Approach to the lower critical dimension of the φ4 theory in the derivative expansion of the functional renormalization group

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We revisit the approach to the lower critical dimension $d_{\rm lc}$ in the Ising-like φ^4 theory within the functional renormalization group by studying the lowest approximation levels in the derivative expansion of the effective average action. Our goal is to assess how the latter, which provides a generic approximation scheme valid across dimensions and found to be accurate in $d \geqslant 2$, is able to capture the long-distance physics associated with the expected proliferation of localized excitations near $d_{\rm lc}$. We show that the convergence of the fixed-point effective potential is nonuniform in the field when $d \to d_{\rm lc}$ with the emergence of a boundary layer around the minimum of the potential. This allows us to make analytical predictions for the value of the lower critical dimension $d_{\rm lc}$ and for the behavior of the critical temperature as $d \to d_{\rm lc}$, which are both found in fair agreement with the known results. This confirms the versatility of the theoretical approach.

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I. INTRODUCTION

Collective behavior characterized by an emergent scale invariance is encountered in a wide variety of physical situations where many degrees of freedom are correlated over long distances. Since its introduction, the Renormalization Group has been the theoretical tool of choice for understanding and describing this phenomenon [1]. It provides a powerful conceptual framework but, exact results being scarce, the search for generic and efficient approximation schemes has been very active from the very beginning [2–5]. One relatively recent line of research starts from an exact formulation of the Renormalization Group, in the form of a functional Renormalization Group (FRG) for scale-dependent generating functionals of correlation functions [6–8], and introduces potentially *nonperturbative* approximations through ansatzes for the scale-dependent generating functional under study. The question we want to address is to what extent such generic approximation schemes are able to describe specific problems in which the long-distance behavior involves strongly nonuniform configurations with localized excitations.

An example of such an approximation scheme within the FRG is the so-called derivative expansion of the effective average action (coarse-grained Gibbs free energy in the language of magnetic systems), which amounts to truncating the functional form of the latter in powers of the external momenta or equivalently in gradients of the fields [9]. The versatility and the effectiveness of the approach have been discussed in several reviews: see Refs. [10,11]. One key advantage of such an approach is that the space dimension d (as well as the number of components of the fields, etc.) can be continuously varied at will, allowing one to describe critical behavior from the upper dimension $d_{\rm uc}$ where spatial fluctuations of the local order parameter are easily tamed and classical (mean-field) exponents are observed down to the lower critical dimension $d_{\rm lc}$ below which fluctuations become so strong that no phase transition is possible.

The derivative-expansion approximation focuses on the long-distance properties and, in terms of coarse-grained configurations of the system, works about uniform configurations. While taking into the account fluctuations around the latter, it does so in a small momentum expansion, thus being best suited for long-wavelength fluctuations. One may therefore wonder if such a scheme is able to capture the physics associated with nonuniform configurations containing, e.g., domain walls, spin waves, or localized defects. The answer appears to be positive in the case of configurations involving extended defects, i.e., defects whose energy scales with the system size but in a subextensive way. For instance, the effect of spin waves and domain walls, which govern the return to convexity of the free energy of an O(N) model in its low-temperature ordered phase when spatial fluctuations are taken into account [10,12–14], or the role of singular avalanche

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events and of scale-free droplet excitations in the critical random-field Ising model [15–17] are all properly accounted for by the truncated derivative expansion even at the lowest orders.

Yet, the jury is still out when the relevant coarse-grained configurations that control the large-scale behavior involve localized excitations such as the kinks and antikinks found in the instanton analysis of the one-dimensional Ising model [18] or in the one-dimensional sine-Gordon model [19,20]. As the approach to the lower critical dimension for systems with a discrete symmetry is expected to be controlled by the proliferation of such localized excitations [21–23], describing the long-distance physics in, say, a model in the Ising universality class such as the φ^4 theory in $d = 1 + \epsilon$ when $\epsilon \to 0$ is thus a more demanding task for the nonperturbative but approximate FRG than describing the $O(N > 2)$ universality class near $d = 2$ [24].

In this paper we investigate how low orders of the derivative expansion in the FRG describe the approach to the lower critical dimension of the scalar φ^4 theory. The lowest order is known as the local potential approximation (LPA) [8] and is clearly unphysical in low dimensions as it predicts $d_{\rm lc} = 2$. Indeed, field renormalization is not accounted for in the LPA, implying that the anomalous dimension of the field is always $\eta = 0$. This then misses a crucial ingredient for investigating Ising criticality in dimensions less than 2. We thus consider the simplest approximation beyond the LPA that incorporates a nontrivial field renormalization and is often referred to as LPA′ [10,11]. Working at this level allows us to provide a detailed analytical treatment of the problem.

We stress that the issue *per se* is not to provide another characterization of the φ^4 theory near $d = 1$, as for instance Bruce and Wallace [21–23] have already developed an efficient approach in terms a specifically tailored droplet theory. The issue is to assess the ability of a *generic* nonperturbative approximation scheme within the FRG to quantitatively describe the long-distance physics of a model across the whole range of space dimensions from $d_{\rm lc}$ to $d_{\rm uc}$ *without a priori knowledge* of the relevant real-space coarse-grained configurations. This also involves the question of the continuity of the critical behavior in the dimensionality of space, which was first investigated by Ballhausen *et al.* [29]. At odds with the latter work we show that convergence of the critical behavior of the φ^4 theory when $d \to d_{\rm lc}$ within the FRG is nonuniform in the field.

The outline of the paper is as follows. In Sec. II we summarize the FRG framework and the derivative expansion scheme for the scalar φ^4 theory and we introduce the LPA′ approximation. We illustrate the approach to $d_{\rm lc}$ by presenting numerical calculations at $d > d_{\rm lc}$, which serve as an illustration and guide for our further analysis. We then show in Sec. III that the convergence of the fixed-point effective potential to the lower critical dimension is nonuniform in the field and involves a boundary layer around the minimum of the potential. We detail the singular perturbation treatment that allows us to find the solution at leading order over the whole range of field. We next present in Sec. IV the results that we obtain for the value the lower critical dimension, which we find close to the exact value $d_{\rm lc} = 1$, as well as for the critical temperature and for the critical exponents as $d \to d_{\rm lc}$. We finally give some concluding remarks and provide additional details on the method and the solution in several appendices.

II. FUNCTIONAL RG, DERIVATIVE EXPANSION, AND THE LPA′

We are interested in the critical behavior of the Ising universality class, which can be represented at a field-theoretical level by a a scalar φ^4 theory,

$$S[\varphi] = \int_x \left(\frac{1}{2}(\partial_\mu \varphi(x))^2 + \frac{r}{2}\varphi(x)^2 + \frac{u}{4!}\varphi(x)^4 \right), \quad (1)$$

where $\int_x \equiv \int d^d x$. To do so, we use the FRG approach which is a modern version of Wilson's RG in which fluctuations are progressively incorporated in the calculation of the partition function of the model through the addition to the action of an infrared (IR) regulator [10],

$$\Delta S_k[\varphi] = \frac{1}{2}\int_{xy} R_k(x-y)\varphi(x)\varphi(y), \quad (2)$$

where R_k is an IR cutoff function that suppresses integration of modes with momenta less than k without altering that of modes with momenta larger than k. Typical choices of R_k will be discussed below. The modified partition function

$$Z_k[J] = \int \mathcal{D}\varphi \exp\left(-S[\varphi] - \Delta S_k[\varphi] + \int_x J(x)\varphi(x)\right) \quad (3)$$

is the scale-dependent generating functional of correlation functions and via a modified Legendre transform,

$$\Gamma_k[\phi] = -\ln Z_k[J] + \int_x J(x)\phi(x) - \Delta S_k[\phi], \quad (4)$$

one can introduce the effective average action $\Gamma_k[\phi]$, with $\phi(x) = \langle \varphi(x) \rangle = \delta \ln Z_k[J]/\delta J(x)$, which is the scale-dependent generating functional of the one-particle irreducible (1-PI) correlation functions. It obeys an exact functional RG equation that describes its evolution with the IR scale k [7],

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\int_{xy} \partial_t R_k(x-y)\left[\left(\Gamma_k^{(2)}[\phi] + R_k\right)^{-1}\right]_{xy}, \quad (5)$$

where $\Gamma_k^{(2)}$ is the second functional derivative of Γ_k and $t = \ln(k/\Lambda)$ with Λ a UV cutoff.

The exact FRG equation in Eq. (5) is a convenient starting point for devising nonperturbative approximation schemes in the form of ansatzes for the functional dependence of the effective average action. One such scheme used to capture the long-distance physics is the derivative expansion in which the Lagrangian associated with Γ_k is expanded in gradients of the fields,

$$\Gamma_k[\phi] = \int_x \left[U_k(\phi(x)) + \frac{1}{2}Z_k(\phi(x))(\partial_x \phi(x))^2 + O(\partial^4) \right]. \quad (6)$$

When inserted in Eq. (5) the above ansatz provides a hierarchy of coupled FRG equations for the functions $U_k(\phi)$, $Z_k(\phi)$, etc., where the field configurations involved are now uniform, i.e., $\phi(x) = \phi$.

Scale invariance associated with criticality is described by a fixed point of the FRG equations once the latter have been cast in a dimensionless form via the use of scaling dimensions. One defines dimensionless quantities φ, u_k, z_k, etc., through

$$\phi = k^{D_\phi}\varphi, \quad U_k(\phi) = k^d u_k(\varphi), \quad Z_k(\phi) = Z_k z_k(\varphi), \tag{7}$$

etc., where the dimension of the field is related to the anomalous dimension η by $D_\phi = (d-2+\eta)/2$ and where the field renormalization constant Z_k goes as $k^{-\eta}$ in the vicinity of the fixed point. [Note that we have used the same notation φ for the bare variable in Eq. (1) and the dimensionless average field, as the former will no longer appear in what follows.]

The hierarchy of FRG equations when expressed in terms of dimensionless quantities takes the form

$$\begin{aligned}\partial_t u_k(\varphi) &= -d u_k(\varphi) + \frac{(d-2+\eta_k)}{2}\varphi u_k'(\varphi) + \beta_u(\varphi;\eta_k),\\ \partial_t z_k(\varphi) &= \eta_k z_k(\varphi) + \frac{(d-2+\eta_k)}{2}\varphi z_k'(\varphi) + \beta_z(\varphi;\eta_k),\end{aligned} \tag{8}$$

etc., where a prime indicates a derivative with respect to the argument of the function; β_u, β_z, etc., are functionals of u_k'', z_k, etc., and are given in Appendix A. The fixed points of the flow equations are reached when $t\to-\infty$ (i.e., $k\to 0$) and the left-hand sides go to zero.

As already stressed, a proper description of the approach to the lower critical dimension should incorporate field renormalization and a nonzero anomalous dimension η. The lowest order of the derivative expansion that achieves this is the so-called LPA′ in which one retains on top of the renormalized potential $U_k(\phi)$ a field independent but scale dependent Z_k. In explicit form, the dimensionless equation for the fixed-point potential is now

$$0 = -d\,u(\varphi) + \frac{d-2+\eta}{2}\varphi u'(\varphi) + 2v_d \ell_0^{(d)}(u''(\varphi);\eta), \tag{9}$$

where $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2)$ and $\ell_0^{(d)}$ is a (strictly positive) dimensionless threshold function which enforces the decoupling of the low-momentum and high-momentum modes; it is defined in terms of the dimensionless IR cutoff function $r(y=q^2/k^2) = R_k(q^2)/(Z_k q^2)$ by

$$\ell_0^{(d)}(w;\eta) = -\frac{1}{2}\int_0^\infty dy\, y^{\frac{d}{2}} \frac{\eta r(y) + 2y r'(y)}{(y[1+r(y)]+w)} \tag{10}$$

and is described in more detail in Appendix B. We have dropped the subscript $k\to 0$ for dimensionless quantities at the fixed point in the above equation to simplify the notation.

Deriving Eq. (9) with respect to the field φ gives an equation for $u'(\varphi)$ from which one extracts the equation for its minima $\pm\varphi_{\rm m}$ [through $u'(\pm\varphi_{\rm m})=0$],

$$0 = \frac{(d-2+\eta)}{2}\varphi_{\rm m} + 2v_d \frac{u'''(\varphi_{\rm m})}{u''(\varphi_{\rm m})}\partial_w \ell_0^{(d)}(w;\eta)|_{w=u''(\varphi_{\rm m})}, \tag{11}$$

and deriving one more time gives an equation for the "squared mass" $u''(\varphi)$. Both equations will be useful below.

In the LPA′ the field renormalization constant Z_k is chosen such that at the minimum of the potential $z_k(\pm\varphi_{\rm m}) = 1$ [10,29]. From Eq. (8) and the explicit form of β_z given in Appendix A one then obtains that

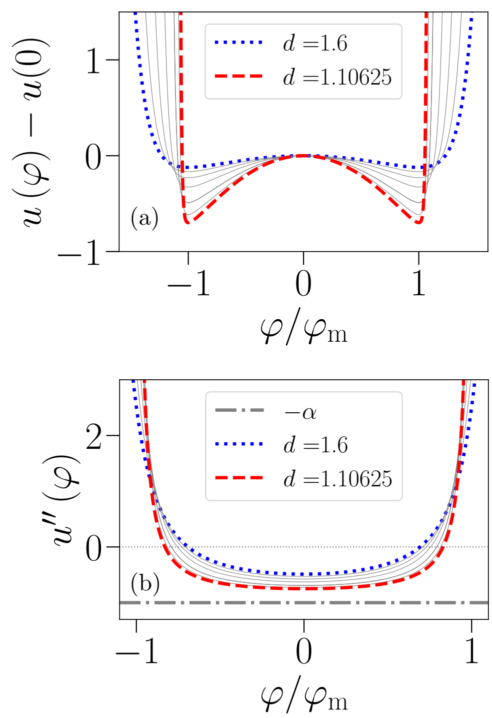

FIG. 1. Dimensionless effective potential $u(\varphi)$ (a) and its second derivative $u''(\varphi)$ (b) at the LPA′ fixed point for several dimensions d between 1.6 and 1.1. We have used the Θ IR cutoff function with $\alpha = 1$ and a numerical resolution of the FRG equations.

$$\eta = \frac{4v_d}{d} u'''(\varphi_{\rm m})^2 m_{4,0}^{(d)}(u''(\varphi_{\rm m});\eta), \tag{12}$$

where $m_{4,0}^{(d)}$ is another (strictly positive) dimensionless threshold function defined by

$$\begin{aligned} m_{4,0}^{(d)}(w;\eta) = \frac{1}{2}\int_0^\infty dy\, y^{\frac{d}{2}} \frac{1+(yr(y))'}{(y[1+r(y)]+w)^4}\Bigg[2\eta(yr(y))' \\ + 4(y^2 r'(y))' - 4\,\frac{y[1+(yr(y))'][\eta r(y) + 2yr'(y)]}{y[1+r(y)]+w}\Bigg]\end{aligned} \tag{13}$$

and discussed in Appendix B. Once a specific form for the dimensionless IR cutoff function $r(y)$ has been chosen, the solution of Eqs. (9)–(13) (together with the appropriate boundary conditions) fully characterizes the LPA′ fixed point. In what follows we will use two much studied forms of $r(y)$:

$$\begin{aligned} r(y) &= \alpha\Theta(1-y)(1-y)/y,\\ r(y) &= \alpha e^{-y}/y, \end{aligned} \tag{14}$$

where Θ is the Heaviside step function and α is a variational parameter of O(1) that can be determined by various forms of optimization [11,30–33]. We will refer to these two choices as Θ and exponential cutoff functions.

We illustrate the results for the fixed point at LPA′ and different choices of cutoff function in Figs. 1 and 2. Figure 1 displays the evolution of the dimensionless potential $u(\varphi)$ and the "square mass" function $u''(\varphi)$ as the space dimension d decreases (for the Θ cutoff function with $\alpha = 1$), and Fig. 2

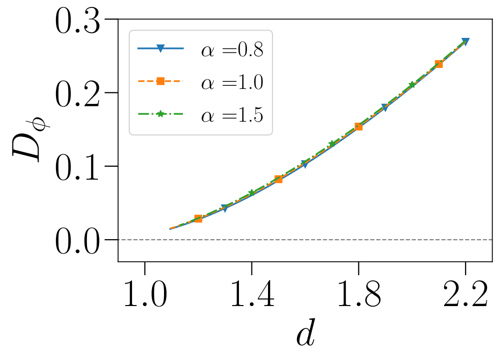

FIG. 2. Variation with the space dimension d of the field scaling dimension $D_\phi = (d-2+\eta)/2$ at the LPA′ fixed point as obtained from a numerical resolution and the exponential IR cutoff function for several values of α. Symbols represent every 30th data point.

that of the field scaling dimension $D_\phi = (d-2+\eta)/2$ (for the exponential cutoff function and several values of α). The fixed-point dimensionless effective potential has a nonconvex shape as a function of the dimensionless field with two symmetric minima, which is typical for the description a critical point [10]. The corresponding dimensionful effective potential is a convex function of the dimensionful field at the critical point, albeit with a singular (power law) behavior. When the lower critical dimension is approached, one expects the minima of the dimensionless effective potential to go to infinity. Note that it is numerically difficult to solve the fixed-point equations for $d \lesssim 1.1$ so that the results presented here as only illustrative and will only serve as a guide for a proper analysis of the limit $d \to d_{\rm lc}$.

To show that a qualitatively similar behavior in low dimension is also expected for higher orders of the derivative expansion, so that the LPA′ level is not atypical, we display in Appendix C the evolution with d of the dimensionless potential and of the field dimension D_ϕ at the second order of the derivative expansion for which the field renormalization is now a full function of the field [see Eq. (8)].

A defining property of the lower critical dimension is that $(d-2+\eta)|_{d\to d_l} \to 0$. This is equivalent to stating that the scaling dimension D_ϕ of the field vanishes (see above and Fig. 2). If the field does not rescale, then its fluctuations along the RG flow remain of order 1 in terms of the dimensionful field and ordering associated with a nonzero dimensionful average field in zero applied source is impossible. We thus find it convenient to define

$$\tilde{\epsilon}(d) = \frac{d-2+\eta}{2(2-\eta)}, \tag{15}$$

which goes to 0 as $d \to d_{\rm lc}$. We use the notation $\tilde{\epsilon}$ to avoid confusion with $\epsilon = d - d_{\rm lc}$ (with $d_{\rm lc} = 1$ in an exact treatment).

Another anticipated feature of the approach to the lower critical dimension is the fact that the propagator of the theory approaches a singularity. Indeed, the lower critical dimension corresponds to the merging of the critical fixed point and the zero-temperature fixed point associated with the symmetry-broken ordered phase, and the return to convexity of the *dimensionful* effective potential along the FRG flow is controlled in the latter by the presence of a singularity in the propagator [10,13,14,34]. In the LPA′ the dimensionless propagator is given by

$$p(y;\varphi) = \frac{1}{y[1+r(y)] + u''(\varphi)} \tag{16}$$

and must of course be positive. With the choice of the Θ cutoff function in Eq. (14) the pole in either in $y = q^2/k^2 = 0$ and $u''(\varphi) = -\alpha$ when $\alpha \leqslant 1$ or in $y = 1$ and $u''(\varphi) = -1$ when $\alpha > 1$. With the exponential cutoff function the pole is either in $y = 0$ and $u''(\varphi) = -\alpha$ when $\alpha \leqslant 1$ or in $y = \ln\alpha$ and $u''(\varphi) = -(1+\ln\alpha)$ when $\alpha > 1$: see Appendix B.

In the following we will investigate in more detail the structure of the fixed-point solution at LPA′ when $d \to d_{\rm lc}$. As the numerical solution becomes harder if not impossible in this limit, progress should be made through an analytical treatment. We stress again that contrary to what happens for the O($N > 2$) models where there are Goldstone modes associated with the breaking of a continuous symmetry which quite straightforwardly imply that $d_{\rm lc} = 2$ [10,11,35,36], the LPA′ or any level of truncation of the derivative expansion within the FRG need not predict the exact value of the lower critical dimension, $d_{\rm lc} = 1$, for the present model with a discrete Z_2 symmetry. The approximate $d_{\rm lc}$ must then be computed.

III. NONUNIFORM CONVERGENCE TO THE LOWER CRITICAL DIMENSION

A. Nonuniform convergence and boundary layer

Consider the LPA′ fixed-point equation for the second derivative of the potential,

$$0 = -u''(\varphi) + \tilde{\epsilon}(d)\varphi u'''(\varphi) + \frac{2v_d}{2-\eta(d)}\partial_\varphi^2 \ell_0^{(d)}(u''(\varphi);\eta(d)), \tag{17}$$

where $\eta = 2 - d + d\tilde{\epsilon} + O(\tilde{\epsilon}^2)$. As the dependence of $\tilde{\epsilon}$ on d is expected to be monotonic, one can study the above equation at fixed $\tilde{\epsilon}$ instead of fixed d, and when $\tilde{\epsilon} \to 0$, which is the limit of interest,

$$0 = -u''(\varphi) + \tilde{\epsilon}\varphi u'''(\varphi) + \frac{2v_d}{d}\partial_\varphi^2 \ell_0^{(d)}(u''(\varphi); 2-d), \tag{18}$$

where $d \equiv d(\tilde{\epsilon}) \to d_{\rm lc}$ and $d_{\rm lc}$ *a priori* unknown. This equation is supplemented by boundary conditions: $u''(\varphi)$ is an even function of the field so that $u'''(0)$ and all odd derivatives in $\varphi = 0$ are equal to 0; furthermore, one expects that $u''(\varphi \to \pm\infty) \to +\infty$.

We are looking for a solution $u''(\varphi;\tilde{\epsilon})$ of Eq. (18) in the limit $\tilde{\epsilon} \to 0$. However, inserting a simple expansion, $u''(\varphi;\tilde{\epsilon}) = u''^{[0]}(\varphi) + \tilde{\epsilon} u''^{[1]}(\varphi) + \cdots$, uniformly valid for all fields does not work when u'' varies very rapidly in a narrow domain so that the second term of the right-hand side of Eq. (18) becomes of the same order as the other two when $\tilde{\epsilon} \to 0$. This is a well-known problem treated by singular perturbation theory [37–39]. Our claim, which we substantiate below, is that this is indeed what happens and that the limit $\tilde{\epsilon} \to 0$ of the solution is actually nonuniform in the field.

From the shape of the fixed-point potential in Fig. 1 one can see that three domains of field values can be distinguished:

(a) the inner region from $\varphi = 0$ to values less than the minima, where the potential and its derivatives seem to be of magnitude of order one [i.e., of $O(\tilde{\epsilon}^0)$];

(b) the large-field region, $|\varphi| \to +\infty$, where $u(\varphi)$ and its derivatives increase rapidly;

(c) the two regions between (a) and (b) near the minima of the potential.

Region (a) should be describable by the $\tilde{\epsilon} = 0$ equation (plus a regular perturbation in $\tilde{\epsilon}$) and is further discussed in Sec. III B. The large-field region (b) corresponds to the situation where the square mass $u''(\varphi)$ is so large that the nontrivial β functions are effectively zero leaving only the scaling part of the equation: here, $0 = -u''(\varphi) + \tilde{\epsilon}\varphi u'''(\varphi)$, which leads to

$$u''(\varphi) \sim |\varphi|^{\frac{1}{\tilde{\epsilon}}} \quad \text{when} \quad \varphi \to \pm\infty. \tag{19}$$

Region (c) in the close vicinity of the minima needs more care and entails a boundary-layer treatment, which is further discussed in Sec. III C. (Note that "interior" or "internal" layer might be a more appropriate terminology because the regions in which the variation of the solution is very fast are away from the boundaries,[40,41] but with this caveat we will nonetheless keep using the more common term "boundary layer.")

A unique global solution valid for all fields is obtained by matching the partial solutions obtained in each domain; this is done the intermediate regions of field over which the solutions are still valid and yet overlap. This so-called "asymptotic matching" procedure or "method of matched asymptotic expansion" is a key element of the singular perturbation treatment [37–39].

The potential $u(\varphi)$ being Z_2 symmetric, we choose to restrict our analysis to positive fields, $\varphi \geqslant 0$.

B. The solution of the $\tilde{\epsilon} = 0$ equation

Consider first the LPA′ equation for $\tilde{\epsilon} = 0$. Introducing for simplicity the notation $w(\varphi) := u''(\varphi)$, one has

$$w(\varphi) = \frac{2v_d}{d}\partial_\varphi^2 \ell_0^{(d)}(w(\varphi); 2-d), \tag{20}$$

where $d = d_{\rm lc}$, which we assume in the following to be strictly less than 2; the initial conditions are $w(0) = w_0$ and all the odd derivatives of w are zero in $\varphi = 0$. Let us also define $\Phi(\varphi) := \ell_0^{(d)}(w(\varphi); 2-d)$. The function $\ell_0^{(d)}(w; 2-d)$ being monotonically decreasing with w, one can invert it and define $w = F(\Phi)$ with F such that $F(\ell_0^{(d)}(w; 2-d)) = w$. Equation (20) can then be rewritten as

$$\partial_\varphi^2 \Phi(\varphi) = \frac{d}{2v_d} F(\Phi(\varphi)), \tag{21}$$

which is the equation of motion of an anharmonic oscillator $\Phi(\varphi)$ with φ playing the role of time and $(d/(2v_d))F(\Phi)$ being the force. The solution for $\Phi(\varphi)$ is a periodic function starting in $\Phi_0 = \ell_0^{(d)}(w_0; 2-d)$ with a velocity $\partial_\varphi\Phi|_0 = 0$. The half-period φ_* corresponds to the first time at which the velocity is again equal to 0. By using the energy balance equation associated with Eq. (21),

$$\int_{\Phi_0}^{\Phi(\varphi)} d\Phi' \frac{d}{2v_d} F(\Phi') = \frac{1}{2}[\partial_\varphi \Phi(\varphi)]^2, \tag{22}$$

one derives that φ_* is obtained from

$$\int_0^{\varphi_*} d\varphi' \partial_{\varphi'}\Phi(\varphi') w(\varphi') = 0. \tag{23}$$

Note that φ_* and the solution $\Phi(\varphi)$ are parametrized by the initial value w_0.

Because of the monotonic relation between Φ and w, the solution for $w(\varphi)$ is also a periodic function of half-period φ_* that oscillates between a minimum value w_0 and a maximum one $w_* = w(\varphi_*)$, the two values being uniquely related. Clearly, this solution cannot be that of the full problem [which is not periodic: see Fig. 1(b)] when φ is close to and larger than φ_*. As alluded to above, a boundary-layer type of solution must then replace the solution of the $\tilde{\epsilon} = 0$ equation. Since $w(\varphi)$ is very large in the close vicinity of the (exact) minimum of the potential, $\varphi_{\rm m}$, a potential matching between the two types of solution must take place for $\varphi < \varphi_{\rm m} \lesssim \varphi_*$, which requires that $w(\varphi_*)$ is very large. In this limit, it can be shown from the properties of the $\tilde{\epsilon} = 0$ solution (see Appendix D) that

$$\varphi_* \sim \sqrt{\ln w_*}. \tag{24}$$

The matching requirement and the constraint it puts on the value of $w(\varphi_0) = w_0$ will be considered in more detail below.

C. The inner solution within the layer

The $\tilde{\epsilon} = 0$ equation ceases to be the proper description when the second term of Eq. (18) becomes of the same order as the other terms, and a new solution must be found. Guided by the numerical solution and by physical intuition, we argue that a new solution takes place within a boundary layer around the minimum $\varphi_{\rm m}$ of the potential. In this region, $w \gg 1$, leading to a considerable simplification of the threshold function $l_0^{(d)}$ and reducing Eq. (18) to

$$0 = -w(\varphi) + \tilde{\epsilon}\varphi w'(\varphi) + \frac{2v_d}{d}\alpha A_d \partial_\varphi^2\left[\frac{1}{w(\varphi)}\right], \tag{25}$$

where we have used that for a large square mass w, Eq. (10) gives

$$\ell_0^{(d)}(w; 2-d) \sim \frac{\alpha A_d}{w} + O\left(\frac{1}{w^2}\right), \tag{26}$$

with

$$A_d = d\int_0^\infty dy y^{\frac{d}{2}}\left[\frac{r(y)}{\alpha}\right]. \tag{27}$$

More details are given in Appendix B. It is then convenient to rescale the field φ by a multiplicative factor $\sqrt{d/(2\alpha v_d A_d)}$, so that Eq. (25) reads

$$0 = -w(\varphi) + \tilde{\epsilon}\varphi w'(\varphi) + \partial_\varphi^2\left[\frac{1}{w(\varphi)}\right], \tag{28}$$

with no explicit dependence on d (which we recall should be taken as $d_{\rm lc}$ in the limit $\tilde{\epsilon} \to 0$ that we consider).

In the vicinity of $\varphi_{\rm m}$ we introduce a rescaled variable $x = (\varphi - \varphi_{\rm m})/\delta(\tilde{\epsilon})$ with $x = O(1)$ as $\tilde{\epsilon} \to 0$. Requiring that the first two terms of Eq. (28) be of the same order of magnitude in $\delta(\tilde{\epsilon})$ ("principle of dominant balance" [38]), one obtains that $\delta(\tilde{\epsilon}) = \tilde{\epsilon}\varphi_{\rm m}$, i.e.,

$$x = \frac{\varphi - \varphi_{\rm m}}{\tilde{\epsilon}\varphi_{\rm m}}, \tag{29}$$

where we assume for now, and check later on, that $\tilde{\epsilon}\varphi_{\rm m} \to 0$ when $\tilde{\epsilon} \to 0$. By also requiring that the third term is of the same order of magnitude in $\delta(\tilde{\epsilon})$ as the first two, one is further led to introduce a function $g(x)$ which is defined by

$$w(\varphi) = \frac{g(x)}{\tilde{\epsilon}\varphi_{\rm m}} \tag{30}$$

and which is of O(1) when x is of O(1). The LPA$'$ equation in the boundary layer can then be expressed as

$$-g(x) + g'(x) + \partial_x^2\left[\frac{1}{g(x)}\right] = 0. \tag{31}$$

This is complemented by the equation for the minimum, Eq. (11), which leads to

$$g'(0) = g(0)^3. \tag{32}$$

The boundary layer equation can be solved in an implicit form by first introducing the auxiliary function

$$X(x) = g(x) + \partial_x\left[\frac{1}{g(x)}\right] \tag{33}$$

which satisfies $X(0) = 0$ because of Eq. (32). Then, one has that

$$g(x) = X'(x), \tag{34}$$

which can be restated as an equation for a function of X, $a(X) := g(x)$, as

$$a'(X) = a(X)(a(X) - X). \tag{35}$$

Solving Eq. (35) gives

$$g(x) = g(0)\frac{e^{-\frac{X(x)^2}{2}}}{\left[1 - \sqrt{\frac{\pi}{2}}g(0) + g(0)\int_{X(x)}^{+\infty} dt\, e^{-\frac{t^2}{2}}\right]}. \tag{36}$$

The interest of the above expression is that it allows us to study the limit $x \to +\infty$ and use matching with the outer solution at large field, $w(\varphi \to \infty) \sim \varphi^{1/\tilde{\epsilon}} \to +\infty$, already derived. (This is the standard method of matched asymptotic expansions used in singular perturbation problems.) Choosing the matching region such that $1/\tilde{\epsilon} \gg x \gg 1$, the latter then imposes that $g(x)$ diverges as $\exp(x)$ at large positive x. From Eqs. (36) and (33) it is straightforward to see that one must have

$$1 - \sqrt{\frac{\pi}{2}}g(0) = 0, \tag{37}$$

which fixes $g(0)$ and via Eq. (32) $g'(0)$.

We plot in Fig. 3 the inverse of $g(x)$, as obtained from the solution of the above equations. It is a monotonically decreasing function. When x is negative and $|x|$ very large, it behaves as

$$\frac{1}{g(x)} \sim \sqrt{2}|x|\sqrt{\ln|x|}\left[1 + \mathrm{O}\left(\frac{\ln(\ln|x|)}{\ln|x|}\right)\right], \tag{38}$$

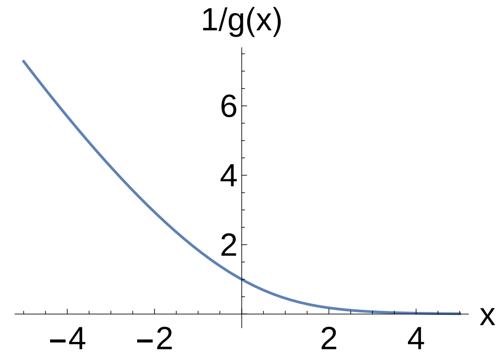

FIG. 3. Inverse of the inner solution $g(x)$ giving the square-mass function within the layer around the minimum of the potential. The latter corresponds to $x = 0$.

an expression which will be useful when considering matching with the outer solution obtained from the equation with $\tilde{\epsilon} = 0$ (see Sec. III B).

D. Matching the inner solution with that of the $\tilde{\epsilon} = 0$ equation

At least at the present LPA$'$ level of approximation the solution $g(x)$ within the layer around $\varphi_{\rm m}$ is fully determined by matching with the outer solution obtained at large field. However, the relation between $\varphi_{\rm m}$ and $\tilde{\epsilon}$ is yet to be determined. It is also crucial to check that matching with the outer solution corresponding to fields less than $\varphi_{\rm m}$ (and to the $\tilde{\epsilon} = 0$ equation) can be enforced, so that a solution can be constructed for all field values.

In Sec. III B we have argued that matching should take place for fields $\varphi < \varphi_{\rm m} \lesssim \varphi_*$, where furthermore the outer solution associated with the $\tilde{\epsilon} = 0$ equation should be such that $1 \ll w(\varphi) \ll w(\varphi_{\rm m}), w_*$. We can choose the matching region where the two solutions overlap such that

$$\varphi_{\rm m} - \varphi = \mathrm{O}((\tilde{\epsilon}\varphi_{\rm m})^a) \text{ with } 0 < a < 1, \tag{39}$$

and, as a result, $x \sim -(\tilde{\epsilon}\varphi_{\rm m})^{1-a}$ is negative and very large. The asymptotic limit of the inner solution is then given by Eq. (38), which implies that

$$w_* \gg w(\varphi) = \mathrm{O}\big(\tilde{\epsilon}\varphi_{\rm m})^{-a}|\ln(\tilde{\epsilon}\varphi_{\rm m})|^{-\frac{1}{2}}\big) \to +\infty. \tag{40}$$

From Eq. (23) one can obtain the relation between w_* and w_0 as

$$\begin{aligned} 0 &= \int_{w_0}^{w_*} dw' w' \partial_{w'}\Phi(w') \\ &= w_*\Phi(w_*) - w_0\Phi(w_0) - \int_{w_0}^{w_*} dw'\Phi(w'), \end{aligned} \tag{41}$$

where, we recall, $\Phi(w) = \ell_0^{(d)}(w; 2-d)$ with the latter given in Eq. (10). To evaluate the quantities in the above equation we split the integral by introducing an intermediate value w_c, which we choose positive and of O(1). Taking into account that w_* diverges and using the property that the function $\Phi(w)$ is monotonically decreasing and asymptotically goes to zero as $\alpha A_d/w + \mathrm{O}(1/w^2)$, we transform Eq. (41) into

$$\alpha A_d \ln w_* = -w_0\Phi(w_0) - \int_{w_0}^{w_c} dw'\Phi(w'), \tag{42}$$

up to O(1) terms. Since w_0 and w_c are both of O(1), this implies that $\Phi(w_0)$ diverges. This can only occur if w_0 approaches the pole of the propagator, which we call w_P and can be either -1, $-\alpha$, or $-(1+\ln\alpha)$ depending on the IR cutoff function and on α [see Eq. (16) and below]. Notwithstanding the precise asymptotic behavior of $\Phi(w) \equiv \ell_0^{(d)}(w; 2-d)$ when the pole is approached (this depends on the IR cutoff function, see Appendix B), the second term of the right-hand side is subdominant compared to the first one and one has

$$\Phi(w_0) \sim \frac{\alpha A_d}{|w_P|} \ln w_* \to +\infty. \tag{43}$$

Matching thus entails that the square mass in zero field $w_0 \to w_P^+$, which, as argued above, is one of the expected hallmarks of the approach to the lower critical dimension. Note that in the limit process w_0 must remain strictly larger than the pole w_P by a quantity that goes to zero with $\tilde{\epsilon}$: this again illustrates the highly singular and nonuniform approach to the lower critical dimension.

To complete the proof, we note that in the chosen matching region, the leading behavior of $w(\varphi)$ in the boundary layer and that corresponding to the $\tilde{\epsilon} = 0$ solution obey the same equation, $-w(\varphi) + \partial_\varphi^2[1/w(\varphi)] = 0$. The difference is in the boundary condition at large field: The $\tilde{\epsilon} = 0$ is limited by φ_* while it is convenient to consider the boundary-layer one up to φ_m. Taking this into account, the solution can then be obtained either as

$$\varphi_m - \varphi(w) = \tilde{\epsilon}\varphi_m|x| \approx \frac{\sqrt{2}}{2} \frac{1}{w\sqrt{\ln\left(\frac{1}{\tilde{\epsilon}\varphi_m w}\right)}} \tag{44}$$

or as

$$\varphi_* - \varphi(w) \approx \frac{\sqrt{2}}{2} \frac{1}{w\sqrt{\ln\left(\frac{w_*}{w}\right)}}. \tag{45}$$

Matching between the two solutions is then enforced at leading order if

$$\begin{aligned} \varphi_* &\sim \varphi_m, \\ w_* &\sim \frac{1}{\tilde{\epsilon}\varphi_m} \sim w_m, \end{aligned} \tag{46}$$

which, since φ_* diverges as $\sqrt{\ln w_*}$ [see Eq. (24) and Appendix D], immediately leads to

$$\begin{aligned} w_m &\sim w_* \sim \frac{1}{\tilde{\epsilon}\sqrt{\ln(\frac{1}{\tilde{\epsilon}})}}, \\ \varphi_m &\sim \varphi_* \sim \sqrt{\ln\left(\frac{1}{\tilde{\epsilon}}\right) + \mathrm{O}\left(\ln\ln\left(\frac{1}{\tilde{\epsilon}}\right)\right)}. \end{aligned} \tag{47}$$

So, as anticipated the location of the minimum of the potential φ_m diverges when $\tilde{\epsilon} \to 0$ but the width of the boundary layer $\tilde{\epsilon}\varphi_m$ goes to 0. This is supported by the numerical resolution of the LPA′ flow equation for values of d approaching as close as possible the lower critical dimension: see Fig. 4. This result is different than the prediction of the previous FRG analysis of the approach to the lower critical dimension within the truncated derivative expansion in Ref. [29]. The latter missed the emergence of the boundary layer near the minimum of the potential, which led to the scaling $\varphi_m \sim 1/\sqrt{\tilde{\epsilon}}$ that does not fit the data as shown in Fig. 4(a).

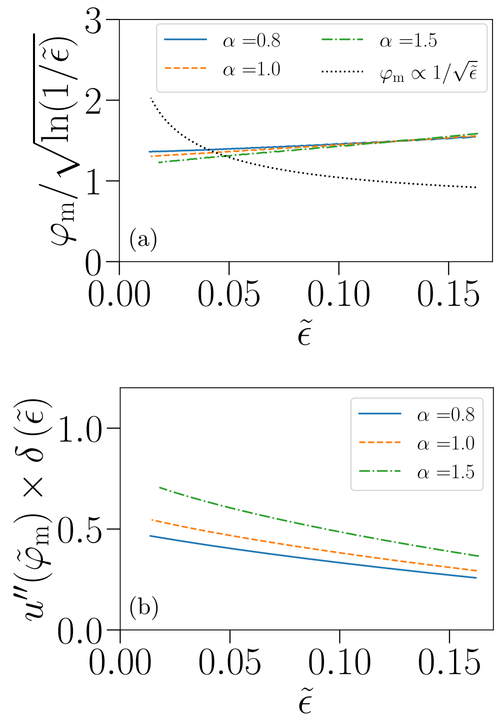

FIG. 4. (a) Location of the minimum of the potential φ_m divided by $\sqrt{\ln(1/\tilde{\epsilon})}$ as a function of $\tilde{\epsilon}$ from numerical calculation for $d > d_{lc}$. (b) Same but for the square mass $w_m \equiv u''(\varphi_m)$ multiplied by $\delta(\tilde{\epsilon}) = \tilde{\epsilon}\sqrt{\ln(1/\tilde{\epsilon})}$. Both plots are consistent with the analytically obtained predictions given in Eq. (47). However, the prediction from Ref. [29] [dashed black line in panel (a)] clearly does not fit the data. The data points in panels (a) and (b) are obtained from the numerical resolution of the LPA′ fixed-point equations for the exponential cutoff functions with several values of the prefactor α.

Collecting all of the above results allows one to build a fixed-point solution $w(\varphi) \equiv u''(\varphi)$ that is valid over the whole range of field values when $\tilde{\epsilon} \to 0$. One can note the peculiar form of the present singular perturbation problem in which neither the initial condition for $w(\varphi)$ in $\varphi = 0$ nor the location of the layer in φ_m are determined *a priori* and must be determined through the matching procedure.

We now discuss the consequences for the LPA′ prediction of the lower critical dimension d_{lc}, the behavior of the critical temperature T_c, and the critical exponents as $\tilde{\epsilon} \to 0$.

IV. RESULTS

A. Determination of the lower critical dimension

To determine the value of the lower critical dimension d_{lc} we consider the last of the LPA′ equations that we have not yet used, i.e., Eq. (12) for the anomalous dimension of the field. This equation involves the square mass $w(\varphi)$ in the boundary layer only since we fix the renormalization point for $z(\varphi)$ at the minimum of the effective potential (see above). At the LPA′ level, the choice of renormalization point determines the value of η, while at the next order of the derivative expansion all choices are equivalent. Choosing the minimum was shown

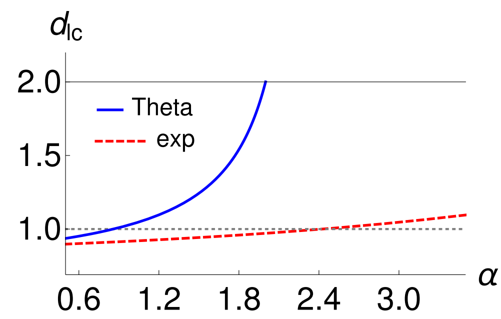

FIG. 5. Lower critical dimension $d_{lc}(\alpha)$ predicted by the solution of the LPA′ with the Θ (upper blue curve) and exponential (lower red curve) IR cutoff functions. The horizontal (gray) dashed line denotes $d = 1$, the exact result.

to be best for convergence within the LPA′ [42,43] because it corresponds to the statistically most preferred configuration of the system. When $\tilde{\epsilon} \to 0$, $w(\varphi_m) \to \infty$, $\eta \to 2 - d$, and the threshold function $m_{4,0}^{(d)}(w;\eta)$ can be replaced in Eq. (12) by its asymptotic form,

$$m_{4,0}^{(d)}(w; 2-d) \sim \frac{\alpha d A_d - \alpha^2 B_d}{w^4}, \tag{48}$$

where A_d is given in Eq. (27) and, as derived in Appendix B,

$$B_d = \frac{3d-2}{2}\int_0^\infty dy y^{\frac{d}{2}}\left[\left(y\frac{r(y)}{\alpha}\right)'\right]^2. \tag{49}$$

As before a prime indicates a derivative with respect to the argument of the function and the IR cutoff functions that we use are defined in Eq. (14).

With the rescaling of the field and the notations introduced in Sec. III C, one can then rewrite Eq. (12) as

$$2 - d = 2\left(d - \alpha\frac{B_d}{A_d}\right)\frac{g'(0)^2}{g(0)^4} = \frac{4}{\pi}\left(d - \alpha\frac{B_d}{A_d}\right), \tag{50}$$

where we have used that the solution within the layer around the minimum satisfies Eqs. (32) and (37). The solution of the above equation gives $d = d_{lc}$.

For the Θ cutoff function, $B_d/A_d = (3d-2)/4$ (see Appendix B), so that we obtain an explicit analytical expression for the lower critical dimension:

$$d_{lc}(\alpha) = 2\frac{\pi - \alpha}{\pi + 4 - 3\alpha}, \tag{51}$$

which for instance predicts $d_{lc} = 1.03419\cdots$ for $\alpha = 1$. Note that the solution derived in the previous section required $d < 2$. This entails that $\alpha < 2$, so that the pole in $\alpha = (\pi + 4)/3$ is not attained. The variation of d_{lc} with α is shown in Fig. 5.

For the exponential cutoff function, one finds that $B_d/A_d = 2^{-(1+d/2)}(3d-2)/4$ (see Appendix B), so that d_{lc} is solution of the implicit equation

$$d_{lc}(\alpha) = 2\frac{\pi - \alpha_*(d)}{\pi + 4 - 3\alpha_*(d)}, \tag{52}$$

with $\alpha_*(d) = 2^{-(1+d/2)}\alpha$. The outcome is plotted in Fig. 5 for a large range of α ($d_{lc} = 2$ is reached for $\alpha = 8$).

We therefore obtain that for a reasonable range of the variational parameter α the predicted lower critical dimension is indeed close to the exact result, $d_{lc} = 1$ (within 10% for the exponential cutoff function with α between 0.5 and 3.5, the result found with the Θ regulator appearing less well-behaved). A spurious dependence of the results on the choice of the IR cutoff function is expected when making approximations to the FRG. As mentioned in Sec. II, one may optimize the choice at a given level of the truncated derivative expansion by various procedures, one of them being the "principle of minimum sensitivity" (PMS) which amounts to choosing the parameters of the IR regulator that lead to an extremum in some computed output such as a critical exponent [11,31,32]. It is clear from Fig. 5 that $d_{lc}(\alpha)$ computed at the LPA′ level with the Θ and the exponential cutoff functions does not display any local minimum, so that one cannot optimize α through a PMS procedure. However, one can require that α stays in a range where d_{lc} does not vary too rapidly, which implies staying away from the pole in Eq. (51) or Eq. (52). An alternative criterion to the PMS could be to require that there is no transition in $d = 1$ [44]. Requiring that $d_{lc} = 1$ leads to $\alpha \approx 0.85$ for the Θ regulator and $\alpha \approx 2.43$ for the exponential regulator. However, this imposes using an exact result known by other means than the FRG, which weakens the significance of the optimization procedure [45].

B. Critical temperature T_c as $\tilde{\epsilon} \to 0$

One of the many defining properties of the lower critical dimension is that the critical temperature T_c goes to zero. This is a bare quantity which is not easily retrieved from the RG flow. However, when it goes to zero, a simple reasoning based on comparing the Boltzmann form of the distribution and the Wilsonian action where the field is rescaled by its value at the minimum of the effective potential suggests that the field at which the effective potential is minimum scales as the square root of the inverse temperature. As the dimension of the field at criticality goes to zero at the lower critical dimension, one therefore expects that $T_c \sim 1/\varphi_m^2$. This is indeed what is found in the correspondence between the Wilsonian dimensionless action of the O($N > 2$) model and the nonlinear sigma model near the lower critical dimension $d = 2$ [11]. Together with Eq. (47), this scaling leads to

$$T_c \sim \frac{1}{\ln\left(\frac{1}{\tilde{\epsilon}}\right)} \to 0, \tag{53}$$

when $\tilde{\epsilon} \to 0$.

Recast in terms of the field dimension $D_\phi = (d - 2 + \eta)/2$ the above expression is equivalent to

$$T_c \propto \frac{1}{\ln\left(\frac{1}{D_\phi}\right)}. \tag{54}$$

This relation is similar to that obtained by Bruce and Wallace from a detailed droplet theory [21,22]. In the latter, the expansion is performed in $\epsilon = d - d_{lc}$ with $d_{lc} = 1$. The outcome is that T_c has a simple expansion in powers of ϵ, $T_c \propto \epsilon + O(\epsilon^2)$, but D_ϕ has instead a singular behavior, with $D_\phi \sim e^{-2/\epsilon}/\epsilon$. Combining the two gives Eq. (54). Note that this relation is not verified by the prediction of Ref. [29].

C. Stability of the fixed point, essential scaling, and correlation-length critical exponent as $\tilde{\epsilon} \to 0$

The stability of the fixed point can be studied by looking at perturbations around it and the resulting eigenvalue equation equation obtained in linear order of the perturbation. For the present LPA′ approximation, after introducing small perturbations around the fixed point as $w_k(\varphi) = w(\varphi) + k^\lambda \delta w(\varphi)$, $\eta_k = \eta + k^\lambda \delta\eta$, $\varphi_{\rm mk} = \varphi_{\rm m} + k^\lambda \delta\varphi_{\rm m}$, etc., with λ an eigenvalue to be determined, the linearized equation for $\delta w(\varphi)$ reads

$$\begin{aligned}\lambda\delta w(\varphi) = &-(2-\eta)\delta w(\varphi) + (2-\eta)\tilde{\epsilon}\varphi\delta w'(\varphi)\\ &+ 2v_d\partial_\varphi^2\big[\partial_w \ell_0^{(d)}(w(\varphi);\eta)\delta w(\varphi)\big]\\ &+\bigg(w(\varphi)+\frac{1}{2}\varphi w'(\varphi)\\ &+2v_d\partial_\varphi^2\big[\partial_\eta \ell_0^{(d)}(w(\varphi);\eta)\big]\bigg)\delta\eta, \end{aligned} \tag{55}$$

and the expressions for $\delta\eta$ and $\delta\varphi_{\rm m}$ are given in Appendix E. We are especially interested in finding the relevant eigenvalue that gives the correlation length exponent ν which is known to diverge at the lower critical dimension in an exact treatment.

As we did for the fixed point, one can attempt a singular perturbation analysis when $\tilde{\epsilon} \to 0$ (and $\eta \to 2-d$) by looking separately at the $\tilde{\epsilon} = 0$ equation for φ of O(1) and at an equation in terms of the scaled variable $x = (\varphi - \varphi_{\rm m})/(\tilde{\epsilon}\varphi_{\rm m})$ near the minimum $\varphi_{\rm m}$. However, one immediately sees that if $\lambda = {\rm O}(\tilde{\epsilon})$ or more generally goes to zero when $\tilde{\epsilon} \to 0$, which is the expected behavior of the relevant eigenvalue(s), working at the leading order in $\tilde{\epsilon}$ does not allow the determination of λ beyond the fact that it starts as 0.

This can be illustrated by considering one eigenvalue that can be exactly obtained together with its eigenfunction. One easily finds that $\lambda = -(2-\eta)\tilde{\epsilon}$ is a solution of Eq. (55) (and of the additional equations given in Appendix E) with

$$\begin{aligned}\delta w(\varphi) &= \delta K\, w'(\varphi),\\ \delta\eta &= 0 = \delta\tilde{\epsilon},\\ \delta\varphi_{\rm m} &= -\delta K,\end{aligned} \tag{56}$$

with δK a constant that can be taken as infinitesimal to linearize the RG flow equations. Note that despite the fact that it corresponds to a relevant direction around the fixed point this eigenvalue is not the one we are interested in because it is associated with an odd (Z_2 antisymmetric) perturbation. We would instead like to determine the relevant eigenvalue associated with an even (Z_2 symmetric) perturbation which gives the correlation-length exponent ν through $1/\nu = -\lambda$. It is nonetheless instructive to study how the exact result for $\lambda = -(2-\eta)\tilde{\epsilon}$ translates into the leading order of the singular perturbation analysis and we trivially find that only $\lambda = 0$ can be obtained by working at the leading order of the $\tilde{\epsilon} = 0$ and of the boundary-layer equations.

This example confirms that eigenvalues going to zero as $\tilde{\epsilon} \to 0$ cannot be determined from the singular perturbation analysis at the leading order. One needs to go to the next order. In the present case this seems a formidable task that we will not undertake. We instead perform a numerical investigation by solving the LPA′ eigenvalue equation, Eq. (55), together with the fixed-point equation at fixed d, trying to reach as low as possible values near the lower critical dimension. As it should, we find that the critical fixed point has two relevant eigendirections: one corresponds to an even eigenfunction and gives the critical exponent ν and the other is equal to $-(2-\eta)\tilde{\epsilon} = -D_\phi$ and is associated with an odd eigenfunction related to the magnetic field (the scaling dimension of the magnetic field is then $d - D_\varphi$). All the other eigenvalues are positive, i.e., irrelevant, and of O(1) when $\tilde{\epsilon} \to 0$, as it should for a critical fixed point. We show $1/\nu$ as obtained for the exponential IR cutoff function and several values of the parameter α in Fig. 6. It is plotted both versus $\tilde{\epsilon}$ and versus the boundary-layer width $\delta(\tilde{\epsilon}) \sim \tilde{\epsilon}\sqrt{\ln(1/\tilde{\epsilon})}$. We observe that $1/\nu$ seems to be heading toward 0 when $\tilde{\epsilon} \to 0$, which is the expected behavior for an essential scaling of the correlation length as one approaches the lower critical dimension. Over the accessible range of dimensions, it appears to do so slower than linearly in $\tilde{\epsilon}$, possibly like $\tilde{\epsilon}\sqrt{\ln(1/\tilde{\epsilon})}$. However, the behavior is not compatible with the prediction of the droplet theory which would be $1/\ln(1/\tilde{\epsilon})$ [21–23]. However, our numerical results may not yet be in the asymptotic regime near $d_{\rm lc}$ and the conclusion should therefore be taken with a grain of salt.

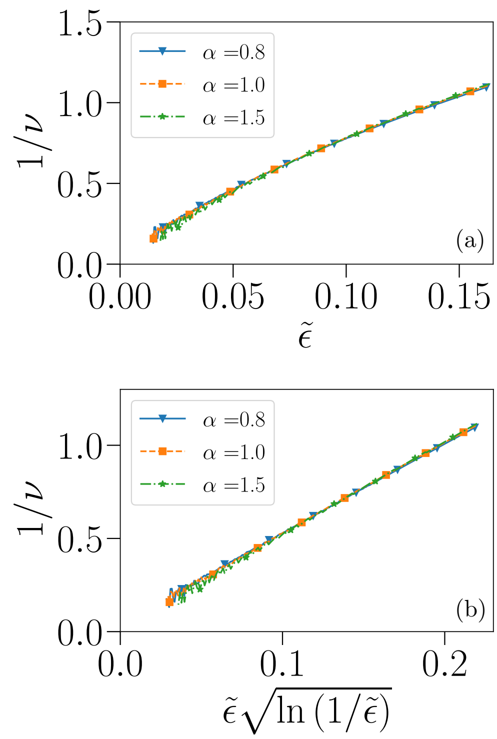

FIG. 6. The relevant eigenvalue $1/\nu$ obtained from the numerical resolution of the LPA′ equations with the exponential IR cutoff function and several values of the parameter α. It is plotted as a function of $\tilde{\epsilon}$ in panel (a) and $\tilde{\epsilon}\sqrt{\ln(1/\tilde{\epsilon})} \sim \delta(\tilde{\epsilon})$ in panel (b). Symbols represent every 30th data point.

V. CONCLUSION

We have presented a functional renormalization group (FRG) description of the approach to the lower critical dimension $d_{\rm lc}$ in a scalar φ^4 theory by using one of the simplest nonperturbative approximation level obtained as a truncation of the derivative expansion, the so-called LPA′. Our purpose is to test how a generic approximation scheme that works across dimensions in a continuous way and has been shown to be accurate in dimensions $d \geqslant 2$, for instance [11], is able to describe dimensions close to the lower critical dimension in a system with a discrete symmetry where it is known that the long-distance physics is controlled by the proliferation of localized excitations (in the present case, droplets that become point-like kinks and antikinks at the lower critical dimension $d_{\rm lc} = 1$ [21–23]). We show that the limit of d going to $d_{\rm lc}$ for the fixed-point effective action is nonuniform in the (average) field, with the emergence of a boundary layer around the minimum of the dimensionless potential. The minimum goes to infinity and the width of the layer goes to zero as $d \to d_{\rm lc}$, at odds with the outcome of an earlier FRG study [29]. The behavior of the critical temperature T_c is compatible with the expected exact results and, although the prediction of $d_{\rm lc}$ is dependent on the infrared regulator used in the FRG, we find it rather close to the exact value $d_{\rm lc} = 1$ for several reasonable choices of IR regulators.

One may wonder whether the description of the approach to the lower critical dimension $d_{\rm lc}$ improves as one considers higher orders of the derivative expansion. The anomalous dimension of the field η is indeed large (it approaches 1), and it seems already quite remarkable that the LPA′ is able to semi-quantitatively capture the critical behavior under such a condition. A first step forward is to check if the scenario found at the LPA′ level is valid at all orders of the truncated derivative expansion. Work is now in progress to investigate the next order, which includes a field-renormalization function in addition to the effective potential. Preliminary results appear to indicate that the same mechanism of a nonuniform convergence to the lower critical dimension with the emergence of a boundary layer around the minimum of the dimensionless potential is also at play. The next order of the derivative expansion also seems to more properly describe the form of divergence of the correlation length exponent ν: see the preliminary results in Appendix C.

As already stressed, our goal is not to provide yet another theoretical description of the approach to the lower critical dimension for systems in the universality class of the Ising model, a question which has been quite well understood for several decades. It is to benchmark a generic nonperturbative but approximate FRG approach to later tackle problems that are still open such as the value of the lower critical dimension of the athermally driven random-field Ising model (RFIM) [48]. The lower critical dimension of the RFIM in equilibrium has been rigorously shown to be $d_{\rm lc} = 2$ [49], but that for the far-from-equilibrium driven RFIM is debated [50–52]. Finally one might also hope that the present solution near the lower critical dimension can suggest new approximation schemes of the FRG that are not necessarily based on the truncation of the derivative expansion.

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APPENDIX A: FRG FLOW EQUATIONS

The β function(al) describing the FRG flow of the dimensionless effective potential $u_k(\varphi)$ is given by [10]

$$\beta_u(\varphi;\eta) = 2v_d\,\ell_0^{(d)}(u''(\varphi);\eta,z(\varphi)), \tag{A1}$$

where $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2)$; $\ell_n^{(d)}$ is a (strictly positive) dimensionless threshold function defined by

$$\ell_n^{(d)}(w;\eta,z) = -\left(\frac{n+\delta_{n,0}}{2}\right)\int_0^\infty dy\,y^{\frac{d}{2}}\,\frac{\eta r(y)+2yr'(y)}{(y[z+r(y)]+w)^{n+1}}, \tag{A2}$$

where the dimensionless infrared cutoff function (or IR regulator) $r(y)$ is obtained from the dimensionful one, $R_k(q^2)$, introduced in Eq. (2) through

$$R_k(q^2) = Z_k k^2 y\, r(y) \quad \text{with} \quad y = \frac{q^2}{k^2}, \tag{A3}$$

with k the running IR cutoff and Z_k the dimensionful field renormalization (such that the running anomalous dimension is defined by $\eta_k = -k\partial_k Z_k$).

From the exact FRG equation for the two-point 1-PI correlation function evaluated for a uniform field configuration one can extract the β functional for the dimensionless field renormalization function $z_k(\varphi)$ [10]

$$\begin{aligned}
\beta_z(\varphi;\eta) = &-\frac{4v_d}{d}u'''(\varphi)^2 m_{4,0}^{(d)}(u''(\varphi);\eta,z(\varphi))\\
&-\frac{8v_d}{d}u'''(\varphi)z'(\varphi)m_{4,0}^{(d+2)}(u''(\varphi);\eta,z(\varphi))\\
&-\frac{4v_d}{d}z'(\varphi)^2 m_{4,0}^{(d+4)}(u''(\varphi);\eta,z(\varphi)) - 2v_d z''(\varphi)\\
&\times \ell_1^{(d)}(u''(\varphi);\eta,z(\varphi))\\
&+4v_d u'''(\varphi)z'(\varphi)\ell_2^{(d)}(u''(\varphi);\eta,z(\varphi))\\
&+2v_d\frac{1+2d}{d}z'(\varphi)^2\ell_2^{(d+2)}(u''(\varphi);\eta,z(\varphi)),
\end{aligned} \tag{A4}$$

where $m_{n,0}^{(d)}$ is another (strictly positive) threshold function defined as

$$\begin{aligned}
m_{n,0}^{(d)}(w;\eta,z) = &\frac{1}{2}\int_0^\infty dy\,y^{\frac{d}{2}}\frac{z+(yr(y))'}{(y[z+r(y)]+w)^n}\bigg[2\eta(yr(y))'\\
&+4(y^2r'(y))' - n\frac{y[z+(yr(y))'][\eta r(y)+2yr'(y)]}{y[z+r(y)]+w}\bigg].
\end{aligned} \tag{A5}$$

To derive Eq. (A4) we have neglected the higher-order terms in Eq. (6) which involve four spatial derivatives: It therefore represents the second order of the derivative expansion which is fully characterized by the two functions $U_k(\phi)$ and $Z_k(\phi)$. Three more functions are required at the order $O(\partial^4)$, etc.

APPENDIX B: PROPERTIES OF THE THRESHOLD FUNCTIONS

The threshold functions introduced in the main text and in Appendix A are strictly positive dimensionless functions that enforce the decoupling of the low-momentum and high-momentum modes [10]. We only consider the LPA′ approximation so that the dimensionless field renormalization function $z(\varphi) \equiv 1$, but this is easily generalizable.

Before discussing some of their generic properties it is illustrative to give their explicit expression for a specific choice of IR cutoff function, $r(y) = \Theta(1-y)(1-y)/y$, which is the Θ cutoff function with $\alpha = 1$ (also called Litim or optimized regulator [30]):

$$\ell_n^{(d)}(w;\eta) = \frac{2(d+2-\eta)}{d(d+2)}\frac{n+\delta_{n,0}}{(1+w)^{n+1}},$$
$$m_{n,0}^{(d)}(w;\eta) = \frac{1}{(1+w)^n}. \tag{B1}$$

One can see that the threshold functions monotonically decrease as w increases, blow up near the pole of the propagator w_P (here $w_P = -1$), and go to zero as power laws when $w \to +\infty$. They are defined for $w > w_P$.

We analyze the behavior of the threshold functions for a generic IR cutoff function $r(y)$ in two limiting cases, when the mass w is large and when it approaches the pole of the propagator, i.e., when $w + \min_y\{y(1+r(y))\} \to 0$.

When the mass $w \to \infty$ and $z = 1$ one easily finds from Eq. (A2) that

$$\ell_n^{(d)}(w;\eta) \sim \frac{A_n^{(d)}(\eta)}{w^{n+1}} + O\left(\frac{1}{w^{n+2}}\right), \tag{B2}$$

where

$$A_n^{(d)}(\eta) = -\left(\frac{n+\delta_{n,0}}{2}\right)\int_0^\infty dy y^{\frac{d}{2}}[\eta r(y) + 2yr'(y)]$$
$$= (n+\delta_{n,0})A_0^{(d)}(\eta), \tag{B3}$$

and $A_0^{(d)}(\eta)$ can be rewritten as

$$A_0^{(d)}(\eta) = \frac{d+2-\eta}{2}\int_0^\infty dy y^{\frac{d}{2}} r(y). \tag{B4}$$

The choices of $r(y)$ that we use in this work are given in Eq. (14) so that $A_0^{(d)}(\eta)$ is proportional to α. When $\eta = 2-d$ this leads to the expression of A_d in Eq. (27).

Similarly, from Eq. (A5) one finds

$$m_{n,0}^{(d)}(w;\eta) \sim \frac{-B^{(d)}(\eta) + dA_0^{(d)}(\eta)}{w^n} + O\left(\frac{1}{w^{n+1}}\right), \tag{B5}$$

with

$$B^{(d)}(\eta) = -\int_0^\infty dy y^{\frac{d}{2}}(yr(y))'[\eta(yr(y))' + 2(y^2r'(y))']$$
$$= \frac{d+2-2\eta}{2}\int_0^\infty dy y^{\frac{d}{2}}[(yr(y))']^2. \tag{B6}$$

When $\eta = 2-d$ one immediately obtains Eq. (49).

All of the above results of course match with the expansion of the expressions in Eq. (B1).

We now turn to the expression of the threshold functions near the pole of the propagator, when $w \to w_P = -\min_y\{y(1+r(y))\}$. Note that the FRG equations are well behaved for $w + \min_y\{y(1+r(y))\} > 0$. The approach to the pole is what controls the return to convexity of the effective potential in the ordered phase [10,13,14] and is therefore important in the vicinity of the lower critical dimension where the critical fixed and the fixed point describing the ordered phase merge.

For the Θ cutoff function and for $z = 1$,

$$\ell_n^{(d)}(w;\eta) = \alpha\frac{n+\delta_{n,0}}{2}\int_0^1 dy y^{\frac{d}{2}-1}\frac{[(2-\eta)+\eta y]}{[w+\alpha+(1-\alpha)y]^{n+1}}, \tag{B7}$$

$$m_{n,0}^{(d)}(w;\eta)$$
$$= \alpha\frac{2-\alpha}{[w+1]^n} - \alpha\eta(1-\alpha)\int_0^1 dy y^{\frac{d}{2}}\frac{1}{[w+\alpha+(1-\alpha)y]^n}$$
$$+ \alpha\frac{n}{2}(1-\alpha)^2\int_0^1 dy y^{\frac{d}{2}}\frac{[(2-\eta)+\eta y]}{[w+\alpha+(1-\alpha)y]^{n+1}}. \tag{B8}$$

The pole of the propagator $1/[w+\alpha+(1-\alpha)y]$ is $w_P = -\alpha$ and is reached in $y = 0$ if $\alpha < 1$; it is $w_P = -1$ and is reached in $y = 1$ if $\alpha > 1$. The case $\alpha = 1$ corresponds to the expressions in Eq. (B1) and the approach to the pole in $w_P = -1$ can be read off directly. The threshold functions generically diverge as inverse power laws of $(w + w_P)$ when w approaches the pole w_P which is either $-\alpha$ or -1. An exception is $\ell_0^{(d)}(w;\eta)$ which behaves as $-\ln(w+1)$ when $\alpha > 1$.

For the exponential cutoff function (and for $z = 1$), a similar behavior is encountered except that the pole is attained either in $y = 0$ and is equal to $w_P = -\alpha$ when $\alpha < 1$ or in $y = \ln\alpha$ and is equal to $w = -(1+\ln\alpha)$ when $\alpha > 1$. The divergence of the threshold function as $w + w_P \to 0^+$ is generically power-law-like.

APPENDIX C: PRELIMINARY RESULTS AT THE SECOND ORDER OF THE DERIVATIVE EXPANSION

We have started to investigate higher orders of the derivative expansion. The next one, i.e., the second order, consists in keeping terms in the effective average action up to the second derivative of the field. At this $O(\partial^2)$ order one ends up with coupled flow equations for the two functions $u_k(\varphi)$ and $z_k(\varphi)$ given by Eqs. (8) with the β function(al)s given by Eqs. (A1) and (A4). The fixed-point equations simply follow. We have obtained some preliminary numerical results showing the behavior of the effective potential as a function of the dimensionless field for low dimensions: see Fig. 7(a). One observes the development of sharper and sharper

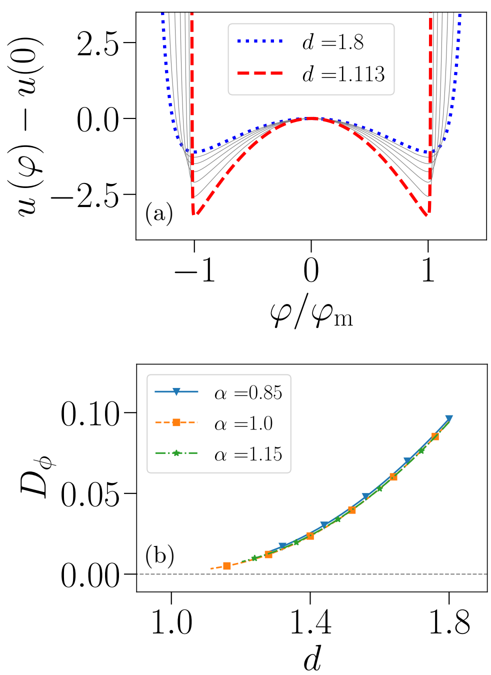

FIG. 7. Dimensionless effective potential $u(\varphi)$ (a) and field scaling dimension $D_\phi = (d-2+\eta)/2$ (b) at the fixed point obtained at the second order of the derivative expansion for several dimensions d between 1.8 and 1.11. We have used the exponential IR cutoff function with $\alpha = 1$ in panel (a) and several values of α in panel (b). In panel (b) symbols represent every 100th data point.

minima as d decreases, similar to what is found for the LPA′ (see Fig. 1) and suggestive of the emergence of a boundary layer in the close vicinity of the minima. Accordingly, the scaling dimension of the field D_ϕ decreases and appears to be heading toward 0 as d approaches some lower critical dimension close to 1. This illustrates that the behavior found at the LPA′ is not singular and may be (qualitatively at least) representative of the truncated derivative expansion. We have also computed the relevant eigenvalue $1/\nu$ for dimensions approaching as much as possible (for the numerical resolution) the lower critical dimension. One can see from Fig. 8 that the

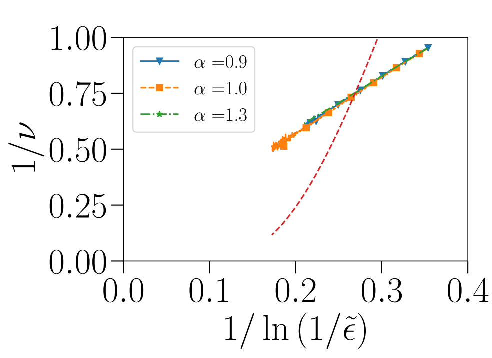

FIG. 8. The relevant eigenvalue $1/\nu$ obtained from the numerical resolution of the ∂^2 equations with the exponential IR cutoff function and several values of the parameter α. It is plotted as a function of $1/\ln(1/\tilde{\epsilon})$. The red dashed line is the LPA′ behavior in $\tilde{\epsilon}\sqrt{\ln(1/\tilde{\epsilon})}$ which obviously does not fit the ∂^2 data. Symbols represent every 100th data point.

behavior of $1/\nu$ seems to be compatible with a dependence on $1/\ln(1/\tilde{\epsilon})$ as $\tilde{\epsilon} \to 0$ and is different than what is obtained at the LPA′ level. It is more in line with the anticipated exact behavior predicted by the droplet theory, $1/\nu \propto 1/\ln(1/\tilde{\epsilon})$ [21–23]. Work is now in progress to analytically solve the fixed-point equation via singular perturbation theory along the lines already used for the LPA′. However, the existence of two coupled differential equations makes the problem much more difficult.

APPENDIX D: FURTHER ANALYSIS OF THE $\tilde{\epsilon} = 0$ SOLUTION

To prove Eq. (24) we start from Eq. (22) where we recall that $F(\Phi) \equiv w(\Phi)$ and $\Phi(w) = \ell_0^{(d)}(w; 2-d)$. From the analysis of the threshold functions in the preceding section, one can infer that $w(\Phi)$ is a monotonically decreasing function that starts from $+\infty$ when $\Phi = 0$ and asymptotically goes to the pole $w_P < 0$ when $\Phi \to +\infty$. In the regime of interest where $w_0 = w(0) \to w_P^+$ and $w_* = w(\varphi_*) \to +\infty$, the relevant range of Φ is from $\Phi_* \sim \alpha A_d/w_* \to 0$ [see Eq. (26)] to $\Phi_0 \sim \alpha(A_d/|w_P|)\ln w_* \to +\infty$ [see Eq. (43)].

Equation (22) can be rewritten as

$$\frac{\partial\Phi}{\partial\varphi} = -\sqrt{\frac{d}{v_d}}\sqrt{-\int_\Phi^{\Phi_0} d\Phi' w(\Phi')}, \tag{D1}$$

with $\Phi(\varphi)$ a monotonically decreasing function between 0 and φ_*. (Note that by definition $\int_{\Phi_*}^{\Phi_0} d\Phi\, w(\Phi) = 0$.) This leads to

$$\varphi(\Phi) = \sqrt{\frac{v_d}{d}}\int_\Phi^{\Phi_0} \frac{d\Phi'}{\sqrt{-\int_{\Phi'}^{\Phi_0} d\Phi'' w(\Phi'')}}. \tag{D2}$$

We define Φ_i and the associated field φ_i such that $w(\Phi_\mathrm{i}) = w(\varphi_\mathrm{i}) = 0$. Then,

$$\varphi_\mathrm{i} = \sqrt{\frac{v_d}{d}}\int_{\Phi_\mathrm{i}}^{\Phi_0} \frac{d\Phi}{\sqrt{-\int_{\Phi}^{\Phi_0} d\Phi' w(\Phi')}}, \tag{D3}$$

where $\Phi_\mathrm{i} = \ell_0^{(d)}(w=0)$ is of O(1) and $w(\Phi)$ is monotonically decreasing and negative in the interval between Φ_i and Φ_0. From the properties of the threshold functions it is easily checked that that $w(\Phi)$ is concave: indeed, its second derivative is $w''(\Phi) = -\partial_w^2\ell_0^{(d)}(w)/[\partial_w\ell_0^{(d)}(w)]^3$ with $\partial_w\ell_0^{(d)}(w) = -\ell_1^{(d)}(w) < 0$ and $\partial_w^2\ell_0^{(d)}(w) = \ell_2^{(d)}(w) > 0$. In consequence, when $\Phi_\mathrm{i} \leqslant \Phi \leqslant \Phi_0$,

$$|w_0| \geqslant -w(\Phi) \geqslant |w_0|\frac{\Phi - \Phi_\mathrm{i}}{\Phi_0 - \Phi_\mathrm{i}}. \tag{D4}$$

When inserted in Eq. (D3), after some elementary algebra and using the asymptotic behavior of Φ_0 when $w_* \to \infty$, this implies that

$$\frac{\sqrt{2}}{2}\pi\sqrt{\frac{\alpha v_d A_d}{d\, w_P^2}}\sqrt{\ln w_*} \geqslant \varphi_\mathrm{i} \geqslant 2\sqrt{\frac{\alpha v_d A_d}{d\, w_P^2}}\sqrt{\ln w_*}, \tag{D5}$$

which proves that $\varphi_\mathrm{i} \sim \sqrt{\ln w_*}$.

To complete the demonstration for all φ's between $\varphi_{\rm i}$ and φ_* we can rewrite Eqs. (D2) and (D3) as

$$\varphi_* - \varphi_{\rm i} = \sqrt{\frac{v_d}{d}} \int_{\Phi_*}^{\Phi_{\rm i}} \frac{d\Phi}{\sqrt{\int_{\Phi_*}^{\Phi} d\Phi' w(\Phi')}}. \tag{D6}$$

By using the properties of the function $w(\Phi)$ we find that $\int_{\Phi_*}^{\Phi} d\Phi' w(\Phi')$ is a monotonically increasing and concave function of Φ for $\Phi \leqslant \Phi_{\rm i}$, which implies that

$$\int_{\Phi_*}^{\Phi} d\Phi' w(\Phi') \geqslant \int_{\Phi_*}^{\Phi_i} d\Phi w(\Phi)\left(\frac{\Phi - \Phi_*}{\Phi_i - \Phi_*}\right). \tag{D7}$$

Then,

$$\begin{aligned}\varphi_* - \varphi_{\rm i} &\leqslant \sqrt{\frac{v_d}{d}} \sqrt{\frac{\Phi_i - \Phi_*}{\int_{\Phi_*}^{\Phi_i} d\Phi w(\Phi)}} \int_{\Phi_*}^{\Phi_{\rm i}} \frac{d\Phi}{\sqrt{\Phi - \Phi_*}} \\ &\leqslant 2\sqrt{\frac{v_d}{d}} \frac{\Phi_i - \Phi_*}{\sqrt{\int_{\Phi_*}^{\Phi_i} d\Phi w(\Phi)}},\end{aligned} \tag{D8}$$

where we recall that $\Phi_{\rm i} = \ell_0^{(d)}(w=0) = O(1)$ and $\Phi_* \sim \alpha A_d/w_* \to 0$. After rewriting $\int_{\Phi_*}^{\Phi_i} d\Phi w(\Phi) = -\int_0^{w_*} dw w \partial_w \ell_0^{(d)}(w)$, integrating by part and using the properties of the threshold function $\ell_0^{(d)}$, one obtains that the integral behaves as $\alpha A_d \ln w_*$ when $w_* \gg 1$. This finally leads to

$$\varphi_* - \varphi_{\rm i} \lesssim 2\sqrt{\frac{v_d}{\alpha d A_d}} \frac{\ell_0^{(d)}(w=0)}{\sqrt{\ln w_*}}, \tag{D9}$$

so that $\varphi_* \sim \varphi_{\rm i} \sim \sqrt{\ln w_*}$, as announced.

APPENDIX E: EIGENVALUE EQUATIONS

The linearized equation for the perturbation of the square-mass function $k^\lambda \delta w(\varphi)$ in Eq. (55) should be complemented by linearized equations for the perturbation of the anomalous dimension $\delta\eta$ and of the minimum of the potential $\delta\varphi_{\rm m}$. That for $\delta\eta$ follows directly from the generalization of Eq. (12) to all scales k where the derivatives of the potential are evaluated at its running minimum $\varphi_{{\rm m}k}$. It reads

$$\begin{aligned}\delta\eta =& \frac{4v_d}{d}\big(2w'(\varphi_{\rm m}) m_{4,0}^{(d)}(w(\varphi_{\rm m});\eta)[\delta w'(\varphi_{\rm m}) + w''(\varphi_{\rm m})\delta\varphi_{\rm m}] \\ &+ w'(\varphi_{\rm m})^2 \partial_w m_{4,0}^{(d)}(w(\varphi_{\rm m});\eta)[\delta w(\varphi_{\rm m}) + w'(\varphi_{\rm m})\delta\varphi_{\rm m}] \\ &+ w'(\varphi_{\rm m})^2 \partial_\eta m_{4,0}^{(d)}(w(\varphi_{\rm m});\eta)\delta\eta\big)\end{aligned} \tag{E1}$$

and does not involve λ explicitly.

The flow equation for the k-dependent minimum which is obtained from that of $u_k'(\varphi)$ as

$$\begin{aligned}\partial_t \varphi_{{\rm m}k} = &-\frac{(d-2+\eta_k)}{2}\varphi_{{\rm m},k} - 2v_d \frac{w_k'(\varphi_{{\rm m}k})}{w_k(\varphi_{{\rm m}k})} \\ &\times \partial_w \ell_0^{(d)}(w;\eta_k)|_{w=w_k(\varphi_{{\rm m}k})}.\end{aligned} \tag{E2}$$

Linearizing then leads to

$$\begin{aligned}\lambda\delta\varphi_{\rm m} = &-\left[\frac{(d-2+\eta)}{2} + 2v_d \ell_1^{(d)}(w(\varphi_{\rm m});\eta)\left(\frac{w'(\varphi_{\rm m})^2}{w(\varphi_{\rm m})^2} - \frac{w''(\varphi_{\rm m})}{w(\varphi_{\rm m})}\right) - 4v_d \ell_2^{(d)}(w(\varphi_{\rm m});\eta)\frac{w'(\varphi_{\rm m})^2}{w(\varphi_{\rm m})}\right]\delta\varphi_{\rm m} \\ &+ 2v_d\left[\ell_1^{(d)}(w(\varphi_{\rm m});\eta)\left(\frac{\delta w'(\varphi_{\rm m})}{w(\varphi_{\rm m})} - \frac{w'(\varphi_{\rm m})\delta w(\varphi_{\rm m})}{w(\varphi_{\rm m})^2}\right) + 2\ell_2^{(d)}(w(\varphi_{\rm m});\eta)\frac{w'(\varphi_{\rm m})\delta w(\varphi_{\rm m})}{w(\varphi_{\rm m})}\right] - \delta\eta\left[\frac{\varphi_{\rm m}}{2}\right. \\ &\left. + 2v_d \partial_\eta \ell_1^{(d)}(w(\varphi_{\rm m});\eta)\frac{w'(\varphi_{\rm m})}{w(\varphi_{\rm m})}\right],\end{aligned} \tag{E3}$$

where we have used the property of the threshold functions that $\partial_w \ell_n^{(d)}(w;\eta) = -(n+1)\ell_{n+1}^{(d)}(w;\eta)$

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