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DEPARTMENT OF MATHEMATICS

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Contributions to asymptotic modelling of the micropolar fluid flow

DOCTORAL DISSERTATION

Supervisor:

prof. dr. sc. Igor Pažanin

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Zahvala

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Sažetak

Cilj ove disertacije je provesti asimptotičku analizu toka mikropolarnog fluida u fizikalno relevantnim tankim domenama (zakrivljeni kanali, cijevi, sustavi cijevi) i režimima (nestacionarni tok, neizotermni tok, tok s nestandardnim rubnim uvjetima). Najprije predlažemo asimptotičku aproksimaciju za neizoterman tok mikropolarnog fluida u tankom zakrivljenom kanalu koju rigorozno opravdavamo dokazujući ocjenu greške. Zatim dokazujemo egzistenciju i jedinstvenost rješenja inicijalno-rubne zadaće za nestacionarni tok u tankoj cijevi s ne-nul rubnim uvjetom za mikrorotaciju. Temeljem dobivenih rezultata te koristeći homogenizaciju i analizu rubnog sloja, izvodimo asimptotički model višeg reda točnosti koji opisuje efektivno ponašanje fluida u cijevi, te izvodimo ocjenu greške. Konačno, opisujemo problem nestacionarnog toka mikropolarnog fluida u sustavu tankih cijevi. Homogenizacijom te analizom rubnog i unutarnjeg sloja izvodimo asimptotički model proizvoljnog reda točnosti, te dobivenu aproksimaciju rigorozno opravdavamo ocjenom greške.

Summary

The goal of this dissertation is to perform the asymptotic analysis of the micropolar fluid flow in physically relevant thin domains (curvilinear channels, pipes, pipe systems) and regimes (unsteady flow, non-isothermal flow, flow with non-standard boundary conditions). First, we propose an asymptotic approximation for the non-isothermal flow of a micropolar fluid in a thin curvilinear channel, which we rigorously justify by proving the error estimates. Then we prove the existence and uniqueness of the solution of the initial-boundary problem for the unsteady flow in a thin pipe with a non-zero boundary condition for microrotation. Based on the obtained results and using homogenization and boundary layer analysis, we derive an asymptotic model of a higher order of accuracy that describes the effective behavior of the fluid in the pipe, and prove the error estimates. Finally, we describe the problem of the unsteady flow of a micropolar fluid in a system of thin pipes. Using homogenization and analysis of the boundary and interior layer, we derive an asymptotic model of arbitrary order of accuracy, and rigorously justify the obtained approximation by proving the error estimates.

Sadržaj

Uvod

U klasičnoj mehanici kontinuuma pretpostavka je da materijal neprekidno i potpuno ispunjava prostor koji zauzima tijelo. Deformacije i gibanja se izučavaju na makroskopskoj razini što znači da ne promatramo ponašanje pojedinih čestica. Takav pristup proučava prosječne vrijednosti fizikalnih veličina od interesa, te omogućava precizno predviđanje gibanja mnogih realnih tijela. S druge strane, u inženjerskoj literaturi ([41, 47, 57, 66]) se pokazalo da se fluidi s kompleksnom mikrostrukturom ne mogu dobro opisati klasičnom teorijom, pogotovo u tankim strukturama. Posljednje se poklapa s intuitivnim očekivanjem da će utjecaj mikrostrukture biti značajan kada je karakteristična dimenzija domene mala.

Sredinom šezdesetih godina 20. stoljeća Eringen u [30] uvodi novu klasu fluida - mikrofluide, koji pokazuju učinke lokalne strukture te mikrogibanja i mikrodeformacija elemenata. Mikrofluidi su izotropni viskozni fluidi koji su u najjednostavnijem slučaju određeni s 22 koeficijenta viskoznosti, te su čak i linearizirane jednadžbe prekomplicirane za primjenu na probleme od interesa.

Iz tog razloga, godine 1966. u [31] Eringen predlaže potklasu mikrofluida zvanu mikropolarni fluidi. Pretpostavimo li da su čestice rigidne te ignorirajući deformacije, odgovarajući matematički model izražava zakone sačuvanja impulsa, mase i angularnog momenta. Za razliku od klasične teorije, materijalne točke mikropolarnih fluida imaju orijentaciju; time umjesto uobičajena tri stupnja slobode imamo šest stupnjeva slobode, tri translacijska i tri rotacijska. Još jedna razlika u usporedbi s klasičnim modelom jest to da tenzor naprezanja nije simetričan, te se dodatno pojavljuje naprezanje sprega.

Fizikalno, mikropolarni fluid je predstavljen velikim brojem malenih sferičnih čestica uniformno raspršenih u viskoznom mediju, te se te čestice mogu rotirati. Uvedena je nova nepoznata funkcija koju zovemo mikrorotacija (tj. polje kutne brzine rotacije čestica), zajedno sa standardnim poljima brzine i tlaka. U skladu s time, Navier-Stokesove jednadžbe bivaju uparene s novom vektorskom jednadžbom koja dolazi od zakona očuvanja kutne količine gibanja te se pojavljuju četiri nove mikrorotacijske viskoznosti. Na ovaj način, mnogi ne-newtonovski fluidi, kao što su tekući kristali, krv, određeni polimerni fluidi te čak i voda u modelima s malim skalama (vidi [73]), mogu se uspješno opisati uparenim sustavom mikropolarnih jednadžbi.

Model mikropolarnog fluida je uvažen od strane inženjerske zajednice te se u literaturi može pronaći mnogo članaka s raznim primjenama: Ahmadi [3] koristi teoriju mikropolarnih fluida kako bi opisao i proučavao suspenziju pri niskim koncentracijama. U [8], Bayada i Lukaszewicz izvode Reynoldsovu jednadžbu za mikropolarne fluide, te raspravljaju o obliku dobivene jednadžbe u ovisnosti o pretpostavkama na viskoznosti i dane podatke. Takhar, Bhargava i Agarwal [94] izučavaju rotirajući disk u koaksijalnom cilindru, dok Chang [21] provodi numeričku analizu karakteristika toka i prijenosa topline kod mješovite konvekcije na vertikalnoj ploči. Ishak, Lok i Pop u [46] provode analizu toka mikropolarnog fluida kroz stagnirajuće područje. Sheikholeslami, Hatami i Ganji [91] razmatraju učinke prijenosa topline na tok mikropolarnog fluida u poroznom kanalu, te Mehmood, Nadeem i Masood [64] numerički i analitički analiziraju tok magneto-mikropolarnog fluida između paralelnih ploča koje su porozne i rotiraju se.

Nadalje, nalazimo velik broj rezultata koji se bave inženjerskim primjenama u biomedicini, posebno u modeliranju krvotoka. Abdullah i Amin [1] te Haghighi i Shahbazi [40] promatraju nelinearni dvodimenzionalni model mikropolarnog fluida u suženoj arteriji, dok Ahmed i Nadeem [4] istražuju učinak metalnih nanočestica na magneto-mikropolarni tok u arteriji sa 6 suženja. Mekheimer i El Kot [65] izučavaju tok magneto-mikropolarnog fluida u vertikalno simetričnoj i horizontalno nesimetričnoj arteriji s blagim suženjem. Boodoo, Bhatt i Comissiong [18] te Chaturani i Upadhya [22] modeliraju krv kao dvofazni fluid; mikropolarne jednadžbe opisuju suspenziju eritrocita dok se uzima da je sloj plazme Newtonovski fluid. U [58], Maddah, Navidbaksh i Atefi promatraju problem pulsirajućeg mikropolarnog fluida u elastičnoj žili. Reddy i Srikanth [90] analiziraju tok mikropolarnog fluida u začepljenoj suženoj arteriji sa slip rubnim uvjetom za brzinu. Zaman, Ali i Anwar Bég [97] numerički rješavaju problem mikropolarnog fluida u suženoj arteriji s poststenotičkom dilatacijom.

Cjelovit pregled matematičke teorije vezane za model mikropolarnog fluida može se pronaći u monografiji [55], dok se pregled dosad izvedenih analitičkih rješenja u inženjerskoj literaturi može pronaći [49]. Nadalje, analiza raznih rješenja i njihove primjene su bile razmatrane u člancima [5, 6] te monografijama [29, 36, 37, 68, 82, 93].

Napomenimo da se u istom razdoblju razvilo nekoliko teorija koje opisuju fluide s mikrostrukturom koje su slične ili identične teoriji mikropolarnih fluida. U [39], Grad izvodi zakone očuvanja mase, linearnog i kutnog momenta, te energije za polarne fluide koristeći pristup statističke termodinamike. Condiff i Dahler [24] izvode jednadžbe slične jednadžbama mikropolarnih fluida sa simetričnim tenzorom naprezanja sprega. Aero, Bulygin i Kuvshinskii [2] razvijaju teoriju asimetrične hidromehanike te izvode model sličan mikropolarnom.

Nadalje, Eringen je uveo nekoliko generalizacija modela mikrofluida, odnosno mikropolarnih fluida. U [32] daje jednadžbe gibanja i konstitutivne jednadžbe za mikropolarne fluide s istezanjem, dok u [33] proširuje teoriju mikrofluida tako da uzima u obzir učinke prijenosa topline. Nadalje, u [34] uspostavlja jednadžbe anizotropnih fluida u okviru mikropolarne teorije, te u [35] izvodi jednadžbe gibanja za termomikropolarne fluide s istezanjem.

Navedimo neke rezultate iz područja mikropolarnih fluida te asimptotičke analize u novijoj literaturi. Stacionarni tok mikropolarnog fluida u tankom kanalu proučavan je u člancima [27, 28]. Generalizacija tih rezultata u slučaju trodimenzionalne cijevi napravljena je 2011. u člancima [74, 75] i u slučaju sustava tankih cijevi 2015. godine [9]. Nestacionarni tokovi newtonovskog fluida proučavani su od strane Panasenka i Pileckasa u seriji članaka objavljenih u zadnjih desetak godina. Kao preduvjet za istraživanje nestacionarnog toka, najprije su izvedeni rezultati egzistencije za generalizirano Poiseuillleovo rješenje u beskonačnom cilindru. Egzistencija i jedinstvenost rješenja u Hölderovim prostorima pokazani su u [86], dok su rezultati za generalizirano Poiseuilleovo rješenje u prostorima Soboljeva doneseni u članku [84]. Nestacionarni linearizirani Navier-Stokesov problem u domeni s cilindričnim izlazima prema beskonačnosti je bio predmet istraživanja članka [83]. Rezultati vezani za stacionarni i nestacionarni Lerayev problem za Newtonovski fluid objedinjeni su i prezentirani u [85]. Koristeći dobivene rezultate egzistencije, jedinstvenosti, kao i eksponencijalnog pada, rigorozno su izvedeni asimptotički modeli višeg reda točnosti za nestacionaran tok newtonovskog fluida u tankoj cijevi [70] te u sustavima cijevi bez rubnog sloja u vremenu [71] kao i u općenitom slučaju [72]. Asimptotički model za nestacionarni tok newtonovskog fluida u tankoj zakrivljenoj cijevi s pomičnom granicom predložen je 2019. godine u [19,20]. Generalizacije navedenih rezultata za slučaj ne-newtonovskog, mikropolarnog fluida moguće je pronaći u člancima [11–13,78].

U uvodnom Poglavlju 1 opisujemo model mikropolarnog fluida. Najprije izvodimo jednadžbe gibanja za širu klasu polarnih fluida iz zakona očuvanja mase, zakona očuvanja linearne količine gibanja, zakona očuvanja kutne količine gibanja te zakona očuvanja energije. Uvodimo dodatne pretpostavke poput izotropije, Fourierovog zakona i nestlačivosti, bez pretpostavki na simetriju tenzora naprezanja. Nakon toga, uvodimo konstitutivne jednadžbe koje daju model mikropolarnog, odnosno termomikropolarnog fluida. Pokazujemo sličnosti i razlike između opisanog te klasičnog modela, te razmatramo prikladne rubne uvjete za mikropolarne jednadžbe.

Poglavlja 2, 3 i 4 centralni su dio ove disertacije i donose originalne rezultate. U Poglavlju 2 promatramo stacionarni tok termomikropolarnog fluida u tankom zakrivljenom kanalu. Tok je određen zadanom razlikom tlakova između krajeva kanala. Prijenos topline se odvija na gornjem rubu kanala, dok je donji rub izoliran. Najprije opisujemo geometriju promatranog kanala, te zatim jednadžbe zapisujemo u bezdimenzionalnom obliku u tankom nedeformiranom kanalu. Koristeći asimptotičku analizu s obzirom na mali parametar ε koji predstavlja širinu kanala računamo asimptotičku aproksimaciju rješenja. Rješenje je dano u eksplicitnom obliku što nam omogućava proučavanje učinaka zakrivljenosti domene i mikropolarnosti fluida na tok u kanalu. Provodimo analizu rubnog sloja za mikrorotaciju kako bismo poboljšali našu aproksimaciju. Konačno, izvodimo ocjene greške ocjenjivanjem razlike između originalnog rješenja i asimptotičke aproksimacije u odgovarajućim normama.

Nadalje, u Poglavlju 3 razmatramo tok nestacionarnog mikropolarnog fluida u tankoj cijevi s dinamičkim rubnim uvjetom opisanim u Odjeljku 1.3.2. Najprije pokazujemo egzistenciju i jedinstvenost rješenja inicijalno-rubnog problema koji opisuje tok. Tada provodimo asimptotičku analizu s obzirom na mali parametar ε koji opisuje promjer cijevi, te konstruiramo asimptotičku aproksimaciju rješenja višeg reda točnosti. Aproksimacija je dana u eksplicitnom obliku, te možemo uočiti učinke mikropolarnosti, dinamičkih rubnih uvjeta te vremenske derivacije na tok fluida. Provedena je detaljna analiza rubnog sloja na krajevima cijevi kako bi aproksimacija zadovoljila rubne uvjete. Predloženi model je rigorozno opravdan izvođenjem ocjena greške u odgovarajućim normama.

U Poglavlju 4 analiziramo nestacionarni tok mikropolarnog fluida u sustavu tankih cijevi. Konstruiramo funkciju koja je proširenje rubnih uvjeta na krajevima cijevi i čija je divergencija jednaka nuli, te diskutiramo dobru postavljenost problema. U cijevima daleko od čvorišta, tok fluida je uvjetovan redom veličine viskoznih koeficijenata u usporedbi s malim parametrom ε koji opisuje debljinu cijevi. Usredotočujemo se na kritični slučaj koji rezultira jakim uparivanjem između brzine i mikrorotacije. Izvodimo asimptotičku aproksimaciju toka fluida u cijevi te provodimo analizu rubnog sloja uz krajeve cijevi, za svaku cijev pojedinačno. Obzirom da očekujemo da se fluid ponaša drugačije u čvorištu, aproksimaciju u cijevima množimo s funkcijom reza koja je jednaka nuli u blizini čvorišta. Time nastaje rezidual u blizini čvorišta koji korigiramo s korektorima unutarnjeg sloja. Kako bi opravdali dobiveni model proizvoljnog reda, izvodimo ocjene greške u odgovarajućim normama.

U Dodatku A dajemo koeficijente asimptotičke aproksimacije izračunate u Poglavlju 3 u eksplicitnom obliku, što nam omogućava proučavanje učinaka dinamičkih rubnih uvjeta na tok.

Konačno, u Dodatku B proučavamo linearan Lerayev problem za stacionarni tok mikropolarnog fluida. Opisujemo neograničenu domenu s cilindričnim otvorima prema beskonačnosti te rješavamo stacionaran linearan mikropolaran problem u prostorima težinskih funkcija, odnosno pokazujemo da postoji jedinstveno rješenje promatranog problema koje eksponencijalno teži k nuli u svakom otvoru. Ovaj rezultat koristimo u Poglavlju 4 u analizi rubnog i unutarnjeg sloja.

Rezultati Poglavlja 2 su objavljeni u [79], rezultati dobre postavljenosti te asimptotička analiza iz Poglavlja 3 su objavljeni u [15], dok su rezultati Poglavlja 4 objavljeni u [80].

Oznake i tehnički rezultati

Neka je $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, tada s $C_0^\infty(\Omega)$ označavamo prostor svih $C^\infty(\Omega)$ funkcija s kompaktnim nosačem u Ω. Za $m \in \mathbb{N}$ i $1 \leq p \leq \infty$ s $W^{m,p}(\Omega)$ označavamo Soboljevljev prostor s konačnom normom

$$\|\boldsymbol{u}\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{|\alpha|=0}^{m} \int_\Omega |D^\alpha \boldsymbol{u}(\boldsymbol{x})|^p d\boldsymbol{x} \right)^{1/p}, & 1 \leq p < \infty, \\ \max\limits_{|\alpha| \leq m} \left(\operatorname*{ess\,supp}_{\boldsymbol{x} \in \Omega} \left| D^\alpha \boldsymbol{u}(\boldsymbol{x}) \right| \right), & p = \infty, \end{cases}$$

gdje je $|\alpha| = \alpha_1 + \cdots + \alpha_n$ i $D_{\boldsymbol{x}}^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$. Posebno, $W^{0,p}(\Omega) = L^p(\Omega)$, te s $W_0^{m,p}(\Omega)$ označavamo zatvarač prostora $C_0^\infty(\Omega)$ u $W^{m,p}(\Omega)$ normi. S $\mathcal{L}_\beta^2(\Omega), \mathcal{W}_\beta^{m,2}(\Omega)$ označavamo težinske prostore funkcija definirane u Dodatku B za sustave polubeskonačnih cijevi.

S $W^{-m,p'}(\Omega_\varepsilon)$ označavamo prostor linearnih neprekidnih funkcionala na $W_0^{m,p}(\Omega)$, te s $W^{m-1/p,p}(\partial\Omega)$ označavamo prostor tragova funkcija iz $W^{m,p}(\Omega)$ na $\partial\Omega$.

Neka je

$$\mathcal{V} = \{\boldsymbol{u} \in C_0^\infty(\Omega) : \operatorname{div} \boldsymbol{u} = 0\},$$

tada s H označavamo zatvarač prostora $\mathcal{V}$ u L^2 normi, te s V zatvarač prostora $\mathcal{V}$ u $W_0^{1,2}$ normi. Vrijede sljedeće karakterizacije prostora H, V (vidi npr. [95]):

$$H = \{\boldsymbol{u} \in L^2(\Omega) : \operatorname{div} \boldsymbol{u} = 0,\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ na } \partial\Omega\},$$

$$V = \{\boldsymbol{u} \in H_0^1(\Omega) : \operatorname{div} \boldsymbol{u} = 0\}.$$

Nadalje, za $1 \leq p \leq \infty$ i Banachov prostor X s $L^p(0,T;X)$ označavamo Bochnerov prostor funkcija s $[0,T]$ u X uz normu

$$\|\boldsymbol{u}\|_{L^p(0,T;X)} = \begin{cases} \left(\displaystyle\int_0^T \left\| \boldsymbol{u}(t) \right\|_X^p dt \right)^{1/p}, & 1 \leq p < \infty, \\ \operatorname*{ess\,supp}_{t \in [0,T]} \left\| \boldsymbol{u}(t) \right\|_X, & p = \infty. \end{cases}$$

Neka je $\boldsymbol{\Omega}_\varepsilon$ tanka domena, to jest kanal visine $\boldsymbol{\varepsilon}$ u slučaju $\boldsymbol{n}=2$ te cijev promjera $\boldsymbol{\varepsilon}$ u slučaju $\boldsymbol{n}=3$, pri čemu je $\boldsymbol{\varepsilon}$ malen pozitivan parametar. Posebno $\boldsymbol{\Omega}_\varepsilon$ može biti sustav kanala ili cijevi koje se sastaju u čvorištu. Označimo s Γ_ε lateralni rub domene $\boldsymbol{\Omega}_\varepsilon$, te s Σ_ε uniju krajeva kanala odnosno cijevi. Tada vrijede sljedeći rezultati:

Lema 0.0.1 (Poincaréova nejednakost)**.** Za sve $\boldsymbol{\varphi}_\varepsilon \in W^{1,2}(\Omega_\varepsilon)$ takve da $\boldsymbol{\varphi}_\varepsilon = 0$ na Γ_ε vrijede sljedeće nejednakosti:

$$\left\|\boldsymbol{\varphi}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \left\|\nabla\boldsymbol{\varphi}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)},$$

$$\left\|\boldsymbol{\varphi}_\varepsilon\right\|_{L^4(\Omega_\varepsilon)} \leq C\varepsilon^{1/2} \left\|\nabla\boldsymbol{\varphi}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}, \ n=2,$$

$$\left\|\boldsymbol{\varphi}_\varepsilon\right\|_{L^4(\Omega_\varepsilon)} \leq C\varepsilon^{1/4} \left\|\nabla\boldsymbol{\varphi}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}, \ n=3.$$

Lema 0.0.2. Neka je $p_\varepsilon \in L^2_0(\Omega_\varepsilon) = \left\{p_\varepsilon \in L^2(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} p_\varepsilon = 0\right\}$, tada problem

$$\operatorname{div} \boldsymbol{d}_\varepsilon = p_\varepsilon \quad \text{u} \quad \Omega_\varepsilon,$$

$$\boldsymbol{d}_\varepsilon = \mathbf{0} \quad \text{na} \quad \partial\Omega_\varepsilon$$

ima barem jedno rješenje $\boldsymbol{d}_\varepsilon \in W^{1,2}_0(\Omega_\varepsilon)$ takvo da

$$\|\nabla \boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon} \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

Lema 0.0.3. Za $\boldsymbol{h}_\varepsilon \in C^1([0,T]; W^{3/2,2}(\Sigma_\varepsilon))$ postoji funkcija $\boldsymbol{h}^{ext}_\varepsilon \in C^1([0,T]; W^{2,2}(\Omega_\varepsilon))$ takva da

$$\operatorname{div} \boldsymbol{h}^{ext}_\varepsilon = 0,$$

$$\boldsymbol{h}^{ext}_\varepsilon = \boldsymbol{h}_\varepsilon \quad \text{na} \quad \Sigma_\varepsilon,$$

$$\boldsymbol{h}^{ext}_\varepsilon = \mathbf{0} \quad \text{na} \quad \Gamma_\varepsilon,$$

te vrijede ocjene

$$\left\|\boldsymbol{h}^{ext}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} + \left\|\frac{\partial \boldsymbol{h}^{ext}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{n-1}{2}},$$

$$\left\|\nabla\boldsymbol{h}^{ext}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} + \left\|\nabla\frac{\partial \boldsymbol{h}^{ext}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{n-3}{2}},$$

$$\left\|\Delta\boldsymbol{h}^{ext}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{n-5}{2}}.$$

Za dokaze gornjih rezultata, vidjeti npr. [9, 56, 59, 71].

1. Model mikropolarnog fluida

Pojedinačne čestice fluida se općenito mogu rotirati, mijenjati oblik, smanjivati ili proširivati, neovisno o toku fluida. Iz tog razloga, Navier-Stokesov model ne opisuje dobro fluide s kompleksnom mikrostrukturom (poput polimera, tekućih kristala, krvi, itd.) čime se javlja potreba za teorijom koja uzima u obzir oblik, deformacije te rotaciju čestica.

Radi svoje jednostavnosti, jedna od popularnijih teorija je Eringenova teorija mikropolarnih fluida. Mikropolarni fluidi se sastoje od krutih sferičnih čestica suspendiranih u viskoznom mediju koje se mogu rotirati neovisno o gibanju fluida, pri čemu ne dolazi do deformacije čestica. Navier-Stokesove jednadžbe su uparene s jednadžbom koja dolazi od zakona očuvanja kutne količine gibanja, te je uvedena nova nepoznanica - mikrorotacija, koja opisuje rotaciju čestica. Pojavljuju se četiri nove viskoznosti, te ako je jedna od njih jednaka nuli (mikrorotacijska viskoznost), mikropolarni model se svodi na klasični model.

Cilj ovog poglavlja je najprije izvesti model šire klase polarnih fluida s nesimetričnim tenzorom naprezanja, te potom model mikropolarnog fluida. U Odjeljku 1.1 izvodimo jednadžbu kontinuiteta iz zakona očuvanja mase i transportnog teorema, koja je ista za obične i polarne fluide. Nadalje, uzimamo u obzir dodatne pojave kod polarnih fluida poput naprezanja sprega i vlastite kutne količine gibanja čestica, te u Odjeljku 1.2 izvodimo jednadžbe gibanja za polarne fluide iz zakona očuvanja linearne i kutne količine gibanja te zakona očuvanja energije, uz pretpostavke izotropije i nestlačivosti. U Odjeljku 1.3 uvodimo konstitutivne jednadžbe koje daju model mikropolarnog fluida te diskutiramo odgovarajuće rubne uvjete. Konačno, u Odjeljku 1.4 zadajemo konstitutivne jednadžbe koje daju model termomikropolarnog fluida, što je daljnja generalizacija mikropolarnog modela koja dopušta prijenos topline.

1.1. Kinematika

Najprije fiksiramo neki pravokutni koordinatni sustav, te promatramo česticu fluida koja u trenutku $t = 0$ ima koordinate $\boldsymbol{X} = (X_1, X_2, X_3)$. Čestica se kreće s fluidom te u kasnijem trenutku t ima koordinate $\boldsymbol{x} = (x_1, x_2, x_3)$. Njih smatramo funkcijom koja ovisi o početnoj koordinati $\boldsymbol{X}$ i vremenu t, to jest

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t). \tag{1.1}$$

Pretpostavljamo da je funkcija $\boldsymbol{x}$ neprekidna i invertibilna, odnosno da postoji inverzna funkcija $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t)$ koja daje početni položaj čestice koja se u trenutku t nalazi na položaju $\boldsymbol{x}$. Dodatno, pretpostavljamo da su funkcije $\boldsymbol{x}$ i $\boldsymbol{X}$ dovoljno glatke te da je Jakobijan transformacije (1.1) dan s

$$J = J(\boldsymbol{X}, t) = \det\left(\frac{\partial x_i}{\partial X_j}\right)$$

pozitivan i konačan.

Svaku funkciju f koja na neki način opisuje stanje fluida u točki $\boldsymbol{x}$ i vremenu t možemo zapisati i preko inicijalnog položaja koristeći transformaciju (1.1):

$$f(\boldsymbol{x}, t) = f(\boldsymbol{x}(\boldsymbol{X}, t), t) =: F(\boldsymbol{X}, t). \tag{1.2}$$

Naglasimo da u daljnjem $\boldsymbol{X}$ smatramo parametrom koji označava danu česticu, a $\boldsymbol{x}$ funkcijom koja ovisi o početnom položaju i vremenu. Brzina $\boldsymbol{u}$ čestice početnog položaja $\boldsymbol{X}$ u trenutku t je dana s

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{U}(\boldsymbol{X}, t) = \frac{d}{dt}\boldsymbol{x}(\boldsymbol{X}, t).$$

Uočimo da u slučaju kada je brzina $\boldsymbol{u}$ zadana, transformacija (1.1) je određena običnom diferencijalnom jednadžbom i početnim uvjetom:

$$\begin{cases} \frac{d}{dt}\boldsymbol{x}(\boldsymbol{X}, t) = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{X}, t), t), \\ \boldsymbol{x}(\boldsymbol{X}, 0) = \boldsymbol{X}. \end{cases}$$

Deriviranjem (1.2) dobivamo sljedeću jednakost:

$$\frac{d}{dt}F(\boldsymbol{X}, t) = \frac{d}{dt}f(\boldsymbol{x}(\boldsymbol{X}, t), t) = \frac{\partial f}{\partial x_i}(\boldsymbol{x}(\boldsymbol{X}, t), t)\frac{dx_i}{dt} + \frac{\partial f}{\partial t}(\boldsymbol{x}(\boldsymbol{X}, t), t).$$

Dakle, vrijedi formula

$$\frac{d}{dt}F(\boldsymbol{X}, t) = \frac{D}{Dt}f(\boldsymbol{x}, t), \tag{1.3}$$

pri čemu je $\frac{D}{Dt}f(\boldsymbol{x},t) = \frac{\partial f}{\partial t}(\boldsymbol{x},t) + \boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla f(\boldsymbol{x},t)$ tzv. materijalna derivacija funkcije f. Formula (1.3) vrijedi i za vektorske funkcije $\boldsymbol{f}$, gdje je materijalna derivacija dana s $\frac{D}{Dt}\boldsymbol{f}(\boldsymbol{x},t) = \frac{\partial \boldsymbol{f}}{\partial t}(\boldsymbol{x},t) + (\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla)\boldsymbol{f}(\boldsymbol{x},t)$. Također, lako je pokazati da za Jakobijan $\boldsymbol{J}$ vrijedi sljedeći identitet:

$$\frac{d}{dt}J(\boldsymbol{X},t) = \operatorname{div}\boldsymbol{u}(\boldsymbol{x}(\boldsymbol{X},t),t)J(\boldsymbol{X},t). \tag{1.4}$$

Sada dokazujemo transportni teorem koji će nam biti potreban za izvođenje jednadžbi polarnog fluida:

Teorem 1.1.1 (Transportni teorem)**.** Neka je $\Omega(t)$ volumen koji se kreće s fluidom, tada vrijedi

$$\frac{d}{dt}\int_{\Omega(t)} f(\boldsymbol{x},t)d\boldsymbol{x} = \int_{\Omega(t)}\left(\frac{\partial f}{\partial t}(\boldsymbol{x},t) + \operatorname{div}(f(\boldsymbol{x},t)\boldsymbol{u}(\boldsymbol{x},t))\right)d\boldsymbol{x}.$$

Dokaz. Promotrimo zamjenu varijabli $\boldsymbol{x}:\Omega(0)\to\Omega(t)$ danu s (1.1), odnosno $\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{X},t)$. Tada vrijedi

$$\int_{\Omega(t)} f(\boldsymbol{x},t)d\boldsymbol{x} = \int_{\Omega(0)} f(\boldsymbol{x}(\boldsymbol{X},t),t)J(\boldsymbol{X},t)d\boldsymbol{X} = \int_{\Omega(0)} F(\boldsymbol{X},t)J(\boldsymbol{X},t)d\boldsymbol{X}. \tag{1.5}$$

Deriviranjem jednakosti (1.5) dobivamo

$$\begin{aligned}\frac{d}{dt}\int_{\Omega(t)} f(\boldsymbol{x},t)d\boldsymbol{x} &= \frac{d}{dt}\int_{\Omega(0)} F(\boldsymbol{X},t)J(\boldsymbol{X},t)d\boldsymbol{X}\\ &= \int_{\Omega(0)}\left(\frac{d}{dt}F(\boldsymbol{X},t)J(\boldsymbol{X},t) + F(\boldsymbol{X},t)\frac{d}{dt}J(\boldsymbol{X},t)\right)d\boldsymbol{X}\\ &= \int_{\Omega(0)}\left(\frac{d}{dt}F(\boldsymbol{X},t)J(\boldsymbol{X},t) + F(\boldsymbol{X},t)\operatorname{div}\boldsymbol{u}(\boldsymbol{x}(\boldsymbol{X},t),t)J(\boldsymbol{X},t)\right)d\boldsymbol{X},\end{aligned} \tag{1.6}$$

gdje smo u zadnjem koraku iskoristili (1.4). Ako uvrstimo jednakost

$$\frac{d}{dt}F(\boldsymbol{X},t) = \frac{\partial f}{\partial t}(\boldsymbol{x}(\boldsymbol{X},t),t) + \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{X},t),t)\nabla f(\boldsymbol{x}(\boldsymbol{X},t),t)$$

u (1.6) i vratimo se nazad na domenu $\Omega(t)$, dobivamo tvrdnju teorema.

■

Fluid ćemo zvati inkompresibilnim ili nestlačivim ako je za svaku domenu $\Omega(0)$ volumen očuvan, odnosno volumen $\Omega(0)$ je jednak volumenu $\Omega(t)$, za svaki trenutak t. Uz $f\equiv 1$, iz Teorema 1.1.1 direktno slijedi da je uvjet inkompresibilnosti ekvivalentan uvjetu

$$\operatorname{div}\boldsymbol{u} = 0.$$

Sada ćemo izvesti jednadžbu kontinuiteta za inkompresibilne fluide iz zakona očuvanja mase. Neka je $\rho = \rho(\boldsymbol{x}, t)$ gustoća fluida u točki $\boldsymbol{x}$ u trenutku t. Tada je masa konačnog volumena $\boldsymbol{\Omega}$ dana s

$$m = \int_{\Omega} \rho(\boldsymbol{x}, t) d\boldsymbol{x}.$$

Zakon očuvanja mase kaže da se masa fluida sadržana u volumenu $\boldsymbol{\Omega}$ ne mijenja kako se $\boldsymbol{\Omega}$ kreće s fluidom, odnosno

$$\frac{d}{dt} \int_{\Omega(t)} \rho(\boldsymbol{x}, t) d\boldsymbol{x} = 0. \tag{1.7}$$

Primjenom Teorema 1.1.1 iz (1.7) zaključujemo

$$\int_{\Omega(t)} \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) \right) d\boldsymbol{x} = 0,$$

odakle zbog proizvoljnosti domene $\boldsymbol{\Omega}(t)$ slijedi

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0. \tag{1.8}$$

Uočimo da iz jednadžbe kontinuiteta (1.8) slijedi da je prostorno homogen fluid inkompresibilan ako i samo ako je gustoća jednaka konstanti. Također, koristeći transportni teorem i jednadžbu kontinuiteta lako se pokazuje općenita formula

$$\frac{d}{dt} \int_{\Omega(t)} \rho f d\boldsymbol{x} = \int_{\Omega(t)} \rho \frac{D}{Dt} f d\boldsymbol{x}. \tag{1.9}$$

1.2. Dinamika

1.2.1. Zakon očuvanja količine gibanja

Pretpostavljamo da dva tipa sile djeluju na volumen $\mathbf{\Omega}$: volumenske sile, koje djeluju na sve čestice tijela (npr. homogeno gravitacijsko ili magnetsko polje), te kontaktne sile koje djeluju na rubnu površinu volumena $\mathbf{\Omega}$.

Neka $\boldsymbol{f}$ označava volumensku silu po jedinici mase, tada je $\rho\boldsymbol{f}$ gustoća dane sile te je njeno djelovanje na volumen $\mathbf{\Omega}$ dano s

$$\int_{\Omega} \rho \boldsymbol{f} \, d\boldsymbol{x}.$$

Nadalje, neka je $\boldsymbol{n}$ vanjska jedinična normala u točki na površini $\partial\mathbf{\Omega}$, te $\boldsymbol{t}_n$ kontaktna sila po jedinici površine, odnosno plošna gustoća kontaktne sile. Tada je djelovanje kontaktne sile na volumen $\mathbf{\Omega}$ dano s

$$\int_{\partial\Omega} \boldsymbol{t}_n \, dS.$$

Cauchyjev princip naprezanja kaže da $\boldsymbol{t}_n$ u svakom trenutku ovisi samo o položaju i orijentaciji elementa površine dS, odnosno vrijedi

$$\boldsymbol{t}_n = \boldsymbol{t}_n(\boldsymbol{x}, t, \boldsymbol{n}).$$

Zakon očuvanja količine gibanja sada kaže da je brzina promjene količine gibanja jednaka sumi svih sila, odnosno vrijedi

$$\frac{d}{dt} \int_{\Omega(t)} \rho \boldsymbol{u} \, d\boldsymbol{x} = \int_{\Omega(t)} \rho \boldsymbol{f} \, d\boldsymbol{x} + \int_{\partial\Omega(t)} \boldsymbol{t}_n \, dS. \tag{1.10}$$

U vidu formule (1.9), zakon očuvanja (1.10) glasi

$$\int_{\Omega(t)} \rho \frac{D\boldsymbol{u}}{Dt} d\boldsymbol{x} = \int_{\Omega(t)} \rho \boldsymbol{f} \, d\boldsymbol{x} + \int_{\partial\Omega(t)} \boldsymbol{t}_n \, dS. \tag{1.11}$$

Koristeći ovu jednadžbu se može pokazati da su naprezanja lokalno u ravnoteži. Posljedica toga je da je normalno naprezanje $\boldsymbol{t}_n$ linearna funkcija u $\boldsymbol{n}$, odnosno može se zapisati u obliku

$$\boldsymbol{t}_n(\boldsymbol{x}, t, \boldsymbol{n}) = \boldsymbol{n}(\boldsymbol{x}, t)\boldsymbol{T}(\boldsymbol{x}, t), \tag{1.12}$$

gdje je $\boldsymbol{T} = \{T_{ij}\}$ matrica koju zovemo tenzor naprezanja. Sada zbog (1.12) i Greenovog teorema jednakost (1.11) možemo zapisati u obliku

$$\int_{\Omega(t)} \rho \frac{D\boldsymbol{u}}{Dt} d\boldsymbol{x} = \int_{\Omega(t)} \left(\rho \boldsymbol{f} + \operatorname{div} \boldsymbol{T}\right) d\boldsymbol{x}.$$

Zbog proizvoljnosti volumena $\Omega(t)$ vrijedi

$$\rho\frac{D\boldsymbol{u}}{Dt} = \rho\boldsymbol{f} + \operatorname{div}\boldsymbol{T}, \tag{1.13}$$

odnosno

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \rho f_i + \frac{\partial T_{ji}}{\partial x_j}, \tag{1.14}$$

čime smo dobili diferencijalni oblik zakona očuvanja količine gibanja. Jednadžba (1.13) opisuje gibanje svakog kontinuuma, dok je tenzor naprezanja $\boldsymbol{T}$ određen tipom fluida koji promatramo. U najjednostavnijem modelu fluida su kontaktne sile okomite na površinu na koju djeluju, to jest oblika su

$$\boldsymbol{t}_n = -p(\boldsymbol{x})\boldsymbol{n},$$

gdje je p tlak. Taj model opisuje naprezanja svih fluida u mirovanju, dok se općenito kod fluida u gibanju pojavljuju i tangencijalna naprezanja te tenzor naprezanja ne mora biti dijagonalan. Tenzor naprezanja se standardno zapisuje u obliku

$$T_{ij} = -p\delta_{ij} + P_{ij},$$

te $\boldsymbol{P} = \{P_{ij}\}$ nazivamo viskoznim tenzorom naprezanja. U klasičnoj mehanici fluida je tenzor naprezanja simetričan, te ćemo takve fluide zvati običnim fluidima.

1.2.2. Zakon očuvanja kutne količine gibanja

U slučaju običnih fluida, iz zakona očuvanja mase i zakona očuvanja gibanja se može izvesti zakon očuvanja kutne količine gibanja

$$\frac{d}{dt}\int_{\Omega(t)}\rho(\boldsymbol{x}\times\boldsymbol{u})d\boldsymbol{x} = \int_{\Omega(t)}\rho(\boldsymbol{x}\times\boldsymbol{f})d\boldsymbol{x} + \int_{\partial\Omega(t)}(\boldsymbol{x}\times\boldsymbol{t}_n)dS. \tag{1.15}$$

Zakon očuvanja kutnog gibanja (1.15) vrijedi ako pretpostavimo da svi zakretni momenti dolaze od makroskopskih sila. Taj zakon više ne vrijedi za polarne fluide, gdje osim volumenske sile $\boldsymbol{f}$ moramo uvesti i zakretni moment po jedinici mase $\boldsymbol{g}$, te osim normalnog naprezanja $\boldsymbol{t}_n$ uvodimo i naprezanja sprega (couple stress) $\boldsymbol{c}_n$. Dodatno, osim momenta količine gibanja kojeg nazivamo vanjskom kutnom količinom gibanja uvodimo i unutrašnju kutnu količinu gibanja po jedinici mase, odnosno spin po jedinici mase $\boldsymbol{l}$. Spin je posljedica toga da u mikropolarnim fluidima čestice mogu imati vlastitu kutnu količinu gibanja.

Zakon očuvanja kutne količine gibanja za polarne fluide sada glasi

$$\frac{d}{dt}\int_{\Omega(t)}\rho(\boldsymbol{l}+\boldsymbol{x}\times\boldsymbol{u})d\boldsymbol{x} = \int_{\Omega(t)}\rho(\boldsymbol{g}+\boldsymbol{x}\times\boldsymbol{f})d\boldsymbol{x} + \int_{\partial\Omega(t)}(\boldsymbol{c}_n+\boldsymbol{x}\times\boldsymbol{t}_n)dS. \tag{1.16}$$

Može se pokazati da kao što se $\boldsymbol{t}_n$ može zapisati u obliku $\boldsymbol{n} \cdot \boldsymbol{T}$, i naprezanje sprega $\boldsymbol{c}_n$ se može zapisati u obliku $\boldsymbol{n} \cdot \boldsymbol{C}$, gdje je $\boldsymbol{C} = \{C_{ij}\}$ tenzor naprezanja sprega. Nadalje, vrijedi identiteta

$$\operatorname{div}(\boldsymbol{x} \times \boldsymbol{T}) = \boldsymbol{x} \times (\operatorname{div} \boldsymbol{T}) + \boldsymbol{T}_x,$$

pri čemu je $\boldsymbol{T}_x = \{\epsilon_{ijk} T_{jk}\}$ i ϵ_{ijk} Levi-Civitin permutacijski simbol. Sada iz (1.16) slijedi

$$\frac{d}{dt} \int_{\Omega(t)} \rho(\boldsymbol{l} + \boldsymbol{x} \times \boldsymbol{u}) d\boldsymbol{x} = \int_{\Omega(t)} (\rho \boldsymbol{g} + \rho \boldsymbol{x} \times \boldsymbol{f} + \operatorname{div} \boldsymbol{C} + x \times \operatorname{div} \boldsymbol{T} + \boldsymbol{T}_x) d\boldsymbol{x}.$$

Diferencijalni oblik zakona očuvanja kutne količine gibanja glasi

$$\rho \frac{D}{Dt}(\boldsymbol{l} + \boldsymbol{x} \times \boldsymbol{u}) = \rho \boldsymbol{g} + \rho \boldsymbol{x} \times \boldsymbol{f} + \operatorname{div} \boldsymbol{C} + x \times (\operatorname{div} \boldsymbol{T}) + \boldsymbol{T}_x. \tag{1.17}$$

S druge strane, vektorskim množenjem $\boldsymbol{x}$ sa zakonom očuvanja količine gibanja (1.13) dobivamo

$$\rho \left(\boldsymbol{x} \times \frac{D\boldsymbol{u}}{Dt} \right) = \rho \frac{D}{Dt}(\boldsymbol{x} \times \boldsymbol{u}) = \rho \boldsymbol{x} \times \boldsymbol{f} + \boldsymbol{x} \times (\operatorname{div} \boldsymbol{T}). \tag{1.18}$$

Oduzimanjem (1.18) od (1.17) dobivamo zakon očuvanja unutrašnje kutne količine gibanja

$$\rho \frac{D\boldsymbol{l}}{Dt} = \rho \boldsymbol{g} + \operatorname{div} \boldsymbol{C} + \boldsymbol{T}_x.$$

Ako shvatimo spin $\boldsymbol{l}$ kao kutnu količinu gibanja čestica fluida, onda ga možemo zapisati u obliku $\boldsymbol{l} = \boldsymbol{I}\boldsymbol{w}$, gdje je $\boldsymbol{I}$ tenzor momenta inercije (odnosno momenta inercije po jedinici mase) te $\boldsymbol{w}$ kutna brzina koja opisuje rotaciju čestica fluida. U ovom radu ćemo promatrati izotropne fluide, dakle tenzor momenta inercije je oblika

$$I_{ij} = I \delta_{ij},$$

gdje I zovemo koeficijentom mikroinercije. Sada zakon očuvanja unutrašnjeg kutnog momenta glasi

$$\rho I \frac{D\boldsymbol{w}}{Dt} = \rho \boldsymbol{g} + \operatorname{div} \boldsymbol{C} + \boldsymbol{T}_x. \tag{1.19}$$

1.2.3. Energetske jednakosti

Prvi zakon termodinamike kaže da je porast unutarnje energije sustava jednak zbroju rada sustava te količini topline dovedene u sustav. Neka je $\boldsymbol{q}$ gustoća toplinskog toka, te neka je $\boldsymbol{E}$ unutarnja energija po jedinici mase. Ako pretpostavimo da se ukupna energija sastoji od kinetičke energije i unutarnje energije, onda prvi zakon termodinamike za izotropne mikropolarne fluide glasi

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega(t)}\rho\left(\frac{1}{2}|\boldsymbol{u}|^2+\frac{I}{2}|\boldsymbol{w}|^2+E\right)d\boldsymbol{x}\\&=\int_{\Omega(t)}(\rho\boldsymbol{f}\cdot\boldsymbol{u}+\rho\boldsymbol{g}\cdot\boldsymbol{w})d\boldsymbol{x}+\int_{\partial\Omega(t)}(\boldsymbol{t}_n\cdot\boldsymbol{u}+\boldsymbol{c}_n\cdot\boldsymbol{w})dS-\int_{\partial\Omega(t)}\boldsymbol{q}\cdot\boldsymbol{n}dS.\end{aligned}\tag{1.20}$$

Koristeći (1.14) jednostavno je pokazati da vrijedi

$$\int_{\partial\Omega(t)}u_iT_{ji}n_jdS=\int_{\Omega(t)}\left(T_{ji}\frac{\partial u_i}{\partial x_j}+\rho u_i\frac{Du_i}{Dt}-\rho f_iu_i\right)d\boldsymbol{x},$$

iz čega slijedi

$$\begin{aligned}\frac{d}{dt}\int_{\Omega(t)}\rho\frac{1}{2}|\boldsymbol{u}|^2d\boldsymbol{x}&=\int_{\Omega(t)}\rho\frac{1}{2}\frac{D}{Dt}|\boldsymbol{u}|^2d\boldsymbol{x}\\&=\rho\int_{\Omega(t)}\boldsymbol{f}\cdot\boldsymbol{u}d\boldsymbol{x}-\int_{\Omega(t)}\boldsymbol{T}\cdot\nabla\boldsymbol{u}d\boldsymbol{x}+\int_{\partial\Omega(t)}\boldsymbol{u}\cdot\boldsymbol{t}_ndS,\end{aligned}\tag{1.21}$$

pri čemu je $\boldsymbol{T}\cdot\nabla\boldsymbol{u}=\{T_{ji}\frac{\partial u_i}{\partial x_j}\}$ Frobeniusov unutarnji produkt matrica. Nadalje, iz (1.19) imamo

$$\int_{\partial\Omega(t)}w_iC_{ji}n_jdS=\int_{\Omega(t)}\left(C_{ji}\frac{\partial w_i}{\partial x_j}+\rho Iw_i\frac{Dw_i}{Dt}-\rho g_iw_i-T_{x,i}w_i\right)d\boldsymbol{x},$$

odnosno

$$\begin{aligned}\frac{d}{dt}\int_{\Omega(t)}\rho\frac{I}{2}|\boldsymbol{w}|^2d\boldsymbol{x}&=\int_{\Omega(t)}\rho\frac{I}{2}\frac{D}{Dt}|\boldsymbol{w}|^2d\boldsymbol{x}\\&=\rho\int_{\Omega(t)}\boldsymbol{g}\cdot\boldsymbol{w}d\boldsymbol{x}-\int_{\Omega(t)}\boldsymbol{C}\cdot\nabla\boldsymbol{w}d\boldsymbol{x}+\int_{\Omega(t)}\boldsymbol{T}_x\cdot\boldsymbol{w}d\boldsymbol{x}+\int_{\partial\Omega(t)}\boldsymbol{w}\cdot\boldsymbol{c}_ndS.\end{aligned}\tag{1.22}$$

Koristeći (1.21), (1.22) te Greenov teorem, iz (1.20) slijedi

$$\int_{\Omega(t)}\left(\rho\frac{DE}{Dt}+\operatorname{div}\boldsymbol{q}-\boldsymbol{T}\cdot\nabla\boldsymbol{u}-\boldsymbol{C}\cdot\nabla\boldsymbol{w}+\boldsymbol{T}_x\cdot\boldsymbol{w}\right)d\boldsymbol{x}=0.$$

Po Fourierovom zakonu kondukcije topline vrijedi

$$\boldsymbol{q}=-\kappa\nabla\theta,$$

gdje je $\kappa\geq 0$ toplinska vodljivost materijala te θ temperatura. Tada je diferencijalni oblik zakona očuvanja energije dan s

$$\rho\frac{DE}{Dt}=\operatorname{div}(k\nabla\theta)+\boldsymbol{T}\cdot\nabla\boldsymbol{u}+\boldsymbol{C}\cdot\nabla\boldsymbol{w}-\boldsymbol{T}_x\cdot\boldsymbol{w}.$$

1.3. Mikropolarni fluidi

1.3.1. Konstitutivne jednadžbe mikropolarnih fluida

Jednadžbe koje opisuju izotropni polarni fluid glase

$$\begin{aligned} \frac{D\rho}{Dt} &= -\rho \operatorname{div} \boldsymbol{u}, \\ \rho \frac{D\boldsymbol{u}}{Dt} &= \operatorname{div} \boldsymbol{T} + \rho \boldsymbol{f}, \\ \rho I \frac{D\boldsymbol{w}}{Dt} &= \operatorname{div} \boldsymbol{C} + \rho \boldsymbol{g} + \boldsymbol{T}_x, \\ \rho \frac{DE}{Dt} &= -\operatorname{div} \boldsymbol{q} + \boldsymbol{T} \cdot \nabla \boldsymbol{u} + \boldsymbol{C} \cdot \nabla \boldsymbol{w} - \boldsymbol{T}_x \cdot \boldsymbol{w}, \end{aligned} \tag{1.23}$$

pri čemu su (1.23) zakon očuvanja mase, zakon očuvanja količine gibanja, zakon očuvanja kutne količine gibanja te zakon očuvanja energije izvedeni u Odjeljcima 1.1 i 1.2. Mikropolarne fluide definiramo kao izotropne polarne fluide gdje su tenzor naprezanja $\boldsymbol{T}$ i tenzor naprezanja sprega $\boldsymbol{C}$ dani sljedećim konstitutivnim jednadžbama:

$$\begin{aligned} T_{ij} &= \left(-p + \lambda \frac{\partial u_k}{\partial x_k}\right) \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \mu_r \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) - 2\mu_r \epsilon_{mij} w_m, \\ C_{ij} &= c_0 \frac{\partial w_k}{\partial x_k} \delta_{ij} + c_d \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) + c_\alpha \left(\frac{\partial w_j}{\partial x_i} - \frac{\partial w_i}{\partial x_j}\right), \end{aligned} \tag{1.24}$$

pri čemu je λ drugi koeficijent viskoznosti, te μ dinamički koeficijent viskoznosti. Koeficijent μ_r je dinamička mikrorotacijska viskoznost, te su c_0, c_α, c_d koeficijenti kutnih viskoznosti. Koristeći drugi zakon termodinamike mogu se izvesti nužna ograničenja na koeficijente:

$$\mu \geq 0, \; 3\lambda + 2\mu \geq 0,$$

$$c_d \geq 0, \; c_\alpha + c_d \geq 0, \; 3c_0 + 2c_d \geq 0, \; |c_d - c_\alpha| \leq c_\alpha + c_d.$$

Uočimo da je simetrični dio tenzora naprezanja $\boldsymbol{T}$ upravo jednak tenzoru naprezanja Newtonovskog fluida

$$\boldsymbol{T}^{(s)} = (-p + \lambda \operatorname{div} \boldsymbol{u}) I + 2\mu \boldsymbol{D},$$

gdje je $\boldsymbol{D}$ tenzor deformacije dan s

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Ako uvrstimo $\boldsymbol{C} = \boldsymbol{0}$, $\boldsymbol{g} = \boldsymbol{w} = \boldsymbol{0}$ u jednadžbe polarnog fluida (1.24) dobivamo upravo Navier-Stokesove jednadžbe. Ovo potvrđuje da je klasični model Navier-Stokesa sadržan kao poseban slučaj u modelu polarnog fluida.

Nadalje, uočimo da se tenzor naprezanja sastoji od simetričnog i antisimetričnog dijela, te da je antisimetričan dio tenzora naprezanja jednak

$$\mu_r \epsilon_{mij} (\operatorname{rot} \boldsymbol{u} - 2\boldsymbol{w})_m.$$

U slučaju kada je mikrorotacija $\boldsymbol{w}$ upravo jednaka vrtložnosti $\frac{1}{2} \operatorname{rot} \boldsymbol{u}$, antisimetričan tenzor naprezanja je jednak nuli te je $\boldsymbol{T}$ jednak tenzoru naprezanja Newtonovskog fluida. Fizikalno, to znači da se rotacija čestica podudara s lokalnim krutim rotacijama fluida te ne dolazi do unutarnjih naprezanja.

Uvrštavanjem tenzora $\boldsymbol{T}$ i $\boldsymbol{C}$ danih s (1.24) u sustav jednadžbi (1.23) dobivamo

$$\begin{aligned} \frac{D\rho}{Dt} &= -\rho \operatorname{div} \boldsymbol{u}, \\ \rho \frac{D\boldsymbol{u}}{Dt} &= -\nabla p + (\lambda + \mu - \mu_r) \nabla \operatorname{div} \boldsymbol{u} + (\mu + \mu_r)\Delta \boldsymbol{u} + 2\mu_r \operatorname{rot} \boldsymbol{w} + \rho \boldsymbol{f}, \\ \rho I \frac{D\boldsymbol{w}}{Dt} &= 2\mu_r (\operatorname{rot} \boldsymbol{u} - 2\boldsymbol{w}) + (c_0 + c_d - c_\alpha) \nabla \operatorname{div} \boldsymbol{w} + (c_\alpha + c_d)\Delta \boldsymbol{w} + \rho \boldsymbol{g}, \\ \rho \frac{DE}{Dt} &= -p \operatorname{div} \boldsymbol{u} + \rho \boldsymbol{\Phi} - \operatorname{div} \boldsymbol{q}, \end{aligned} \tag{1.25}$$

pri čemu je

$$\begin{aligned} \rho \boldsymbol{\Phi} =& \lambda (\operatorname{div} \boldsymbol{u})^2 + 2\mu D \cdot D + 4\mu_r \left(\frac{1}{2} \operatorname{rot} \boldsymbol{u} - w \right)^2 + c_0 (\operatorname{div} \boldsymbol{w})^2 \\ &+ (c_\alpha + c_d) \nabla \boldsymbol{w} \cdot \nabla \boldsymbol{w} + (c_d - c_\alpha) \nabla \boldsymbol{w} \cdot (\nabla \boldsymbol{w})^T \end{aligned}$$

funkcija disipacije mehaničke energije po jedinici mase. U ovom radu ćemo promatrati viskozan i inkompresibilan fluid, odnosno takav da vrijedi $\mu > 0$ i $\operatorname{div} \boldsymbol{u} = 0$. Nadalje, pretpostavljamo da je unutarnja energija po jedinici mase proporcionalna temperaturi, to jest oblika je

$$E = c_r \theta, \;\; c_r > 0.$$

Konačno, sustav (1.25) sada glasi

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho &= 0, \\ \rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \right) &= -\nabla p + (\mu + \mu_r) \Delta \boldsymbol{u} + 2\mu_r \operatorname{rot} \boldsymbol{w} + \rho \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{u} &= 0, \\ \rho I \left(\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{w} \right) + 4\mu_r \boldsymbol{w} &= 2\mu_r \operatorname{rot} \boldsymbol{u} + (c_0 + c_d - c_\alpha) \nabla \operatorname{div} \boldsymbol{w} \\ &+ (c_\alpha + c_d) \Delta \boldsymbol{w} + \rho \boldsymbol{g}, \\ \rho c_r \left(\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta \right) &= 2\mu D \cdot D + 4\mu_r \left(\frac{1}{2} \operatorname{rot} \boldsymbol{u} - w \right)^2 + c_0 (\operatorname{div} \boldsymbol{w})^2 \\ &+ (c_\alpha + c_d) \nabla \boldsymbol{w} \cdot \nabla \boldsymbol{w} + (c_d - c_\alpha) \nabla \boldsymbol{w} \cdot (\nabla \boldsymbol{w})^T + \nabla (\kappa \nabla \theta). \end{aligned} \tag{1.26}$$

U nastavku ovog rada ćemo proučavati više različitih inačica modela (1.26) ovisno o problemu koji ćemo razmatrati, s pridruženim odgovarajućim rubnim i inicijalnim uvjetima.

1.3.2. Rubni uvjeti za mikrorotaciju

U matematičkoj literaturi se najčešće koriste Dirichletovi rubni uvjeti za brzinu $\boldsymbol{u}$ i mikrorotaciju $\boldsymbol{w}$

$$\boldsymbol{u} = \boldsymbol{U}, \ \ \boldsymbol{w} = \boldsymbol{W},$$

gdje su $\boldsymbol{U}, \boldsymbol{W}$ zadane funkcije koje pripadaju prikladnim prostorima te opisuju linearnu i kutnu brzinu točaka ruba domene. Ovi rubni uvjeti se nazivaju relativnim no–slip odnosno no–spin rubnim uvjetima, te odgovaraju slučaju kada su sile na granici koje uzrokuju "lijepljenje" čestica povezano s no–slip uvjetom dovoljno jake da obustave vrtnju čestica u odnosu na rotaciju granice. Dok ovakav rubni uvjet za mikrorotaciju olakšava analizu problema s tehničke strane, nije skroz jasno je li takvo ponašanje čestica fizikalno, te je li ovaj rubni uvjet preveliko ograničenje.

U inženjerskoj literaturi se predlaže nekoliko općenitijih rubnih uvjeta za koje se tvrdi da su fizikalno relevantniji. Aero, Bulygin i Kuvshinskii [2] razmatraju dva granična slučaja. Prvi je slučaj kada je mikrorotacija na rubu jednaka vrtložnosti na rubu uzrokovane linearnim gibanjem ruba

$$\boldsymbol{w} = \frac{1}{2} \operatorname{rot} \boldsymbol{U}. \tag{1.27}$$

Drugi promatrani slučaj je kada se čestice rotiraju neovisno o granici – tada mora vrijediti dinamički rubni uvjet

$$\boldsymbol{C}\boldsymbol{n} = \boldsymbol{0}. \tag{1.28}$$

Slučajevi između dva granična slučaja (1.27) i (1.28) se mogu opisati rubnim uvjetom

$$\boldsymbol{\alpha} \left(\boldsymbol{w} - \frac{1}{2} \operatorname{rot} \boldsymbol{U} \right) = \boldsymbol{C}\boldsymbol{n},$$

pri čemu je $\boldsymbol{\alpha} = \{\alpha_{ij}\}$ koeficijent rotacijskog površinskog trenja. Nadalje, Condiff i Dahler [24] predlažu dinamički rubni uvjet oblika

$$s \left(\boldsymbol{w} - \frac{1}{2} \operatorname{rot} \boldsymbol{U} \right) = \operatorname{rot} \boldsymbol{u} - 2\boldsymbol{w}. \tag{1.29}$$

U graničnom slučaju kada je $s = \infty$, dobivamo $\boldsymbol{w} = \frac{1}{2} \operatorname{rot} \boldsymbol{U}$, odnosno sile na zidu koje uzrokuju relativni no–slip rubni uvjet za brzinu su dovoljno jake da spriječe vrtnju molekula u odnosu na zid. S druge strane, u graničnom slučaju $s = 0$ čestice na zidu se slobodno vrte, te se prosječna kutna brzina čestica podudara s vrtložnosti. U tom slučaju ne dolazi do antisimetričnog naprezanja na zidu. U slučaju granice u mirovanju, rubni uvjet (1.29) možemo zapisati u obliku

$$\boldsymbol{w} = \frac{\alpha}{2}\,\mathrm{rot}\,\boldsymbol{u}, \;\; 0 \le \alpha \le 1. \tag{1.30}$$

Gornji rubni uvjet opisuje moguće slučajeve između dva granična slučaja, kada su utjecaji sile koje uzrokuju no-slip rubni uvjet na zidu te utjecaji mikrostrukture usporedivi, te je mjera interakcije tih utjecaja opisana parametrom α.

Opisani rubni uvjet (1.30) ćemo daljnje proučavati u Poglavlju 2, zajedno s jednadžbama nestacionarnog mikropolarnog fluida. Više o nestandardnim rubnim uvjetima za mikrorotaciju se može pronaći u npr. [25, 51, 67, 87].

1.4. Termomikropolarni fluidi

Godine 1972. Eringen ([33]) je predstavio daljnju generalizaciju mikropolarnih fluida, tzv. termomikropolarne fluide. Konstitutivna jednadžba za tenzor naprezanja sprega $\boldsymbol{C}$ za termomikropolarne fluide je dana s

$$C_{ij} = c_0 \frac{\partial w_k}{\partial x_k} \delta_{ij} + c_d \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right) + c_\alpha \left(\frac{\partial w_j}{\partial x_i} - \frac{\partial w_i}{\partial x_j} \right) + \delta \epsilon_{ijm} \frac{\partial \theta}{\partial x_m}, \tag{1.31}$$

gdje je δ mikropolarna toplinska vodljivost. Nadalje, pretpostavit ćemo da je gustoća ρ konstantna funkcija ρ_0 svugdje osim uz gustoću volumenske sile koja dolazi od sile teže $-\rho g_c \boldsymbol{e}_3$, gdje je jednaka

$$\rho = -\rho_0 \overline{\alpha} \theta.$$

Ovdje $\overline{\alpha}$ označava koeficijent termalne ekspanzije, te je g_c gravitacijsko ubrzanje. Uz ovu pretpostavku, uvrštavanjem $(1.24)_1$ i (1.31) za inkompresibilan fluid dobivamo sustav

$$\rho_0 \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \right) = -\nabla p + (\mu + \mu_r) \Delta \boldsymbol{u} + 2\mu_r \operatorname{rot} \boldsymbol{w} + \rho_0 \overline{\alpha} g_c \theta \boldsymbol{e}_3 + \rho_0 \boldsymbol{f},$$

$$\operatorname{div} \boldsymbol{u} = 0,$$

$$\rho_0 I \left(\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{w} \right) + 4\mu_r \boldsymbol{w} = 2\mu_r \operatorname{rot} \boldsymbol{u} + (c_0 + c_d - c_\alpha) \nabla \operatorname{div} \boldsymbol{w} + (c_\alpha + c_d) \Delta \boldsymbol{w} + \rho_0 \boldsymbol{g},$$

$$\rho_0 c_r \left(\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta \right) = 2\mu D \cdot D + 4\mu_r \left(\frac{1}{2} \operatorname{rot} \boldsymbol{u} - w \right)^2 + c_0 (\operatorname{div} \boldsymbol{w})^2$$

$$+ (c_\alpha + c_d) \nabla \boldsymbol{w} \cdot \nabla \boldsymbol{w} + (c_d - c_\alpha) \nabla \boldsymbol{w} \cdot (\nabla \boldsymbol{w})^T + \delta \operatorname{rot} \boldsymbol{w} \cdot \nabla \theta + \nabla (\kappa \nabla \theta).$$

Konačno, zanemarivanjem svih kvadratnih članova u disipacijskoj funkciji dolazimo do modela termomikropolarnog fluida koji ćemo proučavati u ovome radu:

$$\rho_0 \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \right) = -\nabla p + (\mu + \mu_r) \Delta \boldsymbol{u} + 2\mu_r \operatorname{rot} \boldsymbol{w} + \rho_0 \overline{\alpha} g_c \theta \boldsymbol{e}_3 + \rho_0 \boldsymbol{f},$$

$$\operatorname{div} \boldsymbol{u} = 0,$$

$$\rho_0 I \left(\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{w} \right) + 4\mu_r \boldsymbol{w} = 2\mu_r \operatorname{rot} \boldsymbol{u} + (c_0 + c_d - c_\alpha) \nabla \operatorname{div} \boldsymbol{w} + (c_\alpha + c_d) \Delta \boldsymbol{w} + \rho_0 \boldsymbol{g},$$

$$\rho_0 c_r \left(\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta \right) = \delta \operatorname{rot} \boldsymbol{w} \cdot \nabla \theta + \nabla (\kappa \nabla \theta).$$

Naglasimo da je uparenje tenzora naprezanja sprega s gradijentom temperature rezultiralo pojavom mikrorotacije u jednadžbi provođenja topline.

2. Asimptotička analiza stacionarnog termomikropolarnog fluida u zakrivljenom kanalu

U ovom poglavlju razmatramo generalizaciju modela mikropolarnog fluida opisanog u Poglavlju 1.4, koji uzima u obzir provođenje topline i utjecaj istog na tok fluida.

Inženjerske primjene ovog modela su bile izučavane u nekoliko nedavnih članaka ([23, 44, 45, 88]). Lukaszewicz, Pažanin i Radulović su predložili i rigorozno opravdali model stacionarnog termomikropolarnog toka u tankom nedeformiranom kanalu u [56], što nam je motivacija za razmotriti isti problem u tankom zakrivljenom kanalu. Tok termomikropolarnog fluida u dvodimenzionalnoj domeni je opisan sustavom jednadžbi

$$\rho_0(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\mu+\mu_r)\Delta\boldsymbol{u} + \nabla p = 2\mu_r\nabla^\perp w + \rho_0 g_c\overline{\alpha}\theta\boldsymbol{e}_2 + \rho_0\boldsymbol{f},$$

$$\operatorname{div}\boldsymbol{u} = 0,$$

$$\rho_0 I(\boldsymbol{u}\cdot\nabla w) - \alpha\Delta w + 4\mu_r w = 2\mu_r\operatorname{rot}\boldsymbol{u} + \rho_0 g,$$

$$\rho_0 c_r\boldsymbol{u}\cdot\nabla\theta - \kappa\Delta\theta = \delta\nabla^\perp w\cdot\nabla\theta.$$

Nepoznanice u gornjem sustavu su brzina $\boldsymbol{u} = (u_1, u_2)$, tlak p, mikrorotacija w i temperatura θ. Pozitivne konstante su dinamička Newtonovska viskoznost μ, dinamička mikrorotacijska viskoznost μ_r, konstantna gustoća fluida ρ_0, gravitacijsko ubrzanje g_c, koeficijent termalne ekspanzije $\overline{\alpha}$, mikroinercija I, koeficijent kutne viskoznosti $\alpha = c_\alpha + c_d$, toplinska vodljivost κ i mikropolarna toplinska vodljivost δ. U ovom poglavlju koristimo sljedeću notaciju: $\boldsymbol{e}_2 = (0,1)$, $\operatorname{rot}\boldsymbol{\Phi} = \frac{\partial\Phi_2}{\partial x_1} - \frac{\partial\Phi_1}{\partial x_2}$, $\nabla^\perp\varphi = \left(\frac{\partial\varphi}{\partial x_2}, -\frac{\partial\varphi}{\partial x_1}\right)$, $\boldsymbol{\Phi} = (\Phi_1, \Phi_2)$.

Nadalje, uvodimo sljedeće bezdimenzionalne varijable:

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{\ell}, \quad \hat{\boldsymbol{u}} = \frac{\rho_0 c_r \ell}{\kappa}\boldsymbol{u}, \quad \hat{\boldsymbol{w}} = \frac{\rho_0 c_r \ell^2}{\kappa}\boldsymbol{w},$$
$$\hat{p} = \frac{\rho_0 c_r^2 \ell^2}{\kappa^2} p, \quad \hat{\theta} = \frac{\theta}{\Delta T},$$

gdje je ℓ karakteristična duljina domene, te ΔT karakteristična razlika temperature. Tada promatrane jednadžbe u bezdimenzionalnom obliku glase

$$\begin{aligned}
\frac{1}{Pr}\left((\hat{\boldsymbol{u}} \cdot \nabla)\boldsymbol{u} + \nabla \hat{p}\right) &= \Delta \hat{\boldsymbol{u}} + \frac{N}{1-N}(2\nabla^{\perp}\hat{w} + \Delta \hat{\boldsymbol{u}}) + Ra\hat{\theta}\boldsymbol{e}_2 + \hat{\boldsymbol{f}}, \\
\operatorname{div} \hat{\boldsymbol{u}} &= 0, \\
\frac{M}{Pr}(\hat{\boldsymbol{u}} \cdot \nabla \hat{w}) - L\Delta \hat{w} &= \frac{2N}{1-N}(\operatorname{rot} \hat{\boldsymbol{u}} - 2\hat{w}) + \hat{g}, \\
\hat{\boldsymbol{u}} \cdot \nabla \hat{\theta} - \Delta \hat{\theta} &= D\nabla^{\perp}\hat{w} \cdot \nabla \hat{\theta}.
\end{aligned} \tag{2.1}$$

Bezdimenzionalne konstante koje se pojavljuju u (2.1) su dane s:

$$N = \frac{\mu_r}{\mu + \mu_r}, \quad M = \frac{I}{\ell^2}, \quad L = \frac{\alpha}{\ell^2 \mu}, \quad D = \frac{\delta}{\rho_0 c_r \ell^2},$$
$$Pr = \frac{c_r \mu}{\kappa}, \quad Ra = \frac{\rho_0^2 c_r \overline{\alpha} g_c \Delta T \ell^3}{\mu \kappa}.$$

Bezdimenzionalna konstanta N opisuje vezu između Newtonovskih i mikrorotacijskih viskoznosti, te vrijedi $0 \leq N < 1$. Konstanta M opisuje vezu između mikroinercije i geometrije kanala, parametar L (couple stress parameter) opisuje vezu između svojstva fluida i geometrije kanala, Prandtlov broj Pr opisuje vezu između kinematičke viskoznosti i toplinske difuzivnosti, dok Rayleighov broj Ra opisuje prirodnu konvekciju nastalu zbog sile uzgona.

Poglavlje je organizirano na sljedeći način. U Odjeljku 2.1 formalno opisujemo geometriju zakrivljenog kanala, te promatranim bezdimenzionalnim termomikropolarnim jednadžbama pridružujemo odgovarajuće rubne uvjete. U Odjeljku 2.2 izvodimo asimptotičku aproksimaciju rješenja u eksplicitnom obliku, što nam omogućuje da uočimo efekte zakrivljenosti domene kao i mikropolarnosti fluida. Nadalje, provodimo analizu rubnog sloja pošto regularni dio aproksimacije ne zadovoljava rubne uvjete na krajevima kanala. Konačno, u Odjeljku 2.3 dobiveni asimptotički model rigorozno opravdamo tako da ocjenjujemo razliku između originalnog rješenja i izvedene asimptotičke aproksimacije.

2.1. Postavka problema

Jednadžbe ćemo promatrati u tankom zakrivljenom kanalu danom s:

$$\tilde{\Omega}_\varepsilon = \left\{ (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2 : 0 < \tilde{x}_1 < 1,\ h(\tilde{x}_1) < \tilde{x}_2 < h(\tilde{x}_1) + \varepsilon h_0(\tilde{x}_1) \right\},$$

gdje su funkcije h i h_0 klase $C^2([0,1])$. Uvodimo sljedeće pretpostavke na h i h_0:

$$\begin{gathered} h(0) = 0, \quad h_0(0) = h_0(1), \quad \int_0^1 h_0(x)dx = 1, \\ (\exists c > 0) \quad h_0(x) \geq c > 0,\ \forall x \in [0,1]. \end{gathered} \tag{2.2}$$

Donji i gornji zid kanala dani su s:

$$\tilde{\Gamma}_-^\varepsilon = \left\{ (\tilde{x}_1, h(\tilde{x}_1)) : \tilde{x}_1 \in (0,1) \right\}, \quad \tilde{\Gamma}_+^\varepsilon = \left\{ (\tilde{x}_1, h(\tilde{x}_1) + \varepsilon h_0(\tilde{x}_1)) : \tilde{x}_1 \in (0,1) \right\},$$

dok su krajevi kanala označeni s:

$$\tilde{\Sigma}_i^\varepsilon = \overline{\tilde{\Omega}_\varepsilon} \cap \{\tilde{x}_1 = i\},\ i = 0, 1.$$

Jednadžbe koje opisuju stacionarni tok inkompresibilnog, termomikropolarnog fluida u $\tilde{\Omega}_\varepsilon$ u bezdimenzionalnom obliku glase

$$\begin{aligned} \frac{1}{Pr}((\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon) &= \Delta \boldsymbol{u}_\varepsilon + \frac{N}{1-N}(2\nabla^\perp w_\varepsilon + \Delta \boldsymbol{u}_\varepsilon) + Ra\theta_\varepsilon \boldsymbol{e}_2 + \boldsymbol{f}_\varepsilon, \\ \operatorname{div} \boldsymbol{u}_\varepsilon &= 0, \\ \frac{M}{Pr}(\boldsymbol{u}_\varepsilon \cdot \nabla w_\varepsilon) &= L\Delta w_\varepsilon + \frac{2N}{1-N}(\operatorname{rot} \boldsymbol{u}_\varepsilon - 2w_\varepsilon) + g_\varepsilon, \\ \boldsymbol{u}_\varepsilon \cdot \nabla \theta_\varepsilon &= \Delta\theta_\varepsilon + D\nabla^\perp w_\varepsilon \cdot \nabla\theta_\varepsilon, \end{aligned} \tag{2.3}$$

gdje je $\boldsymbol{u}_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ brzina, w mikrorotacija te p tlak. Sustavu jednadžbi (2.3) pridružujemo sljedeće rubne uvjete:

$$\begin{gathered} u_2^\varepsilon = 0 \text{ na } \tilde{\Sigma}_i^\varepsilon,\ i = 0,1, \quad \boldsymbol{u}_\varepsilon = \boldsymbol{0} \text{ na } \tilde{\Gamma}_\pm^\varepsilon, \\ p_\varepsilon = \frac{1}{\varepsilon^2} Q_i \text{ na } \tilde{\Sigma}_i^\varepsilon,\ i = 0,1, \quad w_\varepsilon = 0 \text{ na } \partial\tilde{\Omega}_\varepsilon, \\ \theta_\varepsilon = \theta_i \text{ na } \tilde{\Sigma}_i^\varepsilon,\ i = 0,1, \quad \frac{\partial\theta_\varepsilon}{\partial\tilde{x}_2} = 0 \text{ na } \tilde{\Gamma}_-^\varepsilon, \quad \frac{\partial\theta_\varepsilon}{\partial\tilde{x}_2} = Nus^\varepsilon(G_T(\tilde{x}_1) - \theta_\varepsilon) \text{ na } \tilde{\Gamma}_+^\varepsilon, \end{gathered} \tag{2.4}$$

gdje je G_T vanjska temperatura za koju pretpostavljamo da je ograničena funkcija. Na ulazu i izlazu kanala smo zadali konstantne tlakove Q_0 i Q_1, odnosno tok je određen razlikom tlakova između krajeva kanala. Na $\tilde{\Gamma}_-^\varepsilon$ smo zadali homogeni Neumannov uvjet za temperaturu θ_ε. Donji zid kanala je izoliran, dok je Robinov rubni uvjet na $\tilde{\Gamma}_+^\varepsilon$ u (2.4)$_3$ zapravo Newtonov zakon hlađenja koji opisuje izmjenu topline između vanjskog sredstva i fluida unutar kanala kroz gornji zid. Bezdimenzionalna konstanta Nus^ε je Nusseltov broj dan kao

$$Nus^\varepsilon = \frac{h_p \ell}{\kappa},$$

gdje je h_p koeficijent prolaska topline. Nusseltov broj opisuje omjer konvekcije i kondukcije, odnosno strujanja i vođenja topline preko granice kanala. Motivirani raznim rezultatima za Newtonovske i mikropolarne tokove fluida (vidi [10, 56, 61–63]), uzimamo sljedeće skaliranje:

$$Nus^\varepsilon = \varepsilon k, \quad k = O(1),$$

što znači da je koeficijent prolaska topline h_p malog reda veličine, što je najzanimljiviji slučaj. Naime, u slučaju skaliranja $Nus^\varepsilon \gg O(\varepsilon)$ temperatura vanjskog sredstva dominira procesom, dok su u slučaju skaliranja $Nus^\varepsilon \ll O(\varepsilon)$ efekti hlađenja gornjeg zida kanala zanemarivi.

Napomena 2.1.1. *Ako vrijedi $\boldsymbol{f}_\varepsilon, g_\varepsilon \in L^2(\tilde{\Omega}_\varepsilon)$, te ako su koeficijenti μ, α i κ dovoljno veliki i razlika tlakova na rubovima kanala dovoljno mala, može se pokazati da problem* (2.3)–(2.4) *ima slabo rješenje $\boldsymbol{u}_\varepsilon, w_\varepsilon \in W^{1,2}(\tilde{\Omega}_\varepsilon)$, $p_\varepsilon, \theta_\varepsilon \in L^2(\tilde{\Omega}_\varepsilon)$, koristeći pristup kao u [55, Poglavlje 2, Teorem 2.4.1] te [61, Odjeljak 2]. Nadalje, radi rubnih uvjeta za tlak te nelinearnosti danog problema, brzina $\boldsymbol{u}_\varepsilon$ je jedinstvena u nekoj kugli $B_{R_0} = \{\boldsymbol{u} \in V_\varepsilon \colon \|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)} \leq R_0\}$, gdje radijus R_0 ostaje ograničen kada $\varepsilon \to 0$, pri čemu je $V_\varepsilon = \{\boldsymbol{u}_\varepsilon \in W^{1,2}(\tilde{\Omega}_\varepsilon) \colon \operatorname{div} \boldsymbol{u}_\varepsilon = 0$ u $\tilde{\Omega}_\varepsilon$, $\boldsymbol{u}_\varepsilon = 0$ na $\tilde{\Gamma}_\pm$ $u_\varepsilon^2 = 0$ na $\tilde{\Sigma}_i^\varepsilon, i = 0, 1\}$ (vidi [42] i [60]). Napomenimo da je dobra postavljenost dvodimenzionalnog Rayleigh–Bénardovog problema za termomikropolarne fluide proučavana u [48].*

Zamjena varijabli kojom prelazimo iz zakrivljenog kanala $\tilde{\Omega}_\varepsilon$ u tanki pravokutnik $\Omega_\varepsilon = (0, 1) \times (0, \varepsilon)$ je dana s difeomorfizmom $\boldsymbol{\Phi} : \Omega_\varepsilon \to \tilde{\Omega}_\varepsilon$:

$$\boldsymbol{\Phi}(x_1, x_2) = (x_1, h_0(x_1)x_2 + h(x_1)).$$

Naglasimo da preslikavanje $\boldsymbol{\Phi}$ ne ovisi o malom parametru ε. Nakon zamjene varijabli, promatrani sustav jednadžbi (2.3)–(2.4) na tankom pravokutniku $\boldsymbol{\Omega}_\varepsilon$ glasi:

$$
\begin{aligned}
&\frac{1}{Pr}((\boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{v}_\varepsilon + \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla q_\varepsilon) = \operatorname{div}(\nabla \boldsymbol{v}_\varepsilon \tilde{\boldsymbol{F}}(\boldsymbol{\Phi})) \\
&+\frac{N}{1-N}(2\nabla\boldsymbol{\Phi}\nabla^\perp z_\varepsilon + \operatorname{div}(\nabla \boldsymbol{v}_\varepsilon \tilde{\boldsymbol{F}}(\boldsymbol{\Phi}))) + Ra\det(\nabla\boldsymbol{\Phi})S_\varepsilon \boldsymbol{e}_2 + \det(\nabla\boldsymbol{\Phi})\boldsymbol{F}_\varepsilon, \\
&\operatorname{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T \boldsymbol{v}_\varepsilon) = 0, \\
&\frac{M}{Pr}(\boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla z_\varepsilon) = L\operatorname{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla z_\varepsilon) + \frac{2N}{1-N}(\operatorname{rot}((\nabla\boldsymbol{\Phi})^T \boldsymbol{v}_\varepsilon) \\
&-2\det(\nabla\boldsymbol{\Phi})z_\varepsilon) + \det(\nabla\boldsymbol{\Phi})G_\varepsilon, \\
&\boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla S_\varepsilon = \operatorname{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla S_\varepsilon) + D\nabla^\perp z_\varepsilon \cdot \nabla S_\varepsilon,
\end{aligned}
\tag{2.5}
$$

te su odgovarajući rubni uvjeti dani s:

$$
\begin{aligned}
&v_2^\varepsilon = 0 \text{ za } x_1 = 0,1, \quad \boldsymbol{v}_\varepsilon = \boldsymbol{0} \text{ za } x_2 = 0,\varepsilon, \\
&q_\varepsilon = \frac{1}{\varepsilon^2}Q_0 \text{ za } x_1 = 0, \quad q_\varepsilon = \frac{1}{\varepsilon^2}Q_1 \text{ za } x_1 = 1, \\
&z_\varepsilon = 0 \text{ za } x_1 = 0,1, \quad z_\varepsilon = 0 \text{ za } x_2 = 0,\varepsilon, \\
&S_\varepsilon = \theta_0 \text{ za } x_1 = 0, \quad S_\varepsilon = \theta_1 \text{ za } x_1 = 1, \\
&\frac{\partial S_\varepsilon}{\partial x_2} = 0 \text{ za } x_2 = 0, \quad \frac{\partial S_\varepsilon}{\partial x_2} = Nus^\varepsilon(G_T(x_1) - S_\varepsilon) \text{ za } x_2 = \varepsilon,
\end{aligned}
\tag{2.6}
$$

gdje je $\boldsymbol{v}_\varepsilon = \boldsymbol{u}_\varepsilon \circ \boldsymbol{\Phi} = (v_1^\varepsilon, v_2^\varepsilon)$ brzina, $q_\varepsilon = p_\varepsilon \circ \boldsymbol{\Phi}$ tlak, $z_\varepsilon = w_\varepsilon \circ \boldsymbol{\Phi}$ mikrorotacija i $S_\varepsilon = \theta_\varepsilon \circ \boldsymbol{\Phi}$ temperatura, $\boldsymbol{F}_\varepsilon = \boldsymbol{f}_\varepsilon \circ \boldsymbol{\Phi}$ i $G_\varepsilon = g_\varepsilon \circ \boldsymbol{\Phi}$ vanjske sile, te $\tilde{\boldsymbol{G}}(\boldsymbol{\Phi}) = \operatorname{cof}(\nabla\boldsymbol{\Phi})$ i $\tilde{\boldsymbol{F}}(\boldsymbol{\Phi}) = (\nabla\boldsymbol{\Phi})^{-1}\operatorname{cof}(\nabla\boldsymbol{\Phi})$. Nadalje, pretpostavljamo da vanjske sile $\boldsymbol{F}_\varepsilon = (F_1^\varepsilon, 0), G_\varepsilon$ ne ovise o varijabli x_2, što je opravdano ako uzmemo u obzir debljinu kanala. U ostatku poglavlja uzimamo skaliranje $F_1^\varepsilon = \frac{1}{\varepsilon^2}F_1$ and $G_\varepsilon = \frac{1}{\varepsilon^2}G$.

Cilj ovog poglavlja je izvesti i rigorozno opravdati model koji opisuje tok fluida dan sustavom jednadžbi (2.3)–(2.4). U tu svrhu, u sljedećem odjeljku konstruiramo asimptotičku aproksimaciju rješenja transformiranog problema (2.5)–(2.6), te nakon toga provodimo analizu rubnog sloja.

2.2. Asimptotička analiza

Uvođenjem brze varijable $y_2 = \frac{x_2}{\varepsilon}$ dobivamo sljedeći sustav jednadžbi na $\Omega = (0,1)^2$:

$$
\begin{aligned}
&\frac{1}{Pr}\Big[h_0(x_1)V_1^\varepsilon\frac{\partial \boldsymbol{V}_\varepsilon}{\partial x_1} - h_0'(x_1)y_2V_1^\varepsilon\frac{\partial \boldsymbol{V}_\varepsilon}{\partial y_2} + \frac{1}{\varepsilon}\left(-h'(x_1)V_1^\varepsilon + V_2^\varepsilon\right)\frac{\partial \boldsymbol{V}_\varepsilon}{\partial y_2} + h_0(x_1)\frac{\partial Q_\varepsilon}{\partial x_1}\mathbf{e}_1\\
&-h_0'(x_1)y_2\frac{\partial Q_\varepsilon}{\partial y_2}\boldsymbol{e}_1 + \frac{1}{\varepsilon}\left(-h'(x_1)\boldsymbol{e}_1 + \boldsymbol{e}_2\right)\frac{\partial Q_\varepsilon}{\partial y_2}\Big] = \frac{1}{1-N}\Big[\frac{\partial}{\partial x_1}\left(h_0(x_1)\frac{\partial \boldsymbol{V}_\varepsilon}{\partial x_1}\right)\\
&-\frac{\partial}{\partial x_1}\left(h_0'(x_1)y_2\frac{\partial \boldsymbol{V}_\varepsilon}{\partial y_2}\right) - \frac{\partial}{\partial y_2}\left(h_0'(x_1)y_2\frac{\partial \boldsymbol{V}_\varepsilon}{\partial x_1}\right) + \frac{\partial}{\partial y_2}\left(\frac{h_0'(x_1)^2}{h_0(x_1)}y_2^2\frac{\partial \boldsymbol{V}_\varepsilon}{\partial y_2}\right)\\
&-\frac{1}{\varepsilon}\frac{\partial}{\partial x_1}\left(h'(x_1)\frac{\partial \boldsymbol{V}_\varepsilon}{\partial y_2}\right) - \frac{1}{\varepsilon}h'(x_1)\frac{\partial^2 \boldsymbol{V}_\varepsilon}{\partial y_2\partial x_1} + 2\frac{1}{\varepsilon}\frac{\partial}{\partial y_2}\left(\frac{h_0'(x_1)h'(x_1)}{h_0(x_1)}y_2\frac{\partial \boldsymbol{V}_\varepsilon}{\partial y_2}\right)\\
&+\frac{1}{\varepsilon^2}\frac{h'(x_1)^2+1}{h_0(x_1)}\frac{\partial^2 \boldsymbol{V}_\varepsilon}{\partial y_2^2}\Big] + \frac{2N}{1-N}\Big[h_0'(x_1)y_2\frac{\partial Z_\varepsilon}{\partial y_2}\boldsymbol{e}_2 - h_0(x_1)\frac{\partial Z_\varepsilon}{\partial x_1}\boldsymbol{e}_2\\
&+\frac{1}{\varepsilon}(\boldsymbol{e}_1 + h'(x_1)\boldsymbol{e}_2)\frac{\partial Z_\varepsilon}{\partial y_2}\Big] + Ra\,h_0(x_1)T_\varepsilon\boldsymbol{e}_2 + \frac{1}{\varepsilon^2}h_0(x_1)F_1\boldsymbol{e}_1,\\
&h_0(x_1)\frac{\partial V_1^\varepsilon}{\partial x_1} - h_0'(x_1)y_2\frac{\partial V_1^\varepsilon}{\partial y_2} + \frac{1}{\varepsilon}\left(-h'(x_1)\frac{\partial V_1^\varepsilon}{\partial y_2} + \frac{\partial V_2^\varepsilon}{\partial y_2}\right) = 0,\\
&\frac{M}{Pr}\Big[h_0(x_1)V_1^\varepsilon\frac{\partial Z_\varepsilon}{\partial x_1} - h_0'(x_1)y_2V_1^\varepsilon\frac{\partial Z_\varepsilon}{\partial y_2} - \frac{1}{\varepsilon}h'(x_1)V_1^\varepsilon\frac{\partial Z_\varepsilon}{\partial y_2} + \frac{1}{\varepsilon}V_2^\varepsilon\frac{\partial Z_\varepsilon}{\partial y_2}\Big]\\
&= L\Big[\frac{\partial}{\partial x_1}\left(h_0(x_1)\frac{\partial Z_\varepsilon}{\partial x_1}\right) - \frac{\partial}{\partial x_1}\left(h_0'(x_1)y_2\frac{\partial Z_\varepsilon}{\partial y_2}\right) - \frac{\partial}{\partial y_2}\left(h_0'(x_1)y_2\frac{\partial Z_\varepsilon}{\partial x_1}\right)\\
&+\frac{\partial}{\partial y_2}\left(\frac{h_0'(x_1)^2}{h_0(x_1)}y_2^2\frac{\partial Z_\varepsilon}{\partial y_2}\right) - \frac{1}{\varepsilon}\frac{\partial}{\partial x_1}\left(h'(x_1)\frac{\partial Z_\varepsilon}{\partial y_2}\right) - \frac{1}{\varepsilon}h'(x_1)\frac{\partial^2 Z_\varepsilon}{\partial y_2\partial x_1}\\
&+2\frac{1}{\varepsilon}\frac{\partial}{\partial y_2}\left(\frac{h_0'(x_1)h'(x_1)}{h_0(x_1)}y_2\frac{\partial Z_\varepsilon}{\partial y_2}\right) + \frac{1}{\varepsilon^2}\frac{h'(x_1)^2+1}{h_0(x_1)}\frac{\partial^2 Z_\varepsilon}{\partial y_2^2}\Big] + \frac{2N}{1-N}\Big[h_0(x_1)\frac{\partial V_2^\varepsilon}{\partial x_1}\\
&-h_0'(x_1)y_2\frac{\partial V_2^\varepsilon}{\partial y_2} - \frac{1}{\varepsilon}\frac{\partial V_1^\varepsilon}{\partial y_2} - \frac{1}{\varepsilon}h'(x_1)\frac{\partial V_2^\varepsilon}{\partial y_2} - 2h_0(x_1)Z_\varepsilon\Big] + \frac{1}{\varepsilon^2}h_0(x_1)G,\\
&h_0(x_1)V_1^\varepsilon\frac{\partial T_\varepsilon}{\partial x_1} - h_0'(x_1)y_2V_1^\varepsilon\frac{\partial T_\varepsilon}{\partial y_2} - \frac{1}{\varepsilon}h'(x_1)V_1^\varepsilon\frac{\partial T_\varepsilon}{\partial y_2} + \frac{1}{\varepsilon}V_2^\varepsilon\frac{\partial T_\varepsilon}{\partial y_2} = \frac{\partial}{\partial x_1}\left(h_0(x_1)\frac{\partial T_\varepsilon}{\partial x_1}\right)\\
&-\frac{\partial}{\partial x_1}\left(h_0'(x_1)y_2\frac{\partial T_\varepsilon}{\partial y_2}\right) - \frac{\partial}{\partial y_2}\left(h_0'(x_1)y_2\frac{\partial T_\varepsilon}{\partial x_1}\right) + \frac{\partial}{\partial y_2}\left(\frac{h_0'(x_1)^2}{h_0(x_1)}y_2^2\frac{\partial T_\varepsilon}{\partial y_2}\right)\\
&-\frac{1}{\varepsilon}\frac{\partial}{\partial x_1}\left(h'(x_1)\frac{\partial T_\varepsilon}{\partial y_2}\right) - \frac{1}{\varepsilon}h'(x_1)\frac{\partial^2 T_\varepsilon}{\partial y_2\partial x_1} + 2\frac{1}{\varepsilon}\frac{\partial}{\partial y_2}\left(\frac{h_0'(x_1)h'(x_1)}{h_0(x_1)}y_2\frac{\partial T_\varepsilon}{\partial y_2}\right)\\
&+\frac{1}{\varepsilon^2}\frac{h'(x_1)^2+1}{h_0(x_1)}\frac{\partial^2 T_\varepsilon}{\partial y_2^2} + D\left(\frac{1}{\varepsilon}\frac{\partial Z_\varepsilon}{\partial y_2}\frac{\partial T_\varepsilon}{\partial x_1} - \frac{1}{\varepsilon}\frac{\partial Z_\varepsilon}{\partial x_1}\frac{\partial T_\varepsilon}{\partial y_2}\right).
\end{aligned}
\tag{2.7}
$$

Pripadni rubni uvjeti glase:

$$\begin{aligned}
V_2^\varepsilon = 0 \text{ za } x_1 = 0,1, \quad & \boldsymbol{V}_\varepsilon = \boldsymbol{0} \text{ za } y_2 = 0,1, \\
Q_\varepsilon = \frac{1}{\varepsilon^2} Q_0 \text{ za } x_1 = 0, \quad & Q_\varepsilon = \frac{1}{\varepsilon^2} Q_1 \text{ za } x_1 = 1, \\
Z_\varepsilon = 0 \text{ za } x_1 = 0,1, \quad & Z_\varepsilon = 0 \text{ za } y_2 = 0,1. \\
T_\varepsilon = \theta_0 \text{ za } x_1 = 0, \quad & T_\varepsilon = \theta_1 \text{ za } x_1 = 1, \\
\frac{\partial T_\varepsilon}{\partial y_2} = 0 \text{ za } y_2 = 0, \quad & \frac{\partial T_\varepsilon}{\partial y_2} = \varepsilon^2 k (G_T(x_1) - T_\varepsilon) \text{ za } y_2 = 1.
\end{aligned} \tag{2.8}$$

Nepoznanice u (2.7)–(2.8) su sljedeće: $\boldsymbol{V}_\varepsilon(x_1, y_2) = \boldsymbol{v}_\varepsilon(x_1, \varepsilon y_2) = (V_1^\varepsilon(x_1, y_2), V_2^\varepsilon(x_1, y_2))$, $Q_\varepsilon(x_1, y_2) = q_\varepsilon(x_1, \varepsilon y_2)$, $Z_\varepsilon(x_1, y_2) = z_\varepsilon(x_1, \varepsilon y_2)$, $T_\varepsilon(x_1, y_2) = S_\varepsilon(x_1, \varepsilon y_2)$. Tražimo rješenje problema (2.7)–(2.8) u obliku asimptotičkog razvoja po malom parametru ε:

$$\begin{aligned}
\boldsymbol{V}_\varepsilon(x_1, y_2) &= \boldsymbol{V}^0(x_1, y_2) + \varepsilon \boldsymbol{V}^1(x_1, y_2) + \varepsilon^2 \boldsymbol{V}^2(x_1, y_2) + \dots, \\
Q_\varepsilon(x_1, y_2) &= \frac{1}{\varepsilon^2} Q^0(x_1, y_2) + \frac{1}{\varepsilon} Q^1(x_1, y_2) + Q^2(x_1, y_2) + \varepsilon Q^3(x_1, y_2) + \dots, \\
Z_\varepsilon(x_1, y_2) &= Z^0(x_1, y_2) + \varepsilon Z^1(x_1, y_2) + \varepsilon^2 Z^2(x_1, y_2) + \dots, \\
T_\varepsilon(x_1, y_2) &= T^0(x_1, y_2) + \varepsilon T^1(x_1, y_2) + \varepsilon^2 T^2(x_1, y_2) + \dots .
\end{aligned} \tag{2.9}$$

2.2.1. Regularni dio razvoja

2.2.1.1 Aproksimacija nultog reda

Sada uvrštavamo (2.9) u sustav jednadžbi (2.7)–(2.8), te prikupljamo članove uz istu potenciju malog parametra ε. Izjednačavanjem članova koji se pojavljuju uz ε^{-3} dobivamo:

$$\frac{1}{\varepsilon^3} : \quad (-h'(x_1)\boldsymbol{e}_1 + \boldsymbol{e}_2) \frac{\partial Q^0}{\partial y_2} = 0,$$

iz čega zaključujemo $Q^0 = Q^0(x_1)$.

Aproksimacija brzine nultog reda $\boldsymbol{V}^0$ dana je sljedećim sustavom jednadžbi:

$$\begin{aligned}
\frac{1}{\varepsilon^2} &: \; -\frac{1}{1-N} \frac{h'(x_1)^2 + 1}{h_0(x_1)} \frac{\partial^2 V_1^0}{\partial y_2^2} = h_0(x_1) F_1 + \frac{1}{Pr} \left(-h_0(x_1) \frac{\partial Q^0}{\partial x_1} + h'(x_1) \frac{\partial Q^1}{\partial y_2} \right), \\
\frac{1}{\varepsilon^2} &: \; -\frac{1}{1-N} \frac{h'(x_1)^2 + 1}{h_0(x_1)} \frac{\partial^2 V_2^0}{\partial y_2^2} = -\frac{1}{Pr} \frac{\partial Q^1}{\partial y_2}, \\
\frac{1}{\varepsilon} &: \; -h'(x_1) \frac{\partial V_1^0}{\partial y_2} + \frac{\partial V_2^0}{\partial y_2} = 0, \\
1 &: \; \boldsymbol{V}^0(x_1, 0) = 0, \quad \boldsymbol{V}^0(x_1, 1) = 0, \\
\frac{1}{\varepsilon^2} &: \; Q^0(0) = Q_0, \quad Q^0(1) = Q_1.
\end{aligned} \tag{2.10}$$

Ako izrazimo derivaciju po y_2 druge komponente brzine preko derivacije po y_2 prve komponente brzine koristeći $(2.10)_3$, pomnožimo $(2.10)_2$ s $h'(x_1)$ i dodamo $(2.10)_1$ dobivamo sljedeću jednakost:

$$-\frac{1}{1-N}\frac{(h'(x_1)^2+1)^2}{h_0(x_1)}\frac{\partial^2 V_1^0}{\partial y_2^2} = h_0(x_1)F_1 - \frac{h_0(x_1)}{Pr}\frac{\partial Q^0}{\partial x_1}. \tag{2.11}$$

Sada iz (2.11), $(2.10)_3$ te rubnih uvjeta $(2.10)_4$ imamo

$$\begin{aligned} V_1^0(x_1,y_2) &= \frac{1-N}{2}\left[\frac{h_0(x_1)}{h'(x_1)^2+1}\right]^2\left(\frac{1}{Pr}\frac{\partial Q^0}{\partial x_1} - F_1\right)(y_2^2-y_2), \\ V_2^0(x_1,y_2) &= \frac{1-N}{2}h'(x_1)\left[\frac{h_0(x_1)}{h'(x_1)^2+1}\right]^2\left(\frac{1}{Pr}\frac{\partial Q^0}{\partial x_1} - F_1\right)(y_2^2-y_2). \end{aligned} \tag{2.12}$$

Aproksimaciju tlaka nultog reda Q^0 možemo odrediti iz uvjeta inkompresibilnosti za korektor brzine prvog reda (vidi $(2.26)_3$):

$$1: \; h_0(x_1)\frac{\partial V_1^0}{\partial x_1} - h_0'(x_1)y_2\frac{\partial V_1^0}{\partial y_2} - h'(x_1)\frac{\partial V_1^1}{\partial y_2} + \frac{\partial V_2^1}{\partial y_2} = 0. \tag{2.13}$$

Integriranjem lijeve strane (2.13) po $[0,a]\times[0,1]$ dobivamo

$$\begin{aligned} &\int_0^a\int_0^1\left(h_0(x_1)\frac{\partial V_1^0}{\partial x_1} - h_0'(x_1)y_2\frac{\partial V_1^0}{\partial y_2} - h'(x_1)\frac{\partial V_1^1}{\partial y_2} + \frac{\partial V_2^1}{\partial y_2}\right)dy_2dx_1 \\ &= \int_0^a\int_0^1\left(\frac{\partial}{\partial x_1}(h_0(x_1)V_1^0) - h_0'(x_1)\frac{\partial}{\partial y_2}(y_2V_1^0)\right)dy_2dx_1 \\ &= \int_0^1 h_0(x_1)V_1^0\Big|_0^a dy_2 - \int_0^a h_0'(x_1)y_2V_1^0\Big|_0^1 dx_1 \\ &= \int_0^1 h_0(a)V_0^1(a,y_2)dy_2 - \int_0^1 h_0(0)V_0^1(0,y_2)dy_2, \end{aligned} \tag{2.14}$$

gdje smo iskoristili $(2.26)_4$ te $(2.10)_4$. Iz (2.13) i (2.14) sada slijedi

$$\int_0^1 h_0(a)V_1^0(a,y_2)dy_2 = C_1, \tag{2.15}$$

gdje je C_1 konstanta koja ne ovisi o a. Uvrštavanjem $(2.12)_1$ u (2.15) dobivamo običnu diferencijalnu jednadžbu

$$C_1 = -\frac{1-N}{12}\frac{h_0(x_1)^3}{(h'(x_1)^2+1)^2}\left(\frac{1}{Pr}\frac{\partial Q^0}{\partial x_1} - F_1\right), \tag{2.16}$$

čije rješenje uz rubne uvjete $Q^0(0)=Q_0, \;\; Q^0(1)=Q_1$ glasi:

$$Q^0(x_1) = Pr\int_0^{x_1}F_1(\xi)d\xi - \frac{12C_1Pr}{1-N}\int_0^{x_1}\frac{(h'(s)^2+1)^2}{h_0(s)^3}ds + C_2. \tag{2.17}$$

Konstante C_1, C_2 u (2.17) su dane s

$$C_1 = \frac{1-N}{12Pr}\left(\int_0^1 \frac{(h'(s)^2+1)^2}{h_0(s)^3}ds\right)^{-1}\left(Q_0 - Q_1 + Pr\int_0^1 F_1(\xi)d\xi\right), \quad C_2 = Q_0. \tag{2.18}$$

Sada iz (2.12) i (2.16) dobivamo aproksimaciju brzine nultog reda:

$$\boldsymbol{V}^0(x_1, y_2) = -\frac{6C_1}{h_0(x_1)}(\boldsymbol{e}_1 + h'(x_1)\boldsymbol{e}_2)(y_2^2 - y_2). \tag{2.19}$$

Nadalje, aproksimacija mikrorotacije nultog reda Z^0 zadovoljava sljedeći sustav jednadžbi:

$$\begin{aligned} \frac{1}{\varepsilon^2} &: \; -L\frac{h'(x_1)^2+1}{h_0(x_1)}\frac{\partial^2 Z^0}{\partial y_2^2} = h_0(x_1)G, \\ 1 &: \quad Z^0(x_1, 0) = 0, \quad Z^0(x_1, 1) = 0. \end{aligned} \tag{2.20}$$

Rješenje problema (2.20) je dano s

$$Z^0(x_1, y_2) = -\frac{1}{2L}\frac{h_0(x_1)^2}{h'(x_1)^2+1}G(x_1)(y_2^2 - y_2). \tag{2.21}$$

Pretpostavimo li dodatno da vrijedi $G(0) = G(1) = 0$, aproksimacija mikrorotacije nultog reda Z^0 dana s (2.21) zadovoljava rubne uvjete na krajevima kanala.

Aproksimacija temperature nultog reda T^0 je dana sustavom

$$\begin{aligned} \frac{1}{\varepsilon^2} &: \qquad -\frac{h'(x_1)^2+1}{h_0(x_1)}\frac{\partial^2 T^0}{\partial y_2^2} = 0, \\ 1 &: \; \frac{\partial T^0}{\partial y_2}(x_1, 0) = 0, \quad \frac{\partial T^0}{\partial y_2}(x_1, 1) = 0. \end{aligned} \tag{2.22}$$

Primijetimo da u ovom trenutku ne možemo odrediti aproksimaciju temperature nultog reda T^0 iz (2.22). Međutim, možemo zaključiti da vrijedi $T^0 = T^0(x_1)$, što će u sljedećem koraku pojednostaviti sustav koji opisuje korektor temperature prvog reda.

Dobiveni sustav jednadžbi za korektor temperature prvog reda T^1 glasi :

$$\begin{aligned} \frac{1}{\varepsilon} &: \; -\frac{h'(x_1)^2+1}{h_0(x_1)}\frac{\partial^2 T^1}{\partial y_2^2} = D\frac{\partial Z^0}{\partial y_2}\frac{\partial T^0}{\partial x_1}, \\ \varepsilon &: \; \frac{\partial T^1}{\partial y_2}(x_1, 0) = 0, \quad \frac{\partial T^1}{\partial y_2}(x_1, 1) = 0. \end{aligned} \tag{2.23}$$

Rješenje problema (2.23) dano je s

$$T^1(x_1, y_2) = \frac{D}{12L}\frac{h_0(x_1)^3}{(h'(x_1)^2+1)^2}G(x_1)\frac{\partial T^0}{\partial x_1}(2y_2^3 - 3y_2^2). \tag{2.24}$$

Naglasimo da zbog pretpostavke $G(0) = G(1) = 0$, korektor temperature T^1 dan s (2.24) također zadovoljava rubne uvjete na krajevima kanala.

U izračunatim izrazima (2.19), (2.21) i (2.24) za aproksimacije brzine i mikrorotacije nultog reda te korektor temperature prvog reda možemo uočiti efekte zakrivljenosti domene na tok fluida kroz prisutnost funkcija h i h_0.

Nadalje, nazire se utjecaj mikrostrukture fluida na aproksimaciju brzine nultog reda $\boldsymbol{V}^0$ kroz prisutnost mikrorotacijske viskoznosti μ_r, o kojoj ovisi parametar N. Međutim, kako bismo uspjeli uhvatiti učinke kutne viskoznosti α koja se pojavljuje u parametru L, trebamo odrediti korektore višeg reda. Također, napomenimo da još nismo odredili aproksimaciju temperature nultog reda T^0.

2.2.1.2 Korektor prvog reda

Izraz za korektor tlaka prvog reda Q^1 dobivamo izravno iz $(2.10)_2$:

$$Q^1(x_1, y_2) = -\frac{12C_1 Pr}{1-N}\frac{(h'(x_1)^2+1)h'(x_1)}{h_0(x_1)^2}y_2 + r_1(x_1), \tag{2.25}$$

gdje je $r_1 = r_1(x_1)$ nepoznata funkcija koju ćemo kasnije odrediti kako bi korektor brzine prvog reda zadovoljio uvjet kompatibilnosti koji dolazi od jednakosti za divergenciju.

Sustav jednadžbi koji određuje korektor brzine prvog reda je dan s

$$\begin{aligned}
\frac{1}{\varepsilon}: \quad & -\frac{1}{1-N}\frac{h'(x_1)^2+1}{h_0(x_1)}\frac{\partial^2 V_1^1}{\partial y_2^2} + \frac{1}{Pr}\left[h_0(x_1)\frac{\partial Q^1}{\partial x_1} - h_0'(x_1)y_2\frac{\partial Q^1}{\partial y_2} - h'(x_1)\frac{\partial Q^2}{\partial y_2}\right] \\
& \quad = \frac{2N}{1-N}\frac{\partial Z^0}{\partial y_2} + \frac{1}{1-N}\left[-\frac{\partial}{\partial x_1}\left(h'(x_1)\frac{\partial V_1^0}{\partial y_2}\right) - h'(x_1)\frac{\partial^2 V_1^0}{\partial y_2 \partial x_1}\right. \\
& \qquad \left. +2\frac{\partial}{\partial y_2}\left(\frac{h_0'(x_1)h'(x_1)}{h_0(x_1)}y_2\frac{\partial V_1^0}{\partial y_2}\right)\right], \\
\frac{1}{\varepsilon}: \quad & -\frac{1}{1-N}\frac{h'(x_1)^2+1}{h_0(x_1)}\frac{\partial^2 V_2^1}{\partial y_2^2} + \frac{1}{Pr}\frac{\partial Q^2}{\partial y_2} = \frac{2N}{1-N}h'(x_1)\frac{\partial Z^0}{\partial y_2} \\
& \quad + \frac{1}{1-N}\left[-\frac{\partial}{\partial x_1}\left(h'(x_1)\frac{\partial V_2^0}{\partial y_2}\right) - h'(x_1)\frac{\partial^2 V_2^0}{\partial y_2 \partial x_1}\right. \\
& \qquad \left. +2\frac{\partial}{\partial y_2}\left(\frac{h_0'(x_1)h'(x_1)}{h_0(x_1)}y_2\frac{\partial V_2^0}{\partial y_2}\right)\right], \\
1: \quad & h_0(x_1)\frac{\partial V_1^0}{\partial x_1} - h_0'(x_1)y_2\frac{\partial V_1^0}{\partial y_2} - h'(x_1)\frac{\partial V_1^1}{\partial y_2} + \frac{\partial V_2^1}{\partial y_2} = 0, \\
\varepsilon: \quad & \boldsymbol{V}^1(x_1, 0) = 0, \ \boldsymbol{V}^1(x_1, 1) = 0, \\
\frac{1}{\varepsilon}: \quad & Q^1(0, y_2) = 0, \quad Q^1(1, y_2) = 0.
\end{aligned} \tag{2.26}$$

Uvrštavanjem izraza za aproksimaciju brzine nultog reda (2.19) u jednakost $(2.26)_3$ dobivamo sljedeću poveznicu između prve i druge komponente korektora brzine:

$$-h'(x_1)\frac{\partial V_1^1}{\partial y_2} + \frac{\partial V_2^1}{\partial y_2} = -6C_1\frac{h_0'(x_1)}{h_0(x_1)}(3y_2^2 - 2y_2). \tag{2.27}$$

Integriranjem (2.27) po y_2 dobivamo

$$V_2^1(x_1, y_2) = h'(x_1)V_1^1(x_1, y_2) - 6C_1\frac{h_0'(x_1)}{h_0(x_1)}(y_2^3 - y_2^2), \tag{2.28}$$

gdje smo iskoristili rubne uvjete $(2.26)_4$. S druge strane, deriviranjem jednakosti (2.27) po y_2 zaključujemo

$$\frac{\partial V_2^1}{\partial y_2^2} = h'(x_1)\frac{\partial V_1^1}{\partial y_2^2} - 12C_1\frac{h_0'(x_1)}{h_0(x_1)}(3y_2 - 1). \tag{2.29}$$

Korektori brzine prvog reda se sada mogu odrediti iz izraza za aproksimaciju brzine nultog reda (2.19), izraza za aproksimaciju mikrorotacije nultog reda (2.21), izraza za korektor tlaka (2.25), relacija (2.28) i (2.29), rubnih uvjeta $(2.26)_4$ i jednadžbi $(2.26)_1$–$(2.26)_2$, te su dani s

$$\begin{aligned} V_1^1(x_1, y_2) &= A_1(x_1)(y_2^3 - y_2^2) + A_2(x_1)(y_2^2 - y_2), \\ V_2^1(x_1, y_2) &= \left(h'(x_1)A_1(x_1) - 6C_1\frac{h_0'(x_1)}{h_0(x_1)}\right)(y_2^3 - y_2^2) + h'(x_1)A_2(x_1)(y_2^2 - y_2), \end{aligned} \tag{2.30}$$

pri čemu je

$$\begin{aligned} A_1(x_1) =& 24C_1\frac{h_0'(x_1)h'(x_1)}{(h'(x_1)^2+1)h_0(x_1)} - 2C_1\frac{(5h'(x_1)^2+2)h''(x_1)}{(h'(x_1)^2+1)^2} \\ &+ \frac{N}{3L}\frac{h_0(x_1)^3}{(h'(x_1)^2+1)^2}G(x_1), \\ A_2(x_1) =& 6C_1\frac{h_0'(x_1)h'(x_1)}{(h'(x_1)^2+1)h_0(x_1)} - C_1\frac{(4h'(x_1)^2+1)h''(x_1)}{(h'(x_1)^2+1)^2} \\ &- \frac{N}{6L}\frac{h_0(x_1)^3}{(h'(x_1)^2+1)^2}G(x_1) + \frac{1-N}{2Pr}\frac{h_0(x_1)^2}{(h'(x_1)^2+1)^2}r_1'(x_1). \end{aligned} \tag{2.31}$$

Gledajući funkcije A_1 i A_2 koje se pojavljuju u izrazu za korektor brzine prvog reda (2.30), dane s (2.31), vidimo utjecaj kutne viskoznosti α kroz pojavljivanje parametra L.

Sada zadajemo funkciju r_1 iz (2.25) na sljedeći način:

$$\begin{aligned} r_1(x_1) =& -\frac{12PrC_3}{1-N}\int_0^{x_1}\frac{(h'(s)^2+1)^2}{h_0(s)^3}ds - \frac{36PrC_1}{1-N}\int_0^{x_1}\frac{h_0'(s)h'(s)(h'(s)^2+1)}{h_0(s)^3}ds \\ &+ \frac{6PrC_1}{1-N}\int_0^{x_1}\frac{(3h'(s)^2+1)h''(s)}{h_0(s)^2}ds, \end{aligned} \tag{2.32}$$

gdje je

$$C_3 = -2C_1 \left(\int_0^1 \frac{(h'(s)^2+1)^2}{h_0(s)^3} ds \right)^{-1} \left(\int_0^1 \frac{h_0'(s)h'(s)(h'(s)^2+1)}{h_0(s)^3} ds \right).$$

Lako je vidjeti da vrijedi $r_1(0) = r_1(1) = 0$. Funkcija r_1 dana s (2.32) je određena tako da V_1^1 zadovoljava uvjet kompatibilnosti

$$\int_0^1 h_0(a) V_1^1(a, y_2) dy_2 = C_3, \quad \text{za sve } a \in [0,1], \tag{2.33}$$

dobiven iz $(2.26)_3$ integriranjem po $[0,a] \times [0,1]$.

U ovom trenutku možemo iz $(2.26)_2$ odrediti korektor tlaka drugog reda Q^2:

$$\begin{aligned} Q^2(x_1,y_2) = &\frac{6C_1 Pr}{1-N} \left(3\frac{(h'(x_1)^2-1)h_0'(x_1)}{h_0(x_1)^2} - \frac{(2h'(x_1)^2-1)h'(x_1)h''(x_1)}{(h'(x_1)^2+1)h_0(x_1)} \right) y_2^2 \\ &+ \frac{6C_1 Pr}{1-N} \left(-2\frac{(3h'(x_1)^2-1)h_0'(x_1)}{h_0(x_1)^2} + \frac{(2h'(x_1)^2-1)h'(x_1)h''(x_1)}{(h'(x_1)^2+1)h_0(x_1)} \right) y_2 \\ &- \frac{12C_3 Pr}{1-N} \frac{(h'(x_1)^2+1)h'(x_1)}{h_0(x_1)^2} y_2 + r_2(x_1), \end{aligned} \tag{2.34}$$

gdje je $r_2 = r_2(x_1)$ proizvoljna funkcija koju odabiremo tako da vrijedi $r_2(0) = r_2(1) = 0$.

Korektor mikrorotacije prvog reda Z^1 rješava sljedeći problem:

$$\begin{aligned} \frac{1}{\varepsilon}: \; &-L\frac{h'(x_1)^2+1}{h_0(x_1)} \frac{\partial^2 Z^1}{\partial y_2^2} = L\left[-\frac{\partial}{\partial x_1}\left(h'(x_1)\frac{\partial Z^0}{\partial y_2} \right) - h'(x_1)\frac{\partial^2 Z^0}{\partial y_2 \partial x_1} \right. \\ &\left. +2\frac{\partial}{\partial y_2}\left(\frac{h_0'(x_1)h'(x_1)}{h_0(x_1)} y_2 \frac{\partial Z^0}{\partial y_2} \right) \right] - \frac{2N}{1-N}\left(\frac{\partial V_1^0}{\partial y_2} + h'(x_1)\frac{\partial V_2^0}{\partial y_2} \right), \\ \varepsilon: \; & Z^1(x_1,0) = 0, \; Z^1(x_1,1) = 0. \end{aligned} \tag{2.35}$$

Lako je odrediti Z^1 iz (2.35), koristeći (2.19) i (2.21):

$$Z^1(x_1,y_2) = B_1(x_1)(y_2^3 - y_2^2) + B_2(x_1)(y_2^2 - y_2), \tag{2.36}$$

pri čemu je

$$\begin{aligned} B_1(x_1) =& -\frac{4C_1 N}{L(1-N)} - \frac{1}{6L}\left(\frac{h_0(x_1)}{h'(x_1)^2+1} \right)^3 h''(x_1)(3h'(x_1)^2-1)G(x_1) \\ &- \frac{1}{3L}\frac{h_0(x_1)^3}{(h'(x_1)^2+1)^2} h'(x_1)G'(x_1), \\ B_2(x_1) =& \frac{2C_1 N}{L(1-N)} - \frac{1}{12L}\left(\frac{h_0(x_1)}{h'(x_1)^2+1} \right)^3 h''(x_1)(3h'(x_1)^2-1)G(x_1) \\ &+ \frac{1}{2L}\left(\frac{h_0(x_1)}{h'(x_1)^2+1} \right)^2 h_0'(x_1)h'(x_1)G(x_1) + \frac{1}{6L}\frac{h_0(x_1)^3}{(h'(x_1)^2+1)^2} h'(x_1)G'(x_1). \end{aligned} \tag{2.37}$$

Korektor temperature drugog reda T^2 određen je sljedećim sustavom jednadžbi:

$$
\begin{aligned}
1 \ :\ -\frac{h'(x_1)^2+1}{h_0(x_1)}\frac{\partial^2 T^2}{\partial y_2^2} &= -h_0(x_1)V_1^0\frac{\partial T^0}{\partial x_1} + h_0(x_1)\frac{\partial^2 T^0}{\partial x_1^2} - \frac{\partial}{\partial x_1}\left(h'(x_1)\frac{\partial T^1}{\partial y_2}\right) \\
&\quad - h'(x_1)\frac{\partial^2 T^1}{\partial y_2 \partial x_1} + 2\frac{\partial}{\partial y_2}\left(\frac{h_0'(x_1)h'(x_1)}{h_0(x_1)}y_2\frac{\partial T^1}{\partial y_2}\right) \\
&\quad + D\left(\frac{\partial Z^1}{\partial y_2}\frac{\partial T^0}{\partial x_1} + \frac{\partial Z^0}{\partial y_2}\frac{\partial T^1}{\partial x_1} - \frac{\partial Z^0}{\partial x_1}\frac{\partial T^1}{\partial y_2}\right), \\
\varepsilon^2 \ : \qquad & \frac{\partial T^2}{\partial y_2}(x_1,0) = 0,\ \ \frac{\partial T^2}{\partial y_2}(x_1,1) = k(G_T(x_1) - T^0).
\end{aligned}
\tag{2.38}
$$

Integriranjem jednakosti $(2.38)_1$ od 0 do 1 po varijabli y_2 te koristeći izraz za korektor temperature prvog reda (2.24) i rubni uvjet $(2.38)_2$ dobivamo sljedeće:

$$
\begin{aligned}
-h_0(x_1)\frac{\partial^2 T^0}{\partial x_1^2} &= \frac{h'(x_1)^2+1}{h_0(x_1)}k(G_T(x_1)-T^0) - C_1\frac{\partial T^0}{\partial x_1} \\
&+G(x_1)\frac{\partial T^0}{\partial x_1}\left(-\frac{D}{12L}\frac{h_0(x_1)^3h''(x_1)(7h'(x_1)^2-1)}{(h'(x_1)^2+1)^3} + \frac{D}{2L}\frac{h_0(x_1)^2h'(x_1)h_0'(x_1)}{(h'(x_1)^2+1)^2}\right) \\
&+\frac{D}{6L}\frac{h_0(x_1)^3h'(x_1)}{(h'(x_1)^2+1)^2}G'(x_1)\frac{\partial T^0}{\partial x_1} + \frac{D}{6L}\frac{h_0(x_1)^3h'(x_1)}{(h'(x_1)^2+1)^2}G(x_1)\frac{\partial^2 T^0}{\partial x_1^2} \\
&+G(x_1)^2\frac{\partial T^0}{\partial x_1}\left(\frac{D^2}{24L^2}\frac{h_0(x_1)^4h_0'(x_1)}{(h'(x_1)^2+1)^3} - \frac{D^2}{20L^2}\frac{h_0(x_1)^5h'(x_1)h''(x_1)}{(h'(x_1)^2+1)^4}\right) \\
&+\frac{D^2}{60L^2}\frac{h_0(x_1)^5}{(h'(x_1)^2+1)^3}G(x_1)G'(x_1)\frac{\partial T^0}{\partial x_1} + \frac{D^2}{120L^2}\frac{h_0(x_1)^5}{(h'(x_1)^2+1)^3}G(x_1)^2\frac{\partial^2 T^0}{\partial x_1^2}.
\end{aligned}
\tag{2.39}
$$

Uz uvjet kompatibilnosti (2.39) zadajemo rubne uvjete $T^0(0) = \theta_0$ i $T^0(1) = \theta_1$, čime dobivamo sljedeći sustav za aproksimaciju tlaka nultog reda T^0:

$$
\begin{aligned}
H_1(x_1)\frac{\partial^2 T^0}{\partial x_1^2} + H_2(x_1)\frac{\partial T^0}{\partial x_1} + H_3(x_1)T^0 &= H_4(x_1), \\
T^0(0) = \theta_0,\quad T^0(1) &= \theta_1,
\end{aligned}
\tag{2.40}
$$

gdje je

$$
\begin{aligned}
H_1(x_1) =& h_0(x_1) + \frac{D}{6L}\frac{h_0(x_1)^3h'(x_1)}{(h'(x_1)^2+1)^2}G(x_1) + \frac{D^2}{120L^2}\frac{h_0(x_1)^5}{(h'(x_1)^2+1)^3}G(x_1)^2, \\
H_2(x_1) =& -C_1 - \frac{D}{12L}\frac{h_0(x_1)^3h''(x_1)}{(h'(x_1)^2+1)^3}(7h'(x_1)^2-1)G(x_1) + \frac{D^2}{60L^2}\frac{h_0(x_1)^5}{(h'(x_1)^2+1)^3}G(x_1)G'(x_1) \\
&+\frac{D}{2L}\frac{h_0(x_1)^2h'(x_1)h_0'(x_1)}{(h'(x_1)^2+1)^2}G(x_1) + \frac{D}{6L}\frac{h_0(x_1)^3h'(x_1)}{(h'(x_1)^2+1)^2}G'(x_1) \\
&+\frac{D^2}{24L^2}\frac{h_0(x_1)^4h_0'(x_1)}{(h'(x_1)^2+1)^3}G(x_1)^2 - \frac{D^2}{20L^2}\frac{h_0(x_1)^5h'(x_1)h''(x_1)}{(h'(x_1)^2+1)^4}G(x_1)^2, \\
H_3(x_1) =& -\frac{h'(x_1)^2+1}{h_0(x_1)}k,\quad H_4(x_1) = -\frac{h'(x_1)^2+1}{h_0(x_1)}kG_T(x_1).
\end{aligned}
\tag{2.41}
$$

Zbog pretpostavki na glatkoću funkcija h i h_0 lako se pokazuje da problem (2.40)–(2.41) ima jedinstveno rješenje. Time je određena aproksimacija temperature nultog reda T^0.

Konačno, korektor temperature drugog reda koji je rješenje sustava (2.38) je dan s

$$T^2(x_1, y_2) = D_1(x_1)y_2^6 + D_2(x_1)y_2^5 + D_3(x_1)y_2^4 + D_4(x_1)y_2^3 + D_5(x_1)y_2^2, \tag{2.42}$$

pri čemu je

$$\begin{aligned}
D_1(x_1) =& -\frac{D^2}{180L^2}\frac{h_0(x_1)^6 h'(x_1)h''(x_1)}{(h'(x_1)^2+1)^5}G(x_1)^2\frac{\partial T^0}{\partial x_1} + \frac{D^2}{180L^2}\frac{h_0(x_1)^6}{(h'(x_1)^2+1)^4}G(x_1)^2\frac{\partial^2 T^0}{\partial x_1^2} \\
&- \frac{D^2}{360L^2}\frac{h_0(x_1)^6}{(h'(x_1)^2+1)^4}G(x_1)G'(x_1)\frac{\partial T^0}{\partial x_1}, \\
D_2(x_1) =& \frac{D^2}{60L^2}\frac{h_0(x_1)^6 h'(x_1)h''(x_1)}{(h'(x_1)^2+1)^5}G(x_1)^2\frac{\partial T^0}{\partial x_1} - \frac{D^2}{60L^2}\frac{h_0(x_1)^6}{(h'(x_1)^2+1)^4}G(x_1)^2\frac{\partial^2 T^0}{\partial x_1^2} \\
&+ \frac{D^2}{120L^2}\frac{h_0(x_1)^6}{(h'(x_1)^2+1)^4}G(x_1)G'(x_1)\frac{\partial T^0}{\partial x_1}, \\
D_3(x_1) =& -\frac{C_1}{2}\frac{h_0(x_1)}{h'(x_1)^2+1}\frac{\partial T^0}{\partial x_1} - \frac{D}{12L}\frac{h_0(x_1)^4 h''(x_1)(5h'(x_1)^2-1)}{(h'(x_1)^2+1)^4}G(x_1)\frac{\partial T^0}{\partial x_1} \\
&+ \frac{D}{6L}\frac{h_0(x_1)^4 h'(x_1)}{(h'(x_1)^2+1)^3}G'(x_1)\frac{\partial T^0}{\partial x_1} + \frac{D}{12L}\frac{h_0(x_1)^4 h'(x_1)}{(h'(x_1)^2+1)^3}G(x_1)\frac{\partial^2 T^0}{\partial x_1^2} \\
&+ \frac{C_1 N}{L(1-N)}\frac{h_0(x_1)}{h'(x_1)^2+1}\frac{\partial T^0}{\partial x_1} - \frac{D^2}{96L^2}\frac{h_0(x_1)^5 h_0'(x_1)}{(h'(x_1)^2+1)^4}G(x_1)^2\frac{\partial T^0}{\partial x_1} \\
&- \frac{D^2}{96L^2}\frac{h_0(x_1)^6}{(h'(x_1)^2+1)^4}G(x_1)G'(x_1)\frac{\partial T^0}{\partial x_1} + \frac{D^2}{96L^2}\frac{h_0(x_1)^6}{(h'(x_1)^2+1)^4}G(x_1)^2\frac{\partial^2 T^0}{\partial x_1^2}, \qquad (2.43) \\
D_4(x_1) =& C_1\frac{h_0(x_1)}{h'(x_1)^2+1}\frac{\partial T^0}{\partial x_1} + \frac{D}{6L}\frac{h_0(x_1)^4 h''(x_1)(5h'(x_1)^2-1)}{(h'(x_1)^2+1)^4}G(x_1)\frac{\partial T^0}{\partial x_1} \\
&- \frac{D}{3L}\frac{h_0(x_1)^3 h'(x_1)h_0'(x_1)}{(h'(x_1)^2+1)^2}G(x_1)\frac{\partial T^0}{\partial x_1} - \frac{D}{3L}\frac{h_0(x_1)^4 h'(x_1)}{(h'(x_1)^2+1)^3}G'(x_1)\frac{\partial T^0}{\partial x_1} \\
&- \frac{D}{6L}\frac{h_0(x_1)^4 h'(x_1)}{(h'(x_1)^2+1)^3}G(x_1)\frac{\partial^2 T^0}{\partial x_1^2} - \frac{10C_1 N}{3L(1-N)}\frac{h_0(x_1)}{h'(x_1)^2+1}\frac{\partial T^0}{\partial x_1}, \\
D_5(x_1) =& \frac{1}{2}k(G_T(x_1)-T^0) - \frac{C_1}{2}\frac{h_0(x_1)}{h'(x_1)^2+1}\frac{\partial T^0}{\partial x_1} + \frac{3C_1 N}{L(1-N)}\frac{h_0(x_1)}{h'(x_1)^2+1}\frac{\partial T^0}{\partial x_1} \\
&+ G(x_1)\frac{\partial T^0}{\partial x_1}\left(-\frac{D}{12L}\frac{h_0(x_1)^4 h''(x_1)(5h'(x_1)^2-1)}{(h'(x_1)^2+1)^4} + \frac{D}{2L}\frac{h_0(x_1)^3 h'(x_1)h_0(x_1)}{(h'(x_1)^2+1)^3}\right) \\
&+ \frac{D}{6L}\frac{h_0(x_1)^4 h'(x_1)}{(h'(x_1)^2+1)^3}G'(x_1)\frac{\partial T^0}{\partial x_1} + \frac{D}{12L}\frac{h_0(x_1)^4 h'(x_1)}{(h'(x_1)^2+1)^3}G(x_1)\frac{\partial^2 T^0}{\partial x_1^2} \\
&+ G(x_1)^2\frac{\partial T^0}{\partial x_1}\left(\frac{D^2}{48L^2}\frac{h_0(x_1)^5 h_0'(x_1)}{(h'(x_1)^2+1)^4} - \frac{D^2}{40L^2}\frac{h_0(x_1)^6 h'(x_1)h''(x_1)}{(h'(x_1)^2+1)^5}\right) \\
&+ \frac{D^2}{120L^2}\frac{h_0(x_1)^6}{(h'(x_1)^2+1)^4}G(x_1)G'(x_1)\frac{\partial T^0}{\partial x_1} + \frac{D^2}{240L^2}\frac{h_0(x_1)^6}{(h'(x_1)^2+1)^4}G(x_1)^2\frac{\partial^2 T^0}{\partial x_1^2}.
\end{aligned}$$

Primijetimo da aproksimacija temperature nultog reda T^0, dana kao rješenje problema (2.40)–(2.41), uvažava učinke kutne viskoznosti α kroz parametar L, kao i učinke mikropolarne toplinske vodljivosti δ kroz parametar D, gdje se spomenuti parametri pojavljuju u izrazima danima u (2.41).

2.2.2. Rubni sloj

Regularni dio asimptotičkog razvoja ima sljedeći oblik:

$$\begin{aligned}
&\boldsymbol{V}^{[1]}_{\varepsilon,reg}(x_1,y_2) = V^0(x_1,y_2) + \varepsilon V^1(x_1,y_2),\\
&Q^{[2]}_{\varepsilon,reg}(x_1,y_2) = \frac{1}{\varepsilon^2}Q^0(x_1) + \frac{1}{\varepsilon}Q^1(x_1,y_2) + Q^2(x_1,y_2),\\
&Z^{[1]}_{\varepsilon,reg}(x_1,y_2) = Z^0(x_1,y_2) + \varepsilon Z^1(x_1,y_2),\\
&T^{[2]}_{\varepsilon,reg}(x_1,y_2) = T^0(x_1) + \varepsilon T^1(x_1,y_2) + \varepsilon^2 T^2(x_1,y_2).
\end{aligned} \tag{2.44}$$

Asimptotička aproksimacija (2.44) izračunata u Odjeljku 2.2.1 zadovoljava rubne uvjete na gornjem i donjem zidu kanala. Štoviše, aproksimacija tlaka nultog reda $Q^{[0]}_{\varepsilon,reg}$, aproksimacija mikrorotacije nultog reda $Z^{[0]}_{\varepsilon,reg}$ i aproksimacija temperature prvog reda $T^{[1]}_{\varepsilon,reg}$ zadovoljavaju i rubne uvjete na krajevima kanala.

Sada uvodimo dodatnu pretpostavku da prve derivacije funkcija h i h_0 iščezavaju u okolini $x_1 = 0$ i $x_1 = 1$. Posljedično, druga komponenta aproksimacije brzine prvog reda $\boldsymbol{V}^{[1]}_{\varepsilon,reg}$ jednaka je nuli kada $x_1 = 0, 1$, iz čega zaključujemo da $\boldsymbol{V}^{[1]}_{\varepsilon,reg}$ zadovoljava rubne uvjete na krajevima kanala. Nadalje, korektori tlaka prvog i drugog reda Q^1 i Q^2 sada isto zadovoljavaju rubne uvjete na krajevima.

Međutim, korektor mikrorotacije prvog reda Z^1 ne zadovoljava rubne uvjete, što znači da trebamo korigirati naš asimptotički razvoj u rubnom sloju na krajevima kanala.

U tu svrhu, uvodimo korektore rubnog sloja mikrorotacije prvog reda u okolini $x_1 = 0$ u obliku

$$\mathcal{Z}_{\varepsilon,bl,0}(x_1,y_2) = \varepsilon \mathcal{Z}^1_{bl,0}\left(\frac{x_1}{\varepsilon}, y_2\right), \tag{2.45}$$

te u okolini $x_1 = 1$ u obliku

$$\mathcal{Z}_{\varepsilon,bl,1}(x_1,y_2) = \varepsilon \mathcal{Z}^1_{bl,1}\left(\frac{x_1-1}{\varepsilon}, y_2\right). \tag{2.46}$$

Korektori rubnog sloja s desne strane (2.45) i (2.46) su definirani na polubeskonačnim kanalima $\mathcal{G}_0 = (0,\infty)\times(0,1)$, odnosno $\mathcal{G}_1 = (-\infty,0)\times(0,1)$. Kako korektori rubnoj sloja ne bi ovisili o parametru ε, funkciju $\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})$ aproksimiramo s njenom vrijednosti u $(0,0)$. Ova aproksimacija će biti opravdana u ocjeni greške u Odjeljku 2.3.

Uvedimo zamjenu varijabli $y_1 = \frac{x_1}{\varepsilon}$ te stavimo $\boldsymbol{G} = 0$. Uvrštavanjem (2.45) u sustav (2.5) i prikupljanjem članova uz istu potenciju od ε dobivamo sljedeći problem za $\mathcal{Z}^1_{bl,0}$:

$$\begin{aligned} \frac{1}{\varepsilon}: &\quad -L\,\mathrm{div}_y(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi}(\boldsymbol{0}))\nabla_y \mathcal{Z}^1_{bl,0}) = 0, \\ \varepsilon: &\quad \mathcal{Z}^1_{bl,0}(0,y_2) + Z^1(0,y_2) = 0, \\ \varepsilon: &\quad \mathcal{Z}^1_{bl,0}(y_1,0) = 0, \quad \mathcal{Z}^1_{bl,0}(y_1,1) = 0. \end{aligned} \tag{2.47}$$

Kako je matrica $\tilde{\boldsymbol{F}}(\boldsymbol{\Phi}(\boldsymbol{0}))$ pozitivno definitna, postoji jedinstveno rješenje $\mathcal{Z}^1_{bl,0} \in W^{1,2}(\mathcal{G}_0)$ problema (2.47). Koristeći standardne metode lako je pokazati da rješenje $\mathcal{Z}^1_{bl,0}$ eksponencijalno teži k nuli kada $|y| \to +\infty$ (vidi npr. [38]). Korektor rubnog sloja $\mathcal{Z}^1_{bl,1}$ na suprotnom kraju kanala se konstruira analogno.

2.2.3. Asimptotička aproksimacija

Asimptotička aproksimacija u tankom pravokutniku $[0,1] \times [0,\varepsilon]$ određena u Odjeljcima 2.2.1 i 2.2.2 sada ima sljedeći oblik:

$$\begin{aligned} \boldsymbol{V}^{[1]}_\varepsilon(x_1,x_2) &= \boldsymbol{V}^{[1]}_{\varepsilon,reg}\left(x_1,\frac{x_2}{\varepsilon}\right) = \boldsymbol{V}^0\left(x_1,\frac{x_2}{\varepsilon}\right) + \varepsilon \boldsymbol{V}^1\left(x_1,\frac{x_2}{\varepsilon}\right), \\ Q^{[2]}_\varepsilon(x_1,x_2) &= Q^{[2]}_{\varepsilon,reg}\left(x_1,\frac{x_2}{\varepsilon}\right) = \frac{1}{\varepsilon^2}Q^0(x_1) + \frac{1}{\varepsilon}Q^1\left(x_1,\frac{x_2}{\varepsilon}\right) + Q^2\left(x_1,\frac{x_2}{\varepsilon}\right), \\ Z^{[1]}_\varepsilon(x_1,x_2) &= Z^{[1]}_{\varepsilon,reg}\left(x_1,\frac{x_2}{\varepsilon}\right) + \mathcal{Z}_{\varepsilon,bl,0}\left(x_1,\frac{x_2}{\varepsilon}\right) + \mathcal{Z}_{\varepsilon,bl,1}\left(x_1,\frac{x_2}{\varepsilon}\right) \\ &= Z^0\left(x_1,\frac{x_2}{\varepsilon}\right) + \varepsilon Z^1\left(x_1,\frac{x_2}{\varepsilon}\right) + \varepsilon \mathcal{Z}^1_{bl,0}\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right) + \varepsilon \mathcal{Z}^1_{bl,1}\left(\frac{x_1-1}{\varepsilon},\frac{x_2}{\varepsilon}\right), \\ T^{[2]}_\varepsilon(x_1,x_2) &= T^{[2]}_{\varepsilon,reg}\left(x_1,\frac{x_2}{\varepsilon}\right) = T^0(x_1) + \varepsilon T^1\left(x_1,\frac{x_2}{\varepsilon}\right) + \varepsilon^2 T^2\left(x_1,\frac{x_2}{\varepsilon}\right), \end{aligned} \tag{2.48}$$

gdje su $\boldsymbol{V}^0$ i $\boldsymbol{V}^1$ dani s (2.19) i (2.30)–(2.31), Q^0, Q^1 i Q^2 s (2.17)–(2.18), (2.25) i (2.34), Z^0 i Z^1 s (2.21) i (2.36)–(2.37), $\mathcal{Z}^1_{bl,0}$ i $\mathcal{Z}^1_{bl,1}$ korektori rubnog sloja mikrorotacije razmatrani u Odjeljku 2.2.2, te su T^0, T^1 i T^2 dani s (2.40)–(2.41), (2.24) i (2.42)–(2.43).

Napomena 2.2.1. *U slučaju kada je $h(x) = 0$ i $h_0(x_1) = 1$ (to jest kada promatramo tok u nedeformiranom kanalu), dobivena asimptotička aproksimacija $(\boldsymbol{V}^{[1]}_\varepsilon, Q^{[2]}_\varepsilon, Z^{[1]}_\varepsilon, T^{[2]}_\varepsilon)$ dana s (2.48) se podudara s onom izvedenom u [56], što je i očekivano. Shodno tome, zaključujemo da asimptotičko rješenje predloženo u ovom poglavlju uzima u obzir zakrivljenost kanala. Naime, u slučaju nedeformiranog kanala druga komponenta aproksimacije* *brzine* $\boldsymbol{V}_{\boldsymbol{\varepsilon}}^{[1]}$ *dane s* $(2.48)_1$ *je jednaka nuli, dok je u slučaju zakrivljenog kanala druga komponenta netrivijalna. Učinke zakrivljenosti također možemo uočiti kroz prisutnost funkcija* $\boldsymbol{h}$ *i* $\boldsymbol{h}_0$ *koje opisuju zakrivljenost kanala.*

Kako bi opravdali dobiveni asimptotički model, u sljedećem odjeljku ocjenjujemo razliku između originalnog rješenja problema (2.5)–(2.6) te dobivenog asimptotičkog rješenja (2.48) u prikladnim normama.

2.3. Rigorozno opravdanje

U ostatku odjeljka koristit ćemo sljedeće ocjene:

$$\left\|\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\right\|_{L^\infty(\Omega_\varepsilon)} \leq C, \quad \left\|\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\right\|_{L^\infty(\Omega_\varepsilon)} \leq C, \tag{2.49}$$

gdje je $C > 0$ konstanta neovisna o ε. Ocjene (2.49) se lako pokazuju koristeći $h, h_0 \in C^2([0,1])$.

Uvodimo sljedeće oznake:

$$\Gamma^\varepsilon = (0,1) \times \{0, \varepsilon\}, \quad \Gamma_+^\varepsilon = (0,1) \times \{\varepsilon\}, \quad \Gamma_-^\varepsilon = (0,1) \times \{0\}, \quad \Sigma_i^\varepsilon = \{i\} \times (0, \varepsilon), \ i = 0, 1.$$

Osim Leme 0.0.1, bit će nam potrebni sljedeći tehnički rezultati:

Lema 2.3.1. Za sve $\boldsymbol{\varphi}_\varepsilon \in W^{1,2}(\Omega_\varepsilon)$ vrijede sljedeće nejednakosti:

$$\|\boldsymbol{\varphi}_\varepsilon\|_{L^2(\Gamma^\varepsilon)} \leq C\varepsilon^{1/2} \|\nabla \boldsymbol{\varphi}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \quad \|\boldsymbol{\varphi}_\varepsilon\|_{L^2(\Sigma_i^\varepsilon)} \leq C \|\nabla \boldsymbol{\varphi}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \ i = 0, 1.$$

Lema 2.3.2. Neka je $\boldsymbol{f}_\varepsilon \in L_0^2(\Omega_\varepsilon)$, tada problem

$$\operatorname{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T \boldsymbol{u}_\varepsilon) = \boldsymbol{f}_\varepsilon \ \text{ u } \ \Omega_\varepsilon,$$
$$\boldsymbol{u}_\varepsilon = 0 \ \text{ na } \ \partial\Omega_\varepsilon$$

ima barem jedno rješenje $\boldsymbol{u}_\varepsilon \in W_0^{1,2}(\Omega_\varepsilon)$ takvo da

$$\|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon} \|\boldsymbol{f}_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

Lema 2.3.3. Postoji $\alpha_1 > 0$ takav da $\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \alpha_1 |\boldsymbol{\xi}|^2$, za sve $\boldsymbol{\xi} \in \mathbb{R}^2$.

Lema 2.3.4. Postoji $\alpha_2 > 0$ takav da $\boldsymbol{A}\tilde{\boldsymbol{F}}(\boldsymbol{\Phi}) \cdot \boldsymbol{A} \geq \alpha_2 |\boldsymbol{A}|^2$, za sve $\boldsymbol{A} \in M_2(\mathbb{R})$.

Dokaz Leme 2.3.1 se može pronaći u [81], dok se Lema 2.3.2 pokazuje slično kao i Lema 0.0.2. Lema 2.3.3 je dokazana u [28, Lema 3.2], dok Lema 2.3.4 slijedi iz Leme 2.3.3 i identitete

$$\boldsymbol{A}\tilde{\boldsymbol{F}}(\boldsymbol{\Phi}) \cdot \boldsymbol{A} = \tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\boldsymbol{A}^T \boldsymbol{e}_1 \cdot \boldsymbol{A}^T \boldsymbol{e}_1 + \tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\boldsymbol{A}^T \boldsymbol{e}_2 \cdot \boldsymbol{A}^T \boldsymbol{e}_2.$$

Sada smo spremni dokazati apriorne ocjene za brzinu, tlak, mikrorotaciju i temperaturu.

2.3.1. Apriorne ocjene

Propozicija 2.3.1 (Apriorne ocjene). Neka je$(\boldsymbol{v}_\varepsilon, q_\varepsilon, z_\varepsilon, S_\varepsilon)$ rješenje problema (2.5)–(2.6). Tada vrijede sljedeće ocjene:

$$\|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\sqrt{\varepsilon}},$$
$$\|q_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq \frac{C}{\sqrt{\varepsilon^3}},$$
$$\|\nabla z_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\sqrt{\varepsilon}},$$
$$\|\nabla S_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon},$$

gdje je C konstanta neovisna o ε.

Dokaz. Mikrorotacija z_ε zadovoljava sljedeće jednadžbe na Ω_ε:

$$\begin{aligned} -L\,\mathrm{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla z_\varepsilon) + \frac{4N}{1-N}\det(\nabla\boldsymbol{\Phi})z_\varepsilon &= -\frac{M}{Pr}\boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla z_\varepsilon \\ &+\frac{2N}{1-N}\,\mathrm{rot}(\nabla\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon) + \det(\nabla\boldsymbol{\Phi})G_\varepsilon, \\ z_\varepsilon = 0 \quad \text{na} \quad \partial\Omega_\varepsilon. \end{aligned} \tag{2.50}$$

Množenjem (2.50) s z_ε i integriranjem po Ω_ε dobivamo

$$\begin{aligned} L\int_{\Omega_\varepsilon}\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla z_\varepsilon\cdot\nabla z_\varepsilon + \frac{4N}{1-N}\int_{\Omega_\varepsilon}\det(\nabla\boldsymbol{\Phi})|z_\varepsilon|^2 &= -\frac{M}{Pr}\int_{\Omega_\varepsilon}\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla z_\varepsilon z_\varepsilon \\ +\frac{2N}{1-N}\int_{\Omega_\varepsilon}\mathrm{rot}((\nabla\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon)z_\varepsilon &+ \int_{\Omega_\varepsilon}\det(\nabla\boldsymbol{\Phi})G_\varepsilon z_\varepsilon. \end{aligned} \tag{2.51}$$

Vrijedi jednakost

$$\begin{aligned} \int_{\Omega_\varepsilon}\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla z_\varepsilon z_\varepsilon &= \frac{1}{2}\int_{\Omega_\varepsilon}\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla(z_\varepsilon)^2 = \frac{1}{2}\int_{\Omega_\varepsilon}\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon\cdot\nabla(z_\varepsilon)^2 \\ &= \frac{1}{2}\int_{\Omega_\varepsilon}\mathrm{div}\left(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon z_\varepsilon^2\right) - \frac{1}{2}\int_{\Omega_\varepsilon}\mathrm{div}\left(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon\right)z_\varepsilon^2 = 0, \end{aligned}$$

pri čemu smo iskoristili $\mathrm{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon) = 0$ te činjenicu da vrijedi $z_\varepsilon = 0$ na $\partial\Omega_\varepsilon$. Ocjenjujemo preostale izraze s desne strane (2.51) koristeći Lemu 0.0.1:

$$\begin{aligned} \int_{\Omega_\varepsilon}\mathrm{rot}((\nabla\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon)z_\varepsilon &\leq C\,\|\nabla\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\,\|z_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\leq C\varepsilon\,\|\nabla\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\,\|\nabla z_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\ \int_{\Omega_\varepsilon)}\det(\nabla\boldsymbol{\Phi})G_\varepsilon z_\varepsilon &\leq C\,\|G_\varepsilon\|_{L^2(\Omega_\varepsilon)}\,\|z_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &= \frac{C}{\varepsilon^2}\,\|G\|_{L^2(\Omega_\varepsilon)}\,\|z_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\leq \frac{C}{\sqrt{\varepsilon}}\,\|\nabla z_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \end{aligned}$$

Također, uočimo da vrijedi

$$\det(\nabla\boldsymbol{\Phi}) = h_0(x_1) \geq c,$$

gdje je c konstanta iz (2.2). Sada iz (2.51) i Leme 2.3.3 za dovoljno mali ε slijedi ocjena

$$\|\nabla z_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \frac{C}{\sqrt{\varepsilon}}. \tag{2.52}$$

Temperatura S_ε zadovoljava

$$\begin{aligned} -\operatorname{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla S_\varepsilon) &= -\boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla S_\varepsilon + D\nabla^\perp z_\varepsilon \cdot \nabla S_\varepsilon, \\ S_\varepsilon = \theta_0 \text{ za } x_1 = 0, &\quad S_\varepsilon = \theta_1 \text{ za } x_1 = 1, \\ \frac{\partial S_\varepsilon}{\partial x_2} = 0 \text{ za } x_2 = 0, &\quad \frac{\partial S_\varepsilon}{\partial x_2} = \varepsilon k(G_T(x_1) - S_\varepsilon) \text{ za } x_2 = \varepsilon. \end{aligned} \tag{2.53}$$

Uvodimo funkciju S_ε^* takvu da $S_\varepsilon^* = 0$ na Σ_0^ε i Σ_1^ε na sljedeći način:

$$S_\varepsilon^* = S_\varepsilon - D_\varepsilon^S, \quad D_\varepsilon^S = \theta_0 + x_1(\theta_1 - \theta_0).$$

Množenjem (2.53) s S_ε^* i integriranjem po Ω_ε dobivamo

$$\begin{aligned} \int_{\Omega_\varepsilon} \tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla S_\varepsilon^* \cdot \nabla S_\varepsilon^* + \varepsilon k \int_{\Gamma_+^\varepsilon} \frac{(h_0'(x_1)x_2 + h'(x_1))^2 + 1}{h_0(x_1)} |S_\varepsilon^*|^2 &= -\int_{\Omega_\varepsilon} \tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla D_\varepsilon^S \cdot \nabla S_\varepsilon^* \\ - \int_{\Omega_\varepsilon} \boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla S_\varepsilon^* S_\varepsilon^* - \int_{\Omega_\varepsilon} \boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla D_\varepsilon^S S_\varepsilon^* &+ D \int_{\Omega_\varepsilon} \nabla^\perp z_\varepsilon \cdot \nabla S_\varepsilon^* S_\varepsilon^* \\ -D(\theta_1 - \theta_0) \int_{\Omega_\varepsilon} \frac{\partial z_\varepsilon}{\partial x_1} S_\varepsilon^* - \int_{\Gamma^\varepsilon} (h_0'(x_1)x_2 + h'(x_1)) \frac{\partial S_\varepsilon^*}{\partial x_1} S_\varepsilon^* \\ -(\theta_1 - \theta_0) \int_{\Gamma^\varepsilon} (h_0'(x_1)x_2 + h'(x_1)) S_\varepsilon^* + \varepsilon k \int_{\Gamma_+^\varepsilon} \frac{(h_0'(x_1)x_2 + h'(x_1))^2 + 1}{h_0(x_1)} G_T S_\varepsilon^* \\ -\varepsilon k \int_{\Gamma_+^\varepsilon} \frac{(h_0'(x_1)x_2 + h'(x_1))^2 + 1}{h_0(x_1)} D_\varepsilon^S S_\varepsilon^*. \end{aligned} \tag{2.54}$$

Vrijedi

$$\begin{aligned} \int_{\Omega_\varepsilon} \boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla S_\varepsilon^* S_\varepsilon^* &= \frac{1}{2}\int_{\Omega_\varepsilon} \boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla (S_\varepsilon^*)^2 = \frac{1}{2}\int_{\Omega_\varepsilon} \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T \boldsymbol{v}_\varepsilon \cdot \nabla (S_\varepsilon^*)^2 \\ &= \frac{1}{2}\int_{\Omega_\varepsilon} \operatorname{div}\left(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T \boldsymbol{v}_\varepsilon (S_\varepsilon^*)^2\right) - \frac{1}{2}\int_{\Omega_\varepsilon} \operatorname{div}\left(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T \boldsymbol{v}_\varepsilon\right)(S_\varepsilon^*)^2 \\ &= \frac{1}{2}\int_{\partial\Omega_\varepsilon} \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T \boldsymbol{v}_\varepsilon \cdot \boldsymbol{n} (S_\varepsilon^*)^2 = 0, \end{aligned}$$

gdje smo iskoristili $\operatorname{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T \boldsymbol{v}_\varepsilon) = 0$ te činjenicu da je $S_\varepsilon^* = 0$ za $x_1 = 0, 1$ te $\boldsymbol{v}_\varepsilon = \boldsymbol{0}$ za $x_2 = 0, \varepsilon$. Nadalje, vrijedi

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla^\perp z_\varepsilon \cdot \nabla S_\varepsilon^* S_\varepsilon^* &= -\frac{1}{2}\int_{\Omega_\varepsilon} \nabla z_\varepsilon \times (\nabla (S_\varepsilon^*)^2) = -\frac{1}{2}\int_{\Omega_\varepsilon} \operatorname{rot}\left(z_\varepsilon (\nabla (S_\varepsilon^*)^2\right) \\ &= -\frac{1}{2}\int_{\partial\Omega_\varepsilon} \boldsymbol{n} \times \left(z_\varepsilon (\nabla (S_\varepsilon^*)^2\right) = 0, \end{aligned}$$

pri čemu koristimo $z_\varepsilon = 0$ na $\partial\Omega_\varepsilon$. Još raspisujemo integralni član na Γ^ε:

$$\begin{aligned}\int_{\Gamma^\varepsilon}(h_0'(x_1)x_2+h'(x_1))\frac{\partial S_\varepsilon^*}{\partial x_1}S_\varepsilon^* =&\frac{1}{2}\int_{\Gamma^\varepsilon}(h_0'(x_1)x_2+h'(x_1))\frac{\partial}{\partial x_1}(S_\varepsilon^*)^2\\ =&\frac{1}{2}\int_{\Gamma^\varepsilon}\frac{\partial}{\partial x_1}\left((h_0'(x_1)x_2+h'(x_1))(S_\varepsilon^*)^2\right)\\ &-\frac{1}{2}\int_{\Gamma^\varepsilon}(h_0''(x_1)x_2+h''(x_1))(S_\varepsilon^*)^2\\ =&-\frac{1}{2}\int_{\Gamma^\varepsilon}(h_0''(x_1)x_2+h''(x_1))(S_\varepsilon^*)^2,\end{aligned}$$

gdje smo iskoristili činjenicu da $h'(0) = h'(1) = 0$ i $h_0'(0) = h_0'(1) = 0$. Sada ocjenjujemo izraze s desne strane jednakosti (2.54) koristeći (2.49) te Leme 0.0.1 i 2.3.1:

$$\begin{aligned}\int_{\Omega_\varepsilon}\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla D_\varepsilon^S\cdot\nabla S_\varepsilon^* \le& C\left\|\nabla D_\varepsilon^S\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla S_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}\\ \le& C\sqrt{\varepsilon}\left\|\nabla S_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)},\\ \int_{\Omega_\varepsilon}\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla D_\varepsilon^S S_\varepsilon^* \le& C\left\|\boldsymbol{v}_\varepsilon\right\|_{L^4(\Omega_\varepsilon)}\left\|S_\varepsilon^*\right\|_{L^4(\Omega_\varepsilon)}\left\|\nabla D_\varepsilon^S\right\|_{L^2(\Omega_\varepsilon)}\\ \le& C\sqrt{\varepsilon^3}\left\|\nabla\boldsymbol{v}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla S_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)},\\ \int_{\Omega_\varepsilon}\frac{\partial z_\varepsilon}{\partial x_1}S_\varepsilon^* \le& \left\|\nabla z_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\left\|S_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}\\ \le& C\varepsilon\left\|\nabla z_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla S_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)},\\ \int_{\Gamma^\varepsilon}(h_0''(x_1)x_2+h''(x_1))(S_\varepsilon^*)^2 \le& C\left\|S_\varepsilon^*\right\|_{L^2(\Gamma^\varepsilon)}^2\\ \le& C\varepsilon\left\|\nabla S_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}^2,\\ \int_{\Gamma^\varepsilon}(h_0'(x_1)x_2+h'(x_1))S_\varepsilon^* \le& \left\|h_0'(x_1)x_2+h'(x_1)\right\|_{L^2(\Gamma^\varepsilon)}\left\|S_\varepsilon^*\right\|_{L^2(\Gamma^\varepsilon)}\\ \le& C\sqrt{\varepsilon}\left\|\nabla S_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)},\\ \varepsilon k\int_{\Gamma_+^\varepsilon}\frac{(h_0'(x_1)x_2+h'(x_1))^2+1}{h_0(x_1)}G_T S_\varepsilon^* \le& C\varepsilon\left\|G_T\right\|_{L^2(\Gamma_+^\varepsilon)}\left\|S_\varepsilon^*\right\|_{L^2(\Gamma_+^\varepsilon)}\\ \le& C\sqrt{\varepsilon^3}\left\|\nabla S_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)},\\ \varepsilon k\int_{\Gamma_+^\varepsilon}\frac{(h_0'(x_1)x_2+h'(x_1))^2+1}{h_0(x_1)}D_\varepsilon^S S_\varepsilon^* \le& C\varepsilon\left\|D_\varepsilon^S\right\|_{L^2(\Gamma_+^\varepsilon)}\left\|S_\varepsilon^*\right\|_{L^2(\Gamma_+^\varepsilon)}\\ \le& C\sqrt{\varepsilon^3}\left\|\nabla S_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}.\end{aligned} \tag{2.55}$$

Sada iz (2.52), (2.54), (2.55) te Leme 2.3.3 za dovoljno mali ε slijedi ocjena

$$\left\|\nabla S_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \le C\sqrt{\varepsilon^3}\left\|\nabla\boldsymbol{v}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} + C\sqrt{\varepsilon}. \tag{2.56}$$

Razmotrimo sada sustav jednadžbi koji zadovoljava brzina $\boldsymbol{v}_\varepsilon$:

$$\begin{gathered}\frac{1}{Pr}((\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{v}_\varepsilon+\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla q_\varepsilon)=\operatorname{div}(\nabla\boldsymbol{v}_\varepsilon\tilde{\boldsymbol{F}}(\boldsymbol{\Phi}))\\ +\frac{N}{1-N}(2\nabla\boldsymbol{\Phi}\nabla^\perp z_\varepsilon+\operatorname{div}(\nabla\boldsymbol{v}_\varepsilon\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})))+Ra\det(\nabla\boldsymbol{\Phi})S_\varepsilon\boldsymbol{e}_2+\det(\nabla\boldsymbol{\Phi})\boldsymbol{F}_\varepsilon,\\ \operatorname{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon)=0,\\ v_2^\varepsilon=0 \text{ za } x_1=0,1,\quad \boldsymbol{v}_\varepsilon=\boldsymbol{0}\text{ za } x_2=0,\varepsilon.\end{gathered}\tag{2.57}$$

Množenjem (2.57) s $\boldsymbol{v}_\varepsilon$ te integriranjem po $\boldsymbol{\Omega}_\varepsilon$ dobivamo

$$\begin{gathered}\frac{1}{1-N}\int_{\Omega_\varepsilon}\nabla\boldsymbol{v}_\varepsilon\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\cdot\nabla\boldsymbol{v}_\varepsilon=\frac{1}{Pr}\frac{1}{\varepsilon^2}Q_0\int_{\Sigma_0^\varepsilon}\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon\cdot\boldsymbol{e}_1\\ -\frac{1}{Pr}\frac{1}{\varepsilon^2}Q_1\int_{\Sigma_0^\varepsilon}\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon\cdot\boldsymbol{e}_1-\frac{1}{Pr}\int_{\Omega_\varepsilon}(\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{v}_\varepsilon\cdot\boldsymbol{v}_\varepsilon\\ +\frac{2N}{1-N}\int_{\Omega_\varepsilon}\nabla\boldsymbol{\Phi}\nabla^\perp z_\varepsilon\cdot\boldsymbol{v}_\varepsilon+Ra\int_{\Omega_\varepsilon}\det(\nabla\boldsymbol{\Phi})S_\varepsilon v_2^\varepsilon+\int_{\Omega_\varepsilon}\det(\nabla\boldsymbol{\Phi})\boldsymbol{F}_\varepsilon\cdot\boldsymbol{v}_\varepsilon.\end{gathered}\tag{2.58}$$

Kao i ranije, može se pokazati identitet

$$\int_{\Omega_\varepsilon}(\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{v}_\varepsilon\cdot\boldsymbol{v}_\varepsilon=0.$$

Također, vrijedi

$$\begin{aligned}Q_0\int_{\Sigma_0^\varepsilon}\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon\cdot\boldsymbol{e}_1-Q_1\int_{\Sigma_0^\varepsilon}\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon\cdot\boldsymbol{e}_1&=\int_{\partial\Omega_\varepsilon}\operatorname{div}\Big((\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon\,(Q_0+x_1(Q_1-Q_0))\Big)\\ &=\int_{\Omega_\varepsilon}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon\,(Q_0+x_1(Q_1-Q_0))\,.\end{aligned}$$

Ocjenjujemo preostale izraze s desne strane (2.58) koristeći (2.49) i Lemu 0.0.1:

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon\,(Q_0+x_1(Q_1-Q_0))&\leq\frac{C}{\varepsilon^2}\left\|Q_0+x_1(Q_1-Q_0)\right\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\ &\leq\frac{C}{\sqrt{\varepsilon}}\|\nabla\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\,,\\ \int_{\Omega_\varepsilon}\nabla\boldsymbol{\Phi}\nabla^\perp z_\varepsilon\cdot\boldsymbol{v}_\varepsilon&\leq C\|\nabla z_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\ &\leq C\varepsilon\|\nabla z_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\,,\\ \int_{\Omega_\varepsilon}\det(\nabla\boldsymbol{\Phi})S_\varepsilon v_2^\varepsilon&\leq C\|S_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\ &\leq C\varepsilon^2\|\nabla S_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\,,\\ \int_{\Omega_\varepsilon}\det(\nabla\boldsymbol{\Phi})\boldsymbol{F}_\varepsilon\cdot\boldsymbol{v}_\varepsilon&\leq\frac{1}{\varepsilon^2}\|\boldsymbol{F}\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\ &\leq\frac{C}{\sqrt{\varepsilon}}\|\nabla\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\,.\end{aligned}\tag{2.59}$$

Iz (2.52), (2.56), (2.58), (2.59), te Leme 2.3.4 zaključujemo

$$\|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\sqrt{\varepsilon}}. \tag{2.60}$$

Pretpostavimo li da za tlak vrijedi normalizirajući uvjet $\int_{\Omega_\varepsilon} q_\varepsilon = 0$ (tlak je određen do na aditivnu konstantu), po Lemi 2.3.2 postoji $\boldsymbol{r}_\varepsilon \in W_0^{1,2}(\Omega_\varepsilon)$ takav da

$$\begin{aligned} \operatorname{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T \boldsymbol{r}_\varepsilon) &= q_\varepsilon \quad \text{u} \quad \Omega_\varepsilon, \\ \|\nabla \boldsymbol{r}_\varepsilon\|_{L^2(\Omega_\varepsilon)} &\leq \frac{C}{\varepsilon} \|q\|_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{2.61}$$

Množenjem $(2.5)_1$ s $\boldsymbol{r}_\varepsilon$ te integriranjem po $\boldsymbol{\Omega}$ dobivamo

$$\begin{aligned} \int_{\Omega_\varepsilon} |q|^2 &= \frac{1}{Pr} \int_{\Omega_\varepsilon} ((\boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{v}_\varepsilon \cdot \boldsymbol{r}_\varepsilon + \frac{1}{1-N} \int_{\Omega_\varepsilon} \nabla \boldsymbol{v}_\varepsilon \tilde{\boldsymbol{F}}(\boldsymbol{\Phi}) \cdot \nabla \boldsymbol{r}_\varepsilon \\ &+ \int_{\Omega_\varepsilon} \frac{2N}{1-N} \nabla \boldsymbol{\Phi} \nabla^\perp z_\varepsilon \cdot \boldsymbol{r}_\varepsilon + Ra \int_{\Omega_\varepsilon} \det(\nabla \boldsymbol{\Phi}) S_\varepsilon \boldsymbol{e}_2 \cdot \boldsymbol{r}_\varepsilon + \int_{\Omega_\varepsilon} \det(\nabla \boldsymbol{\Phi}) \boldsymbol{F}_\varepsilon \cdot \boldsymbol{r}_\varepsilon. \end{aligned} \tag{2.62}$$

Izrazi s desne strane jednakosti (2.62) se mogu ocijeniti koristeći (2.49), Lemu 0.0.1 te ocjene (2.52), (2.56), (2.60) i $(2.61)_2$, čime konačno imamo

$$\|q_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\sqrt{\varepsilon^3}}.$$

■

2.3.2. Ocjene greške

U ovom odjeljku izvodimo ocjenu greške prateći pristup iz [56], gdje ćemo koristiti apriorne ocjene (2.49), Leme 0.0.1, 2.3.1–2.3.4 te svojstva funkcija h i h_0.

U dokazu naglasak stavljamo na ocjenu integralnih članova koji se ne pojavljuju u analizi greške u [56], te uzimamo u obzir da smo u Odjeljku 2.2.2 provodili analizu rubnog sloja koristeći aproksimaciju funkcije $\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})$, što može utjecati na red greške.

Teorem 2.3.1 (Ocjene greške)**.** Neka je $(\boldsymbol{v}_\varepsilon, q_\varepsilon, z_\varepsilon, S_\varepsilon)$ rješenje problema (2.5)–(2.6), te neka je $(\boldsymbol{V}_\varepsilon^{[1]}, Q_\varepsilon^{[2]}, Z_\varepsilon^{[1]}, T_\varepsilon^{[2]})$ asimptotička aproksimacija dana s (2.48). Tada vrijede sljedeće ocjene:

$$\begin{aligned} \left\|\nabla(\boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon^{[1]})\right\|_{L^2(\Omega_\varepsilon)} &\leq C\sqrt{\varepsilon^3}, \\ \left\|q_\varepsilon - Q_\varepsilon^{[2]}\right\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} &\leq \frac{C}{\sqrt{\varepsilon}}, \\ \left\|\nabla(z_\varepsilon - Z_\varepsilon^{[1]})\right\|_{L^2(\Omega_\varepsilon)} &\leq C\sqrt{\varepsilon^3}, \\ \left\|S_\varepsilon - T_\varepsilon^{[2]}\right\|_{H^1(\Omega_\varepsilon)} &\leq C\sqrt{\varepsilon^3}, \end{aligned} \tag{2.63}$$

gdje je $\boldsymbol{C}$ konstanta neovisna o $\boldsymbol{\varepsilon}$.

Dokaz. Najprije uvodimo sljedeće oznake za razliku originalnog rješenja i asimptotičke aproksimacije:

$$\boldsymbol{R}_\varepsilon = \boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon^{[1]}, \quad r_\varepsilon = q_\varepsilon - Q_\varepsilon^{[2]}, \quad \omega_\varepsilon = z_\varepsilon - Z_\varepsilon^{[1]}, \quad K_\varepsilon = S_\varepsilon - T_\varepsilon^{[2]}.$$

Mikrorotacija z_ε zadovoljava sljedeći sustav:

$$\begin{aligned} -L\,\mathrm{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla z_\varepsilon) + \frac{4N}{1-N}\det(\nabla\boldsymbol{\Phi})z_\varepsilon &= -\frac{M}{Pr}(\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla z_\varepsilon) \\ &+\frac{2N}{1-N}\,\mathrm{rot}((\nabla\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon) + \det(\nabla\boldsymbol{\Phi})G_\varepsilon, \\ z_\varepsilon &= 0 \quad \text{na} \quad \partial\Omega_\varepsilon. \end{aligned} \tag{2.64}$$

Aproksimacija mikrorotacije $Z_\varepsilon^{[1]}$ je rješenje sustava jednadžbi

$$\begin{aligned} -L\,\mathrm{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla Z_\varepsilon^{[1]}) &= \frac{1}{\varepsilon^2}h_0(x_1)G - \frac{2N}{1-N}\left(\frac{\partial V_1^0}{\partial x_2} + h'(x_1)\frac{\partial V_2^0}{\partial x_2}\right) \\ &-L\,\mathrm{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla \mathcal{Z}_{\varepsilon,bl,0}) - L\,\mathrm{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla \mathcal{Z}_{\varepsilon,bl,1}) + J^\varepsilon, \\ Z_\varepsilon^{[1]} &= 0 \quad \text{na} \quad \Gamma^\varepsilon, \\ Z_\varepsilon^{[1]} = \eta_0^\varepsilon \quad \text{na} \quad \Sigma_0^\varepsilon, &\quad Z_\varepsilon^{[1]} = \eta_1^\varepsilon \quad \text{na} \quad \Sigma_1^\varepsilon, \end{aligned} \tag{2.65}$$

gdje je $\eta_0^\varepsilon = \varepsilon\mathcal{Z}_{bl,1}^1\left(-\frac{1}{\varepsilon}, \frac{\cdot}{\varepsilon}\right)$, $\eta_1^\varepsilon = \varepsilon\mathcal{Z}_{bl,0}^1\left(\frac{1}{\varepsilon}, \frac{\cdot}{\varepsilon}\right)$, i

$$\begin{aligned} J^\varepsilon = &-L\left[\frac{\partial}{\partial x_1}\left(h_0(x_1)\frac{\partial Z_{\varepsilon,reg}^{[1]}}{\partial x_1}\right) - \frac{\partial}{\partial x_1}\left(h_0'(x_1)x_2\frac{\partial Z_{\varepsilon,reg}^{[1]}}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(h_0'(x_1)x_2\frac{\partial Z_{\varepsilon,reg}^{[1]}}{\partial x_1}\right)\right. \\ &+\frac{\partial}{\partial x_2}\left(\frac{h_0'(x_1)^2}{h_0(x_1)}x_2^2\frac{\partial Z_{\varepsilon,reg}^{[1]}}{\partial x_2}\right) - \varepsilon\frac{\partial}{\partial x_1}\left(h'(x_1)\frac{\partial Z^1}{\partial x_2}\right) - \varepsilon h'(x_1)\frac{\partial^2 Z^1}{\partial x_2\partial x_1} \\ &\left.+2\varepsilon\frac{\partial}{\partial x_2}\left(\frac{h_0'(x_1)h'(x_1)}{h_0(x_1)}x_2\frac{\partial Z^1}{\partial x_2}\right)\right]. \end{aligned}$$

Lako je provjeriti da vrijede sljedeće ocjene:

$$\left\|\eta_0^\varepsilon\right\|_{L^2(\Sigma_0^\varepsilon)} \le C\exp(-\sigma/\varepsilon), \quad \left\|\eta_1^\varepsilon\right\|_{L^2(\Sigma_1^\varepsilon)} \le C\exp(-\sigma/\varepsilon), \quad \|J^\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\sqrt{\varepsilon},$$

gdje su $C, \sigma > 0$ konstante koje ne ovise o ε.

Sada oduzimanjem (2.65) od (2.64) dobivamo sljedeći sustav jednadžbi:

$$\begin{aligned} -L\,\mathrm{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\omega_\varepsilon) + \frac{4N}{1-N}\det(\nabla\boldsymbol{\Phi})\omega_\varepsilon &= -\frac{M}{Pr}\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla\omega_\varepsilon - \frac{M}{Pr}\boldsymbol{R}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla Z_\varepsilon^{[1]} \\ -\frac{M}{Pr}\boldsymbol{V}_\varepsilon^{[1]}\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla Z_\varepsilon^{[1]} &+ \frac{2N}{1-N}\,\mathrm{rot}((\nabla\boldsymbol{\Phi})^T(\boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon^{[0]})) + L\,\mathrm{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\mathcal{Z}_{\varepsilon,bl,0}) \\ &+L\,\mathrm{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\mathcal{Z}_{\varepsilon,bl,1}) + \beta^\varepsilon, \\ \omega_\varepsilon &= 0 \quad \text{na} \quad \Gamma^\varepsilon, \\ \omega_\varepsilon = -\eta_0^\varepsilon \quad \text{na} \quad \Sigma_0^\varepsilon, &\quad \omega_\varepsilon = -\eta_1^\varepsilon \quad \text{na} \quad \Sigma_1^\varepsilon, \end{aligned} \tag{2.66}$$

pri čemu vrijedi $\|\beta^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}$.

Izrazi $\mathrm{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\mathcal{Z}_{\varepsilon,bl,0})$ i $\mathrm{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\mathcal{Z}_{\varepsilon,bl,1})$ koji se pojavljuju s desne strane jednakosti (2.66) dolaze od korektora rubnog sloja te nisu jednaki nuli, ali ćemo pokazati da su dovoljno mali. Uzimajući u obzir (2.47) i regularnost funkcija h i h_0 te koristeći $h_0'(0) = h'(0) = 0$ izvodimo sljedeću ocjenu:

$$
\begin{aligned}
|\,\mathrm{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\mathcal{Z}_{\varepsilon,bl,0})| &= |\,\mathrm{div}((\tilde{\boldsymbol{F}}(\boldsymbol{\Phi}) - \tilde{\boldsymbol{F}}(\boldsymbol{\Phi}(0)))\nabla\mathcal{Z}_{\varepsilon,bl,0})| \\
&= \varepsilon\frac{\partial}{\partial x_1}\left((h_0(x_1) - h_0(0))\frac{\partial\mathcal{Z}^1_{bl,0}}{\partial x_1}\right) - \varepsilon\frac{\partial}{\partial x_1}\left((h_0'(x_1)x_2 + h'(x_1) - h'(0))\frac{\partial\mathcal{Z}^1_{bl,0}}{\partial x_2}\right) \\
&- \varepsilon\frac{\partial}{\partial x_2}\left((h_0'(x_1)x_2 + h'(x_1) - h'(0))\frac{\partial\mathcal{Z}^1_{bl,0}}{\partial x_1}\right) \\
&+ \varepsilon\frac{\partial}{\partial x_2}\left(\left(\frac{h_0'(x_1)^2}{h_0(x_1)}x_2^2 + 2\frac{h_0'(x_1)h'(x_1)}{h_0(x_1)}x_2 + \frac{h'(x_1)^2+1}{h_0(x_1)} - \frac{1}{h_0(0)}\right)\frac{\partial\mathcal{Z}^1_{bl,0}}{\partial x_2}\right) \\
&= \varepsilon\frac{\partial}{\partial x_1}\left(O(x_1^2)\frac{\partial\mathcal{Z}^1_{bl,0}}{\partial x_1}\right) - \varepsilon\frac{\partial}{\partial x_1}\left((h_0'(x_1)x_2 + O(x_1^2))\frac{\partial\mathcal{Z}^1_{bl,0}}{\partial x_2}\right) \\
&- \varepsilon\frac{\partial}{\partial x_2}\left((h_0'(x_1)x_2 + O(x_1^2))\frac{\partial\mathcal{Z}^1_{bl,0}}{\partial x_1}\right) \\
&+ \varepsilon\frac{\partial}{\partial x_2}\left(\left(\frac{h_0'(x_1)^2}{h_0(x_1)}x_2^2 + 2\frac{h_0'(x_1)h'(x_1)}{h_0(x_1)}x_2 + O(x_1^2)\right)\frac{\partial\mathcal{Z}^1_{bl,0}}{\partial x_2}\right),
\end{aligned}
$$

iz čega možemo zaključiti [1]

$$
\big|\big|\,\mathrm{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\mathcal{Z}_{\varepsilon,bl,0})\big|\big|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}. \tag{2.67}
$$

Za rubni sloj na suprotnom kraju kanala $x_1 = 1$ se na analogan način izvodi ista ocjena.

Sada uvodimo novu test funkciju ω_ε^* koja je jednaka nuli na rubu domene $\partial\boldsymbol{\Omega}_\varepsilon$, definiranu s

$$
\omega_\varepsilon^* = \omega_\varepsilon - D_\varepsilon^\omega, \quad D_\varepsilon^\omega = -\eta_0^\varepsilon + x_1(-\eta_1^\varepsilon + \eta_0^\varepsilon).
$$

[1]Istaknimo da se ocjena (2.67) može poboljšati ako uvedemo dodatne pretpostavke na regularnost funkcija h i h_0. Naime, ako pretpostavimo $h_0 \in C^k([0,1])$, $h \in C^{k+1}([0,1])$ i

$$
h_0^{(i)}(0) = 0, \; i = 1, \dots, k-1,
$$
$$
h^{(i)}(0) = 0, \; i = 1, \dots, k,
$$

imamo sljedeću ocjenu:

$$
\big|\big|\,\mathrm{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\mathcal{Z}_{\varepsilon,bl,i})\big|\big|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{k-1/2}, \; i = 0, 1.
$$

Vrijede sljedeće ocjene za D_ε^ω :

$$\left\|D_\varepsilon^\omega\right\|_{L^2(\Omega_\varepsilon)} \leq C\exp(-\sigma/\varepsilon), \ \left\|\nabla D_\varepsilon^\omega\right\|_{L^2(\Omega_\varepsilon)} \leq C\exp(-\sigma/\varepsilon), \\ \left\|D_\varepsilon^\omega\right\|_{L^4(\Omega_\varepsilon)} \leq C\exp(-\sigma/\varepsilon). \tag{2.68}$$

Sada množimo (2.66) s ω_ε^* te integriramo po $\mathbf{\Omega}_\varepsilon$, čime dobivamo jednakost

$$\begin{aligned} L\int_{\Omega_\varepsilon} \tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\omega_\varepsilon^* \cdot \nabla\omega_\varepsilon^* + \frac{4N}{1-N}\int_{\Omega_\varepsilon} \det(\nabla\boldsymbol{\Phi})|\omega_\varepsilon^*|^2 = -L\int_{\Omega_\varepsilon} \tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla D_\varepsilon^\omega \cdot \nabla\omega_\varepsilon^* \\ -\frac{4N}{1-N}\int_{\Omega_\varepsilon} \det(\nabla\boldsymbol{\Phi})D_\varepsilon^\omega\omega_\varepsilon^* - \frac{M}{Pr}\int_{\Omega_\varepsilon} \boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla\omega_\varepsilon^*\omega_\varepsilon^* \\ -\frac{M}{Pr}\int_{\Omega_\varepsilon} \boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla D_\varepsilon^\omega\omega_\varepsilon^* - \frac{M}{Pr}\int_{\Omega_\varepsilon} \boldsymbol{R}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla Z_\varepsilon^{[1]}\omega_\varepsilon^* \\ -\frac{M}{Pr}\int_{\Omega_\varepsilon} \boldsymbol{V}_\varepsilon^{[1]} \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla Z_\varepsilon^{[1]}\omega_\varepsilon^* + \frac{2N}{1-N}\int_{\Omega_\varepsilon} \operatorname{rot}((\nabla\boldsymbol{\Phi})^T(\boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon^{[0]}))\omega_\varepsilon^* \\ +L\int_{\Omega_\varepsilon} \operatorname{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\mathcal{Z}_{\varepsilon,bl,0})\omega_\varepsilon^* + L\int_{\Omega_\varepsilon} \operatorname{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\mathcal{Z}_{\varepsilon,bl,1})\omega_\varepsilon^* + \int_{\Omega_\varepsilon} \beta^\varepsilon\omega_\varepsilon^*. \end{aligned} \tag{2.69}$$

Vidimo da zbog rubnog uvjeta $\omega_\varepsilon^* = 0$ na $\partial\mathbf{\Omega}_\varepsilon$ te činjenice da $\operatorname{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon) = 0$ vrijedi jednakost

$$\begin{aligned} \int_{\Omega_\varepsilon} \boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla\omega_\varepsilon^*\omega_\varepsilon^* &= \frac{1}{2}\int_{\Omega_\varepsilon} \boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla(\omega_\varepsilon^*)^2 = \frac{1}{2}\int_{\Omega_\varepsilon} \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon \cdot \nabla(\omega_\varepsilon^*)^2 \\ &= \frac{1}{2}\int_{\Omega_\varepsilon} \operatorname{div}\left(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon(\omega_\varepsilon^*)^2\right) - \frac{1}{2}\int_{\Omega_\varepsilon} \operatorname{div}\left(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon\right)(\omega_\varepsilon^*)^2 = 0. \end{aligned}$$

Nadalje, vrijedi ocjena

$$\left\|\boldsymbol{V}_\varepsilon^{[1]} \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla Z_\varepsilon^{[1]}\right\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\sqrt{\varepsilon}}.$$

Gornja nejednakost neće dati zadovoljavajuće ocjene, no ona se može poboljšati raspisom nelinearnog člana. Naime, vrijedi

$$\boldsymbol{V}_\varepsilon^{[1]} \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla Z_\varepsilon^{[1]} = \boldsymbol{V}_\varepsilon^{[1]} \cdot \left(h_0(x_1)\frac{\partial Z_\varepsilon^{[1]}}{\partial x_1} - (h_0'(x_1)x_2 + h'(x_1))\frac{\partial Z_\varepsilon^{[1]}}{\partial x_2}, \frac{\partial Z_\varepsilon^{[1]}}{\partial x_2}\right).$$

Posebno, imamo

$$-(h_0'(x_1)x_2 + h'(x_1))V_1^0 + V_2^0 = \frac{6C_1h_0'(x_1)}{h_0(x_1)}x_2\left(\frac{x_2^2}{\varepsilon^2} - \frac{x_2}{\varepsilon}\right),$$

iz čega možemo zaključiti

$$\left\|\boldsymbol{V}_\varepsilon^{[1]} \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla Z_\varepsilon^{[1]}\right\|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}.$$

Preostali izrazi s desne strane (2.69) se mogu ocijeniti koristeći (2.49), (2.67), (2.68), Propoziciju 2.3.1 te Lemu 0.0.1:

$$
\begin{aligned}
\int_{\Omega_\varepsilon} \tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla D_\varepsilon^\omega \cdot \nabla\omega_\varepsilon^* &\le \left\|\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\right\|_{L^\infty(\Omega_\varepsilon)} \left\|\nabla D_\varepsilon^\omega\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\exp(-\sigma/\varepsilon) \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \det(\nabla\boldsymbol{\Phi}) D_\varepsilon^\omega \omega_\varepsilon^* &\le C\varepsilon^2 \left\|\det(\nabla\boldsymbol{\Phi})\right\|_{L^\infty(\Omega_\varepsilon)} \left\|\nabla D_\varepsilon^\omega\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon^2 \exp(-\sigma/\varepsilon) \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla D_\varepsilon^\omega \omega_\varepsilon^* &\le \left\|\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\right\|_{L^\infty(\Omega_\varepsilon)} \|\boldsymbol{v}_\varepsilon\|_{L^4(\Omega_\varepsilon)} \left\|\omega_\varepsilon^*\right\|_{L^4(\Omega_\varepsilon)} \left\|\nabla D_\varepsilon^\omega\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon \|\nabla\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla D_\varepsilon^\omega\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\sqrt{\varepsilon}\exp(-\sigma/\varepsilon) \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \boldsymbol{R}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla Z_\varepsilon^{[1]} \omega_\varepsilon^* &\le \left\|\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\right\|_{L^\infty(\Omega_\varepsilon)} \left\|\nabla Z_\varepsilon^{[1]}\right\|_{L^\infty(\Omega_\varepsilon)} \|\boldsymbol{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \left\|\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon \|\nabla\boldsymbol{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \boldsymbol{V}_\varepsilon^{[1]} \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla Z_\varepsilon^{[1]} \omega_\varepsilon^* &\le \left\|\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \left\|\boldsymbol{V}_\varepsilon^{[1]} \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla Z_\varepsilon^{[1]}\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\sqrt{\varepsilon^3} \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \operatorname{rot}((\nabla\boldsymbol{\Phi})^T(\boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon^{[0]}))\omega_\varepsilon^* &\le C \left\|\nabla(\boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon^{[0]})\right\|_{L^2(\Omega_\varepsilon)} \left\|\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon \left\|\nabla(\boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon^{[1]})\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\quad + C\varepsilon \left\|\nabla(\boldsymbol{V}_\varepsilon^{[1]} - \boldsymbol{V}_\varepsilon^{[0]})\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon \|\nabla\boldsymbol{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} + C\sqrt{\varepsilon^3} \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \operatorname{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\mathcal{Z}_{\varepsilon,bl,0})\omega_\varepsilon^* &\le \left\|\operatorname{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\mathcal{Z}_{\varepsilon,bl,0})\right\|_{L^2(\Omega_\varepsilon)} \left\|\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\sqrt{\varepsilon^3} \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \operatorname{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\mathcal{Z}_{\varepsilon,bl,1})\omega_\varepsilon^* &\le \left\|\operatorname{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla\mathcal{Z}_{\varepsilon,bl,1})\right\|_{L^2(\Omega_\varepsilon)} \left\|\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\sqrt{\varepsilon^3} \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \beta^\varepsilon \omega_\varepsilon^* &\le \|\beta^\varepsilon\|_{L^2(\Omega_\varepsilon)} \left\|\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\sqrt{\varepsilon^3} \left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}.
\end{aligned}
\tag{2.70}
$$

Sada iz (2.69), (2.70), Leme 2.3.3, te činjenice da je $\det(\nabla\boldsymbol{\Phi}) \ge c > 0$, za dovoljno mali parametar ε dobivamo ocjenu

$$\left\|\nabla\omega_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon \left\|\nabla \boldsymbol{R}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} + C\sqrt{\varepsilon^3},$$

odnosno

$$\left\|\nabla\omega_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon \left\|\nabla \boldsymbol{R}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} + C\sqrt{\varepsilon^3}. \tag{2.71}$$

Razmotrimo sada sustav jednadžbi koji rješava temperatura S_ε:

$$\begin{gathered}
-\operatorname{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla S_\varepsilon) = -\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla S_\varepsilon + D\nabla^\perp z_\varepsilon\cdot\nabla S_\varepsilon,\\
S_\varepsilon = \theta_i \text{ na } \Sigma_i^\varepsilon,\ i=0,1,\\
\frac{\partial S_\varepsilon}{\partial x_2} = 0 \text{ za } x_2=0,\quad \frac{\partial S_\varepsilon}{\partial x_2} = \varepsilon k(G_T(x_1)-S_\varepsilon) \text{ za } x_2=\varepsilon,
\end{gathered} \tag{2.72}$$

te sustav jednadžbi koji zadovoljava aproksimacija temperature $T_\varepsilon^{[2]}$:

$$\begin{gathered}
-\operatorname{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla T_\varepsilon^{[2]}) = -h_0(x_1)V_1^0\frac{\partial T^0}{\partial x_1} + D\frac{\partial Z^0}{\partial x_2}\frac{\partial T^0}{\partial x_1} + \varepsilon D\frac{\partial Z^1}{\partial x_2}\frac{\partial T^0}{\partial x_1} + \varepsilon D\frac{\partial Z^0}{\partial x_2}\frac{\partial T^1}{\partial x_1}\\
-\varepsilon D\frac{\partial Z^0}{\partial x_1}\frac{\partial T^1}{\partial x_2} + L^\varepsilon,\\
T_\varepsilon^{[2]} = \theta_i + \varepsilon^2 T^2\left(i,\frac{\cdot}{\varepsilon}\right) \text{ na } \Sigma_i^\varepsilon,\ i=0,1,\\
\frac{\partial T_\varepsilon^{[2]}}{\partial x_2} = 0 \text{ za } x_2=0,\quad \frac{\partial T_\varepsilon^{[2]}}{\partial x_2} = \varepsilon k(G_T(x_1)-T^0) \text{ za } x_2=\varepsilon,
\end{gathered} \tag{2.73}$$

pri čemu je

$$\begin{aligned}
L^\varepsilon = &-\frac{\partial}{\partial x_1}\left(h_0(x_1)\frac{\partial}{\partial x_1}(\varepsilon T^1+\varepsilon^2T^2)\right) + \frac{\partial}{\partial x_1}\left(h_0'(x_1)x_2\frac{\partial}{\partial x_2}(\varepsilon T^1+\varepsilon^2T^2)\right)\\
&+\frac{\partial}{\partial x_2}\left(h_0'(x_1)x_2\frac{\partial}{\partial x_1}(\varepsilon T^1+\varepsilon^2T^2)\right) - \frac{\partial}{\partial x_2}\left(\frac{h_0'(x_1)^2}{h_0(x_1)}x_2^2\frac{\partial}{\partial x_2}(\varepsilon T^1+\varepsilon^2T^2)\right)\\
&+\varepsilon^2\frac{\partial}{\partial x_1}\left(h'(x_1)\frac{\partial T^2}{\partial x_2}\right) + \varepsilon^2\frac{\partial}{\partial x_2}\left(h'(x_1)\frac{\partial T^2}{\partial x_1}\right) - 2\varepsilon^2\frac{\partial}{\partial x_2}\left(\frac{h_0'(x_1)h'(x_1)}{h_0(x_1)}x_2\frac{\partial T^2}{\partial x_2}\right).
\end{aligned}$$

Oduzmimo sada (2.73) od (2.72), čime dobivamo sljedeći sustav za K_ε:

$$\begin{gathered}
-\operatorname{div}(\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla K_\varepsilon) = -\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla K_\varepsilon - \boldsymbol{R}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla T_\varepsilon^{[2]} - (\boldsymbol{V}_\varepsilon^{[1]}-\boldsymbol{V}_\varepsilon^{[0]})\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla T_\varepsilon^{[2]}\\
-\boldsymbol{V}_\varepsilon^{[0]}\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})(\nabla T_\varepsilon^{[2]}-\nabla T_\varepsilon^{[0]}) + D\nabla^\perp z_\varepsilon\cdot\nabla K_\varepsilon + D\nabla^\perp\omega_\varepsilon\cdot\nabla T_\varepsilon^{[2]}\\
+D\nabla^\perp(Z_\varepsilon^{[1]}-Z_\varepsilon^{[0]})\cdot\nabla(T_\varepsilon^{[2]}-T_\varepsilon^{[1]}) + \varepsilon^2 D\frac{\partial Z^1}{\partial x_2}\frac{\partial T^1}{\partial x_1} - \varepsilon^2 D\frac{\partial Z^1}{\partial x_1}\frac{\partial T^1}{\partial x_2} + \varepsilon^2 D\frac{\partial Z^0}{\partial x_2}\frac{\partial T^2}{\partial x_1}\\
-\varepsilon^2 D\frac{\partial Z^0}{\partial x_1}\frac{\partial T^2}{\partial x_2} - L^\varepsilon,\\
K_\varepsilon = \zeta_i^\varepsilon \text{ na } \Sigma_i^\varepsilon,\ i=0,1,\\
\frac{\partial K_\varepsilon}{\partial x_2} = 0 \text{ za } x_2=0,\quad \frac{\partial K_\varepsilon}{\partial x_2} + \varepsilon k K_\varepsilon = \xi_\varepsilon \text{ za } x_2=\varepsilon,
\end{gathered} \tag{2.74}$$

pri čemu je $\zeta_i^\varepsilon = -\varepsilon^2 T^2\left(i, \frac{\cdot}{\varepsilon}\right)$, $i = 0, 1$ i $\xi_\varepsilon = -\varepsilon^2 k(T^1(\cdot, 1) + \varepsilon T^2(\cdot, 1))$.

Vrijede sljedeće ocjene:

$$\begin{aligned} \|L^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon^3}, \ \left\|\zeta_0^\varepsilon\right\|_{L^2(\Sigma_0^\varepsilon)} \leq C\sqrt{\varepsilon^5}, \\ \left\|\zeta_1^\varepsilon\right\|_{L^2(\Sigma_1^\varepsilon)} \leq C\sqrt{\varepsilon^5}, \ \|\xi_\varepsilon\|_{L^2(\Gamma^\varepsilon)} \leq C\varepsilon^2. \end{aligned} \tag{2.75}$$

Uvedimo sada novu test funkciju K_ε^* koja je jednaka nuli na Σ_i^ε, $i = 0, 1$:

$$K_\varepsilon^* = K_\varepsilon - D_\varepsilon^K, \quad D_\varepsilon^K = \zeta_0^\varepsilon + x_1(\zeta_1^\varepsilon - \zeta_0^\varepsilon).$$

Lako je vidjeti da vrijede sljedeće ocjene:

$$\begin{aligned} \left\|\nabla D_\varepsilon^K\right\|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon^3}, \quad \left\|\nabla D_\varepsilon^K\right\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon, \\ \left\|D_\varepsilon^K\right\|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon^5}, \quad \left\|D_\varepsilon^K\right\|_{L^4(\Omega_\varepsilon)} \leq C\sqrt[4]{\varepsilon^9}. \end{aligned} \tag{2.76}$$

Množenjem (2.74) s K_ε^* te integriranjem po $\mathbf{\Omega}_\varepsilon$ dobivamo

$$\begin{aligned} &\int_{\Omega_\varepsilon} \tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla K_\varepsilon^* \cdot \nabla K_\varepsilon^* + \varepsilon k \int_{\Gamma_+^\varepsilon} \frac{(h_0'(x_1)x_2 + h'(x_1))^2 + 1}{h_0(x_1)} |K_\varepsilon^*|^2 \\ &= -\int_{\Omega_\varepsilon} \tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\nabla D_\varepsilon^K \cdot \nabla K_\varepsilon^* - \int_{\Gamma^\varepsilon} (h_0'(x_1)x_2 + h'(x_1)) \frac{\partial K_\varepsilon^*}{\partial x_1} K_\varepsilon^* \\ &- \int_{\Gamma^\varepsilon} (h_0'(x_1)x_2 + h'(x_1)) \frac{\partial D_\varepsilon^K}{\partial x_1} K_\varepsilon^* + \int_{\Gamma_+^\varepsilon} \frac{(h_0'(x_1)x_2 + h'(x_1))^2 + 1}{h_0(x_1)} \xi_\varepsilon K_\varepsilon^* \\ &-\varepsilon k \int_{\Gamma_+^\varepsilon} \frac{(h_0'(x_1)x_2 + h'(x_1))^2 + 1}{h_0(x_1)} D_\varepsilon^K K_\varepsilon^* - \int_{\Omega_\varepsilon} \boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla K_\varepsilon^* K_\varepsilon^* \\ &- \int_{\Omega_\varepsilon} \boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla D_\varepsilon^K K_\varepsilon^* - \int_{\Omega_\varepsilon} \boldsymbol{R}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla T_\varepsilon^{[2]} K_\varepsilon^* \\ &- \int_{\Omega_\varepsilon} (\boldsymbol{V}_\varepsilon^{[1]} - \boldsymbol{V}_\varepsilon^{[0]}) \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla T_\varepsilon^{[2]} K_\varepsilon^* - \int_{\Omega_\varepsilon} \boldsymbol{V}_\varepsilon^{[0]} \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})(\nabla T_\varepsilon^{[2]} - \nabla T_\varepsilon^{[0]}) K_\varepsilon^* \\ &+D \int_{\Omega_\varepsilon} \nabla^\perp z_\varepsilon \cdot \nabla K_\varepsilon^* K_\varepsilon^* + D \int_{\Omega_\varepsilon} \nabla^\perp z_\varepsilon \cdot \nabla D_\varepsilon^K K_\varepsilon^* + D \int_{\Omega_\varepsilon} \nabla^\perp \omega_\varepsilon \cdot \nabla T_\varepsilon^{[2]} K_\varepsilon^* \\ &+D \int_{\Omega_\varepsilon} \nabla^\perp (Z_\varepsilon^{[1]} - Z_\varepsilon^{[0]}) \cdot \nabla (T_\varepsilon^{[2]} - T_\varepsilon^{[1]}) K_\varepsilon^* + \int_{\Omega_\varepsilon} \varepsilon^2 D \frac{\partial Z^1}{\partial x_2} \frac{\partial T^1}{\partial x_1} K_\varepsilon^* \\ &- \int_{\Omega_\varepsilon} \varepsilon^2 D \frac{\partial Z^1}{\partial x_1} \frac{\partial T^1}{\partial x_2} K_\varepsilon^* + \int_{\Omega_\varepsilon} \varepsilon^2 D \frac{\partial Z^0}{\partial x_2} \frac{\partial T^2}{\partial x_1} K_\varepsilon^* - \int_{\Omega_\varepsilon} \varepsilon^2 D \frac{\partial Z^0}{\partial x_1} \frac{\partial T^2}{\partial x_2} K_\varepsilon^* - \int_{\Omega_\varepsilon} L^\varepsilon K_\varepsilon^*. \end{aligned} \tag{2.77}$$

Kao i ranije, lako je pokazati da vrijede sljedeći identiteti:

$$\begin{aligned} \int_{\Omega_\varepsilon} \boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla K_\varepsilon^* K_\varepsilon^* = 0, \quad \int_{\Omega_\varepsilon} \nabla^\perp z_\varepsilon \cdot \nabla K_\varepsilon^* K_\varepsilon^* = 0, \\ \int_{\Gamma^\varepsilon} (h_0'(x_1)x_2 + h'(x_1)) \frac{\partial K_\varepsilon^*}{\partial x_1} K_\varepsilon^* = -\frac{1}{2} \int_{\Gamma^\varepsilon} (h_0''(x_1)x_2 + h''(x_1))(K_\varepsilon^*)^2. \end{aligned}$$

Preostale izraze s desne strane jednakosti (2.77) ocjenjujemo koristeći (2.49), (2.75), (2.76) Leme 0.0.1, 2.3.1 te Propoziciju 2.3.1 na sljedeći način:

$$
\begin{aligned}
\int_{\Omega_\varepsilon} \tilde{\boldsymbol{F}}(\boldsymbol{\Phi}) \nabla D_\varepsilon^K \cdot \nabla K_\varepsilon^* &\leq \left\| \nabla D_\varepsilon^K \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \sqrt{\varepsilon^3} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Gamma^\varepsilon} (h_0'(x_1) x_2 + h'(x_1)) \frac{\partial K_\varepsilon^*}{\partial x_1} K_\varepsilon^* &\leq C \left\| K_\varepsilon^* \right\|_{L^2(\Gamma^\varepsilon)}^2 \\
&\leq C\varepsilon \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}^2, \\
\int_{\Gamma^\varepsilon} (h_0'(x_1) x_2 + h'(x_1)) \frac{\partial D_\varepsilon^K}{\partial x_1} K_\varepsilon^* &\leq C \left\| \nabla D_\varepsilon^K \right\|_{L^2(\Gamma_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Gamma^\varepsilon)} \\
&\leq C\varepsilon^2 \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Gamma_+^\varepsilon} \frac{(h_0'(x_1) x_2 + h'(x_1))^2 + 1}{h_0(x_1)} \xi_\varepsilon K_\varepsilon^* &\leq C \left\| \xi_\varepsilon \right\|_{L^2(\Gamma^\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Gamma^\varepsilon)} \\
&\leq C \sqrt{\varepsilon^5} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \\
\varepsilon k \int_{\Gamma_+^\varepsilon} \frac{(h_0'(x_1) x_2 + h'(x_1))^2 + 1}{h_0(x_1)} D_\varepsilon^K K_\varepsilon^* &\leq C\varepsilon \left\| D_\varepsilon^K \right\|_{L^2(\Gamma^\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Gamma^\varepsilon)} \\
&\leq C\varepsilon^4 \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi}) \nabla D_\varepsilon^K K_\varepsilon^* &\leq \left\| \boldsymbol{v}_\varepsilon \right\|_{L^4(\Omega_\varepsilon)} \left\| \nabla D_\varepsilon^K \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^4(\Omega_\varepsilon)} \\
&\leq C \sqrt{\varepsilon^7} \left\| \nabla \boldsymbol{v}_\varepsilon \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^3 \left\| \nabla \boldsymbol{v}_\varepsilon \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \qquad (2.78) \\
\int_{\Omega_\varepsilon} \boldsymbol{R}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi}) \nabla T_\varepsilon^{[2]} K_\varepsilon^* &\leq \left\| \nabla T_\varepsilon^{[2]} \right\|_{L^\infty(\Omega_\varepsilon)} \left\| \boldsymbol{R}_\varepsilon \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon \left\| \nabla \boldsymbol{R}_\varepsilon \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} (\boldsymbol{V}_\varepsilon^{[1]} - \boldsymbol{V}_\varepsilon^{[0]}) \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi}) \nabla T_\varepsilon^{[2]} K_\varepsilon^* &\leq C \left\| (\boldsymbol{V}_\varepsilon^{[1]} - \boldsymbol{V}_\varepsilon^{[0]}) \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi}) \nabla T_\varepsilon^{[2]} \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq \left\| \left(\boldsymbol{V}_\varepsilon^{[1]} - \boldsymbol{V}_\varepsilon^{[0]} \right) \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi}) \frac{\partial T^0}{\partial x_1} \boldsymbol{e}_1 \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\quad + \left\| \left(\boldsymbol{V}_\varepsilon^{[1]} - \boldsymbol{V}_\varepsilon^{[0]} \right) \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi}) \nabla (T_\varepsilon^{[2]} - T_\varepsilon^{[0]}) \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \sqrt{\varepsilon^5} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \boldsymbol{V}_\varepsilon^{[0]} \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi}) (\nabla T_\varepsilon^{[2]} - \nabla T_\varepsilon^{[0]}) K_\varepsilon^* &\leq \left\| \boldsymbol{V}_\varepsilon^{[0]} \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi}) (\nabla T_\varepsilon^{[2]} - \nabla T_\varepsilon^{[0]}) \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \sqrt{\varepsilon^3} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)},
\end{aligned}
$$

$$
\begin{aligned}
\int_{\Omega_\varepsilon} \nabla^\perp z_\varepsilon \cdot \nabla D_\varepsilon^K K_\varepsilon^* &\leq C \left\| \nabla D_\varepsilon^K \right\|_{L^\infty(\Omega_\varepsilon)} \left\| \nabla z_\varepsilon \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \sqrt{\varepsilon^3} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \nabla^\perp \omega_\varepsilon \cdot \nabla T_\varepsilon^{[2]} K_\varepsilon^* &\leq \left\| \nabla T_\varepsilon^{[2]} \right\|_{L^\infty(\Omega_\varepsilon)} \left\| \nabla \omega_\varepsilon \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \left\| \nabla \omega_\varepsilon \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \nabla^\perp (Z_\varepsilon^{[1]} - Z_\varepsilon^{[0]}) \cdot \nabla (T_\varepsilon^{[2]} - T_\varepsilon^{[1]}) K_\varepsilon^* &\leq \left\| \nabla^\perp (Z_\varepsilon^{[1]} - Z_\varepsilon^{[0]}) \cdot \nabla (T_\varepsilon^{[2]} - T_\varepsilon^{[1]}) \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \sqrt{\varepsilon^5} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \\
\varepsilon^2 D \int_{\Omega_\varepsilon} \frac{\partial Z^1}{\partial x_2} \frac{\partial T^1}{\partial x_1} K_\varepsilon^* &\leq C \varepsilon^2 \left\| \frac{\partial Z^1}{\partial x_2} \frac{\partial T^1}{\partial x_1} \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \sqrt{\varepsilon^5} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \\
\varepsilon^2 D \int_{\Omega_\varepsilon} \frac{\partial Z^1}{\partial x_1} \frac{\partial T^1}{\partial x_2} K_\varepsilon^* &\leq C \varepsilon^2 \left\| \frac{\partial Z^1}{\partial x_1} \frac{\partial T^1}{\partial x_2} \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \sqrt{\varepsilon^5} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \\
\varepsilon^2 D \int_{\Omega_\varepsilon} \frac{\partial Z^0}{\partial x_2} \frac{\partial T^2}{\partial x_1} K_\varepsilon^* &\leq C \varepsilon^2 \left\| \frac{\partial Z^0}{\partial x_2} \frac{\partial T^2}{\partial x_1} \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \sqrt{\varepsilon^5} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \\
\varepsilon^2 D \int_{\Omega_\varepsilon} \frac{\partial Z^0}{\partial x_1} \frac{\partial T^2}{\partial x_2} K_\varepsilon^* &\leq C \varepsilon^2 \left\| \frac{\partial Z^0}{\partial x_1} \frac{\partial T^2}{\partial x_2} \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \sqrt{\varepsilon^5} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} L^\varepsilon K_\varepsilon^* &\leq \left\| L^\varepsilon \right\|_{L^2(\Omega_\varepsilon)} \left\| K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \sqrt{\varepsilon^5} \left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)}.
\end{aligned}
$$

Sada iz (2.77), (2.78) te Leme 2.3.3 za dovoljno mali ε slijedi

$$
\left\| \nabla K_\varepsilon^* \right\|_{L^2(\Omega_\varepsilon)} \leq C \left(\varepsilon \left\| \nabla \boldsymbol{R}_\varepsilon \right\|_{L^2(\omega_\varepsilon)} + \left\| \nabla \omega_\varepsilon \right\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon^3} \right),
$$

odnosno

$$
\left\| \nabla K_\varepsilon \right\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \left\| \nabla \boldsymbol{R}_\varepsilon \right\|_{L^2(\Omega_\varepsilon)} + C\sqrt{\varepsilon^3}, \tag{2.79}
$$

pri čemu smo iskoristili (2.71) i (2.76).

Obratimo sada pažnju na sustav jednadžbi koji zadovoljava brzina $\boldsymbol{v}_\varepsilon$:

$$\begin{aligned}-\frac{1}{1-N}\operatorname{div}(\nabla\boldsymbol{v}_\varepsilon\tilde{\boldsymbol{F}}(\boldsymbol{\Phi}))+\frac{1}{Pr}\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla q_\varepsilon=-\frac{1}{Pr}(\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{v}_\varepsilon+\frac{2N}{1-N}\nabla\boldsymbol{\Phi}\nabla^\perp z_\varepsilon\\ +Ra\det(\nabla\boldsymbol{\Phi})S_\varepsilon\boldsymbol{e}_2+\det(\nabla\boldsymbol{\Phi})\boldsymbol{F}_\varepsilon,\\ \operatorname{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{v}_\varepsilon)=0,\\ v_2^\varepsilon=0 \text{ na } \Sigma_i^\varepsilon,\ i=0,1,\quad \boldsymbol{v}_\varepsilon=\boldsymbol{0} \text{ na } \Gamma^\varepsilon,\\ q_\varepsilon=\frac{1}{\varepsilon^2}Q_i \text{ na } \Sigma_i^\varepsilon,\ i=0,1.\end{aligned}\tag{2.80}$$

Nadalje, aproksimacija brzine prvog reda $\boldsymbol{V}_\varepsilon^{[1]}$ je rješenje sustava

$$\begin{aligned}-\frac{1}{1-N}\operatorname{div}(\nabla\boldsymbol{V}_\varepsilon^{[1]}\tilde{\boldsymbol{F}}(\boldsymbol{\Phi}))+\frac{1}{Pr}\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla Q_\varepsilon^{[2]}=\frac{1}{\varepsilon^2}h_0(x_1)F_1+\frac{1}{Pr}h_0(x_1)\frac{\partial Q^2}{\partial x_1}\boldsymbol{e}_1\\ -\frac{1}{Pr}h_0'(x_1)x_2\frac{\partial Q^2}{\partial x_2}\boldsymbol{e}_1-\frac{2N}{1-N}\left(h_0'(x_1)x_2\frac{\partial Z_\varepsilon^{[0]}}{\partial x_2}-h_0(x_1)\frac{\partial Z_\varepsilon^{[0]}}{\partial x_1}\right)\boldsymbol{e}_2\\ +\frac{2N}{1-N}\nabla\boldsymbol{\Phi}\nabla^\perp Z_\varepsilon^{[0]}+M^\varepsilon,\\ \operatorname{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{V}_\varepsilon^{[1]})=\varepsilon\frac{\partial}{\partial x_1}\left(h_0(x_1)V_1^1\right)-\varepsilon\frac{\partial}{\partial x_2}(h_0'(x_1)x_2V_1^1),\\ V_{\varepsilon,2}^{[1]}=0 \text{ na } \Sigma_i^\varepsilon,\ i=0,1,\quad \boldsymbol{V}_\varepsilon^{[1]}=\boldsymbol{0} \text{ na } \Gamma^\varepsilon,\\ Q_\varepsilon^{[2]}=\frac{1}{\varepsilon^2}Q_0 \text{ na } \Sigma_0^\varepsilon,\quad Q_\varepsilon^{[2]}=\frac{1}{\varepsilon^2}Q_1 \text{ na } \Sigma_1^\varepsilon,\end{aligned}$$

pri čemu je

$$\begin{aligned}M^\varepsilon=-\frac{1}{1-N}\Bigg[\frac{\partial}{\partial x_1}\left(h_0(x_1)\frac{\partial\boldsymbol{V}_\varepsilon^{[1]}}{\partial x_1}\right)-\frac{\partial}{\partial x_1}\left(h_0'(x_1)x_2\frac{\partial\boldsymbol{V}_\varepsilon^{[1]}}{\partial x_2}\right)-\frac{\partial}{\partial x_2}\left(h_0'(x_1)x_2\frac{\partial\boldsymbol{V}_\varepsilon^{[1]}}{\partial x_1}\right)\\ +\frac{\partial}{\partial x_2}\left(\frac{h_0'(x_1)^2}{h_0(x_1)}x_2^2\frac{\partial\boldsymbol{V}_\varepsilon^{[1]}}{\partial x_2}\right)-\varepsilon\frac{\partial}{\partial x_1}\left(h'(x_1)\frac{\partial\boldsymbol{V}^1}{\partial x_2}\right)-\varepsilon h'(x_1)\frac{\partial^2\boldsymbol{V}^1}{\partial x_2\partial x_1}\\ +2\varepsilon\frac{\partial}{\partial x_2}\left(\frac{h_0'(x_1)h'(x_1)}{h_0(x_1)}x_2\frac{\partial\boldsymbol{V}^1}{\partial x_2}\right)\Bigg],\end{aligned}$$

te vrijedi $\|M^\varepsilon\|_{L^2(\Omega_\varepsilon)}\le C\sqrt{\varepsilon}$.

Uočimo da vrijedi sljedeća ocjena

$$\left|\left|\operatorname{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{V}_\varepsilon^{[1]})\right|\right|_{L^2(\Omega_\varepsilon)}=O(\sqrt{\varepsilon^3}),$$

što nije dovoljno dobro kako bismo izveli željenu ocjenu. Iz tog razloga uvodimo korektor divergencije $\boldsymbol{\Psi}_\varepsilon$ kao rješenje idućeg problema:

$$\begin{aligned}\operatorname{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{\Psi}_\varepsilon)=\varepsilon\frac{\partial}{\partial x_1}\left(h_0(x_1)V_1^1\right)-\varepsilon\frac{\partial}{\partial x_2}(h_0'(x_1)x_2V_1^1),\\ \boldsymbol{\Psi}_\varepsilon=\boldsymbol{0}\ \text{ na }\ \partial\Omega_\varepsilon.\end{aligned}\tag{2.81}$$

Koristeći (2.33) te rubni uvjet $(2.26)_4$ vidimo da je zadovoljen nužan uvjet kompatibilnosti:

$$\int_{\Omega_\varepsilon} \frac{\partial}{\partial x_1}\left(h_0(x_1)V_1^1\right) - h_0'(x_1)\frac{\partial}{\partial x_2}(x_2 V_1^1) = \varepsilon \int_0^1 (h_0(1)V_1^1(1,y_2) - h_0(0)V_1^1(0,y_2))dy_2$$
$$-\varepsilon \int_0^1 h_0'(x_1)V_1^1(x_1,1)dx_1 = 0,$$

pa po Lemi 2.3.2 problem (2.81) ima barem jedno rješenje takvo da vrijedi

$$\|\nabla \boldsymbol{\Psi}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}. \tag{2.82}$$

Korigirana aproksimacija brzine sada ima oblik

$$\tilde{\boldsymbol{V}}_\varepsilon^{[1]} = \boldsymbol{V}_\varepsilon^{[1]} - \boldsymbol{\Psi}_\varepsilon,$$

te zadovoljava sljedeći sustav jednadžbi:

$$\begin{aligned}
-\frac{1}{1-N}\operatorname{div}(\nabla\tilde{\boldsymbol{V}}_\varepsilon^{[1]}\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})) + \frac{1}{Pr}\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla Q_\varepsilon^{[2]} &= \frac{1}{\varepsilon^2}h_0(x_1)F_1 + \frac{1}{1-N}\operatorname{div}(\nabla\boldsymbol{\Psi}_\varepsilon\tilde{\boldsymbol{F}}(\boldsymbol{\Phi}))\\
&+\frac{1}{Pr}h_0(x_1)\frac{\partial Q^2}{\partial x_1}\boldsymbol{e}_1 - \frac{1}{Pr}h_0'(x_1)x_2\frac{\partial Q^2}{\partial x_2}\boldsymbol{e}_1 + \frac{2N}{1-N}\nabla\boldsymbol{\Phi}\nabla^\perp Z_\varepsilon^{[0]}\\
&-\frac{2N}{1-N}\left(h_0'(x_1)x_2\frac{\partial Z_\varepsilon^{[0]}}{\partial x_2} - h_0(x_1)\frac{\partial Z_\varepsilon^{[0]}}{\partial x_1}\right)\boldsymbol{e}_2 + M^\varepsilon,\\
\operatorname{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\tilde{\boldsymbol{V}}_\varepsilon^{[1]}) &= 0,\\
\tilde{V}_{\varepsilon,2}^{[1]} = 0 \text{ na } \Sigma_i^\varepsilon,\ i=0,1,\quad \tilde{\boldsymbol{V}}_\varepsilon^{[1]} &= \boldsymbol{0} \text{ na } \Gamma^\varepsilon.
\end{aligned} \tag{2.83}$$

Oduzimanjem (2.83) od (2.80) dobivamo

$$\begin{aligned}
-\frac{1}{1-N}\operatorname{div}(\nabla\tilde{\boldsymbol{R}}_\varepsilon\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})) + \frac{1}{Pr}\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla r_\varepsilon &= -\frac{1}{Pr}(\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\tilde{\boldsymbol{R}}_\varepsilon\\
&-\frac{1}{Pr}(\tilde{\boldsymbol{R}}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\tilde{\boldsymbol{V}}_\varepsilon^{[1]} - \frac{1}{Pr}(\tilde{\boldsymbol{V}}_\varepsilon^{[1]}\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\tilde{\boldsymbol{V}}_\varepsilon^{[1]}\\
-\frac{1}{1-N}\operatorname{div}(\nabla\boldsymbol{\Psi}_\varepsilon\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})) &+ \frac{2N}{1-N}\nabla\boldsymbol{\Phi}\nabla^\perp(z_\varepsilon - Z_\varepsilon^{[0]}) + Ra\,h_0(x_1)S_\varepsilon\boldsymbol{e}_2 + \gamma_\varepsilon,\\
\operatorname{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\tilde{\boldsymbol{R}}_\varepsilon) &= 0,\\
\tilde{R}_2^\varepsilon = 0 \text{ na } \Sigma_i^\varepsilon,\ i=0,1,\quad \tilde{\boldsymbol{R}}_\varepsilon &= \boldsymbol{0} \text{ na } \Gamma^\varepsilon,
\end{aligned} \tag{2.84}$$

pri čemu vrijedi $\|\gamma_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}$. Funkcija $\tilde{\boldsymbol{R}}_\varepsilon = (\tilde{R}_1^\varepsilon, \tilde{R}_2^\varepsilon)$ je razlika između rješenja $\boldsymbol{v}_\varepsilon$ te korigirane asimptotičke aproksimacije, dana s:

$$\tilde{\boldsymbol{R}}_\varepsilon = \boldsymbol{v}_\varepsilon - \tilde{\boldsymbol{V}}_\varepsilon^{[1]} = \boldsymbol{R}_\varepsilon + \boldsymbol{\Psi}_\varepsilon.$$

Promotrimo sljedeći pomoćni problem:

$$\begin{aligned}
\operatorname{div}(\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})^T\boldsymbol{d}_\varepsilon) &= r_\varepsilon + \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} Q_\varepsilon^{[2]},\\
\boldsymbol{d}_\varepsilon &= \boldsymbol{0} \quad \text{na} \quad \partial\Omega_\varepsilon.
\end{aligned} \tag{2.85}$$

Pretpostavimo li da vrijedi normalizirajući uvjet za tlak $\int_{\Omega_\varepsilon} q_\varepsilon = 0$, tada je jasno da desna strana jednakosti $(2.85)_1$ pripada prostoru $L_0^2(\Omega_\varepsilon)$. Dakle, postoji barem jedno rješenje $\boldsymbol{d}_\varepsilon$ problema (2.85) takvo da vrijedi (vidi Lema 2.3.2):

$$\|\nabla \boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le \frac{C}{\varepsilon}\left\| r_\varepsilon + \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} Q_\varepsilon^{[2]}\right\|_{L^2(\Omega_\varepsilon)}. \tag{2.86}$$

Pomnožimo li sada (2.84) s $\boldsymbol{d}_\varepsilon$ te integriramo po Ω_ε, nalazimo

$$\begin{aligned}
\frac{1}{Pr}\int_{\Omega_\varepsilon}\left(r_\varepsilon + \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} Q_\varepsilon^{[2]}\right)^2 &= \frac{1}{1-N}\int_{\Omega_\varepsilon} \nabla \tilde{\boldsymbol{R}}_\varepsilon \tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\cdot\nabla \boldsymbol{d}_\varepsilon \\
+\frac{1}{Pr}\int_{\Omega_\varepsilon}(\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\tilde{\boldsymbol{R}}_\varepsilon\cdot\boldsymbol{d}_\varepsilon &+ \frac{1}{Pr}\int_{\Omega_\varepsilon}(\tilde{\boldsymbol{R}}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\tilde{\boldsymbol{V}}_\varepsilon^{[1]}\cdot\boldsymbol{d}_\varepsilon \\
+\frac{1}{Pr}\int_{\Omega_\varepsilon}(\tilde{\boldsymbol{V}}_\varepsilon^{[1]}\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\tilde{\boldsymbol{V}}_\varepsilon^{[1]}\cdot\boldsymbol{d}_\varepsilon &- \frac{1}{1-N}\int_{\Omega_\varepsilon}\nabla\boldsymbol{\Psi}_\varepsilon\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\cdot\nabla\boldsymbol{d}_\varepsilon \\
-\frac{2N}{1-N}\int_{\Omega_\varepsilon}\nabla\boldsymbol{\Phi}\nabla^\perp(z_\varepsilon - Z_\varepsilon^{[0]})\cdot\boldsymbol{d}_\varepsilon &- Ra\int_{\Omega_\varepsilon} h_0(x_1)S_\varepsilon\boldsymbol{e}_2\cdot\boldsymbol{d}_\varepsilon - \int_{\Omega_\varepsilon}\gamma_\varepsilon\cdot\boldsymbol{d}_\varepsilon.
\end{aligned} \tag{2.87}$$

Četvrti član s desne strane jednakosti (2.87) raspisujemo na sljedeći način:

$$\begin{aligned}
&\int_{\Omega_\varepsilon}(\tilde{\boldsymbol{V}}_\varepsilon^{[1]}\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\tilde{\boldsymbol{V}}_\varepsilon^{[1]}\cdot\boldsymbol{d}_\varepsilon \\
&\qquad = \int_{\Omega_\varepsilon}(\boldsymbol{V}_\varepsilon^{[1]}\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{V}_\varepsilon^{[1]}\cdot\boldsymbol{d}_\varepsilon - \int_{\Omega_\varepsilon}(\boldsymbol{V}_\varepsilon^{[1]}\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{\Psi}_\varepsilon\cdot\boldsymbol{d}_\varepsilon \\
&\qquad - \int_{\Omega_\varepsilon}(\boldsymbol{\Psi}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{V}_\varepsilon^{[1]}\cdot\boldsymbol{d}_\varepsilon + \int_{\Omega_\varepsilon}(\boldsymbol{\Psi}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{\Psi}_\varepsilon\cdot\boldsymbol{d}_\varepsilon.
\end{aligned} \tag{2.88}$$

Nadalje, vrijedi

$$\boldsymbol{a}\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi}\nabla)\boldsymbol{b} = h_0(x_1)a_1\frac{\partial \boldsymbol{b}}{\partial x_1} - h_0'(x_1)x_2a_1\frac{\partial \boldsymbol{b}}{\partial x_2} - h'(x_1)a_1\frac{\partial \boldsymbol{b}}{\partial x_2} + a_2\frac{\partial \boldsymbol{b}}{\partial x_2}.$$

Izraze dobivene u (2.88) ocjenjujemo koristeći (2.86), Lemu 0.0.1 te gornju jednakost:

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon}(\boldsymbol{V}_\varepsilon^{[1]}\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{V}_\varepsilon^{[1]}\boldsymbol{d}_\varepsilon\right| &\le \|\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\left\|(V_1^0+\varepsilon V_1^1)h_0(x_1)\frac{\partial}{\partial x_1}(\boldsymbol{V}^0+\varepsilon\boldsymbol{V}^1)\right\|_{L^2(\Omega_\varepsilon)} \\
&+ \|\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\left\|(-h_0'(x_1)x_2(V_1^0+\varepsilon V_1^1)+\varepsilon(-h'(x_1)V_1^1+V_2^1))\frac{\partial}{\partial x_2}(\boldsymbol{V}^0+\varepsilon\boldsymbol{V}^1)\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\sqrt{\varepsilon^3}\,\|\boldsymbol{\nabla d_\varepsilon}\|_{L^2(\Omega_\varepsilon)}, \\
\left|\int_{\Omega_\varepsilon}(\boldsymbol{V}_\varepsilon^{[1]}\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{\Psi}_\varepsilon\boldsymbol{d}_\varepsilon\right| &\le \|\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\left\|(V_1^0+\varepsilon V_1^1)h_0(x_1)\frac{\partial\boldsymbol{\Psi}_\varepsilon}{\partial x_1}\right\|_{L^2(\Omega_\varepsilon)} \\
&+ \|\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\left\|(-h_0'(x_1)x_2(V_1^0+\varepsilon V_1^1)+\varepsilon(-h'(x_1)V_1^1+V_2^1))\frac{\partial\boldsymbol{\Psi}_\varepsilon}{\partial x_2}\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\sqrt{\varepsilon^3}\,\|\boldsymbol{\nabla d_\varepsilon}\|_{L^2(\Omega_\varepsilon)},
\end{aligned} \tag{2.89}$$

$$\begin{aligned}\left|\int_{\Omega_\varepsilon}(\boldsymbol{\Psi}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{V}_\varepsilon^{[1]}\boldsymbol{d}_\varepsilon\right| &\le C\varepsilon\left\|\nabla\boldsymbol{V}_\varepsilon^{[1]}\right\|_{L^\infty(\Omega_\varepsilon)}\|\boldsymbol{\Psi}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\
&\le C\varepsilon\|\nabla\boldsymbol{\Psi}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\
&\le C\sqrt{\varepsilon^3}\|\nabla\boldsymbol{d}_{\boldsymbol{\varepsilon}}\|_{L^2(\Omega_\varepsilon)},\\
\left|\int_{\Omega_\varepsilon}(\boldsymbol{\Psi}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\boldsymbol{\Psi}_\varepsilon\boldsymbol{d}_\varepsilon\right| &\le \|\boldsymbol{\Psi}_\varepsilon\|_{L^4(\Omega_\varepsilon)}\|\nabla\boldsymbol{\Psi}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{d}_\varepsilon\|_{L^4(\Omega_\varepsilon)}\\
&\le C\varepsilon\|\nabla\boldsymbol{\Psi}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\
&\le C\varepsilon^2\|\nabla\boldsymbol{d}_{\boldsymbol{\varepsilon}}\|_{L^2(\Omega_\varepsilon)}.\end{aligned}$$

Preostali izrazi u (2.87) se mogu ocijeniti koristeći Lemu 0.0.1, Propoziciju 2.3.1, te ocjene (2.49), (2.71) i (2.79):

$$\begin{aligned}\int_{\Omega_\varepsilon}\nabla\tilde{\boldsymbol{R}}_\varepsilon\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\cdot\nabla\boldsymbol{d}_\varepsilon &\le C\left\|\nabla\tilde{\boldsymbol{R}}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)},\\
\int_{\Omega_\varepsilon}(\boldsymbol{v}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\tilde{\boldsymbol{R}}_\varepsilon\cdot\boldsymbol{d}_\varepsilon &\le \|\boldsymbol{v}_\varepsilon\|_{L^4(\Omega_\varepsilon)}\left\|\nabla\tilde{\boldsymbol{R}}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{d}_\varepsilon\|_{L^4(\Omega_\varepsilon)}\\
&\le C\varepsilon\|\nabla\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\left\|\nabla\tilde{\boldsymbol{R}}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\
&\le C\sqrt{\varepsilon}\left\|\nabla\tilde{\boldsymbol{R}}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)},\\
\int_{\Omega_\varepsilon}(\tilde{\boldsymbol{R}}_\varepsilon\cdot\tilde{\boldsymbol{G}}(\boldsymbol{\Phi})\nabla)\tilde{\boldsymbol{V}}_\varepsilon^{[1]}\cdot\boldsymbol{d}_\varepsilon &\le \left\|\tilde{\boldsymbol{R}}_\varepsilon\right\|_{L^4(\Omega_\varepsilon)}\left\|\nabla\tilde{\boldsymbol{V}}_\varepsilon^{[1]}\right\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{d}_\varepsilon\|_{L^4(\Omega_\varepsilon)}\\
&\le C\sqrt{\varepsilon}\left\|\nabla\tilde{\boldsymbol{R}}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)},\\
\int_{\Omega_\varepsilon}\nabla\boldsymbol{\Psi}_\varepsilon\tilde{\boldsymbol{F}}(\boldsymbol{\Phi})\cdot\nabla\boldsymbol{d}_\varepsilon &\le C\|\nabla\boldsymbol{\Psi}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\
&\le C\sqrt{\varepsilon^3}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \qquad (2.90)\\
\int_{\Omega_\varepsilon}\nabla\boldsymbol{\Phi}\nabla^\perp(z_\varepsilon-Z_\varepsilon^{[0]})\cdot\boldsymbol{d}_\varepsilon &\le C\left\|\nabla^\perp\omega_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\
&\quad + C\left\|\nabla^\perp(Z_\varepsilon^{[1]}-Z_\varepsilon^{[0]})\right\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\
&\le C\varepsilon^2\left\|\nabla\tilde{\boldsymbol{R}}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}+C\sqrt{\varepsilon^3}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)},\\
\int_{\Omega_\varepsilon}h_0(x_1)S_\varepsilon\boldsymbol{e}_2\cdot\boldsymbol{d}_\varepsilon &\le C\|K_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}+C\left\|T_\varepsilon^{[2]}\right\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\
&\quad C\varepsilon^2\|\nabla K_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}+C\varepsilon\left\|T_\varepsilon^{[2]}\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\
&\le C\varepsilon^3\left\|\nabla\tilde{\boldsymbol{R}}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}+C\sqrt{\varepsilon^3}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)},\\
\int_{\Omega_\varepsilon}\gamma_\varepsilon\cdot\boldsymbol{d}_\varepsilon &\le \|\gamma_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\
&\le C\sqrt{\varepsilon^3}\|\nabla\boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}.\end{aligned}$$

Sada iz (2.86), (2.87), (2.89) i (2.90) slijedi ocjena

$$\left\| r_\varepsilon + \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} Q_\varepsilon^{[2]} \right\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon} \|\nabla \tilde{\boldsymbol{R}}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\sqrt{\varepsilon}. \tag{2.91}$$

Konačno, množimo (2.84) s $\tilde{\boldsymbol{R}}_\varepsilon$ te integriramo po $\boldsymbol{\Omega}_\varepsilon$ kako bi dobili

$$\begin{aligned} \frac{1}{1-N} \int_{\Omega_\varepsilon} \nabla \tilde{\boldsymbol{R}}_\varepsilon \tilde{\boldsymbol{F}}(\boldsymbol{\Phi}) \cdot \nabla \tilde{\boldsymbol{R}}_\varepsilon &= \frac{1}{1-N} \int_{\Sigma_1^\varepsilon} \nabla \tilde{\boldsymbol{R}}_\varepsilon \tilde{\boldsymbol{F}}(\boldsymbol{\Phi}) \boldsymbol{e}_1 \cdot \tilde{\boldsymbol{R}}_\varepsilon \\ -\frac{1}{1-N} \int_{\Sigma_0^\varepsilon} \nabla \tilde{\boldsymbol{R}}_\varepsilon \tilde{\boldsymbol{F}}(\boldsymbol{\Phi}) \boldsymbol{e}_1 \cdot \tilde{\boldsymbol{R}}_\varepsilon &- \frac{1}{Pr} \int_{\Omega_\varepsilon} (\boldsymbol{v}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi}) \nabla) \tilde{\boldsymbol{R}}_\varepsilon \cdot \tilde{\boldsymbol{R}}_\varepsilon \\ -\frac{1}{Pr} \int_{\Omega_\varepsilon} (\tilde{\boldsymbol{R}}_\varepsilon \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi}) \nabla) \tilde{\boldsymbol{V}}_\varepsilon^{[1]} \cdot \tilde{\boldsymbol{R}}_\varepsilon &- \frac{1}{Pr} \int_{\Omega_\varepsilon} (\tilde{\boldsymbol{V}}_\varepsilon^{[1]} \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi}) \nabla) \tilde{\boldsymbol{V}}_\varepsilon^{[1]} \cdot \tilde{\boldsymbol{R}}_\varepsilon \\ +\frac{1}{1-N} \int_{\Omega_\varepsilon} \nabla \boldsymbol{\Psi}_\varepsilon \tilde{\boldsymbol{F}}(\boldsymbol{\Phi})) \cdot \nabla \tilde{\boldsymbol{R}}_\varepsilon &+ \frac{2N}{1-N} \int_{\Omega_\varepsilon} \nabla \boldsymbol{\Phi} \nabla^\perp (z_\varepsilon - Z_\varepsilon^{[0]}) \cdot \tilde{\boldsymbol{R}}_\varepsilon \\ +Ra \int_{\Omega_\varepsilon} h_0(x_1) S_\varepsilon \boldsymbol{e}_2 \cdot \tilde{\boldsymbol{R}}_\varepsilon &+ \int_{\Omega_\varepsilon} \gamma_\varepsilon \cdot \tilde{\boldsymbol{R}}_\varepsilon. \end{aligned} \tag{2.92}$$

Koristeći $(2.84)_2$ i $(2.84)_3$ izvodimo sljedeću ocjenu:

$$\begin{aligned} \left| \int_{\Sigma_1^\varepsilon} \nabla \tilde{\boldsymbol{R}}_\varepsilon \tilde{\boldsymbol{F}}(\boldsymbol{\Phi}) \boldsymbol{e}_1 \cdot \tilde{\boldsymbol{R}}_\varepsilon \right| &= \left| \int_{\Sigma_1^\varepsilon} \left(h_0(x_1) \frac{\partial \tilde{\boldsymbol{R}}_\varepsilon}{\partial x_1} - (h_0'(x_1)x_2 + h'(x_1)) \frac{\partial \tilde{\boldsymbol{R}}_\varepsilon}{\partial x_2} \right) \cdot \tilde{\boldsymbol{R}}_\varepsilon \right| \\ &= \left| \int_{\Sigma_1^\varepsilon} h_0(x_1) \frac{\partial \tilde{R}_1^\varepsilon}{\partial x_1} \tilde{R}_1^\varepsilon - h_0'(x_1) x_2 \frac{\partial \tilde{R}_1^\varepsilon}{\partial x_2} \tilde{R}_1^\varepsilon - h'(x_1) \frac{\partial \tilde{R}_1^\varepsilon}{\partial x_2} \tilde{R}_1^\varepsilon \right| \\ &= \left| \int_{\Sigma_1^\varepsilon} \frac{\partial \tilde{R}_2^\varepsilon}{\partial x_2} \tilde{R}_1^\varepsilon \right| = \left| \int_0^\varepsilon \tilde{R}_2^\varepsilon(1, x_2) \frac{\partial \tilde{R}_1^\varepsilon}{\partial x_2}(1, x_2) dx_2 \right| = 0. \end{aligned} \tag{2.93}$$

Do istog zaključka za odgovarajući član na Σ_0^ε se dolazi analogno.

Integral

$$\int_{\Omega_\varepsilon} (\tilde{\boldsymbol{V}}_\varepsilon^{[1]} \cdot \tilde{\boldsymbol{G}}(\boldsymbol{\Phi}) \nabla) \tilde{\boldsymbol{V}}_\varepsilon^{[1]} \cdot \tilde{\boldsymbol{R}}_\varepsilon$$

raspisujemo i ocjenjujemo analogno (2.88) i (2.89), dok se preostali članovi ocjenjuju analogno (2.90).

Tada se koristeći (2.92), (2.93) i Lemu 2.3.4 može pokazati da za dovoljno mali $\boldsymbol{\varepsilon}$ vrijedi sljedeća ocjena:

$$\|\nabla \tilde{\boldsymbol{R}}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon^3}.$$

Zbog (2.82) imamo

$$\|\nabla \boldsymbol{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon^3}, \tag{2.94}$$

što nam daje ocjenu $(2.63)_1$. Ocjena $(2.63)_2$ sada slijedi iz (2.91) i (2.94), ocjena $(2.63)_3$ slijedi iz (2.71) i (2.94), te ocjena $(2.63)_4$ slijedi iz (2.79) i (2.94). Time je dovršen dokaz.

■

Napomena 2.3.1. *U ovom odjeljku smo prilagodili dokaz [56, Teorem 4.1] općenitijem slučaju zakrivljenog kanala kako bismo izveli ocjene greške* (2.63) *u Teoremu 2.3.1. Ključne razlike u analizi greške s obzirom na nedeformirani kanal su integralni članovi u* (2.69)*, koji dolaze od korektora rubnog sloja, te integralni članovi u* (2.77)*, koji potječu od Robinovih rubnih uvjeta za temperaturu, a za koje smo izveli zadovoljavajuće ocjene.*

3. Asimptotička analiza nestacionarnog mikropolarnog fluida s dinamičkim rubnim uvjetom za mikrorotaciju

U matematičkoj literaturi je česta upotreba Dirichletovog rubnog uvjeta za mikrorotaciju, kako bi se olakšala analiza promatranog problema. Naime, dokaz dobre postavljenosti (inicijalno–)rubne zadaće postaje tehnički zahtjevan problem u slučaju nestandardnih rubnih uvjeta. S druge strane, razumno je očekivati da će općenitiji rubni uvjeti bolje opisati različite fenomene koji se javljaju zbog postojanja mikrostrukture.

Migun [66] te Kolpashchikov, Migun i Prokhorenko [52] daju prve važne eksperimentalne rezultate koji sugeriraju upotrebu nestandardnih rubnih uvjeta za mikropolarne jednadžbe. Nadalje, Niefer i Kaloni su istraživali kretanje krute sfere u mikropolarnom fluidu pri smicanju u [69], dok je Kirwan [50] proučavao nekoliko različitih rubnih uvjeta za mikrorotaciju.

U novijoj literaturi se mogu pronaći razmatranja mikrorotacijskih rubnih uvjeta u raznim postavkama. Bayada, Benhaboucha i Chambat [7] su proučavali lubrikacijski sloj u tankom kanalu, uz rubne uvjete povezane s viskoznosti na rubu domene (tzv. "boundary viscosity"). Hoffman, Marx i Botkin analiziraju učinke otpora na sfere u mikropolarnim fluidima u [43], dok Deo i Shukla razmatraju laminaran tok mikropolarnog fluida oko sfere u [26]. U [77] Pažanin izvodi efektivan model koji opisuje lubrikaciju s mikropolarnim fluidom, dok Bonnivard, Pažanin i Suárez–Grau [17] izučavaju učinke naboranog ruba na tok mikropolarnog fluida u tankoj domeni.

Motivirani nedavnim rezultatima dobre postavljenosti problema nestacionarnog mikropolarnog toka fluida s ne-nul rubnim uvjetima u dvije dimenzije (vidi [14]), u ovom poglavlju donosimo dokaz dobre postavljenosti u tri dimenzije, te predlažemo efektivni model višeg reda koji opisuje spomenuti tok. Najprije u Odjeljku 3.1 uvodimo sustav mikropolarnih jednadžbi s rubnim uvjetom (1.30) za mikrorotaciju, te zatim u Odjeljku 3.2 dokazujemo egzistenciju i jedinstvenost slabog rješenja. U Odjeljku 3.3 provodimo asimptotičku analizu s obzirom na debljinu cijevi te izvodimo asimptotičku aproksimaciju drugog reda koja uvažava efekte mikrostrukture fluida, vremenske derivacije te nestandardnih rubnih uvjeta. Provedena je detaljna analiza rubnog sloja u blizini krajeva cijevi kako bismo poboljšali točnost aproksimacije. U posljednjem Odjeljku 3.4 izvodimo apriorne ocjene te zatim rigorozno opravdavamo izvedeni model.

3.1. Postavka problema

Promatramo nestacionarni mikropolarni fluid u tankoj cilindričnoj domeni

$$\Omega_\varepsilon = B_\varepsilon \times [0,l] = \varepsilon B(0,1) \times [0,l],$$

gdje je $B(0,1)$ krug radijusa 1 te $\varepsilon \ll 1$ mali pozitivni parametar. Rub domene $S = \partial\Omega_\varepsilon$ je unija tri skupa: dno cilindra S_B, vrh cilindra S_T te lateralni rub S_L.

Linearizirane jednadžbe nestacionarnog mikropolarnog fluida u Ω_ε dane su s

$$\begin{aligned} \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} - \chi\Delta\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon &= a \operatorname{rot} \boldsymbol{w}_\varepsilon + \boldsymbol{f}_\varepsilon, \\ \operatorname{div} \boldsymbol{u}_\varepsilon &= 0, \\ \frac{\partial \boldsymbol{w}_\varepsilon}{\partial t} - \alpha\Delta\boldsymbol{w}_\varepsilon - \beta\nabla\operatorname{div}\boldsymbol{w}_\varepsilon + 2a\boldsymbol{w}_\varepsilon &= a \operatorname{rot} \boldsymbol{u}_\varepsilon + \boldsymbol{g}_\varepsilon. \end{aligned} \tag{3.1}$$

Nepoznanice u (3.1) su brzina $\boldsymbol{u}_\varepsilon$, mikrorotacija $\boldsymbol{w}_\varepsilon$ i tlak p_ε. Gustoća ρ je konstanta postavljena na 1. U (3.1) smo upotrijebili notaciju $\chi = \nu + \nu_r$, $a = 2\nu_r$, $\alpha = c_a + c_d$ te $\beta = c_0 + c_d - c_a$, gdje je ν kinematička Newtonovska viskoznost, ν_r kinematička mikrorotacijska viskoznost, te c_0, c_a i c_d koeficijenti kutnih viskoznosti. Dane funkcije $\boldsymbol{f}_\varepsilon$ i $\boldsymbol{g}_\varepsilon$ opisuju izvore linearnog i angularnog momenta.

Sustav (3.1) uparujemo s nehomogenim Dirichletovim rubnim uvjetom za brzinu na S_B i S_T:

$$\boldsymbol{u}_\varepsilon(x_1,x_2,0,t) = \boldsymbol{u}_\varepsilon(x_1,x_2,l,t) = \boldsymbol{h}_\varepsilon\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon},t\right), \tag{3.2}$$

gdje je $\boldsymbol{h}_\varepsilon$ zadana brzina na krajevima cijevi $x_3 = 0$ te $x_3 = l$.

Na lateralnom rubu cijevi S_L zadajemo no-slip rubni uvjet za brzinu

$$\boldsymbol{u}_\varepsilon = \mathbf{0} \quad \text{na} \quad S_L, \tag{3.3}$$

te na $\partial\Omega_\varepsilon$ zadajemo dinamički rubni uvjet za mikrorotaciju opisan u Odjeljku 1.3.2:

$$\boldsymbol{w}_\varepsilon = \frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \boldsymbol{u}_\varepsilon, \;\; \alpha_0^\varepsilon \in \langle 0, 1] \,. \tag{3.4}$$

Koeficijent α_0^ε koji se pojavljuje u rubnom uvjetu (3.4) opisuje interakciju između promatranog fluida i lateralnog ruba domene S_L. Prema [16], može se definirati preko kinematičke viskoznosti ruba domene ν_b, to jest kao $\alpha_0^\varepsilon = \frac{\nu+\nu_r-\nu_b}{\nu_r}$. U ovom poglavlju ćemo pretpostaviti da je α_0^ε istog reda veličine kao i mali parametar ε, odnosno da je oblika $\alpha_0^\varepsilon = \varepsilon\alpha_0$, $\alpha_0 \in \mathbb{R}$.

Inicijalni uvjeti su dani s

$$\begin{aligned} \boldsymbol{u}_\varepsilon(\boldsymbol{x}, 0) &= \mathbf{0}, \\ \boldsymbol{w}_\varepsilon(\boldsymbol{x}, 0) &= \mathbf{0}. \end{aligned} \tag{3.5}$$

Napomenimo da su uvjeti kompatibilnosti za problem (3.1)–(3.5) zadovoljeni, jer su za brzinu zadani jednaki rubni uvjeti na krajevima cijevi S_B i S_T. Uvodimo oznaku

$$F^*(t) := \int_{B_\varepsilon} h_{\varepsilon,3}\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, t\Big) = \varepsilon^2 \int_B h_3(\boldsymbol{y}', t) = \varepsilon^2 F(t), \tag{3.6}$$

gdje je $\boldsymbol{y}' = \frac{\boldsymbol{x}'}{\varepsilon}$, $\boldsymbol{x}' = (x_1, x_2)$ te $h_{\varepsilon,3}$ treća komponenta od $\boldsymbol{h}_\varepsilon$.

Pretpostavimo da za funkcije $\boldsymbol{f}_\varepsilon$ i $\boldsymbol{g}_\varepsilon$ vrijedi sljedeće: $\boldsymbol{f}_\varepsilon(\boldsymbol{x}', 0, t) = \boldsymbol{f}_\varepsilon(\boldsymbol{x}', l, t)$ i $\boldsymbol{g}_\varepsilon(\boldsymbol{x}', 0, t) = \boldsymbol{g}_\varepsilon(\boldsymbol{x}', l, t)$. Dodatno, pretpostavimo da funkcije $\boldsymbol{f}_\varepsilon(\boldsymbol{x}, t)$, $\boldsymbol{g}_\varepsilon(\boldsymbol{x}, t)$ i $\boldsymbol{h}_\varepsilon(\frac{\boldsymbol{x}'}{\varepsilon}, t)$ iščezavaju u okolini $t = 0$. Iz (3.6) tada slijedi da $F(t)$ također iščezava u $t = 0$. S ovim pretpostavkama se ne pojavljuje rubni sloj u vremenu jer će inicijalni uvjeti (3.5) biti zadovoljeni. S druge strane, morat ćemo provesti analizu rubnog sloja u prostoru jer regularni dio asimptotičkog razvoja neće zadovoljavati rubne uvjete na $x_3 = 0$ i $x_3 = l$ (vidi Odjeljak 3.3.2). Dodatno, napomenimo da promatramo linearizirane jednadžbe jer nelinearni dio mikropolarnih jednadžbi ne pridonosi asimptotičkom razvoju drugog reda.

Cilj sljedećeg odjeljka je pokazati dobru postavljenost problema (3.1)–(3.5).

3.2. Egzistencija i jedinstvenost rješenja

Pretpostavimo li da vrijedi $\boldsymbol{h}_\varepsilon \in C^1([0,T];W^{3/2,2}(S_B)) \cap C^1([0,T];W^{3/2,2}(S_T))$, možemo definirati funkciju $\boldsymbol{h}_\varepsilon^{ext} \in C^1([0,T];W^{2,2}(\Omega_\varepsilon))$ kao rješenje problema (vidi Lemu 0.0.3)

$$\operatorname{div} \boldsymbol{h}_\varepsilon^{ext} = 0 \quad \text{u} \quad \Omega_\varepsilon,$$

$$\boldsymbol{h}_\varepsilon^{ext} = \boldsymbol{0} \quad \text{na} \quad S_L,$$

$$\boldsymbol{h}_\varepsilon^{ext} = \boldsymbol{h}_\varepsilon \quad \text{na} \quad S_B, S_T.$$

Dodatno, za svaki $t \in [0,T]$ vrijede sljedeće ocjene:

$$\begin{aligned} \left\|\boldsymbol{h}_\varepsilon^{ext}\right\|_{L^2(\Omega_\varepsilon)} + \left\|\frac{\partial \boldsymbol{h}_\varepsilon^{ext}}{\partial t}\right\|_{L^2(\Omega_\varepsilon)} &\le C\varepsilon, \\ \left\|\nabla \boldsymbol{h}_\varepsilon^{ext}\right\|_{L^2(\Omega_\varepsilon)} + \left\|\nabla \frac{\partial \boldsymbol{h}_\varepsilon^{ext}}{\partial t}\right\|_{L^2(\Omega_\varepsilon)} &\le C, \\ \left\|\Delta \boldsymbol{h}_\varepsilon^{ext}\right\|_{L^2(\Omega_\varepsilon)} &\le \frac{C}{\varepsilon}. \end{aligned} \tag{3.7}$$

Kako bi pojednostavili slabu formulaciju problema, uvodimo sljedeće oznake:

$$\begin{aligned} ((\boldsymbol{u},\boldsymbol{\varphi}))_u &= \chi \int_{\Omega_\varepsilon} \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\varphi} dx, \\ ((\boldsymbol{w},\boldsymbol{\psi}))_w &= \alpha \int_{\Omega_\varepsilon} \nabla \boldsymbol{w} \cdot \nabla \boldsymbol{\psi} dx + \beta \int_{\Omega_\varepsilon} \operatorname{div}\boldsymbol{w} \operatorname{div}\boldsymbol{\psi} dx + 2a \int_{\Omega_\varepsilon} \boldsymbol{w} \cdot \boldsymbol{\psi} dx, \\ (\boldsymbol{u},\boldsymbol{\varphi}) &= \int_{\Omega_\varepsilon} \boldsymbol{u} \cdot \boldsymbol{\varphi} dx. \end{aligned}$$

gdje su $\boldsymbol{u},\boldsymbol{\varphi} \in V$ i $\boldsymbol{w},\boldsymbol{\psi} \in W_0^{1,2}(\Omega_\varepsilon)$.

Uočimo da su bilinearne forme $((\cdot,\cdot))_u$ i $((\cdot,\cdot))_w$ neprekidne i koercitivne na V, odnosno $W_0^{1,2}(\Omega_\varepsilon)$:

$$\begin{aligned} c_1 \|\boldsymbol{u}\|_V^2 &\le ((\boldsymbol{u},\boldsymbol{u}))_u, \quad \forall \boldsymbol{u} \in V, \\ ((\boldsymbol{u},\boldsymbol{\varphi}))_u &\le c_2 \|\boldsymbol{u}\|_V \|\boldsymbol{\varphi}\|_V, \quad \forall \boldsymbol{u},\boldsymbol{\varphi} \in V, \\ c_3 \|\boldsymbol{w}\|_{W_0^{1,2}(\Omega_\varepsilon)}^2 &\le ((\boldsymbol{w},\boldsymbol{w}))_w, \quad \forall \boldsymbol{w} \in W_0^{1,2}(\Omega_\varepsilon), \\ ((\boldsymbol{w},\boldsymbol{\psi}))_w &\le c_4 \|\boldsymbol{w}\|_{W_0^{1,2}(\Omega_\varepsilon)} \|\boldsymbol{\psi}\|_{W_0^{1,2}(\Omega_\varepsilon)}, \quad \forall \boldsymbol{w},\boldsymbol{\psi} \in W_0^{1,2}(\Omega_\varepsilon), \end{aligned} \tag{3.8}$$

gdje su $c_1, c_2, c_3, c_4 > 0$ konstante neovisne o malom parametru ε.

Nadalje, neka je P ortogonalna projekcija s $L^2(\Omega_\varepsilon)$ na H, te označimo s $\mathcal{A}$ Stokesov operator definiran s

$$\mathcal{A}\boldsymbol{u}_\varepsilon = -P\Delta \boldsymbol{u}_\varepsilon, \ \boldsymbol{u}_\varepsilon \in D(\mathcal{A}),$$

pri čemu je $D(\mathcal{A}) = W^{2,2}(\Omega_\varepsilon) \cap V$. Slično kao u [89, Odjeljak 7] može se pokazati da postoji konstanta C_A koja ne ovisi o ε takva da

$$\|\boldsymbol{u}_\varepsilon\|_{W^{2,2}(\Omega_\varepsilon)} \leq C_A \|\mathcal{A}\boldsymbol{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} . \tag{3.9}$$

Sada formuliramo naš problem u varijacijskom smislu: tražimo par funkcija $(\boldsymbol{v}_\varepsilon, \boldsymbol{z}_\varepsilon)$ takvih da

$$\begin{aligned} &\boldsymbol{v}_\varepsilon \in L^2(0,T;D(\mathcal{A})) \cap C([0,T];V), \quad \frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} \in L^2(0,T;H), \\ &\boldsymbol{z}_\varepsilon \in L^2(0,T;W_0^{1,2}(\Omega_\varepsilon)) \cap C([0,T];L^2(\Omega_\varepsilon)), \quad \frac{\partial \boldsymbol{z}_\varepsilon}{\partial t} \in L^2(0,T;H^{-1}(\Omega_\varepsilon)), \end{aligned} \tag{3.10}$$

vrijede sljedeće jednakosti:

$$\begin{aligned} &\left(\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}, \boldsymbol{\varphi}\right) + ((\boldsymbol{v}_\varepsilon, \boldsymbol{\varphi}))_u = a(\operatorname{rot} \boldsymbol{z}_\varepsilon, \boldsymbol{\varphi}) + \frac{a\alpha_0^\varepsilon}{2}(\operatorname{rot}\operatorname{rot} \boldsymbol{v}_\varepsilon, \boldsymbol{\varphi}) + (\boldsymbol{f}_\varepsilon, \boldsymbol{\varphi}) \\ &\quad - \left(\frac{\partial \boldsymbol{h}_\varepsilon^{ext}}{\partial t}, \boldsymbol{\varphi}\right) - ((\boldsymbol{h}_\varepsilon^{ext}, \boldsymbol{\varphi}))_u + \frac{a\alpha_0^\varepsilon}{2}(\operatorname{rot}\operatorname{rot} \boldsymbol{h}_\varepsilon^{ext}, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in V, \\ &\left\langle \frac{\partial \boldsymbol{z}_\varepsilon}{\partial t}, \boldsymbol{\psi} \right\rangle + ((\boldsymbol{z}_\varepsilon, \boldsymbol{\psi}))_w = a(\operatorname{rot} \boldsymbol{v}_\varepsilon, \boldsymbol{\psi}) + a(\operatorname{rot} \boldsymbol{h}_\varepsilon^{ext}, \boldsymbol{\psi}) + \langle \boldsymbol{g}_\varepsilon, \boldsymbol{\psi}\rangle - \frac{\alpha_0^\varepsilon}{2}\left(\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}, \operatorname{rot} \boldsymbol{\psi}\right) \\ &\quad - \frac{\alpha_0^\varepsilon}{2}\left(\frac{\partial \boldsymbol{h}_\varepsilon^{ext}}{\partial t}, \operatorname{rot} \boldsymbol{\psi}\right) - \frac{\alpha_0^\varepsilon}{2}((\operatorname{rot} \boldsymbol{v}_\varepsilon, \boldsymbol{\psi}))_w - \frac{\alpha_0^\varepsilon}{2}((\operatorname{rot} \boldsymbol{h}_\varepsilon^{ext}, \boldsymbol{\psi}))_w, \quad \forall \boldsymbol{\psi} \in W_0^{1,2}(\Omega_\varepsilon), \end{aligned} \tag{3.11}$$

te su zadovoljeni inicijalni uvjeti

$$\boldsymbol{v}_\varepsilon(\cdot, 0) = \boldsymbol{0}, \quad \boldsymbol{z}_\varepsilon(\cdot, 0) = -\frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \boldsymbol{v}_\varepsilon(\cdot, 0) - \frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \boldsymbol{h}_\varepsilon^{ext}(\cdot, 0) \quad \text{na} \quad \Omega_\varepsilon. \tag{3.12}$$

Tada je s $\boldsymbol{u}_\varepsilon = \boldsymbol{v}_\varepsilon + \boldsymbol{h}_\varepsilon^{ext}$, $\boldsymbol{w}_\varepsilon = \boldsymbol{z}_\varepsilon + \frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \boldsymbol{v}_\varepsilon + \frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \boldsymbol{h}_\varepsilon^{ext}$ dano rješenje problema (3.1)–(3.5). U sljedećem teoremu dokazujemo dobru postavljenost problema (3.10)–(3.12).

Teorem 3.2.1. Neka je $\boldsymbol{f}_\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon))$, $\boldsymbol{g}_\varepsilon \in L^2(0,T;H^{-1}(\Omega_\varepsilon))$. Tada postoji jedinstveno rješenje problema (3.10)–(3.12).

Dokaz. Neka je prostor X definiran kao

$$X = \left\{\boldsymbol{v} : \boldsymbol{v} \in L^2(0,T;D(\mathcal{A})) \cap C([0,T];V), \ \frac{\partial \boldsymbol{v}}{\partial t} \in L^2(0,T;H)\right\},$$

uz normu danu s

$$\|\boldsymbol{v}\|_X = \|\boldsymbol{v}\|_{L^2(0,T;D(\mathcal{A}))} + \|\boldsymbol{v}\|_{C([0,T];V)} + \left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|_{L^2(0,T;H)}.$$

Nadalje, neka je $\tilde{\boldsymbol{v}}_\varepsilon \in X$. Definiramo operatore $\boldsymbol{k} : X \to L^2(0,T;H^{-1}(\Omega_\varepsilon))$, $\boldsymbol{j} : X \to L^2(0,T;H)$ na sljedeći način:

$$\begin{aligned} \langle \boldsymbol{k}(\tilde{\boldsymbol{v}}_\varepsilon), \boldsymbol{\psi}\rangle &= \frac{\alpha_0^\varepsilon}{2}\left(\frac{\partial \tilde{\boldsymbol{v}}_\varepsilon}{\partial t}, \operatorname{rot} \boldsymbol{\psi}\right) + \frac{\alpha_0^\varepsilon}{2}((\operatorname{rot} \tilde{\boldsymbol{v}}_\varepsilon, \boldsymbol{\psi}))_w, \quad \boldsymbol{\psi} \in L^2(0,T;W_0^{1,2}(\Omega_\varepsilon)), \\ (\boldsymbol{j}(\tilde{\boldsymbol{v}}_\varepsilon), \boldsymbol{\varphi}) &= \frac{a\alpha_0^\varepsilon}{2}(\operatorname{rot}\operatorname{rot} \tilde{\boldsymbol{v}}_\varepsilon, \boldsymbol{\varphi}), \quad \boldsymbol{\varphi} \in L^2(0,T;V). \end{aligned}$$

Lako je vidjeti da vrijede sljedeće ocjene:

$$\left\|\boldsymbol{k}(\tilde{\boldsymbol{v}}_\varepsilon)\right\|_{L^2(0,T;H^{-1}(\Omega_\varepsilon))} \le C_k \frac{\alpha_0^\varepsilon}{2} \|\tilde{\boldsymbol{v}}_\varepsilon\|_X , \quad \left\|\boldsymbol{j}(\tilde{\boldsymbol{v}}_\varepsilon)\right\|_{L^2(0,T;H)} \le C_j \frac{\alpha_0^\varepsilon}{2} \|\tilde{\boldsymbol{v}}_\varepsilon\|_X , \tag{3.13}$$

gdje su konstante C_k, C_j neovisne o ε.

Koristeći ocjene (3.8), (3.9) i (3.13) te standardne metode diskretizacije u vremenu (vidi npr. [55]) može se pokazati egzistencija jedinstvenog para funkcija $(\boldsymbol{v}_\varepsilon, \boldsymbol{z}_\varepsilon)$ takvih da

$$\begin{gathered}
\boldsymbol{v}_\varepsilon \in L^2(0,T;D(\mathcal{A})) \cap C([0,T];V), \quad \frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} \in L^2(0,T;H), \\
\boldsymbol{z}_\varepsilon \in L^2(0,T;W_0^{1,2}(\Omega_\varepsilon)) \cap C([0,T];L^2(\Omega_\varepsilon)), \quad \frac{\partial \boldsymbol{z}_\varepsilon}{\partial t} \in L^2(0,T;H^{-1}(\Omega_\varepsilon)),
\end{gathered} \tag{3.14}$$

vrijede jednakosti

$$\begin{aligned}
\left(\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}, \boldsymbol{\varphi}\right) + ((\boldsymbol{v}_\varepsilon, \boldsymbol{\varphi}))_u &= a(\operatorname{rot} \boldsymbol{z}_\varepsilon, \boldsymbol{\varphi}) + (\boldsymbol{j}(\tilde{\boldsymbol{v}}_\varepsilon), \boldsymbol{\varphi}) + (\boldsymbol{f}_\varepsilon, \boldsymbol{\varphi}) \\
&- \left(\frac{\partial \boldsymbol{h}_\varepsilon^{ext}}{\partial t}, \boldsymbol{\varphi}\right) - ((\boldsymbol{h}_\varepsilon^{ext}, \boldsymbol{\varphi}))_u + (\boldsymbol{j}(\boldsymbol{h}_\varepsilon^{ext}), \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in V, \\
\left\langle \frac{\partial \boldsymbol{z}_\varepsilon}{\partial t}, \boldsymbol{\psi} \right\rangle + ((\boldsymbol{z}_\varepsilon, \boldsymbol{\psi}))_w &= a(\operatorname{rot} \boldsymbol{v}_\varepsilon, \boldsymbol{\psi}) + a(\operatorname{rot} \boldsymbol{h}_\varepsilon^{ext}, \boldsymbol{\psi}) + \langle \boldsymbol{g}_\varepsilon, \boldsymbol{\psi} \rangle \\
&- \langle \boldsymbol{k}(\tilde{\boldsymbol{v}}_\varepsilon), \boldsymbol{\psi} \rangle - \langle \boldsymbol{k}(\boldsymbol{h}_\varepsilon^{ext}), \boldsymbol{\psi} \rangle, \quad \forall \boldsymbol{\psi} \in W_0^{1,2}(\Omega_\varepsilon),
\end{aligned} \tag{3.15}$$

te inicijalni uvjeti

$$\boldsymbol{v}_\varepsilon(\cdot,0) = \boldsymbol{0}, \quad \boldsymbol{z}_\varepsilon(\cdot,0) = -\frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \tilde{\boldsymbol{v}}_\varepsilon(\cdot,0) - \frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \boldsymbol{h}_\varepsilon^{ext}(\cdot,0) \ \text{ na } \ \Omega_\varepsilon. \tag{3.16}$$

Nadalje, vrijedi sljedeća ocjena:

$$\begin{aligned}
\|\boldsymbol{v}_\varepsilon\|_X \le & C_S \Bigg(\left\|\boldsymbol{f}_\varepsilon\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \left\|\boldsymbol{g}_\varepsilon\right\|_{L^2(0,T;H^{-1}(\Omega_\varepsilon))} + \left\|\frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \tilde{\boldsymbol{v}}_\varepsilon(\cdot,0)\right\|_{L^2(\Omega_\varepsilon)} \\
& + \left\|\frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \boldsymbol{h}_\varepsilon^{ext}(\cdot,0)\right\|_{L^2(\Omega_\varepsilon)} + \left\|\boldsymbol{j}(\tilde{\boldsymbol{v}}_\varepsilon)\right\|_{L^2(0,T;H)} + \left\|\boldsymbol{k}(\tilde{\boldsymbol{v}}_\varepsilon)\right\|_{L^2(0,T;H^{-1}(\Omega_\varepsilon))} \\
& + \left\|\nabla \boldsymbol{h}_\varepsilon^{ext}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \left\|\frac{\partial \boldsymbol{h}_\varepsilon^{ext}}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \left\|\boldsymbol{j}(\boldsymbol{h}_\varepsilon^{ext})\right\|_{L^2(0,T;H)} \\
& + \left\|\boldsymbol{k}(\boldsymbol{h}_\varepsilon^{ext})\right\|_{L^2(0,T;H^{-1}(\Omega_\varepsilon))} \Bigg),
\end{aligned} \tag{3.17}$$

gdje je C_S konstanta neovisna o ε.

Također, uočimo da vrijedi ocjena

$$\left\|\operatorname{rot} \tilde{\boldsymbol{v}}_\varepsilon(\cdot,0)\right\|_{L^2(\Omega_\varepsilon)} \le C_r \|\tilde{\boldsymbol{v}}_\varepsilon\|_{C([0,T];V)} \le C_r \|\tilde{\boldsymbol{v}}_\varepsilon\|_X . \tag{3.18}$$

Konačno, definiramo preslikavanje $A : X \to X$ s $A(\tilde{\boldsymbol{v}}_\varepsilon) = \boldsymbol{v}_\varepsilon$, te tvrdimo da je A kontrakcija na X za dovoljno mali ε. Neka su $\tilde{\boldsymbol{v}}_\varepsilon^1, \tilde{\boldsymbol{v}}_\varepsilon^2 \in X$, te neka su $(\boldsymbol{v}_\varepsilon^1, \boldsymbol{z}_\varepsilon^1)$ i $(\boldsymbol{v}_\varepsilon^2, \boldsymbol{z}_\varepsilon^2)$ rješenja problema

(3.14)–(3.16) za $\tilde{\boldsymbol{v}}_\varepsilon^1$, odnosno $\tilde{\boldsymbol{v}}_\varepsilon^2$. Onda je $(\hat{\boldsymbol{v}}_\varepsilon, \hat{\boldsymbol{z}}_\varepsilon) = (\boldsymbol{v}_\varepsilon^1 - \boldsymbol{v}_\varepsilon^2, \boldsymbol{z}_\varepsilon^1 - \boldsymbol{z}_\varepsilon^2)$ rješenje problema (3.14)–(3.16) za $\tilde{\boldsymbol{v}}_\varepsilon = \tilde{\boldsymbol{v}}_\varepsilon^1 - \tilde{\boldsymbol{v}}_\varepsilon^2$, uz $\boldsymbol{f}_\varepsilon, \boldsymbol{g}_\varepsilon, \boldsymbol{h}_\varepsilon^{ext} = \boldsymbol{0}$. Tada iz (3.13), (3.17) i (3.18) slijedi ocjena

$$\left\|A(\tilde{\boldsymbol{v}}_\varepsilon^1) - A(\tilde{\boldsymbol{v}}_\varepsilon^2)\right\|_X = \|\hat{\boldsymbol{v}}_\varepsilon\|_X \leq C_S(C_r+C_j+C_k)\frac{\alpha_0^\varepsilon}{2}\left\|\tilde{\boldsymbol{v}}_\varepsilon^1 - \tilde{\boldsymbol{v}}_\varepsilon^2\right\|_X = \varepsilon C_S(C_r+C_j+C_k)\frac{\alpha_0}{2}\left\|\tilde{\boldsymbol{v}}_\varepsilon^1 - \tilde{\boldsymbol{v}}_\varepsilon^2\right\|_X.$$

Dakle, A je kontrakcija za $\varepsilon > 0$ takav da vrijedi

$$\varepsilon C_S\left(C_r + C_j + C_k\right)\frac{\alpha_0}{2} < 1,$$

čime zaključujemo da postoji jedinstveno rješenje problema (3.10)–(3.12).

■

Napomena 3.2.1. *Egzistenciju tlaka p_ε možemo dobiti na standardan način koristeći De Rhamov teorem (vidi npr. [95]).*

Zbog složenosti promatranog problema (3.1)–(3.5) ne možemo pronaći eksplicitno rješenje bez da značajno pojednostavimo model, te je stoga glavni cilj ovog poglavlja pronaći asimptotički razvoj višeg reda točnosti koji dobro aproksimira rješenja problema.

U ostatku poglavlja izvodimo asimptotičku aproksimaciju drugog reda točnosti problema (3.1)–(3.5), te provodimo analizu rubnog sloja.

3.3. Asimptotička analiza

Reskaliranjem domene s obzirom na brzu varijablu $\boldsymbol{y}' = (y_1', y_2') = \frac{\boldsymbol{x}'}{\varepsilon}$ dobivamo sljedeći sustav jednadžbi na $\boldsymbol{\Omega} = B(0,1) \times [0,l]$:

$$\begin{aligned}
&\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} - \frac{\chi}{\varepsilon^2}\Delta_{\boldsymbol{y}'}\tilde{\boldsymbol{u}} - \chi\frac{\partial^2 \tilde{\boldsymbol{u}}}{\partial x_3^2} + \frac{1}{\varepsilon}\nabla_{\boldsymbol{y}'}\tilde{p} + \frac{\partial \tilde{p}}{\partial x_3}\boldsymbol{e}_3 \\
&= a\Big(\frac{1}{\varepsilon}\frac{\partial \tilde{w}_3}{\partial y_2} - \frac{\partial \tilde{w}_2}{\partial x_3}\Big)\boldsymbol{e}_1 + a\Big(\frac{\partial \tilde{w}_1}{\partial x_3} - \frac{1}{\varepsilon}\frac{\partial \tilde{w}_3}{\partial y_1}\Big)\boldsymbol{e}_2 + a\Big(\frac{1}{\varepsilon}\frac{\partial \tilde{w}_2}{\partial y_1} - \frac{1}{\varepsilon}\frac{\partial \tilde{w}_1}{\partial y_2}\Big)\boldsymbol{e}_3 + \tilde{\boldsymbol{f}}, \\
&\frac{1}{\varepsilon}\mathrm{div}_{\boldsymbol{y}'}\tilde{\boldsymbol{u}}_* + \frac{\partial \tilde{u}_3}{\partial x_3} = 0, \\
&\frac{\partial \tilde{\boldsymbol{w}}}{\partial t} - \frac{\alpha}{\varepsilon^2}\Delta_{\boldsymbol{y}'}\tilde{\boldsymbol{w}} - \alpha\frac{\partial^2 \tilde{\boldsymbol{w}}}{\partial x_3^2} - \frac{\beta}{\varepsilon^2}\nabla_{\boldsymbol{y}'}\mathrm{div}_{\boldsymbol{y}'}\tilde{\boldsymbol{w}}_* - \frac{\beta}{\varepsilon}\nabla_{\boldsymbol{y}'}\frac{\partial \tilde{w}_3}{\partial x_3} - \frac{\beta}{\varepsilon}\frac{\partial}{\partial x_3}\mathrm{div}_{\boldsymbol{y}'}\tilde{\boldsymbol{w}}_*\boldsymbol{e}_3 - \beta\frac{\partial^2 \tilde{w}_3}{\partial x_3^2}\boldsymbol{e}_3 \\
&+2a\tilde{\boldsymbol{w}} + a\Big(\frac{1}{\varepsilon}\frac{\partial \tilde{u}_3}{\partial y_2} - \frac{\partial \tilde{u}_2}{\partial x_3}\Big)\boldsymbol{e}_1 + a\Big(\frac{\partial \tilde{u}_1}{\partial x_3} - \frac{1}{\varepsilon}\frac{\partial \tilde{u}_3}{\partial y_1}\Big)\boldsymbol{e}_2 \\
&+a\Big(\frac{1}{\varepsilon}\frac{\partial \tilde{u}_2}{\partial y_1} - \frac{1}{\varepsilon}\frac{\partial \tilde{u}_1}{\partial y_2}\Big)\boldsymbol{e}_3 + \tilde{\boldsymbol{g}},
\end{aligned} \tag{3.19}$$

pri čemu je $\tilde{\boldsymbol{u}}(\boldsymbol{y}', x_3, t) = \boldsymbol{u}_\varepsilon(\varepsilon\boldsymbol{y}', x_3, t)$, $\tilde{\boldsymbol{u}}_* = (\tilde{u}_1, \tilde{u}_2)$, $\tilde{\boldsymbol{w}}(\boldsymbol{y}', x_3, t) = \boldsymbol{w}_\varepsilon(\varepsilon\boldsymbol{y}', x_3, t)$, $\tilde{\boldsymbol{w}}_* = (\tilde{w}_1, \tilde{w}_2)$, $\tilde{p}(\boldsymbol{y}', x_3, t) = p_\varepsilon(\varepsilon\boldsymbol{y}', x_3, t)$, $\tilde{\boldsymbol{f}}(\boldsymbol{y}', x_3, t) = \boldsymbol{f}_\varepsilon(\varepsilon\boldsymbol{y}', x_3, t)$, te $\tilde{\boldsymbol{g}}(\boldsymbol{y}', x_3, t) = \boldsymbol{g}_\varepsilon(\varepsilon\boldsymbol{y}', x_3, t)$.

Koristimo sljedeće oznake za parcijalne diferencijalne operatore:

$$\begin{aligned}
\Delta_{\boldsymbol{y}'}\boldsymbol{v} &= \frac{\partial^2 \boldsymbol{v}}{\partial \hat{y}_1^2} + \frac{\partial^2 \boldsymbol{v}}{\partial \hat{y}_2^2}, \quad \mathrm{div}_{\boldsymbol{y}'}\,\boldsymbol{v}_* = \frac{\partial v_1}{\partial \hat{y}_1} + \frac{\partial v_2}{\partial \hat{y}_2}, \\
\nabla_{\boldsymbol{y}'}\varphi &= \frac{\partial \varphi}{\partial \hat{y}_1}\boldsymbol{e}_1 + \frac{\partial \varphi}{\partial \hat{y}_2}\boldsymbol{e}_2, \quad \mathrm{rot}_{\boldsymbol{y}'}\,\varphi = \frac{\partial \varphi}{\partial \hat{y}_2}\boldsymbol{e}_1 - \frac{\partial \varphi}{\partial \hat{y}_1}\boldsymbol{e}_2,
\end{aligned}$$

pri čemu je $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ kanonska baza lokalnog koordinatnog sustava, $\boldsymbol{v} = (v_1, v_2, v_3)$ te $\boldsymbol{v}_* = (v_1, v_2)$.

Tražimo rješenje (3.19) u obliku asimptotičkog razvoja danog s:

$$\begin{aligned}
\tilde{\boldsymbol{u}}(\boldsymbol{y}', x_3, t) &= \tilde{\boldsymbol{u}}^0(\boldsymbol{y}', x_3, t) + \varepsilon\tilde{\boldsymbol{u}}^1(\boldsymbol{y}', x_3, t) + \varepsilon^2\tilde{\boldsymbol{u}}^2(\boldsymbol{y}', x_3, t) + \ldots, \\
\tilde{p}(\boldsymbol{y}', x_3, t) &= \frac{1}{\varepsilon^2}\tilde{p}^0(x_3, t) + \frac{1}{\varepsilon}\tilde{p}^1(\boldsymbol{y}', x_3, t) + \tilde{p}^2(\boldsymbol{y}', x_3, t) + \varepsilon\tilde{p}^3(\boldsymbol{y}', x_3, t) + \ldots, \\
\tilde{\boldsymbol{w}}(\boldsymbol{y}', x_3, t) &= \tilde{\boldsymbol{w}}^0(\boldsymbol{y}', x_3, t) + \varepsilon\tilde{\boldsymbol{w}}^1(\boldsymbol{y}', x_3, t) + \varepsilon^2\tilde{\boldsymbol{w}}^2(\boldsymbol{y}', x_3, t) + \ldots,
\end{aligned} \tag{3.20}$$

te pretpostavljamo da su dane funkcije $\tilde{\boldsymbol{f}}$, $\tilde{\boldsymbol{g}}$ oblika

$$\begin{aligned}
\tilde{\boldsymbol{f}}(\boldsymbol{y}', x_3, t) &= \frac{1}{\varepsilon^2}\tilde{\boldsymbol{f}}^0(\boldsymbol{y}', x_3, t), \\
\tilde{\boldsymbol{g}}(\boldsymbol{y}', x_3, t) &= \frac{1}{\varepsilon^2}\tilde{\boldsymbol{g}}^0(\boldsymbol{y}', x_3, t).
\end{aligned}$$

Nadalje, pretpostavimo da su funkcije $\tilde{\boldsymbol{f}}$ i $\tilde{\boldsymbol{g}}$ neovisne o $\boldsymbol{y}'$, što je razumna pretpostavka s obzirom na to da promatramo tanku (odnosno dugu) cijev.

3.3.1. Regularni dio razvoja

Uvrštavanjem asimptotičkog razvoja (3.20) u sustav jednadžbi (3.19) te prikupljanjem članova uz iste potencije malog parametra ε, dobivamo rekurzivni niz problema koji opisuju aproksimaciju nultog reda te korektore višeg reda.

3.3.1.1 Aproksimacija nultog reda

Aproksimacija brzine nultog reda $\tilde{\boldsymbol{u}}^0$ je dana kao rješenje sljedećeg problema:

$$\begin{aligned}
\frac{1}{\varepsilon^2}:&\quad -\chi\Delta_{\boldsymbol{y}'}\tilde{\boldsymbol{u}}^0+\nabla_{\boldsymbol{y}'}\tilde{p}^1+\frac{\partial\tilde{p}^0}{\partial x_3}\boldsymbol{e}_3=\tilde{\boldsymbol{f}}^0,\\
\frac{1}{\varepsilon}:&\quad \operatorname{div}_{\boldsymbol{y}'}\tilde{\boldsymbol{u}}^0_*=0,\\
1:&\quad \tilde{\boldsymbol{u}}^0=\boldsymbol{0}\ \text{ na }\ S_L.
\end{aligned}\tag{3.21}$$

Rješenje problema (3.21) je oblika

$$\begin{gathered}
\tilde{\boldsymbol{u}}^0_*=(\tilde{u}^0_1,\tilde{u}^0_2)=(0,0),\quad \tilde{u}^0_3(\boldsymbol{y}',x_3,t)=\frac{1}{4\chi}(1-|\boldsymbol{y}'|^2)\Big(\tilde{f}^0_3(x_3,t)-\frac{\partial\tilde{p}^0}{\partial x_3}(x_3,t)\Big),\\
\tilde{p}^1(\boldsymbol{y}',x_3,t)=\tilde{f}^0_1(x_3,t)y_1+\tilde{f}^0_2(x_3,t)y_2+x_3\tilde{p}^1_{bl,cor}(t),
\end{gathered}\tag{3.22}$$

pri čemu je $\tilde{p}^1_{bl,cor}(t)$ funkcija koja će kasnije biti određena tako da budu zadovoljeni uvjeti kompatibilnosti za korektor rubnog sloja brzine prvog reda (vidi Odjeljak 3.3.2).

Iz jednadžbe za divergenciju imamo jednakost

$$1:\ \operatorname{div}_{\boldsymbol{y}'}\tilde{\boldsymbol{u}}^1_*+\frac{\partial\tilde{u}^0_3}{\partial x_3}=0,$$

čijim integriranjem po $\boldsymbol{B}=B(0,1)$ dobivamo

$$\int_B\operatorname{div}_{\boldsymbol{y}'}\tilde{\boldsymbol{u}}^1_*+\frac{\partial}{\partial x_3}\int_B\tilde{u}^0_3=0,$$

odnosno

$$\int_{\partial B}\tilde{\boldsymbol{u}}^1_*\cdot\boldsymbol{y}'+\frac{\partial}{\partial x_3}\int_B\tilde{u}^0_3=0.\tag{3.23}$$

Sada zbog (3.22) i $(3.28)_3$ iz (3.23) slijedi

$$\frac{\pi}{8\chi}\Big(\tilde{f}^0_3(x_3,t)-\frac{\partial\tilde{p}^0}{\partial x_3}\Big)=F(t),$$

čime dobivamo sljedeći izraz za aproksimaciju tlaka nultog reda $\tilde{p}^0$:

$$\tilde{p}^0(x_3,t)=-\frac{8\chi}{\pi}F(t)x_3+\int_0^{x_3}\tilde{f}^0_3(\xi,t)d\xi+p_0(t),\tag{3.24}$$

gdje je $p_0(t)$ proizvoljna funkcija koja ovisi samo o vremenu t.

Konačno, iz izraza (3.22) i (3.24) dobivamo aproksimaciju brzine nultog reda u klasičnom Poiseuilleovom obliku:

$$\tilde{u}_3^0(\boldsymbol{y}', t) = \frac{2}{\pi}(1 - |\boldsymbol{y}'|^2)F(t). \tag{3.25}$$

Nadalje, sustav jednadžbi koji opisuje aproksimaciju mikrorotacije nultog reda $\tilde{\boldsymbol{w}}^0$ glasi

$$\begin{aligned} \frac{1}{\varepsilon^2}: \quad & -\alpha\Delta_{\boldsymbol{y}'}\tilde{\boldsymbol{w}}^0 - \beta\nabla_{\boldsymbol{y}'}\mathrm{div}_{\boldsymbol{y}'}\tilde{\boldsymbol{w}}_*^0 = \tilde{\boldsymbol{g}}^0, \\ 1 \ : \tilde{\boldsymbol{w}}^0 &= \frac{\alpha_0}{2}\Big(\frac{\partial \tilde{u}_3^0}{\partial y_2}, -\frac{\partial \tilde{u}_3^0}{\partial y_1}, \frac{\partial \tilde{u}_2^0}{\partial y_1} - \frac{\partial \tilde{u}_1^0}{\partial y_2}\Big) \ \text{ na } \ S_L. \end{aligned} \tag{3.26}$$

Uzimajući u obzir dobiveni izraz za aproksimaciju brzine (3.25), rubni uvjet $(3.26)_2$ je jednak

$$\tilde{\boldsymbol{w}}^0 = \Big(-\frac{2\alpha_0 F}{\pi}y_2, \frac{2\alpha_0 F}{\pi}y_1, 0\Big) \ \text{ na } \ S_L.$$

Sada je lako provjeriti da je rješenje problema (3.26) dano s

$$\begin{aligned} \tilde{w}_1^0(\boldsymbol{y}', x_3, t) &= \frac{1}{2(2\alpha+\beta)}(1 - |\boldsymbol{y}'|^2)\tilde{g}_1^0(x_3, t) - \frac{2\alpha_0 F}{\pi}y_2, \\ \tilde{w}_2^0(\boldsymbol{y}', x_3, t) &= \frac{1}{2(2\alpha+\beta)}(1 - |\boldsymbol{y}'|^2)\tilde{g}_2^0(x_3, t) + \frac{2\alpha_0 F}{\pi}y_1, \\ \tilde{w}_3^0(\boldsymbol{y}', x_3, t) &= \frac{1}{4\alpha}(1 - |\boldsymbol{y}'|^2)\tilde{g}_3^0(x_3, t). \end{aligned} \tag{3.27}$$

Vidimo da u izrazima za prve dvije komponente aproksimacije mikrorotacije nultog reda (3.27) možemo uočiti efekte rubnog uvjeta (3.4) kroz pojavu parametra α_0. Međutim, kako bismo mogli uočiti i učinke mikropolarnosti na brzinu fluida, kao i učinke derivacije po vremenu, moramo nastaviti račun te odrediti korektore višeg reda.

3.3.1.2 Korektori prvog reda

Prve dvije komponente korektora brzine prvog reda $\tilde{\boldsymbol{u}}_*^1 = (\tilde{u}_1^1, \tilde{u}_2^1)$ zadovoljavaju sljedeći sustav jednadžbi:

$$\begin{aligned} \frac{1}{\varepsilon}: \ & -\chi\Delta_{\boldsymbol{y}'}\tilde{\boldsymbol{u}}_*^1 + \nabla_{\boldsymbol{y}'}\tilde{p}^2 = \frac{a}{2\alpha}(-y_2, y_1)\tilde{g}_3^0, \\ 1 \ : & \qquad \mathrm{div}_{\boldsymbol{y}'}\tilde{\boldsymbol{u}}_*^1 = 0, \\ \varepsilon \ : & \qquad \tilde{\boldsymbol{u}}_*^1 = \mathbf{0} \ \text{ na } \ S_L, \end{aligned} \tag{3.28}$$

gdje ćemo funkciju $\tilde{p}^2 = x_3\tilde{p}_{bl,cor}^2(t)$ odrediti kasnije kako bi bili zadovoljeni uvjeti kompatibilnosti za korektore rubnog sloja brzine drugog reda (vidi Odjeljak 3.3.2).

Rješenje problema (3.28) je dano s

$$\begin{aligned}\tilde{u}_1^1(\mathbf{y}', x_3, t) &= -\frac{a}{16\chi\alpha}(1 - |\mathbf{y}'|^2)y_2\tilde{g}_3^0(x_3, t),\\ \tilde{u}_2^1(\mathbf{y}', x_3, t) &= \frac{a}{16\chi\alpha}(1 - |\mathbf{y}'|^2)y_1\tilde{g}_3^0(x_3, t).\end{aligned} \tag{3.29}$$

Treća komponenta korektora brzine prvog reda $\tilde{u}_3^1$ je dana sustavom jednadžbi

$$\begin{aligned}\frac{1}{\varepsilon}:\ -\chi\Delta_{\mathbf{y}'}\tilde{u}_3^1 &= \frac{a}{2\alpha+\beta}(-y_1\tilde{g}_2^0 + y_2\tilde{g}_1^0) + \frac{4a\alpha_0 F}{\pi} - \frac{\partial \tilde{f}_1^0}{\partial x_3}y_1 - \frac{\partial \tilde{f}_2^0}{\partial x_3}y_2 - \tilde{p}^1_{bl,cor}(t),\\ \varepsilon:\ \tilde{u}_3^1 &= 0 \ \text{ na } \ S_L.\end{aligned} \tag{3.30}$$

Rješenje problema (3.30) glasi

$$\begin{aligned}\tilde{u}_3^1(\mathbf{y}', x_3, t) =& -\frac{a}{8\chi(2\alpha+\beta)}(1 - |\mathbf{y}'|^2)y_1\tilde{g}_2^0(x_3, t) + \frac{a}{8\chi(2\alpha+\beta)}(1 - |\mathbf{y}'|^2)y_2\tilde{g}_1^0(x_3, t)\\ &+ \frac{a\alpha_0 F(t)}{\pi\chi}(1 - |\mathbf{y}'|^2) - \frac{1}{8\chi}(1 - |\mathbf{y}'|^2)y_1\frac{\partial \tilde{f}_1^0(x_3,t)}{\partial x_3}\\ &- \frac{1}{8\chi}(1 - |\mathbf{y}'|^2)y_2\frac{\partial \tilde{f}_2^0(x_3,t)}{\partial x_3} - \frac{\tilde{p}^1_{bl,cor}(t)}{4\chi}(1 - |\mathbf{y}'|^2).\end{aligned} \tag{3.31}$$

Prve dvije komponente korektora mikrorotacije prvog reda $\tilde{\mathbf{w}}_*^1 = (\tilde{w}_1^1, \tilde{w}_2^1)$ rješavaju sustav jednadžbi

$$\begin{aligned}\frac{1}{\varepsilon}:\ -\alpha\Delta_{\mathbf{y}'}\tilde{\mathbf{w}}_*^1 - \beta\nabla_{\mathbf{y}'}\mathrm{div}_{\mathbf{y}'}\tilde{\mathbf{w}}_*^1 &= \frac{4aF}{\pi}(-y_2, y_1) - \frac{\beta}{2\alpha}(y_1, y_2)\frac{\partial \tilde{g}_3^0(x_3,t)}{\partial x_3},\\ \varepsilon:\ \tilde{\mathbf{w}}_*^1 &= \frac{\alpha_0}{2}\left(\frac{\partial \tilde{u}_3^1}{\partial y_2}, -\frac{\partial \tilde{u}_3^1}{\partial y_1}\right) \ \text{ na } \ S_L,\end{aligned} \tag{3.32}$$

gdje je izraz za $\tilde{u}_3^1$ dan s (3.31). Raspisivanjem rubnog uvjeta $(3.32)_2$ dobivamo

$$\begin{aligned}\tilde{w}_1^1 =& \frac{a\alpha_0}{8\chi(2\alpha+\beta)}y_1y_2\tilde{g}_2^0 + \frac{\alpha_0 a}{16\chi(2\alpha+\beta)}(1 - y_1^2 - 3y_2^2)\tilde{g}_1^0 - \frac{\alpha_0^2 F}{\pi\chi}y_2\\ &+ \frac{\alpha_0}{8\chi}y_1y_2\frac{\partial \tilde{f}_1^0}{\partial x_3} - \frac{\alpha_0}{16\chi}(1 - y_1^2 - 3y_2^2)\frac{\partial \tilde{f}_2^0}{\partial x_3} + \frac{\alpha_0\tilde{p}^1_{bl,cor}(t)}{4\chi}y_2,\\ \tilde{w}_2^1 =& \frac{\alpha_0 a}{16\chi(2\alpha+\beta)}(1 - 3y_1^2 - y_2^2)\tilde{g}_2^0 + \frac{a\alpha_0}{8\chi(2\alpha+\beta)}y_1y_2\tilde{g}_1^0 + \frac{\alpha_0^2 F}{\pi\chi}y_1\\ &+ \frac{\alpha_0}{16\chi}(1 - 3y_1^2 - y_2^2)\frac{\partial \tilde{f}_1^0}{\partial x_3} - \frac{\alpha_0}{8\chi}y_1y_2\frac{\partial \tilde{f}_2^0}{\partial x_3} - \frac{\alpha_0\tilde{p}^1_{bl,cor}(t)}{4\chi}y_1.\end{aligned}$$

Rješenje problema (3.32) je sada dano s

$$
\begin{aligned}
\tilde{w}_1^1(\boldsymbol{y}', x_3, t) = &- \frac{aF(t)}{2\pi\alpha}(1-|\boldsymbol{y}'|^2)y_2 - \frac{\beta}{16\alpha(\alpha+\beta)}(1-|\boldsymbol{y}'|^2)y_1\frac{\partial \tilde{g}_3^0(x_3,t)}{\partial x_3} \\
&+ \frac{a\alpha_0}{8\chi(2\alpha+\beta)}y_1y_2\tilde{g}_2^0(x_3,t) - \frac{\alpha_0 a}{8\chi(2\alpha+\beta)}y_2^2\tilde{g}_1^0(x_3,t) \\
&- (1-|\boldsymbol{y}'|^2)\frac{\alpha_0 a(2\alpha-\beta)}{16\chi(2\alpha+\beta)^2}\tilde{g}_1^0(x_3,t) - \frac{a\alpha_0^2F(t)}{\pi\chi}y_2 + \frac{\alpha_0}{8\chi}y_1y_2\frac{\partial \tilde{f}_1^0(x_3,t)}{\partial x_3} \\
&+ \frac{\alpha_0}{8\chi}y_2^2\frac{\partial \tilde{f}_2^0(x_3,t)}{\partial x_3} + (1-|\boldsymbol{y}'|^2)\frac{\alpha_0(2\alpha-\beta)}{16\chi(2\alpha+\beta)}\frac{\partial \tilde{f}_2^0(x_3,t)}{\partial x_3} \\
&+ \frac{\alpha_0\tilde{p}_{bl,cor}^1(t)}{4\chi}y_2, \\
\tilde{w}_2^1(\boldsymbol{y}', x_3, t) = &\frac{aF(t)}{2\pi\alpha}(1-|\boldsymbol{y}'|^2)y_1 - \frac{\beta}{16\alpha(\alpha+\beta)}(1-|\boldsymbol{y}'|^2)y_2\frac{\partial \tilde{g}_3^0(x_3,t)}{\partial x_3} \\
&- \frac{\alpha_0 a}{8\chi(2\alpha+\beta)}y_1^2\tilde{g}_2^0(x_3,t) - (1-|\boldsymbol{y}'|^2)\frac{\alpha_0 a(2\alpha-\beta)}{16\chi(2\alpha+\beta)^2}\tilde{g}_2^0(x_3,t) \\
&+ \frac{a\alpha_0}{8\chi(2\alpha+\beta)}y_1y_2\tilde{g}_1^0(x_3,t) + \frac{a\alpha_0^2F(t)}{\pi\chi}y_1 - \frac{\alpha_0}{8\chi}y_1^2\frac{\partial \tilde{f}_1^0(x_3,t)}{\partial x_3} \\
&- (1-|\boldsymbol{y}'|^2)\frac{\alpha_0(2\alpha-\beta)}{16\chi(2\alpha+\beta)}\frac{\partial \tilde{f}_1^0(x_3,t)}{\partial x_3} - \frac{\alpha_0}{8\chi}y_1y_2\frac{\partial \tilde{f}_2^0(x_3,t)}{\partial x_3} \\
&- \frac{\alpha_0\tilde{p}_{bl,cor}^1(t)}{4\chi}y_1.
\end{aligned}
\tag{3.33}
$$

Nadalje, treća komponenta korektora mikrorotacije prvog reda $\tilde{w}_3^1$ zadovoljava sustav jednadžbi

$$
\begin{aligned}
\frac{1}{\varepsilon}: \quad & -\alpha\Delta_{\boldsymbol{y}'}\tilde{w}_3^1 = \beta\frac{\partial}{\partial x_3}\mathrm{div}_{\boldsymbol{y}'}\tilde{\boldsymbol{w}}_*^0, \\
\varepsilon: \; & \tilde{w}_3^1 = \frac{\alpha_0}{2}\Big(\frac{\partial \tilde{u}_2^1}{\partial y_1} - \frac{\partial \tilde{u}_1^1}{\partial y_2}\Big) \quad \text{na} \quad S_L.
\end{aligned}
\tag{3.34}
$$

Uzimajući u obzir dobivene izraze za $\tilde{\boldsymbol{w}}_*^0$, $\tilde{u}_1^1$ i $\tilde{u}_2^1$ dane s (3.27) i (3.29), sustav (3.34) glasi

$$
\begin{aligned}
-\alpha\Delta_{\boldsymbol{y}'}\tilde{w}_3^1 &= -\frac{\beta}{2\alpha+\beta}\Big(y_1\frac{\partial \tilde{g}_1^0}{\partial x_3} + y_2\frac{\partial \tilde{g}_2^0}{\partial x_3}\Big), \\
\tilde{w}_3^1 &= \frac{\alpha_0 a}{32\chi\alpha}(1-3y_1^2-y_2^2+1-y_1^2-3y_2^2)\tilde{g}_3^0 \quad \text{na} \quad S_L.
\end{aligned}
\tag{3.35}
$$

Rješenje problema (3.35) je dano s

$$
\tilde{w}_3^1(\boldsymbol{y}', x_3, t) = -\frac{\beta}{8\alpha(2\alpha+\beta)}(1-|\boldsymbol{y}'|^2)\Big(y_1\frac{\partial \tilde{g}_1^0(x_3,t)}{\partial x_3} + y_2\frac{\partial \tilde{g}_2^0(x_3,t)}{\partial x_3}\Big) - \frac{\alpha_0 a}{16\chi\alpha}\tilde{g}_3^0(x_3,t). \tag{3.36}
$$

Uočimo da se u izrazima za korektore brzine i mikrorotacije prvog reda dane s (3.29), (3.31), (3.33) i (3.36) pojavljuje koeficijent mikrorotacijske viskoznosti, pa možemo zaključiti da aproksimacija prvog reda uvažava učinke mikropolarne strukture fluida. Nadalje, kroz prisutnost koeficijenta α_0 u izrazu za treću komponentu korektora brzine prvog reda danog s (3.31) vidimo da aproksimacija brzine prvog reda uvažava učinke rubnog uvjeta (3.4). U idućem odjeljku nastavljamo s računom kako bismo mogli uočiti i učinke derivacije po vremenu.

3.3.1.3 Korektori drugog reda

Prve dvije komponente korektora brzine drugog reda $\tilde{\boldsymbol{u}}_*^2 = (\tilde{u}_1^2, \tilde{u}_2^2)$ su dane kao rješenje sljedećeg problema:

$$\begin{aligned} 1 \ : \ -\chi\Delta_{\boldsymbol{y}'}\tilde{\boldsymbol{u}}_*^2 + \nabla_{\boldsymbol{y}'}\tilde{p}^3 &= a\Big(\frac{\partial \tilde{w}_3^1}{\partial y_2} - \frac{\partial \tilde{w}_2^0}{\partial x_3}, \frac{\partial \tilde{w}_1^0}{\partial x_3} - \frac{\partial \tilde{w}_3^1}{\partial y_1}\Big), \\ \varepsilon \ : \qquad \mathrm{div}_{\boldsymbol{y}'}\tilde{\boldsymbol{u}}_*^2 &= -\frac{\partial \tilde{u}_3^1}{\partial x_3}, \\ \varepsilon^2 \ : \qquad \tilde{\boldsymbol{u}}_*^2 &= \boldsymbol{0} \ \text{ na } \ S_L. \end{aligned} \tag{3.37}$$

Izrazi za $\tilde{w}_1^0, \tilde{w}_2^0, \tilde{u}_3^1$ i $\tilde{w}_3^1$ su dani s (3.27), (3.31) i (3.36), iz čega dobivamo da je desna strana jednakosti $(3.37)_1$ jednaka

$$a\Big(-\frac{\beta}{8\alpha(2\alpha+\beta)}\Big(-2y_1y_2\frac{\partial \tilde{g}_1^0}{\partial x_3} + (1 - y_1^2 - 3y_2^2)\frac{\partial \tilde{g}_2^0}{\partial x_3}\Big) - \frac{1}{2(2\alpha+\beta)}(1-|\boldsymbol{y}'|^2)\frac{\partial \tilde{g}_2^0}{\partial x_3}, \\ \frac{1}{2(2\alpha+\beta)}(1-|\boldsymbol{y}'|^2)\frac{\partial \tilde{g}_1^0}{\partial x_3} + \frac{\beta}{8\alpha(2\alpha+\beta)}\Big((1-3y_1^2-y_2^2)\frac{\partial \tilde{g}_1^0}{\partial x_3} - 2y_1y_2\frac{\partial \tilde{g}_2^0}{\partial x_3}\Big)\Big).$$

Naglasimo da postoji jedinstveno rješenje problema (3.37) (do na konstantu u tlaku), jer je zadovoljen uvjet kompatibilnosti $\int_B \frac{\partial \tilde{u}_3^1}{\partial x_3} = 0$.

Sustav (3.37) možemo zapisati u sljedećem obliku:

$$\begin{aligned} -\chi\Delta_{\boldsymbol{y}'}\tilde{u}_1^2 + \frac{\partial \tilde{p}^3}{\partial y_1} &= A_1y_1^2 + A_2y_1y_2 + A_3y_2^2 + A_4, \\ -\chi\Delta_{\boldsymbol{y}'}\tilde{u}_2^2 + \frac{\partial \tilde{p}^3}{\partial y_2} &= A_5y_1^2 + A_6y_1y_2 + A_7y_2^2 + A_8, \\ \frac{\partial \tilde{u}_1^2}{\partial y_1} + \frac{\partial \tilde{u}_2^2}{\partial y_2} &= A_9y_1^3 + A_{10}y_2^3 + A_9y_1y_2^2 + A_{10}y_1^2y_2 - A_9y_1 - A_{10}y_2, \\ \tilde{u}_1^2 = \tilde{u}_2^2 &= 0 \ \text{ na } \ S_L, \end{aligned} \tag{3.38}$$

gdje su $A_1, \ldots, A_{10}$ dani u Dodatku A (vidi (A.1)).

Rješenje problema (3.38) sada glasi

$$\begin{aligned} \tilde{u}_1^2(\boldsymbol{y}', x_3, t) &= (1-|\boldsymbol{y}'|^2)(B_1y_1^2 + B_2y_1y_2 + B_3y_2^2 + B_4), \\ \tilde{u}_2^2(\boldsymbol{y}', x_3, t) &= (1-|\boldsymbol{y}'|^2)(B_5y_1^2 + B_6y_1y_2 + B_7y_2^2 + B_8), \\ \tilde{p}^3(\boldsymbol{y}', x_3, t) &= M_1y_1^3 + M_2y_2^3 + M_3y_1y_2^2 + M_4y_1^2y_2 + M_5y_1 + M_6y_2, \end{aligned} \tag{3.39}$$

gdje su $B_1, \dots, B_8$ i $M_1, \dots, M_6$ dani u Dodatku A (vidi (A.2)–(A.3)).

Treća komponenta korektora brzine drugog reda $\tilde{u}_3^2$ rješava sustav jednadžbi

$$\begin{aligned} 1 \ : \ -\chi \Delta_{\mathbf{y}'} u_3^2 &= a\Big(\frac{\partial \tilde{w}_2^1}{\partial y_1} - \frac{\partial \tilde{w}_1^1}{\partial y_2}\Big) + \chi \frac{\partial^2 \tilde{u}_3^0}{\partial x_3^2} - \frac{\partial \tilde{u}_3^0}{\partial t} - \tilde{p}_{bl,cor}^2(t), \\ \varepsilon^2 \ : \qquad & \tilde{u}_3^2 = 0 \quad \text{na } S_L, \end{aligned} \tag{3.40}$$

pri čemu su funkcije $\tilde{u}_3^0$, $\tilde{w}_1^1$ i $\tilde{w}_2^1$ dane izrazima (3.25) i (3.33). Desna strana $(3.40)_1$ je jednaka

$$\begin{aligned} a\Bigg(& (1-3y_1^2-y_2^2)\frac{aF}{2\pi\alpha} + \frac{\beta}{8\alpha(\alpha+\beta)} y_1 y_2 \frac{\partial \tilde{g}_3^0}{\partial x_3} - \frac{\alpha_0 a}{4\chi(2\alpha+\beta)} y_1 \tilde{g}_2^0 + y_1 \frac{\alpha_0 a(2\alpha-\beta)}{8\chi(2\alpha+\beta)^2}\tilde{g}_2^0 \\ & + \frac{a\alpha_0}{8\chi(2\alpha+\beta)} y_2 \tilde{g}_1^0 + \frac{\alpha_0^2 F}{\pi\chi} - \frac{\alpha_0}{4\chi} y_1 \frac{\partial \tilde{f}_1^0}{\partial x_3} + y_1 \frac{\alpha_0(2\alpha-\beta)}{8\chi(2\alpha+\beta)}\frac{\partial \tilde{f}_1^0}{\partial x_3} - \frac{\alpha_0}{8\chi} y_2 \frac{\partial \tilde{f}_2^0}{\partial x_3} - \frac{\alpha_0 \tilde{p}_{bl,cor}^1(t)}{4\chi} \\ & + \frac{aF}{2\pi\alpha}(1-y_1^2-3y_2^2) - \frac{\beta}{8\alpha(\alpha+\beta)} y_1 y_2 \frac{\partial \tilde{g}_3^0}{\partial x_3} - \frac{a\alpha_0}{8\chi(2\alpha+\beta)} y_1 \tilde{g}_2^0 + \frac{\alpha_0 a}{4\chi(2\alpha+\beta)} y_2 \tilde{g}_1^0 \\ & - y_2 \frac{\alpha_0 a(2\alpha-\beta)}{8\chi(2\alpha+\beta)^2}\tilde{g}_1^0 + \frac{\alpha_0^2 F}{\pi\chi} - \frac{\alpha_0}{8\chi} y_1 \frac{\partial \tilde{f}_1^0}{\partial x_3} - \frac{\alpha_0}{4\chi} y_2 \frac{\partial \tilde{f}_2^0}{\partial x_3} + y_2 \frac{\alpha_0(2\alpha-\beta)}{8\chi(2\alpha+\beta)}\frac{\partial \tilde{f}_2^0}{\partial x_3} \\ & - \frac{\alpha_0 \tilde{p}_{bl,cor}^1(t)}{4\chi}\Bigg) - \frac{2}{\pi}(1-y_1^2-y_2^2)\frac{\partial F}{\partial t} - \tilde{p}_{bl,cor}^2(t). \end{aligned}$$

Problem (3.40) se sada može zapisati u obliku

$$\begin{aligned} \Delta_{\mathbf{y}'} u_3^2 &= A_{11} y_1^2 + A_{12} y_1 y_2 + A_{13} y_2^2 + A_{14} y_1 + A_{15} y_2 + A_{16}, \\ \tilde{u}_3^2 &= 0 \quad \text{na} \quad S_L, \end{aligned} \tag{3.41}$$

gdje su koeficijenti $A_{11}, \dots, A_{16}$ dani u Dodatku A (vidi (A.4)).

Rješenje problema (3.41) je tada dano s

$$\tilde{u}_3^2 = (|\mathbf{y}'|^2 - 1)(B_9 y_1^2 + B_{10} y_1 y_2 + B_{11} y_2^2 + B_{12} y_1 + B_{13} y_2 + B_{14}), \tag{3.42}$$

gdje su koeficijenti $B_9, \dots, B_{14}$ dani u Dodatku A (vidi (A.5)).

Nadalje, prve dvije komponente korektora mikrorotacije drugog reda $\tilde{\mathbf{w}}_*^2 = (\tilde{w}_1^2, \tilde{w}_2^2)$ rješavaju sustav jednadžbi

$$\begin{aligned} 1 \ : \ -\alpha \Delta_{\mathbf{y}'} \tilde{\mathbf{w}}_*^2 - \beta \nabla_{\mathbf{y}'} \mathrm{div}_{\mathbf{y}'} \tilde{\mathbf{w}}_*^2 &= a\Big(\frac{\partial \tilde{u}_3^1}{\partial y_2}, -\frac{\partial \tilde{u}_3^1}{\partial y_1}\Big) + \alpha \frac{\partial^2 \tilde{\mathbf{w}}_*^0}{\partial x_3^2} + \beta \nabla_{\mathbf{y}'} \frac{\partial \tilde{w}_3^1}{\partial x_3} \\ & \quad - 2a\tilde{\mathbf{w}}_*^0 - \frac{\partial \tilde{\mathbf{w}}_*^0}{\partial t}, \\ \varepsilon^2 \ : \qquad \tilde{\mathbf{w}}_*^2 &= \frac{\alpha_0}{2}\Big(\frac{\partial \tilde{u}_3^2}{\partial y_2} - \frac{\partial \tilde{u}_2^1}{\partial x_3}, \frac{\partial \tilde{u}_1^1}{\partial x_3} - \frac{\partial \tilde{u}_3^2}{\partial y_1}\Big) \quad \text{na} \quad S_L, \end{aligned} \tag{3.43}$$

pri čemu su funkcije $\tilde{\boldsymbol{w}}_*^0, \tilde{u}_1^1, \tilde{u}_2^1, \tilde{u}_3^1, \tilde{w}_3^1$ i $\tilde{u}_3^2$ dane s (3.27), (3.29), (3.31), (3.36) i (3.42). Tada je desna strana jednakosti $(3.43)_1$ jednaka

$$
\begin{aligned}
a&\Bigg(\frac{a}{4\chi(2\alpha+\beta)}y_1y_2\tilde{g}_2^0+\frac{a}{8\chi(2\alpha+\beta)}(1-y_1^2-3y_2^2)\tilde{g}_1^0-\frac{2\alpha_0F}{\pi\chi}y_2\\
&+\frac{1}{4\chi}y_1y_2\frac{\partial\tilde{f}_1^0}{\partial x_3}-\frac{1}{8\chi}(1-y_1^2-3y_2^2)\frac{\partial\tilde{f}_2^0}{\partial x_3}+\frac{\tilde{p}^1_{bl,cor}(t)}{2\chi}y_2,\\
&\frac{a}{8\chi(2\alpha+\beta)}(1-3y_1^2-y_2^2)\tilde{g}_2^0+\frac{a}{4\chi(2\alpha+\beta)}y_1y_2\tilde{g}_1^0+\frac{2\alpha_0F}{\pi\chi}y_1\\
&+\frac{1}{8\chi}(1-3y_1^2-y_2^2)\frac{\partial\tilde{f}_1^0}{\partial x_3}-\frac{1}{4\chi}y_1y_2\frac{\partial\tilde{f}_2^0}{\partial x_3}-\frac{\tilde{p}^1_{bl,cor}(t)}{2\chi}y_1\Bigg)\\
&+\alpha\left(\frac{1}{2(2\alpha+\beta)}(1-y_1^2-y_2^2)\frac{\partial^2\tilde{g}_1^0}{\partial x_3^2},\frac{1}{2(2\alpha+\beta)}(1-y_1^2-y_2^2)\frac{\partial^2\tilde{g}_2^0}{\partial x_3^2}\right)\\
&+\beta\Bigg(-\frac{\beta}{8\alpha(2\alpha+\beta)}(1-3y_1^2-y_2^2)\frac{\partial^2\tilde{g}_1^0}{\partial x_3^2}+\frac{\beta}{4\alpha(2\alpha+\beta)}y_1y_2\frac{\partial^2\tilde{g}_2^0}{\partial x_3^2},\\
&\frac{\beta}{4\alpha(2\alpha+\beta)}y_1y_2\frac{\partial^2\tilde{g}_1^0}{\partial x_3^2}-\frac{\beta}{8\alpha(2\alpha+\beta)}(1-y_1^2-3y_2^2)\frac{\partial^2\tilde{g}_2^0}{\partial x_3^2}\Bigg)\\
&-2a\left(\frac{1}{2(2\alpha+\beta)}(1-y_1^2-y_2^2)\tilde{g}_1^0-\frac{2\alpha_0F}{\pi}y_2,\frac{1}{2(2\alpha+\beta)}(1-y_1^2-y_2^2)\tilde{g}_2^0+\frac{2\alpha_0F}{\pi}y_1\right)\\
-&\left(\frac{1}{2(2\alpha+\beta)}(1-y_1^2-y_2^2)\frac{\partial\tilde{g}_1^0}{\partial t}-\frac{2\alpha_0}{\pi}y_2\frac{\partial F}{\partial t},\frac{1}{2(2\alpha+\beta)}(1-y_1^2-y_2^2)\frac{\partial\tilde{g}_2^0}{\partial t}+\frac{2\alpha_0}{\pi}y_1\frac{\partial F}{\partial t}\right),
\end{aligned}
$$

te se rubni uvjet $(3.43)_2$ svodi na

$$
\begin{aligned}
\tilde{w}_1^2=&\frac{\alpha_0}{2}\Bigg(2y_1^2y_2B_9+y_1^3B_{10}+3y_1y_2^2B_{10}-B_{10}y_1+2y_1^2y_2B_{11}+4y_2^3B_{11}-2y_2B_{11}\\
&+2y_1y_2B_{12}+y_1^2B_{13}+3y_2^2B_{13}-B_{13}+2y_2B_{14}-\frac{a}{16\chi\alpha}y_1\frac{\partial\tilde{g}_3^0}{\partial x_3}\\
&+y_1^3\frac{a}{16\chi\alpha}\frac{\partial\tilde{g}_3^0}{\partial x_3}+y_1y_2^2\frac{a}{16\chi\alpha}\frac{\partial\tilde{g}_3^0}{\partial x_3}\Bigg),\\
\tilde{w}_2^2=&\frac{\alpha_0}{2}\Bigg(-\frac{a}{16\chi\alpha}y_2\frac{\partial\tilde{g}_3^0}{\partial x_3}+\frac{a}{16\chi\alpha}y_1^2y_2\frac{\partial\tilde{g}_3^0}{\partial x_3}+\frac{a}{16\chi\alpha}y_2^3\frac{\partial\tilde{g}_3^0}{\partial x_3}-4y_1^3B_9-2y_1y_2^2B_9\\
&+2y_1B_9-3y_1^2y_2B_{10}-y_2^3B_{10}+y_2B_{10}-2y_1y_2^2B_{11}-3y_1^2B_{12}-y_2^2B_{12}+B_{12}\\
&-2y_1y_2B_{13}-2y_1B_{14}\Bigg).
\end{aligned}
$$

Problem (3.43) sada možemo zapisati u obliku

$$\begin{aligned}
&-\alpha\left(\frac{\partial^2\tilde{w}_1^2}{\partial y_1^2}+\frac{\partial^2\tilde{w}_1^2}{\partial y_2^2}\right)-\beta\left(\frac{\partial^2\tilde{w}_1^2}{\partial y_1^2}+\frac{\partial^2\tilde{w}_2^2}{\partial y_1 y_2}\right)\\
&= C_1y_1^2+C_2y_1y_2+C_3y_2^2+C_4y_1+C_5y_2+C_6,\\
&-\alpha\left(\frac{\partial^2\tilde{w}_2^2}{\partial y_1^2}+\frac{\partial^2\tilde{w}_2^2}{\partial y_2^2}\right)-\beta\left(\frac{\partial^2\tilde{w}_1^2}{\partial y_1 y_2}+\frac{\partial^2\tilde{w}_2^2}{\partial y_2^2}\right)\\
&= C_7y_1^2+C_8y_1y_2+C_9y_2^2+C_{10}y_1+C_{11}y_2+C_{12},\\
&\tilde{w}_1^2 = C_{13}y_1^3+C_{14}y_2^3+C_{15}y_1^2y_2+C_{16}y_1y_2^2+C_{17}y_1^2+C_{18}y_2^2+C_{19}y_1\\
&\quad +C_{20}y_1y_2+C_{21}y_2+C_{22} \quad \text{na} \quad S_L,\\
&\tilde{w}_2^2 = C_{23}y_1^3+C_{24}y_2^3+C_{25}y_1^2y_2+C_{26}y_1y_2^2+C_{27}y_1^2+C_{28}y_2^2+C_{29}y_1\\
&\quad +C_{30}y_1y_2+C_{31}y_2+C_{32} \quad \text{na} \quad S_L,
\end{aligned} \tag{3.44}$$

gdje su koeficijenti $C_1,\dots,C_{32}$ dani u Dodatku A (vidi (A.6)).

Rješenje problema (3.44) je dano s

$$\begin{aligned}
\tilde{w}_1^2(\boldsymbol{y}',x_3,t) =& (|\boldsymbol{y}'|^2-1)(D_1y_1^2+D_2y_1y_2+D_3y_2^2+D_4y_1+D_5y_2+D_6)\\
&+C_{13}y_1^3+C_{16}y_1y_2^2+(1-|\boldsymbol{y}'|^2)y_1N_1+C_{14}y_2^3+C_{15}y_1^2y_2\\
&+(1-|\boldsymbol{y}'|^2)y_2N_2+C_{17}y_1^2+C_{18}y_2^2+C_{20}y_1y_2+(1-|\boldsymbol{y}'|^2)N_3+C_{19}y_1\\
&+C_{21}y_2+C_{22},\\
\tilde{w}_2^2(\boldsymbol{y}',x_3,t) =& (|\boldsymbol{y}'|^2-1)(D_7y_1^2+D_8y_1y_2+D_9y_2^2+D_{10}y_1+D_{11}y_2+D_{12})\\
&+C_{23}y_1^3+C_{26}y_1y_2^2+(1-|\boldsymbol{y}'|^2)y_1N_4+C_{24}y_2^3+C_{25}y_1^2y_2\\
&+(1-|\boldsymbol{y}'|^2)y_2N_5+C_{27}y_1^2+C_{28}y_2^2+C_{30}y_1y_2+(1-|\boldsymbol{y}'|^2)N_6\\
&+C_{29}y_1+C_{31}y_2+C_{32},
\end{aligned} \tag{3.45}$$

gdje su koeficijenti $D_1,\dots,D_{12}$ i $N_1,\dots,N_6$ dani u Dodatku A (vidi (A.7)–(A.8)).

Treća komponenta korektora mikrorotacije drugog reda $\tilde{w}_3^2$ je dana kao rješenje sustava jednadžbi

$$\begin{aligned}
1 \ :& \ -\alpha\Delta_{\boldsymbol{y}'}\tilde{w}_3^2 = -\frac{\partial\tilde{w}_3^0}{\partial t}+\alpha\frac{\partial^2\tilde{w}_3^0}{\partial x_3^2}+\beta\frac{\partial}{\partial x_3}\mathrm{div}_{\boldsymbol{y}'}\tilde{\boldsymbol{w}}_*^1+\beta\frac{\partial^2\tilde{w}_3^0}{\partial x_3^2}-2a\tilde{w}_3^0\\
&\qquad +a\left(\frac{\partial\tilde{u}_2^1}{\partial y_1}-\frac{\partial\tilde{u}_1^1}{\partial y_2}\right),\\
\varepsilon^2 \ :& \qquad \tilde{w}_3^2 = \frac{\alpha_0}{2}\left(\frac{\partial\tilde{u}_2^2}{\partial y_1}-\frac{\partial\tilde{u}_1^2}{\partial y_2}\right) \quad \text{na} \quad S_L.
\end{aligned} \tag{3.46}$$

gdje su izrazi za funkcije $\tilde{w}_3^0$, $\tilde{u}_1^1$, $\tilde{u}_2^1$, $\tilde{\boldsymbol{w}}_*^1$, $\tilde{u}_1^2$ i $\tilde{u}_2^2$ dani s (3.27), (3.29), (3.33) i (3.39). Desna strana jednakosti $(3.46)_1$ je jednaka

$$\begin{aligned}
&-\frac{1}{4\alpha}(1-y_1^2-y_2^2)\frac{\partial \tilde{g}_3^0}{\partial t}+\alpha\frac{1}{4\alpha}(1-y_1^2-y_2^2)\frac{\partial^2\tilde{g}_3^0}{\partial x_3^2}+\beta\frac{\partial}{\partial x_3}\Bigg(-\frac{aF}{2\pi\alpha}(-2y_1y_2)\\
&-\frac{\beta}{16\alpha(\alpha+\beta)}(1-3y_1^2-y_2^2)\frac{\partial\tilde{g}_3^0}{\partial x_3}+\frac{a\alpha_0}{8\chi(2\alpha+\beta)}y_2\tilde{g}_2^0-(-2y_1)\frac{\alpha_0 a(2\alpha-\beta)}{16\chi(2\alpha+\beta)^2}\tilde{g}_1^0\\
&+\frac{\alpha_0}{8\chi}y_2\frac{\partial\tilde{f}_1^0}{\partial x_3}+(-2y_1)\frac{\alpha_0(2\alpha-\beta)}{16\chi(2\alpha+\beta)}\frac{\partial\tilde{f}_2^0}{\partial x_3}+\frac{aF}{2\pi\alpha}(-2y_1y_2)-\frac{\beta}{16\alpha(\alpha+\beta)}(1-y_1^2-3y_2^2)\frac{\partial\tilde{g}_3^0}{\partial x_3}\\
&-(-2y_2)\frac{\alpha_0 a(2\alpha-\beta)}{16\chi(2\alpha+\beta)^2}\tilde{g}_2^0+\frac{a\alpha_0}{8\chi(2\alpha+\beta)}y_1\tilde{g}_1^0-(-2y_2)\frac{\alpha_0(2\alpha-\beta)}{16\chi(2\alpha+\beta)}\frac{\partial\tilde{f}_1^0}{\partial x_3}-\frac{\alpha_0}{8\chi}y_1\frac{\partial\tilde{f}_2^0}{\partial x_3}\Bigg)\\
&+a\left(\frac{a}{16\chi\alpha}(1-3y_1^2-y_2^2)\tilde{g}_3^0+\frac{a}{16\chi\alpha}(1-y_1^2-3y_2^2)\tilde{g}_3^0\right)+\beta\frac{\partial^2}{\partial x_3^2}\frac{1}{4\alpha}(1-y_1^2-y_2^2)\tilde{g}_3^0\\
&-2a\frac{1}{4\alpha}(1-y_1^2-y_2^2)\tilde{g}_3^0,
\end{aligned}$$

dok je rubni uvjet $(3.46)_2$ jednak

$$\begin{aligned}
\tilde{w}_3^2=&\frac{\alpha_0}{2}\big(2y_1B_5-4y_1^3B_5-2y_1y_2^2B_5+y_2B_6-3y_1^2y_2B_6-y_2^3B_6-2y_1y_2^2B_7-2y_1B_8\\
&+2y_1^2y_2B_1-B_2y_1+y_1^3B_2+3y_1y_2^2B_2-2y_2B_3+2y_1^2y_2B_3+4y_2^3B_3+2y_2B_4\big).
\end{aligned}$$

Sada problem (3.46) možemo zapisati u obliku

$$\begin{aligned}
\Delta_{\boldsymbol{y}'}\tilde{w}_3^2&=C_{33}y_1^2+C_{34}y_1y_2+C_{35}y_2^2+C_{36}y_1+C_{37}y_2+C_{38},\\
\tilde{w}_3^2&=C_{39}y_1^3+C_{40}y_2^3+C_{41}y_1^2y_2+C_{42}y_1y_2^2+C_{43}y_1+C_{44}y_2+C_{45}\quad\text{na}\quad S_L,
\end{aligned}\tag{3.47}$$

gdje su koeficijenti $C_{33},\ldots,C_{45}$ dani u Dodatku A (vidi (A.9)).

Konačno, rješenje problema (3.47) je dano s

$$\begin{aligned}
\tilde{w}_3^2(\boldsymbol{y}',x_3,t)=&(|\boldsymbol{y}'|^2-1)(D_{13}y_1^2+D_{14}y_1y_2+D_{15}y_2^2+D_{16}y_1+D_{17}y_2+D_{18})\\
&+C_{39}y_1^3+C_{42}y_1y_2^2+(1-|\boldsymbol{y}'|^2)y_1N_7+C_{40}y_2^3+C_{41}y_1^2y_2\\
&+(1-|\boldsymbol{y}'|^2)y_2N_8+C_{43}y_1+C_{44}y_2+C_{45},
\end{aligned}\tag{3.48}$$

gdje su koeficijenti $D_{13},\ldots,D_{18}$ te N_7,N_8 dani u Dodatku A (vidi (A.10)–(A.11)).

Time je dovršen izvod regularnog dijela asimptotičkog razvoja.

3.3.2. Rubni sloj u prostoru

Regularni dio asimptotičkog razvoja određenog u Odjeljku 3.3.1 je oblika

$$\begin{aligned}
\tilde{\boldsymbol{u}}^{[2]}_{\varepsilon,reg}(\boldsymbol{x},t) &= \tilde{\boldsymbol{u}}^0\Big(\frac{\boldsymbol{x}'}{\varepsilon},t\Big) + \varepsilon\tilde{\boldsymbol{u}}^1\Big(\frac{\boldsymbol{x}'}{\varepsilon},x_3,t\Big) + \varepsilon^2\tilde{\boldsymbol{u}}^2\Big(\frac{\boldsymbol{x}'}{\varepsilon},x_3,t\Big),\\
\tilde{p}^{[3]}_{\varepsilon,reg}(\boldsymbol{x},t) &= \frac{1}{\varepsilon^2}\tilde{p}^0(x_3,t) + \frac{1}{\varepsilon}\tilde{p}^1\Big(\frac{\boldsymbol{x}'}{\varepsilon},x_3,t\Big) + \tilde{p}^2\Big(\frac{\boldsymbol{x}'}{\varepsilon},x_3,t\Big) + \varepsilon\tilde{p}^3\Big(\frac{\boldsymbol{x}'}{\varepsilon},x_3,t\Big),\\
\tilde{\boldsymbol{w}}^{[2]}_{\varepsilon,reg}(\boldsymbol{x},t) &= \tilde{\boldsymbol{w}}^0\Big(\frac{\boldsymbol{x}'}{\varepsilon},x_3,t\Big) + \varepsilon\tilde{\boldsymbol{w}}^1\Big(\frac{\boldsymbol{x}'}{\varepsilon},x_3,t\Big) + \varepsilon^2\tilde{\boldsymbol{w}}^2\Big(\frac{\boldsymbol{x}'}{\varepsilon},x_3,t\Big),
\end{aligned}$$

te je izračunat tako da zadovoljava rubne uvjete na lateralnom rubu cijevi S_L, dok rubne uvjete na krajevima cijevi S_B i S_T još nismo uzeli u obzir. Kako bismo to korigirali te dobili precizniju aproksimaciju, uvodimo korektore rubnog sloja u $x_3 = 0$:

$$\begin{aligned}
\tilde{\boldsymbol{u}}^{[2]}_{\varepsilon,bl,0}(\boldsymbol{x},t) &= \boldsymbol{\mathcal{V}}^0\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3}{\varepsilon},t\Big) + \varepsilon\boldsymbol{\mathcal{V}}^1\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3}{\varepsilon},t\Big) + \varepsilon^2\boldsymbol{\mathcal{V}}^2\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3}{\varepsilon},t\Big),\\
\tilde{p}^{[2]}_{\varepsilon,bl,0}(\boldsymbol{x},t) &= \frac{1}{\varepsilon}\mathcal{P}^0\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3}{\varepsilon},t\Big) + \mathcal{P}^1\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3}{\varepsilon},t\Big) + \varepsilon\mathcal{P}^2\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3}{\varepsilon},t\Big),\\
\tilde{\boldsymbol{w}}^{[2]}_{\varepsilon,bl,0}(\boldsymbol{x},t) &= \boldsymbol{\mathcal{W}}^0\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3}{\varepsilon},t\Big) + \varepsilon\boldsymbol{\mathcal{W}}^1\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3}{\varepsilon},t\Big) + \varepsilon^2\boldsymbol{\mathcal{W}}^2\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3}{\varepsilon},t\Big),
\end{aligned} \tag{3.49}$$

te korektore rubnog sloja u $x_3 = l$:

$$\begin{aligned}
\tilde{\boldsymbol{u}}^{[2]}_{\varepsilon,bl,l}(\boldsymbol{x},t) &= \boldsymbol{\mathcal{Y}}^0\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3-l}{\varepsilon},t\Big) + \varepsilon\boldsymbol{\mathcal{Y}}^1\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3-l}{\varepsilon},t\Big) + \varepsilon^2\boldsymbol{\mathcal{Y}}^2\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3-l}{\varepsilon},t\Big),\\
\tilde{p}^{[2]}_{\varepsilon,bl,l}(\boldsymbol{x},t) &= \frac{1}{\varepsilon}\mathcal{Q}^0\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3-l}{\varepsilon},t\Big) + \mathcal{Q}^1\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3-l}{\varepsilon},t\Big) + \varepsilon\mathcal{Q}^2\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3-l}{\varepsilon},t\Big),\\
\tilde{\boldsymbol{w}}^{[2]}_{\varepsilon,bl,l}(\boldsymbol{x},t) &= \boldsymbol{\mathcal{Z}}^0\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3-l}{\varepsilon},t\Big) + \varepsilon\boldsymbol{\mathcal{Z}}^1\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3-l}{\varepsilon},t\Big) + \varepsilon^2\boldsymbol{\mathcal{Z}}^2\Big(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3-l}{\varepsilon},t\Big).
\end{aligned} \tag{3.50}$$

Korektori rubnog sloja u $x_3 = 0$ s desne strane jednakosti (3.49) su definirani na polubeskonačnoj cijevi $\mathcal{G}_0 = B\times\langle 0,\infty\rangle$, dok su korektori rubnog sloja u $x_3 = l$ dani s (3.50) definirani na $\mathcal{G}_l = B\times\langle -\infty,0\rangle$. Dodatno, uvodimo oznake $\omega = \partial B\times\langle 0,\infty\rangle$ te $\sigma = \partial B\times\langle -\infty,0\rangle$.

Naglasimo da zbog pretpostavke da funkcije $\boldsymbol{f}_\varepsilon$, $\boldsymbol{g}_\varepsilon$ i $\boldsymbol{h}_\varepsilon$ iščezavaju u okolini od $t = 0$ regularni dio aproksimacije zadovoljava homogene inicijalne uvjete te se ne pojavljuje fenomen rubnog sloja u vremenu.

Sada uvodimo zamjenu varijabli $y_3 = \frac{x_3}{\varepsilon}$, stavljamo $\tilde{\boldsymbol{f}} = \tilde{\boldsymbol{g}} = \boldsymbol{0}$ te uvrštavamo korektore rubnog sloja u $x_3 = 0$ dane s (3.49) u mikropolarne jednadžbe (3.1). Prikupljanjem članova uz istu potenciju malog parametra ε dobivamo rekurzivni niz problema koji opisuju naše korektore rubnog sloja. Analogan proces provodimo na suprotnoj strani cijevi $x_3 = l$ s korektorima rubnoj sloja danima s (3.50).

3.3.2.1 Aproksimacija nultog reda

Korektori rubnog sloja brzine i tlaka nultog reda $(\mathcal{V}^0, \mathcal{P}^0)$ u $x_3 = 0$ dani su s

$$\begin{aligned} -\chi\Delta\mathcal{V}^0 + \nabla\mathcal{P}^0 &= \mathbf{0} \quad \text{u} \quad \mathcal{G}_0, \\ \operatorname{div}\mathcal{V}^0 = 0 \quad \text{u} \quad \mathcal{G}_0, &\quad \mathcal{V}^0 = \mathbf{0} \quad \text{na} \quad \omega, \\ \mathcal{V}^0(\boldsymbol{y}', 0, t) = (h_1(\boldsymbol{y}', t), h_2(\boldsymbol{y}', t), &h_3(\boldsymbol{y}', t) - \tilde{u}_3^0(\boldsymbol{y}', t)). \end{aligned} \tag{3.51}$$

Sve tvrdnje o egzistenciji, jedinstvenosti te eksponencijalnom padu rješenja koje ćemo iznijeti u ovom odjeljku se pokazuju istim metodama kao u Dodatku B, koristeći činjenicu da imamo egzistenciju funkcije $\boldsymbol{h}_\varepsilon^{ext}$ koja je proširenje rubnih uvjeta na krajevima cijevi.

Problem (3.51) ima jedinstveno rješenje, jer vrijedi nužan uvjet kompatibilnosti:

$$\int_B \Big(h_3(\boldsymbol{y}', t) - \tilde{u}_3^0(\boldsymbol{y}', t)\Big) = F(t) - F(t) = 0.$$

Nadalje, rješenje $(\mathcal{V}^0, \mathcal{P}^0)$ eksponencijalno teži k nuli kada $y_3 \to +\infty$.

Problem za korektore rubnog sloja brzine i tlaka nultog reda $(\mathcal{Y}^0, \mathcal{Q}^0)$ u $x_3 = l$ je dan s

$$\begin{aligned} -\chi\Delta\mathcal{Y}^0 + \nabla\mathcal{Q}^0 &= \mathbf{0} \quad \text{u} \quad \mathcal{G}_l, \\ \operatorname{div}\mathcal{Y}^0 = 0 \quad \text{u} \quad \mathcal{G}_l, &\quad \mathcal{Y}^0 = \mathbf{0} \quad \text{na} \quad \sigma, \\ \mathcal{Y}^0(\boldsymbol{y}', 0, t) = (h_1(\boldsymbol{y}', t), h_2(\boldsymbol{y}', t), &h_3(\boldsymbol{y}', t) - \tilde{u}_3^0(\boldsymbol{y}', t)). \end{aligned} \tag{3.52}$$

Egzistencija, jedinstvenost te eksponencijalni pad k nuli rješenja $(\mathcal{Y}^0, \mathcal{Q}^0)$ problema (3.52) slijedi analogno kao i za $(\mathcal{V}^0, \mathcal{P}^0)$.

Korektor rubnog sloja mikrorotacije nultog reda $\mathcal{W}^0$ u $x_3 = 0$ je dan kao rješenje problema

$$\begin{aligned} -\alpha\Delta\mathcal{W}^0 - \beta\nabla \operatorname{div}\mathcal{W}^0 &= \mathbf{0} \quad \text{u} \quad \mathcal{G}_0, \\ \mathcal{W}^0 &= \frac{\alpha_0}{2} \operatorname{rot}\mathcal{V}^0 \quad \text{na} \quad \omega, \\ \mathcal{W}^0(\boldsymbol{y}', 0, t) &= \frac{\alpha_0}{2}\left(\frac{\partial\tilde{u}_3^0}{\partial y_2}(\boldsymbol{y}', 0, t), -\frac{\partial\tilde{u}_3^0}{\partial y_1}(\boldsymbol{y}', 0, t), 0\right) \\ &\quad + \frac{\alpha_0}{2}\operatorname{rot}\mathcal{V}^0(\boldsymbol{y}', 0, t) - \tilde{\boldsymbol{w}}^0(\boldsymbol{y}', 0, t). \end{aligned} \tag{3.53}$$

Postoji jedinstveno rješenje problema (3.53). Dodatno, $\mathcal{W}^0$ eksponencijalno teži k nuli kada $y_3 \to +\infty$.

Nadalje, problem za korektor rubnog sloja mikrorotacije nultog reda $\mathcal{Z}^0$ u $x_3 = l$ je dan sustavom

$$
\begin{aligned}
-\alpha\Delta\mathcal{Z}^0 - \beta\nabla\mathrm{div}\mathcal{Z}^0 &= \mathbf{0} \quad \text{u} \quad \mathcal{G}_l, \\
\mathcal{Z}^0 &= \frac{\alpha_0}{2}\,\mathrm{rot}\,\boldsymbol{\mathcal{Y}}^0 \quad \text{na} \quad \sigma, \\
\mathcal{Z}^0(\mathbf{y}',0,t) &= \frac{\alpha_0}{2}\left(\frac{\partial\tilde{u}_3^0}{\partial y_2}(\mathbf{y}',l,t), -\frac{\partial\tilde{u}_3^0}{\partial y_1}(\mathbf{y}',l,t), 0\right) \\
&+\frac{\alpha_0}{2}\,\mathrm{rot}\,\boldsymbol{\mathcal{Y}}^0(\mathbf{y}',l,t) - \tilde{\mathbf{w}}^0(\mathbf{y}',l,t).
\end{aligned}
\tag{3.54}
$$

Egzistencija, jedinstvenost i eksponencijalni pad rješenja $\mathcal{Z}^0$ problema (3.54) slijedi analogno kao i za $\boldsymbol{\mathcal{W}}^0$.

3.3.2.2 Korektori prvog reda

Korektori rubnog sloja brzine i tlaka prvog reda $(\boldsymbol{\mathcal{V}}^1, \mathcal{P}^1)$ u $x_3 = 0$ zadovoljavaju sustav jednadžbi

$$
\begin{aligned}
-\chi\Delta\boldsymbol{\mathcal{V}}^1 + \nabla\mathcal{P}^1 &= a\,\mathrm{rot}\,\boldsymbol{\mathcal{W}}^0 \quad \text{u} \quad \mathcal{G}_0, \\
\mathrm{div}\,\boldsymbol{\mathcal{V}}^1 = 0 \quad \text{u} \quad \mathcal{G}_0, &\quad \boldsymbol{\mathcal{V}}^1 = \mathbf{0} \quad \text{na} \quad \omega, \\
\boldsymbol{\mathcal{V}}^1(\mathbf{y}',0,t) &= -\tilde{\mathbf{u}}^1(\mathbf{y}',0,t).
\end{aligned}
\tag{3.55}
$$

Kako bi problem (3.55) bio dobro postavljen, mora vrijediti uvjet kompatibilnosti

$$
\int_B \tilde{u}_3^1(\mathbf{y}',0,t) = 0. \tag{3.56}
$$

Ako odaberemo korektor tlaka $\tilde{p}^1_{bl,cor}(t)$ iz (3.22) kao

$$
\tilde{p}^1_{bl,cor}(t) = \frac{4a\alpha_0 F(t)}{\pi},
$$

lako je provjeriti da je uvjet kompatibilnosti (3.56) zadovoljen. Dakle, sustav jednadžbi (3.55) ima jedinstveno rješenje koje eksponencijalno teži k nuli kada $y_3 \to +\infty$.

Egzistencija, jedinstvenost i eksponencijalni pad korektora rubnog sloja $(\boldsymbol{\mathcal{Y}}^1, \boldsymbol{\mathcal{Q}}^1)$ na suprotnom kraju cijevi $x_3 = l$ slijedi analogno.

Korektor rubnog sloja mikrorotacije prvog reda $\boldsymbol{\mathcal{W}}^1$ u $x_3 = 0$ je dan sustavom

$$
\begin{aligned}
-\alpha\Delta\boldsymbol{\mathcal{W}}^1 - \beta\nabla\mathrm{div}\boldsymbol{\mathcal{W}}^1 &= a\,\mathrm{rot}\,\boldsymbol{\mathcal{V}}^0 \quad \text{u} \quad \mathcal{G}_0, \\
\boldsymbol{\mathcal{W}}^1 &= \frac{\alpha_0}{2}\,\mathrm{rot}\,\boldsymbol{\mathcal{V}}^1 \quad \text{na} \quad \omega, \\
\boldsymbol{\mathcal{W}}^1(\mathbf{y}',0,t) &= \frac{\alpha_0}{2}\left(\frac{\partial\tilde{u}_3^1}{\partial y_2}(\mathbf{y}',0,t), -\frac{\partial\tilde{u}_3^1}{\partial y_1}(\mathbf{y}',0,t), \frac{\partial\tilde{u}_2^1}{\partial y_1}(\mathbf{y}',0,t) - \frac{\partial\tilde{u}_1^1}{\partial y_2}(\mathbf{y}',0,t)\right) \\
&+\frac{\alpha_0}{2}\boldsymbol{\mathcal{V}}^1(\mathbf{y}',0,t) - \tilde{\mathbf{w}}^1(\mathbf{y}',0,t).
\end{aligned}
\tag{3.57}
$$

Problem (3.57) ima jedinstveno rješenje koje eksponencijalno teži k nuli kada $y_3 \to +\infty$.

Egzistencija, jedinstvenost i eksponencijalni pad korektora rubnog sloja $\mathcal{Z}^1$ na suprotnom kraju cijevi $x_3 = l$ se pokazuje analogno.

3.3.2.3 Korektori drugog reda

Korektori rubnog sloja brzine i tlaka drugog reda $(\boldsymbol{\mathcal{V}}^2, \mathcal{P}^2)$ u $x_3 = 0$ su dani sustavom jednadžbi

$$
\begin{gathered}
-\chi \Delta \boldsymbol{\mathcal{V}}^2 + \nabla \mathcal{P}^2 = \operatorname{rot} \boldsymbol{\mathcal{V}}^1 - \frac{\partial \boldsymbol{\mathcal{V}}^0}{\partial t} \quad \text{u} \quad \mathcal{G}_0, \\
\operatorname{div} \boldsymbol{\mathcal{V}}^2 = 0 \quad \text{u} \quad \mathcal{G}_0, \quad \boldsymbol{\mathcal{V}}^2 = \mathbf{0} \quad \text{na} \quad \omega, \\
\boldsymbol{\mathcal{V}}^2(\boldsymbol{y}', 0, t) = -\tilde{\boldsymbol{u}}^2(\boldsymbol{y}', 0, t).
\end{gathered} \tag{3.58}
$$

Najprije moramo provjeriti vrijedi li nužan uvjet kompatibilnosti za problem (3.58):

$$
\int_B \tilde{u}_3^2(\boldsymbol{y}', 0, t) = 0. \tag{3.59}
$$

Odabiremo korektor tlaka $\tilde{p}^2_{bl,cor}(t)$ iz (3.28) kao

$$
\tilde{p}^2_{bl,cor}(t) = \frac{8\chi}{\pi} \int_B (|\boldsymbol{y}'|^2 - 1)(B_9(t) y_1^2 + B_{10}(t) y_1 y_2 + B_{11}(t) y_2^2 + \tilde{B}_{14}(t)), \tag{3.60}
$$

pri čemu je

$$
\tilde{B}_{14}(t) = -\frac{1}{8\chi\pi}\Big(\frac{a^2 F(t)}{\alpha} + 4\frac{a\alpha_0^2 F(t)}{\chi} - 3\frac{\partial F}{\partial t}(t)\Big) + \frac{a\alpha_0 \tilde{p}^1_{bl,cor}(t)}{8\chi^2}.
$$

S ovako odabranim korektorom tlaka uvjet kompatibilnosti (3.59) je zadovoljen. Uočimo da integral s desne strane (3.60) možemo izračunati eksplicitno, čime dobivamo

$$
\tilde{p}^2_{bl,cor}(t) = \frac{1}{3\pi}\left(\frac{a^2 F(t)}{\alpha} + 6\frac{a^2\alpha_0^2 F(t)}{\chi} - 4\frac{\partial F}{\partial t}(t)\right) - \frac{a\alpha_0 \tilde{p}^1_{bl,cor}(t)}{2\chi}.
$$

Sada problem (3.58) ima jedinstveno rješenje koje eksponencijalno teži k nuli kada $y_3 \to +\infty$.

Egzistencija, jedinstvenost i eksponencijalni pad korektora rubnog sloja $(\boldsymbol{\mathcal{Y}}^2, \boldsymbol{\mathcal{Q}}^2)$ na suprotnom kraju cijevi $x_3 = l$ slijede analogno.

Konačno, problem za korektor rubnog sloja mikrorotacije drugog reda $\boldsymbol{\mathcal{W}}^2$ u $x_3 = 0$ je dan s

$$
\begin{aligned}
-\alpha\Delta\mathcal{W}^2 - \beta\nabla \operatorname{div} \mathcal{W}^2 &= \operatorname{rot} \mathcal{V}^1 - \frac{\partial \mathcal{W}^0}{\partial t} - 2a\mathcal{W}^0 \quad \text{u} \quad \mathcal{G}_0, \\
\mathcal{W}^2 &= \frac{\alpha_0}{2} \operatorname{rot} \mathcal{V}^2 \quad \text{na} \quad \omega, \\
\mathcal{W}^2(\boldsymbol{y}', 0, t) &= \\
\frac{\alpha_0}{2}\Bigg(\frac{\partial \tilde{u}_3^2}{\partial y_2}(\boldsymbol{y}', 0, t) - \frac{\partial \tilde{u}_2^1}{\partial x_3}, &\frac{\partial \tilde{u}_1^1}{\partial x_3}(\boldsymbol{y}', 0, t) - \frac{\partial \tilde{u}_3^2}{\partial y_1}(\boldsymbol{y}', 0, t), \frac{\partial \tilde{u}_2^2}{\partial y_1}(\boldsymbol{y}', 0, t) - \frac{\partial \tilde{u}_1^2}{\partial y_2}(\boldsymbol{y}', 0, t)\Bigg) \\
&+\mathcal{V}^2(\boldsymbol{y}', 0, t) - \tilde{\boldsymbol{w}}^2(\boldsymbol{y}', 0, t).
\end{aligned}
\tag{3.61}
$$

Postoji jedinstveno rješenje problema (3.61), te rješenje eksponencijalno teži k nuli kada $y_3 \to +\infty$.

Egzistencija, jedinstvenost i eksponencijalni pad korektora rubnog sloja $\mathcal{Z}^2$ na suprotnom kraju cijevi $x_3 = l$ slijede analogno.

3.3.3. Asimptotička aproksimacija

Predložena asimptotička aproksimacija određena u ovom poglavlju je dana s

$$
\begin{aligned}
\boldsymbol{u}_\varepsilon^{[2]}(\boldsymbol{x}', x_3, t) =&\tilde{\boldsymbol{u}}_{\varepsilon,reg}^{[2]}(\boldsymbol{x}', x_3, t) + \tilde{\boldsymbol{u}}_{\varepsilon,bl,0}^{[2]}(\boldsymbol{x}', x_3, t) + \tilde{\boldsymbol{u}}_{\varepsilon,bl,l}^{[2]}(\boldsymbol{x}', x_3, t) \\
=&\tilde{\boldsymbol{u}}^0\left(\frac{\boldsymbol{x}'}{\varepsilon}, t\right) + \varepsilon\tilde{\boldsymbol{u}}^1\left(\frac{\boldsymbol{x}'}{\varepsilon}, x_3, t\right) + \varepsilon^2\tilde{\boldsymbol{u}}^2\left(\frac{\boldsymbol{x}'}{\varepsilon}, x_3, t\right) \\
&+ \mathcal{V}^0\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3}{\varepsilon}, t\right) + \varepsilon\mathcal{V}^1\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3}{\varepsilon}, t\right) + \varepsilon^2\mathcal{V}^2\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3}{\varepsilon}, t\right) \\
&+ \mathcal{Y}^0\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3 - l}{\varepsilon}, t\right) + \varepsilon\mathcal{Y}^1\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3 - l}{\varepsilon}, t\right) + \varepsilon^2\mathcal{Y}^2\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3 - l}{\varepsilon}, t\right), \\
p_\varepsilon^{[2]}(\boldsymbol{x}', x_3, t) =&\tilde{p}_{\varepsilon,reg}^{[3]}(\boldsymbol{x}', x_3, t) + \tilde{p}_{\varepsilon,bl,0}^{[2]}(\boldsymbol{x}', x_3, t) + \tilde{p}_{\varepsilon,bl,l}^{[2]}(\boldsymbol{x}', x_3, t) \\
=&\frac{1}{\varepsilon^2}\tilde{p}^0(x_3, t) + \frac{1}{\varepsilon}\tilde{p}^1\left(\frac{\boldsymbol{x}'}{\varepsilon}, x_3, t\right) + \tilde{p}^2\left(\frac{\boldsymbol{x}'}{\varepsilon}, x_3, t\right) + \varepsilon\tilde{p}^3\left(\frac{\boldsymbol{x}'}{\varepsilon}, x_3, t\right) \\
&+ \frac{1}{\varepsilon}\mathcal{P}^0\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3}{\varepsilon}, t\right) + \mathcal{P}^1\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3}{\varepsilon}, t\right) + \varepsilon\mathcal{P}^2\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3}{\varepsilon}, t\right) \\
&+ \frac{1}{\varepsilon}\mathcal{Q}^0\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3 - l}{\varepsilon}, t\right) + \mathcal{Q}^1\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3 - l}{\varepsilon}, t\right) + \varepsilon\mathcal{Q}^2\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3 - l}{\varepsilon}, t\right), \\
\boldsymbol{w}_\varepsilon^{[2]}(\boldsymbol{x}', x_3, t) =&\tilde{\boldsymbol{w}}_{\varepsilon,reg}^{[2]}(\boldsymbol{x}', x_3, t) + \tilde{\boldsymbol{w}}_{\varepsilon,bl,0}^{[2]}(\boldsymbol{x}', x_3, t) + \tilde{\boldsymbol{w}}_{\varepsilon,bl,l}^{[2]}(\boldsymbol{x}', x_3, t) \\
=&\tilde{\boldsymbol{w}}^0\left(\frac{\boldsymbol{x}'}{\varepsilon}, x_3, t\right) + \varepsilon\tilde{\boldsymbol{w}}^1\left(\frac{\boldsymbol{x}'}{\varepsilon}, x_3, t\right) + \varepsilon^2\tilde{\boldsymbol{w}}^2\left(\frac{\boldsymbol{x}'}{\varepsilon}, x_3, t\right) \\
&+ \mathcal{W}^0\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3}{\varepsilon}, t\right) + \varepsilon\mathcal{W}^1\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3}{\varepsilon}, t\right) + \varepsilon^2\mathcal{W}^2\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3}{\varepsilon}, t\right) \\
&+ \mathcal{Z}^0\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3 - l}{\varepsilon}, t\right) + \varepsilon\mathcal{Z}^1\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3 - l}{\varepsilon}, t\right) + \varepsilon^2\mathcal{Z}^2\left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3 - l}{\varepsilon}, t\right),
\end{aligned}
\tag{3.62}
$$

gdje je $(\tilde{\boldsymbol{u}}_{\varepsilon,reg}^{[2]}, \tilde{p}_{\varepsilon,reg}^{[3]}, \tilde{\boldsymbol{w}}_{\varepsilon,reg}^{[2]})$ regularni dio asimptotičke aproksimacije određen u Odjeljku 3.3.1, dok su $(\tilde{\boldsymbol{u}}_{\varepsilon,bl,0}^{[2]}, \tilde{p}_{\varepsilon,bl,0}^{[2]}, \tilde{\boldsymbol{w}}_{\varepsilon,bl,0}^{[2]})$ i $(\tilde{\boldsymbol{u}}_{\varepsilon,bl,l}^{[2]}, \tilde{p}_{\varepsilon,bl,l}^{[2]}, \tilde{\boldsymbol{w}}_{\varepsilon,bl,l}^{[2]})$ korektori rubnog sloja u prostoru u $x_3 = 0$, odnosno $x_3 = l$, određeni u Odjeljku 3.3.2.

Cilj ovog odjeljka bio je konstruirati asimptotičko rješenje koje uzima u obzir sve važne fizikalne procese. Izračunali smo korektore do drugog reda te proveli analizu rubnog sloja u prostoru, čime smo dobili asimptotički model višeg reda točnosti opisan s (3.62). Dani model je pogodan za proučavanje utjecaja mikropolarnosti, dinamičkih rubnih uvjeta, te derivacije po vremenu na tok mikropolarnog fluida u tankoj cijevi.

Asimptotičku aproksimaciju (3.62) je potrebno rigorozno opravdati tako da se dokažu odgovarajuće ocjene greške u prikladnim normama, što ćemo napraviti u sljedećem odjeljku.

3.4. Rigorozno opravdanje

3.4.1. Apriorne ocjene

Teorem 3.4.1 (Apriorne ocjene)**.** Neka je $(\boldsymbol{u}_\varepsilon, \boldsymbol{w}_\varepsilon)$ rješenje problema (3.1)–(3.5), tada vrijede sljedeće ocjene:

$$\begin{aligned} \|\boldsymbol{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\le C, \\ \|\boldsymbol{w}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\nabla \boldsymbol{w}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\le C, \\ \left\|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \|\nabla \boldsymbol{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\Delta \boldsymbol{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\le \frac{C}{\varepsilon}, \\ \|p_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\le \frac{C}{\varepsilon}, \end{aligned} \tag{3.63}$$

gdje je $\boldsymbol{C}$ konstanta neovisna o $\boldsymbol{\varepsilon}$.

Dokaz. Neka je $(\boldsymbol{u}_\varepsilon, \boldsymbol{w}_\varepsilon) = (\boldsymbol{v}_\varepsilon + \boldsymbol{h}_\varepsilon^{ext}, \boldsymbol{z}_\varepsilon + \frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \boldsymbol{v}_\varepsilon + \frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \boldsymbol{h}_\varepsilon^{ext})$ slabo rješenje problema (3.1)–(3.5), odnosno neka $(\boldsymbol{v}_\varepsilon, \boldsymbol{z}_\varepsilon)$ zadovoljava (3.10)–(3.12). Posebno, $\boldsymbol{v}_\varepsilon$ zadovoljava

$$\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} - \chi \Delta \boldsymbol{v}_\varepsilon + \nabla p_\varepsilon = a \operatorname{rot} \boldsymbol{z}_\varepsilon + a\frac{\alpha_0^\varepsilon}{2} \operatorname{rot}\operatorname{rot} \boldsymbol{v}_\varepsilon + \boldsymbol{f}_\varepsilon - \frac{\partial \boldsymbol{h}_\varepsilon^{ext}}{\partial t} + \chi \Delta \boldsymbol{h}_\varepsilon^{ext} + a\frac{\alpha_0^\varepsilon}{2} \operatorname{rot}\operatorname{rot} \boldsymbol{h}_\varepsilon^{ext}.$$

Kako vrijedi $\operatorname{rot}\operatorname{rot} \boldsymbol{v}_\varepsilon = -\Delta \boldsymbol{v}_\varepsilon$ za $\boldsymbol{v}_\varepsilon$ takve da $\operatorname{div} \boldsymbol{v}_\varepsilon = 0$, gornja jednakost glasi

$$\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} - \left(\chi - a\frac{\alpha_0^\varepsilon}{2}\right) \Delta \boldsymbol{v}_\varepsilon + \nabla p_\varepsilon = a \operatorname{rot} \boldsymbol{z}_\varepsilon + \hat{\boldsymbol{f}}_\varepsilon,$$

gdje je

$$\hat{\boldsymbol{f}}_\varepsilon = \boldsymbol{f}_\varepsilon - \frac{\partial \boldsymbol{h}_\varepsilon^{ext}}{\partial t} + \left(\chi - a\frac{\alpha_0^\varepsilon}{2}\right) \Delta \boldsymbol{h}_\varepsilon^{ext}.$$

Djelovanjem operatorom $\boldsymbol{P}$ (vidi Odjeljak 3.2) na gornju jednakost imamo

$$\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} + \left(\chi - a\frac{\alpha_0^\varepsilon}{2}\right) \mathcal{A} \boldsymbol{v}_\varepsilon = a \operatorname{rot} \boldsymbol{z}_\varepsilon + P\hat{\boldsymbol{f}}_\varepsilon, \tag{3.64}$$

pri čemu smo iskoristili $\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}, \operatorname{rot} \boldsymbol{z}_\varepsilon \in \boldsymbol{H}$. Doista, vrijedi $\operatorname{div}\operatorname{rot} \boldsymbol{z}_\varepsilon = 0$ u $\boldsymbol{\Omega}_\varepsilon$, te $\operatorname{rot} \boldsymbol{z}_\varepsilon \cdot \boldsymbol{n} = 0$ na $\partial\boldsymbol{\Omega}_\varepsilon$ slijedi iz $\boldsymbol{z}_\varepsilon = 0$ na $\partial\Omega_\varepsilon$. Množenjem (3.64) s $\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} + \xi \mathcal{A} \boldsymbol{v}_\varepsilon$ te integriranjem po $\boldsymbol{\Omega}_\varepsilon$ dobivamo:

$$\begin{aligned} &\int_{\Omega_\varepsilon} \left|\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}\right|^2 + \frac{1}{2}\left(\chi - a\frac{\alpha_0^\varepsilon}{2}\right) \frac{d}{dt}\int_{\Omega_\varepsilon} |\nabla \boldsymbol{v}_\varepsilon|^2 + \left(\chi - a\frac{\alpha_0^\varepsilon}{2}\right) \xi \int_{\Omega_\varepsilon} |\mathcal{A} \boldsymbol{v}_\varepsilon|^2 \\ &= a\int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{z}_\varepsilon \cdot \frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} + a\xi \int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{z}_\varepsilon \cdot \mathcal{A} \boldsymbol{v}_\varepsilon + \int_{\Omega_\varepsilon} P\hat{\boldsymbol{f}}_\varepsilon \cdot \frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} + \xi \int_{\Omega_\varepsilon} P\hat{\boldsymbol{f}}_\varepsilon \cdot \mathcal{A} \boldsymbol{v}_\varepsilon \\ &\quad - \xi \int_{\Omega_\varepsilon} \frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} \cdot \mathcal{A} \boldsymbol{v}_\varepsilon. \end{aligned} \tag{3.65}$$

Koristeći Youngovu nejednakost, iz (3.65) za dobar odabir parametra ξ slijedi ocjena (vidi npr. [96])

$$\begin{aligned}&\int_{\Omega_\varepsilon}\left|\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}\right|^2+\frac{d}{dt}\int_{\Omega_\varepsilon}|\nabla \boldsymbol{v}_\varepsilon|^2+\int_{\Omega_\varepsilon}|\mathcal{A}\boldsymbol{v}_\varepsilon|^2\\ &\leq C\|\nabla \boldsymbol{z}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}+C\left\|\boldsymbol{f}_\varepsilon\right\|^2_{L^2(\Omega_\varepsilon)}+C\left\|\frac{\partial \boldsymbol{h}^{ext}_\varepsilon}{\partial t}\right\|^2_{L^2(\Omega_\varepsilon)}+C\left\|\Delta \boldsymbol{h}^{ext}_\varepsilon\right\|^2_{L^2(\Omega_\varepsilon)}.\end{aligned}\tag{3.66}$$

Integriranjem (3.66) po t imamo ocjenu

$$\begin{aligned}&\left\|\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))}+\|\nabla \boldsymbol{v}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))}+\|\Delta \boldsymbol{v}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}\\ &\leq C\|\nabla \boldsymbol{z}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}+\frac{C}{\varepsilon},\end{aligned}\tag{3.67}$$

gdje smo iskoristili (3.7) te činjenicu da su norme $\|\Delta \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ i $\|\mathcal{A}\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ ekvivalentne na $D(\mathcal{A})$. Sada uvrštavamo $\boldsymbol{\psi}=\boldsymbol{z}_\varepsilon(\cdot,t)$ u $(3.11)_2$ čime dobivamo

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega_\varepsilon}|\boldsymbol{z}_\varepsilon|^2+\alpha\int_{\Omega_\varepsilon}|\nabla \boldsymbol{z}_\varepsilon|^2+\beta\int_{\Omega_\varepsilon}(\operatorname{div}\boldsymbol{z}_\varepsilon)^2+2a\int_{\Omega_\varepsilon}|\boldsymbol{z}_\varepsilon|^2\\ &=a\int_{\Omega_\varepsilon}\operatorname{rot}\boldsymbol{v}_\varepsilon\cdot\boldsymbol{z}_\varepsilon+a\int_{\Omega_\varepsilon}\operatorname{rot}\boldsymbol{h}^{ext}_\varepsilon\cdot\boldsymbol{z}_\varepsilon+\int_{\Omega_\varepsilon}\boldsymbol{g}_\varepsilon\cdot\boldsymbol{z}_\varepsilon-\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}\cdot\operatorname{rot}\boldsymbol{z}_\varepsilon\\ &-\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\frac{\partial \boldsymbol{h}^{ext}_\varepsilon}{\partial t}\cdot\operatorname{rot}\boldsymbol{z}_\varepsilon-\alpha\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\nabla\operatorname{rot}\boldsymbol{v}_\varepsilon\cdot\nabla \boldsymbol{z}_\varepsilon-\beta\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\operatorname{div}\operatorname{rot}\boldsymbol{v}_\varepsilon\operatorname{div}\boldsymbol{z}_\varepsilon\\ &-2a\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\operatorname{rot}\boldsymbol{v}_\varepsilon\cdot\boldsymbol{z}_\varepsilon-\alpha\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\nabla\operatorname{rot}\boldsymbol{h}^{ext}_\varepsilon\cdot\nabla \boldsymbol{z}_\varepsilon-\beta\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\operatorname{div}\operatorname{rot}\boldsymbol{h}^{ext}_\varepsilon\operatorname{div}\boldsymbol{z}_\varepsilon\\ &-2a\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\operatorname{rot}\boldsymbol{h}^{ext}_\varepsilon\cdot\boldsymbol{z}_\varepsilon.\end{aligned}\tag{3.68}$$

Najprije uočimo da vrijedi $\operatorname{div}\operatorname{rot}\boldsymbol{v}_\varepsilon=\operatorname{div}\operatorname{rot}\boldsymbol{h}^{ext}_\varepsilon=0$. Nadalje, zbog (3.7) i Leme 0.0.1 vrijede ocjene

$$\begin{aligned}&\int_{\Omega_\varepsilon}\operatorname{rot}\boldsymbol{h}^{ext}_\varepsilon\cdot\boldsymbol{z}_\varepsilon\leq C\varepsilon\|\nabla \boldsymbol{z}_\varepsilon\|_{L^2(\Omega_\varepsilon)},\\ &\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\left(\frac{\partial \boldsymbol{h}^{ext}_\varepsilon}{\partial t}\cdot\operatorname{rot}\boldsymbol{z}_\varepsilon+\alpha\nabla\operatorname{rot}\boldsymbol{h}^{ext}_\varepsilon\cdot\nabla \boldsymbol{z}_\varepsilon+2a\operatorname{rot}\boldsymbol{h}^{ext}_\varepsilon\cdot\boldsymbol{z}_\varepsilon\right)\leq C\|\nabla \boldsymbol{z}_\varepsilon\|_{L^2(\Omega_\varepsilon)}.\end{aligned}\tag{3.69}$$

Ostale izraze u (3.68) ocjenjujemo koristeći Lemu 0.0.1 te ocjenu (3.66):

$$\begin{aligned}\int_{\Omega_\varepsilon}\operatorname{rot}\boldsymbol{v}_\varepsilon\cdot\boldsymbol{z}_\varepsilon&\leq\|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{z}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\ &\leq C\varepsilon\|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla \boldsymbol{z}_\varepsilon\|_{L^2(\Omega_\varepsilon)},\\ \int_{\Omega_\varepsilon}\boldsymbol{g}_\varepsilon\cdot\boldsymbol{z}_\varepsilon&\leq\left\|\boldsymbol{g}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{z}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\ &\leq C\varepsilon\left\|\boldsymbol{g}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\nabla \boldsymbol{z}_\varepsilon\|_{L^2(\Omega_\varepsilon)},\end{aligned}\tag{3.70}$$

$$\begin{aligned}\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}\cdot \operatorname{rot}\boldsymbol{z}_\varepsilon &\le C\varepsilon\left\|\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)}\|\nabla \boldsymbol{z}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\ &\le C\varepsilon\|\nabla \boldsymbol{z}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}+C\|\nabla \boldsymbol{z}_\varepsilon\|_{L^2(\Omega_\varepsilon)},\\ \frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\nabla \operatorname{rot}\boldsymbol{v}_\varepsilon\cdot\nabla \boldsymbol{z}_\varepsilon &\le C\varepsilon\|\Delta \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla \boldsymbol{z}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\ &\le C\varepsilon\|\nabla \boldsymbol{z}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}+C\|\nabla \boldsymbol{z}_\varepsilon\|_{L^2(\Omega_\varepsilon)}.\end{aligned}$$

Koristeći Youngovu nejednakost, iz (3.68), (3.69) i (3.70) za dovoljno mali ε slijedi ocjena

$$\frac{d}{dt}\int_{\Omega_\varepsilon}|\boldsymbol{z}_\varepsilon|^2+\int_{\Omega_\varepsilon}|\nabla \boldsymbol{z}_\varepsilon|^2\le C\varepsilon^2\|\nabla \boldsymbol{v}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}+C. \tag{3.71}$$

Integrirajući (3.71) po t dobivamo

$$\sup_{t\in[0,T]}\|\boldsymbol{z}_\varepsilon(\cdot,t)\|^2_{L^2(\Omega_\varepsilon)}+\int_0^T\|\nabla \boldsymbol{z}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\le C\varepsilon^2\|\nabla \boldsymbol{v}_\varepsilon\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))}+C. \tag{3.72}$$

Sada uvrštavamo $\boldsymbol{\varphi}=\boldsymbol{v}_\varepsilon(\cdot,t)$ u $(3.11)_1$, iz čega imamo

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_\varepsilon}|\boldsymbol{v}_\varepsilon|^2+\left(\chi-a\frac{\alpha_0^\varepsilon}{2}\right)\int_{\Omega_\varepsilon}|\nabla \boldsymbol{v}_\varepsilon|^2 &= a\int_{\Omega_\varepsilon}\operatorname{rot}\boldsymbol{z}_\varepsilon\cdot \boldsymbol{v}_\varepsilon-\int_{\Omega_\varepsilon}\frac{\partial \boldsymbol{h}_\varepsilon^{ext}}{\partial t}\cdot \boldsymbol{v}_\varepsilon\\ &+\left(\chi-a\frac{\alpha_0^\varepsilon}{2}\right)\int_{\Omega_\varepsilon}\Delta \boldsymbol{h}_\varepsilon^{ext}\cdot \boldsymbol{v}_\varepsilon+\int_{\Omega_\varepsilon}\boldsymbol{f}_\varepsilon\cdot \boldsymbol{v}_\varepsilon.\end{aligned} \tag{3.73}$$

Iz ocjene (3.7), Leme 0.0.1, jednakosti (3.73) te Youngove nejednakosti slijedi ocjena

$$\frac{d}{dt}\int_{\Omega_\varepsilon}|\boldsymbol{v}_\varepsilon|^2+\int_{\Omega_\varepsilon}|\nabla \boldsymbol{v}_\varepsilon|^2\le C\varepsilon^2\|\nabla \boldsymbol{z}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}+C. \tag{3.74}$$

Sada integriramo (3.74) po t te koristimo ocjenu (3.72) čime dobivamo

$$\sup_{t\in[0,T]}\|\boldsymbol{v}_\varepsilon(\cdot,t)\|^2_{L^2(\Omega_\varepsilon)}+\int_0^T\|\nabla \boldsymbol{v}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\le C. \tag{3.75}$$

Iz (3.67), (3.72) i (3.75) slijede ocjene

$$\begin{aligned}\sup_{t\in[0,T]}\|\boldsymbol{z}_\varepsilon(\cdot,t)\|^2_{L^2(\Omega_\varepsilon)}+\int_0^T\|\nabla \boldsymbol{z}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\le C,\\ \left\|\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))}+\|\nabla \boldsymbol{v}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))}+\|\Delta \boldsymbol{v}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}\le\frac{C}{\varepsilon}.\end{aligned} \tag{3.76}$$

Nadalje, po Lemi 0.0.2 postoji funkcija $\boldsymbol{d}_\varepsilon$ takva da

$$\operatorname{div}\boldsymbol{d}_\varepsilon=p_\varepsilon,$$

$$\boldsymbol{d}_\varepsilon=\boldsymbol{0}\quad\text{na}\quad\partial\Omega_\varepsilon$$

te vrijedi ocjena

$$\|\nabla \boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\le\frac{C}{\varepsilon}\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{3.77}$$

Sada uvrštavanjem $\boldsymbol{\varphi} = \boldsymbol{d}_\varepsilon$ u (3.11)$_1$ dobivamo

$$\begin{aligned}\int_{\Omega_\varepsilon} |p_\varepsilon|^2 &= \int_{\Omega_\varepsilon} \frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} \cdot \boldsymbol{d}_\varepsilon + \left(\chi - a\frac{\alpha_0^\varepsilon}{2}\right) \int_{\Omega_\varepsilon} \nabla \boldsymbol{v}_\varepsilon \cdot \boldsymbol{d}_\varepsilon - a \int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{z}_\varepsilon \cdot \boldsymbol{d}_\varepsilon \\ &+ \int_{\Omega_\varepsilon} \frac{\partial \boldsymbol{h}_\varepsilon^{ext}}{\partial t} \cdot \boldsymbol{d}_\varepsilon - \left(\chi - a\frac{\alpha_0^\varepsilon}{2}\right) \int_{\Omega_\varepsilon} \Delta \boldsymbol{h}_\varepsilon^{ext} \cdot \boldsymbol{d}_\varepsilon - \int_{\Omega_\varepsilon} \boldsymbol{f}_\varepsilon \cdot \boldsymbol{d}_\varepsilon.\end{aligned} \tag{3.78}$$

Desnu stranu jednakosti (3.78) možemo ocijeniti koristeći (3.7), (3.75), (3.76) i (3.77), te tada integriranjem po t konačno imamo

$$\|p_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \le \frac{C}{\varepsilon}. \tag{3.79}$$

Sada ocjene (3.63) slijede iz (3.75), (3.76), (3.79) te (3.7).

■

3.4.2. Ocjene greške

Uvodimo sljedeće oznake za razliku između rješenja problema (3.1)–(3.5) te asimptotičke aproksimacije dane s (3.62):

$$\boldsymbol{D}_\varepsilon = \boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^{[2]}, \quad \boldsymbol{S}_\varepsilon = \boldsymbol{w}_\varepsilon - \boldsymbol{w}_\varepsilon^{[2]}, \quad r_\varepsilon = p_\varepsilon - p_\varepsilon^{[2]}.$$

Glavni rezultat ovog odjeljka može se formulirati na sljedeći način:

Teorem 3.4.2 (Ocjene greške)**.** Vrijede sljedeće ocjene:

$$\begin{aligned}\|\boldsymbol{D}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\nabla \boldsymbol{D}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\le C\varepsilon^3, \\ \|\boldsymbol{S}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\nabla \boldsymbol{S}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\le C\varepsilon^3, \\ \|r_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\le C\varepsilon^2,\end{aligned} \tag{3.80}$$

pri čemu konstanta $C > 0$ ne ovisi o ε.

Dokaz. Asimptotička aproksimacija mikrorotacije $\boldsymbol{w}_\varepsilon^{[2]}$ zadovoljava sljedeći sustav na Ω_ε:

$$\begin{aligned}\frac{\partial \boldsymbol{w}_\varepsilon^{[2]}}{\partial t} - \alpha \Delta \boldsymbol{w}_\varepsilon^{[2]} - \beta \nabla \operatorname{div} \boldsymbol{w}_\varepsilon^{[2]} + 2a \boldsymbol{w}_\varepsilon^{[2]} &= a \operatorname{rot} \boldsymbol{u}_\varepsilon^{[1]} + \boldsymbol{g}_\varepsilon + \boldsymbol{H}_\varepsilon, \\ \boldsymbol{w}_\varepsilon^{[2]} &= \frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \boldsymbol{u}_\varepsilon^{[2]} + \boldsymbol{\lambda}_\varepsilon + \boldsymbol{\eta}_B^\varepsilon \quad \text{na} \quad S_B, \\ \boldsymbol{w}_\varepsilon^{[2]} &= \frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \boldsymbol{u}_\varepsilon^{[2]} + \boldsymbol{\lambda}_\varepsilon + \boldsymbol{\eta}_T^\varepsilon \quad \text{na} \quad S_T, \\ \boldsymbol{w}_\varepsilon^{[2]} &= \frac{\alpha_0^\varepsilon}{2} \operatorname{rot} \boldsymbol{u}_\varepsilon^{[2]} + \boldsymbol{\lambda}_\varepsilon \quad \text{na} \quad S_L, \\ \boldsymbol{w}_\varepsilon^{[2]}(\cdot, 0) &= \boldsymbol{0},\end{aligned} \tag{3.81}$$

pri čemu je

$$\begin{aligned}\boldsymbol{u}_\varepsilon^{[1]}(\boldsymbol{x},t) =&\tilde{\boldsymbol{u}}^0\left(\frac{\boldsymbol{x}'}{\varepsilon},t\right)+\varepsilon\tilde{\boldsymbol{u}}^1\left(\frac{\boldsymbol{x}'}{\varepsilon},x_3,t\right)+\boldsymbol{\mathcal{V}}^0\left(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3}{\varepsilon},t\right)+\varepsilon\boldsymbol{\mathcal{V}}^1\left(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3}{\varepsilon},t\right)\\ &+\boldsymbol{\mathcal{Y}}^0\left(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3-l}{\varepsilon},t\right)+\varepsilon\boldsymbol{\mathcal{Y}}^1\left(\frac{\boldsymbol{x}'}{\varepsilon},\frac{x_3-l}{\varepsilon},t\right),\end{aligned}$$

te vrijede ocjene

$$\begin{aligned}\|\boldsymbol{H}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}&=\mathcal{O}(\varepsilon^2),\\ \left\|\boldsymbol{\eta}_B^\varepsilon\right\|_{L^2(0,T;W^{1/2,2}(S_B))},\left\|\boldsymbol{\eta}_T^\varepsilon\right\|_{L^2(0,T;W^{1/2,2}(S_T))}&=\mathcal{O}(\exp(-\sigma/\varepsilon)).\end{aligned}\tag{3.82}$$

Funkcija $\boldsymbol{\lambda}_\varepsilon=\varepsilon^2\frac{\alpha_0^\varepsilon}{2}\left(\frac{\partial\tilde{u}_2^2}{\partial x_3}-\frac{\partial\tilde{u}_1^2}{\partial x_3}\right)$ je rezidual koji dolazi od dinamičkog rubnog uvjeta, definirana je na cijeloj domeni Ω_ε i vrijede ocjene

$$\begin{aligned}\|\boldsymbol{\lambda}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))},\left\|\frac{\partial\boldsymbol{\lambda}_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))}&=\mathcal{O}(\varepsilon^4),\\ \|\nabla\boldsymbol{\lambda}_\varepsilon\|_{L^2(0,T;L^2(\partial\Omega_\varepsilon))},\left\|\nabla\frac{\boldsymbol{\lambda}_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))}&=\mathcal{O}(\varepsilon^3).\end{aligned}\tag{3.83}$$

Nadalje, iz Leme 0.0.3 te ocjene $(3.82)_2$ zaključujemo da postoji funkcija $\boldsymbol{\eta}_\varepsilon$ takva da $\boldsymbol{\eta}_\varepsilon=\boldsymbol{\eta}_B^\varepsilon$ na S_B, $\boldsymbol{\eta}_\varepsilon=\boldsymbol{\eta}_T^\varepsilon$ na S_T, $\boldsymbol{\eta}_\varepsilon=\boldsymbol{0}$ na S_L te vrijede ocjene

$$\begin{aligned}\left\|\boldsymbol{\eta}_\varepsilon\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))},\left\|\frac{\partial\boldsymbol{\eta}_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))}&=\mathcal{O}(\exp(-\sigma/\varepsilon)),\\ \left\|\nabla\boldsymbol{\eta}_\varepsilon\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))},\left\|\nabla\frac{\partial\boldsymbol{\eta}_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))}&=\mathcal{O}(\exp(-\sigma/\varepsilon)).\end{aligned}\tag{3.84}$$

S druge strane, mikrorotacija $\boldsymbol{w}_\varepsilon$ rješava sustav (vidi (3.1)–(3.5)):

$$\begin{aligned}\frac{\partial\boldsymbol{w}_\varepsilon}{\partial t}-\alpha\Delta\boldsymbol{w}_\varepsilon-\beta\nabla\operatorname{div}\boldsymbol{w}_\varepsilon+2a\boldsymbol{w}_\varepsilon&=a\operatorname{rot}\boldsymbol{u}_\varepsilon+\boldsymbol{g}_\varepsilon,\\ \boldsymbol{w}_\varepsilon&=\frac{\alpha_0^\varepsilon}{2}\operatorname{rot}\boldsymbol{u}_\varepsilon\quad\text{na}\quad\partial\Omega_\varepsilon,\\ \boldsymbol{w}_\varepsilon(\cdot,0)&=\boldsymbol{0}.\end{aligned}\tag{3.85}$$

Oduzimanjem (3.81) od (3.85) dobivamo:

$$\begin{aligned}\frac{\partial\boldsymbol{S}_\varepsilon}{\partial t}-\alpha\Delta\boldsymbol{S}_\varepsilon-\beta\nabla\operatorname{div}\boldsymbol{S}_\varepsilon+2a\boldsymbol{S}_\varepsilon&=a\operatorname{rot}(\boldsymbol{u}_\varepsilon-\boldsymbol{u}_\varepsilon^{[1]})-\boldsymbol{H}_\varepsilon,\\ \boldsymbol{S}_\varepsilon&=\frac{\alpha_0^\varepsilon}{2}\operatorname{rot}\boldsymbol{D}_\varepsilon-\boldsymbol{\lambda}_\varepsilon-\boldsymbol{\eta}_B^\varepsilon\quad\text{na}\quad S_B,\\ \boldsymbol{S}_\varepsilon&=\frac{\alpha_0^\varepsilon}{2}\operatorname{rot}\boldsymbol{D}_\varepsilon-\boldsymbol{\lambda}_\varepsilon-\boldsymbol{\eta}_T^\varepsilon\quad\text{na}\quad S_T,\\ \boldsymbol{S}_\varepsilon&=\frac{\alpha_0^\varepsilon}{2}\operatorname{rot}\boldsymbol{D}_\varepsilon-\boldsymbol{\lambda}_\varepsilon\quad\text{na}\quad S_L,\\ \boldsymbol{S}_\varepsilon(\cdot,0)&=\boldsymbol{0}.\end{aligned}\tag{3.86}$$

Neka je $\boldsymbol{S}_\varepsilon^* = \boldsymbol{S}_\varepsilon - \frac{\alpha_0^\varepsilon}{2}\operatorname{rot}\boldsymbol{D}_\varepsilon + \boldsymbol{\lambda}_\varepsilon + \boldsymbol{\eta}_\varepsilon$. Tada je $\boldsymbol{S}_\varepsilon^* = \boldsymbol{0}$ na $\partial\boldsymbol{\Omega}_\varepsilon$, pa množenjem (3.86) s $\boldsymbol{S}_\varepsilon^*$ te integriranjem po $\boldsymbol{\Omega}_\varepsilon$ dobivamo

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_\varepsilon}|\boldsymbol{S}_\varepsilon^*|^2 + \alpha\int_{\Omega_\varepsilon}|\nabla\boldsymbol{S}_\varepsilon^*|^2 + \beta\int_{\Omega_\varepsilon}(\operatorname{div}\boldsymbol{S}_\varepsilon^*)^2 + 2a\int_{\Omega_\varepsilon}|\boldsymbol{S}_\varepsilon^*|^2 \\
&= a\int_{\Omega_\varepsilon}\operatorname{rot}(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^{[1]})\boldsymbol{S}_\varepsilon^* - \frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\frac{\partial\boldsymbol{D}_\varepsilon}{\partial t}\operatorname{rot}\boldsymbol{S}_\varepsilon^* + \int_{\Omega_\varepsilon}\frac{\partial\boldsymbol{\lambda}_\varepsilon}{\partial t}\boldsymbol{S}_\varepsilon^* + \int_{\Omega_\varepsilon}\frac{\partial\boldsymbol{\eta}_\varepsilon}{\partial t}\boldsymbol{S}_\varepsilon^* \\
&+\alpha\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\nabla\operatorname{rot}\boldsymbol{D}_\varepsilon\nabla\boldsymbol{S}_\varepsilon^* - \alpha\int_{\Omega_\varepsilon}\nabla\boldsymbol{\lambda}_\varepsilon\nabla\boldsymbol{S}_\varepsilon^* - \alpha\int_{\Omega_\varepsilon}\nabla\boldsymbol{\eta}_\varepsilon\nabla\boldsymbol{S}_\varepsilon^* \\
&+\beta\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\operatorname{div}\operatorname{rot}\boldsymbol{D}_\varepsilon\operatorname{div}\boldsymbol{S}_\varepsilon^* - \beta\int_{\Omega_\varepsilon}\operatorname{div}\boldsymbol{\lambda}_\varepsilon\operatorname{div}\boldsymbol{S}_\varepsilon^* - \beta\int_{\Omega_\varepsilon}\operatorname{div}\boldsymbol{\eta}_\varepsilon\operatorname{div}\boldsymbol{S}_\varepsilon^* \\
&+2a\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\operatorname{rot}\boldsymbol{D}_\varepsilon\boldsymbol{S}_\varepsilon^* - 2a\int_{\Omega_\varepsilon}\boldsymbol{\lambda}_\varepsilon\boldsymbol{S}_\varepsilon^* - 2a\int_{\Omega_\varepsilon}\boldsymbol{\eta}_\varepsilon\boldsymbol{S}_\varepsilon^* - \int_{\Omega_\varepsilon}\boldsymbol{H}_\varepsilon\boldsymbol{S}_\varepsilon^*.
\end{aligned} \tag{3.87}$$

Uočimo da vrijedi $\operatorname{div}\operatorname{rot}\boldsymbol{D}_\varepsilon = 0$. Preostale izraze s desne strane jednakosti (3.87) ocjenjujemo koristeći Lemu 0.0.1 te ocjene $(3.82)_1$, (3.83) i (3.84):

$$\begin{aligned}
\int_{\Omega_\varepsilon}\operatorname{rot}(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^{[1]})\boldsymbol{S}_\varepsilon^* &\le C\left\|\nabla(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^{[1]})\right\|_{L^2(\Omega_\varepsilon)}\left\|\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon\left\|\nabla(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^{[2]})\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\quad + C\varepsilon\left\|\nabla(\boldsymbol{u}_\varepsilon^{[2]} - \boldsymbol{u}_\varepsilon^{[1]})\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon\left\|\nabla\boldsymbol{D}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^3\left\|\nabla\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}, \\
\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\frac{\partial\boldsymbol{D}_\varepsilon}{\partial t}\operatorname{rot}\boldsymbol{S}_\varepsilon^* &\le C\varepsilon\left\|\frac{\partial\boldsymbol{D}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon}\frac{\partial\boldsymbol{\lambda}^\varepsilon}{\partial t}\boldsymbol{S}_\varepsilon^* &\le \left\|\frac{\partial\boldsymbol{\lambda}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)}\left\|\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon\left\|\frac{\partial\boldsymbol{\lambda}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon^5\left\|\nabla\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon}\frac{\partial\boldsymbol{\eta}_\varepsilon}{\partial t}\boldsymbol{S}_\varepsilon^* &\le \left\|\frac{\partial\boldsymbol{\eta}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)}\left\|\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon\left\|\frac{\partial\boldsymbol{\eta}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon\exp(-\sigma/\varepsilon)\left\|\nabla\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}, \\
\frac{\alpha_0^\varepsilon}{2}\int_{\Omega_\varepsilon}\nabla\operatorname{rot}\boldsymbol{D}_\varepsilon\nabla\boldsymbol{S}_\varepsilon^* &\le C\varepsilon\left\|\nabla\operatorname{rot}\boldsymbol{D}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon}\nabla\boldsymbol{\lambda}_\varepsilon\nabla\boldsymbol{S}_\varepsilon^* &\le C\left\|\nabla\boldsymbol{\lambda}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon^3\left\|\nabla\boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)},
\end{aligned}$$

$$
\begin{aligned}
\int_{\Omega_\varepsilon} \nabla \boldsymbol{\eta}_\varepsilon \nabla \boldsymbol{S}^*_\varepsilon &\leq C \left\|\nabla \boldsymbol{\eta}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \exp(-\sigma/\varepsilon) \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \operatorname{div} \boldsymbol{\lambda}_\varepsilon \operatorname{div} \boldsymbol{S}^*_\varepsilon &\leq C \left\|\nabla \boldsymbol{\lambda}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^3 \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \operatorname{div} \boldsymbol{\eta}_\varepsilon \operatorname{div} \boldsymbol{S}^*_\varepsilon &\leq C \left\|\nabla \boldsymbol{\eta}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \exp(-\sigma/\varepsilon) \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}, \\
\frac{\alpha_0^\varepsilon}{2} \int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{D}_\varepsilon \boldsymbol{S}^*_\varepsilon &\leq C\varepsilon \left\|\nabla \boldsymbol{D}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \left\|\boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^2 \left\|\nabla \boldsymbol{D}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \boldsymbol{\lambda}_\varepsilon \boldsymbol{S}^*_\varepsilon &\leq \left\|\boldsymbol{\lambda}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \left\|\boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^2 \left\|\nabla \boldsymbol{\lambda}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^5 \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \boldsymbol{\eta}_\varepsilon \boldsymbol{S}^*_\varepsilon &\leq C \left\|\boldsymbol{\eta}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \left\|\boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^2 \left\|\nabla \boldsymbol{\eta}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^2 \exp(-\sigma/\varepsilon) \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \boldsymbol{H}_\varepsilon \boldsymbol{S}^*_\varepsilon &\leq C \left\|\boldsymbol{H}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \left\|\boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon \left\|\boldsymbol{H}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^3 \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}.
\end{aligned}
\tag{3.88}
$$

Koristeći ocjene (3.88) te Youngovu nejednakost, iz (3.87) dobivamo ocjenu

$$
\begin{aligned}
&\frac{d}{dt} \int_{\Omega_\varepsilon} |\boldsymbol{S}^*_\varepsilon|^2 + \int_{\Omega_\varepsilon} |\nabla \boldsymbol{S}^*_\varepsilon|^2 \\
\leq{}& C\varepsilon^2 \left\|\nabla \boldsymbol{D}_\varepsilon\right\|^2_{L^2(\Omega_\varepsilon)} + C\varepsilon^2 \left\|\Delta \boldsymbol{D}_\varepsilon\right\|^2_{L^2(\Omega_\varepsilon)} + C\varepsilon^2 \left\|\frac{\partial \boldsymbol{D}_\varepsilon}{\partial t}\right\|^2_{L^2(\Omega_\varepsilon)} + C\varepsilon^6,
\end{aligned}
$$

odnosno integriranjem po t imamo

$$
\begin{aligned}
\sup_{t\in[0,T]} \left\|\boldsymbol{S}^*_\varepsilon(\cdot,t)\right\|^2_{L^2(\Omega_\varepsilon)} + \int_0^T \left\|\nabla \boldsymbol{S}^*_\varepsilon\right\|^2_{L^2(\Omega_\varepsilon)} &\leq C\varepsilon^2 \int_0^T \left\|\nabla \boldsymbol{D}_\varepsilon\right\|^2_{L^2(\Omega_\varepsilon)} \\
+C\varepsilon^2 \int_0^T \left\|\Delta \boldsymbol{D}_\varepsilon\right\|^2_{L^2(\Omega_\varepsilon)} + C\varepsilon^2 \int_0^T &\left\|\frac{\partial \boldsymbol{D}_\varepsilon}{\partial t}\right\|^2_{L^2(\Omega_\varepsilon)} + C\varepsilon^6.
\end{aligned}
\tag{3.89}
$$

Sljedeće, asimptotička aproksimacija $\boldsymbol{u}_\varepsilon^{[2]}$ zadovoljava

$$\begin{aligned}
\frac{\partial \boldsymbol{u}_\varepsilon^{[2]}}{\partial t} - \chi \Delta \boldsymbol{u}_\varepsilon^{[2]} + \nabla p_\varepsilon^{[2]} = a \operatorname{rot} \boldsymbol{w}_\varepsilon^{[1]} + \boldsymbol{f}_\varepsilon + \boldsymbol{E}_\varepsilon, \\
\operatorname{div} \boldsymbol{u}_\varepsilon^{[2]} = \boldsymbol{\pi}_\varepsilon, \\
\boldsymbol{u}_\varepsilon^{[2]} = \boldsymbol{h}_\varepsilon + \boldsymbol{r}_B^\varepsilon \quad \text{na} \quad S_B, \ \boldsymbol{u}_\varepsilon^{[2]} = \boldsymbol{h}_\varepsilon + \boldsymbol{r}_T^\varepsilon \quad \text{na} \quad S_T, \\
\boldsymbol{u}_\varepsilon^{[2]} = \boldsymbol{0} \quad \text{na} \quad S_L, \\
\boldsymbol{u}_\varepsilon^{[2]}(\cdot, 0) = \boldsymbol{0},
\end{aligned} \tag{3.90}$$

pri čemu vrijede ocjene

$$\begin{aligned}
\|\boldsymbol{E}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} = O(\varepsilon^2), \ \|\boldsymbol{\pi}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} = \left\| \varepsilon^2 \frac{\partial \tilde{u}_3^2}{\partial x_3^2} \right\|_{L^2(0,T;L^2(\Omega_\varepsilon)} = O(\varepsilon^3), \\
\|\boldsymbol{r}_B^\varepsilon\|_{L^2(0,T;W^{1/2,2}(S_B))} = O(\exp(-\sigma/\varepsilon)), \quad \|\boldsymbol{r}_T^\varepsilon\|_{L^2(0,T;W^{1/2,2}(S_T))} = O(\exp(-\sigma/\varepsilon)),
\end{aligned} \tag{3.91}$$

te je

$$\begin{aligned}
\boldsymbol{w}_\varepsilon^{[1]}(\boldsymbol{x}, t) = \tilde{\boldsymbol{w}}^0 \left(\frac{\boldsymbol{x}'}{\varepsilon}, t \right) + \varepsilon \tilde{\boldsymbol{w}}^1 \left(\frac{\boldsymbol{x}'}{\varepsilon}, x_3, t \right) + \mathcal{W}^0 \left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3}{\varepsilon}, t \right) + \varepsilon \mathcal{W}^1 \left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3}{\varepsilon}, t \right) \\
+ \mathcal{Z}^0 \left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3 - l}{\varepsilon}, t \right) + \varepsilon \mathcal{Z}^1 \left(\frac{\boldsymbol{x}'}{\varepsilon}, \frac{x_3 - l}{\varepsilon}, t \right).
\end{aligned}$$

Jer norma od $\boldsymbol{\pi}_\varepsilon$ nije dovoljno mala kako bi postigli željene ocjene, konstruiramo korektor divergencije:

$$\boldsymbol{\Psi}_\varepsilon(\boldsymbol{x}, t) = \varepsilon^3 \sum_{i=1}^{2} \Psi_i \Big(\frac{\boldsymbol{x}'}{\varepsilon}, x_3, t \Big) \mathbf{e}_i, \tag{3.92}$$

pri čemu je funkcija $\boldsymbol{\Psi} = (\Psi_1, \Psi_2)$ rješenje problema

$$\begin{aligned}
\operatorname{div}_{y'} \boldsymbol{\Psi} = \frac{\partial \Psi_1}{\partial y_1} + \frac{\partial \Psi_2}{\partial y_2} = \frac{\partial \tilde{u}_3^2}{\partial x_3^2}, \\
\boldsymbol{\Psi} = \boldsymbol{0} \quad \text{na} \quad \partial B_\varepsilon.
\end{aligned} \tag{3.93}$$

Naglasimo da ovdje varijablu x_3 tretiramo kao parametar. Lako je vidjeti da je zadovoljen nužan uvjet kompatibilnosti $\int_B \frac{\partial \tilde{u}_3^2}{\partial x_3^2} = 0$, čime zaključujemo da postoji rješenje problema (3.92)–(3.93). Posebno, vrijede ocjene

$$\begin{aligned}
\|\nabla \boldsymbol{\Psi}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \left\| \nabla \frac{\partial \boldsymbol{\Psi}_\varepsilon}{\partial t} \right\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C\varepsilon^3, \\
\|\Delta \boldsymbol{\Psi}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C\varepsilon^2.
\end{aligned} \tag{3.94}$$

Uvodimo novu oznaku:

$$\hat{\boldsymbol{u}}_\varepsilon^{[2]} = \boldsymbol{u}_\varepsilon^{[2]} - \boldsymbol{\Psi}_\varepsilon.$$

Nadalje, brzina $\boldsymbol{u}_\varepsilon$ zadovoljava sljedeći sustav:

$$\begin{aligned} \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} - \chi\Delta\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon &= a \operatorname{rot} \boldsymbol{w}_\varepsilon + \boldsymbol{f}_\varepsilon, \\ \operatorname{div} \boldsymbol{u}_\varepsilon &= 0, \\ \boldsymbol{u}_\varepsilon = \boldsymbol{h}_\varepsilon \ \text{ na } \ S_B, \ \boldsymbol{u}_\varepsilon &= \boldsymbol{h}_\varepsilon \ \text{ na } \ S_T, \\ \boldsymbol{u}_\varepsilon &= \boldsymbol{0} \ \text{ na } \ S_L, \\ \boldsymbol{u}_\varepsilon(\cdot, 0) &= \boldsymbol{0}. \end{aligned} \tag{3.95}$$

Označimo $\boldsymbol{D}^*_\varepsilon = \boldsymbol{u}_\varepsilon - \hat{\boldsymbol{u}}^{[2]}_\varepsilon$, tada iz (3.90) i (3.95) dobivamo sljedeći sustav za $\boldsymbol{D}^*_\varepsilon$:

$$\begin{aligned} \frac{\partial \boldsymbol{D}^*_\varepsilon}{\partial t} - \chi\Delta\boldsymbol{D}^*_\varepsilon + \nabla r^\varepsilon &= a \operatorname{rot}(\boldsymbol{w}_\varepsilon - \boldsymbol{w}^{[1]}_\varepsilon) - \boldsymbol{E}^*_\varepsilon, \\ \operatorname{div} \boldsymbol{D}^*_\varepsilon &= 0, \\ \boldsymbol{D}^*_\varepsilon &= \boldsymbol{0} \ \text{ na } \ S_L, \\ \boldsymbol{D}^*_\varepsilon = -\boldsymbol{r}^\varepsilon_B + \boldsymbol{\Psi}_\varepsilon \ \text{ na } \ S_B, \ \boldsymbol{D}^*_\varepsilon &= -\boldsymbol{r}^\varepsilon_T + \boldsymbol{\Psi}_\varepsilon \ \text{ na } \ S_T, \\ \boldsymbol{D}^*_\varepsilon &= \boldsymbol{0} \ \text{ na } \ S_L, \\ \boldsymbol{D}^*_\varepsilon(\cdot, 0) &= \boldsymbol{0}, \end{aligned} \tag{3.96}$$

gdje je $\boldsymbol{E}^*_\varepsilon = \boldsymbol{E}_\varepsilon - \frac{\partial \boldsymbol{\Psi}_\varepsilon}{\partial t} + \chi\Delta\boldsymbol{\Psi}_\varepsilon$. Dodatno, uvodimo $\boldsymbol{\zeta}_\varepsilon$ kao rješenje problema

$$\begin{aligned} \operatorname{div} \boldsymbol{\zeta}_\varepsilon &= 0, \\ \boldsymbol{\zeta}_\varepsilon = -\boldsymbol{r}^\varepsilon_B + \boldsymbol{\Psi}_\varepsilon \ \text{ na } \ S_B, \ \boldsymbol{\zeta}_\varepsilon &= -\boldsymbol{r}^\varepsilon_T + \boldsymbol{\Psi}_\varepsilon \ \text{ na } \ S_T, \\ \boldsymbol{\zeta}_\varepsilon &= \boldsymbol{0} \ \text{ na } \ S_L, \end{aligned}$$

pri čemu vrijede sljedeće ocjene:

$$\begin{aligned} \|\nabla\boldsymbol{\zeta}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \left\|\nabla\frac{\partial \boldsymbol{\zeta}_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\le C\varepsilon^3, \\ \|\Delta\boldsymbol{\zeta}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\le C\varepsilon^2. \end{aligned} \tag{3.97}$$

Neka je $\boldsymbol{D}^{**}_\varepsilon = \boldsymbol{D}^*_\varepsilon - \boldsymbol{\zeta}_\varepsilon$. Tada iz (3.96) imamo da $\boldsymbol{D}^{**}_\varepsilon$ zadovoljava

$$\begin{aligned} \frac{\partial \boldsymbol{D}^{**}_\varepsilon}{\partial t} - \chi\Delta\boldsymbol{D}^{**}_\varepsilon + \nabla r_\varepsilon &= a \operatorname{rot}(\boldsymbol{w}_\varepsilon - \boldsymbol{w}^{[1]}_\varepsilon) - \boldsymbol{E}^{**}_\varepsilon, \\ \operatorname{div} \boldsymbol{D}^{**}_\varepsilon &= 0, \\ \boldsymbol{D}^{**}_\varepsilon &= \boldsymbol{0} \ \text{ na } \ \partial\Omega_\varepsilon, \\ \boldsymbol{D}^{**}_\varepsilon(\cdot, 0) &= \boldsymbol{0}, \end{aligned} \tag{3.98}$$

pri čemu je $\boldsymbol{E}^{**}_\varepsilon = \boldsymbol{E}^*_\varepsilon + \frac{\partial \zeta_\varepsilon}{\partial t} - \chi\Delta\boldsymbol{\zeta}_\varepsilon$. Iz (3.91), (3.94) i (3.97) slijedi ocjena

$$\|\boldsymbol{E}^{**}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \le C\varepsilon^2. \tag{3.99}$$

Istim metodama kao i u Teoremu 3.4.1 možemo pokazati da za rješenje problema (3.98) vrijedi

$$\begin{aligned}
&\left\|\frac{\partial \boldsymbol{D}_\varepsilon^{**}}{\partial t}\right\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + \left\|\Delta \boldsymbol{D}_\varepsilon^{**}\right\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} \\
&\qquad \le C \left\|\nabla(\boldsymbol{w}_\varepsilon - \boldsymbol{w}_\varepsilon^{[1]})\right\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + C\left\|\nabla \boldsymbol{E}_\varepsilon^{**}\right\|^2_{L^2(\Omega_\varepsilon))} \\
&\qquad \le C \left\|\nabla \boldsymbol{S}_\varepsilon\right\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + C\left\|\nabla(\boldsymbol{w}_\varepsilon^{[2]} - \boldsymbol{w}_\varepsilon^{[1]})\right\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + C\varepsilon^4 \\
&\qquad \le C \left\|\nabla \boldsymbol{S}_\varepsilon^{*}\right\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + C\varepsilon^2\left\|\Delta \boldsymbol{D}_\varepsilon^{**}\right\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + C\varepsilon^4,
\end{aligned}$$

gdje smo iskoristili (3.83), (3.84) te (3.99). Za dovoljno mali ε time dobivamo

$$\left\|\frac{\partial \boldsymbol{D}_\varepsilon^{**}}{\partial t}\right\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + \left\|\Delta \boldsymbol{D}_\varepsilon^{**}\right\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} \le C\left\|\nabla \boldsymbol{S}_\varepsilon^{*}\right\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + C\varepsilon^4. \tag{3.100}$$

Sada iz (3.89), (3.94), (3.97) i (3.100) za dovoljno mali ε slijedi ocjena

$$\sup_{t\in[0,T]} \|\boldsymbol{S}_\varepsilon^*(\cdot,t)\|^2_{L^2(\Omega_\varepsilon)} + \int_0^T \|\nabla \boldsymbol{S}_\varepsilon^*\|^2_{L^2(\Omega_\varepsilon)} \le C\varepsilon^2 \int_0^T \left\|\nabla \boldsymbol{D}_\varepsilon^{**}\right\|^2_{L^2(\Omega_\varepsilon)} + C\varepsilon^6. \tag{3.101}$$

Nadalje, množimo (3.98) s $\boldsymbol{D}_\varepsilon^{**}$ i integriramo po $\boldsymbol{\Omega}_\varepsilon$, čime dobivamo:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_\varepsilon} |\boldsymbol{D}_\varepsilon^{**}|^2 + \chi \int_{\Omega_\varepsilon} |\nabla \boldsymbol{D}_\varepsilon^{**}|^2 = a \int_{\Omega_\varepsilon} \operatorname{rot}(\boldsymbol{w}_\varepsilon - \boldsymbol{w}_\varepsilon^{[1]})\boldsymbol{D}_\varepsilon^{**} - \int_{\Omega_\varepsilon} \boldsymbol{E}_\varepsilon^{**}\boldsymbol{D}_\varepsilon^{**}. \tag{3.102}$$

Izraze s desne strane jednakosti (3.102) ocjenjujemo pomoću Teorema 0.0.1 te ocjene (3.99) na sljedeći način:

$$\begin{aligned}
\int_{\Omega_\varepsilon} \operatorname{rot}(\boldsymbol{w}_\varepsilon - \boldsymbol{w}_\varepsilon^{[1]})\boldsymbol{D}_\varepsilon^{**} &\le \left\|\operatorname{rot}(\boldsymbol{w}_\varepsilon - \boldsymbol{w}_\varepsilon^{[1]})\right\|_{L^2(\Omega_\varepsilon)} \left\|\boldsymbol{D}_\varepsilon^{**}\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon \left(\|\nabla \boldsymbol{S}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \left\|\nabla(\boldsymbol{w}_\varepsilon^{[2]} - \boldsymbol{w}_\varepsilon^{[1]})\right\|_{L^2(\Omega_\varepsilon)}\right) \left\|\nabla \boldsymbol{D}_\varepsilon^{**}\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon \left(\left\|\nabla \boldsymbol{S}_\varepsilon^*\right\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^2\right) \left\|\nabla \boldsymbol{D}_\varepsilon^{**}\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \boldsymbol{E}_\varepsilon^{**}\boldsymbol{D}_\varepsilon^{**} &\le \left\|\boldsymbol{E}_\varepsilon^{**}\right\|_{L^2(\Omega_\varepsilon)} \left\|\boldsymbol{D}_\varepsilon^{**}\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon \left\|\boldsymbol{E}_\varepsilon^{**}\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \boldsymbol{D}_\varepsilon^{**}\right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon^3 \left\|\nabla \boldsymbol{D}_\varepsilon^{**}\right\|_{L^2(\Omega_\varepsilon)}.
\end{aligned} \tag{3.103}$$

Sada primjenjujemo ocjene (3.103) te Youngovu nejednakost na desnu stranu nejednakosti (3.102) čime za dovoljno mali ε dobivamo

$$\frac{d}{dt}\int_{\Omega_\varepsilon} |\boldsymbol{D}_\varepsilon^{**}|^2 + \int_{\Omega_\varepsilon} |\nabla \boldsymbol{D}_\varepsilon^{**}|^2 \le C\varepsilon^2 \left\|\nabla \boldsymbol{S}_\varepsilon^*\right\|^2_{L^2(\Omega_\varepsilon)} + C\varepsilon^6. \tag{3.104}$$

Integriranjem (3.104) po t te korištenjem ocjene (3.101), za dovoljno mali ε imamo ocjenu

$$\sup_{t\in[0,T]} \left\| \boldsymbol{D}_{\varepsilon}^{**}(\cdot,t) \right\|^2 + \left\| \nabla \boldsymbol{D}_{\varepsilon}^{**} \right\|^2 \leq C\varepsilon^6. \tag{3.105}$$

Konačno, iz (3.101) i (3.105) imamo

$$\sup_{t\in[0,T]} \|\boldsymbol{S}_{\varepsilon}^{*}(\cdot,t)\|^2_{L^2(\Omega_\varepsilon)} + \int_0^T \|\nabla \boldsymbol{S}_{\varepsilon}^{*}\|^2_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^6. \tag{3.106}$$

Još je potrebno ocijeniti razliku tlakova r_ε. Neka je $\boldsymbol{d}_\varepsilon$ takva da (vidi Lemu 0.0.2)

$$\operatorname{div} \boldsymbol{d}_\varepsilon = r_\varepsilon + \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} p_\varepsilon^{[2]},$$

$$\boldsymbol{d}_\varepsilon = \boldsymbol{0} \quad \text{na} \quad \partial\Omega_\varepsilon$$

te vrijedi ocjena

$$\|\nabla \boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon} \left\| r_\varepsilon + \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} p_\varepsilon^{[2]} \right\|_{L^2(\Omega_\varepsilon)}. \tag{3.107}$$

Množenjem (3.98) s $\boldsymbol{d}_\varepsilon$ te integriranjem po $\boldsymbol{\Omega}_\varepsilon$ imamo

$$\begin{gathered}\int_{\Omega_\varepsilon} \left(r_\varepsilon + \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} p_\varepsilon^{[2]} \right)^2 = \int_{\Omega_\varepsilon} \frac{\partial \boldsymbol{D}_\varepsilon^{**}}{\partial t} \cdot \boldsymbol{d}_\varepsilon + \chi \int_{\Omega_\varepsilon} \nabla \boldsymbol{D}_\varepsilon^{**} \cdot \nabla \boldsymbol{d}_\varepsilon \\ -a \int_{\Omega_\varepsilon} \operatorname{rot}(\boldsymbol{w}_\varepsilon - \boldsymbol{w}_\varepsilon^{[1]}) \cdot \boldsymbol{d}_\varepsilon + \int_{\Omega_\varepsilon} \boldsymbol{E}_\varepsilon^{**} \cdot \boldsymbol{d}_\varepsilon.\end{gathered} \tag{3.108}$$

Desnu stranu jednakosti (3.108) ocjenjujemo koristeći ocjene (3.99), (3.100), (3.105), (3.106) i (3.107). Konačno, integriranjem dobivenog imamo ocjenu

$$\|r_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C\varepsilon^2. \tag{3.109}$$

Sada ocjene (3.80) slijede iz (3.105), (3.106), (3.109) te ocjena (3.83), (3.84), (3.94) i (3.97).

■

4. Asimptotička analiza nestacionarnog mikropolarnog fluida u sustavu cijevi

U inženjerskoj praksi potrebno je opsežno znanje toka fluida u sustavu gdje je više cijevi međusobno povezano tvoreći razgranatu strukturu, poput vodoopskrbnih sustava cijevi ili krvožilnog sustava. Takve cijevi su često jako tanke ili jako dugačke, te se u čvorištima granaju na dvije ili više cijevi s potencijalno različitim profilima. Radi složenosti geometrije proučavane domene, modeliranje toka fluida u sustavu cijevi je zahtjevan problem s ciljem pronalaska pojednostavljenog modela visokog reda točnosti koji bi bio prigodan za numeričke simulacije.

Stacionarni tok Newtonovskog fluida u sustavu tankih cijevi je razmatran u [54, 60], dok su asimptotički modeli za nestacionarni Newtonovski fluid u sustavu tankih cijevi predloženi i rigorozno opravdani u [71, 72]. Nadalje, stacionarni tok mikropolarnog fluida u sustavu tankih cijevi je proučavan u [9], a rezultati za tok u tankoj cijevi u nestacionarnom režimu se mogu pronaći u [12, 78]. U ovom poglavlju je cilj nadopuniti postojeću literaturu pronalaskom i rigoroznim opravdanjem asimptotičkog modela nestacionarnog mikropolarnog toka fluida u sustavu tankih cijevi.

U daljnjem ćemo se ograničiti na proučavanje jednog čvorišta te m ravnih cijevi s kružnim profilom, pri čemu promjeri mogu biti različiti (vidi sliku 4.1). Naglasimo da se analiza ovog slučaja može proširiti i na općenitiji sustav cijevi s više čvorišta.

Poglavlje je organizirano na sljedeći način. Najprije u Odjeljku 4.1 opisujemo promatranu domenu koja se sastoji od m tankih cijevi spojenih u čvorištu. Uvodimo sustav jednadžbi koji opisuje tok nestacionarnog mikropolarnog fluida i zadajemo pripadne rubne i inicijalne uvjete, te potom razmatramo dobru postavljenost proučavanog problema. Kao što je pokazano u [76], asimptotičko ponašanje mikropolarnog toka u cijevi ovisi o redu veličine viskoznih koeficijenata u usporedbi s malim parametrom ε, koji opisuje omjer debljine i duljine cijevi. Iz tog razloga u Odjeljku 4.2 promatrani problem zapisujemo u bezdimenzionalnom obliku te uspoređujemo karakteristične bezdimenzionalne brojeve koji se pojavljuju u jednadžbi očuvanja kutne količine gibanja s malim parametrom ε. Na taj način dobivamo 3 različita modela u cijevima daleko od čvorišta. U ostatku poglavlja promatramo najzanimljiviji model, to jest slučaj karakteriziran jakom uparenosti između brzine i mikrorotacije. Dobivenu aproksimaciju popravljamo u rubnom sloju na krajevima cijevi dodajući korektore rubnog sloja. Nadalje, jer se fluid unutar čvorišta ponaša drugačije nego u cijevima, uvodimo korektore unutarnjeg sloja. U tu svrhu u Dodatku B dokazujemo apstraktni rezultat o eksponencijalnom padu u prostorima težinskih funkcija, po uzoru na [85]. Konačno, u Odjeljku 4.3 predlažemo asimptotičku aproksimaciju rješenja proizvoljnog reda točnosti, te dokazujemo ocjene greške kako bismo rigorozno opravdali predloženi model.

4.1. Postavka problema

Definiramo domenu Ω_ε koja se sastoji od m cijevi Ω_ε^i, $i = 1, 2, \dots, m$ duljine ℓ_i, gdje se cijevi spajaju u čvorištu Ω_ε^0 koje sadrži ishodište O. Cijevi imaju kružne presjeke B_ε^i s potencijalno različitim promjerom, to jest presjek i-te cijevi je oblika

$$B_\varepsilon^i = \varepsilon B^i = \varepsilon\{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| < r_i\}.$$

Označimo s $\{O, (\boldsymbol{e}_k^i)_{k=1,2,3}\}$ lokalne koordinate koje pripadaju i-toj cijevi sustava, za $i = 1, 2, \dots, m$. Tada je i-ta cijev sustava dana s:

$$\Omega_\varepsilon^i = \{(x_1^i, \boldsymbol{x}_*^i) \in \mathbb{R}^3 : 0 < x_1^i < \ell_i,\ \boldsymbol{x}_*^i = (x_2^i, x_3^i) \in B_\varepsilon^i\}.$$

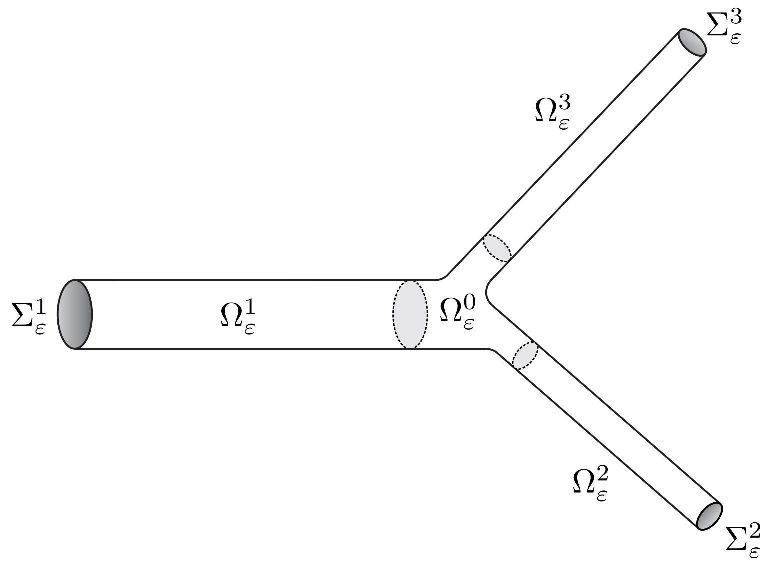
U daljnjem tekstu globalne koordinate označavamo s $\boldsymbol{x}$, a lokalne koordinate s $\boldsymbol{x}^i$. Čvorište u kojem se cijevi spajaju je oblika $\Omega_\varepsilon^0 = \varepsilon\Omega^0$, pri čemu je Ω^0 ograničena C^2 domena. Pretpostavljamo da se svi presjeci cijevi Ω_ε^i nalaze u čvorištu Ω_ε^0. Sada je promatrana domena definirana na sljedeći način:

$$\Omega_\varepsilon = \bigcup_{i=0}^{m} \Omega_\varepsilon^i.$$

Kraj i-te cijevi označavamo s

$$\Sigma_\varepsilon^i = \{(\ell_i, \boldsymbol{x}_*^i) \in \mathbb{R}^3 : \boldsymbol{x}_*^i \in B_\varepsilon^i\},$$

za $i = 1, 2, \ldots, m$, dok je lateralni rub domene Ω_ε dan s $\Gamma_\varepsilon = \partial\Omega_\varepsilon \setminus \left(\cup_{i=1}^m \Sigma_\varepsilon^i\right)$.



Slika 4.1: Opisani sustav cijevi za $m = 3$.

Sustav jednadžbi koji opisuje nestacionaran tok viskoznog mikropolarnog fluida je dan s:

$$\begin{aligned}
\rho\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} + \rho(\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon - (\mu + \mu_r)\Delta\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon &= 2\mu_r \operatorname{rot} \boldsymbol{w}_\varepsilon + \rho \boldsymbol{f}_\varepsilon, \\
\operatorname{div} \boldsymbol{u}_\varepsilon &= 0, \\
\rho I\frac{\partial \boldsymbol{w}_\varepsilon}{\partial t} + \rho I(\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{w}_\varepsilon - (c_\alpha + c_d)\Delta\boldsymbol{w}_\varepsilon - (c_0 + c_d - c_\alpha)\nabla \operatorname{div} \boldsymbol{w}_\varepsilon + 4\mu_r \boldsymbol{w}_\varepsilon & \\
= 2\mu_r \operatorname{rot} \boldsymbol{u}_\varepsilon + \rho \boldsymbol{g}_\varepsilon, &
\end{aligned} \tag{4.1}$$

gdje je $\boldsymbol{u}_\varepsilon(\boldsymbol{x}, t) = (u_1^\varepsilon(\boldsymbol{x}, t), u_2^\varepsilon(\boldsymbol{x}, t), u_3^\varepsilon(\boldsymbol{x}, t))$ brzina, $\boldsymbol{w}_\varepsilon(\boldsymbol{x}, t) = (w_1^\varepsilon(\boldsymbol{x}, t), w_2^\varepsilon(\boldsymbol{x}, t), w_3^\varepsilon(\boldsymbol{x}, t))$ mikrorotacija, te $p_\varepsilon(\boldsymbol{x}, t)$ tlak. Pozitivne konstante u (4.1) su sljedeće: μ je Newtonovska dinamička viskoznost, μ_r je mikrorotacijska dinamička viskoznost, I je koeficijent mikroinercije, c_0, c_α, i c_d su koeficijenti kutnih viskoznosti te ρ gustoća. Izvori linearnog i kutnog momenta su dani s $\boldsymbol{f}_\varepsilon(\boldsymbol{x}, t) = (f_1^\varepsilon(\boldsymbol{x}, t), f_2^\varepsilon(\boldsymbol{x}, t), f_3^\varepsilon(\boldsymbol{x}, t))$ i $\boldsymbol{g}_\varepsilon(\boldsymbol{x}, t) = (g_1^\varepsilon(\boldsymbol{x}, t), g_2^\varepsilon(\boldsymbol{x}, t), g_3^\varepsilon(\boldsymbol{x}, t))$.

Zadajemo sljedeće rubne uvjete:

$$\begin{aligned}
\boldsymbol{u}_\varepsilon &= \boldsymbol{0} \quad \text{na} \quad \Gamma_\varepsilon, \\
\boldsymbol{u}_\varepsilon(\ell_i, \boldsymbol{x}_*^i, t) &= \boldsymbol{h}_\varepsilon^i\left(\frac{\boldsymbol{x}_*^i}{\varepsilon}, t\right) \quad \text{na} \quad \Sigma_\varepsilon^i, \; i = 1, 2, \ldots, m, \\
\boldsymbol{w}_\varepsilon &= \boldsymbol{0} \quad \text{na} \quad \partial\Omega_\varepsilon,
\end{aligned} \tag{4.2}$$

te inicijalne uvjete:

$$\boldsymbol{u}_\varepsilon(\boldsymbol{x},0)=\boldsymbol{0},\quad \boldsymbol{w}_\varepsilon(\boldsymbol{x},0)=\boldsymbol{0}\ \text{ u }\ \Omega_\varepsilon. \tag{4.3}$$

Funkcije $\boldsymbol{h}^i_\varepsilon$ definirane na krajevima cijevi Σ^i_ε su dane u obliku

$$\boldsymbol{h}^i_\varepsilon\left(\frac{\boldsymbol{x}^i_*}{\varepsilon},t\right)=\sum_{j=0}^{J}\varepsilon^{j+2}\boldsymbol{h}^{i,j}\left(\frac{\boldsymbol{x}^i_*}{\varepsilon},t\right),\ i=1,2,\dots,m, \tag{4.4}$$

pri čemu je $\boldsymbol{h}^{i,j}\in C^2([0,T];W^{2,2}(B^i))$, za $i=1,2,\dots,m$, $j=0,1,\dots,J$. Nadalje, neka funkcije $\boldsymbol{h}^{i,j}$ iščezavaju u okolini $t=0$, to jest neka su jednake nuli na $[0,t_*]$, za neki mali $t_*>0$. Dodatno, uvodimo sljedeću notaciju:

$$F^{i,j}_\varepsilon(t):=\int_{B^i_\varepsilon}\boldsymbol{h}^{i,j}\left(\frac{\boldsymbol{x}^i_*}{\varepsilon},t\right)d\boldsymbol{x}^i_*=\varepsilon^2\int_{B^i}\boldsymbol{h}^{i,j}\left(\boldsymbol{y}^i_*,t\right)d\boldsymbol{y}^i_*=\varepsilon^2F^{i,j}(t), \tag{4.5}$$

$i=1,2,\dots,m$, $j=0,1,\dots,J$, gdje je $\boldsymbol{y}^i_*=\frac{\boldsymbol{x}^i_*}{\varepsilon}$. Da bi nužni uvjeti kompatibilnosti bili zadovoljeni, pretpostavljamo da vrijedi sljedeće:

$$\sum_{i=1}^{m}F^{i,j}_\varepsilon(t)=0,\ \forall t\in[0,T],\ j=0,1,\dots,J. \tag{4.6}$$

Kako smo pretpostavili da sve funkcije $\boldsymbol{h}^{i,j}$ iščezavaju u okolini $t=0$ za $i=1,2,\dots,m$, $j=0,1,\dots,J$, također vrijedi $F^{i,j}_\varepsilon(0)=0$ pa je automatski zadovoljen i uvjet kompatibilnosti koji dolazi od inicijalnih uvjeta (4.3). Konačno, pretpostavimo da funkcije $\boldsymbol{f}_\varepsilon$ i $\boldsymbol{g}_\varepsilon$ također iščezavaju blizu $t=0$. Na ovaj način će asimptotička aproksimacija automatski zadovoljavati inicijalne uvjete te time izbjegavamo fenomen rubnog sloja u vremenu. Međutim, pojavljuje se rubni sloj u prostoru koji ćemo analizirati u Odjeljku 4.2.2.

Komentirajmo najprije dobru postavljenost promatranog problema (4.1)–(4.3).

Rubne funkcije $\boldsymbol{h}^i_\varepsilon$ možemo proširiti do funkcije $\boldsymbol{h}^{ext}_\varepsilon\in C^2([0,T];W^{2,2}(\Omega_\varepsilon))$ takve da vrijedi:

$$\operatorname{div}\boldsymbol{h}^{ext}_\varepsilon=0\ \text{ u }\ \Omega_\varepsilon,$$

$$\boldsymbol{h}^{ext}_\varepsilon=0\ \text{ na }\ \Gamma_\varepsilon,$$

$$\boldsymbol{h}^{ext}_\varepsilon=\boldsymbol{h}^i_\varepsilon\ \text{ na }\ \Sigma^i_\varepsilon,\ i=1,2,\dots,m,$$

te su zadovoljene sljedeće nejednakosti za svaki $t\in[0,T]$:

$$\begin{aligned}\left\|\boldsymbol{h}^{ext}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}+\left\|\frac{\partial\boldsymbol{h}^{ext}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)}&\le C\varepsilon^3,\\ \left\|\nabla\boldsymbol{h}^{ext}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}+\left\|\nabla\frac{\partial\boldsymbol{h}^{ext}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)}&\le C\varepsilon^2,\end{aligned} \tag{4.7}$$

gdje je konstanta C neovisna o malom parametru ε. Ovdje smo iskoristili Lemu 0.0.3 te uzeli u obzir oblik funkcija $\boldsymbol{h}_\varepsilon^i$ dan s (4.4).

Slabim rješenjem problema (4.1)–(4.3) smatramo par funkcija $(\boldsymbol{u}_\varepsilon, \boldsymbol{w}_\varepsilon) = (\boldsymbol{v}_\varepsilon + \boldsymbol{h}_\varepsilon^{ext}, \boldsymbol{w}_\varepsilon)$, gdje $(\boldsymbol{v}_\varepsilon, \boldsymbol{w}_\varepsilon)$ pripadaju prostorima

$$\begin{aligned} \boldsymbol{v}_\varepsilon \in L^2(0,T;V) \cap L^\infty(0,T;H), \quad \frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} \in L^2(0,T;H), \\ \boldsymbol{w}_\varepsilon \in L^2(0,T;W_0^{1,2}(\Omega_\varepsilon)) \cap L^\infty(0,T;L^2(\Omega_\varepsilon)), \quad \frac{\partial \boldsymbol{w}_\varepsilon}{\partial t} \in L^2(0,T;L^2(\Omega_\varepsilon)), \end{aligned} \tag{4.8}$$

te zadovoljavaju integralne jednakosti

$$\begin{aligned} &\rho \int_{\Omega_\varepsilon} \frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} \cdot \boldsymbol{\varphi} + \rho \int_{\Omega_\varepsilon} (\boldsymbol{v}_\varepsilon \cdot \nabla)\boldsymbol{v}_\varepsilon \cdot \boldsymbol{\varphi} + \rho \int_{\Omega_\varepsilon} (\boldsymbol{h}_\varepsilon^{ext} \cdot \nabla)\boldsymbol{v}_\varepsilon \cdot \boldsymbol{\varphi} + \rho \int_{\Omega_\varepsilon} (\boldsymbol{v}_\varepsilon \cdot \nabla)\boldsymbol{h}_\varepsilon^{ext} \cdot \boldsymbol{\varphi} \\ &+(\mu + \mu_r) \int_{\Omega_\varepsilon} \nabla \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{\varphi} = 2\mu_r \int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{w}_\varepsilon \cdot \boldsymbol{\varphi} - \rho \int_{\Omega_\varepsilon} \left(\frac{\partial \boldsymbol{h}_\varepsilon^{ext}}{\partial t} + (\boldsymbol{h}_\varepsilon^{ext} \cdot \nabla)\boldsymbol{h}_\varepsilon^{ext} \right) \cdot \boldsymbol{\varphi} \\ &\qquad -(\mu + \mu_r) \int_{\Omega_\varepsilon} \nabla \boldsymbol{h}_\varepsilon^{ext} \cdot \nabla \boldsymbol{\varphi} + \rho \int_{\Omega_\varepsilon} \boldsymbol{f}_\varepsilon \cdot \boldsymbol{\varphi}, \\ &\rho I \int_{\Omega_\varepsilon} \frac{\partial \boldsymbol{w}_\varepsilon}{\partial t} \cdot \boldsymbol{\Psi} + \rho I \int_{\Omega_\varepsilon} (\boldsymbol{v}_\varepsilon \cdot \nabla)\boldsymbol{w}_\varepsilon \cdot \boldsymbol{\Psi} + (c_\alpha + c_d) \int_{\Omega_\varepsilon} \nabla \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{\Psi} \\ &+(c_0 + c_d - c_\alpha) \int_{\Omega_\varepsilon} \operatorname{div} \boldsymbol{w}_\varepsilon \operatorname{div} \boldsymbol{\Psi} + 4\mu_r \int_{\Omega_\varepsilon} \boldsymbol{w}_\varepsilon \cdot \boldsymbol{\Psi} = 2\mu_r \int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{v}_\varepsilon \cdot \boldsymbol{\Psi} \\ &+2\mu_r \int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{h}_\varepsilon^{ext} \cdot \boldsymbol{\Psi} - \rho I \int_{\Omega_\varepsilon} (\boldsymbol{h}_\varepsilon^{ext} \cdot \nabla)\boldsymbol{w}_\varepsilon \cdot \boldsymbol{\Psi} + \rho \int_{\Omega_\varepsilon} \boldsymbol{g}_\varepsilon \cdot \boldsymbol{\Psi}, \end{aligned} \tag{4.9}$$

za sve $\boldsymbol{\varphi} \in V$ i za sve $\boldsymbol{\Psi} \in W_0^{1,2}(\Omega_\varepsilon)$, kao i inicijalne uvjete:

$$\boldsymbol{v}_\varepsilon(\boldsymbol{x}, 0) = \boldsymbol{0}, \quad \boldsymbol{w}_\varepsilon(\boldsymbol{x}, 0) = \boldsymbol{0} \quad \text{u} \quad \Omega_\varepsilon. \tag{4.10}$$

Dobra postavljenost i viša regularnost problema (4.8)–(4.10) za $0 < T < \infty$ je uspostavljena u [55], za $\boldsymbol{f}_\varepsilon, \boldsymbol{g}_\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon))$. U ostatku poglavlja glavni cilj je provesti rigoroznu asimptotičku analizu problema (4.1)–(4.3) s obzirom na mali parametar ε.

4.2. Asimptotička analiza

Označimo s $r > 0$ radijus skupa Ω^0, to jest

$$r = \sup_{x \in \Omega^0} \|x\|_2,$$

te neka je ζ glatka tzv. "cut-off" funkcija takva da je $\zeta(s) = 0$ za $s \leq 1$ te $\zeta(s) = 1$ za $s \geq 2$. Nadalje, definiramo

$$\Omega^i = \{(x_1^i, y_*^i) \in \mathbb{R}^3 : 0 < x_1^i < \ell_i,\ y_*^i \in B^i\}$$

te označimo $\Omega = \cup_{i=0}^m \Omega^i$. U daljnjem pretpostavljamo da se funkcije f_ε i g_ε mogu zapisati na sljedeći način:

$$f_\varepsilon(x,t) = \sum_{i=1}^m \zeta\left(\frac{x_1^i}{r\varepsilon}\right) \tilde{f}_\varepsilon^i\left(\frac{x_*^i}{\varepsilon}, t\right) + f_{\varepsilon,int}\left(\frac{x}{\varepsilon}, t\right),$$
$$g_\varepsilon(x,t) = \sum_{i=1}^m \zeta\left(\frac{x_1^i}{r\varepsilon}\right) \tilde{g}_\varepsilon^i\left(\frac{x_*^i}{\varepsilon}, t\right) + g_{\varepsilon,int}\left(\frac{x}{\varepsilon}, t\right),$$

pri čemu je

$$\tilde{f}_\varepsilon^i = \sum_{j=0}^J \varepsilon^j f_1^{i,j} e_1^i,\ i = 1, 2, \ldots, m,$$

$$\tilde{g}_\varepsilon^i = \sum_{j=0}^J \varepsilon^{j+1} g^{i,j}, \quad g^{i,j} = g_2^{i,j} e_2^i + g_3^{i,j} e_3^i,\ i = 1, 2, \ldots, m,\ j = 0, 1, \ldots, J, \tag{4.11}$$

$$f_{\varepsilon,int} = \sum_{j=0}^J \varepsilon^j f_{int}^j, \quad g_{\varepsilon,int} = \sum_{j=0}^J \varepsilon^{j+1} g_{int}^j.$$

Funkcije u (4.11) pripadaju sljedećim prostorima: $f_1^{i,j}, g_2^{i,j}, g_3^{i,j} \in L^2(0,T; L^2(B^i))$ za $i = 1, 2, \ldots, m,\ j = 0, 1, \ldots, J$ i $f_{int}^j, g_{int}^j \in L^2(0,T; \mathcal{L}_\beta^2(\Omega))$ za $j = 0, 1, \ldots, J$. Ovdje je $\mathcal{L}_\beta^2(\Omega)$ prostor težinskih funkcija definiran u Dodatku B.

Kao u [9], uvodimo zamjenu varijabli

$$\hat{x} = \frac{x}{L}, \quad \hat{t} = \frac{\mu + \mu_r}{\rho L^2} t,$$
$$\hat{u}_\varepsilon = \frac{u_\varepsilon}{U_{ref}}, \quad \hat{w}_\varepsilon = \frac{L}{U_{ref}} w_\varepsilon, \quad \hat{p}_\varepsilon = \frac{L}{(\mu + \mu_r) U_{ref}} p_\varepsilon, \tag{4.12}$$
$$\hat{F}^{i,j} = \frac{F^{i,j}}{U_{ref} L^2}, \quad \hat{f}_\varepsilon = \frac{\rho L^2}{U_{ref}(\mu + \mu_r)} f_\varepsilon, \quad \hat{g}_\varepsilon = \frac{\rho L}{U_{ref}(\mu + \mu_r)} g_\varepsilon,$$

gdje je $L = \max_{i=1,2,\ldots,m} \ell_i$ karakteristična duljina sustava cijevi, te je U_{ref} karakteristična vrijednost brzine. Tada promatrani sustav jednadžbi (4.1) zapisan u bezdimenzionalnom obliku glasi

$$\begin{aligned}
\frac{\partial \hat{\boldsymbol{u}}_\varepsilon}{\partial \hat{t}} + K(\hat{\boldsymbol{u}}_\varepsilon \cdot \nabla)\hat{\boldsymbol{u}}_\varepsilon - \Delta \hat{\boldsymbol{u}}_\varepsilon + \nabla \hat{p}_\varepsilon &= 2N \operatorname{rot} \hat{\boldsymbol{w}}_\varepsilon + \hat{\boldsymbol{f}}_\varepsilon, \\
\operatorname{div} \hat{\boldsymbol{u}}_\varepsilon &= 0, \qquad (4.13)\\
M_\varepsilon \frac{\partial \hat{\boldsymbol{w}}_\varepsilon}{\partial \hat{t}} + KM_\varepsilon(\hat{\boldsymbol{u}}_\varepsilon \cdot \nabla)\hat{\boldsymbol{w}}_\varepsilon - R_{1\varepsilon}\Delta \hat{\boldsymbol{w}}_\varepsilon - R_{2\varepsilon}\nabla \operatorname{div} \hat{\boldsymbol{w}}_\varepsilon + 4N\hat{\boldsymbol{w}}_\varepsilon &= 2N \operatorname{rot} \hat{\boldsymbol{u}}_\varepsilon + \hat{\boldsymbol{g}}_\varepsilon.
\end{aligned}$$

Pozitivne bezdimenzionalne konstante koje se pojavljuju u (4.13) su dane s

$$K = \frac{\rho U_{ref} L}{\mu + \mu_r}, \quad M_\varepsilon = \frac{I}{L^2}, \quad R_{1\varepsilon} = \frac{c_\alpha + c_d}{(\mu + \mu_r)L^2}, \quad R_{2\varepsilon} = \frac{c_0 + c_d - c_\alpha}{(\mu + \mu_r)L^2}, \quad N = \frac{\mu_r}{\mu + \mu_r}.$$

Karakteristični broj M_ε opisuje vezu između mikroinercije i geometrije sustava cijevi, K mjeri koliko dobro viskozne sile nadoknađuju inercijske sile, $R_{1\varepsilon}$ i $R_{2\varepsilon}$ opisuju vezu između viskoznosti i geometrije domene, dok N opisuje vezu između Newtonovske i mikrorotacijske viskoznosti i karakterizira uparenost jednadžbi za brzinu i jednadžbi za mikrorotaciju, te je reda veličine $\mathcal{O}(1)$.

Prema [76], koeficijenti $R_{k\varepsilon}$, $k = 1, 2$ se mogu uspoređivati s malim parametrom ε što nas vodi do tri različita asimptotička ponašanja. U Odjeljku 4.2.1 ćemo izvesti regularni dio asimptotičke aproksimacije za sva tri slučaja.

Napomenimo da ćemo u ostatku poglavlja postaviti gustoću fluida na 1 kako bismo pojednostavili notaciju. Nadalje, pretpostavljamo da vrijedi $M_\varepsilon = \varepsilon^2 M$, gdje je $M = \mathcal{O}(1)$, što je u skladu s opisanim primjenama.

4.2.1. Tok daleko od čvorišta

Sada uvodimo brzu varijablu $\hat{\boldsymbol{y}}^i_* = (\hat{y}^i_2, \hat{y}^i_3) = \left(\frac{\hat{x}^i_2}{\varepsilon}, \frac{\hat{x}^i_3}{\varepsilon}\right)$, te tražimo aproksimaciju rješenja problema (4.13) u i-toj cijevi u obliku $\hat{\boldsymbol{u}}_\varepsilon = \varepsilon^2 \hat{\boldsymbol{u}}^i$, $\hat{\boldsymbol{w}}_\varepsilon = \varepsilon \hat{\boldsymbol{w}}^i$, $\hat{p}_\varepsilon = \hat{p}^i$. Dobivene reskalirane cijevi su neovisne o parametru ε te ih označavamo s

$$\hat{\Omega}^i = \left\{(\hat{x}^i_1, \hat{\boldsymbol{y}}^i_*) \in \mathbb{R}^3 : 0 < \hat{x}^i_1 < \hat{\ell}_i, \ \hat{\boldsymbol{y}}^i_* \in \hat{B}^i\right\},$$

gdje je $\hat{\ell}_i := \frac{\ell_i}{L}$ te $\hat{B}^i = \{\hat{\boldsymbol{y}}^i_* \in \mathbb{R}^2 : |\hat{\boldsymbol{y}}^i_*| < \hat{r}_i\}$, $\hat{r}_i := \frac{r_i}{L}$, za $i = 1, 2, \ldots, m$.

Kako bismo pojednostavili notaciju, izostavit ćemo indeks i kada radimo s bezdimenzionalnim funkcijama. Sada bezdimenzionalne jednadžbe u reskaliranoj cijevi $\hat{\Omega}$ daleko od čvorišta glase:

$$
\begin{aligned}
&\varepsilon^2\frac{\partial \hat{\boldsymbol{u}}}{\partial \hat{t}} + \varepsilon^4 K\hat{u}_1\frac{\partial \hat{\boldsymbol{u}}}{\partial \hat{x}_1} + \varepsilon^3 K\hat{u}_2\frac{\partial \hat{\boldsymbol{u}}}{\partial \hat{y}_2} + \varepsilon^3 K\hat{u}_3\frac{\partial \hat{\boldsymbol{u}}}{\partial \hat{y}_3} - \varepsilon^2\frac{\partial^2 \hat{\boldsymbol{u}}}{\partial \hat{x}_1^2} - \Delta_{\hat{\boldsymbol{y}}_*}\hat{\boldsymbol{u}} + \frac{\partial \hat{p}}{\partial \hat{x}_1}\boldsymbol{e}_1 + \frac{1}{\varepsilon}\nabla_{\hat{\boldsymbol{y}}_*}\hat{p} \\
&\quad = 2N\left[\operatorname{rot}_{\hat{\boldsymbol{y}}_*}\hat{w}_1 + \left(\frac{\partial \hat{w}_3}{\partial \hat{y}_2} - \frac{\partial \hat{w}_2}{\partial \hat{y}_3}\right)\boldsymbol{e}_1 - \varepsilon\frac{\partial \hat{w}_3}{\partial \hat{x}_1}\boldsymbol{e}_2 + \varepsilon\frac{\partial \hat{w}_2}{\partial \hat{x}_1}\boldsymbol{e}_3\right] + \sum_{j=0}^{J}\varepsilon^j \hat{f}_1^j\boldsymbol{e}_1, \\
&\qquad\qquad \varepsilon\operatorname{div}_{\hat{\boldsymbol{y}}_*}\hat{\boldsymbol{u}} + \varepsilon^2\frac{\partial \hat{u}_1}{\partial \hat{x}_1} = 0, \\
&\varepsilon^4 M\frac{\partial \hat{\boldsymbol{w}}}{\partial \hat{t}} + \varepsilon^6 MK\hat{u}_1\frac{\partial \hat{\boldsymbol{w}}}{\partial \hat{x}_1} + \varepsilon^5 MK\hat{u}_2\frac{\partial \hat{\boldsymbol{w}}}{\partial \hat{y}_2} + \varepsilon^5 MK\hat{u}_3\frac{\partial \hat{\boldsymbol{w}}}{\partial \hat{y}_3} - R_{1\varepsilon}\left(\Delta_{\hat{\boldsymbol{y}}_*}\hat{\boldsymbol{w}} + \varepsilon^2\frac{\partial^2 \hat{\boldsymbol{w}}}{\partial \hat{x}_1^2}\right) \\
&\quad -R_{2\varepsilon}\left(\varepsilon^2\frac{\partial^2 \hat{w}_1}{\partial \hat{x}_1^2}\boldsymbol{e}_1 + \varepsilon\frac{\partial}{\partial \hat{x}_1}(\operatorname{div}_{\hat{\boldsymbol{y}}_*}\hat{\boldsymbol{w}})\boldsymbol{e}_1 + \varepsilon\nabla_{\hat{\boldsymbol{y}}_*}\frac{\partial \hat{w}_1}{\partial \hat{x}_1} + \nabla_{\hat{\boldsymbol{y}}_*}\operatorname{div}_{\hat{\boldsymbol{y}}_*}\hat{\boldsymbol{w}}\right) + 4N\varepsilon^2\hat{\boldsymbol{w}} \\
&\quad = 2N\left[\varepsilon^2\operatorname{rot}_{\hat{\boldsymbol{y}}_*}\hat{u}_1 + \varepsilon^2\left(\frac{\partial \hat{u}_3}{\partial \hat{y}_2} - \frac{\partial \hat{u}_2}{\partial \hat{y}_3}\right)\boldsymbol{e}_1 - \varepsilon^3\frac{\partial \hat{u}_3}{\partial \hat{x}_1}\boldsymbol{e}_2 + \varepsilon^3\frac{\partial \hat{u}_2}{\partial \hat{x}_1}\boldsymbol{e}_3\right] + \sum_{j=0}^{J}\varepsilon^{j+2}\hat{\boldsymbol{g}}^j.
\end{aligned}
\tag{4.14}
$$

Koristimo sljedeće oznake za parcijalne diferencijalne operatore:

$$
\begin{aligned}
\Delta_{\hat{\boldsymbol{y}}_*}\boldsymbol{v} &= \frac{\partial^2 \boldsymbol{v}}{\partial \hat{y}_2^2} + \frac{\partial^2 \boldsymbol{v}}{\partial \hat{y}_3^2}, \quad \operatorname{div}_{\hat{\boldsymbol{y}}_*}\boldsymbol{v} = \frac{\partial v_2}{\partial \hat{y}_2} + \frac{\partial v_3}{\partial \hat{y}_3}, \\
\nabla_{\hat{\boldsymbol{y}}_*}\varphi &= \frac{\partial \varphi}{\partial \hat{y}_2}\boldsymbol{e}_2 + \frac{\partial \varphi}{\partial \hat{y}_3}\boldsymbol{e}_3, \quad \operatorname{rot}_{\hat{\boldsymbol{y}}_*}\varphi = \frac{\partial \varphi}{\partial \hat{y}_3}\boldsymbol{e}_2 - \frac{\partial \varphi}{\partial \hat{y}_2}\boldsymbol{e}_3,
\end{aligned}
$$

pri čemu je $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ kanonska baza lokalnog koordinatnog sustava, te $\boldsymbol{v} = (v_1, v_2, v_3)$.

4.2.1.1 1. slučaj

Pretpostavimo da vrijedi $R_{k\varepsilon} \ll O(\varepsilon^2)$ za $k = 1, 2$, to jest da su koeficijenti $R_{k\varepsilon}$ oblika $R_{1\varepsilon} = \alpha\varepsilon^\gamma$, $R_{2\varepsilon} = \beta\varepsilon^\gamma$ za $\gamma \gg 2$, $\alpha, \beta > 0$. Izjednačavanjem članova u (4.14) uz najmanju potenciju parametra ε koja se pojavljuje u jednadžbi, dobivamo sljedeći sustav za aproksimaciju brzine, tlaka i mikrorotacije:

$$
\begin{aligned}
\frac{1}{\varepsilon} &: \quad \nabla_{\hat{\boldsymbol{y}}_*}\hat{p} = 0, \\
1 &: \quad -\Delta_{\hat{\boldsymbol{y}}_*}\hat{\boldsymbol{u}} + \frac{\partial \hat{p}}{\partial \hat{x}_1}\boldsymbol{e}_1 = 2N\left[\operatorname{rot}_{\hat{\boldsymbol{y}}_*}\hat{w}_1 + \left(\frac{\partial \hat{w}_3}{\partial \hat{y}_2} - \frac{\partial \hat{w}_2}{\partial \hat{y}_3}\right)\boldsymbol{e}_1\right] + \hat{f}_1^0\boldsymbol{e}_1, \\
\varepsilon &: \quad \operatorname{div}_{\hat{\boldsymbol{y}}_*}\hat{\boldsymbol{u}} = 0, \\
\varepsilon^2 &: \quad \hat{\boldsymbol{w}} = \frac{1}{2}\left[\operatorname{rot}_{\hat{\boldsymbol{y}}_*}\hat{u}_1 + \left(\frac{\partial \hat{u}_3}{\partial \hat{y}_2} - \frac{\partial \hat{u}_2}{\partial \hat{y}_3}\right)\boldsymbol{e}_1\right] + \frac{1}{4N}\hat{\boldsymbol{g}}^0.
\end{aligned}
\tag{4.15}
$$

Iz (4.15)$_1$ možemo zaključiti $\hat{p} = \hat{p}(\hat{x}_1, \hat{t})$. Nadalje, uvrštavanjem (4.15)$_4$ u (4.15)$_2$ dobivamo sljedeći sustav jednadžbi za $\hat{\boldsymbol{u}}$:

$$
\begin{aligned}
(1-N)\Delta_{\hat{\boldsymbol{y}}_*}\hat{u}_1 - \frac{\partial \hat{p}}{\partial \hat{x}_1} + \frac{1}{2}\left(\frac{\partial \hat{g}_3^0}{\partial \hat{y}_2} - \frac{\partial \hat{g}_2^0}{\partial \hat{y}_3}\right) + \hat{f}_1^0 = 0,\\
-\Delta_{\hat{\boldsymbol{y}}_*}\hat{u}_2 - N\left(\frac{\partial^2 \hat{u}_3}{\partial \hat{y}_2 \partial \hat{y}_3} - \frac{\partial^2 \hat{u}_2}{\partial \hat{y}_3^2}\right) = 0,\\
-\Delta_{\hat{\boldsymbol{y}}_*}\hat{u}_3 - N\left(\frac{\partial^2 \hat{u}_2}{\partial \hat{y}_2 \partial \hat{y}_3} - \frac{\partial^2 \hat{u}_3}{\partial \hat{y}_2^2}\right) = 0,\\
\operatorname{div}_{\hat{\boldsymbol{y}}_*}\hat{\boldsymbol{u}} = 0.
\end{aligned}
\tag{4.16}
$$

Množenjem (4.16)$_2$–(4.16)$_3$ skalarno s $\hat{\mathbf{u}}_* := (\hat{u}_2, \hat{u}_3)$ te integriranjem po $\hat{B}$ dobivamo

$$
\begin{aligned}
0 &= \int_{\hat{B}} \left|\nabla_{\hat{\boldsymbol{y}}_*}\hat{\mathbf{u}}_*\right|^2 + N\int_{\hat{B}}\left(2\frac{\partial \hat{u}_2}{\partial \hat{y}_2}\frac{\partial \hat{u}_3}{\partial \hat{y}_3} - \left(\frac{\partial \hat{u}_2}{\partial \hat{y}_3}\right)^2 - \left(\frac{\partial \hat{u}_3}{\partial \hat{y}_2}\right)^2\right)\\
&= \int_{\hat{B}}(1-N)\left[\left(\frac{\partial \hat{u}_2}{\partial \hat{y}_3}\right)^2 + \left(\frac{\partial \hat{u}_3}{\partial \hat{y}_2}\right)^2\right] + \int_{\hat{B}}\left[\left(\frac{\partial \hat{u}_2}{\partial \hat{y}_2}\right)^2 + \left(\frac{\partial \hat{u}_3}{\partial \hat{y}_3}\right)^2 - 2N\frac{\partial \hat{u}_2}{\partial \hat{y}_2}\frac{\partial \hat{u}_3}{\partial \hat{y}_3}\right].
\end{aligned}
\tag{4.17}
$$

Nadalje, vrijedi

$$
2N\frac{\partial \hat{u}_2}{\partial \hat{y}_2}\frac{\partial \hat{u}_3}{\partial \hat{y}_3} \le N\left(\left(\frac{\partial \hat{u}_2}{\partial \hat{y}_2}\right)^2 + \left(\frac{\partial \hat{u}_3}{\partial \hat{y}_3}\right)^2\right) \le \left(\frac{\partial \hat{u}_2}{\partial \hat{y}_2}\right)^2 + \left(\frac{\partial \hat{u}_3}{\partial \hat{y}_3}\right)^2.
$$

Kako je $N < 1$, vidimo da su oba integrala u drugom redu (4.17) veća ili jednaka nuli, pri čemu je drugi integral jednak nuli ako i samo ako $\frac{\partial \hat{u}_2}{\partial \hat{y}_2} = \frac{\partial \hat{u}_3}{\partial \hat{y}_3} = 0$. Uzimajući u obzir homogene Dirichletove rubne uvjete na $\partial\hat{B}$, iz (4.17) sada možemo zaključiti $\hat{\mathbf{u}}_* = \mathbf{0}$. Dakle, rješenje problema (4.15) je oblika $\hat{\boldsymbol{u}} = \hat{u}_1\boldsymbol{e}_1$, $\hat{\boldsymbol{w}} = \hat{w}_2\boldsymbol{e}_2 + \hat{w}_3\boldsymbol{e}_3$. Pripadni rubni uvjeti su dani s:

$$
\hat{u}_1, \hat{w}_2, \hat{w}_3 = 0 \quad \text{na} \quad \partial\hat{B}.
$$

Naglasimo da možemo odrediti eksplicitno rješenje u slučaju kada funkcije $\hat{f}_1^0$ i $\hat{\boldsymbol{g}}^0$ ovise samo o $\hat{t}$. Naime, iz (4.16)$_1$ možemo zaključiti sljedeće:

$$
\hat{u}_1(\hat{\boldsymbol{y}}_*, \hat{t}) = -\frac{1}{4(1-N)}(\hat{r}_i^2 - |\hat{\boldsymbol{y}}_*|^2)\left(\frac{\partial \hat{p}}{\partial \hat{x}_1} - \hat{f}_1^0\right).
$$

Integriranjem jednadžbe za divergenciju višeg reda dobivamo

$$
\hat{F}^0(\hat{t}) = -\frac{\pi \hat{r}_i^4}{8(1-N)}\left(\frac{\partial \hat{p}}{\partial \hat{x}_1} - \hat{f}_1^0\right),
$$

te je sada rješenje problema (4.15) dano s

$$
\begin{aligned}
\hat{\boldsymbol{u}}(\hat{\boldsymbol{y}}_*, \hat{t}) &= \frac{2}{\pi \hat{r}_i^4}(\hat{r}_i^2 - |\hat{\boldsymbol{y}}_*|^2)\hat{F}^0(\hat{t})\boldsymbol{e}_1,\\
\hat{p}(\hat{x}_1, \hat{t}) &= -\frac{8(1-N)}{\pi \hat{r}_i^4}\hat{F}^0(\hat{t})\hat{x}_1 + \hat{f}_1^0(\hat{t})\hat{x}_1 + \hat{p}_0(\hat{t}),\\
\hat{\boldsymbol{w}}(\hat{\boldsymbol{y}}_*, \hat{t}) &= \frac{2}{\pi \hat{r}_i^4}\hat{F}^0(\hat{t})\left(-\hat{y}_3\boldsymbol{e}_2 + \hat{y}_2\boldsymbol{e}_3\right) + \frac{1}{4N}\hat{\boldsymbol{g}}^0,
\end{aligned}
\tag{4.18}
$$

gdje je $\hat{p}_0$ proizvoljna funkcija koja ovisi samo o vremenu $\hat{t}$. Koristeći (4.12) se vraćamo u dimenzionalnu postavku te iz (4.18) dobivamo aproksimaciju nultog reda u i-toj cijevi Ω^i u sljedećem obliku:

$$\begin{aligned}
\boldsymbol{u}^i_\varepsilon(\boldsymbol{x}^i_*,t) &= \frac{2}{\pi r_i^4}(\varepsilon^2 r_i^2 - |\boldsymbol{x}^i_*|^2)F^{i,0}(t)\boldsymbol{e}^i_1,\\
p^i_\varepsilon(x^i_1,t) &= -\frac{8\mu}{\pi r_i^4}F^{i,0}(t)x^i_1 + f^{i,0}(t)x^i_1 + p^i_0(t),\\
\boldsymbol{w}^i_\varepsilon(\boldsymbol{x}^i_*,t) &= \frac{2}{\pi r_i^4}F^{i,0}(t)(-x^i_3\boldsymbol{e}^i_3 + x^i_2\boldsymbol{e}^i_2) + \frac{\varepsilon}{4\mu_r}\hat{\boldsymbol{g}}^0,
\end{aligned}$$

gdje je p^i_0 proizvoljna funkcija koja ovisi samo o varijabli t. Korektori višeg reda se također mogu eksplicitno odrediti u slučaju kada vanjske funkcije ovise samo o vremenu.

4.2.1.2 2. slučaj

U ovom slučaju razmatramo koeficijente reda veličine $R_{k\varepsilon} \gg O(\varepsilon^2)$ za $k=1,2$, dakle kada su $R_{k\varepsilon}$ oblika $R_{1\varepsilon} = \alpha\varepsilon^\gamma, R_{2\varepsilon} = \beta\varepsilon^\gamma$ za $\gamma \ll 2$, $\alpha, \beta > 0$. Sustav jednadžbi koji opisuje aproksimaciju brzine, tlaka i mikrorotacije je tada dan s:

$$\begin{aligned}
\frac{1}{\varepsilon} &: \quad \nabla_{\hat{\boldsymbol{y}}_*}\hat{p} = 0,\\
1 &: \quad -\Delta_{\hat{\boldsymbol{y}}_*}\hat{\boldsymbol{u}} + \frac{\partial\hat{p}}{\partial\hat{x}_1}\boldsymbol{e}_1 = 2N\left[\mathrm{rot}_{\hat{\boldsymbol{y}}_*}\hat{w}_1 + \left(\frac{\partial\hat{w}_3}{\partial\hat{y}_2} - \frac{\partial\hat{w}_2}{\partial\hat{y}_3}\right)\boldsymbol{e}_1\right] + \hat{f}^0_1\boldsymbol{e}_1,\\
\varepsilon &: \quad \mathrm{div}_{\hat{\boldsymbol{y}}_*}\hat{\boldsymbol{u}} = 0,\\
\varepsilon^\gamma &: \quad -\alpha\Delta_{\hat{\boldsymbol{y}}_*}\hat{\boldsymbol{w}} - \beta\nabla_{\hat{\boldsymbol{y}}_*}\mathrm{div}_{\hat{\boldsymbol{y}}_*}\hat{\boldsymbol{w}} = 0.
\end{aligned} \tag{4.19}$$

Iz $(4.19)_1$ slijedi $\hat{p} = \hat{p}(\hat{x}_1,\hat{t})$, dok iz $(4.19)_4$ možemo zaključiti $\hat{\boldsymbol{w}} = \boldsymbol{0}$. Rješenje problema $(4.19)_2$–$(4.19)_3$ je oblika $\hat{\boldsymbol{u}} = \hat{u}_1\boldsymbol{e}_1$ te zadovoljava jednadžbu

$$-\Delta_{\hat{\boldsymbol{y}}_*}\hat{u}_1 + \frac{\partial\hat{p}}{\partial\hat{x}_1} = \hat{f}^0_1. \tag{4.20}$$

Ako pretpostavimo da $\hat{f}^0_1$ ovisi samo o $\hat{t}$, rješenje od (4.20) je dano s

$$\begin{aligned}
\hat{\boldsymbol{u}}(\hat{\boldsymbol{y}}_*,\hat{t}) &= \frac{2}{\pi\hat{r}_i^4}(\hat{r}_i^2 - |\hat{\boldsymbol{y}}_*|^2)\hat{F}^0(\hat{t})\boldsymbol{e}_1,\\
\hat{p}^0(\hat{x}_1,\hat{t}) &= -\frac{8}{\pi\hat{r}_i^4}\hat{F}^0(\hat{t})\hat{x}_1 + \hat{f}^0_1(\hat{t})\hat{x}_1 + \hat{p}_0(\hat{t}).
\end{aligned}$$

Aproksimacija brzine je jednaka kao u Slučaju 4.2.1.1, dok je aproksimacije tlaka u i-toj (dimenzionalnoj) cijevi Ω^i_ε dana s

$$p^i_\varepsilon(x^i_1,t) = -\frac{8(\mu+\mu_r)}{\pi r_i^4}F^{i,0}(t)x^i_1 + f^{i,0}(t)x^i_1 + p^i_0(t),$$

pri čemu je $p_0^i = p_0^i(t)$ proizvoljna funkcija koja ovisi samo o vremenu t. Kao i u prethodnom slučaju, korektori višeg reda se mogu eksplicitno odrediti kada $\boldsymbol{f}_\varepsilon^i, \boldsymbol{g}_\varepsilon^i$ ovise samo o vremenu.

4.2.1.3 3. slučaj: Jaka uparenost

Sada promatramo kritični slučaj između 1. i 2. slučaja, to jest pretpostavljamo da vrijedi $R_{k\varepsilon} = O(\varepsilon^2)$ za $k = 1, 2$. Ovo smatramo najzanimljivijim slučajem, jer vodi do jake uparenosti između aproksimacija brzine i mikrorotacije. Regularni dio asimptotičkog razvoja rješenja problema (4.13) u bezdimenzionalnoj cijevi $\hat{\Omega}$ tražimo u obliku

$$\hat{\boldsymbol{u}}_\varepsilon^{[J]} = \sum_{j=0}^{J} \varepsilon^{j+2} \hat{\boldsymbol{u}}^j, \quad \hat{\boldsymbol{w}}_\varepsilon^{[J]} = \sum_{j=0}^{J} \varepsilon^{j+1} \hat{\boldsymbol{w}}^j, \quad \hat{p}_\varepsilon^{[J]} = \sum_{j=0}^{J} \varepsilon^{j} \hat{p}^j, \tag{4.21}$$

gdje je

$$\begin{aligned} \hat{\boldsymbol{u}}^j &= \hat{u}_1^j(\hat{\boldsymbol{y}}_*, \hat{t})\boldsymbol{e}_1, \\ \hat{\boldsymbol{w}}^j &= \hat{w}_2^j(\hat{\boldsymbol{y}}_*, \hat{t})\boldsymbol{e}_2 + \hat{w}_3^j(\hat{\boldsymbol{y}}_*, \hat{t})\boldsymbol{e}_3, \\ \hat{p}^j &= -\hat{s}_j(\hat{t})\hat{x}_1 + \hat{q}_j(\hat{t}), \end{aligned} \tag{4.22}$$

za $j = 0, 1, 2, \ldots, J$.

Neka je $R_{1\varepsilon} = \alpha\varepsilon^2$, $R_{2\varepsilon} = \beta\varepsilon^2$, $\alpha, \beta > 0$. Uvrštavanjem asimptotičkog razvoja (4.21) u promatrani bezdimenzionalni sustav (4.13) i izjednačavanjem članova koji se nalaze uz iste potencije parametra ε dolazimo do sljedećeg sustava za aproksimaciju nultog reda $(\hat{\boldsymbol{u}}^0, \hat{\boldsymbol{w}}^0, \hat{p}^0)$:

$$\begin{aligned} \frac{1}{\varepsilon}: &\quad \nabla_{\hat{\boldsymbol{y}}_*} \hat{p}^0 = 0, \\ 1: &\quad -\Delta_{\hat{\boldsymbol{y}}_*} \hat{\boldsymbol{u}}^0 + \frac{\partial \hat{p}^0}{\partial \hat{x}_1}\boldsymbol{e}_1 + \nabla_{\hat{\boldsymbol{y}}_*} \hat{p}^1 = 2N \left[\operatorname{rot}_{\hat{\boldsymbol{y}}_*} \hat{w}_1^0 + \left(\frac{\partial \hat{w}_3^0}{\partial \hat{y}_2} - \frac{\partial \hat{w}_2^0}{\partial \hat{y}_3} \right) \boldsymbol{e}_1 \right] + \hat{f}_1^0 \boldsymbol{e}_1, \\ \varepsilon: &\quad \operatorname{div}_{\hat{\boldsymbol{y}}_*} \hat{\boldsymbol{u}}^0 = 0, \\ \varepsilon: &\quad -\alpha\Delta_{\hat{\boldsymbol{y}}_*} \hat{\boldsymbol{w}}^0 - \beta\nabla_{\hat{\boldsymbol{y}}_*} \operatorname{div}_{\hat{\boldsymbol{y}}_*} \hat{\boldsymbol{w}}^0 + 4N\hat{\boldsymbol{w}}^0 = 2N \left[\operatorname{rot}_{\hat{\boldsymbol{y}}_*} \hat{u}_1^0 + \left(\frac{\partial \hat{u}_3^0}{\partial \hat{y}_2} - \frac{\partial \hat{u}_2^0}{\partial \hat{y}_3} \right) \boldsymbol{e}_1 \right] + \hat{\boldsymbol{g}}^0. \end{aligned} \tag{4.23}$$

Uzimajući u obzir (4.22), problem (4.23) se svodi na

$$\begin{aligned} -\Delta_{\hat{\boldsymbol{y}}_*} \hat{u}_1^0 - \hat{s}_0 &= 2N \left(\frac{\partial \hat{w}_3^0}{\partial \hat{y}_2} - \frac{\partial \hat{w}_2^0}{\partial \hat{y}_3} \right) + \hat{f}_1^0, \\ -\alpha\Delta_{\hat{\boldsymbol{y}}_*} \hat{\boldsymbol{w}}^0 - \beta\nabla_{\hat{\boldsymbol{y}}_*} \operatorname{div}_{\hat{\boldsymbol{y}}_*} \hat{\boldsymbol{w}}^0 + 4N\hat{\boldsymbol{w}}^0 &= 2N \operatorname{rot}_{\hat{\boldsymbol{y}}_*} \hat{u}_1^0 + \hat{\boldsymbol{g}}^0. \end{aligned} \tag{4.24}$$

Dobiveni sustav (4.24) je upotpunjen s rubnim uvjetima

$$\hat{u}_1^0 = 0, \ \hat{\boldsymbol{w}}^0 = \boldsymbol{0} \text{ na } \partial\hat{B}. \tag{4.25}$$

Na sličan način dobivamo sustav jednadžbi koji određuje korektore prvog reda $(\hat{u}_1^1, \hat{\boldsymbol{w}}^1, \hat{p}^1)$:

$$\begin{aligned} -\Delta_{\hat{\boldsymbol{y}}_*} \hat{u}_1^1 - \hat{s}_1 &= 2N \left(\frac{\partial \hat{w}_3^1}{\partial \hat{y}_2} - \frac{\partial \hat{w}_2^1}{\partial \hat{y}_3} \right) + \hat{f}_1^1, \\ -\alpha \Delta_{\hat{\boldsymbol{y}}_*} \hat{\boldsymbol{w}}^1 - \beta \nabla_{\hat{\boldsymbol{y}}_*} \operatorname{div}_{\hat{\boldsymbol{y}}_*} \hat{\boldsymbol{w}}^1 + 4N \hat{\boldsymbol{w}}^1 &= 2N \operatorname{rot}_{\hat{\boldsymbol{y}}_*} \hat{u}_1^1 + \hat{\boldsymbol{g}}^1, \\ \hat{u}_1^1 = 0, \ \hat{\boldsymbol{w}}^1 &= \boldsymbol{0} \ \text{ na } \ \partial \hat{B}, \end{aligned} \tag{4.26}$$

dok korektori višeg reda zadovoljavaju sustav jednadžbi:

$$\begin{aligned} -\Delta_{\hat{\boldsymbol{y}}_*} \hat{u}_1^j - \hat{s}_j &= 2N \left(\frac{\partial \hat{w}_3^j}{\partial \hat{y}_2} - \frac{\partial \hat{w}_2^j}{\partial \hat{y}_3} \right) + \hat{f}_1^j - \frac{\partial \hat{u}_1^{j-2}}{\partial \hat{t}}, \\ -\alpha \Delta_{\hat{\boldsymbol{y}}_*} \hat{\boldsymbol{w}}^j - \beta \nabla_{\hat{\boldsymbol{y}}_*} \operatorname{div}_{\hat{\boldsymbol{y}}_*} \hat{\boldsymbol{w}}^j + 4N \hat{\boldsymbol{w}}^j &= 2N \operatorname{rot}_{\hat{\boldsymbol{y}}_*} \hat{u}_1^j + \hat{\boldsymbol{g}}^j - M \frac{\partial \hat{\boldsymbol{w}}^{j-2}}{\partial \hat{t}}, \\ \hat{u}_1^j = 0, \ \hat{\boldsymbol{w}}^j &= \boldsymbol{0} \ \text{ na } \ \partial \hat{B}, \end{aligned} \tag{4.27}$$

za $j = 2, 3, \dots, J$. Ako vrijedi $\hat{f}_1^j(\cdot, t) \in L^2(\hat{B})$, $\hat{\boldsymbol{g}}^j(\cdot, t) \in L^2(\hat{B})^2$, svi problemi (4.24)–(4.25), (4.26), (4.27) imaju jedinstveno rješenje $\left(\hat{u}_1^j(\cdot, \hat{t}\,), \hat{\boldsymbol{w}}^j(\cdot, \hat{t}\,)\right) \in W_0^{1,2}(\hat{B}) \times W_0^{1,2}(\hat{B})^2$, $j = 0, 1, \dots, J$, takvo da vrijedi

$$\int_{\hat{B}} \hat{u}_1^j(\hat{\boldsymbol{y}}_*, \hat{t}\,) d\hat{\boldsymbol{y}}_* = \hat{F}^j(\hat{t}\,). \tag{4.28}$$

Tvrdnja se lako može pokazati koristeći sličan pristup kao u [92, Odjeljak 2]. Kako bismo osigurali regularnost u vremenu, potrebno je uvesti sljedeće pretpostavke:

$$\begin{aligned} \frac{\partial \hat{f}_1^j}{\partial \hat{t}} \in L^2(\hat{B}), \ \frac{\partial \hat{\boldsymbol{g}}^j}{\partial \hat{t}} \in L^2(\hat{B})^2, \quad j = 0, 1, \dots, J, \\ \frac{\partial^2 \hat{f}_1^{j-2}}{\partial \hat{t}^2} \in L^2(\hat{B}), \ \frac{\partial^2 \hat{\boldsymbol{g}}^{j-2}}{\partial \hat{t}^2} \in L^2(\hat{B})^2, \quad j = 2, 3, \dots, J. \end{aligned}$$

Napomenimo da su funkcije $\hat{s}_j$ u potpunosti određene s $(4.24)_1$, $(4.26)_1$ i $(4.27)_1$. Dodatno, uzimamo $\hat{q}_j(t) = q(t)$, $j = 0, 1, \dots, J$ gdje je $q = q(t)$ proizvoljna funkcija, kako bi makroskopski tlak bio neprekidan u čvorištu.

Ovime zaključujemo određivanje regularnog dijela asimptotičkog razvoja u sva tri slučaja. U ostatku poglavlja ćemo prezentirati analizu rubnog sloja na krajevima cijevi te unutarnjeg sloja blizu čvorišta, u kritičnom slučaju promatranom u 4.2.1.3. Napomenimo da se analogna analiza može provesti i u prva dva slučaja.

4.2.2. Korektori rubnog sloja

Regularni dio asimptotičke aproksimacije rješenja daleko od čvorišta će zadovoljavati inicijalne uvjete zbog uvedene pretpostavke da su funkcije $\boldsymbol{h}^i_\varepsilon, \boldsymbol{f}_\varepsilon, \boldsymbol{g}_\varepsilon$ jednake nuli na nekoj okolini $t = 0$. Nadalje, aproksimacija je određena tako da zadovoljava rubne uvjete na lateralnom rubu domene Γ_ε, no još nismo uzeli u obzir rubne uvjete na krajevima cijevi $\Sigma_i, i = 1, \ldots, m$. Kako bismo to korigirali i poboljšali aproksimaciju, uvodimo korektore rubnog sloja u $\hat{x}^i_1 = \hat{\ell}_i$, $i = 1, 2, \ldots, m$:

$$
\begin{aligned}
\hat{\mathcal{V}}^{i,[J]}_{\varepsilon,bl}(\hat{\boldsymbol{x}}^i, \hat{t}\,) &= \sum_{j=0}^{J} \varepsilon^{j+2} \hat{\mathcal{V}}^{i,j}_{bl} \left(\frac{\hat{x}^i_1 - \hat{\ell}_i}{\varepsilon}, \frac{\hat{\boldsymbol{x}}^i_*}{\varepsilon}, \hat{t} \right), \\
\hat{Q}^{i,[J]}_{\varepsilon,bl}(\hat{\boldsymbol{x}}^i, \hat{t}\,) &= \sum_{j=0}^{J} \varepsilon^{j+1} \hat{Q}^{i,j}_{bl} \left(\frac{\hat{x}^i_1 - \hat{\ell}_i}{\varepsilon}, \frac{\hat{\boldsymbol{x}}^i_*}{\varepsilon}, \hat{t} \right), \\
\hat{\mathcal{Z}}^{i,[J]}_{\varepsilon,bl}(\hat{\boldsymbol{x}}^i, \hat{t}\,) &= \sum_{j=0}^{J} \varepsilon^{j+1} \hat{\mathcal{Z}}^{i,j}_{bl} \left(\frac{\hat{x}^i_1 - \hat{\ell}_i}{\varepsilon}, \frac{\hat{\boldsymbol{x}}^i_*}{\varepsilon}, \hat{t} \right).
\end{aligned}
\tag{4.29}
$$

Korektori rubnog sloja na desnoj strani jednakosti (4.29) su definirani na polubeskonačnoj cijevi $\mathcal{G}^i_\infty = \langle -\infty, 0\rangle \times \hat{B}^i$. Uvedimo zamjenu varijabli $\hat{y}^i_1 = \frac{\hat{x}^i_1 - \hat{\ell}_i}{\varepsilon}$, $\hat{\boldsymbol{y}}_* = \frac{\hat{\boldsymbol{x}}^i_*}{\varepsilon}$ te stavimo $\hat{\boldsymbol{f}}_\varepsilon = \hat{\boldsymbol{g}}_\varepsilon = 0$. Sada uvrštavanjem izraza (4.29) u sustav jednadžbi (4.13) te izjednačavanjem članova uz istu potenciju parametra ε dobivamo rekurzivni niz problema koji određuju korektore rubnog sloja.

Problem koji opisuje korektore rubnog sloja nultog reda u bezdimenzionalnoj i-toj cijevi G^i_∞ je dan s:

$$
\begin{aligned}
-\Delta \hat{\mathcal{V}}^{i,0}_{bl} + \nabla \hat{Q}^{i,0}_{bl} &= 2N \operatorname{rot} \hat{\mathcal{Z}}^{i,0}_{bl}, \\
\operatorname{div} \hat{\mathcal{V}}^{i,0}_{bl} &= 0, \\
-\alpha \Delta \hat{\mathcal{Z}}^{i,0}_{bl} - \beta \nabla \operatorname{div} \hat{\mathcal{Z}}^{i,0}_{bl} + 4N \hat{\mathcal{Z}}^{i,0}_{bl} &= 2N \operatorname{rot} \hat{\mathcal{V}}^{i,0}_{bl}.
\end{aligned}
\tag{4.30}
$$

Pripadni rubni uvjeti glase

$$
\begin{aligned}
\hat{\mathcal{V}}^{i,0}_{bl}(0, \hat{\boldsymbol{y}}^i_*, \hat{t}\,) &= \hat{\boldsymbol{h}}^{i,0}(\hat{\boldsymbol{y}}^i_*, \hat{t}\,) - \hat{\boldsymbol{u}}^{i,0}(\hat{\boldsymbol{y}}^i_*, \hat{t}\,), \\
\hat{\mathcal{Z}}^{i,0}_{bl}(0, \hat{\boldsymbol{y}}^i_*, \hat{t}\,) &= -\hat{\boldsymbol{w}}^{i,0}(\hat{\boldsymbol{y}}^i_*, \hat{t}\,), \\
\hat{\mathcal{V}}^{i,0}_{bl}, \hat{\mathcal{Z}}^{i,0}_{bl} &= \boldsymbol{0} \quad \text{na} \quad \langle -\infty, 0\rangle \times \partial \hat{B}^i,
\end{aligned}
\tag{4.31}
$$

pri čemu je $\hat{\boldsymbol{h}}^{i,0} = \frac{\boldsymbol{h}^{i,0}}{U_{ref}}$. Uvjet kompatibilnosti za problem (4.30)–(4.31) je zadovoljen zbog (4.5) i (4.28):

$$
\int_{\hat{B}^i} \left(\hat{\boldsymbol{h}}^{i,0}(\hat{\boldsymbol{y}}^i_*, \hat{t}\,) - \hat{\boldsymbol{u}}^{i,0}(\hat{\boldsymbol{y}}^i_*, \hat{t}\,) \right) d\hat{\boldsymbol{y}}^i_* = 0.
$$

Neka je funkcija $\boldsymbol{K}_{bl}^0 \in W^{1,2}(\mathcal{G}_\infty^i)$ rješenje problema

$$\begin{aligned} \operatorname{div} \boldsymbol{K}_{bl}^0 &= 0 \quad \text{u} \quad \mathcal{G}_\infty^i, \\ \boldsymbol{K}_{bl}^0 &= \hat{\boldsymbol{h}}^{i,0} - \hat{\boldsymbol{u}}^{i,0} \quad \text{na} \quad \{0\} \times \hat{B}^i, \\ \boldsymbol{K}_{bl}^0 &= \boldsymbol{0} \quad \text{na} \quad \langle -\infty, 0 \rangle \times \partial \hat{B}^i, \end{aligned}$$

takvo da vrijedi $\operatorname{supp} \boldsymbol{K}_{bl}^0 \subset \left\{ \hat{\boldsymbol{y}}^i \in \mathcal{G}_\infty^i : |\hat{\boldsymbol{y}}^i| < 2\hat{r}_i \right\}$ (vidi [71, Appendix]). Slično, postoji funkcija $\boldsymbol{L}_{bl}^0 \in W^{1,2}(\mathcal{G}_\infty^i)$ takva da

$$\begin{aligned} \boldsymbol{L}_{bl}^0 &= 0 \quad \text{u} \quad \{\hat{\boldsymbol{y}}^i \in \mathcal{G}_\infty^i : |\hat{\boldsymbol{y}}^i| \geq 2\hat{r}_i\}, \\ \boldsymbol{L}_{bl}^0 &= -\hat{\boldsymbol{w}}^{i,0} \quad \text{na} \quad \{0\} \times \hat{B}^i, \\ \boldsymbol{L}_{bl}^0 &= \boldsymbol{0} \quad \text{na} \quad \langle -\infty, 0 \rangle \times \partial \hat{B}^i. \end{aligned}$$

Tražimo rješenje problema (4.30)–(4.31) u obliku $\hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,0} = \boldsymbol{\mathcal{R}}_{bl}^{i,0} + \boldsymbol{K}_{bl}^0$, $\hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,0} = \boldsymbol{\mathcal{S}}_{bl}^{i,0} + \boldsymbol{L}_{bl}^0$, gdje $\boldsymbol{\mathcal{R}}_{bl}^{i,0}$ i $\boldsymbol{\mathcal{S}}_{bl}^{i,0}$ zadovoljavaju sljedeći sustav jednadžbi u $\mathcal{G}_\infty^i$:

$$\begin{aligned} -\Delta \boldsymbol{\mathcal{R}}_{bl}^{i,0} + \nabla \mathcal{Q}_{bl}^{i,0} &= 2N \operatorname{rot} \boldsymbol{\mathcal{S}}_{bl}^{i,0} + 2N \operatorname{rot} \boldsymbol{L}_{bl}^0 + \Delta \boldsymbol{K}_{bl}^0, \\ \operatorname{div} \boldsymbol{\mathcal{R}}_{bl}^{i,0} &= 0, \\ -\alpha \Delta \boldsymbol{\mathcal{S}}_{bl}^{i,0} - \beta \nabla \operatorname{div} \boldsymbol{\mathcal{S}}_{bl}^{i,0} + 4N \boldsymbol{\mathcal{S}}_{bl}^{i,0} &= 2N \operatorname{rot} \boldsymbol{\mathcal{R}}_{bl}^{i,0} + 2N \operatorname{rot} \boldsymbol{K}_{bl}^0 + \alpha \Delta \boldsymbol{L}_{bl}^0 \\ &\quad + \beta \nabla \operatorname{div} \boldsymbol{L}_{bl}^0 - 4N \boldsymbol{L}_{bl}^0, \\ \boldsymbol{\mathcal{R}}_{bl}^{i,0}, \boldsymbol{\mathcal{S}}_{bl}^{i,0} &= \boldsymbol{0} \quad \text{na} \quad \partial \mathcal{G}_\infty^i. \end{aligned} \tag{4.32}$$

Za svaku polubeskonačnu cijev $\mathcal{G}_\infty^i$, $i = 1, 2, \ldots, m$ vrijedi Teorem B.0.1 iz Dodatka B za

$$\begin{aligned} \boldsymbol{f} &= 2N \operatorname{rot} \boldsymbol{L}_{bl}^0, \;\; \boldsymbol{g} = 2N \operatorname{rot} \boldsymbol{K}_{bl}^0 - 4N \boldsymbol{L}_{bl}^0, \\ \boldsymbol{F}_j &= \frac{\partial \boldsymbol{K}_{bl}^0}{\partial \hat{y}_j^i}, \;\; \boldsymbol{G}_j = \alpha \frac{\partial \boldsymbol{L}_{bl}^0}{\partial \hat{y}_j^i}, \;\; H_j = \beta \frac{\partial L_{bl,j}^0}{\partial \hat{y}_j^i}, \;\; j = 1, 2, 3. \end{aligned}$$

Dakle, problem (4.32) ima jedinstveno rješenje takvo da eksponencijalno teži k nuli kada $|\hat{\boldsymbol{y}}^i| \to +\infty$, za svaki $i = 1, 2, \ldots, m$.

Problem za korektore rubnog sloja prvog reda na $\mathcal{G}_\infty^i$ je dan s

$$\begin{aligned} -\Delta \hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,1} + \nabla \hat{\mathcal{Q}}_{bl}^{i,1} &= 2N \operatorname{rot} \hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,1}, \\ \operatorname{div} \hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,1} &= 0, \\ -\alpha \Delta \hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,1} - \beta \nabla \operatorname{div} \hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,1} + 4N \hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,1} &= 2N \operatorname{rot} \hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,1}, \\ \hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,1}(0, \hat{\boldsymbol{y}}_*^i, \hat{t}\,) &= \hat{\boldsymbol{h}}^{i,1}(\hat{\boldsymbol{y}}_*^i, \hat{t}\,) - \hat{\boldsymbol{u}}^{i,1}(\hat{\boldsymbol{y}}_*^i, \hat{t}), \\ \hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,1}(0, \hat{\boldsymbol{y}}_*^i, \hat{t}) &= -\hat{\boldsymbol{w}}^{i,1}(\hat{\boldsymbol{y}}_*^i, \hat{t}), \\ \hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,1}, \hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,1} &= \boldsymbol{0} \quad \text{na} \quad \langle -\infty, 0 \rangle \times \partial \hat{B}^i, \end{aligned} \tag{4.33}$$

pri čemu je $\hat{\boldsymbol{h}}^{i,1} = \frac{\boldsymbol{h}^{i,1}}{U_{ref}}$, dok problem za korektore rubnog sloja reda $j = 2, 3, \ldots, J$ glasi

$$\begin{aligned}
-\Delta \hat{\mathcal{V}}_{bl}^{i,j} + \nabla \hat{Q}_{bl}^{i,j} &= 2N \operatorname{rot} \hat{\mathcal{Z}}_{bl}^{i,j} - \frac{\partial \hat{\mathcal{V}}_{bl}^{i,j-2}}{\partial \hat{t}} - K \sum_{k_1+k_2=j-3} (\hat{\mathcal{V}}_{bl}^{i,k_1} \cdot \nabla) \hat{\mathcal{V}}_{bl}^{i,k_2}, \\
\operatorname{div} \hat{\mathcal{V}}_{bl}^{i,j} &= 0, \\
-\alpha \Delta \hat{\mathcal{Z}}_{bl}^{i,j} - \beta \nabla \operatorname{div} \hat{\mathcal{Z}}_{bl}^{i,j} + 4N \hat{\mathcal{Z}}_{bl}^{i,j} &= 2N \operatorname{rot} \hat{\mathcal{V}}_{bl}^{i,j} - M \frac{\partial \hat{\mathcal{Z}}_{bl}^{i,j-2}}{\partial \hat{t}} \\
&\quad - MK \sum_{k_1+k_2=j-3} (\hat{\mathcal{V}}_{bl}^{i,k_1} \cdot \nabla) \hat{\mathcal{Z}}_{bl}^{i,k_2}, \\
\hat{\mathcal{V}}_{bl}^{i,j}(0, \hat{\boldsymbol{y}}_*^i, \hat{t}) &= \hat{\boldsymbol{h}}^{i,j}(\hat{\boldsymbol{y}}_*^i, \hat{t}) - \hat{\boldsymbol{u}}^{i,j}(\hat{\boldsymbol{y}}_*^i, \hat{t}), \\
\hat{\mathcal{Z}}_{bl}^{i,j}(0, \hat{\boldsymbol{y}}_*^i, \hat{t}) &= -\hat{\boldsymbol{w}}^{i,j}(\hat{\boldsymbol{y}}_*^i, \hat{t}), \\
\hat{\mathcal{V}}_{bl}^{i,j}, \hat{\mathcal{Z}}_{bl}^{i,j} &= \mathbf{0} \quad \text{na} \quad \langle -\infty, 0 \rangle \times \partial \hat{B}^i,
\end{aligned} \tag{4.34}$$

gdje je $\hat{\boldsymbol{h}}^{i,j} = \frac{\boldsymbol{h}^{i,j}}{U_{ref}}$, $j = 2, 3, \ldots, m$. Dobra postavljenost i eksponencijalni pad k nuli rješenja problema (4.33) i (4.34) se uspostavlja na isti način kao i za korektore rubnog sloja nultog reda. Napomenimo da pretpostavke na regularnost funkcija $\boldsymbol{h}^{i,j}$, $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots, J$ osiguravaju regularnost u vremenu korektora rubnog sloja.

4.2.3. Korektori unutarnjeg sloja

Regularni dio asimptotičke aproksimacije u i-toj bezdimenzionalnoj cijevi daleko od čvorišta, određen u Odjeljku 4.2.1.3, je oblika

$$\begin{aligned}
\hat{\boldsymbol{u}}_\varepsilon^{i,[J]}(\hat{\boldsymbol{x}}_*^i, \hat{t}) &= \sum_{j=0}^{J} \varepsilon^{j+2} \hat{\boldsymbol{u}}^{i,j} \left(\frac{\hat{\boldsymbol{x}}_*^i}{\varepsilon}, \hat{t} \right) = \sum_{j=0}^{J} \varepsilon^{j+2} \hat{u}_1^{i,j} \left(\frac{\hat{\boldsymbol{x}}_*^i}{\varepsilon}, \hat{t} \right) \boldsymbol{e}_1^i, \\
\hat{\boldsymbol{w}}_\varepsilon^{i,[J]}(\hat{\boldsymbol{x}}_*^i, \hat{t}) &= \sum_{j=0}^{J} \varepsilon^{j+1} \hat{\boldsymbol{w}}^{i,j} \left(\frac{\hat{\boldsymbol{x}}_*^i}{\varepsilon}, \hat{t} \right) \\
&= \sum_{j=0}^{J} \varepsilon^{j+1} \left(\hat{w}_2^{i,j} \left(\frac{\hat{\boldsymbol{x}}_*^i}{\varepsilon}, \hat{t} \right) \boldsymbol{e}_2^i + \hat{w}_3^{i,j} \left(\frac{\hat{\boldsymbol{x}}_*^i}{\varepsilon}, \hat{t} \right) \boldsymbol{e}_3^i \right), \\
\hat{p}_\varepsilon^{i,[J]}(\hat{x}_1^i, \hat{t}) &= \sum_{j=0}^{J} \varepsilon^j \hat{p}^{i,j}(\hat{x}_1^i, \hat{t}) = \sum_{j=0}^{J} \varepsilon^j \left(-\hat{s}_j^i(\hat{t}) \hat{x}_1^i + \hat{q}_j(\hat{t}) \right).
\end{aligned}$$

Korektori rubnog sloja na krajevima cijevi određeni u prethodnom odjeljku su dani u obliku (4.29). S obzirom na dosadašnja razmatranja sustava cijevi (vidi [9, 60, 71]), očekujemo da će ponašanje toka unutar čvorišta $\boldsymbol{\Omega}_\varepsilon^0$ biti bitno različito od onog u cijevima daleko od čvorišta. Shodno tome, množimo aproksimaciju toka u svakoj cijevi s izglađujućom funkcijom koja iščezava u blizini čvorišta $\boldsymbol{\Omega}_\varepsilon^0$. Ovaj produkt stvara rezidual u unutarnjem sloju koji ćemo nadoknaditi s korektorima unutarnjeg sloja oblika

$$\hat{\boldsymbol{V}}^{[J]}_{\varepsilon,int}(\hat{\boldsymbol{x}},t) = \sum_{j=0}^{J} \varepsilon^{j+2}\hat{\boldsymbol{V}}^{j}\left(\frac{\hat{\boldsymbol{x}}}{\varepsilon},\hat{t}\right),$$
$$\hat{Q}^{[J]}_{\varepsilon,int}(\hat{\boldsymbol{x}},t) = \sum_{j=-1}^{J} \varepsilon^{j+1}\hat{Q}^{j}\left(\frac{\hat{\boldsymbol{x}}}{\varepsilon},\hat{t}\right),$$
$$\hat{\boldsymbol{Z}}^{[J]}_{\varepsilon,int}(\hat{\boldsymbol{x}},t) = \sum_{j=0}^{J} \varepsilon^{j+1}\hat{\boldsymbol{Z}}^{j}\left(\frac{\hat{\boldsymbol{x}}}{\varepsilon},\hat{t}\right).$$

Neka je $\hat{r} = \frac{r}{L}$, sada uvrštavamo

$$\hat{\boldsymbol{u}}_\varepsilon = \sum_{i=1}^{m} \zeta\left(\frac{\hat{x}_1^i}{\hat{r}\varepsilon}\right)\hat{\boldsymbol{u}}_\varepsilon^{i,[J]}(\hat{\boldsymbol{x}}_*^i,\hat{t}) + \sum_{i=1}^{m} \zeta\left(\frac{\hat{x}_1^i}{\hat{r}\varepsilon}\right)\hat{\mathcal{V}}^{i,[J]}_{\varepsilon,bl}(\hat{\boldsymbol{x}}^i,\hat{t}) + \hat{\boldsymbol{V}}^{[J]}_{\varepsilon,int}(\hat{\boldsymbol{x}},\hat{t}),$$
$$\hat{p}_\varepsilon = \sum_{i=1}^{m} \zeta\left(\frac{\hat{x}_1^i}{\hat{r}\varepsilon}\right)\hat{p}_\varepsilon^{i,[J]}(\hat{x}_1^i,\hat{t}) + \sum_{i=1}^{m} \zeta\left(\frac{\hat{x}_1^i}{\hat{r}\varepsilon}\right)\hat{\boldsymbol{Q}}^{i,[J]}_{\varepsilon,bl}(\hat{\boldsymbol{x}}^i,\hat{t}) + \hat{Q}^{[J]}_{\varepsilon,int}(\hat{\boldsymbol{x}},\hat{t}),$$
$$\hat{\boldsymbol{w}}_\varepsilon = \sum_{i=1}^{m} \zeta\left(\frac{\hat{x}_1^i}{\hat{r}\varepsilon}\right)\hat{\boldsymbol{w}}_\varepsilon^{i,[J]}(\hat{\boldsymbol{x}}_*^i,\hat{t}) + \sum_{i=1}^{m} \zeta\left(\frac{\hat{x}_1^i}{\hat{r}\varepsilon}\right)\hat{\mathcal{Z}}^{i,[J]}_{\varepsilon,bl}(\hat{\boldsymbol{x}}^i,\hat{t}) + \hat{\boldsymbol{Z}}^{[J]}_{\varepsilon,int}(\hat{\boldsymbol{x}},\hat{t})$$

u sustav jednadžbi (4.13) te prelazimo na brzu varijablu $\hat{\boldsymbol{y}} = \frac{\hat{\boldsymbol{x}}}{\varepsilon}$. Reskalirana bezdimenzionalna domena $\boldsymbol{G}_\infty$ se sastoji od čvorišta $\hat{\boldsymbol{\Omega}}^0 = \frac{1}{L}\boldsymbol{\Omega}^0$ te polubeskonačnih cijevi $\hat{\boldsymbol{\Omega}}^i_\infty = \{(\hat{y}_1,\hat{\boldsymbol{y}}_*^i) \in \mathbb{R}^3 : \hat{y}_1 > 0,\ \hat{\boldsymbol{y}}_*^i \in \hat{B}^i\}$. Izjednačavanjem članova uz $\boldsymbol{\varepsilon}^{-1}$ dobivamo sljedeći problem za $\hat{Q}^{-1}$:

$$\nabla_{\hat{y}}\hat{Q}^{-1} = -\frac{1}{\hat{r}}\sum_{i=1}^{m} \zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{q}_0\boldsymbol{e}_1^i,$$

koji ima rješenje s kompaktnim nosačem

$$\hat{Q}^{-1}(\hat{\boldsymbol{y}},\hat{t}) = -\sum_{i=1}^{m} \zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{q}_0(\hat{t}) + \hat{q}_0(\hat{t}).$$

Nadalje, problem za $(\hat{\boldsymbol{V}}^0,\hat{Q}^0,\hat{\boldsymbol{Z}}^0)$ u G_∞ je dan sustavom jednadžbi

$$\begin{gathered}-\Delta\hat{\boldsymbol{V}}^0 + \nabla\hat{Q}^0 = 2N\operatorname{rot}\hat{\boldsymbol{Z}}^0 + \hat{\boldsymbol{f}}^0_{int} + \boldsymbol{f}^0_{res},\\ \operatorname{div}\hat{\boldsymbol{V}}^0 = z^0_{res},\\ -\alpha\Delta\hat{\boldsymbol{Z}}^0 - \beta\nabla\operatorname{div}\hat{\boldsymbol{Z}}^0 + 4N\hat{\boldsymbol{Z}}^0 = 2N\operatorname{rot}\hat{\boldsymbol{V}}^0 + \hat{\boldsymbol{g}}^0_{int} + \boldsymbol{g}^0_{res},\end{gathered} \tag{4.35}$$

te pripadnim rubnim uvjetima

$$\hat{\boldsymbol{V}}^0, \hat{\boldsymbol{Z}}^0 = \boldsymbol{0} \quad \text{na} \quad \partial G_\infty.$$

Funkcije u (4.35) su dane s

$$
\begin{aligned}
\boldsymbol{f}_{res}^{0} &= \frac{1}{\hat{r}^2}\sum_{i=1}^{m}\zeta''\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(\hat{u}_1^{i,0}\boldsymbol{e}_1^i + \hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,0}\right) + \frac{1}{\hat{r}}\sum_{i=1}^{m}\zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)(\hat{s}_0^i\hat{y}_1^i - \hat{q}_1 - \hat{\boldsymbol{\mathcal{Q}}}_{bl}^{i,0})\boldsymbol{e}_1^i \\
&\quad + \frac{2N}{\hat{r}}\sum_{i=1}^{m}\zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(-\hat{w}_3^{i,0}\boldsymbol{e}_2^i + \hat{w}_2^{i,0}\boldsymbol{e}_3^i - \hat{\mathcal{Z}}_{bl,3}^{i,0}\boldsymbol{e}_2^i + \hat{\mathcal{Z}}_{bl,2}^{i,0}\boldsymbol{e}_3^i\right), \\
z_{res}^{0} &= -\frac{1}{\hat{r}}\sum_{i=1}^{m}\zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(\hat{u}_1^{i,0} + \hat{\mathcal{V}}_{bl,1}^{i,0}\right), \qquad (4.36) \\
\boldsymbol{g}_{res}^{0} &= \frac{2N}{\hat{r}}\sum_{i=1}^{m}\zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(-\hat{\mathcal{V}}_{bl,3}^{i,0}\boldsymbol{e}_2^i + \hat{\mathcal{V}}_{bl,2}^{i,0}\boldsymbol{e}_3^i\right) + \frac{\alpha}{\hat{r}^2}\sum_{i=1}^{m}\zeta''\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(\hat{\boldsymbol{w}}^{i,0} + \hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,0}\right) \\
&\quad + \frac{\beta}{\hat{r}}\sum_{i=1}^{m}\zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(\nabla\hat{\mathcal{Z}}_{bl,1}^{i,0} + (\operatorname{div}\hat{\boldsymbol{w}}^{i,0} + \operatorname{div}\hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,0})\boldsymbol{e}_1^i\right).
\end{aligned}
$$

Uočimo da su funkcije $\boldsymbol{f}_{res}^0, z_{res}^0, \boldsymbol{g}_{res}^0$ iz (4.36) jednake nuli van kugle radijusa $2\hat{r}$. Dodatno, vrijedi $z_{res}^0 \in W_0^{1,2}(\boldsymbol{G}_\infty)$ te je zadovoljen uvjet kompatibilnosti za problem (4.35):

$$
\int_{G_\infty} z_{res}^0 d\hat{\boldsymbol{y}} = -\frac{1}{\hat{r}}\sum_{i=1}^{m}\int_{\hat{r}}^{2\hat{r}}\zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\int_{\hat{B}^i}\left(\hat{u}_1^{i,0} + \hat{\mathcal{V}}_{bl,1}^{i,0}\right)d\hat{\boldsymbol{y}}_*^i d\hat{y}_1^i = C\sum_{i=1}^{m}\hat{F}^{i,0}(\hat{t}\,) = 0,
$$

pri čemu smo iskoristili (4.6) i (4.28). Sada možemo zaključiti da postoji funkcija $\boldsymbol{J}^0 \in W_0^{2,2}(\boldsymbol{G}_\infty \cap \boldsymbol{B}_{2\hat{r}})$ takva da

$$
\operatorname{div}\boldsymbol{J}^0 = z_{res}^0.
$$

Ovdje $\boldsymbol{B}_{2\hat{r}}$ označava kuglu u $\mathbb{R}^3$ radijusa $2\hat{r}$, to jest $\boldsymbol{B}_{2\hat{r}} = \{\hat{\boldsymbol{y}} \in \mathbb{R}^3 : |\hat{\boldsymbol{y}}| < 2\hat{r}\}$. Za više detalja vidi [71, Appendix]. Rješenje problema (4.35) tražimo u obliku $(\hat{\boldsymbol{V}}^0, \hat{\boldsymbol{Z}}^0) = (\boldsymbol{R}^0 + \boldsymbol{J}^0, \hat{\boldsymbol{Z}}^0)$, gdje $\boldsymbol{R}^0$ i $\hat{\boldsymbol{Z}}^0$ zadovoljavaju sustav jednadžbi

$$
\begin{aligned}
-\Delta\boldsymbol{R}^0 + \nabla\hat{Q}^0 &= 2N\operatorname{rot}\hat{\boldsymbol{Z}}^0 + \hat{\boldsymbol{f}}_{int}^0 + \boldsymbol{f}_{res}^0 + \Delta\boldsymbol{J}^0, \\
\operatorname{div}\boldsymbol{R}^0 &= 0, \qquad (4.37) \\
-\alpha\Delta\hat{\boldsymbol{Z}}^0 - \beta\nabla\operatorname{div}\hat{\boldsymbol{Z}}^0 + 4N\hat{\boldsymbol{Z}}^0 &= 2N\operatorname{rot}\boldsymbol{R}^0 + \hat{\boldsymbol{g}}_{int}^0 + \boldsymbol{g}_{res}^0 + 2N\operatorname{rot}\boldsymbol{J}^0,
\end{aligned}
$$

te rubne uvjete

$$
\boldsymbol{R}^0, \hat{\boldsymbol{Z}}^0 = \boldsymbol{0} \quad \text{na} \quad \partial G_\infty. \qquad (4.38)
$$

Za svaki $t > 0$, postoji jedinstveno rješenje problema (4.37)–(4.38) koje teži k nuli kada $|\hat{\boldsymbol{y}}| \to +\infty$ (vidi Dodatak B).

Problem za $(\hat{\boldsymbol{V}}^1, \hat{Q}^1, \hat{\boldsymbol{Z}}^1)$ u G_∞ je dan sustavom jednadžbi

$$
\begin{aligned}
-\Delta\hat{\boldsymbol{V}}^1 + \nabla\hat{Q}^1 &= 2N \operatorname{rot} \hat{\boldsymbol{Z}}^1 + \hat{\boldsymbol{f}}^1_{int} + \boldsymbol{f}^1_{res},\\
\operatorname{div} \hat{\boldsymbol{V}}^1 &= z^1_{res},\\
-\alpha\Delta\hat{\boldsymbol{Z}}^1 - \beta\nabla \operatorname{div} \hat{\boldsymbol{Z}}^1 + 4N\hat{\boldsymbol{Z}}^1 &= 2N \operatorname{rot} \hat{\boldsymbol{V}}^1 + \hat{\boldsymbol{g}}^1_{int} + \boldsymbol{g}^1_{res},
\end{aligned} \tag{4.39}
$$

te rubnim uvjetima

$$
\hat{\boldsymbol{V}}^1, \hat{\boldsymbol{Z}}^1 = \boldsymbol{0} \quad \text{na} \quad \partial G_\infty. \tag{4.40}
$$

Funkcije koje se pojavljuju u (4.39) su dane s

$$
\begin{aligned}
\boldsymbol{f}^1_{res} &= \frac{1}{\hat{r}^2}\sum_{i=1}^m \zeta''\left(\frac{\hat{y}^i_1}{\hat{r}}\right)\left(\hat{u}^{i,1}_1 \boldsymbol{e}^i_1 + \hat{\boldsymbol{\mathcal{V}}}^{i,1}_{bl}\right) + \frac{1}{\hat{r}}\sum_{i=1}^m \zeta'\left(\frac{\hat{y}^i_1}{\hat{r}}\right)(\hat{s}^i_1 y^i_1 - \hat{q}_2 - \hat{\mathcal{Q}}^{i,1}_{bl})\boldsymbol{e}^i_1\\
&\quad + \frac{2N}{\hat{r}}\sum_{i=1}^m \zeta'\left(\frac{\hat{y}^i_1}{\hat{r}}\right)\left(-\hat{w}^{i,1}_3 \boldsymbol{e}^i_2 + \hat{w}^{i,1}_2 \boldsymbol{e}^i_3 - \hat{\mathcal{Z}}^{i,1}_{bl,3}\boldsymbol{e}^i_2 + \hat{\mathcal{Z}}^{i,1}_{bl,2}\boldsymbol{e}^i_3\right),\\
z^1_{res} &= -\frac{1}{\hat{r}}\sum_{i=1}^m \zeta'\left(\frac{\hat{y}^i_1}{\hat{r}}\right)\left(\hat{u}^{i,1}_1 + \hat{\mathcal{V}}^{i,1}_{bl,1}\right),\\
\boldsymbol{g}^1_{res} &= \frac{2N}{\hat{r}}\sum_{i=1}^m \zeta'\left(-\hat{\mathcal{V}}^{i,1}_{bl,3}\boldsymbol{e}^i_2 + \hat{\mathcal{V}}^{i,1}_{bl,2}\boldsymbol{e}^i_3\right) + \frac{\alpha}{\hat{r}^2}\sum_{i=1}^m \zeta''\left(\frac{\hat{y}^i_1}{\hat{r}}\right)\left(\hat{\boldsymbol{w}}^{i,1} + \hat{\boldsymbol{\mathcal{Z}}}^{i,1}_{bl}\right)\\
&\quad + \frac{\beta}{\hat{r}}\sum_{i=1}^m \zeta'\left(\frac{\hat{y}^i_1}{\hat{r}}\right)\left(\nabla\hat{\mathcal{Z}}^{i,1}_{bl,1} + (\operatorname{div}\hat{\boldsymbol{w}}^{i,1} + \operatorname{div}\hat{\boldsymbol{\mathcal{Z}}}^{i,1}_{bl})\boldsymbol{e}^i_1\right).
\end{aligned}
$$

Egzistencija, jedinstvenost te eksponencijalni pad rješenja problema (4.39)–(4.40) se pokazuje na isti način kao i za korektore nultog reda.

Konačno, problem za korektore unutarnjeg sloja višeg reda $(\hat{\boldsymbol{V}}^j, \hat{Q}^j, \hat{\boldsymbol{Z}}^j)$, $j = 2, 3, \ldots, J$ u G_∞ glasi

$$
\begin{aligned}
-\Delta\hat{\boldsymbol{V}}^j + \nabla\hat{Q}^j &= 2N \operatorname{rot} \hat{\boldsymbol{Z}}^j + \hat{\boldsymbol{f}}^j_{int} + \boldsymbol{f}^j_{res},\\
\operatorname{div} \hat{\boldsymbol{V}}^j &= z^j_{res}.\\
-\alpha\Delta\hat{\boldsymbol{Z}}^j - \beta\nabla \operatorname{div} \hat{\boldsymbol{Z}}^j + 4N\hat{\boldsymbol{Z}}^j &= 2N \operatorname{rot} \hat{\boldsymbol{V}}^j + \hat{\boldsymbol{g}}^j_{int} + \boldsymbol{g}^j_{res},
\end{aligned} \tag{4.41}
$$

te su pripadni rubni uvjeti dani s

$$
\hat{\boldsymbol{V}}^j, \hat{\boldsymbol{Z}}^j = \boldsymbol{0} \quad \text{na} \quad \partial G_\infty. \tag{4.42}
$$

Funkcije u (4.41) su definirane s

$$
\begin{aligned}
\boldsymbol{f}^j_{res} =& \frac{1}{\hat{r}^2}\sum_{i=1}^m \zeta''\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(\hat{u}_1^{i,j}\boldsymbol{e}_1^i + \hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,j}\right) + \frac{1}{\hat{r}}\sum_{i=1}^m \zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)(\hat{s}_j^i\hat{y}_1^i - \hat{q}_{j+1} - \hat{\boldsymbol{Q}}_{bl}^{i,j})\boldsymbol{e}_1^i \\
&+ \frac{2N}{\hat{r}}\sum_{i=1}^m \zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(-\hat{w}_3^{i,j}\boldsymbol{e}_2^i + \hat{w}_2^{i,j}\boldsymbol{e}_3^i - \hat{\mathcal{Z}}_{bl,3}^{i,j}\boldsymbol{e}_2^i + \hat{\mathcal{Z}}_{bl,2}^{i,j}\boldsymbol{e}_3^i\right) - \frac{\partial\hat{\boldsymbol{V}}^{j-2}}{\partial\hat{t}} \\
&- K\sum_{k_1+k_2=j-3}\sum_{i=1}^m\Bigg[\frac{1}{\hat{r}}\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{u}_1^{i,k_1}\hat{u}_1^{i,k_2}\boldsymbol{e}_1^i + \frac{1}{\hat{r}}\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\mathcal{V}}_{bl,1}^{i,k_1}\hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,k_2} \\
&+ \zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)-1\right)(\hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,k_1}\cdot\nabla)\hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,k_2} + \left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{u}}^{i,k_1}\cdot\nabla\right)\left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,k_2}\right) \\
&+ \left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,k_1}\cdot\nabla\right)\left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{u}}^{i,k_2}\right) + \left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{u}}^{i,k_1}\cdot\nabla\right)\hat{\boldsymbol{V}}^{k_2} \\
&+ \left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,k_1}\cdot\nabla\right)\hat{\boldsymbol{V}}^{k_2} + (\hat{\boldsymbol{V}}^{k_1}\cdot\nabla)\left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{u}}^{i,k_2}\right) + (\hat{\boldsymbol{V}}^{k_1}\cdot\nabla)\left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,k_2}\right)\Bigg] \\
&- K\sum_{k_1+k_2=j-3}(\hat{\boldsymbol{V}}^{k_1}\cdot\nabla)\hat{\boldsymbol{V}}^{k_2}, \\
z^j_{res} =& -\frac{1}{\hat{r}}\sum_{i=1}^m \zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(\hat{u}_1^{i,j} + \hat{\mathcal{V}}_{bl,1}^{i,j}\right), \qquad (4.43) \\
\boldsymbol{g}^j_{res} =& \frac{2N}{\hat{r}}\sum_{i=1}^m \zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(-\hat{\mathcal{V}}_{bl,3}^{i,j}\boldsymbol{e}_2^i + \hat{\mathcal{V}}_{bl,2}^{i,j}\boldsymbol{e}_3^i\right) + \frac{\alpha}{\hat{r}^2}\sum_{i=1}^m \zeta''\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(\hat{\boldsymbol{w}}^{i,j} + \hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,j}\right) \\
&+ \frac{\beta}{\hat{r}}\sum_{i=1}^m \zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(\nabla\hat{\mathcal{Z}}_{bl,1}^{i,j} + (\operatorname{div}\hat{\boldsymbol{w}}^{i,j} + \operatorname{div}\hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,j})\boldsymbol{e}_1^i\right) - M\frac{\partial\hat{\boldsymbol{Z}}^{j-2}}{\partial\hat{t}} \\
&- MK\sum_{k_1+k_2=j-3}\sum_{i=1}^m\Bigg[\frac{1}{\hat{r}}\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{u}_1^{i,k_1}\hat{\boldsymbol{w}}_1^{i,k_2} + \frac{1}{\hat{r}}\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\zeta'\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\mathcal{V}}_{bl,1}^{i,k_1}\hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,k_2} \\
&+ \zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)-1\right)(\hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,k_1}\cdot\nabla)\hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,k_2} + \left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{u}}^{i,k_1}\cdot\nabla\right)\left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,k_2}\right) \\
&+ \left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,k_1}\cdot\nabla\right)\left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{w}}^{i,k_2}\right) + \left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{u}}^{i,k_1}\cdot\nabla\right)\hat{\boldsymbol{Z}}^{k_2} \\
&+ \left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{\mathcal{V}}}_{bl}^{i,k_1}\cdot\nabla\right)\hat{\boldsymbol{Z}}^{k_2} + (\hat{\boldsymbol{V}}^{k_1}\cdot\nabla)\left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{w}}^{i,k_2}\right) + (\hat{\boldsymbol{V}}^{k_1}\cdot\nabla)\left(\zeta\left(\frac{\hat{y}_1^i}{\hat{r}}\right)\hat{\boldsymbol{\mathcal{Z}}}_{bl}^{i,k_2}\right)\Bigg] \\
&- MK\sum_{k_1+k_2=j-3}(\hat{\boldsymbol{V}}^{k_1}\cdot\nabla)\hat{\boldsymbol{Z}}^{k_2},
\end{aligned}
$$

pri čemu je $q_{j+1} := 0$ kada $j = J$. Svi izrazi u (4.43) imaju kompaktan nosač u kugli radijusa $2\hat{r}$ ili eksponencijalno teže k nuli kada $|\hat{\boldsymbol{y}}| \to +\infty$, što znači da možemo primijeniti rezultate iz Dodatka B na isti način kao i kod korektora nultog reda. Dakle, za svaki $j = 2, 3, \ldots, J$ imamo jedinstveno rješenje problema (4.41)–(4.42) koje eksponencijalno teži k nuli kada $|\hat{\boldsymbol{y}}| \to +\infty$, za svaki $t > 0$. Kao i ranije, za regularnost u vremenu korektora unutarnjeg sloja moramo pretpostaviti sljedeće:

$$\frac{\partial \hat{\boldsymbol{f}}_{int}^{j}}{\partial \hat{t}}, \ \frac{\partial \hat{\boldsymbol{g}}_{int}^{j}}{\partial \hat{t}} \in \mathcal{L}^2_\beta(G_\infty), \quad j = 0, 1, \dots, J,$$
$$\frac{\partial^2 \hat{\boldsymbol{f}}_{int}^{j-2}}{\partial \hat{t}^2}, \ \frac{\partial^2 \hat{\boldsymbol{g}}_{int}^{j-2}}{\partial \hat{t}^2} \in \mathcal{L}^2_\beta(G_\infty), \quad j = 2, 3, \dots, J.$$

4.2.4. Asimptotička aproksimacija

Regularni dio asimptotičke aproksimacije u i-toj cijevi Ω^i_ε, $i = 1, 2, \dots, m$ daleko od čvorišta Ω^0_ε je dan u sljedećem obliku:

$$\boldsymbol{U}_\varepsilon^{i,[J]}(\boldsymbol{x}_*^i, t) = \sum_{j=0}^{J} \varepsilon^{j+2} \boldsymbol{U}^{i,j}\left(\frac{\boldsymbol{x}_*^i}{\varepsilon}, t\right) = \sum_{j=0}^{J} \varepsilon^{j+2} U_1^{i,j}\left(\frac{\boldsymbol{x}_*^i}{\varepsilon}, t\right) \boldsymbol{e}_1^i,$$
$$P_\varepsilon^{i,[J]}(x_1^i, t) = \sum_{j=0}^{J} \varepsilon^{j} P^{i,j}(x_1^i, t) = \sum_{j=0}^{J} \varepsilon^{j} \left(-s_j^i(t) x_1^i + q_j(t)\right),$$
$$\boldsymbol{W}_\varepsilon^{i,[J]}(\boldsymbol{x}_*^i, t) = \sum_{j=0}^{J} \varepsilon^{j+1} \boldsymbol{W}^{i,j}\left(\frac{\boldsymbol{x}_*^i}{\varepsilon}, t\right) = \sum_{j=0}^{J} \varepsilon^{j+1} \left(W_2^{i,j}\left(\frac{\boldsymbol{x}_*^i}{\varepsilon}, t\right) \boldsymbol{e}_2^i + W_3^{i,j}\left(\frac{\boldsymbol{x}_*^i}{\varepsilon}, t\right) \boldsymbol{e}_3^i\right),$$

gdje je

$$U_1^{i,j}\left(\frac{\boldsymbol{x}_*^i}{\varepsilon}, t\right) = U_{ref}\, \hat{u}_1^{i,j}\left(\frac{\hat{\boldsymbol{x}}_*^i}{\varepsilon}, \hat{t}\right),$$
$$s_j^i(t) = \frac{(\mu + \mu_r) U_{ref}}{L^2} \hat{s}_j^i(\hat{t}), \ \ q_j(t) = \frac{(\mu + \mu_r) U_{ref}}{L} \hat{q}_j(\hat{t}),$$
$$W_k^{i,j}\left(\frac{\boldsymbol{x}_*^i}{\varepsilon}, t\right) = \frac{U_{ref}}{L} \hat{w}_k^{i,j}\left(\frac{\hat{\boldsymbol{x}}_*^i}{\varepsilon}, \hat{t}\right), \ k = 2, 3.$$

Funkcije $\hat{u}_1^{i,j}, \hat{s}_j^i, \hat{q}_j, \hat{w}_2^{i,j}, \hat{w}_3^{i,j}$ označavaju regularni dio razvoja u i-toj bezdimenzionalnoj cijevi određen u Odjeljku 4.2.1.3. Korektori rubnog sloja na kraju i-te cijevi Ω^i_ε, $i = 1, 2, \dots, m$ su dani s

$$\mathcal{V}_{\varepsilon,bl}^{i,[J]}(\boldsymbol{x}^i, t) = \sum_{j=0}^{J} \varepsilon^{j+2} \mathcal{V}_{bl}^{i,j}\left(\frac{x_1^i - \ell_i}{\varepsilon}, \frac{\boldsymbol{x}_*^i}{\varepsilon}, t\right),$$
$$\mathcal{Q}_{\varepsilon,bl}^{i,[J]}(\boldsymbol{x}^i, t) = \sum_{j=0}^{J} \varepsilon^{j+1} \varepsilon \mathcal{Q}_{bl}^{i,j}\left(\frac{x_1^i - \ell_i}{\varepsilon}, \frac{\boldsymbol{x}_*^i}{\varepsilon}, t\right),$$
$$\mathcal{Z}_{\varepsilon,bl}^{i,[J]}(\boldsymbol{x}^i, t) = \sum_{j=0}^{J} \varepsilon^{j+1} \mathcal{Z}_{bl}^{i,j}\left(\frac{x_1^i - \ell_i}{\varepsilon}, \frac{\boldsymbol{x}_*^i}{\varepsilon}, t\right),$$

pri čemu

$$\mathcal{V}^{i,j}_{bl}\left(\frac{x^i_1-\ell_i}{\varepsilon},\frac{\boldsymbol{x}^i_*}{\varepsilon},t\right)=U_{ref}\hat{\mathcal{V}}^{i,j}_{bl}\left(\frac{\hat{x}^i_1-\hat{\ell}_i}{\varepsilon},\frac{\hat{\boldsymbol{x}}^i_*}{\varepsilon},\hat{t}\right),$$
$$\mathcal{Q}^{i,j}_{bl}\left(\frac{x^i_1-\ell_i}{\varepsilon},\frac{\boldsymbol{x}^i_*}{\varepsilon},t\right)=\frac{(\mu+\mu_r)U_{ref}}{L}\hat{\mathcal{Q}}^{i,j}_{bl}\left(\frac{\hat{x}^i_1-\hat{\ell}_i}{\varepsilon},\frac{\hat{\boldsymbol{x}}^i_*}{\varepsilon},\hat{t}\right),$$
$$\mathcal{Z}^{i,j}_{bl}\left(\frac{x^i_1-\ell_i}{\varepsilon},\frac{\boldsymbol{x}^i_*}{\varepsilon},t\right)=\frac{U_{ref}}{L}\hat{\mathcal{Z}}^{i,j}_{bl}\left(\frac{\hat{x}^i_1-\hat{\ell}_i}{\varepsilon},\frac{\hat{\boldsymbol{x}}^i_*}{\varepsilon},\hat{t}\right).$$

Ovdje su funkcije $\hat{\mathcal{V}}^{i,j}_{bl},\hat{\mathcal{Q}}^{i,j}_{bl},\hat{\mathcal{Z}}^{i,j}_{bl}$ korektori rubnog sloja na kraju i-te bezdimenzionalne cijevi određeni u Odjeljku 4.2.2. Korektori unutarnjeg sloja definirani na $\boldsymbol{\Omega}_\varepsilon$ su oblika

$$\boldsymbol{V}^{[J]}_{\varepsilon,int}(\boldsymbol{x},t)=\sum_{j=0}^{J}\varepsilon^{j+2}\boldsymbol{V}^j\left(\frac{\boldsymbol{x}}{\varepsilon},t\right),$$
$$Q^{[J]}_{\varepsilon,int}(\boldsymbol{x},t)=\sum_{j=-1}^{J}\varepsilon^{j+1}Q^j\left(\frac{\boldsymbol{x}}{\varepsilon},t\right),$$
$$\boldsymbol{Z}^{[J]}_{\varepsilon,int}(\boldsymbol{x},t)=\sum_{j=0}^{J}\varepsilon^{j+1}\boldsymbol{Z}^j\left(\frac{\boldsymbol{x}}{\varepsilon},t\right),$$

gdje

$$\boldsymbol{V}^j\left(\frac{\boldsymbol{x}}{\varepsilon},t\right)=U_{ref}\hat{\boldsymbol{V}}^j\left(\frac{\hat{\boldsymbol{x}}}{\varepsilon},\hat{t}\right),$$
$$Q^j\left(\frac{\boldsymbol{x}}{\varepsilon},t\right)=\frac{(\mu+\mu_r)U_{ref}}{L}\hat{Q}^j\left(\frac{\hat{\boldsymbol{x}}}{\varepsilon},\hat{t}\right),$$
$$\boldsymbol{Z}^j\left(\frac{\boldsymbol{x}}{\varepsilon},t\right)=\frac{U_{ref}}{L}\hat{\boldsymbol{Z}}^j\left(\frac{\hat{\boldsymbol{x}}}{\varepsilon},\hat{t}\right).$$

Funkcije $\hat{\boldsymbol{V}}^j,\hat{Q}^j,\hat{\boldsymbol{Z}}^j$ su korektori unutarnjeg sloja u bezdimenzionalnom sustavu cijevi određeni u Odjeljku 4.2.3. Konačno, asimptotička aproksimacija toka mikropolarnog fluida u sustavu cijevi $\boldsymbol{\Omega}_\varepsilon$ je dana s

$$\begin{aligned}
\boldsymbol{u}^{approx,[J]}_\varepsilon&=\sum_{i=1}^{m}\zeta\left(\frac{x^i_1}{r\varepsilon}\right)\boldsymbol{U}^{i,[J]}_\varepsilon(\boldsymbol{x}^i_*,t)+\sum_{i=1}^{m}\zeta\left(\frac{x^i_1}{r\varepsilon}\right)\mathcal{V}^{i,[J]}_{\varepsilon,bl}(\boldsymbol{x}^i,t)\\
&\quad+\left(1-\zeta\left(\frac{3|\boldsymbol{x}|}{\ell_{min}}\right)\right)\boldsymbol{V}^{[J]}_{\varepsilon,int}(\boldsymbol{x},t),\\
p^{approx,[J]}_\varepsilon&=\sum_{i=1}^{m}\zeta\left(\frac{x^i_1}{r\varepsilon}\right)P^{i,[J]}_\varepsilon(x^i_1,t)+\sum_{i=1}^{m}\zeta\left(\frac{x^i_1}{r\varepsilon}\right)\mathcal{Q}^{i,[J]}_{\varepsilon,bl}(\boldsymbol{x}^i,t)\\
&\quad+\left(1-\zeta\left(\frac{3|\boldsymbol{x}|}{\ell_{min}}\right)\right)Q^{[J]}_{\varepsilon,int}(\boldsymbol{x},t),\\
\boldsymbol{w}^{approx,[J]}_\varepsilon&=\sum_{i=1}^{m}\zeta\left(\frac{x^i_1}{r\varepsilon}\right)\boldsymbol{W}^{i,[J]}_\varepsilon(\boldsymbol{x}^i_*,t)+\sum_{i=1}^{m}\zeta\left(\frac{x^i_1}{r\varepsilon}\right)\mathcal{Z}^{i,[J]}_{\varepsilon,bl}(\boldsymbol{x}^i,t)\\
&\quad+\left(1-\zeta\left(\frac{3|\boldsymbol{x}|}{\ell_{min}}\right)\right)\boldsymbol{Z}^{[J]}_{\varepsilon,int}(\boldsymbol{x},t),
\end{aligned}\tag{4.44}$$

pri čemu je $\ell_{min} := \min_{i=1,2,\ldots,m} \ell_i$. U razvoju (4.44) smo korektore unutarnjeg sloja pomnožili s izglađujućom funkcijom kako bismo uklonili njihov utjecaj na krajevima cijevi; ovo ćemo opravdati u ocjeni greške.

Cilj idućeg odjeljka je rigorozno opravdati predloženi asimptotički model, odnosno ocijeniti razliku rješenja problema (4.1)–(4.3) i dobivene asimptotičke aproksimacije (4.44) u prikladnoj normi.

4.3. Rigorozno opravdanje

4.3.1. Apriorne ocjene

Teorem 4.3.1 (Apriorne ocjene). Neka je $(\boldsymbol{u}_\varepsilon, \boldsymbol{w}_\varepsilon)$ slabo rješenje problema (4.1)–(4.3). Tada vrijede sljedeće ocjene:

$$\begin{aligned}
\|\boldsymbol{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\leq C\varepsilon^2, \\
\left\|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \left\|\nabla \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\leq C\varepsilon^2, \\
\|\boldsymbol{w}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\nabla \boldsymbol{w}_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\leq C\varepsilon^3, \\
\left\|\frac{\partial \boldsymbol{w}_\varepsilon}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \left\|\nabla \frac{\partial \boldsymbol{w}_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\leq C\varepsilon^3, \\
\|p_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\leq C\varepsilon,
\end{aligned} \tag{4.45}$$

gdje je C konstanta neovisna o ε.

Dokaz. Neka je $(\boldsymbol{u}_\varepsilon, \boldsymbol{w}_\varepsilon) = (\boldsymbol{v}_\varepsilon + \boldsymbol{h}_\varepsilon^{ext}, \boldsymbol{w}_\varepsilon)$ slabo rješenje problema (4.1)–(4.3), odnosno neka $(\boldsymbol{v}_\varepsilon, \boldsymbol{w}_\varepsilon)$ zadovoljava (4.8)–(4.10). Uvrštavanjem $\boldsymbol{\Psi} = \boldsymbol{w}_\varepsilon(\cdot, t)$ u $(4.9)_2$ dobivamo

$$\begin{aligned}
&\frac{I}{2}\frac{d}{dt}\int_{\Omega_\varepsilon} |\boldsymbol{w}_\varepsilon|^2 + (c_\alpha + c_d)\int_{\Omega_\varepsilon} |\nabla \boldsymbol{w}_\varepsilon|^2 + (c_0 + c_d - c_\alpha)\int_{\Omega_\varepsilon} (\operatorname{div} \boldsymbol{w}_\varepsilon)^2 + 4\mu_r \int_{\Omega_\varepsilon} |\boldsymbol{w}_\varepsilon|^2 \\
&\quad = -I\int_{\Omega_\varepsilon} (\boldsymbol{v}_\varepsilon \cdot \nabla)\boldsymbol{w}_\varepsilon \cdot \boldsymbol{w}_\varepsilon - I\int_{\Omega_\varepsilon} (\boldsymbol{h}_\varepsilon^{ext} \cdot \nabla)\boldsymbol{w}_\varepsilon \cdot \boldsymbol{w}_\varepsilon + 2\mu_r \int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{v}_\varepsilon \cdot \boldsymbol{w}_\varepsilon \\
&\quad\quad + 2\mu_r \int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{h}_\varepsilon^{ext} \cdot \boldsymbol{w}_\varepsilon + \int_{\Omega_\varepsilon} \boldsymbol{g}_\varepsilon \cdot \boldsymbol{w}_\varepsilon.
\end{aligned} \tag{4.46}$$

Za funkcije $\boldsymbol{a}_\varepsilon, \boldsymbol{b}_\varepsilon$ takve da $\operatorname{div} \boldsymbol{a}_\varepsilon = 0$ u Ω_ε i $\boldsymbol{a}_\varepsilon, \boldsymbol{b}_\varepsilon = \boldsymbol{0}$ na $\partial\Omega_\varepsilon$ vrijedi

$$\int_{\Omega_\varepsilon} (\boldsymbol{a}_\varepsilon \cdot \nabla)\boldsymbol{b}_\varepsilon \cdot \boldsymbol{b}_\varepsilon = \frac{1}{2}\int_{\Omega_\varepsilon} \boldsymbol{a}_\varepsilon \cdot \nabla|\boldsymbol{b}_\varepsilon|^2 = \frac{1}{2}\int_{\Omega_\varepsilon} \operatorname{div}(\boldsymbol{a}_\varepsilon|\boldsymbol{b}_\varepsilon|^2) - \frac{1}{2}\int_{\Omega_\varepsilon} \operatorname{div} \boldsymbol{a}_\varepsilon|\boldsymbol{b}_\varepsilon|^2 = 0, \tag{4.47}$$

što znači da su prva dva izraza s desne strane jednakosti (4.46) jednaka nuli. Ostale izraze u (4.46) ocjenjujemo koristeći ocjenu $(4.7)_2$ i Lemu 0.0.1:

$$\begin{aligned}
\int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{v}_\varepsilon \cdot \boldsymbol{w}_\varepsilon &\leq \|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\boldsymbol{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon \|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla \boldsymbol{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{h}_\varepsilon^{ext} \cdot \boldsymbol{w}_\varepsilon &\leq \left\|\nabla \boldsymbol{h}_\varepsilon^{ext}\right\|_{L^2(\Omega_\varepsilon)} \|\boldsymbol{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon \left\|\nabla \boldsymbol{h}_\varepsilon^{ext}\right\|_{L^2(\Omega_\varepsilon)} \|\nabla \boldsymbol{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^3 \|\nabla \boldsymbol{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)},
\end{aligned} \tag{4.48}$$

$$\begin{aligned}\int_{\Omega_\varepsilon} \boldsymbol{g}_\varepsilon \cdot \boldsymbol{w}_\varepsilon &\leq \left\|\boldsymbol{g}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \|\boldsymbol{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\leq C\varepsilon \left\|\boldsymbol{g}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \|\nabla \boldsymbol{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\leq C\varepsilon^3 \|\nabla \boldsymbol{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)}.\end{aligned}$$

Sada iz (4.46), (4.48) te Youngove nejednakosti slijedi ocjena

$$\frac{d}{dt}\int_{\Omega_\varepsilon} |\boldsymbol{w}_\varepsilon|^2 + \int_{\Omega_\varepsilon} |\nabla \boldsymbol{w}_\varepsilon|^2 \leq C\varepsilon^2 \|\nabla \boldsymbol{v}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + C\varepsilon^6. \tag{4.49}$$

Integrirajući (4.49) po t dobivamo

$$\sup_{t\in[0,T]} \left\|\boldsymbol{w}_\varepsilon(\cdot,t)\right\|^2_{L^2(\Omega_\varepsilon)} + \int_0^T \|\nabla \boldsymbol{w}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^2 \int_0^T \|\nabla \boldsymbol{v}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + C\varepsilon^6. \tag{4.50}$$

Nadalje, uvrštavamo $\boldsymbol{\varphi} = \boldsymbol{v}_\varepsilon(\cdot,t)$ u $(4.9)_1$, iz čega slijedi

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\int_{\Omega_\varepsilon} |\boldsymbol{v}_\varepsilon|^2 + (\mu+\mu_r)\int_{\Omega_\varepsilon} |\nabla \boldsymbol{v}_\varepsilon|^2 = -\int_{\Omega_\varepsilon} (\boldsymbol{v}_\varepsilon\cdot\nabla)\boldsymbol{v}_\varepsilon\cdot\boldsymbol{v}_\varepsilon - \int_{\Omega_\varepsilon} (\boldsymbol{h}^{ext}_\varepsilon\cdot\nabla)\boldsymbol{v}_\varepsilon\cdot\boldsymbol{v}_\varepsilon \\ -\int_{\Omega_\varepsilon} (\boldsymbol{v}_\varepsilon\cdot\nabla)\boldsymbol{h}^{ext}_\varepsilon\cdot\boldsymbol{v}_\varepsilon + 2\mu_r\int_{\Omega_\varepsilon} \operatorname{rot}\boldsymbol{w}_\varepsilon\cdot\boldsymbol{v}_\varepsilon - \int_{\Omega_\varepsilon} \frac{\partial \boldsymbol{h}^{ext}_\varepsilon}{\partial t}\cdot\boldsymbol{v}_\varepsilon - \int_{\Omega_\varepsilon} (\boldsymbol{h}^{ext}_\varepsilon\cdot\nabla)\boldsymbol{h}^{ext}_\varepsilon\cdot\boldsymbol{v}_\varepsilon \\ -(\mu+\mu_r)\int_{\Omega_\varepsilon} \nabla\boldsymbol{h}^{ext}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon + \int_{\Omega_\varepsilon} \boldsymbol{f}_\varepsilon\cdot\boldsymbol{v}_\varepsilon.\end{gathered} \tag{4.51}$$

Koristeći (4.47) vidimo da su prva dva člana na desnoj strani jednakosti (4.51) jednaka nuli. Koristeći ocjenu (4.7) i Lemu 0.0.1 ocjenjujemo preostale izraze na desnoj strani jednakosti (4.51) na sljedeći način:

$$\begin{aligned}\int_{\Omega_\varepsilon} (\boldsymbol{v}_\varepsilon\cdot\nabla)\boldsymbol{h}^{ext}_\varepsilon\cdot\boldsymbol{v}_\varepsilon &\leq \left\|\nabla\boldsymbol{h}^{ext}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \|\boldsymbol{v}_\varepsilon\|^2_{L^4(\Omega_\varepsilon)} \\ &\leq C\varepsilon^{1/2}\left\|\nabla\boldsymbol{h}^{ext}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \|\nabla\boldsymbol{v}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \\ &\leq C\varepsilon^{5/2}\|\nabla\boldsymbol{v}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}, \\ \int_{\Omega_\varepsilon} \operatorname{rot}\boldsymbol{w}_\varepsilon\cdot\boldsymbol{v}_\varepsilon &\leq \|\nabla\boldsymbol{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\leq C\varepsilon\|\nabla\boldsymbol{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\ \int_{\Omega_\varepsilon} \frac{\partial\boldsymbol{h}^{ext}_\varepsilon}{\partial t}\cdot\boldsymbol{v}_\varepsilon &\leq \left\|\frac{\partial\boldsymbol{h}^{ext}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)}\|\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\leq C\varepsilon\left\|\frac{\partial\boldsymbol{h}^{ext}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\leq C\varepsilon^4\|\nabla\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)},\end{aligned} \tag{4.52}$$

$$\begin{aligned}
\int_{\Omega_\varepsilon} (\boldsymbol{h}_\varepsilon^{ext} \cdot \nabla)\boldsymbol{h}_\varepsilon^{ext} \cdot \boldsymbol{v}_\varepsilon &\leq \left\|\boldsymbol{h}_\varepsilon^{ext}\right\|_{L^4(\Omega_\varepsilon)} \left\|\nabla \boldsymbol{h}_\varepsilon^{ext}\right\|_{L^2(\Omega_\varepsilon)} \|\boldsymbol{v}_\varepsilon\|_{L^4(\Omega_\varepsilon)} \\
&\leq C\varepsilon^{1/2} \left\|\nabla \boldsymbol{h}_\varepsilon^{ext}\right\|_{L^2(\Omega_\varepsilon)}^2 \|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^{9/2} \|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \nabla \boldsymbol{h}_\varepsilon^{ext} \cdot \nabla \boldsymbol{v}_\varepsilon &\leq \left\|\nabla \boldsymbol{h}_\varepsilon^{ext}\right\|_{L^2(\Omega_\varepsilon)} \|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^2 \|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \boldsymbol{f}_\varepsilon \cdot \boldsymbol{v}_\varepsilon &\leq \left\|\boldsymbol{f}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \|\boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon \left\|\boldsymbol{f}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^4 \|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}.
\end{aligned}$$

Vratimo se sada na (4.51). Uzimajući u obzir izvedene ocjene (4.52) te koristeći Youngovu nejednakost dobivamo ocjenu

$$\frac{d}{dt}\int_{\Omega_\varepsilon} |\boldsymbol{v}_\varepsilon|^2 + \int_{\Omega_\varepsilon} |\nabla \boldsymbol{v}_\varepsilon|^2 \leq C\varepsilon^2 \|\nabla \boldsymbol{w}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + C\varepsilon^{5/2} \|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + C\varepsilon^4. \tag{4.53}$$

Integriranjem (4.53) po t i korištenjem nejednakosti (4.50) vidimo da za dovoljno mali ε vrijedi

$$\sup_{t\in[0,T]} \left\|\boldsymbol{v}_\varepsilon(\cdot,t)\right\|_{L^2(\Omega_\varepsilon)}^2 + \int_0^T \|\nabla \boldsymbol{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq C\varepsilon^4. \tag{4.54}$$

Konačno, uvrštavanjem (4.54) u (4.50) dobivamo

$$\sup_{t\in[0,T]} \left\|\boldsymbol{w}_\varepsilon(\cdot,t)\right\|_{L^2(\Omega_\varepsilon)}^2 + \int_0^T \|\nabla \boldsymbol{w}_\varepsilon\|^2 \leq C\varepsilon^6. \tag{4.55}$$

Deriviranjem integralne jednakosti (4.9) i uvrštavanjem $\boldsymbol{\varphi} = \frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}(\cdot,t)$ i $\boldsymbol{\Psi} = \frac{\partial \boldsymbol{w}_\varepsilon}{\partial t}(\cdot,t)$ možemo sličnim metodama kao i gore dobiti sljedeće ocjene[1]:

$$\begin{aligned}
\sup_{t\in[0,T]} \left\|\frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}(\cdot,t)\right\|_{L^2(\Omega_\varepsilon)}^2 + \int_0^T \left\|\nabla \frac{\partial \boldsymbol{v}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)}^2 &\leq C\varepsilon^4, \\
\sup_{t\in[0,T]} \left\|\frac{\partial \boldsymbol{w}_\varepsilon}{\partial t}(\cdot,t)\right\|_{L^2(\Omega_\varepsilon)}^2 + \int_0^T \left\|\nabla \frac{\partial \boldsymbol{w}_\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon)}^2 &\leq C\varepsilon^6.
\end{aligned} \tag{4.56}$$

Ocjene $(4.45)_1$–$(4.45)_4$ sada slijede iz (4.7), (4.54), (4.55) i (4.56). Preostalo je utvrditi postojanje funkcije tlaka te izvesti pripadne ocjene. Promotrimo linearan operator $\mathcal{F}$

[1]Formalno, naša slaba rješenja nisu dovoljno glatka za gornji postupak. Ocjene (4.56) se mogu dobiti izvođenjem apriornih ocjena za Galerkinove aproksimacije i prelaskom na limes (vidi npr. [53]). Ovdje trebamo povećati glatkoću zadanih funkcija, to jest pretpostavljamo dodatno da vrijedi $\frac{\partial \boldsymbol{f}_\varepsilon}{\partial t}, \frac{\partial \boldsymbol{g}_\varepsilon}{\partial t} \in L^2(0,T;L^2(\Omega_\varepsilon))$.

definiran s

$$\mathcal{F}(\boldsymbol{\varphi}) = \int_{\Omega_\varepsilon} \frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} \cdot \boldsymbol{\varphi} + \int_{\Omega_\varepsilon} (\boldsymbol{v}_\varepsilon \cdot \nabla)\boldsymbol{v}_\varepsilon \cdot \boldsymbol{\varphi} + (\mu + \mu_r) \int_{\Omega_\varepsilon} \nabla \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{\varphi} + \int_{\Omega_\varepsilon} (\boldsymbol{h}_\varepsilon^{ext} \cdot \nabla)\boldsymbol{v}_\varepsilon \cdot \boldsymbol{\varphi}$$
$$+ \int_{\Omega_\varepsilon} (\boldsymbol{v}_\varepsilon \cdot \nabla)\boldsymbol{h}_\varepsilon^{ext} \cdot \boldsymbol{\varphi} - 2\mu_r \int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{w}_\varepsilon \cdot \boldsymbol{\varphi} + \int_{\Omega_\varepsilon} \left(\frac{\partial \boldsymbol{h}_\varepsilon^{ext}}{\partial t} + (\boldsymbol{h}_\varepsilon^{ext} \cdot \nabla)\boldsymbol{h}_\varepsilon^{ext} \right) \cdot \boldsymbol{\varphi}$$
$$+ (\mu + \mu_r) \int_{\Omega_\varepsilon} \nabla \boldsymbol{h}_\varepsilon^{ext} \cdot \nabla \boldsymbol{\varphi} - \int_{\Omega_\varepsilon} \boldsymbol{f}_\varepsilon \cdot \boldsymbol{\varphi}.$$

Lako se provjeri da je $\mathcal{F}$ ograničen na $W_0^{1,2}(\Omega_\varepsilon)$ te da vrijedi $\mathcal{F}(\boldsymbol{\varphi}) = 0$ kada $\operatorname{div} \boldsymbol{\varphi} = 0$, dakle postoji funkcija $p_\varepsilon(\cdot, t) \in L_0^2(\Omega_\varepsilon)$ takva da (vidi npr. [38])

$$\mathcal{F}(\boldsymbol{\varphi}) = \int_{\Omega_\varepsilon} p_\varepsilon(\boldsymbol{x}, t) \operatorname{div} \boldsymbol{\varphi}(\boldsymbol{x}) d\boldsymbol{x}, \ \forall \boldsymbol{\varphi} \in W_0^{1,2}(\Omega_\varepsilon). \tag{4.57}$$

Promotrimo sljedeći problem:

$$\begin{aligned} \operatorname{div} \boldsymbol{d}_\varepsilon &= p_\varepsilon \quad \text{u} \quad \Omega_\varepsilon, \\ \boldsymbol{d}_\varepsilon &= \boldsymbol{0} \quad \text{na} \quad \partial\Omega_\varepsilon. \end{aligned} \tag{4.58}$$

Kako su zadovoljeni uvjeti Leme 0.0.2, zaključujemo da problem (4.58) ima rješenje $\boldsymbol{d}_\varepsilon$ takvo da

$$\|\nabla \boldsymbol{d}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon} \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{4.59}$$

Uvrštavanjem $\boldsymbol{\varphi} = \boldsymbol{d}_\varepsilon(\cdot, t)$ u (4.57) dobivamo sljedeću jednakost:

$$\begin{aligned} \int_{\Omega_\varepsilon} |p_\varepsilon|^2 = \int_{\Omega_\varepsilon} \frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} \cdot \boldsymbol{d}_\varepsilon + \int_{\Omega_\varepsilon} (\boldsymbol{v}_\varepsilon \cdot \nabla)\boldsymbol{v}_\varepsilon \cdot \boldsymbol{d}_\varepsilon + (\mu + \mu_r) \int_{\Omega_\varepsilon} \nabla \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{d}_\varepsilon \\ + \int_{\Omega_\varepsilon} (\boldsymbol{h}_\varepsilon^{ext} \cdot \nabla)\boldsymbol{v}_\varepsilon \cdot \boldsymbol{d}_\varepsilon + \int_{\Omega_\varepsilon} (\boldsymbol{v}_\varepsilon \cdot \nabla)\boldsymbol{h}_\varepsilon^{ext} \cdot \boldsymbol{d}_\varepsilon - 2\mu_r \int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{w}_\varepsilon \cdot \boldsymbol{d}_\varepsilon \\ + \int_{\Omega_\varepsilon} \left(\frac{\partial \boldsymbol{h}_\varepsilon^{ext}}{\partial t} + (\boldsymbol{h}_\varepsilon^{ext} \cdot \nabla)\boldsymbol{h}_\varepsilon^{ext} \right) \cdot \boldsymbol{d}_\varepsilon + (\mu + \mu_r) \int_{\Omega_\varepsilon} \nabla \boldsymbol{h}_\varepsilon^{ext} \cdot \nabla \boldsymbol{d}_\varepsilon - \int_{\Omega_\varepsilon} \boldsymbol{f}_\varepsilon \cdot \boldsymbol{d}_\varepsilon. \end{aligned} \tag{4.60}$$

Koristeći ocjene (4.7), (4.54), (4.55), $(4.56)_1$ te Lemu 0.0.1, možemo ocijeniti izraze na desnoj strani jednakosti (4.60). Konačno, integriranjem dobivenog po t te korištenjem ocjene (4.59) zaključujemo

$$\|p_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C\varepsilon.$$

Ovime je dokaz dovršen.

■

4.3.2. Ocjene greške

Uvodimo sljedeće oznake za razliku između originalnog rješenja problema (4.1)–(4.3) te asimptotičke aproksimacije (4.44):

$$\boldsymbol{D}_\varepsilon^{[J]} = \boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^{approx,[J]}, \quad d_\varepsilon^{[J]} = p_\varepsilon - p_\varepsilon^{approx,[J]}, \quad \boldsymbol{S}_\varepsilon^{[J]} = \boldsymbol{w}_\varepsilon - \boldsymbol{w}_\varepsilon^{approx,[J]}.$$

Sada iskazujemo glavni rezultat ovog odjeljka.

Teorem 4.3.2 (Ocjene greške)**.** Vrijede sljedeće ocjene:

$$
\begin{aligned}
\left\|\boldsymbol{D}_\varepsilon^{[J]}\right\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \left\|\nabla \boldsymbol{D}_\varepsilon^{[J]}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\leq C\varepsilon^{J+3},\\
\left\|\frac{\partial \boldsymbol{D}_\varepsilon^{[J]}}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \left\|\nabla \frac{\partial \boldsymbol{D}_\varepsilon^{[J]}}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\leq C\varepsilon^{J+3},\\
\left\|\boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \left\|\nabla \boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\leq C\varepsilon^{J+2},\\
\left\|\frac{\partial \boldsymbol{S}_\varepsilon^{[J]}}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \left\|\nabla \frac{\partial \boldsymbol{S}_\varepsilon^{[J]}}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\leq C\varepsilon^{J+2},\\
\left\|d_\varepsilon^{[J]}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))} &\leq C\varepsilon^{J+2},
\end{aligned}
$$

gdje je C konstanta neovisna o ε.

Dokaz. Aproksimacija mikrorotacije $\boldsymbol{w}_\varepsilon^{approx,[J]}$ zadovoljava sljedeću jednakost:

$$
\begin{aligned}
I\frac{\partial \boldsymbol{w}_\varepsilon^{approx,[J]}}{\partial t} + I(\boldsymbol{u}_\varepsilon^{approx,[J]}\cdot\nabla)\boldsymbol{w}_\varepsilon^{approx,[J]} - (c_\alpha + c_d)\Delta \boldsymbol{w}_\varepsilon^{approx,[J]}&\\
-(c_0 + c_d - c_\alpha)\nabla \operatorname{div} \boldsymbol{w}_\varepsilon^{approx,[J]} + 4\mu_r \boldsymbol{w}_\varepsilon^{approx,[J]} = 2\mu_r \operatorname{rot} \boldsymbol{u}_\varepsilon^{approx,[J]} + \boldsymbol{g}_\varepsilon + \mathcal{R}_\varepsilon^{g,[J]},&\\
\boldsymbol{w}_\varepsilon^{approx,[J]} = \boldsymbol{0} \text{ na } \partial\Omega_\varepsilon,&\\
\boldsymbol{w}_\varepsilon^{approx,[J]}(\boldsymbol{x},0) = \boldsymbol{0} \text{ u } \Omega_\varepsilon,&
\end{aligned}
\tag{4.61}
$$

pri čemu je

$$
\begin{aligned}
\mathcal{R}_\varepsilon^{g,[J]} =& I\sum_{i=1}^m \zeta\left(\frac{x_1^i}{r\varepsilon}\right)\left(\varepsilon^J\frac{\partial \boldsymbol{W}^{i,J-1}}{\partial t} + \varepsilon^{J+1}\frac{\partial \boldsymbol{W}^{i,J}}{\partial t} + \varepsilon^J\frac{\partial \boldsymbol{\mathcal{Z}}_{bl}^{i,J-1}}{\partial t} + \varepsilon^{J+1}\frac{\partial \boldsymbol{\mathcal{Z}}_{bl}^{i,J}}{\partial t}\right)\\
&+ I\varepsilon^J\frac{\partial \boldsymbol{Z}^{J-1}}{\partial t} + I\varepsilon^{J+1}\frac{\partial \boldsymbol{Z}^J}{\partial t}\\
&+ I\sum_{J-2\leq k_1+k_2\leq 2J}\varepsilon^{k_1+k_2+2}\left[\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{U}^{i,k_1}\cdot\nabla\right)\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{W}^{i,k_2}\right)\right.\\
&+ \left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{U}^{i,k_1}\cdot\nabla\right)\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{\mathcal{Z}}_{bl}^{i,k_2}\right)\\
&+ \left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{\mathcal{V}}_{bl}^{i,k_1}\cdot\nabla\right)\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{W}^{i,k_2}\right) + \left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{\mathcal{V}}_{bl}^{i,k_1}\cdot\nabla\right)\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{\mathcal{Z}}_{bl}^{i,k_2}\right)\\
&+ \left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{U}^{i,k_1}\cdot\nabla\right)\boldsymbol{Z}^{k_2} + \left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{\mathcal{V}}_{bl}^{i,k_1}\cdot\nabla\right)\boldsymbol{Z}^{k_2} + (\boldsymbol{V}^{k_1}\cdot\nabla)\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{W}^{i,k_2}\right)\\
&\left.+(\boldsymbol{V}^{k_1}\cdot\nabla)\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{\mathcal{Z}}_{bl}^{i,k_2}\right)\right] + I\sum_{J-2\leq k_1+k_2\leq 2J}\varepsilon^{k_1+k_2+2}(\boldsymbol{V}^{k_1}\cdot\nabla)\boldsymbol{Z}^{k_2} + \mathcal{L}_1.
\end{aligned}
$$

Posljednja funkcija $\mathcal{L}_1$ označava rezidual koji se pojavljuje zbog množenja korektora unutarnjeg sloja s izglađujućom funkcijom $\zeta\left(\frac{3|\boldsymbol{x}|}{\ell_{min}}\right)$, te je njen nosač sadržan u skupu $\{\boldsymbol{x} \in \Omega_\varepsilon : |\boldsymbol{x}| \geq \frac{\ell_{min}}{3}\}$. Kako korektori unutarnjeg sloja eksponencijalno padaju k nuli izvan čvorišta, zaključujemo $\|\mathcal{L}_1\|_{L^2(\Omega_\varepsilon)} \leq C\exp(-\sigma/\varepsilon)$. Sada je jednostavno pokazati da

$$\left\|\mathcal{R}_\varepsilon^{g,[J]}\right\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{J+1}. \tag{4.62}$$

Oduzimanjem $(4.61)_1$ od $(4.1)_3$ dobivamo

$$\begin{aligned} I\frac{\partial \boldsymbol{S}_\varepsilon^{[J]}}{\partial t} - (c_\alpha + c_d)\Delta \boldsymbol{S}_\varepsilon^{[J]} - (c_0 + c_d - c_\alpha)\nabla \operatorname{div} \boldsymbol{S}_\varepsilon^{[J]} + 4\mu_r \boldsymbol{S}_\varepsilon^{[J]} = 2\mu_r \operatorname{rot} \boldsymbol{D}_\varepsilon^{[J]} \\ -I(\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{S}_\varepsilon^{[J]} - I(\boldsymbol{D}_\varepsilon^{[J]} \cdot \nabla)\boldsymbol{w}_\varepsilon^{approx,[J]} - \mathcal{R}_\varepsilon^{g,[J]}, \end{aligned} \tag{4.63}$$

gdje $\boldsymbol{S}_\varepsilon^{[J]}$ zadovoljava sljedeće rubne i inicijalne uvjete:

$$\begin{aligned} \boldsymbol{S}_\varepsilon^{[J]} = \boldsymbol{0} \quad \text{na} \quad \partial\Omega_\varepsilon, \\ \boldsymbol{S}_\varepsilon^{[J]}(\boldsymbol{x}, 0) = \boldsymbol{0} \quad \text{u} \quad \Omega_\varepsilon. \end{aligned}$$

Sada množimo (4.63) s $\boldsymbol{S}_\varepsilon^{[J]}$ te integriramo po Ω_ε čime dobivamo jednakost

$$\begin{aligned} \frac{I}{2}\frac{d}{dt}\int_{\Omega_\varepsilon}\left|\boldsymbol{S}_\varepsilon^{[J]}\right|^2 + (c_\alpha + c_d)\int_{\Omega_\varepsilon}\left|\nabla \boldsymbol{S}_\varepsilon^{[J]}\right|^2 + (c_0 + c_d - c_\alpha)\int_{\Omega_\varepsilon}\left(\operatorname{div} \boldsymbol{S}_\varepsilon^{[J]}\right)^2 \\ +4\mu_r\int_{\Omega_\varepsilon}\left|\boldsymbol{S}_\varepsilon^{[J]}\right|^2 = 2\mu_r\int_{\Omega_\varepsilon}\operatorname{rot} \boldsymbol{D}_\varepsilon^{[J]} \cdot \boldsymbol{S}_\varepsilon^{[J]} - I\int_{\Omega_\varepsilon}(\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{S}_\varepsilon^{[J]} \cdot \boldsymbol{S}_\varepsilon^{[J]} \\ -I\int_{\Omega_\varepsilon}(\boldsymbol{D}_\varepsilon^{[J]} \cdot \nabla)\boldsymbol{w}_\varepsilon^{approx,[J]} \cdot \boldsymbol{S}_\varepsilon^{[J]} - \int_{\Omega_\varepsilon}\mathcal{R}_\varepsilon^{g,[J]} \cdot \boldsymbol{S}_\varepsilon^{[J]}. \end{aligned} \tag{4.64}$$

Kao i u prethodnom odjeljku, lako je pokazati da vrijedi

$$\int_{\Omega_\varepsilon}(\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{S}_\varepsilon^{[J]} \cdot \boldsymbol{S}_\varepsilon^{[J]} = 0.$$

Preostale izraze u (4.64) ocjenjujemo koristeći Lemu 0.0.1 te ocjenu (4.62) :

$$\begin{aligned} \int_{\Omega_\varepsilon}\operatorname{rot} \boldsymbol{D}_\varepsilon^{[J]} \cdot \boldsymbol{S}_\varepsilon^{[J]} &\leq \left\|\nabla \boldsymbol{D}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}\left\|\boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \\ &\leq C\varepsilon\left\|\nabla \boldsymbol{D}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla \boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}, \\ \int_{\Omega_\varepsilon}(\boldsymbol{D}_\varepsilon^{[J]} \cdot \nabla)\boldsymbol{w}_\varepsilon^{approx,[J]} \cdot \boldsymbol{S}_\varepsilon^{[J]} &\leq \left\|\boldsymbol{D}_\varepsilon^{[J]}\right\|_{L^4(\Omega_\varepsilon)}\left\|\nabla \boldsymbol{w}_\varepsilon^{approx,[J]}\right\|_{L^2(\Omega_\varepsilon)}\left\|\boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^4(\Omega_\varepsilon)} \\ &\leq C\varepsilon^{3/2}\left\|\nabla \boldsymbol{D}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla \boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}, \\ \int_{\Omega_\varepsilon}\mathcal{R}_\varepsilon^{g,[J]} \cdot \boldsymbol{S}_\varepsilon^{[J]} &\leq \left\|\mathcal{R}_\varepsilon^{g,[J]}\right\|_{L^2(\Omega_\varepsilon)}\left\|\boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \\ &\leq C\varepsilon\left\|\mathcal{R}_\varepsilon^{g,[J]}\right\|_{L^2(\Omega_\varepsilon)}\left\|\nabla \boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \\ &\leq C\varepsilon^{J+2}\left\|\nabla \boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{4.65}$$

Iz (4.64), ocjena (4.65) te Youngove nejednakosti možemo zaključiti da vrijedi

$$\frac{d}{dt}\left\|\boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}^2 + \left\|\nabla \boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}^2 \le C\varepsilon^2 \left\|\nabla \boldsymbol{D}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}^2 + C\varepsilon^{2J+4}, \tag{4.66}$$

te sada integriranjem (4.66) po t dobivamo

$$\sup_{t\in[0,T]}\left\|\boldsymbol{S}_\varepsilon^{[J]}(\cdot,t)\right\|_{L^2(\Omega_\varepsilon)}^2 + \int_0^T \left\|\nabla \boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}^2 \le C\varepsilon^2 \int_0^T \left\|\nabla \boldsymbol{D}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}^2 + C\varepsilon^{2J+4}. \tag{4.67}$$

Nadalje, aproksimacija brzine $\boldsymbol{u}_\varepsilon^{approx,[J]}$ zadovoljava sustav

$$\begin{aligned}
\frac{\partial \boldsymbol{u}_\varepsilon^{approx,[J]}}{\partial t} + (\boldsymbol{u}_\varepsilon^{approx,[J]}\cdot\nabla)\boldsymbol{u}_\varepsilon^{approx,[J]} - (\mu+\mu_r)\Delta \boldsymbol{u}_\varepsilon^{approx,[J]} + \nabla p_\varepsilon^{approx,[J]}&\\
= 2\mu_r \operatorname{rot} \boldsymbol{w}_\varepsilon^{approx,[J]} + \boldsymbol{f}_\varepsilon + \mathcal{R}_\varepsilon^{f,[J]},&\\
\operatorname{div} \boldsymbol{u}_\varepsilon^{approx,[J]} = -\nabla\zeta\left(\frac{3|\boldsymbol{x}|}{\ell_{min}}\right)\cdot \boldsymbol{V}_{\varepsilon,int}^{[J]},&\\
\boldsymbol{u}_\varepsilon^{approx,[J]} = \boldsymbol{0} \quad \text{na} \quad \Gamma_\varepsilon,&\\
\boldsymbol{u}_\varepsilon^{approx,[J]} = \boldsymbol{h}_\varepsilon^i \quad \text{na} \quad \Sigma_\varepsilon^i,\ i=1,2,\dots,m,&\\
\boldsymbol{u}_\varepsilon^{approx,[J]}(\boldsymbol{x},0) = \boldsymbol{0} \quad \text{u} \quad \Omega_\varepsilon,&
\end{aligned} \tag{4.68}$$

gdje je

$$\begin{aligned}
\mathcal{R}_\varepsilon^{f,[J]} = \sum_{i=1}^m \zeta\left(\frac{x_1^i}{r\varepsilon}\right)&\left(\varepsilon^{J+1}\frac{\partial U_1^{i,J-1}}{\partial t}\boldsymbol{e}_1^i + \varepsilon^{J+2}\frac{\partial U_1^{i,J}}{\partial t}\boldsymbol{e}_1^i + \varepsilon^{J+1}\frac{\partial \boldsymbol{\mathcal{V}}_{bl}^{i,J-1}}{\partial t} + \varepsilon^{J+2}\frac{\partial \boldsymbol{\mathcal{V}}_{bl}^{i,J}}{\partial t}\right)\\
&+ \varepsilon^{J+1}\frac{\partial \boldsymbol{V}^{J-1}}{\partial t} + \varepsilon^{J+2}\frac{\partial \boldsymbol{V}^{J}}{\partial t}\\
&+ \sum_{J-2\le k_1+k_2\le 2J} \varepsilon^{k_1+k_2+3}\Bigg[\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{U}^{i,k_1}\cdot\nabla\right)\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{U}^{i,k_2}\right)\\
&+ \left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{U}^{i,k_1}\cdot\nabla\right)\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{\mathcal{V}}_{bl}^{i,k_2}\right) + \left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{\mathcal{V}}_{bl}^{i,k_1}\cdot\nabla\right)\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{U}^{i,k_2}\right)\\
&+ \left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{\mathcal{V}}_{bl}^{i,k_1}\cdot\nabla\right)\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{\mathcal{V}}_{bl}^{i,k_2}\right) + \left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{U}^{i,k_1}\cdot\nabla\right)\boldsymbol{V}^{k_2}\\
&+ \left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{\mathcal{V}}_{bl}^{i,k_1}\cdot\nabla\right)\boldsymbol{V}^{k_2} + (\boldsymbol{V}^{k_1}\cdot\nabla)\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{U}^{i,k_2}\right)\\
&+ (\boldsymbol{V}^{k_1}\cdot\nabla)\left(\zeta\left(\frac{x_1^i}{r\varepsilon}\right)\boldsymbol{\mathcal{V}}_{bl}^{i,k_2}\right)\Bigg] + \sum_{J-2\le k_1+k_3\le 2J}\varepsilon^{k_1+k_2+2}(\boldsymbol{V}^{k_1}\cdot\nabla)\boldsymbol{V}^{k_2} + \mathcal{L}_2.
\end{aligned}$$

Posljednja funkcija $\mathcal{L}_2$ je rezidual koji se pojavljuje zbog množenja korektora unutarnjeg sloja s funkcijom $\zeta\left(\frac{3|\boldsymbol{x}|}{\ell_{min}}\right)$, te vrijedi ocjena $\|\mathcal{L}_2\|_{L^2(\Omega_\varepsilon)} \le C\exp(-\sigma/\varepsilon)$. Tada je lako vidjeti da

$$\left\|\mathcal{R}_\varepsilon^{f,[J]}\right\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^{J+2}. \tag{4.69}$$

Oduzimanjem (4.68)$_1$ od (4.1)$_1$ dobivamo jednadžbu

$$\frac{\partial \boldsymbol{D}_\varepsilon^{[J]}}{\partial t} - (\mu+\mu_r)\Delta \boldsymbol{D}_\varepsilon^{[J]} + \nabla d_\varepsilon^{[J]} = 2\mu_r \operatorname{rot} \boldsymbol{S}_\varepsilon^{[J]} - (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{D}_\varepsilon^{[J]} - (\boldsymbol{D}_\varepsilon^{[J]} \cdot \nabla)\boldsymbol{u}_\varepsilon^{approx,[J]} - \mathcal{R}_\varepsilon^{f,[J]}. \quad (4.70)$$

Razlika brzine i aproksimacije brzine $\boldsymbol{D}_\varepsilon^{[J]}$ zadovoljava jednadžbu za divergenciju

$$\operatorname{div} \boldsymbol{D}_\varepsilon^{[J]} = \nabla\zeta\left(\frac{3|\boldsymbol{x}|}{\ell_{min}}\right) \cdot \boldsymbol{V}_{\varepsilon,int}^{[J]}$$

te rubne i inicijalne uvjete

$$\boldsymbol{D}_\varepsilon^{[J]} = \mathbf{0} \quad \text{na} \quad \partial\Omega_\varepsilon,$$
$$\boldsymbol{D}_\varepsilon^{[J]}(\boldsymbol{x},0) = \mathbf{0} \quad \text{u} \quad \Omega_\varepsilon.$$

Napomenimo da je uvjet kompatibilnosti za $\mathbf{D}_\varepsilon^{[J]}$ zadovoljen. Doista, $\nabla\zeta\left(\frac{3|\boldsymbol{x}|}{\ell_{min}}\right)$ ima kompaktni nosač sadržan u sredini svake cijevi, te ovisi samo o udaljenosti od ishodišta (to jest o x_1^i u lokalnim koordinatama). Koristeći navedeno, te činjenicu da je fluks funkcije $\mathbf{V}_{\varepsilon,int}^{[J]}$ jednak nuli u svakoj cijevi lako je vidjeti da vrijedi

$$\int_{\Omega_\varepsilon} \nabla\zeta\left(\frac{3|\boldsymbol{x}|}{\ell_{min}}\right) \cdot \mathbf{V}_{\varepsilon,int}^{[J]} = 0.$$

Sada uvodimo novu funkciju $\tilde{\boldsymbol{D}}_\varepsilon^{[J]} = \boldsymbol{D}_\varepsilon^{[J]} - \boldsymbol{\pi}_\varepsilon^{[J]}$, gdje je $\boldsymbol{\pi}_\varepsilon^{[J]}$ rješenje problema

$$\operatorname{div} \boldsymbol{\pi}_\varepsilon^{[J]} = \nabla\zeta\left(\frac{3|\boldsymbol{x}|}{\ell_{min}}\right) \cdot \boldsymbol{V}_{\varepsilon,int}^{[J]} \quad \text{u} \quad \Omega_\varepsilon,$$
$$\boldsymbol{\pi}_\varepsilon^{[J]} = \mathbf{0} \quad \text{na} \quad \partial\Omega_\varepsilon.$$

Jer je nosač funkcije $\nabla\zeta\left(\frac{3|\boldsymbol{x}|}{\ell_{min}}\right)$ sadržan u sredini svake cijevi i jer korektori unutarnjeg sloja eksponencijalno iščezavaju van čvorišta, vrijedi ocjena (vidi [71])

$$\left\|\nabla \boldsymbol{\pi}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} + \left\|\frac{\partial \boldsymbol{\pi}_\varepsilon^{[J]}}{\partial t}\right\|_{L^2(\Omega_\varepsilon)} \leq C\exp(-\sigma/\varepsilon). \quad (4.71)$$

Očito je da je divergencija funkcije $\tilde{\boldsymbol{D}}_\varepsilon^{[J]}$ jednaka nuli te da vrijedi $\tilde{\boldsymbol{D}}_\varepsilon^{[J]} = \mathbf{0}$ na $\partial\Omega_\varepsilon$. Sada jednadžbu (4.70) množimo s $\tilde{\boldsymbol{D}}_\varepsilon^{[J]}$ te integriramo po Ω_ε, čime dobivamo jednakost

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\Omega_\varepsilon} \left|\tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right|^2 + (\mu+\mu_r)\int_{\Omega_\varepsilon} \left|\nabla\tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right|^2 &= 2\mu_r \int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{S}_\varepsilon^{[J]} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]} - \int_{\Omega_\varepsilon} \frac{\partial \boldsymbol{\pi}_\varepsilon^{[J]}}{\partial t} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]} \\
-(\mu+\mu_r)\int_{\Omega_\varepsilon} \nabla \boldsymbol{\pi}_\varepsilon^{[J]} \cdot \nabla\tilde{\boldsymbol{D}}_\varepsilon^{[J]} &- \int_{\Omega_\varepsilon} (\boldsymbol{u}_\varepsilon \cdot \nabla)\tilde{\boldsymbol{D}}_\varepsilon^{[J]} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]} - \int_{\Omega_\varepsilon} (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{\pi}_\varepsilon^{[J]} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]} \qquad (4.72) \\
-\int_{\Omega_\varepsilon} (\tilde{\boldsymbol{D}}_\varepsilon^{[J]} \cdot \nabla)\boldsymbol{u}_\varepsilon^{approx,[J]} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]} &- \int_{\Omega_\varepsilon} (\boldsymbol{\pi}_\varepsilon^{[J]} \cdot \nabla)\boldsymbol{u}_\varepsilon^{approx,[J]} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]} - \int_{\Omega_\varepsilon} \mathcal{R}_\varepsilon^{f,[J]} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]}.
\end{aligned}$$

Izraze s desne strane (4.72) ocjenjujemo koristeći (4.69), (4.71), Lemu 0.0.1 i Teorem 4.3.1:

$$
\begin{aligned}
\int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{S}_\varepsilon^{[J]} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]} &\leq \left\|\nabla \boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \left\|\tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon \left\|\nabla \boldsymbol{S}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \frac{\partial \boldsymbol{\pi}_\varepsilon^{[J]}}{\partial t} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]} &\leq \left\|\frac{\partial \boldsymbol{\pi}_\varepsilon^{[J]}}{\partial t}\right\|_{L^2(\Omega_\varepsilon)} \left\|\tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon \left\|\frac{\partial \boldsymbol{\pi}_\varepsilon^{[J]}}{\partial t}\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon \exp(-\sigma/\varepsilon) \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \nabla \boldsymbol{\pi}_\varepsilon^{[J]} \cdot \nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]} &\leq \left\|\nabla \boldsymbol{\pi}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C \exp(-\sigma/\varepsilon) \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} (\boldsymbol{u}_\varepsilon \cdot \nabla) \tilde{\boldsymbol{D}}_\varepsilon^{[J]} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]} &\leq \|\boldsymbol{u}_\varepsilon\|_{L^4(\Omega_\varepsilon)} \left\|\tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^4(\Omega_\varepsilon)} \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^{1/2} \|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}^2 \\
&\leq C\varepsilon^{5/2} \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}^2, \\
\int_{\Omega_\varepsilon} (\boldsymbol{u}_\varepsilon \cdot \nabla) \boldsymbol{\pi}_\varepsilon^{[J]} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]} &\leq \|\boldsymbol{u}_\varepsilon\|_{L^4(\Omega_\varepsilon)} \left\|\tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^4(\Omega_\varepsilon)} \left\|\nabla \boldsymbol{\pi}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^{1/2} \|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \boldsymbol{\pi}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^{5/2} \exp(-\sigma/\varepsilon) \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} (\tilde{\boldsymbol{D}}_\varepsilon^{[J]} \cdot \nabla) \boldsymbol{u}_\varepsilon^{approx,[J]} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]} &\leq \left\|\tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^4(\Omega_\varepsilon)}^2 \left\|\nabla \boldsymbol{u}_\varepsilon^{approx,[J]}\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^{1/2} \left\|\nabla \boldsymbol{u}_\varepsilon^{approx,[J]}\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}^2 \\
&\leq C\varepsilon^{5/2} \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}^2, \\
\int_{\Omega_\varepsilon} (\boldsymbol{\pi}_\varepsilon^{[J]} \cdot \nabla) \boldsymbol{u}_\varepsilon^{approx,[J]} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]} &\leq \left\|\boldsymbol{\pi}_\varepsilon^{[J]}\right\|_{L^4(\Omega_\varepsilon)} \left\|\tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^4(\Omega_\varepsilon)} \left\|\nabla \boldsymbol{u}_\varepsilon^{approx,[J]}\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^{1/2} \left\|\nabla \boldsymbol{\pi}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \boldsymbol{u}_\varepsilon^{approx,[J]}\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^{5/2} \exp(-\sigma/\varepsilon) \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \mathcal{R}_\varepsilon^{f,[J]} \cdot \tilde{\boldsymbol{D}}_\varepsilon^{[J]} &\leq \left\|\mathcal{R}_\varepsilon^{f,[J]}\right\|_{L^2(\Omega_\varepsilon)} \left\|\tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon \left\|\mathcal{R}_\varepsilon^{f,[J]}\right\|_{L^2(\Omega_\varepsilon)} \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^{J+3} \left\|\nabla \tilde{\boldsymbol{D}}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}.
\end{aligned} \tag{4.73}
$$

Iz (4.72), ocjena (4.73) te Youngove nejednakosti zaključujemo da vrijedi nejednakost

$$\begin{aligned}&\frac{d}{dt}\left\|\boldsymbol{D}_\varepsilon^{[J]}\right\|^2_{L^2(\Omega_\varepsilon)}+\left\|\nabla\boldsymbol{D}_\varepsilon^{[J]}\right\|^2_{L^2(\Omega_\varepsilon)}\\&\le C\varepsilon^2\left\|\nabla\boldsymbol{S}_\varepsilon^{[J]}\right\|^2_{L^2(\Omega_\varepsilon)}+C\varepsilon^{5/2}\left\|\nabla\boldsymbol{D}_\varepsilon^{[J]}\right\|^2_{L^2(\Omega_\varepsilon)}+C\varepsilon^{2J+6}.\end{aligned}\tag{4.74}$$

Sljedeće, integriramo (4.74) po t te koristimo ocjenu (4.67) čime za dovoljno mali ε dobivamo

$$\sup_{t\in[0,T]}\left\|\boldsymbol{D}_\varepsilon^{[J]}(\cdot,t)\right\|^2_{L^2(\Omega_\varepsilon)}+\int_0^T\left\|\nabla\boldsymbol{D}_\varepsilon^{[J]}\right\|^2_{L^2(\Omega_\varepsilon)}\le C\varepsilon^{2J+6}.\tag{4.75}$$

Sada iz (4.67) i (4.75) slijedi

$$\sup_{t\in[0,T]}\left\|\boldsymbol{S}_\varepsilon^{[J]}(\cdot,t)\right\|^2_{L^2(\Omega_\varepsilon)}+\int_0^T\left\|\nabla\boldsymbol{S}_\varepsilon^{[J]}\right\|^2_{L^2(\Omega_\varepsilon)}\le C\varepsilon^{2J+4}.\tag{4.76}$$

Na sličan način se mogu dobiti ocjene

$$\begin{aligned}&\sup_{t\in[0,T]}\left\|\frac{\partial\boldsymbol{D}_\varepsilon^{[J]}}{\partial t}(\cdot,t)\right\|^2_{L^2(\Omega_\varepsilon)}+\int_0^T\left\|\nabla\frac{\partial\boldsymbol{D}_\varepsilon^{[J]}}{\partial t}\right\|^2_{L^2(\Omega_\varepsilon)}\le C\varepsilon^{2J+6},\\&\sup_{t\in[0,T]}\left\|\frac{\partial\boldsymbol{S}_\varepsilon^{[J]}}{\partial t}(\cdot,t)\right\|^2_{L^2(\Omega_\varepsilon)}+\int_0^T\left\|\nabla\frac{\partial\boldsymbol{S}_\varepsilon^{[J]}}{\partial t}\right\|^2_{L^2(\Omega_\varepsilon)}\le C\varepsilon^{2J+4}.\end{aligned}\tag{4.77}$$

Kako bi još ocijenili razliku tlakova, promatramo problem

$$\begin{aligned}\operatorname{div}\boldsymbol{\theta}_\varepsilon^{[J]}&=d_\varepsilon^{[J]}+\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}p_\varepsilon^{approx,[J]},\\\boldsymbol{\theta}_\varepsilon^{[J]}&=\boldsymbol{0}\quad\text{na}\quad\partial\Omega_\varepsilon.\end{aligned}\tag{4.78}$$

Prema Lemi 0.0.2, postoji barem jedno rješenje problema (4.78) takvo da

$$\left\|\nabla\boldsymbol{\theta}_\varepsilon^{[J]}\right\|_{L^2(\Omega_\varepsilon)}\le\frac{C}{\varepsilon}\left\|d_\varepsilon^{[J]}+\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}p_\varepsilon^{approx,[J]}\right\|_{L^2(\Omega_\varepsilon)}.\tag{4.79}$$

Množenjem (4.70) s $\boldsymbol{\theta}_\varepsilon^{[J]}$ te integriranjem po Ω_ε dobivamo

$$\begin{aligned}\int_{\Omega_\varepsilon}\left(d_\varepsilon^{[J]}+\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}p_\varepsilon^{approx,[J]}\right)^2&=\int_{\Omega_\varepsilon}\frac{\partial\boldsymbol{D}_\varepsilon^{[J]}}{\partial t}\cdot\boldsymbol{\theta}_\varepsilon^{[J]}+(\mu+\mu_r)\int_{\Omega_\varepsilon}\nabla\boldsymbol{D}_\varepsilon^{[J]}\cdot\nabla\boldsymbol{\theta}_\varepsilon^{[J]}\\&-2\mu_r\int_{\Omega_\varepsilon}\operatorname{rot}\boldsymbol{S}_\varepsilon^{[J]}\cdot\boldsymbol{\theta}_\varepsilon^{[J]}+\int_{\Omega_\varepsilon}(\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{D}_\varepsilon^{[J]}\cdot\boldsymbol{\theta}_\varepsilon^{[J]}+\int_{\Omega_\varepsilon}(\boldsymbol{D}_\varepsilon^{[J]}\cdot\nabla)\boldsymbol{u}_\varepsilon^{approx,[J]}\cdot\boldsymbol{\theta}_\varepsilon^{[J]}\\&+\int_{\Omega_\varepsilon}\mathcal{R}_\varepsilon^{f,[J]}\cdot\boldsymbol{\theta}_\varepsilon^{[J]}.\end{aligned}\tag{4.80}$$

Izraze na desnoj strani jednakosti (4.80) ocjenjujemo koristeći (4.69), Lemu 0.0.1 i Teorem 4.3.1:

$$\begin{aligned}
\int_{\Omega_\varepsilon} \frac{\partial \boldsymbol{D}_\varepsilon^{[J]}}{\partial t} \cdot \boldsymbol{\theta}_\varepsilon^{[J]} &\le \left\| \frac{\partial \boldsymbol{D}_\varepsilon^{[J]}}{\partial t} \right\|_{L^2(\Omega_\varepsilon)} \left\| \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon \left\| \frac{\partial \boldsymbol{D}_\varepsilon^{[J]}}{\partial t} \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \nabla \boldsymbol{D}_\varepsilon^{[J]} \cdot \nabla \boldsymbol{\theta}_\varepsilon^{[J]} &\le \left\| \nabla \boldsymbol{D}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \operatorname{rot} \boldsymbol{S}_\varepsilon^{[J]} \cdot \boldsymbol{\theta}_\varepsilon^{[J]} &\le \left\| \nabla \boldsymbol{S}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \left\| \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon \left\| \nabla \boldsymbol{S}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} (\boldsymbol{u}_\varepsilon \cdot \nabla) \boldsymbol{D}_\varepsilon^{[J]} \cdot \boldsymbol{\theta}_\varepsilon^{[J]} &\le \| \boldsymbol{u}_\varepsilon \|_{L^4(\Omega_\varepsilon)} \left\| \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^4(\Omega_\varepsilon)} \left\| \nabla \boldsymbol{D}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon^{1/2} \| \nabla \boldsymbol{u}_\varepsilon \|_{L^2(\Omega_\varepsilon)} \left\| \nabla \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla \boldsymbol{D}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon^{5/2} \left\| \nabla \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla \boldsymbol{D}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} (\boldsymbol{D}_\varepsilon^{[J]} \cdot \nabla) \boldsymbol{u}_\varepsilon^{approx,[J]} \cdot \boldsymbol{\theta}_\varepsilon^{[J]} &\le \left\| \boldsymbol{D}_\varepsilon^{[J]} \right\|_{L^4(\Omega_\varepsilon)} \left\| \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^4(\Omega_\varepsilon)} \left\| \nabla \boldsymbol{u}_\varepsilon^{approx,[J]} \right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon^{1/2} \left\| \nabla \boldsymbol{D}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla \boldsymbol{u}_\varepsilon^{approx,[J]} \right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon^{5/2} \left\| \nabla \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla \boldsymbol{D}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)}, \\
\int_{\Omega_\varepsilon} \boldsymbol{\mathcal{R}}_\varepsilon^{f,[J]} \cdot \boldsymbol{\theta}_\varepsilon^{[J]} &\le \left\| \boldsymbol{\mathcal{R}}_\varepsilon^{f,[J]} \right\|_{L^2(\Omega_\varepsilon)} \left\| \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon \left\| \boldsymbol{\mathcal{R}}_\varepsilon^{f,[J]} \right\|_{L^2(\Omega_\varepsilon)} \left\| \nabla \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon^{J+3} \left\| \nabla \boldsymbol{\theta}_\varepsilon^{[J]} \right\|_{L^2(\Omega_\varepsilon)}.
\end{aligned} \tag{4.81}$$

Iz (4.80) te ocjena (4.81) slijedi

$$\begin{aligned}
&\left\| d_\varepsilon^{[J]} + \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} p_\varepsilon^{approx,[J]} \right\|_{L^2(\Omega)}^2 \\
\le C &\left(\left\| \frac{\partial \boldsymbol{D}_\varepsilon}{\partial t} \right\|_{L^2(\Omega)} + \| \nabla \boldsymbol{D}_\varepsilon \|_{L^2(\Omega)} + \varepsilon \| \nabla \boldsymbol{S}_\varepsilon \|_{L^2(\Omega)} + \varepsilon^{J+3} \right) \| \nabla \boldsymbol{\theta}_\varepsilon \|_{L^2(\Omega)}.
\end{aligned} \tag{4.82}$$

Uzimajući u obzir (4.75), (4.76), (4.77) te (4.79), iz (4.82) dobivamo sljedeću ocjenu za d_ε:

$$\left\| d_\varepsilon^{[J]} + \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} p_\varepsilon^{approx,[J]} \right\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^{J+2}.$$

Ovime je dokaz dovršen.

■

A. Koeficijenti asimptotičke aproksimacije rješenja problema (3.1)–(3.5)

Funkcije $A_1, \ldots, A_{10}$ iz (3.38) su dane s:

$$
\begin{aligned}
A_1(x_3,t) &= \frac{a(4\alpha+\beta)}{8\alpha(2\alpha+\beta)} \frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t), \\
A_2(x_3,t) &= \frac{a\beta}{4\alpha(2\alpha+\beta)} \frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t), \\
A_3(x_3,t) &= \frac{a(4\alpha+3\beta)}{8\alpha(2\alpha+\beta)} \frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t), \\
A_4(x_3,t) &= -\frac{a(4\alpha+\beta)}{8\alpha(2\alpha+\beta)} \frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t), \\
A_5(x_3,t) &= -\frac{a(4\alpha+3\beta)}{8\alpha(2\alpha+\beta)} \frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t), \\
A_6(x_3,t) &= -\frac{a\beta}{4\alpha(2\alpha+\beta)} \frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t), \\
A_7(x_3,t) &= -\frac{a(4\alpha+\beta)}{8\alpha(2\alpha+\beta)} \frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t), \\
A_8(x_3,t) &= \frac{a(4\alpha+\beta)}{8\alpha(2\alpha+\beta)} \frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t), \\
A_9(x_3,t) &= -\frac{1}{8\mu}\left(\frac{a}{2\alpha+\beta}\frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t) + \frac{\partial^2 \tilde{f}_1^0}{\partial x_3^2}(x_3,t)\right), \\
A_{10}(x_3,t) &= \frac{1}{8\mu}\left(\frac{a}{2\alpha+\beta}\frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t) - \frac{\partial^2 \tilde{f}_2^0}{\partial x_3^2}(x_3,t)\right).
\end{aligned}
\tag{A.1}
$$

Funkcije $B_1, \ldots, B_8$ iz (3.39) su dane s:

$$
\begin{aligned}
B_1(x_3,t) &= \frac{1}{192\mu}\left(\frac{a(6\alpha+\beta)}{\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t) + 5\frac{\partial^2 \tilde{f}_1^0}{\partial x_3^2}(x_3,t)\right),\\
B_2(x_3,t) &= \frac{1}{48\mu}\left(\frac{a\beta}{\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t) + \frac{\partial^2 \tilde{f}_2^0}{\partial x_3^2}(x_3,t)\right),\\
B_3(x_3,t) &= \frac{1}{192\mu}\left(\frac{a(6\alpha+5\beta)}{\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t) + \frac{\partial^2 \tilde{f}_1^0}{\partial x_3^2}(x_3,t)\right),\\
B_4(x_3,t) &= -\frac{1}{192\mu}\left(\frac{a(6\alpha+\beta)}{\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t) + 5\frac{\partial^2 \tilde{f}_1^0}{\partial x_3^2}(x_3,t)\right),\\
B_5(x_3,t) &= -\frac{1}{192\mu}\left(\frac{a(6\alpha+5\beta)}{\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t) - \frac{\partial^2 \tilde{f}_2^0}{\partial x_3^2}(x_3,t)\right),\\
B_6(x_3,t) &= -\frac{1}{48\mu}\left(\frac{a\beta}{\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t) - \frac{\partial^2 \tilde{f}_1^0}{\partial x_3^2}(x_3,t)\right),\\
B_7(x_3,t) &= -\frac{1}{192\mu}\left(\frac{a(6\alpha+\beta)}{\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t) - 5\frac{\partial^2 \tilde{f}_2^0}{\partial x_3^2}(x_3,t)\right),\\
B_8(x_3,t) &= \frac{1}{192\mu}\left(\frac{a(6\alpha+\beta)}{\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t) - 5\frac{\partial^2 \tilde{f}_2^0}{\partial x_3^2}(x_3,t)\right).
\end{aligned}
\tag{A.2}
$$

Funkcije $M_1, \ldots, M_6$ iz (3.39) su dane s:

$$
\begin{aligned}
M_1(x_3,t) &= -\frac{1}{8}\frac{\partial^2 \tilde{f}_1^0}{\partial x_3^2}(x_3,t),\\
M_2(x_3,t) &= -\frac{1}{8}\frac{\partial^2 \tilde{f}_2^0}{\partial x_3^2}(x_3,t),\\
M_3(x_3,t) &= -\frac{1}{8}\frac{\partial^2 \tilde{f}_1^0}{\partial x_3^2}(x_3,t),\\
M_4(x_3,t) &= -\frac{1}{8}\frac{\partial^2 \tilde{f}_2^0}{\partial x_3^2}(x_3,t),\\
M_5(x_3,t) &= -\frac{a(6\alpha+\beta)}{24\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t) + \frac{1}{6}\frac{\partial^2 \tilde{f}_1^0}{\partial x_3^2}(x_3,t),\\
M_6(x_3,t) &= \frac{a(6\alpha+\beta)}{24\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t) + \frac{1}{6}\frac{\partial^2 \tilde{f}_2^0}{\partial x_3^2}(x_3,t).
\end{aligned}
\tag{A.3}
$$

Funkcije $A_{11}, \dots, A_{16}$ iz (3.41) su dane s:

$$\begin{aligned}
A_{11}(x_3,t) &= \frac{2}{\mu\pi}\left(\frac{a^2F(t)}{\alpha} - \frac{\partial F}{\partial t}(t)\right),\\
A_{12}(x_3,t) &= 0,\\
A_{13}(x_3,t) &= \frac{2}{\mu\pi}\left(\frac{a^2F(t)}{\alpha} - \frac{\partial F}{\partial t}(t)\right),\\
A_{14}(x_3,t) &= \frac{a\alpha_0(\alpha+\beta)}{2\mu^2(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\tilde{g}_2^0(x_3,t) + \frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t)\right),\\
A_{15}(x_3,t) &= -\frac{a\alpha_0(\alpha+\beta)}{2\mu^2(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\tilde{g}_1^0(x_3,t) - \frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t)\right),\\
A_{16}(x_3,t) &= -\frac{2}{\mu\pi}\left(\frac{a^2F(t)}{2\alpha} + \frac{a^2\alpha_0^2F(t)}{\mu} - \frac{\partial F}{\partial t}(t)\right) + \frac{a\alpha_0\tilde{p}^1_{bl,cor}(t)}{2\mu^2} + \frac{1}{\mu}\tilde{p}^2_{bl,cor}(t).
\end{aligned} \tag{A.4}$$

Funkcije $B_9, \dots, B_{14}$ iz (3.42) su dane s:

$$\begin{aligned}
B_9(x_3,t) &= \frac{7A_{11} - A_{13}}{96} = \frac{1}{8\mu\pi}\left(\frac{a^2F(t)}{\alpha} - \frac{\partial F}{\partial t}(t)\right),\\
B_{10}(x_3,t) &= \frac{1}{12}A_{12} = 0,\\
B_{11}(x_3,t) &= \frac{7A_{13} - A_{11}}{96} = \frac{1}{8\mu\pi}\left(\frac{a^2F(t)}{\alpha} - \frac{\partial F}{\partial t}(t)\right),\\
B_{12}(x_3,t) &= \frac{1}{8}A_{14} = \frac{a\alpha_0(\alpha+\beta)}{16\mu^2(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\tilde{g}_2^0(x_3,t) + \frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t)\right),\\
B_{13}(x_3,t) &= \frac{1}{8}A_{15} = -\frac{a\alpha_0(\alpha+\beta)}{16\mu^2(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\tilde{g}_1^0(x_3,t) - \frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t)\right),\\
B_{14}(x_3,t) &= \frac{1}{4}A_{16} + \frac{1}{32}(A_{11} + A_{13})\\
&= -\frac{1}{8\mu\pi}\left(\frac{a^2F(t)}{\alpha} + 4\frac{a^2\alpha_0^2F(t)}{\mu} - 3\frac{\partial F}{\partial t}(t)\right) + \frac{a\alpha_0\tilde{p}^1_{bl,cor}(t)}{8\mu^2}\\
&\quad + \frac{1}{4\mu}\tilde{p}^2_{bl,cor}(t).
\end{aligned} \tag{A.5}$$

Funkcije $C_1, \dots, C_{32}$ iz (3.44) su dane s:

$$\begin{aligned}
C_1(x_3,t) =& \frac{8\mu a - a^2}{8\mu(2\alpha+\beta)}\tilde{g}_1^0(x_3,t) + \frac{3\beta^2 - 4\alpha^2}{8\alpha(2\alpha+\beta)}\frac{\partial^2\tilde{g}_1^0}{\partial x_3^2}(x_3,t) + \frac{a}{8\mu}\frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t)\\
&+ \frac{1}{2(2\alpha+\beta)}\frac{\partial \tilde{g}_1^0}{\partial t}(x_3,t),\\
C_2(x_3,t) =& \frac{a^2}{4\mu(2\alpha+\beta)}\tilde{g}_2^0(x_3,t) + \frac{\beta^2}{4\alpha(2\alpha+\beta)}\frac{\partial^2\tilde{g}_2^0}{\partial x_3^2}(x_3,t) + \frac{a}{4\mu}\frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t),
\end{aligned}$$

$$\begin{aligned}
C_3(x_3,t) =& \frac{8\mu a - 3a^2}{8\mu(2\alpha+\beta)}\tilde{g}_1^0(x_3,t) + \frac{\beta^2-4\alpha^2}{8\alpha(2\alpha+\beta)}\frac{\partial^2 \tilde{g}_1^0}{\partial x_3^2}(x_3,t) + \frac{3a}{8\mu}\frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t) \\
&+ \frac{1}{2(2\alpha+\beta)}\frac{\partial \tilde{g}_1^0}{\partial t}(x_3,t), \\
C_4(x_3,t) =& 0, \\
C_5(x_3,t) =& -\frac{2\alpha_0}{\pi}\left(\frac{a^2F(t)}{\mu} - 2aF(t) - \frac{\partial F}{\partial t}(t)\right) + \frac{a\tilde{p}^1_{bl,cor}(t)}{2\mu}, \\
C_6(x_3,t) =& \frac{a^2 - 8\mu a}{8\mu(2\alpha+\beta)}\tilde{g}_1^0(x_3,t) + \frac{4\alpha^2-\beta^2}{8\alpha(2\alpha+\beta)}\frac{\partial^2 \tilde{g}_1^0}{\partial x_3^2}(x_3,t) - \frac{a}{8\mu}\frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t) \\
&- \frac{1}{2(2\alpha+\beta)}\frac{\partial \tilde{g}_1^0}{\partial t}(x_3,t), \\
C_7(x_3,t) =& \frac{8\mu a - 3a^2}{8\mu(2\alpha+\beta)}\tilde{g}_2^0(x_3,t) + \frac{\beta^2-4\alpha^2}{8\alpha(2\alpha+\beta)}\frac{\partial^2 \tilde{g}_2^0}{\partial x_3^2}(x_3,t) - \frac{3a}{8\mu}\frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t) \\
&+ \frac{1}{2(2\alpha+\beta)}\frac{\partial \tilde{g}_2^0}{\partial t}(x_3,t), \\
C_8(x_3,t) =& \frac{a^2}{4\mu(2\alpha+\beta)}\tilde{g}_1^0(x_3,t) + \frac{\beta^2}{4\alpha(2\alpha+\beta)}\frac{\partial^2 \tilde{g}_1^0}{\partial x_3^2}(x_3,t) - \frac{a}{4\mu}\frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t), \\
C_9(x_3,t) =& \frac{8\mu a - a^2}{8\mu(2\alpha+\beta)}\tilde{g}_2^0(x_3,t) + \frac{3\beta^2-4\alpha^2}{8\alpha(2\alpha+\beta)}\frac{\partial^2 \tilde{g}_2^0}{\partial x_3^2}(x_3,t) - \frac{a}{8\mu}\frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t) \\
&+ \frac{1}{2(2\alpha+\beta)}\frac{\partial \tilde{g}_2^0}{\partial t}(x_3,t), \\
C_{10}(x_3,t) =& \frac{2\alpha_0}{\pi}\left(\frac{a^2F(t)}{\mu} - 2aF(t) - \frac{\partial F}{\partial t}(t)\right) - \frac{a\tilde{p}^1_{bl,cor}(t)}{2\mu}, \\
C_{11}(x_3,t) =& 0, \\
C_{12}(x_3,t) =& \frac{a^2 - 8\mu a}{8\mu(2\alpha+\beta)}\tilde{g}_2^0(x_3,t) + \frac{4\alpha^2-\beta^2}{8\alpha(2\alpha+\beta)}\frac{\partial^2 \tilde{g}_2^0}{\partial x_3^2}(x_3,t) + \frac{a}{8\mu}\frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t) \\
&- \frac{1}{2(2\alpha+\beta)}\frac{\partial \tilde{g}_2^0}{\partial t}(x_3,t), \\
C_{13}(x_3,t) =& \frac{\alpha_0 a}{32\mu\alpha}\frac{\partial \tilde{g}_3^0}{\partial x_3}(x_3,t), \\
C_{14}(x_3,t) =& \frac{\alpha_0}{4\mu\pi}\left(\frac{a^2F(t)}{\alpha} - \frac{\partial F}{\partial t}(t)\right), \\
C_{15}(x_3,t) =& \frac{\alpha_0}{4\mu\pi}\left(\frac{a^2F(t)}{\alpha} - \frac{\partial F}{\partial t}(t)\right), \\
C_{16}(x_3,t) =& \frac{\alpha_0 a}{32\mu\alpha}\frac{\partial \tilde{g}_3^0}{\partial x_3}(x_3,t), \\
C_{17}(x_3,t) =& -\frac{a\alpha_0^2(\alpha+\beta)}{32\mu^2(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\tilde{g}_1^0(x_3,t) - \frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t)\right),
\end{aligned} \tag{A.6}$$

$$C_{18}(x_3,t) = -\frac{3a\alpha_0^2(\alpha+\beta)}{32\mu^2(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\tilde{g}_1^0(x_3,t) - \frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t)\right),$$

$$C_{19}(x_3,t) = -\frac{\alpha_0 a}{32\mu\alpha}\frac{\partial \tilde{g}_3^0}{\partial x_3}(x_3,t),$$

$$C_{20}(x_3,t) = \frac{a\alpha_0^2(\alpha+\beta)}{16\mu^2(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\tilde{g}_2^0(x_3,t) + \frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t)\right),$$

$$C_{21}(x_3,t) = -\frac{\alpha_0}{4\mu\pi}\left(\frac{a^2F(t)}{\alpha} + 2\frac{a^2\alpha_0^2F(t)}{\mu} - 2\frac{\partial F}{\partial t}(t)\right) + \frac{a\alpha_0^2\tilde{p}_{bl,cor}^1(t)}{8\mu^2} + \frac{\alpha_0}{4\mu}\tilde{p}_{bl,cor}^2(t),$$

$$C_{22}(x_3,t) = \frac{a\alpha_0^2(\alpha+\beta)}{32\mu^2(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\tilde{g}_1^0(x_3,t) - \frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t)\right),$$

$$C_{23}(x_3,t) = -\frac{\alpha_0}{4\mu\pi}\left(\frac{a^2F(t)}{\alpha} - \frac{\partial F}{\partial t}(t)\right),$$

$$C_{24}(x_3,t) = \frac{\alpha_0 a}{32\mu\alpha}\frac{\partial \tilde{g}_3^0}{\partial x_3}(x_3,t),$$

$$C_{25}(x_3,t) = \frac{\alpha_0 a}{32\mu\alpha}\frac{\partial \tilde{g}_3^0}{\partial x_3}(x_3,t),$$

$$C_{26}(x_3,t) = -\frac{\alpha_0}{4\mu\pi}\left(\frac{a^2F(t)}{\alpha} - \frac{\partial F}{\partial t}(t)\right),$$

$$C_{27}(x_3,t) = -\frac{3a\alpha_0^2(\alpha+\beta)}{32\mu^2(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\tilde{g}_2^0(x_3,t) + \frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t)\right),$$

$$C_{28}(x_3,t) = -\frac{a\alpha_0^2(\alpha+\beta)}{32\mu^2(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\tilde{g}_2^0(x_3,t) + \frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t)\right),$$

$$C_{29}(x_3,t) = \frac{\alpha_0}{4\mu\pi}\left(\frac{a^2F(t)}{\alpha} + 2\frac{a^2\alpha_0^2F(t)}{\mu} - 2\frac{\partial F}{\partial t}(t)\right) - \frac{a\alpha_0^2\tilde{p}_{bl,cor}^1(t)}{8\mu^2} - \frac{\alpha_0}{4\mu}\tilde{p}_{bl,cor}^2(t),$$

$$C_{30}(x_3,t) = \frac{a\alpha_0^2(\alpha+\beta)}{16\mu^2(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\tilde{g}_1^0(x_3,t) - \frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t)\right),$$

$$C_{31}(x_3,t) = -\frac{\alpha_0 a}{32\mu\alpha}\frac{\partial \tilde{g}_3^0}{\partial x_3}(x_3,t),$$

$$C_{32}(x_3,t) = \frac{a\alpha_0^2(\alpha+\beta)}{32\mu^2(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\tilde{g}_2^0(x_3,t) + \frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t)\right).$$

Funkcije $D_1, \ldots, D_{12}$ iz (3.45) su dane s:

$$\begin{aligned}
D_1(x_3,t) =& \frac{a^2(\alpha+\beta) - 2a\mu(6\alpha+\beta)}{192\mu\alpha(\alpha+\beta)(2\alpha+\beta)} \tilde{g}_1^0(x_3,t) - \frac{a}{192\mu\alpha} \frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t) \\
&+ \frac{6\alpha-5\beta}{192\alpha(2\alpha+\beta)} \frac{\partial^2 \tilde{g}_1^0}{\partial x_3^2}(x_3,t) - \frac{6\alpha+\beta}{192\alpha(\alpha+\beta)(2\alpha+\beta)} \frac{\partial \tilde{g}_1^0}{\partial t}(x_3,t), \\
D_2(x_3,t) =& \frac{2a\mu\beta - a^2(\alpha+\beta)}{48\mu\alpha(\alpha+\beta)(2\alpha+\beta)} \tilde{g}_2^0(x_3,t) - \frac{a}{48\mu\alpha} \frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t) \\
&- \frac{\beta}{48\alpha(2\alpha+\beta)} \frac{\partial^2 \tilde{g}_2^0}{\partial x_3^2}(x_3,t) + \frac{\beta}{48\alpha(\alpha+\beta)(2\alpha+\beta)} \frac{\partial \tilde{g}_2^0}{\partial t}(x_3,t), \\
D_3(x_3,t) =& \frac{5a^2(\alpha+\beta) - 2a\mu(6\alpha+5\beta)}{192\mu\alpha(\alpha+\beta)(2\alpha+\beta)} \tilde{g}_1^0(x_3,t) - \frac{5a}{192\mu\alpha} \frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t) \\
&+ \frac{6\alpha-\beta}{192\alpha(2\alpha+\beta)} \frac{\partial^2 \tilde{g}_1^0}{\partial x_3^2}(x_3,t) - \frac{6\alpha+5\beta}{192\alpha(\alpha+\beta)(2\alpha+\beta)} \frac{\partial \tilde{g}_1^0}{\partial t}(x_3,t), \\
D_4(x_3,t) =& 0, \\
D_5(x_3,t) =& \frac{\alpha_0}{4\pi\alpha} \left(\frac{a^2 F(t)}{\mu} - 2aF(t) - \frac{\partial F}{\partial t}(t) \right) - \frac{a \tilde{p}^1_{bl,cor}(t)}{16\alpha\mu}, \\
D_6(x_3,t) =& \frac{2a\mu(36\alpha^2+36\alpha\beta+\beta^2) - a^2(6\alpha+\beta)(\alpha+\beta)}{192\mu\alpha(\alpha+\beta)(2\alpha+\beta)^2} \tilde{g}_1^0(x_3,t) \\
&+ \frac{a(6\alpha+\beta)}{192\mu\alpha(2\alpha+\beta)} \frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t) - \frac{36\alpha^2-5\beta^2}{192\alpha(2\alpha+\beta)^2} \frac{\partial^2 \tilde{g}_1^0}{\partial x_3^2}(x_3,t) \\
&+ \frac{36\alpha^2+36\alpha\beta+\beta^2}{192\alpha(\alpha+\beta)(2\alpha+\beta)^2} \frac{\partial \tilde{g}_1^0}{\partial t}(x_3,t), \qquad \text{(A.7)} \\
D_7(x_3,t) =& \frac{5a^2(\alpha+\beta) - 2a\mu(6\alpha+5\beta)}{192\mu\alpha(\alpha+\beta)(2\alpha+\beta)} \tilde{g}_2^0(x_3,t) + \frac{5a}{192\mu\alpha} \frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t) \\
&+ \frac{6\alpha-\beta}{192\alpha(2\alpha+\beta)} \frac{\partial^2 \tilde{g}_2^0}{\partial x_3^2}(x_3,t) - \frac{6\alpha+5\beta}{192\alpha(\alpha+\beta)(2\alpha+\beta)} \frac{\partial \tilde{g}_2^0}{\partial t}(x_3,t), \\
D_8(x_3,t) =& \frac{2a\mu\beta - a^2(\alpha+\beta)}{48\mu\alpha(\alpha+\beta)(2\alpha+\beta)} \tilde{g}_1^0(x_3,t) + \frac{a}{48\mu\alpha} \frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t) \\
&- \frac{\beta}{48\alpha(2\alpha+\beta)} \frac{\partial^2 \tilde{g}_1^0}{\partial x_3^2}(x_3,t) + \frac{\beta}{48\alpha(\alpha+\beta)(2\alpha+\beta)} \frac{\partial \tilde{g}_1^0}{\partial t}(x_3,t), \\
D_9(x_3,t) =& \frac{a^2(\alpha+\beta) - 2a\mu(6\alpha+\beta)}{192\mu\alpha(\alpha+\beta)(2\alpha+\beta)} \tilde{g}_2^0(x_3,t) + \frac{a}{192\mu\alpha} \frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t) \\
&+ \frac{6\alpha-5\beta}{192\alpha(2\alpha+\beta)} \frac{\partial^2 \tilde{g}_2^0}{\partial x_3^2}(x_3,t) - \frac{6\alpha+\beta}{192\alpha(\alpha+\beta)(2\alpha+\beta)} \frac{\partial \tilde{g}_2^0}{\partial t}(x_3,t), \\
D_{10}(x_3,t) =& -\frac{\alpha_0}{4\pi\alpha} \left(\frac{a^2 F(t)}{\mu} - 2aF(t) - \frac{\partial F}{\partial t}(t) \right) + \frac{a \tilde{p}^1_{bl,cor}(t)}{16\alpha\mu}, \\
D_{11}(x_3,t) =& 0,
\end{aligned}$$

$$D_{12}(x_3,t) = \frac{2a\mu(36\alpha^2+36\alpha\beta+\beta^2) - a^2(6\alpha+\beta)(\alpha+\beta)}{192\mu\alpha(\alpha+\beta)(2\alpha+\beta)^2}\tilde{g}_2^0(x_3,t) - \frac{a(6\alpha+\beta)}{192\mu\alpha(2\alpha+\beta)}\frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t) - \frac{36\alpha^2-5\beta^2}{192\alpha(2\alpha+\beta)^2}\frac{\partial^2 \tilde{g}_2^0}{\partial x_3^2}(x_3,t) + \frac{36\alpha^2+36\alpha\beta+\beta^2}{192\alpha(\alpha+\beta)(2\alpha+\beta)^2}\frac{\partial \tilde{g}_2^0}{\partial t}(x_3,t).$$

Funkcije $N_1,\dots,N_6$ iz (3.45) su dane s:

$$\begin{aligned}
N_1(x_3,t) &= \frac{\alpha_0 a}{32\mu\alpha}\frac{\partial \tilde{g}_3^0}{\partial x_3}(x_3,t),\\
N_2(x_3,t) &= \frac{\alpha_0}{4\mu\pi}\left(\frac{a^2F(t)}{\alpha} - \frac{\partial F}{\partial t}(t)\right),\\
N_3(x_3,t) &= -\frac{a\alpha_0^2\alpha(\alpha+\beta)}{8\mu^2(2\alpha+\beta)^2}\left(\frac{a}{2\alpha+\beta}\tilde{g}_1^0(x_3,t) - \frac{\partial \tilde{f}_2^0}{\partial x_3}(x_3,t)\right),\\
N_4(x_3,t) &= -\frac{\alpha_0}{4\mu\pi}\left(\frac{a^2F(t)}{\alpha} - \frac{\partial F}{\partial t}(t)\right),\\
N_5(x_3,t) &= \frac{\alpha_0 a}{32\mu\alpha}\frac{\partial \tilde{g}_3^0}{\partial x_3}(x_3,t),\\
N_6(x_3,t) &= -\frac{a\alpha_0^2\alpha(\alpha+\beta)}{8\mu^2(2\alpha+\beta)^2}\left(\frac{a}{2\alpha+\beta}\tilde{g}_2^0(x_3,t) + \frac{\partial \tilde{f}_1^0}{\partial x_3}(x_3,t)\right).
\end{aligned} \tag{A.8}$$

Funkcije $C_{33},\dots,C_{45}$ iz (3.47) su dane s:

$$\begin{aligned}
C_{33}(x_3,t) &= \frac{\alpha+2\beta}{4\alpha(\alpha+\beta)}\frac{\partial^2 \tilde{g}_3^0}{\partial x_3^2}(x_3,t) + \frac{a^2-2a\mu}{4\mu\alpha^2}\tilde{g}_3^0(x_3,t) - \frac{1}{4\alpha^2}\frac{\partial \tilde{g}_3^0}{\partial t}(x_3,t),\\
C_{34}(x_3,t) &= 0,\\
C_{35}(x_3,t) &= \frac{\alpha+2\beta}{4\alpha(\alpha+\beta)}\frac{\partial^2 \tilde{g}_3^0}{\partial x_3^2}(x_3,t) + \frac{a^2-2a\mu}{4\mu\alpha^2}\tilde{g}_3^0(x_3,t) - \frac{1}{4\alpha^2}\frac{\partial \tilde{g}_3^0}{\partial t}(x_3,t),\\
C_{36}(x_3,t) &= -\frac{a\alpha_0\beta}{2\mu(2\alpha+\beta)^2}\frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t) + \frac{\alpha_0\beta}{2\mu(2\alpha+\beta)}\frac{\partial^2 \tilde{f}_2^0}{\partial x_3^2}(x_3,t),\\
C_{37}(x_3,t) &= -\frac{a\alpha_0\beta}{2\mu(2\alpha+\beta)^2}\frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t) - \frac{\alpha_0\beta}{2\mu(2\alpha+\beta)}\frac{\partial^2 \tilde{f}_1^0}{\partial x_3^2}(x_3,t),\\
C_{38}(x_3,t) &= -\frac{2\alpha^2+4\alpha\beta+\beta^2}{8\alpha^2(\alpha+\beta)}\frac{\partial^2 \tilde{g}_3^0}{\partial x_3^2}(x_3,t) - \frac{a^2-4a\mu}{8\mu\alpha^2}\tilde{g}_3^0(x_3,t)\\
&\quad + \frac{1}{4\alpha^2}\frac{\partial \tilde{g}_3^0}{\partial t}(x_3,t),\\
C_{39}(x_3,t) &= \frac{a\alpha_0(\alpha+\beta)}{16\mu\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t),
\end{aligned} \tag{A.9}$$

$$
\begin{aligned}
C_{40}(x_3,t) =& \frac{a\alpha_0(\alpha+\beta)}{16\mu\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t),\\
C_{41}(x_3,t) =& \frac{a\alpha_0(\alpha+\beta)}{16\mu\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t),\\
C_{42}(x_3,t) =& \frac{a\alpha_0(\alpha+\beta)}{16\mu\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t),\\
C_{43}(x_3,t) =& -\frac{\alpha_0}{48\mu}\left(\frac{a(3\alpha+2\beta)}{\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t) - \frac{\partial^2 \tilde{f}_2^0}{\partial x_3^2}(x_3,t)\right),\\
C_{44}(x_3,t) =& -\frac{\alpha_0}{48\mu}\left(\frac{a(3\alpha+2\beta)}{\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t) + \frac{\partial^2 \tilde{f}_1^0}{\partial x_3^2}(x_3,t)\right),\\
C_{45}(x_3,t) =& 0.
\end{aligned}
$$

Funkcije $D_{13},\ldots,D_{18}$ iz (3.48) su dane s:

$$
\begin{aligned}
D_{13}(x_3,t) =& \frac{1}{64\alpha}\left(\frac{\alpha+2\beta}{\alpha+\beta}\frac{\partial^2 \tilde{g}_3^0}{\partial x_3^2}(x_3,t) + \frac{a^2-2a\mu}{\mu\alpha}\tilde{g}_3^0(x_3,t) - \frac{1}{\alpha}\frac{\partial \tilde{g}_3^0}{\partial t}(x_3,t)\right),\\
D_{14}(x_3,t) =& 0,\\
D_{15}(x_3,t) =& \frac{1}{64\alpha}\left(\frac{\alpha+2\beta}{\alpha+\beta}\frac{\partial^2 \tilde{g}_3^0}{\partial x_3^2}(x_3,t) + \frac{a^2-2a\mu}{\mu\alpha}\tilde{g}_3^0(x_3,t) - \frac{1}{\alpha}\frac{\partial \tilde{g}_3^0}{\partial t}(x_3,t)\right),\\
D_{16}(x_3,t) =& -\frac{\alpha_0\beta}{16\mu(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t) - \frac{\partial^2 \tilde{f}_2^0}{\partial x_3^2}(x_3,t)\right),\\
D_{17}(x_3,t) =& -\frac{\alpha_0\beta}{16\mu(2\alpha+\beta)}\left(\frac{a}{2\alpha+\beta}\frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t) + \frac{\partial^2 \tilde{f}_1^0}{\partial x_3^2}(x_3,t)\right),\\
D_{18}(x_3,t) =& \frac{1}{64\alpha^2}\left(-\frac{3\alpha^2+6\alpha\beta+2\beta^2}{\alpha+\beta}\frac{\partial^2 \tilde{g}_3^0}{\partial x_3^2}(x_3,t) - \frac{a^2-6a\mu}{\mu}\tilde{g}_3^0(x_3,t)\right.\\
&\left. +3\frac{\partial \tilde{g}_3^0}{\partial t}(x_3,t)\right).
\end{aligned}
\tag{A.10}
$$

Funkcije N_7, N_8 iz (3.48) su dane s

$$
\begin{aligned}
N_7(x_3,t) &= \frac{a\alpha_0(\alpha+\beta)}{16\mu\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_1^0}{\partial x_3}(x_3,t),\\
N_8(x_3,t) &= \frac{a\alpha_0(\alpha+\beta)}{16\mu\alpha(2\alpha+\beta)}\frac{\partial \tilde{g}_2^0}{\partial x_3}(x_3,t).
\end{aligned}
\tag{A.11}
$$

B. Stacionarni mikropolarni fluid u domeni s otvorima prema beskonačnosti

Neka je $\mathbf{\Omega} \subset \mathbb{R}^3$ domena koja se van ograničenog područja dijeli na m disjunktnih polubeskonačnih cijevi koje su u nekom koordinatnom sustavu $\boldsymbol{x}^i$ dane s

$$\mathbf{\Omega}_i = \{(x_1^i, \boldsymbol{x}_*^i) \in \mathbb{R}^3 : x_1^i > 0,\ \boldsymbol{x}_*^i = (x_2^i, x_3^i) \in \sigma_i\},\ i = 1, 2, \ldots, m,$$

gdje je $\sigma_i \subset \mathbb{R}^2$ ograničena domena. Uvodimo sljedeće oznake:

$$\mathbf{\Omega}_0 = \mathbf{\Omega} \backslash \left(\bigcup_{i=1}^{m} \mathbf{\Omega}_i \right),$$

$$\mathbf{\Omega}_{ik} = \{\boldsymbol{x}^i \in \mathbf{\Omega}_i : x_1^i < k\},$$

$$\mathbf{\Omega}_{(k)} = \mathbf{\Omega}_0 \cup \left(\bigcup_{i=1}^{m} \mathbf{\Omega}_{ik} \right),$$

$$\omega_{ik} = \mathbf{\Omega}_{ik+1} \backslash \mathbf{\Omega}_{ik},\ i = 1, 2, \ldots, m,$$

$$\hat{\omega}_{ik} = \omega_{ik-1} \cup \omega_{ik} \cup \omega_{ik+1},\ i = 1, 2, \ldots, m,$$

gdje je $k \in \mathbb{N}_0$. Neka je $E_{\beta_i}(\boldsymbol{x}) = E_{\beta_i}(x_1^i)$ glatka monotona težinska funkcija pridružena cijevi $\mathbf{\Omega}_i$ takva da

$$\begin{aligned} &E_{\beta_i}(\boldsymbol{x}) > 0,\ E_{\beta_i}(0) = 1, \\ &a_1 \leq E_{-\beta_i}(\boldsymbol{x}) E_{\beta_i}(\boldsymbol{x}) \leq a_2,\ \forall \boldsymbol{x} \in \mathbf{\Omega}_i, \\ &b_1 E_{\beta_i}(k) \leq E_{\beta_i}(\boldsymbol{x}) \leq b_2 E_{\beta_i}(k),\ \forall \boldsymbol{x} \in \omega_{ik}, \\ &|\nabla E_{\beta_i}(\boldsymbol{x})| \leq b_3 \gamma_* E_{\beta_i}(\boldsymbol{x}),\ \forall \boldsymbol{x} \in \mathbf{\Omega}_i, \\ &\lim_{x_1^i \to \infty} E_{\beta_i}(\boldsymbol{x}) = \infty \text{ ako } \beta_i > 0, \end{aligned} \tag{B.1}$$

pri čemu konstante a_1, a_2, b_1, b_2 ne ovise o k, te konstanta b_3 ne ovisi o β_i. U ostatku odjeljka ćemo raditi s težinskom funkcijom $E_{\beta_i}(x_1^i) := \exp(2\beta_i x_1^i)$. Doista, lako je provjeriti da ovako definirana E_{β_i} zadovoljava uvjete (B.1), pri čemu nejednakost (B.1)$_4$ vrijedi za $\gamma_* = |\beta_i|$. Dodatno, definirajmo

$$E_{\boldsymbol{\beta}}(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in \Omega_0, \\ E_{\beta_i}(x_1^i), & \boldsymbol{x} \in \Omega_i, \ i = 1, 2, \dots, m, \end{cases}$$

gdje je $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$.

Nadalje, uvodimo prostore težinskih funkcija $\mathcal{W}_\beta^{l,2}(\Omega)$ kao zatvarače prostora $C_0^\infty(\overline{\Omega})$ u normi danoj s

$$\|\boldsymbol{u}\|_{\mathcal{W}_\beta^{l,2}(\Omega)} = \left(\sum_{|\alpha|=0}^{l} \int_\Omega E_{\boldsymbol{\beta}}(\boldsymbol{x}) |D^\alpha \boldsymbol{u}(\boldsymbol{x})|^2 dx \right)^{1/2},$$

uz oznaku $\mathcal{L}_\beta^2(\Omega) = \mathcal{W}_\beta^{0,2}(\Omega)$. Naglasimo da kada je $\beta_i > 0$, elementi ovih prostora eksponencijalno teže k nuli kada $|\boldsymbol{x}^i| \to +\infty$, $\boldsymbol{x}^i \in \Omega_i$. Prostori težinskih funkcija $\mathcal{W}_\beta^{l,2}(\Omega_{(k)})$, $\mathcal{W}_\beta^{l,2}(\omega_{ik})$ te $\mathcal{W}_\beta^{l,2}(\hat{\omega}_{ik})$ se definiraju analogno. Konačno, definiramo težinsku funkciju

$$E_{\boldsymbol{\beta}}^{(k)}(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in \Omega_0, \\ E_{\beta_i}(x_1^i), & \boldsymbol{x} \in \Omega_{ik}, \ i = 1, 2, \dots, m, \\ E_{\beta_i}(k), & \boldsymbol{x} \in \Omega_i \backslash \Omega_{ik}, \ i = 1, 2, \dots, m. \end{cases}$$

Lako je provjeriti da vrijedi

$$|\nabla E_{\boldsymbol{\beta}}^{(k)}(\boldsymbol{x})| \leq b_3 \gamma_* E_{\boldsymbol{\beta}}^{(k)}(\boldsymbol{x}), \ \forall \boldsymbol{x} \in \Omega \tag{B.2}$$

te

$$E_{\boldsymbol{\beta}_j}^{(k)}(\boldsymbol{x}) \leq E_{\boldsymbol{\beta}_j}(\boldsymbol{x}), \ \forall \boldsymbol{x} \in \Omega, \tag{B.3}$$

kada $\beta_i \geq 0$. Dodatno, vrijedi nejednakost

$$\int_{\Omega_{(k)}} E_{\boldsymbol{\beta}}(\boldsymbol{x}) |\boldsymbol{u}(\boldsymbol{x})|^2 \leq \int_\Omega E_{\boldsymbol{\beta}}^{(k)}(\boldsymbol{x}) |\boldsymbol{u}(\boldsymbol{x})|^2. \tag{B.4}$$

Također, trebat će nam težinska Poincaréova nejednakost (vidi [85], Poglavlje I, Odjeljak 1.1.3, Lema 1.9):

Lema B.0.1. Za svaku funkciju $\boldsymbol{u} \in \mathcal{W}_\beta^{l,2}(\Omega)$ koja je jednaka nuli na $\partial\Omega$ vrijedi sljedeća težinska Poincaréova nejednakost:

$$\int_\Omega E_{\boldsymbol{\beta}}(\boldsymbol{x}) |\boldsymbol{u}(\boldsymbol{x})|^2 d\boldsymbol{x} \leq C_p \int_\Omega E_{\boldsymbol{\beta}}(\boldsymbol{x}) |\nabla \boldsymbol{u}(\boldsymbol{x})|^2 d\boldsymbol{x}.$$

Promotrimo sada linearni mikropolarni sustav jednadžbi na Ω:

$$\begin{aligned}
-\Delta \boldsymbol{u} + \nabla p &= 2N \operatorname{rot} \boldsymbol{w} + \boldsymbol{f} + \sum_{j=1}^{3} \frac{\partial \boldsymbol{F}_j}{\partial x_j}, \\
\operatorname{div} \boldsymbol{u} &= 0, \\
-\alpha \Delta \boldsymbol{w} - \beta \nabla \operatorname{div} \boldsymbol{w} + 4N \boldsymbol{w} &= 2N \operatorname{rot} \boldsymbol{u} + \boldsymbol{g} + \sum_{j=1}^{3} \frac{\partial \boldsymbol{G}_j}{\partial x_j} + \sum_{j=1}^{3} \nabla H_j, \\
\boldsymbol{u}, \boldsymbol{w} &= \boldsymbol{0} \text{ na } \partial\Omega, \\
\int_{\sigma_i} \boldsymbol{u} \cdot n &= 0,
\end{aligned} \tag{B.5}$$

gdje su $\alpha, \beta, N > 0$ te $N < 1$. Slabim rješenjem problema (B.5) smatramo par $(\boldsymbol{u}, \boldsymbol{w}) \in V \times W_0^{1,2}(\Omega)$ takav da su sljedeći integralni identiteti zadovoljeni:

$$\begin{aligned}
\int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\varphi} &= 2N \int_\Omega \operatorname{rot} \boldsymbol{w} \cdot \boldsymbol{\varphi} + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \sum_{j=1}^{3} \int_\Omega \boldsymbol{F}_j \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_j}, \\
\alpha \int_\Omega \nabla \boldsymbol{w} \cdot \nabla \boldsymbol{\psi} + \beta \int_\Omega \operatorname{div} \boldsymbol{w} \operatorname{div} \boldsymbol{\psi} + 4N \int_\Omega \boldsymbol{w} \cdot \boldsymbol{\psi} &= 2N \int_\Omega \operatorname{rot} \boldsymbol{u} \cdot \boldsymbol{\psi} + \int_\Omega \boldsymbol{g} \cdot \boldsymbol{\psi} \\
&\quad - \sum_{j=1}^{3} \int_\Omega \boldsymbol{G}_j \cdot \frac{\partial \boldsymbol{\psi}}{\partial x_j} - \sum_{j=1}^{3} \int_\Omega H_j \operatorname{div} \boldsymbol{\psi},
\end{aligned} \tag{B.6}$$

za sve $\boldsymbol{\varphi} \in V$, $\boldsymbol{\psi} \in W_0^{1,2}(\Omega)$. Sada dokazujemo rezultat koji je nužan za analizu provedenu u Poglavlju 4:

Teorem B.0.1. Neka su $0 < \beta_i \leq \beta_*$, $i = 1, 2, \ldots, m$ i $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{F}_j, \boldsymbol{G}_j, H_j \in \mathcal{L}_\beta^2(\Omega)$, $j = 1, 2, 3$, tada postoji slabo rješenje problema (B.5). Dodatno, postoji funkcija tlaka $p \in L^2_{\text{loc}}(\Omega)$ takva da

$$\int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\varphi} = 2N \int_\Omega \operatorname{rot} \boldsymbol{w} \cdot \boldsymbol{\varphi} + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \sum_{j=1}^{3} \int_\Omega \boldsymbol{F}_j \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_j} + \int_\Omega p \operatorname{div} \boldsymbol{\varphi}, \tag{B.7}$$

za sve $\boldsymbol{\varphi} \in W_0^{1,2}(\Omega)$. Nadalje, pretpostavimo da vrijedi $\partial\Omega \in C^2$ te da $\boldsymbol{F}_j, \boldsymbol{G}_j, H_j$, $j = 1, 2, 3$ imaju kompaktne nosače. Tada za dovoljno mali β_* slabo rješenje $(\boldsymbol{u}, \nabla p, \boldsymbol{w})$ pripada prostoru $\mathcal{W}_\beta^{2,2}(\Omega) \times \mathcal{L}_\beta^2(\Omega) \times \mathcal{W}_\beta^{2,2}(\Omega)$ i zadovoljava ocjenu

$$\begin{aligned}
\|\boldsymbol{u}\|_{\mathcal{W}_\beta^{2,2}(\Omega)} + \|\nabla p\|_{\mathcal{L}_\beta^2(\Omega)} + \|\boldsymbol{w}\|_{\mathcal{W}_\beta^{2,2}(\Omega)} \leq C \Big(& \|\boldsymbol{f}\|_{\mathcal{L}_\beta^2(\Omega)} + \|\boldsymbol{g}\|_{\mathcal{L}_\beta^2(\Omega)} \\
& + \sum_{j=1}^{3} \|\boldsymbol{F}_j\|_{L^2(\Omega)} + \sum_{j=1}^{3} \|\boldsymbol{G}_j\|_{L^2(\Omega)} + \sum_{j=1}^{3} \|H_j\|_{L^2(\Omega)} \Big).
\end{aligned} \tag{B.8}$$

Dokaz. Egzistencija i jedinstvenost slabog rješenja problema (B.5) se može pokazati na standardan način koristeći Lax-Milgramovu lemu. Preostalo je još dokazati eksponencijalni pad rješenja k nuli. Jer smo pretpostavili da funkcije $\boldsymbol{F}_j, \boldsymbol{G}_j, H_j$ imaju kompaktne nosače, one ne utječu na ponašanje toka kada $|\boldsymbol{x}| \to +\infty$. Shodno tome, možemo pretpostaviti $\boldsymbol{F}_j, \boldsymbol{G}_j = \boldsymbol{0},\ H_j = 0,\ j = 1, 2, 3$ u ostatku dokaza.

Uvrstimo sada u jednakost $(\text{B.6})_2$ testnu funkciju $\boldsymbol{\psi}(\boldsymbol{x}) = E_\beta^{(k)}(\boldsymbol{x})\boldsymbol{w}(\boldsymbol{x})$. Tada imamo

$$\begin{aligned}\alpha \int_\Omega E_\beta^{(k)} |\nabla \boldsymbol{w}|^2 + \beta \int_\Omega E_\beta^{(k)} |\operatorname{div} \boldsymbol{w}|^2 + 4N \int_\Omega E_\beta^{(k)} |\boldsymbol{w}|^2 &= -\alpha \int_\Omega \nabla \boldsymbol{w} \cdot (\nabla E_\beta^{(k)} \cdot \boldsymbol{w}) \\ -\beta \int_\Omega \operatorname{div} \boldsymbol{w} \nabla E_\beta^{(k)} \cdot \boldsymbol{w} + 2N \int_\Omega E_\beta^{(k)} \operatorname{rot} \boldsymbol{u} \cdot \boldsymbol{w} &+ \int_\Omega E_\beta^{(k)} \boldsymbol{g} \cdot \boldsymbol{w}.\end{aligned} \tag{B.9}$$

Izraze s desne strane jednakosti (B.9) ocjenjujemo koristeći (B.2), Lemu B.0.1 te Youngovu nejednakost:

$$\begin{aligned}\alpha \left| \int_\Omega \nabla \boldsymbol{w} \cdot (\nabla E_\beta^{(k)} \cdot \boldsymbol{w}) \right| &\leq \alpha \left(\int_\Omega \nabla E_\beta^{(k)} |\nabla \boldsymbol{w}|^2 \right)^{1/2} \left(\int_\Omega \nabla E_\beta^{(k)} |\boldsymbol{w}|^2 \right)^{1/2} \\ &\leq C\gamma_* \int_\Omega E_\beta^{(k)} |\nabla \boldsymbol{w}|^2, \\ \beta \left| \int_\Omega \operatorname{div} \boldsymbol{w} \nabla E_\beta^{(k)} \cdot \boldsymbol{w} \right| &\leq \beta \left(\int_\Omega \nabla E_\beta^{(k)} |\operatorname{div} \boldsymbol{w}|^2 \right)^{1/2} \left(\int_\Omega \nabla E_\beta^{(k)} |\boldsymbol{w}|^2 \right)^{1/2} \\ &\leq C\gamma_* \int_\Omega E_\beta^{(k)} |\nabla \boldsymbol{w}|^2, \\ 2N \left| \int_\Omega E_\beta^{(k)} \operatorname{rot} \boldsymbol{u} \cdot \boldsymbol{w} \right| &\leq 2N \left(\int_\Omega E_\beta^{(k)} |\nabla \boldsymbol{u}|^2 \right)^{1/2} \left(\int_\Omega E_\beta^{(k)} |\boldsymbol{w}|^2 \right)^{1/2} \\ &\leq \frac{N}{2} \int_\Omega E_\beta^{(k)} |\nabla \boldsymbol{u}|^2 + 2N \int_\Omega E_\beta^{(k)} |\boldsymbol{w}|^2, \\ \left| \int_\Omega E_\beta^{(k)} \boldsymbol{g} \cdot \boldsymbol{w} \right| &\leq \left(\int_\Omega E_\beta^{(k)} |\boldsymbol{g}|^2 \right)^{1/2} \left(\int_\Omega E_\beta^{(k)} |\boldsymbol{w}|^2 \right)^{1/2} \\ &\leq C \left(\int_\Omega E_\beta^{(k)} |\boldsymbol{g}|^2 \right)^{1/2} \left(\int_\Omega E_\beta^{(k)} |\nabla \boldsymbol{w}|^2 \right)^{1/2} \\ &\leq C(\gamma_*) \int_\Omega E_\beta^{(k)} |\boldsymbol{g}|^2 + C\gamma_* \int_\Omega E_\beta^{(k)} |\nabla \boldsymbol{w}|^2.\end{aligned} \tag{B.10}$$

Iz (B.9) te (B.10) zaključujemo

$$(\alpha - C\gamma_*) \int_\Omega E_\beta^{(k)} |\nabla \boldsymbol{w}|^2 + 2N \int_\Omega E_\beta^{(k)} |\boldsymbol{w}|^2 \leq \frac{N}{2} \int_\Omega E_\beta^{(k)} |\nabla \boldsymbol{u}|^2 + C(\gamma_*) \int_\Omega E_\beta^{(k)} |\boldsymbol{g}|^2. \tag{B.11}$$

Odaberimo dovoljno mali γ_* tako da $\alpha - C\gamma_* > 0$, onda iz (B.11) slijede ocjene

$$\begin{aligned}2N \int_\Omega E_\beta^{(k)} |\boldsymbol{w}|^2 &\leq \frac{N}{2} \int_\Omega E_\beta^{(k)} |\nabla \boldsymbol{u}|^2 + C \int_\Omega E_\beta^{(k)} |\boldsymbol{g}|^2, \\ \int_\Omega E_\beta^{(k)} |\nabla \boldsymbol{w}|^2 &\leq C \Big(\int_\Omega E_\beta^{(k)} |\nabla \boldsymbol{u}|^2 + \int_\Omega E_\beta^{(k)} |\boldsymbol{g}|^2 \Big).\end{aligned} \tag{B.12}$$

Sada primjenjujemo nejednakosti (B.3) i (B.4) na ocjene (B.12) te puštamo limes $k \to \infty$, čime dobivamo

$$\begin{aligned} 2N \|\boldsymbol{w}\|^2_{\mathcal{L}^2_\beta(\Omega)} &\le \frac{N}{2} \|\nabla \boldsymbol{u}\|^2_{\mathcal{L}^2_\beta(\Omega)} + C \|\boldsymbol{g}\|^2_{\mathcal{L}^2_\beta(\Omega)}, \\ \|\nabla \boldsymbol{w}\|^2_{\mathcal{L}^2_\beta(\Omega)} &\le C\Big(\|\nabla \boldsymbol{u}\|^2_{\mathcal{L}^2_\beta(\Omega)} + \|\boldsymbol{g}\|^2_{\mathcal{L}^2_\beta(\Omega)}\Big). \end{aligned} \tag{B.13}$$

Nadalje, za funkciju $\boldsymbol{u}$ koja je rješenje problema (B.6) postoji funkcija $\boldsymbol{W}^{(k)} \in W^{1,2}_0(\Omega)$ takva da

$$\operatorname{supp} \boldsymbol{W}^{(k)} \subset \bigcup_{i=1}^m \Omega_{ik}$$

i

$$\operatorname{div} \boldsymbol{W}^{(k)}(\boldsymbol{x}) = -\operatorname{div}(E^{(k)}_\beta(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x})).$$

Dodatno, vrijedi nejednakost

$$\begin{aligned} \int_\Omega E^{(k)}_{-\beta}(\boldsymbol{x})|\nabla \boldsymbol{W}^{(k)}(\boldsymbol{x})|^2 dx &\le C\gamma_*^2 \int_\Omega E^{(k)}_\beta(\boldsymbol{x})|\boldsymbol{u}(\boldsymbol{x})|^2 dx \\ &\le C\gamma_*^2 \int_\Omega E^{(k)}_\beta(\boldsymbol{x})|\nabla \boldsymbol{u}(\boldsymbol{x})|^2 dx, \end{aligned} \tag{B.14}$$

gdje je γ_* konstanta iz nejednakosti $(B.1)_4$ i C je konstanta koja ne ovisi o k ni o $\boldsymbol{u}$ (vidi [85], Poglavlje I, Odjeljak 1.1.3, Lema 1.13). Sada uvodimo test funkciju $\boldsymbol{\varphi}(\boldsymbol{x}) = E^{(k)}_\beta(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{W}^{(k)}(\boldsymbol{x})$. Jasno je da vrijedi $\boldsymbol{\varphi} = \boldsymbol{0}$ na $\partial\Omega$ te $\operatorname{div} \boldsymbol{\varphi} = 0$, pa uvrštavanjem $\boldsymbol{\varphi}$ u integralni identitet $(B.6)_1$ dobivamo

$$\begin{aligned} \int_\Omega E^{(k)}_\beta |\nabla \boldsymbol{u}|^2 = &-\int_\Omega \nabla \boldsymbol{u} \cdot (\nabla E^{(k)}_\beta \cdot \boldsymbol{u}) - \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{W}^{(k)} + 2N \int_\Omega E^{(k)}_\beta \operatorname{rot} \boldsymbol{w} \cdot \boldsymbol{u} \\ &+ 2N \int_\Omega \operatorname{rot} \boldsymbol{w} \cdot \boldsymbol{W}^{(k)} + \int_\Omega E^{(k)}_\beta \boldsymbol{f} \cdot \boldsymbol{u} + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{W}^{(k)}. \end{aligned} \tag{B.15}$$

Izraze na desnoj strani jednakosti (B.15) ocjenjujemo koristeći (B.2), Lemu B.0.1, (B.12), (B.14) te Youngovu nejednakost:

$$\begin{aligned} \left| \int_\Omega \nabla \boldsymbol{u} \cdot (\nabla E^{(k)}_\beta \cdot \boldsymbol{u}) \right| &\le \Big(\int_\Omega \nabla E^{(k)}_\beta |\nabla \boldsymbol{u}|^2 \Big)^{1/2} \Big(\int_\Omega \nabla E^{(k)}_\beta |\boldsymbol{u}|^2 \Big)^{1/2} \\ &\le C\gamma_* \int_\Omega E^{(k)}_\beta |\nabla \boldsymbol{u}|^2, \\ \left| \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{W}^{(k)} \right| &\le \Big(\int_\Omega E^{(k)}_\beta |\nabla \boldsymbol{u}|^2 \Big)^{1/2} \Big(\int_\Omega E^{(k)}_{-\beta} |\nabla \boldsymbol{W}^{(k)}|^2 \Big)^{1/2} \\ &\le C\gamma_* \int_\Omega E^{(k)}_\beta |\nabla \boldsymbol{u}|^2, \end{aligned}$$

$$
\begin{aligned}
2N\left|\int_\Omega E_\beta^{(k)} \operatorname{rot} \boldsymbol{w} \cdot \boldsymbol{u}\right| &= 2N\left|\int_\Omega \boldsymbol{w} \cdot \operatorname{rot}(E_\beta^{(k)}\boldsymbol{u})\right| \\
&\le 2N\left|\int_\Omega E_\beta^{(k)}\boldsymbol{w}\cdot\operatorname{rot}\boldsymbol{u}\right| + 2N\left|\int_\Omega \frac{E_\beta^{(k)}}{\partial x_1}\boldsymbol{w}\cdot(-u_3\boldsymbol{e}_2+u_2\boldsymbol{e}_3)\right| \\
&\le 2N\left(\int_\Omega E_\beta^{(k)}|\boldsymbol{w}|^2\right)^{1/2}\left(\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2\right)^{1/2} \\
&\quad + C\gamma_*\left(\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{w}|^2\right)^{1/2}\left(\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2\right)^{1/2} \\
&\le 2N\int_\Omega E_\beta^{(k)}|\boldsymbol{w}|^2 + \frac{N}{2}\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2 \\
&\quad + C\gamma_*\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{w}|^2 + C\gamma_*\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2 \\
&\le N\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2 + C(1+\gamma_*)\int_\Omega E_\beta^{(k)}|\boldsymbol{g}|^2 + C\gamma_*\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2, \\
2N\left|\int_\Omega \operatorname{rot}\boldsymbol{w}\cdot\boldsymbol{W}^{(k)}\right| &\le 2N\left(\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{w}|^2\right)^{1/2}\left(\int_\Omega E_{-\beta}^{(k)}|\boldsymbol{W}^{(k)}|^2\right)^{1/2} \qquad \text{(B.16)} \\
&\le 2N\left(\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{w}|^2\right)^{1/2}\left(\int_\Omega E_{-\beta}^{(k)}|\nabla\boldsymbol{W}^{(k)}|^2\right)^{1/2} \\
&\le C\gamma_*\left(\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{w}|^2\right)^{1/2}\left(\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2\right)^{1/2} \\
&\le C\gamma_*\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2 + C\gamma_*\left(\int_\Omega E_\beta^{(k)}|\boldsymbol{g}|^2\right)^{1/2}\left(\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2\right)^{1/2} \\
&\le C\gamma_*\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2 + C\gamma_*\int_\Omega E_\beta^{(k)}|\boldsymbol{g}|^2, \\
\left|\int_\Omega E_\beta^{(k)}\boldsymbol{f}\cdot\boldsymbol{u}\right| &\le \left(\int_\Omega E_\beta^{(k)}|\boldsymbol{f}|^2\right)^{1/2}\left(\int_\Omega E_\beta^{(k)}|\boldsymbol{u}|^2\right)^{1/2} \\
&\le C\left(\int_\Omega E_\beta^{(k)}|\boldsymbol{f}|^2\right)^{1/2}\left(\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2\right)^{1/2} \\
&\le C(\gamma_*)\int_\Omega E_\beta^{(k)}|\boldsymbol{f}|^2 + C\gamma_*\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2, \\
\left|\int_\Omega \boldsymbol{f}\cdot\boldsymbol{W}^{(k)}\right| &\le \left(\int_\Omega E_\beta^{(k)}|\boldsymbol{f}|^2\right)^{1/2}\left(\int_\Omega E_{-\beta}^{(k)}|\boldsymbol{W}^{(k)}|^2\right)^{1/2} \\
&\le C\left(\int_\Omega E_\beta^{(k)}|\boldsymbol{f}|^2\right)^{1/2}\left(\int_\Omega E_{-\beta}^{(k)}|\nabla\boldsymbol{W}^{(k)}|^2\right)^{1/2} \\
&\le C\gamma_*\left(\int_\Omega E_\beta^{(k)}|\boldsymbol{f}|^2\right)^{1/2}\left(\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2\right)^{1/2} \\
&\le C(\gamma_*)\int_\Omega E_\beta^{(k)}|\boldsymbol{f}|^2 + C\gamma_*\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2.
\end{aligned}
$$

Iz (B.15) i (B.16) slijedi ocjena

$$
(1-N-C\gamma_*)\int_\Omega E_\beta^{(k)}|\nabla\boldsymbol{u}|^2 \le C(\gamma_*)\Big(\int_\Omega E_\beta^{(k)}|\boldsymbol{f}|^2 + \int_\Omega E_\beta^{(k)}|\boldsymbol{g}|^2\Big).
$$

Kako je $N < 1$, možemo odabrati dovoljno mali γ_* tako da vrijedi

$$\int_\Omega E_\beta^{(k)} |\nabla \boldsymbol{u}|^2 \leq C\Big(\int_\Omega E_\beta^{(k)} |\boldsymbol{f}|^2 + \int_\Omega E_\beta^{(k)} |\boldsymbol{g}|^2 \Big). \tag{B.17}$$

Sada možemo iskoristiti nejednakosti (B.3) i (B.4) te pustiti limes $k \to \infty$ čime dobivamo

$$\|\nabla \boldsymbol{u}\|^2_{\mathcal{L}^2_\beta(\Omega)} \leq C \left(\|\boldsymbol{f}\|^2_{\mathcal{L}^2_\beta(\Omega)} + \|\boldsymbol{g}\|^2_{\mathcal{L}^2_\beta(\Omega)} \right), \tag{B.18}$$

te iz (B.13) i (B.18) slijedi

$$\|\nabla \boldsymbol{w}\|^2_{\mathcal{L}^2_\beta(\Omega)} \leq C \left(\|\boldsymbol{f}\|^2_{\mathcal{L}^2_\beta(\Omega)} + \|\boldsymbol{g}\|^2_{\mathcal{L}^2_\beta(\Omega)} \right). \tag{B.19}$$

Poznato je da postoji funkcija $p \in L^2_{\text{loc}}(\Omega)$ takva da vrijedi (B.7) (vidi npr. [38]). Dodatno, vrijedi sljedeća lokalna ocjena:

$$\begin{aligned} &\|\boldsymbol{u}\|_{W^{2,2}(\omega_{ik})} + \|\nabla p\|_{L^2(\omega_{ik})} + \|\boldsymbol{w}\|_{W^{2,2}(\omega_{ik})} \\ &\leq C \left(\|\nabla \boldsymbol{u}\|_{L^2(\hat{\omega}_{ik})} + \|\nabla \boldsymbol{w}\|_{L^2(\hat{\omega}_{ik})} + \|\boldsymbol{f}\|_{L^2(\hat{\omega}_{ik})} + \|\boldsymbol{g}\|_{L^2(\hat{\omega}_{ik})} \right), \end{aligned} \tag{B.20}$$

gdje konstanta C ne ovisi o k (vidi Lema 13. i Lema 15. u [92]). Slično, vrijedi nejednakost

$$\begin{aligned} &\|\boldsymbol{u}\|_{W^{2,2}(\Omega_{(1)})} + \|\nabla p\|_{L^2(\Omega_{(1)})} + \|\boldsymbol{w}\|_{W^{2,2}(\Omega_{(1)})} \\ &\leq C \left(\|\nabla \boldsymbol{u}\|_{L^2(\Omega_{(2)})} + \|\nabla \boldsymbol{w}\|_{L^2(\Omega_{(2)})} + \|\boldsymbol{f}\|_{L^2(\Omega_{(2)})} + \|\boldsymbol{g}\|_{L^2(\Omega_{(2)})} \right). \end{aligned} \tag{B.21}$$

Sada množimo (B.20) s $E_{\beta_i}(k)$, te potom koristimo svojstvo $(\text{B.1})_3$ funkcije $E_{\beta_i}(\boldsymbol{x})$. Sumiranjem dobivenih ocjena po k konačno dobivamo

$$\begin{aligned} &\|\boldsymbol{u}\|_{\mathcal{W}^{2,2}_\beta(\Omega_i)} + \|\nabla p\|_{\mathcal{L}^2_\beta(\Omega_i)} + \|\boldsymbol{w}\|_{\mathcal{W}^{2,2}_\beta(\Omega_i)} \\ &\leq C \left(\|\nabla \boldsymbol{u}\|_{\mathcal{L}^2_\beta(\Omega_i)} + \|\nabla \boldsymbol{w}\|_{\mathcal{L}^2_\beta(\Omega_i)} + \|\boldsymbol{f}\|_{\mathcal{L}^2_\beta(\Omega_i)} + \|\boldsymbol{g}\|_{\mathcal{L}^2_\beta(\Omega_i)} \right). \end{aligned} \tag{B.22}$$

Ocjena (B.8) sada slijedi iz (B.18), (B.19), (B.21) i (B.22).

■

Napomena B.0.1. *Teorem B.0.1 vrijedi i u slučaju kada $m = 1$, to jest kada je Ω polubeskonačna cijev*

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^3 : x_1 > 0,\ \boldsymbol{x}_* \in \sigma\}.$$

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Životopis

Borja Rukavina je rođena 28.2.1997. godine u Zagrebu, gdje završava osnovnu i srednju školu. Diplomirala je 2020. na Matematičkom odsjeku Prirodoslovno-matematičkog fakulteta Sveučilišta u Zagrebu s diplomskim radom *Soboljevljevi prostori i direktna metoda varijacijskog računa* pod mentorstvom izv. prof. dr. sc. Marka Ercega.

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Sudjelovala je na nekoliko domaćih i međunarodnih znanstvenih konferencija, gdje je održala dva priopćenja i prezentirala dva postera. Do sada u koautorstvu ima tri rada:

1. I. Pažanin, M. Radulović i B. Rukavina: *Rigorous derivation of the asymptotic model describing a steady thermomicropolar fluid flow through a curvilinear channel*, Zeitschrift für angewandte Mathematik und Physik 73 (195):1-25, 2022.,

2. M. Beneš, I. Pažanin, M. Radulović i B. Rukavina: *Nonzero boundary condition for the unsteady micropolar pipe flow: Well-posedness and asymptotics*, Applied Mathematics and Computation, 427:127184, 1-22, 2022.,

3. I. Pažanin, M. Radulović i B. Rukavina: *Asymptotic analysis of the nonsteady micropolar fluid flow through a system of thin pipes*, Mathematical Methods in the Applied Sciences, 1-36, 2024.

Izjava o izvornosti rada

Ja, Borja Rukavina, studentica Prirodoslovno-matematičkog fakulteta Sveučilišta u Zagrebu, s prebivalištem na adresi [redacted] matični broj doktoranda I-560/20, ovim putem izjavljujem pod materijalnom i kaznenom odgovornošću da je moj doktorski rad pod naslovom: *Prilozi asimptotičkom modeliranju toka mikropolarnog fluida*, isključivo moje autorsko djelo, koje je u potpunosti samostalno napisano uz naznaku izvora drugih autora i dokumenata korištenih u radu.

U Zagrebu, ____________________

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Borja Rukavina