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DOCTORAL DISSERTATION

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Supervisors:

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Sveučilište u Zagrebu

PRIRODOSLOVNO–MATEMATIČKI FAKULTET

MATEMATIČKI ODSJEK

Tomislav Kralj

O graničnom ponašanju geometrijskih funkcionala konveksnih ljuski slučajnih šetnji

DOKTORSKI RAD

Mentori:

prof. dr. sc. Nikola Sandrić, doc. dr. sc. Stjepan Šebek

Zagreb, 2024.

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SUMMARY

In this dissertation, we study the asymptotic behavior of the convex hull generated by several mutually independent random walks. In the first chapter, we show that the convex hull, appropriately scaled, almost surely converges to the convex hull generated by the corresponding drift vectors and the origin. Then, by applying the continuous mapping theorem, we also demonstrate almost sure convergence of all intrinsic volumes, which implies almost sure convergence of the perimeter. Using a similar argument, we obtain almost sure convergence of the diameter process.

In the next chapter, we move on to explore the distributional limit of the perimeter process. To successfully control the variance, we use the technique of martingale difference sequences and Cauchy's formula. In the end, we obtain a very interesting and intuitive L^2 approximation for the deviation of the perimeter process. Under certain assumptions about the drift vectors of the random walks, we determine the asymptotic behavior of the variance of the perimeter. We can establish a normal distributional limit if this asymptotic variance is positive.

Following this, we focus on the diameter process, where it becomes crucial to note that the mapping that assigns diametral segments to polygons with the unique diametral segment is continuous. Finally, with the additional assumption about the set of drift vectors, we achieve results analogous to those for the perimeter process.

In the final chapter, we will study the convex hull of centroids generated by a single random walk. We open a discussion on generalizing the assumptions made for the observed random walks and provide a detailed simulation study to explore what happens when these assumptions are not satisfied.

Keywords: random walk, central limit theorem, strong law of large numbers, convex hull, perimeter length, diameter

SAŽETAK

U ovoj disertaciji istražujemo granično ponašanje konveksne ljuske definirane pomoću nezavisnih slučajnih šetnji. U prvom poglavlju dokazujemo da konveksna ljuska, prikladno skalirana, gotovo sigurno konvergira prema konveksnoj ljusci definiranoj pripadnim drift vektorima i nulom. Primjenom teorema o neprekidnom preslikavanju, dokazujemo gotovo sigurnu konvergenciju svih intrinzičnih volumena, što implicira gotovo sigurnu konvergenciju opsega. Korištenjem slične argumentacije postiže se gotovo sigurna konvergencija procesa dijametra.

U idućem poglavlju analiziramo distribucijski limes procesa opsega. Kako bismo uspješno kontrolirali varijancu, koristimo tehniku niza martingalnih razlika i Cauchyjevu formulu. Na kraju, dobivamo izuzetno zanimljivu i intuitivnu L^2 aproksimaciju za devijaciju procesa opsega. Pod određenim pretpostavkama o drift vektorima slučajnih šetnji, određujemo granično ponašanje varijance opsega, te ako je ta granična varijanca pozitivna, možemo zaključiti normalni distribucijski limes.

Nakon toga, usmjeravamo se na proces dijametra, gdje se uz prethodno spomenute alate ključnom pokazuje činjenica da je preslikavanje koje poligonima pridružuje dijametralne segmente neprekidno. Na kraju, uz dodatnu pretpostavku o skupu drift vektora, postižu se analogni rezultati kao i za proces opsega.

U završnom poglavlju, promatramo konveksnu ljusku generiranu s centrima mase jedne planarne slučajne šetnje. Nakon toga, otvaramo diskusiju o poopćavanju pretpostavki koje smo postavili za promatrane slučajne šetnje i pružamo simulacijsku studiju koja istražuje posljedice napuštanja tih pretpostavki. Time se osvjetljavaju otvoreni problemi koji proizlaze iz ove disertacije.

Ključne riječi: slučajna šetnja, centralni granični teorem, jaki zakon velikih brojeva, konveksna ljuska, duljina opsega, dijametar

Contents

INTRODUCTION

The concept of random walks is undeniably relevant. Consider fields like the stock market, disease transmission, political campaigns, and wildlife tracking. In these areas, you might observe changes in values like stock prices, infection rates, public support, or geographical locations as random increments. By modeling the named phenomena with random walks, we can predict many key metrics.

We shall pay attention to convex hulls for the following explicit reason — we often do not need the detailed distribution of steps in random walks. Instead, we simplify the problem to more manageable concepts. Convex hulls are an extension of this approach to simplification. By understanding the shape and size of a convex hull, we can estimate the range and scope of random walks.

On the other hand, limit theorems are also quite important. Considering the infinite nature of natural numbers, it becomes pretty intuitive to explore limits to infinity. Take any repeated process and associate it with this concept of infinity. Often, observing these processes for a sufficient duration reveals their limiting behavior.

We will look at existing studies related to random walks and convex hulls and how these two areas intersect. The field of random walks is broad, and it is impossible to cover everything. Therefore, we will focus on a specific segment, notably the fluctuation theory, which seamlessly connects to our study of convex hulls. Later, we will explore how convex hulls and random walks apply in various scientific fields.

The structure of the introduction section is simple: We start with a broad literature overview, consider some possible applications of our theory, and outline the rest of the thesis. The introduction is extensive because it provides a comprehensive background and lays a solid foundation for our results, many of which rely on general or technical aspects of random walk theory that are essential for understanding the main content.

0.1. Literature Overview

Karl Pearson introduced the concept of a *random walk* in 1905 through a letter published in Nature, where he asked for help from the journal's readers [Pea05, p. 294]. Pearson described a scenario involving a man who walks a distance of m yards, then changes direction randomly and continues walking another m yards. This pattern is repeated n times. Pearson was curious about the likelihood of the man ending up within a certain distance from where he started.

Lord Rayleigh answered Pearson's question [Rei05] by mentioning that he had been thinking about a similar issue years earlier in a study on sound wave vibrations [Ray80]. In that study the magnitude was consistent, but the phase could change. In a later edition of Nature, Pearson shared Rayleigh's findings [Pea05, p. 342] and made a humorous comparison, saying it was like trying to find a drunk man who, despite being drunk, would not wander far from where he started.

Louis Bachelier made a connection between these stochastic processes and mathematical finance [Bac00], while Albert Einstein provided a framework for understanding the fluctuating motion of particles in a fluid, later known as *Brownian motion* [Ein05, Bro28].

The establishment of this theory led to widespread research internationally. This included exploring the differences between recurrence and transience in random walks — essentially, whether a state will eventually be revisited indefinitely or only a finite number of times. One notable example was Shizuo Kakutani, who humorously referenced Pearson while exploring a particularly simple random walk [Dur19, p. 191] that a drunk man will eventually find his way home, but a drunk bird may get lost forever. More precisely — while the process tends to be recurrent in two dimensions, it turns out to be transient in three dimensions, where it may not necessarily return to its starting point.

This thesis focuses on the convex hull of random walks, a topic well-explored in previous research. However, discussions about the highest values of random walks, which are closely connected to the idea of the convex hull, came before the formal introduction of this term in the field. These discussions and current understandings can be linked using Cauchy's surface area formula for convex shapes. This formula calculates the surface area by integrating the lengths projected over various angles; see [TV16].

We might also wonder how often a one-dimensional random walk stays on the positive side of its starting point. We express the position of the walk at time n as S_n, and T_n represents the percentage of time spent on the positive side up to that point. A key paper by Lévy, titled *Sur certains processus stochastiques homogènes* [Lév40], had a big impact on this area of study, known as *fluctuation theory*. Lévy demonstrated a principle known as the *arcsine law* for T_n, where the walk followed a simple symmetric pattern:

$$\mathbb{P}(S_{n+1} - S_n = 1) = \mathbb{P}(S_{n+1} - S_n = -1) = \frac{1}{2},$$

and the changes $(S_{n+1} - S_n)_{n \geq 1}$ were independent, starting from $S_0 = 0$. The equation summarizes his findings:

$$\lim_{n \to \infty} \mathbb{P}(T_n < x) = \frac{2}{\pi} \arcsin \sqrt{x},$$

for $x \in (0, 1)$. This means that over time, the walk is more likely to mostly stay on one side of the starting point rather than spending equal time on both sides. In fact, the chance of the walk spending roughly half the time on each side is the least probable outcome.

Significant advancements were made in studying random walks afterward. Initially, Erdős and Kac expanded earlier findings to include walks where the increments had a mean of 0 and a variance of 1 [EK47]. Following this, Sparre Andersen published two papers [And49, And50] that widened the scope further by introducing symmetry conditions. These conditions allowed for the independence of increments to be less strict.

Later, Maruyama and Udagawa independently loosened these conditions even more [Mar51, Uda52]. They only required a central limit theorem to apply to the walks, simplifying the criteria for the previously established generalizations.

More recently, Kabluchko, Vysotsky, and Zaporozhets [KVZ16] have taken Sparre Andersen's theories about symmetric increments and extended them into higher dimensions. This change required redefining what counts as the *positive side* of a walk. In their research, they analyze cases where the origin is included within the convex hull. This is very similar to the work done by Bingham and Doney in 1988 [BD88]. They looked at arcsine laws for Brownian motion across multiple dimensions, defining the *positive side* as scenarios where all components are positive.

Earlier results often relied on combinatorial methods in the study of arcsine laws. However 1949, Chung and Feller introduced generating functions, a more complex analytical tool [CF49]. They used this approach to verify Erdős and Kac's findings regarding the simple symmetric random walk. They also explored further, deriving a remarkable result that offered a different perspective from those proposed by Lévy. They considered a $2n$-step random walk and defined N_{2n} as the count of steps where the position before or after the step is positive — or possibly both. They discovered that if the walk returns to the origin after $2n$ steps, then the probability that exactly $2r$ of those steps (for $r = 0, \ldots, n$) had the walk on the positive side is given by:

$$\mathbb{P}(N_{2n} = 2r \mid S_{2n} = 0) = \frac{1}{n+1}.$$

This result is interesting because it differs from the traditional interpretation of the arcsine law. It suggests that the likelihood of spending any particular portion of time on the positive side is uniform for a walk that returns to its starting point. This means that the probability of having exactly half of the steps on the positive side is the same as the probability of having all or none of the steps on that side.

Lipschutz expanded on Lévy's findings in 1952, adapting the theorem to random walks with a mean equal to 0 and a variance equal to 1 under the condition that they had a finite fourth moment [Lip52]. Shortly after, Baxter introduced his work on Wiener process distributions related to the arcsine law type, acknowledging Erdős and Kac's contributions and including the uniform distribution results from Chung and Feller.

Feller's influential book, first published in 1950, included a comprehensive discussion on coin tossing fluctuations and incorporated this theorem [F^+71, pp. 67-97]. A bit later, there was renewed interest in Chung and Feller's original paper. One important reference involved a study that used similar combinatorial methods as Sparre Andersen to look into the maximum values in random walks [HW16].

During the fifties, a more unified approach to studying these mathematical topics began. Feller's book was published, and Chung and Erdős were improving the Borel-Cantelli lemma to apply it to the zeros and positive terms in simple symmetric random walks [CE52]. Around the same time, Darling studied random walks with symmetric increments and established a theorem discussing the order of walk points. This helped define the distribution for the maximum point and the count of positive walk points [Dar51].

Sparre Andersen explored the dynamics of random walks in two significant papers. The first paper, published in 1953, discussed the initial occurrence of a walk reaching its maximum, the final occurrence of reaching its minimum, and the count of positive steps in the walk [And53]. The following year, Andersen published another paper studying a random walk's convex hull, specifically the convex minorant. This work was part of a broader investigation into random walks' behaviors, or fluctuations [And54].

Research on random walks has greatly developed since the early studies, especially regarding their convex hulls. A key figure in this area is Satya Majumdar, who has written many papers, several of which focus on random walks and, specifically, their convex hulls. Some of his notable works, created with collaborators Mounaix and Schehr, focus on analyzing the first and second maxima of random walks [MMS13, MMS14, MSM16]. Additionally, Majumdar and Schehr revisited similar statistical methods used by Darling six decades earlier to analyze order statistics [SM12].

If we talk about the history of analysis of convex hulls, the fascination with convex shapes and hulls is deeply rooted in history, going back to the ancient works of Archimedes. References to his important work can be found in *The Works of Archimedes*, edited by Heath [H$^+$02], and in Stephen Hawking's *God Created the Integers*, which includes detailed commentary on these topics [Haw07]. Additionally, Gruber notes that Archimedes may have been the first to formally define convexity through the axioms presented in *On the Sphere and the Cylinder* [Gru07, p. 41].

Convex sets are not just theoretical constructs but have practical applications across various science fields, including mathematics. For instance, the convex analysis and optimization field, which extensively uses convex sets and functions, plays a crucial role in solving complex problems like those found in the simplex method of operations research [Roc70, FP93, Sai97]. Beyond mathematics, convex sets are also important in other disciplines. In economics, they help describe equilibrium in consumption [NS08, p. 94], and in ecology, they are used to model species competition [ML64].

Research on the convex hulls of random points in mathematics has seen significant progress over the past century, often driven by seemingly simple problems. One such problem was introduced by Erdős and Szekeres in 1935 [ES35], questioning the minimum number of points required on a plane, arranged so no three are collinear, to ensure a subset forms a convex n-gon. The initial solution to this problem, especially for quadrilaterals and their generalization to n points, was provided by Esther Klein [Bed18], named *happy ending problem.*

Another interesting challenge from this study area is Sylvester's four-point problem, first posed in 1864 in the Educational Times by Sylvester [Syl64] and later elaborated on by Pfiefer in 1989 [Pfi89]. The problem involves showing that the probability that four randomly chosen points on a plane will form a non-convex polygon is $1/4$. This problem's solutions vary based on how the points are randomly selected. However, it has been resolved under conditions where points are uniformly chosen from a finite convex shape, like a circle or a specified polygon. The solution ensures that the probability a point falls within a subset is proportional to that subset's area coverage [Wat65, Woo67].

Rogers raised a question in 1978 about whether two groups of points on a plane could have convex hulls that do not overlap [Rog78]. Following this, Jewell and Romano explored a related problem, focusing on the probability that several arcs of a fixed length, when randomly positioned on a circle, could completely cover it [JR82]. Reitzner revisited Rogers' problem, but with the additional constraint that the sets of points must be within a convex shape [Rei00]. Similarly, Groeneboom tackled a problem related to the one studied by Rényi and Sulanke, discussing the number of vertices on the convex hull of n points, but this time within a convex polygon [Gro12].

Another aspect of convex geometry is the study of the limit shapes of convex polygons, highlighted by the Bárány-Vershik-Sinai results. These researchers independently discovered similar findings [Bár95, Ver94, Sin94]. As it was discussed by Bogachev and Zarbaliev [BZ11], these results deal with the limit shape of a typical convex curve from some set of convex curves. Bárány's initial theorem evaluates the expected shape of convex polygons within the square $[-1,1]^2$ with vertices positioned on the lattice $n^{-1}\mathbb{Z}^2$ as n approaches infinity. If we consider a point x in $[-1,1]^2$, and calculate at each step n, the percentage of convex polygons that contain x inside them, denoted by $\rho_n(x)$, there is a defined shape L such that $\rho_n(x) \to 1$ if x is inside L, and $\rho_n(x) \to 0$ if x is outside L. The defined shape L is the convex set given by:

$$L = \left\{ (x,y) \in \mathbb{R}^2 : \sqrt{1-|x|} + \sqrt{1-|y|} \geq 1 \right\}.$$

However, our focus will primarily investigate deeper into convex hulls generated by random walks. Our exploration of convex hulls spanned by random walks begins by revisiting Sparre Andersen's work from 1954 [And54]. In this foundational paper, referenced by Majumdar et al. in their comprehensive survey [MCRF10], Andersen lays out some of the first rigorous investigations into the structure of these mathematical constructs. While Lévy had previously offered insights into the shape of the curve of Brownian motion, hinting at its convex hull [Lév48], Andersen was the pioneer in providing concrete results.

In his paper, Andersen considers several topics, including the behavior of random variables, such as the time needed to reach a maximum value. However, his significant contribution is the presentation of findings on the number N_n, which represents the count of indices i from 1 to $n-1$ such that S_i matches the largest convex minorant of the one-dimensional random walk $S_0, \ldots, S_n$.

Andersen defines the convex minorant in terms of number sequences. A sequence $x_0, \ldots, x_n$ is considered *convex* if its differences $x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}$ are non-decreasing. He explains that a sequence $a_0, \ldots, a_n$ has a unique, largest convex minorant sequence $x_0, \ldots, x_n$, where $x_0 = a_0$ and $x_n = a_n$, and intermediate values x_i either equal a_i or are interpolated based on the values of a in the following way:

$$x_i = \frac{(k-i)a_j + (i-j)a_k}{k-j},$$

where k is the smallest index greater than i such that $x_k = a_k$ and j is the greatest index smaller than i such that $x_j = a_j$. In Figure 1, we give the insight of the idea. One can observe that $x_3 = a_3$ and $x_7 = a_7$, with, for example, x_5 derived as the interpolated value between a_3 and a_7.

Sparre Andersen explains how N_n (which counts how many points from a random walk lie on the lowest path that still touches all the peaks) is distributed when the steps of the walk (increments) are independent and follow a continuous distribution. He illustrates

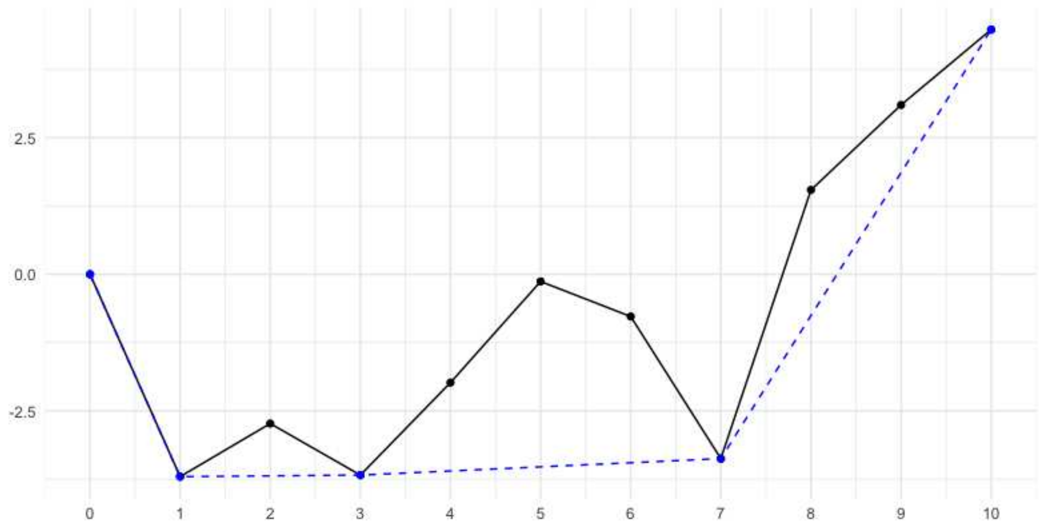


Figure 1: The convex minorant of the time-space diagram of a random walk

this concept using the generating function:

$$G_n(t) := \sum_{m=0}^{n-1} \mathbb{P}(N_n = m) t^m = n^{-1} \prod_{m=1}^{n-1} \left(1 + m^{-1} t\right).$$

This formula is the same as the one used for the sum of $n-1$ Bernoulli random variables, where each variable has a value of 1 with a probability of $1/(i+1)$. Using this formula, Andersen calculates the average number and the variance of points lying on the path:

$$\mathbb{E}(N_n) = \sum_{i=1}^{n-1} \frac{1}{i+1} \sim \log n, \quad \mathbb{V}\mathrm{ar}(N_n) = \sum_{i=1}^{n-1} \frac{i}{(i+1)^2}.$$

These calculations suggest that, on average, about $\log n$ points from the walk will end up on the convex minorant. Due to the symmetry, the concave majorant should also have $\log n$ points on the highest path on average. Since the convex hull is made by joining these two paths, it is expected to have around $2\log n$ points.

Research on convex minorants continued after Andersen's pivotal contributions, focusing primarily on combinatorial methods. Spitzer introduced an innovative approach using cyclic permutations to link the characteristic function of the maximum value in a sequence $S_0, \ldots, S_n$ to the sum of the characteristic functions for each $\max(0, S_i)$ [Spi56]. Brunk later expanded this concept in 1964 [Bru64].

Expanding on these ideas, Goldie studied the convex minorant of a one-dimensional random walk in 1989 [Gol89], using the conditions established by Andersen. Goldie's analysis introduced a new concept where an increment Z_i is considered part of the jth side of the largest convex minorant if exactly j points from S_0 to S_{i-1} match their convex minorant (as described earlier) at all indices from 0 to $i-1$. In this framework, Goldie discovered that the probability of the ith smallest increment starting a new side of the convex minorant is $1/i$, with these events being independent for $i = 1, \dots, n$. However, contrasting these findings, Qiao and Steele in 2002 demonstrated that the concave majorant of a random walk reduces to a single line infinitely often [QS05]. This insight suggests a difference from both Goldie's and Andersen's predictions of approximately $\log n$ faces for a given length of the walk.

Following the work of Andersen and Spitzer, Baxter contributed to the field with his paper on *A combinatorial lemma for complex numbers* [Bax61]. Baxter's focus was on a two-dimensional random walk. He analyzed the scenario where no two vectors, each formed by combining a non-empty sequence of consecutive increments, are parallel. This condition is typically met when the increments are drawn from a continuous distribution. Baxter identified a unique cyclic permutation of the increments $Z_1, \dots, Z_n$ that ensures the random walk remains entirely positive.

Additionally, Baxter studied the structure of the convex hull, noting that any side of the hull is formed by the cumulative addition of a specific subset A of increments. If this subset A contains m increments, then the resulting side of the hull appears in precisely $2(m-1)!(n-m)!$ of the possible permutations of increments, not just the cyclic ones. From these observations, Baxter deduced that typically only two of the increments $Z_1, \dots, Z_n$ contribute directly as edges of the hull. He also confirmed Andersen's expectation of approximately $2\log n$ faces on the hull. Furthermore, Baxter validated the Spitzer-Widom formula, which calculates the expected perimeter length of the convex hull. In a subsequent collaboration 1963 with Barndorff-Nielsen, Baxter extended these findings to higher dimensional spaces [BNB63].

Around the same time, Spitzer and Widom collaborated on studying the expected perimeter length of the convex hull spanned by a planar random walk [SW61]. They framed their exploration using a playful analogy from a later work of Wade and Xu [WX15a]: *Imagine a drunken gardener who drops a seed with each of his n wobbly steps.*

Once the flowers have bloomed, how much fencing would be needed to enclose all the flowers? Spitzer and Widom tackled this question using their solid combinatorial skills combined with Cauchy's surface area formula for convex shapes, which helps calculate the perimeter, L, of any convex shape as:

$$L = \int_0^\pi D(\theta) d\theta, \tag{1}$$

where $D(\theta)$ represents the projection length of the shape along a line at angle θ. Specifically for a random walk, the shape in question is the convex hull, and the projection length can be expressed as:

$$D(\theta) = \max_{0 \le i \le n} S_i \cdot \mathbf{e}_\theta - \min_{0 \le i \le n} S_i \cdot \mathbf{e}_\theta,$$

with $\mathbf{e}_\theta$ being the unit vector in the direction of θ. To derive this, they used a lemma initially attributed to Kac [Kac54], which Dyson had proven. This lemma involves analyzing all permutations of the n increments in a one-dimensional sequence. Let π denote such a permutation of $1, \dots, n$ to $\pi_1, \dots, \pi_n$. The lemma states that:

$$\sum_\pi \left(\max_{0 \le i \le n} S_{\pi_i} - \min_{0 \le i \le n} S_{\pi_i} \right) = \sum_\pi \sum_{i=1}^n \frac{1}{i} \|S_{\pi_i}\|$$

Here, the notation $\sum_\pi$ indicates summation over all permutations. Spitzer and Widom extended this into the two-dimensional space to link the convex hull's perimeter length under each permutation π, denoted L_n^π, to Kac's lemma by modifying the formula to:

$$\sum_\pi L_n^\pi = 2 \sum_\pi \sum_{i=1}^n \frac{1}{i} \|S_{\pi_i}\|$$

This adaptation leads to a remarkable result regarding the expected perimeter length of the convex hull, expressed as:

$$\mathbb{E} L_n = 2 \sum_{i=1}^n \frac{1}{i} \mathbb{E} \|S_i\|.$$

After the significant findings presented by Spitzer and Widom, there was not much further exploration into the perimeter length L_n of convex hulls until Snyder and Steele revisited the topic in 1993 [SS93]. They managed to establish an upper limit for its variance, particularly stating:

$$\mathbb{V}\mathrm{ar}\left(L_n\right) \leq \frac{\pi^2 n}{2}\left(\mathbb{E}\left(\|Z\|^2\right) - \|\mu\|^2\right),$$

assuming that the steps $Z_1,\ldots,Z_n$ of the random walk are independent and identically distributed as Z. Additionally, they demonstrated that if $\mathbb{E}\|Z\|^2$ is finite, then the average perimeter normalized by n converges almost surely to $2\|\mu\|$:

$$n^{-1}L_n \xrightarrow{\text{a.s.}} 2\|\mu\| \text{ as } n \to \infty. \tag{2}$$

They also set bounds for the tail probabilities for $L_n - \mathbb{E}L_n$ but limited this to scenarios where the increments are bounded. Furthermore, they employed Baxter's combinatorial lemma to reconfirm several established results. They introduced new findings. For example, if we denote with $L_n^{(2)}$ the sum of the squares of the face lengths of the convex hull, the expected value of that variable is:

$$\mathbb{E}L_n^{(2)} = 2n\left(\mathbb{E}\left(\|Z\|^2\right) - \|\mu\|^2\right).$$

Steele continued to study combinatorial techniques, contributing to understanding various characteristics of convex hulls. In 2002, he discussed the *Bohnenblust-Spitzer algorithm* [Ste02], which led to further insights into the distribution of such functionals as the number of faces of a convex hull, though not addressing the variance of the perimeter length. In his paper, Steele also noted an interesting geometric relationship suggesting that the expected length of the concave majorant approximates $n\sqrt{1+\mu^2}$ for large n, linking it to the straight-line distance from $(0,0)$ to $(n,\mathbb{E}(S_n)) = (n,n\mu)$.

The investigation to refine the upper bound on the variance of L_n saw significant advancement through the work of Wade and Xu in 2015 [WX15a, WX15b]. They explored cases with a drift ($\|\mu\| > 0$), showing that:

$$n^{-1/2}\left|L_n - \mathbb{E}L_n - 2\left(S_n - \mathbb{E}S_n\right)\cdot\hat{\mu}\right| \to 0, \text{ in } L^2,$$

where $\hat{\mu}$ is the normalized drift vector. This discovery led to an asymptotic expression for the variance:

$$\lim_{n\to\infty} n^{-1}\operatorname{Var} L_n = 4\mathbb{E}\left(((Z-\mu)\cdot\hat{\mu})^2\right).$$

This implies that the order of convergence of the variance in that case is $\mathscr{O}(n)$. They established a basis for a central limit theorem for L_n when there is variance in the direction of the mean. When the upper limit equals zero, Wade and Xu conjectured that the order of variance convergence is of the size $\mathscr{O}(\log n)$. However, in [AKMV20], the authors demonstrated that this conjecture is only partially correct. Specifically, it holds true when $\mathbb{E}\|Z\|^p < \infty$ for $p \geq 3$, but if this condition is satisfied only for $p \in [2,3)$, the variance may grow polynomially.

In their second paper [WX15b], the authors analyzed how the convex hull of a random walk aligns with that of Brownian motion by applying a continuous mapping technique and Donsker's theorem. If we additionally denote by A_n the area of the convex hull spanned by a random walk, they demonstrated that for random walks without drift:

$$n^{-1/2}L_n \xrightarrow{\mathscr{D}} \mathscr{L}\left(\Sigma^{1/2}h_1\right), \quad \text{and} \quad n^{-1}A_n \xrightarrow{\mathscr{D}} \mathscr{A}\left(\Sigma^{1/2}h_1\right) = a_1\sqrt{\det\Sigma},$$

where $\mathscr{L}$ and $\mathscr{A}$ represent the perimeter length and area, respectively, Σ is the covariance matrix of the increments, and h_1 and a_1 are the convex hull and area of a Brownian motion over a unit time. From this, they established that the mean perimeter length in the zero drift scenario converges to:

$$\lim_{n\to\infty} n^{-1/2}\mathbb{E}L_n = 4\mathbb{E}\|Y\|,$$

with Y being a normally distributed vector $\mathscr{N}(0,\Sigma)$. They also determined that the mean area converges to:

$$\lim_{n\to\infty} n^{-1}\mathbb{E}A_n = \frac{\pi}{2}\sqrt{\det\Sigma}.$$

In scenarios where there is a drift ($\|\mu\| > 0$), the hull does not align with that of two-dimensional Brownian motion but rather with the convex hull formed in the space-time diagram of one-dimensional Brownian motion, referred to as $\tilde{h}_1$. They showed that if $\mathbb{E}\|Z\|^p < \infty$ for some $p > 2$, and the variance in the direction perpendicular to the mean,

$\sigma^2_{\mu_\perp}$, is positive, then the mean area's rate of change is given by

$$\lim_{n\to\infty} n^{-3/2}\mathbb{E}A_n = \frac{1}{3}\|\mu\|\sqrt{2\pi\sigma^2_{\mu_\perp}}.$$

The authors also analyzed the variance of these measurements, finding that as n increases, the variance approaches that of the corresponding quantity for Brownian motion. Namely, if $\mu = 0$ and $\mathbb{E}\|Z\|^p < \infty$ for $p > 2$, then

$$\lim_{n\to\infty} n^{-1}\operatorname{Var} L_n = \operatorname{Var}\left(\mathscr{L}\left(\Sigma^{1/2}h_1\right)\right),$$

if $\mu = 0$ and $\mathbb{E}\|Z\|^p < \infty$ for $p > 4$, then

$$\lim_{n\to\infty} n^{-2}\operatorname{Var} A_n = \operatorname{Var}(a_1)\det\Sigma,$$

and if $\|\mu\| > 0$ and $\mathbb{E}\|Z\|^p < \infty$ for $p > 4$, then

$$\lim_{n\to\infty} n^{-3}\operatorname{Var} A_n = \operatorname{Var}\left(\mathscr{A}\left(\tilde{h}_1\right)\right)\|\mu\|^2\sigma^2_{\mu_\perp}.$$

As previously discussed in the first paper, there was no omission in failing to include an equivalent result for L_n when $\|\mu\| > 0$. The critical aspect to understand here is that the limiting scenario in such cases corresponds to the space-time diagram of one-dimensional Brownian motion. This situation leads to scaling of n in the time dimension and $n^{1/2}$ in the spatial dimension, which is why we see $n^{-3/2}$ in the calculations. Unfortunately, this scaling complicates things because just knowing the length of a hull's edge is not enough; the angle of the edge also influences how the length scales. Thus, understanding the perimeter length without additional information about the edge angles is insufficient.

Regarding the perimeter length, Akopyan and Vysotsky in 2016 [AV16] provided a significant finding known as a large deviation result. They demonstrated that the probability $\mathbb{P}(L_n \geq 2cn)$ for $c > \|\mu\|$ decreases exponentially, and a similar exponential decrease occurs for deviations below this threshold. Furthermore, research on the perimeter length and area under certain symmetry conditions and continuous increments has advanced. Thus, in 2021 [AV21], they provided explicit upper and lower bounds for the rate function of the perimeter based on the rate function of the increments for a broad class of distributions, including Gaussian and rotationally invariant ones. For such distributions, large deviations of the perimeter are achieved by trajectories that asymptotically align into line segments, though these may not always be the optimal shape.

In [Vys23], the author demonstrated that the asymptotic shape of the most likely trajectories leading to large deviations in the area of the convex hull of a planar random walk is determined by solving an inhomogeneous anisotropic isoperimetric problem. For increments with a finite Laplace transform, the optimal trajectories are smooth, convex, and satisfy the Euler–Lagrange equation, similar to the isoperimetric problem in the Minkowski plane as solved by Busemann [Bus47] in 1947.

In 2017, Grebenkov, Lanoiselée, and Majumdar [GLM17] explored these scenarios, showing that if the increments have finite variance, the expansion of $\mathbb{E}L_n$ beyond the $n^{1/2}$ term remains constant. Conversely, if the variance is infinite, the order of expansion terms depends on the finite highest moment in the increments' density function. They also found similar patterns for the expansion of $\mathbb{E}A_n$.

If we look in [DCHM16], the authors study, using simulations, the convex hulls formed by the steps of n independent two-dimensional random walks. They analyze the area and perimeter of these hulls. In [RFZ21], the authors derive explicit formulae for the expected volume and the expected number of facets of the convex hull of multiple multidimensional Gaussian random walks. In [RF12], an exact formula is established for the average number of edges on the boundary of the global convex hull of n independent Brownian paths in the plane.

Lo, McRedmond, and Wallace in [LMW18] investigated the asymptotic behavior of functionals related to planar random walks. Their focus included the convex hull and the center of mass process of these random walks. The authors derive several important results under specific assumptions about the random walks. Firstly, they establish a functional law of large numbers, demonstrating that the trajectories of the random walk, when appropriately scaled, converge almost surely to a linear function of time:

$$X_n(t) \to \mu t \quad \text{a.s. in } \left(\mathscr{C}_0^d, \rho_\infty\right)$$

for

$$X_n(t) = \frac{1}{n} S_{\lfloor nt \rfloor} + \frac{nt - \lfloor nt \rfloor}{n} \xi_{\lfloor nt \rfloor + 1},$$

where $\left(\mathscr{C}_0^d, \rho_\infty\right)$ represents the space of continuous d-dimensional functions on the unit interval with $f(0) = 0$, equipped with the supremum metric. A similar result was obtained when they used a different method of interpolating:

$$X_n'(t) := n^{-1} S_{\lfloor nt \rfloor},$$

where they got

$$X_n(t) \to \mu t \quad \text{a.s. in } \left(\mathscr{D}_0^d, \rho_\infty\right),$$

where $\left(\mathscr{D}_0^d, \rho_\infty\right)$ is the space of right-continuous d-dimensional functions with left-hand limits on the unit interval with $f(0) = 0$. The authors further study the maximum functional in one dimension. They show that the maximum of the walk, scaled by n^{-1}, converges almost surely to the positive part of the drift μ:

$$\frac{1}{n} \max_{0 \le k \le n} S_k \to \mu^+ \quad \text{a.s.}$$

Additionally, they derive the central limit theorem for the maximum of the zero-drift random walk in one dimension, establishing that the scaled maximum converges in distribution to the maximum of a standard Brownian motion:

$$\frac{1}{\sqrt{n}} \max_{0 \le k \le n} S_k \xrightarrow{\mathscr{D}} \sigma \sup_{t \in [0,1]} b(t),$$

where $b(t)$ denotes the standard Brownian motion and $\sigma^2 = \mathbb{V}\mathrm{ar}(\xi_1)$. For random walks in higher dimensions, the authors generalize the arcsine law, showing that the proportion of time a zero-drift random walk spends in a given subset of the unit sphere A converges in distribution as follows:

$$\pi_n(A) \xrightarrow{\mathscr{D}} \int_0^1 \mathbf{1}\{b_\Sigma(t) \in A\}\, dt,$$

where

$$\pi_n(A) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\{S_i/\|S_i\| \in A\},$$

and $b_\Sigma(t)$ represents the projection of a Brownian motion onto the sphere. The paper also covers the almost sure convergence of points in the random walk:

$$n^{-1}\{S_0, S_1, \ldots, S_n\} \to I_\mu[0,1], \quad \text{a.s.},$$

and the weak convergence of points when the drift is zero:

$$n^{-1/2}\{S_0, S_1, \ldots, S_n\} \Rightarrow \Sigma^{1/2} b_d[0,1].$$

For the diameter, they establish both almost sure convergence:

$$n^{-1} D_n \to \|\mu\|, \quad \text{a.s.}$$

and the central limit theorem for the diameter in the case $\mu = 0$:

$$n^{-1/2} D_n \xrightarrow{\mathscr{D}} \text{diam}\left(\Sigma^{1/2} b_d[0,1]\right).$$

In terms of the convex hull, the authors show almost sure convergence for random walks with non-zero drift:

$$n^{-1}\,\text{chull}\{S_0, S_1, \ldots, S_n\} \to I_\mu[0,1], \quad \text{a.s.},$$

and weak convergence when the drift is zero:

$$n^{-1/2}\,\text{chull}\{S_0, \ldots, S_n\} \Rightarrow \Sigma^{1/2} h_d.$$

They also derive distributional limits for the mean width, surface area, and volume of the convex hull:

$$n^{-1/2}\mathscr{W}_n \xrightarrow{\mathscr{D}} \mathscr{W}\left(\Sigma^{1/2}h_d\right),$$
$$n^{-(d-1)/2}\mathscr{S}_n \xrightarrow{\mathscr{D}} \mathscr{S}\left(\Sigma^{1/2}h_d\right), \text{ and}$$
$$n^{-d/2}\mathscr{V}_n \xrightarrow{\mathscr{D}} \mathscr{V}\left(\Sigma^{1/2}h_1^d\right) = v_d\sqrt{\det(\Sigma)},$$

and confirmed the almost sure convergence of the perimeter, as it has been done in (2):

$$n^{-1}\mathscr{L}_n \to 2\|\mu\|, \quad \text{a.s.}$$

For the volume, they demonstrate almost sure convergence when $\|\mu\| > 0$:

$$n^{-(d+1)/2}\mathscr{V}_n \xrightarrow{\mathscr{D}} \|\mu\|\sqrt{\det\Sigma_{\mu_\perp}}\tilde{v}_d,$$

where $\tilde{v}_d$ is the volume of $\tilde{h}_d := \text{chull}\,\bar{b}_d[0,1]$ where $\tilde{b}_d[0,1] = \left\{\tilde{b}_d(t) : t \in [0,1]\right\}$ with $\tilde{b}_d(t) = (t, b_{d-1}(t))$, for $t \in [0,1]$, and $\Sigma_{\mu_\perp}$ is the covariance matrix of the increment written in the orthonormal basis where the normed drift vector is the first vector.

Regarding the center of mass process:

$$G_n := \frac{1}{n}\sum_{i=1}^{n} S_i,$$

they prove almost sure convergence of the center of mass:

$$\frac{1}{n}G_{\lfloor nt \rfloor} \xrightarrow{\text{a.s.}} \frac{\mu t}{2},$$

and derive the distributional limit in case $\mu = 0$ as:

$$\frac{1}{\sqrt{n}}G_{\lfloor nt \rfloor} \xrightarrow{\mathscr{D}} \mathscr{N}\left(0, \frac{t\Sigma}{3}\right).$$

They also establish a functional law of large numbers for the center of mass process:

$$\frac{1}{n}\left(G_{\lfloor nt \rfloor}\right)_{t\in[0,1]} \xrightarrow{\text{a.s.}} \frac{1}{2}I_\mu \quad \text{in } \left(\mathscr{D}_0^d, \rho_S\right),$$

where ρ_S is the Skorokhod metric and a functional central limit theorem for the same process:

$$\frac{1}{\sqrt{n}}\left(G_{\lfloor nt\rfloor}\right)_{t\in[0,1]} \Rightarrow \mathscr{G}_P(0,K),$$

where $\mathscr{G}_P(0,K)$ is a Gaussian process with mean 0 and the symmetric covariance function K defined by:

$$K(t_1,t_2) = \begin{cases} t_1\Sigma(3t_2-t_1)/(6t_2), & \text{for } 0<t_1\le t_2; \\ t_2\Sigma/3, & \text{for } t_1=0, t_2\neq 0; \\ 0, & \text{for } t_1=t_2=0. \end{cases}$$

In [MW18], McRedmond and Wade investigate the asymptotic properties of the convex hull formed by the first n steps of planar random walks, focusing on the perimeter length L_n, the diameter D_n, and the overall shape of the convex hull. They establish several key results under various assumptions. For random walks with non-zero mean drift ($\mu \neq 0$), they show that, with probability one, the ratio of the perimeter to the diameter converges to 2:

$$\frac{L_n}{D_n} \to 2 \quad \text{a.s.}$$

This result assumes that $\mathbb{E}\|Z\| < \infty$. In the case of zero drift ($\mu = 0$), they demonstrate that the shape of the convex hull infinitely often approximates any unit-diameter compact convex set K containing the origin:

$$\liminf_{n\to\infty} \rho_H\left(D_n^{-1}\mathscr{H}_n, K\right) = 0 \qquad \text{a.s.}$$

This result requires the additional assumption that $\mathbb{E}\|Z\|^2 < \infty$ and that the covariance matrix Σ is positive definite. For the zero-drift case, they also show that the ratio of the perimeter to the diameter oscillates between 2 and π almost surely:

$$\liminf_{n\to\infty} \frac{L_n}{D_n} = 2 \quad \text{and} \quad \limsup_{n\to\infty} \frac{L_n}{D_n} = \pi \quad \text{a.s.}$$

For a random walk with non-zero mean drift, they prove L^2 convergence of the perimeter, establishing that:

$$n^{-1/2}\left|L_n - 2S_n\cdot\hat{\mu}\right| \to 0 \quad \text{in } L^2,$$

Additionally, they find that the expected perimeter behaves asymptotically as $n + \log n$:

$$\mathbb{E}[L_n] = 2\|\mu\|n + \frac{\sigma^2_{\mu^\perp}}{\|\mu\|}\log n + o(\log n),$$

where $\sigma^2_{\mu^\perp} = \sigma^2 - \sigma^2_\mu$ and $\sigma^2_\mu = \mathbb{E}[(Z-\mu)\cdot\hat{\mu}]^2$. For the diameter D_n in the case of non-zero drift, they prove L^2 convergence:

$$n^{-1/2}\,|D_n - S_n\cdot\hat{\mu}| \to 0 \quad \text{in } L^2.$$

Furthermore, if $\mu \neq 0$, they establish a central limit theorem for the diameter in the regular case $\sigma^2_\mu > 0$ as:

$$\frac{D_n - \mathbb{E}[D_n]}{\sqrt{\mathbb{V}\text{ar}(D_n)}} \xrightarrow{\mathscr{D}} N(0,1),$$

with the variance asymptotically behaving as $\mathbb{V}\text{ar}(D_n) \sim \sigma^2_\mu n$. In the degenerate case where $\sigma^2_\mu = 0$, they show that the diameter's variance converges to a constant, and the errors $D_n - \|\mu\|n$ converge to a rescaled square of a normal distribution:

$$D_n - \|\mu\|n \xrightarrow{\mathscr{D}} \frac{\sigma^2_{\mu^\perp}\zeta^2}{2\|\mu\|},$$

where $\zeta \sim N(0,1)$. This result assumes higher moment conditions, specifically $\mathbb{E}\|Z\|^p < \infty$ for some $p > 2$. If $p > 4$, they proved the variance asymptotic in this case as:

$$\lim_{n\to\infty} \text{Var}\, D_n = \frac{\sigma^4_{\mu_\perp}}{2\|\mu\|^2}.$$

The study of other various properties and functionals related to the convex hulls of random walks has been a fruitful area of research. Kabluchko, Vysotsky, and Zaporozhets have contributed significantly to this field. In one of their studies [KVZ17b], they focused on calculating the expected number of faces on the convex hull formed by random walks. Previously, Vysotsky and Zaporozhets investigated the probability that a multi-dimensional random walk with symmetric increments around the center would contain the origin within its hull [VZ18]. Although their initial findings applied only to two-dimensional cases, they later expanded their proofs to multiple dimensions using a different approach in collaboration with Kabluchko [KVZ17a]. Their techniques provided a robust extension of the Spitzer-Widom formula to higher dimensions.

Prior research has also been done on other characteristics of convex hulls. Khoshnevisan's work [Kho92], for instance, explored a broad range of functionals, Ψ, of the convex hull that sticks to a monotone relationship with respect to the convex hull set and maintained an affine scaling property. He established a law of the iterated logarithm for these functionals, demonstrating that:

$$\limsup_{n\to\infty} \frac{\Psi(\mathscr{H}_n)}{(2n\log\log n)^{\alpha/2}} = c_\Psi \text{ a.s.}$$

where c_Ψ is a constant dependent on the functional chosen, and $\mathscr{H}_n$ is the convex hull spanned by a zero-drift random walk. Furthermore, he proved a corresponding lower bound:

$$\liminf_{n\to\infty} \left(\frac{\log\log n}{n}\right)^{\alpha/2} \Psi(\mathscr{H}_n) = c'_\Psi \text{ a.s.}$$

where α and Ψ remain consistent with the previous definition, but c'_Ψ is a different deterministic constant. Additionally, Kuelbs and Ledoux [KL98] primarily focused on convex hulls related to Brownian motion. Their work also addressed some refinements in the context of random walks, particularly clarifying certain complex scenarios that were not fully resolved by Khoshnevisan's earlier proofs.

In [CSŠW24], the authors discussed the asymptotic properties of geometric functionals associated with the convex hull of a d-dimensional random walk, assuming a non-zero drift ($\mu \neq 0$). They established that there exists a constant

$$\Lambda(d,k,\mathscr{L}_Z)$$

depending on the dimension d, a chosen functional dimension k (where $k \in \{1,2,\ldots,d\}$), and the law $\mathscr{L}_Z$ of the random steps Z. This constant characterizes the almost-sure limit superior of the scaled k-dimensional volume functional $V_k(\mathscr{H}_n)$ of the convex hull $\mathscr{H}_n$ of the first n steps of the walk:

$$\limsup_{n\to\infty} \frac{V_k(\mathscr{H}_n)}{\sqrt{2^{k-1}n^{k+1}(\log\log n)^{k-1}}} = \Lambda(d,k,\mathscr{L}_Z), \quad \text{a.s.}$$

The authors emphasize that, for a genuinely d-dimensional random walk, $\Lambda(d,k,\mathscr{L}_Z) > 0$. While the case $d = 1$ recovers known results, the primary interest lies in dimensions $d \geq 2$.

The authors note that an explicit form for $\Lambda(d,k,\mathscr{L}_Z)$ is generally unavailable, except in the case $k = d$ (the volume functional), where it admits a variational characterization similar to classical isoperimetric problems. They present this formulation:

$$\Lambda(d,d,\mathscr{L}_Z) = \lambda_d \cdot \|\mu\| \cdot \sqrt{\det \Sigma_{\mu^\perp}},$$

where λ_d is defined by the variational formula:

$$\lambda_d := \sup_{f \in U_{d-1}} V_d(H(f)),$$

where $H(f)$ denotes the convex hull of a space-time path. Notably, for $d = 2$, λ_d takes the explicit value $\lambda_2 = \sqrt{3}/6$. Additionally, the authors provide a simplified expression in the case $k = 1$, stating that:

$$\Lambda(d,1,\mathscr{L}_Z) = \|\mu\|.$$

The authors also showed similar results for the convex hull of centroids. They showed that if the covariance matrix is the identity matrix, and $\mu \neq 0$, then:

$$\limsup_{n\to\infty} \frac{A(\mathscr{G}_n)}{n^{3/2}\sqrt{\log\log n}} = \vartheta\|\mu\|, \text{ a.s.},$$

where $\vartheta \in (0,\infty)$, and $A(\mathscr{G}_n)$ is the area of the convex hull spanned by the centroids of a two-dimensional planar random walk.

In [CSŠ22], the authors investigate the convex hulls of a random walk whose steps are in the domain of attraction of a stable law in $\mathbb{R}^d$. The main result is the convergence of the convex hull of appropriately scaled points towards the convex hull of the path of the limiting stable Lévy process X:

$$\frac{\operatorname{chull}\{S_0 - a_0, \dots, S_n - a_n\}}{b_n} \Rightarrow \operatorname{chull} X[0,1].$$

The convergence is proven in the space of all convex and compact subsets of $\mathbb{R}^d$, equipped with the Hausdorff distance. As an application, the authors also showed the convergence of the (expected) intrinsic volumes of these convex hulls:

$$\frac{V_m\left(\operatorname{chull}\{S_0 - a_0, \ldots, S_n - a_n\}\right)}{b_n^m} \xrightarrow{\mathscr{D}} V_m(\operatorname{chull} X[0,1])$$

under mild moment conditions on the random walk, as well as for the Steiner point:

$$\frac{p\left(\operatorname{chull}\{S_0 - a_0, \ldots, S_n - a_n\}\right)}{b_n^m} \xrightarrow{\mathscr{D}} p(\operatorname{chull} X[0,1]).$$

0.2. Possible Applications

Convex hulls of random walks have many uses across different fields. One common use is in ecology, where they help study the home ranges of animals, as shown in research from the late 20th century [Wor87, Wor95]. Luković, Geisel, and Eule [GLM17] expanded on this by looking at convex hulls generated by continuous random walks, comparing these paths to the hunting strategies of Mediterranean seabirds and other predators. Their study also looked at how convex hulls relate to bridges — random walks that return to the starting point — and multiple walks, which model how animals or groups move and return to a fixed spot each night, as discussed in other works [LGE13].

In statistics, a method called convex hull peeling, also known as the onion layer problem, is used to find the centrality of data points. This involves creating a convex hull around a dataset, removing the outermost points that are least central, and repeating the process to sort the data points. Researchers like Brozius [Edd82] and Eddy [Bro89] have studied this technique in depth, focusing on how understanding convex hulls can improve this method's effectiveness. Convex hulls also play a key role in technology and image processing, where algorithms are used to detect the convex hull of objects in images, speeding up computing tasks. This field has seen many important contributions over the years [AT78, MT85, Hus88, Ye95].

In biology and medicine, convex hulls are important for approximating protein surfaces, which helps identify possible uses for specific proteins [Ye95]. Convex hull-based classification algorithms are also used to recognize different proteins [MAH+95] and even to predict the onset of psychosis [BCC+15]. These examples show how useful convex hulls are in many practical applications, suggesting that further study, especially of convex hulls from random walks, could lead to even more discoveries beyond just ecological studies of animal movements.

0.3. Thesis Outline

This dissertation explores the limiting behavior of geometric functionals of convex hulls generated by random walks. We begin by studying the first-order convergence properties of the convex hulls of random walks in Chapter 2. This chapter sets the stage by defining the specific problem of studying the convex hulls of random walks and outlining the mathematical framework used in our analysis. We show that the convex hull, appropriately scaled, almost surely converges to the convex hull generated by the corresponding drift vectors and the origin. This result is fundamental as it provides the basis for further analysis of the geometric functionals of the convex hull.

In Chapter 3, we focus on the perimeter of the convex hull generated by the random walk. By applying martingale difference sequences and Cauchy's formula, we successfully control the variance and derive an L^2 approximation for the deviation of the perimeter process. Under certain assumptions about the drift vectors of the random walks, we determine the asymptotic behavior of the variance of the perimeter. If this asymptotic variance is positive, we can establish a normal distributional limit for the perimeter. The chapter also explores the implications of these results and discusses the conditions under which they hold.

Following the perimeter analysis, we shift our attention to the diameter of the convex hull in Chapter 4. It is crucial to note that, loosely speaking, the mapping that assigns diametral segments to polygons is continuous. This continuity allows us to apply similar techniques used for the perimeter to the diameter process. With additional assumptions about the set of drift vectors, we achieve results analogous to those for the perimeter process. We establish both almost sure convergence and central limit theorems for the diameter under different scenarios. The chapter includes detailed proofs and discussions of the conditions under which these results hold, emphasizing the geometric and probabilistic aspects of the diameter process.

In the final chapter, we study a single random walk's convex hull of centroids and prove the analog results for the perimeter and diameter of such an object. We discuss the generalization of the assumptions made throughout the dissertation and provide a detailed simulation study to explore the consequences of relaxing these assumptions. This simulation study helps us understand the robustness of our results and highlights open problems for future research. We discuss potential generalizations and provide a comprehensive analysis of the simulation results. This chapter serves to open a window to possible extensions of this work.

1. MATHEMATICAL PREREQUISITES

Let's review some essential math topics before starting the central part of this thesis. We will need to understand concepts like measures, metrics, and convexity. We will also touch on some basic probability theory to help us along. For those who want to dive deeper, there are some great books to check out. For metric spaces, you can read [Mag22]. If you are interested in convexity, take a look at [Gru07]. And for a detailed look at probability theory, [Dur19, Gut06] are good choices.

1.1. METRIC SPACES

Firstly, let's recall the definition of a general metric space. Afterward, we will review some important examples of metric spaces and metrics.

Definition 1.1.1. A metric space is a pair $(\mathscr{M}, d)$, consisting of a set $\mathscr{M}$ and a function $d : \mathscr{M} \times \mathscr{M} \to [0, +\infty)$ called a metric. The metric d calculates the distance between any two points in $\mathscr{M}$ satisfying the following properties:

- For any x and y in $\mathscr{M}$, $d(x,y) = 0$ if and only if $x = y$.
- (Symmetry) For any x and y in $\mathscr{M}$, $d(x,y) = d(y,x)$.
- (Triangle Inequality) For any x, y, and z in $\mathscr{M}$, we have

$$d(x,z) \leq d(x,y) + d(y,z).$$

Defining the concepts of open and closed balls is crucial in the theory of metric spaces.

An *open ball* around a point $x \in \mathscr{M}$ with radius ε is the subset

$$B(x,\varepsilon) := \{y \in \mathscr{M} : d(x,y) < \varepsilon\}.$$

A *closed ball* around a point $x \in \mathscr{M}$ with radius ε is the subset

$$\bar{B}(x,\varepsilon) := \{y \in \mathscr{M} : d(x,y) \leq \varepsilon\}.$$

We can quickly induce a topology on a metric space using open balls. Specifically, we say that a subset $A \subseteq \mathscr{M}$ is an *open set* if for every $x \in A$, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq A$. On the other hand, we say that $A \subseteq \mathscr{M}$ is a *closed set* if $\mathscr{M} \setminus A$ is an open set. The family of open subsets form a topological space (see [Mag22, p. 39]). Using that result and De Morgan's laws [Fol99, p. 3], it can be easily shown that an arbitrary intersection of closed sets is a closed set. Therefore, the following objects are well-defined.

Definition 1.1.2. Let $(\mathscr{M},d)$ be a metric space. Let $A \subset \mathscr{M}$. The *interior* of A, denoted by $\mathrm{Int}(A)$, is the union of all open sets U such that $U \subseteq A$:

$$\mathrm{Int}(A) = \bigcup\{U \subseteq \mathscr{M} : U \text{ is open and } U \subseteq A\}.$$

The *closure* of A, denoted by $\mathrm{Cl}(A)$, is the intersection of all closed sets V such that $A \subseteq V$:

$$\mathrm{Cl}(A) = \bigcap\{V \subseteq \mathscr{M} : V \text{ is closed and } A \subseteq V\}.$$

The *boundary* of A, denoted by ∂A, is the set difference $\mathrm{Cl}(A) \setminus \mathrm{Int}(A)$:

$$\partial A = \mathrm{Cl}(A) \setminus \mathrm{Int}(A).$$

Consider two metric spaces $(\mathscr{M}_X, d_X)$ and $(\mathscr{M}_Y, d_Y)$. Let $f : \mathscr{M}_X \to \mathscr{M}_Y$ and let $a \in \mathscr{M}_X$. We say that f is *(pointwise) continuous at the point a* if the following condition is satisfied:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathscr{M}_X, d_X(x,a) < \delta \implies d_Y(f(x), f(a)) < \varepsilon.$$

We can also describe the condition for continuity at a using open balls, making our reasoning more efficient without losing clarity [Mag22, p. 52]:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } f(B(a,\delta)) \subset B(f(a),\varepsilon).$$

A mapping $f : \mathscr{M}_X \to \mathscr{M}_Y$ is said to be *continuous* if it is continuous at each $a \in \mathscr{M}_X$.

Consider the sequence $(x_n)_{n=1}^{\infty}$ in the metric space $(\mathscr{M}, d)$. A point $a \in \mathscr{M}$ is called the *limit* of the sequence $(x_n)_{n=1}^{\infty}$ if for every $\varepsilon > 0$, there exists a number $N \in \mathbb{N}$ such that $x_n \in B(a,\varepsilon)$ for all $n \geq N$. We denote this by:

$$\lim_{n\to\infty} x_n = a$$

or, more informally, by $x_n \to a$. We say that the sequence $(x_n)_{n=1}^{\infty}$ *converges* in $\mathscr{M}$ if it has a limit. If the limit exists, it is unique [Mag22, p. 39], so the definite article 'the' makes sense. We can define the limit $\lim_{x\to a} f(x)$ as follows. For a given $b \in \mathscr{M}_Y$, we say that f has the limit b at the point a, and we write

$$\lim_{x\to a} f(x) = b$$

if the following condition holds:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathscr{M}_X, 0 < d_X(x,a) < \delta \implies d_Y(f(x),b) < \varepsilon.$$

Using the upper definition of a limit, we easily characterize the continuity as follows [Mag22, Prop. 2.24].

Proposition 1.1.3. Function f is continuous at a if and only if:

$$\lim_{x\to a} f(x) = f(a).$$

Finally, the following proposition relates the continuity with the limits of sequences [Mag22, Prop. 2.29].

Proposition 1.1.4. Let $f : \mathscr{M}_X \to \mathscr{M}_Y$. The following conditions are equivalent:

(1) f is continuous.

(2) For every $x \in \mathscr{M}_X$, and for every sequence $(a_n)_{n=1}^{\infty}$ in $\mathscr{M}_X$ such that $\lim_{n\to\infty} a_n = x$, we have $\lim_{n\to\infty} f(a_n) = f(x)$.

An *open covering* of $\mathscr{M}$ is a family $(U_i)_{i\in I}$ of sets, each of which is an open set in $\mathscr{M}$, such that:

$$\mathscr{M} = \bigcup_{i\in I} U_i.$$

A metric space $(\mathscr{M}, d)$ is called *compact* if for every open covering of $\mathscr{M}$, there exists a finite subcovering. In other words, if $(U_i)_{i\in I}$ is a collection of open sets such that $\mathscr{M} = \bigcup_{i\in I} U_i$, then there must be a finite subset $\{i_1, \ldots, i_n\} \subset I$ such that $\mathscr{M} = \bigcup_{k=1}^{n} U_{i_k}$. We say that $M \subseteq \mathscr{M}$ is a *compact subset* if it is a compact space with respect to the relative topology.

We will work with specific metric space types and use special notations for each. Starting with the Euclidean metric on $\mathbb{R}$ - the metric is the absolute value of the difference between any two real numbers. For any $x, y \in \mathbb{R}$, this distance is expressed as $d(x,y) = \rho(x,y) := |x-y|$, where $|x|$ represents the absolute value of a number.

The concept extends to vector distances for spaces of higher dimensions, specifically $\mathbb{R}^d$. In this context, if we have a vector $x = (x_1, x_2, \ldots, x_d)^\top \in \mathbb{R}^d$, we define its Euclidean norm as $\|x\| := \sqrt{x_1^2 + \ldots + x_d^2}$. Given two vectors $x, y \in \mathbb{R}^d$, the Euclidean distance between them is calculated as $d(x,y) = \rho_E(x,y) := \|x-y\|$. The proof of the following proposition can be found in [Mag22, Prop. 4.15]

Proposition 1.1.5. A subset $A \subseteq \mathbb{R}^d$ is a compact set if and only if it is bounded and closed, and if and only if for every sequence $(x_n)_{n=1}^{\infty} \subseteq A$ there exists a convergent subsequence.

Let's introduce two sets of notation that will be useful to us. For the *unit sphere* in $\mathbb{R}^d$, we will use the symbol $\mathbb{S}^{d-1}$. This set is defined as:

$$\mathbb{S}^{d-1} := \left\{ x \in \mathbb{R}^d : \|x\| = 1 \right\}.$$

Additionally, for the *unit ball* in $\mathbb{R}^d$, we will denote it by $\mathbb{B}^d$. The set gives the unit ball:

$$\mathbb{B}^d := \left\{x \in \mathbb{R}^d : \|x\| \leq 1\right\}.$$

Finally, we often need to expand set A by an arbitrary ε. This set will be denoted as:

$$A^{\varepsilon} := \left\{x \in \mathbb{R}^d : \rho_E(x,A) \leq \varepsilon\right\}.$$

We explore the concept of Euclidean distance between a point and a set. For a single point x and a set A, the distance is the smallest possible distance between x and any point y within A, expressed as:

$$d(x,A) = \rho_E(x,A) := \inf_{y \in A} d(x,y).$$

1.2. CONVEX SETS

We begin this section by defining a convex set in $\mathbb{R}^d$. Simply put, a set is convex if it contains every line segment connecting two points within the set.

Definition 1.2.1. A subset $A \subseteq \mathbb{R}^d$ is called convex if it meets the following criterion: for any points x and y in A, and any real number t in the interval $0 \leq t \leq 1$, the point $(1-t)x+ty$ is also in A.

One can easily show that if $(A_i)_{i\in I}$ is a collection of convex sets, then $\bigcap_{i\in I} A_i$ is also a convex set [Sol19, Thm. 3.8]. Given a set A in $\mathbb{R}^d$, its *convex hull*, $\mathrm{chull}(A)$, is defined as the intersection of all convex sets in $\mathbb{R}^d$ that contain A. Since the intersection of convex sets is always convex, $\mathrm{chull}(A)$ is convex, and it is the smallest convex set in $\mathbb{R}^d$ with respect to set inclusion that contains A. For the study of convex hulls, we need the following concept: Let $x_1,\dots,x_n \in \mathbb{R}^d$. Any point x of the form $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$, where $\lambda_1,\dots,\lambda_n \geq 0$ and $\lambda_1 + \cdots + \lambda_n = 1$, is called a *convex combination* of $x_1,\dots,x_n$. The following result is a simple consequence of Carathéodory's theorem (see [Gru07, Thm. 3. 1]).

Proposition 1.2.2. Let $A \subseteq \mathbb{R}^d$ be compact. Then $\mathrm{chull}(A)$ is compact.

Let $\mathscr{K}^d$ denote the collection of all compact and convex subsets of $\mathbb{R}^d$. Similarly, let $\mathscr{K}_0^d$ represent the collection of all compact and convex subsets of $\mathbb{R}^d$ that contains the origin. We define a *polygon* as the convex hull of a finite set of points whose interior is non-empty. If A is a polygon, we denote by $V(A)$ the set of vertices - the smallest possible set of points such that the convex hull of these points coincides with A. Let $\mathscr{P}^d \subseteq \mathscr{K}^d$ and $\mathscr{P}_0^d \subseteq \mathscr{K}_0^d$ denote the collections of polygons within the respective collections of compact and convex sets.

The *support function* of $A \subseteq \mathscr{K}^d$ is function $h_A : \mathbb{R}^d \to \mathbb{R}$, defined by:

$$h_A(u) = \sup\{u \cdot y : y \in A\} \text{ for } u \in \mathbb{R}^d.$$

We need to establish the appropriate metric to construct the metric spaces from these collections. The *Hausdorff metric* ρ_H^d on $\mathscr{K}^d$ is defined as follows:

$$\rho_H^d(A,B) = \max\left\{\max_{x\in A}\min_{y\in B}\|x-y\|, \max_{y\in B}\min_{x\in A}\|x-y\|\right\} \text{ for } A,B\in\mathscr{K}^d.$$

There are equivalent definitions of the Hausdorff metric, which are given in the following proposition (see [Gru07, Prop 6.3]).

Proposition 1.2.3. Let $A,B\in\mathscr{K}$. Then:

- $\rho_H^d(A,B) = \inf\left\{\delta\geq 0 : A\subseteq B^\delta, B\subseteq A^\delta\right\}$.
- $\rho_H^d(A,B) = \max\left\{|h_A(u)-h_B(u)| : u\in\mathbb{S}^{d-1}\right\}$.
- $\rho_H^d(A,B)$ is the maximum distance a point in one of the bodies A or B can have from the other body.

Finally, we have the following metric space.

Proposition 1.2.4. $(\mathscr{K}^d,\rho_H^d)$, $(\mathscr{K}_0^d,\rho_H^d)$, $(\mathscr{P}^d,\rho_H^d)$, and $(\mathscr{P}_0^d,\rho_H^d)$ are metric spaces.

In the following discussion, we may focus only on the case where $d=2$. If that is the case, we will omit the notation for dimension and simply write $\mathscr{K}$, $\mathscr{K}_0$, $\mathscr{P}$, and $\mathscr{P}_0$. The *perimeter* of a set $A\in\mathscr{K}$ is defined as:

$$\mathrm{Per}(A) := \lim_{\varepsilon\to 0^+}\left(\frac{\lambda_2(A^\varepsilon)-\lambda_2(A)}{\varepsilon}\right),$$

where λ_2 is the two-dimensional Lebesgue measure (see Example 1.3.1). The existence of this limit is assured by Steiner's formula (1.1), which represents $\lambda_2(A^\varepsilon)$ as a quadratic polynomial in ε, with coefficients derived from the intrinsic volumes of A:

$$\lambda_2(A^\varepsilon) = \lambda_2(A) + \varepsilon\,\mathrm{Per}(A) + \mathscr{O}\left(\varepsilon^2\right) \text{ as } \varepsilon\to 0^+.$$

On the other hand, the *diameter* of $A\in\mathscr{K}$ is defined as:

$$\mathrm{diam}(A) = \sup_{x,y\in A}\|x-y\|.$$

Working with the general definitions of perimeter and diameter is quite inconvenient. Hence, we use the *Cauchy surface formula*, which relates the projections of the convex sets to the perimeter and diameter. Cauchy's surface area formula is a remarkable and well-known result in integral geometry. It states that the average area of a convex body's projections corresponds to the body's surface area, multiplied by a constant that depends on the dimension. For a convex body $A \in \mathscr{K}^d$ and $u \in \mathbb{S}^{d-1}$, we denote by $A \mid u^{\perp}$ the projection of A onto the $(d-1)$-dimensional subspace of $\mathbb{R}^d$ perpendicular to u.

Theorem 1.2.5 (Cauchy's surface area formula). Let $A \in \mathscr{K}^d$ be a convex and compact set. Then

$$S(A) = \frac{1}{\lambda_{d-1}\left(\mathbb{B}^{d-1}\right)} \int_{\mathbb{S}^{d-1}} \lambda_{d-1}\left(A \mid u^{\perp}\right) \mathrm{d}u.$$

where $S(A)$ denotes the volume of the surface of A, and λ_d represents the d-dimensional Lebesgue measure described in Example 1.3.1.

Cauchy's surface area formula was initially demonstrated by Cauchy for dimensions $n=2$ and $n=3$ in 1841 and 1850, respectively [Cau41, Cau50]. We refer the reader to [TV16] for detailed proof of this assertion, but we find this result so important that we include the proof in the Appendix. If $d=2$, Cauchy's surface area formula reduces to the following.

Corollary 1.2.6. Let $A \in \mathscr{K}_0$ be a convex and compact set. Then:

$$\mathrm{Per}(A) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \lambda_1\left(A \mid u\right) \mathrm{d}u = \int_0^{\pi} \left(\max_{x\in A}(x\cdot \mathbf{e}_\theta) - \min_{x\in A}(x\cdot \mathbf{e}_\theta) \right) \mathrm{d}\theta = \int_0^{2\pi} h_A(\mathbf{e}_\theta)\mathrm{d}\theta,$$

where $\mathbf{e}_\theta$ is a unit vector directed by an angle θ.

Proof. The first equality follows because, in two-dimensional space, orthogonal subspaces are one-dimensional. The second equality follows because:

$$\lambda_1(A|u) = \max_{x\in A}(x\cdot \mathbf{e}_\theta) - \min_{x\in A}(x\cdot \mathbf{e}_\theta),$$

where $\theta \in [0, 2\pi)$ is such that $u = \mathbf{e}_\theta$. With the appropriate parametrization of $\mathbb{S}^1$ (the bijection $u \in \mathbb{S}^1 \leftrightarrow \theta \in [0, 2\pi]$), we have $\pi d\theta = du$ and:

$$\frac{1}{2\pi}\int_{\mathbb{S}^1} \lambda_1 (A \mid u)\,\mathrm{d}u = \frac{1}{2}\int_0^{2\pi} \left(\max_{x\in A}(x\cdot \mathbf{e}_\theta) - \min_{x\in A}(x\cdot \mathbf{e}_\theta) \right) \mathrm{d}\theta.$$

Since

$$\max_{x\in A}(x\cdot \mathbf{e}_\theta) - \min_{x\in A}(x\cdot \mathbf{e}_\theta) = -(\max_{x\in A}(x\cdot \mathbf{e}_{\theta+\pi}) - \min_{x\in A}(x\cdot \mathbf{e}_{\theta+\pi})),$$

we have that:

$$\frac{1}{2}\int_0^{2\pi} \left(\max_{x\in A}(x\cdot \mathbf{e}_\theta) - \min_{x\in A}(x\cdot \mathbf{e}_\theta) \right) \mathrm{d}\theta = \int_0^{\pi} \left(\max_{x\in A}(x\cdot \mathbf{e}_\theta) - \min_{x\in A}(x\cdot \mathbf{e}_\theta) \right) \mathrm{d}\theta.$$

The third equality follows from the fact that for $\theta \in [\pi, 2\pi]$, $\max_{x\in A}(x\cdot \mathbf{e}_\theta) = -\min_{x\in A}(x\cdot \mathbf{e}_\theta)$, and recall that $\max_{x\in A}(x\cdot \mathbf{e}_\theta)$ is precisely $h_A(\mathbf{e}_\theta)$. □

For the diameter, the situation is somewhat simpler. Specifically, the diameter can be obtained as the length of the maximum possible projection of a convex and compact set in the direction of a unit vector $\mathbf{e}_\theta$. The following theorem gives the result for the diameter.

Theorem 1.2.7. Let $A \in \mathscr{K}$ be a convex and compact set. Then

$$\operatorname{diam}(A) = \sup_{0\le\theta\le\pi} \left(\max_{x\in A}(x\cdot \mathbf{e}_\theta) - \min_{x\in A}(x\cdot \mathbf{e}_\theta) \right).$$

Proof. Since A is compact, for each θ there exist points $x, y \in A$ such that

$$\max_{x\in A}(x\cdot \mathbf{e}_\theta) - \min_{x\in A}(x\cdot \mathbf{e}_\theta) = x\cdot \mathbf{e}_\theta - y\cdot \mathbf{e}_\theta = (x-y)\cdot \mathbf{e}_\theta \le \|x-y\|.$$

Thus,

$$\sup_{0\le\theta\le\pi} \left(\max_{x\in A}(x\cdot \mathbf{e}_\theta) - \min_{x\in A}(x\cdot \mathbf{e}_\theta) \right) \le \sup_{x,y\in A} \|x-y\| = \operatorname{diam}(A).$$

It remains to show that

$$\sup_{0\le\theta\le\pi} \left(\max_{x\in A}(x\cdot \mathbf{e}_\theta) - \min_{x\in A}(x\cdot \mathbf{e}_\theta) \right) \ge \operatorname{diam}(A).$$

This is clearly true if A consists of a single point, so let us assume that A contains at least two points. Suppose that the diameter of A is achieved by points $x, y \in A$ such that $x \neq y$,

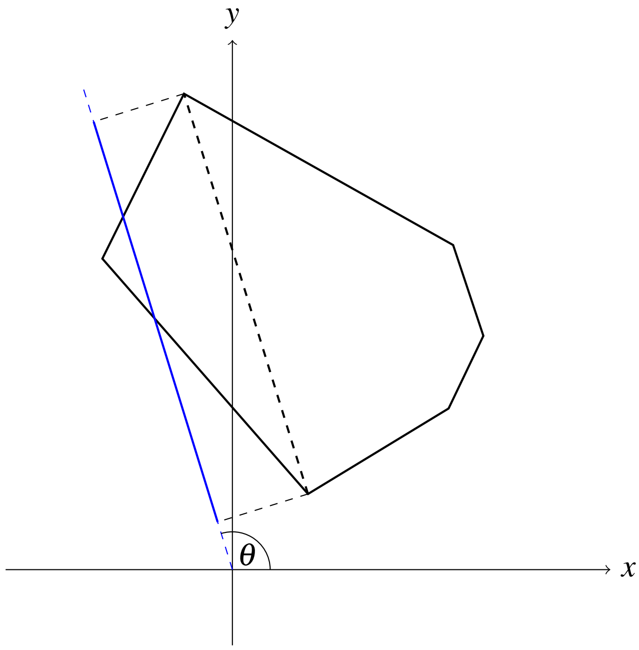


Figure 1.1: Illustration of the Cauchy formula for the diameter.

and let $z = y - x$ such that $\hat{z} := z/\|z\| = \mathbf{e}_{\theta_0}$ for some $\theta_0 \in [0, \pi]$. Then,

$$\begin{aligned}\sup_{0\le\theta\le\pi}\left(\max_{x\in A}(x\cdot\mathbf{e}_\theta) - \min_{x\in A}(x\cdot\mathbf{e}_\theta)\right) &\ge \max_{x\in A}(x\cdot\mathbf{e}_{\theta_0}) - \min_{x\in A}(x\cdot\mathbf{e}_{\theta_0})\\ &\ge y\cdot\mathbf{e}_{\theta_0} - x\cdot\mathbf{e}_{\theta_0}\\ &= z\cdot\hat{z} = \|z\| = \mathrm{diam}(A),\end{aligned}$$

as required. □

The *mean width* and *Steiner point* of a set $A \in \mathscr{K}^d$ are defined as follows:

$$w(A) = \frac{2}{\varpi_d}\int_{\mathbb{S}^{d-1}} s_A(\theta)\sigma(\mathrm{d}\theta) \quad \text{and} \quad p(A) = \frac{1}{\kappa_d}\int_{\mathbb{S}^{d-1}} s_A(\theta)\theta\sigma(\mathrm{d}\theta),$$

where $\sigma(\mathrm{d}\theta)$ represents the surface measure on the unit sphere $\mathbb{S}^{d-1}$, $\varpi_d = \sigma\left(\mathbb{S}^{d-1}\right)$ denotes the total surface measure of $\mathbb{S}^{d-1}$, and $s_A(\theta) := h_A(\mathbf{e}_\theta)$. The term $\kappa_d = \lambda_d\left(\mathbb{B}^d\right)$ is the d-dimensional volume of the unit ball $\mathbb{B}^d$. These quantities are given by $\kappa_d = \pi^{d/2}/\Gamma(1+d/2)$ and $\varpi_d = d\kappa_d$.

The *outer parallel body* of A at a distance $\rho \ge 0$ is defined as $A + \rho\mathbb{B}^d$ (see Minkowski sum in (A.6)). The classical *Steiner formula* [Gru07][Thm. 6.6, Prop. 6.7] expresses the d-dimensional Lebesgue measure of a set A expanded by a ball of radius ρ as a polynomial of degree at most d, with coefficients that are significant geometric quantities. The formula is given by:

$$\lambda_d\left(A+\rho\mathbb{B}^d\right)=\sum_{m=0}^{d}\rho^{d-m}\kappa_{d-m}V_m(A), \tag{1.1}$$

where $V_0(A),\ldots,V_d(A)$ are known as the *intrinsic volumes* of the set A. It is established that $V_0(A)=1$ and $V_1(A)$ is proportional to the mean width of A, specifically:

$$V_1(A)=\frac{d\kappa_d}{2\kappa_{d-1}}w(A). \tag{1.2}$$

For $d=2$, $V_1(A)$ corresponds to half the mean width of A. Additionally, $V_{d-1}(A)$ equals half the surface area of A, and $V_d(A)$ is the d-dimensional volume of A. It is worth noting that the mappings $A\mapsto V_m(A)$ for $m\in\{0,1,\ldots,d\}$ and $A\mapsto p(A)$ are continuous with respect to the Hausdorff distance on compact convex sets, as detailed in [BBS07, Theorem III.1.1] and [Sch13, Lemma 1.8.14].

1.3. PROBABILITY THEORY

Let Ω be a non-empty set. We assume that $\mathscr{F}$ is a σ-field (or σ-algebra), meaning it is a non-empty collection of subsets of Ω that satisfies the following properties:

(i) If $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$.

(ii) If $\{A_i\}$ is a countable sequence of sets in $\mathscr{F}$, then $\bigcup_i A_i \in \mathscr{F}$.

Since $\cap_i A_i = (\cup_i A_i^c)^c$, it follows that a σ-field is also closed under countable intersections. The pair $(\Omega, \mathscr{F})$ is referred to as a *measurable space*, which is a structure where we can define a measure. A *measure* is a nonnegative, countably additive set function $\mu : \mathscr{F} \to [0, +\infty]$ that satisfies:

(i) $\mu(A) \geq 0$ and $\mu(\emptyset) = 0$ for all $A \in \mathscr{F}$.

(ii) If $\{A_i\}$ is a countable sequence of disjoint sets in $\mathscr{F}$, then

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i).$$

If $\mu(\Omega) = 1$, then μ is called a *probability measure*. We denote probability measures by $\mathbb{P}$. A *probability space* is defined as a triple $(\Omega, \mathscr{F}, \mathbb{P})$.

Example 1.3.1 (Lebesgue measure on $\mathbb{R}$)**.** It follows immediately from the definition that if $\mathscr{F}_i$, for $i \in I$, are σ-fields, then $\bigcap_{i \in I} \mathscr{F}_i$ is also a σ-field. From this, we have that if we have a set Ω and a collection $\mathscr{A}$ of subsets of Ω, there exists a smallest σ-field containing $\mathscr{A}$. This is known as the σ-field generated by $\mathscr{A}$ and is denoted by $\sigma(\mathscr{A})$.

Let $\mathscr{R}^d$ denote the Borel sets, which are the smallest σ-field containing the open sets in $\mathbb{R}^d$. Measures on $(\mathbb{R}, \mathscr{R}^1)$ are defined by providing a Stieltjes measure function with the following properties:

(i) F is non-decreasing.

(ii) F is right-continuous, meaning $\lim_{y \to x^+} F(y) = F(x)$.

Associated with each Stieltjes measure function F, there is a unique measure μ on $(\mathbb{R}, \mathscr{R}^1)$ such that $\mu((a,b]) = F(b) - F(a)$. When $F(x) = x$, the resulting measure (after competition and expanding the σ-field to Lebesgue measurable sets) is known as the *Lebesgue measure*, denoted by λ_1. The proof of this statement can be read in [Dur19, Theorem 1.1.4]. This measure can be easily extended to $\mathbb{R}^d$, and we denote it with λ_d (see [Sar02, Note 9.4]). If the dimension is known from the context, we will simply write λ.

A function $X : \Omega \to S$ is called a *random mapping* from measurable space $(\Omega, \mathscr{F})$ to measurable space $(S, \mathscr{S})$ if for every $B \in \mathscr{S}$:

$$X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\} \in \mathscr{F}.$$

If $(S, \mathscr{S}) = (\mathbb{R}^d, \mathscr{R}^d)$ and $d > 1$, then X is referred to as a *random vector*. When $d = 1$, X is called a *random variable*. If $X \geq 0$ is a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$, its *expected value* is defined as

$$\mathbb{E}[X] = \int X(\omega)\, \mathrm{d}\mathbb{P}(\omega),$$

which always exists but may be infinite. To extend this definition to the general case, consider the positive part $x^+ = \max\{x, 0\}$ and the negative part $x^- = \max\{-x, 0\}$ of x. We say that $\mathbb{E}[X]$ exists and set $\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-]$ whenever this difference is meaningful, i.e., when either $\mathbb{E}[X^+] < \infty$ or $\mathbb{E}[X^-] < \infty$. The expected value $\mathbb{E}[X]$ is often called the *mean* of X and denoted by μ. Since $\mathbb{E}[X]$ is defined by integrating X, it inherits all the properties of integrals. If X is a random vector, we say that X is integrable if each component of X is integrable in the upper sense. In this case, the expected vector is called the *drift vector*.

If k is a positive integer, then $\mathbb{E}[X^k]$ is referred to as the *kth moment* of random variable X. If $\mathbb{E}[X^2] < \infty$, the *variance* of X is defined as $\mathbb{V}\mathrm{ar}(X) = \mathbb{E}[(X-\mu)^2]$, where $\mu = \mathbb{E}[X]$. One can observe that $\mathbb{V}\mathrm{ar}(X) \geq 0$. The *covariance matrix* of a d-dimensional random vector X is given by:

$$\mathrm{Cov}(X) := \mathbb{E}\left[(X - \mathbb{E}[X])(X - \mathbb{E}[X])^\intercal\right],$$

where we understand X as a column vector. The usual representation of the covariance matrix is the following:

$$\mathrm{Cov}(X) = \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1,X_2) & \dots & \mathrm{Cov}(X_1,X_d) \\ \mathrm{Cov}(X_2,X_1) & \mathrm{Var}(X_2) & \dots & \mathrm{Cov}(X_2,X_d) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(X_d,X_1) & \mathrm{Cov}(X_d,X_2) & \cdots & \mathrm{Var}(X_d) \end{bmatrix},$$

where X_i is the i-th coordinate of X, and the covariance between two random variables is defined as:

$$\mathrm{Cov}(X_i,X_j) := \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])].$$

Thus, $\mathrm{Cov}(X)$ is a symmetric, positive-semidefinite matrix [Sar02, p. 317-318], since

$$\mathrm{Cov}(X_i,X_j) = \mathrm{Cov}(X_j,X_i).$$

If a variable has a finite k-th moment, then all its lower moments are also finite (see [Ash14, Lem. 5.10.5]). For a nonnegative random variable X and any positive number a, the probability that X is at least a is at most the expectation of X divided by a:

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

This result is known as the *Markov inequality* (see [Gut06, Thm. 1.1]). A direct consequence of the Markov inequality is the *Chebyshev inequality* (see [Gut06, Thm. 1.4]). It states that if X is a random variable with a finite mean μ and finite variance σ^2, and k is a positive real number, then:

$$\mathbb{P}(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}.$$

Finally, for any two random variables X and Y, the following inequality holds:

$$|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]},$$

with equality if and only if $X = \alpha Y$ for some constant $\alpha \in \mathbb{R}$. This result is known as the *Cauchy-Schwarz inequality* (see [Gut06, Thm. 3.1]).

In probability theory, several types of convergence of random variables are considered. Unless otherwise specified, we assume that all random variables are defined on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We say that a sequence of random variables $(X_n)_{n=1}^{\infty}$ *converges almost surely* (a.s.) to a random variable X if:

$$\mathbb{P}\left(\left\{\omega \in \Omega : X(\omega) = \lim_{n\to\infty} X_n(\omega)\right\}\right) = 1.$$

We denote this by $X_n \xrightarrow{\text{a.s.}} X$. Note that this definition is extendable to any metric space, not just $\mathbb{R}$. Such a limit is almost surely unique. We can observe that the above definition can be generalized to the case of a sequence of random elements mapping into a metric space equipped with the Borel σ-field. We say that a sequence of random variables $(X_n)_{n=1}^{\infty}$ *converges in probability* to a random variable X if for every $\varepsilon > 0$ it holds that:

$$\lim_{n\to\infty} \mathbb{P}\left(\{|X_n - X| \geq \varepsilon\}\right) = 0.$$

We denote this by $X_n \xrightarrow{\mathbb{P}} X$. Such a limit is also almost surely unique. Let $1 \leq p < \infty$ and suppose that X_n and X have finite p-th moments. We say that a sequence $(X_n)_{n=1}^{\infty}$ *converges in the L^p norm* to X if:

$$\lim_{n\to\infty} \mathbb{E}\left(|X_n - X|^p\right) = 0.$$

We denote this by $X_n \xrightarrow{L^p} X$. Finally, a sequence $(X_n)_{n=1}^{\infty}$ of real-valued random variables, with cumulative distribution functions $(F_n)_{n=1}^{\infty}$, is said to *converge in distribution* to a random variable X with cumulative distribution function F if:

$$\lim_{n\to\infty} F_n(x) = F(x),$$

for every number $x \in \mathbb{R}$ at which F is continuous. We denote this by $X_n \xrightarrow{\mathscr{D}} X$. Finally, let $\mu, \mu_1, \mu_2, \ldots$ be probability measures on $\mathbb{R}^d$. We say that the sequence $(\mu_n)_{n=1}^{\infty}$ *converges weakly* to μ if the following holds:

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} g \, \mathrm{d}\mu_n = \int_{\mathbb{R}^d} g \, \mathrm{d}\mu$$

for all continuous and bounded functions $g : \mathbb{R}^d \to \mathbb{R}$. When these measures are generated by random variables, this concept of weak convergence is equivalent to convergence in distribution (see [Sar02, Thm 13. 13]). We denote it with $\mu_n \Rightarrow \mu$, or $X_n \Rightarrow X$ if we have the associated random variables/vectors. We have the following relationship between these types of convergence (see [Sar02, Thm. 10.12, Prop 10.21]).

Theorem 1.3.2. The following implications hold:

(i) $X_n \xrightarrow{\text{a.s.}} X \Rightarrow X_n \xrightarrow{\mathbb{P}} X$.

(ii) $X_n \xrightarrow{L^p} X \Rightarrow X_n \xrightarrow{\mathbb{P}} X$ for $1 \leq p < \infty$.

(iii) $X_n \xrightarrow{\mathbb{P}} X \Rightarrow X_n \xrightarrow{\mathscr{D}} X$, where the implication is an equivalence if X is a constant almost surely.

If we have a sequence of random variables $(X_n)_{n=1}^{\infty}$ that converges almost surely to a variable X, under certain conditions, we can also achieve convergence of their corresponding expectations. The *Monotone Convergence Theorem* (see [Dur19, Thm. 1.5.7]) states that if $X_n \geq 0$ are random variables and $X_n \uparrow X$, then

$$\mathbb{E}[X_n] \uparrow \mathbb{E}[X].$$

The *Dominated Convergence Theorem* (see [Dur19, Thm. 1.5.8]) states that if $X_n \to X$ almost surely, $|X_n| \leq Y$ for all n, and Y is integrable, then

$$\mathbb{E}[X_n] \to \mathbb{E}[X].$$

Lastly, consider sequences of random variables $(X_n)_{n=1}^{\infty}$, $(Y_n)_{n=1}^{\infty}$, and $(Z_n)_{n=1}^{\infty}$ that are $\mathbb{P}$-integrable, and suppose that we have almost sure convergence:

$$X := \lim_{n\to\infty} X_n, \quad Y := \lim_{n\to\infty} Y_n, \quad Z := \lim_{n\to\infty} Z_n.$$

Assume that Y and Z are integrable random variables. Additionally, suppose that for all $\omega \in \Omega$ and $n \in \mathbb{N}$:

$$Y_n(\omega) \leq X_n(\omega) \leq Z_n(\omega).$$

Finally, if the following conditions hold:

$$\lim_{n\to\infty} \mathbb{E}[Y_n] = \mathbb{E}[Y] \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}[Z_n] = \mathbb{E}[Z],$$

then:

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X],$$

and the latter expectation is finite. This result is known as *Pratt's lemma* (see [Gut06, Thm. 5.5]). Another important convergence result that we will use is *Slutsky's theorem* (see [Gut06, Thm. 11.4]). This theorem does not connect different types of convergence directly but allows us to consider sums of two random variables and pass the sum through the limit in a specific way.

Theorem 1.3.3 (Slutsky)**.** Let $(Z_n)_{n=1}^{\infty}$ and $(Y_n)_{n=1}^{\infty}$ be sequences of random variables such that $Z_n \xrightarrow{\mathscr{D}} Z$ as $n \to \infty$ and $Y_n \xrightarrow{\mathbb{P}} c$ as $n \to \infty$ for some constant c. Then $Z_n + Y_n \xrightarrow{\mathscr{D}} Z + c$ as $n \to \infty$.

A random walk is defined as the sequence of partial sums of a series of random variables, $(Z_n)_{n=1}^{\infty}$, which are typically assumed to be independent and identically distributed. These random variables are referred to as increments in this context. Below is the formal definition we will use.

Definition 1.3.4 (Random walk)**.** Let $d \in \mathbb{N}$, and suppose that Z and $(Z_n)_{n=1}^{\infty}$ are i.i.d. random vectors in $\mathbb{R}^d$. A *random walk* $(S_n)_{n=0}^{\infty}$ is the sequence of partial sums $S_n := \sum_{i=1}^{n} Z_i$ with $S_0 := \mathbf{0}$.

If we assume that $\mathbb{E}\|Z\| < \infty$, we denote the drift vector of the walk by $\mathbb{E}Z = \boldsymbol{\mu}$, and we will represent this assumption with $(\mathbf{W}(\boldsymbol{\mu}))$. If we additionally require that the increments have a finite second moment, $\mathbb{E}\|Z\|^2 < \infty$, we denote the covariance matrix of the increment Z by Σ, and we will represent this assumption with $(\mathbf{W}(\boldsymbol{\mu}, \Sigma))$.

We will frequently use several classical results of random walks. One fundamental result is *Kolmogorov's Strong Law of Large Numbers* (see [Gut06, Thm. 7.1]), which asserts that the average of the steps in a random walk converges almost surely to the drift vector.

Theorem 1.3.5 (Strong Law of Large Numbers)**.** For a random walk characterized by $(\mathbf{W}(\boldsymbol{\mu}))$, we have:

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} \boldsymbol{\mu}.$$

Additionally, if $f : \mathbb{R}^d \to (\mathscr{M}, d)$ is a continuous mapping from $\mathbb{R}^d$ to a metric space $(\mathscr{M}, d)$, then:

$$f\left(\frac{S_n}{n}\right) \xrightarrow{\text{a.s.}} f(\boldsymbol{\mu}).$$

We denote by $Z \sim \mathscr{N}\left(\mu, \sigma^2\right)$ the *normal distribution* with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. This means Z is a random variable on $\mathbb{R}$ with the probability density function:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

In this context, $\mathbb{E}[Z] = \mu$ and $\mathrm{Var}(Z) = \sigma^2$. The standard normal distribution, denoted by ζ, is the special case where $\mu = 0$ and $\sigma^2 = 1$. The *multivariate normal distribution* in d dimensions is denoted by $Z \sim \mathscr{N}_d(\mu, \Sigma)$, where μ is the d-dimensional drift vector, and Σ is the covariance matrix. When the determinant of the covariance matrix $\det(\Sigma) > 0$, the normal distribution is defined by the density function:

$$f(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right),$$

where x is also a d-dimensional vector. The standard d-dimensional normal random variable has a covariance matrix I_d, the d-dimensional identity matrix, and a mean vector of 0. In the degenerate case where Σ is a $d \times d$ matrix of zeros, the point mass at μ defines the multivariate normal distribution.

Under the condition $(\mathbf{W}(\boldsymbol{\mu}, \Sigma))$, the central limit theorem provides a quantification of the error size as the average step converges (see [Dur19, Thm. 3.4.1]).

Theorem 1.3.6 (Lévy Central Limit Theorem)**.** Given the random walk as defined in Definition 1.3.4 with condition $(\mathbf{W}(\boldsymbol{\mu}, \Sigma))$, the central limit theorem states that

$$\frac{1}{\sqrt{n}}(S_n - n\boldsymbol{\mu}) \xrightarrow{\mathscr{D}} Z,$$

where $Z \sim \mathscr{N}(0, \Sigma)$ is a d-dimensional normal random variable.

The following corollary provides the Central Limit Theorem for the case where we consider functions of random variables. The proof of this theorem can be found in [Dur19, Theorem 3.2.10].

Corollary 1.3.7 (Continuous Mapping Theorem)**.** Let f be a measurable function and let $D_f = \{x : f \text{ is discontinuous at } x\}$. If $X_n \xrightarrow{\mathscr{D}} X$ and $\mathbb{P}(X \in D_f) = 0$, then $f(X_n) \xrightarrow{\mathscr{D}} f(X)$. Additionally, if f is bounded, then $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$.

The following version of the Central Limit Theorem covers the case when the increments in the random walk are not identically distributed (see [Dur19, Thm. 3.4.10]).

Theorem 1.3.8 (Lindeberg-Feller Central Limit Theorem)**.** Consider a sequence of independent random variables $X_{n,m}$ for each n, where $1 \leq m \leq n$ and $\mathbb{E}[X_{n,m}] = 0$. Assume the following conditions hold:

(i) The sum of the variances converges to a positive constant:

$$\sum_{m=1}^{n} \mathbb{E}\left(X_{n,m}^2\right) \to \sigma^2 > 0.$$

(ii) For any $\varepsilon > 0$, the sum of the expected values of squared variables, conditioned on exceeding ε, approaches zero:

$$\lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{E}\left(|X_{n,m}|^2 \cdot \mathbf{1}(\{|X_{n,m}| > \varepsilon\})\right) = 0.$$

Under these conditions, the sum $S_n = X_{n,1} + \cdots + X_{n,n}$ converges in distribution to $\sigma\chi$ as $n \to \infty$, where $\chi \sim \mathscr{N}(0,1)$.

Given are a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a σ-field $\mathscr{F}_0 \subset \mathscr{F}$, and a random variable $X \in \mathscr{F}$ with $\mathbb{E}|X| < \infty$. We define the *conditional expectation* of X given $\mathscr{F}_0$, noted as $\mathbb{E}[X \mid \mathscr{F}_0]$, to be any random variable Y that has

(i) $Y \in \mathscr{F}_0$, i.e., is $\mathscr{F}_0$ measurable

(ii) for all $A \in \mathscr{F}_0$, $\int_A X \mathrm{d}\mathbb{P} = \int_A Y \mathrm{d}\mathbb{P}$

Any Y satisfying (i) and (ii) is said to be a version of $\mathbb{E}[X \mid \mathscr{F}_0]$. The conditional expectation exists and is almost certainly unique (see [Dur19, Section 4.1]). Let $\mathbb{F} = (\mathscr{F}_n)_{n=0}^{\infty}$ be a *filtration*, which is an increasing sequence of σ-fields. A sequence $(X_n)_{n=0}^{\infty}$ is considered *adapted* to $\mathbb{F}$ if each X_n is measurable with respect to $\mathscr{F}_n$ for all $n \geq 0$. If the sequence $(X_n)_{n=0}^{\infty}$ satisfies the following conditions:

(i) $\mathbb{E}\,|X_n| < \infty$,

(ii) $(X_n)_{n=0}^{\infty}$ is adapted to $\mathbb{F}$,

(iii) $\mathbb{E}\,[X_{n+1} \mid \mathscr{F}_n] = X_n$ for all $n \geq 0$,

then X is called a *martingale* with respect to the filtration $\mathbb{F}$. Consider an adapted sequence $(Y_n)_{n=0}^{\infty}$ on a filtered space $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$. The sequence $(Y_n)_{n=0}^{\infty}$ is called a *martingale difference sequence* if it meets the following criteria for all n:

(i) $\mathbb{E}\,|Y_n| < \infty$, and

(ii) $\mathbb{E}\,[Y_n \mid \mathscr{F}_{n-1}] = 0$, almost surely.

By definition, if X_n is a martingale, then the sequence defined by $Y_n = X_n - X_{n-1}$ will form a martingale difference sequence. This explains the terminology used. A *one-dimensional Brownian motion* is a real-valued process $(B_t)_{t=0}^{\infty}$ that possesses the following properties:

(i) For any sequence of times $t_0 < t_1 < \ldots < t_n$, the random variables B_{t_0}, $B_{t_1} - B_{t_0}$, ..., $B_{t_n} - B_{t_{n-1}}$ are independent.

(ii) For any $s, t \geq 0$, the distribution of $B(s+t) - B(s)$ is given by

$$\mathbb{P}(B_{s+t} - B_s \in A) = \int_A \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \mathrm{d}x,$$

for any Borel set A.

(iii) The function $t \mapsto B_t$ is continuous with probability one.

The final theorem we will present is Donsker's theorem, which is the functional equivalent of the Central Limit Theorem. Let

$$S(u) = \begin{cases} S_k & \text{if } u = k \in \mathbb{N} \\ \text{linear on } [k, k+1] & \text{for } k \in \mathbb{N} \end{cases}$$

Essentially, it states that a properly scaled random walk behaves like Brownian motion in the limit. The proof of this statement can be found in [Dur19, Thm. 8.1.4].

Theorem 1.3.9 (Donsker)**.** Let $X_1, X_2, \ldots$ be i.i.d. with a distribution F, which has mean 0 and variance 1, and let $S_n = X_1 + \cdots + X_n$. Then:

$$\frac{S(n\cdot)}{\sqrt{n}} \Rightarrow B_{\cdot},$$

i.e., the associated measures on $C[0,1]$ (endowed with the uniform norm topology) converge weakly.

2. FIRST-ORDER CONVERGENCE

This chapter examines the geometric properties of sets of points created by random walks, focusing on understanding first-order convergence. Knowing these properties helps us better understand how different systems that use random walks behave. Afterward, using the continuity of intrinsic volume operators, we show the law of large numbers for these processes.

2.1. SETTING OF THE PROBLEM

Let $(Z_i^{(k)})_{i=1}^{\infty}$ be m sequences of independent and identically distributed planar random vectors, which are mutually independent, but not necessarily identically distributed, for $k = 1, \ldots, m$. Let also $(S_n^{(k)})_{n=0}^{\infty}$ be the corresponding random walks:

$$S_0^{(k)} := 0, \qquad S_n^{(k)} := \sum_{i=1}^{n} Z_i^{(k)}, \qquad k = 1, \ldots, m.$$

The main objects we focus on in this thesis are the perimeter process

$$L_n := \operatorname{Per}\left(\operatorname{chull}\left\{S_j^{(k)} : 0 \le j \le n,\ k = 1, \ldots, m\right\}\right), \quad n \ge 0,$$

and the diameter process

$$D_n := \operatorname{diam}\left(\operatorname{chull}\left\{S_j^{(k)} : 0 \le j \le n,\ k = 1, \ldots, m\right\}\right), \quad n \ge 0.$$

Here, $\operatorname{Per}(A)$, $\operatorname{diam}(A)$, and $\operatorname{chull}(A)$ stand, respectively, for the perimeter, the diameter, and the convex hull of the set $A \subseteq \mathbb{R}^2$. Notice that the set $\operatorname{chull}\{S_j^{(k)} : 0 \le j \le n,\ k = 1, \ldots, m\}$ is a.s. a polygon.

2.2. LAW OF LARGE NUMBERS

Our first main result is the strong law of large numbers for the set $\mathrm{chull}\{S_j^{(k)} : 0 \leq j \leq n,\ k = 1,\ldots,m\}$. Assuming that sequences $(Z_i^{(k)})_{i=1}^{\infty}$, $k \in \{1,\ldots,m\}$, have finite first moment, and denoting the drift vector of the k-th random walk by $\boldsymbol{\mu}^{(k)} = \mathbb{E}[Z_1^{(k)}]$, we have the following result.

Theorem 2.2.1. In the metric space of convex and compact planar sets endowed with the Hausdorff metric, it holds that

$$n^{-1}\,\mathrm{chull}\left\{S_j^{(k)} : 0 \leq j \leq n,\ k = 1,\ldots,m\right\} \xrightarrow[n\to\infty]{a.s.} \mathrm{chull}\{\mathbf{0},\boldsymbol{\mu}^{(1)},\ldots,\boldsymbol{\mu}^{(m)}\}.$$

Due to continuity of the perimeter and the diameter functionals (see [LMW18, Lemma 5.7 and Lemma 6.7]), from Theorem 2.2.1 it immediately follows that the processes $(L_n)_{n=0}^{\infty}$ and $(D_n)_{n=0}^{\infty}$ converge a.s. to the perimeter, respectively, the diameter of the (possibly degenerate) polygon spanned by $0,\boldsymbol{\mu}^{(1)},\ldots,\boldsymbol{\mu}^{(m)}$, that is,

$$\frac{L_n}{n} \xrightarrow[n\to\infty]{a.s.} \mathrm{Per}\left(\mathrm{chull}\{0,\boldsymbol{\mu}^{(1)},\ldots,\boldsymbol{\mu}^{(m)}\}\right), \tag{2.1}$$

and,

$$\frac{D_n}{n} \xrightarrow[n\to\infty]{a.s.} \mathrm{diam}\left(\mathrm{chull}\{0,\boldsymbol{\mu}^{(1)},\ldots,\boldsymbol{\mu}^{(m)}\}\right). \tag{2.2}$$

Proof of Theorem 2.2.1. Observe first that for sequences $(A_n^{(k)})_{n=0}^{\infty}$, $k \in \{1,\ldots,m\}$, of (closed) subsets in $\mathbb{R}^d$ that converge, respectively, to (closed) sets $A^{(k)}$ (for $k = 1,\ldots,m$), their union converges to the union of the limiting subsets (with respect to ρ_H^d). Namely, it is sufficient to prove that

$$\rho_H^d\left(\bigcup_{k=1}^{m} A_n^{(k)}, \bigcup_{k=1}^{m} A^{(k)}\right) \leq \max_{k\in\{1,\ldots,m\}} \rho_H^d\left(A_n^{(k)},A^{(k)}\right). \tag{2.3}$$

Let

$$\varepsilon = \max_{k\in\{1,\ldots,m\}} \rho_H^d(A_n^{(k)},A^{(k)}).$$

From the definition of the Hausdorff metric, we have that $A_n^{(k)} \subseteq (A^{(k)})^{\varepsilon}$ and $A^{(k)} \subseteq (A_n^{(k)})^{\varepsilon}$

holds for all $k \in \{1, \ldots, m\}$. Consequently,

$$\begin{aligned}\left(\bigcup_{k=1}^{m} A^{(k)}\right)^{\varepsilon} &= (A^{(1)})^{\varepsilon} \cup (A^{(2)})^{\varepsilon} \cup \ldots \cup (A^{(m)})^{\varepsilon} \\ &\supseteq A_n^{(1)} \cup A_n^{(2)} \cup \ldots \cup A_n^{(m)} \\ &= \bigcup_{k=1}^{m} A_n^{(k)}.\end{aligned}$$

Analogously, we deduce

$$\left(\bigcup_{k=1}^{m} A_n^{(k)}\right)^{\varepsilon} \supseteq \bigcup_{k=1}^{m} A^{(k)}.$$

Thus,

$$\rho_H^d\left(\bigcup_{k=1}^{m} A_n^{(k)}, \bigcup_{k=1}^{m} A^{(k)}\right) \leq \varepsilon,$$

which thereby proves (2.3). Next, in [LMW18, Theorem 5.4] it is established that for a single random walk $(S_n)_{n=0}^{\infty}$ with drift $\boldsymbol{\mu}$,

$$n^{-1}\{S_0, S_1, \ldots, S_n\} \xrightarrow[n\to\infty]{a.s.} \{t\boldsymbol{\mu} : t \in [0,1]\}.$$

Hence,

$$n^{-1}\left\{S_j^{(k)} :\ 0 \leq j \leq n\right\} \xrightarrow[n\to\infty]{a.s.} \left\{t\boldsymbol{\mu}^{(k)} :\ t \in [0,1]\right\},$$

for $k \in \{1, \ldots, m\}$, and, by applying (2.3), we establish

$$n^{-1}\left\{S_j^{(k)} :\ 0 \leq j \leq n,\ k = 1, \ldots, m\right\} \xrightarrow[n\to\infty]{a.s.} \left\{t\boldsymbol{\mu}^{(k)} :\ t \in [0,1],\ k = 1, \ldots, m\right\}.$$

Finally, using [LMW18, Lemma 6.1] the result follows. □

Recall that intrinsic volumes $V_1, \ldots, V_d$ are the classical geometric functionals of d-dimensional convex and compact sets. It is known that V_1 is proportional to the mean width of the set, V_{d-1} equals one-half of the surface area of the set, while V_d is the volume of the set. Furthermore, recall that all these functionals are continuous mappings (with respect to the Hausdorff metric), and the ℓ-th intrinsic volume V_ℓ is homogeneous of degree ℓ, that is, $V_\ell(cA) = c^\ell V_\ell(A)$, for any $c \geq 0$. As a consequence of Theorem 2.2.1, we now conclude

$$\frac{V_\ell\left(\text{chull}\left\{S_j^{(k)} :\ 0 \leq j \leq n,\ k = 1,\dots,m\right\}\right)}{n^\ell} \xrightarrow[n\to\infty]{a.s.} V_\ell\left(\text{chull}\left\{\{0\} \cup \left\{\mu^{(k)} : k = 1,\dots,m\right\}\right\}\right).$$

Observe that the above limit is non-trivial if, and only if, there are at least ℓ linearly independent vectors in the set $\{\mu^{(k)} : k = 1,\dots,m\}$. From this, we conclude that in the planar case, we cannot expect a non-trivial limit for the area functional of the convex hull of a single random walk under n^2 scaling.

3. Perimeter process

In this chapter, we explore the limiting behavior of the perimeter process. Our proofs use martingale difference sequences and the Cauchy formula for the perimeter. To simplify the technical details, we focus only on the case of two random walks ($m = 2$). However, we will discuss later how the results change if we generalize this to any number of random walks.

We now introduce some notation that will be used throughout this and the following chapter. For $\theta \in [0, 2\pi)$, we let $\mathbf{e}_\theta = (\cos\theta, \sin\theta)$ be the unit vector pointing in the direction corresponding to this angle. When the sequences $(Z_i^{(k)})_{i=1}^{\infty}$, $k \in \{1, 2\}$, have finite second moment, the associated covariance matrices are denoted by $\mathbf{\Sigma}^{(k)} = \mathbb{E}[(Z_1^{(k)} - \boldsymbol{\mu}^{(k)})(Z_1^{(k)} - \boldsymbol{\mu}^{(k)})^T]$. Expressing drift vectors $\boldsymbol{\mu}^{(k)}$, $k \in \{1, 2\}$, in polar coordinates, we have

$$\boldsymbol{\mu}^{(k)} = \mu^{(k)} \mathbf{e}_{\theta^{(k)}},$$

where $\theta^{(k)} \in [0, 2\pi)$ represents the angle between the drift vector and the positive part of the x-axis, and $\mu^{(k)} \geq 0$ stands for the length of the vector $\boldsymbol{\mu}^{(k)}$. Let $\theta^{(0)} \in [0, 2\pi)$ be an angle satisfying the condition

$$\boldsymbol{\mu}^{(1)} \cdot \mathbf{e}_{\theta^{(0)}} = \boldsymbol{\mu}^{(2)} \cdot \mathbf{e}_{\theta^{(0)}}.$$

In an intuitive sense, $\theta^{(0)}$ is the direction along which the projections of the drift vectors are equal. We also define $\mathbf{e}^{\perp}_{\theta^{(0)}}$, the unit vector perpendicular to this common projection line, subject to the constraint that $\mathbf{e}^{\perp}_{\theta^{(0)}} \cdot \mathbf{e}_{\theta^{(1)}} \geq 0$.

Before stating our remaining main results, we introduce and discuss an assumption that we impose on the drift vectors $\boldsymbol{\mu}^{(1)}$ and $\boldsymbol{\mu}^{(2)}$:

$$\mathbf{0} \notin \{\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}\}. \tag{A1}$$

In the case of a single planar zero-drift random walk, in [WX15b] it has been shown that the process $(L_n)_{n=0}^{\infty}$ (with the classical central limit theorem centering and scaling) has a non-Gaussian distributional limit. Analogously, in the case of two independent planar random walks, we conjecture that if the assumption (A1) is not satisfied, we can again expect a non-Gaussian distributional limit.

3.1. Martingale difference sequence

Let $\mathscr{F}_0 = \{\emptyset, \Omega\}$, and

$$\mathscr{F}_n = \sigma\left(S_j^{(k)} : 0 \le j \le n,\ k = 1,2\right), \qquad n \ge 1,$$

be the information about both random walks up to time n. Further, let $(\tilde{Z}_i^{(1)})_{i=1}^{\infty}$ and $(\tilde{Z}_i^{(2)})_{i=1}^{\infty}$ be independent copies of $(Z_i^{(1)})_{i=1}^{\infty}$ and $(Z_i^{(2)})_{i=1}^{\infty}$, which are also mutually independent. For a fixed $i \ge 1$ the resampled random walk at time i is defined by

$$S_j^{(k,i)} := \begin{cases} S_j^{(k)}, & j < i, \\ S_j^{(k)} - Z_i^{(k)} + \tilde{Z}_i^{(k)}, & j \ge i. \end{cases} \tag{3.1}$$

The corresponding perimeter processes are given as before,

$$L_n^{(i)} := \operatorname{Per}\left(\operatorname{chull}\left\{S_j^{(k,i)} : 0 \le j \le n,\ k = 1,2\right\}\right).$$

In the following lemma we show that

$$\mathscr{L}_{n,i} := \mathbb{E}\left[L_n - L_n^{(i)} \mid \mathscr{F}_i\right], \qquad 1 \le i \le n,$$

is a martingale difference sequence (see [BW21, p. 124]).

Lemma 3.1.1. Let $n \in \mathbb{N}$. Then,

(i) $L_n - \mathbb{E}[L_n] = \sum_{i=1}^{n} \mathscr{L}_{n,i}$,

(ii) $\operatorname{Var}[L_n] = \sum_{i=1}^{n} \mathbb{E}\left[\mathscr{L}_{n,i}^2\right]$, whenever the latter sum is finite.

Proof. Observe that $L_n^{(i)}$ is independent of $Z_i^{(k)}$ for both $k \in \{1,2\}$, so that

$$\mathbb{E}[L_n^{(i)} \mid \mathscr{F}_i] = \mathbb{E}[L_n^{(i)} \mid \mathscr{F}_{i-1}] = \mathbb{E}[L_n \mid \mathscr{F}_{i-1}].$$

Hence, $\mathscr{L}_{n,i}$ can be expressed as

$$\mathscr{L}_{n,i} = \mathbb{E}[L_n \mid \mathscr{F}_i] - \mathbb{E}[L_n \mid \mathscr{F}_{i-1}].$$

Summing over $1 \le i \le n$, we conclude $\sum_{i=1}^{n} \mathscr{L}_{n,i} = \mathbb{E}[L_n \mid \mathscr{F}_n] - \mathbb{E}[L_n \mid \mathscr{F}_0] = L_n - \mathbb{E}[L_n]$, which gives (i). The claim in (ii) follows from the martingale difference property of the sequence $(\mathscr{L}_{n,i})_{i=1}^{n}$. □

3.2. CAUCHY FORMULA FOR PERIMETER

One of the most important contributions to convex analysis is the Cauchy formula for the perimeter (Corollary 1.2.6). For $\theta \in [0, \pi]$, let us define

$$M_n(\theta) := \max_{\substack{0 \le j \le n \\ k=1,2}} \left(S_j^{(k)} \cdot \mathbf{e}_\theta \right), \qquad m_n(\theta) := \min_{\substack{0 \le j \le n \\ k=1,2}} \left(S_j^{(k)} \cdot \mathbf{e}_\theta \right).$$

For a given angle θ, the terms $M_n(\theta)$ and $m_n(\theta)$ denote the maximal and minimal projections, respectively, of the convex hull onto a line passing through the origin and directed by the unit vector $\mathbf{e}_\theta$. Since $S_0^{(k)} = 0$, it is clear that $M_n(\theta) \ge 0$ and $m_n(\theta) \le 0$ a.s. The Cauchy formula expresses the perimeter of the convex set in terms of $M_n(\theta)$ and $m_n(\theta)$:

$$L_n = \int_0^\pi (M_n(\theta) - m_n(\theta))\, \mathrm{d}\theta = \int_0^\pi R_n(\theta) \mathrm{d}\theta,$$

where $R_n(\theta) := M_n(\theta) - m_n(\theta) \ge 0$ is called the *parametrized range function*. Notice that the Cauchy formula for the perimeter can be equivalently stated as

$$L_n = \int_0^{2\pi} M_n(\theta) \mathrm{d}\theta. \tag{3.2}$$

We similarly have that

$$L_n^{(i)} = \int_0^\pi \left(M_n^{(i)}(\theta) - m_n^{(i)}(\theta) \right) \mathrm{d}\theta = \int_0^\pi R_n^{(i)}(\theta) \mathrm{d}\theta$$

with $R_n^{(i)}(\theta) = M_n^{(i)}(\theta) - m_n^{(i)}(\theta)$ and

$$M_n^{(i)}(\theta) := \max_{\substack{0 \le j \le n \\ k=1,2}} \left(S_j^{(k,i)} \cdot \mathbf{e}_\theta \right), \qquad m_n^{(i)}(\theta) := \min_{\substack{0 \le j \le n \\ k=1,2}} \left(S_j^{(k,i)} \cdot \mathbf{e}_\theta \right).$$

We consider the following difference

$$L_n - L_n^{(i)} = \int_0^\pi \left(R_n(\theta) - R_n^{(i)}(\theta) \right) \mathrm{d}\theta = \int_0^\pi \Delta_n^{(i)}(\theta) \mathrm{d}\theta,$$

where $\Delta_n^{(i)}(\theta) := R_n(\theta) - R_n^{(i)}(\theta)$.

We define two random variables for an angle $\theta \in [0, \pi]$. The first random variable represents the last time at which the minimal projections of both the first and the second random walk are achieved. Conversely, the second random variable denotes the first time at which the maximal projections of both random walks are attained. Formally:

$$\underline{J}_{n,k}(\theta) := \max \left\{ \arg\min_{0 \le j \le n} \left(S_j^{(k)} \cdot \mathbf{e}_\theta \right) \right\},$$

and,

$$\overline{J}_{n,k}(\theta) := \min \left\{ \arg\max_{0 \le j \le n} \left(S_j^{(k)} \cdot \mathbf{e}_\theta \right) \right\}.$$

Notice that we record these time instances for each walk individually. For the resampled walks, we analogously define variables $\underline{J}_{n,k}^{(i)}(\theta)$ and $\overline{J}_{n,k}^{(i)}(\theta)$. We further introduce the random variables $\underline{\mathscr{I}}_n(\theta)$ and $\overline{\mathscr{I}}_n(\theta)$, which denote the indices of the random walks ($k = 1$, or $k = 2$) where the minimum and maximum projections are reached, respectively. In the event of a tie, the default choice is $k = 1$. Analogously, we define the variables $\underline{\mathscr{I}}_n^{(i)}(\theta)$ and $\overline{\mathscr{I}}_n^{(i)}(\theta)$. Throughout the subsequent proofs, we frequently require that the variable $\Delta_n^{(i)}(\theta)$ is dominated by an integrable random variable.

Lemma 3.2.1. For any $1 \le i \le n$, we have that

$$\sup_{\theta \in [0,\pi]} |\Delta_n^{(i)}(\theta)| \le 2 \left(\|Z_i^{(1)}\| + \|\tilde{Z}_i^{(1)}\| + \|Z_i^{(2)}\| + \|\tilde{Z}_i^{(2)}\| \right).$$

Proof. Take an arbitrary $\theta \in [0, \pi]$. By definition, we have that

$$M_n(\theta) = S_{\overline{J}_{n,\overline{\mathscr{I}}_n}(\theta)}^{(\overline{\mathscr{I}}_n)} \cdot \mathbf{e}_\theta.$$

Thus,

$$M_n^{(i)}(\theta) \ge S_{\overline{J}_{n,\overline{\mathscr{I}}_n}(\theta)}^{(\overline{\mathscr{I}}_n, i)} \cdot \mathbf{e}_\theta.$$

If $\overline{J}_{n,\overline{\mathscr{I}}_n}(\theta) < i$, then $S_{\overline{J}_{n,\overline{\mathscr{I}}_n}(\theta)}^{(\overline{\mathscr{I}}_n, i)} = S_{\overline{J}_{n,\overline{\mathscr{I}}_n}(\theta)}^{(\overline{\mathscr{I}}_n)}$, so $M_n^{(i)}(\theta) \ge M_n(\theta)$. On the other hand, if

$\overline{J}_{n,\overline{\mathscr{I}}_n}(\theta) \geq i$, we have that

$$S^{(\overline{\mathscr{I}}_n,i)}_{\overline{J}_{n,\overline{\mathscr{I}}_n}(\theta)} = S^{(\overline{\mathscr{I}}_n)}_{\overline{J}_{n,\overline{\mathscr{I}}_n}(\theta)} - (Z^{(\overline{\mathscr{I}}_n)}_i - \tilde{Z}^{(\overline{\mathscr{I}}_n)}_i),$$

so taking a projection in the direction of θ gives us

$$\begin{aligned} M^{(i)}_n(\theta) &\geq S^{(\overline{\mathscr{I}}_n)}_{\overline{J}_{n,\overline{\mathscr{I}}_n}(\theta)} \cdot \mathbf{e}_\theta - (Z^{(\overline{\mathscr{I}}_n)}_i - \tilde{Z}^{(\overline{\mathscr{I}}_n)}_i) \cdot \mathbf{e}_\theta \\ &\geq M_n(\theta) - \left(\|Z^{(1)}_i\| + \|\tilde{Z}^{(1)}_i\| + \|Z^{(2)}_i\| + \|\tilde{Z}^{(2)}_i\| \right). \end{aligned}$$

In both cases, we have the lower bound on $M^{(i)}_n(\theta)$ as follows

$$M^{(i)}_n(\theta) \geq M_n(\theta) - \left(\|Z^{(1)}_i\| + \|\tilde{Z}^{(1)}_i\| + \|Z^{(2)}_i\| + \|\tilde{Z}^{(2)}_i\| \right).$$

Similar arguments can be applied when the original and resampled maximal projections are interchanged, thereby demonstrating that

$$|M^{(i)}_n(\theta) - M_n(\theta)| \leq \|Z^{(1)}_i\| + \|\tilde{Z}^{(1)}_i\| + \|Z^{(2)}_i\| + \|\tilde{Z}^{(2)}_i\|.$$

The same approach can be employed to establish an analogous upper bound on $|m^{(i)}_n(\theta) - m_n(\theta)|$. With this, the assertion of the lemma is verified. □

Before moving on, we show that the convergence in the strong law of large numbers for the perimeter process, presented in (2.1) and (2.2), also holds in L^1 sense.

Corollary 3.2.2. Under the assumptions of Theorem 2.2.1, we have

$$\frac{L_n}{n} \xrightarrow[n\to\infty]{L^1} \operatorname{Per}\left(\operatorname{chull}\{\mathbf{0}, \boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}\} \right).$$

Proof. Using the Cauchy formula from (3.2) we have

$$\begin{aligned} L_n = \int_0^{2\pi} M_n(\theta) d\theta &\leq 2\pi \max_{\substack{0\leq j\leq n \\ k=1,2}} \|S^{(k)}_j\| \\ &\leq 2\pi \max_{k=1,2} \sum_{j=0}^{n} \|Z^{(k)}_j\| \leq 2\pi \sum_{j=0}^{n} \left(\|Z^{(1)}_j\| + \|Z^{(2)}_j\| \right). \end{aligned}$$

Since $(Z_j^{(1)})_{j=1}^{\infty}$ and $(Z_j^{(2)})_{j=1}^{\infty}$ are sequences of i.i.d. random variables, from strong law we have that a.s.,

$$n^{-1}\sum_{j=1}^{n}(\|Z_j^{(1)}\|+\|Z_j^{(2)}\|)\to\mathbb{E}[\|Z_1^{(1)}\|+\|Z_1^{(2)}\|]<\infty,$$

and clearly

$$\mathbb{E}[n^{-1}\sum_{j=1}^{n}(\|Z_j^{(1)}\|+\|Z_j^{(2)}\|)]=\mathbb{E}[\|Z_1^{(1)}\|+\|Z_1^{(2)}\|].$$

Hence, Pratt's lemma implies the claim. □

3.3. Control of extrema

To make the geometric analysis of the problem a little bit more convenient, we may restrict our attention to $\theta^{(1)}, \theta^{(2)} \in [0, \pi]$ such that the projections of their corresponding drift vectors onto the y-axis are equal. This simplification is justifiable due to the geometric properties of the convex hull, which remain unchanged under rotation and reflection operations. After performing these coordinate transformations, we find that we are left with two mutually exclusive scenarios:

(i) The first drift vector lies in the first quadrant, while the second is in the second quadrant. The y-axis effectively separates the two vectors.

(ii) Both drift vectors lie in the first quadrant, with the first vector displaying a smaller angular displacement from the x-axis than the second one.

The described scenarios are illustrated in Figure 3.1. It should be emphasized that while our mathematical manipulations are made to address the first scenario, they are not restrained to it. Transitioning to the second scenario does not demand substantially altering the framework.

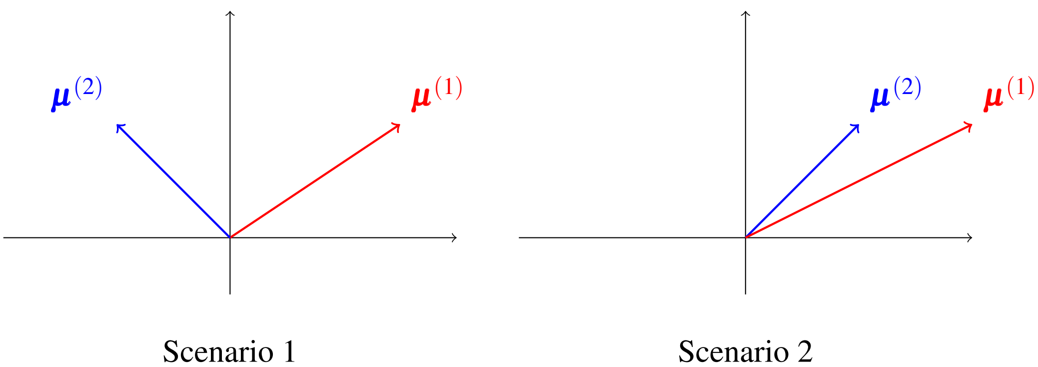


Figure 3.1: Possible positions of the drift vectors.

Observe that $(S_j^{(k)} \cdot \mathbf{e}_\theta)_{j=0}^n$, $k \in \{1, 2\}$, are one-dimensional random walks with means

$$\boldsymbol{\mu}^{(k)} \cdot \mathbf{e}_\theta = \mu^{(k)} \cos(\theta^{(k)} - \theta).$$

For an arbitrary $\varepsilon > 0$, define the following subset of angles $\theta \in [0, \pi]$:

$$\begin{aligned}
\Theta^{\varepsilon}_{(1>2>0)} &= \left\{\theta \in [0,\pi] : \boldsymbol{\mu}^{(1)} \cdot \mathbf{e}_{\theta} - \boldsymbol{\mu}^{(2)} \cdot \mathbf{e}_{\theta} > \varepsilon, \quad \boldsymbol{\mu}^{(2)} \cdot \mathbf{e}_{\theta} > \varepsilon \right\}, \\
\Theta^{\varepsilon}_{(2>1>0)} &= \left\{\theta \in [0,\pi] : \boldsymbol{\mu}^{(2)} \cdot \mathbf{e}_{\theta} - \boldsymbol{\mu}^{(1)} \cdot \mathbf{e}_{\theta} > \varepsilon, \quad \boldsymbol{\mu}^{(1)} \cdot \mathbf{e}_{\theta} > \varepsilon \right\}, \\
\Theta^{\varepsilon}_{(1>0>2)} &= \left\{\theta \in [0,\pi] : \boldsymbol{\mu}^{(1)} \cdot \mathbf{e}_{\theta} > \varepsilon, \quad \boldsymbol{\mu}^{(2)} \cdot \mathbf{e}_{\theta} < -\varepsilon \right\}, \\
\Theta^{\varepsilon}_{(2>0>1)} &= \left\{\theta \in [0,\pi] : \boldsymbol{\mu}^{(2)} \cdot \mathbf{e}_{\theta} > \varepsilon, \quad \boldsymbol{\mu}^{(1)} \cdot \mathbf{e}_{\theta} < -\varepsilon \right\}, \\
\Theta^{\varepsilon}_{(0>2>1)} &= \left\{\theta \in [0,\pi] : \boldsymbol{\mu}^{(2)} \cdot \mathbf{e}_{\theta} - \boldsymbol{\mu}^{(1)} \cdot \mathbf{e}_{\theta} > \varepsilon, \, \boldsymbol{\mu}^{(2)} \cdot \mathbf{e}_{\theta} < -\varepsilon \right\}.
\end{aligned} \tag{3.3}$$

We define these sets in order to divide our domain into segments where we have an information about the dominating drift vector, and positivity or negativity of the projections. In other words, we determine whether we contribute to the minimum or maximum of the projected line with each walk in each region. The subscripts in the set notations indicate what happens in each specific region.

For example, $\Theta^{\varepsilon}_{(1>2>0)}$ is the set of angles on which both drift vectors have a strictly positive projection (greater than some chosen $\varepsilon > 0$), and the first vector has a projection that is larger for at least ε than the projection of the second drift vector. On this set, with high probability, the first walk will contribute to the maximum, and the minimum will be achieved early enough. Similar reasoning can be applied to the rest of the subsets. The Figure 3.2 illustrates this division.

Because of the earlier discussion about rotations and reflections, we do not need to consider the set of angles in $[0, \pi]$ such that the projection of both walks have sufficiently negatively oriented drifts, and the projection of the first walk is sufficiently greater than the projection of the second walk. We write

$$\Theta^{\varepsilon}_{\mathrm{I}} := \Theta^{\varepsilon}_{(1>2>0)} \cup \Theta^{\varepsilon}_{(2>1>0)}, \qquad \Theta^{\varepsilon}_{\mathrm{II}} := \Theta^{\varepsilon}_{(1>0>2)} \cup \Theta^{\varepsilon}_{(2>0>1)}, \qquad \Theta^{\varepsilon}_{\mathrm{III}} := \Theta^{\varepsilon}_{(0>2>1)},$$

and with Θ^{ε} we denote the union of these three sets. For $\gamma \in (0, 1/2)$ and $\varepsilon > 0$ define the event $E_{n,i}(\varepsilon, \gamma)$ with the following:

- for all $\theta \in \Theta^{\varepsilon}_{\mathrm{I}}$, $\underline{J}_{n,\underline{\mathscr{I}}_n(\theta)}(\theta) < \gamma n$, $\overline{J}_{n,\overline{\mathscr{I}}_n(\theta)}(\theta) > (1-\gamma)n$, $\underline{J}^{(i)}_{n,\underline{\mathscr{I}}^{(i)}_n(\theta)}(\theta) < \gamma n$, and $\overline{J}^{(i)}_{n,\overline{\mathscr{I}}^{(i)}_n(\theta)}(\theta) > (1-\gamma)n$,

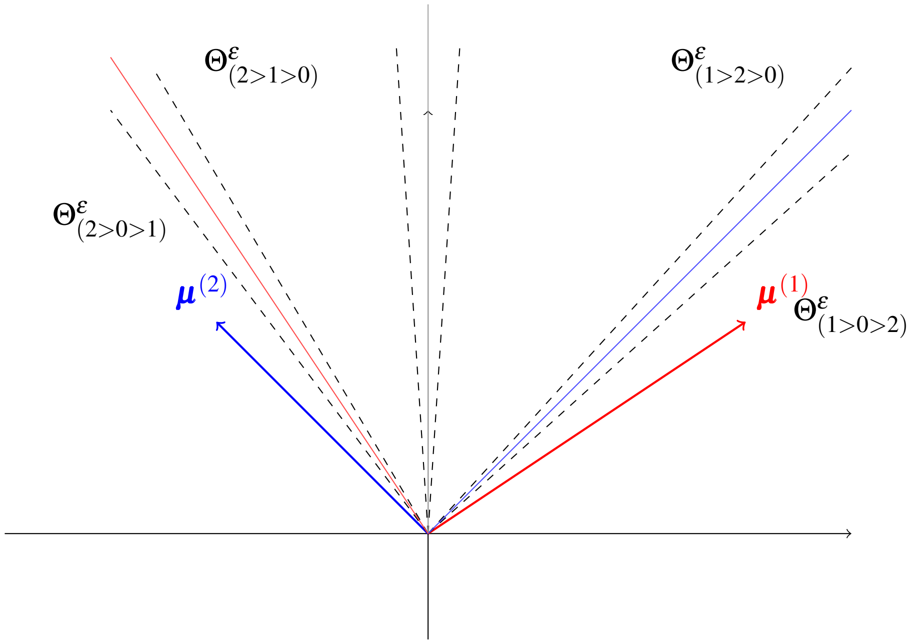


Figure 3.2: Division of angles from $[0,\pi]$.

- for all $\theta \in \Theta^{\varepsilon}_{\mathrm{II}}$, $\underline{J}_{n,\underline{\mathscr{I}}_n(\theta)}(\theta) > (1-\gamma)n$, $\overline{J}_{n,\overline{\mathscr{I}}_n(\theta)}(\theta) > (1-\gamma)n$, $\underline{J}^{(i)}_{n,\underline{\mathscr{I}}^{(i)}_n(\theta)}(\theta) > (1-\gamma)n$, and $\overline{J}^{(i)}_{n,\overline{\mathscr{I}}^{(i)}_n(\theta)}(\theta) > (1-\gamma)n$,
- for all $\theta \in \Theta^{\varepsilon}_{\mathrm{III}}$, $\underline{J}_{n,\underline{\mathscr{I}}_n(\theta)}(\theta) > (1-\gamma)n$, $\overline{J}_{n,\overline{\mathscr{I}}_n(\theta)}(\theta) < \gamma n$, $\underline{J}^{(i)}_{n,\underline{\mathscr{I}}^{(i)}_n(\theta)}(\theta) > (1-\gamma)n$, and $\overline{J}^{(i)}_{n,\overline{\mathscr{I}}^{(i)}_n(\theta)}(\theta) < \gamma n$,
- for all $\theta \in \Theta^{\varepsilon}_{(1>2>0)}$, $\overline{\mathscr{I}}_n(\theta) = \overline{\mathscr{I}}^{(i)}_n(\theta) = 1$,
- for all $\theta \in \Theta^{\varepsilon}_{(2>1>0)}$, $\overline{\mathscr{I}}_n(\theta) = \overline{\mathscr{I}}^{(i)}_n(\theta) = 2$,
- for all $\theta \in \Theta^{\varepsilon}_{(1>0>2)}$, $\overline{\mathscr{I}}_n(\theta) = \overline{\mathscr{I}}^{(i)}_n(\theta) = 1$, and $\underline{\mathscr{I}}_n(\theta) = \underline{\mathscr{I}}^{(i)}_n(\theta) = 2$,
- for all $\theta \in \Theta^{\varepsilon}_{(2>0>1)}$, $\overline{\mathscr{I}}_n(\theta) = \overline{\mathscr{I}}^{(i)}_n(\theta) = 2$, and $\underline{\mathscr{I}}_n(\theta) = \underline{\mathscr{I}}^{(i)}_n(\theta) = 1$,
- for all $\theta \in \Theta^{\varepsilon}_{(0>2>1)}$, $\underline{\mathscr{I}}_n(\theta) = \underline{\mathscr{I}}^{(i)}_n(\theta) = 1$.

The idea behind event $E_{n,i}(\varepsilon,\gamma)$ is that it occurs with very high probability and that we have a good control of $\Delta^{(i)}_n(\theta)$ on that event, namely, for each region, we condition how early or late and on which of the walks the minima and maxima of projections will be achieved. The following proposition establishes our assertion.

Proposition 3.3.1. For any $\gamma \in (0,1/2)$, and any $\varepsilon > 0$, the following hold.

(i) If $i \in I_{n,\gamma} = \{1,\dots,n\} \cap [\gamma n, (1-\gamma)n]$, then, a.s., for any $\theta \in \Theta_{\mathrm{I}}^{\varepsilon}$,

$$\Delta_n^{(i)}(\theta)\mathbf{1}(E_{n,i}(\varepsilon,\gamma)) = \left(Z_i^{(\overline{\mathscr{I}}_n(\theta))} - \tilde{Z}_i^{(\overline{\mathscr{I}}_n(\theta))}\right) \cdot \mathbf{e}_\theta \mathbf{1}(E_{n,i}(\varepsilon,\gamma));$$

for any $\theta \in \Theta_{\mathrm{II}}^{\varepsilon}$ we have

$$\begin{aligned}\Delta_n^{(i)}(\theta)&\mathbf{1}(E_{n,i}(\varepsilon,\gamma)) \\ &= \left(\left(Z_i^{(\overline{\mathscr{I}}_n(\theta))} - \tilde{Z}_i^{(\overline{\mathscr{I}}_n(\theta))}\right) - \left(Z_i^{(\underline{\mathscr{I}}_n(\theta))} - \tilde{Z}_i^{(\underline{\mathscr{I}}_n(\theta))}\right)\right) \cdot \mathbf{e}_\theta \mathbf{1}(E_{n,i}(\varepsilon,\gamma));\end{aligned}$$

while for any $\theta \in \Theta_{\mathrm{III}}^{\varepsilon}$ we have

$$\Delta_n^{(i)}(\theta)\mathbf{1}(E_{n,i}(\varepsilon,\gamma)) = -\left(Z_i^{(\underline{\mathscr{I}}_n(\theta))} - \tilde{Z}_i^{(\underline{\mathscr{I}}_n(\theta))}\right) \cdot \mathbf{e}_\theta \mathbf{1}(E_{n,i}(\varepsilon,\gamma)).$$

(ii) If $\mathbb{E}[\|Z_1^{(k)}\|] < \infty$ for both $k \in \{1,2\}$, and (A1) holds, then

$$\lim_{n\to\infty} \min_{1\le i\le n} \mathbb{P}[E_{n,i}(\varepsilon,\gamma)] = 1.$$

Proof. (i) Suppose that $i \in I_{n,\gamma}$, so $\gamma n \le i \le (1-\gamma)n$. Also, suppose that $\theta \in \Theta_{\mathrm{I}}^{\varepsilon}$. On $E_{n,i}(\varepsilon,\gamma)$, we have that $\underline{J}_{n,\underline{\mathscr{I}}_n(\theta)}(\theta) < i < \overline{J}_{n,\overline{\mathscr{I}}_n(\theta)}(\theta)$ and $\underline{J}^{(i)}_{n,\underline{\mathscr{I}}_n^{(i)}(\theta)}(\theta) < i < \overline{J}^{(i)}_{n,\overline{\mathscr{I}}_n^{(i)}(\theta)}(\theta)$. Therefore, from the definition of the resampled processes (3.1), we can see that it has to be $\underline{J}_{n,\underline{\mathscr{I}}_n(\theta)}(\theta) = \underline{J}^{(i)}_{n,\underline{\mathscr{I}}_n^{(i)}(\theta)}(\theta)$ and moreover $\underline{\mathscr{I}}_n(\theta) = \underline{\mathscr{I}}_n^{(i)}(\theta)$. Thus, it implies that $\underline{J}_{n,\underline{\mathscr{I}}_n(\theta)}(\theta) = \underline{J}^{(i)}_{n,\underline{\mathscr{I}}_n(\theta)}(\theta)$. Hence $m_n(\theta) = m_n^{(i)}(\theta)$. Further, on the event $E_{n,i}(\varepsilon,\gamma)$, it holds that $\overline{\mathscr{I}}_n(\theta) = \overline{\mathscr{I}}_n^{(i)}(\theta)$. Thus, we can express

$$M_n^{(i)}(\theta) = M_n(\theta) + (\tilde{Z}_i^{(\overline{\mathscr{I}}_n(\theta))} - Z_i^{(\overline{\mathscr{I}}_n(\theta))}) \cdot \mathbf{e}_\theta.$$

Therefore, the first equality of (i) follows. For the second equality, take an angle $\theta \in \Theta_{\mathrm{II}}^{\varepsilon}$. On $E_{n,i}(\varepsilon,\gamma)$, we have that $\overline{\mathscr{I}}_n(\theta) = \overline{\mathscr{I}}_n^{(i)}(\theta)$ and $\underline{\mathscr{I}}_n(\theta) = \underline{\mathscr{I}}_n^{(i)}(\theta)$. Hence, similarly as earlier, we obtain that

$$M_n^{(i)}(\theta) = M_n(\theta) + (\tilde{Z}_i^{(\overline{\mathscr{I}}_n(\theta))} - Z_i^{(\overline{\mathscr{I}}_n(\theta))}) \cdot \mathbf{e}_\theta,$$

and similarly

$$m_n^{(i)}(\theta) = m_n(\theta) + (\tilde{Z}_i^{(\underline{\mathscr{I}}_n(\theta))} - Z_i^{(\underline{\mathscr{I}}_n(\theta))}) \cdot \mathbf{e}_\theta,$$

so the claim follows. The third equality (for $\theta \in \Theta_{\mathrm{III}}^{\varepsilon}$) is shown similarly.

(ii) The idea behind the proof of this claim is to show that the probabilities for all eight items in the definition of $E_{n,i}(\gamma,\varepsilon)$ tend to 1 as $n \to \infty$, no matter which $i \in I_{n,\gamma}$ we choose. Let us prove the claim for the first item. The key idea is to simultaneously use the strong law of large numbers (Theorem 1.3.5) for both walks. Take an arbitrary ε_1 such that $0 < \varepsilon_1 < \varepsilon$. There exists a random variable $N := N(\varepsilon_1)$ such that N is finite almost surely and

$$n \geq N \implies \left\| \frac{S_n^{(k)}}{n} - \boldsymbol{\mu}^{(k)} \right\| < \varepsilon_1$$

for both $k \in \{1,2\}$. This implies that, for $\theta \in \Theta_{\mathrm{I}}^{\varepsilon}$, if $n \geq N$, then

$$\left| \frac{S_n^{(k)}}{n} \cdot \mathbf{e}_\theta - \boldsymbol{\mu}^{(k)} \cdot \mathbf{e}_\theta \right| = \left| \frac{S_n^{(k)}}{n} \cdot \mathbf{e}_\theta - \mu^{(k)} \cos(\theta^{(k)} - \theta) \right| \leq \left\| \frac{S_n^{(k)}}{n} - \boldsymbol{\mu}^{(k)} \right\| < \varepsilon_1. \qquad (3.4)$$

For $n \geq N$, we have that

$$S_n^{(k)} \cdot \mathbf{e}_\theta > (\mu^{(k)} \cos(\theta^{(k)} - \theta) - \varepsilon_1)n > (\varepsilon - \varepsilon_1)n.$$

The last term is strictly positive because of the choice of ε_1. Therefore, for any $\theta \in \Theta_{\mathrm{I}}^{\varepsilon}$, we have that $S_n^{(k)} \cdot \mathbf{e}_\theta > 0$ for both $k \in \{1,2\}$ and $n \geq N$. But, recall that $S_0^{(k)} \cdot \mathbf{e}_\theta = 0$, so it gives us that $\underline{J}_{n,\underline{\mathscr{I}}_n(\theta)}(\theta) < N$ for all $\theta \in \Theta_{\mathrm{I}}^{\varepsilon}$. Hence,

$$1 \geq \lim_n \mathbb{P}\left(\bigcap_{\theta \in \Theta_{\mathrm{I}}^{\varepsilon}} \{\underline{J}_{n,\underline{\mathscr{I}}_n(\theta)}(\theta) < \gamma n\} \right) \geq \lim_n \mathbb{P}(N \leq \gamma n) = 1,$$

since N is a.s. finite. Considering the second event, $\overline{J}_{n,\overline{\mathscr{I}}_n(\theta)}(\theta) > (1-\gamma)n$, we have that

$$\max_{0 \leq j \leq (1-\gamma)n} S_j^{(\overline{\mathscr{I}}_n(\theta))} \cdot \mathbf{e}_\theta \leq \max\left\{ \max_{0 \leq j \leq N} S_j^{(\overline{\mathscr{I}}_n(\theta))} \cdot \mathbf{e}_\theta, \max_{N \leq j \leq (1-\gamma)n} S_j^{(\overline{\mathscr{I}}_n(\theta))} \cdot \mathbf{e}_\theta \right\}. \qquad (3.5)$$

For the last term, (3.4) yields

$$\max_{N\leq j\leq(1-\gamma)n} S_j^{(\overline{\mathscr{I}}_n(\theta))}\cdot\mathbf{e}_\theta \leq \max_{0\leq j\leq(1-\gamma)n}\left(\mu^{(\overline{\mathscr{I}}_n(\theta))}\cos(\theta^{(\overline{\mathscr{I}}_n(\theta))}-\theta)+\varepsilon_1\right)j$$
$$\leq \left(\mu^{(\overline{\mathscr{I}}_n(\theta))}\cos(\theta^{(\overline{\mathscr{I}}_n(\theta))}-\theta)+\varepsilon_1\right)(1-\gamma)n.$$

Once again, if $n\geq N$, the inequality (3.4) gives us

$$S_n^{(\overline{\mathscr{I}}_n(\theta))}\cdot\mathbf{e}_\theta > \left(\mu^{(\overline{\mathscr{I}}_n(\theta))}\cos(\theta^{(\overline{\mathscr{I}}_n(\theta))}-\theta)-\varepsilon_1\right)n.$$

The inequality

$$\left(\mu^{(\overline{\mathscr{I}}_n(\theta))}\cos(\theta^{(\overline{\mathscr{I}}_n(\theta))}-\theta)-\varepsilon_1\right)\geq\left(\mu^{(\overline{\mathscr{I}}_n(\theta))}\cos(\theta^{(\overline{\mathscr{I}}_n(\theta))}-\theta)+\varepsilon_1\right)(1-\gamma)$$

holds a.s. if, and only if,

$$\varepsilon_1\leq\frac{\gamma\mu^{(\overline{\mathscr{I}}_n(\theta))}\cos(\theta^{(\overline{\mathscr{I}}_n(\theta))}-\theta)}{2-\gamma}$$

holds a.s. Therefore, we can additionally require that $\varepsilon_1>0$ has been taken such that

$$\varepsilon_1<\frac{\gamma\varepsilon}{2}$$

for the preceding inequality to hold. In that case, for any $\theta\in\Theta_{\mathrm{I}}^{\varepsilon}$, we have that

$$S_n^{(\overline{\mathscr{I}}_n(\theta))}\cdot\mathbf{e}_\theta > \max_{N\leq j\leq(1-\gamma)n} S_j^{(\overline{\mathscr{I}}_n(\theta))}\cdot\mathbf{e}_\theta \quad \text{a.s.}$$

Hence, by (3.5), we have that

$$\mathbb{P}\left[\bigcap_{\theta\in\Theta_{\mathrm{I}}^{\varepsilon}}\left\{\overline{J}_{n,\overline{\mathscr{I}}_n(\theta)}(\theta)>(1-\gamma)n\right\}\right]\geq\mathbb{P}\left[\bigcap_{\theta\in\Theta_{\mathrm{I}}^{\varepsilon}}\left\{S_n^{(\overline{\mathscr{I}}_n(\theta))}\cdot\mathbf{e}_\theta>\max_{0\leq j\leq(1-\gamma)n}S_j^{(\overline{\mathscr{I}}_n(\theta))}\cdot\mathbf{e}_\theta\right\}\right]$$
$$\geq\mathbb{P}\left[N\leq n,\bigcap_{\theta\in\Theta_{\mathrm{I}}^{\varepsilon}}\left\{S_n^{(\overline{\mathscr{I}}_n(\theta))}\cdot\mathbf{e}_\theta>\max_{0\leq j\leq N}S_j^{(\overline{\mathscr{I}}_n(\theta))}\cdot\mathbf{e}_\theta\right\}\right].$$

Additionally, for $n \geq N$, we have that

$$S_n^{(\overline{\mathscr{I}}_n(\theta))}\mathbf{e}_\theta > (\mu^{(\overline{\mathscr{I}}_n(\theta))}\cos(\theta^{(\overline{\mathscr{I}}_n(\theta))} - \theta) - \varepsilon_1)n > (\varepsilon - \frac{\gamma\varepsilon}{2})n = \varepsilon n\left(1 - \frac{\gamma}{2}\right),$$

so we get

$$\mathbb{P}\left[\bigcap_{\theta \in \Theta_{\mathrm{I}}^{\varepsilon}} \left\{\overline{J}_{n,\overline{\mathscr{I}}_n(\theta)}(\theta) > (1-\gamma)n\right\}\right] \geq \mathbb{P}\left[N \leq n, \max_{\substack{0 \leq j \leq N \\ k=1,2}} \|S_j^{(k)}\| \leq \varepsilon n\left(1 - \frac{\gamma}{2}\right)\right].$$

But, since N is finite a.s., we have that $\mathbb{P}(N > n) \to 0$ and

$$\lim_{n\to\infty} \mathbb{P}\left[\max_{\substack{0 \leq j \leq N \\ k=1,2}} \|S_j^{(k)}\| > \varepsilon n\left(1 - \frac{\gamma}{2}\right)\right] = 0.$$

Therefore, we have that

$$\lim_{n\to\infty} \mathbb{P}\left[N \leq n, \max_{\substack{0 \leq j \leq N \\ k=1,2}} \|S_j^{(k)}\| \leq \varepsilon n\left(1 - \frac{\gamma}{2}\right)\right] = 1,$$

so it gives us

$$\lim_{n\to\infty} \mathbb{P}\left[\bigcap_{\theta \in \Theta_{\mathrm{I}}^{\varepsilon}} \left\{\overline{J}_{n,\overline{\mathscr{I}}_n(\theta)}(\theta) > (1-\gamma)n\right\}\right] = 1.$$

This shows the asymptotic probability of the first statement in the first item point of the definition of the set $E_{n,i}(\varepsilon,\gamma)$. Note that the statement of the first item corresponding to the resampled walks can be shown in the same way, given that resampling preserves the underlying distribution. The proofs for the second and third item points are omitted, as they proceed in a completely analogous way as the first item point.

We now proceed with the the fourth item. We focus on the angles belonging to $\Theta^{\varepsilon}_{(1>2>0)}$. It should be noted that the reasoning deployed here can be easily adapted to other cases. We aim to establish that

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcap_{\theta\in\Theta^{\varepsilon}_{(1>2>0)}} \{\overline{\mathscr{I}}_n(\theta) = \overline{\mathscr{I}}^{(i)}_n(\theta) = 1\} \right) = 1.$$

Since $\overline{\mathscr{I}}^{(i)}_n(\theta)$ is identically distributed as $\overline{\mathscr{I}}_n(\theta)$, it is sufficient to prove that

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcap_{\theta\in\Theta^{\varepsilon}_{(1>2>0)}} \{\overline{\mathscr{I}}_n(\theta) = 1\} \right) = 1.$$

Choose $\varepsilon_1 > 0$ such that $2\varepsilon_1 < \boldsymbol{\mu}^{(1)} \cdot \mathbf{e}_\theta - \boldsymbol{\mu}^{(2)} \cdot \mathbf{e}_\theta$, and $\boldsymbol{\mu}^{(2)} \cdot \mathbf{e}_\theta - \varepsilon_1 > 0$ for all $\theta \in \Theta^{\varepsilon}_{(1>2>0)}$, which is possible because of the definition of $\Theta^{\varepsilon}_{(1>2>0)}$. For this collection of one-dimensional walks, we have that

$$n \geq N \implies \left| \frac{S^{(k)}_n}{n} \cdot \mathbf{e}_\theta - \boldsymbol{\mu}^{(k)} \cdot \mathbf{e}_\theta \right| \leq \left\| \frac{S^{(k)}_n}{n} - \boldsymbol{\mu}^{(k)} \right\| < \varepsilon_1$$

for $k \in \{1,2\}$. Hence, we have the following

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{\theta\in\Theta^{\varepsilon}_{(1>2>0)}} \{\overline{\mathscr{I}}_n(\theta) = 1\} \right) &\geq \mathbb{P}\left(N \leq n, \bigcap_{\theta\in\Theta^{\varepsilon}_{(1>2>0)}} \{\overline{\mathscr{I}}_n(\theta) = 1\} \right) \\
&\geq \mathbb{P}\left(N \leq n, \bigcap_{\theta\in\Theta^{\varepsilon}_{(1>2>0)}} \max_{0\leq j\leq n} S^{(1)}_j \cdot \mathbf{e}_\theta \geq \max_{0\leq j\leq n} S^{(2)}_j \cdot \mathbf{e}_\theta \right) \\
&= \mathbb{P}\left(N \leq n, \bigcap_{\theta\in\Theta^{\varepsilon}_{(1>2>0)}} \max_{0\leq j\leq n} \frac{S^{(1)}_j \cdot \mathbf{e}_\theta}{n} \geq \max_{0\leq j\leq n} \frac{S^{(2)}_j \cdot \mathbf{e}_\theta}{n} \right) \\
&= \mathbb{P}\left(N \leq n, \bigcap_{\theta\in\Theta^{\varepsilon}_{(1>2>0)}} \max\left\{ \max_{0\leq j\leq N} \frac{S^{(1)}_j \cdot \mathbf{e}_\theta}{n}, \max_{N<j\leq n} \frac{S^{(1)}_j \cdot \mathbf{e}_\theta}{n} \right\} \geq \right. \\
&\qquad \left. \max\left\{ \max_{0\leq j\leq N} \frac{S^{(2)}_j \cdot \mathbf{e}_\theta}{n}, \max_{N<j\leq n} \frac{S^{(2)}_j \cdot \mathbf{e}_\theta}{n} \right\} \right) \\
&\geq \mathbb{P}\left(N \leq n, \bigcap_{\theta\in\Theta^{\varepsilon}_{(1>2>0)}} \max\left\{ \max_{0\leq j\leq N} \frac{S^{(1)}_j \cdot \mathbf{e}_\theta}{n}, \max_{N<j\leq n} \frac{j}{n}(\boldsymbol{\mu}^{(1)} \cdot \mathbf{e}_\theta - \varepsilon_1) \right\} \geq \right.
\end{aligned}$$

$$\max\left\{\max_{0\le j\le N}\frac{S_j^{(2)}\cdot\mathbf{e}_\theta}{n},\max_{N<j\le n}\frac{j}{n}(\boldsymbol{\mu}^{(2)}\cdot\mathbf{e}_\theta+\varepsilon_1)\right\}\Bigg)\,,$$

where our selection of ε_1 justifies the last inequality. By the dominated convergence theorem, it becomes clear that the final term converges to 1 as $n\to\infty$. Thus, we have proved the asymptotic probability for the fourth item point in the definition of $E_{n,i}(\varepsilon,\gamma)$. The same idea is applied to prove the statements for the remaining four items. Combining all the results, we get (ii). □

Remark 3.3.2. Notice that in the case when the drift vectors are co-linear, see Figure 3.3, the situation is slightly different than the one shown in Figure 3.1. The proof of Proposition 3.3.1 remains the same, the only difference being that some of the subsets of angles $\theta\in[0,\pi]$ that were introduced in (3.3) are empty. In the case when both drift vectors have the same direction, the sets $\Theta^{\varepsilon}_{(1>0>2)}$ and $\Theta^{\varepsilon}_{(2>0>1)}$ are empty, while in the case when drift vectors have opposite directions the sets $\Theta^{\varepsilon}_{(1>2>0)}$, $\Theta^{\varepsilon}_{(2>1>0)}$ and $\Theta^{\varepsilon}_{(0>2>1)}$ are empty. The case when both vectors have the same magnitude and the same orientation is excluded by the assumption (A1). This case is discussed further in Chapter 5.3. Somewhat surprisingly, simulation results suggest that we lose the normality of the distributional limit in this case. We offer a possible explanation for this phenomenon, but the formal proof remains out of our reach. The efforts to extend our results in this direction are currently underway.

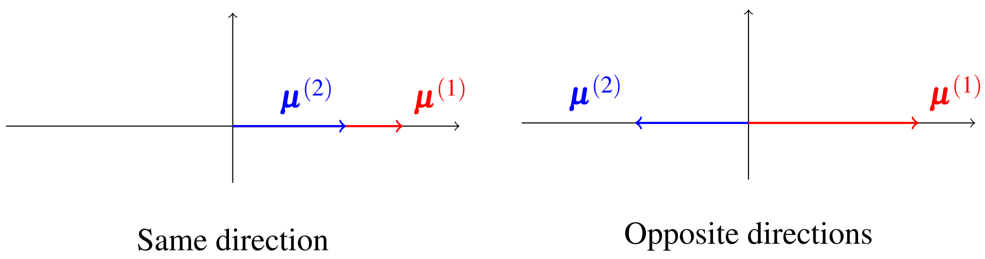


Figure 3.3: Allowed positions of the co-linear drift vectors.

3.4. APPROXIMATION LEMMA FOR PERIMETER

In the following lemma, we compare $\mathscr{L}_{n,i}$ with appropriately centered and projected i-th steps of the walks. The proof of the lemma depends on our earlier assumptions regarding the spatial orientation of the drift vectors relative to the y-axis. Irrespective of the scenario chosen, the outcome remains the same - the only distinction lies in the specific angular regions under consideration and the sequencing of integrals. For clarity, we will explain only the case where the y-axis is situated between the drift vectors.

Lemma 3.4.1. Assume (A1) and $\mathbb{E}[\|Z_1^{(k)}\|] < \infty$ for both $k \in \{1,2\}$. Then, for any $\gamma \in (0,1/2)$, $\varepsilon > 0$, and $i \in I_{n,\gamma}$, we have

$$\begin{aligned}&\left|\mathscr{L}_{n,i} - \left(\left(Z_i^{(1)} - \boldsymbol{\mu}^{(1)}\right)\cdot(\mathbf{e}^{\perp}_{\theta^{(0)}} + \mathbf{e}_{\theta^{(1)}}) + \left(Z_i^{(2)} - \boldsymbol{\mu}^{(2)}\right)\cdot(\mathbf{e}_{\theta^{(2)}} - \mathbf{e}^{\perp}_{\theta^{(0)}})\right)\right| \\ &\le 3\pi(\|Z_i^{(1)}\| + \|Z_i^{(2)}\|)\mathbb{P}\left[E^c_{n,i}(\varepsilon,\gamma) \mid \mathscr{F}_i\right] + 3\pi\mathbb{E}\left[(\|\tilde{Z}_i^{(1)}\| + \|\tilde{Z}_i^{(2)}\|)\mathbf{1}\left(E^c_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \\ &\quad + 3\lambda((\Theta^{\varepsilon})^c)\left(\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + \mathbb{E}\|Z_1^{(1)}\| + \mathbb{E}\|Z_1^{(2)}\|\right),\end{aligned}$$

where λ is Lebesgue measure on $[0,\pi]$ and $(\Theta^{\varepsilon})^c$ is the complement of the set Θ^{ε} in $[0,\pi]$.

Proof. Start from $L_n - L_n^{(i)}$ and take the conditional expectation with respect to $\mathscr{F}_i$. We get

$$\mathscr{L}_{n,i} = \int_0^{\pi} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}(E_{n,i}(\varepsilon,\gamma)) \mid \mathscr{F}_i\right] \mathrm{d}\theta + \int_0^{\pi} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E^c_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta. \tag{3.6}$$

For the second term in (3.6), we have

$$\left|\int_0^{\pi} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E^{\mathrm{c}}_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta\right| \le \int_0^{\pi} \mathbb{E}\left[\left|\Delta_n^{(i)}(\theta)\right|\mathbf{1}\left(E^{\mathrm{c}}_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta, \tag{3.7}$$

and apply the upper bound obtained in Lemma 3.2.1 to get

$$\begin{aligned}
&\int_0^{\pi} \mathbb{E}\left[\left|\Delta_n^{(i)}(\theta)\right| \mathbf{1}\left(E_{n,i}^c(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta \\
&\leq 2\pi\mathbb{E}\left[\left(\left\|Z_i^{(1)}\right\| + \left\|\tilde{Z}_i^{(1)}\right\| + \left\|Z_i^{(2)}\right\| + \left\|\tilde{Z}_i^{(2)}\right\|\right) \mathbf{1}\left(E_{n,i}^c(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \\
&= 2\pi(\|Z_i^{(1)}\| + \|Z_i^{(2)}\|)\mathbb{P}\left[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i\right] + 2\pi\mathbb{E}\left[(\|\tilde{Z}_i^{(1)}\| + \|\tilde{Z}_i^{(2)}\|)\mathbf{1}\left(E_{n,i}^c(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right],
\end{aligned}$$

where we used that $Z_i^{(k)}$ are $\mathscr{F}_i$-measurable with $\mathbb{E}\|Z_i^{(k)}\| < \infty$. Thus, it suffices to show

$$\begin{aligned}
&\left|\int_0^{\pi} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta \right.\\
&\qquad \left. - \left(\left(Z_i^{(1)} - \boldsymbol{\mu}^{(1)}\right)\cdot(\mathbf{e}_{\theta^{(0)}}^{\perp} + \mathbf{e}_{\theta^{(1)}}) + \left(Z_i^{(2)} - \boldsymbol{\mu}^{(2)}\right)\cdot(\mathbf{e}_{\theta^{(2)}} - \mathbf{e}_{\theta^{(0)}}^{\perp})\right)\right|
\end{aligned}$$

is almost surely less or equal than

$$\begin{aligned}
&\pi(\|Z_i^{(1)}\| + \|Z_i^{(2)}\|)\mathbb{P}\left[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i\right] + \pi\mathbb{E}\left[(\|\tilde{Z}_i^{(1)}\| + \|\tilde{Z}_i^{(2)}\|)\mathbf{1}\left(E_{n,i}^c(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \\
&\qquad + 3\lambda((\Theta^{\varepsilon})^c)\left(\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + \mathbb{E}\|Z_1^{(1)}\| + \mathbb{E}\|Z_1^{(2)}\|\right).
\end{aligned}$$

Let us decompose the first integral in (3.6). It can be written as the following sum

$$\begin{aligned}
\int_{[0,\pi]} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta &= \int_{[0,\theta^{(2)}-\pi/2]} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta \\
&+ \int_{[\theta^{(2)}-\pi/2,\pi/2]} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta \\
&+ \int_{[\pi/2,\theta^{(1)}+\pi/2]} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta \\
&+ \int_{[\theta^{(1)}+\pi/2,\pi]} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta.
\end{aligned} \tag{3.8}$$

Denote these integrals with I_1, I_2, I_3 and I_4, respectively. Let us calculate the first integral. For I_1, we have that

$$\begin{aligned}
\int_{[0,\theta^{(2)}-\pi/2]} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta &= \int_{\Theta^{\varepsilon}_{(1>0>2)}} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta \\
&+ \int_{(\Theta^{\varepsilon}_{(1>0>2)})^c} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta,
\end{aligned} \tag{3.9}$$

where the complement $(\Theta^{\varepsilon}_{(1>0>2)})^c$ of $\Theta^{\varepsilon}_{(1>0>2)}$ has been taken with respect to $[0, \theta^{(2)} - \pi/2]$. The second integral is bounded with

$$\left| \int_{(\Theta^{\varepsilon}_{(1>0>2)})^c} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta \right| \\ \leq 2\lambda\left((\Theta^{\varepsilon}_{(1>0>2)})^c\right)\left(\|Z_i^{(1)}\| + \mathbb{E}\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + \mathbb{E}\|Z_i^{(2)}\|\right)$$

by Lemma 3.2.1. An analog bound is obtained for I_2, I_3, and I_4, by replacing $\Theta^{\varepsilon}_{(1>0>2)}$ with $\Theta^{\varepsilon}_{(1>2>0)}$, $\Theta^{\varepsilon}_{(2>1>0)}$, and $\Theta^{\varepsilon}_{(2>0>1)}$ respectively, and by taking complements with respect to $[\theta^{(2)} - \pi/2, \pi/2]$, $[\pi/2, \theta^{(1)} + \pi/2]$, and $[\theta^{(1)} + \pi/2, \pi]$ respectively. Combining these bounds, our problem reduces to showing that

$$\Bigg| \int_{\Theta^{\varepsilon}} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta \\ - \left(\left(Z_i^{(1)} - \boldsymbol{\mu}^{(1)}\right)\cdot\left(\mathbf{e}^{\perp}_{\theta^{(0)}} + \mathbf{e}_{\theta^{(1)}}\right) + \left(Z_i^{(2)} - \boldsymbol{\mu}^{(2)}\right)\cdot\left(\mathbf{e}_{\theta^{(2)}} - \mathbf{e}^{\perp}_{\theta^{(0)}}\right)\right)\Bigg|$$

is almost surely less or equal than

$$\pi(\|Z_i^{(1)}\| + \|Z_i^{(2)}\|)\mathbb{P}\left[E^c_{n,i}(\varepsilon,\gamma) \mid \mathscr{F}_i\right] + \pi\mathbb{E}\left[(\|\tilde{Z}_i^{(1)}\| + \|\tilde{Z}_i^{(2)}\|)\mathbf{1}\left(E^c_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \\ + \lambda\left((\Theta^{\varepsilon})^c\right)\left(\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + \mathbb{E}\|Z_1^{(1)}\| + \mathbb{E}\|Z_1^{(2)}\|\right).$$

By Proposition 3.3.1, we have that the first integral in (3.9) can be rewritten as

$$\int_{\Theta^{\varepsilon}_{(1>0>2)}} \mathbb{E}\left[\Delta_n^{(i)}(\theta)\mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta \\ = \int_{\Theta^{\varepsilon}_{(1>0>2)}} \mathbb{E}\Big[\Big(\Big(Z_i^{(\overline{\mathscr{I}}_n(\theta))} - \tilde{Z}_i^{(\overline{\mathscr{I}}_n(\theta))}\Big) \\ - \Big(Z_i^{(\underline{\mathscr{I}}_n(\theta))} - \tilde{Z}_i^{(\underline{\mathscr{I}}_n(\theta))}\Big)\Big)\cdot\mathbf{e}_{\theta}\mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\Big] \mathrm{d}\theta,$$

while on $E_{n,i}(\varepsilon,\gamma)$ we have that $\overline{\mathscr{I}}_n(\theta) = 1$ and $\underline{\mathscr{I}}_n(\theta) = 2$ for these choices of angles.

Thus, the previous integral is equal to

$$\int_{\Theta^{\varepsilon}_{(1>0>2)}} \mathbb{E}\left[\left(\left(Z_i^{(\overline{\mathscr{I}}_n(\theta))} - \tilde{Z}_i^{(\overline{\mathscr{I}}_n(\theta))}\right) - \left(Z_i^{(\underline{\mathscr{I}}_n(\theta))} - \tilde{Z}_i^{(\underline{\mathscr{I}}_n(\theta))}\right)\right) \cdot \mathbf{e}_\theta \mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta$$
$$= \int_{\Theta^{\varepsilon}_{(1>0>2)}} \mathbb{E}\left[\left(\left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) - \left(Z_i^{(2)} - \tilde{Z}_i^{(2)}\right)\right) \cdot \mathbf{e}_\theta \mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta.$$

Now, we can rewrite it as follows

$$\int_{\Theta^{\varepsilon}_{(1>0>2)}} \mathbb{E}\left[\left(\left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) - \left(Z_i^{(2)} - \tilde{Z}_i^{(2)}\right)\right) \cdot \mathbf{e}_\theta \mathbf{1}\left(E_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta$$
$$= \int_{\Theta^{\varepsilon}_{(1>0>2)}} \mathbb{E}\left[\left(\left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) - \left(Z_i^{(2)} - \tilde{Z}_i^{(2)}\right)\right) \cdot \mathbf{e}_\theta \mid \mathscr{F}_i\right] \mathrm{d}\theta$$
$$- \int_{\Theta^{\varepsilon}_{(1>0>2)}} \mathbb{E}\left[\left(\left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) - \left(Z_i^{(2)} - \tilde{Z}_i^{(2)}\right)\right) \cdot \mathbf{e}_\theta \mathbf{1}\left(E^c_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta.$$

The second integral is negligible since

$$\left|\int_{\Theta^{\varepsilon}_{(1>0>2)}} \mathbb{E}\left[\left(\left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) - \left(Z_i^{(2)} - \tilde{Z}_i^{(2)}\right)\right) \cdot \mathbf{e}_\theta \mathbf{1}\left(E^c_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \mathrm{d}\theta\right|$$
$$\leq \lambda\left(\Theta^{\varepsilon}_{(1>0>2)}\right)\Big((\|Z_i^{(1)}\| + \|Z_i^{(2)}\|)\mathbb{P}[E^c_{n,i}(\varepsilon,\gamma) \mid \mathscr{F}_i]$$
$$+ \mathbb{E}[(\|\tilde{Z}_i^{(1)}\| + \|\tilde{Z}_i^{(2)}\|)\mathbf{1}\left(E^c_{n,i}(\varepsilon,\gamma)\right) \mid \mathscr{F}_i]\Big),$$

while the first one appears as

$$\int_{\Theta^{\varepsilon}_{(1>0>2)}} \mathbb{E}\left[\left(\left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) - \left(Z_i^{(2)} - \tilde{Z}_i^{(2)}\right)\right) \cdot \mathbf{e}_\theta \mid \mathscr{F}_i\right] \mathrm{d}\theta$$
$$= \int_{[0,\theta^{(2)}-\pi/2]} \mathbb{E}\left[\left(\left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) - \left(Z_i^{(2)} - \tilde{Z}_i^{(2)}\right)\right) \cdot \mathbf{e}_\theta \mid \mathscr{F}_i\right] \mathrm{d}\theta$$
$$- \int_{(\Theta^{\varepsilon}_{(1>0>2)})^c} \mathbb{E}\left[\left(\left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) - \left(Z_i^{(2)} - \tilde{Z}_i^{(2)}\right)\right) \cdot \mathbf{e}_\theta \mid \mathscr{F}_i\right] \mathrm{d}\theta.$$

The contribution of the latter integral is small since

$$\left|\int_{(\Theta^{\varepsilon}_{(1>0>2)})^c} \mathbb{E}\left[\left(\left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) - \left(Z_i^{(2)} - \tilde{Z}_i^{(2)}\right)\right) \cdot \mathbf{e}_\theta \mid \mathscr{F}_i\right] \mathrm{d}\theta\right|$$
$$\leq \lambda\left((\Theta^{\varepsilon}_{(1>0>2)})^c\right)\left(\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + \mathbb{E}\|Z_1^{(1)}\| + \mathbb{E}\|Z_1^{(2)}\|\right).$$

Similarly as before, we conclude analogous bounds for I_2, I_3 and I_4, again by replacing $\Theta^{\varepsilon}_{(1>0>2)}$ with $\Theta^{\varepsilon}_{(1>2>0)}$, $\Theta^{\varepsilon}_{(2>1>0)}$, and $\Theta^{\varepsilon}_{(2>0>1)}$ respectively, and $[0,\theta^{(2)}-\pi/2]$ with $[\theta^{(2)}-\pi/2,\pi/2]$, $[\pi/2,\theta^{(1)}+\pi/2]$, and $[\theta^{(1)}+\pi/2,\pi]$ respectively. Putting all this together, and using Proposition 3.3.1, it remains to show that

$$\begin{aligned}
&\int_{[0,\theta^{(2)}-\pi/2]} \mathbb{E}\left[\left(\left(Z_i^{(1)}-\tilde{Z}_i^{(1)}\right)-\left(Z_i^{(2)}-\tilde{Z}_i^{(2)}\right)\right)\cdot \mathbf{e}_\theta \mid \mathscr{F}_i\right] \mathrm{d}\theta \\
&+\int_{[\theta^{(2)}-\pi/2,\pi/2]} \mathbb{E}\left[\left(\left(Z_i^{(1)}-\tilde{Z}_i^{(1)}\right)\right)\cdot \mathbf{e}_\theta \mid \mathscr{F}_i\right] \mathrm{d}\theta \\
&+\int_{[\pi/2,\theta^{(1)}+\pi/2]} \mathbb{E}\left[\left(\left(Z_i^{(2)}-\tilde{Z}_i^{(2)}\right)\right)\cdot \mathbf{e}_\theta \mid \mathscr{F}_i\right] \mathrm{d}\theta \\
&+\int_{[\theta^{(1)}+\pi/2,\pi]} \mathbb{E}\left[\left(\left(Z_i^{(2)}-\tilde{Z}_i^{(2)}\right)-\left(Z_i^{(1)}-\tilde{Z}_i^{(1)}\right)\right)\cdot \mathbf{e}_\theta \mid \mathscr{F}_i\right] \mathrm{d}\theta \\
&=\left(Z_i^{(1)}-\boldsymbol{\mu}^{(1)}\right)\cdot(\mathbf{e}^{\perp}_{\theta^{(0)}}+\mathbf{e}_{\theta^{(1)}})+\left(Z_i^{(2)}-\boldsymbol{\mu}^{(2)}\right)\cdot(\mathbf{e}_{\theta^{(2)}}-\mathbf{e}^{\perp}_{\theta^{(0)}}).
\end{aligned} \tag{3.10}$$

Recall that $\theta^{(0)}$ is $\pi/2$. To calculate the former integrals, use the following notation

$$\mathbb{E}\left[(Z_i^{(k)}-\tilde{Z}_i^{(k)})\cdot \mathbf{e}_\theta \mid \mathscr{F}_i\right]=(Z_i^{(k)}-\mathbb{E}[Z_i^{(k)}])\cdot \mathbf{e}_\theta = R_i^{(k)}\mathbf{e}_{\varphi_i^{(k)}}\cdot \mathbf{e}_\theta,$$

where $R_i^{(k)}$ is the module, and $\mathbf{e}_{\varphi_i^{(k)}}$ is the unit vector of the centered i-th step of the k-th random walk. Therefore, we get

$$\begin{aligned}
&\int_{[0,\theta^{(2)}-\pi/2]} \mathbb{E}\left[\left(\left(Z_i^{(1)}-\tilde{Z}_i^{(1)}\right)-\left(Z_i^{(2)}-\tilde{Z}_i^{(2)}\right)\right)\mathbf{e}_\theta \mid \mathscr{F}_i\right] \mathrm{d}\theta \\
&=\int_{[0,\theta^{(2)}-\pi/2]} \left(R_i^{(1)}\mathbf{e}_{\varphi_i^{(1)}}\cdot \mathbf{e}_\theta - R_i^{(2)}\mathbf{e}_{\varphi_i^{(2)}}\cdot \mathbf{e}_\theta\right)\mathrm{d}\theta \\
&=R_i^{(1)}\left[-\cos(\theta^{(2)}-\varphi_i^{(1)})+\sin(\varphi_i^{(1)})\right]-R_i^{(2)}\left[-\cos(\theta^{(2)}-\varphi_i^{(2)})+\sin(\varphi_i^{(2)})\right].
\end{aligned}$$

The remaining integrals can be expressed in the same way, and they are equal to

$$R_i^{(1)}\left[\cos\left(\varphi_i^{(1)}\right)+\cos\left(\theta^{(2)}-\varphi_i^{(1)}\right)\right], \qquad R_i^{(2)}\left[\cos(\theta^{(1)}-\varphi_i^{(2)})-\cos(\varphi_i^{(2)})\right]$$

and

$$R_i^{(2)}\left[\sin(\varphi_i^{(2)})-\cos(\theta^{(1)}-\varphi_i^{(2)})\right]-R_i^{(1)}\left[\sin(\varphi_i^{(1)})-\cos(\theta^{(1)}-\varphi_i^{(1)})\right],$$

respectively. It is now straightforward to check equality (3.10), which finishes the proof.

□

Remark 3.4.2. Notice that the only changes in the case of co-linear (but not equal) drift vectors is that the domain of the second and third integral in (3.8) reduces to just one point, so the corresponding integrals trivialy vanish.

Let us denote

$$Y_i^{(1)} = \left(Z_i^{(1)} - \boldsymbol{\mu}^{(1)}\right) \cdot (\mathbf{e}_{\theta^{(0)}}^{\perp} + \mathbf{e}_{\theta^{(1)}}), \qquad Y_i^{(2)} = \left(Z_i^{(2)} - \boldsymbol{\mu}^{(2)}\right) \cdot (\mathbf{e}_{\theta^{(2)}} - \mathbf{e}_{\theta^{(0)}}^{\perp}),$$

and

$$W_{n,i} = \mathscr{L}_{n,i} - Y_i^{(1)} - Y_i^{(2)}.$$

In the following lemma we show that the error term $W_{n,i}$ is L^2-negligible under the scaling $\sqrt{n}$.

Lemma 3.4.3. Assume (A1) and $\mathbb{E}\left[\|Z_1^{(k)}\|^2\right] < \infty$ for both $k \in \{1,2\}$. Then,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[W_{n,i}^2\right] = 0.$$

Proof. Take $\varepsilon_1 > 0$, and let $\gamma \in (0,1/2)$ and $\varepsilon > 0$ be small enough (to be specified later). By the definition of $\mathscr{L}_{n,i}$ and $\Delta_n^{(i)}$, with the bound obtained in Lemma 3.2.1, we get

$$\begin{aligned}
|\mathscr{L}_{n,i}| &\le \int_{[0,\pi]} \mathbb{E}[|\Delta_n^{(i)}(\theta)| \mid \mathscr{F}_i]\,\mathrm{d}\theta \\
&\le 2\int_{[0,\pi]} \mathbb{E}[\|Z_i^{(1)}\| + \|\tilde{Z}_i^{(1)}\| + \|Z_i^{(2)}\| + \|\tilde{Z}_i^{(2)}\| \mid \mathscr{F}_i]\mathrm{d}\theta \\
&= 2\pi\left(\|Z_i^{(1)}\| + \mathbb{E}\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + \mathbb{E}\|Z_i^{(2)}\|\right).
\end{aligned}$$

From the definition of $W_{n,i}$ and the triangle inequality, we obtain that

$$\begin{aligned}
|W_{n,i}| &\le |\mathscr{L}_{n,i}| + \left|Y_i^{(1)}\right| + \left|Y_i^{(2)}\right| \\
&\le 2\pi\left(\|Z_i^{(1)}\| + \mathbb{E}\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + \mathbb{E}\|Z_i^{(2)}\|\right) \\
&\quad + 2(\|Z_i^{(1)}\| + \|\mathbb{E}[Z_1^{(1)}]\|) + 2(\|Z_i^{(2)}\| + \|\mathbb{E}[Z_1^{(2)}]\|)
\end{aligned}$$

$$= (2\pi + 2)\left(\|Z_i^{(1)}\| + \mathbb{E}\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + \mathbb{E}\|Z_i^{(2)}\|\right).$$

Hence, under the assumption that $\mathbb{E}[\|Z_1^{(k)}\|^2] < \infty$ for $k \in \{1,2\}$, there is a constant $C_0 > 0$ such that $\mathbb{E}[W_{n,i}^2] \leq C_0$, for all n and i, where C_0 depends only on the distributions of $Z_1^{(1)}$ and $Z_1^{(2)}$. Since $\mathrm{card}(I_{n,\gamma}^c) \leq 2\gamma n$, we have

$$\frac{1}{n}\sum_{i \notin I_{n,\gamma}} \mathbb{E}[W_{n,i}^2] \leq \frac{1}{n} 2\gamma n C_0 = 2\gamma C_0.$$

Recall that $I_{n,\gamma} = \{1,\ldots,n\} \cap [\gamma n, (1-\gamma)n]$. Therefore, we can choose γ small enough so that $2\gamma C_0 < \varepsilon_1$. On the other hand, for $i \in I_{n,\gamma}$, by Lemma 3.4.1 we have that there exists a constant C_1 such that

$$\begin{aligned}|W_{n,i}| \leq & C_1(\|Z_i^{(1)}\| + \|Z_i^{(2)}\|)\mathbb{P}\left[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i\right] + C_1\mathbb{E}\left[(\|\tilde{Z}_i^{(1)}\| + \|\tilde{Z}_i^{(2)}\|)\mathbf{1}\left(E_{n,i}^c(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \\ & + C_1\lambda((\Theta^\varepsilon)^c)\left(\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + \mathbb{E}\|Z_1^{(1)}\| + \mathbb{E}\|Z_1^{(2)}\|\right).\end{aligned}$$

Let us start with bounding the second term. For arbitrary $B_1 > 0$, we have that

$$\begin{aligned}&\mathbb{E}\left[(\|\tilde{Z}_i^{(1)}\| + \|\tilde{Z}_i^{(2)}\|)\mathbf{1}\left(E_{n,i}^c(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \\ &\leq \mathbb{E}\left[(\|\tilde{Z}_i^{(1)}\| + \|\tilde{Z}_i^{(2)}\|)\mathbf{1}\left(\{\|\tilde{Z}_i^{(1)}\| > B_1\} \cup \{\|\tilde{Z}_i^{(2)}\| > B_1\}\right) \mid \mathscr{F}_i\right] + 2B_1\mathbb{P}[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i] \\ &= \mathbb{E}\left[(\|\tilde{Z}_i^{(1)}\| + \|\tilde{Z}_i^{(2)}\|)\mathbf{1}\left(\{\|\tilde{Z}_i^{(1)}\| > B_1\} \cup \{\|\tilde{Z}_i^{(2)}\| > B_1\}\right)\right] + 2B_1\mathbb{P}[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i],\end{aligned}$$

where in the last line we used that

$$(\|\tilde{Z}_i^{(1)}\| + \|\tilde{Z}_i^{(2)}\|)\mathbf{1}(\{\|\tilde{Z}_i^{(1)}\| > B_1\} \cup \{\|\tilde{Z}_i^{(2)}\| > B_1\})$$

is independent of $\mathscr{F}_i$. Because $\mathbb{E}[\|\tilde{Z}_i^{(k)}\|] < \infty$, the dominated convergence theorem gives that

$$\lim_{B_1 \to \infty} \mathbb{E}\left[(\|\tilde{Z}_i^{(1)}\| + \|\tilde{Z}_i^{(2)}\|)\mathbf{1}\left(\{\|\tilde{Z}_i^{(1)}\| > B_1\} \cup \{\|\tilde{Z}_i^{(2)}\| > B_1\}\right)\right] = 0.$$

Thus, we can choose $B_1 = B_1(\varepsilon) > 1$ sufficiently large so that

$$\mathbb{E}\left[(\|\tilde{Z}_i^{(1)}\| + \|\tilde{Z}_i^{(2)}\|)\mathbf{1}\left(\{\|\tilde{Z}_i^{(1)}\| > B_1\} \cup \{\|\tilde{Z}_i^{(2)}\| > B_1\}\right)\right] \leq \lambda((\Theta^\varepsilon)^c).$$

This implies that

$$\mathbb{E}\left[(\|\tilde{Z}_i^{(1)}\|+\|\tilde{Z}_i^{(2)}\|)\mathbf{1}\left(E_{n,i}^c(\varepsilon,\gamma)\right) \mid \mathscr{F}_i\right] \leq \lambda((\Theta^{\varepsilon})^c)+2B_1\mathbb{P}[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i] \quad \text{a.s.},$$

so we have that

$$\begin{aligned}|W_{n,i}| \leq & C_1(\|Z_i^{(1)}\|+\|Z_i^{(2)}\|)\mathbb{P}\left[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i\right]+C_1\left(\lambda((\Theta^{\varepsilon})^c)+2B_1\mathbb{P}[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i]\right)\\ &+C_1\lambda((\Theta^{\varepsilon})^c)\left(\|Z_i^{(1)}\|+\|Z_i^{(2)}\|+\mathbb{E}\|Z_1^{(1)}\|+\mathbb{E}\|Z_1^{(2)}\|\right).\end{aligned}$$

Therefore, there exists a constant $C_2 > 0$ such that

$$|W_{n,i}| \leq C_2(1+\|Z_i^{(1)}\|+\|Z_i^{(2)}\|+\mathbb{E}\|Z_1^{(1)}\|+\mathbb{E}\|Z_1^{(2)}\|)\left(B_1\mathbb{P}[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i]+\lambda((\Theta^{\varepsilon})^c)\right)$$

almost surely. From this, we conclude that there is a constant $C_3 > 0$ such that

$$\begin{aligned}W_{n,i}^2 \leq & C_3(1+\|Z_i^{(1)}\|+\|Z_i^{(2)}\|+\mathbb{E}\|Z_1^{(1)}\|+\mathbb{E}\|Z_1^{(2)}\|)^2\\ &\cdot\left(B_1^2\mathbb{P}[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i]+\lambda((\Theta^{\varepsilon})^c)^2\right).\end{aligned}$$

almost surely. Taking the expectations of both sides, we get

$$\begin{aligned}\mathbb{E}[W_{n,i}^2] \leq & C_3B_1^2\mathbb{E}[(1+\|Z_i^{(1)}\|+\|Z_i^{(2)}\|+\mathbb{E}\|Z_1^{(1)}\|+\mathbb{E}\|Z_1^{(2)}\|)^2\mathbb{P}[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i]]\\ &+C_3\lambda((\Theta^{\varepsilon})^c)^2\mathbb{E}[(1+\|Z_i^{(1)}\|+\|Z_i^{(2)}\|+\mathbb{E}\|Z_1^{(1)}\|+\mathbb{E}\|Z_1^{(2)}\|)^2].\end{aligned}$$

Since $\mathbb{E}[\|Z_i^{(k)}\|^2] < \infty$, there exist a constant $C_4 > 0$ such that

$$C_3\lambda((\Theta^{\varepsilon})^c)^2\mathbb{E}[(1+\|Z_i^{(1)}\|+\|Z_i^{(2)}\|+\mathbb{E}\|Z_1^{(1)}\|+\mathbb{E}\|Z_1^{(2)}\|)^2] \leq C_4\lambda((\Theta^{\varepsilon})^c)^2.$$

Fix $\varepsilon > 0$ sufficiently small such that $C_4\lambda((\Theta^{\varepsilon})^c)^2 < \varepsilon_1$. This is possible since

$$\lim_{\varepsilon\to 0}\lambda((\Theta^{\varepsilon})^c) = 0.$$

Note that this choice also fixes B_1. Thus

$$\mathbb{E}[W_{n,i}^2] \le C_3 B_1^2 \mathbb{E}[(1+\|Z_i^{(1)}\|+\|Z_i^{(2)}\|+\mathbb{E}\|Z_1^{(1)}\|+\mathbb{E}\|Z_1^{(2)}\|)^2 \mathbb{P}[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i]] + \varepsilon_1.$$

To deal with the first term, for $B_2 > 0$ we have that

$$\begin{aligned}
&(1+\|Z_i^{(1)}\|+\|Z_i^{(2)}\|+\mathbb{E}\|Z_1^{(1)}\|+\mathbb{E}\|Z_1^{(2)}\|)^2 \mathbb{P}[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i] \\
&\le (1+2B_2+\mathbb{E}\|Z_1^{(1)}\|+\mathbb{E}\|Z_1^{(2)}\|)^2 \mathbb{P}[E_{n,i}^c(\varepsilon,\gamma) \mid \mathscr{F}_i] \\
&\quad + (1+\|Z_i^{(1)}\|+\|Z_i^{(2)}\|+\mathbb{E}\|Z_1^{(1)}\|+\mathbb{E}\|Z_1^{(2)}\|)^2 \cdot \mathbf{1}\left(\{\|Z_i^{(1)}\| > B_2\} \cup \{\|Z_i^{(2)}\| > B_2\}\right).
\end{aligned}$$

Once again, because $\mathbb{E}[\|Z_i^{(k)}\|^2] < \infty$, the dominated convergence theorem gives us the existence of $B_2 = B_2(\varepsilon_1)$ such that

$$\begin{aligned}
&C_3 B_1^2 \mathbb{E}\left[(1+\|Z_i^{(1)}\|+\|Z_i^{(2)}\|+\mathbb{E}\|Z_1^{(1)}\|+\mathbb{E}\|Z_1^{(2)}\|)^2\right. \\
&\qquad\qquad \left.\cdot \mathbf{1}\left(\{\|Z_i^{(1)}\| > B_2\} \cup \{\|Z_i^{(2)}\| > B_2\}\right)\right] < \varepsilon_1.
\end{aligned}$$

Therefore,

$$\mathbb{E}[W_{n,i}^2] \le 2\varepsilon_1 + C_3 B_1^2 (1+2B_2+\mathbb{E}\|Z_1^{(1)}\|+\mathbb{E}\|Z_1^{(2)}\|)^2 \mathbb{P}[E_{n,i}^c(\varepsilon,\gamma)].$$

Finally, by Proposition 3.3.1, we may find $n_0 \in \mathbb{N}$ sufficiently large such that

$$n \ge n_0 \implies C_3 B_1^2 (1+2B_2+\mathbb{E}\|Z_1^{(1)}\|+\mathbb{E}\|Z_1^{(2)}\|)^2 \mathbb{P}[E_{n,i}^c(\varepsilon,\gamma)] < \varepsilon_1.$$

To conclude, for given $\varepsilon_1 > 0$, we can find $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have that $\mathbb{E}[W_{n,i}^2] \le 3\varepsilon_1$ for all $i \in \mathscr{I}_{n,\gamma}$. Therefore

$$\frac{1}{n} \sum_{i \in I_{n,\gamma}} \mathbb{E}\left[W_{n,i}^2\right] \le 3\varepsilon_1.$$

Combine the estimates for $i \notin I_{n,\gamma}$ and $i \in I_{n,\gamma}$ to get

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[W_{n,i}^2\right] \le 2\gamma C_0 + 3\varepsilon_1 \le 4\varepsilon_1$$

for $n \geq n_0$. Recall that ε_1 was arbitrary, so this completes the proof. □

3.5. CLT FOR PERIMETER

We are now in a position to state and prove the main results (for perimeter) of this thesis. We start with the L^2 approximation result.

Theorem 3.5.1. Assume (A1). Then,

$$n^{-1/2}\left|L_n - \mathbb{E}[L_n] - \sum_{i=1}^{n}\left[(Z_i^{(1)} - \boldsymbol{\mu}^{(1)})\cdot(\mathbf{e}_{\theta^{(0)}}^{\perp} + \mathbf{e}_{\theta^{(1)}}) + (Z_i^{(2)} - \boldsymbol{\mu}^{(2)})\cdot(\mathbf{e}_{\theta^{(2)}} - \mathbf{e}_{\theta^{(0)}}^{\perp})\right]\right|$$

converges to 0 in the L^2 sense as $n \to \infty$.

Intuitively, according to Theorem 2.2.1, the set $\text{chull}\{S_j^{(k)} : 0 \le j \le n,\ k = 1,2\}$ can be approximated (with respect to the Hausdorff metric) by the scaled (possibly degenerate) triangle spanned by the drift vectors $\boldsymbol{\mu}^{(1)}$ and $\boldsymbol{\mu}^{(2)}$. Theorem 3.5.1 analyses the error of this approximation, which is decomposed into three parts. The parts

$$\sum_{i=1}^{n}(Z_i^{(k)} - \boldsymbol{\mu}^{(k)})\cdot\mathbf{e}_{\theta^{(k)}}, \qquad k \in \{1,2\},$$

represent the deviation in the direction of the corresponding drift vectors $\boldsymbol{\mu}^{(k)}$, and the remaining expression,

$$\sum_{i=1}^{n}((Z_i^{(1)} - \boldsymbol{\mu}^{(1)}) - (Z_i^{(2)} - \boldsymbol{\mu}^{(2)}))\cdot\mathbf{e}_{\theta^{(0)}}^{\perp},$$

corresponds to the deviation along the third side of the triangle, the one connecting two drift vectors.

Proof. Note that

$$\mathbb{E}[W_{n,i} \mid \mathscr{F}_{i-1}] = \mathbb{E}[\mathscr{L}_{n,i} \mid \mathscr{F}_{i-1}] - \mathbb{E}[Y_i^{(1)} + Y_i^{(2)} \mid \mathscr{F}_{i-1}] = -\mathbb{E}[Y_i^{(1)} + Y_i^{(2)}],$$

since $\mathscr{L}_{n,i}$ is a martingale difference sequence and $Y_i^{(1)} + Y_i^{(2)}$ is independent of $\mathscr{F}_{i-1}$. By definition, we have that $\mathbb{E}[Y_i^{(k)}] = 0$, for $k \in \{1,2\}$, thus $W_{n,i}$ is also a martingale difference sequence. Using orthogonality, we have that

$$n^{-1}\mathbb{E}\left[\left(\sum_{i=1}^{n} W_{n,i}\right)^2\right] = n^{-1}\sum_{i=1}^{n}\mathbb{E}\left[W_{n,i}^2\right],$$

which, by Lemma 3.4.3, converges to zero, as $n \to \infty$. Hence, we have that

$$n^{-1/2}\sum_{i=1}^{n} W_{n,i} \to 0$$

in L^2. The assertion now follows from Lemma 3.1.1. □

In order to obtain the central limit theorem for the perimeter process, we first have to determine the variance of the limiting normal law.

Theorem 3.5.2. Assume (A1). Then,

$$\lim_{n\to\infty}\frac{\mathbb{V}\mathrm{ar}[L_n]}{n} = \sigma_L^2 \in [0,\infty),$$

where

$$\begin{aligned}\sigma_L^2 =& \mathbb{E}[((Z_1^{(1)} - \boldsymbol{\mu}^{(1)})\cdot \mathbf{e}_{\theta^{(1)}})^2] + \mathbb{E}[((Z_1^{(1)} - \boldsymbol{\mu}^{(1)})\cdot \mathbf{e}^{\perp}_{\theta^{(0)}})^2] + 2(\boldsymbol{\Sigma}^{(1)}\mathbf{e}_{\theta^{(1)}})\cdot \mathbf{e}^{\perp}_{\theta^{(0)}}\\ &+ \mathbb{E}[((Z_1^{(2)} - \boldsymbol{\mu}^{(2)})\cdot \mathbf{e}_{\theta^{(2)}})^2] + \mathbb{E}[((Z_1^{(2)} - \boldsymbol{\mu}^{(2)})\cdot \mathbf{e}^{\perp}_{\theta^{(0)}})^2] - 2(\boldsymbol{\Sigma}^{(2)}\mathbf{e}_{\theta^{(2)}})\cdot \mathbf{e}^{\perp}_{\theta^{(0)}}.\end{aligned}$$

It may be observed that σ_L^2 represents the variance of an individual term in the approximating sum presented in Theorem 3.5.1.

Proof. Denote with :

$$\xi_n = \frac{L_n - \mathbb{E}[L_n]}{\sqrt{n}}, \qquad \text{and} \qquad \zeta_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(Y_i^{(1)} + Y_i^{(2)}\right).$$

Observe that $\mathbb{V}\mathrm{ar}[\zeta_n] = \sigma_L^2$ for all n. From Theorem 3.5.1, $|\xi_n - \zeta_n|$ vanishes in the L^2 norm as $n \to \infty$. Finally, by Cauchy-Schwarz inequality, we have that:

$$\mathbb{E}[(\xi_n - \zeta_n)\,\zeta_n] \le \mathbb{E}\left[(\xi_n - \zeta_n)^2\right]^{1/2}\mathbb{E}\left[\zeta_n^2\right]^{1/2},$$

which implies that:

$$\lim_{n\to\infty}\frac{\mathbb{V}\mathrm{ar}\left[L_n\right]}{n}=\lim_{n\to\infty}\left(\mathbb{E}\left[\left(\xi_n-\zeta_n\right)^2\right]+\mathbb{E}\left[\zeta_n^2\right]+2\mathbb{E}\left[\left(\xi_n-\zeta_n\right)\zeta_n\right]\right)=\sigma_L^2.$$

□

Finally, we state the Central Limit Theorem for the perimeter.

Theorem 3.5.3. Assume (A1), and $\sigma_L^2>0$. Then, for any $x\in\mathbb{R}$,

$$\lim_{n\to\infty}\mathbb{P}\left[\frac{L_n-\mathbb{E}\left[L_n\right]}{\sqrt{\mathbb{V}\mathrm{ar}\left[L_n\right]}}\leq x\right]=\lim_{n\to\infty}\mathbb{P}\left[\frac{L_n-\mathbb{E}\left[L_n\right]}{\sqrt{\sigma_L^2 n}}\leq x\right]=\Phi(x),$$

where Φ stands for the cummulative distribution function of the standard normal distribution.

Proof. We use the same notation as in the proof of Theorem 3.5.2. By the classical central limit theorem for independent and identically distributed random variables (Theorem 1.3.6), we have that

$$\lim_{n\to\infty}\mathbb{P}\left[\frac{\zeta_n}{\sqrt{\sigma_L^2}}\leq x\right]=\Phi(x),\qquad x\in\mathbb{R},$$

where Φ denotes the cumulative distribution function of the standard normal distribution. By Theorem 3.5.1, $|\xi_n-\zeta_n|\to 0$ in probability. Slutsky's theorem (Theorem 1.3.3) now implies

$$\lim_{n\to\infty}\mathbb{P}\left[\frac{L_n-\mathbb{E}\left[L_n\right]}{\sqrt{\sigma_L^2 n}}\leq x\right]=\lim_{n\to\infty}\mathbb{P}\left[\frac{\xi_n}{\sqrt{\sigma_L^2}}\leq x\right]=\Phi(x),\qquad x\in\mathbb{R}.$$

Finally, again by Slutsky's theorem, we have

$$\mathbb{P}\left[\frac{L_n-\mathbb{E}\left[L_n\right]}{\sqrt{\mathrm{Var}\left[L_n\right]}}\leq x\right]=\mathbb{P}\left[\frac{\xi_n\alpha_n}{\sqrt{\sigma_L^2 n}}\leq x\right],$$

where

$$\alpha_n := \sqrt{\frac{\sigma_L^2 n}{\operatorname{Var}[L_n]}} \to 1,$$

as $n \to \infty$, by Theorem 3.5.2. □

Remark 3.5.4. A natural question to ask is: under what conditions will σ_L^2 be strictly positive? This occurs if and only if one of the following holds: either the variance of the projection of the first random walk onto the vector $\mathbf{e}_{\theta^{(0)}}^{\perp} + \mathbf{e}_{\theta^{(1)}}$ is non-zero, or the variance of the projection of the second random walk onto the vector $\mathbf{e}_{\theta^{(2)}} - \mathbf{e}_{\theta^{(0)}}^{\perp}$ is non-zero.

4. DIAMETER PROCESS

We now turn our attention to the asymptotic behavior of the diameter process. First, we will slightly adjust the methodology we have already established for the perimeter process. Similarly, as in the previous chapter, we will develop the distributional limit result only for the case $m = 2$. In the case of the diameter functional, in addition to assumption (A1), we assume the following:

$$\text{the set } \{\|\boldsymbol{\mu}^{(1)}\|, \|\boldsymbol{\mu}^{(2)}\|, \|\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}\|\} \text{ has a unique maximal element.} \tag{A2}$$

Here, $\|x\|$ represents the standard Euclidean norm of $x \in \mathbb{R}^2$. The assumption (A2) allows us to identify the direction of the diameter of the set.

We can observe that assumption (A2) implies (A1). To prove that, suppose that (A1) does not hold. This implies that $\mathbf{0} \in \{\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}\}$. If $\boldsymbol{\mu}^{(1)} = \mathbf{0}$, then we have $\|\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}\| = \|-\boldsymbol{\mu}^{(2)}\| = \|\boldsymbol{\mu}^{(2)}\|$, so (A2) does not hold. We reach the same conclusion if we assume $\boldsymbol{\mu}^{(2)} = \mathbf{0}$. If instead we assume $\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)} = \mathbf{0}$, then it follows that $\|\boldsymbol{\mu}^{(1)}\| = \|\boldsymbol{\mu}^{(2)}\|$, again leading to the same conclusion. Therefore, (A2) implies (A1).

We conjecture that, in the case of the diameter process, we again have a non-Gaussian distributional limit in the case when assumption (A2) is not satisfied (see Section 5.3 for a computer simulation study and discussion that support the conjecture).

4.1. MDS AND CF FOR DIAMETER

This section aims to develop similar tools using martingale difference sequences and the Cauchy formula, but this time for the diameter process. Recall that the diameter process is defined as

$$D_n = \operatorname{diam}\left(\operatorname{chull}\left\{S_j^{(k)} : 0 \leq j \leq n,\ k = 1,2\right\}\right).$$

Similarly, as earlier, we consider the diameter of the convex hull of the resampled processes, and we denote it with

$$D_n^{(i)} := \operatorname{diam}\left(\operatorname{chull}\left\{S_j^{(k,i)} : 0 \leq j \leq n,\ k = 1,2\right\}\right).$$

For $0 \leq i \leq n$, define

$$\mathscr{D}_{n,i} := \mathbb{E}\left[D_n - D_n^{(i)} \mid \mathscr{F}_i\right],$$

which can be interpreted as the expected change in the diameter length of the convex hull, given $\mathscr{F}_i$, on replacing the i-th increment in both random walks. Analogously as in Lemma 3.1.1, we conclude the martingale difference sequence property of the diameter process.

Lemma 4.1.1. Let $n \in \mathbb{N}$. Then,

(i) $D_n - \mathbb{E}[D_n] = \sum_{i=1}^n \mathscr{D}_{n,i}$,

(ii) $\operatorname{Var}[D_n] = \sum_{i=1}^n \mathbb{E}\left[\mathscr{D}_{n,i}^2\right]$, whenever the latter sum is finite.

Denote by $\rho_A(\theta)$ the length of the set A when projected on the line specified by the angle θ:

$$\rho_A(\theta) = \sup_{x \in A}(x \cdot \mathbf{e}_\theta) - \inf_{x \in A}(x \cdot \mathbf{e}_\theta).$$

The Cauchy formula for diameter (Theorem 1.2.7) is then given by

$$\operatorname{diam}(A) = \sup_{\theta \in [0,\pi]} \rho_A(\theta).$$

Recall that $M_n(\theta)$ and $m_n(\theta)$ denote the maximal and minimal projections, respectively, of our convex hull along the direction specified by $\mathbf{e}_\theta$. Using this notation, we have

$$D_n = \sup_{0\le\theta\le\pi} (M_n(\theta) - m_n(\theta)) = \sup_{0\le\theta\le\pi} R_n(\theta). \tag{4.1}$$

Similarly, we represent $D_n^{(i)}$. Once again, we can perform the same linear transformations as for the perimeter case. So, in the further text, we again adopt the assumptions on the positions of the drift vectors presented at the beginning of Section 3.3. Similarly, as in the case of the perimeter process, we show that the convergence in the strong law of large numbers for the diameter process, presented in (2.2), also holds in L^1 sense.

Corollary 4.1.2. Under the assumptions of Theorem 2.2.1, we have

$$\frac{D_n}{n} \xrightarrow[n\to\infty]{L^1} \operatorname{diam}\left(\operatorname{chull}\{\mathbf{0}, \boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}\}\right).$$

Proof. Using the Cauchy formula from (4.1) we have

$$\begin{aligned} D_n = \sup_{0\le\theta\le\pi} R_n(\theta) &\le 2 \sup_{0\le\theta\le\pi} \max_{\substack{0\le j\le n \\ k=1,2}} |S_j^{(k)} \cdot e_\theta| \\ &\le 2 \max_{\substack{0\le j\le n \\ k=1,2}} \|S_j^{(k)}\| \le 2\sum_{j=0}^{n} \left(\|Z_j^{(1)}\| + \|Z_j^{(2)}\|\right). \end{aligned}$$

Using identical arguments as in the proof of Corollary 3.2.2, the claim follows. □

4.2. Control of extrema

Assuming (A2) and that $\|\boldsymbol{\mu}^{(1)}\|$ is the maximum value among the set

$$\{\|\boldsymbol{\mu}^{(1)}\|, \|\boldsymbol{\mu}^{(2)}\|, \|\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}\|\},$$

which can be done without loss of generality; the diameter will stay close to the drift direction of the first random walk. For $\delta > 0$ and $i \in \{1, \ldots, n\}$, we define this event as follows

$$A_{n,i}(\delta) := \left\{\rho_H^1\left(\left\{\theta^{(1)}\right\}, \operatorname*{arg\,max}_{0\leq\theta\leq\pi} R_n(\theta)\right) < \delta\right\} \cap \left\{\rho_H^1\left(\left\{\theta^{(1)}\right\}, \operatorname*{arg\,max}_{0\leq\theta\leq\pi} R_n^{(i)}(\theta)\right) < \delta\right\}.$$

The key observation is that, with high probability, the angle of the diametral segment of the convex hull spanned by these two random walks is close to the angle of the longest side of the triangle spanned by the corresponding drift vectors.

Theorem 4.2.1. Assume (A2), the maximal element of the set from the assumption (A2) is $\|\boldsymbol{\mu}^{(1)}\|$, and $\mathbb{E}[\|Z_i^{(k)}\|] < \infty$ for both $k \in \{1,2\}$. Then, for arbitrary $\delta > 0$,

$$\lim_{n\to\infty} \min_{1\leq i\leq n} \mathbb{P}(A_{n,i}(\delta)) = 1.$$

Before proving Theorem 4.2.1, define $(\mathscr{D}, \rho_H^2) \subseteq (\mathscr{K}, \rho_H^2)$ as the subset of convex and compact sets where the diameter is manifested along a unique segment. More formally, $\mathscr{D}^2$ contains all convex and compact sets A such that the set

$$\operatorname*{arg\,max}_{0\leq\theta\leq\pi} \rho_A(\theta)$$

has exactly one element. For $A \in \mathscr{P}$, by $\mathrm{V}(A)$ we denote the set of its vertices.

Proposition 4.2.2. The function $A \mapsto \operatorname*{arg\,max}_{0\leq\theta\leq\pi} \rho_A(\theta)$ is point-wise continuous on $(\mathscr{P}^2 \cap \mathscr{D}^2, \rho_H^2)$.

Proof. On $(\mathscr{P}^2 \cap \mathscr{D}^2, \rho_H^2)$, we investigate a particular mapping that associates each polygon within the space to the line segment where it attains its diameter. Formally, we focus on the mapping given by

$$(\mathscr{P}^2 \cap \mathscr{D}^2, \rho_H^2) \ni A \mapsto \overline{a_{i_1} a_{i_2}} \in (\mathscr{K}^2, \rho_H^2),$$

where a_{i_1} and a_{i_2} represent the vertices defining the diameter. Our objective is to verify the point-wise continuity of this mapping, that is, we aim to prove that for any given $\varepsilon > 0$, there exists a corresponding $\delta = \delta(\varepsilon, A) > 0$ such that:

$$\forall\, B \in (\mathscr{P}^2 \cap \mathscr{D}^2, \rho_H^2) \quad \text{such that} \quad \rho_H^2(A,B) < \delta \implies \rho_H^2\left(\overline{a_{i_1} a_{i_2}}, \overline{b_{j_1} b_{j_2}}\right) < \varepsilon.$$

Let $\varepsilon > 0$ and $A \in (\mathscr{P}^2 \cap \mathscr{D}^2, \rho_H^2)$ be arbitrarily selected. Observe first that if A is a line segment, than the continuity easily follows from the triangle inequality by taking $\delta = \varepsilon/3$. In what follows we focus on polygons having at least three vertices. For such a polygon, label the vertices as $a_1, \ldots, a_n$ arranged in counterclockwise order. At each vertex, introduce $u_i^{(1)}$ and $u_i^{(2)}$ as the unit vectors oriented along the respective adjacent edges, where $u_i^{(1)}$ is directed towards a_{i-1} and $u_i^{(2)}$ is directed towards a_{i+1} (considering the indices modulo n). Furthermore, let φ_i denote the size of the angle between the vectors $u_i^{(1)}$ and $u_i^{(1)} + u_i^{(2)}$, or $u_i^{(2)}$ and $u_i^{(1)} + u_i^{(2)}$, see Figure 4.1.

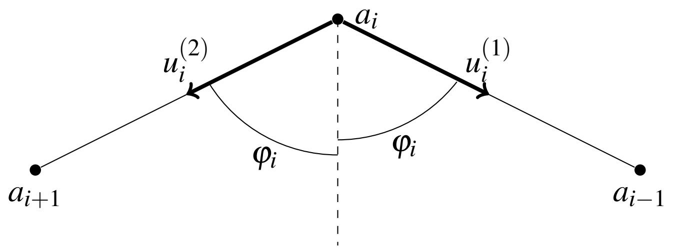


Figure 4.1: Vertex of a polygon with related vectors and angles.

Clearly,

$$\varphi_i \in \left(0, \frac{\pi}{2}\right) \tag{4.2}$$

for all $i \in \{1,\ldots,n\}$. Therefore, we have that

$$\overline{\varphi} := \max\{\varphi_i : i \in \{1,\ldots,n\}\} \in \left(0, \frac{\pi}{2}\right),$$
$$\underline{\varphi} := \min\{\varphi_i : i \in \{1,\ldots,n\}\} \in \left(0, \frac{\pi}{2}\right).$$

Recall that $a_{i_1}, a_{i_2} \in \mathrm{V}(A)$ is the unique pair of vertices for which

$$\operatorname{diam}(A) = \|a_{i_1} - a_{i_2}\|.$$

Therefore, the set

$$\{\|x-y\| : x,y \in V(A)\}$$

is finite, with a unique maximal value. Let $\varepsilon_0 > 0$ be defined as the difference between the two greatest values in this set. For $\delta > 0$ small enough (to be specified later), select an arbitrary polygon $B \in (\mathscr{P}^2 \cap \mathscr{D}^2, \rho_H^2)$ satisfying $\rho_H^2(A,B) < \delta$. Consequently, there exist points b'_{i_1} and b'_{i_2} in ∂B such that

$$\|a_{i_1} - b'_{i_1}\| < \delta, \qquad \text{and} \qquad \|a_{i_2} - b'_{i_2}\| < \delta.$$

However, it is worth noting that b'_{i_1} and b'_{i_2} are not necessarily vertices of B. To find vertices of B that satisfy analogous relations (with modified right hand side) we proceed as follows. Consider two distinct linear optimization problems

$$(OP)_l = \begin{cases} (u_{i_l}^{(1)} + u_{i_l}^{(2)})^T b \to \min \\ b \in B \end{cases}$$

for $l \in \{1,2\}$. Given that B is a convex polygon and the objective function under consideration is also convex, it follows that there exist $b_{i_1}, b_{i_2} \in V(B)$ for which b_{i_l} is the solution to the l-th optimization problem, denoted as $(OP)_l$. Furthermore, b_{i_l} must be situated in a right-angle triangle, one of whose catheti is the segment connecting

$$a_{i_l} + \delta \cdot \frac{u_{i_l}^{(1)} + u_{i_l}^{(2)}}{\|u_{i_l}^{(1)} + u_{i_l}^{(2)}\|}, \qquad \text{and} \qquad a_{i_l} - \frac{\delta}{\sin\varphi_{i_l}} \cdot \frac{u_{i_l}^{(1)} + u_{i_l}^{(2)}}{\|u_{i_l}^{(1)} + u_{i_l}^{(2)}\|}.$$

Figure 4.2 illustrates the reasoning. Hence, the maximal distance between a_{i_l} and b_{i_l} is

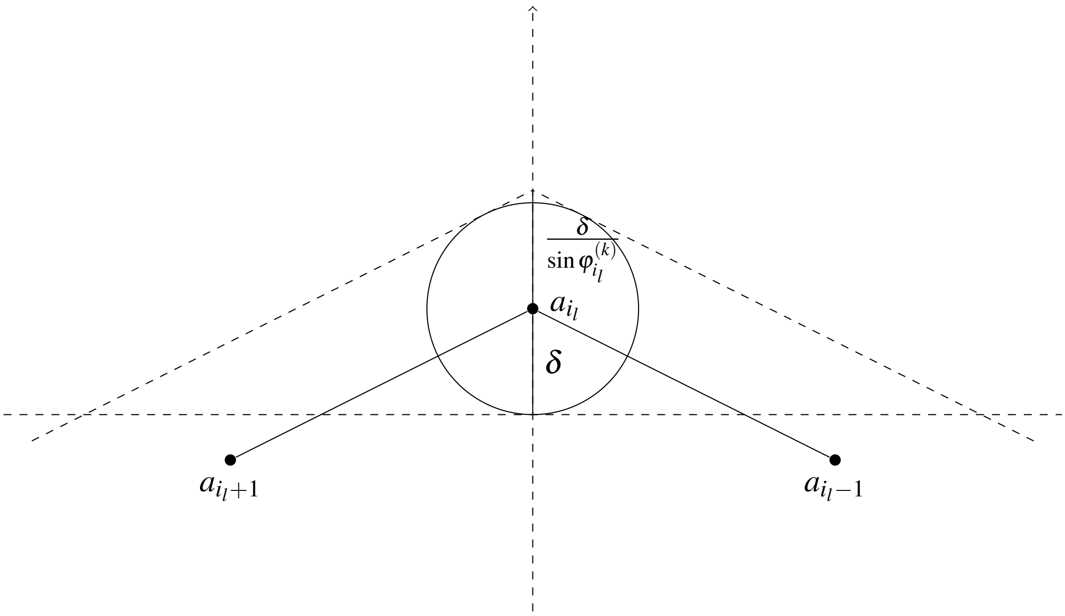


Figure 4.2: Vertex of a polygon with related vectors and angles

bounded by the length of the hypotenuse of the left or right triangle. Therefore, it has to be

$$\|b_{i_l} - a_{i_l}\| \leq \frac{\delta + \frac{\delta}{\sin\varphi_{i_l}}}{\cos\varphi_{i_l}} \leq \frac{2\delta}{\cos\varphi_{i_l}\cdot\sin\varphi_{i_l}} \leq \frac{2\delta}{\cos\overline{\varphi}\cdot\sin\underline{\varphi}}. \tag{4.3}$$

Consequently, by the triangle inequality, we establish a lower bound for the distance between b_{i_1} and b_{i_2} given by

$$\|b_{i_1} - b_{i_2}\| \geq \|a_{i_1} - a_{i_2}\| - \frac{4\delta}{\cos\overline{\varphi}\cdot\sin\underline{\varphi}} = \operatorname{diam}(A) - \frac{4\delta}{\cos\overline{\varphi}\cdot\sin\underline{\varphi}}. \tag{4.4}$$

Thus, it implies that

$$\operatorname{diam}(B) \geq \operatorname{diam}(A) - \frac{4\delta}{\cos\overline{\varphi}\cdot\sin\underline{\varphi}}. \tag{4.5}$$

Next, consider the vertices b_{j_1} and b_{j_2} of B at which the polygon B attains its diameter. Our objective is to demonstrate that one vertex is close to a_{i_1} and the other is close to a_{i_2}. To this end, there must exist points $a_{j_1}, a_{j_2} \in \partial A$ (which are not necessarily vertices) such that

$$\|b_{j_l} - a_{j_l}\| < \delta \tag{4.6}$$

for both $l \in \{1,2\}$. We have the following lower bound on the distance between a_{j_1} and a_{j_2}

$$\|a_{j_1} - a_{j_2}\| \geq \|b_{j_1} - b_{j_2}\| - 2\delta = \mathrm{diam}(B) - 2\delta,$$

by the triangle inequality. Together with (4.5), we get

$$\|a_{j_1} - a_{j_2}\| \geq \mathrm{diam}(A) - \delta \left(\frac{4}{\cos \overline{\varphi} \cdot \sin \underline{\varphi}} + 2 \right) > \mathrm{diam}(A) - \varepsilon_0, \tag{4.7}$$

where the second inequality holds if we choose

$$\delta < \frac{\varepsilon_0}{\left(\dfrac{4}{\cos \overline{\varphi} \cdot \sin \underline{\varphi}} + 2 \right)}.$$

Consider the function $f_0 : \partial A \times \partial A \to \mathbb{R}^+$ defined by $f_0(x,y) = \|x - y\|$. This function is continuous if the domain is equipped with the maximum of the relative norms. Since A obtains its diameter at the unique segment, the relative maxima of f_0 occur at pairs of vertices, with the unique maximal value attained at the pairs (a_{i_1}, a_{i_2}) and (a_{i_2}, a_{i_1}). Given that

$$\mathrm{diam}(A) \geq f_0(a_{j_1}, a_{j_2}) \geq \mathrm{diam}(A) - \delta \left(\frac{4}{\cos \overline{\varphi} \cdot \sin \underline{\varphi}} + 2 \right) > \mathrm{diam}(A) - \varepsilon_0 \tag{4.8}$$

there exists some $\delta_0 > 0$ (which depends on δ) such that the point (a_{j_1}, a_{j_2}) lies within a δ_0-neighborhood of either (a_{i_1}, a_{i_2}) or (a_{i_2}, a_{i_1}) in the relative topology. Without loss of generality, we can assume that (a_{j_1}, a_{j_2}) is in a δ_0-neighborhood of (a_{i_1}, a_{i_2}). Consequently, we obtain that

$$\|a_{j_l} - a_{i_l}\| < \delta_0,$$

for both $l \in \{1,2\}$. Therefore, we get

$$\|b_{j_l} - a_{i_l}\| \le \|b_{j_l} - a_{j_l}\| + \|a_{j_l} - a_{i_l}\| < \delta + \delta_0$$

for both $l \in \{1,2\}$. As δ approaches zero, the quantity $\|a_{j_1} - a_{j_2}\|$ approaches $\mathrm{diam}(A)$, as can be seen from the inequalities in (4.8). Also, as $\delta \to 0^+$, one can see that $\delta_0 \to 0^+$. Given this relationship, we require that $\delta > 0$ is sufficiently small such that δ_0 is also sufficiently small such that $\delta + \delta_0 < \varepsilon$. Hence, it follows that if b_{j_1} and b_{j_2} are the vertices defining the diameter of B we have that

$$\rho_H^2\left(\overline{a_{i_1}a_{i_2}}, \overline{b_{j_1}b_{j_2}}\right) < \delta + \delta_0 < \varepsilon.$$

Therefore, we have successfully demonstrated that the mapping $A \mapsto \overline{a_{i_1}a_{i_2}}$ is point-wise continuous. To complete the proof, note that the function

$$(\mathscr{P} \cap \mathscr{D}, \rho_H^2) \ni A \mapsto \underset{0 \le \theta \le \pi}{\arg\max}\, \rho_A(\theta) \in [0, \pi]$$

can be written as the composition of

$$(\mathscr{P} \cap \mathscr{D}, \rho_H^2) \ni A \mapsto \overline{a_{i_1}a_{i_2}} \in \left(\mathscr{K}, \rho_H^2\right),$$

and

$$\left(\mathscr{K}, \rho_H^2\right) \setminus \left\{\overline{ab} : a, b \in \mathbb{R}^2, a = b\right\} \ni \overline{a_{i_1}a_{i_2}} \mapsto \arctan\left|\frac{\pi_y(a_{i_2}) - \pi_y(a_{i_1})}{\pi_x(a_{i_2}) - \pi_x(a_{i_1})}\right| \in [0, \pi],$$

where π_x and π_y are projections to the x and y axis, respectively, with the understanding that the last expression equals $\pi/2$ if the denominator is zero. Both of these functions are continuous, and therefore, their composition must also be continuous. □

In the following corollary, we show that the function $A \mapsto \arg\max_{0 \le \theta \le \pi} \rho_A(\theta)$ is continuous at points from $\mathscr{P} \cap \mathscr{D}$ in the space $(\mathscr{P}, \rho_H^2)$. We first show the following auxiliary lemma.

Lemma 4.2.3. The set $\mathscr{P} \cap \mathscr{D}$ is dense in $(\mathscr{P}, \rho_H^2)$.

Proof. Let $\varepsilon > 0$. For an arbitrary polygon $A \in (\mathscr{P}, \rho_H^2)$, let θ' be an element of

$$\operatorname{argmax}_{0\leq\theta\leq\pi} \rho_A(\theta).$$

Let the vertices of A be denoted by $a_1, \ldots, a_n$ in a counterclockwise orientation. Assume that a_{i_1} and a_{i_2} are the vertices that correspond to the direction determined by θ'. Note that some of the points $a_{i_l} + \varepsilon \mathbf{e}_{\theta'}$ lie in a direction opposite to the interior of A. Without loss of generality, assume that a_{i_1} is such a point. Consequently, the polygon

$$A_\varepsilon := \operatorname{chull}\{a_1, \ldots, a_{i_1-1}, a_{i_1} + \varepsilon \mathbf{e}_{\theta'}, a_{i_1+1}, \ldots, a_n\},$$

is an element of $\mathscr{P} \cap \mathscr{D}$, and $\rho_H^2(A, A_\varepsilon) = \varepsilon$. The statement is proven given that $\varepsilon > 0$ was chosen arbitrarily. □

Remark 4.2.4. Observe that

$$\operatorname*{arg\,max}_{0\leq\theta\leq\pi} \rho_{A_\varepsilon}(\theta) = \{\theta'\},$$

where θ' corresponds to the angle arbitrarily selected from $\arg\max_{0\leq\theta\leq\pi} \rho_A(\theta)$. Thus, A_ε is an element from $\mathscr{P} \cap \mathscr{D}$ whose unique diameter is attained in the θ' direction and at an arbitrarily small Hausdorff distance from A.

Corollary 4.2.5. The function

$$A \mapsto \operatorname*{arg\,max}_{0\leq\theta\leq\pi} \rho_A(\theta),$$

is continuous at points from $\mathscr{P} \cap \mathscr{D}$ in $(\mathscr{P}, \rho_H^2)$.

Proof. Let $A \in \mathscr{P} \cap \mathscr{D}$ and $\varepsilon > 0$ be arbitrary. By Proposition 4.2.2, there exists $\delta > 0$ satisfying:

$$\forall\, B \in \mathscr{P} \cap \mathscr{D} \quad \text{such that} \quad \rho_H^2(A, B) < \tilde{\delta} \implies |\theta_A - \theta_B| < \varepsilon, \tag{4.9}$$

where θ_A is uniquely determined as $\{\theta_A\} = \arg\max_{0\leq\theta\leq\pi} \rho_A(\theta)$ (the same applies to B). Now, consider a polygon $B \in \mathscr{P}$ with $\rho_H^2(A, B) < \delta/2$, and let $\theta_B \in \arg\max_{0\leq\theta\leq\pi} \rho_B(\theta)$

be arbitrary. According to Lemma 4.2.3 and Remark 4.2.4, we obtain the existence of a polygon $B_{\theta_B} \in \mathscr{P} \cap \mathscr{D}$ satisfying

$$\arg\max_{0 \le \theta \le \pi} \rho_{B_{\theta_B}}(\theta) = \{\theta_B\},$$

with $\rho_H^2(B, B_{\theta_B}) = \delta/2$. It follows that

$$\rho_H^2(A, B_{\theta_B}) \le \rho_H^2(A, B) + \rho_H^2(B, B_{\theta_B}) < \delta/2 + \delta/2 = \delta.$$

Applying equation (4.9), it becomes evident that $|\theta_B - \theta_A| < \varepsilon$. As θ_B was selected arbitrarily, we deduce that

$$\rho_H^1\left(\theta_A, \arg\max_{0 \le \theta \le \pi} \rho_B(\theta)\right) = \sup\left\{|\theta_A - \theta_B| : \theta_B \in \arg\max_{0 \le \theta \le \pi} \rho_B(\theta)\right\} < \varepsilon,$$

thereby confirming the corollary. □

Remark 4.2.6. (i) The density of $\mathscr{P}$ in $(\mathscr{K}, \rho_H^2)$ (see [SW81]) suggests that the previous proof could be adapted to $(\mathscr{K}, \rho_H^2)$ as the domain of interest. Namely, every convex and compact subset of $\mathbb{R}^2$ can be arbitrarily well approximated with the convex polygon, and by Lemma 4.2.3 every convex polygon can be arbitrarily well approximated with the convex polygon with the unique diametrical segment. Hence, if we apply the triangle inequality twice, we would obtain the claimed statement.

(ii) Furthermore, the preceding corollary does not offer insights into the continuity of the mapping when applied to other polygons in $\mathscr{P}$. In fact, it is possible to show with relative ease that the function manifests discontinuities when evaluated on polygons characterized by two or more diametrical segments.

Proof of Theorem 4.2.1. From Theorem 2.2.1, we have that

$$n^{-1} \operatorname{chull}\left\{S_j^{(k)} : 0 \le j \le n, k = 1, 2\right\} \xrightarrow[n\to\infty]{a.s.} \operatorname{chull}\left\{\mathbf{0}, \boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}\right\}. \qquad (4.10)$$

Denote by A the right hand side in (4.10). Because of (A2), we have that $A \in \mathscr{P} \cap \mathscr{D}$. Also, since we assumed that $\|\boldsymbol{\mu}^{(1)}\|$ is the maximal element of the set from the assumption (A2), we find that

$$\arg\max_{0\leq\theta\leq\pi} \rho_A(\theta) = \{\theta^{(1)}\}.$$

Furthermore, denote

$$A_n = n^{-1}\,\text{chull}\left\{S_j^{(k)} : 0 \leq j \leq n,\ k = 1,2\right\}.$$

Using Corollary 4.2.5 with the continuous mapping theorem (Corollary 1.3.7) yields the following

$$\arg\max_{0\leq\theta\leq\pi} \rho_{A_n}(\theta) \xrightarrow[n\to\infty]{a.s.} \theta^{(1)}.$$

It is worth noting that scaling does not impact the direction of the diametrical segment. Consequently, it holds that

$$\arg\max_{0\leq\theta\leq\pi} \rho_{A_n}(\theta) = \arg\max_{0\leq\theta\leq\pi} R_n(\theta).$$

As a result, for a given $\delta > 0$, there exists an almost surely finite random variable N_δ such that:

$$n \geq N_\delta \implies \rho_H^1\left(\left\{\theta^{(1)}\right\}, \arg\max_{0\leq\theta\leq\pi} R_n(\theta)\right) < \delta.$$

Therefore, we get that

$$\mathbb{P}\left(\rho_H^1\left(\left\{\theta^{(1)}\right\}, \arg\max_{0\leq\theta\leq\pi} R_n(\theta)\right) < \delta\right) \geq \mathbb{P}(n \geq N_\delta) \to \mathbb{P}(N_\delta < \infty) = 1. \tag{4.11}$$

It is important to observe that the distribution of

$$\arg\max_{0\leq\theta\leq\pi} R_n^{(i)}(\theta)$$

coincides with the distribution of

$$\arg\max_{0\leq\theta\leq\pi} R_n(\theta).$$

Finally, we have that

$$\min_{1\leq i\leq n} \mathbb{P}\left(A_{n,i}(\delta)\right) \geq 1-2\mathbb{P}\left(\rho_H^1\left(\left\{\theta^{(1)}\right\}, \underset{0\leq\theta\leq\pi}{\arg\max}\, R_n(\theta)\right) \geq \delta\right),$$

which concludes the proof. □

4.3. APPROXIMATION LEMMA FOR DIAMETER

The main goal of this subsection is to prove an analog of Lemma 3.4.1 for the diameter. More precisely, we aim to compare $\mathscr{D}_{n,i}$ with appropriately centered and projected i-th step of the walk with the dominating drift (this can be the first walk, second walk, or the difference walk, but, without loss of generality, as before, we assume that this is the first walk).

As in the case of the perimeter, the proof of the lemma depends on our earlier assumptions regarding the spatial orientation of the drift vectors relative to the y-axis. Again, we only consider the case where the y-axis is situated between the drift vectors. Notice further that the assumption that $\|\boldsymbol{\mu}^{(1)}\|$ is the maximal element of the set

$$\{\|\boldsymbol{\mu}^{(1)}\|, \|\boldsymbol{\mu}^{(2)}\|, \|\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}\|\}$$

positions the drift vector $\boldsymbol{\mu}^{(1)}$ in the region $\Theta^{\varepsilon}_{(1>2>0)}$. Having in mind the previous discussion, it is clear that for a given, sufficiently small, $\delta > 0$, we can select and fix sufficiently small $\varepsilon = \varepsilon(\delta)$ such that

$$(\theta^{(1)} - \delta, \theta^{(1)} + \delta) \subseteq \Theta^{\varepsilon}_{(1>2>0)}.$$

In the following lemma, we describe the behavior of the parameterized range function at this interval.

Lemma 4.3.1. Let $\gamma \in (0, 1/2)$. Then for $\delta > 0$ and $\varepsilon > 0$ from the upper discussion, and any $i \in I_{n,\gamma}$, on $E_{n,i}(\varepsilon, \gamma)$,

$$\left| \sup_{|\theta - \theta^{(1)}| \leq \delta} R_n(\theta) - \sup_{|\theta - \theta^{(1)}| \leq \delta} R_n^{(i)}(\theta) - \left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) \cdot \mathbf{e}_{\theta^{(1)}} \right| \leq 2\delta \left\| Z_i^{(1)} - \tilde{Z}_i^{(1)} \right\|.$$

Proof. We assert that for every i belonging to the set $I_{n,\gamma}$, and for any θ_1 and θ_2 within the interval $\left(\theta^{(1)} - \delta, \theta^{(1)} + \delta\right)$ satisfying $\theta_1 < \theta_2$, the following condition holds on the event $E_{n,i}(\varepsilon,\gamma)$:

$$\inf_{\theta_1\le\theta\le\theta_2}\left(Z_i^{(1)}-\tilde{Z}_i^{(1)}\right)\cdot\mathbf{e}_\theta \le \sup_{\theta_1\le\theta\le\theta_2} R_n(\theta) - \sup_{\theta_1\le\theta\le\theta_2} R_n^{(i)}(\theta) \le \sup_{\theta_1\le\theta\le\theta_2}\left(Z_i^{(1)}-\tilde{Z}_i^{(1)}\right)\cdot\mathbf{e}_\theta. \tag{4.12}$$

Furthermore, it can be easily verified that for any $x\in\mathbb{R}^2$, and any $\theta_1,\theta_2\in\mathbb{R}$, the following holds

$$\left|x\cdot\mathbf{e}_{\theta_1}-x\cdot\mathbf{e}_{\theta_2}\right| \le \|x\|\mid\theta_1-\theta_2\mid.$$

From this, we have that

$$\begin{aligned}\sup_{\theta_1\le\theta\le\theta_2}\left(Z_i^{(1)}-\tilde{Z}_i^{(1)}\right)\cdot\mathbf{e}_\theta &= \sup_{\theta_1\le\theta\le\theta_2}\left(Z_i^{(1)}-\tilde{Z}_i^{(1)}\right)\cdot\left(\mathbf{e}_{\theta^{(1)}}+\mathbf{e}_\theta-\mathbf{e}_{\theta^{(1)}}\right)\\ &\le \left(Z_i^{(1)}-\tilde{Z}_i^{(1)}\right)\cdot\mathbf{e}_{\theta^{(1)}}+2\delta\left\|Z_i^{(1)}-\tilde{Z}_i^{(1)}\right\|.\end{aligned}$$

By analogous argumentation, we conclude

$$\inf_{\theta_1\le\theta\le\theta_2}\left(Z_i^{(1)}-\tilde{Z}_i^{(1)}\right)\cdot\mathbf{e}_\theta \ge \left(Z_i^{(1)}-\tilde{Z}_i^{(1)}\right)\cdot\mathbf{e}_{\theta^{(1)}}-2\delta\left\|Z_i^{(1)}-\tilde{Z}_i^{(1)}\right\|.$$

The assertion of the lemma follows by taking $\theta_1=\theta^{(1)}-\delta$ and $\theta_2=\theta^{(1)}+\delta$. We are left to prove (4.12). Observe that for functions $f,g:\mathbb{R}\to\mathbb{R}$, satisfying $\sup_{\theta\in I}|f(\theta)|<\infty$ and $\sup_{\theta\in I}|g(\theta)|<\infty$, for $I\subseteq\mathbb{R}$,

$$\inf_{\theta\in I}(f(\theta)-g(\theta)) \le \sup_{\theta\in I} f(\theta)-\sup_{\theta\in I} g(\theta) \le \sup_{\theta\in I}(f(\theta)-g(\theta)).$$

In particular, if $I=[\theta_1,\theta_2]$ with $\theta_1,\theta_2\in(\theta^{(1)}-\delta,\theta^{(1)}+\delta)$, we have that

$$\inf_{\theta_1\le\theta\le\theta_2}\left(R_n(\theta)-R_n^{(i)}(\theta)\right) \le \sup_{\theta_1\le\theta\le\theta_2} R_n(\theta)-\sup_{\theta_1\le\theta\le\theta_2} R_n^{(i)}(\theta)$$

and

$$\sup_{\theta_1\le\theta\le\theta_2} R_n(\theta)-\sup_{\theta_1\le\theta\le\theta_2} R_n^{(i)}(\theta) \le \sup_{\theta_1\le\theta\le\theta_2}\left(R_n(\theta)-R_n^{(i)}(\theta)\right).$$

Moreover, on the event $E_{n,i}(\varepsilon,\gamma)$, according to Proposition 3.3.1, for all $\theta\in[\theta_1,\theta_2]$ we have

$$R_n(\theta) - R_n^{(i)}(\theta) = \left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) \cdot \mathbf{e}_\theta.$$

This proves claim (4.12). □

We are now ready to prove the approximation result for $\mathscr{D}_{n,i}$. First, let us denote

$$B_{n,i}(\gamma,\delta) := E_{n,i}(\varepsilon(\delta),\gamma) \cap A_{n,i}(\delta).$$

Lemma 4.3.2. Assume that $\mathbb{E}[\|Z_1^{(k)}\|] < \infty$ for both $k \in \{1,2\}$. Let $\gamma \in (0,1/2)$, and let ε and δ be as in the previous lemma. Then, for any $i \in I_{n,\gamma}$, the following inequality holds a.s.

$$\left|\mathscr{D}_{n,i} - (Z_i^{(1)} - \boldsymbol{\mu}^{(1)}) \cdot \mathbf{e}_{\theta^{(1)}}\right| \leq 3\left(\|Z_i^{(1)}\| + \|Z_i^{(2)}\|\right) \mathbb{P}\left(B_{n,i}^{\mathrm{c}}(\gamma,\delta) \mid \mathscr{F}_i\right)$$
$$+ 2\delta\left(\|Z_i^{(1)}\| + \mathbb{E}\|Z_1^{(1)}\|\right) + 3\mathbb{E}\left[\left(\|\tilde{Z}_i^{(1)}\| + \|\tilde{Z}_i^{(2)}\|\right) \mathbf{1}\left(B_{n,i}^{\mathrm{c}}(\gamma,\delta)\right) \mid \mathscr{F}_i\right].$$

Proof. Given that the random variable $Z_i^{(1)}$ is $\mathscr{F}_i$-measurable and that $\tilde{Z}_i^{(1)}$ is independent of $\mathscr{F}_i$, it follows that

$$\mathscr{D}_{n,i} - \left(Z_i^{(1)} - \boldsymbol{\mu}^{(1)}\right) \cdot \mathbf{e}_{\theta^{(1)}} = \mathbb{E}\left[D_n - D_n^{(i)} - \left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) \cdot \mathbf{e}_{\theta^{(1)}} \mid \mathscr{F}_i\right],$$

from which it follows

$$\left|\mathscr{D}_{n,i} - \left(Z_i^{(1)} - \boldsymbol{\mu}^{(1)}\right) \cdot \mathbf{e}_{\theta^{(1)}}\right| \leq \mathbb{E}\left[\left|D_n - D_n^{(i)} - \left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) \cdot \mathbf{e}_{\theta^{(1)}}\right| \mathbf{1}\left(B_{n,i}(\gamma,\delta)\right) \mid \mathscr{F}_i\right]$$
$$+ \mathbb{E}\left[\left|D_n - D_n^{(i)} - \left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) \cdot \mathbf{e}_{\theta^{(1)}}\right| \mathbf{1}\left(B_{n,i}^{\mathrm{c}}(\gamma,\delta)\right) \mid \mathscr{F}_i\right].$$

From Lemma 3.2.1, we next establish

$$\mathbb{E}\left[\left|D_n - D_n^{(i)} - \left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right) \cdot \mathbf{e}_{\theta^{(1)}}\right| \mathbf{1}\left(B_{n,i}^{\mathrm{c}}(\gamma,\delta)\right) \mid \mathscr{F}_i\right] \leq \mathbb{E}\left[S \cdot \mathbf{1}\left(B_{n,i}^{\mathrm{c}}(\gamma,\delta)\right) \mid \mathscr{F}_i\right],$$

where

$$S = 3\left(\left\|Z_i^{(1)}\right\| + \left\|\tilde{Z}_i^{(1)}\right\|\right) + 2\left(\left\|Z_i^{(2)}\right\| + \left\|\tilde{Z}_i^{(2)}\right\|\right).$$

Now, on the event $A_{n,i}(\delta)$, we have that

$$D_n = \sup_{|\theta-\theta^{(1)}|\leq\delta} R_n(\theta), \qquad \text{and} \qquad D_n^{(i)} = \sup_{|\theta-\theta^{(1)}|\leq\delta} R_n^{(i)}(\theta),$$

and hence, by Lemma 4.3.1, on the event $B_{n,i}(\gamma,\delta)$,

$$\left|D_n - D_n^{(i)} - \left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right)\cdot \mathbf{e}_{\theta^{(1)}}\right| \leq 2\delta \left\|Z_i^{(1)} - \tilde{Z}_i^{(1)}\right\|.$$

Consequently

$$\mathbb{E}\left[\left|D_n - D_n^{(i)} - \left(Z_i^{(1)} - \tilde{Z}_i^{(1)}\right)\cdot \mathbf{e}_{\theta^{(1)}}\right| \mathbf{1}\left(B_{n,i}(\gamma,\delta)\right) \mid \mathscr{F}_i\right] \leq 2\delta \mathbb{E}\left[\left\|Z_i^{(1)}\right\| + \left\|\tilde{Z}_i^{(1)}\right\| \mid \mathscr{F}_i\right].$$

Combining all together, we get the required inequality. □

Let us denote $V_i := (Z_i^{(1)} - \boldsymbol{\mu}^{(1)})\cdot \mathbf{e}_{\theta^{(1)}}$, and $U_{n,i} := \mathscr{D}_{n,i} - V_i$. In the following lemma we show that the error term $U_{n,i}$ is L^2-negligible under the scaling $\sqrt{n}$.

Lemma 4.3.3. Assume (A2) and that $\mathbb{E}[\|Z_1^{(k)}\|^2] < \infty$ for both $k \in \{1,2\}$. Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[U_{n,i}^2\right] = 0.$$

Proof. For a given $\varepsilon \in (0,1)$, let $\gamma \in (0,1/2)$ and $\delta > 0$ be sufficiently small, the specifics of which will be clarified later. From Lemma 3.2.1, we have that

$$|U_{n,i}| \leq 3\left(\|Z_i^{(1)}\| + \mathbb{E}\|Z_1^{(1)}\| + \|Z_i^{(2)}\| + \mathbb{E}\|Z_1^{(2)}\|\right).$$

Thus, $\mathbb{E}\left(U_{n,i}^2\right) \leq C_0$ for all n and all i, and for some constant $C_0 > 0$ whose value depends solely on the distributions of $Z_i^{(k)}$. Therefore, we can conclude that

$$\frac{1}{n}\sum_{i\notin I_{n,\gamma}} \mathbb{E}\left(U_{n,i}^2\right) \leq 2\gamma C_0.$$

We choose and fix $\gamma > 0$ sufficiently small to ensure that $2\gamma C_0 < \varepsilon$. Now, for $i \in I_{n,\gamma}$, Lemma 4.3.2 provides an upper bound on $|U_{n,i}|$. Note that for any constant $C_1^{(k)} > 0$, given that $\tilde{Z}_i^{(k)}$ is independent of $\mathscr{F}_i$, we can conclude the following

$$\mathbb{E}\left[\|\tilde{Z}_i^{(k)}\|\mathbf{1}\left(B_{n,i}^{c}(\gamma,\delta)\right) \mid \mathscr{F}_i\right] \le \mathbb{E}\left[\|Z_i^{(k)}\|\mathbf{1}\left(\|Z_i^{(k)}\| \ge C_1^{(k)}\right)\right] + C_1^{(k)}\mathbb{P}\left[B_{n,i}^{c}(\gamma,\delta) \mid \mathscr{F}_i\right].$$

Given $\varepsilon \in (0,1)$, we can select $C_1 = C_1(\varepsilon) > 0$ sufficiently large such that

$$\mathbb{E}\left[\|Z_i^{(k)}\|\mathbf{1}\left(\|Z_i^{(k)}\| \ge C_1\right)\right] < \varepsilon$$

for both $k \in \{1,2\}$. For the sake of convenience, we also choose $C_1 > 1$ and $C_1 > \mathbb{E}[\|Z_1^{(k)}\|]$ for both $k \in \{1,2\}$. Consequently, by Lemma 4.3.2, we obtain that

$$|U_{n,i}| \le 3\left(\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + 2C_1\right)\mathbb{P}\left[B_{n,i}^{c}(\gamma,\delta) \mid \mathscr{F}_i\right] + 6\varepsilon + 2\delta\left(\|Z_i^{(1)}\| + \mathbb{E}\|Z_1^{(1)}\|\right).$$

Using $\mathbb{P}\left[B_{n,i}^{c}(\gamma,\delta) \mid \mathscr{F}_i\right] \le 1$, $\varepsilon \le 1$, $\delta \le 1$, and the elementary inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ for positive $a,b,c \in \mathbb{R}$, we conclude

$$U_{n,i}^2 \le 27\left(\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + 2C_1\right)^2\mathbb{P}\left[B_{n,i}^{c}(\gamma,\delta) \mid \mathscr{F}_i\right] + 108\varepsilon + 12\delta\left(\|Z_i^{(1)}\| + \mathbb{E}\|Z_1^{(1)}\|\right)^2.$$

By assumption, for a given ε, there is δ small enough such that

$$12\delta\left(\|Z_i^{(1)}\| + \mathbb{E}\|Z_1^{(1)}\|\right)^2 < \varepsilon.$$

We shall fix such $\delta > 0$ for the remainder of the discussion. We then have

$$\mathbb{E}\left(U_{n,i}^2\right) \le 27\mathbb{E}\left[\left(\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + 2C_1\right)^2\mathbb{P}\left[B_{n,i}^{c}(\gamma,\delta) \mid \mathscr{F}_i\right]\right] + 109\varepsilon.$$

Next, for any $C_2 > 0$, we have that

$$\begin{aligned}&\mathbb{E}\left[\left(\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + 2C_1\right)^2\mathbb{P}\left[B_{n,i}^{c}(\gamma,\delta) \mid \mathscr{F}_i\right]\right]\\ &\le C_2^2\mathbb{P}\left(B_{n,i}^{c}(\gamma,\delta)\right) + \mathbb{E}\left[\left(\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + 2C_1\right)^2\mathbf{1}\left(\|Z_i^{(1)}\| + \|Z_i^{(2)}\| + 2C_1 \ge C_2\right)\right],\end{aligned}$$

and using the dominated convergence theorem, it is possible to choose a value C_2 sufficiently large such that the last term is less than $\varepsilon/27$. Consequently, with this choice of C_2, we have

$$\mathbb{E}\left(U_{n,i}^2\right) \leq 110\varepsilon + 27C_2^2\mathbb{P}\left(B_{n,i}^{\mathrm{c}}(\gamma,\delta)\right).$$

By Theorem 4.2.1 and Proposition 3.3.1, we conclude

$$\lim_{n\to\infty} \max_{1\leq i\leq n} \mathbb{P}\left(B_{n,i}^{\mathrm{c}}(\gamma,\delta)\right) = 0,$$

so that, for given $\varepsilon > 0$ (and hence C_1 and C_2) we may select $n_0 \in \mathbb{N}$ sufficiently large so that $\max_{i\in I_{n,\gamma}} \mathbb{E}\left(U_{n,i}^2\right) \leq 111\varepsilon$, for $n \geq n_0$. Consequently,

$$\frac{1}{n}\sum_{i\in I_{n,\gamma}} \mathbb{E}\left(U_{n,i}^2\right) \leq 111\varepsilon,$$

for all $n \geq n_0$. Combining this with the earlier approximation for $i \notin I_{n,\gamma}$, we arrive at

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left(U_{n,i}^2\right) \leq 112\varepsilon,$$

for all $n \geq n_0$. Since $\varepsilon > 0$ was arbitrarily chosen, the conclusion follows. □

4.4. CLT FOR DIAMETER

The proofs of the main results for the diameter in the largest part follow as in the perimeter case. In the proof of Theorem 4.4.1 instead of using Lemma 3.1.1 and Lemma 3.4.3, we employ Lemma 4.1.1 and Lemma 4.3.3.

Theorem 4.4.1. Assume (A2), and that the maximal element of the set from the assumption (A2) is $\|\boldsymbol{\mu}^{(1)}\|$. Then,

$$n^{-1/2}\left|D_n - \mathbb{E}D_n - \left(S_n^{(1)} - \mathbb{E}S_n^{(1)}\right)\cdot \mathbf{e}_{\theta^{(1)}}\right| \xrightarrow[n\to\infty]{L^2} 0.$$

If the maximal element is $\|\boldsymbol{\mu}^{(2)}\|$, our proof follows in the same manner, only interchanging first and second random walk, while in the third scenario we consider the difference of the two walks and the angle $\theta^{(k)}$ is replaced by the angle corresponding to the direction of the vector $\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}$, see Chapter 5.2 for details. The proof of Theorem 4.4.2 follows analogously as the proof of Theorem 3.5.2 with the only difference being in the definition of the sequence $(\zeta_n)_{n=1}^{\infty}$:

$$\zeta_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(Z_i^{(1)} - \boldsymbol{\mu}^{(1)}\right)\cdot \mathbf{e}_{\theta^{(1)}}. \tag{4.13}$$

Theorem 4.4.2. Assume (A2) and that the maximal element of the set from assumption (A2) is $\|\boldsymbol{\mu}^{(1)}\|$. Then,

$$\lim_{n\to\infty}\frac{\mathbb{V}\mathrm{ar}\left[D_n\right]}{n} = \sigma_D^2 \in [0,\infty),$$

where

$$\sigma_D^2 = \mathbb{E}\left[\left(\left(Z_1^{(1)} - \boldsymbol{\mu}^{(1)}\right)\cdot \mathbf{e}_{\theta^{(1)}}\right)^2\right].$$

If the maximal element is not $\|\boldsymbol{\mu}^{(1)}\|$, σ_D^2 is modified as commented above. The proof of the Central Limit Theorem presented in Theorem 4.4.3 remains the same by replacing the constant σ_L with the constant σ_D.

Theorem 4.4.3. Assume (A2) and $\sigma_D^2 > 0$. Then, for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{D_n - \mathbb{E}[D_n]}{\sqrt{\mathbb{V}\mathrm{ar}[D_n]}} \leq x\right] = \lim_{n\to\infty} \mathbb{P}\left[\frac{D_n - \mathbb{E}[D_n]}{\sqrt{\sigma_D^2 n}} \leq x\right] = \Phi(x).$$

Remark 4.4.4. Let's look once again at Theorem 4.4.3. Notice that for σ_D^2 to be strictly positive, it is sufficient, and necessary, that the variance of the projection of the first random walk onto the vector $\mathbf{e}_{\theta^{(1)}}$ is non-zero. When this variance is zero, the walk is characterized by deterministic (rather than random) behavior along this particular direction.

5. DISCUSSION AND SIMULATIONS STUDY

In this chapter, we address some questions that may arise for the reader after the previous chapters and extend the results obtained so far to some similar but very intriguing problems. Therefore, in the first section, we examine the process of centroids and answer whether comparable results can be obtained for the perimeter and diameter of the convex hull generated by the centroids of a random walk.

In the second section, we discuss some assumptions we implicitly made in the previous discussions and also answer the question of what happens if the problem is extended to more than two independent random walks in the plane.

Finally, in the third section, we conduct a simulation study in which we explore the *boundary cases*, that is, cases not covered by the assumption (A1) for the perimeter or by the assumption (A2) for the diameter. It is worth noting that this dissertation does not include formal proof for the boundary cases, representing a potential direction for further research related to this topic.

5.1. HULL OF CENTROIDS

Let $Z, Z_1, Z_2, \ldots$ be independent and identically distributed random vectors such that $\mathbb{E}\|Z\|^2 < \infty$. In this case, we consider only one planar random walk generated by the above increments:

$$S_0 := 0, \quad S_n := \sum_{i=1}^{n} Z_i,$$

thus omitting the walk index in the superscript as done previously (since we only have a single random walk). We study the *process of centroids*, which is defined as follows:

$$G_0 := 0, \quad G_n := \frac{1}{n+1}\sum_{i=1}^{n} S_i.$$

With a very simple algebraic manipulation, we can derive that G_n can be expressed as follows:

$$\begin{aligned} G_n &= \frac{1}{n+1}\sum_{i=1}^{n} S_i = \frac{1}{n+1}\sum_{i=1}^{n}\sum_{m=1}^{i} Z_m \\ &= \frac{1}{n+1}\sum_{m=1}^{n}\sum_{i=m}^{n} Z_m = \frac{1}{n+1}\sum_{m=1}^{n}(n-m+1)Z_m \\ &= \sum_{m=1}^{n}\frac{n-m+1}{n+1}Z_m = \sum_{m=1}^{n}\omega_{n,m}Z_m, \end{aligned}$$

where, for simplicity and ease of notation, we have introduced:

$$\omega_{n,m} := \frac{n-m+1}{n+1}. \tag{5.1}$$

Thus, we can observe that the process of centroids is nothing more than a weighted random walk with the above-described weights. In this section, we aim to provide analogous results for the perimeter and diameter of convex hulls generated by the centroids, specifically:

$$L_n^G := \operatorname{Per}\left(\operatorname{chull}\left\{G_j : 0 \le j \le n\right\}\right), \quad n \ge 0,$$

and

$$D_n^G := \operatorname{diam}\left(\operatorname{chull}\left\{G_j : 0 \le j \le n\right\}\right), \quad n \ge 0.$$

We introduce a simple lemma that describes the behavior of these weights to simplify the later proof.

Lemma 5.1.1. Let the triangular array of weights $(\omega_{n,m})_{1\le m\le n}$ be defined as above in (5.1). Then the following holds:

(i) $0 < \omega_{n,m} < 1$ for every $n \in \mathbb{N}$ and every $1 \le m \le n$.

(ii)

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\omega_{n,m}^2 = \frac{1}{3}.$$

Proof. The first statement, (i), is evident. To prove the second statement, (ii), we proceed as follows:

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\omega_{n,m}^2 &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{n-m+1}{n+1}\right)^2 \\ &= \frac{1}{n(n+1)^2}\sum_{i=1}^{n}(n-m+1)^2 \\ &= \frac{1}{n(n+1)^2}\sum_{i=1}^{n}m^2 \\ &= \frac{1}{n(n+1)^2}\cdot\frac{n(n+1)(2n+1)}{6},\end{aligned}$$

where, in the last equality, we used the formula for the sum of the first n squares. By taking the limit as $n\to\infty$, we obtain the desired result for statement (ii). □

Remark 5.1.2. In the following proof, we will focus on the specific weights that we defined in (5.1) in order to examine exclusively the convex hull of the centroids. However, it is important to note that the subsequent proof is valid for any choice of weights $\omega_{n,m}$ that satisfies the fundamental assumptions outlined in Lemma 5.1.1. Specifically, the only requirements that this sequence must meet are that it is non-negative, uniformly bounded, and that the sum of the squares, scaled by n, converges to some positive value.

The proof principle for the perimeter and diameter of the convex hull generated by the centroids is very similar to the techniques used in the proofs for the perimeter and diameter of the convex hull generated by the original points of a planar random walk (see [WX15a] for the perimeter and [MW18] for the diameter). Without stepping into the details of the analogous proofs, we can state in the following result the L^2 approximations for the deviations of the perimeter and diameter of the convex hull of the centroids. As earlier, we denote by $\boldsymbol{\mu}$ the drift vector of the increment, and by θ_0 the angle corresponding to $\boldsymbol{\mu}$ in polar coordinates.

Theorem 5.1.3. Suppose that $\mathbb{E}[\|Z\|^2]<\infty$ and $\boldsymbol{\mu}\neq 0$. Then, as $n\to\infty$:

(i)

$$n^{-1/2}\left|L_n^G-\mathbb{E}\left[L_n^G\right]-\sum_{m=1}^{n}2\omega_{n,m}\left(Z_m-\mathbb{E}\left[Z\right]\right)\cdot\mathbf{e}_{\theta_0}\right|\to 0,\text{ in } L^2,\text{ and}$$

(ii)

$$n^{-1/2}\left|D_n^G-\mathbb{E}\left[D_n^G\right]-\sum_{m=1}^{n}\omega_{n,m}\left(Z_m-\mathbb{E}\left[Z\right]\right)\cdot\mathbf{e}_{\theta_0}\right|\to 0,\text{ in }L^2.$$

The key step and the fundamental difference compared to the proof of the Central Limit Theorem for the perimeter and diameter processes of the hull generated by the original points lies in demonstrating the Central Limit Theorem for the approximation expressions in Theorem 5.1.3. Recall that in the proof for the perimeter and diameter of the hull generated by the original points; the process was relatively straightforward - the deviation of the perimeter and diameter was approximated by a sum of independent and identically distributed random variables. By assuming these variables have a finite second moment, we directly applied Lévy's version of the Central Limit Theorem (Theorem 1.3.6). In the case of centroids, however, the variables are not identically distributed, so we must consider the Lindeberg-Feller version of the Central Limit Theorem (Theorem 1.3.8).

Theorem 5.1.4. Suppose that $\mathbb{E}[\|Z_1\|^2]<\infty$ and $\boldsymbol{\mu}\neq 0$. Suppose that

$$\sigma_{\mu}^2:=\mathbb{E}\left[\left((Z-\boldsymbol{\mu})\cdot\mathbf{e}_{\theta_0}\right)^2\right]>0.$$

Then for any $x\in\mathbb{R}$:

(i)

$$\lim_{n\to\infty}\mathbb{P}\left[\frac{L_n^G-\mathbb{E}\left[L_n^G\right]}{\sqrt{\mathbb{V}\text{ar}\left[L_n^G\right]}}\leq x\right]=\lim_{n\to\infty}\mathbb{P}\left[\frac{L_n^G-\mathbb{E}\left[L_n^G\right]}{\sqrt{\frac{4}{3}\cdot\sigma_{\mu}^2 n}}\leq x\right]=\Phi(x),\quad\text{and}$$

(ii)

$$\lim_{n\to\infty}\mathbb{P}\left[\frac{D_n^G-\mathbb{E}\left[D_n^G\right]}{\sqrt{\mathbb{V}\text{ar}\left[D_n^G\right]}}\leq x\right]=\lim_{n\to\infty}\mathbb{P}\left[\frac{D_n^G-\mathbb{E}\left[D_n^G\right]}{\sqrt{\frac{1}{3}\cdot\sigma_{\mu}^2 n}}\leq x\right]=\Phi(x),$$

where Φ is the CDF of the standard normal distribution function.

Proof. In the proof, we will only show statement (i) because statement (ii) can be proven in exactly the same way. Let us define:

$$X_{n,m}:=\frac{1}{\sqrt{n}}2\omega_{n,m}\left(Z_m-\mathbb{E}Z\right)\cdot\mathbf{e}_{\theta_0}.$$

We observe that:

$$\begin{aligned}\sum_{m=1}^{n}\mathbb{E}X_{n,m}^2 &= \sum_{m=1}^{n}\mathbb{V}\text{ar}\left(\frac{1}{\sqrt{n}}2\omega_{n,m}\left(Z_m-\mathbb{E}Z\right)\cdot\mathbf{e}_{\theta_0}\right)\\ &= \sum_{m=1}^{n}\frac{1}{n}\cdot 4\cdot\omega_{n,m}^2\cdot\mathbb{V}\text{ar}\left(\left(Z_m-\mathbb{E}Z\right)\cdot\mathbf{e}_{\theta_0}\right)\\ &= \frac{4\sigma_\mu^2}{n}\sum_{m=1}^{n}\omega_{n,m}^2.\end{aligned}$$

By letting $n\to\infty$ and using statement (ii) of Lemma 5.1.1, we obtain:

$$\lim_{n\to\infty}\sum_{m=1}^{n}\mathbb{E}X_{n,m}^2=\frac{4}{3}\sigma_\mu^2. \tag{5.2}$$

Now, let $\varepsilon>0$ be arbitrary. Then we have:

$$\begin{aligned}&\mathbb{E}\left[|X_{n,m}|^2\cdot\mathbf{1}\{|X_{n,m}|>\varepsilon\}\right]\\ &\qquad=\mathbb{E}\left[\frac{1}{n}\cdot 4\omega_{n,m}^2\cdot\left[(Z_m-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right]^2\cdot\mathbf{1}\left\{\frac{1}{\sqrt{n}}\cdot 2\omega_{n,m}\cdot\left|(Z_m-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right|>\varepsilon\right\}\right],\end{aligned}$$

which is equal to

$$\frac{4}{n}\omega_{n,m}^2\mathbb{E}\left(\left[(Z-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right]^2\cdot\mathbf{1}\left\{\left|(Z-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right|>\varepsilon\cdot\frac{\sqrt{n}}{2\omega_{n,m}}\right\}\right).$$

According to Lemma 5.1.1, statement (i), we know that $0<\omega_{n,m}<1$, so we have:

$$\begin{aligned}&\frac{4}{n}\omega_{n,m}^2\mathbb{E}\left(\left[(Z-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right]^2\cdot\mathbf{1}\left\{\left|(Z-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right|>\varepsilon\cdot\frac{\sqrt{n}}{2\omega_{n,m}}\right\}\right)\\ &\qquad\le\frac{4}{n}\omega_{n,m}^2\mathbb{E}\left(\left[(Z-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right]^2\cdot\mathbf{1}\left\{\left|(Z-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right|>\varepsilon\cdot\frac{\sqrt{n}}{2}\right\}\right).\end{aligned}$$

Thus, we have:

$$\begin{aligned}&\sum_{m=1}^{n}\mathbb{E}\left[|X_{n,m}|^2\cdot\mathbf{1}\{|X_{n,m}|>\varepsilon\}\right]\\ &\qquad\le\frac{4}{n}\mathbb{E}\left(\left[(Z-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right]^2\cdot\mathbf{1}\left\{\left|(Z-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right|>\varepsilon\cdot\frac{\sqrt{n}}{2}\right\}\right)\sum_{m=1}^{n}\omega_{n,m}^2,\end{aligned}$$

and again, using statement (ii) of Lemma 5.1.1, we obtain

$$\frac{4}{n}\mathbb{E}\left(\left[(Z-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right]^2\cdot\mathbf{1}\left\{\left|(Z-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right|>\varepsilon\cdot\frac{\sqrt{n}}{2}\right\}\right)\sum_{m=1}^{n}\omega_{n,m}^2$$
$$\leq C_1\cdot\mathbb{E}\left(\left[(Z-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right]^2\cdot\mathbf{1}\left\{\left|(Z-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}\right|>\varepsilon\cdot\frac{\sqrt{n}}{2}\right\}\right),$$

where $C_1<\infty$ is a positive constant which exists due to Lemma 5.1.1, statement (ii). Since $\mathbb{E}([(Z-\mathbb{E}Z)\cdot\mathbf{e}_{\theta_0}]^2)<\infty$, by the Dominated Convergence Theorem, we obtain:

$$\lim_{n\to\infty}\sum_{m=1}^{n}\mathbb{E}\left[|X_{n,m}|^2\cdot\mathbf{1}\{|X_{n,m}|>\varepsilon\}\right]=0. \tag{5.3}$$

We notice that (5.2) and (5.3) are precisely the conditions of the Lindeberg-Feller Central Limit Theorem (Theorem 1.3.8), so we have that:

$$\sum_{m=1}^{n}X_{n,m}\xrightarrow{\mathscr{D}}Z,$$

where $Z\sim\mathcal{N}(0,\frac{4}{3}\sigma_\mu^2)$. Similarly, as in the proofs of Theorem 3.5.3 and Theorem 4.4.3, by using Slutsky's theorem (Theorem 1.3.3), we obtain the desired result. □

5.2. Discussion

This section focuses on specific situations that may have remained unexplained. To begin with, an interesting question arises when we assume that $\|\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}\|$ is the unique maximal element of the set from the assumption (A2). Under this assumption, Theorem 4.4.1 should be restated in the following form:

$$n^{-1/2}\left|D_n - \mathbb{E}D_n - \left(\left(S_n^{(1)} - \mathbb{E}S_n^{(1)}\right) - \left(S_n^{(2)} - \mathbb{E}S_n^{(2)}\right)\right)\cdot \mathbf{e}_\theta\right| \xrightarrow[n\to\infty]{L^2} 0.$$

Here, $\theta \in [0,\pi)$ denotes the angle corresponding to the direction of $\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}$. Consequently, both random walks under consideration contribute to the asymptotic behavior of the diameter in this particular scenario.

Furthermore, the analysis of the diameter of the convex hull spanned by two random walks can be adapted to the more general scenario involving m independent random walks. The asymptotic behavior of the diameter is controlled by the geometric characteristics of the set:

$$\left\{\|x-y\| : x,y \in \left\{\boldsymbol{\mu}^{(k)} : k = 0,\dots,m\right\}\right\},$$

where, for the sake of conventional notation, we have additionally introduced a degenerate random walk $(S_n^{(0)})_{n=0}^{\infty}$ whose increments are almost surely equal to zero. Namely, if this set possesses a unique maximal element, it can be demonstrated that:

$$n^{-1/2}\left|D_n - \mathbb{E}D_n - \left(\left(S_n^{(k_1)} - \mathbb{E}S_n^{(k_1)}\right) - \left(S_n^{(k_2)} - \mathbb{E}S_n^{(k_2)}\right)\right)\cdot \mathbf{e}_\theta\right| \xrightarrow[n\to\infty]{L^2} 0,$$

where $k_1, k_2 \in \{0,\dots,m\}$ are selected such that $\|\boldsymbol{\mu}^{(k_1)} - \boldsymbol{\mu}^{(k_2)}\|$ is the (unique) largest element in the set mentioned above, and θ keeps its role as the angle corresponding to the direction $\boldsymbol{\mu}^{(k_1)} - \boldsymbol{\mu}^{(k_2)}$. Similarly, the present discussion does not readily extend to scenarios involving multiple maximal elements within the set.

On the other hand, the problem of the perimeter of the convex hull generated by multiple independent random walks is much more demanding. To more effectively handle the extrema ($M_n(\theta)$ and $m_n(\theta)$), one might consider using an alternative version of the

Cauchy formula for the perimeter, given by:

$$L_n = \int_0^{2\pi} M_n(\theta)\mathrm{d}\theta. \tag{5.4}$$

Yet, the specific random walks contributing to this integral depend upon the geometric properties of the convex hull formed by their respective drift vectors. If the convex hull of these drift vectors coincides with the convex hull of a subset (of the original set) of drift vectors where all are non-zero, the argument above can be adjusted to arrive at a Gaussian limiting distribution. An example of such a setting is presented in Figure 5.1 on the right graph.

To further clarify the idea behind the proof in this case, let's assume for simplicity that 0 lies in the interior of the convex hull of the drift vectors. Let M represent the set of indices of the walks, counterclockwise ordered, that form the smallest subset of $\left\{\boldsymbol{\mu}^{(k)} : k = 1,\dots,m\right\}$ which generates chull $\left\{\boldsymbol{\mu}^{(k)} : k = 1,\dots,m\right\}$, and reindex it so that it corresponds to the set $\{1,\dots,|M|\}$. The idea is to introduce resampling, similar to the case with two walks, and to break down the integral in (5.4) into several integrals. Within each region, we know which walk *dominates* the others, meaning, more precisely, which walk is most likely to provide the maximum projection for the observed angle θ. In that case, the Cauchy formula can be approximated as follows:

$$n^{-1/2}\left|\int_0^{2\pi} M_n(\theta)\mathrm{d}\theta - \sum_{k=1}^{|M|}\int_{\theta_k}^{\theta_{k+1}} M_n^{(k)}(\theta)\mathrm{d}\theta\right| \xrightarrow[n\to\infty]{L^2} 0,$$

where θ_k represents the angle that marks the boundary between two regions where we can detect the dominant walk (with the additional note that $\theta_{|M|+1} := \theta_1$). By introducing resampling and evaluating the corresponding integral, we can obtain:

$$n^{-1/2}\left|L_n - \mathbb{E}L_n - \sum_{k=1}^{|M|}\sum_{i=1}^{n}\left[(Z_i^{(k)} - \mathbb{E}Z_1^{(k)})(\mathbf{e}_{\theta_{k+1}}^{\perp} - \mathbf{e}_{\theta_k}^{\perp})\right]\right| \xrightarrow[n\to\infty]{L^2} 0, \tag{5.5}$$

where the choice of $\mathbf{e}_{\theta_k}^{\perp}$ is such that $\{\mathbf{e}_{\theta_k}^{\perp}, \mathbf{e}_{\theta_k}\}$ forms a right-handed orthonormal basis for $\mathbb{R}^2$. The intuition is as follows: each walk whose drift vector contributes to the convex hull of the drift vectors will appear twice in the approximation expression (5.5), once for

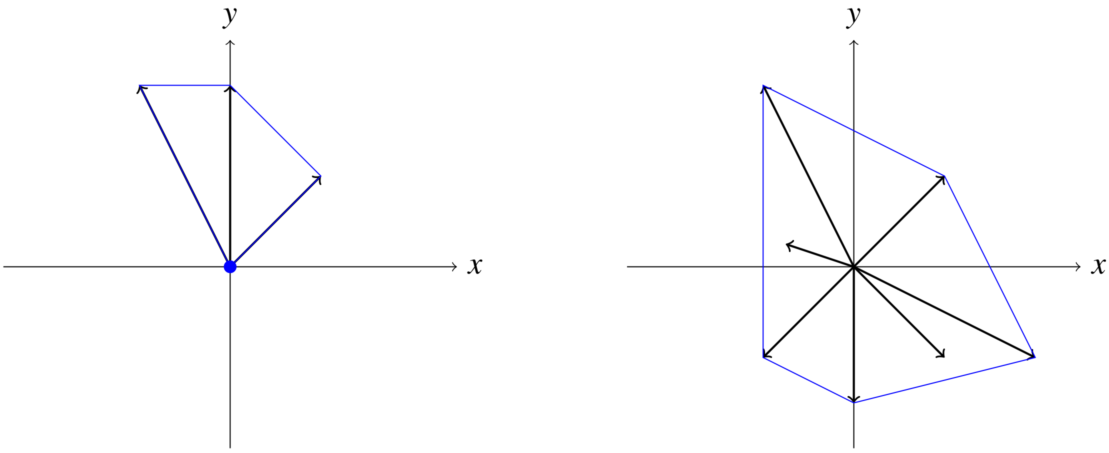


Figure 5.1: Different positions of drift vectors

each adjacent side of the convex hull of the drift vectors.

On the other hand, if there exists a zero-drift random walk and the zero lies on the boundary of the convex hull, a non-Gaussian limit can be expected. The reasoning behind this is relatively straightforward. As we will see in the next section, it was shown in [WX15b] that, in the case of a single random walk with zero drift, the perimeter process has a non-Gaussian limit. If a zero-drift walk is involved and the zero vector is on the edge of the convex hull of the drift vectors, it is likely that this walk will contribute a non-Gaussian component to the overall limit, making the entire limit distribution non-Gaussian. An example of this scenario is depicted in the left graph of Figure 5.1.

5.3. Simulation study

As we already mentioned, in [WX15b], it has been shown that for a single zero-drift planar random walk, $(L_n - \mathbb{E}[L_n])/\sqrt{n}$ converges in distribution to a non-degenerate non-Gaussian limit. We conjecture a similar phenomenon in the case of two planar random walks when the assumption (A1) is not satisfied, that is,

$$\mathbf{0} \in \{\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}\}.$$

In the case when $\{\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}\} = \{\mathbf{0}\}$ this can be proved by completely the same arguments as in [WX15b]. The limiting object can be expressed in terms of the perimeter of the convex hull spanned by two independent planar Brownian motions.

More delicate situation arises when $\mathbf{0} \in \{\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}\} \neq \{\mathbf{0}\}$. In what follows, we provide a simulation study that supports our conjecture, leaving the clarification of our conjecture open. In the case when $\boldsymbol{\mu}^{(1)} \neq \boldsymbol{\mu}^{(2)} = \mathbf{0}$, or $\boldsymbol{\mu}^{(2)} \neq \boldsymbol{\mu}^{(1)} = \mathbf{0}$, our intuition was that, since both walks contribute to the convex hull (see Figure 5.2), distributional limit will not be Gaussian. On one hand, a single (non-degenerate) planar random walk with non-zero drift generates a convex hull whose perimeter has a Gaussian behavior (see [WX15a, Theorem 1.2]). Still, a single zero-drift planar random walk generates a convex hull whose perimeter does not have a Gaussian behavior (see [WX15b, Corollary 2.6 and Proposition 3.7]). This non-Gaussian part affects the convex hull generated by both walks combined. We ran some simulations and the results are shown in Figure 5.3.

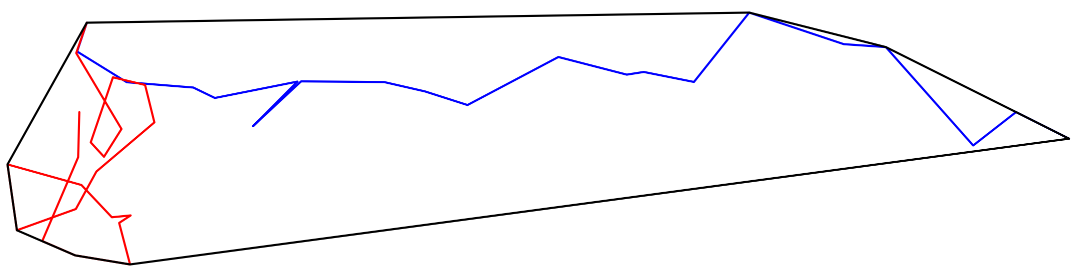


Figure 5.2: The convex hull of two independent planar random walks with parameters $\boldsymbol{\mu}^{(1)} = (1.5, 0)$ (blue), $\boldsymbol{\mu}^{(2)} = (0, 0)$ (red), $\boldsymbol{\Sigma}^{(1)} = \boldsymbol{\Sigma}^{(2)} = I_2$, where I_2 is the two-dimensional identity matrix.

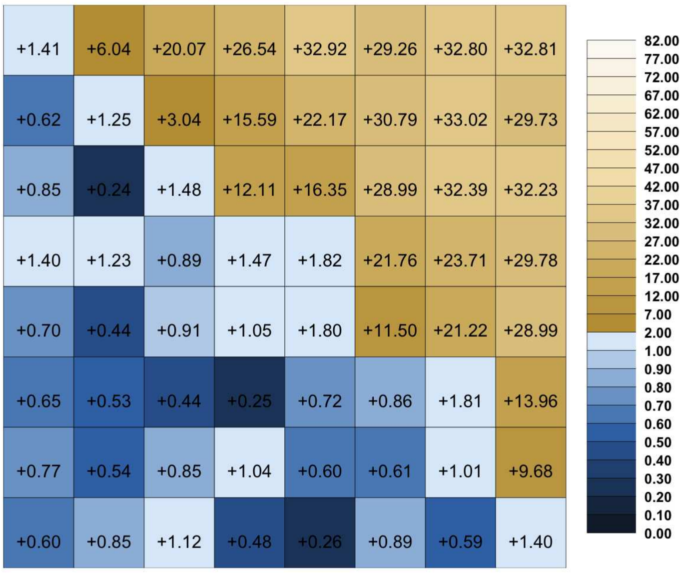


Figure 5.3: Simulation results for the perimeter process – $\boldsymbol{\mu}^{(1)} = (100,0)$, $\boldsymbol{\mu}^{(2)} = (0,0)$.

Since we were simulating two planar random walks, we had some freedom in the design of our simulation study, but we kept everything as simple as possible. Namely, covariance matrices were always multiples of the identity matrix, and the steps of random walks were generated from multivariate normal distribution. To see what happens in the scenario when one of the two walks has a non-zero drift and the other one has a zero drift (illustrated in Figure 5.2), we set one drift vector to $(100,0)$, and the other one, clearly, to $(0,0)$.

As mentioned, the covariance matrices of both walks were always of the shape σI_2 (where I_2 is the two-dimensional identity matrix). We varied the value of σ across all the elements from the set $\{0.1, 0.5, 1, 5, 10, 50, 100, 500\}$. More precisely, for every combination of $\sigma_1, \sigma_2 \in \{0.1, 0.5, 1, 5, 10, 50, 100, 500\}$ we simulated 10^3 random walks with parameters $\boldsymbol{\mu}^{(1)} = (100,0)$, $\boldsymbol{\Sigma}^{(1)} = \sigma_1 I_2$ and $\boldsymbol{\mu}^{(2)} = (0,0)$, $\boldsymbol{\Sigma}^{(2)} = \sigma_2 I_2$. In each of those 10^3 simulations, we simulated 10^4 steps of both random walks, determined the convex hull generated by the trajectories of both walks and then calculated the perimeter of the resulting convex hull. Hence, for each combination of values of σ_1 and σ_2, we had 10^3 realizations of a random variable L_n, for $n = 10^4$. We then tested those 10^3 realizations for normality and calculated the p-value. To gain additional stability of our simulations, we repeated the procedure 5 times and averaged all the p-values obtained.

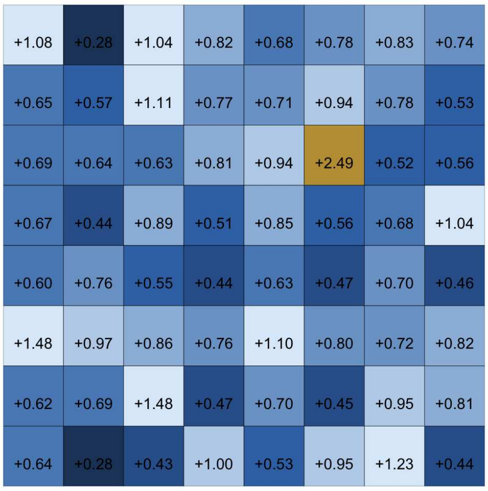
Since we varied the values of σ_1 and σ_2 across 8 different values, we ended up with 8×8 matrix of averaged p-values. We then transformed the matrix elements with the mapping $x \mapsto -\log x$ so that it is easier to present the results. After this transformation, the values in the matrix that were less than or equal to 2 corresponded to p-values big enough to suggest not rejecting the Gaussian distribution hypothesis. Bigger values in the matrix correspond to smaller p-values and point in the direction of non-Gaussian behavior. To stress this difference between values less than or equal to 2 and values bigger than 2, we use different color palettes for those two ranges of values. In Figure 5.3, as in all the figures that follow, the color on the position (i, j) (starting from top left corner) corresponds to the simulation in which σ_1 is equal to the i-th value, and σ_2 to the j-th value from the set $\{0.1, 0.5, 1, 5, 10, 50, 100, 500\}$.

As one can see in Figure 5.3, if the variability of the zero-drift random walk is smaller than or equal to the variability of the random walk with the non-zero drift simulations suggest not to reject the hypothesis of Gaussian distribution. However, we believe that, in this case, the impact of the non-Gaussian part is too small for the test to detect. As soon as the variance of the zero-drift random walk is bigger, the simulations clearly suggest non-Gaussian behavior.

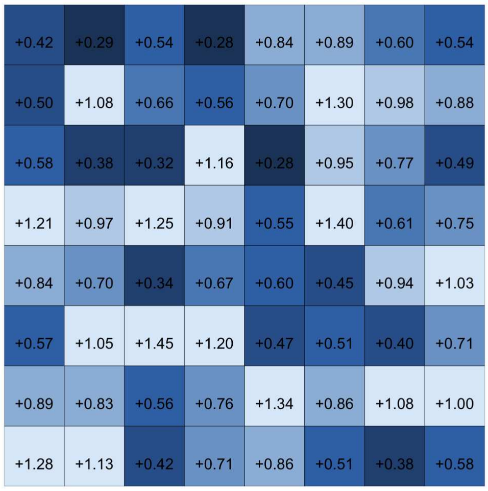
For illustration, we conduct the same experiment in the case when the assumption (A1) is not violated. We can see in Figure 5.4 that the same design of the simulation study as above captures the behavior proven in Theorem 3.5.3.

The last scenario in which assumption (A1) is not satisfied is the one where $\boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} \neq \mathbf{0}$. It was clear to us that our approach to the proof of Theorem 3.5.3 cannot cover this case, but our first impression was that the normality will still hold. Hence, it was somewhat surprising to us when simulations suggested that in this case we again do not have normal behavior (see Figure 5.5). These simulation results motivated the formulation of the assumption (A1) in the present form. Possible justification is that one has to consider the triangle spanned by the drift vectors, and as soon as one of the three sides has length zero, the normality does not hold.

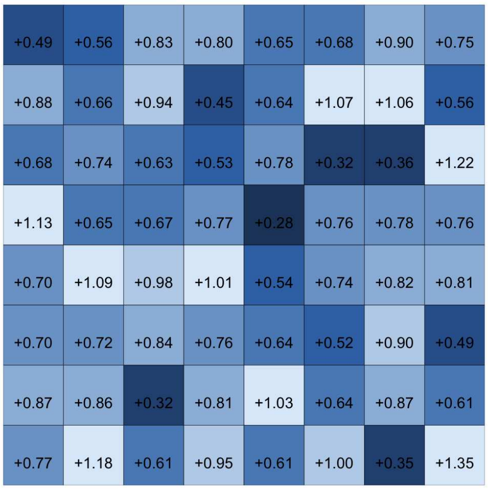
When it comes to the assumption (A2) in the diameter case, we have completely analogous situation as above. We repeated all the experiments as above and got analogous results (see Figure 5.6 and Figure 5.7). In the central limit theorem for the diameter



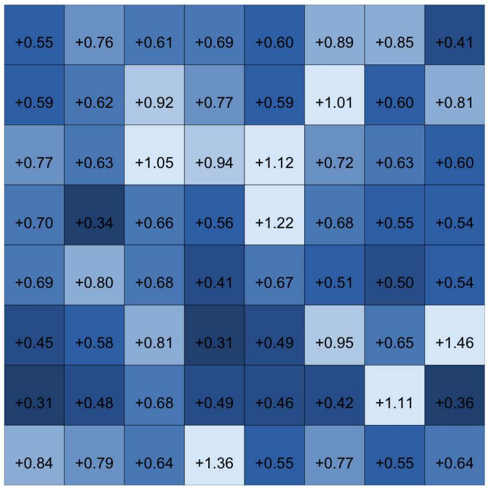
(a) $\boldsymbol{\mu}^{(1)} = (200, 100)$, $\boldsymbol{\mu}^{(2)} = (-100, 100)$



(b) $\boldsymbol{\mu}^{(1)} = (200, 100)$, $\boldsymbol{\mu}^{(2)} = (100, 100)$



(c) $\boldsymbol{\mu}^{(1)} = (200, 0)$, $\boldsymbol{\mu}^{(2)} = (-100, 0)$



(d) $\boldsymbol{\mu}^{(1)} = (200, 0)$, $\boldsymbol{\mu}^{(2)} = (100, 0)$

Figure 5.4: Simulation results (for the perimeter process) illustrating Theorem 3.5.3

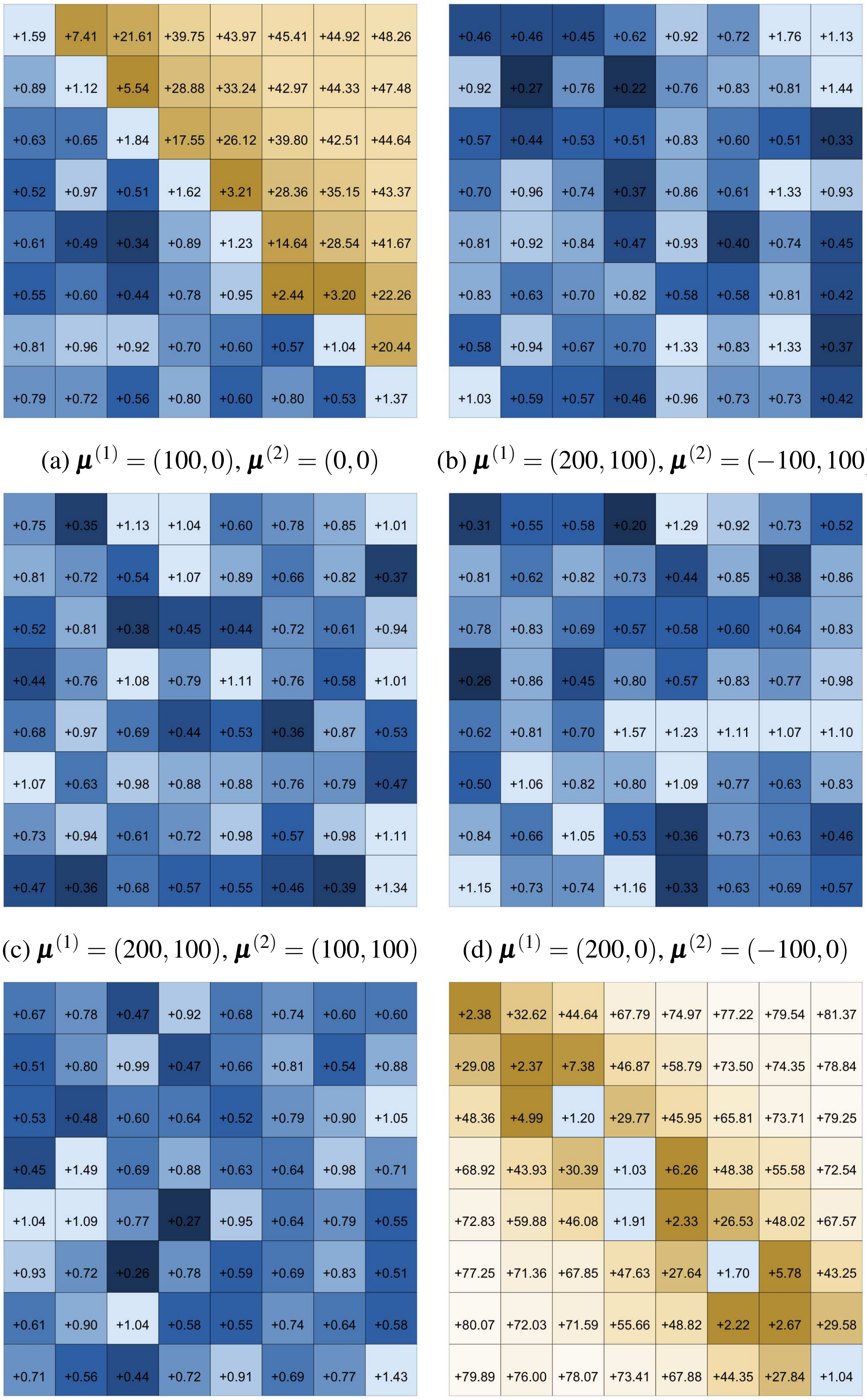
(Theorem 4.4.3), if (A2) is not satisfied, our method of proof does not hold. Nevertheless, our intuition was that the normality should still hold if (A1) is still satisfied. We were quite surprised to see that simulations suggest non-Gaussian behavior when the set $\{\|\boldsymbol{\mu}^{(1)}\|, \|\boldsymbol{\mu}^{(2)}\|, \|\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}\|\}$ does not have a unique maximal element. Those simulation results are shown in Figure 5.7.

One thing that the simulations suggest is that the variability of the walks does not change the limiting behavior of the studied processes (as long as $\sigma_L > 0$ and $\sigma_D > 0$), but it has an effect on simulations. Therefore, it could be that because of a bad simulation study design, we conjectured something that does not hold. Regardless of that, it seems

+1.16	+19.27	+33.08	+48.81	+54.26	+55.62	+58.77	+58.32
+20.30	+0.78	+3.36	+32.63	+41.99	+53.92	+57.15	+58.22
+36.18	+2.53	+0.61	+19.35	+33.81	+51.84	+54.54	+57.96
+51.65	+31.33	+23.45	+0.87	+3.11	+34.19	+43.89	+51.88
+50.94	+45.41	+33.77	+3.29	+1.37	+20.97	+33.20	+50.04
+56.52	+53.64	+51.20	+32.97	+14.23	+1.51	+9.06	+29.16
+59.38	+57.03	+51.63	+40.84	+32.97	+3.42	+0.99	+19.27
+56.87	+58.39	+57.51	+52.32	+52.14	+31.18	+18.49	+1.35

Figure 5.5: Simulation results for the perimeter process – $\boldsymbol{\mu}^{(1)} = (100,0)$, $\boldsymbol{\mu}^{(2)} = (100,0)$.

that scenarios excluded with assumptions (A1) and (A2) require additional work and a different approach, and the efforts to extend our results in this direction are currently underway.



(a) $\boldsymbol{\mu}^{(1)} = (100, 0)$, $\boldsymbol{\mu}^{(2)} = (0, 0)$ (b) $\boldsymbol{\mu}^{(1)} = (200, 100)$, $\boldsymbol{\mu}^{(2)} = (-100, 100)$

(c) $\boldsymbol{\mu}^{(1)} = (200, 100)$, $\boldsymbol{\mu}^{(2)} = (100, 100)$ (d) $\boldsymbol{\mu}^{(1)} = (200, 0)$, $\boldsymbol{\mu}^{(2)} = (-100, 0)$

(e) $\boldsymbol{\mu}^{(1)} = (200, 0)$, $\boldsymbol{\mu}^{(2)} = (100, 0)$ (f) $\boldsymbol{\mu}^{(1)} = (100, 0)$, $\boldsymbol{\mu}^{(2)} = (100, 0)$

Figure 5.6: Simulation results for the diameter process in the same scenarios as the ones analyzed in the context of the perimeter process.

+1.55	+32.06	+47.78	+68.82	+72.27	+80.19	+79.35	+80.29
+32.94	+1.12	+5.51	+44.75	+56.25	+72.33	+76.16	+80.40
+43.48	+6.36	+2.19	+31.11	+48.53	+67.99	+71.12	+79.39
+68.34	+45.83	+28.79	+0.86	+7.19	+42.15	+59.14	+72.11
+73.35	+55.71	+46.41	+7.55	+1.24	+25.51	+41.54	+65.19
+78.23	+72.55	+66.75	+46.43	+30.36	+0.98	+9.45	+43.79
+80.00	+76.01	+73.57	+57.17	+45.21	+3.23	+1.20	+28.41
+78.56	+80.55	+77.39	+73.00	+67.84	+44.79	+30.54	+1.25

(a) $\boldsymbol{\mu}^{(1)} = (100, 200)$, $\boldsymbol{\mu}^{(2)} = (-100, 200)$

+1.67	+0.76	+0.99	+0.70	+1.23	+1.08	+0.42	+0.82
+33.55	+2.69	+1.54	+0.95	+1.13	+0.77	+0.49	+0.47
+45.61	+10.18	+1.15	+0.66	+0.88	+0.99	+0.57	+0.67
+65.94	+47.47	+24.72	+1.97	+2.11	+0.59	+0.39	+0.98
+72.01	+58.33	+44.54	+10.24	+4.09	+0.53	+1.72	+0.59
+78.79	+72.00	+67.55	+48.53	+27.13	+1.43	+1.55	+0.70
+79.70	+75.50	+71.62	+59.95	+47.82	+14.07	+1.68	+0.89
+80.35	+80.41	+78.74	+71.81	+66.53	+45.70	+35.24	+2.71

(b) $\boldsymbol{\mu}^{(1)} = (0, 200)$, $\boldsymbol{\mu}^{(2)} = (200, 100)$

Figure 5.7: Simulation results (for the diameter process) in the case when assumption (A2) is violated, but (A1) is satisfied.

CONCLUSION

In this dissertation, we examined the convex hull spanned by m planar random walks and analyzed its asymptotic behavior, as well as the asymptotic behavior of geometric functionals. After the introduction, literature review, and the mathematical results used in this work, in the second chapter, we proved that the convex hull generated by the first n steps of m independent random walks, scaled by n^{-1}, converges to the convex hull spanned by the corresponding drift vectors. In this case, we made no additional assumptions on the random walks. From this result, we can conclude that all continuous functionals of convex sets (such as perimeter, diameter, area, and intrinsic volumes) almost surely converge to the same functional value of the limit object.

In the continuation of the work, we restricted ourselves to the case when $m = 2$, and in the third chapter, we examined the distributional limit for the perimeter process. The main idea was to construct a sequence of martingale differences using the resampling technique. In this way, we adequately described the approximation of the perimeter deviation, and by applying the classical Levy's Central Limit Theorem (CLT) and Slutsky's Theorem, we showed that, in the case when assumption (A1) is satisfied and when the limiting variance is strictly positive, we obtain a central limit theorem for the perimeter process.

We used a very similar argumentation for the diameter process. The key detail in this process was proving the pointwise continuity of the mapping that assigns to polygons with unique diametrical segments (segments on which the diameter is achieved) that segment, with respect to the Hausdorff metric. Similarly to the perimeter process, if assumption (A2) is satisfied, and if the limiting variance is strictly positive, we obtained a CLT for the diameter process.

In the final chapter, we proved analogous results for the distributional limits of the perimeter and diameter processes of the convex hull spanned by the centroids of a single planar random walk. The main difference compared to the previous proofs was that, instead of the classical Levy's CLT, we had to use the Lindeberg-Feller version. We also commented on possible generalizations of the results for more than two random walks. We provided a simulation study to examine what happens if we abandon assumption (A1) for the perimeter process and assumption (A2) for the diameter process. The study showed that, in such a case, we do not expect a Gaussian distributional limit, which opens a window for new scientific research in this area.

Appendix

In this additional chapter, we will present the proofs of the auxiliary statements referenced in the thesis, which are essential to our discussion. We begin with the proof of Cauchy's surface formula. The *Minkowski sum* of two sets A and B in $\mathbb{R}^d$ is defined as:

$$A+B=\{a+b : a\in A, b\in B\}. \tag{A.6}$$

Then, the surface of K, $S(K)$ is defined as:

$$S(K)=\lim_{\varepsilon\to 0^+}\frac{\lambda_d\left(K+\varepsilon\mathbb{B}^d\right)-\lambda_d(K)}{\varepsilon}.$$

First, we give our definition, which uses the Minkowski sum of sets.

Definition A.1. Let $u, P\subset\mathbb{R}^d$. Define

$$D_u\left(\lambda_d\right)(P)=\lim_{\varepsilon\to 0^+}\frac{\lambda_d(P+\varepsilon u)-\lambda_d(P)}{\varepsilon}.$$

The special case $D_{\mathbb{B}^d}\left(\lambda_d\right)(P)$ gives the surface volume:

$$D_{\mathbb{B}^d}\left(\lambda_d\right)(P)=S(P). \tag{A.7}$$

The foundation of our proof is Minkowski's theorem on mixed volumes, which is discussed in Chapter 5 of [Sch13]. This theorem states that the volume of a Minkowski sum of convex bodies can be expressed as a polynomial in the coefficients of the Minkowski sum, with the polynomial's coefficients depending solely on the convex bodies themselves.

Theorem A.2. Consider convex bodies $K_1, K_2, \ldots, K_m$ in $\mathbb{R}^d$. The volume of their Minkowski sum can be expressed as:

$$\lambda_d\left(\lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_m K_m\right) = \sum \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n} V\left(K_{i_1}, K_{i_2}, \ldots, K_{i_n}\right).$$

Here, the left-hand side represents the Minkowski sum, while the right-hand side involves a sum over all multisets of size n formed from the indices $\{1, 2, \ldots, m\}$. The functions V are nonnegative, symmetric, and depend only on the convex bodies $K_{i_1}, K_{i_2}, \ldots, K_{i_n}$.

From Theorem A.2 it follows that if P and u are convex, then $D_u\left(\lambda_d\right)(P)$ is linear in u.

Lemma A.3. Let $P, u, v \subset \mathbb{R}^d$ be convex bodies, and let $\alpha, \beta \in \mathbb{R}$. Then we have the following relationship:

$$D_{\alpha u + \beta v}\left(\lambda_d\right)(P) = \alpha D_u\left(\lambda_d\right)(P) + \beta D_v\left(\lambda_d\right)(P).$$

Proof. Let's start with the definition:

$$\begin{aligned} D_{\alpha u + \beta v}\left(\lambda_d\right)(P) &= \lim_{\varepsilon \to 0^+} \frac{\lambda_d(P + \varepsilon(\alpha u + \beta v)) - \lambda_d(P)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0^+} \frac{\lambda_d(P + \varepsilon \alpha u + \varepsilon \beta v) - \lambda_d(P)}{\varepsilon}. \end{aligned}$$

According to Theorem A.2, we have:

$$D_{\alpha u + \beta v}\left(\lambda_d\right)(P) = \alpha V(P, P, \ldots, P, u) + \beta V(P, P, \ldots, P, v)$$

where P is repeated $d-1$ times, and V is the function described in Theorem A.2. Similarly, it follows that:

$$\begin{aligned} D_u\left(\lambda_d\right)(P) &= V(P, P, \ldots, P, u) \\ D_v\left(\lambda_d\right)(P) &= V(P, P, \ldots, P, v), \end{aligned}$$

which completes the proof of our Lemma. □

In what follows, we demonstrate that the derivative defined in Definition A.1 effectively computes the projection of a convex body when u represents a line segment of length 1.

Lemma A.4. Consider u as a line segment of length 1 and let $K \subset \mathbb{R}^d$ be a convex set. Then, we have the relationship:

$$\lambda_{d-1}\left(K \mid u^{\perp}\right) = D_u(\lambda_d)(K).$$

Proof. Let L_u represent the collection of lines that are parallel to u. It is important to note that each line l in L_u corresponds to a unique point in the plane orthogonal to u, denoted by $l^{\perp}$. This means that L_u can be considered isomorphic to $\mathbb{R}^{d-1}$, allowing us to define the measure λ_{d-1} on L_u. With this setup, we have:

$$\lambda_d(K) = \int_{l \in L_u} \lambda_1(l \cap K)\,\mathrm{d}\lambda_{d-1}.$$

Additionally, we can express:

$$\lambda_{d-1}\left(K \mid u^{\perp}\right) = \int_{\substack{l \cap K \neq \emptyset \\ l \in L_u}} 1\,\mathrm{d}\lambda_{d-1}.$$

For any $\varepsilon > 0$, the following holds:

$$\lambda_d(K + \varepsilon u) - \lambda_d(K) = \int_{l \in L_u} (\lambda_1(l \cap (K + \varepsilon u)) - \lambda_1(l \cap K))\,\mathrm{d}\lambda_{d-1}.$$

Due to the convexity of K, we know that:

$$\lambda_1(l \cap (K + \varepsilon u)) - \lambda_1(l \cap K) = \varepsilon$$

for all lines l that intersect K, and the difference is zero for lines that do not intersect K. Therefore, the integral simplifies to:

$$\int_{\substack{l \cap K \neq \emptyset \\ l \in L_u}} \varepsilon\,\mathrm{d}\lambda_{d-1} = \varepsilon \lambda_{d-1}\left(K \mid u^{\perp}\right),$$

which leads to:

$$D_u(\lambda_d)(K) = \lim_{\varepsilon \to 0^+} \frac{\lambda_d(K + \varepsilon u) - \lambda_d(K)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{\varepsilon \lambda_{d-1}\left(K \mid u^{\perp}\right)}{\varepsilon} = \lambda_{d-1}\left(K \mid u^{\perp}\right).$$

□

At this point, we are ready to show a concise yet thorough proof of Cauchy's surface area formula.

Proof of Theorem 1.2.5. By applying Lemma A.3, Lemma A.4, and Equation A.7, we can derive the following:

$$
\begin{aligned}
\int_{\mathbb{S}^{d-1}} \lambda_{d-1}\left(K \mid u^{\perp}\right) \mathrm{d}u &= \int_{\mathbb{S}^{d-1}} D_{[0,u]}\left(\lambda_d\right)(K) \mathrm{d}u \\
&= D_{\int_{\mathbb{S}^{d-1}}[0,u] \mathrm{d}u}\left(\lambda_d\right)(K) \\
&= D_{c(n)\mathbb{B}^d}\left(\lambda_d\right)(K) \\
&= c(n) S(K).
\end{aligned}
$$

We finish the proof by choosing K to be the unit ball $\mathbb{B}^d$. □

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Curriculum Vitae

Tomislav Kralj was born on October 8, 1996, in Karlovac. After completing elementary school, he enrolled in the Karlovac Gymnasium in 2011. Upon finishing high school, he began his undergraduate studies in mathematics at the Department of Mathematics, *Faculty of Science, University of Zagreb* in 2015, completing the program in 2018. That same year, he enrolled in the graduate program in Financial and Business Mathematics, which he successfully completed in 2020 with a thesis titled *The Structure of Lévy Processes, Subordinators, and Applications*, under the mentorship of Prof. Zoran Vondraček, earning the distinction *summa cum laude*.

During his studies, he served as a student demonstrator for several courses. He developed a strong interest in probability theory and statistics during the summer between his third and fourth years. Immediately after completing his graduate studies, he enrolled in the doctoral program in mathematics at the *Faculty of Science, University of Zagreb* and began working as a teaching assistant. In this role, he held courses in mathematical analysis, statistical practicum, statistics, probability, and multivariable calculus. Throughout his doctoral studies, he successfully completed the required qualifying exams in probability and analysis and relevant advanced courses.

He actively participated in several prestigious international and domestic conferences during his doctoral studies. In the 2021/2022 academic year, he attended the *Regular Variation and Related Themes* conference at the *Inter-University Centre Dubrovnik, Croatia*. During the 2022/2023 academic year, he took part in the *23rd European Young Statisticians Meeting 2023* in *Ljubljana, Slovenia*, organized by the *Statistical Society of Slovenia*. He also attended the *Random Structures, Applied Probability, and Computation* conference at the *University of Liverpool, United Kingdom*, and additionally participated in the *PMF Doctoral Student Symposium* at the *University of Zagreb, Faculty of Science,* *Croatia*.

In the academic year 2023/2024, he attended the *Two-Dimensional Random Geometry* conference in *Chicago, United States*, hosted by the *Institute for Mathematical and Statistical Innovation*. He also participated in the *Math Mingle: Diverse Dimensions in Mathematics* conference in *Västerås, Sweden*, held at *Mälardalen University*. Another significant event was the *Probability in the North East Workshop* at *Durham University, United Kingdom*. Domestically, he attended the *New Perspectives in the Theory of Extreme Values* conference at the *Inter-University Centre Dubrovnik, Croatia*.

In the 2023/2024 academic year, as a visiting PhD student, he visited Durham University, United Kingdom, where he discussed the boundary cases mentioned in this dissertation with Professor Andrew Wade and Professor Mo Dick Wong. In conclusion, his research work culminated in the article *On the Convex Hull of Two Planar Random Walks*, currently available as a preprint at `https://arxiv.org/abs/2403.17705`. In his free time, he enjoys sports, plays in a music band, and organizes and attends pub quizzes.