Thermodynamics and geometry of black holes in the presence of nonlinear electromagnetic fields

Bokulić, Ana

University of Zagreb
Faculty of Science
Department of Physics

Ana Bokulić

Thermodynamics and geometry of black holes in the presence of nonlinear electromagnetic fields

DOCTORAL DISSERTATION

Zagreb, 2024

University of Zagreb
Faculty of Science
Department of Physics

Ana Bokulić

Thermodynamics and geometry of black holes in the presence of nonlinear electromagnetic fields

DOCTORAL DISSERTATION

Supervisor:
assoc. prof. dr. sc. Ivica Smolić

Zagreb, 2024

Sveučilište u Zagrebu
Prirodoslovno-matematički fakultet
Fizički odsjek

Ana Bokulić

Termodinamika i geometrija crnih rupa u prisutnosti nelinearnih elektromagnetskih polja

DOKTORSKI RAD

Mentor:
izv. prof. dr. sc. Ivica Smolić

Zagreb, 2024.

Supervisor information

Ivica Smolić graduated from the University of Zagreb (Faculty of Science, Department of Physics) in 2004 under the supervision of Prof. Silvio Pallua, with diploma thesis *Lovelock gravitation*, and defended PhD thesis *Hawking radiation, W algebras and anomalies* in 2010, under the supervision of Prof. Silvio Pallua and Prof. Loriano Bonora. He worked as a research and teaching assistant at the Department of Physics (Faculty of Science, University of Zagreb) from 2004 to 2010. After returning from postdoc at SISSA (Trieste, Italy), he has worked as an Assistant Professor (2013–2020) and from 2020 as an Associate Professor at the same department. His research is focused on various aspects of black hole physics (thermodynamics, electrodynamics and no-hair theorems) and some problems in mathematical physics. He has experience of teaching various undergraduate courses (e.g. Mathematical methods in physics, Classical Electrodynamics and Differential geometry in physics), as well as one PhD course (Methods of modern mathematical physics). So far, 17 diploma theses have been defended under his supervision.

Acknowledgements

First and foremost, I am deeply grateful to my supervisor, Assoc. Prof. Ivica Smolić, for all the support, guidance and knowledge he passed on to me during my PhD studies. His dedication and enthusiasm for science have always been an inspiration. I would like to thank our esteemed collaborator, Dr. Tajron Jurić, for invaluable discussions and ideas.

A special thank you goes to my office colleagues; this experience would not be the same without our traditional "trač" meetings. To my friends, thank you for being so reassuring and making this journey much less stressful. I also wish to express gratitude to my parents, Mirjana and Tomislav, sister Ema, and Leo for believing in me and providing unwavering support throughout the years. Finally, I would like to thank Petar for always being there for me and encouraging me in this endeavour.

Abstract

Nonlinear electrodynamics (NLE) is an umbrella term for nonlinear modifications of Maxwell's theory. The first NLE theories appeared as quantum corrections to Maxwell's electrodynamics or in order to cure point charge singularities. Our focus is on NLE fields in the context of gravitational theory, their impact on the laws of black hole thermodynamics and the general geometric properties of spacetimes. Using a perturbative approach, we show that the solution for the Schwarzschild black hole placed in an asymptotically homogeneous test magnetic field in the Born–Infeld and Euler–Heisenberg theories receives NLE corrections. We prove that the well-known results from Maxwell's theory, such as no-soliton theorems and the absence of null electromagnetic fields in static spacetimes, still hold in the NLE case but may be circumvented by stealth field configurations. In the form of several no-go theorems, we summarise the general obstructions that limit the possibility of black hole regularisation using NLE fields. Our results suggest that physically plausible NLE theories do not give rise to singularity-free solutions. Furthermore, we formulate the sufficient conditions that imply the isometry-compatible block-diagonal form of the metric for theories consisting of solely NLE fields and NLE fields combined with scalar fields. Finally, we revisit the laws of black hole thermodynamics with NLE fields and derive the first law by means of the covariant phase space approach. With a special emphasis on the treatment of NLE Lagrangian parameters, we resolve the tension between the generalised Smarr formula and the first law of black hole thermodynamics in a general case.

Keywords: nonlinear electromagnetic fields, black hole electrodynamics, black hole thermodynamics, spacetime singularities, regular black holes

Prošireni sažetak

Ključne riječi: nelinearna elektromagnetska polja, elektrodinamika crnih rupa, termodinamika crnih rupa, prostornovremenski singulariteti, regularne crne rupe

Uvod

Nelinearna elektrodinamika (NLE) pojam je koji obuhvaća raznovrsna poopćenja klasične Maxwellove teorije, a definirana je lagranžijanima koji su glatke funkcije dviju temeljnih elektromagnetskih invarijanti,

$$\mathcal{F} := F_{ab}F^{ab} \quad \text{i} \quad \mathcal{G} := F_{ab}\star F^{ab}. \tag{1}$$

Promatrat ćemo minimalno vezanje NLE lagranžijana $\mathscr{L}(\mathcal{F},\mathcal{G})$ i gravitacijske akcije $\mathscr{L}^{(\mathrm{g})}$, tako da je ukupna 4-forma $\mathbf{L}$ jednaka

$$\mathbf{L} = \frac{1}{16\pi}\left(\mathscr{L}^{(\mathrm{g})} + 4\mathscr{L}(\mathcal{F},\mathcal{G})\right)\boldsymbol{\epsilon}\ . \tag{2}$$

Generalizirane Maxwellove jednadžbe mogu se kompaktno zapisati kao

$$\mathrm{d}\mathbf{F} = 0 \qquad \text{i} \qquad \mathrm{d}{\star}\mathbf{Z} = 4\pi{\star}\mathbf{J}\ , \tag{3}$$

gdje je $\mathbf{J}$ 1-forma električne struje, a $\mathbf{Z}$ je pomoćna 2-forma[1], $\mathbf{Z} := -4\left(\mathscr{L}_\mathcal{F}\mathbf{F} + \mathscr{L}_\mathcal{G}\star\mathbf{F}\right)$. Budući da je tenzor energije i impulsa oblika

$$T_{ab} = -4\mathscr{L}_\mathcal{F} T_{ab}^{(\mathrm{Max})} + \frac{1}{4}T g_{ab}\ , \tag{4}$$

u NLE slučaju moguće je definirati tzv. prikrivena polja (eng. *stealth*) [171] za koja vrijedi $T_{ab} = 0$, ali $F_{ab} \neq 0$. S obzirom na vektorsko polje X^a, možemo definirati električnu i magnetsku 1-formu $\mathbf{E} = -i_X\mathbf{F}$ i $\mathbf{B} = i_X{\star}\mathbf{F}$ te "nelinearne" 1-forme $\mathbf{D} = -i_X\mathbf{Z}$ i $\mathbf{H} = i_X{\star}\mathbf{Z}$. Električni i magnetski naboji definirani su pomoću

[1] Notacija: $\mathscr{L}_X = \partial_X\mathscr{L}$, $\mathscr{L}_{XY} = \partial_Y\partial_X\mathscr{L}$, itd.

Komarovih integrala [96] izvrijednjenih na kompaktnoj, zatvorenoj 2-plohi $\mathcal{S}$,

$$Q_{\mathcal{S}} := \frac{1}{4\pi} \oint_{\mathcal{S}} \star\mathbf{Z} \qquad \text{and} \qquad P_{\mathcal{S}} := \frac{1}{4\pi} \oint_{\mathcal{S}} \mathbf{F} \ . \tag{5}$$

Prve NLE teorije, Euler–Heisenbergova [86] i Born–Infeldova [20, 21], pojavile su se 1930-ih godina u ranim fazama razvoja kvantne teorije polja. U narednim desetljećima konstruirana su brojna poopćenja Maxwellove elektrodinamike, često motivirana traženjem novih rješenja vezanih Einstein-NLE jednadžbi. Osim u kontekstu gravitacijskih teorija, NLE polja relevantna su i u kozmologiji, gdje se pojavljuju kao mogući mehanizam regularizacije početnog singulariteta [48, 31, 65] te objašnjenje ubrzanog širenja svemira [135, 139]. Kompaktni astrofizikalni objekti kao što su magnetari, čija magnetska polja dosežu jačinu od $10^{11}T$ [184], mogu predstavljati pogodno okruženje za testiranje nelinearnih efekata.

Euler–Heisenbergov lagranžijan [86] efektivna je teorija koja uzima u obzir kvantne korekcije klasične Maxwellove elektrodinamike na nivou jedne petlje. Kvantni fenomen koji se može opisati u okviru Euler–Heisenbergove teorije je $\gamma\gamma \to \gamma\gamma$ raspršenje, čije je eksperimentalno opažanje nedavno potvrđeno na LHC-u [1]. U limesu slabih polja, Euler–Heisenbergov lagranžijan poprima oblik

$$\mathscr{L}^{(\mathrm{EH})} = -\frac{1}{4}\,\mathcal{F} + \frac{\alpha^2}{360 m_e^4}\left(4\mathcal{F}^2 + 7\mathcal{G}^2\right) + O(\alpha^3) \ . \tag{6}$$

Born–Infeldov lagranžijan [20, 21],

$$\mathscr{L}^{(\mathrm{BI})} = b^2 \left(1 - \sqrt{1 + \frac{\mathcal{F}}{2b^2} - \frac{\mathcal{G}^2}{16 b^4}} \right) , \tag{7}$$

konstruiran je s ciljem regularizacije singulariteta u električnom polju i energiji točkastog naboja. Kasnije se pojavio i kao efektivna teorija u niskoenergijskim limesima bozonskih teorija struna i supersimetričnih teorija [62, 165]. ModMax lagranžijan [8, 61],

$$\mathscr{L}^{(\mathrm{MM})} = \frac{1}{4}\left(-\mathcal{F}\cosh\gamma + \sqrt{\mathcal{F}^2 + \mathcal{G}^2}\,\sinh\gamma \right) \ , \tag{8}$$

čuva originalne simetrije Maxwellove teorije, konformalnu invarijantnost ($T = 0$) te invarijantnost na SO(2) elektromagnetske rotacije.

Naš je cilj proučiti utjecaj NLE polja na različite aspekte gravitacijske teorije, uključujući termodinamiku crnih rupa, mogućnost regularizacije crnih rupa te poopćenje niza problema iz Einstein-Maxwellove teorije.

Schwarzschildova crna rupa u probnom NLE polju

Waldovo rješenje [189] opisuje rotirajuću, osnosimetričnu crnu rupu uronjenu u asimptotski homogeno magnetsko polje. Takav scenarij relevantan je za astrofizičke crne rupe okružene elektromagnetskim poljima akrecijskih diskova ili drugih galaktičkih objekata. Waldovo rješenje temelji se na Papapetrouovom ansatzu [141] po kojem Killingovo vektorsko polje u funkciji baždarnog potencijala zadovoljava Maxwellove jednadžbe u vakuumu. Pripadna elektromagnetska polja ne utječu na metriku pa ih nazivamo probnima. U slučaju nelinearne elektrodinamike, Waldova konstrukcija ne zadovoljava generalizirane Maxwellove jednadžbe. Budući da nismo uspjeli pronaći način koji bi omogućio pronalaženje egzaktnog rješenja, primijenili smo perturbativni razvoj oko originalnog Waldovog rješenja [16]. Euler–Heisenbergova i Born–Infeldova teorija mogu se prikazati u obliku razvoja s obzirom na konstantu vezanja λ kao

$$\mathscr{L}(\mathcal{F},\mathcal{G}) = -\frac{1}{4}\,\mathcal{F} + \lambda\ell(\mathcal{F},\mathcal{G}) + O(\lambda^2)\ . \tag{9}$$

Baždarni potencijal možemo prikazati kao dominantni "Waldov" član i perturbativnu korekciju v^a, $A_a = K_a + \lambda v_a + O(\lambda^2)$. Jednadžba koju perturbativna korekcija mora zadovoljiti (proizlazi iz (3)) je

$$\mathrm{d}{\star}\mathrm{d}\mathbf{v} = 4(\ell_{\mathcal{F}\mathcal{F}}\,\mathrm{d}\mathcal{F})_0 \wedge {\star}\mathrm{d}\mathbf{K} - 4(\ell_{\mathcal{G}\mathcal{G}}\,\mathrm{d}\mathcal{G})_0 \wedge \mathrm{d}\mathbf{K}\ . \tag{10}$$

Izloženi problem riješit ćemo u Schwarzschildovom prostorvremenu. Kao osnovni ansatz, za baždarni potencijal biramo aksijalno Killingovo vektorsko polje, $K^a = m^a = (\partial/\partial\phi)^a$. Naknadno se se može pokazati da taj odabir uistinu odgovara homogenom magnetskom polju. Pozivajući se na simetrije prostorvremena, jednadžbu (10) možemo riješiti ansatzom oblika $\mathbf{v} = h(r,\theta)\mathrm{d}\phi$. Ukupno rješenje dano je s

$$\mathbf{v} = \frac{(\ell_{\mathcal{F}\mathcal{F}})_0}{4}\,B_\infty^3 M\Big(4(2r-5M)\cos(2\theta) + (M-2r)\big(3+\cos(4\theta)\big)\Big)\,\mathrm{d}\varphi\ . \tag{11}$$

Perturbativna korekcija v^a trne dovoljno brzo, tako da ne mijenja asimptotsko ponašanje osnovog ansatza. Također, v^a ne uvodi električni ili magnetski naboj, što se može provjeriti računom Komarovih integrala. Alternativno, problemu možemo pristupiti pomoću skalarnog magnetskog potencijala, definiranog kao $\mathbf{H} = -\mathrm{d}\Upsilon$. U slučaju kada je električno polje odsutno, magnetske 1-forme $\mathbf{B}$ i $\mathbf{H}$ povezane su relacijom $H[k]_a = -4\mathscr{L}_\mathcal{F} B[k]_a$. Maxwellova jednadžba može se zapisati kao

$$\nabla_a \left(\frac{H[k]^a}{N\mathscr{L}_\mathcal{F}} \right) = 0 \ , \tag{12}$$

te ju potom razvijamo do prvog reda u λ. Magnetski skalarni potencijal također prikazujemo u obliku razvoja, $\Upsilon = \Psi_0 + \lambda\Psi_1 + O(\lambda^2)$, gdje Ψ_0 odgovara originalnom Waldovom rješenju. Ponovno, pogodnim odabirom ansatza možemo pronaći rješenje te pokazati da je konzistentno s prijašnjim pristupom.

Primjenjivost aproksimacije testnog polja ovisi o relevantnim skalama promatranog problema. Može se pokazati da u ovom slučaju postoji raspon energija u kojem su polja dovoljno snažna da do izražaja dolaze nelinearni efekti, ali su svejedno dovoljno slaba da metriku možemo smatrati fiksnom.

Umjesto crne rupe, možemo promatrati neutronsku zvijezdu uronjenu u probno NLE polje, uz prikladno postavljene rubne uvjete. Pretpostavljamo idealizirani model sfernosimetrične i savršeno vodljive neutronske zvijezde. Skalarni potencijal mora zadovoljiti Neumannov rubni uvjet, $n^a \nabla_a \Upsilon = 0$, gdje je n^a normala plohe S koja predstavlja rub zvijezde. Međutim, rješenje jednadžbe (12) dano je kao netrivijalna suma funkcija ovisnih o radijalnoj koordinati, što onemogućuje nametanje rubnih uvjeta.

Odsustvo elektromagnetskih solitona

Poznato je da u Einstein-Maxwellovoj teoriji nije moguće konstruirati strogo stacionarno, asimptotski ravno, regularno rješenje, što je rezultat poznat kao "odsustvo elektromagnetskih solitona" [40]. U širem kontekstu, pozivajući se na regularnost i lokaliziranost, solitoni se mogu smatrati primjerima Wheelerovih geona [196]. Dokazi teorema provode se pomoću Lichnerowiczevog argumenta [120]: ako uspijemo konstruirati nenegativnu veličinu čiji je integral na promatranoj domeni nepozitivan, ta veličina mora identički iščezavati. U nastavku predstavljamo dva teorema koja ograničavaju postojanje NLE solitona [15]. Oba teorema oslanjaju se na pogodno odabrane identitete s divergencijama te nekoliko osnovnih tehničkih pretpostavki. Prvi teorem odnosi se na statična prostorvremena te NLE teorije čiji tenzor energije i impulsa zadovoljava svjetlosni energijski uvjet, $\mathscr{L}_\mathcal{F} \le 0$. Vrijedi za proizvoljne gravitacijske teorije dok god divergencija pripadnog gravitacijskog tenzora iščezava, a vezanje elektromagnetske i gravitacijske akcije je minimalno. Ključni identitet pri dokazu teorema

$$\int_\Sigma \frac{\mathscr{L}_\mathcal{F}}{V} \left(E_a E^a + B_a B^a \right) \hat{\boldsymbol{\epsilon}} = 0 \ , \tag{13}$$

proizlazi iz sume dva vektorska identiteta koju integriramo po prostornoj hiperplohi Σ uz pretpostavljeno odgovarajuće asimptotsko ponašanje polja i pripadnih potencijala. Iz (13) možemo zaključiti da mora vrijediti ili $F_{ab} = 0$ ili je polje prikrivenog tipa. Drugi teorem ne zahtijeva statičnost prostorvremena, već samo strogu stacionarnost. Oslanja se na jači, dominantni energijski uvjet ($\mathscr{L}_{\mathcal{F}} \le 0$ i $T \le 0$) te netrivijalni teorem o pozitivnosti mase [197]. Dokaz teorema ponovno slijedi iz nekoliko vektorskih identiteta koji vode na izraz

$$M = -\frac{1}{16\pi} \int_{\Sigma} \nabla_a W^a \star\mathbf{k} + \frac{1}{2} \int_{\Sigma} T \star\mathbf{k} \,. \tag{14}$$

Prvi član ponovo iščezava zbog rubnih uvjeta, dok drugi po energijskom uvjetu mora biti nepozitivan, što je u kontradikciji s teoremom o pozitivnosti mase. Tada zaključujemo da je prostorvrijeme izometrično prosotrvremenu Minkowskog ili je elektromagnetsko polje prikrivenog tipa. Teoreme možemo parcijalno generalizirati na višedimenzionalne slučajeve te uz prisutnost nabijene materije.

Regularizacija crnih rupa pomoću NLE polja

Budući da su poznati primjeri regularnih crnih rupa koje su rješenja Einstein-NLE jednadžbi [9], prirodno se nametnulo pitanje koja su općenita ograničenja na mogućnost regularizacije uz NLE polja. Prvu sistematičnu analizu za NLE lagranžijane koji su funkcija invarijante $\mathcal{F}$ napravio je Bronnikov [27]. Slijedeći njegov pristup, pokušali smo obuhvatiti širu klasu lagranžijana, one koji ovise o obje elektromagnetske invarijante [18]. "No-go" teoremi ove vrste temelje se na pronalaženju barem jednog neomeđenog skalara zakrivljenosti, što automatski čini prostorvrijeme singularnim [56]. Veza između skalara zakrivljenosti i elektromagnetskih invarijanti može se uspostaviti pomoću Einsteinove jednadžbe,

$$R - 4\Lambda = -8\pi T \,, \tag{15}$$

$$R_{ab}R^{ab} + 2\Lambda(2\Lambda - R) = (8\pi)^2\, T_{ab}T^{ab} \,, \tag{16}$$

gdje je $4\pi^2 T^a{}_b T^b{}_a = \pi^2 T^2 + \mathscr{L}_{\mathcal{F}}^2(\mathcal{F}^2 + \mathcal{G}^2)$. Uzastopne kontrakcije većeg broja tenzora energije i impulsa ne daju nove neovisne kombinacije invarijanti. Ako je prostorvrijeme regularno u smislu omeđenih skalara zakrivljenosti, tada isto mora vrijediti i za $T_{ab}T^{ab}$ i T, odnosno za $\mathscr{L}_{\mathcal{F}}\mathcal{F}$, $\mathscr{L}_{\mathcal{F}}\mathcal{G}$ i T. Budući da rezultati ne ovise o asimptotskom ponašanju prostorvremena, zbog općenitosti možemo zadržati kozmološku konstantu Λ. Najsnažnije ograničenje sadržano je u sljedećem teoremu.

Teorem: Pretpostavimo da je prostorvrijeme statično i sfernosimetrično rješenje Einstein-NLE sustava jednadžbi te da promatrana NLE teorija pripada $\mathcal{FG}$-klasi i poštuje Maxwellov limes za slaba polja[2]*. Tada, za električno nabijene crne rupe, Riccijev skalar R i "kvadrirani" Riccijev tenzor $R_{ab}R^{ab}$ ne mogu oboje biti regularni kada $r \to 0$.*

Ako su prisutna oba naboja ili samo magnetski naboj, ne možemo formulirati univerzalni teorem koji bi obuhvatio sve lagranžijane $\mathcal{FG}$-klase koji poštuju Maxwellov limes za slaba polja. Razlog je taj što invarijante $\mathcal{F}$ i $\mathcal{G}$ ne teže istovremeno u nulu kada $r \to 0$, čime gubimo mogućnost provjere Maxwellovog limesa. Međutim, možemo obuhvatiti neke fizikalno relevantne slučajeve, kao što su kvadratični lagranžijan inspiriran Euler–Heisenbergovim, Born–Infeldov ili ModMax. Svi rezultati sumirani su u tablici ispod.

Pregled "no-go" teorema, ✗ označava slučajeve singularnih prostorvremena

	$Q \neq 0,\ P \neq 0$	$Q = 0,\ P \neq 0$
lagranžijani $\mathcal{F}$-klase	✗	
kvadratični lagranžijani $\mathscr{L} = -\frac{1}{4}\mathcal{F} + a\mathcal{F}^2 + b\mathcal{F}\mathcal{G} + c\mathcal{G}^2$	✗	✗
$\mathscr{L} = -\frac{1}{4}\mathcal{F} + h(\mathcal{G})$	✗	✗
$\mathscr{L} = -\frac{1}{4}\mathcal{F} + a\mathcal{F}^s\mathcal{G}^u,\ s,u \geq 1$	✗	
Born-Infeld lagranžijan	✗	✗
ModMax lagranžijan	✗	✗

Uvjeti integrabilnosti

Metrički tenzor može poprimiti blok-dijagonalni oblik koji je kompatibilan s izometrijama promatranog prostorvremena. Potrebni uvjeti sadržani su u Frobeniusovom teoremu [119] te se odnose na integrabilnost dviju distribucija definiranih Killingovim vektorskim poljima. Distribuciju $\mathscr{D}$ čine Killingova vektorska polja $\{K^a_{(1)}, \dots, K^a_{(n)}\}$ te je integrabilna ako vrijedi $[K_{(i)}, K_{(j)}]^a \in \mathscr{D}$ za sve $i, j \in \{1, \dots, n\}$, što nije strogo ograničenje. Distribuciju $\mathscr{D}^\perp$ čine vektorska polja X^a definirana pomoću $K^{(i)}_a X^a = 0$ za svaki $i \in \{1, \dots, n\}$. Ako uvedemo pomoćnu n-formu $\boldsymbol{\alpha} := \mathbf{K}^{(1)} \wedge \dots \wedge \mathbf{K}^{(n)}$, distribucija $\mathscr{D}^\perp$ je integrabilna ako i samo ako vrijedi $\boldsymbol{\alpha} \wedge \mathrm{d}\mathbf{K}^{(i)} = 0$ za svaki $i \in \{1, \dots, n\}$. Dokazivanje integrabilnosti $\mathscr{D}^\perp$ temelji

[2] $\mathscr{L}_\mathcal{F} \to -1/4$ i $\mathscr{L}_\mathcal{G} \to 0$ kada $(\mathcal{F}, \mathcal{G}) \to (0, 0)$.

se na identitetu

$$\mathrm{d}{\star}(\boldsymbol{\alpha} \wedge \mathrm{d}\mathbf{K}^{(i)}) = 2\,{\star}(\boldsymbol{\alpha} \wedge \mathbf{R}(K_{(i)})) = 16\pi{\star}(\boldsymbol{\alpha} \wedge \mathbf{T}(K_{(i)})). \qquad (17)$$

Ako za danu teoriju uspijemo pokazati $\boldsymbol{\alpha} \wedge \mathbf{T}(K_{(i)}) = 0$, uz dodatne pretpostavke slijedi uvjet integrabilnosti, $\boldsymbol{\alpha} \wedge \mathrm{d}\mathbf{K}^{(i)} = 0$. Za NLE polja dokazali smo sljedeće tvrdnje [17]. Pretpostavimo da m-dimenzionalno prostorvrijeme (M, g_{ab}) sadrži $m-2$ međusobno komutirajuća Killingova vektorska polja $\{K^a_{(1)}, \dots, K^a_{(m-2)}\}$ s nepraznim skupom nultočaka $\mathscr{Z} \subseteq M$. Nadalje, pretpostavimo da je definirana elektromagnetska 2-forma F_{ab} koja nasljeđuje simetrije prostorvremena, $\pounds_{K_{(i)}} F_{ab} = 0$ za svaki i. Tada za teorije čiji su lagranžijani dani s

a) $\mathscr{L}(\mathcal{F}, \mathcal{G})\boldsymbol{\epsilon} + \mu\,\mathbf{A} \wedge \mathbf{F}^{(m-1)/2}$,
b) $\mathscr{L}(\mathcal{F}, \mathcal{G}) - (\mathcal{D}_a\phi)^*(\mathcal{D}^a\phi) - \mathscr{U}(\phi^*\phi)$, uz pretpostavke $\pounds_{K_{(i)}}\phi = 0$ i $\boldsymbol{\alpha} \wedge \mathbf{J} = 0$,
c) $f(\phi)\mathscr{L}(\mathcal{F}, \mathcal{G}) - \frac{1}{2}\nabla_a\phi\,\nabla^a\phi - \mathscr{U}(\phi)$, uz pretpostavku $\pounds_{K_{(i)}}\phi = 0$,

vrijedi $\boldsymbol{\alpha} \wedge \mathbf{T}(K_{(i)}) = 0$ za svaki i na bilo kojem otvorenom skupu koji dijeli rub sa skupom nultočaka $\mathscr{Z}$.

Odsustvo elektromagnetskih polja svjetlosnog tipa

U Einstein-Maxwellovoj teoriji, elektromagnetsko polje svjetlosnog tipa[3] ne može postojati u statičnom prostorvremenu [181]. Predstavljamo poopćenje navedenog teorema, u kojem je generalizacija postignuta u nekoliko aspekata [17]. Maxwellovu elektrodinamiku zamijenit ćemo nelinearnim teorijama te promatrati prostorvremena dimenzije različite od četiri. Dimenzionalno poopćenje dolazi s nekoliko tehničkih poteškoća. Naime, invarijanta $\mathcal{G}$ ostaje skalar samo ako je dimenzija prostorvremena jednaka četiri, što ukazuje na nužnu redefiniciju elektromagnetskih polja svjetlosnog tipa. Ekvivalent svjetlosnih elektromagnetskih polja u višedimenzionalnim slučajevima predstavljaju polja N tipa u Petrovljevoj klasifikaciji [172, 137]. Drugi dio općenitosti očituje se u primjenjivosti na širu klasu gravitacijskih teorija, onih u kojima je pripadni gravitacijski tenzor "neparnog tipa". Naime, skalare možemo tvoriti tako da općeniti tenzor ranga k kontrahiramo sa s Killingovih vektora te $k-s$ vektora koji pripadaju distribuciji $\mathscr{D}^{\perp}$. Ako je s neparan te dobiveni skalar iščezava na integrabilnoj domeni, takav tenzor nazivamo neparnim. Uzevši sve navedene pretpostavke u obzir, uspjeli smo dokazati da niti u NLE slučaju statičnost prostorvremena nije kompatibilna sa svjetlosnim elektromagnetskim poljima, osim ako su ona prikrivenog tipa.

[3] U četverodimenzionalnom prostorvremenu za elektromagnetska polja svjetlosnog tipa mora vrijediti $\mathcal{F} = 0 = \mathcal{G}$.

Zakoni termodinamike uz NLE polja

Termodinamika crnih rupa poveznica je između gravitacije u jakim režimima i kvatnih fenomena te pruža uvid u mikroskopsku prirodu prostorvremena. Klasični zakoni termodinamike oblikom su istovjetni zakonima mehanike crnih rupa [10, 12, 13]. Teorijsko otkriće Hawkingovog zračenja [83] potvrdilo je da nije riječ samo o formalnoj analogiji te uspostavilo fizikalnu vezu između veličina koje opisuju crne rupe i termodinamičkih veličina. Po nultom zakonu mehanike, stacionarne crne rupe imaju konstantnu površinsku gravitaciju κ koja igra ulogu temperature. Definirana je preko $\chi^b \nabla^a \chi_b = -\kappa \chi^a$, gdje je χ^a Killingovo vektorsko polje koje generira horizont. Uz njega je blisko vezan nulti zakon elektrodinamike crnih rupa po kojem su električni i magnetski skalarni potencijali konstantni na horizontu crnih rupa. Prvi zakon mehanike iskazuje očuvanje energije te se za nabijene crne rupe u Einstein–Maxwellovoj teoriji može zapisati kao

$$\delta M = \frac{1}{8\pi} \kappa \delta \mathcal{A} + \Omega_H \delta J + \Phi \delta Q + \Psi \delta P \ , \tag{18}$$

gdje su M i $\mathcal{A}$ masa i površina horizonta crne rupe, Ω_H i J angularna brzina horizonta i angularni moment crne rupe te Q i P električni i magnetski naboj crne rupe. Oblik prvog zakona sugerira identifikaciju površine horizonta $\mathcal{A}$ i termodinamičke entropije. Navedenu tvrdnju podupire drugi zakon mehanike crnih rupa po kojem se površina horizonta ne smanjuje u vremenu, $\delta \mathcal{A} \geq 0$. Za stacionarne crne rupe vrijedi i analogon klasične Gibbs–Duhemove relacije, tzv. Smarrova formula,

$$M = \frac{\kappa}{4\pi} A + 2\Omega_H J + \Phi Q + \Psi P \ . \tag{19}$$

Može se izvesti direktnim, geometrijskim pristupom ili iz prvog zakona procesom "skaliranja veličina".

Budući da je analiza za NLE teorije još uvijek nepotpuna, dokazali smo nulti zakon elektrodinamike uz NLE polja na nekoliko komplementarnih načina. Izveli smo prvi zakon termodinamike koristeći matematički rigorozni formalizam kovarijantnog faznog prostora. Za razliku od postojećih rezultata u literaturi koji se odnose na specifične teorije, naša analiza obuhvaća proizvoljni NLE lagranžijan koji je funkcija obje elektromagnetske invarijante. Na kraju smo izveli generaliziranu Smarrovu formulu kako bismo potvrdili konzistentnost cijelog pristupa [14].

Nulti zakon elektrodinamike crnih rupa

Konstantnost električnog i magnetskog skalarnog potencijala na horizontu crnih rupa može se dokazati na nekoliko načina, ovisno o općenitosti koju želimo postići. Jedan pristup oslanja se na Einsteinovu jednadžbu [151], no njegov je nedostatak taj što nije primjenjiv na generalizirane gravitacijske teorije. Najjednostavniji način dokaza temelji se na postojanju bifurkacijske plohe, međutim, postoje primjeri crnih rupa (npr. ekstremalne) kod kojih takva ploha ne postoji. Treći način bazira se na geometrijskom, Frobeniusovom uvjetu integrabilnosti te vrijedi za cirkularna prostorvremena [168, 169, 11]. Za statična prostorvremena dokaz je moguće provesti samo uz dodatne pretpostavke.

Prvi zakon termodinamike crnih rupa

Formalizam kovarijantnog faznog prostora pristup je blizak Hamiltonovoj mehanici u kojem su očuvane veličine sadržane u rubnim članovima [192, 118, 104, 149]. Dinamika sustava povezana je s predsimplektičkom formom koja je jednaka varijaciji Hamiltonijana. Nakon variranja Einstein–Hilbertove i NLE akcije, identifikacije rubnih članova pomoću Komarovih integrala te uzimajući u obzir nulti zakon elektrodinamike, kao konačan oblik prvog zakona dobivamo [14]

$$\delta M = \frac{\kappa}{8\pi}\,\delta\mathcal{A} + \Omega_{\mathsf{H}}\,\delta J + \Phi_{\mathsf{H}}\delta Q + K_\chi^i\,\delta\beta_i\ , \tag{20}$$

gdje je

$$K_\chi^i := -\frac{1}{4\pi}\int_\Sigma \frac{\partial\mathscr{L}}{\partial\beta_i}\star\boldsymbol{\chi}\ , \tag{21}$$

a β_i parametar NLE lagranžijana. Interpretacija novog člana ovisi o promatranoj teoriji. Na primjer, oslanjajući se na dimenzionalnu analizu, za Born–Infeldovu i Euler–Heisenbergovu teoriju veličina K_χ^i može se shvatiti kao vakuumska polarizacija [72]. Konceptualno drugačiji način izvođenja prvog zakona koji smo također primijenili na NLE slučaj svodi se na promatranje fizikalnog procesa u kojem materija upada u crnu rupu [63].

Generalizirana Smarrova formula

Obzirom da u literaturi postoje oprečni rezultati u vezi prisutnosti dodatnog člana u prvom zakonu, provjerit ćemo konzistentnost s generaliziranom Smarrovom formulom. Prvo poopćenje Smarrove formule za NLE polja [71], izvedeno neovisno o prvom zakonu, ukazuje na prisutnost člana s parametrima NLE lagranžijana. Pokazali smo da se isti rezultat može dobiti iz prvog zakona termodinamike (20) postupkom skaliranja. Ako je $(g_{ab}, \mathbf{A})$ početno rješenje Einstein-NLE jednadžbi, možemo pokazati da reskalirana polja $(\lambda^2 g_{ab}, \lambda\mathbf{A})$ zadovoljavaju iste jednadžbe. Iz te informacije možemo odrediti i konzistentna skaliranja svih veličina koje ulaze u Smarrovu formulu. Konačno, za generaliziranu Smarrovu formulu dobivamo [14]

$$M = \frac{\kappa}{4\pi}\,\mathcal{A} + 2\Omega_{\mathsf{H}} J + \Phi_{\mathsf{H}} Q + \sum_i b_i K^i_\chi \beta_i \ , \tag{22}$$

gdje je b_i faktor skaliranja parametra β_i. Drugo pitanje koje se nameće je koje NLE teorije ostavljaju Smarrovu formulu u linearnom obliku. Predloženi nužni uvjet je dan kao

$$\mathscr{L} = a(\mathscr{L}_{\mathcal{F}}\mathcal{F} + \mathscr{L}_{\mathcal{G}}\mathcal{G}) + b\left(2\mathscr{L}_{\mathcal{F}}\mathscr{L}_{\mathcal{G}}\mathcal{F} + (\mathscr{L}_{\mathcal{G}}^2 - \mathscr{L}_{\mathcal{F}}^2)\mathcal{G}\right) + c\mathcal{G} \ . \tag{23}$$

Od primjera iz literature, navedeni uvjet zadovoljava Maxwellova teorija, ali i power-Maxwell [78] ($\mathscr{L}^{(\mathrm{pM})} = C\mathcal{F}^s$) i ModMax lagranžijani.

Zaključak

Koristeći aproksimaciju testnih polja, izračunali smo perturbativnu NLE korekciju za statični slučaj Waldovog rješenja. Sljedeći korak bilo bi poopćenje na rotirajuću, Kerrovu crnu rupu. Međutim, u tom se slučaju kompliciraju izrazi za elektromagnetske invarijante, što posljedično otežava rješavanje jednadžbe koja definira korekciju. Teoremi o nepostojanju elektromagnetskih solitona vrijede i za NLE polja, a mogu ih zaobići jedino prikrivena polja. Njihova moguća generalizacija leži u oslabljivanju početnih pretpostavki. Navedene teoreme nadopunjuju teoremi o nemogućnosti regularizacije crnih rupa pomoću NLE polja. Naši rezultati ukazuju na to da fizikalno realistične NLE teorije ne mogu ukloniti singularitet kod sfernosimetričnih rješenja. Najvažnije otvoreno pitanje je kako teoremima obuhvatiti još općenitije lagranžijane koji ovise o $\mathcal{F}$ i $\mathcal{G}$, a potencijalno i one koji ovise o derivacijama invarijanti. Definirali smo uvjete koji omogućuju blok-dijagonalizaciju metrike u slučaju NLE teorija te NLE teorija kombiniranih sa skalarnim poljima. Teoremi vrijede za Einstein–Hilbertovu akciju pa se kao moguće poopćenje nameće promatranje modificiranih gravitacijskih teorija. Dokazali smo da teorem o nepostojanju elektromagnetskih polja u statičnom prostorvremenu vrijedi i u NLE slučaju, sve dok polja nisu prikrivenog tipa. Izvedeni prvi zakon termodinamike uz NLE polja sadrži dodatni član s varijacijama parametara lagranžijana. Taj je oblik prvog zakona kompatibilan s ranije izvedenom generaliziranom Smarrovom formulom. Nije sasvim jasno ima li proširenje faznog prostora dodatnim parametrima značajniju fizikalnu interpretaciju. Zbog svoje primjenjivosti u raznim granama fizike, NLE polja će i dalje biti važna stavka budućih istraživanja. U tome bi od pomoći mogla biti neka od ovdje predstavljenih otvorenih pitanja.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Although Maxwell's electrodynamics is a successful theory whose predictions have been precisely verified over the decades, it still has to be tested in different energy regimes. Namely, it is expected that nonlinearities in electromagnetic interactions should reveal themselves at high energy scales. Nonlinear electrodynamics (NLE) is a hypernym that encompasses various nonlinear modifications of classical Maxwell's electrodynamics, usually given by a Lagrangian constructed from two electromagnetic invariants, $F_{ab}F^{ab}$ and $F_{ab}\star F^{ab}$. The two earliest NLE theories, Born–Infeld [20, 21] and Euler–Heisenberg [86] electrodynamics, were formulated in the 1930s and ignited interest due to their unique properties. A vast number of various NLE theories have emerged since then, often intertwined with developments in gravitational physics.

Recent measurements at LHC [1] show evidence for light-by-light scattering, a quantum phenomenon that may be explained only within the framework of nonlinear theories. These results enabled posing stronger constraints on the parameters of NLE Lagrangians [133, 57]. Apart from collider experiments, nonlinear electromagnetic interactions are being tested by a new generation of ultraintense lasers at the Extreme Light Infrastructure [179].

Many different factors, such as their implications for theoretical gravitational physics, cosmology and astrophysics, contributed to the growing interest in NLE theories. For example, it is widely believed that in its early phase, the universe was permeated by strong magnetic fields exceeding 10^{11}T. The strength of these fields exceeds the range of applicability of Maxwell's theory, implying that cosmological models should take nonlinear effects into account [65]. Furthermore, observational cosmological data may disclose whether the accelerated expansion of the universe at its different stages may be attributed to NLE theories [25]. The proposed neutrino tests could shed some light on the influence of NLE fields on supernovae dynamics

[130]. Certain compact astrophysical objects could also represent a suitable domain for testing nonlinearities. Magnetars, a special type of neutron stars, harbour strong magnetic fields, reaching up to 10^{11}T at the surface [184]. The role of nonlinear QED effects on neutron star spin-down, the so-called quantum vacuum friction, is still being debated [156]. Supermassive black holes in the centres of galaxies eject powerful jets, whose formation may be caused by surrounding strong electromagnetic fields.

The venue we shall explore is NLE fields' influence on various aspects of gravitational theory. Some NLE models proved to be successful in removing black hole singularities, however, this prospect has been limited by several constraints [27]. The role of NLE Lagrangian parameters in the formulation of the fundamental laws of black hole thermodynamics has not been completely resolved yet. Furthermore, a number of results from Maxwell's theory are still devoid of their NLE counterparts.

The thesis is organised as follows. First, we lay out the fundamentals of nonlinear electrodynamics in general and in the context of gravitational theory, including an overview of prominent NLE theories and known exact gravitational solutions. Additionally, we reexamine the energy conditions for NLE fields and complement the existing results from the literature. The main part of the thesis addresses a number of separate problems related to the geometric and thermodynamic properties of spacetimes coupled to NLE theories:

- We revisit Wald's solution for a black hole immersed in a homogeneous magnetic field and perturbatively find the lowest order NLE correction in a static case. We also discuss highly conducting stars in the same setting.

- We present a generalisation of the canonical no-soliton result from Einstein-Maxwell theory to a wide class of NLE theories. Additionally, we consider theories with charged matter and different number of spacetime dimensions.

- Building on the previous no-go results, we examine the constraints on black hole regularisation using NLE fields. We extend the existing theorems by considering more general NLE Lagrangians.

- Since it is known that the metric may be brought into a block-diagonal form compatible with the isometries of a given spacetime, we formulate the conditions that ensure such splitting for NLE fields themselves and NLE fields combined with scalar and gauge Chern-Simons terms.

- We present a multifold generalisation of the theorem on the absence of null electromagnetic fields.

- We turn to the laws of black hole thermodynamics with nonlinear electromagnetic fields. After a mathematical introduction to covariant phase space formalism, we derive the first law using two approaches. Besides that, we prove the zeroth law in several different ways and discuss the linearity of the generalised Smarr formula.

Notation and conventions. We will use the "mostly plus" metric signature and the natural system of units with $G = c = 4\pi\epsilon_0 = 1$. Spacetime is defined as an ordered pair $(\mathscr{M}, g_{ab})$ that consists of a connected, smooth manifold $\mathscr{M}$ and a smooth Lorentzian metric g_{ab}. We will denote differential forms by boldface letters with omitted indices, abstract index notation or a combination of both. The volume form will be denoted by $\boldsymbol{\epsilon}$. The result of the contraction of a symmetric tensor S_{ab} with a vector X^a is a 1-form $S_{ab}X^b$, represented by $\mathbf{S}(X)$. The commutator between two vector fields, X^a and Y^a is given by the Lie bracket $[X, Y]^a = X^b\nabla_b Y^a - Y^a\nabla_b X^a$. We will use $\approx$ sign for equalities evaluated on-shell. When referring to a set S, we will denote its interior, boundary and closure by S°, ∂S and $\bar{S}$, respectively, while the $-$ sign will stand for the difference between two sets, $A - B$.

Chapter 2

An overview of NLE

2.1 NLE fields coupled to gravity

As a prelude, we introduce all the basic ingredients of a coupled gravitational-NLE system of equations. We start by giving a precise mathematical definition of an NLE Lagrangian and its relation to the gravitational part of the action. Our next task is to derive modified Maxwell's equations from a general NLE Lagrangian and the energy-momentum tensor of the corresponding NLE theory to obtain a full set of equations. Furthermore, we point out new features otherwise absent in Maxwell's theory.

2.1.1 Fundamentals

Using the electromagnetic 2-form $\mathbf{F}$, we can construct two independent quadratic electromagnetic invariants,

$$\mathcal{F} := F_{ab}F^{ab} \quad \text{and} \quad \mathcal{G} := F_{ab}\star F^{ab}. \tag{2.1}$$

It can be shown, using identities (A.23) and (A.24), that any scalar formed by contracting three or more 2-forms $\mathbf{F}$ or $\star\mathbf{F}$ (for example, $F^a{}_b F^b{}_c F^c{}_a$ or $F^a{}_b \star F^b{}_c F^c{}_a$) can be in fact reduced to a function of two basic invariants $\mathcal{F}$ and $\mathcal{G}$. Thus, with the 2-form $\mathbf{F}$ at our disposal, no new independent invariants can be formed unless we include its covariant derivatives.

Maxwell's Lagrangian density is a linear function of invariant $\mathcal{F}$ only, defined as

$$\mathscr{L}^{(\mathrm{Max})} = -\frac{1}{4}\mathcal{F} \ , \tag{2.2}$$

while we assume that the NLE Lagrangian density is a C^2 function of electromagnetic invariants $\mathcal{F}$ and $\mathcal{G}$. Given an NLE Lagrangian density $\mathscr{L}(\mathcal{F},\mathcal{G})$, we will consider minimal coupling to the gravitational sector, defined with some diffeomorphism covariant Lagrangian density $\mathscr{L}^{(\mathrm{g})}$. Then, the total Lagrangian 4-form is equal to

$$\mathbf{L} = \frac{1}{16\pi}\left(\mathscr{L}^{(\mathrm{g})} + 4\mathscr{L}(\mathcal{F},\mathcal{G})\right)\boldsymbol{\epsilon}\ . \tag{2.3}$$

Partial derivatives of the NLE Lagrangian density $\mathscr{L}$ will be denoted by abbreviations such as $\mathscr{L}_\mathcal{F} = \partial_\mathcal{F}\mathscr{L}$, $\mathscr{L}_\mathcal{G} = \partial_\mathcal{G}\mathscr{L}$, $\mathscr{L}_{\mathcal{F}\mathcal{G}} = \partial_\mathcal{G}\partial_\mathcal{F}\mathscr{L}$, and so on. The corresponding gravitational field equation emanating from (2.3) is

$$E_{ab} = 8\pi T_{ab}\ , \tag{2.4}$$

where E_{ab} is some symmetric, divergence-free gravitational tensor, $\nabla^a E_{ba} = 0$. For Einstein–Hilbert action, we have $\mathscr{L}^{(\mathrm{g})} = R$ and $E_{ab} = G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$. On the right hand side of the gravitational equation (2.4) is the NLE energy-momentum tensor, whose form can be obtained by using variational calculus (see Appendix C)[1]

$$T_{ab} = -\frac{1}{4\pi}\left(\left(\mathscr{L}_\mathcal{G}\mathcal{G} - \mathscr{L}\right)g_{ab} + 4\mathscr{L}_\mathcal{F} F_{ac}F_b{}^c\right)\ . \tag{2.5}$$

Generalised Maxwell's equations emerging from an NLE Lagrangian $\mathscr{L}(\mathcal{F},\mathcal{G})$ are

$$\mathrm{d}\mathbf{F} = 0 \qquad \text{and} \qquad \mathrm{d}{\star}\mathbf{Z} = 4\pi{\star}\mathbf{J}\ , \tag{2.6}$$

where $\mathbf{J}$ is the electric current and $\mathbf{Z}$ is the auxiliary 2-form given by

$$\mathbf{Z} := -4\left(\mathscr{L}_\mathcal{F}\mathbf{F} + \mathscr{L}_\mathcal{G}\,{\star}\mathbf{F}\right). \tag{2.7}$$

The first generalised Maxwell's equation is in fact a topological Bianchi's identity, so it remains unaltered by introducing nonlinear electromagnetic fields. The second one is derived by varying the electromagnetic Lagrangian with respect to the gauge potential A_a, as shown in Appendix C.

In four-dimensional spacetime, another way of expressing the NLE energy-momentum

[1] For a Lagrangian 4-form $\mathbf{L} = \varsigma\left(\mathscr{L}^{(g)} + 4\mathscr{L}^{(\mathrm{em})}\right)\boldsymbol{\epsilon}$, where $\varsigma > 0$ is a normalisation factor, the electromagnetic energy-momentum tensor is defined as

$$T^{(\mathrm{em})}_{ab} := -\frac{1}{8\pi\varsigma}\frac{1}{\sqrt{-g}}\frac{\delta S^{(\mathrm{em})}}{\delta g^{ab}}\ , \quad \text{with} \quad S^{(\mathrm{em})} = 4\varsigma\int \mathscr{L}^{(\mathrm{em})}\boldsymbol{\epsilon}\ .$$

Our choice $\varsigma = 1/(16\pi)$ agrees with, for example, [80, 63], whereas $\varsigma = 1$ normalisation is used in [187].

tensor is to separate it into the "Maxwell" part and the trace part

$$T_{ab} = -4\mathscr{L}_{\mathcal{F}}\widetilde{T}_{ab} + \frac{1}{4}\,T g_{ab}\,, \tag{2.8}$$

where $\widetilde{T}_{ab}$ is Maxwell's energy-momentum tensor[2],

$$\widetilde{T}_{ab} := \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}\,g_{ab}\mathcal{F}\right) \tag{2.9}$$

and the trace T is given by

$$T := g^{ab}T_{ab} = \frac{1}{\pi}\left(\mathscr{L} - \mathscr{L}_{\mathcal{F}}\mathcal{F} - \mathscr{L}_{\mathcal{G}}\mathcal{G}\right)\ . \tag{2.10}$$

Also, using the identity (A.24), the NLE energy-momentum tensor can be written via the auxiliary 2-form Z_{ab},

$$T_{ab} = \frac{1}{4\pi}\left(Z_{ac}F_b{}^c + \mathscr{L}g_{ab}\right)\ . \tag{2.11}$$

One novelty compared to classical Maxwell's electrodynamics are stealth fields, whose properties are summarised in the following definition.

Definition 2.1. *We say that an electromagnetic field is stealth at a point p if the electromagnetic field tensor F_{ab} is nonzero, but the corresponding energy-momentum tensor T_{ab} vanishes at a given point.*

Consequently, such fields do not affect the spacetime metric. In the NLE case, fields are stealth if and only if $T = 0$ and $\mathscr{L}_{\mathcal{F}} = 0$. If we suppose that $T = 0$, $\mathscr{L}_{\mathcal{F}} = 0$ and $F_{ab} \neq 0$, it follows immediately from the form of the energy-momentum tensor (2.8) that $T_{ab} = 0$. Conversely, assuming that $T_{ab} = 0$, we immediately have $T = 0$. Then, we are left with $\mathscr{L}_{\mathcal{F}}\widetilde{T}_{ab} = 0$. By theorem B.4, $F_{ab} \neq 0$ implies $\widetilde{T}_{ab} \neq 0$, so the only possibility left is $\mathscr{L}_{\mathcal{F}} = 0$.

With respect to a vector field X^a with the norm $N = X^a X_a$, we may define the electric and magnetic 1-forms,

$$\mathbf{E} = -i_X\mathbf{F} \quad \text{and} \quad \mathbf{B} = i_X{\star}\mathbf{F}\ . \tag{2.12}$$

Then, the electromagnetic tensor $\mathbf{F}$ can be decomposed with respect to the electric and magnetic fields as

$$-N\mathbf{F} = \mathbf{X} \wedge \mathbf{E} + \star(\mathbf{X} \wedge \mathbf{B})\ , \tag{2.13}$$

[2]We will denote Maxwell's energy-momentum tensor either by $T_{ab}^{(\mathrm{Max})}$ or $\widetilde{T}_{ab}$ to keep notation simpler.

which enables us to express the electromagnetic invariants in terms of the fields,

$$\mathcal{F} = \frac{2}{N}(E_a E^a - B_a B^a) \ , \tag{2.14}$$

$$\mathcal{G} = -\frac{4}{N} E^a B_a \ . \tag{2.15}$$

In the nonlinear case, we can introduce the "nonlinear" electric 1-form $\mathbf{D} = -i_X \mathbf{Z}$ and the nonlinear magnetic 1-form $\mathbf{H} = i_X \star \mathbf{Z}$. They are related to $\mathbf{E}$ and $\mathbf{B}$ via

$$\mathbf{D} = -4\left(\mathscr{L}_\mathcal{F}\,\mathbf{E} - \mathscr{L}_\mathcal{G}\,\mathbf{B}\right) \ , \tag{2.16}$$

$$\mathbf{H} = -4\left(\mathscr{L}_\mathcal{F}\,\mathbf{B} + \mathscr{L}_\mathcal{G}\,\mathbf{E}\right) \ . \tag{2.17}$$

It is often convenient to set $X^a = \xi^a$, where ξ^a is the Killing vector field. Furthermore, if we assume that the electromagnetic fields are symmetry inheriting and generalised source-free Maxwell's equations (2.6) hold, the electric form $\mathbf{E}$ and magnetic form $\mathbf{H}$ are closed,

$$\mathrm{d}\mathbf{E} = (-\pounds_\xi + i_\xi \mathrm{d})\mathbf{F} = 0 \ , \tag{2.18}$$

$$\mathrm{d}\mathbf{H} = (\pounds_\xi - i_\xi \mathrm{d})\star\mathbf{Z} = 0 \ . \tag{2.19}$$

Then, on a simply connected domain, we may introduce the corresponding scalar potentials defined by

$$\mathbf{E} = -\mathrm{d}\Phi \quad \text{and} \quad \mathbf{H} = -\mathrm{d}\Psi \ . \tag{2.20}$$

The question of symmetry inheritance for electromagnetic fields is nontrivial, as examples of electrovacuum spacetimes with symmetry noninheriting electromagnetic fields can be found within both Maxwell's theory [127] and NLE generalisations [171]. The Lie derivative of the electromagnetic 2-form $\mathbf{F}$ can be written as a linear combination $a\star\mathbf{F}+b\mathbf{F}$, where $b = 0$ in Maxwell's theory. Thus, symmetry inheritance comes down to the question of finding sufficient assumptions that force the functions a and b to vanish. Maxwell's theory has been extensively analysed in this regard [127, 200, 199, 42, 152, 186, 185, 181, 46], while the study for the NLE case has been conducted in [11].

The electric and magnetic charges are defined by Komar integrals evaluated over a compact closed 2-surface $\mathcal{S}$,

$$Q_\mathcal{S} := \frac{1}{4\pi}\oint_\mathcal{S} \star\mathbf{Z} \qquad \text{and} \qquad P_\mathcal{S} := \frac{1}{4\pi}\oint_\mathcal{S} \mathbf{F} \ . \tag{2.21}$$

Charges evaluated at infinity are denoted by $Q := Q_{S_\infty}$ and $P := P_{S_\infty}$. When considering source-free Maxwell's equations, the choice of the sphere is irrelevant up to possible technical obstacles, for example, finding a regular coordinate system on the black hole horizon. Note that for a globally well-defined gauge potential $\mathbf{A}$, Stokes' theorem implies $P_{\mathcal{S}} = 0$. Therefore, magnetic charge comes as a consequence of a topologically nontrivial electromagnetic field.

2.1.2 Spherically symmetric spacetimes

As the simplest case, we consider static, spherically symmetric spacetime sourced by some general NLE Lagrangian [18]. The line element can be written as [187]

$$\mathrm{d}s^2 = -\alpha(r)\,\mathrm{d}t^2 + \beta(r)\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) , \tag{2.22}$$

whenever $\nabla_a r \neq 0$. For further convenience, we introduce the abbreviation $w = \sqrt{\alpha(r)\beta(r)}$. In our case, the condition $T^t_{\ t} = T^r_{\ r}$ is satisfied, which is sufficient to take $w = 1$, i.e., $\alpha(r) = 1/\beta(r) = f(r)$, at least for Einstein's field equation [105]. Nevertheless, sometimes it is favourable to keep the function $w(r)$ undetermined because of possible generalisations beyond the Einstein's gravitational theory.

The electromagnetic 2-form $\mathbf{F}$ which inherits the spacetime symmetries can be put in the form

$$\begin{aligned}\mathbf{F} &= -E_r(r)\,\mathrm{d}t\wedge\mathrm{d}r - B_r(r)\star(\mathrm{d}t\wedge\mathrm{d}r) = \\ &= -E_r(r)\,\mathrm{d}t\wedge\mathrm{d}r + \frac{B_r(r)}{w(r)}\,r^2\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi\,,\end{aligned} \tag{2.23}$$

while its Hodge dual is given by

$$\star\mathbf{F} = \frac{E_r(r)}{w(r)}\,r^2\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi + B_r(r)\,\mathrm{d}t\wedge\mathrm{d}r\,. \tag{2.24}$$

To make the notation clearer, we introduce the rescaled electric and magnetic 1-forms,

$$\widetilde{E}_a := \frac{E_a}{w}\ , \quad \widetilde{B}_a := \frac{B_a}{w}\,. \tag{2.25}$$

Then, the electromagnetic invariants are equal to

$$\mathcal{F} = 2(\widetilde{B}_r^2 - \widetilde{E}_r^2) \quad \text{and} \quad \mathcal{G} = 4\widetilde{E}_r\widetilde{B}_r\,. \tag{2.26}$$

Generalised source-free Maxwell's equations are

$$\partial_\mu(\sqrt{-g}\,\star F^{\mu\nu}) = 0 \ , \tag{2.27}$$

$$\partial_\mu\Big(\sqrt{-g}\ (\mathscr{L}_{\mathcal{F}}\, F^{\mu\nu} + \mathscr{L}_{\mathcal{G}}\,\star F^{\mu\nu})\Big) = 0 \ . \tag{2.28}$$

The only nontrivial components in the spherically symmetric case are $\nu = t$, which can be integrated to give

$$\widetilde{B}_r = \frac{P}{r^2}\,, \tag{2.29}$$

$$\mathscr{L}_{\mathcal{F}}\widetilde{E}_r - \mathscr{L}_{\mathcal{G}}\widetilde{B}_r = -\frac{Q}{4r^2}\,, \tag{2.30}$$

where the integration constants are fixed by the definition of Komar integrals (2.21). There are two linearly independent components of Einstein's equation, which, for $w = 1$, read

$$(r(f-1))' = 2r^2\Big(\mathscr{L} + 4(\mathscr{L}_{\mathcal{F}}E_r - \mathscr{L}_{\mathcal{G}}B_r)E_r\Big) \ , \tag{2.31}$$

$$(r^2 f')' = 4r^2\Big(\mathscr{L} - 4(\mathscr{L}_{\mathcal{F}}B_r + \mathscr{L}_{\mathcal{G}}E_r)B_r\Big) \ . \tag{2.32}$$

2.2 NLE Lagrangians

We can systematically categorise NLE Lagrangians into two classes: Lagrangians that depend on invariant $\mathcal{F}$ only belong to the $\mathcal{F}$-class, while Lagrangians that depend on both $\mathcal{F}$ and $\mathcal{G}$ invariants are members of the $\mathcal{F}\mathcal{G}$-class. One could argue that physically reasonable NLE theories should behave as Maxwell's when the fields are weak. Formally, we say that the Lagrangian density obeys the *Maxwellian weak field limit* if $\mathscr{L}_{\mathcal{F}} \to -1/4$ and $\mathscr{L}_{\mathcal{G}} \to 0$ as $(\mathcal{F},\mathcal{G}) \to (0,0)$.

Maxwell's theory exhibits two symmetries, invariance with respect to SO(2) electromagnetic duality rotations and conformal invariance in four dimensions. Generalised electrodynamic theories will not necessarily share these properties, implying that the underlying symmetries will exist only if additional constraints are imposed. In the context of nonlinear electrodynamics, SO(2) rotations correspond to

$$\begin{aligned} Z'_{ab} &= Z_{ab}\cos\theta + \star F_{ab}\sin\theta \ , \\ \star F'_{ab} &= \star F_{ab}\cos\theta - Z_{ab}\sin\theta. \end{aligned} \tag{2.33}$$

Such transformations will not necessarily convert generalised Maxwell's equations into each other due to a nonlinear relation between Z_{ab} and F_{ab}. The question of

SO(2) invariance boils down to finding the condition that leaves Z'_{ab} in the same functional form, i.e., $Z'_{ab}(F') = Z_{ab}(F)$. We recapitulate the derivation of the necessary and sufficient condition following the steps presented in [68]. For an NLE Lagrangian defined as a function of $\mathcal{F}$ and $\mathcal{G}$, we have

$$\frac{\partial \mathscr{L}}{\partial F_{ab}} = \mathscr{L}_{\mathcal{F}} \frac{\partial \mathcal{F}}{\partial F_{ab}} + \mathscr{L}_{\mathcal{G}} \frac{\partial \mathcal{G}}{\partial F_{ab}} \ . \tag{2.34}$$

If we consider F_{ab} and F_{ba} as linearly dependent, i.e.,

$$\frac{\partial F_{cd}}{\partial F_{ab}} = \delta_c{}^a \delta_d{}^b - \delta_c{}^b \delta_d{}^a \ , \tag{2.35}$$

we get

$$\frac{\partial \mathcal{F}}{\partial F_{ab}} = g^{ce} g^{df} \frac{\partial (F_{cd} F_{ef})}{\partial F_{ab}} = 4F^{ab}, \tag{2.36}$$

$$\frac{\partial \mathcal{G}}{\partial F_{ab}} = \frac{1}{2} \epsilon^{cdef} \frac{\partial (F_{cd} F_{ef})}{\partial F_{ab}} = 4\star F^{ab}. \tag{2.37}$$

Using these expressions, we can relate (2.34) to Z^{ab},

$$\frac{\partial \mathscr{L}}{\partial F_{ab}} = -Z^{ab}. \tag{2.38}$$

By varying the Eq. (2.38), we arrive at

$$\frac{1}{2} \delta F_{cd} \frac{\partial}{\partial F_{cd}} \left(\frac{\partial \mathscr{L}}{\partial F_{ab}} \right) = -\delta Z^{ab}, \tag{2.39}$$

since $\delta = \mathrm{d}/\mathrm{d}\theta$. Infinitesimal transformations (2.33) are of the form

$$\delta F_{ab} = \star Z_{ab} \ , \tag{2.40}$$

$$\delta Z_{ab} = \star F_{ab} \ . \tag{2.41}$$

Then, Eq. (2.39) becomes

$$\begin{aligned} \star F^{ab} &= -\frac{1}{2} \star Z_{cd} \frac{\partial^2 \mathscr{L}}{\partial F_{cd} \partial F_{ab}} = \\ &= -\frac{1}{4} \epsilon_{cdef} Z^{ef} \frac{\partial^2 \mathscr{L}}{\partial F_{cd} \partial F_{ab}} = \\ &= \frac{1}{4} \epsilon_{cdef} \frac{\partial \mathscr{L}}{\partial F_{ef}} \frac{\partial^2 \mathscr{L}}{\partial F_{cd} \partial F_{ab}} = \\ &= \frac{1}{8} \frac{\partial}{\partial F_{ab}} \left(\epsilon_{cdef} \frac{\partial \mathscr{L}}{\partial F_{cd}} \frac{\partial \mathscr{L}}{\partial F_{ef}} \right) = \end{aligned}$$

$$= -\frac{1}{4}\frac{\partial}{\partial F_{ab}}\left(\star Z_{cd}\frac{\partial \mathscr{L}}{\partial F_{cd}}\right), \tag{2.42}$$

where we used the symmetries of the Levi-Civita tensor to show that $\epsilon_{cdef} = \epsilon_{efcd}$. After the integration of (2.42) we get the SO(2) invariance condition,

$$\star Z_{ab}Z^{ab} - \mathcal{G} = C \ , \tag{2.43}$$

where C is a constant that necessarily vanishes for Lagrangians obeying the Maxwellian weak field limit.

Conformal transformations are defined as $g_{ab}(x) \to \tilde{g}_{ab}(x) = \Omega^2(x)g_{ab}(x)$, where $\Omega(x)$ is some smooth function. The important consequence of conformal invariance is the vanishing of the trace of the energy-momentum tensor, at least for classical fields [131]. The variation of the action under an infinitesimal conformal transformation, $\delta g^{ab}(x) = -2g^{ab}(x)\delta\Omega(x)$, gives us

$$\delta S = \int \frac{\delta S}{\delta g^{ab}}\delta g^{ab}\boldsymbol{\epsilon} = -\int 2\frac{\delta S}{\delta g^{ab}}g^{ab}\delta\Omega\boldsymbol{\epsilon} = \int \sqrt{-g}T_{ab}g^{ab}\delta\Omega\boldsymbol{\epsilon} \ , \tag{2.44}$$

where in the last step we recognised the definition of the energy-momentum tensor (C.1). If the action is conformally invariant, we have $\delta S = 0$ and it follows that $T = T_{ab}g^{ab} = 0$ since the function $\delta\Omega$ is arbitrary. Conversely, $T = 0$ immediately implies conformal invariance. Note that conformal transformation should be distingushed from conformal isometry associated to a diffeomorphism of a given manifold $\mathscr{M}$ (for details, see Appendix D of [187]).

In the rest of the section, we list the most prominent examples of NLE theories, discuss their motivation and main properties.

2.2.1 $\mathcal{FG}$-class Lagrangians

Euler-Heisenberg Lagrangian

Euler and Heisenberg considered 1-loop QED corrections to classical Maxwell's theory and formulated a nonlinear electromagnetic Lagrangian that describes vacuum polarisation effects [86]. A paradigmatic process described within Euler-Heisenberg effective theory is light by light scattering, $\gamma\gamma \to \gamma\gamma$, recently observed experimentally by the ATLAS Collaboration [1]. We will derive the effective Euler-Heisenberg Lagrangian in the low energy limit [52]. The complete result at the one-loop level, obtained by Euler and Heisenberg, is given in the integral form as

$$\mathscr{L} = \frac{1}{2(2\pi)^2}\int_0^\infty \frac{ds}{s^3}\Big(e^2EBs^2\coth(esE)\cot(esB) -$$

$$- 1 - \frac{e^2}{3}(E^2 - B^2)s^2\Big) e^{is(m_e^2 - i\eta)}, \tag{2.45}$$

where e is the electron charge, m_e its mass and E and B are the electric and magnetic fields, respectively. In the weak field regime, we have

$$\begin{aligned} \coth(eEs)\cot(eBs) \approx &\left(\frac{1}{eEs} + \frac{eEs}{3} + \frac{(eEs)^3}{45} + ...\right) \times \\ &\times \left(\frac{1}{eBs} - \frac{eBs}{3} - \frac{(eBs)^3}{45} + ...\right) . \end{aligned} \tag{2.46}$$

We see that the integral (2.45) is already regularised since the divergent terms proportional to s^{-3} and s^{-1} are cancelled by corresponding counterterms, which leaves us with

$$\begin{aligned} \mathscr{L} &= -\frac{e^4}{360\pi^2}\int_0^\infty dss e^{-is(m_e^2 - i\eta)}(B^4 + 5B^2E^2 + E^4) = \\ &= \frac{\alpha^2}{360 m_e^4}\left(4\mathcal{F}^2 + 7\mathcal{G}^2\right), \end{aligned} \tag{2.47}$$

where $\alpha = e^2/(4\pi)$ is the fine structure constant. In the last step we took the limit in which the regularising parameter η goes to zero. The total effective Lagrangian is given by the Maxwell term and the calculated one-loop correction,

$$\mathscr{L}^{(\mathrm{EH})} = -\frac{1}{4}\mathcal{F} + \frac{\alpha^2}{360 m_e^4}\left(4\mathcal{F}^2 + 7\mathcal{G}^2\right) + O(\alpha^3) . \tag{2.48}$$

Even though the electric field of a point charge in Euler–Heisenberg theory diverges as $r \to 0$, its self energy is finite [43]. Accordance with the Maxwellian weak field limit is obvious from the form of the Lagrangian (2.48).

Born-Infeld Lagrangian

In order to regularise singularities in the electric field and the energy of a point charge appearing within Maxwell's electrodynamics, Max Born proposed a phenomenological $\mathcal{F}$-class NLE Lagrangian [20]

$$\mathscr{L}^{(\mathrm{tBI})} = b^2\left(1 - \sqrt{1 + \frac{\mathcal{F}}{2b^2}}\right) , \tag{2.49}$$

which he subsequently, together with Leopold Infeld, extended to an $\mathcal{F}\mathcal{G}$-class NLE

Lagrangian [21] of the form

$$\mathscr{L}^{(\mathrm{BI})} = b^2 \left(1 - \sqrt{1 + \frac{\mathcal{F}}{2b^2} - \frac{\mathcal{G}^2}{16b^4}} \right) . \tag{2.50}$$

We can prove the finiteness of the electric field and the energy of the point charge explicitly. The electromagnetic field tensor corresponding to a spherically symmetric and static configuration with only electric charge present is

$$\mathbf{F} = -E(r)\mathrm{d}t \wedge \mathrm{d}r \ , \quad \star\mathbf{F} = E(r)r^2 \sin^2\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi \ . \tag{2.51}$$

In the absence of the magnetic field, the derivative of $\mathscr{L}^{(\mathrm{BI})}$ with respect to $\mathcal{F}$ is equal to

$$\mathscr{L}_{\mathcal{F}} = -\frac{1}{4} \frac{1}{\sqrt{1 + \frac{\mathcal{F}}{2b^2}}} = -\frac{1}{4} \frac{1}{\sqrt{1 - \frac{E(r)^2}{b^2}}} \ . \tag{2.52}$$

The 2-form $\mathbf{F}$ (2.51) has to satisfy generalised Maxwell's equations, the second of which becomes

$$\mathrm{d} \left(\frac{E(r)r^2}{\sqrt{1 - E(r)^2/b^2}} \right) = 0 \ , \tag{2.53}$$

and implies that the term in parenthesis is a constant with its value defined by Komar's integral for electric charge (2.21). Finally, we obtain the electric field

$$E(r) = \frac{Q}{\sqrt{r^4 + (Q/b)^2}} \ , \tag{2.54}$$

which is manifestly finite as $r \to 0$. This expression gives a clear physical interpretation of the regularising parameter b as the upper limit of the electric field strength. To calculate the total energy ϵ, defined as $\epsilon = 4\pi \int_0^\infty T^{00} r^2 dr$, first we have to evaluate the relevant component of the energy-momentum tensor,

$$T^{00} = \frac{1}{4\pi} \left(-b^2 \left(1 - \sqrt{1 - E(r)^2/b^2} \right) + \frac{E(r)^2}{\sqrt{1 - E(r)^2/b^2}} \right) \ . \tag{2.55}$$

The solution of the integral is given in terms of gamma functions and can be numerically evaluated,

$$\epsilon = -\frac{\sqrt{b} Q^{3/2}}{2\sqrt{\pi}} \Gamma\left(-\frac{3}{4}\right) \Gamma\left(\frac{5}{4}\right) = 1.236 \ , \tag{2.56}$$

thereby demonstrating that the total self-energy of a point charge is finite.

The interest in Born-Infeld Lagrangian was reignited when it appeared as an effective action in the low energy limit of bosonic string theory and various supersymmetric theories [62, 165]. The string tension α' is inversely related to the Born-Infeld parameter b via $2\pi\alpha' = 1/b$ [183]. It is worth noticing that the Born-Infeld Lagrangian behaves in accordance with the Maxwellian weak field limit and is invariant with respect to electromagnetic duality rotations (2.33).

ModMax Lagrangian

The guiding principle in deriving the novel ModMax Lagrangian [8, 61] was the preservation of the original symmetries of Maxwell's electrodynamics. It is a unique 1-parameter modification of Maxwell's theory that is both conformally invariant and invariant with respect to SO(2) duality transformations,

$$\mathscr{L}^{(\mathrm{MM})} = \frac{1}{4}\left(-\mathcal{F}\cosh\gamma + \sqrt{\mathcal{F}^2+\mathcal{G}^2}\,\sinh\gamma\right) . \tag{2.57}$$

To preserve causality, the dimensionless parameter γ should be nonnegative [8]. Notice that for $\gamma = 0$ we recover Maxwell's Lagrangian.

Although the original derivation of ModMax theory is based on Hamiltonian formalism [8], we will follow a more direct approach [111]. First, we impose the conformal invariance condition, that is, $T = 0$, which for NLE theories becomes

$$\mathscr{L} - \mathscr{L}_{\mathcal{F}}\mathcal{F} - \mathscr{L}_{\mathcal{G}}\mathcal{G} = 0 \ . \tag{2.58}$$

The electromagnetic duality invariance (2.43) condition with constant C set to zero yields

$$16\mathscr{L}_{\mathcal{F}}^2\mathcal{G} - 32\mathscr{L}_{\mathcal{F}}\mathscr{L}_{\mathcal{G}}\mathcal{F} - 16\mathscr{L}_{\mathcal{G}}^2\mathcal{G} = \mathcal{G} \ . \tag{2.59}$$

After multiplying it by $\mathcal{G}$ and eliminating $\mathscr{L}_{\mathcal{G}}$ by using (2.58), we have

$$16\left(\sqrt{\mathcal{F}^2+\mathcal{G}^2}\,\mathscr{L}_{\mathcal{F}} - \mathscr{L}\right)\left(\sqrt{\mathcal{F}^2+\mathcal{G}^2}\,\mathscr{L}_{\mathcal{F}} + \mathscr{L}\right) = \mathcal{G}^2 \ . \tag{2.60}$$

The equation above defines a nonlinear partial differential equation, which can be easily solved with an ansatz of the form

$$\mathscr{L} = \alpha u + \beta v \ , \tag{2.61}$$

where $u = \sqrt{\mathcal{F}^2+\mathcal{G}^2}$ and $v = \mathcal{F}$. New variables u and v are independent everywhere except at $\mathcal{G} = 0$, which is a singular point of the SO(2) invariance condition. That being the case, u and v are well adapted to the problem in question. Since the trace T vanishes, the sought Lagrangian satisfies Euler's theorem for homogeneous functions of degree 1 in variables $\mathcal{F}$ and $\mathcal{G}$. This assertion justifies our choice of ansatz (2.61). Taking into account that $\mathscr{L}_{\mathcal{F}} = \beta + (v/u)\alpha$ and inserting the ansatz back into (2.60), we have

$$16(\alpha^2 - \beta^2)(v^2 - u^2) = u^2 - v^2. \tag{2.62}$$

The solution of the equation above is

$$\alpha = \pm\frac{1}{4}\sinh\gamma, \quad \beta = \pm\frac{1}{4}\cosh\gamma. \tag{2.63}$$

If both α and β are negative, such Lagrangian is unbounded from below, so we discard this solution. The choice $\beta = 1/4$ is not consistent with Maxwell's Lagrangian, which should be restored for $\gamma = 0$. The only possibility left is $\alpha = \frac{1}{4}\sinh\gamma$ and $\beta = -\frac{1}{4}\cosh\gamma$, and eventually, we obtain ModMax Lagrangian (2.57).

2.2.2 $\mathcal{F}$-class Lagrangians

Ayón-Beato-García Lagrangian

For many years after its proposal, the regular Bardeen black hole [9] was devoid of a proper physical interpretation. Eventually, Ayón–Beato and García identified the matter source that gives rise to the Bardeen solution as an $\mathcal{F}$-class NLE theory,

$$\mathscr{L}^{(\mathrm{ABG})} = \frac{3\mu}{g^3}\left(\frac{g\sqrt{2\mathcal{F}}}{2 + g\sqrt{2\mathcal{F}}}\right)^{\frac{5}{2}} . \tag{2.64}$$

Ayón-Beato-García Lagrangian contains two parameters, μ and g, which can *a posteriori* be equated with the black hole mass M and magnetic charge P, respectively.

Power Maxwell Lagrangian

The conformal invariance of Maxwell's theory is lost whenever the number of spacetime dimensions differs from four. Therefore, in order to derive the higher-dimensional analogues of Reissner–Nordström black holes, a new conformally invariant source was proposed in [78]. As its name suggests, the power-Maxwell Lagrangian is equal to the arbitrary power of invariant $\mathcal{F}$,

$$\mathscr{L}^{(\mathrm{pM})} = C\mathcal{F}^s \ , \tag{2.65}$$

where C and s are real constants. The conformal invariance is achieved if s is set to $d/4$, where d stands for the spacetime dimension. In that case, the power-Maxwell energy-momentum tensor

$$T_{ab} = \frac{C}{4\pi}(\mathcal{F}^{d/4}g_{ab} - d\mathcal{F}^{d/4-1}F_{ac}F_b{}^c) \ , \tag{2.66}$$

is indeed traceless, which confirms the former claim. The authors in [78] pointed out that this is the only conformally invariant $\mathcal{F}$-class NLE Lagrangian, a unique solution of the equation $\pi T = 0 = d\mathscr{L} - 4\mathscr{L}_{\mathcal{F}}\mathcal{F}$. On the other hand, by discarding the conformal invariance condition, i.e., by allowing s to attain arbitrary values, one can find a richer variety of black hole solutions [79].

Null-electromagnetic fields in power-Maxwell theory [78, 79] are examples of stealth field solutions [171], and they also belong to a larger family of the so-called universal electromagnetic fields [137, 138, 94]. These configurations got their name due to the fact that they automatically solve equations within various generalisations of Maxwell's theory.

Other NLE Lagrangians

The family of NLE Lagrangians has grown significantly since the appearance of Born–Infeld and Euler–Heisenberg theories during the infant stage of quantum field theory. Motivation often stems from searching for gravitating solutions with intriguing characteristics. For example, rational [115] and exponential [112] $\mathcal{F}$-class Lagrangians render regular magnetically charged black hole solutions, while hyperbolic tangent Lagrangian [5] gives a regular electrically charged black hole, with the caveat that it does not respect the Maxwellian weak field limit. New NLE theories have also been constructed with the purpose of altering the properties of existing cosmological models. Important classes of gravitational solutions can be found analytically within the newly introduced RegMax Lagrangian [178, 75], which is another example of a NLE theory that regularises point charge singularities.

2.3 Exact solutions

Various uniqueness theorems limit the diversity of black hole spacetimes in Einstein-Maxwell theory [103, 126]. These constraints may be circumvented by replacing Maxwell's electrodynamics with its NLE modifications, which enable the construction of novel charged black hole solutions. In the following section, we give an overview of exact solutions sourced by NLE fields, including examples of regular black holes, black holes emerging from well-motivated NLE theories and discuss the implications of NLE fields on cosmological spacetimes.

Bardeen black hole and its generalizations

In an attempt to address the question of how general the formation of black hole singularities is, Bardeen [9] wrote an "ad hoc" ansatz that represents a regular modification of the Schwarzschild black hole,

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{2.67}$$

with

$$f(r) = 1 - \frac{2\mu r^2}{(r^2+g^2)^{3/2}} \ . \tag{2.68}$$

The absence of a Schwarzschild-like curvature singularity can be confirmed by evaluating scalar quantities such as Ricci scalar R, "Ricci squared" $R_{ab}R^{ab}$ and Kretschmann scalar $R_{abcd}R^{abcd}$. The asymptotic expansion of the metric function

$$g_{tt} = -1 + 2\mu/r - 3\mu g^2/r^3 + O(1/r^5)$$

allows one to interpret parameter μ as the black hole's mass M, while the meaning of the regularising parameter g is not immediately clear.

One physical explanation of this solution came in the form of the aforementioned Ayón-Beato-García Largangian [6]. The Bardeen black hole can be derived starting from a standard ansatz representing static and spherically symmetric spacetime (2.22) and the electromagnetic field tensor with only the magnetic part present (2.23),

$$\mathbf{F} = B_r r^2 \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi \ . \tag{2.69}$$

The first Maxwell's equation,

$$\mathrm{d}\mathbf{F} = \frac{\partial(B_r(r)r^2)}{\partial r} \sin\theta \, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi = 0 \ , \tag{2.70}$$

is satisfied if $B_r(r)r^2 = \text{const}$. We may set the arbitrary constant to g and calculate the Komar integral

$$P = \frac{1}{4\pi}\oint_{\mathbb{S}} \mathbf{F} = \frac{g}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin\theta \mathrm{d}\theta \mathrm{d}\phi = g, \tag{2.71}$$

in order to identify g with the magnetic charge P. After inserting the Lagrangian (2.64) into the $G^t_{\ t} = T^t_{\ t}$ component of the gravitational-NLE equation (2.31),

$$(r(f(r)-1))' = 2r^2 \mathscr{L}(\mathcal{F}) = \frac{6Mr^2}{g^6}\left(\frac{Pg}{r^2+Pg}\right)^{5/2} , \tag{2.72}$$

and demanding that $\lim_{r\to\infty} f(r) = 1$, integration of (2.72) gives us

$$f(r) = 1 - \frac{2Mr^2}{(P^2+r^2)^{3/2}} . \tag{2.73}$$

Following the steps from [6], we successfully recovered the Bardeen metric, hence confirming its interrelation with the Ayón-Beato-García Lagrangian.

A regular electrically charged black hole may be constructed by modifying the Bardeen metric,

$$f(r) = 1 - \frac{2\mu r^2}{(r^2+q^2)^{3/2}} + \frac{q^2 r^2}{(r^2+q^2)^2} , \tag{2.74}$$

and evaluating the corresponding NLE Lagrangian as a function of the electromagnetic invariant $\mathcal{F}$ [4]. This construction, however, implicitly uses different Lagrangians in different areas of spacetime, as was pointed out in [27].

ModMax black holes

We are looking into the effects that ModMax theory exerts on Reissner-Nordström-like black holes. The static and spherically symmetric solutions of the Einstein-ModMax equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \left(\cosh\gamma - \sinh\gamma \frac{\mathcal{F}}{\sqrt{\mathcal{F}^2+\mathcal{G}^2}}\right) \widetilde{T}_{\mu\nu} \tag{2.75}$$

were derived in [61].

a) Electrically charged black hole

The electrically charged solution can be obtained without solving the complete equation of motion explicitly. Namely, in spherically symmetric spacetime, $P = 0$ implies the vanishing of the magnetic field (2.29), hence $\mathcal{G} = 0$. The second Maxwell's equation (2.30) returns the electric field,

$$E_r = \frac{Qe^{-\gamma}}{r^2} , \tag{2.76}$$

while ModMax Lagrangian attains the form

$$\mathscr{L}^{(\mathrm{MM})} = -\frac{1}{4}\mathcal{F}(\cosh\gamma + \sinh\gamma) = \frac{Q^2 e^{-\gamma}}{2r^4} \ . \tag{2.77}$$

For the purpose of finding the black hole solution, we can make the identification $\mathscr{L}^{(\mathrm{MM})} = e^{-\gamma}\mathscr{L}^{(\mathrm{Max})} = e^{-\gamma}Q^2/(2r^4)$, where Q is the charge of the Reissner-Nordström black hole. This in turn implies that the charge of the ModMax black hole gets redefined and we can simply read it off from (2.77) as $\widetilde{Q} = Qe^{-\gamma/2}$. Consequently, the metric function remains the same as in the Reissner-Nordström case, provided that we make the substitution $Q \to \widetilde{Q}$,

$$f_e(r) = 1 - \frac{2M}{r} + \frac{Q^2 e^{-\gamma}}{r^2} \ . \tag{2.78}$$

Constant γ can be interpreted as a charge screening factor.

b) Magnetically charged black hole

After setting $E_r = 0$ in (2.23) and evaluating ModMax Lagrangian density as $\mathscr{L}^{(\mathrm{MM})} = -e^{-\gamma}P^2/(2r^4)$, we may solve the Einstein-NLE equation (2.31) to get

$$f_m(r) = 1 - \frac{2M}{r} + \frac{P^2 e^{-\gamma}}{r^2} \ . \tag{2.79}$$

Again, the charge screening effect is apparent.

c) Dyonic black hole

Due to the electromagnetic duality invariance, we may immediately superpose the electric and magnetic solution to obtain

$$f_d(r) = 1 - \frac{2M}{r} + \frac{(Q^2 + P^2) e^{-\gamma}}{r^2} \ . \tag{2.80}$$

Although the mass of the Reissner-Nordström black hole must exceed its charge to avoid the formation of naked singularities, for the ModMax black hole there is no such restriction. The inner and outer horizons of the electrically charged ModMax black hole are defined by

$$r_\pm = M \pm \sqrt{M^2 - (Q^2 + P^2)e^{-\gamma}} \ , \tag{2.81}$$

which implies that the extremal configuration has $M_{ext} = \sqrt{Q^2 + P^2}e^{-\gamma/2}$. For $\gamma > 0$, the charge of the black hole is greater than its mass.

Notice that neither of these black holes is regular in the sense of the absence of curvature singularities.

Born–Infeld black holes

Due to its regularising effect on electrodynamics, it was speculated that Born–Infeld theory might smooth out curvature singularities when coupled to gravitation. Exact black hole solutions found in a series of papers [64, 160, 51, 60, 30, 85] exclude this option and disclose other implications of Born–Infeld theory.

The solution describing the electrically charged Born-Infeld black hole adapted to the standard ansatz (2.22) is given by

$$f(r) = 1 - \frac{2M}{r} + \frac{2b^2r^2}{3}\left(1 - \sqrt{1 + \frac{Q^2}{b^2r^4}}\right) + \frac{4Q^2}{3r^2}\,{}_2F_1\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{Q^2}{b^2r^4}\right) , \tag{2.82}$$

where ${}_2F_1(a, b, c; z)$ is the hypergeometric function. In the asymptotic regime, the first correction to the Reissner-Nordström behaviour is of the order r^{-6} and proportional to the parameter b.

A similar solution in terms of hypergeometric function is obtained for the magnetically charged Born-Infeld black hole,

$$f(r) = 1 - \frac{2M}{r} - \frac{2b^2r^2}{3}\left({}_2F_1\left(-\frac{3}{4}, -\frac{1}{2}, \frac{1}{4}, -\frac{P^2}{b^2r^4}\right) - 1\right) . \tag{2.83}$$

The results can be generalised by considering an arbitrary number of spacetime dimensions [30, 51] or modified gravitational theories [87, 203]. An exhaustive study on thermodynamic properties and phase transitions of (A)dS Born-Infeld black holes has been conducted in [30, 51].

Euler–Heisenberg black holes

The influence of the effective Euler-Heisenberg Lagrangian on nonrotating and asymptotically flat black holes has been studied in [202, 159]. Compared to the standard Maxwell case, black hole quantities such as horizon area and energy acquire QED corrections represented by a series expansion in powers of α. Starting from the metric (2.22) with $w(r) = 1$, we will single out three distinct cases and truncate the results at the leading order of α [159].

a) Electrically charged black holes

The approximate solution of Maxwell's equation (2.30) up to the α^2 order is given by

$$E(r) = \frac{Q}{r^2} - \frac{\alpha^2}{360m_e^4}\frac{64Q^3}{r^6} + O(\alpha^3) . \tag{2.84}$$

Integration of Einstein's equation (2.31) determines the metric function $f(r)$,

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\alpha^2}{360m_e^4}\frac{32Q^4}{5r^6} + O(\alpha^3) \ . \tag{2.85}$$

Euler-Heisenberg Lagrangian introduces a charge-screening effect emerging from the vacuum polarisation. Consequently, the black hole horizon area increases compared to the Reissner-Nordström one, while the total energy decreases [159]. Further generalisation of this solution can be made by including the cosmological constant [122].

b) Magnetically charged black holes

For $E_r = 0$ and magnetic field given by (2.29), the metric function $f(r)$ is equal to

$$f(r) = 1 - \frac{2M}{r} + \frac{P^2}{r^2} - \frac{\alpha^2}{360m_e^4}\frac{32P^4}{5r^6} + O(\alpha^3). \tag{2.86}$$

Again, the horizon area increases in comparison with the magnetically charged Reissner-Nordström black hole, while the total energy is smaller due to the vacuum polarisation effect.

c) Dyonic black holes

With both types of charges present and $\mathscr{L}_\mathcal{G} \neq 0$, the electric and magnetic fields are no longer independent (2.30). Evaluation of the NLE Maxwell's equations up to the α^2 order returns

$$E(r) = \frac{Q}{r^2} - \frac{\alpha^2}{360m_e^4}\frac{64Q^3}{r^6} - \frac{\alpha^2}{360m_e^4}\frac{160QP^2}{r^6} + O(\alpha^3) \ , \tag{2.87}$$

$$B(r) = \frac{P}{r^2} \ , \tag{2.88}$$

while Einstein's equation (2.31) gives us

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{P^2}{r^2} - \frac{\alpha^2}{360m_e^4}\frac{32(P^4 + 5P^2Q^2 + Q^4)}{5r^6} + O(\alpha^3) \ . \tag{2.89}$$

If both charges are equal,we get

$$f(r) = 1 - \frac{2M}{r} + \frac{2Q^2}{r^2} - \frac{\alpha^2}{360m_e^4}\frac{224Q^4}{5r^6} + O(\alpha^3) \ , \tag{2.90}$$

which shows that the QED correction is greater than in the individual electric or magnetic cases because of the combined screening effect on both charges.

Cosmological solutions

In the context of cosmology, some NLE theories have appeared as dark energy mimickers or have succeeded in regularising the initial cosmological singularity. As an illustration, we may take the Friedman–Robertson–Walker model

$$ds^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1+kr^2} + r^2\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) , \tag{2.91}$$

where $a(t)$ is the scale factor and $k \in \{-1, 0, 1\}$, depending on whether the universe is closed, flat or open. The evolution of the scale factor depends on the density ρ and pressure p of the matter permeating the universe,

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) \ . \tag{2.92}$$

Since NLE theories may violate some of the energy conditions, the sign of the combination $\rho + 3p$ is not predetermined, which will be important in discussing the accelerated expansion of the universe.

A singularity will necessarily form at the initial time t_0 whenever $a(t_0) = 0$, which is the case in Maxwell's electrodynamics [48]. In [48] and [31], the authors considered a FRW universe filled by an Euler–Heisenberg-like matter source

$$\mathscr{L} = -\frac{1}{4}\mathcal{F} + \alpha\mathcal{F}^2 + \beta\mathcal{G}^2 \ , \tag{2.93}$$

where α and β are constants. In the simplest case, with the energy-momentum tensor identified as a perfect fluid, the scale factor can be expressed in a closed form

$$a^2(t) \sim \sqrt{(t^2 + \alpha)} \ . \tag{2.94}$$

Since the scale factor is always nonzero, the cosmological singularity is absent. Another concept can be demonstrated in this example. At the early stages, the sum $\rho + 3p$ becomes negative, which is the mechanism responsible for the inflation. The same effect has been noticed for different types of $\mathcal{F}$-class NLE Lagrangians [135, 139]. Apart from highly symmetric FRLW universes, some anisotropic Bianchi spaces with Born-Infeld NLE Lagrangian are also singularity-free [65].

Power-Maxwell black holes

A class of static and spherically symmetric higher-dimensional black holes emerging from the conformally invariant power-Maxwell Lagrangian was derived in [78].

The line element in $d = 4 + 4p$ dimensions with $p \in \mathbb{N}$ is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{4p+2} \ , \tag{2.95}$$

where $d\Omega^2_{4p+2}$ denotes the metric of a unit $(4p+2)$-dimensional sphere. Solving the coupled Einstein-NLE system of equations yields the metric function

$$f(r) = 1 - \frac{A}{r^{4p+1}} + \frac{B}{r^{4p+2}} \ , \tag{2.96}$$

which is split into mass and charge terms, and the radial electric field

$$E = \frac{C}{r^2} \ . \tag{2.97}$$

Curiously, the electric field is independent of the spacetime dimension; moreover, its form agrees with the Reissner-Nordström solution. The unknown constants A, B and C are determined via Komar integrals for mass [110] and charge (2.21). Finally, to ensure that the obtained solution really represents a black hole, the values of the constants have to be further constrained. With the appropriate choice, there are two horizons shielding a curvature singularity at $r = 0$.

Power-Maxwell black holes differ from charged Tangherlini solutions [180], which are higer-dimensional Einstein-Maxwell black holes, in two main points. Tangherlini black holes do not have vanishing scalar curvature as their source is not conformally invariant in $d > 4$, also, the exponent of the charge term in the metric function differs from the one in the power-Maxwell solution.

By dropping the conformal invariance condition, one may find a wide range of black hole spacetimes with different asymptotic behaviour [79]. Among them are solutions that asymptotically approach Minkowski spacetime with various powers of $1/r^n$, non-asymptotically flat solutions that generalise Schwarzschild-(anti)-de-Sitter spacetimes and solutions containing logarithmic dependence in metric coefficients.

Other black hole solutions

The family of charged black hole solutions is proliferating further in parallel with the introduction of new NLE theories. The associated black hole solutions are analysed from different perspectives, including their thermodynamic properties, stability and, as emphasised before, regularity. Another often-adopted recipe for finding new solutions consists of imposing a certain metric ansatz, evaluating the energy-momentum tensor and reconstructing the NLE theory in a coordinate form rather than as a functional of electromagnetic invariants. Unfortunately, the physical significance of Lagrangians engineered in this manner, apart from producing specific solutions, is generally unclear.

2.4 Energy conditions

There is strong experimental evidence that supports the local positivity of energy density and its prevalence over pressure. These ideas are encapsulated by mathematical statements known as energy conditions [45]. We say that the energy-momentum tensor obeys:

a) dominant energy condition (DEC) if $T_{ab}u^a v^b \geq 0$ for all future directed timelike vectors u^a and v^a. Equivalently, $-T^a{}_b v^b$ is future directed and causal for any future directed timelike vector v^a,

b) weak energy condition (WEC) if $T_{ab}v^a v^b \geq 0$ for any timelike vector v^a,

c) strong energy condition (SEC) if $T_{ab}v^a v^b \geq \dfrac{1}{2} T g_{ab} v^a v^b$ for any timelike vector v^a,

d) null energy condition (NEC) if $T_{ab}l^a l^b \geq 0$ for any null vector l^a.

Not all energy conditions are mutually independent but are related by the following implications:

$$\text{DEC} \Rightarrow \text{WEC} \Rightarrow \text{NEC} \Leftarrow \text{SEC}.$$

Maxwell's energy-momentum tensor satisfies DEC and, since it is traceless, SEC. The proof is most easily carried out using spinorial approach. For any pair of spinors κ^A, λ^A and the corresponding pair of future directed null vectors, $k^{AA'} = \kappa^A \overline{\kappa}^{A'}$ and $l^{AA'} = \lambda^A \overline{\lambda}^{A'}$, we have

$$\begin{aligned} T^{(\mathrm{Max})}_{ABA'B'} k^{AA'} \ell^{BB'} &= \frac{1}{2\pi}\, \phi_{AB} \overline{\phi}_{A'B'} \kappa^A \overline{\kappa}^{A'} \lambda^B \overline{\lambda}^{B'} = \\ &= \frac{1}{2\pi} \left| \phi_{AB} \kappa^A \lambda^B \right|^2 \geq 0 \; . \end{aligned} \tag{2.98}$$

Since any future directed causal vector is a sum of a pair of future directed null vectors, it follows that $T^{(\mathrm{Max})}_{ab} u^a v^b \geq 0$ for any pair of future directed causal vectors u^a and v^a. Note that by the implications, Maxwell's energy momentum tensor satisfies all of the energy conditions listed.

Energy conditions are the pillars of many foundational results in gravitational theory, such as the laws of black hole thermodynamics and singularity theorems. For this reason, they have to be closely examined in the case of NLE fields. Complementing the previous results [146, 50], we have proven the following theorem for the NLE-energy momentum tensor [14].

Theorem 2.1. *The NLE energy-momentum tensor, in $\eta = -1$ signature* [3] *satisfies*

- *NEC if and only if $\mathscr{L}_{\mathcal{F}} \le 0$;*
- *DEC if and only if $\mathscr{L}_{\mathcal{F}} \le 0$ and $T \le 0$;*
- *SEC if $\mathscr{L}_{\mathcal{F}} \le 0$ and $T \ge 0$.*

Proof:
Contraction of the energy-momentum tensor (2.8) with two null vectors l^a is

$$T_{ab} l^a l^b = -4\mathscr{L}_{\mathcal{F}} \widetilde{T}_{ab} l^a l^b + \frac{1}{4} T g_{ab} l^a l^b = -4\mathscr{L}_{\mathcal{F}} \widetilde{T}_{ab} l^a l^b. \tag{2.99}$$

Then, recalling the fact that Maxwell's energy-momentum tensor obeys NEC, it immediately follows that for $\mathscr{L}_{\mathcal{F}} \le 0$, T_{ab} satisfies NEC. The same reasoning holds for the "if" direction in DEC and SEC cases. Namely, for the pair of future-directed timelike vectors u^a and v^a we have

$$T_{ab} u^a v^b = -4\mathscr{L}_{\mathcal{F}} \widetilde{T}_{ab} u^a v^b + \frac{1}{4} T g_{ab} u^a v^b \ . \tag{2.100}$$

Since $u^a v_a \le 0$ for future-directed timelike vectors and Maxwell's tensor satisfies DEC, if $\mathscr{L}_{\mathcal{F}} \le 0$ and $T \le 0$, the total NLE energy momentum tensor also obeys DEC. By the analogous arguments, given that $\mathscr{L}_{\mathcal{F}} \le 0$ and $T \ge 0$ hold, we have the following inequality for the future-directed timelike vector v^a,

$$T_{ab} v^a v^b - \frac{1}{2} T g_{ab} v^a v^b \ge -4\mathscr{L}_{\mathcal{F}} \widetilde{T}_{ab} v^a v^b - \frac{1}{4} T g_{ab} v^a v^b \ge 0, \tag{2.101}$$

which demonstrates the validity of SEC.

For the converse direction in the NEC case, we have to prove the existence of a future directed null vector ℓ^a, such that $\widetilde{T}_{ab} \ell^a \ell^b > 0$. By employing spinor representation (the details are presented in Appendix B), we decompose both the electromagnetic spinor as $\phi_{AB} = \alpha_{(A} \beta_{B)}$ and the vector l^a as $\ell^{AA'} = \pm \lambda^A \overline{\lambda}^{A'}$, where the sign is chosen such that l^a is future directed. In the algebraically general case we can choose an auxiliary spinor $\lambda^A = \alpha^A + \beta^A$, such that $\lambda^A \alpha_A \ne 0 \ne \lambda^A \beta_A$, while in the algebraically special case λ^A may be any spinor such that $\lambda^A \alpha_A \ne 0$. In both cases we have $2\pi \widetilde{T}_{ab} \ell^a \ell^b = \left| \phi_{AB} \lambda^A \lambda^B \right|^2 > 0$. Finally, assuming that NEC holds, it follows that $0 \le T_{ab} \ell^a \ell^b = -4\mathscr{L}_{\mathcal{F}} \widetilde{T}_{ab} \ell^a \ell^b$, which implies $\mathscr{L}_{\mathcal{F}} \le 0$.

If the NLE energy-momentum tensor obeys either DEC or SEC, it immediately satisfies NEC and consequently $\mathscr{L}_{\mathcal{F}} \le 0$. It remains to prove that DEC implies

[3] For the discussion on the metric signature in the spinorial approach see Appendix B.

$T \leq 0$, which has already been discussed in [146]. The case when $\mathscr{L}_\mathcal{F} = 0$ is trivial as DEC immediately demands $T \leq 0$, so we assume that $\mathscr{L}_\mathcal{F} < 0$. Using the Newman–Penrose null tetrad (B.19), we may decompose a timelike vector v^a as $v^a = a\ell^a + bn^a + \bar{c}m^a + c\overline{m}^a$ where (a, b, c) are complex numbers. For the sake of simplicity, the normalisation is chosen such that $ab = 1 + |c|^2$ and $v_a v^a = -2$. DEC can be written as $(T^a{}_b v^b)(T_{ac} v^c) \leq 0$, which is after a lengthy calculation reduced to an inequality

$$S + (1 + 2|c|^2)\mathscr{L}_\mathcal{F} T \geq 0, \tag{2.102}$$

where S is a quantity which does not depend on the parameters (a, b, c). We will present the main points of its derivation. The idea is to build the Newman–Penrose tetrad (B.19) from the principal spinors of the symmetric electromagnetic spinor $\phi_{AB} = \alpha_{(A}\beta_{B)}$,

$$\ell^a = \alpha^A \overline{\alpha}^{A'}, \quad n^a = \beta^A \overline{\beta}^{A'}, \quad m^a = \alpha^A \overline{\beta}^{A'}, \quad \overline{m}^a = \beta^A \overline{\alpha}^{A'}. \tag{2.103}$$

Then, we have to calculate the following terms,

$$(T^a{}_b v^b)(T_{ac} v^c) = 16\mathscr{L}_\mathcal{F}^2 \widetilde{T}^a{}_b \widetilde{T}_{ac} v^b v^c - 2\mathscr{L}_\mathcal{F} T \widetilde{T}_{ab} v^a v^b + \frac{1}{16} T^2 v^a v_a. \tag{2.104}$$

From the auxiliary result,

$$\begin{aligned} \widetilde{T}_{ab} v^a &= \frac{1}{8\pi} \left(\alpha_A \beta_B + \alpha_B \beta_A\right) \left(\overline{\alpha}_{A'} \overline{\beta}_{B'} + \overline{\alpha}_{B'} \overline{\beta}_{A'}\right) v^{AA'} = \\ &= \frac{1}{8\pi} \left(\ell_a n_b + m_a \overline{m}_b + \overline{m}_a m_b + n_a \ell_b\right) \left(a\ell^a + bn^a + \bar{c}m^a + c\overline{m}^a\right) = \\ &= \frac{1}{8\pi}(-a\ell_b - bn_b + c\overline{m}_b + \bar{c}m_b), \end{aligned} \tag{2.105}$$

we get

$$\widetilde{T}_{ab} v^a v^b = \frac{1}{4\pi}(ab + |c|^2) = \frac{1}{4\pi}(1 + 2|c|^2) \tag{2.106}$$

and

$$\widetilde{T}^a{}_b \widetilde{T}_{ac} v^b v^c = \frac{1}{32\pi^2}(-ab + |c|^2) = -\frac{1}{32\pi^2}. \tag{2.107}$$

Setting $T > 0$ in (2.102) would lead to a contradiction as we may choose arbitrarily large $|c|$. □

We can examine the implications of theorem 2.1. on two prominent NLE theories. In Born-Infeld theory, we have the following expressions,

$$\mathscr{L}_\mathcal{F}^{(\mathrm{BI})} = -\frac{1}{4\mathcal{W}}, \quad \pi T^{(\mathrm{BI})} = \frac{4b^2(\mathcal{W} - 1) - \mathcal{F}}{4\mathcal{W}}, \tag{2.108}$$

with

$$\mathcal{W} := \sqrt{1 + \frac{\mathcal{F}}{2b^2} - \frac{\mathcal{G}^2}{16b^4}}\ . \tag{2.109}$$

It can be easily seen that $\mathscr{L}_{\mathcal{F}} \leq 0$, so NEC is satisfied. As $2\sqrt{x-y} \neq 2\sqrt{x} \neq x+1$ for nonnegative x and $y \neq x$, we have $2\mathcal{W} \leq 2 + (\mathcal{F}/2b^2)$, implying that $T^{(BI)} \leq 0$. Born-Infeld theory also obeys DEC.

In Euler-Heisenberg theory, we have

$$\mathscr{L}_{\mathcal{F}}^{(\mathrm{EH})} = -\frac{1}{4} + \frac{8\alpha^2}{360 m_e^4}\,\mathcal{F}$$

and

$$\pi T^{(\mathrm{EH})} = -\frac{\alpha^2}{360 m_e^4}\left(4\mathcal{F}^2 + 7\mathcal{G}^2\right). \tag{2.110}$$

We can conclude that Euler-Heisenberg theory obeys both DEC and NEC for electromagnetic fields with $\mathcal{F} \leq 45{m_e}^4/4\alpha^2$. Fields whose strength is above this limit are not of interest, as they exceed the weak field regime in which the effective Euler-Heisenberg Lagrangian is valid.

Furthermore, SEC is satisfied for null electromagnetic fields in both theories. Generally, we can add an arbitrary constant to a NLE Lagrangian such that $\mathscr{L}(0,0) = 0$. Then, if $\mathscr{L}$ is differentiable at the origin of the $\mathcal{F}$–$\mathcal{G}$ plane, the trace of the energy-momentum tensor vanishes for null electromagnetic fields. In other words, null electromagnetic fields in theories obeying $\mathscr{L}_{\mathcal{F}} \leq 0$ immediately satisfy both DEC and SEC.

Chapter 3

Schwarzschild spacetime immersed in NLE fields

3.1 Wald's solution

Electrovacuum uniqueness theorems pioneered by Israel [103] and Mazur [126] are established for black hole configurations that harbour their own electromagnetic fields. Here we are interested in a different scenario in which a black hole is surrounded by external electromagnetic sources. Apart from purely theoretical interest, this setting is relevant for astrophysical black holes surrounded by electromagnetic fields emanating from accretion discs or a wider galactic environment. The behaviour of external electromagnetic fields on a black hole background with a certain degree of symmetry was studied by Wald [189], who derived an analytic form of the electromagnetic field tensor for a rotating black hole placed in an initially uniform magnetic field. We will briefly recapitulate Wald's approach for Maxwell's case before turning to the NLE challenges.

Wald's solution is based on the fact that the Killing vector fields can be used as gauge potential in vacuum spacetimes, as was shown by Papapetrou [141]. To see this, we have to prove that they satisfy source-free Maxwell's equations:

$$\mathrm{d}\mathbf{F} = 0 \quad \text{and} \quad \mathrm{d}{\star}\mathbf{F} = 0 \,. \tag{3.1}$$

Let K^a be a Killing vector field defined on a spacetime (M, g_{ab}) and $\mathbf{F} = \mathrm{d}\mathbf{K}$ a corresponding electromagnetic 2-form. The first Maxwell's equation is immediately satisfied as $\mathbf{F}$ is an exact form. To prove the latter claim, we invoke the Killing lemma which relates the Killing vector fields to the Ricci curvature tensor [187],

$$\mathrm{d}\star\mathrm{d}\mathbf{K} = 2\star\mathbf{R}(K) \ . \tag{3.2}$$

In a vacuum spacetime, the Ricci tensor vanishes and we obtain the second Maxwell's equation (3.1). The electromagnetic fields constructed in this way are called *test fields* since they do not affect the spacetime metric.

Wald imposed several physical constraints that uniquely determine the electromagnetic field tensor $\mathbf{F}$. First, it is assumed that the spacetime admits two mutually commuting Killing vector fields, timelike and axial, denoted respectively by k^a and m^a. Furthermore, suppose that the electric and magnetic fields are symmetry inheriting, therefore implying that $\mathbf{F}$ has to be stationary and axially symmetric. In the asymptotic region, $\mathbf{F}$ has to represent a uniform magnetic field of strength B_0. Matter fields, and consequently, $\mathbf{F}$, have to be regular on the black hole horizon and in the exterior region. As test fields should not alter the charges of a background spacetime, both the electric and magnetic charges, defined by Komar's integrals, have to vanish.

Let us first construct the electromagnetic tensor $\mathbf{F}$ using the axial Killing vector m^a, $\mathbf{F}_m = \mathrm{d}\mathbf{m}$, and check its physical interpretation [189]. As it is stationary and axially symmetric, with $[k, m]^a = 0$, we have

$$\pounds_m \mathbf{F}_m = \pounds_m \mathrm{d}\mathbf{m} = \mathrm{d}\pounds_m \mathbf{m} = 0 \ , \tag{3.3}$$

$$\pounds_k \mathbf{F}_m = \pounds_k \mathrm{d}\mathbf{m} = \mathrm{d}\pounds_k \mathbf{m} = 0 \ . \tag{3.4}$$

The calculation of Komar's charges (2.21) over a sphere at infinity gives us

$$P_\infty = \frac{1}{4\pi} \oint_{\mathcal{S}_\infty} \mathbf{F}_m = 0 \ , \tag{3.5}$$

$$Q_\infty = \frac{1}{4\pi} \oint_{\mathcal{S}_\infty} \star\mathbf{F}_m = \frac{1}{4\pi} \oint_{\mathcal{S}_\infty} \star\mathrm{d}\mathbf{m} = 4J \ , \tag{3.6}$$

where in the last equality we recognise the definition of angular momentum given by the corresponding Komar integral [96]. The electromagnetic tensor generated by the axial Killing vector represents a stationary, axially symmetric electromagnetic field that asymptotically behaves as a uniform magnetic field, with a nonvanishing electric charge.

We can repeat the same procedure for the electromagnetic field tensor defined with respect to the stationary Killing vector field, $\mathbf{F}_k = \mathrm{d}\mathbf{k}$ [189]. Again, the magnetic monopole charge vanishes, but the electric charge is

$$Q_\infty = \frac{1}{4\pi}\oint_{\mathcal{S}_\infty} \star\mathbf{F}_k = \frac{1}{4\pi}\oint_{\mathcal{S}_\infty} \star\mathrm{d}\mathbf{k} = -2M \ , \tag{3.7}$$

where the last equality is the definition of Komar's mass. This electromagnetic tensor differs from the former in two main points: it represents asymptotically vanishing electromagnetic field and the value of the electric charge is altered. Taking into account that both electric and magnetic charges must vanish, the total electromagnetic field tensor is given as the appropriate linear combination of $\mathbf{F}_k$ and $\mathbf{F}_m$

$$\mathbf{F} = \frac{1}{2}\, B_\infty\, (2a\,\mathrm{d}\mathbf{k} + \mathrm{d}\mathbf{m}) \, , \tag{3.8}$$

where the coefficients were adjusted to ensure that the Komar charges are indeed zero. Since $a = J/M$, we have

$$Q_\infty = \frac{1}{4\pi}\oint_{\mathcal{S}_\infty} \star\mathbf{F} = B_\infty(-2aM + 2J) = 0 \ , \tag{3.9}$$

and

$$P_\infty = \frac{1}{4\pi}\oint_{\mathcal{S}_\infty} \mathbf{F} = 0 \ . \tag{3.10}$$

This solution holds for generic axially symmetric, stationary and asymptotically flat black holes. It can be explicitly evaluated for Kerr spacetime [189], in which the rotation of the black hole gives rise to a nonzero electric field, reflecting as nonvanishing invariant $\mathcal{G}$.

The following sections that deal with the NLE case are based on the paper [16].

3.2 NLE case

If we wish to generalise this result to the NLE fields, we encounter several obstacles. Papapetrou's ansatz cannot be used to solve both generalised Maxwell's equations simultaneously. Namely, after setting $\mathbf{F} = \mathrm{d}\mathbf{K}$, the second generalised source-free Maxwell's equation (2.6) is reduced to

$$\mathrm{d}\mathscr{L}_\mathcal{F} \wedge \star\mathbf{F} - \mathrm{d}\mathscr{L}_\mathcal{G} \wedge \mathbf{F} = 0 \ , \tag{3.11}$$

which can be expanded further by noticing that

$$\mathrm{d}\mathscr{L}_\mathcal{F} = \mathscr{L}_{\mathcal{F}\mathcal{F}}\,\mathrm{d}\mathcal{F} + \mathscr{L}_{\mathcal{F}\mathcal{G}}\,\mathrm{d}\mathcal{G} \quad \text{and} \quad \mathrm{d}\mathscr{L}_\mathcal{G} = \mathscr{L}_{\mathcal{G}\mathcal{F}}\,\mathrm{d}\mathcal{F} + \mathscr{L}_{\mathcal{G}\mathcal{G}}\,\mathrm{d}\mathcal{G} \ . \tag{3.12}$$

The equation (3.11) then contains terms proportional to $\mathrm{d}\mathcal{F} \wedge \mathbf{F}$, $\mathrm{d}\mathcal{G} \wedge \mathbf{F}$, $\mathrm{d}\mathcal{F} \wedge \star\mathbf{F}$ and $\mathrm{d}\mathcal{G} \wedge \star\mathbf{F}$, which generally do not vanish. As an illustration, we have

$$\star(\mathrm{d}\mathcal{F} \wedge \star\mathbf{F}) = -i_X\mathbf{F} = -i_X\mathrm{d}\mathbf{K} = (\mathrm{d}i_X - \pounds_X)\mathbf{K} \;, \tag{3.13}$$

with an auxiliary vector field $X^a = \nabla^a\mathcal{F}$. The first term vanishes since

$$i_X\mathbf{K} = K^a\nabla_a\mathcal{F} = \pounds_K\mathcal{F} = 0 \;, \tag{3.14}$$

as well as the Lie derivative of a Killing vector field with respect to X,

$$\pounds_X K^a = -\pounds_K X^a = -\pounds_K\nabla^a\mathcal{F} = -g^{ab}\,\nabla_a\pounds_K\mathcal{F} = 0 \;, \tag{3.15}$$

but the same does not hold for the associated Killing 1-form,

$$\pounds_X K_a = \pounds_X(g_{ab}K^b) = K^b\pounds_X g_{ab} \;. \tag{3.16}$$

It is very improbable that all such terms in equation (3.11) will cancel each other. An alternate idea is to define the electromagnetic field tensor using a rescaled Killing vector field, so that $\mathbf{F} = \mathrm{d}(\psi\mathbf{K})$, where ψ is some auxiliary function. The first generalised Maxwell's equation (2.6) is immediately satisfied, while the second one with the current form set to zero gives

$$\begin{aligned}(\mathscr{L}_\mathcal{F}\star\mathrm{d}\mathbf{K} + \mathrm{d}\mathscr{L}_\mathcal{G}\wedge\mathbf{K})\wedge\mathrm{d}\psi + \mathrm{d}\mathscr{L}_\mathcal{F}\wedge i_K\star\mathrm{d}\psi + \\ +\mathscr{L}_\mathcal{F}(\star\mathrm{d}\pounds_K\psi - (\Box\psi)\star\mathbf{K}) + (\mathrm{d}\mathscr{L}_\mathcal{F}\wedge\star\mathrm{d}\mathbf{K} - \mathrm{d}\mathscr{L}_\mathcal{G}\wedge\mathrm{d}\mathbf{K})\psi = 0 \;.\end{aligned} \tag{3.17}$$

Since both electromagnetic invariants are quadratic in ψ, (3.17) amounts to a highly nonlinear differential equation for ψ. Being unable to find a fruitful approach that would provide an exact solution, we resort to the perturbative scheme, expanding around Wald's solution.

3.2.1 Perturbative approach

Motivated by the examples from the literature, we assume that the NLE Lagrangian density is a function that allows a double Taylor series expansion,

$$\mathscr{L}(\mathcal{F},\mathcal{G}) = \sum_{m,n=0}^{\infty} c_{mn}\,\mathcal{F}^m\mathcal{G}^n, \tag{3.18}$$

with real coefficients c_{mn}. Terms corresponding to c_{00} and c_{01} are non-dynamical, hence these constants can be set to zero. In accordance with the Maxwellian weak field limit, we shall take $c_0 = -1/4$. In a general case, a CP-violating term c_{11} can be present.

Assuming that the NLE Lagrangian may be expanded with respect to the coupling constant λ, we have

$$\mathscr{L}(\mathcal{F}, \mathcal{G}) = -\frac{1}{4}\,\mathcal{F} + \lambda \ell(\mathcal{F}, \mathcal{G}) + O(\lambda^2) \ . \tag{3.19}$$

Our focus will be on two prominent examples, Euler–Heisenberg and Born–Infeld Lagrangians. Regarding the expansion in (3.19), we can make the following identifications for Euler-Heisenberg theory (2.48),

$$\ell^{(\mathrm{EH})} = 4\mathcal{F}^2 + 7\mathcal{G}^2 \quad \text{and} \quad \lambda^{(\mathrm{EH})} = \frac{\alpha^2}{360 m_e^4} \ . \tag{3.20}$$

Born-Infeld theory (2.50) can be represented in the form of Eq. (3.19) after expansion with respect to the coupling constant b,

$$\mathscr{L}^{(\mathrm{BI})} = -\frac{1}{4}\,\mathcal{F} + \frac{1}{32b^2}\left(\mathcal{F}^2 + \mathcal{G}^2\right) + \ldots , \tag{3.21}$$

with

$$\ell^{(\mathrm{BI})} = \mathcal{F}^2 + \mathcal{G}^2 \quad \text{and} \quad \lambda^{(\mathrm{BI})} = \frac{1}{32b^2} \ . \tag{3.22}$$

The similar ansatz can be used for other electromagnetic quantities, so the gauge field A_a may be written as

$$A_a = K_a + \lambda v_a + O(\lambda^2) \ , \tag{3.23}$$

where K_a is the Wald's term and v_a is the sought perturbative correction. The electromagnetic tensor is, by its definition $\mathbf{F} = \mathrm{d}\mathbf{A}$, equal to

$$\mathbf{F} = \mathbf{F}_0 + \lambda\,\mathrm{d}\mathbf{v} + O(\lambda^2) \ , \tag{3.24}$$

with $\mathbf{F}_0 = \mathrm{d}\mathbf{K}$.

This construction must satisfy generalised Maxwell's equations at the $O(\lambda^1)$ order. The first Maxwell's equation is satisfied since

$$\mathrm{d}\mathbf{F} = \mathrm{d}\mathbf{F}_0 + \lambda \mathrm{d}^2\mathbf{v} = 0 \ , \tag{3.25}$$

while the second one becomes

$$\mathrm{d}{\star}\mathbf{Z} = -4\,\mathrm{d}(\mathscr{L}_{\mathcal{F}}\,{\star}\mathbf{F} - \mathscr{L}_{\mathcal{G}}\,\mathbf{F}) = 0\ . \tag{3.26}$$

The terms $\mathscr{L}_{\mathcal{F}}$ and $\mathscr{L}_{\mathcal{G}}$ can be expanded accordingly as

$$\mathscr{L}_{\mathcal{F}} = -\frac{1}{4} + \lambda \ell_{\mathcal{F}} + O(\lambda^2)\ , \quad \mathscr{L}_{\mathcal{G}} = \lambda \ell_{\mathcal{G}} + O(\lambda^2)\ , \tag{3.27}$$

after which we get

$$\mathrm{d}{\star}\mathbf{Z} = \lambda\,(\mathrm{d}{\star}\mathrm{d}\mathbf{v} - 4\,\mathrm{d}\ell_{\mathcal{F}} \wedge {\star}\mathrm{d}\mathbf{K} + 4\,\mathrm{d}\ell_{\mathcal{G}} \wedge \mathrm{d}\mathbf{K}) + O(\lambda^2)\ . \tag{3.28}$$

It remains to simplify the $\mathrm{d}\ell_{\mathcal{F}}$ and $\mathrm{d}\ell_{\mathcal{G}}$ terms, which are the functions of the electromagnetic invariants $\mathcal{F}$ and $\mathcal{G}$,

$$\mathrm{d}\ell_{\mathcal{F}} = \ell_{\mathcal{F}\mathcal{F}}\,\mathrm{d}\mathcal{F} + \ell_{\mathcal{F}\mathcal{G}}\,\mathrm{d}\mathcal{G}\ , \tag{3.29}$$

$$\mathrm{d}\ell_{\mathcal{G}} = \ell_{\mathcal{G}\mathcal{F}}\,\mathrm{d}\mathcal{F} + \ell_{\mathcal{G}\mathcal{G}}\,\mathrm{d}\mathcal{G}\ . \tag{3.30}$$

The electromagnetic invariants can be expanded in the same manner,

$$\mathcal{F} = \mathcal{F}_0 + 2\lambda(\mathrm{d}\mathbf{K})_{ab}(\mathrm{d}\mathbf{v})^{ab} + O(\lambda^2)\ , \tag{3.31}$$

$$\mathcal{G} = \mathcal{G}_0 + 2\lambda({\star}\mathrm{d}\mathbf{K})_{ab}(\mathrm{d}\mathbf{v})^{ab} + O(\lambda^2)\ . \tag{3.32}$$

The obtained master equation for v_a is

$$\mathrm{d}{\star}\mathrm{d}\mathbf{v} = {\star}\mathbf{J}_{\mathrm{eff}}\ , \tag{3.33}$$

with the effective current 1-form $\mathbf{J}_{\mathrm{eff}}$ defined by

$${\star}\mathbf{J}_{\mathrm{eff}} = 4(\ell_{\mathcal{F}\mathcal{F}}\,\mathrm{d}\mathcal{F} + \ell_{\mathcal{F}\mathcal{G}}\,\mathrm{d}\mathcal{G})_0 \wedge {\star}\mathrm{d}\mathbf{K} - 4(\ell_{\mathcal{G}\mathcal{F}}\,\mathrm{d}\mathcal{F} + \ell_{\mathcal{G}\mathcal{G}}\,\mathrm{d}\mathcal{G})_0 \wedge \mathrm{d}\mathbf{K}\ . \tag{3.34}$$

The subscript "0" denotes terms evaluated for Wald's ansatz $\mathbf{F}_0 = \mathrm{d}\mathbf{K}$, which is the solution at the zeroth order.

As a consistency check, we can prove that ${\star}\mathbf{J}_{\mathrm{eff}}$ is a closed form,

$$\begin{aligned} \frac{1}{4}\mathrm{d}{\star}\mathbf{J}_{\mathrm{eff}} = {} & (\ell_{\mathcal{F}\mathcal{F}\mathcal{G}}\mathrm{d}\mathcal{G} \wedge \mathrm{d}\mathcal{F} + \ell_{\mathcal{F}\mathcal{G}\mathcal{F}}\mathrm{d}\mathcal{F} \wedge \mathrm{d}\mathcal{G})_0 \wedge {\star}\mathrm{d}\mathbf{K} - \\ & - (\ell_{\mathcal{G}\mathcal{F}\mathcal{G}}\mathrm{d}\mathcal{G} \wedge \mathrm{d}\mathcal{F} + \ell_{\mathcal{G}\mathcal{G}\mathcal{F}}\mathrm{d}\mathcal{F} \wedge \mathrm{d}\mathcal{G})_0 \wedge \mathrm{d}\mathbf{K} = 0\ . \end{aligned} \tag{3.35}$$

The claim follows since partial derivatives commute and $\mathrm{d}\mathcal{F} \wedge \mathrm{d}\mathcal{G} = -\mathrm{d}\mathcal{G} \wedge \mathrm{d}\mathcal{F}$.

The master equation can be simplified further if we notice that $\ell_{\mathcal{F}\mathcal{G}}$ term vanishes for the model Lagrangians (2.48) and (2.50),

$$\mathrm{d}{\star}\mathrm{d}\mathbf{v} = 4(\ell_{\mathcal{F}\mathcal{F}}\,\mathrm{d}\mathcal{F})_0 \wedge {\star}\mathrm{d}\mathbf{K} - 4(\ell_{\mathcal{G}\mathcal{G}}\,\mathrm{d}\mathcal{G})_0 \wedge \mathrm{d}\mathbf{K}\ . \tag{3.36}$$

One can ask whether the use of test field approximation is justified when dealing with nonlinear electromagnetic fields, whose effects are noticed on higher energy scales. By simple estimations of order of magnitude, we will show that there exists a range of energies in which the electromagnetic field is strong enough to exhibit nonlinear effects, but still weak enough not to alter the spacetime metric. If Einstein's tensor G_{ab} is of order L_g^{-2}, where L_g is the relevant gravitational length scale for the problem, and the energy density of the magnetic field is $B^2/(2\mu_0)$, test field approximation will hold if

$$L_g^{-2} \gg 4\pi G B^2/(c^4\mu_0)\ , \tag{3.37}$$

which was obtained from Einstein's equation. As a characteristic gravitational length scale, we can choose the Schwarzschild radius, $L_g \sim 3(M/M_\odot)\cdot 10^3\mathrm{m}$, where $M_\odot$ is the Solar mass. The condition on the magnetic field strength is $|\mathbf{B}| \ll (M_\odot/M)\cdot 10^{15}\mathrm{T}$, implying that even the strongest known magnetic fields are within the test field regime when the black hole mass is below the order of $10^4 M_\odot$.

3.2.2 Solution in Schwarzschild spacetime

Schwarzschild spacetime represents a static, spherically symmetric solution of the vacuum Einstein's equation. Its metric in (t, r, θ, ϕ) coordinates can be written as [187]

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\ , \tag{3.38}$$

with

$$f(r) = 1 - \frac{2M}{r}\ . \tag{3.39}$$

We consider the general Killing vector field composed as a linear combination of a timelike Killing vector field k^a and an axial Killing vector field m^a

$$K^a = \alpha k^a + \beta m^a\ , \tag{3.40}$$

where α and β are real constants. The electromagnetic invariants calculated at the zeroth order, for $\mathbf{F}_0 = \mathrm{d}\mathbf{K}$, are equal to

$$\mathcal{F}_0 = -\frac{8M^2}{r^4}\,\alpha^2 + 8\left(1 - \frac{2M}{r}\,\sin^2\theta\right)\beta^2 \tag{3.41}$$

and

$$\mathcal{G}_0 = -16M\,\frac{\cos\theta}{r^2}\,\alpha\beta\ . \tag{3.42}$$

In Wald's solution, parameter α was proportional to the angular momentum of a black hole. In the case of a static spacetime, the angular momentum is equal to zero so we may set $\alpha = 0$. This choice will also *a posteriori* prove to be the appropriate one for our problem. Notice that we automatically have $\mathcal{G}_0 = 0$, so with these simplifications, the master equation becomes

$$\mathrm{d}{\star}\mathrm{d}\mathbf{v} = 4\beta(\ell_{\mathcal{F}\mathcal{F}}\,\mathrm{d}\mathcal{F})_0 \wedge {\star}\mathrm{d}\mathbf{m}\ . \tag{3.43}$$

A simple calculation gives us

$$\mathrm{d}\mathcal{F}_0 = \frac{16M}{r^2}\beta^2\sin^2\theta\mathrm{d}r - \frac{32M}{r}\beta^2\sin\theta\,\cos\theta\mathrm{d}\theta\,, \tag{3.44}$$

and finally, using (A.39),

$$\mathrm{d}\mathcal{F}_0 \wedge {\star}\mathrm{d}\mathbf{m} = \frac{32M\beta^2\sin\theta}{r}\,\left(f(r)\sin^2\theta - 2\cos^2\theta\right)\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\theta\ . \tag{3.45}$$

Relying on the symmetries of the spacetime, the appropriate ansatz for v_a is $\mathbf{v} = h(r,\theta)\,\mathrm{d}\varphi$. The solution is given by

$$\mathbf{v} = C\Big(4(2r-5M)\cos(2\theta) + (M-2r)\big(3+\cos(4\theta)\big)\Big)\,\mathrm{d}\varphi\ , \tag{3.46}$$

with an unknown constant C. But, as

$$\mathrm{d}{\star}\mathrm{d}\mathbf{v} = \frac{64C\sin\theta}{r}\,\left(f(r)\sin^2\theta - 2\cos^2\theta\right)\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\theta \tag{3.47}$$

it follows that $C = 2\beta^3M(\ell_{\mathcal{F}\mathcal{F}})_0$. We still have one undetermined constant β, which can be fixed by the boundary conditions. On physical grounds, our solution must respect several conditions. First, we have to check whether it really represents an asymptotically homogeneous magnetic field. Wald's solution evaluated for

Schwarzschild spacetime is given by

$$\mathbf{F}_0 = \frac{1}{2} B_\infty \, \mathrm{d}\mathbf{m} = B_\infty \big(r \sin^2\theta \, \mathrm{d}r \wedge \mathrm{d}\varphi + r^2 \cos\theta \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\varphi \big) \ . \tag{3.48}$$

The homogeneous magnetic field in Minkowski spacetime can be written as $B_\infty \mathrm{d}z = B_\infty \mathrm{d}(r\cos\theta)$, while the electromagnetic field tensor is

$$\mathbf{F}_\infty = B_\infty \big(r \sin^2\theta \, \mathrm{d}r \wedge \mathrm{d}\varphi + r^2 \cos\theta \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\varphi \big) \ , \tag{3.49}$$

which has the same form as (3.48). Since Schwarzschild spacetime is asymptotically flat, the expression (3.48) represents a homogeneous magnetic field. The corrected electromagnetic tensor $\mathbf{F} = \mathbf{F}_0 + \lambda \mathrm{d}\mathbf{v}$ will asymptotically behave as Wald's $\mathbf{F}_0$, provided that

$$\lim_{r\to\infty} \frac{(\mathrm{d}\mathbf{v})_{r\varphi}}{(\mathbf{F}_0)_{r\varphi}} = 0 \quad \text{and} \quad \lim_{r\to\infty} \frac{(\mathrm{d}\mathbf{v})_{\theta\varphi}}{(\mathbf{F}_0)_{\theta\varphi}} = 0 \ , \tag{3.50}$$

which is fulfilled since $\mathrm{d}\mathbf{v}$ is equal to

$$\begin{aligned} \mathrm{d}\mathbf{v} = &- 32\beta^3 M (\ell_{\mathcal{F}\mathcal{F}})_0 \, \big(\sin^4\theta \, \mathrm{d}r \wedge \mathrm{d}\varphi + \\ &+ (2r - 5M + (M - 2r)\cos(2\theta)) \sin\theta \cos\theta \, \mathrm{d}\theta \wedge \mathrm{d}\varphi \big). \end{aligned} \tag{3.51}$$

By comparison with Wald's solution, we can set the normalisation to $\beta = B_\infty/2$, so the final form of perturbative correction v_a is given by

$$\mathbf{v} = \frac{(\ell_{\mathcal{F}\mathcal{F}})_0}{4} \, B_\infty^3 M \Big(4(2r - 5M)\cos(2\theta) + (M - 2r)\big(3 + \cos(4\theta)\big) \Big) \, \mathrm{d}\varphi \ . \tag{3.52}$$

We also demand that the electric and magnetic charges, defined by Komar integrals, remain equal to zero at the $O(\lambda^1)$ level. Since they both vanish for the basic ansatz $\mathbf{F}_0 = \mathrm{d}\mathbf{K}$, the perturbative correction must not contribute to the integrals. The term appearing in the definition of the electric charge is

$$\star\mathbf{Z} = \star\mathbf{F}_0 + \Big(4(-\ell_{\mathcal{F}} \star\mathbf{F} + \ell_{\mathcal{G}} \mathbf{F})_0 + \star\mathrm{d}\mathbf{v} \Big) \lambda + O(\lambda^2) \ . \tag{3.53}$$

Using the fact that $\ell_{\mathcal{F}} = 2\mathcal{F}$, $\ell_{\mathcal{G}} = 2\mathcal{G}$, $\lim_{r\to\infty} \mathcal{F}_0 = 8\beta^2$, $(\star\mathbf{F}_0)_{\theta\varphi} = 0$ and $(\star\mathrm{d}\mathbf{v})_{\theta\varphi} = 0$, the electric charge Q_∞ is zero at the $O(\lambda^1)$ order. The magnetic charge is defined with respect to $\mathrm{d}\mathbf{F} = \lambda \mathrm{d}\mathbf{v}$, while the relevant component that contributes to it is $(\mathrm{d}\mathbf{v})_{\theta\phi}$, given in (3.51). Terms proportional to $\sin(2\theta)$ and $\sin(4\theta)$ vanish after integration over the interval $[0, \pi]$. Hence, the magnetic charge is also unaltered by the perturbative correction.

An alternate way of solving this problem is by introducing the magnetic scalar potential [168, 169, 71]. If the electromagnetic field is symmetry inheriting [199, 200, 127, 42, 152, 186, 185, 181, 46, 11] and the source-free Maxwell's equation $\mathrm{d}{\star}\mathbf{F} = 0$ holds, the magnetic field 1-form is closed,

$$\mathrm{d}\mathbf{B}[K] = \mathrm{d}i_K{\star}\mathbf{F} = (\pounds_K - i_K\mathrm{d})\,{\star}\mathbf{F} = 0\ . \tag{3.54}$$

Then, we can locally write $\mathbf{B} = -\mathrm{d}\Psi$, where Ψ is magnetic scalar potential. Furthermore, if the black hole exterior is simply connected, magnetic scalar potential is globally well-defined. As an illustration of the method, we can calculate the magnetic field for Wald's solution and its corresponding scalar potential. The magnetic field, defined with respect to the timelike Killing vector field $k = \partial/\partial t$ is

$$\mathbf{B}_0[k] = B_\infty\,(\cos\theta\,\mathrm{d}r - rf(r)\sin\theta\,\mathrm{d}\theta)\ , \tag{3.55}$$

and the magnetic scalar potential is

$$\Psi_0 = -B_\infty f(r)\,r\cos\theta\ . \tag{3.56}$$

Note that the scalar potential is constant over the horizon, analogously to the surface gravity κ [168, 169]. In this case, the gauge is chosen such that the potential vanishes at the horizon, $\lim_{r\to 2M}\Psi_0 = 0$, while at the spatial infinity it approaches $\lim_{r\to\infty}\Psi_0 = -B_\infty z$. The NLE case can be treated in the same manner, with the caveat that the magnetic form $\mathbf{B}$ is no longer closed. Nevertheless, we can define a nonlinear 1-form $\mathbf{H}$ (2.17), which is closed since

$$\mathrm{d}\mathbf{H}[K] = \mathrm{d}i_K{\star}\mathbf{Z} = (\pounds_K - i_K\mathrm{d})\,{\star}\mathbf{Z} = 0\ , \tag{3.57}$$

and its associated scalar potential Υ, via $\mathbf{H}[K] = -\mathrm{d}\Upsilon$. A divergence identity that defines the equation for the scalar potential can be derived in a few steps. Starting from the auxiliary expression

$$\begin{aligned}
\delta\mathbf{B} = \delta i_k{\star}\mathbf{F} &= -{\star}\mathrm{d}(\mathbf{F}\wedge\mathbf{k}) = -{\star}(\mathrm{d}\mathbf{F}\wedge\mathbf{k}) - {\star}(\mathbf{F}\wedge\mathrm{d}\mathbf{k}) = \\
&= \frac{1}{N}{\star}(\mathbf{k}\wedge\mathbf{E}\wedge\mathrm{d}\mathbf{k} + {\star}(\mathbf{k}\wedge\mathbf{B})\wedge\mathrm{d}\mathbf{k}) = \frac{1}{N}i_E{\star}(\mathbf{k}\wedge\mathrm{d}\mathbf{k}) - \\
&- \frac{1}{N}(\mathbf{k}\wedge\mathbf{B}|\mathrm{d}\mathbf{k}) = -\frac{1}{N}(\mathbf{B}|i_k\mathrm{d}\mathbf{k}) = \frac{1}{N}(\mathbf{B}|\mathrm{d}N),
\end{aligned} \tag{3.58}$$

where $N = k^a k_a$, $\mathrm{d}\mathbf{F} = 0$ and twist one-form $\boldsymbol{\omega}_k = -{\star}(\mathbf{k}\wedge\mathrm{d}\mathbf{k})$ is set to zero, we have

$$\nabla_a \left(\frac{B[k]^a}{N} \right) = \frac{\nabla^a B_a}{N} - \frac{1}{N^2} B_a (dN)^a = 0. \tag{3.59}$$

If we introduce a differential operator L equal to $L[\Psi] = -r^2 f(r) \nabla^a((\nabla_a \Psi)/N)$ in Schwarzschild spacetime, the equation for axially symmetric potential Ψ is

$$L[\Psi] := f(r) \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \Psi}{\partial \theta} \right) = 0 \ . \tag{3.60}$$

The equation can be solved by separation of variables, $\Psi(r,\theta) = R(r)P(\cos\theta)$. The angular part is a Legendre differential equation, while the radial part is of the form

$$R_\ell(r) = \left(\frac{r}{2M} - 1 \right) \left(a_\ell P'_\ell \left(\frac{r}{M} - 1 \right) + b_\ell Q'_\ell \left(\frac{r}{M} - 1 \right) \right) \ , \tag{3.61}$$

where P_ℓ is a Legendre polynomial and Q_ℓ is a Legendre function of the second kind.

In the NLE case, the equation (3.59) still holds, but it has to be expressed via the nonlinear magnetic field H which is related to the magnetic scalar potential Υ. For a purely magnetic solution, i.e. $k^b F_{ab} = 0$, there exists a simple relation between forms $\mathbf{H}$ and $\mathbf{B}$:

$$H[k]_a = k^b \star Z_{ba} = -4 \mathscr{L}_\mathcal{F} k^b \star F_{ba} = -4 \mathscr{L}_\mathcal{F} B[k]_a \ , \tag{3.62}$$

which gives

$$\nabla_a \left(\frac{H[k]^a}{N \mathscr{L}_\mathcal{F}} \right) = 0 \ . \tag{3.63}$$

Again we expand electromagnetic quantities with respect to the coupling constant λ. We compactly write $\ell = p\mathcal{F}^2 + q\mathcal{G}^2$, where $(p,q) = (4,7)$ for Euler–Heisenberg Lagrangian and $(p,q) = (1,1)$ for Born–Infeld Lagrangian. Since $\mathscr{L}_\mathcal{F} = -1/4 + 2p\mathcal{F}$, the divergence identity (3.63) gives us

$$\nabla_a \left(\frac{H^a}{N} \right) - 16 p \lambda \nabla_a \left(\frac{H_b H^b}{N^2} H^a \right) + O(\lambda^2) = 0 \ . \tag{3.64}$$

The expanded form of magnetic scalar potential

$$\Upsilon = \Psi_0 + \lambda \Psi_1 + O(\lambda^2) \tag{3.65}$$

may be used in the divergence identity (3.64). This gives us a zeroth order equation

$\nabla^a((\nabla_a\Psi_0)/N) = 0$, while at λ order we have

$$\nabla^a\left(\frac{\nabla_a\Psi_1}{N}\right) = 16p\,\nabla^a\left(\frac{(\nabla_b\Psi_0)(\nabla^b\Psi_0)}{N^2}\,\nabla_a\Psi_0\right) + O(\lambda^2). \tag{3.66}$$

Inserting Wald's solution (3.56) into (3.66) yields

$$L[\Psi_1] = 48pMB_\infty^3 f(r)\sin(2\theta)\sin\theta\ . \tag{3.67}$$

The suitable ansatz is of the form

$$\Psi_1(r,\theta) = f(r)\Big(a(r) + b(r)\cos(2\theta)\Big)\cos\theta. \tag{3.68}$$

After discarding the part that grows faster than $O(r^1)$ at spatial infinity, we have

$$\Psi_1(r,\theta) = 4pB_\infty^3\, f(r)\Big(4r - 5M + M\cos(2\theta)\Big)\cos\theta\ . \tag{3.69}$$

It is straightforward but tedious to show that the magnetic field calculated from (3.46) corresponds to the one given by the magnetic scalar potential. Combining the two definitions of field $\mathbf{H}$ provides the relation between $\mathbf{F}$ and Υ,

$$-4\mathscr{L}_{\mathcal{F}}k^b{\star}F_{ba} = -\mathrm{d}\Upsilon. \tag{3.70}$$

We may perform expansion with respect to λ and compare the terms order by order. On the left hand side of (3.70) we have

$$\begin{aligned}
&-4\Big(-\frac{1}{4} + \lambda\ell_{\mathcal{F}}\Big)(k^b({\star}\mathrm{d}\mathbf{m})_{ba} + \lambda k^b({\star}\mathrm{d}\mathbf{v})_{ba}) + O(\lambda^2) = \\
&=k^b({\star}\mathrm{d}\mathbf{m})_{ba} - 4\lambda\ell_{\mathcal{F}}\beta k^b({\star}\mathrm{d}\mathbf{m})_{ba} + \lambda k^b({\star}\mathrm{d}\mathbf{v})_{ba} + O(\lambda^2) = \\
&=k^b({\star}\mathrm{d}\mathbf{m})_{ba} - 8\lambda p\mathcal{F}_0\beta k^b({\star}\mathrm{d}\mathbf{m})_{ba} + \lambda k^b({\star}\mathrm{d}\mathbf{v})_{ba} + O(\lambda^2)
\end{aligned} \tag{3.71}$$

By referring to $(A.39)$ and $(A.41)$, it is not difficult to see that the $O(\lambda^0)$ term is equal to $-\mathrm{d}\Psi_0$, while the term proportional to λ returns $-\mathrm{d}\Psi_1$.

We can analyse the obtained correction from many different angles. For example, we could look at the expansion of the magnetic field defined with respect to the Killing vector field k^a,

$$\mathbf{B}[k] = \mathbf{B}_0[k] + \lambda\,\mathbf{B}_1[k]\ , \quad \mathbf{B}_1[k] := i_k{\star}\mathrm{d}\mathbf{v}\ , \tag{3.72}$$

explicitly, the NLE contribution is

$$\begin{aligned}\mathbf{B}_1[k] =&4M(\ell_{\mathcal{F}\mathcal{F}})_0 B_\infty^3 \Big(f(r) \sin^3\theta \, \mathrm{d}\theta - \\ &- \frac{\cos\theta}{r} \left(2r - 5M + (M-2r)\cos(2\theta) \right) \mathrm{d}r \Big) . \end{aligned} \tag{3.73}$$

However, the magnetic field is an observer-dependent quantity. For example, the static observer moving with the 4-velocity $u^a = k^a/\sqrt{-N}$ would measure the field

$$B[u]^a = \frac{1}{\sqrt{f(r)}} \, B[k]^a \ . \tag{3.74}$$

Therefore, it is better to analyse observer-invariant quantities, such as electromagnetic invariants. The first electromagnetic invariant can be decomposed as $\mathcal{F} = \mathcal{F}_0 + \delta\mathcal{F}$, where $\delta\mathcal{F}$ is the first order correction. To put all the prefactors aside, we introduce the rescaled correction

$$\widehat{\mathcal{F}}_1 := -\frac{1}{16 B_\infty^3 M (\ell_{\mathcal{F}\mathcal{F}})_0} \, (\mathrm{d}m)_{ab} (\mathrm{d}v)^{ab} \ , \tag{3.75}$$

so that

$$\delta\mathcal{F} = -16\lambda B_\infty^4 M (\ell_{\mathcal{F}\mathcal{F}})_0 \, \widehat{\mathcal{F}}_1 + O(\lambda^2) \ . \tag{3.76}$$

Direct calculation gives

$$\widehat{\mathcal{F}}_1 = \frac{1}{r} \, f(r) \sin^4\theta + \frac{\cos^2\theta}{r} \left((3 + f(r)) \sin^2\theta - 2(1 - f(r)) \right) \ . \tag{3.77}$$

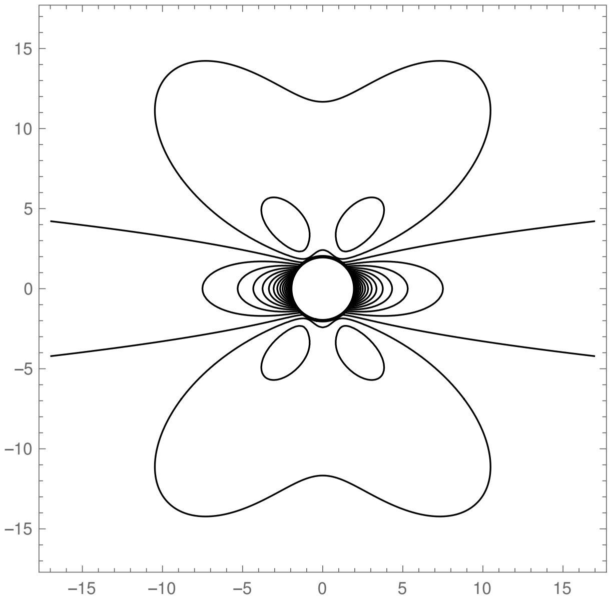
Since $\widehat{\mathcal{F}}_1$ remains bounded as $r \to 2M$, the solution is regular at the black hole horizon. Figure[1] 3.1 shows the contour plots of $\widehat{\mathcal{F}}_1$. In the figure, it can be noticed that there are two local maxima along circles at $(r_c, \theta_\pm)$. Their values can be obtained analytically. From $\partial_r \widehat{\mathcal{F}}_1 = 0$ and $\partial_\theta \widehat{\mathcal{F}}_1 = 0$, we get a system of equations

$$48M - 7r + 4(r + 4M)\cos(2\theta) + 3r\cos(4\theta) = 0 \ , \tag{3.78}$$

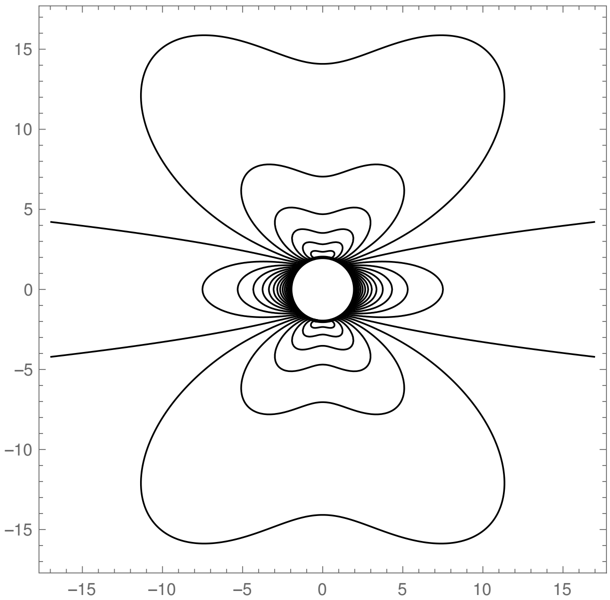
$$(r + 2M + 3r\cos(2\theta))\sin(2\theta) = 0 \ , \tag{3.79}$$

which can be simplified with a substitution $x = \cos(2\theta)$. The solution in the black hole exterior, $r > 2M$ is

$$r_c = \frac{4 + \sqrt{13}}{2} \, M \ , \quad \cos(2\theta_\pm) = \frac{4\sqrt{13} - 19}{9} \ . \tag{3.80}$$



(a) Contour plot of $\widehat{\mathcal{F}}_1$ [16].



(b) Contour plot of rescaled relative correction $8\beta^2\widehat{\mathcal{F}}_1/\mathcal{F}_0$ [16].

Figure 3.1: Contour plots in $r-\theta$ plane with $M=1$. The black circle in the middle represents black hole horizon.

Approximately, these are $r_c \approx 3.8M$, $\theta_+ \approx 60.3°$ and, as $\cos(2(\pi-\theta)) = \cos(2\theta)$, $\theta_- \approx 119.7°$. An intriguing question is whether the astrophysical tests could determine the influence of these maxima on the trajectories of the charged particles around black holes.

If an NLE model contains a c_{11} term in the expansion, but the $\mathcal{G}_0 = 0$ condition is still valid, the additional term appears in the master equation,

$$\mathrm{d}\mathcal{F}_0 \wedge \mathrm{d}\mathbf{m} = 96\beta^2 M \sin^3\theta \cos\theta \, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\varphi \ . \tag{3.81}$$

The new solution can be obtained by a simple modification of ansatz, $A_a = \beta m_a + \lambda(v_a + \widetilde{v}_a)$, which gives

$$\widetilde{\mathbf{v}} = 2Mf(r)(\cos(3\theta) - 9\cos\theta) \, \mathrm{d}t \ . \tag{3.82}$$

This solution does not introduce any additional charges and the magnetic field stays asymptotically homogeneous, but alongside it there is a nonvanishing electric field manifesting as $k^b F_{ab} \neq 0$, even as $r \to \infty$.

3.2.3 Comment on neutron stars

Instead of a black hole, we can consider an idealised neutron star immersed in a test magnetic field. The basic setup of the problem remains the same, up to the boundary conditions which have to be matched at the star's surface. Our model assumes a spherically symmetric, perfectly conducting star. We neglect the fact that the electric conductivity varies between the layers of the star, from the superconducting core to the less conducting outer parts [39, 148].

It is well known that superconducting materials show the Meissner effect, the total expulsion of the external magnetic field. When a superconducting ball of radius R is placed in an external magnetic field of strength B_∞, the total field is a superposition of the external field and the one originating from the induced surface currents, in the form of a dipole field. Junction condition $\mathbf{B} \cdot \hat{\mathbf{n}}|_{r=R} = 0$, continuity of the normal component of the magnetic field, gives the induced magnetic dipole moment, which is in Minkowski spacetime equal to $\mu = -B_\infty R^3/2$.

Let us discuss the junction condition in detail. Suppose that a static spacetime can be foliated by spacelike hypersurfaces Σ, each of which contains a compact spacelike 2-surface $S \subseteq \Sigma$ with a normal n^a, representing the star's boundary. If the norm of the Killing vector N is continuous at S, from the divergence identity (3.59) it follows that the normal component of the magnetic field, $n^a B_a$, also has to be continuous at S. For a superconducting star, magnetic field vanishes in its interior, translating the condition into $n^a B_a = 0$ at S. The scalar potential satisfies the Neumann boundary condition, $n^a \nabla_a \Psi = 0$ at S.

Ginzburg and Ozernoy discussed the problem of the magnetic dipole field in Schwarzschild spacetime [69]. Their solution is equal to the $\ell = 1$ term in the scalar potential,

$$\Psi_{\rm GO}(r,\theta) = \frac{3\mu}{(2M)^2}\Big(1 + f(r) + \frac{r}{M}\, f(r)\ln f(r)\Big)\cos\theta\ . \tag{3.83}$$

The lowest order term in the asymptotic expansion for large r is just a standard dipole potential in Minkowski spacetime,

$$\Psi_{\rm GO}(r,\theta) = \Big(r^{-2} + O(r^{-3})\Big)\mu\cos\theta\ . \tag{3.84}$$

We can superpose (3.84) and Wald's solution as we are considering an asymptotically homogeneous magnetic field,

$$\Psi = \Psi_0 + \Psi_{\rm GO}\ . \tag{3.85}$$

The boundary condition on the surface of a superconducting ball of radius $R > 2M$ is of the Neumann type,

$$\frac{\partial\Psi(R,\theta)}{\partial r} = 0\ , \tag{3.86}$$

from which we infer the induced magnetic dipole moment

$$\mu = \frac{(2M)^2 B_\infty}{3}\left(\frac{3-f(R)}{R} + \frac{1}{M}\,\ln f(R)\right)^{-1}\ . \tag{3.87}$$

In order to compare this result with the flat case, the dipole moment μ can be understood as a function of mass M and expanded in Taylor series around $M = 0$,

$$\mu(M) = -\frac{B_\infty R^3}{2} + \frac{3B_\infty R^2}{4}\, M + O(M^2)\ . \tag{3.88}$$

The two results coincide since $\lim_{M\to 0}\mu(M) = -B_\infty R^3/2$. The linearity of Maxwell's equations enables us to superpose an internal star's magnetic field to this solution.

The nonlinear case demands special care, as we have to adjust the boundary condition. Namely, the relation between 1-forms $\mathbf{B}$ and $\mathbf{H}$ is given by (3.62). If $\mathscr{L}_\mathcal{F}$ is finite at S, the boundary condition $n^a B_a = 0$ at S implies $n^a H_a = 0$. Written with respect to the potential, the boundary condition becomes $n^a \nabla_a \Upsilon = 0$. As the seed solution we may take $\Psi = R(r)\cos\theta$, then the linearized equation for the potential (3.66) becomes

$$L[\Psi_1] = -\frac{4p\sin(2\theta)}{f(r)}\,(\rho_+(r) + \rho_-(r)\cos(2\theta))\ , \tag{3.89}$$

with two auxiliary functions,

$$\rho_{\pm}(r) = \pm\Big(r(r-2M)R'' - 2MR' - 4R\Big)R^2 + \\ + r(r-2M)\Big((r-2M)(3rR'' + 2R') + (-2\pm 4)R\Big)R'^2\ . \tag{3.90}$$

This equation can be solved with the help of *Mathematica* package. However, the solution is given in a rather involved form, as an infinite series with terms consisting of nontrivial functions of a radial coordinate. In this form, imposing the boundary conditions is a daunting task, so this problem remains open.

Chapter 4

NLE fields in strictly stationary spacetimes

4.1 Gravitating solitons

In an attempt to capture the idea of localised, singularity-free and self-consistent objects, Wheeler [196] proposed the existence of *geons*, where the term geon stands for gravitational electromagnetic entities. Geons would emerge as solutions of coupled gravitational and Maxwell's field equations and be sustained by their own gravitational pull. As an illustration of a concept, Wheeler suggested a standing electromagnetic wave wound in a specific toroidal configuration, representing an energy clump held together by its own gravity. Two nontrivial questions, which would either rule out geons or work in favour of their viability, arise: Do they constitute stable configurations and what are the exact solutions representing geons. After a search for electromagnetic [58] and purely gravitational geons [26], the canonical stand is that topologically trivial, vacuum or electrovacuum, *stable* geons do not exist within the framework of general relativity [145]. Therefore, if one insists on the initial definition of a geon as an object composed of self-gravitating standing waves, it seems highly likely that such a structure will be unstable.

The quest for finding geon-like configurations in general relativity is closely related to the classification of the solutions of Einstein's equation. Obtaining the complete picture which would encompass all the solutions is, even under additional constraints, a formidable problem. One can then resort to idealisations in terms of spacetimes admitting symmetries, such as stationary spacetimes. The physical significance of this class of solutions is that time-independent spacetimes represent equilibrium field states. Prominent examples are stationary black hole solutions, whose variety is limited by several uniqueness and no-hair theorems [96, 40, 95, 35, 36].

The proofs of the theorems are based on Lichnerowicz's argument [120]: if the integral of a conveniently constructed non-negative quantity is non-positive over a given domain, the quantity has to be equal to zero. A similar idea may be employed to formulate another type of constraints known as no-soliton theorems. In this context, solitons are defined as stationary, asymptotically flat, everywhere regular solutions of Einstein's equation, thus representing examples of Wheeler's geons in a broader sense. The existence of vacuum solitons in general relativity is prohibited by the argument presented in [3]. However, instances of self-gravitating solitons may be found if massive fields of different spin are coupled to gravity. Solutions representing scalar, Dirac and Proca stars were obtained numerically for both static [91] and rotating cases [90].

Classification can be further simplified by demanding strict stationarity, which is a more stringent condition on the causal structure of the spacetime. It implies that the timelike Killing vector field does not change its causal behaviour throughout spacetime. Static black hole solutions violate this criterion since the Killing vector field corresponding to the stationary isometry becomes spacelike inside the Killing horizons, while the same situation occurs inside the ergoregions of rotating black holes. A well-established fact is the absence of electromagnetic solitons in Einstein-Maxwell theory, precisely, there are no strictly stationary, globally regular, asymptotically flat solutions with a nontrivial electromagnetic field [40]. No-soliton theorems apply also to the case when various scalar fields are added to the Einstein-Maxwell system [88, 89, 166]. The first generalisation of the no-soliton theorem for NLE fields was established for the truncated Born-Infeld and power-Maxwell theories [32]. Our goal is to formulate a broad extension of this result which would cover all NLE Lagrangians belonging to the $\mathcal{FG}$-class. Additionally, we pave the way for dealing with sources represented as charged matter fields and explore the possibilities if the number of spacetime dimensions is different than four. The results of this chapter were presented in [15].

4.2 No-go theorems for NLE fields

We present two no-soliton theorems that apply to slightly different spacetime setups. However, both theorems refer to strictly stationary spacetimes, meaning that spacetimes containing black holes or cosmological horizons are excluded. To prove the theorems, we will invoke two energy conditions: the null energy condition (NEC), which holds if and only if $\mathscr{L}_\mathcal{F} \le 0$ and the dominant energy condition (DEC), valid if and only if $\mathscr{L}_\mathcal{F} \le 0$ and $T \le 0$ [14, 146]. The first theorem holds for an arbitrary gravitational theory whose corresponding field tensor is divergence-free, as long as the coupling between the electromagnetic and gravitational Lagrangian is minimal. Its limitation lies in the fact that it is applicable to static spacetimes only. The second theorem remains valid in nonstatic spacetimes, but its generality is reduced since it relies on Einstein's gravitational field equation and positive energy theorem (for an overview see [47]). Given that the energy density is locally nonnegative, the positive energy theorem states that the total energy associated to the isolated gravitating system is nonnegative, where the zero value coincides with Minkowski spacetime. Shoen and Yau proved it in a series of papers, each of which gradually relaxed the initial assumptions needed for the proof [162, 161, 164, 163]. Witten [197] took another approach based on spinor calculus and presented a simpler proof via the construction of a nonnegative integral which represents the energy of a system. The mathematical technicalities of Witten's proof were further polished by Parker and Taubes [142].

Our two main results are based on several common technical assumptions, listed below.
(1) The spacetime is a four-dimensional smooth, simply connected manifold $\mathscr{M}$ with a smooth Lorentzian metric g_{ab} and a smooth electromagnetic 2-form F_{ab} which are solutions of the gravitational-NLE field equations, with the NLE Lagrangian density $\mathscr{L}$ obeying the Maxwellian weak field limit
(2) The spacetime admits a strictly timelike Killing vector field k^a, meaning that its norm does not change sign throughout the spacetime, precisely $k^a k_a < 0$ on $\mathscr{M}$.
(3) The electromagnetic field is symmetry inheriting, so that $\pounds_k F_{ab} = 0$ [181, 46, 11].
(4) Through each point $p \in \mathscr{M}$ passes at least one complete oriented spacelike hypersurface Σ with induced metric h_{ij} and the associated extrinsic curvature K_{ij}, asymptotically flat so that the following fall-off conditions are met on each of its ends: $1 + k^\alpha k_\alpha = O_\infty(r^{-1})$, $k^\alpha g_{\alpha i} = O_\infty(r^{-1})$, $\gamma_{ij} = O_\infty(r^{-1})$ and $K_{ij} = O_\infty(r^{-2})$ [40, 142, 47]. The electromagnetic 2-form F_{ab} asymptotically behaves as $k^\alpha F_{\alpha i} = O_\infty(r^{-2})$ and $k^\alpha \star F_{\alpha i} = O_\infty(r^{-2})$, so that the associated potentials are of order[1] $O_\infty(r^{-1})$.
The two no-soliton theorems are formulated as follows.

Theorem 4.1. *Suppose that a spacetime with an electromagnetic field satisfies basic assumptions, with the electromagnetic energy-momentum tensor obeying the null energy condition, and where the Killing vector field k^a is hypersurface orthogonal. Then the electromagnetic field is at each point of the spacetime either trivial, $F_{ab} = 0$,*

[1] Following reference [40], we write $f = O(r^{-k})$ when f is of order $O(r^{-k})$ as $r \to \infty$ and $f = O_\infty(r^{-k})$ when $\partial_{i_1} \dots \partial_{i_\ell} f = O(r^{-k-\ell})$ for an arbitrary set of coordinate indices $\{i_1, \dots, i_\ell\}$.

or stealth.

Theorem 4.2. *Suppose that a spacetime with the electromagnetic field satisfies basic assumptions and the gravitational part of the action is the Einstein–Hilbert's with the electromagnetic energy-momentum tensor obeying the dominant energy condition. Then the spacetime is isometric to the Minkowski spacetime (R^4, η_{ab}) and the electromagnetic field is at each point of the spacetime either trivial, $F_{ab} = 0$, or stealth.*

4.2.1 Preliminaries and divergence identities

The theorems apply to spacetimes with a strictly timelike Killing vector field k^a, whose norm is related to the function $V := -k_a k^a > 0$. The associated twist 1-form

$$\boldsymbol{\omega} = -\star(\mathbf{k} \wedge \mathrm{d}\mathbf{k}) \tag{4.1}$$

"measures" the deviation from the case in which k^a is hypersurface orthogonal. Consequently, the twist 1-form vanishes in static spacetimes. The prefactors in the definitions of $\boldsymbol{\omega}$ differ throughout the literature, for example, Heusler [96] introduces a twist 1-form $\tilde{\boldsymbol{\omega}}$ such that $\boldsymbol{\omega} = -2\tilde{\boldsymbol{\omega}}$. Our choice is motivated by its simple form in the abstract index notation, $\omega_a = \epsilon_a{}^{bcd} k_b \nabla_c k_d$, without any additional factors present.

The integral parts of the proofs are appropriately constructed divergence identities. First, we derive an auxilliary result,

$$\begin{aligned} \mathrm{d}\left(\frac{\mathbf{k}}{V}\right) =& \frac{1}{V^2}\left(V\mathrm{d}\mathbf{k} - \mathrm{d}V \wedge \mathbf{k}\right) = -\frac{1}{V^2} i_k(\mathbf{k} \wedge \mathrm{d}\mathbf{k}) = \\ =& -\frac{1}{V^2}\star(\star(\mathbf{k} \wedge \mathrm{d}\mathbf{k}) \wedge \mathbf{k}) = \frac{1}{V^2}\star(\boldsymbol{\omega} \wedge \mathbf{k}) \ . \end{aligned} \tag{4.2}$$

Using the formula above together with the second generalised source-free Maxwell's equation ($\mathrm{d}\star\mathbf{Z} = 0$), we get

$$\begin{aligned} \nabla^a\left(\frac{D_a}{V}\right) &= -\star\mathrm{d}\star\left(-\frac{1}{V} i_k \mathbf{Z}\right) = -\star\mathrm{d}\left(\frac{1}{V}\mathbf{k} \wedge \star\mathbf{Z}\right) = \\ &= -\star\left(\frac{1}{V^2}\star(\boldsymbol{\omega} \wedge \mathbf{k}) \wedge \star\mathbf{Z}\right) = \frac{1}{V^2}(\star\mathbf{Z} \mid \boldsymbol{\omega} \wedge \mathbf{k}) = \\ &= -\frac{1}{V^2}(\boldsymbol{\omega} \mid i_k\star\mathbf{Z}) \ , \end{aligned} \tag{4.3}$$

where the second equality was derived from

$$-\star\, i_k \mathbf{Z} = \star\, i_k \star\star \mathbf{Z} = \mathbf{k} \wedge \star \mathbf{Z}\ . \tag{4.4}$$

Taking everything into account, it follows that

$$\nabla^a \left(\frac{D_a}{V} \right) = -\frac{\omega_a H^a}{V^2}\ . \tag{4.5}$$

An analogous identity can be obtained for the magnetic field,

$$\begin{aligned} \nabla^a \left(\frac{B_a}{V} \right) &= -\star \mathrm{d} \star \left(\frac{1}{V}\, i_k \star \mathbf{F} \right) = -\star \mathrm{d} \left(\frac{1}{V}\, \mathbf{k} \wedge \mathbf{F} \right) = \\ &= -\star \left(\frac{1}{V^2} \star (\boldsymbol{\omega} \wedge \mathbf{k}) \wedge \mathbf{F} \right) = \frac{1}{V^2} (\mathbf{F} \mid \boldsymbol{\omega} \wedge \mathbf{k}) = \\ &= -\frac{1}{V^2} (\boldsymbol{\omega} \mid i_k \mathbf{F})\ , \end{aligned} \tag{4.6}$$

finally,

$$\nabla^a \left(\frac{B_a}{V} \right) = \frac{\omega_a E^a}{V^2}\ . \tag{4.7}$$

Particularly useful identities are the ones involving squares of the fields, derived using (4.5) and (4.7),

$$\nabla^a \left(\frac{\Phi}{V}\, D_a \right) = \frac{4}{V} \left(\mathscr{L}_{\mathcal{F}} E_a E^a - \mathscr{L}_{\mathcal{G}} E_a B^a \right) - \Phi\, \frac{\omega_a H^a}{V^2}\ , \tag{4.8}$$

$$\nabla^a \left(\frac{\Psi}{V}\, B_a \right) = \frac{4}{V} \left(\mathscr{L}_{\mathcal{F}} B_a B^a + \mathscr{L}_{\mathcal{G}} E_a B^a \right) + \Psi\, \frac{\omega_a E^a}{V^2}\ , \tag{4.9}$$

where Φ and Ψ are scalar potentials defined in (2.20). If we focus on Einstein-Hilbert's gravitational action and invoke the Killing lemma [187], $\mathrm{d}\star\mathrm{d}\mathbf{k} = 2\star\mathbf{R}(k)$, the exterior derivative of the twist 1-fom is

$$\mathrm{d}\boldsymbol{\omega} = -\mathrm{d}\star(\mathbf{k} \wedge \mathrm{d}\mathbf{k}) = -\mathrm{d} i_k \star \mathrm{d}\mathbf{k} = 2 i_k \star \mathbf{R}(k) = -2\star(\mathbf{k} \wedge \mathbf{R}(k))\ . \tag{4.10}$$

Einstein's equation provides the relation between the Ricci tensor and the energy-momentum tensor,

$$\mathrm{d}\boldsymbol{\omega} = -2\star(\mathbf{k} \wedge \mathbf{R}(k)) = 64\pi \mathscr{L}_{\mathcal{F}} \star(\mathbf{k} \wedge \mathbf{T}^{(\mathrm{Max})}(k)) = 4\mathbf{E} \wedge \mathbf{H}\ , \tag{4.11}$$

where the last equality is calculated as follows:

$$16\pi \mathscr{L}_{\mathcal{F}} \star(\mathbf{k} \wedge \mathbf{T}^{(\mathrm{Max})}(k)) = 4\mathscr{L}_{\mathcal{F}} \star(\mathbf{k} \wedge i_E \mathbf{F}) = -4\mathscr{L}_{\mathcal{F}} i_k \star i_E \mathbf{F} =$$

$$= 4\mathscr{L}_{\mathcal{F}} i_k(\mathbf{E} \wedge \star\mathbf{F}) = -4\mathscr{L}_{\mathcal{F}}\mathbf{E} \wedge \mathbf{B} = \mathbf{E} \wedge \mathbf{H} \ . \tag{4.12}$$

We may recast this expression using the electromagnetic scalar potentials as

$$\mathrm{d}\boldsymbol{\omega} = -4(\mathrm{d}\Phi \wedge \mathbf{H}) = -4(\mathbf{E} \wedge \mathrm{d}\Psi) \ , \tag{4.13}$$

to obtain two closed 1-forms, $\boldsymbol{\omega} + 4\Phi\mathbf{H}$ and $\boldsymbol{\omega} - 4\Psi\mathbf{E}$. Then, we can introduce two new scalar potentials, U_E and U_H related to the twist 1-form,

$$\boldsymbol{\omega} = -4\Phi\mathbf{H} + \mathrm{d}U_H = 4\Psi\mathbf{E} + \mathrm{d}U_E \ . \tag{4.14}$$

The assumption (1), i.e. the simple connectedness of the manifold $\mathscr{M}$, guarantees that the scalar potentials are globally well-defined. By construction, it immediately follows that $\pounds_k U_E = 0$ and $\pounds_k U_H = 0$. Another fundamental identity,

$$\nabla^a \left(\frac{\omega_a}{V^2}\right) = 0 \ , \tag{4.15}$$

easily follows if one uses the previous result (4.2) and Eq. (A.17),

$$\mathbf{k} \wedge \delta\left(\frac{\boldsymbol{\omega}}{V^2}\right) = -\delta\left(\mathbf{k} \wedge \frac{\boldsymbol{\omega}}{V^2}\right) = \star\mathrm{d}\star\left(\mathbf{k} \wedge \frac{\boldsymbol{\omega}}{V^2}\right) = 0 \ . \tag{4.16}$$

The Eq. (4.15) enables deriving another two divergence identities,

$$\nabla^a \left(U_E \, \frac{\omega_a}{V^2}\right) = \frac{\omega^a \nabla_a U_E}{V^2} \ , \tag{4.17}$$

$$\nabla^a \left(U_H \, \frac{\omega_a}{V^2}\right) = \frac{\omega^a \nabla_a U_H}{V^2} \ . \tag{4.18}$$

4.2.2 Proofs of the theorems

Now that all the main tools are presented, we can apply them to prove the theorems.

Proof of theorem 4.1. The considered domain may be split into two parts depending on the value of $\mathscr{L}_{\mathcal{F}}$, governed by the energy conditions. In this regard, we introduce an auxiliary open set

$$O := \{x \in \mathscr{M} \mid \mathscr{L}_{\mathcal{F}}(x) \neq 0\} \ . \tag{4.19}$$

Due to the null energy condition, $\mathscr{L}_{\mathcal{F}} < 0$ for all $x \in O$ and $\mathscr{L}_{\mathcal{F}} = 0$ for all $y \in \mathscr{M} - O$. The set O is nonempty since the electromagnetic field decays along each end and the Lagrangian density obeys the Maxwellian weak field limit.

The gravitational field equation (2.4) at each point of the set $\mathscr{M} - O$ comes down to $E_{ab} = 2\pi T g_{ab}$. Due to the vanishing divergence of the gravitational tensor, $\nabla^a E_{ab} = 0$, the trace T is constant on each connected component of the interior of the domain, $(\mathscr{M} - O)^\circ$. As we assume staticity, $\boldsymbol{\omega} = 0$ and k^a is a hypersurface orthogonal Killing vector field. Integration is performed over an arbitrary spacelike hypersurface Σ defined in technical assumption (4). Inserting 1-forms $\boldsymbol{\alpha} = \mathbf{D}/V$ and $\boldsymbol{\alpha} = \mathbf{B}/V$, which satisfy $\pounds_k \boldsymbol{\alpha} = 0$ in (A.18), gives us

$$\int_{\partial\Sigma} \frac{1}{V} \star(\mathbf{k} \wedge \mathbf{D}) = 0 \ , \quad \int_{\partial\Sigma} \frac{1}{V} \star(\mathbf{k} \wedge \mathbf{B}) = 0 \ . \tag{4.20}$$

The boundary of Σ may be understood as the "sphere at infinity" and the integral is calculated in the limiting sense. For the 1-forms $\boldsymbol{\alpha} = \Phi\mathbf{D}/V$ and $\boldsymbol{\alpha} = \Psi\mathbf{B}/V$ we get, respectively,

$$\nabla^a \left(\frac{\Phi}{V} D_a \right) = \frac{4}{V} \left(\mathscr{L}_\mathcal{F} E_a E^a - \mathscr{L}_\mathcal{G} E_a B^a \right) \tag{4.21}$$

and

$$\nabla^a \left(\frac{\Psi}{V} B_a \right) = \frac{4}{V} \left(\mathscr{L}_\mathcal{F} B_a B^a + \mathscr{L}_\mathcal{G} E_a B^a \right) \ . \tag{4.22}$$

The sum of these two divergence identities,

$$\nabla^a \left(\frac{\Phi}{V} D_a + \frac{\Psi}{V} B_a \right) = \frac{4}{V} \left(\mathscr{L}_\mathcal{F} E_a E^a + \mathscr{L}_\mathcal{F} B_a B^a \right) \ , \tag{4.23}$$

may be integrated over the hypersurface Σ. The boundary term produced on the left hand side vanishes with the aid of identities (4.20) and the assumed fall-off conditions of the scalar potentials Φ and Ψ, leaving us with

$$\int_\Sigma \frac{\mathscr{L}_\mathcal{F}}{V} \left(E_a E^a + B_a B^a \right) \hat{\boldsymbol{\epsilon}} = 0 \ , \tag{4.24}$$

where $\hat{\boldsymbol{\epsilon}}$ is the induced volume 3-form. Since k^a is a strictly timelike vector field and $k^a E_a = 0 = k_a B^a$, neither E^a nor B^a can be causal. Thus, the integrand is nonpositive on $O \cap \Sigma$ and zero on $(\mathscr{M} - O) \cap \Sigma$. The total integral is zero, so $E^a = 0 = B^a$ and, as[2] $\mathscr{L}(0,0) = 0$, the trace T vanishes on $O \cap \Sigma$. By continuity, the trace is zero also on $\overline{O} \cap \Sigma$ and hence on the whole Σ. To conclude, at each point of the set $O \cap \Sigma$ we have $F_{ab} = 0$, while at each point of $(\mathscr{M} - O) \cap \Sigma$ stealth configurations are also possible besides trivial electromagnetic field.

The generality of the theorem is reflected in the fact that the details of the gravitational action are not specified. The only requirements are that the tensor E_{ab} is divergence-free and the coupling between matter and gravitational sector minimal.

[2] We can always add a constant to the Lagrangian density that renders $\mathscr{L}(0,0) = 0$.

In accordance with the theorem, examples of nontrivial stealth electromagnetic fields in static spacetimes can be found for $\mathscr{L}_{\mathfrak{F}} = 0$ [171].

Proof of theorem 4.2. We start from a conveniently constructed 1-form defined as

$$\mathbf{W} := \frac{U_E + U_H}{V^2}\,\boldsymbol{\omega} + \frac{4}{V}\,(\Phi\mathbf{D} + \Psi\mathbf{B}) \ . \tag{4.25}$$

Its covariant divergence can be written compactly if we notice that

$$\omega_a\omega^a = -4\Phi\omega_a H^a + \omega^a\nabla_a U_H = 4\Psi\omega_a E^a + \omega^a\nabla_a U_E \tag{4.26}$$

and use the basic divergence identities (4.8) and (4.9),

$$\nabla_a W^a = \frac{16}{V}\,\mathscr{L}_{\mathfrak{F}}\,(E_a E^a + B_a B^a) + 2\,\frac{\omega_a\omega^a}{V^2} \ . \tag{4.27}$$

Furthermore, it can be cast in a more suitable form by making use of Einstein's field equation

$$R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2}\,T g_{ab}\right) \ , \tag{4.28}$$

and relation between Maxwell's energy-momentum tensor and electromagnetic fields,

$$8\pi T^{(\mathrm{Max})}_{ab} k^a k^b = E_a E^a + B_a B^a \ . \tag{4.29}$$

Then, we get

$$\frac{4}{V}\,R_{ab}k^a k^b = \frac{32\pi}{V}\left(-4\mathscr{L}_{\mathfrak{F}} T^{(Max)}_{ab} k^a k^b\right) + 8\pi T = -\nabla_a W^a + 2\frac{\omega_a\omega^a}{V^2} + 8\pi T \ , \tag{4.30}$$

which can be related to Heusler's mass formula as follows [95]. We contract $\star\mathrm{d}\boldsymbol{\omega} = 2\,\mathbf{k}\wedge\mathbf{R}(k)$ with k^a and take the Hodge dual,

$$\star i_k \star \mathrm{d}\boldsymbol{\omega} = -2V\star\mathbf{R}(k) - 2R_{ab}k^a k^b \star\mathbf{k} \ . \tag{4.31}$$

On the other hand, we have

$$\star i_k \star \mathrm{d}\boldsymbol{\omega} = \star\star(\mathrm{d}\boldsymbol{\omega}\wedge\mathbf{k}) = \mathbf{k}\wedge\mathrm{d}\boldsymbol{\omega} \ . \tag{4.32}$$

Combination of the two terms returns

$$-\star\mathbf{R}(k) = \frac{R_{ab}k^a k^b}{V}\,\star\mathbf{k} + \frac{1}{2V}\,\mathbf{k}\wedge\mathrm{d}\boldsymbol{\omega} \ . \tag{4.33}$$

The last term in (4.33) can be expressed via

$$-\mathrm{d}\left(\frac{1}{V}\,\mathbf{k}\wedge\boldsymbol{\omega}\right) = -\frac{1}{V^2}\star(\boldsymbol{\omega}\wedge\mathbf{k})\wedge\boldsymbol{\omega} + \frac{1}{V}\mathbf{k}\wedge\mathrm{d}\boldsymbol{\omega} =$$
$$= \frac{1}{V^2}\star i_\omega(\boldsymbol{\omega}\wedge\mathbf{k}) + \frac{1}{V}\mathbf{k}\wedge\mathrm{d}\boldsymbol{\omega} =$$
$$= \frac{\omega_a\omega^a}{V^2}\star\mathbf{k} + \frac{1}{V}\mathbf{k}\wedge\mathrm{d}\boldsymbol{\omega}\ , \tag{4.34}$$

so that we finally get

$$-\star\mathbf{R}(k) = \left(\frac{R_{ab}k^a k^b}{V} - \frac{\omega_a\omega^a}{2V^2}\right)\star\mathbf{k} - \mathrm{d}\left(\frac{1}{2V}\,\mathbf{k}\wedge\boldsymbol{\omega}\right)\ . \tag{4.35}$$

The fall-off properties of the twist 1-form $\boldsymbol{\omega}$ which follow from the technical assumption (4) enable us to write Komar's mass as

$$M = -\frac{1}{4\pi}\int_\Sigma \star\mathbf{R}(k) = \frac{1}{4\pi}\int_\Sigma \left(\frac{R_{ab}k^a k^b}{V} - \frac{\omega_a\omega^a}{2V^2}\right)\star\mathbf{k}\ . \tag{4.36}$$

Finally, combined with equation (4.30) we have

$$M = -\frac{1}{16\pi}\int_\Sigma \nabla_a W^a \star\mathbf{k} + \frac{1}{2}\int_\Sigma T\star\mathbf{k}\ . \tag{4.37}$$

The first term is again a boundary term that vanishes at infinity as a consequence of the imposed fall-off conditions. The trace term is nonpositive due to the assumed dominant energy condition. On the other hand, the positive energy theorem states that $M \geq 0$ and $M = 0$ if and only if the spacetime is Minkowski. In our case, it follows that $M = 0$, which in turn implies $T_{ab} = 0$, signifying that the electromagnetic field is either zero or stealth.

4.3 Further generalisations

After considering a four-dimensional case with only NLE fields present, we explore two possible routes of generalisation. The first possibility is the addition of charged matter, which comes with the difficulty of treating the current term in the generalised Maxwell's equation. As our model, we may choose a complex scalar field ϕ minimally coupled to the electromagnetic field, such that the total Lagrangian density is given by

$$\mathscr{L}^{(\mathrm{tot})} = \mathscr{L}(\mathcal{F},\mathcal{G}) - (\mathcal{D}_a\phi)^*(\mathcal{D}^a\phi) - \mathscr{U}(\phi^*\phi)\ , \tag{4.38}$$

where $\mathcal{D}_a = \nabla_a + iqA_a$ is the covariant gauge derivative and $\mathscr{U}$ the scalar potential. The second generalised Maxwell's equation has a source term,

$$\mathrm{d}{\star}\mathbf{Z} = 4\pi \,{\star}\mathbf{J} \ , \tag{4.39}$$

where the current 1-form reads as

$$J_a = \frac{iq}{4\pi} \left(\phi^* \mathcal{D}_a \phi - \phi (\mathcal{D}_a \phi)^* \right) \ . \tag{4.40}$$

First, we focus on the strictly static case with $\boldsymbol{\omega} = 0$ and rederive the divergence identities taking into account the equation (4.39),

$$\nabla^a \left(\frac{1}{V} \, D_a \right) = \frac{1}{V} \star (\mathbf{k} \wedge \mathrm{d}{\star}\mathbf{Z}) = \frac{4\pi}{V} \star (\mathbf{k} \wedge {\star}\mathbf{J}) = -\frac{4\pi}{V} \, k^a J_a \ , \tag{4.41}$$

$$\nabla^a \left(\frac{\Phi}{V} \, D_a \right) = \frac{4}{V} \left(\mathscr{L}_{\mathcal{F}} E_a E^a - \mathscr{L}_{\mathcal{G}} E_a B^a \right) - \frac{4\pi\Phi}{V} \, k^a J_a \ , \tag{4.42}$$

$$\nabla^a \left(\frac{1}{V} \, B_a \right) = 0 \ . \tag{4.43}$$

There are two major obstacles that prevent us from repeating the same argument as above. An additional term proportional to $\Phi k^a J_a$, which is *a priori* neither positive nor negative definite, appears in the divergence identity (4.42). In order to fix its sign, we assume that the scalar field is symmetry inheriting, $\pounds_k \phi = 0$ and choose a gauge in which $\pounds_k \mathbf{A} = 0$. Taking into account that

$$\mathrm{d}(\Phi + i_k \mathbf{A}) = -\mathbf{E} + (\pounds_k - i_k \mathrm{d})\mathbf{A} = 0 \ , \tag{4.44}$$

and if both Φ and $k^a A_a$ vanish at infinity, we can set $\Phi = -k^a A_a$. Under these assumptions, the "problematic" term becomes nonnegative,

$$\Phi k^a J_a = \frac{(q\Phi)^2}{2\pi} \, \phi^* \phi \geq 0 \ . \tag{4.45}$$

The magnetic scalar potential uncovers another caveat. On the domain with non-vanishing electric current, the magnetic field 1-form $\mathbf{H}$ is no longer closed as

$$\mathrm{d}\mathbf{H} = 4\pi{\star}(\mathbf{k} \wedge \mathbf{J}) \ , \tag{4.46}$$

signifying that the scalar potential cannot be defined in the usual way. Consequently, the proof of the theorem can be carried out only in specific scenarios. For example, if $\mathbf{k} \wedge \mathbf{J} = 0$ (in other words, if the current is proportional to $\mathbf{k}$), the magnetic scalar potential can be introduced in the same manner as before and the divergence identity (4.22) remains valid. The second option is to consider a purely electric system in which $\mathbf{B} = 0$ so that the proof relies on Eq. (4.42).

Whenever one of the conditions above is met, we may repeat the proof and deduce that the electromagnetic field on the domain given by the set O is trivial. For $\mathcal{F}$-class theories we may again use the Eq. (4.42) to conclude that $\mathbf{E} = 0$ on O, however, nothing can be said about the magnetic field $\mathbf{B}$ without invoking one of the aforementioned conditions.

On the interior $(\mathscr{M} - O)^\circ$, the divergence of the gravitational field equation reduces to $\nabla_a T = 4J^b F_{ba}$. From the decomposition $V\mathbf{F} = \mathbf{k} \wedge \mathbf{E} + \star(\mathbf{k} \wedge \mathbf{B})$ and $k^a \nabla_a T = 0$ since all the fields are symmetry inheriting, we can conclude that $V\nabla_a T = 4(k^b J_b)E_a$. Setting $\mathbf{B} = 0$ implies $\mathbf{D} = 0$ on the set $(\mathscr{M} - O)^\circ$, while divergence identities give $k^a J_a = 0$. Finally, we may conclude that the trace T is constant on each connected component of the set $(\mathscr{M} - O)^\circ$. It is not obvious whether the same holds in the more general case when $\mathbf{B} \neq 0$.

One way of evading this no-go theorem is by considering symmetry noninheriting scalar fields, as is the case with spacetimes containing charged boson stars. The scalar field sourcing such solutions is often time dependent, typically of the form $\phi(t,r) = f(r)e^{i\omega t}$, so that $\pounds_k \phi = i\omega\phi$ and the term $\Phi k^a J_a$ generally has no definite sign.

The theorem 4.2 can be readily generalised in the case of Lagrangian density (4.38) if one includes one additional restriction. Its corresponding energy-momentum tensor is equal to

$$T_{ab} = -4\mathscr{L}_\mathcal{F}\widetilde{T}_{ab} + \frac{1}{4}Tg_{ab} + \frac{1}{2\pi}(\mathcal{D}_{(a}\phi)^*\mathcal{D}_{b)}\phi - \frac{1}{4\pi}\big((\mathcal{D}_c\phi)^*(\mathcal{D}^c\phi) + \mathscr{U}(\phi^*\phi)\big)g_{ab}\,, \quad (4.47)$$

therefore, compared to the purely NLE case (4.11), the exterior derivative of $\boldsymbol{\omega}$ contains an extra term,

$$\mathrm{d}\boldsymbol{\omega} = -2\star(\mathbf{k} \wedge \mathbf{R}(k)) = 64\pi\mathscr{L}_\mathcal{F}\star(\mathbf{k} \wedge \mathbf{T}^{(\mathrm{Max})}(k)) + 16\pi(i_k\mathbf{A}) \star (\mathbf{k} \wedge \mathbf{J})\ . \quad (4.48)$$

When deriving the equation (4.48), we again assumed that the scalar field is symmetry inheriting in order to dispose of the $k^a\nabla_a\phi$ terms. If the "spacelike" current is absent, $\mathbf{k} \wedge \mathbf{J} = 0$, one may proceed with the proof in the same manner as before. Notice that the theorem applies also to the symmetry noninheriting complex scalar fields for which $\pounds_k\phi = i\omega\phi$, since

$$2k^b(\mathcal{D}_{(a}\phi)^*\mathcal{D}_{b)}\phi = 4\pi((\omega/q) - k^b A_b)J_a\ . \quad (4.49)$$

Another direction of generalisation lies in considering higher-dimensional theories, with a focus on the spacetimes of dimension $m \geq 5$. By definition, the invariant $\mathcal{G}$ is no longer a scalar when $m \neq 4$ so we only deal with $\mathcal{F}$-class Lagrangians. We will define the twist $(m-3)$-form as $\boldsymbol{\omega} := (-1)^{m+1}\star(\mathbf{k} \wedge \mathrm{d}\mathbf{k})$ in order to keep its form in abstract indices intact,

$$\omega_{a_1 \dots a_{m-3}} = \epsilon_{a_1 \dots a_{m-3}}{}^{bcd} k_b \nabla_c k_d \ . \tag{4.50}$$

The 1-form $\mathbf{D} = -i_k \mathbf{Z}$ can be rewritten as

$$\mathbf{D} = (-1)^m \star(\mathbf{k} \wedge \star\mathbf{Z}) \ , \tag{4.51}$$

so that we get an auxiliary result

$$\nabla^a \left(\frac{D_a}{V} \right) = \frac{1}{(m-2)! V^2} \, (\boldsymbol{\omega} \wedge \mathbf{k})_{a_1 \dots a_{m-2}} \star\mathbf{Z}^{a_1 \dots a_{m-2}} \ . \tag{4.52}$$

The divergence identity in strictly static spacetime becomes

$$\nabla^a \left(\frac{\Phi}{V} \, D_a \right) = \frac{4}{V} \, \mathscr{L}_\mathcal{F} \, E_a E^a \ . \tag{4.53}$$

Relying on the natural fall-off conditions, $\Phi = O(r^{-(m-3)})$ and $\mathbf{D} = O(r^{-(m-2)})$ [136], the proof proceeds as before and $\mathbf{E} = 0$ on the set O. An additional challenge comes with treating the magnetic field forms $\mathbf{B}$ and $\mathbf{H}$. In an m-dimensional spacetime they become $(m-3)$-forms, which makes divergence identities more involved.

Chapter 5

Constraints on singularity resolution by NLE fields

5.1 Singularities in gravitational theory

The first exact solution of Einstein's equation revealed a perplexing feature in the behaviour of the metric. The apparent divergences of the metric components raised a question about their physical interpretation. Taking the Schwarzschild metric as an example, two different types of divergences can be singled out, a removable coordinate singularity representing the black hole horizon and a physical spacetime singularity. This in turn begged for a precise definition and a systematic classification of spacetime singularities. Following the nomenclature given by Ellis and Schmidt [56], the two most common types are non-scalar and scalar singularities. The latter occur in a further non-extendible spacetime and imply "badly behaved" curvature scalars, while the former manifest themselves as geodesically incomplete spacetimes without problematic curvature invariants. In regular spacetime, the curvature scalars must be bounded since they are coordinate-independent, but the converse statement does not hold. The subtlety lies in the fact that the absence of unbounded curvature scalars does not immediately imply the regularity of the spacetime as there are examples of geodesically incomplete spacetimes with vanishing curvature scalars [187]. The problem of determining regularity conditions is even more complex as one can find a geodesically complete spacetime that contains an incomplete nongeodesic timelike curve of bounded acceleration [66].

The existence of the spacetime singularities brought into question the generality of their formation. Namely, the first solutions of Einstein's equation were highly symmetric, thus eliciting the suspicion that their singular behaviour is an artefact of the artificially imposed symmetry. The formulation of Hawking-Penrose singularity theorems [84, 143, 81], whose backbone are energy conditions and certain requirements on the causal structure of the spacetime, disputed this doubt. Although generic, these assumptions are strong enough to imply the existence of incomplete geodesics, consequently confirming that the spacetime singularities are not "by-products" of idealised symmetric solutions.

Before resorting to the quantum extensions of the classical gravitational theory, which are expected to resolve the spacetime singularities [44], there are other options worth exploring. For example, one option is to prove the quantum completeness of the otherwise geodesically incomplete spacetime, which is done by replacing the classical probe with the quantum one [190, 101]. Also, one could utilise the semi-classical backreaction to dress the singularity [73, 38, 37, 106, 97]. Another option is to consider the possibility of regularisation by coupling the classical matter fields to gravitational action. This will be our direction of the investigation, with the matter sector consisting of NLE fields.

Singularities are also encountered in Maxwell's electrodynamics, demonstrating as divergences in the electric field and the self-energy of a point charge. Some of the NLE theories may cure those singularities. For instance, Born–Infeld Lagrangian [21, 20] was constructed with this specific aim, while Euler–Heisenberg Lagrangian [86] removes only the singularity in the energy of a point charge. The electrostatic quantities of a point charge are not regularised within ModMax theory [8], but some of its further modifications are successful in this aspect [114].

Motivated by the examples from electromagnetism, there was a hope that a similar analogy may be established in gravitational theory, i.e. that the regularisation of the spacetime singularities may be achieved by coupling the NLE fields to the gravitational action. This idea blossomed after the proposed regular Bardeen black hole [9] got an interpretation in terms of an NLE Lagrangian [6]. The systematic approach to the problem of regularisation using NLE fields was advised by Bronnikov [27], who gave a general criterion under which a static, spherically symmetric solution sourced by an NLE Lagrangian given as a function of invariant $\mathcal{F}$ and obeying Maxwellian weak field limit will be globally regular. The main result is that black holes endowed with electric charge cannot have a regular centre and regularised black holes can only be found among magnetically charged solutions. This conclusion was supported by many examples of regular magnetically charged black holes emanating from NLE Lagrangians with MWF limit [125, 121, 2, 115, 113, 112] Bronnikov's theorems can be evaded by relaxing the assumptions, precisely, by discarding the MWF limit condition. In that case, even the electrically charged black holes may be regular, due to the presence of a "de Sitter core" (de Sitter behaviour as $r \to 0$)

[53, 29, 7]. Another way of avoiding Bronnikov's constraints is to construct a specific solution with a core simulating a phase transition [28].

Bronnikov's idea is based on examining the behaviour of curvature scalars expressed via electromagnetic invariants and charges. If there exists at least one unbounded curvature invariant, the spacetime is immediately labelled as singular. For this reason, scalar singularities bear enough information to formulate a no-go theorem. Our goal is to extend Bronnikov's theorems to encompass a larger class of NLE Lagrangians, those depending on both invariants $\mathcal{F}$ and $\mathcal{G}$ [18].

5.2 Electromagnetic invariants

Following Bronnikov's approach, we have to infer how many different invariants may be formed out of the contractions of the energy-momentum tensor. Einstein's field equation,

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \ , \tag{5.1}$$

will provide a direct link between calculated matter quantities on one side and curvature invariants on the gravitational side.

The evaluation of relevant electromagnetic invariants can be carried out straightforwardly using spinor calculus [144, 173] (see Appendix B). For notational simplicity, we introduce the shorthand notation $(X^n)^a{}_b := X^a{}_{c_1} X^{c_1}{}_{c_2} \cdots X^{c_{n-1}}{}_b$ for any rank-2 tensor $X^a{}_b$ and $n \in \mathbb{N}$. The trace of the odd number of Maxwell's energy-momentum vanishes since such term is proportional to the contraction of the symmetric spinor ϕ_{AB} with the antisymmetric spinor ϵ_{AB} ,

$$(\widetilde{T}^{2n+1})^a{}_a = 0 \, . \tag{5.2}$$

The trace of the even number of Maxwell's energy-momentum tensors attains a simple form (for a more detailed derivation, see Appendix B),

$$(4\pi)^{2n} (\widetilde{T}^{2n})^a{}_a = \frac{1}{4^{2n-1}} \, (\mathcal{F}^2 + \mathcal{G}^2)^n \, . \tag{5.3}$$

Combining the expressions above, the contraction of two NLE energy-momentum tensors (2.8) can be expressed in terms of two invariants, the trace T and $\mathscr{L}_{\mathcal{F}}^2(\mathcal{F}^2 + \mathcal{G}^2)$,

$$4\pi^2 T^a{}_b T^b{}_a = \pi^2 T^2 + \mathscr{L}_{\mathcal{F}}^2 (\mathcal{F}^2 + \mathcal{G}^2) \, . \tag{5.4}$$

The question is whether a new, independent invariant can be extracted from the consequtive contractions of more NLE energy-momentum tensors. Using the binomial formula and Eq. (2.8), the expression (5.4) can be generalised for the trace of an arbitrary number of NLE energy-momentum tensors,

$$(T^n)^a{}_a = 4(T/4)^n + \sum_{k=1}^{n} \binom{n}{k} 4^{2k-n} (-\mathscr{L}_{\mathcal{F}})^k T^{n-k} (\widetilde{T}^k)^a{}_a \,, \tag{5.5}$$

where the trace term T^n is written separately for clarity. From the derived formula (5.5) it is obvious that no new invariants can be constructed in this manner. All of the contractions reduce to the higher powers of two fundamental invariants, the trace T and $\mathscr{L}_{\mathcal{F}}^2(\mathcal{F}^2 + \mathcal{G}^2)$.

The relation between the curvature invariants and the electromagnetic invariants can be established via Einstein's equation (5.1), resulting in

$$R - 4\Lambda = -8\pi T \,, \tag{5.6}$$

$$R_{ab}R^{ab} + 2\Lambda(2\Lambda - R) = (8\pi)^2 \, T_{ab}T^{ab} \,. \tag{5.7}$$

If the spacetime is regular in the sense of bounded curvature invariants, i.e. Ricci scalar R and Ricci squared $R_{ab}R^{ab}$ are bounded, then the same has to hold for matter invariants, $T_{ab}T^{ab}$ and T. This in turn implies that $\mathscr{L}_{\mathcal{F}}\mathcal{F}$ and $\mathscr{L}_{\mathcal{F}}\mathcal{G}$ also have to stay bounded. Since our arguments do not depend on the asymptotic behaviour of the spacetime, cosmological constant Λ is kept just for the sake of generality.

5.3 Application to the spherically symmetric spacetime

As a test model, we take the most simple, spherically symmetric spacetime. No generality is lost with this choice since a candidate NLE theory should successfully regularise an arbitrary black hole solution, with no additional parameters (for instance, angular momentum) that may be adjusted to aid this cause. The metric of a static and spherically symmetric spacetime can be written as [187]

$$\mathrm{d}s^2 = -\alpha(r)\,\mathrm{d}t^2 + \beta(r)\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\,, \tag{5.8}$$

if $\nabla_a r \neq 0$. We assume that the radial coordinate has a minimum $r = 0$, which we will refer to as a *centre*. Also, we suppose that the function $w(r) = \sqrt{\alpha(r)\beta(r)}$ has no zeros for points with $0 < r < r_w$, where $r_w > 0$. This condition states that there are no horizons at least in some neighbourhood of the centre. Without loss of generality, some specific spacetime configurations such as wormhole or "horn" (infinitely long tube of a fixed finite radius) solutions can be discarded, as guaranteed by Bronnikov's theorem 2 [27]. Namely, the aforementioned theorem states that if the spherically symmetric spacetime is sourced by the energy-momentum tensor which satisfies the condition $T^t{}_t = T^r{}_r$, then the spactime cannot contain a horn and the radial coordinate cannot have a regular minimum. The latter condition excludes wormhole solutions, since in that case the radial coordinate attains a regular and finite minimum at the wormhole throat.

We consider the regularisation of the problematic point, which is the centre, possible if the curvature scalars are bounded as $r \to 0$. The exact meaning of this assertion is defined precisely below.

Definition 5.1. *We say that some scalar $\psi(r)$ is bounded as $r \to 0$ if there is a real constant $M > 0$ and a radius $r_0 > 0$, such that $|\psi(r)| \leq M$ for all $0 < r < r_0$.*

Such definition of boundedness does not impose a very strong constraint, as it may happen that the limit $\lim_{r\to 0} \psi(r)$ does not exist. For example, function of the form $\psi \sim \sin(1/r)$ widely oscillates as $r \to 0$, but stays bounded in the same limit according to the definition above. Therefore, there is a possibility that the invariants in a certain spacetime obey the Definition 5.1, but that does not guarantee that they are well-behaved in a neighbourhood of the centre.

For completeness, we repeat the final form of NLE Maxwell's equations in spherically symmetric spacetime, which were derived in Chapter 2,

$$\widetilde{B}_r = \frac{P}{r^2}\,, \tag{5.9}$$

$$\mathscr{L}_\mathcal{F} \widetilde{E}_r - \mathscr{L}_\mathcal{G} \widetilde{B}_r = -\frac{Q}{4r^2}\,. \tag{5.10}$$

The general idea behind formulating no-go theorems is to assume that both R and $R_{ab}R^{ab}$ are bounded as $r \to 0$, which translates, using Eqs. (5.6), (5.7) and (5.4), into boundedness of $\mathscr{L}_\mathcal{F}\mathcal{F}$, $\mathscr{L}_\mathcal{F}\mathcal{G}$ and T in the same limit. As we will demonstrate, the contradiction stems from the fact that the regularity assumption is often incompatible with the Maxwellian weak field limit. However, it can happen that both R and $R_{ab}R^{ab}$ are bounded, while other curvature invariants diverge. For example, the Kretschmann scalar $R_{abcd}R^{abcd}$ cannot be directly controlled via Einstein's equation and related to the matter fields. Nevertheless, our mild demands are strong enough to pose serious limitations for many physically relevant cases.

5.3.1 Electric case

Since magnetic monopoles are still of theoretical interest only, the most important cases are black holes equipped with electric charge. We present a complete generalisation of Bronnikov's theorem that covers all $\mathcal{FG}$-class Lagrangians with Maxwellian weak field limit.

Theorem 5.1. *Suppose that the spacetime is a static, spherically symmetric solution of the Einstein–NLE field equations with $\mathcal{FG}$-class NLE Lagrangian obeying the Maxwellian weak field limit. Then, in the electrically charged case, that is $P = 0$ and $Q \neq 0$, Ricci scalar R and Ricci squared $R_{ab}R^{ab}$ cannot both remain bounded as $r \to 0$.*

Notice that theorem 5.1 immediately applies to all $\mathcal{F}$-class Lagrangians.

Proof of theorem 5.1. When the magnetic charge is absent, it immediately follows from (5.9) that $B_r = 0$. Then, the second Maxwell's equation (5.10) can be rearranged as

$$\frac{\mathcal{F}}{r^3} = -\frac{8}{Q^2}\,(\mathcal{F}\mathscr{L}_{\mathcal{F}})^2 r\,. \tag{5.11}$$

Assuming that both R and $R_{ab}R^{ab}$ are bounded, the same has to apply for $\mathcal{F}\mathscr{L}_{\mathcal{F}}$, meaning that[1] $\mathcal{F} = o(r^3)$ as $r \to 0$. The invariant $\mathcal{G}$ is identically equal to zero as we have $B_r = 0$. Since $\mathscr{L}_{\mathcal{F}}^2 = -Q^2/(8\mathcal{F}r^4)$, $\mathscr{L}_{\mathcal{F}}$ is unbounded as $r \to 0$ which is in direct contradiction with the assumed Mawellian weak field limit. It is important to stress that the contradiction with the Maxwellian weak field limit is manifest due to a fortunate circumstance: the $r \to 0$ limit coincides with the weak field limit in which both $\mathcal{F}$ and $\mathcal{G}$ approach zero. □

Our conclusion is backed up by the known solutions; electrically charged Born–Infeld [160, 60, 51, 64] and Euler–Hesienberg black holes [202, 159] are not regular.

5.3.2 Dyonic case

The same procedure cannot be applied directly to the dyonic case, in which both $Q \neq 0$ and $P \neq 0$. The main obstacle comes from the fact that the Maxwellian weak field limit does not correspond to the $r \to 0$ limit. This can be easily seen if we rewrite the electromagnetic invariant $\mathcal{F}$ in terms of the other invariant $\mathcal{G}$,

$$\mathcal{F} = 2\left(\frac{P^2}{r^4} - \frac{r^4}{16P^2}\,\mathcal{G}^2\right)\,. \tag{5.12}$$

[1] Throughout the chapter we use Landau's little-o notation to denote the behaviour of a certain quantity as $r \to 0$.

The expression above implies that both invariants $\mathcal{F}$ and $\mathcal{G}$ cannot simultaneously go to zero as we approach the centre and the opportunity to test the Maxwellian weak field limit is lost. Nevertheless, a useful conclusion can be drawn from Eq. (5.12). Namely, if we manage to prove that both $\mathcal{F}$ and $\mathcal{G}$ should remain bounded as $r \to 0$, it immediately leads to a contradiction.

From Maxwell's equations (5.9)-(5.10) and the definition of invariant $\mathcal{G}$ we get

$$\mathscr{L}_{\mathcal{F}} \widetilde{E}_r = \mathscr{L}_{\mathcal{F}} \mathcal{G} \, \frac{r^2}{4P} = \mathscr{L}_{\mathcal{G}} \widetilde{B}_r - \frac{Q}{4r^2} = \frac{1}{r^2} \left(\mathscr{L}_{\mathcal{G}} P - \frac{Q}{4} \right) , \tag{5.13}$$

which can be restated as

$$\frac{1}{r^3} \left(\mathscr{L}_{\mathcal{G}} P - \frac{Q}{4} \right) = \frac{\mathscr{L}_{\mathcal{F}} \mathcal{G}}{4P} \, r \, . \tag{5.14}$$

If the spacetime is regular, $\mathscr{L}_{\mathcal{F}} \mathcal{G}$ and consequently the right hand side remain bounded as $r \to 0$. This in turn fixes the behaviour of $\mathscr{L}_{\mathcal{G}}$,

$$\mathscr{L}_{\mathcal{G}} = \frac{Q}{4P} + o(r^3) \quad \text{as} \quad r \to 0 \, . \tag{5.15}$$

The fact that the invariant $\mathscr{L}_{\mathcal{G}}$ is bounded simplifies proofs in the dyonic case.

First, we revisit Bronnikov's theorem [27] and prove it in a slightly different manner.

Theorem 5.2. *Suppose that the spacetime is a static, spherically symmetric solution of the Einstein–NLE field equations with the $\mathcal{F}$-class NLE Lagrangian. Then, in the dyonic case, that is $P \neq 0$ and $Q \neq 0$, Ricci scalar R and Ricci squared $R_{ab}R^{ab}$ cannot both remain bounded as $r \to 0$.*

Proof of theorem 5.2. If both R and $R_{ab}R^{ab}$ are bounded as $r \to 0$, the same holds for $\mathscr{L}_{\mathcal{F}} \mathcal{F}$ and $\mathscr{L}_{\mathcal{F}} \mathcal{G}$. Maxwell's equation for $\mathcal{F}$-class Lagrangians is

$$\mathscr{L}_{\mathcal{F}} \mathcal{G} \, \frac{r^2}{4P} = \mathscr{L}_{\mathcal{F}} \widetilde{E}_r = -\frac{Q}{4r^2} \, , \tag{5.16}$$

which leads to a contradiction as $\mathscr{L}_{\mathcal{F}} \mathcal{G}$ should remain bounded as $r \to 0$. □

Theorem 5.2 relies only partly on the Maxwellian weak field limit as we do not invoke the condition $\mathscr{L}_{\mathcal{F}} \to -1/4$ as $(\mathcal{F}, \mathcal{G}) \to (0, 0)$, but we have $\mathscr{L}_{\mathcal{G}} = 0$ identically.

Since the Maxwellian weak field limit is not necessarily attained as the radial coordinate approaches the centre, we cannot derive a general constraint valid for all $\mathcal{F}\mathcal{G}$-class Lagrangians using this approach. Instead, we will consider some particular classes of NLE theories. Without loss of generality, $\mathcal{F}\mathcal{G}$-class Lagrangian can be put in the following form

$$\mathscr{L} = -\frac{1}{4}\,\mathcal{F} + h(\mathcal{F},\mathcal{G})\,, \tag{5.17}$$

where h is a C^1-class function.

The theorems can be easily formulated for two specific subclasses of NLE theories, the first of which is valid both for dyonic and strictly magnetically charged solutions.

Theorem 5.3. *Suppose that the spacetime is a static, spherically symmetric solution of the Einstein–NLE field equations with the NLE Lagrangian (5.17), such that $h = h(\mathcal{G})$. Then, given that $P \neq 0$, Ricci scalar R and Ricci squared $R_{ab}R^{ab}$ cannot both remain bounded as $r \to 0$.*

Proof of theorem 5.3. Since $\mathscr{L}_\mathcal{F} = -1/4$, if we demand that $\mathscr{L}_\mathcal{F}\mathcal{F}$ and $\mathscr{L}_\mathcal{F}\mathcal{G}$ are bounded, then $\mathcal{F}$ and $\mathcal{G}$ should also be bounded as $r \to 0$, which leads to a contradiction. □

Similarly to the theorem 5.2, we do not need to invoke the full form of Maxwellian weak field limit. We have identically $\mathscr{L}_\mathcal{F} = -1/4$, but the condition $\mathscr{L}_\mathcal{G} \to 0$ as $(\mathcal{F},\mathcal{G}) \to (0,0)$ is not necessary for the proof.

Theorem 5.4. *Suppose that the spacetime is a static, spherically symmetric solution of the Einstein–NLE field equations with the NLE Lagrangian (5.17), such that $h(\mathcal{F},\mathcal{G}) = a\mathcal{F}^s\mathcal{G}^u$, with a real constant $a \neq 0$ and integers $s,u \geq 1$. Then, in the dyonic case, that is $P \neq 0$ and $Q \neq 0$, Ricci scalar R and Ricci squared $R_{ab}R^{ab}$ cannot both remain bounded as $r \to 0$.*

Proof of theorem 5.4. Explicit evaluation of invariant $\mathscr{L}_\mathcal{F}$ gives us

$$\mathscr{L}_\mathcal{F}\mathcal{F} = -\frac{1}{4}\,\mathcal{F} + sh\ , \tag{5.18}$$

and the second invariant useful for this theory is the trace of the energy-momentum tensor, which is equal to

$$\pi T = (1-s-u)h\ . \tag{5.19}$$

Then, boundedness of T and $\mathscr{L}_\mathcal{F}\mathcal{F}$ imply boundedness of h and $\mathcal{F}$ as $r \to 0$. From

$$\mathscr{L}_\mathcal{G}\mathcal{G} = uh, \tag{5.20}$$

and (5.15), it follows that $\mathcal{G}$ has to be bounded as $r \to 0$ which is in a contradiction with (5.12). □

Motivated by the form of Euler–Heisenberg Lagrangian, we turn to Lagrangians in which the function h is a quadratic polynomial in electromagnetic invariants.

Theorem 5.5. *Suppose that the spacetime is a static, spherically symmetric solution of the Einstein–NLE field equations with the NLE Lagrangian (5.17), such that $h(\mathcal{F},\mathcal{G}) = a\mathcal{F}^2 + b\mathcal{F}\mathcal{G} + c\mathcal{G}^2$, where a, b and c are real constants. Then, in the dyonic case, that is $P \neq 0$ and $Q \neq 0$, Ricci scalar R and Ricci squared $R_{ab}R^{ab}$ cannot both remain bounded as $r \to 0$.*

Proof of theorem 5.5. The derivatives $\mathscr{L}_\mathcal{F}$ and $\mathscr{L}_\mathcal{G}$ define a linear system of equations in $\mathcal{F}$ and $\mathcal{G}$,

$$\mathscr{L}_\mathcal{F} + \frac{1}{4} = 2a\mathcal{F} + b\mathcal{G}\,, \tag{5.21}$$

$$\mathscr{L}_\mathcal{G} = b\mathcal{F} + 2c\mathcal{G}\,. \tag{5.22}$$

Using expression (5.15), we have

$$(b\mathcal{F} + 2c\mathcal{G})\mathscr{L}_\mathcal{F} = \left(\frac{Q}{4P} + o(r^3)\right)\mathscr{L}_\mathcal{F}\,. \tag{5.23}$$

If $\mathcal{F}\mathscr{L}_\mathcal{F}$ and $\mathscr{L}_\mathcal{F}\mathcal{G}$ should remain bounded as $r \to 0$, the same has to be valid for $\mathscr{L}_\mathcal{F}$ itself.

Depending on the determinant of the linear system above, $\Delta = 4ac - b^2$, there are two different cases. First, in the nondegenerate case ($\Delta \neq 0$), boundedness of $\mathscr{L}_\mathcal{F}$ and $\mathscr{L}_\mathcal{G}$ implies boundedness of $\mathcal{F}$ and $\mathcal{G}$ as $r \to 0$, which is an immediate contradiction. In the degenerate case, that is $\Delta = 0$, we have to consider several subcases. If we set $c = 0$, then $b = 0$, and the resulting Lagrangian belongs to the $\mathcal{F}$-class, which is already covered by the theorem 5.2. If $a = 0$, then $b = 0$, and we return to the theorem 5.3. Thus, the only new subcase is $a \neq 0 \neq c$. Multiplying both sides of

$$\mathscr{L}_\mathcal{F} = -\frac{1}{4} + \frac{2a}{b}\mathscr{L}_\mathcal{G} \tag{5.24}$$

by $\mathcal{F}$ and using (5.15) gives us

$$\mathcal{F}\mathscr{L}_\mathcal{F} = \left(-\frac{1}{4} + \frac{aQ}{2bP} + o(r^3)\right)\mathcal{F} \quad \text{as} \quad r \to 0\,. \tag{5.25}$$

If the right hand side does not vanish as $r \to 0$, i.e. $2aQ \neq bP$, we can conclude that $\mathcal{F}$ is bounded as $r \to 0$. Furthermore, from Eq. (5.22) we may deduce that the invariant $\mathcal{G}$ is also bounded as $r \to 0$, which leads to a contradiction.

In the subcase with $2aQ = bP$, equations (5.21)-(5.22) imply

$$P\mathscr{L}_\mathcal{G} - Q\mathscr{L}_\mathcal{F} = \frac{Q}{4}\,. \tag{5.26}$$

Using Eq. (5.14), the invariant $\mathcal{G}$ can be expressed as

$$\mathcal{G} = \frac{4P}{\mathscr{L}_{\mathcal{F}} r^4} \left(P\mathscr{L}_{\mathcal{G}} - \frac{Q}{4} \right) = \frac{4QP}{r^4} \, , \tag{5.27}$$

while the invariant $\mathcal{F}$ via Eq. (5.12) becomes

$$\mathcal{F} = \frac{2}{r^4} \left(P^2 - Q^2 \right) . \tag{5.28}$$

Inserting these expressions into (5.22) gives us

$$\mathscr{L}_{\mathcal{G}} = \frac{2b}{r^4} \left(Q^2 + P^2 \right) . \tag{5.29}$$

The right-hand-side is unbounded as $r \to 0$, which is in a direct contradiction with the Eq. (5.15). □

Two prominent examples of NLE theories not covered by the theorems above are Born–Infeld (2.50) and ModMax (2.57) Lagrangians. The constraints that apply to these theories are summarised in the theorem below.

Theorem 5.6. *Suppose that the spacetime is a static, spherically symmetric solution of the Einstein–NLE field equations with the Born–Infeld or ModMax NLE Lagrangian. Then, given that $P \neq 0$, Ricci scalar R and Ricci squared $R_{ab}R^{ab}$ cannot both remain bounded as $r \to 0$.*

Proof of theorem 5.6. First, we assume that $Q \neq 0$ and consider Born–Infeld Lagrangian. The terms $\mathcal{F}\mathscr{L}_{\mathcal{F}}$ and $\mathscr{L}_{\mathcal{G}}$,

$$\mathcal{F}\mathscr{L}_{\mathcal{F}} = -\frac{1}{4} \frac{\mathcal{F}}{\sqrt{1 + \frac{\mathcal{F}}{2b^2} - \frac{\mathcal{G}^2}{16b^4}}} \, , \tag{5.30}$$

$$\mathscr{L}_{\mathcal{G}} = \frac{1}{16b^2} \frac{\mathcal{G}}{\sqrt{1 + \frac{\mathcal{F}}{2b^2} - \frac{\mathcal{G}^2}{16b^4}}} \, , \tag{5.31}$$

can be interpreted as a system in $\mathcal{G}$, which can be solved in a few steps. Combining (5.30) and (5.31) in order to eliminate the square root gives the relation

$$4b^2 \mathscr{L}_{\mathcal{G}} \mathcal{F} = -(\mathscr{L}_{\mathcal{F}} \mathcal{F}) \mathcal{G} \, , \tag{5.32}$$

and after squaring (5.31), we get

$$\mathcal{G}^2 (1 + 16\mathscr{L}_{\mathcal{G}}^2) = (16b^2 \mathscr{L}_{\mathcal{G}})^2 \left(1 + \frac{\mathcal{F}}{2b^2} \right) . \tag{5.33}$$

The two auxiliary expressions, (5.32) and (5.33), together define a quadratic equation for $\mathcal{G}$ with the solution [2]

$$\mathcal{G} = -16\mathscr{L}_{\mathcal{G}} \frac{\mathcal{F}\mathscr{L}_{\mathcal{F}} \pm W}{1 + 16\mathscr{L}_{\mathcal{G}}^2} \,, \tag{5.34}$$

where W is defined as

$$W := \sqrt{(\mathcal{F}\mathscr{L}_{\mathcal{F}})^2 + b^4(1 + 16\mathscr{L}_{\mathcal{G}}^2)} \,. \tag{5.35}$$

If we demand that $R_{ab}R^{ab}$ and R are bounded, then $\mathcal{G}$ has to be bounded as $r \to 0$. Recalling the Eq. (5.32), we deduce that $\mathcal{F}$ is also bounded as $r \to 0$, which leads to a contradiction. In the purely magnetic case, after evaluating Maxwell's equation (5.10),

$$\left(1 + \frac{P^2}{b^2 r^4}\right) \mathscr{L}_{\mathcal{F}} \widetilde{E}_r = 0 \,, \tag{5.36}$$

we can infer that $\mathscr{L}_{\mathcal{F}} \widetilde{E}_r = 0$ for all points where $r > 0$. We show that the case in which $\mathscr{L}_{\mathcal{F}}$ has zeros on this domain can be excluded from consideration. Since

$$\mathscr{L}_{\mathcal{G}} = -\frac{P}{(br)^2} \mathscr{L}_{\mathcal{F}} \widetilde{E}_r \,, \tag{5.37}$$

it follows that $\mathscr{L}_{\mathcal{G}} = 0$. Given that both the trace T and the invariant $\mathcal{F}\mathscr{L}_{\mathcal{F}}$ should be bounded, the same must hold for the Lagrangian (2.50) itself. Thus, $\mathscr{L}_{\mathcal{F}}$ has no zeros for $r > 0$ and the only possibility left is $\widetilde{E}_r = 0$. Then $\mathcal{G} = 0$ and $\mathcal{F} = 2P^2/r^4$ so that the invariant

$$\mathcal{F}\mathscr{L}_{\mathcal{F}} = -\frac{bP^2}{2r^2\sqrt{P^2 + b^2 r^4}} \,, \tag{5.38}$$

is manifestly unbounded as $r \to 0$.

In the dyonic case of ModMax theory, we have

$$\widetilde{E}_r = \frac{Qe^{-\gamma}}{r^2} \,, \quad \widetilde{B}_r = \frac{P}{r^2} \,. \tag{5.39}$$

Direct evaluation gives us

$$\mathcal{F}\mathscr{L}_{\mathcal{F}} = \frac{Q^2 + P^2}{2e^{\gamma} r^4} \frac{Q^2 - P^2 e^{2\gamma}}{Q^2 + P^2 e^{2\gamma}} \,, \tag{5.40}$$

$$\mathcal{G}\mathscr{L}_{\mathcal{F}} = -\frac{PQ}{r^4} \frac{Q^2 + P^2}{Q^2 + P^2 e^{2\gamma}} \,, \tag{5.41}$$

Both invariants are bounded as $r \to 0$ only in the trivial case, that is $P = 0 = Q$. □

[2] Sign cannot be chosen unambiguously as $\mathcal{F}$, $\mathscr{L}_{\mathcal{F}}$ and $\mathscr{L}_{\mathcal{G}}$ do not determine $\mathcal{G}$ uniquely. However, our result remains unaffected.

Theorem 5.6 implies that static, spherically symmetric, either dyonic or magnetic Born–Infeld and ModMax black holes suffer from unbounded curvature invariants as we approach the centre, meaning that the singularity is still present. If we recall the results from the literature [61, 60, 30, 64], this conclusion comes as no surprise.

5.3.3 Magnetic case

Due to the severe constraints on the electric and dyonic black holes, the only possibility left are strictly magnetically charged solutions, with the caveat that magnetic charge has not yet been observed. A representative example is the regular Bardeen black hole [9], sourced by the reverse-engineered $\mathcal{F}$-class Lagrangian which violates the Maxwellian weak field limit [6]. The family of regular magnetically charged black holes has expanded after Bronnikov [27] noticed under which conditions the central singularity may be absent, even if it is demanded that $\mathcal{F}$-class Lagrangian respects the Maxwellian weak field limit. The argument goes as follows. With the metric function set to $f(r) = 1 - 2M(r)/r$, the relation between the NLE Lagrangian and function $M(r)$ provided via (2.31) returns

$$M(r) = -\int \mathscr{L}(\mathcal{F}) r^2 dr \ , \tag{5.42}$$

where $\mathcal{F} = 2P^2/r^4$. Then, if the limit $\lim_{\mathcal{F}\to\infty} \mathscr{L}(\mathcal{F})$ exists and is finite, the spacetime can be regular as $r \to 0$. The Maxwellian weak field limit guarantees the convergence of the integral $M(\infty)$ obtained by integrating over the full range of radial coordinate, since in the asymptotic region, that is $r \to \infty$, Lagrangian takes the value $\mathscr{L} = -P^2/(2r^4) + O(r^{-5})$.

Before proceeding, we derive two expressions useful for magnetic case. The second NLE Maxwell's equation (5.10) with $Q = 0$, when multiplied by $\widetilde{E}_r$ gives us

$$\left(\frac{P^2}{r^4} - \frac{1}{2}\mathcal{F} \right) \mathscr{L}_\mathcal{F} = \frac{1}{4} \mathscr{L}_\mathcal{G} \mathcal{G} \ , \tag{5.43}$$

while multiplication by $\widetilde{B}_r$ leads to

$$\frac{1}{4} \mathscr{L}_\mathcal{F} \mathcal{G} = \frac{P^2}{r^4} \mathscr{L}_\mathcal{G} \, . \tag{5.44}$$

Previously proved theorems 5.3 and 5.6 apply also to the purely magnetically charged black holes. We reexamine the prominent class of quadratic NLE Lagrangians, appearing in the low energy limits of quantum field theories, this time with the electric charge set to zero.

Theorem 5.7. *Suppose that the spacetime is a static, spherically symmetric solution of the Einstein–NLE field equations with the NLE Lagrangian (5.17), such that* $h(\mathcal{F},\mathcal{G}) = a\mathcal{F}^2 + b\mathcal{F}\mathcal{G} + c\mathcal{G}^2$*, where* a*,* b *and* c *are real constants, such that the ordered pair* $(b,c) \neq (0,0)$*. Then, in the magnetically charged case, that is* $P \neq 0$ *and* $Q = 0$*, Ricci scalar* R *and Ricci squared* $R_{ab}R^{ab}$ *cannot both remain bounded as* $r \to 0$*.*

Proof of theorem 5.7. The proof can be divided into two subcases.

a) Assume that $b = 0$. If $a = 0$, we are back at the theorem 5.3, so we will suppose that $a \neq 0$. From (5.44), we get

$$\left(\mathscr{L}_\mathcal{F} - \frac{8cP^2}{r^4}\right)\mathcal{G} = 0\,. \tag{5.45}$$

At each point where $\mathcal{G} = 0$, we have $\widetilde{E}_r = 0$ and

$$\mathscr{L}_\mathcal{F}\mathcal{F} = \left(-\frac{1}{4} + \frac{4aP^2}{r^4}\right)\frac{2P^2}{r^4}\,, \tag{5.46}$$

while at each point where $\mathcal{G} \neq 0$, we have $\mathscr{L}_\mathcal{F} = 8cP^2/r^4$, while

$$\mathcal{F} = \frac{1}{2a}\left(\frac{1}{4} + \frac{8cP^2}{r^4}\right) \tag{5.47}$$

and

$$\mathscr{L}_\mathcal{F}\mathcal{F} = \left(\frac{1}{4} + \frac{8cP^2}{r^4}\right)\frac{4cP^2}{ar^4}\,. \tag{5.48}$$

So, at either type of points, the invariant $\mathcal{F}\mathscr{L}_\mathcal{F}$ is unbounded as $r \to 0$, in contradiction with the assumption that $R_{ab}R^{ab}$ and R are bounded.

b) Assume that $b \neq 0$. From (5.44), we have

$$\mathcal{F} = \frac{r^4}{4bP^2}\,\mathscr{L}_\mathcal{F}\mathcal{G} - \frac{2c}{b}\,\mathcal{G}\,. \tag{5.49}$$

Inserting the expression above into (5.43) gives us

$$\big(4P^2(b^2 - 4ac) - br^4\mathscr{L}_\mathcal{G}\big)\mathcal{G} = bP^2 + 2r^4\big(b\mathscr{L}_\mathcal{F}\mathcal{F} - a\mathscr{L}_\mathcal{F}\mathcal{G}\big)\,. \tag{5.50}$$

This case is branching further into two subcases. If $b^2 \neq 4ac$, $\mathcal{G}$ is bounded as $r \to 0$ and the contradiction follows from (5.49) which implies that $\mathcal{F}$ should also be bounded. On the other hand, if $b^2 = 4ac$ (notice that the case $c = 0$ is immediately excluded), $\mathscr{L}_\mathcal{F}$ and $\mathscr{L}_\mathcal{G}$ are related through

$$\mathscr{L}_\mathcal{F} = -\frac{1}{4} + \frac{b}{2c}\,\mathscr{L}_\mathcal{G}\,, \tag{5.51}$$

which implies via (5.44) that $\mathscr{L}_{\mathcal{F}}$ is bounded as $r \to 0$. Since

$$\mathscr{L}_{\mathcal{F}}\mathcal{G} = \left(-\frac{1}{4} + \frac{b}{2c}\mathscr{L}_{\mathcal{G}}\right)\mathcal{G}\,, \tag{5.52}$$

the same applies to $\mathcal{G}$. The relation

$$b\mathcal{F} = \mathscr{L}_{\mathcal{G}} - 2c\mathcal{G} \tag{5.53}$$

suggests that $\mathcal{F}$ is also bounded as $r \to 0$ and we again reach a contradiction. $\square$

Theorem 5.7 does not cover quadratic $\mathcal{F}$-class Lagrangian of the form $\mathscr{L}(\mathcal{F}) = -\mathcal{F}/4 + a\mathcal{F}^2$. From Eq. (5.44) we see that one option is to take $\mathcal{G} = 0$ and obtain the contradiction by repeating the proof above. Alternatively, we may set $\mathscr{L}_{\mathcal{F}} = 0$, which translates into $\mathcal{F} = 1/(8a)$. Maxwell's equations are automatically satisfied, with

$$\widetilde{E}_r^2 = \frac{P^2}{r^4} - \frac{1}{16a} \quad \text{and} \quad \widetilde{B}_r = \frac{P}{r^2}\,. \tag{5.54}$$

The Einstein's equation can be written as

$$R_{ab} - \frac{1}{2}Rg_{ab} + \lambda g_{ab} = 0\,, \tag{5.55}$$

where the introduced effective cosmological constant is

$$\lambda := \Lambda + \frac{1}{32a}\,. \tag{5.56}$$

Static and spherically symmetric solutions of Eq. (5.55) are Schwarzschild-(anti-) de Sitter black holes [23, 124]. Although the curvature invariants $R = 4\lambda$ and $R_{ab}R^{ab} = 4\lambda^2$ are constant and thus trivially bounded, the Kretschmann scalar $R_{abcd}R^{abcd}$ suffers from the standard Schwarzschild-like singular behaviour as $r \to 0$. An interesting feature of this solution is that the electromagnetic field enters the field equations as a part of the cosmological constant and manifests itself as a nonvanishing magnetic charge P. In principle, one could glue the solution with $\mathcal{G} = 0$ to the one with $\mathscr{L}_{\mathcal{F}} = 0$ along the overlapping hypersurface $r^4 = 16aP^2$. Unfortunately, this construction carries over the original spacetime irregularities.

The no-go theorems for dyonic and magnetic cases are summarised in Table 5.1.

Table 5.1: A concise overview of no-go theorems with the ✗ sign labelling singular cases.

	Dyonic	Magnetic
$\mathcal{F}$-class Lagrangians	✗	
quadratic Lagrangians $\mathscr{L} = -\frac{1}{4}\mathcal{F} + a\mathcal{F}^2 + b\mathcal{F}\mathcal{G} + c\mathcal{G}^2$	✗	✗
$\mathscr{L} = -\frac{1}{4}\mathcal{F} + h(\mathcal{G})$	✗	✗
$\mathscr{L} = -\frac{1}{4}\mathcal{F} + a\mathcal{F}^s\mathcal{G}^u,\ s, u \geq 1$	✗	
Born-Infeld Lagrangian	✗	✗
ModMax Lagrangian	✗	✗

5.3.4 Neutral case

Finally, we consider the neutral case in which $Q = 0 = P$. From the first Maxwell's equation (5.9), it follows that $\widetilde{B}_r = 0$, while the second one (5.10) is reduced to $\mathscr{L}_\mathcal{F}\widetilde{E}_r = 0$. Hence, at each point, we have either trivial fields or $\mathscr{L}_\mathcal{F} = 0$. In the latter case, the NLE energy-momentum tensor is proportional to the metric, $T_{ab} = (T/4)g_{ab}$, so its contribution is in the form of the effective cosmological constant. The solutions are again Schwarzschild-(anti)-de Sitter black holes, which, by the arguments given at the end of the previous chapter, are not regular in the sense of bounded curvature invariants. This subcase is more of a curiosity since for most of the NLE Lagrangians the function $\mathscr{L}_\mathcal{F}$ does not have zeros [171].

Chapter 6

Spacetime block-diagonalisation with NLE fields

6.1 The problem of integrability and symmetries

Spacetimes with symmetries provide a suitable arena for establishing various uniqueness and black hole no-hair theorems. This mathematically convenient reduction has a physical justification, as near-equilibrium gravitating configurations should correspond to these idealised states. A further simplification is achieved if the symmetries force the metric to attain a block-diagonal form, with coordinates adapted to the corresponding Killing vector fields. We are interested in the conditions that guarantee block diagonalisation of the metric, encapsulated in the Frobenius' theorem. A central element in the statement of the theorem are distributions defined by Killing vector fields $\{K^a_{(1)},\ \dots,\ K^a_{(n)}\}$ on a smooth, orientable m-manifold M with a smooth Lorentzian metric g_{ab}. To set aside the trivial cases, we assume that the Killing vector fields are linearly independent on a nonempty open subset $N \subseteq M$ and that $m \geq 3$ and $n \geq 1$. Linear independence breaks down on the set $M - N$, which consists of, for example, the axis of axial symmetry or bifurcation surfaces of the Killing horizons.

Killing vector fields enable defining two distributions, whose integrability in turn implies block-diagonal metric form. A distribution $\mathscr{D}$, spanned by the n linearly independent vector fields $K^a_{(1)}, \dots, K^a_{(n)}$, is a smooth rank-$n$ subbundle of the tangent bundle TN. By the metric isomorphism, each of the Killing vector fields has an associated 1-form $\mathbf{K}^{(i)}$, whose components are given by $K^{(i)}_a := g_{ab}K^b_{(i)}$. Set of these 1-forms $\{\mathbf{K}^{(1)},\ \dots,\ \mathbf{K}^{(n)}\}$ defines the second distribution $\mathscr{D}^\perp$, a smooth $(m-n)$-subbundle of the tangent bundle TN, such that $\mathscr{D}^\perp|_p := \mathrm{Ker}\,\mathbf{K}^{(1)}|_p \cap \dots \cap \mathrm{Ker}\,\mathbf{K}^{(n)}|_p$, at each point $p \in M$. Hence, the distribution $\mathscr{D}^\perp$ consists of the vector fields X^a

for which $K^{(i)}_a X^a = 0$ at point $p \in M$ for all $i \in \{1, \dots, n\}$. The products of the Killing vector fields and the set of their zeros will be denoted by $\kappa_{ij} := g_{ab} K^a_{(i)} K^b_{(j)}$ and $\mathscr{Z} := \{p \in M \mid (\exists\, i) : K^a_{(i)}|_p = 0\}$, respectively. The Killing vector fields define a decomposable n-form,

$$\boldsymbol{\alpha} := \mathbf{K}^{(1)} \wedge \ldots \wedge \mathbf{K}^{(n)}, \tag{6.1}$$

which is by definition nonzero on the set N.

The problem of the integrability of the distributions comes down to the question of whether there are nonempty immersed submanifolds whose tangent space coincides with a given distribution at each point. The answer is given by the Frobenius' theorem [119]: $\mathscr{D}$ is integrable if and only if it is involutive, meaning that $[K_{(i)}, K_{(j)}]^a \in \mathscr{D}$ for all $i, j \in \{1, \dots, n\}$, while $\mathscr{D}^\perp$ is integrable if and only if $\boldsymbol{\alpha} \wedge \mathrm{d}\mathbf{K}^{(i)} = 0$ for all $i \in \{1, \dots, n\}$. Involutivity of $\mathscr{D}$ does not pose a stringent condition since the commutator of two Killing vector fields is again a Killing vector field and a number of independent Killing vector fields has a maximum value of $m(m+1)/2$ [187]. A stronger condition is commuting of the Killing vector fields, which enables choosing local coordinates in which they take the form $K_{(i)} = \partial/\partial z^i$ for all $i \in \{1, ..., n\}$ [119]. In general, this will not be the case, but a procedure given by Carter [34] and Szabados [176] provides a recipe for constructing commuting Killing vector fields from given ones. Integration of the pull-back of a nonspacelike Killing vector field $K^a_{(1)}$ along the orbits of a spacelike Killing vector field with compact orbits $K^a_{(2)}$ produces a new nonspacelike Killing vector field that commutes with $K^a_{(2)}$. This result is often called upon in stationary and axially symmetric spacetimes, where we assume that the timelike Killing vector field and axial Killing vector field are commuting.

The nontrivial step is proving the integrability of $\mathscr{D}^\perp$. A basic idea [117, 141, 194] relies on the identity

$$\mathrm{d}{\star}(\boldsymbol{\alpha} \wedge \mathrm{d}\mathbf{K}^{(i)}) = 2\,{\star}(\boldsymbol{\alpha} \wedge \mathbf{R}(K_{(i)}))\,, \tag{6.2}$$

which can be derived in a few steps, with the aid of the Killing lemma $\mathrm{d}{\star}\mathrm{d}\mathbf{K} = 2{\star}\mathbf{R}(K)$ [187],

$$\begin{aligned} \mathrm{d}{\star}(\boldsymbol{\alpha} \wedge \mathrm{d}\mathbf{K}^{(i)}) &= \mathrm{d} i_{K_{(n)}} \cdots i_{K_{(1)}} {\star}\mathrm{d}\mathbf{K}^{(i)} = \\ &= (-1)^n i_{K_{(n)}} \cdots i_{K_{(1)}} \mathrm{d}{\star}\mathrm{d}\mathbf{K}^{(i)} = \\ &= (-1)^n 2 i_{K_{(n)}} \cdots i_{K_{(1)}} {\star}\mathbf{R}(K_{(i)}) = \\ &= 2\,{\star}(\boldsymbol{\alpha} \wedge \mathbf{R}(K_{(i)}))\,. \end{aligned}$$

Here we can distinguish two subcases. First, the vacuum Einstein's equation

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0 \,, \tag{6.3}$$

implies that the Ricci tensor is proportional to the metric, precisely $(m-2)R_{ab} = 2\Lambda g_{ab}$ and the $(m-n-2)$-form $\star(\boldsymbol{\alpha} \wedge \mathrm{d}\mathbf{K}^{(i)})$ is closed for each i. In the case when $m = n+2$, the situation is especially simple since $\star(\boldsymbol{\alpha} \wedge \mathrm{d}\mathbf{K}^{(i)})$ is a scalar, constant on each connected component of N. If the set $M - N$ is nonempty, this constant is by continuity zero on each connected component of N with a nonempty boundary and, consequently, $\mathscr{D}^\perp$ is integrable. If there are fewer symmetries in the spacetime ($m > n+2$), the form $\star(\boldsymbol{\alpha} \wedge \mathrm{d}\mathbf{K}^{(i)})$ is no longer a scalar and the integrability of $\mathscr{D}^\perp$ may be established using appropriate divergence identities [96].

Our focus will be on the other scenario, when the metric is a solution of the non-vacuum Einstein's equation

$$R_{ab} = 8\pi T_{ab} + \frac{2\Lambda - 8\pi g^{cd} T_{cd}}{m-2} \, g_{ab} \ . \tag{6.4}$$

Then, the integrability of $\mathscr{D}^\perp$ translates to vanishing of the $(n+1)$-form $\boldsymbol{\alpha} \wedge \mathbf{T}(K_{(i)})$ since

$$\mathrm{d}\star(\boldsymbol{\alpha} \wedge \mathrm{d}\mathbf{K}^{(i)}) = 2\star(\boldsymbol{\alpha} \wedge \mathbf{R}(K_{(i)})) = 16\pi\star(\boldsymbol{\alpha} \wedge \mathbf{T}(K_{(i)})) \ , \tag{6.5}$$

where the relation between R_{ab} and T_{ab} follows from (6.4). Notice that the second term is absent in the final expression since it is of the form $\boldsymbol{\alpha} \wedge \mathbf{K}^{(i)}$.

To illustrate the introduced concepts, we can single out two prominent classes of spacetimes with symmetries. An m-dimensional spacetime is static if it is stationary and the timelike Killing vector field satisfies the Frobenius' condition $\mathbf{k} \wedge \mathrm{d}\mathbf{k} = 0$. A four-dimensional spacetime is circular if it is stationary and axially symmetric, i.e. admits a timelike Killing vector field k^a and an axial Killing vector field m^a, such that $[k,m]^a = 0$ and the Frobenius' conditions $\mathbf{k} \wedge \mathbf{m} \wedge \mathrm{d}\mathbf{k} = 0$ and $\mathbf{k} \wedge \mathbf{m} \wedge \mathrm{d}\mathbf{m} = 0$ are satisfied.

Once the integrability of both distributions has been established and $T_pM = \mathscr{D}|_p \oplus \mathscr{D}^\perp|_p$ at each point p of some open subset of N, then this open set may be covered by local coordinate charts of the form $(U; z^1, \dots, z^n, y^{n+1}, \dots, y^m)$, where $(z^1, \dots, z^n)$ are the coordinates for the integral manifold of $\mathscr{D}$ and $(y^{n+1}, \dots, y^m)$ are the coordinates for the integral manifold of $\mathscr{D}^\perp$. Since $g(\partial/\partial z^i, \partial/\partial y^j) = 0$, the spacetime metric in the matrix representation attains a block-diagonal form. However, this construction is spoiled at each point q where a nonzero vector ℓ^a such that $\ell^a \in \mathscr{D}|_q \cap \mathscr{D}^\perp|_q$ exists. If a vector belongs both to $\mathscr{D}$ and $\mathscr{D}^\perp$, it is a linear combination of Killing vectors $K^a_{(i)}$ and orthogonal to all of them at the same time, thus ℓ^a has to be null. To distinguish the two scenarios, we define the orthogonal-transitive domain of the spacetime as an open set O on which both distributions are integrable.

So far, integrability conditions have been established for a number of theories. In the case of the stationary, axially symmetric four-dimensional solution of the Einstein-Maxwell system, a mild assumption (for instance, the nonempty axis of rotation) is enough to ensure metric block-diagonalisation. As stated earlier, we say that such spacetime is circular, i.e. foliated by 2-surfaces to which the timelike and axial Killing vector fields are orthogonal. On the other hand, a stationary solution of gravitational field equations does not even have to be axially symmetric. Some of the counterexamples are Majumdar-Papapetrou spacetime [140, 123, 77, 132], representing an electrovacuum multi-black hole solution, and black holes endowed with electromagnetic and massive vector fields [154, 153]. Nevertheless, an asymptotically flat and analytic solution of vacuum Einstein's equation will necessarily be axially symmetric, as proven by Hawking's rigidity theorem [82], which was further generalised to higher-dimensional cases and theories beyond general relativity [98, 99, 100].

We analyse the needed integrability conditions for various theories containing NLE fields, including solely NLE Lagrangians and theories in which NLE fields are either minimally or nonminimally coupled to scalar fields [17].

6.2 Integrability imposed by the NLE fields

With the matter sector consisting of purely electromagnetic fields, we consider a threefold generalisation of Maxwell's theory which is performed by replacing Maxwell's Lagrangian with NLE theories while allowing the spacetime dimension to be different than four and by adding the gauge Chern–Simons term (gCS). Precisely, if we restrict the spacetime dimension to $m = 4$, the NLE Lagrangian density may belong to the $\mathcal{FG}$-class, while for $m \neq 4$ we consider $\mathcal{F}$-class Lagrangians only since the invariant $\mathcal{G}$ is no longer a scalar in that case. When the spacetime dimension is odd, an additional gCS term of the form[1] $\mu\,\mathbf{A} \wedge \mathbf{F}^{(m-1)/2}$ may be present in the action, where μ is the coupling constant. Although originally of geometric origin, Chern–Simons terms found their place in many different areas of physics, such as topological quantum field theory, quantum Hall effect or the study of quan-

[1]Notation: $\mathbf{F}^{(m-1)/2} := \underbrace{\mathbf{F} \wedge ... \wedge \mathbf{F}}_{(m-1)/2 \text{ times}}$

tum anomalies. Here we are interested in their effects on the Einstein-NLE field equations and block-diagonalisability of the corresponding metric.

The total electromagnetic Lagrangian with gCS contribution is

$$\mathbf{L}^{(\mathrm{em})} = \mathscr{L}\boldsymbol{\epsilon} + \mu\,\mathbf{A}\wedge\mathbf{F}^{(m-1)/2}\,. \tag{6.6}$$

Variation with respect to the gauge potential $\mathbf{A}$ returns the generalised source-free Maxwell's (gMax) equations

$$\mathrm{d}\mathbf{F} = 0\ , \quad \mathrm{d}{\star}\mathbf{Z} = \frac{m+1}{2}\,\mu\,\mathbf{F}^{(m-1)/2}\,. \tag{6.7}$$

The form $\mathbf{Z}$ in dimensions different than four is given by $\mathbf{Z} := -4\partial_{\mathcal{F}}\mathscr{L}\,\mathbf{F}$. For even m, the right hand side of the second gMax equation (6.7) is zero by definition. The corresponding energy-momentum tensor stays unaltered by the presence of the Chern-Simons term in the electromagnetic action,

$$T_{ab}^{(\mathrm{em})} = \frac{1}{4\pi}\left(Z_{ac}F_b{}^{c} + \mathscr{L}g_{ab}\right). \tag{6.8}$$

Theorem 6.1. *Suppose that the m-dimensional spacetime (M, g_{ab}) admits $m-2$ smooth pairwise commuting Killing vector fields $\{K^a_{(1)}, \dots, K^a_{(m-2)}\}$, with the corresponding nonempty set of zeros $\mathscr{Z} \subseteq M$. Furthermore, suppose that this spacetime contains the electromagnetic 2-form F_{ab} which inherits the symmetries, $\pounds_{K_{(i)}}F_{ab} = 0$ for all i. Then, given that g_{ab} and F_{ab} are solutions of the Einstein-gMax field equations defined above, it follows that $\boldsymbol{\alpha}\wedge\mathbf{T}(K_{(i)}) = 0$ for all i on any open set sharing a boundary with the zero set $\mathscr{Z}$.*

Proof of theorem 6.1. The form $\mathbf{T}(K_{(i)})$ consists of two terms, the nontrivial one being $\boldsymbol{\zeta} := i_E\mathbf{Z}$, while the other is proportional to $K_{(i)}$ and therefore vanishes after taking the wedge product with an n-form α. With the aid of (A.7), for the 1-form $\boldsymbol{\zeta}$ we have

$$\begin{aligned}{\star}(\boldsymbol{\alpha}\wedge\boldsymbol{\zeta}) &= (-1)^n\, i_{K_{(n)}}\cdots i_{K_{(1)}}{\star}\boldsymbol{\zeta} = \\ &= (-1)^{m+n}\, i_{K_{(n)}}\cdots i_{K_{(1)}}({\star}\mathbf{Z}\wedge i_{K_{(i)}}\mathbf{F})\,,\end{aligned} \tag{6.9}$$

so that the integrability depends on the behaviour of the scalar $i_{K_{(i)}}i_{K_{(j)}}\mathbf{F}$ and the $(m-n-2)$-form $i_{K_{(n)}}\cdots i_{K_{(1)}}{\star}\mathbf{Z}$. In a special case when $m = n+2$, we are dealing with two scalar quantities. For a symmetry inheriting form $\mathbf{F}$, $\pounds_{K_{(i)}}\mathbf{F} = 0$ for all i, so that we get

$$\mathrm{d}(i_{K_{(i)}} i_{K_{(j)}} \mathbf{F}) = i_{K_{(i)}} i_{K_{(j)}} \mathrm{d}\mathbf{F} \,, \tag{6.10}$$

$$\mathrm{d}(i_{K_{(n)}} \cdots i_{K_{(1)}} \star\mathbf{Z}) = (-1)^n i_{K_{(n)}} \cdots i_{K_{(1)}} \mathrm{d}\star\mathbf{Z} \,. \tag{6.11}$$

Using the first gMax equation, we may deduce that the scalars $i_{K_{(i)}} i_{K_{(j)}} \mathbf{F}$ are locally constant and vanish on any open set sharing a boundary with the set of the zeros $\mathscr{Z}$. By the second gMax equation, the same holds for the scalar $i_{K_{(n)}} \cdots i_{K_{(1)}} \star\mathbf{Z}$ and $\boldsymbol{\alpha} \wedge \mathbf{T}(K_{(i)}) = 0$ for all i. □

Before treating the multifield cases in which the electromagnetic fields are intertwined with scalar fields, we prove the integrability of scalar fields solely. A large class of theories is covered assuming that the energy-momentum tensor of the scalar field is given by

$$T^{(\mathrm{rs})}_{ab} = F(g_{cd}, \phi, \nabla_c\phi, \dots) \nabla_a\phi \nabla_b\phi + G(g_{cd}, \phi, \nabla_c\phi, \dots) g_{ab} \,, \tag{6.12}$$

where F and G are some real functions. For example, the canonical case is recovered if $F = 1$ and $G = -(\nabla_a\phi\nabla^a\phi)/2 - \mathscr{U}(\phi)$, with a potential function $\mathscr{U}$, while more general choices produce some of the k-essence theories. If a scalar field is symmetry inheriting, $K^a_{(i)} \nabla_a\phi = 0$ for all i, it is not difficult to see that $\boldsymbol{\alpha} \wedge \mathbf{T}(K_{(i)}) = 0$ immediately holds. The same argument applies to the complex scalar fields whose energy-momentum tensor has the following form

$$T^{(\mathrm{cs})}_{ab} = F(g_{cd}, \phi, \phi^*, \dots) \nabla_{(a}\phi^* \nabla_{b)}\phi + G(g_{cd}, \phi, \phi^*, \dots) g_{ab} \,. \tag{6.13}$$

For noninteracting scalar and electromagnetic fields, the integrability easily follows from the given arguments since the total energy-momentum tensor is just a sum of the electromagnetic and scalar parts. Interacting fields pose a more challenging task. First, we consider a theory in which the nonlinear electromagnetic field is coupled to the complex scalar field, given by the following Lagrangian

$$\mathscr{L}^{(\phi,\mathrm{em})} = \mathscr{L}(\mathcal{F}, \mathcal{G}) - (\mathcal{D}_a\phi)^* (\mathcal{D}^a\phi) - \mathscr{U}(\phi^*\phi) \,, \tag{6.14}$$

where $\mathcal{D}_a = \nabla_a + iqA_a$ is a covariant gauge derivative. The NLE Maxwell's equations are of the form

$$\mathrm{d}\mathbf{F} = 0 \quad \text{and} \quad \mathrm{d}\star\mathbf{Z} = 4\pi\star\mathbf{J} \,, \tag{6.15}$$

where the current 1-form is given by

$$J_a = \frac{iq}{4\pi}\left(\phi^* \mathcal{D}_a \phi - \phi(\mathcal{D}_a\phi)^*\right). \tag{6.16}$$

The total energy-momentum tensor is equal to

$$4\pi T_{ab} = Z_{ac}F_b{}^c + \mathscr{L} g_{ab} + 2(\mathcal{D}_{(a}\phi)^* \mathcal{D}_{b)}\phi - \big((\mathcal{D}_c\phi)^*(\mathcal{D}^c\phi) + \mathscr{U}(\phi^*\phi)\big) g_{ab}\,. \tag{6.17}$$

The main complication comes from the current term in the NLE Maxwell's equations, so the theorem will still hold only under an additional assumption.

Theorem 6.2. *Suppose that the m-dimensional spacetime* (M, g_{ab}) *admits* $m-2$ *smooth pairwise commuting Killing vector fields* $\{K^a_{(1)}, \dots, K^a_{(m-2)}\}$*, with the corresponding nonempty set of zeros* $\mathscr{Z} \subseteq M$*. Furthermore, suppose that this spacetime contains complex scalar field* ϕ *and electromagnetic 2-form* F_{ab}*, both of which inherit the spacetime symmetries,* $\pounds_{K_{(i)}}\phi = 0$ *and* $\pounds_{K_{(i)}} F_{ab} = 0$ *for all* i*, and* $\boldsymbol{\alpha} \wedge \mathbf{J} = 0$*. Then, given that* g_{ab}*,* ϕ *and* F_{ab} *are solutions of the Einstein-NLE-Maxwell-scalar field equations defined above, it follows that* $\boldsymbol{\alpha} \wedge \mathbf{T}(K_{(i)}) = 0$ *for all* i *on any open set sharing a boundary with the zero set* $\mathscr{Z}$*.*

Proof of theorem 6.2. The relevant terms contained in the form $\mathbf{T}(K_{(i)})$, i.e. the ones not proportional to any of the Killing vectors $K^a_{(i)}$, are $\zeta_a := Z_{ac}F_b{}^c K^b_{(i)}$ and $2K^b_{(i)}(\mathcal{D}_{(a}\phi)^*\mathcal{D}_{b)}\phi$. The former term is the same one appearing in the theorem 6.1 and can be handled in the identical way, while the latter is proportional to the current J_a since

$$\begin{aligned} 2K^b_{(i)}(\mathcal{D}_{(a}\phi)^*\mathcal{D}_{b)}\phi &= iq\phi(\mathcal{D}_a\phi)^* K^b_{(i)} A_b - iq\phi^*(\mathcal{D}_a\phi)K^b_{(i)}A_b = \\ &= iqK^b_{(i)}A_b(\phi(\mathcal{D}_a\phi)^* - \phi^*\mathcal{D}_a\phi) = \\ &= -4\pi(K^b_{(i)}A_b)J_a\ , \end{aligned} \tag{6.18}$$

where the first equality follows from the symmetry inheritance of the scalar field, $K^a_{(i)}\nabla_a\phi = 0$. Then, if $\boldsymbol{\alpha} \wedge \mathbf{J} = 0$, we also have $\boldsymbol{\alpha} \wedge \mathbf{T}(K_{(i)}) = 0$ for all i. □

Let us briefly comment on the possible relaxation of the assumptions. Since there are hairy black hole solutions harbouring symmetry non-inheriting scalar fields [92], we will consider that case specifically, while keeping the other assumptions unchanged. The only possible complex scalar field hair is of Herdeiro-Radu type, i.e. $\pounds_{K(i)}\phi = i\alpha_i\phi$ with some real constants $\alpha_i \in \mathbb{R}$, for which we have $2K^b_{(i)}(\mathcal{D}_{(a}\phi)^*\mathcal{D}_{b)}\phi = 4\pi((\alpha_i/q) - K^b_{(i)}A_b)J_a$. The "problematic" term is again proportional to the current 1-form, implying that the theorem 6.2 still holds under given conditions and the solution is forced to attain a block-diagonal metric. The analysis for more general forms of symmetry non-inheriting fields needs to be carefully examined, which we leave for future work.

As a second example of the theory containing interacting fields, we take real scalar fields nonminimally coupled to the NLE fields. A prominent example are dilatons [67], whose Lagrangian density is

$$\mathscr{L}^{(\mathrm{dil,em})} = f(\phi)\mathscr{L}(\mathcal{F},\mathcal{G}) - \frac{1}{2}\,\nabla_a\phi\,\nabla^a\phi - \mathscr{U}(\phi)\,, \tag{6.19}$$

where f and $\mathscr{U}$ are some smooth functions of the dilaton field ϕ. The equations governing the electromagnetic and dilaton fields are

$$\mathrm{d}\mathbf{F} = 0\,, \quad \mathrm{d}{\star}(f(\phi)\mathbf{Z}) = 0\,, \tag{6.20}$$

$$\Box\phi - \mathscr{U}'(\phi) + f'(\phi)\mathscr{L}(\mathcal{F},\mathcal{G}) = 0\,. \tag{6.21}$$

The corresponding energy-momentum tensor reads

$$4\pi T_{ab} = 4\pi f(\phi)T_{ab}^{(\mathrm{em})} + \nabla_a\phi\nabla_b\phi - \left(\frac{1}{2}\,\nabla_c\phi\,\nabla^c\phi + \mathscr{U}(\phi)\right) g_{ab}\,. \tag{6.22}$$

Theorem 6.3. *Suppose that the m-dimensional spacetime (M, g_{ab}) admits $m-2$ smooth pairwise commuting Killing vector fields $\{K^a_{(1)},\dots,K^a_{(m-2)}\}$, with the corresponding nonempty set of zeros $\mathscr{Z} \subseteq M$. Furthermore, suppose that this spacetime contains the dilaton field ϕ and the electromagnetic 2-form F_{ab}, both of which inherit the spacetime symmetries, $\pounds_{K_{(i)}}\phi = 0$ and $\pounds_{K_{(i)}}F_{ab} = 0$ for all i. Then, given that g_{ab}, ϕ and F_{ab} are solutions of the Einstein-dilaton-Maxwell field equations defined above, it follows that $\boldsymbol{\alpha} \wedge \mathbf{T}(K_{(i)}) = 0$ for all i on any open set sharing a boundary with the zero set $\mathscr{Z}$.*

Proof of theorem 6.3. Since the dilaton field is by assumption symmetry inheriting, the only nontrivial term is of the form $\zeta_a := f(\phi)Z_{ac}F_b{}^cK^b_{(i)}$. Using equation (6.20), we have

$$\mathrm{d}(i_{K_{(n)}}\cdots i_{K_{(1)}}{\star}f(\phi)\mathbf{Z}) = (-1)^n i_{K_{(n)}}\cdots i_{K_{(1)}}\mathrm{d}{\star}(f(\phi)\mathbf{Z}) = 0\,, \tag{6.23}$$

so that the scalar $i_{K_{(n)}}\cdots i_{K_{(1)}}{\star}f(\phi)\mathbf{Z}$ is constant and zero on each open set sharing a boundary with the set $\mathscr{Z}$. □

6.3 No-go theorem for null electromagnetic fields in static spactimes

In the 4-dimensional spacetime, the null electromagnetic fields are naturally defined as those for which both electromagnetic invariants vanish, i.e. $\mathcal{F} = \mathcal{G} = 0$. A well-established result within Einstein-Maxwell theory is that such fields are absent in the static spacetime. We briefly repeat the simple proof carried out by employing the spinor approach presented in [181]. The metric of a static spacetime can be decomposed into time slices as

$$g = -V^2 dt^2 + h_{ij}(x^k) dx^i dx^j \ , \tag{6.24}$$

with defined hypersurface orthogonal timelike Killing vector field k^a. The gravitational field equation implies $\widetilde{T}_{ti} = 0$, which in a covariant form becomes

$$\widetilde{T}_{ab} k^b = f(x^i) k_a \ , \tag{6.25}$$

where f is some nonpositive function, as guaranteed by the dominant energy condition obeyed by Maxwell's energy-momentum tensor. In the spinor language, equation (6.25) becomes

$$\phi_{AB} \overline{\phi}_{A'B'} k^{BB'} = 2\pi f k_{AA'} \ . \tag{6.26}$$

Contraction with ${\phi^A}_C$ gives us

$$\phi_{AB} {\phi^A}_C \overline{\phi}_{A'B'} k^{BB'} = 2\pi f k_{AA'} {\phi^A}_C \ . \tag{6.27}$$

With the aid of identity (B.10), we get

$$\begin{aligned} \overline{\phi}_{A'B'} k^{BB'} \epsilon_{BC} (\phi_{EF} \phi^{EF}) &= 4\pi f k_{AA'} {\phi^A}_C \ , \\ (\phi_{EF} \phi^{EF}) \overline{\phi}_{A'B'} {k_C}^{B'} &= 4\pi f k_{AA'} {\phi^A}_C \ . \end{aligned} \tag{6.28}$$

For the null electromagnetic fields, it follows that $\phi_{EF} \phi^{EF} = 0$ (see theorem B.3) which leaves us with

$$f k_{AA'} {\phi^A}_C = 0 \ . \tag{6.29}$$

The equation (6.29) will be satisfied if either $f = 0$ or $k_{AA'} {\phi^A}_C = 0$. In the latter case, contraction with $\overline{\phi}^{A'C'}$ gives

$$\phi_{AB} \overline{\phi}_{A'B'} k^{BB'} = 0 \ . \tag{6.30}$$

The same condition holds in the former case, as follows from (6.26). To complete the proof, it remains to show that the timelike character of the Killing vector field is incompatible with null electromagnetic fields. The null electromagnetic spinor can be decomposed in terms of principal spinors as $\phi_{AB} = \alpha_A \alpha_B$, as stated in theorem B.1. Then the expression (6.30) becomes

$$\alpha_A \alpha_B \overline{\alpha}_{A'} \overline{\alpha}_{B'} k^{BB'} = 0 \ , \tag{6.31}$$

while after introducing the corresponding null vector field $l^a = l^{AA'} = \alpha^A \overline{\alpha}^{A'}$, it attains a simple form

$$l_a l_b k^b = 0 \ . \tag{6.32}$$

Assuming that $l^a \neq 0$ and adopting a local inertial coordinate system, $l_b k^b = 0$ implies

$$-k^0 l^0 + \mathbf{k} \cdot \mathbf{l} = l^0(-k^0 + |\mathbf{k}| \cos\theta) = 0 \ , \tag{6.33}$$

where $\cos\theta = \hat{\mathbf{k}} \cdot \hat{\mathbf{l}}$. For a timelike Killing vector k^a we have the following inequalities, $k^0 > |\mathbf{k}| \geq |\mathbf{k}| \cos\theta$, which contradict the equation above. However, null electromagnetic fields can exist in circular spacetimes [74].

We present an extended version of the theorem, where the generalisation is achieved in three aspects [17]. It remains valid for a broader class of both gravitational and electromagnetic theories and also in spacetimes of dimension different from four. Generality in the sense of gravitational theories is achieved by relying on Carter's classification [33] of tensors into even and odd orthogonal-transitive types (shortened as even/odd "o-t tensors"). The idea is to observe the behaviour of scalars formed by contracting the rank-k tensor $T^{a\dots}{}_{b\dots}$ with s Killing vectors $K^a_{(i)}$ and $k-s$ vectors from $\mathscr{D}^\perp$ on the orthogonal-transitive domain O. If all such scalars for which s is even vanish, we say that the tensor $T^{a\dots}{}_{b\dots}$ is of even o-t type. Similarly, if s is odd and all these scalars are zero, the tensor $T^{a\dots}{}_{b\dots}$ is of odd o-t type. The concept can be illustrated by taking Ricci tensor as an example. Contraction of the expression $\boldsymbol{\alpha} \wedge \mathbf{R}(K_{(i)}) = 0$, which is valid on O, with a vector $Y_{(j)} = \partial/\partial y^j$ belonging to $\mathscr{D}^\perp$ implies $R(K_{(i)}, Y_{(j)}) = 0$, thus proving that Ricci tensor is of odd o-t type. Riemann tensor and its covariant derivatives are also odd o-t tensors [201]. As a consequence, field equations of Lovelock gravity and $f(R)$ theories are of the same type [170]. Bach tensor, given by $B_{ab} = 2\nabla_c \nabla_d C^c{}_{ab}{}^d + R_{cd} C^c{}_{ab}{}^d$, where C_{abcd} is Weyl tensor, is a further example of an odd o-t tensor. It emerges as a part of equations of motion in "quadratic gravity" theories [175]. We singled out examples of odd o-t tensors since our result is valid for field equations of this type.

An obstacle encountered in the dimensional generalisation of the theorem comes from the fact that the form $\star\mathbf{F}$ is no longer a 2-form. Consequently, the scalar $\mathcal{G}$ cannot be defined whenever $m \neq 4$ and the definition of null electromagnetic fields has to be replaced accordingly. The suitable extension comes in the form of the N-type fields within the generalised Petrov classification [172, 137].

Definition 6.1. *We say that an electromagnetic field is of type N at a point $p \in M$ if there is a null vector $\ell^a \in T_pM$ such that*

$$i_\ell \mathbf{F} = 0 \quad \textit{and} \quad \boldsymbol{\ell} \wedge \mathbf{F} = 0 \tag{6.34}$$

at this point.

Notice that in four dimensions $\mathbf{F}$ is of type N if and only if it is null (see theorem B.3).

On the tangent space T_pM we can introduce a basis $(\ell^a, n^a, s^a_{(1)}, \ldots, s^a_{(m-2)})$ consisting of two null vectors, ℓ^a and n^a, and spacelike vectors $s^a_{(i)}$, normalised such that $\ell^a n_a = -1$ and $g_{ab} s^a_{(i)} s^b_{(j)} = \delta_{ij}$, while all other products vanish. A naturally associated dual basis of the cotangent space T^*_pM is $(\boldsymbol{\ell}, \mathbf{n}, \mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(m-2)})$. Electromagnetic 2-form of type N can be written as

$$\mathbf{F} = f_i\, \boldsymbol{\ell} \wedge \mathbf{s}^{(i)}\,, \tag{6.35}$$

where f_i are the components in the introduced basis. Let us briefly comment on the uniqueness of this decomposition. From the second condition in (6.34) it follows that $\mathbf{F}$ has to be proportional to $\boldsymbol{\ell}$, while the first one implies it has be wedged by $\mathbf{s}^{(i)}$, due to the normalisation of the basis vectors.

The gravitational field equation sourced by the NLE energy-momentum tensor is

$$\mathcal{E}_{ab} = 8\pi \left(-4\mathscr{L}_\mathcal{F}\, T^{(\mathrm{Max})}_{ab} + \frac{1}{m} \left(g^{cd} T_{cd}\right) g_{ab} - \frac{1}{4\pi m} \mathscr{L}_\mathcal{F} \mathcal{F} (m-4) g_{ab} \right)\,, \tag{6.36}$$

where $\mathcal{E}_{ab}$ is a symmetric tensor of the odd o-t type, possibly belonging to some extended gravitational theory. The additional gCS term may be added to the electromagnetic action since it does not change the energy-momentum tensor, while the modified gMax equations (6.7) are irrelevant for the proof.

Theorem 6.4. *Suppose that the spacetime metric g_{ab} and the electromagnetic field F_{ab} are solutions of the gravitational field equation (6.36) with the odd o-t type tensor $\mathcal{E}_{ab}$. If the metric g_{ab} admits a Killing vector field k^a, hypersurface orthogonal on the open set $O \subseteq M$, then at each point of O where $\mathbf{F} \neq 0$, $\mathbf{F}$ is of type N and $\mathscr{L}_\mathcal{F} \neq 0$, vector k^a cannot be timelike.*

Proof of theorem 6.4. The electromagnetic 2-form $\mathbf{F}$ may be decomposed as

$$-k_a k^a \,\mathbf{F} = \mathbf{k} \wedge \mathbf{E} + \star(\mathbf{k} \wedge \mathbf{B})\,, \tag{6.37}$$

where $\mathbf{E} := -i_k \mathbf{F}$ is the electric 1-form and $\mathbf{B} := i_k \star \mathbf{F}$ is the magnetic (m-3)-form. By the assumptions of the theorem, the timelike Killing vector field k^a is hypersurface orthogonal and the tensor $\mathcal{E}$ is of odd o-t type, implying that

$$\mathbf{k} \wedge \boldsymbol{\mathcal{E}}(k) = 0\ . \tag{6.38}$$

Field equation (6.36) and the decomposition (6.37) allow us to express this condition via electromagnetic fields as

$$\mathscr{L}_{\mathcal{F}}\, \mathbf{B} \wedge \mathbf{E} = 0\,. \tag{6.39}$$

The decomposition (6.35) allows us to write the electric and magnetic forms as

$$\mathbf{E} = (k^a s_a^{(i)}) f_i\, \boldsymbol{\ell} - (k^a \ell_a) f_i\, \mathbf{s}^{(i)}\,, \tag{6.40}$$

$$\mathbf{B} = f_i \star(\boldsymbol{\ell} \wedge \mathbf{s}^{(i)} \wedge \mathbf{k})\,. \tag{6.41}$$

From the definition of the forms $\mathbf{E}$ and $\mathbf{B}$ and taking into account that $k^a E_a = \ell^a E_a = 0$ and $E^a s_a^{(i)} = -(k^a \ell_a) f_i$, we get

$$\star(\mathbf{B} \wedge \mathbf{E}) = i_E \star \mathbf{B} = (-1)^m \Big(\sum_i f_i^2\Big)(k^a \ell_a)\, \boldsymbol{\ell} \wedge \mathbf{k}\,. \tag{6.42}$$

At the points where $\mathscr{L}_{\mathcal{F}} \neq 0$, the equation (6.42) implies $\star(\mathbf{B} \wedge \mathbf{E}) = 0$, while the assumption $\mathbf{F} \neq 0$ translates to $\sum_i f_i^2 \neq 0$. Finally, we can conclude that $(k^a \ell_a)\, \boldsymbol{\ell} \wedge \mathbf{k} = 0$, which is a contradiction for a timelike Killing vector k^a. □

The theorem can be circumvented if the assumptions are relaxed, in particular, if we do not demand that $\mathscr{L}_{\mathcal{F}} \neq 0$. Examples are null electromagnetic stealth fields in power-Maxwell theory which can be found in static spacetimes [171].

Chapter 7

Black hole thermodynamics in the presence of NLE fields

7.1 The laws of black hole mechanics

Black hole thermodynamics represents a meeting point of strong regime gravity and quantum phenomena, along with providing an insight into the microscopic description of spacetime. Although classical laws of black hole mechanics [10] bear resemblance with the basic laws of thermodynamics [12, 13], it was not clear whether this correspondence is just a mathematical formality. The underlying physics behind it was revealed by Hawking's prediction that black holes emit radiation [83], indicating that black hole quantities can be identified with thermodynamic variables.

One relevant quantity that may be assigned to stationary black holes is surface gravity κ defined on the horizon via

$$\chi^a \nabla_a \chi^b = \kappa \chi^b, \tag{7.1}$$

where χ^a is a horizon-generating Killing vector field. The zeroth law of black hole mechanics states that κ attains a constant value over the black hole horizon. It can be proved using various approaches that apply to different scenarios. One possibility is to rely on Einstein's gravitational field equations and dominant energy condition [187]. Another, most elegant proof is based solely on the presence of bifurcate Killing horizons [108]. The third option is to impose additional geometric restrictions, such as integrability conditions that ensure staticity and circularity [150, 96]. The Planck spectrum of emitted particles from the black hole [83] suggests that $\kappa/(2\pi)$ plays the role of the black hole's temperature, thus strengthening the thermodynamic analogy. The zeroth law of black hole electrodynamics, the constancy of electromagnetic scalar potentials over the black hole horizon, may be established using similar methods, at least for Maxwell's theory [168, 169].

Energy conservation is essentially encapsulated in the first law of black hole mechanics, which constrains the mass of stationary black holes upon perturbations. For charged black holes within Maxwell's electrodynamics, it can be brought to a form

$$\delta M = \frac{1}{8\pi}\kappa\delta\mathcal{A} + \Omega_H\delta J + \Phi\delta Q + \Psi\delta P \ , \tag{7.2}$$

where $\mathcal{A}$ stands for the area of the black hole horizon, Ω_H is the angular velocity of the horizon, J is black hole's angular momentum, while Φ, Ψ, Q and P are electric and magnetic scalar potentials and charges, respectively. Again, the form of the first law implies that $\mathcal{A}/4$ should be interpreted as the entropy of the black hole. This conjecture is backed up not only by Hawking's radiation, but also by the second law of black hole mechanics. Given that certain energy conditions hold, it states that the horizon area does not decrease with time, $\delta\mathcal{A} \geq 0$.

Stationary black holes also obey the analogue of the classical Gibbs–Duhem relation, known as the Smarr formula,

$$M = \frac{\kappa}{4\pi}A + 2\Omega_H J + \Phi Q + \Psi P \ . \tag{7.3}$$

It can be derived from the Bardeen-Carter-Hawking mass formula [10], without any reference to the first law. The original derivation for the Kerr-Newman black hole [167] was based on Euler's theorem for homogeneous functions [119]. The starting point was the assumption that the black hole mass $M(\mathcal{A}, J, Q^2)$ is a homogeneous function of degree 1/2 to which one can readily apply the theorem. However, the shortcoming of the Eulerian approach is its inapplicability to a wider range of theories. Namely, in these cases, there is no *a priori* guarantee that the black hole mass will preserve its homogeneity [102]. Once the first law is obtained, one can use it to derive the Smarr formula via the so-called scaling procedure, which we will discuss in detail later.

Over the decades, significant progress has been made in understanding the thermodynamical aspects of Einstein-Maxwell black holes. The central result is Wald's entropy formula [192], which enabled defining entropy as a local, geometric quantity related to the Noether charge. The original Wald's formula holds for general diffeomorphism-invariant Lagrangians, while its later generalisations apply to theories with gravitational Chern–Simons terms present [177, 19]. As far as NLE theories are concerned, thermodynamic analysis is still incomplete and mostly deals with specific scenarios without providing a universal picture.

The early study of black hole thermodynamics with NLE fields [151] contains proof of the zeroth law of black hole electrodynamics via Einstein's field equation, a partial derivation of the first law of black hole thermodynamics and a vague statement that the Smarr formula does not hold. However, as later analyses will show, the presented form of the first law lacks the crucial NLE terms, which also modify the usual Smarr formula. Thermodynamic properties of static and spherically symmetric black holes have been analysed for particular NLE theories such as power-Maxwell in an arbitrary number of dimensions [70], Euler–Heisenberg [122] and Born–Infeld [72, 204, 30]. The Smarr formula for specific black hole solutions has been derived in [24] via scaling arguments, while the same procedure has been applied to general $\mathcal{F}$-class Lagrangians in [59]. The authors in [128, 129] derived the first law for $\mathcal{F}$-class Lagrangians using mathematically well-defined covariant phase space formalism. Their result was devoid of additional NLE terms, in contrast to the first complete generalisation of the Smarr formula for $\mathcal{FG}$-class Lagrangians [71]. Since the first law and the Smarr formula have to be mutually consistent, this conundrum has to be resolved. The derivation of the first law for $\mathcal{F}$-class theories [205] based on the variation of the Bardeen-Carter-Hawking mass formula suggests that these terms play a vital role.

We aim to obtain a consistent framework of thermodynamics with NLE fields, with a clear presentation of all the technical details that are often brushed aside [14]. Special emphasis will be put on the rigorous derivation of the first law for $\mathcal{FG}$-class Lagrangians and the implications of the NLE Lagrangian parameters [14].

7.2 The zeroth law of black hole electrodynamics

The constancy of electric and magnetic scalar potentials over the stationary black hole horizon can be proven in several different ways, depending on the generality we wish to achieve. Analogously to establishing the zeroth law of black hole thermodynamics, one can rely on specific gravitational field equations without going into the details of the underlying geometric setup or consider black holes with particular geometric properties that are independent of the field equations. These techniques have already been successfully applied to Maxwell's electrodynamics [168, 169]. Since the analysis for NLE fields is still incomplete, our aim is to fill in the existing gaps in the literature.

Throughout the section, we will assume that the spacetime $(\mathscr{M}, g_{ab})$ admits a smooth Killing vector field ξ^a and that the electromagnetic field $\mathbf{F}$ is symmetry inheriting, $\pounds_\xi \mathbf{F} = 0$. The introduced assumptions enable us to define 1-forms $\mathbf{E} =$

$-i_\xi \mathbf{F}$ and $\mathbf{H} = i_\xi \star \mathbf{Z}$ (2.17) and their associated scalar potentials Φ and Ψ (2.20). Also, we can define 1-forms $\mathbf{B} = i_\xi \star \mathbf{F}$ and $\mathbf{D} = -i_\xi \mathbf{Z}$ (2.16), which are generally not closed. The electromagnetic invariants are given by the usual expressions,

$$(\xi^a \xi_a)\, \mathcal{F} = 2(E_a E^a - B_a B^a) \;, \tag{7.4}$$

$$(\xi^a \xi_a)\, \mathcal{G} = -4 E_a B^a \;. \tag{7.5}$$

The scalar potentials Ψ and Φ are constant along the orbits of the Killing vector field ξ^a since

$$\pounds_\xi \Phi = -i_\xi \mathbf{E} = 0 \quad \text{and} \quad \pounds_\xi \Psi = -i_\xi \mathbf{H} = 0 \;. \tag{7.6}$$

The constancy of the scalar potentials over the horizon $H[\xi]$ can be easily proven given that expressions

$$\boldsymbol{\xi} \wedge \mathbf{E} \overset{H}{=} 0 \quad \text{and} \quad \boldsymbol{\xi} \wedge \mathbf{H} \overset{H}{=} 0 \tag{7.7}$$

hold under certain assumptions. Contraction of (7.7) with a tangent vector $X^a \in T_p H[\xi]$ results in

$$(\pounds_X \Phi)\, \boldsymbol{\xi} = 0 \quad \text{and} \quad (\pounds_X \Psi)\, \boldsymbol{\xi} = 0 \;. \tag{7.8}$$

At each point where $\boldsymbol{\xi} \neq 0$, we have $\pounds_X \Phi = 0$ and $\pounds_X \Psi = 0$, while at points where $\xi^a = 0$ we have $\mathrm{d}\Phi = 0$ and $\mathrm{d}\Psi = 0$ by construction. There are several approaches that lead to (7.7), which we list below.

a) Einstein's field equation method [151].

Starting from the identity that holds on the black hole horizon [96],

$$R_{ab} \xi^a \xi^b \overset{H}{=} 0 \;, \tag{7.9}$$

and can be converted to the contraction of the energy-momentum tensor using Einstein's equation,

$$\pi T_{ab} \xi^a \xi^b \overset{H}{=} -\mathscr{L}_\mathcal{F} E_a E^a \;, \tag{7.10}$$

we can conclude that the electric field E^a is null at each nondegenerate point[1] of the horizon $H[\xi]$. By definition, we have $\xi^a E_a = 0$, and since any two orthogonal null vectors are necessarily proportional, $\boldsymbol{\xi} \wedge \mathbf{E} = 0$. Then, from (7.4) it follows that the magnetic field B^a is also null on $H[\xi]$, so $\boldsymbol{\xi} \wedge \mathbf{B} = 0$. Once these two relations are established, the same holds for the nonlinear magnetic 1-form, $\boldsymbol{\xi} \wedge \mathbf{H} \overset{H}{=} 0$.

Although simple, the disadvantage of this approach lies in the fact that it cannot be repeated for more general gravitational theories, as the central identity (7.10) is derived from Einstein's field equation.

[1]We say that a point $x \in \mathscr{M}$ is nondegenerate if $\mathscr{L}_\mathcal{F}(x) \neq 0$.

b) Bifurcate horizon method.

The most elegant proof can be performed assuming that the black hole horizon is of bifurcate type. The horizon-generating Killing vector field ξ^a vanishes on the bifurcation surface $\mathcal{B} \subseteq H[\xi]$. Then the potentials Φ and Ψ are constant over the bifurcation surface $\mathcal{B}$ and over each component of the horizon connected to it. However, this proof cannot be applied to all scenarios, as there are examples of horizons that are not of bifurcate type, such as those belonging to the extremal black holes.

c) Frobenius approach [168, 169, 11].

Assume that the spacetime is both stationary and axially symmetric and possesses a mutually commuting timelike Killing vector field k^a and an axial Killing vector field m^a, $[k, m]^a = 0$, which satisfy the Frobenius condition [119]

$$\mathbf{k} \wedge \mathbf{m} \wedge \mathrm{d}\mathbf{k} = 0 = \mathbf{k} \wedge \mathbf{m} \wedge \mathrm{d}\mathbf{m} \ . \tag{7.11}$$

The Killing horizon $H[\chi]$ is generated by the Killing vector field given as a linear combination of vector fields k^a and m^a, defined as $\chi^a = k^a + \Omega_H m^a$, where the constant Ω_H is the angular velocity of the horizon. Several identities hold at the black hole horizon since k^a and m^a are tangent to $H[\chi]$ and χ^a is normal to $H[\chi]$ [96],

$$k_a k^a + \Omega_{\mathsf{H}}\, k_b m^b \overset{H}{=} 0 \ , \tag{7.12}$$

$$k_a m^a + \Omega_{\mathsf{H}}\, m_b m^b \overset{H}{=} 0 \ , \tag{7.13}$$

$$(k_a k^a)(m_a m^a) \overset{H}{=} (k_a m^a)^2 \ . \tag{7.14}$$

We also assume that the electromagnetic field inherits both symmetries, $\pounds_k \mathbf{F} = 0$ and $\pounds_m \mathbf{F} = 0$. After setting $X^a = k^a$ and $Y^a = m^a$ in the auxiliary identity

$$i_X \pounds_Y - i_Y \pounds_X = i_X i_Y \mathrm{d} - \mathrm{d} i_X i_Y + i_{[X,Y]} \ , \tag{7.15}$$

and applying it to $\mathbf{F}$ and $\star\mathbf{Z}$, we can conclude that $F_{ab}k^a m^b$ and $\star Z_{ab} k^a m^b$ are constant. These constants are equal to zero on any connected domain of spacetime containing the points where either k^a or m^a vanish, one of the examples being the rotation axis. It immediately follows that $\star F_{ab} k^a m^b = 0$ on each nondegenerate point of the same domain. These conditions can be restated as

$$\mathbf{k} \wedge \mathbf{m} \wedge \star\mathbf{F} = 0 \quad \text{and} \quad \mathbf{k} \wedge \mathbf{m} \wedge \mathbf{F} = 0 \ , \tag{7.16}$$

since $\mathbf{k} \wedge \mathbf{m} \wedge \star\mathbf{F} = -\mathbf{k} \wedge \star i_m \mathbf{F} = -\star i_k i_m \mathbf{F}$ and $\mathbf{k} \wedge \mathbf{m} \wedge \mathbf{F} = \star i_k i_m \star\mathbf{F}$.

Contracting the expressions (7.16) with $i_m i_k$, together with the aid of equations (7.12)-(7.14), results in

$$\begin{aligned} i_m i_k(\mathbf{k} \wedge \mathbf{m} \wedge \star\mathbf{F}) &= (i_k\mathbf{k})(i_m\mathbf{m})\star\mathbf{F} - (i_k\mathbf{k})\mathbf{m} \wedge i_m\star\mathbf{F} - (i_k\mathbf{m})(i_m\mathbf{k})\star\mathbf{F} + \\ &+ (i_k\mathbf{m})\mathbf{k} \wedge i_m\star\mathbf{F} + (i_m\mathbf{k})\mathbf{m} \wedge i_k\star\mathbf{F} - (i_m\mathbf{m})\mathbf{k} \wedge i_k\star\mathbf{F} = \\ &= -i_m\mathbf{m}(\mathbf{k} \wedge i_k\star\mathbf{F} + \Omega_H\mathbf{k} \wedge i_m\star\mathbf{F} + \Omega_H\mathbf{m} \wedge i_k\star\mathbf{F} + \Omega_H^2\mathbf{m} \wedge i_m\star\mathbf{F}) = \\ &= -(i_m\mathbf{m})\boldsymbol{\chi} \wedge i_\chi\star\mathbf{F} = -(i_m\mathbf{m})\boldsymbol{\chi} \wedge \mathbf{B} = 0 \ , \end{aligned} \tag{7.17}$$

similarly, we have

$$i_m i_k(\mathbf{k} \wedge \mathbf{m} \wedge \mathbf{F}) = (i_m\mathbf{m})\boldsymbol{\chi} \wedge \mathbf{E} = 0 \ , \tag{7.18}$$

and we arrive at (7.7) on each nondegenerate point of the horizon where $m_a m^a \neq 0$. Points at which $m_a m^a = 0$ represent the intersection of the rotation axis and the horizon and the constancy of a potential over the horizon follows directly from the continuity of the potential.

The same strategy cannot be directly applied to the static, but not axially symmetric spacetime with hypersurface orthogonal Killing vector field k^a which satisfies the Frobenius condition $\mathbf{k} \wedge \mathrm{d}\mathbf{k} = 0$. Namely, one would need the relations of the form

$$\mathbf{k} \wedge \star\mathbf{F} = 0 \quad \text{and} \quad \mathbf{k} \wedge \mathbf{Z} = 0 \ , \tag{7.19}$$

which do not hold for the dyonic configurations. However, the same procedure can be repeated for specific subcases with certain simplifications.
(e1) "Purely electric case" in the sense that $\mathbf{B} = 0$. Then, by the expression (7.4), $\mathbf{E}$ is null on the horizon $H[k]$ and the proof follows as in the approach (a).
(e2) "Purely electric case" in the sense that $\mathbf{H} = 0$. In this case, since $\mathbf{H} = i_k\star\mathbf{Z}$, the relation $\mathbf{k} \wedge \mathbf{Z} = 0$ holds. Contracting it with k^a gives

$$\mathscr{L}_\mathcal{F}\, \mathbf{k} \wedge \mathbf{E} - \mathscr{L}_\mathcal{G}\, \mathbf{k} \wedge \mathbf{B} \stackrel{H}{=} 0 \ , \tag{7.20}$$

while $\mathbf{k} \wedge \mathbf{H} = 0$ implies

$$\mathscr{L}_\mathcal{G}\, \mathbf{k} \wedge \mathbf{E} + \mathscr{L}_\mathcal{F}\, \mathbf{k} \wedge \mathbf{B} = 0 \ . \tag{7.21}$$

If $(\mathscr{L}_\mathcal{F})^2 + (\mathscr{L}_\mathcal{G})^2 \neq 0$, the expression (7.7) is valid, thus completing the proof.
(m1) "Purely magnetic case" in the sense that $\mathbf{E} = 0$. Again, (7.4) implies that $\mathbf{B}$ is null on the horizon $H[k]$, and the proof can be performed in the same manner as in (a) approach.

(m2) "Purely magnetic case" in the sense that $\mathbf{D} = 0$. By the definition of $\mathbf{D}$, we immediately have $\mathbf{k} \wedge \star\mathbf{Z} = 0$, which after contraction with k^a reads

$$\mathscr{L}_{\mathcal{G}}\, \mathbf{k} \wedge \mathbf{E} + \mathscr{L}_{\mathcal{F}}\, \mathbf{k} \wedge \mathbf{B} \overset{H}{=} 0 \ . \tag{7.22}$$

Another useful expression is $\mathbf{k} \wedge \mathbf{D} = 0$ which gives

$$\mathscr{L}_{\mathcal{F}}\, \mathbf{k} \wedge \mathbf{E} - \mathscr{L}_{\mathcal{G}}\, \mathbf{k} \wedge \mathbf{B} = 0 \ . \tag{7.23}$$

Whenever $(\mathscr{L}_{\mathcal{F}})^2 + (\mathscr{L}_{\mathcal{G}})^2 \neq 0$, we may deduce (7.7).

For the test electromagnetic fields whose contribution to the gravitational equation may be neglected, any method except approach (a) may be used to carry out the proof.

7.3 The first law of black hole mechanics

There are several different approaches to deriving the first law of black hole thermodynamics, differing in the physical interpretation and the level of mathematical rigour. Adopting the nomenclature from [188], we can make the following classification:

(1) Equilibrium state version.

We are comparing two stationary black hole configurations that are "nearby" in an abstract phase space. It can be subdivided further into two varieties:

(1a) Variation of the Bardeen-Carter-Hawking mass formula [10]

(1b) Covariant phase space formalism [192, 118, 104, 149].

Mathematically precise approach closely related to Hamiltonian mechanics in which conserved quantities are extracted from the boundary terms.

(2) Physical process version [63].

We are considering a physical, quasistatic process in which matter is falling into a black hole.

Our objective is to derive the first law by employing approaches (1b) and (2) to rotating, stationary and axially symmetric black holes within $\mathcal{F}\mathcal{G}$-class NLE theories.

The fundamental assumption underlying the first law is that spacetime is a solution of a coupled Einstein-NLE system with the metric g_{ab} corresponding to a stationary, axially symmetric and asymptotically flat black hole and a symmetry inheriting electromagnetic field $\mathbf{F}$. The spacetime admits two Killing vector fields, $k^a = (\partial/\partial t)^a$, which is timelike at infinity, and axial $m^a = (\partial/\partial\phi)$, with compact orbits. Without loss of generality, it can be assumed that the introduced vectors are mutually commuting, $[k,m]^a = 0$, [34, 176] and satisfy Frobenius conditions (7.11).

One technical difference between the equilibrium state and the physical process version is reflected in the properties of Cauchy surfaces intersecting the black holes. In the former case, we assume that the black hole possesses a Killing horizon $H[\chi]$ which is of a bifurcate type. Precisely, $H[\chi]$ is a pair of null hypersurfaces generated by the Killing vector field $\chi^a = k^a + \Omega_H m^a$, where the constant Ω_H denotes the angular velocity and the horizon's corresponding surface gravity κ is a nonzero constant. It intersects the bifurcation surface $\mathcal{B}$, which is a smooth, compact, embedded 2-surface on which the Killing vector field χ^a vanishes. In the equilibrium state version, the integration of the relevant quantities is performed over a spacelike Cauchy surface $\Sigma \subset \mathscr{M}$, smoothly embedded in $\mathscr{M}$ and possessing a nowhere vanishing normal, whose boundary $\partial\Sigma$ is composed of an asymptotically flat end and bifurcation surface $\mathcal{B} = \Sigma \cap H[\chi]$. Conversely, the derivation of the first law based on the physical process version does not demand the presence of a bifurcation surface. In that case, the setup consists of two spacelike Cauchy surfaces terminating at the horizon.

The quantities describing black holes, the Komar mass $M_{\mathcal{S}}$ and the Komar angular momentum $J_{\mathcal{S}}$ [110], are defined by the integrals over a smooth closed 2-surface $\mathcal{S}$

$$M_{\mathcal{S}} := -\frac{1}{8\pi} \oint_{\mathcal{S}} \star \mathrm{d}\mathbf{k} \qquad \text{and} \qquad J_{\mathcal{S}} := \frac{1}{16\pi} \oint_{\mathcal{S}} \star \mathrm{d}\mathbf{m} \, . \tag{7.24}$$

If the integrals are performed over a sphere at infinity S_∞ (formally, we look at the limit in which the radius of the sphere goes to infinity), we use the symbols $M := M_{S_\infty}$ and $J := J_{S_\infty}$ for the corresponding quantities. In our setting, which is stationary and asymptotically flat spacetime, the ADM and Komar mass coincide [187, 96]. Other two quantities relevant for black hole description are the electric and magnetic charges given by (2.21).

7.3.1 Covariant phase space formalism

Before reviewing the fundamental aspects of covariant phase space formalism, we need to discuss the influence of Lagrangian parameters on the first law of black hole mechanics. Let us assume that the NLE Lagrangian contains a finite number of real parameters, $\{\beta_1, \ldots \beta_n\}$. If we consider these parameters constant under the variations, the obtained version of the first law would not be in agreement with the NLE version of the Smarr formula. This conflict has to be resolved since the generalised Smarr formula can be derived independently of the first law [71]. One of the options is to extend the phase space by including the Lagrangian parameters in a way that they get varied but remain constant on a given spacetime, $\nabla_a \beta_i = 0$. Formally, within the variational procedure, the NLE Lagrangian is understood as a function of both electromagnetic invariants and parameters, $\mathscr{L}(\mathcal{F}, \mathcal{G}; \{\beta_i\})$. This approach is similar to the treatment of the cosmological constant Λ in a thermodynamical context, where it corresponds to the pressure in the $V\mathrm{d}p$ term [107, 116].

The other, more general option is to promote the parameters to spacetime-dependent functions [155], $\nabla_a \beta_i \neq 0$. Using (2.11) together with identities (A.32) and (A.34), we can evaluate the covariant divergence of the energy-momentum tensor,

$$
\begin{aligned}
4\pi \nabla_a T^a{}_b &= \nabla_a \left(Z^{ac} F_{bc} + \mathscr{L} \delta^a{}_b \right) = \\
&= (\nabla_a Z^{ac}) F_{bc} + Z^{ac} (\mathrm{d}F)_{abc} + \sum_{i=1}^{n} \mathscr{L}_{\beta_i} \nabla_b \beta_i \ ,
\end{aligned}
\tag{7.25}
$$

which generally does not vanish on-shell for nonconstant parameters β_i. This signals that one needs to derive the equations of motion for parameters in a given theory. However, we will follow the first approach without pursuing such generalisations.

Given that the above-introduced assumptions are satisfied, we move on to employing the covariant phase space formalism. All dynamical fields, which in our case comprise metric g_{ab} and gauge field $\mathbf{A}$, will be collectively denoted by ϕ, without additional indices. The indices of the coupling parameters β_i will be omitted in arguments of the functions but will be explicitly stated in sums involving variations $\delta\beta_i$. The "variation operator" δ acts on fields ϕ and parameters β_i as

$$
\delta\phi(x) := \left. \frac{\partial \phi(x;\lambda)}{\partial \lambda} \right|_{\lambda=0} \quad \text{and} \quad \delta\beta_i := \frac{\partial \beta_i(\lambda)}{\partial \lambda} \ , \tag{7.26}
$$

where $\phi(x;\lambda)$ and $\beta_i(\lambda)$ are smooth 1-parameter configurations of fields and NLE parameters [187, 118]. The variation of the metric is related to the variation of its inverse,

$$
\delta g_{ab} = -g_{ac} g_{bd} \, \delta g^{cd} \ , \tag{7.27}
$$

while the variation of the volume form is given by (C.3)

$$
\delta \boldsymbol{\epsilon} = -\frac{1}{2} \, \boldsymbol{\epsilon} \, g_{ab} \, \delta g^{ab} \ . \tag{7.28}
$$

After integration by parts, the variation of the Lagrangian 4-form with respect to the dynamical degrees of freedom can generally be expressed as [104]

$$
\delta \mathbf{L}[\phi;\beta] = \mathbf{E}[\phi;\beta] \, \delta\phi + \boldsymbol{\Lambda}^i[\phi;\beta] \, \delta\beta_i + \mathrm{d}\boldsymbol{\Theta}[\phi, \delta\phi; \beta] \ . \tag{7.29}
$$

Field equations are contained in the 4-form $\mathbf{E}$, variation of the Lagrangian with respect to the coupling parameter β_i is denoted by $\mathbf{\Lambda}_i$, while the boundary terms are gathered in the symplectic potential 3-form $\mathbf{\Theta}$. We associate the Noether current 3-form $\mathbf{J}_\xi$ to an arbitrary fixed vector field ξ^a,

$$\mathbf{J}_\xi := \mathbf{\Theta}[\phi, \pounds_\xi \phi; \beta] - i_\xi \mathbf{L}[\phi; \beta] \ . \tag{7.30}$$

The current 3-form is closed on-shell, $\mathrm{d}\mathbf{J}_\xi \approx 0$, since

$$\mathrm{d}\mathbf{J}_\xi = \mathrm{d}\mathbf{\Theta} - \pounds_\xi \mathbf{L} = -\mathbf{E}[\phi; \beta] \, \pounds_\xi \phi - \mathbf{\Lambda}^i[\phi; \beta] \, \pounds_\xi \beta_i \tag{7.31}$$

and $\pounds_\xi \beta_i = 0$. At least locally, there exists a 2-form $\mathbf{Q}_\xi$ such that $\mathbf{J}_\xi \approx \mathrm{d}\mathbf{Q}_\xi$ [191]. This enables us to write the Noether current as

$$\mathbf{J}_\xi = i_\xi \mathbf{C} + \mathrm{d}\mathbf{Q}_\xi \ , \tag{7.32}$$

where $\mathbf{C}$ is a 4-form that vanishes on shell, $\mathbf{C} \approx 0$. In our case, the Lagrangian is a sum of the gravitational and electromagnetic parts, so the 3-form $\mathbf{\Theta}$ and the 2-form $\mathbf{Q}_\xi$ decompose accordingly,

$$\mathbf{\Theta} = \mathbf{\Theta}^{(\mathrm{g})} + \mathbf{\Theta}^{(\mathrm{em})} \quad \text{and} \quad \mathbf{Q}_\xi = \mathbf{Q}_\xi^{(\mathrm{g})} + \mathbf{Q}_\xi^{(\mathrm{em})} \ .$$

The symplectic current 3-form is introduced via two variations δ_1 and δ_2,

$$\boldsymbol{\omega}[\phi, \delta_1\phi, \delta_2\phi; \beta] := \delta_1 \mathbf{\Theta}[\phi, \delta_2\phi; \beta] - \delta_2 \mathbf{\Theta}[\phi, \delta_1\phi; \beta] \ , \tag{7.33}$$

while its integral over a spacelike Cauchy surface Σ defines the presymplectic form Ω_Σ,

$$\Omega_\Sigma[\phi, \delta_1\phi, \delta_2\phi; \beta] := \int_\Sigma \boldsymbol{\omega}[\phi, \delta_1\phi, \delta_2\phi; \beta] \ . \tag{7.34}$$

Here we implicitly assumed that the volume form on Σ is equal to the pullback of $i_{\tilde{n}}\boldsymbol{\epsilon}$, where $\tilde{n}^a$ is a unit, future directed timelike normal vector field on Σ. Using expression (7.32), the variation of the Noether current is given by $\delta\mathbf{J}_\xi = i_\xi \delta\mathbf{C} + \mathrm{d}\delta\mathbf{Q}_\xi$, since $\delta\xi^a = 0$ for a fixed vector field ξ^a. On the other hand, the same variation can be calculated from the initial definition (7.30),

$$\begin{aligned}
\delta\mathbf{J}_\xi &= \delta\mathbf{\Theta}[\phi, \delta\phi; \beta] - i_\xi \delta\mathbf{L}[\phi; \beta] = \\
&= \delta\mathbf{\Theta}[\phi, \delta\phi; \beta] - i_\xi \mathbf{E}[\phi; \beta]\delta\phi - i_\xi \mathbf{\Lambda}[\phi; \beta]\delta\beta_i - i_\xi \mathrm{d}\mathbf{\Theta}[\phi, \delta\phi; \beta] = \\
&= -i_\xi \mathbf{E}[\phi; \beta]\delta\phi + \boldsymbol{\omega}[\phi, \delta\phi, \pounds_\xi\phi; \beta] + \mathrm{d}i_\xi \mathbf{\Theta}[\phi, \delta\phi; \beta] - i_\xi \mathbf{\Lambda}^i[\phi; \beta] \, \delta\beta_i \ ,
\end{aligned} \tag{7.35}$$

where we used Eq. (7.29) in the second line and the definition of $\boldsymbol{\omega}[\phi, \delta\phi, \pounds_\xi\phi; \beta]$ (7.33) in the last step. The combination of the two expressions enables us to extract the symplectic current 3-form,

$$\begin{aligned}\boldsymbol{\omega}[\phi, \delta\phi, \pounds_\xi\phi; \beta] = i_\xi(\mathbf{E}\,\delta\phi + \delta\mathbf{C}) + \\ + \mathrm{d}(\delta\mathbf{Q}_\xi - i_\xi\boldsymbol{\Theta}[\phi, \delta\phi; \beta]) + i_\xi\boldsymbol{\Lambda}^i[\phi; \beta]\,\delta\beta_i \ ,\end{aligned} \tag{7.36}$$

which, after integration over Σ and using Stokes' theorem (see Appendix D), becomes

$$\begin{aligned}\Omega_\Sigma[\phi, \delta\phi, \pounds_\xi\phi; \beta] = \int_\Sigma i_\xi(\mathbf{E}\,\delta\phi + \delta\mathbf{C}) + \\ + \int_{\partial\Sigma}(\delta\mathbf{Q}_\xi - i_\xi\boldsymbol{\Theta}[\phi, \delta\phi; \beta]) - K^i_\xi(\beta)\,\delta\beta_i \ ,\end{aligned} \tag{7.37}$$

where we have introduced the auxiliary functions K^i_ξ,

$$K^i_\xi(\beta) := -\int_\Sigma i_\xi\boldsymbol{\Lambda}^i[\phi; \beta] \ . \tag{7.38}$$

The formal question is whether it is possible to rewrite the last term as a boundary integral. The criterion for determining the exactness of a given differential form is related to the properties of de Rham cohomology groups. As the top compactly supported de Rham cohomology group is trivial for smooth and oriented (both compact and noncompact) manifolds with nonempty boundary (see for example theorems 8.3.10 and 8.4.8 in [195]), the pullback of the $i_\xi\boldsymbol{\Lambda}^i$ to Σ is globally exact, at least if the fields themselves are compactly supported. Then, one can apply Stokes' theorem and convert the K^i_ξ term to an integral over $\partial\Sigma$. However, if Σ is noncompact and fields decay at infinity but are not compactly supported, there is no immediate guarantee that K^i_ξ can be written as a boundary term.

In order to establish the link with Hamiltonian mechanics, summarised in relation

$$\delta H_\xi = \Omega_\Sigma[\phi, \delta\phi, \pounds_\xi\phi; \beta] \ , \tag{7.39}$$

we briefly review its fundamentals. Hamiltonian mechanics is built upon a phase space manifold with local canonical coordinates $s^\mu = (q^1, ..., p_1, ...)$ and a symplectic 2-form $\boldsymbol{\omega}$, which is both closed and nondegenerate. Every smooth function f induces a vector field X_f via $\mathrm{d}f = -i_{X_f}\boldsymbol{\omega}$ such that

$$X_f = (\partial f/\partial p_i)\partial_{q^i} - (\partial f/\partial q^i)\partial_{p_i} \ . \tag{7.40}$$

Hamiltonian H defines the dynamics of the system in the sense that the integral curves of X_H represent its time evolution, and we have $\dot{f} = X_H(f)$ for every function f. Its variation is $\delta H = (\nabla_\mu H)\delta s^\mu = \omega_{\mu\nu}\delta s^\mu \dot{s}^\nu = \omega(\delta s, \dot{s})$, which bears resemblance with (7.39). Without the contributions of $K_\xi^i \delta\beta_i$ terms, Hamiltonian H_ξ conjugate to ξ^a will exist [193] if and only if

$$\int_{\partial\Sigma} i_\xi \boldsymbol{\omega}[\phi, \delta_1\phi, \delta_2\phi] = 0 \tag{7.41}$$

for any two variations, δ_1 and δ_2. The first integral in (7.37) is zero for fields ϕ that satisfy field equations ($\mathbf{E} = 0$) and perturbations $\delta\phi$ that solve the linearized equations of motion ($\delta\mathbf{C} = 0$). If the Hamiltonian exists, one can write the remaining terms as variations of some other forms. As shown in [104], this is possible for the Einstein–Hilbert contribution to the $i_\xi\boldsymbol{\Theta}$ term, which can be written as

$$\int_{\partial\Sigma} i_\xi \boldsymbol{\Theta}^{(\mathrm{g})} = \delta \int_{\partial\Sigma} i_\xi \mathbf{b} \ , \tag{7.42}$$

with the aid of a 3-form $\mathbf{b}$. Lastly, we have to comment on the integrability of the NLE term $K_\xi^i \delta\beta_i$. Mild smoothness assumptions are enough to satisfy the local integrability condition $\partial_{\beta_i} K_\xi^j = \partial_{\beta_j} K_\xi^i$, emerging from $\partial_{\beta_i}\partial_{\beta_j}\mathscr{L} = \partial_{\beta_j}\partial_{\beta_i}\mathscr{L}$. That being the case, we know that $I_\xi(\beta)$ such that $\delta I_\xi = K_\xi^i \delta\beta_i$ exists. If there is only one coupling parameter, I_ξ is a primitive function of K_ξ. Notice that the on-shell Hamiltonian is a purely surface term.

Now we apply the formalism to the geometric scenario described in the introduction of the subsection. First, we promote the smooth vector field ξ^a to a Killing vector field and assume that all the dynamical fields inherit the symmetry, $\pounds_\xi \phi = 0$. Then, we have $\Omega_\Sigma[\phi, \delta\phi, \pounds_\xi\phi; \beta] = 0$, which follows immediately from the definition of the symplectic current (7.33) with $\delta_1 = \delta$ and $\delta_2 = \pounds_\xi$,

$$\begin{aligned} \boldsymbol{\omega}[\phi,\, \delta\phi,\, \pounds_\xi\phi; \beta] = \delta\boldsymbol{\Theta}^{(\mathrm{g})}[\phi,\, \delta\phi,\, \pounds_\xi\phi; \beta] + \delta\boldsymbol{\Theta}^{(\mathrm{em})}[\phi,\, \delta\phi,\, \pounds_\xi\phi; \beta] - \\ - \pounds_\xi\boldsymbol{\Theta}^{(\mathrm{em})}[\phi,\, \delta\phi,\, \pounds_\xi\phi; \beta] - \pounds_\xi\boldsymbol{\Theta}^{(\mathrm{g})}[\phi,\, \delta\phi,\, \pounds_\xi\phi; \beta]. \end{aligned} \tag{7.43}$$

Namely, since 3-forms $\boldsymbol{\Theta}^{(\mathrm{g})}$ and $\boldsymbol{\Theta}^{(\mathrm{em})}$ are constructed out of the symmetry inheriting fields, the last two terms immediately vanish, while explicit calculation for Einstein-Hilbert and NLE Lagrangians will show that the same holds for the first two terms. However, this reasoning demands careful justification in a more general case [104].

The relation (7.36) evaluated on shell reads

$$\delta \oint_{S_\infty} (\mathbf{Q}_\xi - i_\xi \mathbf{b}) - \delta \oint_{\mathcal{B}} (\mathbf{Q}_\xi - i_\xi \mathbf{b}) - K_\xi^i \, \delta\beta_i \approx 0 \; . \tag{7.44}$$

Also, we assume that $\xi^a = \chi^a = k^a + \Omega_H m^a$ and calculate contributions to the boundary integrals.

The gravitational part of the Lagrangian is given by the standard Einstein–Hilbert term, whose variation is presented in detail in Appendix C,

$$\frac{1}{16\pi} \, \delta(R\boldsymbol{\epsilon}) = \frac{1}{16\pi} \, G_{ab} \, \delta g^{ab} \boldsymbol{\epsilon} + \mathrm{d}\mathbf{\Theta}^{(\mathrm{g})} \; , \quad \mathbf{\Theta}^{(\mathrm{g})} := \frac{1}{16\pi} \star \boldsymbol{v} \; , \tag{7.45}$$

where 1-form $\mathbf{v}$ is given by

$$v_a := \nabla^b \delta g_{ab} - g^{cd} \nabla_a \delta g_{cd} \; . \tag{7.46}$$

The explicit evaluation of the current 3-form (7.30) yields

$$\begin{aligned} 16\pi J_{abc} &= \epsilon^d{}_{abc}(v_d - R\xi_d) = \epsilon^d{}_{abc}(\nabla^e \nabla_d \xi_e + \nabla^e \nabla_e \xi_d - 2\nabla_d \nabla_e \xi^e - R\xi_d) = \\ &= \epsilon_{dabc}(\nabla^e \nabla^d \xi_e + \nabla^e \nabla_e \xi^d - 2\nabla^e \nabla^d \xi_e + 2R_e{}^d \xi_e - R\xi^d) = \\ &= 2\epsilon_{dabc}(\nabla_e \nabla^{[e} \xi^{d]} + G_e{}^d \xi_e) \; , \end{aligned} \tag{7.47}$$

from which we can extract the gravitational Noether charge

$$\mathbf{Q}_\xi^{(\mathrm{g})} = -\frac{1}{16\pi} \star \mathrm{d}\boldsymbol{\xi} \; . \tag{7.48}$$

We identify two gravitational contributions over a sphere at infinity as the mass and angular momentum of a black hole, defined respectively by

$$M = \oint_{S_\infty} (\mathbf{Q}_k^{(\mathrm{g})} - i_k \mathbf{b}) \qquad \text{and} \qquad J = -\oint_{S_\infty} \mathbf{Q}_m^{(\mathrm{g})} \; , \tag{7.49}$$

where we took into account that the pullback of $i_m \mathbf{b}$ to a surface tangent to m^a vanishes. The difference in normalisations of Komar integrals for mass and angular momentum is directly related to the absence of the $i_m \mathbf{b}$ term in the integral for the angular momentum [104]. Gravitational contribution on the horizon is the entropy term [192]

$$\delta \oint_{\mathcal{B}} \mathbf{Q}_\xi^{(\mathrm{g})} = \frac{\kappa}{8\pi} \, \delta\mathcal{A} \; , \tag{7.50}$$

where $\mathcal{A}$ is the area of the bifurcation surface $\mathcal{B}$. Altogether, the current form of the first law reads

$$\delta M - \Omega_{\mathsf{H}}\,\delta J + \delta \oint_{S_\infty} \mathbf{Q}_\chi^{(\mathrm{em})} = \frac{\kappa}{8\pi}\,\delta\mathcal{A} + \delta\oint_{\mathcal{B}} \mathbf{Q}_\chi^{(\mathrm{em})} + K_\chi^i\,\delta\beta_i\ . \tag{7.51}$$

In the following subsection, we will confirm that the $i_\xi\mathbf{\Theta}^{(\mathrm{em})}$ term vanishes due to the boundary conditions and gauge choices.

7.3.2 Equilibrium state first law

To obtain the complete form of the first law of black hole thermodynamics, we turn to the calculation of the electromagnetic contributions. After recalling the definition of the trace of the energy-momentum tensor (2.10), the variation of the NLE Lagrangian

$$\delta(\mathscr{L}\boldsymbol{\epsilon}) = \mathscr{L}_{\mathcal{F}}\,\delta\mathcal{F}\,\boldsymbol{\epsilon} + \mathscr{L}_{\mathcal{G}}\,\delta\mathcal{G}\,\boldsymbol{\epsilon} + \mathscr{L}\,\delta\boldsymbol{\epsilon} + \sum_{i=1}^{n}\mathscr{L}_{\beta_i}\,\delta\beta_i\,\boldsymbol{\epsilon} \tag{7.52}$$

can be written as

$$\delta(\mathscr{L}\boldsymbol{\epsilon}) = \mathscr{L}_{\mathcal{F}}\,\delta(\mathcal{F}\boldsymbol{\epsilon}) + \mathscr{L}_{\mathcal{G}}\,\delta(\mathcal{G}\boldsymbol{\epsilon}) + \pi T\delta\boldsymbol{\epsilon} + \sum_{i=1}^{n}\mathscr{L}_{\beta_i}\,\delta\beta_i\,\boldsymbol{\epsilon}\ \ . \tag{7.53}$$

The first term in (7.53) can be expanded further,

$$\mathscr{L}_{\mathcal{F}}\,\delta(\mathcal{F}\boldsymbol{\epsilon}) = 8\pi\mathscr{L}_{\mathcal{F}}T_{ab}^{(\mathrm{Max})}\delta g^{ab}\boldsymbol{\epsilon} - 4\mathscr{L}_{\mathcal{F}}\nabla_a F^{ab}\delta A_b\boldsymbol{\epsilon} + +4\mathscr{L}_{\mathcal{F}}\nabla^a(F_a{}^b\delta A_b)\boldsymbol{\epsilon}\ , \tag{7.54}$$

and contains the standard Maxwellian contribution, up to the $\mathscr{L}_{\mathcal{F}}$ factor. The first term in (7.54) and the third term in (7.53), which are proportional to the variation of the metric, define the NLE energy-momentum tensor,

$$8\pi\mathscr{L}_{\mathcal{F}}T_{ab}^{(\mathrm{Max})}\delta g^{ab}\boldsymbol{\epsilon} + \pi T\,\delta\boldsymbol{\epsilon} = -2\pi T_{ab}\delta g^{ab}\boldsymbol{\epsilon}\ . \tag{7.55}$$

Using the auxiliary expression

$$-\mathscr{L}_{\mathcal{F}}\nabla_a F^{ab}\delta A_b + \mathscr{L}_{\mathcal{F}}\nabla_a(F^{ab}\delta A_b) = -\nabla_a(\mathscr{L}_{\mathcal{F}}F^{ab})\delta A_b + \nabla^a(\mathscr{L}_{\mathcal{F}}F_a{}^b\delta A_b)\ , \tag{7.56}$$

the sum of the first and the third term in (7.53) may be written as

$$\mathscr{L}_{\mathcal{F}}\,\delta(\mathcal{F}\boldsymbol{\epsilon}) + \pi T\delta\boldsymbol{\epsilon} = -2\pi T_{ab}\delta g^{ab}\boldsymbol{\epsilon} - 4\nabla_a(\mathscr{L}_{\mathcal{F}}F^{ab})\delta A_b\boldsymbol{\epsilon} + 4\nabla^a(\mathscr{L}_{\mathcal{F}}F_a{}^b\delta A_b)\boldsymbol{\epsilon}\ . \tag{7.57}$$

With the aid of the identity (A.28), the second term in (7.53) can be written conveniently as

$$\mathscr{L}_{\mathcal{G}}\delta(\mathcal{G}\boldsymbol{\epsilon}) = 4\mathscr{L}_{\mathcal{G}}\left(\nabla_a((\star F^{ab})\,\delta A_b) - (\nabla_a\star F^{ab})\delta A_b\right)\boldsymbol{\epsilon} = \\ = 4\left(\nabla^a(\mathscr{L}_{\mathcal{G}}\,(\star F_a{}^b)\,\delta A_b) - \nabla_a(\mathscr{L}_{\mathcal{G}}\star F^{ab})\delta A_b\right)\boldsymbol{\epsilon}\ . \tag{7.58}$$

Combining all the obtained expressions, we get the final form of the variation of the Lagrangian 4-form

$$\frac{1}{4\pi}\,\delta(\mathscr{L}\boldsymbol{\epsilon}) = \frac{1}{16\pi}\Big(- 8\pi T_{ab}\,\delta g^{ab} + 4(\nabla_a Z^{ab})\delta A_b + 4\sum_i \mathscr{L}_{\beta_i}\,\delta\beta_i\Big)\boldsymbol{\epsilon} + \mathrm{d}\boldsymbol{\Theta}^{(\mathrm{em})}, \tag{7.59}$$

which consists of the energy-momentum tensor term, gauge field equation of motion, additional term proportional to the variation of the NLE Lagrangian parameters and the boundary term. The 3-form Θ can be written compactly as

$$\boldsymbol{\Theta}^{(\mathrm{em})} := \frac{1}{16\pi}\star\boldsymbol{w}\ , \quad w_a = -4Z_a{}^b\,\delta A_b\ , \tag{7.60}$$

where the 1-from $\boldsymbol{w}$ may be represented as $\boldsymbol{w} = -4\star(\star\mathbf{Z}\wedge\delta\mathbf{A})$ in differential form notation. If the electromagnetic field $\mathbf{F}$ is of class $O(r^{-2})$ and perturbation $\delta\mathbf{A}$ of class $O(r^{-1})$ as $r\to\infty$, the 3-form $\boldsymbol{\Theta}^{(\mathrm{em})}$ is irrelevant for the integral at S_∞.

The Noether current 3-form is given by

$$16\pi\mathbf{J}_\xi = \star(\boldsymbol{v}+\boldsymbol{w}) - (R+4\mathscr{L})\star\boldsymbol{\xi}\ . \tag{7.61}$$

In order to rewrite it in a suitable way, we need a series of manipulations. Starting from the identity

$$\nabla^b\nabla_b\xi_a - \nabla_a\nabla^b\xi_b = \mathbf{R}(\xi)_a - (\star\mathrm{d}\star\mathrm{d}\boldsymbol{\xi})_a\ , \tag{7.62}$$

we see that the auxiliary 1-form $\mathbf{v}$ for a variation given by the Lie derivative $\delta = \pounds_\xi$ is equal to

$$\nabla^b\pounds_\xi g_{ab} - g^{cd}\nabla_a\pounds_\xi g_{cd} = 2\mathbf{R}(\xi)_a - (\star\mathrm{d}\star\mathrm{d}\boldsymbol{\xi})_a\ . \tag{7.63}$$

For the same variation, the 1-form $\mathbf{w}$ is proportional to $Z_a{}^b\pounds_\xi A_b$. Hence, we have to find objects containing that contraction in order to recast $\mathbf{w}$ in a more convenient form. One obvious starting point is the electric field 1-form which contains the Lie derivative term,

$$\mathbf{E} = -i_\xi\mathbf{F} = -i_\xi\mathrm{d}\mathbf{A} = -\pounds_\xi\mathbf{A} + \mathrm{d}i_\xi\mathbf{A}\ . \tag{7.64}$$

The electric field can be further contracted with $\mathbf{Z}$, resulting in

$$
\begin{aligned}
4i_E\mathbf{Z} &= -4\star(\star\mathbf{Z}\wedge\mathbf{E}) = 4\star(\star\mathbf{Z}\wedge\pounds_\xi\mathbf{A}) - 4\star(\star\mathbf{Z}\wedge \mathrm{d}i_\xi\mathbf{A}) = \\
&= -\boldsymbol{w} - 4\star\mathrm{d}((i_\xi\mathbf{A})\star\mathbf{Z}) + 4(i_\xi\mathbf{A})\star\mathrm{d}\star\mathbf{Z}\ .
\end{aligned}
\tag{7.65}
$$

On the other hand, the same contraction can be expressed as

$$
4i_E\mathbf{Z} = -16\,(\mathscr{L}_\mathcal{F}\, i_E\mathbf{F} + \mathscr{L}_\mathcal{G}\, i_E\star\mathbf{F}) = 16\pi\mathbf{T}(\xi) - 4\mathscr{L}\boldsymbol{\xi}\ , \tag{7.66}
$$

where the second equality follows since

$$
\begin{aligned}
16\pi\mathbf{T}(\xi) &= -16\mathscr{L}_\mathcal{F} i_E\mathbf{F} + 4\mathscr{L}_\mathcal{F}\mathcal{F}\boldsymbol{\xi} + 4\pi T\boldsymbol{\xi} = \\
&= -16\mathscr{L}_\mathcal{F} i_E\mathbf{F} + 4\mathscr{L}\boldsymbol{\xi} - 4\mathscr{L}_\mathcal{G}\mathcal{G}\boldsymbol{\xi} = \\
&= -16\mathscr{L}_\mathcal{F} i_E\mathbf{F} + 4\mathscr{L}\boldsymbol{\xi} - 16\mathscr{L}_\mathcal{G} i_E\star\mathbf{F}\ .
\end{aligned}
\tag{7.67}
$$

In the last step, we used an auxiliary identity that follows directly from (A.24),

$$
i_E\star\mathbf{F} = \frac{1}{4}\,\mathcal{G}\,\boldsymbol{\xi}\ . \tag{7.68}
$$

Finally, we can write the difference of the two terms in $\mathbf{J}$ as

$$
\boldsymbol{w} - 4\mathscr{L}\boldsymbol{\xi} = -16\pi\mathbf{T}(\xi) - 4\star\mathrm{d}((i_\xi\mathbf{A})\star\mathbf{Z}) + 4(i_\xi\mathbf{A})\star\mathrm{d}\star\mathbf{Z}\ . \tag{7.69}
$$

Next, we are interested in finding a relation between the gauge 1-form $\mathbf{A}$ and scalar potential while taking the gauge freedom into account. Assuming that the electromagnetic field is symmetry inheriting, $\pounds_\xi\mathbf{F} = 0$, and $\mathbf{F} = \mathrm{d}\mathbf{A}_0$ for the initial gauge choice, it does not necessarily hold that $\pounds_\xi\mathbf{A}_0 \neq 0$. Still, $\mathrm{d}\pounds_\xi\mathbf{A}_0$ is a closed form, which can be seen from $\mathrm{d}\pounds_\xi\mathbf{A}_0 = \pounds_\xi\mathbf{F} = 0$. On a simply connected domain, there exists a function α, such that $\pounds_\xi\mathbf{A}_0 = \mathrm{d}\alpha$. We may choose a gauge function λ such that $\pounds_\xi\lambda = -\alpha$. Then the initial and final gauge forms differ by a closed form $\mathbf{A} = \mathbf{A}_0 + \mathrm{d}\lambda$, and we have $\pounds_\xi\mathbf{A} = 0$. Still, even after this procedure, there is residual gauge freedom since $\pounds_\xi(\mathbf{A} + \mathrm{d}\mu) = 0$, as long as the function μ inherits the symmetry, $\pounds_\xi\mu = 0$. Noting that Φ and $-i_\xi\mathbf{A}$ differ by a constant, as follows from

$$
\mathrm{d}(\Phi + i_\xi\mathbf{A}) = -\mathbf{E} + (\pounds_\xi - i_\xi\mathrm{d})\mathbf{A} = 0\ , \tag{7.70}
$$

we may write $\Phi = -i_\xi\mathbf{A} + \Phi_0$, for some $\Phi_0 \in \mathbb{R}$. The final form of the Noether current 3-form is given by

$$\mathbf{J}_\xi = \frac{1}{8\pi}\star(\mathbf{G}(\xi) - 8\pi\mathbf{T}(\xi)) - \frac{\Phi - \Phi_0}{4\pi}\,\mathrm{d}\star\mathbf{Z} + \mathrm{d}(\mathbf{Q}^{(\mathrm{g})}_\xi + \mathbf{Q}^{(\mathrm{em})}_\xi)\ , \tag{7.71}$$

with

$$\mathbf{Q}^{(\mathrm{g})}_\xi = -\frac{1}{16\pi}\star\mathrm{d}\boldsymbol{\xi} \quad \text{and} \quad \mathbf{Q}^{(\mathrm{em})}_\xi = \frac{1}{4\pi}\,(\Phi - \Phi_0)\star\mathbf{Z}\ . \tag{7.72}$$

The 4-form $\mathbf{C}$ is equal to

$$C_{abcd} = \frac{1}{8\pi}\,(G_a{}^e - 8\pi T_a{}^e - 2A_a\nabla_r Z^{re})\epsilon_{ebcd}\ , \tag{7.73}$$

which confirms that the Noether current $\mathbf{J}_\xi$ is closed on-shell, $\mathrm{d}\mathbf{J}_\xi \approx 0$, and $\mathbf{J}_\xi \approx \mathrm{d}\mathbf{Q}_\xi$.

Before evaluating the first law, we have to show that the contribution of the electromagnetic term $\mathbf{Q}^{(\mathrm{em})}_\xi$ does not depend on the choice of gauge. If we choose $\mathbf{A}$ so that the $i_\xi\mathbf{A}$ term does not vanish at the bifurcation surface, we are implicitly using a gauge field that diverges there. The idea is easily illustrated by taking the Reissner–Nordström black hole as an example. Analysis of gauge field 1-form at bifurcation surface is performed by introducing tortoise radial coordinate $\mathrm{d}r_* = \mathrm{d}r/f(r)$, switching to Eddington-Finkelstein coordinates $u = t - r_*$ and $v = t + r_*$ and using them to define Kruskal coordinates $U = -e^{-\kappa u}$ and $V = e^{\kappa v}$. In new coordinates, the Killing horizon is generated by the Killing vector field $k = \kappa\,(V\partial_V - U\partial_U)$. The gauge field which vanishes at infinity is given by

$$\mathbf{A} = -\frac{Q}{r}\,\mathrm{d}t = -\frac{Q}{2\kappa r}\left(\frac{1}{V}\,\mathrm{d}V - \frac{1}{U}\,\mathrm{d}U\right)\ , \tag{7.74}$$

and is manifestly divergent at the bifurcation surface defined by $(U,V) = (0,0)$. In a different gauge,

$$\mathbf{A}' = -\frac{Q}{2\kappa}\left(\frac{1}{r} - \frac{1}{r_+}\right)\left(\frac{1}{V}\,\mathrm{d}V - \frac{1}{U}\,\mathrm{d}U\right)\ , \tag{7.75}$$

where r_+ is the radius of the outer horizon, the gauge field 1-form is regular on the horizon. Our gauge choice will be the one in which $\mathbf{A}$ is finite and smooth at $H[\chi]$ and Φ vanishes at infinity. Then $i_\xi\mathbf{A}|_{\mathcal{B}} = 0$, the constant Φ_0 is equal to the value of the potential over the horizon, so that $-i_\xi\mathbf{A} = \Phi - \Phi_{\mathsf{H}}$ and $i_\xi\mathbf{A}|_\infty = \Phi_{\mathsf{H}}$. The $\mathbf{Q}^{(\mathrm{em})}_\xi$ term makes no contribution at the bifurcation surface, while at infinity it amounts to

$$\delta\oint_{S_\infty}\mathbf{Q}^{(\mathrm{em})}_\xi = -\Phi_{\mathsf{H}}\,\delta Q\ . \tag{7.76}$$

Thus, the final form of the first law of black hole mechanics in the presence of NLE fields is

$$\delta M = \frac{\kappa}{8\pi}\,\delta\mathcal{A} + \Omega_{\mathsf{H}}\,\delta J + \Phi_{\mathsf{H}}\delta Q + K^i_\chi\,\delta\beta_i\ , \tag{7.77}$$

with a new thermodynamic variable K^i_χ conjugate to the NLE parameters β_i,

$$K^i_\chi := -\frac{1}{4\pi}\int_\Sigma \mathscr{L}_{\beta_i} \star\boldsymbol{\chi}\ . \tag{7.78}$$

The obtained formula (7.77) does not contain the often included magnetic term $\Psi_H\delta P$ due to the tacit assumption that the gauge field $\mathbf{A}$ is globally well-defined. The canonical covariant phase space approach does not offer a formal procedure for its inclusion. However, the reference [109] addressed this problem by taking into account contributions from several spacetime patches with changing gauge potential along their edges. The $\Psi_H\delta P$ term appears in mass formula variation procedure (1a) [205, 96], which does not deal with the precise definition of the gauge potential. On the other hand, the authors in [149] argue that the magnetic charge is of a topological nature and therefore invariant under perturbations.

Some of the early treatments of black hole thermodynamics with NLE fields seem inconsistent with the formula (7.77) owing to the absence of the $K^i_\xi\delta\beta_i$ term. For example, in [93] the nonrotating dyonic black hole is obtained from the NLE Lagrangian $\mathscr{L} = \mathscr{L}^{(Max)} + \alpha\mathcal{G}^2$ and the first law is presented in the form $\delta M = \kappa\delta\mathcal{A}/(8\pi) + \Phi_H\delta Q + \Psi_H\delta P$. However, one should take this result with certain reservations, as the variation of the NLE parameter α related to the magnetic charge is kept fixed under the variation.

By establishing a parallel with the analysis of black hole thermodynamics with the cosmological constant [107], the black hole mass in (7.77) can be understood as a generalised enthalpy. It is related to the internal energy $\mathcal{E}$ by means of the Legendre transformation $M = \mathcal{E} + K^i_\chi\beta_i$, so that

$$\delta\mathcal{E} = \frac{\kappa}{8\pi}\,\delta\mathcal{A} + \Omega_{\mathsf{H}}\,\delta J + \Phi_{\mathsf{H}}\,\delta Q - \beta_i\,\delta K^i_\chi\ . \tag{7.79}$$

The physical interpretation of the K^i_ξ quantity is not univocal and depends on the Lagrangian in question. In the case when the coupling parameter β_i is of the same dimension as $\mathcal{F}^{1/2}$, which is also the dimension of the electric field, the corresponding K^i_ξ may be interpreted as an NLE vacuum polarisation. This was already noted for Born–Infeld theory with $\beta = b$ [72] and is also applicable to the Euler–Heisenberg theory where $\beta = m_e^2/\alpha$.

Now we turn to the alternate way of deriving the first law of black hole thermodynamics, the physical process approach.

7.3.3 Physical process first law

In contrast to the "equilibrium state" derivation, which does not address the physical background of the perturbations, here we are looking at the process in which matter is thrown into a black hole. The geometric setting consists of two smooth, spacelike, asymptotically flat Cauchy surfaces, Σ_0 and Σ_1, which, respectively, represent the initial and final states of the process. The two surfaces terminate on the horizon $H[\xi]$ which does not have to be of the bifurcate type. The part of the horizon between $\Sigma_0 \cap H[\xi]$ and $\Sigma_1 \cap H[\xi]$ may be denoted by $\mathcal{H}$. A schematic depiction of the setting is shown in the Figure[2] 7.1.

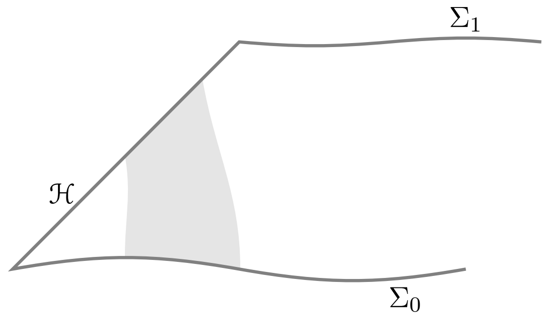


Figure 7.1: Spacelike hypersurfaces Σ_0 and Σ_1, horizon portion $\mathcal{H}$ and infalling matter denoted by gray area [14].

The process starts with the initial stationary black hole, which is then perturbed by adding a small amount of charged matter. After some time, it settles into a final stationary state. The charged matter is composed of fields with compact support which intersects Σ_0 and $H[\xi]$, but is disjoint from both $\Sigma_0 \cap H[\xi]$ and Σ_1. The latter claim follows since we suppose that the matter is initially away from the black hole and that there is no residual matter on the final hypersurface once the process is finished.

The sources consist of the electromagnetic 4-current j^a and two contributions to the energy-momentum tensor, the electromagnetic one T_{ab} and the one unrelated to the electromagnetic fields, $\hat{T}_{ab}$. The gravitational-NLE system of equations is

$$G_{ab} - 8\pi T_{ab} = 8\pi \hat{T}_{ab} \ , \quad \nabla_b Z^{ab} = 4\pi j^a \ . \tag{7.80}$$

We assume that $(g_{ab}, \mathbf{A})$ is a solution of the source-free coupled Einstein-NLE equations and the perturbations $(\delta g_{ab}, \delta\mathbf{A})$ are the solutions of the linearized equations

[2]

with sources $\delta\hat{T}_{ab}$ and δj^a,

$$\delta(G_{ab} - 8\pi T_{ab}) = 8\pi\delta\hat{T}_{ab}\ , \quad \delta(\nabla_b Z^{ab}) = 4\pi\delta j^a\ . \tag{7.81}$$

Variations of the relevant quantities were already calculated within the covariant phase space formalism, with a caveat that additional source terms appear in this approach. Variation of the 4-form $\mathbf{C}$ (7.73) no longer vanishes on-shell but is equal to

$$\delta C_{abcd} \approx \left(\delta\hat{T}_a{}^e + A_a\delta j^e\right)\epsilon_{ebcd}\ , \tag{7.82}$$

since the term $\delta(A_a\nabla_r Z^{re})$ reflects the presence of the sources,

$$\delta(A_a\nabla_r Z^{re}) = (\delta A_a)\nabla_r Z^{re} + A_a\delta\nabla_r Z^{re} = 0 - 4\pi A_a\delta j^e. \tag{7.83}$$

Using the definitions of mass and angular momentum (7.49) together with the assumption that field perturbations vanish at $\Sigma_0 \cap H[\chi]$, after inserting $\xi^a = \chi^a$ into (7.37), we get

$$\delta M - \Omega_{\mathsf{H}}\,\delta J - K_\chi^i\,\delta\beta_i = -\int_{(\Sigma_0,-\hat{\boldsymbol{\epsilon}})} \star\boldsymbol{\alpha}_\chi\ . \tag{7.84}$$

The auxiliary 1-form $\boldsymbol{\alpha}$, which contains sources, is defined by

$$\boldsymbol{\alpha}_\xi := \star(i_\xi\delta\mathbf{C}) = \delta\hat{\mathbf{T}}(\xi) + (i_\xi\mathbf{A})\,\delta\mathbf{j}\ , \tag{7.85}$$

for any Killing vector field ξ^a. The orientation of the hypersurface Σ_0 in (7.84) is opposite of the induced Stokes' orientation $\hat{\boldsymbol{\epsilon}}$ (see the discussion in Appendix D). The 1-form $\boldsymbol{\alpha}$ is conserved in a sense that

$$\mathrm{d}\star\boldsymbol{\alpha}_\xi = \mathrm{d}i_\xi\delta\mathbf{C} = (\pounds_\xi - i_\xi\mathrm{d})\delta\mathbf{C} = 0\ , \tag{7.86}$$

since all fields and perturbations are symmetry inheriting. In the rest of the derivation, we will suppress the additional index on $\boldsymbol{\alpha}$ for notational clarity. Application of the Stokes' theorem (D.6) on a four-dimensional submanifold whose boundaries consist of hypersurfaces Σ_0 and Σ_1, horizon portion $\mathcal{H}$ and a far-away timelike hypersurface S on which perturbations δj^a and $\delta\hat{T}_{ab}$ vanish, leads us to

$$0 = \int_{(\Sigma_0,\hat{\boldsymbol{\epsilon}})} (\tilde{n}^a\alpha_a)\hat{\boldsymbol{\epsilon}} + \int_{(\mathcal{H},\hat{\boldsymbol{\epsilon}})} (-\ell^a\alpha_a)\hat{\boldsymbol{\epsilon}}\ . \tag{7.87}$$

As shall be proven later, a convenient choice of the null vector field ℓ^a is $\ell^a = \zeta^a$, which is a vector field tangent to the affinely parametrized null generators of the unperturbed Killing horizon $H[\xi]$. Now we may shift the integral in (7.84) from the initial hypersurface Σ_0 to the black hole horizon ,

$$
\begin{aligned}
-\int_{(\Sigma_0,-i_n\boldsymbol{\epsilon})} \star\boldsymbol{\alpha} &= -\int_{(\Sigma_0,-i_n\boldsymbol{\epsilon})} (-n^a\alpha_a)(i_n\boldsymbol{\epsilon}) = \\
&= \int_{(\mathcal{H},i_n\boldsymbol{\epsilon})} (\zeta^a\alpha_a)(i_n\boldsymbol{\epsilon}) \ , \qquad (7.88)
\end{aligned}
$$

where we omitted the pullback symbols for notational simplicity. So far we get a partial result of the form

$$
\delta M - \Omega_{\mathsf{H}}\,\delta J - K^i_\chi\,\delta\beta_i = \int_{\mathcal{H}} \zeta^a\alpha_a\,\hat{\boldsymbol{\epsilon}} \ . \qquad (7.89)
$$

This integral has two contributions, one of which will add to the electromagnetic section of the first law, and the other of which corresponds to the area term. Let us first evaluate the electromagnetic part. The gauge choice is chosen such that both Φ and $\mathbf{A}$ vanish at infinity and consequently $\Phi_0 = 0$ and $-i_\xi\mathbf{A} = \Phi_H$ on the horizon. With the positive infalling charge, $\delta Q \geq 0$, we have $\zeta^a\delta j_a \leq 0$ on the horizon since δj^a is on the physical grounds assumed to be causal and future-directed (sign would be reversed for the negatively charged infalling matter). Taking the arguments presented above into account, we get

$$
\delta M - \Omega_{\mathsf{H}}\,\delta J - \Phi_{\mathsf{H}}\,\delta Q - K^i_\chi\,\delta\beta_i = \int_{\mathcal{H}} \zeta^a\chi^b(\delta\hat{T}_{ab})\,\hat{\boldsymbol{\epsilon}} \ . \qquad (7.90)
$$

Now we have to show that the remaining term in (7.90) is indeed the area term. This will be done by employing the Raychaudhuri equation, which governs the expansion of a family of nonintersecting geodesics, collectively called a congruence. As anticipated, we consider a family of horizon-generating, affinely parametrized null geodesics, whose tangent vector field is ζ^a and whose corresponding parameter is denoted by V. Notice that the Killing vector field χ^a satisfies a nonaffinely parametrized geodesic equation $\chi^a\nabla_a\chi^b = \kappa\chi^b$. However, the standard transformation of the form $V = \exp(\kappa v)$ converts the Killing vector field parameter v to an affine parameter V. Then, the two vector fields are related by

$$
\zeta^a = \left(\frac{\partial}{\partial V}\right)^a = \frac{1}{\kappa V}\left(\frac{\partial}{\partial v}\right)^a = \frac{1}{\kappa V}\chi^a. \qquad (7.91)
$$

For extremal black holes with $\kappa = 0$, the Killing vector field χ^a is tangent to the affinely parametrized geodesic horizon generators. The expansion scalar θ has a convenient geometric interpretation; it measures the change in the congruence's cross-sectional area as one moves along the geodesics,

$$\theta = \frac{1}{A}\frac{dA}{dV} \ . \tag{7.92}$$

The Raychaudhuri equation for the null congruence [147]

$$\frac{\mathrm{d}\theta}{\mathrm{d}V} = -\frac{1}{2}\,\theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}\zeta^a\zeta^b \ , \tag{7.93}$$

gets simplified in the case of the stationary background, where the shear tensor σ_{ab} and the expansion θ both vanish. Furthermore, taking into account the Einstein's equation, the Raychaudhuri equation becomes

$$\frac{\mathrm{d}\theta}{\mathrm{d}V} = -8\pi\left(T_{ab} + \hat{T}_{ab}\right)\zeta^a\zeta^b \ . \tag{7.94}$$

To obtain the change in area, we need the perturbed Raychaudhuri equation. Diffeomorphism invariance provides one useful simplification. We may choose a gauge in which the null generators of the unperturbed and perturbed black horizons coincide so that $\delta\zeta^a \sim \zeta^a$ on the horizon. Since $R_{ab}\zeta^a\zeta^b|_H = 0$ [187], the perturbed Raychaudhuri equation [63] reduces to

$$\frac{\mathrm{d}\delta\theta}{\mathrm{d}V} = -8\pi\left(\delta T_{ab} + \delta\hat{T}_{ab}\right)\zeta^a\zeta^b\big|_H \ . \tag{7.95}$$

A closer look at the first term on the right-hand side of (7.95) reveals it consists of three terms,

$$\begin{aligned}\delta T_{ab}\zeta^a\zeta^b = &-4(\delta\mathscr{L}_{\mathcal{F}})T^{(\mathrm{Max})}_{ab}\zeta^a\zeta^b - \\ &- 4\mathscr{L}_{\mathcal{F}}\,\delta T^{(\mathrm{Max})}_{ab}\zeta^a\zeta^b + \frac{1}{4}\,\delta(T g_{ab})\zeta^a\zeta^b \ ,\end{aligned} \tag{7.96}$$

which all vanish on the horizon. The last term is equal to zero since ζ^a is null in both perturbed and unperturbed spacetimes. The other two terms also make no contribution since the electric field is null on the horizon, as was shown while proving the zeroth law,

$$4\pi T^{(\mathrm{Max})}_{ab}\zeta^a\zeta^b|_H = (\kappa V)^{-2}E_aE^a|_H = 0 \ , \tag{7.97}$$

$$4\pi\delta T^{(\mathrm{Max})}_{ab}\zeta^a\zeta^b|_H = (\kappa V)^{-2}\delta(E^aE_a)|_H = 0 \ . \tag{7.98}$$

Thus, the perturbed Raychaudhuri equation (7.95) reduces even further and by rewriting ζ^a in terms of the Killing vector field χ^a, we get

$$\kappa V \frac{\mathrm{d}\delta\theta}{\mathrm{d}V} = -8\pi\zeta^a\chi^b\delta\hat{T}_{ab}\big|_H \ . \tag{7.99}$$

Integration over the $\mathcal{H}$ returns exactly the change in horizon area $\delta\mathcal{A}$

$$\int_{\mathcal{H}} \zeta^a\chi^b(\delta\hat{T}_{ab})\,\hat{\boldsymbol{\epsilon}} = \frac{\kappa}{8\pi}\,\delta\mathcal{A} \ . \tag{7.100}$$

We can elaborate on this fact in a few steps [188]. The volume form on the horizon portion $\mathcal{H}$ can be split into the cross section surface element d^2S and the time part dV. Integration by parts gives us

$$\int d^2S\left(\int_0^\infty V\frac{\mathrm{d}\delta\theta}{\mathrm{d}V}dV\right) = \int d^2S(\delta\theta V)\bigg|_0^\infty - \int d^2S\int_0^\infty \delta\theta dV \tag{7.101}$$

The first term vanishes when evaluated at the lower limit because we have $V=0$, while at the upper limit θ goes to zero since it has to decay faster than $1/V$ as $V\to\infty$ if the final black hole has a finite area. By the definition (7.92), the second term in (7.101) is up to a sign equal to the change in area of the black hole horizon. Finally, we obtain the physical process version of the first law

$$\delta M = \frac{\kappa}{8\pi}\,\delta\mathcal{A} + \Omega_{\mathsf{H}}\,\delta J + \Phi_{\mathsf{H}}\,\delta Q + K^i_\chi\,\delta\beta_i \ , \tag{7.102}$$

which is consistent with the one derived following the equilibrium state approach (7.77).

7.4 Smarr formula

The Smarr formula for stationary and axially symmetric rotating black holes within $\mathcal{F}\mathcal{G}$-class theories can be derived independently of the first law by utilising the Bardeen-Carter-Hawking mass formula. This approach has been employed in [71], yielding an interim result

$$M = \frac{\kappa}{4\pi}\mathcal{A} + 2\Omega_{\mathsf{H}}J + \Phi_{\mathsf{H}}Q_{\mathsf{H}} + \Psi_{\mathsf{H}}P_{\mathsf{H}} + \frac{1}{2}\int_\Sigma T\star\boldsymbol{\chi} \ . \tag{7.103}$$

The key difference between the NLE and Maxwell cases is the presence of the trace term, which is absent in Maxwell's theory but generally exists in the NLE case. Given that the NLE Lagrangian can be written as $\mathscr{L} = \sigma^{-1}f(\sigma\mathcal{F},\sigma\mathcal{G})$, with some parameter σ and a real function f, the trace of the energy-momentum tensor becomes $T = -(\sigma/\pi)\partial_\sigma \mathscr{L}$. One can then, at least formally, interpret the novel NLE term as a product of mutually conjugate thermodynamic variables [71].

Another way of deriving the Smarr formula relies on the first law and its perturbation, which is defined as a path in the phase space of the solutions given by the consistently rescaled fields [174]. This idea has been used in [205] to derive the Smarr relation for $\mathcal{F}$-class theories. We will use the same approach to rederive the Smarr formula for $\mathcal{F}\mathcal{G}$-class theories, which will provide an independent consistency check.

7.4.1 Smarr formula from the first law

Let $(g_{ab}, \mathbf{A})$ denote an initial solution of the gNLE field equations. We are interested in finding rescaled fields $(\lambda^2 g_{ab}, \lambda^\nu \mathbf{A})$, where λ and ν are real constants, chosen such that the family of rescaled field configurations satisfies the same equations.

First, we have to find consistent scalings of the relevant fields entering the Smarr formula. If the metric is rescaled according to $g_{ab} \to \lambda^2 g_{ab}$, it follows that the metric inverse rescales as $g^{ab} \to \lambda^{-2} g^{ab}$, while volume form and the area of the black hole horizon change as $\boldsymbol{\epsilon} \to \lambda^4 \boldsymbol{\epsilon}$ and $\mathcal{A} \to \lambda^2 \mathcal{A}$, respectively. The metric rescaling immediately defines the rules for the curvature tensors,

$$R^a{}_{bcd} \to R^a{}_{bcd}\,,\ R_{ab} \to R_{ab}\,,\ R \to \lambda^{-2} R\,,\ G_{ab} \to G_{ab}.$$

Killing vector k^a is timelike at infinity with normalisation given as $g_{ab}k^a k^b = -1$, so that $k^a \to \lambda^{-1} k^a$ and $\mathbf{k} \to \lambda \mathbf{k}$. The axial Killing vector m^a is normalised along its closed orbits $\mathcal{C}$ as

$$\oint_{\mathcal{C}} \frac{1}{m_a m^a}\,\mathbf{m} = 2\pi, \tag{7.104}$$

implying that $m^a \to m^a$ and $\mathbf{m} \to \lambda^2 \mathbf{m}$. Consistency of the horizon-generating Killing vector field $\chi^a = k^a + \Omega_H m^a$ sets $\Omega_{\mathsf{H}} \to \lambda^{-1}\Omega_{\mathsf{H}}$. The appropriate rule for surface gravity κ follows from the geodesic equation $\chi^b \nabla_b \chi^a = \kappa \chi^a$ and is given by $\kappa \to \lambda^{-1}\kappa$. From the Komar integrals (7.24), we may deduce $M \to \lambda M$ and $J \to \lambda^2 J$. This completes the gravitational sector, so we turn to the electromagnetic quantities.

Starting from the scaling of the gauge field, $\mathbf{A} \to \lambda^\nu \mathbf{A}$, we have $\mathbf{F} \to \lambda^\nu \mathbf{F}$ and $\star\mathbf{F} \to \lambda^\nu \star\mathbf{F}$. Then, the electromagnetic invariants obey $\mathcal{F} \to \lambda^{2(\nu-2)}\mathcal{F}$ and $\mathcal{G} \to \lambda^{2(\nu-2)}\mathcal{G}$. The electric and magnetic 1-forms defined with respect to the Killing vector field χ^a scale as $\mathbf{E} \to \lambda^{\nu-1}\mathbf{E}$ and $\mathbf{B} \to \lambda^{\nu-1}\mathbf{B}$. For their associated scalar potentials, we have $\Phi \to \lambda^{\nu-1}\Phi$ and $\Psi \to \lambda^{\nu-1}\Psi$. The energy-momentum tensor is scale invariant, $T_{ab} \to T_{ab}$, which follows from Einstein's field equation $G_{ab} = 8\pi T_{ab}$. From the expression for the energy-momentum tensor in terms of Lagrangian density (2.11), we see that consistency implies $\mathscr{L} \to \lambda^{-2}\mathscr{L}$. For Maxwell's case, this condition sets $\nu = 1$. Taking this scaling as universal, the same applies to all NLE electromagnetic Lagrangians. Also, it constrains the scalings of the additional parameters in NLE Lagrangians, which will generally be of the form $\beta_i \to \lambda^{b_i}\beta_i$ for some real exponents b_i. For example, in the Born-Infeld theory, we have $b \to \lambda^{-1}b$ and $\alpha \to \lambda\alpha$ in the Euler-Heisenberg theory. Komar charges defined in (2.21) imply $Q \to \lambda Q$ and $P \to \lambda P$, while (7.78) gives $K_i \to \lambda^{1-b_i}K_i$. The scaling exponents are summarised in Table 7.1 below. Note that this is not a necessary but rather a consistent set of scaling transformations that allows us to apply the first law of black hole thermodynamics. The quantities that are varied in the first law of black

Table 7.1: Scaling exponents for various fields and charges appearing in Einstein-NLE theory.

Scaling exponent	
-2	g^{ab}, R, $\mathcal{F}$, $\mathcal{G}$
-1	κ, Ω_{H}
0	${R^a}_{bcd}$, R_{ab}, G_{ab}, $\mathbf{E}$, $\mathbf{B}$, Φ, Ψ
1	M, $\mathbf{k}$, $\mathbf{A}$, $\mathbf{F}$, $\star\mathbf{F}$, Q, P
2	g_{ab}, $\mathbf{m}$, $\mathcal{A}$, J
4	$\boldsymbol{\epsilon}$

hole mechanics depend on the parameter λ and attain a form

$$\mathscr{Q}(\lambda) = \lambda^q \mathscr{Q}(1) \ , \tag{7.105}$$

where q is some scaling exponent. Then, if we denote the original, unperturbed state by the abbreviation $\mathscr{Q} = \mathscr{Q}(1)$, we get the relation between the perturbed and initial quantity,

$$\delta\mathscr{Q} = \frac{\mathrm{d}\mathscr{Q}(\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=1} = q\mathscr{Q} \ . \tag{7.106}$$

Finally, following this approach, we recover the generalised Smarr formula

$$M = \frac{\kappa}{4\pi}\mathcal{A} + 2\Omega_{\mathsf{H}}J + \Phi_{\mathsf{H}}Q + \sum_i b_i K^i_\chi \beta_i \ , \tag{7.107}$$

and confirm the previous, independently obtained result [71]. The only seeming discrepancy is the absence of the magnetic term $\Psi_H P$ in (7.107). It can be attributed to the form of the first law used for the derivation of the generalised Smarr formula (7.77), which is devoid of the magnetic charge term since it emanates from the covariant phase space formalism. A direct derivation done in [71] circumvents the use of the first law, so there is no physical contradiction between the two methods.

One advantage of the scaling procedure stressed in [205] is its generality in the sense that it can treat the NLE Lagrangians with multiple coupling parameters. The only such example that we encountered is the Ayón–Beato–García Lagrangian (2.64) which, as was remarked in [71], can be put in the form $\mathscr{L} = \tilde{\mu}\alpha^{-1}f(\alpha\mathcal{F})$, with $\tilde{\mu} = \mu/g$ and $\alpha = g^2$. Its parameters scale as $\mu \to \lambda\mu$ and $g \to \lambda g$, meaning that the parameter $\tilde{\mu}$ is scale invariant and Ayón–Beato–García Lagrangian is covered by the approach employed in [71]. The authors in [71] also consider more general Lagrangians of the form $\mathscr{L}(\sigma, \mathcal{F}, \mathcal{G}) = \sigma^{-1}\tilde{\mathscr{L}}(\sigma\mathcal{F}, \sigma\mathcal{G})$, with a real parameter σ that scales as $\sigma \to \lambda^2\sigma$. This condition is fulfilled for physically sensible NLE Lagrangians, consisting of Maxwell's term and expansion in the coupling parameter σ in the weak field limit,

$$\mathscr{L} = -\frac{1}{4}\,\mathcal{F} + \sigma\left(c_{20}\mathcal{F}^2 + 2c_{11}\mathcal{F}\mathcal{G} + c_{02}\mathcal{G}^2\right) + O(\sigma^2)\ , \tag{7.108}$$

where c_{ij} are dimensionless constants, irrelevant for the discussion. A simple algebraic manipulation

$$\mathscr{L} = \frac{1}{\sigma}\Big(-\frac{1}{4}\,(\sigma\mathcal{F}) + c_{20}(\sigma\mathcal{F})^2 + 2c_{11}(\sigma\mathcal{F})(\sigma\mathcal{G}) + c_{02}(\sigma\mathcal{G})^2 + O(\sigma^3)\Big) \tag{7.109}$$

brings the Lagrangian to the above-mentioned form.

7.4.2 Linearity of the Smarr formula

Finally, we can inspect the (non-)linearity of the Smarr formula and address the question which NLE theories leave it in the form

$$c_1 M = c_2\kappa\mathcal{A} + c_3\Omega_{\mathsf{H}}J + c_4\Phi_{\mathsf{H}}Q + c_5\Psi_{\mathsf{H}}P + c_6\Phi_{\mathsf{H}}P + c_7\Psi_{\mathsf{H}}Q\ , \tag{7.110}$$

where $\{c_1, \dots, c_7\}$ is the set of some real constants. The idea is to find the terms that would produce the desired products of potentials and charges after the integration of the 3-form $T\star\boldsymbol{\chi}$ over Σ. Our analysis is based on a number of suitably constructed equations,

$$\mathrm{d}(\Phi\star\mathbf{Z}) = -\mathbf{E}\wedge\star\mathbf{Z} = \star i_E\mathbf{Z} = \frac{1}{2}\star\mathbf{R}(\chi) + (2\pi T - \mathscr{L})\star\boldsymbol{\chi}\ , \tag{7.111}$$

$$\mathrm{d}(\Psi\mathbf{F}) = -\mathbf{H}\wedge\mathbf{F} = \frac{1}{2}\star\mathbf{R}(\chi) + \mathscr{L}\star\boldsymbol{\chi}\ , \tag{7.112}$$

$$\mathrm{d}(\Phi\mathbf{F}) = \frac{1}{2}\,i_\chi(\mathbf{F}\wedge\mathbf{F}) = -\frac{1}{4}\,\mathcal{G}\star\boldsymbol{\chi}\ , \tag{7.113}$$

$$\mathrm{d}(\Psi \star\mathbf{Z}) = -\frac{1}{2}\, i_\chi(\star\mathbf{Z}\wedge\star\mathbf{Z}) = 4\left(2\mathscr{L}_\mathcal{F}\mathscr{L}_\mathcal{G}\mathcal{F} + (\mathscr{L}_\mathcal{G}^2 - \mathscr{L}_\mathcal{F}^2)\mathcal{G}\right)\star\boldsymbol{\chi}\ . \tag{7.114}$$

The first equation is derived by contracting the Einstein's equation with χ^a while using the energy-momentum tensor in the form (2.11),

$$R_{ab}\chi^b + 4\pi T\chi_a = 8\pi T_{ab}\chi^b = 2(Z_{ac}F_b{}^c\chi^b + \mathscr{L}\chi_a)\ . \tag{7.115}$$

The second one follows from the contraction of (A.29) with χ^a and combining it with the previous result (7.111),

$$\mathbf{F}\wedge i_\chi\star\mathbf{Z} + i_\chi\mathbf{F}\wedge\star\mathbf{Z} = -2(\mathscr{L}_\mathcal{F}\mathcal{F} + \mathscr{L}_\mathcal{G}\mathcal{G})\star\boldsymbol{\chi} = -2(\mathscr{L} - \pi T)\star\boldsymbol{\chi}\ , \tag{7.116}$$

$$\mathbf{H}\wedge\mathbf{F} - \mathbf{E}\wedge\star\mathbf{Z} = -2(\mathscr{L} - \pi T)\star\boldsymbol{\chi}\ . \tag{7.117}$$

The last two equations follow from contractions of (A.28) and (A.31) with χ^a respectively,

$$i_\chi(\mathbf{F}\wedge\mathbf{F}) = -2\mathbf{E}\wedge\mathbf{F} = -\frac{1}{2}\mathcal{G}\star\boldsymbol{\chi}\ , \tag{7.118}$$

$$i_\chi(\star\mathbf{Z}\wedge\star\mathbf{Z}) = 2\mathbf{H}\wedge\star\mathbf{Z} = 8((\mathscr{L}_\mathcal{F}^2 - \mathscr{L}_\mathcal{G}^2)\mathcal{G} - 2\mathscr{L}_\mathcal{F}\mathscr{L}_\mathcal{G}\mathcal{F})\star\boldsymbol{\chi}\ . \tag{7.119}$$

Without a strict argument, it seems plausible that a necesary condition for the linearity of the Smarr formula is

$$\mathscr{L} = a(\mathscr{L}_\mathcal{F}\mathcal{F} + \mathscr{L}_\mathcal{G}\mathcal{G}) + b\left(2\mathscr{L}_\mathcal{F}\mathscr{L}_\mathcal{G}\mathcal{F} + (\mathscr{L}_\mathcal{G}^2 - \mathscr{L}_\mathcal{F}^2)\mathcal{G}\right) + c\mathcal{G}\ , \tag{7.120}$$

with real constants a, b and c. This form enables one to convert a linear combination of $\star\mathbf{R}(\chi)$ and $T\star\boldsymbol{\chi}$ into a linear combination of $\mathrm{d}(\Phi\star\mathbf{Z})$, $\mathrm{d}(\Psi\mathbf{F})$, $\mathrm{d}(\Phi\mathbf{F})$ and $\mathrm{d}(\Psi\star\mathbf{Z})$, while the remaining terms cancel. Without loss of generality, we can set $c = 0$ as the $\mathcal{G}$ term is nondynamical. The expression (7.120) may be regarded as a nonlinear partial differential equation for the Lagrangian $\mathscr{L}(\mathcal{F},\mathcal{G})$. Unfortunately, we do not know how to obtain the solution in full generality, so we may examine some special cases.

One simplification lies in considering the NLE theories which admit invariance under SO(2) electromagnetic duality rotations, defined by (2.33). The necessary and sufficient condition that ensures this invariance (2.43) translates into the constancy of $2\mathscr{L}_\mathcal{F}\mathscr{L}_\mathcal{G}\mathcal{F} + (\mathscr{L}_\mathcal{G}^2 - \mathscr{L}_\mathcal{F}^2)\mathcal{G} + (\mathcal{G}/16)$. Then, for duality-invariant NLE theories, we may take $b = c = 0$ and deal with the simpler, linear partial differential equation $\mathscr{L} = a(\mathscr{L}_\mathcal{F}\mathcal{F} + \mathscr{L}_\mathcal{G}\mathcal{G})$. Its characteristics in the $\mathcal{F} - \mathcal{G}$ plane, defined by $(\dot{\mathcal{F}},\dot{\mathcal{G}}) = (\mathcal{F},\mathcal{G})$, are just lines through the origin. Along a characteristic, the partial differential equation turns into an ordinary differential equation $a\dot{\mathscr{L}} - \mathscr{L} = 0$. Its general solution on a domain where $\mathcal{F} \neq 0$ is $\mathscr{L} = \mathcal{F}^{1/a} f(\mathcal{G}/\mathcal{F})$, while on a domain where $\mathcal{G} \neq 0$ it attains a similar form, $\mathscr{L} = \mathcal{G}^{1/a} g(\mathcal{F}/\mathcal{G})$, with some differentiable functions f and g. Another prominent class of examples consists of NLE theories with traceless energy-momentum tensor, satisfying the $(a, b, c) = (1, 0, 0)$ case. Its member is ModMax theory (2.57), which can be put in a suitable form by setting $f = -\cosh\gamma(1/4 - \tanh\gamma\sqrt{1 + (\mathcal{G}/\mathcal{F})^2})$. Since the linearity of the Smarr formula for power-Maxwell Lagrangian was confirmed in [71], we have to be able to reconstruct it from our general solution. Indeed, by setting $a = 1/s$ and function f to a constant, we recover precisely the power-Maxwell family of NLE Lagrangians (2.65).

Another physically relevant simplification may be obtained by demanding that the NLE Lagrangian has the Maxwellian weak field limit. Partial derivatives of (7.120) with respect to $\mathcal{F}$ and $\mathcal{G}$ evaluated at $(\mathcal{F}, \mathcal{G}) = (0, 0)$ are

$$-\frac{1}{4} = \mathscr{L}_{\mathcal{F}}(0,0) = -\frac{1}{4}\,a \quad \text{and} \quad 0 = \mathscr{L}_{\mathcal{G}}(0,0) = -\frac{1}{16}\,b \ ,$$

so that we recover the linear case with $(a, b) = (1, 0)$. Again, the solution can be written either as $\mathscr{L} = \mathcal{F} f(\mathcal{G}/\mathcal{F})$ or $\mathscr{L} = \mathcal{G} g(\mathcal{F}/\mathcal{G})$. In the former case, along the lines with $\mathcal{G} = p\mathcal{F}$, where p is a real parameter, we have $\mathscr{L}_{\mathcal{F}} = f(p) - pf'(p)$ and $\mathscr{L}_{\mathcal{G}} = f'(p)$. The Maxwellian weak field limit implies $f(p) = -1/4$ for any $p \in \mathbb{R}$. In the latter case, the analogous reasoning gives us $g(p) = -p/4$ for any $p \in \mathbb{R}$ along the lines defined by $\mathcal{F} = p\mathcal{G}$. Under the assumption that the condition (7.120) holds, the only NLE theory that preserves the linearity of the Smarr formula and simultaneously satisfies the Maxwellian weak field limit is Maxwell's electrodynamics itself.

Chapter 8

Discussion and conclusion

Even though we have resolved several problems related to the properties of spacetimes coupled to NLE theories, each of them has raised new questions, which may outline future investigations.

Using test field approximation, we calculated the first order perturbative NLE correction to the static case of Wald's solution, which represents a black hole surrounded by an external, asymptotically homogeneous magnetic field. The consistency of the approach is confirmed since results obtained either directly from generalised Maxwell's equation or by introducing magnetic scalar potential are mutually agreeable. In principle, the same procedure, albeit with a different boundary condition, can be repeated for a compact, highly conducting star. However, it seems that the corresponding solution cannot be written in a closed form, thus making it unclear how to impose the needed boundary condition. The question is whether one can find a more advantageous approach to this problem. Another further advancement would be to look at the rotating Kerr black hole solution in the same setting. However, in that case the invariants $\mathcal{F}_0$ and $\mathcal{G}_0$ get considerably more involved, and consequently, solving the master equation becomes a highly nontrivial problem.

Our results indicate that a four-dimensional, strictly stationary, regular, and asymptotically flat spacetime cannot support a nontrivial NLE field. The only exception comes in the form of somewhat exotic stealth field solutions, absent in most of the physically significant NLE theories. In the presence of charged matter described by a complex scalar field, generalisations of the theorems are possible if we introduce new assumptions on scalar fields and their corresponding current. These limitations are expected as there are known solitonic-like bosonic star solutions emerging from symmetry non-inheriting scalar fields [91, 90]. The challenge that comes with higher-dimensional cases is the increased rank of the magnetic field form. Naturally arising question is whether the assumptions of the theorems can be further relaxed. We were relying on the simple connectedness of the manifold $\mathscr{M}$ to ensure the existence of scalar potentials Φ, Ψ, U_E and U_H. Without this condition, one would have to adopt either of the following approaches: impose some boundary conditions that guarantee the existence of scalar potentials or construct different divergence identities that do not involve them. The Maxwellian weak field limit assumption, together with the fall-off conditions of metric components and fields, was essential in the elimination of the boundary terms at asymptotic ends. A weaker, but still effective condition, is that partial derivatives $\mathscr{L}_\mathcal{F}$ and $\mathscr{L}_\mathcal{G}$ are well-defined and finite at the origin of the $\mathcal{F}$–$\mathcal{G}$ plane. NLE theories such as power-Maxwell (for powers less than 1) [78, 79] and ModMax [8, 111] do not conform to such behaviour. The notion of asymptotic flatness and appropriate fall-off conditions have to be reexamined in these cases. In the (1+2)-dimensional case, although we still have divergence identity (4.53) and the lower-dimensional positive energy theorem [198] at disposal, the natural logarithmic behaviour of the scalar potential $O(\ln r)$ in the asymptotic region prevents the elimination of the boundary terms.

The presented theorems on the absence of regular solitonic solutions have been complemented by no-go theorems for black hole regularisation using NLE fields, which exclude most of the physically plausible NLE theories as candidates for resolution of singularities. Electrically charged black holes emanating from either $\mathcal{F}$ or $\mathcal{F}\mathcal{G}$-class Lagrangians with Maxwellian weak field limit cannot be regular, so in order to find a singularity-free solution, one has to rely on theoretically proposed magnetic charges. However, even in that case our theorems pose serious limitations to this objective. Examples often encountered in the literature, such as Born–Infeld, ModMax and Euler–Heisenberg-like quadratic Lagrangians, have also proved to be unsuccessful in regularisation, regardless of the charges present. Regular magnetically charged black holes are often found by using *ad hoc* constructed $\mathcal{F}$-class Lagrangians, whose origin is not well-motivated. Alternatively, regular solutions may be generated by choosing a specific metric and explicitly evaluating the associated NLE Lagrangian as a function of coordinates rather than via electromagnetic invariants (e.g.[182]). The main open question left is whether the no-go theorems for dyonic and magnetic cases can be further extended to cover a larger portion of $\mathcal{F}\mathcal{G}$-class Lagrangians. Since our theorems apply to Einstein–Hilbert action, one venue of inquiry could be dealing with some type of modified gravitational action. Taking $f(R)$ gravitational theory as an example [158, 157, 134], the main difficulty is incorporating regularity conditions on higher derivative curvature invariants. Another way of generalising the results is to allow nonminimal coupling between gravitation and NLE fields or include Lagrangians that depend on derivatives of invariants, which may arise from generalised uncertainty principle [22] or noncommutative field theories [73, 76, 41]. When it comes to energy conditions, defined by the signs of the derivative $\mathscr{L}_{\mathcal{F}}$ and the trace T [146, 14], our approach based on the boundedness of curvature invariants remains inconclusive. Although Einstein's field equation relates derivatives of the metric function $f(r)$, $\mathscr{L}_{\mathcal{F}}$ and T, no information can be extracted unless one imposes assumptions about the convexity of the function $f(r)$ "by hand", since it is not obvious which condition would represent the natural choice. NLE theories have also been tested as a way of evading cosmological singularities. In fact, it is possible to obtain regularised FRW universes [48, 31, 135] or Bianchi spaces [65] with various NLE theories. Although the outlook seems better in the context of cosmology, the possible underlying constraints have still not been systematically explored.

By inspecting the integrability of the distribution $\mathcal{D}^{\perp}$, we gave the criteria that guarantee the isometry-compatible block diagonalisation of the metric for general NLE theories, NLE theories with the added Chern-Simons term and NLE theories (non)-minimally coupled to scalar fields. Our results hold for an m-dimensional spacetime, which is a solution of Einstein's field equation and admits $(m-2)$ pairwise commuting Killing vector fields. Moreover, we showed that the theorem on the absence of null fields in static spacetimes remains valid for NLE theories coupled to any odd "o-t" type gravitational field equation in m dimensions, up to stealth field solutions. A further step forward would be to consider the integrability of theories beyond General Relativity or a more careful analysis for symmetry non-inheriting fields. From a phenomenological point of view, new black hole observational data may enable inspecting the deviations from circularity [49, 55, 54].

We reexamined the main building blocks of black hole thermodynamics in which Maxwell's theory is replaced by its NLE generalizations. To derive the first law of black hole thermodynamics with NLE fields, we utilised the previously developed covariant phase space approach and applied it to two conceptually distinct variations, the equilibrium and the physical process versions. The imprint of NLE theories reveals itself as a novel pair of conjugate thermodynamic variables (β_i, K_{ξ}^{i}), consisting of NLE Lagrangian parameters β_i and the K_{ξ}^{i} term which can, by dimensional argument, be interpreted as vacuum polarisation for some theories. Similarly, in an earlier analysis of black hole thermodynamics, the cosmological constant appears as a variable conjugate to volume [107, 116]. The derivation of the first law would not be possible without the auxiliary result, the constancy of the electromagnetic scalar potentials on the horizon. We proved this statement, known as the zeroth law of black hole electrodynamics, in several different complementary ways. Various authors in the literature took opposing stands on the question of whether NLE Lagrangian parameters should be varied in the first law. To resolve the dilemma, one may use the generalised Smarr formula as a guiding principle. Namely, the result we obtained via scaling procedure agrees with the generalised Smarr formula derived independently of the first law [71], thus confirming the necessity of the novel NLE term. Finally, we gave an argument that suggests that the Smarr formula remains linear in Maxwell's theory or for a class of NLE theories which violate Maxwellian weak field limit.

There are several possible generalisations of our results. By dropping the asymptotic flatness condition, we may include the cosmological constant term via the standard procedure presented, for example, in [116, 107]. Covariant phase space formalism may treat modified gravitational theories, as long as the coupling of electromagnetic and gravitational parts is minimal, although the induced corrections may not be easily evaluated [177, 19]. Generalisations for higher or lower dimensional spacetimes can be carried out straightforwardly, provided that one excludes invariant $\mathcal{G}$, since $\mathbf{F}$ and $\star\mathbf{F}$ are 2-forms only in four spacetime dimensions. Dealing with NLE theories nonminimally coupled to gravity or Lagrangians containing derivatives of electromagnetic invariants presents a much greater computational challenge. It is not yet clear if the extension of the phase space by NLE parameters is just an algebraic formality or may be of greater physical significance.

Due to their versatile applicability in numerous areas of physics, NLE theories will continue to inspire further investigations. It can be expected that this pursuit will lead to many interesting developments. The open questions presented here may pave the way for some future endeavours.

Appendix A

Useful identities

A.1 Differential forms

Let $(\mathscr{M}, g_{ab})$ be a smooth m-dimensional manifold with a metric g_{ab} whose signature is s and $\boldsymbol{\omega}$ a p-form. We list the frequently used identities within differential form calculus.

Hodge dual $\star$, contraction with a vector X^a and exterior derivative d are, respectively, defined as

$$(\star\omega)_{a_{p+1}\ldots a_m} = \frac{1}{p!}\,\omega_{a_1\ldots a_p}\epsilon^{a_1\ldots a_p}{}_{a_{p+1}\ldots a_m} \ , \tag{A.1}$$

$$(i_X\omega)_{a_1\ldots a_{p-1}} = X^b\,\omega_{ba_1\ldots a_{p-1}} \ , \tag{A.2}$$

$$(\mathrm{d}\omega)_{a_1\ldots a_{p+1}} = (p+1)\nabla_{[a_1}\omega_{a_2\ldots a_{p+1}]} \ . \tag{A.3}$$

Hodge dual applied twice returns the initial form up to a sign,

$$\star\star\,\boldsymbol{\omega} = (-1)^{p(m-p)+s}\,\boldsymbol{\omega} \ . \tag{A.4}$$

A convenient operation is the so-called "flipping over the Hodge",

$$i_X\star\boldsymbol{\omega} = \star(\boldsymbol{\omega}\wedge\mathbf{X}) \ , \tag{A.5}$$

where $\mathbf{X}$ on the left side denotes the vector X^a and on the right side is its associated 1-form, $X_a = g_{ab}X^b$. Special care has to be taken of the order of the forms when calculating multiple contractions with different vectors. For example, for vectors X^a and Y^a we have

$$i_X i_Y\star\boldsymbol{\omega} = i_X\star(\boldsymbol{\omega}\wedge\mathbf{Y}) = \star(\boldsymbol{\omega}\wedge\mathbf{Y}\wedge\mathbf{X}) \ , \tag{A.6}$$

and in a general case,

$$i_{X_{(1)}} \cdots i_{X_{(n)}} \star \boldsymbol{\omega} = \star(\boldsymbol{\omega} \wedge \mathbf{X}^{(n)} \wedge \ldots \wedge \mathbf{X}^{(1)})\,. \tag{A.7}$$

Lie derivative $\pounds_X$ can be defined via Cartan's formula,

$$i_X \mathrm{d} + \mathrm{d} i_X = \pounds_X . \tag{A.8}$$

Other identities involving Lie derivative can be summarised as follows:

$$\pounds_X \mathrm{d} = \mathrm{d} \pounds_X \ , \tag{A.9}$$

$$\pounds_X i_Y - i_Y \pounds_X = i_{[X,Y]} \ , \tag{A.10}$$

$$\pounds_K \star = \star \pounds_K \ , \tag{A.11}$$

where X^a and Y^a are smooth vector fields and K^a is a smooth Killing vector field.

Notice that the contraction with a vector X^a, exterior derivative d and Lie derivative $\pounds_X$ satisfy the Leibniz rule,

$$i_X(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = i_X \boldsymbol{\alpha} \wedge \boldsymbol{\beta} + (-1)^p \boldsymbol{\alpha} \wedge i_X \boldsymbol{\beta} \ , \tag{A.12}$$

$$\mathrm{d}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = (\mathrm{d}\boldsymbol{\alpha}) \wedge \boldsymbol{\beta} + (-1)^p \boldsymbol{\alpha} \wedge (\mathrm{d}\boldsymbol{\beta}) \ , \tag{A.13}$$

$$\pounds_X(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = \pounds_X \boldsymbol{\alpha} \wedge \boldsymbol{\beta} + \boldsymbol{\alpha} \wedge \pounds_X \boldsymbol{\beta} \ , \tag{A.14}$$

for a p-form $\boldsymbol{\alpha}$ and a q-form $\boldsymbol{\beta}$. The coderivative operator δ acts on a p-form $\boldsymbol{\omega}$ as

$$\delta \boldsymbol{\omega} := (-1)^{m(p+1)+s} \star \mathrm{d} \star \boldsymbol{\omega} \ , \tag{A.15}$$

which in an abstract index notation takes a form

$$\delta \omega_{a_1 \ldots a_{p-1}} = \nabla^b \omega_{b a_1 \ldots a_{p-1}} \ . \tag{A.16}$$

Lie derivative along a Killing vector field K^a can be expressed in terms of the coderivative operator as

$$\pounds_K \boldsymbol{\omega} = \delta(\mathbf{K} \wedge \boldsymbol{\omega}) + \mathbf{K} \wedge \delta \boldsymbol{\omega} \ . \tag{A.17}$$

The expression above is especially useful in the case when $\boldsymbol{\omega}$ is a 1-form that inherits the spacetime symmetries, so that $\pounds_K \boldsymbol{\omega} = 0$. Then, after integration over a smooth hypersurface Σ and application of the generalised Stokes' theorem, we get

$$\int_\Sigma (\delta\boldsymbol{\omega}) \star\mathbf{K} = \int_{\partial\Sigma} \star(\mathbf{K}\wedge\boldsymbol{\omega}) \ , \tag{A.18}$$

where we omitted the pullback symbol for the sake of simplicity. The volume form $\boldsymbol{\epsilon}$ satisfies the following identities:

$$\star 1 = \boldsymbol{\epsilon} \ , \quad \star\boldsymbol{\epsilon} = (-1)^s, \quad i_X\boldsymbol{\epsilon} = \star\mathbf{X} \ , \quad \pounds_X(f\boldsymbol{\epsilon}) = \delta(f\mathbf{X})\boldsymbol{\epsilon} \ , \tag{A.19}$$

where f is a scalar function. For a 1-form $\boldsymbol{\alpha}$ we have $\star\mathrm{d}\star\boldsymbol{\alpha} = -\nabla^a\alpha_a$ and $\mathrm{d}\star\boldsymbol{\alpha} = (-1)^s\nabla^a\alpha_a\boldsymbol{\epsilon}$.

The inner product of two p-forms is defined as

$$(\alpha|\omega) = \frac{1}{p!}\alpha_{a_1..a_p}\omega^{a_1...a_p} \ , \tag{A.20}$$

and admits a number of useful identities:

$$(\alpha|\omega)\boldsymbol{\epsilon} = \boldsymbol{\alpha}\wedge\star\boldsymbol{\omega} = \boldsymbol{\omega}\wedge\star\boldsymbol{\alpha} = (-1)^s(\star\alpha|\star\omega)\boldsymbol{\epsilon} \ , \tag{A.21}$$

$$(X\wedge\gamma|\alpha) = (\gamma|i_X\alpha) \ , \tag{A.22}$$

where $\boldsymbol{\gamma}$ is a $(p-1)$-form.

A.2 Electromagnetic field tensor identities

For any 2-form $\mathbf{F}$, we have two elementary results

$$F_{ac}F^c{}_b - \star F_{ac}\star F^c{}_b = -\frac{1}{2}\,\mathcal{F}g_{ab} \ , \tag{A.23}$$

$$F_{ac}\star F^c{}_b = \star F_{ac}\,F^c{}_b = -\frac{1}{4}\,\mathcal{G}g_{ab} \ . \tag{A.24}$$

The first identity follows directly from rewriting the second term as

$$\star F_{ac}\star F^c{}_b = -\frac{1}{4}F^{ef}F_{gh}g_{db}\epsilon_{cefa}\epsilon^{cghd} = \frac{3!}{4}F^{ef}F_{gh}g_{db}\delta_e{}^{[g}\delta_f{}^h\delta_a{}^{d]} \ , \tag{A.25}$$

and performing all the contractions with products of Kronecker delta tensors. The second one can be derived by applying the similar trick. Starting from $F_{ac} = -\star\star F_{ac}$, we have

$$F_{ac}\star F^c{}_b = -\frac{3!}{8}F_{gh}F_{km}g_{bn}\epsilon^{ghef}\delta_e{}^{[k}\delta_f{}^m\delta_a{}^{n]} \ . \tag{A.26}$$

After proceeding with the calculation, one arrives at the sought identity.

Using the general identity (A.21), we can easily derive the following expressions,

$$\mathbf{F} \wedge \star\mathbf{F} = \frac{1}{2}\,\mathcal{F}\,\boldsymbol{\epsilon}\ , \tag{A.27}$$

$$\mathbf{F} \wedge \mathbf{F} = -\frac{1}{2}\,\mathcal{G}\,\boldsymbol{\epsilon}\ , \tag{A.28}$$

$$\mathbf{F} \wedge \star\mathbf{Z} = -2(\mathcal{F}\mathscr{L}_{\mathcal{F}} + \mathcal{G}\mathscr{L}_{\mathcal{G}})\,\boldsymbol{\epsilon}\ , \tag{A.29}$$

$$\mathbf{F} \wedge \mathbf{Z} = -2(\mathcal{F}\mathscr{L}_{\mathcal{G}} - \mathcal{G}\mathscr{L}_{\mathcal{F}})\,\boldsymbol{\epsilon}\ , \tag{A.30}$$

$$\star\mathbf{Z} \wedge \star\mathbf{Z} = 8\left((\mathscr{L}_{\mathcal{F}}^2 - \mathscr{L}_{\mathcal{G}}^2)\mathcal{G} - 2\mathscr{L}_{\mathcal{F}}\mathscr{L}_{\mathcal{G}}\mathcal{F}\right)\boldsymbol{\epsilon}\ . \tag{A.31}$$

Taking into account that $\mathrm{d}\mathbf{F} = 0$, we have

$$F^{ac}\nabla_a F_{bc} = \frac{1}{4}\,\nabla_b\mathcal{F}\ , \tag{A.32}$$

since

$$\nabla_b(F_{ac}F^{ac}) = 2F^{ac}\nabla_b F_{ac} = 2F^{ac}(\nabla_c F_{ab} + \nabla_a F_{bc}) = 4F^{ac}\nabla_a F_{bc}\ . \tag{A.33}$$

Similarly, as $\star F^{ab}\nabla_c F_{ab} = F^{ab}\nabla_c\star F_{ab}$, it follows that

$$\star F^{ac}\nabla_a F_{bc} = F^{ac}\nabla_a\star F_{bc} = \frac{1}{4}\,\nabla_b\mathcal{G}\ . \tag{A.34}$$

A.3 Schwarzschild spacetime

In Chapter 3, we repeatedly used Hodge duals of 2-forms in Schwarzschild spacetime, so it is convenient to gather them in one place

$$\star(\mathrm{d}t \wedge \mathrm{d}r) = -r^2\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi\ , \quad \star(\mathrm{d}\theta \wedge \mathrm{d}\varphi) = \frac{1}{r^2\sin\theta}\,\mathrm{d}t \wedge \mathrm{d}r \tag{A.35}$$

$$\star(\mathrm{d}t \wedge \mathrm{d}\theta) = \frac{\sin\theta}{f(r)}\,\mathrm{d}r \wedge \mathrm{d}\varphi\ , \quad \star(\mathrm{d}r \wedge \mathrm{d}\varphi) = -\frac{f(r)}{\sin\theta}\,\mathrm{d}t \wedge \mathrm{d}\theta \tag{A.36}$$

$$\star(\mathrm{d}t \wedge \mathrm{d}\varphi) = -\frac{1}{f(r)\sin\theta}\,\mathrm{d}r \wedge \mathrm{d}\theta\ , \quad \star(\mathrm{d}r \wedge \mathrm{d}\theta) = f(r)\sin\theta\,\mathrm{d}t \wedge \mathrm{d}\varphi \tag{A.37}$$

For the 1-form $\mathbf{m}$ associated to the axial Killing vector field, we have

$$\frac{1}{2}\,\mathrm{d}\mathbf{m} = r\sin^2\theta\,\mathrm{d}r \wedge \mathrm{d}\varphi + r^2\cos\theta\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi\ , \tag{A.38}$$

and

$$\frac{1}{2}\star\mathrm{d}\mathbf{m} = \cos\theta\,\mathrm{d}t \wedge \mathrm{d}r - rf(r)\sin\theta\,\mathrm{d}t \wedge \mathrm{d}\theta\ . \tag{A.39}$$

The exterior derivative of $\mathbf{w} = \mathbf{v}/C$, where v^a is given by (3.46), and its Hodge dual are equal to

$$\begin{aligned} \mathrm{d}\mathbf{w} =& -16\sin^4\theta\,\mathrm{d}r\wedge\mathrm{d}\varphi - \\ & -8\sin(2\theta)\Big(2r-5M+(M-2r)\cos(2\theta)\Big)\mathrm{d}\theta\wedge\mathrm{d}\varphi\ , \end{aligned} \tag{A.40}$$

$$\begin{aligned} \star\mathrm{d}\mathbf{w} =& 16f(r)\sin^3\theta\,\mathrm{d}t\wedge\mathrm{d}\theta - \\ & -16\,\frac{\cos\theta}{r^2}\Big(2r-5M+(M-2r)\cos(2\theta)\Big)\mathrm{d}t\wedge\mathrm{d}r\ . \end{aligned} \tag{A.41}$$

Appendix B

Spinors

B.1 Fundamentals and conventions

The fundamental, naturally arising objects in general relativity are tensor fields living on a real, four-dimensional spacetime. However, the theory can be formulated in terms of 2-spinors defined on a 2-dimensional complex vector space [144, 173]. We will denote it by S and its dual, which consists of maps $\omega : S \to \mathbb{C}$, by S^*. To complete the setup, we introduce a complex conjugate dual space $\overline{S}^*$, consisting of antilinear maps $\overline{\omega} : S \to \mathbb{C}$, and finally, a complex conjugate space $\overline{S}$ dual to it. An essential object on spin space is the symplectic structure, a nondegenerate bilinear 2-form defined as

$$[\ ,\ ] : S \times S \to \mathbb{C}\ , \quad [\xi, \phi] = -[\phi, \xi]\ . \tag{B.1}$$

Symplectic structure belongs to $S^* \times S^*$ and is denoted by $\epsilon_{AB} = -\epsilon_{BA}$.

The spin basis for S is built out of two nonzero vectors $o, \imath \in S$ normalised such that $[o, \imath] = 1$. Then each vector $\xi^A \in S$ admits a unique decomposition $\xi^A = \xi^0 o^A + \xi^1 \imath^A$.

The role of the metric tensor as a map that provides a natural isomorphism from tangent space to its dual is now taken by the symplectic structure in an analogous way. It acts on the objects from S and sends them to S^* via $\xi_A = \epsilon_{BA}\xi^B$. Similarly, there is an inverse operation from S^* to S defined as $\xi^A = \epsilon^{AB}\xi_B$. In other words, we use the first index in ϵ_{AB} to lower the indices and the second one in ϵ^{AB} to raise the indices. The elements of $\overline{S}$ are written as $\overline{\xi}^{A'}$ and analogously for lowered indices. Conventionally, $\overline{\epsilon_{AB}}$ is denoted by $\epsilon_{A'B'}$ instead of $\overline{\epsilon}_{A'B'}$. It is normalised according to $\epsilon_{AB}\epsilon^{AB} = 2$ and can be used to split the Levi-Civita tensor as

$$\epsilon_{abcd} = \epsilon_{ABCDA'B'C'D'} = i(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'})\ . \tag{B.2}$$

One should be aware of the differences in conventions used throughout the literature as spinor formalism is usually formulated using the “mostly minus” metric signature. In order to keep track of the signs and cover both scenarios, we introduce $\eta = \text{sgn}(\eta_{00})$. Then, the spacetime metric is equal to spinor $g_{ABA'B'} = \eta\epsilon_{AB}\epsilon_{A'B'}$.

Next, we list a few results that will prove important in future discussions [144, 173].

Lemma B.1. *Any spinor with two indices τ_{AB} may be decomposed as*

$$\tau_{AB} = \tau_{(AB)} + \frac{1}{2}\epsilon_{AB}\tau_C{}^C. \tag{B.3}$$

More generally, the analogous result is valid for spinors with multiple indices.

Theorem B.1. *Let $\tau_{A...B}$ be a totally symmetric spinor. Then there exist univalent spinors $\{\alpha_A, ...\zeta_A\}$ such that*

$$\tau_{A...Z} = \alpha_{(A}...\zeta_{Z)} \ , \tag{B.4}$$

where the spinors $\{\alpha_A, ...\zeta_A\}$ are called principal spinors of τ.

Theorem B.2. *Every real null vector k^a can be written as $k^a = \pm\kappa^A\overline{\kappa}^{A'}$, where the sign determines whether it is future or past directed.*

B.2 Electrodynamics in spinor formalism

In our case, the most important application of the spinor approach is within the electromagnetic theories. The fundamental object, the antisymmetric electromagnetic field tensor F_{ab}, becomes $F_{ab} = F_{ABA'B'} = -F_{BAB'A'}$ in spinor formalism. However, electromagnetism is usually formulated in terms of a symmetric spinor ϕ_{AB} defined by

$$\phi_{AB} = \frac{1}{2}F_{ABC'}{}^{C'} = -\frac{1}{2}F_{BA}{}^{C'}{}_{C'} = \frac{1}{2}F_{BAC'}{}^{C'} = \phi_{BA} \ . \tag{B.5}$$

Our goal is to show that F_{ab} and ϕ_{AB} can be treated on an equal footing since they contain the same information. As a basic consistency check, one may prove that both objects contain the same number of degrees of freedom. In a four-dimensional spacetime, F_{ab} has six independent degrees of freedom due to the antisymmetry, while the symmetric spinor ϕ_{AB} has three independent complex components that in total carry six degrees of freedom.

Using the result of lemma B.1, we can derive the general form of $F_{ABA'B'}$ in terms of ϕ_{AB},

$$F_{ABA'B'} = F_{AB(A'B')} + \epsilon_{A'B'}\phi_{AB} = F_{(AB)(A'B')} + \epsilon_{AB}\overline{\phi}_{A'B'} + \phi_{AB}\epsilon_{A'B'}. \tag{B.6}$$

Due to the antisymmetry, $F_{(AB)(A'B')} = 0$ and the final expression is

$$F_{ABA'B'} = \epsilon_{AB}\overline{\phi}_{A'B'} + \phi_{AB}\epsilon_{A'B'}. \tag{B.7}$$

The dual of F_{ab} is calculated straightforwardly using the spinor equivalent of its standard definition,

$$\begin{aligned}\star F_{ABA'B'} &= \frac{1}{2}\epsilon_{AB}{}^{CD}{}_{A'B'}{}^{C'D'}F_{CDC'D'} = \\ &= i(\epsilon_A{}^C\epsilon_B{}^D\epsilon_{A'}{}^{D'}\epsilon_{B'}{}^{C'} - \epsilon_A{}^D\epsilon_B{}^C\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'})F_{CDC'D'}\ ,\end{aligned} \tag{B.8}$$

finally resulting in

$$\star F_{ABA'B'} = i\left(\epsilon_{AB}\overline{\phi}_{A'B'} - \phi_{AB}\epsilon_{A'B'}\right). \tag{B.9}$$

Lemma B.2. $F_{ABA'B'} = 0$ *iff* $\phi_{AB} = 0$. *Then, it follows that* $F_{ABA'B'} \neq 0$ *iff* $\phi_{AB} \neq 0$.

Proof of lemma B.2. If $\phi_{AB} = 0$, we have $F_{ABA'B'} = 0$ by definition. Conversely, if $F_{ABA'B'} = 0$, by contraction of (B.7) with $\epsilon^{A'B'}$ we get $\phi_{AB} = 0$. □

The term $\phi_{AC}\,\phi_B{}^C$ is antisymmetric in A and B and thus proportional to ϵ_{AB}, as follows from lemma B.1,

$$\phi_{AC}\,\phi_B{}^C = \frac{1}{2}\,\epsilon_{AB}\,\phi_{DC}\phi^{DC}\ . \tag{B.10}$$

The identity (B.10) will be particularly useful in deriving the spinor forms of other electromagnetic quantities. The two quadratic electromagnetic invariants are calculated as follows,

$$\begin{aligned}\mathcal{F} &= \left(\epsilon_{AB}\overline{\phi}_{A'B'} + \phi_{AB}\epsilon_{A'B'}\right)\left(\epsilon^{AB}\overline{\phi}^{A'B'} + \phi^{AB}\epsilon^{A'B'}\right) = \\ &= 2\left(\phi^{AB}\phi_{AB} + \overline{\phi}^{A'B'}\overline{\phi}_{A'B'}\right),\end{aligned} \tag{B.11}$$

where the cross terms vanish due to the contraction of symmetric and antisymmetric tensors, and similarly, for the invariant $\mathcal{G}$, we get

$$\mathcal{G} = -2i\left(\phi^{AB}\phi_{AB} - \overline{\phi}^{A'B'}\overline{\phi}_{A'B'}\right). \tag{B.12}$$

Adopting the standard normalisation of Maxwell's energy-momentum tensor

$$T^{(\mathrm{Max})}_{ab} := -\eta\,\frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}\,g_{ab}F_{cd}F^{cd}\right)\ , \tag{B.13}$$

the first term, with the aid of identity (B.10), becomes

$$\begin{aligned} F_{ac}F_b{}^c &= \eta F_{ACA'C'}F_B{}^C{}_{B'}{}^{C'} = \\ &= \eta\left(-2\phi_{AB}\,\phi_{A'B'} + \frac{1}{2}\,\epsilon_{AB}\,\epsilon_{A'B'}\left(\phi_{CD}\,\phi^{CD} + \overline{\phi}_{C'D'}\,\overline{\phi}^{C'D'}\right)\right)\ . \end{aligned} \tag{B.14}$$

After combining it with the spinor equivalent of the metric tensor and expression (B.11), the final spinor representation of $T^{(\mathrm{Max})}_{ab}$ is

$$T^{(\mathrm{Max})}_{ABA'B'} = \frac{1}{2\pi}\,\phi_{AB}\overline{\phi}_{A'B'}\ , \tag{B.15}$$

manifestly independent of the metric sign convention.

According to theorem B.1 a nontrivial ϕ_{AB} can be decomposed as $\phi_{AB} = \alpha_{(A}\beta_{B)}$. Here, we can discern two distinct cases. If α and β are not proportional, ϕ_{AB} is algebraically general or of type I in the Petrov classification. In the other case, when α and β are proportional, we refer to ϕ_{AB} as algebraically special or of type N. In the latter case, ϕ_{AB} represents a null electromagnetic field ($\mathcal{F} = 0 = \mathcal{G}$), as summarised in the theorem below.

Theorem B.3. *The electromagnetic field F_{ab} is null iff $\phi_{AB}\phi^{AB} = 0$, which corresponds to the type N fields.*

Proof of theorem B.3. If $\phi_{AB}\phi^{AB} = 0$, we immediately have $\mathcal{F} = 0 = \mathcal{G}$. Conversely, if $\mathcal{F} = 0 = \mathcal{G}$, solving the system in $\phi_{AB}\phi^{AB}$ gives a trivial solution. For type N fields we have $\phi_{AB}\phi^{AB} = \alpha_A\alpha_B\alpha^A\alpha^B = 0$. □

In some cases, the spinor approach provides an elegant method of performing otherwise tedious calculations. Its simplicity is illustrated in the proof of the following theorem.

Theorem B.4. *In Maxwell's electrodynamics, there are no stealth electromagnetic fields, $T^{(\mathrm{Max})}_{ab} = 0$ iff $F_{ab} = 0$.*

Proof of theorem B.4. If $F_{ab} = 0$, $T^{(\mathrm{Max})}_{ab}$ immediately vanishes. To prove the other direction of the claim, we suppose that $T^{(\mathrm{Max})}_{ab} = 0$ and $F_{ab} \neq 0$, which also implies that $\phi_{AB} \neq 0$ by lemma B.2. In that case, there exist spinors α^A and β^A for which

$\phi_{AB}\alpha^A\beta^B \neq 0$. Then, we have

$$\frac{1}{2\pi}\phi_{AB}\overline{\phi}_{A'B'}\overline{\alpha}^{A'}\overline{\beta}^{B'} \neq 0 \ , \tag{B.16}$$

which is in a contradiction with the initial assumption that $T^{(\mathrm{Max})}_{ab} = 0$. Therefore, ϕ_{AB} has to be trivial. □

Furthermore, the spinor approach enables relatively simple derivation of a general formula for the consecutive contractions of Maxwell's energy-momentum tensors (5.3). For an illustration, we will consider cases with $n = 1$ and $n = 2$, from which one can already draw a universal rule. Using the identity (B.10), the normalisation of the symplectic structure and the spinor forms of invariants $\mathcal{F}$ (B.11) and $\mathcal{G}$ (B.12), for $n = 1$ we have

$$\begin{aligned}\widetilde{T}_{ab}\widetilde{T}^{ba} &= \frac{1}{(2\pi)^2}\phi_{AB}\overline{\phi}_{A'B'}\phi^{BA}\overline{\phi}^{B'A'} = \\ &= \frac{1}{(2\pi)^2}\left(\phi_{AB}\phi^{AB}\right)\left(\overline{\phi}_{A'B'}\overline{\phi}^{A'B'}\right) = \\ &= \frac{1}{(4\pi)^2}\frac{1}{4}(\mathcal{F}^2 + \mathcal{G}^2) \ ,\end{aligned} \tag{B.17}$$

and for $n = 2$,

$$\begin{aligned}\widetilde{T}_{ab}\widetilde{T}^{bc}\widetilde{T}_{cd}\widetilde{T}^{da} &= \frac{1}{(2\pi)^4}\left(\phi_{AB}\overline{\phi}_{A'B'}\phi^{BC}\overline{\phi}^{B'C'}\right)\left(\phi_{CD}\overline{\phi}_{C'D'}\phi^{DA}\overline{\phi}^{D'A'}\right) = \\ &= \frac{1}{(4\pi)^4}\left(\epsilon_{AC}\left(\phi_{EF}\phi^{EF}\right)\epsilon_{A'C'}\left(\overline{\phi}_{E'F'}\overline{\phi}^{E'F'}\right)\right) \times \\ &\times \left(\epsilon^{AC}\left(\phi_{GH}\phi^{GH}\right)\epsilon^{A'C'}\left(\overline{\phi}_{G'H'}\overline{\phi}^{G'H'}\right)\right) = \\ &= \frac{1}{(4\pi)^4}\frac{1}{4^3}(\mathcal{F}^2 + \mathcal{G}^2)^2 \ .\end{aligned} \tag{B.18}$$

B.3 Newman-Penrose tetrad

For calculational convenience, it would be advantageous if one could use a basis consisting of four null vectors instead of the usual one timelike and three space-like vectors. However, taking into account real vectors only, it is not possible to construct four linearly independent null vectors. The solution to this obstacle lies in considering complex tangent space. The spin basis $\{o^A, \iota^A\}$ enables defining the Newman-Penrose null tetrad composed of vectors $\{l^a, n^a, m^a, \overline{m}^a\}$,

$$l^a = o^A\overline{o}^{A'}, \ \ n^a = \iota^A\overline{\iota}^{A'}, \ \ m^a = o^A\overline{\iota}^{A'}, \ \ \overline{m}^a = \iota^A\overline{o}^{A'}, \tag{B.19}$$

with normalisation (using the "mostly plus" metric signature) given as

$$\begin{aligned} l^a l_a = n^a n_a = m^a m_a = \overline{m}^a \overline{m}_a = 0 \ , \\ l^a n_a = -1 = -m^a \overline{m}_a \ , \\ l^a m_a = l^a \overline{m}_a = n^a m_a = n^a \overline{m}_a = 0 \ . \end{aligned} \tag{B.20}$$

The vectors l^a and n^a are real, while m^a and $\overline{m}^a$ are complex conjugates of each other. The directional derivatives are denoted by

$$D = l^a \nabla_a \ , \ \ \Delta = n^a \nabla_a \ , \ \ \delta = m^a \nabla_a \ , \ \ \overline{\delta} = \overline{m}^a \nabla_a \ . \tag{B.21}$$

The introduced elements provide basic building blocks for rewriting gravitational and Maxwell's equations in Newman-Penrose formalism [144, 173].

Appendix C

Calculation of variations

Starting from a general NLE Lagrangian, which is a smooth function of invariants $\mathcal{F}$ and $\mathcal{G}$, we will derive the corresponding energy-momentum tensor and generalised Maxwell's equations by means of variational procedure.

The energy-momentum tensor is defined with respect to the variation of the action as

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S^{(\mathrm{em})}}{\delta g^{ab}} \ , \tag{C.1}$$

where the electromagnetic action is

$$4\pi S^{(\mathrm{em})} = \int \mathscr{L}^{(\mathrm{em})}(\mathcal{F},\mathcal{G})\sqrt{-g}d^n x \ . \tag{C.2}$$

The variation of the square root of the metric determinant is

$$\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}, \tag{C.3}$$

as $\partial_a(\ln|\mathrm{det}A|) = \sum_{b,c}(A^{-1})_{bc}\partial_a A_{cb}$ for a general invertible matrix A and $\delta(g_{ab}g^{ab}) = 0$.

The variation of the action splits into two terms,

$$\frac{\delta(\mathscr{L}^{(\mathrm{em})}\sqrt{-g})}{\delta g^{ab}} = \frac{\delta\mathscr{L}^{(\mathrm{em})}}{\delta g^{ab}}\sqrt{-g} - \frac{1}{2}\sqrt{-g}g_{ab}\mathscr{L}^{(\mathrm{em})}. \tag{C.4}$$

Since the second term is in the desired form, we will focus on the first one,

$$\frac{\delta\mathscr{L}^{(\mathrm{em})}}{\delta g^{ab}} = \mathscr{L}_{\mathcal{F}}\frac{\delta\mathcal{F}}{\delta g^{ab}} + \mathscr{L}_{\mathcal{G}}\frac{\delta\mathcal{G}}{\delta g^{ab}} \ , \tag{C.5}$$

and calculate variations of the invariants while keeping the terms proportional to

δg^{ab} only,

$$\begin{aligned}
\delta\mathcal{F} &= \delta(F_{ab}F^{ab}) = \delta(F_{ab}F_{cd}g^{ac}g^{bd}) = \\
&= F_{ab}F_{cd}\delta g^{ac}g^{bd} + F_{ab}F_{cd}g^{ac}\delta g^{bd} = \\
&= 2F_{ac}F_{bd}g^{cd}\delta g^{ab} = 2F_{ac}F_{b}{}^{c}\delta g^{ab} \ , \qquad \text{(C.6)}\\
\delta\mathcal{G} &= \delta F_{ab}\star F^{ab} = \delta(F_{ab}\star F_{cd}g^{ac}g^{bd}) = \\
&= F_{ab}\star F_{cd}\delta g^{ac}g^{bd} + F_{ab}\star F_{cd}g^{ac}\delta g^{bd} + \frac{1}{2}g^{ac}g^{bd}F_{ab}F_{ef}\delta\epsilon^{ef}{}_{cd} = \\
&= 2F_{ac}\star F_{bd}g^{cd}\delta g^{ab} + \frac{1}{2}g^{ac}g^{bd}F_{ab}F_{ef}\delta\epsilon^{ef}{}_{cd} = \\
&= \frac{1}{2}\mathcal{G}g_{ab}\delta g^{ab} + \frac{1}{2}g^{ac}g^{bd}F_{ab}F_{ef}\delta\epsilon^{ef}{}_{cd} \ , \qquad \text{(C.7)}
\end{aligned}$$

where we have used the auxiliary identity (A.24) in the last step. Before proceeding further, we will show that the second term in the expression above vanishes. First, we calculate the variation,

$$\begin{aligned}
\delta\epsilon^{ef}{}_{cd} &= \delta(g^{eq}g^{fh}\epsilon_{qhcd}) = \\
&= \delta g^{eq}g^{fh}\epsilon_{qhcd} + g^{eq}\delta g^{fh}\epsilon_{qhcd} + g^{eq}g^{fh}(\delta\sqrt{-g})\varepsilon_{qhcd} = \\
&= \delta g^{eq}g^{fh}\epsilon_{qhcd} + g^{eq}\delta g^{fh}\epsilon_{qhcd} - \frac{1}{2}\sqrt{-g}g^{eq}g^{fh}g_{kl}\delta g^{kl}\varepsilon_{qhcd} = \\
&= \delta g^{eq}g^{fh}\epsilon_{qhcd} + g^{eq}\delta g^{fh}\epsilon_{qhcd} - \frac{1}{2}g^{eq}g^{fh}g_{kl}\delta g^{kl}\epsilon_{qhcd} \ , \qquad \text{(C.8)}
\end{aligned}$$

then the complete term

$$\begin{aligned}
&\frac{1}{2}g^{ac}g^{bd}F_{ab}F_{ef}\left(\delta g^{eq}g^{fh}\epsilon_{qhcd} + g^{eq}\delta g^{fh}\epsilon_{qhcd} - \frac{1}{2}g^{eq}g^{fh}g_{kl}\delta g^{kl}\epsilon_{qhcd}\right) = \\
&= \frac{1}{2}F_{eq}F_{af}g^{ce}g^{dq}g^{fh}\epsilon_{bhcd}\delta g^{ab} + \frac{1}{2}F_{fh}F_{ea}g^{eq}g^{cf}g^{dh}\epsilon_{qbcd}\delta g^{ab} - \\
&- \frac{1}{4}F_{kl}F_{ef}g^{ck}g^{dl}g^{fh}g^{eq}g_{ab}\delta g^{ab}\epsilon_{qhcd} = \\
&= \frac{1}{2}\left(F^{cd}F_{a}{}^{h}\epsilon_{bhcd} + F^{cd}F^{q}{}_{a}\epsilon_{qbcd} - \frac{1}{2}F^{cd}F^{qh}g_{ab}\epsilon_{qhcd}\right)\delta g^{ab} = \\
&= \left(\star F_{bh}F_{a}{}^{h} + \star F_{qb}F^{q}{}_{a} - \frac{1}{2}F^{cd}\star F_{cd}g_{ab}\right)\delta g^{ab} = 0 \ , \qquad \text{(C.9)}
\end{aligned}$$

where we have again used the identity (A.24). Notice that ε_{abcd} in (C.8) denotes the Levi-Civita symbol, while we reserve ϵ_{abcd} for the Levi-Civita tensor, which contains the square root of the metric determinant.

Taking everything into account, the final expression is

$$T_{ab} = -\frac{1}{4\pi}((\mathscr{L}_{\mathcal{G}}\mathcal{G} - \mathscr{L})g_{ab} + 4\mathscr{L}_{\mathcal{F}}F_{ac}F_b{}^c) \ . \tag{C.10}$$

To derive generalised Maxwell's equations, we perform variation of the action with respect to the gauge potential A^a,

$$\frac{\delta(\mathscr{L}^{(\mathrm{em})}\sqrt{-g})}{\delta A^a} = \mathscr{L}_{\mathcal{F}}\frac{\delta(\mathcal{F}\sqrt{-g})}{\delta A^a} + \mathscr{L}_{\mathcal{G}}\frac{\delta(\mathcal{G}\sqrt{-g})}{\delta A^a} \ . \tag{C.11}$$

The variation of the first term is

$$\begin{aligned}\mathscr{L}_{\mathcal{F}}\delta(\mathcal{F}\sqrt{-g}) &= \mathscr{L}_{\mathcal{F}}\delta(F_{ab}F^{ab}\sqrt{-g}) = \mathscr{L}_{\mathcal{F}}\delta(F_{ab}F_{cd}g^{ac}g^{bd}\sqrt{-g}) = \\ &= \mathscr{L}_{\mathcal{F}}\delta((\partial_a A_b - \partial_b A_a)(\partial_c A_d - \partial_d A_c)g^{ac}g^{bd}\sqrt{-g}) = \\ &= \mathscr{L}_{\mathcal{F}}(\partial_a\delta A_b\partial_c A_d + \partial_a A_b\partial_c\delta A_d - \partial_a\delta A_b\partial_d A_c - \\ &- \partial_a A_b\partial_d\delta A_c - \partial_b\delta A_a\partial_c A_d - \partial_b A_a\partial_c\delta A_d + \\ &+ \partial_b\delta A_a\partial_d A_c + \partial_b A_a\partial_d\delta A_c)g^{ac}g^{bd}\sqrt{-g} \ ,\end{aligned} \tag{C.12}$$

which we obtained using the commutation property of variations and partial derivatives. The expression can be rearranged via the Leibniz rule,

$$\partial_a(\mathscr{L}_{\mathcal{F}}\delta A_b\partial_c A_d g^{ac}g^{bd}\sqrt{-g}) - \partial_a(\mathscr{L}_{\mathcal{F}}\partial_c A_d g^{ac}g^{bd}\sqrt{-g})\delta A_b \ , \tag{C.13}$$

which we apply to all of the terms. The total derivative contributions may be discarded and we get

$$\begin{aligned}\mathscr{L}_{\mathcal{F}}\delta(\mathcal{F}\sqrt{-g}) = &- \partial_a(\mathscr{L}_{\mathcal{F}}F_{cd}g^{ac}g^{bd}\sqrt{-g})\delta A_b + \partial_b(\mathscr{L}_{\mathcal{F}}F_{cd}g^{ac}g^{bd}\sqrt{-g})\delta A_a - \\ &- \partial_c(\mathscr{L}_{\mathcal{F}}F_{ab}g^{ac}g^{bd}\sqrt{-g})\delta A_d + \partial_d(\mathscr{L}_{\mathcal{F}}F_{ba}g^{ac}g^{bd}\sqrt{-g})\delta A_c = \\ &= -4\partial_a(\sqrt{-g}\mathscr{L}_{\mathcal{F}}F^{ab})\delta A_b = -4\sqrt{-g}\nabla_a(\mathscr{L}_{\mathcal{F}}F^{ab})\delta A_b = \\ &= 4\sqrt{-g}\nabla^b(\mathscr{L}_{\mathcal{F}}F_{ab})\delta A^a \ ,\end{aligned} \tag{C.14}$$

where in the last step we used the divergence identity for antisymmetric tensors,

$$\sqrt{-g}\nabla_b F^{ab} = \partial_b(\sqrt{-g}F^{ab}) \ . \tag{C.15}$$

The analogous calculation can be carried out for the second term in (C.11),

$$\begin{aligned}\mathscr{L}_{\mathcal{G}}\delta(\mathcal{G}\sqrt{-g}) &= \mathscr{L}_{\mathcal{G}}\delta(F_{ab}{\star}F^{ab}\sqrt{-g}) = \mathscr{L}_{\mathcal{G}}\delta(F_{ab}{\star}F_{cd}g^{ac}g^{bd}\sqrt{-g}) = \\ &= \frac{1}{2}\mathscr{L}_{\mathcal{G}}\delta(F_{ab}F_{ef}g^{ac}g^{bd}\epsilon^{ef}{}_{cd}\sqrt{-g}) =\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\mathscr{L}_{\mathcal{G}}\delta((\partial_a A_b - \partial_b A_a)(\partial_e A_f - \partial_f A_e)\epsilon^{ef}{}_{cd} g^{ac} g^{bd}\sqrt{-g}) = \\
&= \frac{1}{2}\mathscr{L}_{\mathcal{G}}(\partial_a \delta A_b \partial_e A_f + \partial_a A_b \partial_e \delta A_f - \partial_a \delta A_b \partial_f A_e - \\
&- \partial_a A_b \partial_f \delta A_e - \partial_b \delta A_a \partial_e A_f - \partial_b A_a \partial_e \delta A_f + \\
&+ \partial_b \delta A_a \partial_f A_e + \partial_b A_a \partial_f \delta A_e)\epsilon^{ef}{}_{cd} g^{ac} g^{bd}\sqrt{-g} \ .
\end{aligned}
\tag{C.16}
$$

After using the Lebniz rule, we have

$$
\begin{aligned}
\mathscr{L}_{\mathcal{G}}\delta(\mathcal{G}\sqrt{-g}) &= -\frac{1}{2}\partial_a(\mathscr{L}_{\mathcal{G}}\epsilon^{ef}{}_{cd} g^{ac} g^{bd}\sqrt{-g}F_{ef})\delta A_b + \\
&+ \frac{1}{2}\partial_e(\mathscr{L}_{\mathcal{G}}\epsilon^{ef}{}_{cd} g^{ac} g^{bd}\sqrt{-g}F_{ba})\delta A_f + \\
&+ \frac{1}{2}\partial_f(\mathscr{L}_{\mathcal{G}}\epsilon^{ef}{}_{cd} g^{ac} g^{bd}\sqrt{-g}F_{ab})\delta A_e - \\
&- \frac{1}{2}\partial_b(\mathscr{L}_{\mathcal{G}}\epsilon^{ef}{}_{cd} g^{ac} g^{bd}\sqrt{-g}F_{fe})\delta A_a = \\
&= -\partial_a(\mathscr{L}_{\mathcal{G}}\star F^{ab}\sqrt{-g})\delta A_b + \partial_b(\mathscr{L}_{\mathcal{G}}\star F^{ab}\sqrt{-g})\delta A_a - \\
&- \partial_e(\mathscr{L}_{\mathcal{G}}\star F^{ef}\sqrt{-g})\delta A_f + \partial_f(\mathscr{L}_{\mathcal{G}}\star F^{ef}\sqrt{-g})\delta A_e = \\
&= -4\partial_a(\mathscr{L}_{\mathcal{G}}\star F^{ab}\sqrt{-g})\delta A_b = -4\sqrt{-g}\nabla_a(\mathscr{L}_{\mathcal{G}}\star F^{ab})\delta A_b = \\
&= 4\sqrt{-g}\nabla^b(\mathscr{L}_{\mathcal{G}}\star F_{ab})\delta A^a \ .
\end{aligned}
\tag{C.17}
$$

Putting all the terms together, the total equation is

$$
-4\nabla^a(\mathscr{L}_{\mathcal{F}}F_{ab} + \mathscr{L}_{\mathcal{G}}\star F_{ab}) = 0 \ , \tag{C.18}
$$

which in the language of the differential forms becomes,

$$
\delta(\mathscr{L}_{\mathcal{F}}\mathbf{F} + \mathscr{L}_{\mathcal{G}}\star\mathbf{F}) = 0 \ . \tag{C.19}
$$

It can be written more compactly by introducing 2-form $\mathbf{Z}$ (2.7) and taking into account the relation between the exterior derivative and coderivative operators (A.15),

$$
\mathrm{d}\star\mathbf{Z} = 0 \ . \tag{C.20}
$$

So far we considered only the source-free case, now we will add the current-gauge coupling term of the form $A^a J_a$ to the initial NLE Lagrangian. After performing a simple variation of the additional term, the generalised Maxwell's equation with source J^a becomes

$$
-4\nabla^a(\mathscr{L}_{\mathcal{F}}F_{ab} + \mathscr{L}_{\mathcal{G}}\star F_{ab}) + 4\pi J_a = 0 \ . \tag{C.21}
$$

In differential form notation,

$$\mathrm{d}\star\mathbf{Z} = 4\pi\star\mathbf{J}\ . \tag{C.22}$$

In order to derive the boundary term (7.45) needed for the extraction of the conserved quantities, we calculate the variation of classical gravitational action,

$$\mathscr{L}^{(\mathrm{grav})} = \frac{1}{16\pi} R\sqrt{-g}\ . \tag{C.23}$$

The variation consists of three terms:

$$\frac{\delta\mathscr{L}^{(\mathrm{grav})}}{\delta g_{ab}} = \frac{\delta(\sqrt{-g}g^{cd}R_{cd})}{\delta g_{ab}} = \frac{\delta\sqrt{-g}}{\delta g_{ab}}R + \sqrt{-g}\frac{\delta g^{cd}}{\delta g_{ab}}R_{cd} + \sqrt{-g}g^{cd}\frac{\delta R_{cd}}{\delta g_{ab}}\ , \tag{C.24}$$

where the first two terms produce Einstein's equation,

$$\frac{1}{2}\sqrt{-g}g^{ab}R - \sqrt{-g}g^{ac}g^{bd}R_{cd} = -\sqrt{-g}\left(R^{ab} - \frac{1}{2}g^{ab}R\right) = -\sqrt{-g}G^{ab}. \tag{C.25}$$

It remains to show that the last term is a total derivative. The variation of the Ricci tensor can be expressed in terms of variations of Christoffel symbols

$$\begin{aligned}\delta R_{ab} &= \delta(\partial_c\Gamma^c_{ba} - \partial_a\Gamma^c_{bc} + \Gamma^c_{cd}\Gamma^d_{ab} - \Gamma^c_{ad}\Gamma^d_{cb}) = \\ &= \partial_c\delta\Gamma^c_{ba} - \delta\Gamma^c_{ad}\Gamma^d_{cb} + \Gamma^c_{cd}\delta\Gamma^d_{ab} - (\partial_a\delta\Gamma^c_{bc} + \Gamma^c_{ad}\delta\Gamma^d_{cb} - \delta\Gamma^c_{cd}\Gamma^d_{ab}) + \\ &+ \Gamma^d_{ac}\delta\Gamma^c_{bd} - \Gamma^d_{ac}\delta\Gamma^c_{bd} = \nabla_c\delta\Gamma^c_{ab} - \nabla_b\delta\Gamma^c_{ac}\ .\end{aligned} \tag{C.26}$$

To proceed, we need an auxiliary result

$$\begin{aligned}2\delta\Gamma^c_{ab} &= \delta g^{cd}(\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab}) + g^{cd}(\partial_a\delta g_{db} + \partial_b\delta g_{da} - \partial_d\delta g_{ab}) = \\ &= g^{cd}(\partial_a\delta g_{db} + \partial_b\delta g_{da} - \partial_d\delta g_{ab} - 2\delta g_{de}\Gamma^e_{ab}) = \\ &= g^{cd}(\partial_a\delta g_{db} - \Gamma^e_{ab}\delta g_{de} - \Gamma^e_{ad}\delta g_{eb} + \partial_b\delta g_{da} - \Gamma^e_{bd}\delta g_{ea} - \Gamma^e_{ba}\delta g_{ed} - \\ &- \partial_d\delta g_{ab} + \Gamma^e_{da}\delta g_{eb} + \Gamma^e_{bd}\delta g_{ea}) = \\ &= g^{cd}(\nabla_a\delta g_{db} + \nabla_b\delta g_{da} - \nabla_d\delta g_{ab})\ .\end{aligned} \tag{C.27}$$

Notice that even though the Christoffel symbol itself is not a tensor, its variation is. Finally, we have

$$g^{ab}\delta R_{ab} = \nabla^c\nabla^b\delta g_{cb} - g^{ab}\nabla^c\nabla_c\delta g_{ab} = \nabla^c(\nabla^b\delta g_{cb} - g^{ab}\nabla_c\delta g_{ab}) = \nabla^c v_c\ . \tag{C.28}$$

Appendix D

Stokes' theorem on Lorentzian manifolds

Let $\mathscr{M}$ be an orientable smooth m-manifold with boundary $\partial\mathscr{M}$ and an inclusion operator $\imath : \partial\mathscr{M} \hookrightarrow \mathscr{M}$. Orientation on $\mathscr{M}$ is determined by the choice of a nowhere vanishing volume form $\boldsymbol{\epsilon}$. The induced orientation on the boundary is defined via inclusion as $\hat{\boldsymbol{\epsilon}} = \imath^*(i_N\boldsymbol{\epsilon})$, where N^a is the outward pointing nonvanishing vector field on $\partial\mathscr{M}$. Stokes' theorem [119] states that the integral of a smooth, compactly supported $(m-1)$-form $\boldsymbol{\alpha}$ over the boundary is equal to the integral of its exterior derivative over the whole $\mathscr{M}$,

$$\int_{(\mathscr{M},\boldsymbol{\epsilon})} \mathrm{d}\boldsymbol{\alpha} = \int_{(\partial\mathscr{M},\hat{\boldsymbol{\epsilon}})} \imath^*\boldsymbol{\alpha} \ . \tag{D.1}$$

In this form, Stokes' theorem makes no reference to any additional structure on the manifold, such as metric or connection. However, in the case of (pseudo)-Riemannian manifolds, it admits a few calculationally practical results. Suppose that $\mathscr{M}$ is a smooth manifold of Lorentzian type and $\mathscr{N} \subseteq \mathscr{M}$ its embedded compact m-dimensional submanifold with boundary $\partial\mathscr{N}$. The inclusion operator $\jmath : \partial\mathscr{N} \hookrightarrow \mathscr{N}$, together with an outward pointing, nonvanishing vector field n^a, defines the induced orentation on $\partial\mathscr{N}$ as $\hat{\boldsymbol{\epsilon}} = \jmath^*(i_n\boldsymbol{\epsilon})$. Using Stokes' theorem, we have

$$\int_{(\mathscr{N},\boldsymbol{\epsilon})} (\nabla_a v^a)\,\boldsymbol{\epsilon} = \int_{(\mathscr{N},\boldsymbol{\epsilon})} \mathrm{d}i_v\boldsymbol{\epsilon} = \int_{(\partial\mathscr{N},\hat{\boldsymbol{\epsilon}})} \jmath^*(i_v\boldsymbol{\epsilon}) \ , \tag{D.2}$$

for any smooth vector field v^a on $\mathscr{N}$.

We will apply the theorem to a concrete scenario in which the boundary of $\mathscr{N}$ consists of two spacelike hypersurfaces Σ and Σ', a timelike hypersurface S and a null hypersurface H, representing a portion of a black hole horizon,

$$\partial\mathscr{N} = \Sigma \cup \Sigma' \cup S \cup H \ ,$$

illustrated in Figure[1] D.1.

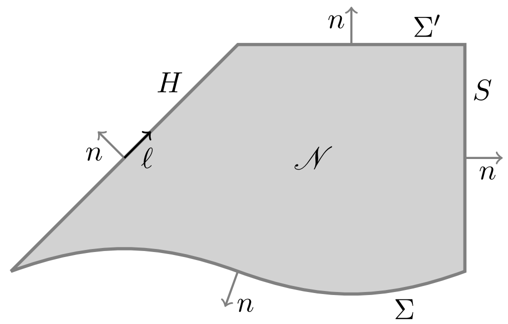


Figure D.1: A schematic representation of submanifold $\mathscr{N}$. Its boundary consists of four parts (spacelike hypersurfaces Σ and Σ', timelike hypersurface S, null hypersurface H) and n^a is the corresponding outward pointing vector field [14].

The decomposition of the volume form $\boldsymbol{\epsilon}$ on the non-null and null parts of the boundary, respectively, is performed as follows:

(i) We assume that the normalization of n^a is given by $n^a n_a = \pm 1$. Adopting the convention from [187], we introduce an auxiliary vector field $\tilde{n}^a := (n^b n_b)\, n^a$, so that $\tilde{n}^a$ is outward oriented for spacelike n^a and inward for timelike n^a. Then $\mathbf{n} \wedge i_n \boldsymbol{\epsilon} = f\boldsymbol{\epsilon}$, for some function f, and contraction with n^a implies the decomposition

$$\boldsymbol{\epsilon} = (n^a n_a)\, \mathbf{n} \wedge i_n \boldsymbol{\epsilon} = \tilde{\mathbf{n}} \wedge i_n \boldsymbol{\epsilon} \ . \tag{D.3}$$

(ii) The null part of the boundary is generated by the future directed vector field ℓ^a, while the future directed null vector field n^a plays the role of the outward pointing vector field on H. If we define the normalisation by $n^a l_a = -1$, we have $\ell \wedge i_n \boldsymbol{\epsilon} = f\boldsymbol{\epsilon}$ for some function f, which finally leads to

$$\boldsymbol{\epsilon} = -\boldsymbol{\ell} \wedge i_n \boldsymbol{\epsilon} \ . \tag{D.4}$$

These decompositions imply

$$\jmath^*(i_v \boldsymbol{\epsilon}) = \begin{cases} (\tilde{n}_a v^a)\hat{\boldsymbol{\epsilon}} & \text{on non-null part of } \partial\mathscr{N} \\ -(\ell_a v^a)\hat{\boldsymbol{\epsilon}} & \text{on null part of } \partial\mathscr{N} \end{cases} \tag{D.5}$$

and the boundary integral can be split as

$$\int_{\mathscr{N}} (\nabla_a v^a)\, \boldsymbol{\epsilon} = \int_{\Sigma} (\tilde{n}_a v^a) \hat{\boldsymbol{\epsilon}} + \int_{\Sigma'} (\tilde{n}_a v^a) \hat{\boldsymbol{\epsilon}} + \int_{S} (\tilde{n}_a v^a) \hat{\boldsymbol{\epsilon}} + \int_{H} (-\ell_a v^a) \hat{\boldsymbol{\epsilon}} \;, \tag{D.6}$$

where the orientation of each component of the boundary $\partial\mathscr{N}$ is fixed by the induced Stokes' orientation $\hat{\boldsymbol{\epsilon}}$. The vector field ℓ^a is not uniquely determined since it can be rescaled, $\ell^a \to \ell'^a = \lambda \ell^a$, for some positive real function λ. In order to preserve the normalisation, vector n^a has to be redefined as $n'^a = \lambda^{-1} n^a$ and $\hat{\boldsymbol{\epsilon}}' = \jmath^*(i_{n'}\boldsymbol{\epsilon})$. Nevertheless, the integrand above is invariant to these ambiguities because $\ell_a v^a \, \hat{\boldsymbol{\epsilon}} = \ell'_a v^a \, \hat{\boldsymbol{\epsilon}}'$

Curriculum vitae

Ana Bokulić was born on the 27th of May 1995 in Zagreb, Croatia. In 2014, she enrolled in a research-oriented study of physics at the Faculty of Science, University of Zagreb. In 2019 she defended her master's thesis *Ergoregions* under the supervision of Asst. Prof. Dr. Sc. Ivica Smolić with a total grade Magna cum laude. In the same year, she enrolled in a doctoral study programme at the Faculty of Science, University of Zagreb. Since 2019, she has been employed as a teaching and research assistant in the theoretical physics division of particles and fields at the Faculty of Science, University of Zagreb. Her teaching experience includes holding several exercise classes: General Physics I and II (2019-2020), General Relativity (2020-2023), Advanced Gravity (2020-2023), Differential Geometry (2020-), Advanced Physics Lab (2021-2022) and Classical Electrodynamics (2022-).

Conferences and schools

2021

The 1st Electronic Conference on Universe (Online), talk: *Immersing the Schwarzschild black hole in test nonlinear electromagnetic fields*

2022

Strings, Cosmology, & Gravity Student Conference 2022 (Amsterdam, Netherlands), poster: *Nonlinear electromagnetic fields in strictly stationary spacetimes*

Quantum aspects of Spacetime and Gravity (Zagreb, Croatia), invited talk: *No-go theorems for gravitating nonlinear electromagnetic fields*

XV Black Holes Workshop (Lisbon, Portugal), talk: *Can nonlinear electromagnetic fields regularize black hole singularities?*

2023

SIGRAV International School 2023 - School of Applied Quantum Gravity (Online participation)

BritGrav23 (Southampton, UK), talk: *Can nonlinear electromagnetic fields cure*

black hole singularities?
13th Central European Relativity Seminar (Stockholm, Sweden), talk: *Can nonlinear electromagnetic fields cure black hole singularities?*

List of publications

- A. Bokulić and I. Smolić. Schwarzschild spacetime immersed in test nonlinear electromagnetic fields. *Class. Quantum Grav.*, **37**(5):055004, 2020.
- A. Bokulić, T. Jurić, and I. Smolić. Black hole thermodynamics in the presence of nonlinear electromagnetic fields. *Phys. Rev. D*, **103**(12):124059, 2021.
- A. Bokulić, T. Jurić, and I. Smolić. Nonlinear electromagnetic fields in strictly stationary spacetimes. *Phys. Rev. D*, **105**(2):024067, 2022.
- A. Bokulić, T. Jurić, and I. Smolić. Constraints on singularity resolution by nonlinear electrodynamics. *Phys. Rev. D*, **106**(6):064020, 2022.
- A. Bokulić and I. Smolić. Generalizations and challenges for the spacetime block-diagonalization. *Class. Quantum Grav.*, **40**(16):165010, 2023.

Bibliography

[1] M. Aaboud et al. Evidence for light-by-light scattering in heavy-ion collisions with the ATLAS detector at the LHC. *Nature Phys.*, **13**(9):852–858, 2017.

[2] A. Ali and K. Saifullah. Asymptotic magnetically charged non-singular black hole and its thermodynamics. *Phys. Lett.*, **B792**:276–283, 2019.

[3] M. T. Anderson. On Stationary Vacuum Solutions to the Einstein Equations. *Annales Henri Poincare*, **1**:977, 2000.

[4] E. Ayón-Beato and A. García. Regular Black Hole in General Relativity Coupled to Nonlinear Electrodynamics. *Phys. Rev. Lett.*, **80**:5056–5059, 1998.

[5] E. Ayón-Beato and A. García. New regular black hole solution from nonlinear electrodynamics. *Phys. Lett.*, **B464**:25, 1999.

[6] E. Ayón-Beato and A. García. The Bardeen model as a nonlinear magnetic monopole. *Phys. Lett.*, **B493**:149–152, 2000.

[7] E. Babichev, C. Charmousis, A. Cisterna, and M. Hassaine. Regular black holes via the Kerr-Schild construction in DHOST theories. *JCAP*, **06**:049, 2020.

[8] I. Bandos, D. Lechner, K. Sorokin, and P. K. Townsend. A non-linear duality-invariant conformal extension of Maxwell's equations. *Phys. Rev. D*, **102**:121703, 2020.

[9] J. M. Bardeen. Non-singular General Relativistic Gravitational Collapse. In *Proceeding of the International Conference GR5*, pages 174–175. Tbilisi, 1968.

[10] J. M. Bardeen, B. Carter, and S. W. Hawking. The Four Laws of Black hole Mechanics. *Commun. Math. Phys.*, **31**:161–170, 1973.

[11] I. Barjašić, L. Gulin, and I. Smolić. Nonlinear electromagnetic fields and symmetries. *Phys. Rev.*, D **95**(12):124037, 2017.

[12] J. D. Bekenstein. Black holes and entropy. *Phys. Rev. D*, **7**:2333–2346, 1973.

[13] J. D. Bekenstein. Generalized second law of thermodynamics in black hole physics. *Phys. Rev. D*, **9**:3292–3300, 1974.

[14] A. Bokulić, T. Jurić, and I. Smolić. Black hole thermodynamics in the presence of nonlinear electromagnetic fields. *Phys. Rev. D*, **103**(12):124059, 2021.

[15] A. Bokulić, T. Jurić, and I. Smolić. Nonlinear electromagnetic fields in strictly stationary spacetimes. *Phys. Rev. D*, **105**(2):024067, 2022.

[16] A. Bokulić and I. Smolić. Schwarzschild spacetime immersed in test nonlinear electromagnetic fields. *Class. Quantum Grav.*, **37**(5):055004, 2020.

[17] A. Bokulić and I. Smolić. Generalizations and challenges for the spacetime block-diagonalization. *Class. Quantum Grav.*, **40**(16):165010, 2023.

[18] A. Bokulić, I. Smolić, and T. Jurić. Constraints on singularity resolution by nonlinear electrodynamics. *Phys. Rev. D*, **106**(6):064020, 2022.

[19] L. Bonora, M. Cvitan, P. Dominis Prester, S. Pallua, and I. Smolić. Gravitational Chern–Simons Lagrangians and black hole entropy. *JHEP*, **07**:085, 2011.

[20] M. Born. On the Quantum Theory of the Electromagnetic Field. *Proc. R. Soc. A*, **143**:410–437, 1934.

[21] M. Born and L. Infeld. Foundations of the New Field Theory. *Proc. R. Soc. A*, **144**:425–451, 1934.

[22] P. Bosso, S. Das, and V. Todorinov. Quantum field theory with the generalized uncertainty principle I: Scalar electrodynamics. *Annals Phys.*, **422**:168319, 2020.

[23] W. Boucher, G. W. Gibbons, and Gary T. Horowitz. Uniqueness theorem for anti–de Sitter spacetime. *Phys. Rev. D*, **30**:2447–2451, 1984.

[24] N. Bretón. Smarr's formula for black holes with non-linear electrodynamics. *Gen. Rel. Grav.*, **37**:643–650, 2005.

[25] N. Bretón, R. Lazkoz, and A. Montiel. Observational constraints on electromagnetic Born–Infeld cosmology. *JCAP*, **1210**:013, 2012.

[26] D. R. Brill and J. B. Hartle. Method of the self-consistent field in general relativity and its application to the gravitational geon. *Phys. Rev.*, **135**:B271–B278, 1964.

[27] K. A. Bronnikov. Regular magnetic black holes and monopoles from nonlinear electrodynamics. *Phys. Rev. D*, **63**:044005, 2001.

[28] A. Burinskii and S. R. Hilderbrandt. New type of regular black holes and particlelike solutions from nonlinear electrodynamics. *Phys. Rev. D*, **65**:104017, 2002.

[29] M. Cadoni, M. Oi, and A. P. Sanna. Effective models of nonsingular quantum black holes. *Phys. Rev. D*, **106**:024030, 2022.

[30] R.-G. Cai, D.-W. Pang, and A. Wang. Born–Infeld black holes in (A)dS spaces. *Phys. Rev. D*, **70**:124034, 2004.

[31] C. S. Camara, M. R. de Garcia Maia, J. C. Carvalho, and J. A. S. Lima. Nonsingular FRW cosmology and nonlinear electrodynamics. *Phys. Rev. D*, **69**:123504, 2004.

[32] L.-M. Cao, Y. Peng, and J. Xu. Lichnerowicz-Type Theorems for Self-gravitating Systems with Nonlinear Electromagnetic Fields. *Phys. Rev. D*, **90**(2):024046, 2014.

[33] B. Carter. Killing horizons and orthogonally transitive groups in space-time. *J. Math. Phys.*, **10**:70–81, 1969.

[34] B. Carter. The Commutation Property of a Stationary, Axisymmetric System. *Commun. Math. Phys.*, **17**:233–238, 1970.

[35] B. Carter. Black Hole Equilibrium States. In *Black Holes*. Gordon and Breach, New York, 1973.

[36] B. Carter. Republication of: Black hole equilibrium states Part II. General theory of stationary black hole states. *Gen. Relativ. Gravit.*, **42**:653–744, 2010.

[37] M. Casals, A. Fabbri, C. Martínez, and J. Zanelli. Quantum dress for a naked singularity. *Phys. Lett. B*, **760**:244–248, 2016.

[38] M. Casals, A. Fabbri, C. Martínez, and J. Zanelli. Quantum Backreaction on Three-Dimensional Black Holes and Naked Singularities. *Phys. Rev. Lett.*, **118**(13):131102, 2017.

[39] N. Chamel and P. Haensel. Physics of Neutron Star Crusts. *Living Rev. Rel.*, **11**:10, 2008.

[40] P. T. Chruściel, J. L. Costa, and M. Heusler. Stationary Black Holes: Uniqueness and Beyond. *Living Rev.Rel.*, 15:7, 2012.

[41] M. Ćirić Dimitrijević, N. Konjik, and A. Samsarov. Noncommutative scalar quasinormal modes of the Reissner–Nordström black hole. *Class. Quantum Grav.*, **35**(17):175005, 2018.

[42] B. Coll. Sur l'invariance du champ électromagnétique dans un espace-temps d'Einstein-Maxwell admettant un groupe d'isométries. *C. R. Acad. Sci. (Paris) A*, **280**:1773–1776, 1975.

[43] C. V. Costa, D. G. Gitman, and A. E. Shabad. Finite field-energy of a point charge in QED. *Phys. Scr.*, **90**(7):074012, 2015.

[44] K. Crowther and S. De Haro. Four Attitudes Towards Singularities in the Search for a Theory of Quantum Gravity. 2021. (arXiv preprint gr-qc/2112.08531).

[45] E. Curiel. A Primer on Energy Conditions. *Einstein Stud.*, **13**:43–104, 2017.

[46] M. Cvitan, P. Dominis Prester, and I. Smolić. Does three dimensional electromagnetic field inherit the spacetime symmetries? *Class. Quantum Grav.*, **33**:077001, 2016.

[47] S. Dain. Positive energy theorems in General Relativity. In *Springer Handbook of Spacetime*, pages 363–380. 2014.

[48] V. A. De Lorenci, R. Klippert, M. Novello, and J. M. Salim. Nonlinear electrodynamics and FRW cosmology. *Phys. Rev.D*, **65**, 2002.

[49] H. Delaporte, A. Eichhorn, and A. Held. Parameterizations of black-hole spacetimes beyond circularity. *Class. Quantum Grav.*, **39**(13):134002, 2022.

[50] V. I. Denisov, E. E. Dolgaya, V. A. Sokolov, and I. P. Denisova. Conformal invariant vacuum nonlinear electrodynamics. *Phys. Rev. D*, **96**(3):036008, 2017.

[51] T. K. Dey. Born–Infeld black holes in the presence of a cosmological constant. *Phys. Lett. B*, 595:484–490, 2004.

[52] G. V. Dunne. *Heisenberg–Euler effective Lagrangians: Basics and extensions*, pages 445–522. World Scientific, 2004.

[53] I. Dymnikova. Regular electrically charged vacuum structures with de Sitter center in Nonlinear Electrodynamics coupled to General Relativity. *Class. Quantum Grav.*, **21**:4417–4429, 2004.

[54] A. Eichhorn and A. Held. From a locality-principle for new physics to image features of regular spinning black holes with disks. *JCAP*, **05**:073, 2021.

[55] A. Eichhorn and A. Held. Image features of spinning regular black holes based on a locality principle. *Eur. Phys. J. C*, **81**(10):933, 2021.

[56] G. F. R. Ellis and B. G. Schmidt. Singular Space-Times. *Gen. Rel. Grav*, **8**:915–953, 1977.

[57] J. Ellis, N. E. Mavromatos, and T. You. Light-by-Light Scattering Constraint on Born–Infeld Theory. *Phys. Rev. Lett.*, **118**(26):261802, 2017.

[58] F. J. Ernst. Linear and toroidal geons. *Rev. Mod. Phys.*, **29**:496, 1957.

[59] Z.-Y. Fan and X. Wang. Construction of Regular Black Holes in General Relativity. *Phys. Rev. D*, **94**(12):124027, 2016.

[60] S. Fernando and D. Krug. Charged black hole solutions in Einstein–Born–Infeld gravity with a cosmological constant. *Gen. Rel. Grav.*, **35**:129–137, 2003.

[61] D. Flores-Alfonso, B. A. González-Morales, R. Linares, and M. Maceda. Black holes and gravitational waves sourced by non-linear duality rotation-invariant conformal electromagnetic matter. *Phys. Lett. B*, **812**:136011, 2021.

[62] E. S. Fradkin and A. A. Tseytlin. Nonlinear Electrodynamics from Quantized Strings. *Phys. Lett.*, **163B**:123–130, 1985.

[63] S. Gao and R. M. Wald. "Physical process version" of the first law and the generalized second law for charged and rotating black holes. *Phys. Rev. D*, **64**:084020, 2001.

[64] D. A. García, I. H. Salazar, and J. F. Plebański. Type-D solutions of the Einstein and Born–Infeld nonlinear-electrodynamics equations. *Nuovo Cimento B Serie*, **84**:65–90, 1984.

[65] R. García-Salcedo and N. Bretón. Singularity-free Bianchi spaces with non-linear electrodynamics. *Class. Quantum Grav.*, **22**:4783–4801, 2005.

[66] R. Geroch. What is a singularity in general relativity? *Ann. Phys.*, **48**:562–540, 1968.

[67] G. W. Gibbons and K. Maeda. Black Holes and Membranes in Higher Dimensional Theories with Dilaton Fields. *Nucl. Phys. B*, **298**:741–775, 1988.

[68] G. W. Gibbons and D. A. Rasheed. Electric-magnetic duality rotations in non-linear electrodynamics. *Nucl. Phys. B*, **454**:185–206, 1995.

[69] V. L. Ginzburg and L. M. Ozernoy. On gravitational collapse of magnetic stars. *JETP*, **47**:1030–1040, 1964. [Sov. Phys. JETP **20** (1965) 689–696].

[70] H. A. González, M. Hassaïne, and C. Martínez. Thermodynamics of charged black holes with a nonlinear electrodynamics source. *Phys. Rev. D*, **80**:104008, 2009.

[71] L. Gulin and I. Smolić. Generalizations of the Smarr formula for black holes with nonlinear electromagnetic fields. *Class. Quantum Grav.*, **35**(2):025015, 2018.

[72] S. Gunasekaran, D. Kubizňák, and R. B. Mann. Extended phase space thermodynamics for charged and rotating black holes and Born–Infeld vacuum polarization. *JHEP*, **11**:110, 2012.

[73] K. S. Gupta, T. Jurić, A. Samsarov, and I. Smolić. Noncommutativity and the Weak Cosmic Censorship. *JHEP*, **10**:170, 2019.

[74] M. Gürses. Some solutions of stationary, axially-symmetric gravitational field equations. *J. Math. Phys.*, **18**:2356–2359, 1977.

[75] T. Hale, D Kubizňák, O. Svítek, and T. Tahamtan. Solutions and basic properties of regularized Maxwell theory. *Phys. Rev. D*, **107**:124031, 2023.

[76] E. Harikumar, T. Jurić, and S. Meljanac. Electrodynamics on κ-Minkowski space-time. *Phys. Rev. D*, **84**:085020, 2011.

[77] J. B. Hartle and S. W. Hawking. Solutions of the Einstein–Maxwell equations with many black holes. *Commun. Math. Phys.*, **26**:87–101, 1972.

[78] M. Hassaïne and C. Martínez. Higher-dimensional black holes with a conformally invariant Maxwell source. *Phys. Rev. D*, **75**:027502, 2007.

[79] M. Hassaïne and C. Martínez. Higher-dimensional charged black holes solutions with a nonlinear electrodynamics source. *Class. Quantum Grav.*, **25**:195023, 2008.

[80] S. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time.* Cambridge University Press, Cambridge England New York, 1973.

[81] S. W. Hawking. Singularities in the Universe. *Phys. Rev. Lett.*, **17**:444, 1966.

[82] S. W. Hawking. Black holes in general relativity. *Commun. Math. Phys.*, **25**:152–166, 1972.

[83] S. W. Hawking. Particle Creation by Black Holes. *Commun. Math. Phys.*, **43**:199–220, 1975. [Erratum: Commun.Math.Phys. 46, 206 (1976)].

[84] S. W. Hawking and R. Penrose. The singularities of gravitational collapse and cosmology. *Proc. Roy. Soc. Lond. A.*, **314**:529–548, 1970.

[85] A. He, J. Tao, P. Wang, Y. Xue, and L. Zhang. Effects of Born-Infeld electrodynamics on black hole shadows. *Eur. Phys. J. C*, **82**:683, 2022.

[86] W. Heisenberg and H. Euler. Folgerungen aus der Diracschen Theorie des Positrons. *Z. Phys.*, **98**:714–732, 1936.

[87] S. H. Hendi, B. E. Panah, and S. Panahiyan. Einstein-Born-Infeld-Massive Gravity: adS-Black Hole Solutions and their Thermodynamical properties. *JHEP*, **11**:157, 2015.

[88] C. A. R. Herdeiro and J. M. S. Oliveira. On the inexistence of solitons in Einstein–Maxwell-scalar models. *Class. Quantum Grav.*, **36**(10):105015, 2019.

[89] C. A. R. Herdeiro and J. M. S. Oliveira. On the inexistence of self-gravitating solitons in generalised axion electrodynamics. *Phys. Lett. B*, **800**:135076, 2020.

[90] C. A. R. Herdeiro, I. Perapechka, E. Radu, and Ya. Shnir. Asymptotically flat spinning scalar, Dirac and Proca stars. *Phys. Lett. B*, **797**:134845, 2019.

[91] C. A. R. Herdeiro, A. M. Pombo, and E. Radu. Asymptotically flat scalar, Dirac and Proca stars: discrete vs. continuous families of solutions. *Phys. Lett. B*, **773**:654–662, 2017.

[92] C. A. R. Herdeiro and E. Radu. Kerr black holes with scalar hair. *Phys. Rev. Lett.*, **112**:221101, 2014.

[93] C. A. R. Herdeiro and E. Radu. Black hole scalarization from the breakdown of scale invariance. *Phys. Rev. D*, **99**(8):084039, 2019.

[94] S. Hervik, M. Ortaggio, and V. Pravda. Universal electromagnetic fields. *Class. Quant. Grav.*, **35**(17):175017, 2018.

[95] M. Heusler. A mass bound for spherically symmetric black hole space-times. *Class. Quantum Grav.*, **12**:779–790, 1995.

[96] M. Heusler. *Black Hole Uniqueness Theorems*. Cambridge University Press, Cambridge New York, 1996.

[97] S. Hofmann and M. Schneider. Classical versus quantum completeness. *Phys. Rev. D*, **91**(12):125028, 2015.

[98] S. Hollands and A. Ishibashi. On the 'Stationary Implies Axisymmetric' Theorem for Extremal Black Holes in Higher Dimensions. *Commun. Math. Phys.*, **291**:403–441, 2009.

[99] S. Hollands and A. Ishibashi. Black hole uniqueness theorems in higher dimensional spacetimes. *Class. Quantum Grav.*, **29**:163001, 2012.

[100] S. Hollands, A. Ishibashi, and H. S. Reall. A stationary black hole must be axisymmetric in effective field theory. *Commun. Math. Phys.*, **401**:2757–2791, 2023.

[101] G. T. Horowitz and D. Marolf. Quantum probes of space-time singularities. *Phys. Rev. D*, **52**:5670–5675, 1995.

[102] SQ. Hu, XM. Kuang, and Y. C. Ong. A Note on Smarr Relation and Coupling Constants. *Gen. Relativ. Gravit.*, **51**(5):55, 2019.

[103] W. Israel. Event horizons in static electrovac space-times. *Commun. math. Phys.*, **8**:245–260, 1968.

[104] V. Iyer and R. M. Wald. Some properties of Noether charge and a proposal for dynamical black hole entropy. *Phys. Rev. D*, **50**:846–864, 1994.

[105] T. Jacobson. When is g(tt) g(rr) = -1? *Class. Quantum Grav.*, **24**:5717–5719, 2007.

[106] T. Jurić. Quantum space and quantum completeness. *JHEP*, **05**:007, 2018.

[107] D. Kastor, S. Ray, and J. Traschen. Enthalpy and the Mechanics of AdS Black Holes. *Class. Quantum Grav.*, **26**:195011, 2009.

[108] B. S. Kay and R. M. Wald. Theorems on the Uniqueness and Thermal Properties of Stationary, Nonsingular, Quasifree States on Space-Times with a Bifurcate Killing Horizon. *Phys.Rept.*, **207**:49–136, 1991.

[109] J. Keir. Stability, Instability, Canonical Energy and Charged Black Holes. *Class. Quantum Grav.*, **31**(3):035014, 2014.

[110] A. Komar. Positive-Definite Energy Density and Global Consequences for General Relativity. *Phys. Rev.*, **129**(4):1873, 1963.

[111] B. P. Kosyakov. Nonlinear electrodynamics with the maximum allowable symmetries. *Phys. Lett. B*, **810**:135840, 2020.

[112] S. I. Kruglov. Black hole as a magnetic monopole within exponential nonlinear electrodynamics. *Ann. Phys.*, **378**:59–70, 2017.

[113] S. I. Kruglov. Magnetically charged black hole in framework of nonlinear electrodynamics model. *Int. J. Mod. Phys. A*, **33**:1850023, 2018.

[114] S. I. Kruglov. On generalized ModMax model of nonlinear electrodynamics. *Phys. Lett.*, B822:136633, 2021.

[115] S. I. Kruglov. Regular model of magnetized black hole with rational nonlinear electrodynamics. *Int. J. Mod. Phys. A*, **36**:2150158, 2021.

[116] D. Kubizňák, R. B. Mann, and M. Teo. Black hole chemistry: thermodynamics with Lambda. *Class. Quantum Grav.*, **34**(6):063001, 2017.

[117] W. Kundt and M. Trümper. Orthogonal decomposition of axi-symmetric stationary spacetimes. *Z. Phys.*, **192**:419–422, 1966.

[118] J. Lee and R. M. Wald. Local symmetries and constraints. *J. Math. Phys.*, **31**:725–743, 1990.

[119] J. M. Lee. *Introduction to Smooth Manifolds*. Springer, New York, 2003.

[120] A. Lichnerowicz. *Théories Relativistes de la Gravitation et de l'Électromagnétisme*. Masson et Cie, Paris, 1955.

[121] M. S. Ma. Magnetically charged regular black hole in a model of nonlinear electrodynamics. *Ann. Phys.*, **362**:529–537, 2015.

[122] D. Magos and N. Bretón. Thermodynamics of the Euler–Heisenberg-AdS black hole. *Phys. Rev. D*, **102**(8):084011, 2020.

[123] S. D. Majumdar. A class of exact solutions of Einstein's field equations. *Phys. Rev.*, **72**:390–398, 1947.

[124] A. K. M. Masood-ul Alam and W. Yu. Uniqueness of de Sitter and Schwarzschild–de Sitter spacetimes. *Comm. Analys. Geom.*, **23**:377–387, 2015.

[125] J. Matyjasek, D. Tryniecki, and M. Klimek. Regular black holes in an asymptotically de Sitter universe. *Mod. Phys.Lett. A*, **23**:3377–3392, 2009.

[126] P. O. Mazur. Proof of uniqueness of the Kerr-Newman black hole solution. *J. Phys. A: Math. Gen*, **15**(3173), 1982.

[127] H. Michalski and J. Wainwright. Killing vector fields and the Einstein-Maxwell field equations in General relativity. *Gen. Relativ. Gravit.*, **6**:289–318, 1975.

[128] O. Mišković and R. Olea. Conserved charges for black holes in Einstein–Gauss–Bonnet gravity coupled to nonlinear electrodynamics in AdS space. *Phys. Rev. D*, **83**:024011, 2011.

[129] O. Mišković and R. Olea. Quantum Statistical Relation for black holes in nonlinear electrodynamics coupled to Einstein–Gauss–Bonnet AdS gravity. *Phys. Rev. D*, **83**:064017, 2011.

[130] H. J. Mosquera Cuesta, G. Lambiase, and J. P. Pereira. Probing nonlinear electrodynamics in slowly rotating spacetimes through neutrino astrophysics. *Phys. Rev. D*, **95**(2):025011, 2017.

[131] V. F. Mukhanov and S. Winitzki. *Introduction to Quantum Effects in Gravity.* Cambridge University Press, Cambridge, 2007.

[132] R. C. Myers. Higher-dimensional black holes in compactified space-times. *Phys. Rev. D*, **35**:455, 1987.

[133] P. Niau Akmansoy and L. G. Medeiros. Constraining nonlinear corrections to Maxwell electrodynamics using $\gamma\gamma$ scattering. *Phys. Rev.*, D **99**(11):115005, 2019.

[134] S. Nojiri and S. D. Odintsov. Regular multihorizon black holes in modified gravity with nonlinear electrodynamics. *Phys. Rev. D*, **96**(10):104008, 2017.

[135] M. Novello, E. Goulart, J. M. Salim, and S. E. Perez Bergliaffa. Cosmological Effects of Nonlinear Electrodynamics. *Class. Quantum Gravity*, **24**:3021, 2007.

[136] M. Ortaggio. Asymptotic behavior of Maxwell fields in higher dimensions. *Phys. Rev. D*, **90**(12):124020, 2014.

[137] M. Ortaggio and V. Pravda. Electromagnetic fields with vanishing scalar invariants. *Class. Quantum Grav.*, **33**(11):115010, 2016.

[138] M. Ortaggio and V. Pravda. Electromagnetic fields with vanishing quantum corrections. *Phys. Lett. B*, **779**:393–395, 2018.

[139] A. Övgün, G. Leon, J. Magaña, and K. Jusufi. Falsifying cosmological models based on a non-linear electrodynamics. *Eur. Phys. J. C*, **78**:462, 2018.

[140] A. Papapetrou. A Static Solution of the Equations of the Gravitational Field for an Arbitrary Charge Distribution. *Proc. Roy. Irish Acad. A*, **51**:191–204, 1947.

[141] A. Papapetrou. Champs gravitationnels stationnaires à symétrie axiale. *Ann. Inst. H. Poincaré Phys. Theor.*, **4**(2):83–105, 1966.

[142] T. Parker and C. H. Taubes. On Witten's Proof of the Positive Energy Theorem. *Commun. Math. Phys.*, **84**:223, 1982.

[143] R. Penrose. Gravitational Collapse and Space-Time Singularities. *Phys. Rev. Lett.*, **14**:57, 1965.

[144] R. Penrose and W. Rindler. *Spinors and space-time, Volume 1.* Cambridge University Press, Cambridge Cambridgeshire New York, 1986.

[145] G. P. Perry and F. I. Cooperstock. Stability of gravitational and electromagnetic geons. *Class. Quantum Grav.*, **16**(6):1889–1916, 1999.

[146] J. Plebański. Lectures on Non-linear Electrodynamics. 1970. Nordita.

[147] E. Poisson. *A Relativist's Toolkit.* Cambridge University Press, Cambridge, 2004.

[148] A.Y. Potekhin. The physics of neutron stars. *Phys.-Uspekhi*, **53**(12):1235–1256, 2010.

[149] K. Prabhu. The First Law of Black Hole Mechanics for Fields with Internal Gauge Freedom. *Class. Quantum Grav.*, **34**(3):035011, 2017.

[150] I. Rácz and R. M. Wald. Global extensions of space-times describing asymptotic final states of black holes. *Class. Quantum Grav.*, **13**:539–553, 1996.

[151] D. A. Rasheed. Non-Linear Electrodynamics: Zeroth and First Laws of Black Hole Mechanics. 1997. (unpublished arXiv preprint hep-th/9702087).

[152] J. R. Ray and E. L. Thompson. Spacetime symmetries and the complexion of the electromagnetic field. *J. Math. Phys.*, **16**(2):345–346, 1975.

[153] S. A. Ridgway and E. J. Weinberg. Are All Static Black Hole Solutions Spherically Symmetric? *Gen. Rel. Grav.*, **27**:1017–1021, 1995.

[154] S. A. Ridgway and E. J. Weinberg. Static black hole solutions without rotational symmetry. *Phys. Rev. D*, **52**:3440–3456, 1995.

[155] Á. Rincón, E. Contreras, P. Bargueño, B. Koch, G. Panotopoulos, and A. Hernández-Arboleda. Scale dependent three-dimensional charged black holes in linear and non-linear electrodynamics. *Eur. Phys. J. C*, **77**(7):494, 2017.

[156] P. Ripoche and J. Heyl. QED effects are negligible for neutron-star spin-down. *Phys. Rev.*, D **99**(8):083004, 2019.

[157] M. E. Rodrigues, J. C. Fabris, E. L. B. Junior, and G. T. Marques. Generalisation for regular black holes on general relativity to $f(R)$ gravity. *Eur. Phys. J. C*, **76**(5):250, 2016.

[158] M. E. Rodrigues, E. L. B. Junior, G. T. Marques, and V. T. Zanchin. Regular black holes in $f(R)$ gravity coupled to nonlinear electrodynamics. *Phys. Rev. D*, **94**(2):024062, 2016. [Addendum: Phys.Rev.D 94, 049904 (2016)].

[159] R. Ruffini, Y.-B. Wu, and S.-S. Xue. Einstein–Euler–Heisenberg Theory and charged black holes. *Phys. Rev. D*, **88**:085004, 2013.

[160] I. H. Salazar, A. García, and J. Plebański. Duality rotations and type D solutions to Einstein equations with nonlinear electromagnetic sources. *J. Math. Phys.*, **28**:2171–2181, 1987.

[161] R. Schoen and S.-T. Yau. On the Proof of the Positive Mass Conjecture in General Relativity. *Commun. Math. Phys.*, **65**:45–76, 1979.

[162] R. Schoen and S.-T. Yau. Positivity of the Total Mass of a General Space-Time. *Phys. Rev. Lett.*, **43**:1457–1459, 1979.

[163] R. Schoen and S.-T. Yau. Proof of the Positive Mass Theorem. II. *Commun. Math. Phys.*, **79**:231–260, 1981.

[164] R. Schoen and S.-T. Yau. The Energy and the Linear Momentum of Space-Times in General Relativity. *Commun. Math. Phys.*, **79**:47–51, 1981.

[165] N. Seiberg and E. Witten. String theory and noncommutative geometry. *JHEP*, **09**:032, 1999.

[166] T. Shiromizu, S. Ohashi, and R. Suzuki. A no-go on strictly stationary spacetimes in four/higher dimensions. *Phys. Rev. D*, **86**:064041, 2012.

[167] L. Smarr. Mass formula for Kerr black holes. *Phys. Rev. Lett.*, **30**:71–73, 1973. [Erratum: Phys. Rev. Lett. **30** (1973) 521].

[168] I. Smolić. Killing Horizons as Equipotential Hypersurfaces. *Class. Quantum Grav.*, **29**:207002, 2012.

[169] I. Smolić. On the various aspects of electromagnetic potentials in spacetimes with symmetries. *Class. Quantum Grav.*, **31**:235002, 2014.

[170] I. Smolić. Constraints on the symmetry noninheriting scalar black hole hair. *Phys. Rev. D*, **95**(2):024016, 2017.

[171] I. Smolić. Spacetimes dressed with stealth electromagnetic fields. *Phys. Rev. D*, **97**(8):084041, 2018.

[172] L. M. Sokołowski, F. Occhionero, M. Litterio, and L. Amendola. Classical Electromagnetic Radiation in Multidimensional Space-Times. *Annals Phys.*, **225**:1–47, 1993.

[173] J. Stewart. *Advanced General Relativity.* Cambridge University Press, Cambridge England New York, 1993.

[174] D. Sudarsky and R. M. Wald. Mass formulas for stationary Einstein–Yang–Mills black holes and a simple proof of two staticity theorems. *Phys. Rev. D*, **47**:5209–5213, 1993.

[175] R. Švarc, J. Podolský, V. Pravda, and A. Pravdová. Exact black holes in quadratic gravity with any cosmological constant. *Phys. Rev. Lett.*, **121**(23):231104, 2018.

[176] L. B. Szabados. Commutation Properties of Cyclic and Null Killing Symmetries. *J. Math. Phys.*, **28**:2688, 1987.

[177] Y. Tachikawa. Black hole entropy in the presence of Chern–Simons terms. *Class. Quantum Grav.*, **24**:737–744, 2007.

[178] T. Tahamtan. On the compatibility of nonlinear electrodynamics models with Robinson–Trautman geometry. *Phys. Rev. D*, **103**:064052, 2021.

[179] K. A. Tanaka et al. Current status and highlights of the ELI-NP research program. *Matter Radiat. at Extremes*, **5**(2):024402, 2020.

[180] F. R. Tangherlini. Schwarzschild field in n dimensions and the dimensionality of space problem. *Il Nuovo Cimento*, **27**:636–651, 1963.

[181] P. Tod. Conditions for nonexistence of static or stationary, Einstein-Maxwell, non-inheriting black-holes. *Gen. Relativ. Gravit.*, **39**:111–127, 2007.

[182] B. Toshmatov, Z. Stuchlík, and B. Ahmedov. Generic rotating regular black holes in general relativity coupled to nonlinear electrodynamics. *Phys. Rev. D*, **95**:084037, 2017.

[183] A. A. Tseytlin. *Born–Infeld action, supersymmetry and string theory*, pages 417–452. World Scientific, 8 1999.

[184] R. Turolla, S. Zane, and A. Watts. Magnetars: the physics behind observations. A review. *Rept. Prog. Phys.*, **78**(11):116901, 2015.

[185] J. Wainwright and P. A. E. Yaremovicz. Symmetries and the Einstein-Maxwell Field Equations. The Null Field Case. *Gen. Relativ. Gravit.*, **7**:595–608, 1976.

[186] J. Wainwright and P. E. A. Yaremovicz. Killing Vector Fields and the Einstein-Maxwell Field Equations with Perfect Fluid Source. *Gen. Relativ. Gravit.*, **7**:345–359, 1976.

[187] R. Wald. *General Relativity.* University of Chicago Press, Chicago, 1984.

[188] R. Wald. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics.* University of Chicago Press, Chicago, 1994.

[189] R. M. Wald. Black hole in a uniform magnetic field. *Phys. Rev. D*, **10**:1680–1685, 1974.

[190] R. M. Wald. Dynamics in nonglobally hyperbolic, static space-times. *J. Math. Phys.*, **21**:2802–2805, 1980.

[191] R. M. Wald. On identically closed forms locally constructed from a field. *J.Math.Phys.*, **31**:2378–2384, 1990.

[192] R. M. Wald. Black hole entropy is the Noether charge. *Phys. Rev. D*, **48**(8):3427–3431, 1993.

[193] R. M. Wald and A. Zoupas. A General definition of "conserved quantities" in general relativity and other theories of gravity. *Phys. Rev. D*, **61**:084027, 2000.

[194] G. Weinstein. N-black hole stationary and axially symmetric solutions of the Einstein–Maxwell equations. *Commun. Part. Diff. Eq.*, **21**:1389–1430, 1996.

[195] S. Weintraub. *Differential Forms: Theory and Practice.* Elsevier, Oxford, UK, 2014.

[196] J. A. Wheeler. Geons. *Phys. Rev.*, **97**:511–536, 1955.

[197] E. Witten. A Simple Proof of the Positive Energy Theorem. *Commun. Math. Phys.*, **80**:381–402, 1981.

[198] W. W.-Y. Wong. A positive mass theorem for two spatial dimensions. 2012. (unpublished arXiv preprint gr-qc/1202.6279).

[199] M. L. Woolley. On the role of the Killing tensor in the Einstein-Maxwell theory. *Comm. Math. Phys.*, **33**(2):135–144, 1973.

[200] M. L. Woolley. The structure of groups of motions admitted by Einstein-Maxwell space-times. *Comm. Math. Phys.*, **31**(1):75–81, 1973.

[201] Y. Xie, J. Zhang, H. O. Silva, C. de Rham, H. Witek, and N. Yunes. Square Peg in a Circular Hole: Choosing the Right Ansatz for Isolated Black Holes in Generic Gravitational Theories. *Phys. Rev. Lett.*, **126**(24):241104, 2021.

[202] H. Yajima and T. Tamaki. Black hole solutions in Euler–Heisenberg theory. *Phys. Rev. D*, **63**:064007, 2001.

[203] K. Yang, B. M. Gu, S. W. Wei, and Y. X. Liu. Born–Infeld black holes in 4D Einstein–Gauss–Bonnet gravity. *Eur. Phys. J. C*, **80**:662, 2020.

[204] W. Yi-Huan. Energy and first law of thermodynamics for Born–Infeld-anti-de-Sitter black hole. *Chin. Phys.*, **B19**:090404, 2010.

[205] Y. Zhang and S. Gao. First law of black hole mechanics in nonlinear electrodynamic theory and its application to Bardeen black holes. *Class. Quantum Grav.*, **35**(14):145007, 2018.