Trace anomalies from matter models in curved spacetime

Štemberga, Tamara

UNIVERSITY OF ZAGREB
FACULTY OF SCIENCE
DEPARTMENT OF PHYSICS

Tamara Štemberga

TRACE ANOMALIES FROM MATTER MODELS IN CURVED SPACETIME

DOCTORAL DISSERTATION

Zagreb, 2018

UNIVERSITY OF ZAGREB
FACULTY OF SCIENCE
DEPARTMENT OF PHYSICS

Tamara Štemberga

TRACE ANOMALIES FROM MATTER MODELS IN CURVED SPACETIME

DOCTORAL DISSERTATION

Supervisor: doc.dr.sc. Maro Cvitan

Zagreb, 2018

SVEUČILIŠTE U ZAGREBU
PRIRODOSLOVNO-MATEMATIČKI FAKULTET
FIZIČKI ODSJEK

Tamara Štemberga

ANOMALIJE TRAGA IZ MODELA MATERIJE U ZAKRIVLJENOM PROSTORU

DOKTORSKI RAD

Mentor: doc.dr.sc. Maro Cvitan

Zagreb, 2018

Acknowledgements

I would like to express my sincere gratitude to my supervisor, Maro Cvitan for the guidance and support during my research. Also, I would also like to thank my collaborators Loriano Bonora, Predrag Dominis Prester, Antonio Duarte Pereira, Stefano Giaccari, Bruno Lima de Souza, Silvio Pallua, Mateo Paulišić, Ivica Smolić and Ivan Vuković for many stimulating discussions.

Trace anomalies from matter models in curved spacetime

Abstract

Using the effective action approach we deal with two main topics: the trace anomaly in chiral theory and higher spin effective actions.

First, we recalculate the odd-parity trace anomaly for Weyl fermion and consider possible contributions from tadpole and seagull terms in the Feynman diagram approach with dimensional regularization. Introduction of an axial symmetric tensor, in addition to the usual gravitational metric, allows us to use Dirac fermions which are coupled not only to the usual metric but also to the additional axial tensor. We obtain the trace anomaly for Majorana and Weyl fermions in two suitable limits of such a general configuration. We also compute non-perturbatively the odd-parity trace anomaly in a theory of a Dirac fermion field coupled to a metric-axial-tensor background, using Schwinger-DeWitt heat kernel technique with two different regularizations: dimensional and ζ-function regularization. We find that in theories with chiral fermions coupled to curved background the trace of the energy-momentum tensor at one-loop gets a contribution from the Pontryagin density with an imaginary coefficient. We also find that for Majorana and Dirac fermions the odd-parity part of the trace anomaly vanishes as expected.

Second, we analyze the effective actions obtained in both massless and massive scalar and fermion model coupled to higher spin sources (external fields) via conserved currents. We are focused on two-point correlators so that the constructed one-loop effective action contains only the quadratic terms and the relevant equations of motion for the sources we obtain are the linearized ones. We show that our results can be expressed in a geometric form, that is, in terms of covariant generalized Jacobi tensors. In 3d we also consider the odd-parity sector where we find a generalization of Pope-Townsend Chern-Simons-like action. Moreover, we formulate the worldline quantization of a massive fermion model coupled to external higher spin sources. We find that the regularized effective action obtained in this way is endowed with an L_∞ symmetry.

Keywords: effective actions, trace anomalies, Pontryagin density, higher spins, L_∞ symmetry

Anomalije traga iz modela materije u zakrivljenom prostoru

Prošireni sažetak

Kada opisujemo fundamentalne interakcije u fizici, simetrije i pridruženi zakoni očuvanja igraju glavnu ulogu. Može se dogoditi da, nakon što kvantiziramo teoriju, zakon očuvanja koji je vrijedio na klasičnom nivou, na kvantnom nivou više ne vrijedi. Tada kažemo da je teorija anomalna. Kvantne anomalije mogu biti bezopasne ili štetne. Bezopasne anomalije (na primjer kiralna anomalija koja objašnjava raspad piona na dva fotona) imaju fzikalne posljedice, dok štetne anomalije (na primjer kiralna baždarna anomalija) narušavaju konzistentnost teorije pa se stoga koriste za isključenje teorija.

Koristeći metodu efektivne akcije bavit ćemo se neparnom anomalijom traga tenzora-energije impulsa u 4d u teoriji s kiralnim fermionima u zakrivljenom prostoru te Diracovim fermionima u MAT gravitaciji. Pokazat ćemo da je neparni dio anomalije traga dan s Pontryaginovom gustoćom s imaginarnim koeficijentom što ukazuje na lom unitarnosti i narušenje konzistencije teorije (jer tenzor energije-impulsa postaje imaginaran). To sugerira da se ova anomalija može koristiti kao selektivni kriterij za razne teorije.

Zanimaju nas i modeli materije vezani na polja višeg spina. Dok je teorija slobodnih bezmasenih polja višeg spina većeg od dva konzistentna, postoje ozbiljna ograničenja u obliku "no-go" teorema za opis njihovih interakcija, osobito u ravnom prostor-vremenu.

Kao prvi korak prema našem cilju da analiziramo anomalije u modelima materije koji interagiraju s poljima višeg spina, potrebno je vidjeti koji je oblik efektivne akcije dobivene integrirajući mikroskopsko polje materije (fermionsko ili skalarno) u teoriji u kojoj je mikroskopsko polje vezano na polja višeg spina putem očuvanih struja. Usredotočit ćemo se na kvadratni dio efektivne akcije te doznati da su dobivene akcije nelokalne. Nakon oduzimanja konačnog broja kontračlanova iz efektivne akcije, pokazuje se da ova metoda predstavlja alat za dobivanje informacije o dinamici viših spinova. To ukazuje da bismo u ovom pristupu, računanjem korelatora višeg reda mogli doznati više o nelinearnoj strukturi viših spinova. Na ovaj način možemo dobiti i uvid u to kako su "no-go" teoremi povezani s našim slučajem, tj. predstavljaju li ograničenja ili ih zaobilazimo.

Naš drugi pristup temelji se na kvantizaciji svjetske linije fermionskog modela vezanog na polja višeg spina. U ovom pristupu dobivamo egzaktnu baždarnu transformaciju pa postoji mogućnost da dobivena efektivna akcija bude baždarno invarijantna bez dodavanja kontračlanova. U slučaju da nema generaliziranih anomalija difeomorfizama, pronalazimo da efektivna akcija posjeduje L_∞ simetriju. To sugerira da integriranje L_∞ algebre možemo koristiti za pronalaženje mogućih kandidata za teorije viših spinova.

Efektivne akcije i simetrije

Fundamentalni objekt u kvantnoj teoriji polja je particijska funkcija koja je generator svih korelacijskih funkcija. Definiramo je sa

$$Z[\varphi] = \int \mathcal{D}\phi e^{iS[\phi,\varphi]} \tag{1}$$

gdje je φ vanjsko klasično polje kao na primjer spin 1 polje A_μ, spin 2 polje $h_{\mu\nu}$ ili općenito polje višeg spina-s $\varphi_{\mu_1 \dots \mu_s}$. Pretpostavljamo da je klasična akcija $S[\phi, \varphi]$ suma slobodne akcije $S_0[\phi]$ za neko polje materije ϕ te interakcije $S_{int}[\phi,\varphi]$: $S[\phi,\varphi]=S_0[\phi]+S_{int}[\phi,\varphi]$. Za danu particijsku funkciju $Z[\varphi]$ uvodimo efektivnu akciju $W[\varphi]=-i\ln Z[\varphi]$ koja generira sve povezane korelacijske funkcije. Pretpostavimo li da je interakcija oblika

$$S_{int}[\phi,\varphi]=\sum_s\int d^dx\varphi_{\mu_1\dots\mu_s}j^{\mu_1\dots\mu_s}(x). \tag{2}$$

gdje su $j^{\mu_1\dots\mu_s}(x)$ očuvane struje (na ljusci mase) minimalno vezane na vanjsko polje spina s $\varphi_{\mu_1\dots\mu_s}$, tada efektivna akcija postaje:

$$iW[\varphi]=iW[0]+\sum_{n=1}^{\infty}\sum_{s_1,\dots,s_n}\frac{i^n}{n!}\int\prod_{i=1}^{n}d^dx_i\varphi^{\mu_{11}\dots\mu_{1s_1}}(x_1)\dots\varphi^{\mu_{n1}\dots\mu_{ns_n}}(x_n)$$
$$\times\langle 0|\mathcal{T}j_{\mu_{11}\dots\mu_{1s_1}}(x_1)\dots j_{\mu_{n1}\dots\mu_{ns_n}}(x_n)|0\rangle_c \tag{3}$$

Jednopetljeni korelator na jednu točku za $j_{\mu_1\dots\mu_s}$ definira se kao $\langle\langle j_{\mu_1\dots\mu_s}(x)\rangle\rangle=\frac{\delta W[\varphi]}{\delta\varphi^{\mu_1\dots\mu_s}(x)}$.

Simetrije klasične teorije. Ako je klasična akcija S, koja opisuje polje materije $\phi(x)$ vezano na baždarno polje A_μ, invarijantna na baždarnu transformaciju $\delta A_\mu=\partial_\mu\lambda$, struja $j^\mu(x)=\frac{\delta S}{\delta A_\mu(x)}$ biti će očuvana $\partial_\mu j^\mu(x)=0$.

Slično, za polje materije $\phi(x)$ vezano na gravitaciju, klasična akcija invarijantna je na difeomorfizme $\delta_\xi g_{\mu\nu}(x)=\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu$ pa je tenzor energije-impulsa kovarijantno očuvan $\nabla^\mu T_{\mu\nu}(x)=0$. Osim toga, akcija je invarijantna i na Weylove transformacije $\delta_\omega g_{\mu\nu}(x)=2\omega(x)g_{\mu\nu}(x)$ u bezmasenom slučaju što implicira da trag tenzora energije-impulsa iščezava $T^\mu_\mu=0$.

Nadalje, za polje materije $\phi(x)$ vezano na polje višeg spina $\varphi_{\mu_1\dots\mu_s}$, $s>2$, ako je akcija invarijantna na baždarnu transformaciju u najnižem redu $\delta\varphi_{\mu_1\dots\mu_s}=\partial_{(\mu_1}\Lambda_{\mu_2\dots\mu_s)}$, tada je struja $j_{\mu_1\dots\mu_s}(x)=\frac{\delta S}{\delta\varphi^{\mu_1\dots\mu_s}(x)}$ očuvana na ljusci mase $\partial^{\mu_1}j_{\mu_1\dots\mu_s}(x)=0$. Povrh toga, u limesu $m\to 0$, ako je teorija invarijantna na generalizirane Weylove transformacije $\delta\varphi_{\mu_1\dots\mu_s}=\eta_{(\mu_1\mu_2}\omega_{\mu_3\dots\mu_s)}$, trag struje $j_{\mu_1\dots\mu_s}$ iščezava $\eta^{\mu_1\mu_2}j_{\mu_1\dots\mu_s}(x)=0$.

Simetrije kvantne teorije. Ako kvantna teorija posjeduje iste simetrije kao i klasična teorija, kvantna efektivna akcija W biti će invarijantna na iste baždarne transformacije kao i klasična akcija. Za polje spina 1 korelator na jednu točku struje $\langle\langle j^\mu(x)\rangle\rangle$ poštuje Wardov identitet za baždarnu invarijantnost

$$\partial_\mu\langle\langle j^\mu(x)\rangle\rangle=0 \tag{4}$$

Nadalje, za polje spina 2 imamo Wardov identitet za invarijantnost na difeomorfizme te Wardov identitet za Weylovu invarijantnost:

$$\nabla^\mu\langle\langle T_{\mu\nu}(x)\rangle\rangle=0,\qquad\langle\langle T^\mu_\mu(x)\rangle\rangle=0 \tag{5}$$

Sličan kovarijantan zakon očuvanja trebao bi biti zapisan i za $s>2$ struje, ali često ćemo se zadovoljiti i s najnižim netrivijalnim redom za koji se zakon očuvanja reducira na

$$\partial^{\mu_1}\langle\langle j_{\mu_1\dots\mu_s}(x)\rangle\rangle=0 \tag{6}$$

Konačno, Wardov identitet za generalizirane Weylove transformacije je

$$\eta^{\mu_1\mu_2}\langle\langle j_{\mu_1\ldots\mu_s}(x)\rangle\rangle = 0 \tag{7}$$

U slučaju kada kvantna teorija ne poštuje iste simetrije kao i klasična teorija, Wardovi identiteti su narušeni i tada kažemo da je teorija posjeduje anomaliju.

Anomalija traga

Pri opisu fundamentalnih interakcija u fizici, simetrije i pripadni zakoni očuvanja igraju važnu ulogu. Može se dogoditi da simetrija klasične teorije nije simetrija efektivne akcije u kvantnoj teoriji i tada teorija posjeduje anomaliju [1]-[3]. U ovom radu fokusiramo se na anomaliju traga za Weylove fermione vezane na gravitaciju. Ovu anomaliju još nazivamo i Weylova anomalija ili konformna anomalija. Oblik anomalije traga ovisi o dimenziji prostorvremena i uvjetima konzistencije (Wess-Zumino). U 4 dimenzije anomalija traga sadrži Weylovu, Eulerovu (Gauss-Bonnet) i Pontryaginovu gustoću [4]-[12]:

$$\langle\langle T^\mu_\mu(x)\rangle\rangle = a\,E + c\,W^2 + e\,P \tag{8}$$

gdje je posebno

$$P = \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,R_{\mu\nu}{}^{\alpha\beta}\,R_{\rho\sigma\alpha\beta} \tag{9}$$

Dok Weylova i Eulerova gustoća čuvaju CP (nabojna konjugacija i paritet), Pontryaginova gustoća narušava CP. Koeficijenti a, c and e ovise o teoriji [7, 13, 14] . Mi ćemo se fokusirati na koeficijent e uz neparni dio anomalije.

Jedan slučaj gdje se Pontryaginova gustoća može javiti je u teoriji s kiralnim fermionima koji interagiraju s gravitacijom [15]-[21] u 4d. Akcija je

$$S = \int d^4x\,\sqrt{|g|}\,i\overline{\psi_L}\gamma^\mu\left(\nabla_\mu + \frac{1}{2}\omega_\mu\right)\psi_L \tag{10}$$

dok je metrika $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ gdje je $h_{\mu\nu}$ mala preturbacija oko ravnog prostora. U originalnom računu [15] polje ψ je redefinirano $\psi \to |g|^{\frac{1}{4}}\psi$. Račun anomalije traga baziran je na Feynmanovim dijagramima i dimenzionalnoj regularizaciji. Slijedeći [19], u ovom radu predstaviti ćemo detaljniji račun neparnog djela anomalije traga. Prije svega, ne redefiniramo polje ψ te razmatramo postojanje dodatnih neiščezavajućih dijagrama. Ispostavlja se da samo korelator na 3 točke (trokutni dijagram) doprinosi. Eksplicitan račun trokutnog dijagrama daje h^2 član u razvoju Pontryaginove gustoće

$$\langle\langle T^\mu_\mu\rangle\rangle = -\frac{3i}{768\pi^2}P$$

Moramo još provjeriti i očuvanje tenzora energije-impulsa. Pokazuje se da anomalija na difeomorfizme ne iščezava. Da bismo je pokratili uvodimo kontračlan $\mathcal{C} = -\frac{1}{2}\int \omega\, h^\mu_\mu\,\mathcal{A}_0$ gdje je $\mathcal{A}_0 = \frac{i}{768\pi^2}P$. Na taj način anomalija traga postaje

$$\langle\langle T^\mu_\mu\rangle\rangle = \frac{i}{768\pi^2}P \tag{11}$$

Time se potkrepljuje rezultat iz [15]. Za desne fermione koeficijent e ima suprotan predznak $e_R = -\frac{i}{768\pi^2}$.

Neparni dio anomalije traga za Weylove fermione često je prihvaćen sa sumnjom, a razlog tome je tvrdnja da su bezmaseni Majoranini i Weylovi fermioni isti jer njihove klasične akcije izgledaju isto u dvokomponentnoj notaciji. Ako je ova tvrdnja istinita i na kvantnom nivou, anomalija traga za Weylove fermione ne postoji. S druge strane, ne smijemo zaboraviti da je centralni objekt u kvantnoj teoriji, kada razgovaramo o anomalijama, integralna mjera koja nije ista za Majoranine i Weyove fermione. Jedan način na koji možemo pokazati da one nisu iste je eksplicitni račun anomalije traga. Neparni dio anomalije traga za Majorana fermione iščezava dok je za Weylove fermione zadan Pontyaginovom gustoćom.

Da bismo učvrstili naš rezultat i izbjegli probleme s integralnom mjerom, uvodimo MAT gravitaciju gdje povrh obične metrike $g_{\mu\nu}$, uvodimo i aksijalni metrički tenzor $f_{\mu\nu}$: $G_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu}$. Ideja je ugraditi naš sustav u šire okruženje te na taj način omogućiti formulaciju problema pomoću Diracovih fermiona. Akcija je tada

$$S = \int d^4x \, i\overline{\psi}\sqrt{|\bar{G}|}\gamma^a \hat{E}_a^\mu \left(\partial_\mu + \frac{1}{2}\Omega_\mu\right)\psi \tag{12}$$

Akcija je invarijantna na difeomorfizme $\delta_\Xi G_{\mu\nu} = \mathcal{D}_\mu \Xi_\nu + \mathcal{D}_\nu \Xi_\mu$ s parametrom $\Xi^\mu = \xi^\mu + \gamma_5 \zeta^\mu$ i na Weylove transformacije $\delta_\omega G_{\mu\nu} = 2\omega G_{\mu\nu}$ s parametrom ω te aksijalne Weylove transformacije $\delta_\eta G_{\mu\nu} = 2\gamma_5 \eta G_{\mu\nu}$ s parametrom η. Sada postoje dva očuvana tenzora energije-impulsa $T_\mu^\mu(x)$ i $T_{5\mu}{}^\mu(x)$. Račun anomalije traga pomoću Feynamovih dijagrama i dimenzionalne regularizacije daje

$$\begin{aligned}\langle\langle T_\mu^\mu(x)\rangle\rangle &= \frac{i}{768\pi^2}\epsilon^{\mu\nu\lambda\rho}\mathcal{R}^{(1)\sigma\tau}_{\mu\nu}\mathcal{R}^{(2)}_{\lambda\rho\sigma\tau} \\ \langle\langle T_{5\mu}{}^\mu(x)\rangle\rangle &= \frac{i}{1536\pi^2}\,\epsilon^{\mu\nu\lambda\rho}\left(\mathcal{R}^{(1)\sigma\tau}_{\mu\nu}\mathcal{R}^{(1)}_{\lambda\rho\sigma\tau} + \mathcal{R}^{(2)\sigma\tau}_{\mu\nu}\mathcal{R}^{(2)}_{\lambda\rho\sigma\tau}\right)\end{aligned} \tag{13}$$

gdje su $\mathcal{R}^{(1)}_{\mu\nu\rho\lambda}$ i $\mathcal{R}^{(2)}_{\mu\nu\rho\lambda}$ redom, obični i aksijalni dio Riemannovog tenzora.

Neparni dio anomalije traga za lijevi Weylov fermion dobiva se u limesu $h_{\mu\nu} \to \frac{h_{\mu\nu}}{2}$, $f_{\mu\nu} \to \frac{h_{\mu\nu}}{2}$ glasi

$$\langle\langle T_\mu^\mu\rangle\rangle = \frac{i}{768\pi^2}P \tag{14}$$

S druge strane, za desni Weylov fermion koristimo limes $h_{\mu\nu} \to \frac{h_{\mu\nu}}{2}$, $k_{\mu\nu} \to -\frac{h_{\mu\nu}}{2}$ te u ovom slučaju anomalija mijenja predznak. Nadalje, neparni dio anomalije traga za Diracov fermion (ili Majorana ako ψ zadovoljava uvjet realnosti) dobiva se u limesu $h_{\mu\nu} \to h_{\mu\nu}$, $f_{\mu\nu} \to 0$. Anomalija u ovom slučaju iščezava.

Isti rezultat može se dobiti i neperturbativno koristeći Schwinger-DeWittovu metodu zajedno s dvije različite regularizacije: dimenzionalnom i regularizacijom pomoću ζ-funkcije, kao što je pokazano u [20]. Definiramo amplitudu

$$\langle\widehat{x}, \widehat{s}|\widehat{x}', 0\rangle = \langle\widehat{x}|e^{i\widehat{\mathfrak{F}}\widehat{s}}|\widehat{x}'\rangle \tag{15}$$

gdje je $\widehat{\mathcal{F}} = \overline{\overline{\nabla}}_\mu \overline{\widehat{g}}^{\mu\nu} \overline{\overline{\nabla}}_\nu - \frac{1}{4}\overline{\overline{R}}$ i koja zadovoljava sljedeću diferencijalnu jednadžbu

$$i\frac{\partial}{\partial \widehat{s}}\langle \widehat{x}, \widehat{s}|\widehat{x}', 0\rangle = -\widehat{\mathcal{F}}_{\hat{x}}\langle \widehat{x}, \widehat{s}|\widehat{x}', 0\rangle \tag{16}$$

Koristeći anstatz

$$\langle \widehat{x}, \widehat{s}|\widehat{x}', 0\rangle = -\lim_{m\to 0}\frac{i}{16\pi^2}\frac{\sqrt{\widehat{D}(\widehat{x}, \widehat{x}')}}{\widehat{s}^2} e^{i\left(\frac{\widehat{\sigma}(\widehat{x},\widehat{x}')}{2\widehat{s}} - m^2\widehat{s}\right)}\widehat{\Phi}(\widehat{x}, \widehat{x}', \widehat{s}) \tag{17}$$

gdje je $\widehat{\Phi}(\widehat{x}, \widehat{x}', \widehat{s}) = \sum_{n=0}^{\infty}\widehat{a}_n(\widehat{x}, \widehat{x}')(i\widehat{s})^n$ s rubnim uvjetom $[\widehat{a}_0] = 1$, dobiva se rekurzivna relacija za koeficijente $\widehat{a}_n$:

$$(n+1)\widehat{a}_{n+1} + \widehat{\nabla}^\mu \widehat{a}_{n+1}\widehat{\nabla}_\mu\widehat{\sigma} - \frac{1}{\sqrt{\widehat{D}}}\widehat{\nabla}^\mu\widehat{\nabla}_\mu\left(\sqrt{\widehat{D}}\widehat{a}_n\right) + \left(\frac{1}{4}\widehat{R} - m^2\right)\widehat{a}_n = 0 \tag{18}$$

Uz ovu relaciju možemo odrediti koeficijente $\widehat{a}_n$ u limesu $\widehat{x} \to \widehat{x}'$ što označavamo s $[\widehat{a}_n]$. Za neparni dio anomalije traga u 4d relevantan je koeficijent $[\widehat{a}_2]$

$$[\widehat{a}_2]|_{odd} = \frac{1}{48}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}$$

Da bismo izračunali anomaliju traga potreban nam je i regulator da bismo eliminirali divergencije u koincidentnim točkama. Kao što smo već spomenuli koristimo dimenzionalnu i regularizaciju pomoću ζ-funkcije. U dimenzionalnoj regularizaciji, $d = 4$ i za $m = 0$ efektivna akcija glasi

$$\widehat{L} \;=\; \frac{1}{16\pi^2}\left(\frac{1}{d-4} - \frac{3}{4}\right)\int d^d\widehat{x}\,\mathrm{tr}\left([\widehat{\mathrm{a}}_2]|_{\mathrm{m}=0}\sqrt{\widehat{\mathrm{g}}}\right) + \widehat{\mathrm{L}}_{\mathrm{R}} \tag{19}$$

gdje je

$$\widehat{L}_R = \frac{i}{64\pi^2}\mathrm{tr}\int_0^\infty d\widehat{\mathrm{s}}\,\ln(4\pi\mathrm{i}\mu^2\widehat{\mathrm{s}})\sqrt{\widehat{\mathrm{g}}}\frac{\partial^3}{\partial(\mathrm{i}\widehat{\mathrm{s}})^3}\left(\mathrm{e}^{-\mathrm{i}m^2\widehat{\mathrm{s}}}[\widehat{\Phi}(\widehat{\mathrm{x}}, \widehat{\mathrm{x}}, \widehat{\mathrm{s}})]\right)\Big|_{\mathrm{m}\to 0} \tag{20}$$

Goli dio akcije je invarijantan na Weylove transformacije $\delta_{\widehat{\omega}}\widehat{L} = 0$ dok renormalizirani dio $\widehat{L}_R$ definira tenzor energije-impulsa $\frac{2}{\sqrt{\widehat{g}}}\frac{\delta}{\delta\widehat{g}_{\mu\nu}}\widehat{L}_R = \langle\langle\widehat{T}^{\mu\nu}\rangle\rangle$.U prikladnom limesu, za neparni dio anomalije traga opet dobivamo (14), čime potkrepljujemo rezultat iz [15, 19]. Isto možemo potvrditi i korištenjem regularizacije pomoću ζ-funkcije. Povrh toga, pokazujemo da se opisana metoda može proširiti na MAT gravitaciju.

Važno je primijetiti da je koeficijent uz Pontryaginovu gustoću imaginaran. Imaginarni tenzor energije-impulsa može slomiti unitarnost te narušiti konzistentnost teorije. To sugerira da ovu anomaliju koristimo kao selektivni kriterij za razne modele. Naime, ako u nekoj teoriji postoji balans lijevih i desnih kiralnih fermiona neparni dio anomalije se pokrati te problem anomalije tada ne postoji. Napomenimo i da Pontryaginova gustoća iščezava u nekim slučajevima kao što su FRW ili Schwarzschildova geometrija.

Jedan od važnih ishoda ovog pristupa je i sama MAT gravitacija koja se može samostalno proučavati kao novi bimetrički model.

Efektivne akcije i polja višeg spina

Da bismo konstruirali konzistentnu kvantnu teoriju gravitacije i materije, ideja je koristiti beskonačan broj polja višeg spina. Jedan primjer te ideje je teorija (super)struna gdje se u spektru javlja beskonačan toranj polja višeg spina [22, 23]. Još jedan primjer teorije s beskonačno mnogo polja viših spina je i Vasilievljeva teorija [24]-[27] . Moguće je da ovo nisu jedini primjeri, ali tada se postavlja pitanje: koji zahtjevi moraju biti zadovoljeni da bi teorija viših spinova imala smisla?

U ovom radu predstaviti ćemo ideju započetu u [28] te u [29]-[33] gdje koristimo pristup efektivne akcije da bismo odredili linearnu klasičnu dinamiku polja višeg spina. Interakcija masivnog skalarnog ili fermionskog polje s poljima višeg spina $\varphi^{\mu_1\cdots\mu_s}$ dana je putem očuvanih $j_{\mu_1\ldots\mu_s}$, $S_{int} \sim \sum_s \int d^dx\varphi^{\mu_1\cdots\mu_s} j_{\mu_1\ldots\mu_s}$. Da bismo analizirali dinamiku polja višeg spina potreban nam je kvadratni dio efektivne akcije što znači da su jednadžbe gibanja linearizirane. Efektivnu akciju dobivamo računanjem korelatora na 2 točke očuvanih struja pomoću Feynmanovih dijagrama i metode koju su uveli Davydychev i suradnici, [34]-[36]. Kao što smo prethodno spomenuli, ideja je uvesti beskonačno polja viših spinova u teoriju. Zato razmatramo i korelatore dvije struje za bilo koji spin vezane na polja koji mogu doprinositi efektivnoj akciji. Ove korelatore zovemo mješoviti ili ne-dijagonalni.

Važno je napomenuti i da očuvane struje nisu jedinstvene te da njihov oblik utječe na oblik efektivne akcije. Uglavnom ćemo se usredotočiti na dva specifična izbora koje nazivamo jednostavne struje i struje čiji trag iščezava. Jednostavne struje su

$$j^{\mathrm{s}}_{\mu_1\ldots\mu_s} = i^s\varphi^\dagger \left(\overleftrightarrow{\partial}_\mu\right)^s \varphi\,, \quad j^{\mathrm{f}}_{\mu_1\ldots\mu_s} = i^{s-1}\bar{\psi}\gamma_\mu \left(\overleftrightarrow{\partial}_\mu\right)^{s-1} \psi \tag{21}$$

dok su struje bez traga dane kao posebna linearna kombinacija prethodnih struja. Njihov trag iščezava u limesu kada masa ide u nulu (slučaj generalizirane Weylove invarijantnosti). Definirane su sa:

$$j^{\mathrm{st}}_{\mu_1\ldots\mu_s} = \sum_{l=0}^{\lfloor\frac{s}{2}\rfloor} a^{\mathrm{s}}_{s,l} \left(\Box\pi_{\mu\mu}\right)^l \tilde{j}^{\mathrm{s}}_{\mu_1\ldots\mu_{s-2l}}\,, \quad j^{\mathrm{ft}}_{\mu_1\ldots\mu_s} = \sum_{l=0}^{\lfloor\frac{s-1}{2}\rfloor} a^{\mathrm{f}}_{s,l} \left(\Box\pi_{\mu\mu}\right)^l \tilde{j}^{\mathrm{f}}_{\mu_1\ldots\mu_{s-2l}} \tag{22}$$

gdje je

$$a^{\mathrm{s}}_{s,l} = \frac{(-1)^l s!\,\Gamma\left(s+\frac{d-3}{2}-l\right)}{2^{2l}l!(s-2l)!\,\Gamma\left(s+\frac{d-3}{2}\right)}\,, \quad a^{\mathrm{f}}_{s,l} = \frac{(-1)^l (s-1)!\,\Gamma\left(s+\frac{d-3}{2}-l\right)}{2^{2l}l!(s-2l-1)!\,\Gamma\left(s+\frac{d-3}{2}\right)} \tag{23}$$

Amplituda za struje dva različita spina s_1 i s_2 za struje čiji trag iščezava može biti zapisana kao linearna kombinacija amplituda za jednostavne struje.

Analizom općenitog oblika očuvanog korelatora na 2 točke doznajemo da se isti mogu zapisti pomoću projektora $\pi_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}$, na sljedeći način

$$\sum_{l=0}^{\lfloor s/2\rfloor} a_l \pi^{s-2l}_{\mu\nu}\pi^l_{\mu\mu}\pi^l_{\nu\nu} \tag{24}$$

gdje su koeficijenti a_l funkcije impulsa k i mase m. Iako je zapis pomoću projektora jako prikladan, informacija o geometriji naših rezultata zadana je implicitno na ovaj način.

Najjednostavniji način za formulaciju slobodne bezmasene teorije viših spinova je pomoću Fronsdalovog tenzora [37, 38]

$$\mathcal{F} \equiv \Box\varphi - \partial\,\partial\cdot\varphi + \partial^2\varphi' = 0 \tag{25}$$

Fronsdalova jednadžba invarijantna je na lokalne baždarne transformacije $\delta\varphi = \partial\Lambda$ s parametrom $\Lambda \equiv \Lambda_{\mu_1\cdots\mu_{s-1}}$, samo ako je parametar Λ ograničen $\Lambda' = 0$. Ovo ograničenje možemo izbjeći ako uvedemo generalizaciju $\mathcal{F}^{(n)}$ Fronsdalovog diferencijalnog operatora [39]-[41], koji je baždarno invarijantan za n dovoljno velik. Operator $\mathcal{F}^{(n)}$ zadan je rekurzivno

$$\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + \frac{1}{(n+1)(2n+1)}\frac{\partial^2}{\Box}\mathcal{F}^{(n)\prime} - \frac{1}{n+1}\frac{\partial}{\Box}\partial\cdot\mathcal{F}^{(n)} \tag{26}$$

sa $\mathcal{F}^{(0)} = \Box\varphi$ i $\mathcal{F}^{(1)} \equiv \mathcal{F} = \Box\varphi - \partial\partial\cdot\varphi + \partial^2\varphi'$. Obzirom na to da rezultate izražavamo pomoću projektora, operatori $\mathcal{F}^{(n)}$ nisu prikladni za našu analizu jer su nelokalni i neočuvani (njihova divergencija ne iščezava). Za naše svrhe važan je generalizirani Einsteinov tenzor

$$\mathcal{G}^{(n)} = \sum_{p=0}^{n}(-1)^p\frac{(n-p)!}{2^p n!}\,\eta^p\,\mathcal{F}^{(n)[p]}, \quad \text{gdje je} \quad \begin{cases} s = 2n & s \quad \text{even} \\ s = 2n-1 & s \quad \text{odd} \end{cases} \tag{27}$$

Divergencija $\mathcal{G}^{(n)}$ je nula te su neograničene jednadžbe gibanja za φ

$$\mathcal{G}^{(n)} = 0 \tag{28}$$

U [30] pokazujemo da se bilo koja jednadžba gibanja može izraziti pomoću generaliziranog Einsteinovog tenzora i njegovih tragova.

Da bismo izrazili efektivnu akciju u geometrijskom obliku, uvodimo generalizirani Jacobijev tenzor $\mathcal{R}_{\mu_1,\ldots\mu_s\nu_1\ldots\nu_s} = \partial_{\mu_1}\ldots\partial_{\mu_s}\varphi_{\nu_1\ldots\nu_s}\big|_{\text{antisimetriziran u svim }(\mu_j,\nu_j)}$ (generalizacija Riemannovog tenzora), koji je povezan sa $\mathcal{F}^{(n)}$ na sljedeći način:

$$\mathcal{F}^{(n)} = \begin{cases} \frac{1}{\Box^{n-1}}\mathcal{R}^{(s)[n]} & s = 2n \\ \frac{1}{\Box^{n-1}}\partial\cdot\mathcal{R}^{(s)[n-1]} & s = 2n-1 \end{cases} \tag{29}$$

Bilo koju akciju ili jednadžbu gibanja možemo izraziti pomoću generaliziranih Jacobijevih tenzora tako da ovisnost o generaliziranim Einsteinovim tenzorima zamijenimo s ovisnošću o $\mathcal{F}^{(n)}$, te ovisnost o $\mathcal{F}^{(n)}$ zamijenimo s ovisnošću o Jacobijevim tenzorima.

Da bi naša opća zapažanja bila konkretnija promatramo sljedeće eksplicitne primjere. Najjednostavni primjeri su bezmaseni skalarni i fermionski model s jednostavnim i strujama bez traga. Posebno, eksplicitnim računom korelatora s dvije struje doznajemo da, za struje čiji je trag nula, i sam korelator ima svojstvo da je njegov trag nula. Doznajemo i da, u ovom slučaju, ne-dijagonalni korelatori iščezavaju.

Općenito, bezmaseni slučaj ne sadržava potpunu informaciju pa stoga koristimo masivni skalarni i fermionski model. Iako možemo izračunati izraze za korelatore dvije struje u općenitoj dimenziji, rezultati su izraženi pomoću hipergeometrijskih funkcija iz kojih je teško iščitati efektivnu akciju. Stoga ih često razvijamo u red oko infracrvenog (IR) $\left(\frac{k}{m}\to 0\right)$ i ultraljubičastog (UV) $\left(\frac{m}{k}\to 0\right)$ područja. Ovaj razvoj dopušta nam da izdvojimo informaciju o dinamici izvora.

U IR sektoru, za više spinove nalazimo članove koji nisu očuvani te narušavaju Wardove identitete. Spomenuti članovi su lokalni te ih možemo eliminirati tako da od akcije oduzmemo konačan broj prikladnih kontračlanova. Postoji jedno važno opažanje vezano uz oduzimanje lokalnih kontračlanova. Naime, za spin-1 i spin-2 znamo kovarijantni oblik minimalnog vezanja pa u tim slučajevima nije potrebno oduzimati kontračlanove, jer formalizam perturbativne teorije polja se automatski brine za kovarijantnost, pod uvjetom da u obzir uzmemo i korelator na jednu točku povrh korelatora na dvije točke. Ovaj primjer pokazuje i da dimenzionalna regularizacija daje kovarijantne izraze (bez oduzimanja rukom) kao što je npr. pokazano u [42] za skalarnu teoriju koja interagira s gravitacijom.

Spomenimo neke od rezultata. Na primjer, vodeći član u IR u fermionskom modelu je univerzalan

$$\tilde{T}_{\mu_1\ldots\mu_s\nu_1\ldots\nu_s} \overset{IR}{\sim} im^{d-4+2\lfloor\frac{s}{2}\rfloor}\Gamma\left(2-\frac{d}{2}-\lfloor s/2\rfloor\right)k^{2\lfloor\frac{s+1}{2}\rfloor}\pi_{\mu\nu}^{s \bmod 2}\left(\pi_{\mu\nu}^2-\pi_{\mu\mu}\pi_{\nu\nu}\right)^{\lfloor\frac{s}{2}\rfloor}+\cdots$$

pa pripadna jednadžba gibanja glasi

$$\langle\langle j_{\mu_1\ldots\mu_s}\rangle\rangle \sim m^{d-4+2\lfloor\frac{s}{2}\rfloor}\Gamma\left(2-\frac{d}{2}-\lfloor s/2\rfloor\right)\Box^{\lfloor\frac{s-1}{2}\rfloor}G_{\mu_1\ldots\mu_s}+\ldots \tag{30}$$

Posebno, za spin 1, dominantni član efektivne akcije u IR je uobičajena Maxwellova akcija

$$W \sim m^{d-4}\int \mathrm{d}x\, F_{\mu\nu}F^{\mu\nu}$$

Za spin 2, efektivna akcija je suma kozmološke konstante, Einstein-Hilbertove akcije i Weylove gustoće $\mathcal{W}^2 = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$ (konformna invarijanta u 4d).

$$W \;\sim\; m^d\int \mathrm{d}^dx\sqrt{g}\times\left[\Gamma\left(-\frac{d}{2}\right)-\frac{\Gamma\left(1-\frac{d}{2}\right)}{24m^2}R-\frac{\Gamma\left(2-\frac{d}{2}\right)}{80m^4}\mathcal{W}^2+\ldots\right] \tag{31}$$

Za spin 3 efektivna akcija dana je kao generalizacija Maxwellove akcije

$$W \sim m^{d-2}\int \mathrm{d}x\left((\mathcal{F}_{\mu\nu\lambda})^2-(\mathcal{F}'_\mu)^2\right)+\ldots$$

Povrh toga, treba provjeriti da su IR i UV limesi efektivne akcije dobro definirani. U IR sektoru nalazimo članove koji su divergentni za $m\to\infty$. Spomenuti članovi su lokalni i obuhvaćaju neočuvane članove pa biramo shemu u kojoj ih eliminiramo tako da od akcije oduzmemo konačan broj prikladnih kontračlanova. Općenito, efektivna akcija tada je dana pomoću Fronsdalovog kinetičkog operatora, [37, 38], u nelokalnoj formi uvedenoj u [39]-[41].

Nadalje, u 3d fermionskom modelu postoji i neparni sektor gdje za ne-dijagonalne korelatore struja čiji trag iščezava općeniti izraz za spinove $s_1\times s_2$, $s_2>s_1$ glasi

$$\tilde{T}_{\mu_1\ldots\mu_{s_1}\nu_1\ldots\nu_{s_2}} = (-1)^{\frac{s_1+s_2}{2}}\frac{imk^{s_1+s_2-3}}{2^{s_2+1}}\pi_{\nu\nu}^{\frac{s_2-s_1}{2}}\sum_{l=0}^{\lfloor\frac{s_1}{2}\rfloor}\frac{(-1)^l\Gamma(s_1-l)}{2^{2l}l!\Gamma(s_1-2l)}\pi_{\mu\mu}^l\pi_{\nu\nu}^l\pi_{\mu\nu}^{s_1-2l-1}\epsilon_{\sigma\mu\nu}k^\sigma \tag{32}$$

Za jednake spinove, ova akcija promatrana je u [43, 44] te nedavno u [45]-[52].

Dakle, počevši od slobodne kvantne teorije polja vezane na vanjska polja višeg spina putem očuvanih struja, nalazimo da efektivna akcija, dobivena integriranjem mikroskopskog polja, sadrži informaciju o klasičnoj dinamici viših spinova. Kako se zadržavamo na korelatorima na dvije točke, efektivna akcija je kvadratna, dok su jednadžbe gibanja linearne u polju.

Napomenimo ponovno da za više spinove znamo samo lineariziranu verziju interakcije i baždarne transformacije i stoga nalazimo narušenje Wardovog identiteta. Da bismo zadovoljili Wardove identitete dovoljno je od efektivne akcije oduzeti konačan broj lokalnih kontračlanova. Očekujemo da bismo za očuvanje (bez oduzimanja kontračlanova) trebali znati potpuni oblik kovarijantnog minimalnog vezanja i baždarne transformacije. U tu svrhu, u [33] promatrali smo kvantizaciju svjetske linije koja se temelji na Weylovoj kvantizaciji čestice u kvantnoj mehanici. Početna točka je slobodna fermionska teorija vezana na vanjske izvore. Zatim koristimo Weylovu kvantizaciju. Potpuna akcija izražena je kao očekivana vrijednost operatora

$$S = \langle\overline{\psi}| - \gamma\cdot(\widehat{P} - \widehat{H}) - m|\psi\rangle \tag{33}$$

Općeniti kvantni operator $\widehat{O}$ može se uz Weylovo preslikavanje predstaviti pomoću simbola $O(x, p)$

$$\widehat{O} = \int d^dx\, d^dy \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\, O(x,p)\, e^{ik\cdot(x-\widehat{X})-iy\cdot(p-\widehat{P})} \tag{34}$$

tako da operator $\widehat{H}$ ima simbol $h(x, p)$

$$h^\mu(x,p) = \sum_{n=0}^{\infty} \frac{1}{n!}\, h_{(s)}^{\mu\mu_1\ldots\mu_n}(x)\, p_{\mu_1}\ldots p_{\mu_n} \tag{35}$$

gdje je $s = n + 1$ spin i $h_{(s)}^{\mu\mu_1\ldots\mu_n}$ je simetrični tenzor. Simetrično tenzorsko polje $h^{\mu\mu_1\ldots\mu_n}$ je linearno vezano na očuvanu struju višeg spina

$$J^{(s)}_{\mu\mu_1\ldots\mu_{s-1}}(x) = \frac{i^{s-1}}{(s-1)!}\frac{\partial}{\partial z^{(\mu_1}}\cdots\frac{\partial}{\partial z^{\mu_{s-1}}}\overline{\psi}\left(x+\frac{z}{2}\right)\gamma_{\mu)}\psi\left(x-\frac{z}{2}\right)\Big|_{z=0}. \tag{36}$$

Akcija (33) invarijantna je na transformaciju

$$\delta_\varepsilon h^\mu(x,p) = \partial_x^\mu\varepsilon(x,p) - i[h^\mu(x,p) \overset{*}{,} \varepsilon(x,p)] \equiv \mathcal{D}_x^{*\mu}\varepsilon(x,p) \tag{37}$$

gdje $*$ označava Moyalov produkt. Sljedeće, promatramo regulariziranu efektivnu akciju:

$$W_{reg}[h,\epsilon] = -N\int_\epsilon^\infty \frac{dt}{t}\,\mathrm{Tr}\Big[e^{-t\widehat{G}}\Big] \tag{38}$$

gdje je $\widehat{G} = -\gamma\cdot(\widehat{P} - \widehat{H}) - m$. Ideja je razviti efektivnu akciju perturbativno

$$W[h] = \sum_{n=1}^{\infty}\frac{1}{n!}\int\prod_{i=1}^{n} d^dx_i\,\frac{d^dp_i}{(2\pi)^d}\,\mathcal{W}^{(n)}_{\mu_1,\ldots,\mu_n}(x_1,p_1,\ldots,x_n,p_n,\epsilon)\,h^{\mu_1}(x_1,p_1)\ldots h^{\mu_n}(x_n,p_n)$$

te na taj način dobivamo izraze za amplitude $\mathcal{W}^{(n)}_{\mu_1,\ldots,\mu_n}(x_1,p_1,..,x_n,p_n,\epsilon)$ (slične Feynmanovim dijagramima). Jednadžba gibanja je

$$
\begin{aligned}
\mathcal{F}_\mu(x,p) \;\equiv\; & \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} d^d x_i \, \frac{d^d p_i}{(2\pi)^d} \, \mathcal{W}^{(n+1)}_{\mu,\mu_1\ldots,\mu_n}(x,p,x_1,p_1,\ldots,x_n,p_n,\epsilon) \\
& \times h^{\mu_1}(x_1,p_1)\ldots h^{\mu_n}(x_n,p_n) = 0
\end{aligned}
$$

Važna prednost gore opisane procedure je da daje potpuni oblik baždarne transformacije što ima dalekosežne posljedice: možemo pokazati da cijela akcija ima L_∞ simetriju [53].

U L_∞-algebri imamo graduirani vektorski prostor $X = \bigoplus_i X_i$ gdje je X_i vektorski prostor, $i = \ldots, 1, 0, -1, \ldots$, sa stupnjem i te multilinearnim preslikavanjima L_j, $j = 1, 2, \ldots$, sa stupnjem $d_j = j - 2$. Vektore iz X označavamo s $x_1, x_2, \ldots$ a njihov stupanj je $\mathrm{x}_i = deg(x_i)$. Preslikavanja L_j zadovoljavaju sljedeće kvadratne identitete:

$$
\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} (-1)^{\sigma} \epsilon(\sigma;x)\, L_j(L_i(x_{\sigma(1)},\ldots,x_{\sigma(i)}),x_{\sigma(i+1)},\ldots,x_{\sigma(n)}) = 0 \tag{39}
$$

gdje σ označava permutaciju dok je $\epsilon(\sigma;x)$ Koszulov predznak.

U našem slučaju, zbog strukture efektivne akcije i jednadžbe gibanja, biti će nam potrebna samo tri vektorska prostora X_0, X_{-1}, X_{-2} te kompleks

$$
X_0 \xrightarrow{L_1} X_{-1} \xrightarrow{L_1} X_{-2} \xrightarrow{L_1} 0 \tag{40}
$$

Stupnjevi su sljedeći: $\varepsilon \in X_0$, $h^\mu \in X_{-1}$ te $\mathcal{F}_\mu \in X_{-2}$. Eksplicitnom provjerom L_∞ relacija (39) može se pokazati da je na ovaj način generirana L_∞ algebra.

L_∞ relacije mogu se interpretirati kao Wardovi identiteti. U dokazu L_∞ simetrije pretpostavili smo da nema anomalija, ali tu pretpostavku treba provjeriti eksplicitnim računom amplituda. Lom Wardovih identiteta na kvantnom nivou indicirao bi da je teorija anomalna što ukazuje na moguće prepreke u konstrukciji teorije viših spinova. S druge strane, ako nema anomalija, L_∞ algebra može se koristiti za pronalaženje teorija koje automatski zadovoljavaju L_∞ relacije i baždarnu invarijantnost za više spinova, što otvara novi pristup za istraživanje modela viših spinova.

Ključne riječi: efektivne akcije, anomlije traga, Pontryaginova gustoća, viši spinovi, L_∞ simetrija

Contents

Chapter 1

Introduction

When describing the fundamental interactions in physics, symmetries and related conservation laws play a crucial role. A symmetry is a transformation of the fields that leaves the classical action invariant. However, it may happen that, once we quantize the theory, a classically valid conservation law is violated. We call such a theory anomalous. Anomalies can be harmful or harmless. Harmful anomalies, such as chiral gauge anomalies, spoil the consistency of the theory and can be used as a selective criterion for theories. On the other hand, harmless anomalies have physical consequences, such as Adler-Bell-Jackiw anomaly which explains the pion decay to two photons.

Using the effective action approach, we will deal with odd parity trace anomalies in 4d in chiral fermion theory coupled to curved background and Dirac fermion in metric-axial-tensor (MAT) gravity . We will show in several ways that the odd parity part of the trace anomaly is given with Pontryagin density with imaginary coefficient, which indicates breakdown of unitarity and hence spoils the consistency of the theory. This suggests that we can use this anomaly as an exclusion criterion.

We are also interested in matter models coupled to higher spin fields. However, while the theory of free massless fields of spin higher than two is consistent, their interactions pose a challenge, that is, there exist several "no-go" theorems which impose serious restrictions on interacting theories, particularly in flat spacetime.

As the first step toward our goal to analyze anomalies in matter models coupled to higher spin fields, it is important to see what is the form of effective actions obtained by integrating out microscopic matter fields (scalar or fermion) in a theory in which a microscopic field is coupled to higher spin fields via conserved currents. We will focus on the quadratic part of the effective action and find that they are nonlocal. In this approach, after subtraction of finite number of local counterterms, we gain information about dynamics of higher spins. This indicates that, using this approach to compute higher-point correlators, we could acquire information about non-linear structure of higher spin theory. In this way we could also gain insight on how the "no-go" theorems are connected with our case, whether they pose restrictions or they are circumvented.

Our second approach is based on worldline quantization of the fermion field coupled to higher spin fields. In this way we obtain the exact gauge transformation and hence the effective action has prospective to be gauge invariant without subtraction of counterterms. If there are no generalized diff-anomalies, the effective action admits L_∞ symmetry. This indicates that integrating L_∞ algebra could be used to determine possible candidates for higher spin theories.

1.1 Trace anomaly

If a symmetry of a classical action is not a symmetry of an effective action in quantum field theory, we say that such a theory is anomalous. An introduction to anomalies can be found in the following textbooks [1]-[3].

In general, in fermionic field theory, we can divide the anomalies into two groups: *split* and *non-split* anomalies. Aspects of split and non-split anomalies are discussed in [54, 55]. Split anomalies have an opposite sign for opposite fermion chiralities, while the non-split anomalies have the same sign for opposite chiralities. As an example of the split anomalies let us mention the consistent chiral gauge or gravity anomalies. They occur only in theories with chiral imbalance. These anomalies are harmful and spoil the consistency of theory. As a consequence, they have been used as an exclusion criterion. On the other hand, as an example of non-split anomalies let us mention the covariant gauge or gravity anomalies, such as the anomaly that explains the decay of a π^0 into two γ's or the Kimura-Delbourgo-Salam anomaly [56]-[58]. In this thesis we will focus on the *trace anomalies*, also referred to as Weyl anomaly or conformal anomaly. Regarding the trace anomalies, the even-parity part falls into the non-split category, while the odd-parity part of the trace anomaly is split.

The appearance of even parity part of the trace anomaly was first discussed in [4], see also [5] for a general form of trace anomaly in various dimensions and [6] for arbitrary spin. One can follow general algorithm for the construction of gravitational axial and conformal anomalies for arbitrary spin [7]. Even trace anomaly can be applied to Hawking effect, gravitational instantons, asymptotic freedom and Weinberg asymptotic safety, see [8]. There exists a vast literature on even trace anomalies in 4d, mostly old [59]-[78], but also recent, such as [79, 80] related to renormalization of 3-pt correlators of energy-momentum tensor and conserved currents, and [81] where the Feynamn diagram approach was used to compute the conformal anomaly for spin-$\frac{1}{2}$ fermions, denoting a renewed interest in the subject.

The form of trace anomaly is determined by the dimension of spacetime and the consistency conditions. In particular, the most general form of the trace anomaly in four dimensions contains squares of the curvature and d'Alambertian of Ricci scalar. Using cohomological analysis, it was found that the trace anomaly can be written in terms of Weyl density, Euler (Gauss-Bonnet) density, d'Alambertian of Ricci scalar and also the Pontryagin density, see [9]-[12]. The d'Alambertian of the Ricci scalar is trivial in the sense that it can be removed by adding local counterterm to the action. Also Weyl and Euler density preserve CP (charge conjugation and parity) and hence belong to even parity part of the trace anomaly, while the Pontryagin density violates CP and belongs to the odd parity part. Recently, trace anomalies gained on popularity due to conformal field theories and their relation to the AdS/CFT correspondence. In [82, 83] the appearence of the Pontryagin anomaly was discussed in context of AdS/CFT correspondence.

We will focus our attention to the parity-odd part of the trace anomaly in 4d, see [15]-[21], [84, 85]. One model where the Pontryagin density can appear in the trace of the energy-momentum tensor is the theory of chiral fermions. In fact, the coupling between gravity and matter is given by the metric and energy-momentum tensor and it is important to note that the energy-momentum tensor for two fermions with two different chiralities is different. This difference can emerge in the form of an anomaly, in particular the trace anomaly.

In this thesis we will present a continuation of work done in [15]-[17], namely [19, 20] (see also [21]) where we revisit the computation of the odd part of the trace anomaly in the theory of chiral fermions. Following [19] we will present a more detailed derivation of parity odd trace anomaly using Feynman diagrams approach together with dimensional regularization. First, contrary to [15], we will not redefine the fermion field. Secondly, we will take into account possible contributions from the tadpole and seagull terms. In this way we confirm the result of [15].

Furthermore, motivated by Bardeen's method for computation of chiral gauge anomalies [86], we will introduce metric-axial-tensor gravity (for short MAT). The main idea is to embed our system in a larger framework and to formulate our problem in terms of Dirac fermions instead of Weyl fermions. We will couple Dirac fermion to the usual metric $g_{\mu\nu}$ and an axial symmetric tensor $f_{\mu\nu}$.

Let us briefly explain the main problem with chiral fermions and the reason behind the introduction of MAT. In anomaly calculations the most important part is played by the integral measure. However, in the case of chiral fermions the definition of the measure presents a problem because the Dirac operator for a Weyl fermion contains a chiral projector. We introduce MAT to avoid problems related to fermion integral measure, and instead we are allowed to work with Dirac fermions for which the integral measure is well defined. Note that, throughout the calculation, it is necessary to preserve the information about the definite chirality of the fermion field. We repeat the calculation of parity-odd trace anomaly in this new setup and we derive the anomaly for Dirac, Majorana and Weyl fermion in specific limits (which we call collapsing limits) and confirm our previous result.

The limitation of this derivation is that it is perturbative, that is, we compute only the lowest order of the odd trace anomaly and we then covariantize it. This is of course permitted provided we are convinced that there are no diff-anomalies. With a MAT background this verification is exceedingly complex and in this thesis and in [19] we content ourselves with an analogous but simpler verification carried out in [17]. Instead, there is a method that guarantees that diffeomorphisms are respected throughout the derivation: DeWitt's method, [13, 14]. Our aim here is to combine DeWitt's with Bardeen's method. This will require a introduction on the so-called hypercomplex calculus, which is the appropriate framework for MAT gravity. Hypercomplex analysis in physical problems was introduced and used in [87]-[93]. Following [20], we show that the same result for parity odd trace can be obtained non-perturbatively by means of heat kernel and using two different regularizations: dimensional regularization and ζ-function regularization.

Finally, although we do not use it here, we should mention the method recently devised in [94], where a fifth dimension is introduced as a regulator.

It turns out that the odd-parity part of the trace anomaly comes with an imaginary coefficient. It was pointed already in [15] that imaginary energy-momentum tensor might break unitarity and thus spoil the consistency of the theory. This observation suggests that we could use this anomaly as a selective criterion for the theories.

1.2 Effective actions in higher spin theories

It is a common belief that, to construct a consistent quantum theory of gravity and matter, we need an infinite number of fields. One example which supports this idea is (super)string theory, where an infinite tower of higher-spin excitations appears [22, 23]. One more example of higher spin theory with an infinitely many higher spin fields is the Vasiliev theory [24]-[27] which exist in a four-dimensional and lower space-time. Very likely these are not the only possibilities. But then a question arises: what are the requirements to be satisfied in order for these theories to make sense?

The theory of higher spins dates back to 1936 when Dirac tried to generalize his spin-$\frac{1}{2}$ equation [95]. In 1939 Fierz and Pauli [96] systematized the study of massive higher spin fields through Lorentz covariance and energy positivity and in 1974 Singh and Hagen in [97, 98] constructed the Lagrangian formulation of Fierz and Pauli equations. A few years later, Fronsdal in [37, 38] considered the massless limit of Singh-Hagen Lagrangian and found that the equation of motion is invariant under gauge transformations only if the gauge parameter is traceless. In [39]-[41] Francia and Sagnotti constructed the free field equations which are unconstrained and nonlocal for spin larger than two.

Here we will present a program started in [28] and continued in [29, 30] (see also [31, 32]) where we used the effective action approach to determine the classical dynamics of the higher spin fields. Higher spin fields appear naturally in the one-loop effective action of the simplest free theories in any dimension and it is possible to make contact with the literature on classical higher spin theories, [39]-[41], [99]-[109]. Sources of inspiration for our approach has been Sakharov method of induced gravity [110], higher spin theories in 3d [111]-[116] and also [117]-[119]. The idea of exploring the one-loop effective action is far from new: the list of works which may have some overlap with our program includes [119]-[128].

We start by coupling a massive scalar and fermion theory to higher spin fields via conserved currents. Next, to analyze the dynamics of higher spin fields we need the quadratic part of the effective action (linearized equations of motion). We obtain the latter by computing the 2-point correlator of our conserved currents using Feynman diagrams and a method introduced by Davydychev and collaborators, [34]-[36]. Even though we will often consider only 2-point correlators of currents with equal spins, as we previously mentioned, in higher spin theory an infinite tower of spins appears. This suggests that we should also consider the correlator of any two currents coupled to fields that can enter the action. We refer to these correlators as mixed or non-diagonal. One more important thing to note is that the conserved currents are not unique and their form affects the form of the effective action. We will mostly focus on two specific choices, we call them "simple" and "traceless". We will demonstrate several examples for scalar and fermion field theories with different choices of currents.

To prepare the ground, we will start with a consideration of the general form of conserved 2-point correlators and learn that they can be represented in terms of projectors which make the conservation obvious. We also consider a form of traceless correlators. Now, even though the projectors are extremely convenient, the geometrical content of the results remains hidden in this way. For this reason we turn to the formulation of our results in terms of geometrical objects - Jacobi tensors.

To make our general observations more concrete, we have to turn to explicit examples. The easiest examples are the massless scalar and fermion model where we are able to derive some very general results. In particular, we compute the 2-point correlator for simple and traceless currents. We find that the correlators for traceless currents are themselves traceless. In general, in the massless case we do not get all the information we can extract from the massive theory and to make sure we get a complete information we must use massive models.

Using massive scalar and fermion models we derive general expressions for 2-pt correlators in a general dimension, however, these results are given in terms of hypergeometric functions and are not particularly "readable". Because of that we often expand the results around IR $\left(\frac{k}{m} \to 0\right)$ and around UV $\left(\frac{m}{k} \to 0\right)$ in a specific dimension: $d = 3, 4$. The expansion in powers of mass allows us to single out the dynamics of the sources and we will refer to it as *tomography*.

There is another reason why we use UV and IR expansions: we have to check that the IR and UV limits of the one-loop effective action are well defined. In the IR sector we find terms which are divergent in the limit $m \to \infty$. There are also terms which are non-conserved and violate Ward identity. These terms are local and can be subtracted by adding a finite number of local counterterms to the action. In this way, for spin-1 we find the Maxwell action, for spin-2 Einstein-Hilbert and for spin-3 the effective action is based on the corresponding linearized Fronsdal kinetic operator, [37, 38], in the nonlocal form introduced by Francia and Sagnotti, [39]-[41]. In view of constructing a covariant action for higher spins, this result is promising. It suggests that integrating out scalar or fermion fields (or any other field by which one can form conserved currents) can be a prospective way to analyze the dynamics of higher spin fields.

Also, in 3d in fermion model we can consider the odd parity sector which emerges from the parity-breaking fermion mass term, and we find a generalization of Pope and Townsend's Chern-Simons-like action in the case when mixed higher-spin fields are taken into consideration. In the case of equal spins, this is the action considered in [43, 44] and recently discussed by a number of authors, see e.g. [45]-[52].

We previously mentioned that different choices of currents lead to different effective actions. In particular, we discuss diagonalization of our 2-point correlators, that is, the possibility of vanishing off-diagonal correlators for appropriate choice of coefficients in the currents.

There is one more important point related to local subtractions. We already mentioned that we find several violations of Ward identities, but since the terms which violate conservation are local, we can recover conservation by subtracting appropriate local counterterms from the action. We recall that for spin-1 and spin-2 we know the covariant form of minimal coupling. In these cases we show that we do not have to subtract local counterterms, because the perturbative field theory formalism already automatically takes care of covariance, provided one takes into account not only the two-point bubble diagrams but also tadpole and seagulls. This exercise also shows that dimensional regularization gives manifestly covariant expressions (without subtractions by hand) as was e.g. done in [42] for scalar matter coupled to gravity.

The example of spin-2 shows that the gauge transformation is not linear, in fact, it is crucial to consider the complete gauge transformation to show that the theory respects

Ward identities. In contrast, for spin 3 and higher we have only the linearized version of gauge transformation and as a consequence our Ward identity is not satisfied. The reason is that seagull diagrams are related to the additional terms in the initial action, beyond the minimal model we start with (a scalar or fermion field minimally coupled to a background field). Conservation (without subtractions) requires the presence of such additional terms and constraints their form and their coefficients. Hence, when we consider higher spin backgrounds, this observation may be used in order to determine the form of the additional action terms. This goes in the direction of constructing an off-shell covariant model.

So, to avoid subtractions, we should know the full form of gauge transformation and covariant minimal coupling. In this regard, in [33] we considered the worldline quantization method of a fermion model which is based on the Weyl quantization of a particle in quantum mechanics. The literature on the worldline quantization is large. Here we refer in particular to the calculation of effective actions via the worldline quantization in relation to higher spin theories, [117, 118, 128]. The first elaboration of this method is given in [129], to which many others followed, see for instance [130]-[138].

The main idea in worldline quantization is to replace the field dependence on the position and the field derivatives by the corresponding position and momentum operators, respectively, and we rely on the Weyl quantization for the latter. We define the effective action and expand it perturbatively. In this way we obtain the expressions for the amplitudes, which are similar to Feynman diagram approach.

A peculiar thing about this procedure it that it comes with the precise form of the gauge symmetry. This has a outstanding consequence: it is possible show that the full (not only the local part of) effective action in the fermion model accommodates (curved) L_∞ symmetry. The latter is a symmetry that characterizes many (classical) field theories, including closed string field theory. This fact first appeared in [139, 140], see also [141], as a particular case of strongly homotopic algebras [142, 143]. L_∞ describes other field theories as well [144], such as gauge field theories [145]-[147], Chern-Simons theories, Einstein gravity and double field theory [53]. For other, more recent applications, see [148]-[150].

We interpret L_∞ relations as Ward identities. Breakdown of these relations at the quantum level would suggest the presence of anomalies. Possible obstructions in construction of the higher spin theories may appear in the form of anomalies in this approach. If there are no generalized diff-anomalies, integrating L_∞ algebra, that is determining theories which satisfy L_∞ relations and higher spin gauge invariance, is a prospective way to investigate higher spin models.

1.3 Organization of the thesis

The thesis is organized as follows.

In chapter 2 we introduce the notion of the effective action and discuss its symmetries. We also discuss the general form of the trace anomaly and we review the properties of massless Weyl and Majorana fermions in 4d.

Chapter 3 follows [19]. We reconsider the computation of the anomaly given in [15]. We calculate the trace anomaly, but here we do not redefine the fermion field and we consider possible tadpole and seagull terms. We complete this chapter with the discussion of Ward identity for diffeomorphisms and some final remarks on the the odd trace anomaly.

Chapter 4 is based on [19]. We introduce the MAT (metric-axial-tensor) gravity, and we couple it to Dirac fermions. Afterwards, we give a derivation of the trace anomalies in this formalism and we compute the collapsing limits for Dirac, Weyl and Majorana fermions.

Chapter 5 is based on [20]. We give a brief introduction to axial-complex numbers and axial-complex analysis. We also present the axial-complex analysis of geodesics in an axial-complex space: we define normal coordinates, the world function and the coincidence limit, the VVM determinant and the parallel displacement matrix for tensors and for spinors. Even though the (pseudo)Riemannian geometry of an axial-complex space is already introduced in the previous chapter, for this chapter is practical to partially change the notation. We formulate the theory of Dirac fermions in a MAT background, define the ordinary energy-momentum tensor and its axial partner and analyze their classical Ward identities with respect to ordinary and axial diffeormorphisms and Weyl transformations. We also define the 'square' of the Dirac operator, a central object for the application of the Schwinger-DeWitt method. Next we describe this method and derive the relevant heat kernel coefficients. We use these results to the non-perturbative computation of the odd part of the trace anomalies tor the two energy-momentum tensors with two different regularizations: the dimensional and ζ-function regularization. Finally, we compute the collapsing limit for Weyl fermion and show that the two anomalies collapse to a single one and, as expected, correspond to the odd trace anomaly already calculated in [15, 17] and [19].

In chapter 6 we give a short introduction to higher spin theories and related "no-go" theorems.

In chapter 7 we introduce the massive scalar and fermion model. This chapter is based general observations related to 2-point functions given in [29, 30]. We discuss universal form of equations of motion and show how to geometrize our results, that is how to express them in terms of Jacobi tensors. Next, we give a short summary of Davydychev's method to compute one-loop Feynman diagramsand summarize the results in 3d worked out in [28]. Finally, we give general guidelines for calculations. Next we turn to calculations of 2-pt functions.

Chapters 8 and 9 follow main results from [29, 30]. We analyze massless scalar and fermion models for simple and traceless currents and we find some general expressions for any spin and any dimension. We also consider the one-loop scalar and fermion massive model two-point functions of simple currents and their IR and UV expansion (tomography) in 3 and 4 dimensions. We also produce the expressions for two-point correlators of spin 1, 2, 3 currents in any dimensions. Next we show some examples of mixed correlators in fermion model in various dimensions and give their UV and IR expansions. We also discuss the issue of tadpole and seagull terms and how they guarantee covariance without subtractions in the case of spin 1 and 2. Furthermore, we try to diagonalize our results, that is, we try to find the form of currents for which the mixed spins correlators vanish.

Last two chapters are based on [33]. In chapter 10 we carry out the worldline quantization for free Dirac fermions coupled to external sources (the case of a scalar field is given in [128]) and give expressions for the amplitudes. In chapter 11 we reveal the L_∞ structure of the related effective action.

Section 12 is devoted to the conclusion and discussion of our results.

Chapter 2

Effective actions, symmetries and anomalies

To analyze our matter models and the existence of anomalies within them we use the effective action approach. In this chapter we introduce main definitions which we will use throughout this thesis, such as the partition function and quantum effective action. Moreover, we will discuss symmetries and associated conservation laws for both classical and quantum actions in gauge theory, gravity and a general spin-s theory. We conclude that, if the emergent Ward identities are violated, the theory is anomalous.

Next we focus on a specific type of anomalies, the trace anomalies in matter models. We first discuss a general form of the trace anomaly given by Wess-Zumino consistency conditions. It turns out that there are three possible terms which can contribute to the anomaly: Weyl, Euler and Pontryagin density. The coefficients of these terms depend on the theory in question. Our focus will be on the coefficient of the parity-odd part - Pontryagin density. One possible model in which such a term does not vanish is a theory of a chiral fermion (for example left-handed Weyl fermion) coupled to curved background. Let us mention that there is a common misconception that a Weyl fermion is the same as massless Majorana fermion at both classical and quantum level. While the odd-parity part of the trace anomaly for massless Majorana certainly vanishes, this is not the case for Weyl fermion [15]. Because of this, to prepare the ground for the calculation of the parity-odd trace anomaly, we first discuss fermions in $4d$, in particular, we focus on the similarities and differences between massless Majorana and Weyl fermions.

2.1 Effective action

Let us start with the main definitions, see [1]. Fundamental object in quantum field theory is the partition function. The partition function $Z[\varphi]$ is the generating function of all correlation functions. It can be written as

$$Z[\varphi] = \int \mathcal{D}\phi e^{iS[\phi,\varphi]} \tag{2.1}$$

where φ is some external (classical) field such as spin 1 field A_μ, spin 2 field $h_{\mu\nu}$ or higher spin fields $\varphi_{\mu_1\ldots\mu_s}$. We assume that the classical action $S[\phi,\varphi]$ is a sum of the free action $S_0[\phi]$ for some field ϕ and the interaction $S_{int}[\phi,\varphi]$:

$$S[\phi,\varphi] = S_0[\phi] + S_{int}[\phi,\varphi] \tag{2.2}$$

Next we expand the partition function

$$Z[\varphi] = \sum_{n=0}^{\infty} \sum_{s_1,\ldots,s_n} \frac{1}{n!} \int \prod_{i=0}^{n} d^d x_i \varphi_{\mu_{i1}\ldots\mu_{is_i}}(x_i) \frac{\delta^n Z[\varphi]}{\delta\varphi_{\mu_{11}\ldots\mu_{1s_1}}(x_1)\ldots\delta\varphi_{\mu_{n1}\ldots\mu_{nsn}}(x_n)}\Big|_{\varphi=0} \tag{2.3}$$

where $\frac{\delta^n Z[\varphi]}{\delta\varphi_{\mu_{11}\ldots\mu_{1s_1}}(x_1)\ldots\delta\varphi_{\mu_{n1}\ldots\mu_{nsn}}(x_n)}\Big|_{\varphi=0}$ is the correlation function. Given the partition function $Z[\varphi]$ we can introduce the effective action $W[\varphi]$

$$Z[\varphi] = e^{iW[\varphi]} \qquad \Rightarrow \qquad iW[\varphi] = \ln Z[\varphi] \tag{2.4}$$

The effective action is the generating function for all connected correlation functions. The expansion of the effective action for the external source $\varphi_{\mu_1\ldots\mu_s}$ is

$$iW[\varphi] = \sum_{n=0}^{\infty} \sum_{s_1,\ldots,s_n} \frac{1}{n!} \int \prod_{i=0}^{n} d^d x_i \varphi_{\mu_{i1}\ldots\mu_{is_i}}(x_i) \frac{\delta^n (iW[\varphi])}{\delta\varphi_{\mu_{11}\ldots\mu_{1s_1}}(x_1)\ldots\delta\varphi_{\mu_{n1}\ldots\mu_{ns_n}}(x_n)}\Big|_{\varphi=0} \tag{2.5}$$

where $\frac{\delta^n (iW[\varphi])}{\delta\varphi_{\mu_{11}\ldots\mu_{1s_1}}(x_1)\ldots\delta\varphi_{\mu_{n1}\ldots\mu_{nsn}}(x_n)}\Big|_{\varphi=0}$ is the connected correlation function.

Let us now assume that we constructed on-shell conserved currents $j^{\mu_1\ldots\mu_s}(x)$ and let us couple them minimally to spin-s external fields $\varphi_{\mu_1\ldots\mu_s}$. We can write interaction as

$$S_{int}[\phi,\varphi] = \sum_s \int d^d x \varphi_{\mu_1\ldots\mu_s} j^{\mu_1\ldots\mu_s}(x) \tag{2.6}$$

The n-point correlation function then reads

$$\frac{\delta^n Z[\varphi]}{\delta\varphi_{\mu_{11}\ldots\mu_{1s_1}}(x_1)\ldots\delta\varphi_{\mu_{n1}\ldots\mu_{ns_n}}(x_n)}\Big|_{\varphi=0} = i^n\langle 0|\mathcal{T} j^{\mu_{11}\ldots\mu_{1s_1}}(x_1)\ldots j^{\mu_{n1}\ldots\mu_{ns_n}}(x_n)|0\rangle \quad (2.7)$$

while the n-point connected correlation function becomes

$$\frac{\delta^n (iW[\varphi])}{\delta\varphi_{\mu_{11}\ldots\mu_{1s_1}}(x_1)\ldots\delta\varphi_{\mu_{n1}\ldots\mu_{ns_n}}(x_n)}\Big|_{\varphi=0} = i^n\langle 0|\mathcal{T} j^{\mu_{11}\ldots\mu_{1s_1}}(x_1)\ldots j^{\mu_{n1}\ldots\mu_{ns_n}}(x_n)|0\rangle_c \quad (2.8)$$

For example, 1-point correlator is the same as 1-point connected correlator

$$\begin{aligned} i\langle 0|j^{\mu_{11}\cdots\mu_{1s}}(x_1)|0\rangle &= \frac{\delta Z[\varphi]}{\delta\varphi_{\mu_{11}\ldots\mu_{1s}}(x_1)}\Big|_{\varphi=0} = \frac{\delta e^{iW[\varphi]}}{\delta\varphi_{\mu_{11}\ldots\mu_{1s}}(x_1)}\Big|_{\varphi=0} \\ &= \frac{\delta(iW[\varphi])}{\delta\varphi_{\mu_{11}\ldots\mu_{1s}}(x_1)} e^{iW[\varphi]}\Big|_{\varphi=0} = i\langle 0|j^{\mu_{11}\cdots\mu_{1s}}(x_1)|0\rangle_c \quad (2.9) \end{aligned}$$

while for the 2-point correlator we get

$$\begin{aligned} & i^2\langle 0|\mathcal{T} j^{\mu_{11}\cdots\mu_{1s_1}}(x_1) j^{\mu_{21}\cdots\mu_{2s_2}}(x_2)|0\rangle = \frac{\delta^2 Z[\varphi]}{\delta\varphi_{\mu_{11}\ldots\mu_{1s_1}}(x_1)\delta\varphi_{\mu_{21}\ldots\mu_{2s_2}}(x_2)}\Big|_{\varphi=0} \\ = &\ \frac{\delta^2 e^{iW[\varphi]}}{\delta\varphi_{\mu_{11}\ldots\mu_{1s_1}}(x_1)\delta\varphi_{\mu_{21}\ldots\mu_{2s_2}}(x_2)}\Big|_{\varphi=0} \\ = &\ \left(\frac{\delta^2(iW[\varphi])}{\delta\varphi_{\mu_{11}\ldots\mu_{1s_1}}(x_1)\delta\varphi_{\mu_{21}\ldots\mu_{2s_2}}(x_2)} + \frac{\delta(iW[\varphi])}{\delta\varphi_{\mu_{11}\ldots\mu_{1s_1}}(x_1)}\frac{\delta(iW[\varphi])}{\delta\varphi_{\mu_{21}\ldots\mu_{2s_2}}(x_2)}\right) e^{iW[\varphi]}\Big|_{\varphi=0} \\ = &\ i^2\langle 0|\mathcal{T} j^{\mu_{11}\cdots\mu_{1s_1}}(x_1) j^{\mu_{21}\cdots\mu_{2s_2}}(x_2)|0\rangle_c + i^2\langle 0|j^{\mu_{11}\cdots\mu_{1s_1}}(x_1)|0\rangle_c\langle 0|j^{\mu_{21}\cdots\mu_{2s_2}}(x_2)|0\rangle_c \\ = &\ i^2\langle 0|\mathcal{T} j^{\mu_{11}\cdots\mu_{1s_1}}(x_1) j^{\mu_{21}\cdots\mu_{2s_2}}(x_2)|0\rangle_c + i^2\langle 0|j^{\mu_{11}\cdots\mu_{1s_1}}(x_1)|0\rangle\langle 0|j^{\mu_{21}\cdots\mu_{2s_2}}(x_2)|0\rangle \end{aligned}$$

Altogether, the connected 2-point correlator can be expressed as

$$\begin{aligned} & \langle 0|\mathcal{T} j^{\mu_{11}\cdots\mu_{1s_1}}(x_1) j^{\mu_{21}\cdots\mu_{2s_2}}(x_2)|0\rangle_c \\ & = \langle 0|\mathcal{T} j^{\mu_{11}\cdots\mu_{1s_1}}(x_1) j^{\mu_{21}\cdots\mu_{2s_2}}(x_2)|0\rangle - \langle 0|j^{\mu_{11}\cdots\mu_{1s_1}}(x_1)|0\rangle\langle 0|j^{\mu_{21}\cdots\mu_{2s_2}}(x_2)|0\rangle \quad (2.10) \end{aligned}$$

Finally, we can write the effective action as:

$$\begin{aligned} iW[\varphi] = iW[0] + \sum_{n=1}^{\infty}\sum_{s_1,\ldots,s_n}\frac{i^n}{n!}\int\prod_{i=1}^{n} d^d x_i \varphi^{\mu_{11}\cdots\mu_{1s_1}}(x_1)\ldots\varphi^{\mu_{n1}\cdots\mu_{ns_n}}(x_n) \\ \times\langle 0|\mathcal{T} j_{\mu_{11}\ldots\mu_{1s_1}}(x_1)\ldots j_{\mu_{n1}\ldots\mu_{ns_n}}(x_n)|0\rangle_c. \quad (2.11) \end{aligned}$$

where we separated the constant term. The full one-loop 1-pt correlator for $j_{\mu_1\cdots\mu_s}$ is given as a variation of the effective action with respect to the source and it reads

$$\langle\!\langle j_{\mu_1\cdots\mu_s}(x)\rangle\!\rangle = \frac{\delta W[\varphi]}{\delta\varphi^{\mu_1\cdots\mu_s}(x)} = \sum_{n=0}^{\infty}\sum_{s_1,\ldots,s_n}\frac{i^n}{n!}\int\prod_{i=1}^{n}d^dx_i\varphi^{\mu_{11}\cdots\mu_{1s_1}}(x_1)\ldots\varphi^{\mu_{n1}\cdots\mu_{ns_n}}(x_n)$$
$$\times\langle 0|\mathcal{T}j_{\mu_1\cdots\mu_s}(x)j_{\mu_{11}\cdots\mu_{1s_1}}(x_1)\ldots j_{\mu_{n1}\cdots\mu_{ns_n}}(x_n)|0\rangle_c. \quad (2.12)$$

2.2 Symmetries of a classical theory

Let us start with a simple example of the classical action S that describes some matter field $\phi(x)$ coupled to gauge field A_μ. If the action is invariant under gauge transformation

$$\delta A_\mu = \partial_\mu\lambda \quad (2.13)$$

where λ is the parameter,

$$\delta S = \int d^dx\frac{\delta S}{\delta A_\mu}\delta A_\mu = \int d^dx j^\mu(x)\partial_\mu\lambda = -\int d^dx\partial_\mu j^\mu(x)\lambda = 0 \quad (2.14)$$

the current $j^\mu(x) = \frac{\delta S}{\delta A_\mu(x)}$ will be conserved since the above equation holds for any λ

$$\partial_\mu j^\mu(x) = 0 \quad (2.15)$$

Next, consider classical action S that describes matter field $\phi(x)$ coupled to curved background. The classical action is invariant under diffeomorphisms (general coordinate transformations) and Weyl transformations (for massless theory). For coordinate transformations $x^\mu \to x'^\mu(x)$ the metric transforms as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}(x) \quad (2.16)$$

For infinitesimal transformations $x^\mu \to x'^\mu = x^\mu - \xi^\mu$, the variation of the metric is given as Lie derivative of the metric in the direction of ξ

$$\delta_\xi g_{\mu\nu}(x) = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu \quad (2.17)$$

If the action is invariant under diff-transformations

$$\delta_\xi S = \int d^d x \frac{\delta S}{\delta g^{\mu\nu}} \delta_\xi g^{\mu\nu} = -\int d^d x \sqrt{g} T_{\mu\nu}(x) \nabla^\mu \xi^\nu = \int d^d x \sqrt{g} \xi^\nu \nabla^\mu T_{\mu\nu}(x) = 0 \quad (2.18)$$

since the above equation holds for any parameter ξ, the energy-momentum tensor must be covariantly conserved

$$\nabla^\mu T_{\mu\nu}(x) = 0 \quad (2.19)$$

The energy momentum tensor is defined as

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}, \qquad T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}} \quad (2.20)$$

Furthermore, let us consider Weyl transformations

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = e^{2\omega(x)} g_{\mu\nu}(x) \quad (2.21)$$

which in the infinitesimal form read

$$\delta_\omega g_{\mu\nu}(x) = 2\omega(x) g_{\mu\nu}(x) \quad (2.22)$$

Weyl invariance of the action

$$\delta_\omega S = \int d^d x \frac{\delta S}{\delta g^{\mu\nu}} \delta_\omega g^{\mu\nu} = \int d^d x \sqrt{g} \omega(x) T^\mu_\mu(x) = 0 \quad (2.23)$$

implies tracelesness of the energy-momentum tensor

$$T^\mu_\mu = 0 \quad (2.24)$$

Finally, consider classical action S which describes matter field $\phi(x)$ coupled to some higher spin field $\varphi_{\mu_1 \ldots \mu_s}$, $s > 2$. If the action is invariant under gauge transformation (to the lowest order)

$$\delta \varphi_{\mu_1 \ldots \mu_s} = \partial_{(\mu_1} \Lambda_{\mu_2 \ldots \mu_s)} \quad (2.25)$$

then the current $j_{\mu_1\ldots\mu_s}(x) = \frac{\delta S}{\delta\varphi^{\mu_1\cdots\mu_s}(x)}$ is conserved on-shell:

$$\partial^{\mu_1} j_{\mu_1\ldots\mu_s}(x) = 0 \tag{2.26}$$

In the limit $m \to 0$, we can also have invariance under the local transformations

$$\delta\varphi_{\mu_1\ldots\mu_s} = \eta_{(\mu_1\mu_2}\omega_{\mu_3\ldots\mu_s)} \tag{2.27}$$

which are usually referred to as (generalized) Weyl transformations. These transformations induce tracelessness of the currents $j_{\mu_1\ldots\mu_s}$ in any couple of indices:

$$\eta^{\mu_1\mu_2} j_{\mu_1\ldots\mu_s}(x) = 0 \tag{2.28}$$

2.3 Symmetries of a quantum theory

Let us now consider the quantum theory. If the quantum theory possesses the same symmetries as a classical theory the quantum effective action will be invariant under infinitesimal transformations. We start with the effective action $W[A]$, where A is the spin-1 field. If the action is invariant under gauge transformation $\delta A_\mu = \partial_\mu\lambda$ with λ parameter,

$$\delta W = \int d^d x \frac{\delta W}{\delta A_\mu}\delta A_\mu = \int d^d x \langle\!\langle j^\mu(x)\rangle\!\rangle \partial_\mu\lambda = -\int d^d x \partial_\mu \langle\!\langle j^\mu(x)\rangle\!\rangle \lambda = 0 \tag{2.29}$$

the 1-point correlator of the current $\langle\!\langle j^\mu(x)\rangle\!\rangle = \frac{\delta W}{\delta A_\mu(x)}$ will be conserved since the above equation holds for any parameter λ

$$\partial_\mu \langle\!\langle j^\mu(x)\rangle\!\rangle = 0 \tag{2.30}$$

The above equation represents the Ward identity for gauge invariance.

Next, we treat the effective action $W[g]$ where g is the metric (spin-2). If this action is invariant under diff-transformations $\delta_\xi g_{\mu\nu}(x) = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$

$$\delta_\xi W = \int d^d x \frac{\delta W}{\delta g^{\mu\nu}}\delta_\xi g^{\mu\nu} = -\int d^d x \sqrt{g} \langle\!\langle T_{\mu\nu}(x)\rangle\!\rangle \nabla^\mu\xi^\nu = \int d^d x \sqrt{g}\, \xi^\nu \nabla^\mu \langle\!\langle T_{\mu\nu}(x)\rangle\!\rangle = 0$$

the 1-point correlator of the energy-momentum tensor must be covariantly conserved

$$\nabla^{\mu}\langle\langle T_{\mu\nu}(x)\rangle\rangle = 0 \tag{2.31}$$

Furthermore, Weyl invariance of the effective action

$$\delta_{\omega}W = \int d^{d}x \frac{\delta W}{\delta g^{\mu\nu}(x)}\delta_{\omega}g^{\mu\nu} = \int d^{d}x\sqrt{g}\omega(x)\langle\langle T^{\mu}_{\mu}(x)\rangle\rangle = 0 \tag{2.32}$$

implies tracelesness of the 1-point correlator of the energy-momentum tensor

$$\langle\langle T^{\mu}_{\mu}(x)\rangle\rangle = 0 \tag{2.33}$$

Expressions (2.31) and (2.33) correspond to Ward identities for diff- and Weyl invariance.

A similar covariant conservation as (2.31) should be written also for the $s > 2$ currents, but we will often content ourselves with the lowest non-trivial order in which the conservation law reduces to

$$\partial^{\mu_1}\langle\langle j_{\mu_1\ldots\mu_s}(x)\rangle\rangle = 0 \tag{2.34}$$

For 1-point correlator we can also write the tracelessness condition in the limit $m \to 0$

$$\eta^{\mu_1\mu_2}\langle\langle j_{\mu_1\ldots\mu_s}(x)\rangle\rangle = 0 \tag{2.35}$$

In case it is not possible to retain classical symmetries at the quantum level we say that the theory is anomalous. The next section we devote to the discussion of the anomalies and their general form.

2.4 Wess-Zumino consistency conditions

To determine a general form of an anomaly we can use cohomological analysis. It turns out that potential candidates for the anomaly satisfy Wess-Zumino consistency conditions. Here we will mostly discuss the Weyl anomaly, based on [10]-[12].

Let us consider a classical theory invariant under some symmetry group G with gauge parameters λ^a. Let us denote generic fields of the theory with φ_i, $i = 1, \ldots, N$ and let the local transformation law be

$$\varphi_i(x) \to \varphi_i(x) + \delta_\lambda \varphi_i(x) \tag{2.36}$$

As we already mentioned, the classical action S is invariant under (2.36)

$$\delta_\lambda S = 0 \tag{2.37}$$

where

$$\delta_\lambda = \int d^d x \sum_i \delta_\lambda \varphi_i(x) \frac{\delta}{\delta \varphi_i(x)} \tag{2.38}$$

The variation of the 1-loop effective action gives Ward identity

$$\delta_\lambda W = \mathcal{A}_\lambda \tag{2.39}$$

where $\mathcal{A}_\lambda$ is a local functional of the fields linear in parameter λ. If we can eliminate $\mathcal{A}_\lambda$ by subtracting a local counter-term $\mathcal{C}$ from the effective action so that

$$\mathcal{A}_\lambda = \delta_\lambda \mathcal{C} \tag{2.40}$$

then

$$\delta_\lambda (W - \mathcal{C}) = 0 \tag{2.41}$$

we obtain the classical Ward identity. On the other hand, if we cannot find such a counter-term then the classical conservation law is broken at 1-loop and $\mathcal{A}_\lambda$ is an anomaly.

Let us now turn the anomaly problem to the cohomology problem. Inspired by the BRST formalism we:

- promote gauge parameters λ^a to anticommuting fields (Fadeev Popov ghosts)

- for λ^a assume the transformation law

$$\lambda^a(x) \to \lambda^a(x) + \delta_\lambda \lambda^a(x) \tag{2.42}$$

with a particular choice of $\delta_\lambda \lambda^a$

- modify the operator

$$\delta_\lambda = \int d^d x \sum_i \delta_\lambda \chi_i(x) \frac{\delta}{\delta \chi_i(x)} \tag{2.43}$$

where χ_i now represents all fields in the theory including ghosts.

There is a particular choice of $\delta_\lambda \lambda^a(x)$ for which the operator δ_λ defined in (2.43) becomes *nilpotent*

$$\delta_\lambda^2 = 0 \tag{2.44}$$

we call this operator the *coboundary operator* corresponding to the symmetry G. The Ward identity now becomes

$$\delta_\lambda W = \mathcal{A}_\lambda \tag{2.45}$$

with δ_λ defined in (2.43). Now $\mathcal{A}_\lambda$ satisfies the Wess-Zumino consistency condition

$$\delta_\lambda \mathcal{A}_\lambda = 0 \tag{2.46}$$

We call $\mathcal{A}_\lambda$ a *cocycle*. Furthermore, if there exists a term $\mathcal{C}$ so that we can write

$$\mathcal{A}_\lambda = \delta_\lambda \mathcal{C} \tag{2.47}$$

then we call $\mathcal{A}_\lambda$ a *coboundary*. If this is not true for any $\mathcal{C}$ then $\mathcal{A}_\lambda$ is a *non-trivial cocycle - anomaly*. Cocycles split into classes and each class is defined by a cocycle modulo all coboundaries. These classes form *cohomology groups*.

From now on we will focus on an example where a symmetry group G consists of diffemorphisms and Weyl symmetry. We will see that the anomaly in this case satisfies also a cross-consistency condition which gives further restrictions on the form of the anomaly. Moreover, we will see that it is possible to completely eliminate diff-anomaly (or Weyl-anomaly) by subtracting a suitable counterterm from the action. In this case only Weyl anomaly remains.

Let us start with diffeomorphisms. Let us denote with $\xi_\mu(x)$ the parameter of infinitesimal diff-transformations which act on the metric as

$$\delta_\xi \, g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{2.48}$$

Next we promote the gauge parameter ξ^μ to anticommuting field with the transformation law

$$\delta_\xi \xi^\mu = \xi^\nu \partial_\nu \xi^\mu \tag{2.49}$$

The variational operator is

$$\delta_\xi = \int d^d x \sum_i \delta_\xi \chi_i \frac{\delta}{\delta \chi_i} \tag{2.50}$$

where χ_i stands for all the fields in the theory including ghosts. We choose the transformation law of the ghost ξ^μ so that the operator δ_ξ is nilpotent

$$\delta_\xi^2 = 0 \tag{2.51}$$

Let us now consider Weyl transformations. Let $\omega(x)$ parametrise infinitesimal Weyl transformations which act on the metric as

$$\delta_\omega \, g_{\mu\nu} = 2\omega(x) g_{\mu\nu} \tag{2.52}$$

where $\omega(x)$ is some generic positive function. Now we promote the gauge parameter $\omega(x)$ to an anticommuting field with the transformation law

$$\delta_\omega \omega(x) = 0 \tag{2.53}$$

Next we define

$$\delta_\omega = \int d^d x \sum_i \delta_\omega \chi_i \frac{\delta}{\delta \chi_i} \tag{2.54}$$

where the transformation law for $\omega(x)$ is such that δ_ω is nilpotent

$$\delta_\omega^2 = 0 \tag{2.55}$$

If a classical theory is invariant under Weyl or diff-transformations we can write a corresponding Ward identity and check if we get an anomaly at the quantum level.

We can also simultaneously include both Weyl and diff-invariance in the theory at the classical level. We need two more transformation laws

$$\delta_\xi \omega(x) = \xi^\mu \partial_\mu \omega, \qquad \delta_\omega \xi^\mu = 0 \tag{2.56}$$

Furthermore, we assume that ω and ξ^μ are anticommuting with each other. Nilpotent coboundary operator

$$(\delta_\omega + \delta_\xi)^2 = 0 \tag{2.57}$$

now defines a coupled cohomological problem. Altogether we have

$$\delta_\omega^2 = 0, \qquad \delta_\xi^2 = 0, \qquad \delta_\omega \delta_\xi + \delta_\xi \delta_\omega = 0 \tag{2.58}$$

For a classical theory which is invariant under Weyl and diff- transformations we have

$$\delta_\xi S = 0, \qquad \delta_\omega S = 0 \tag{2.59}$$

On the other hand, in quantum theory, the Ward identity for 1-loop effective action is

$$\begin{aligned}
\delta_\xi W &= \mathcal{A}_\xi = -\int d^d x \xi^\nu \nabla^\mu T_{\mu\nu} \\
\delta_\omega W &= \mathcal{A}_\omega = \int d^d x 2\omega T_\mu^\mu
\end{aligned} \tag{2.60}$$

The anomaly satisfies the consistency conditions

$$\delta_\omega \mathcal{A}_\omega = 0, \qquad \delta_\xi \mathcal{A}_\xi = 0 \tag{2.61}$$

$\mathcal{A}_\xi$ is a cocycle of δ_ξ, while $\mathcal{A}_\omega$ is a cocycle of δ_ω. Since the classical theory is invariant under both Weyl and diff- transformations the generic cocycle of the coboundary operator $\delta_\omega + \delta_\xi$ is $\mathcal{A}_\omega + \mathcal{A}_\xi$. We also get a cross-consistency condition

$$\delta_\omega \mathcal{A}_\xi + \delta_\xi \mathcal{A}_\omega = 0 \tag{2.62}$$

If a pair $\mathcal{A}_\omega$ and $\mathcal{A}_\xi$ is such that there exists a local term $\mathcal{C}$ satisfying

$$\mathcal{A}_\omega = \delta_\omega \, \mathcal{C} \quad \text{and} \quad \mathcal{A}_\xi = \delta_\xi \, \mathcal{C} \tag{2.63}$$

then such anomaly pair is considered to be trivial as it can be cancelled by adding the local term C to the quantum action. The condition which identifies the anomaly is that for any $\mathcal{C}$

$$\mathcal{A}_\omega + \mathcal{A}_\xi \neq (\delta_\omega + \delta_\xi)\mathcal{C} \tag{2.64}$$

Note that in general both $\mathcal{A}_\xi$ and $\mathcal{A}_\omega$ are nonvanishing, however, by subtraction of an appropriate counter-term we can restore covariance of the quantum theory.

$$\mathcal{A}_\xi \quad \rightarrow \quad A_\xi - \delta_\xi C = 0 \tag{2.65}$$

$$\mathcal{A}_\omega \quad \rightarrow \quad A_\omega + \delta_\omega C \Rightarrow \langle\langle T^\mu{}_\mu \rangle\rangle \neq 0 \tag{2.66}$$

In this case the theory has only Weyl anomaly.

2.5 General form of trace anomaly

In this section we will discuss a general form of the trace anomaly [6]-[12]. For a review see [151].

Let us assume that the theory is covariant at the quantum level

$$\mathcal{A}_\xi = 0 \tag{2.67}$$

It follows from consistency conditions that the trace anomaly must be invariant under both diffeomorphisms and Weyl transformations

$$\delta_\xi \mathcal{A}_\omega = 0, \qquad \delta_\omega \mathcal{A}_\omega = 0 \tag{2.68}$$

Possible terms in the anomaly, by dimensional analysis, must have dimension four in 4d. Moreover, because of diff-invariance, the anomaly must be constructed from diff-invariant objects such as Riemann tensor, Ricci tensor and Ricci scalar. Mentioned objects have dimension 2, which means that in 4d we can construct the trace anomaly from squares of Riemann tensor or d'Alambertian of Ricci scalar. Recall that the trace of energy-momentum tensor at the quantum level in general is not vanishing. Possible terms are:

$$\langle\langle T^\mu_\mu(x)\rangle\rangle = aR_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + bR_{\mu\nu}R^{\mu\nu} + cR^2 + d\Box R + e\,\varepsilon^{\mu\nu\rho\sigma}\,R_{\mu\nu}{}^{\alpha\beta}\,R_{\rho\sigma\alpha\beta} \tag{2.69}$$

The d'Alambertian of Ricci scalar can be subtracted by a local counterterm (Weyl variation of R^2) and hence it is not a true anomaly. From consistency conditions we get

$$a + b + 3c = 0 \tag{2.70}$$

that only two of the three constants a, b, c are independent. Usually, we write the trace anomaly in terms of

- Euler density: $E = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2$
- Weyl density: $\mathcal{W}^2 = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$
- Pontryagin density: $P = \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,R_{\mu\nu}{}^{\alpha\beta}\,R_{\rho\sigma\alpha\beta}$

General form of the trace of energy-momentum tensor therefore is

$$\langle\langle T^\mu_\mu(x)\rangle\rangle = a\,E + c\,\mathcal{W}^2 + e\,P \tag{2.71}$$

Coefficients a, c and e depend on the theory and are well known for various matter types [7, 13]. The coefficient e is the one we would like to study in detail for chiral models.

2.6 Dirac, Majorana and Weyl fermions in 4d.

We would like to devote this section to fermions in 4d. In particular, we will focus on a discussion of the statement that a massless Majorana fermion is the same as a Weyl fermion. If this is true at both classical and quantum level, there is no chance for an odd parity trace anomaly to exist. On the other hand this statement is not undisputed. Our aim here is to examine classical and quantum differences between the two types of fermions and show that there is no a priori uncontroversial evidence that the relevant statement is true. Therefore it is necessary to leave the last word to explicit computations, such as the one for odd parity trace anomaly. We will start with a review on the properties of Dirac, Majorana and Weyl fermions, based on [152].

2.6.1 Majorana fermions

We start with a few basic facts about fermions in 4d. We call a *fermion field* $\psi(x)$ any solution of the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{2.72}$$

where γ^μ denotes a set of 4×4 matrices which we call *Dirac matrices* (or γ-matrices). Dirac matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2.73}$$

where metric $g_{\mu\nu}$ has mostly - signature, and

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0$$

One possible solution to Dirac equation is the real solution. Majorana found a representation of γ-matrices for which the Dirac equation is real. In this representation γ-matrices are real, and we will denote them with $\tilde{\gamma}$. Let us write down Majorana representation of

$$\tilde{\gamma}^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \tilde{\gamma}^1 = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \quad \tilde{\gamma}^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad \tilde{\gamma}^3 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \tag{2.74}$$

where σ^i are Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.75}$$

With this choice of γ-matrices, as a solution of the Dirac equation we get a Majorana field $\tilde{\psi}$ which satisfies a *reality condition*:

$$\tilde{\psi} = \tilde{\psi}^* \tag{2.76}$$

Note that Majorana representation is not unique. If we have two choices of Dirac matrices they are related by an unitary transformation. This means that a general solution for γ-matrices can be obtained using Majorana representation so that

$$\gamma^\mu = U \tilde{\gamma}^\mu U^\dagger \tag{2.77}$$

where U is a unitary matrix. It follows that, if $\tilde{\psi}$ is a solution of Dirac equation in Majorana representation, then ψ is a solution to Dirac equation in a general representation

$$\psi = U \tilde{\psi} \tag{2.78}$$

Let us now see how does the Majorana reality condition look in this other representation. We can rewrite

$$U^\dagger \psi = (U^\dagger \psi)^* \tag{2.79}$$

so that

$$\psi = U U^T \psi^* \tag{2.80}$$

Usually, instead of matrix U, we use another unitary matrix C defined by

$$U U^T = \gamma_0 C \tag{2.81}$$

with properties

$$\gamma_\mu^T = -C^{-1}\gamma_\mu C, \qquad CC^* = -1, \qquad CC^\dagger = 1 \tag{2.82}$$

Now we can introduce the notion of *Lorentz-covariant conjugate* $\hat{\psi}$

$$\hat{\psi} = \gamma_0 C \psi^* \tag{2.83}$$

The reality condition (2.80) now becomes

$$\psi = \hat{\psi} \tag{2.84}$$

Above we introduced Lorentz-covariant conjugate $\hat{\psi}$. Let us now explain the reason for that name. We start from a 4-component Dirac fermion ψ. Under Lorentz it transforms as

$$\psi(x) \to \psi'(x') = \exp\left[-\frac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}\right]\psi(x)\,, \tag{2.85}$$

for $x'^\mu = (e^\lambda)^\mu{}_\nu\, x^\nu$, where $\Sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$ are the Lorentz generators. Now, the Majorana reality condition makes sense only if it holds in any reference frame. To prove that this is true we must show that $\hat{\psi}$ and ψ transform in the same way under Lorentz transformations. We take a complex conjugate of (2.85) and multiply with $\gamma_0 C$. It turns out that if ψ transforms like (2.85), then

$$\hat{\psi}(x) \to \hat{\psi}'(x') = \exp\left[-\frac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}\right]\hat{\psi}(x)\,, \tag{2.86}$$

The fact that $\hat{\psi}$ transforms in the same way as ψ is the reason why we call $\hat{\psi}$ Lorentz-covariant conjugate.

2.6.2 Helicity and chirality

Let us now introduce two concepts: helicity and chirality. Helicity of a particle is defined as a projection of the spin along the direction of motion of the particle. For a particle with momentum $\vec{p}$ helicity is:

$$h = \frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|} \tag{2.87}$$

where $\vec{\Sigma}$ denotes the spin

$$\Sigma^i = \frac{i}{2}\epsilon^{ijk}\Sigma_{jk} \tag{2.88}$$

where $i, j, k = 1, 2, 3$. Eigenvalues of helicity h are ± 1. An eigenstate with helicity -1 we call right-handed, while an eigenstate with helicity $+1$ we call left-handed.

Since helicity commutes with the Dirac Hamiltonian, it follows that helicity is a conserved quantity for a free Dirac particle. However, helicity is not Lorentz invariant for massive particles. If we imagine a fermion with spin and momentum in the same direction, its helicity will be $+1$. On the other hand, let us now imagine a second observer, which is moving faster than the particle in the first reference frame. For this observer the particle is moving in the other direction, and since the spin does not change, its helicity is -1. For massless particles, since they are traveling at the speed of light, helicity is Lorentz invariant. All observers agree on the value of helicity for a massless particle.

Let us now discuss chirality (handedness) of a particle. Chirality of a particle is associated to the matrix γ_5 defined as:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{2.89}$$

which anticommutes with γ-matrices:

$$\{\gamma_5, \gamma_\mu\} = 0 \tag{2.90}$$

Properties of γ_5 are

$$\gamma_5^\dagger = \gamma_5, \qquad (\gamma_5)^2 = 1, \qquad C^{-1}\gamma_5 C = \gamma_5^T$$

which ensure that the matrices

$$P_L = \frac{1+\gamma_5}{2}, \qquad P_R = \frac{1-\gamma_5}{2}$$

behave like projection matrices on fermion field. We are now in position to write a generic fermion field as a sum

$$\psi = \psi_L + \psi_R \tag{2.91}$$

where ψ_L and ψ_R are left-handed and right-handed projections of ψ defined by

$$\psi_L = P_L\psi, \qquad \psi_R = P_R\psi \tag{2.92}$$

The eigenvalues of γ_5 are ± 1

$$\gamma_5\psi_L = +\psi_L, \qquad \gamma_5\psi_R = -\psi_R \tag{2.93}$$

Note that chirality is a Lorentz invariant quantity, but it is not conserved since γ_5 does not commute with the Hamiltonian. To be precise, γ_5 does not commute with the mass term in the Hamiltonian. For a massless fermion both helicity and chirality are well defined.

2.6.3 Weyl fermions

Previously we were searching for real solutions of the Dirac equation. Let us now focus on the search for the solutions of the Dirac equation which satisfy a chirality constraint:

$$\begin{aligned} &\gamma_5\psi_L = +\psi_L \quad \text{for left-handed fermion} \\ &\gamma_5\psi_R = -\psi_R \quad \text{for right-handed fermion} \end{aligned} \tag{2.94}$$

A solution which is eigenvector of the chirality matrix γ_5 is called a Weyl fermion. Here we use chiral (Weyl) representation of Dirac matrices

$$\tilde{\gamma}^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \tilde{\gamma}_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.95}$$

In this representation γ_5 is diagonal, so that the projectors become

$$P_L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.96}$$

It follows that, in chiral representation, a generic Dirac field ψ (4-component) can be written as

$$\psi = \begin{pmatrix} \omega_t \\ \omega_b \end{pmatrix}, \tag{2.97}$$

where ω_t and ω_b are 2-component spinors. Right-handed field has only the top two components ω_t, while the left-handed field has only the bottom 2-components ω_b:

$$\psi_R = \begin{pmatrix} \omega_t \\ 0 \end{pmatrix}, \quad \psi_L = \begin{pmatrix} 0 \\ \omega_b \end{pmatrix}, \tag{2.98}$$

The Lagrangian for left-handed field can now be written as

$$L_L = i\omega_b^\dagger \sigma^\mu \partial_\mu \omega_b \tag{2.99}$$

where $\sigma^\mu = (1, \vec{\sigma})$. The left-handed Weyl fermion is a solution of

$$i\sigma^\mu \partial_\mu \omega_b = 0 \tag{2.100}$$

For right-handed Weyl fields we have

$$L_R = i\omega_t^\dagger \bar{\sigma}^\mu \partial_\mu \omega_t \tag{2.101}$$

where $\bar{\sigma}^\mu = (1, -\vec{\sigma})$. The right-handed Weyl fermion is a solution of

$$i\bar{\sigma}^\mu \partial_\mu \omega_t = 0 \tag{2.102}$$

2.6.4 Dirac fermions from Weyl fermions

Here we want to show how to represent Dirac fermion using Weyl fermions. Since Dirac fermion is in general massive we must include both left and right chirality. Dirac field can be constructed from two independent (say left) Weyl fields ψ_{1L} and ψ_{2L}

$$\psi = \psi_{1L} + \widehat{\psi_{2L}} \tag{2.103}$$

Note that the Dirac field is, in contrast to Majorana and Weyl, completely unconstrained solution to Dirac equation. Let us see if it is possible to impose both chirality and reality conditions in the same time. In other words we want to see if it is possible for a fermion field to be Weyl and Majorana in the same time. To see that this is not possible, let use Majorana representation of Dirac matrices where Majorana field is real. Weyl fermion, on the other side, satisfies

$$\gamma_5\psi_{L,R} = \pm\psi_{L,R} \tag{2.104}$$

now, in Majorana representation, γ_5 is purely imaginary and hence the above equation cannot be satisfied by a real field $\psi_{L,R}$. We conclude that Majorana cannot be Weyl at the same time.

2.6.5 Majorana fermions from Weyl fermions

Just like Dirac fermion, Majorana fermion can be massive. To represent a Majorana fermion using Weyl fermions we must include both chiralities. In addition, the combination of left and right Weyl fermion now must satisfy Majorana reality condition. A left chiral fermion satisfies

$$(1-\gamma_5)\psi_L = 0 \tag{2.105}$$

Let us now take complex conjugate and multiply with $\gamma_0 C$

$$\gamma_0 C(1-\gamma_5^*)\psi_L^* = 0 \tag{2.106}$$

Since γ_5 is hermitian $\gamma_5^* = \gamma_5^T$ and

$$C^{-1}\gamma_5 C = \gamma_5^T \tag{2.107}$$

we conclude

$$\gamma_0 C(1-\gamma_5^*)\psi_L^* = (1+\gamma_5)\gamma_0 C\psi_L^* = (1+\gamma_5)\widehat{\psi_L} \tag{2.108}$$

that $\widehat{\psi_L}$ is a right-handed Weyl fermion. We can write Majorana fermion in terms of Weyl fermion as

$$\psi = \psi_L + \widehat{\psi_L} \tag{2.109}$$

We can also rewrite Majorana fermion in terms of Weyl fermions using 2-component notation. Majorana condition is

$$\psi = \begin{pmatrix} \omega_t \\ \omega_b \end{pmatrix} = \begin{pmatrix} i\sigma^2 \omega_b^* \\ -i\sigma^2 \omega_t^* \end{pmatrix} = \hat{\psi} \tag{2.110}$$

where we used

$$\gamma_0 C = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \tag{2.111}$$

in chiral representation. This means that we can write Majorana field as

$$\psi = \begin{pmatrix} \omega_t \\ -i\sigma^2 \omega_t^* \end{pmatrix} = \begin{pmatrix} i\sigma^2 \omega_b^* \\ \omega_b \end{pmatrix} = \hat{\psi} \tag{2.112}$$

In terms of 2-component spinors massive Dirac equation splits into:

$$\begin{aligned} i\bar{\sigma}^\mu \partial_\mu \omega_t &= m\omega_b \\ i\sigma^\mu \partial_\mu \omega_b &= m\omega_t \end{aligned} \tag{2.113}$$

which means that Majorana field satisfies

$$\begin{aligned} \bar{\sigma}^\mu \partial_\mu \omega_t &= -m\sigma^2 \omega_t^* \\ \sigma^\mu \partial_\mu \omega_b &= m\sigma^2 \omega_b^* \end{aligned} \tag{2.114}$$

Let us now focus on the statement that there is a one-to-one correspondence between the components of a Weyl spinor and those of a Majorana spinor in such a way that the Lagrangians in two-component notation look the same. We start with the analysis of the

Majorana Lagrangian in 2-component notation:

$$\begin{aligned} L_M &= \frac{1}{2}\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \\ &= \frac{i}{2}\left[\omega_t\bar{\sigma}^\mu\partial_\mu\omega_t - \partial_\mu\omega_t^\dagger\bar{\sigma}^\mu\omega_t - m\left(\omega_t^T\sigma^2\omega_t - \omega_t^\dagger\sigma^2\omega_t^*\right)\right] \end{aligned} \quad (2.115)$$

where we used the right-handed field to express the Lagrangian. Now let us take a look at the Lagrangian for right-handed Weyl fermion. We can split it

$$\begin{aligned} L_W &= i\omega_t^\dagger\bar{\sigma}^\mu\partial_\mu\omega_t \\ &= \frac{i}{2}\omega_t^\dagger\bar{\sigma}^\mu\partial_\mu\omega_t - \frac{i}{2}\partial_\mu\omega_t^\dagger\bar{\sigma}^\mu\omega_t + \frac{i}{2}\partial_\mu\left(\omega_t^\dagger\bar{\sigma}^\mu\omega_t\right) \end{aligned} \quad (2.116)$$

ignoring the total derivative, we get

$$L_W = \frac{i}{2}\left[\omega_t^\dagger\bar{\sigma}^\mu\partial_\mu\omega_t - \partial_\mu\omega_t^\dagger\bar{\sigma}^\mu\omega_t\right] \quad (2.117)$$

By comparison we see that the Lagrangians L_M and L_W are the same in the massless case. Even though in the massless case these two Lagrangians are indistinguishable, we must keep in mind that representations of Lorentz group for Majorana and Weyl fermion are different. Weyl field is a part of chiral representation $\left(\frac{1}{2},0\right)$ or $\left(0,\frac{1}{2}\right)$ while Majorana is a part of $\left(\frac{1}{2},0\right)\otimes\left(0,\frac{1}{2}\right)$ constrained with reality condition.

2.6.6 Charge conjugation, parity and CP

Charge conjugation $\mathcal{C}$ is an operation on the fields that replaces all fields with complex conjugates. For a fermion field, charge conjugation must must be Lorentz covariant, otherwise the action would not be Lorentz invariant. Charge conjugation operation on the field ψ reads

$$\mathcal{C}\psi\mathcal{C}^{-1} = \eta_C\hat{\psi} \quad (2.118)$$

where η_C is a phase which, for simplicity, we set equal to 1. Let us recall the properties of a Weyl fermion $\psi_L = P_L\psi$ under charge conjugation. Since P_L is a constant matrix the operation of charge conjugation acts only on fields

$$\mathcal{C}\psi_L\mathcal{C}^{-1} = P_L\eta_C\hat{\psi} = P_L\hat{\psi} = \hat{\psi}_L \tag{2.119}$$

Let us now consider Lorentz-covariant conjugate $\widehat{\psi_L}$

$$\widehat{\psi_L} = \gamma_0 C(\psi_L)^* = \gamma_0 C P_L^* \psi^* \tag{2.120}$$

Now use the fact that γ_5 is hermitian and $CP_L^T = P_L C$ to write

$$\widehat{\psi_L} = \gamma_0 P_L C\psi^* = P_R\gamma_0 C\psi^* = P_R\hat{\psi} = \hat{\psi}_R \tag{2.121}$$

It follows that Lorentz-covariant conjugate of ψ_L is a right-handed fermion and its charge conjugate is left-handed.

The parity operation is a spacetime transformation that maps $(t, \vec{x})$ to $(t, -\vec{x})$. Under a parity transformation momentum changes sign, and spin remains the same so that the helicity of a particle changes. Since helicity and chirality coincide for massless particles, chirality changes as well. The parity operation is defined by

$$\mathcal{P}\psi_L(t, \vec{x})\mathcal{P}^{-1} = \eta_P\gamma_0\psi_R(t, -\vec{x}) \tag{2.122}$$

where η_P is a phase.

If we consider CP, the action of a Majorana fermion is obviously invariant under it. For a Weyl fermion we have

$$\mathcal{CP}\psi_L(t, \vec{x})(\mathcal{CP})^{-1} = \gamma_0\widehat{\psi}_L(t, -\vec{x}) = \gamma_0 P_R\hat{\psi}(t, -\vec{x}) = \gamma_0\hat{\psi}_R(t, -\vec{x}) \tag{2.123}$$

Applying CP to the Weyl action one gets

$$\begin{aligned}
\mathcal{CP}\left(\int i\overline{\psi_L}\gamma^\mu\partial_\mu\psi_L\right)(\mathcal{CP})^{-1} &= \int i\overline{\hat{\psi}_R}(t, -\vec{x})\gamma^{\mu\dagger}\partial_\mu\hat{\psi}_R(t, -\vec{x}) \\
&= \int i\overline{\hat{\psi}_R}(t, \vec{x})\gamma^\mu\partial_\mu\hat{\psi}_R(t, \vec{x})
\end{aligned} \tag{2.124}$$

But one can easily prove that

$$\int i\overline{\hat{\psi}_R}(t,\vec{x})\gamma^\mu\partial_\mu\hat{\psi}_R(t,\vec{x}) = \int i\overline{\psi_L}(x)\gamma^\mu\partial_\mu\psi_L(x) \tag{2.125}$$

Therefore the action for a Weyl fermion is CP invariant. It is also, separately, T invariant, and, so, CPT invariant.

Now let us go to the quantum interpretation of the field ψ_L. It's plane wave expansion is

$$\psi_L(x) = \int dp\ \left(a(p)u_L(p)e^{-ipx} + b^\dagger(p)v_L(p)e^{ipx}\right) \tag{2.126}$$

where u_L, v_L are fixed and independent left-handed spinors. The interpretation is: $b^\dagger(p)$ creates a left-handed particle while $a(p)$ destroys a left-handed particle with negative helicity (because of the opposite momentum). However eqs.(2.123, 2.124) force us to identify the latter with a right-handed antiparticle: C maps particles to antiparticles, while P invert helicities, so CP maps left-handed particles to right-handed antiparticles.

2.6.7 Comments on massless Majorana and Weyl fermions

The evident difference between massless Majorana and Weyl fermions is that they belong to two different representations of the Lorentz group, irreducible to each other (in 4d there cannot exist a spinor that is simultaneously Majorana and Weyl).

Next, the reason why they are sometimes considered as a unique object is due to the fact that we can establish a one-to-one correspondence between the components of a Weyl spinor and of a Majorana spinor so that the Lagrangian in two-component notation looks the same. But, if the action is the same for both Weyl and Majorana, how can there be any differences?

In general, the action does not contain the complete information. In the quantum theory a crucial role is played by the functional measure, which is very likely to be different for Weyl and Majorana fermions. This is the decisive point for the anomalies. The path integral of a free Dirac fermion is interpreted as the determinant of the massless Dirac operator $\not{D} = i\not{\partial} + \not{V}$ (where V denotes any potential), i.e. the (suitably regularized) product of its eigenvalues. A similar interpretation holds also for a massless Majorana fermion.

For a Weyl fermion the matter is not so straightforward. The Dirac operator anticommutes with γ_5 and hence it maps a left-handed spinor to a right-handed one. Therefore, the eigenvalue problem is not well defined for $\not{D}_L = \not{D} P_L$ so that the determinant is ill-defined. Another idea is to replace $\det \not{D}_{\rm L}$ with $\left(\det\left(\not{D}_{\rm L}^{\dagger}\not{D}_{\rm L}\right)\right)^{\frac{1}{2}}$, but in this case we have an undetermined overall phase factor. This problem has been known for a long time[1]. There is a few ways to overcome this problem. One way is to use a perturbative approach (Feynman diagram technique) in a chiral fermion theory. This is the method used in [15, 17]. We will revisit it below. The second way is based on Dirac fermions, [54, 55, 86], (i.e. with the ordinary Dirac path integral measure), where we recover the chiral fermion theory as a special limit. Finally, let us mention [94], where a fifth dimension is introduced as a regulator, although we do not use it here.

The above arguments lead toward the conclusion that massless Majorana and Weyl fermions, notwithstanding some similarities, may really be different objects. It is important to avoid a priori conclusions, but rather develop both hypotheses and compare the final results. This said, we should find properties that differentiate Weyl and massless Majorana fermions. For this reason, in the next chapter we show that one such property is the parity odd Weyl anomaly, which is zero for a massless Majorana fermion, while it equals the Pontryagin density for a Weyl fermion. On the other hand, the even parity trace anomaly is the same for both.

[1] In particular, since Fujikawa method holds when we have both chiralities present in the theory one cannot use it for chiral theories. This problem has been discussed in detail in [55] where it is shown that the original Fujikawa method cannot reproduce the non-Abelian consistent chiral anomalies, but only the covariant ones in chirally symmetric theories. We cannot expect to be able to reproduce the odd parity trace anomaly in a left-handed theory, because the latter belongs to the same class as the non-Abelian consistent chiral anomalies (split anomalies). This observation applies to [85], where, using Fujikawa method and Pauli-Villars regularization, the authors obtain a vanishing odd trace anomaly which seems to contradict our result below. Using a Dirac fermion path integral measure introduces both chiralities, even though formally the action itself is declared to be the Weyl one. For this particular anomaly what matters is that only one chirality is involved through all the steps, including the path integral measure. Bearing this in mind, the result of [85] applies to Dirac and Majorana fermions and is in fact consistent with ours.

Chapter 3

Odd parity trace anomaly in chiral theories

In this chapter we reconsider the calculation of the odd trace anomaly in chiral fermion theories in a 4d curved background given in [15]. The motivation for this is to give a more complete and detailed calculation of the trace anomaly. In particular, in [15, 16], as well as in [17], tadpoles and seagull diagrams were neglected. In ordinary (non-chiral) theories coupled to gravity such diagrams can contribute in a form of local terms to the effective action, and they help to restore conservation, which otherwise would be violated by local terms, see [153]. Instead, we find in [19], that these diagrams do not contribute for the parity odd diagrams in a chiral theory, and do not change the final result of [15]. However, they should be taken into account and evaluated. Moreover, in contrast to [15], here we do not redefine the fermion field[1]. As a consequence, the energy-momentum tensor is different from the energy-momentum tensor in [15], that is, it contains an additional term from the $\sqrt{|g|}$ in the action. This additional term gives a contribution to both the trace anomaly and the diff-anomaly. However, subtraction of the appropriate counterterm from the effective action cancels the diff-anomaly and in the same time produces the same trace anomaly as in [15]. In this chapter we closely follow [19].

[1] In [15] the fermion field was redefined $\psi \to (|g|)^{\frac{1}{4}}\psi$.

3.1 Odd parity trace anomaly in chiral theories

The model considered in [15] was a left-handed Weyl spinor coupled to external gravity in 4d. The action is

$$S = \int d^4x \sqrt{|g|}\, i\overline{\psi_L}\gamma^\mu \left(\nabla_\mu + \frac{1}{2}\omega_\mu\right)\psi_L \tag{3.1}$$

where $\gamma^\mu = e^\mu_a\gamma^a$ ($\mu, \nu, ...$ are world indices, $a, b, ...$ are flat indices), ∇ is the covariant derivative with respect to the world indices and ω_μ is the spin connection:

$$\omega_\mu = \omega_\mu{}^{ab}\Sigma_{ab}$$

Finally $\psi_L = \frac{1+\gamma_5}{2}\psi$. Classically the energy-momentum tensor

$$T^{\mu\nu} = -\frac{i}{4}\overline{\psi_L}\gamma^\mu\overset{\leftrightarrow}{\nabla}{}^\nu\psi_L + (\mu \leftrightarrow \nu) \tag{3.2}$$

is both conserved on shell and traceless.

From (3.1) we can extract the (simplified) Feynman rules as follows. The action (3.1) can be written as

$$S = \int d^4x \sqrt{|g|}\, \left[\frac{i}{2}\overline{\psi_L}\gamma^\mu\overset{\leftrightarrow}{\partial}_\mu\psi_L - \frac{1}{4}\epsilon^{\mu abc}\omega_{\mu ab}\overline{\psi_L}\gamma_c\gamma_5\psi_L\right] \tag{3.3}$$

where it is understood that the derivative applies to ψ_L and $\overline{\psi_L}$ only, and we used the relation $\{\gamma^a, \Sigma^{bc}\} = i\,\epsilon^{abcd}\gamma_d\gamma_5$. Expanding

$$e^a_\mu = \delta^a_\mu + \chi^a_\mu + ..., \qquad e^\mu_a = \delta^\mu_a + \hat{\chi}^\mu_a + ..., \qquad \text{and} \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.4}$$

and inserting these expansions in the defining relations $e^a_\mu e^\mu_b = \delta^a_b$, $g_{\mu\nu} = e^a_\mu e^b_\nu\eta_{ab}$, one finds

$$\hat{\chi}^\mu_\nu = -\chi^\mu_\nu \qquad \text{and} \qquad h_{\mu\nu} = 2\,\chi_{\mu\nu}. \tag{3.5}$$

Expanding accordingly the spin connection

$$\omega_{\mu ab} = e_{\nu a}(\partial_\mu e^\nu_b + e^\sigma{}_b\Gamma_\sigma{}^\nu{}_\mu), \qquad \Gamma_\sigma{}^\nu{}_\mu = \frac{1}{2}\eta^{\nu\lambda}(\partial_\sigma h_{\lambda\mu} + \partial_\mu h_{\lambda\sigma} - \partial_\lambda h_{\sigma\mu}) + ...$$

after some algebra one gets

$$\omega_{\mu ab}\,\epsilon^{\mu abc} = -\frac{1}{4}\epsilon^{\mu abc}\,\partial_\mu h_{a\lambda}\,h_b^\lambda + ... \tag{3.6}$$

Therefore, up to second order the action, by incorporating $(|g|)^{\frac{1}{4}}$ in the ψ field, can be written as

$$S \approx \int d^4x\,\left[\frac{i}{2}(\delta_a^\mu - \frac{1}{2}h_a^\mu)\overline{\psi_L}\gamma^a\overleftrightarrow{\partial}_\mu\psi_L + \frac{1}{16}\epsilon^{\mu abc}\,\partial_\mu h_{a\lambda}\,h_b^\lambda\,\bar{\psi}_L\gamma_c\gamma_5\psi_L\right]$$

The free action is

$$S_{free} = \int d^4x\,\frac{i}{2}\overline{\psi_L}\gamma^a\overleftrightarrow{\partial}_a\psi_L \tag{3.7}$$

and the lowest interaction terms are

$$S_{int} \;=\; \int d^4x\left[-\frac{i}{4}h_a^\mu\,\overline{\psi_L}\gamma^a\overleftrightarrow{\partial}_\mu\psi_L + \frac{1}{16}\epsilon^{\mu abc}\,\partial_\mu h_{a\lambda}\,h_b^\lambda\,\bar{\psi}_L\gamma_c\gamma_5\psi_L\right] \tag{3.8}$$

Retaining only the above terms of the action of (3.8), the Feynman rules are as follows (momenta are ingoing and *the external gravitational field is assumed to be* $h_{\mu\nu}$). The fermion propagator is

$$P: \qquad \frac{i}{\not{p} + i\epsilon} \tag{3.9}$$

The two-fermion-one-graviton vertex is

$$V_{ffh}: \qquad -\frac{i}{8}\left[(p+p')_\mu\gamma_\nu + (p+p')_\nu\gamma_\mu\right]\frac{1+\gamma_5}{2} \tag{3.10}$$

The two-fermion-two-graviton vertex (V^ϵ_{ffhh}) is

$$V^\epsilon_{ffhh}: \qquad \frac{1}{64}t_{\mu\nu\mu'\nu'\kappa\lambda}(k-k')^\lambda\gamma^\kappa\frac{1+\gamma_5}{2} \tag{3.11}$$

where

$$t_{\mu\nu\mu'\nu'\kappa\lambda} = \eta_{\mu\mu'}\epsilon_{\nu\nu'\kappa\lambda} + \eta_{\nu\nu'}\epsilon_{\mu\mu'\kappa\lambda} + \eta_{\mu\nu'}\epsilon_{\nu\mu'\kappa\lambda} + \eta_{\nu\mu'}\epsilon_{\mu\nu'\kappa\lambda} \tag{3.12}$$

3.1.1 Complete expansion

The previous action (3.1) is a simplified one. It disregards the measure $\sqrt{|g|}$, which is incorporated in the fermion field ψ. In a more complete approach one should take into account tadpole and seagull terms and reinsert $\sqrt{|g|}$ in the action. Some of these, in principle, might be relevant for the trace anomaly. To this end we need the complete expansion in $h_{\mu\nu}$ up to order three of the action, more precisely,

$$
\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \\
g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu} + (h^2)^{\mu\nu} + \dots \\
e^{\mu}_{a} &= \delta^{\mu}_{a} - \frac{1}{2}h^{\mu}_{a} + \frac{3}{8}(h^2)^{\mu}_{a} - \frac{5}{16}(h^3)^{\mu}_{a} + \dots \\
e^{a}_{\mu} &= \delta^{a}_{\mu} + \frac{1}{2}h^{a}_{\mu} - \frac{1}{8}(h^2)^{a}_{\mu} + \frac{1}{16}(h^3)^{a}_{\mu} + \dots \\
\sqrt{|g|} &= 1 + \frac{1}{2}(\mathrm{tr\,h}) + \frac{1}{8}(\mathrm{tr\,h})^2 - \frac{1}{4}(\mathrm{tr\,h^2}) - \frac{1}{8}(\mathrm{tr\,h})(\mathrm{tr\,h^2}) + \frac{1}{48}(\mathrm{trh})^3 + \frac{1}{6}(\mathrm{trh^3}) + \dots
\end{aligned}
\tag{3.13}
$$

and

$$
\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu h^{\lambda}_{\nu} + \partial_\nu h^{\lambda}_{\mu} - \partial^{\lambda} h_{\mu\nu}\right) - \frac{1}{2}(h - h^2)^{\lambda\rho}\left(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu}\right) \tag{3.14}
$$

In this approximation the spin connection is

$$
\begin{aligned}
\omega^{ab}_{\mu} = {} & \frac{1}{2}\left(\partial^b h^a_\mu - \partial^a h^b_\mu\right) + \frac{1}{4}\left(h^{\sigma a}\partial_\sigma h^b_\mu - h^{\sigma b}\partial_\sigma h^a_\mu + h^{b\sigma}\partial^a h_{\sigma\mu} - h^{a\sigma}\partial^b h_{\sigma\mu}\right) \\
& -\frac{1}{8}\left(h^{a\sigma}\partial_\mu h^b_\sigma - h^{b\sigma}\partial_\mu h^a_\sigma\right) \\
& +\frac{1}{8}\left((h^2)^{a\lambda}\partial_\mu h^b_\lambda - (h^2)^{b\lambda}\partial_\mu h^a_\lambda\right) + \frac{3}{16}\left((h^2)^{a\lambda}\partial^b h_{\mu\lambda} - (h^2)^{b\lambda}\partial^a h_{\mu\lambda}\right) \\
& -\frac{3}{16}\left((h^2)^{a\lambda}\partial_\lambda h^b_\mu - (h^2)^{b\lambda}\partial_\lambda h^a_\mu\right) + \frac{1}{8}\left(h^{a\rho}h^{b\lambda} - h^{b\rho}h^{a\lambda}\right)\partial_\lambda h_{\mu\rho} + \dots
\end{aligned}
\tag{3.15}
$$

Up to third order in h the action is

$$\begin{aligned}
S &= \int d^4x \Big[\frac{i}{2}\overline{\psi_L}\gamma^m \overleftrightarrow{\partial}_m\psi_L - \frac{i}{4}\overline{\psi_L}h_a^m\gamma^a\overleftrightarrow{\partial}_m\psi_L + \frac{3i}{16}\overline{\psi_L}(h^2)_a^m\gamma^a\overleftrightarrow{\partial}_m\psi_L - \frac{5i}{32}\overline{\psi_L}(h^3)_a^m\gamma^a\overleftrightarrow{\partial}_m\psi_L \\
&\quad -\frac{1}{16}\epsilon^{mabc}\overline{\psi_L}\gamma_c\gamma_5\psi_L\left(h_m^\sigma\partial_a h_{b\sigma} + (h^2)_m^\sigma\partial_b h_{a\sigma} - h_m^\rho h_a^\sigma\partial_\sigma h_{\rho b} - \frac{1}{2}h_m^\rho\partial_a h_{\rho\sigma}h_c^\sigma\right) \qquad (3.16)\\
&\quad +\frac{1}{2}(\mathrm{tr h})\left(\frac{\mathrm{i}}{2}\overline{\psi_\mathrm{L}}\gamma^\mathrm{m}\overleftrightarrow{\partial}_\mathrm{m}\psi_\mathrm{L} - \frac{\mathrm{i}}{4}\overline{\psi_\mathrm{L}}\mathrm{h}_\mathrm{a}^\mathrm{m}\gamma^\mathrm{a}\overleftrightarrow{\partial}_\mathrm{m}\psi_\mathrm{L} + \frac{3\mathrm{i}}{16}\overline{\psi_\mathrm{L}}(\mathrm{h}^2)_\mathrm{a}^\mathrm{m}\gamma^\mathrm{a}\overleftrightarrow{\partial}_\mathrm{m}\psi_\mathrm{L}\right.\\
&\qquad \left. -\frac{1}{16}\,\epsilon^{mabc}\overline{\psi_L}\gamma_c\gamma_5\psi_L h_m^\sigma\partial_a h_{b\sigma}\right)\\
&\quad +\left(\frac{1}{8}(\mathrm{tr\,h})^2 - \frac{1}{4}(\mathrm{tr\,h}^2)\right)\left(\frac{i}{2}\overline{\psi_L}\gamma^m\overleftrightarrow{\partial}_m\psi_L - \frac{i}{4}\overline{\psi_L}h_a^m\gamma^a\overleftrightarrow{\partial}_m\psi_L\right)\\
&\quad +\left(-\frac{1}{8}(\mathrm{tr\,h})(\mathrm{tr\,h}^2) + \frac{1}{48}(\mathrm{trh})^3 + \frac{1}{6}(\mathrm{trh}^3)\right)\frac{i}{2}\overline{\psi_L}\gamma^m\overleftrightarrow{\partial}_m\psi_L + \dots\Big]
\end{aligned}$$

The propagator (3.9) comes from the first term of the first line in the RHS of (3.16). The vertex V_{ffh} comes from the second term, while V^ϵ_{ffhh} originates from the first term in the second line of (3.16). There are many other vertices of the type $V_{ffh}, V_{ffhh}, V_{ffhhh}$. It is important to single out which may be relevant to trace anomalies.

The Ward identity for Weyl invariance, in absence of anomalies, is:

$$\mathfrak{T}(x) \equiv g_{\mu\nu}(x)\langle\!\langle T^{\mu\nu}(x)\rangle\!\rangle = \langle\!\langle T_\mu^\mu(x)\rangle\!\rangle + h_{\mu\nu}(x)\langle\!\langle T^{\mu\nu}(x)\rangle\!\rangle = 0 \qquad (3.17)$$

Writing

$$\begin{aligned}
\langle\!\langle T^{\mu\nu}(x)\rangle\!\rangle &= \langle 0|T^{\mu\nu}_{(0)}(x)|0\rangle \qquad (3.18)\\
&\quad + \sum_{n=1}^\infty \frac{1}{2^n n!}\int\prod_{i=0}^n dx_i\, h_{\mu_1\nu_1}(x_1)\dots h_{\mu_n\nu_n}(x_n)\mathcal{T}^{\mu\nu\mu_1\nu_1\dots\mu_n\nu_n}(x,x_1,\dots,x_n),
\end{aligned}$$

order by order in h, eq.(3.17) breaks down to

$$\begin{aligned}
\mathfrak{T}^{(0)}(x) &\equiv \langle 0|T_{(0)\mu}{}^\mu(x)|0\rangle = 0 \qquad (3.19)\\
\mathfrak{T}^{(1)}(x) &\equiv \mathcal{T}_\mu^{\mu\mu_1\nu_1}(x,x_1) + 2\delta(x-x_1)\langle 0|T^{\mu_1\nu_1}_{(0)}(x)|0\rangle = 0 \qquad (3.20)\\
\mathfrak{T}^{(2)}(x) &\equiv \mathcal{T}_\mu^{\mu\mu_1\nu_1\mu_2\nu_2}(x,x_1,x_2) + 2\delta(x-x_1)\mathcal{T}^{\mu_1\nu_1\mu_2\nu_2}(x,x_2)\\
&\quad +2\delta(x-x_2)\mathcal{T}^{\mu_2\nu_2\mu_1\nu_1}(x,x_1) = 0 \qquad (3.21)
\end{aligned}$$

where

$$T^{\mu\nu}_{(0)} = 2\frac{\delta S}{\delta h_{\mu\nu}(x)}\bigg|_{h=0} = -\frac{i}{4}\left(\overline{\psi_L}\gamma^\mu \overset{\leftrightarrow}{\partial^\nu}\psi_L + \mu \leftrightarrow \nu\right) + \frac{i}{2}\eta_{\mu\nu}\,\overline{\psi_L}\gamma^m \overset{\leftrightarrow}{\partial}_m\psi_L, \tag{3.22}$$

$$\begin{aligned}\mathcal{T}^{\mu\nu\mu_1\nu_1}(x,x_1) &= i\langle 0|\mathcal{T}T^{\mu\nu}_{(0)}(x)T^{\mu_1\nu_1}_{(0)}(x_1)|0\rangle - \eta^{\mu_1\nu_1}\delta(x-x_1)\langle 0|T^{\mu\nu}_{(0)}(x)|0\rangle \\ &\quad +4\langle 0|\frac{\delta^2 S}{\delta h_{\mu\nu}(x)\delta h_{\mu_1\nu_1}(x_1)}|0\rangle \end{aligned}\tag{3.23}$$

and

$$\begin{aligned}&\mathcal{T}^{\mu\nu\mu_1\nu_1\mu_2\nu_2}(x,x_1,x_2)\\ &= -\langle 0|\mathcal{T}T^{\mu\nu}_{(0)}(x)T^{\mu_1\nu_1}_{(0)}(x_1)T^{\mu_2\nu_2}_{(0)}(x_2)|0\rangle + 4i\langle 0|\mathcal{T}T^{\mu\nu}_{(0)}(x)\frac{\delta^2 S}{\delta h_{\mu_1\nu_1}(x_1)\delta h_{\mu_2\nu_2}(x_2)}|0\rangle\\ &-i\eta^{\mu_1\nu_1}\delta(x-x_1)\langle 0|\mathcal{T}T^{\mu\nu}_{(0)}(x)T^{\mu_2\nu_2}_{(0)}(x_2)|0\rangle - i\eta^{\mu_2\nu_2}\delta(x-x_2)\langle 0|\mathcal{T}T^{\mu\nu}_{(0)}(x)T^{\mu_1\nu_1}_{(0)}(x_1)|0\rangle\\ &+4i\langle 0|\mathcal{T}T^{\mu_1\nu_1}_{(0)}(x_1)\frac{\delta^2 S}{\delta h_{\mu\nu}(x)\delta h_{\mu_2\nu_2}(x_2)}|0\rangle + 4i\langle 0|\mathcal{T}T^{\mu_2\nu_2}_{(0)}(x_2)\frac{\delta^2 S}{\delta h_{\mu_1\nu_1}(x_1)\delta h_{\mu\nu}(x)}|0\rangle\\ &+\left(\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} + \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1} + \eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\right)\delta(x-x_1)\delta(x-x_2)\langle 0|T^{\mu\nu}_{(0)}(x)|0\rangle\\ &-4\eta^{\mu_1\nu_1}\delta(x-x_1)\langle 0|\frac{\delta^2 S}{\delta h_{\mu\nu}(x)\delta h_{\mu_2\nu_2}(x_2)}|0\rangle - 4\eta^{\mu_2\nu_2}\delta(x-x_2)\langle 0|\frac{\delta^2 S}{\delta h_{\mu\nu}(x)\delta h_{\mu_1\nu_1}(x_1)}|0\rangle\\ &+8\langle 0|\frac{\delta^3 S}{\delta h_{\mu\nu}(x)\delta h_{\mu_1\nu_1}(x_1)h_{\mu_2\nu_2}(x_2)}|0\rangle\end{aligned}\tag{3.24}$$

The functional derivatives of S with respect to h are understood to be evaluated at $h = 0$.

In the sequel we will need the explicit expressions of vertices, up to order two in h (for a derivation of Feynman rules see [19]). Beside (3.10) and (3.11) we have:

$$V'_{ffh} \quad : \quad \frac{i}{4}\eta_{\mu\nu}(\not p + \not p')P_L \tag{3.25}$$

$$\begin{aligned}V'_{ffhh} \quad : \quad &\frac{3i}{64}\Big[\left((p+p')_\mu\gamma_{\mu'}\eta_{\nu\nu'} + (p+p')_\mu\gamma_{\nu'}\eta_{\nu\mu'} + \{\mu\leftrightarrow\nu\}\right)\\ &\quad + \left((p+p')_{\mu'}\gamma_\mu\eta_{\nu\nu'} + (p+p')_{\mu'}\gamma_\nu\eta_{\mu\nu'} + \{\mu'\leftrightarrow\nu'\}\right)\Big]P_L\end{aligned}\tag{3.26}$$

$$\begin{aligned}V''_{ffhh} \quad : \quad &-\frac{i}{16}\Big[\eta_{\mu\nu}\left((p+p')_{\mu'}\gamma_{\nu'} + \{\mu'\leftrightarrow\nu'\}\right)\\ &\quad +\eta_{\mu'\nu'}\left((p+p')_\mu\gamma_\nu + \{\mu\leftrightarrow\nu\}\right)\Big]P_L\end{aligned}\tag{3.27}$$

$$V'''_{ffhh} \quad : \quad \frac{i}{8}(\not p + \not p')(\eta_{\mu\nu}\eta_{\mu'\nu'} - \eta_{\mu\nu'}\eta_{\mu'\nu} - \eta_{\mu\mu'}\eta_{\nu\nu'})P_L \tag{3.28}$$

So far we have been completely general. From now on we consider only the odd part of the correlators, that is only correlators linear in $\epsilon_{\mu\nu\lambda\rho}$.

To start with, consider $\langle 0|T_{(0)\mu}{}^{\mu}(x)|0\rangle$, to which only a tadpole can contribute, but its odd part vanishes because we cannot construct a scalar using ϵ and η. For the same reason also $\langle 0|T^{\mu\nu}_{(0)}(x)|0\rangle$ vanishes.

The two-point function $\langle 0|\mathcal{T}T^{\mu\nu}_{(0)}(x)T^{\mu_1\nu_1}_{(0)}(x_1)|0\rangle$ also must vanish, because in momentum space it must be a 4-tensor linear in ϵ and formed with η and the momentum k: there is no such tensor, symmetric in $\mu \leftrightarrow \nu$, $\mu_1 \leftrightarrow \nu_1$ and $(\mu,\nu) \leftrightarrow (\mu_1,\nu_1)$.

As for the terms $\langle 0|\frac{\delta^2 S}{\delta h_{\mu\nu}(x)\delta h_{\mu_1\nu_1}(x_1)}|0\rangle$ they might also produce nonvanishing contribution from tadpoles diagram, but like in the previous case it is impossible to satisfy the combinatorics.

In conclusion (3.19) and (3.20) are identically satisfied, while (3.21) becomes

$$
\begin{aligned}
&\mathfrak{T}^{(2)}(x) = \mathcal{T}_{\mu}^{\mu\mu_1\nu_1\mu_2\nu_2}(x,x_1,x_2)\\
&= \eta_{\mu\nu}\Big(- \langle 0|\mathcal{T}T^{\mu\nu}_{(0)}(x)T^{\mu_1\nu_1}_{(0)}(x_1)T^{\mu_2\nu_2}_{(0)}(x_2)|0\rangle + 4i\langle 0|\mathcal{T}T^{\mu\nu}_{(0)}(x)\frac{\delta^2 S}{\delta h_{\mu_1\nu_1}(x_1)\delta h_{\mu_2\nu_2}(x_2)}|0\rangle\\
&+4i\langle 0|\mathcal{T}T^{\mu_1\nu_1}_{(0)}(x_1)\frac{\delta^2 S}{\delta h_{\mu\nu}(x)\delta h_{\mu_2\nu_2}(x_2)}|0\rangle + 4i\langle 0|\mathcal{T}T^{\mu_2\nu_2}_{(0)}(x_2)\frac{\delta^2 S}{\delta h_{\mu_1\nu_1}(x_1)\delta h_{\mu\nu}(x)}|0\rangle\\
&+8\langle 0|\frac{\delta^3 S}{\delta h_{\mu\nu}(x)\delta h_{\mu_1\nu_1}(x_1)h_{\mu_2\nu_2}(x_2)}|0\rangle\Big)
\end{aligned}
\tag{3.29}
$$

To proceed further, we focus now on the terms containing the second derivative of S. Looking at (3.16) we see that there are several such terms. We argue now that those among them that do not contain the ϵ tensor, although the gamma trace algebra may generate an ϵ tensor, cannot contribute to the odd trace anomaly. The vertices corresponding to such terms have two fermion and two graviton legs, that is, they are of the type V_{ffhh}. By Fourier transform, we associate an incoming e^{ipx} plane wave to one fermion and an outgoing $e^{-ip'x}$ one to the other, while we associate two incoming plane waves e^{ik_1x}, e^{ik_2x} to the two gravitons. Since none of them contain derivatives of h, the vertex will depend at most on $q = k_1 + k_2$, not on $k_1 - k_2$, see for instance the vertex coming from the third term in the first line of (3.16), i.e. $V^{'}_{ffhh}$. This being so, the contributions from the terms related to the second derivative of S in (3.29) via such vertices, and linear in ϵ, must vanish, because it is impossible to form a 4-tensor symmetric in $\mu_1 \leftrightarrow \nu_1$, $\mu_2 \leftrightarrow \nu_2$ and $(\mu_1,\nu_1) \leftrightarrow (\mu_2\nu_2)$ with ϵ, η and q_μ.

It follows that only the contribution with the vertex V^{ϵ}_{ffhh} might contribute non trivially to the odd trace anomaly. Looking at the form of V^{ϵ}_{ffhh}, it is clear that the two terms in the third line of (3.29) give vanishing contribution because the contraction of μ with ν becomes a (vanishing) contraction of the t tensor, (3.12). The second term in the second line vanishes as well, an to prove that, we have to introduce a dimensional regulator and use Feynman parametrization (for details see [19]).

Next, let us consider the fourth line of (3.29). These are seagull terms, with three external graviton lines attached to the same point of a fermion loop. The gamma trace algebra cannot generate an ϵ tensor from all such terms, except of course the second term in the second line and the one in the fourth line. Therefore we can exclude all the former from our consideration. As for the latter the relevant vertex has two fermion legs, with the usual momenta p and p', and three graviton legs, with incoming momenta k_1, k_2, k_3 and labels $\mu_1, \nu_1, \mu_2, \nu_2$ and μ_3, ν_3, respectively. Its expression for the second term in the second line of (3.29) is

$$\sim \epsilon_{\mu_2\mu_3\lambda\rho} k_3^{\lambda} \gamma^{\rho} \eta_{\mu_1\nu_3} \eta_{\nu_1\nu_2} \tag{3.30}$$

symmetrized in $\mu_1 \leftrightarrow \nu_1$, $\mu_2 \leftrightarrow \nu_2$, $\mu_3 \leftrightarrow \nu_3$, and with respect to the exchange of any two couples (μ_i, ν_i). The seagull term is therefore proportional to

$$\int d^4p \, \frac{p^{\rho}}{p^2}$$

which vanishes. As for the term in the fourth line of (3.29), one comes to similar conclusions.

In summary, the odd trace anomaly receives contributions only from

$$\begin{aligned} \mathcal{T}^{(2)}(x) &= \mathcal{T}_{\mu}^{\mu\mu_1\nu_1\mu_2\nu_2}(x, x_1, x_2) \qquad (3.31)\\ &= \eta_{\mu\nu}\Big(- \langle 0|\mathcal{T} T^{\mu\nu}_{(0)}(x) T^{\mu_1\nu_1}_{(0)}(x_1) T^{\mu_2\nu_2}_{(0)}(x_2)|0\rangle + 4i \langle 0|\mathcal{T} T^{\mu\nu}_{(0)}(x) \frac{\delta^2 S}{\delta h_{\mu_1\nu_1}(x_1)\delta h_{\mu_2\nu_2}(x_2)}|0\rangle\Big) \end{aligned}$$

This result looks very much like the starting point of [15], i.e. it seems to reduce to the same contributions, i.e. the triangle diagram and bubble diagram (which turned out to vanish), but there is an important modification: the $T^{\mu\nu}_{(0)}(x)$ is different from the free energy-momentum tensor in [15], the definition (3.22) contains an additional piece (the second). It is not hard to show that the second term in the RHS of (3.31) vanishes also when taking account of this modification. As for the three point function in the first term of (3.31)

- we obtain of course the same result as in [15] when the calculation is made with three vertices V_{ffh}: $P - V_{ffh}$-P-V_{ffh}-P-V_{ffh} (this calculation is repeated in [19]);
- it is 0 when the second or third vertices are replaced by V'_{ffh},
- and it is -4 times the result of [15] if the first vertex is replaced by V'_{ffh}, i.e. P-V'_{ffh}-P-V_{ffh}-P-V_{ffh} .
- When we replace more than one vertex V_{ffh} with V'_{ffh} we get 0.

So the overall result of (3.31) is $(1 - 4 = -3)$ times the end result for the trace anomaly in [15].

We will see below that this modification of the anomaly must be canceled in order to guarantee conservation. Let us call the lowest order integrated anomaly, obtained in [15], $\mathcal{A}_\omega = -\int \omega\, \mathcal{A}_0$. Then the new addition equals $-4\mathcal{A}_\omega$. By adding to the effective action the term $\mathcal{C} = -\int \frac{1}{2}\mathrm{trh}\, \mathcal{A}_0$ we exactly cancel this additional unwanted piece. We will verify that this counterterm cancels an analogous anomalous term in the Ward identity of the diffeomorphisms, anomalous term which is generated by the same diagram P-V'_{ffh}-P-V_{ffh}-P-V_{ffh} which is the cause of the additional term in question in the trace anomaly.

In conclusion, the only relevant term for the odd trace anomaly is the P-V_{ffh}-P-V_{ffh}-P-V_{ffh} one. This is the term computed first in [15], which gives rise to the Pontryagin anomaly. It should be remarked that in the odd trace anomaly calculation there are no contributions from tadpole and seagull terms.

3.1.2 Odd trace anomaly for Dirac and Majorana fermions

The action for a Dirac fermion is the same as in (3.16) with ψ_L everywhere replaced by the Dirac fermion ψ. In order to evaluate the odd trace anomaly we remark that an odd contribution in (3.24) can come only from the terms in (3.16) that contain the ϵ tensor. Since these terms contain γ_5, upon tracing the gamma matrix part, either they give 0 or another ϵ tensor. In the latter case they produce an even contribution to the trace anomaly, which does not concern us here. In conclusion the odd trace anomaly, in the case of a Dirac fermion, vanishes.

When the fermion are Majorana the conclusion does not change. The simplest way to see it is to use the Majorana representation for the gamma matrices. Then ψ has four real components, and the only change with respect to the Dirac case is that in the path integral we integrate over real fermion fields instead of complex ones, while all the rest remains unchanged.

3.2 Conservation of the energy-momentum tensor

As already anticipated above, trace anomalies are strictly connected with diffeomorphism anomalies. In 4d the so-called Einstein-Lorentz anomalies are absent, but there may appear other anomalous terms in the Ward identity of the diffeomorphisms. The latter together with a Weyl anomaly partner form a cocycle of the joint diff+Weyl cohomology, see [10, 11]. Usually, by adding a local counterterm to the effective action, one can restore diffeomorphism invariance. In the present case, odd parity trace anomaly, the analysis of such possible anomalies was carried out in a simplified form in [17]. In this section we wish to complete that analysis by considering also tadpoles and seagull terms.

If we take into account the tadpole and seagull terms in the conservation law one has to take into account also the VEV of the energy-momentum tensor. Let us set

$$\langle 0|T^{\mu\nu}_{(0)}(x)|0\rangle = \langle 0|T^{\mu\nu}_{(0)}(0)|0\rangle = \Theta^{\mu\nu} = A\eta^{\mu\nu} \tag{3.32}$$

The Ward identity is

$$\nabla_\mu \langle\!\langle T^{\mu\nu}(x)\rangle\!\rangle = \partial_\mu \langle\!\langle T^{\mu\nu}(x)\rangle\!\rangle + \Gamma^\mu_{\mu\lambda}\langle\!\langle T^{\lambda\nu}(x)\rangle\!\rangle + \Gamma^\nu_{\mu\lambda}\langle\!\langle T^{\mu\lambda}(x)\rangle\!\rangle = 0 \tag{3.33}$$

because $\langle\!\langle T^{\mu\nu}(x)\rangle\!\rangle \equiv \frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g_{\mu\nu}(x)}$. To first order in $h_{\mu\nu}$ we have

$$\begin{aligned} \Gamma^\nu_{\mu\lambda}(x) &\approx \frac{1}{2}\left(\partial_\mu h^\nu_\lambda + \partial_\lambda h^\nu_\mu - \partial^\nu h_{\mu\lambda}\right) \\ \Gamma^\mu_{\mu\lambda}(x) &\approx \frac{1}{2}\partial_\lambda h^\mu_\mu \end{aligned} \tag{3.34}$$

Now we use (3.18, 3.22, 3.23, 3.24). To the 0-th order in h (3.33) implies

$$\partial_\mu \langle 0|T^{\mu\nu}(x)|0\rangle = 0 \tag{3.35}$$

To get the WI to first order one must differentiate (3.33) with respect to $h_{\mu\nu}$. One has

$$\frac{\delta h_{\mu\nu}(x)}{\delta h_{\lambda\rho}(y)} = \frac{1}{2}\left(\delta^\lambda_\mu \delta^\rho_\nu + \delta^\lambda_\nu \delta^\rho_\mu\right)\delta(x-y) \tag{3.36}$$

Differentiating the first term on the RHS of (3.33) one gets the ordinary divergence of the two-point function. Then

$$\begin{aligned}
\frac{\delta \Gamma^\mu_{\mu\lambda}(x)}{\delta h_{\mu_1\nu_1}(y)} &= \frac{1}{2}\eta^{\mu_1\nu_1}\partial^x_\lambda \delta(x-y) &(3.37)\\
\frac{\delta \Gamma^\nu_{\mu\lambda}(x)}{\delta h_{\mu_1\nu_1}(y)} &= \frac{1}{4}\left(\partial_\mu \delta(x-y)\ (\delta^{\nu_1}_\lambda \eta^{\mu_1\nu} + \delta^{\mu_1}_\lambda \eta^{\nu_1\nu}) + \partial_\lambda \delta(x-y)\ \left(\delta^{\mu_1}_\mu \eta^{\nu\nu_1} + \delta^{\nu_1}_\mu \eta^{\nu\mu_1}\right)\right.\\
&\quad \left. -\, \partial^\nu \delta(x-y)\ \left(\delta^{\nu_1}_\lambda \delta^{\mu_1}_\mu + \delta^{\mu_1}_\lambda \delta^{\nu_1}_\mu\right)\right) &(3.38)
\end{aligned}$$

Putting everything together one finds

$$\begin{aligned}
&\partial^x_\mu \mathcal{T}^{\mu\nu\mu_1\nu_1}(x,y) + \frac{1}{2}\eta^{\mu_1\nu_1}\partial^x_\lambda \delta(x-y)\Theta^{\lambda\nu} &(3.39)\\
&+\frac{1}{2}\left(\partial^x_\lambda \delta(x-y)\eta^{\mu_1\nu}\Theta^{\lambda\nu_1} + \partial^x_\lambda \delta(x-y)\eta^{\nu_1\nu}\Theta^{\lambda\mu_1} - \partial^{x\,\nu}\delta(x-y)\Theta^{\mu_1\nu_1}\right)\\
&= i\partial^x_\mu \langle 0|\mathcal{T} T^{\mu\nu}_{(0)}(x) T^{\mu_1\nu_1}_{(0)}(y)|\rangle + 4\partial^x_\mu \langle 0|\frac{\delta^2 S}{\delta h_{\mu\nu}(x)\delta h_{\mu_1\nu_1}(y)}|0\rangle\\
&+\partial^x_\lambda \delta(x-y)\eta^{\mu_1\nu}\Theta^{\lambda\nu_1} + \partial^x_\lambda \delta(x-y)\eta^{\nu_1\nu}\Theta^{\lambda\mu_1} - \partial^{x\,\nu}\delta(x-y)\Theta^{\mu_1\nu_1} = 0.
\end{aligned}$$

We have already noted that, for what concerns the odd part, all the terms in the RHS vanish. Therefore conservation is guaranteed up to second order in h.

The order three Ward identity has a rather cumbersome expression, in particular it contains various terms linear in $\Theta^{\mu\nu}$, see equation (62) in [19]. Since they do not contribute to the odd part of the identity we drop them altogether. Furthermore, the two point functions $\langle 0|\mathcal{T} T^{\mu\nu}_{(0)}(x) T^{\lambda\rho}_{(0)}(y)|0\rangle$ cannot contribute to the odd part because the combinatorics of the ϵ and η tensor plus an external momentum does not allow it. Next the VEV's of second and third derivative of S with respect to h cannot contribute with a tadpole term: if we look at (3.16) and focus on the vertices that can give an odd parity contribution, i.e. those containing the ϵ tensor, we notice that they depend linearly on the external momenta (not on the fermion momenta); therefore, in a tadpole term, the momentum integrand can only be linear in the internal momentum p^μ, and thus vanishes.

Therefore, as far as the odd part is concerned, the remaining terms are:

$$
\begin{aligned}
&-\partial^x_\mu \langle 0|\mathcal{T} T^{\mu\nu}_{(0)}(x) T^{\mu_1\nu_1}_{(0)}(x_1) T^{\mu_2\nu_2}_{(0)}(x_2)|0\rangle + 4i\partial^x_\mu \langle 0|\mathcal{T} T^{\mu\nu}_{(0)}(x) \frac{\delta^2 S}{\delta h_{\mu_1\nu_1}(x_1)\delta h_{\mu_2\nu_2}(x_2)}|0\rangle \\
&+4i\partial^x_\mu \langle 0|\mathcal{T} T^{\mu_2\nu_2}_{(0)}(x_2) \frac{\delta^2 S}{\delta h_{\mu\nu}(x)\delta h_{\mu_1\nu_1}(x_1)}|0\rangle + 4i\partial^x_\mu \langle 0|\mathcal{T} T^{\mu_1\nu_1}_{(0)}(x_1) \frac{\delta^2 S}{\delta h_{\mu\nu}(x)\delta h_{\mu_2\nu_2}(x_2)}|0\rangle \\
&= 0. \qquad (3.40)
\end{aligned}
$$

The last three terms on the LHS can be shown to vanish. The proof is not as simple as the previous ones. One has to push the calculations one step further, introduce a dimensional regulator and use Feynman parametrization (see [19] for details). The integration over the relevant parameter can easily be shown to vanish. What remains to be verified is therefore

$$
\partial^x_\mu \langle 0|\mathcal{T} T^{\mu\nu}_{(0)}(x) T^{\mu_1\nu_1}_{(0)}(x_1) T^{\mu_2\nu_2}_{(0)}(x_2)|0\rangle = 0. \qquad (3.41)
$$

Let us consider the term generated by the diagram $P - V'_{ffh} - P - V_{ffh} - P - V_{ffh}$. We have already calculated it above, it equals $-\partial^x_\nu \mathcal{A}(x)$, where $\mathcal{A}(x)$ is the unintegrated Weyl anomaly calculated in [15]. So conservation is violated by this term. Adding to the action the term $\mathcal{C} = -\int \frac{1}{2}\mathrm{trh}\, \omega\, \mathcal{A}_0$, as we have anticipated above, we get the diff variation

$$
\delta_\xi \mathcal{C} = -\int \partial_\nu \xi^\nu\, \mathcal{A} = \int \xi^\nu \partial_\nu \mathcal{A} \qquad (3.42)
$$

which exactly cancels this anomaly[2].

Next we have to consider the diagram $P - V_{ffh} - P - V'_{ffh} - P - V_{ffh}$ and $P - V_{ffh} - P - V_{ffh} - P - V'_{ffh}$. In the on-shell case, $k_1^2 = 0 = k_2^2$, these contributions can be shown to vanish. It is enough to take formula (3.18) of [15]. The first diagram corresponds to contracting this formula with k_1^μ or k_1^ν. It is easy to see that such a contraction vanishes. The second diagram corresponds to contracting the same formula with $k_2^{\mu'}$ or $k_2^{\nu'}$, which again vanishes. Therefore, at least in the on-shell case these diagrams do not contribute.

In conclusion we have to verify (3.41) for the triangle diagram $P - V_{ffh} - P - V_{ffh} - P - V_{ffh}$ (and the crossed one). This is what was already done in [15, 17].

[2] Concerning the signs remember that there is a relative - sign between the unintegrated Diff and trace anomalies

3.2.1 On-shell, off-shell and locality

In [15, 17] the following integrals were used in order to compute the relevant Feynman diagram

$$\int \frac{d^4p}{(2\pi)^4}\int \frac{d^\delta \ell}{(2\pi)^\delta}\frac{p^2}{(p^2+\ell^2+\Delta)^3} = \frac{1}{(4\pi)^2}\left(-\frac{2}{\delta}-\gamma+\log(4\pi)-\log\Delta\right)$$
$$\int \frac{d^4p}{(2\pi)^4}\int \frac{d^\delta \ell}{(2\pi)^\delta}\frac{p^4}{(p^2+\ell^2+\Delta)^3} = \frac{\Delta}{2(4\pi)^2}\left(-\frac{2}{\delta}-\gamma+4+\log(4\pi)-\log\Delta\right) \tag{3.43}$$

and

$$\int \frac{d^4p}{(2\pi)^4}\int \frac{d^\delta \ell}{(2\pi)^\delta}\frac{\ell^2}{(p^2+\ell^2+\Delta)^3} = -\frac{1}{2(4\pi)^2}$$
$$\int \frac{d^4p}{(2\pi)^4}\int \frac{d^\delta \ell}{(2\pi)^\delta}\frac{\ell^2 p^2}{(p^2+\ell^2+\Delta)^3} = \frac{1}{(4\pi)^2}\Delta \tag{3.44}$$

where $\Delta = u(1-u)k_1^2+v(1-v)k_2^2+2uv\,k_1k_2$, u, v are Feynman parameters, and δ is the dimensional regulator: $d = 4+\delta$.

The odd trace anomaly comes from the term

$$-\frac{1}{128}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^\delta \ell}{(2\pi)^\delta}\,\mathrm{tr}\left(\frac{\not p+\not\ell}{\mathrm{p}^2-\ell^2}(2\mathrm{p}-\mathrm{k}_1)_\lambda\gamma_\rho\right.$$
$$\left.\times\ \frac{\not p+\not\ell-\not k_1}{(p-k_1)^2-\ell^2}(2p-2k_1-k_2)_\alpha\gamma_\beta\frac{\not p+\not\ell-\not q}{(p-q)^2-\ell^2}\,\not\ell\,\frac{\gamma_5}{2}\right) \tag{3.45}$$

see also [15, 17]. This requires the two integrals (3.44), which must be further integrated on v from 0 to $1-u$ and on u from 0 to 1. The integrations over the Feynman parameters are elementary and lead to the result

$$\mathcal{T}^{\mu}_{\mu\alpha\beta\lambda\rho}(k_1,k_2) = \frac{1}{192(4\pi)^2}k_1^\sigma k_2^\tau\left(t_{\lambda\rho\alpha\beta\sigma\tau}(k_1^2+k_2^2+k_1k_2)-t^{(21)}_{\lambda\rho\alpha\beta\sigma\tau}\right) \tag{3.46}$$

We report this result here to stress the fact that the terms contained in it are contact terms and thus lead to a local anomaly. In [17] we remarked that the piece proportional to $(k_1^2+k_2^2)$ disappears on shell, and off-shell corresponds to a trivial anomaly.

To compute the conservation law (3.41) we need also the integrals (3.43). It is evident from the form of their RHS's that integrating on u and v will lead to non-contact terms, and non-local expressions for the odd diff anomaly. However if we put k_1 and k_2 on shell things change. The contact terms have been discussed in [17]. They can be eliminated by subtracting local counterterms without spoiling the trace anomaly. As for the noncontact terms they are polynomials of k_1 and k_2 multiplied by $\log k_1{\cdot}k_2$. All such terms are listed in Appendix E of [17]. They look non-local. However, using the Fourier transform

$$\begin{aligned}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}e^{i(k_1(x-z)+k_2(y-z))}\log{(k_1+k_2)^2}\\ = -\frac{1}{4\pi^2}\delta^{(4)}(x-y)\Box_z\left(\frac{1}{(x-z)^2}\log\frac{(x-z)^2}{4}\right),\end{aligned} \tag{3.47}$$

one can show that they give a vanishing contribution when inserted into the effective action, because of the on shell condition $\Box h_{\mu\nu}=0$ (De Donder gauge, see Appendix 3.A). On the other hand, when k_1 and k_2 are off shell, the anomaly looks nonlocal. This is a surprise because we are used to think of anomalies as local expressions. But we have learned from [29] and from the higher spins analysis that when higher spins are involved (including the metric) covariance generally requires to sacrifice locality. However the ensuing non-locality is a gauge artifact. By imposing a suitable gauge choice, locality can be restored. As an example see eq.(8.21) and others in [29].

3.3 Comments on the Pontryagin trace anomaly

Let us add some comments on the Pontryagin trace anomaly. A non-trivial property is that it belongs to the family of chiral anomalies characterized by having opposite coefficients for opposite chiralities - split anomalies. This anomaly did not appear for the first time in [15]. The possibility of its existence due to its Wess-Zumino consistency was pointed out in [12] and, although somewhat implicitly, its existence was implied by [7]. A similar anomaly was found in a different contest (originating from an antisymmetric tensor field) in the framework of an AdS/CFT in [82, 83], where a possible conflict with unitarity was pointed out. The same risk has been pointed out, from a different viewpoint, in [15]. In general it seems that its presence signals some kind of difficulty in properly defining the theory. Very likely for this reason the existence of the Pontryagin trace anomaly for chiral fermions is still considered controversial and objections have been raised against it. Such objections are often reducible to the credence that Weyl fermions are equivalent to massless Majorana fermions.

One more important observation is that in conformal field theory in 4d the three-point functions of the energy-momentum tensor cannot have an odd part, so how can an anomaly arise from the regularization of a vanishing bare correlator? The answer to this question is given in [17]: an anomaly can arise as a simple quantum effect; we have shown other examples of correlators which do not arise from the regularization of nonvanishing bare correlators, [18]. The crucial criterion is consistency.

Finally, we have stressed above that the crucial ingredient in the anomalies computation is the functional integral measure and we have also pointed out the issues connected with the latter for chiral fermions. Here we used a Feynman diagram technique, assuming that it reproduces the correct path integral measure. Although this must be the case, because the relevant Feynman diagrams (with chiral propagators and chiral vertices) are different from those for Dirac or Majorana fermions, it is fair to say that we do not have a direct proof of it. However, there is a way to avoid any residual doubts concerning the path integral measure. It relies in the analogue of the method used by Bardeen, [86], for chiral gauge anomalies, see also [54]. In such an approach one uses Dirac fermions (and, consequently, the ordinary Dirac measure) and recovers the chiral fermion theory by taking a specific limit. To this approach is devoted the next chapter.

Appendices

3.A de Donder gauge

To simplify the anomaly calculation, in the section above (and in [15]) we used

$$k_1^2 = k_2^2 = 0 \tag{3.48}$$

This means that we are putting the external lines on-shell. In other words, the above equation is telling us that the external fields satisfy the EOM of gravity $R_{\mu\nu} = 0$ which in linearized form reads

$$\Box h_{\mu\nu} = \partial_\mu \partial_\lambda h^\lambda_\nu + \partial_\nu \partial_\lambda h^\lambda_\mu - \partial_\mu \partial_\nu h' \tag{3.49}$$

where h' denotes the trace of $h_{\mu\nu}$. Now, we can choose the de Donder gauge

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0 \tag{3.50}$$

which at the linearized level can be written as

$$\partial_\mu h^\mu_\lambda - \frac{1}{2}\partial_\lambda h' = 0 \tag{3.51}$$

Using the de Donder gauge, the EOM of gravity at linearized level is

$$\Box h_{\mu\nu} = 0 \tag{3.52}$$

In momentum space this becomes $k_1^2 = k_2^2 = 0$.

Chapter 4

Metric-Axial tensor gravity

In previous chapters we mentioned problems related to the path integral measure with Weyl fermions. To avoid these issues, we will rely on the method inspired by Bardeen, [86] for chiral gauge anomalies, see also [54]. In this approach the idea is to construct a model where one uses Dirac fermions (and, consequently, the ordinary Dirac measure). Transferring this technique in the context of trace anomalies for chiral fermions, requires, in addition to the usual metric $g_{\mu\nu}$, the introduction of an axial tensor $f_{\mu\nu}$. This second tensor couples axially to Dirac fermions. We call this model metric-axial gravity, or for short MAT. In this way, we are able to derive the trace anomalies for Dirac, Majorana and Weyl fermions as particular limits of the general case. This chapter is based on [19].

4.1 Bardeen's method

This section is a short review of Bardeen's method to derive gauge anomalies, [86]. This method enables us to calculate covariant and consistent anomalies in a unique model by coupling Dirac fermions to an axial potential A, in addition to the usual vector potential V. The anomalies one obtains in this way satisfy the Wess-Zumino consistency conditions, but depend on two potentials.

We consider a theory of Dirac fermions coupled to two non-Abelian (vector V_μ and axial A_μ) gauge potentials, both valued in a Lie algebra with anti-hermitean generators T^a, with $[T^a, T^b] = f^{abc}T^c$. The action is

$$S[V, A] = i \int d^4x \, \overline{\psi} \left(\partial\!\!\!/ + V\!\!\!/ + \gamma_5 A\!\!\!/ \right) \psi \tag{4.1}$$

It is invariant under two sets of gauge transformations

$$\begin{cases} V_\mu \longrightarrow V_\mu + D_{V\mu}\alpha \\ A_\mu \longrightarrow A_\mu + [A_\mu, \alpha], \\ \psi \longrightarrow (1-\alpha)\psi \end{cases} \qquad \begin{cases} V_\mu \longrightarrow V_\mu + [A_\mu, \beta] \\ A_\mu \longrightarrow A_\mu + D_{V\mu}\beta \\ \psi \longrightarrow (1+\gamma_5\beta)\psi \end{cases} \tag{4.2}$$

where $D_{V\mu} = \partial_\mu + [V_\mu, \ \cdot\ ]$ and $\alpha = \alpha^a(x)T^a, \beta = \beta^a(x)T^a$. As a consequence there are two covariantly conserved currents, $j_\mu = j^a_\mu T^a$ and $j_{5\mu} = j^a_{5\mu}T^a$, where

$$j^a_\mu = \overline{\psi}\gamma_\mu T^a \psi, \qquad j^a_{5\mu} = \overline{\psi}\gamma_\mu\gamma_5 T^a\psi \tag{4.3}$$

In the one-loop quantum theory it is impossible to preserve both conservations. The most one can do is to preserve, for instance, the vector one

$$[D^\mu_V j_\mu]^a + [A^\mu, j_{5\mu}]^a = 0 \tag{4.4}$$

while the axial conservation becomes anomalous:

$$\begin{aligned}[D^\mu_V j_{5\mu}]^a + [A^\mu, j_\mu]^a \;\; &= \;\; \frac{1}{4\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}\left[\mathrm{T^a}\left(\frac{1}{4}\mathrm{F^{\mu\nu}_V F^{\lambda\rho}_V} + \frac{1}{12}\mathrm{F^{\mu\nu}_A F^{\lambda\rho}_A} - \frac{1}{6}\mathrm{F^{\mu\nu}_V A^\lambda A^\rho}\right.\right. \\ &\quad \left.\left. - \frac{1}{6}A^\mu A^\nu F^{\lambda\rho}_V - \frac{2}{3}A^\mu F^{\nu\lambda}_A A^\rho - \frac{1}{3}A^\mu A^\nu A^\lambda A^\rho\right)\right] \end{aligned} \tag{4.5}$$

where $F^{\mu\nu}_V = \partial^\mu V^\nu - \partial^\nu V^\mu + [V^\mu, V^\nu] + [A^\mu, A^\nu]$, and $F^{\mu\nu}_A = \partial^\mu A^\nu - \partial^\nu A^\mu + [V^\mu, A^\nu] + [A^\mu, V^\nu]$. From this expression we can derive two results in particular. Setting $A_\mu = 0$ we get the covariant anomaly

$$[D^\mu_V j_{5\mu}]^a = \frac{1}{16\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}\left(\mathrm{T^a F^{\mu\nu}_V F^{\lambda\rho}_V}\right) \tag{4.6}$$

Taking the collapsing limit $V \to \frac{V}{2}, A \to \frac{V}{2}$, and adding (4.4) to (4.5) we get the consistent non-Abelian gauge anomaly

$$[D_{V\mu} j^\mu_L]^a = \frac{1}{24\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}\left[\mathrm{T^a}\partial^\mu\left(\mathrm{V^\nu}\partial^\lambda \mathrm{V^\rho} + \frac{1}{2}\mathrm{V^\nu V^\lambda V^\rho}\right)\right] \tag{4.7}$$

where $j_{L\mu} = \overline{\psi}_L\gamma_\mu\psi_L$, with $\psi_L = \frac{1+\gamma_5}{2}\psi$.

4.2 Metric-Axial-Tensor Gravity

4.2.1 Axial metric

We use the symbols $g_{\mu\nu}$, $g^{\mu\nu}$ and e_μ^a, e_a^μ in the usual sense of metric and vierbein and their inverses. Then we introduce the formal writing[1]

$$G_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu} \tag{4.8}$$

where f is a symmetric tensor. Their background values are $\eta_{\mu\nu}$ and 0, respectively, so that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad f_{\mu\nu} = k_{\mu\nu} \tag{4.9}$$

In matrix notation the inverse of G, G^{-1}, is defined by

$$G^{-1} = \hat{g} + \gamma_5 \hat{f}, \qquad G^{-1}G = 1, \qquad \hat{G}^{\mu\lambda} G_{\lambda\nu} = \delta_\nu^\mu \tag{4.10}$$

which implies

$$\hat{g} f + \hat{f} g = 0, \qquad \hat{g} g + \hat{f} f = 1. \tag{4.11}$$

That is

$$\hat{f} = -\hat{g} f g^{-1}, \qquad \hat{g} = \left(g - f g^{-1} f\right)^{-1} \tag{4.12}$$

So

$$\hat{g} = (1 - g^{-1} f g^{-1} f)^{-1} g^{-1}, \qquad \hat{f} = -(1 - g^{-1} f\, g^{-1} f)^{-1} g^{-1} f\, g^{-1} \tag{4.13}$$

[1] We use at times the suggestive terminology axial-complex for an expression like $G_{\mu\nu}$, axial-real for $g_{\mu\nu}$ and axial-imaginary for $f_{\mu\nu}$. This alludes to a geometrical interpretation, which is however not necessary to expand on in this context.

Keeping up to second order terms:

$$
\begin{aligned}
g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu}_{\lambda} h^{\lambda\nu} + \dots \\
\hat{g}^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu}_{\lambda} h^{\lambda\nu} + k^{\mu}_{\lambda} k^{\lambda\nu} + \dots \\
\hat{f}^{\mu\nu} &= -k^{\mu\nu} + h^{\mu}_{\lambda} k^{\lambda\nu} + k^{\mu}_{\lambda} h^{\lambda\nu} + \dots
\end{aligned}
\tag{4.14}
$$

4.2.2 MAT vierbein

Likewise for the vierbein one writes

$$
E^a_\mu = e^a_\mu + \gamma_5 c^a_\mu, \qquad \hat{E}^\mu_a = \hat{e}^\mu_a + \gamma_5 \hat{c}^\mu_a \tag{4.15}
$$

This implies

$$
\eta_{ab}\left(e^a_\mu e^b_\nu + c^a_\mu c^b_\nu\right) = g_{\mu\nu}, \qquad \eta_{ab}\left(e^a_\mu c^b_\nu + e^a_\nu c^b_\mu\right) = f_{\mu\nu} \tag{4.16}
$$

Moreover, from $\hat{E}^\mu_a E^a_\nu = \delta^\mu_\nu$,

$$
\hat{e}^\mu_a c^a_\nu + \hat{c}^\mu_a e^a_\nu = 0, \qquad \hat{e}^\mu_a e^a_\nu + \hat{c}^\mu_a c^a_\nu = \delta^\mu_\nu, \tag{4.17}
$$

one gets

$$
\hat{e}^\mu_a = \left(\frac{1}{1 - e^{-1}c\, e^{-1}c} e^{-1}\right)^\mu_a, \qquad \hat{c}^\mu_a = -\left(e^{-1}c \frac{1}{1 - e^{-1}c\, e^{-1}c} e^{-1}ce^{-1}\right)^\mu_a \tag{4.18}
$$

In accord with (4.9) we have

$$
\begin{aligned}
e^a_\mu &= \delta^a_\mu + \frac{1}{2}h^a_\mu - \frac{1}{8}\left(hh + kk\right)^a_\mu + \frac{1}{16}(h^3 + khk + hk^2 + k^2h)^a_\mu + \dots \\
\hat{e}^\mu_a &= \delta^\mu_a - \frac{1}{2}h^\mu_a + \frac{3}{8}\left(hh + kk\right)^\mu_a - \frac{5}{16}(h^3 + khk + hk^2 + k^2h)^\mu_a + \dots \\
c^a_\mu &= \frac{1}{2}k^a_\mu - \frac{1}{8}\left(hk + kh\right)^a_\mu + \frac{1}{16}(k^3 + hkh + h^2k + kh^2)^a_\mu + \dots \\
\hat{c}^\mu_a &= -\frac{1}{2}k^\mu_a + \frac{1}{16}\left(hk + kh\right)^\mu_a - \frac{5}{16}(k^3 + hkh + h^2k + kh^2)^\mu_a + \dots
\end{aligned}
\tag{4.19}
$$

or

$$E^a_\mu = \delta^a_\mu + \frac{1}{2}h^a_\mu - \frac{1}{8}(hh+kk)^a_\mu + \gamma_5\left(\frac{1}{2}k^a_\mu - \frac{1}{8}(hk+kh)^a_\mu\right) + \dots \tag{4.20}$$

$$\hat{E}^\mu_a = \delta^\mu_a - \frac{1}{2}h^\mu_a + \frac{3}{8}(hh+kk)^\mu_a - \gamma_5\left(\frac{1}{2}k^\mu_a - \frac{3}{8}(hk+kh)^\mu_a\right) + \dots$$

4.2.3 Christoffel and Riemann

The ordinary Christoffel symbols are

$$\gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \tag{4.21}$$

The MAT Christoffel symbols are defined in a similar way

$$\begin{aligned}\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}\hat{G}^{\lambda\rho}\left(\partial_\mu G_{\rho\nu} + \partial_\nu G_{\rho\mu} - \partial_\rho G_{\mu\nu}\right) \\ &= \frac{1}{2}\Big(\hat{g}^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) + \hat{f}^{\lambda\rho}\left(\partial_\mu f_{\rho\nu} + \partial_\nu f_{\rho\mu} - \partial_\rho f_{\mu\nu}\right)\Big) \\ &\quad + \frac{1}{2}\gamma_5\Big(\hat{g}^{\lambda\rho}\left(\partial_\mu f_{\rho\nu} + \partial_\nu f_{\rho\mu} - \partial_\rho f_{\mu\nu}\right) + \hat{f}^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)\Big) \\ &\equiv \Gamma^{(1)\lambda}_{\mu\nu} + \gamma_5\Gamma^{(2)\lambda}_{\mu\nu}\end{aligned} \tag{4.22}$$

Up to order two in h and k these become

$$\begin{aligned}\Gamma^{(1)\lambda}_{\mu\nu} &= \frac{1}{2}\Big(\partial_\mu h^\lambda_\nu + \partial_\nu h^\lambda_\mu - \partial^\lambda h_{\mu\nu} \\ &\quad - h^{\lambda\rho}\left(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}\right) - k^{\lambda\rho}\left(\partial_\mu k_{\nu\rho} + \partial_\nu k_{\mu\rho} - \partial_\rho k_{\mu\nu}\right)\Big) + \dots\end{aligned} \tag{4.23}$$

$$\begin{aligned}\Gamma^{(2)\lambda}_{\mu\nu} &= \frac{1}{2}\Big(\partial_\mu k^\lambda_\nu + \partial_\nu k^\lambda_\mu - \partial^\lambda k_{\mu\nu} \\ &\quad - h^{\lambda\rho}\left(\partial_\mu k_{\nu\rho} + \partial_\nu k_{\mu\rho} - \partial_\rho k_{\mu\nu}\right) - k^{\lambda\rho}\left(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}\right)\Big) + \dots\end{aligned} \tag{4.24}$$

Proceeding the same way one can define the MAT Riemann tensor via ${\mathcal{R}_{\mu\nu\lambda}}^\rho$:

$$\begin{aligned}{\mathcal{R}_{\mu\nu\lambda}}^\rho &= -\partial_\mu\Gamma^\rho_{\nu\lambda} + \partial_\nu\Gamma^\rho_{\mu\lambda} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} + \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda} \\ &= -\partial_\mu\Gamma^{(1)\rho}_{\nu\lambda} + \partial_\nu\Gamma^{(1)\rho}_{\mu\lambda} - \Gamma^{(1)\rho}_{\mu\sigma}\Gamma^{(1)\sigma}_{\nu\lambda} + \Gamma^{(1)\rho}_{\nu\sigma}\Gamma^{(1)\sigma}_{\mu\lambda} - \Gamma^{(2)\rho}_{\mu\sigma}\Gamma^{(2)\sigma}_{\nu\lambda} + \Gamma^{(2)\rho}_{\nu\sigma}\Gamma^{(2)\sigma}_{\mu\lambda} \\ &\quad + \gamma_5\Big(-\partial_\mu\Gamma^{(2)\rho}_{\nu\lambda} + \partial_\nu\Gamma^{(2)\rho}_{\mu\lambda} - \Gamma^{(1)\rho}_{\mu\sigma}\Gamma^{(2)\sigma}_{\nu\lambda} + \Gamma^{(1)\rho}_{\nu\sigma}\Gamma^{(2)\sigma}_{\mu\lambda} - \Gamma^{(2)\rho}_{\mu\sigma}\Gamma^{(1)\sigma}_{\nu\lambda} + \Gamma^{(2)\rho}_{\nu\sigma}\Gamma^{(1)\sigma}_{\mu\lambda}\Big) \\ &\equiv {\mathcal{R}^{(1)}_{\mu\nu\lambda}}^\rho + \gamma_5{\mathcal{R}^{(2)}_{\mu\nu\lambda}}^\rho\end{aligned} \tag{4.25}$$

The MAT spin connection is introduced in analogy

$$\Omega_\mu^{ab} = E_\nu^a \left(\partial_\mu \hat{E}^{\nu b} + \hat{E}^{\sigma b}\Gamma_{\sigma\mu}^\nu\right) = \Omega_\mu^{(1)ab} + \gamma_5 \Omega_\mu^{(2)ab} \tag{4.26}$$

where

$$\Omega_\mu^{(1)ab} = e_\nu^a \left(\partial_\mu \hat{e}^{\nu b} + \hat{e}^{\sigma b}\Gamma_{\sigma\mu}^{(1)\nu} + \hat{c}^{b\sigma}\Gamma_{\sigma\mu}^{(2)\nu}\right) + c_\nu^a \left(\partial_\mu \hat{c}^{\nu b} + \hat{e}^{\sigma b}\Gamma_{\sigma\mu}^{(2)\nu} + \hat{c}^{b\sigma}\Gamma_{\sigma\mu}^{(1)\nu}\right) \tag{4.27}$$

$$\Omega_\mu^{(2)ab} = e_\nu^a \left(\partial_\mu \hat{c}^{\nu b} + \hat{e}^{\sigma b}\Gamma_{\sigma\mu}^{(2)\nu} + \hat{c}^{b\sigma}\Gamma_{\sigma\mu}^{(1)\nu}\right) + c_\nu^a \left(\partial_\mu \hat{e}^{\nu b} + \hat{e}^{\sigma b}\Gamma_{\sigma\mu}^{(1)\nu} + \hat{c}^{b\sigma}\Gamma_{\sigma\mu}^{(2)\nu}\right) \tag{4.28}$$

4.2.4 Transformations. Diffeomorphisms

Under diffeomorphisms, $\delta x^\mu = \xi^\mu$, the Christoffel symbols transform as tensors except for one non-covariant piece

$$\delta_\xi^{(n.c.)} \gamma_{\mu\nu}^\lambda = \partial_\mu \partial_\nu \xi^\lambda \tag{4.29}$$

The same happens for the MAT Christoffel symbols

$$\delta_\xi^{(n.c.)} \Gamma_{\mu\nu}^\lambda = \partial_\mu \partial_\nu \xi^\lambda \tag{4.30}$$

This means in particular that $\Gamma_{\mu\nu}^{(2)\lambda}$ is a tensor.

It is more convenient to introduce also axial diffeomorphisms and use the following compact notation. The axially-extended (AE) diffeomorphisms are defined by

$$x^\mu \to x^\mu + \Xi^\mu, \qquad \Xi^\mu = \xi^\mu + \gamma_5 \zeta^\mu \tag{4.31}$$

Since operationally these transformations act the same way as the usual diffeomorphisms, it is easy to obtain for the non-covariant part

$$\delta^{(n.c.)} \Gamma_{\mu\nu}^\lambda = \partial_\mu \partial_\nu \Xi^\lambda \tag{4.32}$$

We can also write

$$\delta_\Xi G_{\mu\nu} = \mathcal{D}_\mu \Xi_\nu + \mathcal{D}_\nu \Xi_\mu \tag{4.33}$$

where $\Xi_\mu = G_{\mu\nu}\Xi^\nu$.

In components one easily finds

$$\begin{aligned}
\delta_\xi g_{\mu\nu} &= \xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\lambda\mu} \qquad (4.34)\\
\delta_\xi f_{\mu\nu} &= \xi^\lambda \partial_\lambda f_{\mu\nu} + \partial_\mu \xi^\lambda f_{\lambda\nu} + \partial_\nu \xi^\lambda f_{\lambda\mu}\\
\delta_\zeta g_{\mu\nu} &= \zeta^\lambda \partial_\lambda f_{\mu\nu} + \partial_\mu \zeta^\lambda f_{\lambda\nu} + \partial_\nu \zeta^\lambda f_{\lambda\mu} \qquad (4.35)\\
\delta_\zeta f_{\mu\nu} &= \zeta^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \zeta^\lambda g_{\lambda\nu} + \partial_\nu \zeta^\lambda g_{\lambda\mu}
\end{aligned}$$

Summarizing

$$\begin{aligned}
&\delta_\xi^{(n.c.)} \Gamma_{\mu\nu}^{(1)\lambda} = \partial_\mu \partial_\nu \xi^\lambda, \qquad \delta_\xi^{(n.c.)} \Gamma_{\mu\nu}^{(2)\lambda} = 0\\
&\delta_\zeta^{(n.c.)} \Gamma_{\mu\nu}^{(1)\lambda} = 0, \qquad \delta_\zeta^{(n.c.)} \Gamma_{\mu\nu}^{(2)\lambda} = \partial_\mu \partial_\nu \zeta^\lambda
\end{aligned} \qquad (4.36)$$

and the overall Riemann and Ricci tensors are tensor, and the Ricci scalar $\mathcal{R}$ is a scalar. But also $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$, separately, have the same tensorial properties.

4.2.5 Transformations. Weyl transformations

There are two types of Weyl transformations. The first is the obvious one

$$G_{\mu\nu} \longrightarrow e^{2\omega} G_{\mu\nu}, \qquad \hat{G}^{\mu\nu} \to e^{-2\omega} \hat{G}^{\mu\nu} \qquad (4.37)$$

and

$$E_\mu^a \longrightarrow e^\omega E_\mu^a, \qquad \hat{E}_a^\mu \to e^{-\omega} \hat{E}_a^\mu \qquad (4.38)$$

This leads to the usual relations

$$\Gamma_{\mu\nu}^\lambda \longrightarrow \Gamma_{\mu\nu}^\lambda + \partial_\mu \omega\, \delta_\nu^\lambda + \partial_\nu \omega\, \delta_\mu^\lambda - \partial_\rho \omega\, \hat{G}^{\lambda\rho} G_{\mu\nu} \qquad (4.39)$$

and

$$\Omega_\mu^{ab} \longrightarrow \Omega_\mu^{ab} + \left(E_\mu^a \hat{E}^{\sigma b} - E_\mu^b \hat{E}^{\sigma a} \right) \partial_\sigma \omega \qquad (4.40)$$

For infinitesimal ω this implies

$$\delta_\omega g_{\mu\nu} = 2\omega\, g_{\mu\nu}, \qquad \delta_\omega f_{\mu\nu} = 2\omega\, f_{\mu\nu} \tag{4.41}$$
$$\delta^{(0)}_\omega h_{\mu\nu} = 2\omega\eta_{\mu\nu}, \qquad \delta^{(1)}_\omega h_{\mu\nu} = 2\omega h_{\mu\nu}, \ldots$$
$$\delta^{(0)}_\omega k_{\mu\nu} = 0, \qquad \delta^{(1)}_\omega k_{\mu\nu} = 2\omega k_{\mu\nu}, \ldots$$

The second type of Weyl transformation is the axial one

$$G_{\mu\nu} \longrightarrow e^{2\gamma_5\eta} G_{\mu\nu}, \qquad \hat{G}^{\mu\nu} \to e^{-2\gamma_5\eta}\hat{G}^{\mu\nu} \tag{4.42}$$

and

$$E^a_\mu \longrightarrow e^{\gamma_5\eta} E^a_\mu, \qquad \hat{E}^\mu_a \to e^{-\gamma_5\eta}\hat{E}^\mu_a \tag{4.43}$$

This leads to

$$\Gamma^\lambda_{\mu\nu} \longrightarrow \Gamma^\lambda_{\mu\nu} + \gamma_5 \left(\partial_\mu\eta\, \delta^\lambda_\nu + \partial_\nu\eta\, \delta^\lambda_\mu - \partial_\rho\eta\, \hat{G}^{\lambda\rho} G_{\mu\nu}\right) \tag{4.44}$$

and

$$\Omega^{ab}_\mu \longrightarrow \Omega^{ab}_\mu + \gamma_5 \left(E^a_\mu \hat{E}^{\sigma b} - E^b_\mu \hat{E}^{\sigma a}\right) \partial_\sigma\eta \tag{4.45}$$

Eq.(4.42) implies

$$g_{\mu\nu} \longrightarrow \cosh(2\eta)\, g_{\mu\nu} + \sinh(2\eta)\, f_{\mu\nu}, \qquad f_{\mu\nu} \longrightarrow \cosh(2\eta)\, f_{\mu\nu} + \sinh(2\eta)\, g_{\mu\nu} \tag{4.46}$$

which, for infinitesimal η becomes

$$\delta_\eta g_{\mu\nu} = 2\eta\, f_{\mu\nu}, \qquad \delta^{(0)}_\eta h_{\mu\nu} = 0, \qquad \delta^{(1)}_\eta h_{\mu\nu} = 2\eta\, k_{\mu\nu}, \quad \ldots$$
$$\delta_\eta f_{\mu\nu} = 2\eta\, g_{\mu\nu}, \qquad \delta^{(0)}_\eta k_{\mu\nu} = 2\eta\, \eta_{\mu\nu}, \qquad \delta^{(1)}_\eta k_{\mu\nu} = 2\eta\, h_{\mu\nu}, \quad \ldots \tag{4.47}$$

4.2.6 Volume density

The ordinary density $\sqrt{|g|}$ is replaced with

$$\sqrt{|G|} = \sqrt{\det(\mathrm{G})} = \sqrt{\det(\mathrm{g} + \gamma_5 \mathrm{f})} \tag{4.48}$$

The expression in the RHS has to be understood as a formal Taylor expansion in terms of the axial-complex variable $g + \gamma_5 f$. This means

$$\begin{aligned} \mathrm{tr}\ln(\mathrm{g}+\gamma_5\mathrm{f}) &= \mathrm{tr}\ln\mathrm{g} + \mathrm{tr}\ln\left(1+\gamma_5(\mathrm{g}^{-1}\mathrm{f})\right) \\ &= \mathrm{tr}\ln\mathrm{g} + \frac{1}{2}\mathrm{tr}\ln\left(1-(\mathrm{g}^{-1}\mathrm{f})^2\right) + \gamma_5\,\mathrm{tr}\,\mathrm{arcth}(\mathrm{g}^{-1}\mathrm{f}) \\ &= \frac{1+\gamma_5}{2}\mathrm{tr}\ln(\mathrm{g}+\mathrm{f}) + \frac{1-\gamma_5}{2}\mathrm{tr}\ln(\mathrm{g}-\mathrm{f}) \end{aligned} \tag{4.49}$$

It follows that

$$\begin{aligned} \sqrt{|G|} &= e^{\frac{1}{2}\mathrm{tr}\ln(\mathrm{g}+\gamma_5\mathrm{f})} = e^{\frac{1}{2}\left(\frac{1+\gamma_5}{2}\mathrm{tr}\ln(\mathrm{g}+\mathrm{f}) + \frac{1-\gamma_5}{2}\mathrm{tr}\ln(\mathrm{g}-\mathrm{f})\right)} \\ &= \frac{1}{2}\left(\sqrt{\det(\mathrm{g}+\mathrm{f})} + \sqrt{\det(\mathrm{g}-\mathrm{f})}\right) + \frac{\gamma_5}{2}\left(\sqrt{\det(\mathrm{g}+\mathrm{f})} - \sqrt{\det(\mathrm{g}-\mathrm{f})}\right) \end{aligned} \tag{4.50}$$

$\sqrt{|G|}$ has the basic property that, under diffeomorphisms,

$$\delta_\xi \sqrt{|G|} = \xi^\lambda \partial_\lambda \sqrt{|G|} + \sqrt{|G|}\,\partial_\lambda \xi^\lambda \tag{4.51}$$

This is a volume density, and has the following properties

$$\sqrt{|G|} \to e^{4\omega}\sqrt{|G|}, \sqrt{|G|} \to e^{4\eta\gamma_5}\sqrt{|G|}, \tag{4.52}$$

under Weyl and axial-Weyl transformations, respectively. Moreover

$$\frac{1}{\sqrt{|G|}}\partial_\nu\sqrt{|G|} = \frac{1}{2}\hat{G}^{\mu\lambda}\partial_\nu G_{\mu\lambda} = \Gamma^\mu_{\mu\nu} \tag{4.53}$$

4.3 Axial fermion theories

From the above it is evident that the action for a fermion field in interaction with MAT cannot be written in the classical form $\int d^4x \sqrt{|g|}\, \overline{\psi}\mathcal{O}\psi$, as in the case of ordinary gravity, where $\mathcal{O}$ is the usual operatorial kinetic operator in the presence of gravity, because in the MAT case $\sqrt{|G|}$ contains the γ_5 matrix. Instead, $\sqrt{|G|}$ must be inserted between $\overline{\psi}$ and ψ. Moreover we have to take into account that the kinetic operator contains a γ matrix that anticommutes with γ_5. Thus, for instance, using $\mathcal{D}_\lambda G_{\mu\nu} = 0$ and $(\mathcal{D}_\lambda + \frac{1}{2}\Omega_\lambda)E = 0$, where $\mathcal{D} = \partial + \Gamma$, one gets

$$\overline{\psi}\gamma^a \hat{E}^\mu_a \left(\partial_\mu + \frac{1}{2}\Omega_\mu\right)\psi = \overline{\psi}(\bar{D}_\mu + \frac{1}{2}\bar{\Omega}_\mu)\gamma^a \hat{E}^\mu_a \psi \tag{4.54}$$

where a bar denotes axial-complex conjugation, i.e. a sign reversal in front of each γ_5 contained in the expression, for instance $\bar{\Omega}_\mu = \Omega^{(1)}_\mu - \gamma_5\Omega^{(2)}_\mu$. The reader should be aware that, in particular, a concise notation like $\mathcal{D}_\mu\gamma^\lambda$ is ambiguous. The MAT fermion action is now

$$\begin{aligned} S &= \int d^4x\, i\overline{\psi}\sqrt{|\bar{G}|}\gamma^a \hat{E}^\mu_a \left(\partial_\mu + \frac{1}{2}\Omega_\mu\right)\psi \\ &= \int d^4x\, i\overline{\psi}\sqrt{|\bar{G}|}\gamma^a(\hat{e}^\mu_a + \gamma_5\hat{c}^\mu_a)\left(\partial_\mu + \frac{1}{2}\left(\Omega^{(1)}_\mu + \gamma_5\Omega^{(2)}_\mu\right)\right)\psi \\ &= \int d^4x\, \overline{\psi}\sqrt{|\bar{G}|}(\hat{e}^\mu_a - \gamma_5\hat{c}^\mu_a)\left[\frac{i}{2}\gamma^a \overset{\leftrightarrow}{\partial}_\mu + \frac{i}{4}\left(\gamma^a\Omega_\mu + \bar{\Omega}_\mu\gamma^a\right)\right]\psi \\ &= \int d^4x\, \overline{\psi}\sqrt{|\bar{G}|}(\hat{e}^\mu_a - \gamma_5\hat{c}^\mu_a)\left[\frac{i}{2}\gamma^a \overset{\leftrightarrow}{\partial}_\mu - \frac{1}{4}\epsilon^{abcd}\left(\Omega^{(1)}_{\mu bc}\gamma_d\gamma_5 + \Omega^{(2)}_{\mu bc}\gamma_d\right)\right]\psi \end{aligned} \tag{4.55}$$

where it is understood that ∂_μ applies only to ψ or $\overline{\psi}$, as indicated, and $\bar{G}$ denotes the axial-complex conjugate. To obtain this one must use (4.53) and (4.54).

4.3.1 Classical Ward identities

Let us consider AE diffeomorphisms first, (4.31). It is not hard to prove that the action (4.55) is invariant under these transformations. Now, define the full MAT energy-momentum tensor by means of

$$\mathbf{T}^{\mu\nu} = \frac{2}{\sqrt{|G|}}\frac{\overset{\leftarrow}{\delta} S}{\delta G_{\mu\nu}} \tag{4.56}$$

This formula needs a comment, since $\sqrt{|G|}$ contains γ_5. To give a meaning to it we understand that the operator $\frac{2}{\sqrt{|G|}}\frac{\overleftarrow{\delta}}{\delta G_{\mu\nu}}$ in the RHS acts on the operatorial expression, say $\mathcal{O}\sqrt{|G|}$, which is inside the scalar product, i.e. $\overline{\psi}\mathcal{O}\sqrt{|G|}\psi$. Moreover the functional derivative acts from the right of the action. Now the conservation law under diffeorphisms is

$$\begin{aligned} 0 = \delta_{\Xi} S &= \int \overline{\psi}\frac{\overleftarrow{\delta}\,\mathcal{O}}{\delta G_{\mu\nu}}\delta G_{\mu\nu}\psi = \int \overline{\psi}\frac{\overleftarrow{\delta}\,\mathcal{O}}{\delta G_{\mu\nu}}\left(\mathcal{D}_\mu \Xi_\nu + \mathcal{D}_\nu \Xi_\mu\right)\psi \\ &= -2\int \overline{\psi}\frac{\overleftarrow{\delta}\,\mathcal{O}}{\delta G_{\mu\nu}}\overleftarrow{\mathcal{D}}_\mu \Xi_\nu \psi \end{aligned} \tag{4.57}$$

where $\mathcal{D}$ acts (from the right) on everything except the parameter Ξ_ν. Differentiating with respect to the arbitrary parameters ξ^μ and ζ^ν we obtain two conservation laws involving the two tensors

$$T^{\mu\nu} = 2\overline{\psi}\frac{\overleftarrow{\delta}\,\mathcal{O}}{\delta G_{\mu\nu}}\psi \tag{4.58}$$

$$T_5^{\mu\nu} = 2\overline{\psi}\frac{\overleftarrow{\delta}\,\mathcal{O}}{\delta G_{\mu\nu}}\gamma_5\psi \tag{4.59}$$

At the lowest order the latter are given by eqs. (4.81),(4.82) below.

Repeating the same derivation for the axial complex Weyl transformation one can prove that, assuming for the fermion field the transformation rule

$$\psi \to e^{-\frac{3}{2}(\omega+\gamma_5\eta)}\psi, \tag{4.60}$$

(4.55) is invariant and obtain the Ward identity

$$0 = \int \overline{\psi}\frac{\overleftarrow{\delta}\,\mathcal{O}}{\delta G_{\mu\nu}} G_{\mu\nu}\,(\omega + \gamma_5\eta)\psi \tag{4.61}$$

We obtain in this way two WI's

$$T^{\mu\nu} g_{\mu\nu} + T_5^{\mu\nu} f_{\mu\nu} = 0, \tag{4.62}$$

$$T^{\mu\nu} f_{\mu\nu} + T_5^{\mu\nu} g_{\mu\nu} = 0, \tag{4.63}$$

4.3.2 A simplified version

A simplified approach to the trace anomaly calculation consists first in absorbing $\sqrt{|G|}$ in ψ by setting $\Psi = |G|^{\frac{1}{4}}\psi$ and thereby assuming the transformation properties

$$\delta_\Xi \Psi = \Xi^\mu \partial_\mu \Psi + \frac{1}{2} D_\mu \Xi^\mu \Psi \tag{4.64}$$

for AE diffeomorphisms, and

$$\delta_{\omega+\gamma_5}\Psi = e^{\frac{1}{2}\omega+\gamma_5\eta}\Psi, \tag{4.65}$$

for axial-complex Weyl transformations.

To arrive at an expanded action one uses (4.9,4.19), up to second order, and finds

$$\begin{aligned}
\Omega_\mu^{(1)ab} \;=\;& \frac{1}{2}\left(\partial^b h_\mu^a - \partial^a h_\mu^b\right) + \frac{1}{4}\left(h^{\sigma a}\partial_\sigma h_\mu^b - h^{\sigma b}\partial_\sigma h_\mu^a + h^{b\sigma}\partial^a h_{\sigma\mu} - h^{a\sigma}\partial^b h_{\sigma\mu}\right)\\
&-\frac{1}{8}\left(h^{a\sigma}\partial_\mu h_\sigma^b - h^{b\sigma}\partial_\mu h_\sigma^a\right) - \frac{1}{8}\left(k^{a\sigma}\partial_\mu k_\sigma^b - k^{b\sigma}\partial_\mu k_\sigma^a\right)\\
&+\frac{1}{4}\left(k^{\sigma a}\partial_\sigma k_\mu^b - k^{\sigma b}\partial_\sigma k_\mu^a + k^{b\sigma}\partial^a k_{\sigma\mu} - k^{a\sigma}\partial^b k_{\sigma\mu}\right) + \ldots
\end{aligned} \tag{4.66}$$

and

$$\begin{aligned}
\Omega_\mu^{(2)ab} \;=\;& \frac{1}{2}\left(\partial^b k_\mu^a - \partial^a k_\mu^b\right) + \frac{1}{4}\left(h^{\sigma a}\partial_\sigma k_\mu^b - h^{\sigma b}\partial_\sigma k_\mu^a + h^{b\sigma}\partial^a k_{\sigma\mu} - h^{a\sigma}\partial^b k_{\sigma\mu}\right)\\
&-\frac{1}{8}\left(h^{a\sigma}\partial_\mu k_\sigma^b - h^{b\sigma}\partial_\mu k_\sigma^a\right) - \frac{1}{8}\left(k^{a\sigma}\partial_\mu h_\sigma^b - k^{b\sigma}\partial_\mu h_\sigma^a\right)\\
&+\frac{1}{4}\left(k^{\sigma a}\partial_\sigma h_\mu^b - k^{\sigma b}\partial_\sigma h_\mu^b + k^{b\sigma}\partial^a h_{\sigma\mu} - k^{a\sigma}\partial^b h_{\sigma\mu}\right) + \ldots
\end{aligned} \tag{4.67}$$

In particular

$$\epsilon^{\mu abc}\Omega_{\mu ab}^{(1)} = -\frac{1}{4}\epsilon^{\mu abc}\left(h_a^\sigma \partial_b h_{\mu\sigma} + k_a^\sigma \partial_b k_{\mu\sigma}\right) + \ldots \tag{4.68}$$

$$\epsilon^{\mu abc}\Omega_{\mu ab}^{(2)} = -\frac{1}{4}\epsilon^{\mu abc}\left(h_a^\sigma \partial_b k_{\mu\sigma} + k_a^\sigma \partial_b h_{\mu\sigma}\right) + \ldots \tag{4.69}$$

Up to order two in h and k we have

$$
\begin{aligned}
S &= \int d^4x\,\overline{\psi}|\bar{G}|^{\frac{1}{4}}(\hat{e}_a^{\mu}-\gamma_5\hat{c}_a^{\mu})\left[\frac{i}{2}\gamma^a\overset{\leftrightarrow}{\partial}_{\mu}-\frac{1}{4}\epsilon^{abcd}\left(\Omega^{(1)}_{\mu bc}\gamma_d\gamma_5+\Omega^{(2)}_{\mu bc}\gamma_d\right)\right]|G|^{\frac{1}{4}}\psi \\
&= \int d^4x\left[\frac{i}{2}\overline{\Psi}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\Psi-\frac{i}{4}\overline{\Psi}(h_a^{\mu}-\gamma_5k_a^{\mu})\gamma^a\overset{\leftrightarrow}{\partial}_{\mu}\Psi\right. \\
&\quad+\frac{3i}{16}\overline{\Psi}\left((k^2)_a^{\mu}+(h^2)_a^{\mu}-\gamma_5(hk+kh)_a^{\mu}\right)\gamma^a\overset{\leftrightarrow}{\partial}_{\mu}\Psi \\
&\quad+\frac{1}{16}\epsilon^{\mu abc}\overline{\Psi}\left((h_a^{\sigma}\partial_bh_{\mu\sigma}+k_a^{\sigma}\partial_bk_{\mu\sigma})\gamma_c\gamma_5+(h_a^{\sigma}\partial_bk_{\mu\sigma}+k_a^{\sigma}\partial_bh_{\mu\sigma})\gamma_c\right)\Psi \\
&\quad\left.+\frac{1}{8}\epsilon^{abcd}\overline{\Psi}(h_a^{\mu}-\gamma_5k_a^{\mu})\left(\partial_ch_{b\mu}\gamma_d\gamma_5+\partial_ck_{b\mu}\gamma_d\right)\Psi\right]+\ldots \\
&= \int d^4x\left[\frac{i}{2}\overline{\Psi}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\Psi-\frac{i}{4}\overline{\Psi}(h_a^{\mu}-\gamma_5k_a^{\mu})\gamma^a\overset{\leftrightarrow}{\partial}_{\mu}\Psi\right. \\
&\quad+\frac{3i}{16}\overline{\Psi}\left((k^2)_a^{\mu}+(h^2)_a^{\mu}-\gamma_5(hk+kh)_a^{\mu}\right)\gamma^a\overset{\leftrightarrow}{\partial}_{\mu}\Psi \\
&\quad\left.-\frac{1}{16}\epsilon^{\mu abc}\overline{\Psi}\left((h_a^{\sigma}\partial_bh_{\mu\sigma}+k_a^{\sigma}\partial_bk_{\mu\sigma})\gamma_c\gamma_5+(h_a^{\sigma}\partial_bk_{\mu\sigma}+k_a^{\sigma}\partial_bh_{\mu\sigma})\gamma_c\right)\Psi\right]+\ldots
\end{aligned}
\tag{4.70}
$$

Here we do not report explicitly the terms cubic in h and k: they contains three powers of h and/or k multiplied by $\overline{\Psi}\gamma_{\mu}\Psi$ or $\overline{\Psi}\gamma_{\mu}\gamma_5\Psi$ and possibly by the ϵ tensor. They contain one single derivative, applied to either h,k or Ψ. These cubic terms will not affect our results.

4.3.3 Feynman rules

For a derivation of the Feynman rules in this case see [19]. The fermion propagator is

$$\frac{i}{\not{p}+i\epsilon} \tag{4.71}$$

The two-fermion-h-graviton vertex is (V_{ffh}):

$$-\frac{i}{8}\left[(p+p')_{\mu}\gamma_{\nu}+(p+p')_{\nu}\gamma_{\mu}\right] \tag{4.72}$$

The axial two-fermion-k-graviton vertex is (V_{ffk}):

$$-\frac{i}{8}\left[(p+p')_{\mu}\gamma_{\nu}+(p+p')_{\nu}\gamma_{\mu}\right]\gamma_5 \tag{4.73}$$

(p incoming, p' outgoing).

There are 6 2-fermion-2-graviton vertices:

$$
\begin{aligned}
V^{(1)}_{ffhh}: \quad & \frac{3i}{64}\Big[\left((p+p')_\mu\gamma_{\mu'}\eta_{\nu\nu'}+(p+p')_\mu\gamma_{\nu'}\eta_{\nu\mu'}+\{\mu\leftrightarrow\nu\}\right)\\
& +\left((p+p')_{\mu'}\gamma_\mu\eta_{\nu\nu'}+(p+p')_{\mu'}\gamma_\nu\eta_{\mu\nu'}+\{\mu'\leftrightarrow\nu'\}\right)\Big] \qquad (4.74)\\
V^{(2)}_{ffkk}: \quad & \frac{3i}{64}\Big[\left((p+p')_\mu\gamma_{\mu'}\eta_{\nu\nu'}+(p+p')_\mu\gamma_{\nu'}\eta_{\nu\mu'}+\{\mu\leftrightarrow\nu\}\right)\\
& +\left((p+p')_{\mu'}\gamma_\mu\eta_{\nu\nu'}+(p+p')_{\mu'}\gamma_\nu\eta_{\mu\nu'}+\{\mu'\leftrightarrow\nu'\}\right)\Big] \qquad (4.75)\\
V^{(3)}_{ffhk}: \quad & \frac{3i}{64}\Big[\left((p+p')_\mu\gamma_{\mu'}\eta_{\nu\nu'}+(p+p')_\mu\gamma_{\nu'}\eta_{\nu\mu'}+\{\mu\leftrightarrow\nu\}\right)\\
& +\left((p+p')_{\mu'}\gamma_\mu\eta_{\nu\nu'}+(p+p')_{\mu'}\gamma_\nu\eta_{\mu\nu'}+\{\mu'\leftrightarrow\nu'\}\right)\Big]\gamma_5 \qquad (4.76)\\
V^{(1)\epsilon}_{ffhh}: \quad & \frac{1}{64}\,t_{\mu\nu\mu'\nu'\kappa\lambda}\,(k-k')^\lambda\gamma^\kappa\,\gamma_5 \qquad (4.77)\\
V^{(2)\epsilon}_{ffkk}: \quad & \frac{1}{64}\,t_{\mu\nu\mu'\nu'\kappa\lambda}\,(k-k')^\lambda\gamma^\kappa\,\gamma_5 \qquad (4.78)\\
V^{(3)\epsilon}_{ffhk}: \quad & \frac{1}{64}\,t_{\mu\nu\mu'\nu'\kappa\lambda}\,(k-k')^\lambda\gamma^\kappa \qquad (4.79)
\end{aligned}
$$

where t is the tensor (3.12). The graviton momenta k, k' are incoming.

As anticipated above, we dispense from writing down the vertices with three h, k legs. For the purposes of this calculation it is possible to dispose of them with a general argument, without entering detailed calculations.

4.3.4 Trace anomalies - a simplified derivation

We will now derive the odd parity trace anomalies in the model (4.70), by considering only the triangle diagram contributions and disregarding tadpoles and seagull terms. We will justify later on this simplified procedure.

The overall effective action is

$$
\begin{aligned}
W[h,k] \;=\; & W[0]+\sum_{n,m=0}^{\infty}\frac{i^{m+n-1}}{2^{n+m}n!m!}\int\prod_{i=1}^{n}dx_i h_{\mu_i\nu_i}(x_i)\prod_{j=1}^{m}dy_j k_{\lambda_j\rho_j}(y_j)\\
& \cdot\langle 0|\mathcal{T}T^{\mu_1\nu_1}(x_1)\dots T^{\mu_n\nu_n}(x_n)T_5^{\lambda_1\rho_1}(y_1)\dots T_5^{\lambda_m\rho_m}(y_m)|0\rangle \qquad (4.80)
\end{aligned}
$$

where, in the simplified version of this section, the T operator in the time-ordered amplitudes refer to the classical ones, i.e.

$$T^{\mu\nu} \equiv T^{\mu\nu}_{(0,0)} = -\frac{i}{4}\left(\overline{\psi}\gamma^{\mu}\overset{\leftrightarrow}{\partial^{\nu}}\psi + \mu \leftrightarrow \nu\right), \tag{4.81}$$

$$T_5^{\mu\nu} \equiv T^{\mu\nu}_{5(0,0)} = \frac{i}{4}\left(\overline{\psi}\gamma_5\gamma^{\mu}\overset{\leftrightarrow}{\partial^{\nu}}\psi + \mu \leftrightarrow \nu\right), \tag{4.82}$$

The quantum Ward identities for the Weyl and axial Weyl symmetry are obtained by replacing the classical energy-momentum tensor expressions with the one-loop one-point functions in (4.62) and (4.63)

$$\mathcal{T}(x) \equiv \langle\langle T^{\mu\nu}\rangle\rangle g_{\mu\nu} + \langle\langle T_5^{\mu\nu}\rangle\rangle f_{\mu\nu} = 0, \qquad \text{i.e.} \qquad \langle\langle T^{\mu}_{\mu}\rangle\rangle + \ldots = 0 \tag{4.83}$$

$$\mathcal{T}_5(x) \equiv \langle\langle T^{\mu\nu}\rangle\rangle f_{\mu\nu} + \langle\langle T_5^{\mu\nu}\rangle\rangle g_{\mu\nu} = 0, \qquad \text{i.e.} \qquad \langle\langle T^{\mu}_{5\mu}\rangle\rangle + \ldots = 0 \tag{4.84}$$

In the present simplified setup the relevant one-loop one-point functions are

$$\begin{aligned}\langle\langle T^{\mu\nu}(x)\rangle\rangle &= \sum_{n,m=0}^{\infty}\frac{i^{m+n}}{2^{n+m}n!m!}\int\prod_{i=1}^{n}dx_i h_{\mu_i\nu_i}(x_i)\prod_{j=1}^{m}dy_j k_{\lambda_j\rho_j}(y_j)\\ &\cdot\langle 0|\mathcal{T}T^{\mu\nu}(x)T^{\mu_1\nu_1}(x_1)\ldots T^{\mu_n\nu_n}(x_n)T_5^{\lambda_1\rho_1}(y_1)\ldots T_5^{\lambda_m\rho_m}(y_m)|0\rangle \quad (4.85)\\ \langle\langle T_5^{\mu\nu}(x)\rangle\rangle &= \sum_{n,m=0}^{\infty}\frac{i^{m+n}}{2^{n+m}n!m!}\int\prod_{i=1}^{n}dx_i h_{\mu_i\nu_i}(x_i)\prod_{j=1}^{m}dy_j k_{\lambda_j\rho_j}(y_j)\\ &\cdot\langle 0|\mathcal{T}T_5^{\mu\nu}(x)T^{\mu_1\nu_1}(x_1)\ldots T^{\mu_n\nu_n}(x_n)T_5^{\lambda_1\rho_1}(y_1)\ldots T_5^{\lambda_m\rho_m}(y_m)|0\rangle \quad (4.86)\end{aligned}$$

In particular for the trace anomalies, at level $\mathcal{O}(h^2, hk, k^2)$, we have

$$\begin{aligned}\langle\langle T^{\mu}_{\mu}(x)\rangle\rangle^{(2)} &= -\frac{1}{8}\int dx_1dx_2h_{\mu_1\nu_1}(x_1)h_{\mu_2\nu_2}(x_2)\langle 0|\mathcal{T}T^{\mu}_{\mu}(x)T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)|0\rangle\\ &-\frac{1}{4}\int dx_1dyh_{\mu_1\nu_1}(x_1)k_{\lambda\rho}(y)\langle 0|\mathcal{T}T^{\mu}_{\mu}(x)T^{\mu_1\nu_1}(x_1)T_5^{\lambda\rho}(y)|0\rangle \quad (4.87)\\ &-\frac{1}{8}\int dy_1dy_2k_{\lambda_1\rho_1}(y_1)k_{\lambda_2\rho_2}(y_2)\langle 0|\mathcal{T}T^{\mu}_{\mu}(x)T_5^{\lambda_1\rho_1}(y_1)T_5^{\lambda_2\rho_2}(y_2)|0\rangle\\ \langle\langle T_{5\mu}{}^{\mu}(x)\rangle\rangle^{(2)} &= -\frac{1}{8}\int dx_1dx_2h_{\mu_1\nu_1}(x_1)h_{\mu_2\nu_2}(x_2)\langle 0|\mathcal{T}T_{5\mu}{}^{\mu}(x)T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)|0\rangle\\ &-\frac{1}{4}\int dxdyh_{\mu_1\nu_1}(x_1)k_{\lambda\rho}(y)\langle 0|\mathcal{T}T_{5\mu}{}^{\mu}(x)T^{\mu_1\nu_1}(x_1)T_5^{\lambda\rho}(y)|0\rangle \quad (4.88)\\ &-\frac{1}{8}\int dy_1dy_2k_{\lambda_1\rho_1}(y_1)k_{\lambda_2\rho_2}(y_2)\langle 0|\mathcal{T}T_{5\mu}{}^{\mu}(x)T_5^{\lambda_1\rho_1}(y_1)T_5^{\lambda_2\rho_2}(y_2)|0\rangle\end{aligned}$$

It is clear that only the terms containing an odd number of T_5 will contribute to the odd parity trace anomaly.

The three-point functions (4.87,4.88) are given by the ordinary triangle diagrams. All such diagrams give the same contribution

$$\sim \left(k_1 \cdot k_2 \, t_{\mu\nu\mu'\nu'\lambda\rho} - t^{(21)}_{\mu\nu\mu'\nu'\lambda\rho}\right) k_1^\lambda k_2^\rho \tag{4.89}$$

where

$$t^{(21)}_{\mu\nu\mu'\nu'\kappa\lambda} = k_{2\mu} k_{1\mu'} \epsilon_{\nu\nu'\kappa\lambda} + k_{2\nu} k_{1\nu'} \epsilon_{\mu\mu'\kappa\lambda} + k_{2\mu} k_{1\nu'} \epsilon_{\nu\mu'\kappa\lambda} + k_{2\nu} k_{1\mu'} \epsilon_{\mu\nu'\kappa\lambda} \tag{4.90}$$

Upon Fourier-anti-transforming and replacing in (4.87) and in (4.88) we get:

$$\begin{aligned}
\langle\!\langle T^\mu_\mu(x) \rangle\!\rangle^{(2)} &= -2N \epsilon^{\mu\nu\lambda\rho} \left(\partial_\mu \partial_\sigma h^\tau_\nu \, \partial_\lambda \partial_\tau k^\sigma_\rho - \partial_\mu \partial_\sigma h^\tau_\nu \, \partial_\lambda \partial^\sigma k_{\tau\rho}\right) && (4.91)\\
\langle\!\langle T_{5\mu}{}^\mu(x) \rangle\!\rangle^{(2)} &= -2N \Big[\frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \left(\partial_\mu \partial_\sigma h^\tau_\nu \, \partial_\lambda \partial_\tau h^\sigma_\rho - \partial_\mu \partial_\sigma h^\tau_\nu \, \partial_\lambda \partial^\sigma h_{\tau\rho}\right) && (4.92)\\
&\quad + \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \left(\partial_\mu \partial_\sigma k^\tau_\nu \, \partial_\lambda \partial_\tau k^\sigma_\rho - \partial_\mu \partial_\sigma k^\tau_\nu \, \partial_\lambda \partial^\sigma k_{\tau\rho}\right) \Big]
\end{aligned}$$

where $N = \frac{i}{768\pi^2}$ is the constant that appears in front of the Pontryagin anomaly in [15]. Covariantizing these expressions we get

$$\begin{aligned}
\Theta^\mu_\mu &\equiv \int \omega \langle\!\langle T^\mu_\mu(x) \rangle\!\rangle = N \int \omega \epsilon^{\mu\nu\lambda\rho} \mathcal{R}^{(1)\sigma\tau}_{\mu\nu} \mathcal{R}^{(2)}_{\lambda\rho\sigma\tau} && (4.93)\\
\Theta_{5\mu}{}^\mu &\equiv \int \eta \langle\!\langle T_{5\mu}{}^\mu(x) \rangle\!\rangle = \frac{N}{2} \int \eta \, \epsilon^{\mu\nu\lambda\rho} \left(\mathcal{R}^{(1)\sigma\tau}_{\mu\nu} \mathcal{R}^{(1)}_{\lambda\rho\sigma\tau} + \mathcal{R}^{(2)\sigma\tau}_{\mu\nu} \mathcal{R}^{(2)}_{\lambda\rho\sigma\tau}\right) && (4.94)
\end{aligned}$$

The important remark is now that the odd parity trace anomaly, in an ordinary theory of Weyl fermions, can be calculated using the above theory of Dirac fermions coupled to MAT gravity and setting at the end $h_{\mu\nu} \to \frac{h_{\mu\nu}}{2}, k_{\mu\nu} \to \frac{h_{\mu\nu}}{2}$ and $\omega = \eta$, for left-handed Weyl fermions, and $h_{\mu\nu} \to \frac{h_{\mu\nu}}{2}, k_{\mu\nu} \to -\frac{h_{\mu\nu}}{2}$ for right-handed ones. We will refer to these as *collapsing limits*.

4.3.5 What happens when $h_{\mu\nu} \to \frac{h_{\mu\nu}}{2}, k_{\mu\nu} \to \frac{h_{\mu\nu}}{2}$.

Let us show that in the collapsing limit $h_{\mu\nu} \to \frac{h_{\mu\nu}}{2}, k_{\mu\nu} \to \frac{h_{\mu\nu}}{2}$ we have the following results:

$$\Gamma^{(1)\lambda}_{\mu\nu} \to \frac{1}{2} \gamma^\lambda_{\mu\nu}, \qquad \Gamma^{(2)\lambda}_{\mu\nu} \to \frac{1}{2} \gamma^\lambda_{\mu\nu} \tag{4.95}$$

This is evident in the approximate expressions (4.23,4.24), but it can be proved in general. To order n in the expansion of h and k of $\Gamma^{(1)\lambda}_{\mu\nu}$ we are going to have a first term of order n in h alone, then $\begin{pmatrix} n \\ 2 \end{pmatrix}$ of order $n-2$ in h and order 2 in k, then $\begin{pmatrix} n \\ 4 \end{pmatrix}$ of order $n-4$ in h and order 4 in k, and so on, up to order $[n/2]$ in h. In the collapsing limit, all these terms collapse to the first term of order n in h divided by 2^n. In total they are

$$\sum_{k=0}^{[n/2]} \begin{pmatrix} n \\ 2k \end{pmatrix} = 2^{n-1} \tag{4.96}$$

Therefore they give the order n term in h of $\gamma^{\lambda}_{\mu\nu}$ divided by 2. A similar proof holds for $\Gamma^{(2)\lambda}_{\mu\nu}$.

Looking at the definition (4.25) of the curvatures $\mathcal{R}^{(1)}_{\mu\nu\lambda}{}^{\rho}$ and $\mathcal{R}^{(2)}_{\mu\nu\lambda}{}^{\rho}$ one easily sees that in the collapsing limit

$$\mathcal{R}^{(1)}_{\mu\nu\lambda}{}^{\rho} \to \frac{1}{2} R_{\mu\nu\lambda}{}^{\rho}, \qquad \mathcal{R}^{(2)}_{\mu\nu\lambda}{}^{\rho} \to \frac{1}{2} R_{\mu\nu\lambda}{}^{\rho}, \tag{4.97}$$

where $R_{\mu\nu\lambda}{}^{\rho}$ is the curvature of $g_{\mu\nu}$. In a similar way, using (4.66, 4.67), one can show that

$$\Omega^{(1)ab}_{\mu} \to \frac{1}{2}\omega^{ab}_{\mu}, \qquad \Omega^{(2)ab}_{\mu} \to \frac{1}{2}\omega^{ab}_{\mu} \tag{4.98}$$

Notice also that in the collapsing limit

$$\begin{aligned} g_{\mu\nu} + f_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} + k_{\mu\nu} \to g_{\mu\nu} \\ g_{\mu\nu} - f_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} - k_{\mu\nu} \to \eta_{\mu\nu} \end{aligned} \tag{4.99}$$

so that

$$\sqrt{|G|} \to \frac{1-\gamma_5}{2} + \frac{1+\gamma_5}{2}\sqrt{|g|}, \tag{4.100}$$

and

$$E^a_m \to \delta^a_m \frac{1-\gamma_5}{2} + e^a_m \frac{1+\gamma_5}{2}, \qquad \hat{E}^m_a \to \delta^m_a \frac{1-\gamma_5}{2} + \hat{e}^m_a \frac{1+\gamma_5}{2}. \tag{4.101}$$

From the above follows that the action (4.70) tends to

$$S = \int d^4x\, i\overline{\Psi}\gamma^a \hat{E}_a^m(\partial_m + \frac{1}{2}\Omega_m)\Psi \tag{4.102}$$
$$\longrightarrow \int d^4x \left[i\overline{\Psi}\gamma^m \frac{1-\gamma_5}{2}\partial_m\Psi + i\overline{\Psi}\gamma^a \hat{e}_a^m \left(\partial_m + \frac{1}{2}\omega_m\right)\frac{1+\gamma_5}{2}\Psi\right]$$

As for the opposite handedness one notices that, if $h_{\mu\nu} \to \frac{h_{\mu\nu}}{2}, k_{\mu\nu} \to -\frac{h_{\mu\nu}}{2}$, we have

$$\Omega_\mu^{(1)ab} \to \frac{1}{2}\omega_\mu^{ab}, \qquad \Omega_\mu^{(2)ab} \to -\frac{1}{2}\omega_\mu^{ab} \tag{4.103}$$

and in (4.101) the sign in front of γ_5 is reversed. Therefore the limiting action is

$$S' = \int d^4x \left[i\overline{\Psi}\gamma^a \frac{1+\gamma_5}{2}\partial_a\Psi + i\overline{\Psi}\gamma^a \hat{e}_a^m \left(\partial_m + \frac{1}{2}\omega_m\right)\frac{1-\gamma_5}{2}\Psi\right] \tag{4.104}$$

We recall that γ^a is the flat (non-dynamical) gamma matrix.

Concerning the energy-momentum tensor, from the definitions (4.58,4.59), in the collapsing limit both $T^{\mu\nu}$ and $T_5^{\mu\nu}$ become

$$T'^{\mu\nu}(x) = 4\frac{\delta S'}{\delta h_{\mu\nu}(x)} \tag{4.105}$$

As a consequence (4.83) and (4.84) collapse to the same expression

$$\mathcal{T}(x) \rightarrow \langle\langle T'^{\mu\nu}\rangle\rangle g_{\mu\nu} \equiv \mathcal{T}'(x) \tag{4.106}$$
$$\mathcal{T}_5(x) \rightarrow \langle\langle T'^{\mu\nu}\rangle\rangle g_{\mu\nu} \equiv \mathcal{T}'(x) \tag{4.107}$$

that is, there is only one trace Ward identity.

4.3.6 The Pontryagin anomaly

As pointed out above the odd parity trace anomaly in an ordinary theory of Weyl fermions can be calculated, to first order, using the above theory of Dirac fermions coupled to MAT gravity and calculating the collapsing limit of the Weyl anomaly for a Dirac fermion coupled to MAT gravity. The collapsing limit of the relevant action reproduces the action for Weyl fermions

$$S' = \int d^4x\sqrt{|g|}\left[\frac{i}{2}\overline{\psi}_L\gamma^m\overset{\leftrightarrow}{\partial}_m\psi_L - \frac{i}{4}\omega^{\mu abc}\overline{\psi_L}\gamma_c\gamma_5\psi_L\right] \tag{4.108}$$

up to a right-handed kinetic term, which is however harmless due to the presence of the P_L projector in the vertices. Inserting the replacements into either (4.93) or (4.94) we find

$$\mathcal{T}'(x) = \frac{N}{4}\,\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu}{}^{\sigma\tau}R_{\lambda\rho\sigma\tau} \tag{4.109}$$

This is not yet the correct result for one must take into account the different combinatorics in (4.80) and in

$$W[h] = W[0] + \sum_{n=0}^{\infty}\frac{i^{n-1}}{2^n n!}\int\prod_{i=1}^{n}dx_i h_{\mu_i\nu_i}(x_i)\langle 0|\mathcal{T}T^{\mu_1\nu_1}(x_1)\dots T^{\mu_n\nu_n}(x_n)|0\rangle \tag{4.110}$$

which is appropriate for (4.108)[2]. This amounts to multiplying (4.109) by a factor of 2. Therefore, finally the anomaly is

$$\mathcal{T}(x) = \frac{N}{2}\,\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu}{}^{\sigma\tau}R_{\lambda\rho\sigma\tau} \tag{4.111}$$

which is the already found Pontrygin anomaly, [15].

In the case of right-handed fermions the anomaly is the same, but with reversed sign. Thus the odd trace anomaly for Dirac fermions vanishes. This is confirmed by the following subsection.

4.3.7 Odd trace anomaly in the Dirac and Majorana case

From the results (4.93,4.94) we can draw other conclusions. The action (4.55) reduces to the usual Dirac action if we set $f_{\mu\nu} = 0$, and to the Majorana action if ψ satisfies the Majorana condition. From (4.93) we have the confirmation that the odd trace anomaly

[2]The factor $\frac{1}{2^n}$ in the RHS must be properly interpreted. When inserting the results for the n-point functions in (4.110), one should recall that the vertex (4.72) contains already a $\frac{1}{2}$ factor in it with respect to the energy-momentum tensor: symbolically we could write $V_{ffh} = \frac{1}{2}\tilde{T}$, where $\tilde{T}$ is the Fourier transform of the energy-momentum tensor with fields replaced by corresponding plane waves. A simple practical recipe is to just forget factor $\frac{1}{2^n}$ in (4.110), as was done, in [15]. The same holds also for the formula (4.80).

of these theories vanishes. But we also see that in both cases there is an anomaly in the axial energy-momentum tensor.

$$\Theta_{5\mu}{}^{\mu} = \frac{N}{2} \int \eta \, \epsilon^{\mu\nu\lambda\rho} \, R_{\mu\nu}{}^{\sigma\tau} R_{\lambda\rho\sigma\tau} \tag{4.112}$$

for the Dirac case, and $\frac{1}{2}$ of it in the Majorana case. This is a new result and it is the analog in the trace case of the Kimura-Delbourgo-Salam anomaly for the axial current.

4.4 Odd trace anomalies (the complete calculation)

Let us now justify the assumption made above, that only triangle diagrams provide a nonvanishing contribution to the odd trace anomaly. The complete calculation requires taking into account all the tadpoles and seagull terms that arise from the action (4.55).

4.4.1 Trace Ward indentity

We need to expand Ward identity (4.83, 4.84) in series of h and k. (expanded version is written down in [19]). Since we are interested only in the odd terms, we will drop all the terms that are even or vanish (the vev of $T^{\mu\nu}_{(0,0)}(x)$ and $T^{\mu\nu}_{5(0,0)}(x)$, the two-point functions of the energy-momentum tensors, as well as the vev of the second and third derivatives of S). In this way the WI's get simplified as follows

$$\begin{aligned}
\mathcal{T}_{(1,1)}(x, x_1, y_1) &\equiv \mathcal{T}_{(1,1)}{}_{\mu}^{\mu\mu_1\nu_1\lambda_1\rho_1}(x, x_1, y_1) = 0 && (4.113)\\
\mathcal{T}_{(2,0)}(x, x_1, x_2) &\equiv \mathcal{T}_{(2,0)}{}_{\mu}^{\mu\mu_1\nu_1\mu_2\nu_2}(x, x_1, x_2) = 0 && (4.114)\\
\mathcal{T}_{(0,2)}(x, y_1, y_2) &\equiv \mathcal{T}_{(0,2)}{}_{\mu}^{\mu\lambda_1\rho_1\lambda_2\rho_2}(x, y_1, y_2) = 0 && (4.115)\\
&\dots
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}_{5(1,1)}(x, x_1, y_1) &\equiv \mathcal{T}_{5(1,1)}{}_{\mu}^{\mu\mu_1\nu_1\lambda_1\rho_1}(x, x_1, y_1) = 0 && (4.116)\\
\mathcal{T}_{5(2,0)}(x, x_1, x_2) &\equiv \mathcal{T}_{5(2,0)}{}_{\mu}^{\mu\mu_1\nu_1\mu_2\nu_2}(x, x_1, x_2) = 0 && (4.117)\\
\mathcal{T}_{5(0,2)}(x, y_1, y_2) &\equiv \mathcal{T}_{5(0,2)\mu}{}^{\mu\lambda_1\rho_1\lambda_2\rho_2}(x, y_1, y_2) = 0 && (4.118)\\
&\dots
\end{aligned}$$

These are the Ward identities in the absence of anomalies, but we expect the rhs's of all these identities to be different from zero at one-loop. The odd parity anomaly can be present only in the rhs of (4.113, 4.117) and (4.118): the remaining two cannot contain the ϵ tensor linearly. After such a repeated trimming, the relevant WI for our purposes are (4.113, 4.117) and (4.118), and the terms that need to be closely scrutinized are

$$\begin{aligned}
\mathcal{T}^{\mu\nu\mu_1\nu_1\lambda_1\rho_1}_{(1,1)}(x,x_1,y_1) &= -\langle 0|\mathcal{T}T^{\mu\nu}_{(0,0)}(x)T^{\mu_1\nu_1}_{(0,0)}(x_1)T^{\lambda_1\rho_1}_{5(0,0)}(y_1)|0\rangle \\
&\quad +4i\langle 0|\mathcal{T}T^{\lambda\rho_1}_{5(0,0)}(y_1)\frac{\delta^2 S}{\delta h_{\mu\nu}(x)\delta h_{\mu_1\nu_1}(x_1)}|0\rangle + 4i\langle 0|\mathcal{T}T^{\mu_1\nu_1}_{(0,0)}(x_1)\frac{\delta^2 S}{\delta k_{\lambda_1\rho_1}(y_1)\delta h_{\mu\nu}(x)}|0\rangle \\
&\quad +4i\langle 0|\mathcal{T}T^{\mu\nu}_{(0,0)}(x)\frac{\delta^2 S}{\delta k_{\lambda_1\rho_1}(y_1)\delta k_{\mu_1\nu_1}(x_1)}|0\rangle, \qquad (4.119) \\
\mathcal{T}^{\lambda\rho\mu_1\nu_1\mu_2\nu_2}_{5(2,0)}(x,x_1,x_2) &= -\langle 0|\mathcal{T}T^{\lambda\rho}_{5(0,0)}(x)T^{\mu_1\nu_1}_{(0,0)}(x_1)T^{\mu_2\nu_2}_{(0,0)}(x_2)|0\rangle \\
&\quad +4i\langle 0|\mathcal{T}T^{\mu_1\nu_1}_{(0,0)}(x_1)\frac{\delta^2 S}{\delta k_{\lambda\rho}(x)\delta h_{\mu_2\nu_2}(x_2)}|0\rangle + 4i\langle 0|\mathcal{T}T^{\mu_2\nu_2}_{(0,0)}(x_2)\frac{\delta^2 S}{\delta h_{\mu_1\nu_1}(x_1)\delta k_{\lambda\rho}(x)}|0\rangle \\
&\quad +4i\langle 0|\mathcal{T}T^{\lambda\rho}_{5(0,0)}(x)\frac{\delta^2 S}{\delta h_{\mu_1\nu_1}(x_1)\delta h_{\mu_2\nu_2}(x_2)}|0\rangle \qquad (4.120) \\
\mathcal{T}^{\lambda\rho\lambda_1\rho_1\lambda_2\rho_2}_{5(0,2)}(x,y_1,y_2) &= -\langle 0|\mathcal{T}T^{\lambda\rho}_{5(0,0)}(x)T^{\lambda_1\rho_1}_{5(0,0)}(y_1)T^{\lambda_2\rho_2}_{5(0,0)}(y_2)|0\rangle \\
&\quad +4i\langle 0|\mathcal{T}T^{\lambda_1\rho_1}_{5(0,0)}(y_1)\frac{\delta^2 S}{\delta k_{\lambda\rho}(x)\delta k_{\lambda_2\rho_2}(y_2)}|0\rangle + 4i\langle 0|\mathcal{T}T^{\lambda_2\rho_2}_{5(0,0)}(y_2)\frac{\delta^2 S}{\delta k_{\lambda_1\rho_1}(y_1)\delta k_{\lambda\rho}(x)}|0\rangle \\
&\quad +4i\langle 0|\mathcal{T}T^{\lambda\rho}_{5(0,0)}(x)\frac{\delta^2 S}{\delta k_{\lambda_1\rho_1}(y_1)\delta k_{\lambda_2\rho_2}(y_2)}|0\rangle \qquad (4.121)
\end{aligned}$$

The terms above that contain the second derivative of S are bubble diagrams where one vertex has two external h and/or k graviton lines. These diagrams are similar to those already met above and in [15], and can be shown to similarly vanish, see [19] for details. Therefore we are left with

$$\mathfrak{T}_{(1,1)}(x,x_1,y_1) = -\langle 0|\mathcal{T}T_{(0,0)\mu}{}^{\mu}(x)T^{\mu_1\nu_1}_{(0,0)}(x_1)T^{\lambda_1\rho_1}_{5(0,0)}(y_1)|0\rangle \qquad (4.122)$$

$$\mathfrak{T}_{5(2,0)}(x,x_1,x_2) = -\langle 0|\mathcal{T}T_{5(0,0)}{}^{\lambda}_{\lambda}(x)T^{\mu_1\nu_1}_{(0,0)}(x_1)T^{\mu_2\nu_2}_{(0,0)}(x_2)|0\rangle \qquad (4.123)$$

$$\mathfrak{T}_{5(0,2)}(x,y_1,y_2) = -\langle 0|\mathcal{T}T_{5(0,0)}{}^{\lambda}_{\lambda}(x)T^{\lambda_1\rho_1}_{5(0,0)}(y_1)T^{\lambda_2\rho_2}_{5(0,0)}(y_2)|0\rangle \qquad (4.124)$$

which are the intermediate results already obtained above. From this point on the calculation proceeds as in section 4.3.4.

Chapter 5

A non-perturbative approach to split anomalies

In the previous chapter we introduced a new model of modified gravity, metric-axial-tensor gravity, where beside the usual metric, we introduced an additional symmetric tensor to interact axially with fermions. Recall that in the previous chapter, the approach was perturbative, we calculated the Feynman diagrams at the lowest significant order and then covariantized the result. This is of course permitted, provided we are sure that there are no diff-anomalies. Unfortunately, this verification is extremely complicated with a MAT background, and so we limited to an analogous but simpler verification carried out in [17]. However, we have to guarantee that diffeomorphism invariance is not broken throughout the derivation. This can be done with DeWitt's method, [13, 14], which is based on point-splitting. Since the point-splitting is along a geodesic, this guarantees covariance under diffeomorphisms. We will need a regularization in order to get rid of divergences. Note that this method requires a formulation of MAT more accurate than in [19] and in previous chapter. For this reason we introduce an appropriate framework for MAT gravity, the so-called hypercomplex calculus [20]. We define all necessary ingredients so that they are compatible with MAT gravity. In particular, we define a 'square' Dirac operator, which respects the axially extended diffeomorphisms. The result for a fermion of specific handedness is obtained by taking the appropriate smooth collapsing limit. We will use two different regularization methods: the dimensional and the ζ-function regularization, which give identical results. The latter agree with the perturbative results previously obtained in [15, 17, 19]. In this chapter we closely follow [20].

5.1 Axial-complex analysis

Axial-complex numbers are defined by

$$\hat{a} = a_1 + \gamma_5 a_2 \tag{5.1}$$

where a_1 and a_2 are real numbers. Arithmetic is defined in the obvious way. We can define a conjugation operator

$$\overline{\hat{a}} = a_1 - \gamma_5 a_2 \tag{5.2}$$

We will denote by $\mathcal{AC}$ the set axial-complex numbers, by $\mathcal{AR}$ the set of axial-complex numbers with $a_2 = 0$ (the axial-real numbers) and by $\mathcal{AI}$ the set of axial-complex numbers with $a_1 = 0$ (the axial-imaginary numbers). We can define a (pseudo)norm

$$(a, a) = \hat{a}\overline{\hat{a}} = a_1^2 - a_2^2 \tag{5.3}$$

This determines an axial-light-cone with all the related problems. In general, whenever possible, we will keep away from it by considering the case $|a_1| > |a_2|$. Alternatively we will use an axial-Wick-rotation (analogous to the Wick rotation for the Minkowski spacetime light-cone) $a_2 \to ia_2$. Whenever we resort to it explicit mention will be made.

Introducing the chiral projectors $P_\pm = \frac{1\pm\gamma_5}{2}$, we can also write

$$\hat{a} = a_+ P_+ + a_- P_-, \qquad a_\pm = a_1 \pm a_2 \tag{5.4}$$

We will consider functions $\hat{f}(\hat{x})$ of the axial-complex variable

$$\widehat{x} = x_1 + \gamma_5 x_2 \tag{5.5}$$

from $\mathcal{AC}$ to $\mathcal{AC}$, which are axial-analytic, i.e. admit a Taylor expansion, and actually identify the functions with their expansions. Using the property of the projectors it is easy to see that

$$\hat{f}(\hat{x}) = P_+\hat{f}(x_+) + P_-\hat{f}(x_-) = \frac{1}{2}\Big(\hat{f}(x_+) + \hat{f}(x_-)\Big) + \frac{\gamma_5}{2}\Big(\hat{f}(x_+) - \hat{f}(x_-)\Big) \tag{5.6}$$

In the same way we will consider functions from $\mathcal{AC}^4$ to $\mathcal{AC}$, with analogous properties.

$$\hat{f}(\hat{x}^\mu) = P_+\hat{f}(x_+^\mu) + P_-\hat{f}(x_-^\mu) = \frac{1}{2}\left(\hat{f}(x_+^\mu) + \hat{f}(x_-^\mu)\right) + \frac{\gamma_5}{2}\left(\hat{f}(x_+^\mu) - \hat{f}(x_-^\mu)\right) \tag{5.7}$$

with $\mu = 0, 1, 2, 3$, and

$$\widehat{x}^\mu = x_1^\mu + \gamma_5 x_2^\mu \tag{5.8}$$

are the axial-complex coordinates. Axial-complex numbers and analysis are a particular case of pseudo-complex or hyper-complex numbers and analysis, [92, 93].

Derivatives are defined in the obvious way:

$$\frac{\partial}{\partial \hat{x}^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^\mu} + \gamma_5 \frac{\partial}{\partial x_2^\mu}\right), \qquad \frac{\partial}{\partial \bar{\hat{x}}^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^\mu} - \gamma_5 \frac{\partial}{\partial x_2^\mu}\right) \tag{5.9}$$

Notice that for axial-analytic functions

$$\frac{d}{d\hat{x}} = \frac{\partial}{\partial x_1} \equiv \frac{\partial}{\partial \hat{x}}, \tag{5.10}$$

whereas $\frac{\partial}{\partial \bar{\hat{x}}}\widehat{f}(\hat{x}) = 0$.

As for integrals, since we will always have to do with rapidly decreasing functions at infinity, we define

$$\int d\hat{x}\, \widehat{f}(\hat{x})$$

as the rapidly decreasing primitive $\widehat{g}(\hat{x})$ of $\widehat{f}(\hat{x})$. Therefore the property

$$\int d\hat{x}\, \frac{\partial}{\partial \hat{x}^\mu}\hat{f}(\hat{x}) = 0 \tag{5.11}$$

follows immediately. As a consequence of (5.10) it follows that, for an axial-analytic function,

$$\int d\hat{x}\, \widehat{f}(\hat{x}) = \int dx_1\, \widehat{f}(\hat{x}) \tag{5.12}$$

and we can define definite integrals such as

$$\int_{\hat{a}}^{\hat{b}} d\hat{x}\, \widehat{f}(\hat{x}) = \widehat{g}(\hat{b}) - \widehat{g}(\hat{a}) \tag{5.13}$$

In this axial-spacetime we introduce an axial-Riemannian geometry as follows. The main formulas have already appeared in 4.2, although in a somewhat different notation. An important difference with 4.2 is that, there, all the quantities where functions of x^μ. Here, and throughout this chapter they are functions of $\hat{x}^\mu$ unless otherwise specified. Consequently, the main changes in notation are

$$\begin{array}{llll}
G_{\mu\nu} \longrightarrow \widehat{g}_{\mu\nu}, & \hat{G}^{\mu\nu} \longrightarrow \widehat{g}^{\mu\nu}, & \hat{g} \longrightarrow \tilde{g}, & \hat{f} \longrightarrow \tilde{f} \\
E^a_\mu \longrightarrow \widehat{e}^a_\mu, & \hat{E}^\mu_a \longrightarrow \widehat{e}^\mu_a, & \hat{e}^\mu_a \longrightarrow \tilde{e}^\mu_a, & \hat{c}^\mu_a \longrightarrow \tilde{c}^\mu_a \\
\gamma^\lambda_{\mu\nu} \longrightarrow \Gamma^\lambda_{\mu\nu}, & \Gamma^\lambda_{\mu\nu} \longrightarrow \widehat{\Gamma}^\lambda_{\mu\nu}, & \Omega^{ab}_\mu \longrightarrow \widehat{\Omega}^{ab}_\mu, & \Xi^\mu \longrightarrow \widehat{\xi}^\mu \\
\mathcal{R} \longrightarrow \widehat{R}, & \mathcal{R}^{(1,2)} \longrightarrow \widehat{R}^{(1,2)} & &
\end{array}$$

Starting from a metric $\widehat{g}_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu}$, the Christoffel symbols are defined by

$$\widehat{\Gamma}^\lambda_{\mu\nu} \;=\; \frac{1}{2}\widehat{g}^{\lambda\rho}\left(\frac{\partial}{\partial\widehat{x}^\mu}\widehat{g}_{\rho\nu} + \frac{\partial}{\partial\widehat{x}^\nu}\widehat{g}_{\mu\rho} - \frac{\partial}{\partial\widehat{x}^\rho}\widehat{g}_{\mu\nu}\right) \tag{5.14}$$

They split as follows

$$\widehat{\Gamma}^\mu_{\nu\lambda} = \Gamma^{(1)\mu}_{\nu\lambda} + \gamma_5 \Gamma^{(2)\mu}_{\nu\lambda} \tag{5.15}$$

and are such that the metricity condition is satisfied

$$\frac{\partial}{\partial\hat{x}^\mu}\widehat{g}_{\nu\lambda} = \widehat{\Gamma}^\rho_{\mu\nu}\,\widehat{g}_{\rho\lambda} + \widehat{\Gamma}^\rho_{\mu\lambda}\,\widehat{g}_{\nu\rho}, \tag{5.16}$$

which, in $\mathcal{AR}^4$, takes the form

$$\frac{\partial}{\partial\hat{x}^\mu}g_{\nu\lambda} = \Gamma^{(1)\rho}_{\mu\nu}\, g_{\rho\lambda} + \Gamma^{(1)\rho}_{\mu\lambda}\, g_{\nu\rho} + \Gamma^{(2)\rho}_{\mu\nu}\, f_{\rho\lambda} + \Gamma^{(2)\rho}_{\mu\lambda}\, f_{\nu\rho} \tag{5.17}$$

$$\frac{\partial}{\partial\hat{x}^\mu}f_{\nu\lambda} = \Gamma^{(1)\rho}_{\mu\nu}\, f_{\rho\lambda} + \Gamma^{(1)\rho}_{\mu\lambda}\, f_{\nu\rho} + \Gamma^{(2)\rho}_{\mu\nu}\, g_{\rho\lambda} + \Gamma^{(2)\rho}_{\mu\lambda}\, g_{\nu\rho} \tag{5.18}$$

5.2 MAT geodesics

Let us set

$$\widehat{\Gamma}^{\mu}_{\nu\lambda} = \Gamma^{(1)\mu}_{\nu\lambda} + \gamma_5 \Gamma^{(2)\mu}_{\nu\lambda} \tag{5.19}$$

The equation for MAT geodesics is

$$\ddot{\widehat{x}}^{\mu} + \widehat{\Gamma}^{\mu}_{\nu\lambda} \dot{\widehat{x}}^{\nu} \dot{\widehat{x}}^{\lambda} = 0 \tag{5.20}$$

where a dot denotes derivative with respect to an axial-affine parameter $t = t_1 + \gamma_5 t_2$. For axial-real and axial-imaginary components this means

$$\ddot{x}_1^{\mu} + \Gamma^{(1)\mu}_{\nu\lambda}(\dot{x}_1^{\nu}\dot{x}_1^{\lambda} + \dot{x}_2^{\nu}\dot{x}_2^{\lambda}) + \Gamma^{(2)\mu}_{\nu\lambda}(\dot{x}_1^{\nu}\dot{x}_2^{\lambda} + \dot{x}_2^{\nu}\dot{x}_1^{\lambda}) = 0 \tag{5.21}$$

$$\ddot{x}_2^{\mu} + \Gamma^{(1)\mu}_{\nu\lambda}(\dot{x}_1^{\nu}\dot{x}_2^{\lambda} + \dot{x}_2^{\nu}\dot{x}_1^{\lambda}) + \Gamma^{(2)\mu}_{\nu\lambda}(\dot{x}_1^{\nu}\dot{x}_1^{\lambda} + \dot{x}_2^{\nu}\dot{x}_2^{\lambda}) = 0 \tag{5.22}$$

These geodesic equations can be obtained as equations of motion from the action

$$\widehat{S} = \int d\hat{t} \sqrt{\widehat{g}_{\mu\nu} \dot{\widehat{x}}^{\mu} \dot{\widehat{x}}^{\nu}} = S_1 + \gamma_5 S_2 \tag{5.23}$$

where $\widehat{g}_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu}$.

The action takes values in $\mathcal{AC}$. For instance, setting the proper time $\hat{\tau} = \tau_1 + \gamma_5 \tau_2$,

$$\widehat{S}[\widehat{x}] = \int d\hat{\tau} \left(\widehat{g}_{\mu\nu} \dot{\widehat{x}}^{\mu} \dot{\widehat{x}}^{\nu} \right)^{\frac{1}{2}} \tag{5.24}$$

But unlike [92, 93] we require the action principle to be specified by $\delta\widehat{S}[\widehat{x}] = 0$.

Taking the variation of $S[\widehat{x}]$ with respect to $\delta\widehat{x} = \delta x_1 + \gamma_5 \delta x_2$, with

$$\begin{aligned}
\delta\widehat{g}_{\mu\nu} &= \frac{\partial \widehat{g}_{\mu\nu}}{\partial \widehat{x}^{\lambda}} \delta\widehat{x}^{\lambda} \\
\delta g_{\mu\nu} &= \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x_1^{\lambda}} + \frac{\partial f_{\mu\nu}}{\partial x_2^{\lambda}} \right) \delta x_1^{\lambda} + \left(\frac{\partial f_{\mu\nu}}{\partial x_1^{\lambda}} + \frac{\partial g_{\mu\nu}}{\partial x_2^{\lambda}} \right) \delta x_2^{\lambda} = \frac{\partial g_{\mu\nu}}{\partial x_1^{\lambda}} \delta x_1^{\lambda} + \frac{\partial f_{\mu\nu}}{\partial x_1^{\lambda}} \delta x_2^{\lambda} \\
\delta f_{\mu\nu} &= \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x_1^{\lambda}} + \frac{\partial f_{\mu\nu}}{\partial x_2^{\lambda}} \right) \delta x_2^{\lambda} + \left(\frac{\partial f_{\mu\nu}}{\partial x_1^{\lambda}} + \frac{\partial g_{\mu\nu}}{\partial x_2^{\lambda}} \right) \delta x_1^{\lambda} = \frac{\partial g_{\mu\nu}}{\partial x_1^{\lambda}} \delta x_2^{\lambda} + \frac{\partial f_{\mu\nu}}{\partial x_1^{\lambda}} \delta x_1^{\lambda}
\end{aligned} \tag{5.25}$$

we get the equation of motion

$$\widehat{g}_{\mu\rho}\ddot{\widehat{x}}^{\rho} + \widehat{\Gamma}^{\rho}_{\nu\lambda}\,\widehat{g}_{\mu\rho}\,\dot{\widehat{x}}^{\mu}\dot{\widehat{x}}^{\nu} = 0, \qquad \text{i.e.} \qquad \ddot{\widehat{x}}^{\mu} + \widehat{\Gamma}^{\mu}_{\nu\lambda}\dot{\widehat{x}}^{\nu}\dot{\widehat{x}}^{\lambda} = 0 \tag{5.26}$$

Let us rewrite

$$\begin{aligned} \sqrt{\widehat{g}_{\mu\nu}\dot{\widehat{x}}^{\mu}\dot{\widehat{x}}^{\nu}} &= \sqrt{A+\gamma_5 B}, \\ A &= g_{\mu\nu}\left(\dot{x}_1^{\mu}\dot{x}_1^{\nu} + \dot{x}_2^{\mu}\dot{x}_2^{\nu}\right) + 2f_{\mu\nu}\dot{x}_1^{\mu}\dot{x}_2^{\nu}, \\ B &= f_{\mu\nu}\left(\dot{x}_1^{\mu}\dot{x}_1^{\nu} + \dot{x}_2^{\mu}\dot{x}_2^{\nu}\right) + 2g_{\mu\nu}\dot{x}_1^{\mu}\dot{x}_2^{\nu}, \end{aligned} \tag{5.27}$$

so that we have

$$\begin{aligned} \widehat{S}[\widehat{x}] &= \int d\hat{\tau}\sqrt{\widehat{g}_{\mu\nu}\dot{\widehat{x}}^{\mu}\dot{\widehat{x}}^{\nu}} \\ &= \frac{1}{2}\left[\int d\tau_1\left(\sqrt{A+B}+\sqrt{A-B}\right) + \int d\tau_2\left(\sqrt{A+B}-\sqrt{A-B}\right)\right] \\ &+ \frac{\gamma_5}{2}\left[\int d\tau_1\left(\sqrt{A+B}-\sqrt{A-B}\right) + \int d\tau_2\left(\sqrt{A+B}+\sqrt{A-B}\right)\right] \end{aligned} \tag{5.28}$$

Varying this action with respect to δx^{λ} we obtain the same equation of motion (5.26). This is due to (5.12) and to the fact that, the action is an analytic function of $\widehat{x}$, so that the variation with respect to $\delta\widehat{x}^{\lambda}$ is the same as the variation of δx_1^{λ}.

Eventually we will set $x_2 = 0$ everywhere, but it is very convenient to keep the axial-analytic notation as far as possible.

5.2.1 Geodetic interval and distance

The quantity

$$\widehat{E} = E_1 + \gamma_5 E_2 = \frac{1}{2}\widehat{g}_{\mu\nu}\dot{\widehat{x}}^{\mu}\dot{\widehat{x}}^{\nu} \tag{5.29}$$

is conserved as a function of $\hat{t}$. Since $\widehat{g}_{\mu\nu}\dot{\widehat{x}}^{\mu}\dot{\widehat{x}}^{\nu}$ is constant for geodesics, we can write for the arc length parameter $\widehat{s}$

$$\frac{d\widehat{s}}{d\hat{t}} = \sqrt{\widehat{g}_{\mu\nu}\dot{\widehat{x}}^{\mu}\dot{\widehat{x}}^{\nu}}, \tag{5.30}$$

and

$$\widehat{s}-\widehat{s}' = \int_{\hat{t}'}^{\hat{t}} d\hat{\tau}\,\sqrt{2\widehat{E}} = \sqrt{2\widehat{E}}\,(\hat{t}-\hat{t}'). \tag{5.31}$$

$\widehat{s}-\widehat{s}'$ is the axial arc length along the geodesic between $\widehat{x}$ and $\widehat{x}'$. The half square of it is called the *world function* and it is denoted

$$\widehat{\sigma}(\widehat{x},\widehat{x}') = \frac{1}{2}(\widehat{s}-\widehat{s}')^2 = \widehat{E}(\hat{t}-\hat{t}')^2 = (\hat{t}-\hat{t}')\int_{\hat{t}'}^{\hat{t}} \widehat{E}d\hat{\tau} \tag{5.32}$$

The main properties are

$$\widehat{\sigma}_{;\mu} = \widehat{\partial}_\mu\widehat{\sigma} = (\hat{t}-\hat{t}')\widehat{g}_{\mu\nu}\dot{\widehat{x}}^\nu \equiv -\widehat{g}_{\mu\nu}\widehat{y}^\nu \tag{5.33}$$

$\widehat{y}^\mu$ are the *normal coordinates* based at $\widehat{x}$. Using (5.32,5.33) one can see that

$$\frac{1}{2}\widehat{\sigma}_{;\mu}\widehat{\sigma}_{;}{}^{\mu} = \widehat{\sigma} \tag{5.34}$$

The subscript $_{;\mu}$ means the covariant derivative with respect to $\widehat{x}^\mu$, while $_{;\mu'}$ means the covariant derivative with respect to $\widehat{x}'^{\mu'}$.

Remark 1. $\widehat{\sigma} = \sigma_1 + \gamma_5\sigma_2$, but notice that, even when we set $x_2 = 0$, we cannot infer that $\sigma_2 = 0$. This descends from eq.(5.30). Looking at (5.28), we see that B does not vanish even when $x_2^\nu = 0$. As a consequence the axial-imaginary part of (5.27) does not vanish, so the axial-imaginary part of eq.(5.30) will not automatically vanish either.

5.2.2 Normal coordinates

Normal coordinates can be defined based at x or at x':

$$\begin{aligned} \widehat{y}^{\mu'}(\widehat{x}',\widehat{x}) &= (\hat{t}-\hat{t}')\frac{d\widehat{x}^{\mu'}}{d\hat{t}'} && (5.35)\\ \widehat{y}^{\mu}(\widehat{x},\widehat{x}') &= (\hat{t}'-\hat{t})\frac{d\widehat{x}^{\mu}}{d\hat{t}} && (5.36) \end{aligned}$$

The tangent vector $\frac{d\widehat{x}^\mu}{d\hat{t}}$ to the geodesic at $\hat{x}$ satifies

$$\frac{D}{d\hat{t}}\frac{d\widehat{x}^\mu}{d\hat{t}} = \frac{d^2\widehat{x}^\mu}{d\hat{t}^2} + \widehat{\Gamma}^\mu_{\nu\lambda}\frac{d\widehat{x}^\nu}{d\hat{t}}\frac{d\widehat{x}^\lambda}{d\hat{t}} = 0 \tag{5.37}$$

and an analogous equation at $\hat{x}'$. Now we can write

$$\begin{aligned}\widehat{y}^{\mu'}{}_{;\nu}(\hat{x}',\hat{x})\widehat{y}^{\nu}(\hat{x},\hat{x}') &= (\hat{t}'-\hat{t})\widehat{y}^{\mu'}{}_{;\nu}(\widehat{x}',\widehat{x})\frac{d\widehat{x}^{\nu}(\hat{t})}{d\hat{t}} \\ &= (\hat{t}'-\hat{t})\frac{d}{d\hat{t}}\widehat{y}^{\mu'}(\widehat{x}',\widehat{x}) = (\hat{t}'-\hat{t})\frac{d\widehat{x}^{\mu'}(\hat{t}')}{d\hat{t}'} = -\widehat{y}^{\mu'}(\widehat{x}',\widehat{x}) \quad (5.38)\end{aligned}$$

It is useful to determine the coincidence limit $\widehat{x}' \to \widehat{x}$. We use the notation:

$$[\ldots] = \lim_{\widehat{x}'\to\widehat{x}}(\ldots) \tag{5.39}$$

Dividing by $\hat{t}-\hat{t}'$ the second and fourth terms and taking the coincidence limit, one gets

$$[\widehat{y}^{\mu'}{}_{;\nu}]\frac{d\widehat{x}^{\nu}}{d\hat{t}} = \frac{d\widehat{x}^{\mu}}{d\hat{t}} \qquad \to \qquad [\widehat{y}^{\mu'}{}_{;\nu}] = \delta^{\mu}_{\nu} \tag{5.40}$$

where $[X]$ denotes the result of the coincidence limit of the quantity X. In a similar way one can prove

$$\left[\widehat{y}^{\mu'}{}_{;\nu'}\right]\frac{d\widehat{x}^{\nu}}{d\hat{t}} = -\frac{d\widehat{x}^{\mu}}{d\hat{t}} \qquad \to \qquad \left[\widehat{y}^{\mu'}{}_{;\nu'}\right] = -\delta^{\mu}_{\nu} \tag{5.41}$$

$$[\widehat{y}^{\mu}{}_{;\nu}]\frac{d\widehat{x}^{\nu}}{d\hat{t}} = -\frac{d\widehat{x}^{\mu}}{d\hat{t}} \qquad \to \qquad [\widehat{y}^{\mu}{}_{;\nu}] = -\delta^{\mu}_{\nu} \tag{5.42}$$

$$[\widehat{y}^{\mu}{}_{;\nu'}]\frac{d\widehat{x}^{\nu}}{dt} = \frac{d\widehat{x}^{\mu}}{dt} \qquad \to \qquad [\widehat{y}^{\mu}{}_{;\nu'}] = \delta^{\mu}_{\nu} \tag{5.43}$$

From (5.38) we get

$$\widehat{y}^{\mu'}{}_{;\nu}\,\widehat{y}^{\nu} + \widehat{y}^{\mu'} = 0 \tag{5.44}$$

In a similar way one derives also

$$\widehat{y}^{\mu'}{}_{;\nu'}\,\widehat{y}^{\nu'} + \widehat{y}^{\mu'} = 0 \tag{5.45}$$

$$\widehat{y}^{\mu}{}_{;\nu'}\,\widehat{y}^{\nu'} + \widehat{y}^{\mu} = 0 \tag{5.46}$$

$$\widehat{y}^{\mu}{}_{;\nu}\,\widehat{y}^{\nu} + \widehat{y}^{\mu} = 0 \tag{5.47}$$

For instance, differentiating (5.45) with respect to $\widehat{x}^{\lambda'}$, one gets

$$\widehat{y}^{\mu'}{}_{;\nu'\lambda'}\,\widehat{y}^{\nu'} + \widehat{y}^{\mu'}{}_{;\nu'}\,\widehat{y}^{\nu'}{}_{;\lambda'} + \widehat{y}^{\mu'}{}_{;\lambda'} = 0$$

taking the coincidence limit, and using (5.41), one finds an identity, because $[\widehat{y}^{\mu'}] = 0$. Differentiating another time with respect to $\widehat{x}^{\rho'}$ one gets

$$[\widehat{y}^{\mu'}{}_{;\lambda'\rho'}] = 0 \tag{5.48}$$

Differentiating again with respect to $\widehat{x}^{\tau'}$ and using the Bianchi identity for $\widehat{R}^{\mu}{}_{\lambda\rho\tau} = R^{(1)\mu}{}_{\lambda\rho\tau} + \gamma_5 R^{(2)\mu}{}_{\lambda\rho\tau}$, one finds

$$[\widehat{y}^{\mu'}{}_{;\lambda'\rho'\tau'}] = \frac{1}{3}\left(\widehat{R}^{\mu}{}_{\rho\lambda\tau} + \widehat{R}^{\mu}{}_{\tau\lambda\rho}\right) \tag{5.49}$$

and, in a similar way,

$$\left[\widehat{y}^{\mu'}{}_{;\lambda\rho\tau}\right] = \frac{1}{3}\left(\widehat{R}^{\mu}{}_{\lambda\rho\tau} + \widehat{R}^{\mu}{}_{\rho\lambda\tau}\right) \tag{5.50}$$

$$[\widehat{y}^{\mu}{}_{;\lambda\rho\tau}] = \frac{1}{3}\left(\widehat{R}^{\mu}{}_{\tau\lambda\rho} + \widehat{R}^{\mu}{}_{\rho\lambda\tau}\right) \tag{5.51}$$

5.2.3 Coincidence limits of $\widehat{\sigma}$

Covariantly differentiating (5.34) we get

$$\widehat{\sigma}_{;\nu} = \widehat{\sigma}_{;\mu\nu}\widehat{\sigma}_{;}{}^{\mu} \tag{5.52}$$

In the coincidence limit $[\widehat{\sigma}_{;\nu}] = 0$. Therefore (5.52) is trivial in the coincidence limit. Differentiating the first and last member of (5.33) we get

$$\widehat{\sigma}_{;\mu\lambda} = -\widehat{g}_{\mu\nu}\,\widehat{y}^{\nu}{}_{;\lambda} \tag{5.53}$$

Using (5.42) one gets

$$[\widehat{\sigma}_{;\mu\lambda}] = \widehat{g}_{\mu\lambda} \tag{5.54}$$

Similarly

$$[\widehat{\sigma}_{;\mu\lambda'}] = -\widehat{g}_{\mu\lambda} \tag{5.55}$$

Differentiating (5.52) once more one gets

$$\widehat{\sigma}_{;\nu\lambda} = \widehat{\sigma}_{;\mu\nu\lambda}\,\widehat{\sigma}_{;}{}^{\mu} + \widehat{\sigma}_{;\mu\nu}\,\widehat{\sigma}^{\mu}_{;\lambda}$$

which, in the coincidence limit, using the previous results, yields an identity. Differentiating it again

$$\widehat{\sigma}_{;\nu\lambda\rho} = \widehat{\sigma}_{;\mu\nu\lambda\rho}\,\widehat{\sigma}_{;}{}^{\mu} + \widehat{\sigma}_{;\mu\nu\lambda}\,\widehat{\sigma}^{\mu}_{;\rho} + \widehat{\sigma}_{;\mu\nu\rho}\,\widehat{\sigma}^{\mu}_{;\lambda} + \widehat{\sigma}_{;\mu\nu}\,\widehat{\sigma}_{;}{}^{\mu}{}_{\lambda\rho} \tag{5.56}$$

In the coincidence limit this becomes

$$[\widehat{\sigma}_{;\nu\lambda\rho}] = [\widehat{\sigma}_{;\rho\nu\lambda}] + [\widehat{\sigma}_{;\lambda\nu\rho}] + [\widehat{\sigma}_{;\nu\lambda\rho}] \tag{5.57}$$

Since $\widehat{\sigma}$ is a biscalar we have

$$[\widehat{\sigma}_{;\nu\lambda\rho}] = [\widehat{\sigma}_{;\nu\rho\lambda}] + \widehat{R}_{\rho\lambda\nu}{}^{\tau}[\widehat{\sigma}_{;\tau}] = [\widehat{\sigma}_{;\rho\nu\lambda}] \tag{5.58}$$

Therefore

$$[\widehat{\sigma}_{;\rho\nu\lambda}] = [\widehat{\sigma}_{;\lambda\nu\rho}] = [\widehat{\sigma}_{;\nu\lambda\rho}] = 0 \tag{5.59}$$

Differentiating (5.56) once more and taking the coincidence limit one gets

$$[\widehat{\sigma}_{;\nu\lambda\rho\tau}] = -\frac{1}{3}\left(\widehat{R}_{\nu\tau\lambda\rho} + \widehat{R}_{\nu\rho\lambda\tau}\right) \equiv \widehat{S}_{\nu\lambda\rho\tau} \tag{5.60}$$

where $\widehat{R}_{\nu\tau\lambda\rho} = \widehat{g}_{\nu\mu}\widehat{R}^{\mu}{}_{\tau\lambda\rho}$. Differentiating once more

$$[\widehat{\sigma}_{;\nu\lambda\rho\sigma\tau}] = \frac{3}{4}\left(\widehat{S}_{\nu\lambda\sigma\tau;\rho} + \widehat{S}_{\nu\lambda\sigma\rho;\tau} + \widehat{S}_{\nu\lambda\tau\rho;\sigma}\right) \tag{5.61}$$

We will need also the coincidence limits of tensors covariantly differentiated with respect to a primed index ν'. In general

$$[t_{\mu_1\ldots\mu_k;\nu'}] = [t_{\mu_1\ldots\mu_k}]_{;\nu} - [t_{\mu_1\ldots\mu_k;\nu}] \tag{5.62}$$

So

$$[\widehat{\sigma}_{;\mu\nu'}] = [\widehat{\sigma}_{;\mu}]_{;\nu} - [\widehat{\sigma}_{;\mu\nu}] = -\widehat{g}_{\mu\nu} \tag{5.63}$$

$$[\widehat{\sigma}_{;\mu\nu'\lambda}] = [\widehat{\sigma}_{;\mu\lambda\nu'}] = [\widehat{\sigma}_{;\mu\lambda}]_{;\nu} - [\widehat{\sigma}_{;\mu\lambda\nu}] = 0 \tag{5.64}$$

$$[\widehat{\sigma}_{;\mu\nu'\lambda\rho}] = [\widehat{\sigma}_{;\mu\lambda\rho\nu'}] = [\widehat{\sigma}_{;\mu\lambda\rho}]_{;\nu} - [\widehat{\sigma}_{;\mu\lambda\rho\nu}] = -[\widehat{\sigma}_{;\mu\lambda\rho\nu}] = -\widehat{S}_{\mu\lambda\rho\nu} \tag{5.65}$$

$$[\widehat{\sigma}_{;\mu\nu'\lambda\rho\sigma}] = [\widehat{\sigma}_{;\mu\lambda\rho\sigma\nu'}] = [\widehat{\sigma}_{;\mu\lambda\rho\sigma}]_{;\nu} - [\widehat{\sigma}_{;\mu\lambda\rho\sigma\nu}] = \frac{1}{4}\widehat{S}_{\mu\lambda\rho\sigma;\nu} - \frac{3}{4}\left(\widehat{S}_{\mu\lambda\nu\rho;\sigma} + \widehat{S}_{\mu\lambda\sigma\nu;\rho}\right) \tag{5.66}$$

Similarly, one obtains

$$\begin{aligned}
[\widehat{\sigma}_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu}{}_{\rho}{}^{\rho}] &= -\frac{8}{5}R_{;\mu}{}^{\mu} + \frac{4}{15}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} - \frac{4}{15}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho} \\
[\widehat{\sigma}_{;\mu\nu}{}^{\nu}{}_{\rho}{}^{\rho\mu}] &= -\left[\widehat{\sigma}_{;\mu}{}^{\mu'}{}_{\nu}{}^{\nu}{}_{\rho}{}^{\rho}\right] = \frac{2}{5}R_{;\mu}{}^{\mu} - \frac{1}{15}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} - \frac{4}{15}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho}
\end{aligned}$$

5.2.4 Van Vleck-Morette determinant

The Van Vleck-Morette determinant in MAT is defined by

$$\widehat{D}(\widehat{x}, \widehat{x}') = \det(-\widehat{\sigma}_{;\mu\nu'}) \tag{5.67}$$

$\widehat{D}(\widehat{x}, \widehat{x}')$ is a bidensity of weight 1 both at $\widehat{x}$ and $\widehat{x}'$. Later on we will need a bidensity of weight 0:

$$\widehat{\Delta}(\widehat{x}, \widehat{x}') = \frac{1}{\sqrt{\widehat{g}(\widehat{x})}}\widehat{D}(\widehat{x}, \widehat{x}')\frac{1}{\sqrt{\widehat{g}(\widehat{x}')}} \tag{5.68}$$

The VVM determinant also satisfies (for 4 dimensions)

$$(\widehat{D}(\widehat{x}, \widehat{x}')\widehat{\sigma}^{;\mu})_{;\mu} = 4\widehat{D}(\widehat{x}, \widehat{x}') \tag{5.69}$$

In the coincidence limit

$$[\widehat{\Delta}^{\frac{1}{2}}_{;\lambda}] = [\widehat{g}^{-\frac{1}{4}}(\widehat{x})\sqrt{\widehat{D}(\widehat{x}, \widehat{x}')}\frac{1}{2}\left(\widehat{\sigma}^{-1\mu\nu'}\widehat{\sigma}_{;\mu\nu'\lambda}\right)\widehat{g}^{-\frac{1}{4}}(\widehat{x}')] = \frac{1}{2}[\widehat{\sigma}^{\mu}_{;\mu\lambda}] = 0 \tag{5.70}$$

We need to compute the covariant derivatives of $\widehat{\sigma}^{-1\mu\nu'} \equiv \{\widehat{\sigma}^{-1}_{;\mu\nu'}\}$. The latter is defined as

$$\widehat{\sigma}^{-1\mu\nu'}\widehat{\sigma}_{;\nu'\lambda} = \delta^{\mu}_{\lambda} \tag{5.71}$$

Differentiating this relation once, twice and thrice one gets

$$\begin{aligned}
[\widehat{\sigma}^{-1\mu\nu'}{}_{;\lambda}] &= 0, && (5.72)\\
\left[\widehat{\sigma}^{-1}{}_{\mu\lambda';\rho\sigma}\right] &= -[\widehat{\sigma}_{;\mu'\lambda\rho\sigma}] = [\widehat{\sigma}_{;\lambda\rho\sigma\mu}] = \widehat{S}_{\lambda\rho\sigma\mu} && (5.73)\\
\left[\widehat{\sigma}^{-1}{}_{\mu\lambda';\rho\sigma\tau}\right] &= -[\widehat{\sigma}_{;\lambda\mu'\rho\sigma\tau}] = \frac{1}{4}\widehat{S}_{\mu\rho\sigma\tau;\lambda} - \frac{3}{4}\left(\widehat{S}_{\mu\rho\lambda\sigma;\tau} + \widehat{S}_{\mu\rho\tau\lambda;\sigma}\right) && (5.74)
\end{aligned}$$

Differentiating once more one gets

$$[\widehat{\Delta}^{\frac{1}{2}}_{;\lambda\rho}] = \frac{1}{6}\widehat{g}^{\mu\nu}\left(\widehat{R}_{\mu\nu\lambda\rho} + \widehat{R}_{\mu\lambda\nu\rho}\right) = \frac{1}{6}\widehat{g}^{\mu\nu}\widehat{g}_{\mu\sigma}\widehat{R}^{\sigma}{}_{\lambda\nu\rho} = \frac{1}{6}\left(R^{(1)}_{\lambda\rho} + \gamma_5 R^{(2)}_{\lambda\rho}\right) \tag{5.75}$$

and

$$[\widehat{\Delta}^{\frac{1}{2}}_{;\lambda\rho\sigma}] = \frac{1}{12}\left(\widehat{R}_{\lambda\rho;\sigma} + \widehat{R}_{\rho\sigma;\lambda} + \widehat{R}_{\sigma\lambda;\rho}\right) \tag{5.76}$$

Finally

$$[\widehat{\Delta}^{\frac{1}{2}}_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu}] = +\frac{1}{5}\widehat{R}_{;\mu}{}^{\mu} + \frac{1}{36}\widehat{R}^2 - \frac{1}{30}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} + \frac{1}{30}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho} \tag{5.77}$$

5.2.5 The geodetic parallel displacement matrix

The geodetic parallel displacement matrix $\widehat{G}^{\mu}{}_{\nu'}(\widehat{x}, \widehat{x}')$ is needed in order to parallel displace vectors from one end to the other of the geodetic interval. It is defined by

$$[\widehat{G}^{\mu}{}_{\nu'}] = \delta^{\mu}_{\nu}, \qquad \widehat{G}^{\mu}{}_{\nu';\lambda}\widehat{\sigma}^{;\lambda} = 0 \tag{5.78}$$

The second condition means that the covariant derivative of $\widehat{G}^{\mu}{}_{\nu'}$ vanishes in directions parallel to the geodesic. Since tangents to the geodesics are self-parallel, it follows that

$$
\begin{aligned}
&\widehat{G}_{\mu}{}^{\nu'}\,\widehat{\sigma}_{;\nu'} = -\sigma_{;\mu}, \qquad \widehat{\sigma}_{;\mu}\,\widehat{G}^{\mu}{}_{\nu'} = -\widehat{\sigma}_{;\nu'} \\
&\widehat{G}_{\mu\nu'} = \widehat{G}_{\nu'\mu}, \qquad \widehat{\sigma}_{;}{}^{\lambda'}\widehat{G}^{\mu}{}_{\nu';\lambda'} = 0 \\
&\widehat{G}_{\mu}{}^{\nu'}\widehat{G}_{\nu'}{}^{\lambda} = \delta^{\lambda}_{\mu}
\end{aligned}
\tag{5.79}
$$

The analogous parallel displacement for spinors is $I(x, x')$: the object $I(x, x')\psi(x')$ is the spinor $\psi(x)$ obtained by parallel displacement of $\psi(x')$ along the geodesic from x' to x. It is a bispinor quantity satisfying

$$
\widehat{\sigma}_{;}{}^{\mu}\widehat{I}_{;\mu} = 0, \qquad [\widehat{I}] = \mathbf{1} \tag{5.80}
$$

and $\mathbf{1}$ is the identity matrix in the spinor space. Differentiating (5.80) once we get $[\widehat{I}_{;\mu}] = 0$. Differentiating twice we get

$$
[\widehat{I}_{;(\mu\nu)}] = 0, \tag{5.81}
$$

while

$$
\widehat{I}(x, x')_{;\mu\nu} - \widehat{I}(x, x')_{;\nu\mu} = -\frac{1}{2}(d\widehat{\Omega} + \widehat{\Omega}\widehat{\Omega})_{\mu\nu}\widehat{I}(x, x') = -\frac{1}{2}\widehat{\mathcal{R}}_{\mu\nu}I(x, x') \tag{5.82}
$$

where $\widehat{\mathcal{R}}_{\mu\nu} = \widehat{R}_{\mu\nu}{}^{ab}\Sigma_{ab}$. So

$$
[\widehat{I}(x, x')_{;[\mu,\nu]}] = [\widehat{I}(x, x')_{;\mu\nu}] = -\frac{1}{4}\widehat{\mathcal{R}}_{\mu\nu} \tag{5.83}
$$

Proceeding with the differentiations of (5.80) we find

$$
[\widehat{I}_{;\nu\lambda\rho}] + [\widehat{I}_{;\lambda\nu\rho}] + [\widehat{I}_{;\rho\lambda\nu}] = 0 \tag{5.84}
$$

Now

$$
[\widehat{I}_{;\nu\lambda\rho}] - [\widehat{I}_{;\nu\rho\lambda}] = \frac{1}{2}\widehat{\mathcal{R}}_{\rho\lambda}[\widehat{I}_{;\nu}] = 0 \tag{5.85}
$$

and

$$3[\widehat{I}_{;\nu\lambda\rho}] = \frac{1}{2}\widehat{\nabla}_\rho\widehat{\mathcal{R}}_{\lambda\nu} + \frac{1}{2}\widehat{\nabla}_\lambda\widehat{\mathcal{R}}_{\rho\nu} \tag{5.86}$$

In particular

$$[\widehat{I}_{;\nu}{}^{\nu}{}_{\rho}] = \frac{1}{6}\widehat{\nabla}^\nu\widehat{\mathcal{R}}_{\rho\nu} \tag{5.87}$$

Differentiating (5.80) once more with respect to x^σ, using (5.60) and then contracting with $\widehat{g}^{\nu\lambda}\widehat{g}^{\sigma\rho}$ we find, after simplifying,

$$[\widehat{I}_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu}] + [\widehat{I}_{;\mu\nu}{}^{\nu\mu}] = 0 \tag{5.88}$$

A contraction with $\widehat{g}^{\nu\sigma}\widehat{g}^{\lambda\rho}$ gives:

$$[\widehat{I}_{;\mu\nu}{}^{\nu\mu}] + 2[\widehat{I}_{;\mu\nu}{}^{\mu\nu}] + [\widehat{I}_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu}] = 0 \tag{5.89}$$

Using (5.82), we get

$$[\widehat{I}_{;\sigma\rho\mu\nu}] = [\widehat{\nabla}_\nu\widehat{\nabla}_\mu(\widehat{I}_{;\sigma\rho})] = -\frac{1}{2}\widehat{\mathcal{R}}_{\sigma\rho;\mu\nu} + \frac{1}{8}\widehat{\mathcal{R}}_{\sigma\rho}\widehat{\mathcal{R}}_{\mu\nu} + [\widehat{I}_{;\rho\sigma\mu\nu}] \tag{5.90}$$

Contracting with $\widehat{g}^{\mu\sigma}\widehat{g}^{\nu\rho}$ gives

$$[\widehat{I}_{;\mu\nu}{}^{\mu\nu}] = 0 + \frac{1}{8}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu} + [\widehat{I}_{;\mu\nu}{}^{\nu\mu}] \tag{5.91}$$

since by Walker's identity

$$\widehat{\nabla}_\rho\widehat{\nabla}_\lambda\widehat{\mathcal{R}}^{\rho\lambda} = 0 \tag{5.92}$$

Finally, by using (5.88), (5.89), one gets

$$[\widehat{I}_{;\nu}{}^{\nu}{}_{\rho}{}^{\rho}] = \frac{1}{8}\widehat{\mathcal{R}}_{\rho\lambda}\widehat{\mathcal{R}}^{\rho\lambda} \tag{5.93}$$

5.3 Fermions in MAT background

The action of a fermion interacting with a metric and an axial tensor is

$$\begin{aligned}\widehat{S} &= \int d^4\widehat{x}\left(i\overline{\psi}\sqrt{\overline{\widehat{g}}}\gamma^a\widehat{e}_a^\mu\left(\partial_\mu+\frac{1}{2}\widehat{\Omega}_\mu\right)\psi\right)(\widehat{x}) \\ &= \int d^4\widehat{x}\left(i\overline{\psi}\sqrt{\overline{\widehat{g}}}(\tilde{e}_a^\mu-\gamma_5\tilde{c}_a^\mu)\left[\frac{1}{2}\gamma^a\overset{\leftrightarrow}{\partial}_\mu\psi+\frac{i}{4}\gamma_d\epsilon^{dabc}\widehat{\Omega}_{\mu bc}\gamma_5\right]\psi\right)(\widehat{x})\end{aligned} \tag{5.94}$$

It must be noticed that this action takes axial-real values[1]. The field $\psi(\widehat{x})$ can be understood, classically, as a series of powers of $\widehat{x}$ applied to constant spinors on their right and the symmetry transformations act on it from the left. The analogous definitions for $\psi^\dagger$ are obtained via hermitean conjugation. In the second line it is stressed that the action contains also an axial part. It is understood that $\partial_\mu = \frac{\partial}{\partial\widehat{x}^\mu}$ applies only to ψ or $\overline{\psi}$, as indicated, and $\overline{\widehat{g}}$ denotes, as usual, the axial-complex conjugate of $\widehat{g}$.

5.3.1 A more precise formula for the energy-momentum tensor

In our calculation a more explicit formula of the energy-momentum tensor is needed than in the previous chapter. The energy-momentum tensor is defined by

$$\mathbf{T}^{\mu\nu} = \frac{2}{\sqrt{\widehat{g}}}\frac{\overleftarrow{\delta}\,\widehat{S}}{\delta\widehat{g}_{\mu\nu}} = \frac{1}{2}\left(\mathbf{T}_a^\mu\widehat{e}^{a\nu}+\mathbf{T}_a^\nu\widehat{e}^{a\mu}\right) \tag{5.95}$$

where

$$\mathbf{T}_a^\mu = \frac{1}{\sqrt{|\widehat{g}|}}\frac{\overleftarrow{\delta}\,\widehat{S}}{\delta\widehat{e}_\mu^a} \tag{5.96}$$

Let us prove first that the functional derivative of $\widehat{\Omega}_m$ does not contribute to the energy-momentum tensor. Consider the general variational formula

$$\begin{aligned}\delta\widehat{\Omega}_\mu^{bc} &= \frac{1}{2}\widehat{e}^{b\nu}\left(\widehat{\nabla}_\mu(\delta\widehat{e}_\nu^c)-\widehat{\nabla}_\nu(\delta\widehat{e}_\mu^c)\right)-\frac{1}{2}\widehat{e}^{c\nu}\left(\widehat{\nabla}_\mu(\delta\widehat{e}_\nu^b)-\widehat{\nabla}_\nu(\delta\widehat{e}_\mu^b)\right) \\ &\quad +\frac{1}{2}\widehat{e}^{b\nu}\widehat{e}^{c\lambda}\left(\widehat{\nabla}_\lambda(\delta\widehat{e}_\nu^e)-\widehat{\nabla}_\nu(\delta\widehat{e}_\lambda^e)\right)\widehat{e}_{e\mu}\end{aligned} \tag{5.97}$$

[1]One could consider also an axial complex action, but for our purposes this is a useless complication. That is why we use the notation ψ instead of $\widehat{\psi}$.

where $\widehat{\nabla}$ denotes the covariant derivative such that $\widehat{\nabla}_\mu \widehat{e}^a_\lambda = 0$. After some algebra one gets

$$\gamma_d \, \epsilon^{dabc} \, \widehat{e}^\mu_a \, \delta\widehat{\Omega}_{\mu bc} = \gamma_d \, \epsilon^{dabc} \, \widehat{e}^\mu_a \widehat{e}^\nu_b \nabla_\mu \delta e_{c\nu} \tag{5.98}$$

Now use this and

$$\frac{\delta \widehat{e}^a_\mu(x)}{\delta \widehat{e}^b_\nu(y)} = \delta^a_b \delta^\nu_\mu \delta(x,y)$$

and insert them into the definition (5.95). The relevant contribution is

$$\begin{aligned} \mathbf{T}^{\lambda\rho}_\Omega &= \frac{1}{2}\left(\mathbf{T}^\lambda_a \widehat{e}^{a\rho} + \mathbf{T}^\rho_a \widehat{e}^{a\lambda}\right)_\Omega \\ &\equiv \frac{1}{8}\int \overline{\psi}\gamma_d \epsilon^{dabc} \widehat{e}^\mu_a \left(\frac{\delta\widehat{\Omega}_{\mu bc}}{\delta \widehat{e}^e_\lambda} \widehat{e}^{e\rho} + \frac{\delta\widehat{\Omega}_{\mu bc}}{\delta \widehat{e}^e_\rho} \widehat{e}^{e\lambda} \right) \gamma_5 \psi \\ &= \frac{1}{8}\int \overline{\psi}\gamma_d \epsilon^{dabc} \widehat{e}^\mu_a \left(\widehat{e}^\lambda_b \, \widehat{e}^\rho_c \widehat{\nabla}_\mu \delta(x,y) + \widehat{e}^\rho_b \, \widehat{e}^\lambda_c \widehat{\nabla}_\mu \delta(x,y)\right) \gamma_5 \psi = 0 \end{aligned} \tag{5.99}$$

Therefore the only contribution to the energy-momentum tensor comes from the variation of the first $\widehat{e}^\mu_a$ factor in (5.94). The result is

$$\mathbf{T}^{\lambda\rho} \;=\; -\frac{i}{2}\overline{\psi}\widehat{\gamma}^\lambda \widehat{g}^{\rho\mu}\left(\partial_\mu + \frac{1}{2}\widehat{\Omega}_\mu\right) + (\lambda \leftrightarrow \rho) = -\frac{i}{2}\overline{\psi}\widehat{\gamma}^\lambda \widehat{\nabla}^\rho \psi + (\lambda \leftrightarrow \rho) \tag{5.100}$$

where $\widehat{\gamma}^\lambda = \gamma^a \widehat{e}^\lambda_a$.

It is useful to write it as a trace

$$\mathbf{T}^{\lambda\rho}(x) = \frac{i}{2}\mathrm{tr}\left(\eta\widehat{\gamma}^{(\lambda}\widehat{\nabla}^{\rho)}\psi(\mathrm{x})\psi^\dagger(\mathrm{x})\right) = \frac{\mathrm{i}}{4}\mathrm{tr}\left(\eta\widehat{\gamma}^{(\lambda}[\widehat{\nabla}^{\rho)}\psi(\mathrm{x}), \psi^\dagger(\mathrm{x})]\right) \tag{5.101}$$

where $\eta \equiv \gamma_0$, the flat gamma matrix. The commutator is interpreted as

$$[\widehat{\nabla}^\rho \psi, \psi^\dagger](x) = \frac{1}{2}\lim_{x'\to x}\left([\widehat{\nabla}^\rho\psi(x), \psi^\dagger(x')] + [\widehat{\nabla}^\rho\psi(x'), \psi^\dagger(x)]\right) \tag{5.102}$$

Inserting (5.101) in the path integral it becomes

$$\langle\!\langle \mathbf{T}^{\lambda\rho}(x)\rangle\!\rangle = \frac{i}{8}\lim_{x'\to x} \mathrm{tr}\left(\eta\widehat{\gamma}^{(\lambda}\left(\widehat{\mathbb{S}}^{(1);\rho)}(\mathrm{x},\mathrm{x}') - \widehat{\mathbb{S}}^{(1);\rho')}(\mathrm{x},\mathrm{x}')\right)\right) \tag{5.103}$$

where $\widehat{\mathbb{S}}^{(1)}$ is the Hadamard function

$$\widehat{\mathbb{S}}^{(1)}(x,x') = \langle\langle[\psi(x), \psi^\dagger(x')]\rangle\rangle \tag{5.104}$$

This leads to Christensen's method, [66, 67], to compute the energy-momentum tensor and related quantities, such as trace anomalies. We will not pursue this point of view here although it could be done. It is in fact strictly connected with the main approach we will follow later on, which we consider simpler. They are both based on fermion propagators such as $\widehat{\mathbb{S}}^{(1)}(x,x')$. A discussion of fermion propagators and their properties in a MAT background is presented in Appendix 5.A.

5.3.2 The Dirac operator and its inverse

In the action (5.94) the Dirac operator is

$$\widehat{F} = i\widehat{\gamma}\cdot\widehat{\nabla} = i\widehat{\gamma}^\mu\widehat{\nabla}_\mu = i\gamma^a\widehat{e}^\mu_a\widehat{\nabla}_\mu \equiv \gamma^a\,\widehat{F}_a \tag{5.105}$$

where the $\widehat{\nabla}$ operator is, schematically, $\widehat{D} + \frac{1}{2}\widehat{\Omega}$ and satisfies $\widehat{\nabla}_\mu \widehat{e}^a_\nu = 0$.

Under AE diffeomorphisms ψ transforms as: $\delta_{\hat{\xi}}\psi = \widehat{\xi}\cdot\partial\psi$, while

$$\delta_{\hat{\xi}}\left(i\widehat{\gamma}\cdot\widehat{\nabla}\psi\right) = \overline{\widehat{\xi}}\cdot\partial\left(i\widehat{\gamma}\cdot\widehat{\nabla}\psi\right) \tag{5.106}$$

Under AE Weyl transformation $\widehat{F}$ transform as

$$\delta_{\hat{\omega}}\widehat{F} = -\frac{1}{2}\gamma^a\{\widehat{F}_a, \widehat{\omega}\} \tag{5.107}$$

and it has the following hermiticity property

$$\widehat{F}^\dagger = \eta\widehat{F}\eta \tag{5.108}$$

where $\eta = \gamma_0$ and γ_0 is the nondynamical (flat) gamma matrix. To obtain (5.108) use $\widehat{\Omega}^\dagger = -\eta\overline{\widehat{\Omega}}^\dagger\eta$, etc.

Integrating out the fermion field in (5.94) means, roughly speaking, evaluating the determinant of the Dirac operator $\widehat{F}$. This is however not what we need. First, because the log of the determinant is formally the trace of the log of $\widehat{F}$; taking this trace means integrating over spacetime and tracing over the gamma matrices: this would suppress any explicit γ_5 dependence and, thus, any axial splitting. Second, because $\widehat{F}$ is local, while, in order to exploit a coincidence limit (in order to guarantee covariance), we need a bilocal quantity. This quantity exists, it is the inverse of $\widehat{F}$: the fermion propagator. The Schwinger-DeWitt method is based on it. Let us explain this approach, adapting it to MAT.

One starts from the propagator

$$\widehat{G}(\widehat{x},\widehat{x}') = \langle 0|\mathcal{T}\psi(\widehat{x})\psi^\dagger(\widehat{x}')|0\rangle \tag{5.109}$$

which satisfies

$$i\sqrt{\widehat{g}}\eta\,\widehat{\gamma}^\mu\widehat{\nabla}_\mu\widehat{G}(\widehat{x},\widehat{x}') = -\mathbf{1}\delta(\widehat{x},\widehat{x}') \tag{5.110}$$

where $\mathbf{1}$ is the unit matrix in the spinor space. $\widehat{G}$ is not yet what we need. The Schwinger-DeWitt method requires a quadratic operator and, in addition, we must get rid of the γ matrices, except γ_5. This is achieved with the ansatz

$$\widehat{G}(x,x') = -i\overline{\widehat{\gamma}}^\mu\overline{\widehat{\nabla}}_\mu\overline{\widehat{\mathcal{G}}}(x,x')\eta^{-1} \tag{5.111}$$

Remark 2. Why the ansatz (5.111). In ordinary gravity, from the diff invariance of the fermion action, we can extract the transformation rule

$$\delta_\xi\left(i\gamma^\mu\nabla_\mu\psi\right) = \xi\cdot\partial\left(i\gamma\cdot\nabla\psi\right) \tag{5.112}$$

while $\delta_\xi\psi = \xi\cdot\partial\psi$. Therefore it makes sense to apply $\gamma\cdot\nabla$ to $\gamma\cdot\nabla\psi$, because the latter transforms as ψ. This allows us to define the square of the Dirac operator:

$$F^2\psi = \left(i\gamma\cdot\nabla\right)^2\psi \tag{5.113}$$

It is not possible to repeat the same thing for MAT because of (5.106), from which we see that $\left(i\widehat{\gamma}\cdot\widehat{\nabla}\psi\right)$ does not transform like ψ, and an expression like $\left(i\widehat{\gamma}\cdot\widehat{\nabla}\right)^2\psi$ would break general covariance. Noting that

$$\delta_{\widehat{\xi}}\left(i\overline{\widehat{\gamma}}\cdot\overline{\widehat{\nabla}}\psi\right) = \widehat{\xi}\cdot\partial\left(i\overline{\widehat{\gamma}}\cdot\overline{\widehat{\nabla}}\psi\right) \tag{5.114}$$

when $\delta_{\overline{\overline{\xi}}}\psi = \overline{\overline{\xi}}\cdot\partial\psi$, we will consider instead the *covariant* quadratic operator

$$\left(i\overline{\overline{\gamma}}\cdot\overline{\overline{\nabla}}\right)\left(i\widehat{\gamma}\cdot\widehat{\nabla}\right)\psi \tag{5.115}$$

Let us quote next a few useful identities.

$$\overline{\overline{\widehat{\nabla}}}_\mu\widehat{\gamma}_\nu - \widehat{\gamma}_\nu\widehat{\nabla}_\mu = \gamma^a\left(\partial_\mu\,\widehat{e}_{a\nu} - \widehat{\Gamma}^\lambda_{\mu\nu}\widehat{e}_{a\lambda} + \frac{1}{2}\widehat{\Omega}_{\mu ab}\,\widehat{e}^b_\nu\right) = 0 \tag{5.116}$$

because of metricity, and

$$\overline{\overline{\widehat{\nabla}}}_\mu\gamma^a - \gamma^a\widehat{\nabla}_\mu = 0 \tag{5.117}$$

The axial conjugate relation holds as well. Therefore

$$\widehat{\gamma}^\mu\widehat{\nabla}_\mu\,\overline{\overline{\widehat{\gamma}}}^\nu\overline{\overline{\widehat{\nabla}}}_\nu = \gamma^a\gamma^b\overline{\overline{\widehat{e}}}^\mu_a\overline{\overline{\widehat{e}}}^\nu_b\overline{\overline{\widehat{\nabla}}}_\mu\overline{\overline{\widehat{\nabla}}}_\nu = \eta^{ab}\overline{\overline{\widehat{e}}}^\mu_a\overline{\overline{\widehat{e}}}^\nu_b\overline{\overline{\widehat{\nabla}}}_\mu\overline{\overline{\widehat{\nabla}}}_\nu + \Sigma^{ab}\overline{\overline{\widehat{e}}}^\mu_a\overline{\overline{\widehat{e}}}^\nu_b[\overline{\overline{\widehat{\nabla}}}_\mu,\overline{\overline{\widehat{\nabla}}}_\nu] \tag{5.118}$$

On the other hand, when acting on a (bi-)spinor quantity

$$\Sigma^{ab}\overline{\overline{\widehat{e}}}^\mu_a\overline{\overline{\widehat{e}}}^\nu_b[\overline{\overline{\widehat{\nabla}}}_\mu,\overline{\overline{\widehat{\nabla}}}_\nu] = \frac{1}{8}\gamma^a\gamma^b\gamma^c\gamma^d\widehat{R}_{abcd} = -\frac{1}{4}\widehat{R}_{\mu\nu\lambda\rho}\widehat{g}^{\mu\lambda}\widehat{g}^{\nu\rho} = -\frac{1}{4}\widehat{R} \tag{5.119}$$

where use is made of

$$\widehat{R}_{abcd} = \widehat{e}^\mu_a\widehat{e}^\nu_b\widehat{e}^\lambda_c\widehat{e}^\rho_d\widehat{R}_{\mu\nu\lambda\rho}. \tag{5.120}$$

Now replacing (5.111) into (5.110) and using the above we get

$$\sqrt{|\overline{\overline{\widehat{g}}}|}\left(\overline{\overline{\widehat{\nabla}}}_\mu\overline{\overline{\widehat{g}}}^{\mu\nu}\overline{\overline{\widehat{\nabla}}}_\nu - \frac{1}{4}\overline{\overline{\widehat{R}}}\right)\overline{\overline{\widehat{\mathcal{G}}}}(\widehat{x},\widehat{x}') = -\mathbf{1}\delta(\widehat{x},\widehat{x}') \tag{5.121}$$

The differential operator acting on $\overline{\overline{\widehat{\mathcal{G}}}}$ will be denoted by $\overline{\overline{\widehat{\mathcal{F}}}}_{\hat{g}}$. In compact operator notation

$$\overline{\overline{\widehat{\mathcal{F}}}}_{\hat{g}}\overline{\overline{\widehat{\mathcal{G}}}}_{\hat{g}} = -\mathbf{1}, \tag{5.122}$$

with $\langle\widehat{x}|\widehat{\mathcal{G}}_{\hat{g}}|\widehat{x}'\rangle = \overline{\overline{\widehat{\mathcal{G}}}}_{\hat{g}}(\widehat{x},\widehat{x}')$.

As a consequence of (5.108) we have

$$\left[\sqrt{\widehat{g}}\left(\widehat{\nabla}_\mu \widehat{g}^{\mu\nu}\widehat{\nabla}_\nu - \frac{1}{4}\widehat{R}\right)\right]^\dagger = \eta\left[\sqrt{|\widehat{g}|}\left(\widehat{\nabla}_\mu \widehat{g}^{\mu\nu}\widehat{\nabla}_\nu - \frac{1}{4}\widehat{R}\right)\right]\eta \tag{5.123}$$

or

$$\left(\widehat{\mathcal{F}}_{\hat{g}}\right)^\dagger = \eta\,\widehat{\mathcal{F}}_{\hat{g}}\,\eta \tag{5.124}$$

We shall refer often to the related operator

$$\widehat{\mathcal{F}} = \frac{1}{\sqrt{\widehat{g}}}\,\widehat{\mathcal{F}}_{\hat{g}}, \qquad \widehat{\mathcal{F}}^\dagger = \eta\,\widehat{\mathcal{F}}\eta \tag{5.125}$$

and to its inverse $\widehat{\mathcal{G}}$: $\widehat{\mathcal{F}}\widehat{\mathcal{G}} = -\mathbf{1}$.

Remark 3. The operator $\widehat{\mathcal{F}}$ is the main intermediate result of this chapter. It is natural to assume that its inverse $\widehat{\mathcal{G}}$ exists. There is no reason to believe that it does not, because, the differential operator $\widehat{\mathcal{F}}$ (after a Wick rotation) can be defined as an axial-elliptic operator, at least under reasonable conditions on the axial tensor $f_{\mu\nu}$. In fact its quadratic part can be cast in the form $-\partial_i A_{ij}(x)\partial_j$, where A_{ij} is an invertible matrix and its dominating part is symmetric and positive definite. However, no doubt, it would be desirable to have a mathematical (possibly constructive) proof of the existence of $\widehat{\mathcal{G}}$. In Appendix C we discuss this issue and, following [13], we give some arguments in this direction.

5.4 The Schwinger proper time method

From now on, for practical reasons, we drop the bar symbol of axial conjugation. At the end we will axially-conjugate the result.

Let us define the point-to-point amplitude

$$\langle \widehat{x}, \widehat{s} | \widehat{x}', 0\rangle = \langle \widehat{x} | e^{i\widehat{\mathcal{F}}\widehat{s}} | \widehat{x}'\rangle \tag{5.126}$$

which satisfies the (heat kernel) differential equation

$$i\frac{\partial}{\partial \widehat{s}}\langle \widehat{x}, \widehat{s} | \widehat{x}', 0\rangle = -\widehat{\mathcal{F}}_{\hat{x}}\langle \widehat{x}, \widehat{s} | \widehat{x}', 0\rangle \equiv K(\widehat{x}, \widehat{x}', \widehat{s}) \tag{5.127}$$

where $\widehat{\mathcal{F}}_{\hat{x}}$ is the differential operator

$$\widehat{\mathcal{F}}_{\widehat{x}} = \widehat{\nabla}_\mu \widehat{g}^{\mu\nu} \widehat{\nabla}_\nu - \frac{1}{4}\widehat{R} \tag{5.128}$$

Then we make the ansatz

$$\langle \widehat{x}, \widehat{s} | \widehat{x}', 0\rangle = -\lim_{m\to 0} \frac{i}{16\pi^2} \frac{\sqrt{\widehat{D}(\widehat{x},\widehat{x}')}}{\widehat{s}^2} e^{i\left(\frac{\widehat{\sigma}(\widehat{x},\widehat{x}')}{2\widehat{s}} - m^2\widehat{s}\right)} \widehat{\Phi}(\widehat{x},\widehat{x}',\widehat{s}) \tag{5.129}$$

where $\widehat{D}(\widehat{x},\widehat{x}')$ is the VVM determinant and $\widehat{\sigma}$ is the world function (see above). $\widehat{\Phi}(\widehat{x},\widehat{x}',\widehat{s})$ is a function to be determined. It is useful to introduce also the mass parameter m, which we will eventually set to zero. In the limit $\widehat{s} \to 0$ the RHS of (5.129) becomes the definition of a delta function multiplied by $\widehat{\Phi}$. More precisely, since it must be $\langle \widehat{x}, 0 | \widehat{x}', 0\rangle = \delta(\widehat{x},\widehat{x}')$, and

$$\lim_{\widehat{s}\to 0} \frac{i}{4\pi^2} \frac{\sqrt{\widehat{D}(\widehat{x},\widehat{x}')}}{\widehat{s}^2} e^{i\left(\frac{\widehat{\sigma}(\widehat{x},\widehat{x}')}{2\widehat{s}} - m^2\widehat{s}\right)} = \sqrt{|\widehat{g}(\widehat{x})|}\ \delta(\widehat{x},\widehat{x}'), \tag{5.130}$$

we must have

$$\lim_{\widehat{s}\to 0} \widehat{\Phi}(\widehat{x},\widehat{x}',\widehat{s}) = \mathbf{1} \tag{5.131}$$

Eq.(5.127) becomes an equation for $\widehat{\Phi}(\widehat{x},\widehat{x}',\widehat{s})$. Using (5.34) and (5.69), after some algebra one gets

$$i\frac{\partial \widehat{\Phi}}{\partial \widehat{s}} + \frac{i}{\widehat{s}} \widehat{\nabla}^\mu \widehat{\Phi} \widehat{\nabla}_\mu \widehat{\sigma} + \frac{1}{\sqrt{\widehat{D}}} \widehat{\nabla}^\mu \widehat{\nabla}_\mu \left(\sqrt{\widehat{D}}\widehat{\Phi}\right) - \left(\frac{1}{4}\widehat{R} - m^2\right)\widehat{\Phi} = 0 \tag{5.132}$$

Now we expand

$$\widehat{\Phi}(\widehat{x},\widehat{x}',\widehat{s}) = \sum_{n=0}^{\infty} \widehat{a}_n(\widehat{x},\widehat{x}')(i\widehat{s})^n \tag{5.133}$$

with the boundary condition $[\widehat{a}_0] = 1$. The $\widehat{a}_n$ must satisfy the recursive relations:

$$(n+1)\widehat{a}_{n+1} + \widehat{\nabla}^\mu \widehat{a}_{n+1} \widehat{\nabla}_\mu \widehat{\sigma} - \frac{1}{\sqrt{\widehat{D}}} \widehat{\nabla}^\mu \widehat{\nabla}_\mu \left(\sqrt{\widehat{D}}\widehat{a}_n\right) + \left(\frac{1}{4}\widehat{R} - m^2\right)\widehat{a}_n = 0 \tag{5.134}$$

Using these relations and the coincidence results of section 3.3, 3.4 and 3.5, it is possible to compute each coefficient a_n at the coincidence limit.

5.4.1 Computing $\widehat{a}_n$

In this subsection we wish to compute $[\widehat{a}_1]$ and $[\widehat{a}_2]$, which will be needed later on. We start from (5.134) for $n = -1$.:

$$\widehat{\nabla}^\mu \widehat{a}_0\, \sigma_{;\mu} = 0, \qquad \text{with} \qquad [\widehat{a}_0] = \mathbf{1}, \tag{5.135}$$

which implies that

$$\widehat{a}_0(\widehat{x}, \widehat{x}') = \widehat{I}(\widehat{x}, \widehat{x}'). \tag{5.136}$$

Replacing this inside (5.134) for $n = 0$ one gets

$$\widehat{a}_1(\widehat{x}, \widehat{x}') + \widehat{\nabla}^\mu \widehat{\sigma} \nabla_\mu \widehat{a}_1(\widehat{x}, \widehat{x}') - \frac{1}{\sqrt{\widehat{\Delta}}} \widehat{\nabla}^\mu \widehat{\nabla}_\mu \left(\sqrt{\widehat{\Delta}}\, \widehat{I}(\widehat{x}, \widehat{x}') \right) + \left(\frac{1}{4} \widehat{R} - m^2 \right) \widehat{I}(\widehat{x}, \widehat{x}') = 0, \tag{5.137}$$

which implies

$$[\widehat{a}_1] = \left(-\frac{1}{12} \widehat{R} + m^2 \right) \mathbf{1} \tag{5.138}$$

Moreover differentiating (5.137) with respect to ∇_λ and taking the coincidence limit:

$$2[\widehat{\nabla}_\lambda \widehat{a}_1] \;\; = \;\; \frac{1}{4} \widehat{R}_{;\lambda} \mathbf{1} - [\sqrt{\widehat{\Delta}}_{;\mu}{}^{\mu}{}_{\lambda} \widehat{I} + \widehat{\nabla}_\lambda \widehat{\nabla}^\mu \widehat{\nabla}_\mu \widehat{I}]$$

so

$$[\widehat{\nabla}_\lambda \widehat{a}_1] \;\; = \;\; \left(\frac{1}{12} \widehat{\mathcal{R}}_{\lambda\nu;}{}^{\nu} - \frac{1}{24} \widehat{R}_{;\lambda} \right) \mathbf{1}. \tag{5.139}$$

Next we have

$$[\widehat{\nabla}^\lambda \widehat{\nabla}_\lambda \left(\widehat{a}_1 + \widehat{\nabla}^\mu \widehat{\sigma}\, \widehat{\nabla}_\mu \widehat{a}_1 \right)] = 3[\widehat{\nabla}^\lambda \widehat{\nabla}_\lambda \widehat{a}_1]$$

so that

$$[\widehat{\nabla}^{\lambda}\widehat{\nabla}_{\lambda}\widehat{a}_1] = \frac{1}{3}\left[\widehat{\nabla}^{\lambda}\widehat{\nabla}_{\lambda}\left(\frac{1}{\sqrt{\widehat{\Delta}}}\widehat{\nabla}^{\mu}\widehat{\nabla}_{\mu}\left(\sqrt{\widehat{\Delta}}\,\widehat{I}\right) - \left(\frac{1}{4}\widehat{R} - m^2\right)\widehat{I}\right)\right] \tag{5.140}$$

$$= \frac{1}{3}\left(-\frac{1}{20}\widehat{R}_{;\mu}{}^{\mu} - \frac{1}{30}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} + \frac{1}{30}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho} + \frac{1}{8}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right) \tag{5.141}$$

Finally

$$[\widehat{a}_2] = \frac{1}{2}\left[\widehat{\nabla}^{\lambda}\widehat{\nabla}_{\lambda}\widehat{a}_1 - \left(\frac{1}{12}\widehat{R} - m^2\right)\widehat{a}_1\right] \tag{5.142}$$

$$= \frac{1}{2}m^4 - \frac{1}{12}m^2\widehat{R} + \frac{1}{288}\widehat{R}^2 - \frac{1}{120}\widehat{R}_{;\mu}{}^{\mu} - \frac{1}{180}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} + \frac{1}{180}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho} + \frac{1}{48}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}$$

We recall that $\widehat{\mathcal{R}}_{\mu\nu} = \widehat{R}_{\mu\nu}{}^{ab}\Sigma_{ab}$.

5.5 The odd trace anomaly

We are now ready to compute that odd parity trace anomaly. Beside the point-splitting, which we have used above, we need a regulator to get rid of the infinities at coincident point. We will use two regularizations: the dimensional and zeta function ones.

5.5.1 Schwinger-DeWitt and dimensional regularization

We start again from the Dirac operator (5.105). We have defined above the covariant square

$$\widehat{\mathcal{F}} = -\overline{\widehat{F}}\widehat{F} \tag{5.143}$$

We identify the effective action for Dirac fermions with

$$\widehat{W} = -\frac{i}{2}\mathrm{Tr}\left(\ln\,\widehat{\mathcal{F}}\right) \tag{5.144}$$

The trace Tr includes also the spacetime integration. The AE Weyl variation of (5.144) is given by

$$\delta_{\widehat{\omega}}\widehat{W} = \frac{i}{2}\mathrm{Tr}\left(\widehat{\mathcal{G}}\,\delta_{\widehat{\omega}}\widehat{\mathcal{F}}\right) \tag{5.145}$$

where

$$\widehat{\mathcal{F}}\widehat{\mathcal{G}} = -1 \tag{5.146}$$

So we can write

$$\delta_{\widehat{\omega}}\widehat{W} = \delta_{\widehat{\omega}}\left(-\frac{1}{2}\int_0^\infty \frac{d\widehat{s}}{i\widehat{s}} e^{i\widehat{\mathcal{F}}\,\widehat{s}}\right) = -\frac{1}{2}\mathrm{Tr}\left(\int_0^\infty d\widehat{s}\, e^{i\widehat{\mathcal{F}}\,\widehat{s}}\delta_{\widehat{\omega}}\widehat{\mathcal{F}}\right). \tag{5.147}$$

It follows that, as far as the variation with respect to axial-Weyl transform is concerned, the effective action can be represented as

$$\widehat{W} = -\frac{1}{2}\int_0^\infty \frac{d\widehat{s}}{i\widehat{s}} e^{i\widehat{\mathcal{F}}\widehat{s}} + \mathrm{const} \equiv \widehat{L} + \mathrm{const} \tag{5.148}$$

where $\widehat{L}$ is the relevant effective action

$$\widehat{L} = \int d^d\widehat{x}\, \widehat{L}(\widehat{x}) \tag{5.149}$$

which can be written as

$$\widehat{L}(\widehat{x}) = -\frac{1}{2}\mathrm{tr}\int_0^\infty \frac{\mathrm{d}\widehat{\mathrm{s}}}{\mathrm{i}\widehat{\mathrm{s}}}\widehat{\mathrm{K}}(\widehat{\mathrm{x}}, \widehat{\mathrm{x}}', \widehat{\mathrm{s}}) \tag{5.150}$$

where the kernel $\widehat{K}$ is defined by

$$\widehat{K}(\widehat{x}, \widehat{x}', \widehat{s}) = e^{i\widehat{\mathcal{F}}\,\widehat{s}}\delta(\widehat{x}, \widehat{x}') \tag{5.151}$$

Inserted in $\delta_{\widehat{\omega}}\widehat{W}$, under the symbol Tr, it means integrating over x after taking the limit $x' \to x$. So, looking at (5.129), in dimension d,

$$\widehat{K}(\widehat{x}, \widehat{x}, \widehat{s}) = \frac{i}{(4\pi i\widehat{s})^{\frac{d}{2}}}\sqrt{\widehat{g}}\, e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})] \tag{5.152}$$

A specification is in order at this point. For the heat kernel method to work a Riemannian metric is required. Therefore at this stage we Wick-rotate the metric, so that the operator $\widehat{\mathcal{F}}$ becomes axial-elliptic. This operation is understood from now on. After calculating the anomaly we will return to the Lorentz signature.

5.5.2 Analytic continuation in d

The purpose now is to analytically continue in d. But we can do this only for dimensionless quantities. We therefore multiply $\widehat{L}$ by μ^{-d}, where μ is a mass parameter. We have for a Dirac fermion

$$\frac{\widehat{L}(x)}{\mu^d} = -\frac{i}{2}(4\pi\mu^2)\mathrm{tr}\int_0^\infty \mathrm{d}\widehat{\mathrm{s}}\,(4\pi\mathrm{i}\mu^2\widehat{\mathrm{s}})^{-\frac{\mathrm{d}}{2}-1}\sqrt{\widehat{\mathrm{g}}}\mathrm{e}^{-\mathrm{im}^2\widehat{\mathrm{s}}}[\widehat{\Phi}(\widehat{\mathrm{x}},\widehat{\mathrm{x}},\widehat{\mathrm{s}})] \tag{5.153}$$

where tr denotes the trace over gamma matrices. Now we make the assumption that

$$\lim_{s\to\infty} e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})] = 0 \tag{5.154}$$

As a consequence we can integrate by parts

$$\begin{aligned}
\frac{\widehat{L}(x)}{\mu^d} &= \frac{i}{d}\mathrm{tr}\int_0^\infty \mathrm{d}\widehat{\mathrm{s}}\frac{\partial}{\partial(\mathrm{i}\widehat{\mathrm{s}})}(4\pi\mathrm{i}\mu^2\widehat{\mathrm{s}})^{-\frac{\mathrm{d}}{2}}\sqrt{\widehat{\mathrm{g}}}\mathrm{e}^{-\mathrm{im}^2\widehat{\mathrm{s}}}[\widehat{\Phi}(\widehat{\mathrm{x}},\widehat{\mathrm{x}},\widehat{\mathrm{s}})] &(5.155)\\
&= -\frac{i}{d}\mathrm{tr}\int_0^\infty \mathrm{d}\widehat{\mathrm{s}}\,(4\pi\mathrm{i}\mu^2\widehat{\mathrm{s}})^{-\frac{\mathrm{d}}{2}}\sqrt{\widehat{\mathrm{g}}}\frac{\partial}{\partial(\mathrm{i}\widehat{\mathrm{s}})}\left(\mathrm{e}^{-\mathrm{im}^2\widehat{\mathrm{s}}}[\widehat{\Phi}(\widehat{\mathrm{x}},\widehat{\mathrm{x}},\widehat{\mathrm{s}})]\right)\\
&= \frac{2i}{d(2-d)4\pi\mu^2}\mathrm{tr}\int_0^\infty \mathrm{d}\widehat{\mathrm{s}}\,(4\pi\mathrm{i}\mu^2\widehat{\mathrm{s}})^{1-\frac{\mathrm{d}}{2}}\sqrt{\widehat{\mathrm{g}}}\frac{\partial^2}{\partial(\mathrm{i}\widehat{\mathrm{s}})^2}\left(\mathrm{e}^{-\mathrm{im}^2\widehat{\mathrm{s}}}[\widehat{\Phi}(\widehat{\mathrm{x}},\widehat{\mathrm{x}},\widehat{\mathrm{s}})]\right)\\
&= -\frac{4i}{d(2-d)(4-d)}\frac{1}{(4\pi\mu^2)^2}\mathrm{tr}\int_0^\infty \mathrm{d}\widehat{\mathrm{s}}\,(4\pi\mathrm{i}\mu^2\widehat{\mathrm{s}})^{2-\frac{\mathrm{d}}{2}}\sqrt{\widehat{\mathrm{g}}}\frac{\partial^3}{\partial(\mathrm{i}\widehat{\mathrm{s}})^3}\left(\mathrm{e}^{-\mathrm{im}^2\widehat{\mathrm{s}}}[\widehat{\Phi}(\widehat{\mathrm{x}},\widehat{\mathrm{x}},\widehat{\mathrm{s}})]\right)
\end{aligned}$$

Next we use

$$[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})] = 1 + [\widehat{a}_1]i\widehat{s} + [\widehat{a}_2](i\widehat{s})^2 + \ldots \tag{5.156}$$

and, around $d=2$, we use $\frac{1}{d(2-d)} = \frac{1}{2}\left(\frac{1}{d-2} - \frac{1}{d}\right)$ and in the third line of (5.155) we use

$$(4\pi i\mu^2 s)^{1-\frac{d}{2}} = 1 - \frac{d-2}{2}\ln(4\pi i\mu^2 s) + \ldots$$

Then we differentiate once $[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]$, and the remaining derivation we get rid of by integrating by parts. Finally one gets

$$\begin{aligned}
\widehat{L}(\widehat{x}) &= \frac{1}{4\pi}\left(\frac{1}{d-2} - \frac{1}{2}\right)\mathrm{tr}\left(([\widehat{\mathrm{a}}_1] - \mathrm{m}^2)\sqrt{\widehat{\mathrm{g}}}\right) &(5.157)\\
&\quad -\frac{i}{8\pi}\mathrm{tr}\int_0^\infty \mathrm{d}\widehat{\mathrm{s}}\,\ln(4\pi\mathrm{i}\mu^2\widehat{\mathrm{s}})\sqrt{\widehat{\mathrm{g}}}\frac{\partial^2}{\partial(\mathrm{i}\widehat{\mathrm{s}})^2}\left(\mathrm{e}^{-\mathrm{im}^2\widehat{\mathrm{s}}}[\widehat{\Phi}(\widehat{\mathrm{x}},\widehat{\mathrm{x}},\widehat{\mathrm{s}})]\right)
\end{aligned}$$

Around $d=4$ we use $\frac{1}{d(d-2)(d-4)} \approx \frac{1}{8}\left(\frac{1}{d-4}-\frac{3}{4}\right)$. With reference to the last line of (5.155), we differentiate twice $[\widehat{\Phi}(x,x,s)]$ and integrate by parts the third derivative. The result is

$$\begin{aligned}
\widehat{L}(\widehat{x}) \quad \approx \quad & \frac{1}{32\pi^2}\left(\frac{1}{d-4}-\frac{3}{4}\right) \mathrm{tr}\left(\mathrm{m}^4 - 2\mathrm{m}^2[\widehat{\mathrm{a}}_1] + 2[\widehat{\mathrm{a}}_2]\right)\sqrt{\widehat{\mathrm{g}}} \qquad (5.158)\\
& + \frac{i}{64\pi^2}\mathrm{tr}\int_0^\infty d\widehat{\mathrm{s}}\, \ln(4\pi \mathrm{i}\mu^2\widehat{\mathrm{s}})\sqrt{\widehat{\mathrm{g}}}\frac{\partial^3}{\partial(\mathrm{i}\widehat{\mathrm{s}})^3}\left(\mathrm{e}^{-\mathrm{im}^2\widehat{\mathrm{s}}}[\widehat{\Phi}(\widehat{\mathrm{x}},\widehat{\mathrm{x}},\widehat{\mathrm{s}})]\right)
\end{aligned}$$

The last line depends explicitly on the parameter μ and represent a nonlocal part.

5.5.3 The anomaly

Let us take the variation of (5.158) with respect to $\widehat{\omega} = \omega + \gamma_5 \eta$. Recall that

$$\begin{aligned}
\delta_{\widehat{\omega}}\sqrt{\widehat{g}} &= d\,\widehat{\omega}\,\sqrt{\widehat{g}} && (5.159)\\
\delta_{\widehat{\omega}}\widehat{R} &= -2\widehat{\omega}\,\widehat{R} - 2(d-1)\widehat{\Box}\widehat{\omega} && (5.160)\\
\delta_{\widehat{\omega}}\widehat{R}_{\mu\nu\lambda}{}^{\rho} &= -\delta^\rho_\nu \widehat{D}_\mu\widehat{D}_\lambda\widehat{\omega} + \delta^\rho_\mu\widehat{D}_\nu\widehat{D}_\lambda\widehat{\omega} + \widehat{D}_\mu\widehat{D}_\sigma\widehat{\omega}\,\widehat{g}^{\rho\sigma}\widehat{g}_{\nu\lambda} - \widehat{D}_\nu\widehat{D}_\sigma\widehat{\omega}\,\widehat{g}^{\rho\sigma}\widehat{g}_{\mu\lambda} && (5.161)
\end{aligned}$$

From these follows, for instance,

$$\begin{aligned}
\delta_{\widehat{\omega}}\left(\sqrt{\widehat{g}}\widehat{R}^2\right) &= (d-4)\sqrt{\widehat{g}}\,\widehat{\omega}\,\widehat{R}^2 - 4(d-1)\widehat{R}\sqrt{\widehat{g}}\,\widehat{\Box}\widehat{\omega} && (5.162)\\
\delta_{\widehat{\omega}}\left(\sqrt{\widehat{g}}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu}\right) &= (d-4)\widehat{\omega}\sqrt{\widehat{g}}\,\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} + 2(2-d)\sqrt{\widehat{g}}\,\widehat{R}^{\mu\nu}\widehat{D}_\mu\widehat{D}_\nu\widehat{\omega} - 2\sqrt{\widehat{g}}\,\widehat{R}\widehat{\Box}\widehat{\omega}\\
&= (d-4)\widehat{\omega}\sqrt{\widehat{g}}\,\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} - d\sqrt{\widehat{g}}\,\widehat{R}\widehat{\Box}\widehat{\omega} && (5.163)\\
\delta_{\widehat{\omega}}\left(\sqrt{\widehat{g}}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho}\right) &= (d-4)\widehat{\omega}\sqrt{\widehat{g}}\,\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho} - 8\sqrt{\widehat{g}}\,\widehat{R}^{\mu\nu}\widehat{D}_\mu\widehat{D}_\nu\widehat{\omega}\\
&= (d-4)\widehat{\omega}\sqrt{\widehat{g}}\,\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho} - 4\sqrt{\widehat{g}}\,\widehat{R}\widehat{\Box}\widehat{\omega} && (5.164)\\
\delta_{\widehat{\omega}}\left(\sqrt{\widehat{g}}\widehat{\Box}\widehat{R}\right) &= (d-4)\widehat{\omega}\sqrt{\widehat{g}}\,\widehat{\Box}\widehat{R} + (d-6)\sqrt{\widehat{g}}\,\partial_\mu\widehat{\omega}\,\partial^\mu\widehat{R} - 2\sqrt{\widehat{g}}\,\widehat{R}\,\widehat{\Box}\,\widehat{\omega}\\
&\quad -2(d-1)\sqrt{\widehat{g}}\,\widehat{\Box}^2\,\widehat{\omega}\\
&= 0 && (5.165)\\
\delta_{\widehat{\omega}}\mathrm{tr}\left(\sqrt{\widehat{\mathrm{g}}}\,\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right) &= (d-4)\mathrm{tr}\left(\widehat{\omega}\sqrt{\widehat{\mathrm{g}}}\,\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right) + 4\,\mathrm{tr}\left(\sqrt{\widehat{\mathrm{g}}}\,\widehat{\mathrm{R}}^{\mu\nu}\widehat{\mathrm{D}}_\mu\widehat{\mathrm{D}}_\nu\widehat{\omega}\right)\\
&= (d-4)\mathrm{tr}\left(\widehat{\omega}\sqrt{\widehat{\mathrm{g}}}\,\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right) + 2\,\mathrm{tr}\left(\sqrt{\widehat{\mathrm{g}}}\,\widehat{\mathrm{R}}\widehat{\Box}\widehat{\omega}\right) && (5.166)
\end{aligned}$$

In the first line of (5.158) one can ignore m^2 or m^4 terms (either one sets $m=0$ or they can be subtracted because they are trivial). The second line (5.158) does not contain singularities when $d \to 4$: it contains either vanishing or finite terms in this limit. Let us denote the second line by $\widehat{L}_R$.

$$\widehat{L} = \frac{1}{16\pi^2}\left(\frac{1}{d-4} - \frac{3}{4}\right)\int d^d\widehat{x}\,\mathrm{tr}\left([\widehat{\mathrm{a}}_2]|_{\mathrm{m}=0}\sqrt{\widehat{\mathrm{g}}}\right) + \widehat{\mathrm{L}}_{\mathrm{R}} \tag{5.167}$$

We now act with $\delta_{\widehat{\omega}} = \int d^d\widehat{x}\, 2\mathrm{tr}\left(\widehat{\omega}\,\widehat{\mathrm{g}}_{\mu\nu}\frac{\delta}{\delta\widehat{\mathrm{g}}_{\mu\nu}}\right)$[2]. From (5.159)-(5.163) it follows that

$$\delta_{\widehat{\omega}}\mathrm{tr}\left(\sqrt{\widehat{\mathrm{g}}}\,[\widehat{\mathrm{a}}_2]|_{\mathrm{m}=0}\right) = (\mathrm{d}-4)\mathrm{tr}\left(\sqrt{\widehat{\mathrm{g}}}\,\widehat{\omega}\,[\widehat{\mathrm{a}}_2]|_{\mathrm{m}=0}\right) - \frac{\mathrm{d}-4}{120}\mathrm{tr}\left(\sqrt{\widehat{\mathrm{g}}}\,\widehat{\mathrm{R}}\widehat{\square}\widehat{\omega}\right) \tag{5.168}$$

The second piece can be canceled e.g. by a counterterm proportional to $\mathrm{tr}\left(\sqrt{\widehat{\mathrm{g}}}\widehat{\mathrm{R}}^2\right)$. Using the fact that the bare part of the action is Weyl invariant $\delta_{\widehat{\omega}}\widehat{L} = 0$ and that the renormalised part $\widehat{L}_R$ defines the (quantum) energy momentum tensor $\frac{2}{\sqrt{\widehat{g}}}\frac{\delta}{\delta\widehat{g}_{\mu\nu}}\widehat{L}_R = \widehat{\Theta}^{\mu\nu}$ we get

$$\int d^d\widehat{x}\,\mathrm{tr}\left(\widehat{\omega}\sqrt{\widehat{\mathrm{g}}}\,\widehat{\mathrm{g}}_{\mu\nu}\widehat{\Theta}^{\mu\nu}\right) = -\frac{1}{16\pi^2}\int d^d\widehat{x}\mathrm{tr}\left(\sqrt{\widehat{\mathrm{g}}}\,\widehat{\omega}\,[\widehat{\mathrm{a}}_2]|_{\mathrm{m}=0}\right) \tag{5.169}$$

where the $d-4$ factor in (5.168) canceled the pole $\frac{1}{d-4}$ in (5.167).

Clearly, the odd parity anomaly can come only from the term $\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}$ contained in $[\widehat{a}_2]$, with a coefficient of $\frac{1}{32\pi^2}$ (for Majorana fermions, $\times 2$ for Dirac fermions). For the odd part we have

$$\int d^d\widehat{x}\,\mathrm{tr}\sqrt{\widehat{\mathrm{g}}}\,\widehat{\omega}\,\widehat{\mathcal{T}} = -\frac{1}{768\pi^2}\int \mathrm{d}^4\mathrm{x}\,\mathrm{tr}\sqrt{\widehat{\mathrm{g}}}\,\widehat{\omega}\,\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\Big|_{\mathrm{odd}} \tag{5.170}$$

where we denoted $\widehat{\mathcal{T}} = \widehat{g}_{\mu\nu}\widehat{\Theta}^{\mu\nu} = \widehat{g}_{\mu\nu}\langle\!\langle \widehat{T}^{\mu\nu}\rangle\!\rangle$. The (odd parity) coefficient of ω defines $\mathcal{T}$ and the (odd parity) coefficient of η defines $\mathcal{T}_5$. Setting $\widehat{\mathcal{T}} = \mathcal{T} + \gamma_5\mathcal{T}_5$ one obtains in this way[3]

$$\mathcal{T} = -\frac{1}{4}\frac{1}{768\pi^2}\mathrm{tr}\left(\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right)\Big|_{\mathrm{odd}} = \frac{1}{4}\frac{2\mathrm{i}}{768\pi^2}\epsilon^{\mu\nu\lambda\rho}\mathrm{R}^{(1)}_{\mu\nu\alpha\beta}\mathrm{R}^{(2)\alpha\beta}_{\lambda\rho} \tag{5.171}$$

$$\mathcal{T}_5 = -\frac{1}{4}\frac{1}{768\pi^2}\mathrm{tr}\left(\gamma_5\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right)\Big|_{\mathrm{odd}} = \frac{1}{4}\frac{\mathrm{i}}{768\pi^2}\epsilon^{\mu\nu\lambda\rho}\left(\mathrm{R}^{(1)}_{\mu\nu\alpha\beta}\mathrm{R}^{(1)\alpha\beta}_{\lambda\rho} + \mathrm{R}^{(2)}_{\mu\nu\alpha\beta}\mathrm{R}^{(2)\alpha\beta}_{\lambda\rho}\right) \tag{5.172}$$

In the last step we have Wick-rotated back the result: this is the origin of the i in the

[2]In MAT case, $\widehat{g}_{\mu\nu}$ also has two spinor indices, so that $\omega\, g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} \to \widehat{\omega}_{AB}\,\widehat{g}_{\mu\nu\,BC}\frac{\delta}{\delta\widehat{g}_{\mu\nu\,AC}}$. Since in our case γ^5 is symmetric, we have $\widehat{a}_{AB} = \widehat{a}_{BA}$ and we can write $\delta_{\widehat{\omega}}$ as $\int d^d\widehat{x}\, 2\mathrm{tr}\left(\widehat{\omega}\,\widehat{\mathrm{g}}_{\mu\nu}\frac{\delta}{\delta\widehat{\mathrm{g}}_{\mu\nu}}\right)$.

[3]Here we changed the convention for Levi-Civita tensor with respect to [20], that is, we use $\epsilon^{0123} = 1$.

anomaly coefficient. At this point we can safely set $x_2^\mu = 0$ everywhere.

5.5.4 ζ-function regularization

Given a differential operator A in analogy with the Riemann ζ function, the expression A^{-z}, for complex z, is called ζ function regularization of A:

$$\zeta(z, A) = A^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty dt\, t^{z-1}\, e^{-tA} \tag{5.173}$$

We will apply this representation to the operator $\widehat{\mathcal{F}}(\widehat{x}, \widehat{x})$:

$$(\widehat{\mathcal{F}}(\widehat{x}))^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty dt\, t^{z-1}\, \langle \widehat{x}| e^{-t\widehat{\mathcal{F}}} |\widehat{x}\rangle \tag{5.174}$$

where $\langle \widehat{x}| e^{-t\widehat{\mathcal{F}}} |\widehat{x}\rangle$ means the coincidence limit of $\langle \widehat{x}| e^{-t\widehat{\mathcal{F}}} |\widehat{x}'\rangle$. Eq.(5.174) is not quite correct because only dimensionless quantities can be raised to an arbitrary power. Moreover the object of interest will be $\widehat{\mathcal{G}}$, rather than $\widehat{\mathcal{F}}$. Thus we introduce again the mass parameter μ and shift from t to $i\widehat{s}\mu$.

$$\zeta(\widehat{x}, z) \equiv (\mu^2 \widehat{\mathcal{G}}(\widehat{x}, \widehat{x}))^z = \frac{1}{\Gamma(z)} \int_0^\infty (i\mu^2) d\widehat{s}\, (i\widehat{s}\mu^2)^{z-1}\, \langle x| e^{i\widehat{s}\widehat{\mathcal{F}}} |\widehat{x}\rangle \tag{5.175}$$

Finally we replace $\langle \widehat{x}| e^{i\widehat{s}\widehat{\mathcal{F}}} |\widehat{x}\rangle$ with $\widehat{K}(\widehat{x}, \widehat{x}, \widehat{s})$ in eq.(5.152). The result is

$$\zeta(\widehat{x}, z) = (\mu^2 \widehat{\mathcal{G}}(\widehat{x}, \widehat{x}))^z = \frac{i}{\Gamma(z)} \frac{\mu^d}{(4\pi)^{\frac{d}{2}}} \sqrt{g} \int_0^\infty (i\mu^2) d\widehat{s}\, (i\widehat{s}\mu^2)^{z-1-\frac{d}{2}}\, e^{-im^2\widehat{s}} [\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})] \tag{5.176}$$

which can be rewritten as

$$\begin{aligned} \zeta(\widehat{x}, z) = (\mu^2 \widehat{\mathcal{G}}(\widehat{x}, \widehat{x}))^z \quad &= \quad -\frac{i}{\Gamma(z)} \frac{\mu^{d-4}}{(4\pi)^{\frac{d}{2}}} \frac{\sqrt{g}}{(z-\frac{d}{2})(z-\frac{d}{2}+1)(z-\frac{d}{2}+2)} \\ &\times \quad \int_0^\infty d(i\widehat{s})\, (i\widehat{s}\mu^2)^{z-\frac{d}{2}+2} \frac{\partial^3}{\partial(i\widehat{s})^3} \left(e^{-im^2\widehat{s}} [\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})] \right) \end{aligned} \tag{5.177}$$

This is well defined for $d = 4$ at $z = 0$

$$\zeta(\widehat{x}, 0) = \frac{i\sqrt{g}}{2(4\pi)^2} \left[\frac{\partial^2}{\partial(i\widehat{s})^2} \left(e^{-im^2\widehat{s}} [\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})] \right) \right]_{\widehat{s}=0} \tag{5.178}$$

Now, differentiating (5.173) with respect to z and evaluating at $z=0$, we get formally

$$\frac{d}{dz}\zeta(z,A)|_{z=0} = -\mathrm{Tr}\ln A \tag{5.179}$$

This suggest the procedure to regularize $\widehat{W}$ (which is the trace of a log). More precisely

$$\widehat{W} \to \widehat{W}_\zeta = -\frac{i}{2}\zeta'(0), \qquad \text{where} \qquad \zeta(z) = \int \mathrm{tr}\,\zeta(\widehat{\mathrm{x}},\mathrm{z})\mathrm{d}^{\mathrm{d}}\widehat{\mathrm{x}} \tag{5.180}$$

As a consequence for $d=4$:

$$\begin{aligned}\widehat{L}_\zeta(x) \;&=\; \frac{1}{64\pi^2}(\gamma+\frac{3}{2}-ln(4\pi))\sqrt{\widehat{g}}\,\mathrm{tr}\left(2[\widehat{\mathrm{a}}_2\widehat{(\mathrm{x})}]-2\mathrm{m}^2[\widehat{\mathrm{a}}_1(\widehat{\mathrm{x}})]+\mathrm{m}^4\right) \\ &\quad -\frac{i}{64\pi^2}\sqrt{\widehat{g}}\int_0^\infty d\widehat{s}\,\ln(4\pi i\mu^2\widehat{s})\frac{\partial^3}{\partial(i\widehat{s})^3}\left(e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]\right)\end{aligned} \tag{5.181}$$

Now, suppose that the operator A, under a symmetry transformation with parameter ϵ, transforms as

$$\delta_\epsilon A = \{A,\epsilon\}. \tag{5.182}$$

Then

$$\delta_\epsilon \mathrm{Tr}A^{-z} = -2z\mathrm{Tr}\left(A^{-z}\epsilon\right) = -2z\mathrm{Tr}\left(\zeta(z,A)\epsilon\right) \tag{5.183}$$

Since the relevant result is obtained by differentiating with respect to z and setting $z=0$, once the functional is regularized, the anomalous part of the effective action is extremely easy to derive:

$$\widehat{L}_{\mathcal{A}} = -2\mathrm{Tr}\left(\zeta(0,A)\epsilon\right) \tag{5.184}$$

Let us return to the our problem. The operator to be regulated is $\widehat{\mathfrak{F}} = \widehat{\mathfrak{F}}_{\hat{x}}$. Its AE Weyl transformation is

$$\begin{aligned}\delta_{\widehat{\omega}}\widehat{\mathfrak{F}} \;&=\; -2\widehat{\omega}\,\widehat{\mathfrak{F}} + \left(\overline{\widehat{\gamma}}^\mu\widehat{\gamma}^\nu + \widehat{g}^{\mu\nu}\right)\partial_\nu\widehat{\omega}\widehat{\nabla}_\mu + \frac{3}{2}\widehat{\Box}\widehat{\omega} \\ &=\; -2\widehat{\omega}\,\widehat{\mathfrak{F}} + \widehat{\mathfrak{F}}\left[\frac{1}{\widehat{\mathfrak{F}}}\left(\left(\overline{\widehat{\gamma}}^\mu\widehat{\gamma}^\nu + \widehat{g}^{\mu\nu}\right)\partial_\nu\widehat{\omega}\widehat{\nabla}_\mu + \frac{3}{2}\widehat{\Box}\widehat{\omega}\right)\right]\end{aligned}$$

$\widehat{\mathcal{G}}(\widehat{x}, \widehat{x})$ is the inverse of $\widehat{\mathcal{F}}$ and its transformation is similar:

$$\delta_{\hat{\omega}}\widehat{\mathcal{G}} = 2\,\widehat{\mathcal{G}}\,\widehat{\omega} + \widehat{\mathcal{G}}\left[\left(\left(\overline{\widehat{\gamma}}^{\mu}\widehat{\gamma}^{\nu} + \widehat{g}^{\mu\nu}\right)\partial_{\nu}\widehat{\omega}\widehat{\nabla}_{\mu} + \frac{3}{2}\widehat{\Box}\widehat{\omega}\right)\widehat{\mathcal{G}}\right]$$

The first piece in the RHS reproduces exactly the mechanism in (5.183). The second is a nonlocal term of the effective action; it does not concern us here and we drop it. As noticed above this procedure does not lead directly to the anomaly. It rather gives the anomalous part of the effective action, i.e. the anomaly integrated with the insertion of $\sqrt{\widehat{g}}$:

$$\begin{aligned} \widehat{L}_{\mathcal{A}}(\widehat{\omega}) &= -i\mathrm{Tr}\left(\widehat{\omega}\,\zeta(\widehat{x}, 0)\right) \qquad (5.185)\\ &= i\,\mathrm{Tr}\left(\frac{\sqrt{\widehat{g}}}{2(4\pi)^2}\left[\frac{\partial^2}{\partial(i\widehat{s})^2}\left(e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})]\right)\right]_{s=0}\widehat{\omega}\right)\\ &= i\,\mathrm{Tr}\left(\frac{\sqrt{\widehat{g}}}{2(4\pi)^2}\left(2[\widehat{a}_2(\widehat{x})] - 2m^2[\widehat{a}_1(\widehat{x})] + m^4\right)\widehat{\omega}\right) \end{aligned}$$

Now, proceeding as before, we differentiate with respect to $\widehat{\omega}$ and strip off $\sqrt{\widehat{g}}$, multiply back $\widehat{\omega}$ and obtain the true integrated anomaly. This leads to the same results as above.

5.5.5 The collapsing limit

After computing the trace anomalies (5.171) and (5.172) of a Dirac fermion coupled to a metric and an axial symmetric tensor, we are now interested in returning to the original problem, that is the trace anomaly of a Weyl tensor in an chiral fermion theory coupled to ordinary gravity. To this end we take the collapsing limit. In [19] the latter was defined as $h_{\mu\nu} \to \frac{h_{\mu\nu}}{2}, k_{\mu\nu} \to \frac{h_{\mu\nu}}{2}$, with $h_{\mu\nu}$ and $k_{\mu\nu}$ both infinitesimal. Here we do not put such a limitation. The collapsing limit is defined by making the replacements

$$g_{\mu\nu} \to \eta_{\mu\nu} + \frac{h_{\mu\nu}}{2} \qquad , \qquad f_{\mu\nu} \to \frac{h_{\mu\nu}}{2} \qquad (5.186)$$

in the previous formulas, with finite $h_{\mu\nu}$. With this choice one has

$$\hat{g}_{\mu\nu} = \frac{1}{2}(1-\gamma^5)\,\eta_{\mu\nu} + \frac{1}{2}(1+\gamma^5)\,G_{\mu\nu} \qquad , \qquad G_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} \qquad (5.187)$$

From this we see that the right-handed part couples to the flat metric, while the left-handed part couples to the (generic) metric $G_{\mu\nu}$. As a consequence we have also

$$\widehat{e}_m^a \to \delta_m^a \frac{1-\gamma_5}{2} + e_m^a \frac{1+\gamma_5}{2}, \qquad \widehat{e}_a^m \to \delta_a^m \frac{1-\gamma_5}{2} + e_a^m \frac{1+\gamma_5}{2}, \tag{5.188}$$

as well as

$$\sqrt{\widehat{g}} \to \frac{1-\gamma_5}{2} + \frac{1+\gamma_5}{2}\sqrt{G}, \tag{5.189}$$

Similarly for the Christoffel symbols

$$\Gamma_{\mu\nu}^{(1)\lambda} \to \frac{1}{2}\Gamma_{\mu\nu}^{\lambda}, \qquad \Gamma_{\mu\nu}^{(2)\lambda} \to \frac{1}{2}\Gamma_{\mu\nu}^{\lambda}, \tag{5.190}$$

for the spin connections

$$\Omega_\mu^{(1)ab} \to \frac{1}{2}\omega_\mu^{ab}, \qquad \Omega_\mu^{(2)ab} \to \frac{1}{2}\omega_\mu^{ab}, \tag{5.191}$$

and for the curvatures

$$R_{\mu\nu\lambda}^{(1)\ \ \rho} \to \frac{1}{2}R_{\mu\nu\lambda}{}^{\rho}, \qquad R_{\mu\nu\lambda}^{(2)\ \ \rho} \to \frac{1}{2}R_{\mu\nu\lambda}{}^{\rho}, \tag{5.192}$$

where all the quantities on the RHS of these limits are built with the metric $G_{\mu\nu}$.

As a consequence, the action (5.94) becomes

$$\widehat{S} \longrightarrow S' = \int d^4x \left[i\overline{\psi}\gamma^a \frac{1-\gamma_5}{2}\partial_a\psi + \int d^4x \sqrt{G}\, i\overline{\psi}\gamma^a e_a^\mu \left(\partial_\mu + \frac{1}{2}\omega_\mu\right)\frac{1+\gamma_5}{2}\psi\right] \tag{5.193}$$

where γ^a is the flat (non-dynamical) gamma matrix while the vierbein e_a^μ and the connection ω_μ are compatible with the metric $G_{\mu\nu}$. Up to the term that represents a decoupled right-handed fermion in the flat spacetime, the action S' is the action of a left-handed Weyl fermion coupled to the ordinary gravity.

In the collapsing limit we have

$$\mathcal{T}(x) = \mathcal{T}_5(x) = \frac{1}{16}\frac{2i}{768\pi^2}\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta} \tag{5.194}$$

The integrated anomaly (5.170) corresponding to $\widehat{S}$ thus becomes

$$\begin{aligned}\int d^d\widehat{x}\,\mathrm{tr}\sqrt{\widehat{\mathrm{g}}}\,\widehat{\omega}\,\widehat{\mathcal{T}} &= \int d^dx\,\sqrt{G}\,(\omega+\eta)\,(\mathcal{T}+\mathcal{T}_5)\,\mathrm{trP}_+ + \int \mathrm{d^dx}\,(\omega-\eta)\,(\mathcal{T}-\mathcal{T}_5)\,\mathrm{trP}_- \\ &= 4\int d^dx\,\sqrt{G}\,\omega_+\,\mathcal{T} \qquad (5.195)\end{aligned}$$

where we used $\mathrm{trP}_+ = 2$, $\mathcal{T}-\mathcal{T}_5 = 0$ and set $\omega_+ = \omega+\eta$. Notice that due to (5.187) the transformation property of $G_{\mu\nu}$ is $G_{\mu\nu} \to e^{2\omega_+}G_{\mu\nu}$. To extract an anomaly of the left fermion of the effective action corresponding to (5.193) we take its Weyl variation with respect to the metric $G_{\mu\nu}$

$$\int d^dx\,\sqrt{G}\,\omega_+\,\mathcal{T}' \qquad (5.196)$$

where we denoted $\mathcal{T}' = G_{\mu\nu}\Theta'^{\mu\nu} = G_{\mu\nu}\langle\!\langle T'^{\mu\nu}\rangle\!\rangle$. Comparing (5.195) and (5.196) we get

$$\mathcal{T}'(x) = \frac{i}{1536\pi^2}\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta} \qquad (5.197)$$

If we instead of (5.186) take the following collapsing limit

$$g_{\mu\nu} \to \eta_{\mu\nu} + \frac{h_{\mu\nu}}{2} \qquad , \qquad f_{\mu\nu} \to -\frac{h_{\mu\nu}}{2} \qquad (5.198)$$

then one obtains

$$\hat{g}_{\mu\nu} = \frac{1}{2}(1-\gamma^5)\,G_{\mu\nu} + \frac{1}{2}(1+\gamma^5)\,\eta_{\mu\nu} \qquad , \qquad G_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} \qquad (5.199)$$

Now the left handed part is coupled to the flat metric and right handed part to generic curved metric. We can now repeat the arguments from above and obtain the Pontryagin Weyl anomaly for right-handed Weyl fermion

$$\mathcal{T}'(x) = -\frac{i}{1536\pi^2}\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta}. \qquad (5.200)$$

The relative minus sign with respect to left-handed case is because of the opposite sign in front of γ_5 matrix in the defining relation for projectors $P_\pm$.

Appendices

5.A Green's functions

In the text we have assumed the existence of the propagator $\widehat{\mathcal{G}}$, the inverse of $\widehat{\mathcal{F}}$. In this Appendix we discuss this question by comparing it with the ordinary case, as discussed in [13]. First we review the approach of [13] in the ordinary gravity case. Then we explain the modifications required in the MAT case. We consider the case of a stationary metric and axial-metric background. We will assume eventually that the results hold also for nonstationary case, provided the background varies mildly in time.

In this Appendix the flat gamma matrices are understood to be the Majorana ones, that is, they are purely imaginary, together with γ_5: $\gamma_0 \equiv \eta$ and γ_5 are antisymmetric, while γ_i, $i = 1, 2, 3$ are symmetric.

5.A.1 A summary of Green's functions

Let us give first a short review of ordinary fermionic propagators, see [13, 14, 66, 67]. We start from

$$G(x, x') = \langle 0|\mathcal{T}\psi(x)\psi^\dagger(x')|0\rangle \tag{5.201}$$

This is not the standard Feynman Green function

$$S_F(x, x') = \langle 0|\mathcal{T}\psi(x)\bar{\psi}(x')|0\rangle \tag{5.202}$$

The two are related by $S_F(x, x') = G(x, x')\eta$

Other Green functions are the advanced, $G^+(x, x')$, and retarded, $G^-(x, x')$; the positive and negative frequency Green functions, $G^{(+)}(x, x')$ and $G^{(-)}(x, x')$, respectively; and the principal value Green function $\bar{G}(x,x') = \frac{1}{2}\left(G^{+}(x,x') + G^{-}(x,x')\right)$. The definitions depends only on the contour of integration of p^0 in the momentum space representation, while for the rest they are the same. The important relation in this context is

$$G(x,x') = \bar{G}(x,x') + \frac{i}{2}G^{(1)}(x,x'), \qquad G^{(1)} = i\left(G^{(+)} - G^{(-)}\right) \tag{5.203}$$

For real fermions $\bar{G}(x,x')$ and $G^{(1)}(x,x')$ are real. So they represent the real and imaginary part of $G(x,x')$. $G^{(1)}(x,x')$ can be represented as

$$G^{(1)}(x,x') = \langle 0|[\psi(x), \psi^{\dagger}(x')]|0\rangle \equiv \mathcal{S}^{(1)}(x,x') \tag{5.204}$$

The Feynman propagator satisfies the equation

$$i\sqrt{g}\eta\left(\gamma^{\mu}\nabla_{\mu} + m\right) G(x,x') = -\mathbf{1}\delta(x,x') \tag{5.205}$$

and $\mathbf{1}$ is the identity matrix in the spinor space. Both sides of (5.205) transform as a bispinor density, i.e. like $\sqrt{g}\gamma_0\psi(x)$ at x and as $\psi^{\dagger}(x')$ at x'. Instead

$$i\sqrt{g}\eta\left(\gamma^{\mu}\nabla_{\mu} + m\right) G^{(1)}(x,x') = 0 \tag{5.206}$$

The approach of [66, 67] is based essentially on $G^{(1)}$.

Now let us make the ansatz

$$G(x,x') = -i\left(\gamma^{\mu}\nabla_{\mu} - m\right) \mathcal{G}(x,x')\eta^{-1} \tag{5.207}$$

Inserting this into (5.205) one gets

$$\sqrt{g}\left(\nabla_{\mu}g^{\mu\nu}\nabla_{\nu} - \left(m^2 + \frac{1}{4}R\right)\right)\mathcal{G}(x,x') = -\mathbf{1}\delta(x,x') \tag{5.208}$$

Now we represent (5.208) as

$$\int dx''\mathcal{F}(x,x'')\mathcal{G}(x'',x') = -\mathbf{1}\delta(x,x') \tag{5.209}$$

or, in operator form,

$$\mathcal{F}\,\mathcal{G} = -1 \tag{5.210}$$

(understanding $\langle x|\mathcal{G}|x'\rangle = \mathcal{G}(x,x')$, etc.), where

$$\mathcal{F}(x,x') = \sqrt{g}\left(\nabla_\mu g^{\mu\nu}\nabla_\nu - \left(m^2 + \frac{1}{4}R\right)\right)\mathbf{1}\delta(x,x') \tag{5.211}$$

and the function and derivatives in the RHS are understood to be evaluated at x. Alternatively we represent (5.208) as

$$\mathcal{F}_x\,\mathcal{G}(x,x') = -\mathbf{1}\delta(x,x') \tag{5.212}$$

where $\mathcal{F}_x$ is the differential operator acting on $\mathbf{1}\delta(x,x')$ in the RHS of (5.211).

5.A.2 Properties of $\mathcal{F}$

The operator $\mathcal{F}$ in (5.208) is not selfadjoint. In fact

$$\mathcal{F}^\dagger = \gamma_0\mathcal{F}\gamma_0 \tag{5.213}$$

This implies that the construction of a Green's function is not straightforward. In a stationary background a propagator is constructed out of modes which are stationary eigenfunctions (plane waves, at least asymptotically) with real frequencies. Given the Dirac equation

$$i(\gamma^\mu\nabla_\mu + m)u = 0 \tag{5.214}$$

by suitably fixing the gauge for diffeomorphisms, one can always define a complete set of eigenfunctions with real frequencies, symbolically $u_+ = \chi e^{-i\omega t}, u_- = \lambda e^{i\omega t}$, so that (understanding the indices and integration over the space momenta)

$$\psi = u_+ a + u_- a^\dagger \tag{5.215}$$

where $a, a^\dagger$ are annihilation, creation operators (see chapter 19 of [14]).

In the same way one can infer the existence of an analogous complete set of solutions, say v_+, v_- of

$$i(\gamma^\mu \nabla_\mu - m)v = 0 \tag{5.216}$$

Now, even if $\mathfrak{F}$ is not self-adjoint, we can construct the following operator

$$\mathcal{F} = \begin{pmatrix} 0 & \mathfrak{F} \\ \mathfrak{F}^\dagger & 0 \end{pmatrix} \tag{5.217}$$

which is self-adjoint, and whose inverse is

$$\mathcal{G} = \begin{pmatrix} 0 & \mathfrak{G}^\dagger \\ \mathfrak{G} & 0 \end{pmatrix} \tag{5.218}$$

The mode solutions of $\mathcal{F}$ are

$$\begin{pmatrix} 0 \\ u_+ \end{pmatrix}, \quad \begin{pmatrix} 0 \\ u_- \end{pmatrix}, \quad \begin{pmatrix} \gamma_0 v_+ \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \gamma_0 v_- \\ 0 \end{pmatrix} \tag{5.219}$$

which have all real frequencies. It follows that we can construct the Feynman propagator of $\mathcal{F}$. Following the argument of [14], end of chapter 20, it has the form

$$\mathcal{F}^{-1} = \begin{pmatrix} 0 & -\frac{i}{\mathfrak{F}^\dagger + i\epsilon} \\ -\frac{1}{\mathfrak{F}+i\epsilon} & 0 \end{pmatrix} \tag{5.220}$$

Comparing with (5.218) we get

$$\mathfrak{G} = -\frac{1}{\mathfrak{F} + i\epsilon} \tag{5.221}$$

5.A.3 Existence of mode functions

The existence of mode functions, i.e. solutions of the Dirac equation (5.214) of the type $u = \chi e^{i\omega t}$ with real ω, in a stationary background, is the basis for the existence of propagators. In [14] the problem is discussed as follows. One shows that one can cast

(5.214) in the form

$$Fu = 0, \qquad F = \frac{1}{2}\left\{B^\mu, \frac{\partial}{\partial x^\mu}\right\} - C \tag{5.222}$$

where

$$B^\mu = i\eta\gamma^\mu, \qquad C = -\frac{i}{4}\eta\{\gamma^\mu, \omega_\mu\} \tag{5.223}$$

The important thing is that, in the Majorana representation of the γ matrices, B^μ is a symmetric matrix, while C is antisymmetric, and they are both purely imaginary. By choosing the gauge $e_0^0 = 1, e_0^i = 0$ for the vierbein e, the operator F becomes

$$F = \frac{1}{2}\left\{B, \frac{\partial}{\partial t}\right\} - \mathcal{C} \tag{5.224}$$

where

$$B = i, \qquad \mathcal{C} = C - \frac{1}{2}\left\{B^i, \frac{\partial}{\partial x^i}\right\} \tag{5.225}$$

Again while B is symmetric imaginary with $-iB$ being positive definite, $\mathcal{C}$ is antisymmetric imaginary. Plugging the ansatz $u_A = \chi_A e^{-i\omega_A t}$ into $Fu = 0$ one gets the eigenvalue equation

$$(\mathcal{C} + i\omega_A B)\chi_A = 0 \tag{5.226}$$

Due to the abovementioned propertis of B and $\mathcal{C}$, one can find eigenvalues and eigenvectors. The eigenvalues ω_A can be taken real and positive.

5.A.4 What changes when the background is MAT

In this case the analogue of (5.213) is

$$\widehat{\mathfrak{F}}^\dagger = \eta\,\widehat{\mathfrak{F}}\,\eta \tag{5.227}$$

But as above we can proceed to construct the operator

$$\widehat{\mathcal{F}} = \begin{pmatrix} 0 & \widehat{\mathfrak{F}} \\ \widehat{\mathfrak{F}}^{\dagger} & 0 \end{pmatrix} \tag{5.228}$$

which is self-adjoint, and whose inverse is

$$\widehat{\mathcal{G}} = \begin{pmatrix} 0 & \widehat{\mathfrak{G}}^{\dagger} \\ \overline{\widehat{\mathfrak{G}}} & 0 \end{pmatrix}. \tag{5.229}$$

Using the same argument as above we can conclude that

$$\widehat{\mathfrak{G}} = -\frac{1}{\widehat{\mathfrak{F}} + i\epsilon} \tag{5.230}$$

The only delicate point in reaching this conclusion is the solutions of

$$i\,\widehat{\gamma}^{\mu}\widehat{\nabla}_{\mu}\, u = 0 \tag{5.231}$$

Eq.(5.214) is real, since the gamma matrices are purely imaginary. But, in (5.231), the presence of γ_5 poses a problem. In a representation in which the gamma matrices are purely imaginary, the γ_5 is also imaginary, thus eq.(5.231) is complex, and, based on the analogy with the previous subsection, one cannot be sure a priori that there are real frequency solutions. However we notice that the operator $\eta\widehat{F}$ is self-adjoint. This remark lends us a way out.

Another crucial point is the gauge fixing, so that one can end up with something analogue to (5.225), in which $-iB$ is positive definite. As we saw above, this is obtained by choosing in particular $e_0^0 = 1, e_0^i = 0$. In MAT the coefficient of γ^0 is $\widehat{e}_0^{\mu}$, which contains also $\gamma_5 c_0^{\mu}$. We shall choose $c_0^{\mu} = 0$. As a consequence the analogue of $Fu = 0$ is $\widehat{F}\widehat{u} = 0$ where

$$\hat{F} = \frac{1}{2}\left\{\widehat{B}, \frac{\partial}{\partial t}\right\} - \widehat{\mathcal{C}} \tag{5.232}$$

where $\widehat{B} = B$, i.e. symmetric and such that $-iB$ is positive definite. As for $\widehat{\mathcal{C}}$, it can be written as

$$\widehat{\mathcal{C}} = \widehat{\mathcal{C}}_a + \widehat{\mathcal{C}}_s \tag{5.233}$$

where $\widehat{\mathcal{C}}_a$ is imaginary antisymmetric and does not contain γ_5, while $\widehat{\mathcal{C}}_s$ is real, linear in γ_5 and symmetric. However altogether it is self-adjoint.

Plugging the ansatz $\widehat{u}_A = \widehat{\chi}_A e^{-i\omega_A t}$ into $\eta\widehat{F}\widehat{u} = 0$ one gets the equation

$$(\widehat{\mathcal{C}} - \omega_A)\widehat{\chi}_A = 0 \tag{5.234}$$

which is an eigenvalue equation for $\widehat{\mathcal{C}}$. Since the latter is self-adjoint we know there exists a complete set of eigenfunctions. This is what we need.

So the remaining question is: is the choice $c_0^\mu = 0$ permitted? In order to see this one has to check that the defining equations (4.15,4.16) for the axial-complex vierbein and the like in Appendix B are still valid. Now, suppose the ordinary gauge fixed vierbein satisfies such defining equation (which they do in [13]). Then we can set the axial-imaginary vierbein c and c^{-1} to 0, while preserving the defining relations. In other words, there is a large gauge freedom, and in particular we can choose $c_0^\mu = 0$.

Chapter 6

Higher spin theories

One interesting problem in quantum field theory is the construction of interacting quantum field theories with massless higher spin ($s > 2$) fields in flat spacetime. Reasons to study higher spins are diverse. First, while free HS theories are fine, once we try to turn on the interactions we find various inconsistencies in the form of "no-go" theorems [154]-[157], see [158]-[160] for a review. We review some of the possible obstacles which one could stumble upon: Weinberg, Aragone-Desser and Weinberg-Witten theorem. On the other hand, consistent theory of interacting higher spin fields (involving an infinite tower of higher spin fields) has been constructed by Vasiliev [24]-[27] in the framework of 4d AdS background.

Moreover, in open string theory we have an infinite tower of massive higher spin excitations where the mass is given by

$$M^2 \sim T(s+1) \sim \frac{1}{\alpha'}(s+1)$$

In the above formula $T \sim \frac{1}{\alpha'}$ is the tension of the string and s is the spin. In the tensionless limit of the theory, $\alpha' \to \infty$ the mass of the higher spin fields goes to zero. The dynamics of higher spin excitations is very important for better understanding of the quantum properties of string theory. Furthermore, there is a conjecture which states that string theory describes a broken phase of higher spin gauge theory [161]-[169]. Similar to Higgs mechanism that provides fundamental particles with mass, there is a possibility that a similar mechanism could generate massive states in string theory. For this reason, it is important to get a better understanding of higher spin gauge theory.

For motivational purposes, we will finish this chapter with a quick tour through the higher spin history.

6.1 No-go theorems

There are different "no-go" theorems putting serious constraints interacting higher spin theories, especially in flat space-time (see e.g. [158]-[160] and references within). We will review Weinberg theorem [154], Aragone-Desser theorem [156] and Weinberg-Witten theorem [157].

6.1.1 Weinberg theorem

Weinberg in 1964 showed, using S-matrix approach, that there are no consistent long-range interactions mediated by massless bosons with $s > 2$, see [154]. Let us consider S-matrix element with N external fields of momenta p_i, $i = 1, \dots, N$ and one massless spin-s field with momentum q and polarization vector $\epsilon^{\mu_1 \dots \mu_s}(q)$. We will assume soft limit $q \to 0$. The structure of the diagram for emission of soft spin-s field from the particle line with momentum p_i is

$$S(p_1, \dots, p_N, q, \epsilon) = \frac{g^i}{p^i \cdot q} p^i_{\mu_1} \dots p^i_{\mu_s} \epsilon^{\mu_1 \dots \mu_s}(q) S_{hard}(p_1, \dots, p_N) \tag{6.1}$$

where g^i is the coupling constant and S_{hard} describes the hard process. We used $q \to 0$ and the fact that both field i and spin-s field are on-shell. We get similar contribution from diagrams in which the spin-s field is attached to a different field. We still have to perform summation over all N fields since the full amplitude consists of contributions from all N fields. The total matrix element factorizes in the soft limit:

$$S(p_1, \dots, p_N, q, \epsilon) = \sum_{i=1}^{N} \frac{g^i}{p^i \cdot q} p^i_{\mu_1} \dots p^i_{\mu_s} \epsilon^{\mu_1 \dots \mu_s}(q) S_{hard}(p_1, \dots, p_N) \tag{6.2}$$

The polarization tensor $\epsilon^{\mu_1 \dots \mu_s}$ is transverse and traceless:

$$q_{\mu_1} \epsilon^{\mu_1 \dots \mu_s}(q) = 0, \qquad \eta_{\mu_1 \mu_2} \epsilon^{\mu_1 \dots \mu_s}(q) = 0 \tag{6.3}$$

It has more components than the physical polarizations of the massless field. We can eliminate this redundancy by demanding that the S-matrix element is independent of spurious polarizations. That is, we demand that the S-matrix element vanishes for

$$\epsilon^{\mu_1 \dots \mu_s}_{(spur)}(q) = q^{(\mu_1} \eta^{\mu_2 \dots \mu_s)}(q) \tag{6.4}$$

where $\eta^{\mu_1 \dots \mu_{s-1}}(q)$ is transverse and traceless

$$q_{\mu_1} \eta^{\mu_1 \dots \mu_{s-1}}(q) = 0, \qquad \eta^{\mu\mu_3 \dots \mu_{s-1}}_{\mu} = 0 \tag{6.5}$$

Spurious states decouple for any p^i_μ if

$$\sum_{i=1}^{N} q^i p^i_{\mu_1} \dots p^i_{\mu_{s-1}} = 0 \tag{6.6}$$

For generic momenta p^i this equation has solution only in two cases:

- For $s = 1$ (photon) the above equation becomes

$$\sum_{i=1}^{N} q^i = 0 \tag{6.7}$$

  This is the conservation of charge.

- For $s = 2$ (graviton) we have

$$\sum_{i=1}^{N} q^i p^i_\mu = 0 \tag{6.8}$$

  which is satisfied only if $g^i = \kappa$. This gives us the equivalence principle which says that all particles interact with gravitons with equal strength κ. We are left with

$$\sum_{i=1}^{N} p^i_\mu = 0 \tag{6.9}$$

  which represents energy-momentum conservation.

For $s > 2$ there is no solution for the above equation. Only $s \leq 2$ fields can give rise to long-distance interactions. Note that this argument does not rule out massless bosons with $s > 2$, it just says that there are no long-range interactions. There is still possibility for $s > 2$ massless fields to mediate short-range interactions. Massless higher spin fields can exist, but their coupling g^i in low energy limit $q \to 0$ vanishes. In [170, 171] the authors showed that long-range interactions with fermionic higher spin exist up to $s < \frac{5}{2}$.

6.1.2 Aragone-Desser theorem

Aragone and Desser in 1979 showed that higher spin fields cannot consistently interact with gravity, see [156]. They proved this by attempting to explicitly couple spin $\frac{5}{2}$ field to gravity.

Let us consider interaction of spin-$\frac{5}{2}$ with gravity up to quadratic order. Spin-$\frac{5}{2}$ is described by tensor-spinor ψ_{ab} an we couple it minimally to vielbein e^μ_a

$$S = \int d^4xe \left(-\frac{1}{2}\bar{\psi}_{ab}\not{D}\psi_{ab} - \bar{\psi}_{ab}\gamma_b\not{D}\gamma_c\psi_{ca} + 2\bar{\psi}_{ab}\gamma_b D_c\psi_{ca} + \frac{1}{4}\bar{\psi}_{aa}\not{D}\psi_{bb} - \bar{\psi}_{aa}D_b\gamma_c\psi_{bc} \right)$$

where e is square root of metric determinant $e = \sqrt{g}$. The field ψ_{ab} gives a redundant description of spin-$\frac{5}{2}$ field. The redundancy is removed by gauge invariance

$$\delta\psi_{ab} = \partial_a\epsilon_b + \partial_b\epsilon_a, \qquad \gamma^a\epsilon_a = 0 \tag{6.10}$$

To covariantize, we replace partial derivatives with covariant derivatives

$$\delta\psi_{ab} = D_a\epsilon_b + D_b\epsilon_a, \qquad \gamma^a\epsilon_a = 0 \tag{6.11}$$

The action transforms under this gauge transformation as

$$\delta S = -4\int d^4xe\bar{\epsilon}_a\gamma_b\psi_{cd}R^{abcd} \tag{6.12}$$

We conclude that the action is invariant only in flat spacetime where Riemann tensor vanishes $R^{abcd} = 0$. This means that gauge modes decouple only in the free theory.

This theorem rests on the Lagrangian formalism. This means that there is one major implicit assumption: locality. Consequently, introducing non-locality in the Lagrangian could avoid the difficulties.

6.1.3 Weinberg-Witten theorem

Finally, let us mention one more "no-go" theorem. Weinberg and Witten, using S-matrix approach, proved that a theory which allows a construction of a conserved Lorentz covariant energy-momentum tensor cannot contain massless particles of spin $s > 1$, see [157] (for a review see [172]). It states that no massless higher spin field can consistently interact with gravity in flat spacetime. The statement of the theorem goes as follows: "*A theory that allows the construction of a conserved Lorentz covariant energy-momentum tensor $T_{\mu\nu}$ for which $\int d^3xT^{0\nu}$ is the energy-momentum 4-vector cannot contain massless particles of spin $s > 2$.*"

Let us analyze the scattering of massless fields off soft gravitons. We assume that p is the initial momentum of the spin-s field, and that the final momentum is $p' = p + q$. The graviton is off-shell with momentum q. The S-matrix element we are interested in is:

$$\langle \pm s, p'|T_{\mu\nu}| \pm s, p\rangle \tag{6.13}$$

where $\pm s$ denotes the polarization of the spin-s field. In the soft limit $q \to 0$ the S-matrix element is determined by the equivalence principle

$$\langle \pm s, p'|T_{\mu\nu}| \pm s, p\rangle = p_\mu p_\nu \tag{6.14}$$

where we used the normalization $\langle p|p'\rangle = 2p_0(2\pi)^3\delta(\vec{p} - \vec{p}')$.

On the other hand, to show that the matrix element vanishes for $s > 1$ we choose a Lorentz frame in which

$$p = (|\vec{p}|, \vec{p}), \qquad p' = (\vec{p}, -\vec{p}) \tag{6.15}$$

this is always possible for $q^2 = 0$ because in that case $p + p'$ is timelike and by Poincaré covariance we can choose a Lorentz frame in which $p + p'$ has no spatial component. Let us now consider rotation $R(\theta)$ by an angle θ around the $\vec{p}$ direction. The one-particle states under this transformation become

$$\begin{aligned} | \pm s, p\rangle &\to e^{\pm i\theta s}| \pm s, p\rangle \\ | \pm s, p'\rangle &\to e^{\mp i\theta s}| \pm s, p'\rangle \end{aligned} \tag{6.16}$$

where the difference in sign comes from the fact that $R(\theta)$ is a rotation for $+\theta$ around $\vec{p}$ but $-\theta$ around $\vec{p}'$. Matrix element becomes:

$$e^{\pm 2i\theta s}\langle \pm s, p'|T_{\mu\nu}| \pm s, p\rangle = R(\theta)^{\mu}_{\rho}R(\theta)^{\nu}_{\sigma}\langle \pm s, p'|T_{\rho\sigma}| \pm s, p\rangle \tag{6.17}$$

Rotation matrix $R(\theta)$ has eigenvalues $e^{i\theta}$, 1 and $e^{-i\theta}$. Therefore, the above equation requires the matrix element to vanish unless $2s = 0, 1, 2$. Now, since we assumed that the energy-momentum tensor is Lorentz covariant, the matrix element has to vanish in all frames and for all p and p' for which $(p' - p)^2 = q^2 = 0$.

Note that this theorem does not apply to theories which do not have a Lorentz covariant energy-momentum tensor (like gravity). In other words if we want gauge invariance we must sacrifice Lorentz covariance.

Regardless of the "no-go" theorems, there are significant higher spin results: free fields can be constructed in the same manner as in lower spin cases (see, e.g. [173]). A few cubic interaction terms have been constructed in the literature (see [99]-[106]). And most notably, a fully consistent covariant higher spin theory, which includes an infinite tower of higher spin fields, in AdS background has been constructed by Vasiliev and collaborators [24]-[27]. Note that "no-go" theorems are mostly based on the S-matrix approach. In Vasiliev theory such "no-go" theorems are evaded because in AdS there is no genuine S-matrix.

6.2 History of higher spins

In this section we will make a quick review of higher spin theory throughout history, see [159, 161, 174, 175] and references therein. It is often stated that the theory of higher spins dates back to 1936 when Dirac tried to generalize his spin-$\frac{1}{2}$ equation [95]. In 1939 Fierz and Pauli [96] systematized the study of massive higher spin fields through Lorentz covariance and energy positivity. It took a long time before Singh and Hagen in 1974 in [97, 98] constructed the Lagrangian formulation of Fierz and Pauli equations. A few years later, Fronsdal [37, 38] investigated the massless limit of Singh-Hagen Lagrangian and found that, for the equation of motion to be invariant under gauge transformation, the gauge parameter must be constrained. Later on, Francia and Sagnotti found the unconstrained Fronsdal equations. We will restrict our historical tour to bosonic higher spin fields since they are the focus of this thesis.

6.2.1 Fierz-Pauli-Dirac

As already mentioned, Fierz and Pauli in their study of higher spins [96] required Lorentz invariance and energy positivity. Due to Wigner's work [176] on representations of Poincaré group and Bergman's and Wigner's work [177] on relativistic field equations, the positivity requirement was replaced by the condition that the one-particle states carry a unitary representation of Poincaré group. The symmetric rank-s tensor then satisfies

$$(\Box - m^2)\phi_{\mu_1 \ldots \mu_s} = 0 \tag{6.18}$$

$$\partial^{\mu_1}\phi_{\mu_1 \ldots \mu_s} = 0 \tag{6.19}$$

$$\eta^{\mu_1 \mu_2}\phi_{\mu_1 \ldots \mu_s} = 0 \tag{6.20}$$

Total symmetry of the higher spin field $\phi_{\mu_1 \ldots \mu_s}$ ensures that the field transforms in a desired representation. The first equation says that the Klein-Gordon equation must be satisfied, which we can see from the first Casimir invariant C_1. The transversality condition ensures that we are propagating the appropriate number of degrees of freedom. Casimir invariant C_2 requires that all lower spin values are eliminated and this is achieved by imposing the transversality condition. This condition is necessary for the energy to be positive definite. The third condition above, the tracelessness condition ensures that massive field representations are irreducible. Number of independent components of symmetric rank-s tensor $\phi_{\mu_1 \ldots \mu_s}$ is

$$\binom{d+s-1}{s} \tag{6.21}$$

Tracelessness condition removes $\binom{d+s-3}{s-2}$ components while the transversality condition eliminates $\binom{d+s-2}{s-1}$. However, we must be careful, because its trace part has already been included in the tracelessness condition. So we must add $\binom{d+s-4}{s-3}$ to avoid double counting. The total number of degrees of freedom is

$$\begin{aligned} \binom{d+s-1}{s} &- \binom{d+s-3}{s-2} - \binom{d+s-2}{s-1} + \binom{d+s-4}{s-3} \\ &= \binom{d+s-4}{s} + 2\binom{d+s-4}{s-1} \end{aligned} \tag{6.22}$$

6.2.2 Singh-Hagen

Singh and Hagen in [97] constructed a Lagrangian formulation for spin-s fields that gave the correct Fierz-Pauli conditions. The Singh-Hagen Lagrangian for integer spin can be written in terms of symmetric traceless tensor fields of rank s, s-2, s-3,... 0. Let us start with a simple example of spin-1:

$$L_{spin-1} = -\frac{1}{2}(\partial_\mu \phi_\nu)^2 - \frac{1}{2}(\partial \cdot \phi)^2 - \frac{m^2}{2}(\phi_\mu)^2 \tag{6.23}$$

where $\partial\phi$, $\partial \cdot \phi$ and ϕ' ($\phi^{[p]}$) denote gradient, divergence and trace (p-th trace) of the higher spin field. The corresponding equation of motion is the Proca equation

$$\Box\phi_\mu - \partial_\mu(\partial \cdot \phi) - m^2\phi_\mu = 0 \tag{6.24}$$

Taking the divergence of this equation we get the Fierz-Pauli transversality condition

$$\partial \cdot \phi = 0 \tag{6.25}$$

together with Klein-Gordon equation for ϕ_μ

$$\Box\phi_\mu - m^2\phi_\mu = 0 \tag{6.26}$$

Let us now turn to the generalization of the above result for spin-2 field. The Lagrangian for traceless field $\phi_{\mu\nu}$ is

$$L_{spin-2} = -\frac{1}{2}(\partial_\mu \phi_{\nu\rho})^2 + \frac{\alpha}{2}(\partial \cdot \phi_\mu)^2 - \frac{m^2}{2}(\phi_{\mu\nu})^2 \tag{6.27}$$

where we introduced constant α instead of 1. The equation of motion is

$$\Box\phi_{\mu\nu} - \frac{\alpha}{2}\left(\partial_\mu \partial \cdot \phi_\nu + \partial_\nu \partial \cdot \phi_\mu - \frac{2}{d}\eta_{\mu\nu}\partial \cdot \partial \cdot \phi\right) - m^2\phi_{\mu\nu} = 0 \tag{6.28}$$

Taking the divergence of this equation gives

$$\left(1 - \frac{\alpha}{2}\right)\Box\partial \cdot \phi_\nu + \alpha\left(\frac{1}{d} - \frac{1}{2}\right)\partial_\nu \partial \cdot \partial \cdot \phi - m^2 \partial \cdot \phi_\nu = 0 \tag{6.29}$$

In the above equation we used the fact that the field $\phi_{\mu\nu}$ is traceless. Note that in spin-2 case it is not possible to immediately get the transversality condition like for spin-1. We can get rid of some terms in the above equation by setting $\alpha = 2$, however we would still have to require $\partial \cdot \partial \cdot \phi = 0$ to obtain the Fierz-Pauli constraint. Because of that, let us proceed in the following way. Introduce an auxiliary field π so that the the condition $\partial \cdot \partial \cdot \phi = 0$ becomes a consequence of the field equations. To the original Lagrangian we add the Lagrangian L_π for the auxiliary field π

$$L_\pi = \pi \partial \cdot \partial \cdot \phi + c_1 (\partial_\mu \pi)^2 + c_2 \pi^2 \tag{6.30}$$

where c_1 and c_2 are constants which we still have to determine. The equations of motion for field $\phi_{\mu\nu}$ and π are

$$\begin{aligned} \phi_{\mu\nu} \quad &: \quad \Box \phi_{\mu\nu} - \left(\partial_\mu \partial \cdot \phi_\nu + \partial_\nu \partial \cdot \phi_\mu - \frac{2}{d} \eta_{\mu\nu} \partial \cdot \partial \cdot \phi \right) - m^2 \phi_{\mu\nu} + \partial_\mu \partial_\nu \pi - \frac{1}{d} \eta_\mu \nu \Box \pi = 0 \\ \pi \quad &: \quad \partial \cdot \partial \cdot \phi + 2(c_2 - c_1 \Box) \pi = 0 \end{aligned} \tag{6.31}$$

where $\alpha = 2$ is already used. Taking the divergence of the first equation twice

$$\left[(2-d)\Box - dm^2 \right] \partial \cdot \partial \cdot \phi + (d-1) \Box^2 \pi = 0 \tag{6.32}$$

The last equation together with the equation of motion for π can be regarded as a linear homogeneous system of equations in variables $\partial \cdot \partial \cdot \phi$ and π. The associated determinant is

$$\Delta = -2dm^2 c_2 + 2((2-d)c_2 + dm^2 c_1)\Box - (2(2-d)c_1 - (D-1))\Box^2 \tag{6.33}$$

This system of equation has a solution if the determinant does not vanish. We also require that the determinant does not depend on the D'Alambertian $\Box$. The determinant will be proportional to m^2. Due to these requirements we get constraints on constants c_1 and c_2

$$c_1 = \frac{d-1}{d(d-2)}, \qquad c_2 = \frac{m^2 d(d-1)}{2(d-2)^2}, \qquad d > 2 \tag{6.34}$$

The obtained solution is exactly the transversality condition together with the condition that the auxiliary field vanishes. Altogether we have

$$\pi = 0 \qquad \partial \cdot \partial \cdot \phi = 0 \tag{6.35}$$

$$\partial \cdot \phi_\nu = 0 \qquad (\Box - m^2)\phi_{\mu\nu} = 0 \tag{6.36}$$

One can follow a similar procedure for fields with spin $s > 2$. In that case $(s-1)$ auxiliary fields is needed to obtain Fierz-Pauli conditions.

6.2.3 Fronsdal

Let us now follow Fronsdal's approach [37, 38] and take $m \to 0$ limit of Sing-Hagen Lagrangian. We will see that in this particular limit, only the spin-s and the spin-(s-2) auxiliary fields remain and the rest of auxiliary fields decouple.

For $s = 2$, the limit $m \to 0$ of Singh-Hagen Lagrangian reads

$$L_{spin-2} = -\frac{1}{2}(\partial_\mu \phi_{\nu\rho})^2 + (\partial \cdot \phi_\mu)^2 + \pi \partial \cdot \partial \cdot \phi + \frac{d-1}{2(d-2)}(\partial_\mu \pi)^2 \tag{6.37}$$

The corresponding equations of motion are

$$\begin{aligned} \phi_{\mu\nu} \ &: \ \Box\phi_{\mu\nu} - \left(\partial_\mu \partial \cdot \phi_\nu + \partial_\nu \partial \cdot \phi_\mu - \frac{2}{d}\eta_{\mu\nu}\partial \cdot \partial \cdot \phi\right) + \partial_\mu \partial_\nu \pi - \frac{1}{d}\eta_\mu \nu \Box \pi = 0 \\ \pi \ &: \ \partial \cdot \partial \cdot \phi - \frac{d-1}{d-2}\Box\pi = 0 \end{aligned} \tag{6.38}$$

Next, let us introduce $\varphi_{\mu\nu}$, a new field which is a combination of $\phi_{\mu\nu}$ and π

$$\varphi_{\mu\nu} = \phi_{\mu\nu} + \frac{1}{d-2}\eta_{\mu\nu}\pi \tag{6.39}$$

and the equation of motion then becomes

$$\mathcal{F}_{\mu\nu} = \Box\varphi_{\mu\nu} - (\partial_\mu \partial \cdot \varphi_\nu + \partial_\nu \partial \cdot \varphi_\mu) + \partial_\mu \partial_\nu \varphi' = 0 \tag{6.40}$$

This is the linearized Einstein equation where the Fronsdal tensor $\mathcal{F}_{\mu\nu}$ is just linearized Ricci tensor $R_{\mu\nu}$. The Lagrangian is now

$$L_{spin-2} = -\frac{1}{2}(\partial_\mu \varphi_{\nu\rho})^2 + (\partial \cdot \varphi_\mu)^2 + \frac{1}{2}(\partial_\mu \varphi')^2 + \varphi' \partial \cdot \partial \cdot \phi \tag{6.41}$$

and it is invariant under gauge transformation

$$\delta\varphi_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu \tag{6.42}$$

This Lagrangian would give the Einstein equation

$$\mathcal{F}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{F}' = 0 \tag{6.43}$$

which, when combined with its trace $\mathcal{F}' = 0$ implies

$$\mathcal{F}_{\mu\nu} = 0 \tag{6.44}$$

Let us now try to generalize Fronsdal equation to spin-3 fields

$$\mathcal{F}_{\mu\nu\rho} = \Box\varphi_{\mu\nu\rho} - (\partial_\mu \partial \cdot \varphi_{\nu\rho} + perms) + (\partial_\mu \partial_\nu \varphi'_\rho + perms) = 0 \tag{6.45}$$

with gauge transformation

$$\delta\varphi_{\mu\nu\rho} = \partial_\mu \Lambda_{\nu\rho} + \partial_\nu \Lambda_{\rho\mu} + \partial_\rho \Lambda_{\mu\nu} \tag{6.46}$$

where Λ is a rank-2 tensor. Note that Fronsdal tensor is not immediately invariant under this transformation

$$\delta\mathcal{F}_{\mu\nu\rho} = 3\partial_\mu \partial_\nu \partial_\rho \Lambda' \tag{6.47}$$

Fronsdal tensor is invariant if the gauge parameter is constrained

$$\Lambda' = 0 \tag{6.48}$$

This condition on gauge parameter is quite strange and unnatural and we would like to avoid it. One approach to rewrite the Fronsdal equation in an unconstrained form is by introducing a rank-$(s-3)$ compensator field α which compensates for the non-vanishing term in (6.47). Second way to avoid constrained gauge parameter is to construct non-local equation of motion and Lagrangian.

But before we continue with the study of unconstrained Fronsdal equation, let us first describe general Fronsdal formulation for any spin. We can write

$$\mathcal{F}_{\mu_1\ldots\mu_s} = \Box\varphi_{\mu_1\ldots\mu_s} - (\partial_{\mu_1}\partial\cdot\varphi_{\mu_2\ldots\mu_s} + perms) + (\partial_{\mu_1}\partial_{\mu_2}\varphi'_{\mu_1\ldots\mu_s} + perms) = 0 \tag{6.49}$$

To simplify the notation we will omit the indices so that we will write for completely symmetric rank-s tensor field $\varphi \equiv \varphi_{\mu_1\cdots\mu_s}$. We also write $\partial^p\varphi$ for p-th gradient, $\partial^p\cdot\varphi$ for p-th divergence and $\varphi^{[p]}$ for p-th trace. Now Fronsdal equation can be written as

$$\mathcal{F} = \Box\varphi - \partial\,\partial\cdot\varphi + \partial^2\varphi' = 0 \tag{6.50}$$

In this expression standard higher spin conventions from [39, 107, 108] are assumed.[1] The Fronsdal equation (6.50) is invariant under local transformations parametrized by traceless completely symmetric rank-$(s-1)$ tensor fields $\Lambda \equiv \Lambda_{\mu_1\cdots\mu_{s-1}}$

$$\delta\varphi = \partial\Lambda \tag{6.51}$$

with

$$\Lambda' = 0 \tag{6.52}$$

We call this constraint on gauge parameter first Fronsdal constraint.

However, there is one more condition needed for the Lagrangian

$$L = \varphi\cdot\left(\mathcal{F} - \frac{1}{2}\eta\mathcal{F}'\right) \tag{6.53}$$

to be invariant. The variation of the Lagrangian is, up to total derivative,

$$\delta L = \delta\varphi\cdot\left(\mathcal{F} - \frac{1}{2}\eta\mathcal{F}'\right) = -s\Lambda\left(\partial\cdot\mathcal{F} - \frac{1}{2}\partial\mathcal{F}'\right) + \frac{s}{2}\Lambda'\partial\cdot\mathcal{F}' \tag{6.54}$$

The third term vanishes because of first Fronsdal condition and we are left with

$$\delta L = -s\Lambda\left(\partial\cdot\mathcal{F} - \frac{1}{2}\partial\mathcal{F}'\right) \tag{6.55}$$

[1] Conventions assume symmetrization over free indices with minimal number of terms and without any symmetry factors. Also, a prime denotes contraction of a pair of indices, so, e.g., $\varphi' \equiv \varphi_{\mu_1\cdots\mu_{s-2}} = \eta^{\mu_{s-1}\mu_s}\varphi_{\mu_1\cdots\mu_s}$ is a completely symmetric rank-$(s-2)$ tensor field.

To calculate $\left(\partial \cdot \mathcal{F} - \frac{1}{2}\partial\mathcal{F}'\right)$ we use

$$\begin{aligned}
\partial \cdot \mathcal{F} &= \Box\partial\varphi' - \partial\partial \cdot \partial \cdot \varphi + \partial^2\partial \cdot \varphi && (6.56)\\
\partial\mathcal{F}' &= 2\Box\partial\varphi' - 2\partial\partial \cdot \partial \cdot \varphi + 3\partial^3\varphi'' + 2\partial^2\partial \cdot \varphi && (6.57)
\end{aligned}$$

As it tuns out, Fronsdal operator satisfies the anomalous Bianchi identity

$$\partial \cdot \mathcal{F} - \frac{1}{2}\partial\mathcal{F}' = -\frac{3}{2}\partial^3\varphi'' \tag{6.58}$$

For $s \geq 4$ the Lagrangian is gauge invariant only if the field φ is subjected to the Fronsdal second condition

$$\varphi'' = 0 \tag{6.59}$$

From the Lagrangian (6.53) together with the two Frondal conditions (6.52) and (6.59) we get the equation of motion

$$\mathcal{F} - \frac{1}{2}\eta\mathcal{F}' = 0 \tag{6.60}$$

We can also introduce the Fronsdal-Einstein tensor

$$\mathcal{G} = \mathcal{F} - \frac{1}{2}\eta\mathcal{F}' \tag{6.61}$$

and write the Lagrangian as $L = \varphi\mathcal{G}$.

Let us now determine number of degrees of freedom for the constrained Fronsdal theory. A symmetric rank-s tensor which is double traceless has $\binom{d+s-1}{s} - \binom{d+s-5}{s-4}$ independent components. Furthermore, Fronsdal tensor $\mathcal{F}$ is gauge invariant under the condition $\Lambda' = 0$ and hence we can remove $\binom{d+s-2}{s-1} - \binom{d+s-4}{s-3}$ by imposing the de Donder gauge

$$\partial \cdot \varphi - \frac{1}{2}\partial\varphi' = 0 \tag{6.62}$$

which reduces the Fronsdal equation to

$$\Box\varphi = 0 \tag{6.63}$$

Now we see that φ really describes massless field. However, de Donder gauge does not completely fix the gauge since

$$\delta\left(\partial\cdot\varphi - \frac{1}{2}\partial\varphi'\right) = \Box\Lambda \tag{6.64}$$

Because of that, we still have freedom to gauge away $\binom{d+s-2}{s-1} - \binom{d+s-4}{s-3}$. Altogether, we have

$$\binom{d+s-3}{s} - \binom{d+s-5}{s-2} \tag{6.65}$$

degrees of freedom.

6.3 Unconstrained Fronsdal equation

Let us now give a brief overview of work done by Francia and Sagnotti on unconstrained Fronsdal equations [39, 40, 41]. The fact that we need to impose the conditions

$$\Lambda' = 0 \quad \text{and} \quad \varphi'' = 0 \tag{6.66}$$

for Fronsdal theory to be invariant under gauge transformation $\delta\varphi = \partial\Lambda$ is a sign that the theory is incomplete. For that reason let us rewrite the Fronsdal equation in an unconstrained form by introducing a rank-$(s-3)$ compensator field α transforming on (unconstrained) gauge transformations (6.51) as $\delta\alpha = \Lambda'$, in the following way

$$\mathcal{F} = \partial^3\alpha \tag{6.67}$$

This equation is invariant under the unconstrained gauge transformations (6.51) because the variation of α exactly cancels the variation of the Fronsdal tensor.

Let us now present the second way to construct free higher spin gauge theory with unconstrained gauge parameters and fields. Let us start with spin-3 case where $\delta\mathcal{F}_{\mu\nu\rho} =$

$3\partial_\mu\partial_\nu\partial_\rho\Lambda'$, the idea is to build a non-local operator $\mathcal{F}_{NL}$ that transforms like Fronsdal operator $\mathcal{F}$. The combination $\mathcal{F} - \mathcal{F}_{NL}$ will then be gauge invariant without imposing any additional constraints. The candidates for $\mathcal{F}_{NL}$ are

$$\begin{aligned}
&\frac{1}{3\Box}\left(\partial_\mu\partial_\nu\mathcal{F}'_\rho + \partial_\nu\partial_\rho\mathcal{F}'_\mu + \partial_\rho\partial_\mu\mathcal{F}'_\nu\right)\\
&\frac{1}{3\Box}\left(\partial_\mu\partial\cdot\mathcal{F}_{\nu\rho} + \partial_\nu\partial\cdot\mathcal{F}'_{\rho\mu} + \partial_\rho\partial\cdot\mathcal{F}'_{\mu\nu}\right)\\
&\frac{1}{\Box^2}\partial_\mu\partial_\nu\partial_\rho\partial\cdot\mathcal{F}'
\end{aligned}\tag{6.68}$$

The first two candidates actually coincide by means of Bianchi identity. Now it seems that we are left with two possibilities for gauge invariant equations

$$\begin{aligned}
&\mathcal{F}_{\mu\nu\rho} - \frac{1}{3\Box}\left(\partial_\mu\partial_\nu\mathcal{F}'_\rho + \partial_\nu\partial_\rho\mathcal{F}'_\mu + \partial_\rho\partial_\mu\mathcal{F}'_\nu\right) = 0\\
&\mathcal{F}_{\mu\nu\rho} - \frac{1}{\Box^2}\partial_\mu\partial_\nu\partial_\rho\partial\cdot\mathcal{F}' = 0
\end{aligned}\tag{6.69}$$

but these two equations can be turned one into another using their traces.

Generalizing to higher spins, we can write the analogue $\mathcal{F}^{(n)}$ of the Fronsdal differential operator in terms of the recursive equation

$$\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + \frac{1}{(n+1)(2n+1)}\frac{\partial^2}{\Box}\mathcal{F}^{(n)\prime} - \frac{1}{n+1}\frac{\partial}{\Box}\partial\cdot\mathcal{F}^{(n)} \tag{6.70}$$

with $\mathcal{F}^{(0)} = \Box\varphi$. So, in particular, $\mathcal{F}^{(1)} \equiv \mathcal{F} = \Box\varphi - \partial\partial\cdot\varphi + \partial^2\varphi'$ is the original Fronsdal operator. Gauge transformation of $\mathcal{F}^{(n)}$ is

$$\delta\mathcal{F}^{(n)} = (2n+1)\frac{\partial^{2n+1}}{\Box^{n-1}}\Lambda^{[n]} \tag{6.71}$$

the n-th trace of gauge parameter vanishes for $n > \frac{s-1}{2}$ and the operator $\mathcal{F}^{(n)}$ with n that satisfies this condition is gauge invariant without any constraints. The corresponding Bianchi identity is anomalous

$$\partial\cdot\mathcal{F}^{(n)} - \frac{1}{2n}\partial\mathcal{F}^{(n)\prime} = -\left(1+\frac{1}{2n}\right)\frac{\partial^{2n+1}}{\Box^{n-1}}\varphi^{[n+1]} \tag{6.72}$$

unless the $(n+1)$-th trace of the gauge field vanishes, which happens for $n > \frac{s}{2} - 1$.

Taking successive traces of the above relation gives us

$$\partial \cdot \mathcal{F}^{(n)[p]} - \frac{1}{2(n-p)}\partial\mathcal{F}^{(n)[p+1]} = 0, \quad \text{for } p \leq n-1 \tag{6.73}$$

However, the connection with our results cannot be in terms of the tensor $\mathcal{F}^{(n)}$, because the latter does not satisfy a conservation law, while our results will be conserved two-point functions (see bellow). To make the connection one constructs the Einstein-like tensor

$$\mathcal{G}^{(n)} = \sum_{p=0}^{n}(-1)^p \frac{(n-p)!}{2^p n!}\, \eta^p\, \mathcal{F}^{(n)[p]} \tag{6.74}$$

where the superscript in square bracket denotes the number of time $\mathcal{F}^{(n)}$ has been traced, and η is the Minkowski metric. The association of n with the spin s is as follows:

$$\begin{cases} s = 2n & s \quad \text{even} \\ s = 2n-1 & s \quad \text{odd} \end{cases}$$

The $\mathcal{G}^{(n)}$ tensor is divergenceless

$$\partial \cdot \mathcal{G}^{(n)} = 0 \tag{6.75}$$

The free (unconstrained) linearized equations of motion for φ are

$$\mathcal{G}^{(n)} = 0 \tag{6.76}$$

It can be shown that such an equation can be cast in local Lagrangian form, provided one introduces auxiliary fields (compensators).

Chapter 7

One loop effective actions and higher spins

We will approach higher spin theories with the induced gravity method [110]. In this chapter, we introduce the necessary ingredients to study the effective actions of a scalar and fermion theory coupled to classical sources using symmetric conserved currents. It is important to note that there is an infinite choice for conserved currents, here we will use two types: simple currents and a particular linear combination of them which becomes traceless in the massless limit. Since we will mainly focus on the quadratic part of the effective action, the main object we will be dealing with is the 2-point correlator of currents. To give a motivation for what is following, we summarise the results in $3d$ case obtained in [28].

We expect that the 2-pt functions of symmetric conserved currents are conserved and we exclude the presence of anomalies. As a consequence, the 2-pt functions can be expressed in terms of projectors [29]. Expressions in terms of a projection operator are very convenient because they make the conservation obvious. But, in this way, the geometrical content of the resulting equations of motion or the effective action remains implicit. For this reason, we rewrite general expressions in terms of generalized Jacobi tensors, see [30].

Finally, we describe our method to compute 2-point functions and give some general directions for their calculation.

7.1 Free field theory models

Here we limit ourselves to two type of models, the free scalar and free fermion, although it is possible to extend the analysis to other models. By the first we mean the complex scalar theory defined by the Lagrangian

$$L = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \tag{7.1}$$

in any dimension. On shell the current

$$j_\mu = i \left(\phi^\dagger \partial_\mu \phi - \partial_\mu \phi^\dagger \phi \right) \tag{7.2}$$

is conserved. We can couple it to a gauge field via the action term $\int d^d x \, A^\mu(x) j_\mu(x)$. In the case $s = 2$ the conserved current is the energy-momentum tensor and the external source is the metric fluctuation $h_{\mu\nu}$, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In this case the action is the integral of (7.1) multiplied by $\sqrt{g}$.

But, of course we can define infinitely many completely symmetric (on shell) conserved currents, of which (7.2) is only the simplest example:

$$j^{\mathrm{s}}_{\mu_1 \ldots \mu_s} = i^s \phi^\dagger \overleftrightarrow{\partial}_{\mu_1} \ldots \overleftrightarrow{\partial}_{\mu_s} \phi \tag{7.3}$$

They couple minimally to external spin s fields, $\varphi^{\mu_1 \ldots \mu_s}$. The on-shell current conservation implies (to the lowest order) invariance under the gauge transformations (2.25)

$$\delta \varphi_{\mu_1 \ldots \mu_s} = \partial_{(\mu_1} \Lambda_{\mu_2 \ldots \mu_s)} \tag{7.4}$$

where round brackets stand for symmetrization.

The free fermion model is represented by a Dirac fermion coupled to a gauge field. The action is

$$S[A] = \int d^d x \, \left[i \bar{\psi} \gamma^\mu D_\mu \psi - m \bar{\psi} \psi \right], \quad D_\mu = \partial_\mu + A_\mu \tag{7.5}$$

where $A_\mu = A^a_\mu(x) T^a$ and T^a are the generators of a gauge algebra in a given representation determined by ψ. We will use the antihermitean convention, so that $[T^a, T^b] = f^{abc} T^c$, and the normalization $\mathrm{tr}(T^a T^b) = -\delta^{ab}$. The current

$$j^a_\mu(x) = i\bar{\psi}\gamma_\mu T^a \psi \tag{7.6}$$

is (classically) covariantly conserved on shell as a consequence of the gauge invariance of the action (7.5)

$$(Dj)^a = (\partial^\mu \delta^{ac} + f^{abc} A^{b\mu}) j^c_\mu = 0 \tag{7.7}$$

The next example involves the coupling to gravity

$$S[h] = \int d^d x\, e\left[i\bar{\psi} E^\mu_a \gamma^a \nabla_\mu \psi - m\bar{\psi}\psi\right], \quad \nabla_\mu = \partial_\mu + \frac{1}{2}\omega_{\mu bc}\Sigma^{bc}, \quad \Sigma^{bc} = \frac{1}{4}\left[\gamma^b, \gamma^c\right]. \tag{7.8}$$

The corresponding energy momentum tensor

$$T^{(g)}_{\mu\nu} = \frac{i}{4}\bar{\psi}\left(\gamma_\mu \overset{\leftrightarrow}{\nabla}_\nu + \gamma_\nu \overset{\leftrightarrow}{\nabla}_\mu\right)\psi. \tag{7.9}$$

is covariantly conserved on shell as a consequence of the diffeomorphism invariance of the action. In the massless limit, the action is invariant under Weyl transformations and because of that the energy momentum tensor becomes traceless. If we expand the metric around the flat spacetime, $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$, then, contrary to spin-1 case, interaction is not linear in the gauge field $h_{\mu\nu}$. If we limit our analysis only to the linear term, it is given by coupling the flat space energy-momentum tensor

$$T_{\mu\nu} = \frac{i}{4}\bar{\psi}\left(\gamma_\mu \overset{\leftrightarrow}{\partial}_\nu + \gamma_\nu \overset{\leftrightarrow}{\partial}_\mu\right)\psi. \tag{7.10}$$

to the metric fluctuation $h_{\mu\nu}$.

Similarly to the gauge field and the metric, we can couple the fermion ψ to a new external spin-3 source $b_{\mu\nu\lambda}$ by adding to (7.15) the term

$$\int d^d x j_{\mu\nu\lambda} b^{\mu\nu\lambda} \tag{7.11}$$

with the choice of current

$$j_{\mu_1\mu_2\mu_3} \quad = \quad i^2 \bar{\psi}\gamma_{\mu_1} \overset{\leftrightarrow}{\partial}_{\mu_2} \overset{\leftrightarrow}{\partial}_{\mu_3} \psi \tag{7.12}$$

Due to the (on shell) current conservation this coupling is invariant (to lowest order) under the infinitesimal gauge transformations

$$\delta b_{\mu\nu\lambda} = \partial_{(\mu}\Lambda_{\nu\lambda)} \tag{7.13}$$

In the limit $m \to 0$, if we also have invariance under the generalized Weyl transformations

$$\delta b_{\mu\nu\lambda} = \Lambda_{(\mu}\eta_{\nu\lambda)} \tag{7.14}$$

we can induce tracelessness of the current $j_{\mu\nu\lambda}$ in any couple of indices. In that case the form of the current is more complicated than (7.12). We will come back to this point shortly.

We notice that to lowest order in the external sources the relevant action, in all cases above, takes the form of the free action + a linear interaction term such as (7.11). We make the identification $\varphi_\mu = A_\mu$, $\varphi_{\mu\nu} \sim h_{\mu\nu}$, $\varphi_{\mu\nu\lambda} \sim b_{\mu\nu\lambda}$, with the obvious exception of the non-Abelian field in (7.5). However, for simplicity, we will often consider just the Abelian case.[1]

In general, we can couple the fermions to more general fields. Consider the free action

$$S_0 = \int d^3x \, \left[i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \right], \tag{7.15}$$

and the spin-s conserved current

$$j^{\rm f}_{\mu_1\ldots\mu_s} \;\; = \;\; i^{s-1}\bar{\psi}\gamma_{\mu_1} \overleftrightarrow{\partial}_{\mu_2} \ldots \overleftrightarrow{\partial}_{\mu_s} \psi \tag{7.16}$$

Our goal is to compute the effective action for the external source fields at the quadratic order. Inspired by [22]-[27], we will introduce an infinite set of higher spin fields so that the generic linearized interaction we consider is:

$$S_{int} = \sum_s \int d^dx j_{\mu_1\ldots\mu_s}\varphi^{\mu_1\ldots\mu_s} \tag{7.17}$$

[1] Also note that the nonlinearity present in spin-2 case, which is forced by the consistency requirements, is a signal that we should expect the same for higher-spin fields.

In both scalar and fermion cases, let us repeat that the effective action is given by (2.11)

$$W[\varphi, s] = W[0] + \sum_{n=1}^{\infty} \sum_{s_1,\ldots,s_n} \frac{i^{n-1}}{n!} \int \prod_{i=1}^{n} d^d x_i \varphi^{\mu_{11}\ldots\mu_{1s_1}}(x_1)\ldots\varphi^{\mu_{n1}\ldots\mu_{ns_n}}(x_n)$$
$$\times \langle 0|\mathcal{T} j_{\mu_{11}\ldots\mu_{1s_1}}(x_1)\ldots j_{\mu_{n1}\ldots\mu_{ns_n}}(x_n)|0\rangle. \quad (7.18)$$

In particular $\varphi_\mu = A_\mu, \varphi_{\mu\nu} = \frac{1}{4}h_{\mu\nu}$ and $j_{\mu\nu} = 2T_{\mu\nu}$ with $\varphi_{\mu\nu\lambda} = b_{\mu\nu\lambda}$. The full one-loop 1-pt correlator for $j_{\mu_1\ldots\mu_s}$ is given by (2.12)

$$\langle\langle j_{\mu_1\ldots\mu_s}(x)\rangle\rangle = \frac{\delta W[\varphi, s]}{\delta\varphi^{\mu_1\ldots\mu_s}(x)} = \sum_{n=0}^{\infty} \sum_{s_2,\ldots,s_n} \frac{i^n}{n!} \int \prod_{i=2}^{n} d^d x_i \varphi^{\mu_{21}\ldots\mu_{2s_2}}(x_2)\ldots\varphi^{\mu_{n1}\ldots\mu_{ns_n}}(x_n)$$
$$\times \langle 0|\mathcal{T} j_{\mu_1\ldots\mu_s}(x) j_{\mu_{21}\ldots\mu_{2s_2}}(x_2)\ldots j_{\mu_{n1}\ldots\mu_{ns_n}}(x_n)|0\rangle. \quad (7.19)$$

To compute the effective action up to quadratic order we need the two-point functions

$$\langle 0|\mathcal{T} j_{\mu_1\ldots\mu_{s_1}}(x)\, j_{\nu_1\ldots\nu_{s_2}}(y)|0\rangle \quad (7.20)$$

or their Fourier transforms

$$\tilde{T}_{\mu_1\ldots\mu_{s_1}\nu_1\ldots\nu_{s_2}}(k) = \langle 0|\mathcal{T}\tilde{j}_{\mu_1\ldots\mu_{s_1}}(k)\, \tilde{j}_{\nu_1\ldots\nu_{s_2}}(-k)|0\rangle \quad (7.21)$$

In the sequel we compute them by using the Feynman diagram technique. For all two-point functions the only relevant diagram is the bubble diagram with one spin s line of ingoing momentum k and one with the same outgoing momentum and one scalar or fermion circulating in the internal loop.

Warning. One must be careful when applying the previous formulas for generating functions. If the correlator $\langle 0|\mathcal{T} j_{\mu_{11}\ldots\mu_{1s}}(x_1)\cdots j_{\mu_{n1}\ldots\mu_{ns}}(x_n)|0\rangle$ in (2.11) is meant to denote the n-th point-function calculated by using Feynman diagrams, a factor i^n is already included in the diagram themselves and so it should be dropped in (2.11). When the current is the energy-momentum tensor an additional precaution is necessary: the factor $\frac{i^{n-1}}{n!}$ must be replaced by $\frac{i^{n-1}}{2^n n!}$. The factor $\frac{1}{2^n}$ is motivated by the fact that when we expand the action

$$S[\eta + h] = S[\eta] + \int d^d x \frac{\delta S}{\delta g^{\mu\nu}}\Big|_{g=\eta} h^{\mu\nu} + \cdots,$$

the factor $\frac{\delta S}{\delta g^{\mu\nu}}\Big|_{g=\eta} = \frac{1}{2}T_{\mu\nu}$. Another consequence of this fact will be that the presence of vertices with one graviton in Feynman diagrams will correspond to insertions of the operator $\frac{1}{2}T_{\mu\nu}$ in correlation functions.

Recall that scalar and fermion currents are given by

$$j^{\mathrm{s}}_{\mu_1\ldots\mu_s} = i^s \varphi^\dagger \left(\overleftrightarrow{\partial}_\mu\right)^s \varphi\,, \quad j^{\mathrm{f}}_{\mu_1\ldots\mu_s} = i^{s-1}\bar\psi\gamma_\mu \left(\overleftrightarrow{\partial}_\mu\right)^{s-1}\psi \tag{7.22}$$

(For fermions in case $s=0$ we use $j^{\mathrm{f}}_{s=0} = \bar\psi\psi$.) These currents will be henceforth referred to as *simple* currents. In the fermionic case the two point correlator is

$$\tilde T^{\mathrm{f}}_{\mu_1\ldots\mu_{s_1}\nu_1\ldots\nu_{s_2}}(k) \;=\; -\int \frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}\left(\frac{i}{\not p - m}\gamma_\sigma \frac{i}{\not p - \not k - m}\gamma_\tau\right) V^\sigma_{\mu_1\ldots\mu_{s_1}} V^\tau_{\nu_1\ldots\nu_{s_2}} \tag{7.23}$$

whereas in the scalar case it is

$$\tilde T^{\mathrm{s}}_{\mu_1\ldots\mu_{s_1}\nu_1\ldots\nu_{s_2}}(k) \;=\; \int \frac{d^dp}{(2\pi)^d}\,\frac{1}{(p^2-m^2)((p-k)^2-m^2)} V_{\mu_1\ldots\mu_{s_1}} V_{\nu_1\ldots\nu_{s_2}} \tag{7.24}$$

with the Feynman vertices for fermions and scalars respectively

$$V^\sigma_{\mu_1\ldots\mu_s} = i\,\delta^\sigma_\mu\,(2p_\mu - k_\mu)^{s-1}\,, \quad V_{\mu_1\ldots\mu_s} = i\,(2p_\mu - k_\mu)^s \tag{7.25}$$

In addition, some general formulas are easy to write in terms of particular linear combination of the previous currents which become traceless in the massless case (case of generalized Weyl invariance). *Traceless* currents can be defined in the following way:

$$j^{\mathrm{st}}_{\mu_1\ldots\mu_s} = \sum_{l=0}^{\lfloor\frac{s}{2}\rfloor} a^{\mathrm{s}}_{s,l}\,(\Box\pi_{\mu\mu})^l\,\tilde J^{\mathrm{s}}_{\mu_1\ldots\mu_{s-2l}}\,, \quad j^{\mathrm{ft}}_{\mu_1\ldots\mu_s} = \sum_{l=0}^{\lfloor\frac{s-1}{2}\rfloor} a^{\mathrm{f}}_{s,l}\,(\Box\pi_{\mu\mu})^l\,\tilde J^{\mathrm{f}}_{\mu_1\ldots\mu_{s-2l}} \tag{7.26}$$

where

$$a^{\mathrm{s}}_{s,l} = \frac{(-1)^l s!\,\Gamma\left(s+\frac{d-3}{2}-l\right)}{2^{2l}l!(s-2l)!\,\Gamma\left(s+\frac{d-3}{2}\right)}\,, \quad a^{\mathrm{f}}_{s,l} = \frac{(-1)^l (s-1)!\,\Gamma\left(s+\frac{d-3}{2}-l\right)}{2^{2l}l!(s-2l-1)!\,\Gamma\left(s+\frac{d-3}{2}\right)} \tag{7.27}$$

It is easy to see that amplitudes for two general spins s_1 and s_2 for the traceless currents can be written as linear combinations of the amplitudes (7.24) and (7.25) of the simple currents (7.22)

$$\tilde{T}^{\mathrm{st}}_{\mu_1\ldots\mu_{s_1}\nu_1\ldots\nu_{s_2}} = \sum_{l=0}^{\lfloor\frac{s_1}{2}\rfloor}\sum_{k=0}^{\lfloor\frac{s_2}{2}\rfloor} a^{\mathrm{s}}_{s_1,l}a^{\mathrm{s}}_{s_2,k}\left(k^2\eta_{\mu\mu}-k^2_{\mu}\right)^l\left(k^2\eta_{\nu\nu}-k^2_{\nu}\right)^k\tilde{T}^{\mathrm{s}}_{\mu_1\ldots\mu_{s_1-2l}\nu_1\ldots\nu_{s_2-2k}}$$
$$\tilde{T}^{\mathrm{ft}}_{\mu_1\ldots\mu_{s_1}\nu_1\ldots\nu_{s_2}} = \sum_{l=0}^{\lfloor\frac{s_1-1}{2}\rfloor}\sum_{k=0}^{\lfloor\frac{s_2-1}{2}\rfloor} a^{\mathrm{f}}_{s_1,l}a^{\mathrm{f}}_{s_2,k}\left(k^2\eta_{\mu\mu}-k^2_{\mu}\right)^l\left(k^2\eta_{\nu\nu}-k^2_{\nu}\right)^k\tilde{T}^{\mathrm{f}}_{\mu_1\ldots\mu_{s_1-2l}\nu_1\ldots\nu_{s_2-2k}}$$

Before we start with the analysis of the results for 2pt correlators coming from Feynman diagrams, we should prepare the ground with a general analysis of their expected structure. We argued in chapter 2 that the full one-loop conservation law for the spin s current is

$$\partial^{\mu_1}\langle\!\langle j_{\mu_1\ldots\mu_s}(x)\rangle\!\rangle = 0 \tag{7.28}$$

From the spin-2 example we know that a covariant conservation law should be written also for the higher spin currents, but for $s > 2$ we will satisfy ourselves with the lowest nontrivial order given by the above equation. Using this conservation law, in the next section, we will determine a general form of our 2-pt correlators.

7.2 Universal equations of motion and conserved structures for spin s

Our starting point is the 2-pt functions of symmetric conserved currents. We expect them to be conserved, i.e. we expect to find 0 if we contract any index with the external momentum k. We exclude the presence of anomalies. In fact we will come across also some non-conservations, but they can be fixed by subtracting local counterterms. This aspect of our analysis is interesting in itself, but we will illustrate it later on in detail. For the time being we ignore this fact and suppose that all 2-pt functions we deal with are conserved. We will also write a general form of the traceless 2-pt function.

7.2.1 Conserved even-parity structures

The form of the conserved structures is universal, in a sense that is does not depend on the dimension d of spacetime. They can be easily constructed by means of the projector

$$\pi_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \tag{7.29}$$

Conservation is a consequence of the transversality property

$$k^\mu \pi_{\mu\nu} = 0 \tag{7.30}$$

The name for the projector is justified by the property

$$\pi_{\mu\nu}\pi^\nu_\lambda = \pi_{\mu\lambda} \tag{7.31}$$

For equal spin s, the 2pt correlator can be written in terms of the following structures:

$$\begin{aligned}
\tilde{A}^{(s)}_{0,\mu_1\ldots\mu_s\nu_1\ldots\nu_s}(k) &= \pi^s_{\mu\nu} && (7.32)\\
\tilde{A}^{(s)}_{1,\mu_1\ldots\mu_s\nu_1\ldots\nu_s}(k) &= \pi^{s-2}_{\mu\nu}\pi_{\mu\mu}\pi_{\nu\nu} && (7.33)\\
\ldots \quad & \quad \ldots\ldots &&\\
\tilde{A}^{(s)}_{l,\mu_1\ldots\mu_s\nu_1\ldots\nu_s}(k) &= \pi^{s-2l}_{\mu\nu}\pi^l_{\mu\mu}\pi^l_{\nu\nu} && (7.34)\\
\ldots \quad & \quad \ldots\ldots &&
\end{aligned}$$

There are $\lfloor s/2 \rfloor$ independent such terms. Let us set

$$\tilde{E}_{\mu_1\ldots\mu_s\nu_1\ldots\nu_s}(k) = \sum_{l=0}^{\lfloor s/2 \rfloor} a_l \tilde{A}_{l,\mu_1\ldots\mu_s,\nu_1\ldots\nu_s}(k) \tag{7.35}$$

where a_l are arbitrary constants and $\tilde{E}^{(s)}_{\mu_1\ldots\mu_s\nu_1\ldots\nu_s}(k)$ are conserved tensors. This is the most general conserved structure for spin s.

Let us give a proof by induction that a conserved structure can be written in terms of products of π alone. In the lowest case (spin 1), the most general Lorentz covariant (dimensionless) conserved even structure can be written in terms of $\eta_{\mu\nu}$ and $\frac{k_\mu k_\nu}{k^2}$. Imposing conservation the result is $\sim \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} = \pi_{\mu\nu}$. In the same way one can prove the property for the case $s = 2$. Now we suppose that the proposition is true for s. So it is true for the combination $T^{(s)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s}(k) = \tilde{E}^{(s)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s}(k) = \sum_{l=0}^{[s/2]} a_l \tilde{A}^{(s)}_{l,\mu_1\dots\mu_s\nu_1\dots\nu_s}$, meaning that $k^{\mu_i} T^{(s)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s} = 0$ for any $i, i = 1, \dots s$. In order to construct $T^{(s+1)}_{\mu_1\dots\mu_{s+1}\nu_1\dots\nu_{s+1}}$ we can multiply $T^{(s)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s}$ by $\eta_{\mu\nu}$ or $\frac{k_\mu k_\nu}{k^2}$ or multiplying $T^{(s-1)}$ by $\eta_{\mu\mu}\eta_{\nu\nu}$, $\frac{k_\mu k_\mu}{k^2}\eta_{\nu\nu}, \eta_{\mu\mu}\frac{k_\nu k_\nu}{k^2}$ or by $\frac{k_\mu k_\mu}{k^2}\frac{k_\nu k_\nu}{k^2}$, because the construction is in steps of 2. So we can have only

$$\begin{aligned} T^{(s+1)}_{\mu_1\dots\mu_{s+1}\nu_1\dots\nu_{s+1}} &= a_1 \eta_{\mu\nu} T^{(s)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s} + a_2 \eta_{\mu\mu}\eta_{\nu\nu} T^{(s-1)}_{\mu_1\dots\mu_{s-1}\nu_1\dots\nu_{s-1}} + b_1 \frac{k_\mu k_\nu}{k^2} T^{(s)} \\ &+ b_2 \frac{k_\mu k_\mu}{k^2}\eta_{\nu\nu} T^{(s-1)}_{\mu_1\dots\mu_{s-1}\nu_1\dots\nu_{s-1}} + b_3 \eta_{\mu\mu}\frac{k_\nu k_\nu}{k^2} T^{(s-1)}_{\mu_1\dots\mu_{s-1}\nu_1\dots\nu_{s-1}} \\ &+ b_4 \frac{k_\mu k_\mu}{k^2}\frac{k_\nu k_\nu}{k^2} T^{(s-1)}_{\mu_1\dots\mu_{s-1}\nu_1\dots\nu_{s-1}} \end{aligned} \tag{7.36}$$

Now applying k^μ to this expression we find that conservation requires $a_1 = -b_1, a_2 = -b_2 = -b_3 = b_4$. So that (7.36) becomes

$$T^{(s+1)}_{\mu_1\dots\mu_{s+1}\nu_1\dots\nu_{s+1}} = a\,\pi_{\mu\nu} T^{(s)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s} + b\,\pi_{\mu\mu}\pi_{\nu\nu} T^{(s-1)}_{\mu_1\dots\mu_{s-1}\nu_1\dots\nu_{s-1}} \tag{7.37}$$

with arbitrary a and b.

By Fourier anti-transforming and inserting into (2.11), one can construct the effective action corresponding to (7.35) multiplied by k^2 for the spin s field $\varphi_{\mu_1\dots\mu_s}$ as follows

$$S_E \sim \int d^d x\, \varphi^{\mu_1\dots\mu_s} \Box E(\partial)_{\mu_1\dots\mu_s,\nu_1\dots\nu_s} \varphi^{\nu_1\dots\nu_s} \tag{7.38}$$

where $E(\partial)$ is the formal Fourier transform of $\tilde{E}(k)$, i.e. the same expression with k_μ replaced by $-i\partial_\mu$. The equation of motion is of course

$$\Box E(\partial)_{\mu_1\dots\mu_s,\nu_1\dots\nu_s} \varphi^{\nu_1\dots\nu_s} = 0 \tag{7.39}$$

After canonical normalization, it depends on $\lfloor s/2 \rfloor - 1$ arbitrary constants. This is the most general linearized equation of motion for a completely symmetric spin s field.

For correlators of currents with two different spins s_1 and s_2, $s_2 > s_1$ the general structure is

$$\pi_{\nu\nu}^{\frac{s_2 - s_1}{2}} \sum_{l=0}^{\lfloor \frac{s_1}{2} \rfloor} a_l \tilde{A}_{l,\mu_1\dots\mu_{s_1},\nu_1\dots\nu_{s_1}}(k) \tag{7.40}$$

provided that both s_1 and s_2 are either even or odd.

From $\tilde{E}^{(s)}(k)$ we can obtain the most general traceless combination, by taking the trace of (7.35) and imposing it to vanish. The resulting equation is the recurrence relation

$$a_l = -\frac{(s-2l+2)(s-2l+1)}{2l(2(s-l-1)+d-1)}a_{l-1} \tag{7.41}$$

Setting $a_0 = 1$ the solution is

$$a_l = \frac{(-1)^l}{2^{2l}l!}\frac{s!}{(s-2l)!}\frac{\Gamma\left(s+\frac{d-3}{2}-l\right)}{\Gamma\left(s+\frac{d-3}{2}\right)} \tag{7.42}$$

Replacing this in (7.35) we obtain a traceless conserved structure. In turn this gives rise to a traceless equation of motion.

7.2.2 Conserved odd parity structures

It is easy to obtain also all the odd parity structures. The spin 1 odd parity conserved Lorentz structure (linear in k) can only be

$$\tilde{C}^{(1)}_{0,\mu\nu}(k) = \epsilon_{\mu\nu\lambda}k^{\lambda} \tag{7.43}$$

It is easy to realize that, for higher spin, the ϵ tensor can only appear in the form $\epsilon_{\mu\nu\lambda}k^{\lambda}$ in every single term, thus it can be factored out. What remains is an even spin structure of one order less. So the most general odd conserved Lorentz structure will be a combination of

$$\begin{aligned}
\tilde{C}^{(s)}_{0,\mu_1\ldots\mu_s,\nu_1\ldots\nu_s}(k) &= \epsilon_{\mu\nu\lambda}k^{\lambda}\tilde{A}^{(s-1)}_{0,\mu_1\ldots\mu_{s-1},\nu_1\ldots\nu_{s-1}}(k)\\
\tilde{C}^{(s)}_{1,\mu_1\ldots\mu_s,\nu_1\ldots\nu_s}(k) &= \epsilon_{\mu\nu\lambda}k^{\lambda}\tilde{A}^{(s-1)}_{1,\mu_1\ldots\mu_{s-1},\nu_1\ldots\nu_{s-1}}(k)\\
\ldots &\quad \ldots\\
\tilde{C}^{(s)}_{l,\mu_1\ldots\mu_s,\nu_1\ldots\nu_s}(k) &= \epsilon_{\mu\nu\lambda}k^{\lambda}\tilde{A}^{(s-1)}_{l,\mu_1\ldots\mu_{s-1},\nu_1\ldots\nu_{s-1}}(k)\\
\ldots &\quad \ldots
\end{aligned} \tag{7.44}$$

where $A^{(0)}_0 = 1$, by definition. Let us define

$$\tilde{O}_{\mu_1\ldots\mu_s\nu_1\ldots\nu_s}(\partial) = \sum_{l=0}^{\lfloor s/2\rfloor} c_l\tilde{C}_{l,\mu_1\ldots\mu_s\nu_1\ldots\nu_s}(\partial) \tag{7.45}$$

The odd parity action is supposed to be local (and higher derivative)

$$S_O = \int d^d x \, \varphi^{\mu_1 \dots \mu_s} \Box^{s-1} O(\partial)_{\mu_1 \dots \mu_s \nu_1 \dots \nu_s} \varphi^{\nu_1 \dots \nu_s} \tag{7.46}$$

Therefore the odd equation of motion is

$$\Box^{s-1} O_{\mu_1 \dots \mu_s \nu_1 \dots \nu_s}(\partial) \varphi^{\nu_1 \dots \nu_s} = 0 \tag{7.47}$$

For correlators of currents with two different spins s_1 and s_2, $s_2 > s_1$ the general structure is

$$\pi_{\nu\nu}^{\frac{s_2 - s_1}{2}} \sum_{l=0}^{\lfloor \frac{s_1}{2} \rfloor} c_l \tilde{C}_{l, \mu_1 \dots \mu_{s_1}, \nu_1 \dots \nu_{s_1}}(k) \tag{7.48}$$

under the condition that both s_1 and s_2 are either even or odd.

The tracelessness condition (for spin $s > 1$) implies a recursion relation for the coefficients c_l:

$$c_l = -\frac{(s - 2l + 1)(s - 2l)}{2l(2(s - l - 2) + d + 1)} c_{l-1} \tag{7.49}$$

Setting $c_0 = 1$ the solution is:

$$c_l = \frac{(-1)^l}{2^{2l} l!} \frac{(s-1)!}{(s - 2l - 1)!} \frac{\Gamma\left(s + \frac{d-3}{2} - l\right)}{\Gamma\left(s + \frac{d-3}{2}\right)} \tag{7.50}$$

7.3 Geometry in effective actions

The most important point of our approach will be the connection between the on-shell conservation of the initial free field theory current and the gauge invariance of the minimal coupling term with the higher spin field. This, in turn, induces a gauge invariance of the linearized higher spin effective action (or covariance of the corresponding equation of motion). This invariance is left implicit if we write our results in terms of projectors. To make it explicit, we can express our results in terms of covariant 'geometric' tensors constructed out of the symmetric higher spin fields. In this section we would like to make connection with such a geometrization program.

Eq.(7.35) can be easily translated into a corresponding differential operator by Fourier anti-transforming

$$E^{(s)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s}(\partial) = \sum_{l=0}^{\lfloor s/2\rfloor} a_l A^{(s)}_{l,\mu_1\dots\mu_s\nu_1\dots\nu_s}(\partial) \tag{7.51}$$

These are the types of differential operators that appear in the EA's acting on the spin s field $\varphi_{\mu_1\dots\mu_s}$. The corresponding equation of motion will take the following form

$$E^{(s)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s}(\partial)\varphi^{\nu_1\dots\nu_s} = 0 \tag{7.52}$$

multiplied by a function of $\Box$ and m^2.

The purpose of this section is to rewrite the equations such as (7.52) in the geometrical form of [39, 40, 41]. For this purpose, let us introduce the *symbol* of $\mathcal{G}^{(n)}_{\mu_1\dots\mu_s}$, $\tilde{\mathcal{G}}^{(n)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s}(k)$, as follows. We Fourier transform it and replace the Fourier transform of φ, $\tilde{\varphi}$, with s symmetric indices $\nu_1\dots\nu_s$. Finally we define

$$\mathcal{G}^{(n)}_{\mu_1\dots\mu_s} \equiv \mathcal{G}^{(n)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s}(\partial)\,\varphi^{\nu_1\dots\nu_s} \tag{7.53}$$

Then the connection between (6.76) and (7.52) is given by

$$\frac{1}{k^2}\,\tilde{\mathcal{G}}^{(n)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s}(k) = \sum_{l=0}^{\lfloor s/2\rfloor}(-1)^l \begin{pmatrix}\lfloor s/2\rfloor\\ l\end{pmatrix} \tilde{A}^{(s)}_{l,\mu_1\dots\mu_s\nu_1\dots\nu_s}(k), \tag{7.54}$$

which corresponds to a particular choice of the coefficients a_l in (7.35). In index notation, and using formalism of π-projectors, generalized Einstein tensor reads

$$\frac{1}{k^2}\tilde{\mathcal{G}}_{\mu_1\dots\mu_s} \;=\; \pi_{\mu\nu}^{s \bmod 2}\left(\pi^2_{\mu\nu} - \pi_{\mu\mu}\pi_{\nu\nu}\right)^{\lfloor\frac{s}{2}\rfloor}\varphi^{\nu_1\dots\nu_s} \tag{7.55}$$

Of course we are interested not only in the relation (7.54), but in expressing all the $\tilde{A}^{(s)}_{l,\mu_1\dots\mu_s\nu_1\dots\nu_s}(k)$ in terms of the $\tilde{\mathcal{G}}^{(n)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s}(k)$. To do so we have to take the successive traces of (7.54). We have, for instance

$$\tilde{\mathcal{G}}^{(n)\prime}_{\mu_1\dots\mu_{s-2}\nu_1\dots\nu_s} = -2\lfloor s/2\rfloor(2\lfloor s/2\rfloor + D - 4)\tilde{\mathcal{G}}^{(n-1)}_{\mu_1\dots\mu_{s-2}\nu_1\dots\nu_{s-2}}\,\pi_{\nu\nu} \tag{7.56}$$

In general

$$\tilde{\mathcal{G}}^{(n)[p]}_{\mu_1\dots\mu_{s-2p}\nu_1\dots\nu_s} = (-2)^p \frac{(2\lfloor s/2\rfloor + D - 4)!!(\lfloor s/2\rfloor)!}{(2\lfloor s/2\rfloor + D - 2p - 4)!!(\lfloor s/2\rfloor - p)!} \tilde{\mathcal{G}}^{(n-p)}_{\mu_1\dots\mu_{s-2p}\nu_1\dots\nu_{s-2p}}\, \pi^p_{\nu\nu} \quad (7.57)$$

and

$$\tilde{\mathcal{G}}^{(n)[n]}_{\nu_1\dots\nu_s} = (-2)^n \frac{(2\lfloor s/2\rfloor + D - 4)!!(\lfloor s/2\rfloor)!}{(D-4)!!} \tilde{\mathcal{G}}^{(0)}_{\nu_1\dots\nu_s}\, \pi^n_{\nu\nu} \quad (7.58)$$

for s even, with $\tilde{\mathcal{G}}^{(0)} = k^2$, and

$$\tilde{\mathcal{G}}^{(n)[n-1]}_{\mu\nu_1\dots\nu_s} = (-2)^{n-1} \frac{(2\lfloor s/2\rfloor + D - 4)!!(\lfloor s/2\rfloor)!}{(D-4)!!} \tilde{\mathcal{G}}^{(1)}_{\mu\nu}\, \pi^{n-1}_{\nu\nu} \quad (7.59)$$

for s odd, with $\tilde{\mathcal{G}}^{(1)}_{\mu\nu} = k^2\pi_{\mu\nu}$.

Now, using (7.54), one can write

$$\pi^s_{\mu\nu} \equiv \tilde{A}^{(s)}_{0,\mu_1\dots\mu_s\nu_1\dots\nu_s}(k) = \frac{1}{k^2}\tilde{\mathcal{G}}^{(n)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s}(k) + \sum_{l=0}^{\lfloor s/2\rfloor - 1} (-1)^l \begin{pmatrix} \lfloor s/2\rfloor \\ l+1 \end{pmatrix} \pi^{l+1}_{\mu\mu}\pi^{s-2l-2}_{\mu\nu}\pi^{l+1}_{\nu\nu} \quad (7.60)$$

for even s, and a similar expression for odd s. Now the strategy consists in repeating the same step for the second line in (7.60), by using (7.56) and successively (7.58). The end result is

$$k^2\pi^s_{\mu\nu} = \sum_{p=0}^{\lfloor s/2\rfloor} \left(-\frac{1}{2}\right)^p \frac{(2\lfloor s/2\rfloor + D - 2p - 4)!!}{p!(2\lfloor s/2\rfloor + D - 4)!!} \pi^p_{\mu\mu}\, \tilde{\mathcal{G}}^{(n)[p]}_{\mu_1\dots\mu_{s-2p}\nu_1\dots\nu_s}(k) \quad (7.61)$$

In a similar way one can obtain

$$\begin{aligned} &k^2\pi^{s-2l}_{\mu\nu}\pi^l_{\mu\mu}\pi^l_{\nu\nu} \quad (7.62)\\ &= \begin{pmatrix} \lfloor s/2\rfloor \\ l \end{pmatrix}^{-1} \sum_{p=l}^{\lfloor s/2\rfloor} \left(-\frac{1}{2}\right)^p \begin{pmatrix} p \\ l \end{pmatrix} \frac{(2\lfloor s/2\rfloor + D - 2p - 4)!!}{p!(2\lfloor s/2\rfloor + D - 4)!!} \pi^p_{\mu\mu}\, \tilde{\mathcal{G}}^{(n)[p]}_{\mu_1\dots\mu_{s-2p}\nu_1\dots\nu_s}(k) \end{aligned}$$

In conclusion, any expression of the type (7.35), i.e. any conserved structure, can be expressed in terms of the generalized Einstein symbols $\tilde{\mathcal{G}}^{(n)}(k, n_1, n_2)$ and its traces. Thus any effective action (or any equation of motion) we obtain from our models, by integrating out matter, can be expressed in terms of the generalized Einstein tensor $\mathcal{G}^{(n)}$ and its traces preceded by a function of $\Box$ and the mass m^2 of the model, with suitable multiples of the projector operator acting on the traces. Using (6.74) one can replace the dependence on $\mathcal{G}^{(n)}$ of such expressions with the dependence on $\mathcal{F}^{(n)}$. The geometrization program can be completed by introducing the generalized Jacobi tensors $\mathcal{R}_{\mu_1,\dots\mu_s\nu_1\dots\nu_s}$ (one of the possible generalizations of the 4d Riemann tensor, [43, 178]) by means of

$$\mathcal{R}^{(s)}_{\mu_1\dots\mu_s\nu_1\dots\nu_s} = \sum_{l=0}^{s}(-1)^l \partial_\mu^{s-l}\,\partial_\nu^l\,\varphi_{\mu_1\dots\mu_l\nu_{l+1}\dots\nu_{s-l}} \tag{7.63}$$

The tensors $\mathcal{R}^{(s)}$ are connected to the $\mathcal{F}^{(n)}$ as follows:

$$\mathcal{F}^{(n)} = \begin{cases} \frac{1}{\Box^{n-1}}\mathcal{R}^{(s)[n]} & s = 2n \\ \frac{1}{\Box^{n-1}}\partial\cdot\mathcal{R}^{(s)[n-1]} & s = 2n-1 \end{cases} \tag{7.64}$$

where the traces in square brackets refer to the first set of indices. In this way we can express any effective action or any equation of motion in terms of $\mathcal{R}^{(s)}$ and traces (in the second set of indices) thereof. Further formulations of equations of motion that are local and include mixed symmetry cases can be found in [179, 180].

Since above we have referred to [39]-[41], let us clarify the context in which our results are derived and point out the differences with the spirit of [39]-[41],[107, 108]. In these papers the initial purpose was to write down a generalization of the Fronsdal equations for higher spin in such a way as to avoid the constraints needed in the original formulation of [37, 38]. The authors of [39]-[41] chose to sacrifice locality in favour of an unconstrained gauge symmetry. The typical (linearized) non-local equation of motion one obtains in this way is (6.76). It can be shown that such an equation can be cast in Lagrangian form, provided one introduces auxiliary fields (compensators). Therefore one can say that the nonlocality of (6.76) is a gauge artifact, with no physical implication. However equations of motion invariant under unrestricted gauge symmetry are far from unique. There actually exist several families of them depending on arbitrary parameters (by the way, this is evident by reversing the argument above and starting from the generic operator (7.52), instead of the completely fixed one (7.54). These are all equally valid as long as the field φ is considered in isolation and the linearized equation of motion is the free one, (6.76). However, if the spin s system is minimally coupled to a conserved current the question arises as to whether the propagating degrees of freedom are the truly physical ones, i.e. those corresponding to the appropriate little group representation for massless fields. The authors of [107, 108] were able to prove that there exist only one choice for the Einstein-like tensor which is Lagrangian and satisfies such a physicality condition.

Such 'physical' Einstein tensors do not correspond, in general, to the kinetic operators we will find in our effective actions below. This is not surprising, as our main goal is covariance: our purpose is to arrive at a covariant effective action with respect to a completely unfolded gauge symmetry. In a logical development the next step will be to introduce auxiliary fields to eliminate nonlocalities. Following this we would need to gauge-fix the action and introduce appropriate ghosts to produce the physical propagators. At that point would the problem handled by [107, 108] come to the surface. However, we would like to recall that our immediate prospect is to construct the linearized covariant effective action in preparation for the analysis of the three-point function.

7.4 The general method

In this section we illustrate the method to compute the 2-pt functions with Feynman diagrams. The method to obtain the results below is largely based on the approach of Davydychev and collaborators, [34]-[36]. To compute the diagrams explicitly we use a Mathematica code [181]. The integrals we have to compute are of the general form

$$\tilde{J}_{\mu_1\dots\mu_p}(d;\alpha,\beta;q_1,q_2,m) = \int \frac{d^dp}{(2\pi)^d} \frac{p_{\mu_1}\dots p_{\mu_p}}{\left((p+q_1)^2-m^2\right)^\alpha \left((p+q_2)^2-m^2\right)^\beta} \tag{7.65}$$

where, eventually, $q_1 = 0, q_2 = -k$. We will use the method invented by [34]-[36] to reduce the tensor integral to a sum of scalar ones

$$\begin{aligned}\tilde{J}_{\mu_1\dots\mu_p}(d;\alpha,\beta,\gamma;q_1,q_2,m) = &\sum_{\substack{\lambda,\kappa_1,\kappa_2\\ 2\lambda+\sum\kappa_i=p}} \left(-\frac{1}{2}\right)^\lambda (4\pi)^{p-\lambda} \left\{[\eta]^\lambda\,[q_1]^{\kappa_1}\,[q_2]^{\kappa_2}\right\}_{\mu_1\dots\mu_p}\\ &\times (\alpha)_{\kappa_1}(\beta)_{\kappa_2}\,\tilde{I}^{(2)}(d+2(p-\lambda);\alpha+\kappa_1,\beta+\kappa_2;q_1,q_2,m),\end{aligned} \tag{7.66}$$

where the symbol $\left\{[\eta]^\lambda\,[q_1]^{\kappa_1}\dots[q_N]^{\kappa_N}\right\}_{\mu_1\dots\mu_M}$ stands for the complete symmetrization of the objects inside the curly brackets, for example

$$\{\eta q_1\}_{\mu_1\mu_2\mu_3} = \eta_{\mu_1\mu_2}q_{1\mu_3} + \eta_{\mu_1\mu_3}q_{1\mu_2} + \eta_{\mu_2\mu_3}q_{1\mu_1}.$$

The basic integral is now the scalar one

$$\tilde{I}^{(2)}(d;\alpha,\beta;q_1,q_2,m) = \int \frac{d^dp}{(2\pi)^d} \frac{1}{\left((p+q_1)^2-m^2\right)^{\alpha}\left((p+q_2)^2-m^2\right)^{\beta}} \tag{7.67}$$

For instance, the bubble integral for the $s=1$ current in the scalar model

$$\tilde{J}_{\mu\nu}(k) = \int \frac{d^dp}{(2\pi)^d} \frac{(2p-k)_\mu(2p-k)_\nu}{(p^2-m^2)((p-k)^2-m^2)} \tag{7.68}$$

reduces to

$$\begin{aligned} \tilde{J}_{\mu\nu}(m,k) \;&=\; -\frac{8\pi}{(2\pi)^{d+2}}\eta_{\mu\nu}\,\tilde{I}^{(2)}(d+2;1,1) + 8\frac{(4\pi)^2}{(2\pi)^{d+4}}k_\mu k_\nu\,\tilde{I}^{(2)}(d+4;1,3) \\ &+\frac{16\pi}{(2\pi)^{d+2}}k_\mu k_\nu\,\tilde{I}^{(2)}(d+2;1,2) + \frac{1}{(2\pi)^d}k_\mu k_\nu\,\tilde{I}^{(2)}(d;1,1) \end{aligned} \tag{7.69}$$

The integral $\tilde{I}^{(2)}(d;\alpha,\beta;k,m)$ can be cast into the form of a hypergeometric series

$$\begin{aligned} \tilde{I}^{(2)}_{IR}(d;\alpha,\beta;k,m) \;&=\; 2^{-d}\pi^{-d/2}i^{1-d}\left(-m^2\right)^{-\alpha-\beta+\frac{d}{2}}\frac{\Gamma\left(-\frac{d}{2}+\alpha+\beta\right)}{\Gamma(\alpha+\beta)} \\ &\times\, {}_3F_2\left(\begin{matrix}\alpha,\beta,-\frac{d}{2}+\alpha+\beta \\ \frac{\alpha+\beta}{2},\frac{\alpha+\beta+1}{2}\end{matrix}\middle|\frac{k^2}{4m^2}\right) \end{aligned} \tag{7.70}$$

This representation is valid for large m compared to k. When m is small compared to k another representation is available

$$\begin{aligned} \tilde{I}^{(2)}_{UV}(d;\alpha,\beta;k,m) \;&=\; 2^{-d}\pi^{-d/2}i^{1-d}\left(k^2\right)^{-\alpha-\beta+\frac{d}{2}}\left\{\frac{\left(\Gamma\left(\frac{d}{2}-\alpha\right)\Gamma\left(\frac{d}{2}-\beta\right)\Gamma\left(-\frac{d}{2}+\alpha+\beta\right)\right)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d-\alpha-\beta)}\right. \\ &\times\, {}_3F_2\left(\begin{matrix}-\frac{d}{2}+\alpha+\beta,\frac{-d+\alpha+\beta+1}{2},\frac{-d+\alpha+\beta+2}{2} \\ -\frac{d}{2}+\alpha+1,-\frac{d}{2}+\beta+1\end{matrix}\middle|\frac{4m^2}{k^2}\right) \\ &+\left(-\frac{m^2}{k^2}\right)^{\frac{d}{2}-\alpha}\frac{\Gamma\left(\alpha-\frac{d}{2}\right)}{\Gamma(\alpha)}\,{}_3F_2\left(\begin{matrix}\beta,\frac{-\alpha+\beta+1}{2},\frac{-\alpha+\beta+2}{2} \\ \frac{d}{2}-\alpha+1,-\alpha+\beta+1\end{matrix}\middle|\frac{4m^2}{k^2}\right) \\ &\left.+\left(-\frac{m^2}{k^2}\right)^{\frac{d}{2}-\beta}\frac{\Gamma\left(\beta-\frac{d}{2}\right)}{\Gamma(\beta)}\,{}_3F_2\left(\begin{matrix}\alpha,\frac{\alpha-\beta+1}{2},\frac{\alpha-\beta+2}{2} \\ -\beta+\frac{d}{2}+1,\alpha-\beta+1\end{matrix}\middle|\frac{4m^2}{k^2}\right)\right\} \end{aligned} \tag{7.71}$$

In the sequel we consider also massless models. The relevant results can be obtained from the massive models by taking the $m \to 0$ limit. But they can also be obtained by setting $m = 0$ from the very beginning. In such a case the basic integral is

$$\begin{aligned} \tilde{I}^{(2)}(d;\alpha,\beta;q_1,q_2,0) &= \int \frac{d^dp}{(2\pi)^d} \frac{1}{((p+q_1)^2)^\alpha\,((p+q_2)^2)^\beta} \\ &= 2^{-d}\pi^{-d/2}i^{1-d}(k^2)^{\frac{d}{2}-\alpha-\beta} \frac{\Gamma\left(\frac{d}{2}-\alpha\right)\Gamma\left(\frac{d}{2}-\beta\right)\Gamma\left(\alpha+\beta-\frac{d}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d-\alpha-\beta)} \end{aligned} \tag{7.72}$$

7.5 An appetizer in 3d

Let us start with a motivational example. In [28] it was calculated, in particular, the two-point function of the current j^a in the fermion model as well as its IR and UV limits. In the parity violating part it was found a well-known result: when Fourier antitransformed and inserted in the generating function of the effective action (2.11) it gives rise to the linearized version of the gauge Chern-Simons action in 3d (which is in fact conformal invariant). For the two-point correlator of the energy-momentum tensor for the fermion model, and proceeding the same way, the linearized version of the gravity Chern-Simons action was found. Something that was also known before, [121]. Repeating the same thing for the spin 3 traceless current above it was found a previously unknown result: the UV limit in particular leads to a linearized action that corresponds to a spin 3 Chern-Simons generalization postulated long ago by Pope and Townsend, see [43, 44, 178, 182].

These were the results found in the parity odd part ([28] is mostly interested in the latter). But the even parity parts of the two-point correlators have perhaps even more interesting interpretations, so let us briefly analyze the parity even parts of the linearized effective actions obtained from 2-point current correlators in the free massive Dirac fermion quantum field theory in 3d in [28].

7.5.1 Spin one and two - parity even sectors

The UV limit of the two-point function of the j^a currents are nonlocal conformal correlators, according to expectations, see [114]. The same is true for the energy-momentum tensor two-point function. But now let us focus on the IR limits. According to [28], for the j^a current two-point function, for large m we have

$$\tilde{T}^{ab(even)}_{\mu\nu}(k) = -\frac{i}{12\pi}\frac{1}{|m|}\delta^{ab}k^2\pi_{\mu\nu} \tag{7.73}$$

This term is local. Fourier anti-transforming it and inserting it into (2.11) it gives rise to the action

$$S \sim \frac{1}{|m|} \int d^3x \, \left(A^a_\mu \partial^\mu \partial^\nu A^a_\nu - A^a_\nu \Box A^{a\nu} \right) \tag{7.74}$$

which is the lowest term in the expansion of the YM action

$$S_{YM} = -\frac{1}{g_{YM}} \int d^3x \, \mathrm{Tr} \left(F_{\mu\nu} F^{\mu\nu} \right) \tag{7.75}$$

where $g_{YM} \sim |m|$.

Now let us go to the IR limit of the even part of the 2pt energy-momentum tensor correlator. Eq.(3.36) of [28] says

$$\langle T_{\mu_1\mu_2}(k) T_{\nu_1\nu_2}(-k) \rangle^{IR}_{even} = -\frac{i|m|}{96\pi} k^2 \left(\pi_{\mu_1\nu_1} \pi_{\mu_2\nu_2} - \pi_{\mu_1\mu_2} \pi_{\nu_1\nu_2} \right) \tag{7.76}$$

This is a local expression multiplied by $|m|$. In fact Fourier anti-transforming it and inserting it into (2.19) it gives rise to the action

$$S \sim |m| \int d^3x \left(-2 \partial_\mu h^{\mu\lambda} \partial_\nu h^\nu_\lambda - 2h \, \partial_\mu \partial_\nu h^{\mu\nu} - h^{\mu\nu} \Box h_{\mu\nu} + h \Box h \right) \tag{7.77}$$

which is the linearized Einstein-Hilbert action:

$$S_{EH} = \frac{1}{2\kappa} \int d^3x \, \sqrt{g} \, R \tag{7.78}$$

where $\kappa \sim \frac{1}{|m|}$.

These results for spin-1 and -2 are known have been known for a long time, see for instance [110]. Now, we ask the same question for the 2pt correlator of the 3-current. What action, if any, does it represent for the external source field?

7.5.2 Linearized equations for spin 3 in parity even sector

Before presenting results in 3d, let us briefly recall chapter 6 and the status of the linearized equations for the massless spin 3 field described by the completely symmetric field $\varphi_{\mu\nu\lambda}$. Historically the first formulation of equations for the *unconstrained* free massless spin 3 field was given by Fronsdal [37, 38]

$$\mathcal{F}_{\mu\nu\lambda} \equiv \Box\varphi_{\mu\nu\lambda} - (\partial_\mu \partial\cdot\varphi_{\nu\lambda} + \text{perm.}) + (\partial_\mu\partial_\nu\varphi'_\lambda + \text{perm.}) = 0 \tag{7.79}$$

Under the gauge variation (7.13), $\delta\varphi_{\mu\nu\lambda} = \partial_\mu\Lambda_{\nu\lambda} + \text{perm.}$, the Fronsdal kinetic tensor transforms as $\delta\mathcal{F}_{\mu\nu\lambda} = 3\partial_\mu\partial_\nu\partial_\lambda\Lambda'$. It follows that the Fronsdal equation is invariant only on *restricted* gauge transformations satisfying $\Lambda' = 0$ (this requirement holds for all higher spins). Also, the Fronsdal tensor is not divergence-free, $\partial \cdot \mathcal{F} \neq 0$, so one cannot directly couple the spin 3 field to a conserved (i.e., divergence-free) current using the Fronsdal equation. As we construct effective actions and corresponding equations for the higher spin fields by (minimally) coupling to conserved currents, it is obvious that Fronsdal's formalism is not suited for our purposes.

The formulation appropriate for our purposes was proposed in [39, 40, 41], and analyzed in more detail in [107] (for a review, see [108]). It was shown that there is a one parameter class of equations for unconstrained spin 3 field, which are order 2 in derivatives, fully gauge invariant, and ready to be coupled to the external conserved current. These equations are most elegantly expressed by using gauge invariant linearized spin 3 Riemann tensor defined by

$$R_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} = \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\,\varphi_{\nu_1\nu_2\nu_3} \quad (\,\text{antisymmetrised in all } (\mu_j,\nu_j)\,) \tag{7.80}$$

The spin 3 equations are parametrized by real number a and given by

$$\mathcal{G}(a)_{\mu\nu\lambda} \equiv \mathcal{A}(a)_{\mu\nu\lambda} - \eta_{\underline{\lambda\nu}}\,\mathcal{A}(a)'_{\underline{\mu}} = 0 \tag{7.81}$$

$$\mathcal{A}(a)_{\mu\nu\lambda} \equiv \frac{1}{\Box}\,\partial\cdot R'_{\underline{\mu\nu\lambda}} + a\,\frac{\partial_{\underline{\nu}}\partial_{\underline{\lambda}}}{\Box^2}\,\partial\cdot R''_{\underline{\mu}} \tag{7.82}$$

where spin 3 Ricci tensors are defined by

$$\begin{aligned} R'_{\mu\nu\rho\sigma} &\equiv \eta^{\alpha\beta}\,R_{\mu\nu\rho\alpha\sigma\beta} = 2\partial_{[\mu}\mathcal{F}_{\nu]\rho\sigma} \\ R''_{\mu\nu} &\equiv \eta^{\rho\sigma}\,R'_{\mu\nu\rho\sigma} = 2\partial_{[\mu}\mathcal{F}'_{\nu]} \end{aligned} \tag{7.83}$$

while their divergences are defined by[2]

$$\partial\cdot R'_{\mu\nu\lambda}=\partial_\alpha R'^{\alpha}{}_{\mu\nu\lambda}\qquad,\qquad \partial\cdot R''_{\mu}=\partial_\alpha R''^{\alpha}{}_{\mu} \tag{7.84}$$

What is the difference between equations with different a? First of all, it can be shown that regardless the value of a, the free field equation (7.81)-(7.82) is equivalent to Fronsdal equation (7.79). They start to differ when interactions are introduced. Note that equations (for any a) are non-local. From the purely mathematical side, the equation for $a=0$ plays a special role because it is the least singular on-shell[3], and because of this it was originally promoted in [39, 40, 41]. However, it was later shown in [107] that equations with different parameters a propagate different set of excitations when coupled to a conserved external current $j_{\mu\nu\lambda}$,

$$\mathcal{G}(a)=j\qquad,\qquad \partial\cdot j=0 \tag{7.85}$$

In particular, it was shown that only equation with $a=1/2$ propagates *spin 3 massless excitations and nothing else*, if one does not introduce additional constraints on φ or j. For $a=1/2$ the tensor $\mathcal{A}$ can be also written as

$$\mathcal{A}(1/2)=\mathcal{F}-\frac{\partial^3}{\Box^2}\,\partial\cdot\mathcal{F}' \tag{7.86}$$

Let us emphasize that this by itself does not mean that the equation with $a=1/2$ is the "right one" to be used for the consistent coupling to the dynamical matter.

The non-locality of equations (7.81)-(7.81) can be 'cured' by multiplying with $\Box^r$ with r large enough. It is obvious that the equation with $a=0$ is special in that $r=1$ already does the job, while for $a\neq 0$ one needs $r=2$. In this way one cures non-locality, but the price paid is that equations become higher-derivative (order 4 for $a=0$ and order 6 for $a\neq 0$). This opens up an additional question when one considers coupling to the conserved current j: should we do this as in (7.85), or should we couple the current in

[2] The Riemann tensor symmetries guarantee that the definitions for Ricci's and corresponding divergences (after symmetrization is taken into account) are essentially unique, in the sense that different choices for contracting indexes can differ only by a sign, or are vanishing [43].

[3] In momentum space the on-shell condition is $k^2=0$.

the local way,

$$\Box^r \mathcal{G}(a) = j \qquad , \qquad \partial \cdot j = 0 \tag{7.87}$$

with r large enough?

The moral of the above analysis is that, due to several reasons, there is a large degeneracy in formulating equations of motion for the free massless spin 3 field, and it is not obvious that all formulations can be used as a basis for constructing consistent interacting quantized theories. It would be advantageous to know which formulation(s) are more promising, before embarking into such enterprise. We shall now argue that the induced action method may give us a hint.

In section 3.2.4 of [28] it was shown that the parity even part of the spin 3 two-point traceless current correlator for a massive Dirac fermion in 3d is given by

$$\widetilde{\mathcal{T}}^{(even)}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(k) = \tau_b\left(\frac{k^2}{m^2}\right)|k|^5\pi_{\mu_1\mu_2}\pi_{\mu_3\nu_1}\pi_{\nu_2\nu_3} + \tau_b'\left(\frac{k^2}{m^2}\right)|k|^5\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3} \tag{7.88}$$

where τ_b and τ_b' are form factors presented in [28]. From (2.12) it follows that the linearized effective equation in momentum space for the background spin 3 field minimally coupled to a conserved current in free QFT with massive Dirac field in 3d, is given by

$$\widetilde{T}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(k)\,\widetilde{\varphi}^{\nu_1\nu_2\nu_3}(k) = \langle\!\langle \widetilde{j}^{(3)}_{\mu_1\mu_2\mu_3}(k)\rangle\!\rangle \qquad , \qquad k\cdot\widetilde{j}^{(3)}(k) = 0 \tag{7.89}$$

The form factors contain branch-cuts, which means that this equation is strongly nonlocal. There are two independent conserved structures present in (7.88), and consequently in (7.89), which is directly connected with the one-parameter degeneracy introduced in (7.82).

In the IR region ($|k^2|/m^2 < 4$) the form factors are analytic, as expected, and the equation is weakly nonlocal (infinite sum of local terms) when expanded around $|k|/m = 0$. Using the expansions of form factors from [28], we obtain that the leading term in the IR is given by

$$\widetilde{T}^{(even)}_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(k) \sim |m|\,k^4\left(\pi_{\mu_1\mu_2}\pi_{\mu_3\nu_1}\pi_{\nu_2\nu_3} - \pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}\right) \tag{7.90}$$

Observe that this is *the lowest derivative conserved local expression*, which is *unique*.

Now, putting (7.90) into (7.89) and Fourier antitransforming, we obtain for the linearized induced equation in the coordinate space

$$|m|\, G_{\mu\nu\rho}(x) \sim \langle\!\langle j^{(3)}_{\mu_1\mu_2\mu_3}(x) \rangle\!\rangle \qquad , \qquad \partial \cdot j^{(3)} = 0 \tag{7.91}$$

where G is the conserved symmetric local tensor linear in φ, which is 4th-order in derivatives. As there is a unique such tensor, we can conclude (without doing any calculations) that it must be proportional to $\Box\, \mathcal{G}(0)$, with $\mathcal{G}(0)$ defined in (7.81)-(7.82). Explicitly written,

$$G_{\mu\nu\lambda} = \partial_\alpha F^\alpha{}_{(\mu\nu\lambda)} \tag{7.92}$$

where

$$F_{\alpha\mu\nu\lambda} \equiv R'_{\alpha\mu\nu\lambda} - \frac{1}{2} R''_{\alpha\mu} \eta_{\nu\lambda} = 2\partial_{[\alpha} \left(\mathcal{F}_{\mu]\nu\lambda} - \frac{1}{2} \mathcal{F}'_{\mu]} \eta_{\nu\lambda} \right) \tag{7.93}$$

The result (7.91)-(7.93) is, in some sense, natural. First of all, it is the lowest derivative linear local parity invariant equation satisfying unrestricted gauge invariance and conservation condition. Also, the equation is of the same form as in spin 1 case, and we can identify the tensor F as spin 3 Maxwell tensor, while G appears to be spin 3 Riemann tensor (it is the lowest derivative local conserved gauge invariant parity even rank-3 tensor).[4]

Let us connect these result with the known constructions, reviewed above. It is obvious that our result (7.91)-(7.93) is the same as (7.87) with $a = 0$ and $r = 1$, i.e., the obtained expression is a local version of the equation proposed in [39, 40, 41]. As we already mentioned, this equation does not propagate only spin 3 massless excitations, unless the conserved spin 3 current of the Dirac theory has some special properties which takes care of the redundant modes.

Let us now briefly comment the UV limit ($m/|k| \to 0$). After subtracting IR divergent terms (for a full explanation of this issue, see below), form factors in the UV limit tend to constants, which gives rise to a non-local correlator. However one of the subleading

[4] Conventions for naming objects in higher-spin metric-like formalism is notorious for its inconsistency. In the literature different objects are called Ricci tensor and Riemann tensor. We believe that our conventions are natural generalizations of spin 1 and 2 cases.

terms gives a combination of the two conserved quantities

$$A \ : \ k^2\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3} \quad \text{and} \quad B \ : \ k^2\pi_{\mu_1\mu_2}\pi_{\mu_3\nu_1}\pi_{\nu_2\nu_3} \tag{7.94}$$

which is not the same combination as the one present in IR limit (7.90). So, the corresponding induced linearized equation is also different. Expanding (7.88) in the UV we obtain the traceless combination $A - \frac{3}{4}B$, with coefficients corresponding to (7.42) for $d = 3$ and $s = 3$, for which the equation of motion is

$$\begin{aligned} &\Box\varphi_{\mu\nu\lambda} - 3\partial_\mu\partial\cdot\varphi_{\nu\lambda} + \frac{3}{4}\partial_\mu\partial_\nu\varphi'_\lambda - \frac{3}{4}\frac{1}{\Box}\partial_\mu\partial_\nu\partial_\lambda\partial\cdot\varphi' - \frac{1}{4}\frac{1}{\Box^2}\partial_\mu\partial_\nu\partial_\lambda\partial\cdot\partial\cdot\partial\cdot\varphi \\ &+\frac{9}{4}\frac{1}{\Box}\partial_\mu\partial_\nu\partial\cdot\varphi_\lambda - \frac{3}{4}\eta_{\mu\nu}\Box\varphi'_\lambda + \frac{3}{4}\eta_{\mu\nu}\partial_\lambda\partial\cdot\varphi' + \frac{3}{4}\eta_{\mu\nu}\partial\cdot\partial\cdot\varphi_\lambda - \frac{3}{4}\eta_{\mu\nu}\frac{1}{\Box^2}\partial_\lambda\partial\cdot\partial\cdot\partial\cdot\varphi = 0 \end{aligned} \tag{7.95}$$

In conclusion, we see that our simple analysis, based solely on the classification of possible conserved structures, recovers the Francia-Sagnotti analysis and gives an efficient method for analyzing higher spin actions. But, we emphasize that the induced action method, out of many possibilities, picks particular equations which are already coupled to particular external currents.

Comment. The previous results are limited to 3d and to the lowest spins. They are nevertheless enough to stir our interest and motivate a more in depth analysis. It is also clear enough that equations in the coordinate space are not always the best fit to generalizations to higher spins. Writing down the actions and equations of motion in the explicit form used so far becomes rapidly unwieldy with increasing spins and dimensions. Because of that, we simply use the projector (7.29).

7.6 Guidelines for the calculations

In the next two chapters we do explicit calculations and mostly focus on results for two-point functions (bubble diagrams formed by two internal scalar or fermion lines and two vertices) in the scalar and fermion model in different dimensions.

We will start with spin-1 and spin-2 fields coupled to scalar and fermion model. In contrast to higher spin ($s > 2$) fields, for $s \leq 2$ we know the full covariant action (beyond linear order). As a consequence, in the initial action we have additional terms, additional with respect to the minimal couplings (symbolically $\int j\varphi$), which are on-shell covariant, but off-shell non-covariant. One of the crucial steps in our program is clearly implementing off-shell gauge covariance of the initial models, that is adding to the minimal couplings in the relevant actions the terms that render them off-shell covariant, at least to the lowest order in a perturbative approach to the gauge symmetry. We know such additional terms exactly in the case of spin 1 and spin 2 and in these cases, perturbative field theory formalism already automatically takes care of satisfied Ward identities provided one takes into account not only the two-point bubble diagrams but also other diagrams such as tadpole and seagull ones, [120, 153]. Although this is a rather well-known fact, we would like to show it in detail here for spin 1 and 2 as a guide for the more challenging higher spin cases. We will show the role of tadpole and seagull terms in the Ward identities for two-point functions of spin 1 and 2 respectively, and their origin in the various terms of the initial actions. For completeness, we analyze the full structure of the relevant two-point functions and, in particular, their IR and UV expansions, as well as their contributions to the effective actions.

The same is not as easy for higher spin currents. In generic spin current correlators we will find violation of Ward identities. Such violations come in a form of local terms and we can recover conservation by subtracting local counterterms from the effective action. Besides the non-conserved (or better said non-transverse) terms for higher spins, for any spin we also find terms that diverge in the IR limit $m \to \infty$. Fortunately these terms are finite in number and easy to identify by expanding the amplitude near the IR and the UV. All the IR divergent terms are also local.

The Feynman diagram method is the most convenient for our purposes, but it is nevertheless one out of many. In fact, even within it there are different possibilities or schemes. We expect that our results may depend on such schemes, but also to find a criterion to extract the scheme independent part. In most cases this is conservation and finiteness. In particular, by suitably choosing the scheme we will be able, for instance, to obtain both finiteness and conservation in our models.

It is possible to subtract all the terms that diverge in the IR, which include, in particular, all the nonconserved ones and recover both conservation and finiteness in the IR. In this process a particular attention has to be paid to the terms of order 0 in m, in even dimensions. In some cases they are local and conserved, and appear both in the IR and the UV. Even in this case we follow the attitude of subtracting the IR term from the corresponding UV one, on the assumption that physical information is contained in the difference between the UV and the IR, not in their absolute values. Finally it should be added that the resulting IR and UV expansions are both convergent.

Even dimensional models present an additional problem concerning their regularization. For odd d works by itself as a complete regulator in carrying out the integrals generated by the Feynman diagrams. This is not true for even d. The way out is well-known, we will set $d = 4+\varepsilon$. Another difference we will come across with, which is related to this, is the appearance of log terms in the form factors. We will again expand the two-point functions in powers of m near the IR and UV limits. In almost all the two-point correlators and, therefore, in all the one-loop effective actions, we will find non-conserved terms and terms that diverge in the IR $m \to \infty$, like in the odd dimensional case, but we will find also ε-divergent terms. Our general attitude is to recover both conservation and finiteness in the IR. This is possible because all the nonconserved and all divergent terms in the IR, as well as all ε-divergent terms, are local. We will therefore subtract all the terms that diverge in the IR and in ε. They include, in particular, all the nonconserved ones.

There remains however an ambiguity. Beside divergent and/or nonconserved terms, in the case of m^0 we meet also finite contributions, both in the IR and in the UV. Also for these terms we subtract the IR from the UV contribution, on the assumption that it is this difference that contains the physical information.

A few more remarks regarding the notation. For conciseness, we use a simplified notation, taken from the literature on higher spin fields: the same repeated subscript, say $\mu \dots \mu$ repeated s times, stand for s completely symmetrized labels. Sometimes we will instead of $\mu \dots \mu$ repeated s times use simply μ^s. To somewhat abbreviate the following formulas, at times we use the compact notation

$$\Pi^{(2)}_{\mathfrak{a},\,\mu^2\nu^2}(k) \quad = \quad \pi^2_{\mu\nu} + \mathfrak{a}\pi_{\mu\mu}\pi_{\nu\nu}, \tag{7.96}$$

$$\Pi^{(3)}_{\mathfrak{a},\,\mu^3\nu^3}(k) \quad = \quad \pi^3_{\mu\nu} + \mathfrak{a}\pi_{\mu\mu}\pi_{\mu\nu}\pi_{\nu\nu}, \tag{7.97}$$

where $\mathfrak{a}$ is some constant. Finally, contrary to ([28]), the latter is $k \equiv |k| = \sqrt{k^2}$. The calculations in the sequel are mainly carried out using a Mathematica code [181].

Chapter 8

Scalar models

In this chapter we consider a scalar theory coupled to spin-s fields via conserved currents, and we closely follow [29] and [30]. The method we use is the perturbative approach based on Feynman diagrams and dimensional regularization.

We start by considering the massless case for the scalar model, i.e. we set $m = 0$ in the action, and derive the relevant two-point functions for simple and traceless currents in any dimension. These results are based on the scalar integral (7.72). For traceless currents the amplitude is itself traceless, and this amplitude vanishes for currents with two different spins. We also compute 1-point functions (tadpoles) for general spin s and general dimension d. Tadpole diagrams vanish in the massless case.

Next we present results for a scalar theory coupled to spin 1, 2 and 3 fields. In general, results for our correlators will be given in terms of hypergeometric functions. Since it is quite hard to extract information from these general expressions, we turn to their IR and UV expansions for $d = 3, 4$ (for expansions in $d = 5, 6$ see [29]). In the UV $\mathcal{O}_{UV}(m^0) - \mathcal{O}_{IR}(m^0) - \mathcal{O}_{IR}(\log(m))$ terms exactly coincide with the massless results.

For spin 1 and 2 we know full form of the interaction and because of that, beside the 2-point function (bubble diagram formed by two internal scalar lines and two vertices) we include seagull and tadpole diagrams as well. By explicit computation we show that Ward identities are satisfied. For spin 3 the situation is not so simple because in this case we know only the linear coupling and the linear form of gauge transformation. As a consequence we will find several violations of Ward identities in a form of finite number of local non-conserved (non-transverse) terms. In all of these cases, besides the non-conserved terms, we also find terms that diverge in the IR limit $m \to \infty$. These terms are also finite in number and local. We easily identify them by expanding the amplitude near the IR and the UV for specific dimension. Our prescription to extract physical information is such that we subtract all the terms that diverge in the IR (these terms include all the nonconserved ones) by subtracting a finite number of counterterms from the effective action. In this way we recover both conservation and finiteness in the IR. We demonstrate how this particular scheme works, not just in the higher spin case, but also for spin 1 and 2.

In this model we also give a general expression for the conserved part of the 2-point function for general spin s and general dimension d.

The final part of the chapter is devoted to diagonalization of our results, that is, the possibility of vanishing off-diagonal correlators for appropriate choice of currents. It turns out that there is an infinite number of non-conserved terms in the off-diagonal correlators one should cancel, and hence the diagonalization is not possible when we choose the currents of the form (7.26). One more example we consider is the case of traceless local currents where we are able to diagonalize our results by appropriate choice of coefficients in the currents and by subtraction of finite number of counterterms.

8.1 Massless model

Here we will present some general results for massless case. Let us start with mixed correlators of scalar simple currents (7.2). General expression for spin $s_1 \times s_2$, $s_2 > s_1$

$$\begin{aligned}\tilde{T}_{\mu_1\ldots\mu_{s_1}\nu_1\ldots\nu_{s_2}} &= (-1)^{\frac{s_1+s_2}{2}} \frac{\left(2\lfloor\frac{s_2+1}{2}\rfloor-1\right)!!\left(2\lfloor\frac{s_2+1}{2}\rfloor\right)!!2^{4-2d-\frac{s_1+s_2}{2}}\pi^{\frac{3}{2}-\frac{d}{2}}\left(k^2\right)^{d/2+\frac{s_1+s_2}{2}-2}}{\left(2\lfloor\frac{s_2}{2}\rfloor-2\lfloor\frac{s_1}{2}\rfloor\right)!!\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d+s_1+s_2-1}{2}\right)} \\ &\times\pi_{\nu\nu}^{\frac{s_2-s_1}{2}}\sum_{l=0}^{\lfloor\frac{s_1}{2}\rfloor}\frac{s_1!(s_2-s_1)!!}{2^{\frac{l(l+1)}{2}}(s_1-2l)!(s_2-s_1+2l)!!}\pi_{\mu\mu}^{l}\pi_{\nu\nu}^{l}\pi_{\mu\nu}^{s_1-2l} \qquad (8.1)\end{aligned}$$

Next, we use traceless currents (traceless in the limit $m \to 0$), that is (7.26) with coefficients (7.27). General expression for spin $s \times s$

$$
\begin{aligned}
\tilde{T}_{\mu_1\ldots\mu_s\nu_1\ldots\nu_s} &= (-1)^s \frac{2^{4-2d-s}\pi^{\frac{3}{2}-\frac{d}{2}} s!\,(k^2)^{d/2+s-2}}{(-1+e^{i\pi d})\,\Gamma\left(\frac{d+2s-1}{2}\right)} \sum_{l=0}^{\lfloor\frac{s}{2}\rfloor} (-1)^l a_l \pi^l_{\mu\mu}\pi^l_{\nu\nu}\pi^{s-2l}_{\mu\nu} \\
&= (-1)^s \frac{2^{4-2d-s}\pi^{\frac{3}{2}-\frac{d}{2}} s!\,(k^2)^{d/2+s-2}}{(-1+e^{i\pi d})\,\Gamma\left(\frac{d+2s-1}{2}\right)} \pi^s_{\mu\nu} \\
&\quad \times {}_2F_1\left(\frac{1-s}{2}, -\frac{s}{2}, \frac{5-d-2s}{2}, \frac{\pi_{\mu\mu}\pi_{\nu\nu}}{\pi^2_{\mu\nu}}\right)
\end{aligned}
\tag{8.2}
$$

Traceless currents give traceless amplitude in the massless limit, that is, coefficient a_l corresponds to (7.42), the coefficient appearing in the traceless amplitude. In this case mixed spin correlators vanish.

8.2 Tadpoles

Let us also write down the tadpole diagram contributions for any dimension and any spin. In this chapter we will need only spin 1 and 2 tadpoles. The tadpole contribution actually vanishes for odd spins, as we will shortly see. Tadpoles (1-point function) are defined with

$$
\Theta^{\mu_1\ldots\mu_s}(x) \equiv \Theta^{\mu^s} = \frac{\delta\left(i\,W[\varphi]\right)}{\delta\varphi_{\mu_1\ldots\mu_s}(x)}\bigg|_{\varphi=0} \tag{8.3}
$$

Tadpoles with scalar current for any spin s and any dimension d are given by

$$
\tilde{\Theta}^{\mathrm{s}}_{\mu^s} = \int \frac{d^dp}{(2\pi)^d} V^{(s)}_{\mu\ldots\mu} \frac{i}{p^2-m^2} = -\int \frac{d^dp}{(2\pi)^d} (2p)^s_\mu \frac{1}{p^2-m^2} \tag{8.4}
$$

Next we use

$$
p^s_\mu = \begin{cases} \frac{(s-1)!!}{\prod_{i=0}^{\frac{s}{2}-1}(d+2i)} (p^2)^{\frac{s}{2}} \eta^{\frac{s}{2}}_{\mu\mu} & s \text{ even} \\ 0, & s \text{ odd} \end{cases} \tag{8.5}
$$

so that for even spin s we get

$$
\tilde{\Theta}^{\mathrm{s}}_{\mu^s} = -\frac{2^s(s-1)!!}{\prod_{i=0}^{\frac{s}{2}-1}(d+2i)} \eta^{\frac{s}{2}}_{\mu\mu} \int \frac{d^dp}{(2\pi)^d} \frac{p^s}{p^2-m^2} \tag{8.6}
$$

To evaluate the integral we use

$$\int \frac{d^dp}{(2\pi)^d}\frac{(p^2)^n}{(p^2-\Delta^2)^m} = (-1)^{n-m} i\frac{\Gamma\left(\frac{d}{2}+n\right)\Gamma\left(m-n-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)\Gamma\left(m\right)}\Delta^{\frac{d}{2}+n-m} \tag{8.7}$$

Altogether we have

$$\tilde{\Theta}^{\rm s}_{\mu^s} = \begin{cases} i(-1)^{\frac{s}{2}}2^{\frac{s}{2}-d}(s-1)!!\ \pi^{-\frac{d}{2}}m^{d+s-2}\Gamma\left(1-\frac{d}{2}-\frac{s}{2}\right)\eta_{\mu\mu}^{\frac{s}{2}}, & s \text{ even} \\ 0, & s \text{ odd} \end{cases} \tag{8.8}$$

8.3 Spin 1

This case is well known and simple, but it is excellent for pedagogical purposes. Let us start by writing the action for the scalar QED model

$$S = \int \mathrm{d}^dx \left[D_\mu\varphi^\dagger D^\mu\varphi - m^2\varphi^\dagger\varphi\right] \tag{8.9}$$

where $D_\mu = \partial_\mu - i\,A_\mu$. The full covariant action is

$$S = \int \mathrm{d}x \left[\partial_\mu\varphi^\dagger\partial^\mu\varphi + i\,A_\mu\left(\varphi^\dagger\partial^\mu\varphi - \partial^\mu\varphi^\dagger\varphi\right) + A_\mu A^\mu\varphi^\dagger\varphi - m^2\varphi^\dagger\varphi\right] \tag{8.10}$$

In the scalar model the scalar-scalar-photon vertex is (7.25) and we also have scalar-scalar-photon-photon vertex (coming from $\int d^dxA^\mu A_\mu\varphi^\dagger\varphi$ term in Lagrangian)

$$V^{\mu\nu}_{sspp}(p,p') \quad : \quad 2i\eta^{\mu\nu} \tag{8.11}$$

One-loop conservation which for spin 1 is (2.30), so that the Ward identity for the two-point function in momentum space can be written as

$$k_\mu\tilde{T}^{\mu\nu}(k) = 0 \tag{8.12}$$

The two-point function for the massive scalar in any dimension d for spin $s=1$ is

$$\tilde{T}^{\mu\nu}(k) = -2^{1-d}\,i\,\pi^{-d/2}m^{d-2}\Gamma\left(1-\frac{d}{2}\right)\left(\,{}_2F_1\left[1,1-\frac{d}{2};\frac{3}{2};\frac{k^2}{4m^2}\right]\pi^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right) \tag{8.13}$$

The theory is quadratic in the external photon field A we also have a seagull diagram (which is obtained by joining with a unique a fermion line the two fermion legs of the vertex (8.11) for which we obtain

$$\tilde{T}^{\mu\nu}_{(s)}(k) = 2^{1-d} i\pi^{-\frac{d}{2}} m^{d-2} \Gamma\left(1 - \frac{d}{2}\right) \eta^{\mu\nu} \tag{8.14}$$

After combining (8.13) and (8.14) we can write down the full 2-point function

$$\tilde{T}^{\mu\nu}(k) = 2^{1-d} i\pi^{-\frac{d}{2}} m^{d-2} \Gamma\left(1 - \frac{d}{2}\right) \left(1 - {}_2F_1\left[1, 1 - \frac{d}{2}; \frac{3}{2}; \frac{k^2}{4m^2}\right]\right) \pi^{\mu\nu}, \tag{8.15}$$

which is conserved. Expanding the two-point function (8.15) in the IR gives

$$\tilde{T}^{\mu\nu}(k) \quad = \quad -2^{-d} i\, m^{d-4} \pi^{-\frac{d}{2}} \sum_{n=0}^{\infty} \frac{m^{-2n}\Gamma\left(2 + n - \frac{d}{2}\right)}{2^n (2n+3)!!} k^{2n+2} \pi^{\mu\nu} \tag{8.16}$$

Using the IR expansion together with (2.12), the one-loop 1-point function now reads

$$\langle\langle J^\mu \rangle\rangle = -2^{-d} m^{d-4} \pi^{-\frac{d}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n\, m^{-2n}\Gamma\left(2 + n - \frac{d}{2}\right)}{2^n (2n+3)!!} \Box^n \partial_\nu F^{\mu\nu} \tag{8.17}$$

The dominating term in the IR corresponds to the Maxwell equation. The dominating term for the effective action in the IR region

$$W \overset{\text{IR}}{=} -\frac{2^{-d}}{3} m^{d-4} \pi^{-\frac{d}{2}} \Gamma\left(2 - \frac{d}{2}\right) \int \mathrm{d}^d x F_{\mu\nu} F^{\mu\nu} \tag{8.18}$$

gives Maxwell action.

The leading order term in the UV (term m^0 corresponds to (B.13) from [29])

$$\tilde{T}^{\mu\nu}(k) \overset{UV}{=} -\frac{2^{3-2d}\, \pi^{\frac{3}{2}-\frac{d}{2}} (k^2)^{\frac{d}{2}-1}}{(-1 + e^{i\pi d})\, \Gamma\left(\frac{d+1}{2}\right)} \pi^{\mu\nu} \tag{8.19}$$

Hence, the effective action in the UV is

$$W \overset{UV}{=} -i \frac{(-1)^{\frac{d}{2}} 2^{3-2d}\, \pi^{\frac{3}{2}-\frac{d}{2}}}{(-1 + e^{i\pi d})\, \Gamma\left(\frac{d+1}{2}\right)} \int d^d x F^{\mu\nu} \Box^{\frac{d}{2}-2} F_{\mu\nu} \tag{8.20}$$

8.3.1 3d msm; spin 1 tomography

Even though we just showed that for spin-1 current in the scalar model Ward identities are satisfied, we will also show what happens if we know the interaction only up to the linear order. In this case we only have the bubble diagram. We demonstrate our scheme to extract physical information from the amplitude by expanding it in the IR and UV and subtracting the divergent and nonconserved terms from the effective action. The exact 2-pt correlator for $s = 1$ in 3d is obtained by putting $d = 3$ in (8.13)

$$\begin{aligned}\tilde{T}(k)_{\mu\nu} &= \frac{i}{8\pi k^3}\left(-4m^2\coth^{-1}\left(\frac{2m}{k}\right) + 2km + k^2\coth^{-1}\left(\frac{2m}{k}\right)\right)k_\mu k_\nu \\ &\quad + \frac{i}{8\pi k}\left(4m^2\coth^{-1}\left(\frac{2m}{k}\right) + 2km - k^2\coth^{-1}\left(\frac{2m}{k}\right)\right)\eta_{\mu\nu} \end{aligned} \tag{8.21}$$

We can expand (8.21) in power of $\frac{k}{m}$ (IR) or of $\frac{m}{k}$ (UV). In the sequel we will consider only the minimal model with a linear coupling and because of that we will find violation of Ward identities. We remove the non-conserved terms by subtracting appropriate countertems from the effective action. In the IR case we find

$$\mathcal{O}(m): \quad \frac{im}{2\pi}\eta_{\mu\nu} \tag{8.22}$$

$$\mathcal{O}(m^{-1}): \quad -\frac{ik^2}{24\pi m}\pi_{\mu\nu} \tag{8.23}$$

$$.... \quad ....$$

while the even powers of m vanish. The first is a (non-conserved and divergent in the IR limit) local term $\sim \eta_{\mu\nu}$, which must be subtracted away. The other terms are all conserved and proportional to the conserved structure $\pi_{\mu\nu}$.

The UV expansion is instead

$$\mathcal{O}(m^0): \quad -\frac{k}{16}\pi_{\mu\nu} \tag{8.24}$$

$$\mathcal{O}(m): \quad \frac{im}{2\pi k^2}k_\mu k_\nu \tag{8.25}$$

$$\mathcal{O}(m^2): \quad -\frac{m^2}{4k}\pi_{\mu\nu} \tag{8.26}$$

$$.... \quad ....$$

In fact we have $\mathcal{O}(m^{2n}) = 0$ for $n \geq 2$. The only nonvanishing terms with even powers of

m are $\mathcal{O}(m^0), \mathcal{O}(m^2)$. For these terms see the comment below.

Except (8.25) the other terms are conserved and proportional to the projector $\pi_{\mu\nu}$. The terms proportional to $\pi_{\mu\nu}$ are all non-local in the UV, and local in the IR, in particular (8.23) is local and corresponds to the Maxwell action in 3d.

The two nonconserved terms are (8.22) in the IR and (8.25) in the UV. The first is local and the second is nonlocal, but their divergence is the same and local:

$$-\frac{i}{2\pi}k_\nu$$

This means that we can cancel it by subtracting a local term, $\sim m \int d^3x\, A^2$. This amounts to subtracting the IR contribution (which is local) from the UV one. Indeed we get

$$\mathcal{O}_{UV}(m) - \mathcal{O}_{IR}(m) = -\frac{im}{2\pi}\,\pi_{\mu\nu} \tag{8.27}$$

So the term of order m in the UV and IR conjure up to reform again the same conserved structure as all the other terms. Taking the UV and IR limits splits apart this conserved structure. The conclusion is that, up to a local term we can view the effective action as a sum of infinite many terms, all proportional to $\pi_{\mu\nu}$ with coefficients proportional to various monomials of m and k. In compact form:

$$\frac{i}{8\pi k}\left(4m^2\coth^{-1}\left(\frac{2m}{k}\right) - 2km - k^2\coth^{-1}\left(\frac{2m}{k}\right)\right)\pi_{\mu\nu} \tag{8.28}$$

8.3.2 4d msm: spin 1 tomography

Let us repeat the same procedure as above for $d = 4$. We will focus on the power of m expansions again. However, as previously mentioned, we have to consider also $\log(m)$ and $\frac{1}{\varepsilon}$ factors. In the IR the nonvanishing terms are

$$\begin{aligned}
\mathcal{O}(m^2): &\quad -\frac{im^2}{8\pi^2}\Big(\gamma - 1 - \log(4\pi) + 2\log(m) + \frac{2}{\varepsilon}\Big)\eta_{\mu\nu} && (8.29)\\
\mathcal{O}(\log(m)): &\quad \frac{i\log(m)}{24\pi^2}k^2\pi_{\mu\nu} && (8.30)\\
\mathcal{O}(m^0): &\quad \frac{ik^2}{48\pi^2}\Big(\gamma - \log(4\pi) + \frac{2}{\varepsilon}\Big)\pi_{\mu\nu} && (8.31)\\
\mathcal{O}(m^{-2}): &\quad -\frac{ik^4}{480\pi^2 m^2}\pi_{\mu\nu} && (8.32)\\
\dots\dots &\quad \dots\dots
\end{aligned}$$

These coefficients are conserved except $\mathcal{O}(m^2)$. All the odd powers of m vanish.

In the UV we find the following nonvanishing terms:

$$\mathcal{O}(m^0): \qquad -i\frac{k^2}{144\pi^2}\Big(8-3\gamma-\log\left(\frac{1}{64\pi^3}\right)-3\log\left(-k^2\right)-\frac{6}{\varepsilon}\Big)\,\pi_{\mu\nu} \tag{8.33}$$

$$\begin{aligned}\mathcal{O}(m^2): \qquad & -\frac{im^2}{24\pi^2k^2}\Bigg(\left(-3\log\left(-\frac{k^2}{m^2}\right)+3\right)k_\mu k_\nu \\ & +k^2\left(3(-2+\gamma-\log(4\pi))+3\log\left(-k^2\right)+\frac{6}{\varepsilon}\right)\eta_{\mu\nu}\Bigg)\end{aligned} \tag{8.34}$$

$$\mathcal{O}(m^4): \qquad -i\frac{m^4}{16\pi^2k^2}\Big(-2\log\left(-\frac{k^2}{m^2}\right)-3\Big)\pi_{\mu\nu} \tag{8.35}$$

$$\ldots. \qquad \ldots.$$

All odd powers of m vanish. The even powers are conserved except (8.34). Subtracting from the latter the analogous (local) non-conserved term in the IR we find a conserved term

$$\mathcal{O}_{UV}(m^2)-\mathcal{O}_{IR}(m^2)=-\frac{im^2}{8\pi^2}\Big(2\log\left(-\frac{k^2}{m^2}\right)-1\Big)\pi_{\mu\nu} \tag{8.36}$$

The $\mathcal{O}(\log(m))$ term is divergent in the IR, and the $\mathcal{O}(m^0)$ is divergent in the $\varepsilon\to 0$ limit. Luckily they are local and can be subtracted with the following result:

$$\mathcal{O}_{UV}(m^0)-\mathcal{O}_{IR}(m^0)-\mathcal{O}_{IR}(\log(m))=-\frac{ik^2}{144\pi^2}\left(-3\log\left(-\frac{k^2}{m^2}\right)+8\right)\pi_{\mu\nu} \tag{8.37}$$

This term corresponds to the Maxwell action.

8.4 Spin 2

Let us now consider the action of a scalar field φ in a curved space ($g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$) with a scalar curvature coupling

$$S=\int d^dx\sqrt{g}\left(g^{\mu\nu}\partial_\mu\varphi^\dagger\partial_\nu\varphi-m^2\varphi^\dagger\varphi+\xi R\varphi^\dagger\varphi\right) \tag{8.38}$$

Let us redefine $\phi = g^{\frac{1}{4}}\varphi$. The expansion of the action in the external field h is

$$
\begin{aligned}
S &= \int d^dx \Bigg[\eta^{\mu\nu}\partial_\mu\phi^\dagger\partial_\nu\phi - m^2\phi^\dagger\phi + h^{\mu\nu}\left(\frac{1}{4}\phi^\dagger \overleftrightarrow{\partial}_\mu \overleftrightarrow{\partial}_\nu \phi + \left(\xi - \frac{1}{4}\right)(\partial_\mu\partial_\nu - \Box\eta_{\mu\nu})\phi^\dagger\phi\right) \\
&\quad + h^{\mu\sigma}h^\nu_\sigma\partial_\mu\phi^\dagger\partial_\nu\phi + \frac{1}{16}h\Box h\phi^\dagger\phi + \left(-\frac{\xi}{4} + \frac{1}{8}\right)\partial_\mu h\partial^\mu h\phi^\dagger\phi - 2\xi h^{\mu\nu}\partial_\nu\partial_\lambda h^\lambda_\mu\phi^\dagger\phi \\
&\quad + \xi h^{\mu\nu}\Box h_{\mu\nu}\phi^\dagger\phi - \xi\partial_\nu h^{\mu\nu}\partial_\lambda h^\lambda_\mu\phi^\dagger\phi + \frac{3}{4}\xi\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu}\phi^\dagger\phi - \frac{1}{2}\xi\partial_\lambda h^{\mu\nu}\partial_\nu h^\lambda_\mu\phi^\dagger\phi \\
&\quad \left(\xi - \frac{1}{4}\right)h^{\mu\nu}\partial_\mu\partial_\nu h\phi^\dagger\phi + \left(\xi - \frac{1}{4}\right)\partial_\mu h\partial_\nu h^{\mu\nu}\phi^\dagger\phi\Bigg]
\end{aligned}
\tag{8.39}
$$

The scalar-scalar-graviton vertex is:

$$
V^{\mu\mu}_{ssh}(p,p') \;:\; -\frac{i}{4}(p^\mu + p'^\mu)^2 - i\left(\xi - \frac{1}{4}\right)\left((p'^\mu - p^\mu)^2 - \eta^{\mu\mu}(p'-p)^2\right) \tag{8.40}
$$

which reduces to (7.25) for $\xi = \frac{1}{4}$ and there is a vertex with two scalars and two gravitons:

$$
\begin{aligned}
V^{\mu\mu\nu\nu}_{sshh}(p,p',k,k') \;:\; & i\eta^{\mu\nu}\left(p'^\mu p^\nu + p^\mu p'^\nu\right) - i\Big[\left(\xi - \frac{1}{4}\right)\left(\eta^{\mu\mu}k^\nu k^\nu + \eta^{\nu\nu}k^\mu k^\mu\right) \\
& + 2\left(\xi\eta^{\mu\nu}\eta^{\mu\nu} + \frac{1}{16}\eta^{\mu\mu}\eta^{\nu\nu}\right)k^2 - 4\xi\eta^{\mu\nu}k^\mu k^\nu\Big] \\
& - i\Big[\left(\left(\frac{1}{4} - \frac{\xi}{2}\right)\eta^{\mu\mu}\eta^{\nu\nu} + \frac{3}{2}\xi\eta^{\mu\nu}\eta^{\mu\nu}\right)k\cdot k' \\
& + \left(\xi - \frac{1}{4}\right)\left(\eta^{\mu\mu}k^\nu k'^\nu + \eta^{\nu\nu}k^\mu k'^\mu\right) - 2\xi\eta^{\mu\nu}k^\mu k'^\nu - \xi\eta^{\mu\nu}k^\nu k'^\mu\Big]
\end{aligned}
\tag{8.41}
$$

The full conservation law of the energy-momentum tensor is (2.31), and hence, the Ward identity for one-point function is

$$
\partial_\mu\Theta^{\mu\mu}(x) = 0 \tag{8.42}
$$

while for two-point correlator we have

$$
\begin{aligned}
\partial_\mu T^{\mu\mu\nu\nu}(x,y) &= \frac{1}{2}\eta^{\nu\nu}\delta(x-y)\partial_\mu\Theta^{\mu\mu}(x) + \frac{1}{2}\Theta^{\nu\nu}(x)\partial^\mu\delta(x-y) \\
&\quad - \partial_\mu\left(\delta\left(x-y\right)\Theta^{\mu\nu}\left(x\right)\right)\eta^{\mu\nu}
\end{aligned}
\tag{8.43}
$$

From (8.8), it follows that the tadpole contribution is $\tilde{\Theta}^{\mu\mu}(k) = \tilde{\Theta}\,\eta^{\mu\mu}$ where $\tilde{\Theta}$ is a constant. The Ward identity in momentum space is now

$$k_\mu \tilde{T}^{\mu\mu\nu\nu}(k) = \left[-k^\nu \eta^{\mu\nu} + \frac{1}{2} k^\mu \eta^{\nu\nu} \right] \tilde{\Theta} \tag{8.44}$$

Taking the result for the tadpole diagram (8.8) for $s = 2$ we have

$$\tilde{\Theta}^{\mu\mu} = 2^{-d-1}\, i\, \pi^{-d/2} m^d \Gamma\left(-\frac{d}{2}\right) \eta^{\mu\mu} \tag{8.45}$$

while the contribution from the seagull term is

$$\begin{aligned} \tilde{T}^{\mu\mu\nu\nu}_{(s)}(k) &= -2^{-4-d}\, i\pi^{-d/2} m^{d-2} \Gamma\left(-\frac{d}{2}\right) \\ &\times \left(dk^2(1-4\xi)\eta^{\mu\mu}\eta^{\nu\nu} + 4\eta^{\mu\nu}\eta^{\mu\nu}\left(4m^2 - dk^2\xi\right) + 8d\xi\eta^{\mu\nu}k^\mu k^\nu \right) \end{aligned} \tag{8.46}$$

Furthermore, the transverse part of the bubble diagram reads

$$\begin{aligned} \tilde{T}^{\mu\mu\nu\nu}_t(k) = &-\frac{1}{3d\left(d^2-1\right)k^4} i 2^{-d-2} e^{-\frac{1}{2}i\pi d} \pi^{-d/2} (-m^2)^{d/2} m^{-2} \Gamma\left(1 - \frac{d}{2}\right) \\ &\Big[\Big(12\left(d^2-1\right)k^4 m^2 \left(8\xi^2 - 8\xi + 1\right) + d\left(d^2-1\right)k^6\left(24\xi^2 - 1\right) \\ &+ 24dk^2m^4(3-8\xi) - 192k^2m^4\xi + 96m^6 \\ &+ \left(-6k^4m^2\left(d^2(1-4\xi)^2 + d(8\xi-2) - 2\left(8\xi^2 - 8\xi + 1\right)\right)\right. \\ &\left.+24k^2m^4(d(8\xi-2)+8\xi) - 96m^6\right)\, {}_2F_1\left[1, -\frac{d}{2}; -\frac{1}{2}; \frac{k^2}{4m^2}\right] \Big) \pi^{\mu\mu}\pi^{\nu\nu} \\ &+ \Big(-12d^2k^4m^2 + d\left(d^2-1\right)k^6 + 48dk^2m^4 - 96k^2m^4 + 12k^4m^2 + 192m^6 \\ &-12m^2\left(k^2 - 4m^2\right)^2\, {}_2F_1\left[1, -\frac{d}{2}; -\frac{1}{2}; \frac{k^2}{4m^2}\right] \Big) \pi^{\mu\nu}\pi^{\mu\nu} \Big] \end{aligned} \tag{8.47}$$

The expansion of the transverse part $\tilde{T}^{\mu\mu\nu\nu}_t(k)$ in the IR is

$$\begin{aligned} \tilde{T}^{\mu\mu\nu\nu}_t(k) &= 2^{-3-d} i\, m^{d-4} \pi^{-\frac{d}{2}} k^4 \sum_{n=0}^{\infty} \frac{m^{-2n}\Gamma\left(2 + n - \frac{d}{2}\right)}{2^n\,(2n+5)!!} k^{2n} \\ &\times \left(\pi^{\mu\nu}\pi^{\mu\nu} + \frac{a(n,\xi)}{2} \pi^{\mu\mu}\pi^{\nu\nu} \right) \end{aligned} \tag{8.48}$$

where $a(n, \xi)$ is a constant

$$a(n,\xi) = (2n+5)(2n+3)(4\xi-1)^2 + 2(2n+5)(4\xi-1) + 1 \tag{8.49}$$

The non-transverse part of the bubble diagram is

$$\begin{aligned}\tilde{T}^{\mu\mu\nu\nu}_{nt}(k) &= \frac{2^{-4-d}}{3}\, i\,\pi^{-d/2} m^{d-2}\Gamma\left(-\frac{d}{2}\right) \\ &\left(\eta^{\mu\nu}\eta^{\mu\nu}\left(24m^2-2dk^2\right)+4\,d\,\eta^{\mu\nu}k^\mu k^\nu+2d\,(6\xi-1)\eta^{\mu\mu}k^\nu k^\nu\right. \\ &\left.+\eta^{\nu\nu}\left(\eta^{\mu\mu}\left(dk^2(5-24\xi)+12m^2\right)+2d(6\xi-1)k^\mu k^\mu\right)\right) \qquad (8.50)\end{aligned}$$

The seagull diagram and the non-transverse part of 2-pt function together give

$$\begin{aligned}\tilde{T}^{\mu\mu\nu\nu}_{(s)}(k)+\tilde{T}^{\mu\mu\nu\nu}_{nt}(k) &= -2^{-d-2}i\pi^{-d/2}m^d\Gamma\left(-\frac{d}{2}\right)\left(2\eta^{\mu\nu}\eta^{\mu\nu}-\eta^{\mu\mu}\eta^{\nu\nu}\right) \qquad (8.51)\\ &+2^{-d-1}i\pi^{-d/2}m^{d-2}\left(\xi-\frac{1}{6}\right)\Gamma\left(1-\frac{d}{2}\right)k^2\left(\pi^{\mu\nu}\pi^{\mu\nu}-\pi^{\mu\mu}\pi^{\nu\nu}\right)\end{aligned}$$

Taking formulas (8.45), (8.46), (8.47) and (8.50) and substituting them in (8.44) we can see that the Ward identity is satisfied for any dimension d.

The one-loop 1-point correlator of the energy-momentum tensor

$$\begin{aligned}\langle\!\langle T^{\mu\mu}(x)\rangle\!\rangle &= -2^{-d}\,m^d\pi^{-\frac{d}{2}}\left[\Gamma\left(-\frac{d}{2}\right)g^{\mu\mu}-\frac{2\Gamma\left(1-\frac{d}{2}\right)}{m^2}\left(\xi-\frac{1}{6}\right)G^{\mu\mu}\right. \\ &+\sum_{n=2}^{\infty}\frac{(-1)^n\,m^{-2n}\Gamma\left(n-\frac{d}{2}\right)}{2^n(2n+1)!!}\Box^{n-2} \\ &\left.\times\left(-2\Box G^{\mu\mu}+\left(1-\frac{a(n,\xi)}{2}\right)\left(\eta^{\mu\mu}\Box-\partial^\mu\partial^\mu\right)R\right)\right]+O(h^2) \quad (8.52)\end{aligned}$$

is covariantly conserved. For the effective action in the IR we obtain

$$\begin{aligned}W[h] &\overset{\text{IR}}{=} 2^{-d}m^d\pi^{-\frac{d}{2}}\int d^dx\sqrt{g}\left[\Gamma\left(-\frac{d}{2}\right)-\frac{\Gamma\left(1-\frac{d}{2}\right)}{2m^2}\left(\xi-\frac{1}{6}\right)R\right. \\ &\left.+\frac{\Gamma\left(2-\frac{d}{2}\right)}{120m^4}\left(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}+\frac{a(0,\xi)}{2}R^2\right)+\ldots\right]+O(h^3) \qquad (8.53)\end{aligned}$$

For $\xi=\frac{1}{6}$ (the conformal case) the third term in the expansion is proportional to

$$W[h] \propto m^{d-4}\int d^dx\sqrt{g}\left(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}-\frac{1}{3}R^2\right) \qquad (8.54)$$

We can use the Gauss-Bonnet theorem

$$R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}-4R_{\mu\nu}R^{\mu\nu}+R^2=\text{total derivative} \qquad (8.55)$$

to write the divergent part of the effective action in $d = 4$ as a Weyl square density

$$W[h] \overset{\text{IR}}{=} -\frac{1}{16\pi^2\varepsilon}\int d^4x\sqrt{g}\left(m^4 + \frac{1}{30}\mathcal{W}^2\right) + O(h^3) \tag{8.56}$$

In the massless case (m^0 is the dominating term in the UV) we have

$$\tilde{T}^{\mu\mu\nu\nu}(k) \overset{\text{UV}}{=} \frac{2^{-1-2d}\,\pi^{\frac{3}{2}-\frac{d}{2}}(k^2)^{\frac{d}{2}}}{(-1+e^{i\pi d})\,\Gamma\left(\frac{d+3}{2}\right)}\left(\pi^{\mu\nu}\pi^{\mu\nu} + \frac{b(d,\xi)}{2}\pi^{\mu\mu}\pi^{\nu\nu}\right) \tag{8.57}$$

where

$$b(d,\xi) = (d^2-1)(4\xi-1)^2 + 2(d+1)(4\xi-1) + 1 \tag{8.58}$$

The effective action in the UV now becomes

$$W[h] \overset{UV}{=} (-1)^{\frac{d}{2}}\frac{2^{-2-2d+\lfloor\frac{d}{2}\rfloor}\pi^{\frac{3}{2}-\frac{d}{2}}}{(-1+e^{i\pi d})\Gamma\left(\frac{d+3}{2}\right)}\int \mathrm{d}^dx\,\left(R^{\mu\nu\lambda\rho}\Box^{\frac{d}{2}-2}R_{\mu\nu\lambda\rho} + \frac{b(d,\xi)}{2}R\Box^{\frac{d}{2}-2}R\right)$$

After we use (8.55) and put $\xi = \frac{1}{6}$ in 4d we will again get the Weyl square density

$$W[h] \overset{UV}{=} \int \mathrm{d}^dx\,\mathcal{W}^2 \tag{8.59}$$

8.4.1 3d msm: spin 2 tomography

Just as for spin-1, we showed that for spin-2 in the scalar model Ward identities are satisfied. However, we will also show what happens if we knew the interaction only up to the linear order. We demonstrate our scheme to extract physical information from the two-point function by expanding it in the IR and UV and subtracting the divergent and nonconserved terms from the effective action. We consider the 2-point correlator with the currents (7.22). The result is given as a sum of (8.47) and (8.50) with $\xi = \frac{1}{4}$. Expanding in the IR we find that all the even powers vanish. Moreover, the $\mathcal{O}(m^3)$ and $\mathcal{O}(m)$ terms are non-conserved, while the other terms are all conserved and proportional to the same structure $\Pi^{(2)}_{\frac{1}{2},\mu^2\nu^2}(k)$.

In the UV, we have $\mathcal{O}(m^{2m}) = 0$ for $m \geq 3$. The only nonvanishing terms with even powers of m are $\mathcal{O}(m^0), \mathcal{O}(m^2), \mathcal{O}(m^4)$ (again, about these terms, see the comment below). All the terms are conserved except $\mathcal{O}(m)$ and $\mathcal{O}(m^3)$. But putting together the analogous non-conserved terms in the UV and IR (that is, subtracting the local IR terms from the (nonlocal) UV ones) we recover conservation.

$$\mathcal{O}_{UV}(m) - \mathcal{O}_{IR}(m) = \frac{imk^2}{3\pi}\Pi^{(2)}_{\frac{1}{2}\,\mu^2\nu^2}(k) \tag{8.60}$$

$$\mathcal{O}_{UV}(m^3) - \mathcal{O}_{IR}(m^3) = -\frac{4im^3}{3\pi}\Pi^{(2)}_{\frac{1}{2}\,\mu^2\nu^2}(k) \tag{8.61}$$

Up to local terms, the effective action is a sum of infinite many terms, all proportional to the same conserved structure (8.61) with coefficients proportional to various monomials of m and k. They form a convergent series both in the IR and in the UV. In compact form:

$$\frac{i}{48\pi k}\left(48m^4\coth^{-1}\left(\frac{2m}{k}\right) + 2km\left(5k^2-12m^2\right) - 24k^2m^2\coth^{-1}\left(\frac{2m}{k}\right)\right.$$
$$\left. +3k^4\coth^{-1}\left(\frac{2m}{k}\right)\right)\Pi^{(2)}_{\frac{1}{2},\mu^2\nu^2}(k) \tag{8.62}$$

It should be noticed that the massless model case gives the result:

$$\tilde{T}(k)_{\mu\mu\nu\nu} = -\frac{k^3}{32}\Pi^{(2)}_{\frac{1}{2}\,\mu^2\nu^2}(k) \tag{8.63}$$

This is conserved *but not traceless*, which is not surprising because a scalar massless model in $d \geq 3$ is not conformally invariant in this case.

Eq.(8.60) is conserved. It does not coincide with the linearized Einstein-Hilbert action (in particular it is nonlocal), but this is simply a nonlocal version of the same, in the same sense as we have already seen for spin 3 and higher in section 7.5.

8.4.2 4d msm: spin 2 tomography

Let us repeat the above procedure for $d = 4$. We again consider the 2-point correlator with the currents (7.22). The result is given as a sum of (8.47) and (8.50) with $\xi = \frac{1}{4}$ and $d = 4$. In the IR the odd powers of m vanish. The terms $\mathcal{O}(m^4)$ and $\mathcal{O}(m^2)$ are not conserved, the logarithmic term is conserved but divergent in the IR, the m^0 term is divergent in the limit $\varepsilon \to 0$. They all must be subtracted. The remaining terms are conserved and proportional to $\Pi^{(2)}_{\frac{1}{2},\mu^2\nu^2}(k)$.

In the UV all the odd powers of m vanish. Term $\mathcal{O}(m^0)$ and all terms with even m

power larger than 4 are conserved, while $\mathcal{O}(m^2)$ and $\mathcal{O}(m^4)$ are not. According to our prescription we have to subtract not only $\mathcal{O}_{IR}(m^2)$ and $\mathcal{O}_{IR}(m^4)$, but also $\mathcal{O}_{IR}(m^0)$ and $\mathcal{O}_{IR}(\log(m))$. We obtain

$$\mathcal{O}_{UV}(m^4) - \mathcal{O}_{IR}(m^4) = -\frac{im^4}{8\pi^2}\left(2\log\left(-\frac{k^2}{m^2}\right) - 1\right)\Pi^{(2)}_{\frac{1}{2},\mu^2\nu^2}(k) \tag{8.64}$$

$$\mathcal{O}_{UV}(m^2) - \mathcal{O}_{IR}(m^2) = \frac{im^2}{36\pi^2}k^2\left(3\log\left(-\frac{k^2}{m^2}\right) - 5\right)\Pi^{(2)}_{\frac{1}{2},\mu^2\nu^2}(k) \tag{8.65}$$

and

$$\mathcal{O}_{UV}(m^0) - \mathcal{O}_{IR}(m^0) - \mathcal{O}_{IR}(\log(m)) = \frac{i}{1800\pi^2}k^4\left(-15\log\left(-\frac{k^2}{m^2}\right) + 46\right)\Pi^{(2)}_{\frac{1}{2},\mu^2\nu^2}(k) \tag{8.66}$$

They are all conserved. (8.65) contains a nonlocal linearized version of the EH eom.

8.5 Spin 3

In this case we do not know the full covariant theory and we must satisfy ourselves with only linear coupling of spin-3 to the current (7.22). Two-point function is

$$\begin{aligned}
\tilde{T}_{\mu^3\nu^3}(k) = & -\frac{1}{35}i2^{-d}\pi^{-d/2}k^6 m^{d-4}\Gamma\left(2-\frac{d}{2}\right)\,{}_2F_1\left(1, 2-\frac{d}{2}; \frac{9}{2}; \frac{k^2}{4m^2}\right) \\
& \times\pi_{\mu\nu}\left(2\pi_{\mu\nu}^2 + 3\pi_{\mu\mu}\pi_{\nu\nu}\right) \\
& -3i2^{3-d}\pi^{-d/2}m^{d+2}\Gamma\left(-\frac{d}{2}-1\right)\eta_{\mu\nu}\left(2{\eta_{\mu\nu}}^2 + 3\eta_{\mu\mu}\eta_{\nu\nu}\right) \\
& +i2^{2-d}\pi^{-d/2}m^d\Gamma\left(-\frac{d}{2}\right)\left(3\eta_{\mu\mu}\eta_{\mu\nu}{k_\nu}^2 + 3\left(2{\eta_{\mu\nu}}^2 + \eta_{\mu\mu}\eta_{\nu\nu}\right)k_\mu k_\nu\right. \\
& \left.+\,\eta_{\mu\nu}\left(3\eta_{\nu\nu}{k_\mu}^2 - k^2\left(2{\eta_{\mu\nu}}^2 + 3\eta_{\mu\mu}\eta_{\nu\nu}\right)\right)\right) \\
& -\frac{1}{5}i2^{1-d}\pi^{-d/2}m^{d-2}\Gamma\left(1-\frac{d}{2}\right)\left(3\eta_{\nu\nu}k_\nu{k_\mu}^3 + 3\eta_{\mu\nu}\left(3{k_\nu}^2 - k^2\eta_{\nu\nu}\right){k_\mu}^2\right. \\
& +3k_\nu\left(\eta_{\mu\mu}{k_\nu}^2 - k^2\left(2{\eta_{\mu\nu}}^2 + \eta_{\mu\mu}\eta_{\nu\nu}\right)\right)k_\mu \\
& \left.+k^2\eta_{\mu\nu}\left(k^2\left(2{\eta_{\mu\nu}}^2 + 3\eta_{\mu\mu}\eta_{\nu\nu}\right) - 3\eta_{\mu\mu}{k_\nu}^2\right)\right)
\end{aligned} \tag{8.67}$$

Let us now demonstrate how to draw out information from the two-point function by expanding it in the IR and UV and subtracting the divergent and nonconserved terms from the effective action.

8.5.1 3d msm: spin 3 tomography

For the 3-spin current, in the IR, the coefficients of even powers in m vanish, while the negative odd powers are all proportional to the conserved structure $\Pi^{(3)}_{\frac{3}{2},\mu^3\nu^3}(k)$. The terms $\mathcal{O}(m^5), \mathcal{O}(m^3), \mathcal{O}(m)$ are local and non-conserved.

In the UV, the terms $\mathcal{O}(m^{2n})$ with $n \geq 4$ vanish. All terms are conserved, except $\mathcal{O}(m), \mathcal{O}(m^3), \mathcal{O}(m^5)$. Proceeding as above we subtract from the non-conserved terms in the UV the homogeneous local non-conserved terms in the IR and obtain conserved terms:

$$\mathcal{O}_{UV}(m) - \mathcal{O}_{IR}(m) = -\frac{imk^4}{5\pi}\Pi^{(3)}_{\frac{3}{2},\mu^3\nu^3}(k) \tag{8.68}$$

$$\mathcal{O}_{UV}(m^3) - \mathcal{O}_{IR}(m^3) = \frac{4im^3}{3\pi}k^2\Pi^{(3)}_{\frac{3}{2},\mu^3\nu^3}(k) \tag{8.69}$$

$$\mathcal{O}_{UV}(m^5) - \mathcal{O}_{IR}(m^5) = -\frac{16im^5}{5\pi}\Pi^{(3)}_{\frac{3}{2},\mu^3\nu^3}(k) \tag{8.70}$$

In compact form, after subtractions, the 2-pt correlator is:

$$\frac{i}{480\pi k}\left(960m^6\coth^{-1}\left(\frac{2m}{k}\right) - 480km^5 - 720k^2m^4\coth^{-1}\left(\frac{2m}{k}\right) + 320\left(k^2\right)^{3/2}m^3 \right.$$
$$\left. +180k^4m^2\coth^{-1}\left(\frac{2m}{k}\right) - 66k^4km - 15k^6\coth^{-1}\left(\frac{2m}{k}\right)\right)\Pi^{(3)}_{\frac{3}{2},\mu^3\nu^3}(k) \tag{8.71}$$

The term (8.69) gives rise to an equation of motion, which is the nonlocal version of the Fronsdal spin 3 equation of motion.

8.5.2 4d msm: spin 3 tomography

The scheme is the same as above. In the IR the odd powers of m vanish. The even powers m^{2n} with $n \leq 0$ are conserved together with the term proportional to $\log(m)$. The terms $\mathcal{O}_{IR}(m^2), \mathcal{O}_{IR}(m^6), \mathcal{O}_{IR}(m^6)$ are not conserved. Of course $\mathcal{O}(\log(m))$ diverges in the IR, while the term $\mathcal{O}_{IR}(m^0)$ diverges for $\varepsilon \to 0$. According to our prescription all these terms, which are local, have to be subtracted from the effective action.

In the UV the odd m power terms vanish. The even powers of order $2, 4, 6$ are not conserved, but

$$\begin{aligned}\mathcal{O}_{UV}(m^0) - \mathcal{O}_{IR}(m^0) &- \mathcal{O}_{IR}(\log(m)) \\ &= -\frac{ik^6}{29400\pi^2}\left(-105\log\left(-\frac{k^2}{m^2}\right)+352\right)\Pi^{(3)}_{\frac{3}{2},\mu^3\nu^3}(k)\end{aligned} \tag{8.72}$$

and

$$\mathcal{O}_{UV}(m^2) - \mathcal{O}_{IR}(m^2) = \frac{im^2k^4}{300\pi^2}\Pi^{(3)}_{\frac{3}{2},\mu^3\nu^3}(k)\left((31-15\log\left(-\frac{k^2}{m^2}\right)\right) \tag{8.73}$$

$$\mathcal{O}_{UV}(m^4) - \mathcal{O}_{IR}(m^4) = -\frac{im^4k^2}{24\pi^2}\Pi^{(3)}_{\frac{3}{2},\mu^3\nu^3}(k)\left((7-6\log\left(\frac{k^2}{m^2}\right)\right) \tag{8.74}$$

$$\mathcal{O}_{UV}(m^6) - \mathcal{O}_{IR}(m^6) = \frac{im^6}{12\pi^2}\Pi^{(3)}_{\frac{3}{2},\mu^3\nu^3}(k)\left((1-6\log\left(-\frac{k^2}{m^2}\right)\right) \tag{8.75}$$

$$\ldots \qquad \ldots$$

are all conserved. Eq.(8.74) is related to a nonlocal version of the spin 3 Fronsdal equation.

8.6 msm: higher spin currents

This scheme repeats itself for higher spin currents. For spin 4 there are 4 non-conserved terms in the IR and 4 in the UV, while the others are conserved or 0. Subtracting the IR non-conserved terms from the corresponding UV ones all the non-vanishing terms turn out to be proportional to the conserved structure:

$$\frac{1}{3}\pi^4_{\mu\nu} + \frac{1}{8}\pi^2_{\mu\mu}\pi^2_{\nu\nu} + \pi_{\mu\mu}\pi^2_{\mu\nu}\pi_{\nu\nu} \tag{8.76}$$

For example, in 3d all terms with even powers of m vanish, except m^0, m^2, m^4, m^6, m^8.

For spin 5 there are 5 non-conserved terms in the IR and 5 in the UV, while the others are conserved or 0. Subtracting the IR non-conserved terms from the corresponding UV ones all the nonvanishing terms turn out to be proportional to the conserved structure:

$$\pi^5_{\mu\nu} + \frac{15}{8}\pi^2_{\mu\mu}\pi_{\mu\nu}\pi^2_{\nu\nu} + 5\pi_{\mu\mu}\pi^3_{\mu\nu}\pi_{\nu\nu} \tag{8.77}$$

For example, in 3d all terms with even powers of m vanish, except $m^0, m^2, m^4, m^6, m^8, m^{10}$.

Comment 1. As we have seen above, any conserved structure is connected to a (non-local) higher spin field equation of motion. In particular eqs.(8.23) and (8.60) are conserved structures which represent the linearized Maxwell and Einstein-Hilbert actions, respectively, the second one in a nonlocal version. Eq.(8.69) is non-local and gives rise to a variant of the non-local Fronsdal equation discussed in sec.7.5. It is clear that any two-point correlator structure can be uniquely related to a given (linearized) equation of motion. The structure of the 2pt-functions conform to the general discussion in sec.7.2.

It is remarkable that the conserved structures that appear in the above expansions are always the same for any fixed 2pt correlator. As we will see this is not the case for the effective field action originating from a fermion model.

Comment 2.

It is interesting to compare the $\mathcal{O}(m^0)$ results with the massless model case, obtained via (7.72). In particular, in the massless case for spin 1 in 3d we get

$$-\frac{1}{16}k\pi_{\mu\nu} \tag{8.78}$$

for spin 2

$$\frac{k^3}{32}\Pi^{(2)}_{\frac{1}{2},\mu^2\nu^2}(k) \tag{8.79}$$

and for spin 3

$$-\frac{k^5}{64}\Pi^{(3)}_{\frac{3}{2},\mu^3\nu^3}(k) \tag{8.80}$$

These correlators are non-local and coincide with the $\mathcal{O}_{UV}(m^0)$ terms evaluated above. Similarly, all other $O_{UV}(m^0)$ terms coincide with the expressions obtained in the massless limit in section 8.1 for simple currents for appropriate spin s and dimension d.

8.6.1 Scalar model - simple currents - general

In the scalar model it is particularly simple to find a general expression for the conserved 2-point correlators. Omitting the non-conserved terms, we find: General expression for spin $s_1 \times s_2$, $s_2 > s_1$

$$\begin{aligned}\tilde{T}_{\mu_1\ldots\mu_{s_1}\nu_1\ldots\nu_{s_2}} &= (-1)^{\frac{s_1+s_2}{2}}\, i\, \frac{2^{\lfloor\frac{s_1}{2}\rfloor-d} m^{d-4}\pi^{-\frac{d}{2}} s_1!(s_2-s_1)!!\left(2\lfloor\frac{s_2+1}{2}\rfloor-1\right)!!\left(\lfloor\frac{s_2}{2}\rfloor\right)!}{\left(2\lfloor\frac{s_1+s_2}{2}\rfloor+1\right)!!\left(\lfloor\frac{s_2}{2}\rfloor-\lfloor\frac{s_1}{2}\rfloor\right)!} \\ &\times\Gamma\left(2-\frac{d}{2}\right){}_2F_1\left(1,2-\frac{d}{2},\frac{s_1+s_2+3}{2},\frac{k^2}{4m^2}\right) \\ &\times\pi_{\nu\nu}^{\frac{s_2-s_1}{2}}\sum_{l=0}^{\lfloor\frac{s_1}{2}\rfloor}\frac{1}{2^{\frac{l(l+1)}{2}}(s_1-2l)!(s_2-s_1+2l)!!}\pi_{\mu\mu}^l\pi_{\nu\nu}^l\pi_{\mu\nu}^{s_1-2l} \end{aligned} \tag{8.81}$$

8.7 Diagonalization

We demonstrated in this chapter that different choices of currents lead to different effective actions. In particular, now we wonder whether is possible to make a choice of currents for which the mixed correlators vanish which may simplify our analysis.

8.7.1 "Local" currents

One can start with a general form of spin-s current

$$j^{(s)}_{\mu_1\ldots\mu_s} = i^s\sum_{l=0}^{\lfloor\frac{s}{2}\rfloor} a_l^{(s)}\left(\Box\pi_{\mu\mu}\right)^l\,\phi^\dagger\left(\overleftrightarrow{\partial}_\mu\right)^{s-2l}\phi \tag{8.82}$$

where $a_l^{(s)}$ are some numerical coefficients and we can choose $a_0^{(s)}=1$. Now, is there a choice of coefficients $a_l^{(s)}$ for which the 2-point correlators with mixed scalar currents vanish?

Amplitude 0x2. In fact, we can use a more general current

$$\begin{aligned} j^{(2)}_{\mu^2} &= i^2\left(\phi^\dagger\left(\overleftrightarrow{\partial}_\mu\right)^2\phi + a_1^{(2)}\Box\pi_{\mu\mu}\left(\phi^\dagger\phi\right)\right. \\ &\left.+a_1'^{(2)}\eta_{\mu\mu}(\Box+m^2)\phi^\dagger\phi + b_2^{(2)}\eta_{\mu\mu}\phi^\dagger(\Box+m^2)\phi\right) \end{aligned} \tag{8.83}$$

where we added terms such as $(\Box+m^2)$ (vanishes on-shell). Due to hermiticity of the currents we have $a_2'^{(2)} = a_1'^{(2)}$. The conserved part of 0x2 amplitude is

$$\begin{aligned}\tilde{T}^{\rm t}_{\nu^2} &= \frac{i2^{-d}\pi^{-d/2}m^{d-4}\Gamma\left(2-\frac{d}{2}\right)}{3}k^2\pi_{\nu\nu} \\ &\times\left(\left(-1+a_1^{(2)}\right){}_2F_1\left(1,2-\frac{d}{2};\frac{5}{2};\frac{k^2}{4m^2}\right)+2a_1^{(2)}\,{}_2F_1\left(2,2-\frac{d}{2};\frac{5}{2};\frac{k^2}{4m^2}\right)\right)\end{aligned} \tag{8.84}$$

The conserved part of the correlator with mixed spins 0x2 vanishes for

$$a_1^{(2)} = \frac{{}_2F_1\left(1, 2-\frac{d}{2}; \frac{5}{2}; \frac{k^2}{4m^2}\right)}{{}_2F_1\left(1, 2-\frac{d}{2}; \frac{5}{2}; \frac{k^2}{4m^2}\right) + 2\,{}_2F_1\left(2, 2-\frac{d}{2}; \frac{5}{2}; \frac{k^2}{4m^2}\right)} \tag{8.85}$$

Values of $a_1^{(2)}$ in specific dimensions:

$$\begin{array}{ll} d=3 & \dfrac{m\left(\frac{k}{\coth^{-1}\left(\frac{2m}{k}\right)} - 2m\right)}{k^2} + \dfrac{1}{2} \\ d=4+\epsilon & \dfrac{1}{3} + \epsilon\left(\dfrac{4m^2\left(k - m\sqrt{4-\frac{k^2}{m^2}}\csc^{-1}\left(\frac{2m}{k}\right)\right)}{3k^3} - \dfrac{1}{9}\right) + \ldots \end{array}$$

The coefficient $a_1^{(2)}$ is a function of momenta and mass. Since this coefficient enters the definition of the current, it defines the coupling to the source. If we write the coefficient $a_1^{(2)}$ in powers of the momentum k^2, we get an interaction with infinite number of higher derivative terms.

The non-conserved (non-transverse) part

$$T^{\text{nt}}_{\nu_2} = -i2^{1-d}(1+a_1'^{(2)})\pi^{-d/2}m^{d-2}\Gamma\left(1-\frac{d}{2}\right)\eta_{\nu\nu} \tag{8.86}$$

vanishes for $a_1'^{(2)} = -1$. Terms such as $(\Box + m^2)$ in the current contribute only to the non-conserved part and behave as counterterms.

Amplitude 1x3 Again we can add to the spin-3 current terms such as $(\Box + m^2)$ which vanish on-shell

$$\begin{aligned} j^{(3)}_{\mu^3} = \; & i^3\left(\phi^\dagger\left(\overleftrightarrow{\partial}_\mu\right)^3\phi + a_1^{(3)}\Box\pi_{\mu\mu}\left(\phi^\dagger\left(\overleftrightarrow{\partial}_\mu\right)\phi\right) + a_1'^{(3)}\eta_{\mu\mu}(\Box+m^2)\partial_\mu\phi^\dagger\phi\right. \\ & \left.+a_2'^{(3)}\eta_{\mu\mu}(\Box+m^2)\phi^\dagger\partial_\mu\phi + a_3'^{(3)}\eta_{\mu\mu}\phi^\dagger(\Box+m^2)\partial_\mu\phi + a_4'^{(3)}\eta_{\mu\mu}\partial_\mu\phi^\dagger(\Box+m^2)\phi\right) \end{aligned} \tag{8.87}$$

Due to hermiticity of the currents we have $a_3'^{(3)} = -a_1'^{(3)}$ and $a_4'^{(3)} = -a_2'^{(3)}$. The conserved part of 1x3 amplitude is

$$\begin{aligned}\tilde{T}^{\mathrm{t}}_{\mu\nu^3} &= -\frac{1}{15}i2^{-d}\pi^{-d/2}m^{d-4}\Gamma\left(2-\frac{d}{2}\right)k^4\pi_{\mu\nu}\pi_{\nu\nu} \\ &\times\left(3(a_1^{(3)}-1)\,{}_2F_1\left(1,2-\frac{d}{2};\frac{7}{2};\frac{k^2}{4m^2}\right)+2a_1^{(3)}\,{}_2F_1\left(2,2-\frac{d}{2};\frac{7}{2};\frac{k^2}{4m^2}\right)\right)\end{aligned} \tag{8.88}$$

The conserved part of the correlator with mixed spins 1x3 vanishes for

$$a_1^{(3)} = \frac{3\,{}_2F_1\left(1,2-\frac{d}{2};\frac{7}{2};\frac{k^2}{4m^2}\right)}{3\,{}_2F_1\left(1,2-\frac{d}{2};\frac{7}{2};\frac{k^2}{4m^2}\right)+2\,{}_2F_1\left(2,2-\frac{d}{2};\frac{7}{2};\frac{k^2}{4m^2}\right)} \tag{8.89}$$

The non-conserved part is

$$\begin{aligned}\tilde{T}^{\mathrm{nt}}_{\mu\nu^3} &= -i2^{1-d}\pi^{-d/2}m^{d}\Gamma\left(-\frac{d}{2}\right)\eta_{\mu\nu}\eta_{\nu\nu}(a_1'^{(3)}-a_2'^{(3)}-6) \\ &+i2^{1-d}\pi^{-d/2}m^{d-2}\Gamma\left(1-\frac{d}{2}\right)\left((1-a_1^{(3)})\eta_{\mu\nu}k^2\pi_{\nu\nu}+(a_1'^{(3)}-1)k_\mu k_\nu\eta_{\nu\nu}\right)\end{aligned} \tag{8.90}$$

The m^d term vanishes for $a_2'^{(3)} = a_1'^{(3)} - 6$ and we can choose $a_1'^{(3)} = 1$ to cancel the second term in m^{d-2} term. Then we have $a_2'^{(3)} = -5$. However, note that m^{d-2} non-conserved term depends on $a_1^{(3)}$. This coefficient, once expanded in powers of momenta, brings infinite number of non-conserved terms. The number of counterterms which cancel non-conserved terms should be finite, and we conclude that it is not possible to diagonalize the 2-pt correlators within this simple model. A similar conclusion follows for all other higher spin off-diagonal correlators.

In the massless limit all non-diagonal terms vanish for (7.27), that is for the choice of coefficient for traceless scalar currents given in (7.27). In this case only the correlators for currents of equal spins are non-vanishing an they are given by (8.2).

8.7.2 Traceless non-local currents

One more idea is to construct currents which are traceless even in the massive case. It is enough to use simple currents to write down a general form of current is now

$$\bar{j}^{\mathrm{s}}_{\mu^s} = \sum_{l=0}^{\lfloor\frac{s}{2}\rfloor} b_l^{(s)}\pi^l_{\mu\mu}j^{\mathrm{s}[l]}_{\mu^{s-2l}} \tag{8.91}$$

where $j^{\rm s}_{\mu^s}$ is a simple scalar current (7.22) and its l-th trace reads

$$j^{\rm s[l]}_{\mu^{s-2l}} = i^s \frac{s!}{(s-2l)!} \phi^\dagger \left(\overleftrightarrow{\partial}_\mu\right)^{s-2l} \left(\overleftrightarrow{\partial}\right)^{2l} \phi \tag{8.92}$$

and $b_l^{(s)}$ are numerical coefficients. These currents are nonlocal (we have appearance of terms such as $\frac{\partial^2}{\Box}$ and their powers). If we impose tracelessness of the currents (on-shell) we get a recurrence relation for the coefficients:

$$b_l^{(s)} = -\frac{1}{2l(d-3+2s-2l)} b_{l-1} \tag{8.93}$$

We can choose $b_0^{(s)} = 1$ so that the coefficient $b_l^{(s)}$ reads:

$$b_l^{(s)} = \frac{(-1)^l \left(s-k-1+\frac{d-3}{2}\right)!}{2^{2k} k! \left(s-1+\frac{d-3}{2}\right)!} \tag{8.94}$$

It turns out that the conserved parts of all mixed-spin correlators vanish for this exact choice of coefficients. The conserved part of the amplitude with equal spin currents (8.91) and coefficients (8.94) for general spin s and general dimension is

$$\tilde{T}_{\mu^s\nu^s} = \frac{i(-1)^s 2^{-d} \pi^{-d/2} s! m^{d-4}}{(2s+1)!!} \Gamma\left(2-\frac{d}{2}\right) {}_2F_1\left(1, 2-\frac{d}{2}; s+\frac{3}{2}; \frac{k^2}{4m^2}\right) k^{2s} \sum_{l=0}^{\lfloor \frac{s}{2} \rfloor} a_l \pi^l_{\mu\mu} \pi^l_{\nu\nu} \pi^{s-2l}_{\mu\nu}$$

where the coefficient a_l is

$$a_l = \frac{(-1)^l s! \, \Gamma\left(s+\frac{d-3}{2}-l\right)}{2^{2l} l! (s-2l)! \, \Gamma\left(s+\frac{d-3}{2}\right)} \tag{8.95}$$

corresponds to the coefficient for the traceless amplitude (7.42). This amplitude is more compactly written as

$$\begin{aligned} T^{\rm t}_{\mu^s\nu^s} &= \frac{i(-1)^s 2^{-d} \pi^{-d/2} s! m^{d-4}}{(2s+1)!!} \Gamma\left(2-\frac{d}{2}\right) {}_2F_1\left(1, 2-\frac{d}{2}; s+\frac{3}{2}; \frac{k^2}{4m^2}\right) \\ &\quad \times k^{2s} \pi^s_{\mu\nu} \, {}_2F_1\left(\frac{1-s}{2}, -\frac{s}{2}; \frac{1}{2}(-d-2s+5); \frac{\pi_{\mu\mu}\pi_{\nu\nu}}{\pi^2_{\mu\nu}}\right) \end{aligned} \tag{8.96}$$

However, we are still left with the non-conserved part.

Amplitude 0x2. The non-conserved part of the amplitude is

$$\tilde{T}^{\text{nt}}_{\nu^2} = \frac{i2^{-d}d\pi^{-d/2}m^{d-2}\Gamma\left(-\frac{d}{2}\right)}{k^2}\left(k^2(2b_1^{(2)}d+1)\eta_{\nu\nu} - 2b_1^{(2)}dk_\nu^2\right) \tag{8.97}$$

Notice that it is non-local, and hence it cannot be canceled by a counterterm. Similarly, all non-conserved parts of higher mixed-spin correlators are non-local and cannot be canceled. The nonconserved parts of the correlators with equal spin currents are also non-local and cannot be canceled.

Notice that there is one way to avoid nonlocality. We can, instead of

$$S_{int} \sim \sum_s \int d^d x j_{\mu_1\ldots\mu_s}\varphi^{\mu_1\ldots\mu_s} \tag{8.98}$$

use a higher derivative coupling

$$S_{int} \sim \sum_s \int d^d x j_{\mu_1\ldots\mu_s}\Box^n\varphi^{\mu_1\ldots\mu_s} \sim \sum_s \int d^d x \Box^n j_{\mu_1\ldots\mu_s}\varphi^{\mu_1\ldots\mu_s} \tag{8.99}$$

To get rid of nonlocality it is enough to put $n = \lfloor\frac{s}{2}\rfloor$. In that case all amplitudes should be multiplied by $(k^2)^{\lfloor\frac{s_1}{2}\rfloor+\lfloor\frac{s_2}{2}\rfloor}$. In that case the nonconserved part becomes local and we can subtract it by a finite number of counterterms.

Chapter 9

Fermion models

In this chapter we consider a fermion theory coupled to spin-s fields via conserved currents. Here we closely follow [29] and [30]. The analysis is similar to the scalar case given in the previous chapter. We will again use the perturbative approach based on Feynman diagrams and dimensional regularization.

First, we consider the massless case for the fermion model and compute the relevant two-point functions for simple and traceless currents in any dimension. For traceless currents the amplitude is again traceless, and the contribution for mixed-spins correlators vanishes. We also compute 1-point functions (tadpoles) for general spin s and general dimension d and we find that the latter vanishes in the massless case.

Next we show results for a fermion theory coupled to spin 1, 2 and 3 fields. In the fermion case, the results for correlators are again given in terms of hypergeometric functions, and because of that, we turn to their IR and UV expansions for $d = 3, 4$ (for expansions in $d = 5, 6$ see [29]). In the UV $\mathcal{O}_{UV}(m^0) - \mathcal{O}_{IR}(m^0) - \mathcal{O}_{IR}(\log(m))$ terms exactly coincide with the massless results.

For spin 1 and 2, just as in the scalar case, we know full form of the interaction and so, beside the bubble diagram we also include seagull and tadpole diagrams. We show that Ward identities are satisfied in this case. For spin 3, instead, we know only the linear coupling and the linear form of gauge transformation. As a consequence we find several violations of Ward identities which come in a form of finite number of local terms. Beside the non-conserved terms, we also find terms that diverge in the IR. These terms are also finite in number and local and they include all the nonconserved ones. Our prescription to extract physical information is such that we subtract all the terms that diverge in the

IR by subtracting a finite number of counterterms from the effective action. In this way we recover both conservation and finiteness in the IR. We demonstrate how this scheme works for both higher spin case and for spin 1 and 2.

We also give an example of mixed spin correlator with spins 3 and 5: the full amplitude and its expansions in UV and IR in $d = 3, 4$ (for expansions in $d = 5, 6$ see [30]). In the odd parity sector, for traceless currents, we find a generalization of the linearized action proposed by Pope and Townsend, [44], for conformal higher spin fields.

The final part of the chapter is devoted to diagonalization of our results, that is, the possibility of vanishing off-diagonal correlators for appropriate choice of coefficients in the currents. It turns out that the diagonalization is not possible with the choice of currents (7.26). One more example we consider is the case of traceless local currents where we are able to diagonalize our results by appropriate choice of coefficients in the currents and by subtraction of finite number of counterterms.

9.1 Massless model

In the massless case for simple currents (7.16) we do not have a general expression. Here are some examples of the amplitudes:

$$\text{Spin } 0\times 0 \;:\; \tilde{T} = \frac{2^{3-2d+\lfloor\frac{d}{2}\rfloor}\pi^{\frac{3}{2}-\frac{d}{2}}\left(k^2\right)^{d/2-2}}{\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d-1}{2}\right)} \tag{9.1}$$

$$\text{Spin } 0\times 2 \;:\; \tilde{T}_{\nu^2} = 0 \tag{9.2}$$

$$\text{Spin } 1\times 1 \;:\; \tilde{T}_{\mu\nu} = -\frac{2^{2-2d+\lfloor\frac{d}{2}\rfloor}\pi^{\frac{3}{2}-\frac{d}{2}}\left(k^2\right)^{d/2-1}}{\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d+1}{2}\right)}(d-2)\pi_{\mu\nu} \tag{9.3}$$

$$\text{Spin } 1\times 3 \;:\; \tilde{T}_{\mu\nu^3} = \frac{2^{1-2d+\lfloor\frac{d}{2}\rfloor}\pi^{\frac{3}{2}-\frac{d}{2}}\left(k^2\right)^{d/2}}{\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d+3}{2}\right)}(d-2)\pi_{\mu\nu}\pi_{\nu\nu} \tag{9.4}$$

$$\text{Spin } 2\times 2 \;:\; \tilde{T}_{\mu^2\nu^2} = \frac{2^{1-2d+\lfloor\frac{d}{2}\rfloor}(d-1)\pi^{\frac{3}{2}-\frac{d}{2}}\left(k^2\right)^{d/2}}{\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d+3}{2}\right)}\left((d-1)\pi_{\mu\nu}^2-\pi_{\mu\mu}\pi_{\nu\nu}\right) \tag{9.5}$$

$$\text{Spin } 2\times 4 \;:\; \tilde{T}_{\mu^2\nu^4} = -\frac{3\cdot 2^{-2d+\lfloor\frac{d}{2}\rfloor}\pi^{\frac{3}{2}-\frac{d}{2}}\left(k^2\right)^{d/2+1}}{\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d+5}{2}\right)}\pi_{\nu\nu}\left((d-1)\pi_{\mu\nu}^2-\pi_{\mu\mu}\pi_{\nu\nu}\right) \tag{9.6}$$

$$\text{Spin } 3\times 3 \;:\; \tilde{T}_{\mu^3\nu^3} = -\frac{2^{-2d+\lfloor\frac{d}{2}\rfloor}\pi^{\frac{3}{2}-\frac{d}{2}}\left(k^2\right)^{d/2+1}}{\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d+5}{2}\right)}\pi_{\mu\nu}\left(2d\pi_{\mu\nu}^2+(d-6)\pi_{\mu\mu}\pi_{\nu\nu}\right) \tag{9.7}$$

$$\begin{aligned}\text{Spin } 3\times 5 \;:\; \tilde{T}_{\mu^3\nu^5} &= \frac{3\cdot 2^{-1-2d+\lfloor\frac{d}{2}\rfloor}\pi^{\frac{3}{2}-\frac{d}{2}}\left(k^2\right)^{d/2+2}}{\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d+7}{2}\right)}\\ &\times\pi_{\mu\nu}\pi_{\nu\nu}\left(4d\pi_{\mu\nu}^2+(d-10)\pi_{\mu\mu}\pi_{\nu\nu}\right)\end{aligned} \tag{9.8}$$

$$\text{Spin } 4\times 4 \quad : \quad \tilde{T}_{\mu^4\nu^4} = \frac{3\cdot 2^{-1-2d+\lfloor \frac{d}{2}\rfloor}\pi^{\frac{3}{2}-\frac{d}{2}}\left(k^2\right)^{d/2+2}}{\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d+7}{2}\right)} \times\left(2(d+1)\pi_{\mu\nu}^4+3(d-3)\pi_{\mu\nu}^2\pi_{\mu\mu}\pi_{\nu\nu}-3\pi_{\mu\mu}^2\pi_{\nu\nu}^2\right) \tag{9.9}$$

$$\text{Spin } 4\times 6 \quad : \quad \tilde{T}_{\mu^4\nu^6} = -\frac{5\cdot 3\cdot 2^{-2-2d+\lfloor \frac{d}{2}\rfloor}\pi^{\frac{3}{2}-\frac{d}{2}}\left(k^2\right)^{d/2+3}}{\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d+9}{2}\right)} \times\pi_{\nu\nu}\left(4(d+1)\pi_{\mu\nu}^4+3(d-5)\pi_{\mu\nu}^2\pi_{\mu\mu}\pi_{\nu\nu}-3\pi_{\mu\mu}^2\pi_{\nu\nu}^2\right) \tag{9.10}$$

Next, we use traceless currents (traceless in the limit $m\to 0$), that is (7.26) with coefficients (7.27). General expression for spin $s\times s$, $s>0$

$$\begin{aligned}\tilde{T}_{\mu_1\ldots\mu_s\nu_1\ldots\nu_s} &= (-1)^s\frac{2^{3-2d-s+\lfloor \frac{d}{2}\rfloor}\pi^{\frac{3}{2}-\frac{d}{2}}(s-1)!(d-3+s)\left(k^2\right)^{d/2+s-2}}{\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d+2s-1}{2}\right)} \\ &\quad \times\sum_{l=0}^{\lfloor \frac{s}{2}\rfloor}(-1)^l a_l\pi_{\mu\mu}^l\pi_{\nu\nu}^l\pi_{\mu\nu}^{s-2l} \\ &= (-1)^s\frac{2^{3-2d-s+\lfloor \frac{d}{2}\rfloor}\pi^{\frac{3}{2}-\frac{d}{2}}(s-1)!(d-3+s)\left(k^2\right)^{d/2+s-2}}{\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d+2s-1}{2}\right)}\pi_{\mu\nu}^s \\ &\quad \times{}_2F_1\left(\frac{1-s}{2},-\frac{s}{2},\frac{5-d-2s}{2},\frac{\pi_{\mu\mu}\pi_{\nu\nu}}{\pi_{\mu\nu}^2}\right)\end{aligned} \tag{9.11}$$

where the coefficient a_l corresponds to (7.42), the coefficient appearing in the traceless amplitude. In this case mixed spin correlators vanish.

9.2 Tadpoles

For convenience let us write down the tadpole diagram contributions for any dimension and any spin. In this chapter we will need only spin 1 and 2 tadpoles. The tadpole contribution actually vanishes for odd spins, as we will shortly see.

Tadpoles with fermion current for any spin and any dimension

$$\tilde{\Theta}^{\mathrm{f}}_{\mu^s} = -\int\frac{d^dp}{(2\pi)^d}\mathrm{tr}\left(V^{\mathrm{f}}_{\mu^s}\frac{i}{\not{p}-m}\right) = 2^{\lfloor \frac{d}{2}\rfloor+s-1}\int\frac{d^dp}{(2\pi)^d}\frac{p_\mu^s}{p^2-m^2} \tag{9.12}$$

Next we use (8.5) together with (8.7) to obtain the tadpole

$$\tilde{\Theta}_{\mu_1\ldots\mu_s} = \begin{cases} i(-1)^{\frac{s}{2}-1}2^{\frac{s}{2}+\lfloor \frac{d}{2}\rfloor-d-1}(s-1)!!\,\pi^{-\frac{d}{2}}m^{d+s-2}\Gamma\left(1-\frac{d}{2}-\frac{s}{2}\right)\eta_{\mu\mu}^{\frac{s}{2}}, & s \text{ even, } s>0 \\ 0, & s \text{ odd}\end{cases} \tag{9.13}$$

and spin 0

$$\tilde{J} = -i2^{\lfloor \frac{d}{2} \rfloor - d}\, \pi^{-\frac{d}{2}} m^{d-1} \Gamma\left(1 - \frac{d}{2}\right) \tag{9.14}$$

9.3 Spin 1

The action for the theory of fermions interacting with gauge field can be written as

$$S = \int \mathrm{d}x \left[\bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi\right] \tag{9.15}$$

where $D_\mu = \partial_\mu - i\,A_\mu$. There is one fermion-fermion-photon vertex

$$V^\mu_{ffp} \quad : \quad i\gamma^\mu \tag{9.16}$$

From the one-loop conservation law (2.30), we get the Ward identity for the two-point function in momentum space

$$k_\mu \tilde{T}^{\mu\nu}(k) = 0 \tag{9.17}$$

9.3.1 Even parity part

In the case of fermions coupled to gauge field the tadpole diagram vanishes, while the seagull is zero because the theory is linear in the gauge field. The only contribution we get from the 2-pt correlator ((11.7) from [29]) which in the momentum space reads

$$\begin{aligned} \tilde{T}^{\mu\nu}(k) &= \frac{2^{-d+\lfloor \frac{d}{2} \rfloor}\, i\, \pi^{-\frac{d}{2}} m^{d-2}}{4m^2 - k^2} \Gamma\left(1 - \frac{d}{2}\right) \\ &\times \left(-4m^2 + {}_2F_1\left[1, 1 - \frac{d}{2}; \frac{3}{2}; \frac{k^2}{4m^2}\right]\left(4m^2 + (d-2)k^2\right)\right)\pi^{\mu\nu} \end{aligned} \tag{9.18}$$

Since the 2-point correlator can be expressed in terms of the projector, it satisfies Ward identity (9.17). We can expand the two-point correlator in the IR region

$$\tilde{T}^{\mu\nu}(k) = -2^{1-d+\lfloor \frac{d}{2} \rfloor} i\, m^{d-2} \pi^{-\frac{d}{2}} \sum_{n=1}^{\infty} \frac{n\, m^{-2n} \Gamma\left(1 + n - \frac{d}{2}\right)}{2^n (2n+1)!!} k^{2n} \pi^{\mu\nu} \tag{9.19}$$

Using the Fourier transform of (9.19) in the one-loop 1-point function (2.12) we get

$$\langle\!\langle j^\mu(x)\rangle\!\rangle = 2^{1-d+\lfloor\frac{d}{2}\rfloor} m^{d-2}\pi^{-\frac{d}{2}}\sum_{n=1}^{\infty}\frac{(-1)^n n\, m^{-2n}\Gamma\left(1+n-\frac{d}{2}\right)}{2^n(2n+1)!!}\Box^{n-1}\partial_\nu F^{\mu\nu} \tag{9.20}$$

The one-loop 1-point correlator satisfies (2.30). Using the same expansion in the IR (9.19) for the effective action (2.11) we obtain

$$\begin{aligned} W &= 2^{-1-d+\lfloor\frac{d}{2}\rfloor} m^{d-2}\pi^{-\frac{d}{2}}\sum_{n=1}^{\infty}\frac{(-1)^n n\, m^{-2n}\Gamma\left(1+n-\frac{d}{2}\right)}{2^n(2n+1)!!}\int \mathrm{d}^d x F_{\mu\nu}\Box^{n-1}F^{\mu\nu} \\ &\overset{\mathrm{IR}}{=} -\frac{2^{-2-d+\lfloor\frac{d}{2}\rfloor}}{3} m^{d-4}\pi^{-\frac{d}{2}}\Gamma\left(2-\frac{d}{2}\right)\int \mathrm{d}^d x F_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{9.21}$$

So, in the IR region (large m) we get the Maxwell action.

Furthermore, the dominating term in the UV ($O(m^0)$) of (9.18) corresponds to the massless case (B.2) from [29]

$$\tilde{T}^{\mu\nu}(k) \overset{UV}{=} -\frac{2^{2-2d+\lfloor\frac{d}{2}\rfloor}\,\pi^{\frac{3}{2}-\frac{d}{2}}(d-2)}{\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d+1}{2}\right)}(k^2)^{\frac{d}{2}-1}\pi^{\mu\nu} \tag{9.22}$$

The effective action in the UV is then

$$W \overset{UV}{=} \frac{(-1)^{\frac{d}{2}}2^{1-2d+\lfloor\frac{d}{2}\rfloor}\,\pi^{\frac{3}{2}-\frac{d}{2}}(d-2)}{\left(-1+e^{i\pi d}\right)\Gamma\left(\frac{d+1}{2}\right)}F^{\mu\nu}\Box^{\frac{d}{2}-2}F_{\mu\nu} \tag{9.23}$$

9.3.2 Odd parity part

For the analysis of the odd parity correlators we will restrict ourselves to $d=3$. The odd part of the two-point correlator is non-vanishing only in $3d$ and it is given by

$$\tilde{T}_o^{\mu\nu}(k) = \frac{m}{2\pi k}\mathrm{ArcCoth}\left(\frac{2m}{k}\right)\epsilon^{\mu\nu\lambda}k_\lambda \tag{9.24}$$

The expansion of (9.24) in the IR reads

$$\tilde{T}_o^{\mu\nu}(k) = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{k^{2n}m^{-2n}}{2^{2(n+1)}(2n+1)}\epsilon^{\mu\nu\lambda}k_\lambda \tag{9.25}$$

Using the IR expansion in (2.12), the odd part of the one-loop 1-point correlator is now

$$\langle\!\langle j^\mu(x)\rangle\!\rangle = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n m^{-2n}}{2^{2n+3}(2n+1)}\epsilon^{\mu\nu\lambda}\Box^n F_{\lambda\nu} \tag{9.26}$$

and just like the even parity part satisfies (9.17). The effective action in the IR (the dominating term)

$$W \stackrel{IR}{=} \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}\int d^3x\, A_\mu\partial_\nu A_\lambda + \dots \tag{9.27}$$

corresponds to Chern-Simons term in 3d

$$S_{CS} = \frac{1}{8\pi}\int d^3x\, Tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \tag{9.28}$$

9.3.3 3d mfm: spin 1 tomography

The case of a gauge field interacting with fermions is a textbook example, but it is perfect to show how to extract the relevant information from the correlators. In 3d the amplitude is

$$\begin{aligned} T_{\mu\nu}(k) &= \frac{i}{8\pi k}\Bigg(-\left(4m^2\coth^{-1}\left(\frac{2m}{k}\right) - 2km + k^2\coth^{-1}\left(\frac{2m}{k}\right)\right)\pi_{\mu\nu} \\ &\quad +4im\coth^{-1}\left(\frac{2m}{k}\right)\epsilon_{\lambda\mu\nu}k^\lambda\Bigg) \end{aligned} \tag{9.29}$$

and is conserved without any subtraction. Expanding, the term

$$\mathcal{O}(m^0) : \frac{1}{4\pi}\epsilon_{\lambda\mu\nu}k^\lambda \tag{9.30}$$

corresponds to the linearized Chern-Simons action, and the term

$$\mathcal{O}(m^{-1}) : -\frac{i}{12\pi m}\,k^2\,\pi_{\mu\nu} \tag{9.31}$$

in the IR corresponds to the Maxwell action.

9.3.4 4d mfm: spin 1 tomography

Here we repeat the same procedure as above in the case of $d = 4$. In even dimensions we must be careful and use $d = 4 + \varepsilon$. Similarly to the scalar case, we demonstrate how our scheme to extract physical information from the amplitude works. Again we expand the amplitude in the IR and UV. In fermion model, spin-1 example is particularly simple because the full amplitude is conserved and consequently there is no need for subtraction of nonconserved terms. However, we will find divergent terms in the IR and subtract them from the effective action.

The m-power expansion in the IR is as follows

$$\begin{aligned}
\mathcal{O}(\log(m)) : &\qquad \frac{i\log(m)}{6\pi^2}k^2\pi_{\mu\nu} && (9.32)\\
\mathcal{O}(m^0) : &\qquad \frac{i}{12\pi^2}(\gamma - \log(4\pi) + \frac{2}{\varepsilon})k^2\pi_{\mu\nu} && (9.33)\\
\mathcal{O}(m^{-2}) : &\qquad -\frac{ik^4}{60\pi^2 m^2}\pi_{\mu\nu} && (9.34)\\
.... &\qquad ....
\end{aligned}$$

All odd powers of m vanish. The above terms are all conserved. The term $\mathcal{O}(\log(m))$ is divergent in the IR and $\mathcal{O}(m^0)$ is divergent in ε.

In the UV all odd powers of m vanish, while

$$\begin{aligned}
\mathcal{O}(m^0) : &\qquad \frac{ik^2}{36\pi^2}\Big(\frac{6}{\varepsilon} - 5 + 3\gamma + 3i\pi - \log\left(16\pi^2\right)\Big)\pi_{\mu\nu} && (9.35)\\
\mathcal{O}(m^2) : &\qquad -\frac{im^2}{2\pi^2}\pi_{\mu\nu} && (9.36)\\
\mathcal{O}(m^4) : &\qquad -i\frac{m^4}{4\pi^2k^2}\Big(2\log\left(-\frac{k^2}{m^2}\right) + 1\Big)\pi_{\mu\nu} && (9.37)\\
.... &\qquad ....
\end{aligned}$$

All the terms are conserved. But, subtracting from them the corresponding local terms in the IR we get

$$\mathcal{O}_{UV}(m^0) - \mathcal{O}_{IR}(m^0) - \mathcal{O}_{IR}(\log(m)) = \frac{i}{36\pi^2}\left(3\log\left(-\frac{k^2}{m^2}\right) - 5\right)k^2\,\pi_{\mu\nu} \qquad (9.38)$$

Clearly (9.38) reproduces the Maxwell action.

9.4 Spin 2

Let us consider the free fermion theory in a generic dimension d

$$S = \int d^dx \sqrt{|g|} \left[i\overline{\psi} E_a^m \gamma^a \left(\partial_m + \frac{1}{2}\Omega_m \right) \psi - m\bar{\psi}\psi \right] \tag{9.39}$$

where E_a^m is the inverse vierbein. From now on we will set $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a small perturbation around flat background. Using the following expansions

$$\begin{aligned} g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu} + (h^2)^{\mu\nu} + \dots, & \sqrt{|g|} &= 1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu} + \dots, \\ e_a^\mu &= \delta_a^\mu - \frac{1}{2}h_a^\mu + \frac{3}{8}(h^2)_a^\mu + \dots, & e_\mu^a &= \delta_\mu^a + \frac{1}{2}h_\mu^a - \frac{1}{8}(h^2)_\mu^a + \dots \end{aligned} \tag{9.40}$$

we can expand the parity even part of the action (9.39) in powers of h:

$$\begin{aligned} S_e = \int d^dx \Big[& \frac{i}{2}\overline{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi - m\bar{\psi}\psi + \frac{1}{2}h\left(\frac{i}{2}\overline{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi - m\bar{\psi}\psi \right) - \frac{i}{4}\overline{\psi} h_a^\mu \gamma^a \overleftrightarrow{\partial}_\mu \psi \\ & + \frac{1}{8}\left(h^2 - 2h^{\alpha\beta}h_{\alpha\beta} \right) \left(\frac{i}{2}\overline{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi - m\bar{\psi}\psi \right) \\ & - \frac{i}{8} h\overline{\psi} h_a^\mu \gamma^a \overleftrightarrow{\partial}_\mu \psi + \frac{3i}{16}\overline{\psi}(h^2)_a^\mu \gamma^a \overleftrightarrow{\partial}_\mu \psi + \dots \Big] \end{aligned} \tag{9.41}$$

There is one fermion-fermion-graviton vertex

$$V_{ffh}^{\mu\mu}(p,p') \; : \; -\frac{i}{4}(p+p')^\mu \gamma^\mu + \frac{i}{4}\eta^{\mu\mu}(\not{p} + \not{p}' - 2m) \tag{9.42}$$

and one vertex with two fermions and two gravitons:

$$\begin{aligned} V_{ffhh}^{\mu\mu\nu\nu}(p,p') \; : \; & \frac{3i}{16}\left((p+p')^\mu \gamma^\nu \eta^{\mu\nu} + (p+p')^\nu \gamma^\mu \eta^{\mu\nu} \right) \\ & + \frac{i}{8}(\not{p} + \not{p}' - 2m)\left(\eta^{\mu\mu}\eta^{\nu\nu} - 2\eta^{\mu\nu}\eta^{\mu\nu} \right) \\ & - \frac{i}{8}\left((p+p')^\mu \gamma^\mu \eta^{\nu\nu} + (p+p')^\nu \gamma^\nu \eta^{\mu\mu} \right) \end{aligned} \tag{9.43}$$

We can also expand the odd parity part of the action in 3d(the latter contains a part proportional to the completely antisymmetric symbol). We restrict ourselves to 3d because only in this case can we get a non-vanishing contribution to the effective action and 1-point correlator.

$$S_o \;=\; \frac{1}{16}\int d^3x\,\epsilon^{abc}\partial_a h_{b\sigma}h_c^{\sigma}\overline{\psi}\psi \tag{9.44}$$

The relevant vertex with two fermions and two gravitons is

$$V_{\epsilon,ffhh}^{\mu\mu\nu\nu} \;:\; \frac{1}{16}\,\eta^{\mu\nu}\epsilon^{\mu\nu\lambda}\,(k-k')_\lambda \tag{9.45}$$

Furthermore, for spin 2, the energy-momentum tensor is defined with $\langle\langle T^{\mu\nu}(x)\rangle\rangle = \frac{2}{\sqrt{g}}\frac{\delta W}{\delta h_{\mu\nu}(x)}$. The full conservation law of the energy-momentum tensor is (2.31). Hence, the Ward identity for one-point function is

$$\partial_\mu \Theta^{\mu\mu}(x) = 0 \tag{9.46}$$

while for two-point correlator we have

$$\begin{aligned}\partial_\mu T^{\mu\mu\nu\nu}(x,y) \;&=\; \frac{1}{2}\eta^{\nu\nu}\delta(x-y)\partial_\mu\Theta^{\mu\mu}(x) + \frac{1}{2}\Theta^{\nu\nu}(x)\partial^\mu\delta(x-y)\\ &\quad -\partial_\mu\left(\delta\left(x-y\right)\Theta^{\mu\nu}\left(x\right)\right)\eta^{\mu\nu}\end{aligned} \tag{9.47}$$

The tadpole contribution is $\tilde{\Theta}^{\mu\mu}(k) = \tilde{\Theta}\,\eta^{\mu\mu}$ where $\tilde{\Theta}$ is a constant. The Ward identity in momentum space is then

$$k_\mu \tilde{T}^{\mu\mu\nu\nu}(k) = \left[-k^\nu\eta^{\mu\nu} + \frac{1}{2}k^\mu\eta^{\nu\nu}\right]\tilde{\Theta} \tag{9.48}$$

9.4.1 Even parity part

The tadpole contribution is now

$$\tilde{\Theta}^{\mu\mu}(k) = -2^{-2-d+\lfloor\frac{d}{2}\rfloor}\,i\,m^d\pi^{\frac{d}{2}}\Gamma\left(-\frac{d}{2}\right)\eta^{\mu\mu} = \tilde{\Theta}\,\eta^{\mu\mu} \tag{9.49}$$

where $\tilde{\Theta}$ is a constant. Since the theory of gravity is non-linear we have a contribution from the seagull term, which can be written as

$$\tilde{T}_{(s)}^{\mu\mu\nu\nu}(k) = 2^{-3-d+\lfloor\frac{d}{2}\rfloor}\,i\,m^d\pi^{\frac{d}{2}}\Gamma\left(-\frac{d}{2}\right)\left(3\eta^{\mu\nu}\eta^{\mu\nu} - 2\eta^{\mu\mu}\eta^{\nu\nu}\right) \tag{9.50}$$

The bubble diagram contributes two parts, the transverse (conserved) part,

$$\begin{aligned}\tilde{T}_t^{\mu\mu\nu\nu}(k) &= -\frac{1}{d(d+1)k^2} 2^{-2-d+\lfloor \frac{d}{2} \rfloor}\, i\, m^d \pi^{\frac{d}{2}} \Gamma\left(1-\frac{d}{2}\right) \\ & \left[\left(-8m^2+(d+1)k^2+{}_2F_1\left[1,-\frac{d}{2},\frac{1}{2},\frac{k^2}{4m^2}\right](8m^2+(d-1)k^2)\right)\pi^{\mu\nu}\pi^{\mu\nu}\right. \\ & \left.+\left(-4m^2+(d+1)k^2+{}_2F_1\left[1,-\frac{d}{2},\frac{1}{2},\frac{k^2}{4m^2}\right](4m^2-k^2)\right)\pi^{\mu\mu}\pi^{\nu\nu}\right] \qquad (9.51)\end{aligned}$$

whose expansion in the IR is

$$\tilde{T}_t^{\mu\mu\nu\nu}(k) = -2^{-3-d+\lfloor \frac{d}{2} \rfloor} i\, m^d \pi^{-\frac{d}{2}} \sum_{n=1}^{\infty} \frac{m^{-2n}\Gamma\left(n-\frac{d}{2}\right)}{2^n(2n+1)!!} k^{2n} \left((2n-1)\pi^{\mu\nu}\pi^{\mu\nu} - \pi^{\mu\mu}\pi^{\nu\nu}\right), \qquad (9.52)$$

and the non-transverse (non-conserved) part

$$\tilde{T}_{nt}^{\mu\mu\nu\nu}(k) = -2^{-3-d+\lfloor \frac{d}{2} \rfloor}\, i\, m^d \pi^{\frac{d}{2}} \Gamma\left(-\frac{d}{2}\right)(\eta^{\mu\nu}\eta^{\mu\nu} - \eta^{\mu\mu}\eta^{\nu\nu}). \qquad (9.53)$$

Taking formulas (9.49), (9.50), (9.51) and (9.53) and substituting them in (9.48) we can see that the Ward identity is satisfied for any dimension d.

The one-loop 1-point function (energy-momentum tensor) now becomes

$$\begin{aligned}\langle\!\langle T^{\mu\mu}(x)\rangle\!\rangle &= -2^{-1-d+\lfloor \frac{d}{2} \rfloor}\, m^d \pi^{-\frac{d}{2}} \left[\Gamma\left(-\frac{d}{2}\right) g^{\mu\mu} + \sum_{n=1}^{\infty} \frac{(-1)^n\, m^{-2n}\Gamma\left(n-\frac{d}{2}\right)}{2^{n+1}(2n+1)!!}\right. \\ &\times \left.\left((2n-1)\Box^{n-1}G^{\mu\mu} + (n-1)\Box^{n-2}(\eta^{\mu\mu}\Box - \partial^\mu\partial^\mu)R\right)\right] + O(h^2) \qquad (9.54)\end{aligned}$$

where $G_{\mu\mu} = R_{\mu\mu} - \frac{1}{2}\eta_{\mu\mu}R$ is the Einstein tensor. The energy-momentum tensor is clearly divergence free. For the effective action in the IR we obtain (in the even parity sector)

$$\begin{aligned}W &\overset{\text{IR}}{=} -2^{-1-d+\lfloor \frac{d}{2} \rfloor} m^d \pi^{-\frac{d}{2}} \int \mathrm{d}^d x \sqrt{g} \times \left[\Gamma\left(-\frac{d}{2}\right) - \frac{\Gamma\left(1-\frac{d}{2}\right)}{24m^2} R\right. \\ & \left. -\frac{\Gamma\left(2-\frac{d}{2}\right)}{80m^4}\left(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right) + \ldots\right] + O(h^3) \qquad (9.55)\end{aligned}$$

The first term is a cosmological constant term and the second is the linearized Einstein-Hilbert action. The third term (m^0 term in $d = 4$) is the Weyl density $\mathcal{W}^2 = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$ (conformal invariant in 4d).

The dominating term in the UV ($O(m^0)$ term corresponds to (B.3) from [29]) of the transverse part $\tilde{T}_t^{\mu\mu\nu\nu}(k)$ is

$$\tilde{T}_t^{\mu\mu\nu\nu}(k) \overset{UV}{=} \frac{2^{-3-2d+\lfloor \frac{d}{2} \rfloor} \pi^{\frac{3}{2}-\frac{d}{2}} (k^2)^{\frac{d}{2}}}{(-1+e^{i\pi d}) \Gamma\left(\frac{d+3}{2}\right)} ((d-1)\pi^{\mu\nu}\pi^{\mu\nu} - \pi^{\mu\mu}\pi^{\nu\nu}) \tag{9.56}$$

The effective action in the UV is then

$$\begin{aligned} W &\overset{UV}{=} (-1)^{\frac{d}{2}} \frac{2^{-4-2d+\lfloor \frac{d}{2} \rfloor} \pi^{\frac{3}{2}-\frac{d}{2}}}{(-1+e^{i\pi d})\Gamma\left(\frac{d+3}{2}\right)} \int \mathrm{d}^d x \sqrt{g} \Big[(d-4) R_{\mu\nu\lambda\rho} \square^{\frac{d}{2}-2} R^{\mu\nu\lambda\rho} \\ &+6 \left(R_{\mu\nu\lambda\rho} \square^{\frac{d}{2}-2} R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}\square^{\frac{d}{2}-2} R^{\mu\nu} + \frac{1}{3} R \square^{\frac{d}{2}-2} R \right) + \dots \Big] + O(h^3) \end{aligned} \tag{9.57}$$

which for $d = 4$ reproduces Weyl density as expected.

9.4.2 Odd parity part

In 3d the contribution from the seagull diagram with vertex (9.45) becomes

$$\tilde{T}_{(s,o)}^{\mu\mu\nu\nu}(k) = -\frac{m^2}{16\pi} \eta^{\mu\nu} \epsilon^{\mu\nu\lambda} k_\lambda \tag{9.58}$$

The odd part of the two-point correlator is non-vanishing only in $3d$ (the vertex is (9.42)). The transverse part can be written as

$$\tilde{T}_{t,o}^{\mu\mu\nu\nu}(k) = -\frac{m}{64\pi k} \left((k^2 - 4m^2) \mathrm{ArcCoth}\left(\frac{2m}{k}\right) + 2mk \right) \pi^{\mu\nu} \epsilon^{\mu\nu\lambda} k_\lambda \tag{9.59}$$

and the expansion of $\tilde{T}_{t,o}^{\mu\mu\nu\nu}(k)$ in the IR is

$$\tilde{T}_{t,o}^{\mu\mu\nu\nu}(k) = -\frac{1}{64\pi} \sum_{n=0}^{\infty} \frac{k^{2(n+1)} m^{-2n}}{4^{2n}(4(n+1)^2 - 1)} \pi^{\mu\nu} \epsilon^{\mu\nu\lambda} k_\lambda \tag{9.60}$$

The odd non-transverse part reads

$$\tilde{T}_{nt,o}^{\mu\mu\nu\nu}(k) = \frac{m^2}{16\pi} \eta^{\mu\nu} \epsilon^{\mu\nu\lambda} k_\lambda \tag{9.61}$$

and can be canceled by the seagull contribution (9.58). So, only the transverse odd part remains. The odd part of the one-loop 1-pt function (energy-momentum tensor)

$$\langle\!\langle T^{\mu\mu}(x) \rangle\!\rangle = \frac{1}{32\pi} \sum_{n=0}^{\infty} \frac{(-1)^n m^{-2n}}{4^{2n}(4(n+1)^2 - 1)} \square^n C^{\mu\mu} \tag{9.62}$$

where $C^{\mu\mu}$ is the linearized Cotton tensor for $d = 3$

$$C_{\mu\nu} = \epsilon_{\mu}{}^{\tau\rho}\partial_{\tau}\left(R_{\rho\nu} - \frac{1}{d-1}g_{\nu\rho}R\right) = \frac{1}{2}\epsilon_{\mu}{}^{\rho\tau}\partial_{\tau}\left(\Box h_{\nu\rho} - \partial_{\lambda}\partial_{\nu}h^{\lambda}_{\rho}\right) + O(h^2) \tag{9.63}$$

The effective action in the IR (the dominating term)

$$W \overset{IR}{=} -\frac{1}{384\pi}\epsilon^{\mu\nu\lambda}\int d^3x\, h_{\nu\nu}\left(\partial_{\lambda}\partial^{\mu}\partial^{\nu}h_{\mu\mu} - \partial_{\lambda}\Box h^{\nu}_{\mu}\right) + O(h^3) \tag{9.64}$$

corresponds to gravitational Chern-Simons term in 3d

$$S_{gCS} = \frac{1}{192\pi}\epsilon^{\mu\nu\lambda}\int d^3x\,\left(\partial_{\mu}\omega_{\nu}{}^{ab}\omega_{\lambda ba} + \frac{2}{3}\omega_{\mu a}{}^{b}\omega_{\nu b}{}^{c}\omega_{\lambda c}{}^{a}\right) \tag{9.65}$$

9.4.3 3d mfm: spin 2 even part tomography

Just as for spin-1, we showed that for spin-2 in the fermion model Ward identities are satisfied. However, similarly to the scalar case, we will also show what happens if we knew the interaction only up to the linear order. We demonstrate our scheme to draw out physical information from the amplitude by expanding it in the IR and UV and subtracting the divergent and nonconserved terms from the effective action. We consider the correlator of two spin 2 currents (7.22). For the spin 2 current, in the IR (all formulas below have to be multiplied by the factor $\frac{1}{16}$ if we use the energy-momentum tensor instead of the current $j_{\mu\mu}$). All even powers of m vanish. The $\mathcal{O}(m^3)$ term is not conserved, while the other terms are all conserved and proportional to *different combinations* of the two conserved structures.

In the UV all terms are conserved except $\mathcal{O}(m^3)$. But putting together the analogous non-conserved term in the UV and IR (that is subtracting the local IR term from the (nonlocal) UV one) we recover conservation. Moreover, according to our general prescription the term $\mathcal{O}(m)$ in the IR is divergent and it should be subtracted. Altogether we have

$$\begin{aligned} \mathcal{O}_{UV}(m) - \mathcal{O}_{IR}(m) &= \frac{imk^2}{6\pi}\Pi^{(2)}_{-1,\mu^2\nu^2}(k) && (9.66)\\ \mathcal{O}_{UV}(m^3) - \mathcal{O}_{IR}(m^3) &= \frac{2im^3}{3\pi}\Pi^{(2)}_{1,\mu^2\nu^2}(k) && (9.67)\end{aligned}$$

Eq. (9.66) is the linearized and *local* version of the EH equation of motion (see sec.7.5).

Once again, up to local terms, the effective action is a sum of infinite many terms, which form a convergent series both in the IR and in the UV, all of them proportional to various combinations of the conserved structures with coefficients proportional to various monomials of m and k.

9.4.4 3d mfm: spin 2 odd part tomography

In the IR (all formulas below have to be multiplied by the factor $\frac{1}{16}$ for the correlator of two energy-momentum tensors) all odd powers of m vanish. The $\mathcal{O}(m^2)$ term is not conserved, while the other terms are all conserved and proportional to the unique odd conserved structure $\epsilon_{\lambda\mu\nu}k^\lambda\pi_{\mu\nu}$.

In the UV the only nonconserved tem is $\mathcal{O}(m^2)$, but

$$\mathcal{O}_{UV}(m^2) - \mathcal{O}_{IR}(m^2) \;\; = \;\; \frac{m^2}{\pi}\epsilon_{\lambda\mu\nu}k^\lambda\pi_{\mu\nu} \tag{9.68}$$

is. In summary, after subtracting $\mathcal{O}_{IR}(m^2)$ the odd 2-pt correlator is:

$$-\frac{m}{4\pi k}\left(4m^2\coth^{-1}\left(\frac{2m}{k}\right) - 2km - k^2\coth^{-1}\left(\frac{2m}{k}\right)\right)\epsilon_{\lambda\mu\nu}k^\lambda\pi_{\mu\nu} \tag{9.69}$$

The term

$$\mathcal{O}(m^0): \qquad -\frac{k^2}{12\pi}\epsilon_{\lambda\mu\nu}k^\lambda\pi_{\mu\nu} \tag{9.70}$$

give rise to the linearized Chern-Simons action as discussed in [28].

9.4.5 4d mfm: spin 2 tomography

Let us repeat the same procedure in $d=4$. In the IR the odd powers of m vanish. The $\mathcal{O}(m^4)$ term is not conserved, while terms m^0, m^2 are conserved but are divergent in the limit $\varepsilon \to 0$.The logarithmic term is conserved but it is divergent in the IR. They all must be subtracted. The remaining terms are conserved.

In the UV all the odd powers of m vanish. All terms with even m power larger than 4, as well as $\mathcal{O}(log(m))$, are conserved, while $\mathcal{O}(m^0), \mathcal{O}(m^2)$ and $\mathcal{O}(m^4)$ are not. According to our prescription we have to subtract not only $\mathcal{O}_{IR}(m^0), \mathcal{O}_{IR}(m^2)$ and $\mathcal{O}_{IR}(m^4)$, but also $\mathcal{O}_{IR}(\log(m))$. We obtain

$$\begin{aligned}\mathcal{O}_{UV}(m^4)-\mathcal{O}_{IR}(m^4) &= \frac{im^4}{8\pi^2}\left(2\log\left(-\frac{k^2}{m^2}\right)-5\right){\pi_{\mu\nu}}^2 \\ &+\frac{im^4}{8\pi^2}\left(2\log\left(-\frac{k^2}{m^2}\right)-1\right)\pi_{\mu\mu}\pi_{\nu\nu}\end{aligned} \tag{9.71}$$

$$\begin{aligned}\mathcal{O}_{UV}(m^2)-\mathcal{O}_{IR}(m^2) &= \frac{im^2k^2}{36\pi^2}\left(3\log\left(-\frac{k^2}{m^2}\right)+1\right){\pi_{\mu\nu}}^2 \\ &-\frac{im^2k^2}{36\pi^2}\left(3\log\left(-\frac{k^2}{m^2}\right)-5\right)\pi_{\mu\mu}\pi_{\nu\nu}\end{aligned} \tag{9.72}$$

$$\begin{aligned}\mathcal{O}_{UV}(m^0)-\mathcal{O}_{IR}(m^0)\,-\,\mathcal{O}_{IR}(\log(m)) &= \frac{i}{1800\pi^2}k^4 \\ \times\left(9\left(-5\log\left(-\frac{k^2}{m^2}\right)+12\right){\pi_{\mu\nu}}^2-\right.&\left.\left(-15\log\left(-\frac{k^2}{m^2}\right)+46\right)\pi_{\mu\mu}\pi_{\nu\nu}\right)\end{aligned} \tag{9.73}$$

They are all conserved. Eq. (9.72) contains a nonlocal linearized version of the Einstein-Hilbert equation of motion.

9.5 Spin 3

For spin-3 fermion current we use (7.22) (instead, in [29], we used traceless current (7.27) in $d=3$). The two-point function reads

$$\begin{aligned}\tilde{T}_{\mu^3\nu^3}(k) &= -\frac{2^{\lfloor\frac{d}{2}\rfloor+2-d}i}{15}\pi^{-d/2}k^4m^{d-2}\Gamma\left(1-\frac{d}{2}\right){}_2F_1\left(2,1-\frac{d}{2};\frac{7}{2};\frac{k^2}{4m^2}\right)\pi^3_{\mu\nu} \\ &-\frac{2^{\lfloor\frac{d}{2}\rfloor-d}i}{3}\pi^{-d/2}k^2m^{d-2}\Gamma\left(1-\frac{d}{2}\right)\pi_{\mu\nu}\pi_{\mu\mu}\pi_{\nu\nu} \\ &\times\left({}_2F_1\left(1,1-\frac{d}{2};\frac{5}{2};\frac{k^2}{4m^2}\right)-2\,{}_2F_1\left(2,1-\frac{d}{2};\frac{5}{2};\frac{k^2}{4m^2}\right)\right) \\ &+i2^{\lfloor\frac{d}{2}\rfloor+4-d}\pi^{-d/2}m^{d+2}\Gamma\left(-\frac{d}{2}-1\right)\eta_{\mu\nu}\left({\eta_{\mu\nu}}^2+2\eta_{\mu\mu}\eta_{\nu\nu}\right) \\ &-i2^{\lfloor\frac{d}{2}\rfloor+4-d}\pi^{-d/2}m^{d}\Gamma\left(-\frac{d}{2}\right)\left(-\eta_{\mu\nu}\eta_{\nu\nu}\pi_{\mu\mu}k^2+\eta_{\mu\mu}\eta_{\mu\nu}{k_\nu}^2+\eta_{\mu\mu}\eta_{\nu\nu}k_\mu k_\nu\right) \\ &+\frac{m}{48\pi k}\left(10k^3m+3\left(k^2-4m^2\right)^2\coth^{-1}\left(\frac{2m}{k}\right)-24km^3\right) \\ &\times\left(2{\pi_{\mu\nu}}^2+\pi_{\mu\mu}\pi_{\nu\nu}\right)\epsilon_{\lambda\mu\nu}k^\lambda\delta_{d,3} \\ &+\frac{4m^4}{3\pi}\left(2{\eta_{\mu\nu}}^2+\eta_{\mu\mu}\eta_{\nu\nu}\right)\epsilon_{\lambda\mu\nu}k^\lambda\delta_{d,3} \\ &+\frac{m^2}{3\pi}\left(-2k^2{\eta_{\mu\nu}}^2-\eta_{\nu\nu}\pi_{\mu\mu}k^2+\eta_{\mu\mu}{k_\nu}^2+4\eta_{\mu\nu}k_\mu k_\nu\right)\epsilon_{\lambda\mu\nu}k^\lambda\delta_{d,3}\end{aligned} \tag{9.74}$$

Expansion of the even conserved part in the IR

$$\tilde{T}_{\mu^3\nu^3}(k) = -\sum_{n=0}^{\infty} \frac{i}{(2n+5)!!} 2^{\lfloor\frac{d}{2}\rfloor+3-d-n} m^{d-4-2n} \pi^{-\frac{d}{2}} \Gamma\left(1+n-\frac{d}{2}\right) k^{2n+4}$$
$$\times \pi_{\mu\nu}\left((n-2)\pi_{\mu\nu}^2 + 2(n+1)\pi_{\mu\mu}\pi_{\nu\nu}\right) \tag{9.75}$$

The leading order is proportional to

$$\tilde{T}_{\mu^3\nu^3}(k) \sim m^{d-4}\Gamma\left(1-\frac{d}{2}\right) k^4 \pi_{\mu\nu}\left(\pi_{\mu\nu}^2 - \pi_{\mu\mu}\pi_{\nu\nu}\right) \tag{9.76}$$

and we get

$$\langle\langle j_{\mu_1\mu_2\mu_3}(x)\rangle\rangle \sim m^{d-4}\Gamma\left(1-\frac{d}{2}\right) \Box\, \mathcal{G}_{\mu_1\mu_2\mu_3}(x) \tag{9.77}$$

where $\tilde{\mathcal{G}}_{\mu_1\ldots\mu_3}$ is the generalized Einstein tensor (7.55).

In what follows we show how to draw out information from the two-point function by expanding it in the IR and UV. We again use the scheme in which we subtract the divergent and nonconserved terms from the effective action.

9.5.1 3d mfm: spin 3 even part tomography

Here the procedure is analogous to the scalar case. In this case one must subtract the local terms $\mathcal{O}(m^5), \mathcal{O}(m^3)$ in the IR, because they are not conserved. Moreover, we also must subtract $\mathcal{O}(m^1)$ because it diverges in the IR.

$$\mathcal{O}_{UV}(m^5) - \mathcal{O}_{IR}(m^5): \quad \frac{32im^5}{15\pi}\,\Pi^{(3)}_{2,\,\mu^3\nu^3}(k) \tag{9.78}$$

$$\mathcal{O}_{UV}(m^3) - \mathcal{O}_{IR}(m^3): \quad -\frac{4im^3k^2}{3\pi}\pi_{\mu\mu}\pi_{\mu\nu}\pi_{\nu\nu} \tag{9.79}$$

$$\mathcal{O}_{UV}(m^1) - \mathcal{O}_{IR}(m^1): \quad -\frac{2imk^4}{15\pi}\,\Pi^{(3)}_{-1,\,\mu^3\nu^3}(k) \tag{9.80}$$

After that the effective action becomes

$$-\frac{i}{480\pi k}\Bigg(960m^6\coth^{-1}\left(\frac{2m}{k}\right)-480km^5-240k^2m^4\coth^{-1}\left(\frac{2m}{k}\right) \tag{9.81}$$

$$+80k^3m^3-60k^4m^2\coth^{-1}\left(\frac{2m}{k}\right)-34k^5m+15k^6\coth^{-1}\left(\frac{2m}{k}\right)\Bigg){\pi_{\mu\nu}}^3$$

$$-\frac{i}{960\pi k}\Bigg(2880m^6\coth^{-1}\left(\frac{2m}{k}\right)-1440km^5-1680k^2m^4\coth^{-1}\left(\frac{2m}{k}\right)+720k^3m^3$$

$$+300k^4m^2\coth^{-1}\left(\frac{2m}{k}\right)-98k^5m-15k^6\coth^{-1}\left(\frac{2m}{k}\right)\Bigg)\pi_{\mu\mu}\pi_{\mu\nu}\pi_{\nu\nu}$$

Eq.(9.79) is related to a nonlocal version of the spin 3 Fronsdal equation.

9.5.2 3d mfm: spin 3 odd part tomography

One must subtract the local terms $\mathcal{O}(m^4), \mathcal{O}(m^2)$ in the IR, which are not conserved.

$$\mathcal{O}_{UV}(m^4)-\mathcal{O}_{IR}(m^4): \qquad -\frac{8m^4}{3\pi}\,\Pi^{(2)}_{\frac{1}{2},\mu^2\nu^2}(k)\,\epsilon_{\lambda\mu\nu}k^\lambda \tag{9.82}$$

$$\mathcal{O}_{UV}(m^2)-\mathcal{O}_{IR}(m^2): \qquad \frac{2m^2k^2}{3\pi}\,\Pi^{(2)}_{\frac{1}{2},\mu^2\nu^2}(k)\,\epsilon_{\lambda\mu\nu}k^\lambda \tag{9.83}$$

After that the effective action becomes:

$$-\frac{m}{24\pi k}\left((48m^4-24m^2k^2+3k^4)\coth^{-1}\left(\frac{2m}{k}\right)-24km^3+10k^3m\right)\epsilon_{\lambda\mu\nu}k^\lambda\,\Pi^{(2)}_{\frac{1}{2},\mu^2\nu^2}(k)$$

9.5.3 4d mfm: spin 3 tomography

The scheme is the same as above. In the IR the odd power of m vanish. The even powers m^{2n} with $n<0$ are conserved together with the term proportional to $\log(m)$. The terms $\mathcal{O}_{IR}(m^0), \mathcal{O}_{IR}(m^2), \mathcal{O}_{IR}(m^6)$ and $\mathcal{O}_{IR}(m^6)$ are not conserved. Of course $\mathcal{O}(\log(m))$ diverges in the IR, while the term $\mathcal{O}_{IR}(m^0)$ diverges for $\varepsilon\to 0$. According to our prescription all these terms, which are local, have to be subtracted from the effective action. In the UV the odd m power terms vanish. The even power of order $2,4,6$ are not conserved, but again

$$\begin{aligned}
\mathcal{O}_{UV}(m^0) - \mathcal{O}_{IR}(m^0) - \mathcal{O}_{IR}(\log(m)) &= -\frac{ik^6}{22050\pi^2}\left(-210\log\left(-\frac{k^2}{m^2}\right)+599\right){\pi_{\mu\nu}}^3 \\
&+ \frac{ik^6}{44100\pi^2}\left(-105\log\left(-\frac{k^2}{m^2}\right)+457\right)\pi_{\mu\nu}\pi_{\mu\mu}\pi_{\nu\nu}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{O}_{UV}(m^2) - \mathcal{O}_{IR}(m^2) &= -\frac{im^2k^4}{225\pi^2}\left(15\log\left(-\frac{k^2}{m^2}\right)-16\right){\pi_{\mu\nu}}^3 \qquad (9.84)\\
&+ \frac{im^2k^4}{450\pi^2}\left(30\log\left(-\frac{k^2}{m^2}\right)-77\right)\pi_{\mu\nu}\pi_{\mu\mu}\pi_{\nu\nu} \\
\mathcal{O}_{UV}(m^4) - \mathcal{O}_{IR}(m^4) &= \frac{im^4k^2}{3\pi^2}{\pi_{\mu\nu}}^3 - \frac{im^4k^2}{12\pi^2}\left(2\log\left(-\frac{k^2}{m^2}\right)-3\right)\pi_{\mu\nu}\pi_{\mu\mu}\pi_{\nu\nu} \qquad (9.85)\\
\mathcal{O}_{UV}(m^6) - \mathcal{O}_{IR}(m^6) &= \frac{im^6}{9\pi^2}\left(6\log\left(-\frac{k^2}{m^2}\right)-7\right){\pi_{\mu\nu}}^3 \qquad (9.86)\\
&+ \frac{im^6}{9\pi^2}\left(12\log\left(-\frac{k^2}{m^2}\right)-5\right)\pi_{\mu\nu}\pi_{\mu\mu}\pi_{\nu\nu} \\
&\quad .... \qquad ....
\end{aligned}$$

are all conserved. Eq.(9.85) is related to a nonlocal version of the spin 3 Fronsdal equation.

9.6 Correlators

We also made a systematic collection of results for the massive case concerning all types of two-point correlators, including the mixed ones, for symmetric currents of spin up to 5 and in dimension $3 \leq d \leq 6$. Since the volume of these formulas is rather big it is moved to the ancillary file [30]. A part of this material is nevertheless kept here in the main text: sections 9.6.1 and 9.6.2 contain some representative calculations.

For even d, we use $d \to d + \varepsilon$ and expand around ε. For odd d this is not necessary. It is convenient to use the following shorthand notation

$$L_n = \frac{2}{\varepsilon} + \log\left(\frac{m^2}{4\pi}\right) + \gamma - \sum_{k=1}^{n}\frac{1}{k} \qquad (9.87)$$

as well as

$$K = \log\left(-\frac{k^2}{m^2}\right), \qquad P = \frac{2}{\varepsilon} + \log\left(-\frac{k^2}{4\pi}\right) + \gamma \qquad (9.88)$$

We see that there is a relationship $P = K + L_0$. Furthermore we define

$$T = -\frac{2i\coth^{-1}\left(\frac{2m}{k}\right)}{\pi}, \qquad S = \sqrt{4m^2 - k^2}\csc^{-1}\left(\frac{2m}{k}\right) \tag{9.89}$$

It turns out that T is useful in even dimensions d and S is useful in odd. The branches of the functions T and S are chosen such that the IR and UV expansions are

$$T \overset{IR}{=} -\frac{ik}{\pi m} - \frac{ik^3}{12\pi m^3} - \frac{ik^5}{80\pi m^5} + \dots \tag{9.90}$$

$$S \overset{IR}{=} k - \frac{k^3}{12m^2} - \frac{k^5}{120m^4} + \dots \tag{9.91}$$

$$T \overset{UV}{=} 1 - \frac{4im}{\pi k} - \frac{16im^3}{3\pi k^3} - \frac{64im^5}{5\pi k^5} + \dots \tag{9.92}$$

$$S \overset{UV}{=} \frac{kK}{2} - \frac{m^2(1+K)}{k} + \frac{m^4(1-2K)}{2k^3} + \frac{m^6(5-6K)}{3k^5} + \dots \tag{9.93}$$

In the following two sections we list the results for fermions for mixed spin 3 - spin 5 amplitudes for dimensions 3 and 4. Section 4.1 contains the full transverse analytic expressions of the correlators. Section 4.2 contains the UV and IR expansions of the latter.

9.6.1 Fermion amplitudes for spins 3x5

Fermions, spin 3 x 5, dimension 3:

$$\begin{aligned}
\tilde{T}^{\text{f,t}}_{\mu^3\nu^5;3\text{D}} = k^8\pi^3_{\mu\nu}\pi_{\nu\nu}&\left(\frac{i}{4\pi}\left(-\frac{3}{16}\frac{m}{k^2} + \frac{3}{4}\frac{m^3}{k^4} - \frac{31}{3}\frac{m^5}{k^6} + 20\frac{m^7}{k^8}\right)+\right.\\
&\left.+T\left(-\frac{3}{256}\frac{1}{k} + \frac{1}{16}\frac{m^2}{k^3} + \frac{3}{8}\frac{m^4}{k^5} - 3\frac{m^6}{k^7} + 5\frac{m^8}{k^9}\right)\right)+\\
&+k^8\pi_{\mu\mu}\pi_{\mu\nu}\pi^2_{\nu\nu}\left(\frac{i}{4\pi}\left(\frac{7}{64}\frac{m}{k^2} + \frac{155}{48}\frac{m^3}{k^4} - \frac{47}{4}\frac{m^5}{k^6} + 15\frac{m^7}{k^8}\right)+\right.\\
&\left.+T\left(\frac{7}{1024}\frac{1}{k} - \frac{9}{64}\frac{m^2}{k^3} + \frac{33}{32}\frac{m^4}{k^5} - \frac{13}{4}\frac{m^6}{k^7} + \frac{15}{4}\frac{m^8}{k^9}\right)\right)+\\
&+k^6(k\cdot\epsilon)_{\mu\nu}\pi^2_{\mu\nu}\pi_{\nu\nu}\left(\frac{1}{\pi}\left(\frac{3}{4}\frac{m^2}{k^2} - \frac{8}{3}\frac{m^4}{k^4} + 4\frac{m^6}{k^6}\right)+\right.\\
&\left.+iT\left(\frac{1}{16}\frac{m}{k} - \frac{3}{4}\frac{m^3}{k^3} + 3\frac{m^5}{k^5} - 4\frac{m^7}{k^7}\right)\right)+\\
&+k^6(k\cdot\epsilon)_{\mu\nu}\pi_{\mu\mu}\pi^2_{\nu\nu}\left(\frac{1}{\pi}\left(-\frac{1}{16}\frac{m^2}{k^2} - \frac{2}{3}\frac{m^4}{k^4} + \frac{m^6}{k^6}\right)+\right.\\
&\left.+iT\left(\frac{1}{64}\frac{m}{k} - \frac{3}{16}\frac{m^3}{k^3} + \frac{3}{4}\frac{m^5}{k^5} - \frac{m^7}{k^7}\right)\right)
\end{aligned} \tag{9.94}$$

$$
\begin{aligned}
\tilde{T}^{\text{f,nt}}_{\mu^3\nu^5;3\text{D}} = &\left(k_\mu k_\nu^3 \eta_{\mu\mu}\eta_{\nu\nu} + k_\mu^2 k_\nu^2 \eta_{\mu\nu}\eta_{\nu\nu}\right)\left(-\frac{8i}{3\pi}m^3\right) + \left(k_\nu^4 \eta_{\mu\mu}\eta_{\mu\nu} + k_\mu^3 k_\nu \eta_{\nu\nu}^2\right)\left(-\frac{4i}{3\pi}m^3\right) + \\
&+ \left(k_\mu k_\nu \eta_{\mu\mu}\eta_{\nu\nu}^2 + k_\mu^2 \eta_{\mu\nu}\eta_{\nu\nu}^2\right)\left(\frac{i}{\pi}\left(\frac{4}{3}k^2 m^3 - \frac{32}{5}m^5\right)\right) + \\
&+ k_\nu^2 \eta_{\mu\mu}\eta_{\mu\nu}\eta_{\nu\nu}\left(\frac{i}{\pi}\left(\frac{8}{3}k^2 m^3 - \frac{64}{5}m^5\right)\right) + k_\mu k_\nu \eta_{\mu\nu}^2 \eta_{\nu\nu}\left(-\frac{64i}{5\pi}m^5\right) + \\
&+ k_\nu^2 \eta_{\mu\nu}^3\left(-\frac{64i}{15\pi}m^5\right) + \eta_{\mu\mu}\eta_{\mu\nu}\eta_{\nu\nu}^2\left(\frac{i}{\pi}\left(-\frac{4}{3}k^4 m^3 + \frac{32}{5}k^2 m^5 - \frac{64}{5}m^7\right)\right) + \\
&+ \eta_{\mu\nu}^3 \eta_{\nu\nu}\left(\frac{i}{5\pi}\left(\frac{64}{3}k^2 m^5 - \frac{512}{7}m^7\right)\right) + k_\mu^2 k_\nu^2 (k\cdot\epsilon)_{\mu\nu}\eta_{\nu\nu}\left(-\frac{1}{\pi}m^2\right) + \\
&+ k_\mu k_\nu^3 (k\cdot\epsilon)_{\mu\nu}\eta_{\mu\nu}\left(-\frac{2}{\pi}m^2\right) + k_\nu^2 (k\cdot\epsilon)_{\mu\nu}\eta_{\mu\mu}\eta_{\nu\nu}\left(-\frac{8}{3\pi}m^4\right) + \\
&+ k_\mu^2 (k\cdot\epsilon)_{\mu\nu}\eta_{\nu\nu}^2\left(-\frac{4}{3\pi}m^4\right) + k_\mu k_\nu (k\cdot\epsilon)_{\mu\nu}\eta_{\mu\nu}\eta_{\nu\nu}\left(\frac{1}{3\pi}\left(6k^2 m^2 - 32m^4\right)\right) + \\
&+ k_\nu^2 (k\cdot\epsilon)_{\mu\nu}\eta_{\mu\nu}^2\left(\frac{1}{3\pi}\left(3k^2 m^2 - 16m^4\right)\right) + \\
&+ (k\cdot\epsilon)_{\mu\nu}\eta_{\mu\mu}\eta_{\nu\nu}^2\left(\frac{1}{\pi}\left(\frac{4}{3}k^2 m^4 - \frac{16}{5}m^6\right)\right) + \\
&+ (k\cdot\epsilon)_{\mu\nu}\eta_{\mu\nu}^2\eta_{\nu\nu}\left(\frac{1}{\pi}\left(-k^4 m^2 + \frac{16}{3}k^2 m^4 - \frac{64}{5}m^6\right)\right) \qquad (9.95)
\end{aligned}
$$

Fermions, spin 3 x 5, dimension 4:

$$
\begin{aligned}
\tilde{T}^{\text{f,t}}_{\mu^3\nu^5;4\text{D}} = &\, k^8 \pi_{\mu\nu}^3 \pi_{\nu\nu}\left(\frac{i}{7\pi^2}\left(\left(\frac{1937}{14175} - \frac{2L_0}{45}\right) + \left(-\frac{1622}{1575} + \frac{2L_0}{5}\right)\frac{m^2}{k^2} - \frac{32}{15}\frac{m^4}{k^4} + \right.\right.\\
&\quad \left. + \frac{2432}{135}\frac{m^6}{k^6} - \frac{256}{9}\frac{m^8}{k^8}\right) + \\
&\quad \left. + \frac{iS}{3\pi^2}\left(-\frac{4}{105}\frac{1}{k} + \frac{4}{15}\frac{m^2}{k^3} + \frac{16}{35}\frac{m^4}{k^5} - \frac{704}{105}\frac{m^6}{k^7} + \frac{256}{21}\frac{m^8}{k^9}\right)\right) + \\
&+ k^8 \pi_{\mu\mu}\pi_{\mu\nu}\pi_{\nu\nu}^2\left(\frac{i}{\pi^2}\left(\left(-\frac{1231}{132300} + \frac{L_0}{420}\right) + \left(\frac{258}{1225} - \frac{2L_0}{35}\right)\frac{m^2}{k^2} - \frac{104}{105}\frac{m^4}{k^4} + \right.\right.\\
&\quad \left. + \frac{128}{45}\frac{m^6}{k^6} - \frac{64}{21}\frac{m^8}{k^8}\right) + \\
&\quad \left. + \frac{iS}{\pi^2}\left(\frac{1}{210}\frac{1}{k} - \frac{11}{105}\frac{m^2}{k^3} + \frac{4}{5}\frac{m^4}{k^5} - \frac{272}{105}\frac{m^6}{k^7} + \frac{64}{21}\frac{m^8}{k^9}\right)\right) \qquad (9.96)
\end{aligned}
$$

$$\tilde{T}^{\text{f,nt}}_{\mu^3\nu^5;4\text{D}} = \left(k_\mu k_\nu^3\eta_{\mu\mu}\eta_{\nu\nu} + k_\mu^2 k_\nu^2\eta_{\mu\nu}\eta_{\nu\nu}\right)\left(\frac{iL_2}{\pi^2}m^4\right) + \left(k_\nu^4\eta_{\mu\mu}\eta_{\mu\nu} + k_\mu^3 k_\nu\eta_{\nu\nu}^2\right)\left(\frac{iL_2}{2\pi^2}m^4\right) +$$
$$+ \left(k_\mu k_\nu\eta_{\mu\mu}\eta_{\nu\nu}^2 + k_\mu^2\eta_{\mu\nu}\eta_{\nu\nu}^2\right)\left(\frac{i}{2\pi^2}\left(-L_2k^2m^4 + 4L_3m^6\right)\right) +$$
$$+ k_\nu^2\eta_{\mu\mu}\eta_{\mu\nu}\eta_{\nu\nu}\left(\frac{i}{\pi^2}\left(-L_2k^2m^4 + 4L_3m^6\right)\right) + k_\mu k_\nu\eta_{\mu\nu}^2\eta_{\nu\nu}\left(\frac{4iL_3}{\pi^2}m^6\right) +$$
$$+ k_\nu^2\eta_{\mu\nu}^3\left(\frac{4iL_3}{3\pi^2}m^6\right) + \eta_{\mu\mu}\eta_{\mu\nu}\eta_{\nu\nu}^2\left(\frac{i}{2\pi^2}\left(L_2k^4m^4 - 4L_3k^2m^6 + 7L_4m^8\right)\right) +$$
$$+ \eta_{\mu\nu}^3\eta_{\nu\nu}\left(\frac{i}{3\pi^2}\left(-4L_3k^2m^6 + 12L_4m^8\right)\right) \tag{9.97}$$

9.6.2 Expansions in UV and IR for fermions for spins 3x5

Fermions, spin 3 x 5, dimension 3:

$$\begin{aligned}
\tilde{T}^{\text{f,t,UV}}_{\mu^3\nu^5;3\text{D}} &= k^8\pi_{\mu\nu}^3\pi_{\nu\nu}\left(-\frac{3}{256}\frac{1}{k} + \frac{1}{16}\frac{m^2}{k^3} + \frac{3}{8}\frac{m^4}{k^5} - \frac{64i}{15\pi}\frac{m^5}{k^6} - 3\frac{m^6}{k^7} + \dots\right) \\
&+ k^8\pi_{\mu\mu}\pi_{\mu\nu}\pi_{\nu\nu}^2\left(\frac{7}{1024}\frac{1}{k} - \frac{9}{64}\frac{m^2}{k^3} + \frac{4i}{3\pi}\frac{m^3}{k^4} + \frac{33}{32}\frac{m^4}{k^5} - \frac{32i}{5\pi}\frac{m^5}{k^6} - \frac{13}{4}\frac{m^6}{k^7} + \dots\right) \\
&+ k^6(k\cdot\epsilon)_{\mu\nu}\pi_{\mu\nu}^2\pi_{\nu\nu}\left(\frac{i}{16}\frac{m}{k} + \frac{1}{\pi}\frac{m^2}{k^2} - \frac{3i}{4}\frac{m^3}{k^3} - \frac{16}{3\pi}\frac{m^4}{k^4} + 3i\frac{m^5}{k^5} + \frac{64}{5\pi}\frac{m^6}{k^6} + \dots\right) \\
&+ k^6(k\cdot\epsilon)_{\mu\nu}\pi_{\mu\mu}\pi_{\nu\nu}^2\left(\frac{i}{64}\frac{m}{k} - \frac{3i}{16}\frac{m^3}{k^3} - \frac{4}{3\pi}\frac{m^4}{k^4} + \frac{3i}{4}\frac{m^5}{k^5} + \frac{16}{5\pi}\frac{m^6}{k^6} + \dots\right)
\end{aligned} \tag{9.98}$$

$$\begin{aligned}
\tilde{T}^{\text{f,t,IR}}_{\mu^3\nu^5;3\text{D}} &= k^8\pi_{\mu\nu}^3\pi_{\nu\nu}\left(\frac{i}{\pi}\left(-\frac{4}{35}\frac{m}{k^2} + \frac{2}{315}\frac{1}{m} + \frac{1}{4620}\frac{k^2}{m^3} + \frac{1}{60060}\frac{k^4}{m^5} + \dots\right)\right) \\
&+ k^8\pi_{\mu\mu}\pi_{\mu\nu}\pi_{\nu\nu}^2\left(\frac{i}{5\pi}\left(\frac{4}{7}\frac{m}{k^2} - \frac{1}{84}\frac{1}{m} - \frac{1}{5544}\frac{k^2}{m^3} - \frac{1}{192192}\frac{k^4}{m^5} + \dots\right)\right) \\
&+ k^6(k\cdot\epsilon)_{\mu\nu}\pi_{\mu\nu}^2\pi_{\nu\nu}\left(\frac{1}{\pi}\left(\frac{1}{5}\frac{m^2}{k^2} + \frac{1}{35} + \frac{1}{1260}\frac{k^2}{m^2} + \frac{1}{18480}\frac{k^4}{m^4} + \dots\right)\right) \\
&+ k^6(k\cdot\epsilon)_{\mu\nu}\pi_{\mu\mu}\pi_{\nu\nu}^2\left(\frac{1}{\pi}\left(-\frac{1}{5}\frac{m^2}{k^2} + \frac{1}{140} + \frac{1}{5040}\frac{k^2}{m^2} + \frac{1}{73920}\frac{k^4}{m^4} + \dots\right)\right)
\end{aligned} \tag{9.99}$$

Fermions, spin 3 x 5, dimension 4:

$$\begin{aligned}
\tilde{T}^{\mathrm{f,t,UV}}_{\mu^3\nu^5;4\mathrm{D}} &= k^8\pi^3_{\mu\nu}\pi_{\nu\nu}\left(\frac{i}{\pi^2}\left(\left(\frac{1937}{99225}-\frac{2P}{315}\right)+\left(-\frac{494}{3675}+\frac{2P}{35}\right)\frac{m^2}{k^2}-\frac{2}{5}\frac{m^4}{k^4}\right.\right.\\
&\qquad\left.\left.+\left(\frac{22}{9}-\frac{4K}{3}\right)\frac{m^6}{k^6}+\dots\right)\right)\\
&+ k^8\pi_{\mu\mu}\pi_{\mu\nu}\pi^2_{\nu\nu}\left(\frac{i}{\pi^2}\left(\left(-\frac{1231}{132300}+\frac{P}{420}\right)+\left(\frac{1513}{7350}-\frac{2P}{35}\right)\frac{m^2}{k^2}+\right.\right.\\
&\qquad\left.\left.+\left(-\frac{53}{60}+\frac{K}{2}\right)\frac{m^4}{k^4}+(2-2K)\frac{m^6}{k^6}+\dots\right)\right)
\end{aligned}\tag{9.100}$$

$$\begin{aligned}
\tilde{T}^{\mathrm{f,t,IR}}_{\mu^3\nu^5;4\mathrm{D}} &= k^8\pi^3_{\mu\nu}\pi_{\nu\nu}\left(\frac{i}{5\pi^2}\left(\left(-\frac{2}{7}+\frac{2L_0}{7}\right)\frac{m^2}{k^2}-\frac{2L_0}{63}+\frac{1}{462}\frac{k^2}{m^2}+\frac{1}{9009}\frac{k^4}{m^4}+\dots\right)\right)\\
&+ k^8\pi_{\mu\mu}\pi_{\mu\nu}\pi^2_{\nu\nu}\left(\frac{i}{35\pi^2}\left((2-2L_0)\frac{m^2}{k^2}+\frac{L_0}{12}-\frac{1}{396}\frac{k^2}{m^2}-\frac{1}{20592}\frac{k^4}{m^4}+\dots\right)\right)
\end{aligned}\tag{9.101}$$

9.7 Parity-odd part

In this section we focus on the parity-odd part streaming from the mixed 2-point correlators in 3d. We will look at UV an IR leading terms in the expansion of the full correlator and find generalized expressions for dimension d and two higher spin fields s_1 and s_2. The general expression for dominating term in the correlator of two simple fermion currents for spin $s_1 \times s_2$, $s_2 > s_1$ (7.16) in the UV is

$$\begin{aligned}
\tilde{T}_{\mu_1\dots\mu_{s_1}\nu_1\dots\nu_{s_2}} &= (-1)^{\frac{s_1+s_2}{2}}\frac{\left(2\lfloor\frac{s_2-1}{2}\rfloor\right)!!\left(s_1+s_2-2\lfloor\frac{s_1-1}{2}\rfloor-3\right)!!mk^{s_1+s_2-3}}{2^2(s_1+s_2-2)!!\left(2\lfloor\frac{s_2-1}{2}\rfloor-2\lfloor\frac{s_1-1}{2}\rfloor\right)!!}\\
&\times\pi_{\nu\nu}^{\frac{s_2-s_1}{2}}\epsilon_{\sigma\mu\nu}k^\sigma\sum_{l=0}^{\lfloor\frac{s_1}{2}\rfloor-1}\frac{(s_1-1)!(s_2-s_1)!!}{2^{\frac{l(l+1)}{2}}(s_1-2l-1)!(s_2-s_1+2l)!!}\pi^l_{\mu\mu}\pi^l_{\nu\nu}\pi^{s-2l-1}_{\mu\nu}
\end{aligned}\tag{9.102}$$

In the IR instead we have a general expression for spin $s_1 \times s_2$, $s_2 > s_1$

$$\begin{aligned}
\tilde{T}_{\mu_1\dots\mu_{s_1}\nu_1\dots\nu_{s_2}} &= (-1)^{\frac{s_1+s_2}{2}-1}\frac{\left(2\lfloor\frac{s_2-1}{2}\rfloor\right)!!\left(s_1+s_2-2\lfloor\frac{s_1-1}{2}\rfloor-3\right)!!k^{s_1+s_2-2}}{2^2\pi(s_1+s_2-1)!!\left(2\lfloor\frac{s_2-1}{2}\rfloor-2\lfloor\frac{s_1-1}{2}\rfloor\right)!!}\\
&\times\pi_{\nu\nu}^{\frac{s_2-s_1}{2}}\epsilon_{\sigma\mu\nu}k^\sigma\sum_{l=0}^{\lfloor\frac{s_1}{2}\rfloor-1}\frac{(s_1-1)!(s_2-s_1)!!}{2^{\frac{l(l+1)}{2}}(s_1-2l-1)!(s_2-s_1+2l)!!}\pi^l_{\mu\mu}\pi^l_{\nu\nu}\pi^{s-2l-1}_{\mu\nu}
\end{aligned}\tag{9.103}$$

For traceless currents (traceless in the limit $m \to 0$) we use (7.26) with coefficients

(7.27). General expression for dominating term in the UV of 2-point correlators with traceless currents for spin $s_1 \times s_2$, $s_2 > s_1$ is

$$\begin{aligned}
\tilde{T}_{\mu_1 \ldots \mu_{s_1} \nu_1 \ldots \nu_{s_2}} &= (-1)^{\frac{s_1+s_2}{2}} \frac{imk^{s_1+s_2-3}}{2^{s_2+1}} \pi_{\nu\nu}^{\frac{s_2-s_1}{2}} \sum_{l=0}^{\lfloor \frac{s_1}{2} \rfloor} \frac{(-1)^l \Gamma(s_1 - l)}{2^{2l} l! \Gamma(s_1 - 2l)} \pi_{\mu\mu}^l \pi_{\nu\nu}^l \pi_{\mu\nu}^{s_1-2l-1} \epsilon_{\sigma\mu\nu} k^\sigma \quad (9.104) \\
&= (-1)^{\frac{s_1+s_2}{2}} \frac{imk^{s_1+s_2-3}}{2^{s_2+1}} \pi_{\nu\nu}^{\frac{s_2-s_1}{2}} \pi_{\mu\nu}^{s_1-1} \, {}_2F_1 \left(\frac{1-s_1}{2}, -\frac{s_1}{2}, 1-s_1, \frac{\pi_{\mu\mu}\pi_{\nu\nu}}{\pi_{\mu\nu}^2} \right) \epsilon_{\sigma\mu\nu} k^\sigma
\end{aligned}$$

This formula is a straightforward generalization of the linearized action proposed long ago by Pope and Townsend, [44], for conformal higher spin fields.

For completeness let us give also some examples of the expressions for the correlators with traceless currents ((7.26) with coefficients (7.27)) in the IR, even though we are not able to write a general expression. Also, for spin $0 \times 2n$ full amplitudes are zero. Dominating terms in the IR:

$$\text{Spin } 1 \times 1 : \tilde{T}_{\mu\nu} = \frac{1}{4\pi} \epsilon_{\sigma\mu\nu} k^\sigma \tag{9.105}$$

$$\text{Spin } 1 \times 3 : \tilde{T}_{\mu\nu^3} = -\frac{ik^2}{48\pi} \pi_{\nu\nu} \epsilon_{\sigma\mu\nu} k^\sigma \tag{9.106}$$

$$\text{Spin } 2 \times 2 : \tilde{T}_{\mu^2\nu^2} = -\frac{k^2}{12\pi} \pi_{\mu\nu} \epsilon_{\sigma\mu\nu} k^\sigma \tag{9.107}$$

$$\text{Spin } 2 \times 4 : \tilde{T}_{\mu^2\nu^4} = \frac{k^4}{120\pi} \pi_{\nu\nu} \pi_{\mu\nu} \epsilon_{\sigma\mu\nu} k^\sigma \tag{9.108}$$

$$\text{Spin } 3 \times 3 : \tilde{T}_{\mu^3\nu^3} = \frac{k^4}{960\pi} \left(32\pi_{\mu\nu}^2 - 9\pi_{\mu\mu}\pi_{\nu\nu} \right) \epsilon_{\sigma\mu\nu} k^\sigma \tag{9.109}$$

$$\text{Spin } 3 \times 5 : \tilde{T}_{\mu^3\nu^5} = -\frac{k^6}{26880\pi} \pi_{\nu\nu} \left(96\pi_{\mu\nu}^2 - 25\pi_{\mu\mu}\pi_{\nu\nu} \right) \epsilon_{\sigma\mu\nu} k^\sigma \tag{9.110}$$

$$\text{Spin } 4 \times 4 : \tilde{T}_{\mu^4\nu^4} = -\frac{k^6}{1680\pi} \pi_{\mu\nu} \left(24\pi_{\mu\nu}^2 - 3\pi_{\mu\mu}\pi_{\nu\nu} \right) \epsilon_{\sigma\mu\nu} k^\sigma \tag{9.111}$$

$$\text{Spin } 4 \times 6 : \tilde{T}_{\mu^4\nu^6} = \frac{k^8}{40320\pi} \pi_{\mu\nu} \pi_{\nu\nu} \left(64\pi_{\mu\nu}^2 - 33\pi_{\mu\mu}\pi_{\nu\nu} \right) \epsilon_{\sigma\mu\nu} k^\sigma \tag{9.112}$$

$$\text{Spin } 5 \times 5 : \tilde{T}_{\mu^5\nu^5} = \frac{k^8}{322560\pi} \left(2048\pi_{\mu\nu}^4 - 1632\pi_{\mu\nu}^2 \pi_{\mu\mu}\pi_{\nu\nu} + 147\pi_{\mu\mu}^2 \pi_{\nu\nu}^2 \right) \epsilon_{\sigma\mu\nu} k^\sigma \tag{9.113}$$

9.8 Diagonalization

Just like for scalars in the previous chapter, in this chapter we again showed that different choices of currents lead to different effective actions. Let us now see if we can choose currents so that the mixed correlators vanish.

9.8.1 "Local" currents

We can write a general form of spin - s current

$$j^{\mathrm{f}}_{\mu_1\ldots\mu_s} = i^{s-1}\sum_{l=0}^{\lfloor\frac{s}{2}\rfloor} a_l^{(s)}\left(\Box\pi_{\mu\mu}\right)^l\left(\bar{\psi}\gamma_\mu\left(\overleftrightarrow{\partial}_\mu\right)^{s-2l-1}\psi\right) \tag{9.114}$$

where $a_l^{(s)}$ are numerical coefficients with $a_0^{(s)} = 1$. For $s \neq 0$ and we can introduce also spin-0 current $j^{\mathrm{f}}_{\mu^0} = \bar{\psi}\psi$.

Amplitude 0x2. We use the current

$$j^{\mathrm{f}}_{\mu^2} = i\left(\bar{\psi}\gamma_\mu\left(\overleftrightarrow{\partial}_\mu\right)^2\psi + a_1^{(2)}\Box\pi_{\mu\mu}\left(\bar{\psi}\psi\right)\right) \tag{9.115}$$

where in this case, contrary to scalar case (see chapter 8.7.1), we neglected the terms such as $(\Box + m^2)$. These terms vanish on-shell, contribute only to the nonconserved part and they effectively behave like counterterms. The conserved part of 0x2 amplitude is

$$\begin{aligned}\tilde{T}^{\mathrm{t}}_{\nu^2} &= \frac{i2^{-d+\lfloor\frac{d}{2}\rfloor}\pi^{-d/2}m^{d-3}\Gamma\left(1-\frac{d}{2}\right)}{3}k^2\pi_{\nu\nu} \\ &\times\left(-\frac{1}{2}\left(d-2\right)\,{}_2F_1\left(1,2-\frac{d}{2};\frac{5}{2};\frac{k^2}{4m^2}\right)+3(d-1)ma_1^{(2)}\,{}_2F_1\left(1,1-\frac{d}{2};\frac{3}{2};\frac{k^2}{4m^2}\right)\right)\end{aligned} \tag{9.116}$$

and the conserved part of the correlator with mixed spins 0x2 vanishes for

$$a_1^{(2)} = \frac{(d-2){}_2F_1\left(1,2-\frac{d}{2};\frac{5}{2};\frac{k^2}{4m^2}\right)}{6(d-1){}_2F_1\left(1,1-\frac{d}{2};\frac{3}{2};\frac{k^2}{4m^2}\right)} \tag{9.117}$$

Just like in the scalar case, the coefficient $a_1^{(2)}$ is a function of momenta and mass. If we expand it in powers of the momentum k^2 , we get an interaction with infinite number of higher derivative terms. The non-conserved part

$$\tilde{T}^{\mathrm{nt}}_{\nu^2} = -i2^{-d+\lfloor\frac{d}{2}\rfloor}d\pi^{-d/2}m^{d-1}\Gamma\left(-\frac{d}{2}\right)\eta_{\nu\nu} \tag{9.118}$$

is local and can be canceled by a counterterm. Similar conclusions can be drawn for spin 1x3 amplitude. For spin 4 (and higher), the non-conserved part of the correlator again depends on the coefficients $a_l^{(4)}$. This coefficient, once expanded in powers of momenta, brings infinite number of non-conserved terms. Moreover, for spin 4 (and higher) there is no choice of coefficients $a_1^{(4)}$ and $a_2^{(4)}$ for which the conserved part of the correlators with mixed spins 0x4 and 2x4 vanishes. We again conclude that it is not possible to diagonalize the 2-pt correlators.

In the massless limit all non-diagonal terms vanish for (7.27), that is for the choice of coefficient for traceless scalar currents given in (7.27). In this case only the correlators for currents of equal spins are non-vanishing and they are given by (9.11).

9.8.2 Traceless non-local currents

Just like in the scalar case, we construct on-shell traceless currents. We write down a general form of current

$$\bar{j}^{\mathrm{f}}_{\mu^s} = \sum_{l=0}^{\lfloor \frac{s}{2} \rfloor} b_l^{(s)} \pi^l_{\mu\mu} j^{\mathrm{f}\,[l]}_{\mu_{s-2l}} \tag{9.119}$$

where $j^{\mathrm{f}}_{\mu^s}$ is a simple fermion current (7.22). The l-th trace of the simple current reads

$$\begin{aligned} j^{(s)[l]}_{\mu_{s-2l}} &= i^{s-1}\frac{2k(s-1)!}{(s-2l)!}\bar{\psi}\gamma_\nu \overset{\leftrightarrow}{\partial}{}^{\nu} \left(\overset{\leftrightarrow}{\partial}_\mu\right)^{s-2l} \left(\overset{\leftrightarrow}{\partial}\right)^{2(l-1)} \psi \\ &\quad + i^{s-1}\frac{(s-1)!}{(s-2l-1)!}\bar{\psi}\gamma_\mu \left(\overset{\leftrightarrow}{\partial}_\mu\right)^{s-2l-1} \left(\overset{\leftrightarrow}{\partial}\right)^{2l} \psi \end{aligned} \tag{9.120}$$

and $b_l^{(s)}$ are numerical coefficients. We also use spin zero current $J^{(s)}_{\mu_0} = \bar{\psi}\psi$. If we impose tracelessness (on-shell) we get a recurrence relation:

$$b_l^{(s)} = -\frac{1}{2l(d-3+2s-2l)} b_{l-1}^{(s)} \tag{9.121}$$

We choose $b_0^{s)} = 1$, so that the coefficient $b_l^{(s)}$ reads:

$$b_l^{(s)} = \frac{(-1)^l \left(s-l-1+\frac{d-3}{2}\right)!}{2^{2l} l! \left(s-1+\frac{d-3}{2}\right)!} \tag{9.122}$$

For this exact choice of coefficients, the conserved parts of all mixed-spin correlators vanish. The conserved part of the amplitude with equal spin currents (9.119) and coefficients

(9.122) for general spin s is proportional to

$$\tilde{T}_{\mu^s\nu^s} \sim k^{2s} \sum_{l=0}^{\lfloor \frac{s}{2} \rfloor} a_l \pi^l_{\mu\mu} \pi^l_{\nu\nu} \pi^{s-2l}_{\mu\nu} \tag{9.123}$$

where the coefficient a_l, just like in the scalar case, corresponds to the coefficient for the traceless amplitude (7.42). Let us give some examples:

$$\begin{aligned}
\text{Spin } 2\times 2: \; \tilde{T}_{\mu^2\nu^2} &= -\frac{i2^{-d+\lfloor \frac{d}{2} \rfloor}\pi^{-d/2}k^2 m^{d-2}\Gamma\left(1-\frac{d}{2}\right)}{3}\left(\pi^2_{\mu\nu} - \frac{1}{(d-1)}\pi_{\mu\mu}\pi_{\nu\nu}\right) \\
&\times \left({}_2F_1\left(1, 1-\frac{d}{2}; \frac{5}{2}; \frac{k^2}{4m^2}\right) - 2\,{}_2F_1\left(2, 1-\frac{d}{2}; \frac{5}{2}; \frac{k^2}{4m^2}\right)\right) \\
\text{Spin } 3\times 3: \; \tilde{T}_{\mu^3\nu^3} &= -\frac{i}{15}2^{2-d+\lfloor \frac{d}{2} \rfloor}\pi^{-d/2}k^4\pi_{\mu\nu}m^{d-4}\left(\pi^2_{\mu\nu} - \frac{3}{(d+1)}\pi_{\mu\mu}\pi_{\nu\nu}\right) \\
&\times {}_2F_1\left(2, 1-\frac{d}{2}; \frac{7}{2}; \frac{k^2}{4m^2}\right)
\end{aligned}$$

Again, similarly to scalars, the non-conserved parts of the amplitudes does not vanish. These terms are non-local, and because of that they cannot be canceled by a counterterm. To avoid nonlocality, we can, instead of

$$S_{int} \sim \sum_s \int d^dx J_{\mu_1\ldots\mu_s}\varphi^{\mu_1\ldots\mu_s} \tag{9.124}$$

use a higher derivative coupling

$$S_{int} \sim \sum_s \int d^dx J_{\mu_1\ldots\mu_s}\Box^n\varphi^{\mu_1\ldots\mu_s} \sim \sum_s \int d^dx \Box^n J_{\mu_1\ldots\mu_s}\varphi^{\mu_1\ldots\mu_s} \tag{9.125}$$

It is enough to put $n = \lfloor \frac{s}{2} \rfloor$. Then, all amplitudes should be multiplied by $(k^2)^{\lfloor \frac{s_1}{2} \rfloor + \lfloor \frac{s_2}{2} \rfloor}$. In that case the nonconserved parts become local and we can subtract them by a finite number of counterterms.

Chapter 10

Worldline quantization of a fermion model

In this chapter we will turn our attention to another quantization method, the worldline quantization method and apply it to a free Dirac fermion coupled to external sources [33]. Similar computation for the scalar model has already been worked out in [128]. The advantage of this method is that it gives the exact form of the higher spin gauge symmetry.

In particular, we will determine the expression for the effective action, by expanding it in a perturbative series, and determine the generalized equations of motion. This procedure will allow us, in the next chapter, to show that this setup of the theory accommodates an L_∞ algebra. In this chapter we closely follow [33].

10.1 Fermion linearly coupled to higher spin fields

Let us consider a free fermion theory

$$S_0 = \int d^d x\, \overline{\psi}(i\gamma\cdot\partial - m)\psi, \tag{10.1}$$

coupled to external sources. We second-quantize it using the Weyl quantization method for a particle worldline. The full action is expressed as an expectation value of operators as follows

$$S = \langle\overline{\psi}| - \gamma\cdot(\widehat{P} - \widehat{H}) - m|\psi\rangle \tag{10.2}$$

Here $\widehat{P}_\mu$ is the momentum operator whose symbol is the classical momentum p_μ. $\widehat{H}$ is an operator whose symbol is $h(x,p)$, where

$$h^\mu(x,p) = \sum_{n=0}^{\infty} \frac{1}{n!}\, h_{(s)}^{\mu\mu_1\dots\mu_n}(x)\, p_{\mu_1}\dots p_{\mu_n} \tag{10.3}$$

$s = n+1$ is the spin and the tensors are assumed to be symmetric. We recall that a quantum operator $\widehat{O}$ can be represented with a symbol $O(x,p)$ through the Weyl map

$$\widehat{O} = \int d^dx\, d^dy \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\, O(x,p)\, e^{ik\cdot(x-\widehat{X})-iy\cdot(p-\widehat{P})} \tag{10.4}$$

where $\widehat{X}$ is the position operator. Next we insert this into the RHS of (10.2), where we also insert two completenesses $\int d^dx|x\rangle\langle x|$, and make the identification $\psi(x) = \langle x|\psi\rangle$. Expressing S in terms of symbols we find

$$\begin{aligned} S &= S_0 + \int \frac{d^dq}{(2\pi)^d}\, d^dx\, d^dz\, e^{iq\cdot z}\, \overline{\psi}\Big(x+\frac{z}{2}\Big)\, \gamma\cdot h(x,q)\, \psi\Big(x-\frac{z}{2}\Big) \qquad (10.5)\\ &= S_0 + \sum_{n=0}^{\infty}\int d^dx\, \frac{i^n}{n!}\frac{\partial}{\partial z^{\mu_1}}\dots\frac{\partial}{\partial z^{\mu_n}}\overline{\psi}\Big(x+\frac{z}{2}\Big)\, \gamma_\mu h^{\mu\mu_1\dots\mu_n}(x)\, \psi\Big(x-\frac{z}{2}\Big)\Big|_{z=0}\\ &= S_0 + \sum_{s=1}^{\infty}\int d^dx\, j_{\mu_1\dots\mu_s}(x)\, h_{(s)}^{\mu_1\dots\mu_s}(x) \end{aligned}$$

The symmetric tensor field $h^{\mu\mu_1\dots\mu_n}$ is linearly coupled to the HS (higher spin) current

$$j_{\mu\mu_1\dots\mu_{s-1}}(x) = \frac{i^{s-1}}{(s-1)!}\frac{\partial}{\partial z^{(\mu_1}}\dots\frac{\partial}{\partial z^{\mu_{s-1}}}\overline{\psi}\left(x+\frac{z}{2}\right)\gamma_{\mu)}\psi\left(x-\frac{z}{2}\right)\Big|_{z=0}. \tag{10.6}$$

For instance, for $s=1$ and $s=2$ one obtains

$$j_\mu = \overline{\psi}\gamma_\mu\psi \tag{10.7}$$

$$j_{\mu\mu_1} = \frac{i}{2}\left(\partial_{(\mu_1}\overline{\psi}\gamma_{\mu)}\psi - \overline{\psi}\gamma_{(\mu}\partial_{\mu_1)}\psi\right) \tag{10.8}$$

and we see that these currents correspond to simple fermion currents given in (7.22). The HS currents are on-shell conserved in the free theory (10.1)

$$\partial_\mu j^{\mu\mu_1\cdots\mu_{s-1}} = 0 \tag{10.9}$$

which is a consequence of invariance of $S_0[\psi]$ on global (rigid) transformations

$$\delta_n\psi(x) = -\frac{(-i)^{n+1}}{n!}\,\varepsilon_{(n)}^{\mu_1\cdots\mu_n}\partial_{\mu_1}\ldots\partial_{\mu_n}\psi(x) \tag{10.10}$$

We shall next show that for the full action (10.5) this extends to the local symmetry. The consequence is that the currents are still conserved, with the HS covariant derivative substituting ordinary derivative in (10.9).

Notice that these currents are conserved even without symmetrizing μ with the other indices. But in the sequel we will suppose that they are symmetric.

10.2 Symmetries

The action (10.2) is trivially invariant under the operation

$$S = \langle\overline{\psi}|\widehat{O}\widehat{O}^{-1}\widehat{G}\widehat{O}\widehat{O}^{-1}|\psi\rangle \tag{10.11}$$

where $\widehat{G} = -\gamma\cdot(\widehat{P}-\widehat{H}) - m$. So it is invariant under

$$\widehat{G} \longrightarrow \widehat{O}^{-1}\widehat{G}\widehat{O}, \qquad |\psi\rangle \longrightarrow \widehat{O}^{-1}|\psi\rangle \tag{10.12}$$

Writing $\widehat{O} = e^{-i\widehat{E}}$ we easily find the infinitesimal version.

$$\delta|\psi\rangle = i\widehat{E}|\psi\rangle, \qquad \delta\langle\overline{\psi}| = -i\langle\overline{\psi}|\widehat{E}, \tag{10.13}$$

and

$$\delta\widehat{G} = i[\widehat{E}\,,\widehat{G}] = i[\gamma\cdot(\widehat{P}-\widehat{H})\,,\widehat{E}] = \gamma\cdot\delta\widehat{H} \tag{10.14}$$

Let the symbol of $\widehat{E}$ be $\varepsilon(x,p)$, then the symbol of $[i\gamma\cdot\widehat{P},\widehat{E}]$ is

$$\int d^dy\langle x-\frac{y}{2}|[i\gamma\cdot\widehat{P},\widehat{E}]|x+\frac{y}{2}\rangle\, e^{iy\cdot p} \tag{10.15}$$

An easy way to make this explicit is to use the fact that the symbol of the product of two operators is given by the Moyal product of the symbols. Thus

$$\begin{aligned}\mathrm{Symb}\big([\gamma\cdot\widehat{P},\widehat{E}]\big) &= [\gamma\cdot p \overset{*}{,} \varepsilon(x,p)] = \gamma\cdot p\, e^{-\frac{i}{2}\overrightarrow{\partial_x}\cdot\overleftarrow{\partial_p}}\varepsilon(x,p) - \varepsilon(x,p)\, e^{\frac{i}{2}\overleftarrow{\partial_x}\cdot\overrightarrow{\partial_p}}\gamma\cdot p \\ &= -i\gamma\cdot\partial_x\varepsilon(x,p) \end{aligned} \tag{10.16}$$

Similarly

$$\mathrm{Symb}\big([\widehat{H}^\mu,\widehat{E}]\big) = [h^\mu(x,p) \overset{*}{,} \varepsilon(x,p)] \tag{10.17}$$

where $[a \overset{*}{,} b] \equiv a * b - b * a$. Therefore, in terms of symbols,

$$\delta_\varepsilon h^\mu(x,p) = \partial_x^\mu\varepsilon(x,p) - i[h^\mu(x,p) \overset{*}{,} \varepsilon(x,p)] \equiv \mathcal{D}_x^{*\mu}\varepsilon(x,p) \tag{10.18}$$

where we introduced the covariant derivative defined by

$$\mathcal{D}_x^{*\mu} = \partial_x^\mu - i[h^\mu(x,p) \overset{*}{,} \quad] \tag{10.19}$$

This will be referred to hereafter as HS transformation, and the corresponding symmetry HS symmetry.

The transformations of ψ are somewhat different. They can also be expressed as Moyal product of symbols

$$\delta_\varepsilon\tilde{\psi}(x,p) = i\varepsilon(x,p) * \tilde{\psi}(x,p) \tag{10.20}$$

provided we use the partial Fourier transform

$$\tilde{\psi}(x,p) = \int d^dy\,\psi\left(x - \frac{y}{2}\right) e^{iy\cdot p}. \tag{10.21}$$

and finally we antitransform back the result. Alternatively we can proceed as follows. We compute

$$\begin{aligned}\langle x|\widehat{E}|\psi\rangle &= \int \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d} d^dx' d^dy'\ \varepsilon(x',p)\,\langle x|e^{ik\cdot(x'-\widehat{X})-iy'\cdot(p-\widehat{P})}|\psi\rangle \\ &= \int \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d} d^dx' d^dy'\ \varepsilon(x',p')\, e^{ik\cdot(x'-x)-iy'\cdot p}\langle x|e^{iy'\widehat{P}}|\psi\rangle e^{-\frac{i}{2}y'\cdot k}\end{aligned} \tag{10.22}$$

Next we insert a momentum completeness $\int d^dq|q\rangle\langle q|$ to evaluate $\langle x|e^{iy'\widehat{P}}|\psi\rangle$ and subsequently a coordinate completeness to evaluate $\langle q|\psi\rangle$ using the standard relation $\langle x|p\rangle = e^{ip\cdot x}$. Then we produce two delta functions by integrating over k and q. In this way we get rid of two coordinate integrations. Finally we arrive at

$$\begin{aligned}
\delta_\varepsilon\psi(x) &= i\langle x|\widehat{E}|\psi\rangle = i\int \frac{d^dp}{(2\pi)^d}\, d^dz\, \varepsilon\Big(x+\frac{z}{2},p\Big)\, e^{-ip\cdot z}\,\psi(x+z) \qquad (10.23)\\
&= i\sum_{n=0}^{\infty}\int \frac{d^dp}{(2\pi)^d}\, d^dz\, \frac{e^{-ip\cdot z}}{n!}(-i\partial_z)^n\cdot\Big(\varepsilon_{(n)}\Big(x+\frac{z}{2}\Big)\,\psi(x+z)\Big)\\
&= \sum_{n=0}^{\infty}\frac{i}{n!}(-i\partial_z)^n\cdot\Big(\varepsilon_{(n)}\Big(x+\frac{z}{2}\Big)\,\psi(x+z)\Big)\Big|_{z=0}\\
&= i\varepsilon_{(0)}(x)\,\psi(x)+\varepsilon^\mu_{(1)}(x)\,\partial_\mu\psi(x)+\frac{1}{2}\partial_\mu\varepsilon^\mu_{(1)}(x)\,\psi(x)\\
&\quad -\frac{i}{2}\left(\varepsilon^{\mu\nu}_{(2)}\,\partial_\mu\partial_\nu\psi+\partial_\mu\varepsilon^{\mu\nu}_{(2)}\,\partial_\nu\psi+\frac{1}{4}\partial_\mu\partial_\nu\varepsilon^{\mu\nu}_{(2)}\,\psi\right)(x)+\dots
\end{aligned}$$

where a dot denotes the contraction of upper and lower indices. The first method leads to the same result.

Now we want to understand the conservation law ensuing from the HS symmetry of the interacting classical action (10.5)

$$0=\delta_\varepsilon S[\psi,h]=\int d^dx\left(\frac{\delta S}{\delta\psi(x)}\,\delta_\varepsilon\psi(x)+\delta_\varepsilon\overline{\psi}(x)\frac{\delta S}{\delta\overline{\psi}(x)}+\int d^dp\,\frac{\delta S}{\delta h^\mu(x,p)}\,\delta_\varepsilon h^\mu(x,p)\right)$$

Now we evaluate this expression on the classical solution, in which case the first two terms vanish (remember that h is the background field). We are left with

$$0=\int d^dx\int d^dp\,J_\mu(x,p)\,\delta_\varepsilon h^\mu(x,p) \qquad (\mathrm{on-shell}) \qquad (10.24)$$

where

$$J_\mu(x,p)\equiv\int d^dz\,e^{ip\cdot z}\,\overline{\psi}\Big(x+\frac{z}{2}\Big)\,\gamma_\mu\,\psi\Big(x-\frac{z}{2}\Big) \qquad (10.25)$$

Using (10.18), partially integrating and using the following property of the Moyal product

$$\int d^dx\int d^dp\,a(x,p)[b(x,p)\,\overset{*}{,}\,c(x,p)]=\int d^dx\int d^dp\,[a(x,p)\,\overset{*}{,}\,b(x,p)]\,c(x,p) \qquad (10.26)$$

we obtain

$$0 = \int d^d x \int d^d p \, \varepsilon(x,p) \, \mathcal{D}_x^{*\mu} J_\mu(x,p) \qquad \text{(on} - \text{shell)} \tag{10.27}$$

From this follows the conservation law in the classical interacting theory

$$\mathcal{D}_x^{*\mu} J_\mu(x,p) = 0 \qquad \text{(on} - \text{shell)} \tag{10.28}$$

Using the $*$-Jacobi identity (it holds also for the Moyal product, because it is associative) one can easily get

$$\begin{aligned} (\delta_{\varepsilon_2}\delta_{\varepsilon_1} - \delta_{\varepsilon_1}\delta_{\varepsilon_2}) \, h^\mu(x,p) &= i \left(\partial_x [\varepsilon_1 \stackrel{*}{,} \varepsilon_2](x,p) - i[h^\mu(x,p) \stackrel{*}{,} [\varepsilon_1 \stackrel{*}{,} \varepsilon_2](x,p)]] \right) \\ &= i \, \mathcal{D}_x^{*\mu} [\varepsilon_1 \stackrel{*}{,} \varepsilon_2](x,p) \end{aligned} \tag{10.29}$$

We see that the HS ε-transform is of the Lie algebra type.

10.3 Perturbative expansion of the effective action

In this subsection we work out (heuristic) rules, similar to the Feynman ones, to compute n-point amplitudes in the above fermion model. The purpose is to reproduce formulas similar to those of [128] for the scalar case. We would like to point out, however, that this is not strictly necessary: the good old Feynman rules are anyhow a valid alternative.

We start from the representation of the effective action as trace-logarithm of a differential operator:

$$W[h] = N \, \mathrm{Tr}[\ln \widehat{G}] \tag{10.30}$$

and use a well-known mathematical formula to regularize it

$$W_{reg}[h, \epsilon] = -N \int_\epsilon^\infty \frac{dt}{t} \, \mathrm{Tr}\left[e^{-t\widehat{G}} \right] \tag{10.31}$$

where ϵ is an infrared regulator. The crucial factor is therefore

$$K[g|t] \equiv \mathrm{Tr}\left[e^{-t\widehat{G}} \right] = \mathrm{Tr}\left[e^{t(\gamma \cdot (\widehat{P} - \widehat{H}) + m)} \right], \tag{10.32}$$

known as the heat kernel, where g is the symbol of $\widehat{G}$. The trace Tr includes both an integration over the momenta and tr, the trace over the gamma matrices,

$$K[g|t] = e^{mt} \int \frac{d^d p}{(2\pi)^d} \, \mathrm{tr}\langle p| e^{t\gamma\cdot(\widehat{P}-\widehat{H})}|p\rangle \tag{10.33}$$

Next we expand

$$e^{t\gamma\cdot(\widehat{P}-\widehat{H})} = e^{t\,\gamma\cdot\widehat{P}} \sum_{n=0}^{\infty} (-1)^n \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n \, \gamma\cdot\widehat{H}(\tau_1)\,\gamma\cdot\widehat{H}(\tau_2)\ldots\gamma\cdot\widehat{H}(\tau_n)$$

where $\gamma\cdot\widehat{H}(\tau) = e^{-\tau\,\gamma\cdot\widehat{P}}\gamma\cdot\widehat{H}\,e^{\tau\,\gamma\cdot\widehat{P}}$. We have

$$\langle p|\gamma\cdot\widehat{H}(\tau)|q\rangle = e^{-\tau\,\gamma\cdot p}\langle p|\gamma\cdot\widehat{H}|q\rangle\, e^{\tau\,\gamma\cdot q} \tag{10.34}$$

Using a formula analogous to (10.22) for $\widehat{H}$ and inserting completenesses one finds

$$\begin{aligned} \langle p|\gamma\cdot\widehat{H}|q\rangle &= \int d^d x \int d^d y \frac{d^d k}{(2\pi)^d}\frac{d^d p'}{(2\pi)^d}\,\gamma\cdot h(x,p')\langle p|e^{ik\cdot(x-\widehat{X})-iy\cdot(p'-\widehat{P})}|q\rangle \qquad (10.35)\\ &= \int d^d x \,\gamma\cdot h(x,\partial_u) e^{i(q-p)\cdot x + u\cdot\frac{p+q}{2}}\Big|_{u=0} \end{aligned}$$

Therefore

$$\langle p|\gamma\cdot\widehat{H}(\tau)|q\rangle = \int d^d x\, e^{-\tau\,\gamma\cdot p}\,\gamma\cdot h(x,\partial_u)\,e^{\tau\,\gamma\cdot q}\,e^{i(q-p)\cdot x + u\cdot\frac{p+q}{2}}\Big|_{u=0} \tag{10.36}$$

Using this we can write

$$\begin{aligned} \mathrm{Tr}\left[e^{-t\widehat{G}}\right] &= e^{mt}\sum_{n=0}^{\infty}(-1)^n \int \prod_{i=1}^{n}\frac{d^d p_i}{(2\pi)^d}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n \\ &\times\ \mathrm{tr}\Big(e^{t\,\gamma\cdot p_n}\langle p_n|\gamma\cdot\widehat{H}(\tau_1)|p_1\rangle\langle p_1|\gamma\cdot\widehat{H}(\tau_2)|p_2\rangle\ldots\langle p_{n-1}|\gamma\cdot\widehat{H}(\tau_n)|p_n\rangle\Big) \\ &= e^{mt}\sum_{n=0}^{\infty}(-1)^n \int \prod_{i=1}^{n} d^d x_i \frac{d^d p_i}{(2\pi)^d}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n \\ &\times\ \mathrm{tr}\Big(e^{(t-\tau_1)\,\gamma\cdot p_n}\gamma^{\mu_1}e^{(\tau_1-\tau_2)\,\gamma\cdot p_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}}e^{(\tau_{n-1}-\tau_n)\,\gamma\cdot p_{n-1}}\gamma^{\mu_n}e^{\tau_n\gamma\cdot p_n}\Big) \\ &\times\ \prod_{j=1}^{n} e^{ip_j\cdot\left(x_j - x_{j+1} - i\frac{u_{j+1}+u_j}{2}\right)}\ h_{\mu_1}\left(x_1,\overleftarrow{\partial}_{u_1}\right)\ldots h_{\mu_n}\left(x_n,\overleftarrow{\partial}_{u_n}\right)\Big|_{u_j=0} \qquad (10.37) \end{aligned}$$

where $x_{n+1} = x_1$. Now we can factor out in $K[g,t]$ the terms $h_{\mu_1}\left(x_1,\overleftarrow{\partial}_{u_1}\right)\ldots h_{\mu_n}\left(x_n,\overleftarrow{\partial}_{u_n}\right)$, and write

$$K[g|t] = \sum_{n=0}^{\infty} \langle\langle K^{(n)\mu\ldots\mu}(t)|h_\mu^{\otimes n}\rangle\rangle \tag{10.38}$$

where the double brackets means integration of the x_i and derivation with respect to the u_i. In turn $K^{(n)\mu\ldots\mu}(t)$ can be written more explicitly as

$$K^{\mu_1\ldots\mu_n}(x_1,u_1,\ldots,x_n,u_n|t) = e^{tm}\int\prod_{j=1}^{n}\frac{d^dp_j}{(2\pi)^d}e^{ip_j\cdot\left(x_j-x_{j+1}-i\frac{u_{j+1}+u_j}{2}\right)}\widetilde{K}^{\mu_1\ldots\mu_n}(p_1,\ldots,p_n|t) \tag{10.39}$$

where we symmetrized

$$\begin{aligned}\widetilde{K}^{\mu_1\ldots\mu_n}(p_1,\ldots,p_n|t) &= \frac{(-1)^n}{n}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\ldots\int_0^{\tau_{n-1}}d\tau_n \\ &\times \operatorname{tr}\Big(\gamma^{\mu_1}e^{(\tau_1-\tau_2)\,\gamma\cdot p_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}}e^{(\tau_{n-1}-\tau_n)\,\gamma\cdot p_{n-1}}\gamma^{\mu_n}e^{(\tau_n-\tau_1)\,\gamma\cdot p_n}e^{t\gamma\cdot p_n} \\ &\quad+\gamma^{\mu_2}e^{(\tau_1-\tau_2)\,\gamma\cdot p_2}\gamma^{\mu_3}\ldots\gamma^{\mu_n}e^{(\tau_{n-1}-\tau_n)\,\gamma\cdot p_n}\gamma^{\mu_1}e^{(\tau_n-\tau_1)\,\gamma\cdot p_1}e^{t\gamma\cdot p_1} \\ &\quad\vdots \\ &\quad+\gamma^{\mu_n}e^{(\tau_1-\tau_2)\,\gamma\cdot p_n}\gamma^{\mu_1}\ldots\gamma^{\mu_{n-2}}e^{(\tau_{n-1}-\tau_n)\,\gamma\cdot p_{n-2}}\gamma^{\mu_{n-1}}e^{(\tau_n-\tau_1)\,\gamma\cdot p_{n-1}}e^{t\gamma\cdot p_{n-1}}\Big)\end{aligned} \tag{10.40}$$

Note that in the above equation, for $n=0$, there is no need for symmetrization and hence there is no $\frac{1}{n}$ term.

Now, the nested integral can be rewritten in the following way

$$\begin{aligned}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\ldots\int_0^{\tau_{n-1}}d\tau_n &= \int_0^t d\sigma_1\int_0^{t-\sigma_1}d\sigma_2\int_0^{t-\sigma_1-\sigma_2}d\sigma_3\ldots\int_0^{t-\sigma_1-\ldots-\sigma_{n-1}}d\sigma_n \\ &= \int_0^\infty d\sigma_1\int_0^\infty d\sigma_2\ldots\int_0^\infty d\sigma_n\,\theta(t-\sigma_1-\ldots-\sigma_n)\end{aligned} \tag{10.41}$$

where $\sigma_i = \tau_{i-1}-\tau_i$, with $\tau_0 = t$. Notice that defining $\sigma_0 = t-\sigma_1-\ldots-\sigma_n$ we can identify $\sigma_0 = \tau_n$. Next one uses the following representation of the Heaviside function

$$\theta(t) = \lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi i}\,\frac{e^{i\omega t}}{\omega-i\epsilon} = \lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{i\omega t}\int_0^\infty d\sigma_0\,e^{-i\sigma_0(\omega-i\epsilon)} \tag{10.42}$$

The ω integration has to be understood as a contour integration. Using this in (10.41) we obtain

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t} \int_0^{\infty} d\sigma_0 \int_0^{\infty} d\sigma_1 \ldots \int_0^{\infty} d\sigma_n\, e^{-i(\sigma_0+\ldots+\sigma_n)(\omega - i\epsilon)}$$

Replacing this inside (10.40) we get

$$\begin{aligned} \widetilde{K}^{\mu_1\cdots\mu_n}(p_1,\ldots,p_n|t) &= \frac{(-1)^n}{n} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t} \int_0^{\infty} d\sigma_0 \int_0^{\infty} d\sigma_1 \ldots \int_0^{\infty} d\sigma_n \qquad (10.43) \\ \times \mathrm{tr}\Big[& \gamma^{\mu_1} e^{\sigma_2(\gamma\cdot p_1 - i\omega')} \gamma^{\mu_2} \ldots \gamma^{\mu_{n-1}} e^{\sigma_n(\gamma\cdot p_{n-1} - i\omega')} \gamma^{\mu_n} e^{(\sigma_0+\sigma_1)(\gamma\cdot p_n - i\omega')} \\ & \ldots \\ + & \gamma^{\mu_n} e^{\sigma_2(\gamma\cdot p_n - i\omega')} \gamma^{\mu_1} \ldots \gamma^{\mu_{n-2}} e^{\sigma_n(\gamma\cdot p_{n-2} - i\omega')} \gamma^{\mu_{n-1}} e^{(\sigma_0+\sigma_1)(\gamma\cdot p_{n-1} - i\omega')} \Big] \end{aligned}$$

where $\omega' = \omega - i\epsilon$ and ϵ in the exponents allows us to perform the integrals[1], the result being

$$\begin{aligned} \widetilde{K}^{\mu_1\cdots\mu_n}(p_1,\ldots,p_n|t) &= \frac{(-1)^n}{n} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t} \qquad (10.44) \\ &\times \mathrm{tr}\Big[\gamma^{\mu_1} \frac{-1}{\not{p}_1 - i\omega'} \gamma^{\mu_2} \ldots \gamma^{\mu_{n-1}} \frac{-1}{\not{p}_{n-1} - i\omega'} \gamma^{\mu_n} \frac{1}{(\not{p}_n - i\omega')^2} \\ & \quad \ldots \\ &+ \gamma^{\mu_n} \frac{-1}{\not{p}_n - i\omega'} \gamma^{\mu_1} \ldots \gamma^{\mu_{n-2}} \frac{-1}{\not{p}_{n-2} - i\omega'} \gamma^{\mu_{n-1}} \frac{1}{(\not{p}_{n-1} - i\omega')^2} \Big] \end{aligned}$$

We remark that $\frac{1}{(\not{p}-i\omega')^2} = \frac{\partial}{\partial(i\omega)} \frac{1}{\not{p}-i\omega'}$. Integrating by parts we can simplify (10.44)

$$\widetilde{K}^{\mu_1\cdots\mu_n}(p_1,\ldots,p_n|t) = \frac{t}{n} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, \mathrm{tr}\Big[\gamma^{\mu_1} \frac{1}{\not{p}_1 - i\omega'} \gamma^{\mu_2} \ldots \frac{1}{\not{p}_{n-1} - i\omega'} \gamma^{\mu_n} \frac{1}{\not{p}_n - i\omega'} \Big] \qquad (10.45)$$

We can also include the factor e^{tm} in (10.39) in a new kernel $\widetilde{K}^{\mu_1\cdots\mu_n}(p_1,\ldots,p_n|m,t)$ which has the same form as $\widetilde{K}^{\mu_1\cdots\mu_n}(p_1,\ldots,p_n|t)$ with all the $\not{p}_i$ replaced by $\not{p}_i + m$:

$$K^{\mu_1\cdots\mu_n}(x_1,u_1,\ldots,x_n,u_n|t) = \prod_{j=1}^{n} e^{ip_j\cdot\left(x_j - x_{j+1} - i\frac{u_{j+1}+u_j}{2}\right)}\, \widetilde{K}^{\mu_1\cdots\mu_n}(p_1,\ldots,p_n|m,t) \qquad (10.46)$$

[1]This is evident with the Majorana representation of the gamma matrices, because in such a case the term $\gamma\cdot p$ in the exponent is purely imaginary, the gamma matrices being imaginary. This term therefore gives rise to oscillatory contributions, much like the $i\omega$ term.

where

$$
\begin{aligned}
\widetilde{K}^{\mu_1\cdots\mu_n}(p_1,\ldots,p_n|m,t) &= \frac{(-1)^n}{n}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{i\omega t} \qquad (10.47)\\
\times \quad & \mathrm{tr}\Big[\gamma^{\mu_1}\frac{-1}{\not p_1+m-i\omega'}\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}}\frac{-1}{\not p_{n-1}+m-i\omega'}\gamma^{\mu_n}\frac{1}{(\not p_n+m-i\omega')^2}\\
+ \quad & \gamma^{\mu_n}\frac{-1}{\not p_n+m-i\omega'}\gamma^{\mu_1}\ldots\gamma^{\mu_{n-2}}\frac{-1}{\not p_{n-2}+m-i\omega'}\gamma^{\mu_{n-1}}\frac{1}{(\not p_{n-1}+m-i\omega')^2}\Big]\\
= \quad & \frac{t}{n}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{i\omega t}\,\mathrm{tr}\Big[\gamma^{\mu_1}\frac{1}{\not p_1+m-i\omega'}\gamma^{\mu_2}\ldots\frac{1}{\not p_{n-1}+m-i\omega'}\gamma^{\mu_n}\frac{1}{\not p_n+m-i\omega'}\Big]
\end{aligned}
$$

Integrating further as in the scalar model case, [128], is not possible at this stage because of the gamma matrices. One has to proceed first to evaluate the trace over the latter. Using (10.37) we can write the regularized effective action as

$$
\begin{aligned}
W_{reg}[h,\epsilon] &= -N\int_{\epsilon}^{\infty}\frac{dt}{t}\,e^{mt}\sum_{n=0}^{\infty}\int\prod_{i=1}^{n}d^dx_i\,\frac{d^dp_i}{(2\pi)^d}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\ldots\int_0^{\tau_{n-1}}d\tau_n\\
&\quad\times\mathrm{tr}\left(e^{(t-\tau_1)\cdot p_n}\gamma^{\mu_1}e^{(\tau_1-\tau_2)\cdot p_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}}e^{(\tau_{n-1}-\tau_n)\cdot p_{n-1}}\gamma^{\mu_n}e^{\tau_n\gamma\cdot p_n}\right)\\
&\quad\times\prod_{j=1}^{n}e^{ip_j\cdot(x_j-x_{j+1})}\;h_{\mu_1}\left(x_1,\frac{p_1+p_n}{2}\right)\ldots h_{\mu_n}\left(x_n,\frac{p_{n-1}+p_n}{2}\right)\\
&= -N\int_{\epsilon}^{\infty}dt\sum_{n=0}^{\infty}\frac{1}{n}\int\prod_{i=1}^{n}d^dx_i\,\frac{d^dp_i}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{i\omega t}\\
&\quad\times\mathrm{tr}\Big[\gamma^{\mu_1}\frac{1}{\not p_1+m-i\omega'}\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}}\frac{1}{\not p_{n-1}+m-i\omega'}\gamma^{\mu_n}\frac{1}{\not p_n+m-i\omega'}\Big]\\
&\quad\times\prod_{j=1}^{n}e^{ip_j\cdot(x_j-x_{j+1})}\;h_{\mu_1}\Big(x_1,\frac{p_1+p_n}{2}\Big)\ldots h_{\mu_n}\Big(x_n,\frac{p_{n-1}+p_n}{2}\Big) \qquad (10.48)
\end{aligned}
$$

10.4 Ward identities and generalized EoM

The general formula for the effective action is

$$
W[h]=\sum_{n=1}^{\infty}\frac{1}{n!}\int\prod_{i=1}^{n}d^dx_i\,\frac{d^dp_i}{(2\pi)^d}\,\mathcal{W}^{(n)}_{\mu_1,\ldots,\mu_n}(x_1,p_1,\ldots,x_n,p_n,\epsilon)\,h^{\mu_1}(x_1,p_1)\ldots h^{\mu_n}(x_n,p_n) \qquad (10.49)
$$

where we have discarded the constant 0-point contribution, as we will do hereafter. The effective action can be calculated by various methods, of which (10.48) is a particular example. In the latter case the amplitudes are given by

$$
\begin{aligned}
\mathcal{W}^{(n)}_{\mu_1,\dots,\mu_n}(x_1,p_1,\ \dots\ ,x_n,p_n,\epsilon) = -N\frac{n!}{n}\int_\epsilon^\infty dt \int \prod_{i=1}^n \frac{d^d q_i}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{i\omega t} \\
\times\ \ \mathrm{tr}\Big[\gamma^{\mu_1}\frac{1}{\not{q}_1+m-i\omega'}\gamma^{\mu_2}\dots\gamma^{\mu_{n-1}}\frac{1}{\not{q}_{n-1}+m-i\omega'}\gamma^{\mu_n}\frac{1}{\not{q}_n+m-i\omega'}\Big] \\
\times\ \ \prod_{j=1}^n e^{iq_j\cdot(x_j-x_{j+1})}\,\delta\Big(p_1-\frac{q_1+q_n}{2}\Big)\dots\delta\Big(p_n-\frac{q_{n-1}+q_n}{2}\Big) \qquad (10.50)
\end{aligned}
$$

We stress once more, however, that the regularized effective action (10.49) may not be derived only via (10.50), that is via the procedure of section 2.2. It could as well be obtained by means of the ordinary Feynman diagrams.

This amplitude has cyclic symmetry. When saturated with the corresponding h's, as in (10.49), it gives the level n effective action. Here we would like to investigate some general consequences of the invariance of the general effective action under the HS symmetry, codified by eq. (10.18), assuming for the $\mathcal{W}^{(n)}$ the same cyclic symmetry as (10.50). The invariance of the effective action under (10.18) is expressed as

$$
\begin{aligned}
0 &= \delta_\varepsilon W[h] \qquad (10.51)\\
&= \sum_{n=1}^\infty \frac{1}{(n-1)!}\int \prod_{i=1}^n d^d x_i\,\frac{d^d p_i}{(2\pi)^d} \\
&\quad \times \mathcal{W}^{(n)}_{\mu_1,\dots,\mu_n}(x_1,p_1,\dots,x_n,p_n,\epsilon)\,\delta_\varepsilon h^{\mu_1}(x_1,p_1)\dots h^{\mu_n}(x_n,p_n)\\
&= \sum_{n=1}^\infty \frac{1}{(n-1)!}\int \prod_{i=1}^n d^d x_i\,\frac{d^d p_i}{(2\pi)^d} \\
&\quad \times \mathcal{W}^{(n)}_{\mu_1,\dots,\mu_n}(x_1,p_1,\dots,x_n,p_n)\,\mathcal{D}_x^{*\mu_1}\varepsilon(x_1,p_1)\,h^{\mu_2}(x_2,p_2)\dots h^{\mu_n}(x_n,p_n)
\end{aligned}
$$

Hereafter we assume that the HS symmetry is not anomalous and that there is a regularization procedure leading to a HS invariant effective action. The question of whether the particular effective action (10.48) satisfies (10.51) requires an explicit calculation of (10.50) and is left to future work.

In order to expose the L_∞ structure we need the equations of motion (EoM). Here we can talk of *generalized equations of motion*. They are obtained by varying $W[h,\epsilon]$ with respect to $h^\mu(x,p)$:

$$
\frac{\delta}{\delta h^\mu(x,p)}W[h]=0 \qquad (10.52)
$$

Then, expanding in p, we obtain the generalized EoM's for the components $h^{\mu_1\cdots\mu_n}(x)$. The most general EoM is therefore

$$\mathcal{F}_\mu(x,p) = 0 \tag{10.53}$$

where

$$\begin{aligned}\mathcal{F}_\mu(x,p) \;\; &\equiv \;\; \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} d^d x_i \, \frac{d^d p_i}{(2\pi)^d} \, \mathcal{W}^{(n+1)}_{\mu,\mu_1\cdots,\mu_n}(x,p,x_1,p_1,\ldots,x_n,p_n,\epsilon) \\ &\qquad \times h^{\mu_1}(x_1,p_1)\ldots h^{\mu_n}(x_n,p_n)\end{aligned}$$

Integrating by parts (10.51) and using (10.26) we obtain the off-shell equation

$$\mathcal{D}_x^{*\mu}\, \mathcal{F}_\mu(x,p) \equiv \partial_x^\mu \mathcal{F}_\mu(x,p) - i[h^\mu(x,p) \overset{*}{,} \mathcal{F}_\mu(x,p)] = 0 \tag{10.54}$$

Taking the variation of this equation with respect to (10.18) we get

$$0 = \delta_\varepsilon(\mathcal{D}_x^{*\mu}\, \mathcal{F}_\mu(x,p)) = \mathcal{D}_x^{*\mu}\; (\delta_\varepsilon \mathcal{F}_\mu(x,p)) - i[\mathcal{D}_x^{*\mu}\varepsilon \overset{*}{,} \mathcal{F}_\mu(x,p)] \tag{10.55}$$

From (10.54) and (10.55) one can deduce

$$\delta_\varepsilon \mathcal{F}_\mu(x,p) = i[\varepsilon(x,p) \overset{*}{,} \mathcal{F}_\mu(x,p)] \tag{10.56}$$

Now that we have determined the formula for the effective action and the generalized equations of motion, in the next chapter, we will show that this theory has L_∞ symmetry.

Chapter 11

L_∞ structure of higher spins

The procedure described in the previous chapter comes with a bonus, the precise form of the gauge symmetry. This has a outstanding consequence: it enables us to demonstrate L_∞ symmetry of the full effective action $W[h]$ obtained by integrating out a fermion field coupled to the higher spin fields. In this chapter we closely follow [33].

Let us mention that in the first part of this chapter we introduce a simplification: we neglect the generalized cosmological constant term $\mathcal{W}^{(1)}$. In the final part of this chapter, we complete the analysis of L_∞ symmetry of the fermion model with the presence of generalized cosmological constant term, that is, we show that such effective action admits curved L_∞ symmetry, see [184].

11.1 L_∞ symmetry of higher spin effective actions

In this section we will uncover the L_∞ symmetry of the $W[h]$. To this end we use the general transformation properties derived in the previous subsection, notably eqs. (10.53), (10.56), beside (10.18). We will also introduce a simplification, we will neglect the generalized cosmological term $\mathcal{W}^{(1)}$. The expansion of the effective action (10.49) is in essence an expansion around a flat background. Using standard regularizations we get that, in general, the effective action contains term linear in HS fields, which gives constant contribution to EoM's of even-spin HS fields of the form $c(s,\epsilon)\,(\eta_{\mu\mu})^{s/2}$, where $c(s,\epsilon)$ are scheme dependent coefficients which need to be renormalized. As this term is a generalization of the lowest-order contribution of the cosmological constant term expanded around flat spacetime, we shall call the part of the effective action that contains the full linear term and is invariant on HS transformations, *generalized cosmological constant* term. As a flat background is not a solution when the generalized cosmological constant term is present, consistency requires that we take this term out of an effective action (or, in other words, renormalize the cosmological constant to zero). This will be assumed from now on. Technically, this means that we now assume that the sum in (10.49) starts from $n = 2$, and the sum in (10.54) starts from $n = 1$, while all other relations from subsection 10.4 are the same.

To start with let us recall that an L_∞ structure characterizes closed string field theory[1]. This fact first appeared in [139], see also [141], as a particular case of a general mathematical structure called strongly homotopic algebras (or SH algebras), see the introduction for physicists [142, 143]. It became later evident that this kind of structure characterizes not only closed string field, but other field theories as well [144], in particular gauge field theories [145], Chern-Simons theories [138], Einstein gravity and double field theory [53]. For other, more recent applications, see [148, 149].

For the L_∞-algebra we closely follow the notation and definitions of [53]. L_∞-algebras (also referred to as strong homotopy Lie algebras) are generalization of Lie algebras. In L_∞-algebra we have a graded vector space

$$X = \bigoplus_i X_i \tag{11.1}$$

where X_i, $i = \ldots, 1, 0, -1, \ldots$ is a set of vector spaces, with degree i and multilinear maps (products) among them L_j, $j = 1, 2, \ldots$, with degree $d_j = j - 2$. It follows $deg(L_1) = -1$, $deg(L_2) = 0$, $deg(L_3) = 1$. To denote vectors in X we use notation $x_1, x_2, \ldots$. Each of these vectors has a definite degree $\mathrm{x}_i = deg(x_i)$. The degree of a map L_j acting on a collection of entries

$$deg(L_j(x_1, x_2, ..., x_j)) = j - 2 + \sum_{i=1}^{j} deg(x_i) \tag{11.2}$$

The properties of the mappings L_i under permutation are defined in [53]. The mappings L_j are defined to be graded commutative. For instance

$$L_2(x_1, x_2) \;\; = \;\; -(-1)^{\mathrm{x}_1 \mathrm{x}_2} L_2(x_2, x_1) \tag{11.3}$$

[1]Open string field theory is instead characterized by an A_∞ structure, see [53] and references therein.

In general

$$L_n(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = (-1)^\sigma \epsilon(\sigma; x) L_n(x_1, x_2, \ldots, x_n) \tag{11.4}$$

where σ denotes a permutation of the entries so that $(-1)^\sigma$ gives a positive sign if the permutation is even and a negative sign if the permutation is odd, and $\epsilon(\sigma; x)$ is the Koszul sign. To define it consider an algebra with product $x_i \wedge x_j = (-1)^{\mathrm{x}_i \mathrm{x}_j} x_j \wedge x_i$, then $\epsilon(\sigma; x)$ is defined by the relation

$$x_1 \wedge x_2 \wedge \ldots \wedge x_n = \epsilon(\sigma; x)\, x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \ldots \wedge x_{\sigma(n)} \tag{11.5}$$

It is worth noting that if all the $\mathrm{x_i}$'s are odd $(-1)^\sigma \epsilon(\sigma; x) = 1$.

Multilinear maps L_j satisfy the following quadratic identities:

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_\sigma (-1)^\sigma \epsilon(\sigma; x)\, L_j(L_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0 \tag{11.6}$$

In this formula $n \leq 1$ denotes a number of input vectors. The sum over permutations σ is a sum over "unshuffles" so that the entries are partially ordered $\sigma(1) < \ldots < \sigma(i)$, $\sigma(i+1) < \ldots < \sigma(n)$ We will schematically write this relation as

$$\sum_{i+j=n+1} (-1)^{i(j-1)} L_i l_j \tag{11.7}$$

In our case, due to the structure of the effective action and the equation of motion, we will need only three spaces X_0, X_{-1}, X_{-2} and the complex

$$X_0 \xrightarrow{L_1} X_{-1} \xrightarrow{L_1} X_{-2} \xrightarrow{L_1} 0 \tag{11.8}$$

The degree assignment is as follows: $\varepsilon \in X_0$, $h^\mu \in X_{-1}$ and $\mathcal{F}_\mu \in X_{-2}$.

The product L_i are defined as follows. We first define the maps ℓ_i

$$\delta_\varepsilon h = \ell_1(\varepsilon) + \ell_2(\varepsilon, h) - \frac{1}{2}\ell_3(\varepsilon, h, h) - \frac{1}{3!}\ell_4(\varepsilon, h, h, h) + \ldots \tag{11.9}$$

Therefore, in our case,

$$\begin{aligned}\ell_1(\varepsilon)^\mu &= \partial_x^\mu \varepsilon(x,p) \\ \ell_2(\varepsilon,h)^\mu &= -i[h^\mu(x,p) \overset{*}{,} \varepsilon(x,p)] = -\ell_2(h,\varepsilon)^\mu \\ \ell_j(\varepsilon,h,...,h)^\mu &= 0 \qquad , \qquad j \geq 3\end{aligned} \tag{11.10}$$

For these entries, i.e. $\varepsilon, (\varepsilon,h), (\varepsilon,h,h), \ldots$ we set $L_i = \ell_i$. From the above we can extract $L_2(\varepsilon,\varepsilon) \equiv \ell_2(\varepsilon,\varepsilon)$. We have

$$\begin{aligned}(\delta_{\varepsilon_1}\delta_{\varepsilon_2} - \delta_{\varepsilon_2}\delta_{\varepsilon_1})\, h^\mu &= \delta_{\varepsilon_1}\left(\ell_1(\varepsilon_2) + \ell_2(\varepsilon_2,h)\right) - \delta_{\varepsilon_2}\left(\ell_1(\varepsilon_1) + \ell_2(\varepsilon_1,h)\right) \\ &= \delta_{\varepsilon_1}\left(\ell_2(\varepsilon_2,h)\right) - \delta_{\varepsilon_2}\left(\ell_2(\varepsilon_1,h)\right) \\ &= \ell_2(\varepsilon_2,\delta_{\varepsilon_1}h) - \ell_2(\varepsilon_1,\delta_{\varepsilon_2}h) = \ell_2(\varepsilon_2,\ell_1(\varepsilon_1)) - \ell_2(\varepsilon_1,\ell_1(\varepsilon_2)) + \mathcal{O}(h)\end{aligned} \tag{11.11}$$

Now, the L_∞ relation (11.6) involving L_1 and L_2 is

$$L_1(L_2(x_1,x_2)) = L_2(L_1(x_1),x_2) - (-1)^{\mathrm{x}_1\mathrm{x}_2} L_2(L_1(x_2),x_1) \tag{11.12}$$

for two generic elements of x_1, x_2 of degree $\mathrm{x}_1, \mathrm{x}_2$, respectively. If we wish to satisfy it we have to identify

$$(\delta_{\varepsilon_1}\delta_{\varepsilon_2} - \delta_{\varepsilon_2}\delta_{\varepsilon_1})\, h = -\ell_1(\ell_2(\varepsilon_1,\varepsilon_2)) + \mathcal{O}(h) \tag{11.13}$$

By comparing this with (10.29) we obtain

$$\ell_2(\varepsilon_1,\varepsilon_2) = i\,[\varepsilon_1 \overset{*}{,} \varepsilon_2] \tag{11.14}$$

The next step is to determine L_3. It must satisfy, in particular, the L_∞ relation

$$\begin{aligned}0 \;=\;& L_1(L_3(x_1,x_2,x_3)) \\ +\;& L_3(L_1(x_1),x_2,x_3) + (-1)^{\mathrm{x}_1} L_3(x_1,L_1(x_2),x_3) + (-1)^{\mathrm{x}_1+\mathrm{x}_2} L_3(x_1,x_2,L_1(x_3)) \\ +\;& L_2(L_2(x_1,x_2),x_3) + (-1)^{(\mathrm{x}_1+\mathrm{x}_2)\mathrm{x}_3} L_2(L_2(x_3,x_1),x_2) + (-1)^{(\mathrm{x}_2+\mathrm{x}_3)\mathrm{x}_1} L_2(L_2(x_2,x_3),x_1)\end{aligned} \tag{11.15}$$

We define first the ℓ_i with only h entries. They are given by the generalized EoM:

$$\mathcal{F} = \ell_1(h) - \frac{1}{2}\ell_2(h,h) - \frac{1}{3!}\ell_3(h,h,h) + \dots \tag{11.16}$$

Let us write $\mathcal{F}_\mu$, (10.53) in compact form as

$$\mathcal{F}_\mu = \sum_{n=1}^{\infty} \frac{1}{n!} \langle\!\langle \mathcal{W}_\mu^{(n+1)}, h^{\otimes n} \rangle\!\rangle \tag{11.17}$$

then

$$\begin{aligned} \ell_n(h,\dots,h) &= (-1)^{\frac{n(n-1)}{2}} \langle\!\langle \mathcal{W}_\mu^{(n+1)}, h^{\otimes n} \rangle\!\rangle \qquad (11.18) \\ &= (-1)^{\frac{n(n-1)}{2}} \int \prod_{i=1}^{n} d^d x_i \frac{d^d p_i}{(2\pi)^d} \mathcal{W}^{(n+1)}_{\mu,\mu_1\dots,\mu_n}(x,p,x_1,p_1,\dots,x_n,p_n) \\ &\quad \times h^{\mu_1}(x_1,p_1)\dots h^{\mu_n}(x_n,p_n) \end{aligned}$$

in particular,

$$\ell_1(h) = \langle\!\langle \mathcal{W}_\mu^{(2)}, h \rangle\!\rangle = \int d^d x_i \frac{d^d p_i}{(2\pi)^d} \mathcal{W}^{(2)}_{\mu,\mu_1}(x,p,x_1,p_1) h^{\mu_1}(x_1,p_1) \tag{11.19}$$

Notice that $\mathcal{W}^{(n+1)}_{\mu,\mu_1\dots,\mu_n}$ is not symmetric in the exchange of its indices. In fact it has only a cyclic symmetry.

Furthermore, let us unfold (10.56). On one side we have

$$\begin{aligned} \delta_\varepsilon \mathcal{F}_\mu &= \sum_{n=1}^{\infty} \frac{1}{n!} \Big(\sum_{i=1}^{n} \langle\!\langle \mathcal{W}^{(n+1)}_{\mu\mu_1\dots\mu_i\dots\mu_n}, h^{\mu_1}\dots \partial_x^{\mu_i}\varepsilon \dots h^{\mu_n} \rangle\!\rangle \qquad (11.20) \\ &\quad -i \sum_{i=1}^{n} \langle\!\langle \mathcal{W}^{(n+1)}_{\mu\mu_1\dots\mu_i\dots\mu_n}, h^{\mu_1}\dots [h^{\mu_i} \overset{*}{,} \varepsilon]\dots h^{\mu_n} \rangle\!\rangle \Big) \end{aligned}$$

On the other side

$$i[\varepsilon \overset{*}{,} \mathcal{F}_\mu] = i \sum_{n=1}^{\infty} \frac{1}{n!} [\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}_\mu^{(n+1)}, h^{\otimes n} \rangle\!\rangle] \tag{11.21}$$

The two must be equal order by order in h. Thus we have

$$\begin{aligned} i[\varepsilon \stackrel{*}{,} \langle\!\langle \mathcal{W}^{(n+1)}_{\mu} , h^{\otimes n} \rangle\!\rangle] \;&=\; \frac{1}{n+1}\sum_{i=1}^{n+1} \langle\!\langle \mathcal{W}^{(n+2)}_{\mu\mu_1\dots\mu_i\dots\mu_{n+1}} , h^{\mu_1}\dots \partial_x^{\mu_i}\varepsilon \dots h^{\mu_{n+1}} \rangle\!\rangle \\ &\quad -i \sum_{i=1}^{n} \langle\!\langle \mathcal{W}^{(n+1)}_{\mu\mu_1\dots\mu_i\dots\mu_n} , h^{\mu_1}\dots [h^{\mu_i} \stackrel{*}{,} \varepsilon]\dots h^{\mu_n} \rangle\!\rangle \end{aligned} \tag{11.22}$$

This is the Ward identity for the symmetry (10.18).

In order to verify the L_∞ relations we have to know products l_i for different entries. Following [53] we define, for instance,

$$2L_2(h_1, h_2) \;=\; \ell_2(h_1 + h_2, h_1 + h_2) - \ell_2(h_1, h_1) - \ell_2(h_2, h_2) \tag{11.23}$$

which is equivalent to

$$L_2(h_1, h_2) \;=\; \frac{1}{2}\left(\ell_2(h_1, h_2) + \ell_2(h_2, h_1)\right) \tag{11.24}$$

Similarly

$$L_3(h_1, h_2, h_3) = \frac{1}{6}\left(\ell_3(h_1, h_2, h_3) + \mathrm{perm}(h_1, h_2, h_3)\right) \tag{11.25}$$

In general, when we have a non-symmetric n-linear function f_n of the variable h we can generate a symmetric function F_n linearly dependent on each of n variables $h_1, \dots, h_n$ through the following process

$$\begin{aligned} &F_n(h_1, \dots, h_n) \\ &= \frac{1}{n!}\Big(f_n(h_1 + \dots + h_n) - \Big[f_n(h_1 + \dots + h_{n-1}) + f_n(h_1 + \dots + h_{n-2} + h_n) \\ &\quad + \dots + f_n(h_2 + \dots + h_n)\Big] + \Big[f_n(h_1 + \dots + h_{n-2}) + \dots + f_n(h_3 + \dots + h_n)\Big] + \dots \\ &\quad + (-1)^{n-k}\Big[f_n(h_1 + \dots + h_k) + \dots + f_n(h_{n-k+1} + \dots + h_n)\Big] + \dots \\ &\quad + (-1)^{n-1}\Big[f_n(h_1) + \dots + f_n(h_n)\Big] \end{aligned} \tag{11.26}$$

We shall define $L_n(h_1, \dots, h_n)$ by using this formula: replace F_n with L_n and f_n with ℓ_n, the latter being given by (11.18). We shall see that beside $L_n(h_1, \dots, h_n)$, (11.10) and (11.14) the only nonvanishing objects defining the L_∞ algebra of the HS effective action are

$$L_2(\varepsilon, E) = i[\varepsilon \stackrel{*}{,} E] \tag{11.27}$$

where E represents $\mathcal{F}_\mu$ or any of its homogeneous pieces.

In the rest of this section we shall prove that L_n defined in this way generate an L_∞ algebra.

Note that in the previous chapter we assumed that higher spin symmetry is not anomalous and that the higher spin effective action is invariant under gauge transformations. To confirm that the effective action is indeed invariant under higher spin transformation one should explicitly compute (10.50).

11.2 Proof of the L_∞ relations

11.2.1 Relation $L_1^2 = 0$, degree -2

Now let us verify the remaining L_∞ relations. The first is $L_1^2 \equiv \ell_1^2 = 0$. [2]

Let us start from $\ell_1(\ell_1(\varepsilon))$. We recall that $\ell_1(\varepsilon) = \partial_x \varepsilon(x,p)$ and belongs to X_{-1}. Now

$$\ell_1(h) = \langle\!\langle \mathcal{W}_\mu^{(2)}, h \rangle\!\rangle \tag{11.28}$$

Replacing h with $\partial_x \varepsilon(x,p)$ corresponds to taking the variation of the lowest order in h of $\mathcal{F}_\mu$ with respect to h, i.e. with respect to (10.18). On the other hand the variation of $\mathcal{F}_\mu$ is given by (10.56) and is linear in $\mathcal{F}_\mu$. Therefore, since $\ell_1(\partial_x \varepsilon(x,p))$ is order 0 in h it must vanish. In fact it does, which corresponds to the gauge invariance of the EoM to the lowest order in h. This case corresponds to setting $n = 0$ in (11.22).

Next let us consider $\ell_1(\ell_1(h))$. It has degree -3, so it is necessarily 0 since $X_{-3} = 0$.

[2] We remark that if the generalized cosmological constant term (see end of sec. 10.4) is non-vanishing, then $\ell_1^2 \neq 0$. In this case an enlarged version of L_∞, called *curved* L_∞, is necessary.

11.2.2 Relation $L_1L_2 = L_2L_1$, degree -1

Next, we know $\ell_2(\varepsilon_1,\varepsilon_2), \ell_2(\varepsilon,h)$ and $\ell_2(h_1,h_2)$, and we have to verify $L_1L_2 = L_2L_1$. The latter is written explicitly in (11.12) and takes the form

$$\begin{aligned} \ell_1(\ell_2(\varepsilon,h)) &= L_2(\ell_1(\varepsilon),h)+L_2(\varepsilon,\ell_1(h)) \\ &= \frac{1}{2}\Big(\ell_2(\ell_1(\varepsilon),h)+\ell_2(h,\ell_1(\varepsilon))\Big)+L_2(\varepsilon,\ell_1(h)) \end{aligned} \tag{11.29}$$

where we used (11.24). More explicitly (11.29) writes

$$-i\ell_1([h \overset{*}{,} \varepsilon])_\mu = \frac{1}{2}\Big(\ell_2(\partial^x\varepsilon,h)+\ell_2(h,\partial^x\varepsilon)\Big)_\mu + L_2(\varepsilon,\langle\!\langle \mathcal{W}^{(2)}_\mu, h\rangle\!\rangle) \tag{11.30}$$

i.e.

$$i\langle\!\langle \mathcal{W}^{(2)}_{\mu\nu}\,,\, [h^\nu \overset{*}{,} \varepsilon])\rangle\!\rangle = \frac{1}{2}\Big(\langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}\,,\, \partial^\nu_x\varepsilon\, h^\lambda\rangle\!\rangle + \langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}\,,\, h^\nu\, \partial^\lambda_x\varepsilon\rangle\!\rangle\Big) - L_2(\varepsilon,\langle\!\langle \mathcal{W}^{(2)}_\mu, h\rangle\!\rangle) \tag{11.31}$$

Setting $n = 1$ in (11.22) gives precisely (11.31) provided

$$L_2(\varepsilon,\langle\!\langle \mathcal{W}^{(2)}_\mu, h\rangle\!\rangle) = i[\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(2)}_\mu\,,\, h\rangle\!\rangle] \tag{11.32}$$

The quantity $\mathcal{F}^{(1)} = \langle\!\langle \mathcal{W}^{(2)}_\mu\,,\, h\rangle\!\rangle$ is the lowest order piece of the EoM (of degree -2), see (11.17). So we can say

$$L_2(\varepsilon,\mathcal{F}^{(1)}) \equiv \ell_2(\varepsilon,\mathcal{F}^{(1)}) = i[\varepsilon \overset{*}{,} \mathcal{F}^{(1)}] \tag{11.33}$$

In general,

$$\ell_2(\varepsilon,\mathcal{F}) = i[\varepsilon \overset{*}{,} \mathcal{F}] \tag{11.34}$$

The next relation to be verified is

$$L_1(L_2(h_1,h_2)) = L_2(L_1(h_1),h_2) - L_2(h_1,L_1(h_2)) \tag{11.35}$$

The entries of L_2 on the rhs have degree -3, so they must vanish. On the other hand $L_2(h_1,h_2)$ on the lhs has degree -2, and is mapped to degree -3 by L_1. So it is consistent to equate both sides to 0. In particular we can set $L_2(\mathcal{F}^{(1)}, h) = 0$ (and, more generally, $L_2(X_{-2}, h) = 0$).

11.2.3 Relation $L_3L_1 + L_2L_2 + L_1L_3 = 0$, degree 0

First we should evaluate $L_3(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Its degree is 1, therefore it exits the complex. Is it consistent to set it to 0? The relevant L_∞ relation is

$$\begin{aligned} 0 &= \ell_1(L_3(x_1, x_2, x_3)) \\ &+ L_3(\ell_1(x_1), x_2, x_3) + (-1)^{\mathrm{x}_1} L_3(x_1, \ell_1(x_2), x_3) + (-1)^{\mathrm{x}_1+\mathrm{x}_2} L_3(x_1, x_2, \ell_1(x_3)) \\ &+ L_2(L_2(x_1, x_2), x_3) + (-1)^{(\mathrm{x}_1+\mathrm{x}_2)\mathrm{x}_3} L_2(L_2(x_3, x_1), x_2) + (-1)^{(\mathrm{x}_2+\mathrm{x}_3)\mathrm{x}_1} L_2(L_2(x_2, x_3), x_1) \end{aligned} \tag{11.36}$$

In our case the second line equals $\partial_x L_3(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Thus if we set $L_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) = 0$, the first two lines vanish. Using (11.14), we see that the third line is nothing but the $*$-Jacobi identity:

$$[\varepsilon_1 \overset{*}{,} [\varepsilon_2 \overset{*}{,} \varepsilon_3]] + [\varepsilon_2 \overset{*}{,} [\varepsilon_3 \overset{*}{,} \varepsilon_1]] + [\varepsilon_3 \overset{*}{,} [\varepsilon_1 \overset{*}{,} \varepsilon_2]] = 0 \tag{11.37}$$

From (11.10) we also know that $L_3(\varepsilon, h_1, h_2) \equiv \ell_3(\varepsilon, h_1, h_2) = 0$. Following [53] we will set also $L_3(\varepsilon_1, \varepsilon_2, h) = 0$, $L_3(\varepsilon_1, \varepsilon_2, \mathcal{F}^{(1)}) = 0$. Therefore

$$L_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) = 0, \quad L_3(\varepsilon, h_1, h_2) = 0, \quad L_3(\varepsilon_1, \varepsilon_2, h) = 0, \quad L_3(\varepsilon_1, \varepsilon_2, \mathcal{F}^{(1)}) = 0 \tag{11.38}$$

Let us consider next the entries $\varepsilon_1, \varepsilon_2, h$. The terms of the first two lines in (11.15) vanish due to (11.38). The last line is

$$\begin{aligned} &\ell_2(\ell_2(\varepsilon_1, \varepsilon_2), h) + \ell_2(\ell_2(h, \varepsilon_1), \varepsilon_2) + \ell_2(\ell_2(\varepsilon_2, h), \varepsilon_1) \\ &= [h^\mu \overset{*}{,} [\varepsilon_1 \overset{*}{,} \varepsilon_2]] - [[h^\mu \overset{*}{,} \varepsilon_1] \overset{*}{,} \varepsilon_2] + [[h^\mu \overset{*}{,} \varepsilon_2] \overset{*}{,} \varepsilon_1] \end{aligned} \tag{11.39}$$

which vanishes due to $*$-Jacobi identity.

Now we consider the entries ε, h_1, h_2. Plugging them into (11.15), the first line vanishes because of (11.38). The rest is

$$
\begin{aligned}
0 \;=\;& \frac{1}{6}\Big(\ell_3(\ell_1(\varepsilon), h_1, h_2) + \mathrm{perm}_3\Big)\\
&+L_3(\varepsilon, \ell_1(h_1), h_2) - L_3(\varepsilon, h_1, \ell_1(h_2))\\
&+\frac{1}{2}\Big(\ell_2(\ell_2(\varepsilon, h_1), h_2) + \ell_2(h_2, \ell_2(\varepsilon, h_1)) - \ell_2(\ell_2(h_2, \varepsilon), h_1)\\
&-\ell_2(h_1, \ell(h_2, \varepsilon)) + \ell_2(\ell_2(h_1, h_2), \varepsilon) + \ell_2(\ell_2(h_2, h_1), \varepsilon)\Big) \qquad (11.40)
\end{aligned}
$$

where perm_3 means the permutation of the three entries of ℓ_3. Writing down explicitly the first line, it takes the form

$$
\frac{1}{6}\left(\ell_3(\ell_1(\varepsilon), h_1, h_2) + \mathrm{perm}_3\right) = -\frac{1}{6}\left(\langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}\,,\, \partial^\nu_x\varepsilon\, h_1^\lambda\, h_2^\rho\rangle\!\rangle + \mathrm{perm}_3\right) \qquad (11.41)
$$

The last two lines of (11.40) give

$$
\begin{aligned}
&\ell_2(\ell_2(\varepsilon, h_1), h_2) + \ell_2(h_2, \ell_2(\varepsilon, h_1)) - \ell_2(\ell_2(h_2, \varepsilon), h_1) - \ell_2(h_1, \ell_2(h_2, \varepsilon)) + \ell_2(\ell_2(h_1, h_2), \varepsilon)\\
&+\ell_2(\ell_2(h_2, h_1), \varepsilon) = +i\Big(\langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}\,,\, [h_1^\nu \overset{*}{,} \varepsilon]h_2^\lambda\rangle\!\rangle + \langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}\,,\, h_2^\lambda[h_1^\nu \overset{*}{,} \varepsilon]\rangle\!\rangle + \langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}\,,\, [h_2^\nu \overset{*}{,} \varepsilon]h_1^\lambda\rangle\!\rangle\\
&+\langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}\,,\, h_1^\lambda[h_2^\nu \overset{*}{,} \varepsilon]\rangle\!\rangle + [\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}\,,\, h_1^\nu\, h_2^\lambda\rangle\!\rangle] + [\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}\,,\, h_2^\lambda\, h_1^\nu\rangle\!\rangle]\Big) \qquad (11.42)
\end{aligned}
$$

Summing the rhs's of (11.41) and (11.42) one gets, apart from the second line, (11.40) expressed in terms of the expressions appearing in the rhs of (11.22) with entries h_1, h_2, instead of one single h. Now let us consider (11.22) for $n = 2$, i.e.

$$
\begin{aligned}
i[\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}\,,\, h^\nu\, h^\lambda\rangle\!\rangle] \;=\;& \frac{1}{3}\langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}\,,\, \partial^\nu_x\varepsilon h^\lambda h^\rho + h^\nu\partial^\lambda_x\varepsilon h^\rho + h^\nu\, h^\lambda\partial^\rho_x\varepsilon\rangle\!\rangle \qquad (11.43)\\
&-i\,\langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}\,,\, [h^\nu \overset{*}{,} \varepsilon]h^\lambda + h^\nu[h^\lambda \overset{*}{,} \varepsilon]\rangle\!\rangle.
\end{aligned}
$$

This can be read as

$$
\begin{aligned}
-i[\varepsilon \overset{*}{,} \ell_2(h, h)] \;=\;& -\frac{1}{3}\Big(\ell_3(\partial_x\varepsilon, h, h) + \ell_3(h, \partial_x\varepsilon, h) + \ell_3(h, h, \partial_x\varepsilon)\Big)\\
&+i\ell_2(h, [h \overset{*}{,} \varepsilon]) + i\ell_2([h \overset{*}{,} \varepsilon], h) \qquad (11.44)
\end{aligned}
$$

Now we consider the same equation obtained by replacing h with $h_1 + h_2$ according to the symmetrization procedure in (11.23). We get in this way the symmetrized equation

$$
\begin{aligned}
&-i[\varepsilon \overset{*}{,} \ell_2(h_1, h_2)] - i[\varepsilon \overset{*}{,} \ell_2(h_2, h_1)] \\
&= -\frac{1}{3}\Big(\ell_3(\partial_x \varepsilon, h_1, h_2) + \ell_3(\partial_x \varepsilon, h_2, h_1) + \ell_3(h_1, \partial_x \varepsilon, h_2) \\
&+\ell_3(h_2, \partial_x \varepsilon, h_1) + \ell_3(h_1, h_2, \partial_x \varepsilon) + \ell_3(h_2, h_1, \partial_x \varepsilon)\Big) \\
&+i\ell_2(h_1, [h_2 \overset{*}{,} \varepsilon]) + i\ell_2(h_2, [h_1 \overset{*}{,} \varepsilon]) + i\ell_2([h_1 \overset{*}{,} \varepsilon], h_2) + i\ell_2([h_2 \overset{*}{,} \varepsilon], h_1)
\end{aligned} \tag{11.45}
$$

This is the same as the sum of the first, third and fourth lines of (11.40), or, alternatively, the sum of the rhs's of (11.41) and (11.42). Thus (11.40) is satisfied if the two remaining terms in the second line vanish. They are all of the type $L_3(\varepsilon, h, \mathcal{F}^{(1)})$ and we can assume that such types of terms vanish. So, beside (11.38) we have

$$
L_3(\varepsilon, h, E) = -L_3(\varepsilon, E, h) = 0 \tag{11.46}
$$

where E represent $\mathcal{F}_\mu$ or anything in X_{-2}.

The relation with entries $\varepsilon_1, \varepsilon_2$ and E is nontrivial and has to be verified. Consider again (11.15) with entries $\varepsilon_1, \varepsilon_2$ and E. Due to (11.38), (11.46) the relation (11.15) reduces to the last line:

$$
\begin{aligned}
&\ell_2(\ell_2(\varepsilon_1, \varepsilon_2), E) + \ell_2(\ell_2(E, \varepsilon_1), \varepsilon_2) + \ell_2(\ell_2(\varepsilon_2, E), \varepsilon_1) \qquad (11.47)\\
&= i\ell_2([\varepsilon_1 \overset{*}{,} \varepsilon_2], E) + i\ell_2([E \overset{*}{,} \varepsilon_1], \varepsilon_2) + i\ell_2([\varepsilon_2 \overset{*}{,} E], \varepsilon_1) \\
&= +[E \overset{*}{,} [\varepsilon_1 \overset{*}{,} \varepsilon_2]] - [[E \overset{*}{,} \varepsilon_1] \overset{*}{,} \varepsilon_2] - [[\varepsilon_2 \overset{*}{,} E] \overset{*}{,} \varepsilon_1]
\end{aligned}
$$

which vanishes because of the $*$-Jacobi identity.

11.2.4 Relation $L_1L_4 - L_2L_3 + L_3L_2 - L_4L_1 = 0$, degree 1

The L_∞ relation to be proved at degree 1 is

$$
\begin{aligned}
& L_1(L_4(x_1,x_2,x_3,x_4)) \\
& -L_2(L_3(x_1,x_2,x_3),x_4) + (-1)^{\mathrm{x}_3\mathrm{x}_4} L_2(L_3(x_1,x_2,x_4),x_3) \\
& +(-1)^{(1+\mathrm{x}_1)\mathrm{x}_2} L_2(x_2, L_3(x_1,x_3,x_4)) - (-1)^{\mathrm{x}_1} L_2(x_1, L_3(x_2,x_3,x_4)) \\
& +L_3(L_2(x_1,x_2),x_3,x_4) + (-1)^{1+\mathrm{x}_2\mathrm{x}_3} L_3(L_2(x_1,x_3),x_2,x_4) \\
& +(-1)^{\mathrm{x}_4(\mathrm{x}_2+\mathrm{x}_3)} L_3(L_2(x_1,x_4),x_2,x_3) \\
& -L_3(x_1,L_2(x_2,x_3),x_4) + (-1)^{\mathrm{x}_3\mathrm{x}_4} L_3(x_1,L_2(x_2,x_4),x_3) + L_3(x_1,x_2,L_2(x_3,x_4)) \\
& -L_4(L_1(x_1),x_2,x_3,x_4) - (-1)^{\mathrm{x}_1} L_4(x_1,L_1(x_2),x_3,x_4) \\
& -(-1)^{\mathrm{x}_1+\mathrm{x}_2} L_4(x_1,x_2,L_1(x_3),x_4) - (-1)^{\mathrm{x}_1+\mathrm{x}_2+\mathrm{x}_4} L_4(x_1,x_2,x_3,L_1(x_4)) = 0
\end{aligned} \tag{11.48}
$$

We have

$$
L_4(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)=0, \quad L_4(\varepsilon_1,\varepsilon_2,\varepsilon_3,h)=0, \quad L_4(\varepsilon_1,\varepsilon_2,h_1,h_2)=0, \quad L_4(\varepsilon,h_1,h_2,h_3)=0 \tag{11.49}
$$

Arguing the same way as for $L_3(\varepsilon_1,\varepsilon_2,\varepsilon_3)=0$, $L_4(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$ has a positive degree and so the first equality vanishes. The second equality also has positive degree and hence it must vanish. The fourth has been proven above, see (11.10). The other is an ansatz to be checked by consistency.

The relation (11.48) with three ε entries and one h is trivial as a consequence of (11.38) and (11.49). The same happens in the case of two ε entries and two h, as a consequence again of (11.38) and (11.49).

Now let us consider the case of one ε and three h's. Plugging them into (11.48) here is what we get in terms of ℓ_i's (only the nonzero terms are written down)

$$
\begin{aligned}
0 \;=\; & -\frac{1}{6}\Big(\ell_2(\varepsilon,\ell_3(h_1,h_2,h_3)) + \mathrm{perm}_3\Big) \\
& +\frac{1}{6}\Big(\ell_3(\ell_2(\varepsilon,h_1),h_2,h_3) + \ell_3(\ell_2(\varepsilon,h_2),h_1,h_3) + \ell_3(\ell_2(\varepsilon,h_3),h_1,h_2) + \mathrm{perm}_3\Big) \\
& -\frac{1}{4!}\Big(\ell_4(\ell_1(\varepsilon),h_1,h_2,h_3) + \mathrm{perm}_4\Big) \\
& -L_4(\varepsilon,\ell_1(h_1),h_2,h_3) + L_4(\varepsilon,h_1,\ell_1(h_2),h_3) - L_4(\varepsilon,h_1,h_2,\ell_1(h_3))
\end{aligned} \tag{11.50}
$$

where $\mathrm{perm}_3, \mathrm{perm}_4$ refer to the permutations of the ℓ_3, ℓ_4 entries, respectively. Disregarding for the moment the last line, which is of type $L_4(\varepsilon, E, h, h)$, this equation becomes

$$\begin{aligned} 0 \;=\; & \frac{i}{6}\Big([\varepsilon \stackrel{*}{,} \langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}\,,\, h_1^\nu h_2^\lambda h_3^\rho \rangle\!\rangle] + \mathrm{perm}(h_1, h_2, h_3) \\ & + \langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}\,,\, [h_1^\nu \stackrel{*}{,} \varepsilon] h_2^\lambda h_3^\rho \rangle\!\rangle + \mathrm{perm}([h_1 \stackrel{*}{,} \varepsilon], h_2, h_3) \\ & + \langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}\,,\, [h_2^\nu \stackrel{*}{,} \varepsilon] h_1^\lambda h_3^\rho \rangle\!\rangle + \mathrm{perm}([h_2 \stackrel{*}{,} \varepsilon], h_1, h_3) \\ & + \langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}\,,\, [h_3^\nu \stackrel{*}{,} \varepsilon] h_1^\lambda h_2^\rho \rangle\!\rangle + \mathrm{perm}([h_3 \stackrel{*}{,} \varepsilon], h_1, h_2)\Big) \\ & - \frac{1}{4!}\Big(\langle\!\langle \mathcal{W}^{(5)}_{\mu\nu\lambda\rho\sigma}\,,\, \partial_x^\nu \varepsilon\, h_1^\lambda h_2^\rho h_3^\sigma \rangle\!\rangle + \mathrm{perm}(\partial_x \varepsilon, h_1, h_2, h_3)\Big) \end{aligned} \tag{11.51}$$

For comparison let us go back to (11.22) with $n = 3$. It writes

$$\begin{aligned} & i[\varepsilon \stackrel{*}{,} \langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}\,,\, h^\nu h^\lambda h^\rho \rangle\!\rangle] \\ & = \frac{1}{4}\langle\!\langle \mathcal{W}^{(5)}_{\mu\nu\lambda\rho\sigma}\,,\, \partial_x^\nu \varepsilon h^\lambda h^\rho h^\sigma + h^\nu \partial_x^\lambda \varepsilon h^\rho h^\sigma + h^\nu h^\lambda \partial_x^\rho \varepsilon h^\sigma + h^\nu h^\lambda h^\rho \partial_x^\sigma \varepsilon \rangle\!\rangle \\ & - i\langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}\,,\, [h^\nu \stackrel{*}{,} \varepsilon] h^\lambda h^\rho + h^\nu [h^\lambda \stackrel{*}{,} \varepsilon] h^\rho + h^\nu h^\lambda [h^\rho \stackrel{*}{,} \varepsilon] \rangle\!\rangle \end{aligned} \tag{11.52}$$

If now we transform the LHS of this equation to a trilinear function of h_1, h_2, h_3 according to the recipe (11.26), we obtain precisely eq. (11.51). As a consequence we are forced to set

$$L_4(\varepsilon, E, h, h) = L_4(\varepsilon, h, E, h) = L_4(\varepsilon, h, h, E) = 0 \tag{11.53}$$

Considering the entries $\varepsilon, \varepsilon, E, h$ in (11.48) one can show that

$$L_4(\varepsilon, \varepsilon, E, h) = 0 \tag{11.54}$$

for consistency. Using this and evaluating (11.48) with entries $\varepsilon, \varepsilon, h, h$, one can see that the third ansatz in (11.49) is justified.

11.2.5 Relation $L_1 L_n + \ldots \pm L_n L_1 = 0$, degree $n - 3$

The general L_∞ relation is (11.6). As the $n = 4$ example shows, for $n \geq 4$ it is consistent to set the values of L_n to zero except when all the entries have degree -1. Schematically, out of (11.6), the only nontrivial relation is

$$-L_n(\varepsilon, L_{n-1}(h,\dots,h)) + L_{n-1}(L_2(\varepsilon,h),h,\dots,h) + (-1)^{n-1}L_n(L_1(\varepsilon),h,\dots,h) = 0 \tag{11.55}$$

Written in explicit form in terms of ℓ_n, it is

$$\begin{aligned}
&-\frac{1}{(n-1)!}\Big(\ell_2(\varepsilon,\ell_{n-1}(h_1,\dots,h_{n-1})) + \mathrm{perm}_{n-1}\Big) \\
&+\frac{1}{(n-1)!}\Big(\ell_{n-1}(\ell_2(\varepsilon,h_1),h_2,\dots,h_{n-1}) + \ell_{n-1}(\ell_2(\varepsilon,h_2),h_1,\dots,h_{n-1}) + \dots \\
&\qquad\qquad + \ell_{n-1}(\ell_2(\varepsilon,h_{n-1}),h_1,\dots,h_{n-2}) + \mathrm{perm}_{n-1}\Big) \\
&+\frac{(-1)^{n-1}}{n!}\Big(\ell_n(\ell_1(\varepsilon),h_1,\dots,h_{n-1}) + \mathrm{perm}_n\Big) = 0
\end{aligned} \tag{11.56}$$

In order to obtain this it is essential to remark that, for entries of degree -1, the factor $(-1)^\sigma\epsilon(\sigma;x)$ in (11.6) is 1.

Using now the definition (11.18) and simplifying, (11.56) becomes

$$\begin{aligned}
&-i\Big([\varepsilon \stackrel{*}{,} \langle\!\langle \mathcal{W}^{(n)}_{\mu\nu_1\dots\nu_{n-1}}, h_1^{\nu_1}\dots h_{n-1}^{\nu_{n-1}}\rangle\!\rangle] + \mathrm{perm}_{n-1}\Big) \\
&+i\Big(\langle\!\langle \mathcal{W}^{(n)}_{\mu\nu_1\dots\nu_{n-1}}, [\varepsilon \stackrel{*}{,} h_1^{\nu_1}]h_2^{\nu_2}\dots h_{n-1}^{\nu_{n-1}}\rangle\!\rangle + \mathcal{W}^{(n)}_{\mu\nu_1\dots\nu_{n-1}}, [\varepsilon \stackrel{*}{,} h_2^{\nu_1}]h_1^{\nu_2}\dots h_{n-1}^{\nu_{n-1}}\rangle\!\rangle \\
&\qquad\qquad + \dots + \mathcal{W}^{(n)}_{\mu\nu_1\dots\nu_{n-1}}, [\varepsilon \stackrel{*}{,} h_{n-1}^{\nu_1}]h_1^{\nu_2}\dots h_{n-2}^{\nu_{n-1}}\rangle\!\rangle + \mathrm{perm}_{n-1}\Big) \\
&+\frac{1}{n}\Big(\langle\!\langle \mathcal{W}^{(n+1)}_{\mu\nu_1\dots\nu_n}, \partial_x^{\nu_1}\varepsilon\, h_1^{\nu_2}h_2^{\nu_3}\dots h_{n-1}^{\nu_n}\rangle\!\rangle + \mathrm{perm}_n\Big) = 0
\end{aligned} \tag{11.57}$$

where perm_{n-1} means the permutations of $h_1,\dots,h_{n-1}$, and perm_n means the permutations of $h_1,\dots,h_{n-1}$ *and* $\partial_x\varepsilon$.

Now, from (11.22) we get

$$\begin{aligned}
&i[\varepsilon \stackrel{*}{,} \langle\!\langle \mathcal{W}^{(n)}_{\mu\nu_1\dots\nu_{n-1}}, h^{\mu_1}\dots h^{\mu_{n-1}}\rangle\!\rangle] - i\sum_{i=1}^{n-1}\langle\!\langle \mathcal{W}^{(n)}_{\mu\mu_1\dots\mu_i\dots\mu_{n-1}}, h^{\mu_1}\dots[\varepsilon \stackrel{*}{,} h^{\mu_i}]\dots h^{\mu_{n-1}}\rangle\!\rangle \\
&-\frac{1}{n}\sum_{i=1}^{n}\langle\!\langle \mathcal{W}^{(n+1)}_{\mu\mu_1\dots\mu_i\dots\mu_n}, h^{\mu_1}\dots\partial_x^{\mu_i}\varepsilon\dots h^{\mu_n}\rangle\!\rangle = 0
\end{aligned} \tag{11.58}$$

If now we transform the LHS of this equation to a multilinear function of $h_1,\dots,h_{n-1}$ according to the recipe (11.26), we obtain precisely (11.57). This completes the proof of the n-th L_∞ relation.

11.3 Curved L_∞ algebra

So far, in this chapter, we assumed that $\mathcal{W}_\mu^{(1)} = 0$. In many cases, however this is not true and we have a cosmological constant term. Here we extend the L_∞ structure of the fermion model to curved L_∞ algebra, see [184]. When cosmological constant term is present $\mathcal{W}_\mu^{(1)} \neq 0$, we have to introduce an additional 'product' L_0, besides the L_n of [33] and in the previous sections. The algebra in this case is called *curved* L_∞. We define L_0 by setting

$$L_0 = \mathcal{W}_\mu^{(1)} \tag{11.59}$$

Both sides of this equation have degree -2, because of the fact that the degree of products L_n is $n-2$. Now L_1 is not nilpotent. In this case, the defining property $L_1^2 = 0$ of the L_∞ algebra is modified as follows

$$L_1\left(L_1(v)\right) + L_2\left(L_0, v\right) = 0 \tag{11.60}$$

where $v \in X = X_0 \oplus X_{-1} \oplus X_{-2}$. This relation is nontrivial only when $v \in X_0$, i.e. when v is ε. We can see that by degree counting. Now using eq.(11.34), and recalling that $L_1(\varepsilon)^\mu(x,u) = \partial_x^\mu \varepsilon(x,u)$ and $L_1(h)_\mu = \langle\langle \mathcal{W}_\mu^{(2)}, h \rangle\rangle$, this equation becomes

$$i[\mathcal{W}_\mu^{(1)}, \varepsilon] + \langle\langle \mathcal{W}_{\mu\nu}^{(2)} h^\nu \rangle\rangle = 0 \tag{11.61}$$

This corresponds to the case $n = 0$ of (11.22). All the other L_∞ relations remain unchanged. For instance, the relation

$$L_3 L_0 - L_2 L_1 + L_1 L_2 = 0 \tag{11.62}$$

is not a priori excluded by the degree counting, however we have proved that $L_3(E, *, *) = 0$ is consistent for E of degree -2.

L_0 is called the *curvature* of the curved L_∞ algebra.

Chapter 12

Conclusion

In this chapter we will discuss our results, give final concluding remarks and give guidelines for future research.

12.1 Comments on the Pontryagin anomaly

We were dealing with odd part of the trace anomaly of a Weyl fermion coupled to curved background. To confirm the result of [15] we used several methods. First, we reconsidered the calculation of [15] and gave a more complete analysis of the latter by including the tadpole and seagull diagrams and came to a conclusion that they do not change the final result, see [19]. We checked trace and diff-Ward identities and we conclude that the parity-odd part of the trace anomaly is given by Pontryagin density which comes from the triangle diagram. In this way we obtain only the lowest order term of the anomaly. To obtain the full anomaly we covariantize the result.

The problem with Weyl fermions lies in the definition of the path integral measure, or better said, a lack of a well defined path integral measure. Let us recall that the path integral measure of a free Dirac fermion can be interpreted as a determinant of the Dirac operator $\not{D}$, that is, the product of its eigenvalues. We come to a similar deduction for a Majorana fermion. However, for Weyl fermion the situation is a bit more complicated. If we choose for the Dirac operator $\not{D}_L = \not{D}P_L$, since Dirac operator anticommutes with γ_5, it maps left-handed fermion to right-handed one, and as a consequence the eigenvalue problem is not well defined in this case. Another idea is to replace $\not{D}_L$ with $\not{D}_L^\dagger \not{D}_L$, but in this case we face a problem of undetermined overall phase factor.

Bearing this in mind, and inspired by Bardeen's method, we propose a solution to this problem following [19]. The main idea is to embed our system in a larger setup: metric-axial-tensor (MAT) gravity. Beside the usual metric $g_{\mu\nu}$ we introduced an additional axial metric $f_{\mu\nu}$ and let them interact with Dirac fermions. Since in this framework we are allowed to use Dirac instead of Weyl fermions, we are able to bypass the problem of the integral measure. Again, using Feynman diagram approach together with dimensional regularization we were able to confirm that the theory of chiral fermions coupled to curved background indeed contains a nonvanishing parity-odd part of the trace anomaly. We obtain the result by taking the collapsing limit $h_{\mu\nu} \to \frac{h_{\mu\nu}}{2}, f_{\mu\nu} \to \frac{h_{\mu\nu}}{2}$ (or $h_{\mu\nu} \to \frac{h_{\mu\nu}}{2}, f_{\mu\nu} \to -\frac{h_{\mu\nu}}{2}$ for the opposite handedness) in the final result. This limit is smooth and we have not found any singularities. Along the way, by taking the suitable collapsing limit ($h_{\mu\nu} \to h_{\mu\nu}, f_{\mu\nu} \to 0$) we proved that for Majorana and Dirac fermions the parity-odd part of the trace anomaly vanishes. Let us mention one more time that with Feynman diagram method we obtain only the lowest order contribution to the anomaly, and the full anomaly is then reconstructed by covariantization. This is correct only if the diffeomorphisms are not broken by the regularization procedure, however, we did not check Ward identities for diffeomorphisms in MAT background. The computation of the latter is extremely complicated in this case.

Instead, we choose to use another method - DeWitt point-splitting method. In this method covariance under diffeomorphisms is guaranteed because the point-splitting is along a geodesic. We showed that the heat kernel method can be extended to MAT gravity. Finally, by taking the appropriate collapsing limits, we again confirm the previous results.

We can conclude that all mentioned methods give the same result: The left-handed Weyl fermion coupled to curved background admits a parity-odd part of the trace anomaly given in terms of Pontyagin density, while the parity-odd part for Majorana and Dirac fermion vanishes, as expected. For right-handed fermion the overall sign of the anomaly is switched.

Let us also mention that a negative result was obtained in [85]. The authors found a vanishing parity-odd contribution to the trace anomaly using Fujikawa method and Pauli-Villars regularization. However, with this method one introduces both chiralities through the path integral measure, even though the action is describing a Weyl fermion. In the anomaly calculation it is essential to avoid mixing of chiralities during the computation. It necessary to keep only one chirality throughout every step of the calculation. That being said, the result of [85] applies to Dirac and Majorana fermions and it is consistent with our results.

Let us mention some characteristics and consequences of the Pontryagin anomaly. Note that in Lorentzian metric, the Pontryagin density comes with an imaginary coefficient. This means that the trace of the energy momentum tensor becomes purely imaginary and as a consequence the Hamiltonian density becomes complex. As long as we are in the effective field theory regime, this is not a problem. On the other hand, if we quantize gravity, in this case unitarity would be broken. This suggests that we should use this anomaly as a selective criterion for theories, because the Pontryagin trace anomaly is present only in theories with chiral imbalance. Let us point out that Pontryagin density vanishes in some particular geometries such as FRW or Schwarzschild.

One important outcome of our computation is the MAT gravity itself, which can be studied on its own as a new bimetric model.

12.2 Comments on effective actions in higher spin theories

Let us give some concluding remarks about the effective action approach to higher spin theories. Our idea was to extract information about the dynamics of the higher spin fields from the quadratic part of the effective action. We coupled a free massive fermion and scalar theory to various external sources using conserved currents and subsequently we used these currents to compute the 2-point correlator. Since we focus on 2-point functions the effective action is quadratic and the equations of motion are linear in the external field. Let us just mention that the choice of currents is not unique, however, we used two particular forms: the simplest symmetric conserved current and a current which becomes traceless in the massless limit.

We expressed our results in terms of conserved structures which turn out to be extremely practical because they make the conservation of the correlators obvious. Our currents are conserved on-shell and as a consequence, the effective action inherits off-shell gauge invariance. Our gauge transformation is linear and the associated parameters are unconstrained as in [39]-[41], [107]-[109]. This motivated us to express our results in the geometric language of [43, 178].

We analysed several examples. To warm up, we started with a massless scalar and fermion model coupled to higher spin fields using simple and traceless currents. In the case of traceless currents, in the parity even sector, we found traceless correlators which in turn give conformal theories.

Next, we coupled massive scalar and fermion model to spin $s = 1, 2, 3$ external fields. One important issue we stumbled upon are the non-conserved and divergent terms in the IR expansion of the 2-point correlators. We found that these terms are local and their number is finite. To extract physical information from the amplitudes, we choose a particular scheme: we subtract all divergent terms (which include non-conserved terms) in the IR from the UV. That is, we subtract a finite number of local counterterms from the action to recover finiteness and conservation. We showed that, for spin 1 and 2, in general subtractions are not necessary, provided we know the full form of minimal coupling and gauge transformation above linear level. For spin 3 or higher we do not know the full gauge transformation and the full interaction of scalar and fermion fields with higher spin fields, and hence, the subtractions are unavoidable. To be precise, for spin 1 and 2, we introduced additional local terms to the interaction so that the effective action is gauge invariant without any subtractions. In this case, the additional terms enter Ward identity in a form of tadpole and seagull terms. Of course, this is not a surprise, because the fully off-shell covariant theories are well known for QED and gravity coupled to scalars and fermions.

Expanding our results in IR and UV for $d = 3, 4$ (for $d = 5, 6$ see [29]) we found that the effective action of any background field is based on the corresponding linearized Fronsdal kinetic operator given in [37, 38], in the nonlocal form introduced by Francia and Sagnotti in [39, 40, 41]. In particular, for the scalar model in both 3d and 4d we find

- for spin 1, Maxwell equation (8.27, 8.37),
- for spin 2, nonlocal version of Einstein-Hilbert (8.60, 8.65)
- and for spin 3, nonlocal Fronsdal operator (8.69, 8.74).

Moreover, for the fermion model

- for spin 1 we found Maxwell equation (8.23, 9.38),

- for spin 2 in 3d we obtained local version of Einstein-Hilbert (9.66) while in 4d we got nonlocal version of Einstein-Hilbert (9.72)
- and for spin 3 we got nonlocal version of the Fronsdal equation (9.79, 9.85).

Besides the correlators of equal spin currents, we also presented some examples of mixed spin correlators. We expect that presence of these terms is necessary in higher spin theories, main motivation for this being the argument that for a consistent higher spin theory we need infinitely many higher spin fields. All of these fields interact with our fermion or scalar model and in turn give a contribution to the mixed spin correlators.

Let us point out one more result. In 3d, upon integrating out the fermion field, we find also parity-odd kinetic terms. In particular, for the traceless currents, in the UV limit mixed spin generalization of a conformal higher spin action (9.104) found in [43, 44]. Recently, in [50]-[52] have been discussed supersymmetric generalizations pointing out dualities and extension to massive higher spin fields.

We also discuss diagonalization of our 2-point correlators, that is, the possibility of vanishing off-diagonal correlators for a particular choice of coefficients in the currents. It turns out that the diagonalization is not possible with the currents (7.26) neither in scalar nor in the fermion case the reason being an infinite number of non-conserved terms, see eq. (8.90). One more example we consider is the case of traceless local currents (traceless even in the massive case) where we are able to diagonalize our results by appropriate choice of coefficients in the currents and by subtraction of finite number of counterterms.

Note that throughout the thesis we have been dealing only with 2-point correlators of higher spin currents. The next logical step would be to compute higher-point correlators which could give us some insight on the non-linear structure of the higher spin fields or we could find obstacles which forbid higher spin couplings.

As we previously mentioned, we do not know the form of gauge transformation beyond the linear order for higher spin fields. Because of that, we turned to the the worldline quantization of a Dirac fermion field coupled to higher spin external sources (scalar case is already worked out in [128]). The advantage of this procedure is that it comes with the exact form of gauge transformation. In this new framework, we gave the perturbative expansion of the effective action (very similar to Feynman diagram approach) and determined the generalized equations of motion.

This allowed us to show that our full one-loop effective action possess a L_∞ symmetry.

We also showed how to realize curved L_∞ algebra in this model. Although we do not give here an explicit proof, the same holds also for the effective action obtained by integrating out a scalar field coupled to higher spin fields. The proof in the scalar case is actually easier, because the corresponding $\mathcal{W}^{(n)}$'s are automatically symmetric, see [128].

In L_∞ symmetry the equation of motion plays the fundamental role, which means that symmetry is dynamical (for an early formulation in this sense, see [183]). of perturbative field theories [53]. For our purposes, we give L_∞ a try to construct higher spin theories by integrating out matter fields.

Our interpretation of L_∞ relations between correlators is that they play a role of Ward identities. To expose L_∞ symmetry we assumed there is no generalized diff-anomalies, however, one has to check that there is no anomalies by explicit calculation. Breakdown of these relations at the quantum level would suggest the presence of anomalies, in other words, possible obstructions in construction of the higher spin theories may appear in the form of anomalies in our approach.

If there is no generalized diff-anomalies, L_∞ algebra could be used to find theories which automatically satisfy L_∞ relations and higher spin gauge invariance. This opens up a new approach to analyze higher spin models.

12.3 Summary

Let us sumarize our main results. First, in [19]-[21] we recalculated the parity odd trace anomaly in 4d in three ways: with Weyl fermions without field redefinition $\psi \to |g|^{\frac{1}{4}}\psi$, using MAT gravity with Dirac fermions and using Schwinger-DeWitt proper-time method where we extended the heat kernel method to MAT gravity. We find that parity-odd part of the trace anomaly is given by Pontryagin density in 4D which supports the statement that Weyl and massless Majorana are not the same beyond classical level. Pontryagin anomaly appears with imaginary coefficient $e = \pm\frac{i}{768\pi^2}$ which could break unitarity. This suggests that this anomaly could be used as a selective criterion for theories.

Second, starting from free quantum theory coupled to external higher spin sources via conserved currents, we find that the effective action, obtained by integrating out the microscopic field, contains information about classical dynamics of sources, see [29]-[32]. We were dealing with 1-pt and 2-pt correlators and consequently the one-loop effective action is quadratic while the equations of motion are linearized. For higher spin fields, after subtraction of finite number of local non-conserved terms, we find gauge invariant effective actions. Next, we used worldline quantization of fermion field coupled to higher spin sources, see [33]. This method comes with the exact form of gauge transformation which enables us to show that the full one-loop effective action has L_∞ symmetry provided there are no generalized diff-anomalies.

Curriculum Vitae

Tamara Štemberga was born on August 29th, 1990 in Pula, Croatia. In 2014 she obtained her Master's degree in Physics at the Department of Physics, Faculty of Science, University of Zagreb, on the subject "Gravitational Chern-Simons terms" under the supervision of doc.dr.sc. Maro Cvitan. That same year she enrolled in doctoral studies in the field of elementary particle physics. Since 2015 she is a research and teaching assistant at the Department of Physics, Faculty of Science, University of Zagreb, working in the group led by doc.dr.sc Maro Cvitan and funded by HRZZ grant "Gravity and parity violation". Her teaching activities include participation in auditory and practical exercises for courses: Fundamentals of Programming, Physical cosmology, General theory of relativity and Advanced Gravity.

List of publications

- L. Bonora, M. Cvitan, P.D. Prester, S. Giaccari, B. L. de Souza, T. Štemberga, *One-loop effective actions and higher spins*, Journal of High Energy Physics **1612** (2016) 084
- L. Bonora, M. Cvitan, P. D. Prester, A. D. Pereira, S. Giaccari, T. Štemberga, *Axial gravity, massless fermions and trace anomalies*, The European Physical Journal C **77** (2017) no. 8, 511
- L. Bonora, M. Cvitan, P. D. Prester, S. Giaccari, T. Štemberga, *One-loop effective actions and higher spins. Part II*, Journal of High Energy Physics **1801** (2018), 080
- L. Bonora, M. Cvitan, P. D. Prester, S. Giaccari, M. Paulišić, T. Štemberga, *Worldline quantization of field theory, effective actions and L_∞ structure*, Journal of High Energy Physics **1804** (2018) 095
- L. Bonora, M. Cvitan, P. D. Prester, S. Giaccari, M. Paulišić, T. Štemberga, *Axial gravity: a non-perturbative approach to split anomalies*, The European Physical Journal C **78** (2018) no. 8, 652

Conference proceedings

- L. Bonora, M. Cvitan, P. D. Prester, S. Giaccari, B. L. de Souza, T. Štemberga, *Massive Dirac field in 3D and induced equations for higher spin fields*, Physical and Mathematical Aspects of Symmetries (2017) 293-298
- P. D. Prester, L. Bonora, M. Cvitan, S. Giaccari, T. Štemberga, *Induced actions for higher spin fields*, Journal of Physics: Conference Series 1051 (1), 012008 (2018)
- L. Bonora, M. Cvitan, P. Dominis Prester, A. D. Pereira, S. Giaccari, T. Štemberga, *Pontryagin trace anomaly*, EPJ Web of Conferences 182, 02100 (2018)

Bibliography

[1] R. A. Bertlmann, *Anomalies in quantum field theory*, Oxford Science Publications, Oxford U.K. (1996).

[2] K. Fujikawa and H. Suzuki, *Path integrals and quantum anomalies*, Oxford Science Publications, Oxford U.K. (2004).

[3] F. Bastianelli and P. Van Nieuwenhuizen, *Path integrals and anomalies in curved space*, Cambridge University Press, Cambridge U.K. (2009).

[4] D. M. Capper and M. J. Duff, *Trace Anomalies in Dimensional Regularization*, Nuovo Cimento **23A** (1974) 173.

[5] S. Deser, M. J. Duff and C. J. Isham, *Non-local Conformal Anomalies*, Nucl. Phys. B **111** (1976) 45.

[6] S.M.Christensen and M.J.Duff, *Axial and conformal anomalies for arbitrary spin in gravity and supergravity*, Phys.Lett. **76B** (1978) 571.

[7] S. M. Christensen and M. J. Duff, *New gravitational index theorems and super theorems*, Nucl.Phys. **B154** (1979) 301.

[8] M. J. Duff, *Observations on Conformal Anomalies*, Nucl. Phys. B **125** (1977) 334.

[9] L. Bonora, P. Cotta-Ramusino and C. Reina, *Conformal Anomaly and Cohomology*, Phys. Lett. B **126** (1983) 305.

[10] L. Bonora, P. Pasti and M. Tonin, *Gravitational and Weyl anomalies*, Phys.Lett. **149B** (1985), 346.

[11] L. Bonora, P. Pasti and M. Tonin, *The anomaly structure of theories with external gravity* J.Math.Phys. **27** (1986), 2259.

[12] L. Bonora, M. Bregola and P. Pasti, *Weyl cocycles*, Class. Quant. Gravity **3** (1986) 635.

[13] B. S. DeWitt, *Dynamical theory of groups and fields*, Gordon and Breach, New York, 1965.

[14] B. S. DeWitt, *Global approach to quantum field theory*, vol.I and II.

[15] L. Bonora, S. Giaccari and B. Lima de Souza, *Trace anomalies in chiral theories revisited,* JHEP **1**407, 117 (2014) [arXiv:1403.2606 [hep-th]].

[16] L. Bonora, S. Giaccari and B. L. D. Souza, *Revisiting Trace Anomalies in Chiral Theories,* Springer Proc. Math. Stat. **111**, 3 (2014).

[17] L. Bonora, A. D. Pereira and B. L. de Souza, *Regularization of energy-momentum tensor correlators and parity-odd terms,* JHEP **1506**, 024 (2015) [arXiv:1503.03326 [hep-th]].

[18] L. Bonora and B. L. de Souza, *Pure contact term correlators in CFT,* Proc. 18th Bled Workshop "What Comes Beyond Standard Models", Bled 2015. arXiv:1511.06635 [hep-th].

[19] L. Bonora, M. Cvitan, P. Dominis Prester, A. Duarte Pereira, S. Giaccari and T. Štemberga, *Axial gravity, massless fermions and trace anomalies,* Eur. Phys. J. C **77** (2017) no.8, 511 [arXiv:1703.10473 [hep-th]].

[20] L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari, M. Paulišić and T. Štemberga, *Axial gravity: a non-perturbative approach to split anomalies,* Eur. Phys. J. C **78** (2018) no.8, 652 [arXiv:1807.01249[hep-th]].

[21] L. Bonora, M. Cvitan, P. Dominis Prester, A. D. Pereira, S. Giaccari and T. Štemberga, *Pontryagin trace anomaly*, EPJ Web of Conferences 182, 02100 (2018).

[22] J. M. Maldacena and G. L. Pimentel, *On graviton non-Gaussianities during inflation,* JHEP **1109** (2011) 045, [arXiv:1104.2846 [hep-th]].

[23] X. O. Camanho, J. D. Edelstein, J. Maldacena and A. Zhiboedov, *Causality Constraints on Corrections to the Graviton Three-Point Coupling,* JHEP **1602** (2016) 020 [arXiv:1407.5597 [hep-th]].

[24] M. A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions,* Phys. Lett. B **243** (1990) 378.

[25] M. A. Vasiliev, *Properties of equations of motion of interacting gauge fields of all spins in (3+1)-dimensions,* Class. Quant. Grav. **8** (1991) 1387.

[26] M. A. Vasiliev, *Algebraic aspects of the higher spin problem,* Phys. Lett. B **257** (1991) 111.

[27] M. A. Vasiliev, *More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions,* Phys. Lett. B **285** (1992) 225.

[28] L. Bonora, M. Cvitan, P. Dominis Prester, B. Lima de Souza and I. Smolić, *Massive fermion model in 3d and higher spin currents,* JHEP **1605** (2016) 072 [arXiv:1602.07178 [hep-th]].

[29] L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari, B. Lima de Souza and T. Štemberga, *One-loop effective actions and higher spins,* JHEP **1612** (2016) 084 arXiv:1609.02088 [hep-th].

[30] L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari, and T. Štemberga, *One-loop effective actions and higher spins. II*, JHEP **1801** (2018) 080, [arXiv:1709.01738 [hep-th]].

[31] L. Bonora, M. Cvitan, P. D. Prester, S. Giaccari, B. L. de Souza and T. Štemberga, *Massive Dirac field in 3D and induced equations for higher spin fields*, Physical and Mathematical Aspects of Symmetries, 293-298.

[32] P. D. Prester, L. Bonora, M. Cvitan, S. Giaccari and T. Štemberga, *Induced actions for higher spin fields*, Journal of Physics: Conference Series 1051 (1), 012008.

[33] L. Bonora , M. Cvitan , P. Dominis Prester , S. Giaccari , M. Paulišić and T. Štemberga, *Worldline quantization of field theory, effective actions and L_∞ structure,* JHEP **1804** (2018) 095 [arXiv:1802.02968 [hep-th]].

[34] E. E. Boos and A. I. Davydychev. *A Method of evaluating massive Feynman integrals*, Theor. Math. Phys., **89** (1991) 1052 [Teor. Mat. Fiz.89,56(1991)].

[35] A. I. Davydychev, *A Simple formula for reducing Feynman diagrams to scalar integrals*, Phys. Lett. **B263** (1991) 107.

[36] A. I. Davydychev, *Recursive algorithm of evaluating vertex type Feynman integrals*, J. Phys., A25 (1992) 5587.

[37] C. Fronsdal, *Massless Fields with Integer Spin,* Phys. Rev. D **18** (1978) 3624.

[38] J. Fang and C. Fronsdal, *Massless Fields with Half Integral Spin,* Phys. Rev. D **18** (1978) 3630.

[39] D. Francia and A. Sagnotti, *On the geometry of higher spin gauge fields,* Class. Quant. Grav. **20** (2003) S473 [Comment. Phys. Math. Soc. Sci. Fenn. **166** (2004) 165] [PoS JHW **2003** (2003) 005] [hep-th/0212185].

[40] D. Francia and A. Sagnotti, *Free geometric equations for higher spins,* Phys. Lett. B **543** (2002) 303 [hep-th/0207002].

[41] D. Francia and A. Sagnotti, *Minimal local Lagrangians for higher-spin geometry*, Phys. Lett. B **624** (2005) 93-104, [hep-th/0507144].

[42] D. M. Capper, *"On Quantum Corrections To The Graviton Propagator,"* Nuovo Cim. A **25** (1975) 29.

[43] T. Damour and S. Deser, *'Geometry' of Spin 3 Gauge Theories*, Annales Poincare Phys. Theor. **47** (1987) 277.

[44] C. N. Pope and P. K. Townsend, *Conformal higher spins in (2+1) dimensions*, Phys.Lett. **B225** (1989) 245.

[45] M. Henneaux, S. Hortner and A. Leonard, *"Higher Spin Conformal Geometry in Three Dimensions and Prepotentials for Higher Spin Gauge Fields,"* JHEP **1601** (2016) 073, [arXiv:1511.07389 [hep-th]].

[46] H. Linander and B. E. W. Nilsson, *The non-linear coupled spin 2 - spin 3 Cotton equation in three dimensions,* JHEP **1607**, 024 (2016), [arXiv:1602.01682 [hep-th]].

[47] T. Basile, R. Bonezzi and N. Boulanger, *The Schouten tensor as a connection in the unfolding of 3D conformal higher-spin fields,* JHEP **1704**, 054 (2017), [arXiv:1701.08645 [hep-th]].

[48] E. A. Bergshoeff, O. Hohm and P. K. Townsend, *On Higher Derivatives in 3D Gravity and Higher Spin Gauge Theories,* Annals Phys. **325**, 1118 (2010), [arXiv:0911.3061 [hep-th]].

[49] B. E. W. Nilsson, *Towards an exact frame formulation of conformal higher spins in three dimensions,* JHEP **1509**, 078 (2015), [arXiv:1312.5883 [hep-th]].

[50] S. M. Kuzenko and D. X. Ogburn, *Off-shell higher spin N=2 supermultiplets in three dimensions,* Phys. Rev. D **94** (2016) no.10, 106010, [arXiv:1603.04668 [hep-th]].

[51] S. M. Kuzenko, *Higher spin super-Cotton tensors and generalisations of the linear-chiral duality in three dimensions,* Phys. Lett. B **763** (2016) 308, [arXiv:1606.08624 [hep-th]].

[52] S. M. Kuzenko and M. Tsulaia, *Off-shell massive N=1 supermultiplets in three dimensions,* Nucl. Phys. B **914** (2017) 160, [arXiv:1609.06910 [hep-th]].

[53] O. Hohm and B. Zwiebach, L_∞ *Algebras and Field Theory,* Fortsch. Phys. **65** (2017) no.3-4, 1700014 doi:10.1002/prop.201700014 [arXiv:1701.08824 [hep-th]].

[54] A. Andrianov, L. Bonora, *Finite-mode regularization of the fermion functional integral.I*, Nucl.Phys. **B233** (1984), 232.

[55] A. Andrianov, L. Bonora, *Finite-mode regularization of the fermion functional integral.II*, Nucl.Phys. **B233** (1984), 247.

[56] T. Kimura, *Divergence of Axial-Vector Current in the Gravitational Field*, Prog. Theor. Phys. **42** (1969) 1191.

[57] R. Delbourgo and A.Salam *PCAC anomalies and Gravitation* preprint IC/72/86.

[58] R. Delbourgo and A.Salam *The gravitational correction to PCAC*, Phys.Lett. **40B** (1972) 381.

[59] D. M. Capper and M. J. Duff, *Trace Anomalies in Dimensional Regularization*, Nuovo Cim. **23 A** (1974) 173.

[60] D. M. Capper and M. J. Duff, *Conformal anomalies and the renormalizability problem in quantum gravity*, Phys. Lett. **53 A** (1975) 361.

[61] S. Deser, M. J. Duff and C. J. Isham, *Non-local conformal anomalies*, Nucl. Phys. B **111** (1976) 45.

[62] C. Bernard and A. Duncan, *Regularization and renormalization of quantum field theory in curved space-time*, Ann. Phys. **107** (1977) 201.

[63] L. S. Brown, *Stress-tensor trace anomaly ina a gravitational metric: scalar fields*, Phys. Rev. **D 15** (1977) 1469.

[64] L. S. Brown and J. P. Cassidy, *Stress-tensor trace anomaly in a gravitational metric: General theory, Maxwell field*, Phys. Rev. **D 15** (1977) 2810.

[65] L. S. Brown and J. P. Cassidy, *Stress tensors and their trace anomalies in conformally flat space-time*, Phys. Rev. **D 16** (1977) 1712.

[66] S. M. Christensen, *Vacuum expectation value of the stress tensor in an arbitrary curved background: The covariant point-separation method*, Phys. Rev. **D 14** (1976) 2490.

[67] S. M. Christensen, *Regularization, renormalization and covariant geodesic point separation*, Phys. Rev. **D17** (1978) 946.

[68] S. L. Adler, J. Lieberman, Y. J. Ng, *Regularization of the stress-energy tensor for vector and scalar particles propagating in a general background metric*, Ann. Phys. **106** (1977) 209.

[69] M. J. Duff, *Observations on conformal anomalies*, Nucl. Phys. **B125** (1977) 334.

[70] J. S. Dowker and R. Critchley. *Stress-tensor conformal anomaly for scalar, spinor, and vector fields*, Phys. Rev. **D 16** (1977) 3390.

[71] H. -S. Tsao, *Conformal anomaly in a general background metric*, Phys. Lett. **68B** (1977) 79.

[72] A. Vilenkin, *Pauli-Villars Regularization and Trace Anomalies*, Nuovo Cim. **A44** (1978) 441.

[73] R. M. Wald, *Axiomatic renormalization of the stress tensor of a conformally invariant field in conformally flat spacetimes*, Ann. Phys. **110** (1978) 472.

[74] R. M. Wald, *Trace anomaly of a conformally invariant quantum field in curved space-time*, Phys. Rev. **D 17** (1978) 1477.

[75] H. Osborn, *Weyl consistency conditions and a local renormalization group equation for general renormalizable field theoriees*, Nucl. Phys. **B363** (1991) 486.

[76] H. Osborn and A. C. Petkos, *Implications of conformal invariance in field theories for general dimensions,* Annals Phys. **231** (1994) 311 [hep-th/9307010].

[77] M. J. Duff, *Twenty years of the Weyl anomaly,* Class. Quant. Grav. **11** (1994) 1387 [hep-th/9308075].

[78] S. Deser and A. Schwimmer, *Geometric classification of conformal anomalies in arbitrary dimensions,* Phys. Lett. B **309** (1993) 279 [hep-th/9302047].

[79] A. Bzowski, P. McFadden and K. Skenderis, *Renormalised 3-point functions of stress tensors and conserved currents in CFT,* arXiv:1711.09105 [hep-th].

[80] H. Godazgar and H. Nicolai, *A rederivation of the conformal anomaly for spin-1/2,* Class. Quant. Grav. **35** (2018) no.10, 105013 [arXiv:1801.01728 [hep-th]].

[81] C. Coriano and M. M. Maglio, *Renormalization, Conformal Ward Identities and the Origin of a Conformal Anomaly Pole,* Phys. Lett. B **781** (2018) 283 [arXiv:1802.01501 [hep-th]].

[82] Y. Nakayama, *CP-violating CFT and trace anomaly,* Nucl. Phys. B **859** (2012) 288.

[83] Y. Nakayama, *On the realization of impossible anomalies,* arXiv:1804.02940 [hep-th].

[84] S. Mauro and I. L. Shapiro, *Anomaly-induced effective action and Chern-Simons modification of general relativity,* Phys. Lett. B **746** (2015) 372 [arXiv:1412.5002 [gr-qc]].

[85] F. Bastianelli and R. Martelli, *On the trace anomaly of a Weyl fermion,* JHEP **1611** (2016) 178 doi:10.1007/JHEP11(2016)178 [arXiv:1610.02304 [hep-th]].

[86] W.A. Bardeen, *Anomalous Ward Identities in Spinor Field Theories*, Phys. Rev. **184** 1848 (1969).

[87] A . Crumeyrolle, *Variétés différentiables à coordonnées hypercomplexes. Application à une géométrisation et à une généralisation de la théorie d'Einstein-Schrodinger.* Ann. de la Fac. des Sciences de Toulouse, 4^{e} série, **26** (1962) 105.

[88] A . Crumeyrolle, *Sur quelques interprétations physiques et théoriques des equations du champ unitaire d'Einstein-Schödinger,* Riv. Mat. Univ. Parma (2) **5** (1964) 85.

[89] R. -L. Clerc, *Résolution des équations aux connexions du cas antisymétrique de la théorie unitaire hypercomplexe. Application à un principe variationnel*, Ann. de l'I.H.P. Section A, **12** no. 4, (1970) 343.

[90] R. -L. Clerc, *Équations de champ symétriques et équations du mouvement sur une variété pseudo-riemannienne à connexion non symétrique*, Ann. de l'I.H.P. Section A, **17** no. 3, (1972) 227.

[91] C. Mantz and T. Prokopec, *Hermitian Gravity and Cosmology,* arXiv:0804.0213 [gr-qc].

[92] P. O. Hess and W. Greiner, *Pseudo-complex General Relativity,* Int. J. Mod. Phys. E **18**, 51 (2009) [arXiv:0812.1738 [gr-qc]].

[93] P. O. Hess and W. Greiner, *Pseudo-Complex Field Theory,* Int. J. Mod. Phys. E **16**, 1643 (2007) [arXiv:0705.1233 [hep-th]].

[94] D. M. Grabowska and D. B. Kaplan, *Chiral solution to the Ginsparg-Wilson equation,* Phys. Rev. D **94** (2016) no.11, 114504 [arXiv:1610.02151 [hep-lat]].

[95] P. A. M. Dirac, *Relativistic wave equations*, Proc. Roy. Soc. Lond. **155A**, 447 (1936).

[96] M. Fierz and W. Pauli, *On relativistic wave equations for particles of arbitrary spin in an electromagnetic field*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences (1939): 211-232.

[97] L. P. S. Singh and C. R. Hagen, *Lagrangian formulation for arbitrary spin. 1. The boson case.* Phys. Rev. D **9**, 898 (1974).

[98] L. P. S. Singh and C. R. Hagen, *Lagrangian formulation for arbitrary spin. 2. The fermion case,* Phys. Rev. D **9** (1974) 910.

[99] *Higher-Spin Gauge Theories*, Proceedings of the First Solvay Workshop, held in Brussels on May 12-14, 2004, eds. R. Argurio, G. Barnich, G. Bonelli and M. Grigoriev (Int. Solvay Institutes, 2006).

[100] D. Sorokin, *Introduction to the classical theory of higher spins*, AIP Conf. Proc. **767**, 172 (2005) [hep-th/0405069].

[101] D. Francia and A. Sagnotti, *Higher-spin geometry and string theory*, J. Phys. Conf. Ser. **33**, 57 (2006) [hep-th/0601199].

[102] A. Fotopoulos and M. Tsulaia, *Gauge Invariant Lagrangians for Free and Interacting Higher Spin Fields. A Review of the BRST formulation*, Int. J. Mod. Phys. A **24**, 1 (2009) [arXiv:0805.1346 [hep-th]].

[103] C. Iazeolla, *On the Algebraic Structure of Higher-Spin Field Equations and New Exact Solutions*, arXiv:0807.0406 [hep-th].

[104] A. Campoleoni, *Metric-like Lagrangian Formulations for Higher-Spin Fields of Mixed Symmetry*, Riv. Nuovo Cim. **33**, 123 (2010) [arXiv:0910.3155 [hep-th]].

[105] A. Sagnotti, *Higher Spins and Current Exchanges*, PoS CORFU **2011**, 106 (2011) [arXiv:1002.3388 [hep-th]].

[106] D. Francia, *Low-spin models for higher-spin Lagrangians*, Prog. Theor. Phys. Suppl. **188**, 94 (2011) [arXiv:1103.0683 [hep-th]].

[107] D. Francia, J. Mourad and A. Sagnotti, *Current Exchanges and Unconstrained Higher Spins,* Nucl. Phys. **B 773** (2007) 203 [hep-th/0701163].

[108] D. Francia, *On the Relation between Local and Geometric Lagrangians for Higher spins,* J. Phys. Conf. Ser. **222** (2010) 012002 [arXiv:1001.3854 [hep-th]].

[109] D. Francia, *Generalised connections and higher-spin equations,* Class. Quant. Grav. **29** (2012) 245003 [arXiv:1209.4885 [hep-th]].

[110] A. D. Sakharov, *Vacuum quantum fluctuations in curved space and the theory of gravitation,* Sov. Phys. Dokl. **12** (1968) 1040 [Dokl. Akad. Nauk Ser. Fiz. **177** (1967) 70] [Sov. Phys. Usp. **34** (1991) 394] [Gen. Rel. Grav. **32** (2000) 365].

[111] J. Maldacena and A. Zhiboedov, *Constraining Conformal Field Theories with A Higher Spin Symmetry* J. Phys. A **46** (2013) 214011 [arXiv:1112.1016 [hep-th]].

[112] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a slightly broken higher spin symmetry,* Class. Quant. Grav. **30** (2013) 104003. [arXiv:1204.3882 [hep-th]].

[113] A. Campoleoni, *Higher Spins in D = 2 + 1,* Subnucl. Ser. **49** (2013) 385 [arXiv:1110.5841 [hep-th]].

[114] C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski and N. Seiberg, *Comments on Chern-Simons Contact Terms in Three Dimensions*, JHEP **1209**, 091 (2012) [arXiv:1206.5218 [hep-th]].

[115] S. Giombi, S. Prakash and X. Yin, *A note on CFT correlators in Three Dimensions*, JHEP **1307**, 105 (2013) [arXiv:1104.4317 [hep-th]].

[116] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia and X. Yin, *Chern-Simons Theory with Vector Fermion Matter,* Eur. Phys. J. C **72**, 2112 (2012) [arXiv:1110.4386 [hep-th]].

[117] A. Y. Segal, *Conformal higher spin theory,* Nucl. Phys. B **664** (2003) 59 [hep-th/0207212].

[118] A. Y. Segal, *Point particle in general background fields vsersus gauge theories of traceless symmetric tensors,* Int. J. Mod. Phys. A **18** (2003) 4999 [hep-th/0110056].

[119] F. S. Gama, J. R. Nascimento and A. Y. Petrov, *Derivative expansion and the induced Chern-Simons term in N=1, d=3 superspace,* Phys. Rev. D **93**, no. 4, 045015 (2016) [arXiv:1511.05471 [hep-th]].

[120] R. Jackiw and S. Templeton, *How Superrenormalizable Interactions Cure their Infrared Divergences,* Phys. Rev. D **23** (1981) 2291.

[121] I. Vuorio, *Parity Violation and the Effective Gravitational Action in Three-dimensions,* Phys. Lett. B **175** (1986) 176.

[122] T. Appelquist, M. J. Bowick, E. Cohler and L. C. R. Wijewardhana, *Chiral Symmetry Breaking in (2+1)-dimensions,* Phys. Rev. Lett. **55** (1985) 1715.

[123] T. W. Appelquist, M. J. Bowick, D. Karabali and L. C. R. Wijewardhana, *Spontaneous Chiral Symmetry Breaking in Three-Dimensional QED,* Phys. Rev. D **33** (1986) 3704.

[124] T. Appelquist, M. J. Bowick, D. Karabali and L. C. R. Wijewardhana, *Spontaneous Breaking of Parity in (2+1)-dimensional QED,* Phys. Rev. D **33** (1986) 3774.

[125] K. S. Babu, A. K. Das and P. Panigrahi, *Derivative Expansion and the Induced Chern-simons Term at Finite Temperature in (2+1)-dimensions,* Phys. Rev. D **36** (1987) 3725.

[126] R. Delbourgo and A. B. Waites, *Induced parity violation in odd dimensions,* Austral. J. Phys. **47** (1994) 465.

[127] G. V. Dunne, *Aspects of Chern-Simons theory,* hep-th/9902115.

[128] X. Bekaert, E. Joung and J. Mourad, *Effective action in a higher-spin background,* JHEP **1102**, 048 (2011), [arXiv:1012.2103 [hep-th]].

[129] M. J. Strassler, *Field theory without Feynman diagrams: One loop effective actions,* Nucl. Phys. B **385** (1992) 145 [hep-ph/9205205].

[130] M. G. Schmidt and C. Schubert, *The Worldline path integral approach to Feynman graphs,* hep-ph/9412358.

[131] M. G. Schmidt and C. Schubert, *Worldline Green functions for multiloop diagrams,* Phys. Lett. B **331** (1994) 69 doi:10.1016/0370-2693(94)90944-X [hep-th/9403158].

[132] C. Schubert, *An Introduction to the worldline technique for quantum field theory calculations,* Acta Phys. Polon. B **27** (1996) 3965 [hep-th/9610108].

[133] E. D'Hoker and D. G. Gagne, *Worldline path integrals for fermions with general couplings,* Nucl. Phys. B **467** (1996) 297 [hep-th/9512080].

[134] F. Bastianelli and A. Zirotti, *Worldline formalism in a gravitational background,* Nucl. Phys. B **642** (2002) 372 [hep-th/0205182].

[135] F. Bastianelli, O. Corradini and A. Zirotti, *BRST treatment of zero modes for the worldline formalism in curved space,* JHEP **0401** (2004) 023 [hep-th/0312064].

[136] F. Bastianelli, P. Benincasa and S. Giombi, *Worldline approach to vector and antisymmetric tensor fields,* JHEP **0504** (2005) 010 [hep-th/0503155].

[137] P. Dai and W. Siegel, *Worldline Green Functions for Arbitrary Feynman Diagrams,* Nucl. Phys. B **770** (2007) 107 [hep-th/0608062].

[138] R. Bonezzi, *Induced Action for Conformal Higher Spins from Worldline Path Integrals,* Universe **3** (2017) no.3, 64 [arXiv:1709.00850 [hep-th]].

[139] M. R. Gaberdiel and B. Zwiebach, *Tensor constructions of open string theories. 1: Foundations,* Nucl. Phys. B **505** (1997) 569 [hep-th/9705038].

[140] B. Zwiebach, *Oriented open - closed string theory revisited,* Annals Phys. **267** (1998) 193 [hep-th/9705241].

[141] H. Kajiura and J. Stasheff, *Homotopy algebras inspired by classical open-closed string field theory,* Commun. Math. Phys. **263** (2006) 553 [math/0410291 [math-qa]].

[142] T. Lada and J. Stasheff, *Introduction to SH Lie algebras for physicists,* Int. J. Theor. Phys. **32** (1993) 1087 [hep-th/9209099].

[143] T. Lada and M. Markl, *Strongly homotopy Lie algebras,* hep-th/9406095.

[144] G. Barnich, R. Fulp, T. Lada and J. Stasheff, *The sh Lie structure of Poisson brackets in field theory,* Commun. Math. Phys. **191** (1998) 585 [hep-th/9702176].

[145] A. M. Zeitlin, *Homotopy Lie Superalgebra in Yang-Mills Theory,* JHEP **0709** (2007) 068 [arXiv:0708.1773 [hep-th]].

[146] A. M. Zeitlin, *String field theory-inspired algebraic structures in gauge theories,* J. Math. Phys. **50** (2009) 063501 [arXiv:0711.3843 [hep-th]].

[147] A. M. Zeitlin, *Conformal Field Theory and Algebraic Structure of Gauge Theory,* JHEP **1003** (2010) 056 [arXiv:0812.1840 [hep-th]].

[148] D. Gaiotto, G. W. Moore and E. Witten, *Algebra of the Infrared: String Field Theoretic Structures in Massive $\mathcal{N} = (2,2)$ Field Theory In Two Dimensions,* arXiv:1506.04087 [hep-th].

[149] R. Blumenhagen, M. Fuchs and M. Traube, *W algebras are L_∞ algebras,* JHEP **1707** (2017) 060 [arXiv:1705.00736 [hep-th]].

[150] R. Blumenhagen, M. Fuchs and M. Traube, *On the Structure of Quantum L_∞ algebras,* JHEP **1710** (2017) 163 [arXiv:1706.09034 [hep-th]].

[151] M. J. Duff *Twenty years of the Weyl anomaly.* In: Class. Quant. Grav. **11** (1994), pp. 1387-1404. arXiv: hep-th/9308075 [hep-th].

[152] P. B. Pal, *Dirac, Majorana and Weyl fermions*, arXiv:1006.1718 [hep-th].

[153] L. E. Parker, *Quantum field theory in curved spacetime: quantized fields and gravity*, Cambridge University Press, 2009.

[154] S. Weinberg, *Photons and gravitons in S-matrix theory: derivation of charge conservation and equality of gravitational and inertial mass*, Phys. Rev. **135** B1049 (1964): B1049.

[155] S. R. Coleman, J. Mandula, Phys. Rev. **159** (1967) 1251-1256.

[156] C. Aragone and S. Deser, *Consistency problems of hypergravity*, Phys. Lett. B **86** (1979): 161-163.

[157] S. Weinberg and E. Witten, *Limits on massless particles*, Phys. Lett. B **96** (1980): 59-62.

[158] X. Bekaert, N. Boulanger and P. Sundell, *How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples,* Rev. Mod. Phys. **84**, 987 (2012) [arXiv:1007.0435 [hep-th]].

[159] R. Rahman, *Higher Spin Theory - Part I*,[arXiv:1307.3199[hep-th]].

[160] M. Porrati, *Universal Limits on Massless High-Spin Particles*, Phys. Rev. D **78**, 065016 (2008) [arXiv:0804.4672 [hep-th]].

[161] A. Sagnotti, *Notes on Strings and Higher Spins*, J. Phys. A **46**, 214006 (2013) [arXiv:1112.4285 [hep-th]].

[162] D. J. Gross and P. F. Mende, *The High-Energy Behavior of String Scattering Amplitudes*, Phys. Lett. B 197, 129 (1987).

[163] D. J. Gross and P. F. Mende *String Theory Beyond the Planck Scale*, Nucl. Phys. B **303**, 407 (1988).

[164] D. J. Gross, *High-Energy Symmetries of String Theory*, Phys. Rev. Lett. **60**, 1229 (1988).

[165] D. Amati, M. Ciafaloni and G. Veneziano, *Superstring Collisions at Planckian Energies*, Phys. Lett. B **197**, 81 (1987).

[166] D. Amati, M. Ciafaloni and G. Veneziano, *Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions*, Int. J. Mod. Phys. A **3**, 1615 (1988).

[167] D. Amati, M. Ciafaloni and G. Veneziano, *Can Space-Time Be Probed Below the String Size?*, Phys. Lett. B **216**, 41 (1989).

[168] E. S. Fradkin and M. A. Vasiliev, *On The Gravitational Interaction Of Massless Higher Spin Fields*, Phys. Lett. B **189**, 89 (1987).

[169] E. S. Fradkin and M. A. Vasiliev, *Cubic Interaction In Extended Theories Of Massless Higher Spin Fields*, Nucl. Phys. B **291**, 141 (1987).

[170] M. T. Grisaru and H. N. Pendleton, *Soft Spin 3/2 Fermions Require Gravity And Supersymmetry*, Phys. Lett. B **67**, 323 (1977).

[171] M. T. Grisaru, H. N. Pendleton and P. van Nieuwenhuizen, *Supergravity and the S matrix* Phys. Rev. D **15**, 996 (1977).

[172] F. Loebbert, *The Weinberg-Witten theorem on massless particles: An Essay*, Annalen Phys. **17** (2008) 803-829.

[173] S. Weinberg, *Quantum Theory of Fields, Vol. 1*, (1995).

[174] N. Bouatta, G. Compere, A. Sagnotti, *An Introduction to Free Higher-Spin Fields*, arXiv:hep-th/0409068.

[175] P. Kessel, *The Very Basics of Higher-Spin Theory*, arXiv:1702.03694 [hep-th].

[176] E. P. Wigner, *On Unitary Representations Of The Inhomogeneous Lorentz Group*, Annals Math. **40**, 149 (1939) [Nucl. Phys. Proc. Suppl. 6, 9 (1989)].

[177] V. Bargmann and E. P. Wigner, *Group Theoretical Discussion Of Relativistic Wave Equations,* Proc. Nat. Acad. Sci. **34**, 211 (1948).

[178] B. de Wit and D. Z. Freedman, *Systematics of Higher Spin Gauge Fields,* Phys. Rev. D **21** (1980) 358.

[179] X. Bekaert and N. Boulanger, *"On geometric equations and duality for free higher spins,"* Phys. Lett. B **561** (2003) 183, [hep-th/0301243].

[180] X. Bekaert and N. Boulanger, *"Tensor gauge fields in arbitrary representations of GL(D,R). II. Quadratic actions,"* Commun. Math. Phys. **271** (2007) 723, [hep-th/0606198].

[181] B. L. de Souza, https://github.com/blimasouza/DavyFeyn

[182] M. P. Blencowe, *A consistent interacting massless higher-spin field theory in D=2+1* Class.Quant.Grav. **6** (1989) 443.

[183] F.A.Berends, G.J.H. Burgers and H. Van Dam, *On the theoretical problems in constructing interactions involving higher-spin massless particles*, Nucl. Phys. **B260** (1985) 295-322.

[184] L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari and T. Štemberga, *HS in flat spacetime. The effective action method*, arXiv: 1811.04847 [hep-th].