Non-invertible symmetries and higher categories

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Master Thesis

Non-invertible symmetries and higher categories

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Sažetak

Ovaj diplomski rad izlaže moderan tretman simetrija u kvantnim teorijama polja koji je rezultat nedavnih napredovanja u razumijevanju simetrija iz nove perspektive u kojoj su simetrije dane topološkim operatorima teorije. Rad izlaže osnovne pojmove teorije kategorije korištene u SymTFT konstrukciji kao i osnovne aspekte topoloških kvantnih teorija polja (TQFT) koji su potrebani za svladavanje ove teme bez oslanjanja na mašineriju teorije struna i konformalnih teorija polja (CFT) koja je prisutna u većini radova na ovu temu. Konačno, korišteni su primjeri Chern-Simonsove i BF teorije kako bi se demonstrirala SymTFT konstrukcija.
Ključne riječi: SymTFT, simetrije, topologija, TQFT, orbifold, kompaktifikacija, generalizirani naboji, prošireni operatori, topološki operatori, defekti, stog teorije, kategorija, fuzijska kategorija

Non-invertible symmetries and higher categories

Abstract

This Master's Thesis describes the modern treatment of symmetries in quantum field theories which is the result of recent developments and a change in perspective on symmetries in which symmetries are given by topological operators of the theory. It outlines the basic notions of category theory used in the SymTFT construction as well as the basic aspects of Topological Quantum Field Theories (TQFTs) needed to tackle the subject without relying heavily on the string theory and Conformal Field Theory (CFT) machinery which is usually present in other papers on this subject. Lastly, it uses the Chern-Simons and BF theory to provide elementary examples of the SymTFT construction at work.

Keywords: SymTFT, symmetries, topology, TQFT, orbifold, compactification, generalized charges, extended operators, topological operators, defects, stacking, category, fusion categories

Contents

1 Introduction

Recent developments in the mathematical physics community have led to a new framework used to study symmetries of quantum field theories. Following the realization that operators implementing the symmetries are in a one-to-one correspondance with the topological operators of the theory in [6], there has been a series of generalizations in terms of the so-called higher-form symmetries, categorical symmetries and non-invertible symmetries, all of which can be studied under the recent construction called the SymTFT which is implemented as an extended topological quantum field theory that with the careful choice of boundary conditions implementing the dimensional reduction mechanism and the orbifold construction retrieve the theory with the desired symmetry structure. Additionally, the SymTFT construction provides maps between dual theories as well as the anomaly inflow for anomalous theories, as well as a way to construct non-invertible symmetries of theories. Many of the tools used in the construction originate from work on conformal theories [7] [14] [15] [19] such as the fusion algebras of line operators and orbifold constructions. However, this thesis uses their generalizations in the framework of topological quantum field theories of Witten and Atiyah [8] giving an almost self-contained overview neccessary for introduction of the SymTFT construction which is the main goal of this thesis. The thesis is structured as follows.

Chapter 2 details the aspects of category theory needed for the subject. We start with introduction of basic categorical definitions and constructions with provided examples. Following that, we define the type of categories which will be of interest. We finish with the notion of higher categories.

Chapter 3 deals with topological quantum field theories. We introduce the Chern-Simons and BF theories. Then we introduce the axiomatic system of Atiyah given in [8]. We introduce the mechanism of dimensional reduction and we detail the extended topological quantum field theories, extended operators and boundary conditions on them.

Chapter 4 studies symmetries. We show symmetries are implemented by topological operators, introduce higher-form symmetries and provide an example with the Maxwell theory.

Chapter 5 starts with the orbifold construction which we motivate using the

Dijkgraaf-Witten model. Following that we consider boundary conditions of a theory and introduce the SymTFT construction. We end the chapter and the thesis with several examples.

2 Categories

This chapter introduces the core concepts required to understand the main construction of this paper, the SymTFT. It contains the basic definitions and constructions in category theory with a special affinity towards tensor categories and homotopy theory. The relevant references for this chapter are [1] and [3] for general theory of categories, [5] for enriched categories and homotopy theory, [2] and [4] for monoidal and tensor categories.

2.1 Categories

Definition 2.1 (Category) *Let C be a collection. Suppose that for every $x, y \in C$ there exists another collection, denoted by $C(x, y)$. Furthermore, suppose that for every $x, y, z \in C$ there exists a map $\circ : C(x, y) \times C(y, z) \to C(x, z)$. The above data is then called a* **category***(or equivalently in the context of higher category theory, a* **1-category***) if the following conditions hold true:*

- *For each $x \in C$ there exists id_x in $C(x, x)$ such that for all $y \in C, f \in C(x, y), g \in C(y, x)$ $f \circ id_x = f$ and $id_x \circ g = g$,*
- *For all $x, y, z, w \in C, f \in C(x, y), g \in C(y, z), h \in C(z, w)$ $h \circ (g \circ f) = (h \circ g) \circ f$.*

When talking about an abstract category, we denote it with letters written in script. For example, the category in this definition is denoted by $\mathscr{C}$. When talking about specific concrete categories they will be denoted with the use of bold fonts such as $\mathbf{Set}, \mathbf{sSet}, \mathbf{Grp}$ for categories of sets, simplicial sets and groups respectively. Elements of the collection C are then referred to as **objects** *of the category and denoted as $\mathrm{ob}(\mathscr{C})$. Elements of the collections $C(x, y)$ are referred to as* **morphisms** *from x to y in category $\mathscr{C}$ denoted by $f \in \mathscr{C}(x, y)$. Aditionally, when talking about all of the morphisms of a category, we use the collection of all morphisms denoted by $\mathrm{mor}(\mathscr{C})$.*

The fundamental example of a category is **Set** whose objects are sets, morphisms functions between sets and composition the usual composition of functions. A less trivial example would be the category of groups, $\mathbf{Grp}$ whoose objects are groups, morphisms group homomorphisms and composition given by the usual composition of functions. Notice that a selection has been made for morphisms of the category

Grp. After all, we would require the codomain of a map to be a group in this case and an arbitrary function from a group will not necessitate that. In other words, when defining a category, there is a need to introduce the restriction on maps such that by applying the map, the structure of the object is preserved.

The word collection is a technicallity since without it we would quickly run into some consistency problems. For example, the set of all sets is not a well-defined object. In order to approach this topic rigorously there is a need to have a more general notion than that of a set. This is commonly achieved through the usage of collections/classes or by specifying the Grothendieck universe in which the data of the category belongs to. A category in which all morphisms form a set is called a **locally small** category. Additionally, if the collection of objects also forms a set, such a category is then called **small**. For our purposes, we may work freely just with the notion of locally small categories and think about collections of objects and morphisms simply as sets.

Although they most often will be, morphisms of a category need not be functions. A basic example of such a category is the category defined by a partial ordering on some set S, denoted by $\mathbf{Poset}\left[S\right]$, whose objects are elements of S, and for which the morphisms are given by:

$$\mathbf{Poset}\left[S\right](x,y) = \begin{cases} \{1\} & \text{, if } x \leq y, \\ \emptyset & \text{, otherwise.} \end{cases} \tag{2.1}$$

At this point, we introduce the diagrammatic notation of category theory. In it, we denote $f \in \mathscr{C}(x,y)$ as $x \xrightarrow{f} y$. That is to say, we may use weighted directed graphs (which we will refer to as **diagrams**) where objects are vertices and morphisms are labeled edges to represent categories and different expressions about them. A path on such a diagram then represents the composition of morphisms that label the edges of the path. In this notation, the category $\mathbf{Poset}\left[\{1,2,3\}\right]$ is given by the following diagram:

$$\begin{tikzcd} 1 \arrow[loop left, "\leq = id_1"] \arrow[r, "\leq"] \arrow[rr, bend right, "\leq"'] & 2 \arrow[loop above, "\leq = id_2"] \arrow[r, "\leq"] & 3 \arrow[loop right, "\leq = id_3"] \end{tikzcd},$$

where the notation $\leq$ has been used suggestively to hint at the meaning of the morphisms in such a category. In practice, unless they are specifically needed in the context, the identity morphisms are usually assumed and left out of such diagrams. Also, unless otherwise specified, we will assume that the drawn diagrams of 1-categories are commutative i.e. if we specify two endpoints, then any two paths between such endpoints are equivalent.

Definition 2.2 (Dual category) *Let $\mathscr{C}$ be a category. We define its* **dual** *category, denoted by $\mathscr{C}^{op}$ as the category whose objects are object of $\mathscr{C}$, $\mathrm{ob}(\mathscr{C}^{op}) = \mathrm{ob}(\mathscr{C})$, whose morphisms are $\mathscr{C}^{op}(x,y) = \mathscr{C}(y,x)$ and composition $\cdot : \mathscr{C}^{op}(x,y) \times \mathscr{C}^{op}(y,z) \to \mathscr{C}^{op}(x,z)$ is given by $(f,g) \mapsto g \cdot f = f \circ g$. Diagramatically speaking it corresponds to reversing the direction of all arrows/morphisms hence the dual category is often also referred to as the* **opposite** *category.*

As usual, when a structure is introduced in mathematics, along come the notions of its substructure and structure preserving maps.

Definition 2.3 *Let $f \in C(x,y)$. f is called a* **monomorphism** *if for any $g,h \in C(z,x)$ $f \circ g = f \circ h$ implies $g = h$. an* **epimorphism** *if for any $g,h \in C(y,z)$ $g \circ f = h \circ f$ implies $g = h$. and an* **isomorphism** *if there exists $g \in C(y,x)$ such that $g \circ f = id_x$ and $f \circ g = id_y$ in which case g is called an inverse of f and is denoted as $g = f^{-1}$. Additionally, when f is an isomorphism, we say that x and y are isomorphic and denote it by $x \cong y$.*

Definition 2.4 (Functor) *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A map $F : \mathrm{ob}(\mathscr{C}) \to \mathrm{ob}(\mathscr{D})$ along with maps $\mathscr{C}(x,y) \to \mathscr{D}(F(x),F(y))$, $f \mapsto F(f)$ for all $x,y \in \mathrm{ob}(\mathscr{C})$ is called a* **(covariant) functor** *if the following conditions are met:*

- $F(id_x) = id_{F(x)}$ *for all $x \in \mathrm{ob}(\mathscr{C})$, and*
- $F(g \circ f) = F(g) \circ F(f)$ *for all $f \in \mathscr{C}(x,y), g \in \mathscr{C}(y,z)$.*

It is simply deonted by $F : \mathscr{C} \to \mathscr{D}$ where the symbol is used for both the map between objects and for the maps between morphisms. A **contravariant functor** *$\mathscr{C} \to \mathscr{D}$ is a covariant functor $\mathscr{C}^{op} \to \mathscr{D}$. Is is equivalent to reversing the composition on the right side in the above condition.*

Trivial examples of functors are the so-called **forgetful** functors. For example $F : \mathbf{Grp} \to \mathbf{Set}$ which maps a group to its underlaying set forgetting the group structure on it. Consider now a map $\mathscr{C}(x,-)$, which maps $y \mapsto \mathscr{C}(x,y)$. If it were a functor it would have to map $\mathscr{C}(y,z) \ni f \mapsto \mathscr{C}(x,f) : \mathscr{C}(x,y) \to \mathscr{C}(x,z)$. By taking $\mathscr{C}(x,f) = f \circ -$ we see that the the following diagram commutes:

$$\begin{tikzcd}
x \arrow[rr, "h"] \arrow[dr, "\mathscr{C}(x,f)(h)"] \arrow[dd, "\mathscr{C}(x,g)\circ\mathscr{C}(x,f)(h)"'] & & y \arrow[dl, "f"] \\
& z \arrow[dl, "g"] & \\
w & &
\end{tikzcd},$$

for all $x, y, z, w \in \mathrm{ob}(\mathscr{C})$, $h \in \mathscr{C}(x,y)$, $f \in \mathscr{C}(y,z)$ and $g \in \mathscr{C}(z,w)$. By explicit construction we have defined a covariant functor $\mathscr{C}(x,-) : \mathscr{C} \to \mathbf{Set}$. Similarly, we can define a contravariant functor denoted by $\mathscr{C}(-,y) : \mathscr{C}^{op} \to \mathbf{Set}$ by $\mathrm{ob}(\mathscr{C}) \ni x \mapsto \mathscr{C}(x,y)$ and $\mathscr{C}(x,z) \ni f \mapsto \mathscr{C}(f,y) = - \circ f : \mathscr{C}(z,y) \to \mathscr{C}(x,y)$. These two functors are commonly known as **Yoneda embeddings**.

We will also use the notation $1_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ for the functor which acts as the identity on objects and morphisms.

Definition 2.5 *Let $F : \mathscr{C} \to \mathscr{D}$ be a (covariant) functor. We say that F is* **faithful** *(respectvely,* **full***) if for all $x, y \in \mathrm{ob}(\mathscr{C})$ the mapping $\mathscr{C}(x,y) \to \mathscr{D}(F(x), F(y))$ is injective (respectively, surjective). If F is both full and faithful, we say that it is* **fully faithful**. *We say that F is* **essentially surjective** *if for every $y \in \mathrm{ob}(\mathscr{D})$ there exists $x \in \mathrm{ob}(\mathscr{C})$ such that $F(x)$ is isomorphic to y.*

Definition 2.6 (Product category) *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Then we define a* **product category***, denoted by $\mathscr{C} \times \mathscr{D}$, by setting $\mathrm{ob}(\mathscr{C} \times \mathscr{D}) = \mathrm{ob}(\mathscr{C}) \times \mathrm{ob}(\mathscr{D})$ and $\mathscr{C} \times \mathscr{D}((x_1, y_1), (x_2, y_2)) = \mathscr{C}(x_1, x_2) \times \mathscr{D}(y_1, y_2)$ for all $x_1, x_2 \in \mathrm{ob}(\mathscr{C}), y_1, y_2 \in \mathrm{ob}(\mathscr{D})$. Composition is given component-wise.*

Definition 2.7 (Bifunctor) *Let $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ be categories. A functor $\mathscr{C} \times \mathscr{D} \to \mathscr{E}$ is then called a* **bifunctor**.

A noteworthy example of a bifunctor is $\mathscr{C}(-,-) : \mathscr{C}^{op} \times \mathscr{C} \to \mathbf{Set}$.

Definition 2.8 (Natural transformation) *Suppose $\mathscr{C}$ and $\mathscr{D}$ are categories and $F, G : \mathscr{C} \to \mathscr{D}$ functors between them. Let $\alpha = \{F(x) \to G(x)\}_{x \in \mathrm{ob}(\mathscr{C})}$ be a family of mor-*

phisms in $\mathscr{D}$. If the following diagram commutes for every $f \in \mathscr{C}(x, y)$:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ G(x) & \xrightarrow[G(f)]{} & G(y) \end{array},$$

then we say that α is a **natural transformation** *from F to G, denoted as $\alpha : F \implies G$. Furthermore, the morphisms α_x are referred to as* **components** *of α. Naturally, since they are made from morphisms, these transformations can be composed component-wise. Suppose now $\beta : G \implies H$ is another natural transformation for some functor $H : \mathscr{C} \to \mathscr{D}$. Then we define the* **vertical composition** *$\beta \circ \alpha$ by setting $(\beta \circ \alpha)_x = \beta_x \circ \alpha_x$. We have now actually defined another category, denoted by $[\mathscr{C}, \mathscr{D}]$, whose objects are functors from $\mathscr{C}$ to $\mathscr{D}$ and whose morphisms are the natural transformations between those functors.*

Definition 2.9 (Equivalence of categories) *Suppose $F \in \mathrm{ob}([\mathscr{C}, \mathscr{D}])$ and $G \in \mathrm{ob}([\mathscr{D}, \mathscr{C}])$. If there exist $\eta \in [\mathscr{C}, \mathscr{C}](1_{\mathscr{C}}, G \circ F)$ and $\epsilon \in [\mathscr{D}, \mathscr{D}](F \circ G, 1_{\mathscr{D}})$ isomorphisms, then we say that F and G are* **equivalences of categories** *$\mathscr{C}$ and $\mathscr{D}$. Additionally, if such functors exist, we say that $\mathscr{C}$ and $\mathscr{D}$ are* **equivalent** *and we denote it by $\mathscr{C} \cong \mathscr{D}$.*

2.2 Universal Constructions

The power of category theory lies in finding non-trivial equivalences between categories and identifying prominent objects which exhibit special properties often in the form on uniqness or isomorphisms. Many such objects are defined in terms of universal properties and constructions on them which are the main focus of this section.

Definition 2.10 *Let $\mathscr{C}$ be a category. We say that an object $e \in \mathrm{ob}(\mathscr{C})$ is* **initial** *if for every $x \in \mathrm{ob}(\mathscr{C})$, the set $\mathscr{C}(e, x)$ contains exactly one element, i.e. there is only one morphism from e to x. On the other side, we say that e is* **terminal** *if the set $\mathscr{C}(x, e)$ contains only one element for every $x \in \mathrm{ob}(\mathscr{C})$.*

Definition 2.11 *Suppose $\mathscr{C}, \mathscr{D}, \mathscr{E}$ are categories and $S : \mathscr{C} \to \mathscr{E}$ and $T : \mathscr{D} \to \mathscr{E}$ are functors. We define the* **comma category** *$(S \downarrow T)$ as follows. Objects are triples (c, d, f) where $c \in \mathrm{ob}(\mathscr{C}), d \in \mathrm{ob}(\mathscr{D})$ and $f \in \mathscr{E}(S(c), T(d))$. Morphisms are pairs*

$(g, h) \in (S \downarrow T)((c, d, f), (c', d', f'))$ where $g \in \mathscr{C}(c, c')$ and $h \in \mathscr{D}(d, d')$ such that the diagram

$$\begin{tikzcd} S(c) \arrow[r, "S(g)"] \arrow[d, "f"'] & S(c') \arrow[d, "f'"] \\ T(d) \arrow[r, "T(h)"'] & T(d') \end{tikzcd}$$

commutes. Composition is given component-wise.

We will denote the category with a single object and single morphism (the identity over that object) as $*$. Notice that a functor from that category corresponds to picking an object, and so we will often use the notation $x \equiv x(*)$ for a functor $x : * \to \mathscr{C}$.

Definition 2.12 *Let $\mathscr{C}$ and $\mathscr{D}$ be categories, $d \in \mathrm{ob}(\mathscr{D})$ and $F : \mathscr{C} \to \mathscr{D}$ a functor. Suppose $(d, c, f : d \to F(c))$ is an initial object in $(d \downarrow F)$. Then for any other object (d, c', f') in $(d \downarrow F)$ there exists a unique morphism (id_d, g) such that the diagram*

$$\begin{tikzcd} d \arrow[r, "f"] \arrow[dr, "f'"'] & F(c) \arrow[d, "F(g)"] \\ & F(c') \end{tikzcd}$$

commutes. We say that f is a **universal morphism from d to F**. *Additionally, we say that c and f satisfy a* **universal property**. *Analogously, if $(c, d, f : F(c) \to d)$ is a terminal object in $(F \downarrow d)$ we say that f is a* **universal morphism from F to d**.

Definition 2.13 *Let $\mathscr{C}$ be a category. We define the* **diagonal functor** $\Delta : \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ *which maps $c \mapsto (c, c)$ and $f \mapsto (f, f)$ for $c, d \in \mathrm{ob}(\mathscr{C})$ and $f \in \mathscr{C}(c, d)$.*

Consider now the comma category $(\Delta \downarrow (x, y))$, more precisely the terminal object $(z, (x, y), u : (z, z) \to (x, y))$ in it. By definition, for any other object $((w, w), (x, y), f : w \to (x, y))$, there exists a unique morphism $(f', id_{(x,y)})$ such that $f = u \circ (f', f')$. Taking $u = (u_1, u_2)$ and $f = (f_1, f_2)$ we have the following commuting diagram:

$$\begin{tikzcd} & w \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ x & z \arrow[l, "u_1"] \arrow[r, "u_2"'] \arrow[u, dashed, "\exists! f'"'] & y \end{tikzcd} .$$

We call such an object z, if it exists, a product of x and y and denote it as $x \times y \equiv z$ and the morphism $\pi \equiv u$ projections.

This procedure generalizes by considering similar functors on different categories.

Definition 2.14 *Let $\mathscr{C}$ and $\mathscr{I}$ be categories (we ask that $\mathscr{I}$ be a small category). Consider the functor $\Delta : \mathscr{C} \to [\mathscr{I}, \mathscr{C}]$ which maps objects c in $\mathscr{C}$ to the* **constant functor** *$\Delta_c : \mathscr{I} \to \mathscr{C}$ (which maps $\mathrm{ob}(\mathscr{I}) \ni x \mapsto c$ and $\mathrm{mor}(\mathscr{I}) \ni f \mapsto id_c$) and morphisms $f : c \to c'$ to natural transformations Δ_f whose components are $(\Delta_f)_x : \Delta_c(x) \to \Delta_{c'}(x) = f$. Suppose $F : \mathscr{I} \to \mathscr{C}$ is a functor (as an object in $[\mathscr{I}, \mathscr{C}]$). The* **limit** *of F , if it exists, is the universal morphism from Δ to F, we denote the underlying object in $\mathscr{C}$ as $\lim_{\leftarrow \mathscr{I}} F$. Analogously, the* **colimit** *of F is a universal morphism from F to Δ and we denote the underlying object in $\mathscr{C}$ as $\lim_{\rightarrow \mathscr{I}} F$.*

Consider a category with two objects, the identity morphisms and two morphism from one object to another, A limit over such a category is called **an equalizer**. In the category **Set**, the equalizer is the subset of the domain of two functions on which they take the same value, hence the name. A special case is the equalizer of the pair of morphisms $(f, 0)$, where 0 is a constant function returning 0, in that case the limit is referred to as the **kernel** and noted as $\mathrm{Ker}(f)$. Its dual, the colimit is referred to as the **cokernel** and noted as $\mathrm{Coker}(f)$.

Consider a functor $F : \mathscr{C} \to \mathscr{D}$. If there is a universal morphism from d to F for every d in $\mathscr{D}$. Then for each d, d' in $\mathscr{D}$ we have the corresponding $(d, c, u : d \to F(c))$ and $(d', c', u' : d' \to F(c'))$ which are initial in $(d \downarrow F)$ and $(d' \downarrow F)$ respectively. By the universal property, given a morphism $h : d \to d'$, for the composition $h \circ u' : d \to F(c')$ there is a unique $g : c \to c'$ such that the diagram

$$\begin{array}{ccc} d & \xrightarrow{u} & F(c) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle F(g)} \\ d' & \xrightarrow{u'} & F(c') \end{array}$$

commutes. Notice that this defines a functor $G : \mathscr{D} \to \mathscr{C}$ and that the underlying maps of the universal object then define components of a natural transformation $\eta : 1_{\mathscr{D}} \implies G \circ F$. We say that (F, G) form a pair of **adjoint functors**, denoted as $F \dashv G$ and we refer to the natural transformation η as the **unit**. Analogously, by watching the terminal objects we obtain the same functor with now a natural transformation $\epsilon : F \circ G \implies 1_{\mathscr{C}}$ referred to as the **co-unit**.

Proposition 2.15 *Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be adjoint functors. We have a* *natural isomorphism*

$$\mathscr{D}(F(c), d, \cong)\mathscr{C}(c, G(d)). \tag{2.2}$$

Consider a set y. Taking the product with it defines a functor $(- \times y)$ which maps $x \mapsto x \times y$. This functor has an adjoint, the Yoneda embedding $\mathbf{Set}(-, y)$, and by the above proposition we have

$$\mathbf{Set}(x \times y, z) \cong \mathbf{Set}(x, \mathbf{Set}(y, z)),$$

which tells us that if we fix a variable of a function with two variables we get a function with a single variable. Consider now the adjoint of the diagonal functors from the definition of limits.

Proposition 2.16 *Let $\mathscr{I}$ be a small category and $\mathscr{C}$ a category which has all limits of shape $\mathscr{I}$. Then we have an adjunction*

$$[\mathscr{I}, \mathscr{C}](\Delta_c, F) \cong \mathscr{C}(c, \lim_{\leftarrow \mathscr{I}} F),$$

where the $\lim_{\leftarrow \mathscr{I}}$ is a functor mapping functors $F : \mathscr{I} \to \mathscr{C} \mapsto \lim_{\leftarrow \mathscr{I}} F \in \mathrm{ob}(\mathscr{C})$.

Consider now a category $\mathscr{D}$ whose objects are pairs of groups (N, G) such that $N \triangleleft G$ is a normal subgroup, and whose morphisms from (N, G) to (H, K) are a group homomorphisms $f : G \to K$ such that $f(N) = \{id_K\}$. Consider a functor $F : \mathbf{Grp} \to \mathscr{D}$ sending $G \mapsto (1, G)$ where 1 is a trivial group. By the above considerations, the functor adjoint to F, U is generated by universal morphisms of $((N, G) \downarrow F)$. Given a morphism $f : (N, G) \to F(K) = (1, K)$, we have by the universal property

$$\begin{tikzcd}
(N, G) \arrow[r, "\eta_{(N,G)}"] \arrow[dr, "f"'] & (1, U(N, G)) \arrow[d, dashed, "\exists! f'"] \\
& (1, K)
\end{tikzcd}.$$

One such functor maps $(N, G) \mapsto G/N$, and from the uniqueness of the universal morphism follows that this is the only one, upto isomorphisms that is.

Theorem 2.17 (Yoneda) *Let $\mathscr{C}$ be a locally small category. Then we have a natural isomorphism*

$$[\mathscr{C}^{op}, \mathbf{Set}](\mathscr{C}(-, c), F) \cong F(c),$$

for $c \in \mathrm{ob}(\mathscr{C})$ *and* $F : \mathscr{C}^{op} \to \mathbf{Set}$ *a functor. Similarly, we have a natural isomorphism*

$$[\mathscr{C}, \mathbf{Set}](\mathscr{C}(c, -), G) \cong G(c),$$

for $G : \mathscr{C} \to \mathbf{Set}$.

We refer to the choice of the Yoneda embeddings in the Yoneda lemma as **representations** of their respected functors. Universal properties are given by these representations.

Proposition 2.18 *Let* $\mathscr{I}$ *be a small category, and* $\mathscr{C}$ *a category with limits of shape* $\mathscr{I}$ *then the Yoneda embedding preserves limits,*

$$\mathscr{C}(c, \lim_{\leftarrow \mathscr{I}} F) \cong \lim_{\leftarrow \mathscr{I}} \mathscr{C}(c, F),$$

where $F : \mathscr{I} \to \mathscr{I}$ *is a functor and* $c \in \mathrm{ob}(\mathscr{C})$.

Before we end this chapter we will consider a special universal property, that of the tensor product. Let k be a field. We then have the category of vector spaces over k and linear maps between them, $\mathbf{Vec}_k$. We restrict ourselves to its subcategory of finite dimensional vector spaces. Vector space on the product of spaces is well defined, but the morphisms on in the category are linear on the entire space and we are interested in functions on the product space which would be, along with being linear on the entire space, linear on each of the variables, we are looking for bilinear functions. Consequently we define a functor Bilin$(U, V, -) : \mathbf{Vec}_k \to \mathbf{Set}$ that maps the vector space W to the set containing all bilinear functions $U \times V \to W$. We then define the tensor product being the element $U \otimes V$ representing the functor Bilin$(U, V, -)$.By Yoneda, given a morphism $f : U \times V \to W$ we have the associated $f' : U \otimes V \to W$ bilinear function such that the diagram

$$\begin{tikzcd} \mathbf{Vec}_k(U \otimes V, U \otimes V) \arrow[r, "\cong"] \arrow[d, "f' \circ -"'] & \text{Bilin}(U, V, U \otimes V) \arrow[d, "f' \circ -"] \\ \mathbf{Vec}_k(U \otimes V, W) \arrow[r, "\cong"'] & \text{Bilin}(U, V, W) \end{tikzcd}$$

commutes. Evaluating the diagram on $id_{U \otimes V}$ we arrive at the associated universal property of the tensor product,

$$
\begin{tikzcd}
U \otimes V \arrow[r, "\otimes"] \arrow[dr, "f"'] & U \otimes V \arrow[d, "\exists! f'"] \\
& U
\end{tikzcd}
$$

where $\otimes$ is the universal morphism which is intial in the associated comma category given by the category whose object are all bilinear functions given by Bilin$(U, V, -)$ with fixed U and V and morphisms are commutative triangles between them. The vector space $U \otimes V$ is constructed as a vector space by the quotient with the image of the $\otimes$ map analogously to the quotient group construction above.

2.3 Tensor Categories

The theme of this subsection are additional structures on vector spaces. We provide categorical equivalents of properties of vector spaces and algebras on them and along the way introduce some of these structures which will be of interest in subsequent chapters, as well as important results which can be found (along with their proofs), in [4]. We start with several definitions.

Definition 2.19 *Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is an* **additive category** *if for each $x, y \in \mathrm{ob}(\mathscr{C})$ we have that $\mathscr{C}(x, y)$ is an abelian group, there is a an object $0 \in \mathrm{ob}(\mathscr{C})$ such that $\mathscr{C}(0, 0) = 0$ and if there exists a bifunctor $\oplus : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called the* **direct sum** *along with projections $p_1 : x \oplus y \to x$, $p_2 : x \oplus y \to y$ and inclusions $i_1 : x \to x \oplus y$ and $i_2 : y \to x \oplus y$ such that*

$$
\begin{aligned}
p_1 \circ i_1 &= id_x, \\
p_2 \circ i_2 &= id_y, \\
i_1 \circ p_1 + i_2 \circ p_2 &= id_y,
\end{aligned}
$$

for all $x, y, z \in \mathrm{ob}(\mathscr{C})$. If for every $f \in \mathscr{C}(x, y)$ there is a sequence $K \xrightarrow{k} x \xrightarrow{i} I \xrightarrow{j} y \xrightarrow{c} C$

such that

$$
\begin{aligned}
j \circ i &= f, \\
(K, k) &= \mathit{Ker}(f), \\
(C, c) &= \mathit{Coker}(f), \\
(I, i) &= \mathit{Coker}(k), \\
(I, j) &= \mathit{Ker}(c),
\end{aligned}
$$

then we say tht $\mathscr{C}$ is an **abelian category**. *We say that a functor $F : \mathscr{C} \to \mathscr{D}$ between additive categories $\mathscr{C}$ and $\mathscr{D}$ is an* **additive functor** *if the associated map $\mathscr{C}(x, y) \to \mathscr{D}(F(x), F(y))$ is a homomorphism of abelian groups.*

Definition 2.20 *Let k be a field. We say that a category $\mathscr{C}$ is* **k-linear** *if for every $x, y \in \mathrm{ob}(\mathscr{C})$, $\mathscr{C}(x, y)$ has the structure of a vector space over k such that the composition of morphisms i k-linear.*

Definition 2.21 *Let $\mathscr{C}$ be an abelian category and $x, y \in \mathrm{ob}(\mathscr{C})$. x is called a* **subobject** *of y if there is a monomorphism $x \to y$. x is called* **simple** *if 0 and x are its only subobjects. An object is called* **semisimple** *if it can be written as a direct sum of simple objects. Finally, we say that $\mathscr{C}$ is* **semisimple** *if every object in it is semisimple.*

Definition 2.22 *Let $\mathscr{C}, \mathscr{D}$ be abelian categories. An additive functor $F : \mathscr{C} \to \mathscr{D}$ is called* **left (respectively, right) exact** *if for eny short exact sequence $0 \to x \to y \to z \to 0$ in $\mathscr{C}$, the sequence $0 \to F(x) \to F(y) \to F(z)$ (respectively, $F(x) \to F(y) \to F(z) \to 0$) is exact. We say that F is* **exact** *if it is both left and right exact.*

These might seem like exhaustive definitions, but they really only ensure that we have the direct sum, the kernel, image and cokernel well defined in our categories. Notice that $\mathbf{Vec}_k$ is such a category.

Definition 2.23 (Deligne's tensor product) *Let $\mathscr{C}$ and $\mathscr{D}$ be locally finite k-linear abelian categories.* **Deligne's tensor product** *$\mathscr{C} \boxtimes \mathscr{D}$ is an abelian k-linear category satisfying the following universal propery: given right exact bilinear functors $\boxtimes : \mathscr{C} \times \mathscr{D} \to \mathscr{C} \boxtimes \mathscr{D}$ (mapping $(c, d) \mapsto c \boxtimes d$ for all $c \in \mathrm{ob}(\mathscr{C}), d \in \mathrm{ob}(\mathscr{D})$) and $F : \mathscr{C} \times \mathscr{D} \to \mathscr{E}$ there exists a unique right exact functor $\hat{F} : \mathscr{C} \times \mathscr{D} \to \mathscr{E}$ such that $\hat{F} \circ \boxtimes = F$.*

Notice the resemblance to the universal property of the usual tensor product on vector spaces.

Proposition 2.24 *Deligne's tensor product $\mathscr{C} \boxtimes \mathscr{D}$ exists and the $\boxtimes$ functor is exact in both variables satisfying*

$$\mathscr{C}(x_1, y_1) \otimes \mathscr{D}(x_2, y_2) \cong \mathscr{C} \boxtimes \mathscr{D}(x_1 \boxtimes x_2, y_1 \boxtimes y_2).$$

Definition 2.25 (Monoidal category) *A* **monoidal category** *is a sextuple $(\mathscr{C}, \otimes, 1, \alpha, \rho, \lambda)$ comprised of a category $\mathscr{C}$, a bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called the* **tensor product***, a natural isomorphism $(- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$ called the* **associator***, an object $1 \in \mathrm{ob}(\mathscr{C})$ called a* **unit***, the left and right* **unit isomorphisms***, $\lambda : 1 \otimes - \xrightarrow{\sim} -$ and $\rho : - \otimes 1 \xrightarrow{\sim} -$ respectively, such that the following diagrams:*

$$\begin{tikzcd}
 & (x \otimes y) \otimes (z \otimes w) \arrow[dr, "\alpha_{x,y,z\otimes w}"] & \\
((x \otimes y) \otimes z) \otimes w \arrow[ur, "\alpha_{x\otimes y,z,w}"] \arrow[d, "\alpha_{x,y,z}\otimes id_w"'] & & x \otimes (y \otimes (z \otimes w)) \\
(x \otimes (y \otimes z)) \otimes w \arrow[rr, "\alpha_{x,y\otimes z,w}"'] & & x \otimes ((y \otimes z) \otimes w) \arrow[u, "id_x\otimes\alpha_{y,z,w}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "\alpha_{x,1,y}"] \arrow[dr, "\rho_x\otimes id_y"'] & & x \otimes (1 \otimes y) \arrow[dl, "id_x\otimes\lambda_y"] \\
 & x \otimes y &
\end{tikzcd}$$

commute for all $x, y, z, w \in \mathrm{ob}(\mathscr{C})$.

We swiftly follow this definition by some well-known examples. A basic example is the one of **Set** where we take the tensor product to be the usual Cartesian product of sets. Consider now the following two sets , $\{1, 2\}$ and $\{1\}$, and their Cartesian product, $\{(1, 1), (1, 2)\}$. We have an obvious bijection $\{1, 2\} \to \{1, 2\} \times \{1\}$ given by $x \mapsto (x, 1)$ making those two sets isomorphic, and from that it is immediately obvious unit corresponds to the set with a single element, $1 = \{*\}$. Another example is $\mathbf{Vec}_k$ where the tensor product is given by the standard tensor product of vector spaces which we have seen in the last subchapter. We often encode symmetries using groups, more specifically, their actions. For example, imagine an equilateral triangle where we labeled the vertices as 0,1,2 in a clockwise order. We can then represent the rotation of the triangle by 120° clockwise as a function $\{0,1,2\} \to \{0,1,2\}$ which maps $x \mapsto (x+1) \bmod 3$. We can do the same for other transformations forming the cyclic group of the triangle, In effect, we have defined a function $\mathbb{Z}_3 \times \{0,1,2\} \to \{0,1,2\}$ as $(x,y) \mapsto (x+y) \bmod 3$. We call such functions, which encode how symmetries transform objects, actions.

Definition 2.26 *Let $(G,\cdot)$ be a group and S a set. We say that S is a* **left G-set** *if there exists a map $\phi : G \times S \to S$ such that*

$$\phi(g_1, \phi(g_2, s)) = \phi(g_1 \cdot g_2, s),$$

for all $g_1, g_2 \in G, s \in S$. We call the map ϕ a **left G-action** *on S. Additionally, when the map ϕ is clear from the context, often the following notation is used, $\phi(g,s) = g.s$ where the above condition is then written as $g_1.(g_2.s) = (g_1 \cdot g_2).s$. We analogously define a* **right G-action** *with a map $S \times G \to S$.*

More often than not, we are interested in a special kind of actions, namely, the ones where the set S from the above definition has the structure of a vector space.

Definition 2.27 *Let $(G,\cdot)$ be a group and $V \in \mathrm{ob}(\mathbb{V}_k)$ a vector space. A map $\rho : G \to$ End(V) such that*

$$\rho(g_1 \cdot g_2) = \rho(g_1)\rho(g_2),$$

for all $g_1, g_2 \in G$ is then called a **representation***. In that case we say that V is a* **(left) G-module***. We also have an associated left G-action given as $g.v = \rho(g)v$ for all $v \in V$.*

These representations of a group define a monoidal category where the tensor product is given by the induced representation on the tensor product vector space. Now, since End(V) is also a vector space, we often run into expressions of the form $g_1.v + \alpha g_2.v$ which makes the following structure interesting.

Definition 2.28 *Suppose $(R,+,\cdot,1)$ is a unital ring. A* **left R-module** *A consists of an* *abelian group $(A, \star)$ along with a map $* : R \times A \to A$ such that we have*

$$r * (a \star b) = (r * a) \star (r * b),$$
$$(r + s) * a = (r * a) \star (s * a),$$
$$(r \cdot s) * a = r * (s * a),$$
$$1 * a = a,$$

for all $a, b \in A, r, s \in R$. In practice, often the same symbols are uses, $\cdot$ for $*$ and $+$ for $\star$, and furthermore $\cdot$ is often omitted so that the above expressions read

$$r(a + b) = ra + rb,$$
$$(r + s)a = ra + sa,$$
$$(rs)a = r(sa),$$
$$1a = a.$$

Again, there is an analogous definition for a **right R-module**. *Suppose now S is another unital ring. We call an abelian group which is both a left R-module and a right S-module an* **(R, S)-bimodule**.

We can then think of group representations as modules over the group ring $k[G]$ which allow us to write $g_1.v + \alpha g_2.v = (g_1 + \alpha g_2).v$ and thus a more general structure to analyze symmetries defined on a vector space and this is obviously of keen interest when we think of the Hilbert space of states of our quantum system. Similarly, for commutative untial ring R we have a monoidal category $R - \mathbf{mod}$ of left R-modules. For an associative unital ring A the category $A - \mathbf{bimod}$ of (A, A)-bimodules is a monoidal category with the tensor product being the tensor product $\otimes_A$ over A and the unit given by A seen as a bimodule over itself. Finally, for each category $\mathscr{C}$, the category of its endofunctors, $\mathbf{End}(\mathscr{C}) \equiv [\mathscr{C}, \mathscr{C}]$ is a monoidal category with functor composition acting as the tensor product.

Let us quickly notice something important before continuing. In ordinary monoids, the product is associative. However, we remember when we would take tensor product of groups this order of operations would come into play in terms of Wigner 3j symbols. We address this in the following definition and a theorem.

Definition 2.29 *Suppose $\mathscr{C} = (\mathscr{C}, \otimes, 1, \alpha, \rho, \lambda)$ is a monoidal category. We say that $\mathscr{C}$ is a* **strict** *monoidal category if the isomorphisms α, ρ and λ are identities. In that case we have equalities*

$$(x \otimes y) \otimes z = x \otimes (y \otimes z),$$
$$x \otimes 1 = 1 \otimes x = x,$$

for all $x, y, z \in \mathrm{ob}(\mathscr{C})$.

Theorem 2.30 *Every monoidal category is equivalent to a strict monoidal category.*

This is a strong statement. However, it doesn't mean that we can always forget about parenthesis in our monoidal categories as remember our notion of equivalences of categories is a lot weaker as we don't require the existance of an isomorphism between categories. A prototypical example of a strict category is that of $\mathbf{End}(\mathscr{C})$. We follow this by functors between monoidal categories.

Definition 2.31 *Suppose $(\mathscr{C}, \otimes, 1, \alpha, \rho, \lambda)$ and $(\mathscr{D}, \boxtimes, 1', \alpha', \rho', \lambda')$ are two monoidal categories categories. A* **monoidal functor** *from $\mathscr{C}$ to $\mathscr{D}$ is a pair (F, J) of a functor $F : \mathscr{C} \to \mathscr{D}$ and a natural isomorphism $J_{x,y} : F(x) \boxtimes F(y) \to F(x \otimes y)$ such that $F(1) \cong 1'$ and the diagram*

$$\begin{tikzcd}
(F(x) \boxtimes F(y)) \boxtimes F(z) \arrow[r, "\alpha'_{F(x),F(y),F(z)}"] \arrow[d, "J_{x,y} \boxtimes id_{F(z)}"'] & F(x) \boxtimes (F(y) \boxtimes F(z)) \arrow[d, "id_{F(x)} \boxtimes J_{y,z}"] \\
F(x \otimes y) \boxtimes F(z) \arrow[d, "J_{x \otimes y, z}"'] & F(x) \boxtimes F(y \otimes z) \arrow[d, "J_{x, y \otimes z}"] \\
F((x \otimes y) \otimes z) \arrow[r, "F(\alpha_{x,y,z})"'] & F(x \otimes (y \otimes z))
\end{tikzcd}$$

commutes for all $x, y, z \in \mathrm{ob}(\mathscr{C})$.

We now list several examples of monoidal functors. An obvious one is the forgetful functor $\mathbf{Rep}(G) \to \mathbf{Vec}_k$. Given a unital k-algebra A, we have a monoidal functor $F : A - \mathbf{bimod} \to \mathbf{End}(A - \mathbf{mod})$ which maps $M \mapsto (M \otimes_A -)$.

Proposition 2.32 *The functor $F : A{-}\mathbf{bimod} \to \mathbf{End}(A{-}\mathbf{mod})$ defines an equivalence of monoidal categories $A - \mathbf{bimod} \cong \mathbf{End}_{re}(A - \mathbf{mod})$ where $\mathbf{End}_{re}(A - \mathbf{mod})$ is the subcategory of right exact endofunctors of $A - \mathbf{mod}$.*

Before we continue with properties of vector spaces, there is one final generalization of actions left, that of module categories.

Definition 2.33 *Let* $\mathscr{C} = (\mathscr{C}, \otimes, 1, \alpha, \rho, \lambda)$ *be a monoidal category. A* **left module category** *over* $\mathscr{C}$ *is a category* $\mathscr{M}$ *equipped with a bifunctor* $\otimes : \mathscr{C} \times \mathscr{M} \to \mathscr{M}$ *called an* **action** *and a natural isomorphisms* $m_{x,y,m} : (x \otimes y) \otimes m \xrightarrow{\sim} x \otimes (y \otimes m)$ *and* $\lambda_m : 1 \otimes m \xrightarrow{\sim} m$ *for all* $x, y \in \mathrm{ob}(\mathscr{C}), m \in \mathrm{ob}(\mathscr{M})$ *such that the diagrams*

$$\begin{tikzcd}
 & (x \otimes y) \otimes (z \otimes m) \arrow[dr, "m_{x,y,z\otimes m}"] & \\
((x \otimes y) \otimes z) \otimes m \arrow[ur, "m_{x\otimes y,z,m}"] \arrow[d, "\alpha_{x,y,z}\otimes id_m"'] & & x \otimes (y \otimes (z \otimes m)) \\
(x \otimes (y \otimes z)) \otimes m \arrow[rr, "m_{x,y\otimes z,m}"] & & x \otimes ((y \otimes z) \otimes m) \arrow[u, "id_x \otimes m_{y,z,m}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
(x \otimes 1) \otimes m \arrow[rr, "m_{x,1,m}"] \arrow[dr, "\rho_x \otimes id_m"'] & & x \otimes (1 \otimes m) \arrow[dl, "id_x \otimes \lambda_m"] \\
 & x \otimes m &
\end{tikzcd}$$

commute for all $x, y, z \in \mathrm{ob}(\mathscr{C}), m \in \mathrm{ob}(\mathscr{M})$*.* **Right module categories** *are defined analogously.*

Notice the resemblance to the diagrams in the definition of the monoidal category 2.25. Similar to how we had actions associated to representations we have the following correspondance.

Proposition 2.34 *There is a one-to-one correspondance between left* $\mathscr{C}$*-modules* $\mathscr{M}$ *and monoidal functors* $F : \mathscr{C} \to \mathbf{End}(\mathscr{M})$*.*

Modules over categories will be the main structure of our interest in later chapters. We continue with our order of business introducting properties of vector spaces in their categorical setting.

Vector spaces have their duals which are captured by rigid categories whose definition follows suit.

Definition 2.35 *Let* $(\mathscr{C}, \otimes, 1, \alpha, \rho, \lambda)$ *be a monoidal category and* $x \in \mathrm{ob}(\mathscr{C})$*. An object* $x^* \in \mathrm{ob}(\mathscr{C})$ *is said to be a* **left dual** *of* x *if there exist morphsms* $ev_x : x^* \otimes x \to 1$ *and* *coev*$_x : 1 \to x \otimes x^*$, *called* **evaluation** *and* **coevaluation***, such that the diagrams*

$$\begin{tikzcd}
 & (x \otimes x^*) \otimes x \arrow[rr, "\alpha_{x,x^*,x}"] & & x \otimes (x^* \otimes x) \arrow[dr, "id_x \otimes ev_x"] & \\
1 \otimes x \arrow[ur, "coev_x \otimes id_x"] \arrow[rr, "\lambda_x"] & & x & & x \otimes 1 \arrow[ll, "\rho_x"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & x^* \otimes (x \otimes x^*) \arrow[rr, "\alpha^{-1}_{x^*,x,x^*}"] & & (x^* \otimes x) \otimes x^* \arrow[dr, "ev_x \otimes id_{x^*}"] & \\
x^* \otimes 1 \arrow[ur, "id_{x^*} \otimes coev_x"] \arrow[rr, "\rho_{x^*}"] & & x^* & & 1 \otimes x^* \arrow[ll, "\lambda_{x^*}"']
\end{tikzcd}$$

commute. Similarly, we say that $^*x \in \mathrm{ob}(\mathscr{C})$ *is the* **right dual** *of* x *if there exist morphisms ev*$'_x : x \otimes^a stx \to 1$ *and coev*$'_x : 1 \to^a stx \otimes x$ *such that the diagrams*

$$\begin{tikzcd}
 & x \otimes (^*x \otimes x) \arrow[rr, "\alpha^{-1}_{x,x^*,x}"] & & (x \otimes^* x) \otimes x \arrow[dr, "ev'_x \otimes id_x"] & \\
x \otimes 1 \arrow[ur, "id_x \otimes coev'_x"] \arrow[rr, "\rho_{x^*}"] & & x & & 1 \otimes x \arrow[ll, "\lambda_x"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & (^*x \otimes x) \otimes^* x \arrow[rr, "\alpha_{^*x,x,^*x}"] & & ^*x \otimes (x \otimes^* x) \arrow[dr, "id_{^*x} \otimes ev'_x"] & \\
1 \otimes^* x \arrow[ur, "coev'_x \otimes id_{^*x}"] \arrow[rr, "\lambda_{^*x}"] & & ^*x & & ^*x \otimes 1 \arrow[ll, "\rho_{^*x}"']
\end{tikzcd}$$

commute.

If an object has a left or right dual then this dual is unique upto a unique isomorphism. Furthermore, if $x, y \in \mathrm{ob}(\mathscr{C})$ have a (left) dual, so does their tensor product, $(x \otimes y)^* = y^* \otimes x^*$. Additionally we have the following adjoint functors, $(x^* \otimes -) \dashv (x \otimes -)$ and $(- \otimes x) \dashv (- \otimes x^*)$ giving rise to the natural isomorphism $\mathscr{C}(x^*, y) \cong \mathscr{C}(1, x \otimes y)$.

Definition 2.36 *Let* $\mathscr{C}$ *be a monoidal category. We say that* $\mathscr{C}$ *is* **rigid** *if every object has left and right duals.*

Naturally, the category of finite dimensional vector spaces is a rigid category where the evaluation map is given by contraction, $(x, y) \mapsto y(x) = \langle y|x \rangle$, and coevaluation by the partition of unity, $\alpha \mapsto \sum_x \alpha x^* \otimes x = \sum_x \alpha | x \rangle \langle x |$. Similarly, $\mathbf{Rep}(G)$ is another example of a rigid category. We have a correspondance between left(right)module structures on rigid categories with right(left) module sturcture on the dual category. Suppose $\mathscr{C}$ is a left $\mathscr{C}$-module category, then $\mathscr{M}^{op}$ has the structure of a right $\mathscr{C}$-module category with the action $\odot$ given by $m \odot x \equiv x^* \otimes m$ for $x \in \mathrm{ob}(\mathscr{C}), m \in \mathrm{ob}(\mathscr{M})$.

Definition 2.37 *Let $\mathscr{C}$ be a locally finite k-linear abelian rigid monoidal category. If the bifunctor $\otimes$ is bilinear on morphisms, we say that $\mathscr{C}$ is a* **multitensor category**. *Additionally if $\mathbf{End}(1) \cong k$, we say $\mathscr{C}$ is a* **tensor category**. *A* **multifusion category** *is a finite semisimple multitensor category. A* **fusion category** *is a multifusion category such that $\mathbf{End}(1) \cong k$.*

The category of finite dimensional vector spaces over k, Vec_k is a fusion category. In a multifusion category, the $\otimes$ bifunctor is exact in both variables.

Proposition 2.38 *Deligne's tensor product $\mathscr{C} \boxtimes \mathscr{D}$ is a (multi)tensor category and a (multi)fusion category respectively if so are $\mathscr{C}$ and $\mathscr{D}$.*

Definition 2.39 *Let $\mathscr{C}$ be a rigid monoidal category, $x, y \in \mathrm{ob}(\mathscr{C})$ and $a \in \mathscr{C}(x, x^{**})$. We define the* **left (quantum) trace** $Tr_L(a) \equiv 1 \xrightarrow{coev_x} x \otimes x^* \xrightarrow{a \otimes id_{x^*}} x^{**} \otimes x \xrightarrow{ev_{x^*}} 1$. *Similarly, for $b \in \mathscr{C}(y, {}^{**}y)$ we define the* **right (quantum) trace** $Tr_R(b) \equiv 1 \xrightarrow{coev_{^*y}} {}^*y \otimes y \xrightarrow{id_{^*y} \otimes b} {}^*y \otimes {}^{**}y \xrightarrow{ev_{^{**}y}} 1$.

Proposition 2.40 *Suppose $\mathscr{C}$ is a rigid monoidal category. Let $a \in \mathscr{C}(x, x^{**},,)b \in \mathscr{C}(y, y^{**})$ and $c \in \mathscr{C}(x, x)$. We have the following properties:*

$$
\begin{aligned}
Tr_L(a) &= Tr_R(a^*), \\
Tr_L(a \oplus b) &= Tr_L(a) + Tr_L(b) \text{ (in additive categories)}, \\
Tr_L(a \otimes b) &= Tr_L(a) Tr_L(b), \\
Tr_L(a \circ c) &= Tr_L(c^{**} \circ a), \\
Tr_R(a \circ c) &= Tr_R({}^{**}c \circ a).
\end{aligned}
$$

Definition 2.41 *Let $\mathscr{C}$ be a rigid monoidal category. A* **pivotal structure** *on $\mathscr{C}$ is an isomorphism of monoidal functors $a_x : x \xrightarrow{\sim} x^{**}$. A rigid monoidal category equipped* *with a pivotal structure is said to be* **pivotal**. *Given a pivotal structure a, we define the* **dimension** *of $x \in \mathrm{ob}(\mathscr{C})$ as*

$$\mathit{dim}_a(x) = \mathit{Tr}_L(a_x).$$

We say that a pivotal structure a is **spherical** *if $\mathit{dim}_a(x) = \mathit{dim}_a(x^*)$ for all $x \in \mathrm{ob}(\mathscr{C})$. In that case we also say that $\mathscr{C}$ is* **spherical**.

Theorem 2.42 *Let $\mathscr{C}$ be a spherical category, $x \in \mathrm{ob}(\mathscr{C})$ and $f \in \mathscr{C}(x, x)$. Then $\mathit{Tr}_L(a_x \circ f) = \mathit{Tr}_R(f \circ a_x^{-1})$.*

Of special interest will be module categories over (spherical) fusion categories which have an abelian structure.A good way to think about this is the generalization of group representations as $k[G]$-modules.

Definition 2.43 *Let $\mathscr{C}$ be a multitensor category over k. A* **left $\mathscr{C}$-module category** *is a locally finite abelian category $\mathscr{M}$ over k that is a left $\mathscr{C}$-module.*

Proposition 2.44 *There is a one-to-one correspondance between left $\mathscr{C}$-module structures on $\mathscr{M}$ and tensor functors $\mathscr{C} \to \mathbf{End}_{le}(\mathscr{M})$.*

Proposition 2.45 *Given two left $\mathscr{C}$-modules $\mathscr{M}_1$ and $\mathscr{M}_2$ over a multitensor category $\mathscr{C}$, the category $\mathscr{M}_1 \oplus \mathscr{M}_2$ given by direct sum over objects and morphisms is again a left $\mathscr{C}$-module.*

Definition 2.46 *Let $\mathscr{C}, \mathscr{D}$ be multitensor categories. A (C, D)***-bimodule category** *is a $(\mathscr{C} \boxtimes \mathscr{D}^{op})$-module category.*

The last piece of the puzzle is that of the center and the additional structure we encounter on it.

Definition 2.47 *Let $\mathscr{C}$ be a monoidal category. We define the* **center** *of the category $\mathscr{C}$, $\mathcal{Z}(\mathscr{C})$ as follows. Object of $\mathcal{Z}(\mathscr{C})$ are pairs (x, γ) where $x \in \mathrm{ob}(\mathscr{C})$ and γ is a natural isomorphism $\gamma_y : y \otimes x \xrightarrow{\sim} x \otimes y$ for all $y \in \mathrm{ob}(\mathscr{C})$ such that the diagram*

$$\begin{tikzcd}
z \otimes (x \otimes y) \arrow[r, "\alpha^{-1}_{z,x,y}"] & (z \otimes x) \otimes y \arrow[d, "\gamma_z \otimes id_y"] \\
z \otimes (y \otimes x) \arrow[u, "id_z \otimes \gamma_y"] \arrow[d, "\alpha^{-1}{z,y,x}"'] & (x \otimes z) \otimes y \\
(z \otimes y) \otimes x \arrow[r, "\gamma_{z \otimes y}"'] & x \otimes (z \otimes y) \arrow[u, "\alpha^{-1}_{x,z,y}"']
\end{tikzcd}$$

commutes for all $y, z \in \mathrm{ob}(\mathscr{C})$. Morphisms in $\mathcal{Z}(\mathscr{C})$ from (x, γ) to (x', γ') are morphisms $f \in \mathscr{C}(x, x')$ such that $(f \otimes id_y) \circ \gamma_y = \gamma'_y \circ (id_y \otimes f)$ for all $y \in \mathrm{ob}(\mathscr{C})$.

This center, which is a generalization of center in a ring, has the structure of a monoidal category with the following tensor product:

$$(x, \gamma) \otimes (x', \gamma') \equiv (x \otimes x', \hat{\gamma}),$$

where $\hat{\gamma}$ is a natural isomorphism whose components are defined as a composition

$$\hat{\gamma}_y \equiv y \otimes (x \otimes x') \xrightarrow{\alpha^{-1}_{y,x,x'}} (y \otimes x) \otimes x' \xrightarrow{\gamma_y \otimes id_{x'}} (x \otimes y) \otimes x' \xrightarrow{\alpha_{x,y,x'}} x \otimes (y \otimes x') \circ$$
$$\circ\, x \otimes (y \otimes x') \xrightarrow{id_x \otimes \gamma'_y} x \otimes (x' \otimes y) \xrightarrow{\alpha^{-1}_{x,x',y}} (x \otimes x') \otimes y,$$

and the unit is given as $(1, \rho^{-1} \circ \lambda)$.

Proposition 2.48 *Let $\mathscr{C}$ be a monoidal category. If $\mathscr{C}$ is rigid then so is its center $\mathcal{Z}(\mathscr{C})$. Similarly, if $\mathscr{C}$ is pivotal, so is its center. If it is a tensor category, so is its center.*

Proposition 2.49 *We have an equivalence $\mathcal{Z}(\mathscr{C}^{op}) \cong \mathcal{Z}(\mathscr{C})$.*

We have an obvious forgetful functor $F : \mathcal{Z}(\mathscr{C}) \to \mathscr{C}, (x, \gamma) \mapsto x$.

Proposition 2.50 *The forgetful functor $\mathcal{Z}(\mathscr{C}) \to \mathscr{C}$ is surjective.*

$\mathscr{C}$ is a $(\mathscr{C}, \mathscr{C})$-bimodule where $(x \boxtimes y) \otimes z \equiv x \otimes y \otimes z$.

Proposition 2.51 *There is an equivalence $\mathcal{Z}(\mathscr{C}) \cong \mathbf{End}((\mathscr{C}, \mathscr{C}) - \mathbf{bimod})$ between the center and endofunctor category of the category $(\mathscr{C}, \mathscr{C})$-bimodules.*

Definition 2.52 *A* **braiding** *on a monoidal category $\mathscr{C}$ is a natural isomorphism $c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x$. such that the diagrams*

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "\alpha_{x,y,z}"] \arrow[d, "c_{x,y} \otimes id_z"'] & x \otimes (y \otimes z) \arrow[r, "c_{x,y\otimes z}"] & (y \otimes z) \otimes x \arrow[d, "\alpha_{y,z,x}"] \\
(y \otimes x) \otimes z \arrow[r, "\alpha_{y,x,z}"'] & y \otimes (x \otimes z) \arrow[r, "id_y \otimes c_{x,z}"'] & y \otimes (z \otimes x)
\end{tikzcd}$$

and

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, "\alpha^{-1}_{x,y,z}"] \arrow[d, "id_x \otimes c_{y,z}"'] & (x \otimes y) \otimes z \arrow[r, "c_{x\otimes y,z}"] & z \otimes (x \otimes y) \arrow[d, "\alpha^{-1}_{z,x,y}"] \\
x \otimes (z \otimes y) \arrow[r, "\alpha^{-1}_{x,z,y}"'] & (x \otimes z) \otimes y \arrow[r, "c_{x,z\otimes y}"'] & (z \otimes x) \otimes y
\end{tikzcd}$$

commute for all $x, y, z \in \mathrm{ob}(\mathscr{C})$. In that case we say that $\mathscr{C}$ is **braided**. *We denote the category equipped with the braiding $c'_{x,y} \equiv c^{-1}_{y,x}$ as $\mathscr{C}^{rev}$ and call it the* **reverse** *category and the braiding $c'_{x,y}$ the* **reverse braiding**. *Finally, we say that a braided monoidal category is* **symmetric** *if*

$$c_{y,x} \circ c_{x,y} = id_{x \otimes y},$$

for all $x, y \in \mathrm{ob}(\mathscr{C})$.

Proposition 2.53 *$\mathcal{Z}(\mathscr{C})$ is a braided monoidal category with braiding*

$$c_{(x,\gamma),(x',\gamma')} \equiv \gamma'_x.$$

Proposition 2.54 *We have the equivalences $\mathcal{Z}(\mathscr{C}^{op}) \cong \mathcal{Z}(\mathscr{C})^{rev}$ and $\mathcal{Z}(\mathscr{C} \boxtimes \mathscr{C}^{op}) \cong \mathcal{Z}(\mathscr{C}) \boxtimes \mathcal{Z}(\mathscr{C})^{rev}$.*

Definition 2.55 *A* **twist** *on a braided rigid monoidal category $\mathscr{C}$ is a natural isomorphism θ such that*

$$\theta_{x \otimes y} = (\theta_x \otimes \theta_y) \circ c_{y,x} \circ c_{x,y},$$

for all $x, y \in \mathrm{ob}()$. If $(\theta_x)^* = \theta_{x^*}$ then we say that the twist defines a **ribbon structure** *and we call $\mathscr{C}$ a* **ribbon category**.

2.4 Higher Categories

So far, we have considered what are known as ordinary or 1-categories. The notion of a higher category is not yet well defined as there are multiple working definitions depending on the context. However, some things common among different definitions. Let us first start with what is known as a 2-category.

The concept revolves around the idea that our morphisms might have additional structure, beyond the set. Consider for example the category $\mathbf{Vec}_k$. The set $\mathbf{Vec}_k(U, V)$ of morphisms from U to V by itself has the structure of a vector space. We know this from linear algebra. This makes us consider morphisms between morphisms. But we have already seen those, namely, the natural transformations. And this is what a 2-category is, We have objects, morphisms and morphisms between morphisms. There is a small caveat. Higher categories come in two flavors, strict and weak. Strict categories are those where composition has to be associative identically. Weak categories are where they only need to be composited associativelly only upto natural isomorphisms. Since we are interested in monoidal categories, our flavor of higher categories is that of categories enriched in other monoidal categories.

Definition 2.56 *Let $\mathscr{C} = (\mathscr{C}, \otimes, 1)$ be a symmetric monoidal category. a $\mathscr{C}$***-category** *$\underline{\mathscr{D}}$ consists of a collection of objects $\mathrm{ob}(\underline{\mathscr{D}})$, for each pair of objects x, y, a* **hom-object** *$\underline{\mathscr{D}}(x,y)$ in $\mathscr{C}$, for each object x, a morphism $id_x : 1 \to \underline{\mathscr{D}}(x,x)$ in $\mathscr{C}$, and for each triple x, y, z, a morphism $\circ : \underline{\mathscr{D}}(y,z) \otimes \underline{\mathscr{D}}(x,y) \to \underline{\mathscr{D}}(x,z)$ in $\mathscr{C}$ such that the diagrams*

$$\begin{tikzcd}
\underline{\mathscr{D}}(z,w) \otimes \underline{\mathscr{D}}(y,z) \otimes \underline{\mathscr{D}}(x,y) \arrow[r, "1\otimes\circ"] \arrow[d, "\circ\otimes 1"'] & \underline{\mathscr{D}}(z,w) \otimes \underline{\mathscr{D}}(x,z) \arrow[d, "\circ"] \\
\underline{\mathscr{D}}(y,w) \otimes \underline{\mathscr{D}}(x,y) \arrow[r, "\circ"'] & \underline{\mathscr{D}}(x,w)
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathscr{D}}(x,y) \otimes 1 \arrow[r, "id_{\underline{\mathscr{D}}(x,y)}\otimes id_x"] \arrow[dr, "\cong"'] & \underline{\mathscr{D}}(x,y) \otimes \underline{\mathscr{D}}(x,x) \arrow[d, "\circ"] \\
 & \underline{\mathscr{D}}(x,y)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\underline{\mathscr{D}}(y,y) \otimes \underline{\mathscr{D}}(x,y) \arrow[d, "\circ"] & 1 \otimes \underline{\mathscr{D}}(x,y) \arrow[l, "id_y\otimes id_{\underline{\mathscr{D}}(x,y)}"'] \arrow[dl, "\cong"] \\
\underline{\mathscr{D}}(x,y) &
\end{tikzcd}$$

commute for all $x, y, z, w \in \mathrm{ob}(\underline{\mathscr{D}})$.

A category enriched over $\mathbf{Set}$ is a 1-category. A strict 2-category is then a category enriched over $\mathbf{Cat}$ where the monoidal structure is given by the product of categories. There is no reason to stop. We then define a strict **n-category** as a category enriched over an (n-1)-category. We define a (∞, n)**-category** as a ∞-category where k-morphisms for k$\geq$ n are isomorphisms. We are interested in weak (∞, n)-categories.

Given an (∞, n)-category $\mathscr{C}$ and an object $x \in \mathrm{ob}(\mathscr{C})$, we have that $\mathrm{End}_{\mathscr{C}}(x) = \mathscr{C}(x,x)$ is a monoidal $(\infty, n-1)$-category. Given a monoidal (∞, n)-category $\mathscr{C}$ we can then canonically assign to it a monoidal $(\infty, n-1)$ category $\Omega\mathscr{C} = \mathrm{End}_{\mathscr{C}}(1)$ called the **loop space** of $\mathscr{C}$. By extension we can then canonically assign to it the $(\infty, n-k)$-category $\Omega^k\mathscr{C}$. On the other hand, to each monoidal (∞, n)-category $\mathscr{C}$ we can assign a $(\infty, n+1)$-category $B\mathscr{C}$ called the **classifying space** which has a single object and

$\mathscr{C}$ as the ∞-category of morphisms.

Lemma 2.57 *Let $\mathscr{C}$ be a symmetric monoidal (∞, n)-category and $\mathscr{D}$ a symmetric monoidal $(\infty, n+1)$-category. Then*

$$[B\mathscr{C}, \mathscr{D}] \cong [\mathscr{C}, \Omega\mathscr{D}].$$

3 Topological Quantum Field Theory

This chapter introduces the concept of Topological Quantum Field Theories (TQFTs). We start with a motivational example based on original historic usage of such theories, namely, caluclation of topological invariants of knots, which is still widely used in the theory of quantum computing. Following that we touch the two main examples of Chern-Simons and BF theories based on the treatment of the subject in [11]. Afterwards, we delve into the axiomatic approach to TQFTs of Atiyah [8] which we motivate using quantum mechanical systems and the path integral formalism. Beyond that, we look at generalizations including extended operators and how they relate to boundaries of manifolds. Finally we define the Extended TQFT used in the final construction of this thesis.

3.1 Knots and Links

Consider a set of fields $\phi = \{\phi_i\}_{i \in I}$ defined on a d-dimensional Riemannian (smooth) manifold Ω endowed with the metric $g_{\mu\nu}$ and an action functional $S[\phi]$ of those fields. A theory is said to be **topological** if all the correlation functions of observables $\mathcal{O} = \{\mathcal{O}_j\}_{j \in J}$ are invariant under the variations of the metric $g_{\mu\nu}$, that is to say,

$$\frac{\delta}{\delta g_{\mu\nu}} \left\langle \mathcal{O}_{j_1} \dots \mathcal{O}_{j_p} \right\rangle = 0$$

where

$$\left\langle \mathcal{O}_{j_1} \dots \mathcal{O}_{j_p} \right\rangle = \int_\Omega \mathcal{D}\phi \mathcal{O}_{j_1}[\phi] \dots \mathcal{O}_{j_p}[\phi] \exp(-S[\phi]).$$

Notice the exponent, we assume the action on Euclidean spacetime obtained by Wick rotation. Additionally, the adjective smooth in regards to a manifold is implied in the continuation. Such theories were first considered by Witten [10] when discussing supersymmetric quantum field theories but have found their first applications in the computation of knot invariants.

Consider the action defined on a 3-dimensional manifold Ω,

$$S_{CS}[A] = \frac{k}{4\pi} \int_\Omega \left[A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right],$$

where A is a $SU(2)$ gauge field and k is an integer. The theory defined by this action is referred to as a Chern-Simons theory, at least, one incarnation of it. Other similar theories obtained from characteristic classes of principle bundles with different gauge groups and dimensions of base manifolds are also called Chern-Simons theories. For example, choosing a $U(1)$ gauge symmetry, we obtain the abelian version of the Chern-Simons theory as

$$S_{CS}[A] = \frac{k}{4\pi} \int_\Omega A \wedge \mathrm{d}A.$$

These theories are metric independant and their partition function is gauge invariant. They are the prototypical example of topological quantum field theories. We define the following operator

$$W(\gamma : x \to y) = \mathrm{Tr} \left[\mathcal{P} \exp \left(\int_\gamma A \right) \right],$$

where γ is a curve from the space-time point x to y and $\mathcal{P}$ stands for path ordering. We call such an operator, which transforms as

$$W(\gamma : x \to y) \mapsto U(x)^{-1} W(\gamma : x \to y) U(y)$$

under the gauge transformations

$$A_\mu(x) \mapsto U^{-1}(x) A_\mu(x) U(x) + U^{-1}(x) \partial_\mu U(x),$$

Wilson line operator, often referred to simply as a **Wilson line**. Consider the infinitesimal line segment. The Wilson line is then given by

$$W(\gamma : x \to x + \mathrm{d}x) = 1 + A_\mu \mathrm{d}x^\mu$$

which transforms as

$$\begin{aligned}W(\gamma : x \to x + \mathrm{d}x) \mapsto\ & U^{-1}(x)W(\gamma : x \to x + \mathrm{d}x)U(x + \mathrm{d}x) \\ =\ & U^{-1}(x)[1 + A_\mu(x)\mathrm{d}x^\mu]U(x + \mathrm{d}x) \\ =\ & U^{-1}(x)[1 + A_\mu(x)\mathrm{d}x^\mu][U(x) + \mathrm{d}x^\mu \partial_\mu U(x)] \\ =\ & 1 + [U^{-1}(x)A_\mu(x)U(x) + U^{-1}(x)\partial_\mu U(x)]\mathrm{d}x^\mu.\end{aligned}$$

We see that such an operator is gauge-invariant and hence an observable of the theory. Given the Wilson line operators of loops, E. Witten showed in [9] that the correlator of such Wilson lines computes a knot invariant known as the Jones polynomial for the choice of the manifold $\Omega = S^3$,

$$V(\gamma_1, \dots, \gamma_n) = \frac{\int_{S^3} \mathcal{D}A W(\gamma_1) \dots W(\gamma_n) \exp(iS_{CS}[A])}{\int_{S^3} \mathcal{D}A \exp(iS_{CS}[A])}.$$

Subsequently, Chern-Simons theory has been found to describe the integer quantum Hall effect directly and was later used to model the fractional quantum Hall effect.

Before we continue, we give another example of a topological quantum field theory. **BF theories** are topological gauge theories defined by the action

$$S = \int_\Omega \mathrm{Tr}[B_{n-2} \wedge F(A)]$$

for non-abelian gauge symmetry and

$$S = \int_\Omega B_p \wedge \mathrm{d}A_{d-p-1}$$

in the case of an abelian gauge symmetry. In the above actions defined on the d-dimensional closed oriented manifold Ω, the rank of the form is indicated by their subscript, $F(A)$ is the curvature of the connection 1-form A. These theories, abelian version in particular, are interesting because they give us a generalization of the observables and the topological invariants their correlators give encountered in the Chern-Simons theory. Observables of BF theories, known as **Wilson surfaces** are given by

$$\exp\left(\int_{\partial\Sigma_{p+1}} B\right)$$

and

$$\exp\left(\int_{\partial\Sigma_{d-p}} A\right).$$

The correlators of these observables compute the so-called **linking number** (see section 6.2.2 of [11]) of $\partial\Sigma_{p+1}$ and $\partial\Sigma_{d-p}$ manifolds where Σ_{p+1} is a $(p+1)$-dimensional oriented manifold and Σ_{d-p} a $(d-p)$-dimensional oriented manifold. This linking number will be of significance when we study symmetries in the subsequent chapters.

3.2 Axiomatic Approach

Lack of rigorous definition of the path integral has led to the pursuit of an alternative which could be used in its place. The main strategy being to take a rigorous mathematical object which contains all the desired properties of a path integral and to express the theory in terms of such an object. This was first done by a set of axioms for conformal theories by G. Segal, [7], and was later generalized to the case of topological theories by M. F. Atiyah [8]. Before we introduce Atiyah's axiomatic approach to TQFTs, let us first discuss some of the desired properties.

Consider a partition function of a quantum field theory given by the path integral

$$Z_\Omega = \int_\Omega \mathcal{D}\phi \exp(-S_\Omega[\phi])$$

defined on the manifold Ω. In a slightly more suggestive way, we may write the path integral as

$$Z_\Omega = \int_{C_\Omega} \exp(-S_\Omega[\phi])\mathrm{d}\mu_\Omega[\phi],$$

Where C_Ω denotes the space of fields on Ω and μ_Ω the measure on C_Ω. If Ω has a boundary, $\partial\Omega \neq \emptyset$, then Z_Ω is a function of a field on the boundary $\partial\Omega$. By setting

$$C_\Omega(\psi) = \{\phi \in C_\Omega : \phi|_{\partial\Omega} = \psi\},$$

we can write

$$Z_\Omega(\psi) = \int_{C_\Omega(\psi)} \exp(-S_\Omega[\phi])\mathrm{d}\mu_\Omega(\phi).$$

Z_Ω then, interpreted as a function of fields on the boundary, then belongs to a vector space, namely, the Hilbert space of the theory which we will denote as $\mathcal{H}_{\partial\Omega}$. A field theory is then formulated on spacetimes of dimension d and $d-1$. The basic objects are the Hilbert space given from a $(d-1)$-dimensional manifold and the path integral,

$$\begin{aligned}\Sigma &\mapsto \mathcal{H}_\Sigma,\\ \Omega &\mapsto Z_\Omega \in \mathcal{H}_{\partial\Omega}.\end{aligned}$$

Given a diffeomorphism of spacetime $f : \Omega' \to \Omega$ we have an induced diffeomorphism on the boundary, $\partial f : \partial\Omega' \to \partial\Omega$ which induces a map $(\partial f)_* : \mathcal{H}_{\partial\Omega'} \to \mathcal{H}_{\partial\Omega}$ such that

$$(\partial f)_*(Z_{\Omega'}) = Z_\Omega.$$

Similarly, given a disjoint union of manifolds we expect the following

$$\begin{aligned}\mathcal{H}_{\Sigma_1\sqcup\Sigma_2} &\cong \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}\\ Z_{\Omega_1\sqcup\Omega_2} &= Z_{\Omega_1} \otimes Z_{\Omega_2}\end{aligned}$$

In the case of $d = 1$. We retrieve the usual quantum mechanic systems where we take the spacetime Ω to be the time dimension. Given an interval $[0, t]$, it maps to the evolution operator of the system, $U(0, t)$, more precisely, its trace. We notice then by swapping the orientation of the boundaries we expect to get the inverse, $\mathrm{Tr}[U(t, 0)] = \mathrm{Tr}[U(0, t)^\dagger] = \mathrm{Tr}[U(0, t)]^*$. We require the following property,

$$\begin{aligned}\mathcal{H}_{\overline{\Sigma}} &\cong \mathcal{H}^*_\Sigma,\\ Z_{\overline{\Omega}} &= Z^*_\Omega,\end{aligned}$$

where the overline $\overline{\Sigma}$ means we take the manifold with the reveresed orientation, and $\mathcal{H}^*_\Sigma$ denotes the space dual to $\mathcal{H}_\Sigma$. Supposing now the above interval such that both the boundary components are facing inwards we obtain the map

$$\mathcal{H}^* \otimes \mathcal{H} \to \mathbb{C}, \quad \langle\psi_1| \otimes |\psi_2\rangle \mapsto \langle\psi_1|U(0, t)|\psi_2\rangle\,.$$

Also note that in the limit $t \to 0$, this becomes the usual inner product on $\mathcal{H}$. We come to the last property by observing what happens when we cut the manifold Ω along a codimension 1 submanifold Σ. This creates a manifold Ω' whose boundary is now $\partial\Omega' = \partial\Omega \sqcup \Sigma \sqcup \overline{\Sigma}$ creating a path integral

$$Z_{\Omega'} \in \mathcal{H}_{\partial\Omega'} \cong \mathcal{H}_{\partial\Omega} \otimes \mathcal{H}_{\Sigma} \otimes \mathcal{H}_{\Sigma}^*.$$

We can now take the partial trace by contracting the Hilbert spaces of the cut sections using the evaluation map (inner product), and so we assert

$$Z_{\Omega} = \mathrm{Tr}_{\Sigma} Z_{\Omega'}.$$

Before we are ready to give the Atiyah's definition of topological quantum field theories, we introduce the category $\mathbf{Bord}_d$.

Definition 3.1 ($\mathbf{Bord}_d$) *Objects in $\mathbf{Bord}_d$ are oriented closed (d-1)-dimensional real manifolds. Suppose E and F are tow such manifolds. A* **bordism** *$E \to F$ is a triple (M, ι_i, ι_o) where M is an oriented compact d-dimensional manifold with a boundary, $\iota_i : E \to M$ and $\iota_o : F \to M$ are smooth maps with image in ∂M such that $\bar{\iota}_i \sqcup \iota_o : \overline{E} \sqcup F \to \partial M$ is an orientation-preserving diffeomorphism, where $\overline{E}$ denotes E with the opposite orientation and $\sqcup$ is the disjoint union. We define an equivalence relation between bordisms. Given two bordisms $(M, \iota_i, \iota_o), (M', \iota_i', \iota_o') : E \to F$, we say they are equivalent if there exists an orientation preserving diffeomorphism $\phi : M \to M'$ such that the diagram*

$$\begin{tikzcd}
 & M \arrow[dd, "\phi"] & \\
E \arrow[ur, "\iota_i"] \arrow[dr, "\iota_i'"'] & & F \arrow[ul, "\iota_o"'] \arrow[dl, "\iota_o'"] \\
 & M' &
\end{tikzcd}$$

commutes. Morphisms in $\mathbf{Bord}_d$ are given by equivalence classes of bordisms between objects. Composition of morphisms $M_1 : E \to F$ and $M_2 : F \to G$ is given by gluing M_1 and M_2 along F. $\mathbf{Bord}_d$ has the structure of a monoidal category taking the disjoint union $\sqcup$ as the tensor product and the empty set $\emptyset$ (seen as a (d-1)-dimensional manifold) to be the unit. Finally, the cannonical diffeomorphism $E \sqcup F \xrightarrow{\sim} F \sqcup E$ gives $\mathbf{Bord}_d$ a symmetric braiding. Orientation reversal gives a rigid structure on $\mathbf{Bord}_d$.

Definition 3.2 (Topological Quantum Field Theory) *A d-dimensional oriented closed topological quantum field theory (TQFT) is a symmetric monoidal functor $Z : \mathbf{Bord}_d \to \mathbf{Vec}_k$.*

Consider the cylinder $E \times [0,1]$ in $\mathbf{Bord}_d$. As a morphism $\overline{E} \sqcup E \to \emptyset$ it is mapped under Z to the evaluation map $Z(E)^* \otimes_k Z(E) \to k$. As a morphism $\emptyset \to E \sqcup \overline{E}$ it is mapped to the coevaluation map $k \to Z(E) \otimes Z(E)^*$. As a map $E \to E$ we have $Z(E \times [0,1]) = id_{Z(E)}$. Similarly, $E \times S^1$, seen as a composition of evaluation and coevaluation maps, it is mapped to $Z(E \times S^1) = \dim_k(Z(E))$. Finally we see that by cutting a manifold we have the isomorphism

$$Z(\partial\Omega) \otimes Z(\Sigma) \otimes Z(\Sigma)^* \xrightarrow{id_{Z(\partial\Omega)} \otimes \mathrm{ev}_{Z(\Sigma)}} Z(\partial\Omega) \otimes k \xrightarrow{\sim} Z(\partial\Omega)$$

implementing the cutting property.

We finish this subsection by motivating the next one. Suppose now $d = 1$. This theory is generated by tensor products of oriented 0-manifolds, $\bullet^+$ and $\bullet^-$. Morphisms are comprised of oriented lines connecting such points. Given a vector space $\mathcal{H}$ we can construct the 1-dimensional TQFT as $Z(\bullet^+) = \mathcal{H}$ and $Z(\bullet^-) = \mathcal{H}^*$. In general, we have

$$Z((\bullet^+)^{\sqcup m} \sqcup (\bullet^-)^{\sqcup n}) = \mathcal{H}^{\otimes m} \otimes (\mathcal{H}^*)^{\otimes n}.$$

Additionally, we can represent any morphism as a composition and tensor product of morphisms given by combinations given by the choice of orientations of the boundary component of the interval, as well as the twist map. These are the generators of the category.

For $d = 2$, our objects are now given by oriented S^1 spheres. Similarly, these theories are also finitely generated. Its generators are the so-called cap given by the half sphere S^2 and the pair of pants morphism obtained from cutting the disk from the cilinder morphism $S^1 \times I$.

In higher dimensions topological theories have a much richer structure as there is no longer a finite number of generators of the theory. However, there is another way in which we have a richer structure even by only looking at a 2-dimensional theory.

In the last chapter we discussed categories we saw a particular functor on $\mathbf{Set}$ which was given by taking a set $X \in \mathbf{Set}$, $(- \times X) : \mathbf{Set} \to \mathbf{Set}$. We now have a similar functor.

Proposition 3.3 (Dimensional reduction) *Let $Z : \mathbf{Bord}_d \to \mathbf{Vec}_k$ be a TQFT and X be a closed, compact, oriented r-manifold such that $r < n$. Then $(- \times X)$ is a symmetric monoidal functor $\mathbf{Bord}_{d-r} \to \mathbf{Bord}_d$ and we have a $(d-r)$-dimensional TQFT Z_{red}*

called the **(dimensionally) reduced** *theory given by composition,*

$$Z_{red} : \mathbf{Bord}_{d-r} \to \mathbf{Vec}_k, \quad Z_{red}(M : E \to F) \equiv Z(M \times X : E \times X \to F \times X).$$

Using dimensional reduction we can construct operators on our vector spaces using manifolds of lower dimensions. Considering a cylinder $I \times S^1$ in a 2-dimensional theory by dimensional reduction it represents a morphism on vector spaces depending on the orientation of the components of the boundary. Additionally, a closed d-dimensional manifold as a bordism corresponds to an element of the field k, a closed $(d-1)$-manifold corresponds to a vector space, and by dimensional reduction a closed $(d-2)$-manifold corresponds to a morphism between vector spaces. We might then consider then an alternative category in which we allow lower dimensional manifolds with a boundary $\mathbf{Bord}_2^\partial$ and endow it with a higer categorical structure.

3.3 Extended Topological Quantum Field Theory

There have been several variations of generalizations of the Atiyah's TQFT axioms trying to capture the structure on observables and to incorporate manifolds with boundaries. These vary by the additional structure desired, such as graded vector spaces or module structure in the target category [25], or additional structure on bordisms such as defects [26] [27] or group actions [28]. In this subsection we generalize the $\mathbf{Bord}_d$ category to define the extended topological quantum field theories.

Definition 3.4 *For $0 \leq k \leq n$, we define the symmetric monoidal (∞, k)-category $\mathbf{Cob}_k(n)$ as*

$$\mathbf{Cob}_k(n) \equiv \Omega^{n-k}\mathbf{Bord_n}$$

Definition 3.5 *An* n**-dimensional TQFT extended down to codimension** k **with moduli level** m *is a symmetric monoidal functor*

$$Z : \mathbf{Cob}_k(n) \to (m+k) - \mathbf{Vec}_k.$$

An n-dimensional TQFT extended to codimension 1 with moduli level 0 then corresponds to the Atiyah's definition 3.2

Definition 3.6 *The Picard ∞-groupoid Pic($\mathscr{C}$) of a symmetric monoidal (∞, n)-category $\mathscr{C}$ is defined as the ∞-category whose k-morphisms are the invertible k-morphisms of $\mathscr{C}$ for any k.*

Definition 3.7 *An n-dimensional TQFT extended down to codimension k with moduli level m $Z : \mathbf{Cob}_k(n) \to (m+k) - \mathbf{Vec}_k$ is said to be* **invertible** *if it factors as*

$$\begin{tikzcd} \mathbf{Cob}_k(n) \arrow[r, "Z"] \arrow[dr] & (m+k) - \mathbf{Vec}_k \\ & \mathit{Pic}((m+k) - \mathbf{Vec}_k) \arrow[u] \end{tikzcd}$$

Definition 3.8 *An* n**-dimensional anomaly** *is an invertible TQFT of moduli level 1,*

$$W : \mathbf{Cob}_k(n) \to \mathit{Pic}((k+1) - \mathbf{Vec}_k) \hookrightarrow (k+1) - \mathbf{Vec}_k$$

Definition 3.9 *Let $W : \mathbf{Cob}_k(n) \to \mathit{Pic}((k+1) - \mathbf{Vec}_k) \hookrightarrow (k+1) - \mathbf{Vec}_k$ be an n-dimensional anomaly. An* **anomalous n-dimensional extended TQFT with anomaly** *W is a morphism of n-dimensional TQFTs with moduli level 1*

$$Z_W : 1 \to W,$$

where 1 is the trivial TQFT mapping objects to the monoidal unit and all morphisms to the identity.

Definition 3.10 *Let $Z : \mathbf{Cob}_k(n) \to (k+m) - \mathbf{Vec}_k$ be an n-dimensional TQFT extended to codimension k with moduli level m. A* **boundary condition** *for Z is a symmetric monoidal extension*

$$\begin{tikzcd} \mathbf{Cob}^{\partial}_k(n) \arrow[r, "\overline{Z}"] & (k+m) - \mathbf{Vec}_k \\ \mathbf{Cob}_k(n) \arrow[u, "i"] \arrow[ur, "Z"'] & \end{tikzcd}$$

Theorem 3.11 *Let $Z : \mathbf{Bord}_n \to n - \mathbf{Vec}_k$ be a fully extended TQFT with moduli level 0. Then there is an equivalence*

$$\{\textit{boundary conditions for } Z\} \cong Z(\bullet^+).$$

Consider now a d-dimensional fully extended theory Z, a d-dimensional manifold X, and a small sphere centered around $x \in X$ of small radius ε, $S_\varepsilon(x)$ called its link. We refer to the vector space to which this sphere is mapped to, $Z(S_\varepsilon(x))$, as the space of **point operators** at x. Note that there exists the isomorphism $Z(S^{d-1}) \cong Z(S_\varepsilon(x))$, called the **state-operator correspondance** as we can think of the elements of a general sphere as states of the system. However this isomorphism is not a canonical one. Correlators are then constructed from a closed d-dimensional manifold by cutting out balls of small radius over the points in which the operators are defined giving a bordism

$$X \setminus \bigcup_{i=1}^{n} B_\varepsilon(x_i) : \coprod_{i=1}^{n} S_\varepsilon(x_i) \to \emptyset.$$

Considering now a connected 1-dimensional submanifold L of X, we could similarly look at a normal sphere bundle over L, $S_\varepsilon(L) \to L$. At each point $x \in L$ we have an isomorphism $Z(S^{d-1}) \cong Z(S_\varepsilon(K)_x)$ as the link to L at x. Similarly, all $Z(S_\varepsilon(L))$ are isomorphic to $Z(S^{d-2})$. We call $Z(S^{d-2})$ the category of **line operators** of the theory. Interpreting the 1-dimensional submanifolds as the Wick rotated worldlines of particles, morphisms $x \otimes y \to z$ then paramatrize the fusion of particles hence why the products of line operators are called fusion in field theoretic setting and their categories fusion categories.

Going to higher dimensions, we retrieve the notion of the so-called **extended operators** of a theory. Given a d-dimensional extended TQFT Z with a target category $\mathscr{C}$ and an n-dimensional manifold X, let M be an l-dimensional submanifold of X and $B \to M$ a normal bundle with the fibre B^{n-l}. The extended operator of dimension l is then a vector in $\mathscr{C}(1, Z(\partial B))$. At every point $x \in M$, we have as the link the sphere $S^{n-l-1} \cong \partial B_x$ such that by dimensional reduction along S^{n-l-1} $Z(S^{n-l-1}) \in \Omega^{n-l-1}\mathscr{C}$ defines the $(d-n-l+1)$-dimensional extended TQFT.

If X has a boundary, the link of the boundary ∂X in X is a point. From the above theorem, $Z(\bullet)$ is the collection of boundary conditions for Z. However, these boundary conditions have a categorical structure.

Consider a d-dimensional extended TQFT on a manifold of the form $M \times N$ where N is an oriented but not necessarily closed $(r+1)$-manifold. By dimensional reduction we can describe this theory by an effective reduced $(r+1)$-dimensional TQFT on N. We can then assign to M the r-category of boundary conditions of the reduced theory.

In the case that $M = S^{d-r-1}$, the category corresponds to the category of extended operators of dimension r. The monoidal structure of this category is given by a pair of pants bordism $S^{n-r-1} \sqcup S^{n-r-1} \to S^{n-r-1}$ which we refer to as the **fusion product**. More on this construction can be found in [21], [22] and [23].

4 Symmetries

Until recently, symmetries in quantum theories have been explored almost exclusively through the use of Wigner's theorem. In [6], a realization has been made that symmetries are in a one-to-one correspondance with topological operators defined on the theory which has shifted the study of symmetries to the study of all topological operators one can construct. This chapter starts with a short summary of the usual treatment of symmetries in field theories followed by its restatement in the language of differential forms which is more convenient for generalizations. Following that, we extend the treatment of symmetries to line operators and look at the example of Maxwell theory such as in chapter 82 of [18] and chapter 2 of [20]. We end this chapter with the statement of the p-form symmetries.

4.1 Charges

In the field-theoretic formalism, the main approach to symmetries is through the use of Nother's theorem relating continuous symmetries to their associated conserved quantities. First, a short summary of the usual way this is approached (which can be found in [16] [17] and [18]) in order to introduce the object of our interest. For a classical action in D dimensions on a collection of fields $\phi = \{\phi_i\}_{i\in I}$ in the Lagrangian formalism,

$$S[\phi] = \int \mathrm{d}^D x \mathcal{L}(\phi),$$

we can write its variation with respect to the parameters ε_i of the infinitesimal transformation

$$\phi \mapsto \phi' = \phi + \varepsilon_i \delta\phi_i \tag{4.1}$$

as

$$\delta S_{\varepsilon_i} = \int \mathrm{d}^D x J_i^\mu \partial_\mu \varepsilon_i \tag{4.2}$$

for some J_i^μ. This transformation is then a symmetry if the parameters ε_i are constants in which case they are referred to as global parameters and the transformation as a global symmetry. Using Gauss's theorem we obtain:

$$\begin{aligned}\delta S_{\varepsilon_i} &= \int \mathrm{d}^D x \left[\partial_\mu (J_i^\mu \varepsilon_i) - \partial_\mu J_i^\mu \varepsilon_i\right] \\ &= \int_{\text{boundary}} J_i^\mu \varepsilon_i - \int \mathrm{d}^D x \partial_\mu J_i^\mu \varepsilon_i\end{aligned}$$

Assuming the integral on the boundary vanishes and that the transformation is indeed a global symmetry (the variation of the action vanishes), we obtain the following condition from the above integral:

$$\partial_\mu J_i^\mu = 0, \tag{4.3}$$

and we see that the J_i^μ are actually conserved currents which motivates the definition of Noether's charges as:

$$Q_i \equiv \int \mathrm{d}^{D-1} x J_i^0.$$

These Noether charges are the generators of infinitesimal transformations in the sense that in a classical field theory the variation of the field is given in terms of the Poisson bracket,

$$\delta \phi_i = \{\phi, Q_i\}.$$

In a quantum theory, we have the Wigner theorem and so we know that given a symmetry there needs to be an operator on the Hilbert space which commutes with the Hamiltonian and which by Wigner is either linear and unitary or anti-linear and anti-unitary. For continuous symmetries, these operators are constructed from the Noether charges as

$$U = \exp(i\varepsilon_i Q_i)$$

which act on fields as $\phi \mapsto \phi' = U\phi U^\dagger$.

The quantum version of the current conservation 4.3 is given by the so-called Ward identities. Consider the partition function of the theory given by the path integral

$$Z = \int \mathcal{D}\phi \exp(iS[\phi]).$$

Given an arbitrary product of fields, $\Phi \equiv \prod_{j=1}^{N} \phi(x_j)$, we have the following correlation function:

$$\langle \Phi \rangle = \frac{1}{Z} \int \mathcal{D}\phi \Phi \exp(iS[\phi]).$$

Assuming the infinitesimal version of the global symmetry 4.1 we have

$$\begin{aligned}
\langle \Phi \rangle &= \frac{1}{Z} \int \mathcal{D}\phi' \Phi' \exp(iS[\phi']) \\
&= \frac{1}{Z} \int \mathcal{D}\phi J \left[\prod_{j=1}^{N} (\phi + \varepsilon_i \delta\phi_i(x_j)) \right] \exp(iS[\phi] + i\delta S[\phi]) \\
&= \frac{1}{Z} \int \mathcal{D}\phi J \left[\Phi + \sum_{j=1}^{N} \phi(x_1) \dots \varepsilon_i \delta\phi_i(x_j) \dots \phi(x_N) \right] (1 + i\delta S[\phi]) \exp(iS[\phi]),
\end{aligned}$$

where J is the Jacobian of the change of the integral measure which might occur in an anomalous theory. After parametrizing the Jacobian as

$$J = 1 + i \int \mathrm{d}^D x \varepsilon_i \mathcal{O}_i$$

we obtain from the above expression:

$$\begin{aligned}
\langle \Phi \rangle = \frac{1}{Z} \int \mathcal{D}\phi \Bigg[\Phi + \sum_{j=1}^{N} \phi(x_1) \dots \varepsilon_i \delta\phi_i(x_j) \dots \phi(x_N) + \\
+ i\Phi \delta S[\phi] + i\Phi \int \mathrm{d}^D x \varepsilon_i \mathcal{O}_i + \dots \Bigg] \exp(iS[\phi]).
\end{aligned}$$

After we substitute the variation of the action in the above expression with 4.2 we have:

$$0 = \int \mathrm{d}^D x \varepsilon_i \left[\sum_{j=1}^{N} \delta^{(D)}(x - x_j) \langle \phi(x_1) \dots \delta\phi_i(x_j) \dots \phi(x_N) \rangle - i\partial_\mu \langle J_i^\mu \Phi \rangle + i \langle \mathcal{O}_i \Phi \rangle \right],$$

which reduces to the Ward identities in an anomaly free theory,

$$\partial_\mu \langle J_i^\mu \Phi \rangle = -i \sum_{j=1}^{N} \delta^{(D)}(x - x_j) \langle \phi(x_1) \dots \delta\phi_i(x_j) \dots \phi(x_N) \rangle .$$

So far we have used the usual treatment of symmetries in which our theories have been defined on all of $\mathbb{R}^D$ endowed with a Lorentzian metric. However, what we really want is to look at any manifold, especially the ones with a boundary. In order to facilitate this, we first rephrase the equations above in the language of differential forms. Consider a 1-form $J = J_\mu \mathrm{d}x^\mu$. Its Hodge dual is given as

$$\begin{aligned}\star J &= J_\mu \star (\mathrm{d}x^\mu) \\ &= \frac{1}{(D-1)!} J_\mu g^{\mu\nu} \epsilon_{\nu\mu_1\ldots\mu_{D-1}} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_{D-1}}.\end{aligned}$$

By applying the differential to the dual form we obtain

$$\begin{aligned}\mathrm{d} \star J &= \mathrm{d}\frac{1}{(D-1)!} J_\mu g^{\mu\nu} \epsilon_{\nu\mu_2\ldots\mu_D} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_{D-1}} \\ &= \frac{1}{(D-1)!} \partial_\rho J_\mu g^{\mu\nu} \epsilon_{\nu\mu_1\ldots\mu_{D-1}} \mathrm{d}x^\rho \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_{D-1}} \\ &= \frac{1}{(D-1)!} \partial_\rho J_\mu g^{\mu\nu} \epsilon_{\nu\mu_1\ldots\mu_{D-1}} \mathrm{sgn}(\det[g]) \sqrt{|\det[g]|} \epsilon^{\rho\mu_1\ldots\mu_{D-1}} \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^{D-1} \\ &= \frac{1}{(D-1)!} \partial_\rho J_\mu g^{\mu\nu} \sqrt{|\det[g]|} (D-1)! \delta^\rho_\nu \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^{D-1} \\ &= \partial_\mu J^\mu \sqrt{|\det[g]|} \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^{D-1} \\ &= \partial_\mu J^\mu \mathrm{d}^D x\end{aligned}$$

which matches 4.3 upto the volume form, and so the current conservation is written as

$$\mathrm{d} \star J = 0. \tag{4.4}$$

Suppose we have a D-dimensional manifold Ω with a boundary $\partial\Omega = \Sigma$. Consider the following integral:

$$\begin{aligned}\int_\Omega \langle \mathrm{d} \star J \Phi \rangle &= \int_\Sigma \langle \star J \Phi \rangle \\ &= \int_\Omega \left\langle \partial_\mu J^\mu \sqrt{|g|} \mathrm{d}x^0 \wedge \ldots \mathrm{d}x^{D-1} \Phi \right\rangle \\ &= \int_\Omega \mathrm{d}^D x \partial_\mu \langle J^\mu \Phi \rangle \\ &= -i \sum_{j=1}^{N} \int_\Omega \mathrm{d}^D x \delta^{(D)}(x - x_j) \langle \phi(x_1) \ldots \delta\phi(x_j) \ldots \phi(x_N) \rangle\end{aligned}$$

where we used the Stokes' theorem in the first line and the Ward identities in the last one. For a correlator with a single field in Φ we have:

$$\int_\Omega \langle \mathrm{d} \star J\phi(y)\rangle = -i \int_\Omega \mathrm{d}^D x \delta^{(D)}(x-y) \langle \delta\phi(y)\rangle$$

Taking the manifold $\Omega = [y^0 - \varepsilon, y^0 + \varepsilon] \times \mathbb{R}^{D-1}$ where $\Sigma = \partial\Omega = \{y^0 - \varepsilon\} \times \mathbb{R}^{D-1} \cup \{y^0 + \varepsilon\} \times \mathbb{R}^{D-1}$ gives:

$$\begin{aligned}\int_\Omega \langle \mathrm{d} \star J\phi(y)\rangle &= \int_\Sigma \langle \star J\phi(y)\rangle \\ &= \int \mathrm{d}^{D-1}x \left\langle J^0\Big|_{x^0=y^0-\varepsilon} \phi(y)\right\rangle + \int \mathrm{d}^{D-1}x \left\langle J^0\Big|_{x^0=y^0+\varepsilon} \phi(y)\right\rangle \\ &= \big\langle Q(y^0+\varepsilon)\phi(y)\big\rangle - \big\langle \phi(y) Q(y^0-\varepsilon)\big\rangle\end{aligned}$$

where the sign flips due to time ordering of the correlator. In the limit $\varepsilon \to 0$ this expression becomes the equal-time commutator and we have the infinitesimal version of the variation of the field:

$$\langle [Q, \phi(y)]\rangle = -i \langle \delta\phi(y)\rangle .$$

This motivates our new definition for the charge in the language of differential forms as

$$Q(\Sigma) = \int_\Sigma \star J \tag{4.5}$$

In a general manifold we now have

$$\begin{aligned}\langle Q(\Sigma)\phi(x)\rangle &= -i \int_\Omega \mathrm{d}^D x \delta^{(D)}(x-y) \langle \delta\phi(y)\rangle \\ &= -i\mathrm{Link}(\Sigma, y) \langle \delta\phi(y)\rangle\end{aligned}$$

where we introduced a new symbol

$$\mathrm{Link}(\Sigma, y) \equiv \int_\Omega \mathrm{d}^D x \delta^{(D)}(x-y) \tag{4.6}$$

which evaluates to 1 or 0 depending on if the point y is inside of the manifold Ω or not.

Consider now a smooth deformation of the original manifold $\Omega \mapsto \Omega \cup \Omega'$ such that y is not contained in Ω'. We have the following:

$$
\begin{aligned}
\langle Q(\Sigma \cup \partial\Omega')\phi(y)\rangle &= \int_{\Omega\cup\Omega'} \langle \mathrm{d} \star J\phi(y)\rangle \\
&= \int_{\Omega} \langle \mathrm{d} \star J\phi(y)\rangle + \int_{\Omega'} \langle \mathrm{d} \star J\phi(y)\rangle \\
&= \langle Q(\Sigma)\phi(y)\rangle + 0,
\end{aligned}
$$

where the second integral evaluates to zero because y is not contained in Ω'.

We now see that the value of the charge is invariant to smooth deformations of the manifold as long as its boundary does not cross a point in which we have a field in the correlator, or in the language of TQFTs, as long as the boundary doesn't cross field insertions. In that sense we say that the charge operator is topological in nature. This observation, that global symmetries are implemented by topological operators, is what triggered the recent developments and generalizations of the notion of symmetries in which we study the topological operators that exist in the theory and treat them as symmetries. This point of view lead to the notions of higher-form symmetries, categorical symmetries as well as to a framework which allows the study of non-invertible symmetries.

This argumentation was made for infinitesimal transformations. Under large transformations we expect, in the case of usual group/invertible symmetries, the existance of a unitary topological operator implementing the symmetry. Given a group G and a representation ρ which acts on fields we have the following equation for a group element $g \in G$:

$$
\langle U(g,\Sigma)\phi(y)\rangle = \rho(g)\,\langle \phi(y)\rangle\,, \tag{4.7}
$$

where $U(g,\Sigma) = \exp(i\theta_i Q_i(\Sigma))$. Such operators can be constructed even in the case of discrete, finite, symmetries when there is no conserved Noether's current.

4.2 Higher-form symmetries

When we looked at TQFTs we found operators (Wilson lines) which were not only defined on a point but rather on a curve and so before we can take the SymTFT construction head on, we still have to do some generalization of these symmetry operators to see how they act on observables supported on a higher-dimensional manifold and not just a point. Differential forms make this process relatively straight forward.

In the last subchapter we had an ordinary global symmetry transformation parametrized by a scalar parameter ε. That parameter was constant on the entire manifold and so we can think of it as a closed 0-form ($\mathrm{d}\varepsilon = 0$). In this perspective we rewrite the variation of the action 4.2 as

$$\delta S = \int_{\Omega} \star J \wedge \mathrm{d}\varepsilon, \tag{4.8}$$

where we promote ε to a local parameter (no longer closed form) where the conservation of current 4.4 is retrieved after an integration by parts and enforced through equations of motion. In this context we refer to such symmetries as 0-form symmetries. Before we generalize this to the case of line operators, we need to introduce another perspective on these parameters which will facilitate the generalization of the linking symbol we introduced in the last chapter.

Another way we can now understand the action of the topological operator implementing the symmetry on fields is by deforming the manifold Ω such that its boundary crosses the field insertion in the correlator thereby making the link have a non-zero value. In this perspective, the action of the operator is parametrized by this manifold. Furthermore, this manifold and parameter of the global symmetry ε are intimately connected, namely, ε is the Poincaré dual form A.16 of the boundary of the manifold, $\partial\Omega$.

Consider now that our transformation parameter was actually a 1-form, $\xi_1(\Sigma_{D-2}) = \xi_\mu \mathrm{d}x^\mu$ dual to a (D-2)-dimensional submanifold $\Sigma_{D-2} = \partial\Omega_{D-1}$. We expect the variation of the action to now take the form of:

$$\delta S = \int_{\Omega_D} \star J \wedge \mathrm{d}\xi_1, \tag{4.9}$$

where J now has to be a 2-form, and the statement that this is a global symmetry is equivalent to the condition $\mathrm{d}\xi_1 = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu = 0$. The conservation equation 4.4

now reads:

$$
\begin{aligned}
\mathrm{d}\star J &= \mathrm{d}\left[\frac{1}{2!}J_{\mu\nu}\star(\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu})\right]\\
&= \mathrm{d}\left[\frac{1}{2!}J_{\mu\nu}\frac{1}{(D-2)!}g^{\mu\rho}g^{\nu\sigma}\epsilon_{\rho\sigma\mu_2\ldots\mu_{D-1}}\mathrm{d}x^{\mu_2}\wedge\ldots\wedge\mathrm{d}x^{\mu_{D-1}}\right]\\
&= \frac{1}{2!}\partial_{\alpha}J_{\mu\nu}g^{\mu\rho}g^{\nu\sigma}(D-2)!\epsilon_{\rho\sigma\mu_2\ldots\mu_{D-1}}\mathrm{d}x^{\alpha}\wedge\mathrm{d}x^{\mu_2}\wedge\ldots\wedge\mathrm{d}x^{\mu_{D-1}}\\
&= \frac{1}{2!}\partial_{\alpha}J^{\rho\sigma}(\delta^{\alpha}_{\rho}-\delta^{\alpha}_{\sigma})\mathrm{d}^{D-1}x\\
&= \partial_{\mu}J^{\mu\nu}\mathrm{d}^{D-1}x,
\end{aligned}
$$

where we used the fact that $J^{\mu\nu}=-J^{\nu\mu}$. Assuming the Stokes' theorem holds, we can write the variation of the action as

$$
\delta S=-\int\mathrm{d}^{D}x\xi_{\nu}\partial_{\mu}J^{\mu\nu}.
$$

In order to transform a line operator, we need to transform it along every point of its support, namely, it transforms as

$$
W[\gamma]\to W[\gamma]'=W[\gamma]+\int_{\gamma}\xi_1(\partial\Omega_{D-1})\delta W[\gamma]. \tag{4.10}
$$

From the correlator of the line operator,

$$
\begin{aligned}
\langle W[\gamma]\rangle &= \int\mathcal{D}\phi W[\gamma]\exp(iS[\phi])\\
&= \int\mathcal{D}\phi' W'[\gamma]\exp(iS[\phi'])\\
&= \int\mathcal{D}\phi\left[W[\gamma]+\int_{\gamma}\xi_1(\Sigma_{D-2})\delta W[\gamma]\right](1+\delta S)\exp(iS[\phi]),
\end{aligned}
$$

assuming invariance to the transformation and that the theory has no anomalies, we obtain the following:

$$
\begin{aligned}
-i\int \mathcal{D}\phi W[\gamma]\delta S\exp(iS[\phi]) &= i\int \mathcal{D}\phi W[\gamma]\int \mathrm{d}^D x\xi_\nu \partial\mu J^{\mu\nu} \\
&= i\int \mathrm{d}^D x\xi_\nu \langle \partial\mu J^{\mu\nu}\rangle \\
&= i\int \xi_1 \wedge \langle \mathrm{d}\star J\rangle \\
&= \int \mathcal{D}\phi \int_\gamma \xi_1[\Sigma_{D-2}]\delta W[\gamma]\exp(iS[\phi]) \\
&= \int_\gamma \mathrm{d}y^\nu \xi_\nu(y)\langle \delta W[\gamma]\rangle \\
&= \int \mathrm{d}^D x\xi_\nu(x)\int_\gamma \delta^{(D)}(x-y)\mathrm{d}y^\nu \langle \delta W[\gamma]\rangle ,
\end{aligned}
$$

from which we can read the associated Ward identity for the line operator to be

$$
\langle \partial_\mu J^{\mu\nu} W[\gamma]\rangle = -i\int_\gamma \mathrm{d}y^\nu \delta^{(D)}(x-y)\langle \delta W[\gamma]\rangle .
$$

The equation above also gives us a generalization of our linking constant. If we were to think about a general, compact, (D-1)-dimensional manifold Ω_{D-1} over which we compute the integral above, we might swap the form ξ_1 with the (oriented) volume form of the manifold $d\Omega_{D-1}$ and we have

$$
\begin{aligned}
\int_{\Omega_{D-1}} (d\Omega_{D-1})_\nu \langle \partial_\mu J^{\mu\nu} W[\gamma]\rangle &= \int_{\partial\Omega_{D-1}} \langle \star J W[\gamma]\rangle \\
&= \langle Q(\partial\Omega_{D-1})W[\gamma]\rangle \\
&= -i\int_{\Omega_{D-1}} (d\Omega_{D-1})_\nu \int_\gamma \mathrm{d}y^\nu \delta^{(D)}(x-y)\langle \delta W[\gamma]\rangle \\
&= -i\text{Link}(\partial\Omega_{D-1},\gamma)\langle \delta W[\gamma]\rangle .
\end{aligned}
$$

Further generalization is straightforward. Given a (q+1)-form conserved current J, we construct the conserved charge by integrating the conservation equation 4.4 over a (D-q)-dimensional submanifold Ω_{D-q} with a boundary $\Sigma_{D-q-1} = \partial\Omega_{D-q}$. By Stokes' theorem we then have

$$
Q(\Sigma_{D-q-1}) \equiv \int_{\Sigma_{D-q-1}} \star J = \int_{\Omega_{D-q}} \mathrm{d}\star J. \tag{4.11}
$$

We identify the Poincaré dual of the Σ_{D-q-1} as the parameter of the transformation acting on operators $O_q[M_q]$ supported on a q-dimensional manifold M_q. We say that the operator is charged under such a symmetry/transformation if it links nontrivially to the charge operator, that is to say, if the link in the following equation has a non-zero value:

$$\langle Q(\Sigma_{D-q-1})O_q[M_q]\rangle = -i\text{Link}(\Sigma_{D-q-1}, M_q)\,\langle \delta O_q[M_q]\rangle\,, \tag{4.12}$$

where the link is given as

$$\text{Link}(\Sigma_{D-q-1}, M_q) = \int_{\Omega_D} \text{PD}(\Omega_{D-q}) \wedge \text{dPD}(N_{q+1}), \tag{4.13}$$

where $\partial N_{q+1} = M_q$, and PD stands for the Poincaré dual form of the manifold. For large transformations, we obtain the unitary operators by exsponentiation,

$$U_q(\Sigma_{D-q-1}) = \exp(i\theta Q(\Sigma_{D-q-1})), \tag{4.14}$$

which then act on charged operators of the theory supported on q-dimensional manifolds, $O_q[M_q]$, as

$$U_q(\Sigma_{D-q-1})O[M_q] = \exp(i\theta\text{Link}(\Sigma_{D-q-1}, M_q))O_q[M_q]. \tag{4.15}$$

4.3 Maxwell

Lastly, we introduce the example of the Maxwell theory in d dimensions which is a U(1) gauge theory given by the action

$$S[A] = \frac{-1}{2e^2}F \wedge \star F, \tag{4.16}$$

where $F_{\mu\nu} = \partial_\mu A\nu - \partial_\nu A_\mu$. This theory has extended gauge-invariant operators, namely, the Wilson lines. It also has the higher symmetries we discussed in this chapter. Since $F = \mathrm{d}A$ we see that $\mathrm{d}F = 0$. Likewise, we have the equations of motion of the theory, $\mathrm{d} \star F = 0$ which give us two conserved currents from which we construct topological operators. Using F as a Noether current, we have a (d-3)-form symmetry referred to as the **magnetic** symmetry implemented by the operators

$$U_g^{(m)}(\Sigma_2) = \exp\left[i\theta \int_{\Sigma_2} F\right], \text{ where } g = e^{i\theta} \in U(1). \tag{4.17}$$

Similarly, using $\star F$ as a Noether current, which is a (d-2)-form, we have a 1-form symmetry, reffered to as the **electric** symmetry which is implemented by the operators

$$U_g^{(e)}(\Sigma_{d-2}) = \exp\left[i\theta \int_{\Sigma_{d-2}} \star F\right], \text{ where } g = e^{i\theta} \in U(1). \tag{4.18}$$

Wilson lines are charged under the electric symmetry and are labeled by their charge q,

$$W_q(\gamma) = \exp\left[2\pi i q \int_\gamma A\right], \tag{4.19}$$

which comes from 4.18 operators acting on them. This theory is a good example because in it field insertions $\phi(x)$ of matter fields charged under the U(1) symmetry we add to the theory are not gauge-invariant operators. Under a gauge transofrmation

$$A(x) \mapsto A(x) - \frac{\mathrm{d}\lambda(x)}{2\pi},$$

the field insertion transforms as

$$\phi(x) \mapsto e^{iq\lambda(x)}\phi(x).$$

Consider now a Wilson line inserted on γ such that $\partial\gamma = x$. This Wilson line transforms as

$$W_q[\gamma] = \exp\left[2\pi i q \int_\gamma A\right] \mapsto \exp\left[-iq \int_{\partial\gamma} \mathrm{d}\lambda\right] W_q[\gamma] = e^{iq\lambda(x)} W_q[\gamma],$$

and so we see that the product $\phi(x)W_q[\gamma]$ is invariant under such transformation. We say that $\phi(x)$ is **not a genuine** local operator as it needs to be attached to a higher dimensional operator to be gauge-invariant and hence an observable of the theory.

Using multiple field insertions leads to an identification of Wilson lines, $W_p \simeq W_{p+nq}$, for $p \in \mathbb{Z}$ which breaks the U(1) symmetry to its $\mathbb{Z}_q$ subgroup leaving us with what is known as a discrete gauge theory. This effect is known as screening and we will return to it and the operators charged under the magnetic symmetry in the next chapter once we tackle dualities.

5 SymTFT

This chapter introduces the SymTFT construction. We start with the orbifold construction, its relation to gauging and dualities in theories. Following that, we relate d-dimensional quantum field theories (as boundary/relative theories) to TQFTs in one dimension higher where we show the relation between charged operators of the TQFT and the boundary theory. Finally, we use the transformations/operations on theories developed in this chapter (gauging, stacking and compactification) to state the SymTFT construction and provide explicit examples.

5.1 Orbifolds and Gauging

In this subsection we take a closer look at gauge theories from the TQFT perspective. Gauge symmetries signify a redundancy of our model in which different configurations of fields correspond to the same state of the system. This leads to overcounting in the path integral which we must account for by counting over the equivalence classes of gauge fields rather than all of the configurations, that is to say, we need to modify the expression

$$\langle A_2|U|A_1\rangle = \int_{A|_{\Sigma_1}=A_1}^{A|_{\Sigma_2}=A_2} \mathcal{D}A \exp(iS[A])$$

to run over equivalence classes of connections. We start with an example, the so-called Dijkgraaf-Witten model in d dimensions which is a gauge theory with a finite gauge group. In such a case, there are no smooth deformations along the fiber hence there can be only one connection defined on it which is then neccessarily flat. Let G be our finite group of gauge symmetries. Given a closed (d-1)-manifold Σ, we define the collection of fields on it to be

$$F_\Sigma = \{\text{principal G-bundles } P \to \Sigma\},$$

as this now corresponds to the collection of all possible G-connections on Σ. F_Σ has a structure of a category whose morphisms are given by G-equivariant maps between principal G-bundles. As they are all over the same base manifold, these maps are neccessarily isomorphisms thereby making the category a groupoid. It is now easy to define the gauge equivalent classes of bundles as the isomorphism classes of the category F_Σ, $\mathcal{F}_\Sigma \equiv F_\Sigma / \sim$. From this we can identify the Hilbert space of the manifold Σ as

$$Z(\Sigma) \equiv \{\text{functions } \mathcal{F}_\Sigma \to \mathbb{C}\}$$

equipped with the vector space structure by point-wise addition and the usual inner product of functions. To fully specify a TQFT, we need to define how d-dimensional manifolds are mapped. Suppose Ω is a d-dimensional manifold with a boundary. Taking $\mathcal{F}_\Omega$ to be the isomorphism classes of principal G-bundles over Ω and $[Q] \in \mathcal{F}_{\partial\Omega}$, we define

$$Z(\Omega)(Q) \equiv \sum_{[P]\in\mathcal{F}_\Omega : P|_{\partial\Omega}=Q} \frac{1}{|\text{Aut}(P)|} \in \mathbb{C},$$

where we sum over the equivalence classes with representatives which restrict to f on the boundary weighting each representative by the rank of its automorphism group to account for multiple counts.

Consider a point $x \in \Omega$. Given a principal G-bundle $P \to \Omega$, we have a free and transitive right G-action $\triangleleft$ on the fiber P_x. Parallel transport along loops around x induces a group homomorphism $\phi : \pi_1(\Omega, x) \to G$ giving the holonomy structure subject to the selection of the point in the fiber $p \in P_x$, $p \mapsto p \triangleleft g \implies \phi(\gamma) = g$ for some loop γ and $g \in G$. This homomorphism completely determines the bundle P. Given a point $p \triangleleft h$ for some $h \in G$, we obtain the holonomy of the loop γ as $p \triangleleft h \mapsto p \triangleleft h \triangleleft h^{-1}gh \implies \phi(\gamma) = h^{-1}gh$ which induces a right G-action on the space of group homomorphisms $\pi_1(\Omega, x) \to G$ as

$$(\phi.g)(\gamma) = g^{-1}\phi(\gamma)g.$$

We see then, if two homomorphisms are related by the action, their principle G-bundles are isomorphic and so we have

$$\text{Aut}(P) = \text{St}(\phi),$$

that is to say, automorphism group of a principle bundle corresponds to the stabilizer of the holonomy homomorphism defined on the bundle giving us the following decomposition of the partition function,

$$\begin{aligned} Z(\Omega) &= \sum_{\text{orbits}} \frac{1}{\mathsf{St}(\phi)} \\ &= \frac{1}{|G|} \sum_{\text{orbits}} |\mathsf{Orb}(\phi)|. \end{aligned}$$

A theory defined in this way corresponds to the trivial form of the orbifold construction. Strictly speaking, it is the procedure of promoting a background field B_{p+1} of a p-form symmetry to a dynamical $(p+1)$-form gauge field b_{p+1}. It is a transformation between theories. In general, given a d-dimensional TQFT Z and a finite p-form symmetry group G, we obtain a TQFT $Z_{/G}$,

$$Z_{/G} \sim \sum_{[B_{p+1}]} I[B_{p+1}] Z[B_{p+1}],$$

up to normalization where the parameter I is referred to as **discrete torsion** where in the case of a nontrivial value we say that the theory is **twisted**. In order for this to be well-defined both the partition function and the parameter I need to be gauge-invariant. This transformation has certain interesting properties. The gauged group admits a $(d-p-2)$-form symmetry given by the Pontryagin dual group $\hat{G}$ called the **dual symmetry**. Given a discrete non-abelian group, its higher-dimensional irreducible representations generate non-invertible topological operators, which can be obtained by repeated fusion of invertible operators from the decomposition to irreducible representations in their fusion category.

Gauging the dual symmetry returns the original theory,

$$Z_{/G/\hat{G}} = Z.$$

5.2 SymTFT

Given a d-dimensional extended TQFT Z, and a d-dimensional manifold of the form $M \times I$, by dimensional reduction along I we obtain the $(d-1)$-category of extended operators on M which corresponds to the Drinfeld center $\mathcal{Z}(\mathscr{C})$ of the $(d-1)$-category of the extended operators of the theory, $\mathscr{C}$. Given a choice of boundary condition on

M and orientation we have a canonical structure of a left or right $\mathcal{Z}(\mathscr{C})$-module given by the composition of the tensor product with the forgetful functor $F : \mathcal{Z}(\mathscr{C}) \to \mathscr{C}$:

$$\mathcal{Z}(\mathscr{C}) \otimes \mathscr{C} \cong \mathcal{Z}(\mathscr{C}) \xrightarrow{F\otimes 1_{\mathscr{C}}} \mathscr{C} \otimes \mathscr{C} \xrightarrow{\otimes} \mathscr{C}$$

In that case we say that this left(right) $\mathcal{Z}(\mathscr{C})$-module is a **left(right) Z-module**.

Notice that by combining the left and right $\mathcal{Z}(\mathscr{C})$-modules, $\mathcal{Z}(\mathscr{C}) \boxtimes \mathcal{Z}(\mathscr{C})^{rev} \cong \mathcal{Z}(\mathscr{C} \boxtimes \mathscr{C}^{op})$ by 2.54 we obtain a bimodule structure and by using the monoidal structure on endofunctors of bimodule category we have an equivalence 2.51 with $\mathcal{Z}(\mathscr{C})$.

Definition 5.1 (SymTFT) *Let Z be a $(d+1)$-dimensional topological field theory and R a right Z-module. Let W be a d-dimensional field theory. A (Z, R)-module structure on W is pair (L, ϕ) where L is a left Z-module and ϕ an isomorphism*

$$\phi : R \boxtimes L \xrightarrow{\sim} W.$$

We refer to the (Z, R)-module as the **sandwich** *and to Z as the* **SymTFT**.

In practice, by evaluating the theory Z on a cylinder with boundary conditions defining R and L we obtain the isomorphism ϕ by dimensional reduction along the interval. Gauging of the theory, transformation between dual theories as well as anomaly inflow is realized through this mechanism by selecting appropriate boundary conditions. Under the interpretation of the elements of the boundary as states and the limit in which the interval goes to a point, this isomorphism is often expressed in terms of the evaluation map of the theory and takes the form of an inner product of states defined by boundary conditions.

5.3 *Examples*

Consider a quantum field theory with $\mathbb{Z}_N$ 0-form symmetry on d-manifold M_d with a partition function $Z[A]$ where A is the background fileld associated to the symmetry. The SymTFT of such a theory is given by the 1-form BF theory with the action

$$S = \frac{iN}{2\pi} \int_{M_{d+1}} \mathrm{d}a_1 \wedge b_{d-1},$$

on $M_{d+1} = M_d \times [0,1]$. The observables of this theory are the Wilson lines given by the field a_1 and surfaces of the field b_{d-1},

$$W_n(\gamma) = \exp\left(in \int_\gamma a_1\right), \quad V_m(\Sigma) = \exp\left(im \int_\Sigma b_{d-1}\right), \quad n, m \in \mathbb{Z}_N.$$

Since there is a one-to-one correspondance between boundary conditions of the theory on M_d and the vectors of the Hilbert space $\mathcal{H}_{M_d}$, we can think of boundary conditions in terms of the states of the states of the system and more specifically we can then think of the basis on which our topological operators act. We consider the basis $|D_A\rangle$ in which the Wilson lines are diagonal and we refer to these boundary conditions as **Dirichlet** boundary conditions. Analogously the basis $|N_B\rangle$ diagonalising the Wilson surfaces is called **Neumann** boundary conditions.

$$W_n(\gamma)\,|D_A\rangle = \exp\left(in \int_\gamma A\right)|D_A\rangle\,, \quad V_m(\Sigma)\,|N_B\rangle = \exp\left(im \int_\Sigma B\right)|N_B\rangle$$

These two basis, a_1 and b_{d-1} being conjugate variables, are connected by a Fourier transform,

$$|N_B\rangle = \frac{1}{\sqrt{|H^1(M_d, \mathbb{Z}_N)|}} \sum_{A \in H^1(M_d, 2\pi\mathbb{Z}/N)} \exp\left(\frac{iN}{2\pi} \int_{M_d} A \wedge B\right)|D_A\rangle\,.$$

The original quantum field theory is represented as the state

$$\langle QFT| = \sum_{B \in H^{d-1}(M_d, 2\pi\mathbb{Z}_N/N)} Z[A]\,\langle D_A|$$

From this we have

$$\langle QFT|D_A\rangle = Z[A],$$
$$\langle QFT|N_B\rangle = Z_{/\mathbb{Z}_N}[B],$$

where we recognize the gauged theory by the orbifold construction.

We can also construct the boundary conditions explicitly from the action. Consider the action of a $(d+1)$-dimensional theory,

$$S = \frac{i}{2\pi} \int_{M_{d+1}} \mathrm{d}a_1 \wedge b_{d-2},$$

where a_1 is an $\mathbb{R}$-valued 1-form gauge field and b_{d-2} an $\mathbb{R}$-valued $(d-2)$-form gauge field. Variation of the action then has the boundary component

$$\delta S|_{\partial M_{d+1}} = \frac{-i}{2\pi}\int_{\partial M_{d+1}} \delta a_1 \wedge b_{d-2}.$$

Under infinitesimal gauge transformations

$$a_1 \mapsto a_1 + \mathrm{d}\lambda, \quad b_{d-2} \mapsto b_{d-2} + \mathrm{d}\lambda_{d-3},$$

there is the following boundary term

$$\delta_{gauge} S|_{\partial M_{d+1}} = \frac{i}{2\pi}\int_{\partial M_{d+1}} \mathrm{d}a \wedge \lambda_{d-3}.$$

Topological gauge invariant boundary conditions are given by setting either a_1 be a flat gauge field on the boundary or setting b_{d-2} to zero on the boundary. In the first case, the boundary variation of a_1 is given by its gauge transformation $\mathrm{d}\lambda$ thus for the variation on the boundary to vanish we require from the above equation $\mathrm{d}b_{d-2}|_{\partial M_{d+1}} = 0$. As a result, the corresponding Wilson surface operators are trivial on the boundary. Suppose $M_{d+1} = M_d \times [0,1]$ and we set $a_1 = A_L$ on $M_d \times \{0\}$ and $a_1 = A_R$ on $M_d \times \{1\}$. We can now rewrite the action as

$$S = \frac{i}{2\pi}\int_{M_{d+1}} a_1 \wedge \mathrm{d}b_{d-2} + \frac{i}{2\pi}\int_{M_d} (A_L \wedge b_L - A_R \wedge b_R),$$

where $b_{L,R} = b_{d-2}|_{M_d \times \{0,1\}}$. By integrating out the gauge field a_1 we arrive at the following:

$$\langle A_L | A_R \rangle = \int \mathcal{D}b_{d-2}\delta(\mathrm{d}b_{d-2}) \exp\left[\frac{i}{2\pi}\int_{M_d} (A_L \wedge b_L - A_R \wedge b_L)\right] = \delta([A_L - A_R]).$$

From this we see that the Wilson lines of a_1 are diagonalized on the boundary by these boundary conditions/states,

$$\exp\left(i\int_\gamma a_1\right)|A\rangle = \exp\left(i\int_\gamma A\right)|A\rangle\,.$$

The other boundary conditions diagonalize the Wilson surfaces defined by the b_{d-2} field.

Consider the 5-dimensional theory given by the following BF-action

$$S = \frac{1}{2\pi} \int_{M_5} a_2 \wedge \mathrm{d}b_2,$$

where a_2 and b_2 are $\mathbb{R}$-valued differential forms. The equations of motion of this theory are

$$\mathrm{d}a_2 = 0, \quad \mathrm{d}b_2 = 0,$$

making them closed forms. Classical solutions are then given by $a_2, b_2 \in H^2(M_5, \mathbb{R})$. Observables of this theory are

$$W(M_2)_\alpha \equiv \exp\left(i\alpha \oint_{M_2} a_2\right), \quad V(M_2)_\beta \equiv \exp\left(i\beta \oint_{M_2} b_2\right),$$

obeying the following commutation relation

$$W(M_2)_\alpha V(N_2)_\beta = \exp(2\pi i\alpha\beta \text{Link}(M_2, N_2)) V(N_2)_\beta W(M_2)_\alpha$$

for $M_2, N_2 \in H_2(M_5, \mathbb{R})$ generating a Heisenberg group. This is the SymTFT for the Maxwell theory from Chapter 4. We now recognize the magnetic symmetry as the dual of the electric under gauging.

6 Summary and outlook

6.1 Summary

In Chapter 2, we have given a short introduction to category theory and constructions used in its studies, we have also introduced the special kind of categories which we have used to describe the physics of the topological quantum field theories in the subsequent chapters. Chapter 3 was the introduction to topological qunatum field theories, its categorical framework in the axiomatic system of Atiyah. We have also intorduced the concept of dimensional reduction as a functor which we have used to find the categorical structure of operators of the theory and introduced the extended topological field theory as well as boundary conditions on it. In Chapter 4 we looked at global symmetries and the notion that the operators implementing the symmetry are topological. From this we have generalized this using the framework of differential forms to symmetry operators supported on higher-dimensional manifolds such as the ones we encounter in topological field theories. In chapter 5, We motivated the orbifold construction on an example of a Dijkgraaf-Witten model and introduced the SymTFT construction as a dimensional reduction of an extended topological quantum field theory along an interval which imposes boundary conditions on the theory. We finished with examples on BF theories and boundary conditions on them.

6.2 Outlook

The SymTFT offers a new perspective on symmetries in which they can be studied seperately from a specific theory. Recently, this construction has be used to describe a variety of effects, from holography in string theory [**?**], spontaneous symmetry breaking [32] [33] and anomalies and non-invertible symmetries [34]. SymTFT offers new avenues of research by looking at how different theories are connected by a change of boundary conditions on the sandwich and the relations between different theories of the same symmetry. Finally, so far this framework has largely been confined to the study of finite symmetries and the consequences of continuous symmetries have been of recent focus in the scientific community [33].

Appendices

Appendix A Differential Forms

This chapter lists the definitions and conventions used on differential forms.

A.1 Kronecker delta

Given an n-dimensional vector space, we define the **Kronecker delta tensor** as a type (1, 1) tensor whose component value is

$$\delta^i_j = \begin{cases} 1 & \text{, when } i = j, \\ 0 & \text{, when } i \neq j \end{cases}. \tag{A.1}$$

From it, for $p \in \mathbb{N}$ we also define the **generalized Kronecker delta tensor** as a type (p, p) tensor as

$$\delta^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_p} = \text{det}[\delta^{\mu_i}_{\nu_j}]_{ij} = p! \delta^{[\mu_1}_{\nu_1} \dots \delta^{\mu_p]}_{\nu_p}. \tag{A.2}$$

If a portion of the indices match, we can reduce the order using the following identity:

$$\delta^{\mu_1 \dots \mu_s \mu_{s+1} \dots \mu_p}_{\nu_1 \dots \nu_s \mu_{s+1} \dots \mu_p} = \frac{(n-s)!}{(n-p)!} \delta^{\mu_1 \dots \mu_s}_{\nu_1 \dots \nu_s}. \tag{A.3}$$

A.2 Levi-Civita

Suppose $\pi \in S_n$ is a permutation of $\{1, \dots, n\}$ for some $n \in \mathbb{N}$. We define the generalized **Levi-Civita symbol** as

$$\varepsilon^{\pi(1) \dots \pi(n)} = \text{sgn}(\pi). \tag{A.4}$$

Levi-Civita and Kronecker tensors are connected using the following identities:

$$\varepsilon_{\mu_1 \dots \mu_n} \varepsilon^{\nu_1 \dots \nu_n} = \delta^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_n} \tag{A.5}$$

$$\varepsilon_{\mu_1 \dots \mu_n} \varepsilon^{\mu_1 \dots \mu_s \nu_{s+1} \dots \nu_n} = s! \delta^{\nu_{s+1} \dots \nu_n}_{\mu_{s+1} \dots \mu_n} \tag{A.6}$$

$$\varepsilon_{\mu_1 \dots \mu_n} \varepsilon^{\mu_1 \dots \mu_n} = n!. \tag{A.7}$$

On a pseudo-Riemannian manifold of dimension n equipped with a metric g we also define the **covariant Levi-Civita tensor (Riemannian volume form)** as

$$\epsilon_{\mu_1 \dots \mu_n} = \sqrt{|\det[g]|}\varepsilon_{\mu_1 \dots \mu_n}. \tag{A.8}$$

The contravariant version is given by raising the indices using the metric,

$$\epsilon^{\mu_1 \dots \mu_n} = \sqrt{|\det[g]|}\varepsilon^{\mu_1 \dots \mu_n} = \frac{\mathrm{sgn}(\det[g])}{\sqrt{|\det[g]|}}\varepsilon_{\mu_1 \dots \mu_n}. \tag{A.9}$$

A.3 Differential Forms

Suppose M is a D-dimensional smooth manifold with a boundary $\Sigma = \partial M$. We use the following conventions. A p-form $\omega \in \Omega^p(M)$ has the form:

$$\omega = \frac{1}{p!}\omega_{\mu_1 \dots \mu_p} \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p}. \tag{A.10}$$

Let $\eta \in \Omega^q(M)$ be a q-form. We have a so-called wedge product of forms, $\wedge : \Omega^p(M) \times \Omega^q(M) \to \Omega^{(p+q)}(M)$ for all $p + q \leq D$ given as:

$$(\omega \wedge \eta) = \frac{1}{p!q!}\omega_{\mu_1 \dots \mu_p}\eta_{\nu_1 \dots \nu_q} \mathrm{d}x^{\mu_1} \wedge \dots \mathrm{d}x^{\mu_p} \wedge \mathrm{d}x^{\nu_1} \wedge \dots \wedge \mathrm{d}x^{\nu_q}. \tag{A.11}$$

This product has the following property:

$$\omega \wedge \eta = (-1)^{pq}\eta \wedge \omega. \tag{A.12}$$

We also have a map, called the **exterior derivative**, $\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)$ which acts on p-forms as

$$\mathrm{d}\omega = \frac{1}{p!}\partial_\mu \omega_{\mu_1 \dots \mu_p} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p}. \tag{A.13}$$

. We define the **Hodge dual** of a form as

$$\star(\mathrm{d}x_1^{\mu} \wedge \dots \wedge \mathrm{d}x_p^{\mu}) = \frac{1}{(D-p)!}\epsilon^{\mu_1 \dots \mu_p}{}_{\mu_{p+1} \dots \mu_D} \mathrm{d}x^{\mu_{p+1}} \wedge \dots \wedge \mathrm{d}x^{\mu_D} \tag{A.14}$$

$$\star\omega = \frac{1}{p!}\omega_{\mu_1 \dots \mu_p} \star (\mathrm{d}x_1^{\mu} \wedge \dots \wedge \mathrm{d}x^{\mu_p}). \tag{A.15}$$

To a smooth submanifold of dimension p, Σ_p we assign its **Poincaré dual** (D-p)-form $\mathrm{PD}(\Sigma_p) \equiv \xi_{D-p}$ whose components are

$$\xi_{\mu_{p+1}\dots\mu_D}(x) \equiv \frac{1}{p!}\int_{\Sigma_p} \epsilon_{\mu_1\dots\mu_D}\delta^{(D)}(x-y)\mathrm{d}y^{\mu_1}\wedge\dots\wedge\mathrm{d}y^{\mu_p}. \tag{A.16}$$

and is implemented as the following constraint:

$$\int_{\Sigma_p}\omega = \int_{\Omega_D}\omega\wedge\mathrm{PD}(\Sigma_p). \tag{A.17}$$

This form is closed if Σ_p has no boundary. Finally, we have the Stokes' theorem:

Theorem A.1 (Stokes) *Let ω be a smooth (D-1)-form with compact support on an oriented, D-dimensional manifold-with-boundary Ω, where $\partial\Omega$ is given the induced orientation.Then*

$$\int_{\Omega}\mathrm{d}\omega = \int_{\partial\Omega}\omega. \tag{A.18}$$

7 Prošireni sažetak

7.1 Uvod

Promjenom perspektive s kojom gledamo na simetrije fizičkih teorija shvaćajući skup topoloških operatora teorije kao njene simetrije dolazi do raznih generalizacija u obliku simetrija viših formi [31], kategoričkih i neinvertibilnih simetrija [32]. Neinvertibilne simetrije su poznate iz formalizma konformalnih teorija polja u obliku fuzijskih algebri [15] [14], ali su ostale relativno nezamijećene do nedavne pojave SymTFT konstrukcije [25] koje omogućavaju promatranje simetrije nezavisno od teorije i u višim dimenzijama. Ovaj diplomski rad koristi Atiyahov kategorički opis topoloških kvantnih teorija polja [8] kako bi uveo SymTFT konstrukciju.

7.2 Kategorije

Osnovni alat potreban za daljnje razumijevanje su kategorije.

Definition 7.1 *Neka je C klasa. Pretpostavimo kako za svaki $x, y \in C$ postoji njima asocirana klasa koju označimo $C(x, y)$. Nadalje, pretpostavimo kako za svaki $x, y, z \in C$ postoji preslikavanje $\circ : C(x, y) \times C(y, z) \to C(x, z)$. Tada za C kažemo da je* **kategorija** *ako:*

- *Za svaki $x \in C$ postoji $id_x \in C(x, x)$ takav da za sve $y \in C, f \in C(x, y), g \in C(y, x)$, $f \circ id_x = f$ i $id_x \circ g = g$,*
- *Za sve $x, y, z, w \in C, f \in C(x, y), g \in C(y, z), h \in C(z, w)$ vrijedi $h \circ (g \circ f) = (h \circ g) \circ f$.*

Elemente klase C nazivamo **objektima** *kategorije a preslikavanja u klasama $C(x, y)$ morfizmima. Kategoriju C^{op} dobivenu zamjenom $C^{op}(x, y) = C(y, x)$ za sve $x, y \in C$ nazivamo* **dual** *od C.*

Također definiramo preslikavanja koja čuvaju kategoričku strukturu.

Definition 7.2 *Neka su $\mathscr{C}$ i $\mathscr{D}$ kategorije. Za preslikavanje $F : \mathrm{ob}(\mathscr{C}) \to \mathrm{ob}(\mathscr{D})$ zajedno s preslikavanjima $\mathscr{C}(x, y) \to \mathscr{D}(F(x), F(y))$, $f \mapsto F(f)$ za sve $x, y \in \mathrm{ob}(\mathscr{C})$ kažemo da je* **(kovarijantni) funktor** *ako vrijede:*

- *$F(id_x) = id_{F(x)}$ za sve $x \in \mathrm{ob}(\mathscr{C})$, i*

- $F(g \circ f) = F(g) \circ F(f)$ *za sve* $f \in \mathscr{C}(x,y), g \in \mathscr{C}(y,z)$.

Ovako definiran funktor označavamo kao $F : \mathscr{C} \to \mathscr{D}$ *gdje koristimo isti simbol za preslikavanje objekata i morfizama.* **Kontravarijanti funktor** $\mathscr{C} \to \mathscr{D}$ *is a kovarijantni funktor* $\mathscr{C}^{op} \to \mathscr{D}$.

Od posebnog interesa su nam takozvane monoidalne kategorije i strukture na njima.

Definition 7.3 **Monoidalna kategorija** *je uređena šestorka* $(\mathscr{C}, \otimes, 1, \alpha, \rho, \lambda)$ *koju čine kategorija* $\mathscr{C}$*, bifunktor* $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ *zvan* **tenzorski produkt***, a prirodni isomorfizam* $(- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$ *zvan* **asocijator***, objekt* $1 \in \mathrm{ob}(\mathscr{C})$ *zvan a* **jedinica***, lijevi i desni* **izomorfizmi jedinice***,* $\lambda : 1 \otimes - \xrightarrow{\sim} -$ *and* $\rho : - \otimes 1 \xrightarrow{\sim} -$ *takvi da dijagrami*

$$\begin{tikzcd}
 & (x \otimes y) \otimes (z \otimes w) \arrow[dr, "\alpha_{x,y,z\otimes w}"] & \\
((x \otimes y) \otimes z) \otimes w \arrow[ur, "\alpha_{x\otimes y,z,w}"] \arrow[d, "\alpha_{x,y,z}\otimes id_w"'] & & x \otimes (y \otimes (z \otimes w)) \\
(x \otimes (y \otimes z)) \otimes w \arrow[rr, "\alpha_{x,y\otimes z,w}"'] & & x \otimes ((y \otimes z) \otimes w) \arrow[u, "id_x \otimes \alpha_{y,z,w}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "\alpha_{x,1,y}"] \arrow[dr, "\rho_x \otimes id_y"'] & & x \otimes (1 \otimes y) \arrow[dl, "id_x \otimes \lambda_y"] \\
 & x \otimes y &
\end{tikzcd}$$

komutiraju za sve $x, y, z, w \in \mathrm{ob}(\mathscr{C})$.

Definition 7.4 *Neka je* $\mathscr{C} = (\mathscr{C}, \otimes, 1, \alpha, \rho, \lambda)$ *monoidalna kategorija. A* **lijevi modul** *od* $\mathscr{C}$ *is je kategorija* $\mathscr{M}$ *sa bifunktorom* $\otimes : \mathscr{C} \times \mathscr{M} \to \mathscr{M}$ *zvanim* **akcija** *i prirodnim isomorfizmima* $m_{x,y,m} : (x \otimes y) \otimes m \xrightarrow{\sim} x \otimes (y \otimes m)$ *i* $\lambda_m : 1 \otimes m \xrightarrow{\sim} m$ *za sve* $x, y \in \mathrm{ob}(\mathscr{C}), m \in \mathrm{ob}(\mathscr{M})$ *tako da dijagrami*

$$\begin{tikzcd}
 & (x \otimes y) \otimes (z \otimes m) \arrow[dr, "m_{x,y,z\otimes m}"] & \\
((x \otimes y) \otimes z) \otimes m \arrow[ur, "m_{x\otimes y,z,m}"] \arrow[d, "\alpha_{x,y,z}\otimes id_m"'] & & x \otimes (y \otimes (z \otimes m)) \\
(x \otimes (y \otimes z)) \otimes m \arrow[rr, "m_{x,y\otimes z,m}"] & & x \otimes ((y \otimes z) \otimes m) \arrow[u, "id_x \otimes m_{y,z,m}"']
\end{tikzcd}$$

i

$$\begin{tikzcd}[column sep=small]
(x \otimes 1) \otimes m \arrow[rr, "m_{x,1,m}"] \arrow[dr, "\rho_x \otimes id_m"'] & & x \otimes (1 \otimes m) \arrow[dl, "id_x \otimes \lambda_m"] \\
& x \otimes m &
\end{tikzcd}$$

komutiraju za sve $x, y, z \in \mathrm{ob}(\mathscr{C}), m \in \mathrm{ob}(\mathscr{M})$*. Desni moduli su definirani analogno.*

Definition 7.5 *Neka je* $\mathscr{C}$ *monoidalna kategorija. Definiramo* **centar** *kategorije* $\mathscr{C}$*,* $\mathcal{Z}(\mathscr{C})$ *na sljedeći način. Objekti* $\mathcal{Z}(\mathscr{C})$ *su uređeni parovi* (x, γ) *gdje je* $x \in \mathrm{ob}(\mathscr{C})$ *i* γ *je a prirodni izomorfizam* $\gamma_y : y \otimes x \xrightarrow{\sim} x \otimes y$ *za sve* $y \in \mathrm{ob}(\mathscr{C})$ *tako da dijagram*

$$\begin{tikzcd}
z \otimes (x \otimes y) \arrow[r, "\alpha^{-1}_{z,x,y}"] & (z \otimes x) \otimes y \arrow[d, "\gamma_z \otimes id_y"] \\
z \otimes (y \otimes x) \arrow[u, "id_z \otimes \gamma_y"] \arrow[d, "\alpha^{-1}{z,y,x}"'] & (x \otimes z) \otimes y \\
(z \otimes y) \otimes x \arrow[r, "\gamma_{z \otimes y}"'] & x \otimes (z \otimes y) \arrow[u, "\alpha^{-1}_{x,z,y}"']
\end{tikzcd}$$

komutira za sve $y, z \in \mathrm{ob}(\mathscr{C})$*. Morfizmi u* $\mathcal{Z}(\mathscr{C})$ *od* (x, γ) *do* (x', γ') *su morfizmi* $f \in \mathscr{C}(x, x')$ *takvi da* $(f \otimes id_y) \circ \gamma_y = \gamma'_y \circ (id_y \otimes f)$ *za sve* $y \in \mathrm{ob}(\mathscr{C})$*.*

7.3 Topološke kvantne teorije polja

Definiramo kategoriju bordizama.

Definition 7.6 ($\mathbf{Bord}_d$) *Objekti u* $\mathbf{Bord}_d$ *su orientirane zatvorene (d-1)-dimenzionalne realne mnogostrukosti. Pretpostavimo da su* E *i* F *takve mnogostrukosti.* **Bordizamm** $E \to F$ *je uređena trojka* (M, ι_i, ι_o) *gdje je* M *orijentirana kompaktna d-dimenzionalna mnogostrukost s rubom,* $\iota_i : E \to M$ *i* $\iota_o : F \to M$ *su glatka preslikavanja sa slikom u* ∂M *takva da* $\bar{\iota}_i \sqcup \iota_o : \overline{E} \sqcup F \to \partial M$ *je difeomorfizam koji čuva orijentaciju gdje* $\overline{E}$ *označava* E *sa suprotnim odabirom orijentacije i* $\sqcup$ *je disjunktna unija. Definiramo relaciju ekvivalencije na bordizmima. Neka su* $(M, \iota_i, \iota_o), (M', \iota'_i, \iota'_o) : E \to F$ *dva bordizma. Kažemo da su ekvivalentni ako postoji difeomorfizam koji čuva orijentaciju*

$\phi : M \to M'$ takav da dijagram

$$\begin{tikzcd} & M \arrow[dd, "\phi"] & \\ E \arrow[ur, "\iota_i"] \arrow[dr, "\iota_i'"'] & & F \arrow[ul, "\iota_o"'] \arrow[dl, "\iota_o'"] \\ & M' & \end{tikzcd}$$

komutira. Morfizmi u $\mathbf{Bord}_d$ su klase ekvivalencije bordizama između objekata. Kompozicija morfizama $M_1 : E \to F$ i $M_2 : F \to G$ je dana lijepljenjem M_1 i M_2 duž F. $\mathbf{Bord}_d$ ima strukturu monoidalne kategorije odabirom disjunktne unije $\sqcup$ kao tenzorskog produkta i praznog skupa $\emptyset$ (kao $(d-1)$-mnogostrukost) kao jedinice.

Definition 7.7 *d-dimenzionalna topološka kvantna teorija polja (TQFT) je simetrični monoidalni funktor $Z : \mathbf{Bord}_d \to \mathbf{Vec}_k$.*

Proposition 7.8 *Neka je $Z : \mathbf{Bord}_d \to \mathbf{Vec}_k$ TQFT i X zatvorena, kompaktna, orijentirana r-mnogostrukost takva da $r < n$. Tada je $(- \times X)$ simetrični monoidalni funktor $\mathbf{Bord}_{d-r} \to \mathbf{Bord}_d$ te imamo $(d-r)$-dimenzionalni TQFT Z_{red} zvanu* **(dimenzionalni) reducirana/kompaktificirana** *teorija dana kompozicijom,*

$$Z_{red} : \mathbf{Bord}_{d-r} \to \mathbf{Vec}_k, \quad Z_{red}(M : E \to F) \equiv Z(M \times X : E \times X \to F \times X).$$

Kompaktifikacijom proširene topološke kvantne teorije polja koja ima strukturu više kategorije dobivamo kategorije proširenih operatora na njima.

7.4 Simetrije

U teoriji definiranoj na D-dimenzionalnoj mnogostrukosti sačuvana $(q+1)$-forma J simetrije teorije, $\mathrm{d} \star J = 0$, daje operator naboja

$$Q(\Sigma_{D-q-1}) = \int_{\Sigma_{D-q-1}} \star J,$$

gdje se mnogostrukost Σ_{D-q-1} identificira s Poincaréovim dualom parametra infinitezimalnih transformacija. Naboj djeluje na proširene operatore teorije kao

$$\langle Q(\Sigma_{D-q-1}) O_q[M_q] \rangle = -i \mathrm{Link}(\Sigma_{D-q-1}, M_q) \langle \delta O_q[M_q] \rangle .$$

Eksponencijalnom mapom stvara se topološki operator za velike transformacije,

$$U_q(\Sigma_{D-q-1}) = \exp(i\theta Q(\Sigma_{D-q-1})).$$

7.5 SymTFT

Nametanjem rubnih uvjeta topološkoj kvantnoj teoriji na cilindru dobivaju se strukture lijevih i desnih modula centra $\mathcal{Z}(\mathscr{C})$ fuzijske kategorije $\mathscr{C}$ proširenih operatora, tako uređenu teoriju nazivamo SymTFT. Kompaktifikacijom se dobiva niže-dimenzionalna teorija koja ovisi o odabiru topološhih rubnih uvjeta.

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