Sferno simetrično trodimenzionalno nestacionarno gibanje mikropolarnog kompresibilnog viskoznog fluida

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PRIRODOSLOVNO-MATEMATIČKI FAKULTET

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SFERNO SIMETRIČNO TRODIMENZIONALNO NESTACIONARNO GIBANJE MIKROPOLARNOG KOMPRESIBILNOG VISKOZNOG FLUIDA

DOKTORSKI RAD

Zagreb, 2014.

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Temelj ove disertacije dugogodišnja je suradnja s prof. dr. sc. Nerminom Mujaković, započeta još na prvoj godini studija matematike kada me je svojim izvrsnim predavanjima usmjerila ka području matematičke analize. Neizmjerno sam joj zahvalan na tome što me je uočila i poticala, a kasnije i nesebično vodila kroz proces istraživanja vezan uz ovu disertaciju. Veliko hvala i mojem drugom mentoru, prof. dr. sc. Zvonimiru Tuteku na pomoći, korisnim savjetima i sugestijama.

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Sadržaj

1 Uvod u problematiku istraživanja

Svaki materijal (kruto tijelo, tekućina ili plin) koji se susreće u prirodi i tehnici sastoji se od malih čestica (npr. molekula) odvojenih prazninama. Precizno matematičko modeliranje materijala, uzimajući u obzir praznine između čestica prilično je složeno. Zbog toga se promatra idealizirani model (tzv. kontinuum) koji pretpostavlja da su čestice unutar materijala neprekidno distribuirane i ispunjavaju čitavo područje u kojem se nalaze, što omogućava da se materijal može rastaviti na infinitezimalno male dijelove koji zadržavaju svojstva materijala.

Klasična mehanika kontinuuma promatra deformiranje i gibanje materijalnog tijela samo na makrorazini, dok zanemaruje mikrogibanja i mikrodeformacije svake pojedine čestice. U današnjoj znanosti proučavaju se tvari ili pak smjese tvari koje klasična mehanika kontinuuma ne može opisati na zadovoljavajući način, kao što su primjerice tekući kristali, krv, oblaci dima, i sl. Ono što je zajedničko navedenim materijalima je mikrostruktura koja se ne može zanemariti. Kontinuum sa mikrostrukturom, tj. kontinuum kod kojeg se promatraju gibanja i deformacije kako na makro, tako i na mikro razini zovemo mikrokontinuum. Teoriju mikrokontinuma razvio je sredinom šezdesetih godina prošlog stoljeća Ahmed Cemal Eringen ([Eri64], [Eri66], [Eri99]).

Kod klasičnog kontinuuma infinitezimalnoj materijalnoj čestici pridružena je njena prostorna pozicija u određenom vremenskom trenutku. Kako bi opisao mikrodeformacije i mikrogibanja Eringen materijalnoj točki, čiji je položaj u euklidskom prostoru $\mathbb{R}^3$ određen nekim vektorom pridružuje novi vektor koji sadrži informacije o orijentaciji i deformaciji točke na mikrorazini, što za posljedicu ima uvođenje pojma mikrodeformacijskog tenzora koji se često naziva direktor.

Eringenov direktor u najopćenitijem slučaju sadrži devet nezavisnih komponenti, tri za mikrorotaciju i šest za mikrodeformaciju. Ako u konstitutivne jednadžbe uvedemo svih devet komponenti bez ograničenja govorimo o mikromorfnom kontinuumu. Mikromorfni kontinuum je tako praktički univerzalan u opisivanju materijala, međutim, zbog svoje složenosti, nema praktičnu primjenu pa se proučavaju njegovi specijalni slučajevi. Jedan od najpoznatijih specijalnih slučajeva mikromorfnog kontinuuma je upravo mikropolaran kontinuum koji je i predmet razmatranja ovog rada. Kod mikropolarnog kontinuuma su direktori ortonormalni i kruti, odnosno ne dozvoljavaju se mikrodeformacije vec samo mikrorotacije čestica kontinuuma.

U ovom se radu mikropolarni kontinuum promatra u vidu izotropnog kompresibilnog mikropolarnog fluida. Mikropolarni fluid ima široku primjenu primjerice kod modeliranja tekucih kristala, fluida sa magnetnim svojstvima, oblaka prašine, smoga ili pak nekih bioloških fluida.

Matematička analiza modela mikropolarnih fluida počela se istraživati sedamdesetih godina prošlog stoljeća ([GR77], [RRI83], [Sav73], [För71], [Sav76]). Teorija mikropolarnog fluida u inkompresibilnom slučaju relativno je dobro istražena i većina rezultata sistematizirana je u monografiji [Luk99]. Međutim u kompresibilnom slučaju teorija mikropolarnog fluida tek se počela razvijati, posebice u slučaju modela koji uključuju temperaturu fluida. Do sada najviše rezultata ima u istraživanjima izentropnih, odnosno barotropnih modela ([Zha13], [AH09], [DC09], [CP06]).

Konstitutivne jednadžbe izentropnog fluida ne sadrže temperaturu fluida već samo gustoću što jednadžbu koja opisuje temperaturu fluida čini nezavisnom od preostalih jednadžbi modela. U ovom radu promatra se politropni idealni fluid kod kojeg je temperatura sadržana u jednoj od konstitutivnih jednadžbi fluida, što dovodi do toga da se temperatura osim u jednadžbi koja opisuje zakon očuvanja energije, pojavljuje i u jednadžbi koja opisuje zakon očuvanja momenta te model postaje složeniji.

Preciznije, predmet istraživanja ovog rada je model kompresibilnog, viskoznog i toplinski provodljivog mikropolarnog fluida koji je u termodinamičkom smislu idealan i politropan, a razvila ga je Nermina Mujaković ([Muj98b]). U [Muj98b] je dokazana egzistencija generaliziranog rješenja modela s homogenim rubnim uvjetima za brzinu, mikrorotaciju i toplinski fluks lokalno u vremenu kao i jedinstvenost generaliziranog rješenja u jedodimenzionalnom slučaju. U [Muj98a] dokazana je egzistencija generaliziranog rješenja i globalno po vremenu. Za isti model razmatran je problem regularnosti rješenja ([Muj01]), stabilizacije ([Muj05a]) kao i Cauchyijev problem ([Muj05b],[Muj06], [Muj10]). Također je razmatran i problem s nehomogenim rubnim uvjetima ([Muj07], [Muj08]). S numeričkog stanovišta model je tretiran u radovima [MD07b] i [MD07a].

Različite probleme za opisani model fluida razmatraju i Chen, Qin, Wang i Hu ([Min12], [Che11], [QWH12]). Qin u [QWH12] dokazima egzistencije pristupa metodom polugrupa te egzistenciju dokazuje za proizvoljne početne podatke. Svi spomenuti autori istražuju samo jednodimenzionalni slučaj. U trodimenzionalnom slučaju razmatra se samo kompresibilan izentropni fluid (primjerice u [CP06] i [JZZD13]).

Ovaj rad bavi se istim modelom koji je razmatran u [Muj98b], ali se ovdje istražuje trodimenzionalni model u sferno simetričnom slučaju. Sferno simetričan model klasičnog fluida razmatran je primjerice u [Hof92], [Jia96], [FYB95] i [Yan00]. Navedeni radovi poslužili su kao temelj za razmatranje ovog nešto kompliciranijeg modela, koji kao razmatranu veličinu uključuje i mikrorotaciju. Dokazu egzistencije u ovom radu pristupamo kao u [Muj98b] Faedo-Galerkinovom metodom.

Glavni rezultati disertacije su sljedeći:

1. Izvod trodimenzionalnog sferno simetričnog modela viskoznog kompresibilnog termo-provodljivog mikropolarnog fluida koji je u termodinamičkom smislu idealan i politropan.

2. Dokaz Teorema egzistencije generaliziranog rješenja lokalno po vremenu za inicijalno-rubni problem definiran s homogenim rubnim uvjetima za brzinu, mikrorotaciju i toplinski fluks, a koji opisuje nestacionarno gibanje razmatranog fluida izmedu dvije termički izolirane sferne stjenke.

3. Dokaz Teorema jedinstvenosti generaliziranog rješenja za opisani inicijalno-rubni problem.

4. Dokaz Teorema egzistencije generaliziranog rješenja globalno po vremenu, odnosno egzistencije rješenja na vremenskoj domeni $[0, T]$, gdje je $T > 0$ konačno i proizvoljno.

Disertacija je struktuirana na sljedeći način: U drugom poglavlju opisuju se prostori funkcija koji će biti korišteni u radu te se daje pregled neophodnog matematičkog aparata. U trećem poglavlju izvodi se opisani model te se isti zapisuje u Lagrangeovoj deskripciji. U četvrtom poglavlju sistematiziraju se glavni rezultati: Teorem egzistencije lokalnog rješenja, Teorem jedinstvenost rješenja te Teorem egzistencije globalnog rješenja. Dokazi spomenutih teorema nalaze se u u petom, šestom i sedmom poglavlju.

Prvi dio rezultata, odnosno izvod modela i Teorem egzistencije lokalnog rješenja, već je objavljen u [DM12]. U trenutku predaje disertacije rad u kojem se dokazuje Teorem jedinstvenosti rješenja je prihvaćen za objavu u časopisu *Boundary value problems*, a rad u kojem se opisuje egzistencija globalnog rješenja nalazi se na recenziji u istom časopisu.

2 Pomoćni rezultati

U ovom poglavlju navodimo neke rezultate iz realne i funkcionalne analize. Spominjemo one prostore funkcija i njihova svojstva koji se u radu koriste te znanja o egzistenciji rješenja iz teorije običnih diferencijalnih jednadžbi. Također navodimo više važnih nejednakosti koje koristimo u dokazima.

2.1 Prostori funkcija

U daljnjem tekstu sa U ćemo označavati dovoljno regularan, otvoren i ograničen skup u euklidskom prostoru R^n, pri čemu treba imati na umu da u našem radu koristimo specijalan slučaj skupa U kada je on zapravo interval $]0,1[$.

Definicije prostora i njihova svojstva uglavnom preuzimamo iz [Eva98], [RR04], [CM12] i [AF03].

Prostor realnih neprekidnih funkcija na skupu U označavat ćemo sa $\mathrm{C}(U)$, a prostor realnih funkcija s neprekidnom k-tom derivacijom ($k \in [0,\infty]$) sa $\mathrm{C}^k(U)$. Prostori $\mathrm{C}(\overline{U})$ i $\mathrm{C}^k(\overline{U})$ su Banachovi prostori sa normama

$$\|u\|_{\mathrm{C}(\overline{U})} = \max_{x\in\overline{U}} |u(x)| \tag{2.1}$$

i

$$\|u\|_{\mathrm{C}^k(\overline{U})} = \sum_{i=0}^{k} \max_{x\in\overline{U}} |u^{(i)}(x)|, \tag{2.2}$$

gdje je $\overline{U}$ zatvarač skupa U. Prostor $\mathrm{C}_c^k(U)$ sastoji se od svih funkcija iz $\mathrm{C}^k(U)$ sa kompaktnim nosačem u U. Prostor $\mathrm{C}_c^\infty(U)$ često se označava sa $\mathfrak{D}(U)$ i zove prostorom test funkcija.

Definicija 2.1. *Neka je $u : U \to R$ Lebesque izmjeriva funkcija. Za funkciju u kažemo da pripada prostoru $\mathrm{L}^p(U)$, $p \in [1, \infty[$ ako je*

$$\|u\|_{\mathrm{L}^p(U)} = \left(\int\limits_U |u|^p dx \right)^{\frac{1}{p}} < \infty, \tag{2.3}$$

a prostoru $\mathrm{L}^\infty(U)$ ako je

$$\|u\|_{\mathrm{L}^\infty(U)} = \text{ess sup}_U |u| < \infty. \tag{2.4}$$

Napomenimo da su prostori $\mathrm{L}^p(U)$ separabilni i refleksivni Banachovi prostori ako je $p \neq \{1, \infty\}$. Prostor $\mathrm{L}^1(U)$ je separabilan ali nije refleksivan, dok prostor $\mathrm{L}^\infty(U)$ nije ni separabilan ni refleksivan.

Prostor $\mathrm{L}^2(U)$ kojeg u radu najćešće koristimo Hilbertov je prostor sa skalarnim umnoškom

$$\langle u, v \rangle := \int\limits_U u(x)v(x)dx, \quad u, v \in \mathrm{L}^2(U). \tag{2.5}$$

U nastavku podrazumijevao da je norma $\| \cdot \|$ bez oznake prostora, norma u prostoru $\mathrm{L}^2(U)$.

Vrijedi:

Teorem 2.1. *Prostor $\mathrm{C}_c^\infty(U) = \mathfrak{D}(U)$ je gust u $\mathrm{L}^2(U)$.*

U cilju uvođenja prostora Soboljeva nužno nam je prvo navesti definiciju slabe derivacije.

Definicija 2.2. *Neka su $u, v \in \mathrm{L}^1_{loc}(U)$ i α multiindeks. Kažemo da je v slaba derivacija od u reda α, pišemo $D^\alpha u = v$, ako je*

$$\int\limits_U u D^\alpha \varphi = (-1)^{|\alpha|} \int\limits_U v \varphi dx \tag{2.6}$$

za sve test funkcije $\varphi \in \mathfrak{D}(U)$.

Definicija 2.3. *Neka je funkcija $u \in \mathrm{L}^p(U)$ takva da za svaki multiindeks α, $|\alpha| \leq m$, $m \in N$ postoji slaba derivacija $D^\alpha u$. Za funkciju u kažemo da pripada prostoru $\mathrm{W}^{m,p}(U)$, $p \in [1, \infty[$ ako je*

$$\|u\|_{\mathrm{W}^{m,p}(U)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{\mathrm{L}^p(U)} \right)^{\frac{1}{p}} < \infty. \tag{2.7}$$

Prostor $\mathrm{W}^{m,2}(U)$ označava se s $\mathrm{H}^m(U)$. Prostore $\mathrm{W}^{m,p}(U)$ zovemo prostorima Soboljeva.

Prostori Soboljeva su Banachovi. Za $p \neq \infty$ su separabilni, a za $p \neq \{1, \infty\}$ refleksivni. Prostori $\mathrm{H}^k(U)$ su Hilbertovi.

Potrebna su nam svojstva prostora $\mathrm{H}^m(U)$ navedena u sljedećem teoremu ([AF03]).

Teorem 2.2. *Za $U \subset R^n$ vrijede ulaganja* *(i)* $\mathrm{H}^m(U) \hookrightarrow \mathrm{H}^{m'}(U)$ *za* $m \geq m'$,

(ii) $\mathrm{H}^m(U) \hookrightarrow \mathrm{C}(\overline{U})$ *za* $m > \frac{n}{2}$,

(iii) $\mathrm{H}^m(U) \hookrightarrow \mathrm{C}^k(\overline{U})$ *za* $m - k > \frac{n}{2}$,

(iv) $\mathrm{H}^m(U) \hookrightarrow \mathrm{L}^2(U)$ *za* $m \leq \frac{n}{2}$.

Ulaganja (ii)-(iv) su i kompaktna.

Koristimo i tzv. evolucijske prostore na koje se odnose sljedeće definicije.

Definicija 2.4. *Neka je X realni Banachov prostor sa normom $\|\cdot\|$ te neka je $\mathbf{u} : [0,T] \to X$ jako izmjeriva funkcija*[1]. *Za funkciju $\mathbf{u}$ kažemo da pripada prostoru $\mathrm{L}^p(0,T;X)$, $p \in [1,\infty[$ ako je*

$$\|u\|_{\mathrm{L}^p(0,T;X)} = \left(\int_0^T \|u(t)\|^p dx \right)^{\frac{1}{p}} < \infty, \tag{2.8}$$

a prostoru $\mathrm{L}^\infty(0,T;X)$ ako je

$$\|u\|_{\mathrm{L}^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\| < \infty. \tag{2.9}$$

Definicija 2.5. *Neka je X realni Banachov prostor sa normom $\|\cdot\|$. Za funkciju $\mathbf{u} : [0,T] \to X$ kažemo da pripada prostoru $\mathrm{C}(0,T;X)$ ako je*

$$\|\mathbf{u}\|_{\mathrm{C}(0,T;X)} = \max_{t\in[0,T]} \|\mathbf{u}(t)\| < \infty. \tag{2.10}$$

Dakle, prostor $\mathrm{C}(0,T;X)$ je prostor neprekidnih funkcija na $[0,T]$.

Navedeni evolucijski prostori su Banachovi prostori s obzirom na navedene norme.

Budući ćemo tražiti da naš razmatrani sustav diferencijalnih jednadžbi bude zadovoljen u smislu distribucija nužno je navesti neke osnovne pojmove iz teorije distribucija.

Definicija 2.6. *Dualni prostor prostora test funkcija zovemo prostorom distribucija i označavamo sa $\mathfrak{D}'(U)$.*

Svaka realna lokalno integrabilna funkcija $u \in \mathrm{L}^1_{loc}(U)$ identificira se sa distribucijom $\mathcal{T}_u$ na sljedeći način

$$\mathcal{T}_u(\varphi) := \langle u, \varphi \rangle = \int_U u\varphi dx. \tag{2.11}$$

Distribucije nastale identifikacijom s klasičnim funkcijama zovu se regularne distribucije.

Kako bi mogli uvesti pojam vremenske derivacije funkcija iz prostora $\mathrm{L}^p(0,T;X)$ kojem će pripadati rješenje našeg problema, nužno je uvesti pojam vektorske distribucije. Definiciju i svojstva vektorskih distribucija preuzimamo iz [Trö10] i [LM72].

[1] ili izmjeriva u Bochnerovu smislu

Definicija 2.7. *Neka je X Banachov prostor. Svako neprekidno linearno preslikavanje $\mathcal{T}$: $\mathfrak{D}\left(]0,T[\right) \to X$ zovemo vektorskom distribucijom na $]0,T[$.*

Prostor vektorskih distribucija na $]0,T[$ označavamo s $\mathfrak{D}'\left(0,T;X\right)$.

Način identificiranja funkcija iz prostora $\mathrm{L}^p(0,T;X)$ i vektorskih distribucija dan je u sljedećoj propoziciji.

Propozicija 2.1. *Neka je $u \in \mathrm{L}^1_{loc}(0,T;X)$. Preslikavanje $\mathcal{T}_u : \mathfrak{D}\left(]0,T[\right) \to X$ definirano sa*

$$\mathcal{T}_u(\varphi) := \int_0^T u(t)\varphi(t)dt \tag{2.12}$$

je vektorska distribucija na $]0,T[$.

Definirajmo sada derivaciju vektorske distribucije.

Definicija 2.8. *Neka je $f \in \mathfrak{D}'\left(]0,T[;X\right)$ i neka je m nenegativan cijeli broj. Preslikavanje*

$$\varphi \mapsto (-1)^m f\left(\frac{d^m\varphi}{dt^m}\right), \quad \varphi \in \mathfrak{D}\left(]0,T[\right) \tag{2.13}$$

zovemo distribucijskom derivacijom vektorske distribucije m-tog reda i označavamo sa $\frac{d^m f}{dt^m}$.

Primjetimo da je distribucijska derivacija vektorske distribucije također distribucija. Ako je X prostor funkcija varijable x, npr. $X = \mathrm{L}^2(U)$ tada se vektorska distribucija $u \in \mathrm{L}^1_{loc}(0,T;X)$ identificira s funkcijom $u(x,t)$. Često se sa $u(t)$ označava funkcija $x \mapsto u(x,t)$ za s.s. t. U tom se slučaju distribucijska derivacija $\frac{du}{dt}$ identificira s parcijalnom derivacijom $\frac{\partial u}{\partial t}$ funkcije u iz prostora $\mathfrak{D}'\left(U\times]0,T[\right)$.

Sada možemo uvesti generalizaciju gornjih prostora.

Definicija 2.9. *Neka su X i Y realni Banachovi prostor sa normama $\|\cdot\|_X$ i $\|\cdot\|_Y$ te neka je $\mathbf{u} : [0,T] \to X$ jako izmjeriva funkcija. Za funkciju $\mathbf{u}$ kažemo da pripada Banachovu prostoru $\mathrm{W}^{m,p}(0,T;X,Y)$ ako je*

$$\|\mathbf{u}\|_{\mathrm{W}^{m,p}(0,T;X,Y)} = \left(\int_0^T \left(\|\mathbf{u}(t)\|_X^p + \|\mathbf{u}^{(m)}(t)\|_Y^p\right) dt\right)^{\frac{1}{p}} < \infty. \tag{2.14}$$

Prostor $\mathrm{W}^{m,p}(0,T;X,Y)$ sastoji se od funkcija iz prostora $\mathrm{L}^p(0,T;X)$ čije su derivacije (do uključujući reda m) u distribucijskom smislu u prostoru $\mathrm{L}^p(0,T;Y)$. Često se koriste i sljedeće oznake:

$$\mathrm{W}^{m,p}(0,T;X,X) = \mathrm{W}^{m,p}(0,T;X), \tag{2.15}$$

$$\mathrm{W}^{m,2}(0,T;X,Y) = \mathrm{H}^m(0,T;X,Y), \tag{2.16}$$

$$\mathrm{W}^{m,2}(0,T;X) = \mathrm{H}^m(0,T;X). \tag{2.17}$$

Ako je X Hilbertov prostor, prostor $\mathrm{H}^m(0,T;X)$ je također Hilbertov sa skalarnim produktom definiranim s

$$\langle \mathbf{u},\mathbf{v}\rangle_{\mathrm{H}^m(0,T;X)} := \int_0^T \left(\langle \mathbf{u}(t),\mathbf{v}(t)\rangle_X + \left\langle \mathbf{u}^{(m)}(t),\mathbf{v}^{(m)}(t)\right\rangle_X \right) dt. \tag{2.18}$$

Imamo sljedeći važan rezultat

Teorem 2.3. *Vrijedi ulaganje*

$$\mathrm{H}^1(0,T;X,Y) \hookrightarrow \mathrm{C}\left(0,T;[X,Y]_{\frac{1}{2}}\right)$$

gdje je $[X,Y]_{\frac{1}{2}}$ interpolacijski prostor prostora X i Y indeksa $\frac{1}{2}$.

U radu se susrećemo sa funkcijama iz prostora $\mathrm{H}^1\left(0,T;\mathrm{H}^2\left(]0,1[\right),\mathrm{L}^2\left(]0,1[\right)\right)$ koje prema navedenom teoremu pripadaju i prostoru

$$\mathrm{C}\left(0,T;\left[\mathrm{H}^2\left(]0,1[\right),\mathrm{L}^2\left(]0,1[\right)\right]_{\frac{1}{2}}\right) = \mathrm{C}\left(0,T;\mathrm{H}^1\left(]0,1[\right)\right). \tag{2.19}$$

Navedimo sada nekoliko svojstava funkcionalnih prostora ([Rek80]) koja će nam omogućiti razmatranje svojstava tragova funkcija.

Teorem 2.4 (Gelfandova trojka)**.** *Neka je V Banachov, a H Hilbertov prostor. Neka su V' i H' njihovi duali te neka je H identificiran sa svojim dualom H'. Ako je $V \subseteq H$ vrijedi struktura tzv. Gelfandove trojke, tj.*

$$V \hookrightarrow H \hookrightarrow V' \tag{2.20}$$

pri čemu $\hookrightarrow$ označava kanonsko ulaganje, odnosno preslikavanje $v \in V \mapsto v \in H$. Nadalje, ako je ulaganje $V \hookrightarrow H$ gusto, gusto je i ulaganje $H' \hookrightarrow V'$.

Hilbertov prostor iz Gelfandove trojke često se naziva *pivotni prostor*.

U radu koristimo sljedeće važno svojstvo Gelfandove trojke.

Teorem 2.5. *Vrijedi ulaganje*

$$\mathrm{H}^1(0,T;V,V') \hookrightarrow \mathrm{C}(0,T;H) \tag{2.21}$$

gdje su V, H i V' iz Teorema 2.4.

Ovaj teorem opravdava uvođenje tragova u H. Napomenimo da je u radu važna primjena Greenove formule koju iskazujemo sljedećim teoremom.

Teorem 2.6 (Greenova formula)**.** *Neka su $u,v \in \mathrm{H}^1(0,T;V,V')$. Vrijedi*

$$\int_0^T \langle u'(t),v(t)\rangle dt + \int_0^T \langle v'(t),u(t)\rangle dt = \langle u(T),v(T)\rangle - \langle u(0),v(0)\rangle. \tag{2.22}$$

2.2 Slabe i jake konvergencije

Dokaz egzistencije generaliziranog rješenja problema kojeg ćemo razmatrati lokalno po vremenu temelji se na analizi niza aproksimativnih rješenja. Iz dobivenih uniformnih ocjena toga niza zaključujemo njegovu konvergenciju u nekom slabom ili jakom smislu u različitim prostorima funkcija. Stoga ovdje navodimo definicije slabe i jake konvergencije ([Dob10], [RR04]).

Definicija 2.10. *Neka je X Banachov prostor. Prostor svih omeđenih linearnih funkcionala na X zovemo dualnim prostorom prostora X i označavamo s X'.*

Neka je X Banachov prostor, a X' njegov dual. Promatramo dvije topologije na X:

(i) jaku topologiju generiranu normom i

(ii) slabu topologiju generiranu polunormama

$$p_f(x) = |f(x)|\,,\ f \in X'. \tag{2.23}$$

Na prostoru X' promatramo tzv. slabu* topologiju generiranu polunormama

$$p_x(f) = |f(x)|\,,\ x \in X. \tag{2.24}$$

Konvergencije u smislu slabe i slabe* topologije karakterizirane su sljedećim lemema:

Lema 2.1. *Niz $(x_n) \in X$ konvergira u slaboj topologiji (odnosno konvergira slabo) ka $x \in X$ (što označavamo s $x_n \rightharpoonup x$) ako i samo ako vrijedi*

$$f(x_n) \to f(x), \quad \forall f \in X'. \tag{2.25}$$

Lema 2.2. *Niz $(f_n) \in X'$ konvergira u slaboj* topologiji (odnosno konvergira *slabo) ka $f \in X'$ (što označavamo s $f_n \overset{*}{\rightharpoonup} f$) ako i samo ako vrijedi*

$$f_n(x) \to f(x), \quad \forall x \in X. \tag{2.26}$$

Iz definicija slabe i slabe* topologije slijede sljedeća važna svojstva:

1. Svaki jako konvergentan niz je i slabo konvergentan.

2. Slabo konvergentan niz u X' je i slabo* konvergentan u X'. Obrat vrijedi ako je X refleksivan prostor.

3. Slabi* limes je jedinstven prema svojoj definiciji. Prema Hahn-Banachovu teoremu jedinstven je i slabi limes.

Uniformne ocjene niza aproksimativnih rješenja našeg problema dovode do rezultata koji su posljedica sljedećih teorema.

Teorem 2.7 (Alaoglu). *Neka je X separabilan Banachov prostor i neka je (f_n) omeđen niz u X'. Tada (f_n) ima slabo* konvergentan podniz u X'.*

Teorem 2.8. *Neka je X refleksivan Banachov prostor i neka je (x_n) omeđen niz u X. Tada (x_n) ima slabo konvergentan podniz u X.*

Za dokaz egzistencije rješenja našeg lokalnog problema od velikog je značaja i Arzela-Ascolijev teorem za koji nam je potreban pojam ekvineprekidnosti niza funkcija.

Definicija 2.11. *Neka je (f_m) niz realnih funkcija definiranih na $U \subset R^n$ te neka je $\mathbf{x} \in U$. Niz (f_m) zovemo ekvineprekidnim u $\mathbf{x}$ ako za svako $\varepsilon > 0$ postoji $\delta > 0$, neovisan o m, tako da za $\mathbf{y} \in U$ vrijedi*

$$\|\mathbf{y} - \mathbf{x}\|_{R^n} < \delta \Longrightarrow |f_m(\mathbf{y}) - f_m(\mathbf{x})| < \varepsilon. \tag{2.27}$$

Teorem 2.9 (Arzela-Ascoli). *Neka je (f_m) niz realnih funkcija definiranih na kompaktnom podskupu $S \subset R^n$ te neka je (f_m) ekvineprekidan za sve $\mathbf{x} \in S$. Ako postoji konstanta M takva da je $|f_m(\mathbf{x})| \leq M$, za sve $m \in N$ i za sve $\mathbf{x} \in S$ onda postoji podniz niza (f_m) koji konvergira uniformno na S.*

Napomenimo da je uniformna konvergencija iz Arzela-Ascolijeva teorema ekvivalentna jakoj konvergenciji u prostoru $\mathrm{C}(S)$.

2.3 Pregled korištenih nejednakosti

Teorem 2.10 (Jensenova nejednakost - diskretni oblik, [MPF93]). *Neka je φ realna konveksna funkcija i neka su x_i, $i = 1, ..., n$ iz domene funkcije φ, a a_i, $i = 1, ...n$ pozitivne konstante (težine). Tada vrijedi nejednakost*

$$\varphi\left(\frac{\sum_{i=1}^n a_i x_i}{\sum_{i=1}^n a_i}\right) \leq \frac{\sum_{i=1}^n a_i \varphi(x_i)}{\sum_{i=1}^n a_i}. \tag{2.28}$$

U ovom će radu od posebnog značaja biti slučaj diskretnog oblika Jensenove nejednakosti kada je $\varphi(x) = x^2$ te $a_i = 1$, $i = 1, ..., n$, odnosno nejednakost

$$\left(\sum_{i=1}^n x_i\right)^2 \leq n \sum_{i=1}^n x_i^2. \tag{2.29}$$

Teorem 2.11 (Jensenova nejednakost - integralni oblik, [MPF93]). *Neka je φ realna konveksna funkcija te neka je $\varphi : [a, b] \to \mathbf{R}$ integrabilna funkcija. Tada vrijedi nejednakost*

$$\varphi\left(\int_a^b f(x)dx\right) \leq \frac{1}{b-a} \int_a^b \varphi\left((b-a)f(x)\right)dx. \tag{2.30}$$

Teorem 2.12 (Cauchy–Schwarz, [MPF93])**.** *Neka su* $\mathbf{x}$ *i* $\mathbf{y}$ *dva vektora u nekom unitarnom prostoru. Tada vrijedi nejednakost*

$$|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\cdot\|\mathbf{y}\|\,. \tag{2.31}$$

Ovdje je od značaja sljedeći oblik Cauchy-Schwarzove nejednakosti

$$\left|\int_a^b f(x)g(x)dx\right|^2 \leq \int_a^b |f(x)|^2\,dx\cdot\int_a^b |g(x)|^2\,dx, \tag{2.32}$$

gdje su f i g kvadratno-integrabilne funkcije. Ako u ovu nejednakost uvrstimo da je $g(x) = 1$ dobijamo sljedeću nejednakost

$$\left|\int_a^b f(x)dx\right|^2 \leq (b-a)\int_a^b |f(x)|^2\,dx. \tag{2.33}$$

Teorem 2.13 (Hölder, [Eva98])**.** *Neka su* $u \in \mathrm{L}^p(U)$ *i* $v \in \mathrm{L}^q(U)$*, pri čemu je* $1 \leq p,q \leq \infty$ *i* $\frac{1}{p}+\frac{1}{q} = 1$*. Tada vrijedi nejednakost*

$$\int_U |uv|dx \leq \|u\|_{\mathrm{L}^p(U)}\,\|v\|_{\mathrm{L}^q(U)}\,. \tag{2.34}$$

Teorem 2.14 (Gagliardo-Ladyzhenskaya, [LSU88])**.** *Neka je dana funkcija* $u : U \to R$ *koja zadovoljava jedno od sljedeća dva svojstva:*

1. $u(x) \in \mathrm{W}_0^{1,p}(U)$,
2. $u(x) \in \mathrm{W}^{1,p}(U)$, $\int_U u(x)dx = 0$,

za $p \geq 1$*. Neka je nadalje* $r \geq 1$, $q \in [r,\infty]$*. Tada postoji pozitivna konstanta* C *(neovisna o funkciji* u*) takva da vrijedi nejednakost*

$$\|u\|_{\mathrm{L}^q(U)} \leq C\|u\|_{\mathrm{L}^r(U)}^{1-\alpha}\left\|\frac{\partial u}{\partial x}\right\|_{\mathrm{L}^p(U)}^{\alpha} \tag{2.35}$$

pri čemu je

$$\alpha = \left(\frac{1}{r}-\frac{1}{q}\right)\left(1-\frac{1}{p}+\frac{1}{r}\right)^{-1}. \tag{2.36}$$

U radu smo koristili tvrdnju teorema za funkciju u iz prostora $H_0^1\,(]0,1[)$ uzimajući za $p = 2$, $r = 2$ i $q = \infty$, pa (2.35) postaje

$$|u|^2 \leq C\|u\|\left\|\frac{\partial u}{\partial x}\right\|. \tag{2.37}$$

Kada funkcija $\frac{\partial u}{\partial x}$ zadovoljava neki od navedenih uvjeta dobivamo i nejednakost

$$\left|\frac{\partial u}{\partial x}\right|^2 \leq C\left\|\frac{\partial u}{\partial x}\right\|\left\|\frac{\partial^2 u}{\partial x^2}\right\|. \tag{2.38}$$

Teorem 2.15 (Friedrichs-Poincaré, [Rek80]). *Neka funkcija u zadavoljava uvjete Teorema 2.14 za $p = 2$. Tada postoji pozitivna konstanta C takva da vrijedi nejednakost*

$$\|u\| \leq C \left\| \frac{\partial u}{\partial x} \right\|. \tag{2.39}$$

Ako i derivacija $\dfrac{\partial u}{\partial x}$ funkcije u zadovoljava spomenute uvjete vrijedi

$$\|u\| \leq C \left\| \frac{\partial^2 u}{\partial x^2} \right\|. \tag{2.40}$$

Uvrštavanjem nejednakosti (2.39) u (2.37) odmah dobivamo i nejednakost

$$|u| \leq C \left\| \frac{\partial u}{\partial x} \right\| \tag{2.41}$$

koja također vrijedi za funkciju u koja zadovoljava uvjete Teorema 2.14.

U radu koristimo sljedeću nejednakost Gronwall-Bellmanova tipa.

Teorem 2.16 (Gronwall-Čandirov, [Sev03], [AKM90]). *Neka je y nenegativna funkcija definirana na intevalu $[0, T]$ i neka vrijedi nejednakost*

$$y(t) \leq C + \int\limits_0^t [A(\tau)y(\tau) + B(\tau)]\, d\tau, \tag{2.42}$$

gdje je C pozitivna konstanta, a A i B funkcije koje pripadaju prostoru $\mathrm{L}^1(]0, T[)$. Tada s.s. na $[0, T]$ vrijedi nejednakost

$$y(t) \leq \exp\left\{ \int\limits_0^t A(\tau) d\tau \right\} \left[C + \int\limits_0^t B(\tau) \exp\left\{ \int\limits_0^\tau -A(s) ds \right\} d\tau \right]. \tag{2.43}$$

Teorem 2.17 (Youngova nejednakost, [MPF93]). *Neka je f neprekidna i rastuća funkcija na segmentu $[0, c]$, gdje je $c > 0$. Ako je $f(0) = 0$, $a \in [0, c]$ te $b \in [0, f(c)]$ tada vrijedi nejednakost*

$$\int\limits_0^a f(x) dx + \int\limits_0^b f^{-1}(x) dx \geq ab. \tag{2.44}$$

Jednakost vrijedi ako i samo ako je $b = f(a)$.

Uzmemo li da je $f(x) = x^{p-1},\ p > 1$ dobivamo sljedeću posljedicu Youngove nejednakosti koja se u ovom radu vrlo često koristi.

Korolar 2.1. *Neka su $a, b \geq 0,\ p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1$. Tada vrijedi nejednakost*

$$\frac{1}{p} a^p + \frac{1}{q} b^q \geq ab. \tag{2.45}$$

Posebno za $p = q = 2$ *imamo*

$$\frac{1}{2}a^2 + \frac{1}{2}b^2 \geq ab, \tag{2.46}$$

dok za $a = \varepsilon x$ *i* $b = \varepsilon^{-1} y$ *dobivamo*

$$\frac{1}{p}\varepsilon^p x^p + \frac{1}{q}\varepsilon^{-q} y^q \geq ab. \tag{2.47}$$

Koristit ćemo i sljedeću posljedicu nejednakosti (2.45) koju dobivamo ako uzmemo da je $b = 1$ i koja glasi

$$a \leq C\left(1 + a^p\right), \tag{2.48}$$

gdje je $C = \max\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\}$, $p > 1$.

2.4 Normalni sustav običnih diferencijalnih jednadžbi

Dokaz egzistencije rješenja našeg problema temelji se na činjenici da postoji rješenje normalnog sustava običnih diferencijalnih jednadžbi na nekom dovoljno malenom vremenskom intervalu.

Prema [AO08], [Arn92] i [Pet54] normalni sustav diferencijalnih jednadžbi prvog reda je oblika

$$\begin{aligned} u_1' &= g_1(x, u_1, ..., u_n) \\ u_2' &= g_2(x, u_1, ..., u_n) \\ &\dots \\ u_n' &= g_n(x, u_1, ..., u_n) \end{aligned} \tag{2.49}$$

sa početnim uvjetima

$$u_1(x_0) = u_1^0,\ u_2(x_0) = u_2^0, ...,\ u_n(x_0) = u_n^0, \tag{2.50}$$

gdje su $u_1, ...u_n : I \to R$ nepoznate funkcije, $g_1, ..., g_n$ zadane funkcije na $I \times E$, $E \subset \mathbf{R}^n$, te $x_0 \in I$.

Teorem egzistencije i jedinstvenosti rješenja opisanog sustava običnih diferencijalnih jednadžbi možemo formulirati na sljedeći način.

Teorem 2.18 (Cauchy-Picard)**.** *Neka je*

$$\Delta = \left\{(x, u_1, ..., u_n) \in I \times E : |x - x_0| \leq a, |u_i - u_i^0| \leq b, i = 1, ..., n\right\}$$

i neka su

(i) $g_1, ..., g_n$ *neprekidne funkcije na skupu* Δ*,*

(ii) $|g_i(x, u_1, ..., u_n)| \leq M$*,* $i = 1, ..., n$*,*

(iii) $|g_i(x, u_1, ..., u_n) - g_i(x, \overline{u}_1, ..., \overline{u}_n)| \leq L\left(|u_1 - \overline{u}_1| + |u_n - \overline{u}_n|\right)$, $i = 1, ..., n$ *(Lipschitzovo svojstvo)*

Tada na intervalu $]x_0 - h, x_0 + h[$, $h = \min\{a, bM^{-1}\}$ *postoji rješenje*

$$(u_1(x), ..., u_n(x)),\ x \in]x_0 - h, x_0 + h[\tag{2.51}$$

problema (2.49)-(2.50) i ono je jedinstveno.

Može se pokazati da je rješenje iz Cauchy-Picardova teorema klase C^∞ na području egzistencije.

Također koristimo svojstvo maksimalnog rješenja diferencijalne jednadžbe preuzetog iz [AO08]. Promatramo početni problem

$$y' = f(x, y),\ y(x_0) = y_0 \tag{2.52}$$

pri čemu se pretpostavlja da je funkcija $f(x, y)$ neprekidna na domeni $D \subset R^2$ koja sadržava točku (x_0, y_0). Sa J ćemo označiti interval egzistencije (ne nužno jedinstvenog) rješenja problema (2.52).

Definicija 2.12. *Rješenje $r(x)$ problema (2.52) zove se maksimalno rješenje ako za proizvoljno rješenje $y(x)$ problema (2.52) vrijedi $y(x) \leq r(x)$ za svaki $x \in J$.*

Teorem 2.19. *Neka je $f(x, y)$ neprekidna na domeni D tako da problem (2.52) ima rješenje na intervalu J te neka je $r(x)$ maksimalno rješenje problema (2.52). Također, neka je $y(x)$ rješenje diferencijalne nejednadžbe*

$$y'(x) \leq f(x, y(x)) \tag{2.53}$$

na intevalu J. Tada nejednakost $y(x_0) \leq y_0$ povlači nejednakost $y(x) \leq r(x)$ za sve $x \in J$.

3 Izvod modela

3.1 Zakoni očuvanja

U mehanici kontinuuma fluid se poistovjećuje sa područjem $\Omega \subset \mathbf{R}^3$, pri čemu je Ω proizvoljan otvoren i povezan skup. Predmet proučavanja mehanike kontinuuma su mjerljiva svojstva fluida, odnosno gustoća, brzina i temperatura te u slučaju mikropolarnog fluida dodatno i mikrorotacija. Sva navedena svojstva ispravno je promatrati kao srednje vrijednosti po infinitezimalno malim volumenima i ona su povezana skupom dinamičkih jednadžbi koje opisuju gibanje fluida, a koje nazivamo zakonima očuvanja ([Chi09], [Gra07], [Luk99]). Zakoni očuvanja su:

1. zakon očuvanja mase,
2. zakon očuvanja momenta,
3. zakon očuvanja angularnog momenta te
4. zakon očuvanja energije.

Zakon očuvanja mase

Osnovna veličina koja se veže uz opisivanje stanja fluida je *gustoća* koja predstavlja masu po jedinici volumena. Sa stanovišta mehanike kontinuuma gustoća je skalarna nenegativna funkcija $(\mathbf{x}, t) \mapsto \rho(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$, $t \geq 0$. Zakon očuvanja mase unutar područja Ω može se iskazati na sljedeći način

$$\frac{d}{dt} \int_{\Omega} \rho dV = - \int_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS \tag{3.1}$$

Vektorsko polje $\mathbf{v}$ u izrazu (3.1) označava fluks polja ρ kroz rub područja Ω, odnosno polje brzina. Prema tome, jednadžba (3.1) govori da je brzina promjene mase područja Ω jednaka protoku mase kroz rub područja Ω.

Koristeći se teoremom o divergenciji, izraz (3.1) može se napisati u obliku

$$\frac{d}{dt}\int\limits_{\Omega}\rho dV = -\int\limits_{\Omega}\operatorname{div}(\rho\mathbf{v})dV \tag{3.2}$$

odakle slijedi tzv. lokalna forma zakona o očuvanju mase

$$\frac{d\rho}{dt} + \operatorname{div}(\mathbf{v}\rho) = 0. \tag{3.3}$$

Jednakost (3.3) možemo zapisati i na sljedeći način

$$\dot{\rho} + \rho \operatorname{div}\mathbf{v} = 0, \tag{3.4}$$

gdje je

$$\dot{\rho} = \frac{d\rho}{dt} + \mathbf{v}\cdot\nabla\rho \tag{3.5}$$

materijalna derivacija gustoće po vremenu.

Jednadžbu (3.4) zovemo i jednadžbom kontinuiteta.

Zakon očuvanja momenta

Mehanika kontinuuma razlikuje dvije vrste sila - volumne i kontaktne. Djelovanje volumnih sila distribuira se na sve točke područja Ω, dok se djelovanje kontaktnih sila distribuira samo po rubu područja Ω. Kontaktne sile često se opisuju normalnim naprezanjem. Sa $\mathbf{f}$ ćemo označiti ukupnu volumnu silu koja djeluje na jedinicu mase, a s $\mathbf{t_n}$ normalno naprezanje. Zakon očuvanja momenta dan je izrazom

$$\int\limits_{\Omega}\mathbf{F}dV = \int\limits_{\Omega}\rho\mathbf{f}dV + \int\limits_{\partial\Omega}\mathbf{t_n}dS. \tag{3.6}$$

gdje je $\mathbf{F}$ ukupna sila koja djeluje na fluid po jedinici volumena. Jednakost (3.6) govori da su sve sile koje djeluju na fluid u međusobnoj ravnoteži, što se naziva i D'Alambertov princip.

U skladu s Cauchyjevim načelom, normalno naprezanje može se iskazati na sljedeći način

$$\mathbf{t_n}(\mathbf{x},t) = \mathbf{n}(\mathbf{x},t)\cdot\mathbf{T}(\mathbf{x},t), \tag{3.7}$$

gdje je $\mathbf{T}(\mathbf{x},t)$ tenzor naprezanja, a $\mathbf{n}(\mathbf{x},t)$ vanjska normala. Kako je $\mathbf{F}=\rho\dot{\mathbf{v}}$ uz primjenu teorema o divergenciji iz (3.6) i (3.7) dobivamo

$$\int\limits_{\Omega}\rho\dot{\mathbf{v}}dV = \int\limits_{\Omega}\rho\mathbf{f}dV + \int\limits_{\Omega}\operatorname{div}\mathbf{T}dV. \tag{3.8}$$

odakle slijedi lokalna forma zakona o očuvanju momenta

$$\rho\dot{\mathbf{v}} = \operatorname{div}\mathbf{T} + \rho\mathbf{f}. \tag{3.9}$$

U radu pretpostavljamo da na fluid ne djeluju volumne sile, odnosno da je $\mathbf{f} = \mathbf{0}$, pa u našem slučaju zakon očuvanja momenta postaje

$$\rho\dot{\mathbf{v}} = \operatorname{div}\mathbf{T}. \tag{3.10}$$

Zakon očuvanja angularnog momenta

Zakon očuvanja momenta motivira i zakon očuvanja angularnog (kinetičkog) momenta, znajući da se angularni moment definira kao djelovanje polja $\rho(\mathbf{x} \times \mathbf{v})$. Iz (3.6) tako dobivamo jednakost

$$\frac{d}{dt}\int_{\Omega} \rho(\mathbf{x} \times \mathbf{v})dV = \int_{\Omega} \rho(\mathbf{x} \times \mathbf{f})dV + \int_{\partial\Omega} \mathbf{x} \times \mathbf{t_n} dS. \tag{3.11}$$

Međutim, zakon (3.11) vrijedi samo u slučaju klasičnog fluida, odnosno ako pretpostavimo da svi obrtni momenti dolaze zbog makroskopskih sila. U slučaju mikropolarnog fluida, uz volumnu silu $\mathbf{f}$ moramo uzeti u obzir i obrtni moment $\mathbf{g}$, a uz normalno naprezanje i naprezanje sprega (couple stress) $\mathbf{c_n}$. Ukupni angularni moment će se sada sastojati od ranije spomenutog angularnog momenta $\rho(\mathbf{x} \times \mathbf{v})$ i unutarnjeg angularnog momenta koji ćemo označiti s $\rho\mathbf{l}$, gdje je $\mathbf{l}$ unutarnji angularni moment. Zakon očuvanja (3.11) sada možemo pisati u formi

$$\frac{d}{dt}\int_{\Omega} \rho(\mathbf{l} + \mathbf{x} \times \mathbf{v})dV = \int_{\Omega} \rho(\mathbf{g} + \mathbf{x} \times \mathbf{f})dV + \int_{\partial\Omega} (\mathbf{c_n} + \mathbf{x} \times \mathbf{t_n})dS. \tag{3.12}$$

Analogno (3.7) za naprezanje sprega $\mathbf{c_n}$ stavimo da vrijedi

$$\mathbf{c_n}(\mathbf{x}, t) = \mathbf{n}(\mathbf{x}, t) \cdot \mathbf{C}(\mathbf{x}, t), \tag{3.13}$$

gdje je $\mathbf{C}(\mathbf{x}, t)$ tenzor naprezanja sprega, a $\mathbf{n}(\mathbf{x}, t)$ vanjska normala.

Kako bi došli do lokalne forme zakona očuvanja angularnog momenta koristimo jednakost

$$\operatorname{div}(\mathbf{x} \times \mathbf{T}) = \mathbf{x} \times \operatorname{div}\mathbf{T} + \mathbf{T}_x, \tag{3.14}$$

pri čemu je $\mathbf{T}_x$ vektor s komponentama

$$(\mathbf{T}_x)_i = \varepsilon_{ijk}\mathbf{T}_{jk}, \tag{3.15}$$

gdje je ε_{ijk} Levi-Civitin alternirajući simbol uz pretpostavku Einsteinove notacije sumiranja.

Koristeći (3.15) te teorem o divergenciji, kao i kod (3.3) i (3.9) iz (3.12) dobivamo lokalni oblik zakona očuvanja angularnog momenta

$$\rho\frac{D}{Dt}(\mathbf{l} + \mathbf{x} \times \mathbf{v}) = \rho\mathbf{g} + \rho\mathbf{x} \times \mathbf{f} + \operatorname{div}\mathbf{C} + x \times \operatorname{div}\mathbf{T} + \mathbf{T}_x. \tag{3.16}$$

gdje diferencijalni operator $\frac{D}{Dt}$ označava materijalnu derivaciju. Uvrštavanjem (3.9) u (3.16) dolazimo do konačne forme zakona očuvanja angularnog momenta

$$\rho\dot{\mathbf{l}} = \operatorname{div}\mathbf{C} + \rho\mathbf{g} + \mathbf{T}_x. \tag{3.17}$$

Kao i kod (3.7) te (3.13) unutarnji angularni moment $\mathbf{l}$ zapisati ćemo u obliku

$$\mathbf{l}(\mathbf{x},t) = \boldsymbol{\omega}(\mathbf{x},t)\cdot\mathbf{I}(\mathbf{x},t), \tag{3.18}$$

gdje je $\mathbf{I}(\mathbf{x},t)$ mikroinercijski tenzor, a vektor $\boldsymbol{\omega}(\mathbf{x},t)$ predstavlja mikrorotaciju.

U ovom radu pretpostavljamo da je fluid izotropan, odnosno da njegova svojstva ne ovise o smjeru zbog čega je mikroinercijski tenzor definiran izrazom

$$\mathbf{I}_{ik} = j_I\delta_{ik}, \tag{3.19}$$

gdje δ_{ik} označava Kroneckerovu deltu, a $j_I > 0$ je konstanta koju zovemo mikrorotacijski koeficijent. Prema tome za izotropan mikropolaran fluid (3.18) postaje

$$\mathbf{l}(\mathbf{x},t) = j_I\boldsymbol{\omega}(x,t). \tag{3.20}$$

Kao što smo pretpostavili da na fluid ne djeluju volumne sile, pretpostavit ćemo da nema djelovanja ni vanjskog obrtnog momenta, tj. da je $\mathbf{g} = 0$, pa iz (3.20) i (3.17) dobivamo konačnu formu zakona očuvanja angularnog momenta

$$\rho j_I\dot{\boldsymbol{\omega}} = \operatorname{div}\mathbf{C} + \mathbf{T}_x. \tag{3.21}$$

Zakon očuvanja energije

Prema prvom zakonu termodinamike porast ukupne energije (promatramo samo kinetičku i unutarnju energiju) unutar tijela jednak je sumi prenesene topline i rada koje tijelo učini. Prema tome vrijedi

$$\begin{aligned}\frac{d}{dt}\int\limits_{\Omega}\rho\left(\frac{\mathbf{v}^2}{2} + j_I\frac{\boldsymbol{\omega}^2}{2} + E\right)dV = &\int\limits_{\Omega}\rho\left(\rho\mathbf{v}\cdot\mathbf{f} + \rho\omega\cdot\mathbf{g}\right)dV \\ &+ \int\limits_{\partial\Omega}\mathbf{t_n}\cdot\mathbf{v}dS + \int\limits_{\partial\Omega}\mathbf{c_n}\cdot\boldsymbol{\omega}dS - \int\limits_{\partial\Omega}\mathbf{q}\cdot\mathbf{n}dS,\end{aligned} \tag{3.22}$$

gdje $\mathbf{q}$ označava toplinski fluks, a E specifičnu unutarnju energiju. Prema prethodno navedenim zakonima očuvanja momenta i angularnog momenta iz (3.22) zaključujemo da vrijedi

$$\rho\dot{E} = -\operatorname{div}\mathbf{q} + \mathbf{T} : \nabla\mathbf{v} + \mathbf{C} : \nabla\boldsymbol{\omega} - \mathbf{T}_x\cdot\boldsymbol{\omega}. \tag{3.23}$$

U (3.23) sa $\mathbf{A} : \mathbf{B}$ označen je tzv. skalarni produkt tenzora $\mathbf{A}$ i $\mathbf{B}$, odnosno

$$\mathbf{A} : \mathbf{B} = \operatorname{tr}\mathbf{A}^T\mathbf{B}. \tag{3.24}$$

3.2 Konstitutivne jednadžbe mikropolarnog fluida

Prema [Luk99] tenzori $\mathbf{T}$ i $\mathbf{C}$, odnosno konstitutivne jednadžbe mikropolarnog fluida definirane su formulama:

$$\mathbf{T} = (-p + \lambda \operatorname{div} \mathbf{v})\mathbf{I} + 2\mu \operatorname{sym} \nabla \mathbf{v} - 2\mu_r \operatorname{skw} \nabla \mathbf{v} - 2\mu_r \boldsymbol{\omega}_{skw}, \tag{3.25}$$

$$\mathbf{C} = c_0 \operatorname{div} \boldsymbol{\omega} \mathbf{I} + 2c_d \operatorname{sym} \nabla \boldsymbol{\omega} - 2c_a \operatorname{skw} \nabla \boldsymbol{\omega}, \tag{3.26}$$

gdje $\operatorname{sym} \mathbf{A}$ označava simetrični dio tenzora $\mathbf{A}$, odnosno

$$\operatorname{sym} \mathbf{A} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right), \tag{3.27}$$

a $\operatorname{skw} \mathbf{A}$ antisimetrični dio tenzora $\mathbf{A}$, tj.

$$\operatorname{skw} \mathbf{A} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right). \tag{3.28}$$

Tensor $\boldsymbol{\omega}_{skw}$ je antisimetrični tenzor vektora $\boldsymbol{\omega}$ definiran sa

$$(\boldsymbol{\omega}_{skw})_{ij} = \varepsilon_{mij} \boldsymbol{\omega}_m. \tag{3.29}$$

uz iste oznake kao u (3.15).

U (3.25) skalarno polje p označava tlak, $\mathbf{I}$ je jedinična matrica, a λ i μ su koeficijenti viskoznosti za koje vrijede nejednakosti

$$\mu \geq 0, \qquad 3\lambda + 2\mu \geq 0. \tag{3.30}$$

Konstante μ_r, c_0, c_d i c_a u izrazima (3.25) i (3.26) su koeficijenti mikroviskoznosti i za njih vrijede sljedeća svojstva

$$\mu_r \geq 0, \quad c_d \geq 0, \quad 3c_0 + 2c_d \geq 0, \quad |c_d - c_a| \leq c_d + c_a. \tag{3.31}$$

Također ćemo pretpostaviti da je fluid idealan i politropan, tj. da vrijedi

$$p = R\rho\theta, \tag{3.32}$$

$$E = c_v \theta, \tag{3.33}$$

gdje je skalarno polje θ apsolutna temperatura, a R i c_v su pozitivne konstante pri čemu konstantu c_v nazivamo specifičnom toplinom. Uvažava se i Fourierov zakon

$$\mathbf{q} = -k\nabla\theta \tag{3.34}$$

pri čemu je k pozitivna konstanta koju zovemo toplinski konduktivitet.

3.3 Postavka modela. Početni i rubni uvjeti

Uzimajući u obzir navedene zakone očuvanja (3.4), (3.10), (3.21) i (3.23) dobivamo sljedeći sustav jednadžbi

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \tag{3.35}$$

$$\rho \dot{\mathbf{v}} = \operatorname{div} \mathbf{T}, \tag{3.36}$$

$$\rho j_I \dot{\boldsymbol{\omega}} = \operatorname{div} \mathbf{C} + \mathbf{T}_x, \tag{3.37}$$

$$\rho \dot{E} = - \operatorname{div} \mathbf{q} + \mathbf{T} : \nabla \mathbf{v} + \mathbf{C} : \nabla \boldsymbol{\omega} - \mathbf{T}_x \cdot \boldsymbol{\omega}, \tag{3.38}$$

sa svojstvima

$$\mathbf{T} = (-p + \lambda \operatorname{div} \mathbf{v})\mathbf{I} + 2\mu \operatorname{sym} \nabla \mathbf{v} - 2\mu_r \operatorname{skw} \nabla \mathbf{v} - 2\mu_r \boldsymbol{\omega}_{skw}, \tag{3.39}$$

$$\mathbf{C} = c_0 \operatorname{div} \boldsymbol{\omega} \mathbf{I} + 2c_d \operatorname{sym} \nabla \boldsymbol{\omega} - 2c_a \operatorname{skw} \nabla \boldsymbol{\omega}, \tag{3.40}$$

$$p = R\rho\theta, \tag{3.41}$$

$$E = c_v \theta, \tag{3.42}$$

$$\mathbf{q} = -k \nabla \theta. \tag{3.43}$$

Sustav (3.35)-(3.43) ćemo razmatrati na području

$$Q_T = \Omega \times]0, T[\tag{3.44}$$

gdje je $T > 0$ proizvoljno, a

$$\Omega = \{\mathbf{x} \in \mathbf{R}^3, a < |\mathbf{x}| < b\},\ a > 0, \tag{3.45}$$

predstavlja domenu omeđenu sa dvije koncentrične sfere radijusa a i b te rubom

$$\partial\Omega = \{\mathbf{x} \in \mathbf{R}^3, |\mathbf{x}| = a \text{ ili } |\mathbf{x}| = b\}. \tag{3.46}$$

Pretpostavljamo da vrijede sljedeći početni uvjeti

$$\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}), \tag{3.47}$$

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \tag{3.48}$$

$$\boldsymbol{\omega}(\mathbf{x}, 0) = \boldsymbol{\omega}_\mathbf{0}(\mathbf{x}), \tag{3.49}$$

$$\theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \tag{3.50}$$

za $\mathbf{x} \in \Omega$, te rubni uvjeti

$$\mathbf{v}|_{\partial\Omega} = 0, \tag{3.51}$$

$$\boldsymbol{\omega}|_{\partial\Omega} = 0, \tag{3.52}$$

$$\left.\frac{\partial \theta}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \tag{3.53}$$

za $0 < t < T$, gdje je $\mathbf{n}$ vektor vanjske normale.

Rubni uvjeti (3.51)-(3.53) fizikalno definiraju tok fluida između dvije čvrste toplinski izolirane stijenke.

Opisani model do sada je razmatran isključivo u jednodimenzionalnom slučaju, npr. u [Muj98b]. U ovom radu isti model se razmatra u trodimenzionalnom slučaju, ali uz pretpostavku da problem zadovoljava svojstvo sferne simetrije tj. da početne funkcije i traženo rješenje ovise samo o vremenskoj varijabli t i prostornoj varijabli $r = |\mathbf{x}|$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

3.4 Sferno simetrični model

U ovom poglavlju model (3.35)-(3.43), (3.47)-(3.53) prevodimo u sferno simetrični oblik. U skladu s time pretpostavimo najprije da su početni uvjeti (3.47) sferno simetrični, tj. da vrijedi

$$\rho_0(\mathbf{x}) = \rho_0(r), \tag{3.54}$$

$$\mathbf{v}_0(\mathbf{x}) = \frac{\mathbf{x}}{r} v_0(r), \tag{3.55}$$

$$\boldsymbol{\omega}_0(\mathbf{x}) = \frac{\mathbf{x}}{r} \omega_0(r), \tag{3.56}$$

$$\theta_0(\mathbf{x}) = \theta_0(r). \tag{3.57}$$

Rješenje $(\rho, \mathbf{v}, \boldsymbol{\omega}, \theta)$ problema (3.35)-(3.43), (3.47)-(3.51) tražimo u obliku

$$\rho(\mathbf{x}, t) = \rho(r, t), \tag{3.58}$$

$$\mathbf{v}(\mathbf{x}, t) = v(r, t)\frac{\mathbf{x}}{r}, \tag{3.59}$$

$$\boldsymbol{\omega}(\mathbf{x}, t) = \omega(r, t)\frac{\mathbf{x}}{r}, \tag{3.60}$$

$$\theta(\mathbf{x}, t) = \theta(r, t). \tag{3.61}$$

Uzimajući u obzir (3.58)-(3.61), sustav (3.35)-(3.43) očito će poprimiti jednostavniji oblik. Prvo ćemo transformirati jednakosti (3.39) i (3.40).

Kako je

$$\operatorname{div} \frac{\mathbf{x}}{r} = \frac{2}{r} \tag{3.62}$$

lako se dokaže da vrijedi

$$\operatorname{div} \mathbf{v} = \frac{\partial v}{\partial r} + \frac{2}{r} v. \tag{3.63}$$

Uzimajući u obzir da je

$$v_i(\mathbf{x}, t) = \frac{v(r, t)}{r} x_i, i = 1, 2, 3 \tag{3.64}$$

gdje su v_i komponente vektorskog polja $\mathbf{v}$, a x_i komponente vektora $\mathbf{x}$, dobivamo da je gradijent polja $\mathbf{v}$ oblika

$$\nabla \mathbf{v} = \frac{v}{r}\mathbf{I} + \left(\frac{\partial v}{\partial r}\frac{1}{r^2} - \frac{v}{r^3}\right) \mathbf{x} \otimes \mathbf{x}, \tag{3.65}$$

gdje je $\mathbf{I}$ jedinična matrica, a $\otimes$ oznaka za tenzorski produkt vektora. Iz (3.65) lako se vidi da je $\nabla\mathbf{v}$ simetričan tenzor, odnosno da vrijedi

$$\nabla\mathbf{v} = (\nabla\mathbf{v})^T. \tag{3.66}$$

Iz (3.29) zaključujemo da vrijedi

$$\boldsymbol{\omega}_{skw} = \frac{\omega}{r}\mathbf{x}_{skw}. \tag{3.67}$$

Analogno formulama (3.63)-(3.66) za vektorsko polje $\boldsymbol{\omega}$ dobivamo da vrijedi

$$\operatorname{div}\boldsymbol{\omega} = \frac{\partial\omega}{\partial r} + \frac{2}{r}\omega, \tag{3.68}$$

$$\nabla\boldsymbol{\omega} = \frac{\omega}{r}\mathbf{I} + \left(\frac{\partial\omega}{\partial r}\frac{1}{r^2} - \frac{\omega}{r^3}\right)\mathbf{x}\otimes\mathbf{x}, \tag{3.69}$$

$$\nabla\boldsymbol{\omega} = (\nabla\boldsymbol{\omega})^T. \tag{3.70}$$

Uvrštavanjem (3.63), (3.65), (3.66) i (3.67) u (3.39), te sređivanjem, tenzor $\mathbf{T}$ postaje

$$\mathbf{T} = \left(-p + \lambda\left(\frac{\partial v}{\partial r} + 2\frac{v}{r}\right) + 2\mu\frac{v}{r}\right)\mathbf{I} + 2\mu\left(\frac{\partial v}{\partial r}\frac{1}{r^2} - \frac{v}{r^3}\right)\mathbf{x}\otimes\mathbf{x} - 2\mu_r\frac{\omega}{r}\mathbf{x}_{skw}, \tag{3.71}$$

dok uvrštavanjem (3.68)-(3.70) u (3.40) dobivamo

$$\mathbf{C} = \left(c_0\frac{\partial\omega}{\partial r} + 2c_0\frac{\omega}{r} + 2c_d\frac{\omega}{r}\right)\mathbf{I} + 2c_d\left(\frac{\partial\omega}{\partial r}\frac{1}{r^2} - \frac{\omega}{r^3}\right)\mathbf{x}\otimes\mathbf{x}. \tag{3.72}$$

Sada ćemo u sferno simetričnom obliku zapisati i preostale jednadžbe sustava (3.35)-(3.43). Za određivanje divergencija tenzora $\mathbf{T}$ i $\mathbf{C}$ koristimo sljedeća svojstva:

$$\operatorname{div}\mathbf{I} = 0, \tag{3.73}$$

$$\operatorname{div}(\mathbf{x}\otimes\mathbf{x}) = 4\mathbf{x}, \tag{3.74}$$

$$\operatorname{div}\mathbf{x}_{skw} = 0, \tag{3.75}$$

$$\operatorname{div}(\varphi\mathbf{U}) = \varphi\operatorname{div}\mathbf{U} + \mathbf{U}\nabla\varphi \tag{3.76}$$

gdje je φ proizvoljno skalarno, a $\mathbf{U}$ proizvoljno tenzorsko polje. Za divergenciju tenzora $\mathbf{T}$ dobivamo

$$\operatorname{div}\mathbf{T} = \left(-\frac{\partial p}{\partial r} + \lambda\left(\frac{\partial^2 v}{\partial r^2} + 2\frac{\partial v}{\partial r}\frac{1}{r} - 2\frac{v}{r^2}\right) + 2\mu\left(\frac{\partial^2 v}{\partial r^2} + 2\frac{\partial v}{\partial r}\frac{1}{r} - 2\frac{v}{r^2}\right)\right)\frac{\mathbf{x}}{r}. \tag{3.77}$$

Analogno, za divergenciju tenzora $\mathbf{C}$ slijedi

$$\operatorname{div}\mathbf{C} = \left(c_0\omega_{rr} + 2c_0\left(\frac{\omega_r}{r} - \frac{\omega}{r^2}\right) + 2c_d\left(\omega_{rr} + 2\frac{\omega_r}{r} - 2\frac{\omega}{r^2}\right)\right)\frac{\mathbf{x}}{r}. \tag{3.78}$$

Kako vektor $\mathbf{T}_x$, prema (3.15), ima oblik

$$\mathbf{T}_x = (T_{23} - T_{32}, T_{31} - T_{13}, T_{12} - T_{21}) \tag{3.79}$$

vidimo da on ne sadrži dijagonalne elemente tenzora $\mathbf{T}$. Uzimajući u obzir da su prva dva sumanda u izrazu (3.71) simetrični tenzori, te uvrštavajući (3.71) u (3.79), zaključujemo da je

$$\mathbf{T}_x = -4\mu_r \frac{\omega}{r}\mathbf{x}. \tag{3.80}$$

Iz (3.65) i (3.71), kao i iz (3.69) i (3.72) za skalarne umnoške tenzora $\mathbf{T}$ i $\mathbf{C}$ respektivno sa gradijentima $\nabla\mathbf{v}$ i $\nabla\boldsymbol{\omega}$, dobivamo

$$\mathbf{T} : \nabla\mathbf{v} = \left(-p + \lambda\left(v_r + 2\frac{v}{r}\right) + 2\mu\frac{v}{r}\right)\left(v_r + 2\frac{v}{r}\right) + 2\mu v_r\left(v_r - \frac{v}{r}\right), \tag{3.81}$$

$$\mathbf{C} : \nabla\boldsymbol{\omega} = \left(c_0\omega_r + 2c_0\frac{\omega}{r} + 2c_d\frac{\omega}{r}\right)\left(\omega_r + 2\frac{\omega}{r}\right) + 2c_d\omega_r\left(\omega_r - \frac{\omega}{r}\right), \tag{3.82}$$

dok iz (3.80) i (3.60) slijedi

$$\mathbf{T}_x \cdot \boldsymbol{\omega} = -4\mu_r\omega^2. \tag{3.83}$$

Za materijalne derivacije funkcija ρ, $\mathbf{v}$, $\boldsymbol{\omega}$ i θ u sferno simetričnom obliku sada imamo

$$\dot{\rho} = \rho_t + v\rho_r, \tag{3.84}$$

$$\dot{\mathbf{v}} = (v_t + vv_r)\frac{\mathbf{x}}{r}, \tag{3.85}$$

$$\dot{\boldsymbol{\omega}} = (\omega_t + v\omega_r)\frac{\mathbf{x}}{r}, \tag{3.86}$$

$$\dot{\theta} = \theta_t + v\theta_r. \tag{3.87}$$

Uvrštavanjem (3.63), (3.77), (3.78), (3.80)-(3.87), u sustav (3.35)-(3.43) i sređivanjem dobivenih jednadžbi, dolazimo do sljedećeg sustava jednadžbi

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial r}(v\rho) + \frac{2\rho}{r}v = 0, \tag{3.88}$$

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right) = -R\frac{\partial}{\partial r}(\rho\theta) + (\lambda + 2\mu)\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r} + 2\frac{v}{r}\right), \tag{3.89}$$

$$\rho j_I\left(\frac{\partial\omega}{\partial t} + v\frac{\partial\omega}{\partial r}\right) = -4\mu_r\omega + (c_0 + 2c_d)\frac{\partial}{\partial r}\left(\frac{\partial\omega}{\partial r} + 2\frac{\omega}{r}\right), \tag{3.90}$$

$$\begin{aligned}\rho c_v\left(\frac{\partial\theta}{\partial t} + v\frac{\partial\theta}{\partial r}\right) &= k\left(\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial\theta}{\partial r}\right) - R\rho\theta\left(\frac{\partial v}{\partial r} + 2\frac{v}{r}\right)\\ &+(\lambda + 2\mu)\left(\frac{\partial v}{\partial r} + 2\frac{v}{r}\right)^2 - 4\mu\frac{v}{r}\left(2\frac{\partial v}{\partial r} + \frac{v}{r}\right) +\\ (c_0 + 2c_d)&\left(\frac{\partial\omega}{\partial r} + 2\frac{\omega}{r}\right)^2 - 4c_d\frac{\omega}{r}\left(2\frac{\partial\omega}{\partial r} + \frac{\omega}{r}\right) + 4\mu_r\omega^2.\end{aligned} \tag{3.91}$$

Početni uvjeti (3.47)-(3.50) sada postaju

$$\rho(r, 0) = \rho_0(r), \tag{3.92}$$

$$v(r, 0) = v_0(r), \tag{3.93}$$

$$\omega(r, 0) = \omega_0(r), \tag{3.94}$$

$$\theta(r,0)=\theta_0(r), \tag{3.95}$$

za $r\in]a,b[$, dok rubni uvjeti (3.51)-(3.53) poprimaju oblik

$$v(a,t)=v(b,t)=0, \tag{3.96}$$

$$\omega(a,t)=\omega(b,t)=0, \tag{3.97}$$

$$\frac{\partial\theta}{\partial r}(a,t)=\frac{\partial\theta}{\partial r}(b,t)=0, \tag{3.98}$$

za $t\in]0,T[$. Primjetimo da smo trodimenzionalni problem zadan na prostornoj domeni (3.45) preveli u jednodimenzionalni problem na domeni $]a,b[$, gdje su a i b radijusi rubnih sfera iz (3.46).

3.5 Lagrangeova deskripcija

U prethodnom poglavlju sve su veličine (gustoća, brzina, mikrorotacija i temperatura) dane u tzv. Eulerovoj deskripciji, tj. one su opisane kao funkcije vremenske koordinate t i prostorne koordinate $r\in[a,b]$, pri čemu r predstavlja poziciju materijalne točke u trenutku t. Prateći ideje iz [AKM90], [Jia96] i [CK02], u cilju dobivanja jednostavnijih jednadžbi, problem (3.88)-(3.98) prevodimo iz Eulerove deskripcije u tzv. Lagrangeovu deskripciju. U Lagrageovoj deskripciji sve su veličine opisane kao funkcije vremenske koordinate t i prostorne koordinate $\xi\in[a,b]$, gdje je ξ početna pozicija razmatrane materijalne točke.

Eulerove koordinate (r,t) i Lagrangeove koordinate (ξ,t) povezane su relacijom

$$r(\xi,t)=r_0(\xi)+\int\limits_0^t\tilde{v}(\xi,t)d\tau \tag{3.99}$$

gdje je $\tilde{v}(\xi,t):=v(r(\xi,t),t)$ i $r_0(\xi)=r(\xi,0)$.

Neka je funkcija η definirana sa

$$\eta(\xi)=\int\limits_a^\xi\rho_0(s)s^2ds. \tag{3.100}$$

Primjetimo da će inverz η^{-1} postojati ako je $\rho_0(s)>0$ za sve $s\in[a,b]$. Navedeno svojstvo funkcije ρ_0 bit će kasnije navedeno kao pretpostavka teorema egzistencije.

Korištenjem jednadžbe (3.88) i relacije (3.99) dobivamo da vrijedi

$$\frac{\partial}{\partial t}\int\limits_a^{r(\xi,t)}\rho(s)s^2ds=0 \tag{3.101}$$

pa integriranjem preko $[0,t]$ slijedi

$$\int\limits_a^{r(\xi,t)}\rho(s)s^2ds=\int\limits_a^\xi\rho_0(s)s^2ds=\eta(\xi). \tag{3.102}$$

Označimo nadalje

$$\eta(b) = L. \tag{3.103}$$

Budući je $\eta(a) = 0$ problem je sada definiran na prostornoj domeni $[0, L]$.

Kako bi dodatno pojednostavili problem uvest ćemo koordinatu x, $0 \le x \le 1$ sa

$$x = L^{-1}\eta(\xi) \tag{3.104}$$

i definirati nove funkcije $\rho(x,t)$, $v(x,t)$, $\omega(x,t)$ i $\theta(x,t)$ preko funkcija $\tilde{\rho}(\xi,t)$, $\tilde{v}(\xi,t)$, $\tilde{\omega}(\xi,t)$ i $\tilde{\theta}(\xi,t)$ zadanim u Lagrangeovim koordinatama na sljedeći način:

$$\rho(x,t) = \tilde{\rho}\left(\eta^{-1}(xL), t\right), \tag{3.105}$$

$$v(x,t) = \tilde{v}\left(\eta^{-1}(xL), t\right), \tag{3.106}$$

$$\omega(x,t) = \tilde{\omega}\left(\eta^{-1}(xL), t\right), \tag{3.107}$$

$$\theta(x,t) = \tilde{\theta}\left(\eta^{-1}(xL), t\right), \tag{3.108}$$

Pomoću funkcije r zadane sa (3.99) dobivamo funkciju

$$r(x,t) = r\left(\eta^{-1}(xL), t\right). \tag{3.109}$$

Početne funkcije sada postaju

$$\rho_0(x) = \rho_0\left(\eta^{-1}(xL)\right), \tag{3.110}$$

$$v_0(x) = v_0\left(\eta^{-1}(xL)\right), \tag{3.111}$$

$$\omega_0(x) = \omega_0\left(\eta^{-1}(xL)\right), \tag{3.112}$$

$$\theta_0(x) = \theta_0\left(\eta^{-1}(xL)\right), \tag{3.113}$$

$$r_0(x) = r_0\left(\eta^{-1}(xL)\right) = \eta^{-1}(xL). \tag{3.114}$$

Sada ćemo sustav (3.88)-(3.91) iskazati u novom koordinatnom sustavu. Primijetimo da vrijede jednakosti[1]

$$\frac{\partial r(x,t)}{\partial x} = \frac{L}{\rho(x,t) r^2(x,t)}, \tag{3.115}$$

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial f(r,t)}{\partial t} + v(r,t)\frac{\partial f(r,t)}{\partial r}, \tag{3.116}$$

$$\frac{\partial f(x,t)}{\partial x} = \frac{\partial f(r,t)}{\partial r} \cdot \frac{\partial r(x,t)}{\partial x} = \frac{L}{\rho(x,t) r^2(x,t)}\frac{\partial f(r,t)}{\partial r} \tag{3.117}$$

iz kojih odmah dobivamo

$$\frac{\partial \rho}{\partial t} = -\frac{1}{L}\rho^2 \frac{\partial}{\partial x}\left(r^2 v\right) \tag{3.118}$$

što predstavlja jednadžbu (3.88) u novim koordinatama x i t.

[1] u nastavu s f označavamo proizvoljnu funkciju, u našem slučaju ρ, v, ω ili θ.

Kako bi jednadžbe (3.89) i (3.90) iskazali u novoj koordinati koristimo sljedeće jednakosti

$$\frac{\partial v}{\partial r} + 2\frac{v}{r} = \frac{1}{L}\rho\frac{\partial}{\partial x}\left(r^2 v\right), \tag{3.119}$$

$$\frac{\partial \omega}{\partial r} + 2\frac{\omega}{r} = \frac{1}{L}\rho\frac{\partial}{\partial x}\left(r^2 \omega\right). \tag{3.120}$$

pa dobivamo

$$\frac{\partial v}{\partial t} = r^2\frac{\partial}{\partial x}\left(-\frac{R}{L}\rho\theta + \frac{\lambda+2\mu}{L^2}\rho\frac{\partial}{\partial x}\left(r^2 v\right)\right) \tag{3.121}$$

i

$$\frac{\partial \omega}{\partial t} = -\frac{4\mu_r}{j_I}\frac{\omega}{\rho} + r^2\frac{\partial}{\partial x}\left(\frac{c_0+2c_d}{L^2}\rho\frac{\partial}{\partial x}\left(r^2\omega\right)\right). \tag{3.122}$$

Jednadžbu (3.91) transformiramo korištenjem jedakosti

$$\frac{v}{r}\left(2\frac{\partial v}{\partial r} + \frac{v}{r}\right) = \frac{1}{L}\rho\frac{\partial}{\partial x}\left(r v^2\right), \tag{3.123}$$

$$\frac{\omega}{r}\left(2\frac{\partial \omega}{\partial r} + \frac{\omega}{r}\right) = \frac{1}{L}\rho\frac{\partial}{\partial x}\left(r \omega^2\right), \tag{3.124}$$

$$\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial\theta}{\partial r} = \frac{1}{L^2}\rho\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial\theta}{\partial x}\right) \tag{3.125}$$

te dobivamo model u Lagrangeovoj deskripciji:

$$\frac{\partial\rho}{\partial t} = -\frac{1}{L}\rho^2\frac{\partial}{\partial x}\left(r^2 v\right), \tag{3.126}$$

$$\frac{\partial v}{\partial t} = -\frac{R}{L}r^2\frac{\partial}{\partial x}(\rho\theta) + \frac{\lambda+2\mu}{L^2}r^2\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}\left(r^2 v\right)\right), \tag{3.127}$$

$$\rho\frac{\partial\omega}{\partial t} = -\frac{4\mu_r}{j_I}\omega + \frac{c_0+2c_d}{j_I L^2}r^2\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}\left(r^2\omega\right)\right), \tag{3.128}$$

$$\begin{aligned}\rho\frac{\partial\theta}{\partial t} &= \frac{k}{c_v L^2}\rho\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial\theta}{\partial x}\right) - \frac{R}{c_v L}\rho^2\theta\frac{\partial}{\partial x}\left(r^2 v\right) + \frac{\lambda+2\mu}{c_v L^2}\left[\rho\frac{\partial}{\partial x}\left(r^2 v\right)\right]^2 \\ &-\frac{4\mu}{c_v L}\rho\frac{\partial}{\partial x}\left(r v^2\right) + \frac{c_0+2c_d}{c_v L^2}\left[\rho\frac{\partial}{\partial x}\left(r^2\omega\right)\right]^2 - \frac{4c_d}{c_v L}\rho\frac{\partial}{\partial x}\left(r\omega^2\right) + \frac{4\mu_r}{c_v}\omega^2,\end{aligned} \tag{3.129}$$

$$\rho(x,0) = \rho_0(x), \tag{3.130}$$

$$v(x,0) = v_0(x), \tag{3.131}$$

$$\omega(x,0) = \omega_0(x), \tag{3.132}$$

$$\theta(x,0) = \theta_0(x), \tag{3.133}$$

$$v(0,t) = v(1,t) = 0, \tag{3.134}$$

$$\omega(0,t) = \omega(1,t) = 0, \tag{3.135}$$

$$\frac{\partial\theta}{\partial x}(0,t) = \frac{\partial\theta}{\partial x}(1,t) = 0, \tag{3.136}$$

za $x \in]0,1[$, $t \in]0,T[$, $T > 0$.

Primjetimo još da vrijedi

$$r(x,t) = r_0(x) + \int\limits_0^t v(x,\tau)d\tau,\ (x,t) \in]0,1[\times[0,T[\tag{3.137}$$

te je

$$\frac{\partial r(x,t)}{\partial t} = v(x,t), \tag{3.138}$$

pa uzevši da je $t = 0$ i integrirajući preko $]0,x[$ dobivamo

$$r_0(x) = \left(a^3 + 3L\int\limits_0^x \frac{1}{\rho_0(y)}dy\right)^{\frac{1}{3}},\ x \in]0,1[, \tag{3.139}$$

gdje je $a > 0$ polumjer manje rubne sfere iz (3.46).

4 Glavni rezultati

Glavni cilj ovog rada je istražiti egzistenciju i jedinstvenost generaliziranog rješenja problema (3.126)-(3.136), pri čemu se egzistencija promatra lokalno i globalno po vremenu. Generalizirano rješenje uvodimo sljedećom definicijom.

Definicija 4.1. *Generalizirano rješenje problema (3.126)-(3.136) na području* $Q_T =]0,1[\times]0,T[$, $T > 0$ *je funkcija*

$$(x,t) \mapsto (\rho, v, \omega, \theta)(x,t), \quad (x,t) \in Q_T, \tag{4.1}$$

gdje je

$$\rho \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(]0,1[)) \cap \mathrm{H}^1(Q_T)\ ,\ \inf_{Q_T} \rho > 0\ , \tag{4.2}$$

$$v, \omega, \theta \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(]0,1[)) \cap \mathrm{H}^1(Q_T) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(]0,1[)), \tag{4.3}$$

koja zadovoljava jednadžbe (3.126)-(3.129) skoro svuda na Q_T *te uvjete (3.130)-(3.136) u smislu tragova.*

Prema teoremima ulaganja i interpolacije funkcijskih prostora iz (4.2) i (4.3) zaključujemo da će za funkcije ρ, v, ω i θ vrijediti sljedeća svojstva:

$$\rho \in \mathrm{L}^\infty(0,T;\mathrm{C}([0,1])) \cap \mathrm{C}(0,T;\mathrm{L}^2(]0,1[)), \tag{4.4}$$

$$v, \omega, \theta \in \mathrm{L}^2(0,T;\mathrm{C}^{(1)}([0,1])) \cap \mathrm{C}(0,T;\mathrm{H}^1(]0,1[)), \tag{4.5}$$

$$v, \omega, \theta \in \mathrm{C}(\overline{Q}_T). \tag{4.6}$$

Temeljem navedenih svojstava zaključujemo da je rješenje iz Definicije 4.1 ujedno i jako rješenje opisanog problema.

Pretpostavljamo da su početna gustoća i početna temperatura iz prostora $\mathrm{H}^1(]0,1[)$. Za početnu gustuću i početnu temperaturu pretpostavljamo i da su ograničene odozdo, odnosno da vrijedi

$$\rho_0(x) \geq m, \quad \theta_0(x) \geq m \quad \text{for } x \in]0,1[, \tag{4.7}$$

gdje je $m \in \mathbb{R}^+$. Za funkcije v_0 i ω_0 zahtjevamo pripadnost prostoru $\mathrm{H}_0^1(]0,1[)$.

Kako je prostor $\mathrm{H}^1(]0,1[)$ uložen u prostor $\mathrm{C}([0,1])$ zaključujemo da postoji konstanta $M \in \mathbb{R}^+$ tako da vrijedi

$$\rho_0(x),\ |v_0(x)|,\ |\omega_0(x)|,\ \theta_0(x) \leq M,\ x \in [0,1]. \tag{4.8}$$

Primijetimo sada da funkcija r_0 definirana s (3.139) pripada prostoru $\mathrm{H}^2(]0,1[)$, a kako je prostor $\mathrm{H}^2(]0,1[)$ uložen u prostor $\mathrm{C}^{(1)}([0,1])$ imamo da je

$$r_0 \in C^{(1)}([0,1]), \tag{4.9}$$

i zaključujemo da vrijedi

$$0 < a \leq r_0(x) \leq M, \tag{4.10}$$

$$0 < a_1 \leq r_0'(x) \leq M_1,\ x \in [0,1], \tag{4.11}$$

gdje su $a_1 = LM^{-3}$ i $M_1 = L(ma^2)^{-1}$, dok je konstanta a polumjer manje sfere u postavljenoj domeni na početku rada.

Kao što smo u uvodu naveli, glavni rezultati ovog rada iskazani su kroz tri teorema pri, čemu prvi teorem govori o egzistenciji generaliziranog rješenja lokalno po vremenu, drugi teorem odnosi se na jedinstvenost tog rješenja, dok u u trećem teoremu dokazujemo egzistenciju generaliziranog rješenja globalno po vremenu.

Teorem 4.1. *Neka funkcije*

$$\rho_0, \theta_0 \in \mathrm{H}^1(]0,1[) \tag{4.12}$$

zadovoljavaju uvjete (4.7) te neka je

$$v_0, \omega_0 \in \mathrm{H}_0^1(]0,1[). \tag{4.13}$$

Tada postoji T_0, $0 < T_0 \leq T$, takav da problem (3.126)-(3.136) ima generalizirano rješenje na području $Q_0 = Q_{T_0}$, sa svojstvom

$$\theta > 0 \ \text{ na } \overline{Q}_0. \tag{4.14}$$

Također, za funkciju r vrijedi

$$r \in \mathrm{L}^\infty(0,T;\mathrm{H}^2(]0,1[)) \cap \mathrm{H}^2(Q_0) \cap \mathrm{C}(\overline{Q}_0), \tag{4.15}$$

$$\frac{a}{2} \leq r \leq 2M \text{ na } \overline{Q}_0. \tag{4.16}$$

Teorem 4.2. *Neka početne funkcije ρ_0, v_0, ω_0 θ_0 zadovoljavaju uvjete iz prethodnog teorema. Tada problem (3.126)-(3.136) na području Q_T ima najviše jedno generalizirano rješenje $(\rho, v, \omega, \theta)$ sa svojstvom*

$$\theta > 0 \ \text{ in } \overline{Q}_T. \tag{4.17}$$

Bitno je napomenuti da dokaz Teorema 4.2 ne ovisi o veličini intervala vremenske varijable zbog čega se u iskazu ovog teorema uzima da je $T_0 = T$.

Teorem 4.3. *Neka početne funkcije ρ_0, v_0, ω_0 i θ_0 zadovoljavaju uvjete iz Teorema 4.1. Tada za svako $T \in \mathbb{R}^+$ postoji generalizirano rješenje problema (3.126)-(3.136) na području Q_T sa svojstvom*

$$\theta > 0 \ \text{ na } \overline{Q}_T. \tag{4.18}$$

5 Dokaz egzistencije lokalnog rješenja

U ovom poglavlju dokazujemo lokalnu egzistenciju generaliziranog rješenja problema (3.126)-(3.136), odnosno dokazujemo Teorem 4.1.

Dokaz teorema baziran je na Faedo-Galerkinovoj metodi. Najprije za svako $n \in \mathbb{N}$ definiramo aproksimativni problem, a potom konstruiramo niz aproksimativnih rješenjate za koji izvodimo apriorne ocjene uniformne po n. Do tih apriornih ocjena dolazimo koristeći se tehnikom Kazhikova [AKM90] koju je za problem mikropolarnog fluida prilagodila Mujaković u [Muj98b]. Pomoću dobivenih ocjena i teorije slabo kompaktnih nizova formiramo slabo konvergentan podniz aproksimativnih rješenja u različitim prostorima funkcija i dokazujemo da je limes tog podniza rješenje problema (3.126)-(3.136) na $]0,1[\times]0,T_0[$ za dovoljno maleno T_0, $0 < T_0 \leq T$. Na kraju provjeravamo zadovoljava li dobiveni limes sve tvrdnje Teorema 4.1.

5.1 Aproksimativna rješenja

Namjera nam je naći lokalno generalizirano rješenje problema (3.126)-(3.136) kao limes aproksimativnih rješenja

$$(\rho^n, v^n, \omega^n, \theta^n),\ n \in \mathbb{N}, \tag{5.1}$$

čija će konstrukcija biti objašnjena u nastavku.

Najprije uvodimo aproksimacije v^n i r^n funkcija v i r sa

$$v^n(x,t) = \sum_{i=1}^{n} v_i^n(t) \sin(\pi i x), \tag{5.2}$$

$$r^n(x,t) = r_0(x) + \int_0^t v^n(x,\tau) d\tau, \tag{5.3}$$

gdje je $r_0(x)$ definiran sa (3.139) a v_i^n, $i = 1, 2, ..., n$ su nepoznate dovoljno glatke funkcije definirane na nekom intervalu $[0, T_n]$, $T_n \leq T$.

Funkcija ρ^n definirana je kao rješenje problema

$$\frac{\partial \rho^n}{\partial t} + L^{-1}(\rho^n)^2 \frac{\partial}{\partial x}\left((r^n)^2 v^n\right) = 0, \tag{5.4}$$

$$\rho^n(x, 0) = \rho_0(x). \tag{5.5}$$

Slično kao u [Muj98b] funkcija ρ^n može se pisati u obliku

$$\rho^n(x, t) = \frac{L\rho_0(x)}{L + \rho_0(x) \dfrac{\partial}{\partial x} \displaystyle\int_0^t (r^n)^2 v^n d\tau}. \tag{5.6}$$

Zbog glatkoće funkcija r^n i v^n dobivamo da je funkcija ρ^n neprekidna na pravokutniku $[0, 1] \times [0, T_n]$ te vrijedi

$$\rho^n(x, 0) = \rho_0(x) \geq m > 0. \tag{5.7}$$

Sada možemo zaključiti da postoji takav T_n, $0 < T_n \leq T$ tako da je

$$\rho^n(x, t) > 0, \tag{5.8}$$

za $(x, t) \in [0, 1] \times [0, T_n]$.

Na sličan način uvesti ćemo aproksimacije funkcija ω i θ koje označavamo sa ω^n i θ^n, respektivno

$$\omega^n(x, t) = \sum_{j=1}^{n} \omega_j^n(t) \sin(\pi j x), \tag{5.9}$$

$$\theta^n(x, t) = \sum_{k=0}^{n} \theta_k^n(t) \cos(\pi k x), \tag{5.10}$$

gdje su ω_j^n i θ_k^n ponovno nepoznate dovoljno glatke funkcije definirane na intervalu $[0, T_n]$, $T_n \leq T$.

Iz konstrukcije aproksimativnih funkcija jasno je da vrijede rubni uvjeti

$$v^n(0, t) = v^n(1, t) = 0, \tag{5.11}$$

$$\omega^n(0, t) = \omega^n(1, t) = 0, \tag{5.12}$$

$$\frac{\partial \theta^n}{\partial x}(0, t) = \frac{\partial \theta^n}{\partial x}(1, t) = 0, \tag{5.13}$$

za $t \in]0, T_n[$.

U skladu sa Faedo-Galerkinovom metodom promatramo sljedeće aproksimativne uvjete:

$$\begin{aligned} &\int_0^1 \left[\frac{\partial v^n}{\partial t} + \frac{R}{L}(r^n)^2 \frac{\partial}{\partial x}(\rho^n \theta^n) \right. \\ &\left. - \frac{\lambda + 2\mu}{L^2}(r^n)^2 \frac{\partial}{\partial x}\left(\rho^n \frac{\partial}{\partial x}\left((r^n)^2 v^n\right)\right)\right] \sin(\pi i x) dx = 0, \end{aligned} \tag{5.14}$$

$$\begin{aligned} &\int_0^1 \left[\frac{\partial \omega^n}{\partial t} + \frac{4\mu_r}{\mathrm{j_I}} \frac{\omega^n}{\rho^n} - \right. \\ &\left. -\frac{c_0 + 2c_d}{\mathrm{j_I} L^2} (r^n)^2 \frac{\partial}{\partial x} \left(\rho^n \frac{\partial}{\partial x} \left((r^n)^2 \omega^n \right) \right) \right] \sin(\pi j x) dx = 0, \end{aligned} \tag{5.15}$$

$$\begin{aligned} \int_0^1 \Bigg[\frac{\partial \theta^n}{\partial t} &- \frac{k}{c_v L^2} \frac{\partial}{\partial x} \left((r^n)^4 \rho^n \frac{\partial \theta^n}{\partial x} \right) + \frac{R}{c_v L} \rho^n \theta^n \frac{\partial}{\partial x} \left((r^n)^2 v^n \right) \\ &- \frac{\lambda + 2\mu}{c_v L^2} \rho^n \left[\frac{\partial}{\partial x} \left((r^n)^2 v^n \right) \right]^2 + \frac{4\mu}{c_v L} \frac{\partial}{\partial x} \left(r^n (v^n)^2 \right) \\ &- \frac{c_0 + 2c_d}{c_v L^2} \rho^n \left[\frac{\partial}{\partial x} \left((r^n)^2 \omega^n \right) \right]^2 + \frac{4c_d}{c_v L} \frac{\partial}{\partial x} \left(r^n (\omega^n)^2 \right) \\ &- \frac{4\mu_r}{c_v} \frac{(\omega^n)^2}{\rho^n} \Bigg] \cos(\pi k x) dx = 0 \end{aligned} \tag{5.16}$$

za $i, j = 1, ..., n$, $k = 0, 1, ..., n$.

Nadalje definiramo v_0^n, ω_0^n i θ_0^n sa

$$v_0^n(x) = \sum_{i=1}^{n} v_{0i} \sin(\pi i x), \tag{5.17}$$

$$\omega_0^n(x) = \sum_{j=1}^{n} \omega_{0j} \sin(\pi j x), \tag{5.18}$$

$$\theta_0^n(x) = \sum_{k=0}^{n} \theta_{0k} \cos(\pi k x), \tag{5.19}$$

pri čemu su v_{0i}, ω_{0j} i θ_{0k} Fourierovi koeficijenti u razvoju funkcija v_0, ω_0 i θ_0, pa vrijedi

$$v_{0i} = 2 \int_0^1 v_0(x) \sin(\pi i x) dx, \quad i = 1, ..., n, \tag{5.20}$$

$$\omega_{0j} = 2 \int_0^1 \omega_0(x) \sin(\pi j x) dx, \quad j = 1, ..., n, \tag{5.21}$$

$$\theta_{00} = \int_0^1 \theta_0(x) dx, \quad \theta_{0k} = 2 \int_0^1 \theta_0(x) \cos(\pi k x) dx, \quad k = 1, ..., n. \tag{5.22}$$

Početne uvjete za v^n, ω^n i θ^n uzimamo u obliku

$$v^n(x, 0) = v_0^n(x), \tag{5.23}$$

$$\omega^n(x, 0) = \omega_0^n(x), \tag{5.24}$$

$$\theta^n(x,0) = \theta_0^n(x). \tag{5.25}$$

Definirajmo funkcije z_m^n, λ_{pq}^n μ_{slg}^n sa

$$z_m^n(t) = \int_0^t v_m^n(\tau)d\tau, \quad m = 1, ..., n, \tag{5.26}$$

$$\lambda_{pq}^n(t) = \int_0^t z_p^n(\tau)v_q^n(\tau)d\tau, \quad p, q = 1, ..., n, \tag{5.27}$$

$$\mu_{slg}^n(t) = \int_0^t z_l^n(\tau)z_s^n(\tau)v_g^n(\tau)d\tau, \quad s, l, g = 1, ..., n. \tag{5.28}$$

Imamo

$$r^n(x,t) = r_0(x) + \sum_{m=1}^{n} z_m^n(t)\sin(\pi m x), \tag{5.29}$$

$$\begin{aligned}\rho^n(x,t) = L\rho_0(x)\Bigg[L + \rho_0\frac{\partial}{\partial x}\Bigg[r_0^2(x)\sum_{i=1}^{n} z_i^n(t)\sin(\pi i x)\\ +2r_o(x)\sum_{i,j=1}^{n}\lambda_{ij}^n(t)\sin(\pi i x)\sin(\pi j x)\\ +\sum_{i,j,k=1}^{n}\mu_{ijk}^n(t)\sin(\pi i x)\sin(\pi j x)\sin(\pi k x)\Bigg]\Bigg]^{-1},\end{aligned} \tag{5.30}$$

gdje su $r_0(x)$ i $\rho_0(x)$ poznate funkcije.

Uzevši u obzir (5.2), (5.9), (5.10), (5.26)-(5.30), iz (5.14)-(5.16) proizlazi za

$$\{(v_i^n, \omega_j^n, \theta_k^n, z_m^n, \lambda_{pq}^n, \mu_{slg}^n) : i, j, m, p, q, s, l, g = 1, ..., n, \ k = 0, 1, ..., n\} \tag{5.31}$$

sljedeći Cauchyjev problem:

$$\dot{v}_i^n(t) = \phi_i^n(v_1^n, ...v_n^n, \omega_1^n, ..., \omega_n^n, \theta_0^n, \theta_1^n, ..., \theta_n^n, z_1^n, ..., z_n^n, \lambda_{11}^n, ..., \lambda_{nn}^n, \mu_{111}^n, ..., \mu_{nnn}^n), \tag{5.32}$$

$$\dot{\omega}_j^n(t) = \Psi_j^n(v_1^n, ...v_n^n, \omega_1^n, ..., \omega_n^n, \theta_0^n, \theta_1^n, ..., \theta_n^n, z_1^n, ..., z_n^n, \lambda_{11}^n, ..., \lambda_{nn}^n, \mu_{111}^n, ..., \mu_{nnn}^n), \tag{5.33}$$

$$\dot{\theta}_k^n(t) = \lambda_k\Pi_k^n(v_1^n, ...v_n^n, \omega_1^n, ..., \omega_n^n, \theta_0^n, \theta_1^n, ..., \theta_n^n, z_1^n, ..., z_n^n, \lambda_{11}^n, ..., \lambda_{nn}^n, \mu_{111}^n, ..., \mu_{nnn}^n), \tag{5.34}$$

$$\dot{z}_m^n(t) = v_m^n, \tag{5.35}$$

$$\dot{\lambda}_{pq}^n(t) = z_p^n \cdot v_q^n, \tag{5.36}$$

$$\dot{\mu}_{slg}^n(t) = z_s^n \cdot z_l^n \cdot v_g^n, \tag{5.37}$$

$$v_i^n(0) = v_{0i}, \ \omega_j^n(0) = \omega_{0j}, \ \theta_k^n(0) = \theta_{0k}, \tag{5.38}$$

$$z_m^n(0) = 0, \ \lambda_{pq}^n(0) = 0, \ \mu_{slg}^n(0) = 0. \tag{5.39}$$

Ovdje je $\lambda_0 = 1, \lambda_k = 2$ za $k = 1, 2, ..., n$ i

$$\Phi_i^n = 2\int_0^1 \left[\frac{\lambda+2\mu}{L^2}(r^n)^2\frac{\partial}{\partial x}\left(\rho^n\frac{\partial}{\partial x}\left((r^n)^2v^n\right)\right) - \frac{R}{L}(r^n)^2\frac{\partial}{\partial x}(\rho^n\theta^n)\right]\sin(\pi i x)dx, \tag{5.40}$$

$$\Psi_j^n = 2\int_0^1 \left[\frac{c_0+2c_d}{\mathrm{j_I}L^2}(r^n)^2\frac{\partial}{\partial x}\left(\rho^n\frac{\partial}{\partial x}\left((r^n)^2\omega^n\right)\right) - \frac{4\mu_r}{\mathrm{j_I}}\frac{\omega^n}{\rho^n}\right]\sin(\pi j x)dx, \tag{5.41}$$

$$\begin{aligned}\Pi_k^n = \int_0^1 \Bigg[&\frac{k}{c_vL^2}\frac{\partial}{\partial x}\left((r^n)^4\rho^n\frac{\partial\theta^n}{\partial x}\right) - \frac{R}{c_vL}\rho^n\theta^n\frac{\partial}{\partial x}\left((r^n)^2v^n\right)\\ &+\frac{\lambda+2\mu}{c_vL^2}\rho^n\left[\frac{\partial}{\partial x}\left((r^n)^2v^n\right)\right]^2 - \frac{4\mu}{c_vL}\frac{\partial}{\partial x}\left(r^n(v^n)^2\right)\\ &+\frac{c_0+2c_d}{c_vL^2}\rho^n\left[\frac{\partial}{\partial x}\left((r^n)^2\omega^n\right)\right]^2 - \frac{4c_d}{c_vL}\frac{\partial}{\partial x}\left(r^n(\omega^n)^2\right) + \frac{4\mu_r}{c_v}\frac{(\omega^n)^2}{\rho^n}\Bigg]\cos(\pi k x)dx.\end{aligned} \tag{5.42}$$

Primijetimo da funkcije sa desnih strana diferencijalnih jednadžbi (5.32)-(5.37) zadovoljavaju uvjete Cauchy-Picardova teorema, pa lako možemo zaključiti da vrijedi sljedeća lema.

Lema 5.1. *Za svako $n \in \mathbb{N}$ postoji takav $T_n,\ 0 < T_n \leq T$, tako da Cauchyjev problem (5.32)-(5.39) ima jedinstveno rješenje definirano na $[0, T_n]$. Funkcije v^n, ω^n i θ^n definirane izrazima (5.2), (5.9) i (5.10) pripadaju prostoru $\mathrm{C}^\infty(\overline{Q}_n), Q_n =]0,1[\times]0,T_n[$ i zadovoljavaju uvjete (5.23)-(5.25).*

Iz izraza (5.29) i (5.30) lako se može zaključiti da je

$$\rho^n \in C(\bar{Q}_n), \tag{5.43}$$

i

$$r^n \in C^{(1)}(\overline{Q}_n). \tag{5.44}$$

Također, za te funkcije dobivamo i sljedeće ocjene.

Lema 5.2. *Postoji takav $T_n, 0 < T_n \leq T$ tako da na domeni $\overline{Q}_n$ funkcije ρ^n, r^n i $\dfrac{\partial r^n}{\partial x}$ zadovoljavaju uvjete*

$$\frac{m}{2} \leq \rho^n(x,t) \leq 2M, \tag{5.45}$$

$$\frac{a}{2} \leq r^n(x,t) \leq 2M, \tag{5.46}$$

$$\frac{a_1}{2} \leq \frac{\partial r^n}{\partial x}(x,t) \leq 2M_1. \tag{5.47}$$

Konstante m, a, a_1, M i M_1 uvedene su izrazima (3.139), (4.7), (4.8), (4.10) i (4.11) .

Dokaz. Vrijedi

$$\rho^n(x,0) = \rho_0(x) \leq M. \tag{5.48}$$

Kako je funkcija $\rho^n(x,t)$ neprekidna po varijabli t na nekom $Q_n = [0,1]\times[0,T_n]$, zbog svojstva funkcija neprekidnih na segmentu iz (5.48) možemo zaključiti da postoji takav T_n, da za $t \in [0,T_n]$ vrijedi

$$\rho^n(x,t) \leq 2M. \tag{5.49}$$

Na isti način iz (4.7) dobivamo ocjenu

$$\frac{m}{2} \leq \rho^n(x,t). \tag{5.50}$$

Analognim zaključivanjem i primjenom ocjena (4.10) i (4.11) iz (5.44) lako dobivamo (5.46) i (5.47).

□

5.2 Svojstva aproksimativnih rješenja

U nastavku navodimo niz svojstava (tj. međuodnosa) funkcija r_n, ρ_n, v^n, ω^n, θ^n kao i njihovih derivacija po varijabli x. Ocjene će se razmatrati na segmentu $[0,T_n]$, gdje je T_n takav da vrijede tvrdnje Lema 5.1 i 5.2.

Sa $C > 0$ nadalje označavamo konstantu neovisue o $n \in \mathbb{N}$ i koja na različitim mjestima može poprimati različite vrijednosti. Vrijednosti konstante C uglavnom ovise o normama početnih funkcija i unaprijed zadanoj vrijednosti T.

Kao što je već rečeno koristimo sljedeću oznaku

$$\|f\| = \|f\|_{L^2(]0,1[)}.$$

Primjetimo da iz definicija (5.17), (5.18) i (5.19) lako dobivamo da je

$$\|v_0^n\| \leq \|v_0\|, \tag{5.51}$$

$$\|\omega_0^n\| \leq \|\omega_0\|, \tag{5.52}$$

$$\|\theta_0^n\| \leq \|\theta_0\|. \tag{5.53}$$

Dokažimo sada sljedećih nekoliko nejednakosti za norme aproksimativnih funkcija i njihovih derivacija.

Lema 5.3. *Za svako $t \in [0,T_n]$ vrijedi nejednakost*

$$\left\|\frac{\partial^2 r^n}{\partial x^2}(t)\right\|^2 \leq C\left(1 + \int_0^t \left\|\frac{\partial^2 v^n}{\partial x^2}(\tau)\right\|^2 d\tau\right). \tag{5.54}$$

Dokaz. Deriviranjem izraza (5.3) dobivamo

$$\frac{\partial^2 r^n}{\partial x^2} = r_0''(x) + \int_0^t \frac{\partial^2 v^n}{\partial x^2} d\tau, \tag{5.55}$$

pa vrijedi nejednakost

$$\left|\frac{\partial^2 r^n}{\partial x^2}\right|^2 \leq |r_0''(x)|^2 + T\int_0^t \left|\frac{\partial^2 v^n}{\partial x^2}\right|^2 d\tau. \tag{5.56}$$

Uzimajući u obzir da r_0 pripada prostoru $H^2\left(]0,1[\right)$ i integrirajući izraz (5.56) preko $]0,1[$ odmah slijedi (5.54). □

Lema 5.4. *Za svako $t \in [0, T_n]$ vrijedi nejednakost*

$$\|\omega^n(t)\|^2 + \int_0^t \left(\|\omega^n(\tau)\|^2 + \left\|\frac{\partial}{\partial x}(r_n^2\omega^n)(\tau)\right\|^2 d\tau \right) \leq C. \tag{5.57}$$

Dokaz. Množenjem jednadžbe (5.15) sa ω_j^n te sumiranjem po $j = 1, ..., n$ uz primjenu parcijalne integracije dobivamo

$$\frac{1}{2}\frac{d}{dt}\|\omega^n(t)\|^2 + \frac{4\mu_r}{j_I}\int_0^1 \frac{(\omega^n)^2}{\rho^n}dx + \frac{c_0+2c_d}{j_I L^2}\int_0^1 \rho^n \left[\frac{\partial}{\partial x}\left((r^n)^2\omega^n\right)\right]^2 dx = 0. \tag{5.58}$$

Nakon integriranja po $[0,t], 0 < t \leq T_n$ i uzimanja u obzir (5.24) i (5.52) dobivamo

$$\begin{aligned} &\frac{1}{2}\|\omega^n(t)\|^2 + \frac{4\mu_r}{j_I}\int_0^t\int_0^1 \frac{(\omega^n)^2}{\rho^n}dxd\tau \\ &+\frac{c_0+2c_d}{j_I L^2}\int_0^t\int_0^1 \rho^n \left[\frac{\partial}{\partial x}\left((r^n)^2\omega^n\right)\right]^2 dxd\tau = \frac{1}{2}\|\omega_0^n\|^2 \leq \frac{1}{2}\|\omega_0\|^2. \end{aligned} \tag{5.59}$$

Primjenom ocjene (5.45) na gornji izraz odmah dobivamo (5.57).

□

Lema 5.5. *Za svako $t \in [0, T_n]$ vrijedi nejednakost*

$$\left|\int_0^1 \theta^n(x,t)dx\right| \leq C\left(1 + \left\|\frac{\partial v^n}{\partial x}(t)\right\|^2\right). \tag{5.60}$$

Dokaz. Množenjem jednadžbe (5.14) sa $c_v^{-1}v_i^n$, sumiranjem po $i = 1, ..., n$, primjenom parcijalne integracije te zbrajanjem s jednadžbom (5.16), stavljajući $k = 0$, slijedi

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2c_v}\|v^n(t)\|^2 + \int_0^1 \theta^n(x,t)dx\right) = \\ &\frac{c_0+2c_d}{c_v L^2}\int_0^1 \rho^n \left[\frac{\partial}{\partial x}\left((r^n)^2\omega^n\right)\right]^2 dx + \frac{4\mu_r}{c_v}\int_0^1 \frac{(\omega^n)^2}{\rho^n}dx. \end{aligned} \tag{5.61}$$

Primjenom (5.23) i (5.25) te integriranjem preko $[0,t], 0 < t \leq T_n$, odmah dobivamo

$$\begin{aligned} \left| \frac{1}{2c_v}\|v^n(t)\|^2 + \int_0^1 \theta^n(x,t)dx \right| &= \\ \left| \int_0^t \left(\frac{c_0 + 2c_d}{c_v L^2} \int_0^1 \rho^n \left[\frac{\partial}{\partial x}\left((r^n)^2\omega^n\right) \right]^2 dx + \frac{4\mu_r}{c_v} \int_0^1 \frac{(\omega^n)^2}{\rho^n} dx \right) dt \right. \\ \left. + \frac{1}{2c_v}\|v_0^n(t)\|^2 + \int_0^1 \theta_0^n(x)dx \right|, \end{aligned} \tag{5.62}$$

pa primjenjujući ocjenu (5.45) dobivamo

$$\begin{aligned} \left| \frac{1}{2c_v}\|v^n(t)\|^2 + \int_0^1 \theta^n(x,t)dx \right| &\leq \\ C \int_0^t \left(\|\omega^n(t)\|^2 + \left\| \frac{\partial}{\partial x}\left((r^n)^2\omega^n\right)(\tau) \right\|^2 \right) d\tau + \frac{1}{2c_v}\|v_0^n(t)\|^2 + \|\theta_0^n(t)\|. \end{aligned} \tag{5.63}$$

Uzevši u obzir (5.57), (5.51) i (5.53) jasno je da vrijedi

$$\left| \frac{1}{2c_v}\|v^n(t)\|^2 + \int_0^1 \theta^n(x,t)dx \right| \leq C, \tag{5.64}$$

tj. da je

$$\left| \int_0^1 \theta^n(x,t)dx \right| \leq C\left(1 + \|v^n(t)\|^2\right). \tag{5.65}$$

Primjenom nejednakosti Friedrichs-Poincarea na funkciju v^n u (5.65) odmah dobivamo tvrdnju leme. □

Lema 5.6. *Za $(x,t) \in \overline{Q}_n$ vrijedi nejednakost*

$$|\theta^n(x,t)| \leq C\left(1 + \left\|\frac{\partial \theta^n}{\partial x}(t)\right\| + \left\|\frac{\partial v^n}{\partial x}(t)\right\|^2\right). \tag{5.66}$$

Dokaz. Neka je $t \in [0, T_n]$ fiksan ali proizvoljan. Kako je funkcija $\theta^n(x,t)$ neprekidna po varijabli x na segmentu $[0,1]$ postoje takvi $x_1(t), x_2(t) \in [0,1]$ tako da vrijedi

$$m_n(t) = \min_{x\in[0,1]} \theta^n(x,t) = \theta^n(x_1(t),t), \tag{5.67}$$

$$M_n(t) = \max_{x\in[0,1]} \theta^n(x,t) = \theta^n(x_2(t),t). \tag{5.68}$$

Prema svojstvima određenog integrala zaključujemo da je

$$\theta^n(x,t) - m_n(t) = \int\limits_{x_1}^{x} \frac{\partial \theta^n}{\partial x}(y,t)dy \le \int\limits_0^1 \left| \frac{\partial \theta^n}{\partial x}(x,t) \right| dx \tag{5.69}$$

pa primjenom Hölderove nejednakosti dobivamo

$$\theta^n(x,t) - m_n(t) \le \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|. \tag{5.70}$$

Iz (5.70) lako zaključujemo da je

$$\theta^n(x,t) \le \left\| \frac{\partial \theta^n}{\partial x}(t) \right\| + m_n(t) \le \left\| \frac{\partial \theta^n}{\partial x}(t) \right\| + \left| \int\limits_0^1 \theta^n(x,t)dx \right|. \tag{5.71}$$

Koristeći funkciju M_n na isti način dobivamo nejednakost

$$\theta^n(x,t) \ge - \left\| \frac{\partial \theta^n}{\partial x}(t) \right\| - \left| \int\limits_0^1 \theta^n(x,t)dx \right| \tag{5.72}$$

i zaključujemo da je

$$|\theta^n(x,t)| \le \left\| \frac{\partial \theta^n}{\partial x}(t) \right\| + \left| \int\limits_0^1 \theta^n(x,t)dx \right|. \tag{5.73}$$

Primjenom rezultata Leme 5.5 slijedi tvrdnja (5.66). □

Lema 5.7. *Za svako $t \in [0, T_n]$ vrijedi nejednakost*

$$\left\| \frac{\partial \rho^n}{\partial x}(t) \right\|^2 \le C \left(1 + \left(\int\limits_0^t \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^2 \right). \tag{5.74}$$

Dokaz. Deriviranjem funkcije (5.6) po varijabli x dobivamo

$$\frac{\partial \rho^n(x,t)}{\partial x} = (\rho^n(x,t))^2 \left(\frac{\rho_0'(x)}{\rho_0^2(x)} + \frac{1}{L} \frac{\partial^2}{\partial x^2} \int\limits_0^t (r^n)^2 v^n d\tau \right) \tag{5.75}$$

te primjenom ocjena (5.45)-(5.47) i (4.7) zaključujemo da vrijedi

$$\left| \frac{\partial \rho^n}{\partial x} \right| \le C \left(|\rho_0'(x)| + \int\limits_0^t \left(|v^n| + |v^n| \left| \frac{\partial^2 r^n}{\partial x^2} \right| + \left| \frac{\partial v^n}{\partial x} \right| + \left| \frac{\partial^2 v^n}{\partial x^2} \right| \right) d\tau \right). \tag{5.76}$$

Kvadriranjem (5.76) i primjenom Cauchy-Schwarzove nejednakosti na desnu stranu dobivenog izraza dobivamo

$$\begin{gathered} \left| \frac{\partial \rho^n}{\partial x} \right|^2 \le C \left(|\rho_0'(x)| + \int\limits_0^t |v^n|^2 d\tau + \right. \\ \left. \int\limits_0^t |v^n|^2 d\tau \cdot \int\limits_0^t \left| \frac{\partial^2 r^n}{\partial x^2} \right|^2 d\tau + \int\limits_0^t \left| \frac{\partial v^n}{\partial x} \right|^2 d\tau + \int\limits_0^t \left| \frac{\partial^2 v^n}{\partial x^2} \right|^2 d\tau \right). \end{gathered} \tag{5.77}$$

Integriranjem (5.77) preko $]0,1[$ i primjenom nejednakosti Friedrichs-Poincarea na funkciju v^n zaključujemo da vrijedi

$$\left\|\frac{\partial\rho^n}{\partial x}(t)\right\|^2 \leq C\left(1+\int_0^1\left\|\frac{\partial^2 v^n}{\partial x^2}\right\|^2 d\tau\cdot\int_0^t\left\|\frac{\partial^2 r^n}{\partial x^2}\right\|^2 d\tau+\int_0^t\left\|\frac{\partial^2 v^n}{\partial x^2}\right\|^2 d\tau\right) \tag{5.78}$$

iz čega pomoću Youngove nejednakosti i (5.54) odmah dobivamo tvrdnju leme. □

Lema 5.8. *Vrijedi nejednakost*

$$\begin{aligned} \frac{d}{dt}\left(\left\|\frac{\partial v^n}{\partial x}(t)\right\|^2+\left\|\frac{\partial\omega^n}{\partial x}(t)\right\|^2+\left\|\frac{\partial\theta^n}{\partial x}(t)\right\|^2\right)+ \\ B\left(\left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\|^2+\left\|\frac{\partial^2\omega^n}{\partial x^2}(t)\right\|^2+\left\|\frac{\partial^2\theta^n}{\partial x^2}(t)\right\|^2\right)\leq \\ C\left(1+\left\|\frac{\partial v^n}{\partial x}(t)\right\|^{16}+\left\|\frac{\partial\omega^n}{\partial x}(t)\right\|^{16}+\left\|\frac{\partial\theta^n}{\partial x}(t)\right\|^{16}+\left(B\int_0^t\left\|\frac{\partial^2 v^n}{\partial x^2}(\tau)\right\|^2 d\tau\right)^8\right) \end{aligned} \tag{5.79}$$

za $t\in[0,T_n]$, *gdje je*

$$B=\frac{ma^4}{32}\min\left\{\frac{\lambda+2\mu}{L^2},\frac{c_0+2c_d}{j_I L^2},\frac{k}{c_v L^2}\right\}. \tag{5.80}$$

Dokaz. Množenjem jednadžbi (5.14), (5.15) i (5.16) respektivno sa $(\pi i)^2 v_i^n$, $(\pi j)^2\omega_j^n$ i $(\pi k)^2\theta_k^n$, uzimanjem u obzir (5.2), (5.9) i (5.10), zbrajanjem po $i,j,k=1,2,...,n$ te zbrajanjem dobivenih jednadžbi imamo

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\left\|\frac{\partial v^n}{\partial x}\right\|^2+\left\|\frac{\partial\omega^n}{\partial x}\right\|^2+\left\|\frac{\partial\theta^n}{\partial x}\right\|^2\right)+\frac{\lambda+2\mu}{L^2}\int_0^1\rho^n(r^n)^4\left(\frac{\partial^2 v^n}{\partial x^2}\right)^2 dx \\ +\frac{c_0+2c_d}{j_I L^2}\int_0^1\rho^n(r^n)^4\left(\frac{\partial^2\omega^n}{\partial x^2}\right)^2 dx+ \\ \frac{k}{c_v L^2}\int_0^1\rho^n(r^n)^4\left(\frac{\partial^2\theta^n}{\partial x^2}\right)^2 dx=\sum_{p=1}^{28} I_p(t), \end{aligned} \tag{5.81}$$

gdje je

$$I_1=-\frac{2(\lambda+2\mu)}{L^2}\int_0^1 (r^n)^3\frac{\partial\rho^n}{\partial x}\frac{\partial r^n}{\partial x}v^n\frac{\partial^2 v^n}{\partial x^2}dx, \tag{5.82}$$

$$I_2=-\frac{2(\lambda+2\mu)}{L^2}\int_0^1 \rho^n\,(r^n)^2\left(\frac{\partial r^n}{\partial x}\right)^2 v^n\frac{\partial^2 v^n}{\partial x^2}dx, \tag{5.83}$$

$$I_3=-\frac{2(\lambda+2\mu)}{L^2}\int_0^1 \rho^n\,(r^n)^3\frac{\partial^2 r^n}{\partial x^2}v^n\frac{\partial^2 v^n}{\partial x^2}dx, \tag{5.84}$$

$$I_4 = -\frac{4(\lambda+2\mu)}{L^2}\int_0^1 \rho^n (r^n)^3 \frac{\partial r^n}{\partial x}\frac{\partial v^n}{\partial x}\frac{\partial^2 v^n}{\partial x^2}dx, \tag{5.85}$$

$$I_5 = -\frac{\lambda+2\mu}{L^2}\int_0^1 (r^n)^4 \frac{\partial \rho^n}{\partial x}\frac{\partial v^n}{\partial x}\frac{\partial^2 v^n}{\partial x^2}dx, \tag{5.86}$$

$$I_6 = \frac{R}{L}\int_0^1 (r^n)^2 \theta^n \frac{\partial \rho^n}{\partial x}\frac{\partial^2 v^n}{\partial x^2}dx, \tag{5.87}$$

$$I_7 = \frac{R}{L}\int_0^1 (r^n)^2 \rho^n \frac{\partial \theta^n}{\partial x}\frac{\partial^2 v^n}{\partial x^2}dx, \tag{5.88}$$

$$I_8 = \frac{4\mu_r}{j_I}\int_0^1 \frac{\omega^n}{\rho^n}\frac{\partial^2 \omega^n}{\partial x^2}dx, \tag{5.89}$$

$$I_9 = -\frac{2(c_0+2c_d)}{j_I L^2}\int_0^1 (r^n)^3 \frac{\partial \rho^n}{\partial x}\frac{\partial r^n}{\partial x}\omega^n\frac{\partial^2 \omega^n}{\partial x^2}dx, \tag{5.90}$$

$$I_{10} = -\frac{2(c_0+2c_d)}{j_I L^2}\int_0^1 \rho^n (r^n)^2 \left(\frac{\partial r^n}{\partial x}\right)^2 \omega^n\frac{\partial^2 \omega^n}{\partial x^2}dx, \tag{5.91}$$

$$I_{11} = -\frac{2(c_0+2c_d)}{j_I L^2}\int_0^1 \rho^n (r^n)^3 \frac{\partial^2 r^n}{\partial x^2}\omega^n\frac{\partial^2 \omega^n}{\partial x^2}dx, \tag{5.92}$$

$$I_{12} = -\frac{4(c_0+2c_d)}{j_I L^2}\int_0^1 \rho^n (r^n)^3 \frac{\partial r^n}{\partial x}\frac{\partial \omega^n}{\partial x}\frac{\partial^2 \omega^n}{\partial x^2}dx, \tag{5.93}$$

$$I_{13} = -\frac{c_0+2c_d}{j_I L^2}\int_0^1 (r^n)^4 \frac{\partial \rho^n}{\partial x}\frac{\partial \omega^n}{\partial x}\frac{\partial^2 \omega^n}{\partial x^2}dx, \tag{5.94}$$

$$I_{14} = -\frac{4k}{c_v L^2}\int_0^1 \rho^n (r^n)^3 \frac{\partial r^n}{\partial x}\frac{\partial \theta^n}{\partial x}\frac{\partial^2 \theta^n}{\partial x^2}dx, \tag{5.95}$$

$$I_{15} = -\frac{k}{c_v L^2}\int_0^1 (r^n)^4 \frac{\partial \rho^n}{\partial x}\frac{\partial \theta^n}{\partial x}\frac{\partial^2 \theta^n}{\partial x^2}dx, \tag{5.96}$$

$$I_{16} = \frac{2R}{c_v L}\int_0^1 \rho^n r^n \frac{\partial r^n}{\partial x}v^n\theta^n\frac{\partial^2 \theta^n}{\partial x^2}dx, \tag{5.97}$$

$$I_{17} = \frac{R}{c_v L}\int_0^1 \rho^n (r^n)^2 \theta^n \frac{\partial v^n}{\partial x}\frac{\partial^2 \theta^n}{\partial x^2}dx, \tag{5.98}$$

$$I_{18} = -\frac{4(\lambda + 2\mu)}{c_v L^2} \int\limits_0^1 \rho^n \left(r^n\right)^2 \left(\frac{\partial r^n}{\partial x}\right)^2 \left(v^n\right)^2 \frac{\partial^2 \theta^n}{\partial x^2} dx, \tag{5.99}$$

$$I_{19} = -\frac{4(\lambda + 2\mu)}{c_v L^2} \int\limits_0^1 \rho^n \left(r^n\right)^3 \frac{\partial r^n}{\partial x} v^n \frac{\partial v^n}{\partial x} \frac{\partial^2 \theta^n}{\partial x^2} dx, \tag{5.100}$$

$$I_{20} = -\frac{\lambda + 2\mu}{c_v L^2} \int\limits_0^1 \rho^n \left(r^n\right)^4 \left(\frac{\partial v^n}{\partial x}\right)^2 \frac{\partial^2 \theta^n}{\partial x^2} dx, \tag{5.101}$$

$$I_{21} = \frac{4\mu}{c_v L} \int\limits_0^1 \left(v^n\right)^2 \frac{\partial r^n}{\partial x} \frac{\partial^2 \theta^n}{\partial x^2} dx, \tag{5.102}$$

$$I_{22} = \frac{8\mu}{c_v L} \int\limits_0^1 r^n v^n \frac{\partial v^n}{\partial x} \frac{\partial^2 \theta^n}{\partial x^2} dx, \tag{5.103}$$

$$I_{23} = -\frac{4(c_0 + 2c_d)}{c_v L^2} \int\limits_0^1 \rho^n \left(r^n\right)^2 \left(\frac{\partial r^n}{\partial x}\right)^2 (\omega^n)^2 \frac{\partial^2 \theta^n}{\partial x^2} dx, \tag{5.104}$$

$$I_{24} = -\frac{4(c_0 + 2c_d)}{c_v L^2} \int\limits_0^1 \rho^n \left(r^n\right)^3 \frac{\partial r^n}{\partial x} \frac{\partial \omega^n}{\partial x} \omega^n \frac{\partial^2 \theta^n}{\partial x^2} dx, \tag{5.105}$$

$$I_{25} = -\frac{c_0 + 2c_d}{c_v L^2} \int\limits_0^1 \rho^n \left(r^n\right)^4 \left(\frac{\partial \omega^n}{\partial x}\right)^2 \frac{\partial^2 \theta^n}{\partial x^2} dx, \tag{5.106}$$

$$I_{26} = \frac{4c_d}{c_v L} \int\limits_0^1 \left(\omega^n\right)^2 \frac{\partial r^n}{\partial x} \frac{\partial^2 \theta^n}{\partial x^2} dx, \tag{5.107}$$

$$I_{27} = \frac{8c_d}{c_v L} \int\limits_0^1 r^n \omega^n \frac{\partial \omega^n}{\partial x} \frac{\partial^2 \theta^n}{\partial x^2} dx, \tag{5.108}$$

$$I_{28} = -\frac{4\mu_r}{c_v} \int\limits_0^1 \frac{(\omega^n)^2}{\rho^n} \frac{\partial^2 \theta^n}{\partial x^2} dx. \tag{5.109}$$

Sada koristeći rezultate prethodnih lema ocjenjujemo integrale $I_1 - I_{28}$. Tako primjenom Cauchy-Swarzove nejednakosti i ocjena (5.46) i (5.47), dobivamo

$$\begin{aligned} |I_1| = \frac{2(\lambda + 2\mu)}{L^2} \left| \int\limits_0^1 \left(r^n\right)^3 \frac{\partial \rho^n}{\partial x} \frac{\partial r^n}{\partial x} v^n \frac{\partial^2 v^n}{\partial x^2} dx \right| \leq \\ C \max_{x \in [0,1]} \left|v^n(x,t)\right| \left\| \frac{\partial \rho^n}{\partial x}(t) \right\| \left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|. \end{aligned} \tag{5.110}$$

Primjenom nejednakosti Friedrichs-Poincarea za funkciju v^n iz (5.110) slijedi

$$|I_1| \leq C \left\| \frac{\partial v^n}{\partial x}(t) \right\| \left\| \frac{\partial \rho^n}{\partial x}(t) \right\| \left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\| \tag{5.111}$$

što nakon primjene Youngove nejednakosti postaje

$$|I_1| \leq \varepsilon \left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|^2 + C \left(\left\| \frac{\partial v^n}{\partial x}(t) \right\|^4 + \left\| \frac{\partial \rho^n}{\partial x}(t) \right\|^4 \right) \tag{5.112}$$

gdje je $\varepsilon > 0$ proizvoljan parametar. Sada uvrstimo (5.74) u (5.112) i primjenimo Youngovu nejednakost te dobivamo

$$|I_1| \leq \varepsilon \left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^{16} + \left(\int_0^t \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^8 \right). \tag{5.113}$$

Za ocjenu ostalih integrala koristimo još ocjene (5.45) i (5.66) te analognim postupkom dobivamo

$$|I_2| \leq \varepsilon \left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^{16} \right), \tag{5.114}$$

$$|I_3| \leq \varepsilon \left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^{16} + \left(\int_0^t \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^8 \right), \tag{5.115}$$

$$|I_4| \leq \varepsilon \left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^{16} \right), \tag{5.116}$$

$$|I_5| \leq \varepsilon \left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^{16} + \left(\int_0^t \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\| d\tau \right)^8 \right), \tag{5.117}$$

$$\begin{aligned} |I_6| \leq \varepsilon \left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|^2 + \\ C \left(1 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^{16} + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^{16} + \left(\int_0^t \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^8 \right), \end{aligned} \tag{5.118}$$

$$|I_7| \leq \varepsilon \left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^{16} \right), \tag{5.119}$$

$$|I_8| \leq \varepsilon \left\| \frac{\partial^2 \omega^n}{\partial x^2}(t) \right\|^2 + C, \tag{5.120}$$

$$|I_9| \leq \varepsilon \left\| \frac{\partial^2 \omega^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^{16} + \left(\int_0^t \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^8 \right), \tag{5.121}$$

$$|I_{10}| \leq \varepsilon \left\| \frac{\partial^2 \omega^n}{\partial x^2}(t) \right\|^2 + C, \tag{5.122}$$

$$|I_{11}| \le \varepsilon \left\| \frac{\partial^2 \omega^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^{16} + \left(\int_0^t \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^8 \right), \tag{5.123}$$

$$|I_{12}| \le \varepsilon \left\| \frac{\partial^2 \omega^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^{16} \right), \tag{5.124}$$

$$|I_{13}| \le \varepsilon \left\| \frac{\partial^2 \omega^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^{16} + \left(\int_0^t \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^8 \right), \tag{5.125}$$

$$|I_{14}| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^{16} \right), \tag{5.126}$$

$$|I_{15}| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^{16} + \left(\int_0^t \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^8 \right), \tag{5.127}$$

$$|I_{16}| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^{16} + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^{16} \right), \tag{5.128}$$

$$|I_{17}| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^{16} + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^{16} \right), \tag{5.129}$$

$$|I_{18}| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^{16} \right), \tag{5.130}$$

$$|I_{19}| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^{16} \right), \tag{5.131}$$

$$|I_{20}| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + \varepsilon \left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^{16} \right), \tag{5.132}$$

$$|I_{21}| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^{16} \right), \tag{5.133}$$

$$|I_{22}| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^{16} \right), \tag{5.134}$$

$$|I_{23}| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^{16} \right), \tag{5.135}$$

$$|I_{24}| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^{16} \right), \tag{5.136}$$

$$|I_{25}| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + \varepsilon \left\| \frac{\partial^2 \omega^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^{16} \right), \tag{5.137}$$

$$|I_{26}| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^{16} \right), \tag{5.138}$$

$$|I_{27}| \leq \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^{16} \right), \tag{5.139}$$

$$|I_{28}| \leq \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^{16} \right). \tag{5.140}$$

Uvrštavanjem (5.113)-(5.140) u (5.81) uz izbor dovoljno malog parametra ε i primjenom ocjena (5.45)-(5.47) dobivamo tvrdnju leme. $\square$

5.3 Apriorne ocjene

Cilj ovog poglavlja je odrediti takav $T_0, 0 < T_0 \leq T$ tako da postoji rješenje $(\rho^n, v^n, \omega^n, \theta^n)$ problema (5.32)-(5.39) definirano na $[0, T_0]$ za svako $n \in \mathbb{N}$. U tu svrhu za funkcije ρ^n, v^n, ω^n i θ^n definirane Lemama 5.1 i 5.2 izvodimo apriorne ocjene uniformne po $n \in \mathbb{N}$.

Lema 5.9. *Postoji takav T_0, $(0 < T_0 \leq T)$ tako da za sve $n \in \mathbb{N}$ Cauchyjev problem (5.32)-(5.39) ima jedinstveno rješenje na intervalu $[0, T_0]$. Štoviše, funkcije v^n, ω^n, θ^n, ρ^n i r^n zadovoljavaju nejednakosti*

$$\begin{aligned} &\max_{t\in[0,T_0]} \left(\left\| \frac{\partial v^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 \right) + \\ B &\int_0^{T_0} \left(\left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 + \left\| \frac{\partial^2 \omega^n}{\partial x^2}(\tau) \right\|^2 + \left\| \frac{\partial^2 \theta^n}{\partial x^2}(\tau) \right\|^2 \right) d\tau \leq C, \end{aligned} \tag{5.141}$$

$$\frac{a}{2} \leq r^n(x,t) \leq 2M, \tag{5.142}$$

$$\frac{a_1}{2} \leq \frac{\partial r^n}{\partial x}(x,t) \leq 2M_1, \tag{5.143}$$

$$\frac{m}{2} \leq \rho^n(x,t) \leq 2M, \quad (x,t) \in \overline{Q}_0, \quad Q_0 = Q_{T_0}, \tag{5.144}$$

$$\max_{t\in[0,T_0]} \left\| \frac{\partial \rho^n}{\partial x}(t) \right\| \leq C, \tag{5.145}$$

$$\max_{t\in[0,T_0]} \left\| \frac{\partial^2 r^n}{\partial x^2}(t) \right\| \leq C, \tag{5.146}$$

pri čemu su a, a_1, m i M definirane sa (4.7) i (4.8)-(4.10).

Dokaz. Definirajmo funkciju

$$y_n(t) = \left\| \frac{\partial v^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 + B \int_0^t \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 d\tau. \tag{5.147}$$

Prema Lemi 5.8 očito je da funkcija y_n zadovoljava diferencijalnu nejednadžbu

$$\dot{y}_n(t) \leq C \left(1 + y_n^8(t) \right). \tag{5.148}$$

Neka je konstanta $\overline{C}$ definirana s

$$\overline{C} = \left\| \frac{dv_0}{dx} \right\|^2 + \left\| \frac{d\omega_0}{dx} \right\|^2 + \left\| \frac{d\theta_0}{dx} \right\|^2 \tag{5.149}$$

i neka je y rješenje sljedećeg Cauchyjevog problema

$$\dot{y}(t) = C\left(1 + y^8(t)\right), \tag{5.150}$$

$$y(0) = \overline{C}. \tag{5.151}$$

Kako je

$$y_n(0) = \left\| \frac{dv_0^n}{dx} \right\|^2 + \left\| \frac{d\omega_0^n}{dx} \right\|^2 + \left\| \frac{d\theta_0^n}{dx} \right\|^2 \leq \overline{C}. \tag{5.152}$$

prema svojstvu maksimalnog rješenja obične diferencijalne jednadžbe[1] zaključujemo da vrijedi

$$y_n(t) \leq y(t),\ t \in [0, T'[\tag{5.153}$$

gdje je $[0, T'[$, $0 < T' \leq T$ interval egzistencije rješenja problema (5.150)-(5.151).

Neka je T_0 iz intervala $]0, T']$. Iz (5.153) i (5.147) dobivamo

$$\begin{aligned} &\max_{t\in[0,T_0]} \left(\left\| \frac{\partial v^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 \right) + \\ &\int_0^{T_0} \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 d\tau \leq \max_{t\in[0,T_0]} y(t) \leq C. \end{aligned} \tag{5.154}$$

Sada koristeći taj rezultat iz (5.79) dobivamo

$$\begin{aligned} &\frac{d}{dt}\left(\left\| \frac{\partial v^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 \right) \\ &+B\left(\left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|^2 + \left\| \frac{\partial^2 \omega^n}{\partial x^2}(t) \right\|^2 + \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 \right) \leq C. \end{aligned} \tag{5.155}$$

Integrirajući (5.79) preko $[0, t]$, $0 < t \leq T_0$ i koristeći ocjene (5.152) i (5.155) odmah dobivamo (5.141).

Pokažimo sada ocjene (5.142) i (5.143). Te ocjene izvodimo temeljem jednakosti (5.3), odnosno baziramo se na ocjenama za funkciju v^n. Koristimo sljedeće nejednakosti dobivene primjenom nejednakosti Gagliardo-Ladizhenskaye i Friedrichs-Poincarea za funkciju v^n:

$$|v^n(x,t)| \leq 2\left\| \frac{\partial v^n}{\partial x}(t) \right\|, \quad \left\| \frac{\partial v^n}{\partial x}(t) \right\| \leq 2\left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|, \quad \left| \frac{\partial v^n}{\partial x}(t) \right| \leq 2\left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|. \tag{5.156}$$

Pomoću (5.156) i (5.141) primjenom Hölderove nejednakosti dolazimo do sljedećih ocjena

$$\int_0^{T_0} |v^n(x,t)|\, d\tau \leq 4\int_0^{T_0} \left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\| d\tau \leq 4(CB^{-1})^{\frac{1}{2}} T_0^{\frac{1}{2}}, \tag{5.157}$$

[1] Teorem 2.19 u drugom poglavlju

$$\int\limits_0^{T_0} \left|\frac{\partial v^n}{\partial x}(t)\right| d\tau \leq 2\int\limits_0^{T_0} \left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\| d\tau \leq 2(CB^{-1})^{\frac{1}{2}} T_0^{\frac{1}{2}} \tag{5.158}$$

gdje su C i B konstante iz (5.141). Iz (5.3) dobivamo

$$|r_0(x)| - \int\limits_0^{T_0} |v^n(x,t)|\, d\tau \leq |r^n(x,t)| \leq |r_0(x)| + \int\limits_0^{T_0} |v^n(x,t)|\, d\tau. \tag{5.159}$$

Uvrštavanjem (5.157) u (5.159), izborom adekvatne vrijednosti T_0 i primjenom ocjena (4.8)-(4.10) dobivamo ocjenu (5.142). Ocjenu (5.143) dobivamo analogno iz nejednakosti

$$|r_0'(x)| - \int\limits_0^{T_0} \left|\frac{\partial v^n}{\partial x}(t)\right| d\tau \leq \left|\frac{\partial r^n}{\partial x}(x,t)\right| \leq |r_0'(x)| + \int\limits_0^{T_0} \left|\frac{\partial v^n}{\partial x}(t)\right| d\tau. \tag{5.160}$$

koju dobivamo deriviranjem izraza (5.3) te ocjene (5.158).

Sada ćemo dokazati ocjenu (5.144). Iz (5.6) dobivamo

$$\frac{Lm}{L + M\dfrac{\partial}{\partial x}\displaystyle\int\limits_0^t (r^n)^2 v^n d\tau} \leq \rho^n(x,t) \leq \frac{LM}{L - M\left|\dfrac{\partial}{\partial x}\displaystyle\int\limits_0^t (r^n)^2 v^n d\tau\right|}. \tag{5.161}$$

Primijetimo da zbog (5.157), (5.158), (5.142) i (5.143) vrijedi

$$\begin{aligned} \left|\frac{\partial}{\partial x}\int\limits_0^t (r^n)^2 v^n d\tau\right| &= \left|\int\limits_0^t 2r^n \frac{\partial r^n}{\partial x} v^n d\tau + \int\limits_0^t (r^n)^2 \frac{\partial v^n}{\partial x} d\tau\right| \\ &\leq 8(CB^{-1})^{\frac{1}{2}} T_0^{\frac{1}{2}} M(4M_1 + M). \end{aligned} \tag{5.162}$$

Uvrštavanjem (5.162) u (5.161) uz izbor adekvatnog T_0 odmah dobivamo (5.144).

Da bi vrijedile ocjene (5.142)-(5.144) prema navedenom mora biti

$$T_0 < \min\left\{T', \frac{a^2 B}{64C}, \frac{a_1^2 B}{16C}, \left(\frac{LB^{\frac{1}{2}}}{16M^2(4M_1+M)C^{\frac{1}{2}}}\right)^2\right\}. \tag{5.163}$$

Ocjene (5.145)-(5.146) dobivamo neposrednom primjenom ocjene (5.141) na (5.74) i (5.54). Još ostaje za pokazati da je rješenje problema (5.32)-(5.39) definirano na $[0, T_0]$. Primijetimo da je zbog (5.2)

$$\left\|\frac{\partial v^n}{\partial x}\right\|^2 = \sum_{i=1}^n (v_i^n)^2 \frac{\pi i}{2} \geq \sum_{i=1}^n (v_i^n)^2. \tag{5.164}$$

Analogno iz (5.9) i (5.10) slijedi

$$\left\|\frac{\partial \omega^n}{\partial x}\right\|^2 \geq \sum_{i=1}^n (\omega_i^n)^2, \tag{5.165}$$

$$\left\|\frac{\partial \theta^n}{\partial x}\right\|^2 \geq \sum_{i=1}^n (\theta_i^n)^2. \tag{5.166}$$

Također je

$$\left(\sum_{i=1}^{n}\left(|v_i^n(t)|+|\omega_i^n(t)|+|\theta_i^n(t)|\right)\right)^2 \leq C\left(\sum_{i=1}^{n}\left(|v_i^n(t)|^2+|\omega_i^n(t)|^2+|\theta_i^n(t)|^2\right)\right). \quad (5.167)$$

Uvrštavanjem (5.164)-(5.166) u (5.167) i koristeći (5.141) lako dobivamo da za $t \in [0, T_0]$ vrijedi

$$\sum_{i=1}^{n}\left(|v_i^n(t)|+|\omega_i^n(t)|+|\theta_i^n(t)|\right) \leq C. \quad (5.168)$$

Još ostaje za ocijeniti $|\theta_0^n(t)|$. Integriranjem (5.10) preko $[0, 1]$ dobivamo

$$\int_0^1 \theta^n(x,t)dx = \theta_0^n(t) \quad (5.169)$$

što uz korištenje (5.60) i (5.141) dovodi do zaključka

$$|\theta_0^n(t)| \leq C \quad (5.170)$$

čime je dokaz leme završen. □

Primjetimo još da iz (5.168) i (5.170) slijedi

$$\max_{t\in[0,T_0]}\left(\|v^n(t)\|^2+\|\omega^n(t)\|^2+\|\theta^n(t)\|^2\right) \leq C, \quad (5.171)$$

dok iz (5.54), (5.74) i (5.66) dobivamo

$$\max_{t\in[0,T_0]}\left\|\frac{\partial^2\rho^n}{\partial x^2}(t)\right\| \leq C, \quad (5.172)$$

$$\max_{(x,t)\in\overline{Q}_0}|\theta^n(t)| \leq C. \quad (5.173)$$

Preostaje nam još dobiti ocjene vremenskih derivacija funkcija v^n, ω^n, θ^n, ρ^n i r^n u cilju zaključivanja ograničenosti niza $(r^n, \rho^n, v^n, \omega^n, \theta^n)$ u odgovarajućim prostorima.

Lema 5.10. *Neka je T_0 definiran Lemom 5.9. Tada za sve $n \in \mathbb{N}$ vrijede nejednakosti*

$$\int_0^{T_0}\left(\left\|\frac{\partial v^n}{\partial t}(\tau)\right\|^2+\left\|\frac{\partial \omega^n}{\partial t}(\tau)\right\|^2+\left\|\frac{\partial \theta^n}{\partial t}(\tau)\right\|^2\right)d\tau \leq C, \quad (5.174)$$

$$\max_{t\in[0,T_0]}\left\|\frac{\partial\rho^n}{\partial t}(t)\right\| \leq C \quad (5.175)$$

$$\max_{t\in[0,T_0]}\left\|\frac{\partial r^n}{\partial t}(t)\right\| \leq C \quad (5.176)$$

$$\max_{t\in[0,T_0]}\left\|\frac{\partial^2 r^n}{\partial x\partial t}(t)\right\| \leq C, \quad (5.177)$$

$$\int_0^{T_0}\left\|\frac{\partial^2 r^n}{\partial t^2}(\tau)\right\|^2 d\tau \leq C. \quad (5.178)$$

Dokaz. Množenjem (5.14) sa $\frac{dv_i^n}{dt}(t)$, zbrajanjem po $i = 1, 2, ..., n$ te korištenjem (5.142)-(5.144) dobivamo

$$\begin{aligned}\left\|\frac{\partial v^n}{\partial t}(t)\right\|^2 &= -\frac{R}{L}\int_0^1 (r^n)^2 \frac{\partial}{\partial x}(\rho^n\theta^n)\frac{\partial v^n}{\partial t}dx+ \\ &\frac{\lambda+2\mu}{L^2}\int_0^1 (r^n)^2\frac{\partial}{\partial x}\left(\rho^n\frac{\partial}{\partial x}\left((r^n)^2 v^n\right)\right)\frac{\partial v^n}{\partial t}dx \leq \\ C\Bigg(&\max_{(x,t)\in\overline{Q}_0}|\theta^n(x,t)|\left\|\frac{\partial \rho^n}{\partial x}(t)\right\|\left\|\frac{\partial v^n}{\partial t}(t)\right\| + \left\|\frac{\partial \theta^n}{\partial x}(t)\right\|\left\|\frac{\partial v^n}{\partial t}(t)\right\| + \\ &\max_{(x,t)\in\overline{Q}_0}|v^n(x,t)|\left\|\frac{\partial \rho^n}{\partial x}(t)\right\|\left\|\frac{\partial v^n}{\partial t}(t)\right\| + \|v^n(t)\|\left\|\frac{\partial v^n}{\partial t}(t)\right\| + \\ &\max_{(x,t)\in\overline{Q}_0}|v^n(x,t)|\left\|\frac{\partial^2 r^n}{\partial x^2}(t)\right\|\left\|\frac{\partial v^n}{\partial t}(t)\right\| + \left\|\frac{\partial v^n}{\partial t}(t)\right\|\left\|\frac{\partial v^n}{\partial x}(t)\right\| + \\ &\max_{(x,t)\in\overline{Q}_0}\left|\frac{\partial v^n}{\partial x}(x,t)\right|\left\|\frac{\partial \rho^n}{\partial x}(t)\right\|\left\|\frac{\partial v^n}{\partial t}(t)\right\| + \left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\|\left\|\frac{\partial v^n}{\partial x}(t)\right\|\Bigg).\end{aligned} \tag{5.179}$$

Korištenjem (5.171), (5.146), (5.141), (5.156) i primjenom Youngove nejednakosti dobivamo

$$\left\|\frac{\partial v^n}{\partial t}(t)\right\|^2 \leq C\left(1 + \left\|\frac{\partial^2 v^n}{\partial x^2}\right\|^2\right) + \varepsilon\left\|\frac{\partial v^n}{\partial t}(t)\right\|^2. \tag{5.180}$$

Uzimajući u obzir (5.141) za dovoljno malen $\varepsilon > 0$ iz (5.180) dobivamo

$$\int_0^{T_0}\left\|\frac{\partial v^n}{\partial t}(\tau)\right\|^2 d\tau \leq C. \tag{5.181}$$

Množenjem (5.15) sa $\frac{d\omega_i^n}{dt}(t)$, zbrajanjem po $i = 1, 2, ..., n$ kao i kod (5.179) imamo

$$\begin{aligned}\left\|\frac{\partial \omega^n}{\partial t}(t)\right\|^2 &= -\frac{4\mu_r}{j_I}\int_0^1 \frac{\omega^n}{\rho^n}\frac{\partial \omega^n}{\partial t}dx+ \\ \frac{c_0+2c_d}{L^2}\int_0^1 (r^n)^2\frac{\partial}{\partial x}\left(\rho^n\frac{\partial}{\partial x}\left((r^n)^2\omega^n\right)\right)\frac{\partial \omega^n}{\partial t}dx &\leq C\Bigg(\|\omega^n(t)\|\left\|\frac{\partial \omega^n}{\partial t}(t)\right\| + \\ &\max_{(x,t)\in\overline{Q}_0}|\omega^n(x,t)|\left\|\frac{\partial \rho^n}{\partial x}(t)\right\|\left\|\frac{\partial \omega^n}{\partial t}(t)\right\| + \|\omega^n(t)\|\left\|\frac{\partial \omega^n}{\partial t}(t)\right\| + \\ &\max_{(x,t)\in\overline{Q}_0}|\omega^n(x,t)|\left\|\frac{\partial^2 r^n}{\partial x^2}(t)\right\|\left\|\frac{\partial \omega^n}{\partial t}(t)\right\| + \left\|\frac{\partial \omega^n}{\partial t}(t)\right\|\left\|\frac{\partial \omega^n}{\partial x}(t)\right\| + \\ &\max_{(x,t)\in\overline{Q}_0}\left|\frac{\partial \omega^n}{\partial x}(x,t)\right|\left\|\frac{\partial \rho^n}{\partial x}(t)\right\|\left\|\frac{\partial \omega^n}{\partial t}(t)\right\| + \left\|\frac{\partial^2 \omega^n}{\partial x^2}(t)\right\|\left\|\frac{\partial \omega^n}{\partial x}(t)\right\|\Bigg)\end{aligned} \tag{5.182}$$

te dobivamo

$$\left\|\frac{\partial \omega^n}{\partial t}(t)\right\|^2 \leq C\left(1 + \left\|\frac{\partial^2 \omega^n}{\partial x^2}\right\|^2\right) + \varepsilon\left\|\frac{\partial \omega^n}{\partial t}(t)\right\|^2. \tag{5.183}$$

odnosno

$$\int_0^{T_0} \left\| \frac{\partial \omega^n}{\partial t}(\tau) \right\|^2 d\tau \leq C. \tag{5.184}$$

Množenjem (5.16) sa $\frac{d\theta_k^n}{dt}(t)$, zbrajanjem po $k = 0, 1, 2, ..., n$ kao i kod (5.179) i (5.182) dobivamo

$$\begin{aligned}
\left\| \frac{\partial \theta^n}{\partial t}(t) \right\|^2 = \frac{k}{c_v L^2} \int_0^1 \frac{\partial}{\partial x}\left((r^n)^4 \rho^n \frac{\partial \theta^n}{\partial x} \right) \frac{\partial \theta^n}{\partial t} dx - \\
\frac{R}{c_v L} \int_0^1 \rho^n \theta^n \frac{\partial}{\partial x}\left((r^n)^2 v^n \right) \frac{\partial \theta^n}{\partial t} dx + \frac{\lambda + 2\mu}{c_v L^2} \int_0^1 \rho^n \left[\frac{\partial}{\partial x}\left((r^n)^2 v^n \right) \right]^2 \frac{\partial \theta^n}{\partial t} dx - \\
\frac{4\mu}{c_v L} \int_0^1 \frac{\partial}{\partial x}\left(r^n (v^n)^2 \right) \frac{\partial \theta^n}{\partial t} dx + \frac{c_0 + 2c_d}{c_v L^2} \int_0^1 \rho^n \left[\frac{\partial}{\partial x}\left((r^n)^2 \omega^n \right) \right]^2 \frac{\partial \theta^n}{\partial t} dx - \\
\frac{4c_d}{c_v L} \int_0^1 \frac{\partial}{\partial x}\left(r^n (\omega^n)^2 \right) \frac{\partial \theta^n}{\partial t} dx + \frac{4\mu_r}{c_v} \int_0^1 \frac{(\omega^n)^2}{\rho^n} \frac{\partial \theta^n}{\partial t} dx \leq C \left(\left\| \frac{\partial \theta^n}{\partial x}(t) \right\| \left\| \frac{\partial \theta^n}{\partial t}(t) \right\| + \right. \\
\max_{(x,t) \in \overline{Q}_0} \left| \frac{\partial \theta^n}{\partial x}(x,t) \right| \left\| \frac{\partial \rho^n}{\partial x}(t) \right\| \left\| \frac{\partial \theta^n}{\partial t}(t) \right\| + \left\| \frac{\partial \theta^n}{\partial t}(t) \right\| \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\| + \\
\max_{(x,t) \in \overline{Q}_0} |\theta^n(x,t)| \, \|v^n(t)\| \left\| \frac{\partial \theta^n}{\partial t}(t) \right\| + \max_{(x,t) \in \overline{Q}_0} |\theta^n(x,t)| \left\| \frac{\partial v^n}{\partial x}(t) \right\| \left\| \frac{\partial \theta^n}{\partial t}(t) \right\| + \\
\|v^n(t)\| \left\| \frac{\partial \theta^n}{\partial t}(t) \right\| + \max_{(x,t) \in \overline{Q}_0} |v^n(x,t)| \left\| \frac{\partial v^n}{\partial x}(t) \right\| \left\| \frac{\partial \theta^n}{\partial t}(t) \right\| + \\
\max_{(x,t) \in \overline{Q}_0} \left| \frac{\partial v^n}{\partial x}(x,t) \right| \left\| \frac{\partial v^n}{\partial x}(t) \right\| \left\| \frac{\partial \theta^n}{\partial t}(t) \right\| + \max_{(x,t) \in \overline{Q}_0} |v^n(x,t)| \, \|v^n(t)\| \left\| \frac{\partial \theta^n}{\partial t}(t) \right\| + \\
+ \max_{(x,t) \in \overline{Q}_0} |\omega^n(x,t)| \, \|\omega^n(t)\| \left\| \frac{\partial \theta^n}{\partial t}(t) \right\| + \max_{(x,t) \in \overline{Q}_0} |\omega^n(x,t)| \left\| \frac{\partial \omega^n}{\partial x}(t) \right\| \left\| \frac{\partial \theta^n}{\partial t}(t) \right\| + \\
\left. \max_{(x,t) \in \overline{Q}_0} \left| \frac{\partial \omega^n}{\partial x}(x,t) \right| \left\| \frac{\partial \omega^n}{\partial x}(t) \right\| \left\| \frac{\partial \theta^n}{\partial t}(t) \right\| \right)
\end{aligned} \tag{5.185}$$

i zaključujemo da vrijedi

$$\int_0^{T_0} \left\| \frac{\partial \theta^n}{\partial t}(\tau) \right\|^2 d\tau \leq C. \tag{5.186}$$

čime je ocjena (5.174) dokazana.

Iz (5.4) slijedi nejednakost

$$\left\| \frac{\partial \rho^n}{\partial t}(t) \right\|^2 \leq C \left(\|v^n\|^2 + \|v^n\|^2 \left\| \frac{\partial v^n}{\partial x}(t) \right\| + \left\| \frac{\partial v^n}{\partial x}(t) \right\|^2 \right). \tag{5.187}$$

Primjenom ocjena (5.141) i (5.171) na desnu stranu u (5.187) odmah dobivamo (5.175).

Ocjene (5.176)-(5.178) dobivamo deriviranjem izraza (5.3), uzimanjem normi i primjenom ocjena (5.141) i (5.171). □

Propozicija 5.1. *Neka je T_0 definiran Lemom 5.9. Tada za niz $(r^n, \rho^n, v^n, \omega^n, \theta^n)$ vrijede sljedeća svojstva:*

(i) Niz (r^n) je omeđen u prostorima $\mathrm{L}^\infty(Q_0)$, $\mathrm{L}^\infty(0, T_0; \mathrm{H}^2(]0, 1[))$ i $\mathrm{H}^2(Q_0)$,

(ii) Niz $\left(\dfrac{\partial r^n}{\partial x}\right)$ je omeđen u prostoru $\mathrm{L}^\infty(Q_0)$,

(iii) Niz (ρ^n) je omeđen u prostorima $\mathrm{L}^\infty(Q_0)$, $\mathrm{L}^\infty(0, T_0; \mathrm{H}^1(]0, 1[))$ i $\mathrm{H}^1(Q_0)$,

(iv) Nizovi (v^n), (ω^n) i (θ^n) su omeđeni u prostorima $\mathrm{L}^\infty(0, T_0; \mathrm{H}^1(]0, 1[))$, $\mathrm{H}^1(Q_0)$ kao i u $\mathrm{L}^2(0, T_0; \mathrm{H}^2(]0, 1[))$.

Dokaz. Omeđenost nizova je posljedica prethodnih lema kako slijedi.

(i) Iz (5.142) slijedi omeđenost u prostoru $\mathrm{L}^\infty(Q_0)$. Iz (5.142), (5.145) i (5.146) slijedi omeđenost u prostoru $\mathrm{L}^\infty(0, T_0; \mathrm{H}^2(]0, 1[))$, dok omeđenost u $\mathrm{H}^2(Q_0)$ slijedi iz (5.142), (5.142), (5.143), (5.146), (5.176), (5.177) i (5.178).

(ii) Tvrdnja slijedi iz (5.143).

(iii) Omeđenost u prostoru $\mathrm{L}^\infty(Q_0)$ je posljedica ocjene (5.144). Iz (5.144) i (5.145) slijedi omeđenost u prostoru $\mathrm{L}^\infty(0, T_0; \mathrm{H}^1(]0, 1[))$, dok je omeđenost u prostoru $\mathrm{H}^1(Q_0)$ posljedica ocjena (5.144), (5.145) i (5.175).

(iv) Omeđenost u prostorima $\mathrm{L}^\infty(0, T_0; \mathrm{H}^1(]0, 1[))$ i $\mathrm{L}^2(0, T_0; \mathrm{H}^2(]0, 1[))$ slijedi iz (5.171) i (5.141), dok omeđenost u prostoru $\mathrm{H}^1(Q_0)$ slijedi iz (5.171), (5.141) i (5.174).

□

5.4 Dokaz Teorema 4.1

Neka je $T_0 \in \mathbb{R}^+$ definiran Lemom 5.9. Teorem 4.1 je posljedica sljedećih lema.

Lema 5.11. *Postoji funkcija*

$$r \in \mathrm{L}^\infty(0, T_0; \mathrm{H}^2(]0, 1[)) \cap \mathrm{H}^2(Q_0) \cap \mathrm{C}(\overline{Q}_0) \tag{5.188}$$

i podniz niza[2] (r^n) *sa sljedećim svojstvima:*

$$r^n \overset{*}{\rightharpoonup} r \text{ u } \mathrm{L}^\infty(0, T_0; \mathrm{H}^2(]0, 1[)), \tag{5.189}$$

$$r^n \rightharpoonup r \text{ u } \mathrm{H}^2(Q_0), \tag{5.190}$$

$$r^n \to r \text{ u } \mathrm{C}(\overline{Q}_0), \tag{5.191}$$

[2]zbog jednostavnosti i dalje označen s (r^n)

$$\frac{\partial r^n}{\partial x} \to \frac{\partial r}{\partial x} \text{ u } \mathrm{C}(\overline{Q}_0). \tag{5.192}$$

Funkcija r zadovoljava isljedeće uvjete:

$$\frac{a}{2} \leq r \leq 2M \quad \textit{in} \;\; \overline{Q}_0, \tag{5.193}$$

$$r(x,0) = r_0(x), \quad x \in [0,1], \tag{5.194}$$

gdje je r_0 definirano s (3.139).

Dokaz. Tvrdnje (5.189) i (5.190) direktna su posljedica Propozicije 5.1 (i) u skladu s Teoremima (2.7) i (2.8) iz drugog poglavlja.

Za dokaz jake konvergencija u prostoru $\mathrm{C}(\overline{Q}_0)$ koristimo Arzela-Ascolijev teorem. Pokažimo najprije ekvineprekidnost niza (r^n). Neka $(x,t),\ (x',t')$ pripadaju području $\overline{Q}_0$. Imamo

$$|r^n(x,t) - r^n(x',t')| \leq |r^n(x,t) - r^n(x',t)| + |r^n(x',t) - r^n(x',t')|. \tag{5.195}$$

Prema (5.143) zaključujemo

$$|r^n(x,t) - r^n(x',t)| \leq \int_{x'}^{x} \left|\frac{\partial r^n}{\partial x}(\xi,t)\right| d\xi \leq C|x-x'|. \tag{5.196}$$

Koristeći (5.3) i primjenjujući nejednakosti Friedrichs-Poincarea na funkciju v^n te ocjenu (5.141), dobivamo

$$\begin{aligned} |r^n(x',t) - r^n(x',t')| &\leq \int_{t'}^{t} \left|\frac{\partial r^n}{\partial t}(x',\tau)\right| d\tau = \int_{t'}^{t} |v^n(x',\tau)| d\tau \leq \\ & 2\int_{t}^{t'} \left\|\frac{\partial v^n}{\partial x}(\tau)\right\| d\tau \leq C|t-t'|. \end{aligned} \tag{5.197}$$

Uvrštavanjem (5.196) i (5.197) u (5.195) dobivamo ekvineprekidnost niza (r^n) pa sa (5.142) imamo zadovoljene sve pretpostavke Arzela-Ascolijeva što povlači konvergenciju podniza (5.191).

Pokažimo sada ekvineprekidnost niza $\left(\dfrac{\partial r^n}{\partial x}\right)$. Neka su opet $(x,t),\ (x',t') \in \overline{Q}_0$. Imamo

$$\left|\frac{\partial r^n}{\partial x}(x,t) - \frac{\partial r^n}{\partial x}(x',t')\right| \leq \left|\frac{\partial r^n}{\partial x}(x,t) - \frac{\partial r^n}{\partial x}(x',t)\right| + \left|\frac{\partial r^n}{\partial x}(x',t) - \frac{\partial r^n}{\partial x}(x',t')\right|. \tag{5.198}$$

Sada, primjenom Caucyh-Schwarzove nejednakosti i ocjene (5.146) zaključujemo da vrijedi

$$\begin{aligned} \left|\frac{\partial r^n}{\partial x}(x',t) - \frac{\partial r^n}{\partial x}(x',t)\right| &\leq \int_{x'}^{x} \left|\frac{\partial^2 r^n}{\partial x^2}(\xi,t)\right| d\xi \leq \\ \left\|\frac{\partial^2 r^n}{\partial x^2}(t)\right\| |x-x'|^{1/2} &\leq C|x-x'|^{1/2}. \end{aligned} \tag{5.199}$$

Istim postupkom uz primjenu ocjene (5.177) dobivamo

$$\begin{aligned} \left|\frac{\partial r^n}{\partial x}(x',t) - \frac{\partial r^n}{\partial x}(x',t)\right| \leq \int\limits_{t'}^{t} \left|\frac{\partial^2 r^n}{\partial x \partial t}(\xi,t)\right| d\xi \leq \\ \left\|\frac{\partial^2 r^n}{\partial x \partial t}(t)\right\| |x-x'|^{1/2} \leq C|x-x'|^{1/2} \end{aligned} \tag{5.200}$$

pa slijedi konvergencija (5.192).

Pokažimo još tvrdnje (5.193) i (5.194). Zbog (5.191) i (5.142) zaključujemo da vrijedi

$$\frac{a}{2} - \varepsilon < r^n(x,t) - \varepsilon < r(x,t) < r^n(x,t) + \varepsilon < 2M + \varepsilon, \tag{5.201}$$

za $(x,t) \in \overline{Q}_0$ i za svako $\varepsilon > 0$. Iz (5.201) lako zaključujemo da vrijedi (5.193). Iz (5.191) slijedi

$$\lim_{n\to\infty} \max_{x\in[0,1]} |r^n(x,0) - r(x,0)| = \max_{x\in[0,1]} |r_0(x) - r(x,0)| = 0 \tag{5.202}$$

što povlači (5.194).

□

Lema 5.12. *Postoji funkcija*

$$\rho \in \mathrm{L}^\infty(0,T_0;\mathrm{H}^1(]0,1[)) \cap \mathrm{H}^1(Q_0) \cap \mathrm{C}(\overline{Q}_0) \tag{5.203}$$

i podniz[3] *niza* (ρ^n) *sa sljedećim svojstvima:*

$$\rho^n \overset{*}{\rightharpoonup} \rho \textit{ u } \mathrm{L}^\infty(0,T_0;\mathrm{H}^1(]0,1[)), \tag{5.204}$$

$$\rho^n \rightharpoonup \rho \textit{ u } \mathrm{H}^1(Q_0), \tag{5.205}$$

$$\rho^n \to \rho \textit{ u } \mathrm{C}(\overline{Q}_0). \tag{5.206}$$

Funkcija ρ *zadovoljava uvjete*

$$\frac{m}{2} \leq \rho(x,t) \leq 2M \quad \textit{in} \;\; \overline{Q}_0, \tag{5.207}$$

$$\rho(x,0) = \rho_0(x), \quad x \in [0,1]. \tag{5.208}$$

Dokaz. Uzevši u obzir Propoziciju 5.1, ocjene (5.141)-(5.145) i Arzelà-Ascolijev teorem analogno kao u dokazu prethodne leme dobivamo svojstva (5.204)-(5.208). □

Lema 5.13. *Postoje funkcije*

$$v,\omega,\theta \in \mathrm{L}^\infty(0,T_0;\mathrm{H}^1(]0,1[)) \cap \mathrm{H}^1(Q_0) \cap \mathrm{L}^2(0,T_0;\mathrm{H}^2(]0,1[)) \tag{5.209}$$

i podnizovi[4] (v^n,ω^n,θ^n) *tako da*

$$(v^n,\omega^n,\theta^n) \overset{*}{\rightharpoonup} (v,\omega,\theta) \textit{ u } (\mathrm{L}^\infty(0,T_0;\mathrm{H}^1(]0,1[)))^3, \tag{5.210}$$

[3]zbog jednostavnosti i dalje označen s (ρ^n)
[4]zbog jednostavnosti i dalje označeni s (v^n,ω^n,θ^n)

$$(v^n, \omega^n, \theta^n) \rightharpoonup (v, \omega, \theta) \text{ u } (\mathrm{H}^1(Q_0))^3, \tag{5.211}$$

$$(v^n, \omega^n, \theta^n) \rightharpoonup (v, \omega, \theta) \text{ u } (\mathrm{L}^2(0, T_0; \mathrm{H}^2(]0,1[)))^3, \tag{5.212}$$

$$(v^n, \omega^n, \theta^n) \to (v, \omega, \theta) \text{ u } (\mathrm{L}^2(Q_0))^3, \tag{5.213}$$

Funkcije v, ω i θ zadovoljavaju uvjete

$$v(x,0) = v_0(x), \tag{5.214}$$

$$\omega(x,0) = \omega_0(x), \tag{5.215}$$

$$\theta(x,0) = \theta_0(x), \tag{5.216}$$

za $x \in [0,1]$,

$$v(0,t) = v(1,t) = 0, \tag{5.217}$$

$$\omega(0,t) = \omega(1,t) = 0, \tag{5.218}$$

za $t \in [0, T_0]$,

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0, \tag{5.219}$$

s.s. u $]0, T_0[$.

Dokaz. Tvrdnje (5.210)-(5.212) direktna su posljedica Propozicije 5.1 (iv) u skladu s Teoremima (2.7) i (2.8) iz drugog poglavlja. Tvrdnja (5.213) posljedica je kompaktnog ulaganja prostora $\mathrm{H}^1(Q_0) \hookrightarrow \mathrm{L}^2(Q_0)$. Preostaje nam dokazati svojstva (5.214)-(5.219).

Pokažimo najprije da vrijedi početni uvjet (5.214). Neka je φ funkcija iz prostora $C^\infty([0,T_0])$ takva da je $\varphi(0) \neq 0$ dok u okolini točke T_0 vrijedi $\varphi(t) = 0$ te neka je $u \in H^1(]0,1[)$. Primijetimo da za funkciju $\psi(x,t) = \varphi(t)u(x)$ vrijedi

$$\psi, \frac{\partial \psi}{\partial x} \in \mathrm{L}^2(0, T_0; \mathrm{H}^1(]0,1[)). \tag{5.220}$$

Primjenom Greenove formule za funkcije v^n i v dobivamo jednakosti:

$$\begin{aligned}\int_0^{T_0}\int_0^1 \frac{\partial v^n}{\partial t}(x,t)u(x)\varphi(t)dxdt + \int_0^{T_0}\int_0^1 v^n(x,t)u(x)\frac{d\varphi}{dt}(t)dxdt = \\ -\varphi(0)\int_0^1 v_0^n(x)u(x)dx,\end{aligned} \tag{5.221}$$

$$\begin{aligned}\int_0^{T_0}\int_0^1 \frac{\partial v}{\partial t}(x,t)u(x)\varphi(t)dxdt + \int_0^{T_0}\int_0^1 v(x,t)u(x)\frac{d\varphi}{dt}(t)dxdt = \\ -\varphi(0)\int_0^1 v(x,0)u(x)dx.\end{aligned} \tag{5.222}$$

Ako $n \to \infty$ u (5.221), usporedba s (5.222) daje

$$\int_0^1 v(x,0)u(x)dx = \int_0^1 v_0(x)u(x)dx, \quad \forall u \in H^1(]0,1[) \tag{5.223}$$

odnosno

$$v(x,0) = v_0(x), \quad x \in [0,1]. \tag{5.224}$$

Analognim postupkom dokazujemo da vrijede početni uvjeti (5.215) i (5.216).

Pokažimo sada da vrijede rubni uvjeti (5.217). Neka je sada φ funkcija iz prostora $C^\infty([0,1])$ takva da je $\varphi(0) \neq 0$ dok u okolini točke 1 vrijedi $\varphi(x) = 0$ te neka je $u \in H^1(]0,T_0[)$. Vrijedi

$$\begin{aligned}\int_0^{T_0}\int_0^1 \frac{\partial v^n}{\partial x}(x,t)u(t)\varphi(x)dxdt + \int_0^{T_0}\int_0^1 v^n(x,t)u(t)\frac{d\varphi}{dx}(x)dxdt = \\ -\varphi(0)\int_0^{T_0} v^n(0,t)u(t)dt = 0,\end{aligned} \tag{5.225}$$

$$\begin{aligned}\int_0^{T_0}\int_0^1 \frac{\partial v}{\partial x}(x,t)u(t)\varphi(x)dxdt + \int_0^{T_0}\int_0^1 v(x,t)u(t)\frac{d\varphi}{dx}(x)dxdt = \\ -\varphi(0)\int_0^{T_0} v(0,t)u(t)dt.\end{aligned} \tag{5.226}$$

Opet, stavljajući da $n \to \infty$ u (5.225) i usporedbom s (5.226) dobivamo

$$\int_0^{T_0} v(0,t)u(t)dt = 0, \quad \forall u \in H^1(]0,T_0[) \tag{5.227}$$

odnosno

$$v(0,t) = 0, \quad t \in [0,T_0]. \tag{5.228}$$

Jednakost $\omega(0,t) = 0$ dokazujemo analogno. Jednakosti $v(1,t) = 0$ i $\omega(1,t) = 0$ dokazujemo na isti način ali uzimamo da je funkcija φ takva da je $\varphi(1) \neq 0$ dok u okolini točke 0 vrijedi $\varphi(x) = 0$. Za dokaz jednakosti (5.219) koristimo funkcije φ i u iz istih prostora kao i u dokazu (5.217) i (5.218) ali sada promatramo jednakosti

$$\begin{aligned}\int_0^{T_0}\int_0^1 \frac{\partial^2 \theta^n}{\partial x^2}(x,t)u(t)\varphi(x)dxdt + \int_0^{T_0}\int_0^1 \frac{\partial \theta^n}{\partial x}(x,t)u(t)\frac{d\varphi}{dx}(x)dxdt = \\ -\varphi(0)\int_0^{T_0} \frac{\partial \theta^n}{\partial x}(0,t)u(t)dt = 0,\end{aligned} \tag{5.229}$$

$$\int_0^{T_0}\int_0^1 \frac{\partial^2\theta}{\partial x^2}(x,t)u(t)\varphi(x)dxdt + \int_0^{T_0}\int_0^1 \frac{\partial\theta}{\partial x}(x,t)u(t)\frac{d\varphi}{dx}(x)dxdt = -\varphi(0)\int_0^{T_0}\frac{\partial\theta}{\partial x}(0,t)u(t)dt, \tag{5.230}$$

te zaključujemo da vrijedi

$$\frac{\partial\theta}{\partial x}(x,t) = 0 \tag{5.231}$$

s.s. u $]0,T_0[$. Analognim postupkom dokazujemo drugu jednakost u (5.219). □

Lema 5.14. *Funkcije r, ρ, v, ω i θ definirane Lemama 5.11, 5.12 i 5.13 zadovoljavaju jednadžbe (3.126)-(3.129) s.s. u Q_0.*

Dokaz. Neka je $(r^n,\rho^n,v^n,\omega^n,\theta^n)$ podniz definiran Lemama 5.11, 5.12 i 5.13. Uzevši u obzir (5.205), (5.211) i konvergencije (5.191), (5.192), (5.206) te (5.213) dokazujemo da funkcije r, ρ, v, ω θ zadovoljavaju polazni sustav jednadžbi.

Odmah uočavamo da vrijedi konvergencija

$$\int_0^{T_0}\int_0^1 \frac{\partial\rho^n}{\partial t}\varphi(x,t)dxdt \to \int_0^{T_0}\int_0^1 \frac{\partial\rho}{\partial t}\varphi(x,t)dxdt. \tag{5.232}$$

gdje je φ test funkcija iz prostora $\mathfrak{D}(Q_0)$. Pokažimo sada da je

$$\int_0^{T_0}\int_0^1 (\rho^n)^2\frac{\partial}{\partial x}\left((r^n)^2v^n\right)\varphi dxdt \to \int_0^{T_0}\int_0^1 \rho^2\frac{\partial}{\partial x}\left(r^2v\right)\varphi dxdt. \tag{5.233}$$

Uočimo da vrijedi

$$(\rho^n)^2\frac{\partial}{\partial x}\left((r^n)^2v^n\right) = 2(\rho^n)^2r^n\frac{\partial r^n}{\partial x}v^n + (\rho^n)^2(r^n)^2\frac{\partial v^n}{\partial x}, \tag{5.234}$$

$$\rho^2\frac{\partial}{\partial x}\left(r^2v\right) = 2\rho^2r\frac{\partial r}{\partial x}v + \rho^2r^2\frac{\partial v}{\partial x}. \tag{5.235}$$

Korištenjem navedenih konvergencija zaključujemo da vrijedi

$$\begin{aligned} &\left|\int_0^{T_0}\int_0^1 \left[(\rho^n)^2r^n\frac{\partial r^n}{\partial x}v^n - \rho^2r\frac{\partial r}{\partial x}v\right]\varphi dxdt\right| \le \\ &C\max_{(x,t)\in\overline{Q}_0}\left|(\rho^n)^2-\rho^2\right|\|v^n\| + C\max_{(x,t)\in\overline{Q}_0}|r^n-r|\,\|v^n\| + \\ &C\max_{(x,t)\in\overline{Q}_0}\left|\frac{\partial r^n}{\partial x}-\frac{\partial r}{\partial x}\right|\|v^n\| + C\,\|v^n-v\|, \end{aligned} \tag{5.236}$$

$$\left|\int_0^{T_0}\int_0^1 \left[(\rho^n)^2(r^n)^2\frac{\partial v^n}{\partial x}-\rho^2 r\frac{\partial v}{\partial x}\right]\varphi dxdt\right| \leq$$
$$C\max_{(x,t)\in\overline{Q}_0}\left|(\rho^n)^2-\rho^2)\right|\left\|\frac{\partial v^n}{\partial x}\right\|+C\max_{(x,t)\in\overline{Q}_0}\left|(r^n)^2-r^2\right|\left\|\frac{\partial v^n}{\partial x}\right\|+ \tag{5.237}$$
$$C\left|\int_0^{T_0}\int_0^1\left(\frac{\partial v^n}{\partial x}-\frac{\partial v}{\partial x}\right)\varphi dxdt\right|.$$

Sada je jasno da funkcije ρ i v zadovoljavaju prvu jednadžbu sustava. Pokažimo sada da je zadovoljena i druga jednadžba sustava. Očito je

$$\int_0^{T_0}\int_0^1 \frac{\partial v^n}{\partial t}\sin(\pi ix)\varphi(t)dxdt \to \int_0^{T_0}\int_0^1 \frac{\partial v}{\partial t}\sin(\pi ix)\varphi(t)dxdt \tag{5.238}$$

za $\varphi\in\mathfrak{D}(]0,T_0[)$. Pokažimo konvergenciju

$$\int_0^{T_0}\int_0^1 (r^n)^2\frac{\partial}{\partial x}\left(\rho^n\frac{\partial}{\partial x}\left((r^n)^2v^n\right)\right)\sin(\pi ix)\varphi(t)dxdt \to$$
$$\int_0^{T_0}\int_0^1 r^2\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}\left(r^2v\right)\right)\sin(\pi ix)\varphi(t)dxdt \tag{5.239}$$

kad $n\to\infty$, $\varphi\in\mathfrak{D}(]0,T_0[)$.

Uočimo da vrijedi

$$(r^n)^2\frac{\partial}{\partial x}\left(\rho^n\frac{\partial}{\partial x}\left((r^n)^2v^n\right)\right)=2(r^n)^3\frac{\partial\rho^n}{\partial x}\frac{\partial r^n}{\partial x}v^n+2(r^n)^2\rho^n\left(\frac{\partial r^n}{\partial x}\right)^2v^n+$$
$$2(r^n)^3\rho^n\frac{\partial^2 r^n}{\partial x^2}v^n+4(r^n)^3\rho^n\frac{\partial r^n}{\partial x}\frac{\partial v^n}{\partial x}+(r^n)^4\frac{\partial\rho^n}{\partial x}\frac{\partial v^n}{\partial x}+(r^n)^4\rho^n\frac{\partial^2 v^n}{\partial x^2} \tag{5.240}$$

$$r^2\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}\left(r^2v\right)\right)=2r^3\frac{\partial\rho}{\partial x}\frac{\partial r}{\partial x}v+2r^2\rho\left(\frac{\partial r}{\partial x}\right)^2v+$$
$$2r^3\rho\frac{\partial^2 r}{\partial x^2}v+4r^3\rho\frac{\partial r}{\partial x}\frac{\partial v}{\partial x}+r^4\frac{\partial\rho}{\partial x}\frac{\partial v}{\partial x}+r^4\rho\frac{\partial^2 v}{\partial x^2} \tag{5.241}$$

Korištenjem prethodno utvrđenih konvergencija zaključujemo da je

$$\left|\int_0^{T_0}\int_0^1\left[(r^n)^3\frac{\partial\rho^n}{\partial x}\frac{\partial r^n}{\partial x}v^n-r^3\frac{\partial\rho}{\partial x}\frac{\partial r}{\partial x}v\right]\sin(\pi ix)\varphi(t)dxdt\right|\leq$$
$$C\max_{(x,t)\in\overline{Q}_0}\left|(r^n)^3-r^3\right|\left\|\frac{\partial\rho^n}{\partial x}(t)\right\|\|v^n(t)\|+$$
$$C\left\|\frac{\partial v^n}{\partial x}(t)\right\|\left|\int_0^{T_0}\int_0^1\left(\frac{\partial\rho^n}{\partial x}-\frac{\partial\rho}{\partial x}\right)\sin(\pi ix)\varphi(t)dxdt\right| \tag{5.242}$$
$$C\max_{(x,t)\in\overline{Q}_0}\left|\frac{\partial r^n}{\partial x}-\frac{\partial r}{\partial x}\right|\left\|\frac{\partial\rho}{\partial x}(t)\right\|\|v^n(t)\|++C\left\|\frac{\partial\rho(t)}{\partial x}\right\|\|(v^n-v)(t)\|,$$

$$\left|\int_0^{T_0}\int_0^1 \left[(r^n)^2\rho^n\left(\frac{\partial r^n}{\partial x}\right)^2 v^n - r^2\rho\left(\frac{\partial r}{\partial x}\right)^2 v\right]\sin(\pi i x)\varphi(t)dxdt\right| \leq$$
$$C\max_{(x,t)\in\overline{Q}_0}\left|(r^n)^2-r^2\right|\|v^n(t)\| + C\max_{(x,t)\in\overline{Q}_0}|\rho^n-\rho|\|v^n(t)\| + \tag{5.243}$$
$$C\max_{(x,t)\in\overline{Q}_0}\left|\left(\frac{\partial r^n}{\partial x}\right)^2-\left(\frac{\partial r}{\partial x}\right)^2\right|\|v^n(t)\| + C\|(v^n-v)(t)\|,$$

$$\left|\int_0^{T_0}\int_0^1 \left[(r^n)^3\rho^n\frac{\partial^2 r^n}{\partial x^2}v^n - r^3\rho\frac{\partial^2 r}{\partial x^2}v\right]\sin(\pi i x)\varphi(t)dxdt\right| \leq$$
$$C\max_{(x,t)\in\overline{Q}_0}\left|(r^n)^3-r^3\right|\left\|\frac{\partial^2 r^n}{\partial x^2}(t)\right\|\|v^n(t)\|$$
$$+C\max_{(x,t)\in\overline{Q}_0}|\rho^n-\rho|\left\|\frac{\partial^2 r^n}{\partial x^2}(t)\right\|\|v^n(t)\| + \tag{5.244}$$
$$C\left\|\frac{\partial v^n}{\partial x}(t)\right\|\left|\int_0^{T_0}\int_0^1\left(\frac{\partial^2 r^n}{\partial x^2}-\frac{\partial^2 r}{\partial x^2}\right)\sin(\pi i x)\varphi(t)dxdt\right| +$$
$$C\left\|\frac{\partial^2 r(t)}{\partial x^2}\right\|\|(v^n-v)(t)\|,$$

$$\left|\int_0^{T_0}\int_0^1 \left[(r^n)^3\rho^n\frac{\partial r^n}{\partial x}\frac{\partial v^n}{\partial x} - r^3\rho\frac{\partial r}{\partial x}\frac{\partial v}{\partial x}\right]\sin(\pi i x)\varphi(t)dxdt\right| \leq$$
$$C\max_{(x,t)\in\overline{Q}_0}\left|(r^n)^3-r^3\right|\left\|\frac{\partial v^n}{\partial x}(t)\right\| +$$
$$C\max_{(x,t)\in\overline{Q}_0}|\rho^n-\rho|\left\|\frac{\partial v^n}{\partial x}(t)\right\| + \tag{5.245}$$
$$C\max_{(x,t)\in\overline{Q}_0}\left|\frac{\partial r^n}{\partial x}-\frac{\partial r}{\partial x}\right|\left\|\frac{\partial v^n}{\partial x}(t)\right\| + C\left|\int_0^{T_0}\int_0^1\left(\frac{\partial v^n}{\partial x}-\frac{\partial v}{\partial x}\right)\sin(\pi i x)\varphi(t)dxdt\right|,$$

$$\left|\int_0^{T_0}\int_0^1 \left[(r^n)^4\frac{\partial \rho^n}{\partial x}\frac{\partial v^n}{\partial x} - r^4\frac{\partial \rho}{\partial x}\frac{\partial v}{\partial x}\right]\sin(\pi i x)\varphi(t)dxdt\right| \leq$$
$$C\max_{(x,t)\in\overline{Q}_0}\left|(r^n)^4-r^4\right|\left\|\frac{\partial v^n}{\partial x}(t)\right\| +$$
$$C\left\|\frac{\partial v^n}{\partial x}(t)\right\|\left|\int_0^{T_0}\int_0^1\left(\frac{\partial \rho^n}{\partial x}-\frac{\partial \rho}{\partial x}\right)\sin(\pi i x)\varphi(t)dxdt\right| + \tag{5.246}$$
$$C\left\|\frac{\partial \rho}{\partial x}(t)\right\|\left|\int_0^{T_0}\int_0^1\left(\frac{\partial v^n}{\partial x}-\frac{\partial v}{\partial x}\right)\sin(\pi i x)\varphi(t)dxdt\right|,$$

$$\left|\int_0^{T_0}\int_0^1 \left[(r^n)^4\rho^n\frac{\partial^2 v^n}{\partial x^2} - r^4\rho\frac{\partial^2 v}{\partial x^2}\right]\sin(\pi i x)\varphi(t)dxdt\right| \leq$$
$$C\max_{(x,t)\in\overline{Q}_0}\left|(r^n)^4 - r^4\right|\left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\| + C\max_{(x,t)\in\overline{Q}_0}|\rho^n-\rho|\left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\| + \tag{5.247}$$
$$C\left|\int_0^{T_0}\int_0^1\left(\frac{\partial^2 v^n}{\partial x} - \frac{\partial^2 v}{\partial x}\right)\sin(\pi i x)\varphi(t)dxdt\right|,$$

odakle slijedi (5.239).

Za dokaz konvergencije

$$\int_0^{T_0}\int_0^1 (r^n)^2\frac{\partial}{\partial x}(\rho^n\theta^n)\sin(\pi i x)\varphi(t)dxdt \to \int_0^{T_0}\int_0^1 r^2\frac{\partial}{\partial x}(\rho\theta)\sin(\pi i x)\varphi(t)dxdt \tag{5.248}$$

kad $n\to\infty$, $\varphi\in\mathfrak{D}(]0,T_0[)$ promatramo jednakosti

$$(r^n)^2\frac{\partial}{\partial x}(\rho^n\theta^n) = (r^n)^2\frac{\partial\rho^n}{\partial x}\theta^n + (r^n)^2\rho^n\frac{\partial\theta^n}{\partial x} \tag{5.249}$$

$$r^2\frac{\partial}{\partial x}(\rho\theta) = r^2\frac{\partial\rho}{\partial x}\theta + r^2\rho\frac{\partial\theta}{\partial x} \tag{5.250}$$

te izvodimo ocjene kao u dokazu prethodne konvergencije.

Analognim postupkom dokazujemo da funkcije r, ρ, v, ω i θ zadovoljavaju treću i četvrtu jednadžbu polaznog sustava. $\square$

Primjetimo da zbog (5.212) i (5.3) lako zaključujemo da funkcija r definirana Lemom 5.11 ima oblik

$$r(x,t) = r_0(x) + \int_0^t v(x,t)d\tau, \quad (x,t)\in\overline{Q}_0 \tag{5.251}$$

pri čemu je funkcija v iz Leme 5.13.

Lema 5.15. *Postoji takav T_0, $0 < T_0 \leq T$ tako da funkcija θ definirana Lemom 5.190 zadovoljava uvjet*

$$\theta > 0 \textit{ in } \overline{Q}_0. \tag{5.252}$$

Dokaz. Kako je $\theta\in C(\overline{Q}_0)$, zaključujemo da za svako $\varepsilon > 0$ postoji T_0, $T_0\leq T$, tako da za $(x,t)\in\overline{Q}_0$ vrijedi

$$|\theta(x,t)-\theta(x,0)| = |\theta(x,t)-\theta_0(x)| < \varepsilon,$$

$$\theta(x,t) > \theta_0(x) - \varepsilon \geq m - \varepsilon.$$

$\square$

Tvrdnja Teorema 4.1 direktna je posljedica navedenih lema.

6 Dokaz jedinstvenosti rješenja

Sada, kada znamo da naš problem (3.126)-(3.136) ima generalizirano rješenje sa svojstvom (4.14) na području $Q_{T_0} =]0,1[\times]0,T_0[$ za neko $T_0 > 0$, ima smisla dokazivati da je to rješenje i jedinstveno na istoj domeni.

Dokaz koji provodimo ne ovisi o širini intervala egzistencije rješenja problema (3.126)-(3.136) zbog čega ćemo zbog jednostavnosti staviti da je $T_0 = T$. U dokazu koristimo neka svojstva funkcija r, v, ω i θ na domeni Q_T koja odmah i navodimo.

Lema 6.1. *Za funkciju r definiranu sa (3.137) vrijedi ocjena*

$$r(x,t) \geq a, \quad (x,t) \in Q_T, \tag{6.1}$$

gdje je $a > 0$ polumjer manje rubne sfere iz polaznog problema.

Dokaz. Vrijedi

$$\frac{\partial}{\partial t}\left(r^2\frac{\partial r}{\partial x} - \frac{L}{\rho}\right) = 2r\frac{\partial r}{\partial t}\frac{\partial r}{\partial x} + r^2\frac{\partial^2 r}{\partial x \partial t} - L\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right). \tag{6.2}$$

Koristeći (3.126) i (3.138) slijedi

$$\frac{\partial}{\partial t}\left(r^2\frac{\partial r}{\partial x} - \frac{L}{\rho}\right) = 2r\frac{\partial r}{\partial x}v + r^2\frac{\partial v}{\partial x} - \frac{\partial}{\partial x}\left(r^2 v\right) = 0. \tag{6.3}$$

Integriranjem preko $]0,t[$ odmah dobivamo

$$r^2\frac{\partial r}{\partial x} - \frac{L}{\rho} = r_0^2 r_0'(x) - \frac{L}{\rho_0}. \tag{6.4}$$

Zbog pozitivnosti funkcija ρ i $r_0'(x)$ iz (6.4) zaključujemo da vrijedi

$$\frac{\partial r}{\partial x}(x,t) > 0, \tag{6.5}$$

za sve $(x,t) \in Q_T$. Prema tome funkcija r je rastuća funkcija po varijabli x, pa je

$$r(x,t) \geq r(0,t) = a, \quad (x,t) \in Q_T, \tag{6.6}$$

čime je lema dokazana. □

Primijetimo da smo sa (6.1) dobili precizniju ocjenu za r u odnosu na onu koju već imamo u (4.16). Funkcija r, za koju već vrijedi (4.15), zbog teorema o ulaganju pripada prostoru $\mathrm{L}^\infty(0,T;\mathrm{C}^1([0,1]))$ iz čega zaključujemo da postoji konstanta $C \in \mathbb{R}^+$ takva da je

$$r(x,t) \leq C, \tag{6.7}$$

$$\frac{\partial r}{\partial x}(x,t) \leq C, \tag{6.8}$$

za svako $(x,t) \in Q_T$.

Zbog pripadanja prostorima (4.4) i (4.6) očigledno je da postoji konstanta $C \in \mathbb{R}^+$ takva da je

$$\rho(x,t) \leq C, \tag{6.9}$$

$$|v(x,t)| \leq C, \tag{6.10}$$

$$|\omega(x,t)| \leq C, \tag{6.11}$$

$$|\theta(x,t)| \leq C \tag{6.12}$$

na području $\overline{Q}_T$.

Nadalje, integriranjem jednadžbe (3.126) preko $[0,t]$ i uvažavanjem početnog uvjeta (3.130) zaključujemo da funkciju ρ možemo pisati u obliku

$$\rho(x,t) = \frac{\rho_0(x)}{1 + L^{-1}\rho_0(x) \int\limits_0^t \frac{\partial}{\partial x}\left(r^2(x,\tau)v(x,\tau)\right) d\tau}. \tag{6.13}$$

Koristeći se svojstvima (4.5) i (4.12) iz (6.13) lako zaključujuemo da je funkcija ρ neprekidna na $\overline{Q}_T$.

6.1 Formiranje pomoćnog sustava jednadžbi

Sada ćemo zbog jednostavnosti umjesto funkcije gustoće ρ uvesti funkciju specifičnog volumena $u = \rho^{-1}$. Kako je gustoća strogo pozitivna jasno je da specifični volumen ima smisla.

Pretpostavimo sada da su $(u_i, v_i, \omega_i, \theta_i)$, $i = 1,2$ dva različita generalizirana rješenja problema (3.126)-(3.136) u domeni Q_T sa svojstvom (4.14). Svakoj funkciji v_i pridružena je funkcija

$$r_i(x,t) = r_0(x) + \int\limits_0^t v_i(x,\tau) d\tau, \quad i = 1,2. \tag{6.14}$$

U nastavku uvodimo funkcije $u = u_1 - u_2$, $v = v_1 - v_2$, $\omega = \omega_1 - \omega_2$, $\theta = \theta_1 - \theta_2$ i $r = r_1 - r_2$. Iz (6.14) je očito da funkcija r ima oblik

$$r(x,t) = \int_0^t v(x,\tau)d\tau. \tag{6.15}$$

Nakon određenih izračuna na sustavu jednadžbi (3.126)-(3.129) dobivamo da funkcija (u, v, ω, θ) na domeni Q_T zadovoljava sljedeći sustav jednadžbi:

$$\frac{\partial u}{\partial t} = \frac{1}{L}\frac{\partial}{\partial x}\left(r_1^2 v\right) + \frac{1}{L}\frac{\partial}{\partial x}\left(r(r_1+r_2)v_2\right), \tag{6.16}$$

$$\begin{aligned}\frac{\partial v}{\partial t} &= \alpha r_1^2 \frac{\partial}{\partial x}\left[\frac{1}{u_1}\frac{\partial}{\partial x}(r_1^2 v) + \frac{1}{u_1}\frac{\partial}{\partial x}(r(r_1+r_2)v_2) - \frac{u}{u_1u_2}\frac{\partial}{\partial x}(r_2^2 v_2)\right] + \\ \alpha r(r_1+r_2)&\frac{\partial}{\partial x}\left[\frac{1}{u_2}\frac{\partial}{\partial x}(r_2^2v_2)\right] - \beta r_1^2\frac{\partial}{\partial x}\left[\frac{\theta}{u_1} - \frac{\theta_2 u}{u_1u_2}\right] - \beta r(r_1+r_2)\frac{\partial}{\partial x}\left(\frac{\theta_2}{u_2}\right),\end{aligned} \tag{6.17}$$

$$\begin{aligned}j_I\frac{\partial \omega}{\partial t} &= \gamma r_1^2\frac{\partial}{\partial x}\left[\frac{1}{u_1}\frac{\partial}{\partial x}(r_1^2\omega) + \frac{1}{u_1}\frac{\partial}{\partial x}(r(r_1+r_2)\omega_2) - \frac{u}{u_1u_2}\frac{\partial}{\partial x}(r_2^2\omega_2)\right] + \\ &\gamma r(r_1+r_2)\frac{\partial}{\partial x}\left[\frac{1}{u_2}\frac{\partial}{\partial x}(r_2^2\omega_2)\right] - \delta u_1\omega - \delta\omega_2 u,\end{aligned} \tag{6.18}$$

$$\begin{aligned}c_v\frac{\partial\theta}{\partial t} &= \nu\frac{\partial}{\partial x}\left[\frac{r_1^4}{u_1}\frac{\partial\theta}{\partial x} + \frac{r(r_1+r_2)(r_1^2+r_2^2)}{u_1}\frac{\partial\theta_2}{\partial x} - \frac{ur_2^4}{u_1u_2}\frac{\partial\theta_2}{\partial x}\right] \\ &+\frac{\alpha}{u_2}\left[\frac{\partial}{\partial x}(r_1^2v_1) + \frac{\partial}{\partial x}(r_2^2v_2)\right]\cdot\left[\frac{\partial}{\partial x}(r(r_1+r_2)v_1) + \frac{\partial}{\partial x}(r_2^2 v)\right] \\ &-\alpha\frac{u}{u_1u_2}\left[\frac{\partial}{\partial x}(r_1^2v_1)\right]^2 - \frac{\beta}{u_1}\frac{\partial}{\partial x}(r_1^2v_1)\left[\theta - \frac{\theta_2}{u_2}u\right] \\ &-\beta\frac{\theta_2}{u_2}\left[\frac{\partial}{\partial x}(r(r_1+r_2)v_1) + \frac{\partial}{\partial x}(r_2^2 v)\right] - d\frac{\partial}{\partial x}\left[rv_1^2 + r_2(v_1+v_2)v\right] \\ &-\gamma\frac{u}{u_1u_2}\left[\frac{\partial}{\partial x}(r_1^2\omega_1)\right]^2 + \\ &\frac{\gamma}{u_2}\left[\frac{\partial}{\partial x}(r_1^2\omega_1) + \frac{\partial}{\partial x}(r_2^2\omega_2)\right]\cdot\left[\frac{\partial}{\partial x}(r(r_1+r_2)\omega_1) + \frac{\partial}{\partial x}(r_2^2\omega)\right] \\ &-h\frac{\partial}{\partial x}\left[r\omega_1^2 + r_2(\omega_1+\omega_2)\omega\right] + \delta\left[(\omega_1+\omega_2)\omega u_1 + \omega_2^2 u\right],\end{aligned} \tag{6.19}$$

gdje zbog jednostavnosti uzimamo $\alpha = \dfrac{\lambda+2\mu}{L^2}$, $\beta = \dfrac{R}{L}$, $\gamma = \dfrac{c_0+2c_d}{L^2}$, $\delta = 4\mu_r$, $\nu = \dfrac{k}{L^2}$, $d = \dfrac{4\mu}{L^2}$ i $h = \dfrac{4c_d}{L}$. Primjetimo da su α, β, γ, δ, ν, d i h pozitivne konstante.

Sada dobivamo sljedeće početne i rubne uvjete:

$$u(x,0) = 0, \tag{6.20}$$

$$v(x,0) = 0, \tag{6.21}$$

$$\omega(x,0) = 0, \tag{6.22}$$

$$\theta(x,0) = 0, \tag{6.23}$$

$$v(0,t) = v(1,t) = 0, \tag{6.24}$$

$$\omega(0,t) = \omega(1,t) = 0, \tag{6.25}$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0, \tag{6.26}$$

za $x \in]0,1[$, $t \in]0,T[$. Primjetimo da za funkciju r vrijedi

$$r(x,0) = 0, \; x \in]0,1[. \tag{6.27}$$

6.2 Dokaz Teorema 4.2

Dokaz teorema o jedinstvenosti rješenja baziran je na dobivanju četiri nejednakosti koje se odnose na ograničenost funkcija u, v, ω i θ. Te četiri nejednakosti dokazujemo u sljedećim lemama.

Lema 6.2. *Postoji konstanta $C > 0$ takva da vrijedi*

$$\|u(t)\|^2 \leq C \int_0^t \left\| \frac{\partial v}{\partial x}(\tau) \right\|^2 d\tau \tag{6.28}$$

za svaki $t \in]0,T[$.

Dokaz. Kvadriranjem (6.15), integriranjem preko $]0,1[$ i primjenom Jensenove nejednakosti dobivamo

$$\|r(t)\|^2 \leq C \int_0^t \|v(t)\|^2 \, d\tau. \tag{6.29}$$

Deriviranjem (6.15) po varijabli x te primjenom istog postupka kao i za (6.29) imamo

$$\left\| \frac{\partial r}{\partial x}(t) \right\|^2 \leq C \int_0^t \left\| \frac{\partial v}{\partial x}(\tau) \right\|^2 d\tau. \tag{6.30}$$

Sada pomnožimo jednadžbu (6.16) sa u i integriramo preko $]0,1[$. Dobivamo sljedeću jednakost

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 &= \frac{2}{L}\int_0^1 r_1 \frac{\partial r_1}{\partial x} vu dx + \frac{1}{L}\int_0^1 r_1^2 \frac{\partial v}{\partial x} u dx \\ + \frac{1}{L}\int_0^1 \frac{\partial r}{\partial x}(r_1 + r_2) v_2 u dx &+ \frac{1}{L}\int_0^1 r \frac{\partial}{\partial x}(r_1 + r_2) v_2 u dx + \frac{1}{L}\int_0^1 r(r_1 + r_2)\frac{\partial v_2}{\partial x} u dx. \end{aligned} \tag{6.31}$$

Kako vrijedi

$$\left| \frac{2}{L}\int_0^1 r_1 \frac{\partial r_1}{\partial x} vu dx \right| \leq C \max_{x\in[0,1]} \left| r_1 \frac{\partial r_1}{\partial x} \right| \int_0^1 |vu| \, dx, \tag{6.32}$$

primjenom Youngove nejednakosti odmah slijedi

$$\left|\frac{2}{L}\int_0^1 r_1\frac{\partial r_1}{\partial x}vudx\right| \leq C\left(\|v(t)\|^2+\|u(t)\|^2\right). \tag{6.33}$$

Koristeći nejednakosti Friedrichs-Poincaréa za funkciju v u (6.33) nadalje imamo

$$\left|\frac{2}{L}\int_0^1 r_1\frac{\partial r_1}{\partial x}vudx\right| \leq C\left(\left\|\frac{\partial v}{\partial x}(t)\right\|^2+\|u(t)\|^2\right). \tag{6.34}$$

Koristeći ocjene (6.7), (6.8), (6.10), (6.29) i (6.30) na preostale integrale desne strane u (6.31) te nejednakost Gagliardo-Ladyzhenskaye, analognim zaključivanjem dobivamo sljedeće ocjene:

$$\left|\frac{1}{L}\int_0^1 r_1^2\frac{\partial v}{\partial x}udx\right| \leq C\max_{x\in[0,1]}|r_1|\int_0^1\left|\frac{\partial v}{\partial x}u\right|dx \leq C\left(\left\|\frac{\partial v}{\partial x}(t)\right\|^2+\|u(t)\|^2\right), \tag{6.35}$$

$$\begin{aligned}\left|\frac{1}{L}\int_0^1 \frac{\partial r}{\partial x}(r_1+r_2)v_2udx\right| &\leq C\max_{x\in[0,1]}|(r_1+r_2)v_2|\int_0^1\left|\frac{\partial r}{\partial x}u\right|dx \leq \\ C\left(\left\|\frac{\partial r}{\partial x}(t)\right\|^2+\|u(t)\|^2\right) &= C\left(\int_0^t\left\|\frac{\partial v}{\partial x}(\tau)\right\|^2d\tau+\|u(t)\|^2\right),\end{aligned} \tag{6.36}$$

$$\begin{aligned}\left|\frac{1}{L}\int_0^1 r\frac{\partial}{\partial x}(r_1+r_2)v_2udx\right| &\leq C\max_{x\in[0,1]}\left|\frac{\partial}{\partial x}(r_1+r_2)v_2\right|\int_0^1|ru|\,dx \leq \\ C\left(\|r(t)\|^2+\|u(t)\|^2\right) &= C\left(\int_0^t\|v(\tau)\|^2d\tau+\|u(t)\|^2\right) \leq \\ &C\left(\int_0^t\left\|\frac{\partial v}{\partial x}(\tau)\right\|^2d\tau+\|u(t)\|^2\right)\end{aligned} \tag{6.37}$$

$$\begin{aligned}\left|\frac{1}{L}\int_0^1 r(r_1+r_2)\frac{\partial v_2}{\partial x}udx\right| &\leq C\max_{x\in[0,1]}|r_1+r_2|\int_0^1\left|r\frac{\partial v_2}{\partial x}u\right|dx \leq \\ &C\left(\max_{x\in[0,1]}\left|\frac{\partial v_2}{\partial x}\right|^2\|u(t)\|^2+\|r(t)\|^2\right) = \\ &C\left(\max_{x\in[0,1]}\left|\frac{\partial v_2}{\partial x}\right|^2\|u(t)\|^2+\int_0^t\|v(\tau)\|^2d\tau\right) \leq \\ &C\left(\max_{x\in[0,1]}\left|\frac{\partial v_2}{\partial x}\right|^2\|u(t)\|^2+\int_0^t\left\|\frac{\partial v}{\partial x}(\tau)\right\|^2d\tau\right) \leq \\ &C\left(\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\|u(t)\|^2+\int_0^t\left\|\frac{\partial v}{\partial x}(\tau)\right\|^2d\tau\right).\end{aligned} \tag{6.38}$$

Uvrštavanjem (6.34)-(6.38) u (6.31) i integriranjem preko $]0,t[$ dobivamo nejednakost

$$\|u(t)\|^2 \leq C\int_0^t \left[\left(1+\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\right)\|u(\tau)\|^2 + \left\|\frac{\partial v}{\partial x}(\tau)\right\|^2\right] d\tau. \tag{6.39}$$

Primjenom inkluzije (4.3) za funkciju v_2 te korištenjem Gronwallove nejednakosti iz (6.39) odmah slijedi (6.28). □

Lema 6.3. *Postoji konstanta $C > 0$ takva da vrijedi*

$$\|v(t)\|^2 + \int_0^t \left\|\frac{\partial v}{\partial x}(\tau)\right\|^2 d\tau \leq C\int_0^t \|\theta(\tau)\|^2 d\tau \tag{6.40}$$

za svaki $t\in]0,T[$.

Dokaz. Množenjem jednadžbe (6.17) sa v i integriranjem preko $]0,1[$ dobivamo

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \alpha\int_0^1 \frac{r_1^4}{u_1}\left(\frac{\partial v}{\partial x}\right)^2 dx &= -4\alpha\int_0^1 \frac{r_1^3}{u_1}\frac{\partial r_1}{\partial x} v\frac{\partial v}{\partial x}dx \\
-4\alpha\int_0^1 \frac{r_1^2}{u_1}\left(\frac{\partial r_1}{\partial x}\right)^2 v^2 dx &- \alpha\int_0^1 \frac{1}{u_1}\frac{\partial r}{\partial x}(r_1+r_2)v_2\frac{\partial}{\partial x}(r_1^2 v)dx \\
-\alpha\int_0^1 \frac{1}{u_1} r\frac{\partial}{\partial x}(r_1+r_2)v_2\frac{\partial}{\partial x}(r_1^2 v)dx &- \alpha\int_0^1 \frac{1}{u_1} r(r_1+r_2)\frac{\partial v_2}{\partial x}\frac{\partial}{\partial x}(r_1^2 v)dx \\
+\alpha\int_0^1 \frac{u}{u_1 u_2}\frac{\partial}{\partial x}(r_2^2 v_2)\frac{\partial}{\partial x}(r_1^2 v)dx &- \alpha\int_0^1 \frac{1}{u_1}\frac{\partial}{\partial x}(r_2^2 v_2)\frac{\partial r}{\partial x}(r_1+r_2)v dx \\
-\alpha\int_0^1 \frac{1}{u_1}\frac{\partial}{\partial x}(r_2^2 v_2) r\frac{\partial}{\partial x}(r_1+r_2)v dx &- \alpha\int_0^1 \frac{1}{u_1}\frac{\partial}{\partial x}(r_2^2 v_2) r(r_1+r_2)\frac{\partial v}{\partial x}dx \\
+\beta\int_0^1 \frac{\theta}{u_1}\frac{\partial}{\partial x}(r_1^2 v)dx - \beta\int_0^1 \frac{\theta_2}{u_1 u_2} u\frac{\partial}{\partial x}(r_1^2 v)dx &+ \beta\int_0^1 \frac{\theta_2}{u_2}\frac{\partial r}{\partial x}(r_1+r_2)v dx \\
+\beta\int_0^1 \frac{\theta_2}{u_2} r\frac{\partial}{\partial x}(r_1+r_2)v dx &+ \beta\int_0^1 \frac{\theta_2}{u_2} r(r_1+r_2)\frac{\partial v}{\partial x}dx.
\end{aligned} \tag{6.41}$$

Koristeći se nejednakošću Gagliardo-Ladyzhenskaye te ocjenama (6.7), (6.8) i (6.10) lako zaključujemo da vrijede sljedeće nejednakosti koje nam trebaju u dokazu leme.

$$\left|\frac{\partial}{\partial x}(r_2^2 v_2)\right| \leq C\left(1+\left|\frac{\partial v_2}{\partial x}\right|\right) \leq C\left(1+\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|\right), \tag{6.42}$$

$$\left|\frac{\partial}{\partial x}(r_1^2 v)\right| \leq C\left(|v|+\left|\frac{\partial v}{\partial x}\right|\right). \tag{6.43}$$

Postupkom sličnim onom u dokazu Leme 6.2, korištenjem nejednakosti (6.29), (6.30), (6.42) i (6.43), nejednakosti Friedrichs-Poincaréa i Gagliardo-Ladyzhenskaye za funkciju v, ocjena (6.7), (6.8), (6.9), (6.10) i (6.12) te Youngove nejedankosti s parametrom $\varepsilon > 0$ dobivamo ocjene integrala na desnoj strani jednakosti (6.41).

$$\left|4\alpha\int_0^1 \frac{r_1^3}{u_1}\frac{\partial r_1}{\partial x}v\frac{\partial v}{\partial x}dx\right| \leq C\left\|v\right\|\left\|\frac{\partial v}{\partial x}\right\| \leq \varepsilon\left\|\frac{\partial v}{\partial x}\right\|^2 + C\left\|v\right\|^2, \tag{6.44}$$

$$\left|4\alpha\int_0^1 \frac{r_1^2}{u_1}\left(\frac{\partial r_1}{\partial x}\right)^2 v^2dx\right| \leq C\left\|v\right\|^2, \tag{6.45}$$

$$\begin{gathered}\left|\alpha\int_0^1 \frac{1}{u_1}\frac{\partial r}{\partial x}(r_1+r_2)v_2\frac{\partial}{\partial x}(r_1^2v)dx\right| \leq C\left(\int_0^1\left|\frac{\partial r}{\partial x}\right||v|\,dx + \int_0^1\left|\frac{\partial r}{\partial x}\right|\left|\frac{\partial v}{\partial x}\right|dx\right) \leq \\ C\left(\left\|\frac{\partial r}{\partial x}\right\|^2 + \|v\|^2\right) + \varepsilon\left\|\frac{\partial v}{\partial x}\right\|^2 = \varepsilon\left\|\frac{\partial v}{\partial x}\right\|^2 + C\left(\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau + \|v\|^2\right),\end{gathered} \tag{6.46}$$

$$\begin{gathered}\left|\alpha\int_0^1 \frac{1}{u_1}r\frac{\partial}{\partial x}(r_1+r_2)v_2\frac{\partial}{\partial x}(r_1^2v)dx\right| \leq C\int_0^1|r|\left(|v| + \left|\frac{\partial v}{\partial x}\right|\right)dx \leq \\ C\left(\|r\|^2 + \|v\|^2\right) + \varepsilon\left\|\frac{\partial v}{\partial x}\right\|^2 = \varepsilon\left\|\frac{\partial v}{\partial x}\right\|^2 + C\left(\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau + \|v\|^2\right),\end{gathered} \tag{6.47}$$

$$\begin{gathered}\left|\alpha\int_0^1 \frac{1}{u_1}r(r_1+r_2)\frac{\partial v_2}{\partial x}\frac{\partial}{\partial x}(r_1^2v)dx\right| \leq C\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|\int_0^1|r|\left(|v| + \left|\frac{\partial v}{\partial x}\right|\right)dx \leq \\ C\left(\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\|v\|^2 + \|r\|^2\left(1 + \left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\right)\right) + \varepsilon\left\|\frac{\partial v}{\partial x}\right\|^2 \leq \\ \varepsilon\left\|\frac{\partial v}{\partial x}\right\|^2 + C\left(\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\|v\|^2 + \left(1 + \left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\right)\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau\right),\end{gathered} \tag{6.48}$$

$$\begin{gathered}\left|\alpha\int_0^1 \frac{u}{u_1u_2}\frac{\partial}{\partial x}(r_2^2v_2)\frac{\partial}{\partial x}(r_1^2v)dx\right| \leq C\left(1 + \left\|\frac{\partial^2 v_2}{\partial x^2}\right\|\right)\int_0^1|u|\left(|v| + \left|\frac{\partial v}{\partial x}\right|\right)dx \leq \\ C\left(1 + \left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\right)\left(\|v\|^2 + \|u\|^2\right) + \varepsilon\left\|\frac{\partial v}{\partial x}\right\|^2,\end{gathered} \tag{6.49}$$

$$\left|\alpha\int_0^1 \frac{1}{u_1}\frac{\partial}{\partial x}(r_2^2 v_2)\frac{\partial r}{\partial x}(r_1+r_2)v dx\right| \leq C\left(1+\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|\right)\int_0^1 |v|\left|\frac{\partial r}{\partial x}\right| dx \leq$$
$$C\left(\left\|\frac{\partial r}{\partial x}\right\|^2+\left(1+\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\right)\|v\|^2\right) \leq \tag{6.50}$$
$$C\left(\int_0^t \left\|\frac{\partial v}{\partial x}\right\|^2 d\tau+\left(1+\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\right)\|v\|^2\right),$$

$$\left|\alpha\int_0^1 \frac{1}{u_1}\frac{\partial}{\partial x}(r_2^2 v_2) r\frac{\partial}{\partial x}(r_1+r_2)v dx\right| \leq C\left(1+\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|\right)\int_0^1 |r||v| dx \leq$$
$$C\left(\left(1+\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\right)\|v\|^2+\int_0^t \left\|\frac{\partial v}{\partial x}\right\|^2 d\tau\right), \tag{6.51}$$

$$\left|\alpha\int_0^1 \frac{1}{u_1}\frac{\partial}{\partial x}(r_2^2 v_2) r(r_1+r_2)\frac{\partial v}{\partial x} dx\right| \leq C\left(1+\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|\right)\int_0^1 |r|\left|\frac{\partial v}{\partial x}\right| dx \leq$$
$$\varepsilon\left\|\frac{\partial v}{\partial x}\right\|^2+C\left(1+\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\right)\int_0^t \left\|\frac{\partial v}{\partial x}\right\|^2 d\tau, \tag{6.52}$$

$$\left|\beta\int_0^1 \frac{\theta}{u_1}\frac{\partial}{\partial x}(r_1^2 v) dx\right| \leq C\int_0^1 |\theta|\left(|v|+\left|\frac{\partial v}{\partial x}\right|\right) dx \leq C\left(\|\theta\|^2+\|v\|^2\right)+\varepsilon\left\|\frac{\partial v}{\partial x}\right\|^2, \tag{6.53}$$

$$\left|\beta\int_0^1 \frac{\theta_2}{u_1 u_2} u\frac{\partial}{\partial x}(r_1^2 v) dx\right| \leq C\int_0^1 |u|\left(|v|+\left|\frac{\partial v}{\partial x}\right|\right) dx \leq$$
$$C\left(\|u\|^2+\|v\|^2\right)+\varepsilon\left\|\frac{\partial v}{\partial x}\right\|^2, \tag{6.54}$$

$$\left|\beta\int_0^1 \frac{\theta_2}{u_2}\frac{\partial r}{\partial x}(r_1+r_2)v dx\right| \leq C\int_0^1 \left|\frac{\partial r}{\partial x}\right||v| dx \leq C\left(\|v\|^2+\int_0^t \left\|\frac{\partial v}{\partial x}\right\|^2 d\tau\right), \tag{6.55}$$

$$\left|\beta\int_0^1 \frac{\theta_2}{u_2} r\frac{\partial}{\partial x}(r_1+r_2)v dx\right| \leq C\int_0^1 |r|\,|v| dx \leq C\left(\|v\|^2+\int_0^t \left\|\frac{\partial v}{\partial x}\right\|^2 d\tau\right), \tag{6.56}$$

$$\left|\beta\int_0^1 \frac{\theta_2}{u_2} r(r_1+r_2)\frac{\partial v}{\partial x} dx\right| \leq C\int_0^1 |r|\left|\frac{\partial v}{\partial x}\right| dx \leq \varepsilon\left\|\frac{\partial v}{\partial x}\right\|^2+C\int_0^t \left\|\frac{\partial v}{\partial x}\right\|^2 d\tau. \tag{6.57}$$

Uvrštavanjem (6.44)-(6.57) u (6.41) i integriranjem preko $]0,t[$ dobivamo

$$\|v(t)\|^2+(1-9\varepsilon)\int_0^t \left\|\frac{\partial v}{\partial x}\right\|^2 d\tau \leq C\int_0^t \left[\left(1+\left\|\frac{\partial^2 v_2}{\partial x^2}(\tau)\right\|^2\right)\cdot\right.$$
$$\left.\left(\|u(\tau)\|^2+\|v(\tau)\|^2+\int_0^\tau \left\|\frac{\partial v}{\partial x}(s)\right\|^2 ds\right)+\|\theta(\tau)\|^2\right] d\tau. \tag{6.58}$$

Uzevši u obzir (6.28) za dovoljno mali $\varepsilon > 0$, (6.58) postaje

$$\begin{aligned} \|v(t)\|^2 + \int_0^t \left\|\frac{\partial v}{\partial x}\right\|^2 d\tau \leq \\ C\int_0^t \left[\left(1+\left\|\frac{\partial^2 v_2}{\partial x^2}(\tau)\right\|^2\right)\left(\|v(\tau)\|^2 + \int_0^\tau \left\|\frac{\partial v}{\partial x}(s)\right\|^2 ds\right) + \|\theta(\tau)\|^2\right] d\tau. \end{aligned} \tag{6.59}$$

Primjenom Gronwallove nejednakosti i svojstva (4.3) za funkciju v_2, iz (6.59) slijedi

$$\|v(t)\|^2 + \int_0^t \left\|\frac{\partial v}{\partial x}(\tau)\right\|^2 d\tau \leq C\int_0^t \|\theta(\tau)\|^2 d\tau. \tag{6.60}$$

□

Lema 6.4. *Postoji konstanta $C > 0$ takva da vrijedi*

$$\|\omega(t)\|^2 + \int_0^t \left\|\frac{\partial \omega}{\partial x}(\tau)\right\|^2 d\tau \leq C\int_0^t \|\theta(\tau)\|^2 d\tau \tag{6.61}$$

za svaki $t \in]0, T[$.

Dokaz. Kako funkcije ω_1, ω_2 i ω imaju ista svojstva i pripadaju istim prostorima kao i funkcije v_1, v_2 i v koristimo se analognim postupkom kao i u dokazu Leme 6.3. Množenjem jednadžbe (6.18) sa ω i integriranjem preko $]0, 1[$ dobivamo

$$\begin{aligned} \frac{j_I}{2}\frac{d}{dt}\|\omega(t)\|^2 + \gamma\int_0^1 \frac{r_1^4}{u_1}\left(\frac{\partial\omega}{\partial x}\right)^2 dx = -4\gamma\int_0^1 \frac{r_1^3}{u_1}\frac{\partial r_1}{\partial x}\omega\frac{\partial \omega}{\partial x}dx \\ -4\gamma\int_0^1 \frac{r_1^2}{u_1}\left(\frac{\partial r_1}{\partial x}\right)^2\omega^2 dx - \gamma\int_0^1 \frac{1}{u_1}\frac{\partial r}{\partial x}(r_1+r_2)\omega_2\frac{\partial}{\partial x}(r_1^2\omega)dx \\ -\gamma\int_0^1 \frac{1}{u_1}r\frac{\partial}{\partial x}(r_1+r_2)\omega_2\frac{\partial}{\partial x}(r_1^2\omega)dx - \gamma\int_0^1 \frac{1}{u_1}r(r_1+r_2)\frac{\partial\omega_2}{\partial x}\frac{\partial}{\partial x}(r_1^2\omega)dx \\ +\gamma\int_0^1 \frac{u}{u_1u_2}\frac{\partial}{\partial x}(r_2^2\omega_2)\frac{\partial}{\partial x}(r_1^2\omega)dx - \gamma\int_0^1 \frac{1}{u_2}\frac{\partial}{\partial x}(r_2^2\omega_2)\frac{\partial r}{\partial x}(r_1+r_2)\omega dx \\ -\gamma\int_0^1 \frac{1}{u_2}\frac{\partial}{\partial x}(r_2^2\omega_2)r\frac{\partial}{\partial x}(r_1+r_2)\omega dx - \gamma\int_0^1 \frac{1}{u_2}\frac{\partial}{\partial x}(r_2^2\omega_2)r(r_1+r_2)\frac{\partial\omega}{\partial x}dx \\ -\delta\int_0^1 u_1\omega^2 dx - \delta\int_0^1 \omega_2 u\omega dx. \end{aligned} \tag{6.62}$$

Uočimo najprije da vrijede nejednakosti

$$\left|\frac{\partial}{\partial x}(r_2^2\omega_2)\right| \leq C\left(1+\left|\frac{\partial\omega_2}{\partial x}\right|\right) \leq C\left(1+\left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|\right), \tag{6.63}$$

$$\left|\frac{\partial}{\partial x}(r_1^2\omega)\right| \leq C\left(|\omega| + \left|\frac{\partial\omega}{\partial x}\right|\right). \tag{6.64}$$

Kao i u dokazu Leme 6.3, korištenjem nejednakosti (6.29), (6.30), (6.63) i (6.64), nejednakosti Friedrichs-Poincaréa i Gagliardo-Ladyzhenskaye za funkciju ω, ocjena (6.7), (6.8), (6.9) i (6.11) te Youngove nejedankosti s parametrom $\varepsilon > 0$ dobivamo ocjene integrala s desne strane jednakosti (6.91).

$$\left|4\gamma\int_0^1 \frac{r_1^2}{u_1}\left(\frac{\partial r_1}{\partial x}\right)^2\omega^2 dx\right| \leq C\left\|\omega\right\|^2, \tag{6.65}$$

$$\left|4\gamma\int_0^1 \frac{r_1^3}{u_1}\frac{\partial r_1}{\partial x}\omega\frac{\partial\omega}{\partial x}dx\right| \leq \varepsilon\left\|\frac{\partial\omega}{\partial x}\right\|^2 + C\left\|\omega\right\|^2, \tag{6.66}$$

$$\begin{gathered}\left|\gamma\int_0^1 \frac{1}{u_1}\frac{\partial r}{\partial x}(r_1+r_2)\omega_2\frac{\partial}{\partial x}(r_1^2\omega)dx\right| \leq C\left(\int_0^1\left|\frac{\partial r}{\partial x}\right||\omega|\,dx + \int_0^1\left|\frac{\partial r}{\partial x}\right|\left|\frac{\partial\omega}{\partial x}\right|dx\right) \leq \\ C\left(\left\|\frac{\partial r}{\partial x}\right\|^2 + \|\omega\|^2\right) + \varepsilon\left\|\frac{\partial\omega}{\partial x}\right\|^2 \leq \varepsilon\left\|\frac{\partial\omega}{\partial x}\right\|^2 + C\left(\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau + \|\omega\|^2\right),\end{gathered} \tag{6.67}$$

$$\begin{gathered}\left|\gamma\int_0^1 \frac{1}{u_1}r\frac{\partial}{\partial x}(r_1+r_2)\omega_2\frac{\partial}{\partial x}(r_1^2\omega)dx\right| \leq C\int_0^1|r|\left(|\omega| + \left|\frac{\partial\omega}{\partial x}\right|\right)dx \leq \\ C\left(\|r\|^2 + \|\omega\|^2\right) + \varepsilon\left\|\frac{\partial\omega}{\partial x}\right\|^2 \leq \varepsilon\left\|\frac{\partial\omega}{\partial x}\right\|^2 + C\left(\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau + \|\omega\|^2\right),\end{gathered} \tag{6.68}$$

$$\begin{gathered}\left|\gamma\int_0^1 \frac{1}{u_1}r(r_1+r_2)\frac{\partial\omega_2}{\partial x}\frac{\partial}{\partial x}(r_1^2\omega)dx\right| \leq \\ C\left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|\int_0^1|r|\left(|\omega| + \left|\frac{\partial\omega}{\partial x}\right|\right)dx \leq \varepsilon\left\|\frac{\partial\omega}{\partial x}\right\|^2 + \\ C\left(\left(1 + \left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|^2\right)\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau + \|\omega\|^2\right),\end{gathered} \tag{6.69}$$

$$\begin{gathered}\left|\gamma\int_0^1 \frac{u}{u_1u_2}\frac{\partial}{\partial x}(r_2^2\omega_2)\frac{\partial}{\partial x}(r_1^2\omega)dx\right| \leq C\left(1 + \left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|\right)\int_0^1|u|\left(|\omega| + \left|\frac{\partial\omega}{\partial x}\right|\right)dx \leq \\ C\left(\left(1 + \left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|^2\right)\|u\|^2 + \|\omega\|^2\right) + \varepsilon\left\|\frac{\partial\omega}{\partial x}\right\|^2,\end{gathered} \tag{6.70}$$

$$\left|\gamma\int_0^1 \frac{1}{u_2}\frac{\partial}{\partial x}(r_2^2\omega_2)\frac{\partial r}{\partial x}(r_1+r_2)\omega dx\right| \le C\left(1+\left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|\right)\int_0^1 |\omega|\left|\frac{\partial r}{\partial x}\right| dx \le$$
$$C\left[\left(1+\left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|^2\right)\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau + \|\omega\|^2\right], \tag{6.71}$$

$$\left|\gamma\int_0^1 \frac{1}{u_1}\frac{\partial}{\partial x}(r_2^2\omega_2)r\frac{\partial}{\partial x}(r_1+r_2)\omega dx\right| \le C\left(1+\left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|\right)\int_0^1 |r||\omega| dx \le$$
$$C\left[\left(1+\left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|^2\right)\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau + \|\omega\|^2\right], \tag{6.72}$$

$$\left|\gamma\int_0^1 \frac{1}{u_2}\frac{\partial}{\partial x}(r_2^2\omega_2)r(r_1+r_2)\frac{\partial\omega}{\partial x}dx\right| \le C\left(1+\left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|\right)\int_0^1 |r|\left|\frac{\partial\omega}{\partial x}\right| dx \le$$
$$\varepsilon\left\|\frac{\partial\omega}{\partial x}\right\|^2 + C\left(1+\left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|^2\right)\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau, \tag{6.73}$$

$$\left|\delta\int_0^1 u_1\omega^2 dx\right| \le C\|\omega\|^2, \tag{6.74}$$

$$\left|\delta\int_0^1 \omega_2 u\omega dx\right| \le C\left(\|u\|^2+\|\omega\|^2\right), \tag{6.75}$$

Korištenjem ocjene (6.28) za funkciju u, ocjene (6.1) za funkciju r_1 te svojstva (6.9) za funkciju u_1^{-1}, nakon uvrštavanja (6.65)-(6.75) u (6.62) i integriranja preko $]0,t[$ dobivamo

$$\|\omega(t)\|^2 + (1-6\varepsilon)\int_0^t\left\|\frac{\partial\omega}{\partial x}\right\|^2 d\tau \le$$
$$C\int_0^t\left[\|\omega(\tau)\|^2 + \int_0^\tau\left\|\frac{\partial\omega}{\partial x}(s)\right\|^2 ds + \left(1+\left\|\frac{\partial^2\omega_2}{\partial x^2}(\tau)\right\|^2\right)\int_0^\tau\left\|\frac{\partial v}{\partial x}(s)\right\|^2 ds\right]d\tau. \tag{6.76}$$

Uzimanjem dovoljno malog ε i primjenom Gronwallove nejednakosti na (6.76) dobivamo

$$\|\omega(t)\|^2 + \int_0^t\left\|\frac{\partial\omega}{\partial x}\right\|^2 d\tau \le C\int_0^t\left(1+\left\|\frac{\partial^2\omega_2}{\partial x^2}(\tau)\right\|^2\right)\int_0^\tau\left\|\frac{\partial v}{\partial x}(s)\right\|^2 ds d\tau \le$$
$$C\int_0^t\left\|\frac{\partial v}{\partial x}(s)\right\|^2 ds\int_0^t\left(1+\left\|\frac{\partial^2\omega_2}{\partial x^2}(\tau)\right\|^2\right)d\tau. \tag{6.77}$$

Sada, korištenjem inkluzije (4.3) za funkciju ω_2, iz (6.77) slijedi

$$\|\omega(t)\|^2 + \int_0^t\left\|\frac{\partial\omega}{\partial x}(\tau)\right\|^2 d\tau \le C\int_0^t\left\|\frac{\partial v}{\partial x}(\tau)\right\|^2 d\tau \tag{6.78}$$

za svako $t \in]0, T[$. Primjenom nejednakosti (6.40) na (6.78) odmah dobivamo (6.61) čime je lema dokazana.

□

Lema 6.5. *Postoji konstanta $C > 0$ takva da vrijedi*

$$\|\theta(t)\|^2 + \int_0^t \left\| \frac{\partial \theta}{\partial x}(\tau) \right\|^2 d\tau \leq C \int_0^t \|\theta(\tau)\|^2 d\tau \tag{6.79}$$

za svako $t \in]0, T[$.

Dokaz. Množenjem jednadžbe (6.19) sa θ, nakon integriranja preko $]0, 1[$, korištenjem formule parcijalne integracije i rubnih uvjeta za derivacije funkcija θ, θ_1 i θ_2 dobivamo

$$\begin{aligned}
\frac{c_v}{2}\frac{d}{dt}\|\theta(t)\|^2 + \nu \int_0^1 \frac{r_1^2}{u_1}\left(\frac{\partial \theta}{\partial x}\right)^2 dx = -\nu \int_0^1 \frac{r(r_1+r_2)(r_1^2+r_2^2)}{u_1}\frac{\partial \theta_2}{\partial x}\frac{\partial \theta}{\partial x} dx \\
+\nu \int_0^1 \frac{u}{u_1 u_2} r_2^4 \frac{\partial \theta_2}{\partial x}\frac{\partial \theta}{\partial x} dx - \alpha \int_0^1 \frac{u}{u_1 u_2}\left[\frac{\partial}{\partial x}(r_1^2 v_1)\right]^2 \theta dx + \\
\alpha \int_0^1 \frac{1}{u_2}\left[\frac{\partial}{\partial x}(r_1^2 v_1) + \frac{\partial}{\partial x}(r_2^2 v_2)\right]\frac{\partial}{\partial x}(r(r_1+r_2)v_1)\theta dx + \\
\alpha \int_0^1 \frac{1}{u_2}\left[\frac{\partial}{\partial x}(r_1^2 v_1) + \frac{\partial}{\partial x}(r_2^2 v_2)\right]\frac{\partial}{\partial x}(r_2^2 v)\theta dx - \\
\beta \int_0^1 \frac{1}{u_1}\frac{\partial}{\partial x}(r_1^2 v_1)\theta^2 dx + \beta \int_0^1 \frac{1}{u_1}\frac{\partial}{\partial x}(r_1^2 v_1)\frac{\theta_2}{u_2} u\theta dx - \\
\beta \int_0^1 \frac{\theta_2}{u_2}\frac{\partial}{\partial x}(r(r_1+r_2)v_1)\theta dx - \beta \int_0^1 \frac{\theta_2}{u_2}\frac{\partial}{\partial x}(r_2^2 v)\theta dx + \\
d \int_0^1 \left(r v_1^2 + r_2(v_1+v_2)v\right)\frac{\partial \theta}{\partial x} dx - \gamma \int_0^1 \frac{u}{u_1 u_2}\left[\frac{\partial}{\partial x}(r_1^2 \omega_1)\right]^2 \theta dx + \\
\gamma \int_0^1 \frac{1}{u_2}\left[\frac{\partial}{\partial x}(r_1^2 \omega_1) + \frac{\partial}{\partial x}(r_2^2 \omega_2)\right]\frac{\partial}{\partial x}(r(r_1+r_2)\omega_1)\theta dx + \\
\gamma \int_0^1 \frac{1}{u_2}\left[\frac{\partial}{\partial x}(r_1^2 \omega_1) + \frac{\partial}{\partial x}(r_2^2 \omega_2)\right]\frac{\partial}{\partial x}(r_2^2 \omega)\theta dx + h \int_0^1 r\omega_1^2 \frac{\partial \theta}{\partial x} dx + \\
h \int_0^1 r_2 \omega(\omega_1+\omega_2)\frac{\partial \theta}{\partial x} dx + \delta \int_0^1 (\omega_1+\omega_2)u_1 \omega \theta dx + \delta \int_0^1 \omega_2^2 u \theta dx.
\end{aligned} \tag{6.80}$$

Koristeći se istim ocjenama i nejednakostima kao u prethodnim lemama, rezultatom Leme 6.2 te uz primjenu nejednakosti Gagliardo-Ladyzhenskaye za funkciju $\frac{\partial \theta_2}{\partial x}$ dobivamo ocjene integrala na desnoj strani jednakosti (6.80). Bez navođenja detalja u izvođenju tih ocjena dobivamo:

$$\left|\nu\int_0^1 \frac{r(r_1+r_2)(r_1^2+r_2^2)}{u_1}\frac{\partial\theta_2}{\partial x}\frac{\partial\theta}{\partial x}dx\right| \leq \varepsilon\left\|\frac{\partial\theta}{\partial x}\right\|^2 + C\left\|\frac{\partial^2\theta_2}{\partial x^2}\right\|^2\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau, \tag{6.81}$$

$$\begin{aligned}\left|\nu\int_0^1 \frac{u}{u_1u_2}r_2^4\frac{\partial\theta_2}{\partial x}\frac{\partial\theta}{\partial x}dx\right| &\leq \varepsilon\left\|\frac{\partial\theta}{\partial x}\right\|^2 + C\left\|\frac{\partial^2\theta_2}{\partial x^2}\right\|^2\|u\|^2 \leq \\ &\varepsilon\left\|\frac{\partial\theta}{\partial x}\right\|^2 + C\left\|\frac{\partial^2\theta_2}{\partial x^2}\right\|^2\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau,\end{aligned} \tag{6.82}$$

$$\begin{aligned}\left|\alpha\int_0^1 \frac{u}{u_1u_2}\left[\frac{\partial}{\partial x}(r_1^2v_1)\right]^2\theta dx\right| &\leq C\left(1+\left\|\frac{\partial^2 v_1}{\partial x^2}\right\|^2\right)\left(\|u\|^2+\|\theta\|^2\right) \leq \\ &C\left(1+\left\|\frac{\partial^2 v_1}{\partial x^2}\right\|^2\right)\left(\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau + \|\theta\|^2\right),\end{aligned} \tag{6.83}$$

$$\begin{aligned}&\left|\alpha\int_0^1 \frac{1}{u_2}\left[\frac{\partial}{\partial x}(r_1^2v_1)+\frac{\partial}{\partial x}(r_2^2v_2)\right]\frac{\partial}{\partial x}(r(r_1+r_2)v_1)\theta dx\right| \leq \\ &C\left(1+\left\|\frac{\partial^2 v_1}{\partial x^2}\right\|^2+\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\right)\left(\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau + \|\theta\|^2\right)\end{aligned} \tag{6.84}$$

$$\begin{aligned}&\left|\alpha\int_0^1 \frac{1}{u_2}\left[\frac{\partial}{\partial x}(r_1^2v_1)+\frac{\partial}{\partial x}(r_2^2v_2)\right]\frac{\partial}{\partial x}(r_2^2v)\theta dx\right| \leq \\ &C\left(1+\left\|\frac{\partial^2 v_1}{\partial x^2}\right\|^2+\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\right)\|\theta\|^2 + C\left(\|v\|^2+\left\|\frac{\partial v}{\partial x}\right\|^2\right) \leq \\ &C\left(1+\left\|\frac{\partial^2 v_1}{\partial x^2}\right\|^2+\left\|\frac{\partial^2 v_2}{\partial x^2}\right\|^2\right)\|\theta\|^2 + C\left\|\frac{\partial v}{\partial x}\right\|^2,\end{aligned} \tag{6.85}$$

$$\left|\beta\int_0^1 \frac{1}{u_1}\frac{\partial}{\partial x}(r_1^2v_1)\theta^2 dx\right| \leq C\left(1+\left\|\frac{\partial^2 v_1}{\partial x^2}\right\|^2\right)\|\theta\|^2, \tag{6.86}$$

$$\left|\beta\int_0^1 \frac{1}{u_1}\frac{\partial}{\partial x}(r_1^2v_1)\frac{\theta_2}{u_2}u\theta dx\right| \leq C\left[\left(1+\left\|\frac{\partial^2 v_1}{\partial x^2}\right\|^2\right)\|\theta\|^2+\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau\right], \tag{6.87}$$

$$\left|\beta\int_0^1 \frac{\theta_2}{u_2}\frac{\partial}{\partial x}(r(r_1+r_2)v_1)\theta dx\right| \leq C\left[\left(1+\left\|\frac{\partial^2 v_1}{\partial x^2}\right\|^2\right)\|\theta\|^2+\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau\right], \tag{6.88}$$

$$\left|\beta\int\limits_0^1 \frac{\theta_2}{u_2}\frac{\partial}{\partial x}(r_2^2 v)\theta dx\right| \le C\left(\|\theta\|^2+\|v\|^2+\left\|\frac{\partial v}{\partial x}\right\|^2\right) \le C\left(\|\theta\|^2+\left\|\frac{\partial v}{\partial x}\right\|^2\right), \tag{6.89}$$

$$\begin{aligned}\left|d\int\limits_0^1 \left(rv_1^2+r_2(v_1+v_2)v\right)\frac{\partial\theta}{\partial x}dx\right| \le \varepsilon\left\|\frac{\partial\theta}{\partial x}\right\|^2+C\left(\|v\|^2+\int\limits_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau\right) \le \\ \varepsilon\left\|\frac{\partial\theta}{\partial x}\right\|^2+C\left(\left\|\frac{\partial v}{\partial x}\right\|^2+\int\limits_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau\right),\end{aligned} \tag{6.90}$$

$$\begin{aligned}\left|\gamma\int\limits_0^1 \frac{u}{u_1u_2}\left[\frac{\partial}{\partial x}(r_1^2\omega_1)\right]^2\theta dx\right| \le C\left(1+\left\|\frac{\partial^2\omega_1}{\partial x^2}\right\|^2\right)\left(\|u\|^2+\|\theta\|^2\right) \le \\ C\left(1+\left\|\frac{\partial^2\omega_1}{\partial x^2}\right\|^2\right)\left(\int\limits_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau+\|\theta\|^2\right),\end{aligned} \tag{6.91}$$

$$\begin{aligned}\left|\gamma\int\limits_0^1 \frac{1}{u_2}\left[\frac{\partial}{\partial x}(r_1^2\omega_1)+\frac{\partial}{\partial x}(r_2^2\omega_2)\right]\frac{\partial}{\partial x}(r(r_1+r_2)\omega_1)\theta dx\right| \le \\ C\left(1+\left\|\frac{\partial^2\omega_1}{\partial x^2}\right\|^2+\left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|^2\right)\left(\int\limits_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau+\|\theta\|^2\right),\end{aligned} \tag{6.92}$$

$$\begin{aligned}\left|\gamma\left[\frac{\partial}{\partial x}(r_1^2\omega_1)+\frac{\partial}{\partial x}(r_2^2\omega_2)\right]\frac{\partial}{\partial x}(r_2^2\omega)\theta dx\right| \le \\ C\left(1+\left\|\frac{\partial^2\omega_1}{\partial x^2}\right\|^2+\left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|^2\right)\|\theta\|^2+C\left(\left\|\frac{\partial\omega}{\partial x}\right\|^2+\|\omega\|^2\right) \le \\ C\left(1+\left\|\frac{\partial^2\omega_1}{\partial x^2}\right\|^2+\left\|\frac{\partial^2\omega_2}{\partial x^2}\right\|^2\right)\|\theta\|^2+C\left\|\frac{\partial\omega}{\partial x}\right\|^2,\end{aligned} \tag{6.93}$$

$$\left|h\int\limits_0^1 r\omega_1^2\frac{\partial\theta}{\partial x}dx\right| \le \varepsilon\left\|\frac{\partial\theta}{\partial x}\right\|^2+C\int\limits_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau, \tag{6.94}$$

$$\left|h\int\limits_0^1 r_2\omega(\omega_1+\omega_2)\frac{\partial\theta}{\partial x}dx\right| \le \varepsilon\left\|\frac{\partial\theta}{\partial x}\right\|^2+C\|\omega\|^2 \le \varepsilon\left\|\frac{\partial\theta}{\partial x}\right\|^2+C\left\|\frac{\partial\omega}{\partial x}\right\|^2, \tag{6.95}$$

$$\left|\delta\int\limits_0^1 (\omega_1+\omega_2)u_1\omega\theta dx dx\right| \le C\left(\|\omega\|^2+\|\theta\|^2\right) \le C\left(\left\|\frac{\partial\omega}{\partial x}\right\|^2+\|\theta\|^2\right), \tag{6.96}$$

$$\left|\delta\int\limits_0^1 \omega_2^2 u\theta dx\right| \le C\left(\|u\|^2+\|\theta\|^2\right) \le C\left(\int\limits_0^t\left\|\frac{\partial v}{\partial x}\right\|^2 d\tau+\|\theta\|^2\right). \tag{6.97}$$

Sada ponovno primjenjujemo ocjene (6.1) i (6.9).Uvrštavanjem (6.81)-(6.97) u (6.80) i integriranjem preko $]0, t[$ dobivamo

$$\begin{aligned} \|\theta\|^2 + (1-5\varepsilon) \int_0^t \left\| \frac{\partial \theta}{\partial x}(\tau) \right\|^2 d\tau \leq C \int_0^t A(\tau) \left(\int_0^\tau \left\| \frac{\partial v}{\partial x}(s) \right\|^2 + \|\theta(s)\|^2 \right) d\tau + \\ C \left(\int_0^t \left\| \frac{\partial \omega}{\partial x}(\tau) \right\|^2 d\tau + \int_0^t \left\| \frac{\partial v}{\partial x}(\tau) \right\|^2 d\tau \right), \end{aligned} \tag{6.98}$$

gdje je

$$\begin{aligned} A(\tau) = 1 + \left\| \frac{\partial^2 \theta_2}{\partial x^2}(\tau) \right\|^2 + \left\| \frac{\partial^2 v_1}{\partial x^2}(\tau) \right\|^2 + \\ \left\| \frac{\partial^2 v_2}{\partial x^2}(\tau) \right\|^2 + \left\| \frac{\partial^2 \omega_1}{\partial x^2}(\tau) \right\|^2 + \left\| \frac{\partial^2 \omega_2}{\partial x^2}(\tau) \right\|^2. \end{aligned} \tag{6.99}$$

Izborom dovoljno malog parametra $\varepsilon > 0$, korištenjem svojstva (6.61), (6.40) i (4.3) dobivamo

$$\begin{aligned} \|\theta\|^2 + \int_0^t \left\| \frac{\partial \theta}{\partial x}(\tau) \right\|^2 d\tau \leq \\ C \int_0^t A(\tau) \left(\|\theta(\tau)\|^2 + \int_0^\tau \left\| \frac{\partial \theta}{\partial x}(s) \right\|^2 \right) d\tau + C \int_0^t \|\theta(\tau)\|^2 \, d\tau. \end{aligned} \tag{6.100}$$

Primjenom Gronwallove nejednakosti na (6.100) konačno dobivamo

$$\|\theta(t)\|^2 + \int_0^t \left\| \frac{\partial \theta}{\partial x}(\tau) \right\|^2 d\tau \leq C \int_0^t \|\theta(\tau)\|^2 d\tau. \tag{6.101}$$

čime je lema dokazana. □

Sada lako zaključujemo da tvrdnja Teorema 4.2 vrijedi. Naime, ako ponovno primjenimo Gronwallovu nejednakost na (6.101) odmah dobivamo jednakost

$$\theta = 0. \tag{6.102}$$

Uvrštavanjem (6.102) u (6.40) i (6.61) zaključujemo da je

$$v = 0, \quad \omega = 0. \tag{6.103}$$

Na kraju, iz (6.28) dobivamo

$$u = 0 \tag{6.104}$$

čime je Teorem 4.2 dokazan.

7 Dokaz egzistencije globalnog rješenja

U prethodna dva poglavlja pokazali smo da generalizirano rješenje problema (3.126)-(3.136) postoji na domeni Q_{T_0}, za dovoljno malo $T_0 > 0$ te da je generalizirano rješenje jedinstveno.

U ovom poglavlju dokazujemo egzistenciju generaliziranog rješenja globalno po vremenu, odnosno dokazujemo da rješenje postoji na domeni Q_T, za proizvoljno $T > 0$.

Slično kao u [Muj98a] i [AKM90] dokaz egzistencije globalnog rješenja dobivamo bazirajući se na sljedećoj propoziciji.

Propozicija 7.1. *Neka je $T \in \mathbf{R}^+$ i neka je funkcija*

$$(x,t) \mapsto (\rho, v, \omega, \theta)(x,t), \quad (x,t) \in Q_T \tag{7.1}$$

generalizirano rješenje problema (3.126)-(3.136) na domeni $Q_{T'}$, $T' < T$ te neka je $\theta > 0$ u $\overline{Q}_{T'}$. Tada je (7.1) generalizirano rješenje istog problema i na domeni Q_T te vrijedi nejednakost $\theta > 0$ na $\overline{Q}_T$.

Primjenjujući pretpostavke Propozicije 7.1 na funkciju (7.1) u daljnjem tekstu dobivamo rezultate koji za posljedicu imaju tvrdnju Propozicije 7.1.

Nužno nam je, kao prvo, dobiti neka svojstva funkcije (7.1) kao rješenja problema (3.126)-(3.136) na domeni $\overline{Q}_{T'}$ koja će nam omogućiti izvođenje globalnih apriornih ocjena na Q_T, a time i dokaza Teorema 4.3 u skladu sa tvrdnjom Propozicije 7.1.

Ako drugačije nije navedeno sve konstante u nastavku ovisit će samo o početnim podacima i broju T. Označavat ćemo ih s C ili C_i gdje je $i = 1, 2, ...$, a na različitim mjestima mogu poprimati različite vrijednosti. Prilikom izvođenja ocjena, osim već spomenutih referenci, koristimo se i idejama iz članka [Jia96].

7.1 Neka svojstva rješenja

Prema Lemi 6.1 lako zaključujemo da vrijedi

$$r(x,t) \geq a, \quad (x,t) \in \overline{Q}_{T'} \tag{7.2}$$

gdje je a definiran s (3.139). Također, kao i kod (6.13) lako zaključujuemo da je funkcija ρ neprekidna na $\overline{Q}_{T'}$.

Lema 7.1. *Neka je A konstanta definirana s*

$$A = \int_0^1 \frac{1}{\rho_0(x)} dx. \tag{7.3}$$

Za funkciju ρ i svako $T' \in]0, T[$ vrijedi

$$\int_0^1 \frac{1}{\rho(x,t)} dx = A, \quad t \in [0, T']. \tag{7.4}$$

Također postoji funkcija $g : [0, T'] \to [0, 1]$ takva da je

$$\rho(g(t), t) = A^{-1}, \quad t \in [0, T']. \tag{7.5}$$

Dokaz. Dijeljenjem (3.126) s ρ^2, integriranjem preko $]0, 1[$ te uvažavanjem rubnih uvjeta (3.134) za funkciju v, dobivamo

$$\frac{d}{dt} \int_0^1 \frac{1}{\rho(x,t)} \,\mathrm{d}x = 0, \tag{7.6}$$

odnosno

$$\int_0^1 \frac{1}{\rho(x,t)} \,\mathrm{d}x = C, \quad t \in [0, T']. \tag{7.7}$$

Uvrštavanjem $t = 0$ u (7.7) zaključujemo da vrijedi (7.4). Primjenom teorema o srednjoj vrijednosti funkcije na (7.4) zaključujemo da postoji funkcija g sa svojstvom (7.5). □

Da bismo kasnije u radu dobili nužnu ograničenost funkcija ρ i θ uvodimo funkcije Kazhikhova (koje označavamo s $Y(t)$ i $B(x,t)$) prilagođene našem problemu. Postupak je sličan onom u [AKM90].

Lema 7.2. *Za funkciju ρ na području $Q_{T'}$ vrijedi*

$$\rho(x,t) = \frac{\rho_0(x) \cdot Y(t) \cdot B(x,t)}{1 + \dfrac{R}{\lambda + 2\mu} \rho_0(x) \displaystyle\int_0^t \theta(x,\tau) \cdot Y(\tau) \cdot B(x,\tau) \, d\tau} \tag{7.8}$$

gdje su

$$Y(t) = \frac{1}{A\rho_0(g(t))} \exp\left\{ \frac{R}{\lambda + 2\mu} \int_0^t \rho(g(t),\tau)\theta(g(t),\tau)\, d\tau \right\} \tag{7.9}$$

i

$$B(x,t) = \exp\left\{ -\frac{L}{\lambda + 2\mu} \int_{g(t)}^{x} \int_0^t r^{-2}(y,\tau) \frac{\partial v(y,\tau)}{\partial t}\, d\tau\, dy \right\} \tag{7.10}$$

funkcije Kazhikhova. (Konstanta A i funkcija g određene su u Lemi 7.1).

Dokaz. Ako jednadžbu (3.126) zapišemo u obliku

$$\frac{1}{L}\rho \frac{\partial}{\partial x}\left(r^2 v\right) = -\frac{\partial}{\partial t} \ln \rho \tag{7.11}$$

i uvrstimo u (3.127) dobivamo

$$r^{-2}\frac{\partial v}{\partial t} = -\frac{R}{L}\frac{\partial}{\partial x}(\rho\theta) - \frac{\lambda + 2\mu}{L}\frac{\partial^2}{\partial x \partial t}(\ln \rho). \tag{7.12}$$

Integriranjem (7.12) preko $[0,t],\ t \in]0,T']$, slijedi

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{\lambda + 2\mu}{L} \ln \rho + \frac{R}{L} \int_0^t \rho(x,\tau)\theta(x,\tau) d\tau \right) = \\ &\frac{\lambda + 2\mu}{L} \frac{\partial}{\partial x} \ln \rho_0(x) - \int_0^t r^{-2}(x,\tau) \frac{\partial v(x,\tau)}{\partial t} d\tau. \end{aligned} \tag{7.13}$$

Neka je sada $t \in]0,T']$ fiksan. Daljnjim integriranjm (7.13) preko $[g(t),x],\ x \in]0,1[$ dobivamo

$$\begin{aligned} &\frac{\lambda + 2\mu}{L} \ln \rho + \frac{R}{L} \int_0^t \rho(x,\tau)\theta(x,\tau) d\tau = \\ &\frac{\lambda + 2\mu}{L} \ln \rho(g(t),t) + \frac{R}{L} \int_0^t \rho(g(t),\tau)\theta(g(t),\tau) d\tau \\ &+\frac{\lambda + 2\mu}{L} \ln \frac{\rho_0(x)}{\rho_0(g(t))} - \int_{g(t)}^{x} \int_0^t r^{-2}(y,\tau) \frac{\partial v(y,\tau)}{\partial t} d\tau dy. \end{aligned} \tag{7.14}$$

što možemo, u skladu s (7.9) i (7.10) zapisati jednostavnije u obliku

$$\rho(x,t) \exp\left\{ \frac{R}{\lambda + 2\mu} \int_0^t \rho(x,\tau)\theta(x,\tau) d\tau \right\} = \rho_0(x) \cdot Y(t) \cdot B(x,t). \tag{7.15}$$

Množenjem (7.15) sa $\dfrac{R}{\lambda+2\mu}\theta$ te ponovnim integriranjem preko $[0,t]$ imamo

$$\exp\left\{\frac{R}{\lambda+2\mu}\int_0^t \rho(x,\tau)\theta(x,\tau)d\tau\right\}= \\ 1+\frac{R}{\lambda+2\mu}\rho_0(x)\int_0^t \theta(x,\tau)\cdot Y(\tau)\cdot B(x,\tau)d\tau \tag{7.16}$$

što uvrštavamo u (7.15) i odmah dobivamo (7.8). □

Sada postupamo slično kao u [Jia96]. Izvodimo ocjenu za funkciju $U(x,t)$ definiranu sa

$$U(x,t)=\frac{v^2}{2}+j_I\frac{\omega^2}{2}+R\psi\left(\frac{1}{\rho}\right)+c_v\psi(\theta), \tag{7.17}$$

pri čemu je $\psi(x)=x-\ln x-1$. Primjetimo da je ψ nenegativna konveksna funkcija.

Lema 7.3. *Za $t\in[0,T']$ postoji konstanta $C\in\mathbb{R}^+$ takva da vrijedi*

$$\int_0^1 U(x,t)dx+\int_0^t\int_0^1\left[\frac{k}{L^2}\frac{r^4\rho}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\lambda+\frac{2}{3}\mu\right)\frac{\rho}{\theta}\left(\frac{\partial}{\partial x}\left(r^2v\right)\right)^2+\right. \\ \left.\left(c_0+\frac{2}{3}c_d\right)\frac{\rho}{\theta}\left(\frac{\partial}{\partial x}\left(r^2\omega\right)\right)^2\right]dx\,d\tau\le C. \tag{7.18}$$

Dokaz. Uočimo da množenjem jednadžbi (3.126), (3.127), (3.128) i (3.129) respektivno s $R\left(-\dfrac{1}{\rho^2}+\dfrac{1}{\rho}\right)$, v, $j_I\omega\rho^{-1}$ i $c_v\left(1-\dfrac{1}{\theta}\right)\rho^{-1}$, njihovim zbrajanjem i integriranjem preko $]0,1[$ dobivamo

$$\int_0^1\frac{\partial}{\partial t}U dx+\frac{\lambda+2\mu}{L^2}\int_0^1\frac{\rho}{\theta}\left[\frac{\partial}{\partial x}\left(r^2v\right)\right]^2dx+\frac{c_0+2c_d}{L^2}\int_0^1\frac{\rho}{\theta}\left[\frac{\partial}{\partial x}\left(r^2\omega\right)\right]^2dx \\ +\frac{k}{L^2}\int_0^1\frac{r^4\rho}{\theta^2}\left[\frac{\partial\theta}{\partial x}\right]^2dx+4\mu_r\int_0^1\frac{\omega^2}{\rho\theta}dx= \\ \frac{4\mu}{L}\int_0^1\frac{1}{\theta}\frac{\partial}{\partial x}\left(rv^2\right)dx+\frac{4c_d}{L}\int_0^1\frac{1}{\theta}\frac{\partial}{\partial x}\left(r\omega^2\right)dx. \tag{7.19}$$

Za dokaz leme ključne su jednakosti koje navodimo u nastavku.

Koristeći (3.115) lako se pokazuje da vrijedi

$$\frac{\partial}{\partial x}\left(rv^2\right)=\frac{2}{r}v\frac{\partial}{\partial x}\left(r^2v\right)-\frac{3Lv^2}{\rho r^2} \tag{7.20}$$

i

$$\frac{\partial}{\partial x}\left(r\omega^2\right)=\frac{2}{r}\omega\frac{\partial}{\partial x}\left(r^2\omega\right)-\frac{3L\omega^2}{\rho r^2}. \tag{7.21}$$

Primjenom (7.20) i (7.21) dobivamo

$$\begin{gathered}\frac{\lambda+2\mu}{L^2}\rho\left[\frac{\partial}{\partial x}\left(r^2 v\right)\right]^2-\frac{4\mu}{L}\frac{\partial}{\partial x}\left(rv^2\right)= \\ \rho\left(\lambda+\frac{2}{3}\mu\right)\frac{1}{L^2}\left[\frac{\partial}{\partial x}\left(r^2 v\right)\right]^2+4\mu\rho\left[\frac{1}{\sqrt{3}L}\frac{\partial}{\partial x}\left(r^2 v\right)-\sqrt{3}\frac{v}{\rho r}\right]^2\end{gathered} \tag{7.22}$$

i

$$\begin{gathered}\frac{c_2+2c_d}{L^2}\rho\left[\frac{\partial}{\partial x}\left(r^2 \omega\right)\right]^2-\frac{4c_d}{L}\frac{\partial}{\partial x}\left(r\omega^2\right)= \\ \rho\left(c_0+\frac{2}{3}c_d\right)\frac{1}{L^2}\left[\frac{\partial}{\partial x}\left(r^2 \omega\right)\right]^2+4c_d\rho\left[\frac{1}{\sqrt{3}L}\frac{\partial}{\partial x}\left(r^2 \omega\right)-\sqrt{3}\frac{\omega}{\rho r}\right]^2.\end{gathered} \tag{7.23}$$

Uvrštavanjem (7.22) i (7.23) u (7.19) i integriranjem preko $[0,t]$ dobivamo nejednakost

$$\begin{aligned}\int_0^1 U(x,t)dx &+\left(\lambda+\frac{2}{3}\mu\right)\frac{1}{L^2}\int_0^t\int_0^1\frac{\rho}{\theta}\left[\frac{\partial}{\partial x}\left(r^2 v\right)\right]^2 dxd\tau \\ &+\left(c_0+\frac{2}{3}c_d\right)\frac{1}{L^2}\int_0^t\int_0^1\frac{\rho}{\theta}\left[\frac{\partial}{\partial x}\left(r^2 \omega\right)\right]^2 dxd\tau \\ &+\frac{k}{L^2}\int_0^t\int_0^1\frac{r^4\rho}{\theta^2}\left[\frac{\partial\theta}{\partial x}\right]^2 dxd\tau\le\int_0^1 U(x,0)dx.\end{aligned} \tag{7.24}$$

Koristeći (4.12) i (4.13) lako zaključujemo da vrijedi

$$\int_0^1 U(x,0)\mathrm{d}x\le C\left(1+\|(\rho_0,v_0,\omega_0,\theta_0)\|^2_{\mathrm{L}^2(]0,1[)^4}\right)\le C \tag{7.25}$$

čime je tvrdnja leme dokazana. □

Lema 7.4. *Neka su α_1 i α_2 dva pozitivna rješenja jednadžbe*

$$\psi(x)=Cc_v^{-1}, \tag{7.26}$$

gdje je C konstanta iz ocjene (7.18), c_v konstanta iz jednadžbe 3.33*, a ψ funkcija iz prethodne leme. Tada vrijedi nejednakost*

$$\alpha_1\le\int_0^1\theta(x,t)dx\le\alpha_2 \tag{7.27}$$

i postoji funkcija $a:[0,T']\to[0,1]$ takva da vrijedi

$$\alpha_1\le\theta(a(t),t)\le\alpha_2. \tag{7.28}$$

Dokaz. Iz (7.18) odmah zaključujemo da vrijedi nejednakost

$$\int_0^1 (\theta - \ln\theta - 1)(x,t)dx \le Cc_v^{-1}. \tag{7.29}$$

Kako je funkcija ψ konveksna, možemo primijeniti Jensenovu nejednakost te zaključujemo da iz (7.29) slijedi

$$\int_0^1 \theta(x,t)dx - \ln\int_0^1 \theta(x,t)dx - 1 \le Cc_v^{-1}. \tag{7.30}$$

Iz (7.30) odmah dobivamo nejednakost (7.27) iz koje primjenom teorema o srednjoj vrijednosti funkcije slijedi tvrdnja (7.28). □

Lema 7.5. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $(x,t) \in Q_{T'}$ vrijedi*

$$\left|\int_{g(t)}^{x}\int_0^t r^{-2}(y,\tau)\frac{\partial v(y,\tau)}{\partial t}\,d\tau\,dy\right| \le C \tag{7.31}$$

pri čemu je funkcija g definirana je s (7.5).

Dokaz. Primjenjujući parcijalnu integraciju dobivamo da vrijedi

$$\begin{aligned}\left|\int_{g(t)}^{x}\int_0^t r^{-2}(y,\tau)\frac{\partial v(y,\tau)}{\partial t}d\tau dy\right| &\le \left|\int_{g(t)}^{x} r^{-2}(y,t)v(y,t)dy\right| \\ &+\left|\int_{g(t)}^{x} r_0^{-2}(y)v_0(y)dy\right| + 2\left|\int_{g(t)}^{x}\int_0^t r^{-3}(y,\tau)v^2(y,\tau)d\tau dy\right|.\end{aligned} \tag{7.32}$$

Sada primjenom Cauchy-Schwarzove nejednakosti i ocjene (7.2) slijedi

$$\begin{aligned}&\left|\int_{g(t)}^{x}\int_0^t r^{-2}(y,\tau)\frac{\partial v(y,\tau)}{\partial t}\,\mathrm{d}\tau\,\mathrm{d}y\right| \le \\ C&\left(\left|\int_{g(t)}^{x} v^2(y,t)\,\mathrm{d}y\right|^{\frac{1}{2}} + \left|\int_{g(t)}^{x} v_0^2(y)\,\mathrm{d}y\right|^{\frac{1}{2}} + \left|\int_{g(t)}^{x}\int_0^t v^2(y,\tau)\,\mathrm{d}\tau\,\mathrm{d}y\right|\right)\end{aligned} \tag{7.33}$$

pa zbog (7.63) i (4.13) odmah dobivamo (7.31). □

Neposredna posljedica Leme 7.5 je sljedeći rezultat.

Korolar 7.1. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $(x,t) \in Q_{T'}$ vrijedi*

$$C^{-1} \le B(x,t) \le C \tag{7.34}$$

pri čemu je funkcija B definirana s (7.10).

Sada ćemo pokazati ograničenost funkcije $Y(t)$.

Lema 7.6. *Postoje konstante $C_1, C_2 \in \mathbb{R}^+$ takve da za $t \in [0, T']$ vrijedi*

$$C_1 \leq Y(t) \leq C_2 \tag{7.35}$$

pri čemu je funkcija Y definirana s (7.9).

Dokaz. Iz (7.8) dobivamo

$$\frac{Y(t)}{\rho(x,t)} = \frac{1}{B(x,t)} \left(\frac{1}{\rho_0(x)} + \frac{R}{\lambda + 2\mu} \int_0^t \theta(x,\tau) \cdot Y(\tau) \cdot B(x,\tau) d\tau \right). \tag{7.36}$$

Nakon integriranja preko $]0,1[$ te primjene (7.4) i (7.34), odmah slijede nejednakosti

$$Y(t) \geq A^{-1} \left(C^{-1} A + \frac{R}{C^2(\lambda + 2\mu)} \int_0^t Y(\tau) \int_0^1 \theta(x,\tau) dx d\tau \right) \tag{7.37}$$

i

$$Y(t) \leq A^{-1} \left(CA + \frac{RC^2}{\lambda + 2\mu} \int_0^t Y(\tau) \int_0^1 \theta(x,\tau) dx d\tau \right). \tag{7.38}$$

Zbog pozitivnosti funkcija Y i θ iz (7.37) odmah dobivamo da postoji konstanta $C_1 \in \mathbb{R}^+$ takva da je

$$Y(t) \geq C_1, \quad t \in [0, T'], \tag{7.39}$$

a primjenjujući Gronwallovu nejednakost na (7.38) i koristeći ocjenu (7.27) za funkciju θ zaključujemo da je

$$Y(t) \leq C_2, \quad t \in [0, T'], \quad C_2 \in \mathbf{R}^+, \tag{7.40}$$

čime je lema dokazana. □

Sljedećim lemama izvodimo vezu između maksimalne gustoće i minimalne temperature, odnosno minimalne gustoće i maksimalne temperature.

Lema 7.7. *Za funkcije m_ρ, M_ρ, m_θ i M_θ definirane s*

$$m_\rho(t) = \min_{[0,1]} \rho(x,t), \tag{7.41}$$

$$M_\rho(t) = \max_{[0,1]} \rho(x,t), \tag{7.42}$$

$$m_\theta(t) = \min_{[0,1]} \theta(x,t), \tag{7.43}$$

$$M_\theta(t) = \max_{[0,1]} \theta(x,t), \tag{7.44}$$

za $t \in [0, T']$, postoje pozitivne konstante C_1 i C_2 takve da vrijede nejednakosti

$$M_\rho(t) \leq C_1 \left(1 + \int_0^t m_\theta(\tau)\, d\tau \right)^{-1} \tag{7.45}$$

i

$$m_\rho(t) \geq C_2 \left(1 + \int\limits_0^t M_\theta(\tau)\, d\tau\right)^{-1}. \tag{7.46}$$

Dokaz. Koristeći svojstvo (4.7) za funkciju ρ_0, te ocjene (7.34) i (7.35) iz (7.8) odmah dobivamo navedene nejednakosti. □

Uvedimo sada funkcije I_1 i I_2 definirane na intervalu $[0, T']$ formulama

$$I_1(t) = \int\limits_0^1 r^4 \rho \left(\frac{\partial \theta}{\partial x}\right)^2 dx, \tag{7.47}$$

$$I_2(t) = \int\limits_0^t I_1(\tau) d\tau. \tag{7.48}$$

Slično kao u [AKM90] vrijedi lema.

Lema 7.8. *Za svako $\varepsilon > 0$ postoji konstanta $C_\varepsilon > 0$, takva da za svako $t \in [0, T']$ vrijedi nejednakost*

$$M_\theta^2(t) \leq \varepsilon I_1(t) + C_\varepsilon \left(1 + I_2(t)\right). \tag{7.49}$$

Dokaz. Definirajmo funkciju Ψ na $\overline{Q}_{T'}$ formulom

$$\Psi(x,t) = \theta(x,t) - \int\limits_0^1 \theta(\xi, t) d\xi. \tag{7.50}$$

Primjenom teorema o srednjoj vrijednosti funkcije zaključujemo da postoji $x(t) \in [0, 1]$ tako da je

$$\Psi(x(t), t) = 0. \tag{7.51}$$

Sada imamo

$$|\Psi(x,t)|^{\frac{3}{2}} = \int\limits_{x(t)}^x \frac{\partial}{\partial \xi} |\Psi(\xi,t)|^{\frac{3}{2}} d\xi = \frac{3}{2} \int\limits_{x(t)}^x |\Psi(\xi,t)|^{\frac{1}{2}} \frac{\partial}{\partial \xi} \left(\Psi(\xi,t)\right) \operatorname{sign}\left(\Psi(\xi,t)\right) d\xi. \tag{7.52}$$

Iz (7.52) slijedi nejednakost

$$\begin{gathered} |\Psi(x,t)|^{\frac{3}{2}} \leq \frac{3}{2} \int\limits_0^1 |\Psi(\xi,t)|^{\frac{1}{2}} \left|\frac{\partial \Psi(\xi,t)}{\partial \xi}\right| d\xi = \\ \frac{3}{2} \int\limits_0^1 r^{-2}(\xi,t) \rho^{-\frac{1}{2}}(\xi,t) \, |\Psi(\xi,t)|^{\frac{1}{2}} \, r^2(\xi,t) \rho^{\frac{1}{2}}(\xi,t) \left|\frac{\partial \Psi(\xi,t)}{\partial \xi}\right| d\xi. \end{gathered} \tag{7.53}$$

Primjenom Cauchy-Schwarzove nejednakosti na desnu stranu od (7.53) dobivamo

$$|\Psi(x,t)|^{\frac{3}{2}} \leq C \left(\int\limits_0^1 \frac{|\Psi(\xi,t)|\, d\xi}{r^4(\xi,t)\rho(\xi,t)}\right)^{\frac{1}{2}} \left(\int\limits_0^1 r^4(\xi,t)\rho(\xi,t) \left|\frac{\partial \Psi(\xi,t)}{\partial \xi}\right|^2 d\xi\right)^{\frac{1}{2}}. \tag{7.54}$$

Uzimajući u obzir (7.2), (7.46), (7.47) i jednakost

$$\frac{\partial \Psi(\xi,t)}{\partial \xi} = \frac{\partial \theta(\xi,t)}{\partial \xi}. \tag{7.55}$$

iz (7.54) nadalje dobivamo da je

$$|\Psi(x,t)| \leq C \left(1 + \int_0^t M_\theta(\tau) d\tau\right)^{\frac{1}{3}} (I_1(t))^{\frac{1}{3}}, \tag{7.56}$$

pa koristeći (7.27) zaključujemo da vrijedi

$$M_\theta(t) \leq C \left[1 + \left(1 + \int_0^t M_\theta(\tau) d\tau\right)^{\frac{1}{3}} (I_1(t))^{\frac{1}{3}}\right]. \tag{7.57}$$

Kvadriranjem (7.57) i primjenom Youngove nejednakosti dobivamo

$$M_\theta^2(t) \leq C + \varepsilon I_1(t) + C_\varepsilon \left(1 + \int_0^t M_\theta(\tau) d\tau\right)^2 \tag{7.58}$$

gdje je $\varepsilon > 0$ proizvoljno mali, odakle slijedi nejednakost

$$M_\theta^2(t) \leq C + \varepsilon I_1(t) + C_\varepsilon \int_0^t M_\theta^2(\tau) d\tau \tag{7.59}$$

koju možemo zapisati u obliku

$$M_\theta^2(t) - \varepsilon I_1(t) \leq C + C_\varepsilon \int_0^t \left(M_\theta^2(\tau) - \varepsilon I_1(t) + \varepsilon I_1(t)\right) d\tau \tag{7.60}$$

te primjenom Gronwallove nejednakosti odmah dobivamo tvrdnju leme. □

7.2 Globalne apriorne ocjene i dokaz Teorema 4.3

Sada pomoću dobivenih rezultata dokazujemo da je funkcija (7.1) generalizirano rješenje problema (3.126)-(3.136) na domeni Q_T, pri čemu je T kao u Propoziciji 7.1.

Lema 7.9. *Vrijede inkluzije*

$$v, \omega \in \mathrm{L}^\infty \left(0, T; \mathrm{L}^2 \left(]0,1[\right)\right), \tag{7.61}$$

$$\theta \in \mathrm{L}^\infty \left(0, T; \mathrm{L}^1 \left(]0,1[\right)\right). \tag{7.62}$$

Dokaz. Inkluzija (7.61) posljedica je ocjene

$$\|v(t)\|^2 + \|\omega(t)\|^2 \leq C \tag{7.63}$$

koja slijedi neposredno iz (7.18) dok je (7.62) posljedica ocjene

$$\|\theta(t)\|_{\mathrm{L}^1(]0,1[)} \leq C \tag{7.64}$$

koju dobivamo iz (7.28). □

Lema 7.10. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $t \in [0,T]$ vrijedi*

$$m_\theta(t) \geq C \tag{7.65}$$

pri čemu je funkcija m_θ definirana s (7.43).

Dokaz. Primijetimo da vrijedi

$$-\frac{k}{c_v L^2}\frac{1}{\theta^2}\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial\theta}{\partial x}\right) = \frac{k}{c_v L^2}\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial}{\partial x}\left(\frac{1}{\theta}\right)\right) - \frac{2k}{c_v L^2}\frac{\rho r^4}{\theta^3}\left(\frac{\partial\theta}{\partial x}\right)^2. \tag{7.66}$$

Množenjem jednadžbe (3.129) sa $-\theta^{-2}\rho^{-1}$ i primjenom jednakosti (7.66) imamo

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{\theta}\right) = \frac{k}{c_v L^2}\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial}{\partial x}\left(\frac{1}{\theta}\right)\right) - \frac{2k}{c_v L^2}\frac{\rho r^4}{\theta^3}\left(\frac{\partial\theta}{\partial x}\right)^2 + \frac{R}{c_v L}\frac{\rho}{\theta}\frac{\partial}{\partial x}\left(r^2 v\right)\\ -\frac{1}{\theta^2}\left\{\frac{\lambda+2\mu}{c_v L^2}\rho\left[\frac{\partial}{\partial x}\left(r^2 v\right)\right]^2 - \frac{4\mu}{c_v L}\frac{\partial}{\partial x}\left(r v^2\right)\right\}\\ -\frac{1}{\theta^2}\left\{\frac{c_2+2c_d}{c_v L^2}\rho\left[\frac{\partial}{\partial x}\left(r^2\omega\right)\right]^2 - \frac{4c_d}{c_v L}\frac{\partial}{\partial x}\left(r\omega^2\right)\right\} - \frac{4\mu_r}{c_v}\frac{\omega^2}{\rho\theta^2}.\end{aligned} \tag{7.67}$$

Koristeći svojstvo pozitivnosti funkcija ρ i θ, te jednakosti (7.22) i (7.23) iz (7.67) dobivamo

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{\theta}\right) \leq \frac{k}{c_v L^2}\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial}{\partial x}\left(\frac{1}{\theta}\right)\right) +\\ \frac{R}{c_v L}\frac{\rho}{\theta}\frac{\partial}{\partial x}\left(r^2 v\right) - \frac{\rho}{\theta^2}\left(\lambda+\frac{2}{3}\mu\right)\frac{1}{c_v L^2}\left[\frac{\partial}{\partial x}\left(r^2 v\right)\right]^2\end{aligned} \tag{7.68}$$

odnosno

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{\theta}\right) \leq \frac{k}{c_v L^2}\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial}{\partial x}\left(\frac{1}{\theta}\right)\right) + \frac{R^2}{4c_v\left(\lambda+\frac{2}{3}\mu\right)}\rho\\ -\frac{\rho}{\theta^2}\left[\frac{1}{L}\sqrt{\frac{\lambda+\frac{2}{3}\mu}{c_v}}\frac{\partial}{\partial x}\left(r^2 v\right) - \frac{R}{2\sqrt{c_v}\sqrt{\lambda+\frac{2}{3}\mu}}\theta\right]^2.\end{aligned} \tag{7.69}$$

Koristeći (7.69) lako zaključujemo da vrijedi

$$\frac{\partial}{\partial t}\left(\frac{1}{\theta}\right) \leq \frac{k}{c_v L^2}\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial}{\partial x}\left(\frac{1}{\theta}\right)\right) + \frac{R^2}{4c_v\left(\lambda+\frac{2}{3}\mu\right)}\rho. \tag{7.70}$$

Sada koristimo tehniku sličnu onoj uvedenoj u [Ali79]. Množenjem (7.70) s $p\theta^{-p+1}$, $p \geq 2$, dobivamo

$$\frac{\partial}{\partial t}\left(\frac{1}{\theta^p}\right) \leq \frac{kp}{c_v L^2}\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial}{\partial x}\left(\frac{1}{\theta}\right)\right)\left(\frac{1}{\theta}\right)^{p-1} + \frac{R^2 p}{4c_v\left(\lambda+\frac{2}{3}\mu\right)}\rho\left(\frac{1}{\theta}\right)^{p-1} \tag{7.71}$$

što integriranjem preko $]0,1[$ uz primjenu parcijalne integracije i rubnih uvjeta (3.136) daje

$$\begin{aligned}\frac{d}{dt}\left\|\frac{1}{\theta(t)}\right\|^p_{\mathrm{L}^p(]0,1[)} \leq & -\frac{kp(p-1)}{c_v L^2}\int_0^1 \frac{r^4\rho}{\theta^{p-2}}\left[\frac{\partial}{\partial x}\left(\frac{1}{\theta}\right)\right]^2 dx \\ & + \frac{R^2 p}{4c_v\left(\lambda+\frac{2}{3}\mu\right)}\int_0^1 \rho\left(\frac{1}{\theta}\right)^{p-1} dx,\end{aligned} \tag{7.72}$$

odnosno

$$\frac{d}{dt}\left\|\frac{1}{\theta(t)}\right\|^p_{\mathrm{L}^p(]0,1[)} \leq \frac{R^2 p}{4c_v\left(\lambda+\frac{2}{3}\mu\right)}\int_0^1 \rho\left(\frac{1}{\theta}\right)^{p-1} dx. \tag{7.73}$$

Primjenom Hölderove nejdnakosti na integral s desne strane u (7.73) dobivamo

$$\frac{d}{dt}\left\|\frac{1}{\theta(t)}\right\|^p_{\mathrm{L}^p(]0,1[)} \leq \frac{R^2 p}{4c_v\left(\lambda+\frac{2}{3}\mu\right)}\|\rho(t)\|_{\mathrm{L}^p(]0,1[)}\left\|\frac{1}{\theta(t)}\right\|^{p-1}_{\mathrm{L}^p(]0,1[)} \tag{7.74}$$

iz čega slijedi

$$\frac{d}{dt}\left\|\frac{1}{\theta(t)}\right\|_{\mathrm{L}^p(]0,1[)} \leq \frac{R^2}{4c_v\left(\lambda+\frac{2}{3}\mu\right)}\|\rho(t)\|_{\mathrm{L}^p(]0,1[)}. \tag{7.75}$$

Integriranjem (7.75) preko $]0,t[$, $t \in]0,T[$ dobivamo

$$\left\|\frac{1}{\theta(t)}\right\|_{\mathrm{L}^p(]0,1[)} \leq \left\|\frac{1}{\theta(0)}\right\|_{\mathrm{L}^p(]0,1[)} + \frac{R^2}{4c_v\left(\lambda+\frac{2}{3}\mu\right)}\int_0^t \|\rho(\tau)\|_{\mathrm{L}^p(]0,1[)}\, d\tau. \tag{7.76}$$

Stavljajući u (7.76) limes kad $p \to \infty$ i korištenjem (7.43), (4.7) i (7.45) dobivamo nejednakost

$$\left(m_\theta(t)^{-1}\right) \leq m^{-1} + C\int_0^t \left(1+\int_0^\tau m_\theta(s)ds\right)^{-1} d\tau. \tag{7.77}$$

Promotrimo sada funkciju $t \mapsto y(t)$ definiranu na $]0,T'[$ formulom

$$y(t) = m^{-1} + C\int_0^t \left(1+\int_0^\tau m_\theta(s)ds\right)^{-1} d\tau. \tag{7.78}$$

Primjetimo da vrijedi

$$y'(t) = C\left(1+\int_0^t m_\theta(s)ds\right)^{-1} \tag{7.79}$$

i
$$y''(t) = -C^{-1}(y'(t))^2 m_\theta(t). \tag{7.80}$$

Koristeći (7.80) zaključujemo da (7.77) možemo pisati u obliku diferencijalne nejednadžbe
$$-\frac{y''y}{(y')^2} \geq C^{-1} \tag{7.81}$$

ili u obliku
$$\frac{d}{dt}\left(\frac{y}{y'}\right) \geq 1 + C^{-1}. \tag{7.82}$$

Integriranjem (7.82) preko $]0,t[$, $t \in]0,T[$ dobivamo
$$\frac{y}{y'} \geq (1 + C^{-1})t + m^{-1}C^{-1}. \tag{7.83}$$

Ponovnim integriranjem (7.83) dobivamo da postoji $C \in \mathbf{R}^+$ takav da je
$$y(t) \leq C, \quad t \in [0,T], \tag{7.84}$$

pa iz (7.77) odmah dobivamo (7.65). □

Iz (7.65) i (7.45) zaključujemo da vrijedi sljedeće svojstvo funkcije ρ.

Korolar 7.2. *Vrijedi inkluzija*
$$\rho \in \mathrm{L}^\infty(Q_T). \tag{7.85}$$

Sada ćemo pokazati omeđenost gustoće odozdo.

Lema 7.11. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $t \in [0,T]$ vrijedi*
$$m_\rho(t) \geq C. \tag{7.86}$$

Dokaz. Korištenjem Cauchy-Schwarzove nejednakosti i (7.28) imamo
$$\begin{gathered}\left|\sqrt{\theta(x,t)} - \sqrt{\theta(a(t),t)}\right| \leq C \int_{a(t)}^{x} \frac{1}{\sqrt{\theta}}\left|\frac{\partial\theta}{\partial x}\right| dy \leq \\ C\left(\int_0^1 \frac{r^4\rho}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 \frac{\theta}{r^4\rho} dx\right)^{\frac{1}{2}}\end{gathered} \tag{7.87}$$

što primjenom (7.27) i (7.2) prelazi u
$$\left|\sqrt{\theta(x,t)} - \sqrt{\theta(a(t),t)}\right| \leq C\left(\int_0^1 \frac{r^4\rho}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 dx\right)^{\frac{1}{2}} \frac{1}{\sqrt{m_\rho}}. \tag{7.88}$$

Kvadriranjem (7.88) i korištenjem ocjene (7.28) dobivamo nejednakost
$$\theta(x,t) \leq C\left(1 + \left(\int_0^1 \frac{r^4\rho}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 dx\right)\frac{1}{m_\rho}\right). \tag{7.89}$$

Primjetimo da nejednakost (7.46) možemo zapisati u obliku

$$\frac{1}{m_\rho} \leq C \left(1 + \int_0^t M_\theta(x,t) d\tau \right). \tag{7.90}$$

pa uvrštavanjem (7.89) u (7.90) slijedi

$$\frac{1}{m_\rho} \leq C \left(1 + \int_0^t \int_0^1 \frac{r^4 \rho}{\theta^2} \left(\frac{\partial \theta}{\partial x} \right)^2 dx \frac{1}{m_\rho} d\tau \right). \tag{7.91}$$

odakle primjenom Gronwallove nejednakosti i ocjene (7.18) odmah dobivamo tvrdnju leme. □

Lema 7.12. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $t \in [0,T]$ vrijedi*

$$\int_0^1 \left(\Phi^2 + v^4 + \omega^4 \right) dx + I_2 \leq C, \tag{7.92}$$

pri čemu je funkcija Φ definirana sa

$$\Phi = \frac{1}{2} v^2 + \frac{1}{2} j_I \omega^2 + c_v \theta. \tag{7.93}$$

Dokaz. Množenjem jednadžbi (3.127), (3.128) i (3.129) respektivno s v, $j_I \omega \rho^{-1}$ i $c_v \rho^{-1}$ i njihovim zbrajanjem dobivamo

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= -\frac{R}{L} r^2 v \frac{\partial}{\partial x}(\rho\theta) + \frac{\lambda + 2\mu}{L^2} r^2 v \frac{\partial}{\partial x} \left(\rho \frac{\partial}{\partial x} \left(r^2 v \right) \right) \\ &+ \frac{c_0 + 2c_d}{L^2} r^2 \omega \frac{\partial}{\partial x} \left(\rho \frac{\partial}{\partial x} \left(r^2 \omega \right) \right) + \frac{k}{L^2} \frac{\partial}{\partial x} \left(r^4 \rho \frac{\partial \theta}{\partial x} \right) - \frac{R}{L} \rho\theta \frac{\partial}{\partial x} \left(r^2 v \right) \\ &+ \frac{\lambda + 2\mu}{L^2} \rho \left[\frac{\partial}{\partial x} \left(r^2 v \right) \right]^2 - \frac{4\mu}{L} \frac{\partial}{\partial x} \left(r v^2 \right) + \frac{c_0 + 2c_d}{L^2} \rho \left[\frac{\partial}{\partial x} \left(r^2 \omega \right) \right]^2 - \frac{4c_d}{L} \frac{\partial}{\partial x} \left(r\omega^2 \right) \end{aligned} \tag{7.94}$$

što množenjem s Φ, integriranjem preko $]0,1[$, primjenom parcijalne integracije, rubnih uvjeta i jednakosti (3.115) postaje

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \Phi^2(x,t) dx &= -\int_0^1 \left(\frac{\lambda + 2\mu}{L^2} \rho r^4 \frac{\partial \Phi}{\partial x} + \frac{2\lambda}{L} r v^2 - \frac{R}{L} \rho\theta r^2 v + \frac{2c_o}{L} r \omega^2 \right. \\ &\left. + \left(\frac{c_0 + 2c_d}{L^2} - j_I \frac{\lambda + 2\mu}{L^2} \right) \rho r^4 \omega \frac{\partial \omega}{\partial x} + \left(\frac{k}{L^2} - c_v \frac{\lambda + 2\mu}{L^2} \right) r^4 \rho \frac{\partial \theta}{\partial x} \right) \frac{\partial \Phi}{\partial x} dx. \end{aligned} \tag{7.95}$$

Iz (7.95) slijedi nejednakost

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \Phi^2(x,t) \mathrm{d}x + \frac{\lambda + 2\mu}{L^2} \int_0^1 \rho r^4 \left(\frac{\partial \Phi}{\partial x} \right)^2 dx \\ &+ \left(\frac{k}{L^2} - c_v \frac{\lambda + 2\mu}{L^2} \right) \int_0^1 r^4 \rho \frac{\partial \theta}{\partial x} \frac{\partial \Phi}{\partial x} dx \leq \frac{2\lambda}{L} \int_0^1 r v^2 \left| \frac{\partial \Phi}{\partial x} \right| dx + \frac{R}{L} \int_0^1 \rho\theta r^2 \left| v \frac{\partial \Phi}{\partial x} \right| dx \\ &\frac{2c_0}{L} \int_0^1 r \omega^2 \left| \frac{\partial \Phi}{\partial x} \right| \mathrm{d}x + \left| \frac{c_0 + 2c_d}{L^2} - j_I \frac{\lambda + 2\mu}{L^2} \right| \int_0^1 \rho r^4 \left| \omega \frac{\partial \omega}{\partial x} \frac{\partial \Phi}{\partial x} \right| dx, \end{aligned} \tag{7.96}$$

koja primjenom Youngove nejednakosti postaje

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Phi^2(x,t)\mathrm{d}x + \int_0^1 \rho r^4 \left[\frac{\lambda+2\mu}{L^2}\left(\frac{\partial\Phi}{\partial x}\right)^2 \mathrm{d}x + \left(\frac{k}{L^2} - c_v\frac{\lambda+2\mu}{L^2}\right)\frac{\partial\theta}{\partial x}\frac{\partial\Phi}{\partial x}dx\right] \leq$$
$$C_1\frac{\varepsilon^{-1}}{2}\int_0^1 \left((\rho r^2)^{-1}v^4 + \rho\theta^2 v^2 + (\rho r^2)^{-1}\omega^4 + \rho r^4\omega^2\left(\frac{\partial\omega}{\partial x}\right)^2\right)dx \quad (7.97)$$
$$+2C_1\varepsilon\int_0^1 \rho r^4\left(\frac{\partial\Phi}{\partial x}\right)^2,$$

pri čemu je $\varepsilon > 0$ proizvoljno.

U nastavku ćemo koristiti sljedeću algebarsku nejednakost

$$(A-B)(a+b+c)^2 + (C-A)c(a+b+c) \geq$$
$$(C-3B)c^2 - \left(2B + \frac{(A-2B+C)^2}{4B}\right)a^2 - \frac{1}{2B}\left[(A-B)^2 + \frac{(A-2B+C)^2}{2}\right]b^2 \quad (7.98)$$

gdje je A, B, C, a, b, $c \in \mathbb{R}$ i $A > B > 0$.

Stavimo u (7.98) $a = v\frac{\partial v}{\partial x}$, $b = j_I\omega\frac{\partial\omega}{\partial x}$, $c = c_v\frac{\partial\theta}{\partial x}$, $A = \frac{\lambda+2\mu}{L^2}$, $B = 2C_1\varepsilon$, $C = \frac{k}{c_v L^2}$, pri čemu ε biramo tako da vrijedi $A - B > 0$ i $C - 3B > 0$. Označimo zbog jednostavnosti $D = C - 3B$ i

$$C_2 = \max\left\{2B + \frac{(A-2B+C)^2}{4B},\right.$$
$$\left.\frac{1}{2B}\left[(A-B)^2 + \frac{(A-2B+C)^2}{2}\right] + C_1\frac{\varepsilon^{-2}}{2}, C_1\frac{\varepsilon^{-2}}{2}\right\}. \quad (7.99)$$

Primjenom (7.98) u (7.97) dobivamo

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Phi^2(x,t)dx + D\int_0^1 \rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2 dx \leq$$
$$C_2\int_0^1\left((\rho r^2)^{-1}v^4 + \rho\theta^2v^2 + (\rho r^2)^{-1}\omega^4 + \rho r^4 v^2\left(\frac{\partial v}{\partial x}\right)^2 + \rho r^4\omega^2\left(\frac{\partial\omega}{\partial x}\right)^2\right)dx. \quad (7.100)$$

Pomoću ocjena (7.85), (7.86) i (7.2) zaključujemo da postoji pozitivna konstanta C_3 takva da vrijedi

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Phi^2(x,t)dx + D\int_0^1 \rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2 dx \leq$$
$$C_3\int_0^1\left(v^4 + \theta^2v^2 + \omega^4 + r^4v^2\left(\frac{\partial v}{\partial x}\right)^2 + r^4\omega^2\left(\frac{\partial\omega}{\partial x}\right)^2\right)dx. \quad (7.101)$$

Sada jednadžbu (3.127) množimo sa v^3 i integriramo preko $]0,1[$. Primjenom parcijalne integracije, svojstva (3.115) i rubnih uvjeta za funkciju v dobivamo

$$\begin{aligned} \frac{1}{4}\int_0^1 \frac{\partial v^4}{\partial t}dx &= 2R\int_0^1 r^{-1}\theta v^3 dx + \frac{3R}{L}\int_0^1 \rho r^2 v^2\theta\frac{\partial v}{\partial x}dx \\ &-\frac{\lambda+2\mu}{L^2}\int_0^1 \left[4L^2\frac{v^4}{r^2\rho} + 8Lrv^3\frac{\partial v}{\partial x} + 3\rho r^4 v^2\left(\frac{\partial v}{\partial x}\right)^2\right]dx, \end{aligned} \tag{7.102}$$

što primjenom Youngove nejednakosti postaje

$$\begin{aligned} \frac{1}{4}\int_0^1 \frac{\partial v^4}{\partial t}dx \leq& -3\frac{\lambda+2\mu}{L^2}\int_0^1 \rho r^4 v^2\left(\frac{\partial v}{\partial x}\right)^2 dx + R\int_0^1 r^{-2}v^4 dx + R\int_0^1 \theta^2 v^2 dx \\ &+\frac{3R\epsilon}{2L}\int_0^1 \rho r^4 v^2\left(\frac{\partial v}{\partial x}\right)^2 dx + \frac{3R}{2L\epsilon}\int_0^1 \rho v^2\theta^2 dx \\ &-\frac{\lambda+2\mu}{L^2}\int_0^1 \rho r^4 v^2\left(\frac{\partial v}{\partial x}\right)^2 dx - 20(\lambda+2\mu)\int_0^1 (\rho r^2)^{-1}v^4 dx \end{aligned} \tag{7.103}$$

Stavljajući $\epsilon = \dfrac{2}{3}\dfrac{\lambda+2\mu}{LR}$ u (7.103) dobivamo nejednakost

$$\begin{aligned} \frac{1}{4}\int_0^1 \frac{\partial v^4}{\partial t}dx \leq& -3\frac{\lambda+2\mu}{L^2}\int_0^1 \rho r^4 v^2\left(\frac{\partial v}{\partial x}\right)^2 dx + R\int_0^1 r^{-2}v^4 dx + R\int_0^1 \theta^2 v^2 dx \\ &+\frac{9}{4}\frac{R^2}{\lambda+2\mu}\int_0^1 \rho v^2\theta^2 dx - 20(\lambda+2\mu)\int_0^1 (\rho r^2)^{-1}v^4 dx. \end{aligned} \tag{7.104}$$

Koristeći ocjene (7.85), (7.86) i (7.2) iz (7.104) zaključujemo da postoje pozitivne konstante C_4 i C_5 takve da vrijedi

$$\frac{1}{4}\int_0^1 \frac{\partial v^4}{\partial t}\mathrm{d}x + C_4\int_0^1 r^4 v^2\left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x \leq C_5\int_0^1 \left(v^4+\theta^2 v^2\right)\mathrm{d}x. \tag{7.105}$$

Analognim postupkom primjenjenim na jednadžbi (3.128) zaključujemo da postoje pozitivne konstante C_6 i C_7 takve da vrijedi

$$\frac{1}{4}\int_0^1 \frac{\partial \omega^4}{\partial t}dx + C_6\int_0^1 r^4\omega^2\left(\frac{\partial \omega}{\partial x}\right)^2 dx \leq C_7\int_0^1 \omega^4 dx. \tag{7.106}$$

Nakon množenja (7.105) s $C_3C_4^{-1}$ i (7.106) s $C_3C_6^{-1}$ te njihovim zbrajanjem s (7.101) dobivamo

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\int\limits_0^1\left(\Phi^2(x,t)+\frac{C_3C_4^{-1}}{2}v^4+\frac{C_3C_6^{-1}}{2}\omega^4\right)dx+D\int\limits_0^1\rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2dx\leq\\ C_3\int\limits_0^1\left(\left(1+C_5C_4^{-1}\right)\left(v^4+\theta^2v^2\right)+\left(1+C_7C_6^{-1}\right)\omega^4\right)dx\end{gathered}\tag{7.107}$$

odnosno

$$\begin{gathered}\frac{d}{dt}\left[\int\limits_0^1\left(\Phi^2(x,t)+\frac{C_3C_4^{-1}}{2}v^4+\frac{C_3C_6^{-1}}{2}\omega^4\right)dx+2D\int\limits_0^t\int\limits_0^1\rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2dx\,\mathrm{d}\tau\right]\\ \leq 2C_3\int\limits_0^1\left(\left(1+C_5C_4^{-1}\right)\left(v^4+\theta^2v^2\right)+\left(1+C_7C_6^{-1}\right)\omega^4\right)dx.\end{gathered}\tag{7.108}$$

Za daljnji postupak potrebna nam je sljedeća nejednakost koja slijedi iz (7.49)

$$\int\limits_0^1\theta^2v^2dx\leq C_8\epsilon\int\limits_0^1 r^4\rho\left(\frac{\partial\theta}{\partial x}\right)^2dx+C_\epsilon\left(1+\int\limits_0^t\int\limits_0^1\rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2dxd\tau\right).\tag{7.109}$$

Nakon uvrštavanja (7.109) u (7.108) uzimajući da je

$$\epsilon=\frac{D}{(1+C_5C_4^{-1})\cdot 2C_3C_8}\tag{7.110}$$

dobivamo

$$\begin{gathered}\frac{d}{dt}\int\limits_0^1\left(\Phi^2(x,t)+\frac{C_3C_4^{-1}}{2}v^4+\frac{C_3C_6^{-1}}{2}\omega^4+D\int\limits_0^t\rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2d\tau\right)dx\\ \leq 2C_3\int\limits_0^1\left(\left(1+C_5C_4^{-1}\right)v^4+\left(1+C_7C_6^{-1}\right)\omega^4+\right.\\ \left.\left(1+C_5C_4^{-1}\right)C_\epsilon\int\limits_0^t\rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2d\tau\right)dx+C_9\end{gathered}\tag{7.111}$$

što jednostavnije možemo zapisati u obliku

$$\begin{gathered}\frac{d}{dt}\int\limits_0^1\left(\Phi^2(x,t)+\frac{C_3C_4^{-1}}{2}v^4+\frac{C_3C_6^{-1}}{2}\omega^4+D\int\limits_0^t\rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2d\tau\right)dx\\ \leq C_{10}\int\limits_0^1\left(\Phi^2(x,t)+\frac{C_3C_4^{-1}}{2}v^4+\frac{C_3C_6^{-1}}{2}\omega^4+D\int\limits_0^t\rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2d\tau\right)\mathrm{d}x+1.\end{gathered}\tag{7.112}$$

Primjenom Gronwallove nejednakosti na (7.112) odmah dobivamo tvrdnju leme. □

Iz (7.92) i (7.49) zaključujemo da vrijedi sljedeće svojstvo funkcije M_θ.

Korolar 7.3. *Postoji konstanta $C \in \mathbb{R}^+$ takva da vrijedi*

$$\|M_\theta\|_{\mathrm{L}^2(]0,T[)} \leq C. \tag{7.113}$$

Posljedice ocjene (7.92) su i inkluzije

$$\Phi \in \mathrm{L}^\infty\left(0,T;\mathrm{L}^2\left(]0,1[\right)\right), \tag{7.114}$$

$$\theta \in \mathrm{L}^\infty\left(0,T;\mathrm{L}^2\left(]0,1[\right)\right). \tag{7.115}$$

U cilju dobivanja ograničenosti prostornih derivacija funkcija v i ω potrebna nam je sljedeća lema.

Lema 7.13. *Vrijede nejednakosti*

$$\left[\frac{\partial}{\partial x}\left(r^2 v\right)\right]^2 \geq \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 - Cv^2 \tag{7.116}$$

i

$$\left[\frac{\partial}{\partial x}\left(r^2 \omega\right)\right]^2 \geq \frac{1}{2}\left(\frac{\partial \omega}{\partial x}\right)^2 - C\omega^2 \tag{7.117}$$

gdje je C pozitivna konstanta.

Dokaz. Očito je

$$\left(\frac{\partial}{\partial x}\left(r^2 v\right)\right)^2 \geq 4r^3\frac{\partial r}{\partial x}v\frac{\partial v}{\partial x} + r^4\left(\frac{\partial v}{\partial x}\right)^2. \tag{7.118}$$

Korištenjem jednakosti (3.115), ocjena (7.2) i (7.85) nejednakost (7.118) možemo napisati u obliku

$$\left(\frac{\partial}{\partial x}\left(r^2 v\right)\right)^2 \geq -C\left|v\frac{\partial v}{\partial x}\right| + \left(\frac{\partial v}{\partial x}\right)^2. \tag{7.119}$$

Primjenom Youngove nejednakosti na izraz $\left|v\dfrac{\partial v}{\partial x}\right|$ odmah dobivamo (7.116). Nejednakost (7.117) dokazuje se analogno. □

Lema 7.14. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $t \in [0,T]$ vrijedi*

$$\|v(t)\|^2 + \int_0^t \left\|\frac{\partial v}{\partial x}(\tau)\right\|^2 \leq C. \tag{7.120}$$

Dokaz. Množenjem jednadžbe (3.127) s v, integriranjem preko $]0,1[$ i primjenom parcijalne integracije dobivamo

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \frac{\lambda+2\mu}{L^2}\int_0^1 \rho\left(\frac{\partial}{\partial x}\left(r^2 v\right)\right)^2 dx = \frac{R}{L}\int_0^1 \rho\theta\frac{\partial}{\partial x}\left(r^2 v\right)dx. \tag{7.121}$$

Primjenjujući Youngovu nejednakost uz korištenje dovoljno malenog $\varepsilon > 0$ te ocjena (7.85) i (7.86) dobivamo □

$$\frac{d}{dt}\|v(t)\|^2 + \int_0^1 \left(\frac{\partial}{\partial x}\left(r^2 v\right)\right)^2 dx \leq CM_\theta^2. \tag{7.122}$$

Uvrštavanjem (7.116) u (7.122), integriranjem preko $]0,t[$ i korištenjem ocjena (7.63) i (7.113) odmah dobivamo (7.120).

Lema 7.15. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $t \in [0,T]$ vrijedi*

$$\|\omega(t)\|^2 + \int_0^t \left\|\frac{\partial \omega}{\partial x}(\tau)\right\|^2 \leq C. \tag{7.123}$$

Dokaz. Množenjem jednadžbe (3.128) s $\rho^{-1}\omega$, integriranjem preko $]0,1[$ i primjenom parcijalne integracije dobivamo

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|^2 + \frac{c_0+2c_d}{j_I L^2}\int_0^1 \rho\left(\frac{\partial}{\partial x}\left(r^2\omega\right)\right)^2 dx = \frac{4\mu_r}{j_I}\int_0^1 \frac{\omega^2}{\rho}dx. \tag{7.124}$$

Primjenom (7.86) i (7.117) te integriranjem preko $]0,t[$ odmah dobivamo (7.123). □

Sada možemo zaključiti da vrijedi

$$v \in \mathrm{L}^2\left(0,T;\mathrm{H}^1\left(]0,1[\right)\right), \tag{7.125}$$

i

$$\omega \in \mathrm{L}^2\left(0,T;\mathrm{H}^1\left(]0,1[\right)\right). \tag{7.126}$$

Pokažimo sada omeđenost funkcije r odozgo.

Lema 7.16. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za sve $(x,t) \in \overline{Q}_T$ vrijedi*

$$r(x,t) \leq C. \tag{7.127}$$

Dokaz. Iz (3.137) i (3.139) te nejednakosti Gagliardo-Ladyzhenskaye za funkciju v zaključujemo da vrijedi

$$r(x,t) \leq C\left(1 + \int_0^t \|v\|^{\frac{1}{2}}\left\|\frac{\partial v}{\partial x}\right\|^{\frac{1}{2}}\right)dx. \tag{7.128}$$

Koristeći (7.63) i Youngovu nejednakost dobivamo

$$r(x,t) \leq C\left(1 + \int_0^t \left\|\frac{\partial v}{\partial x}\right\|^2\right)dx \tag{7.129}$$

iz čega primjenom ocjene (7.120) slijedi tvrdnja leme. □

Iz (7.86), (7.92) i (7.2) zaključujemo da je

$$\frac{\partial \theta}{\partial x} \in \mathrm{L}^2\left(Q_T\right). \tag{7.130}$$

što zajedno sa (7.92) daje

$$\theta \in \mathrm{L}^2\left(0, T; \mathrm{H}^1\left(]0,1[\right)\right). \tag{7.131}$$

Lema 7.17. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $t \in]0,T[$ vrijedi*

$$\left\|\frac{\partial \rho}{\partial x}(t)\right\| \leq C. \tag{7.132}$$

Dokaz. Definirajmo funkciju φ na Q_T sa

$$\varphi(x,t) = \frac{-L}{\lambda+2\mu}\int_0^t r^{-2}(y,\tau)\frac{\partial v(y,\tau)}{\partial t}\,\mathrm{d}\tau. \tag{7.133}$$

Parcijalnom integracijom i korištenjem ocjene (7.2) dobivamo

$$|\varphi| \leq C\left(|v| + |v_0| + \int_0^t v^2 d\tau\right). \tag{7.134}$$

Kvadriranjem (7.134), integriranjem preko $]0,1[$ i primjenom svojstva (4.12) zaključujemo da vrijedi

$$\|\varphi(t)\|^2 \leq C\left(1 + \|v\|^2 + \int_0^t\int_0^1 v^4 dx d\tau\right) \tag{7.135}$$

odakle primjenom ocjena (7.63) i (7.92) slijedi

$$\|\varphi(t)\| \leq C, \tag{7.136}$$

za $t \in]0,T[$.

Primjetimo da za funkciju B iz (7.10) vrijedi

$$\frac{\partial B(x,t)}{\partial x} = B(x,t)\varphi(x,t). \tag{7.137}$$

Deriviranjem (7.8) po varijabli x te primjenom (7.133) i (7.137) dobivamo

$$\frac{\partial \rho}{\partial x} = \rho\varphi - \rho^2 Y^{-1}B^{-1}\left[\frac{d}{dx}\left(\frac{1}{\rho_0}\right) + \frac{RL}{\lambda+2\mu}\int_0^t BY\left(\frac{\partial\theta}{\partial x} + \theta\varphi\right)d\tau\right]. \tag{7.138}$$

Korištenjem (7.34), (7.35) i (7.85) iz (7.138) dobivamo

$$\left|\frac{\partial \rho}{\partial x}\right| \leq C\left(|\varphi| + \left|\frac{d}{dx}\left(\frac{1}{\rho_0}\right)\right| + \int_0^t\left(\left|\frac{\partial\theta}{\partial x}\right| + |\theta\varphi|\right)d\tau\right), \tag{7.139}$$

što kvadriranjem i integriranjem preko $]0,1[$ postaje

$$\left\|\frac{\partial \rho}{\partial x}\right\|^2 \leq C\left(\|\varphi\|^2 + \int_0^1 \frac{1}{\rho_0^4}\left(\rho_0'\right)^2 dx + \int_0^t\int_0^1\left(\frac{\partial\theta}{\partial x}\right)^2 dx d\tau + \int_0^t M_\theta^2\|\varphi\|^2 d\tau\right) \tag{7.140}$$

odakle, primjenom (4.7), (4.12), (7.130), (7.136) i (7.113), slijedi tvrdnja leme. □

Koristeći (7.132) i (7.85) lako zaključujemo da je

$$\rho \in \mathrm{L}^{\infty}\left(0, T; \mathrm{H}^{1}\left(]0,1[\right)\right). \tag{7.141}$$

Lema 7.18. *Postoji konstanta $C \in \mathbb{R}^{+}$ takva da za $t \in]0,T[$ vrijedi*

$$\left\|\frac{\partial v}{\partial x}(t)\right\|^{2}+\int_{0}^{t}\left\|\frac{\partial^{2} v}{\partial x^{2}}(\tau)\right\|^{2} d\tau \leq C. \tag{7.142}$$

Dokaz. Množenjem jednadžbe (3.127) s $\frac{\partial^{2} v}{\partial x^{2}}$, integriranjem preko $]0,1[$, korištenjem jednakosti (3.115), parcijalne integracije i rubnih uvjeta (3.134) dobivamo

$$\frac{1}{2} \frac{d}{dt}\left\|\frac{\partial v}{\partial x}(t)\right\|^{2}+\frac{\lambda+2\mu}{L^{2}} \int_{0}^{1} r^{4} \rho\left(\frac{\partial^{2} v}{\partial x^{2}}\right)^{2} dx=\sum_{k=1}^{5} I_{k}(t) \tag{7.143}$$

gdje je

$$I_{1}=\frac{R}{L} \int_{0}^{1} r^{2} \theta \frac{\partial^{2} v}{\partial x^{2}} \frac{\partial \rho}{\partial x} dx, \tag{7.144}$$

$$I_{2}=\frac{R}{L} \int_{0}^{1} r^{2} \rho \frac{\partial^{2} v}{\partial x^{2}} \frac{\partial \theta}{\partial x} dx, \tag{7.145}$$

$$I_{3}=2(\lambda+2\mu) \int_{0}^{1} \frac{v}{r^{2} \rho} \frac{\partial^{2} v}{\partial x^{2}} dx, \tag{7.146}$$

$$I_{4}=-\frac{\lambda+2\mu}{L^{2}} \int_{0}^{1} r^{4} \frac{\partial \rho}{\partial x} \frac{\partial v}{\partial x} \frac{\partial^{2} v}{\partial x^{2}} dx, \tag{7.147}$$

$$I_{5}=-\frac{4(\lambda+2\mu)}{L} \int_{0}^{1} r \frac{\partial v}{\partial x} \frac{\partial^{2} v}{\partial x^{2}} dx. \tag{7.148}$$

Sada ćemo ocijeniti integrale $I_1 - I_5$. Pomoću (7.127) i Hölderove nejednakosti dobivamo

$$|I_{1}| \leq C M_{\theta}\left\|\frac{\partial^{2} v}{\partial x^{2}}\right\|\left\|\frac{\partial \rho}{\partial x}\right\|, \tag{7.149}$$

Primjenom ocjene (7.132) i Youngove nejednakosti na (7.149) slijedi

$$|I_{1}| \leq \varepsilon\left\|\frac{\partial^{2} v}{\partial x^{2}}\right\|^{2}+C M_{\theta}^{2}. \tag{7.150}$$

Analogno imamo

$$|I_{2}| \leq \varepsilon\left\|\frac{\partial^{2} v}{\partial x^{2}}\right\|^{2}+C\left\|\frac{\partial \theta}{\partial x}\right\|^{2}, \tag{7.151}$$

$$|I_{3}| \leq \varepsilon\left\|\frac{\partial^{2} v}{\partial x^{2}}\right\|^{2}+C\|v\|^{2}. \tag{7.152}$$

Primjenom (7.127) i nejednakosti Gagliardo-Ladyzhenskaye za funkciju $\dfrac{\partial v}{\partial x}$ dobivamo

$$|I_4| \leq C \left\|\frac{\partial v}{\partial x}\right\|^{\frac{1}{2}} \left\|\frac{\partial^2 v}{\partial x^2}\right\|^{\frac{1}{2}} \int\limits_0^1 \left|\frac{\partial \rho}{\partial x}\frac{\partial^2 v}{\partial x^2}\right| dx \tag{7.153}$$

što primjenom Hölderove nejednakosti daje

$$|I_4| \leq C \left\|\frac{\partial v}{\partial x}\right\|^{\frac{1}{2}} \left\|\frac{\partial^2 v}{\partial x^2}\right\|^{\frac{3}{2}} \left\|\frac{\partial \rho}{\partial x}\right\|. \tag{7.154}$$

Korištenjem ocjene (7.132) i Youngove nejednakosti iz (7.154) dobivamo

$$|I_4| \leq \varepsilon \left\|\frac{\partial^2 v}{\partial x^2}\right\|^2 + C \left\|\frac{\partial v}{\partial x}\right\|^2. \tag{7.155}$$

Istim postupkom kao i kod drugih integrala imamo

$$|I_5| \leq \varepsilon \left\|\frac{\partial^2 v}{\partial x^2}\right\|^2 + C \left\|\frac{\partial v}{\partial x}\right\|^2. \tag{7.156}$$

Uvrštavanjem (7.150)-(7.152), (7.155) i (7.156) u (7.143) uz uzimanje dovoljno malog parametra $\varepsilon > 0$ dobivamo

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial v}{\partial x}(t)\right\|^2 + C_1 \left\|\frac{\partial^2 v}{\partial x^2}\right\|^2 \leq C_2 \left(1 + M_\theta^2 + \|v\|^2 + \left\|\frac{\partial \theta}{\partial x}\right\|^2 + \left\|\frac{\partial v}{\partial x}\right\|^2\right), \tag{7.157}$$

što integriranjem preko $]0,t[$ te korištenjem (4.13), (7.130) i (7.120) daje tvrdnju leme. $\square$

Sada iz (7.63), (7.120) i (7.142) zaključujemo da je

$$v \in \mathrm{L}^2\left(0, T; \mathrm{H}^2\left(]0,1[\right)\right) \cap \mathrm{L}^\infty\left(0, T; \mathrm{H}^1\left(]0,1[\right)\right). \tag{7.158}$$

Lema 7.19. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $t \in]0,T[$ vrijedi*

$$\left\|\frac{\partial \omega}{\partial x}(t)\right\|^2 + \int\limits_0^t \left\|\frac{\partial^2 \omega}{\partial x^2}(\tau)\right\|^2 d\tau \leq C. \tag{7.159}$$

Dokaz. Množenjem jednadžbe (3.127) s $\rho^{-1}\dfrac{\partial^2 \omega}{\partial x^2}$ i istim postupkom kao u dokazu prethodne leme dobivamo jednakost

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial \omega}{\partial x}(t)\right\|^2 + \frac{c_0 + 2c_d}{j_I L^2}\int\limits_0^1 r^4 \rho \left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 dx = \sum_{k=1}^4 I_k(t) \tag{7.160}$$

gdje je

$$I_1 = \frac{4\mu_r}{j_I}\int\limits_0^1 \frac{\omega}{\rho}\frac{\partial^2 \omega}{\partial x^2} dx, \tag{7.161}$$

$$I_2 = 2\frac{c_0 + 2c_d}{j_I} \int_0^1 \frac{\omega}{r^2\rho} \frac{\partial^2 \omega}{\partial x^2} dx, \tag{7.162}$$

$$I_3 = -\frac{c_0 + 2c_d}{j_I L^2} \int_0^1 r^4 \frac{\partial \rho}{\partial x} \frac{\partial \omega}{\partial x} \frac{\partial^2 \omega}{\partial x^2} dx, \tag{7.163}$$

$$I_4 = -\frac{4(c_0 + 2c_d)}{j_I L} \int_0^1 r \frac{\partial \omega}{\partial x} \frac{\partial^2 \omega}{\partial x^2} dx. \tag{7.164}$$

Analogno, kao i u dokazu Leme 7.18 imamo

$$|I_1| \leq C \left\| \frac{\partial^2 \omega}{\partial x^2} \right\| \|\omega\| \leq \varepsilon \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 + C \|\omega\|^2, \tag{7.165}$$

$$|I_2| \leq \varepsilon \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 + C \|\omega\|^2, \tag{7.166}$$

$$|I_3| \leq \varepsilon \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 + C \left\| \frac{\partial \omega}{\partial x} \right\|^2 \left\| \frac{\partial \rho}{\partial x} \right\|^4, \tag{7.167}$$

$$|I_4| \leq \varepsilon \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 + C \|\omega\|^2. \tag{7.168}$$

Uvrštavanjem (7.165)-(7.168) u (7.160) za dovoljno mali parametar $\varepsilon > 0$, dobivamo da vrijedi

$$\frac{1}{2}\frac{d}{dt} \left\| \frac{\partial \omega}{\partial x}(t) \right\|^2 + C_1 \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 \leq C_2 \left(1 + \|\omega\|^2 + \left\| \frac{\partial \omega}{\partial x} \right\|^2 \right), \tag{7.169}$$

odakle integriranjem preko $]0, t[$, uz primjenu (4.13), (7.130) i (7.123) slijedi (7.159). $\square$

Koristeći (7.63), (7.123) i (7.159) sada zaključujemo da vrijedi inkluzija

$$\omega \in \mathrm{L}^2\left(0, T; \mathrm{H}^2\left(]0,1[\right)\right) \cap \mathrm{L}^\infty\left(0, T; \mathrm{H}^1\left(]0,1[\right)\right). \tag{7.170}$$

Lema 7.20. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $t \in]0, T[$ vrijedi*

$$\left\| \frac{\partial \theta}{\partial x}(t) \right\|^2 + \int_0^t \left\| \frac{\partial^2 \theta}{\partial x^2}(\tau) \right\|^2 d\tau \leq C. \tag{7.171}$$

Dokaz. Množenjem jednadžbe (3.129) s $\rho^{-1}\dfrac{\partial^2 \theta}{\partial x^2}$ te istim postupkom kao i u dokazu Lema 7.18 i 7.19 dobivamo da vrijedi

$$\frac{1}{2}\frac{d}{dt} \left\| \frac{\partial \theta}{\partial x}(t) \right\|^2 + \frac{k}{c_v L^2} \int_0^1 r^4 \rho \left(\frac{\partial^2 \theta}{\partial x^2} \right)^2 dx = \sum_{k=1}^{11} I_k(t) \tag{7.172}$$

gdje je

$$I_1 = -\frac{4k}{c_v L} \int_0^1 r \frac{\partial \theta}{\partial x} \frac{\partial^2 \theta}{\partial x^2} dx, \tag{7.173}$$

$$I_2 = -\frac{k}{c_v L^2} \int\limits_0^1 r^4 \frac{\partial \rho}{\partial x} \frac{\partial \theta}{\partial x} \frac{\partial^2 \theta}{\partial x^2} dx, \tag{7.174}$$

$$I_3 = \frac{2R}{c_v} \int\limits_0^1 \frac{\theta v}{r} \frac{\partial^2 \theta}{\partial x^2} dx, \tag{7.175}$$

$$I_4 = \frac{R}{c_v L} \int\limits_0^1 r^2 \rho \theta \frac{\partial v}{\partial x} \frac{\partial^2 \theta}{\partial x^2} dx, \tag{7.176}$$

$$I_5 = -\frac{4(\lambda + \mu)}{c_v} \int\limits_0^1 \frac{v^2}{r^2 \rho} \frac{\partial^2 \theta}{\partial x^2} dx, \tag{7.177}$$

$$I_6 = -\frac{4\lambda}{c_v L} \int\limits_0^1 r v \frac{\partial v}{\partial x} \frac{\partial^2 \theta}{\partial x^2} dx, \tag{7.178}$$

$$I_7 = -\frac{\lambda + 2\mu}{c_v L^2} \int\limits_0^1 r^4 \rho \left(\frac{\partial v}{\partial x} \right)^2 \frac{\partial^2 \theta}{\partial x^2} dx, \tag{7.179}$$

$$I_8 = -\frac{4(c_0 + c_d)}{c_v} \int\limits_0^1 \frac{\omega^2}{r^2 \rho} \frac{\partial^2 \theta}{\partial x^2} dx, \tag{7.180}$$

$$I_9 = -\frac{4c_0}{c_v L} \int\limits_0^1 r \omega \frac{\partial \omega}{\partial x} \frac{\partial^2 \theta}{\partial x^2} dx, \tag{7.181}$$

$$I_{10} = -\frac{c_0 + 2c_d}{c_v L^2} \int\limits_0^1 r^4 \rho \left(\frac{\partial \omega}{\partial x} \right)^2 \frac{\partial^2 \theta}{\partial x^2} dx, \tag{7.182}$$

$$I_{11} = -\frac{4\mu_r}{c_v} \int\limits_0^1 \frac{\omega^2}{\rho} \frac{\partial^2 \theta}{\partial x^2} dx. \tag{7.183}$$

Napravimo sada ocjene integrala $I_1 - I_{11}$. Korištenjem (7.127), Hölderove i Youngove nejednakosti dobivamo

$$|I_1| \leq C \left\| \frac{\partial \theta}{\partial x} \right\| \left\| \frac{\partial^2 \theta}{\partial x^2} \right\| \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + C \left\| \frac{\partial \theta}{\partial x} \right\|^2. \tag{7.184}$$

Primjenom ocjene (7.127) i nejednakosti Gagliardo-Ladyzhenskaye za funkciju $\dfrac{\partial \theta}{\partial x}$ imamo

$$|I_2| \leq C \left\| \frac{\partial \theta}{\partial x} \right\|^{\frac{1}{2}} \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^{\frac{1}{2}} \int\limits_0^1 \left| \frac{\partial \rho}{\partial x} \frac{\partial^2 \theta}{\partial x^2} \right| dx, \tag{7.185}$$

što primjenom Hölderove i Youngove nejednakosti daje

$$|I_2| \leq C \left\| \frac{\partial \theta}{\partial x} \right\|^{\frac{1}{2}} \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^{\frac{3}{2}} \left\| \frac{\partial \rho}{\partial x} \right\| \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + C \left\| \frac{\partial \theta}{\partial x} \right\|^2. \tag{7.186}$$

Sličnim postupkom dobivamo

$$|I_3| \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + CM_\theta^2 \|v\|^2 \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + CM_\theta^2, \tag{7.187}$$

$$|I_4| \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + CM_\theta^2 \left\| \frac{\partial v}{\partial x} \right\|^2 \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + CM_\theta^2, \tag{7.188}$$

$$|I_5| \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + C \left\| v^2 \right\|^2 \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + C, \tag{7.189}$$

$$|I_6| \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + C \left(\|v\|^2 + \left\| \frac{\partial v}{\partial x} \right\|^6 \right) \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + C, \tag{7.190}$$

$$|I_7| \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + C \left(\left\| \frac{\partial^2 v}{\partial x^2} \right\|^2 + \left\| \frac{\partial v}{\partial x} \right\|^6 \right) \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + C \left(1 + \left\| \frac{\partial^2 v}{\partial x^2} \right\|^2 \right). \tag{7.191}$$

$$|I_8| \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + C, \tag{7.192}$$

$$|I_9| \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + C, \tag{7.193}$$

$$|I_{10}| \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + C \left(1 + \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 \right). \tag{7.194}$$

$$|I_{11}| \leq \varepsilon \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + C. \tag{7.195}$$

Uvrštavanjem dobivenih ocjena u (7.172) uz uzimanje dovoljno malog parametra ε, slijedi

$$\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial \theta}{\partial x}(t) \right\|^2 + C_1 \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 \leq C_2 \left(1 + M_\theta^2 + \left\| \frac{\partial \theta}{\partial x} \right\|^2 + \left\| \frac{\partial^2 v}{\partial x^2} \right\|^2 + \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 \right). \tag{7.196}$$

odakle integriranjem preko $]0, t[$, kao i u dokazu prethodnih lema, dobivamo (7.171). □

Iz (7.131), (7.92) i (7.171) dobivamo da je

$$\theta \in \mathrm{L}^2 \left(0, T; \mathrm{H}^2 \left(]0, 1[\right)\right) \cap \mathrm{L}^\infty \left(0, T; \mathrm{H}^1 \left(]0, 1[\right)\right). \tag{7.197}$$

Još nam preostaje pokazati ograničenost derivacija funkcija ρ, v, ω i θ po vremenskoj varijabli.

Lema 7.21. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $t \in]0, T[$ vrijedi*

$$\int_0^t \left\| \frac{\partial \rho}{\partial t}(\tau) \right\|^2 d\tau \leq C. \tag{7.198}$$

Dokaz. Kvadriranjem jednadžbe (3.126), integriranjem preko $]0,1[$, primjenom (7.2), (7.127) i (7.85) dobivamo

$$\left\|\frac{\partial\rho}{\partial t}\right\|^2 = \frac{1}{L^2}\int_0^1 \rho^4\left[\frac{\partial}{\partial x}\left(r^2 v\right)\right]^2 dx \le C\left(\|v\|^2 + \left\|\frac{\partial v}{\partial x}\right\|^2\right), \tag{7.199}$$

što integriranjem preko $]0,t[$ uz korištenje (7.63) i (7.142) daje (7.198). □

Lema 7.22. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $t \in]0,T[$ vrijedi*

$$\int_0^t \left\|\frac{\partial v}{\partial t}(\tau)\right\|^2 d\tau \le C. \tag{7.200}$$

Dokaz. Kvadriranjem jednadžbe (3.127), primjenom (3.115) i integriranjem preko $]0,1[$ zaključujemo da vrijedi nejednakost

$$\begin{aligned}\left\|\frac{\partial v}{\partial t}\right\|^2 \le C\int_0^1 &\left(r^4\rho^2\left(\frac{\partial\theta}{\partial x}\right)^2 + r^4\theta^2\left(\frac{\partial\rho}{\partial x}\right)^2 + \frac{v^2}{r^4\rho^2} + \right.\\ &\left. r^8\left(\frac{\partial\rho}{\partial x}\right)^2\left(\frac{\partial v}{\partial x}\right)^2 + r^2\left(\frac{\partial v}{\partial x}\right)^2 + r^8\rho^2\left(\frac{\partial^2 v}{\partial x^2}\right)^2\right)dx.\end{aligned} \tag{7.201}$$

Primjenom ocjena (7.2), (7.127) za funkciju r i (7.85), (7.86) i (7.132) za funkciju ρ nejednakost (7.201) prelazi u

$$\left\|\frac{\partial v}{\partial t}\right\|^2 \le C\left(1 + M_\theta^2 + \left\|\frac{\partial\theta}{\partial x}\right\|^2 + \|v\|^2 + \left\|\frac{\partial v}{\partial x}\right\|^2 + \left\|\frac{\partial^2 v}{\partial x^2}\right\|^2\right). \tag{7.202}$$

Integriranjem nejednakosti (7.202) preko $]0,t[$, primjenom ocjena (7.113) i (7.142) te inkluzije (7.131) zaključujemo da vrijedi (7.200). □

Lema 7.23. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $t \in]0,T[$ vrijedi*

$$\int_0^t \left\|\frac{\partial\omega}{\partial t}(\tau)\right\|^2 d\tau \le C. \tag{7.203}$$

Dokaz. Množenjem jednadžbe (3.128) s ρ^{-1}, analogno kao u dokazu prethodne leme dobivamo

$$\begin{aligned}\left\|\frac{\partial\omega}{\partial t}\right\|^2 \le C\int_0^1 &\left(\frac{\omega^2}{\rho^2} + \frac{\omega^2}{r^4\rho^2} + r^8\left(\frac{\partial\rho}{\partial x}\right)^2\left(\frac{\partial\omega}{\partial x}\right)^2 + \right.\\ &\left. r^2\left(\frac{\partial\omega}{\partial x}\right)^2 + r^8\rho^2\left(\frac{\partial^2\omega}{\partial x^2}\right)^2\right)dx,\end{aligned} \tag{7.204}$$

odnosno

$$\left\|\frac{\partial\omega}{\partial t}\right\|^2 \le C\left(1 + \|\omega\|^2 + \left\|\frac{\partial\omega}{\partial x}\right\|^2 + \left\|\frac{\partial^2\omega}{\partial x^2}\right\|^2\right), \tag{7.205}$$

odakle integriranjem preko $]0,t[$ dobivamo tvrdnju leme. □

Lema 7.24. *Postoji konstanta $C \in \mathbb{R}^+$ takva da za $t \in]0, T[$ vrijedi*

$$\int_0^t \left\| \frac{\partial \theta}{\partial t}(\tau) \right\|^2 d\tau \leq C. \tag{7.206}$$

Dokaz. Analognim postupkom kao i u dokazu prethodne dvije leme iz (3.129) dobivamo

$$\begin{aligned} \left\| \frac{\partial \theta}{\partial t} \right\|^2 \leq C \int_0^1 \Bigg(& r^8 \rho^2 \left(\frac{\partial^2 \theta}{\partial x^2} \right)^2 + r^2 \left(\frac{\partial \theta}{\partial x} \right)^2 + r^8 \left(\frac{\partial \rho}{\partial x} \right)^2 \left(\frac{\partial \theta}{\partial x} \right)^2 + \frac{\theta^2 v^2}{r^2} + \\ & r^4 \rho^2 \theta^2 \left(\frac{\partial v}{\partial x} \right)^2 + \frac{v^4}{r^4 \rho^2} + r^2 v^2 \left(\frac{\partial v}{\partial x} \right)^2 + r^8 \rho^2 \left(\frac{\partial v}{\partial x} \right)^4 + \frac{\omega^4}{r^4 \rho^2} + \\ & r^2 \omega^2 \left(\frac{\partial \omega}{\partial x} \right)^2 + r^8 \rho^2 \left(\frac{\partial \omega}{\partial x} \right)^4 + \frac{\omega^4}{\rho^2} \Bigg) dx, \end{aligned} \tag{7.207}$$

te zaključujemo da vrijedi

$$\begin{aligned} \left\| \frac{\partial \theta}{\partial t} \right\|^2 \leq C \Bigg(1 + \int_0^1 \Bigg(& \left(\frac{\partial^2 \theta}{\partial x^2} \right)^2 + \left(\frac{\partial \theta}{\partial x} \right)^2 + \left(\frac{\partial \rho}{\partial x} \right)^2 \left(\frac{\partial \theta}{\partial x} \right)^2 + \theta^2 v^2 + \\ & \theta^2 \left(\frac{\partial v}{\partial x} \right)^2 + v^2 \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^4 + \omega^2 \left(\frac{\partial \omega}{\partial x} \right)^2 + \left(\frac{\partial \omega}{\partial x} \right)^4 \Bigg) dx \Bigg). \end{aligned} \tag{7.208}$$

Primjenom nejednakosti Gagliardo-Ladyzhenskaye i (7.63), (7.208) postaje

$$\begin{aligned} \left\| \frac{\partial \theta}{\partial t} \right\|^2 \leq C \Bigg(& 1 + M_\theta^2 + \left\| \frac{\partial \theta}{\partial x} \right\|^2 + \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + \left\| \frac{\partial \theta}{\partial x} \right\| \left\| \frac{\partial^2 \theta}{\partial x^2} \right\| \left\| \frac{\partial \rho}{\partial x} \right\|^2 + \\ & \left\| \frac{\partial v}{\partial x} \right\|^3 \|v\| + \left\| \frac{\partial \omega}{\partial x} \right\|^3 \|\omega\| + \left\| \frac{\partial v}{\partial x} \right\|^3 \left\| \frac{\partial^2 v}{\partial x^2} \right\| + \left\| \frac{\partial \omega}{\partial x} \right\|^3 \left\| \frac{\partial^2 \omega}{\partial x^2} \right\| \Bigg), \end{aligned} \tag{7.209}$$

Koristeći (7.132), (7.63), (7.142), (7.159) i (7.171) te Youngovu nejednakost dobivamo

$$\left\| \frac{\partial \theta}{\partial t} \right\|^2 \leq C \left(1 + M_\theta^2 + \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 + \left\| \frac{\partial^2 v}{\partial x^2} \right\| + \left\| \frac{\partial^2 \omega}{\partial x^2} \right\| \right), \tag{7.210}$$

odakle integriranjem preko $]0, t[$ kao i u dokazu Lema 7.22 i 7.22 slijedi (7.206). □

Koristeći rezultate zadnje četiri leme lako zaključujemo da je

$$\rho, v, \omega, \theta \in \mathrm{H}^1 (Q_T). \tag{7.211}$$

Sada iz (7.141), (7.125), (7.126), (7.131), (7.158), (7.197), (7.170) i (7.211) te ocjene (7.86) u skladu s Propozicijom 7.1 slijedi tvrdnja Teorema 4.3.

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Sažetak

Predmet istraživanja disertacije je sferno simetrični trodimenzionalni model kompresibilnog viskoznog izotropnog i toplinski provodljivog mikropolarnog fluida koji je u termodinamičkom smislu savršen i politropan.

U prvom dijelu temeljem konstitutivnih jednadžbi za opisani fluid te zakona očuvanja izvodi se u Eulerovoj deskripciji matematički model toka promatranog fluida izmedu dvije termički izolirane koncentrične čvrste sferne stijenke u trodimenzionalnom euklidskom prostoru, a zatim se uz pretpostavku sferne simetrije rješenja formira inicijalno-rubni problem sa dvije varijable u Lagrangeovoj deskripciji, s homogenim rubnim uvjetima za brzinu, mikrorotaciju i toplinski fluks te dovoljno glatkim početnim funkcijama.

U drugom dijelu rada korištenjem Faedo-Galerkinove metode dokazuje se egzistencija generaliziranog rješenja opisanog inicijalno-rubnog problema lokalno po vremenu.

U sljedećem dijelu rada pokazuje se jedinstvenost generaliziranog rješenja, dok se u završnom dijelu rada dokazuje i egzistencija rješenja za svaki konačni vremenski interval.

Summary

The subject of the thesis is spherically symmetric three dimensional model of the compressible viscous isotropic and heat-conducting micropolar fluid that is in thermodynamical sense perfect and polytropic.

In the first part of the thesis based on constitutive equations for described fluid as well as the balance laws we derive mathematical model of the fluid flow in Eulerian description for the flow between two concentric spherical thermoinsulated solid walls in three dimensional Euclidean space. Using the assumptions of spherical symmetry we derive initial-boundary problem with two variables in Lagrangian description with homogeneous boundary conditions for velocity, microrotation and heat flux for sufficiently smooth initial functions.

In the second part of the work using the Faedo-Galerkin method we prove the existence of the generalized solution of described initial-boundary problem locally in time.

In the next part of the work we prove the uniqueness of the generalized solution, and in the final part there is the proof of the existence of the solution for each finite time.

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Objavio je osam znanstvenih i devet stručnih radova, te sudjelovao s priopćenjem na osam međunarodnih i četiri domaća znanstvena skupa. Održao je veći broj stručnih predavanja i autor je jednog sveučilišnog udžbenika.

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1. Mujaković, N., Dražić, I., *3-D flow of a compressible viscous micropolar fluid with spherical symmetry: uniqueness of a generalized solution*, Boundary value problems, prihvaćeno za objavljivanje

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