one invited talk, three presentations, and three posters. He also presented on seven occasions in departmental seminars. He was one of the 21 students selected for the Master's program in Mathematics at TIFR, CAM after successfully competing in a national-level exam (2017) held in two levels. He received a Master's fellowship for the same. He was a part of the scientific project, "Microlocal Defect Tools in Partial Differential Equations", funded by the Croatian Science Foundation (HRZZ) under the leadership of Prof. Nenad Antonić. Currently, he is working as a teaching assistant at the Department of Mathematics, Faculty of Science, University of Zagreb, Croatia.

List of publications

- M. Erceg, S.K. Soni, *Classification of classical Friedrichs differential operators: One-dimensional scalar case,* Commun. Pure Appl. Anal., 2022, 21 (10) : 3499-3527.

  `https://www.aimsciences.org/article/doi/10.3934/cpaa.2022112`

- M. Erceg, S.K. Soni, *The von Neumann extension theory for abstract Friedrichs operators,* `https://arxiv.org/abs/2312.09618`

- M. Erceg, S.K. Soni, *Friedrichs systems on an interval,* `http://arxiv.org/abs/2401.11941`

IZJAVA O IZVORNOSTI RADA

Ja, Sandeep Kumar Soni, doktorand na Matematičkom odsjeku Prirodoslovno–matematičkog fakulteta Sveučilišta u Zagrebu,

- prebivalište: [redacted]
- JMBAG: [redacted]
- matični broj doktoranda: [redacted]
- matični broj znanstvenika: [redacted]

ovim putem izjavljujem pod materijalnom i kaznenom odgovornošću da je moj doktorski rad pod naslovom: Classification of boundary conditions for Friedrichs systems, isključivo moje autorsko djelo (na hrvatskom: Klasifikacija rubnih uvjeta za Friedrichsove sustave), koje je u potpunosti samostalno napisano uz naznaku izvora drugih autora i dokumenata korištenih u radu.

U Zagrebu,

Sandeep Kumar Soni