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STOHASTIČKI MODELI U TRANSFORMIRANOM VREMENU: FRAKCIJSKE PEARSONOVE DIFUZIJE I ODGOĐENI AUTOREGRESIVNI PROCESI U NEPREKIDNOM VREMENU

DOKTORSKI RAD

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Summary

In this thesis we study time-changed stochastic models via inverse of the standard stable subordinator. Moreover, we study two types of such models: fractional Pearson diffusions and delayed continuous-time autoregressive processes.

In Chapter 2 we review the connection between fractional calculus and stochastic processes, i.e. connection between fractional differential equations and time-changed stochastic processes.

Chapter 3 gives a review on modern theory of diffusion processes with emphasize on the corresponding infinitesimal generator and its spectral properties. Moreover, we define the family of Pearson diffusions and give an overview of their properties, including spectral structure of the corresponding infinitesimal generator and spectral representation of transition densities of Pearson diffusions.

In Chapter 4 we define fractional Pearson diffusions. We give an overview of known results regarding non-heavy-tailed fractional Pearson diffusions and prove several results regarding heavy-tailed fractional Pearson diffusions. In particular, we establish spectral representations of their transition densities, strong solutions of fractional Cauchy problems which involve corresponding infinitesimal generator, and we show that they exhibit long-range dependence property.

Next in Chapter 5, we give a general framework in our setting on how to construct time-changed stochastic process, which as the weak limit in J_1 topology has a desired diffusion process. Based on this result, we define correlated continuous time random walks which as the limiting processes have fractional Pearson diffusions.

In the last Chapter 6 we define delayed Lévy-driven continuous-time autoregressive processes, where we study their correlation and distributional properties.

Keywords: Pearson diffusions, fractional Pearson diffusions, spectral representation of transition density, correlation structure, correlated continuous time random walks, urn-scheme models, delayed continuous-time autoregressive processes, Caputo fractional derivative, Mittag-Leffler function, inverse of the standard stable subordinator;

Sažetak

Prva osoba koja je uvela ideju slučajnih procesa u transformiranom vremenu korištenjem subordinatora, tj. transformaciju slučajnog procesa u novi slučajni proces putem slučajnog vremena dobivenog subordinatorom, bio je Bochner 1949. godine. To ujedno prestavlja početke stohastičkih modela u transformiranom vremenu. U posljednjih nekoliko desetljeća postoji snažan interes za stohastičke modele u transformiranom vremenu koji uključuju inverz standardnog stabilnog subordinatora umjesto samog subordinatora. Takvi modeli su interesantni jer mogu opisati periode vremena kada proces miruje. Također postoji snažna veza između frakcionalnog računa i slučajnih procesa dobivenih putem takvog slučajnog vremena. Naime, pokazuje se da vremenski-promijenjeni slučajni procesi, odnosno stohastički modeli u transformiranom vremenu, imaju funkcije gustoće koje rješavaju odgovarajuće frakcionalne diferencijalne jednadžbe. S druge strane, može se pokazati da su takvi stohastički modeli u transformiranom vremenu granični procesi odgovarajućih (koreliranih) slučajnih šetnji u neprekidnom vremenu. Inače, slučajne šetnje u neprekidnom vremenu su često korišten alat u statističkoj fizici, gdje se koriste kao model gibanja čestica. Stoga, takvi modeli povezuju frakcionalne diferencijalne jednadžbe, odgođene slučajne procese i (korelirane) slučajne šetnje u neprekidnom vremenu i mogu biti korisni u raznim područjima. U ovom radu, proučavaju se dvije vrste stohastičkih modela u transformiranom vremenu: frakcijske Pearsonove difuzije i odgođeni autoregresivni procesi u neprekidnom vremenu.

U prvom dijelu rada analizirat će se frakcijske Pearsonove difuzije, tj. Pearsonove difuzije u transformiranom vremenu putem inverza standardnog stabilnog subordinatora. Eksplicitno će se izračunati spektralna reprezentacija prijelaznih funkcija gustoće frakcijskih Pearsonovih difuzija s teškim repovima i jaka rješenja odgovarajućih vremenski - frakcionalnih Kolmogorovljevih jednadžbi unazad s pripadnim početnim uvjetom. Nadalje, na temelju korelacijske strukture frakcijskih Pearsonovih difuzija pokazat će se da su to stohastički modeli s dugoročnom zavisnošću. Također, uspostavit će se stohastičke diferencijalne jednadžbe koje opisuju frakcijske Pearsonove difuzije.

U sljedećem koraku dokazat će se konvergencija specifično definiranih koreliranih slučajnih šetnji u neprekidnom vremenu prema frakcijskim Pearsonovim difuzijama. Konkretno, pokazat ćemo da se frakcijske Pearsonove difuzije koje nemaju teške repove mogu dobiti kao granični proces koreliranih slučajnih šetnji u neprekidnom vremenu koje su konstruirane i

motivirane poznatim modelima urni: Laplace-Bernoullijev i Wright-Fisherov model urni. S druge strane korelirane slučajne šetnje u neprekidnom vremenu koje kao granični proces imaju frakcijske Pearsonove difuzije s teškim repovima, nisu konstruirane na temelju nekog konkretnog modela. Dakle, Pearsonove difuzije u transformiranom vremenu pokazat će se kao stohastički model čije funkcije gustoće rješavaju odgovarajuće vremenski - frakcionalne Kolmogorovljeve jednadžbe unazad, a s druge strane su granični procesi odgovarajućih koreliranih slučajnih šetnji u neprekidnom vremenu. Na taj način, frakcijske Pearsonove difuzije se mogu interpretirati kao stohastički, frakcionalni i fizikalni model.

U drugom dijelu rada razmatraju se odgođeni autoregresivni procesi u neprekidnom vremenu, pri čemu je pogonski proces Lévyjev proces, odnosno autoregresivni procesi u neprekidnom vremenu s pogonskim Lévyjevim procesom, koje je odgođeno inverzom standardnog stabilnog subordinatora. Na temelju generalnih i asimptotskih svojstava Mittag-Lefflerovih funkcija, bit će izračunata korelacijska struktura odgođenih autoregresivnih procesa u neprekidnom vremenu, a na temelju kojih će se ustvrditi da i ovi stohastički modeli u transformiranom vremenu imaju dugoročnu zavisnost. Također, bit će izvedena određena distribucijska svojstva.

Ključne riječi: Pearsonove difuzije, frakcijske Pearsonove difuzije, spektralna reprezentacija prijelazne funkcije gustoće, korelacijska struktura, korelirane slučajne šetnje u neprekidnom vremenu, modeli urni, odgođeni autoregresivni procesi u neprekidnom vremenu, Caputova frakcionalna derivacija, Mittag-Lefflerova funkcija, inverz standardnog stabilnog subordinatora;

Contents

Chapter 1

Introduction

Historical roots and motivation

The first person to introduce the idea of time-changed stochastic processes via subordination, i.e. transformation of a stochastic process via random time change obtained by a subordinator, was Bochner in 1949 (see Bochner (1949)). Formally, for a stochastic process $\{X(t),\ t \geq 0\}$, let $\{T(t),\ t \geq 0\}$ denote the time change, i.e. a non–negative, non–decreasing stochastic process (usually independent of process $\{X(t),\ t \geq 0\}$). Then the corresponding time-changed stochastic process is $\{Y(t),\ t \geq 0\}$, where

$$Y(t) := X(T(t)),\ t \geq 0. \tag{1.1}$$

Some authors refer the time change process $\{T(t),\ t \geq 0\}$ as a business time or a stochastic clock, while the process $\{Y(t),\ t \geq 0\}$ is said to evolve in operational time. Feller introduced a subordinated process $X(T(t))$ for a Markov process $X(t)$ with $T(t)$ being a process with independent increments and he referred $T(t)$ as "randomized operational time" (see Feller (1966)). Few decades later the idea of random time change was applied in mathematical finance (Clark (1973)), where the time-changed diffusions were used to model financial time series on speculative markets.

In the last few decades there is a strong interest in time-changed stochastic models, which include inverse subordinator, instead of subordinator, as the time change stochastic process. One example of such models are time-fractional models. In finance, these models can describe delays between trades (Scalas (2006)), interest rate data for developing countries (Janczura et al. (2011)), and has been used to develop the Black-Scholes formalism in this context (see Magdziarz (2009), Stanislavsky (2009)). In hydrology, time-fractional models can capture behaviour such as sticking and trapping of contaminant particles in a porous medium or river flow (Chakraborty et al. (2009), Schumer et al. (2003)). Moreover, there is a deep connection between fractional calculus and stochastic processes obtained by such random time change.
In 1695 Leibnitz wrote a curious letter to L'Hôpital in which he asked *"Can the meaning*

of derivatives with integer order be generalized to derivatives with non-integer orders?" Soon after their integer-order cousins, fractional derivatives were invented by Leibnitz. In 1905 Einstein published his now famous paper Einstein (1905), in which the connection between diffusion equation

$$\frac{\partial}{\partial t}p(x,t) = \frac{\partial^2}{\partial x^2}p(x,t),$$

Brownian motion $\{W(t),\ t \geq 0\}$ and a simple random walk is established. On the other hand, if the partial derivative in time is replaced with fractional partial derivative of order $0 < \alpha < 1$, one arrives at the time-fractional diffusion equation

$$\frac{\partial^\alpha}{\partial t^\alpha}p(x,t) = \frac{\partial^2}{\partial x^2}p(x,t) \tag{1.2}$$

which governs the time-changed Brownian motion $\{W(E(t)),\ t \geq 0\}$, where $\{E(t),\ t \geq 0\}$ is the inverse of the standard α-stable subordinator. Time-changed Brownian motion is the limit of a continuous time random walk with a power-law (of order α) waiting times between particle jumps (see Meerschaert & Scheffler (2004) for more details). Therefore, inverse of the α-stable subordinator provides a probabilistic model for time-fractional differential equations, where the time-fractional derivative is of order α. In statistical physics, time-fractional derivative appears in the equation for a continuous time random walk limit, and reflects random waiting times between particle jumps (see Meerschaert & Scheffler (2004), Metzler & Klafter (2000, 2004)). In other words, fractional calculus serves as a bridge between various scientific areas including probability, differential equations and statistical physics. For a systematic read on these connections see Meerschaert & Sikorskii (2011).

Motivated by these connections, in this thesis several time-changed stochastic models are studied. Inverse of stable subordinator with stability index $0 < \alpha < 1$ is used as the base ingredient for time change process $T(t)$ in (1.1), since it can be seen as a probabilistic model for time-fractional differential equations. If not stated otherwise, it is assumed that the stability index is $0 < \alpha < 1$. For the outer process $\{X(t),\ t \geq 0\}$ in (1.1), Pearson diffusions and continuous-time autoregressive processes are considered. For time-changed stochastic models based on Pearson diffusions, both probabilistic and fractional models are obtained, and therefore in this thesis these models are referred as fractional Pearson diffusions. In the case of time-changed continuous-time autoregressive processes, only probabilistic models are established, and therefore in the thesis are referred as delayed continuous-time autoregressive processes. In particular, in the thesis we study fractional Pearson diffusions, their related correlated continuous time random walks and delayed Lévy-driven continuous-time autoregressive processes, and therefore provide several tractable stochastic models.

Motivation for studying fractional Pearson diffusions lies in tractability of the Pearson diffusions which have applicable class of stationary distributions. In 1930's Kolmogorov was first to introduce this family of diffusions (see Kolmogorov (1931)) by characterizing corresponding infinitesimal parameters, i.e. drift and diffusion via their corresponding stationary distributions $g(x)$ which satisfy the famous Pearson differential equation of the form

$$\frac{g'(x)}{g(x)} = \frac{c_0 x + c_1}{b_2 x^2 + b_1 x + b_0}, \tag{1.3}$$

first introduced by Pearson in 1914 (see Pearson (1914)). Pearson distributions, i.e. family of distributions which satisfy equation (1.3) include normal, gamma, beta, Fisher-Snedecor, reciprocal gamma and Student distribution. The last three distributions are heavy-tailed and therefore are of special interest in stochastic modeling. Step forward in tractability of Pearson diffusions was done by Wong in 1964 (see Wong (1964)) where spectral representation of transition densities of five specially parameterized Pearson diffusions were obtained. In the view of statistical tractability, Forman and Sorensen (Forman & Sørensen (2008)) made significant progress. In particular, the problem of parameter estimation for Pearson diffusions is considered via martingale estimation equations. It is proved that the proposed estimators are consistent and asymptotically normal and explicit expressions for the elements of the limiting covariance matrix are given. Recently, Avram et al. made significant developments regarding spectral and statistical analysis of heavy-tailed Pearson diffusions, i.e. Fisher-Snedecor, reciprocal gamma and Student diffusions, and several related papers were published (Avram et al. (2011, 2012, 2013a,b), Leonenko & Šuvak (2010a,b)). In particular, spectral representation of transition densities of heavy-tailed Pearson diffusions are explicitly calculated, which is the starting point in spectral analysis of the fractional counterparts of these diffusions. Also, general method of moments estimators of parameters of the corresponding Pearson diffusions are obtained, while taking into account spectrum of the infinitesimal generator of the related diffusions. On the other hand, non-heavy-tailed Pearson diffusions, i.e. Ornstein-Uhlenbeck (Uhlenbeck & Ornstein (1930)), Cox-Ingersoll-Ross (Cox et al. (1985)) and Jacobi diffusions are well known and studied. Spectral representation of their transition densities are known and reflect the simple structure of corresponding spectrum of their infinitesimal generators (cf. Karlin & Taylor (1981a)). Non-heavy-tailed fractional Pearson diffusions have recently been studied in Leonenko et al. (2013a) and Leonenko et al. (2013b), where spectral representation of transition densities and correlation structure, as well as strong solutions of related fractional Cauchy problems of these fractional diffusions are obtained. Moreover, it is proved that these non-Markov processes are long-range dependent with a correlation function that falls off like a power law, whose exponent equals the order of the involved fractional derivative. Motivated by these results, in this thesis in a similar manner we study heavy-tailed fractional Pearson diffusions, but with crucially different techniques.

Concept of continuous time random walks was first proposed in 1965 by Montroll and Weiss (see Montroll & Weiss (1965)) and developed further in Scher & Lax (1973), Klafter & Silbey (1980) and Hilfer & Anton (1995). Continuous time random walks have applications in many areas such as finance (for a review see Scalas (2006)), hydrology (for a review see Berkowitz et al. (2006)) and medicine, where it is used as a model for the migration of cancer cells (see Fedotov & Iomin (2007)). On the other hand, sometimes it is convenient to approximate continuous time random walks with the corresponding limiting processes (i.e. with a Lévy process or a diffusion). In particular, regarding correlated continuous time random walks and their weak convergence to a concrete diffusion, not much is known. One of the simplest urn-scheme models is the Bernoulli-Laplace urn-scheme model which was first studied by Laplace (Laplace (1812)) and later by Markov (Markov (1915)). In their work, they found strong connection between this urn-scheme model and Ornstein-Uhlenbeck process with specific parameters by only heuristical arguments. Another well known urn-scheme model is the Wright-Fisher model which describes gene mutations (in some genetic pool) over time, strongly influencing selection and sampling forces in the corresponding population and it is named after S. Wright and R. Fisher. There are several different versions of this model, and in the thesis the scheme described in the book Karlin & Taylor (1981*a*) is used. In the aforementioned book there are examples of this model for which weak converge to Cox-Ingersoll-Ross or Jacobi diffusion with specific parameters is described. Again, only heuristical arguments are given. In this thesis we establish formal weak convergence results for all fractional Pearson diffusions. In particular, for non-heavy-tailed fractional Pearson diffusions, i.e. fractional Ornstein-Uhlenbeck, Cox-Ingersoll-Ross and Jacobi diffusion, we prove weak convergence of correlated continuous time random walks, based on several models, including these two aforementioned urn-scheme models. On the other hand, weak convergence results for heavy-tailed fractional Pearson diffusions, i.e. fractional reciprocal gamma, Fisher-Snedecor and Student diffusions are established based on the correlated continuous time random walks without specific model motivation.

Continuous-time autoregressive processes are continuous counterparts of well known discrete autoregressive processes. Recently, interest for delayed continuous-time autoregressive processes is increasing, e.g. see Gajda et al. (2016) and Wyłomańska & Gajda (2016), where correlation properties and applications to real data are given. In particular, in Wyłomańska & Gajda (2016) codifference structure for delayed continuous-time autoregressive process of low order is examined, as well as the simulation and estimation procedures. In Gajda et al. (2016) delayed continuous-time autoregressive process of low order is used to model technical data, i.e. it is used to model behavior of particular mechanical system. Motivated by the work in Gajda et al. (2016) and Wyłomańska & Gajda (2016) we study correlation properties of the Lévy-driven continuous-time autoregressive process delayed by the inverse of the standard stable subordinator and examine the long-range dependence of the process.

Overview

In Chapter 2 we explain the connection between fractional calculus and stochastic processes, which is then in the following sections established for specific time-changed stochastic models. Chapter 3 gives a review on general theory of diffusion processes. In particular, family of Pearson diffusions is defined with the corresponding properties, including spectral representation of their transition densities which is the starting point for Chapter 4, where fractional Pearson diffusions are defined via inverse of the standard stable subordinator. Moreover, we establish results of these time-changed Pearson diffusions in the form of spectral representation of "transition densities" of heavy-tailed fractional Pearson diffusions, strong solutions of fractional Cauchy problems which involve corresponding infinitesimal generator, correlation structure and corresponding stochastic differential equations driven by the time-changed Brownian motion.
Next, in Chapter 5 we define correlated continuous time random walks which as the limiting processes have fractional Pearson diffusions. In particular, construction of correlated continuous time random walks related to non-heavy-tailed fractional Pearson diffusions are motivated by several famous urn-scheme models, namely Laplace-Bernoulli and Wright-Fisher urn schemes.
In Chapter 6 we define delayed Lévy-driven continuous-time autoregressive processes, and for such processes of order p, emphasizing low orders, we explicitly calculate correlation structure and show that it decays as a power law, i.e. we show that delayed Lévy-driven continuous-time autoregressive processes are long-range dependent processes. Moreover, we establish some results regarding their distributional properties.

Chapter 2

Fractional calculus and stochastic processes

In this section, connection between fractional calculus, stochastic processes and statistical physics is explained in details. In particular, it is explained that the time-changed stochastic models, with time change process being inverse of the standard stable subordinator, have governing time-space fractional differential equations. On the other hand, it is explained how continuous time random walks have these time-changed stochastic models as their scaling limits.

2.1 Fractional derivative

The operator

$$\mathbb{D}_x^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(y)\,dy}{(x-y)^{\alpha-n+1}}, \quad x>0, \quad n-1 \le \alpha < n,\, n \in \mathbb{N}, \tag{2.1}$$

where $\Gamma(\cdot)$ is the standard Gamma function (see e.g. Olver et al. (2010)), is called Riemann-Liouville fractional derivative of order α, while the operator

$$\partial_x^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(y)\,dy}{(x-y)^{\alpha-n+1}}, \quad x>0, \quad n-1 \le \alpha < n,\, n \in \mathbb{N} \tag{2.2}$$

is called Caputo fractional derivative of order α.

These operators are well defined e.g. for infinitely differentiable functions on $\mathbb{R}^+$ with a compact support (for details see e.g. Kilbas et al. (2006)). There are many different versions of fractional derivatives of Riemann-Liouville and Caputo type. In this thesis (2.1) and (2.2) are preferred since we will only consider fractional derivatives in time, i.e. for functions $f(t)$ defined on $t \ge 0$. In general for functions defined on $\mathbb{R}$, Riemann-Liouville and Caputo fractional derivative take the forms

$$\mathbb{D}_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int\limits_{-\infty}^{x} \frac{f(y)\,dy}{(x-y)^{\alpha-n+1}}, \quad x \in \mathbb{R}$$

and

$$\partial_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int\limits_{-\infty}^{x} \frac{f^{(n)}(y)\,dy}{(x-y)^{\alpha-n+1}}, \quad x \in \mathbb{R},$$

respectively. Notice that the forms (2.1) and (2.2) are equivalent with the last two forms, respectively, if one sets $f(x) = 0$ for $x < 0$. There are many other types of fractional derivatives, such as Grunwald-Letnikov and Hadamard types. For general treatment on fractional derivatives and their properties we refer to Kilbas et al. (2006) and Podlubny (1998).

From probabilistic point of view of special interest is the case when $0 < \alpha < 1$. This connection is further explained in Section 2.1. In this case fractional derivatives (2.1) and (2.2) reduce to

$$\mathbb{D}_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int\limits_{0}^{x} \frac{f(y)\,dy}{(x-y)^{-\alpha}} \tag{2.3}$$

and

$$\partial_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int\limits_{0}^{x} \frac{f'(y)\,dy}{(x-y)^{-\alpha}}. \tag{2.4}$$

In particular, these fractional derivatives are well defined e.g. in the space of continuously differentiable and bounded functions. Several important properties of fractional derivatives (2.3) and (2.4) of order $0 < \alpha < 1$ are addressed here. Laplace transform (LT) of a function f is defined as

$$\mathcal{L}f(s) = \tilde{f}(s) := \int\limits_{0}^{\infty} e^{-st} f(t)\,dt.$$

In particular, if it exists, Laplace transform of Riemann-Liouville fractional derivative (2.3) is of the form

$$s^\alpha \tilde{f}(s), \tag{2.5}$$

while Laplace transform of Caputo fractional derivative (2.4) is of the form

$$s^\alpha \tilde{f}(s) - s^{\alpha-1} f(0). \tag{2.6}$$

In applications, Laplace transform (2.6) is preferred over (2.5), since it incorporates the initial condition with physically appropriate interpretation. For this reason, Caputo fractional derivative is often preferred in applications over it's Riemann-Liouville counterpart. Riemann-Liouville and Caputo fractional derivatives (2.5) and (2.6) of a function f coincide if $f(0) = 0$ since they are related by

$$\partial_x^\alpha f(x) = \mathbb{D}_x^\alpha f(x) - f(0)\frac{x^{-\alpha}}{\Gamma(1-\alpha)}.$$

Example 2.1 Let $p > 0$ and $f(t) = t^p$ defined on $t \geq 0$. Using the Laplace transform method, one can easily calculate fractional derivatives of Riemann-Liouville and Caputo type:

$$\tilde{f}(s) = \int_0^\infty e^{-st} t^p \, dt = \frac{1}{s^{p+1}} \int_0^\infty e^{-u} u^p \, du = \frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0, \tag{2.7}$$

where in the last equality definition of Gamma function is used. Now from (2.5) it follows that Riemann-Liouville fractional derivative has LT

$$s^\alpha \tilde{f(s)} = s^{\alpha-p-1}\Gamma(p+1).$$

Finally, inverting the obtained LT yields

$$\mathbb{D}_t^\alpha \left(t^p\right) = t^{p-\alpha}\frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}.$$

Since $f(0) = 0$, Caputo and Riemann-Liouville fractional derivative conicide, i.e.

$$\partial_t^\alpha \left(t^p\right) = \mathbb{D}_t^\alpha \left(t^p\right) = t^{p-\alpha}\frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}. \tag{2.8}$$

On the other hand, notice how for $\alpha = 1$ and $p \in \mathbb{N}$ (2.8) recoveres usual derivative of power function

$$\frac{d}{dt}t^p = pt^{p-1}.$$

Example 2.2 Let $c \in \mathbb{R}$ and $f(t) = c$ defined on $t \geq 0$. Since

$$\tilde{f}(s) = \int_0^\infty e^{-st} c \, dt = \frac{c}{s}$$

it follows that LT of Riemann-Liouville fractional derivative is

$$s^\alpha \tilde{f(s)} = s^{\alpha-1}c.$$

Therefore, from (2.8) it follows

$$\mathbb{D}_t^\alpha c = \frac{ct^{-\alpha}}{\Gamma(1-\alpha)}.$$

On the other hand, LT of Caputo fractional derivative is

$$s^\alpha \tilde{f(s)} - s^{\alpha-1}f(0) = s^\alpha \frac{c}{s} - s^{\alpha-1}c = 0,$$

which yields

$$\partial_t^\alpha c = 0.$$

Therefore, Caputo fractional derivative has yet another preferred property over its Riemann-Liouville counterpart, i.e. Caputo fractional derivative of a constant vanishes just like in the case of the usual derivative.

It is well known that the eigenfunction of the usual derivative, i.e. operator $\frac{d}{dt}$ is the function $f(t) = e^{\lambda t}$, where $\lambda \in \mathbb{C}$ is the corresponding eigenvalue. In other words, $f(t) = e^{\lambda t}$ is the solution of the eigenvalue problem

$$\frac{d}{dt} f(t) = \lambda f(t).$$

The eigenfunction of the Caputo fractional derivative of order $0 < \alpha < 1$, i.e. operator ∂_t^α is the function $f(t) = \mathcal{E}_\alpha(\lambda t^\alpha)$, where

$$\mathcal{E}_\alpha(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(1+\alpha j)}, \quad z \in \mathbb{C} \tag{2.9}$$

is the famous Mittag-Leffler function, introduced by Gösta Magnus Mittag-Leffler in 1902, and defined by the series that converge absolutely for every complex z (see Gorenflo et al. (2016) and references therein). To illustrate this, differentiate term-by-term and apply (2.8):

$$\begin{aligned}
\partial_t^\alpha f(t) &= \partial_t^\alpha \left(\sum_{j=0}^{\infty} \frac{\lambda^j t^{\alpha j}}{\Gamma(1+\alpha j)} \right) \\
&= \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(1+\alpha j)} \frac{\Gamma(1+\alpha j)}{\Gamma(1+\alpha j-\alpha)} t^{\alpha j-\alpha} \\
&= \lambda \sum_{j=1}^{\infty} \frac{(\lambda t^\alpha)^{j-1}}{\Gamma(1+\alpha(j-1))} = \lambda f(t).
\end{aligned}$$

Another way to see this, apply LT together with (2.7) to obtain

$$\begin{aligned}
\mathcal{L}\left(\mathcal{E}_\alpha(\lambda t^\alpha)\right) &= \mathcal{L}\left(\sum_{j=0}^{\infty} \frac{(\lambda t^\alpha)^j}{\Gamma(1+\alpha j)} \right) \\
&= \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(1+\alpha j)} \frac{\Gamma(1+\alpha j)}{s^{\alpha j+1}} \\
&= s^{-1} \sum_{j=0}^{\infty} \left(\lambda s^{-\alpha}\right)^j = \frac{s^{\alpha-1}}{s^\alpha - \lambda}, \quad s^\alpha > |\lambda|.
\end{aligned}$$

Next, from (2.6) it follows

$$\begin{aligned}\mathcal{L}\{\partial_t^\alpha \mathcal{E}_\alpha(\lambda t^\alpha)\} &= s^\alpha \mathcal{L}\{\mathcal{E}_\alpha(\lambda t^\alpha)\} - s^{\alpha-1}\mathcal{E}_\alpha(0)\\ &= s^\alpha \frac{s^{\alpha-1}}{s^\alpha-\lambda} - s^{\alpha-1}\cdot 1\\ &= \lambda \frac{s^{\alpha-1}}{s^\alpha-\lambda}.\end{aligned}$$

Finally, inverting the LT gives

$$\partial_t^\alpha \mathcal{E}_\alpha(\lambda t^\alpha) = \lambda \mathcal{E}_\alpha(\lambda t^\alpha).$$

Taking LT of this infinite series can indeed be justified, for the full proof and technical details see (Meerschaert & Sikorskii 2011, Remark 2.11., pages 36-38).
Notice that for $\alpha = 1$ Mittag-Leffler function recovers the exponential function, i.e. $\mathcal{E}_1(z) = e^z$. On the other hand, since $\mathcal{E}_\alpha(0) = 1$ Caputo and Riemann-Liouville fractional derivative of this function do not coincide. In particular,

$$\begin{aligned}\mathbb{D}_t^\alpha \mathcal{E}_\alpha(\lambda t^\alpha) &= \partial_t^\alpha \mathcal{E}_\alpha(\lambda t^\alpha) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\\ &= \lambda \mathcal{E}_\alpha(\lambda t^\alpha) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.\end{aligned}$$

Regarding eigenfunctions of Riemann-Liouville and Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ see e.g. Grigoletto et al. (2018), while for more details on eigenfunction $f(t) = \mathcal{E}_\alpha(\lambda t^\alpha)$ of the Caputo fractional derivative of order $0 < \alpha < 1$ and its role in fractional differential equations we refer to Mainardi & Gorenflo (2000).

2.2 Continuos time random walks

Here we outline the continuous time random walk model with proper physical interpretation. Let $\{Y_i,\, i \in \mathbb{N}\}$ and $\{G_i,\, i \in \mathbb{N}\}$ be sequences of random variables which are independent and identically distributed (iid) as Y and G, respectively. Moreover, let Y_n and G_n be independent for each n. Define the corresponding random walks $S(n) = Y_1 + Y_2 + \cdots Y_n$ and $T(n) = G_1 + G_2 + \cdots G_n$, where random walk $S(n)$ gives location of a randomly selected particle after n jumps, with Y_i being iid particle jumps, while G_i are iid waiting times between particle jumps, so that particle arrives at location $S(n)$ at time $T(n)$. Let

$$N(t) = \max\{n \geq 0\colon\ T(n) \leq t\}$$

be the number of particle jumps by time $t \geq 0$ with $T(0) = 0$. Then $S(N(t))$ is called continuous time random walk (CTRW) which gives the location of a particle by time

$t \geq 0$. CTRW together with the corresponding long-time stochastic limit illuminates the connection between fractional calculus, probability and statistical physics, since the governing equation for the stochastic model in the CTRW scaling limit involves fractional derivative in time, while CTRW itself provides physical interpretation of a random particle movement. Formal illustration requires definition of the limit-involved stochastic processes together with the convergence procedure.

Let $\{W_i,\, i \in \mathbb{N}\}$ be a sequence of random variables iid as W. A random variable W belongs to the domain of attraction of nondegenerate random variable Z and we write $W \in \mathrm{DOA}(Z)$ if

$$a_n\left(W_1 + W_2 + \cdots W_n\right) - b_n \Rightarrow Z$$

for some $a_n > 0$ and $b_n \in \mathbb{R}$, where " $\Rightarrow$ " means convergence in distribution. If $W \in \mathrm{DOA}(Z)$ it is well known that the random variable Z must be either normal or stable with index $0 < \alpha < 2$ (see e.g., (Meerschaert & Sikorskii 2011, Chapter 4)), i.e. the random walk (after proper scaling)

$$S(n) = W_1 + W_2 + \cdots W_n$$

can only have normal or stable limit. Recall that a random variable Z is said to have a stable distribution if for any positive numbers A and B, there is a positive number C and a real number D such that

$$AZ_1 + BZ_2 \stackrel{d}{=} CZ + D, \tag{2.10}$$

where Z_1, Z_2 and Z are iid random variables, and " $\stackrel{d}{=}$ " denotes equality in distribution. Moreover, for any stable random variable Z there is a number $\alpha \in \langle 0,\, 2]$ such that number C in (2.10) satisfies

$$C^\alpha = A^\alpha + B^\alpha.$$

The number α is called the index of stability or characteristic exponent. A random variable Z with index of stability α is called α-stable random variable. Furthermore, characteristic function has the following form

$$\mathbb{E}[e^{i\theta Z}] = \begin{cases} \exp\{-\sigma^\alpha\,|\theta|^\alpha \left(1 - i\beta\,\mathrm{sign}(\theta)\tan\frac{\pi\alpha}{2}\right) + i\mu\theta\} & \text{if } \alpha \neq 1, \\ \exp\{-\sigma\,|\theta| \left(1 + i\beta\frac{2}{\pi} sign(\theta)\ln|\theta|\right) + i\mu\theta\} & \text{if } \alpha = 1, \end{cases}$$

where $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 < \beta < 1$ and $\mu \in \mathbb{R}$ are unique parameters, and therefore for stable random variable Z it is denoted by $Z \sim S_\alpha(\sigma,\, \beta,\, \mu)$ (for more details see Samorodnitsky & Taqqu (1994) and references therein). In particular, stable distribution with index of stability $\alpha = 2$ recovers the normal distribution.

In order to study CTRW together with the corresponding limit process in terms of their trajectories, appropriate space and topology needs to be introduced. Therefore, let

$\mathbb{D}([0, +\infty\rangle\,; S)$ denote the set of right continuous functions with left limits defined on $[0, +\infty\rangle$ with values in S, i.e. the set of càdlàg functions. In particular, when $S = \mathbb{R}$, we simply write $\mathbb{D}\,[0, +\infty\rangle$. In 1956. A. V. Skorokhod introduced several relevant topologies for this space of functions (see Skorokhod (1956)), now known as J_1, J_2, M_1 and M_2 topologies. In particular, J_1 topology is the strongest and most relevant for stochastic processes which have isolated jumps such as in a random walk. In J_1 topology, convergence $x_n(t) \to x(t)$ in $\mathbb{D}[0, T]$ holds if for increasing continuous functions $\lambda_n : [0, T] \to [0, T]$ such that $\lambda_n(0) = 0$ and $\lambda_n(T) = T$, which can be seen as transforms in the time axis,

$$\lim_{n\to\infty} \underset{0\le t\le T}{sup} |\lambda_n(t) - t| = 0$$

implies

$$\lim_{n\to\infty} \underset{0\le t\le T}{sup} |x_n(\lambda_n(t)) - x(t)| = 0.$$

Furthemore, $x_n(t) \to x(t)$ in $\mathbb{D}\,[0, +\infty\rangle$ if $x_n(t) \to x(t)$ in $\mathbb{D}[0, T]$ for every continuity point $T > 0$ of function $x(t)$.

Moreover, Skorokhod proved that if $Y \in \mathrm{DOA}(A)$, where A is β-stable random variable $(0 < \beta \le 2)$, i.e. if

$$a_n S(n) - b_n = a_n\,(Y_1 + Y_2 + \cdots Y_n) - b_n \Rightarrow A$$

for some $a_n > 0$ and $b_n \in \mathbb{R}$, then

$$a_n S(\lfloor nt \rfloor) - b_n t \Rightarrow A(t) \quad \text{in} \quad \mathbb{D}\,[0, +\infty\rangle \tag{2.11}$$

with J_1 topology (see Skorokhod (1957)), where $A(t)$ is either Brownian motion or a β-stable Lévy process. Recall that the CTRW $S(N(t))$ is defined via two random walks $S(n) = Y_1 + Y_2 + \cdots Y_n$ and $T(n) = G_1 + G_2 + \cdots G_n$, where Y_i are iid particle jumps and G_i are iid waiting times between particle jumps. In order to illustrate the underlying CTRW limit, let Y_1 and G_1 be in the domain of attraction of β-stable and α-stable random variables $(0 < \alpha < 1)$, respectively, so that $A(t)$ is β-stable Lévy process with $a_n = n^{-1/\beta}$ and $b_n = 0$. Now, (2.11) reduces to

$$n^{-1/\beta} S(\lfloor nt \rfloor) \Rightarrow A(t) \quad \text{in} \quad \mathbb{D}\,[0, +\infty\rangle\,. \tag{2.12}$$

Next, in order to obtain the limiting process of the CTRW $S(N(t))$, limiting behaviour of $N(t)$ in J_1 topology needs to be introduced. It turns out (see (Meerschaert & Sikorskii 2011, Chapter 4.4) and references therein)

$$n^{-\alpha} N(nt) \Rightarrow E(t) \quad \text{in} \quad \mathbb{D}\,[0, +\infty\rangle \tag{2.13}$$

with J_1 topology, where $E(t)$ is inverse of the standard α-stable subordinator (see subsection 2.3 for precise definition and details). On the other hand, in the CTRW model we assume Y_n and G_n are independent and therefore (2.11) together with (2.13) yields

$$(n^{-1/\beta}S(\lfloor nt\rfloor),\, n^{-\alpha}N(nt)) \Rightarrow (A(t),\, E(t)) \quad \text{in} \quad \mathbb{D}\,[0,\, +\infty\rangle \times \mathbb{D}\,[0,\, +\infty\rangle$$

with J_1 topology. Finally, one shows that normalized and rescaled CTRW have time-changed process in the limit, i.e.

$$n^{-\alpha/\beta}S(N(\lfloor nt\rfloor)) = (n^{\alpha})^{-1/\beta}S(n^{\alpha}n^{-\alpha}N(\lfloor nt\rfloor)) \approx (n^{\alpha})^{-1/\beta}S(n^{\alpha}E(t)) \Rightarrow A(E(t)) \quad (2.14)$$

in $\mathbb{D}\,[0,\, +\infty\rangle$ with M_1 topology. Formal proof can be found in Meerschaert & Scheffler (2004). In fact, convergence (2.14) holds in the stronger J_1 topology as well (for details see Straka & Henry (2011)).

2.3 Inverse of the standard stable subordinator

Recall that a stochastic process $X = \{X(t),\, t \geq 0\}$ is called Lévy process if

1. $X(0) = 0$ a.s.
2. for all $0 \leq s \leq t$, $X(t) - X(s) \stackrel{d}{=} X(t-s)$ (stationary increments)
3. for all $0 \leq u \leq s \leq t$, $X(t) - X(s)$ and $X(u)$ are independent (independent increments)
4. X is càdlàg process, i.e. $t \mapsto X(t)$ is almost surely right continuous with left limits.

In particular, X is called standard α-stable Lévy process if the second requirement is replaced with

2*. for all $0 \leq s \leq t$, $X(t) - X(s) \stackrel{d}{=} S_{\alpha}((t-s)^{1/\alpha},\, \beta,\, 0)$, $\quad 0 < \alpha \leq 2,\ -1 \leq \beta \leq 1$.

Notice that if $\alpha = 2$, then standard stable Lévy process is the standard Brownian motion.

Moreover, a stochastic process is called subordinator if it is a.s. increasing Lévy process. In particular, α-stable subordinator $\{D(u),\, u \geq 0\}$ with index $0 < \alpha < 1$ has Laplace transform

$$\mathbb{E}[e^{-sD(u)}] = \exp\{-uC\Gamma(1-\alpha)s^{\alpha}\}, \quad s > 0.$$

If $C = 1/\Gamma(1-\alpha)$, i.e. if

$$\mathbb{E}[e^{-sD(u)}] = \exp\{-us^{\alpha}\}, \quad s > 0$$

we say that $\{D(u), u \geq 0\}$ is standard α-stable subordinator. Inverse process of the standard α-stable subordinator, $\{E(t), t \geq 0\}$ is defined via

$$E(t) = \inf\{u > 0 : D(u) > t\}. \tag{2.15}$$

This stochastic process is non-Markovian, non-decreasing and corresponds to the first passage time of the standard α-stable subordinator strictly above level t. Moreover it is self-similar with exponent α, i.e.

$$\{E(ct)\} \stackrel{d}{=} \{c^{\alpha} E(t)\},$$

where equality is in finite dimensional distributions. For every t random variable $E(t)$ has a density $f_t(\cdot)$ with Laplace transform (see Bingham (1971))

$$\mathbb{E}[e^{-sE(t)}] = \int_0^{\infty} e^{-sx} f_t(x)dx = \mathcal{E}_{\alpha}(-st^{\alpha}), \quad s > 0. \tag{2.16}$$

Therefore, LT (2.16) connects inverse of the standard α-stable subordinator and the Mittag-Leffler function. Recall that the function $f(t) = \mathcal{E}_{\alpha}(\lambda t^{\alpha})$ is an eigenfunction for the Caputo fractional derivative ∂_t^{α} with eigenvalue λ (see Section 2.1). The density of $E(t)$ is related to the density of the standard stable subordinator $D(u)$ as follows. With g_{α} denoting the density of $D(1)$

$$f_t(u) = \frac{t}{\alpha} u^{-1-1/\alpha} g_{\alpha}(tu^{-1/\alpha}), \quad u > 0, \tag{2.17}$$

see Meerschaert & Scheffler (2004).

2.4 Governing equation for the continuous time random walk limit

Let $p(x, t)$ denote the density of the outer process $A(t)$ in the CTRW limit (2.14). Then simple conditioning argument shows that the CTRW limit process $A(E(t))$ has density

$$m(x, t) = \int_0^{\infty} p(x, u) f_t(u)\, du,$$

where f_t is the density of $E(t)$ (2.17). Moreover, governing equation for the outer process $A(t)$ is

$$\frac{\partial}{\partial u} p(x, u) = Da \frac{\partial^{\beta}}{\partial x^{\beta}} p(x, u) + Db \frac{\partial^{\beta}}{\partial (-x)^{\beta}} p(x, u), \tag{2.18}$$

where D, a and b are parameters such that $a + b = 1$ and defined via β-stable law in the outer process. Moreover, fractional derivative $\frac{\partial^{\beta}}{\partial(-x)^{\beta}} f(x)$ is defined so that it has Fourier transform $(-ik)^{\beta} \hat{f}(k)$ (here $\hat{f}(k)$ denotes Fourier transform of the function f), while the time change process $E(t)$ has governing equation of the form

$$\partial_t^\alpha f_t(u) = -\frac{\partial}{\partial u} f_t(u).$$

On the other hand, density of the CTRW limit $A(E(t))$ solves the differential equation

$$\partial_t^\alpha m(x,t) = Da\frac{\partial^\beta}{\partial x^\beta} m(x,t) + Db\frac{\partial^\beta}{\partial(-x)^\beta} m(x,t). \tag{2.19}$$

In order to obtain these governing equations, Fourier-Laplace transform methods are employed (see (Meerschaert & Sikorskii 2011, Chapter 4.5)). Recall that the outer process $A(u)$ is self similar with index $1/\beta$, while the time change process $E(t)$ is self similar with index α, so that the CTRW limit process $A(E(t))$ is self similar with index α/β.

Remark 2.1. Notice that for $\beta = 2$ the outer process $A(u)$ reduces to Brownian motion, while governing equations (2.18) and (2.19) reduce to

$$\frac{\partial}{\partial u} p(x,u) = D\frac{\partial^2}{\partial x^2} p(x,u)$$

and

$$\partial_t^\alpha m(x,t) = D\frac{\partial^2}{\partial x^2} m(x,t).$$

In general, Brownian motion is self-similar with index $1/2$, while time-changed Brownian motion via inverse of α-stable subordinator $E(t)$ is self similar with index $\alpha/2$. Since $0 < \alpha < 1$ it turns out that time-changed Brownian motion spreads slower then the Brownian motion and therefore yields the so called sub-diffusive phenomena.

For these reasons, stochastic process $\{E(t),\, t \geq 0\}$ defined in (2.15) is a desirable probabilistic model for a time change which enables sub-diffusive phenomena. In Section 4 and 5 it is used to define fractional Pearson diffusions, while in Section 6 it is used to define delayed continuous-time autoregressive processes. In this section, connection between fractional calculus, CTRWs and inverse of stable subordinator is explained. In particular, time-changed Lévy process emerges in the scaling CTRW limit, while governing equation for this process involves fractional derivatives in time and space. In this thesis, similar connection for fractional Pearson diffusions is established. We define correlated CTRWs which have fractional Pearson diffusions as weak limits in J_1 topology. Governing equations for fractional (time-changed) Pearson diffusions are obtained via Caputo time-fractional Kolmogorov backward and forward equations, where the solutions to the corresponding fractional Cauchy problems are obtained through spectral methods, involving Mittag-Leffler function as the eigenfunctions of Caputo fractional derivative in time.

Chapter 3

Pearson diffusions

In this section family of Pearson diffusions is defined with an overview of corresponding properties, including spectral representation of their transition densities. The Pearson diffusions form a flexible and statistically tractable family of diffusions, which includes famous Ornstein-Uhlenbeck and Cox-Ingersoll-Ross process. Their invariant distributions belong to the Pearson family of distributions. On the other hand, basic reason for their tractability is that their moments and some conditional moments can be calculated explicitly and corresponding infinitesimal generator maps polynomials into polynomials of (at most) the same degree.

3.1 Diffusions

In this section we give a review on general theory of diffusion processes, emphasizing the importance of infinitesimal parameters, infinitesimal generator and its corresponding spectrum, which leads to the categorization of diffusions into three categories: Spectral category I., II. and III., systematically introduced in Linetsky (2004), heavily referencing to 10 spectral categories given in Fulton et al. (2005).

3.1.1 Definition

A stochastic process is said to be a diffusion process if it is a continuous time stochastic process which possesses strong Markov property and has a.s. continuous sample paths. There are several ways to describe a diffusion.
Transition probability of a Markov process $\{X(t),\, t \geq 0\}$ is the probability

$$P(B,t;y,s) = P(X(t) \in B | X(s) = y), \quad 0 \leq s < t,$$

while the function $p(x,t;y,s)$ which satisfies the relation

$$P(B,t;y,s) = \int_B p(x,t;y,s)\, dx$$

is called the transition density of the Markov process $\{X(t),\, t \geq 0\}$. A continuous time Markov process with values in $\mathbb{R}$ and transition density $p(x,t;y,s)$ is called a diffusion if the following limits exist for all $\epsilon > 0$, $s \geq 0$ and $x \in \mathbb{R}$:

$$\lim_{t \downarrow s} \frac{1}{t-s} \int\limits_{|x-y|>\epsilon} p(x,t;y,s)\, dx = 0, \tag{3.1}$$

$$\lim_{t \downarrow s} \frac{1}{t-s} \int\limits_{|x-y|<\epsilon} (x-y)p(x,t;y,s)\, dx = \mu(y,s), \tag{3.2}$$

$$\lim_{t \downarrow s} \frac{1}{t-s} \int\limits_{|x-y|<\epsilon} (x-y)^2 p(x,t;y,s)\, dx = \sigma^2(y,s), \tag{3.3}$$

where the limits $\mu(y,s)$ and $\sigma(y,s)$ are well defined continuous functions of y and s, and are called drift parameter and diffusion parameter, respectively.

Remark 3.1. In particular, condition (3.1) ensures almost sure continuity of diffusion sample paths, while conditions (3.2) and (3.3) are equivalent with

$$\lim_{t \downarrow s} \frac{1}{t-s} \mathbb{E}[X(t) - X(s) | X(s) = y] = \mu(y,s) \tag{3.4}$$

and

$$\lim_{t \downarrow s} \frac{1}{t-s} \mathbb{E}[(X(t) - X(s))^2 | X(s) = y] = \sigma^2(y,s), \tag{3.5}$$

respectively. Therefore, (3.4) yields interpretation of $\mu(y,s)$ as instantaneous rate of change in the mean of the diffusion process assuming $X(s) = y$, and therefore it is sometimes referred to as infinitesimal mean or expected infinitesimal displacement of diffusion process, while (3.5) yields interpretation of $\sigma^2(y,s)$ as instantaneous rate of change in the variance of the diffusion process assuming $X(s) = y$, and it is sometimes referred to as infinitesimal variance of diffusion process.

In particular, if the transition density depends only on the time difference $(t-s)$ we say that diffusion process is time-homogeneous and we simply write $p(x,t-s;y)$ instead of $p(x,t;y,s)$. Moreover without loss of generality, we may assume that $s = 0$ and write $p(x,t;y)$ for the transition density. In this thesis only time-homogeneous diffusion processes are studied and therefore this notation is used. It is well known that governing equations for the diffusion process are Kolmogorov backward and forward equation which both incorporate the infinitesimal parameters, i.e. transition density $p(x,t;y)$ satisfy the following partial differential equations (PDEs) with initial point-source condition $p(x,0;\, y) = \delta(x-y)$:

- Kolmogorov backward equation:

$$\frac{\partial p(x,t;y)}{\partial t} = \mu(y)\frac{\partial p(x,t;y)}{\partial y} + \frac{\sigma^2(y)}{2}\frac{\partial^2 p(x,t;y)}{\partial y^2}, \tag{3.6}$$

where the future state x is constant, and the equation describes the "backward evolution" of the diffusion.

- Kolmogorov forward equation or Fokker-Planck equation:

$$\frac{\partial p(x,t;y)}{\partial t} = -\frac{\partial}{\partial x}\left(\mu(x)p(x,t;y)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(\sigma^2(x)p(x,t;y)\Big), \tag{3.7}$$

where the current state y is constant, and the equation describes the "forward evolution" of the diffusion.

Moreover, if stationary distribution of a time-homogeneous diffusion process exist, it can be obtained or calculated via Kolmogorov forward equation. In particular, if there exists a solution $\mathfrak{f}(x)$ of the time-independent Kolmogorov forward equation

$$-\frac{d}{dx}\left(\mu(x)\,\mathfrak{f}(x)\right) + \frac{1}{2}\frac{d^2}{dx^2}\Big(\sigma^2(x)\,\mathfrak{f}(x)\Big) = 0, \tag{3.8}$$

it will be the stationary distribution of the corresponding diffusion with infinitesimal parameters $\mu(x)$ and $\sigma(x)$. On the other hand, if the equation (3.8) doesn't have a solution, stationary distribution won't exist. If the solution exists, the following procedure involving equation (3.8) reveals the stationary distribution:

- integration of the equation (3.8) with respect to the variable x leads to

$$\frac{d}{dx}\left(\frac{\sigma^2(x)}{2}\,\mathfrak{f}(x)\right) - \mu(x)\,\mathfrak{f}(x) = \frac{1}{2}C_1, \tag{3.9}$$

where C_1 is a real constant;

- multiplication of the equation (3.9) by integration factor

$$\mathfrak{s}(x) = \exp\Bigg\{-2\int^{x}\frac{\mu(y)}{\sigma^2(y)}dy\Bigg\}, \tag{3.10}$$

also called the scale density, leads to equation

$$\frac{d}{dx}\Big(\mathfrak{s}(x)\,\sigma^2(x)\,\mathfrak{f}(x)\Big) = C_1\mathfrak{s}(x); \tag{3.11}$$

- integration of the equation (3.11) with respect to the variable x leads to

$$\mathfrak{f}(x) = \frac{\mathfrak{m}(x)}{2}\left(C_1 \int^{x} \mathfrak{s}(y)dy + C_2\right), \tag{3.12}$$

  where

$$\mathfrak{m}(x) = \frac{2}{\sigma^2(x)\,\mathfrak{s}(x)} \tag{3.13}$$

  is the so called speed density.

- constants C_1 and C_2 are determined via conditions $\mathfrak{f}(x) \geq 0$ and $\int_{\mathbb{R}} \mathfrak{f}(x)\,dx = 1$, ensuring that $\mathfrak{f}(x)$ is a probability density function.

In particular, if the speed density is integrable on the diffusion state space $I = \langle l,\, r\rangle$, $-\infty \leq l < r \leq \infty$, i.e. if

$$\int_l^r \mathfrak{m}(x) = C < \infty,$$

then

$$\mathfrak{f}(x) = \frac{\mathfrak{m}(x)}{C}\mathrm{I}_{\langle l,\,r\rangle}(x),\ x \in \mathbb{R}.$$

For details, we refer to Karlin & Taylor (1981*a*).

3.1.2 Stochastic differential equation

Beside the Kolmogorov forward and backward PDEs, diffusions can be defined as solutions of the stochastic differential equations (SDEs) with Brownian motion as the governing process. Let us consider autonomous stochastic differential equation

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t), \quad X_0 = Y,\ 0 \leq t \leq T, \tag{3.14}$$

where $\{W(t),\, t \geq 0\}$ is the standard Brownian motion adapted to the filtration $\{\mathcal{F}_t,\, t \geq 0\}$, usually called the driving process of the SDE (3.14), while random variable Y is assumed to be independent of $\{W(t),\, t \geq 0\}$. SDE (3.14) is interpreted as a stochastic integral equation

$$X(t) = X_0 + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dW(s), \quad 0 \leq t \leq T, \tag{3.15}$$

where the integrals on the right-hand side of equation (3.15) are Riemann integral and Itô stochastic integral, respectively. Therefore, solution to this stochastic integral equation is obviously a stochastic process. One can distinguish two types of solutions of the SDE (3.14), strong and weak solutions. Stochastic process $X = \{X(t),\, 0 \leq t \leq T\}$ is said to be a strong solution to the SDE (3.14) if

- X is adapted to the driving process of the SDE (3.14), i.e. at the time t it is a function of $B(s)$, $s \leq t$,

- stochastic integral equation (3.15) is well defined, i.e. Riemann and Itô stochastic integrals in (3.15) are well defined,
- X is a function of the driving Brownian sample path and of $\mu(x)$ and $\sigma(x)$.

Under certain conditions on $\mu(x)$, $\sigma(x)$ and X_0, there is a pathwise unique strong solution of the SDE (3.14). In general, for two solutions X and $\tilde{X}$ of the SDE (3.14) we say they are pathwise unique if

$$P\left(\sup_{0\leq t\leq T}\left|X(t)-\tilde{X}(t)\right|>0\right)=0.$$

In particular, if conditions

(1) Measurability: $\mu(x)$ and $\sigma(x)$ are L^2-measurable functions regarding the variable x;

(2) Lipschitz condition: there exists a constant $K>0$ such that

$$|\mu(x)-\mu(y)|\leq K\,|x-y|\,,$$

$$|\sigma(x)-\sigma(y)|\leq K\,|x-y|\,,$$

for all $x,\, y\in\mathbb{R}$;

(3) Linear growth bound: there exists a constant $K>0$ such that

$$|\mu(x)|^2\leq K^2\left(1+x^2\right),$$

$$|\sigma(x)|^2\leq K^2\left(1+x^2\right),$$

for all $x\in\mathbb{R}$;

(4) Initial value: random variable $X_0=Y$ is $\mathcal{F}_0$ measurable and $\mathbb{E}[X_0]^2<\infty$

are met, according to (Kloeden & Platen 1995, Theorem 4.5.3) SDE (3.14) has a pathwise unique strong solution $\{X(t),\, 0\leq t\leq T\}$. Moreover, if the coefficients $\mu(x)$ and $\sigma(x)$ in the SDE (3.14) are continuous and conditions $(2)-(4)$ hold, solution $\{X(t),\, 0\leq t\leq T\}$ of the SDE (3.14) for any fixed initial value X_0 is a diffusion process on $[0,\, T]$ with drift parameter $\mu(x)$ and diffusion parameter $\sigma(x)$ (Kloeden & Platen 1995, Theorem 4.6.1).

In general conditions $(1)-(4)$ are strong and can be weakened. In particular, condition (2) can be weakened in a sense that it holds for possibly different constants K_N for $|x|\,,\,|y|\leq N$ and for each $N>0$. This enlarges the class of admissible parameters since for parameters in C^1 the Mean value theorem implies this weakened condition. Since Pearson diffusions which we consider in this thesis satisfy these (weakened) conditions we won't go further into details (for more information see Kloeden & Platen (1995)).

Furthermore, if the solution of SDE (3.14) X has invariant density $f(x)$ and initial value X_0 has the same density $f(x)$, then the solution X is a strictly stationary time-homogenous Markov process.
On the other hand, weak solution of the SDE (3.14) is a solution for which the coefficients $\mu(x)$ and $\sigma^2(x)$, but not the Brownian motion $\{W(t),\, t \geq 0\}$, are specified. This provides equivalence in probability law of the solutions, but not the sample path equivalence. This kind of a solution exist e.g. if conditions $(1) - (3)$ are met and of course, every strong solution is also a weak solution, but not visa versa. Moreover, usage of weak solutions is justified if one wants to determine only distributional characteristics of the solutions, such as probability density, moments and covariance structure. In this thesis, both distributional and sample path properties are studied and therefore, when a solution of SDE is mentioned, strong solution is assumed.

3.1.3 Classification of boundaries of diffusion state space

Scale and speed densities (3.10) and (3.13) are closely related to the behaviour of a diffusion at the corresponding boundaries. In general, there are three important classification schemes for boundaries l and r of the diffusion state space:

- the so-called Feller's classification scheme, where the central point of view is attainability and possibility of starting the diffusion from a specific boundary (for details see Karlin & Taylor (1981a), Linetsky (2004));
- the so called oscillatory/non-oscillatory (O/NO) classification scheme, where the central point of view is oscillation of zeros of solutions of the corresponding Sturm-Liouville equation (3.22) in the neighborhood of the boundaries (for details see Linetsky (2004) and Fulton et al. (2005));
- the so-called Weyl's limit-point/limit-circle classification scheme, where the central point of view is the square integrability of solutions of the Sturm-Liouville equation in the neighborhood of the boundaries (for details see Fulton et al. (2005)).

In this thesis, relevant classification schemes are Feller's and O/NO classification scheme and therefore we give a brief overview of these schemes.

Feller's classification scheme

The speed measure, closely related to the speed density (3.13) is defined via

$$M[x,\, y] = \int_x^y \mathfrak{m}(z)\, dz, \;\; [x,\, y] \subset \langle l,\, r\rangle\,,$$

while scale measure, closely related to the scale density (3.10) is defined via

$$S[x,\, y] = \int\limits_{x}^{y} \mathfrak{s}(z)\, dz,\;\; [x,\, y] \subset \langle l,\, r\rangle\, .$$

Next, related to the speed measure, for arbitrary $\epsilon \in \langle l,\, r\rangle$ let

$$\Sigma_l = \int\limits_{l}^{\epsilon} M[z,\, \epsilon]\, \mathfrak{s}(z)\, dz, \quad \Sigma_r = \int\limits_{\epsilon}^{r} M[\epsilon,\, z]\, \mathfrak{s}(z)\, dz.$$

Boundary $e \in \{l, r\}$ is said to be attainable if $\Sigma_e < \infty$ and unattainable if $\Sigma_e = \infty$. Moreover, left boundary l is said to be attracting if $S\,\langle l,\, y] = \lim_{x\downarrow l} S[x,\, y] < \infty$ independently of $y \in \langle l, r\rangle$, while right boundary l is said to be attracting if $S\,[x,\, r\rangle = \lim_{y\uparrow r} S[x,\, y] < \infty$ independently of $x \in \langle l, r\rangle$. According to (Karlin & Taylor 1981a, Lemma 6.3)

- $S\,\langle l,\, y] = \infty$ implies $\Sigma_l = \infty$, i.e. if the left boundary is not attracting, it is not attainable;
- $\Sigma_l < \infty$ implies $S\,\langle l,\, y] < \infty$, i.e. attainable left boundary is always attracting;
- $S\,[x,\, r\rangle = \infty$ implies $\Sigma_r = \infty$, i.e. if the right boundary is not attracting, it is not attainable;
- $\Sigma_r < \infty$ implies $S\,[x,\, r\rangle < \infty$, i.e. attainable right boundary is always attracting.

Let

$$I_l = \int\limits_{l}^{\epsilon} S\,\langle l,\, z]\, \mathfrak{m}(z)\, dz, \quad I_r = \int\limits_{\epsilon}^{r} S\,[z,\, r\rangle\, \mathfrak{m}(z)\, dz,$$

$$J_l = \int\limits_{l}^{\epsilon} S\,[z,\, \epsilon]\, \mathfrak{m}(z)\, dz, \quad J_r = \int\limits_{\epsilon}^{r} S\,[\epsilon,\, z]\, \mathfrak{m}(z)\, dz,$$

where ϵ is arbitrary point from the diffusion state space I. According to Feller's boundary classification scheme, the boundary $e \in \{l, r\}$ is said to be:

- regular if $I_e < \infty$ and $J_e < \infty$ (the diffusion process can both enter and leave from the regular boundary);
- entrance if $I_e = \infty$ and $J_e < \infty$ (the entrance boundary can never be reached from the interior of the state space);
- exit if $I_e < \infty$ and $J_e = \infty$ (such boundary is always an absorbing point for the diffusion process);
- natural if $I_e = \infty$ and $J_e = \infty$ (the diffusion process cannot start from such a boundary nor reach it in a finite expected time).

For details, see Karlin & Taylor (1981a) and references therein. Next, before we give a short overview of O/NO classification scheme we need to define the so called infinitesimal generator.

3.1.4 Infinitesimal generator

In general, the theory of semigroups gives an elegant treatment of various differential equations, including time-fractional differential equations. For an overview on theory of semigroups we refer to Pazy (1983). Let $\mathbb{B}$ be a Banach space. A family of linear operators $\{T_t,\, t \geq 0\}$ on a Banach space $\mathbb{B}$ is called a semigroup if

$$T_0 f = f, \forall f \in \mathbb{B}, \quad T_{t+s} = T_t T_s.$$

Moreover, we say T_t is

- bounded if, for each $t \geq 0$ there exists some $M_t > 0$ such that

$$\|T_t f\| \leq M_t \|f\|, \; \forall f \in \mathbb{B},$$

- strongly continuous if $\lim_{t\to 0} \|T_t f - f\| = 0, \; \forall f \in \mathbb{B}$.

The generator of the semigroup T_t is a linear operator defined by

$$\mathcal{L} f(x) = \lim_{t\to 0} \frac{T_t f(x) - T_0 f(x)}{t - 0} = \lim_{t\to 0} \frac{T_t f(x) - f(x)}{t}, \tag{3.16}$$

which can be seen as the abstract derivative of the semigroup evaluated at $t = 0$. The limit (3.16) is of course taken in the underlying Banach space norm. If T_t is a strongly continuous, bounded semigroup (C_0 semigroup), the generator (3.16) always exists and the corresponding domain is

$$D(\mathcal{L}) = \{f \in \mathbb{B} : \mathcal{L} f \text{ exists}\}.$$

According to (Pazy 1983, Corollary I.2.5), $D(\mathcal{L})$ is dense in the underlying Banach space $\mathbb{B}$, meaning that for any $f \in \mathbb{B}$ we can find an approximating sequence of functions $f_n \in D(\mathcal{L})$ such that $\lim_{n\to\infty} \|f_n - f\| = 0$.

In the theory of diffusion processes, a particular semigroup plays a central role. Suppose we work on the space of real valued, bounded and continuous functions on I, denoted by $\mathcal{C}_b(I)$, equipped with the norm $\|f\|_\infty = \sup_{x\in I} |f(x)|$. For the diffusion process $\{X(t),\, t \geq 0\}$ with state space $I = \langle l,\, r\rangle$, infinitesimal parameters $\mu(x)$ and $\sigma(x)$ and transition density

$$p(x, t; y) = \frac{\partial}{\partial x} P(X(t) \leq y \,|\, X(0) = x),$$

the family of transition operators $\{T_t,\ t \geq 0\}$, where

$$T_t f(x) = \int_l^r f(y)p(x,t;y)\,dy = \mathbb{E}[f(X(t)) \,|\, X(0) = x] \tag{3.17}$$

forms a C_0 semigroup. Moreover, its a contraction, i.e. $\|T_t f\|_\infty \leq \|f\|_\infty$, $\forall t \geq 0$. In particular, generator of the semigroup (3.17)

$$\hat{\mathcal{G}}f(x) = \lim_{t \downarrow 0} \frac{T_t f(x) - f(x)}{t}, \tag{3.18}$$

is called the infinitesimal generator. Reason for the name becomes clear at once since:

$$\mathbb{E}[f(X(h)) - f(X(0)) \,|\, X(0) = x] = T_h f(x) - f(x) = h\hat{\mathcal{G}}f(x) + o(h).$$

Therefore, infinitesimal generator describes the expected movement of the diffusion process in an infinitesimal time interval. Moreover, infinitesimal generator is a closed, negative semidefinite, self-adjoint linear operator. The self-adjointness property is provided under certain boundary conditions regarding asymptotic behavior near boundaries l and r of the diffusion state space. Therefore, according to McKean (1956), domain of the infinitesimal generator is

$$D(\hat{\mathcal{G}}) = \{f \in \mathcal{C}_b(I) \cap C^2(I) : \hat{\mathcal{G}}f \in \mathcal{C}_b(I) \text{ and } f \text{ satisfies boundary conditions at } l \text{ and } r\},$$

where $C^2(I)$ denotes the space of twice continuously differentiable functions on I, while the appropriate boundary conditions at boundary $e \in \{l, r\}$ are:

- if e is regular or exit the boundary condition is given by

$$\lim_{x \to e} f(x) = 0; \tag{3.19}$$

- if e is entrance the boundary condition is given by

$$\lim_{x \to e} \frac{f'(x)}{\mathfrak{s}(x)} = 0; \tag{3.20}$$

- if e is natural, there are no boundary conditions needed as long as $f,\ \hat{\mathcal{G}}f \in \mathcal{C}_b(I)$.

In many situations a more suitable underlying Hilbert space is L^2 space. In particular, for $L^2(I,\, \mathfrak{m})$, the space of all real-valued functions on I which are square integrable with respect to the speed density $\mathfrak{m}(x)$ and with the corresponding inner product

$$(f, g) = \int_l^r f(x)g(x)\mathfrak{m}(x)\,dx,$$

C_0 semigroup (3.17) restricted to $\mathcal{C}_b(I) \cap L^2(I,\, \mathfrak{m})$ extends uniquely to a C_0 semigroup in $L^2(I,\, \mathfrak{m})$ with the corresponding infinitesimal generator $\mathcal{G}$ (see McKean (1956) for details). Moreover, the new infinitesimal generator $\mathcal{G}$ is closed, generally unbounded, negative semidefinite, self-adjoint differential operator with domain

$$D(\mathcal{G}) = \{f \in L^2(I,\, \mathfrak{m}) \cap C^2(I) : \mathcal{G}f \in L^2(I,\, \mathfrak{m}) \text{ and } f \text{ satisfies boundary conditions at } l \text{ and } r\},$$

where the corresponding boundary conditions at boundary $e \in \{l, r\}$ are:

- if e is regular or exit boundary, the boundary condition is given by (3.19);
- if e is entrance, the boundary condition is given by (3.20);
- if e is natural boundary, then depending on the integrability of the speed density $\mathfrak{m}(x)$ we have:
  - if the speed density $\mathfrak{m}(x)$ is integrable near e, i.e. if

    $$\left| \int_e^{\epsilon} \mathfrak{m}(x)\, dx \right| < \infty, \;\; \epsilon \in I,$$

    then the boundary condition at e is given by (3.20);
  - if the speed density $\mathfrak{m}(x)$ is not integrable near boundary e, then the square integrability of f with respect to speed density $\mathfrak{m}$ near e, i.e.

    $$\left| \int_e^{\epsilon} f^2(x)\, \mathfrak{m}(x)\, dx \right| < \infty, \;\; \epsilon \in I,$$

    implies that the boundary condition at e is given by (3.19).

In this thesis we work with only one-dimensional diffusions, for which the following explicit representation of $\mathcal{G}$ in terms of the corresponding infinitesimal parameters or scale and speed density

$$\mathcal{G}f(x) = \mu(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x) = \frac{1}{\mathfrak{m}(x)} \left(\frac{f'(x)}{\mathfrak{s}(x)} \right)', \;\; x \in I, \tag{3.21}$$

is valid as long as $\mu \in C^1(I)$ and $\sigma \in C^2(I)$, $\sigma(x) > 0$, $\forall x \in I$. Moreover, knowing the nature of the spectrum of infinitesimal generator $\mathcal{G}$ is very important. In particular, one of the central objects for the diffusion process is the corresponding transition density, which is unfortunately rarely known in the explicit form, but if the qualitative nature of the spectrum of infinitesimal generator $\mathcal{G}$ is known, in some cases one can obtain the spectral representation of the transition density. This spectral representation is given according to the decomposition of the spectrum of the infinitesimal generator $\mathcal{G}$, on the discrete and

the essential part. This is closely related to the spectral theory.
Recall, a family of bounded linear operators $\{\varepsilon(B),\, B \in \mathfrak{B}(\mathbb{R})\}$ with underlying separable Hilbert space $\mathcal{H}$ and Borel σ-field $\mathfrak{B}(\mathbb{R})$ is called spectral measure, projection-valued measure or resolution of the identity if

- every operator $\varepsilon(B)$ is an orthogonal projector, i.e. $\varepsilon^2(B) = \varepsilon(B)$ and $\varepsilon^*(B) = \varepsilon(B)$;
- $\varepsilon(\emptyset) = 0$, $\varepsilon(\mathbb{R}) = I$ where I here denotes identity operator on Hilbert space $\mathcal{H}$;
- $\varepsilon(B) = \sum\limits_{n=1}^{\infty} \varepsilon(B_n)$, $B_i \cap B_j = \emptyset$, $i \neq j$, $B = \cup_{i=1}^{\infty} B_n$;
- $\varepsilon(B_1 \cap B_2) = \varepsilon(B_1)\,\varepsilon(B_2)$.

Moreover, there is a one-one correspondence between self-adjoint operators and spectral measures, i.e.

$$H \mapsto \{\varepsilon(B),\, B \in \mathfrak{B}(\mathbb{R})\},$$
$$\{\varepsilon(B),\, B \in \mathfrak{B}(\mathbb{R})\} \mapsto H = \int_{\mathbb{R}} \lambda \varepsilon(d\lambda),$$

where H is a self-adjoint operator on the Hilbert space $\mathcal{H}$ and $\{\varepsilon(B),\, B \in \mathfrak{B}(\mathbb{R})\}$ is the spectral measure. In general, real number λ is an element of the spectrum $\sigma(H)$ of a self-adjoint operator H if and only if

$$\varepsilon(\langle \lambda - \delta,\, \lambda + \delta \rangle) \neq 0,\ \ \forall \delta > 0.$$

In particular, we say $\lambda \in \sigma(H)$ is

- in the discrete spectrum of H, denoted by $\sigma_d(H)$, if and only if the range of $\varepsilon(\langle \lambda - \delta,\, \lambda + \delta \rangle)$ is finite-dimensional for some $\delta > 0$;
- in the essential spectrum of H, denoted by $\sigma_e(H)$, if and only if the range of $\varepsilon(\langle \lambda - \delta,\, \lambda + \delta \rangle)$ is infinite-dimensional for all $\delta > 0$.

Therefore, this gives a decomposition of the spectrum into two disjoint parts:

$$\sigma(H) = \sigma_d(H) \cup \sigma_e(H),$$

where $\sigma_d(H)$ is not necessarily closed, but $\sigma_e(H)$ is always closed (for details on spectral theory we refer to Reed & Simon (1980) and Linetsky (2004)).

Oscillatory/non-oscillatory classification scheme

O/NO classification scheme is primarily concerned in determining oscillation of zeros of the solutions of Sturm-Liouville (SL) equation near boundaries l and r of the diffusion state space $I = \langle l,\, r \rangle$:

$$(-\mathcal{G})f(x) = \lambda f(x), \tag{3.22}$$

where $\lambda \geq 0$ is a real spectral parameter and $\mathcal{G}$ is the infinitesimal generator (3.21). On the other hand, differential operator $(-\mathcal{G})$ is called Sturm-Liouville operator and therefore the name for the equation.

For a given real parameter $\lambda \geq 0$, the equation (3.22) is called oscillatory (O) at boundary $e \in \{l, r\}$ if and only if every solution has infinitely many zeros clustering at e, otherwise it is called non-oscillatory (NO) at boundary e. This classification is mutually exclusive for a fixed λ, but can vary for different values of λ. According to (Linetsky 2004, Theorem 1.) or (Fulton et al. 2005, Theorem 1.), a boundary $e \in \{l, r\}$ of the SL equation (3.22) belongs to one and only one of the following two classes:

- equation (3.22) is NO at boundary e for all real λ and e is called NO boundary.
- there exists a real number $\Lambda \geq 0$ such that the equation (3.22) is O at boundary e for all $\lambda > \Lambda$ and NO at e for all $\lambda \leq \Lambda$. Therefore, boundary e is called O/NO boundary with cutoff Λ. Moreover, O/NO boundary e can be either O or NO for $\lambda = \Lambda > 0$, and it is always NO for $\lambda = \Lambda = 0$.

Based on the O/NO classification scheme, spectrum of the SL operator $(-\mathcal{G})$ (and therefore the spectrum of the infinitesimal generator $\mathcal{G}$) can be classified into three possible categories (see (Linetsky 2004, Theorem 2.)):

(i) Spectral category I.
If both the left and right boundary l and r of the equation (3.22) are NO, then the spectrum of SL operator $(-\mathcal{G})$ is simple, non-negative and purely discrete.

(ii) Spectral category II.
If one of the boundaries of the equation (3.22) is NO and the other is O/NO with cutoff $\Lambda \geq 0$, then the spectrum of SL operator $(-\mathcal{G})$ is simple and non-negative. The essential spectrum $\sigma_e(-\mathcal{G}) \subset [\Lambda, \infty\rangle$ is nonempty and Λ is the lowest point of the essential spectrum. If the equation (3.22) is NO at the O/NO boundary for $\lambda = \Lambda \geq 0$ then there is a finite set of simple eigenvalues in $[0, \Lambda]$ (it may be empty). On the other hand, if the equation (3.22) is O at the O/NO boundary for $\lambda = \Lambda \geq 0$, then there is an infinite sequence of simple eigenvalues in $[0, \Lambda\rangle$ clustering at Λ.

(iii) Spectral category III.
If for the equation (3.22), left boundary l is O/NO with cutoff $\Lambda_1 \geq 0$ and right boundary r is O/NO with cutoff $\Lambda_2 \geq 0$, then the essential spectrum $\sigma_e(\mathcal{G}) \subset [\min\{\Lambda_1, \Lambda_2\}, \infty\rangle$ is nonempty and $\min\{\Lambda_1, \Lambda_2\}$ is the lowest point of the essential spectrum. Below the $\max\{\Lambda_1, \Lambda_2\}$, spectrum is simple (has multiplicity one), while above $\max\{\Lambda_1, \Lambda_2\}$, spectrum is not simple (has multiplicity two). If the equation (3.22) is NO for $\lambda = \min\{\Lambda_1, \Lambda_2\} \geq 0$, then there is a finite set of simple eigenvalues in $[0, \min\{\Lambda_1, \Lambda_2\}]$ (it may be empty). On the other hand, if the equation (3.22) is

O for $\lambda = \min\{\Lambda_1, \Lambda_2\} > 0$, then there is an infinite sequence of simple eigenvalues in $[0, \min\{\Lambda_1, \Lambda_2\})$ clustering at $\min\{\Lambda_1, \Lambda_2\}$.

Moreover, according to Linetsky (2004), there is a connection between Feller's and O/NO classification schemes. In particular, regular, entrance and exit boundaries are always NO, while natural boundaries can be either NO or O/NO with cutoff $\Lambda \geq 0$. In order to properly connect natural boundaries from Feller's classification scheme and NO or O/NO boundaries, SL equation (3.22) is usually transformed into a more suitable Liouville normal form or one-dimensional Schrödinger equation:

$$-g''(u) + Q(u)g(u) = \lambda g(u), \tag{3.23}$$

where $Q(u)$ is called the potential function and it is defined by the expression

$$Q(u) = \frac{h''(u)}{h(u)},$$

with

$$u = u(x) = \int \sqrt{\mathfrak{m}(x)\mathfrak{s}(x)}\,dx,\ \ h(u) = \sqrt[4]{\frac{\mathfrak{m}(x(u))}{\mathfrak{s}(x(u))}},\ \ x(u) = u^{-1}(x),$$

and $g(u) = h(u)f(x(u))$, where $f(x)$ is the solution of the corresponding SL equation (3.22). This transformation can be carried though, provided that infinitesimal parameters $\mu \in C^1(I)$, $\sigma \in C^2(I)$. According to Fulton et al. (2005), O/NO classification scheme is invariant to the presented Liouville transformation, i.e. O/NO classification of boundaries of equations (3.22) and (3.23) are the same. Moreover, (Linetsky 2004, Theorem 3.) provides sufficient conditions in order to describe and connect natural boundaries with NO and O/NO boundaries.

3.1.5 Spectral representation of transition density

Spectral category I.

As explained in the previous section, in this category, when there is no O/NO boundaries, the spectrum of Sturm-Liouville operator $(-\mathcal{G})$ is simple, non-negative and purely discrete, consisting of infinite sequence of eigenvalues

$$\{\lambda_n,\, n \in \mathbb{N}_0\},\ \ \lambda_0 > \lambda_1 > \lambda_2 > \cdots,\ \ \lim_{n\to\infty} \lambda_n = \infty.$$

Corresponding eigenfunctions

$$\{\varphi_n(x),\, n \in \mathbb{N}_0\},$$

i.e. solutions of the corresponding eigenvalue problem for the SL operator $(-\mathcal{G})$

$$(-\mathcal{G})\varphi_n(x) = \lambda_n \varphi_n(x),$$

are assumed to be normalized with respect to the speed density $\mathfrak{m}(x)$:

$$\int_l^r \varphi_n(x)\varphi_m(x)\mathfrak{m}(x)\, dx = \delta_{mn},$$

where δ_{mn} is the standard Kronecker's symbol. Now, spectral representation of the transition density $p(x,t;y)$ of the diffusion process belonging to this spectral category is given by

$$p(x,t;y) = \mathfrak{m}(x) \sum_{n=0}^{\infty} e^{-\lambda_n} \varphi_n(x)\varphi_n(y), \;\; x, y \in I, \;\; t \geq 0. \tag{3.24}$$

For more details and examples of spectral representations of the transition densities of the diffusion processes belonging to spectral category I. see (Linetsky 2004, Section 5.1) and (Karlin & Taylor 1981*a*, Section 15.13.).

Spectral category II.

Let us assume that left boundary l is NO and the right boundary r O/NO natural boundary with cutoff $\Lambda \geq 0$ of the SL equation (3.22). Recall the structure of the spectrum of the SL operator $(-\mathcal{G})$:

$$\sigma(-\mathcal{G}) = \sigma_d(-\mathcal{G}) \cup \sigma_e(-\mathcal{G}),$$

where $\sigma_d(-\mathcal{G}) \subset [0,\, \Lambda\rangle$ is the discrete spectrum, while $\sigma_e(-\mathcal{G}) = [\Lambda,\, \infty\rangle$ is the essential spectrum of the SL operator $(-\mathcal{G})$. In particular, if the right boundary r is NO for $\lambda = \Lambda \geq 0$, then the SL operator $(-\mathcal{G})$ has a finite set of simple eigenvalues in $[0,\, \Lambda]$ (it may be empty) and if right boundary r is O for $\lambda = \Lambda \geq 0$, then the SL operator $(-\mathcal{G})$ has an infinite sequence of simple eigenvalues in $[0,\, \Lambda\rangle$ clustering near Λ.
Now, spectral representation of the transition density $p(x,t;y)$ of the diffusion process belonging to this spectral category is given by

$$p(x,t;y) = \mathfrak{m}(x) \left(\sum_{n=0}^{N} e^{-\lambda_n} \varphi_n(x)\varphi_n(y) + \int\limits_{\Lambda}^{\infty} e^{-\lambda t} \varphi(x,\lambda)\varphi(y,\lambda) d\rho_{ac}(\lambda) \right), \;\; x, y \in I, \;\; t \geq 0, \tag{3.25}$$

where

- λ_n, $n \in \{0, 1, 2, \ldots, N\}$ are eigenvalues and $\varphi(n)$, $n \in \{0, 1, 2, \ldots, N\}$ are the corresponding eigenfunctions normalized with respect to the speed density $\mathfrak{m}(x)$;
- $\varphi(x, \lambda)$ is non-trivial solution of the SL equation (3.22) which is square-integrable with respect to the speed density $\mathfrak{m}(x)$ in the neighborhood of the left boundary l for all $\lambda \geq 0$, satisfies the appropriate boundary condition at left boundary l and

such that $\varphi(x,\lambda)$ and $\varphi'(x,\lambda)$ are continuous functions of both variable x and λ, as well as entire in λ for each fixed $x \in I$, while $\rho_{ac}(\lambda)$ is the spectral function which is absolutely continuous on $\langle \Lambda,\, \infty \rangle$ and normalized relative to $\varphi(x,\lambda)$.

For details, we refer to (Linetsky 2004, Section 5.2.).

Spectral category III.

Let us assume that left boundary l is O/NO natural boundary with cutoff $\Lambda_1 \geq 0$ and right boundary r is O/NO natural boundary with cutoff $\Lambda_2 \geq 0$ of the SL equation (3.22), and without loss of generality let us assume $\Lambda_1 < \Lambda_2$. Recall the structure of the spectrum of the SL operator $(-\mathcal{G})$:

$$\sigma(-\mathcal{G}) = \sigma_d(-\mathcal{G}) \cup \sigma_e(-\mathcal{G}),$$

where $\sigma_d(-\mathcal{G}) \subset [0,\, \Lambda_1]$ is the discrete spectrum, while $\sigma_e(-\mathcal{G}) = [\Lambda_1,\, \infty\rangle$ is the essential spectrum of the SL operator $(-\mathcal{G})$. In particular if the SL equation (3.22) is NO for $\lambda = \Lambda_1$, then there is a finite set of simple eigenvalues in $[0,\, \Lambda_1]$ (it may be empty) and if the SL equation (3.22) is O for $\lambda = \Lambda_1$, then there is an infinite sequence of simple eigenvalues in $[0,\, \Lambda_1\rangle$ clustering near Λ_1. Below the Λ_2, spectrum is simple (has multiplicity one), while above Λ_2, spectrum is not simple (has multiplicity two). Now, spectral representation of the transition density $p(x,t;y)$ of the diffusion process belonging to this spectral category is given by

$$\begin{aligned} p(x,t;y) =& \mathfrak{m}(x) \sum_{n=0}^{N} e^{-\lambda_n} \varphi_n(x)\varphi_n(y) + \mathfrak{m}(x) \int_{\Lambda_1}^{\Lambda_2} e^{-\lambda t} \varphi(x,\lambda)\varphi(y,\lambda) d\rho_{ac}(\lambda) + \\ &+ \mathfrak{m}(x) \int_{\Lambda_2}^{\infty} e^{-\lambda t} \sum_{i,j=1}^{2} f_i(x,\lambda) f_j(y,\lambda) d\rho_{ac,\,ij}(\lambda), \;\; x,y \in I, \;\; t \geq 0, \end{aligned} \tag{3.26}$$

where

- λ_n, $n \in \{0,1,2,\ldots,N\}$ are eigenvalues and $\varphi(n)$, $n \in \{0,1,2,\ldots,N\}$ are the corresponding eigenfunctions normalized with respect to the speed density $\mathfrak{m}(x)$;
- $\lambda \in \sigma_e(-\mathcal{G})$, $\Lambda_1 < \lambda < \Lambda_2$ and $\varphi(x,\lambda)$ is the corresponding solution of the SL equation (3.22), while $\rho_{ac}(\lambda)$ is the spectral function which is continuous on this part of the spectrum and normalized relative to $\varphi(x,\lambda)$ with respect to the speed density $\mathfrak{m}(x)$;
- $\lambda \in \sigma_e(-\mathcal{G})$, $\Lambda_2 < \lambda$ and $f_1(x,\lambda)$ and $f_2(x,\lambda)$ are the solutions of the initial boundary value problem

$$(-\mathcal{G}) f(x,\lambda) = \lambda f(x,\lambda), \;\; \lambda \geq 0, \;\; x \in I,$$

which satisfy the corresponding initial conditions

$$f_1(x_0,\lambda)=1,\ \ \frac{f_1'(x_0,\lambda)}{\mathfrak{s}(x_0)}=0,\ \ f_2(x_0,\lambda)=0,\ \ \frac{f_2'(x_0,\lambda)}{\mathfrak{s}(x_0)}=1$$

for arbitrary $x_0 \in I$. The boundary conditions ensure that the Wronskian

$$W_\lambda(f_1,f_2)=f_1'(x,\lambda)f_2(x,\lambda)-f_1(x,\lambda)f_2'(x,\lambda)$$

of the solutions $f_1(x,\lambda)$ and $f_2(x,\lambda)$ equals one. These boundary conditions are related to the Weyl's limit-point/limit-circle classification scheme of the SL equation (3.22) and in some cases aren't necessary (see Fulton et al. (2005)).

For details, we refer to (Linetsky 2004, Section 5.3.).

3.2 Pearson diffusions

In this section we define the family of Pearson diffusions with overview of necesarry properties for further sections.

Recall that the family of all continuous distributions which satisfy the so-called Pearson differential equation

$$\frac{\mathfrak{g}'(x)}{\mathfrak{g}(x)}=\frac{c_0x+c_1}{b_2x^2+b_1x+b_0}=\frac{c(x)}{b(x)} \tag{3.27}$$

is called the Pearson family of distributions. In a series of papers (Pearson (1895), Pearson (1901) and Pearson (1916)) K. Pearson introduced continuous distributions which satisfy the Pearson differential equation (3.27). In particular, in Pearson (1895) Pearson identified four types of continuous distributions: type I (generalized beta distribution), type II (symmetric case of the generalized beta distribution), type III (gamma, chi-squared and exponential distributions) and type IV (Cauchy and skewed Student distribution), while Normal distribution was already known as type V. In the following paper, Pearson (1901) Pearson introduced the new type V (reciprocal gamma distribution) and type VI (Fisher-Snedecor and beta-prime distributions), while in the paper Pearson (1916) he introduced type VII (Student's T-distribution).

Based on the following continuous distributions: normal, gamma, beta, Fisher-Snedecor, reciprocal gamma and Student, which all satisfy the Pearson differential equation (3.27), we will define the so-called Pearson diffusions with these invariant distributions. Moreover, according to the tail behaviour of marginal distribution, we will classify them as either heavy-tailed, or non-heavy-tailed Pearson distributions. In general, we say that a random variable X has a heavy-tailed distribution if

$$e^{\gamma x}P(X>x)\to\infty,\ \ x\to\infty,\ \ \gamma>0,$$

i.e. if the distribution of the tail is not exponentially bounded. In particular:

- normal, gamma and beta distributions are non-heavy-tailed;
- Fisher-Snedecor, reciprocal gamma and Student distributions are heavy-tailed.

Connection between Pearson family of distributions and corresponding time-homogenous diffusions was first established by A. Kolmogorov in his paper Kolmogorov (1931). In particular, Kolmogorov observed the Fokker-Planck equation (now known as Kolmogorov forward equation):

$$\frac{\partial p(x,t;y)}{\partial t} = -\frac{\partial}{\partial x}\left(A(x)p(x,t;y)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(B(x)p(x,t;y)\right)$$

with corresponding linear drift and at most quadratic squared diffusion parameter, i.e. with $A(x) = a_0 + a_1 x$ and $B(x) = B_0 + B_1 x + B_2 x^2$. Moreover, if there is a unique solution $\mathfrak{g}(x)$ of the time independent Fokker-Planck equation

$$-\frac{d}{dx}\left(A(x)\mathfrak{g}(x)\right) + \frac{1}{2}\frac{d^2}{dx^2}\left(B(x)\mathfrak{g}(x)\right) = 0,$$

then it must be the invariant distribution of the diffusion process with linear drift $A(x)$ and diffusion parameter $\sqrt{B(x)}$. According to Kolmogorov (1931), the time independent Fokker-Planck equation reduces to

$$\frac{\mathfrak{g}'(x)}{\mathfrak{g}(x)} = \frac{A(x) - B'(x)}{B(x)} = \frac{(A_1 - 2B_2)x + (A_0 - B_1)}{B_0 + B_1 x + B_2 x^2} \tag{3.28}$$

which is clearly differential equation of Pearson type, i.e. of type (3.27). Polynomials which appear in numerator and denominator in ODE (3.28) are fully determined by the linear drift $A(x)$ and the diffusion parameter $\sqrt{B(x)}$. Therefore, this provides one-to-one correspondence between infinitesimal parameters of the diffusion process and Pearson differential equation. Based on this one-to-one correspondence Kolmogorov (Kolmogorov (1931)) defined the class of diffusion processes which have invariant distributions from Pearson family and today, this class of diffusions is known as the class of Pearson diffusions.

The modern definition of Pearson diffusions is provided by Forman & Sørensen (2008), where Pearson diffusion $\{X(t),\, t \geq 0\}$ is defined as a unique ergodic and stationary solution of the stochastic differential equation

$$dX(t) = -\theta(X(t) - \mu) + \sqrt{2\theta k(b_2 X^2(t) + b_1 X(t) + b_0)}\, dW(t),\ k > 0,\ t \geq 0, \tag{3.29}$$

where

- the drift parameter $\mu(x) = -\theta(x - \mu)$ coincides with $A(x)$ in the equation (3.28);

- the squared diffusion parameter $\sigma^2(x) = 2\theta k(b_2x^2 + b_1x + b_0) = 2\theta kb(x)$ coincides with $B(x)$ in the equation (3.28);
- $\theta > 0$ is a scaling of time parameter determining how fast the diffusion process $\{X(t),\, t \geq 0\}$ moves to the corresponding mean;
- parameter $\mu \in \mathbb{R}$ is the mean of the invariant distribution;
- parameters $\mu, b_0,\, b_1,\, b_2 \in \mathbb{R}$ determine the state space of the diffusion, as well as the shape of the invariant distribution. In particular, the state space $I = \langle l,\, r\rangle$ of the Pearson diffusion is defined so that $\sigma^2(x) = 2\theta k(b_2x^2 + b_1x + b_0) > 0$ for all $x \in I$. Moreover, parameters b_0, b_1 and b_2 are not all simultaneously equal to zero.

Corresponding scale and speed densities are given via

$$\mathfrak{s}(x) = \exp\left\{\int^{x} \frac{y-\mu}{k(b_2y^2 + b_1y + b_0)}\, dy\right\}, \quad \mathfrak{m}(x) = \frac{1}{\theta\, k\, \mathfrak{s}(x)(b_2x^2 + b_1x + b_0)},$$

and since the invariant distribution $\mathfrak{g}(x)$ is proportional to the speed density $\mathfrak{m}(x)$, it follows that $\mathfrak{g}(x)$ satisfies Pearson differential equation (3.28) with $A_0 = \theta\mu$, $A_1 = -\theta$, $B_2 = 2\theta kb_2$, $B_1 = 2\theta kb_1$ and $B_0 = 2\theta kb_0$, i.e. it satisfies ODE

$$\frac{\mathfrak{g}'(x)}{\mathfrak{g}(x)} = -\frac{(k^{-1} + 2b_2)x + (b_1 - \mu k^{-1})}{b_2x^2 + b_1x + b_0}.$$

Notice that the diffusion process which satisfies SDE (3.29) has invariant distribution $\mathfrak{g}(x)$, which satisfies Pearson differential equation (3.28) and therefore is in agreement with the Kolmogorov original definition (Kolmogorov (1931)).
According to Bibby et al. (2005), based on a desired marginal distribution and the correlation structure, one can construct the corresponding diffusion process. In particular, if a probability density function $\mathfrak{g}(x)$ satisfies the condition

(A) $\mathfrak{g}(x)$ is continuous, bounded, positive on some interval $\langle l,\, r\rangle$, zero outside the interval, has expectation μ and finite variance,

then the following is valid:

1. If

$$\sigma^2(x) = \frac{2\theta}{\mathfrak{g}(x)} \int_{l}^{x} (\mu - y)\mathfrak{g}(y)\, dy,\ \ \theta > 0,\ \ \mu \in \mathbb{R},$$

   then $\sigma^2(x) > 0$, $\forall x \in \langle l,\, r\rangle$ and SDE

$$dX(t) = -\theta(X(t) - \mu) + \sigma(x)\, dW(t),\ \ k > 0,\ \ t \geq 0 \tag{3.30}$$

   has a unique weak mean-reverting Markovian solution;

2. The diffusion process $\{X(t),\, t \geq 0\}$ which solves SDE (3.30) has invariant density $\mathfrak{g}(x)$ and it is ergodic, i.e. it satisfies

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(X(t))\, dt = \int_l^r f(x)\mathfrak{g}(x)\, dx = \mathbb{E}[f(X(t))]$$

for all bounded measurable functions $f : \mathbb{R} \to \mathbb{R}$;

3. The function $\sigma^2(x)$ is integrable with respect to the function $\mathfrak{g}(x)$, i.e.

$$\int_l^r \sigma^2(x)\mathfrak{g}(x)\, dx < \infty \tag{3.31}$$

and

$$\mathbb{E}[X(t+s)|X(s) = x] = xe^{-\theta t} + \mu(1 - e^{-\theta t}).$$

Moreover, if the inital random variable X_0 has probability density $\mathfrak{g}(x)$, then the diffusion process $\{X(t),\, t \geq 0\}$ is stationary and the autocorrelation function is given by

$$\mathrm{Corr}(X(s+t),\, X(s)) = e^{-\theta t},\;\; 0 \leq s \leq t < \infty;$$

4. If $-\infty < l$ or $r < \infty$ then the diffusion process which solves SDE (3.30) is the only ergodic diffusion with drift parameter $-\theta(x-\mu)$ and invariant density $\mathfrak{g}(x)$. On the other hand, if the state space of the diffusion is $\mathbb{R}$, it is the only ergodic diffusion with drift parameter $-\theta(x-\mu)$ and invariant density $\mathfrak{g}(x)$ for which the condition (3.31) is satisfied.

For the proof see (Bibby et al. 2005, Theorem 2.1.).

In general, any distribution $\mathfrak{p}(x)$ from the Pearson family satisfy the condition (A) (after imposing some parameter restrictions in the heavy-tailed cases, namely to ensure existence of mean and variance) so the diffusion process which solves (3.30) coincides with the diffusion process which solves SDE (3.29). Therefore, properties 1. – 4. are all valid for Pearson diffusions. Moreover, a stationary and ergodic weak solution of SDE (3.29) is established. In fact, we will show that Pearson diffusions can be established as strong solutions of SDE (3.29). This is necessary because beside distributional properties, we will investigate some sample path properties. Since every strong solution of a SDE is also a weak solution, all established properties will still be valid.

Furthermore, the class of Pearson diffusions is closed under translations and scale transformations, i.e. if $\{X(t),\, t \geq 0\}$ is a Pearson diffusion, then $\{\tilde{X}(t),\, t \geq 0\}$, where $\tilde{X}(t) = \alpha X(t) + \beta,\; \alpha \neq 0,\; \beta \in \mathbb{R}$, is also a Pearson diffusion. Therefore, Pearson diffusions can be categorized into six subfamilies, according to the degree of the polynomial $b(x)$ and, in the quadratic case $b(x) = b_2x^2 + b_1x + b_0$, according to the sign of its leading coefficient b_2 and the sign of its discriminant Δ (see e.g. Avram et al. (2012)):

- constant $b(x)$ - Ornstein-Uhlenbeck (OU) process with normal stationary distribution;
- linear $b(x)$ - Cox-Ingersoll-Ross (CIR) process with gamma stationary distribution;
- quadratic $b(x)$ with $b_2 < 0$ - Jacobi diffusion with beta stationary distribution;
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta > 0$ - Fisher-Snedecor (FS) diffusion with the Fisher-Snedecor stationary distribution;
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta = 0$ - reciprocal gamma (RG) diffusion with reciprocal gamma stationary distribution;
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta < 0$ - Student diffusion (ST) with the Student stationary distribution.

In the next two subsections, we review the most important properties of these subfamilies. According to the tail behaviour of invariant distribution of the corresponding Pearson diffusion, we further distinguish two major families of Pearson diffusions:

- non-heavy-tailed Pearson diffusions: OU, CIR and Jacobi diffusion;
- heavy-tailed Pearson diffusions: FS, RG, and Student diffusion.

Most recent development in general description, spectral properties and statistical inference of heavy-tailed Pearson diffusions has been done in a series of papers Avram et al. (2011, 2012, 2013*a*,*b*), Leonenko & Šuvak (2010*b*,*a*).

3.2.1 Non-heavy-tailed Pearson diffusions

For a general description and spectral properties of non-heavy-tailed Pearson diffusions we refer to (Karlin & Taylor 1981*a*, Section 15.13.), Wong (1964), Linetsky (2004) and Linetsky (2007). Moreover, see (Avram et al. 2012, Appendix 3.11.) for a list of their eigenvalues, eigenfunctions and spectral representations of transition densities.

Ornstein-Uhlenbeck process

Ornstein-Uhlenbeck process (Uhlenbeck & Ornstein (1930)) $\{X(t),\, t \geq 0\}$, also known as Vasicek model (Vasicek (1977)) is the solution of the SDE

$$dX(t) = -\theta(X(t) - \mu)dt + \sqrt{2\theta\sigma^2}dW(t), \quad \theta > 0,\, t \geq 0. \tag{3.32}$$

The corresponding invariant distribution is normal distribution with probability density function

$$\mathfrak{n}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \; x \in \mathbb{R}, \tag{3.33}$$

where $\mu \in \mathbb{R}$ is mean and $\sigma^2 > 0$ variance of the invariant distribution. Infinitesimal drift and diffusion parameters of OU process are given by

$$\mu(x) = -\theta(x-\mu), \quad \sigma(x) = \sqrt{2\theta\sigma^2}, \tag{3.34}$$

while infinitesimal generator is given by

$$\mathcal{G}f(x) = -\theta(x-\mu)f'(x) + \theta\sigma^2 f''(x), \; x \in \mathbb{R}. \tag{3.35}$$

The scale and speed densities of the OU process are

$$\mathfrak{s}(x) = e^{\frac{(x-\mu)^2}{2\sigma^2}}, \quad \mathfrak{m}(x) = \frac{1}{\theta\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \tag{3.36}$$

which gives alternative form of the infinitesimal generator (3.35)

$$\mathcal{G}f(x) = \theta\sigma^2 e^{\frac{(x-\mu)^2}{2\sigma^2}} \frac{d}{dx}\left(e^{-\frac{(x-\mu)^2}{2\sigma^2}} f'(x)\right), \; x \in \mathbb{R}.$$

Moreover, the state space of OU process is $I = \mathbb{R}$ with boundaries $l = -\infty$ and $r = \infty$. In particular, according to the Feller's and O/NO classification scheme, both boundaries are NO natural for all admissible values of parameters of the invariant density (3.33) and since the speed density $\mathfrak{m}(x)$ is integrable near both boundaries, the appropriate boundary conditions are

$$\lim_{x\to-\infty} \frac{f'(x)}{\mathfrak{s}(x)} = \lim_{x\to\infty} \frac{f'(x)}{\mathfrak{s}(x)} = 0. \tag{3.37}$$

Therefore, recall that the domain of the OU infinitesimal generator is

$$D(\mathcal{G}) = \{f \in L^2(I, \mathfrak{m}) \cap C^2(I) : \mathcal{G}f \in L^2(I, \mathfrak{m}) \text{ and } f \text{ satisfies boundary conditions (3.37)}\}.$$

Since both boundaries are NO natural, OU diffusion belongs to the spectral category I and therefore, the spectrum of the Sturm-Liouville operator $(-\mathcal{G})$ is simple, non-negative and purely discrete, consisting of infinite sequence of eigenvalues

$$\lambda_n = \theta n, \; n \in \mathbb{N}_0, \tag{3.38}$$

with corresponding eigenfunctions, normalized orthogonal Hermite polynomials $H_n(x)$, given by the Rodrigues formula

$$H_n(x) = (-1)^n \frac{\sigma^n}{\sqrt{n!}} e^{\frac{(x-\mu)^2}{2\sigma^2}} \frac{d^n}{dx^n} \left(e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right), \ n \in \mathbb{N}_0. \tag{3.39}$$

Ornstein-Uhlenbeck process is one of the few diffusion processes for which transition density is known in explicit form:

$$p(x,t;x_0) = \frac{1}{\sigma\sqrt{2\pi(1-e^{-2\theta t})}} \exp\left\{ - \frac{\left(x-\mu-(x_0-\mu)e^{-\theta t}\right)^2}{2\sigma^2(1-e^{-2\theta t})} \right\}. \tag{3.40}$$

Nevertheless, transition density $p(x,t;x_0)$ admits spectral representation, reflecting simple structure of the spectrum $\sigma(-\mathcal{G})$:

$$p(x,t;x_0) = \mathfrak{n}(x) \sum_{n=0}^{\infty} e^{-\lambda_n t} H_n(x) H_n(x_0), \quad x, x_0 \in \mathbb{R}, \tag{3.41}$$

where eigenvalues λ_n and corresponding eigenfunctions $H_n(x)$ are given by (3.38) and (3.39), respectively (cf. Karlin & Taylor (1981*a*), page 332. and Wong (1964)).

Cox-Ingersoll-Ross process

Cox-Ingersoll-Ross process (Cox et al. (1985)) $\{X(t),\, t \geq 0\}$, also known as the square-root diffusion (Feller (1951)), is the solution of the SDE

$$dX(t) = -\theta \left(X(t) - \frac{b}{a} \right) dt + \sqrt{\frac{2\theta}{a} X(t)} dW(t), \quad \theta > 0,\, t \geq 0. \tag{3.42}$$

The corresponding invariant distribution is gamma distribution with probability density function

$$\mathfrak{g}(x) = \frac{a^b}{\Gamma(b)} x^{b-1} e^{-ax} \mathrm{I}_{\langle 0,\infty \rangle}(x),\, x \in \mathbb{R}, \tag{3.43}$$

where $a > 0$ is scale and $b > 0$ is the shape parameter of the invariant distribution. Moreover, mean and variance are

$$\mathbb{E}[X_t] = \frac{b}{a}, \quad Var(X_t) = \frac{b}{a^2}, \ a > 0, \ b > 0. \tag{3.44}$$

Infinitesimal drift and diffusion parameters of CIR process are given by

$$\mu(x) = -\theta \left(x - \frac{b}{a} \right), \quad \sigma(x) = \sqrt{\frac{2\theta}{a} x}, \tag{3.45}$$

while infinitesimal generator is given by

$$\mathcal{G}f(x) = -\theta\left(x - \frac{b}{a}\right)f'(x) + \frac{\theta}{a}xf''(x),\ x \in \langle 0,\, \infty\rangle\,. \tag{3.46}$$

The scale and speed densities of the OU process are

$$\mathfrak{s}(x) = x^{-b}e^{ax}, \quad \mathfrak{m}(x) = \frac{a}{\theta}x^{b-1}e^{-ax}, \tag{3.47}$$

which gives alternative form of the infinitesimal generator (3.46)

$$\mathcal{G}f(x) = \frac{\theta}{a}x^{1-b}e^{ax}\frac{d}{dx}\left(x^b e^{-ax}f'(x)\right),\ x \in \langle 0,\, \infty\rangle\,.$$

Moreover, the state space of CIR process is $I = \langle 0,\, \infty\rangle$ with boundaries $l = 0$ and $r = \infty$. In particular, according to the Feller's classification scheme, left boundary 0 is regular for $b \in \langle 0,\, 1\rangle$, and entrance for $b \geq 1$, while right boundary ∞ is natural for $b > 0$. On the other hand, according to O/NO classification scheme both boundaries are NO for all $b > 0$. Since the speed density $\mathfrak{m}(x)$ is integrable near right boundary ∞, the appropriate boundary conditions are

- for $b \in \langle 0,\, 1\rangle$:
$$\lim_{x\to 0} f(x) = 0, \quad \lim_{x\to\infty}\frac{f'(x)}{\mathfrak{s}(x)} = 0; \tag{3.48}$$
- for $b \geq 1$:
$$\lim_{x\to 0}\frac{f'(x)}{\mathfrak{s}(x)} = \lim_{x\to\infty}\frac{f'(x)}{\mathfrak{s}(x)} = 0. \tag{3.49}$$

Therefore, recall that the domain of the CIR infinitesimal generator is (e.g. for $b \geq 1$)

$$D(\mathcal{G}) = \{f \in L^2(I,\, \mathfrak{m}) \cap C^2(I) : \mathcal{G}f \in L^2(I,\, \mathfrak{m}) \text{ and } f \text{ satisfies boundary conditions (3.49)}\}.$$

Since both boundaries are NO, CIR diffusion belongs to the spectral category I and therefore, the spectrum of the Sturm-Liouville operator $(-\mathcal{G})$ is simple, non-negative and purely discrete, consisting of infinite sequence of eigenvalues

$$\lambda_n = \theta n,\ n \in \mathbb{N}_0, \tag{3.50}$$

with corresponding eigenfunctions, normalized orthogonal Laguerre polynomials $L_n(x)$, given by the Rodrigues formula

$$L_n(x) = (-1)^n\sqrt{\frac{\Gamma(b)}{n!\Gamma(n+b)}}x^{1-b}e^{ax}\frac{d^n}{dx^n}\left(x^{n+b-1}e^{-ax}\right),\ n \in \mathbb{N}_0. \tag{3.51}$$

Just like Ornstein-Uhlenbeck process, Cox-Ingersoll-Ross process has transition density which is known in explicit form:

$$p(x,t;x_0) = \left(\frac{x}{x_0}\right)^{\frac{b-1}{2}} \frac{a}{(1-e^{-\theta t})\Gamma(b)} \exp\left\{\frac{\theta(b-1)t}{2} - ax - \frac{a(x+x_0)}{e^{\theta t}-1}\right\} I_{b-1}\left(\frac{a\sqrt{xx_0}}{\sinh(0.5\theta t)}\right), \tag{3.52}$$

where $I_{b-1}(\cdot)$ is the modified Bessel function of the first kind (see e.g. Olver et al. (2010)). Nevertheless, transition density $p(x,t;x_0)$ also admits spectral representation, reflecting simple structure of the spectrum $\sigma(-\mathcal{G})$:

$$p(x,t;x_0) = \mathfrak{g}(x) \sum_{n=0}^{\infty} e^{-\lambda_n t} L_n(x) L_n(x_0), \quad x, x_0 \in \langle 0, \infty\rangle, \tag{3.53}$$

where eigenvalues λ_n and corresponding eigenfunctions $L_n(x)$ are given by (3.50) and (3.51), respectively (cf. Karlin & Taylor (1981a), page 334. and Wong (1964)).

Jacobi diffusion

Jacobi diffusion $\{X(t),\, t \geq 0\}$ is the solution of the SDE

$$dX(t) = -\theta\left(X(t) - \frac{a}{a+b}\right)dt + \sqrt{\frac{2\theta}{a+b} X(t)(1-X(t))} dW(t), \quad \theta > 0,\, t \geq 0. \tag{3.54}$$

The corresponding invariant distribution is beta distribution with probability density function

$$\mathfrak{b}(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} \mathrm{I}_{[0,\,1]}(x),\; x \in \mathbb{R}, \tag{3.55}$$

where

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is the standard beta function (see e.g. Olver et al. (2010)), and $a > 0$ and $b > 0$ are the shape parameters of the invariant distribution. Moreover, mean and variance are

$$\mathbb{E}[X_t] = \frac{a}{a+b}, \quad Var(X_t) = \frac{ab}{(a+b)^2(a+b-1)},\; a > 0,\; b > 0. \tag{3.56}$$

Infinitesimal drift and diffusion parameters of Jacobi diffusion are given by

$$\mu(x) = -\theta\left(x - \frac{a}{a+b}\right), \quad \sigma(x) = \sqrt{\frac{2\theta}{a+b} x(1-x)}, \tag{3.57}$$

while infinitesimal generator is given by

$$\mathcal{G}f(x) = -\theta\left(x - \frac{a}{a+b}\right) f'(x) + \frac{\theta}{a+b} x(1-x) f''(x),\; x \in [0,\,1]. \tag{3.58}$$

The scale and speed densities of the Jacobi diffusion are

$$\mathfrak{s}(x) = x^{-a}(1-x)^{-b}, \quad \mathfrak{m}(x) = \frac{a+b}{\theta} x^{a-1}(1-x)^{b-1}, \tag{3.59}$$

which gives alternative form of the infinitesimal generator (3.58)

$$\mathcal{G}f(x) = \frac{\theta}{a+b} x^{1-a}(1-x)^{1-b} \frac{d}{dx}\left(x^a(1-x)^b f'(x)\right), \; x \in [0,\, 1]\,.$$

Moreover, the state space of Jacobi diffusion is $I = [0,\, 1]$ with boundaries $l = 0$ and $r = 1$. In particular, according to the Feller's classification scheme, left boundary 0 is regular for $a \in \langle 0,\, 1\rangle$, and entrance for $a \geq 1$, while right boundary 1 is regular for $b \in \langle 0,\, 1\rangle$, and entrance for $b \geq 1$. On the other hand, according to O/NO classification scheme both boundaries are NO for all $a > 0$, $b > 0$. Therefore, the appropriate boundary conditions are

- for $a,\, b \in \langle 0,\, 1\rangle$:
$$\lim_{x\to 0} f(x) = \lim_{x\to 1} f(x) = 0; \tag{3.60}$$
- for $a \in \langle 0,\, 1\rangle$ and $b \geq 1$:
$$\lim_{x\to 0} f(x) = \lim_{x\to 1} \frac{f'(x)}{\mathfrak{s}(x)} = 0; \tag{3.61}$$
- for $a \geq 1$ and $b \in \langle 0,\, 1\rangle$:
$$\lim_{x\to 0} \frac{f'(x)}{\mathfrak{s}(x)} = \lim_{x\to 1} f(x) = 0; \tag{3.62}$$
- for $a, b \geq 1$:
$$\lim_{x\to 0} \frac{f'(x)}{\mathfrak{s}(x)} = \lim_{x\to 1} \frac{f'(x)}{\mathfrak{s}(x)} = 0. \tag{3.63}$$

Therefore, recall that the domain of the Jacobi infinitesimal generator is (e.g. for $a,\, b \geq 1$)

$$D(\mathcal{G}) = \{f \in L^2(I,\, \mathfrak{m}) \cap C^2(I) : \mathcal{G}f \in L^2(I,\, \mathfrak{m}) \text{ and } f \text{ satisfies boundary conditions (3.63)}\}.$$

Since both boundaries are NO, Jacobi diffusion belongs to the spectral category I and therefore, the spectrum of the Sturm-Liouville operator $(-\mathcal{G})$ is simple, non-negative and purely discrete, consisting of infinite sequence of eigenvalues

$$\lambda_n = \frac{\theta}{a+b} n(n+a+b-1), \; n \in \mathbb{N}_0, \tag{3.64}$$

with corresponding eigenfunctions, normalized orthogonal Jacobi polynomials $J_n(x)$, given by the Rodrigues formula

$$J_n(x) = (-1)^n \sqrt{\frac{B(a,b)B^{-1}(a+n,b+n)}{n!(n+a+b+1)_n}} x^{1-a}(1-x)^{1-b} \frac{d^n}{dx^n}\left(x^{n+a-1}(1-x)^{n+b-1}\right), \; n \in \mathbb{N}_0, \tag{3.65}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the standard Pochhammer symbol.
Therefore, transition density $p(x,t;x_0)$ admits spectral representation, reflecting simple structure of the spectrum $\sigma(-\mathcal{G})$:

$$p(x,t;x_0) = \mathfrak{b}(x) \sum_{n=0}^{\infty} e^{-\lambda_n t} J_n(x) J_n(x_0), \quad x, x_0 \in [0,\, 1]\,, \tag{3.66}$$

where eigenvalues λ_n and corresponding eigenfunctions $J_n(x)$ are given by (3.64) and (3.65), respectively (cf. Karlin & Taylor (1981*a*), page 335. and Wong (1964)).

Remark 3.2. The presented parametrization of the Jacobi diffusion is not the only possible. In particular, in Leonenko et al. (2013*b*) the Jacobi diffusion is parameterized with the state space $[-1,\, 1]$. In Section 4.2 we present the results regarding fractional counterpart of the Jacobi diffusion with this particular parametrization. With respect to this parametrization, the SDE for Jacobi diffusion takes the following form:

$$dX(t) = -\theta\left(X(t) - \frac{b-2}{a+b+2}\right) + \sqrt{\frac{2\theta}{a+b-2}(1-X(t)^2)}dW(t), \; \theta > 0, \; t \geq 0, \tag{3.67}$$

with invariant beta distribution, but with a different normalizing constant:

$$\mathfrak{b}(x) = \frac{\Gamma(a+b+2)}{\Gamma(b+1)\Gamma(a+1)2^{a+b+1}}(1-x)^a(1+x)^b\, \mathrm{I}_{\langle -1,\,1\rangle}(x), \;\; x \in \mathbb{R}, \tag{3.68}$$

where $a > -1$ and $b > -1$ are the shape parameters of the invariant distribution.
Corresponding eigenvalues are

$$\lambda_n = n\theta(n+a+b+1)/(a+b+2), \; n \in \mathbb{N}, \tag{3.69}$$

while corresponding eigenfunctions are normalized Jacobi polynomials of the form

$$J_n(x) = K_n(2^n n!)^{-1}(-1)^n(1-x)^{-a}(1+x)^{-b}\frac{d^n}{dx^n}\left((1-x)^{a+n}(1+x)^{b+n}\right), \tag{3.70}$$

with normalizing constant

$$K_n = \sqrt{\frac{2^{a+b+1}}{2n+a+b+1}\frac{\Gamma(n+a+1)\Gamma(n+b+1)}{\Gamma(n+1)\Gamma(n+a+b+1)}}.$$

3.2.2 Heavy-tailed Pearson diffusions

For a general description and spectral properties of heavy-tailed Pearson diffusions we refer to Avram et al. (2012) and references therein. In particular, Avram et al. (2012) contains a list of eigenvalues, eigenfunctions and spectral representations of transition densities of corresponding heavy-tailed Pearson diffusions.

Fisher-Snedecor diffusion

Fisher-Snedecor diffusion $\{X(t),\, t \geq 0\}$ is the solution of the SDE

$$dX(t) = -\theta\left(X(t) - \frac{\beta}{\beta-2}\right)dt + \sqrt{\frac{4\theta}{\gamma(\beta-2)}X(t)(\gamma X(t)+\beta)}dW(t), \quad \theta > 0,\, t \geq 0. \tag{3.71}$$

The corresponding invariant distribution is Fisher-Snedecor distribution with probability density function

$$\mathfrak{fs}(x) = \frac{\gamma^{\frac{\gamma}{2}}\beta^{\frac{\beta}{2}}}{B\left(\frac{\gamma}{2},\frac{\beta}{2}\right)}\frac{x^{\frac{\gamma}{2}-1}}{(\gamma x+\beta)^{\frac{\gamma}{2}+\frac{\beta}{2}}}\ \mathrm{I}_{\langle 0,\infty\rangle}(x),\ x \in \mathbb{R}, \tag{3.72}$$

where $\gamma > 0$ and $\beta > 0$ are shape parameters of the invariant distribution. Moreover, $\beta > 4$ ensures existence of the mean and variance:

$$\mathbb{E}[X_t] = \frac{\beta}{\beta-2}, \quad \mathrm{Var}(X_t) = \frac{2\beta^2(\alpha+\beta-2)}{\alpha(\beta-2)^2(\beta-4)},\ \beta > 4. \tag{3.73}$$

Infinitesimal drift and diffusion parameters of FS diffusion are given by

$$\mu(x) = -\theta\left(x - \frac{\beta}{\beta-2}\right), \quad \sigma(x) = \sqrt{\frac{4\theta}{\gamma(\beta-2)}x(\gamma x+\beta)}, \tag{3.74}$$

while infinitesimal generator is given by

$$\mathcal{G}f(x) = -\theta\left(x - \frac{\beta}{\beta-2}\right)f'(x) + \frac{2\theta}{\gamma(\beta-2)}x(\gamma x+\beta)f''(x),\ x \in \langle 0,\,\infty\rangle\,. \tag{3.75}$$

The scale and speed densities of the FS diffusion are

$$\mathfrak{s}(x) = x^{-\frac{\gamma}{2}}(\gamma x+\beta)^{\frac{\gamma}{2}+\frac{\beta}{2}-1}, \quad \mathfrak{m}(x) = \frac{\gamma(\beta-2)}{2\theta}x^{\frac{\gamma}{2}-1}(\gamma x+\beta)^{-\frac{\gamma}{2}-\frac{\beta}{2}}, \tag{3.76}$$

which gives alternative form of the infinitesimal generator (3.75)

$$\mathcal{G}f(x) = \frac{2\theta}{\gamma(\beta-2)}\,x^{1-\frac{\gamma}{2}}(\gamma x+\beta)^{\frac{\gamma}{2}+\frac{\beta}{2}}\,\frac{d}{dx}\,\left(x^{\frac{\gamma}{2}}(\gamma x+\beta)^{1-\frac{\gamma}{2}-\frac{\beta}{2}}f'(x)\right),\ x \in \langle 0,\,\infty\rangle\,.$$

Moreover, the state space of FS diffusion is $I = \langle 0, \infty\rangle$ with boundaries $l = 0$ and $r = \infty$. In particular, according to the Feller's classification scheme, left boundary 0 is regular for $\gamma \in \langle 0, 2\rangle$, and entrance for $\gamma \geq 2$, while right boundary ∞ is natural for $\gamma > 0$, $\gamma \notin \{2m, m \in \mathbb{N}\}$. On the other hand, according to O/NO classification the left boundary 0 is NO for all $\gamma > 0$, $\gamma \notin \{2m, m \in \mathbb{N}\}$, while right boundary ∞ is O/NO with unique positive cutoff

$$\Lambda = \frac{\theta\beta^2}{8(\beta-2)}. \tag{3.77}$$

Moreover, left boundary ∞ is NO for $\lambda \leq \Lambda$, and O for $\lambda > \Lambda$. Since the speed density $\mathfrak{m}(x)$ is integrable over whole state space $\langle 0, \infty\rangle$, i.e. since

$$\int_0^\infty \mathfrak{m}(x)\,dx = \frac{\gamma(\beta-2)}{2\theta}\gamma^{-\frac{\gamma}{2}}\beta^{-\frac{\beta}{2}}B(\gamma/2,\beta/2) < \infty, \tag{3.78}$$

the appropriate boundary conditions are

- for $\gamma \in \langle 0, 2\rangle$:

$$\lim_{x\to 0} f(x) = 0, \quad \lim_{x\to\infty}\frac{f'(x)}{\mathfrak{s}(x)} = 0; \tag{3.79}$$

- for $\gamma \geq 2$, $\gamma \notin \{2m, m \in \mathbb{N}\}$:

$$\lim_{x\to 0}\frac{f'(x)}{\mathfrak{s}(x)} = \lim_{x\to\infty}\frac{f'(x)}{\mathfrak{s}(x)} = 0. \tag{3.80}$$

Therefore, recall that the domain of the FS infinitesimal generator is (e.g. for $\gamma \geq 2$, $\gamma \notin \{2m, m \in \mathbb{N}\}$)

$$D(\mathcal{G}) = \{f \in L^2(I, \mathfrak{m}) \cap C^2(I) : \mathcal{G}f \in L^2(I, \mathfrak{m}) \text{ and } f \text{ satisfies boundary conditions (3.80)}\}. \tag{3.81}$$

Moreover, FS diffusion belongs to the spectral category II and therefore, the spectrum of the Sturm-Liouville operator $(-\mathcal{G})$ is simple and is given by

$$\sigma(-\mathcal{G}) = \sigma_d(-\mathcal{G}) \cup \sigma_e(-\mathcal{G}),$$

where $\sigma_d(-\mathcal{G}) \subset [0, \Lambda\rangle$ and $\sigma_e(-\mathcal{G}) = [\Lambda, \infty\rangle$.
In particular, $\sigma_d(-\mathcal{G})$ is consisted of finite set of simple eigenvalues

$$\lambda_n = \frac{\theta}{\beta-2}n(\beta-2n), \quad n \in \left\{0, 1, \ldots, \left\lfloor\frac{\beta}{4}\right\rfloor\right\}, \quad \beta > 2, \tag{3.82}$$

with corresponding eigenfunctions, orthogonal Fisher-Snedecor polynomials $F_n(x)$, given by the Rodrigues formula

$$\widetilde{F}_n(x) = x^{1-\frac{\gamma}{2}} (\gamma x+\beta)^{\frac{\gamma}{2}+\frac{\beta}{2}} \frac{d^n}{dx^n} \left\{2^n x^{\frac{\gamma}{2}+n-1} (\gamma x+\beta)^{n-\frac{\gamma}{2}-\frac{\beta}{2}}\right\}, \quad n \in \left\{0,1,\dots,\left\lfloor\frac{\beta}{4}\right\rfloor\right\}, \quad \beta > 2. \tag{3.83}$$

Moreover, the normalized Fisher-Snedecor polynomials are

$$F_n(x) = K_n \widetilde{F}_n(x), \tag{3.84}$$

where

$$K_n = (-1)^n \sqrt{\frac{B(\frac{\gamma}{2},\frac{\beta}{2})}{n!(-1)^n(2\beta)^{2n} B\left(\frac{\gamma}{2}+n,\frac{\beta}{2}-2n\right)} \frac{\Gamma\left(n-\frac{\beta}{2}\right)}{\Gamma\left(2n-\frac{\beta}{2}\right)}} =$$

$$= (-1)^n \sqrt{\frac{B(\frac{\gamma}{2},\frac{\beta}{2})}{n!(2\beta)^{2n} B\left(\frac{\gamma}{2}+n,\frac{\beta}{2}-2n\right)} \left[\prod_{k=1}^{n}\left(\frac{\beta}{2}+k-2n\right)\right]^{-1}}$$

is the normalizing constant. On the other hand, $\lambda \in \sigma_e(-\mathcal{G})$ can be parametrized as

$$\lambda = \Lambda + \frac{2\theta k^2}{\beta-2} = \frac{2\theta}{\beta-2}\left(\frac{\beta^2}{16}+k^2\right), \quad \beta > 2, \quad k \geq 0.$$

The spectral representation of the transition density of the FS diffusion with parameters $\gamma > 2$ (ensuring the ergodicity), $\gamma \notin \{2(m+1),\, m \in \mathbb{N}\}$, and $\beta > 2$ consists of two parts

$$p_1(x,t;x_0) = p_d(x,t;x_0) + p_c(x,t;x_0), \tag{3.85}$$

and therefore reflects the nature of the spectrum of SL operator $(-\mathcal{G})$. The discrete part of the spectral representation is

$$p_d(x,t;x_0) = \mathfrak{fs}(x) \sum_{n=0}^{\lfloor\frac{\beta}{4}\rfloor} e^{-\lambda_n t} F_n(x_0) F_n(x),$$

where $\mathfrak{fs}(\cdot)$ is the invariant density (3.72), eigenvalues λ_n are given by (3.82) and the normalized FS polynomials are given by (3.84). The continuous part of the spectral representation is given in terms of the elements λ of the essential part of the spectrum of the operator $(-\mathcal{G})$:

$$p_c(x,t;x_0) = \mathfrak{fs}(x) \frac{1}{\pi} \int\limits_{\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} e^{-\lambda t} a(\lambda) f_1(x_0,-\lambda) f_1(x,-\lambda)\, d\lambda,$$

where

$$k(\lambda) = -i\sqrt{\frac{\beta^2}{16} - \frac{\lambda(\beta-2)}{2\theta}}, \quad a(\lambda) = k(\lambda)\left|\frac{B^{\frac{1}{2}}\left(\frac{\gamma}{2},\frac{\beta}{2}\right)\Gamma\left(-\frac{\beta}{4}+ik(\lambda)\right)\Gamma\left(\frac{\gamma}{2}+\frac{\beta}{4}+ik(\lambda)\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(1+2ik(\lambda)\right)}\right|^2. \tag{3.86}$$

Function f_1 is a solution of the Sturm-Liouville equation $\mathcal{G}f(x) = -\lambda f(x)$, $\lambda > 0$, and is given by

$$f_1(x,-\lambda) = {}_2F_1\left(-\frac{\beta}{4}+\sqrt{\frac{\beta^2}{16}-\frac{\lambda(\beta-2)}{2\theta}}, -\frac{\beta}{4}-\sqrt{\frac{\beta^2}{16}-\frac{\lambda(\beta-2)}{2\theta}}; \frac{\gamma}{2}; -\frac{\gamma}{\beta}x\right), \tag{3.87}$$

where ${}_2F_1$ is the Gauss hypergeometric function, a special case of generalized hypergeometric function ${}_pF_q$ with $p = 2$ and $q = 1$, see Slater (1966) or Olver et al. (2010). For more details on FS diffusion and the proof of spectral representation of the transition density (3.85) we refer to Avram et al. (2013*b*).

Reciprocal gamma diffusion

Reciprocal gamma diffusion $\{X(t),\, t \geq 0\}$ is the solution of the SDE

$$dX(t) = -\theta\left(X(t) - \frac{\gamma}{\beta-1}\right)dt + \sqrt{\frac{2\theta}{\beta-1}X(t)^2}\,dW(t), \quad \theta > 0,\, t \geq 0. \tag{3.88}$$

The corresponding invariant distribution is reciprocal gamma distribution with probability density function

$$\mathfrak{rg}(x) = \frac{\gamma^\beta}{\Gamma(\beta)}\, x^{-\beta-1} e^{-\frac{\gamma}{x}} \mathrm{I}_{\langle 0,\infty\rangle}(x),\ x \in \mathbb{R}, \tag{3.89}$$

where $\gamma > 0$ is the scale and $\beta > 0$ is the shape parameter of the invariant distribution. Moreover, $\beta > 2$ ensures existence of the mean and variance:

$$\mathbb{E}[X_t] = \frac{\gamma}{\beta-1}, \quad \mathrm{Var}(X_t) = \frac{\gamma^2}{(\beta-1)^2(\beta-2)},\ \beta > 2. \tag{3.90}$$

Infinitesimal drift and diffusion parameters of RG diffusion are given by

$$\mu(x) = -\theta\left(x - \frac{\gamma}{\beta-1}\right), \quad \sigma(x) = \sqrt{\frac{2\theta}{\beta-1}x^2}, \tag{3.91}$$

while infinitesimal generator is given by

$$\mathcal{G}f(x) = -\theta\left(x - \frac{\gamma}{\beta-1}\right) f'(x) + \frac{\theta}{\beta-1}\, x^2 f''(x),\ x \in \langle 0, \infty\rangle. \tag{3.92}$$

The scale and speed densities of the RG diffusion are

$$\mathfrak{s}(x) = x^{\beta-1} e^{\frac{\gamma}{x}}, \quad \mathfrak{m}(x) = \frac{\beta-1}{\theta}\, x^{-\beta-1} e^{-\frac{\gamma}{x}}, \tag{3.93}$$

which gives alternative form of the infinitesimal generator (3.92)

$$\mathcal{G}f(x) = \frac{\theta}{\beta-1}\, x^{\beta+1} e^{\frac{\gamma}{x}}\, \frac{d}{dx}\left(x^{-\beta+1} e^{-\frac{\gamma}{x}} f'(x)\right),\ x \in \langle 0,\, \infty\rangle\,.$$

Moreover, the state space of RG diffusion is $I = \langle 0,\, \infty\rangle$ with boundaries $l = 0$ and $r = \infty$. In particular, for $\gamma > 0$ and $\beta > 1$ and according to the Feller's classification scheme, left boundary 0 is entrance, while right boundary ∞ is natural. On the other hand, according to O/NO classification the left boundary 0 is NO, while right boundary ∞ is O/NO with unique positive cutoff

$$\Lambda = \frac{\theta\beta^2}{4(\beta-1)}. \tag{3.94}$$

Moreover, left boundary ∞ is NO for $\lambda \le \Lambda$, and O for $\lambda > \Lambda$. Since for $\gamma > 0$, $\beta > 1$ the speed density $\mathfrak{m}(x)$ is integrable on the state space $\langle 0,\, \infty\rangle$, i.e. since

$$\int_0^\infty \mathfrak{m}(x)\, dx = \frac{\beta-1}{\theta}\frac{\Gamma(\gamma)}{\gamma^\beta} < \infty, \tag{3.95}$$

the appropriate boundary conditions are:

$$\lim_{x\to 0} \frac{f'(x)}{\mathfrak{s}(x)} = \lim_{x\to\infty} \frac{f'(x)}{\mathfrak{s}(x)} = 0. \tag{3.96}$$

Therefore, recall that the domain of the RG infinitesimal generator is

$$D(\mathcal{G}) = \{f \in L^2(I,\, \mathfrak{m}) \cap C^2(I) : \mathcal{G}f \in L^2(I,\, \mathfrak{m}) \text{ and } f \text{ satisfies boundary conditions (3.96)}\}. \tag{3.97}$$

Moreover, RG diffusion belongs to the spectral category II and therefore, the spectrum of the Sturm-Liouville operator $(-\mathcal{G})$ is simple and is given by

$$\sigma(-\mathcal{G}) = \sigma_d(-\mathcal{G}) \cup \sigma_e(-\mathcal{G}),$$

where $\sigma_d(-\mathcal{G}) \subset [0,\, \Lambda\rangle$ and $\sigma_e(-\mathcal{G}) = [\Lambda,\, \infty\rangle$.
In particular, $\sigma_d(-\mathcal{G})$ is consisted of finite set of simple eigenvalues

$$\lambda_n = \frac{\theta}{\beta-1}\, n(\beta - n), \quad n \in \left\{0, 1, \ldots, \left\lfloor \frac{\beta}{2} \right\rfloor\right\}, \quad \beta > 1, \tag{3.98}$$

with corresponding eigenfunctions, orthogonal Bessel polynomials $B_n(x)$, given by the Rodrigues formula

$$\widetilde{B}_n(x) = x^{\beta+1} e^{\frac{\gamma}{x}} \frac{d^n}{dx^n} \left(x^{2n-(\beta+1)} e^{-\frac{\gamma}{x}}\right), \quad n \in \left\{0, 1, \dots, \left\lfloor \frac{\beta}{2} \right\rfloor \right\}, \quad \beta > 1. \tag{3.99}$$

Moreover, the normalized Bessel polynomials are

$$B_n(x) = K_n \, \widetilde{B}_n(x), \tag{3.100}$$

where

$$K_n = \frac{(-1)^n}{\gamma^n} \sqrt{\frac{(\beta-2n)\Gamma(\beta)}{\Gamma(n+1)\Gamma(\beta-n+1)}} = \frac{(-1)^n}{\gamma^n} \sqrt{\frac{\Gamma(\beta)}{n!\Gamma(\beta-2n)} \left(\prod_{k=0}^{n-1} (\beta-n-k) \right)^{-1}}$$

is the normalizing constant. On the other hand, $\lambda \in \sigma_e(-\mathcal{G})$ can be parametrized as

$$\lambda = \Lambda + \frac{\theta k^2}{\beta-1} = \frac{\theta}{\beta-1} \left(\frac{\beta^2}{4} + k^2 \right), \quad \beta > 1, \quad k \geq 0.$$

The spectral representation of the transition density of the RG diffusion consists of two parts:

$$p_1(x,t;x_0) = p_d(x,t;x_0) + p_c(x,t;x_0), \tag{3.101}$$

and therefore reflects the nature of the spectrum of SL operator $(-\mathcal{G})$. The discrete part of the spectral representation is

$$p_d(x,t;x_0) = \mathfrak{rg}(x) \sum_{n=0}^{\left\lfloor \frac{\beta}{2} \right\rfloor} e^{-\lambda_n t} \, B_n(x_0) \, B_n(x),$$

where $\mathfrak{rg}(\cdot)$ is the invariant density (3.89), eigenvalues λ_n are given by (3.98) and normalized Bessel polynomials are given by (3.100). The continuous part of the spectral representation is given in terms of the elements λ of the essential part of the spectrum of the operator $(-\mathcal{G})$

$$p_c(x,t;x_0) = \mathfrak{rg}(x) \frac{1}{4\pi} \int_{\frac{\theta\beta^2}{4(\beta-1)}}^{\infty} e^{-\lambda t} \, b(\lambda) \, \psi(x,-\lambda) \, \psi(x_0,-\lambda) \, d\lambda,$$

where

$$b(\lambda) = \frac{\gamma^{-\beta-1}}{k(\lambda)} \left| \frac{\Gamma^{\frac{1}{2}}(\beta) \, \Gamma\left(-\frac{\beta}{2} + ik(\lambda)\right)}{\Gamma\left(2ik(\lambda)\right)} \right|^2, \qquad k(\lambda) = -i\sqrt{\frac{\beta^2}{4} - \frac{\lambda(\beta-1)}{\theta}}. \tag{3.102}$$

The function

$$\psi(x,-\lambda) = \gamma^{\frac{\beta+1}{2}}\,{}_2F_0\left(-\frac{\beta}{2} + ik(\lambda), -\frac{\beta}{2} - ik(\lambda);; -\frac{x}{\gamma}\right) \tag{3.103}$$

is the solution of the Sturm-Liouville equation $\mathcal{G}f(x) = -\lambda f(x)$, $\lambda > 0$, where ${}_2F_0$ is the special case of the generalized hypergeometric function ${}_pF_q$ with $p = 2$ and $q = 0$, see Slater (1966) or Olver et al. (2010). For more details on RG diffusion and the proof of spectral representation of the transition density (3.101) we refer to Leonenko & Šuvak (2010*a*). See also Wong (1964).

Student diffusion

Student diffusion $\{X(t),\, t \geq 0\}$ is the solution of the SDE

$$dX(t) = -\theta\left(X(t) - \mu\right)dt + \sqrt{\frac{2\theta\delta^2}{\nu-1}\left(1 + \left(\frac{X(t)-\mu}{\delta}\right)^2\right)}\,dW(t), \quad \theta > 0,\, t \geq 0. \tag{3.104}$$

The corresponding invariant distribution is symmetric scaled Student distribution with probability density function

$$\mathfrak{st}(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\delta\sqrt{\pi}\Gamma(\frac{\nu}{2})}\left(1 + \left(\frac{x-\mu}{\delta}\right)^2\right)^{-\frac{\nu+1}{2}}, \; x \in \mathbb{R}, \tag{3.105}$$

where $\delta > 0$ is scale parameter, $\mu \in \mathbb{R}$ is location parameter, while $\nu > 1$ is degree of freedom of the invariant distribution. Moreover, $\nu > 2$ ensures the existence of the mean and variance:

$$\mathbb{E}[X_t] = \mu, \quad Var(X_t) = \frac{\delta^2}{\nu-2}. \tag{3.106}$$

Infinitesimal drift and diffusion parameters of ST diffusion are given by

$$\mu(x) = -\theta\left(x-\mu\right), \quad \sigma(x) = \sqrt{\frac{2\theta\delta^2}{\nu-1}\left(1 + \left(\frac{x-\mu}{\delta}\right)^2\right)}, \tag{3.107}$$

while infinitesimal generator is given by

$$\mathcal{G}f(x) = -\theta\left(x-\mu\right)\,f'(x) + \frac{\theta\delta^2}{\nu-1}\left(1 + \left(\frac{x-\mu}{\delta}\right)^2\right)f''(x), \; x \in \mathbb{R}. \tag{3.108}$$

The scale and speed densities of the ST diffusion are

$$\mathfrak{s}(x) = \left(1 + \left(\frac{x-\mu}{\delta}\right)^2\right)^{\frac{\nu-1}{2}}, \quad \mathfrak{m}(x) = \frac{\nu-1}{\theta\delta^2}\left(1 + \left(\frac{x-\mu}{\delta}\right)^2\right)^{-\frac{\nu+1}{2}}, \tag{3.109}$$

which gives alternative form of the infinitesimal generator (3.108)

$$\mathcal{G}f(x) = \frac{\theta\delta^2}{\nu - 1}\left(1 + \left(\frac{x-\mu}{\delta}\right)^2\right)^{\frac{\nu+1}{2}} \frac{d}{dx}\left(\left(1 + \left(\frac{x-\mu}{\delta}\right)^2\right)^{\frac{1-\nu}{2}} f'(x)\right), \ x \in \mathbb{R}.$$

Moreover, the state space of ST diffusion is $I = \mathbb{R}$ with boundaries $l = -\infty$ and $r = \infty$. In particular, for $\nu > 1$, $\delta > 0$ and $\mu \in \mathbb{R}$ and according to the Feller's classification scheme, both boundaries are natural. On the other hand, according to O/NO classification both boundaries are O/NO with unique positive cutoffs

$$\Lambda_1 = \Lambda_2 = \Lambda = \frac{\theta\nu^2}{4(\nu-1)}. \tag{3.110}$$

Moreover, both boundaries are NO for $\lambda \leq \Lambda$, and O for $\lambda > \Lambda$. Since for $\nu > 1$, $\delta > 0$ and $\mu \in \mathbb{R}$ the speed density $\mathfrak{m}(x)$ is integrable on the state space $\mathbb{R}$, i.e. since

$$\int\limits_{-\infty}^{\infty} \mathfrak{m}(x)\, dx = \frac{\nu-1}{\theta\delta}\frac{\delta\sqrt{\pi}\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})} < \infty, \tag{3.111}$$

the appropriate boundary conditions are:

$$\lim_{x\to-\infty}\frac{f'(x)}{\mathfrak{s}(x)} = \lim_{x\to\infty}\frac{f'(x)}{\mathfrak{s}(x)} = 0. \tag{3.112}$$

Therefore, recall that the domain of the ST infinitesimal generator is

$$D(\mathcal{G}) = \{f \in L^2(I, \mathfrak{m}) \cap C^2(I) : \mathcal{G}f \in L^2(I, \mathfrak{m}) \text{ and } f \text{ satisfies boundary conditions (3.112)}\}.$$

Moreover, ST diffusion belongs to the spectral category III and therefore, the spectrum of the Sturm-Liouville operator $(-\mathcal{G})$ is not simple and is given by

$$\sigma(-\mathcal{G}) = \sigma_d(-\mathcal{G}) \cup \sigma_e(-\mathcal{G}),$$

where $\sigma_d(-\mathcal{G}) \subset [0, \Lambda\rangle$ and $\sigma_e(-\mathcal{G}) = [\Lambda, \infty\rangle$.
In particular, $\sigma_d(-\mathcal{G})$ is consisted of finite set of simple eigenvalues

$$\lambda_n = \frac{\theta}{\nu-1} n(\nu - n), \quad n \in \left\{0, 1, \ldots, \left\lfloor\frac{\nu}{2}\right\rfloor\right\}, \quad \nu > 1, \tag{3.113}$$

with corresponding eigenfunctions, orthogonal Routh-Romanovski polynomials $R_n(x)$, given by the Rodrigues formula

$$\widetilde{R}_n(x) = \delta^{2n}\left(1 + \left(\frac{x-\mu}{\delta}\right)^2\right)^{\frac{\nu+1}{2}} \frac{d^n}{dx^n}\left(1 + \left(\frac{x-\mu}{\delta}\right)^2\right)^{n-\frac{\nu+1}{2}}, \quad n \in \left\{0, 1, \ldots, \left\lfloor\frac{\nu}{2}\right\rfloor\right\}, \quad \nu > 1. \tag{3.114}$$

Moreover, the normalized Routh-Romanovski polynomials are

$$R_n(x) = K_n \, \widetilde{R}_n(x), \tag{3.115}$$

where

$$K_n = \frac{(-1)^n}{\delta^n} \sqrt{\frac{\Gamma(\nu - 2n + 1)\Gamma(\frac{\nu}{2})\Gamma(\frac{\nu+1}{2} - n)}{n!\Gamma(\nu - n + 1)\Gamma(\frac{\nu+1}{2})\Gamma(\frac{\nu}{2} - n)}}$$

is the normalizing constant. On the other hand, $\lambda \in \sigma_e(-\mathcal{G})$ can be parametrized as

$$\lambda = \Lambda + \frac{\theta k^2}{\nu - 1} = \frac{\theta}{\nu - 1}\left(\frac{\nu^2}{4} + k^2\right), \quad \nu > 1, \quad k \geq 0.$$

For more details on ST diffusion we refer to Leonenko & Šuvak (2010*b*). See also Wong (1964).

3.2.3 Pearson diffusions as strong solutions of stochastic differential equations

In the last two subsections, the Pearson family of diffusions was introduced via their corresponding stochastic differential equations. Recall that, according to (Bibby et al. 2005, Theorem 2.1) (see Section 3.2), Pearson diffusions are unique weak, ergodic and mean-reverting solutions of their corresponding SDEs, assuming the existence of mean and variance of corresponding invariant distributions.

On the other hand, recall the conditions (1) – (4) for existence of a strong solution of a SDE (see section 3.1.2). These conditions are not satisfied for all Pearson diffusions. In particular, Lipschitz condition (2) regarding diffusion parameter is not satisfied for CIR and FS diffusion, i.e. there is no such constant $K > 0$ such that

$$|\sigma(x) - \sigma(y)| \leq K\,|x - y|\,, \ \forall x, \, y \in \langle 0, \, \infty \rangle \,.$$

However, this condition can weakened in a sense that it holds for possibly different constants K_N for $|x|\,,\ |y| \leq N$ and for each $N > 0$. Since diffusion parameters of CIR and FS diffusion are in $C^1(I)$, the Mean value theorem implies this weakened condition and therefore, their corresponding SDEs admit a strong solution.

Alternatively, according to (Ait-Sahalia 1996, page 415, assumption A1 (i) and (ii)), global Lipschitz and linear growth conditions, i.e. conditions (2) – (3) can be replaced with conditions

- the drift coefficient $\mu(x)$ and the diffusion coefficient $\sigma(x)$ are continuously differentiable in x and $\sigma^2(x)$ is strictly positive on the whole diffusion state space,
- the integral of the speed density $\mathfrak{m}(x)$ of the diffusion process converges at both

boundaries of the diffusion state space,

in order for SDE to admit a unique strong solution. Clearly, infinitesimal parameters of the Pearson diffusion satisfy the first condition, while the second condition is met since the invariant distribution $\mathfrak{f}(x)$ of the corresponding Pearson diffusion is just a normalized speed density $\mathfrak{m}(x)$, i.e. there exist a constant M such that

$$\mathfrak{f}(x) = \frac{\mathfrak{m}(x)}{M},$$

and therefore the speed density is integrable on the whole state space of the diffusion process.
To conclude, for each Pearson diffusion, corresponding SDE admits a pathwise unique strong, ergodic and mean-reverting solution. Moreover, the solution is stationary if the initial value X_0 has probability density equal to the invariant density of the corresponding Pearson diffusion. Therefore, if not stated otherwise, such solutions for Pearson diffusions are assumed.

Chapter 4

Fractional Pearson diffusions

In this section, fractional Pearson diffusions (fPDs) are defined as a stochastic model for time-fractional Cauchy problems of Caputo type, involving the corresponding infinitesimal generator. In particular, spectral representation of transition densities, stationary distributions, correlation structure and SDEs for fractional Pearson diffusions are established. Section 4.2 contains results regarding non-heavy-tailed fractional Pearson diffusions, which have been established in Leonenko et al. (2013*b*), while Section 4.3 contains new results regarding heavy-tailed fractional Pearson diffusions belonging to spectral category II. Moreover, Sections 4.4 and 4.5 contain results regarding correlation structure and SDE representation of fractional Pearson diffusions.
Let $X = \{X(t),\, t \geq 0\}$ be the Pearson diffusion solving SDE (3.29) and let $\{E(t),\, t \geq 0\}$ be inverse of the standard α-stable subordinator, where $0 < \alpha < 1$ (see section 2.3). Moreover, we assume that the time change process $\{E(t),\, t \geq 0\}$ is independent of the process X. Then, fractional Pearson diffusion $\{X_\alpha(t),\, t \geq 0\}$ is defined as the time-changed Pearson diffusion via inverse of the stable subordinator $\{E(t),\, t \geq 0\}$, i.e. by

$$X_\alpha(t) := X(E(t)),\ t \geq 0. \tag{4.1}$$

In particular, $\{X_\alpha(t),\, t \geq 0\}$ is

- fractional Ornstein-Uhlenbeck diffusion, if X is OU diffusion defined via SDE (3.32);

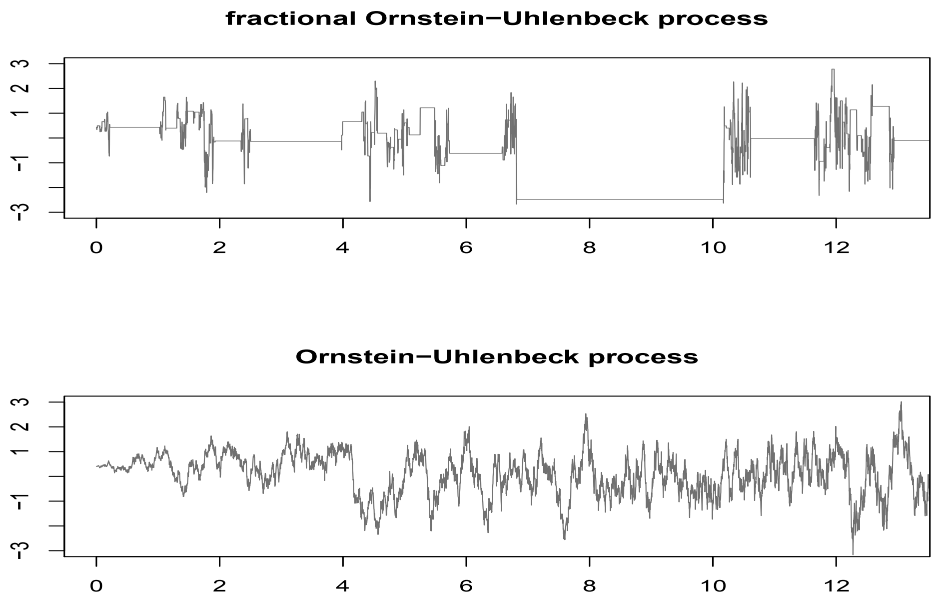

Figure 4.1: Sample paths of the fractional/non-fractional OU process with parameters $\mu = 0$, $\sigma^2 = 1$, $\theta = 0.01$ and $\alpha = 0.7$, based on 10000 points with inital state $X_0 = 0.4$.

- fractional Cox-Ingersoll Ross diffusion, if X is CIR diffusion defined via SDE (3.42);

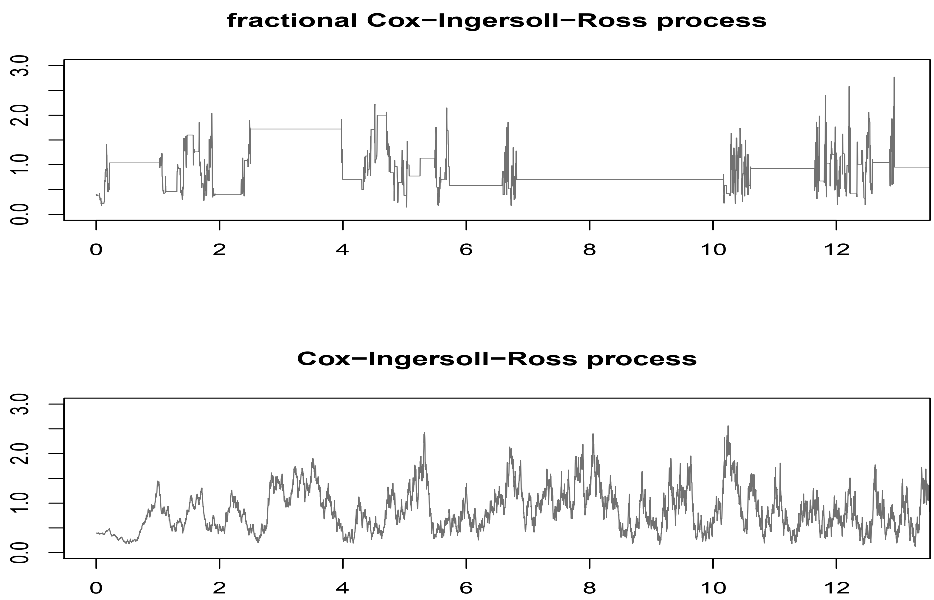

Figure 4.2: Sample paths of the fractional/non-fractional CIR process with parameters $a = 4$, $b = 4$, $\theta = 0.01$ and $\alpha = 0.7$, based on 10000 points with inital state $X_0 = 0.4$.

- fractional Jacobi diffusion, if X is Jacobi diffusion defined via SDE (3.54);

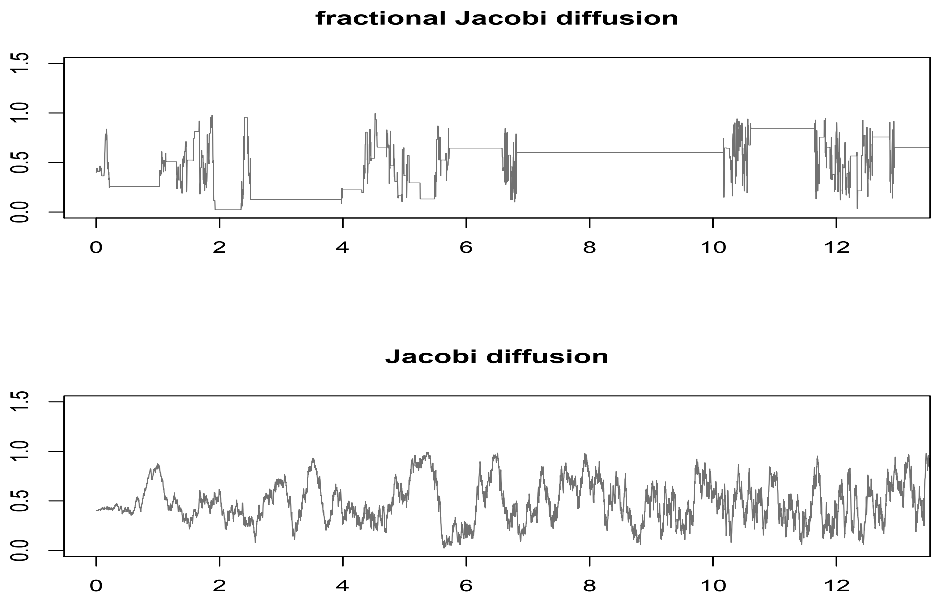

Figure 4.3: Sample paths of the fractional/non-fractional Jacobi diffusion with parameters $a = 2$, $b = 2$, $\theta = 0.01$ and $\alpha = 0.7$, based on 10000 points with inital state $X_0 = 0.4$.

- fractional Fisher-Snedecor diffusion, if X is FS diffusion defined via SDE (3.71);

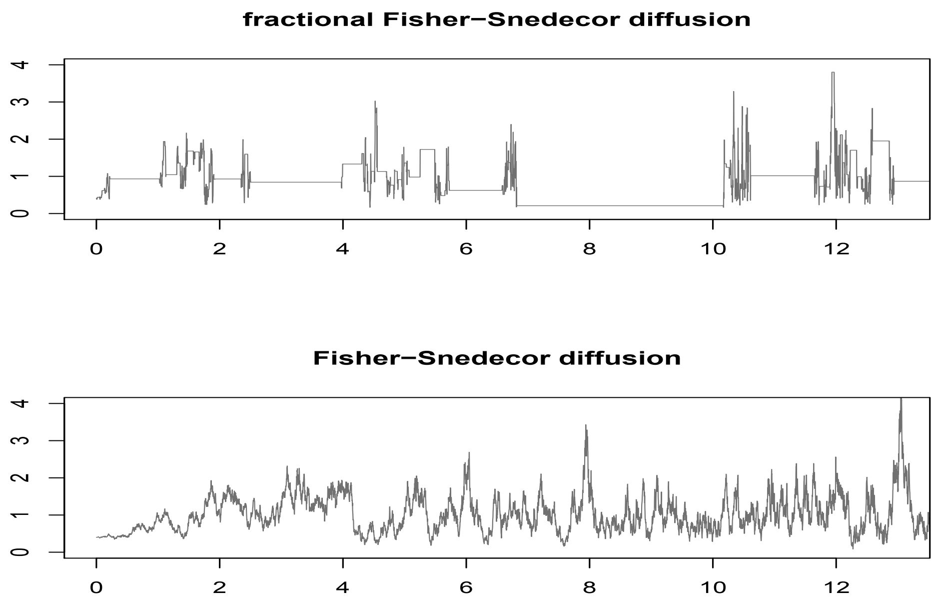

Figure 4.4: Sample paths of the fractional/non-fractional FS diffusion with parameters $\gamma = 10$, $\beta = 20$, $\theta = 0.01$ and $\alpha = 0.7$, based on 10000 points with inital state $X_0 = 0.4$.

- fractional reciprocal gamma diffusion, if X is RG diffusion defined via SDE (3.88);

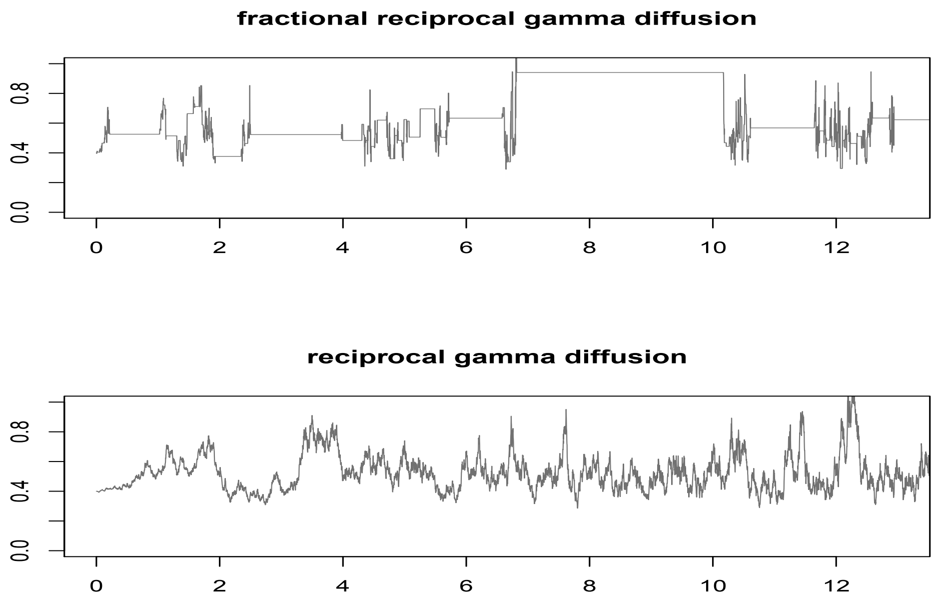

Figure 4.5: Sample paths of the fractional/non-fractional RG diffusion with parameters $\gamma = 10$, $\beta = 20$, $\theta = 0.01$ and $\alpha = 0.7$, based on 10000 points with inital state $X_0 = 0.4$.

- fractional Student diffusion, if X is ST diffusion defined via SDE (3.104);

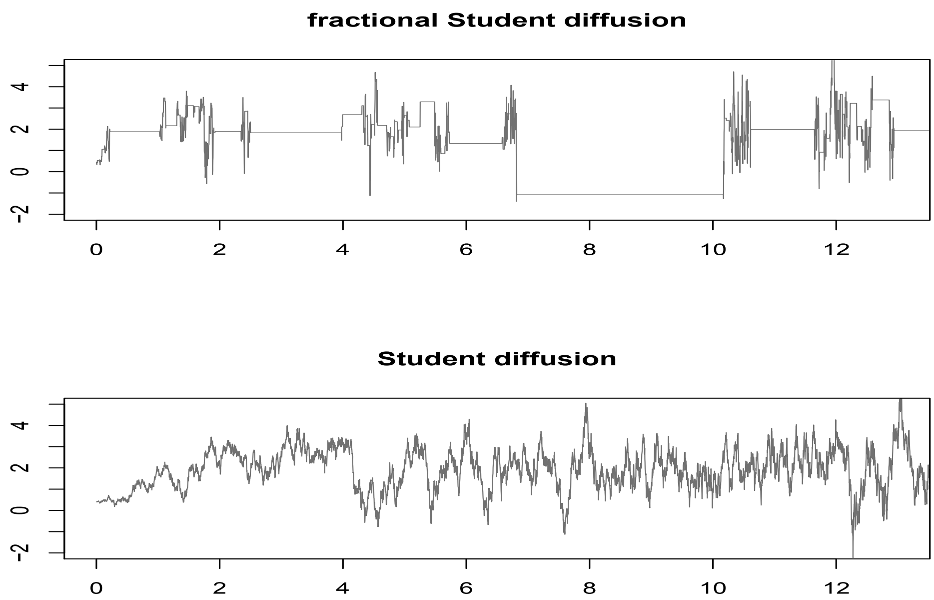

Figure 4.6: Sample paths of the fractional/non-fractional ST diffusion with parameters $\mu = 2$, $\delta = 3$, $\nu = 9$, $\theta = 0.01$ and $\alpha = 0.7$, based on 10000 points with inital state $X_0 = 0.4$.

If not stated otherwise, this parametrization for fractional Pearson diffusions is assumed. Since $E(t)$ rests for periods of time with non-exponential distribution, the process $\{X_\alpha(t),\, t \geq 0\}$ is non-Markovian and non-stationary (for more details see e.g. Meerschaert

& Scheffler (2004)). Corresponding "transition density" $p_\alpha(x,t;y)$ satisfies

$$P(X_\alpha(t) \in B | X_\alpha(0) = y) = \int_B p_\alpha(x,t;y)dx, \tag{4.2}$$

where B is any Borel subset of $\langle l, r\rangle$.

Remark 4.1. Notice how $p_\alpha(x,t;y)$ is not a transition density in a classical sense, i.e. it doesn't describe the underlying non-Markovian process completely. Nevertheless, due to analogy to the diffusion processes we will refer to this function as to the transition density of the fractional diffusion process.

4.1 Fractional Cauchy problems

Consider the fractional Cauchy problem involving the negative differential operator $\mathcal{G}$:

$$\partial_t^\alpha q(y,t) = \mathcal{G}q(y,t), \quad q(y,0) = g(y), \tag{4.3}$$

where ∂_t^α is the Caputo fractional derivative of order $0 < \alpha < 1$ (see Section 2.1).

The ordinary (non-fractional) Cauchy problem for the operator $\mathcal{G}$ is

$$\frac{\partial q(y,t)}{\partial t} = \mathcal{G}q(y,t), \quad q(y,0) = g(y). \tag{4.4}$$

As described in Leonenko et al. (2013*b*), separation of variables approach can be used to find heuristical solutions to (4.3) and (4.4) in the form $q(y,t) = T(t)\varphi(y)$, where functions T and φ may depend on x and α. Then for (4.3) we have

$$\frac{1}{T(t)}\partial_t^\alpha T(t) = \frac{\mathcal{G}\varphi(y)}{\varphi(y)},$$

assuming that T and φ do not vanish. The last equation obviously holds if and only if both sides are equal to a constant denoted $-\lambda$ (so that $\lambda > 0$), leading to two equations:

$$\partial_t^\alpha T(t) = -\lambda T(t) \tag{4.5}$$

and

$$\mathcal{G}\varphi = -\lambda\varphi. \tag{4.6}$$

In the case of non-fractional Cauchy problem (4.4), (4.6) is the same, while (4.5) is replaced by

$$\frac{dT(t)}{dt} = -\lambda T(t). \tag{4.7}$$

Recall that the equations (4.5) and (4.7) are eigenvalue problems for operators ∂_t^α and d/dt, and have well-known strong solutions, $\mathcal{E}_\alpha(-\lambda t)$ and $\exp\{-\lambda t\}$, respectively. In other words, $\mathcal{E}_\alpha(-\lambda t)$ and $\exp\{-\lambda t\}$ are eigenfunctions for the corresponding operators (see Section 2.1).

Regarding the space part, both fractional and non-fractional Cauchy problems lead to an eigenvalue problem for the operator $-\mathcal{G}$. Therefore, if quantitative nature of the spectrum $\sigma(-\mathcal{G})$ is available, i.e. if one can identify corresponding eigenvalues and eigenfunctions, heuristic arguments can lead to the spectral representation of the solution. For illustration, assume that the spectrum $\sigma(-\mathcal{G})$ is purely discrete, consisting of infinitely many simple eigenvalues $\{\lambda_n,\, n \in \mathbb{N}\}$ with corresponding eigenfunctions $\{\varphi_n,\, n \in \mathbb{N}\}$. Then it is clear that

$$q_n(y,t) = e^{-\lambda_n t}\varphi_n(y)$$

and

$$q_{n,\alpha}(y,t) = \mathcal{E}_\alpha(-\lambda_n t)\varphi_n(y)$$

are solutions of the equations in Cauchy problems (4.4) and (4.3), respectively. By heuristic arguments, infinite sum of these solutions is again a solution, i.e.

$$q(y,t) = \sum_{n=0}^{\infty} b_n\, e^{-\lambda_n t}\varphi_n(y) \tag{4.8}$$

and

$$q_\alpha(y,t) = \sum_{n=0}^{\infty} b_{n,\alpha}\, \mathcal{E}_\alpha(-\lambda_n t)\varphi_n(y) \tag{4.9}$$

are solutions of the Cauchy problems (4.4) and (4.3), respectively, where b_n and $b_{n,\alpha}$ are constants depending on the initial value of the corresponding Cauchy problems.

At the first look, it is clear that the crucial difference in the solutions (4.8) and (4.9) lies in the variable t. In particular, for the solution of fractional Cauchy problem, the role of the exponential function $e^{-\lambda_n t}$ is taken by the Mittag-Leffler function $\mathcal{E}_\alpha(-\lambda_n t)$. Since the Mittag-Leffler function $\mathcal{E}_\alpha(-\lambda t)$ falls off like a power law $t^{-\alpha}$ (see Section 6.3), i.e. since

$$\mathcal{E}_\alpha(-\lambda t^\alpha) \sim \frac{1}{\lambda\Gamma(1-\alpha)t^\alpha},\ t \to \infty,$$

it yields a slower decay then the traditional exponential decay in the solution of non-fractional Cauchy problem as $t \to \infty$. Also, since $\mathcal{E}_1(-\lambda_n t) = \exp\{-\lambda_n t\}$, for $\alpha = 1$ the solution (4.9) reduces to (4.8) and non-fractional Cauchy problem is recovered.

In particular, if the differential operator in (4.3) is infinitesimal generator $\mathcal{G}$ of the corresponding diffusion process, this Cauchy problem reduces to the time-fractional Kolmogorov backward equation with predefined initial condition. Cauchy problems of this type for non-heavy-tailed Pearson diffusions were first investigated in Leonenko et al. (2013*b*). Strong solutions of such fractional Cauchy problems were established, as well as the stochastic model for the governing equations - the so called fractional Pearson diffusions, i.e. time-

changed Pearson diffusions via inverse of the standard stable subordinator. Since the structure of the spectrum of non-heavy-tailed Pearson diffusions is simple and purely discrete, results regarding the corresponding fractional Cauchy problems resemble the structure of (4.9). In this thesis, similar results are obtained for heavy-tailed fractional Pearson diffusions. In particular, strong solutions of time-fractional Cauchy problems of Caputo type, involving the corresponding infinitesimal generator, as well as their corresponding stochastic model - fractional RG and FS diffusions, reflecting the fact that these diffusions belong to the spectral category II, are established.
In this thesis, relevant fractional Cauchy problems for fractional Pearson diffusions are

- time-fractional Cauchy problem of Caputo type, involving time-fractional Kolmogorov backward equation:

$$\partial_t^\alpha u(y;t) = \mathcal{G}u(y;t) = \mu(y)\frac{\partial u(y;t)}{\partial y} + \frac{1}{2}\sigma^2(y)\frac{\partial^2 u(y;t)}{\partial y^2}, \quad u(y;0) = g(y); \tag{4.10}$$

- time-fractional Cauchy problem of Caputo type, involving time-fractional Kolmogorov forward equation:

$$\partial_t^\alpha u(x;t) = -\frac{\partial}{\partial x}\left(\mu(x)u(x;t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(\sigma^2(x)u(x;t)\Big), \quad u(x;0) = g(x). \tag{4.11}$$

Remark 4.2. Just like Kolmogorov backward and forward equations are governing equations for Pearson diffusions, we will show that their time-fractional counterparts are governing equations for fractional Pearson diffusions. In other words, just like transition densities of Pearson diffusions solve KBE and KFE, transition densities of fractional Pearson diffusions solve their time-fractional counterparts.

4.2 Non-heavy-tailed fractional Pearson diffusions

In this section, results regarding non-heavy-tailed fractional Pearson diffusions are presented and all results can be found in Leonenko et al. (2013*b*). All results in this section are based on spectral representation of transition densities of non-heavy-tailed PDs, properties of corresponding eigenfunctions/orthonormal polynomials and asymptotics of Mittag-Leffler function.

4.2.1 Spectral representation of the transition densities

For the class of non-heavy-tailed fractional Pearson diffusions, i.e. for fractional OU, CIR and Jacobi diffusion, with corresponding invariant density $\mathfrak{p}$ and system of orthonormal polynomials $\{Q_n,\, n \in \mathbb{N}\}$, corresponding transition density is

$$p_\alpha(x,t;x_0) = \mathfrak{p}(x) \sum_{n=0}^{\infty} \mathcal{E}_\alpha(-\lambda_n t^\alpha) Q_n(x_0) Q_n(x),\;\; x,\, x_0 \in I,\;\; t \geq 0, \tag{4.12}$$

where

- in the OU case, invariant density $\mathfrak{p}$ is normal density (3.33), while orthonormal polynomials Q_n are normalized Hermite polynomials (3.39) with corresponding eigenvalues (3.38);
- in the CIR case, invariant density $\mathfrak{p}$ is gamma density (3.43), while orthonormal polynomials Q_n are normalized Laguerre polynomials (3.51) with corresponding eigenvalues (3.50);
- in the Jacobi case, invariant density $\mathfrak{p}$ beta density (3.68), while orthonormal polynomials Q_n are normalized Jacobi polynomials (3.70) with corresponding eigenvalues (3.69).

For the proof see (Leonenko et al. 2013*b*, Lemma 3.1.).

Remark 4.3. Comparing transition densities (4.12) of non-heavy-tailed fPDs with transition densities (3.41), (3.53) and (3.66) of non-heavy-tailed PDs, resembles the conclusion from the Section 4.1, i.e. in the fractional case, in spectral representation of the transition density, the role of the exponential function $e^{-\lambda_n t}$ is taken by the Mittag-Leffler function $\mathcal{E}_\alpha(-\lambda_n t)$. Therefore, it is expected that the fractional Pearson diffusion is a stochastic model for corresponding fractional Cauchy problem involving infinitesimal generator. This connection for non-heavy-tailed fPDs is formally established in the next Section 4.2.2.

4.2.2 Strong solutions of time-fractional Kolmogorov equations

Consider the fractional Cauchy problem

$$\partial_t^\alpha u(y;t) = \mathcal{G}u(y;t) = \mu(y)\frac{\partial u(y;t)}{\partial y} + \frac{1}{2}\sigma^2(y)\frac{\partial^2 u(y;t)}{\partial y^2}, \quad u(y;0) = g(y), \tag{4.13}$$

where $\mathcal{G}$ is the infinitesimal generator of corresponding non-heavy-tailed Pearson diffusion and initial condition function g satisfies conditions

- $g \in L^2(I, \mathfrak{p})$, where $\mathfrak{p}$ is invariant density of the corresponding non-heavy-tailed Pearson diffusion with state space I;
- $\sum\limits_n g_n Q_n$, where $g_n = \int\limits_I g(x) Q_n(x) \mathfrak{p}(x)\, dx$, converges to g uniformly on finite intervals $[y_1,\, y_2] \subset I$.

Then, fractional Cauchy problem (4.13) has a strong (classical) solution

$$u_\alpha(y;t) = \int\limits_I p_\alpha(x,t;y)g(x)\,dx = \sum_{n=0}^{\infty} \mathcal{E}_\alpha(-\lambda_n t^\alpha) Q_n(y) g_n,$$

where λ_n and Q_n are the eigenvalues and eigenfunctions of corresponding OU, CIR or Jacobi diffusion.
For the proof see (Leonenko et al. 2013*b*, Theorem 3.2).
Next, consider a fractional Cauchy problem

$$\partial_t^\alpha u(x;t) = -\frac{\partial}{\partial x}\left(\mu(x)u(x;t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x)u(x;t)\right), \quad u(x;0) = f(x), \tag{4.14}$$

where $\mu(x)$ and $\sigma^2(x)$ are infinitesimal parameters of corresponding non-heavy-tailed Pearson diffusion and initial condition function f satisfies conditions

- $f/\mathfrak{p} \in L^2(I,\mathfrak{p})$, where $\mathfrak{p}$ is invariant density of the corresponding non-heavy-tailed Pearson diffusion with state space I;
- $\sum\limits_n f_n Q_n$, where $f_n = \int\limits_I f(y)Q_n(y)\,dy$, converges to $f/\mathfrak{p}$ uniformly on finite intervals $[y_1,\, y_2] \subset I$.

Then, fractional Cauchy problem (4.14) has a strong (classical) solution

$$u_\alpha(x;t) = \int\limits_I p_\alpha(x,t;y)f(y)\,dy = \mathfrak{p}(y)\sum_{n=0}^{\infty} \mathcal{E}_\alpha(-\lambda_n t^\alpha) Q_n(x) f_n,$$

where λ_n and Q_n are the eigenvalues and eigenfunctions of corresponding OU, CIR or Jacobi diffusion.
For the proof see (Leonenko et al. 2013*b*, Theorem 3.3).

Remark 4.4. If the initial function f is also a probability density of $X_\alpha(0)$, where $\{X_\alpha(t),\, t \geq 0\}$ is the corresponding non-heavy-tailed fPD, then the solution of the fractional Cauchy problem (4.14) is the probability density of $X_\alpha(t)$.

4.2.3 Stationary distributions

Let $\{X_\alpha(t),\, t \geq 0\}$ be the non-heavy-tailed fPD such that $X_\alpha(0)$ has probability density f which satisfies conditions given for the fractional Cauchy problem (4.14). Then the density $p_\alpha(x;t)$ of $X_\alpha(t)$ converges to the stationary distribution of corresponding non-fractional Pearson diffusion, i.e.

$$p_\alpha(x;t) = \int\limits_I p_\alpha(x,t;y)f(y) \to \mathfrak{p}(x),\ t \to \infty,$$

where $\mathfrak{p}(x)$ is

- invariant normal distribution (3.33) in the fractional OU case;
- invariant gamma distribution (3.43) in the fractional CIR case;
- invariant beta distribution (3.68) in the fractional Jacobi case.

For the proof see (Leonenko et al. 2013*b*, Theorems 4.6 - 4.8).
Therefore, non-heavy-tailed fractional Pearson diffusions have the same stationary distributions as their non-fractional counterparts.

4.3 Heavy-tailed fractional Pearson diffusions

In this section, new results for heavy-tailed fractional Pearson diffusions are established. In particular, spectral representation of transition densities, stationary distributions and strong solutions of time-fractional Cauchy problem (4.10), for fractional reciprocal gamma and Fisher-Snedecor diffusion are proven. On the other hand, only L^2 solutions of the time-fractional Cauchy problem (4.11) are derived. Moreover, correlation structure and SDE interpretation for all fractional Pearson diffusions are given.

4.3.1 Spectral representation of transition densities

Here we prove Theorems 4.1 and 4.2 which provide spectral representation of transition densities of fractional RG and fractional FS diffusion. These results are crucial and key ingredients for results to follow.

Theorem 4.1 The transition density of the fractional RG diffusion is given by

$$p_\alpha(x,t;x_0) = \sum_{n=0}^{\lfloor \frac{\beta}{2} \rfloor} \mathfrak{rg}(x)\, B_n(x)\, B_n(x_0)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) + \frac{\mathfrak{rg}(x)}{4\pi} \int_\Lambda^\infty \mathcal{E}_\alpha(-\lambda t^\alpha)\, b(\lambda)\, \psi(x,-\lambda)\, \psi(x_0,-\lambda)\, d\lambda, \tag{4.15}$$

where $\mathfrak{rg}(x)$ is given by (3.89), Bessel polynomials B_n are given by (3.100), the solution of the Sturm-Liouville equation ψ is given by (3.103) with $b(\lambda)$ given by (3.102) and $\Lambda = \frac{\theta\beta^2}{4(\beta-1)}$ is the cutoff (3.94).

Proof. Since the Pearson diffusion $X(t)$ is independent of the time change $E(t)$, using spectral representation of reciprocal gamma diffusion (3.101) and Laplace transform of $E(t)$ (2.16) (see section 2.3) together with the Fubini argument, we have

$$\begin{aligned} P(X_\alpha(t) \in B | X_\alpha(0) = x_0) &= \int_0^\infty P(X_1(\tau) \in B | X_1(0) = x_0)\, f_t(\tau)\, d\tau \\ &= \int_0^\infty \int_B p_1(x,\tau;x_0)\, f_t(\tau)\, dx\, d\tau \\ &= \int_B \int_0^\infty (p_d(x,\tau;x_0) + p_c(x,\tau;x_0))\, f_t(\tau)\, d\tau\, dx \end{aligned} \tag{4.16}$$

$$= \int_B \left[\int_0^\infty \sum_{n=0}^{\lfloor \frac{\beta}{2} \rfloor} \mathfrak{rg}(x)\, B_n(x_0)\, B_n(x)\, e^{-\lambda_n \tau}\, f_t(\tau)\, d\tau + \frac{\mathfrak{rg}(x)}{4\pi} \int_0^\infty \int_\Lambda^\infty e^{-\lambda\tau}\, f_t(\tau)\, b(\lambda)\, \psi(x,-\lambda)\, \psi(x_0,-\lambda)\, d\lambda\, d\tau \right] dx$$

$$= \int_B \left[\sum_{n=0}^{\lfloor \frac{\beta}{2} \rfloor} \mathfrak{rg}(x)\, B_n(x)\, B_n(x_0)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) + \frac{\mathfrak{rg}(x)}{4\pi} \int_\Lambda^\infty \mathcal{E}_\alpha(-\lambda t^\alpha)\, b(\lambda)\, \psi(x,-\lambda)\, \psi(x_0,-\lambda)\, d\lambda \right] dx. \tag{4.17}$$

The change of the order of integration in (4.16) is justified by the Fubini-Tonelli Theorem (Rudin 1987, Theorem 8.8 (a)) since functions p_1 and f_t are non-negative. In contrast, change of the order of integration in (4.17) cannot be justified by the Fubini-Tonelli Theorem since the integrand

$$g(\lambda,\tau) = e^{-\lambda\tau}\, f_t(\tau)\, b(\lambda)\, \psi(x,-\lambda)\, \psi(x_0,-\lambda)$$

is not necessarily non-negative. To justify this step using the Fubini Theorem (Rudin 1987, Theorem 8.8 (b-c)), below we show that

$$\int_\Lambda^\infty \int_0^\infty |g(\lambda,\tau)|\, d\tau\, d\lambda < \infty. \tag{4.18}$$

Let

$$h(\lambda) = \mathcal{E}_\alpha(-\lambda t^\alpha)\, b(\lambda)\, \psi(x,-\lambda)\, \psi(x_0,-\lambda).$$

Since

$$\begin{aligned}
\int_\Lambda^\infty \int_0^\infty |g(\lambda,\tau)| d\tau d\lambda &= \int_\Lambda^\infty \int_0^\infty e^{-\lambda\tau} f_t(\tau) |\, b(\lambda)\, \psi(x,-\lambda)\, \psi(x_0,-\lambda)|\, d\tau\, d\lambda \\
&= \int_\Lambda^\infty \mathcal{E}_\alpha(-\lambda t^\alpha) |b(\lambda)\, \psi(x,-\lambda)\, \psi(x_0,-\lambda)|\, d\lambda \\
&= \int_\Lambda^\infty |h(\lambda)|\, d\lambda\,,
\end{aligned}$$

we need to show that

$$\int_\Lambda^\infty |h(\lambda)|\, d\lambda < \infty.$$

According to Slater (1960) or Buchholz (1969)

$$\psi(x,-\lambda) = {}_2F_0\left(-\frac{\beta}{2} + ik(\lambda), -\frac{\beta}{2} - ik(\lambda);\, ;\, -\frac{x}{\gamma}\right) = \left(\frac{\gamma}{x}\right)^{-\frac{\beta+1}{2}} e^{\frac{\gamma}{2x}}\, \mathcal{W}_{\frac{\beta+1}{2},\, ik(\lambda)}\left(\frac{\gamma}{x}\right),$$

where $\mathcal{W}$ is given by

$$\mathcal{W}_{\frac{\beta+1}{2},ik(\lambda)}\left(\frac{\gamma}{x}\right)=\frac{\pi}{\sin\left(2i\pi k(\lambda)\right)}\left(\frac{\mathcal{M}_{\frac{\beta+1}{2},-ik(\lambda)}\left(\frac{\gamma}{x}\right)}{\Gamma\left(-\frac{\beta}{2}+ik(\lambda)\right)}-\frac{\mathcal{M}_{\frac{\beta+1}{2},ik(\lambda)}\left(\frac{\gamma}{x}\right)}{\Gamma\left(-\frac{\beta}{2}-ik(\lambda)\right)}\right), \tag{4.19}$$

and $\mathcal{M}$ is the Whittaker function.

From (Buchholz 1969, p. 94, Equation (1))

$$\mathcal{M}_{\frac{\beta+1}{2},ik(\lambda)}\left(\frac{\gamma}{x}\right)=\frac{\left(\frac{\gamma}{x}\right)^{\frac{1}{2}+ik(\lambda)}}{\Gamma\left(1+2ik(\lambda)\right)}\left(1+O(|2ik(\lambda)|^{-1})\right),\ \lambda\to\infty. \tag{4.20}$$

Using (4.19) together with

$$\Gamma\left(2ik(\lambda)\right)\,\Gamma\left(1-2ik(\lambda)\right)=\frac{\pi}{\sin\left(2i\pi k(\lambda)\right)}$$

we obtain

$$\left|\mathcal{W}_{\frac{\beta+1}{2},ik(\lambda)}\left(\frac{\gamma}{x}\right)\right|\leq\left|\Gamma\left(2ik(\lambda)\right)\,\Gamma\left(1-2ik(\lambda)\right)\right|\left(\frac{\left|\mathcal{M}_{\frac{\beta+1}{2},-ik(\lambda)}\left(\frac{\gamma}{x}\right)\right|}{\left|\Gamma\left(-\frac{\beta}{2}+ik(\lambda)\right)\right|}+\frac{\left|\mathcal{M}_{\frac{\beta+1}{2},ik(\lambda)}\left(\frac{\gamma}{x}\right)\right|}{\left|\Gamma\left(-\frac{\beta}{2}-ik(\lambda)\right)\right|}\right).$$

Now, (4.20) implies

$$\begin{aligned}\left|\mathcal{W}_{\frac{\beta+1}{2},ik(\lambda)}\left(\frac{\gamma}{x}\right)\right|\leq\left(\frac{\gamma}{x}\right)^{\frac{1}{2}}&\left(\frac{1}{\left|\Gamma\left(1+2ik(\lambda)\right)\right|\left|\Gamma\left(-\frac{\beta}{2}+ik(\lambda)\right)\right|}+\frac{1}{\left|\Gamma\left(1-2ik(\lambda)\right)\right|\left|\Gamma\left(-\frac{\beta}{2}-ik(\lambda)\right)\right|}\right)\\&\times\left|\Gamma\left(2ik(\lambda)\right)\,\Gamma\left(1-2ik(\lambda)\right)\right|\left(1+O(|2ik(\lambda)|^{-1})\right),\ \lambda\to\infty.\end{aligned}$$

It follows that as $\lambda\to\infty$

$$\begin{aligned}|h(\lambda)|\leq&\left(\frac{\gamma^2}{xx_0}\right)^{-\frac{\beta}{2}}e^{\frac{\gamma}{2}\left(\frac{1}{x}+\frac{1}{x_0}\right)}\left(\frac{1}{\left|\Gamma\left(1+2ik(\lambda)\right)\right|\left|\Gamma\left(-\frac{\beta}{2}+ik(\lambda)\right)\right|}+\frac{1}{\left|\Gamma\left(1-2ik(\lambda)\right)\right|\left|\Gamma\left(-\frac{\beta}{2}-ik(\lambda)\right)\right|}\right)^2\\&\times\mathcal{E}_\alpha(-\lambda t^\alpha)\,\frac{1}{|k(\lambda)|}\left|\frac{\Gamma^{\frac{1}{2}}(\beta)\,\Gamma\left(-\frac{\beta}{2}+ik(\lambda)\right)}{\Gamma\left(2ik(\lambda)\right)}\right|^2\left|\Gamma\left(2ik(\lambda)\right)\,\Gamma\left(1-2ik(\lambda)\right)\right|^2\left(1+O(|2ik(\lambda)|^{-1})\right).\end{aligned}$$

Since

$$\Gamma(x+iy)\sim\sqrt{2\pi}\cdot|y|^{x-\frac{1}{2}}\cdot e^{-\pi\frac{|y|}{2}},\ \ |y|\to\infty, \tag{4.21}$$

and from (Simon 2014, Equation (6.8)) for $0<\alpha<1$ we have

$$\frac{1}{1+\Gamma(1-\alpha)\,\lambda\,t^\alpha}\leq\mathcal{E}_\alpha(-\lambda t^\alpha)\leq\frac{1}{1+\Gamma(1+\alpha)^{-1}\,\lambda\,t^\alpha}, \tag{4.22}$$

it yields

$$|h(\lambda)| \leq \left(\frac{\gamma^2}{xx_0}\right)^{-\frac{\beta}{2}} e^{\frac{\gamma}{2}\left(\frac{1}{x}+\frac{1}{x_0}\right)} \frac{1}{1+\Gamma\left(1+\alpha\right)^{-1}\lambda t^{\alpha}} \frac{\Gamma(\beta)}{|k(\lambda)|} \left(1+O(|2ik(\lambda)|^{-1})\right), \ \lambda \to \infty$$

and therefore

$$|h(\lambda)| = O(\lambda^{-\frac{3}{2}}) \text{ as } \lambda \to \infty.$$

Finally, according to (Olver et al. 2010, p. 335), since $\lambda \mapsto \psi(x, \lambda)$ is an entire function for a fixed x, $\lambda \mapsto |h(\lambda)|$ is also an entire function. This verifies (4.18) and completes the proof of the spectral representation of the transition density of the fractional RG diffusion. □

Theorem 4.2 The transition density of fractional FS diffusion is given by

$$p_\alpha(x,t;x_0) = \sum_{n=0}^{\lfloor\frac{\beta}{4}\rfloor} \mathfrak{fs}(x)\, F_n(x_0)\, F_n(x)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) + \frac{\mathfrak{fs}(x)}{\pi} \int_{\Lambda}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, a(\lambda)\, f_1(x_0,-\lambda), f_1(x,-\lambda)\, d\lambda, \tag{4.23}$$

where $\mathfrak{fs}(x)$ is given by (3.72), the FS polynomials F_n are given by (3.84), function f_1 is given by (3.87) with $a(\lambda)$ given by (3.86) and $\Lambda = \frac{\theta\beta^2}{8(\beta-2)}$ is the cutoff (3.77).

Proof. Since the FS diffusion $X(t)$ is independent of the time change $E(t)$, using spectral representation of Fisher-Snedecor diffusion (3.85) and Laplace transform of $E(t)$ (2.16) together with the Fubini argument, we have:

$$\begin{aligned} P(X_\alpha(t) \in B | X_\alpha(0) = x_0) &= \int_0^\infty P(X_1(\tau) \in B | X_1(0) = x_0)\, f_t(\tau)\, d\tau \\ &= \int_0^\infty \int_B p_1(x,\tau;x_0)\, f_t(\tau)\, dx\, d\tau \\ &= \int_B \int_0^\infty (p_d(x,\tau;x_0) + p_c(x,\tau;x_0))\, f_t(\tau)\, d\tau\, dx \end{aligned} \tag{4.24}$$

$$\begin{aligned} &= \int_B \left[\int_0^\infty \sum_{n=0}^{\lfloor\frac{\beta}{4}\rfloor} \mathfrak{fs}(x)\, F_n(x_0)\, F_n(x)\, e^{-\lambda_n \tau}\, f_t(\tau)\, d\tau + \frac{\mathfrak{fs}(x)}{\pi} \int_0^\infty \int_\Lambda^\infty e^{-\lambda\tau}\, f_t(\tau)\, a(\lambda) f_1(x_0,-\lambda)\, f_1(x,-\lambda)\, d\lambda\, d\tau \right] dx \\ &= \int_B \left[\sum_{n=0}^{\lfloor\frac{\beta}{4}\rfloor} \mathfrak{fs}(x)\, F_n(x_0)\, F_n(x)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) + \frac{\mathfrak{fs}(x)}{\pi} \int_\Lambda^\infty \mathcal{E}_\alpha(-\lambda t^\alpha)\, a(\lambda)\, f_1(x_0,-\lambda)\, f_1(x,-\lambda)\, d\lambda \right] dx. \end{aligned} \tag{4.25}$$

Change of the order of integration in (4.24) is justified by the Fubini-Tonelli Theorem since functions p_1 and f_t are non-negative. Change of the order of integration in (4.25) cannot be justified by the Fubini-Tonelli Theorem as in (4.24) since the integrand

$$g(\lambda,\tau) = e^{-\lambda\tau}\, f_t(\tau)\, a(\lambda)\, f_1(x_0,-\lambda)\, f_1(x,-\lambda)$$

is not necessarily non-negative. In order to use the Fubini Theorem, we need to show that

$$\int_{\Lambda}^{\infty}\int_{0}^{\infty} |g(\lambda,\tau)|\, d\tau\, d\lambda < \infty. \tag{4.26}$$

Let

$$h(\lambda) = \mathcal{E}_\alpha(-\lambda t^\alpha)\, a(\lambda)\, f_1(x_0,-\lambda)\, f_1(x,-\lambda).$$

Since

$$\begin{aligned}\int_{\Lambda}^{\infty}\int_{0}^{\infty} |g(\lambda,\tau)|\, d\tau\, d\lambda &= \int_{\Lambda}^{\infty}\int_{0}^{\infty} e^{-\lambda\tau}\, f_t(\tau)\, |a(\lambda)\, f_1(x_0,-\lambda)\, f_1(x,-\lambda)|\, d\tau\, d\lambda \\ &= \int_{\Lambda}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, |a(\lambda)\, f_1(x_0,-\lambda)\, f_1(x,-\lambda)|\, d\lambda, \\ &= \int_{\Lambda}^{\infty} |h(\lambda)|\, d\lambda\end{aligned}$$

we need to show that

$$\int_{\Lambda}^{\infty} |h(\lambda)|\, d\lambda < \infty.$$

From (Erdelyi 1981, p. 77, Equation (17)),

$$\begin{aligned}f_1(x,-\lambda) = &\frac{\Gamma\left(1+\frac{\beta}{4}+ik(\lambda)\right)\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\gamma}{2}+\frac{\beta}{4}+ik(\lambda)\right)}\cdot 2^{-\frac{\beta}{2}-1}\cdot\left(1-e^{\xi}\right)^{-\frac{\gamma}{2}+\frac{1}{2}}\cdot\left(1+e^{\xi}\right)^{\frac{\gamma}{2}+\frac{\beta}{2}-\frac{1}{2}} \\ &\cdot (ik(\lambda))^{-\frac{1}{2}}\cdot\left(e^{\xi(\frac{\beta}{4}+ik(\lambda))}+e^{i\pi(\frac{\gamma}{2}-\frac{1}{2})}\cdot e^{\xi(\frac{\beta}{4}-ik(\lambda))}\right)\left(1+O(|k(\lambda)|^{-1})\right),\ \lambda\to\infty,\end{aligned}$$

where $e^{\xi} = 1+\frac{2\gamma}{\beta}x+\sqrt{\frac{4\gamma}{\beta}x(1+\frac{\gamma}{\beta}x)}$.
It follows that

$$\begin{aligned}|f_1(x,-\lambda)| \le &\left|\frac{\Gamma\left(1+\frac{\beta}{4}+ik(\lambda)\right)\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\gamma}{2}+\frac{\beta}{4}+ik(\lambda)\right)}\right| 2^{-\frac{\beta}{2}-1}\cdot|1-e^{\xi}|^{-\frac{\gamma}{2}+\frac{1}{2}}\cdot|1+e^{\xi}|^{\frac{\gamma}{2}+\frac{\beta}{2}-\frac{1}{2}} \\ &\times|k(\lambda)|^{-\frac{1}{2}}\cdot e^{\xi\frac{\beta}{4}}\left(1+|e^{i\pi(\frac{\gamma}{2}-\frac{1}{2})}|\right)\left(1+O(|k(\lambda)|^{-1})\right),\ \lambda\to\infty.\end{aligned}$$

Now we have

$$\begin{aligned}|h(\lambda)| \le\ &\mathcal{E}_\alpha(-\lambda t^\alpha)\left|\frac{B^{\frac{1}{2}}\left(\frac{\gamma}{2},\frac{\beta}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right|^2\left|\frac{\Gamma\left(-\frac{\beta}{4}+ik(\lambda)\right)\Gamma\left(1+\frac{\beta}{4}+ik(\lambda)\right)}{\Gamma\left(1+2ik(\lambda)\right)}\right|^2 2^{-\beta-2}\cdot|(1-e^{\xi})(1-e^{\xi_0})|^{-\frac{\gamma}{2}+\frac{1}{2}} \\ &\times\left|(1+e^{\xi})(1+e^{\xi_0})\right|^{\frac{\gamma}{2}+\frac{\beta}{2}-\frac{1}{2}}\cdot e^{(\xi+\xi_0)\frac{\beta}{4}}\left(1+|e^{i\pi(\frac{\gamma}{2}-\frac{1}{2})}|\right)^2\left(1+O(|k(\lambda)|^{-1})\right),\ \lambda\to\infty.\end{aligned}$$

Using (4.21) and (4.22) we obtain

$$|h(\lambda)| \leq \frac{1}{1+\Gamma(1+\alpha)^{-1}\,\lambda\, t^{\alpha}} \cdot B\left(\frac{\gamma}{2}, \frac{\beta}{2}\right) \cdot |k(\lambda)|^{-1}\, 2^{-\beta-2} \cdot |(1-e^{\xi})(1-e^{\xi_0})|^{-\frac{\gamma}{2}+\frac{1}{2}}$$
$$\times \left|(1+e^{\xi})(1+e^{\xi_0})\right|^{\frac{\gamma}{2}+\frac{\beta}{2}-\frac{1}{2}} \cdot e^{(\xi+\xi_0)\frac{\beta}{4}} \left(1+|e^{i\pi(\frac{\gamma}{2}-\frac{1}{2})}|\right)^2 \left(1+O(|k(\lambda)|^{-1})\right),\ \lambda \to \infty.$$

Now, it follows that

$$|h(\lambda)| = O(\lambda^{-\frac{3}{2}}),\ \text{as}\ \lambda \to \infty.$$

Finally, according to (Erdelyi 1981, p. 68), $\lambda \mapsto f_1(x,\lambda)$ is an entire function for a fixed x and so $\lambda \mapsto |h(\lambda)|$ is also an entire function. This verifies (4.26) and completes the proof of the spectral representation of the transition density of the fractional FS diffusion. □

Remark 4.5. Just like in the non-heavy-tailed case, comparing transition densities (4.15) and (4.23) of fractional RG and FS diffusion, with transition densities (3.101) and (3.85) of RG and FS diffusion, respectively, one concludes that in the fractional case, in spectral representation of the transition density, the role of the exponential function $e^{-\lambda t}$ is taken by the Mittag-Leffler function $\mathcal{E}_\alpha(-\lambda t)$. Moreover, that makes the only difference in the representations and again indicates slower decay in the fractional case in comparance to the non-fractional case (cf. Section 4.1). In the next section, fractional RG and FS diffusion are established as a stochastic model for corresponding fractional Cauchy problem involving infinitesimal generator, i.e. fractional Cauchy problem (4.10).

Remark 4.6. Recall that beside the RG and FS diffusion, there is another heavy-tailed Pearson diffusion: Student diffusion. It would be natural to establish similar result regarding fractional Student diffusion, but since the non-fractional case of this diffusion doesn't have known exact spectral representation of the transition density, this is not possible. In particular, continuous part of the spectral representation of transition density provides only theoretical form, without exact normalizing constants, and therefore techniques used in the proofs for fractional RG and FS diffusion cannot be recovered.

4.3.2 Strong solutions of time-fractional Kolmogorov equations

Based on the proved results regarding spectral representation of transition densities of fractional RG and FS diffusions, more precisely, based on spectral representations (4.15) and (4.23), we establish solutions of fractional Cauchy problems (4.10) and (4.11), which associate time-fractional Kolmogorov backward and forward equations.

To establish the main result of this section, we need the following Lemma.

Lemma 4.3 For the reciprocal gamma and Fisher-Snedecor diffusions, the family of operators

$$T_t\, g(y) = \mathbb{E}[g(X(t))\,|\,X(0)=y], \quad t \geq 0$$

forms a strongly continuous bounded (C_0) semigroup on the space of bounded continuous functions g on $[0,\infty\rangle$ vanishing at infinity.

Proof. The proof of this Lemma is the same as for the non-heavy-tailed diffusions considered in Leonenko et al. (2013*b*). We provide it here for completeness. The semigroup property follows from the Chapman-Kolmogorov equation for the reciprocal gamma and Fisher-Snedecor diffusions, and uniform boundedness of the semigroup on the above Banach space of continuous functions with the supremum norm follows from (Friedman 1975, Theorem 3.4). Therefore, the family of operators $\{T(t),\, t \geq 0\}$ forms a uniformly bounded semigroup on the respective Banach space of continuous functions, with the supremum norm. Next, we show the pointwise continuity of the semigroup. For any fixed $y \in \langle 0,\infty\rangle$

$$\begin{aligned}
T_t\, g(y) - g(y) &= \int_l^L p_1(x,t;y)(g(x)-g(y))dx \\
&= \int_{|x-y|\leq\epsilon\cap\langle 0,\infty\rangle} p_1(x,t;y)(g(x)-g(y))dx \\
&\qquad + \int_{|x-y|>\epsilon\cap\langle 0,\infty\rangle} p_1(x,t;y)(g(x)-g(y))dx \\
&\leq \sup_{|x-y|\leq\epsilon\cap\langle 0,\infty\rangle} |g(x)-g(y)| \int_{|x-y|\leq\epsilon\cap\langle 0,\infty\rangle} p_1(x,t;y)dx \\
&\qquad + C\int_{|x-y|>\epsilon\cap\langle 0,\infty\rangle} p_1(x,t;y)dx
\end{aligned}$$

since the function g is bounded. Since $\int_{|x-y|>\epsilon\cap\langle 0,\infty\rangle} p_1(x,t;y)dx \to 0$ as $t \to 0$ for any $\epsilon > 0$ (see Karlin & Taylor (1981*a*), p. 158), the second term in the above expression tends to zero as $t \to 0$. The first term is bounded by $\sup_{|x-y|\leq\epsilon\cap\langle 0,\infty\rangle} |g(x)-g(y)|$, which tends to zero as $\epsilon \to 0$ because of the continuity of g. Pointwise continuity then implies strong continuity in view of (Rogers & Williams 1994, Lemma 6.7).

□

The next result gives the strong solutions to the fractional Cauchy problem associated with the time-fractional backward Kolmogorov equation for fractional RG and FS diffusion.

Theorem 4.4 For any g from the domain of the generator $\mathcal{G}$ specified in (3.81) for RG and (3.81) for FS case, a strong (classical) solution to the fractional Cauchy problem (4.10) is given by

$$q_\alpha(t;y) = \int_0^\infty p_\alpha(x,t;y)g(x)dx, \tag{4.27}$$

where the transition density p_α is given by equation (4.15) in the reciprocal gamma case and by equation (4.23) in the Fisher-Snedecor case.

Proof. The proof of this Theorem consists of several steps. First, according to Lemma 4.3 and (Arendt et al. 2011, Proposition 3.1.9)

$$q(y;t) = T_t\, g(y) = \mathbb{E}[g(X(t))\,|\,X(0)=y],\ t \geq 0$$

solves the non-fractional Cauchy problem

$$\frac{\partial q(y,t)}{\partial t} = \mathcal{G}q(y,t), \quad q(y,0) = g(y).$$

Second, strong continuity of the semigroup in the Banach space of continuous functions with the supremum norm and Theorem 3.1 in Baeumer & Meerschaert (2001) show that

$$S_t\, g(y) = \int_0^\infty T_u\, g(y)\, f_t(u)\, du, \tag{4.28}$$

where f_t is the density of the inverse stable subordinator E_t given by (2.17), solves the fractional Cauchy problem (4.10) for any g from the domain of the generator $\mathcal{G}$.

Third, since

$$\begin{aligned} S_t\, g(y) &= \int_0^\infty T_u\, g(y)\, f_t(u)\, du \\ &= \int_0^\infty \mathbb{E}[g(X(t)) \,|\, X(0) = y]\, f_t(u)\, du \\ &= \mathbb{E}[g(X(E(t))) \,|\, X(E(0)) = y] \\ &= \mathbb{E}[g(X_\alpha(t)) \,|\, X_\alpha(0) = y], \end{aligned}$$

where $E(0) = 0$ almost surely, and

$$\mathbb{E}[g(X_\alpha(t)) \,|\, X_\alpha(0) = y] = \int_0^\infty p_\alpha(x,t;y)\, g(x)\, dx,$$

a strong solution to (4.10) is given by (4.27).

□

Remark 4.7. The explicit expressions for strong solutions (4.27) of the fractional Cauchy problem associated with the infinitesimal generators of the reciprocal gamma and Fisher-Snedecor diffusions are:

$$\begin{aligned} u_{\mathfrak{rg}}(t;y) = &\sum_{n=0}^{\lfloor \frac{\beta}{2} \rfloor} B_n(y)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) \int_0^\infty B_n(x)\, \mathfrak{rg}(x)\, g(x)\, dx \\ &+ \int_0^\infty \frac{\mathfrak{rg}(x)\, g(x)}{4\pi} \int_{\frac{\theta\beta^2}{4(\beta-1)}}^\infty \mathcal{E}_\alpha(-\lambda t^\alpha)\, b(\lambda)\, \psi(x,-\lambda)\, \psi(y,-\lambda)\, d\lambda\, dx \end{aligned}$$

and

$$u_{\mathfrak{fs}}(t;y) = \sum_{n=0}^{\lfloor \frac{\beta}{4} \rfloor} F_n(y)\mathcal{E}_\alpha(-\lambda_n t^\alpha) \int_0^\infty F_n(x)\,\mathfrak{fs}(x)\,g(x)\,dx$$
$$+ \frac{1}{\pi}\int_0^\infty \mathfrak{fs}(x)\,g(x) \int_{\frac{\theta\beta^2}{8(\beta-2)}}^\infty \mathcal{E}_\alpha(-\lambda t^\alpha)a(\lambda)f_1(y,-\lambda)f_1(x,-\lambda)d\lambda\,dx.$$

The explicit expressions for strong solutions of the Cauchy problem for the fractional Fokker-Planck equations were obtained in Leonenko et al. (2013*b*) for all three non-heavy-tailed fractional Pearson diffusions using their spectral properties. Since the structure of the spectrum for the reciprocal gamma and Fisher-Snedecor diffusions are much more complex than in the non-heavy-tailed cases, strong solutions of Cauchy problems associated with the fractional Fokker-Planck equation are not presented here. Below we state the result on the L^2 solutions. Proving that these are also strong solutions that hold pointwise remains an open problem at this time.

Theorem 4.5 The fractional Cauchy problem (4.11), i.e. the Cauchy problem

$$\partial_t^\alpha q(x,t) = -\frac{\partial}{\partial x}\left(\mu(x)q(x,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x)q(x,t)\right),\ q(x,0) = f(x), \tag{4.29}$$

where f is twice continuously differentiable function that vanishes at zero and has a compact support, is solved by

$$q(x,t) = \int_0^\infty p_\alpha(x,t;y)f(y)dy. \tag{4.30}$$

The transition density p_α is given by equation (4.15) in the reciprocal gamma case and by equation (4.23) in the Fisher-Snedecor case, and the solution is in the following sense: for every $t > 0$, $q(x,t)$ given by (4.30) satisfies (4.29), and the equality holds in the space of functions $\{q(\cdot,t) \in L^2(\langle 0,\infty\rangle)\}$.

Proof. Infinitesimal generator of the RG and FS diffusions with corresponding domains (3.97) and (3.81), respectively, and with state space $I = \langle 0,\infty\rangle$ is self-adjoint differential operator (see Section 3.1). We consider the space $L^2(\langle 0,\infty\rangle)$ without the weight $\mathfrak{p}$, and the generator defined on a subset of its domain, namely on the set of functions $f \in L^2(\langle 0,\infty\rangle) \cap C^2(\langle 0,\infty\rangle)$ such that f vanishes at 0 and has compact support.

The Fokker-Planck operator

$$\mathcal{L}f(x) = -\frac{\partial}{\partial x}\left(\mu(x)f(x)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x)f(x)\right)$$

is adjoint to the infinitesimal generator $\mathcal{G}$ of the diffusion on this subspace of $L^2(\langle 0,\infty\rangle)$. Recall that the semigroup T_t is a C_0-semigroup in this space as well as in the space of continuous functions with the supremum norm. From (Pazy 1983, Corollary 10.6), the adjoint semigroup T_t^*

$$T_t^* f(x) = \int_0^\infty p_1(x,t;y) f(y) dy$$

is the C_0-semigroup as well, and its generator is the Fokker-Planck operator.

Since $T_t^* f$ solves the non-fractional Cauchy problem for the Fokker-Planck equation (Arendt et al. 2011, Proposition 3.1.9), just like in the proof of Theorem 4.4, the application of Theorem 3.1 in Baeumer & Meerschaert (2001) completes the proof. □

4.3.3 Stationary distributions

We now show that as $t \to \infty$, the distribution of $X_\alpha(t)$, where $\{X_\alpha(t),\, t \geq 0\}$ is either fractional RG or fractional FS diffusion, approaches the stationary distribution of the non-fractional RG or FS Pearson diffusion, respectively.

Theorem 4.6 Let $\{X_\alpha(t),\, t \geq 0\}$ be the fractional reciprocal gamma or Fisher-Snedecor diffusion and let $p_\alpha(x,t)$ be the density of $X_\alpha(t)$. Assume that $X_\alpha(0)$ has a twice continuously differentiable density f that vanishes at infinity. Then

$$p_\alpha(x,t) \to \mathfrak{p}(x),\ \ t \to \infty,$$

where $\mathfrak{p}(\cdot)$ is the stationary density of the non-fractional reciprocal gamma or Fisher-Snedecor diffusion, respectively.

Proof. Using the definition of the transition density $p_\alpha(x,t;y)$ (4.2), we have

$$p_\alpha(x,t) = \int_0^\infty p_\alpha(x,t;y) f(y) dy$$

and therefore it suffices to prove that

$$p_\alpha(x,t;y) \to \mathfrak{p}(x) \quad \text{as} \quad t \to \infty$$

for fixed x and y. This together with the fact that $f(y)$ and $p_\alpha(x,t;y)$ are density functions then yields

$$\int_0^\infty p_\alpha(x,t;y) f(y) dy \to \mathfrak{p}(x) \int_0^\infty f(y) dy = \mathfrak{p}(x) \text{ as } t \to \infty.$$

We treat the reciprocal gamma and Fisher-Snedecor cases separately.

Reciprocal gamma diffusion

Since $\lambda_0 = 0$, it follows that

$$\begin{aligned} p_\alpha(x,t;y) &= \sum_{n=0}^{\lfloor \frac{\beta}{2} \rfloor} \mathfrak{rg}(x)\, B_n(x)\ B_n(y)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) + \frac{\mathfrak{rg}(x)}{4\pi} \int\limits_{\frac{\theta\beta^2}{4(\beta-1)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, b(\lambda)\, \psi(x,-\lambda)\, \psi(y,-\lambda)\, d\lambda \\ &= \mathfrak{rg}(x) + \sum_{n=1}^{\lfloor \frac{\beta}{2} \rfloor} \mathfrak{rg}(x)\, B_n(x)\, B_n(y)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) + \frac{\mathfrak{rg}(x)}{4\pi} \int\limits_{\frac{\theta\beta^2}{4(\beta-1)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, b(\lambda)\, \psi(x,-\lambda)\, \psi(y,-\lambda)\, d\lambda. \end{aligned}$$

For a constant c such that

$$c \geq \frac{4(\beta-1)}{\theta\beta^2 t^\alpha} + \Gamma(1-\alpha) \geq \frac{1}{\lambda t^\alpha} + \Gamma(1-\alpha) > 0$$

from (4.22) we obtain

$$\frac{1}{c\lambda t^\alpha} \leq \frac{1}{1+\Gamma(1-\alpha)\lambda t^\alpha} \leq \mathcal{E}_\alpha(-\lambda t^\alpha) \leq \frac{1}{1+\Gamma(1+\alpha)^{-1}\lambda t^\alpha}.$$

Now it follows that

$$\frac{1}{t^\alpha} \int\limits_{\frac{\theta\beta^2}{4(\beta-1)}}^{\infty} \frac{1}{c\lambda}\, b(\lambda)\, \psi(x,-\lambda)\, \psi(y,-\lambda)\, d\lambda \leq \int\limits_{\frac{\theta\beta^2}{4(\beta-1)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, b(\lambda)\, \psi(x,-\lambda)\, \psi(y,-\lambda)\, d\lambda,$$

$$\int\limits_{\frac{\theta\beta^2}{4(\beta-1)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, b(\lambda)\, \psi(x,-\lambda)\, \psi(y,-\lambda)\, d\lambda \leq \frac{1}{t^\alpha} \int\limits_{\frac{\theta\beta^2}{4(\beta-1)}}^{\infty} \frac{\Gamma(1+\alpha)}{\lambda}\, b(\lambda)\, \psi(x,-\lambda)\, \psi(y,-\lambda)\, d\lambda.$$

Letting $t \to \infty$ yields

$$\int\limits_{\frac{\theta\beta^2}{4(\beta-1)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, b(\lambda)\, \psi(x,-\lambda)\, \psi(y,-\lambda)\, d\lambda \to 0\,,\ t \to \infty$$

and

$$\sum_{n=1}^{\lfloor \frac{\beta}{2} \rfloor} \mathfrak{rg}(x)\, B_n(x)\, B_n(y)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) \to 0,\ t \to \infty.$$

Therefore

$$p_\alpha(x,t;y) \to \mathfrak{rg}(x), \quad x > 0 \text{ as } t \to \infty.$$

Fisher-Snedecor diffusion

Since $\lambda_0 = 0$, it follows that

$$\begin{aligned} p_\alpha(x,t;y) &= \sum_{n=0}^{\lfloor \frac{\beta}{4} \rfloor} \mathfrak{fs}(x)\, F_n(y)\, F_n(x)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) + \frac{\mathfrak{fs}(x)}{\pi} \int_{\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, a(\lambda)\, f_1(y,-\lambda)\, f_1(x,-\lambda)\, d\lambda \\ &= \mathfrak{fs}(x) + \sum_{n=1}^{\lfloor \frac{\beta}{4} \rfloor} \mathfrak{fs}(x)\, F_n(y)\, F_n(x)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) + \frac{\mathfrak{fs}(x)}{\pi} \int_{\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, a(\lambda)\, f_1(y,-\lambda)\, f_1(x,-\lambda)\, d\lambda. \end{aligned}$$

Let c be a constant such that

$$c \geq \frac{8(\beta-2)}{\theta\beta^2 t^\alpha} + \Gamma(1-\alpha) \geq \frac{1}{\lambda t^\alpha} + \Gamma(1-\alpha) > 0.$$

From (4.22) we obtain

$$\frac{1}{c\lambda t^\alpha} \leq \frac{1}{1+\Gamma(1-\alpha)\lambda t^\alpha} \leq \mathcal{E}_\alpha(-\lambda t^\alpha) \leq \frac{1}{1+\Gamma(1+\alpha)^{-1}\lambda t^\alpha}.$$

Therefore,

$$\frac{1}{t^\alpha} \int_{\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} \frac{1}{c\lambda}\, a(\lambda)\, f_1(y,-\lambda)\, f_1(x,-\lambda)\, d\lambda \leq \int_{\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, a(\lambda)\, f_1(y,-\lambda)\, f_1(x,-\lambda)\, d\lambda,$$

$$\int_{\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, a(\lambda)\, f_1(y,-\lambda)\, f_1(x,-\lambda)\, d\lambda \leq \frac{1}{t^\alpha} \int_{\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} \frac{\Gamma(1+\alpha)}{\lambda}\, a(\lambda)\, f_1(y,-\lambda)\, f_1(x,-\lambda)\, d\lambda.$$

Letting $t \to \infty$ yields

$$\int_{\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, a(\lambda)\, f_1(y,-\lambda)\, f_1(x,-\lambda)\, d\lambda \;\to\; 0, \text{ as } t \to \infty$$

and

$$\sum_{n=1}^{\lfloor \frac{\beta}{4} \rfloor} \mathfrak{fs}(x)\, F_n(y)\, F_n(x)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) \to 0, \text{ as } t \to \infty.$$

Therefore

$$p_\alpha(x,t;y) \to \mathfrak{fs}(x), \quad x > 0 \text{ as } t \to \infty.$$

□

4.4 Correlation structure of fractional Pearson diffusions

Assume that $\{X(t),\, t \geq 0\}$ is a stationary Pearson diffusion and that its parameters are such that the stationary distribution has finite second moment. Then the correlation function of $X(t)$ is given by

$$\text{Corr}\,[X(t),\, X(s)] = \exp(-\theta|t-s|), \tag{4.31}$$

where $\theta > 0$ is the autocorrelation parameter. Since the autocorrelation function (4.31) falls off exponentially, Pearson diffusions exhibit short-range dependence.
In general, let $\{X(t),\, t \geq 0\}$ be the non-stationary stochastic process with the correlation function $\text{Corr}(X(t), X(s))$ which satisfies

$$\text{Corr}(X(t), X(s)) \sim c(s)\, t^{-d},\ t \to \infty,$$

i.e.,

$$\lim_{t\to\infty} \frac{\text{Corr}(X(t), X(s))}{t^{-d}} = c(s),$$

for a fixed $s > 0$, some constant $c(s) > 0$ and $d > 0$.
We say that $\{X(t),\, t \geq 0\}$ has the long-range dependence property if $d \in \langle 0, 1\rangle$ and the short-range dependence property if $d \in \langle 1, 2\rangle$.

Remark 4.8. This definition is further used in Section 6, in order to establish long-range dependence property for the delayed continuous-time autoregressive processes.

Next, we show that fractional Pearson diffusions have this property.
We say that fractional Pearson diffusion $\{X_\alpha(t),\, t \geq 0\}$ is in the steady state if it starts from invariant distribution from corresponding non-fractional Pearson diffusion with the probability density $\mathfrak{p}$. Then the correlation function of $X_\alpha(t) = X(E(t))$ is given by

$$\text{Corr}\,[X_\alpha(t),\, X_\alpha(s)] = \mathcal{E}_\alpha(-\theta t^\alpha) + \frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)} \int_0^{s/t} \frac{\mathcal{E}_\alpha(-\theta t^\alpha(1-z)^\alpha)}{z^{1-\alpha}} dz \tag{4.32}$$

for $t \geq s > 0$.

The proof of this fact for non-heavy-tailed fractional Pearson diffusions is given in (Leonenko et al. 2013*a*, Theorem 3.1). The proof does not depend on the type of invariant Pearson distribution, and therefore the same proof can be repeated for all three heavy-tailed fractional Pearson diffusions, provided that the tails are not too heavy so that the second moment of the corresponding heavy-tailed Pearson distribution exists. In particular, required parameter restrictions are:

- in the fractional RG diffusion case, restriction on the parameters of density (3.89) ensuring the existence of second moment is $\beta > 2$;

- in the fractional FS diffusion case, restriction on the parameters of density (3.72) ensuring the existence of second moment is $\beta > 4$;
- in the fractional ST diffusion case, restriction on the parameters of density (3.105) ensuring the existence of second moment is $\nu > 2$.

In particular, the autocorrelation function (4.32) falls off like power law with exponent $\alpha \in \langle 0, 1\rangle$, i.e. for any fixed $s > 0$

$$\mathrm{Corr}[X_\alpha(t), X_\alpha(s)] = \frac{1}{t^\alpha \Gamma(1-\alpha)} \left(\frac{1}{\theta} + \frac{s^\alpha}{\Gamma(1+\alpha)} \right) (1 + o(1)) \text{ as } t \to \infty.$$

Therefore, unlike non-fractional Pearson diffusions, their fractional analogues are long-range dependent processes.

4.5 Fractional Pearson diffusions as solutions of stochastic differential equations

The fractional Pearson diffusions defined via a time-change of the ordinary (non-fractional) Pearson diffusions satisfy the special types of SDEs considered in Kobayashi (2011). Let $\{\mathcal{F}_t, t \geq 0\}$ be the natural filtration associated with the Brownian motion from equation (3.29). Since $E(t)$ has almost surely continuous sample paths, for any $t > 0$ $[E(t-), E(t)]$ contains only one point, and the Brownian motion and the solution of SDE (3.29) $X(t)$ are in synchronization with the time change $E(t)$ as defined in Kobayashi (2011). This synchronization is key to the stochastic calculus for the semimartingale $B(E(t))$ with respect to filtration $\{\mathcal{F}_{E(t)}, t \geq 0\}$. See also Magdziarz & Schilling (2015) for the discussion on martingale properties of $B(E(t))$ and other processes obtained as time-changes of the Brownian motion using inverse of subordinators.

Specialized to fractional Pearson diffusions, according to the duality Theorem (Kobayashi 2011, Theorem 4.2, part (1)), when the process $X(t)$ satisfies SDE (3.29) with the initial condition $X(0) = X_0$, the time-changed process $X_\alpha(t) = X(E(t))$ satisfies the SDE

$$dX_\alpha(t) = \mu(X_\alpha(t))dE(t) + \sigma(X_\alpha(t))dB(E(t)) \tag{4.33}$$

with the initial condition $X_\alpha(0) = X_0$.

Further, when μ and σ^2 are polynomials of the first and second degree, respectively, as specified in (3.29), (Kobayashi 2011, Lemma 4.1) ensures the existence and uniqueness of the strong solution of (4.33), giving another possible definition of the fractional Pearson diffusions as solutions of this SDE. The solution can be represented in the following integral form:

$$X_\alpha(t) = X(E(t)) = \int_0^{E(t)} (a_0 + a_1 X(s))\, ds + \int_0^{E(t)} \sqrt{2(b_0 + b_1 X(s) + b_2 (X(s))^2)} dB(s).$$

The integrals in this representation are the Lebesgue and the Itô integrals under the continuous time change $t \mapsto E(t)$, in light of the change-of-variable formula from (Kobayashi 2011, Theorem 3.1). This representation could be useful for simulating paths of fractional Pearson diffusions. The discrete schemes for the underlying densities and their error bounds can be found in Kelbert et al. (2016). For similar approaches to obtaining solutions of such SDEs we refer to Scalas & Viles (2014).

Chapter 5

Correlated continuous time random walks and fractional Pearson diffusions

In this Section we define correlated CTRWs, some of which are based on famous Bernoulli-Laplace and Wright-Fisher urn-scheme models and prove their weak convergence in J_1 topology to the class of fractional Pearson diffusions.

5.1 Correlated continuous time random walks

In Section 2 connection between fractional calculus, statistical physics and probability is established. Recall that the CTRW $S(N(t))$ is defined via two random walks $S(n) = Y_1 + Y_2 + \cdots Y_n$ and $T(n) = G_1 + G_2 + \cdots G_n$, where Y_i are iid particle jumps and G_i are iid waiting times between particle jumps, where Y_1 and G_1 are in the domain of attraction of β-stable and α-stable random variables, respectively, where $0 < \alpha < 1,\ 0 < \beta \leq 2$ and

$$N(t) = \max\{n \geq 0\colon T(n) \leq t\}.$$

Moreover, if e.g. the following convergence is satisfied

$$n^{-1/\beta} S(\lfloor nt \rfloor) \Rightarrow A(t),\ n \to \infty \tag{5.1}$$

in $\mathbb{D}\,[0, +\infty\rangle$ with J_1 topology, where $\{A(t),\, t \geq 0\}$ is a β-stable Lévy process, then

$$n^{-\alpha/\beta} S(N(\lfloor nt \rfloor)) \Rightarrow A(E(t)),\ n \to \infty \tag{5.2}$$

in $\mathbb{D}\,[0, +\infty\rangle$ with J_1 topology (see Section 2.1). Moreover, time-changed stochastic process $\{A(E(t)),\, t \geq 0\}$ has governing equation (2.19):

$$\partial_t^\alpha m(x,t) = Dp \frac{\partial^\beta}{\partial x^\beta} m(x,t) + Dq \frac{\partial^\beta}{\partial (-x)^\beta} m(x,t),$$

i.e. density of $A(E(t))$

$$m(x,t) = \int_0^\infty p(x,u) f_t(u)\, du,$$

where $f_t(u)$ is the density of $E(t)$ given by (2.17) and $p(x,t)$ is the density of β-stable Lévy process $\{A(t),\, t \geq 0\}$, solves this fractional equation. Therefore, time-changed stochastic process $\{A(E(t)),\, t \geq 0\}$ is a stochastic solution (model) for time-space fractional differential equation (2.19), as well as a scaling limit of the CTRW $S(N(t))$.

Several other Lévy processes can also be obtained as the CTRWs limits by employing trinagular arrays (see Meerschaert & Scheffler (2008)). In general, when particle jumps Y_i and waiting times G_i are independent, CTRW is called decoupled. In this thesis, only decoupled CTRWs are considered (for more information on decoupled CTRWs see Meerschaert & Scalas (2006), while for coupled CTRWs see Germano et al. (2009)). Moreover, when particle jumps are correlated, we emphasize this by saying we observe the correlated CTRW. The case of correlated jumps given by the stationary linear process was considered in Meerschaert et al. (2009), where the outer process in the limit was either a stable Lévy process or a linear fractional stable motion, depending on the strength of the dependence in the particle jump sequence.

In this thesis, specific correlated and decoupled CTRWs are considered in order to obtain fractional diffusion process in the scaling limit. In particular, particle jumps are modeled by a suitably chosen Markov chain $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$, rather then with a random walk $S(n)$, and therefore particle jumps are correlated. Based on this Markov chain, continuous-time stochastic process $\{X^{(n)}(t),\, t \geq 0\}$ is defined:

$$X^{(n)}(t) := H^{(n)}\left(\lfloor h_n^{-1} t \rfloor\right),$$

where $(h_n,\, n \in \mathbb{N})$ is a specific sequence of positive reals tending to zero as $n \to \infty$ (see Section 5.3 for details). If instead of (5.1) we can prove the weak convergence

$$X^n \Rightarrow X,\ n \to \infty \tag{5.3}$$

in $\mathbb{D}([0, +\infty\rangle\,; S)$ with J_1 topology, where $\{X(t),\, t \geq 0\}$ is a diffusion process with state space S, then instead of the time-changed Lévy process as in (5.2), the correlated CTRW has time-changed diffusion process in the limit, i.e.

$$X^{(n)}\left(n^{-1} N\left(n^{1/\alpha} t\right)\right) \Rightarrow X(E(t)) \tag{5.4}$$

in $\mathbb{D}([0, +\infty\rangle\,; S)$ with J_1 topology. In the first step we establish weak convergence (5.3) to Pearson diffusions, based on suitably chosen Markov chains (for OU diffusion see Section 5.4, for CIR and Jacobi diffusion see Sections 5.5 and 5.6, and for heavy-tailed Pearson diffusions - RG, FS and ST diffusions see Section 5.7). Next, based on these results in Section 5.8 we establish correlated CTRWs which as the scaling limits have fractional Pearson diffusions. In other words, we prove weak convergence (5.4) to the fractional Pearson diffusions, based on suitably chosen Markov chains from aforementioned sections.

5.2 Historical roots and motivation

Many processes observed in science can be mathematically described by the discrete-time Markov urn-scheme models. One of the most famous urn-scheme models is the Wright-Fisher model for gene mutations in a population from which a rich class of the limiting processes could be obtained. On the other hand, one of the simplest urn-scheme models is the Bernoulli-Laplace urn-scheme model, named after D. Bernoulli and P. S. Laplace, the pioneers of probability theory, and later studied in depth by Markov (1915). Historical review of this model and its connection with OU process can be found in Jacobsen (1996), which we now present. This model considers two urns, urn A containing j balls and urn B containing k balls. Of the total $j+k$ balls in two urns, suppose that r balls are white, and $(j+k-r)$ are black. At time $n \in \mathbb{N}$ one ball is drawn randomly from urn A and another from urn B. The ball drawn from urn A is placed into urn B, the ball drawn from urn B is placed into urn A. Let X_n be the number of white balls in urn B at time $n \in \mathbb{N}$. Then $(k-X_n)$ is the number of black balls in urn B, $(r-X_n)$ is the number of white balls in urn A and $(j-r+X_n)$ is the number of black balls in urn A at time $n \in \mathbb{N}$. Therefore, $(X_n,\, n \in \mathbb{N})$ is a homogeneous discrete-time Markov chain with state space $S = \{\max\{0, r-j\}, \dots, \min\{k, r\}\}$ and the following transition probabilities:

$$p_{x,y} = P\left(X_{n+1} = y | X_n = x\right) = \begin{cases} \dfrac{(r-x)(k-x)}{jk} & , \quad y = x+1 \\ \dfrac{(r-x)x + (j-r+x)(k-x)}{jk} & , \quad y = x \\ \dfrac{(j-r+x)x}{jk} & , \quad y = x-1 \\ 0 & , \quad \text{otherwise.} \end{cases}$$

In his book, Laplace (1812) worked with a particular case of this model, in which each urn contains n balls and also n out of total $2n$ balls are black. In this setting he defined a discrete time Markov chain $\{Z_r^{(n)},\, r \geq 0\}$ with the state space $\{0, 1, 2, \dots, n\}$, where n is the number of balls in urn A and r is the number of draws, with transition probabilities

$$p_{x,x+1} = \left(1 - \frac{x}{n}\right)^2, \quad p_{x,x} = 2\frac{x}{n}\left(1 - \frac{x}{n}\right), \quad p_{x,x-1} = \left(\frac{x}{n}\right)^2 \quad \text{and 0 otherwise.}$$

Laplace was interested in finding the heat kernel of this irreducible, reversible and ergodic Markov chain with stationary/ergodic distribution

$$\pi = \{\pi_0, \cdots, \pi_n\}, \quad \pi_i = \frac{\binom{n}{i}\binom{n}{n-i}}{\binom{2n}{n}}.$$

By denoting $z_{x,r}$ the probability that there are x white balls in urn A after r draws, he deduced the following partial second-order difference equation:

$$z_{x,r+1} = \left(\frac{x+1}{n}\right)^2 z_{x+1,r} + 2\frac{x}{n}\left(1-\frac{x}{n}\right) z_{x,r} + \left(1-\frac{x-1}{n}\right) z_{x-1,r}. \tag{5.5}$$

After that, instead of determining the generating function for z, he approximated the solution of the equation (5.5) by introducing the new space variable μ and new time variable r' and obtained the following relation:

$$x = \frac{1}{2}(n+\mu\sqrt{n}), \quad r = nr'.$$

Then, he introduced the function

$$U(\mu, r') := z_{x,r}$$

of space and time, and heuristically deduced the following relations describing the changes of state in the transformed Markov chain:

$$\begin{aligned} z_{x+1,r} &= z_{x,r} + \frac{\partial z_{x,r}}{\partial x} + \frac{1}{2}\frac{\partial^2 z_{x,r}}{\partial x^2} \\ z_{x-1,r} &= z_{x,r} - \frac{\partial z_{x,r}}{\partial x} + \frac{1}{2}\frac{\partial^2 z_{x,r}}{\partial x^2} \\ z_{x,r+1} &= z_{x,r} + \frac{\partial z_{x,r}}{\partial r}. \end{aligned}$$

By another purely heuristic argument, he claimed that function $U(\mu, r')$ satisfies the second-order differential equation

$$\frac{\partial U}{\partial r'} = -\frac{\partial}{\partial \mu}(-2\mu U) + \frac{1}{2}\frac{\partial^2}{\partial \mu^2}(2U) = 2U + 2\mu\frac{\partial U}{\partial \mu} + \frac{\partial^2 U}{\partial \mu^2}, \tag{5.6}$$

which is a special case of the Fokker-Planck or Kolmogorov forward equation governing the OU process:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left(-\theta(x-\mu)p(x,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(2\theta\sigma^2 p(x,t)\right).$$

In particular, equation (5.6) is the governing equation for the OU diffusion with infinitesimal mean $\mu(x) = -2x$ and infinitesimal variance $\sigma^2(x) = 2$. While rigorous proofs were not provided, this work had the first mention of the forward equation for the OU diffusion, even though the underlying process wasn't known back in the day.

A century later, Markov (Markov (1915)) considered a more general model. In his scheme there are also two urns, urn A containing n balls and urn B containing n_1 balls. Out of total $(n+n_1)$ balls, there are $(n+n_1)p$ white and $(n+n_1)q$ black balls ($0 < p < 1$, $q = 1-p$). Similar to the Laplace's scheme, by denoting probability that there are x

white balls in the urn A after r draws by $z_{x,r}$, Markov obtained the following difference equation:

$$\begin{aligned} z_{x,r+1} =& \frac{x+1}{n} \cdot \frac{n_1 q - np + x + 1}{n_1} \cdot z_{x+1,r} + \frac{n-x+1}{n} \cdot \frac{(n+n_1)p - x + 1}{n_1} \cdot z_{x-1,r} \\ &+ \left(\frac{x}{n} \cdot \frac{(n+n_1)p - x}{n_1} + \frac{n-x}{n} \cdot \frac{n_1 q - np + x}{n_1} \right) \cdot z_{x,r}. \end{aligned} \tag{5.7}$$

Next, Markov introduced new space variable μ and new time variable ρ and obtained the following relation:

$$x = np + \mu \frac{1}{\Delta\mu}, \ r\left(\frac{1}{n} + \frac{1}{n_1}\right) = 2\rho, \text{ where } \Delta\mu = \sqrt{\frac{n+n_1}{2pqnn_1}},$$

demanding that the ratio of number of balls in urn A and urn B remains constant at all times, i.e. $n_1\alpha = n$ for some $\alpha > 0$. Obviously, for $\alpha = 1$ and $p = q = \frac{1}{2}$ this model reduces to the previously described Laplace's urn scheme.

Finally, the difference equation (5.7) was, again by pure heuristics, approximated by the corresponding Fokker-Planck equation. In order to do so, Markov defined the space and time dependent function $U(\mu, \rho) := z_{x,r}$, smooth enough for obtaining approximations

$$\begin{aligned} z_{x+1,r} &= U(\mu + \Delta\mu, \rho) \approx U + \Delta\mu \frac{\partial U}{\partial \mu} + \frac{1}{2}(\Delta\mu)^2 \frac{\partial^2 U}{\partial \mu^2} \\ z_{x-1,r} &= U(\mu - \Delta\mu, \rho) \approx U - \Delta\mu \frac{\partial U}{\partial \mu} + \frac{1}{2}(\Delta\mu)^2 \frac{\partial^2 U}{\partial \mu^2} \\ z_{x,r+1} &= U(\mu, \rho + \frac{1+\alpha}{2n}) \approx U + \frac{1+\alpha}{2n} \frac{\partial U}{\partial \rho}, \end{aligned}$$

precise up to the order $o(n^{-1})$.

By combining these approximations with the new space and time transformations and putting it into (5.7), Markov obtained the second order differential equation

$$\frac{\partial U}{\partial \rho} = 2U + 2\mu \frac{\partial U}{\partial \mu} + \frac{\partial^2 U}{\partial \mu^2},$$

the Fokker-Planck equation for the OU process with infinitesimal mean $\mu(x) = -2x$ and infinitesimal variance $\sigma^2(x) = 2$, completely coinciding with Laplace's result. In Section 5.4 we formally establish this result. In fact, we extend the result to the generally parametrized OU process.

We now briefly discuss the Wright-Fisher urn scheme, named after S. Wright and R. Fisher. Wright-Fisher urn scheme is a model that describes gene mutations (in some genetic pool) over time, strongly influencing selection and sampling forces in the corresponding population. There are several different versions of this model, and we use the scheme described in Karlin & Taylor (1981*a*).

Suppose that in a population of size n each individual is either of type A or type a. Let i be the number of A-types in the population. Therefore, the remaining $(n-i)$ population members are a-types. The next generation of a population is produced depending on the influence of mutation, selection and sampling forces. Once born, individual of A-type can mutate in a-type with probability α and individual of a-type can mutate in A-type with probability β. Taking into account parental population comprised of i A-types and $(n-i)$ a-types, the expected fraction of A-types after mutation is

$$\frac{i}{n}(1-\alpha)+\left(1-\frac{i}{n}\right)\beta,$$

while the expected fraction of a-types is

$$\frac{i}{n}\alpha+\left(1-\frac{i}{n}\right)(1-\beta).$$

Survival ability of each type is modeled by parameter s so that the ratio of A-types over a-types is equal to $1+s$, meaning that A-type is selectively superior to a-type. Then the expected fraction of mature A-types before reproduction is

$$p_i=\frac{(1+s)\left[i(1-\alpha)+(n-i)\beta\right]}{(1+s)\left[i(1-\alpha)+(n-i)\beta\right]+\left[i\alpha+(n-i)(1-\beta)\right]}. \tag{5.8}$$

The last assumption of this model is that the composition of the next generation is determined through n binomial trials, where the probability of producing an A-type in each trial is p_i. This model, tracking the number of A-types in population over time, can be described by the discrete-time Markov chain $\{G_r^n,\ r\in\mathbb{N}_0\}$ with the state space $\{0,1,2,\dots,n\}$ and binomial transition probabilities

$$p_{ij}=\binom{n}{j}p_i^j(1-p_i)^{n-j}. \tag{5.9}$$

This model is parametrized by α, β, and $s\in[0,1]$. Depending on their values, different limiting diffusions could be obtained. In Section 5.5 we present two different settings that lead to generally parametrized Jacobi and CIR diffusions. The procedures of constructing these diffusions as limits of some suitably selected discrete-time Markov chain (urn-scheme) are based on examples given in (Karlin & Taylor 1981*a*, p. 176-183), where only heuristic arguments are given for special cases of the limiting Jacobi and CIR diffusions.

From this review of historical facts, we see that the connection between discrete-time Markov chains in urn-scheme models and some limiting continuous-time stochastic processes has been brought up in the literature long time ago. Today, the conjectures of Laplace and Markov could be rigorously proved by modern techniques of analysis and probability theory, involving the convergence of evolution operators of discrete time

Markov chains. In this thesis we derive such connections between several urn-scheme models and the corresponding diffusions. More precisely, for appropriately chosen discrete time Markov chains, we derive the corresponding limiting Pearson diffusions, and then define the corresponding correlated CTRWs and their fractional Pearson diffusion limits.

5.3 General framework for construction of Markov chains and related diffusions

In this section we explain the general idea on how to construct Markov chain which will lead to the desired diffusion process in the limit. In Subsection 5.3.1 we explain the necesarry technicalities, while in Subsection 5.3.2 we give concrete steps on how to construct generally parametrized diffusion via Markov chain in our setting.

5.3.1 Transition operators of discrete-time Markov chains

General theory needed for diffusion approximation of discrete-time Markov chains includes transition kernels and transition operators. Let μ be an arbitrary probability kernel on a measurable space $(S, \mathcal{S})$. The associated transition operator T is defined as

$$Tf(x) = (Tf)(x) = \int \mu(x, dy) f(y), \quad x \in S, \tag{5.10}$$

where $f \colon S \to \mathbb{R}$ is assumed to be measurable and either bounded or nonnegative. From the approximation of f by simple functions and the monotone convergence argument, it follows that Tf is again a measurable function on S. Furthermore, T is a positive contraction operator: $0 \le f \le 1$ implies that $0 \le Tf \le 1$. For more details on transition operators and their importance to the study of Markov processes, we refer to (Kallenberg 2002, Chapter 19).

Consider the Banach space of bounded continuous functions on space S with the supremum norm. For a closed operator $\mathcal{A}$ with domain $\mathcal{D}$, a core for $\mathcal{A}$ is a linear subspace $D \subset \mathcal{D}$ such that the restriction $\mathcal{A}|_D$ has closure $\mathcal{A}$. In that case, $\mathcal{A}$ is clearly uniquely determined by its restriction $\mathcal{A}|_D$. Suitable core is important in order to technically establish connection between desired Markov chains and their limiting diffusions. We work with $C_c^3(S)$ as a core of the diffusion infinitesimal generator, but in general not all diffusions have it as its core. Theorems 1.6 and 2.1 from (Ethier & Kurtz 2009, Section 8) ensure that $C_c^\infty(S)$ (and therefore $C_c^3(S)$ as well) can be referred to as the core for all six Pearson diffusions for which we establish results regarding CTRWs limits. In particular, Jacobi diffusion satisfies conditions of Theorem 1.6, while other Pearson diffusions satisfy conditions of Theorem 2.1.

The main technical tool used for obtaining the non-fractional Pearson diffusion as the scaling limit of a suitably chosen Markov chain with known transition operator is Theorem

19.28 from Kallenberg (2002) which we now state.

Theorem 5.1 Let $\{Y^{(n)},\, n \in \mathbb{N}\}$ be a sequence of discrete-time Markov chains on S with transition operators $\{U_n,\, n \in \mathbb{N}\}$. Consider a Feller process X on S with semigroup T_t and generator $\mathcal{A}$. Fix a core D for the generator $\mathcal{A}$, and assume that $(h_n,\, n \in \mathbb{N})$ is the sequence of positive reals tending to zero as $n \to \infty$. Let

$$A_n = h_n^{-1}(U_n - I), \quad T_{n,t} = U_n^{\lfloor t/h_n \rfloor}, \quad X_t^n = Y^n(\lfloor t/h_n \rfloor).$$

Then the following statements are equivalent:

a) If $f \in D$, there exist some $f_n \in Dom(A_n)$ with $f_n \to f$ and $A_n f_n \to \mathcal{A}f$ as $n \to \infty$

b) $T_{n,t} \to T_t$ strongly for each $t > 0$

c) $T_{n,t} f_n \to T_t f$ for each $f \in C_0$, uniformly for bounded $t > 0$

d) if $X^{(n)}(0) \Rightarrow X(0)$ in S, then $X^n \Rightarrow X$ in the Skorokhod space $\mathbb{D}([0, +\infty\rangle\,; S)$ with the J_1 topology.

The proof can be found in (Kallenberg 2002, Theorem 19.28, page 387).

5.3.2 General approach to diffusion approximation via Markov chains

We start with the sequence of Markov chains $\{N^{(n)}(r),\, r \in \mathbb{N},\, n \in \mathbb{N}\}$. For a fixed $n \in \mathbb{N}$, $\{N^{(n)}(r),\, r \in \mathbb{N}\}$ is a Markov chain with state space $S_n \subseteq \mathbb{N}_0$ and transition probabilities $p_{ij},\ i, j \in S_n$. On the other hand, let $X = \{X(t),\, t \geq 0\}$ be the desired diffusion process with state space S, i.e. solution of the SDE

$$dX(t) = \mu(X(t))\, dt + \sqrt{\sigma^2(X(t))}\, dW(t), \quad t \geq 0, \quad x \in S,$$

with the infinitesimal generator

$$\mathcal{A}f(x) = \mu(x) f'(x) + \frac{1}{2}\sigma^2(x) f''(x), \quad f \in C_c^3(S). \tag{5.11}$$

First, one needs to connect starting points $N^{(n)}(0) = i \in S_n$ with $X(0) = x \in S$, i.e. we want to connect state space of the starting Markov chain and the state space of the desired diffusion process. Define strictly monotonic function $g_n : S \to \mathbb{R}$, such that

$$i = \lfloor g_n(x) \rfloor \tag{5.12}$$

for n large enough and

$$\lim_{n\to\infty} \left\| g_n^{-1}(i+1) - g_n^{-1}(i) \right\|_\infty = 0.$$

According to the state space S of the desired diffusion process X, one constructs new Markov chain $\{H^{(n)}(r),\, r \in \mathbb{N}\}$ via state space transformation

$$H^{(n)}(r) = g_n^{-1}\left(N^{(n)}(r)\right) \tag{5.13}$$

with state space $g_n^{-1}(S_n)$. The transition operator T_n of the Markov chain $\{H^{(n)}(r),\ n \in \mathbb{N}\}$ is given by

$$T_n f\left(g_n^{-1}(i)\right) = \sum_{j=0}^{n} p_{ij} f\left(g_n^{-1}(i)\right). \tag{5.14}$$

Now we define the operator

$$A_n := h_n^{-1}(T_n - I), \quad f_n \in Dom(A_n), \quad f_n(x) := f\left(g_n^{-1}(i)\right), \quad f \in C_c^3(S), \tag{5.15}$$

where $(h_n,\, n \in \mathbb{N})$ is sequence of positive reals tending to zero as $n \to \infty$.
Finally, define continuous-time stochastic process $\{X^{(n)}(t),\, t \geq 0\}$ via time-change in the Markov chain

$$X^{(n)}(t) := H^{(n)}\left(\lfloor h_n^{-1} t \rfloor\right). \tag{5.16}$$

The next theorem gives sufficient conditions for obtaining the diffusion process $\{X(t),\, t \geq 0\}$ as the limit of the time-changed stochastic process $\{X^{(n)}(t),\, t \geq 0\}$.

Theorem 5.2 Let $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$, be the Markov chain defined by (5.13) with the transition operator (5.14). Let $X^n = \{X^{(n)}(t),\, t \geq 0\}$, for each $n \in \mathbb{N}$, be its corresponding time-changed process, with the time change (5.16). Let the operators $(A_n,\, n \in \mathbb{N})$ be defined by (5.15). If

$$\begin{aligned}
\mu_n(x) :&= h_n^{-1} \sum_{j=0}^{n} p_{ij}\left(g_n^{-1}(j) - g_n^{-1}(i)\right), \quad \sigma_n^2(x) := h_n^{-1} \sum_{j=0}^{n} p_{ij}\left(g_n^{-1}(j) - g_n^{-1}(i)\right)^2, \\
R_n(x) :&= h_n^{-1} \sum_{j=0}^{n} p_{ij} \frac{(g_n^{-1}(j) - g_n^{-1}(i))^3}{3!} f'''(\zeta), \quad |\zeta - g_n^{-1}(i)| < |g_n^{-1}(j) - g_n^{-1}(i)|
\end{aligned} \tag{5.17}$$

have uniform limits

$$\lim_{n\to\infty} \|\mu_n - \mu\|_\infty = \lim_{n\to\infty} \left\|\sigma_n^2 - \sigma^2\right\|_\infty = \lim_{n\to\infty} \|R_n\|_\infty = 0, \tag{5.18}$$

where μ and σ^2 are infinitesimal parameters given in (5.11), then

$$X^n \Rightarrow X \text{ in } \mathbb{D}([0, +\infty\rangle\,; S),$$

where $X = \{X(t),\, t \geq 0\}$ is a diffusion process with state space S and infinitesimal generator $\mathcal{A}$ given by (5.11).

Proof. First, we prove statement $a)$ of Theorem 5.1, i.e. we show that infinitesimal generator (5.11) can be approximated by operator A_n defined in (5.15). According to the definition of function $\lfloor\cdot\rfloor$

$$\lfloor g_n(x)\rfloor \leq g_n(x) < \lfloor g_n(x)\rfloor + 1,$$

i.e.

$$i \leq g_n(x) < i+1. \tag{5.19}$$

Now, let g_n be monotone increasing function (monotone decreasing case is analogous). Using this with (5.19) gives

$$g_n^{-1}(i) \leq g_n^{-1}(g_n(x)) < g_n^{-1}(i+1)$$

so that

$$|g_n^{-1}(i) - x| < \left|g_n^{-1}(i+1) - g_n^{-1}(i)\right|.$$

Last inequality implies

$$\lim_{n\to\infty}\left\|g_n^{-1}(i) - x\right\|_\infty \leq \lim_{n\to\infty}\left\|g_n^{-1}(i+1) - g_n^{-1}(i)\right\|_\infty = 0. \tag{5.20}$$

Therefore, for $f \in C_c^3(S)$,

$$\lim_{n\to\infty}\|f_n - f\|_\infty = \lim_{n\to\infty}\sup_{x\in S}|f_n(x) - f(x)| = \lim_{n\to\infty}\sup_{x\in S}\left|f(g_n^{-1}(i)) - f(x)\right| = 0.$$

Since

$$\begin{aligned} A_n f(g_n^{-1}(i)) &= h_n^{-1}\left[\sum_{j=0}^{n} p_{ij} f\left(g_n^{-1}(j)\right) - f\left(g_n^{-1}(i)\right)\right] \\ &= h_n^{-1}\sum_{j=0}^{n} p_{ij}\left[f\left(g_n^{-1}(j)\right) - f\left(g_n^{-1}(i)\right)\right], \end{aligned}$$

Taylor formula for function f around $g_n^{-1}(i)$ with mean-value form of the remainder yields

$$\begin{aligned} A_n f(g_n^{-1}(i)) =& h_n^{-1}\sum_{j=0}^{n} p_{ij}\left(g_n^{-1}(j) - g_n^{-1}(i)\right) f'\left(g_n^{-1}(i)\right) + h_n^{-1}\sum_{j=0}^{n} p_{ij}\frac{\left(g_n^{-1}(j) - g_n^{-1}(i)\right)^2}{2!} f''\left(g_n^{-1}(i)\right) \\ &+ h_n^{-1}\sum_{j=0}^{n} p_{ij}\frac{\left(g_n^{-1}(j) - g_n^{-1}(i)\right)^3}{3!} f'''(\zeta), \end{aligned} \tag{5.21}$$

where ζ is a real number such that $|\zeta - g_n^{-1}(i)| < |g_n^{-1}(j) - g_n^{-1}(i)|$.
Therefore (5.21) reduces to

$$A_n f(g_n^{-1}(i)) = \mu_n(x) f'\left(g_n^{-1}(i)\right) + \frac{\sigma_n^2(x)}{2} f''\left(g_n^{-1}(i)\right) + R_n(x).$$

Now, triangle inequality gives

$$
\begin{aligned}
\|A_n f_n - \mathcal{A} f\|_\infty &= \sup_{x\in S} |A_n f_n(x) - \mathcal{A} f(x)| = \sup_{x\in S} \left| A_n f(g_n^{-1}(i)) - \mathcal{A} f(x) \right| \\
&\le \sup_{x\in S} \left| \mu_n(x) f'\left(g_n^{-1}(i)\right) - \mu(x) f'(x) \right| + \sup_{x\in S} \left| \frac{\sigma_n^2(x)}{2} f''\left(g_n^{-1}(i)\right) - \frac{\sigma^2(x)}{2} f''(x) \right| \\
&+ \sup_{x\in S} |R_n(x)| . \qquad (5.22)
\end{aligned}
$$

For $f \in C_c^3(S)$, (5.20) implies

$$
\lim_{n\to\infty} \|f_n' - f'\|_\infty = \lim_{n\to\infty} \|f_n'' - f''\|_\infty = 0 \tag{5.23}
$$

and uniform limits (5.18) and (5.23) together with (5.22) yields

$$
\lim_{n\to\infty} \|A_n f_n - \mathcal{A} f\|_\infty = 0.
$$

Therefore, we have proved statement a) of the Theorem 5.1, and since

$$
X^{(n)}(0) \Rightarrow X(0) \iff \lim_{n\to\infty} \left\| g_n^{-1}(i) - x \right\|_\infty = 0,
$$

using equivalence of statements a) and d) in Theorem 5.1 immediately it follows

$$
X^n \Rightarrow X \text{ in } \mathbb{D}([0, +\infty\rangle\,; S).
$$

□

Remark 5.1. The limiting diffusion process obtained this way is clearly not stationary, since we demand the initial value $X(0) = x$ of the diffusion process to be connected to the initial value of the Markov chain by (5.12).

Remark 5.2. In all cases considered in this thesis, function $g_n : S \to \mathbb{R}$, is affine function of the form

$$
g_n(x) = a_n x + b_n,
$$

where $(a_n,\, n \in \mathbb{N})$ and $(b_n,\, n \in \mathbb{N})$ are sequences of real numbers such that

$$
\lim_{n\to\infty} \left\| g_n^{-1}(i+1) - g_n^{-1}(i) \right\|_\infty = \lim_{n\to\infty} \frac{1}{a_n} = 0.
$$

and

$$
i = \lfloor g_n(x) \rfloor
$$

for n large enough.

Moreover, μ_n, σ_n^2 and R_n reduces to

$$\mu_n(x) = \frac{h_n^{-1}}{a_n}\sum_{j=0}^{n} p_{ij}\,(j-i)\,, \quad \sigma_n^2(x) = \frac{h_n^{-1}}{a_n^2}\sum_{j=0}^{n} p_{ij}\,(j-i)^2\,,$$
$$R_n(x) = \frac{h_n^{-1}}{a_n^3}\sum_{j=0}^{n} p_{ij}\frac{(j-i)^3}{3!}f'''(\zeta), \quad \left|\zeta - g_n^{-1}(i)\right| < \left|\frac{j-i}{a_n}\right|. \tag{5.24}$$

Remark 5.3. Some starting Markov chains considered in this thesis have transition probabilities of the form

$$p_{i,i+1} > 0, \quad p_{i,i-1} > 0, \quad p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}, \quad 0 \text{ otherwise.}$$

For such Markov chains, (5.24) further reduces to

$$\mu_n(x) = \frac{h_n^{-1}}{a_n}\left(p_{i,i+1} - p_{i,i-1}\right), \quad \sigma_n^2(x) = \frac{h_n^{-1}}{a_n^2}\left(p_{i,i+1} + p_{i,i-1}\right),$$
$$R_n(x) = \frac{h_n^{-1}}{6a_n^3}\left(p_{i,i+1} - p_{i,i-1}\right)f'''(\zeta), \quad \left|\zeta - g_n^{-1}(i)\right| < \left|\frac{j-i}{a_n}\right|. \tag{5.25}$$

This procedure makes manipulations in the state space and time change easier in order to obtain the desired diffusion.

5.4 Bernoulli-Laplace urn scheme: Markov chain for the Ornstein-Uhlenbeck process

In this section, we define the Bernoulli-Laplace urn-scheme model regarding early ideas of Laplace and Markov summarized in Section 5.2, with crucial changes in space and time transformations in order to obtain OU diffusion with general parameters. The urn scheme consists of two urns, A and B, each containing n balls. Furthermore, n out of total $2n$ balls are black. At each step one ball is randomly chosen from each urn. The ball drawn from urn A is then placed into urn B and the ball drawn from urn B is placed into urn A. We are interested in the number of white balls in the urn A after $r \in \mathbb{N}$ draws.
Let for each $n \in \mathbb{N}$, $\{Z_r^{(n)},\, r \in \mathbb{N}_0\}$ be the Markov chain with the state space $\{0, 1, 2, \ldots, n\}$, where n is the number of balls in urn A, r is the number of draws and $Z_r^{(n)}$ is the number of white balls in the urn A after r draws. The transition probabilities for this Markov chain are as follows:

$$p_{i,i+1} = \left(1 - \frac{i}{n}\right)^2, \quad p_{i,i} = 2\frac{i}{n}\left(1 - \frac{i}{n}\right), \quad p_{i,i-1} = \left(\frac{i}{n}\right)^2, \quad 0 \text{ otherwise.} \tag{5.26}$$

Recall that the generally parametrized Ornstein-Uhlenbeck process $X = \{X(t),\, t \geq 0\}$ is defined as the solution of the SDE (3.32) with corresponding infinitesimal generator

$$\mathcal{A}f(x) = -\theta(x-\mu)f'(x) + \theta\sigma^2 f''(x), \quad f \in C_c^3(\mathbb{R}), \tag{5.27}$$

where $\theta > 0$, $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. Define the function $g_n : \mathbb{R} \to \mathbb{R}$,

$$g_n(x) = \frac{1}{2}\left(n + (ax+b)\sqrt{n}\right), \quad a \neq 0, \quad b \in \mathbb{R}.$$

We assume initial states of the Markov chain $\{Z^{(n)}(r),\, r \in \mathbb{N}_0\}$ and OU diffusion $X = \{X(t),\, t \geq 0\}$ are given by $Z^{(n)}(0) = i$ and $X(0) = x$ respectively, where

$$i = i(x) = \lfloor g_n(x) \rfloor = \left\lfloor \frac{1}{2}\left(n + (ax+b)\sqrt{n}\right)\right\rfloor, \; x \in \mathbb{R}$$

and n is always large enough so that $i(x)$ is in the state space of Markov chain $\{Z^{(n)}(r),\; r \in \mathbb{N}_0\}$. Moreover, we assume that the initial Markov chain $\{Z^{(n)}(r),\; r \in \mathbb{N}_0\}$ never starts from states 0 and n. Notice that the initial state is a function of x, but we will use notation i for simplicity.

For each $n \in \mathbb{N}$, we define the new Markov chain $\{H^{(n)}(r),\, r \in \mathbb{N}\}$ where

$$H^{(n)}(r) = g_n^{-1}(Z^{(n)}(r)) = \frac{1}{a\sqrt{n}}\left(2Z^{(n)}(r) - n - b\sqrt{n}\right), \tag{5.28}$$

with state space $\left\{\frac{1}{a\sqrt{n}}\left(-n - b\sqrt{n}\right), \frac{1}{a\sqrt{n}}\left(2 - n - b\sqrt{n}\right), \cdots \frac{1}{a\sqrt{n}}\left(n - b\sqrt{n}\right)\right\}$. The transition operator T_n of the Markov chain $\{H^{(n)}(r),\, n \in \mathbb{N}\}$ is given by

$$\begin{aligned} T_n f\left(\frac{2i - n - b\sqrt{n}}{a\sqrt{n}}\right) &= \sum_{j=0}^{n} p_{ij} f\left(\frac{2j - n - b\sqrt{n}}{a\sqrt{n}}\right) \\ &= p_{i,i-1}\, f\left(\frac{2(i-1) - n - b\sqrt{n}}{a\sqrt{n}}\right) + p_{i,i}\, f\left(\frac{2i - n - b\sqrt{n}}{a\sqrt{n}}\right) + \\ &\quad + p_{i,i+1}\, f\left(\frac{2(i+1) - n - b\sqrt{n}}{a\sqrt{n}}\right). \end{aligned} \tag{5.29}$$

Now we define the operator

$$A_n := \frac{\theta}{2} n(T_n - I), \quad f_n \in Dom(A_n), \quad f_n(x) := f\left(g_n^{-1}(i)\right) = f\left(\frac{2i - n - b\sqrt{n}}{a\sqrt{n}}\right) \tag{5.30}$$

where $\theta > 0$ and $f \in C_c^3(\mathbb{R})$. By the following scaling of time in $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$ we obtain the corresponding continuous-time stochastic process $\{X^{(n)}(t),\, t \geq 0\}$:

$$X^{(n)}(t) := H^{(n)}\left(\left\lfloor \frac{\theta}{2} nt \right\rfloor\right). \tag{5.31}$$

The next theorem states that the generally parametrized OU diffusion can be obtained as the limiting process of the time-changed processes $\{X^{(n)}(t),\, t \geq 0\}$.

Theorem 5.3 Let $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$, be the Markov chain defined by (5.28) with the transition operator (5.29). Let $X^n = \{X^{(n)}(t),\, t \geq 0\}$, for each $n \in \mathbb{N}$, be its corresponding time-changed process, with the time-change (5.31). Let the operators $(A_n,\, n \in \mathbb{N})$ be defined by (5.30). Then

$$X^n \Rightarrow X \text{ in } \mathbb{D}\left[0,\, +\infty\right\rangle,$$

where $X = \{X(t),\, t \geq 0\}$ is the OU diffusion with the infinitesimal generator $\mathcal{A}$ given by (5.27), and

$$\mu = -\frac{b}{a}, \quad \sigma^2 = \frac{1}{2a^2}.$$

Proof. First, notice that function g_n satisfies conditions given in Section 5.3, i.e. function g_n is strictly monotonic and

$$\lim_{n\to\infty} \left\| g_n^{-1}(i+1) - g_n^{-1}(i) \right\|_\infty = \lim_{n\to\infty} \left| \frac{2}{a\sqrt{n}} \right| = 0.$$

Taking into account Remark 5.3, state space transformation (5.28) together with the time scale $h_n^{-1} = \theta n/2$ yields

$$\mu_n(x) = \frac{\theta}{a}\sqrt{n}\left(p_{i,i+1} - p_{i,i-1}\right), \quad \sigma_n^2(x) = \frac{2\theta}{a^2}\left(p_{i,i+1} + p_{i,i-1}\right),$$

$$R_n(x) = \frac{2\theta}{3a^3\sqrt{n}}\left(p_{i,i+1} - p_{i,i-1}\right) f'''(\zeta), \quad \left|\zeta - \frac{2i - n - b\sqrt{n}}{a\sqrt{n}}\right| < \left|\frac{2}{a\sqrt{n}}(j-i)\right|.$$

Next, by substituting transition probabilities (5.26) in the above expressions it follows

$$\begin{aligned} \mu_n(x) &= \frac{\theta}{a}\sqrt{n}\left(\left(1 - \frac{i}{n}\right)^2 - \left(\frac{i}{n}\right)^2\right) \\ &= \frac{\theta\sqrt{n}}{a}\left(1 - \frac{2i}{n}\right) = \theta\left(\frac{n-2i}{a\sqrt{n}}\right), \end{aligned} \tag{5.32}$$

$$\begin{aligned} \sigma_n^2(x) &= \frac{2\theta}{a^2}\left(1 - \frac{2i}{n} + \frac{2i^2}{n^2}\right) \\ &= \frac{2\theta}{a^2}\left(\frac{n-2i}{n} + 2\left(\frac{i}{n}\right)^2\right), \end{aligned} \tag{5.33}$$

$$
\begin{aligned}
|R_n(x)| &\leq \left| \frac{2\theta}{3a^3\sqrt{n}} \left(\left(1 - \frac{i}{n}\right)^2 - \left(\frac{i}{n}\right)^2 \right) \right| K \\
&= \left| \frac{2\theta}{3a^3\sqrt{n}} \left(1 - \frac{2i}{n}\right) \right| K \\
&= \left| \frac{2\theta}{3a^3} \left(\frac{n - 2i}{a\, n\sqrt{n}} \right) \right| K,
\end{aligned}
\tag{5.34}
$$

where K is a constant such that $|f'''(\zeta)| \leq K$. Since $i = \lfloor g_n(x) \rfloor$, it follows

$$
\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{n - 2i}{a\sqrt{n}} + \left(x + \frac{b}{a} \right) \right| = 0, \quad \lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{i}{n} - \frac{1}{2} \right| = 0. \tag{5.35}
$$

Now, using (5.32), (5.33), (5.34) together with (5.35) and the fact that $f \in C_c^3(\mathbb{R})$ yields

$$
\lim_{n\to\infty} \|\mu_n - \mu\|_\infty = 0, \quad \lim_{n\to\infty} \left\| \sigma_n^2 - \sigma^2 \right\|_\infty = 0, \quad \lim_{n\to\infty} \|R_n\|_\infty = 0, \tag{5.36}
$$

where

$$
\mu(x) = -\theta \left(x + \frac{b}{a} \right), \quad \sigma^2(x) = \frac{\theta}{a^2}.
$$

By re-parametrizing

$$
\mu = -\frac{b}{a}, \quad \sigma^2 = \frac{1}{2a^2} \tag{5.37}
$$

it follows

$$
\mu(x) = -\theta \left(x - \mu \right), \quad \sigma^2(x) = 2\theta\sigma^2. \tag{5.38}
$$

Notice how this re-parametrization ensures that parameters of OU diffusion satisfy

$$
\theta > 0, \quad \mu \in \mathbb{R}, \quad \sigma^2 > 0,
$$

since $a \neq 0,\ b \in \mathbb{R}$. Now, comparing the obtained limits (5.38) with (5.27) we see that the limits coincide with the infinitesimal parameters of the OU diffusion. Since (5.36) holds, as a direct consequence of Theorem 5.2 we obtain $X^n \Rightarrow X$ in $\mathbb{D}\,[0, +\infty\rangle$, where X is the generally parametrized OU diffusion. □

5.5 Wright-Fisher urn-scheme

In this section we present two different versions of the Wright-Fisher model that lead to generally parametrized Jacobi and CIR diffusions.

5.5.1 Markov chain for the Cox-Ingersoll-Ross diffusion

Recall that the generally parametrized Cox-Ingersoll-Ross diffusion $X = \{X(t),\, t \geq 0\}$ is defined as the solution of the SDE (3.42) with corresponding infinitesimal generator

$$\mathcal{A}f(x) = -\theta\left(x - \frac{b}{a}\right)f'(x) + \frac{\theta}{a}xf''(x), \quad f \in C_c^3([0,\, +\infty\rangle), \tag{5.39}$$

where $\theta > 0$, $a > 0$ and $b > 0$.
For the Wright-Fisher model, we assume there is only mutation of the order

$$\alpha = \frac{a}{2n^d}, \quad \beta = \frac{b}{2n}, \quad 0 < d < 1, \quad 0 < a,\ b < \infty, \quad s = 0,$$

so that expected fraction of A-types (5.8) becomes

$$p_i = \frac{i}{n}\left(1 - \frac{a}{2n^d}\right) + \left(1 - \frac{i}{n}\right)\frac{b}{2n}. \tag{5.40}$$

Define the function $g_n : [0,\, +\infty\rangle \to \mathbb{R}$,

$$g_n(x) = n^d x.$$

We assume initial states of the Markov chain $\{G^{(n)}(r),\, r \in \mathbb{N}_0\}$ and CIR diffusion $X = \{X(t),\, t \geq 0\}$ are given by $G^{(n)}(0) = i$ and $X(0) = x$ respectively, where

$$i = i(x) = \lfloor g_n(x) \rfloor = \left\lfloor n^d x \right\rfloor,\ x \in [0,\, +\infty\rangle$$

and n is always large enough so that $i(x)$ is in the state space of Markov chain $\{G^{(n)}(r),\ r \in \mathbb{N}_0\}$. Notice that the initial state is a function of x, but we will use notation i for simplicity.

For each $n \in \mathbb{N}$, we define the new Markov chain $\{H^{(n)}(r),\, r \in \mathbb{N}\}$ where

$$H^{(n)}(r) = g_n^{-1}(G^{(n)}(r)) = \frac{G^{(n)}(r)}{n^d}, \tag{5.41}$$

with state space $\left\{0,\, \frac{1}{n^d},\, \frac{2}{n^d}, \cdots n^{1-d}\right\}$. The transition operator T_n of the Markov chain $\{H^{(n)}(r),\, n \in \mathbb{N}\}$ is given by

$$T_n f\left(\frac{i}{n^d}\right) = \sum_{j=0}^{n} p_{ij} f\left(\frac{i}{n^d}\right) \tag{5.42}$$

where p_{ij} is defined in (5.9) and p_i in (5.40). Now we define the operator

$$A_n := \frac{2\theta}{a} n^d (T_n - I), \quad f_n \in Dom(A_n), \quad f_n(x) := f\left(g_n^{-1}(i)\right) = f\left(\frac{i}{n^d}\right) \tag{5.43}$$

where $\theta > 0$ and $f \in C_c^3([0, +\infty\rangle)$. By the following scaling of time in $\{H^{(n)}(r), r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$ we obtain the corresponding continuous-time stochastic process $\{X^{(n)}(t), t \geq 0\}$:

$$X^{(n)}(t) := H^{(n)}\left(\left\lfloor \frac{2\theta}{a} n^d t \right\rfloor\right). \tag{5.44}$$

The next theorem states that the generally parametrized CIR diffusion can be obtained as the limiting process of the time-changed processes $\{X^{(n)}(t), t \geq 0\}$.

Theorem 5.4 Let $\{H^{(n)}(r), r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$, be the Markov chain defined by (5.41) with the transition operator (5.42). Let $X^n = \{X^{(n)}(t), t \geq 0\}$, for each $n \in \mathbb{N}$, be its corresponding time-changed process, with the time change (5.44). Let the operators $(A_n, n \in \mathbb{N})$ be defined by (5.43). Then

$$X^n \Rightarrow X \text{ in } \mathbb{D}([0, +\infty\rangle\,;\,[0, +\infty\rangle),$$

where $X = \{X(t), t \geq 0\}$ is the CIR diffusion with the infinitesimal generator $\mathcal{A}$ given by (5.39).

Proof. First, notice that function g_n satisfies conditions given in Section 5.3, i.e. function g_n is strictly monotonic and

$$\lim_{n\to\infty} \left\| g_n^{-1}(i+1) - g_n^{-1}(i) \right\|_\infty = \lim_{n\to\infty} \left| \frac{1}{n^d} \right| = 0.$$

Taking into account Remark 5.2, state space transformation (5.41) together with the time scale $h_n^{-1} = 2\theta n^d / a$ yields

$$\mu_n(x) = \frac{2\theta}{a} \sum_{j=0}^{n} p_{ij}\,(j-i)\,, \quad \sigma_n^2(x) = \frac{2\theta}{a\,n^d} \sum_{j=0}^{n} p_{ij}\,(j-i)^2\,,$$

$$R_n(x) = \frac{\theta}{3an^{2d}} \sum_{j=0}^{n} p_{ij}\,(j-i)^3\, f'''(\zeta), \quad \left|\zeta - \frac{i}{n^d}\right| < \left|\frac{1}{n^d}\,(j-i)\right|.$$

Next, by substituting transition probabilities (5.9) in the above expressions it follows

$$\begin{aligned} \mu_n(x) &= \frac{2\theta}{a} E[G_1^n - G_0^n | G_0^n = i] \\ &= \frac{2\theta}{a}\,(np_i - i)\,, \end{aligned} \tag{5.45}$$

$$
\begin{aligned}
\sigma_n^2(x) &= \frac{2\theta}{a}\frac{1}{n^d}E[(G_1^n - G_0^n)^2\,|G_0^n = i] \\
&= \frac{2\theta}{a}\frac{1}{n^d}\left(np_i(1-p_i) + n^2p_i^2 - 2nip_i + i^2\right) \\
&= \frac{2\theta}{a}\frac{1}{n^d}\left(np_i(1-p_i) + (np_i - i)^2\right), \qquad (5.46)
\end{aligned}
$$

$$
\begin{aligned}
|R_n(x)| &\leq \left|\frac{\theta}{3a\,n^{2d}}E[(G_1^n - G_0^n)^3\,|G_0^n = i]\right| K \\
&= \left|\frac{\theta}{3a\,n^{2d}}\left(np_i(1-3p_i+2p_i^2) + 3np_i(np_i - i)(1-p_i) + (np_i - i)^3\right)\right| K \\
&= \left|\frac{\theta}{3a}\left(\frac{np_i}{n^d}\frac{(1-3p_i+2p_i^2)}{n^d} + \frac{3np_i}{n^d}\frac{(np_i-i)(1-p_i)}{n^d} + \frac{(np_i-i)^3}{n^{2d}}\right)\right| K, \qquad (5.47)
\end{aligned}
$$

where K is a constant such that $|f'''(\zeta)| \leq K$. Since $i = \lfloor g_n(x) \rfloor$, it follows

$$
\lim_{n\to\infty}\sup_{x\in[0,\,+\infty\rangle}\left|(np_i - i) - \frac{(-ax+b)}{2}\right| = 0, \quad \lim_{n\to\infty}\sup_{x\in[0,\,+\infty\rangle}\left|\frac{np_i}{n^d} - x\right| = 0, \quad \lim_{n\to\infty}\sup_{x\in[0,\,+\infty\rangle} p_i = 0. \qquad (5.48)
$$

Now, using (5.45), (5.46), (5.47) together with (5.48) and the fact that $f \in C_c^3([0,\,+\infty\rangle)$ yields

$$
\lim_{n\to\infty}\|\mu_n - \mu\|_\infty = 0, \quad \lim_{n\to\infty}\left\|\sigma_n^2 - \sigma^2\right\|_\infty = 0, \quad \lim_{n\to\infty}\|R_n\|_\infty = 0, \qquad (5.49)
$$

where

$$
\mu(x) = -\theta\left(x - \frac{b}{a}\right), \quad \sigma^2(x) = \frac{2\theta}{a}x. \qquad (5.50)
$$

Now, comparing the obtained limits (5.50) with (5.39) we see that the limits coincide with the infinitesimal parameters of the CIR diffusion. Since (5.49) holds, as a direct consequence of Theorem 5.2 we obtain $X^n \Rightarrow X$ in $\mathbb{D}([0,\,+\infty\rangle\,;\,[0,\,+\infty\rangle)$, where X is the generally parametrized CIR diffusion. □

5.5.2 Markov chain for the Jacobi diffusion

Recall that the generally parametrized Jacobi diffusion $X = \{X(t),\, t \geq 0\}$ is defined as the solution of the SDE (3.54) with corresponding infinitesimal generator

$$
\mathcal{A}f(x) = -\theta\left(x - \frac{a}{a+b}\right)f'(x) + \frac{\theta}{a+b}x(1-x)f''(x), \quad f \in C_c^3([0,\,1]), \qquad (5.51)
$$

where $\theta > 0$, $a > 0$ and $b > 0$.

For the Wright-Fisher model we assume that there are no survival abilities ($s = 0$) and that probability of mutation of each type (A to a and a to A) is proportional to the size of the population, i.e.

$$\alpha = \frac{b}{2n}, \quad \beta = \frac{a}{2n}, \quad s = 0, \quad a, b > 0.$$

Therefore, the expected fraction of mature A-types (5.8) becomes

$$p_i = \frac{i}{n}\left(1 - \frac{b}{2n}\right) + \left(1 - \frac{i}{n}\right)\frac{a}{2n}. \tag{5.52}$$

Define the function $g_n : [0,\, 1] \to \mathbb{R}$,

$$g_n(x) = nx.$$

We assume initial states of the Markov chain $\{G^{(n)}(r),\, r \in \mathbb{N}_0\}$ and Jacobi diffusion $X = \{X(t),\, t \geq 0\}$ are given by $G^{(n)}(0) = i$ and $X(0) = x$, respectively, where

$$i = i(x) = \lfloor g_n(x) \rfloor = \lfloor nx \rfloor\,, \;\; x \in [0,\, 1]$$

and n is always large enough so that $i(x)$ is in the state space of Markov chain $\{G^{(n)}(r),\;\, r \in \mathbb{N}_0\}$. Notice that the initial state is a function of x, but we will use notation i for simplicity.

For each $n \in \mathbb{N}$, we define the new Markov chain $\{H^{(n)}(r),\, r \in \mathbb{N}\}$, where

$$H^{(n)}(r) = g_n^{-1}(G^{(n)}(r)) = \frac{G^{(n)}(r)}{n}, \tag{5.53}$$

with state space $\left\{0,\, \frac{1}{n},\, \frac{2}{n}, \cdots, 1\right\}$. The transition operator T_n of the Markov chain $(H^{(n)}(r),\, n \in \mathbb{N})$ is given by

$$T_n f\left(\frac{i}{n}\right) = \sum_{j=0}^{n} p_{ij} f\left(\frac{i}{n}\right) \tag{5.54}$$

where p_{ij} is defined in (5.9) and p_i in (5.52). Now we define the operator

$$A_n := \frac{2\theta}{a+b} n(T_n - I), \quad f_n \in Dom(A_n), \quad f_n(x) := f\left(g_n^{-1}(i)\right) = f\left(\frac{i}{n}\right) \tag{5.55}$$

where $\theta > 0$ and $f \in C_c^3\left([0,\, 1]\right)$. By the following scaling of time in $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$ we obtain the corresponding continuous-time process $\{X^{(n)}(t),\, t \geq 0\}$:

$$X^{(n)}(t) := H^{(n)}\left(\left\lfloor \frac{2\theta}{a+b} nt \right\rfloor\right). \tag{5.56}$$

The next theorem states that the generally parametrized Jacobi diffusion can be obtained as the limiting process of the time-changed processes $\{X^{(n)}(t),\, t \geq 0\}$.

Theorem 5.5 Let $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$, be the Markov chain defined by (5.53) with the transition operator (5.54). Let $X^n = \{X^{(n)}(t),\, t \geq 0\}$, for each $n \in \mathbb{N}$, be its corresponding time-changed process, with the time change (5.56). Let the operators $(A_n,\, n \in \mathbb{N})$ be defined by (5.55). Then

$$X^n \Rightarrow X \text{ in } \mathbb{D}([0,\, +\infty\rangle\,;\,[0,\, 1]),$$

where $X = \{X(t),\, t \geq 0\}$ is the Jacobi diffusion with the infinitesimal generator $\mathcal{A}$ given by (5.66).

Proof. First, notice that function g_n satisfies conditions given in Section 5.3, i.e. function g_n is strictly monotonic and

$$\lim_{n\to\infty} \left\| g_n^{-1}(i+1) - g_n^{-1}(i) \right\|_\infty = \lim_{n\to\infty} \left| \frac{1}{n} \right| = 0.$$

Taking into account Remark 5.2, state space transformation (5.53) together with the time scale $h_n^{-1} = 2\theta n/(a+b)$ yields

$$\mu_n(x) = \frac{2\theta}{a+b} \sum_{j=0}^{n} p_{ij}\,(j-i)\,, \quad \sigma_n^2(x) = \frac{2\theta}{(a+b)n} \sum_{j=0}^{n} p_{ij}\,(j-i)^2\,,$$

$$R_n(x) = \frac{\theta}{3(a+b)n^2} \sum_{j=0}^{n} p_{ij}\,(j-i)^3\, f'''(\zeta), \quad \left|\zeta - \frac{i}{n}\right| < \left|\frac{1}{n}\,(j-i)\right|.$$

Next, by substituting transition probabilities (5.9) together with (5.52) in the above expressions it follows

$$\begin{aligned} \mu_n(x) &= \frac{2\theta}{(a+b)} E[G_1^n - G_0^n | G_0^n = i] \\ &= \frac{2\theta}{(a+b)}\,(np_i - i)\,, \end{aligned} \tag{5.57}$$

$$\begin{aligned} \sigma_n^2(x) &= \frac{2\theta}{(a+b)n} E[(G_1^n - G_0^n)^2\, | G_0^n = i] \\ &= \frac{2\theta}{(a+b)n} \left(np_i(1-p_i) + n^2p_i^2 - 2nip_i + i^2 \right) \\ &= \frac{2\theta}{(a+b)} \left(p_i - p_i^2 + \frac{(np_i - i)^2}{n} \right), \end{aligned} \tag{5.58}$$

$$|R_n(x)| \leq \left| \frac{\theta}{3(a+b)n^2} E[(G_1^n - G_0^n)^3\, | G_0^n = i] \right| K$$

$$
\begin{aligned}
&= \left| \frac{\theta}{3(a+b)n^2} \left(np_i(1-3p_i+2p_i^2) + 3np_i(np_i-i)(1-p_i) + (np_i-i)^3 \right) \right| K \\
&= \left| \frac{\theta}{3(a+b)} \left(\frac{p_i(1-3p_i+2p_i^2)}{n} + \frac{3p_i(np_i-i)(1-p_i)}{n} + \frac{(np_i-i)^3}{n^2} \right) \right| K, \quad (5.59)
\end{aligned}
$$

where K is a constant such that $|f'''(\zeta)| \leq K$. Since $i = \lfloor g_n(x) \rfloor$, it follows

$$
\lim_{n\to\infty} \sup_{x\in[0,\,1]} \left| (np_i - i) - \frac{(-(a+b)x+a)}{2} \right| = 0, \quad \lim_{n\to\infty} \sup_{x\in[0,\,1]} \left| \frac{(np_i-i)^2}{n} \right| = 0, \quad \lim_{n\to\infty} \sup_{x\in[0,\,1]} p_i = x. \tag{5.60}
$$

Now, using (5.57), (5.58), (5.59) together with (5.60) and the fact that $f \in C_c^3([0,\,1])$ yields

$$
\lim_{n\to\infty} \|\mu_n - \mu\|_\infty = 0, \quad \lim_{n\to\infty} \left\|\sigma_n^2 - \sigma^2\right\|_\infty = 0, \quad \lim_{n\to\infty} \|R_n\|_\infty = 0, \tag{5.61}
$$

where

$$
\mu(x) = -\theta \left(x - \frac{a}{a+b} \right), \quad \sigma^2(x) = \frac{2\theta}{a+b} x(1-x). \tag{5.62}
$$

Now, comparing the obtained limits (5.62) with (5.51) we see that the limits coincide with the infinitesimal parameters of the Jacobi diffusion. Since (5.61) holds, as a direct consequence of Theorem 5.2 we obtain $X^n \Rightarrow X$ in $\mathbb{D}([0,\,+\infty\rangle\,;\,[0,\,1])$, where X is the generally parametrized Jacobi diffusion. □

5.6 Ehrenfest-Brillouin model

In this Section, motivated by applications in economics, particle physics and genetics, we present the discrete-time birth-and-death Markov chain which have the Jacobi diffusion as the scaling limit. Just like in Section 5.5, the limiting Jacobi diffusion is generally parametrized, however it gives another model as possible interpretation.

5.6.1 Ehrenfest-Brillouin Markov chain

The dynamics of this model, in which n objects move within N categories according to prescribed transition probabilities, could be viewed as the generalization of the famous Ehrenfest's model. In particular, Markov chain which describes the Ehrenfest model has state space $S = \{-n, -n+1, \ldots, n-1, n\}$ and transition probabilities

$$
p_{i,i+1} = \frac{n-i}{2n}, \quad p_{i,i-1} = \frac{n+i}{2n}, \quad 0 \text{ otherwise.}
$$

Interpretation of this model is as follows. Consider two boxes, A and B, where box A has k balls, and box B has $2n-k$ balls. Of total $2n$ balls, one is randomly selected and moved to the opposite box. Each selection represents a transition of a process which could be used to model certain physical systems (see Karlin & Taylor (1981*b*), p. 51). In the Ehrenfest-Brillouin model, the destruction mechanism is the same as in Ehrenfest's model, but the creation mechanism is more general and more complex than in the Ehrenfest's case. Here we give a brief overview of the facts on model dynamics, according to Garibaldi & Scalas (2010), inheriting the notation.

To explain the destruction-creation mechanism, consider a population of n objects that could be interpreted as particles in a physical system, genes in applications in genetics or agents in economics models. The state of the system is given by the occupation number vector

$$\mathbf{n} = (n_1, \ldots, n_i, \ldots, n_N), \quad n_k \geq 0, \quad \forall k \in \{1, \ldots N\}, \quad \sum_{k=1}^{N} n_k = n.$$

Obviously, the state space is the set of N-tuples with non-negative components summing up to n, denoted here as S_N^n. The dynamics of the system observed here is simple: the state of the system in one step changes from initial state $\mathbf{n} = (n_1, \ldots, n_i, \ldots, n_k, \ldots, n_N)$ to the final state $\mathbf{n}_i^k = (n_1, \ldots, n_i - 1, \ldots, n_k + 1, \ldots, n_N)$. This change of state could be viewed as the two-component transition:

- the destruction of the object on the ith coordinate (category) in the initial state $\mathbf{n}$ (the "Ehrenfest's term"), resulting in the state vector

$$\mathbf{n}_i = (n_1, \ldots, n_i - 1, \ldots, n_k, \ldots, n_N),$$

  which happens with probability

$$P(\mathbf{n}_i|\mathbf{n}) = \frac{n_i}{n}$$

- the creation of the object in the kth coordinate (category) given the state vector $\mathbf{n}_i$, resulting in the final state vector $\mathbf{n}_i^k$, with probability

$$P(\mathbf{n}_i^k|\mathbf{n}_i) = \frac{\alpha_k + n_k - \delta_{k,i}}{\alpha + n - 1},$$

  where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is the vector of parameters such that $\sum_{k=1}^{N} \alpha_k = \alpha$ and $\delta_{k,i}$ is the usual Kronecker's delta symbol, taking value 1 when $k = i$ and zero otherwise.

Interpretation of parameter α_i is related to the probability of accommodation on the coordinate (category) i if it is empty. In Garibaldi & Scalas (2010) two interesting cases are discussed. In the first case all α_i are negative. Then the population size is limited by $|\alpha|$ and categories by $|\alpha_i|$. In this case the transition probability is

$$P(\mathbf{n}_i^k|\mathbf{n}) = P(\mathbf{n}_i|\mathbf{n}) \cdot P(\mathbf{n}_i^k|\mathbf{n}_i) = \frac{n_i}{n} \cdot \frac{\alpha_k + n_k - \delta_{k,i}}{\alpha + n - 1}. \tag{5.63}$$

In the second case, all $\alpha_i > 0$. Then, starting from initial state $\mathbf{n}$ by repeated application of the previous transition probabilities, each state from the state space S_N^n can be reached with positive probability, meaning that the Ehrenfest-Brillouin Markov chain is irreducible. Finiteness of the state space together with the irreducibility implies that Ehrenfest-Brillouin Markov chain is recurrent, and therefore it has a unique invariant measure $\pi(\mathbf{n})$. Furthermore, the transition probability doesn't exclude the case $k = j$, so this Markov chain is aperiodic. It implies that the invariant measure $\pi(\mathbf{n})$ is the equilibrium distribution as well. The standard procedure recovers the N-dimensional Pólya distribution

$$\pi(\mathbf{n}) = Pólya(\mathbf{n}; \alpha) = \frac{n!}{\alpha^{[n]}} \prod_{i=1}^{N} \frac{\alpha_i^{[n_i]}}{n_i!}, \quad \sum_{i=1}^{N} n_i = n, \quad \sum_{i=1}^{N} \alpha_i = \alpha, \tag{5.64}$$

$$\alpha^{[n]} = \alpha \cdot (\alpha + 1) \cdot \ldots \cdot (\alpha + n - 1),$$

as the equilibrium distribution (see (Garibaldi & Scalas 2010, page 175)). This distribution comprises some famous multivariate distributions of quantum physics:

- if all $\alpha_i > 0$, the special case of equilibrium distribution (5.64) for $\alpha_i = 1$ and $\alpha = N$ is the Bose-Einstein distribution;
- if all $\alpha_i < 0$, (5.64) is the N-dimensional hypergeometric distribution whose special case, for $\alpha_i = -1$ and $\alpha = -N$ is the Fermi-Dirac distribution;
- as $|\alpha| \to \infty$, the limit of (5.64) is the multinomial distribution whose symmetric case is known as the Maxwell-Boltzmann distribution.

An important observation, directly connecting one particular case of this model to Jacobi diffusion, is that in case of two categories from the invariant Pólya distribution $Pólya(k, n-k; 1/2, 1/2)$ the distribution of the ratio k/n is the Beta distribution (3.55) with $a = 1/2$ and $b = 1/2$. For more details we refer to (Garibaldi & Scalas 2010, Section 7.3).

One example of the Ehrenfest-Brillouin Markov chain is the taxation-redistribution economics model, see (Garibaldi & Scalas 2010, page 212), where n coins are redistributed among N agents. A taxation is a step in which coin is randomly taken out of the set of n coins (destruction) and a redistribution is a step in which the coin is given to one of N agents (creation). The (destruction) probability of selecting one coin belonging to the ith agent is n_i/n, while in the redistribution step there are several possible schemes, e.g. favoring the agents already having many coins or those having few coins. For example, if it is assumed that the probability of giving the coin taken from agent i to agent j is proportional to $(w_j + n_j)$, where n_j is the wealth of jth agent and w_j is the corresponding weight, then depending on the choice of the weight different equilibrium distributions could be obtained. In this general framework one could assume that the transition probability is of the following form:

$$P(\mathbf{n}_i^j|\mathbf{n}) = \frac{n_i}{n} \cdot \frac{w_j + n_j - \delta_{i,j}}{w+n-1}, \quad w = \sum_{i=1}^{N} w_i.$$

If no agent is favored in this scheme, then $w_j = \alpha$ for all $j \in \{1, \dots, N\}$, and therefore

$$P(\mathbf{n}_i^j|\mathbf{n}) = \frac{n_i}{n} \cdot \frac{\alpha + n_j - \delta_{i,j}}{N\alpha + n - 1}, \tag{5.65}$$

which is exactly the Ehrenfest-Brillouin model with unary moves. For more details on the taxation-redistribution model see (Garibaldi & Scalas 2010, Section 8.2), while (Garibaldi & Scalas 2010, Section 8.3) contains more applications of the Ehrenfest-Brillouin model to economics.

5.6.2 Markov chain for the Jacobi diffusion

In this subsection we use the margin of the two-dimensional Ehrenfest-Brillouin Markov chain from Subsection 5.6.1 to construct a transformed and rescaled Markov chain converging to the Jacobi diffusion.

Recall that the generally parametrized Jacobi diffusion $X = \{X(t),\, t \geq 0\}$ is defined as the solution of the SDE (3.54) with corresponding infinitesimal generator

$$\mathcal{A}f(x) = -\theta\left(x - \frac{a}{a+b}\right) f'(x) + \frac{\theta}{a+b} x(1-x)f''(x), \quad f \in C_c^3([0,\, 1]), \tag{5.66}$$

where $\theta > 0$, $a > 0$ and $b > 0$. For each $n \in \mathbb{N}$, denote by $\{G^{(n)}(r),\, r \in \mathbb{N}_0\}$ the marginal Ehrenfest-Brillouin Markov chain with the state space $\{0, 1, 2, \dots, n\}$. The transition probabilities for this Markov chain are as follows:

$$p_{i,i+1} = \frac{n-i}{n} \cdot \frac{a+i}{a+b+n-1}, \quad p_{i,i-1} = \frac{i}{n} \cdot \frac{b+n-i}{a+b+n-1}, \quad p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}, \tag{5.67}$$

0 otherwise, where $a > 0$, $b > 0$. In light of the taxation-redistribution model with uniformly weighted agents (with weight α), these transition probabilities could be interpreted in terms of the number of coins belonging to agent 1 in time t. If we start with i coins, $p_{i,i+1}$ is the probability that a randomly chosen coin, out from the set of $(n-i)$ coins belonging to other agents, is redistributed to agent 1; $p_{i,i-1}$ is the probability that a randomly chosen coin, out of i coins belonging to agent 1, is redistributed to one of the other agents; $p_{i,i}$ is the probability that reflects agent 1 invariance to the coin "destruction-creation".

Define the function $g_n : [0,\, 1] \to \mathbb{R}$,

$$g_n(x) = nx.$$

We assume initial states of the Markov chain $\{G^{(n)}(r),\, r \in \mathbb{N}_0\}$ and Jacobi diffusion $X = \{X(t),\, t \geq 0\}$ are given by $G^{(n)}(0) = i$ and $X(0) = x$, respectively, where

$$i = i(x) = \lfloor g_n(x) \rfloor = \lfloor nx \rfloor \,,\; x \in [0,\, 1]$$

and n is always large enough so that $i(x)$ is in the state space of Markov chain $\{G^{(n)}(r),\; r \in \mathbb{N}_0\}$. Moreover, we assume that the initial Markov chain $\{G^{(n)}(r),\, r \in \mathbb{N}_0\}$ never starts from states 0 and n. Notice that the initial state is a function of x, but we will use notation i for simplicity.

For each $n \in \mathbb{N}$, we define the new Markov chain $\{H^{(n)}(r),\, r \in \mathbb{N}\}$, where

$$H^{(n)}(r) = g_n^{-1}(G^{(n)}(r)) = \frac{G^{(n)}(r)}{n}, \tag{5.68}$$

with state space $\left\{0,\, \frac{1}{n},\, \frac{2}{n}, \cdots, 1\right\}$. The transition operator T_n of the Markov chain $\{H^{(n)}(r),\, n \in \mathbb{N}\}$ is given by

$$T_n f\left(\frac{i}{n}\right) = \sum_{j=0}^{n} p_{ij} f\left(\frac{j}{n}\right) = p_{i,i-1}\, f\left(\frac{i-1}{n}\right) + p_{i,i}\, f\left(\frac{i}{n}\right) + p_{i,i+1}\, f\left(\frac{i+1}{n}\right) \tag{5.69}$$

where p_{ij} is defined in (5.67).
Now we define the operator

$$A_n := \frac{\theta}{a+b} n^2 (T_n - I), \quad f_n \in Dom(A_n), \quad f_n(x) := f\left(g_n^{-1}(i)\right) = f\left(\frac{i}{n}\right) \tag{5.70}$$

where $\theta > 0$ and $f \in C_c^3\left([0,\, 1]\right)$. By the following scaling of time in $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$ we obtain the corresponding continuous-time stochastic process $\{X^{(n)}(t),\, t \geq 0\}$:

$$X^{(n)}(t) := H^{(n)}\left(\left\lfloor \frac{\theta}{a+b} n^2 t \right\rfloor\right). \tag{5.71}$$

The next theorem states that the generally parametrized Jacobi diffusion can be obtained as the limiting process of the time-changed processes $\{X^{(n)}(t),\, t \geq 0\}$.

Theorem 5.6 Let $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$, be the Markov chain defined by (5.68) with the transition operator (5.69). Let $X^n = \{X^{(n)}(t),\, t \geq 0\}$, for each $n \in \mathbb{N}$, be its corresponding time-changed process, with the time change (5.71). Let the operators $(A_n,\, n \in \mathbb{N})$ be defined by (5.70). Then

$$X^n \Rightarrow X \text{ in } \mathbb{D}([0,\, +\infty\rangle\, ;\, [0,\, 1]),$$

where $X = \{X(t),\, t \geq 0\}$ is the Jacobi diffusion with the infinitesimal generator $\mathcal{A}$ given by (5.66).

Proof. First, notice that function g_n satisfies conditions given in Section 5.3, i.e. function g_n is strictly monotonic and

$$\lim_{n\to\infty} \left\| g_n^{-1}(i+1) - g_n^{-1}(i) \right\|_\infty = \lim_{n\to\infty} \left| \frac{1}{n} \right| = 0.$$

Taking into account Remark 5.3, state space transformation (5.68) together with the time scale $h_n^{-1} = \theta n^2/(a+b)$ yields

$$\mu_n(x) = \frac{\theta}{a+b} n \left(p_{i,i+1} - p_{i,i-1}\right), \quad \sigma_n^2(x) = \frac{\theta}{a+b} \left(p_{i,i+1} + p_{i,i-1}\right),$$

$$R_n(x) = \frac{\theta}{6(a+b)n} \left(p_{i,i+1} - p_{i,i-1}\right) f'''(\zeta), \quad \left|\zeta - \frac{i}{n}\right| < \left|\frac{1}{n}(j-i)\right|.$$

Next, by substituting transition probabilities (5.67) in the above expressions it follows

$$\begin{aligned} \mu_n(x) &= \frac{\theta}{(a+b)} n \left(\frac{n-i}{n} \cdot \frac{a+i}{a+b+n-1} - \frac{i}{n} \cdot \frac{b+n-i}{a+b+n-1} \right) \\ &= \frac{\theta}{(a+b)} \left(\frac{a - \frac{i}{n}(a+b)}{\frac{a}{n} + \frac{b}{n} + 1 - \frac{1}{n}} \right), \end{aligned} \tag{5.72}$$

$$\begin{aligned} \sigma_n^2(x) &= \frac{\theta}{(a+b)} \left(\frac{n-i}{n} \cdot \frac{a+i}{a+b+n-1} + \frac{i}{n} \cdot \frac{b+n-i}{a+b+n-1} \right) \\ &= \frac{\theta}{(a+b)} \left((1 - \frac{i}{n}) \cdot \frac{\frac{a}{n} + \frac{i}{n}}{\frac{a}{n} + \frac{b}{n} + 1 - \frac{1}{n}} + \frac{i}{n} \cdot \frac{\frac{b}{n} + 1 - \frac{i}{n}}{\frac{a}{n} + \frac{b}{n} + 1 - \frac{1}{n}} \right), \end{aligned} \tag{5.73}$$

$$\begin{aligned} |R_n(x)| &\le \left| \frac{\theta}{6(a+b)n} \left(\frac{n-i}{n} \cdot \frac{a+i}{a+b+n-1} - \frac{i}{n} \cdot \frac{b+n-i}{a+b+n-1} \right) \right| K \\ &= \left| \frac{\theta}{6(a+b)} \frac{1}{n} \left(\frac{a - \frac{i}{n}(a+b)}{\frac{a}{n} + \frac{b}{n} + 1 - \frac{1}{n}} \right) \right| K, \end{aligned} \tag{5.74}$$

where K is a constant such that $|f'''(\zeta)| \le K$. Since $i = \lfloor g_n(x) \rfloor$, it follows

$$\lim_{n\to\infty} \sup_{x\in[0,\,1]} \left| \frac{i}{n} - x \right| = 0. \tag{5.75}$$

Now, using (5.72), (5.73), (5.74) together with (5.75) and the fact that $f \in C_c^3([0,\,1])$ yields

$$\lim_{n\to\infty} \|\mu_n - \mu\|_\infty = 0, \quad \lim_{n\to\infty} \left\|\sigma_n^2 - \sigma^2\right\|_\infty = 0, \quad \lim_{n\to\infty} \|R_n\|_\infty = 0, \tag{5.76}$$

where

$$\mu(x) = -\theta \left(x - \frac{a}{a+b} \right), \quad \sigma^2(x) = \frac{2\theta}{a+b} x(1-x). \tag{5.77}$$

Now, comparing the obtained limits (5.77) with (5.66) we see that the limits coincide with the infinitesimal parameters of the Jacobi diffusion. Since (5.76) holds, as a direct consequence of Theorem 5.2 we obtain $X^n \Rightarrow X$ in $\mathbb{D}([0, +\infty\rangle\,; [0, 1])$, where X is the generally parametrized Jacobi diffusion. $\square$

5.7 Markov chains for heavy-tailed Pearson diffusions

In this section, we construct discrete-time Markov chains with scaling limits being heavy-tailed Pearson diffusions.

5.7.1 Markov chain for the Student diffusion

Recall that the generally parametrized Student diffusion $X = \{X(t),\, t \geq 0\}$ is defined as the solution of the SDE (3.104) with corresponding infinitesimal generator

$$\mathcal{A}f(x) = -\theta\,(x-\mu)\,f'(x) + \frac{\theta\delta^2}{\nu-1}\left(1+\left(\frac{x-\mu}{\delta}\right)^2\right)f''(x), \quad f \in C_c^3(\mathbb{R}). \tag{5.78}$$

where $\theta > 0$, $\mu \in \mathbb{R}$, $\nu > 1$ and $\delta > 0$. Let $\{Z^{(n)}(r),\, r \in \mathbb{N}\}$ be the Markov chain with state space $\{0, 1, 2, \ldots, n\}$ and transition probabilities

$$p_{0,1} = 1, \quad p_{n,n-1} = 1,$$
$$p_{i,i+1} = \frac{1}{2c}\left(1-\frac{2i}{n}\right)^2 + \frac{1}{n}\left(1-\frac{i}{n}\right)^2, \quad p_{i,i-1} = \frac{1}{2c}\left(1-\frac{2i}{n}\right)^2 + \frac{1}{n}\left(\frac{i}{n}\right)^2, \quad p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1} \tag{5.79}$$

and 0 otherwise, where $0 < d < 1,\ c > 1$ and n large enough, ensuring $p_{i,i+1} + p_{i,i-1} < 1$. This Markov chain is clearly irreducible since each state can be reached with positive probability. Finiteness of the state space $\{0, 1, 2, \ldots, n\}$ with the irreducibility implies that the Markov chain is also reccurent, which again implies it has a unique (up to a constant) invariant measure. Furthermore, finiteness of the state space implies this Markov chain has unique stationary distribution π:

$$\pi(n) = \pi(0) = \left(2 + \frac{2cn^3}{n(n-2)^2+2c}\left(1 + \sum_{x=2}^{n-1}\frac{\prod_{k=1}^{x-1}[n(n-2k)^2 + 2c(n-k)^2]}{\prod_{k=2}^{x}[n(n-2k)^2+2ck^2]}\right)\right)^{-1},$$

$$\pi(x) = \frac{2cn^3}{n(n-2)^2+2c}\cdot\frac{\prod_{k=1}^{x-1}[n(n-2k)^2 + 2c(n-k)^2]}{\prod_{k=2}^{x}[n(n-2k)^2+2ck^2]}\cdot\pi(0), \quad x \in \{1, 2, 3, \ldots, n-1\}.$$

On the other hand, Markov chain is periodic, since states 0 and n have periods of 2.

Define the function $g_n : \mathbb{R} \to \mathbb{R}$,

$$g_n(x) = \frac{1}{2}\left(n + (ax+b)\sqrt{n}\right), \quad a > 0, \quad b \in \mathbb{R}.$$

We assume initial states of the Markov chain $\{Z^{(n)}(r),\, r \in \mathbb{N}_0\}$ and Student diffusion $X = \{X(t),\, t \geq 0\}$ are given by $Z^{(n)}(0) = i$ and $X(0) = x$, respectively, where

$$i = i(x) = \lfloor g_n(x) \rfloor = \left\lfloor \frac{1}{2}\left(n + (ax+b)\sqrt{n}\right) \right\rfloor, \; x \in \mathbb{R}.$$

and n is always large enough so that $i(x)$ is in the state space of Markov chain $\{Z^{(n)}(r),\ r \in \mathbb{N}_0\}$. Moreover, we assume that the initial Markov chain $\{Z^{(n)}(r),\ r \in \mathbb{N}_0\}$ never starts from states 0 and n. Notice that the initial state is a function of x, but we will use notation i for simplicity.

For each $n \in \mathbb{N}$, we define the new Markov chain $\{H^{(n)}(r),\, r \in \mathbb{N}\}$ where

$$H^{(n)}(r) = g_n^{-1}(Z^{(n)}(r)) = \frac{1}{a\sqrt{n}}\left(2Z^{(n)}(r) - n - b\sqrt{n}\right), \tag{5.80}$$

with state space $\left\{\frac{1}{a\sqrt{n}}\left(-n - b\sqrt{n}\right), \frac{1}{a\sqrt{n}}\left(2 - n - b\sqrt{n}\right), \cdots \frac{1}{a\sqrt{n}}\left(n - b\sqrt{n}\right)\right\}$. The transition operator T_n of the Markov chain $\{H^{(n)}(r),\, n \in \mathbb{N}\}$ is given by

$$\begin{aligned} T_n f\left(\frac{2i - n - b\sqrt{n}}{a\sqrt{n}}\right) &= \sum_{j=0}^{n} p_{ij} f\left(\frac{2j - n - b\sqrt{n}}{a\sqrt{n}}\right) \\ &= p_{i,i-1}\, f\left(\frac{2(i-1) - n - b\sqrt{n}}{a\sqrt{n}}\right) + p_{i,i}\, f\left(\frac{2i - n - b\sqrt{n}}{a\sqrt{n}}\right) + \\ &\quad + p_{i,i+1}\, f\left(\frac{2(i+1) - n - b\sqrt{n}}{a\sqrt{n}}\right) \end{aligned} \tag{5.81}$$

where p_{ij} is defined in (5.79).
Now we define the operator

$$A_n := \frac{\theta}{2} n^2 (T_n - I), \quad f_n \in Dom(A_n), \quad f_n(x) := f\left(g_n^{-1}(i)\right) = f\left(\frac{2i - n - b\sqrt{n}}{a\sqrt{n}}\right), \tag{5.82}$$

where $\theta > 0$ and $f \in C_c^3(\mathbb{R})$. By the following scaling of time in $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$ we obtain the corresponding continuous-time stochastic process $\{X^{(n)}(t),\, t \geq 0\}$:

$$X^{(n)}(t) := H^{(n)}\left(\left\lfloor \frac{\theta}{2} n^2 t \right\rfloor\right). \tag{5.83}$$

The next theorem states that the ST diffusion can be obtained as the limiting process of the time-changed processes $\{X^{(n)}(t),\, t \geq 0\}$.

Theorem 5.7 Let $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$, be the Markov chain defined by (5.80) with the transition operator (5.81). Let $X^n = \{X^{(n)}(t),\, t \geq 0\}$, for each $n \in \mathbb{N}$, be its corresponding time-changed process, with the time change (5.83). Let the operators $(A_n,\, n \in \mathbb{N})$ be defined by (5.82). Then

$$X^n \Rightarrow X \text{ in } \mathbb{D}\,[0,\, +\infty\rangle\,,$$

where $X = \{X(t),\, t \geq 0\}$ is the ST diffusion with the infinitesimal generator $\mathcal{A}$ given by (5.78), and

$$\mu = -\frac{b}{a}, \quad \nu = c + 1, \quad \delta = \frac{1}{a}\sqrt{\frac{c}{2}}.$$

Proof. First, notice that function g_n satisfies conditions given in Section 5.3, i.e. function g_n is strictly monotonic and

$$\lim_{n\to\infty} \left\| g_n^{-1}(i+1) - g_n^{-1}(i) \right\|_\infty = \lim_{n\to\infty} \left| \frac{2}{a\sqrt{n}} \right| = 0.$$

Taking into account Remark 5.3, state space transformation (5.80) together with the time scale $h_n^{-1} = \theta n^2/2$ yields

$$\mu_n(x) = \frac{\theta}{a} n\sqrt{n}\left(p_{i,i+1} - p_{i,i-1}\right), \quad \sigma_n^2(x) = \frac{2\theta}{a^2} n\left(p_{i,i+1} + p_{i,i-1}\right),$$

$$R_n(x) = \frac{2\theta}{3a^3}\sqrt{n}\left(p_{i,i+1} - p_{i,i-1}\right) f'''(\zeta), \quad \left|\zeta - \frac{2i - n - b\sqrt{n}}{a\sqrt{n}}\right| < \left|\frac{2}{a\sqrt{n}}\,(j-i)\right|.$$

Next, by substituting transition probabilities (5.79) in the above expressions it follows

$$\begin{aligned}\mu_n(x) &= \frac{\theta}{a} n\sqrt{n}\left(\frac{1}{2c}\left(1-\frac{2i}{n}\right)^2 + \frac{1}{n}\left(1-\frac{i}{n}\right)^2 - \frac{1}{2c}\left(1-\frac{2i}{n}\right)^2 - \frac{1}{n}\left(\frac{i}{n}\right)^2\right) \\ &= \frac{\theta\sqrt{n}}{a}\left(1-\frac{2i}{n}\right) = \theta\left(\frac{n-2i}{a\sqrt{n}}\right),\end{aligned} \tag{5.84}$$

$$\begin{aligned}\sigma_n^2(x) &= \frac{2\theta}{a^2} n\left(\frac{1}{2c}\left(1-\frac{2i}{n}\right)^2 + \frac{1}{n}\left(1-\frac{i}{n}\right)^2 + \frac{1}{2c}\left(1-\frac{2i}{n}\right)^2 + \frac{1}{n}\left(\frac{i}{n}\right)^2\right) \\ &= 2\theta\left(\frac{1}{c}\left(\frac{n-2i}{a\sqrt{n}}\right)^2 + \frac{1}{a^2}\left(1-\frac{i}{n}\right)^2 + \frac{1}{a^2}\left(\frac{i}{n}\right)^2\right),\end{aligned} \tag{5.85}$$

$$|R_n(x)| \leq \left|\frac{2\theta}{3a^3}\sqrt{n}\left(\frac{1}{2c}\left(1-\frac{2i}{n}\right)^2 + \frac{1}{n}\left(1-\frac{i}{n}\right)^2 - \frac{1}{2c}\left(1-\frac{2i}{n}\right)^2 - \frac{1}{n}\left(\frac{i}{n}\right)^2\right)\right| K$$

$$
\begin{aligned}
&= \left| \frac{2\theta}{3a^3} \left(\frac{1}{\sqrt{n}} \left(1 - \frac{i}{n}\right)^2 - \frac{1}{\sqrt{n}} \left(\frac{i}{n}\right)^2 \right) \right| K \\
&= \left| \frac{2\theta}{3a^3} \left(\frac{n-2i}{n\sqrt{n}} \right) \right| K, \qquad (5.86)
\end{aligned}
$$

where K is a constant such that $|f'''(\zeta)| \leq K$. Since $i = \lfloor g_n(x) \rfloor$, it follows

$$
\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{n-2i}{a\sqrt{n}} + \left(x + \frac{b}{a}\right) \right| = 0, \quad \lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{i}{n} - \frac{1}{2} \right| = 0. \qquad (5.87)
$$

Now, using (5.84), (5.85), (5.86) together with (5.87) and the fact that $f \in C_c^3(\mathbb{R})$ yields

$$
\lim_{n\to\infty} \|\mu_n - \mu\|_\infty = 0, \quad \lim_{n\to\infty} \left\|\sigma_n^2 - \sigma^2\right\|_\infty = 0, \quad \lim_{n\to\infty} \|R_n\|_\infty = 0, \qquad (5.88)
$$

where

$$
\mu(x) = -\theta \left(x + \frac{b}{a}\right), \quad \sigma^2(x) = 2\theta \left(\frac{1}{c} \left(x + \frac{b}{a}\right)^2 + \frac{1}{2a^2} \right).
$$

By re-parametrizing

$$
\mu = -\frac{b}{a}, \quad \nu = c + 1, \quad \delta = \frac{1}{a}\sqrt{\frac{c}{2}} \qquad (5.89)
$$

it follows

$$
\mu(x) = -\theta\,(x - \mu)\,, \quad \sigma^2(x) = \frac{2\theta\delta^2}{\nu - 1} \left(1 + \left(\frac{x-\mu}{\delta}\right)^2 \right). \qquad (5.90)
$$

Now, comparing the obtained limits (5.90) with (5.78) we see that the limits coincide with the infinitesimal parameters of the ST diffusion. Since (5.88) holds, as a direct consequence of Theorem 5.2 we obtain $X^n \Rightarrow X$ in $\mathbb{D}\,[0, +\infty\rangle$, where X is the generally parametrized ST diffusion.

□

Remark 5.4. Notice how re-parametrization (5.89) ensures parameters of ST diffusion satisfy

$$
\theta > 0, \quad \mu \in \mathbb{R}, \quad \nu > 2, \quad \delta > 0,
$$

since $a > 0$, $b \in \mathbb{R}$ and $c > 1$. In general, parameter ν can be any real number larger then 1, but the obtained $\nu > 2$ ensures that invariant Student distribution has finite second moment.

Remark 5.5. It is well known that the Student distribution (3.105), for high degrees of freedom ν, can be approximated by the Normal distribution (3.33). In a similar manner, if we let $c \to \infty$ in transition probabilities (5.79), they reduce to the transition probabilities (5.26) of the famous Bernoulli-Laplace urn-scheme model which leads to the OU diffusion. On the other hand, by taking into account (5.89), infinitesimal parameters (5.90) of ST diffusion reduce to the infinitesimal parameters of OU process as $c \to \infty$ (and therefore

$\nu \to \infty$, $\delta \to \infty$). Therefore, by letting $c \to \infty$ the correlated CTRW which leads to the ST diffusion, reduces to the correlated CTRW which leads to the OU process.

5.7.2 Markov chains for the Fisher-Snedecor and reciprocal gamma diffusion

First, we define starting Markov chain which will lead to FS and RG diffusion with appropriately chosen parameters. Let $\{G^{(n)}(r),\ r \in \mathbb{N}\}$ be the Markov chain with state space $\{0,\ 1,\ 2,\ \dots,\ n\}$ and transition probabilities

$$p_{0,1} = 1, \quad p_{n,n-1} = 1,$$
$$p_{i,i+1} = \left(\frac{i}{n}\right)^2 \frac{a^*}{n^d} + \frac{b^*}{n^2} + \frac{c^* i}{n^2}, \quad p_{i,i-1} = \frac{a^* i + d^*}{n^2}\frac{i}{n^d} + \frac{c^* i}{n^2}, \quad p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}, \tag{5.91}$$

and 0 otherwise, where $0 < d < 1,\ a^* \geq 0,\ b^* \geq 0,\ c^* \geq 0,\ d^* \geq 0$. This Markov chain is clearly irreducible since each state can be reached with positive probability. Finiteness of the state space $\{0,\ 1,\ 2,\ \dots,\ n\}$ with the irreducibility implies that the Markov chain is also reccurent, which again implies that it has a unique (up to a constant) invariant measure. Furthermore, finiteness of the state space implies this Markov chain has unique stationary distribution π:

$$\begin{aligned}\pi(0) \quad = \quad & \left(1 + \frac{n^{d+2}}{a^* + b^* + c^* n^d}\left(1 + \frac{(n-1)^2 a^* + n^d(b^* + c^*(n-1))}{n^{d+2}} \frac{\prod_{k=1}^{x-1}\left[a^* k^2 + c^* n^d k + b^* n^d\right]}{\prod_{k=2}^{x}\left[a^* k^2 + (c^* n^d + d^*)k\right]}\right.\right. \\ & \left.\left. + \sum_{x=2}^{n-1} \frac{\prod_{k=1}^{x-1}\left[a^* k^2 + c^* n^d k + b^* n^d\right]}{\prod_{k=2}^{x}\left[a^* k^2 + (c^* n^d + d^*)k\right]}\right)\right)^{-1},\end{aligned}$$

$$\pi(x) = \frac{n^{d+2}}{a^* + b^* + c^* n^d} \cdot \frac{\prod_{k=1}^{x-1}\left[a^* k^2 + c^* n^d k + b^* n^d\right]}{\prod_{k=2}^{x}\left[a^* k^2 + (c^* n^d + d^*)k\right]} \cdot \pi(0), \quad x \in \{1, 2, 3, \dots, n-1\}$$

$$\pi(n) = \pi(n-1) \cdot \left[\left(\frac{n-1}{n}\right)^2 \frac{a^*}{n^d} + \frac{b^*}{n^2} + \frac{c^*(n-1)}{n^2}\right].$$

On the other hand, Markov chain is periodic, since states 0 and n have periods of 2.

Fisher-Snedecor diffusion

Recall that the generally parametrized Fisher-Snedecor diffusion $X = \{X(t),\, t \geq 0\}$ is defined as the solution of the SDE (3.71) with corresponding infinitesimal generator

$$\mathcal{A}f(x) = -\theta\left(x - \frac{\beta}{\beta-2}\right)f'(x) + \frac{2\theta}{\gamma(\beta-2)}x(\gamma x+\beta)f''(x), \quad f \in C_c^3([0,\, +\infty\rangle), \tag{5.92}$$

where $\theta > 0$, $\beta > 2$ and $\gamma > 0$. Let $\{G^{(n)}(r),\ r \in \mathbb{N}\}$ be the Markov chain with state space $\{0,\, 1,\, 2,\, \ldots,\, n\}$ and transition probabilities (5.91) with parameters

$$a^* = a, \quad b^* = a+b, \quad c^* = c, \quad d^* = b, \quad a > 0, \quad b > 0, \quad c > 0,$$

i.e.

$$p_{0,1} = 1, \quad p_{n,n-1} = 1,$$
$$p_{i,i+1} = \left(\frac{i}{n}\right)^2 \frac{a}{n^d} + \frac{a+b}{n^2} + \frac{ci}{n^2}, \quad p_{i,i-1} = \frac{ai+b}{n^2}\frac{i}{n^d} + \frac{ci}{n^2}, \quad p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}, \tag{5.93}$$

and 0 otherwise, where n is large enough, ensuring $p_{i,i+1} + p_{i,i-1} < 1$. Define the function $g_n : [0,\, +\infty\rangle \to \mathbb{R}$,

$$g_n(x) = n^d x.$$

We assume initial states of the Markov chain $\{G^{(n)}(r),\ r \in \mathbb{N}_0\}$ and Fisher-Snedecor diffusion $X = \{X(t),\, t \geq 0\}$ are given by $G^{(n)}(0) = i$ and $X(0) = x$, respectively, where

$$i = i(x) = \lfloor g_n(x) \rfloor = \left\lfloor n^d x \right\rfloor, \ x \in [0,\, +\infty\rangle$$

and n is always large enough so that $i(x)$ is in the state space of Markov chain $\{G^{(n)}(r),\ r \in \mathbb{N}_0\}$. Moreover, we assume that the initial Markov chain $\{G^{(n)}(r),\, r \in \mathbb{N}_0\}$ never starts from states 0 and n. Notice that the initial state is a function of x, but we will use notation i for simplicity. For each $n \in \mathbb{N}$, we define the new Markov chain $\{H^{(n)}(r),\ r \in \mathbb{N}\}$ with the state space $\{0, 1/n^d, \ldots, 1/n^{d-1}\}$

$$H^{(n)}(r) = g_n^{-1}(G^{(n)}(r)) = \frac{G^{(n)}(r)}{n^d}. \tag{5.94}$$

The transition operator T_n of the Markov chain $\{H^{(n)}(r),\ n \in \mathbb{N}\}$ is given by

$$T_n f\left(\frac{i}{n^d}\right) = \sum_{j=0}^{n} p_{ij} f\left(\frac{j}{n^d}\right) = p_{i,i-1}\, f\left(\frac{i-1}{n^d}\right) + p_{i,i}\, f\left(\frac{i}{n^d}\right) + p_{i,i+1}\, f\left(\frac{i+1}{n^d}\right) \tag{5.95}$$

where p_{ij} is defined in (5.93).
Now we define the operator

$$A_n := n^{2+d}(T_n - I), \quad f_n \in Dom(A_n), \quad f_n(x) := f\left(g_n^{-1}(i)\right) = f\left(\frac{i}{n^d}\right), \tag{5.96}$$

where $f \in C_c^3\left([0, +\infty\rangle\right)$ and by the following scaling of time in $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$ we obtain the corresponding continuous-time stochastic process $\{X^{(n)}(t),\, t \geq 0\}$:

$$X^{(n)}(t) := H^{(n)}\left(\lfloor n^{2+d}t \rfloor\right). \tag{5.97}$$

The next theorem states that the FS diffusion can be obtained as the limiting process of the time-changed processes $\{X^{(n)}(t),\, t \geq 0\}$.

Theorem 5.8 Let $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$, be the Markov chain defined by (5.94) with the transition operator (5.95). Let $X^n = \{X^{(n)}(t),\, t \geq 0\}$, for each $n \in \mathbb{N}$, be its corresponding time-changed process, with the time change (5.97). Let the operators $(A_n,\, n \in \mathbb{N})$ be defined by (5.96). Then

$$X^n \Rightarrow X \text{ in } \mathbb{D}([0, +\infty\rangle\,;\,[0, +\infty\rangle),$$

where $X = \{X(t),\, t \geq 0\}$ is the FS diffusion with the infinitesimal generator $\mathcal{A}$ given by (5.92), and

$$\theta = b, \quad \beta = 2\left(\frac{b}{a} + 1\right), \quad \gamma = \frac{2\,(a+b)}{c}.$$

Proof. First, notice that function g_n satisfies conditions given in Section 5.3, i.e. function g_n is strictly monotonic and

$$\lim_{n\to\infty} \left\| g_n^{-1}(i+1) - g_n^{-1}(i) \right\|_\infty = \lim_{n\to\infty} \left| \frac{1}{n^d} \right| = 0.$$

Taking into account Remark 5.3, state space transformation (5.94) together with the time scale $h_n^{-1} = n^{2+d}$ yields

$$\mu_n(x) = n^2\left(p_{i,i+1} - p_{i,i-1}\right), \quad \sigma_n^2(x) = n^{2-d}\left(p_{i,i+1} + p_{i,i-1}\right),$$

$$R_n(x) = \frac{1}{6} n^{2-2d}\left(p_{i,i+1} - p_{i,i-1}\right) f'''(\zeta), \quad \left|\zeta - \frac{i}{n^d}\right| < \left|\frac{j-i}{n^d}\right|.$$

Next, by substituting transition probabilities (5.93) in the above expressions it follows

$$\begin{aligned} \mu_n(x) &= n^2\left(\left(\frac{i}{n}\right)^2 \frac{a}{n^d} + \frac{a+b}{n^2} + \frac{ci}{n^2} - \frac{ai+b}{n^2}\frac{i}{n^d} - \frac{ci}{n^2}\right) \\ &= a + b - b\frac{i}{n^d}, \end{aligned} \tag{5.98}$$

$$\begin{aligned}\sigma_n^2(x) &= n^{2-d}\left(\left(\frac{i}{n}\right)^2\frac{a}{n^d}+\frac{a+b}{n^2}+\frac{ci}{n^2}+\frac{ai+b}{n^2}\frac{i}{n^d}+\frac{ci}{n^2}\right)\\ &= 2a\left(\frac{i}{n^d}\right)^2+2c\frac{i}{n^d}+\frac{a+b}{n^d}+b\frac{i}{n^{2d}},\end{aligned} \tag{5.99}$$

$$\begin{aligned}|R_n(x)| &\le \left|\frac{n^{2-2d}}{6}\left(\left(\frac{i}{n}\right)^2\frac{a}{n^d}+\frac{a+b}{n^2}+\frac{ci}{n^2}-\frac{ai+b}{n^2}\frac{i}{n^d}-\frac{ci}{n^2}\right)\right|K\\ &= \left|\frac{1}{6}\left(\frac{a+b}{n^{2d}}-b\frac{i}{n^{3d}}\right)\right|K,\end{aligned} \tag{5.100}$$

where K is a constant such that $|f'''(\zeta)| \le K$. Since $i = \lfloor g_n(x) \rfloor$, it follows

$$\lim_{n\to\infty}\sup_{x\in[0,\,+\infty\rangle}\left|\frac{i}{n^d}-x\right| = 0. \tag{5.101}$$

Now, using (5.98), (5.99), (5.100) together with (5.101) and the fact that $f \in C_c^3([0,\,+\infty\rangle)$ yields

$$\lim_{n\to\infty}\|\mu_n-\mu\|_\infty = 0, \quad \lim_{n\to\infty}\left\|\sigma_n^2-\sigma^2\right\|_\infty = 0, \quad \lim_{n\to\infty}\|R_n\|_\infty = 0, \tag{5.102}$$

where

$$\mu(x) = a+b-bx, \quad \sigma^2(x) = 2ax^2+2cx.$$

By re-parametrizing

$$\theta = b, \quad \beta = 2\left(\frac{b}{a}+1\right), \quad \gamma = \frac{2\,(a+b)}{c} \tag{5.103}$$

it follows

$$\mu(x) = -\theta\left(x-\frac{\beta}{\beta-2}\right), \quad \sigma^2(x) = \frac{4\theta}{\gamma\,(\beta-2)}x\,(\gamma x+\beta)\,. \tag{5.104}$$

Notice how re-parametrization (5.103) ensures the generality of parameters of FS diffusion, i.e.

$$\theta > 0, \quad \beta > 2, \quad \gamma > 0$$

since $a > 0,\ b > 0,\ c > 0$. Now, comparing the obtained limits (5.104) with (5.92) we see that the limits coincide with the infinitesimal parameters of the FS diffusion. Since (5.102) holds, as a direct consequence of Theorem 5.2 we obtain $X^n \Rightarrow X$ in $\mathbb{D}([0,\,+\infty\rangle\,;\,[0,\,+\infty\rangle)$, where X is the generally parametrized FS diffusion. □

Reciprocal gamma diffusion

Recall that the generally parametrized reciprocal gamma diffusion $X = \{X(t),\, t \geq 0\}$ is defined as the solution of the SDE (3.88) with corresponding infinitesimal generator

$$\mathcal{A}f(x) = -\theta\left(x - \frac{\gamma}{\beta-1}\right)f'(x) + \frac{1}{2}\frac{2\theta}{\beta-1}x^2 f''(x), \quad f \in C_c^3([0,\, +\infty\rangle). \tag{5.105}$$

Let $\{G^{(n)}(r),\ r \in \mathbb{N}\}$ be the Markov chain with state space $\{0,\, 1,\, 2,\, \ldots,\, n\}$ and transition probabilities (5.91) with parameters

$$a^* = a, \quad b^* = c, \quad c^* = 0, \quad d^* = b, \quad a > 0, \quad b > 0, \quad c > 0,$$

i.e.

$$p_{0,1} = 1, \quad p_{n,n-1} = 1,$$
$$p_{i,i+1} = \left(\frac{i}{n}\right)^2 \frac{a}{n^d} + \frac{c}{n^2}, \quad p_{i,i-1} = \frac{ai+b}{n^2}\frac{i}{n^d}, \quad p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}, \tag{5.106}$$

and 0 otherwise, where n is large enough, ensuring $p_{i,i+1} + p_{i,i-1} < 1$. Define the function $g_n : [0,\, +\infty\rangle \to \mathbb{R}$,

$$g_n(x) = n^d x.$$

We assume initial states of the Markov chain $\{G^{(n)}(r),\ r \in \mathbb{N}_0\}$ and reciprocal gamma diffusion $X = \{X(t),\, t \geq 0\}$ are given by $G^{(n)}(0) = i$ and $X(0) = x$, respectively, where

$$i = i(x) = \lfloor g_n(x) \rfloor = \left\lfloor n^d x \right\rfloor,\ x \in [0,\, +\infty\rangle$$

and n is always large enough so that $i(x)$ is in the state space of Markov chain $\{G^{(n)}(r),\ r \in \mathbb{N}_0\}$. Moreover, we assume that the initial Markov chain $\{G^{(n)}(r),\, r \in \mathbb{N}_0\}$ never starts from states 0 and n. Notice that the initial state is a function of x, but we will use notation i for simplicity. For each $n \in \mathbb{N}$, we define the new Markov chain $\{H^{(n)}(r),\ r \in \mathbb{N}\}$ with the state space $\{0, 1/n^d, \ldots, 1/n^{d-1}\}$

$$H^{(n)}(r) = g_n^{-1}(G^{(n)}(r)) = \frac{G^{(n)}(r)}{n^d}. \tag{5.107}$$

The transition operator T_n of the Markov chain $\{H^{(n)}(r),\ n \in \mathbb{N}\}$ is given by

$$T_n f\left(\frac{i}{n^d}\right) = \sum_{j=0}^{n} p_{ij} f\left(\frac{j}{n^d}\right) = p_{i,i-1}\, f\left(\frac{i-1}{n^d}\right) + p_{i,i}\, f\left(\frac{i}{n^d}\right) + p_{i,i+1}\, f\left(\frac{i+1}{n^d}\right), \tag{5.108}$$

where p_{ij} is defined in (5.106).
Now we define the operator

$$A_n := n^{2+d}(T_n - I), \quad f_n \in Dom(A_n), \quad f_n(x) := f\left(g_n^{-1}(i)\right) = f\left(\frac{i}{n^d}\right), \tag{5.109}$$

where $f \in C_c^3([0, +\infty\rangle)$ and by the following scaling of time in $\{H^{(n)}(r), r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$ we obtain the corresponding continuous-time stochastic process $\{X^{(n)}(t), t \geq 0\}$:

$$X^{(n)}(t) := H^{(n)}\left(\lfloor n^{2+d}t \rfloor\right). \tag{5.110}$$

The next theorem states that the RG diffusion can be obtained as the limiting process of the time-changed processes $\{X^{(n)}(t), t \geq 0\}$.

Theorem 5.9 Let $\{H^{(n)}(r), r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$, be the Markov chain defined by (5.107) with the transition operator (5.108). Let $X^n = \{X^{(n)}(t), t \geq 0\}$, for each $n \in \mathbb{N}$, be its corresponding time-changed process, with the time change (5.110). Let the operators $(A_n, n \in \mathbb{N})$ be defined by (5.109). Then

$$X^n \Rightarrow X \text{ in } \mathbb{D}([0, +\infty\rangle\,; [0, +\infty\rangle),$$

where $X = \{X(t), t \geq 0\}$ is the RG diffusion with the infinitesimal generator $\mathcal{A}$ given by (5.105), and

$$\theta = b, \quad \beta = \frac{b}{a} + 1, \quad \gamma = \frac{c}{a}.$$

Proof. First, notice that function g_n satisfies conditions given in Section 5.3, i.e. function g_n is strictly monotonic and

$$\lim_{n\to\infty} \left\| g_n^{-1}(i+1) - g_n^{-1}(i) \right\|_\infty = \lim_{n\to\infty} \left| \frac{1}{n^d} \right| = 0.$$

Taking into account Remark 5.3, state space transformation (5.107) together with the time scale $h_n^{-1} = n^{2+d}$ yields

$$\mu_n(x) = n^2 \left(p_{i,i+1} - p_{i,i-1}\right), \quad \sigma_n^2(x) = n^{2-d}\left(p_{i,i+1} + p_{i,i-1}\right),$$

$$R_n(x) = \frac{1}{6} n^{2-2d} \left(p_{i,i+1} - p_{i,i-1}\right) f'''(\zeta), \quad \left|\zeta - \frac{i}{n^d}\right| < \left|\frac{j-i}{n^d}\right|.$$

Next, by substituting transition probabilities (5.106) in the above expressions it follows

$$\begin{aligned} \mu_n(x) &= n^2 \left(\left(\frac{i}{n}\right)^2 \frac{a}{n^d} + \frac{c}{n^2} - \frac{ai+b}{n^2}\frac{i}{n^d} \right) \\ &= c - b\frac{i}{n^d}, \end{aligned} \tag{5.111}$$

$$\begin{aligned}\sigma_n^2(x) &= n^{2-d}\left(\left(\frac{i}{n}\right)^2\frac{a}{n^d}+\frac{c}{n^2}+\frac{ai+b}{n^2}\frac{i}{n^d}\right)\\ &= 2a\left(\frac{i}{n^d}\right)^2+\frac{c}{n^d}+b\frac{i}{n^{2d}},\end{aligned}\tag{5.112}$$

$$\begin{aligned}|R_n(x)| &\leq \frac{K}{6}\left|n^{2-2d}\left(\left(\frac{i}{n}\right)^2\frac{a}{n^d}+\frac{c}{n^2}-\frac{ai+b}{n^2}\frac{i}{n^d}\right)\right|\\ &= \frac{K}{6}\left|\left(\frac{c}{n^{2d}}-b\frac{i}{n^{3d}}\right)\right|,\end{aligned}\tag{5.113}$$

where K is a constant such that $|f'''(\zeta)| \leq K$. Since $i = \lfloor g_n(x)\rfloor$, it follows

$$\lim_{n\to\infty}\sup_{x\in[0,\,+\infty\rangle}\left|\frac{i}{n^d}-x\right| = 0.\tag{5.114}$$

Now, using (5.111), (5.112), (5.113) together with (5.114) and the fact that $f \in C_c^3([0,\,+\infty\rangle)$ yields

$$\lim_{n\to\infty}\|\mu_n-\mu\|_\infty = 0, \quad \lim_{n\to\infty}\left\|\sigma_n^2-\sigma^2\right\|_\infty = 0, \quad \lim_{n\to\infty}\|R_n\|_\infty = 0,\tag{5.115}$$

where

$$\mu(x) = c - bx, \quad \sigma^2(x) = 2ax^2.$$

By re-parametrizing

$$\theta = b, \quad \beta = \frac{b}{a}+1, \quad \gamma = \frac{c}{a}\tag{5.116}$$

it follows

$$\mu(x) = -\theta\left(x-\frac{\gamma}{\beta-1}\right), \quad \sigma^2(x) = \frac{2\theta}{\beta-1}x^2.\tag{5.117}$$

Notice how re-parametrization (5.116) ensures the generality of parameters of RG diffusion, i.e.

$$\theta > 0, \quad \beta > 1, \quad \gamma > 0$$

since $a > 0,\ b > 0,\ c > 0$. Now, comparing the obtained limits (5.117) with (5.105) we see that the limits coincide with the infinitesimal parameters of the RG diffusion. Since (5.115) holds, as a direct consequence of Theorem 5.2 we obtain $X^n \Rightarrow X$ in $\mathbb{D}([0,\,+\infty\rangle\,;\,[0,\,+\infty\rangle)$, where X is the generally parametrized RG diffusion.

□

5.8 Fractional Pearson diffusions as the correlated continuous time random walk limits

Suppose that $\{T(r),\, r \in \mathbb{N}_0\}$, where $T(0) = 0$, $T(r) = G_1 + \ldots + G_r$, is the random walk where $G_r \geq 0$ are iid waiting times between particle jumps that are independent of the Markov chain $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$ (see Section 5.3.2). We assume G_1 is in the domain of attraction of the α-stable distribution with index $0 < \alpha < 1$, and that the waiting time of the Markov chain until its r-th move is described by $G(r)$. Let

$$N(t) = \max\{r \geq 0\colon T(r) \leq t\} \tag{5.118}$$

be the number of jumps up to time $t \geq 0$. Then the continuous time stochastic process

$$\{H^{(n)}(N(t)),\, t \geq 0\},$$

where $H^{(n)}(N(t))$ is the state of the Markov chain at time $t \geq 0$, is the correlated CTRW process.

Next theorem is the main ingredient to connect our correlated CTRWs with their limits, i.e. fractional Pearson diffusions.

Theorem 5.10 Let $\{A(t),\, t \geq 0\}$ be the weak limit of $\{A^{(n)}(t),\, t \geq 0\}$, where both processes are càdlàg, i.e. let

$$A^{(n)} \Rightarrow A \text{ in } \mathbb{D}([0, +\infty\rangle\,; S)$$

with J_1 topology, where S is the state space for the process A. Let $\{N(t),\, t \geq 0\}$ be the renewal process defined in (5.118), and $\{E(t),\, t \geq 0\}$ be the inverse of the standard α-stable subordinator $\{D(t),\, t \geq 0\}$ with $0 < \alpha < 1$. Then

$$A^{(n)}\left(n^{-1}N\left(n^{1/\alpha}t\right)\right) \Rightarrow A(E(t)), \quad n \to \infty \tag{5.119}$$

in the Skorokhod space $\mathbb{D}([0, +\infty\rangle\,; S)$ with J_1 topology.

Proof. From (Meerschaert & Sikorskii 2011, Section 4.4)

$$n^{-\frac{1}{\alpha}}T(\lceil nt\rceil) \Rightarrow D(t), \quad n \to \infty$$

in the sense of finite dimensional distributions, where $\{D(t),\, t \geq 0\}$ is standard α-stable subordinator. Since α-stable subordinator $D(t)$ is a Lévy process, it follows that $D(t)$ is continuous in probability. Since the sample paths of the process $T(\lceil nt\rceil)$ are monotone non-decreasing, (Bingham 1971, Theorem 3) yields

$$n^{-\frac{1}{\alpha}}T(\lceil nt\rceil) \Rightarrow D(t), \quad n\to\infty \tag{5.120}$$

in the Skorokhod space $\mathbb{D}([0, +\infty\rangle\,;\,[0, +\infty\rangle)$ with J_1 topology. Assumption (5.119) together with (5.120) yields

$$\left(A^{(n)}(t), n^{-\frac{1}{\alpha}}T(\lceil nt\rceil)\right) \Rightarrow (A(t), D(t))\,, \quad n\to\infty, \tag{5.121}$$

in the product space $\mathbb{D}([0, +\infty\rangle \times [0, +\infty\rangle\,;\,S\times[0, +\infty\rangle)$ with J_1 topology. Following the notation from Straka & Henry (2011) let

- $h = (v,\sigma) \in \mathbb{D}([0, +\infty\rangle \times [0, +\infty\rangle\,;\,S\times[0, +\infty\rangle)$,
- $v \in \mathbb{D}([0, +\infty\rangle\,;\,S)$, $\sigma \in \mathbb{D}([0, +\infty\rangle\,;\,[0, +\infty\rangle)$,

and $D_u, D_\uparrow$ and $D_{\uparrow\uparrow}$ be sets of all such h which have unbounded, non-decreasing and increasing σ, respectively. As shown in Straka & Henry (2011) sets

$$D_{\uparrow,u} = D_\uparrow \cap D_u,\;\; D_{\uparrow\uparrow,u} = D_{\uparrow\uparrow}\cap D_u$$

are Borel measurable.

Introduce the function

$$\Psi : D_{\uparrow,u} \mapsto \mathbb{D}([0, +\infty\rangle\,;\,S),\;\; \Psi(h) = v\circ\sigma^{-1}.$$

From (Straka & Henry 2011, Proposition 2.3) function Ψ is continuous in $D_{\uparrow\uparrow,u}$.

Note that $\left(A^{(n)}(t), n^{-\frac{1}{\alpha}}T(\lceil nt\rceil)\right)$, $(A(t), D(t))$ are in the domain of function Ψ, i.e.

$$\left(A^{(n)}(t), n^{-\frac{1}{\alpha}}T(\lceil nt\rceil)\right),\;\, (A(t), D(t)) \in D_{\uparrow,u}.$$

Also, since the standard α-stable subordinator $\{D(t),\, t\geq 0\}$ is strictly increasing, it follows

$$(A(t), D(t)) \in D_{\uparrow\uparrow,u}. \tag{5.122}$$

Observe that for the generalized inverses we have

$$(n^{-\frac{1}{\alpha}}T(\lceil nt\rceil))^{-1} = n^{-1}N(n^{\frac{1}{\alpha}}t),\;\; (D(t))^{-1} = E(t).$$

Since the function Ψ is continuous at $D_{\uparrow\uparrow,u}$,

$$\Psi\left(A^{(n)}(t),\, n^{-\frac{1}{\alpha}}T(\lceil nt\rceil)\right) = A^{(n)}(n^{-1}N(n^{\frac{1}{\alpha}}t))$$

and

$$\Psi\left(A(t), D(t)\right) = X(E(t)),$$

using (5.121) and (5.122) it follows

$$A^{(n)}(n^{-1}N(n^{\frac{1}{\alpha}}t)) \Rightarrow A(E(t)),\ n \to \infty$$

in the space $\mathbb{D}([0, +\infty\rangle\,; S)$ with J_1 topology. $\square$

Remark 5.6. The proof of Theorem 5.10 uses the approach from the proof of (Straka & Henry 2011, Theorem 3.6) but in our case the first component is a Markov chain, rather then a random walk.

Corollary 5.11 Let $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$ be the Markov chain defined by (5.28). Let $\{X^{(n)}(t),\, t \geq 0\}$ be the corresponding rescaled Markov chain given by (5.31). Let $\{N(t),\, t \geq 0\}$ be the renewal process defined in (5.118), and $\{E(t),\, t \geq 0\}$ be the inverse of the standard α-stable subordinator $\{D(t),\, t \geq 0\}$ with $0 < \alpha < 1$. Then

$$X^{(n)}\left(n^{-1}N\left(n^{1/\alpha}t\right)\right) \Rightarrow X(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}\,[0, +\infty\rangle$ with J_1 topology, where $\{X(t),\, t \geq 0\}$ is Ornstein-Uhlenbeck diffusion with generator

$$\mathcal{A}f(x) = -\theta(x-\mu)f'(x) + \theta\sigma^2 f''(x), \quad f \in C_c^3(\mathbb{R}).$$

Proof. Stochastic processes $\{X^{(n)}(t),\, t \geq 0\}$ and $\{X(t),\, t \geq 0\}$ are both càdlàg and Theorem 5.3 implies

$$X^{(n)} \Rightarrow X \text{ in } \mathbb{D}\,[0, +\infty\rangle\,.$$

Now, simply apply Theorem 5.10 to obtain the desired result. $\square$

Corollary 5.12 Let $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$ be the Markov chain defined by (5.41). Let $\{X^{(n)}(t),\, t \geq 0\}$ be the corresponding rescaled Markov chain given by (5.44). Let $\{N(t),\, t \geq 0\}$ be the renewal process defined in (5.118), and $\{E(t),\, t \geq 0\}$ be the inverse of the standard α-stable subordinator $\{D(t),\, t \geq 0\}$ with $0 < \alpha < 1$. Then

$$X^{(n)}\left(n^{-1}N\left(n^{1/\alpha}t\right)\right) \Rightarrow X(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}([0, +\infty\rangle\,; [0, +\infty\rangle)$ with J_1 topology, where $\{X(t),\, t \geq 0\}$ is Cox-Ingersoll-Ross diffusion with generator

$$\mathcal{A}f(x) = -\theta\left(x - \frac{b}{a}\right)f'(x) + \frac{\theta}{a}xf''(x), \quad f \in C_c^3([0, +\infty\rangle).$$

Proof. Stochastic processes $\{X^{(n)}(t),\, t \geq 0\}$ and $\{X(t),\, t \geq 0\}$ are both càdlàg and Theorem 5.4 implies

$$X^{(n)} \Rightarrow X \text{ in } \mathbb{D}([0, +\infty\rangle\,;\,[0, +\infty\rangle).$$

Now, simply apply Theorem 5.10 to obtain the desired result. $\square$

Corollary 5.13 Let $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$ be the Markov chain defined by (5.53). Let $\{X^{(n)}(t),\, t \geq 0\}$ be the corresponding rescaled Markov chain given by (5.56). Let $\{N(t),\, t \geq 0\}$ be the renewal process defined in (5.118), and $\{E(t),\, t \geq 0\}$ be the inverse of the standard α-stable subordinator $\{D(t),\, t \geq 0\}$ with $0 < \alpha < 1$. Then

$$X^{(n)}\left(n^{-1}N\left(n^{1/\alpha}t\right)\right) \Rightarrow X(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}([0, +\infty\rangle\,;\,[0, 1])$ with J_1 topology, where $\{X(t),\, t \geq 0\}$ is Jacobi diffusion with generator

$$\mathcal{A}f(x) = -\theta\left(x - \frac{a}{a+b}\right)f'(x) + \frac{\theta}{a+b}x(1-x)f''(x), \quad f \in C_c^3([0, 1]).$$

Proof. Stochastic processes $\{X^{(n)}(t),\, t \geq 0\}$ and $\{X(t),\, t \geq 0\}$ are both càdlàg and Theorem 5.5 implies

$$X^{(n)} \Rightarrow X \text{ in } \mathbb{D}([0, +\infty\rangle\,;\,[0, 1]).$$

Now, simply apply Theorem 5.10 to obtain the desired result. $\square$

Corollary 5.14 Let $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$ be the Markov chain defined by (5.68). Let $\{X^{(n)}(t),\, t \geq 0\}$ be the corresponding rescaled Markov chain given by (5.71). Let $\{N(t),\, t \geq 0\}$ be the renewal process defined in (5.118), and $\{E(t),\, t \geq 0\}$ be the inverse of the standard α-stable subordinator $\{D(t),\, t \geq 0\}$ with $0 < \alpha < 1$. Then

$$X^{(n)}\left(n^{-1}N\left(n^{1/\alpha}t\right)\right) \Rightarrow X(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}([0, +\infty\rangle\,;\,[0, 1])$ with J_1 topology, where $\{X(t),\, t \geq 0\}$ is Jacobi diffusion with generator

$$\mathcal{A}f(x) = -\theta\left(x - \frac{a}{a+b}\right)f'(x) + \frac{\theta}{a+b}x(1-x)f''(x), \quad f \in C_c^3([0, 1]).$$

Proof. Stochastic processes $\{X^{(n)}(t),\, t \geq 0\}$ and $\{X(t),\, t \geq 0\}$ are both càdlàg and Theorem 5.6 implies

$$X^{(n)} \Rightarrow X \text{ in } \mathbb{D}([0, +\infty\rangle\,;\,[0, 1]).$$

Now, simply apply Theorem 5.10 to obtain the desired result. $\square$

Corollary 5.15 Let $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$ be the Markov chain defined by (5.80). Let $\{X^{(n)}(t),\, t \geq 0\}$ be the corresponding rescaled Markov chain given by (5.83). Let $\{N(t),\, t \geq 0\}$ be the renewal process defined in (5.118), and $\{E(t),\, t \geq 0\}$ be the inverse of the standard α-stable subordinator $\{D(t),\, t \geq 0\}$ with $0 < \alpha < 1$. Then

$$X^{(n)}\left(n^{-1}N\left(n^{1/\alpha}t\right)\right) \Rightarrow X(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}\left[0, +\infty\right\rangle$ with J_1 topology, where $\{X(t),\, t \geq 0\}$ is Student diffusion with generator

$$\mathcal{A}f(x) = -\theta\,(x-\mu)\,f'(x) + \frac{1}{2}\frac{2\theta\delta^2}{\nu-1}\left(1+\left(\frac{x-\mu}{\delta}\right)^2\right)f''(x), \quad f \in C_c^3(\mathbb{R}).$$

Proof. Stochastic processes $\{X^{(n)}(t),\, t \geq 0\}$ and $\{X(t),\, t \geq 0\}$ are both càdlàg and Theorem 5.7 implies

$$X^{(n)} \Rightarrow X \text{ in } \mathbb{D}\left[0, +\infty\right\rangle.$$

Now, simply apply Theorem 5.10 to obtain the desired result. □

Corollary 5.16 Let $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$ be the Markov chain defined by (5.94). Let $\{X^{(n)}(t),\, t \geq 0\}$ be the corresponding rescaled Markov chain given by (5.97). Let $\{N(t),\, t \geq 0\}$ be the renewal process defined in (5.118), and $(E(t),\, t \geq 0)$ be the inverse of the standard α-stable subordinator $\{D(t),\, t \geq 0\}$ with $0 < \alpha < 1$. Then

$$X^{(n)}\left(n^{-1}N\left(n^{1/\alpha}t\right)\right) \Rightarrow X(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}(\left[0, +\infty\right\rangle; \left[0, +\infty\right\rangle)$ with J_1 topology, where $\{X(t),\, t \geq 0\}$ is Fisher-Snedecor diffusion with generator

$$\mathcal{A}f(x) = -\theta\left(x-\frac{\beta}{\beta-2}\right)f'(x) + \frac{1}{2}\frac{4\theta}{\gamma(\beta-2)}x(\gamma x+\beta)f''(x), \quad f \in C_c^3(\left[0, +\infty\right\rangle).$$

Proof. Stochastic processes $\{X^{(n)}(t),\, t \geq 0\}$ and $\{X(t),\, t \geq 0\}$ are both càdlàg and Theorem 5.8 implies

$$X^{(n)} \Rightarrow X \text{ in } \mathbb{D}(\left[0, +\infty\right\rangle; \left[0, +\infty\right\rangle).$$

Now, simply apply Theorem 5.10 to obtain the desired result. □

Corollary 5.17 Let $\{H^{(n)}(r),\, r \in \mathbb{N}_0\}$ be the Markov chain defined by (5.107). Let $\{X^{(n)}(t),\, t \geq 0\}$ be the corresponding rescaled Markov chain given by (5.110). Let $\{N(t),\, t \geq 0\}$ be the renewal process defined in (5.118), and $\{E(t),\, t \geq 0\}$ be the inverse of the standard α-stable subordinator $\{D(t),\, t \geq 0\}$ with $0 < \alpha < 1$. Then

$$X^{(n)}\left(n^{-1}N\left(n^{1/\alpha}t\right)\right) \Rightarrow X(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}([0, +\infty\rangle\,; [0, +\infty\rangle)$ with J_1 topology, where $\{X(t),\, t \geq 0\}$ is reciprocal gamma diffusion with generator

$$\mathcal{A}f(x) = -\theta\left(x - \frac{\gamma}{\beta - 1}\right)f'(x) + \frac{1}{2}\frac{2\theta}{\beta - 1}x^2 f''(x), \quad f \in C_c^3([0, +\infty\rangle).$$

Proof. Stochastic processes $\{X^{(n)}(t),\, t \geq 0\}$ and $\{X(t),\, t \geq 0\}$ are both càdlàg and Theorem 5.9 implies

$$X^{(n)} \Rightarrow X \text{ in } \mathbb{D}([0, +\infty\rangle\,; [0, +\infty\rangle).$$

Now, simply apply Theorem 5.10 to obtain the desired result. □

Chapter 6

Delayed continuous-time autoregressive processes

In this Section we define the Lévy-driven continuous-time autoregressive processes delayed via inverse of the standard α-stable subordinator. Moreover, correlation structure and distributional properties of such processes are studied. Therefore, beside the fractional Pearson diffusions, we give an alternative model for "trapping events", i.e. time periods when observed process rests. Unlike in the case of fractional Pearson diffusions, fractional model is not established here. In particular, Section 6.1 contains preliminary facts regarding Lévy-driven continuous-time autoregressive processes with corresponding definition of its delayed counterpart. In Section 6.2 we explicitly derive the correlation structure for Lévy-driven continuous-time autoregressive processes of order p, with special interest in low orders. Next, in Section 6.3 we show that these processes are long-range dependent, while in Section 6.4 we examine their distributional properties.

6.1 Delayed Lévy-driven continuous-time autoregressive processes

Continuous-time autoregressive process of order p, CAR(p) process, can be symbolically represented in analogy to discrete case with equation:

$$dX^{p-1}(t) + \alpha_1 X^{p-1}(t)dt + \ldots + \alpha_p X(t)dt = \sigma dW(t), \ t \geq 0,$$

where driving process $\{W(t),\, t \geq 0\}$ is the standard Brownian motion.
In this thesis, we focus on Lévy-driven CAR(p) process, i.e. CAR(p) process with Lévy process as the driving process. Reason for the usage of such processes is the rich class of non-Gaussian and heavy-tailed marginal distributions of underlying process, due to usage of Lévy process instead of Brownian motion as the driving process. Brockwell made such extensions for the Lévy-driven CARMA(p,q) processes, giving necessary and sufficient conditions for such process to be weakly and strictly stationary, as well as the explicit form of the corresponding cumulant generating function with several examples (see Brockwell

(2001*b*), Brockwell & Marquardt (2005)).
Moreover, we focus on dCAR(p) process, i.e. Lévy-driven CAR(p) process delayed by the inverse of the standard stable subordinator for low degrees of p. We will be able to give explicit calculations and formulas for the correlation structure and distributional properties, which makes it a more trackable process then the general case.
Formal definition of the Lévy-driven CAR(p) process is as follows. Let us introduce p-variate process

$$\boldsymbol{S}(t) := [X(t), X^1(t), \ldots, X^{p-1}(t)]^T, \ \ p \in \mathbb{N} \tag{6.1}$$

which satisfies SDE

$$d\boldsymbol{S}(t) - A\boldsymbol{S}(t)dt = \boldsymbol{e}dL(t), \ t \geq 0, \tag{6.2}$$

where $\{L(t),\, t \geq 0\}$ is Lévy process such that $\mathbb{E}L(1)^2 < \infty$,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_p & -\alpha_{p-1} & -\alpha_{p-2} & \cdots & -\alpha_1 \end{bmatrix}, \ \boldsymbol{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and

$$\boldsymbol{S}(0) \text{ is independent of the driving Lévy process } \{L(t),\, t \geq 0\}. \tag{6.3}$$

In particular, if $p = 1$ then $A = -\alpha_1$.
Moreover, solution of the SDE (6.2) satisfies

$$\boldsymbol{S}(t) = e^{A(t-s)}\boldsymbol{S}(s) + \int_s^t e^{A(t-u)}\boldsymbol{e}dL(u), \ t > s \geq 0 \tag{6.4}$$

(see Brockwell (2001*b*)). Then, Lévy-driven CAR(p) process is defined as $\{X(t),\, t \geq 0\}$, the first component of the process (6.1). Here the process $\{\boldsymbol{S}(t),\, t \geq 0\}$ is the strictly stationary solution of (6.4), and which satisfies (6.3).
Additionally, we will assume that the driving process $\{L(t),\, t \geq 0\}$ in SDE (6.2) is the second-order Lévy processes which satisfy

$$\mathbb{E}L(t) = \mu t, \ \ Var(L(t)) = \sigma^2 t, \ \ t \geq 0, \ \text{ for some real constants } \mu,\, \sigma^2. \tag{6.5}$$

Lévy-driven CAR(p) processes with this additional assumption are referred to as second-order Lévy-driven CAR(p) processes by Brockwell (see Brockwell (2001*b*)). Since we will only consider such processes, we will simply refer to them as Lévy-driven CAR(p) processes.
Necessary and sufficient conditions for weak stationarity of Lévy-driven CAR(p) process are given via Proposition 1. in Brockwell & Marquardt (2005). For process $\{\boldsymbol{S}(t),\, t \geq 0\}$

to be weakly stationary it is both necessary and sufficient that all eigenvalues of matrix A have strictly negative real parts and

$$\boldsymbol{S}(0) \text{ has mean and covariance matrix of } \int_0^\infty e^{Au}\boldsymbol{e}dL(u).$$

On the other hand, necessary and sufficient conditions for strict stationarity of Lévy-driven CAR(p) process are given via Proposition 2. in Brockwell & Marquardt (2005). For process $\{\boldsymbol{S}(t),\, t \geq 0\}$ to be strictly stationary it is both necessary and sufficient that all eigenvalues of matrix A have strictly negative real parts and

$$\boldsymbol{S}(0) \overset{d}{=} \int_0^\infty e^{Au}\boldsymbol{e}dL(u).$$

Now it follows that the same conditions are necessary and sufficient for Lévy-driven CAR(p) processes $\{X(t),\, t \geq 0\}$ to be weakly and strictly stationary.
Eigenvalues of matrix A are the roots of the characteristic equation

$$C(\lambda) = \lambda^p + \alpha_1\lambda^{p-1} + \alpha_2\lambda^{p-2} + \dots \alpha_{p-1}\lambda + \alpha_p = 0. \tag{6.6}$$

When $p = 1$ or $p = 2$, characteristic roots of equation (6.6) have negative real parts if and only if all coefficients in the same equation are positive. From here we assume that conditions for strict stationarity of process $\{X(t),\, t \geq 0\}$ are fulfilled.
Since stationary Lévy-driven CAR(p) process has the same autocovariance structure as usual stationary CAR(p) process, it follows that autocovariance function of the stationary Lévy-driven CAR(p) process is of the form

$$Cov(X(t), X(s)) = \sum_{\lambda:\, C(\lambda)=0} \frac{\sigma^2}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}\frac{(z-\lambda)^m e^{z|t-s|}}{C(z)C(-z)}\right]\Bigg|_{z=\lambda}$$

where m is the multiplicity of the root λ of the equation (6.6). If the roots are distinct, last formula simplifies to

$$Cov(X(t), X(s)) = \sum_{\lambda:\, C(\lambda)=0} \frac{\sigma^2 e^{\lambda|t-s|}}{C'(\lambda)C(-\lambda)}. \tag{6.7}$$

For details, see e.g. Brockwell (2001a). We also use some ideas from Scalas & Viles (2014). Therefore, stationary Lévy-driven CAR(1) process has autocorrelation function (ACF) of the form

$$\mathrm{Corr}(X(t), X(s)) = e^{-\alpha_1|t-s|}, \tag{6.8}$$

where $\alpha_1 > 0$ in order to have the stationarity of the process.
In the case of stationary Lévy-driven CAR(2) process (6.6) becomes

$$z^2 + \alpha_1 z + \alpha_2 = 0, \tag{6.9}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ again for the stationarity of the process. Depending on the sign of the discriminant $D = \alpha_1^2 - 4\alpha_2$ of the equation (6.9), we will have three cases for the ACF of the stationary Lévy-driven CAR(2) process:

- $D > 0$ - **the over-damped case**

$$\mathrm{Corr}(X(t), X(s)) = \frac{-\lambda_2 e^{\lambda_1|t-s|} + \lambda_1 e^{\lambda_2|t-s|}}{\lambda_1 - \lambda_2}, \tag{6.10}$$

  where λ_1, λ_2 are two distinct real roots of the equation (6.9).

- $D = 0$ - **the critically-damped case**

$$\mathrm{Corr}(X(t), X(s)) = \left(1 + \frac{\alpha_1}{2}|t-s|\right) e^{-\frac{\alpha_1}{2}|t-s|}, \tag{6.11}$$

  where $\lambda_1 = \lambda_2 = -\frac{\alpha_1}{2}$ is the double real root of the equation (6.9).

- $D < 0$ - **the under-damped case**

$$\mathrm{Corr}(X(t), X(s)) = \frac{-\bar{\lambda} e^{\lambda|t-s|} + \lambda e^{\bar{\lambda}|t-s|}}{\lambda - \bar{\lambda}}, \tag{6.12}$$

  where $\lambda = a + bi$, $\bar{\lambda} = a - bi$ are two distinct complex roots of the equation (6.9) and $a < 0$, $b > 0$. Notice that

$$a = -\frac{\alpha_1}{2}, \ \ b = \sqrt{\alpha_2 - \frac{\alpha_1^2}{4}}.$$

  ACF (6.12) can also be written in the following form

$$\mathrm{Corr}(X(t), X(s)) = \left(\cos\left(b(t-s)\right) - \frac{a}{b}\sin\left(b|t-s|\right)\right) e^{a|t-s|}. \tag{6.13}$$

Let $\{X(t),\, t \geq 0\}$ be the stationary Lévy-driven CAR(p) process. Then, the delayed Lévy-driven CAR(p) process (dCAR(p) process) $\{X_\alpha(t),\, t \geq 0\}$ is defined via a non-Markovian time-change $E(t)$ independent of $X(t)$:

$$X_\alpha(t) := X(E(t)), \quad t \geq 0,$$

where $E(t) = \inf\{x > 0 : D(x) > t\}$ is the inverse of the standard α-stable subordinator. Notice how this definition is fully analogous to that of fractional Pearson diffusion. In particular, they share the same time-change process, while the difference lies in the outer process, which is Lévy-driven CAR(p) process in the case of dCAR(p) process, and Pearson diffusion in the case of fPD (see Section 4). Therefore, the defined process is non-Markovian and non-stationary.

Recall that the density $f_t(\cdot)$ of $E(t)$ is given by

$$f_t(x) = \frac{t}{\alpha} x^{-1-\frac{1}{\alpha}} g_\alpha(tx^{-1/\alpha}), \tag{6.14}$$

where $g_\alpha(\cdot)$ is probability density of standard stable subordinator $D(1)$ (see Section 2.3), while the corresponding Laplace transform is

$$\mathbb{E}[e^{-sE(t)}] = \int_0^\infty e^{-sx} f_t(x) dx = \mathcal{E}_\alpha(-st^\alpha),\ \ s > 0, \tag{6.15}$$

where $\mathcal{E}_\alpha(\cdot)$ is the Mittag-Leffler function (see (2.9)).

Unlike in previous sections, in order to establish some results regarding correlation structure of dCAR(p) processes, we need (6.15) to be valid for complex s. To see that (6.15) is indeed valid for any complex s, notice that for any $0 < \alpha < 1$ and any complex s

$$\alpha \mathcal{E}_\alpha(-s) = \int_0^\infty e^{-sx} x^{-1-1/\alpha} g_\alpha(x^{-1/\alpha}) dx \tag{6.16}$$

(see Theorem 2.10.2, Zolotarev (1986)). Now combining (6.14) and (6.16) we directly obtain (6.15).

Moreover, in some cases in this section we use two-parametric Mittag-Leffler function, which is defined as

$$\mathcal{E}_{\alpha,\beta}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)},$$

where $\alpha,\ \beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\ Re(\beta) > 0$.

This function was first studied by Wiman in 1905 (see Wiman (1905)). Notice when $\beta = 1$, two-parametric Mittag-Leffler function reduces to classical Mittag-Leffler function $\mathcal{E}_\alpha(z)$. For details regarding Mittag-Leffler functions we refer to Gorenflo et al. (2016), Popov & Sedletskii (2011).

6.2 Correlation structure of delayed Lévy-driven continuous-time autoregressive processes

In this section we compute formulas for the correlation structure of dCAR(p) processes, with special interest in low orders. The correlation function (CF) of the dCAR(p) process $\{X_\alpha(t),\, t \geq 0\}$ where $0 < \alpha < 1$ is of the form

$$\mathrm{Corr}[X_\alpha(t), X_\alpha(s)] = \mathrm{Corr}[X(E(t)), X(E(s))] = \int_0^\infty \int_0^\infty Corr[X(u), X(v)] H(du, dv), \tag{6.17}$$

where the last integral is the Lebesgue-Stieltjes integral with respect to the bivariate distribution function $H(u, v) := \mathbb{P}(E(t) \leq u, E(s) \leq v)$ of the process $\{E(t),\, t \geq 0\}$.
In order to compute the last integral we use the idea of bivariate integration by parts (see Lemma 2.2, Gill et al. (1995))

$$\begin{aligned}\int_0^\infty \int_0^\infty F(u,v) H(du,dv) &= \int_0^\infty \int_0^\infty H([u,\infty] \times [v,\infty]) F(du,dv) + \int_0^\infty H([u,\infty] \times (0,\infty]) F(du,0) \\ &+ \int_0^\infty H((0,\infty] \times [v,\infty]) F(0,dv) + F(0,0) H((0,\infty] \times (0,\infty]).\end{aligned} \tag{6.18}$$

This approach was exploited for calculating the correlation structure of the fractional Pearson diffusions, i.e. time-changed (delayed) Pearson diffusions via inverse of the standard α-stable subordinator (see Leonenko et al. (2013a)). Recall that the fractional Pearson diffusion, i.e. the process $\{Y_\alpha(t),\, t \geq 0\}$ has correlation structure of the form (see Section 4.4)

$$\mathrm{Corr}(Y_\alpha(t), Y_\alpha(s)) = \mathcal{E}_\alpha(-\theta t^\alpha) + \frac{\alpha \theta t^\alpha}{\Gamma(1+\alpha)} \int_0^{s/t} \frac{\mathcal{E}_\alpha(-\theta t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz, \tag{6.19}$$

which we use in the following results.

Remark 6.1. Notice that the integral representation (6.17) for CF of the general delayed stochastic process, depends only on the CF of the non-delayed process $\{X(t),\, t \geq 0\}$ (i.e. the outer stationary process) and the bivariate distribution $H(u, v)$ of the process $\{E(t),\, t \geq 0\}$. So if two non-delayed processes have the same CF, their delayed counterparts will have the same CF as well.

The next theorem provides a general formula for correlation structure of the dCAR(p) process for which the corresponding characteristic equation (6.6) has distinct roots. In the case of non-distinct roots, extended techniques must be used (see Theorem 6.5).

Theorem 6.1 Let $\{X_p(t),\, t \geq 0\}$ be the stationary Lévy-driven CAR(p) process defined in Section 6.1 with the autocovariance function given by (6.7). Then the correlation function of the corresponding dCAR(p) process $\{X_\alpha(t),\, t \geq 0\}$ is given by

$$Corr(X_\alpha(t), X_\alpha(s)) = \frac{\sum\limits_{\lambda: C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1} \left[\mathcal{E}_\alpha(\lambda t^\alpha) - \frac{\alpha\lambda t^\alpha}{\Gamma(1+\alpha)} \int\limits_0^{s/t} \frac{\mathcal{E}_\alpha(\lambda t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz\right]}{\sum\limits_{\lambda: C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1}}, \tag{6.20}$$

where $t \geq s > 0$.

Proof.

$$\begin{aligned} Corr(X_\alpha(t), X_\alpha(s)) &= Corr(X_p(E(t)), X_p(E(s))) \\ &= \int\limits_0^\infty \int\limits_0^\infty Corr(X_p(t), X_p(s)) H(du,\, dv) \\ &= \frac{1}{\sum\limits_{\lambda:\, C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1}} \int\limits_0^\infty \int\limits_0^\infty \sum_{\lambda:\, C(\lambda)=0} \frac{e^{\lambda|t-s|}}{C'(\lambda)C(-\lambda)} H(du,\, dv) \\ &= \frac{1}{\sum\limits_{\lambda:\, C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1}} \sum_{\lambda:\, C(\lambda)=0} \frac{1}{C'(\lambda)C(-\lambda)} \int\limits_0^\infty \int\limits_0^\infty e^{\lambda|u-v|} H(du,\, dv), \end{aligned} \tag{6.21}$$

where the integral after the first equality is a Lebesgue-Stieltjes integral with respect to the bivariate distribution function $H(u,v) = \mathbb{P}(E(t) \leq u, E(s) \leq v)$ of the process $\{E(t),\, t \geq 0\}$. Since integrands in (6.21) have the same form as the ACF of the stationary Pearson diffusion, from (6.19) (i.e. Theorem 3.1., Leonenko et al. (2013*a*)) the result immediately follows. $\square$

Remark 6.2. The last theorem is also valid for complex eigenvalues. To see this, notice that the Laplace transform of the density of random variable $E(t)$ (6.15) is valid for any complex number s and procedure from Leonenko et al. (2013*a*) is valid for complex eigenvalues as well.

6.2.1 Delayed Lévy-driven continuous-time autoregressive process of order $p = 1$

Corollary 6.2 Let $\{X_1(t),\, t \geq 0\}$ be the stationary Lévy-driven CAR(1) process defined in Section 6.1 with the correlation function given by (6.8). Then the correlation function of the corresponding dCAR(1) process $\{X_\alpha(t),\, t \geq 0\}$ is given by

$$\operatorname{Corr}(X_\alpha(t), X_\alpha(s)) = \mathcal{E}_\alpha(-\alpha_1 t^\alpha) + \frac{\alpha\alpha_1 t^\alpha}{\Gamma(1+\alpha)} \int_0^{s/t} \frac{\mathcal{E}_\alpha(-\alpha_1 t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz, \tag{6.22}$$

where $t \geq s > 0$.

Proof. In this case, characteristic equation is of the form

$$C(z) = z + \alpha_1 = 0,$$

where $\alpha_1 > 0$. Now simply apply Theorem 6.1 and the result follows. □

Remark 6.3. If in the outer process, stationary Lévy-driven CAR(1) process is replaced with the usual stationary CAR(1) process (i.e. if the driving process is Brownian motion), the outer process becomes the Ornstein-Uhlenbeck process (one of the six Pearson diffusions), while the corresponding dCAR(1) process becomes the fractional Ornstein-Uhlenbeck process.

6.2.2 Delayed Lévy-driven continuous-time autoregressive process of order $p = 2$

The over-damped case

Corollary 6.3 Let $\{X_2(t),\, t \geq 0\}$ be the stationary Lévy-driven CAR(2) process defined in section 6.1 with the correlation function given by (6.10). Then the correlation function of the corresponding dCAR(2) process $\{X_\alpha(t),\, t \geq 0\}$ is given by

$$\begin{aligned}\operatorname{Corr}(X_\alpha(t), X_\alpha(s)) &= \frac{\lambda_1 \mathcal{E}_\alpha(\lambda_2 t^\alpha) - \lambda_2 \mathcal{E}_\alpha(\lambda_1 t^\alpha)}{\lambda_1 - \lambda_2} \\ &+ \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \frac{\alpha t^\alpha}{\Gamma(1+\alpha)} \int_0^{s/t} \frac{\mathcal{E}_\alpha(\lambda_1 t^\alpha (1-z)^\alpha) - \mathcal{E}_\alpha(\lambda_2 t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz,\end{aligned} \tag{6.23}$$

where $t \geq s > 0$.

Proof. In this case, characteristic equation is of the form

$$C(z) = z^2 + \alpha_1 z + \alpha_2 = 0,$$

where $\alpha_1,\, \alpha_2 > 0$, $D = \alpha_1^2 - 4\alpha_2 > 0$, while the corresponding roots are λ_1 and λ_2. Now simply apply Theorem 6.1 and the result follows. □

The under-damped case

Corollary 6.4 Let $\{X_2(t),\, t \geq 0\}$ be the stationary Lévy-driven CAR(2) process defined in section 6.1 with the correlation function given by (6.12). Then the correlation function of the corresponding dCAR(2) process $\{X_\alpha(t),\, t \geq 0\}$ is given by

$$\begin{aligned}\mathrm{Corr}(X_\alpha(t), X_\alpha(s)) = {} & \frac{\lambda \mathcal{E}_\alpha(\bar{\lambda} t^\alpha) - \bar{\lambda}\mathcal{E}_\alpha(\lambda t^\alpha)}{\lambda - \bar{\lambda}} + \\ & + \frac{\lambda\bar{\lambda}}{\lambda - \bar{\lambda}} \frac{\alpha t^\alpha}{\Gamma(1+\alpha)} \int_0^{s/t} \frac{\mathcal{E}_\alpha(\lambda t^\alpha (1-z)^\alpha) - \mathcal{E}_\alpha(\bar{\lambda} t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz, \end{aligned} \tag{6.24}$$

where $t \geq s > 0$.

Proof. In this case, characteristic equation is of the form

$$C(z) = z^2 + \alpha_1 z + \alpha_2 = 0,$$

where $\alpha_1,\, \alpha_2 > 0$, $D = \alpha_1^2 - 4\alpha_2 < 0$, while the corresponding roots are λ and $\bar{\lambda}$. Now simply apply Theorem 6.1 and the result follows. □

The critically-damped case

Theorem 6.5 Let $\{X_2(t),\, t \geq 0\}$ be the stationary Lévy-driven CAR(2) process defined in section 6.1 with the correlation function given by (6.11). Then the correlation function of the corresponding dCAR(2) process $\{X_\alpha(t),\, t \geq 0\}$ is given by

$$\begin{aligned}\mathrm{Corr}(X_\alpha(t), X_\alpha(s)) = {} & \frac{\alpha_1}{2\alpha} t^\alpha \mathcal{E}_{\alpha,\alpha}\left(-\frac{\alpha_1}{2} t^\alpha\right) + \mathcal{E}_\alpha\left(-\frac{\alpha_1}{2} t^\alpha\right) \\ & + \frac{\alpha_1^2\, t^{2\alpha}}{4\Gamma(1+\alpha)} \int_{z=0}^{s/t} z^{\alpha-1}(1-z)^\alpha \mathcal{E}_{\alpha,\alpha}\left(-\frac{\alpha_1}{2} t^\alpha (1-z)^\alpha\right) dz, \end{aligned} \tag{6.25}$$

where $t \geq s > 0$.

Proof.

$$\begin{aligned}\mathrm{Corr}(X_\alpha(t), X_\alpha(s)) = \mathrm{Corr}(X_2(E(t)), X_2(E(s))) &= \int_0^\infty \int_0^\infty Corr(X(t), X(s)) H(du,\, dv) \\ &= \int_0^\infty \int_0^\infty \left(1 + \frac{\alpha_1}{2}|u-v|\right) e^{-\frac{\alpha_1}{2}|u-v|} H(du,\, dv), \end{aligned} \tag{6.26}$$

where the last integral is a Lebesgue-Stieltjes integral with respect to the bivariate distribution function $H(u, v) = \mathbb{P}(E(t) \leq u, E(s) \leq v)$ of the process $\{E(t),\, t \geq 0\}$.

Let $F(u,v)=\left(1+\frac{\alpha_1}{2}|u-v|\right)e^{-\frac{\alpha_1}{2}|u-v|}$. Following bivariate integration by parts approach as in Leonenko et al. (2013a), i.e. bivariate integration by parts formula (6.18) we obtain

$$\begin{aligned}\int_0^\infty\int_0^\infty F(u,\,v)H(du,dv) &= \int_0^\infty\int_0^\infty \mathbb{P}(E(t)\geq u, E(s)\geq v)F(du,dv)+\int_0^\infty \mathbb{P}(E(t)\geq u)F(du,0)+\\ &+\int_0^\infty \mathbb{P}(E(s)\geq v)F(0,dv)+1\\ &= I_1+I_2+I_3+1. \end{aligned}\tag{6.27}$$

Since $F(du,v)=f_v(u)du$ for $v\geq 0$ where

$$f_v(u)=-\frac{\alpha_1^2}{4}(u-v)e^{-\frac{\alpha_1}{2}(u-v)}I(u>v)-\frac{\alpha_1^2}{4}(u-v)e^{-\frac{\alpha_1}{2}(v-u)}I(u\leq v),$$

using (6.15), it follows

$$\begin{aligned}I_2 &= \int_0^\infty \mathbb{P}(E(t)\geq u)F(du,0)=\int_0^\infty \mathbb{P}(E(t)\geq u)\left(-\frac{\alpha_1^2}{4}ue^{-\frac{\alpha_1}{2}u}\right)du\\ &= \frac{\alpha_1}{2}e^{-\frac{\alpha_1}{2}u}u\mathbb{P}(E(t)\geq u)\Big|_0^\infty-\int_0^\infty \frac{\alpha_1}{2}e^{-\frac{\alpha_1}{2}u}\left(\mathbb{P}(E(t)\geq u)-uf_t(u)\right)du\\ &= \int_0^\infty \frac{\alpha_1}{2}e^{-\frac{\alpha_1}{2}u}uf_t(u)du-\int_0^\infty \frac{\alpha_1}{2}e^{-\frac{\alpha_1}{2}u}\mathbb{P}(E(t)\geq u)du\\ &= \int_0^\infty \frac{\alpha_1}{2}e^{-\frac{\alpha_1}{2}u}uf_t(u)du+\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}t^\alpha\right)-1. \end{aligned}\tag{6.28}$$

Similarly,

$$I_3=\int_0^\infty \mathbb{P}(E(s)\geq v)F(0,dv)=\int_0^\infty \frac{\alpha_1}{2}e^{-\frac{\alpha_1}{2}v}vf_s(v)dv+\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}s^\alpha\right)-1.\tag{6.29}$$

Now, (6.27) reduces to

$$\int_0^\infty\int_0^\infty F(u,\,v)H(du,dv)=I_1+\int_0^\infty \frac{\alpha_1}{2}e^{-\frac{\alpha_1}{2}u}u\left(f_s(u)+f_t(u)\right)du+\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}t^\alpha\right)+\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}s^\alpha\right)-1.$$

Since $F(du,dv)=h(u,v)dudv$ where

$$\begin{aligned}h(u,v) &= \left(\frac{\alpha_1^2}{4}e^{-\frac{\alpha_1}{2}(u-v)}-\frac{\alpha_1^3}{8}(u-v)e^{-\frac{\alpha_1}{2}(u-v)}\right)I(u>v)+\\ &+\left(\frac{\alpha_1^2}{4}e^{-\frac{\alpha_1}{2}(v-u)}-\frac{\alpha_1^3}{8}(v-u)e^{-\frac{\alpha_1}{2}(v-u)}\right)I(u\leq v)\end{aligned}$$

and the process $\{E(t),\, t \geq 0\}$ is nondecreasing it follows that for $u \leq v$

$$\mathbb{P}(E(t) \geq u, E(s) \geq v) = P(E(s) \geq v).$$

Write

$$I_1 = I_1^{(a)} + I_2^{(b)} + I_3^{(c)},$$

where

$$\begin{aligned}
I_1^{(a)} &= \int\limits_{u<v} \mathbb{P}(E(t) \geq u, E(s) \geq v) F(du, dv) = \int\limits_{u<v} \mathbb{P}(E(s) \geq v) F(du, dv) \\
I_1^{(b)} &= \int\limits_{u=v} \mathbb{P}(E(t) \geq u, E(s) \geq v) F(du, dv) = \int\limits_{u=v} \mathbb{P}(E(s) \geq v) F(du, dv) \\
I_1^{(c)} &= \int\limits_{u>v} \mathbb{P}(E(t) \geq u, E(s) \geq v) F(du, dv).
\end{aligned}$$

Once again, using integration by parts and (6.15) we obtain

$$\begin{aligned}
I_1^{(a)} &= \frac{\alpha_1^2}{4} \int\limits_{v=0}^{\infty} \int\limits_{u=0}^{v} \mathbb{P}(E(s) \geq v) e^{-\frac{\alpha_1}{2}(v-u)} du dv - \frac{\alpha_1^3}{8} \int\limits_{v=0}^{\infty} \int\limits_{u=0}^{v} \mathbb{P}(E(s) \geq v)(v-u) e^{-\frac{\alpha_1}{2}(v-u)} du dv \\
&= \frac{\alpha_1^2}{4} \int\limits_{0}^{\infty} \mathbb{P}(E(s) \geq v) v e^{-\frac{\alpha_1}{2} v} dv \\
&= 1 - \int\limits_{0}^{\infty} \frac{\alpha_1}{2} e^{-\frac{\alpha_1}{2} v} v f_s(v) dv - \mathcal{E}_\alpha \left(-\frac{\alpha_1}{2} s^\alpha \right).
\end{aligned} \tag{6.30}$$

Notice that $I_1^{(a)} = -I_3$.
Since function $f_v(u)du = F(du, v)$ does not have a jump at $u = v$ it follows

$$I_1^{(b)} = \int\limits_{u=v} \mathbb{P}(E(s) \geq v) F(du, dv) = 0. \tag{6.31}$$

Next,

$$\begin{aligned}
I_1^{(c)} = \frac{\alpha_1^2}{4} \int\limits_{v=0}^{\infty} \mathbb{P}(E(t) \geq u, E(s) \geq v) \int\limits_{u=v}^{\infty} e^{-\frac{\alpha_1}{2}(u-v)} du dv - \\
- \frac{\alpha_1^3}{8} \int\limits_{v=0}^{\infty} \mathbb{P}(E(t) \geq u, E(s) \geq v) \int\limits_{u=v}^{\infty} (u-v) e^{-\frac{\alpha_1}{2}(u-v)} du dv.
\end{aligned} \tag{6.32}$$

From Leonenko et al. (2013a), page 741 we have

$$\mathbb{P}(E(t) \geq u, E(s) \geq v) = \int\limits_{y=0}^{s} \frac{\alpha}{y} v f_y(v) \int\limits_{x=0}^{t-y} \frac{\alpha}{x} (u-v) f_x(u-v) dx dy.$$

Now using this expression together with Fubini theorem in (6.32) we obtain

$$
\begin{aligned}
I_1^{(c)} = & \frac{\alpha_1^2}{4} \int\limits_{y=0}^{s} \frac{\alpha}{y} \int\limits_{x=0}^{t-y} \frac{\alpha}{x} \int\limits_{v=0}^{\infty} v f_y(v) \int\limits_{u=v}^{\infty} (u-v) f_x(u-v) e^{-\frac{\alpha_1}{2}(u-v)} du\, dv\, dx\, dy \\
& - \frac{\alpha_1^3}{8} \int\limits_{0}^{s} \frac{\alpha}{y} \int\limits_{x=0}^{t-y} \frac{\alpha}{x} \int\limits_{v=0}^{\infty} v f_y(v) \int\limits_{u=v}^{\infty} (u-v)^2 f_x(u-v) e^{-\frac{\alpha_1}{2}(u-v)} du\, dv\, dx\, dy.
\end{aligned}
$$

Since

$$
\int\limits_{u=v}^{\infty} (u-v) f_x(u-v) e^{-\frac{\alpha_1}{2}(u-v)} du = \int\limits_{0}^{\infty} z f_x(z) e^{-\frac{\alpha_1}{2} z} dz, \tag{6.33}
$$

$$
\int\limits_{u=v}^{\infty} (u-v)^2 f_x(u-v) e^{-\frac{\alpha_1}{2}(u-v)} du = \int\limits_{0}^{\infty} z^2 f_x(z) e^{-\frac{\alpha_1}{2} z} dz \tag{6.34}
$$

and

$$
\int\limits_{v=0}^{\infty} v f_y(v) dv = \mathbb{E}[E(s)] = \frac{y^\alpha}{\Gamma(1+\alpha)},
$$

(see Baeumer & Meerschaert (2007), Eq. (9)) it follows

$$
\begin{aligned}
I_1^{(c)} = & \frac{\alpha_1^2 \alpha^2}{4\Gamma(1+\alpha)} \int\limits_{y=0}^{s} \frac{1}{y^{1-\alpha}} \int\limits_{x=0}^{t-y} \frac{1}{x} \int\limits_{0}^{\infty} z f_x(z) e^{-\frac{\alpha_1}{2} z} dz\, dx\, dy \\
& - \frac{\alpha_1^3 \alpha^2}{8\Gamma(1+\alpha)} \int\limits_{y=0}^{s} \frac{1}{y^{1-\alpha}} \int\limits_{x=0}^{t-y} \frac{1}{x} \int\limits_{0}^{\infty} z^2 f_x(z) e^{-\frac{\alpha_1}{2} z} dz\, dx\, dy.
\end{aligned} \tag{6.35}
$$

As in Leonenko et al. (2013a), we proceed by expanding $e^{-\frac{\alpha_1}{2} z}$ in (6.33) and (6.34) to obtain

$$
\int\limits_{0}^{\infty} z f_x(z) e^{-\frac{\alpha_1}{2} z} dz = -\frac{2}{\alpha_1} \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} x^\alpha\right)^j j}{\Gamma(1+\alpha j)}
$$

and

$$
\int\limits_{0}^{\infty} z^2 f_x(z) e^{-\frac{\alpha_1}{2} z} dz = \frac{4}{\alpha_1^2} \left(\sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} x^\alpha\right)^j j^2}{\Gamma(1+\alpha j)} - \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} x^\alpha\right)^j j}{\Gamma(1+\alpha j)} \right).
$$

On the other hand,

$$
\frac{d}{dx} \mathcal{E}_\alpha \left(-\frac{\alpha_1}{2} x^\alpha \right) = \frac{\alpha}{x} \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} x^\alpha\right)^j j}{\Gamma(1+\alpha j)} \tag{6.36}
$$

and

$$
\frac{d^2}{dx^2} \mathcal{E}_\alpha \left(-\frac{\alpha_1}{2} x^\alpha \right) = -\frac{\alpha}{x^2} \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} x^\alpha\right)^j j}{\Gamma(1+\alpha j)} + \frac{\alpha^2}{x^2} \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} x^\alpha\right)^j j^2}{\Gamma(1+\alpha j)},
$$

which implies

$$\int_0^\infty z f_x(z) e^{-\frac{\alpha_1}{2}z} dz = -\frac{2x}{\alpha\alpha_1}\frac{d}{dx}\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}x^\alpha\right) \tag{6.37}$$

and

$$\int_0^\infty z^2 f_x(z) e^{-\frac{\alpha_1}{2}z} dz = \frac{4}{\alpha_1^2}\left[\frac{x^2}{\alpha^2}\frac{d^2}{dx^2}\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}x^\alpha\right) + \left(\frac{x}{\alpha^2}-\frac{x}{\alpha}\right)\frac{d}{dx}\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}x^\alpha\right)\right].$$

Using these expressions in (6.35) we obtain

$$\begin{aligned} I_1^{(c)} &= -\frac{\alpha_1\alpha}{2\Gamma(1+\alpha)}\int_{y=0}^{s}\frac{1}{y^{1-\alpha}}\int_{x=0}^{t-y}\frac{d}{dx}\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}x^\alpha\right)dx\,dy \\ &- \frac{\alpha_1\alpha^2}{2\Gamma(1+\alpha)}\int_{y=0}^{s}\frac{1}{y^{1-\alpha}}\int_{x=0}^{t-y}\left[\frac{x}{\alpha^2}\frac{d^2}{dx^2}\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}x^\alpha\right) + \left(\frac{1}{\alpha^2}-\frac{1}{\alpha}\right)\frac{d}{dx}\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}x^\alpha\right)\right]dx\,dy \\ &= -\frac{\alpha_1}{2\Gamma(1+\alpha)}\int_{y=0}^{s}\frac{1}{y^{1-\alpha}}\int_{x=0}^{t-y}\left[x\frac{d^2}{dx^2}\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}x^\alpha\right) + \frac{d}{dx}\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}x^\alpha\right)\right]dx\,dy \end{aligned}$$

Since

$$x\frac{d^2}{dx^2}\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}x^\alpha\right) + \frac{d}{dx}\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}x^\alpha\right) = \left(x\frac{d}{dx}\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}x^\alpha\right)\right)'$$

it follows

$$I_1^{(c)} = -\frac{\alpha_1}{2\Gamma(1+\alpha)}\int_{y=0}^{s}\frac{1}{y^{1-\alpha}}\left[x\frac{d}{dx}\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}x^\alpha\right)\Bigg|_{x=t-y}\right]dy.$$

Using the definition of the two-parametric Mittag-Leffler function $\mathcal{E}_{\alpha,\beta}(\cdot)$ together with (6.36), after straightforward calculations we obtain

$$\left[x\frac{d}{dx}\mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}x^\alpha\right)\right]\Bigg|_{x=z} = -\frac{\alpha_1}{2}z^\alpha\mathcal{E}_{\alpha,\alpha}\left(-\frac{\alpha_1}{2}z^\alpha\right) \tag{6.38}$$

so that

$$I_1^{(c)} = \frac{\alpha_1^2}{4\Gamma(1+\alpha)}\int_{y=0}^{s} y^{\alpha-1}(t-y)^\alpha\mathcal{E}_{\alpha,\alpha}\left(-\frac{\alpha_1}{2}(t-y)^\alpha\right)dy.$$

Substituting $y = t\,z$ in the last integral it follows

$$I_1^{(c)} = \frac{\alpha_1^2\,t^{2\alpha}}{4\Gamma(1+\alpha)}\int_{z=0}^{s/t} z^{\alpha-1}(1-z)^\alpha\mathcal{E}_{\alpha,\alpha}\left(-\frac{\alpha_1}{2}t^\alpha(1-z)^\alpha\right)dz. \tag{6.39}$$

On the other hand, (6.37) and (6.38) imply that (6.28) reduces to

$$I_2 = \frac{\alpha_1}{2\alpha}t^\alpha\mathcal{E}_{\alpha,\alpha}\left(-\frac{\alpha_1}{2}t^\alpha\right) + \mathcal{E}_\alpha\left(-\frac{\alpha_1}{2}t^\alpha\right) - 1. \tag{6.40}$$

Finally, combining together (6.29), (6.30), (6.31), (6.39) and (6.40) we obtain

$$\begin{aligned}
\operatorname{Corr}(X_\alpha(t), X_\alpha(s)) &= I_1 + I_2 + I_3 + 1 = I_1^{(a)} + I_1^{(b)} + I_1^{(c)} + I_2 + I_3 + 1 \\
&= \frac{\alpha_1}{2\alpha} t^\alpha \mathcal{E}_{\alpha,\,\alpha}\left(-\frac{\alpha_1}{2} t^\alpha\right) + \mathcal{E}_\alpha\left(-\frac{\alpha_1}{2} t^\alpha\right) \\
&+ \frac{\alpha_1^2\, t^{2\alpha}}{4\Gamma(1+\alpha)} \int\limits_{z=0}^{s/t} z^{\alpha-1}(1-z)^\alpha \mathcal{E}_{\alpha,\,\alpha}\left(-\frac{\alpha_1}{2} t^\alpha (1-z)^\alpha\right) dz. \qquad (6.41)
\end{aligned}$$

□

Remark 6.4. When $t = s$, it must be true that $\operatorname{Corr}(X_\alpha(t), X_\alpha(s)) = 1$.

For $t = s$ (6.39) becomes

$$\begin{aligned}
I_1^{(c)} &= \frac{\alpha_1^2\, t^{2\alpha}}{4\Gamma(1+\alpha)} \int\limits_{z=0}^{1} z^{\alpha-1}(1-z)^\alpha \mathcal{E}_{\alpha,\,\alpha}\left(-\frac{\alpha_1}{2} t^\alpha (1-z)^\alpha\right) dz \\
&= -\frac{\alpha_1 t^\alpha}{2\Gamma(1+\alpha)} \int\limits_{z=0}^{1} z^{\alpha-1} \left[x \frac{d}{dx} \mathcal{E}_\alpha\left(-\frac{\alpha_1}{2} x^\alpha\right)\right]\Bigg|_{x=t(1-z)} dz \\
&= -\frac{\alpha_1 t^\alpha \alpha}{2\Gamma(1+\alpha)} \int\limits_{z=0}^{1} z^{\alpha-1} \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} t^\alpha (1-z)^\alpha\right)^j j}{\Gamma(1+\alpha j)} dz \\
&= -\frac{\alpha_1 t^\alpha \alpha}{2\Gamma(1+\alpha)} \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} t^\alpha\right)^j j}{\Gamma(1+\alpha j)} \int\limits_{z=0}^{1} z^{\alpha-1}(1-z)^{\alpha j} dz. \qquad (6.42)
\end{aligned}$$

Since formula for the beta density yields

$$\int\limits_0^x y^{a-1}(x-y)^{b-1} dy = B(a,\, b) x^{a+b-1}$$

where $B(a,\, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\ a > 0,\, b > 0$, (6.42) reduces to

$$\begin{aligned}
I_1^{(c)} &= -\frac{\alpha_1 t^\alpha}{2} \frac{\alpha\Gamma(\alpha)}{\Gamma(1+\alpha)} \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} t^\alpha\right)^j j}{\Gamma(1+\alpha(j+1))} \\
&= \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} t^\alpha\right)^{j+1} j}{\Gamma(1+\alpha(j+1))} \\
&= \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} t^\alpha\right)^{j+1} (j+1)}{\Gamma(1+\alpha(j+1))} - \sum_{j=0}^{\infty} \frac{\left(-\frac{\alpha_1}{2} t^\alpha\right)^{j+1}}{\Gamma(1+\alpha(j+1))} \\
&= \frac{t}{\alpha} \frac{d}{dt} \mathcal{E}_\alpha\left(-\frac{\alpha_1}{2} t^\alpha\right) - \mathcal{E}_\alpha\left(-\frac{\alpha_1}{2} t^\alpha\right) + 1 \\
&= -\frac{\alpha_1}{2\alpha} t^\alpha \mathcal{E}_{\alpha,\alpha}\left(-\frac{\alpha_1}{2} t^\alpha\right) - \mathcal{E}_\alpha\left(-\frac{\alpha_1}{2} t^\alpha\right) + 1.
\end{aligned}$$

Now, from (6.41) it follows $\mathrm{Corr}(X_\alpha(t), X_\alpha(s)) = 1$.

Remark 6.5. For dCAR(p) process of order $p > 2$ such that the non-distinct roots of the corresponding characteristic equation (6.6) have the highest multiplicity $m = 2$, one can expect similar correlation structure as in Theorem 6.5. On the other hand, with higher multiplicities ($m > 2$), higher derivatives of Mittag-Leffler function should appear in the correlation structure, and case-by-case analysis is expected.

6.3 Long-range dependence property

In this section we show that dCAR(p) processes are long-range dependent processes, emphasizing low orders.

Recall that for non-stationary stochastic process $\{X(t),\, t \geq 0\}$ we say it has long-range dependence property if

$$\mathrm{Corr}(X(t), X(s)) \sim c(s)\, t^{-d},\ t \to \infty,$$

for a fixed $s > 0$, some constant $c(s) > 0$ and $d \in \langle 0, 1\rangle$ (see Section 4.4, where we first used this definition to show that fractional Pearson diffusions exhibit long-range dependence property).

Before we proceed, we need some technical results regarding Mittag-Leffler functions. First notice that

$$\mathcal{E}_\alpha(\theta t^\alpha) \sim -\frac{1}{\theta\Gamma(1-\alpha)t^\alpha},\ t \to \infty, \tag{6.43}$$

where θ is a complex number such that $\mathrm{Re}\,\theta < 0$ and $0 < \alpha < 1$ (see Theorem 1.4., Podlubny (1998)). Since

$$\mathcal{E}_{\alpha,\alpha}(\theta t^\alpha) \sim O(|\theta t^\alpha|^{-2}),\ t \to \infty$$

(again, see Theorem 1.4., Podlubny (1998)) from (6.38) and (6.43) immediately after applying L'Hospital's rule for complex valued functions (see Carter (1958)) it follows

$$\mathcal{E}_{\alpha,\alpha}(\theta t^\alpha) \sim \frac{\alpha}{\theta^2\Gamma(1-\alpha)t^{2\alpha}},\ t \to \infty. \tag{6.44}$$

On the other hand, if θ is a complex number such that $\mathrm{Re}\,\theta < 0$, $0 < \alpha < 1$ and C a real constant, then (see Theorem 1.6., Podlubny (1998))

$$|\mathcal{E}_\alpha(\theta t^\alpha)| \leq \frac{C}{1+|\theta|t^\alpha},\ t > 0, \tag{6.45}$$

$$|\mathcal{E}_{\alpha,\alpha}(\theta t^\alpha)| \leq \frac{C}{1+|\theta|t^\alpha},\ t > 0. \tag{6.46}$$

Next, we prove two lemmas needed for the proof of the long-range dependence for dCAR(p) processes.

Lemma 6.6 Let θ be a complex number such that $\mathrm{Re}\,\theta < 0$. If $0 < \alpha < 1$ and $t \geq s > 0$ then

$$\int_0^{s/t} \frac{\mathcal{E}_\alpha(\theta t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz \sim -\frac{1}{\theta\alpha\Gamma(1-\alpha)} \frac{s^\alpha}{t^{2\alpha}}, \; t \to \infty.$$

Proof. By change of variable $z = s/ty$ we have

$$\int_0^{s/t} \frac{\mathcal{E}_\alpha(\theta t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz = \left(\frac{s}{t}\right)^\alpha \int_0^1 \frac{\mathcal{E}_\alpha(\theta t^\alpha (1-sy/t)^\alpha)}{y^{1-\alpha}} dy.$$

From (6.45) we see that last integrand is bounded with $g(y) = C/y^{1-\alpha}$ and $\int_0^1 g(y) < \infty$, so by using Lebesgue's dominated convergence theorem together with (6.43) we obtain

$$\begin{aligned}\left(\frac{s}{t}\right)^\alpha \int_0^1 \frac{\mathcal{E}_\alpha(\theta t^\alpha (1-sy/t)^\alpha)}{y^{1-\alpha}} dy &\sim -\left(\frac{s}{t}\right)^\alpha \frac{1}{\theta\Gamma(1-\alpha)t^\alpha} \int_0^1 y^{\alpha-1} dy, \; t \to \infty \\ &= -\frac{1}{\theta\alpha\Gamma(1-\alpha)} \frac{s^\alpha}{t^{2\alpha}}, \; t \to \infty.\end{aligned}$$

□

Lemma 6.7 Let θ be a complex number such that $\mathrm{Re}\,\theta < 0$. If $0 < \alpha < 1$ and $t \geq s > 0$ then

$$\int_{z=0}^{s/t} z^{\alpha-1}(1-z)^\alpha \mathcal{E}_{\alpha,\alpha}\left(\theta t^\alpha (1-z)^\alpha\right) dz \sim \frac{1}{\theta^2\Gamma(1-\alpha)} \frac{s^\alpha}{t^{3\alpha}}, \; t \to \infty.$$

Proof. Once again, by change of variable $z = s/ty$ we have

$$\int_{z=0}^{s/t} z^{\alpha-1}(1-z)^\alpha \mathcal{E}_{\alpha,\alpha}\left(\theta t^\alpha (1-z)^\alpha\right) dz = \left(\frac{s}{t}\right)^\alpha \int_0^1 y^{\alpha-1}\left(1-sy/t\right)^\alpha \mathcal{E}_{\alpha,\alpha}\left(\theta t^\alpha (1-sy/t)^\alpha\right) dy.$$

From (6.46) we see that last integrand is bounded with $g(y) = Cy^{\alpha-1}$ and $\int_0^1 g(y) < \infty$, so by using Lebesgue's dominated convergence theorem together with (6.44) we obtain

$$\begin{aligned}\left(\frac{s}{t}\right)^\alpha \int_0^1 y^{\alpha-1}\left(1-sy/t\right)^\alpha \mathcal{E}_{\alpha,\alpha}\left(\theta t^\alpha (1-sy/t)^\alpha\right) dy &\sim \left(\frac{s}{t}\right)^\alpha \frac{\alpha}{\theta^2\Gamma(1-\alpha)t^{2\alpha}} \int_0^1 y^{\alpha-1}\left(1-sy/t\right)^{-\alpha} dy \\ &\sim \left(\frac{s}{t}\right)^\alpha \frac{\alpha}{\theta^2\Gamma(1-\alpha)t^{2\alpha}} \int_0^1 y^{\alpha-1} dy \\ &= \frac{1}{\theta^2\Gamma(1-\alpha)} \frac{s^\alpha}{t^{3\alpha}}, \; t \to \infty.\end{aligned}$$

□

Theorem 6.8 Let $\{X_\alpha(t),\, t \geq 0\}$ be the dCAR(p) process as in Theorem 6.1 with corresponding correlation function (6.20). Then stochastic process $\{X_\alpha(t),\, t \geq 0\}$ has the long-range dependence property, i.e. for a fixed $s > 0$

$$Corr(X_\alpha(t), X_\alpha(s)) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \left(-\frac{\sum\limits_{\lambda:C(\lambda)=0} (\lambda C'(\lambda)C(-\lambda))^{-1}}{\sum\limits_{\lambda:C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1}} + \frac{s^\alpha}{\Gamma(1+\alpha)} \right),\ t \to \infty.$$

Proof. Since distinct roots λ of equation (6.6) have negative real parts, using (6.43) together with Lemma 6.6 it follows

$$\begin{aligned} Corr(X_\alpha(t), X_\alpha(s)) &= \frac{\sum\limits_{\lambda:C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1} \left[\mathcal{E}_\alpha(\lambda t^\alpha) - \frac{\alpha\lambda t^\alpha}{\Gamma(1+\alpha)} \int\limits_0^{s/t} \frac{\mathcal{E}_\alpha(\lambda t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz \right]}{\sum\limits_{\lambda:C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1}} \\ &\sim \frac{\sum\limits_{\lambda:C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1} \left[-\frac{1}{\lambda\Gamma(1-\alpha)t^\alpha} + \frac{\alpha\lambda t^\alpha}{\Gamma(1+\alpha)} \cdot \frac{1}{\lambda\alpha\Gamma(1-\alpha)} \frac{s^\alpha}{t^{2\alpha}} \right]}{\sum\limits_{\lambda:C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1}},\ t \to \infty \\ &= \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \left(-\frac{\sum\limits_{\lambda:C(\lambda)=0} (\lambda C'(\lambda)C(-\lambda))^{-1}}{\sum\limits_{\lambda:C(\lambda)=0} (C'(\lambda)C(-\lambda))^{-1}} + \frac{s^\alpha}{\Gamma(1+\alpha)} \right),\ t \to \infty. \end{aligned}$$

□

Corollary 6.9 Let $\{X_\alpha(t),\, t \geq 0\}$ be the dCAR(1) process as in Corollary 6.2 with corresponding correlation function (6.22). Then stochastic process $\{X_\alpha(t),\, t \geq 0\}$ has the long-range dependence property, i.e. for a fixed $s > 0$

$$\mathrm{Corr}(X_\alpha(t), X_\alpha(s)) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \left(\frac{1}{\alpha_1} + \frac{s^\alpha}{\Gamma(1+\alpha)} \right),\ t \to \infty.$$

Proof. In this case, characteristic equation is of the form

$$C(z) = z + \alpha_1 = 0,$$

where $\alpha_1 > 0$. Now simply apply Theorem 6.8 and the result follows. □

Corollary 6.10 Let $\{X_\alpha(t),\, t \geq 0\}$ be the dCAR(2) process in the over-damped case, i.e. as in Corollary 6.3 with corresponding correlation function (6.23). Then stochastic process $\{X_\alpha(t),\, t \geq 0\}$ has the long-range dependence property, i.e. for a fixed $s > 0$

$$\mathrm{Corr}(X_\alpha(t), X_\alpha(s)) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \left(-\frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2} + \frac{s^\alpha}{\Gamma(1+\alpha)} \right),\ t \to \infty.$$

Proof. In this case, characteristic equation is of the form

$$C(z) = z^2 + \alpha_1 z + \alpha_2 = 0,$$

where α_1, $\alpha_2 > 0$, $D = \alpha_1^2 - 4\alpha_2 > 0$, while the corresponding roots are λ_1 and λ_2. Now simply apply Theorem 6.8 and the result follows. □

Corollary 6.11 Let $\{X_\alpha(t),\, t \geq 0\}$ be the dCAR(2) process in the under-damped case, i.e. as in Corollary 6.4 with corresponding correlation function (6.24). Then stochastic process $\{X_\alpha(t),\, t \geq 0\}$ has the long-range dependence property, i.e. for a fixed $s > 0$

$$\mathrm{Corr}(X_\alpha(t), X_\alpha(s)) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \left(-\frac{\lambda + \bar{\lambda}}{\lambda \bar{\lambda}} + \frac{s^\alpha}{\Gamma(1+\alpha)} \right),\; t \to \infty.$$

Proof. In this case, characteristic equation is of the form

$$C(z) = z^2 + \alpha_1 z + \alpha_2 = 0,$$

where α_1, $\alpha_2 > 0$, $D = \alpha_1^2 - 4\alpha_2 < 0$, while the corresponding roots are λ and $\bar{\lambda}$. Now simply apply Theorem 6.8 and the result follows. □

Theorem 6.12 Let $\{X_\alpha(t),\, t \geq 0\}$ be the dCAR(2) process in the critically-damped case, i.e. as in Theorem 6.5 with corresponding correlation function (6.25). Then stochastic process $\{X_\alpha(t),\, t \geq 0\}$ has the long-range dependence property, i.e. for a fixed $s > 0$

$$\mathrm{Corr}(X_\alpha(t), X_\alpha(s)) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \left(\frac{4}{\alpha_1} + \frac{s^\alpha}{\Gamma(1+\alpha)} \right),\; t \to \infty.$$

Proof. Since $\alpha_1 > 0$, using (6.43), (6.44) together with Lemma 6.7 for $\theta = -\alpha_1/2$ it follows

$$\begin{aligned}
\mathrm{Corr}(X_\alpha(t), X_\alpha(s)) &= \frac{\alpha_1}{2\alpha} t^\alpha \mathcal{E}_{\alpha,\alpha}\left(-\frac{\alpha_1}{2} t^\alpha\right) + \mathcal{E}_\alpha\left(-\frac{\alpha_1}{2} t^\alpha\right) \\
&\quad + \frac{\alpha_1^2\, t^{2\alpha}}{4\Gamma(1+\alpha)} \int\limits_{z=0}^{s/t} z^{\alpha-1}(1-z)^\alpha \mathcal{E}_{\alpha,\alpha}\left(-\frac{\alpha_1}{2} t^\alpha (1-z)^\alpha\right) dz \\
&\sim \frac{\alpha_1}{2\alpha} t^\alpha \cdot \frac{4\alpha}{\alpha_1^2 \Gamma(1-\alpha) t^{2\alpha}} + \frac{2}{\alpha_1 \Gamma(1-\alpha) t^\alpha} + \frac{\alpha_1^2\, t^{2\alpha}}{4\Gamma(1+\alpha)} \cdot \frac{4}{\alpha_1^2 \Gamma(1-\alpha)} \frac{s^\alpha}{t^{3\alpha}} \\
&= \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \left(\frac{4}{\alpha_1} + \frac{s^\alpha}{\Gamma(1+\alpha)} \right),\; t \to \infty.
\end{aligned}$$

□

6.4 Distribution of delayed Lévy-driven continuous-time autoregressive processes

Let

$$p(x,t) := \frac{d}{dx}\mathbb{P}\left(X(t) \le x\right)$$

denote the density of the Lévy-driven CAR(p) process $\{X(t),\ t \ge 0\}$, and like in previous sections, let

$$f_t(x) = \frac{d}{dx}\mathbb{P}\left(E(t) \le x\right)$$

denote the density of the inverse of the standard α-stable subordinator $\{E(t),\ t \ge 0\}$. Then for the density of the dCAR(p) process $\{X_\alpha(t),\ t \ge 0\}$

$$q(x,t) := \frac{d}{dx}\mathbb{P}\left(X_\alpha(t) \le x\right) = \frac{d}{dx}\mathbb{P}\left(X(E(t)) \le x\right)$$

the following representation is valid

$$q(x,t) = \int_0^\infty p(x,s) f_t(s) ds. \tag{6.47}$$

To see this, since $X(t)$ and $E(t)$ are independent, using conditional argument yields

$$\begin{aligned}\mathbb{P}\left(X(E(t)) \le x\right) &= \mathbb{E}\left[\mathbb{P}\left(X(E(t)) \le x\right)|E(t)\right] \\ &= \int_0^\infty \mathbb{P}(X(u) \le x | E(t) = u) f_t(u) du \\ &= \int_0^\infty \mathbb{P}(X(u) \le x) f_t(u) du.\end{aligned}$$

After differentiating (which can be justified by the dominated convergence theorem) we arrive at (6.47).
Since the density $f_t(x)$ of the process $\{E(t),\ t \ge 0\}$ is given via (6.14), it is clear that once we know the density $p(x,t)$ of the process $\{X(t),\ t \ge 0\}$, we can calculate the density $q(x,t)$ of the dCAR(p) process $\{X_\alpha(t),\ t \ge 0\}$ via (6.47).

Example 6.1 Let us consider the non-stationary CAR(1) process with $L(t) = W(t)$, i.e. driven by the standard Brownian motion. SDE (6.2) reduces to

$$dX(t) + \alpha_1 X(t) dt = dW(t).$$

Therefore CAR(1) process reduces to the well known Ornstein-Uhlenbeck process (see

Section 3.2) with transition density (cf. Karlin & Taylor (1981a), page 332)

$$p(x,t;x_0)=\frac{1}{\sqrt{2\pi(2\alpha_1)^{-1}(1-e^{1-2\alpha_1 t})}}\exp\left\{-\frac{x-x_0e^{-\alpha_1 t}}{(\alpha_1)^{-1}(1-e^{-2\alpha_1 t})}\right\}. \tag{6.48}$$

If we denote probability density of the initial distribution of CAR(1) process with p_0, then the density of CAR(1) process is given by

$$p(x,t)=\int\limits_{\mathbb{R}} p_0(x_0)p(x,t;x_0)dx_0,$$

where the transition density $p(x,t;x_0)$ is given by (6.48). Now, density of the corresponding dCAR(1) process, i.e. expression (6.47) becomes

$$q(x,t)=\int\limits_{\mathbb{R}} p_0(x_0)\left(\int\limits_0^\infty p(x,s;x_0)f_t(s)ds\right)dx_0,$$

where p_0 is the initial distribution of the non-stationary CAR(1) process, f_t is the probability density of the inverse of the stable subordinator (6.14) and the transition density of CAR(1) process $p(x,s;x_0)$ is given by (6.48).

However, in this thesis, we consider only stationary Lévy-driven CAR(p) process $\{X(t),\ t\geq 0\}$. If $m(x)$ denotes its probability density, then from (6.47) it is clear that the density of corresponding dCAR(p) process stays the same over all time, i.e. it has the probability density $m(x)$. Therefore, density of the dCAR(p) process is the same as the density of the corresponding stationary Lévy-driven CAR(p) process.

Stationary Lévy-driven CAR(p) process $\{X(t),\ t\geq 0\}$ has cumulant generating function (cgf) for $\{X(t_1),\, X(t_2),\, \ldots, X(t_n),\ 0<t_1<t_2<\cdots<t_n\}$ (see Brockwell (2001b), Brockwell & Marquardt (2005))

$$\begin{aligned}\ln\mathbb{E}\left[exp(i\theta_1X(t_1)+\cdots+i\theta_nX(t_n))\right]=&\int\limits_0^\infty \xi\left(\sum_{i=1}^n\theta_i\boldsymbol{b}^Te^{A(t_i+u)}\boldsymbol{e}\right)du+\int\limits_0^{t_1}\xi\left(\sum_{i=1}^n\theta_i\boldsymbol{b}^Te^{A(t_i-u)}\boldsymbol{e}\right)du\\&+\int\limits_{t_1}^{t_2}\xi\left(\sum_{i=2}^n\theta_i\boldsymbol{b}^Te^{A(t_i-u)}\boldsymbol{e}\right)+\ldots+\int\limits_{t_{n-1}}^{t_n}\xi\left(\theta_n\boldsymbol{b}^Te^{A(t_n-u)}\boldsymbol{e}\right)du,\end{aligned} \tag{6.49}$$

where $\boldsymbol{b}=[1,0,\ldots 0]^T,\ \boldsymbol{b}\in\mathbb{R}^p$, the characteristic function of the driving Lévy process $\{L(t),\ t\geq 0\}$ of the CAR(p) process

$$\phi_t(\theta):=\mathbb{E}\left[e^{i\theta L(t)}\right]=e^{t\xi(\theta)},$$

where
$$\xi(\theta) = i\theta m - \frac{1}{2}\theta^2 s^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\theta x} - 1 - i\theta x\, \mathbb{I}_{|x|<1}\right) \nu(dx),$$
for some $m \in \mathbb{R}$, $s \geq 0$ and Lévy measure ν. In particular, marginal distribution of $X(t)$ (and therefore of $X_\alpha(t)$ as well) has cgf

$$\ln \mathbb{E}\left[exp(i\theta X(t))\right] = \int_0^\infty \xi\left(\theta \boldsymbol{b}^T e^{Au} \boldsymbol{e}\right) du. \tag{6.50}$$

In our setting, (6.50) reduces to cases:

- stationary Lévy-driven CAR(1) process with the correlation function given by (6.8)
$$\ln \mathbb{E}\left[exp(i\theta X(t))\right] = \int_0^\infty \xi\left(\theta\, e^{-\alpha_0 u}\right) du$$
- stationary Lévy-driven CAR(2) process with the correlation function given by (6.10)
$$\ln \mathbb{E}\left[exp(i\theta X(t))\right] = \int_0^\infty \xi\left(\theta\, \frac{e^{\lambda_1 u} - e^{\lambda_2 u}}{\lambda_1 - \lambda_2}\right) du$$
- stationary Lévy-driven CAR(2) process with the correlation function given by (6.11)
$$\ln \mathbb{E}\left[exp(i\theta X(t))\right] = \int_0^\infty \xi\left(\theta\, u e^{-\frac{\alpha_1}{2}u}\right) du$$
- stationary Lévy-driven CAR(2) process with the correlation function given by (6.12)
$$\ln \mathbb{E}\left[exp(i\theta X(t))\right] = \int_0^\infty \xi\left(\theta\, \frac{e^{\lambda u} - e^{\bar{\lambda} u}}{\lambda - \bar{\lambda}}\right) du.$$

Example 6.2 Let us consider the stationary Lévy-driven CAR(1) process with the driving process being compound Poisson process with finite jump-rate λ and bilateral exponential jump size distribution with probability density $f(x) = \beta/2e^{-\beta|x|}$, while corresponding characteristic exponent is of the form

$$\xi(\theta) = -\frac{\lambda\theta^2}{\beta^2 + \theta^2}.$$

Then, marginal distribution of the corresponding dCAR(1) process has cumulant generating function of the form

$$\ln \mathbb{E}\left[exp(i\theta X_\alpha(t))\right] = \int\limits_0^\infty \xi\left(\theta\, e^{-\alpha_0 u}\right) du = -\frac{\lambda}{2\alpha_0} \ln\left(1 + \frac{\theta^2}{\beta^2}\right),$$

which shows that corresponding dCAR(1) process has marginals distributed as the difference between two independent gamma distributed random variables with exponent $\lambda/(2\alpha_0)$ and scale parameter β.

Many examples regarding distribution of stationary Lévy-driven CAR(p) process can be found in Barndorff-Nielsen & Shephard (2001), Brockwell (2001b) and Brockwell & Marquardt (2005).

Bibliography

Ait-Sahalia, Y. (1996), 'Testing continuous-time models of the spot interest rate', *The Review of Financial Studies* **9**(2), 385–426.

Arendt, W., Batty, C., Hieber, M. & Neubrander, F. (2011), *Vector-valued Laplace Transforms and Cauchy Problems*, 2 edn, Birkhäuser.

Avram, F., Leonenko, N. N. & Šuvak, N. (2011), 'Parameter estimation for Fisher Snedecor diffusion', *Statistics* **45**(1), 27–42.

Avram, F., Leonenko, N. N. & Šuvak, N. (2012), 'Hypothesis testing for Fisher Snedecor diffusion', *Journal of Statistical Planning and Inference* **142**(8), 2308–2321.

Avram, F., Leonenko, N. N. & Šuvak, N. (2013*a*), 'On spectral analysis of heavy-tailed Kolmogorov-Pearson diffusions', *Markov Processes and Related Fields* **19**(2), 249–298.

Avram, F., Leonenko, N. N. & Šuvak, N. (2013*b*), 'Spectral representation of transition density of Fisher Snedecor diffusion', *Stochastics* **85**(2), 346–369.

Baeumer, B. & Meerschaert, M. M. (2001), 'Stohastic solutions for fractional Cauchy problems', *Fractional Calculus and Applied Analysis* **4**, 481–500.

Baeumer, B. & Meerschaert, M. M. (2007), 'Fractional diffusion with two time scales', *Physica A: Statistical Mechanics and its Applications* **373**, 237 – 251.

Barndorff-Nielsen, O. E. & Shephard, N. (2001), 'Non–Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics', *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **63**(2), 167–241.

Berkowitz, B., Cortis, A., Dentz, M. & Scher, H. (2006), 'Modeling non-Fickian transport in geological formations as a continuous time random walk', *Reviews of Geophysics* **44**(2).

Bibby, B. M., Skovgaard, I. M. & Sørensen, M. (2005), 'Diffusion-type models with given marginal distribution and autocorrelation function', *Bernoulli* **11**(2), 191–220.

Bingham, N. H. (1971), 'Limit theorems for occupation times of Markov processes', *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **17**(1), 1–22.

Bochner, S. (1949), 'Diffusion equation and stochastic processes', *Proceedings of the National Academy of Sciences* **35**(7), 368–370.

Brockwell, P. J. (2001*a*), 'Continuous-time ARMA processes', *Handbook of Statistics* **19**, 249–276.

Brockwell, P. J. (2001*b*), 'Lévy-driven carma processes', *Annals of the Institute of Statistical Mathematics* **53**(1), 113–124.

Brockwell, P. J. & Marquardt, T. (2005), 'Lévy-driven and fractionally integrated ARMA processes with continuous time parameter', *Statistica Sinica* pp. 477–494.

Buchholz, H. (1969), *The Confluent Hypergeometric Function: with Special Emphasis on its Applications*, Springer Tracts in Natural Philosophy 15, 1 edn, Springer-Verlag Berlin Heidelberg.

Carter, D. (1958), 'L'Hospital's rule for complex-valued functions', *The American Mathematical Monthly* **65**(4), 264–266.

Chakraborty, P., Meerschaert, M. M. & Lim, C. Y. (2009), 'Parameter estimation for fractional transport: A particle-tracking approach', *Water Resources Research* **45**(10).

Clark, P. K. (1973), 'A subordinated stochastic process model with finite variance for speculative prices', *Econometrica: Journal of the Econometric Society* **40**(1), 135–155.

Cox, J., Ingersoll, J. & Ross, S. (1985), 'A theory of the term structure of interest rates', *Econometrica* **53**(2), 385–407.

Einstein, A. (1905), 'On the motion of small particles suspended in liquids at rest required by the molecular-kinetic theory of heat', *Annalen der physik* **17**, 549–560.

Erdelyi, A. (1981), *Higher Transcendental Functions, Volume II*, Krieger Pub Co.

Ethier, S. N. & Kurtz, T. G. (2009), *Markov Processes: Characterization and Convergence*, John Wiley & Sons.

Fedotov, S. & Iomin, A. (2007), 'Migration and proliferation dichotomy in tumor-cell invasion', *Physical Review Letters* **98**(11), 118101.

Feller, W. (1951), 'Two singular diffusion problems', *Annals of mathematics* pp. 173–182.

Feller, W. (1966), *An Introduction to Probability Theory and Its Applications*, Vol. II of *Wiley series in Probability and Mathematical Statistics*, 2ed. edn, Wiley.

Forman, J. & Sørensen, M. (2008), 'The Pearson diffusions: A class of statistically tractable diffusion processes', *Scandinavian Journal of Statistics* **35**(3), 438–465.

Friedman, A. (1975), *Stochastic Differential Equations and Applications*, Vol. 1, Acedemic Press, New York.

Fulton, C. T., Pruess, S. & Xie, Y. (2005), 'The automatic classification of sturm-liouville problems', *J. Appl. Math. Comput* **124**, 149–186.

Gajda, J., Wyłomańska, A. & Zimroz, R. (2016), 'Subordinated continuous-time AR processes and their application to modeling behavior of mechanical system', *Physica A: Statistical Mechanics and its Applications* **464**, 123–137.

Garibaldi, U. & Scalas, E. (2010), *Finitary Probabilistic Methods in Econophysics*, Cambridge University Press.

Germano, G., Politi, M., Scalas, E. & Schilling, R. L. (2009), 'Stochastic calculus for uncoupled continuous-time random walks', *Phys. Rev. E* **79**, 066102.

Gill, R. D., van der Laan, M. J. & Wellner, J. A. (1995), 'Inefficient estimators of the bivariate survival function for three models', *Annales de l'I.H.P. Probabilités et Statistiques* **31**(3), 545–597.

Gorenflo, R., Kilbas, A. A., Mainardi, F. & Rogosin, S. V. (2016), *Mittag-Leffler Functions, Related Topics and Applications*, Springer.

Grigoletto, E. C., de Oliveira, E. C. & de Figueiredo Camargo, R. (2018), 'Linear fractional differential equations and eigenfunctions of fractional differential operators', *Computational and Applied Mathematics* **37**(2), 1012–1026.

Hilfer, R. & Anton, L. (1995), 'Fractional master equations and fractal time random walks', *Physical Review E* **51**(2), R848–R851.

Jacobsen, M. (1996), 'Laplace and the origin of the Ornstein-Uhlenbeck process', *Bernoulli* **2**(3), 271–286.

Janczura, J., Orzeł, S. & Wyłomańska, A. (2011), 'Subordinated α-stable Ornstein–Uhlenbeck process as a tool for financial data description', *Physica A: Statistical Mechanics and its Applications* **390**(23-24), 4379–4387.

Kallenberg, O. (2002), *Foundations of Modern Probability*, Springer Series in Statistics. Probability and Its Applications., 2nd edn, Springer.

Karlin, S. & Taylor, H. (1981*a*), *A Second Course in Stochastic Processes*, Academic Press.

Karlin, S. & Taylor, H. M. (1981*b*), *A First Course in Stochastic Processes*, Academic Press.

Kelbert, M., Konakov, V. & Menozzi, S. (2016), 'Weak error for continuous time markov chains related to fractional in time P(I)DEs', *Stochastic Processes and their Applications* **126**(4), 1145 – 1183.

Kilbas, A. A., Srivastava, H. M. & Trujillo, J. J. (2006), *Theory and Applications of Fractional Differential Equations, Volume 204 (North-Holland Mathematics Studies)*, Elsevier, Amsterdam.

Klafter, J. & Silbey, R. (1980), 'Derivation of the continuous-time random-walk equation', *Physical Review Letters* **44**(2), 55–58.

Kloeden, P. E. & Platen, E. (1995), *Numerical Solution of Stochastic Differential Equations*, Springer.

Kobayashi, K. (2011), 'Stochastic calculus for a time-changed semimartingale and the associated stochastic differential equations', *Journal of Theoretical Probability* **24**(3), 789–820.

Kolmogorov, A. (1931), 'Über die analytischen methoden in der wahrscheinlichkeitsrechnung', *Mathematische Annalen* **104**(1), 415–458.

Laplace, P. (1812), *Théorie Analytique des Probabilités*, Paris, Ve. Courcier.

Leonenko, N. N., Meerschaert, M. M. & Sikorskii, A. (2013*a*), 'Correlation structure of fractional Pearson diffusions', *Computers & Mathematics with Applications* **66**(5), 737 – 745.

Leonenko, N. N., Meerschaert, M. M. & Sikorskii, A. (2013*b*), 'Fractional Pearson diffusions', *Journal of Mathematical Analysis and Applications* **403**(2), 532–546.

Leonenko, N. N. & Šuvak, N. (2010*a*), 'Statistical inference for reciprocal gamma diffusion process', *Journal of Statistical Planning and Inference* **140**(1), 30 – 51.

Leonenko, N. N. & Šuvak, N. (2010*b*), 'Statistical inference for Student diffusion process', *Stochastic Analysis abd Applictions* **28**(6), 972–1002.

Linetsky, V. (2004), 'The spectral decomposition of the option value', *International Journal of Theoretical and Applied Finance* **7**(03), 337–384.

Linetsky, V. (2007), 'Spectral methods in derivatives pricing', *Handbooks in Operations Research and Management Science* **15**, 223–299.

Magdziarz, M. (2009), 'Black-Scholes formula in subdiffusive regime', *Journal of Statistical Physics* **136**(3), 553–564.

Magdziarz, M. & Schilling, R. (2015), 'Asymptotic properties of Brownian motion delayed by inverse subordinators', *Proceedings of the American Mathematical Society* **143**, 4485–4501.

Mainardi, F. & Gorenflo, R. (2000), 'On Mittag-Leffler-type functions in fractional evolution processes', *Journal of Computational and Applied Mathematics* **118**(1-2), 283–299.

Markov, A. A. (1915), 'On a problem of Laplace', *Bulletin de l'Académie Impériale des Sciences, Petrograd, VI Série* **9**, 87–104.

McKean, H. P. (1956), 'Elementary solutions for certain parabolic partial differential equations', *Transactions of the American Mathematical Society* **82**(2), 519–548.

Meerschaert, M. M., Nane, E. & Xiao, Y. (2009), 'Correlated continuous time random walks', *Statistics & Probability Letters* **79**(9), 1194–1202.

Meerschaert, M. M. & Scalas, E. (2006), 'Coupled continuous time random walks in finance', *Physica A: Statistical Mechanics and its Applications* **370**(1), 114 – 118.

Meerschaert, M. M. & Scheffler, H. (2004), 'Limit theorems for continuous-time random walks with infinite mean waiting times', *Journal of Applied Probability* (3), 623–638.

Meerschaert, M. M. & Scheffler, H.-P. (2008), 'Triangular array limits for continuous time random walks', *Stochastic Processes and their Applications* **118**(9), 1606–1633.

Meerschaert, M. M. & Sikorskii, A. (2011), *Stochastic Models for Fractional Calculus*, De Gruyter.

Metzler, R. & Klafter, J. (2000), 'The random walk's guide to anomalous diffusion: a fractional dynamics approach', *Physics Reports* **339**(1), 1 – 77.

Metzler, R. & Klafter, J. (2004), 'The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics', *Journal of Physics A* **37**(1), 161–208.

Montroll, E. W. & Weiss, G. H. (1965), 'Random walks on lattices. ii', *Journal of Mathematical Physics* **6**(2), 167–181.

Olver, F., Lozier, D., Boisvert, R. & Clark, C. (2010), *NIST Handbook of Mathematical Functions*, 1 pap/cdr edn, Cambridge University Press.

Pazy, A. (1983), *Semigroups of Linear Operator and Applications to Partial Differential Equations*, Springer-Verlag.

Pearson, K. (1895), 'Contributions to the mathematical theory of evolution, ii: Skew variation in homogeneous material', *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **186**, 343–414.

Pearson, K. (1901), 'Mathematical contributions to the theory of evolution, x: Supplement to a memoir on skew variation', *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **197**, 443–459.

Pearson, K. (1914), *Tables for Statisticians and Biometricians, Part I*, Biometrics Laboratory, University College London.

Pearson, K. (1916), 'Mathematical contributions to the theory of evolution, xix: Second supplement to a memoir on skew variation', *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **216**, 429–457.

Podlubny, I. (1998), *Fractional Differential Equations: an Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications*, Vol. 198, Elsevier.

Popov, A. Y. & Sedletskii, A. M. (2011), 'Distribution of roots of Mittag-Leffler functions', *Contemporary Mathematics. Fundamental Directions* **40**, 3–171.

Reed, M. & Simon, B. (1980), *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, London.

Rogers, L. & Williams, D. (1994), *Diffusions, Markov Processes, and Martingales*, Vol. 1, John Wiley & Sons.

Rudin, W. (1987), *Real and Complex Analysis*, 3 edn, McGraw-Hill.

Samorodnitsky, G. & Taqqu, M. (1994), *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance (Stochastic Modeling Series)*, 1 edn, Chapman and Hall/CRC.

Scalas, E. (2006), Five years of continuous-time random walks in econophysics, *in* 'The complex Networks of Economic Interactions', Springer, pp. 3–16.

Scalas, E. & Viles, N. (2014), 'A functional limit theorem for stochastic integrals driven by a time-changed symmetric α-stable Lévy process', *Stochastic Processes and their Applications* **124**(1), 385–410.

Scher, H. & Lax, M. (1973), 'Stochastic transport in a disordered solid. i. theory', *Physical Review B* **7**(10), 4491–4502.

Schumer, R., Benson, D. A., Meerschaert, M. M. & Baeumer, B. (2003), 'Fractal mobile/immobile solute transport', *Water Resources Research* **39**(10).

Simon, T. (2014), 'Comparing Fréchet and positive stable laws', *Electronic Journal of Probability* **19**(16), 1–25.

Skorokhod, A. V. (1956), 'Limit theorems for stochastic processes', *Theory of Probability & Its Applications* **1**(3), 261–290.

Skorokhod, A. V. (1957), 'Limit theorems for stochastic processes with independent increments', *Theory of Probability & Its Applications* **2**(2), 138–171.

Slater, L. J. (1960), *Confluent Hypergeometric Functions*, Cambridge University Press.

Slater, L. J. (1966), *Generalized Hypergeometric Functions*, Cambridge University Press.

Stanislavsky, A. A. (2009), 'Black-Scholes model under subordination', *Physica A* **18**(1), 469–474.

Straka, P. & Henry, B. (2011), 'Lagging and leading coupled continuous time random walks, renewal times and their joint limits', *Stochastic Processes and their Applications* **121**(2), 324 – 336.

Uhlenbeck, G. & Ornstein, L. (1930), 'On the theory of the Brownian motion', *Physical Review* **36**, 823–841.

Vasicek, O. (1977), 'An equilibrium characterization of the term structure', *Journal of financial economics* **5**(2), 177–188.

Wiman, A. (1905), 'Über den Fundamentalsatz in der Teorie der Funktionen $E_a(x)$', *Acta Mathematica* **29**(1), 191–201.

Wong, E. (1964), 'The construction of a class of stationary Markoff processes', *Sixteen Symposium in Applied mathematics - Stochastic Processes in mMthematical Physics and Engineering* (16), 264–276.

Wyłomańska, A. & Gajda, J. (2016), 'Stable continuous-time autoregressive process driven by stable subordinator', *Physica A: Statistical Mechanics and its Applications* **444**, 1012–1026.

Zolotarev, V. M. (1986), *One-dimensional Stable Distributions*, Vol. 65, American Mathematical Soc.

Curriculum vitae

Ivan Papić was born on September 22, 1989 in Osijek, Croatia. After finishing high school in Osijek, he enroled in the undergraduate program in Mathematics at the Department of Mathematics, J. J. Strossmayer University of Osijek. In October 2013, he obtained his master's degree Financial and Business Mathematics at the Department of Mathematics, University of Osijek. In November 2013, he enroled Croatian doctoral program in Mathematics at the Department of Mathematics, Faculty of Science at the University of Zagreb. From May 2014 he has been employed as a teaching assistant at the Department of Mathematics, University of Osijek. During his career he visited the School of Mathematics at the University of Cardiff, UK as a visiting researcher at several occasions and also participated in other programmes of professional development. He attended four international conferences so far, giving a talk at three of them and poster presentation at one. He was one of the collaborators in two scientific projects, "Fractional Pearson Diffusions" and "Stochastic models with long-range dependence" at University of Osijek under the leadership of Nenad Šuvak. He received a scientific award from United Kingdom Association of Alumni and Friends of Croatian Universities. In 2016 he was granted Erasmus programme scholarship for staff mobility.

List of publications

Journal Publications:

1. Graovac, N., Papić, I., Merdić, E. (2015), Pupil's Diet-Related Attitudes to Healthy Lifestyle, *Journal of Environmental Science and Engineering A*, 4, 651-664.

2. Leonenko, N. N., Papić, I., Sikorskii, A., Šuvak, N. (2017), Heavy-tailed fractional Pearson diffusions, *Stochastic Processes and their Applications*, 127/11, 3512-3535.

3. Leonenko, N. N., Papić, I., Sikorskii, A., Šuvak, N. (2018), Correlated continuous time random walks and fractional Pearson diffusions, *Bernoulli*, 24/4B, 3603-3627.

4. Leonenko, N. N., Papić, I., Sikorskii, A., Šuvak, N. (2018), Ehrenfest-Brillouin-type correlated continuous time random walk and fractional Jacobi diffusion, *Theory of Probability and Mathematical Statistics*, *accepted for publication*

5. Kulik, A., Leonenko, N. N., Papić, I., Šuvak, N. (2018), Non-stationary Fisher-Snedecor diffusion, *submitted*

6. Leonenko, N. N., Papić, I. (2018), Correlation properties of continuous-time autoregressive processes delayed by the inverse of the stable subordinator, *submitted*

Refereed Proceedings:

1. Leonenko, N. N., Papić, I., Sikorskii, A., Šuvak, N. (2017), Theoretical and simulation results on heavy-tailed fractional Pearson diffusions, *Proceedings of the 20th European Young Statisticians Meeting*, Wiklund T. (ed.), Uppsala University, Department of Mathematics, 95-103.

Others:

1. Jankov Maširević, D., Papić, I. (2012), Tri klasična problema, *Osječki matematički list* 12, 11-19.