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Jordanovi homomorfizmi strukturnih matričnih algebri

DOKTORSKI RAD

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Summary

Jordan homomorphisms between two associative algebras are linear maps which preserve squares. This dissertation studies Jordan homomorphisms between structural matrix algebras (SMAs), which are unital subalgebras of M_n (the algebra of all complex square matrices of order n) spanned by some set of matrix units. The first objective of the dissertation is to completely describe the form of all Jordan embeddings between two SMAs using concepts introduced by Coelho's description of SMA automorphisms. Secondly, continuing the work of Molnar and Šemrl on rank-one preservers of the algebra of upper-triangular matrices $\mathcal{T}_n$, it considers several natural linear preserver problems on SMAs and puts them into the broader context of Jordan embeddings on SMAs. The third part of the dissertation contains a full extension of the well-known nonlinear preserver results of Petek and Šemrl (on M_n and $\mathcal{T}_n$) in the context of SMAs. More precisely, a characterization of SMAs $\mathcal{A} \subseteq M_n$ is obtained with the property that all injective continuous commutativity and spectrum preserving map $\mathcal{A} \to M_n$ are necessarily Jordan embeddings.

Keywords: Jordan homomorphism, structural matrix algebra, rank preserver, spectrum preserver, commutativity preserver

Sažetak

Jordanovi homomorfizmi između dviju asocijativnih algebri su linearna preslikavanja koja čuvaju kvadrate. U ovoj disertaciji se proučavaju Jordanovi homomorfizmi između strukturnih matričnih algebri (SMA), tj. unitalnih podalgebri od M_n (algebre svih kompleksnih kvadratnih matrica reda n) razapetih nekim skupom matričnih jedinica. Prvi cilj disertacije je u potpunosti opisati formu svih Jordanovih ulaganja između dviju SMA koristeći koncepte uvedene u Coelhinom opisu automorfizama SMA. Nastavljajući rad Molnara i Šemrla na preserverima ranga jedan algebre gornjetrokutastih matrica $\mathcal{T}_n$, u drugom dijelu disertacije se promatra nekoliko prirodnih linearnih preservera na SMA i stavlja ih se u širi kontekst Jordanovih ulaganja na SMA. Treći dio disertacije sadrži potpuno proširenje dobro poznatog preserverskog rezultata Petek i Šemrla u kontekstu SMA. Preiznije, dobivena je karakterizacija SMA $\mathcal{A} \subseteq M_n$ takvih da su sva injektivna neprekidna preslikavanja $\mathcal{A} \to M_n$ koja čuvaju komutativnost i spektar nužno Jordanova ulaganja.

Ključne riječi: Jordanov homomorfizam, strukturna matrična algebra, preserver ranga, preserver spektra, preserver komutativnosti

Contents

CHAPTER 1

Introduction

1.1 General introduction

The theory of preserver problems is an active research area in linear algebra and functional analysis, in particular matrix and operator theory. The problems concern maps between spaces of matrices or linear operators on a vector space (esp. Hilbert or Banach space), having certain properties which usually include preserving some of the algebraic, analytical or topological structure of the spaces they act on. The objective is to shed light on the general form of such maps and provide a characterization in terms of simpler properties, if possible. The resulting theorems often have a particularly elegant, simple and attractive form.

First relevant results appeared as early as the turn of the 20th century, in the area of linear preserver problems, in particular those concerning linear maps between matrix algebras preserving some basic matrix properties. Frobenius [21] proved that every linear endomorphism $\phi : M_n \to M_n$ (where M_n stands for the algebra of all $n \times n$ complex matrices) which preserves the determinant (i.e. satisfies $\det \phi(X) = \det X$ for all matrices $A \in M_n$) is necessarily of the form $X \mapsto MXN$ or $X \mapsto MX^tN$ where $M, N \in M_n$ are invertible matrices satisfying $\det(MN) = 1$ and X^t stands for the transpose of X. The statement of this theorem is in some sense typical as it characterizes linear preservers (in this case determinant preservers) in terms of a simple class of maps (in this case left and right multiplications by some invertible matrices and their compositions with the transposition map). Many other theorems feature a similar class of maps; we present some of them in Chapter 2. Among the most studied linear preserver problems are those dealing with spectrum preservers (more generally, invertibility preservers) and commutativity preservers, which are particularly interesting for their connection with Lie homomorphisms. Starting with the seminal theorem of Frobenius, linear preserver theory distinguishes itself by especially elegant results. Furthermore, even relatively deep results are often proved by surprisingly elementary techniques which are cleverly combined and modified depending on the particularities of each problem. Linear preserver theory has considerable applications even outside pure mathematics: Wigner-Uhlhorn theorem (see e.g. [49]) and related results form a cornerstone of the mathematical foundation of quantum mechanics ([54]).

The study of linear spectrum preservers leads to another interesting class of maps called Jordan homomorphisms which originate from ring theory. Jordan homomorphisms are defined as linear maps between algebras which preserve squares (i.e. satisfy $\phi(a^2) = \phi(a)^2$ for all elements a). A similar definition exists for Jordan homomorphisms between rings. Jordan homomorphisms also serve as morphisms in the category of Jordan algebras, a class of nonassociative algebras analogous to Lie algebras with a commutative product instead of an anticommutative one. Jordan algebras are frequently encountered throughout physics, in particular in quantum mechanics. A result of Herstein [30] from 1957, later refined in [51], states that surjective Jordan homomorphisms onto a prime ring are necessarily multiplicative or antimultiplicative. In the particular case of Jordan homomorphisms on M_n, this yields the result that nonzero Jordan homomorphisms $M_n \to M_n$ are implemented as $X \mapsto TXT^{-1}$ or $X \mapsto TX^tT^{-1}$ for some invertible matrix $T \in M_n$. A similar conclusion for Jordan embeddings holds for the algebra $\mathcal{T}_n$ of upper-triangular matrices [43, Corollary 4].

Moving on from M_n and $\mathcal{T}_n$, it is natural to consider unital subalgebras of M_n which are spanned by some set of matrix units E_{ij}. Such algebras were first introduced in the literature by Van Wyk in [57] under the name *structural matrix algebras* (abbreviated as SMAs). Incidentally, a simple argument (see Proposition 3.1.1) shows that structural matrix algebras are precisely the subalgebras of M_n which contain all diagonal matrices.SMAs, and somewhat more general incidence algebras, have been studied in many papers, such as [2, 3, 4, 14, 15, 12, 13, 17, 18, 22, 23, 50, 57]. The automorphisms of SMAs are completely described in Coelho's important paper [17] from 1993, which also supplies necessary and sufficient conditions on SMAs such that all automorphisms are inner. The paper cleverly combines the methods of abstract algebra, such as the semidirect product of groups and Wedderburn's principal theorem, with the purely combinatorial techniques of graph theory. It serves as both the model and the main inspiration for an important part of this dissertation. The later paper [2] reproduces the aforementioned automorphism description using a different approach. This newer proof is based on the useful observation that every SMA can be conjugated by a permutation matrix to obtain a subalgebra of a block upper-triangular algebra. These block upper-triangular algebras are also known as parabolic algebras for reasons related to Lie group theory, and appear in papers such as [1, 56].

Regarding Jordan homomorphisms of SMAs, the most important result is again due to Akkurt et. al. [3, Case 2] which states that for any SMA $\mathcal{A}$ which is, up to permutation similarity, contained in a parabolic algebra where each diagonal block has size larger than 1, and an arbitrary ring $\mathcal{B}$, we have that a Jordan homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is necessarily a sum of a homomorphism and an antihomomorphism. Furthermore, [3, Case 1] extends the previous result of Benkovič [8, Theorem 4.1] which describes Jordan homomorphisms of triangular algebras.

Circling back to preserver problems, many attempts have been made to characterize Jordan homomorphisms via preserving properties, especially on matrix algebras. By the famous Gleason-Kahane-Żelazko theorem [28, 34] a linear functional on a unital complex Banach algebra $\mathcal{A}$ is a character (i.e. a nonzero algebra homomorphism) if and only if it maps every element to a complex number belonging to its spectrum. It is also easily shown that a linear functional is a character if and only if it preserves squares. Inspired by these results, Kaplansky in his famous lecture notes [35] in 1970 formulated the following problem (known as Kaplansky problem): If $\phi : \mathcal{A} \to \mathcal{B}$ is a linear unital map between complex unital Banach algebras which shrinks spectrum, i.e. it satisfies $\sigma(\phi(a)) \subseteq \phi(a)$ for all $a \in \mathcal{A}$, is ϕ necessarily a Jordan homomorphism? It is widely known (and stated by Kaplansky himself) that this problem has a negative answer (see e.g. [7]), so the true form of his question was what additional conditions should be imposed onto the algebras $\mathcal{A}$, $\mathcal{B}$ and the map ϕ to force a positive answer. For example, Aupetit famously conjectured that the Kaplansky problem has a positive answer under additional assumptions that the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are semisimple and that ϕ is surjective. This problem is still widely open, surprisingly even for C^*-algebras [11].

Only much more recently the first results on non-linear preserver problems appeared in the literature. A particularly important result in this line of research is due to Petek and Šemrl [45, 47] who characterize Jordan homomorphisms on $M_n, (n \geq 3)$ as (not necessarily linear) continuous spectrum and commutativity preservers. In particular, this result and the techniques used to prove it serve as the main starting point for a large portion of research pertaining to this dissertation. Petek went further and made a complete description of continuous spectrum and commutativity preservers $\mathcal{T}_n \to \mathcal{T}_n$ (where $\mathcal{T}_n \subseteq M_n$ is the subalgebra of all upper-triangular matrices). If one additionally assumes injectivity, the result is precisely a characterization of Jordan embeddings $\mathcal{T}_n \to \mathcal{T}_n$.

1.2 Our results

Now we circle back to the novel results which we intend to present in this dissertation.

In Chapter 3 after defining structural matrix algebras, we present a preparatory original result on intrinsic simultaneous diagonalization of a commuting family of diagonalizable matrices within an SMA, which will be used at several occasions throughout the dissertation.

Chapter 4 presents our first major result, Theorem 4.2.4, which provides a complete description of Jordan embeddings $\mathcal{A} \to M_n$ where $\mathcal{A} \subseteq M_n$ is a structural matrix algebra. This result easily implies Corollary 4.2.14, which describes Jordan automorphisms of SMA and thus directly expands on the already mentioned Coelho's description of $\operatorname{Aut}(\mathcal{A})$ where $\mathcal{A}$ is an SMA. The statement of Theorem 4.2.4 itself heavily relies on the concept of transitive maps, a term introduced by [17] and referenced later by [2]. However, methods used are much more elementary than in [17], and are more reminiscent of some of the older papers on Jordan homomorphisms such as [30].

Chapter 5 deals with several simple linear preserver problems on structural matrix algebras. Namely, now being aware of the general form of Jordan embeddings $\mathcal{A} \to M_n$ where $\mathcal{A} \subseteq M_n$ is a SMA from Chapter 3, we were able to formulate two linear preserver problems whose conclusions make use of the given description. Theorem 5.1.7 regards rank-one preserving unital linear maps $\mathcal{A} \to M_n$ where $\mathcal{A} \subseteq M_n$ is a SMA and proves that these maps are necessarily Jordan embeddings. The converse is false in general, but the same theorem states an algebraic condition for when exactly are Jordan embeddings rank-one preserving. The unitality condition might seem innocuous but is actually an essential ingredient in the result. In contrast with the case when $\mathcal{A} = M_n$ or $\mathcal{T}_n$, linear rank-one preserving maps $\mathcal{A} \to M_n$ are not necessarily rank-preserving, even when assuming injectivity. In view of this, Theorem 5.2.5 and Corollary 5.2.7 provide a complete description of rank preservers $\mathcal{A} \to M_n$ and place them in the broader context of Jordan embeddings which were described before. The proof of this result builds heavily on the rank-one result and hence fits in nicely with all previous results. This chapter ends with a few simple remarks about linear determinant preservers $\mathcal{A} \to M_n$ where $\mathcal{A} \subseteq M_n$ is a SMA.

Chapter 6 contains the final part of the dissertation which concerns nonlinear preserver problems. More specifically, it builds on the already mentioned Petek and Šemrl's result [44, 45, 47] as well as its continuation [24] by Gogić, Petek and the author, concerning block upper-triangular matrices. We extend this result in Theorem 6.2.2: we obtain a complete characterization of SMAs $\mathcal{A} \subseteq M_n$ such that all injective continuous commutativity and spectrum preserving map $\mathcal{A} \to M_n$ are necessarily Jordan embeddings. We had to add the injectivity assumption, as it turns out it is necessary unless the algebra $\mathcal{A}$ is not semisimple.

Original results from Chapters 3–5 are taken mostly from the preprint [25], while Chapter 6 is sourced from the preprint [26]. Following the author's personal preferences, the dissertation was written with the aim of keeping the text firmly within the scope of linear algebra. Moreover, this was also done with hopes of fitting into the broader style of papers from the area of linear preservers, which prides itself on the aesthetic value of its results along with a minimalist and elegant exposition of proofs.

1.3 Acknowledgements

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Chapter 2

Preliminaries

2.1 Notation

We begin this section by introducing some general notation and terminology. Let $\mathcal{A}$ be an algebra.

- If $\mathcal{A}$ is unital, then $1_{\mathcal{A}}$ denotes the unity in $\mathcal{A}$ and $\mathcal{A}^{\times}$ denotes set of all invertible elements of $\mathcal{A}$.
- $[a, b] = ab - ba$ denotes the commutator of $a, b \in \mathcal{A}$.
- For $a, b \in \mathcal{A}$ by $a \leftrightarrow b$ we denote the fact that a and b commute, i.e. $ab = ba$.
- For $a, b \in \mathcal{A}$ by $a \perp b$ we denote the fact that a and b are orthogonal, i.e. $ab = ba = 0$. In this sense, for $S \subseteq \mathcal{A}$, by $S^{\perp}$ we denote the set $\{a \in \mathcal{A} : a \perp x, \text{ for all } x \in S\}$.
- $Z(\mathcal{A})$ denotes the centre of $\mathcal{A}$.
- For $S \subseteq \mathcal{A}$, by $S' = \{a \in \mathcal{A} : ax = xa, \text{ for all } x \in S\}$ we denote the commutant of S in $\mathcal{A}$.
- $\operatorname{Aut}(\mathcal{A})$ denotes the set of all automorphisms of $\mathcal{A}$.
- If $S \subseteq \mathcal{A}$ and $a \in \mathcal{A}$, then aS denotes the set $\{ax : x \in S\}$. Other expressions such as Sa, aSa, aSb for $b \in \mathcal{A}$ and so on are defined similarly.

Let $n \in \mathbb{N}$.

- As usual, by $M_n := M_n(\mathbb{C})$ we denote the set of all $n \times n$ complex matrices and by $M_{m,n}$ ($m \in \mathbb{N}$) the set of all $m \times n$ complex matrices. More generally, $M_{m,n}(S)$ denotes the set of all $m \times n$ matrices with entries in the set S.
- $\mathcal{T}_n$ and $\mathcal{D}_n$ denote the sets of all upper-triangular and diagonal matrices of M_n, respectively.
- Following [24], for $p, k_1, \ldots, k_p \in \mathbb{N}$ such that $k_1 + \cdots + k_p = n$, $\mathcal{A}_{k_1,\ldots,k_p}$ denotes the corresponding *block upper-triangular subalgebra* of M_n, i.e.

$$\mathcal{A}_{k_1,\ldots,k_p} := \begin{bmatrix} M_{k_1,k_1} & M_{k_1,k_2} & \cdots & M_{k_1,k_p} \\ 0 & M_{k_2,k_2} & \cdots & M_{k_2,k_p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{k_p,k_p} \end{bmatrix}. \tag{2.1.1}$$

- For $A, B \in M_n$ we denote by $A \leftrightarrow B$ the fact that A and B commute, i.e. $AB = BA$.
- For $A, B \in M_n$ we say that A and B are orthogonal (and write $A \perp B$) if $AB = BA = 0$.
- For $A \in M_n$ and $1 \leq i, j \leq n$, by $A_{ij} \in \mathbb{C}$ we denote the element of A at the position (i, j). We also write this fact as $A = [A_{ij}]_{1 \leq i,j \leq n} = [A_{ij}]_{i,j=1}^{n}$.
- For $A \in M_n$, by $\sigma(A)$ we denote the spectrum of A. Unless stated otherwise, the

spectrum is considered as a set, not a multiset.

- For $A \in M_n$ by $r(A)$ we denote the rank of A.
- For $A \in M_n$, by $R(A)$ we denote the image of A, while by $N(A)$ we denote the nullspace of A.
- For $A \in M_n$, by $k_A(x) = \det(xI - A)$ we denote the characteristic polynomial of A, while by $m_A(x)$ we denote the minimal polynomial of A.
- We denote by Δ_n the full diagonal relation $\{(i,i) : 1 \le i \le n\}$ on $[1,n]$.
- We denote by $\operatorname{diag}(\lambda_1, \dots, \lambda_n) \in \mathcal{D}_n$ the diagonal matrix with complex numbers $\lambda_1, \dots, \lambda_n$ on the diagonal, in this order. We also extend this notation to block-diagonal matrices i.e. $\operatorname{diag}(X_1, \dots, X_k)$ denotes the matrix with matrices $X_1, \dots, X_k$ as blocks on the diagonal, in that order, and zeroes elsewhere. Furthermore, it can also denote subalgebras of the matrix algebra. For example, $\operatorname{diag}(M_2, M_3)$ stands for the subalgebra of M_5 consisting of all matrices of the form $\operatorname{diag}(A, B)$ where $A \in M_2$ and $B \in M_3$.
- For $1 \le i, j \le n$ we denote by $E_{ij} \in M_n$ the standard matrix unit with 1 at the position (i,j) and 0 elsewhere.
- For vectors $u, v \in \mathbb{C}^n$ by $u \parallel v$ we denote the statement that the set $\{u, v\}$ is linearly dependent. Similar notation is used for matrices.
- For any permutation $\pi \in S_n$ (where, as usual, S_n denotes the symmetric group), by

$$R_\pi := \sum_{k=1}^{n} E_{k\pi(k)} \tag{2.1.2}$$

we denote the permutation matrix in M_n associated to π.

When describing the action of certain functions, sometimes we shall omit the argument entirely. For example, we will denote the matrix transposition by $(\cdot)^t$ and for $A, B \in M_n$ the matrix multiplication function $X \mapsto AXB$ will be denoted by $A(\cdot)B$.

Following [37], if P is a logical expression, by $[P]$ we denote its Iverson bracket, which is defined as 1 if P is true and 0 if P is false.

2.2 Algebras

In the context of this dissertation, by an *algebra* we shall refer to an associative algebra over $\mathbb{C}$, unless explicitly stated otherwise. To reiterate what was said in the introduction, consequentially some definitions and theorems which hold for algebras over other fields (or even for rings) will thus be stated more restrictively than is necessary, but this will have little bearing on the new results themselves. On the positive side, this will also streamline the prerequisites slightly in order to remain within of the scope of the new results. In a similar vein, an *ideal* of an algebra refers exclusively to two-sided ideals.

We start with a few elementary definitions from [9]. An algebra $\mathcal{A}$ is said to be

(i) *simple* if it possesses no proper ideals.
(ii) *prime* if all $a, b \in \mathcal{A}$ satisfy the implication $a\mathcal{A}b = 0 \implies a = 0$ or $b = 0$.
(iii) *semiprime* if all $a \in \mathcal{A}$ satisfy the implication $a\mathcal{A}a = 0 \implies a = 0$.

Note that $(i) \implies (ii) \implies (iii)$.

The unit of a unital algebra $\mathcal{A}$ will be denoted by $1_{\mathcal{A}}$ by default. For a unital algebra $\mathcal{A}$, we define the *centre* of $\mathcal{A}$ as

$$Z(\mathcal{A}) := \{a \in \mathcal{A} : ax = xa, \text{ for all } x \in \mathcal{A}\}.$$

We say that a unital algebra $\mathcal{A}$ is *central* if $Z(\mathcal{A}) = \mathbb{C}1_{\mathcal{A}}$.

For algebras $\mathcal{A}$ and $\mathcal{B}$, by $\mathcal{A} \oplus \mathcal{B}$ we refer to their external direct sum, as defined in [9, p. xxviii].

An automorphism $\phi \in \mathrm{Aut}(\mathcal{A})$ of a unital algebra $\mathcal{A}$ is said to be *inner* if there exists $a \in \mathcal{A}^\times$ such that $\phi(x) = axa^{-1}$ for all $x \in \mathcal{A}$. Otherwise, ϕ is said to be an *outer* automorphism.

The following elementary result is crucial for motivating much of our work on Jordan homomorphisms:

Theorem 2.2.1 (Skolem-Noether, [9, Theorem 1.30])**.** *Every automorphism of a finite dimensional central simple algebra is inner.*

In particular, every automorphism of the full matrix algebra M_n is inner. This fact can also be shown directly; for a very short and elegant argument see [48, Theorem 1.1].

If $\mathcal{A}$ is finite-dimensional, the set $\mathcal{A}^\times$ is path-connected in $\mathcal{A}$. Namely, for every $A \in \mathcal{A}^\times$ by finiteness of the spectrum we can take an appropriate branch of the logarithm to conclude that $A = \exp B$ for some $B \in \mathcal{A}$. Then $t \mapsto \exp(tB)$ is a (continuous) path from I to A within $\mathcal{A}$.

Let $\mathcal{A}$ be a finite-dimensional unital algebra. Recall that for a fixed $a \in \mathcal{A}$, the evaluation

$$\mathbb{C}[x] \to \mathcal{A}, \qquad f \mapsto f(a)$$

is an algebra homomorphism and there exists a unique monic polynomial $m_a \in \mathbb{C}[x]$ (called that *minimal polynomial*) of minimal degree which annihilates a. One easily shows that $f(a) = 0$ for some $f \in \mathbb{C}[x]$ if and only if $m_a \mid f$.

Lemma 2.2.2. *Let $\mathcal{A}$ be a finite-dimensional unital algebra. The zeroes of m_a are precisely the elements of $\sigma(a)$.*

Proof. Fix $\lambda \in \mathbb{C}$ and notice that λ is a zero of the polynomial $m_a(x) - m_a(\lambda) \in \mathbb{C}[x]$. Therefore, there exists $g \in \mathbb{C}[x]$ such that

$$m_a(x) - m_a(\lambda) = (x - \lambda)g(x).$$

Evaluation at a yields

$$-m_a(\lambda)1 = (a - \lambda 1)g(a).$$

From here we conclude $m_a(\lambda) = 0 \iff a - \lambda 1 \notin \mathcal{A}^\times$. Indeed:

- If $m_a(\lambda) \neq 0$, then it immediately follows $a - \lambda 1 \in \mathcal{A}^\times$.
- If $m_a(\lambda) = 0$ but $a - \lambda 1 \in \mathcal{A}^\times$, then it would follow that $g(a) = 0$, which is a contradiction since $\deg g < \deg m_a$.

□

If the minimal polynomial of an element $a \in \mathcal{A}$ splits into distinct linear factors (i.e. $m_a(x) = \prod_{\lambda \in \sigma(a)}(x - \lambda)$ by Lemma 2.2.2), then the evaluation satisfies this property:

$$f|_{\sigma(a)} \equiv g|_{\sigma(a)} \iff f(a) = g(a), \qquad \text{for all } f, g \in \mathbb{C}[x].$$

Indeed, it suffices to show that $f|\sigma(a) = 0$ is equivalent to $f(a) = 0$. This follows from Lemma 2.2.2 since

$$f|_{\sigma(a)} = 0 \iff (x - \lambda) \mid f, \forall \lambda \in \sigma(a) \iff m_a \mid f \iff f(a) = 0.$$

Lemma 2.2.3. *Let $\mathcal{A}$ be a finite-dimensional unital algebra. For an element $a \in \mathcal{A}$ it is equivalent:*
(i) The minimal polynomial of a splits into distinct linear factors.
(ii) There exists a nonempty family $\mathcal{P} \subseteq \mathcal{A}$ of nonzero mutually orthogonal idempotents such that $a \in \operatorname{span} \mathcal{P}$.

Proof. $\boxed{(i) \implies (ii)}$ By Lemma 2.2.2, the minimal polynomial of a is equal to

$$m_a(x) = \prod_{\lambda \in \sigma(a)} (x - \lambda).$$

For each $\lambda \in \sigma(a)$ let $f_\lambda \in \mathbb{C}[x]$ be the unique polynomial of degree $< |\sigma(a)|$ with the property that $f_\lambda(\mu) = [\lambda = \mu]$ for all $\mu \in \sigma(a)$. For $\lambda, \mu \in \sigma(a)$, the relations

$$f_\lambda \neq 0, \qquad f_\lambda f_\mu = [\lambda = \mu] f_\lambda$$

hold true on $\sigma(a)$, which implies that $\{f_1(a), \dots, f_k(a)\}$ is a family of mutually orthogonal nonzero idempotents. Furthermore, we have

$$\sum_{\lambda \in \sigma(a)} f_\lambda \equiv 1$$

on $\sigma(a)$, so $\sum_{\lambda \in \sigma(a)} f_\lambda(a) = 1$. Finally, for each $\lambda \in \sigma(a)$ we have $(x - \lambda) f_\lambda(x)|_{\sigma(a)} = 0$ so

$$0 = (a - \lambda) f_\lambda(a) \implies a f_\lambda(a) = \lambda f_\lambda(a).$$

Therefore, we have

$$\sum_{\lambda \in \sigma(a)} \lambda f_\lambda(a) = \sum_{\lambda \in \sigma(a)} a f_\lambda(a) = a,$$

which completes the proof.

$\boxed{(ii) \implies (i)}$ Suppose that $a = \sum_{i=1}^k \lambda_i p_i$ where $p_1, \dots, p_k$ are pairwise orthogonal nonzero idempotents. One easily shows that for each polynomial $f \in \mathbb{C}[x]$ we have

$$f(a) = \sum_{i \in [1,k]} f(\lambda_i) p_i.$$

In particular, if we consider $g(x) = \prod_{i=1}^k (x - \lambda_i) \in \mathbb{C}[x]$, then $g(a) = 0$ so $m_a \mid q$, which implies that m_a splits into linear factors. □

Lemma 2.2.4. *Let V be a complex vector space. Then each nonempty set $S \subseteq V$ possesses a finite subset $S_0 \subseteq S$ such that $\operatorname{span} S_0 = \operatorname{span} S$.*

Proof. Let $\{v_1, \dots, v_n\} \subseteq \operatorname{span} S$ be a basis for $\operatorname{span} S$. By definition of the linear span, for each $1 \leq j \leq n$, there exists a finite subset $S_j \subseteq S$ such that $v_j \in \operatorname{span} S_j$. We claim that

$$S_0 := \bigcup_{1 \leq j \leq n} S_j \subseteq S$$

spans $\operatorname{span} S$. Indeed, $\operatorname{span} S_0 \subseteq \operatorname{span} S$ is clear, while the converse inclusion follows from the fact that $\operatorname{span} S_0$ contains a basis $\{v_1, \dots, v_n\}$ for $\operatorname{span} S$. □

Lemma 2.2.5. *Let $\mathcal{A}$ be a finite-dimensional unital algebra. Suppose that $\mathcal{F} \subseteq \mathcal{A}$ is a family such that m_a splits into distinct linear factors for all $a \in \mathcal{F}$. It is equivalent:*

(i) $\mathcal{F}$ is a commutative family.

(ii) There exists a nonempty family $\mathcal{P} \subseteq \mathcal{A}$ of nonzero mutually orthogonal idempotents such that $\mathcal{F} \subseteq \operatorname{span} \mathcal{P}$.

Proof. $\boxed{(i) \implies (ii)}$ Suppose first that $\mathcal{F}$ is a finite family, i.e. $\mathcal{F} = \{a_1, \dots, a_k\}$. For each $1 \le j \le k$ and $\lambda \in \sigma(a_j)$, let $f_{j,\lambda} \in \mathbb{C}[x]$ be the unique polynomial of degree $< |\sigma(a_j)|$ such that $f_j(\mu) = [\mu = \lambda]$ for all $\mu \in \sigma(a_j)$. For each $(\lambda_1, \dots, \lambda_k) \in \sigma(a_1) \times \cdots \times \sigma(a_k)$ define

$$p_{(\lambda_1,\dots,\lambda_k)} := f_{1,\lambda_1}(a_1) \cdots f_{k,\lambda_k}(a_k) \in \mathcal{A}.$$

For each $1 \le j \le k$, from the proof of Lemma 2.2.3 we know that

$$\{f_{j,\lambda_j}(a_j) : \lambda_j \in \sigma(a_j)\}$$

is a set of mutually orthogonal nonzero idempotents which sum up to 1. By the commutativity of $\{a_1, \dots, a_k\}$, it follows directly that

$$\mathcal{P} := \{p_{(\lambda_1,\dots,\lambda_k)} : (\lambda_1, \dots, \lambda_k) \in \sigma(a_1) \times \cdots \times \sigma(a_k)\} \setminus \{0\}$$

is a family of mutually orthogonal nonzero idempotents which sum up to 1. Furthermore, also from the proof of Lemma 2.2.2 we conclude that for each $1 \le j \le k$ and $(\lambda_1, \dots, \lambda_k) \in \sigma(a_1) \times \cdots \times \sigma(a_k)$ we obtain

$$(a - \lambda_j) p_{(\lambda_1,\dots,\lambda_k)} = 0.$$

Therefore, for each $1 \le j \le k$ we have

$$a_j = \sum_{(\lambda_1,\dots,\lambda_k) \in \sigma(a_1) \times \cdots \times \sigma(a_k)} p_{(\lambda_1,\dots,\lambda_k)} a_j = \sum_{(\lambda_1,\dots,\lambda_k) \in \sigma(a_1) \times \cdots \times \sigma(a_k)} \lambda_j p_{(\lambda_1,\dots,\lambda_k)}$$

which proves the claim.

Now, let $\mathcal{F}$ be an infinite family. By Lemma 2.2.4 we can choose a finite subset $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $\operatorname{span} \mathcal{F}_1 = \operatorname{span} \mathcal{F}$. By the finite case, $\mathcal{F}_1$ satisfies (ii). Since the property (ii) remains true when passing to the linear span, we conclude that $\operatorname{span} \mathcal{F}_1 = \operatorname{span} \mathcal{F}$ also satisfies (ii), so $\mathcal{F}$ does as well.

$\boxed{(ii) \implies (i)}$ Follows from the fact that $\mathcal{P}$ is itself a commutative family. □

Remark 2.2.6. If $\mathcal{A} = M_n$, the properties (i) and (ii) are also equivalent to the fact that the family $\mathcal{F}$ is simultaneously diagonalizable, i.e. that there exists $S \in M_n^\times$ such that $\mathcal{F} \subseteq S \mathcal{D}_n S^{-1}$. Indeed, if $\mathcal{F}$ is contained in the span of some nonempty family of mutually orthogonal nonzero idempotents $\mathcal{P}$, then one easily shows that

$$\mathbb{C}^n = \dot{\sum_{P \in \mathcal{P}}} R(P)$$

is a direct sum and that each $R(P)$ is contained in an eigenspace of some $Q \in \mathcal{P}$. This yields a simultaneous eigenbasis for the entire $\mathcal{P}$, and hence for the entire $\mathcal{F}$ as well.

This argument essentially proves the following classical result:

Theorem 2.2.7 ([31, Theorem 1.3.21]). *Let $\mathcal{F} \subseteq M_n$ be a family of diagonalizable matrices. Then $\mathcal{F}$ is simultaneously diagonalizable if and only if $\mathcal{F}$ is a commuting family.*

2.3 Matrices

As usual, we will frequently identify vectors $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ as column-matrices

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

and $x^t = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$ as row-matrices.

As any matrix $A = (A_{ij})_{ij} \in M_n$ can be understood as a map $\{1, \dots, n\}^2 \to \mathbb{C}, (i, j) \mapsto A_{ij}$, we consider its support $\operatorname{supp} A$ as the set of all indices $(i, j) \in \{1, \dots, n\}^2$ such that $A_{ij} \neq 0$. We also say that A is supported in a set $S \subseteq \{1, \dots, n\}^2$ if $\operatorname{supp} A \subseteq S$.

Recall that every rank-one matrix $A \in M_n$ can be written in the form $A = uv^* = [u_i\overline{v_j}]_{1 \le i,j \le n}$ for some nonzero vectors $u, v \in \mathbb{C}^n$. The converse is clearly true as well; every such matrix is rank-one. This decomposition is unique up to scalar multiplication. Namely, we have $R(A) = \operatorname{span}\{u\}$ and $N(A)^\perp = \operatorname{span}\{v\}$. For nonzero vectors $u_1, u_2, v_1, v_2 \in \mathbb{C}^n$ it follows that

$$u_1 v_1^* = u_2 v_2^* \implies u_1 \parallel u_2 \text{ and } v_1 \parallel v_2.$$

Furthermore, a rank-one matrix $A = uv^*$ is idempotent if and only if $\langle u, v \rangle = 1$ and nilpotent if and only if $u \perp v$ (in this case we clearly have $A^2 = 0$).

The characteristic polynomial of a rank-one matrix $A = uv^*$ is of the form $k_A(x) = x^{n-1}(x - \lambda)$ where $\lambda = \operatorname{Tr} A = \langle u, v \rangle$. The minimal polynomial is $m_A(x) = x(x - \lambda)$, which implies that every non-nilpotent rank-one matrix is diagonalizable.

To say a few words about some concrete rank-one matrices, for a vector $v \in \mathbb{C}^n$ and $1 \le i \le n$, the matrix $e_i v^t$ is a matrix having the vector v as its i-th row and having all other rows equal to zero. Similarly, ve_i^t is a matrix having the vector v as its i-th column and having all other columns equal to zero.

Lemma 2.3.1. *Suppose that $D \in \mathcal{D}_n$ has distinct elements on the diagonal. Then $\{D\}' = \mathcal{D}_n$.*

Proof. Clearly $\mathcal{D}_n \subseteq \{D\}'$. Conversely, suppose that $A \in M_n$ commutes with D. For each $1 \le i \neq j \le n$ we have

$$D_{ii} A_{ij} = \sum_{k=1}^{n} D_{ik} A_{kj} = (DA)_{ij} = (AD)_{ij} = \sum_{k=1}^{n} A_{ik} D_{kj} = A_{ij} D_{jj}.$$

Since $D_{ii} \neq D_{jj}$, it follows that $A_{ij} = 0$. We conclude $A \in \mathcal{D}_n$. $\square$

Throughout the dissertation we will freely use all results regarding block-matrix operations, as nicely laid out in [53, Section 2].

Let $A \in M_n$ and $S \subseteq [1, n]$.

- When $S \neq [1, n]$, denote by $A^{\flat S} \in M_{n-|S|}$ the matrix obtained from A by deleting all rows i and columns j where $i, j \in S$. We also formally allow $A^{\flat \emptyset} = A$.

- Denote by $A^{\sharp S} \in M_{n+|S|}$ the matrix obtained from A by adding zero rows and columns so that $(A^{\sharp S})^{\flat S} = A$.

By using block-matrix multiplication, it is not difficult to verify that $(\cdot)^{\flat S} : M_n \to M_{n-|S|}$ and $(\cdot)^{\sharp S} : M_n \to M_{n+|S|}$ are algebra homomorphisms. We also extend this notation to sets of matrices by applying the respective operation elementwise.

Proposition 2.3.2. *Suppose that $A \in \mathcal{T}_n$ is a matrix with distinct elements $\lambda_1, \dots, \lambda_n$ on the diagonal (in that order). Then there exists a unique $S \in \mathcal{T}_n^\times$ with only ones on the diagonal such that $A = S \operatorname{diag}(\lambda_1, \dots, \lambda_n) S^{-1}$.*

Proof. Define $S \in \mathcal{T}_n^\times$ explicitly as

$$S_{ii} = 1, \qquad S_{ij} = \sum_{s \geq 0} \sum_{i < k_1 < \cdots < k_s < j} \frac{A_{ik_1} A_{k_1 k_2} \cdots A_{k_s j}}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_{k_1}) \cdots (\lambda_j - \lambda_{k_s})}, \qquad 1 \leq i < j \leq n$$

(note that the $s = 0$ case corresponds to the summand $\frac{A_{ij}}{\lambda_j - \lambda_i}$). Let us verify that the above S satisfies $A = SDS^{-1}$ i.e. $AS = SD$. Clearly AS and SD are both upper triangular and their diagonals are $\lambda_1, \dots, \lambda_n$. Fix $1 \leq i < j \leq n$. We have

$$(SD)_{ij} = \sum_{k=1}^{n} S_{ik} D_{kj} = \lambda_j S_{ij}.$$

On the other hand, we have

$$\begin{aligned}
(AS)_{ij} &= \sum_{k=1}^{n} A_{ik} S_{kj} \\
&= \sum_{i \leq k \leq j} A_{ik} S_{kj} \\
&= A_{ii} S_{ij} + \sum_{i < k < j} A_{ik} S_{kj} + A_{ij} S_{jj} \\
&= \lambda_i \sum_{s \geq 0} \sum_{i < k_1 < \cdots < k_s < j} \frac{A_{ik_1} A_{k_1 k_2} \cdots A_{k_s j}}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_{k_1}) \cdots (\lambda_j - \lambda_{k_s})} \\
&\quad + (\lambda_j - \lambda_i) \sum_{i < k < j} \sum_{s \geq 0} \sum_{k < k_1 < \cdots < k_s < j} \frac{A_{ik} A_{kk_1} A_{k_1 k_2} \cdots A_{k_s j}}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)(\lambda_j - \lambda_{k_1}) \cdots (\lambda_j - \lambda_{k_s})} \\
&\quad + A_{ij} \\
&= (\lambda_i + (\lambda_j - \lambda_i)) \sum_{s \geq 1} \sum_{i < k_1 < \cdots < k_s < j} \frac{A_{ik_1} A_{k_1 k_2} \cdots A_{k_s j}}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_{k_1}) \cdots (\lambda_j - \lambda_{k_s})} \\
&\quad + \lambda_i \frac{A_{ij}}{\lambda_j - \lambda_i} + A_{ij} \\
&= \lambda_j \sum_{s \geq 1} \sum_{i < k_1 < \cdots < k_s < j} \frac{A_{ik_1} A_{k_1 k_2} \cdots A_{k_s j}}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_{k_1}) \cdots (\lambda_j - \lambda_{k_s})} + \frac{\lambda_j A_{ij}}{\lambda_j - \lambda_i} \\
&= \lambda_j \sum_{s \geq 0} \sum_{i < k_1 < \cdots < k_s < j} \frac{A_{ik_1} A_{k_1 k_2} \cdots A_{k_s j}}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_{k_1}) \cdots (\lambda_j - \lambda_{k_s})} \\
&= \lambda_j S_{ij}
\end{aligned}$$

which proves the existence. Now we prove uniqueness. Suppose that

$$A = T \operatorname{diag}(\lambda_1, \dots, \lambda_n) T^{-1}$$

for some $T \in \mathcal{T}_n^\times$ with ones on the diagonal. In particular, we have

$$S \operatorname{diag}(\lambda_1, \dots, \lambda_n) S^{-1} = T \operatorname{diag}(\lambda_1, \dots, \lambda_n) T^{-1} \implies T^{-1}S \leftrightarrow \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

so by Lemma 2.3.1 it follows that $T^{-1}S$ is a diagonal matrix. But T^{-1} and S have both only ones on the diagonal so their product $T^{-1}S$ has so as well. Therefore $T^{-1}S = I$ so $T = S$. □

Remark 2.3.3. This explicit formula for S will come in handy later, but note that the mere existence of S can also be shown with a simple inductive argument. Namely, let $x \in \mathbb{C}^n$ be an eigenvector for A such that $x_n = 1$. Note that such an eigenvector certainly exists, since otherwise all eigenvectors for A would be contained in the subspace $\operatorname{span}\{e_1, \dots, e_{n-1}\}$, but this would be a contradiction with the fact that A possesses a basis of eigenvectors by virtue of being diagonalizable. It is easy to show that $Ax = \lambda_n x$.

Consider the invertible matrix

$$S := \begin{bmatrix} e_1 & \cdots & e_{n-1} & x \end{bmatrix} \in \mathcal{T}_n^\times.$$

The inverse is given by

$$S^{-1} = \begin{bmatrix} e_1 & \cdots & e_{n-1} & (-x + 2e_n) \end{bmatrix} \in \mathcal{T}_n^\times$$

and we have $S^{-1}AS = \operatorname{diag}(B, \lambda_n)$ where $B \in \mathcal{T}_{n-1}$ is the upper left corner of the matrix A. Now, B has distinct eigenvalues $\lambda_1, \dots, \lambda_{n-1}$ so we are able to apply the inductive hypothesis to B and obtain an upper-triangular invertible matrix $T \in \mathcal{T}_n^\times$ such that $T^{-1}BT = \operatorname{diag}(\lambda_1, \dots, \lambda_{n-1})$. We conclude that

$$(\underbrace{S \operatorname{diag}(T, 1)}_{\in \mathcal{T}_n^\times})^{-1} A (S \operatorname{diag}(T, 1)) = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

which closes the proof.

Remark 2.3.4. Note also that if an upper-triangular matrix $A \in \mathcal{T}_n$ diagonalizes in $\mathcal{T}_n$ as $A = TDT^{-1}$ for some $T \in \mathcal{T}_n^\times$ and $D \in \mathcal{D}_n$, then D is precisely the diagonal of A. We point out the contrast with the M_n case, where for a diagonalizable matrix $A \in M_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$, and an arbitrary permutation $\pi \in S_n$, there exists an invertible matrix $S \in M_n^\times$ such that $A = S \operatorname{diag}(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)}) S^{-1}$.

The algebra $\mathcal{T}_n$ is clearly not simple; for instance the set of all strictly-upper triangular matrices is a nontrivial ideal. Even though Theorem 2.2.1 hence does not apply to $\mathcal{T}_n$, it nevertheless does hold true that all automorphisms of $\mathcal{T}_n$ are inner ([36]).

Since transposition is not a well-defined antiautomorphism $\mathcal{T}_n \to \mathcal{T}_n$, we consider the matrix

$$J := \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \in M_n$$

and the corresponding map

$$\odot : M_n \to M_n, \qquad X^{\odot} := JX^tJ^{-1} \tag{2.3.1}$$

It is not difficult to see that $X^{\odot}$ is obtained by mirroring the matrix X along its secondary diagonal. Explicitly, for each $1 \le i, j \le n$ we have $E_{ij}^{\odot} = E_{n+1-j,n+1-i}$. It follows that the restriction of $\odot$ to $\mathcal{T}_n$ is an antiautomorphism of $\mathcal{T}_n$, and we will indeed use it as a sort of a canonical antiautomorphism on $\mathcal{T}_n$, in place of transposition. Also, for a block upper-triangular algebra $\mathcal{A} \subseteq M_n$ we denote by $\mathcal{A}^{\odot} \subseteq M_n$ the image of $\mathcal{A}$ under the map $X \mapsto X^{\odot}$. Then $\mathcal{A}^{\odot}$ is the block upper-triangular algebra obtained from $\mathcal{A}$ by reversing the sizes of the diagonal blocks.

We also state the following useful lemma concerning particular upper-triangular families of pairwise orthogonal rank-one idempotent matrices.

Lemma 2.3.5. *Let $S(y) := I + e_1y^t \in \mathcal{T}_n$, where $y = \sum_{i=2}^n y_ie_i \in \mathbb{C}^n$. Then $S(y)^{-1} = S(-y)$ and*

$$S(y)^{-1}E_{ii}S(y) = \begin{cases} E_{11} + e_1y^t, & \text{if } i = 1, \\ E_{ii} - y_iE_{1i}, & \text{if } 1 < i \le n. \end{cases} \tag{2.3.2}$$

Proof. It is easy to check that $S(y)^{-1} = S(-y)$. The straightforward calculation

$$\begin{aligned} S(y)^{-1}E_{11}S(y) &= (S(-y)e_1)(e_1^tS(y)) = (e_1 - (y^te_1)e_1)(e_1 + y)^t \\ S(y)^{-1}E_{ii}S(y) &= (S(-y)e_i)(e_i^tS(y)) = (e_i - (y^te_i)e_1)e_i^t, \quad i \ne 1, \end{aligned}$$

provides (2.3.2). □

The next class of algebras we consider are subalgebras $\mathcal{A}$ of M_n which contain $\mathcal{T}_n$. As it turns out, these algebras are precisely the *block upper-triangular subalgebras* of M_n (see [56]). More specifically, any such algebra is of the form

$$\mathcal{A}_{k_1,\dots,k_r} := \begin{bmatrix} M_{k_1,k_1} & M_{k_1,k_2} & \cdots & M_{k_1,k_r} \\ 0 & M_{k_2,k_2} & \cdots & M_{k_2,k_r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{k_r,k_r} \end{bmatrix} \tag{2.3.3}$$

for some $r, k_1, \dots, k_r \in \mathbb{N}$ such that $k_1 + \cdots + k_r = n$. These algebras also appear in the literature under the name *parabolic algebras* (see e.g. [1, 56]), a term coming from the theory of Lie groups. It is interesting to note that the block upper-triangular algebras $\mathcal{A}_{1,n-1}$ and $\mathcal{A}_{n-1,1}$ are exactly (up to similarity) the unital strict subalgebras of M_n of maximal dimension (see [1]).

Lemma 2.3.6. *The map*

$$\odot|_{\mathcal{A}_{k_1,\dots,k_r}}^{\mathcal{A}_{k_r,\dots,k_1}} : \mathcal{A}_{k_1,\dots,k_r} \to \mathcal{A}_{k_r,\dots,k_1}$$

is a well-defined algebra antiisomorphism.

Proof. It suffices to show that $JX^tJ \in \mathcal{A}_{k_r,\dots,k_1}$ for all $X \in \mathcal{A}_{k_1,\dots,k_r}$. Indeed, for each

$0 \le s \le r-1$, $1 \le i \le k_1 + \cdots + k_{s+1}$ and $k_1 + \cdots + k_s + 1 \le j \le n$ we have

$$JE_{ij}^t J = E_{n+1-j,n+1-i} \in \mathcal{A}_{k_r,\ldots,k_1}$$

since $k_r + \cdots + k_s + 1 \le n+1-i \le n$ and $1 \le n+1-j \le k_r + \cdots + k_{s+1}$. □

Remark 2.3.7. Consider the map $k_{(\cdot)} : M_n \to \mathbb{C}_{\le n}[x]$ which maps a matrix A to its characteristic polynomial k_A. Then this map is continuous with respect to the standard topologies on M_n and $\mathbb{C}_{\le n}[x]$ as finite-dimensional complex vector spaces. It is not difficult to check that a sequence of polynomials $(p_j)_{j=1}^\infty$ in $\mathbb{C}_{\le n}[x]$ converges to $p \in \mathbb{C}_{\le n}[x]$ (in the standard topology of $\mathbb{C}_{\le n}[x]$) if and only if $p_j \to p$ pointwise.

Suppose $A_j \to A$ in M_n. Then for each fixed $x \in \mathbb{C}$ we have

$$k_{A_j}(x) = \det(A_j - xI) \xrightarrow{j\to\infty} \det(A - xI) = k_A(x)$$

by the continuity of the determinant $\det : M_n \to \mathbb{C}$. It follows $k_{A_j} \to k_A$ pointwise and hence in $\mathbb{C}_{\le n}[x]$ as well.

2.4 Properties of general Jordan homomorphisms

If $\mathcal{A}$ is an algebra, for $a, b \in \mathcal{A}$ we define their *Jordan product* as $a \circ b := ab + ba$. The normalizing multiplicative factor of $\frac{1}{2}$ is sometimes included in the definition, especially when encountered in physics. It is not difficult to check that the vector space $\mathcal{A}$ equipped with the above defined binary multiplication operation $\circ : \mathcal{A}^2 \to \mathcal{A}$ forms a nonassociative algebra, which we shall denote by $(\mathcal{A}, \circ)$. Furthermore, $(\mathcal{A}, \circ)$ is in fact a so-called *Jordan algebra* ([33]). In general, a Jordan algebra $\mathfrak{A}$ is a nonassociative algebra whose multiplication satisfies these two properties:

- $ab = ba$ for all $a, b \in \mathfrak{A}$ (commutativity),
- $(ab)(aa) = a(b(aa))$ for all $a, b \in \mathfrak{A}$ (Jordan identity).

Both the definition of Jordan algebras and the aforementioned method of inducing a Jordan algebra from an associative algebra immediately reminds one of Lie algebras. A notable difference is worth pointing out: it is well-known that every Lie algebra can be naturally embedded in a Lie algebra which arises from an associative algebra (namely, this is the universal enveloping algebra construction). A similar result does not hold true for Jordan algebras. In fact, those Jordan algebras which are embeddable into a Jordan algebra which arises from an associative algebra are known in the literature as *special Jordan algebras*, and have found considerable use in physics in the context of the Jordan formalism of quantum mechanics (for a brief introduction see e.g. [42]). On the other hand, those Jordan algebras which are not embeddable into an associative algebra in this way are known as *exceptional Jordan algebras*. Their existence is commonly seen as a drawback or even a fundamental flaw of the Jordan formalism. A well-known example is the algebra $H_3(\mathbb{O})$ of all 3×3 self-adjoint matrices over the octonions equipped with the multiplication $a \circ b := \frac{1}{2}(ab + ba)$, known as the Albert algebra [5].

Lemma 2.4.1. *Let $\mathcal{A}$ be a simple algebra. Then $(\mathcal{A}, \circ)$ is a simple Jordan algebra.*

Proof. Suppose that $\mathcal{I} \ne \{0\}, \mathcal{A}$ is a nontrivial ideal of $(\mathcal{A}, \circ)$. Fix $a, b \in \mathcal{I}$. One easily verifies that for all $x \in \mathcal{A}$ holds

$$[a \circ b, x] = \underbrace{[a, x] \circ b}_{\in \mathcal{I}} + \underbrace{a \circ [b, x]}_{\in \mathcal{I}} \in \mathcal{I}.$$

We also have $a \circ b \in \mathcal{I}$ and then $(a \circ b) \circ x \in \mathcal{I}$. In particular, we have

$$2(a \circ b)x = [a \circ b, x] + (a \circ b) \circ x \in \mathcal{I}$$

so $(a \circ b)x \in \mathcal{I}$ for all $x \in \mathcal{A}$.

For all $x, y \in \mathcal{A}$ we have

$$\mathcal{I} \ni y \circ ((a \circ b)x) = y(a \circ b)x + \underbrace{(a \circ b)xy}_{\in \mathcal{I}}$$

so it follows that $y(a \circ b)x \in \mathcal{I}$. The simplicity of $\mathcal{A}$ implies that $a \circ b = 0$, as otherwise $\mathcal{A}(a \circ b)\mathcal{A}$ would be a nontrivial ideal of the algebra $\mathcal{A}$.

Now let $a \in \mathcal{I}$ be arbitrary. By the above consideration we have $0 = a \circ a = 2a^2$ so it follows that $a^2 = 0$. Furthermore, for all $x \in \mathcal{A}$ we have $a \circ x \in \mathcal{I}$ and hence

$$0 = a \circ (a \circ x) = a^2 x + x a^2 + 2axa = 2axa.$$

It follows that $a\mathcal{A}a = \{0\}$. Since $\mathcal{A}$ is simple, in particular it is semiprime so it follows that $a = 0$. Therefore $\mathcal{I} = \{0\}$, which is a contradiction. We conclude that $(\mathcal{A}, \circ)$ is simple. □

A *Jordan homomorphism* between algebras $\mathcal{A}, \mathcal{B}$ is a linear map $\phi : \mathcal{A} \to \mathcal{B}$ such that

$$\phi(a \circ b) = \phi(a) \circ \phi(b), \qquad \text{for all } a, b \in \mathcal{A}. \tag{2.4.1}$$

Jordan homomorphisms between rings are defined as additive maps which satisfy the same condition. In the case of algebras, Jordan homomorphisms $\mathcal{A} \to \mathcal{B}$ can be understood as Jordan algebra homomorphisms $(\mathcal{A}, \circ) \to (\mathcal{B}, \circ)$. Assuming linearity, the $\circ$-preserving condition (2.4.1) is equivalent to a simpler condition

$$\phi(a^2) = \phi(a)^2, \qquad \text{for all } a \in \mathcal{A}.$$

(we say that ϕ *preserves squares*). Indeed, if a linear map $\phi : \mathcal{A} \to \mathcal{B}$ preserves $\circ$, then for all $a \in \mathcal{A}$ we have

$$2\phi(a^2) = \phi(2a^2) = \phi(a \circ a) = \phi(a) \circ \phi(a) = 2\phi(a) \implies \phi(a)^2 = \phi(a).$$

Conversely, if ϕ preserves squares, we have

$$\begin{aligned}
&= \phi(a)^2 + 2\phi(a \circ b) + \phi(b)^2 \\
&= \phi(a^2) + \phi(2(ab + ba)) + \phi(b^2) \\
&= \phi((a + b)^2) \\
&= \phi(a + b)^2 \\
&= (\phi(a) + \phi(b))^2 \\
&= \phi(a)^2 + 2(\phi(a)\phi(b) + \phi(b)\phi(a)) + \phi(b)^2
\end{aligned}$$

$$= \phi(a)^2 + 2\phi(a) \circ \phi(b) + \phi(b)^2$$

so $\phi(a \circ b) = \phi(a) \circ \phi(b)$. By analysing the proof, it is clear that the same equivalence holds for Jordan homomorphisms between 2-torsion-free rings. Furthermore, the above simple calculation is reminiscent of the fact that characters of a unital algebras are characterised by unitality and the square-preserving property (see e.g. [27]). In contrast, Jordan homomorphisms between unital algebras need not be unital.

We state a few basic properties of general Jordan homomorphisms, taken directly from [25, Lemma 3.1], proofs of which are elementary and can be found in [32].

Lemma 2.4.2. *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a Jordan homomorphism between algebras $\mathcal{A}$ and $\mathcal{B}$. We have:*

(a) $\phi(aba) = \phi(a)\phi(b)\phi(a)$ for all $a, b \in \mathcal{A}$.
(b) $\phi(abc + cba) = \phi(a)\phi(b)\phi(c) + \phi(c)\phi(b)\phi(a)$ for all $a, b, c \in \mathcal{A}$.
(c) $\phi([[a, b], c]) = [[\phi(a), \phi(b)], \phi(c)]$, for all $a, b, c \in \mathcal{A}$.
(d) $\phi([a, b]^2) = [\phi(a), \phi(b)]^2$ for all $a, b \in \mathcal{A}$.
(e) $\phi(a^k) = \phi(a)^k$ for all $a \in \mathcal{A}$ and $k \in \mathbb{N}$. In particular, $\phi(p(a)) = p(\phi(a))$ for all $a \in \mathcal{A}$ and polynomials $p \in \mathbb{C}[x]$ such that $p(0) = 0$.
(f) For every $a \in \mathcal{A}$ and an idempotent $p \in \mathcal{A}$ such that $[p, a] = 0$ we have $\phi(pa) = \phi(p)\phi(a) = \phi(a)\phi(p)$.

Proof. (a) We have

$$a \circ (a \circ b) = a(ab + ba) + (ab + ba)a = a^2b + 2aba + ba^2 = 2aba + a^2 \circ b$$

so applying ϕ yields

$$\begin{aligned} 2\phi(a)\phi(b)\phi(a) + \phi(a)^2 \circ \phi(b) &= \phi(a) \circ (\phi(a) \circ \phi(b)) \\ &= \phi(a \circ (a \circ b)) \\ &= 2\phi(aba) + \phi(a)^2 \circ \phi(b). \end{aligned}$$

It follows $\phi(aba) = \phi(a)\phi(b)\phi(a)$.

(b) We have

$$(a + c)b(a + c) = aba + abc + cba + cbc$$

so applying ϕ yields

$$\begin{aligned} &\phi(a)\phi(b)\phi(a) + \phi(a)\phi(b)\phi(c) + \phi(c)\phi(b)\phi(a) + \phi(c)\phi(b)\phi(c) \\ &\quad = (\phi(a) + \phi(c))\phi(b)(\phi(a) + \phi(c)) \\ &\quad \overset{(a)}{=} \phi(a + c)\phi(b)\phi(a + c) \\ &\quad = \phi(aba) + \phi(abc + cba) + \phi(cbc). \end{aligned}$$

It follows $\phi(abc + cba) = \phi(a)\phi(b)\phi(c) + \phi(c)\phi(b)\phi(a)$.

(c) One easily verifies

$$[[a, b], c] = abc + cba - (bac + cab)$$

and hence

$$\begin{aligned} \phi([[a, b], c]) &= \phi(abc + cba - (bac + cab)) \\ &\overset{(b)}{=} \phi(a)\phi(b)\phi(c) + \phi(c)\phi(b)\phi(a) - (\phi(b)\phi(a)\phi(c) + \phi(c)\phi(a)\phi(b)) \end{aligned}$$

$$= [[\phi(a), \phi(b)], \phi(c)].$$

(d) Note that

$$[a, b]^2 = (ab - ba)^2 = a(bab) + (bab)a - ab^2a - ba^2b = a \circ (bab) - ab^2a - ba^2b$$

so we conclude

$$\begin{aligned}\phi([a, b]^2) &= \phi(a \circ (bab)) - \phi(ab^2a) - \phi(ba^2b)\\ &= \phi(a) \circ \phi(bab) - \phi(a)\phi(b)^2\phi(a) - \phi(b)\phi(a)^2\phi(b)\\ &= [\phi(a), \phi(b)]^2.\end{aligned}$$

(e) We prove the claim by induction on k. The claim is evidently true for $k \in \{1, 2\}$. Suppose that for some $k \geq 3$ satisfies $\phi(a^j) = \phi(a)^j$ for all $1 \leq j \leq k$. Then for $k + 1$ we have

$$\begin{aligned}\phi(a^{k+1}) &= \phi(aa^{k-1}a) \overset{(a)}{=} \phi(a)\phi(a^{k-1})\phi(a) \overset{\text{inductive hypothesis}}{=} \phi(a)\phi(a)^{k-1}\phi(a)\\ &= \phi(a)^{k+1}.\end{aligned}$$

(f) Clearly, $\phi(p)$ is again an idempotent and we have

$$0 = \phi([[p, a], p]) \overset{(c)}{=} [[\phi(p), \phi(a)], \phi(p)] = 2\phi(p)\phi(a)\phi(p) - \phi(a)\phi(p) - \phi(p)\phi(a).$$

Left-multiplication by $\phi(p)$ yields

$$\phi(a)\phi(p) = \phi(p)\phi(a)\phi(p),$$

while right-multiplication by $\phi(p)$ yields

$$\phi(p)\phi(a) = \phi(p)\phi(a)\phi(p).$$

On the other hand, $pa = ap$ implies

$$\phi(p)\phi(a)\phi(p) = \phi(pap) = \phi(pa) = \phi(ap)$$

and hence $\phi(pa) = \phi(p)\phi(a) = \phi(a)\phi(p)$.

□

Lemma 2.4.3. *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a Jordan homomorphism of an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ such that the image of ϕ has trivial commutant in $\mathcal{B}$ (meaning, every element of $\phi(\mathcal{A})'$ is either zero or a scalar multiple of the unity $1_{\mathcal{B}}$, if it exists). Then ϕ preserves commutativity.*

Proof. Suppose that $a, b \in \mathcal{A}$ satisfy $[a, b] = 0$. Then for all $x \in \mathcal{A}$ we have

$$[[a, b], x] = 0 \implies 0 = \phi([[a, b], x]) \overset{\text{Lemma 2.4.2 (c)}}{=} [[\phi(a), \phi(b)], \phi(x)].$$

It follows that $[\phi(a), \phi(b)] \in \phi(\mathcal{A})'$. If $\mathcal{B}$ is not unital, it immediately follows $[\phi(a), \phi(b)] = 0$. Otherwise, this commutator is of the form $[\phi(a), \phi(b)] = \lambda 1_{\mathcal{B}}$ for some scalar $\lambda \in \mathbb{C}$.

On the other hand, we have

$$\lambda^2 1_{\mathcal{B}} = [\phi(a), \phi(a)]^2 \overset{\text{Lemma 2.4.2 (d)}}{=} \phi([a,b]^2) = 0$$

so $\lambda = 0$ and hence $[\phi(a), \phi(b)] = 0$. □

Example 2.4.4 ([10, Example 7.21])**.** Consider the unital algebra

$$\mathcal{A} := \left\{ \begin{bmatrix} a & b & -c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix} : a,b,c,d \in \mathbb{C} \right\} \subseteq M_4$$

and the map

$$\phi : \mathcal{A} \to \mathcal{A}, \qquad \phi\left(\begin{bmatrix} a & b & -c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix} \right) := \begin{bmatrix} a & b & -d & c \\ 0 & a & 0 & d \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix}.$$

One easily verifies that ϕ is a unital Jordan homomorphism. However, ϕ does not preserve commutativity. Namely, we have $E_{12} + E_{34} \perp E_{14}$ but

$$\phi(E_{12} + E_{34})\phi(E_{14}) = (E_{12} + E_{34})(-E_{13} + E_{24}) = E_{14},$$
$$\phi(E_{14})\phi(E_{12} + E_{34}) = (-E_{13} + E_{24})(E_{12} + E_{34}) = -E_{14}.$$

Lemma 2.4.5 ([52, Prop. 1.3] and [20, Theorem 2.5])**.** *Suppose $\mathcal{A}$ and $\mathcal{B}$ unital algebras. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a Jordan homomorphism such that $1_{\mathcal{B}}$ is in the image of ϕ. Then ϕ is unital and preserves inverses in the sense that for every $a \in \mathcal{A}^\times$ we have $\phi(a) \in \mathcal{B}^\times$ and $\phi(a^{-1}) = \phi(a)^{-1}$.*

Proof. Suppose that $a_0 \in \mathcal{A}$ satisfies $\phi(a_0) = 1_{\mathcal{B}}$. Then

$$2 \cdot 1_{\mathcal{B}} = \phi(a_0 + a_0) = \phi(1_{\mathcal{A}} \circ a_0) = \phi(1_{\mathcal{A}}) \circ \phi(a_0) = \phi(1_{\mathcal{A}}) \circ 1_{\mathcal{B}} = 2\phi(1_{\mathcal{A}})$$

which implies $\phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ so ϕ is unital. Let $a \in \mathcal{A}^\times$ be arbitrary. Then

$$\phi(a) = \phi(aa^{-1}a) \overset{\text{Lemma 2.4.2 (a)}}{=} \phi(a)\phi(a^{-1})\phi(a).$$

If we set $p_1 := \phi(a)\phi(a^{-1})$ and $p_2 := \phi(a^{-1})\phi(a)$, it is immediate that p_1 and p_2 are idempotents in $\mathcal{B}$. Furthermore, we have

$$p_1 + p_2 = \phi(a) \circ \phi(a^{-1}) = 2\phi(a \circ a^{-1}) = 2\phi(1_{\mathcal{A}}) = 2 \cdot 1_{\mathcal{B}}$$

and hence $2 \cdot 1_{\mathcal{B}} - p_1 = p_2$ is idempotent as well. It follows that $2(p_1 - 1_{\mathcal{B}}) = 0$ so we conclude $p_1 = 1_{\mathcal{B}}$ and then $p_2 = 1_{\mathcal{B}}$ as well. Therefore, $\phi(a)$ is invertible in $\mathcal{B}$, with the inverse being equal to $\phi(a^{-1})$. □

Remark 2.4.6. An immediate consequence of this lemma is that all linear unital Jordan homomorphisms between unital algebras are spectrum preserving.

In any case, Jordan algebras gave rise to the study of Jordan homomorphisms in the context of associative rings and algebras. Multiplicative and antimultiplicative maps are immediate examples of such maps. However, in general there are also other examples, for instance consider

$$\phi : M_2 \oplus M_2 \to M_2 \oplus M_2, \qquad \phi(A, B) = (A, B^t).$$

A general way for constructing Jordan homomorphisms is the following. Let $\mathcal{A}$ be an algebra, $p \in \mathcal{A}$ central idempotent, and $\psi, \eta : \mathcal{A} \to \mathcal{A}$ a homomorphism and an antihomomorphism respectively. Then

$$\mathcal{A} \to \mathcal{A}, \qquad x \mapsto p\psi(x) + (1-p)\eta(x)$$

is a Jordan homomorphism which in general is not multiplicative or antimultiplicative. Still, this map does preserve commutativity (c.f. Lemma 2.4.3). One of the main problems in the theory of Jordan homomorphisms is determining under which assumptions on algebras $\mathcal{A}$ and $\mathcal{B}$ can we conclude that every Jordan homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ (possibly satisfying some extra conditions such as surjectivity) is either multiplicative or antimultiplicative. More generally, the question is whether one can express all such Jordan homomorphisms as a suitable combination of ring homomorphisms and antihomomorphisms. This question goes a long way back. We present a few of the early results here. The results presented in the context of rings all assume that the rings in question are 2-torsion-free. We will apply the results in the case of algebras over $\mathbb{C}$ so this technicality does not particularly concern us.

Theorem 2.4.7 ([32, Theorem 2])**.** *Let $\phi : \mathcal{R} \to \mathcal{S}$ be a Jordan homomorphism from a ring $\mathcal{R}$ to an integral domain $\mathcal{S}$. Then ϕ is a homomorphism or an antihomomorphism.*

Theorem 2.4.8 ([32, Theorem 7])**.** *Let $M_n(\mathcal{R})$ be the ring of $n \times n$ matrices, $n \geq 2$, with entries in some unital ring $\mathcal{R}$. Let $\phi : M_n(\mathcal{R}) \to \mathcal{S}$ be a Jordan homomorphism to an arbitrary ring $\mathcal{S}$. Then ϕ is a sum of a homomorphism and an antihomomorphism.*

Theorem 2.4.9 ([30, Theorem H], [51])**.** *Let $\phi : \mathcal{R} \to \mathcal{S}$ be a Jordan epimorphism from a ring $\mathcal{R}$ to a prime ring $\mathcal{S}$. Then ϕ is a homomorphism or an antihomomorphism.*

By combining the aforementioned result of Herstein with the well-known fact that all automorphisms of M_n are inner (Theorem 2.2.1), one obtains this fundamental result (see e.g. [45]):

Theorem 2.4.10. *All nonzero Jordan endomorphisms ϕ of M_n are precisely maps of the form*

$$\phi(\cdot) = T(\cdot)T^{-1} \qquad \text{or} \qquad \phi(\cdot) = T(\cdot)^t T^{-1} \tag{2.4.2}$$

for some invertible matrix $T \in M_n^\times$.

Proof. As an alternative to the above abstract argument, here we present an elementary direct proof of this fact, which is in actuality a simpler variant of the proof of Lemma 4.2.3. It also showcases a few other techniques we shall encounter later. Firstly, ϕ, being a Jordan endomorphism of M_n, is in fact an endomorphism of the Jordan algebra $(M_n, \circ)$. Since M_n is a simple algebra, by Lemma 2.4.1 the Jordan algebra $(M_n, \circ)$ is also simple. It follows that ϕ is a bijective map.

We prove the theorem by induction on n. For $n = 1$ the statement is clear. So suppose it holds for all k, $1 \leq k < n$. Note that by Lemma 2.4.2 (f), $\{\phi(E_{11}), \ldots, \phi(E_{nn})\}$ is a set of mutually orthogonal idempotents so by Theorem 2.2.7 there exists $S \in M_n^\times$ such that $\phi(E_{ii}) = SE_{ii}S^{-1}$ for all $1 \le i \le n$.

By passing to the map $S^{-1}\phi(\cdot)S$, without loss of generality we can assume that $\phi(E_{ii}) = E_{ii}$ for all $1 \le i \le n$.

Lemma 2.4.2 (f), we have $\phi(\{E_{11}\}^\perp) \subseteq \{E_{11}\}^\perp$ so it makes sense to define a map $\psi : \mathcal{A}_{n-1} \to M_{n-1}$ with the relation

$$\phi\left(\begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & \psi(X) \end{bmatrix}, \qquad X \subseteq M_{n-1}.$$

It is easy to show that ψ is a Jordan embedding so we apply the induction hypothesis to conclude that there exists an invertible $T \in M_{n-1}$ such that either $\psi(X) = TXT^{-1}$ for all $X \in \mathcal{A}_{n-1}$ or, $\psi(X) = TX^tT^{-1}$ for all $X \in \mathcal{A}_{n-1}$. We have $\psi(E_{ii}) = E_{ii}$ for all $1 \le i \le n-1$ so $T \leftrightarrow \{E_{ii} : 1 \le i \le n-1\}$ from where it follows that T is a diagonal matrix.

By passing to the map $\operatorname{diag}(1, T)^{-1}\phi(\cdot)\operatorname{diag}(1, T)$, we can further take T to be identity matrix in M_{n-1}. After that, by passing to the map $\phi(\cdot)^t$ if necessary, we can assume that ψ is the identity map on M_n.

Fix $1 \le i \ne j \ne n$. By Lemma 2.4.2 (b) we have

$$\phi(E_{ij}) = \phi(E_{ii}E_{ij}E_{jj} + E_{jj}E_{ij}E_{ii}) = E_{ii}\phi(E_{ij})E_{jj} + E_{jj}\phi(E_{ij})E_{ii}.$$

Therefore, $\phi(E_{ij})$ is supported in $\{(i,j),(j,i)\}$ so there exist scalars $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$ such that

$$\phi(E_{ij}) = \alpha_{ij}E_{ij} + \beta_{ij}E_{ji}.$$

Furthermore, we have

$$0 = \phi(E_{ij}^2) = \phi(E_{ij})^2 = \alpha_{ij}\beta_{ij}(E_{ii} + E_{jj})$$

so exactly one of α_{ij} and β_{ij} is equal to zero 0 (not both because of injectivity).

We claim that $\phi(E_{1j}) = \alpha_{1j}E_{1j}$ for all $2 \le j \le n$. If not, first suppose that $\phi(E_{1k}) = \beta_{1k}E_{k1}$ for some $2 \le k \le n-1$. Then $\phi(E_{1k} \circ E_{kn}) = \phi(E_{1n})$ but $\phi(E_{1k}) \circ \phi(E_{kn}) = \beta_{1k}E_{k1} \circ E_{kn} = 0$, a contradiction. Secondly, the case $\phi(E_{1n}) = \beta_{1n}E_{n1}$ can be eliminated by comparing $\phi(E_{1n} \circ E_{1,n-1}) = 0$ and $\phi(E_{1n}) \circ \phi(E_{1,n-1}) = \beta_{1n}E_{n1} \circ E_{1,n-1} = \beta_{1n}E_{n,n-1}$. By a diagonal similarity implemented by $\operatorname{diag}(1, \alpha_{12}, \ldots, \alpha_{1n}) \in \mathcal{D}_n^\times$, we can achieve that $\phi(E_{1j}) = E_{1j}$ for every $2 \le j \le n$.

For every $2 \le j \le n$, we have that $\phi(E_{1j} \circ E_{j1}) = E_{11} + E_{jj}$, so $E_{1j} \circ (\alpha_{j1}E_{j1} + \beta_{j1}E_{1j}) = \alpha_{j1}(E_{11} + E_{jj})$ gives $\alpha_{j1} = 1$ and hence $\beta_{j1} = 0$. By linearity it follows that ϕ is the identity map which closes the proof. □

In view of the canonical automorphism of $\mathcal{T}_n$ introduced in 2.3.1, Molnar and Šemrl arrived at the following description of Jordan automorphisms of $\mathcal{T}_n$.

Theorem 2.4.11 ([43, Corollary 4]). *Every Jordan automorphism $\phi : \mathcal{T}_n \to \mathcal{T}_n$ is of the form*

$$\phi(\cdot) = T(\cdot)T^{-1} \qquad \text{or} \qquad \phi(\cdot) = T(\cdot)^{\odot}T^{-1} \tag{2.4.3}$$

for some invertible matrix $T \in \mathcal{T}_n^\times$.

Note that in contrast with the M_n case, assuming injectivity here is crucial since there exist nontrivial non-injective Jordan endomorphisms of $\mathcal{T}_n$. A typical example is the map

$$\phi : \mathcal{T}_n \to \mathcal{T}_n, \qquad \phi(X) = \operatorname{diag}(X_{11}, \dots, X_{nn})$$

which maps an upper-triangular matrix X into its diagonal.

In view of Theorems 2.4.10 and 2.4.11, for an algebra $\mathcal{A}$ with a fixed antiautomorphism $f : \mathcal{A} \to \mathcal{A}$ it makes sense to speak of *inner Jordan automorphisms* when referring to maps from $\operatorname{Aut}(\mathcal{A})$ and their compositions with f. We can therefore say that all Jordan automorphisms of M_n and $\mathcal{T}_n$ are inner. The same will be true for block upper-triangular algebras (a consequence of our Corollary 4.2.15), but not for general structural matrix algebras.

2.5 Linear preserver theory

The theory started with the result of Frobenius [21] from 1897 which completely describes linear determinant preservers of the algebra M_n of $n \times n$ complex matrices:

Theorem 2.5.1 (Frobenius)**.** *Let $\phi : M_n \to M_n$ be a linear map which preserves determinant. Then there exist invertible matrices $A, B \in M_n^\times$ satisfying $\det(AB) = 1$ such that*

$$\phi = A(\cdot)B, \qquad \text{or} \qquad \phi = A(\cdot)^t B.$$

We now showcase a few more simple linear preserver problems on M_n, all taken from the survey paper [38].

Theorem 2.5.2. *(i) [19] Let $\phi : M_n \to M_n$ be a linear singularity preserver. Then there exist invertible matrices $A, B \in M_n^\times$ such that ϕ is of the form*

$$\phi = A(\cdot)B, \qquad \text{or} \qquad \phi = A(\cdot)^t B.$$

(ii) [40, Theorem 2.1] Let $\phi : M_n \to M_n$ be a linear invertibility preserver. Then there an invertible matrices $A, B \in M_n^\times$ such that ϕ is of the form

$$\phi = A(\cdot)B, \qquad \text{or} \qquad \phi = A(\cdot)^t B.$$

(iii) [40] Let $\phi : M_n \to M_n$ be a linear spectrum preserver. Then there exists an invertible matrix $A \in M_n^\times$ such that ϕ is of the form

$$\phi = A(\cdot)A^{-1}, \qquad \text{or} \qquad \phi = A(\cdot)^t A^{-1}.$$

(iv) [41] Let $\phi : M_n \to M_n$ be a linear rank-one preserver. Then there exists invertible matrices $A, B \in M_n^\times$ such that ϕ is of the form

$$\phi = A(\cdot)B, \qquad \text{or} \qquad \phi = A(\cdot)^t B.$$

(v) [29] Let $\phi : M_n \to M_n$ be a linear similarity preserver. Then there exist an invertible matrix $A \in M_n^\times$ and $a, b \in \mathbb{C}$ such that

$$\phi = aA(\cdot)A^{-1} + b(\operatorname{Tr}(\cdot))I, \qquad \text{or} \qquad \phi = aA(\cdot)^t A^{-1} + b(\operatorname{Tr}(\cdot))I$$

or there exists a fixed $B \in M_n$ such that $\phi = (\operatorname{Tr}(\cdot))B$.

(vi) Let $\phi : M_n \to M_n$ be a linear map which preserves unitary matrices. Then there exist unitary matrices $U, V \in M_n^\times$ such that

$$\phi = U(\cdot)V, \qquad \text{or} \qquad \phi = U(\cdot)^t V.$$

(vii) Let $\phi : M_n \to M_n$ be a linear map which preserves the spectral norm. Then there exist unitary matrices $U, V \in M_n^\times$ such that

$$\phi = U(\cdot)V, \qquad \text{or} \qquad \phi = U(\cdot)^t V.$$

In particular, (ii) and (iii) imply that the unital linear rank (or rank-one) preservers $\phi : M_n \to M_n$ are precisely the Jordan automorphisms of M_n. Analogous linear preserver problems as above were considered in the case of the triangular algebra $\mathcal{T}_n$. We already mentioned the important paper [43] by Šemrl and Molnar in the context of Jordan automorphisms of $\mathcal{T}_n$. The same paper also presents the following result in the context of rank-one preservers, which we paraphrase here:

Theorem 2.5.3. *[43, Theorem 1] Let $\phi : \mathcal{T}_n \to M_n$ be a linear rank-one preserver. Then there exist invertible matrices $A, B \in M_n^\times$ such that*

$$\phi = A(\cdot)B, \qquad \text{or} \qquad \phi = A(\cdot)^t B$$

or there exists an antilinear map $f : \mathcal{T}_n \to \mathbb{C}^n$ which is nonzero on every rank-one matrix, and a nonzero vector $x \in \mathbb{C}^n$ such that

$$\phi = xf(\cdot)^*,$$

or there exists a linear map $f : \mathcal{T}_n \to \mathbb{C}^n$ which is nonzero on every rank-one matrix, and a nonzero vector $y \in \mathbb{C}^n$ such that

$$\phi = f(\cdot)y^*.$$

Such a map ϕ is clearly not necessarily a rank-preserver (as was the case for maps $M_n \to M_n$). However, this conclusion does follow if we additionally assume ϕ to be, for instance, injective. Another possibility is to assume unitality, in which case we obtain that ϕ is precisely a Jordan embedding $\mathcal{T}_n \to M_n$.

Knowing this, and the structure of Jordan embeddings of SMAs, our goal was to formulate a similar result regarding linear unital rank or rank-one preservers $\phi : \mathcal{A} \to M_n$ where $\mathcal{A} \subseteq M_n$ is an SMA (see Chapter 3). In the case of rank-one preservers, unitality indeed turned out to be an indispensable assumption since without it they can behave strangely. For instance, consider the map

$$\phi : \mathcal{A} \to \mathcal{A}, \qquad \phi\left(\begin{bmatrix} x_{11} & 0 & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{11} + x_{33} \end{bmatrix}.$$

Then ϕ preserves rank-one matrices, but maps the identity matrix to a rank-two matrix. Hence this map does not fit into any of the conclusions of Theorem 2.5.3. With unitality, however, we obtained Theorem 5.1.7 which shows that linear unital rank-one preservers $\phi : \mathcal{A} \to M_n$ are necessarily Jordan embeddings. In contrast with the case of $\mathcal{T}_n$, the converse does not hold in general, although we managed find an algebraic condition which characterizes those maps for which it does hold using an algebraic condition. The basic techniques used in the proof of Theorem 5.1.7 indeed bear some similarity to the ones used by Šemrl and Molnar in Theorem 2.5.3.

We also state again the Kaplansky problem, which was already mentioned in the introduction:

Problem 2.5.4 ([35]). For which complex unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is every linear unital invertibility preserving map $\phi : \mathcal{A} \to \mathcal{B}$ (perhaps under some extra assumptions) necessarily a Jordan homomorphism?

Note that it is equivalent to assume that ϕ shrinks spectrum, i.e. that $\sigma(\phi(a)) \subseteq \phi(a)$ for all $a \in \mathcal{A}$. Theorem 2.5.2 (ii) precisely yields the positive answer for when $\mathcal{A} = \mathcal{B} = M_n$.

We state the Kaplansky problem because it serves as one of the main motivations for our nonlinear preserver problems.

2.6 Nonlinear preserver theory

As already mentioned in the introduction,, we would like to distinguish the following nonlinear preserver problem which elegantly characterizes Jordan automorphisms of M_n:

Theorem 2.6.1 (Šemrl). *Let $\phi : M_n \to M_n, n \geq 3$ be a continuous map which preserves commutativity and spectrum. Then there exists an invertible matrix $T \in M_n$ such that ϕ is of the form* (2.4.2)*.*

A precursor to this result was first formulated in [45] and it assumed its current optimal form a decade later in [47], where it is shown by counterexamples that all assumptions are indispensable. The paper [47] proved this result using a consequence of the Fundamental theorem of projective geometry, while the initial paper [45] uses methods entirely within the scope of elementary linear algebra. It is precisely this elementary approach which served as a starting point for our results from Chapter 6.

Tatjana Petek continued her research on this problem by considering maps satisfying the same assumptions as in Theorem 2.6.1 on the algebra of upper-triangular matrices $\mathcal{T}_n$, hence arriving at the complete description of those maps [44, Theorem 1]. In particular, when including surjectivity as an assumption, in [44, Corollary 3] she obtained a characterization of Jordan automorphisms of the upper-triangular algebra $\mathcal{T}_n$. It is easy to see that the same holds true if instead of surjectivity one assumes injectivity, thus motivating our assumptions in Chapter 6.

Chapter 3

Structural matrix algebras

3.1 Definition and basic properties

By a **quasi-order** on $[1, n]$ we mean a reflexive transitive relation ρ on $[1, n]$. We define several auxiliary relations on $[1, n]$ as

$$\rho^{\times} := \rho \setminus \Delta_n, \qquad \rho^t := \{(y, x) : (x, y) \in \rho\}, \qquad \overline{\rho} := \rho \cap \rho^t.$$

Obviously, ρ^t is also a quasi-order on $[1, n]$, which we refer to as the *reverse quasi-order* of ρ. On the other hand, $\overline{\rho}$ is moreover an equivalence relation. For a quasi-order ρ we define the subspace of M_n by

$$\mathcal{A}_\rho := \{A \in M_n : \operatorname{supp} A \subseteq \rho\} = \operatorname{span}\{E_{ij} : (i, j) \in \rho\}.$$

It is easy to see that $\mathcal{A}_\rho$ is in fact a unital subalgebra of M_n. Following [57], we refer to $\mathcal{A}_\rho$ as a *structural matrix algebra (SMA) defined by the quasi-order* ρ.

The term "structural" can be intuitively interpreted in this sense: the only conditions imposed on matrices X from a SMA $\mathcal{A} \subseteq M_n$ are of the form $x_{ij} = 0$. No other nontrivial relations are present, i.e. only the structure of X is proscribed.

Obviously, not all unital subalgebras of M_n are SMAs. For example, one can consider algebras of the form $T\mathcal{A}T^{-1}$ where $\mathcal{A} \subseteq M_n$ is an SMA and $T \in M_n^{\times}$. It is perhaps slightly less obvious whether every unital subalgebra of M_n can be always be conjugated to an SMA. The answer is negative when $n \geq 2$, as it is easy to see that for every such algebra $\mathcal{B} := T\mathcal{A}T^{-1}$, the cardinality of a maximal set of mutually orthogonal idempotents of $\mathcal{B}$ is precisely n. Namely, $\{TE_{ii}T^{-1} : 1 \leq i \leq n\}$ is an example of such a set, and a set of mutually orthogonal idempotents of larger cardinality is impossible even in M_n. As a consequence of this fact, here we discuss some concrete examples:

- Consider a nilpotent matrix $N \in M_n, n \geq 2$ and the unital algebra

  $$\mathcal{A} := \operatorname{span}\{N^k : k \geq 0\} \subseteq M_n$$

  generated by N. Then $\mathcal{A}$ is not conjugated to an SMA. Indeed, for every polynomial $p = \sum_{k=0}^{\infty} \alpha_k x^k \in \mathbb{C}[x]$ we have

  $$\sigma(p(N)) = p(\sigma(N)) = \{\alpha_0\}$$

  so we conclude that the only nonzero idempotent of $\mathcal{A}$ is I.

- Consider the unital algebra

$$\mathcal{A} := \left\{ \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & a & b & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & b & a & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} : a, b \in \mathbb{C} \right\} \subseteq M_5.$$

Then $\mathcal{A}$ is a central algebra not conjugated to a SMA in M_5. Indeed, it is easy to verify that for all $A \in \mathcal{A}$ the spectrum of A (as a multiset) is given by precisely the diagonal elements of A, and that the product of diagonals of $A, B \in \mathcal{A}$ can be calculated elementwise. It follows that all possible diagonals of rank-one idempotents in $\mathcal{A}$ are

$$(1,0,0,0,0), (0,0,1,0,0), (0,0,0,0,1).$$

If $\mathcal{P} \subseteq \mathcal{A}$ were a set of mutually orthogonal (rank-one) idempotents with cardinality 5, it would follow that at least two elements $P, Q \in \mathcal{P}$ would have the same diagonal, and hence couldn't be orthogonal. It follows that the set

$$\{E_{11}, E_{22} + E_{44}, E_{33}, E_{55}\}$$

is a maximal set of mutually orthogonal idempotents, and has cardinality 4.
On the other hand, $\mathcal{A}$ is (algebra) isomorphic to the SMA

$$\mathcal{B} := \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \subseteq M_4$$

via the isomorphism

$$\phi : \mathcal{A} \to \mathcal{B}, \qquad \phi\left(\begin{bmatrix} x_{11} & 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 \\ 0 & 0 & x_{33} & 0 & 0 \\ 0 & 0 & x_{34} & x_{22} & x_{45} \\ 0 & 0 & 0 & 0 & x_{55} \end{bmatrix} \right) := \begin{bmatrix} x_{11} & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & x_{45} \\ 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{55} \end{bmatrix}$$

which has the additional property $\phi(\mathcal{A} \cap \mathcal{D}_5) = \mathcal{D}_4$.

SMAs have a simple characterization within the class of all unital subalgebras of M_n:

Proposition 3.1.1. *Let $\mathcal{A} \subseteq M_n$ be a unital subalgebra. The following statements are equivalent:*
(i) $\mathcal{A}$ is an SMA.
(ii) $\mathcal{A}$ contains all diagonal matrices.
(iii) $\mathcal{A}$ contains a diagonal matrix with distinct diagonal entries.

Proof. The implications $(i) \implies (ii) \implies (iii)$ are trivial.

$\boxed{(iii) \implies (ii)}$ Suppose that $D \in \mathcal{D}_n \cap \mathcal{A}$ has distinct diagonal entries. Since the minimal polynomial of D has degree n, it follows that the set $\{I, D, \dots, D^{n-1}\} \subseteq \mathcal{A}$ is linearly independent, and hence a basis for $\mathcal{D}_n$. Thus, $\mathcal{D}_n \subseteq \mathcal{A}$.

$\boxed{(ii) \implies (i)}$ Suppose that $\mathcal{D}_n \subseteq \mathcal{A}$. Define a subset $\rho \subseteq [1,n]^2$ as

$$\rho := \bigcup_{A \in \mathcal{A}} \operatorname{supp} A.$$

One easily shows that ρ is a quasi-order. We claim that $\mathcal{A} = \mathcal{A}_\rho$. Indeed, let $(i,j) \in \rho$ be arbitrary. By definition, there exists $A \in \mathcal{A}$ such that $A_{ij} \neq 0$. We have

$$\mathcal{A} \ni E_{ii} A E_{jj} = A_{ij} E_{ij} \implies E_{ij} \in \mathcal{A}.$$

We conclude that $\mathcal{A}_\rho \subseteq \mathcal{A}$. On the other hand, for any $A \in \mathcal{A}$ we have $\operatorname{supp} A \subseteq \rho$ and hence $A \in \mathcal{A}_\rho$. Therefore, $\mathcal{A} = \mathcal{A}_\rho$ is an SMA.

□

Let $\rho, \rho' \subseteq [1,n]^2$ be quasi-orders. A permutation $\pi \in S_n$ is said to be a (ρ,ρ')-*increasing* if

$$(\pi(i), \pi(j)) \in \rho', \qquad \text{for all } (i,j) \in \rho.$$

Such a permutation π gives rise to an algebra embedding

$$\mathcal{A}_\rho \to \mathcal{A}_{\rho'}, \qquad E_{ij} \mapsto E_{\pi(i)\pi(j)}, \qquad (i,j) \in \rho. \tag{3.1.1}$$

This map can be expressed precisely as $R_\pi(\cdot)R_\pi^{-1}$ where $R_\pi \in M_n$ is the associated permutation matrix (2.1.2). Moreover, for any permutation $\pi \in S_n$, from (3.1.1) it is clear that the automorphism $R_\pi(\cdot)R_\pi^{-1}$ of M_n restricts to an embedding $\mathcal{A}_\rho \to \mathcal{A}_{\rho'}$ if and only if π is (ρ,ρ')-increasing.

Furthermore, if $|\rho| = |\rho'|$, which is equivalent to $\dim \mathcal{A}_\rho = \dim \mathcal{A}_{\rho'}$, a (ρ,ρ')-increasing permutation $\pi \in S_n$ is in fact a *quasi-order isomorphism*, i.e.

$$(i,j) \in \rho \iff (\pi(i), \pi(j)) \in \rho', \qquad \text{for all } 1 \leq i,j \leq n.$$

Indeed, the above is equivalent to $R_\pi \mathcal{A}_\rho R_\pi^{-1} = \mathcal{A}_{\rho'}$, which follows from $R_\pi \mathcal{A}_\rho R_\pi^{-1} \subseteq \mathcal{A}_{\rho'}$ and the equality of dimensions.

Lemma 3.1.2. *Let $\rho \subseteq [1,n]^2$ be a quasi-order and $\pi \in S_n$. Then $R_\pi \in \mathcal{A}_\rho^\times$ if and only if π fixes the equivalence classes of ρ.*

Proof. Suppose that $R_\pi \in \mathcal{A}_\rho^\times$ and let $1 \leq i \leq n$. We have $(i, \pi(i)) \in \operatorname{supp} R_\pi \subseteq \rho$. Furthermore, clearly $R_\pi^t = R_\pi^{-1} \in \mathcal{A}_\rho^\times$ as well so $(\pi(i), i) \in \operatorname{supp} R_\pi^t \subseteq \rho$. We conclude $(i, \pi(i)) \in \overline{\rho}$ so π maps i into an element of the same $\overline{\rho}$-class.

Conversely, suppose that a permutation $\pi \in S_n$ fixes the equivalence classes of $\overline{\rho}$. This precisely means that $\operatorname{supp} R_\pi \subseteq \overline{\rho}$, but since $\overline{\rho} \subseteq \rho$, it follows that $R_\pi \in \mathcal{A}_\rho^\times$. □

Lemma 3.1.3. *Let $(X, \preceq)$ be a poset of cardinality $n \in \mathbb{N}$. Then there exists an increasing bijection $f : (X, \preceq) \to (\{1, \ldots, n\}, \leq)$. Then there exists a bijection $f : X \to \{1, \ldots, n\}$ such that for all $x, y \in X$ we have that $x \preceq y$ implies $f(x) \leq f(y)$.*

Proof. We prove the claim by induction on n. The claim is trivially true for $n = 1$. Assume that the claim is true for some $n \in \mathbb{N}$ and let $(X, \preceq)$ be a poset of cardinality $n+1$. Recall that X necessarily possesses a $\preceq$-maximal element $x_0 \in X$. By the inductive hypothesis applied to $X\setminus\{x_0\}$, there exists an increasing bijection $f : (X\setminus\{x\}, \preceq) \to (\{1, \dots, n\}, \leq)$. Define a map

$$g : (X, \preceq) \to (\{1, \dots, n+1\}, \leq), \qquad g(x) = \begin{cases} n+1, & \text{if } x = x_0, \\ f(x), & \text{if } x \in X \setminus \{x\}. \end{cases}$$

Then g is clearly bijective, we claim that it is increasing. Fix $x, y \in X$ such that $x \preceq y$; we wish to show $g(x) \leq g(y)$. If both $x, y \in X \setminus \{x\}$, then this is true since f is increasing. If $x = x_0$, then necessarily $y = x_0$ as well and hence $g(x) = g(y)$. If $y = x_0$, then $g(x) \leq n+1 = g(x_0)$. □

Next, given a quasi-order ρ, we define the equivalence relation $\overline{\rho}$ on the set $\{1, \dots, n\}$ as

$$(i,j) \in \overline{\rho} \overset{\text{def}}{\iff} (i,j), (j,i) \in \rho.$$

We provide an explicit argument for the following fact, which can be deduced from [2, p. 432].

Lemma 3.1.4. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA. There exists a permutation $\pi \in S_n$ such that*

$$R_\pi \mathcal{A}_\rho R_\pi^{-1} = \begin{bmatrix} M_{m_1,m_1} & M_{m_1,m_2}(\mathbb{K}) & \cdots & M_{m_1,m_p}(\mathbb{K}) \\ 0 & M_{m_2,m_2} & \cdots & M_{m_2,m_p}(\mathbb{K}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{m_p,m_p} \end{bmatrix} \tag{3.1.2}$$

for some $p, m_1, \dots, m_p \in \mathbb{N}$ such that $m_1 + \dots + m_p = n$, where for any $1 \leq i < j \leq n$, $M_{m_i,m_j}(\mathbb{K})$ is either zero or M_{m_i,m_j}.

Proof. On the quotient set $[1,n]/\overline{\rho}$ we define the relation $\preceq$ in the following way: for $1 \leq i,j \leq n$ the corresponding $\overline{\rho}$-classes satisfy

$$[i]_{\overline{\rho}} \preceq [j]_{\overline{\rho}} \overset{\text{def}}{\iff} (i,j) \in \rho.$$

Note that $\preceq$ is well-defined. Indeed, suppose that $i' \in [i]_{\overline{\rho}}$ and $j' \in [j]_{\overline{\rho}}$ are arbitrary. Then we claim that

$$(i,j) \in \rho \iff (i',j') \in \rho.$$

Supposing the former, by definition of $\overline{\rho}$ we obtain

$$(i',i), (i,j), (j,j') \in \rho \implies (i',j') \in \rho.$$

The converse is similar. Further, one easily checks that $\preceq$ is in fact a partial order on the quotient set $[1,n]/\overline{\rho}$. Now let

$$[1,n]/\overline{\rho} = \{[r_1]_{\overline{\rho}}, \dots, [r_p]_{\overline{\rho}}\}, \qquad r_1, \dots, r_p \in [1,n]$$

be an ordering of the quotient set which respects the partial order $\preceq$ in the sense that $[r_i]_{\overline{\rho}} \preceq [r_j]_{\overline{\rho}}$ implies $i \leq j$ (its existence can be easily shown by an inductive argument).

For each $1 \leq k \leq p$ denote the elements of the class $[r_k]_{\overline{\rho}}$ explicitly as

$$[r_k]_{\overline{\rho}} = \{r_{k,1}, \dots, r_{k,m_k}\}$$

for some $m_k \in \mathbb{N}$. Define a permutation $\pi \in S_n$ by

$$\pi(r_{k,j}) := m_1 + \cdots + m_{k-1} + j, \qquad \text{for } 1 \le k \le p \text{ and } 1 \le j \le m_k$$

(here we formally set $m_0 := 0$). Define a new quasi-order ρ' on $[1, n]$ with the following condition:

$$(i, j) \in \rho' \overset{\text{def}}{\iff} (\pi^{-1}(i), \pi^{-1}(j)) \in \rho.$$

The permutation π is (ρ, ρ')-increasing by definition, so clearly $\mathcal{A}_{\rho'} = R_\pi \mathcal{A}_\rho R_\pi^{-1}$ (as $|\rho'| = |\rho|$). We now prove (3.1.2), which is equivalent to

$$\text{diag}(M_{m_1}, \ldots, M_{m_p}) \subseteq \mathcal{A}_{\rho'} \subseteq \mathcal{A}_{m_1,\ldots,m_p}, \tag{3.1.3}$$

where $\mathcal{A}_{m_1,\ldots,m_p}$ is the block upper-triangular algebra (as defined in (2.3.3)). Let $1 \le k, l \le p$ be arbitrary and let

$$(i, j) \in (m_1 + \cdots + m_{k-1}, m_1 + \cdots + m_k] \times (m_1 + \cdots + m_{l-1}, m_1 + \cdots + m_l]$$

be arbitrary. We have

$$(i, j) \in \rho' \iff (\underbrace{\pi^{-1}(i)}_{\in [r_k]_{\overline{\rho}}}, \underbrace{\pi^{-1}(j)}_{\in [r_l]_{\overline{\rho}}}) \in \rho \iff [r_k]_{\overline{\rho}} \preceq [r_l]_{\overline{\rho}}.$$

The latter depends only on k and l, implies $k \le l$, and is certainly true when $k = l$. This establishes both inclusions in (3.1.3).

□

Remark 3.1.5. Let ρ be a quasi-order on $[1, n]$. On the same set $[1, n]$, we define a new relation $\approxeq_0 := \rho \cup \rho^t$. Explicitly:

$$i \approxeq_0 j \iff ((i, j) \in \rho \text{ or } (j, i) \in \rho).$$

Let $\approxeq$ be the transitive closure of $\approxeq_0$. Then $\approxeq$ is an equivalence relation on $[1, n]$ and one easily checks that the centre of the SMA $\mathcal{A}_\rho$ is given by

$$\begin{aligned} Z(\mathcal{A}_\rho) &= \{\text{diag}(\lambda_1, \ldots, \lambda_n) \in \mathcal{D}_n : \lambda_i = \lambda_j \text{ for all } (i, j) \in \rho^\times\} \\ &= \{\text{diag}(\lambda_1, \ldots, \lambda_n) \in \mathcal{D}_n : (\forall 1 \le i, j \le n)(i \approxeq j \implies \lambda_i = \lambda_j)\}. \end{aligned}$$

Indeed, let $D \in Z(\mathcal{A}_\rho)$. In particular $D \leftrightarrow \mathcal{D}_n$, which implies $D \in \mathcal{D}_n$, If we denote $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$, we have

$$\begin{aligned} i \approxeq_0 j &\implies (i, j) \in \rho \text{ or } (j, i) \in \rho \implies D \leftrightarrow E_{ij} \text{ or } D \leftrightarrow E_{ji} \\ &\implies \lambda_i = \lambda_j. \end{aligned}$$

By transitivity of $\approxeq$ we conclude $i \approxeq j \implies \lambda_i = \lambda_j$ as well.

Conversely, suppose that $D = \text{diag}(\lambda_1, \ldots, \lambda_n) \in \mathcal{D}_n$ is constant on equivalence classes of $\approxeq$. We claim that $D \leftrightarrow \mathcal{A}_\rho$. Clearly $D \leftrightarrow \mathcal{D}_n$. Let $(i, j) \in \rho^\times$ be arbitrary. In particular we have $i \approxeq_0 j$ so $\lambda_i = \lambda_j$, which in turn implies $D \leftrightarrow E_{ij}$. We conclude that $D \leftrightarrow \text{span}\{E_{ij} : (i, j) \in \rho\} = \mathcal{A}_\rho$ which finishes the proof.

Denote by $\mathcal{Q}$ (or $\mathcal{Q}_\rho$) the quotient set of the equivalence relation $\approx$, i.e.

$$\mathcal{Q} := [1, n]/\approx . \tag{3.1.4}$$

Clearly, we have $|\mathcal{Q}| = \dim Z(\mathcal{A}_\rho)$. For any equivalence class $C \in \mathcal{Q}$ we define an idempotent

$$P_C := \sum_{i \in C} E_{ii} \in \mathcal{D}_n.$$

Obviously, $\operatorname{Tr} P_C = |C|$ and $P_C \in Z(\mathcal{A}_\rho)$. In fact $(P_C)_{C\in\mathcal{Q}}$ is a basis for $Z(\mathcal{A}_\rho)$. Furthermore, by definition of the quotient set, it follows that $(P_C)_{C\in\mathcal{Q}}$ is a mutually orthogonal family and $\sum_{C\in\mathcal{Q}} P_C = I$.

Next, for each $C \in \mathcal{Q}$ denote by $\pi_C : C \to \{1, \ldots, |C|\}$ the unique strictly increasing bijection and consider the quasi-order

$$\rho_C := \{(\pi_C(i), \pi_C(j)) : (i, j) \in (C \times C) \cap \rho\}$$

on $\{1, \ldots, |C|\}$. Then $\mathcal{A}_{\rho_C} \subseteq M_{|C|}$ is an SMA which is easily shown to be obtained from $\mathcal{A}_\rho$ by deleting all rows and columns not in C. Therefore, $\mathcal{A}_{\rho_C} \cong P_C\mathcal{A}_\rho$. Furthermore, each $\mathcal{A}_{\rho_C}$ is a central SMA (i.e. $Z(\mathcal{A}_{\rho_C})$ consists only of the scalar multiples of the identity) and there exists an algebra isomorphism

$$\mathcal{A}_\rho \cong \bigoplus_{C\in\mathcal{Q}} \mathcal{A}_{\rho_C}.$$

We refer to this fact as the *central decomposition* of $\mathcal{A}_\rho$.

Example 3.1.6. We illustrate the central decomposition on a concrete example. Consider the quasi-order

$$\rho := \Delta \cup \{(1,3),(1,6),(2,8),(2,9),(3,6),(4,3),(4,6),(5,7),(8,2),(8,9),(9,2),(9,8)\}$$

on $\{1, \ldots, 9\}$. The respective SMA is given by

$$\mathcal{A} = \begin{bmatrix} * & 0 & * & 0 & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & * & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & * & 0 & 0 & 0 & 0 & 0 & * & * \end{bmatrix} \subseteq M_9.$$

Direct computation shows that its centre is three-dimensional and given by

$$Z(\mathcal{A}) = \{\operatorname{diag}(a, c, a, a, b, a, b, c, c) : a, b, c \in \mathbb{C}\}.$$

The quotient set with respect to the equivalence relation $\approx$ is given by

$$\mathcal{Q} = \{\{1,3,4,6\}, \{2,8,9\}, \{5,7\}\}$$

and the respective central projections are given by

$$\begin{aligned} P_{\{1,3,4,6\}} &= \operatorname{diag}(1,0,1,1,0,1,0,0,0), \\ P_{\{2,8,9\}} &= \operatorname{diag}(0,1,0,0,0,0,0,1,1), \\ P_{\{5,7\}} &= \operatorname{diag}(0,0,0,0,1,0,1,0,0). \end{aligned}$$

The central summands are obtained by deleting respective rows and columns:

$$\mathcal{A}_{\{1,3,4,6\}} = \mathcal{A}_\rho^{\flat\{1,3,4,6\}^c} = \begin{bmatrix} * & * & 0 & * \\ 0 & * & 0 & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix},$$

$$\mathcal{A}_{\{2,8,9\}} = \mathcal{A}_\rho^{\flat\{2,8,9\}^c} = \mathcal{T}_2, \quad \mathcal{A}_{\{5,7\}} = \mathcal{A}_\rho^{\flat\{5,7\}^c} = M_3.$$

3.2 Intrinsic diagonalization

We now state our first preparatory result, regarding the intrinsic diagonalization of matrices within SMAs. Besides showcasing an interesting property of these algebras, it will be applied in an essential way in the description of Jordan embeddings between SMAs, as well as their Jordan automorphisms (Corollaries 4.2.13 and 4.2.14, respectively).

Theorem 3.2.1. *Let $\mathcal{A} = \mathcal{A}_\rho \subseteq M_n$ be an SMA and let $\mathcal{F} \subseteq \mathcal{A}$ be a commuting family of diagonalizable matrices. Then there exists $S \in \mathcal{A}^\times$ such that $S^{-1}\mathcal{F}S \subseteq \mathcal{D}_n$.*

Before proving Theorem 3.2.1 we consider a special case when $\mathcal{A}$ is contained in $\mathcal{T}_n$ and the family $\mathcal{F}$ consists of mutually orthogonal idempotents.

Lemma 3.2.2. *Suppose that $\mathcal{P} \subseteq \mathcal{T}_n$ is a family of mutually orthogonal nonzero idempotents such that $\sum_{P\in\mathcal{P}} P = I$. Define a matrix $T \in \mathcal{T}_n^\times$ with $T_{ij} := P_{ij}$ $(1 \le i \le j \le n)$, where $P \in \mathcal{P}$ is the unique idempotent such that $P_{jj} = 1$.*

(a) We have $T^{-1}\mathcal{P}T \subseteq \mathcal{D}_n$.

(b) If in addition $\mathcal{P} \subseteq \mathcal{A}_\rho$ for some SMA $\mathcal{A}_\rho \subseteq \mathcal{T}_n$, then $T \in \mathcal{A}_\rho^\times$.

Proof.

(a) Fix $P \in \mathcal{P}$ and let $D_P \in \mathcal{D}_n$ be the diagonal of P. Fix $1 \le i \le j \le n$ and let $Q \in \mathcal{P}$ be the unique idempotent such that $Q_{jj} = 1$. We have

$$(TD_P)_{ij} = \sum_{i\le k\le j} T_{ik}(D_P)_{kj} = T_{ij}P_{jj} = \begin{cases} P_{ij}, & \text{if } P_{jj} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, we have

$$\begin{aligned} (PT)_{ij} &= \sum_{i\le k\le j} P_{ik}T_{kj} = \sum_{i\le k<j} P_{ik}T_{kj} + P_{ij}T_{jj} = \sum_{i\le k<j} P_{ik}Q_{kj} + P_{ij} \\ &= (PQ)_{ij} - P_{ij}Q_{jj} + P_{ij} = (PQ)_{ij} = \begin{cases} P_{ij}, & \text{if } P_{jj} = 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

This closes the proof of $TD_P = PT$.

(b) The statement follows directly from

$$\operatorname{supp} T \subseteq \bigcup_{P\in\mathcal{P}} \operatorname{supp} P \subseteq \rho.$$

□

Remark 3.2.3. Let $A \in M_n$ is a diagonalizable matrix. By an elementary linear algebra argument, there exist a unique family $\{P_\lambda : \lambda \in \sigma(A)\}$ of mutually orthogonal nonzero idempotents (spectral idempotents of A) such that

$$A = \sum_{\lambda\in\sigma(A)} \lambda P_\lambda, \qquad \sum_{\lambda\in\sigma(A)} P_\lambda = I.$$

For each $\lambda \in \sigma(A)$ let $p_\lambda \in \mathbb{C}[x]$ be the unique polynomial of degree $< |\sigma(A)|$ such that

$$p_\lambda(x) = \begin{cases} 1, & \text{if } x = \lambda, \\ 0, & \text{if } x \in \sigma(A) \setminus \{\lambda\}. \end{cases}$$

Then it is well-known (and easy to verify) that $p_\lambda(A) = P_\lambda$ for all $\lambda \in \sigma(A)$. In particular, all spectral projections of a diagonalizable matrix $A \in M_n$ are contained in the unital subalgebra $\mathbb{C}[A]$ of M_n (generated by A). Therefore, any unital (Jordan) subalgebra $\mathcal{A}$ of M_n containing A also contains all spectral projections of A.

Proof of Theorem 3.2.1. By an elementary argument, it suffices to prove the statement when $\mathcal{F}$ is finite (otherwise we conduct the argument on the basis of the linear span of $\mathcal{F}$, which is again a finite commuting subset of $\mathcal{A}$ consisting of diagonalizable matrices). For simplicity, we can further assume that $|\mathcal{F}| = 2$, so that $\mathcal{F} = \{A, B\}$, as the general case follows analogously.

Case 1. We prove the statement when $\mathcal{A} \subseteq \mathcal{T}_n$.

Let $\{P_\lambda : \lambda \in \sigma(A)\}$ and $\{Q_\mu : \mu \in \sigma(B)\}$ be the families of spectral projections of A and B, respectively. By Remark 3.2.3, both families are in fact contained in $\mathcal{A}$.

Since $A \leftrightarrow B$, we have $P_\lambda \leftrightarrow Q_\mu$ for all $\lambda \in \sigma(A)$ and $\mu \in \sigma(B)$ (also by Remark 3.2.3). Using this, it is easy to show that

$$\mathcal{P} := \{P_\lambda Q_\mu : \lambda \in \sigma(A), \mu \in \sigma(B)\} \setminus \{0\} \subseteq \mathcal{A}$$

is a family of mutually orthogonal nonzero idempotents such that $\sum_{R\in\mathcal{P}} R = I$. Also, both A and B are linear combinations of elements of $\mathcal{P}$. We can apply Lemma 3.2.2 to $\mathcal{P}$ to obtain $T \in \mathcal{A}^\times$ which diagonalizes the entire family $\mathcal{P}$, and consequently A and B as well.

Case 2. We prove the statement when

$$\operatorname{diag}(M_{k_1}, \ldots, M_{k_p}) \subseteq \mathcal{A} \subseteq \mathcal{A}_{k_1,\ldots,k_p}$$

(where $\mathcal{A}_{k_1,\ldots,k_p}$ is the corresponding block upper-triangular algebra, as defined by (2.1.1)).

Proof. Denote by $X_1, Y_1 \in M_{k_1}, \ldots, X_p, Y_p \in M_{k_p}$ the diagonal blocks of A and B, respectively. Since $A \leftrightarrow B$, for each $1 \le j \le p$ we have $X_j \leftrightarrow Y_j$ so by the Schur triangularization (see e.g. [46, Theorem 40.5]) we can choose $U_j \in M_{k_j}^\times$ such that $U_j X_j U_j^{-1}, U_j Y_j U_j^{-1} \in \mathcal{T}_{k_j}$. Then

$$U := \operatorname{diag}(U_1, \dots, U_p) \in \operatorname{diag}(M_{k_1}, \dots, M_{k_p}) \subseteq \mathcal{A}$$

and

$$UAU^{-1}, UBU^{-1} \in \mathcal{A} \cap \mathcal{T}_n.$$

It follows that $\{UAU^{-1}, UBU^{-1}\}$ is a commuting family of diagonalizable matrices, so we can apply Case 1 on the SMA $\mathcal{A} \cap \mathcal{T}_n$ to obtain $S \in (\mathcal{A} \cap \mathcal{T}_n)^\times$ such that

$$UAU^{-1}, UBU^{-1} \in S\mathcal{D}_n S^{-1}$$

or equivalently $A, B \in R\mathcal{D}_n R^{-1}$, where $R := U^{-1}S \in \mathcal{A}^\times$. □

Case 3. Now assume that $\mathcal{A}$ is a general SMA.

Proof. From Lemma 3.1.4 we know that there exists a permutation matrix $R \in M_n^\times$ and a block upper-triangular subalgebra $\mathcal{A}_{k_1,\dots,k_p} \subseteq M_n$ such that the SMA $R\mathcal{A}R^{-1}$ satisfies

$$\operatorname{diag}(M_{k_1}, \dots, M_{k_p}) \subseteq R\mathcal{A}R^{-1} \subseteq \mathcal{A}_{k_1,\dots,k_p}.$$

Since $A, B \in \mathcal{A}$, we have $RAR^{-1}, RBR^{-1} \in R\mathcal{A}R^{-1}$ so by Case 2, there exists $S \in (R\mathcal{A}R^{-1})^\times = R\mathcal{A}^\times R^{-1}$ such that

$$RAR^{-1}, RBR^{-1} \in S\mathcal{D}_n S^{-1}.$$

It follows that

$$A, B \in (R^{-1}SR)R^{-1}\mathcal{D}_n R(R^{-1}SR)^{-1},$$

where $R^{-1}SR \in \mathcal{A}^\times$. The proof is now complete. □

□

Remark 3.2.4. Not all unital subalgebras $\mathcal{A} \subseteq M_n$ possess the intrinsic diagonalization property in the sense of Theorem 3.2.1. For instance, fix an arbitrary diagonalizable matrix $T \in M_n \setminus \mathcal{D}_n$ and let $\mathcal{A} := \mathbb{C}[T] \subseteq M_n$. Then $\mathcal{A}$ is a unital commutative subalgebra of M_n (its dimension is equal to the degree of the minimal polynomial of T), so trivially a matrix $A \in \mathcal{A}$ is of the form $A = SDS^{-1}$ for some $S \in \mathcal{A}^\times$ and $D \in \mathcal{D}_n$ if and only if A itself is diagonal. However, this is clearly not true for the matrix $T \in \mathcal{A}$.

Furthermore, let

$$T := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \in \mathcal{T}_3$$

and

$$\mathcal{A} := \mathbb{C}[T] = \left\{ \begin{bmatrix} x & 0 & \frac{1}{2}(z-x) \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} : x, y, z \in \mathbb{C} \right\} \subseteq \mathcal{T}_3.$$

One easily verifies that the algebra $\mathcal{A}$ is conjugated to $\mathcal{D}_3$ (e.g. $\mathcal{A} = (T + E_{11})\mathcal{D}_3(T + E_{11})^{-1}$). In particular, the property of being conjugated to an SMA is not sufficient to ensure the intrinsic diagonalization in the sense of Theorem 3.2.1.

In particular, every SMA is a subalgebra of a parabolic algebra up to conjugation with a permutation matrix (note that the diagonal blocks are always fully present).

CHAPTER 4

Jordan embeddings of structural matrix algebras

4.1 Algebra embeddings

Let ρ be a quasi-order on $[1,n]$. Following [17], we say that a map $g : \rho \to \mathbb{C}^\times$ is *transitive* if

$$g(i,j)g(j,k) = g(i,k), \qquad \text{for all } (i,j),(j,k) \in \rho.$$

Note that necessarily $g|_{\Delta_n} \equiv 1$, so it suffices to verify the above condition for $(i,j),(j,k) \in \rho^\times$. We say that a transitive map $g : \rho \to \mathbb{C}^\times$ is *trivial* if there exists a map $s : [1,n] \to \mathbb{C}^\times$ such that g *separates through* s, that is

$$g(i,j) = \frac{s(i)}{s(j)}, \qquad \text{for all } (i,j) \in \rho \tag{4.1.1}$$

(again, it suffices to stipulate this for all $(i,j) \in \rho^\times$).

Example 4.1.1. Consider the quasi-order ρ on $[1,4]$ given by

$$\rho := \Delta_4 \cup \{(1,3),(1,4),(2,3),(2,4)\}.$$

The map

$$g : \rho \to \mathbb{C}^\times, \qquad g(i,j) = \begin{cases} 1, & \text{if } (i,j) \neq (1,4), \\ 2, & \text{if } (i,j) = (1,4). \end{cases}$$

is easily shown to be a transitive map which is not trivial.

Remark 4.1.2. If a quasi-ordered set $([1,n],\rho)$ has a smallest or a largest element, then all transitive maps $g : \rho \to \mathbb{C}^\times$ are trivial. Indeed, suppose for example that $\rho^{-1}(k) = [1,n]$ for some $k \in [1,n]$ and let $g : \rho \to \mathbb{C}^\times$ be a transitive map. Define $s : [1,n] \to \mathbb{C}^\times$ by $s(i) := g(i,k)$. For each $(i,j) \in \rho$ we have

$$g(i,j)g(j,k) = g(i,k) \implies g(i,j)s(j) = s(i) \implies g(i,j) = \frac{s(i)}{s(j)}$$

which implies that g separates through s.

Every transitive map g induces an automorphism $g^* : \mathcal{A}_\rho \to \mathcal{A}_\rho$ given on the basis of matrix units as

$$g^*(E_{ij}) = g(i,j)E_{ij}, \qquad \text{for all } (i,j) \in \rho.$$

Triviality of a transitive map reflects on the induced map in a natural way:

Lemma 4.1.3. *[17, Lemma 4.10] Let $g : \rho \to \mathbb{C}^\times$ be a transitive map. Then g is trivial if and only if there exists $T \in M_n^\times$ such that the induced automorphism $g^* : \mathcal{A}_\rho \to \mathcal{A}_\rho$ is of the form $g^*(\cdot) = T(\cdot)T^{-1}$. In this case, T is in fact a diagonal matrix, so that g^* is an inner automorphism of $\mathcal{A}_\rho$.*

Proof. Suppose that g is trivial and separates through a map $s : [1, n] \to \mathbb{C}^\times$. Then one easily verifies that $D = \operatorname{diag}(s(1), \dots, s(n)) \in \mathcal{D}_n^\times$ satisfies $g^*(\cdot) = D(\cdot)D^{-1}$. Conversely, suppose that $g^*(\cdot) = T(\cdot)T^{-1}$ for some $T \in M_n^\times$. Then $\Delta_n = g^*(\Delta_n) = T\Delta_n T^{-1}$ so $T \leftrightarrow \Delta_n$ and hence $T \in \mathcal{D}_n$. Now $s : [1, n] \to \mathbb{C}^\times$ given by $s(i) := T_{ii}$ is easily seen to satisfy (4.1.1) so g is trivial. □

We now state the aforementioned description of all automorphisms of SMAs, which was first obtained by Coelho in [17, Theorem C] and later streamlined by Akkurt et al. as:

Theorem 4.1.4. *[2, Theorem 2.2 (Factorization Theorem)] Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA. A map $\phi : \mathcal{A}_\rho \to \mathcal{A}_\rho$ is an algebra automorphism if and only if there exists an invertible matrix $T \in \mathcal{A}_\rho^\times$, a transitive map $g : \rho \to \mathbb{C}^\times$ and a (ρ, ρ)-increasing permutation $\pi \in S_n$ such that*

$$\phi(\cdot) = (TR_\pi)g^*(\cdot)(TR_\pi)^{-1}.$$

Remark 4.1.5. In Theorem 4.1.4, the three parameters T, g, π which represent an automorphism $\phi : \mathcal{A}_\rho \to \mathcal{A}_\rho$ are in general not unique. In fact, there is a significant degree of redundancy, which is clear from Coelho's group theoretic formulation of this result ([17, Theorem C]). To illustrate the issue, consider the quasi-order

$$\rho := \Delta_3 \cup \{(1, 2), (1, 3), (2, 3), (3, 2)\}$$

and the corresponding SMA

$$\mathcal{A}_\rho = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

Let $\phi = \mathrm{id} : \mathcal{A}_\rho \to \mathcal{A}_\rho$ be the trivial automorphism. We have

$$\phi(\cdot) = (T_1 R_{\pi_1})g_1^*(\cdot)(T_1 R_{\pi_1})^{-1} = (T_2 R_{\pi_2})g_2^*(\cdot)(T_2 R_{\pi_2})^{-1},$$

where

- π_1 is the identity, $T_1 = I$ and $g_1 : \rho \to \mathbb{C}^\times$ is the constant map 1.
- π_2 is the transposition $2 \leftrightarrow 3$, $T_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $g_2 : \rho \to \mathbb{C}^\times$ is given by

$$g_2(i, j) = \begin{cases} 2, & \text{if } (i, j) \in \{(1, 2), (1, 3)\}, \\ 1, & \text{if } (i, j) \in \rho \setminus \{(1, 2), (1, 3)\}. \end{cases}$$

We explicitly state the following simple generalization of Theorem 4.1.4, as it motivates our later result regarding Jordan embeddings between SMAs (Corollary 4.2.13).

Corollary 4.1.6. *Let $\mathcal{A}_\rho, \mathcal{A}_{\rho'} \subseteq M_n$ be SMAs. Then $\mathcal{A}_\rho$ embeds (as an algebra) into $\mathcal{A}_{\rho'}$ if and only if there exists a (ρ,ρ')-increasing permutation $\pi \in S_n$. Furthermore, if $\phi : \mathcal{A}_\rho \to \mathcal{A}_{\rho'}$ is an algebra embedding, then there exists an invertible matrix $T \in \mathcal{A}_{\rho'}^\times$, a (ρ,ρ')-increasing permutation $\pi \in S_n$ and a transitive map $g : \rho \to \mathbb{C}^\times$ such that*

$$\phi(\cdot) = (TR_\pi)g^*(\cdot)(TR_\pi)^{-1}.$$

Proof. If there exists a (ρ,ρ')-increasing permutation $\pi \in S_n$ then, following the discussion around (3.1.1), the map $R_\pi(\cdot)R_\pi^{-1}$ defines the algebra embedding $\mathcal{A}_\rho \to \mathcal{A}_{\rho'}$.

Conversely, assume that $\phi : \mathcal{A}_\rho \to \mathcal{A}_{\rho'}$ is an algebra embedding. Note that

$$\{\phi(E_{11}), \dots, \phi(E_{nn})\} \subseteq \mathcal{A}_{\rho'}$$

is a set of mutually orthogonal idempotents so by Theorem 3.2.1 (in fact, Lemma 3.2.2 suffices) there exists $S \in \mathcal{A}_{\rho'}^\times$ and a permutation $\pi \in S_n$ such that

$$\phi(E_{ii}) = SE_{\pi(i)\pi(i)}S^{-1} = (SR_\pi)E_{ii}(SR_\pi)^{-1}, \qquad \text{for all } 1 \le i \le n.$$

For each $(i,j) \in \rho^\times$ we have

$$\phi(E_{ij}) = \phi(E_{ii}E_{ij}E_{jj}) = ((SR_\pi)E_{ii}(SR_\pi)^{-1})\phi(E_{ij})((SR_\pi)E_{jj}(SR_\pi)^{-1})$$

and hence $\phi(E_{ij}) = ((SR_\pi)(g(i,j)E_{ij}))(SR_\pi)^{-1})$ for some nonzero scalar $g(i,j) \in \mathbb{C}^\times$. Multiplicativity of ϕ directly implies that the map

$$g : \rho \to \mathbb{C}^\times, \qquad (i,j) \mapsto \begin{cases} g(i,j), & \text{if } (i,j) \in \rho^\times, \\ 1, & \text{if } i = j \end{cases}$$

is transitive. We conclude $\phi = (SR_\pi)g^*(\cdot)(SR_\pi)^{-1}$. For all $X \in \mathcal{A}_\rho$ we have

$$R_\pi g^*(X)R_\pi^{-1} = (S^{-1}\phi(X)S) \in \mathcal{A}_{\rho'}.$$

By surjectivity of g^*, it follows that $R_\pi \mathcal{A}_\rho R_\pi^{-1} \subseteq \mathcal{A}_{\rho'}$ so π is (ρ,ρ')-increasing. □

We showcase an interesting consequence of Corollary 4.1.6 in the particular case of block upper-triangular algebras:

Proposition 4.1.7. *Let $\mathcal{A}, \mathcal{B} \subseteq M_n$ be block upper-triangular subalgebras. Then $\mathcal{A}$ algebra-embeds into $\mathcal{B}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$.*

Proof. Denote $\mathcal{A} = \mathcal{A}_\rho = \mathcal{A}_{k_1,\dots,k_p}$ and $\mathcal{B} = \mathcal{A}_{\rho'} = \mathcal{A}_{l_1,\dots,l_q}$. By Corollary 4.1.6, we know that $\mathcal{A}$ algebra-embeds into $\mathcal{B}$ if and only if there exists a (ρ,ρ')-increasing permutation $\pi \in S_n$. We claim that the latter is equivalent to $\rho \subseteq \rho'$. Since the $\boxed{\Longleftarrow}$ direction is trivial, we focus on $\boxed{\Longrightarrow}$ which we show via induction on n. If $n = 1$, then clearly $\mathcal{A} = \mathcal{B} = M_1$. Suppose therefore that $n \ge 2$, that the statement holds for all pairs of block upper-triangular algebras in M_{n-1}, and that $\pi \in S_n$ is (ρ,ρ')-increasing. Notice that for each $j \in [k_1 + \cdots + k_{p-1} + 1, n]$ we have

$$\rho^{-1}(j) = [1,n] \implies (\rho')^{-1}(\pi(j)) = [1,n] \implies \pi(j) \in [l_1 + \ldots + l_{q-1} + 1, n]. \tag{4.1.2}$$

Let $\sigma \in S_n$ be the transposition $n \leftrightarrow \pi(n)$. We claim that the composition $\sigma \circ \pi \in S_n$ is also

(ρ, ρ')-increasing. Indeed, suppose that $(i, j) \in \rho$. We wish to show $(\sigma(\pi(i)), \sigma(\pi(j))) \in \rho'$. Clearly, we only need to examine the cases where i or j is in $\{n, \pi^{-1}(n)\}$.

- Suppose $j \in \{n, \pi^{-1}(n)\}$. Then $\sigma(\pi(n)) \in \{n, \pi(n)\}$ and $(\rho')^{-1}(n) = (\rho')^{-1}(\pi(n)) = [1, n]$ so $(\sigma(\pi(i)), \sigma(\pi(j))) \in \rho'$ no matter what $\sigma(\pi(i))$ is.
- Suppose $i \in \{n, \pi^{-1}(n)\}$. Then $(\pi(i), \pi(j)) \in \rho'$ and $\pi(i) \in \{n, \pi(n)\} \subseteq [l_1 + \ldots + l_{q-1} + 1, n]$ together imply $\pi(j) \in [l_1 + \ldots + l_{q-1} + 1, n]$ as well. Therefore, $\sigma(\pi(j))$ belongs to the same interval so, since $\sigma(\pi(i)) \in \{n, \pi(n)\}$, it follows

$$(\sigma(\pi(i)), \sigma(\pi(j))) \in [l_1 + \ldots + l_{q-1} + 1, n]^2 \subseteq \rho'.$$

Now, notice that $(\sigma \circ \pi)(n) = n$ so $(\sigma \circ \pi)|_{[1,n-1]} \in S_{n-1}$ is a $(\rho \cap [1, n-1]^2, \rho' \cap [1, n-1]^2)$-increasing permutation. Since the quasi-orders $\rho \cap [1, n-1]^2$ and $\rho' \cap [1, n-1]^2$ clearly again correspond to block upper-triangular subalgebras, by the induction hypothesis we conclude $\rho \cap [1, n-1]^2 \subseteq \rho' \cap [1, n-1]^2$. Now (4.1.2) allows us to conclude $\rho \subseteq \rho'$. □

To end this subsection, we also mention Coelho's characterization of SMAs which admit only inner automorphisms.

Theorem 4.1.8. *[17, Theorem D] Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA. Then every automorphism $\mathcal{A}_\rho \to \mathcal{A}_\rho$ is inner if and only if*

(i) every transitive map $g : \rho \to \mathbb{C}^\times$ is trivial,

(ii) every quasi-order automorphism π of ρ fixes the equivalence classes π of $\overline{\rho}$ (this is equivalent to $R_\pi \in \mathcal{A}_\rho^\times$ by Lemma 3.1.2).

Proof. $\boxed{\Longrightarrow}$ (i) follows from Lemma 4.1.3. To prove (ii), let π be a quasi-order automorphism of ρ. Then $R_\pi(\cdot)R_\pi^{-1} \in \operatorname{Aut}(\mathcal{A}_\rho)$ so by the assumption, there exists $T \in \mathcal{A}_\rho^\times$ such that $R_\pi(\cdot)R_\pi^{-1} = T(\cdot)T^{-1}$. Lemma 2.3.1 implies that $R_\pi \in T\mathcal{D}_n \subseteq \mathcal{A}_\rho$ which is the desired conclusion.

$\boxed{\Longleftarrow}$ Let $\phi \in \operatorname{Aut}(\mathcal{A}_\rho)$ be arbitrary. By Theorem 4.1.4, there exist an invertible matrix $A \in \mathcal{A}_\rho^\times$, a transitive map $g : \rho \to \mathbb{C}^\times$ and a (ρ, ρ)-increasing permutation $\pi \in S_n$ such that

$$\phi(\cdot) = Ag^*(R_\pi(\cdot)R_\pi^{-1})A^{-1}.$$

By (i) and Lemma 4.1.3, there exists a diagonal matrix $D \in \mathcal{D}_n^\times$ such that $g^*(\cdot) = D(\cdot)D^{-1}$. By (ii), we have $R_\pi \in \mathcal{A}_\rho^\times$. It follows that

$$\phi(\cdot) = \underbrace{(ADR_\pi)}_{\in \mathcal{A}_\rho^\times}(\cdot)(ADR_\pi)^{-1}$$

so ϕ is an inner automorphism. □

4.2 Jordan embeddings

Now we are ready to describe the general form of Jordan embeddings $\phi : \mathcal{A} \to \mathcal{B}$ between SMAs $\mathcal{A}$ and $\mathcal{B}$ of M_n. As expected, the transitive maps will play a role of similar importance in the description of all Jordan embeddings and rank(-one) preservers $\mathcal{A} \to M_n$. Their appearance displays the relative complexity of (Jordan) algebraic and preserver theory on SMAs when compared to M_n, $\mathcal{T}_n$ or the block upper-triangular subalgebras (see [24]). On the other hand, permutation matrices appear only when we restrict the codomain to $\mathcal{B}$, i.e. in the description of Jordan embeddings $\mathcal{A} \to \mathcal{B}$.

We first start with the special case when $\mathcal{B} = M_n$.

Lemma 4.2.1. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA and let $\phi : \mathcal{A}_\rho \to M_n$ be a Jordan homomorphism such that $\phi(E_{ij}) \neq 0$ for all $(i,j) \in \rho$. Then there exists an invertible matrix $S \in M_n^\times$ such that $\phi(D) = SDS^{-1}$ for all $D \in \mathcal{D}_n$.*

Proof. By Lemma 2.4.2 (c) we conclude that $\phi(E_{11}), \dots, \phi(E_{nn})$ is a family of mutually orthogonal nonzero idempotents. Therefore, there exists $S \in M_n^\times$ such that

$$\phi(E_{kk}) = SE_{kk}S^{-1}, \qquad 1 \leq k \leq n.$$

The claim follows by linearity. □

Lemma 4.2.2. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA and let $\phi : \mathcal{A}_\rho \to M_n$ be a Jordan homomorphism such that $\phi(E_{ij}) \neq 0$ for all $(i,j) \in \rho$ and $\phi|_{\mathcal{D}_n}$ is the identity. Then for every $(i,j) \in \rho^\times$ there exist scalars $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$, exactly one of which is zero, such that $\phi(E_{ij}) = \alpha_{ij}E_{ij} + \beta_{ij}E_{ji}$.*

Proof. By Lemma 2.4.2 (b) we have

$$\phi(E_{ij}) = \phi(E_{ii}E_{ij}E_{jj} + E_{jj}E_{ij}E_{ii}) = E_{ii}\phi(E_{ij})E_{jj} + E_{jj}\phi(E_{ij})E_{ii}.$$

Therefore, $\phi(E_{ij})$ is supported in $\{(i,j),(j,i)\}$ so there exist scalars $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$ such that

$$\phi(E_{ij}) = \alpha_{ij}E_{ij} + \beta_{ij}E_{ji}.$$

Furthermore, we have

$$0 = \phi(E_{ij}^2) = \phi(E_{ij})^2 = \alpha_{ij}\beta_{ij}(E_{ii} + E_{jj})$$

so exactly one of α_{ij} and β_{ij} is equal to zero 0 (as $\phi(E_{ij}) \neq 0$ for all $(i,j) \in \rho$, by the assumption). □

Lemma 4.2.3. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA and let $\phi : \mathcal{A}_\rho \to M_n$ be a Jordan homomorphism such that $\phi(E_{ij}) \neq 0$ for all $(i,j) \in \rho$ and $\phi|_{\mathcal{D}_n}$ is the identity. Define*

$$\rho_M^\phi := \{(i,j) \in \rho : \phi(E_{ij}) \parallel E_{ij}\}, \qquad \rho_A^\phi := \{(i,j) \in \rho : \phi(E_{ij}) \parallel E_{ji}\}.$$

We have

(a) $\rho_M^\phi \cup \rho_A^\phi = \rho$ and $\rho_M^\phi \cap \rho_A^\phi = \Delta_n$.

(b) ρ_M^ϕ and ρ_A^ϕ are quasi-orders on $[1,n]$ and there exist transitive maps $g : \rho_M^\phi \to \mathbb{C}^\times$ and $h : \rho_A^\phi \to \mathbb{C}^\times$ such that the restrictions

$$\phi|_{\mathcal{A}_{\rho_M^\phi}} : \mathcal{A}_{\rho_M^\phi} \to M_n, \qquad \phi|_{\mathcal{A}_{\rho_A^\phi}} : \mathcal{A}_{\rho_A^\phi} \to M_n$$

are equal to $g^*(\cdot)$ and $h^*(\cdot)^t$, respectively. In particular, the maps $\phi|_{\mathcal{A}_{\rho_M^\phi}}$ and $\phi|_{\mathcal{A}_{\rho_A^\phi}}$ are multiplicative and antimultiplicative, respectively.

(c) Suppose $(i,j) \in \rho^\times$. Then

$$\rho(i) \cup \rho^{-1}(i) \cup \rho(j) \cup \rho^{-1}(j) \subseteq \rho_M^\phi \qquad \text{or} \qquad \rho(i) \cup \rho^{-1}(i) \cup \rho(j) \cup \rho^{-1}(j) \subseteq \rho_A^\phi.$$

(d) Let $P \in \mathcal{D}_n$ be a diagonal idempotent defined by

$$P_{ii} = 1 \iff \text{there exists } j \in [1,n] \setminus \{i\} \text{ such that } (i,j) \in \rho_M^\phi \text{ or } (j,i) \in \rho_M^\phi.$$

Then $P \in Z(\mathcal{A}_\rho)$, $PX \in \mathcal{A}_{\rho_M^\phi}$ and $(I-P)X \in \mathcal{A}_{\rho_A^\phi}$ for all $X \in \mathcal{A}_\rho$.

(e) Suppose that $(i,j),(j,k) \in \rho^\times$. Then either $(i,j),(j,k) \in \rho_M^\phi$ or $(i,j),(j,k) \in \rho_A^\phi$.

Proof. Following the notation from Lemma 4.2.2, throughout the proof for each $(i,j) \in \rho_M^\phi$ by $\alpha_{ij} \in \mathbb{C}^\times$ we denote the unique nonzero scalar such that $\phi(E_{ij}) = \alpha_{ij}E_{ij}$, while for each $(i,j) \in \rho_A$ by $\beta_{ij} \in \mathbb{C}^\times$ we denote the unique nonzero scalar such that $\phi(E_{ij}) = \beta_{ij}E_{ji}$.

(a) Clearly, for all $1 \le i \le n$ we have $\phi(E_{ii}) = E_{ii}$ so (i,i) is contained in both ρ_M^ϕ and ρ_A^ϕ. It follows that $\Delta_n \subseteq \rho_M^\phi \cap \rho_A^\phi$. On the other hand, let $(i,j) \in \rho^\times$ be arbitrary. Lemma 4.2.2 directly implies that either $(i,j) \in \rho_M^\phi$ or $(i,j) \in \rho_A^\phi$. We conclude $\rho_M^\phi \cup \rho_A^\phi = \rho$ and $\rho_M^\phi \cap \rho_A^\phi \subseteq \Delta_n$.

(b) We first prove that ρ_M^ϕ is a quasi-order and that there exists a transitive map $g : \rho_M^\phi \to \mathbb{C}^\times$ such that $\phi|_{\mathcal{A}_{\rho_M^\phi}} = g^*(\cdot)$. The reflexivity of ρ_M^ϕ follows from (a). Suppose that $(i,j),(j,k) \in \rho_M^\phi$. We show $(i,k) \in \rho_M^\phi$ and $\alpha_{ik} = \alpha_{ij}\alpha_{jk}$. Both of these are clear if $i = j$ or $j = k$ so we can further assume $(i,j),(j,k) \in (\rho_M^\phi)^\times$. We have

$$\begin{aligned}\phi(E_{ik}) + \delta_{ki}E_{jj} &= \phi(E_{ik} + \delta_{ki}E_{jj}) \\ &= \phi(E_{ij}E_{jk} + E_{jk}E_{ij}) = \phi(E_{ij})\phi(E_{jk}) + \phi(E_{jk})\phi(E_{ij}) \\ &= \alpha_{ij}\alpha_{jk}(E_{ik} + \delta_{ki}E_{jj}),\end{aligned}$$

where, as usual, δ_{ij} denotes the Kronecker delta symbol. If $i = k$, then $(i,i) \in \rho_M^\phi$ is trivial and the above relation reduces to

$$E_{ii} + E_{jj} = \alpha_{ij}\alpha_{ji}(E_{ii} + E_{jj})$$

which implies $\alpha_{ij}\alpha_{ji} = 1$. On the other hand, if $i \ne k$, then the above relation reduces to

$$\phi(E_{ik}) = \alpha_{ij}\alpha_{jk}E_{ik}$$

which first implies $(i,k) \in \rho_M^\phi$ and then $\alpha_{ik} = \alpha_{ij}\alpha_{jk}$. It follows directly that the map $g : \rho_M^\phi \to \mathbb{C}^\times, g(i,j) := \alpha_{ij}$ is transitive and $\phi|_{\mathcal{A}_{\rho_M^\phi}} = g^*(\cdot)$.

To show the second claim, consider the map $\phi^t : \mathcal{A}_\rho \to M_n$, given by $X \mapsto \phi(X)^t$. Obviously, ϕ^t satisfies the same properties as ϕ and $\rho_A^\phi = \rho_M^{\phi^t}$. By the first part of the proof it follows that ρ_A^ϕ is a quasi-order and that there exists a transitive map $h : \rho_M^{\phi^t} \to \mathbb{C}^\times$ such that $\phi^t|_{\mathcal{A}_{\rho_M^{\phi^t}}} = h^*(\cdot)$. Then clearly $\phi|_{\mathcal{A}_{\rho_A^\phi}} = h^*(\cdot)^t$.

(c) For concreteness assume that $(i,j) \in \rho_M^\phi$. Let $(j,k) \in \rho^\times$ be arbitrary. We claim that $(j,k) \in \rho_M^\phi$. Assume the contrary, that $(j,k) \in \rho_A^\phi$. Then we have

$$\begin{aligned}\phi(E_{ik}) + \delta_{ik}E_{jj} &= \phi(E_{ik} + \delta_{ik}E_{jj}) \\ &= \phi(E_{ij}E_{jk} + E_{jk}E_{ij}) \\ &= \phi(E_{ij})\phi(E_{jk}) + \phi(E_{jk})\phi(E_{ij}) \\ &= \alpha_{ij}\beta_{jk}(E_{ij}E_{kj} + E_{kj}E_{ij}) \\ &= 0\end{aligned}$$

which is a contradiction. Therefore, $(j,k) \in \rho_M^\phi$, and consequently $(i,k) \in \rho_M^\phi$ by (b). It follows that $\rho(i) \cup \rho(j) \subseteq \rho_M^\phi$. The proof of $\rho^{-1}(i) \cup \rho^{-1}(j) \subseteq \rho_M^\phi$ is analogous.

(d) It suffices to show that for all $(i,j) \in \rho$ we have $P \leftrightarrow E_{ij}$, $PE_{ij} \in \mathcal{A}_{\rho_M^\phi}$ and $(I-P)E_{ij} \in \mathcal{A}_{\rho_A^\phi}$. Since all three claims are trivially true when $i=j$, fix $(i,j) \in \rho^\times$. By (a), we consider two cases:
- If $(i,j) \in \rho_M^\phi$, then $P_{ii} = P_{jj} = 1$ by definition, so
$$PE_{ij} = \underbrace{E_{ij}}_{\in \mathcal{A}_{\rho_M^\phi}} = E_{ij}P \implies (I-P)E_{ij} = 0 \in \mathcal{A}_{\rho_A^\phi}$$
which establishes all three claims.
- If $(i,j) \in \rho_A^\phi$, then $P_{ii} = P_{jj} = 0$ as a consequence of (c), so
$$PE_{ij} = \underbrace{0}_{\in \mathcal{A}_{\rho_M^\phi}} = E_{ij}P \implies (I-P)E_{ij} = E_{ij} \in \mathcal{A}_{\rho_A^\phi}$$
which establishes all three claims.

(e) This is a direct consequence of (c).

□

Theorem 4.2.4. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA and let $\phi : \mathcal{A}_\rho \to M_n$ be a Jordan homomorphism such that $\phi(E_{ij}) \neq 0$ for all $(i,j) \in \rho$. Then there exists an invertible matrix $S \in M_n^\times$, a central idempotent $P \in Z(\mathcal{A}_\rho)$, and a transitive map $g : \rho \to \mathbb{C}^\times$ such that*
$$\phi(\cdot) = S(Pg^*(\cdot) + (I-P)g^*(\cdot)^t)S^{-1}.$$
In particular, ϕ is injective (i.e. a Jordan embedding).

Proof. By Lemma 4.2.1 there exists $S \in M_n^\times$ such that $\phi(D) = SDS^{-1}$ for all $D \in \mathcal{D}_n$. By passing onto the map $S^{-1}\phi(\cdot)S$ which satisfies the same properties, without loss of generality we assume that $\phi|_{\mathcal{D}_n}$ is the identity.

Let $g : \rho_M^\phi \to \mathbb{C}^\times$ and $h : \rho_A^\phi \to \mathbb{C}^\times$ be transitive maps from Lemma 4.2.2 (b). Define a map
$$f : \rho \to \mathbb{C}^\times, \qquad f(i,j) = \begin{cases} g(i,j), & \text{if } (i,j) \in \rho_M^\phi, \\ h(i,j), & \text{if } (i,j) \in \rho_A^\phi. \end{cases}$$
Lemma 4.2.2 (d) directly implies that f is a transitive map. Clearly, $f^*|_{\mathcal{A}_{\rho_M^\phi}} = g^*$ and $f^*|_{\mathcal{A}_{\rho_A^\phi}} = h^*$.

Let $P \in Z(\mathcal{A}_\rho)$ be the central idempotent defined in Lemma 4.2.2 (c). As $PX \in \mathcal{A}_{\rho_M^\phi}$ and $(I-P)X \in \mathcal{A}_{\rho_A^\phi}$ for all for $X \in \mathcal{A}_\rho$, we have
$$\begin{aligned} \phi(X) &= \phi(PX) + \phi((I-P)X) = g^*(PX) + h^*((I-P)X)^t \\ &= f^*(PX) + f^*((I-P)X)^t = f^*(P)f^*(X) + f^*(X)^t f^*(I-P)^t \\ &= Pf^*(X) + (I-P)f^*(X)^t. \end{aligned}$$

It remains to show the injectivity of ϕ. Let $X \in \mathcal{A}_\rho$ be a matrix such that $\phi(X) = 0$.

By left-multiplying the expression

$$0 = \phi(X) = Pf^*(X) + (I-P)f^*(X)^t$$

by P and $I-P$ respectively, we conclude $Pf^*(X) = (I-P)f^*(X)^t = 0$. Since $I-P \in Z(\mathcal{A}_\rho) \subseteq \mathcal{D}_n$, the latter equality can be transposed to yield $(I-P)f^*(X) = 0$. Overall, we obtain

$$0 = Pf^*(X) + (I-P)f^*(X) = f^*(X),$$

which implies $X = 0$ by the injectivity of f^*. We conclude that ϕ is injective. □

Corollary 4.2.5. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA. The following conditions are equivalent:*

(i) Every Jordan embedding $\mathcal{A}_\rho \to M_n$ is multiplicative or antimultiplicative.

(ii) The quotient set $\mathcal{Q}$ defined by (3.1.4) contains at most one class $C \in \mathcal{Q}$ with $|C| \geq 2$.

Proof. $\boxed{(i) \implies (ii)}$ We prove the contrapositive. Suppose that there exist $C_1, C_2 \in \mathcal{Q}$, $C_1 \neq C_2$ such that $|C_1|, |C_2| \geq 2$. Let

$$P := \sum_{i \in C_1} E_{ii} \in \mathcal{D}_n.$$

By Remark 3.1.5, P is a central projection and therefore, the map

$$\phi : \mathcal{A}_\rho \to M_n, \qquad \phi(\cdot) = P(\cdot) + (I-P)(\cdot)^t$$

is a Jordan embedding which is neither multiplicative nor antimultiplicative. Indeed, choose some $i, j \in C_2$ such that $(i,j) \in \rho^\times$. Then

$$\phi(E_{ij}E_{jj}) = \phi(E_{ij}) = E_{ji} \neq 0 = E_{ji}E_{jj} = \phi(E_{ij})\phi(E_{jj})$$

shows that ϕ is not multiplicative. That ϕ is not antimultiplicative can be shown in a similar way by choosing elements of C_1.

$\boxed{(ii) \implies (i)}$ Suppose that there exists $C \in \mathcal{Q}$, $|C| \geq 1$ such that

$$\mathcal{Q} = \{C\} \cup \{\{i\} : i \in C^c\}.$$

Let $\phi : \mathcal{A}_\rho \to M_n$ be a Jordan embedding. By Theorem 4.2.4 there exists an invertible matrix $S \in M_n^\times$, a central idempotent $P \in Z(\mathcal{A}_\rho)$, and a transitive map $g : \rho \to \mathbb{C}^\times$ such that

$$\phi(\cdot) = S(Pg^*(\cdot) + (I-P)g^*(\cdot)^t)S^{-1}.$$

Notice that for all $i \in [1,n] \setminus C$ we have

$$E_{ii}X = E_{ii}X^t, \qquad \text{for all } X \in \mathcal{A}_\rho.$$

Recall from Remark 3.1.5 that both P and $I-P$ are sums of E_{ii} for $i \in [1,n] \setminus C$ and $P_C = \sum_{j \in C} E_{jj}$. Therefore, for C and each $i \in C^c$ there is a map $(\cdot)^\circ_C, (\cdot)^{\circ_i} \in \{\mathrm{id}, (\cdot)^t\}$ such that

$$\phi(\cdot) = S\left(\sum_{i \in C^c} E_{ii}g^*(\cdot)^{\circ_i} + P_C g^*(\cdot)^{\circ_C}\right)S^{-1}$$

$$= S\left(\sum_{i\in C^c} E_{ii} g^*(\cdot)^{\circ_C} + P_C g^*(\cdot)^{\circ_C}\right) S^{-1}$$
$$= S g^*(\cdot)^{\circ_C} S^{-1}.$$

We conclude that ϕ is multiplicative or antimultiplicative (depending on whether $\circ_C$ is the identity map or the transposition map). □

Remark 4.2.6. Note that if $\phi|_{\mathcal{D}_n}$ is the identity, then the invertible matrix S from Theorem 4.2.4 is necessarily diagonal and hence the conjugation $S(\cdot)S^{-1}$ can be absorbed into the induced map g^*.

Now we discuss how the central decomposition from Remark 3.1.5 extends to Jordan homomorphisms.

Proposition 4.2.7. *Let $\mathcal{A}_\rho \subseteq M_n$ be a SMA and let $\phi : \mathcal{A}_\rho \to M_n$ be a Jordan homomorphism such that $\phi(E_{ij}) \neq 0$ for all $(i, j) \in \rho$ and such that $\phi|_{\mathcal{D}_n}$ is the identity map. For each $C \in \mathcal{Q}$ the map*

$$\phi_C : \mathcal{A}_C \to M_{|C|}, \qquad \phi_C(X) = \phi(X^{\sharp C^c})^{\flat C^c}$$

is a Jordan embedding on a central SMA and hence is of the form $g^*(\cdot)^\circ$ for some transitive map $g : \rho_C \to \mathbb{C}^\times$ and a map $\circ \in \{\mathrm{id}, \cdot^t\}$. Furthermore, for all $X \in \mathcal{A}_\rho$ we have

$$\phi(X) = \sum_{C\in\mathcal{Q}} \phi_C(X^{\flat C^c})^{\sharp C^c}. \tag{4.2.1}$$

Proof. Note that ϕ_C is obtained as the composition of maps

$$\mathcal{A}_C \to P_C\mathcal{A}_\rho, \qquad X \mapsto X^{\sharp C^c},$$

$$P_C\mathcal{A}_\rho \to P_C M_n P_C, \qquad X \mapsto \phi(X),$$

and

$$P_C M_n P_C \to M_{|C|}, \qquad X \mapsto X^{\flat C^c}.$$

The first and the last map are multiplicative isomorphisms, while the second one is multiplicative or antimultiplicative. It is well defined since for all $X \in \mathcal{A}_\rho$ we have

$$\phi(P_C X P_C) = \phi(P_C(P_C X P_C)P_C) \overset{\text{Lemma 2.4.2 (a)}}{=} \phi(P_C)\phi(P_C X P_C)\phi(P_C)$$
$$= P_C \phi(P_C X P_C) P_C.$$

Let $X \in \mathcal{A}_\rho$. We have

$$\phi(X) = \sum_{C\in\mathcal{Q}} \phi(P_C X P_C)$$
$$= \sum_{C\in\mathcal{Q}} \phi((X^{\flat C^c})^{\sharp C^c})$$
$$= \sum_{C\in\mathcal{Q}} \phi_C(X^{\flat C^c})^{\sharp C^c}.$$

This proves (4.2.1). □

Remark 4.2.8. Let $\mathcal{A}_\rho \subseteq M_n$ be a SMA and let $\phi : \mathcal{A}_\rho \to M_n$ be a Jordan homomorphism such that $\phi(E_{ij}) \neq 0$ for all $(i,j) \in \rho$. By Lemma 4.2.1 there exists $S \in M_n^\times$ such that $S^{-1}\phi(\cdot)S$ satisfies the same properties and $S^{-1}\phi(\cdot)S|_{\mathcal{D}_n}$ is the identity. Then Proposition 4.2.7 applies to this new map. We obtain that for each $C \in \mathcal{Q}$ there exists a transitive map $g_C : \rho_C \to \mathbb{C}^\times$ and a map $\circ_C \in \{\mathrm{id}, \cdot^t\}$ such that for all $X \in \mathcal{A}_\rho$ we have

$$\phi(X) = S\left(\sum_{C\in\mathcal{Q}} \left((g_C)^*(X^{\flat C^c})^{\circ_C}\right)^{\sharp C^c}\right)S^{-1}, \qquad \text{for all } X \in \mathcal{A}_\rho.$$

We can define a global $g : \rho \to \mathbb{C}^\times$ by setting

$$g(i,j) = g_C(\pi_C(i), \pi_C(j)) \qquad \text{for all } (i,j) \in \rho \cap (C \times C)$$

where π_C is taken from Remark 3.1.5. It is easy to check that g is a transitive map. The induced map $g^* : \mathcal{A}_\rho \to M_n$ satisfies $(g_C)^* = (g^*)_C$ in the sense of Proposition 4.2.7. Indeed, for $C \in \mathcal{Q}$ and $(i,j) \in \rho \cap (C \times C)$ we have

$$(g^*)_C(E_{\pi_C(i)\pi_C(j)}) = g^*(E^{\sharp C^c}_{\pi_C(i)\pi_C(j)})^{\flat C^c} = g^*(E_{ij})^{\flat C^c} = g(i,j)E_{\pi_C(i)\pi_C(j)}.$$

In particular, for all $X \in \mathcal{A}_\rho$ we have

$$g^*(P_C X) = g_C^*(X^{\flat C^c})^{\sharp C^c}.$$

Furthermore, it is easy to verify that the map $(\cdot)^{\sharp C^c}$ commutes with transposition so for all $X \in \mathcal{A}_\rho$ we obtain

$$\begin{aligned}
\phi(X) &= S\left(\sum_{C\in\mathcal{Q}} \left((g_C)^*(X^{\flat C^c})^{\circ_C}\right)^{\sharp C^c}\right)S^{-1} \\
&= S\left(\sum_{C\in\mathcal{Q}} \left((g_C)^*(X^{\flat C^c})^{\sharp C^c}\right)^{\circ_C}\right)S^{-1} \\
&= S\left(\sum_{C\in\mathcal{Q}} g^*(P_C X)^{\circ_C}\right)S^{-1} \\
&= S\left(\sum_{C\in\mathcal{Q}} (P_C g^*(X))^{\circ_C}\right)S^{-1} \\
&= S\left(\sum_{C\in\mathcal{Q}} P_C g^*(X)^{\circ_C}\right)S^{-1}.
\end{aligned}$$

We can now reproduce the form from Theorem 4.2.4 by setting

$$P := \sum_{\substack{C\in\mathcal{Q} \text{ such} \\ \text{that } \circ_C = \mathrm{id}}} P_C \in \mathcal{D}_n$$

and noticing that P is central, by Remark 3.1.5. This allows us to obtain the simplest form for ϕ. Indeed, for all $X \in \mathcal{A}_\rho$ we have:

$$\begin{aligned}
\phi(X) &= S\left(\sum_{C\in\mathcal{Q}} P_C g^*(X)^{\circ_C}\right) S^{-1} \\
&= S\left(\left(\sum_{\substack{C\in\mathcal{Q} \text{ such} \\ \text{that } \circ_C=\mathrm{id}}} P_C\right) g^*(X) + \left(\sum_{\substack{C\in\mathcal{Q} \text{ such} \\ \text{that } \circ_C=\cdot^t}} P_C\right) g^*(X)^t\right) S^{-1} \\
&= S(Pg^*(X) + (I-P)g^*(X)^t)S^{-1}.
\end{aligned}$$

If $\mathcal{A}$ is an SMA which satisfies the properties stated in Corollary 4.2.5, we shall say it satisfies the **multiplicative–antimultiplicative property (MAMP)**. From the proof of Theorem 4.2.4, it can also be seen that if $\mathcal{A} \subseteq M_n$ is a SMA satisfying MAMP, then every Jordan embedding $\phi : \mathcal{A} \to M_n$ is precisely of the form

$$\phi(\cdot) = Tg^*(\cdot)T^{-1}, \qquad \text{or} \qquad \phi(\cdot) = Tg^*(\cdot)^t T^{-1}$$

for some invertible matrix $T \in M_n^\times$ and a transitive map $g : \rho \to \mathbb{C}^\times$.

Remark 4.2.9. Let ρ be a quasi-order on $[1, n]$. We define a relation $\sim_0$ on $\rho^\times$ as

$$(i,j) \sim_0 (k,l) \overset{\text{def}}{\iff} \{i,j\} \cap \{k,l\} \neq \emptyset.$$

Clearly, $\sim_0$ is reflexive and symmetric. Let $\sim$ be the transitive closure of $\sim_0$; then $\sim$ is an equivalence relation. Then the SMA $\mathcal{A}_\rho$ satisfies MAMP if and only $\sim$ posseses only a single equivalence class. Indeed, in view of Corollary 4.2.5, it suffices to show that there is at most one class $C \in \mathcal{Q}$ with $|C| \geq 2$ if and only if $|\rho^\times / \sim| = 1$.

$\boxed{\implies}$ Let $C \in \mathcal{Q}$ be the only class with $|C| \geq 2$. Let $(i,j), (k,l) \in \rho^\times$ be arbitrary; we wish to prove $(i,j) \sim (k,l)$. We necessarily have $i,j,k,l \in C$ and therefore $i \approx k$. Assume first that $i \approx_0 k$. Then if $i \neq k$, we have

$$(i,j) \sim_0 ((i,k) \text{ or } (k,i)) \sim_0 (k,l),$$

while if $i = k$, we have

$$(i,j) = (k,j) \sim_0 (k,l).$$

By transitivity we inductively arrive at the same conclusion when $k \approx i$. We conclude that there is exactly one equivalence class of $\sim$.

$\boxed{\impliedby}$ Suppose that $|\rho^\times / \sim| = 1$. If $\mathcal{Q}$ contains only singleton classes, the claim holds. Suppose that $C \in \mathcal{Q}$ is a class with $|C| \geq 2$. We claim that C contains all elements of $[1, n]$ not already contained in some singleton class; in particular, it will follow that C is the only class with cardinality ≥ 2. It is easy to see that we can choose distinct $i, j \in C$ such that $i \approx_0 j$. Without loss of generality assume that $(i,j) \in \rho$. If $(k,l) \in \rho^\times$ satisfies $(i,j) \sim_0 (k,l)$, we readily obtain that $k, l \in C$. By transitivity, we inductively conclude that the same assertion holds with the weaker assumption of $(i,j) \sim (k,l)$. Now the desired claim follows since $(i,j) \sim \rho^\times$.

Corollary 4.2.10. *Let $\mathcal{A}_\rho \subseteq M_n$ be a SMA. The following conditions are equivalent:*

(i) Every Jordan embedding $\mathcal{A}_\rho \to M_n$ is extensible to a Jordan automorphism of M_n.

(ii) $\mathcal{A}_\rho$ satisfies MAMP and every transitive map $g : \rho \to \mathbb{C}^\times$ is trivial.

Proof. From Corollary 4.2.5 and the discussion immediately after it, it is clear that when $\mathcal{A}$ satisfies MAMP, every Jordan embedding $\mathcal{A} \to M_n$ is of the form $T(\cdot)T^{-1}$ or $T(\cdot)^t T^{-1}$ for some invertible matrix $T \in M_n^\times$ if and only if every algebra automorphism $g^* \in \mathrm{Aut}(\mathcal{A})$ induced by a transitive map $g : \rho \to \mathbb{C}^\times$ is also of the form $S(\cdot)S^{-1}$ for some invertible matrix $S \in M_n^\times$. By Lemma 4.1.3, the latter condition is equivalent to the fact that every transitive map $g : \rho \to \mathbb{C}^\times$ is trivial. □

Remark 4.2.11. The two conditions of Corollary 4.2.10 (ii) are logically independent. Namely, the SMA

$$\begin{bmatrix} * & 0 & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \subseteq M_4$$

satisfies MAMP, but not every transitive map is trivial (see Example 4.1.1), while the SMA $\mathrm{diag}(M_2, M_2) \subseteq M_4$ does not satisfy MAMP but every transitive map is indeed trivial (see the semisimple case in [17]).

Example 4.2.12. If an SMA $\mathcal{A}_\rho \subseteq M_n$ contains an entire row or an entire column, then it satisfies the conditions of Corollary 4.2.10. This is precisely the situation when the quasi-ordered set $([1, n], \rho)$ has a minimal or a maximal element. Indeed, for the sake of simplicity suppose that $\mathcal{A}_\rho$ contains the n-th column. Then for any $(i, j), (k, l) \in \rho^\times$ we have

$$(i, j) \sim_0 (i, n) \sim_0 (k, n) \sim_0 (k, l)$$

and therefore $(i, j) \sim (k, l)$. We conclude $|\rho^\times / \sim| = 1$ so by Remark 4.2.9 the SMA $\mathcal{A}_\rho$ satisfies MAMP. The fact that all transitive maps are trivial follows from Remark 4.1.2.

On the other hand, the converse is not true in general. Consider

$$\rho = \Delta_5 \cup \{(1, 4), (1, 5), (3, 2), (3, 4), (3, 5), (4, 5), (5, 4)\}.$$

Then the SMA

$$\mathcal{A}_\rho = \begin{bmatrix} * & 0 & 0 & * & * \\ 0 & * & 0 & 0 & 0 \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}$$

satisfies the conditions of Corollary 4.2.10, but does not contain a full row or a full column. Indeed, we have

$$\begin{array}{l} (5, 4) \sim_0 (3, 4) \sim_0 (1, 4) \sim_0 (1, 5) \sim_0 (3, 5) \sim_0 (4, 5) \\ \wr_0 \\ (3, 2) \end{array}$$

so $|\rho^\times / \sim| = 1$. Suppose now that $g : \rho \to \mathbb{C}^\times$ is a transitive map. Define $s : \{1, \dots, 5\} \to \mathbb{C}^\times$ as

$$s(1) = g(1, 5), \qquad s(2) = \frac{g(3, 5)}{g(3, 2)}, \qquad s(3) = g(3, 5), \qquad s(4) = g(4, 5), \qquad s(5) = 1.$$

We need to show $g(i, j) = \frac{s(i)}{s(j)}$ for all $(i, j) \in \rho$. This is clear when $j = 5$ by the definition of s. When $j = 4$ we have

$$g(i,4) = \frac{g(i,5)}{g(4,5)} = \frac{\frac{s(i)}{s(5)}}{\frac{s(4)}{s(5)}} = \frac{s(i)}{s(4)}, \qquad \text{for all } i \in \{1,3,4,5\}.$$

The only remaining case is $(i,j) = (3,2)$ where we have

$$g(3,2) = \frac{g(3,5)}{\frac{g(3,5)}{g(3,2)}} = \frac{s(3)}{s(2)}.$$

We conclude that g is trivial.

Using Theorem 3.2.1 we can now prove the following consequences of Theorem 4.2.4 regarding Jordan embeddings between two SMAs. Before stating them, let us introduce the following auxiliary notation. Let ρ be a quasi-order on $[1,n]$ and consider a subset $\mathcal{U} \subseteq [1,n]$ expressible as a union of some equivalence classes of $\approx$ (i.e. $\mathcal{U}$ is a union of some classes of the quotient set $\mathcal{Q}_\rho$). Denote $\mathcal{U}^c := [1,n] \setminus \mathcal{U}$ and define a relation on $[1,n]$ by

$$\rho^{\mathcal{U}} := (\rho \cap (\mathcal{U} \times \mathcal{U})) \cup \left(\rho^t \cap (\mathcal{U}^c \times \mathcal{U}^c)\right), \tag{4.2.2}$$

which is easily seen to be a quasi-order.

Corollary 4.2.13. *Let $\mathcal{A}_\rho, \mathcal{A}_{\rho'} \subseteq M_n$ be SMAs. Then $\mathcal{A}_\rho$ Jordan-embeds into $\mathcal{A}_{\rho'}$ if and only if there exists a subset $\mathcal{U} \subseteq [1,n]$ expressible as a union of classes of $\mathcal{Q}_\rho$, and a $(\rho^{\mathcal{U}}, \rho')$-increasing permutation $\pi \in S_n$. Furthermore, if $\phi : \mathcal{A}_\rho \to \mathcal{A}_{\rho'}$ is a Jordan embedding, then there exists an invertible matrix $S \in \mathcal{A}_{\rho'}^\times$, a central idempotent $P \in Z(\mathcal{A}_\rho)$, a transitive map $g : \rho \to \mathbb{C}^\times$, and a $(\rho^{\mathcal{U}}, \rho')$-increasing permutation $\pi \in S_n$, where $\mathcal{U} := \{1 \le i \le n : (i,i) \in \operatorname{supp} P\}$, such that*

$$\phi(\cdot) = (SR_\pi)(Pg^*(\cdot) + (I-P)g^*(\cdot)^t)(SR_\pi)^{-1}.$$

Proof. $\boxed{\implies}$ Suppose that $\phi : \mathcal{A}_\rho \to \mathcal{A}_{\rho'}$ is a Jordan embedding. By Theorem 4.2.4, there exists an invertible matrix $T \in M_n^\times$, a central idempotent $P \in Z(\mathcal{A}_\rho)$ and a transitive map $g : \rho \to \mathbb{C}^\times$ such that

$$\phi(\cdot) = T(Pg^*(\cdot) + (I-P)g^*(\cdot)^t)T^{-1}.$$

Denote $\Lambda_n := \operatorname{diag}(1,\dots,n) \in \mathcal{D}_n$. Obviously, the matrix $\phi(\Lambda_n) = T\Lambda_n T^{-1}$ has eigenvalues $1,\dots,n$ so by Theorem 3.2.1, there exists $S \in \mathcal{A}_{\rho'}^\times$ and a permutation $\pi \in S_n$ such that $\phi(\Lambda_n) = (SR_\pi)\Lambda_n(SR_\pi)^{-1}$. We have

$$T\Lambda_n T^{-1} = (SR_\pi)\Lambda_n(SR_\pi)^{-1} \implies (SR_\pi)^{-1}T \leftrightarrow \Lambda_n$$

and hence we conclude that there exists a diagonal matrix $D \in \mathcal{D}_n^\times$ such that

$$T = SR_\pi D = S\underbrace{(R_\pi D R_\pi^{-1})}_{\in \mathcal{D}_n^\times} R_\pi.$$

By absorbing $R_\pi D R_\pi^{-1}$ into S, without loss of generality we can write $T = SR_\pi$ and therefore

$$\phi(\cdot) = (SR_\pi)(Pg^*(\cdot) + (I-P)g^*(\cdot)^t)(SR_\pi)^{-1}.$$

Note that, by Remark 3.1.5, $\mathcal{U} = \{1 \le i \le n : (i,i) \in \operatorname{supp} P\}$ is a union of certain classes of $\mathcal{Q}_\rho$. Consider the quasi-order $\rho^{\mathcal{U}}$ from (4.2.2) with respect to this $\mathcal{U}$. We in particular obtain $R_\pi \mathcal{A}_{\rho^{\mathcal{U}}} R_\pi^{-1} \subseteq \mathcal{A}_{\rho'}$, thus concluding that π is $(\rho^{\mathcal{U}}, \rho')$-increasing.

$\boxed{\Longleftarrow}$ We have $R_\pi \mathcal{A}_{\rho^{\mathcal{U}}} R_\pi^{-1} \subseteq \mathcal{A}_{\rho'}$ for some subset $\mathcal{U} \subseteq [1,n]$ expressible as a union of classes of $\mathcal{Q}_\rho$. Let $P \in Z(\mathcal{A}_\rho)$ be a central idempotent corresponding to $\mathcal{U}$, i.e. $P := \sum_{i\in\mathcal{U}} E_{ii}$. It easily follows that

$$R_\pi(P(\cdot) + (I-P)(\cdot)^t)R_\pi^{-1}$$

is a Jordan embedding $\mathcal{A}_\rho \to \mathcal{A}_{\rho'}$. □

By plugging $\rho' = \rho$ into Corollary 4.2.13, we obtain the description of Jordan automorphisms of SMAs:

Corollary 4.2.14. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA. A map $\phi : \mathcal{A}_\rho \to \mathcal{A}_\rho$ is a Jordan automorphism if and only if there exists an invertible matrix $S \in \mathcal{A}_\rho^\times$, a central idempotent $P \in Z(\mathcal{A}_\rho)$, a transitive map $g : \rho \to \mathbb{C}^\times$, and a $(\rho^{\mathcal{U}}, \rho)$-increasing permutation $\pi \in S_n$, where $\mathcal{U} := \{1 \le i \le n : (i,i) \in \operatorname{supp} P\}$, such that*

$$\phi(\cdot) = (SR_\pi)(Pg^*(\cdot) + (I-P)g^*(\cdot)^t)(SR_\pi)^{-1}.$$

In the special case of block upper-triangular algebras, the description of Jordan embeddings attains a much simpler form:

Corollary 4.2.15. *Let $\mathcal{A}$ and $\mathcal{B}$ be block-upper-triangular subalgebras of M_n. Suppose that $\phi : \mathcal{A} \to \mathcal{B}$ is a Jordan embedding. Then one of the following is true:*
(a) $\mathcal{A} \subseteq \mathcal{B}$ and there exists $T \in \mathcal{B}^\times$ such that $\phi(\cdot) = T(\cdot)T^{-1}$,
(b) $\mathcal{A}^\odot \subseteq \mathcal{B}$ and there exists $T \in \mathcal{B}^\times$ such that $\phi(\cdot) = T(\cdot)^\odot T^{-1}$.

Proof. Denote $\mathcal{A} = \mathcal{A}_\rho = \mathcal{A}_{k_1,\dots,k_p}$ and $\mathcal{B} = \mathcal{A}_{\rho'} = \mathcal{A}_{l_1,\dots,l_q}$. Since all block upper-triangular subalgebras are central, $\mathcal{U}$ from Corollary 4.2.13 can be either $[1,n]$ or $\emptyset$. Suppose $\mathcal{U} = [1,n]$. Then $\rho^{\mathcal{U}} = \rho$. Since all transitive maps $g : \rho \to \mathbb{C}$ are trivial (Remark 4.1.2), there exists an invertible matrix $S \in \mathcal{B}^\times$ and a (ρ, ρ')-increasing permutation $\pi \in S_n$ such that

$$\phi(\cdot) = (SR_\pi)(\cdot)(SR_\pi)^{-1}.$$

We need to show that $R_\pi \in \mathcal{B}$, i.e. that $(i, \pi(i)) \in \rho'$ for all $i \in [1,n]$. Similarly as in the proof of Proposition 4.1.7 we obtain that $\pi(n) \in [l_1 + \cdots + l_{q-1} + 1, n]$. Let $\sigma \in S_n$ denote the transposition $n \leftrightarrow \pi(n)$. Then as in the proof of Proposition 4.1.7 we conclude that $\sigma \circ \pi$ is (ρ, ρ')-increasing. Moreover, we have $R_{\sigma\circ\pi} = R_\sigma R_\pi$ and $R_\sigma \in \mathcal{B}^\times$ so without loss of generality by passing to $\sigma \circ \pi$ we can assume $\pi(n) = n$. Since $(n,n) \in \rho'$, this allows us to conduct an inductive argument as in Proposition 4.1.7 to conclude that $(i, \pi(i)) \in \rho'$ for all $i \in [1, n-1]$. This completes the proof of this case.

Suppose now that $\mathcal{U} = \emptyset$. Then $\rho^{\mathcal{U}} = \rho^t$ and ϕ is an antihomomorphism. This case follows from the previous one by considering the Jordan embedding $\phi((\cdot)^\odot) : \mathcal{A}^\odot \to \mathcal{B}$. □

Corollary 4.2.16. *Let $\mathcal{A} = \mathcal{A}_{k_1,\dots,k_p}$ and $\mathcal{B} = \mathcal{A}_{l_1,\dots,l_q}$ be block upper-triangular subalgebras of M_n.*

(a) $\mathcal{A}$ *and* $\mathcal{B}$ *are algebra-isomorphic if and only if* $(k_1, \dots, k_p) = (l_1, \dots, l_q)$.
(b) $\mathcal{A}$ *and* $\mathcal{B}$ *are algebra-antiisomorphic if and only if* $(k_1, \dots, k_p) = (l_q, \dots, l_1)$.
(c) $\mathcal{A}$ *and* $\mathcal{B}$ *are Jordan-isomorphic if and only if*

$$(k_1, \dots, k_p) = (l_1, \dots, l_q) \qquad \text{or} \qquad (k_1, \dots, k_p) = (l_q, \dots, l_1).$$

Proof. We write the proofs only for the nontrivial implications.

(a) Let $\phi : \mathcal{A} \to \mathcal{B}$ be an algebra isomorphism. Then ϕ is a multiplicative Jordan isomorphism and Corollary 4.2.15 implies $\mathcal{A} \subseteq \mathcal{B}$. As the same argument applies to the (multiplicative) map ϕ^{-1}, we obtain $\mathcal{A} = \mathcal{B}$, i.e. $(k_1, \dots, k_p) = (l_1, \dots, l_q)$.
(b) For an algebraic antiisomorphism $\phi : \mathcal{A} \to \mathcal{B}$ which is antimultiplicative, Corollary 4.2.15 implies $\mathcal{A}^{\odot} \subseteq \mathcal{B}$. Applying the same argument to ϕ^{-1} yields $\mathcal{B}^{\odot} \subseteq \mathcal{A}$. Since the map $\odot$ is involutory, we have $\mathcal{B} \subseteq \mathcal{A}^{\odot}$ and therefore $\mathcal{A}^{\odot} = \mathcal{B}$ (which is equivalent to $\mathcal{A} = \mathcal{B}^{\odot}$). Consequently, $(k_1, \dots, k_p) = (l_q, \dots, l_1)$.
(c) By Corollary 4.2.15 a Jordan isomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is multiplicative or antimultiplicative, so the arguments in (a) and (b) can be applied.

□

Note that the above conclusions do not follow if we merely assume that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as vector spaces. More precisely, we have:

Proposition 4.2.17. *Let* $\mathcal{A}_{k_1,\dots,k_p}$ *and* $\mathcal{A}_{l_1,\dots,l_q}$ *be parabolic subalgebras of* M_n. *Then* $\mathcal{A}_{k_1,\dots,k_p}$ *and* $\mathcal{A}_{l_1,\dots,l_q}$ *are isomorphic as vector spaces if and only if*

$$k_1^2 + \cdots + k_p^2 = l_1^2 + \cdots + l_q^2.$$

Proof. Using the standard notation for power sums and elementary symmetric polynomials, we have

$$\begin{aligned}
\dim \mathcal{A}_{k_1,\dots,k_p} &= k_1(k_1 + \cdots + k_p) + k_2(k_2 + \cdots + k_p) + \cdots + k_{p-1}(k_{p-1} + k_p) + k_p^2 \\
&= k_1^2 + \cdots + k_p^2 + \prod_{1 \le i < j \le p} k_i k_j \\
&= p_2(k_1, \dots, k_p) + e_2(k_1, \dots, k_p) \\
&= p_2(k_1, \dots, k_p) + \frac{1}{2}(p_1(k_1, \dots, k_p)^2 - p_2(k_1, \dots, k_n)) \\
&= \frac{1}{2}(n^2 + p_2(k_1, \dots, k_p))
\end{aligned}$$

and of course the same holds for $\mathcal{A}_{l_1,\dots,l_q}$ so the result follows. □

In particular, $\mathcal{A}_{k_1,\dots,k_p} \cong \mathcal{A}_{l_1,\dots,l_q}$ does not imply $p = q$. Namely, we have

$$\dim \mathcal{A}_{4,1,1} = \dim \mathcal{A}_{3,3} = 27.$$

Even when $p = q$, the subalgebras do not have to be algebra-isomorphic e.g.

$$\dim \mathcal{A}_{1,4,6,7} = \dim \mathcal{A}_{2,3,5,8} = 213.$$

Chapter 5

Linear preserver problems on SMAs

5.1 Rank-one preservers

5.1.1 On transitive maps and rank-one preservers

We start this section by examining the connection between transitive maps and rank-one preservers on SMAs. We first state two elementary and well-known facts regarding the representation of rank-one matrices.

Lemma 5.1.1.
(a) A matrix $A \in M_n$ is rank-one if and only if it can be represented as $A = uv^*$ for some nonzero vectors $u, v \in \mathbb{C}^n$.
(b) Suppose that a rank-one matrix $A \in M_n$ has two distinct representations $A = u_1v_1^* = u_2v_2^*$ where $u_1, u_2, v_1, v_2 \in \mathbb{C}^n$. Then $u_1 \parallel u_2$ and $v_1 \parallel v_2$.
(c) Let $u_1, u_2, v_1, v_2 \in \mathbb{C}^n$ be nonzero vector such that $u_1v_1^* + u_2v_2^*$ is a rank-one matrix. Then $u_1 \parallel u_2$ or $v_1 \parallel v_2$.

As before, we make use of relations $\approxeq_0$ and $\approxeq$ (which are defined in Remark 3.1.5).

Lemma 5.1.2. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA and let $g : \rho \to \mathbb{C}^\times$ be a transitive map. Suppose that*

$$g|_{\rho \cap [1,n-1]^2} \equiv 1.$$

If $\approxeq \subseteq [1, n-1]^2$ denotes the equivalence relation corresponding to the quasi-order $\rho \cap [1, n-1]^2$, then we have the equivalence:

$$g \text{ is trivial} \iff \begin{cases} (\forall i, j \in (\rho^\times)^{-1}(n))(i \approxeq j \implies g(i, n) = g(j, n)), \\ (\forall i, j \in (\rho^\times)(n))(i \approxeq j \implies g(n, i) = g(n, j)). \end{cases}$$

Proof. $\boxed{\implies}$ Suppose that g is trivial and separates through the map $s : [1, n] \to \mathbb{C}^\times$. Note that for all $(i, j) \in \rho \cap [1, n]^2$ we have

$$1 = g(i, j) = \frac{s(i)}{s(j)} \implies s(i) = s(j).$$

In particular, for all $1 \le i, j \le n - 1$ we conclude

$$i \approxeq_0 j \implies s(i) = s(j)$$

and then inductively

$$i \approx j \implies s(i) = s(j). \tag{5.1.1}$$

Let $i, j \in (\rho^\times)^{-1}(n)$ such that $i \approx j$. We claim that $g(i,n) = g(j,n)$. We have

$$g(i,n) = \frac{s(i)}{s(n)} \overset{(5.1.1)}{=} \frac{s(j)}{s(n)} = g(j,n).$$

This proves the first implication. The second implication is proved similarly.

$\boxed{\Longleftarrow}$ Suppose that the two implications hold true. Define $s : [1,n] \to \mathbb{C}^\times$ by first setting $s(n) := 1$. For $1 \le j \le n-1$, let $[j]$ be the equivalence class of $\approx$ containing j, and set

$$s(j) := \begin{cases} g(i,n), & \text{if } i \in [j] \cap (\rho^\times)^{-1}(n), \\ \frac{1}{g(n,i)}, & \text{if } i \in [j] \cap (\rho^\times)(n), \\ 1, & \text{if } [j] \cap (\rho^\times)^{-1}(n) = [j] \cap (\rho^\times)(n) = \emptyset. \end{cases}$$

Note that s is well-defined. Indeed, if $i_1, i_2 \in [j] \cap (\rho^\times)^{-1}(n)$, we have $i_1 \approx i_2$ and therefore $g(i_1,n) = g(i_2,n)$. The case $i_1, i_2 \in [j] \cap (\rho^\times)(n)$ is similar. Suppose now that $i_1 \in [j] \cap (\rho^\times)^{-1}(n)$ and also $i_2 \in [j] \cap (\rho^\times)(n)$. Then $(i_1,n),(n,i_2) \in \rho$ imply $(i_1,i_2) \in \rho$ and hence by transitivity

$$g(i_1,n)g(n,i_2) = g(i_1,i_2) = 1 \implies g(i_1,n) = \frac{1}{g(n,i_2)},$$

which is exactly what we wanted to show.

In particular, s is constant on each equivalence class of $\approx$. Now we prove that g separates through s. If $(i,j) \in \rho \cap [1,n-1]^2$, then clearly $i \approx_0 j$ and hence $s(i) = s(j)$ which implies

$$g(i,j) = 1 = \frac{s(i)}{s(j)}.$$

If $(j,n) \in \rho^\times$, then $s(j) = g(j,n)$ by definition and therefore

$$g(j,n) = \frac{s(j)}{s(n)}.$$

Similarly we cover the case $(n,j) \in \rho^\times$. □

Lemma 5.1.3. *Let $X \in M_n$. Then X has rank one if and only if $X \neq 0$ and for all $1 \le i,j,k,l \le n$ with $i \neq k$, $j \neq l$ holds*

$$\begin{vmatrix} X_{ij} & X_{il} \\ X_{kj} & X_{kl} \end{vmatrix} = 0.$$

Proof. See for example the section on Minors and cofactors in [46]. □

Let ρ be a quasi-order on $[1,n]$. Let $1 \le i,j,k,l \le n$. We say that ordered pairs $(i,j),(i,l),(k,j),(k,l) \in \rho$ form a *rectangle* of the SMA $\mathcal{A}_\rho$ if $i \neq k$, $j \neq l$.

Lemma 5.1.4. *Let ρ be a quasi-order on $[1,n]$ and let $g : \rho \to \mathbb{C}^\times$ be a transitive map. Then the induced map $g^* : \mathcal{A}_\rho \to M_n$ is a rank-one preserver if and only if for every* *rectangle* $(i,j),(i,l),(k,j),(k,l) \in \rho$ *we have*

$$\begin{vmatrix} g(i,j) & g(i,l) \\ g(k,j) & g(k,l) \end{vmatrix} = 0. \tag{5.1.2}$$

Proof. $\boxed{\implies}$ Suppose that g^* is a rank-one preserver and let $(i,j),(i,l),(k,j),(k,l) \in \rho$ be a rectangle of $\mathcal{A}_\rho$. The matrix

$$g^*(E_{ij} + E_{il} + E_{kj} + E_{kl}) = g(i,j)E_{ij} + g(i,l)E_{il} + g(k,j)E_{kj} + g(k,l)E_{kl}$$

has rank one, which by Lemma 5.1.3 implies the desired result.

$\boxed{\impliedby}$ Conversely, suppose that g satisfies the stated condition and let $X \in \mathcal{A}_\rho$ be a rank-one matrix. We wish to prove that $g^*(X)$ is a rank-one matrix. Since clearly $g^*(X) \neq 0$, by Lemma 5.1.3 it remains to verify the determinant condition. Fix $1 \le i,j,k,l \le n$ where $i \neq k$, $j \neq l$. If $\{(i,j),(i,l),(k,j),(k,l)\} \subsetneq \rho$, then the submatrix $\begin{bmatrix} X_{ij} & X_{il} \\ X_{kj} & X_{kl} \end{bmatrix}$ has at least one zero-row or zero-column, so the same holds for $\begin{bmatrix} g^*(X)_{ij} & g^*(X)_{il} \\ g^*(X)_{kj} & g^*(X)_{kl} \end{bmatrix}$. On the other hand, if $(i,j),(i,l),(k,j),(k,l)$ is a rectangle of ρ, we have

$$\begin{vmatrix} X_{ij} & X_{il} \\ X_{kj} & X_{kl} \end{vmatrix} = 0 \implies X_{ij}X_{kl} = X_{il}X_{kj}$$

and therefore

$$\begin{vmatrix} g^*(X)_{ij} & g^*(X)_{il} \\ g^*(X)_{kj} & g^*(X)_{kl} \end{vmatrix} = \begin{vmatrix} g(i,j)X_{ij} & g(i,l)X_{il} \\ g(k,j)X_{kj} & g(k,l)X_{kl} \end{vmatrix} = \begin{vmatrix} g(i,j) & g(i,l) \\ g(k,j) & g(k,l) \end{vmatrix} (X_{ij}X_{kl}) \overset{(5.1.2)}{=} 0,$$

which completes the proof. □

Corollary 5.1.5. *Let* $\mathcal{A}_\rho \subseteq M_n$ *be an SMA without rectangles. Suppose that* $g : \rho \to \mathbb{C}^\times$ *is a transitive map. Then the the induced map* $g^* : \mathcal{A}_\rho \to M_n$ *is a rank-one preserver.*

Lemma 5.1.6. *Let* $\mathcal{A}_\rho \subseteq M_n$ *be an SMA. For every central idempotent* $P \in Z(\mathcal{A}_\rho)$ *and* $X \in \mathcal{A}_\rho$ *we have*

$$r(X) = r(PX + (I-P)X^t).$$

Proof. It is easy to show that if two matrices $A, B \in M_n$ are supported on mutually disjoint sets of rows or columns, then

$$r(A+B) = r(A) + r(B).$$

Having this in mind, since $Z(\mathcal{A}_\rho) \subseteq \mathcal{D}_n$, we have $(I-P)^t = I - P \in Z(\mathcal{A}_\rho)$. We obtain

$$\begin{aligned} r(X) &= r(PX + (I-P)X) = r(PX) + r((I-P)X) = r(PX) + r((I-P)X^t) \\ &= r(PX + (I-P)X^t). \end{aligned}$$

□

5.1.2 Main result

We are now ready to prove the main result of this section.

Theorem 5.1.7. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA.*
(a) Let $\phi : \mathcal{A}_\rho \to M_n$ be a linear unital map preserving rank-one matrices. Then ϕ is a Jordan embedding.
(b) Let $\phi : \mathcal{A}_\rho \to M_n$ be a Jordan homomorphism which satisfies $\phi(E_{ij}) \neq 0$ for all $(i,j) \in \rho$. Then ϕ is a rank-one preserver if and only if the associated transitive map $g : \rho \to \mathbb{C}^\times$ obtained from Theorem 4.2.4 satisfies

$$\begin{vmatrix} g(i,j) & g(i,l) \\ g(k,j) & g(k,l) \end{vmatrix} = 0$$

for every rectangle $(i,j),(i,l),(k,j),(k,l)$ of $\mathcal{A}_\rho$.

The following remark justifies the claim of Theorem 5.1.7 (b).

Remark 5.1.8. If $\phi : \mathcal{A}_\rho \to M_n$ is a Jordan embedding, then the associated transitive map $g : \rho \to \mathbb{C}^\times$ obtained from Theorem 4.2.4 is not unique. However, if $g, h : \rho \to \mathbb{C}^\times$ are two such maps, then there exists a function $s : [1,n] \to \mathbb{C}^\times$ such that $h(i,j) = \frac{s(i)}{s(j)} g(i,j)$ for all $(i,j) \in \rho$. Indeed, suppose that

$$S(Pg^*(\cdot) + (I-P)g^*(\cdot)^t)S^{-1} = T(Qh^*(\cdot) + (I-Q)h^*(\cdot)^t)T^{-1}$$

for some $S,T \in M_n^\times$, central idempotents $P,Q \in Z(\mathcal{A}_\rho)$. Denote $\Lambda_n := \operatorname{diag}(1,\dots,n) \in \mathcal{D}_n$ and note that

$$\begin{aligned} S\Lambda_n S^{-1} &= S(Pg^*(\Lambda_n) + (I-P)g^*(\Lambda_n)^t)S^{-1} = T(Qh^*(\Lambda_n) + (I-Q)h^*(\Lambda_n)^t)T^{-1} \\ &= T\Lambda_n T^{-1}, \end{aligned}$$

which implies $T^{-1}S \leftrightarrow \Lambda_n$ and hence $S = TD$ for some diagonal matrix $D \in \mathcal{D}_n^\times$. By plugging this in and cancelling T from both sides, we obtain

$$D(P(g^*(\cdot) + (I-P)g^*(\cdot)^t)D^{-1} = Qh^*(\cdot) + (I-Q)h^*(\cdot)^t.$$

Now let $(i,j) \in \rho^\times$ be arbitrary. Since P and Q are central, by Remark 3.1.5 we certainly have $P_{ii} = P_{jj}$ and $Q_{ii} = Q_{jj}$. We consider four cases:

- If $P_{ii} = P_{jj} = Q_{ii} = Q_{jj} = 1$, then we obtain

$$\frac{D_{ii}}{D_{jj}} g(i,j)E_{ij} = D(g(i,j)E_{ij})D^{-1} = h(i,j)E_{ij} \implies h(i,j) = \frac{D_{ii}}{D_{jj}} g(i,j).$$

- If $P_{ii} = P_{jj} = 1$ and $Q_{ii} = Q_{jj} = 0$, then we obtain

$$\frac{D_{ii}}{D_{jj}} g(i,j)E_{ij} = D(g(i,j)E_{ij})D^{-1} = (h(i,j)E_{ij})^t = h(i,j)E_{ji},$$

  which is a contradiction.
- The same argument also shows that the case $P_{ii} = P_{jj} = 0$ and $Q_{ii} = Q_{jj} = 1$ is also not possible.
- If $P_{ii} = P_{jj} = Q_{ii} = Q_{jj} = 0$, then we obtain

$$\frac{D_{jj}}{D_{ii}} g(i,j)E_{ji} = D(g(i,j)E_{ij})^t D^{-1} = (h(i,j)E_{ij})^t = h(i,j)E_{ji}$$

which implies $h(i,j) = \frac{D_{jj}}{D_{ii}} g(i,j)$. Overall, if we define $s : [1,n] \to \mathbb{C}^\times$ by

$$s(i) := \begin{cases} D_{ii}, & \text{if } P_{ii} = 1, \\ \frac{1}{D_{ii}}, & \text{otherwise,} \end{cases}$$

we conclude

$$h(i,j) = \frac{s(i)}{s(j)} g(i,j), \qquad \text{for all } (i,j) \in \rho.$$

In particular,

$$\begin{vmatrix} h(i,j) & h(i,l) \\ h(k,j) & h(k,l) \end{vmatrix} = \begin{vmatrix} \frac{s(i)}{s(j)} g(i,j) & \frac{s(i)}{s(l)} g(i,l) \\ \frac{s(k)}{s(j)} g(k,j) & \frac{s(k)}{s(l)} g(k,l) \end{vmatrix} = \underbrace{\frac{s(i)s(k)}{s(j)s(l)}}_{\neq 0} \begin{vmatrix} g(i,j) & g(i,l) \\ g(k,j) & g(k,l) \end{vmatrix},$$

which shows that the condition from Theorem 5.1.7 (b) is unambiguously defined (i.e. it is independent of the choice of the particular transitive map).

Proof of Theorem 5.1.7. First we prove (a). By Lemma 5.1.1, for each $(i,j) \in \rho$ we can choose $u_{ij}, v_{ij} \in \mathbb{C}^n$ such that $\phi(E_{ij}) = u_{ij} v_{ij}^*$.

Claim 5.1.8.1.

(a) For all $1 \leq i \leq n$ and distinct $j,k \in \rho(i)$ we have either $u_{ij} \parallel u_{ik}$ or $v_{ij} \parallel v_{ik}$.

(b) For all $1 \leq i \leq n$ we have

$$\dim \operatorname{span}\{u_{ij} : j \in \rho(i)\} = 1 \qquad \text{or} \qquad \dim \operatorname{span}\{v_{ij} : j \in \rho(i)\} = 1.$$

If $(\rho^\times)(i)$ is nonempty, then the disjunction is exclusive.

(c) For all $1 \leq i \leq n$ we have the implications
- $\dim \operatorname{span}\{u_{ij} : j \in \rho(i)\} = 1 \implies \{v_{ij} : j \in \rho(i)\}$ is linearly independent in $\mathbb{C}^n$.
- $\dim \operatorname{span}\{v_{ij} : j \in \rho(i)\} = 1 \implies \{u_{ij} : j \in \rho(i)\}$ is linearly independent in $\mathbb{C}^n$.

These are rowwise versions; the columnwise versions of the claims also hold true:

(a') For all $1 \leq j \leq n$ and distinct $i,k \in \rho^{-1}(j)$ we have either $u_{ij} \parallel u_{kj}$ or $v_{ij} \parallel v_{kj}$.

(b') For all $1 \leq j \leq n$ we have

$$\dim \operatorname{span}\{u_{ij} : i \in \rho^{-1}(j)\} = 1 \qquad \text{or} \qquad \dim \operatorname{span}\{v_{ij} : i \in \rho^{-1}(j)\} = 1.$$

If $(\rho^\times)^{-1}(j)$ is nonempty, then the disjunction is exclusive.

(c') For all $1 \leq i \leq n$ we have the implications
- $\dim \operatorname{span}\{u_{ij} : i \in \rho^{-1}(j)\} = 1 \implies \{v_{ij} : i \in \rho^{-1}(j)\}$ is linearly independent in $\mathbb{C}^n$.
- $\dim \operatorname{span}\{v_{ij} : i \in \rho^{-1}(j)\} = 1 \implies \{u_{ij} : i \in \rho^{-1}(j)\}$ is linearly independent in $\mathbb{C}^n$.

Proof. We only prove (a), (b) and (c), as the proofs of (a'), (b') and (c') are analogous (or one can just pass to the map $\phi(\cdot)^t$).

(a) The matrices $\phi(E_{ij})$ and $\phi(E_{ik})$ are rank-one and their sum $\phi(E_{ij} + E_{ik})$ is rank-one as well so Lemma 5.1.1 (c) applies. Suppose now that $u_{ij} \parallel u_{ik}$ and $v_{ij} \parallel v_{ik}$. Let

$\alpha, \beta \in \mathbb{C}^\times$ such that $u_{ik} = \alpha u_{ij}$ and $v_{ik} = \beta v_{ij}$. Then

$$\phi(\alpha\overline{\beta}E_{ij} - E_{ik}) = \alpha\overline{\beta}u_{ij}v_{ij}^* - u_{ik}v_{ik}^* = \alpha\overline{\beta}u_{ij}v_{ij}^* - \alpha\overline{\beta}u_{ij}v_{ij}^* = 0,$$

which is a contradiction.

(b) For a fixed $1 \leq i \leq n$, we need to show that the same option from (a) arises for all $1 \leq j \neq k \leq n$ such that $(i,j),(i,k) \in \rho$. Suppose the contrary, for example that $1 \leq j,k,l \leq n$ are distinct indices such that $(i,j),(i,k),(i,l) \in \rho$ and $u_{ij} \parallel u_{ik} \nparallel u_{il}$. Then by (a) it follows that $v_{ij} \nparallel v_{ik} \parallel v_{il}$ and hence we have $u_{ij} \nparallel u_{il}$ and $v_{ij} \nparallel v_{il}$, which is a contradiction with (a). This proves the first part of the claim. To prove the second part, assume that $(i,j) \in \rho^\times$, but both dimensions are 1. Then $u_{ii} \parallel u_{ij}$ and $v_{ii} \parallel v_{ij}$, which is a contradiction with (a).

(c) We will prove the first claim as the second one is very similar. Fix $1 \leq i \leq n$ and assume $\dim \operatorname{span}\{u_{ij} : j \in \rho(i)\} = 1$. For each $j \in \rho(i)$ we can choose a scalar $\lambda_{ij} \in \mathbb{C}^\times$ such that $u_{ij} = \lambda_{ij}u_{ii}$. Then we have

$$\phi(E_{ij}) = u_{ij}v_{ij}^* = (\lambda_{ij}u_{ii})v_{ij}^* = u_{ii}(\overline{\lambda_{ij}}v_{ij})^*,$$

so we can replace v_{ij} with $\overline{\lambda_{ij}}v_{ij}$ and assume that $\phi(E_{ij}) = u_{ii}v_{ij}^*$ for all $j \in \rho(i)$. Let $\alpha_{ij} \in \mathbb{C}, j \in \rho(i)$ be scalars such that

$$\sum_{j \in \rho(i)} \alpha_{ij}v_{ij} = 0.$$

Then

$$\phi\left(\sum_{j\in\rho(i)} \alpha_{ij}E_{ij}\right) = u_{ii}\sum_{j\in\rho(i)} \alpha_{ij}v_{ij}^* = 0.$$

This implies $\alpha_{ij} = 0$ for all $j \in \rho(i)$, as otherwise the rank-one matrix $\sum_{j\in\rho(i)} \alpha_{ij}E_{ij}$ is in the kernel of ϕ.

□

Claim 5.1.8.2. The sets $\{u_{11}, \ldots, u_{nn}\}$ and $\{v_{11}, \ldots, v_{nn}\}$ are linearly independent. Furthermore, they satisfy the orthogonality relations

$$\langle u_{ii}, v_{jj}\rangle = \delta_{ij}, \qquad 1 \leq i,j \leq n.$$

Proof. We have

$$I = \phi(I) = \sum_{i=1}^n u_{ii}v_{ii}^* = \begin{bmatrix} u_{11} & \cdots & u_{nn}\end{bmatrix}\begin{bmatrix} v_{11} & \cdots & v_{nn}\end{bmatrix}^*.$$

In particular, the matrices $\begin{bmatrix} u_{11} & \cdots & u_{nn}\end{bmatrix}$ and $\begin{bmatrix} v_{11} & \cdots & v_{nn}\end{bmatrix}$ are invertible. We also have

$$[\delta_{ij}]_{i,j=1}^n = I = \begin{bmatrix} v_{11} & \cdots & v_{nn}\end{bmatrix}^*\begin{bmatrix} u_{11} & \cdots & u_{nn}\end{bmatrix} = [v_{ii}^*u_{jj}]_{i,j=1}^n = [\langle u_{jj}, v_{ii}\rangle]_{i,j=1}^n.$$

□

Denote

$$\mathcal{U}_R := \{1 \le i \le n : \dim \operatorname{span}\{u_{ij} : j \in \rho(i)\} = 1\},$$
$$\mathcal{V}_R := \{1 \le i \le n : \dim \operatorname{span}\{v_{ij} : j \in \rho(i)\} = 1\}.$$

By Claim 5.1.8.1 (b), it is clear that

$$\mathcal{U}_R \cup \mathcal{V}_R = \{1, \ldots, n\}, \qquad \mathcal{U}_R \cap \mathcal{V}_R = \{1 \le i \le n : (\rho^\times)(i) = \emptyset\}. \tag{5.1.3}$$

Similarly, for the columns, we define

$$\mathcal{U}_C := \{1 \le j \le n : \dim \operatorname{span}\{u_{ij} : i \in \rho^{-1}(j)\} = 1\},$$

$$\mathcal{V}_C := \{1 \le j \le n : \dim \operatorname{span}\{v_{ij} : i \in \rho^{-1}(j)\} = 1\}$$

and we can make similar observations as above to conclude

$$\mathcal{U}_C \cup \mathcal{V}_C = \{1, \ldots, n\}, \qquad \mathcal{U}_C \cap \mathcal{V}_C = \{1 \le j \le n : \rho^{-1}(j) = \emptyset\}. \tag{5.1.4}$$

Claim 5.1.8.3. We have

$$\rho^\times \subseteq ((\mathcal{U}_R \setminus \mathcal{V}_R) \times (\mathcal{V}_C \setminus \mathcal{U}_C)) \sqcup ((\mathcal{V}_R \setminus \mathcal{U}_R) \times (\mathcal{U}_C \setminus \mathcal{V}_C)),$$

where $\sqcup$ stands for the disjoint union.

Proof. Let $(i, j) \in \rho^\times$. Since $(\rho^\times)(i)$ and $(\rho^\times)^{-1}(j)$ are both nonempty, by (5.1.3) and (5.1.4) we have

$$i \in (\mathcal{U}_R \setminus \mathcal{V}_R) \sqcup (\mathcal{V}_R \setminus \mathcal{U}_R) \quad \text{and} \quad j \in (\mathcal{U}_C \setminus \mathcal{V}_C) \sqcup (\mathcal{V}_C \setminus \mathcal{U}_C).$$

Suppose by way of contradiction that $(i, j) \in (\mathcal{U}_R \setminus \mathcal{V}_R) \times (\mathcal{U}_C \setminus \mathcal{V}_C)$. It follows $u_{ii} \parallel u_{ij} \parallel u_{jj}$, which contradicts Claim 5.1.8.2. The case $(i, j) \in (\mathcal{V}_R \setminus \mathcal{U}_R) \times (\mathcal{V}_C \setminus \mathcal{U}_C)$ is similarly shown to be impossible. $\square$

Claim 5.1.8.4. Suppose that $(i, j) \in \rho^\times$. If there exists $(j, k) \in \rho^\times$ then

$$(i, j) \in ((\mathcal{U}_R \setminus \mathcal{V}_R) \times (\mathcal{U}_R \setminus \mathcal{V}_R)) \sqcup ((\mathcal{V}_R \setminus \mathcal{U}_R) \times (\mathcal{V}_R \setminus \mathcal{U}_R)).$$

Proof. As in the proof of Claim 5.1.8.3, first note that $i, j \in (\mathcal{U}_R \setminus \mathcal{V}_R) \sqcup (\mathcal{V}_R \setminus \mathcal{U}_R)$. We shall assume that $i \in \mathcal{U}_R \setminus \mathcal{V}_R$ as the other case is similar. Assume first that $k \neq i$. Then by Claim 5.1.8.3 from $(i, k) \in \rho^\times$ it follows

$$i \in \mathcal{U}_R \setminus \mathcal{V}_R \implies k \in \mathcal{V}_C \setminus \mathcal{U}_C \implies j \in \mathcal{U}_R \setminus \mathcal{V}_R.$$

Now assume that $k = i$. Then $(i, i), (i, j), (j, i), (j, j) \in \rho$ so the matrix

$$\phi(E_{ii} + E_{ij} + E_{ji} + E_{jj}) = u_{ii}v_{ii}^* + u_{ij}v_{ij}^* + u_{ji}v_{ji}^* + u_{jj}v_{jj}^*$$

has rank one. By (5.1.3), $j \in \mathcal{U}_R \setminus \mathcal{V}_R$ or $j \in \mathcal{V}_R \setminus \mathcal{U}_R$ so by way of contradiction suppose the latter. Then there exist scalars $\alpha, \beta \in \mathbb{C}^\times$ such that $u_{ij} = \alpha u_{ii}$ and $v_{ji} = \beta v_{jj}$. Therefore

$$u_{ii}v_{ii}^* + u_{ij}v_{ij}^* + u_{ji}v_{ji}^* + u_{jj}v_{jj}^* = u_{ii}(v_{ii} + \overline{\alpha} v_{ij})^* + (\overline{\beta} u_{ji} + u_{jj})v_{jj}^*$$

so by Lemma 5.1.1 (c) it follows that

$$u_{ii} \parallel (\overline{\beta} u_{ji} + u_{jj}) \qquad \text{or} \qquad (v_{ii} + \overline{\alpha} v_{ij}) \parallel v_{jj}.$$

By Claim 5.1.8.3, since $(j,i) \in \rho^{\times}$, from $j \in \mathcal{V}_R \setminus \mathcal{U}_R$ we obtain $i \in \mathcal{U}_C \setminus \mathcal{V}_C$ and therefore

$$u_{ii} \parallel \left(\overline{\beta} \underbrace{u_{ji}}_{\parallel u_{ii}} + u_{jj} \right)$$

is a contradiction with Claim 5.1.8.2. Similarly, from $i \in \mathcal{U}_R \setminus \mathcal{V}_R$ we obtain $j \in \mathcal{V}_C \setminus \mathcal{U}_C$ and therefore

$$\left(v_{ii} + \overline{\alpha} \underbrace{v_{ij}}_{\parallel v_{jj}} \right) \parallel v_{jj}$$

is a contradiction with Claim 5.1.8.2. □

Claim 5.1.8.5. Suppose that two nonzero vectors $v, w \in \mathbb{C}^n$ satisfy $v \parallel w$. Then

$$v = \frac{\langle v, w \rangle}{\|w\|^2} w = \frac{\|v\|^2}{\langle w, v \rangle} w.$$

Proof. An easy computation. □

Claim 5.1.8.6. Suppose that $(p,q), (p,s), (r,q), (r,s) \in \rho$ form a rectangle in $\mathcal{A}_\rho$. Then

$$\langle u_{pq}, u_{ps} \rangle \langle u_{rs}, u_{rq} \rangle \langle v_{rq}, v_{pq} \rangle \langle v_{ps}, v_{rs} \rangle = \|u_{pq}\|^2 \|u_{rs}\|^2 \|v_{pq}\|^2 \|v_{rs}\|^2.$$

Proof. By definition we have $p \neq r$ and $q \neq s$ which implies that all sets $(\rho^{\times})(p)$, $(\rho^{\times})(r)$, $(\rho^{\times})^{-1}(q)$, $(\rho^{\times})^{-1}(s)$ are nonempty, so by (5.1.3) we may assume that $p \in \mathcal{U}_R \setminus \mathcal{V}_R$ (as the other case $p \in \mathcal{V}_R \setminus \mathcal{U}_R$ is similar). We first show that $r \in \mathcal{U}_R \setminus \mathcal{V}_R$ as well. If $r = q$, then in particular $r \neq s$ so $(p,r), (r,s) \in \rho^{\times}$ and hence $r \in \mathcal{U}_R \setminus \mathcal{V}_R$ follows from Claim 5.1.8.4. Assume $r \neq q$.

- If $p = q$, then $r \neq p \neq s$ and therefore $(r,p), (p,s) \in \rho^{\times}$ so by Claim 5.1.8.4 it follows that $r \in \mathcal{U}_R \setminus \mathcal{V}_R$.
- If $p \neq q$, then

$$p \in \mathcal{U}_R \setminus \mathcal{V}_R \overset{\text{Claim 5.1.8.3}}{\Longrightarrow} q \in \mathcal{V}_C \setminus \mathcal{U}_C \overset{\text{Claim 5.1.8.3}}{\Longrightarrow} r \in \mathcal{U}_R \setminus \mathcal{V}_R.$$

Putting it all together, since at least one of $(p,q), (r,q)$ is in $\rho^{\times}$, and similarly for $(p,s), (r,s)$, we have

$$p, r \in \mathcal{U}_R \setminus \mathcal{V}_R \overset{\text{Claim 5.1.8.3}}{\Longrightarrow} q, s \in \mathcal{V}_C \setminus \mathcal{U}_C$$

and hence, by Claim 5.1.8.5,

$$p \in \mathcal{U}_R \setminus \mathcal{V}_R \implies u_{ps} = \frac{\langle u_{ps}, u_{pq} \rangle}{\|u_{pq}\|^2} u_{pq}, \qquad r \in \mathcal{U}_R \setminus \mathcal{V}_R \implies u_{rq} = \frac{\langle u_{rq}, u_{rs} \rangle}{\|u_{rs}\|^2} u_{rs},$$

$$q \in \mathcal{V}_C \setminus \mathcal{U}_C \implies v_{rq} = \frac{\langle v_{rq}, v_{pq} \rangle}{\|v_{pq}\|^2} v_{pq}, \qquad s \in \mathcal{V}_C \setminus \mathcal{U}_C \implies v_{rs} = \frac{\|v_{rs}\|^2}{\langle v_{ps}, v_{rs} \rangle} v_{ps}.$$

The matrix

$$\begin{aligned}\phi(E_{pq}+E_{ps}+E_{rq}+E_{rs}) &= u_{pq}v_{pq}^*+u_{ps}v_{ps}^*+u_{rq}v_{rq}^*+u_{rs}v_{rs}^*\\ &= u_{pq}\left(v_{pq}+\frac{\langle u_{pq},u_{ps}\rangle}{\|u_{pq}\|^2}v_{ps}\right)^*+u_{rs}\left(\frac{\langle u_{rs},u_{rq}\rangle}{\|u_{rs}\|^2}v_{rq}+v_{rs}\right)^*\\ &= u_{pq}\left(v_{pq}+\frac{\langle u_{pq},u_{ps}\rangle}{\|u_{pq}\|^2}v_{ps}\right)^*+u_{rs}\left(\frac{\langle u_{rs},u_{rq}\rangle}{\|u_{rs}\|^2}\frac{\langle v_{rq},v_{pq}\rangle}{\|v_{pq}\|^2}v_{pq}+\frac{\|v_{rs}\|^2}{\langle v_{ps},v_{rs}\rangle}v_{ps}\right)^*\end{aligned}$$

has rank one. Since $q\in\mathcal{V}_C\setminus\mathcal{U}_C$ and $r\in\mathcal{U}_R\setminus\mathcal{V}_R$, by Claim 5.1.8.1 (a) we have $u_{pq}\not\parallel u_{rq}\parallel u_{rs}$. Claim 5.1.1 (c) now yields

$$\left(v_{pq}+\frac{\langle u_{pq},u_{ps}\rangle}{\|u_{pq}\|^2}v_{ps}\right)\parallel\left(\frac{\langle u_{rs},u_{rq}\rangle}{\|u_{rs}\|^2}\frac{\langle v_{rq},v_{pq}\rangle}{\|v_{pq}\|^2}v_{pq}+\frac{\|v_{rs}\|^2}{\langle v_{ps},v_{rs}\rangle}v_{ps}\right).$$

In particular, we have

$$0=\begin{vmatrix}1 & \frac{\langle u_{pq},u_{ps}\rangle}{\|u_{pq}\|^2}\\ \frac{\langle u_{rs},u_{rq}\rangle}{\|u_{rs}\|^2}\frac{\langle v_{rq},v_{pq}\rangle}{\|v_{pq}\|^2} & \frac{\|v_{rs}\|^2}{\langle v_{ps},v_{rs}\rangle} =\end{vmatrix}=\frac{\|v_{rs}\|^2}{\langle v_{ps},v_{rs}\rangle}-\frac{\langle u_{rs},u_{rq}\rangle}{\|u_{rs}\|^2}\frac{\langle v_{rq},v_{pq}\rangle}{\|v_{pq}\|^2}\frac{\langle u_{pq},u_{ps}\rangle}{\|u_{pq}\|^2}$$

which is exactly what we desired to show. □

The next claim proves that ϕ is a Jordan homomorphism.

Claim 5.1.8.7. For all $(i,j),(k,l)\in\rho$ we have

$$\phi(E_{ij}\circ E_{kl})=\phi(E_{ij})\circ\phi(E_{kl}). \tag{5.1.5}$$

Proof. We will assume throughout that $i\in\mathcal{U}_R$ (as the case $i\in\mathcal{V}_R$ can be treated similarly). We have several cases to consider.

(1°) Suppose $i=l=k=j$. Then

$$\phi(E_{ii}\circ E_{ii})=2\phi(E_{ii})=2u_{ii}v_{ii}^*\overset{\text{Claim 5.1.8.2}}{=}2u_{ii}(v_{ii}^*u_{ii})v_{ii}^*=2\phi(E_{ii})^2=\phi(E_{ii})\circ\phi(E_{ii})$$

so (5.1.5) holds.

(2°) Suppose $i=l\neq k=j$. Then

$$\phi(E_{ij}\circ E_{ji})=\phi(E_{ii})+\phi(E_{jj})=u_{ii}v_{ii}^*+u_{jj}v_{jj}^*.$$

On the other hand,

$$\begin{aligned}\phi(E_{ij})\phi(E_{ji})+\phi(E_{ji})\phi(E_{ij}) &= u_{ij}(v_{ij}^*u_{ji})v_{ji}^*+u_{ji}(v_{ji}^*u_{ij})v_{ij}^*\\ &= \langle u_{ji},v_{ij}\rangle u_{ij}v_{ji}^*+\langle u_{ij},v_{ji}\rangle u_{ji}v_{ij}^*\end{aligned}$$

Since $(i,j),(j,i)\in\rho^\times$, by (5.1.3) we have $i\in\mathcal{U}_R\setminus\mathcal{V}_R$ and hence

$$i\in\mathcal{U}_R\setminus\mathcal{V}_R\overset{\text{Claim 5.1.8.4}}{\Longrightarrow}j\in\mathcal{U}_R\setminus\mathcal{V}_R\implies u_{ji}\parallel u_{jj}\overset{\text{Claim 5.1.8.5}}{\Longrightarrow}u_{ji}=\frac{\langle u_{ji},u_{jj}\rangle}{\|u_{jj}\|^2}u_{jj},$$

$$j \in \mathcal{U}_R \setminus \mathcal{V}_R \overset{\text{Claim 5.1.8.3}}{\Longrightarrow} i \in \mathcal{V}_C \setminus \mathcal{U}_C \implies v_{ji} \parallel v_{ii} \overset{\text{Claim 5.1.8.5}}{\Longrightarrow} v_{ji} = \frac{\langle v_{ji}, v_{ii}\rangle}{\|v_{ii}\|^2} v_{ii},$$
$$i \in \mathcal{U}_R \setminus \mathcal{V}_R \overset{\text{Claim 5.1.8.3}}{\Longrightarrow} j \in \mathcal{V}_C \setminus \mathcal{U}_C \implies v_{ij} \parallel v_{jj} \overset{\text{Claim 5.1.8.5}}{\Longrightarrow} v_{ij} = \frac{\langle v_{ij}, v_{jj}\rangle}{\|v_{jj}\|^2} v_{jj}.$$

We also have

$$i \in \mathcal{U}_R \setminus \mathcal{V}_R \implies u_{ij} \parallel u_{ii} \overset{\text{Claim 5.1.8.5}}{\Longrightarrow} u_{ij} = \frac{\langle u_{ij}, u_{ii}\rangle}{\|u_{ii}\|^2} u_{ii}.$$

Therefore,

$$\begin{aligned}
\langle u_{ji}, v_{ij}\rangle u_{ij} v_{ji}^* &= \left\langle \frac{\langle u_{ji}, u_{jj}\rangle}{\|u_{jj}\|^2} u_{jj}, \frac{\langle v_{ij}, v_{jj}\rangle}{\|v_{jj}\|^2} v_{jj} \right\rangle \left(\frac{\langle u_{ij}, u_{ii}\rangle}{\|u_{ii}\|^2} u_{ii}\right) \left(\frac{\langle v_{ji}, v_{ii}\rangle}{\|v_{ii}\|^2} v_{ii}\right)^* \\
&\overset{\text{Claim 5.1.8.2}}{=} \frac{\langle u_{ji}, u_{jj}\rangle \langle v_{jj}, v_{ij}\rangle \langle u_{ij}, u_{ii}\rangle \langle v_{ii}, v_{ji}\rangle}{\|u_{jj}\|^2 \|v_{jj}\|^2 \|u_{ii}\|^2 \|v_{ii}\|^2} u_{ii} v_{ii}^* \\
&= u_{ii} v_{ii}^*,
\end{aligned}$$

where the last equality follows by conjugation from Claim 5.1.8.6 for $p = q = i$ and $r = s = j$. By exchanging i and j in the same way we obtain

$$\langle u_{ij}, v_{ji}\rangle u_{ji} v_{ij}^* = u_{jj} v_{jj}^*.$$

This proves (5.1.5).

(3°) Suppose $k = j$ but $i \neq l$. Then $(i,j), (j,l) \in \rho$ so $(i,l) \in \rho^\times$ (in particular, $i \in \mathcal{U}_R \setminus \mathcal{V}_R$ by (5.1.3)). We need to prove that

$$\phi(E_{ij}E_{jl} + E_{jl}E_{ij}) = \phi(E_{il}) = u_{il} v_{il}^*$$

equals

$$\begin{aligned}
\phi(E_{ij})\phi(E_{jl}) + \phi(E_{jl})\phi(E_{ij}) &= u_{ij}(v_{ij}^* u_{jl}) v_{jl}^* + u_{jl}(v_{jl}^* u_{ij}) v_{ij}^* \\
&= \langle u_{jl}, v_{ij}\rangle u_{ij} v_{jl}^* + \langle u_{ij}, v_{jl}\rangle u_{jl} v_{ij}^*.
\end{aligned}$$

For the time being, we additionally assume that $j \neq l$. First note that we always have $j \in \mathcal{U}_R \setminus \mathcal{V}_R$. Indeed, if $i = j$, then this is obvious, while if $i \neq j$ then $(i,j), (j,l) \in \rho^\times$, so this follows from Claim 5.1.8.4. In any case, since $(j,l) \in \rho^\times$, Claim 5.1.8.3 also implies $l \in \mathcal{V}_C \setminus \mathcal{U}_C$. We now have

$$i \in \mathcal{U}_R \setminus \mathcal{V}_R \implies u_{ij} \parallel u_{ii}, \qquad l \in \mathcal{V}_C \setminus \mathcal{U}_C \implies v_{jl} \parallel v_{ll}$$

and hence $\langle u_{ij}, v_{jl}\rangle = 0$ by Claim 5.1.8.2. Therefore, the second term of the right hand size is zero. Now we focus on the first term.

We have

$$j \in \mathcal{U}_R \setminus \mathcal{V}_R \implies u_{jl} \parallel u_{jj}, \qquad l \in \mathcal{V}_C \setminus \mathcal{U}_C \implies v_{jl} \parallel v_{il},$$

$$i \in \mathcal{U}_R \backslash \mathcal{V}_R \implies u_{ij} \parallel u_{il}, \qquad (i = j \text{ or } (i \neq j \overset{\text{Claim 5.1.8.3}}{\Longrightarrow} j \in \mathcal{V}_C \backslash \mathcal{U}_C)) \implies v_{ij} \parallel v_{jj}$$

and therefore (again invoking Claims 5.1.8.5 and 5.1.8.2),

$$\begin{aligned}\langle u_{jl}, v_{ij}\rangle u_{ij}v_{jl}^* &= \left\langle \frac{\langle u_{jl}, u_{jj}\rangle}{\|u_{jj}\|^2}u_{jj}, \frac{\langle v_{ij}, v_{jj}\rangle}{\|v_{jj}\|^2}v_{jj}\right\rangle \left(\frac{\langle u_{ij}, u_{il}\rangle}{\|u_{il}\|^2}u_{il}\right)\left(\frac{\langle v_{jl}, v_{il}\rangle}{\|v_{il}\|^2}v_{il}\right)^* \\ &= \frac{\langle u_{jl}, u_{jj}\rangle\langle v_{jj}, v_{ij}\rangle\langle u_{ij}, u_{il}\rangle\langle v_{il}, v_{jl}\rangle}{\|u_{jj}\|^2\|v_{jj}\|^2\|u_{il}\|^2\|v_{il}\|^2}u_{il}v_{il}^* \\ &= u_{il}v_{il}^*,\end{aligned}$$

where the last equality is obtained by conjugating the equality from Claim 5.1.8.6 for $(p, q) = (i, l)$ and $(r, s) = (j, j)$. This proves (5.1.5).

It remains to consider the case $j = l$ so we are looking at $(i, l), (l, l)$. We need to prove that

$$\phi(E_{il}E_{ll} + E_{ll}E_{il}) = \phi(E_{il}) = u_{il}v_{il}^*$$

equals

$$\begin{aligned}\phi(E_{il})\phi(E_{ll}) + \phi(E_{ll})\phi(E_{il}) &= u_{il}(v_{il}^*u_{ll})v_{ll}^* + u_{ll}(v_{ll}^*u_{il})v_{il}^* \\ &= \langle u_{ll}, v_{il}\rangle u_{il}v_{ll}^* + \langle u_{il}, v_{ll}\rangle u_{ll}v_{il}^*.\end{aligned} \tag{5.1.6}$$

Since $i \in \mathcal{U}_R \setminus \mathcal{V}_R$, we have $u_{il} \parallel u_{ii}$ and thus $\langle u_{il}, v_{ll}\rangle = 0$ by Claim 5.1.8.2, rendering the second term of (5.1.6) zero. We have

$$i \in \mathcal{U}_R \setminus \mathcal{V}_R \overset{\text{Claim 5.1.8.3}}{\Longrightarrow} l \in \mathcal{V}_C \setminus \mathcal{U}_C \implies v_{ll} \parallel v_{il}$$

and therefore (by Claims 5.1.8.5 and 5.1.8.2), the first term of (5.1.6) is

$$\langle u_{ll}, v_{il}\rangle u_{il}v_{ll}^* = \left\langle u_{ll}, \frac{\langle v_{il}, v_{ll}\rangle}{\|v_{ll}\|^2}v_{ll}\right\rangle u_{il}\left(\frac{\|v_{ll}\|^2}{\langle v_{il}, v_{ll}\rangle}v_{il}\right)^* = u_{il}v_{il}^*$$

which proves (5.1.5).

(4°) Suppose $i = l$ but $k \neq j$. By the commutativity of the Jordan product, this case reduces to (3°) by exchanging E_{ij} and E_{kl}.

(5°) Suppose $k \neq j$ and $i \neq l$ (so, as before, $i \in \mathcal{U}_R \setminus \mathcal{V}_R$). Then the left hand side of (5.1.5) is zero, while the right hand side equals

$$\begin{aligned}\phi(E_{ij})\phi(E_{kl}) + \phi(E_{kl})\phi(E_{ij}) &= u_{ij}(v_{ij}^*u_{kl})v_{kl}^* + u_{kl}(v_{kl}^*u_{ij})v_{ij}^* \\ &= \langle u_{kl}, v_{ij}\rangle u_{ij}v_{kl}^* + \langle u_{ij}, v_{kl}\rangle u_{kl}v_{ij}^*.\end{aligned}$$

Note that

$$(i = j \text{ or } (i \neq j \overset{\text{Claim 5.1.8.3}}{\Longrightarrow} j \in \mathcal{V}_C \setminus \mathcal{U}_C)) \implies v_{ij} \parallel v_{jj}. \tag{5.1.7}$$

Suppose that $k \in \mathcal{U}_R$. Then

$$(k = l \text{ or } (k \neq l \implies k \in \mathcal{U}_R \setminus \mathcal{V}_R \overset{\text{Claim 5.1.8.3}}{\Longrightarrow} l \in \mathcal{V}_C \setminus \mathcal{U}_C)) \implies v_{kl} \parallel v_{ll}.$$

We also have

$$i \in \mathcal{U}_R \implies u_{ii} \parallel u_{ij}, \qquad k \in \mathcal{U}_R \implies u_{kk} \parallel u_{kl},$$

so $\langle u_{kl}, v_{ij}\rangle = 0$ and $\langle u_{ij}, v_{kl}\rangle = 0$ by Claim 5.1.8.2.

Suppose now $k \in \mathcal{V}_R \setminus \mathcal{U}_R$. Then in particular $i \neq k$. Assume first that $j \neq l$. We have

$$i \in \mathcal{U}_R \implies u_{ii} \parallel u_{ij}, \qquad k \in \mathcal{V}_R \implies v_{kl} \parallel v_{kk},$$

$$(k = l \text{ or } (k \neq l \overset{\text{Claim 5.1.8.3}}{\implies} l \in \mathcal{U}_C \setminus \mathcal{V}_C)) \implies u_{ll} \parallel u_{kl}$$

so by (5.1.7) we have $\langle u_{kl}, v_{ij}\rangle = \langle u_{ij}, v_{kl}\rangle = 0$ by Claim 5.1.8.2. Now assume $j = l$. Then $i \neq l = j$ so $(i,j),(k,j) \in \rho^\times$ and therefore

$$i \in \mathcal{U}_R \setminus \mathcal{V}_R \overset{\text{Claim 5.1.8.3}}{\implies} j \in \mathcal{V}_C \setminus \mathcal{U}_C \overset{\text{Claim 5.1.8.3}}{\implies} k \in \mathcal{U}_R \setminus \mathcal{V}_R,$$

which is a contradiction. Therefore $j = l$ is impossible.

In either case, this proves $\phi(E_{ij})\phi(E_{kl}) + \phi(E_{kl})\phi(E_{ij}) = 0$.

□

Claim 5.1.8.8. ϕ is injective.

Proof. ϕ is a Jordan homomorphism which clearly satisfies $\phi(E_{ij}) \neq 0$ for all $(i,j) \in \rho$. Now Theorem 4.2.4 directly implies the claim. □

This concludes the proof of (a). Now we prove (b). In view of Lemma 5.1.4, it suffices to prove that ϕ is a rank-one preserver if and only if g^* is a rank-one preserver. Theorem 4.2.4 implies that there exists an invertible matrix $T \in M_n^\times$, a central idempotent $P \in Z(\mathcal{A}_\rho)$, and a transitive map $g : \rho \to \mathbb{C}^\times$ such that

$$\phi(\cdot) = T(Pg^*(\cdot) + (I-P)g^*(\cdot)^t)T^{-1}.$$

For each $X \in \mathcal{A}_\rho$ we have

$$\begin{aligned} r(X) = r(\phi(X)) &= r(T(Pg^*(X) + (I-P)g^*(X)^t)T^{-1}) \\ &= r(Pg^*(X) + (I-P)g^*(X)^t) \\ &\overset{\text{Lemma 5.1.6}}{=} r(g^*(X)) \end{aligned}$$

This implies that $\phi(X)$ is a rank-one matrix if and only if $g^*(X)$ is. Since $X \in \mathcal{A}_\rho$ was an arbitrary rank-one matrix, the claim follows. □

Remark 5.1.9. For a concrete example showing that the converse of Theorem 5.1.7 (a) does not hold in general, consider the map from Example 4.1.1. On the other hand, by direct computations, one easily shows that the converse of Theorem 5.1.7 (a) does hold if $n \leq 3$. Moreover, for general $n \in \mathbb{N}$ and an SMA $\mathcal{A}_\rho \subseteq M_n$, the same holds true if we only assume $|C| \leq 3$ for all $C \in \mathcal{Q}$ (following the notation from Chapter 3). It can be easily shown that this condition, combined with the central decomposition of $\mathcal{A}_\rho$ from Remark 3.1.5, implies that all transitive maps $g : \rho \to \mathbb{C}^\times$ are necessarily trivial. Moreover, using Lemma 5.1.6, it turns out that in this case all Jordan embeddings $\mathcal{A}_\rho \to M_n$ are in fact rank preservers, which will be a topic of the next subsection.

5.1.3 When does the converse of Theorem 5.1.7 hold?

Lemma 5.1.10. *Let ρ be a quasi-order on $[1,n]$ with $1 \leq n \leq 3$. Then all transitive maps $g : \rho \to \mathbb{C}^\times$ are trivial.*

Proof.
- If $n = 1$, the claim is obvious.
- If $n = 2$, then $\mathcal{A}_\rho$ is either $\mathcal{D}_2$, or contains an entire row or an entire column. The first case is clear and the second one follows from Remark 4.1.2.
- Suppose $n = 3$. Then $\rho \cap \{1, 2\}$ is a quasi-order on $\{1, 2\}$ so $g|_{\rho\cap\{1,2\}} : \rho \cap \{1, 2\} \to \mathbb{C}^\times$ is a trivial transitive map. Without losing generality we can assume that $g|_{\rho\cap\{1,2\}} \equiv 1$ so that we are in the position to use Lemma 5.1.2. If $(1, 2), (1, 3) \in \rho$, then $\mathcal{A}_\rho$ contains the entire third column so g is trivial by Remark 4.1.2. If at most one of $(1, 2), (1, 3)$ is in ρ, then the desired condition is trivially fulfilled. Similarly we argue for the elements in the third row.

□

Lemma 5.1.11. *Let $\mathcal{A}_\rho \subseteq M_4$ be a central SMA. Suppose that a transitive map $g : \rho \to \mathbb{C}^\times$ has the property that the induced map $g^* : \mathcal{A}_\rho \to M_4$ is a rank-one preserver. Then g is trivial.*

Proof. $\rho \cap \{1, 2, 3\}$ is a quasi-order on $\{1, 2, 3\}$ so by Lemma 5.1.10 we can without loss of generality assume that $g|_{\rho\cap\{1,2,3\}} \equiv 1$ so that we are in the position to use Lemma 5.1.2. We will discuss elements in the fourth column; the discussion for the fourth row is analogous. Assume that some distinct $i, k \in (\rho^\times)^{-1}(4)$ satisfy $i \approxeq k$, where $\approxeq$ denotes the respective relation on $\rho \cap \{1, 2, 3\}$. We need to show $g(i, 4) = g(k, 4)$. Since we are working on $\{1, 2, 3\}$, by unraveling the definition of $i \approxeq k$ we arrive at these two options:

- Suppose that $i \approxeq_0 k$. Then $(i, k) \in \rho$ or $(k, i) \in \rho$; without loss of generality assume the former. Then $(i, k), (i, 4), (k, k), (k, 4)$ is a rectangle in $\mathcal{A}_\rho$ and therefore by Lemma 5.1.4 we have

$$0 = \begin{vmatrix} g(i,k) & g(i,4) \\ g(k,k) & g(k,4) \end{vmatrix} = \begin{vmatrix} 1 & g(i,4) \\ 1 & g(k,4) \end{vmatrix} \implies g(i,4) = g(k,4).$$

- Suppose that there exists $j \in \{1, 2, 3\} \setminus \{i, k\}$ such that $i \approxeq_0 j \approxeq_0 k$. If $(i, j), (j, k) \in \rho$ or $(j, i), (k, j) \in \rho$, then by transitivity of ρ we arrive at $i \approxeq_0 k$, which was the first case. If $(i, j), (k, j) \in \rho$, then $(i, j), (i, 4), (k, j), (k, 4)$ form a rectangle in $\mathcal{A}_\rho$ and therefore by Lemma 5.1.4 we have

$$0 = \begin{vmatrix} g(i,j) & g(i,4) \\ g(k,j) & g(k,4) \end{vmatrix} = \begin{vmatrix} 1 & g(i,4) \\ 1 & g(k,4) \end{vmatrix} \implies g(i,4) = g(k,4).$$

  Finally, suppose that $(j, i), (j, k) \in \rho$. Then by transitivity from $(j, k), (k, 4) \in \rho$ we obtain $(j, 4) \in \rho$ and clearly $j \approxeq_0 i, k$ so the previous cases allow us to conclude

$$g(i, 4) = g(j, 4) = g(k, 4).$$

□

Claim 5.1.11.1. Let $\mathcal{A}_\rho \subseteq M_5$ be a central SMA. Suppose that a transitive map $g : \rho \to \mathbb{C}^\times$ has the property that the induced map $g^* : \mathcal{A}_\rho \to M_5$ is a rank-one preserver. Then g is trivial.

Proof. The proof of this assertion is completely analogous to the proof of Lemma 5.1.11. Notice that $(g|_{\rho\cap\{1,\dots,4\}})^*$ is a rank-preserver so we can use Lemma 5.1.11 and without loss of generality assume that $g|_{\rho\cap\{1,2,3,4\}} \equiv 1$ so that we are in the position to use Lemma 5.1.2. We will discuss elements in the fifth column; the discussion for the fifth row is analogous. Assume that distinct $i, k \in (\rho^\times)^{-1}(5)$ satisfy $i \approx k$, where $\approx$ denotes the equivalence relation on $\{1,2,3,4\}$ with respect to the quasi-order $\rho \cap \{1,2,3,4\}$. We need to show $g(i,5) = g(k,5)$. Since we are working on $\{1,2,3,4\}$, by unraveling the definition of $i \approx k$ we arrive at these options:

- Suppose that $i \approx_0 k$. Then $(i,k) \in \rho$ or $(k,i) \in \rho$; without loss of generality assume the former. Then $(i,k),(i,5),(k,k),(k,5)$ is a rectangle in $\mathcal{A}_\rho$ and therefore by Lemma 5.1.4 we have
$$0 = \begin{vmatrix} g(i,k) & g(i,5) \\ g(k,k) & g(k,5) \end{vmatrix} = \begin{vmatrix} 1 & g(i,5) \\ 1 & g(k,5) \end{vmatrix} \implies g(i,5) = g(k,5).$$
- Suppose that there exists $j \in \{1,2,3,4\} \setminus \{i,k\}$ such that $i \approx_0 j \approx_0 k$. If $j \in \pi_5^2(\rho^\times)$, then by the previous case it follows that
$$g(i,5) = g(j,5) = g(k,5).$$
This situation is obtained from transitivity when $(j,i) \in \rho$ or $(j,k) \in \rho$. Therefore it only remains to consider $(i,j),(k,j) \in \rho$. Then we observe that the ordered pairs $(i,j),(i,5),(k,j),(k,5)$ form a rectangle on $\mathcal{A}_\rho$ and therefore by Lemma 5.1.4 we have
$$0 = \begin{vmatrix} g(i,j) & g(i,5) \\ g(k,j) & g(k,5) \end{vmatrix} = \begin{vmatrix} 1 & g(i,5) \\ 1 & g(k,5) \end{vmatrix} \implies g(i,5) = g(k,5).$$
- Suppose that there exist distinct $j_1, j_2 \in \{1,2,3,4\} \setminus \{i,k\}$ such that $i \approx_0 j_1 \approx_0 j_2 \approx_0 k$. If $j_1 \in (\rho^\times)^{-1}(5)$ then by the previous two cases it follows that
$$g(i,5) = g(j_1,5) = g(k,5).$$
Similarly when $j_2 \in (\rho^\times)^{-1}(5)$. At least one of these two cases arises from transitivity when $(j_1,i) \in \rho$ or $(j_2,k) \in \rho$. It remains to consider the case $(i,j_1),(k,j_2) \in \rho$ and analyse cases with respect to $j_1 \approx_0 j_2$. If $(j_1,j_2) \in \rho$, then by transitivity $(i,j_2) \in \rho$ and therefore $i \approx_0 j_2 \approx_0 k$ so we are in the previous case. Similarly, if $(j_2,j_1) \in \rho$, then by transitivity $(k,j_1) \in \rho$ and therefore $i \approx_0 j_1 \approx_0 k$ so we are in the previous case. Overall we conclude $g(i,5) = g(k,5)$.

□

In the context of the previous three claims, we conclude:

Lemma 5.1.12. *Let $\mathcal{A}_\rho \subseteq M_n$ be a central SMA with $1 \le n \le 5$. Suppose that a transitive map $g : \rho \to \mathbb{C}^\times$ has the property that the induced map $g^* : \mathcal{A}_\rho \to M_n$ is a rank-one preserver. Then g is trivial.*

Corollary 5.1.13. *Let $\mathcal{A}_\rho \subseteq M_n$ be a SMA such that $|C| \le 5$ for all $C \in \mathcal{Q}$. Then every unital linear rank-one preserver $\phi : \mathcal{A}_\rho \to M_n$ is of the form*
$$\phi(\cdot) = S(P(\cdot) + (I-P)(\cdot)^t)S^{-1}$$
for some central idempotent $P \in \mathcal{D}_n$ and $S \in M_n^\times$.

Proof. By Theorem 5.1.7, ϕ is a Jordan embedding. In view of Lemma 4.2.1, without loss of generality we can assume that $\phi|_{\mathcal{D}_n}$ is the identity map. Now we can use Proposition 4.2.7 and Remark 4.2.8 to conclude that all transitive maps $g_C : \mathcal{A}_C \to \mathbb{C}^\times$ for $C \in \mathcal{Q}$ are trivial so the result follows. □

In particular, by Corollary 5.2.7 it will follow that ϕ is a rank-preserver.

Example 5.1.14. Claim 5.1.12 is false when $n \geq 6$. A counterexample is given by the "staircase algebra", a central SMA $\mathcal{A}_\rho \subseteq M_n$ defined by the quasi-order

$$\rho := \begin{cases} \Delta_n \cup \{(i-1,i),(i+1,i) : 2 \leq i \leq n-2 \text{ even}\} \cup \{(n,2)\}, & \text{if } n \text{ is odd,} \\ \Delta_n \cup \{(i-1,i),(i+1,i) : 2 \leq i \leq n-2 \text{ even}\} \cup \{(1,n),(n-1,n)\}, & \text{if } n \text{ is even.} \end{cases}$$

Consider the transitive map $g : \rho \to \mathbb{C}^\times$ given by

$$g(i,j) := \begin{cases} \alpha, & \text{if } (i,j) = (1,n), \\ \beta, & \text{if } (i,j) = (n-1,n), \\ 1, & \text{otherwise} \end{cases}$$

if n is even, and

$$g(i,j) := \begin{cases} \alpha, & \text{if } (i,j) = (n,2), \\ \beta, & \text{if } (i,j) = (n,n-1), \\ 1, & \text{otherwise} \end{cases}$$

if n is odd, where $\alpha, \beta \in \mathbb{C}^\times, \alpha \neq \beta$. Then the induced map $g^* : \mathcal{A}_\rho \to \mathcal{A}_\rho$ is a rank-one preserver (by Claim 5.1.4) but nontrivial (by Lemma 5.1.2). Visually, for $n = 6, 8$ these algebras and maps are given by

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & \beta \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

while for $n = 7, 9$ they are given by

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \alpha & 0 & 0 & 0 & \beta & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & \beta & 1 \end{bmatrix}.$$

Now we state a sufficient condition for the converse of Theorem 5.1.7 (and the later Theorem 5.2.5):

Corollary 5.1.15. *Let $\mathcal{A}_\rho \subseteq M_n$ be a SMA such that $|C| \leq 3$ for all $C \in \mathcal{Q}$. Then every Jordan homomorphism $\phi : \mathcal{A}_\rho \to M_n$ such that $\phi(E_{ij}) \neq 0$ for all $(i,j) \in \rho$, is a rank preserver.*

Proof. The proof is very similar to the proof of Corollary 5.1.13, but relies on Lemma 5.1.10 instead of Lemma 5.1.11. □

Example 5.1.16. Let $n \geq 4$ and consider the quasi-order

$$\rho = \Delta_n \cup \{(1, n-1), (1, n), (2, n-1), (2, n)\}.$$

Then the map

$$g : \rho \to \mathbb{C}^\times, \qquad g(i,j) = \begin{cases} 2, & \text{if } (i,j) = (1,n), \\ 1, & \text{otherwise} \end{cases}$$

is transitive so the induced map $g^* : \mathcal{A}_\rho \to M_n$ is a Jordan embedding. However, by Lemma 5.1.4 it is not even a rank-one preserver. This SMA $\mathcal{A}_\rho$ satisfies

$$\mathcal{Q} = \{\{1, 2, n-1, n\}\} \cup \{\{i\} : 3 \leq i \leq n-2\}$$

so it has a class of cardinality 4.

5.2 Rank and determinant preservers

5.2.1 Rank preservers

As previously announced, in this section for an arbitrary SMA $\mathcal{A}_\rho \subseteq M_n$ we fully describe the form of all rank preservers $\mathcal{A}_\rho \to M_n$ (Theorem 5.2.5). We start with the following lemma.

Lemma 5.2.1. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA. Assume that $g : \rho \to \mathbb{C}^\times$ is a transitive map such that*

$$g|_{\rho \cap [1.n-1]^2} \equiv 1$$

and that the induced map $g^* : \mathcal{A}_\rho \to M_n$ is a rank preserver. Then g is trivial.

To motivate the proof of Lemma 5.2.1, which is rather technical, we first illustrate it on a concrete example.

Example 5.2.2. Consider the SMA $\mathcal{A}_\rho \subseteq M_{10}$ given by

$$\mathcal{A}_\rho = \begin{bmatrix} (1,1) & (1,2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1,10) \\ 0 & (2,2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (3,2) & (3,3) & (3,4) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (4,4) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (5,4) & (5,5) & (5,6) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (6,6) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (7,6) & (7,7) & (7,8) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (8,8) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (9,8) & (9,9) & (9,10) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (10,10) \end{bmatrix}.$$

If $\approx$ denotes the equivalence relation on $\{1, \ldots, 9\}$ with respect to the quasi-order $\rho \cap \{1, \ldots, 9\}^2$, then by Lemma 5.1.2 we have to show the implications

$$\begin{cases} (\forall i, j \in (\rho^\times)^{-1}(10))(i \approx j \implies g(i, 10) = g(j, 10)), \\ (\forall i, j \in (\rho^\times)(10))(i \approx j \implies g(10, i) = g(10, j)). \end{cases}$$

The second implication is vacuously true, and to prove the first one we only have to consider $1, 9 \in (\rho^\times)^{-1}(10)$. Their equivalence $1 \approx 9$ (in $\rho \cap \{1, \dots, 9\}^2$) manifests through the chain of length 8 given by

$$(1, 2), (3, 2), (3, 4), (5, 4), (5, 6), (7, 6), (7, 8), (9, 8).$$

Consider the following matrix $A \in \mathcal{A}_\rho$, which has ± 1 at those exact positions:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, the first four nonzero columns of A are linearly independent, while the last one is precisely the sum of all the previous ones. Therefore, $r(A) = 4$ and hence the rank of

$$g^*(A) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g(1,10) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -g(9,10) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is 4 as well. Now it is easy to arrive at $g(1, 10) = g(9, 10)$, which is the desired conclusion.

In general, every pair $a \approx b$ for which the implications from Lemma 5.1.2 apply requires us to produce a similar chain of consecutive positions which establishes the equivalence. Hence, first we prove this auxiliary lemma:

Lemma 5.2.3. *Suppose ρ is a quasi-order on $[1, n]$ and assume that distinct $a, b \in [1, n]$ satisfy $a \approx b$ (where $\approx$ corresponds to ρ). Then there exist $m \in \mathbb{N}$ and distinct $a = i_0, i_1, \dots, i_{m-1}, i_m = b \in [1, n]$ such that at least one of the following is true:*

1. $(a, i_1), (i_2, i_1), (i_2, i_3), (i_4, i_3), \dots, (i_{m-1}, i_{m-2}), (i_{m-1}, b) \in \rho^\times$,
2. $(a, i_1), (i_2, i_1), (i_2, i_3), (i_4, i_3), \dots, (i_{m-2}, i_{m-1}), (b, i_{m-1}) \in \rho^\times$,
3. $(i_1, a), (i_1, i_2), (i_3, i_2), (i_3, i_4), \dots, (i_{m-1}, i_{m-2}), (i_{m-1}, b) \in \rho^\times$,
4. $(i_1, a), (i_1, i_2), (i_3, i_2), (i_3, i_4), \dots, (i_{m-2}, i_{m-1}), (b, i_{m-1}) \in \rho^\times$.

Proof. By definition of $\approx$, there exists $m \in \mathbb{N}$ and $a = i_0, i_1, \dots, i_{m-1}, i_m = b \in [1, n]$ such that

$$a = i_0 \approx_0 i_1 \approx_0 i_2 \approx_0 \cdots \approx i_{m-1} \approx_0 i_m = b.$$

We further assume that m is chosen to be minimal (in particular, the elements are distinct). Certainly $m \geq 1$ since $a \neq b$ so i_1 exists.

By definition of $a \approxeq_0 i_1$, we have $(a, i_1) \in \rho$ or $(i_1, a) \in \rho$. First we assume the former. Define a sequence $(s_j)_{j=1}^m$ of length m in $[1, n]^2$ by

$$s_j = \begin{cases} (i_{j-1}, i_j), & \text{if } j \text{ is odd,} \\ (i_j, i_{j-1}), & \text{if } j \text{ is even.} \end{cases}$$

Note that $(s_j)_{j=1}^m$ is precisely a sequence of the form (1) or (2) if m is odd or even, respectively. It therefore suffices to prove that $s_j \in \rho$ for all $1 \leq j \leq m$. We prove this fact by induction on j. For $j = 1$ we have $s_1 = (a, i_1) \in \rho$ by assumption. Assume that $m \geq 2$ and that $s_j \in \rho$ for some $1 \leq j \leq m - 1$. We aim to prove $s_{j+1} \in \rho$. If j is even, then by the inductive hypothesis, we have $s_j = (i_j, i_{j-1}) \in \rho$. Since $i_j \approxeq_0 i_{j+1}$, by definition we have $(i_j, i_{j+1}) \in \rho$ or $(i_{j+1}, i_j) \in \rho$. If the latter is true, by transitivity we would conclude $(i_{j+1}, i_{j-1}) \in \rho$, and hence

$$a = i_0 \approxeq_0 \cdots \approxeq_0 i_{j-1} \approxeq_0 i_{j+1} \approxeq_0 \cdots \approxeq_0 i_m = b$$

which contradicts the minimality of m. Therefore, $s_{j+1} = (i_j, i_{j+1}) \in \rho$, which is what we wanted to show. On the other hand, if j is odd, then by the inductive hypothesis we have $s_j = (i_{j-1}, i_j) \in \rho$. Since $i_j \approxeq_0 i_{j+1}$, by definition we have $(i_j, i_{j+1}) \in \rho$ or $(i_{j+1}, i_j) \in \rho$, but the former would, by transitivity, again contradict the minimality of m. Therefore, we conclude $s_{j+1} = (i_{j+1}, i_j) \in \rho$, thus completing the inductive step.

The other case $(i_1, a) \in \rho$ follows from the above argument applied on the reverse quasi-order ρ^t, yielding a sequence of the form (3) or (4). □

Proof of Lemma 5.2.1. If $\approxeq$ denotes the equivalence relation on $[1, n-1]$ with respect to the quasi-order $\rho \cap [1.n-1]^2$, then by Lemma 5.1.2 we have to show the implications

$$\begin{cases} (\forall i, j \in (\rho^\times)^{-1}(n))(i \approxeq j \implies g(i, n) = g(j, n)), \\ (\forall i, j \in (\rho^\times)(n))(i \approxeq j \implies g(n, i) = g(n, j)). \end{cases}$$

We will show the first one as the second one will then follow by considering the reverse quasi-order ρ^t and the corresponding algebra $\mathcal{A}_{\rho^t}$.

Suppose that distinct $a, b \in (\rho^\times)^{-1}(n)$ satisfy $a \approxeq b$. Then by Lemma 5.2.3 applied on the quasi-order $\rho \cap \{1, \ldots, 9\}^2$ there exists $m \in \mathbb{N}$ and distinct $a = i_0, i_1, \ldots, i_{m-1}, i_m = b \in [1, n-1]$ such that at least one of the conditions $(1) - (4)$ is true.

Assume first that $m = 1$. Then $(a, b) \in \rho$ or $(b, a) \in \rho$. In the first case we notice that $(a, b), (a, n), (b, b), (b, n)$ form a rectangle in $\mathcal{A}_\rho$ so by Lemma 5.1.4 it follows

$$0 = \begin{vmatrix} g(a, b) & g(a, n) \\ g(b, b) & g(b, n) \end{vmatrix} = \begin{vmatrix} 1 & g(a, n) \\ 1 & g(b, n) \end{vmatrix} = g(b, n) - g(a, n).$$

In the second case we proceed similarly with the rectangle $(b, a), (b, n), (a, a), (a, n)$. Therefore in the remainder of the proof we can assume that $m \geq 2$.

Assume that we are in the case (2). In particular, m is even. Consider the matrix $A \in \mathcal{A}_\rho$ given by

$$A := (E_{i_0 i_1} + E_{i_2 i_1}) - (E_{i_2 i_3} + E_{i_4 i_3}) + \cdots + (-1)^{\frac{m}{2}-1}(E_{i_{m-2} i_{m-1}} + E_{i_m i_{m-1}}) + E_{an} + (-1)^{\frac{m}{2}-1} E_{bn}.$$

Note that A has rank exactly equal to $\frac{m}{2}$. Namely, the $\frac{m}{2}$ columns $i_1, i_3, \ldots, i_{m-1}$ are linearly independent, while its n-th column is their sum:

$$\sum_{\substack{0 \leq j \leq m-2 \\ j \text{ even}}} \underbrace{(-1)^{\frac{j}{2}}(e_{i_j} + e_{i_{j+2}})}_{i_{j+1}\text{-th column of } A} = \underbrace{e_a + (-1)^{\frac{m}{2}-1} e_b}_{n\text{-th column of } A}.$$

The map g^* maps the matrix A to the matrix

$$\begin{aligned} g^*(A) = (E_{i_0 i_1} + E_{i_2 i_1}) - (E_{i_2 i_3} + E_{i_4 i_3}) + \cdots + (-1)^{\frac{m}{2}-1}(E_{i_{m-2} i_{m-1}} + E_{i_m i_{m-1}}) \\ + g(a,n) E_{an} + (-1)^{\frac{m}{2}-1} g(b,n) E_{bn} \end{aligned}$$

which, by assumption, also has rank $\frac{m}{2}$. Hence there exist scalars $\alpha_0, \alpha_2, \ldots, \alpha_{m-2} \in \mathbb{C}$ such that

$$\sum_{\substack{0 \leq j \leq m-2 \\ j \text{ even}}} \alpha_j \underbrace{(-1)^{\frac{j}{2}}(e_{i_j} + e_{i_{j+2}})}_{i_{j+1}\text{-th column of } g^*(A)} = \underbrace{g(a,n)e_a + (-1)^{\frac{m}{2}-1} g(b,n) e_b}_{n\text{-th column of } g^*(A)}.$$

Comparing coefficients of e_{i_j} yields

$$\begin{cases} i_0 = a: & \alpha_0 = g(a,n), \\ i_j \text{ for even } 2 \leq j \leq m-2: & (-1)^{\frac{j-2}{2}} \alpha_{j-2} + (-1)^{\frac{j}{2}} \alpha_j = 0 \implies \alpha_j = \alpha_{j-2}, \\ i_m = b: & (-1)^{\frac{m-2}{2}} \alpha_{m-2} = (-1)^{\frac{m}{2}-1} g(b,n) \implies \alpha_{m-2} = g(b,n). \end{cases}$$

Inductively it follows

$$g(a,n) = \alpha_0 = \alpha_2 = \cdots = \alpha_{m-2} = g(b,n),$$

which is precisely what we desired.

Assume that we are in the case (1). Then by transitivity $(i_{m-1}, b), (b, n) \in \rho^\times$ implies $(i_{m-1}, n) \in \rho^\times$. Then a and i_{m-1} are connected by a sequence of the form (2) and hence $g(a,n) = g(i_{m-1}, n)$. Furthermore, we have $i_{m-1} \approxeq_0 b$ so by the $m = 1$ case it follows $g(i_{m-1}, n) = g(b,n)$.

Cases (3) and (4) are treated similarly.

□

Lemma 5.2.4. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA and let $g : \rho \to \mathbb{C}^\times$ be a transitive map. Suppose that the induced map $g^* : \mathcal{A}_\rho \to M_n$ is a rank preserver. Then g is trivial.*

Proof. We prove the claim by induction on n. For $n = 1$ the claim is clear. Suppose that $n \geq 2$ and that the claim holds for all SMAs contained in M_{n-1}. As the automorphism

$$(g|_{\rho \cap [1,n-1]^2})^* : \mathcal{A}_{\rho \cap [1,n-1]^2} \to \mathcal{A}_{\rho \cap [1,n-1]^2}$$

induced by the transitive map $g|_{\rho \cap [1,n-1]^2}$ coincides with the restriction of g^* on $\mathcal{A}_{\rho \cap [1,n-1]^2}$, which is a rank preserver, by the induction hypothesis $g|_{\rho \cap [1,n-1]^2}$ is trivial. Hence, there exists a map $s : [1, n-1] \to \mathbb{C}^\times$ such that

$$g(i,j) = \frac{s(i)}{s(j)}, \qquad \text{for all } (i,j) \in \rho \cap [1, n-1]^2.$$

Additionally set $s(n) := 1$ and define a new transitive map

$$h : \rho \to \mathbb{C}^\times, \qquad h(i,j) := \frac{s(j)}{s(i)} g(i,j).$$

Note that $h|_{\rho\cap[1,n-1]^2} \equiv 1$. Furthermore, if we denote $D := \operatorname{diag}(s(1), \dots, s(n)) \in \mathcal{D}_n^\times$, then one easily verifies that

$$h^* = D^{-1} g^*(\cdot) D$$

so h is also a rank-one preserver. Lemma 5.2.1 implies that h is trivial, which implies that g is trivial as well. □

Theorem 5.2.5. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA. A map $\phi : \mathcal{A}_\rho \to M_n$ is a linear unital rank preserver if and only if there exists an invertible matrix $T \in M_n^\times$ and a central idempotent $P \in Z(\mathcal{A}_\rho)$ such that*

$$\phi(\cdot) = T\left(P(\cdot) + (I-P)(\cdot)^t\right)T^{-1}. \tag{5.2.1}$$

Proof. $\boxed{\Longrightarrow}$ Suppose that $\phi : \mathcal{A}_\rho \to M_n$ is a linear unital rank preserver. By Theorem 5.1.7 (a) it follows that ϕ is a Jordan embedding. Theorem 4.2.4 then implies that there exists an invertible matrix $T \in M_n^\times$, a central idempotent $P \in Z(\mathcal{A}_\rho)$, and a transitive map $g : \rho \to \mathbb{C}^\times$ such that

$$\phi(\cdot) = S(Pg^*(\cdot) + (I-P)g^*(\cdot)^t)S^{-1}.$$

By a similar argument as in the proof of Theorem 5.1.7 (b) we obtain that $r(X) = r(\phi(X)) = r(g^*(X))$ for each $X \in \mathcal{A}_\rho$. It follows that g^* is a rank preserver, so by Lemma 5.2.4 we conclude that the transitive map g is trivial. By Lemma 4.1.3, the induced map g^* is of the form $g^* = D(\cdot)D^{-1}$ for some $D \in \mathcal{D}_n^\times$. Consider $\Gamma \in \mathcal{D}_n^\times$ given by

$$\Gamma_{jj} := \begin{cases} D_{jj}, & \text{if } (j,j) \in \operatorname{supp} P, \\ \frac{1}{D_{jj}}, & \text{if } (j,j) \notin \operatorname{supp} P. \end{cases}$$

For all $X \in \mathcal{A}_\rho$ we have

$$\begin{aligned} \phi(X) &= S\left(Pg^*(X) + (I-P)g^*(X)^t\right)S^{-1} \\ &= S\left(PDXD^{-1} + (I-P)D^{-1}X^tD\right)S^{-1} \\ &= S\left(D(PX)D^{-1} + D^{-1}((I-P)X^t)D\right)S^{-1} \\ &= S\left(\Gamma(PX)\Gamma^{-1} + \Gamma((I-P)X^t)\Gamma^{-1}\right)S^{-1} \\ &= T(PX + (I-P)X^t)T^{-1} \end{aligned}$$

where $T := S\Gamma \in \mathcal{A}_\rho^\times$.

$\boxed{\Longleftarrow}$ Suppose that $\phi : \mathcal{A}_\rho \to M_n$ is of the form (5.2.1) for some invertible matrix $T \in M_n^\times$ and a central idempotent $P \in Z(\mathcal{A}_\rho)$. Then, by Lemma 5.1.6, for all $X \in \mathcal{A}_\rho$ we have

$$r(\phi(X)) = r(PX + (I-P)X^t) = r(X),$$

so that ϕ is a rank preserver.

□

Remark 5.2.6.

(a) In Lemmas 5.2.1 and 5.2.4 and in Theorem 5.2.5 it actually suffices to assume that the respective map preserves the rank of matrices up to $\max\left\{1, \left\lfloor \frac{n}{2} \right\rfloor - 1\right\}$. Indeed, let $1 \leq a, b \leq n-1$ and $m \in \mathbb{N}$ be as in the beginning of the proof of Lemma 5.2.1. To prove the result, we use the fact that g^* is a rank-one preserver, while if $m \geq 2$ then we additionally use that g^* preserves rank $\frac{m}{2}$. Since $i_0, i_1, \ldots, i_m \in \{1, \ldots, n-1\}$ are distinct, it follows that $m + 1 \leq n - 1$ so we arrive at $\frac{m}{2} \leq \frac{n}{2} - 1$.

(b) As a consequence of (a), every unital rank-one preserver $\phi : \mathcal{A}_\rho \to M_n$ where $n \leq 5$, is necessarily a rank preserver. In fact, we already showed in Corollary 5.1.13 that for general $n \in \mathbb{N}$ and $\mathcal{A}_\rho \subseteq M_n$ the same result holds whenever $|C| \leq 5$ for all $C \in \mathcal{Q}$.

Corollary 5.2.7. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA. A map $\phi : \mathcal{A}_\rho \to M_n$ is a linear rank preserver if and only if there exist invertible matrices $S, T \in M_n^\times$ and a central idempotent $P \in Z(\mathcal{A}_\rho)$ such that*

$$\phi(\cdot) = S\left(P(\cdot) + (I - P)(\cdot)^t\right)T.$$

Proof. In either direction it follows that the matrix $\phi(I) \in M_n$ is invertible. The equivalence follows by applying Theorem 5.2.5 to the unital map $\phi(I)^{-1}\phi(\cdot)$. □

Example 5.2.8. Corollary 5.2.7 cannot be further strengthened by assuming that $\phi : \mathcal{A}_\rho \to M_n$ is only a rank-k preserver for all $1 \leq k \leq n-1$. Indeed, let $\mathcal{A}_\rho = \mathcal{D}_n$ and consider the map $\phi : \mathcal{D}_n \to M_n$ defined by

$$\operatorname{diag}(x_{11}, \ldots, x_{nn}) \mapsto \begin{bmatrix} x_{11} + x_{nn} & x_{nn} & \cdots & x_{nn} & 0 \\ x_{nn} & x_{22} + x_{nn} & \cdots & x_{nn} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{nn} & x_{nn} & \cdots & x_{(n-1)(n-1)} + x_{nn} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Clearly, $\phi(I)$ is a singular matrix, so in particular ϕ is not of the form outlined in Corollary 5.2.7. Further, ϕ preserves rank of all singular matrices (hence ranks up to $n-1$). Indeed, let $X \in \mathcal{D}_n$ be a singular matrix. We need to prove that $r(\phi(X)) = r(X)$. This is clear if $x_{nn} = 0$ so assume that $x_{nn} \neq 0$. By conjugating by a permutation matrix we can further assume that $x_{11} = 0$. By subtracting the first row from rows $2, \ldots, n-1$, we conclude $\phi(X)$ has the same rank as

$$\begin{bmatrix} x_{nn} & x_{nn} & \cdots & x_{nn} & 0 \\ 0 & x_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_{(n-1)(n-1)} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \sim \operatorname{diag}(x_{nn}, x_{22}, \ldots x_{(n-1)(n-1)}, 0) \sim X,$$

where $\sim$ denotes the matrix equivalence.

5.2.2 Remarks about determinant preservers

Note that (algebra) automorphisms of a unital subalgebra $\mathcal{A} \subseteq M_n$ are not necessarily determinant preservers. Indeed, consider the subalgebra

$$\mathcal{A} := \{\operatorname{diag}(x, x, y) : x, y \in \mathbb{C}\} \subseteq M_3$$

and the automorphism $\phi \in \operatorname{Aut}(\mathcal{A})$ given by

$$\phi(\operatorname{diag}(x, x, y)) := \operatorname{diag}(y, y, x).$$

On the other hand, SMAs do satisfy this property. We establish this fact in the remainder of this short subsection.

Lemma 5.2.9. *Let $\mathcal{S} \subseteq [1, n]$ be a subset and suppose that $X \in M_n$ satisfies*

$$\operatorname{supp} X \subseteq (\mathcal{S} \times \mathcal{S}) \cup (\mathcal{S}^c \times \mathcal{S}^c).$$

Define an idempotent $P := \sum_{i \in \mathcal{S}} E_{ii} \in M_n$. Then

$$\det(PX + (I - P)X^t) = \det X.$$

Proof. Define the subgroup

$$S_n(\mathcal{S}) := \{\sigma \in S_n : \sigma(\mathcal{S}) = \mathcal{S}\} \leq S_n$$

of all permutations $\sigma \in S_n$ which fix the sets $\mathcal{S}$ and $\mathcal{S}^c$ (this is usually called the stabilizer subgroup). Since

$$\operatorname{supp}(PX + (I - P)X^t) \subseteq (\mathcal{S} \times \mathcal{S}) \cup (\mathcal{S}^c \times \mathcal{S}^c),$$

when calculating $\det(PX + (I - P)X^t)$, it suffices to sum over permutations from $S_n(\mathcal{S})$. We have

$$\begin{aligned}
\det(PX + (I - P)X^t) &= \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \prod_{1 \leq i \leq n} (PX + (I - P)X^t)_{i\sigma(i)} \\
&= \sum_{\sigma \in S_n(\mathcal{S})} (\operatorname{sgn} \sigma) \prod_{1 \leq i \leq n} (PX + (I - P)X^t)_{i\sigma(i)} \\
&= \sum_{\sigma \in S_n(\mathcal{S})} (\operatorname{sgn} \sigma) \left(\prod_{i \in \mathcal{S}} (PX + (I - P)X^t)_{i\sigma(i)} \right) \\
&\quad \cdot \left(\prod_{i \in \mathcal{S}^c} (PX + (I - P)X^t)_{i\sigma(i)} \right) \\
&= \sum_{\sigma \in S_n(\mathcal{S})} (\operatorname{sgn} \sigma) \left(\prod_{i \in \mathcal{S}} X_{i\sigma(i)} \right) \left(\prod_{i \in \mathcal{S}^c} X_{\sigma(i)i} \right).
\end{aligned}$$

Notice now that every $\sigma \in S_n(\mathcal{S})$ can be uniquely written as a composition $\sigma = \sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$ of two permutations $\sigma_1, \sigma_2 \in S_n(\mathcal{S})$ such that $\sigma_1|_{\mathcal{S}}$ and $\sigma_2|_{\mathcal{S}^c}$ act as the identity. Having this in mind, we continue the calculation.

$$\det(PX + (I - P)X^t) = \sum_{\substack{\sigma_1, \sigma_2 \in S_n(\mathcal{S}), \\ \sigma_1|_{\mathcal{S}} = \operatorname{id}_{\mathcal{S}}, \\ \sigma_2|_{\mathcal{S}^c} = \operatorname{id}_{\mathcal{S}^c}}} (\operatorname{sgn} \sigma_1)(\operatorname{sgn} \sigma_2)) \left(\prod_{i \in \mathcal{S}} X_{i\sigma_1(i)} \right) \left(\prod_{i \in \mathcal{S}^c} X_{\sigma_2(i)i} \right)$$

$$
\begin{aligned}
&= \sum_{\substack{\sigma_1,\sigma_2\in S_n(\mathcal{S}),\\ \sigma_1|_{\mathcal{S}}=\mathrm{id}_{\mathcal{S}},\\ \sigma_2|_{\mathcal{S}^c}=\mathrm{id}_{\mathcal{S}^c}}} (\operatorname{sgn}\sigma_1)(\operatorname{sgn}\sigma_2^{-1}) \left(\prod_{i\in\mathcal{S}} X_{i\sigma_1(i)}\right)\left(\prod_{i\in\mathcal{S}^c} X_{i\sigma_2^{-1}(i)}\right)\\
&= \sum_{\substack{\sigma_1,\sigma_2\in S_n(\mathcal{S}),\\ \sigma_1|_{\mathcal{S}}=\mathrm{id}_{\mathcal{S}},\\ \sigma_2|_{\mathcal{S}^c}=\mathrm{id}_{\mathcal{S}^c}}} (\operatorname{sgn}\sigma_1)(\operatorname{sgn}\sigma_2) \left(\prod_{i\in\mathcal{S}} X_{i\sigma_1(i)}\right)\left(\prod_{i\in\mathcal{S}^c} X_{i\sigma_2(i)}\right)\\
&= \sum_{\sigma\in S_n(\mathcal{S})} (\operatorname{sgn}\sigma) \prod_{1\le i\le n} X_{i\sigma_2(i)}\\
&= \sum_{\sigma\in S_n} (\operatorname{sgn}\sigma) \prod_{1\le i\le n} X_{i\sigma_2(i)}\\
&= \det X
\end{aligned}
$$

Here we used the fact that for $\sigma_2 \in S_n$ such that $\sigma_2|_{\mathcal{S}} = \mathrm{id}_{\mathcal{S}}$, we have

$$\{X_{\sigma_2(i)i} : i \in \mathcal{S}^c\} = \{X_{i\sigma_2(i)} : i \in \mathcal{S}^c\}$$

and that σ_2^{-1} has the same property $\sigma_2^{-1}|_{\mathcal{S}} = \mathrm{id}_{\mathcal{S}}$. □

Lemma 5.2.10. *Let ρ be a quasi-order on $[1, n]$ and let $g : \rho \to \mathbb{C}^\times$ be a transitive map. Then the induced map $g^* : \mathcal{A}_\rho \to M_n$ is a determinant preserver.*

Proof. Denote by

$$S_n^\rho := \{\sigma \in S_n : (i, \sigma(i)) \in \rho, \forall 1 \le i \le n\} \le S_n$$

the subgroup of all permutations $\sigma \in S_n$ such that the graph of the function σ is contained in ρ. Let $X \in \mathcal{A}_\rho$ be arbitrary. Since X and $g^*(X)$ are supported on ρ, we have

$$
\begin{aligned}
\det g^*(X) &= \sum_{\sigma\in S_n} (\operatorname{sgn}\sigma) \prod_{1\le i\le n} g^*(X)_{i\sigma(i)}\\
&= \sum_{\sigma\in S_n} (\operatorname{sgn}\sigma) \prod_{1\le i\le n} g(i,\sigma(i)) X_{i\sigma(i)}\\
&= \sum_{\sigma\in S_n^\rho} (\operatorname{sgn}\sigma) \left(\prod_{1\le i\le n} g(i,\sigma(i))\right)\left(\prod_{1\le i\le n} X_{i\sigma(i)}\right).
\end{aligned}
$$

To prove $\det g^*(X) = \det X$, for a fixed $\sigma \in S_n^\rho$ we need to show that $\prod_{1\le i\le n} g(i, \sigma(i)) = 1$. We have

$$
\begin{aligned}
\prod_{1\le i\le n} g(i,\sigma(i)) &= \left(\left(\prod_{1\le i\le n} g(i,\sigma(i))\right)\left(\prod_{1\le i\le n} g(\sigma(i),\sigma^2(i))\right)\right)^{\frac{1}{2}}\\
&= \left(\prod_{1\le i\le n} g(i,\sigma(i))g(\sigma(i),\sigma^2(i))\right)^{\frac{1}{2}}\\
&\overset{\text{transitivity of } g}{=} \left(\prod_{1\le i\le n} g(i,\sigma^2(i))\right)^{\frac{1}{2}}
\end{aligned}
$$

$$= \left(\prod_{1\le i\le n} g(i,\sigma^4(i))\right)^{\frac{1}{4}}$$
$$\vdots$$
$$= \left(\prod_{1\le i\le n} g(i,\sigma^{2^k}(i))\right)^{\frac{1}{2^k}}$$

for any $k \in \mathbb{N}$. By choosing $k \in \mathbb{N}$ large enough such that 2^k exceeds the order of σ, we obtain that the last expression is equal to 1, which is what we wanted to show. □

Proposition 5.2.11. *Let $\mathcal{A}_\rho \subseteq M_n$ be a SMA and let $\phi : \mathcal{A}_\rho \to M_n$ be a Jordan embedding. Then ϕ is a determinant preserver.*

Proof. By Theorem 4.2.4, there exists an invertible matrix $S \in M_n^\times$, a central idempotent $P \in Z(\mathcal{A}_\rho)$, and a transitive map $g : \rho \to \mathbb{C}^\times$ such that

$$\phi(\cdot) = S(Pg^*(\cdot) + (I-P)g^*(\cdot)^t)S^{-1}.$$

For all $X \in \mathcal{A}_\rho$ note that

$$\operatorname{supp} X \subseteq ((\operatorname{supp} P) \times (\operatorname{supp} P)) \cup ((\operatorname{supp} P)^c \times (\operatorname{supp} P)^c)$$

and therefore

$$\begin{aligned}
\det \phi(X) &= \det(S(Pg^*(X) + (I-P)g^*(X)^t)S^{-1}) \\
&= \det(Pg^*(X) + (I-P)g^*(X)^t) \\
&\overset{\text{Lemma 5.2.9}}{=} \det g^*(X) \\
&\overset{\text{Lemma 5.2.10}}{=} \det X.
\end{aligned}$$

□

Remark 5.2.12. There is a shorter, less computational proof of the previous fact which we outline here. In view of Lemma 3.1.4, one can assume that $\mathcal{A}_\rho$ satisfies

$$\operatorname{diag}(M_{k_1}, \ldots, M_{k_p}) \subseteq \mathcal{A}_\rho \subseteq \mathcal{A}_{k_1,\ldots,k_p}$$

for some block upper-triangular subalgebra $\mathcal{A}_{k_1,\ldots,k_p} \subseteq M_n$. Then, by Lemma 4.2.1, one can assume that $\phi|_{\mathcal{D}_n}$ is the identity map. Then similarly as in the proof of Proposition 4.2.7, it follows that ϕ maps each diagonal block of $\mathcal{A}_\rho$ into itself. On the other hand, the restriction of ϕ onto each diagonal block M_{k_j} is a Jordan embedding of M_{k_j} and hence a determinant preserver. Since the determinant of $\phi(X)$ is completely determined by the action of ϕ on the diagonal blocks of the matrix $X \in \mathcal{A}_\rho$, it follows $\det \phi(X) = \det X$.

Example 5.2.13. In contrast with the M_n case (Theorem 2.5.1), linear determinant preservers $\mathcal{A} \to M_n$ on a general SMA $\mathcal{A} \subseteq M_n$ are not necessarily Jordan homomorphisms. For example, any linear map $\phi : \mathcal{T}_n \to M_n$ which acts as identity on the diagonal and satisfies $\phi(E_{ij}) \parallel E_{ij}$ for all $1 \le i < j \le n$ clearly preserves the determinant, but is nowhere close to a Jordan homomorphism in general. Another such example is any map of the form $X \mapsto SXT$ for any $S, T \in M_n^\times$ with $\det(ST) = 1$ and $T \ne S^{-1}$, though this is less interesting since it is not a unital example.

Chapter 6

Nonlinear preserver problems on SMAs

6.1 Groundwork

Lemma 6.1.1. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA. Define*

$$\mathcal{R} := \{A \in \mathcal{A}_\rho : A \text{ is a rank-one non-nilpotent}\}.$$

Then

$$\overline{\mathcal{R}} = \{ab^* \in M_n : a, b \in \mathbb{C}^n, \exists 1 \le k \le n \text{ such that } ae_k^*, e_k b^* \in \mathcal{A}_\rho\} \subseteq \mathcal{A}_\rho.$$

In particular, $\overline{\mathcal{R}}$ contains all matrices in $\mathcal{A}_\rho$ supported in a single row or a single column.

Proof. Note that

$$\mathcal{R} = \{uv^* \in \mathcal{A}_\rho : u, v \in \mathbb{C}^n, v^* u \neq 0\}.$$

$\boxed{\subseteq}$ By the lower-semicontinuity of the rank, clearly any nonzero element $A \in \overline{\mathcal{R}}$ has rank one and hence is of the form $A = ab^*$ for some nonzero vectors $a, b \in \mathbb{C}^n$. Since $A \in \mathcal{A}_\rho$, we have

$$(\operatorname{supp} a) \times (\operatorname{supp} b) = \operatorname{supp} A \subseteq \rho.$$

For the sake of concreteness, we assume that M_n is equipped with the norm

$$\|X\|_\infty := \max_{1 \le i,j \le n} |X_{ij}|.$$

Denote

$$\mu := \min_{(i,j) \in \operatorname{supp} A} |A_{ij}| > 0.$$

By the assumption, there exist $uv^* \in \mathcal{R}$ (where $u, v \in \mathbb{C}^n, v^* u \neq 0$) such that $\|uv^* - A\|_\infty < \mu$. In particular, for each $(i,j) \in \operatorname{supp} A$ we have

$$|A_{ij}| - |(uv^*)_{ij}| \le |A_{ij} - (uv^*)_{ij}| < \mu \implies |(uv^*)_{ij}| > |A_{ij}| - \mu \ge 0$$

so $u_i \overline{v}_j = (uv^*)_{ij} \neq 0$. It follows

$$(\operatorname{supp} a) \times (\operatorname{supp} b) = \operatorname{supp} A \subseteq \operatorname{supp}(uv^*) = (\operatorname{supp} u) \times (\operatorname{supp} v)$$

which implies $\operatorname{supp} a \subseteq \operatorname{supp} u$ and $\operatorname{supp} b \subseteq \operatorname{supp} v$. Since $v^* u \neq 0$, we can choose some

$1 \le k \le n$ such that $u_k\overline{v}_k \neq 0$. Then $k \in (\operatorname{supp} u) \cap (\operatorname{supp} v)$ and therefore

$$\begin{aligned}((\operatorname{supp} a) \times \{k\}) \cup (\{k\} \times (\operatorname{supp} b)) &\subseteq ((\operatorname{supp} u) \times \{k\}) \cup (\{k\} \times (\operatorname{supp} v)) \\ &\subseteq (\operatorname{supp} u) \times (\operatorname{supp} v) \subseteq \rho.\end{aligned}$$

In particular, we have

$$\operatorname{supp}(ae_k^*) = (\operatorname{supp} a) \times \{k\} \subseteq \rho \implies ae_k^* \in \mathcal{A}_\rho$$

and

$$\operatorname{supp}(e_k b^*) = \{k\} \times (\operatorname{supp} b) \subseteq \rho \implies e_k b^* \in \mathcal{A}_\rho.$$

$\boxed{\supseteq}$ Clearly $0 \in \overline{\mathcal{R}}$. Suppose now that $ab^* \in \mathcal{A}_\rho$ for some nonzero vectors $a, b \in \mathbb{C}^n$ such that $ae_k^*, e_k b^* \in \mathcal{A}_\rho$ for some $1 \le k \le n$. We claim that $ab^* \in \overline{\mathcal{R}}$. If $b^*a \neq 0$, then clearly $ab^* \in \mathcal{R} \subseteq \overline{\mathcal{R}}$ so assume $b^*a = 0$.

For $\varepsilon > 0$ consider

$$A_\varepsilon := (a + \varepsilon e_k)(b + \varepsilon e_k)^* \in M_n.$$

We have

$$A_\varepsilon = ab^* + \varepsilon(ae_k^* + e_k b^*) + \varepsilon^2 E_{kk} \in \mathcal{A}_\rho$$

and clearly $\lim_{\varepsilon \to 0} A_\varepsilon = ab^*$. Furthermore,

$$(b + \varepsilon e_k)^*(a + \varepsilon e_k) = b^*a + \varepsilon(e_k^* a + b^* e_k) + \varepsilon^2 e_k e_k^* = \varepsilon(a_k + \overline{b_k}) + \varepsilon^2$$

which is nonzero when $\varepsilon \neq -(a_k + \overline{b_k})$, implying that $A_\varepsilon \in \mathcal{R}$ for such ε. This completes the proof.

Finally, suppose that a matrix $A \in \mathcal{A}_\rho$ is supported in a single row $j \in [1, n]$. Then there exists a vector $b \in \mathbb{C}^n$ such that $A = e_j b^*$. We have $e_j b^*, e_j e_j^* = E_{jj} \in \mathcal{A}_\rho$ so $A \in \overline{\mathcal{R}}$. The case when a matrix is supported in a single column is treated by a similar argument. $\square$

The next example shows that for a general SMA $\mathcal{A}_\rho \subseteq M_n$, the set $\mathcal{R}$ does not need to be dense in the set of all rank-one matrices in $\mathcal{A}_\rho$.

Example 6.1.2. Consider the quasi-order

$$\rho := \Delta_4 \cup \{(1,3), (1,4), (2,3), (2,4)\}$$

on $[1, 4]$ and the corresponding SMA

$$A_\rho = \begin{bmatrix} * & 0 & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \subseteq \mathcal{T}_4.$$

Then

$$A := \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{A}_\rho$$

is a rank-one matrix such that $A \notin \overline{\mathcal{R}}$, as for any $0 < \varepsilon < 1$, the $\|\cdot\|_\infty$-ball $B(A,\varepsilon)$ in $\mathcal{A}_\rho$ does not intersect $\mathcal{R}$. Indeed, if $B \in B(A,\varepsilon)$ is a diagonalizable matrix in $\mathcal{A}_\rho$, then there exists some $j \in [1,4]$ such that $([1,2]\times[3,4]) \cup \{(j,j)\} \subseteq \operatorname{supp} B$. Hence, B cannot be of rank-one.

Alternatively, note that $A = (e_1+e_2)(e_3+e_4)^*$ and

$$(e_1+e_2)e_1^*, (e_1+e_2)e_2^*, e_3(e_3+e_4)^*, e_4(e_3+e_4)^* \notin \mathcal{A}_\rho.$$

Therefore, A does not satisfy the condition of Lemma 6.1.1 and thus $A \notin \overline{\mathcal{R}}$.

Remark 6.1.3. In fact, given a quasi-order ρ on $[1,n]$, one easily sees that $\mathcal{R}$ is dense in the set of all rank-one matrices in $\mathcal{A}_\rho$ if and only if for all subsets $S,T \subseteq [1,n]$ we have

$$S \times T \subseteq \rho \implies \exists k \in [1,n] \text{ such that } (S\times\{k\}) \cup (\{k\}\times T) \subseteq \rho.$$

It is not difficult to check that this condition is fulfilled for all block upper-triangular subalgebras of M_n.

Lemma 6.1.4. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA. Then the set*

$$\{S\operatorname{diag}(\lambda_1,\ldots,\lambda_n)S^{-1} : S \in \mathcal{A}_\rho^\times, \lambda_1,\ldots,\lambda_n \textit{ pairwise distinct}\}$$

is dense in $\mathcal{A}_\rho$.

Proof.

Case 1. First we consider the case when $\operatorname{diag}(M_{k_1},\ldots,M_{k_p}) \subseteq \mathcal{A}_\rho \subseteq \mathcal{A}_{k_1,\ldots,k_p}$. Let $A \in \mathcal{A}_\rho$ be arbitrary and let $\varepsilon > 0$. By applying the Schur triangularization on each diagonal block, we obtain a unitary block-diagonal matrix $U \in \mathcal{A}_\rho^\times$ such that $U^*AU \in \mathcal{T}_n$. Let $\Theta \in \mathcal{T}_n$ be a matrix which is identical to U^*AU outside the diagonal, while its diagonal $D \in \mathcal{D}_n$ consists of pairwise distinct complex numbers $\Theta_{11},\ldots,\Theta_{nn}$ such that

$$\sum_{k\in[1,n]} |(U^*AU)_{kk} - \Theta_{kk}|^2 < \varepsilon^2.$$

Clearly, $\operatorname{supp}\Theta \subseteq \operatorname{supp}(U^*AU) \cup \Delta_n \subseteq \rho$, so $\Theta \in \mathcal{A}_\rho$. If $\|\cdot\|_F$ denotes the Frobenius norm on M_n, we have

$$\|A - U\Theta U^*\|_F = \|U^*AU - \Theta\|_F = \left(\sum_{k\in[1,n]} |(U^*AU)_{kk} - \Theta_{kk}|^2\right)^{\frac{1}{2}} < \varepsilon.$$

Since $U\Theta U^* \in \mathcal{A}_\rho$ has n distinct eigenvalues, it remains to apply Theorem 3.2.1.

Case 2. Now we consider the general case. By Lemma 3.1.4, there exists a permutation $\pi \in S_n$ such that

$$\operatorname{diag}(M_{k_1},\ldots,M_{k_p}) \subseteq R_\pi \mathcal{A}_\rho R_\pi^{-1} \subseteq \mathcal{A}_{k_1,\ldots,k_p},$$

where $R_\pi \in M_n^\times$ is defined by (2.1.2). By Case 1, the set

$$\mathcal{S} := \{S\operatorname{diag}(\lambda_1,\ldots,\lambda_n)S^{-1} : S \in (R_\pi\mathcal{A}_\rho R_\pi^{-1})^\times = R_\pi \mathcal{A}_\rho^\times R_\pi^{-1}, \lambda_1,\ldots,\lambda_n \in \mathbb{C} \text{ p. d.}\}$$

is dense in $R_\pi \mathcal{A}_\rho R_\pi^{-1}$, which immediately implies that $R_\pi^{-1}\mathcal{S}R_\pi$, which equals

$$\{(R_\pi^{-1}SR_\pi)\operatorname{diag}(\lambda_{\pi^{-1}(1)},\dots,\lambda_{\pi^{-1}(n)})(R_\pi^{-1}SR_\pi)^{-1} : S \in R_\pi\mathcal{A}_\rho^\times R_\pi^{-1}, \lambda_1,\dots,\lambda_n \in \mathbb{C} \text{ p. d.}\}$$
$$= \{T\operatorname{diag}(\mu_1,\dots,\mu_n)T^{-1} : T \in \mathcal{A}_\rho^\times, \mu_1,\dots,\mu_n \in \mathbb{C} \text{ p. d.}\},$$

is dense in $\mathcal{A}_\rho$ (where p. d. abbreviates "pairwise distinct").

□

6.2 Main results

We say that an SMA $\mathcal{A}_\rho \subseteq M_n$ is *2-free* if $|C| \neq 2$ for all $C \in \mathcal{Q}$.

Proposition 6.2.1. *Let $\mathcal{A}_\rho \subseteq M_n, n \geq 3$ be a 2-free SMA and let $\phi : \mathcal{A}_\rho \to M_n$ be an injective continuous commutativity and spectrum preserver. Then there exists $S \in M_n^\times$, a transitive map $g : \rho \to \mathbb{C}^\times$ and quasi-orders $\rho_M^\phi, \rho_A^\phi \subseteq \rho$ such that $\rho_M^\phi \cup \rho_A^\phi = \rho, \rho_M^\phi \cap \rho_A^\phi = \Delta_n$ and*

$$\phi(E_{ij}) = \begin{cases} g(i,j)SE_{ij}S^{-1}, & \text{if } (i,j) \in \rho_M^\phi, \\ g(i,j)SE_{ji}S^{-1}, & \text{if } (i,j) \in \rho_A^\phi. \end{cases}$$

Proof. In view of Theorem 2.6.1, we may assume throughout that $\mathcal{A}_\rho \subsetneq M_n$.

Claim 6.2.1.1. ϕ preserves characteristic polynomial.

Proof. ϕ clearly preserves characteristic polynomial on the set

$$\{S\operatorname{diag}(\lambda_1,\dots,\lambda_n)S^{-1} : S \in \mathcal{A}_\rho^\times, \lambda_1,\dots,\lambda_n \in \mathbb{C} \text{ pairwise distinct}\}$$

so the claim follows by the continuity of ϕ and of the characteristic polynomial. □

Claim 6.2.1.2. Without loss of generality we can assume $\phi(\Lambda_n) = \Lambda_n$ and hence that ϕ acts as the identity map on $\mathcal{D}_n$.

Proof. Since the matrix $\phi(\Lambda_n) \in M_n$ is diagonalizable with eigenvalues $1,\dots,n$, there exists an $S \in M_n^\times$ such that $\phi(\Lambda_n) = S\Lambda_n S^{-1}$. By passing to the map $S^{-1}\phi(\cdot)S$, we can assume $\phi(\Lambda_n) = \Lambda_n$. Fix an arbitrary $D \in \mathcal{D}_n$. We have $D \leftrightarrow \Lambda_n$ and hence $\phi(D) \leftrightarrow \phi(\Lambda_n) = \Lambda_n$, since $D \leftrightarrow \Lambda_n$. We conclude $\phi(D) \in \mathcal{D}_n$. The same argument from [47, Lemma 2.1] now gives that $\phi(D) = D$. For completeness, we include it here. Assume first that the diagonal entries of D are all distinct, and denote them by $\lambda_1,\dots,\lambda_n$. Choose continuous paths $f_k : [0,1] \to \mathbb{C}, 1 \leq k \leq n$ from k to λ_k such that for all $t \in [0,1]$ the values $f_1(t),\dots,f_n(t)$ are all distinct. To be explicit, for each $1 \leq k \leq n$ and any path

$$\alpha_k : [0,1] \to (\mathbb{C} \setminus \{1,\dots,n,\lambda_1,\dots,\lambda_n\}) \cup \{k,\lambda_k\}$$

from k to λ_k (which exists by path-connectedness), we can define

$$f_k(t) := \begin{cases} k, & \text{if } t \in \left[0, \frac{k-1}{n}\right], \\ \alpha_k\left(n\left(t - \frac{k-1}{n}\right)\right), & \text{if } t \in \left[\frac{k-1}{n}, \frac{k}{n}\right], \\ \lambda_k, & \text{if } t \in \left[\frac{k}{n}, 1\right]. \end{cases}$$

Denote

$$d := \min_{t\in[0,1]} \Big\{|f_i(t) - f_j(t)| : 1 \le i, j \le n\Big\} > 0.$$

Notice that the set

$$\begin{aligned}\mathcal{S} &= \{t \in [0,1] : \phi(\operatorname{diag}(f_1(t), \dots, f_n(t))) \ne \operatorname{diag}(f_1(t), \dots, f_n(t))\} \\ &= \{t \in [0,1] : \|\phi(\operatorname{diag}(f_1(t), \dots, f_n(t))) - \operatorname{diag}(f_1(t), \dots, f_n(t))\|_\infty \ge d\}\end{aligned}$$

is both open and closed in $[0,1]$. Since $0 \notin \mathcal{S}$, by the connectedness of $[0,1]$ it follows that $\mathcal{S} = \emptyset$. In particular, for $t = 1$ we get

$$\begin{aligned}\phi(\operatorname{diag}(\lambda_1, \dots, \lambda_n)) &= \phi(\operatorname{diag}(f_1(1), \dots, f_n(1))) = S \operatorname{diag}(f_1(1), \dots, f_n(1)) S^{-1} \\ &= S \operatorname{diag}(\lambda_1, \dots, \lambda_n) S^{-1}.\end{aligned}$$

As the diagonal matrices with distinct eigenvalues are dense in $\mathcal{D}_n$ and ϕ is continuous, the claim follows for all $D \in \mathcal{D}_n$. □

Claim 6.2.1.3. For each $S \in \mathcal{A}_\rho^\times$ there exists $T \in M_n^\times$ such that

$$\phi(SDS^{-1}) = TDT^{-1}, \qquad \text{for all } D \in \mathcal{D}_n.$$

Proof. For a fixed $S \in \mathcal{A}_\rho^\times$ there exists $T \in M_n^\times$ such that $\phi(S\Lambda_n S^{-1}) = T\Lambda_n T^{-1}$. Now we can apply the Claim 6.2.1.2 to the map $T^{-1}\phi(S(\cdot)S^{-1})T$ which satisfies the same properties as ϕ, as well as $\Lambda_n \mapsto \Lambda_n$. □

Claim 6.2.1.4. Fix $k \in \{1, 2, \dots, n\}$. Let $S \in \mathcal{A}_\rho^\times$ and $T \in M_n^\times$ be invertible matrices such that $\phi(SE_{ii}S^{-1}) = TE_{ii}T^{-1}$ for all $i \in [1,n] \setminus \{k\}$. Then $\phi(SE_{kk}S^{-1}) = TE_{kk}T^{-1}$.

Proof. We know that $\phi(S\Lambda_n S^{-1}) = P\Lambda_n P^{-1}$ for some invertible $P \in M_n^\times$, which by Claim 6.2.1.3 implies that $\phi(SDS^{-1}) = PDP^{-1}$ for every diagonal matrix $D \in \mathcal{D}_3$. It follows that $TE_{ii}T^{-1} = PE_{ii}P^{-1}$ whenever $i \in [1,n] \setminus \{k\}$ and so

$$\phi(SE_{kk}S^{-1}) = PE_{kk}P^{-1} = I - \sum_{i\in[1,n]\setminus\{k\}} PE_{ii}P^{-1} = I - \sum_{i\in[1,n]\setminus\{k\}} TE_{ii}T^{-1} = TE_{kk}T^{-1}.$$

□

Claim 6.2.1.5. Let $A, B \in \mathcal{A}_\rho$ be two diagonalizable matrices such that $A \perp B$. Then $\phi(A) \perp \phi(B)$.

Proof. Follows directly from Theorem 3.2.1 and Claim 6.2.1.3. □

Claim 6.2.1.6.

(a) Let $A, B \in \mathcal{A}_\rho$ be diagonalizable matrices such that $A \leftrightarrow B$. Then $\phi(\alpha A + \beta B) = \alpha\phi(A) + \beta\phi(B)$ for all $\alpha, \beta \in \mathbb{C}$.

(b) ϕ is a homogeneous map.

Proof. (a) follows directly from Theorem 3.2.1 and Claim 6.2.1.3, while (b) follows from (a), Lemma 6.1.4 and the continuity of ϕ. □

Claim 6.2.1.7. ϕ maps every nonzero matrix from $\overline{\mathcal{R}}$ to a rank-one matrix.

Proof. If $A \in \mathcal{R}$, then A is diagonalizable in $\mathcal{A}_\rho$ (Theorem 3.2.1) so the claim follows directly from Claim 6.2.1.3.

Now suppose that $A \in \overline{\mathcal{R}}$ and let $(A_k)_{k=1}^\infty$ be a sequence of matrices in $\mathcal{R}$ such that $A_k \to A$. By continuity we have $\phi(A_k) \to \phi(A)$ and then by lower semicontinuity of the rank we conclude that $\phi(A)$ has rank one. □

Claim 6.2.1.8. Suppose that nonzero matrices $A_1, A_2 \in \overline{\mathcal{R}}$ satisfy $A_1 \perp A_2$. Then $\phi(A_1) \perp \phi(A_2)$.

Proof. Suppose first that $A_1 \parallel A_2$. Then $A_2 = \alpha A_1$ for some $\alpha \in \mathbb{C}^\times$ so $A_1 \perp A_2$ implies $A_1^2 = 0$. By Claim 6.2.1.7, $\phi(A_1)$ is a rank-one nilpotent so $\phi(A_1)^2 = 0$. By homogeneity of ϕ, we conclude $\phi(A_1) \perp \alpha\phi(A_1) = \phi(\alpha A_1) = \phi(A_2)$, as desired.

Suppose now $A_1 \nparallel A_2$. In view of Claim 6.2.1.7, for $j = 1, 2$ denote $\phi(A_j) = x_j y_j^*$ for some nonzero vectors $x_j, y_j \in \mathbb{C}^n$. Since $A_1 \leftrightarrow A_2$, we obtain

$$(y_1^* x_2) x_1 y_2^* = (x_1 y_1^*)(x_2 y_2^*) = (x_2 y_2^*)(x_1 y_1^*) = (y_2^* x_1) x_2 y_1^*.$$

If $y_1^* x_2 = y_2^* x_1 = 0$, it follows $\phi(A_1) \perp \phi(A_2)$, as desired. Assume therefore $y_1^* x_2, y_2^* x_1 \neq 0$. Then $x_1 y_2^* \parallel x_2 y_1^*$ so $x_1 \parallel x_2$ and $y_1 \parallel y_2$. It follows $\phi(A_1) = x_1 y_1^* \parallel x_2 y_2^* = \phi(A_2)$, so by injectivity of ϕ it follows $A_1 \parallel A_2$. This contradicts our assumption that $A_1 \nparallel A_2$ so the proof is concluded. □

Claim 6.2.1.9. We have $\phi(E_{ij}) \parallel E_{ij}$ or $\phi(E_{ij}) \parallel E_{ji}$ for all $(i,j) \in \rho^\times$.

Proof. By Lemma 6.1.1, note that all matrix units of $\mathcal{A}_\rho$ are contained in $\overline{\mathcal{R}}$. For each $(i,j) \in \rho^\times$ we have $E_{ij} \perp E_{kk}$ for all $k \in [1,n] \setminus \{i,j\}$ so by Claim 6.2.1.8 we obtain

$$\operatorname{supp} \phi(E_{ij}) \subseteq \{i,j\} \times \{i,j\}.$$

Via a direct computation we now show that for each $1 \le i \le n$ we have

$$|\rho(i)| \ge 3 \implies (\phi(E_{ij}) \parallel E_{ij}, \forall j \in \rho(i)) \text{ or } (\phi(E_{ij}) \parallel E_{ji}, \forall j \in \rho(i)).$$

Indeed, if $(i,j), (i,k) \in \rho^\times$ for $j \neq k$, then there exist scalars such that

$$\begin{aligned}
\phi(E_{ij}) &= \alpha_{ii} E_{ii} + \alpha_{ij} E_{ij} + \alpha_{ji} E_{ji} + \alpha_{jj} E_{jj}, \\
\phi(E_{ik}) &= \beta_{ii} E_{ii} + \beta_{ik} E_{ik} + \beta_{ki} E_{ki} + \beta_{kk} E_{kk}.
\end{aligned}$$

Since $E_{ij} \perp E_{ik}$, by invoking Claim 6.2.1.8, we obtain

$$\begin{aligned}
0 = \phi(E_{ij})\phi(E_{ik}) &= \alpha_{ii}\beta_{ii} E_{ii} + \alpha_{ii}\beta_{ik} E_{ik} + \alpha_{ji}\beta_{ii} E_{ji} + \alpha_{ji}\beta_{ik} E_{jk}, \\
0 = \phi(E_{ik})\phi(E_{ij}) &= \alpha_{ii}\beta_{ii} E_{ii} + \alpha_{ij}\beta_{ii} E_{ij} + \alpha_{ij}\beta_{ki} E_{kj} + \alpha_{ii}\beta_{ki} E_{ki}.
\end{aligned}$$

Suppose that $\beta_{ii} \neq 0$. Then $\alpha_{ii}\beta_{ii} = \alpha_{ij}\beta_{ii} = \alpha_{ji}\beta_{ii} = 0$ imply $\alpha_{ii} = \alpha_{ij} = \alpha_{ji} = 0$ and hence $\phi(E_{ij}) = \alpha_{jj} E_{jj}$, which contradicts injectivity (as $\phi(\alpha_{jj} E_{jj}) = \alpha_{jj} E_{jj}$). We run into a similar contradiction when assuming $\alpha_{ii} \neq 0$ so we conclude $\alpha_{ii} = \beta_{ii} = 0$. Since $\phi(E_{ij})$ and $\phi(E_{ik})$ are rank-one nilpotents, we have

$$0 = \phi(E_{ij})^2 = \alpha_{ij}\alpha_{ji} E_{ii} + \alpha_{ij}\alpha_{jj} E_{ij} + \alpha_{jj}\alpha_{ji} E_{ji} + (\alpha_{ij}\alpha_{ji} + \alpha_{jj}^2) E_{jj}$$

and

$$0 = \phi(E_{ik})^2 = \beta_{ik}\beta_{ki} E_{ii} + \beta_{ik}\beta_{kk} E_{ik} + \beta_{kk}\beta_{ki} E_{ki} + (\beta_{ik}\beta_{ki} + \beta_{kk}^2) E_{kk}.$$

We first conclude $\alpha_{jj} = \beta_{kk} = 0$ and then $\alpha_{ij} = 0$ or $\alpha_{ji} = 0$ (but not both since otherwise we would have $\phi(E_{ij}) = 0$) and similarly $\beta_{ik} = 0$ or $\beta_{ki} = 0$ but not both. If $\alpha_{ij} \neq 0$, then $\phi(E_{ij}) = \alpha_{ij}E_{ij}$ and $\alpha_{ij}\beta_{ki} = 0$ implies $\beta_{ki} = 0$ from which we conclude $\phi(E_{ik}) = \beta_{ik}E_{ik}$. Similarly, if $\alpha_{ji} \neq 0$, we conclude $\phi(E_{ij}) = \alpha_{ji}E_{ji}$ and $\phi(E_{ik}) = \beta_{ki}E_{ki}$.

An analogous argument shows that for each $1 \le j \le n$ we have

$$\left|\rho^{-1}(j)\right| \ge 3 \implies (\phi(E_{ij}) \parallel E_{ij}, \forall i \in \rho^{-1}(j)) \text{ or } (\phi(E_{ij}) \parallel E_{ji}, \forall i \in \rho^{-1}(j)).$$

It remains to consider the case $(i,j) \in \rho^\times$ when $|\rho(i)| = 2$ (or $|\rho^{-1}(j)| = 2$). For concreteness, assume $\rho(i) = \{i,j\}$ for some $j \in [1,n] \setminus \{i\}$. Since, by assumption, $\mathcal{A}_\rho$ is 2-free, clearly there exists some $k \in [1,n] \setminus \{i,j\}$ such that $k \approxeq_0 i$ or $k \approxeq_0 j$. The possibilities $(i,k) \in \rho$ and $(j,k) \in \rho$ can be excluded, as they would lead to $k \in \rho(i)$, which is false. The possibilities which remain are $(k,i) \in \rho$ or $(k,j) \in \rho$, so in either case we can assume $(k,j) \in \rho$. Then $i,j,k \in \rho^{-1}(j)$ and hence $|\rho^{-1}(j)| \ge 3$, which allows us to reach the desired conclusion that $\phi(E_{ij}) \parallel E_{ij}$ or $\phi(E_{ij}) \parallel E_{ji}$. □

In view of Claim 6.2.1.9, for each $(i,j) \in \rho$, denote by $g(i,j) \in \mathbb{C}^\times$ the unique scalar such that

$$\phi(E_{ij}) = g(i,j)E_{ij} \qquad \text{or} \qquad \phi(E_{ij}) = g(i,j)E_{ji}.$$

In this manner we obtain a function $g : \rho \to \mathbb{C}^\times$ whose transitivity we intend to show in the remainder of the proof. As, by assumption, $\phi|_{\mathcal{D}_n}$ is the identity map, it is immediate that $g|_{\Delta_n} \equiv 1$. Define

$$(6.2.1) \qquad \rho_M^\phi := \{(i,j) \in \rho : \phi(E_{ij}) \parallel E_{ij}\}, \qquad \rho_A^\phi := \{(i,j) \in \rho : \phi(E_{ij}) \parallel E_{ji}\}.$$

Clearly, $\rho_M^\phi \cup \rho_A^\phi = \rho$ and $\rho_M^\phi \cap \rho_A^\phi = \Delta_n$.

Claim 6.2.1.10. Suppose $(i,j) \in \rho^\times$. Then

$$(\{i\} \times \rho(i)) \cup (\rho^{-1}(i) \times \{i\}) \cup (\{j\} \times \rho(j)) \cup (\rho^{-1}(j) \times \{j\}) \subseteq \rho_M^\phi$$

or

$$(\{i\} \times \rho(i)) \cup (\rho^{-1}(i) \times \{i\}) \cup (\{j\} \times \rho(j)) \cup (\rho^{-1}(j) \times \{j\}) \subseteq \rho_A^\phi.$$

Proof. For concreteness suppose that $(i,j) \in \rho_M^\phi$, as the other case is similar.

- If $k \in \rho^\times(i)$, then $E_{ij} \perp E_{ik}$ and Claim 6.2.1.8 imply $(i,k) \in \rho_M^\phi$.
- If $k \in (\rho^\times)^{-1}(j)$, then $E_{ij} \perp E_{kj}$ and Claim 6.2.1.8 imply $(k,j) \in \rho_M^\phi$.
- Let $k \in (\rho^\times)^{-1}(i)$. If $k \neq j$, then by transitivity we obtain $(k,j) \in \rho^\times$ and hence $(k,j) \in \rho_M^\phi$ and then $(k,i) \in \rho_M^\phi$, by the previous two cases. On the other hand, the case $k = j$ follows from the injectivity and homogeneity of ϕ.
- Let $k \in \rho^\times(j)$. The case $k = i$ again follows from the injectivity and homogeneity of ϕ. If $k \neq i$, then by transitivity we obtain $(i,k) \in \rho^\times$ and hence $(i,k) \in \rho_M^\phi$ and then $(j,k) \in \rho_M^\phi$, by the first two cases.

□

Claim 6.2.1.11. Suppose that $X \in \mathcal{A}_\rho$ satisfies $\operatorname{supp} X \subseteq S \times S$ for some $S \subseteq [1,n]$. Then $\operatorname{supp} \phi(X) \subseteq S \times S$.

Proof. By applying Claim 6.1.4 on the matrix $X^{\flat S^c} \in \mathcal{A}_\rho^{\flat S^c}$, by the continuity of ϕ it suffices to assume that X is a diagonalizable matrix. Now the assertion follows from $X \perp E_{kk}$ for all $k \in [1,n] \setminus S$ and Claims 6.2.1.2 and 6.2.1.8. □

Claim 6.2.1.12. Let $S \subseteq [1, n]$. The map

$$\psi : \mathcal{A}_\rho^{\flat S^c} \to M_{|S|}, \qquad X \mapsto \phi(X^{\sharp S^c})^{\flat S^c}$$

is an injective continuous commutativity and spectrum preserver.

Proof. In view of Claim 6.2.1.11 it makes sense to consider the maps

$$\begin{aligned}
&\psi_1 : \mathcal{A}_\rho^{\flat S^c} \to \{X \in \mathcal{A}_\rho : \operatorname{supp} X \subseteq S \times S\}, \qquad X \mapsto X^{\sharp S^c},\\
&\psi_2 : \{X \in \mathcal{A}_\rho : \operatorname{supp} X \subseteq S \times S\} \to \{X \in M_n : \operatorname{supp} X \subseteq S \times S\}, \qquad X \mapsto \phi(X),\\
&\psi_3 : \{X \in M_n : \operatorname{supp} X \subseteq S \times S\} \to M_{|S|}, \qquad X \mapsto X^{\flat S^c}.
\end{aligned}$$

Note that $\psi = \psi_3 \circ \psi_2 \circ \psi_1$. Since ψ_1 and ψ_3 are algebra isomorphisms, it follows that ψ is an injective continuous commutativity preserver. Finally, for each $X \in \mathcal{A}_\rho^{\flat S^c}$ and we have the equality of polynomials

$$\begin{aligned}
(-x)^{n-|S|} k_{\psi(X)}(x) &= (-x)^{n-|S|} k_{\phi(X^{\sharp S^c})^{\flat S^c}}(x) \overset{\text{Claim 6.2.1.11}}{=} k_{\phi(X^{\sharp S^c})}(x) \overset{\text{Claim 6.2.1.1}}{=} k_{X^{\sharp S^c}}(x)\\
&= (-x)^{n-|S|} k_X(x),
\end{aligned}$$

which implies $k_{\psi(X)} = k_X$ so ψ is a spectrum preserver. □

Claim 6.2.1.13. Suppose that $n = 3$ and that $\rho(i) = \rho^{-1}(j) = [1, 3]$ for some distinct $1 \leq i, j \leq 3$. Then $\rho = \rho_M^\phi$ and ϕ is the identity map, or $\rho = \rho_A^\phi$ and ϕ acts as the transposition map.

Proof. We start by making several reductions. Let $\pi \in S_3$ be the permutation such that $\pi(i) = 1$ and $\pi(j) = 3$. By passing to the map

$$\phi(P^{-1}(\cdot)P) : P\mathcal{A}_\rho P^{-1} \to M_3$$

which is again an injective continuous spectrum and commutativity preserver (and acts as the identity map on $\mathcal{D}_3$), we may assume that $i = 1$ and $j = 3$, i.e. $\mathcal{A}_\rho \supseteq \mathcal{T}_3$. Therefore, we have $\mathcal{A}_\rho \in \{\mathcal{T}_3, \mathcal{A}_{1,2}, \mathcal{A}_{2,1}, M_3\}$. When $\mathcal{A}_\rho = M_3$, the assertion follows from Theorem 2.6.1. On the other hand, if we prove the result for $\mathcal{A}_\rho = \mathcal{A}_{1,2}$, the result for $\mathcal{A}_\rho = \mathcal{A}_{2,1}$ will follow by considering the map $\phi((\cdot)^t)^t$. Therefore, without loss of generality, we can assume $\mathcal{A}_\rho \in \{\mathcal{T}_3, \mathcal{A}_{1,2}\}$. By Claim 6.2.1.10 it easily follows that $\rho = \rho_M^\phi$ or $\rho = \rho_A^\phi$. Without loss of generality assume the former (otherwise we pass to the map $\phi(\cdot)^t$). Finally, by passing to the map

$$\operatorname{diag}(1, g(1,2), g(1,3))\phi(\cdot)\operatorname{diag}(1, g(1,2), g(1,3))^{-1} : \mathcal{A} \to M_3$$

we can assume that $g(1, 2) = g(1, 3) = 1$, i.e. that $\phi(E_{1j}) = E_{1j}$ for all $j \in [1, 3]$.

We show that $\phi(A) = A$ for all non-nilpotent rank-one matrices $A \in \mathcal{A}_\rho$. Every rank-one matrix in $\mathcal{A}_\rho$ is either of the form

$$(a)\begin{bmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{or} \qquad (b)\begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

We start with case (a). Let $y = (0, y_2, y_3)$ and

$$R_1(\alpha, y) := \begin{bmatrix} \alpha & y_2 & y_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In view of Lemma 2.3.5, if $\alpha = 1$, we write $R_1(1, y) = S(y)^{-1}E_{11}S(y)$. Let

$$\begin{aligned} R_2 &:= S(y)^{-1}E_{22}S(y) = E_{22} - y_2E_{12} \\ R_3 &:= S(y)^{-1}E_{33}S(y) = E_{33} - y_3E_{13}. \end{aligned}$$

Moreover, $R_2 \perp E_{13}, E_{33}$ and commutes with E_{13}, while $R_3 \perp E_{12}, E_{22}$ and commutes with E_{12}. Correspondingly, by Claim 6.2.1.8, the same holds for $\phi(R_2)$ and $\phi(R_3)$. Hence there exist continuous functions $f_j : \mathbb{C} \to \mathbb{C}$, such that $\phi(R_j) = E_{jj} - f_j(y_j)E_{1j}$, $j = 2, 3$. Since $\phi(E_{jj}) = E_{jj}$ we have $f_j(0) = 0$, $j = 2, 3$. Let $f(y) := (0, f_2(y_2), f_3(y_3))$. By Lemma 2.3.5, for $j = 2, 3$ we have

$$\phi(R_j) = S(f(y))^{-1}E_{jj}S(f(y)).$$

By Claim 6.2.1.4 we have that

$$\phi(R_1(1, y)) = S(f(y))^{-1}E_{11}S(f(y)) = R_1(1, f(y)).$$

From

$$\phi(\alpha E_{11} + y_jE_{1j}) = \alpha\phi\left(E_{11} + \frac{y_j}{\alpha}E_{1j}\right) = \alpha E_{11} + \alpha f_j\left(\frac{y_j}{\alpha}\right)E_{1j},$$

the homogeneity of ϕ, the continuity of f_j, and

$$y_jE_{1j} = \lim_{\alpha\to 0}\phi(\alpha E_{11} + y_jE_{1j})$$

we get

$$\lim_{\alpha\to 0}\alpha f_j\left(\frac{y_j}{\alpha}\right) = y_j, \qquad j = 2, 3.$$

Similarly,

$$\begin{aligned} \phi(R_1(0, y)) &= \phi\left(\lim_{\alpha\to 0}R_1(\alpha, y)\right) = \alpha\phi\left(\lim_{\alpha\to 0}R_1\left(1, \frac{y}{\alpha}\right)\right) = R_1\left(0, \lim_{\alpha\to 0}\alpha f\left(\frac{y}{\alpha}\right)\right) \\ &= R_1(0, y). \end{aligned}$$

For arbitrary nonzero $t \in \mathbb{C}$ let us next consider the matrix $E_{33} - tE_{23} \in \mathcal{A}_\rho$, which is orthogonal to E_{11}. By setting $z = (0, 1, t)$, $E_{33} - tE_{23}$ commutes with $R_1(0, z) = E_{12} + tE_{13} = \phi(R_1(0, z))$. An easy computation yields that there exist complex numbers p_t and q_t, not both equal to zero at any nonzero t, such that

$$\phi(E_{33} - tE_{23}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -tp_z & -tq_t \\ 0 & p_t & q_t \end{bmatrix}.$$

We use this observation with the relation $R(1, y_2, y_3) \perp E_{33} - y_3/y_2E_{23}$, $y_2 \neq 0$, (we have chosen $t = y_3/y_2$) and thus obtain that $[1\ f_2(y_2)\ f_3(y_3)]\,[0\ -y_3/y_2\ 1]^t = 0$ when y_2, y_3 are both nonzero, giving $f_2(y_2)/y_2 = f_3(y_3)/y_3 = c$ for some nonzero complex scalar c. Recall that by our reduction $\phi(E_{12}) = E_{12}$. By the continuity and the homogeneity of ϕ we have

$$E_{12} = \phi(E_{12}) = \lim_{\alpha\to 0} \phi(\alpha E_{11} + E_{12}) = \lim_{\alpha\to 0} \alpha R\left(1, \frac{c}{\alpha}, 0\right) = \lim_{\alpha\to 0} R(\alpha, c, 0) = cE_{12}.$$

Hence $c = 1$. The cases when $y_2 = 0$ or $y_3 = 0$ follow by density, which finishes the proof of case (a).

Let us proceed with the case (b). We can write any such rank-one (possibly nilpotent) matrix as ab^* where a, $b \in \mathbb{C}^3$ and $b \perp e_1$. Suppose $a \notin \operatorname{span}\{e_1\}$ (otherwise, we are in the case (a)). By Claim 6.2.1.7, we have $\phi(ab^*) = uv^*$ for some nonzero vectors $u, v \in \mathbb{C}^3$ such that $b^*a = v^*u$. Obviously, ab^* commutes with $e_1(a^\perp)^*$ for any vector $a^\perp$ orthogonal to a. From (a) we already know that $\phi(e_1(a^\perp)^*) = e_1(a^\perp)^*$. So,

$$(v^*e_1)u(a^\perp)^* = uv^*e_1(a^\perp)^* = e_1(a^\perp)^*uv^* = ((a^\perp)^*u)e_1v^*. \tag{6.2.2}$$

We claim that $v \perp e_1$ and $u \perp a^\perp$. Suppose the contrary. Then, since both $u(a^\perp)^*$ and e_1v^* are nonzero, (6.2.2) shows that both v^*e_1 and $(a^\perp)^*u$ are zero or both nonzero. If they are nonzero, we have $u \in \operatorname{span}\{e_1\}$, which conflicts injectivity. Therefore, $v \perp e_1$ and $u \perp a^\perp$ and so, $u \in (\{a\}^\perp)^\perp = \mathbb{C}a$. Without loss of any generality, we can assume $u = a$, and the scalar factor can be absorbed in v. So far we have obtained that $\phi(ab^*) = av^* \in \mathcal{A}$ for some vector $v \perp e_1$ which depends on a and b. On the other hand, ab^* commutes with $(b^\perp)c^*$, where $b^\perp \perp b$ and $c \in \{e_1, a\}^\perp$ (it is possible that c can be chosen only in the way that $(b^\perp)c^*$ is nilpotent and so, we can use only commutativity preserving). From the previous argument, we know that $\phi(b^\perp c^*) = b^\perp w^*$ for some $w \neq 0$. Then $\phi(ab^*) = av^* \leftrightarrow b^\perp w^* = \phi(b^\perp c^*)$. This gives

$$(v^*b^\perp)aw^* = av^*b^\perp w^* = b^\perp w^*av^* = (w^*a)b^\perp v^*.$$

At this point assume that ab^* is not nilpotent, i.e. b is not orthogonal to a. Then $b^\perp$ (defined up to a scalar factor) and a are linearly independent, hence $v^*b^\perp = 0$. This implies that $v = \beta b$ for some scalar $\beta \neq 0$. From the spectrum preserving property, comparing the traces, we get that $\beta = 1$, and we are done.

By density, to show that ϕ is the identity map, it suffices to prove that ϕ is the identity map on the set of matrices in $\mathcal{A}$ with 3 distinct eigenvalues. By Theorem 3.2.1, for every matrix A of this kind there exists a matrix $S \in \mathcal{A}^\times$ and a diagonal matrix $D \in \mathcal{D}_n$ such that $A = SDS^{-1}$. Applying Step 1 for the map $X \mapsto \phi(SXS^{-1})$ we get that there exists $T \in M_3^\times$ such that

$$\phi(SDS^{-1}) = TDT^{-1}, \qquad \text{for all } D \in \mathcal{D}_3.$$

By Claim 6.2.1.3 we also have

$$SE_{jj}S^{-1} = \phi(SE_{jj}S^{-1}) = TE_{jj}T^{-1}, \qquad 1 \le j \le 3,$$

so by linearity $TDT^{-1} = SDS^{-1}$ and consequently,

$$\phi(SDS^{-1}) = TDT^{-1} = SDS^{-1}$$

for all diagonal matrices $D \in \mathcal{D}_3$. We conclude that ϕ is the identity map. □

We return to the proof of the proposition. The reflexivity of ρ_M^ϕ and ρ_A^ϕ is immediate from Claim 6.2.1.2, while their transitivity follows from Claim 6.2.1.10. To begin proving that g is a transitive map, first notice that Claim 6.2.1.10 implies

$$(i,j),(j,k)\in\rho \implies (i,j),(j,k)\in\rho_M^\phi \quad \text{or} \quad (i,j),(j,k)\in\rho_A^\phi. \tag{6.2.3}$$

Therefore, it suffices to show that $g|_{\rho_M^\phi}$ and $g|_{\rho_A^\phi}$ are transitive maps. We focus on ρ_M^ϕ as the proof for ρ_A^ϕ is analogous. Suppose that $(i,j),(j,k)\in\rho_M^\phi$. We show that $g(i,j)g(j,k)=g(i,k)$. This is obvious if $i=j$ or $j=k$ so assume further that $(i,j),(j,k)\in(\rho_M^\phi)^\times$. We first focus on the case $i\neq k$. By deleting l-th row and column in $\mathcal{A}_\rho$ where $l\in[1,n]\setminus\{i,j,k\}$, the elements which remain are

$$\begin{bmatrix} (i,i) & (i,j) & (i,k) \\ * & (j,j) & (j,k) \\ * & * & (k,k) \end{bmatrix}.$$

Therefore, $\mathcal{A}_\rho^{\flat(\{i,j,k\}^c)}\subseteq M_3$ contains $\mathcal{T}_3$. Hence, in view of Claim 6.2.1.12 we can apply Claim 6.2.1.13 to the map

$$\psi:\mathcal{A}_\rho^{\flat(\{i,j,k\}^c)}\to M_3,\qquad X\mapsto \phi(X^{\sharp(\{i,j,k\}^c)})^{\flat(\{i,j,k\}^c)}$$

to conclude that ψ is an algebra homomorphism or an antihomomorphism. We have

$$\psi|_{\mathcal{D}_3}=\mathrm{id},\qquad \psi(E_{12})=g(i,j)E_{12},\qquad \psi(E_{13})=g(i,k)E_{13},\qquad \psi(E_{23})=g(j,k)E_{23},$$

whence ψ is a multiplicative map. In particular, we obtain

$$g(i,k)E_{13}=\psi(E_{13})=\psi(E_{12}E_{23})=\psi(E_{12})\psi(E_{23})=g(i,j)g(j,k)E_{13},$$

so $g(i,k)=g(i,j)g(j,k)$.

It remains to consider the easier case $i=k$. Since $\mathcal{A}_\rho$ is 2-free, there exists some $l\in[1,n]\setminus\{i,j\}$ such that $i\approxeq_0 l$ or $j\approxeq_0 l$. Without loss of generality suppose the former. If $(i,l)\in\rho^\times$, then from $(j,i)\in\rho_M^\phi$ via (6.2.3) we conclude $(i,l)\in(\rho_M^\phi)^\times$. Now from $(j,i),(i,l)\in(\rho_M^\phi)^\times$ by the previous case if follows $(j,l)\in(\rho_M^\phi)^\times$ and

$$g(j,l)=g(j,i)g(i,l),\qquad g(i,l)=g(i,j)g(j,l).$$

We obtain

$$g(i,j)g(j,i)=\frac{g(j,l)}{g(i,l)}\cdot\frac{g(i,l)}{g(j,l)}=1$$

as desired. On the other hand, if $(l,i)\in\rho^\times$ then similarly we obtain

$$g(l,j)=g(l,i)g(i,j),\qquad g(l,i)=g(l,j)g(j,i),$$

again yielding $g(i,j)g(j,i)=1$. □

Theorem 6.2.2. *Let $\mathcal{A}_\rho\subseteq M_n$ be an SMA. Then the following two statements are equivalent:*

(i) For each $(i,j)\in\rho^\times$ we have

$$|(\rho(i)\cup\rho^{-1}(i))\cap(\rho(j)\cup\rho^{-1}(j))|\geq 3.$$

(ii) Every continuous injective commutativity and spectrum preserver $\phi:\mathcal{A}_\rho\to M_n$ is

necessarily a Jordan embedding.

Remark 6.2.3. Note that for $(i, j) \in \rho^\times$, the condition (i) of Theorem 6.2.2 (i) is equivalent to

$$(\rho(i) \cup \rho^{-1}(i)) \cap (\rho(j) \cup \rho^{-1}(j)) \not\subseteq \{i, j\}.$$

In particular, (i) implies that $\mathcal{A}_\rho$ is 2-free.

Proof of Theorem 6.2.2. First we consider the $n = 3$ case.

Claim 6.2.3.1. An SMA $\mathcal{A}_\rho \subseteq M_3$ distinct from $\mathcal{D}_3$ satisfies (i) if and only if there exist distinct $i, j \in [1, 3]$ such that $\rho(i) = \rho^{-1}(j) = [1, 3]$.

Proof. $\boxed{\Longleftarrow}$ By the assumption, we have

$$\rho(i) \cup \rho^{-1}(i) = \rho(j) \cup \rho^{-1}(j) = [1, 3].$$

Denote $k \in [1, 3]\setminus\{1, 2\}$. Since $k \in \rho(i)$ and $k \in \rho^{-1}(j)$, we also have $\rho(k)\cup\rho^{-1}(k) = [1, 3]$. Therefore, $\mathcal{A}_\rho$ satisfies (i).

$\boxed{\Longrightarrow}$ Since $\mathcal{A}_\rho \neq \mathcal{D}_3$, choose some $(i, j) \in \rho^\times$. By (i), there exists $k \in [1, 3] \setminus \{i, j\}$ such that

$$k \in (\rho(i) \cup \rho^{-1}(i)) \cap (\rho(j) \cup \rho^{-1}(j)).$$

We consider two cases:

- If $k \in \rho(j)$, then from $(i, j), (j, k) \in \rho$ it follows $(i, k) \in \rho$ and hence $\rho(i) = \rho^{-1}(k) = [1, 3]$.
- If $k \in \rho^{-1}(j)$, then from $(i, j), (k, j) \in \rho$ it follows $\rho^{-1}(j) = [1, 3]$. Moreover, since $k \in \rho(i) \cup \rho^{-1}(i)$ we obtain $(i, k) \in \rho$ or $(k, i) \in \rho$, which implies $\rho(i) = [1, 3]$ or $\rho(k) = [1, 3]$, respectively.

□

$\boxed{(i) \implies (ii)}$ Since $\mathcal{A}_\rho$ is 2-free, in view of Proposition 6.2.1 (and Claim 6.2.1.6) without loss of generality by passing to the map $S^{-1}\phi((g^*)^{-1}(\cdot))S$ we can assume that

$$\phi(E_{ij}) = \begin{cases} E_{ij}, & \text{if } (i, j) \in \rho_M^\phi, \\ E_{ji}, & \text{if } (i, j) \in \rho_A^\phi \end{cases} \tag{6.2.4}$$

where ρ_M^ϕ and ρ_A^ϕ are quasi-orders defined in (6.2.1).

Claim 6.2.3.2. Every rank-one matrix in $\mathcal{A}_\rho$ is entirely contained in $\mathcal{A}_{\rho_M^\phi}$ or in $\mathcal{A}_{\rho_A^\phi}$.

Proof. Assume $ab^* \in \mathcal{A}_\rho$ for some nonzero vectors $a, b \in \mathbb{C}^n$. We have

$$(\operatorname{supp} a) \times (\operatorname{supp} b) \subseteq \rho.$$

Let $(i, j), (k, l) \in ((\operatorname{supp} a) \times (\operatorname{supp} b)) \cap \rho^\times$ be distinct but otherwise arbitrary. We need to show that $(i, j), (k, l) \in \rho_M^\phi$ or $(i, j), (k, l) \in \rho_A^\phi$. If $j = k$ or $i = l$, this follows from (6.2.3). Otherwise, since $\{i, k\} \times \{j, l\} \subseteq \rho$, we have

$$\begin{cases} (i, l) \in \rho^\times \implies (i, j), (i, l), (k, l) \in \rho^\times, \\ (k, j) \in \rho^\times \implies (i, j), (k, j), (k, l) \in \rho^\times, \end{cases}$$

so in either case we obtain the desired assertion from Claim 6.2.1.10. □

Claim 6.2.3.3. ϕ acts as the identity on $\mathcal{R} \cap \mathcal{A}_{\rho_M^\phi}$, and as transposition on $\mathcal{R} \cap \mathcal{A}_{\rho_A^\phi}$.

Proof. For concreteness assume that $ab^* \in \mathcal{R} \cap \mathcal{A}_{\rho_M^\phi}$ for some (nonzero) vectors $a, b \in \mathbb{C}^n$, as the other case is similar. In view of Claim 6.2.1.7, denote $\phi(ab^*) = xy^*$ for some nonzero $x, y \in \mathbb{C}^n$. As $b^*a \neq 0$, we can fix some

$$j \in (\operatorname{supp} a) \cap (\operatorname{supp} b).$$

Let $i \in [1, n] \setminus \{j\}$ be arbitrary and consider the matrix

$$A := (\overline{b_j} e_i - \overline{b_i} e_j)(\overline{a_i} e_j - \overline{a_j} e_i)^*.$$

Note that

$$(\overline{b_j} e_i - \overline{b_i} e_j) e_i^* = \begin{cases} \overline{b_j} E_{ii} - \overline{b_i} E_{ji}, & \text{if } i \in \operatorname{supp} b, \\ \overline{b_j} E_{ii}, & \text{if } i \notin \operatorname{supp} b, \end{cases}$$

$$e_i(\overline{a_i} e_j - \overline{a_j} e_i)^* = \begin{cases} a_i E_{ij} - a_j E_{ii}, & \text{if } i \in \operatorname{supp} a, \\ -a_j E_{ii}, & \text{if } i \notin \operatorname{supp} a. \end{cases}$$

In all cases, these matrices belong to $\mathcal{A}_\rho$, so by Lemma 6.1.1 we have $A \in \overline{\mathcal{R}}$ and in particular $A \in \mathcal{A}_\rho$.

We claim that $\phi(A) = A$. Consider the following cases:

- If $i \notin (\operatorname{supp} a) \cup (\operatorname{supp} b)$, then $A = -a_j \overline{b_j} E_{ii}$ and hence $\phi(A) = A$ by the homogeneity (Claim 6.2.1.6).
- Suppose that $i \in \operatorname{supp} a$. Then $(i, j) \in \operatorname{supp} A \subseteq \{i, j\} \times \{i, j\}$, so $(i, j) \in \rho^\times$. By (i) there exists $k \in [1, n] \setminus \{i, j\}$ such that

  $$k \in (\rho(i) \cup \rho^{-1}(i)) \cap (\rho(j) \cup \rho^{-1}(j)).$$

  It is immediate that

  $$\{i, j, k\} \subseteq (\rho(i) \cup \rho^{-1}(i)) \cap (\rho(j) \cup \rho^{-1}(j)) \cap (\rho(k) \cup \rho^{-1}(k)),$$

  which implies that the SMA $\mathcal{A}_\rho^{\flat(\{i,j,k\}^c)} \subseteq M_3$ satisfies (i). Furthermore, since

  $$(i, j) \in (\operatorname{supp} a) \times (\operatorname{supp} b) = \operatorname{supp}(ab^*) \subseteq \rho_M^\phi,$$

  by Claim 6.2.1.10 we conclude that

  $$(\{i, j, k\} \times \{i, j, k\}) \cap \rho \subseteq \rho_M^\phi.$$

  In particular, by (6.2.4), ϕ acts as the identity on all matrix units supported in $(\{i, j, k\} \times \{i, j, k\}) \cap \rho$. We can now invoke Claim 6.2.1.12 and the $n = 3$ case (which was already covered) to conclude that the map

  $$\psi : \mathcal{A}_\rho^{\flat(\{i,j,k\}^c)} \to M_3, \qquad X \mapsto \phi(X^{\sharp(\{i,j,k\}^c)})^{\flat(\{i,j,k\}^c)}$$

  is a Jordan embedding. Since ψ acts as the identity on all matrix units of $\mathcal{A}_\rho^{\flat(\{i,j,k\}^c)}$, we conclude that ψ is the identity map. In particular, Claim 6.2.1.11 implies that $\phi(A) = A$.

- Suppose that $i \in \operatorname{supp} b$. Then $(j,i) \in \rho^\times$ so by the symmetry of our assumption (i), the exact same discussion as above yields $\phi(A) = A$.

Now notice that $ab^* \perp A$ so by Claim 6.2.1.8 we obtain

$$xy^* = \phi(ab^*) \perp \phi(A) = A \implies (\overline{a_i}e_j - \overline{a_j}e_i)^* x = y^*(\overline{b_j}e_i - \overline{b_i}e_j) = 0.$$

Overall, it follows

$$x \perp \{\overline{a_i}e_j - \overline{a_j}e_i : i \in [1,n] \setminus \{j\}\}, \qquad y \perp \{\overline{b_j}e_i - \overline{b_i}e_j : i \in [1,n] \setminus \{j\}\}.$$

In fact, these sets are bases for $\{a\}^\perp$ and $\{b\}^\perp$ respectively. We conclude $x \parallel a$ and $y \parallel b$, which implies $\phi(ab^*) \parallel ab^*$. Equating the traces yields $\phi(ab^*) = ab^*$. ☐

Claim 6.2.3.4. ϕ acts as the identity on $\mathcal{A}_{\rho_M^\phi}$, and as transposition on $\mathcal{A}_{\rho_A^\phi}$.

Proof. For the sake of variety, we prove the second claim, as the first one is similar. By Claim 6.2.1.3, for each $S \in \mathcal{A}_{\rho_A^\phi}^\times$ there exists $T \in M_n^\times$ such that

$$\phi(SDS^{-1}) = TDT^{-1}, \qquad \text{for all } D \in \mathcal{D}_n.$$

In particular, for all $j \in [1,n]$ we have

$$TE_{jj}T^{-1} = \phi(\underbrace{SE_{jj}S^{-1}}_{\in \mathcal{R} \cap \mathcal{A}_{\rho_A^\phi}}) \overset{\text{Claim 6.2.3.3}}{=} (SE_{jj}S^{-1})^t.$$

Hence, by the linearity of the maps $T(\cdot)T^{-1}$ and $(S(\cdot)S^{-1})^t$, for all $D \in \mathcal{D}_n$ we have

$$\phi(SDS^{-1}) = TDT^{-1} = (SDS^{-1})^t.$$

The Claim now follows by the continuity of ϕ from Lemma 6.1.4 applied to $\mathcal{A}_{\rho_A^\phi}$. ☐

Claim 6.2.3.5. Let $P \in \mathcal{D}_n$ be a diagonal idempotent defined by

$$P_{ii} = 1 \iff \text{there exists } j \in [1,n] \setminus \{i\} \text{ such that } (i,j) \in \rho_M^\phi \text{ or } (j,i) \in \rho_M^\phi.$$

Then $P \in Z(\mathcal{A}_\rho)$, $PX \in \mathcal{A}_{\rho_M^\phi}$ and $(I-P)X \in \mathcal{A}_{\rho_A^\phi}$ for all $X \in \mathcal{A}_\rho$.

Proof. A variant of this argument (when ϕ is assumed to be a Jordan homomorphism) already appears in Lemma 4.2.3 and a similar proof applies here. Indeed, it suffices to show that for all $(i,j) \in \rho$ we have

$$P \leftrightarrow E_{ij}, \qquad PE_{ij} \in \mathcal{A}_{\rho_M^\phi}, \qquad (I-P)E_{ij} \in \mathcal{A}_{\rho_A^\phi}. \tag{6.2.5}$$

Since all three claims are trivially true when $i = j$, fix $(i,j) \in \rho^\times$. We consider two separate cases:

- If $(i,j) \in \rho_M^\phi$, then $P_{ii} = P_{jj} = 1$ by definition, so

$$PE_{ij} = \underbrace{E_{ij}}_{\in \mathcal{A}_{\rho_M^\phi}} = E_{ij}P \implies (I-P)E_{ij} = 0 \in \mathcal{A}_{\rho_A^\phi},$$

which establishes (6.2.5).

- If $(i,j) \in \rho_A^\phi$, then by Claim 6.2.1.10, $P_{ii} = P_{jj} = 0$. Hence,

$$PE_{ij} = \underbrace{0}_{\in \mathcal{A}_{\rho_M^\phi}} = E_{ij}P \implies (I-P)E_{ij} = E_{ij} \in \mathcal{A}_{\rho_A^\phi},$$

so again (6.2.5) follows.

□

Claim 6.2.3.6. Let $P \in Z(\mathcal{A}_\rho)$ be the idempotent from Claim 6.2.3.5. Then for all $X \in \mathcal{A}_\rho$ we have

$$\phi(X) = PX + (I-P)X^t.$$

In particular, ϕ is a Jordan embedding.

Proof. Fix a diagonalizable matrix $X \in \mathcal{A}_\rho$. Since the idempotent P is central, from Theorem 3.2.1 it easily follows that PX and $(I-P)X$ are both (in fact, simultaneously) diagonalizable. By Claim 6.2.3.5, we have $PX \in \mathcal{A}_{\rho_M^\phi}$, $(I-P)X \in \mathcal{A}_{\rho_A^\phi}$ and $PX \perp (I-P)X$ so by Claims 6.2.1.6 and 6.2.3.4, we have

$$\begin{aligned}\phi(X) &= \phi(PX + (I-P)X) = \phi(PX) + \phi((I-P)X) = PX + ((I-P)X)^t \\ &= PX + (I-P)X^t.\end{aligned}$$

It follows that the continuous maps ϕ and $P(\cdot) + (I-P)(\cdot)^t$ coincide on the set of all diagonalizable matrices in $\mathcal{A}_\rho$, so Lemma 6.1.4 implies their overall equality. □

$\boxed{(ii) \Longleftarrow (i)}$ Suppose that an SMA $\mathcal{A}_\rho \subseteq M_n$ fails to satisfy (i). Then by Remark 6.2.3 there exists $(r,s) \in \rho^\times$ such that

$$(\rho(r) \cup \rho^{-1}(r)) \cap (\rho(s) \cup \rho^{-1}(s)) = \{r,s\}.$$

We consider two separate cases:

Case 1. Suppose that $(s,r) \in \rho$. Then by the transitivity of ρ it easily follows

$$\rho(r) = \rho^{-1}(r) = \rho(s) = \rho^{-1}(s) = \{r,s\}. \tag{6.2.6}$$

Define a rank-two idempotent

$$P := E_{rr} + E_{ss} \in \mathcal{A}_\rho.$$

Note that P is central. Indeed, fix $(i,j) \in \rho$ and note that $i \in \{r,s\}$ if and only if $j \in \{r,s\}$. On the other hand, we clearly have

$$PE_{ij} = \begin{cases} E_{ij}, & \text{if } i \in \{r,s\}, \\ 0, & \text{if } i \notin \{r,s\}, \end{cases} \qquad E_{ij}P = \begin{cases} E_{ij}, & \text{if } j \in \{r,s\}, \\ 0, & \text{if } j \notin \{r,s\}, \end{cases}$$

so $P \leftrightarrow E_{ij}$. If follows that

$$\mathcal{A}_\rho = P\mathcal{A}_\rho \oplus (I-P)\mathcal{A}_\rho$$

is an inner direct sum of algebras. Moreover, by (6.2.6), $P\mathcal{A}_\rho$ is isomorphic to M_2 so let $\psi : M_2 \to M_2$ be a slight modification of the nonlinear counterexample due to [45, Example 7] (in order to ensure its injectivity). Let $f : [0,+\infty\rangle \to \mathbb{S}^1$ be any nonconstant continuous map such that $\lim_{t\to+\infty} f(t) = 1$ (concretely, we can choose $f(t) := e^{\frac{i}{t+1}}$). Define $\psi : M_2 \to M_2$ by

$$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) := \begin{cases} \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}, & \text{if } b = 0, \\ \begin{bmatrix} \dfrac{a}{c\, f\left(\left|\frac{c}{b}\right|\right)} & b\, f\left(\left|\frac{c}{b}\right|\right) \\ & d \end{bmatrix}, & \text{otherwise.} \end{cases}$$

Then ψ is a nonlinear injective continuous spectrum and commutativity preserver (and hence not a Jordan homomorphism). Indeed:

- We claim that ψ is continuous. Let $(A_n)_{n\in\mathbb{N}}, A_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$ be a sequence in M_2 such that $A_n \to A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2$. We need to show that $\psi(A_n) \to \psi(A)$.

  If $b \neq 0$, for large enough $n \in \mathbb{N}$ we have $b_n \neq 0$ nd hence

  $$\psi(A_n) = \begin{bmatrix} \dfrac{a_n}{c_n\, f\left(\left|\frac{c_n}{b_n}\right|\right)} & b_n\, f\left(\left|\frac{c_n}{b_n}\right|\right) \\ & d_n \end{bmatrix} \to \begin{bmatrix} \dfrac{a}{c\, f\left(\left|\frac{c}{b}\right|\right)} & b\, f\left(\left|\frac{c}{b}\right|\right) \\ & d \end{bmatrix} = \psi(A).$$

  On the other hand, if $b = 0$, then we decompose the sequence $(A_n)_{n\in\mathbb{N}}$ into two subsequences; suppose that $(A'_n)_{n\in\mathbb{N}}$ satisfies $b'_n \neq 0$, while $(A''_n)_n$ satisfies $b''_n = 0$ (if either of the two subsequences turns out to be finite, we simply omit the corresponding part of the following argument). For the second subsequence, we have simply

  $$\psi(A''_n) = A''_n \to A = \psi(A).$$

  Let us focus on the first subsequence. We have $b'_n \to b = 0$. If $c \neq 0$, in particular we have

  $$\lim_{n\to\infty} f\left(\left|\frac{c'_n}{b'_n}\right|\right) = f\left(\lim_{n\to\infty}\left|\frac{c'_n}{b'_n}\right|\right) = f\left(\left|\frac{c}{0}\right|\right) = f(+\infty) = 1.$$

  On the other hand, $f(\cdot) \in \mathbb{S}^1$ implies $b'_n\, f\left(\left|\frac{c'_n}{b'_n}\right|\right) \to 0$. We conclude

  $$\psi(A'_n) = \begin{bmatrix} \dfrac{a'_n}{c'_n\, f\left(\left|\frac{c'_n}{b'_n}\right|\right)} & b'_n\, f\left(\left|\frac{c'_n}{b'_n}\right|\right) \\ & d'_n \end{bmatrix} \to \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} = A = \psi(A).$$

  If, however, $c = 0$, then $f(\cdot) \in \mathbb{S}^1$, $b'_n \to b = 0$ and $c'_n \to c = 0$ implies $c'_n\, \overline{f\left(\left|\frac{c'_n}{b'_n}\right|\right)} \to 0$. We conclude

  $$\psi(A'_n) = \begin{bmatrix} \dfrac{a'_n}{c'_n\, f\left(\left|\frac{c'_n}{b'_n}\right|\right)} & b'_n\, f\left(\left|\frac{c'_n}{b'_n}\right|\right) \\ & d'_n \end{bmatrix} \to \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = A = \psi(A).$$

  Therefore, $\psi(A'_n) \to \psi(A)$ i $\psi(A''_n) \to \psi(A)$ so we conclude $\psi(A_n) \to \psi(A)$. This shows the continuity of ψ.

- We claim that ψ preserves commutativity. Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in$

M_2 satisfy $A \leftrightarrow A'$. We need to show that $\psi(A) \leftrightarrow \psi(A')$. If $b = b' = 0$, then this is clear. If $b, b' \neq 0$, then we have

$$\begin{bmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = AA' \tag{6.2.7}$$

$$= A'A = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a'a+b'c & a'b+b'd \\ c'a+d'c & c'b+d'd \end{bmatrix}.$$

In particular, it follows $b'c = bc'$ and hence

$$\frac{c}{b} = \frac{c'}{b'} \implies f\left(\left|\frac{c}{b}\right|\right) = f\left(\left|\frac{c'}{b'}\right|\right) =: f.$$

Now the desired conclusion follows by a direct calculation:

$$\begin{aligned}
\psi(A)\psi(A') &= \begin{bmatrix} a & bf \\ c\overline{f} & d \end{bmatrix}\begin{bmatrix} a' & b'f \\ c'\overline{f} & d' \end{bmatrix} \\
&= \begin{bmatrix} aa'+bc' & (ab'+bd')f \\ (ca'+dc')\overline{f} & cb'+dd' \end{bmatrix} \\
&= \begin{bmatrix} a'a+b'c & (a'b+b'd)f \\ (c'a+d'c)\overline{f} & c'b+d'd \end{bmatrix} \\
&= \begin{bmatrix} a' & b'f \\ c'\overline{f} & d' \end{bmatrix}\begin{bmatrix} a & bf \\ c\overline{f} & d \end{bmatrix} \\
&= \psi(A')\psi(A).
\end{aligned}$$

Suppose now that $b = 0$ and $b' \neq 0$. Then (6.2.7) implies $b'c = bc' = 0$ so by $b' \neq 0$ we conclude $c = 0$. We have

$$\begin{bmatrix} aa' & ab' \\ dc' & dd' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = AA' = A'A = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a'a & b'd \\ c'a & d'd \end{bmatrix}.$$

If we again denote $f := f\left(\left|\frac{c'}{b'}\right|\right)$, a direct calculation shows

$$\begin{aligned}
\psi(A)\psi(A') &= \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} a' & b'f \\ c'\overline{f} & d' \end{bmatrix} \\
&= \begin{bmatrix} aa' & ab'f \\ dc'\overline{f} & dd' \end{bmatrix} \\
&= \begin{bmatrix} a'a & b'df \\ c'a\overline{f} & d'd \end{bmatrix} \\
&= \begin{bmatrix} a' & b'f \\ c'\overline{f} & d' \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \\
&= \psi(A')\psi(A).
\end{aligned}$$

The final case of $b \neq 0$ and $b' = 0$ follows by swapping the roles of A and A'.
Now it is easy to see that the map

$$\phi : \mathcal{A}_\rho \to M_n, \qquad \phi(X) := \underbrace{\psi(X^{\flat\{r,s\}^c})^{\sharp\{r,s\}^c}}_{\in P\mathcal{A}_\rho} + \underbrace{(X^{\flat\{r,s\}})^{\sharp\{r,s\}}}_{=(I-P)X}$$

satisfies the same properties as ψ.

Case 2. Suppose that $(s,r) \notin \rho$. By the assumption and the transitivity of ρ it easily follows that

$$\rho^{-1}(r) = \{r\}, \qquad \rho(s) = \{s\}. \tag{6.2.8}$$

Each $X \in \mathcal{A}_\rho$ can be written as

$$X = X^{\ominus} + X_{rs}E_{rs},$$

where

$$X^{\ominus} := X - X_{rs}E_{rs} \in \mathcal{A}_\rho.$$

Consider the map

$$f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \qquad f(u,v) := \begin{cases} v, & \text{if } |u| \le |v|, \\ v\left|\frac{v}{u}\right|, & \text{if } |u| > |v|. \end{cases}$$

It is straightforward to check that f is continuous, homogeneous, and that $f(u,\cdot) : \mathbb{C} \to \mathbb{C}$ is injective for each fixed $u \in \mathbb{C}$. Define a map

$$\phi : \mathcal{A}_\rho \to \mathcal{A}_\rho, \qquad \phi(X) := X^{\ominus} + f(X_{ss} - X_{rr}, X_{rs})E_{rs},$$

that is

$$\phi(X)_{ij} = \begin{cases} X_{ij}, & \text{if } (i,j) \ne (r,s), \\ f(X_{ss} - X_{rr}, X_{rs}), & \text{if } (i,j) = (r,s). \end{cases}$$

We claim that ϕ is a continuous injective commutativity and spectrum preserver, but not a linear map (and hence not a Jordan homomorphism).

- The continuity of ϕ follows directly from the continuity of f.
- Using the Laplace expansion along the s-th row and the r-th column, we obtain

  $$k_X(x) = (x - X_{rr})(x - X_{ss})k_{X^{\flat\{r,s\}}}(x) = k_{X^{\ominus}}(x).$$

  In particular, the spectrum of X is equal to the spectrum of $X^{\ominus}$. As $\phi(X)^{\ominus} = X^{\ominus}$, we conclude that ϕ is a spectrum preserver.
- Suppose that $\phi(X) = \phi(Y)$ for some $X, Y \in \mathcal{A}_\rho$. Immediately we obtain
  (i) $X^{\ominus} = Y^{\ominus}$,
  (ii) $f(X_{ss} - X_{rr}, X_{rs}) = f(Y_{ss} - Y_{rr}, Y_{rs})$,
  whence we conclude $f(X_{ss} - X_{rr}, X_{rs}) = f(X_{ss} - X_{rr}, Y_{rs})$. This implies $X_{rs} = Y_{rs}$, since $f(X_{ss} - X_{rr}, \cdot)$ is injective. Therefore, $X = Y$, so ϕ is injective.
- Suppose that $X, Y \in \mathcal{A}_\rho$. We have

  $$\begin{aligned} XY &= (X^{\ominus} + X_{rs}E_{rs})(Y^{\ominus} + Y_{rs}E_{rs}) \\ &= X^{\ominus}Y^{\ominus} + Y_{rs}X^{\ominus}E_{rs} + X_{rs}E_{rs}Y^{\ominus} \\ &= X^{\ominus}Y^{\ominus} + Y_{rs}\left(\sum_{i \in \rho^{-1}(r)} X_{ir}E_{ir}\right)E_{rs} + X_{rs}E_{rs}\left(\sum_{i \in \rho(s)} Y_{si}E_{si}\right) \\ &\overset{(6.2.8)}{=} X^{\ominus}Y^{\ominus} + (X_{rr}Y_{rs} + X_{rs}Y_{ss})E_{rs}. \end{aligned}$$

Moreover, $(XY)^{\ominus} = X^{\ominus}Y^{\ominus}$, as $E_{rr}(X^{\ominus}Y^{\ominus})E_{ss} = 0$. Indeed, if $E_{rr}E_{ij}E_{kl}E_{ss} \neq 0$ for some $(i,j),(k,l) \in \rho \setminus \{(r,s)\}$, then $i = r$, $j = k$ and $l = s$, which implies that $j \in \rho(r) \cap \rho^{-1}(s) = \{r,s\}$; a contradiction. Similarly,

$$YX = Y^{\ominus}X^{\ominus} + (Y_{rr}X_{rs} + Y_{rs}X_{ss})E_{rs}, \qquad (YX)^{\ominus} = Y^{\ominus}X^{\ominus}.$$

Hence,

$$X \leftrightarrow Y \iff \begin{cases} X^{\ominus} \leftrightarrow Y^{\ominus}, \\ (X_{ss} - X_{rr})Y_{rs} = (Y_{ss} - Y_{rr})X_{rs}. \end{cases} \tag{6.2.9}$$

Assume now $X \leftrightarrow Y$. Since $\phi(X)^{\ominus} = X^{\ominus}$ and $\phi(Y)^{\ominus} = Y^{\ominus}$, to show that $\phi(X) \leftrightarrow \phi(Y)$, it remains to verify that

$$(X_{ss} - X_{rr})f(Y_{ss} - Y_{rr}, Y_{rs}) = (Y_{ss} - Y_{rr})f(X_{ss} - X_{rr}, X_{rs}).$$

By the homogeneity of f, this is equivalent to

$$f((X_{ss}-X_{rr})(Y_{ss}-Y_{rr}),(X_{ss}-X_{rr})Y_{rs}) = f((X_{ss}-X_{rr})(Y_{ss}-Y_{rr}),(Y_{ss}-Y_{rr})X_{rs}),$$

which is true by (6.2.9). We conclude that ϕ preserves commutativity.

- That ϕ is not an additive map follows from

$$\phi(2E_{rr} + E_{rs}) = 2E_{rr} + \frac{1}{2}E_{rs}, \qquad \phi(E_{rs}) = E_{rs}, \qquad \phi(2E_{rr} + 2E_{rs}) = 2E_{rr} + 2E_{rs}.$$

The proof of Theorem 6.2.2 is now complete. □

Note that, in the contrast to the previous cases of M_n, $\mathcal{T}_n$, and general block upper-triangular subalgebras of M_n, Jordan embeddings $\mathcal{A}_\rho \to M_n$ are not necessarily multiplicative or antimultiplicative, even when $\mathcal{A}_\rho$ satisfies the condition (i) of Theorem 6.2.2.

Example 6.2.4. Consider the quasi-order

$$\rho := ([1,3] \times [1,3]) \cup ([4,6] \times [4,6])$$

on $[1,6]$. Then clearly $\mathcal{A}_\rho = \operatorname{diag}(M_3, M_3)$ satisfies the condition (i) of Theorem 6.2.2. However, the Jordan automorphism ϕ of $\mathcal{A}_\rho$ given by $\phi(\operatorname{diag}(X,Y)) := \operatorname{diag}(X, Y^t)$ is obviously neither multiplicative nor antimultiplicative.

One may also wonder whether an SMA $\mathcal{A}_\rho \subseteq M_n$ which satisfies the condition (i) of Theorem 6.2.2 necessarily admits only trivial transitive maps or, more generally, if all Jordan embeddings $\mathcal{A}_\rho \to M_n$ are necessarily rank-one preservers, as it is the case when $\mathcal{A}_\rho$ is M_n, $\mathcal{T}_n$ or an arbitrary block-upper triangular subalgebra od M_n. This is not true in general, as the following example shows.

Example 6.2.5. Consider the quasi-order

$$\rho := \Delta_7 \cup ([1,3] \times [4,7]) \cup \{(1,3),(4,5),(6,7)\}$$

on $[1,7]$ and the corresponding SMA:

$$\mathcal{A}_\rho := \begin{bmatrix} * & 0 & * & * & * & * & * \\ 0 & * & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix} \subseteq \mathcal{T}_7.$$

We have

$$\begin{aligned} &\rho(1) \cup \rho^{-1}(1) = \rho(3) \cup \rho^{-1}(3) = \{1,3,4,5,6,7\}, \\ &\rho(2) \cup \rho^{-1}(2) = \{2,4,5,6,7\}, \\ &\rho(4) \cup \rho^{-1}(4) = \rho(5) \cup \rho^{-1}(5) = \{1,2,3,4,5\}, \\ &\rho(6) \cup \rho^{-1}(6) = \rho(7) \cup \rho^{-1}(7) = \{1,2,3,6,7\}. \end{aligned}$$

One easily checks that $\mathcal{A}_\rho$ satisfies (i) of Theorem 6.2.2 and that the map

$$g : \rho \to \mathbb{C}^\times, \qquad g(i,j) := \begin{cases} 2, & \text{if } (i,j) \in \{(2,4),(2,5)\}, \\ 1, & \text{otherwise} \end{cases}$$

is transitive map. On the other hand, g is nontrivial. Indeed, if g separates through the map $s : [1,7] \to \mathbb{C}^\times$, we obtain

$$\frac{s(1)}{s(4)} = g(1,4) = 1, \qquad \frac{s(1)}{s(6)} = g(1,6) = 1 \qquad \Longrightarrow \qquad s(4) = s(1) = s(6),$$

so

$$2 = g(2,4) = \frac{s(2)}{s(4)} = \frac{s(2)}{s(6)} = g(2,6) = 1,$$

which is a contradiction.
Further, as for the induced algebra automorphism g^* of $\mathcal{A}_\rho$ we have

$$g^*(E_{14} + E_{16} + E_{24} + E_{26}) = E_{14} + E_{16} + 2E_{24} + E_{26},$$

it is clear that g^* does not preserve rank-one matrices.

Remark 6.2.6. Suppose that $\mathcal{A} \subseteq M_n$ is a subalgebra and let $S \in M_n^\times$. Clearly, every injective continuous commutativity and spectrum preserver $\phi : \mathcal{A} \to M_n$ is a Jordan embedding if and only if the same holds for maps $\phi : S^{-1}\mathcal{A}S \to M_n$.

Remark 6.2.7. The statement of Theorem 6.2.2 does not have to include $n \geq 3$. Indeed, the $n = 1$ case is trivial, while the only SMA in M_2 which satisfies (i) is $\mathcal{D}_2$.

Remark 6.2.8. For an SMA $\mathcal{A}_\rho \subseteq M_n, n \geq 3$ which satisfies (i) of Theorem 6.2.2, we discuss the necessity of all assumptions in (ii).

- Spectrum preserving is necessary to assume for all SMAs $\mathcal{A}_\rho$. Indeed, let D be the open unit disk in $\mathbb{C}$. Define a map $g : D \to D$ by

$$g(z) = \frac{1-3z}{3-z}.$$

It is easy to check that g is a holomorphic bijection (actually, it is an involution). Consider the map $\phi : \mathcal{A}_\rho \to M_n$ given by

$$\phi(X) = g\left(\frac{X}{1 + \|X\|}\right),$$

where $\|\cdot\|$ denotes the spectral norm. The map ϕ is well-defined, as for each $X \in \mathcal{A}_\rho$ the matrix $\frac{X}{1+\|X\|}$ has norm < 1 and hence its spectrum is contained in D, at which point we can apply g using the holomorphic functional calculus. Using the properties of the holomorphic functional calculus, we conclude that ϕ is continuous and preserves commutativity. Moreover, since the map $X \mapsto \frac{X}{1+\|X\|}$ is injective, via the application of $g^{-1} = g$ we conclude that ϕ is injective. However, ϕ is clearly not linear as

$$\phi(0) = g(0) = \frac{1}{3}I.$$

- Commutativity preserving is necessary to assume for all SMAs $\mathcal{A}_\rho$. Indeed, when $\mathcal{A}_\rho \neq \mathcal{D}_n$, this follows by considering the map $\phi : \mathcal{A}_\rho \to M_n$,

$$\phi(X) := \operatorname{diag}(1, \dots, 1, e^{\det X}, 1, \dots, 1)X \operatorname{diag}(1, \dots, 1, e^{-\det X}, 1, \dots, 1),$$

where both $e^{\det X}$ and $e^{-\det X}$ stand at some position $i \in [1, n]$ for which $\rho^\times(i) \neq \emptyset$. Then ϕ is clearly continuous and preserves the spectrum. Moreover, since $\phi(X)$ is similar to X, we can see that ϕ is a bijection (its inverse is given by $Y \mapsto f(Y)^{-1}Yf(Y)$. However, ϕ is not linear:

$$\phi(I + E_{12}) = I + eE_{12} \neq I + E_{12} = \phi(I) + \phi(E_{12}).$$

On the other hand, when $\mathcal{A}_\rho = \mathcal{D}_n$, we can consider the map $\phi : \mathcal{D}_n \to M_n$ given by

$$\phi(X) := X + X_{11}E_{2n},$$

which is clearly a continuous injective spectrum preserver, but not a Jordan homomorphism.

- Continuity is necessary to assume for all SMAs $\mathcal{A}_\rho$. As in [45], consider the map $\phi : \mathcal{A}_\rho \to M_n$ given by

$$\phi(X) = \begin{cases} \operatorname{diag}(\lambda_2, \lambda_1, \dots, \lambda_n), & \text{if } X = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \text{ and all } \lambda_i \text{ are distinct,} \\ X, & \text{otherwise} \end{cases}$$

which is bijective, spectrum and commutativity preserving but clearly not continuous.

- Injectivity is necessary to assume if and only if $\mathcal{A}_\rho$ is not semisimple (note that by [17], $\mathcal{A}_\rho$ is semisimple if and only if ρ is symmetric). Indeed, suppose that $\mathcal{A}_\rho$ is semisimple and $\phi : \mathcal{A}_\rho \to M_n$ is a continuous commutativity and spectrum preserver. In view of Remark 6.2.6 and Lemma 3.1.4, we can assume that $\mathcal{A}_\rho = \operatorname{diag}(M_{k_1}, \dots, M_{k_p})$ where, by (i), each $k_j \neq 2$. By analysing the proofs of Claims 6.2.1.1, 6.2.1.2, 6.2.1.3 and 6.2.1.5 we observe that they do not require the injectivity of the map ϕ. Therefore, without loss of generality, one can assume that ϕ acts as the identity map on $\mathcal{D}_n$. Further, using Claim 6.2.1.5 and a standard density argument, it easily follows that ϕ maps each diagonal block of $\mathcal{A}_\rho$ into itself. Therefore, for each $1 \le j \le k$ the map ϕ restricts to a continuous spectrum and commutativity preserver $\phi_j : M_{k_j} \to M_{k_j}$. If $k_j = 1$, clearly ϕ_j is the identity, while if $k_j \ge 3$ we apply Theorem 2.6.1 to ϕ_j to conclude that it acts as the identity or as the transposition map. Putting everything back together, it follows that ϕ is a Jordan embedding. Now, on the other hand, suppose that $\mathcal{A}_\rho$ is not semisimple. Again, in view of Remark 6.2.6 and Lemma 3.1.4, we can assume that $\mathcal{A}_\rho$ is in the block upper-triangular form with at least one nonzero entry in the strict upper triangle. Then a map $\phi : \mathcal{A}_\rho \to M_n$ which maps a matrix $X \in \mathcal{A}_\rho$ to a matrix $\phi(X)$ having the same diagonal blocks but zeroes everywhere else, is an example of a non-injective unital Jordan homomorphism (in particular, ϕ is a continuous spectrum and commutativity preserver).

Example 6.2.9. Note that if $\mathcal{A}_\rho \subseteq M_n, n \ge 3$ is an SMA with the property that there exist distinct $i, j \in [1, n]$ such that $\rho(i) = \rho^{-1}(j) = [1, n]$, then $\mathcal{A}_\rho$ satisfies (i). Indeed, for each $r, s \in [1, n]$ we have

$$i, j \in (\rho(r) \cup \rho^{-1}(r)) \cap (\rho(s) \cup \rho^{-1}(s))$$

which proves (i) except when $r, s \in \{i, j\}$, in which case (i) follows from

$$(\rho(i) \cup \rho^{-1}(i)) \cap (\rho(j) \cup \rho^{-1}(j)) = [1, n]$$

and $n \ge 3$. An important class of such algebras are the ubiquitous block upper-triangular subalgebras which satisfy $\rho(1) = \rho^{-1}(n) = [1, n]$.

When we further assume that $\mathcal{B} = \mathcal{A}$, so that $\phi : \mathcal{A} \to \mathcal{A}$, we can relax the spectrum preserving assumption to spectrum shrinking ($\sigma(\phi(X)) \subseteq \sigma(X)$ for all $X \in \mathcal{A}$). More precisely, we obtain the following result (similarly as in [55], the proof relies on the invariance of domain theorem).

Remark 6.2.10. Suppose that $\mathcal{A}_\rho \subseteq M_n$ is an SMA which satisfies (i). Then every $\phi : \mathcal{A} \to \mathcal{A}$ continuous injective commutativity preserving spectrum shrinking ($\sigma(\phi(X)) \subseteq \sigma(X)$ for all $X \in \mathcal{A}_\rho$) map is necessarily Jordan embedding. Indeed, similarly as in [55], the proof of this fact relies on the invariance of domain theorem to show that ϕ in fact preserves the characteristic polynomial. Namely, by the invariance of domain theorem, the image $\mathcal{R} = \phi(\mathcal{A})$ is an open set in M_n and $\phi|^{\mathcal{R}} : \mathcal{A} \to \mathcal{R}$ is a homeomorphism. Let $\mathcal{E}$ denote the set of all matrices in M_n with n distinct eigenvalues. As $\mathcal{E}$ is dense in M_n, $\mathcal{E} \cap \mathcal{R}$ is dense in $\mathcal{R}$.

Now, since ϕ shrinks spectrum, its inverse $(\phi|^{\mathcal{R}})^{-1}$ expands spectrum. In particular, the restriction $(\phi|^{\mathcal{R}})^{-1}|_{\mathcal{E} \cap \mathcal{R}}$ preserves characteristic polynomial. Since the characteristic polynomial $k_\cdot : M_n \to \mathbb{C}_{\le n}[x]$ is a continuous map (Remark 2.3.7), we conclude that the continuous maps

$$\mathcal{R} \to \mathbb{C}_{\le n}[x] : \qquad X \mapsto k_{(\phi|^{\mathcal{R}})^{-1}(X)}, \qquad \text{and} \qquad X \mapsto k_X$$

are equal on the dense set $\mathcal{E} \cap \mathcal{R}$. Hence, they are equal everywhere so $(\phi|^{\mathcal{R}})^{-1}$ preserves characteristic polynomial. The same follows for ϕ, of course.

Conclusion

In this dissertation, we studied structural matrix algebras, their Jordan embeddings and some characterizations thereof using preserver properties.

In the first part of the dissertation we have successfully provided a complete description of Jordan embeddings $\mathcal{A} \to M_n$ where $\mathcal{A} \subseteq M_n$ is an SMA. In particular we can read out that each such map is a sum of an (algebra) homomorphism and an antihomomorphism. The description involves transitive maps in an essential way, a concept used by Coelho in her description of the automorphism group $\operatorname{Aut}(\mathcal{A})$ so our results can be interpreted as a direct continuation of her work. For SMAs $\mathcal{A}, \mathcal{B} \subseteq M_n$, we also gave a criterion for when does $\mathcal{A}$ Jordan-embed into $\mathcal{B}$, as well as a description of Jordan embeddings $\mathcal{A} \to \mathcal{B}$. A preliminary result which was used in the proof which stands out is the result concerning the intrinsic diagonalization of a family of diagonalizable matrices $\mathcal{F}$ in an SMA $\mathcal{A}$. Via counterexamples we have shown that this kind of intrinsic characterization cannot be expected to hold in general, at least without suitable modifications.

In the second part of the dissertation, we solved two natural linear preserver problems for maps $\mathcal{A} \to M_n$ where $\mathcal{A} \subseteq M_n$ is an SMA. The first one shows that unital linear rank-one maps $\mathcal{A} \subseteq M_n$ are necessarily Jordan embeddings (and hence of the form outlined in the first part). This conclusion serves as a direct generalization of rank-one preserver results obtained earlier by Marcus-Purves (on M_n) and Molnar-Šemrl (on $\mathcal{T}_n$). We have also provided a criterion for when does the converse hold (i.e. which Jordan embeddings $\mathcal{A} \subseteq M_n$ are rank-one preservers). By counterexamples, we have shown that omitting the unitality assumption can result in maps which exhibit strange behaviour. The second linear preserver problem concerns linear rank preservers. Essentially, we obtained that unital linear rank preservers $\mathcal{A} \to M_n$, where $\mathcal{A} \subseteq M_n$ is an SMA, are precisely Jordan embedding with the property that the corresponding transitive map (from the first part of the dissertation) is trivial. We have also made an observation that it actually suffices to assume that the map preserves rank up to $\frac{n}{2} - 1$ to ensure the same conclusion. We were also able to give a simple description of linear rank preservers without assuming unitality.

The third and final part of the dissertation concerns a particular nonlinear preserver problem which was originally studied by Petek and Šemrl on M_n and $\mathcal{T}_n$. We have extended this result to the class of SMAs. More precisely, we have given a simple necessary and sufficient condition on an SMA $\mathcal{A} \subseteq M_n$ such that all injective continuous spectrum and commutativity preservers $\mathcal{A} \to M_n$ are precisely Jordan embeddings.

An open area for further research would be to try and extend some of these results to a wider class of subalgebras of M_n (or ideally all of them). A substantial generalization in this direction would likely have to include radically more advanced techniques, since most of the existing preserver results we built upon heavily relied on the use of matrix units E_{ij}. In view of this, extending the use of such elementary methods beyond the class of SMAs seems somewhat unlikely.

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Curriculum Vitae

Mateo Tomašević was born on the 11th of July 1995 in Zagreb, where he attended elementary school and later The Fifth Gymnasium. During high-school, he participated in national competitions in mathematics, logic, physics and chemistry, as well as the International Chemistry Olympiad 2014. In autumn of the same year, he commenced studying mathematics at the Department of Mathematics of the Faculty of Science in Zagreb. Simultaneously he participated in and completed courses at the Department of Physics as well as the Faculty of Electrical Engineering and Computing. In 2017 he began studying Theoretical mathematics, again at the Department of Mathematics. During the entirety of his higher education, he was a student tutor for the courses *Linear Algebra 1* and *2*, *Discrete Mathematics*, and *Measure and Integral*. After completing the all courses, he received an award for exceptional success by the Council of the Department of Mathematics. He was receiving the City of Zagreb scholarships, as well as a scholarship from the University of Zagreb. In autumn 2019, he graduated summa cum laude under the supervisor Assoc. Prof. Ilja Gogić, with the diploma thesis titled *On the epimorphic image of the centre of a C^*-algebra*. He begins working on his doctorate degree under the same supervisor. At the same time, he begins his employment as a teaching assistant at the Department of Mathematics, as a part of the Division of Mathematical Analysis. He completed introductory doctorate courses in *Analysis*, *Geometry and Topology*, and advanced courses in *Nonlinear Fourier Analysis*, *Representation Theory of Lie Groups and Algebras* and *Concrete Mathematics*. He is a member of the Functional Analysis Seminar, as part of which he held four lectures. His advisor, Prof. Gogić, directs him into the research area of preservers in linear algebra and functional analysis, and in 2021 connects him with the leading world expert in the area, Prof. Peter Šemrl, from the University of Ljubljana. A product of this cooperation was Mateo's first research paper:

- M. Tomašević, *A nonlinear preserver problem on positive matrices*, Linear Algebra Appl. **666** (2023), 96–113. https://doi.org/10.1016/j.laa.2023.02.022

He continues working on his doctoral dissertation in the same area, with emphasis on Jordan homomorphisms and structural matrix algebras, as directed by Prof. Gogić. The two of them together wrote two papers and one preprint:

- I. Gogić, T. Petek, M. Tomašević, *Characterizing Jordan embeddings between block upper-triangular subalgebras via preserving properties*, Linear Algebra Appl. **704** (2025), 192–217. https://doi.org/10.1016/j.laa.2024.10.005
- I. Gogić, M. Tomašević, *Jordan embeddings and linear rank preservers of structural matrix algebras*, Linear Algebra Appl. **707** (2025), 1–48. https://doi.org/10.1016/j.laa.2024.11.013
- I. Gogić, M. Tomašević, *An extension of Petek-Šemrl preserver theorems for Jordan embeddings of structural matrix algebras*, http://arxiv.org/abs/2411.11092

The first paper was a collaboration with Prof. Tatjana Petek (Prof. Šemrl's former doctoral student and also a renowned expert in the area of preservers).