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FRAKTALNA ANALIZA NEOMEĐENIH SKUPOVA U EUKLIDSKIM PROSTORIMA I LAPIDUSOVE ZETA FUNKCIJE

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University of Zagreb
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FRACTAL ANALYSIS OF UNBOUNDED SETS IN EUCLIDEAN SPACES AND LAPIDUS ZETA FUNCTIONS

DOCTORAL THESIS

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prof. dr. sc. Darko Žubrinić

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Summary

In this thesis, we consider *relative fractal drums* and their corresponding *Lapidus fractal zeta functions*, as well as a generalization of this notions to the case of *unbounded sets at infinity*. Relative fractal drums themselves are a generalization of the notion of a bounded subset in an Euclidean space. Here, we continue the ongoing research into their properties and the higher-dimensional theory of their fractal zeta functions and *complex dimensions* which started as a collaboration between M. L. Lapidus and D. Žubrinić in 2009 with the later addition of the author of this thesis.

The theory of complex dimensions is already well developed for *fractal strings*; that is, for fractal subsets of the real line. The complex dimensions of a relative fractal drum are defined as poles of a meromorphic continuation of its corresponding *distance* or *tube zeta function*. Complex dimensions of a relative fractal drum generalize, in a way, the notion its *box* (or *Minkowski*) *dimension*. More precisely, under some mild conditions, the value of the box dimension of a relative fractal drum is a pole of its corresponding fractal zeta function with maximal real part. Moreover, the residue computed at this pole is closely related to its *Minkowski content*.

Here we derive important results which further justify the notion of 'complex dimensions' and connect it to fractal properties of a given relative fractal drum. More precisely, we establish *fractal tube formulas* for a class of relative fractal drums which express their *relative tube function*; that is, the Lebesgue measure of their relative δ-neighborhood for small values of δ, as a sum over the residues of their fractal zeta function. These formulas are given with or without an error term and hold pointwise or distributionally depending on the growth properties of the corresponding fractal zeta function. The importance of these formulas is that they show how the complex dimensions are related to the asymptotic development of the relative tube function of a given relative fractal drum. As an application we derive a *Minkowski measurability* criterion for a large class of relative fractal drums. Furthermore, we also show that the complex dimensions of a relative fractal drum are, as expected, invariant to the dimension of the ambient space.

We introduce a further generalization of the theory of complex dimensions to the context of *unbounded sets at infinity* which can be used as a new approach of applying fractal analysis to unbounded subsets in Euclidean spaces. This is done for unbounded sets of finite Lebesgue measure by introducing the notions of *Minkowski content at infinity* and

Minkowski (or *box*) *dimension at infinity* which describe their fractal properties. Furthermore, we proceed by introducing an appropriate *Lapidus* (or *distance*) *zeta function at infinity* and show that it is well connected with the fractal properties of unbounded sets. We proceed by constructing interesting examples of quasiperiodic sets at infinity with arbitrary number (even infinite) of quasiperiods that exhibit complex fractal behavior.

We also address the natural question which arises when dealing with unbounded sets and their fractal properties; that is, establish results about the fractal properties of their images under the one-point compactification and under the geometric inversion. Furthermore, we also investigate fractal properties of unbounded sets of infinite Lebesgue measure by introducing notions of the parametric *ϕ-shell Minkowski content at infinity* and the corresponding parametric *ϕ-shell Minkowski* (or *box*) *dimension at infinity* and we establish results connecting these notions with the distance zeta function at infinity.

Finally we demonstrate how fractal analysis of unbounded sets via the geometric inversion may be applied to investigate bifurcations of dynamical systems occurring at infinity.

Keywords: fractal set, relative fractal drum, fractal zeta functions, distance zeta function, tube zeta function, shell zeta function, geometric zeta function of a fractal string, Minkowski content, Minkowski measurability, upper box (or Minkowski) dimension, complex dimensions of a fractal set, holomorphic and meromorphic functions, abscissa of convergence, quasiperiodic function, quasiperiodic set, order of quasiperiodicity, Mellin transform, fractal tube formula, Hopf bifurcation, polynomial vector field

Sažetak

U ovoj disertaciji bavimo se *relativnim fraktalnim bubnjevima* i njihovim *fraktalnim zeta funkcijama Lapidusovog tipa*, kao i generalizacijama ovih pojmova za slučaj neomeđenih skupova u beskonačnosti. Relativni fraktalni bubnjevi su sami po sebi generalizacija pojma omeđenog skupa u Euklidskom prostoru. Ovdje nastavljamo istraživanje njihovih svojstava i višedimenzionalne teorije njihovih fraktalnih zeta funkcija te pripadajućih *kompleksnih dimenzija* koje je započeto suradnjom M. L. Lapidusa i D. Žubrinića 2009. godine a kojoj se autor disertacije pridružio nešto kasnije.

Teorija kompleksnih dimenzija već je vrlo dobro razvijena za slučaj *fraktalnih struna*, odnosno, fraktalnih podskupova realnog pravca. Kompleksne dimenzije relativnog fraktalnog bubnja definirane su kao polovi meromorfnog proširenja pripadajuće *razdaljinske* ili *cijevne zeta funkcije*. Na određeni način kompleksne dimenzije relativnog fraktalnog bubnja generaliziraju pojam njegove *box dimenzije* (ili *dimenzije Minkowskog*). Preciznije, uz neke blage uvjete, vrijednost box dimenzije relativnog fraktalnog bubnja jest pol njegove pripadajuće fraktalne zeta funkcije s maksimalnom vrijednošću realnog dijela. Štoviše, reziduum u tom polu usko je povezan sa *sadržajem Minkowskog* danog relativnog fraktalnog bubnja.

U ovoj radnji izvodimo važne rezultate koji donose daljnje opravdanje pojma 'kompleksnih dimenzija' i povezuju ga s fraktalnim svojstvima danog relativnog fraktalnog bubnja. Preciznije, kao rezultat dobivamo *fraktalne cijevne formule* za klasu relativnih fraktalnih bubnjeva koje izražavaju njihovu *relativnu cijevnu funkciju*, odnosno, Lebesgueovu mjeru njihove relativne δ-okoline za male vrijednosti δ, kao sumu po reziduumima njihove fraktalne zeta funkcije. Te formule su dane s greškom ili bez greške i vrijede po točkama ili distribucijski ovisno svojstvima rasta pripadajuće fraktalne zeta funkcije. Važnost ovih formula je u tome što pokazuju kako su kompleksne dimenzije povezane s asimptotikom relativne cijevne funkcije danog relativnog fraktalnog bubnja. Kao primjenu izvodimo kriterij za *Minkowskivljevu izmjerivost* velike klase relativnih fraktalnih bubnjeva. Nadalje, očekivano, pokazujemo da su kompleksne dimenzije danog relativnog fraktalnog bubnja invarijantne u odnosu na dimenziju ambijentnog prostora.

U nastavku radnje uvodimo generalizaciju teorije kompleksnih dimenzija u kontekstu *neomeđenih skupova u beskonačnosti* koja može poslužiti kao novi pristup primjeni fraktalne analize na neomeđene skupove u Euklidskim prostorima. U slučaju neomeđenih skupova konačne Lebesgueove mjere, generalizaciju provodimo uvođenjem pojmova *sadržaja Minkowskog u beskonačnosti* i *box dimenzije u beskonačnosti* (ili *dimenzije Minkowskog u beskonačnosti*) koji opisuju njihova fraktalna svojstva. Nadalje, uvodimo i pripadajuću *Lapidusovu* (ili *razdaljinsku*) *zeta funkciju u beskonačnosti* te pokazujemo da je dobro povezana s fraktalnim svojstvima neomeđenih skupova. Nastavljamo s konstrukcijom zanimljivih primjera kvaziperiodičkih skupova u beskonačnosti s proizvoljnim brojem (moguće i beskonačnim) kvaziperioda koji posjeduju složena fraktalna svojstva.

Također se bavimo i prirodnim pitanjem koje se postavlja prilikom istraživanja neomeđenih skupova i njihovih fraktalnih svojstava, u vidu pronalaženja rezultata koji ih povezuju s fraktalnim svojstvima njihovih slika po jednotočkovnoj kompaktifikaciji i po geometrijskoj inverziji. Nadalje, također istražujemo i fraktalna svojstva neomeđenih skupova beskonačne Lebesgueove mjere uvođenjem pojmova parametarskog *ϕ-omotačkog sadržaja Minkowskog u beskonačnosti* i pripadajuće parametarske *ϕ-omotačke dimenzije Minkowskog u beskonačnosti* (ili *ϕ-omotačke box dimenzije u beskonačnosti*) te izvodimo rezultate koji povezuju ove pojmove s razdaljinskom zeta funkcijom u beskonačnosti.

Naposljetku, demonstriramo kako se fraktalna analiza neomeđenih skupova preko geometrijske inverzije može primijeniti u istraživanju bifurkacija dinamičkih sustava koje se događaju u beskonačnosti.

Ključne riječi: fraktalni skup, relativni fraktalni bubanj, fraktalna zeta funkcija, razdaljinska zeta funkcija, cijevna zeta funkcija, omotačka zeta funkcija, geometrijska zeta funkcija fraktalne strune, sadržaj Minkowskog, Minkowskivljeva izmjerivost, gornja box (ili Minkowskivljeva) dimenzija, kompleksne dimenzije fraktalnog skupa, holomorfne i meromorfne funkcije, abscisa konvergencije, kvaziperidička funkcija, kvaziperiodički skup, red kvaziperiodičnosti, Mellinova transformacija, fraktalna cijevna formula, Hopfova bifurkacija, polinomijalno vektorsko polje

Contents

Chapter 1

Introduction

1.1 Motivation and Goals

In this dissertation we will provide a potentially useful extension of the theory of fractal zeta functions of bounded subsets and, more generally, of relative fractal drums in Euclidean spaces which was developed in [LapRaŽu1]. The theory of [LapRaŽu1] is in itself a significant extension of the theory of fractal zeta functions for fractal strings which has been developed by M. L. Lapidus and his collaborators in the last two decades. Fractal strings can be viewed as objects that are generated by certain fractal sets on the real line. More precisely, for $A \subset \mathbb{R}$ of zero Lebesgue measure, a fractal string is defined as the sequence of the finite lengths of the complementary intervals of the set A; see [Lap–vFr1–3] and the relevant references therein. In this thesis, when referring to a fractal set, we actually mean any nonempty bounded subset A of the N-dimensional Euclidean space $\mathbb{R}^N$, with $N \geq 1$. The attribute 'fractal' actually means that the basic tool, when studying the set A, is the notion of fractal dimension. As it turns out, the one[1] that best suits this theory is the (upper) box dimension (also called the Minkowski dimension, Bouligand dimension, limit capacity, etc.). Furthermore, the value of the Minkowski content of a bounded subset A of $\mathbb{R}^N$ is also referred to as a 'fractal property' and can be used as one of the equivalent ways to define the box dimension. More precisely, for a bounded subset A of $\mathbb{R}^N$ and $0 \leq r \leq N$ we denote its r-dimensional Minkowski content by

$$\mathcal{M}^r(A) := \lim_{\delta \to 0^+} \frac{|A_\delta|}{\delta^{N-r}}, \tag{1.1.1}$$

whenever this limit exists as a value in $[0, \infty]$. Here, $|\cdot|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$ and

$$A_\delta := \{x \in \mathbb{R}^N : d(x, A) < \delta\} \tag{1.1.2}$$

[1]There are several different notions of fractal dimension, e.g., Hausdorff dimension, packing dimension, etc. (see [Fal1]).

is the *δ-neighborhood* (or the *δ-parallel set*) of A with $d(x, A) := \inf\{|x - a| : a \in A\}$ being the Euclidean distance from x to A. The set A is said to be Minkowski measurable (of dimension r) if $\mathcal{M}^r(A)$ exists and satisfies $0 < \mathcal{M}^r(A) < \infty$. We point out here that there is no difference if we work with the closed δ-parallel set of A, that is, with $\overline{A_\delta} = \{x \in \mathbb{R}^N : d(x, A) \leq \delta\}$ instead of A_δ. This is a consequence of a nontrivial result that for every bounded subset A of $\mathbb{R}^N$ we have that $|\overline{A_\delta}| = |A_\delta|$, which was proved by Stachó in [Sta].[2]

It has been of considerable interest in the past to determine whether a set A is Minkowski measurable. One of the motivations is Mandelbrot's suggestion in [Man2] to use the Minkowski content as a characteristic for the texture of sets (see [Man1, §X]). Mandelbrot called the quantity $1/\mathcal{M}^r(A)$ the *lacunarity* of the set A and made an observation that for subsets of $\mathbb{R}^N$ small lacunarity corresponds to spatial homogeneity of the set, i.e., the set has small, uniformly distributed holes. On the other hand, large lacunarity corresponds to clustering of the set and large holes between different clusters. More on this can be found in [BedFi, Fr, Lap–vFr1] and in [Lap–vFr3, §12.1.3].

Particular attention to the notion of Minkowski content arose in connection to the (modified) Weyl–Berry conjecture[3] (see the formulation in [Lap1]) which was proved for subsets of $\mathbb{R}$ in 1993 by M. L. Lapidus and C. Pomerance [LapPo2]. This conjecture relates the spectral asymptotics of the Laplacian on a bounded open set and the Minkowski content of its boundary. A crucial part of this result was the characterization of Minkowski measurability of bounded subsets of $\mathbb{R}$ obtained in [LapPo2].[4] In particular, this led to an important reformulation of the Riemann hypothesis in terms of an inverse spectral problem for fractal strings; see [LapMa].

In the dimension one, the Minkowski content of a set A is completely determined by the fractal string which it generates. Furthermore, the particular geometric arrangement of the complementary intervals is irrelevant, which is in sharp contrast to the case of the Hausdorff measure. Fractal strings themselves have become an independent object of study and exhibit numerous applications to, for example, spectral geometry and number theory; see [Lap–vFr1–3] and the references therein. Furthermore, they made possible the introduction and development of a rigorous theory of complex dimensions. For this theory in a variety of situations see, for example [Lap6], [ElLapMRo], [HeLap], [Lap1–5], [Lap7–10], [LapLéRo], [LapMa], [LapPe1–3], [LapPeWi1–2], [LapPo1–3], [HerLap], [LapRo], [LapLu] and the relevant references therein.

The theory of complex dimensions has been generalized to higher-dimensions in the

[2] Equivalently, for the boundary of A_δ we have $|\partial A_\delta| = 0$. Note that, there exists a subset U of $\mathbb{R}^N$ such that $|\partial U| > 0$. Furthermore, since every unbounded set $A \subseteq \mathbb{R}^N$ can be partitioned into a countable union of bounded subsets, $|\partial A_\delta| = 0$ also holds in this case.

[3] For the original Weyl–Berry conjecture and its physical applications see Berry's papers [Berr1–2]. Furthermore, early mathematical work on this conjecture and its applications can be found in [BroCar, FlVa, Lap1, Lap3, LapPo2, LapPo3]. For a more extensive list of later work see [Lap–vFr3, §12.5].

[4] A new proof of this is given in [Fal2] and more recently in [RatWi2].

research monograph [LapRaŽu1] and in [LapRaŽu2–8]; that is, to the case of arbitrary compact subsets in Euclidean spaces of any dimension. The fractal zeta function on which this generalization is based was introduced in 2009 by M. L. Lapidus and its definition was inspired by a work of D. Žubrinić on singular sets of some spaces of functions (see [Žu1–3, Žu5]). For another higher-dimensional generalization of the theory of complex dimensions see [LapRoŽu], where the notion of *box-counting zeta function* is introduced.

More specifically, for a given bounded subset $A \subseteq \mathbb{R}^N$ we define its *distance zeta function* as

$$\zeta_A(s;\delta) := \int_{A_\delta} d(x,A)^{s-N}\,\mathrm{d}x, \tag{1.1.3}$$

for $\operatorname{Re} s$ sufficiently large and a fixed $\delta > 0$. It turns out that the dependence of ζ_A on δ is inessential since in the theory of complex dimensions we are, generally, interested in the poles of a meromorphic extension of $\zeta_A(s;\delta)$. Since for $\delta_1, \delta_2 > 0$ the difference $\zeta_A(s;\delta_1) - \zeta_A(s;\delta_2)$ is an entire function (see [LapRaŽu1]), we conclude that changing the δ does not affect the poles of a (possible) meromorphic extension of the distance zeta function in any way. We also point out that, without loss of generality, we could assume that A is an arbitrary compact subset of $\mathbb{R}^N$, since replacing A with $\overline{A}$ does not change the distance zeta function. Indeed, we have that $A_\delta = \big(\overline{A}\big)_\delta$ and $d(x,A) = d(x,\overline{A})$.

One generalization of the notion of Minkowski content and box dimension can be made for objects that we call *relative fractal drums*. A relative fractal drum is an ordered pair of subsets (A,Ω) of $\mathbb{R}^N$ such that Ω is Lebesgue measurable and of finite N-dimensional Lebesgue measure.[5] Furthermore, for a relative fractal drum (A,Ω) in $\mathbb{R}^N$ and $r \in \mathbb{R}$ we denote its r-dimensional relative Minkowski content by

$$\mathcal{M}^r(A,\Omega) = \lim_{\delta\to 0^+} \frac{|A_\delta \cap \Omega|}{\delta^{N-r}}, \tag{1.1.4}$$

whenever this limit exists as a value in $[0,\infty]$. Now, by using this notion, we can define the *relative box dimension of* (A,Ω) in a standard way, with values in $[-\infty, N]$. The novelty here is that we now let $r \in \mathbb{R}$, which is not a coincidence, since there exist relative fractal drums with negative box dimension as was demonstrated in [LapRaŽu1]. Furthermore, for a relative fractal drum (A,Ω) in $\mathbb{R}^N$, one defines its relative distance zeta function as

$$\zeta_A(s,\Omega) := \int_{\Omega} d(x,A)^{s-N}\,\mathrm{d}x, \tag{1.1.5}$$

for $\operatorname{Re} s$ sufficiently large. This, in turn, allows one to develop a theory of complex dimensions of relative fractal drums in a much the same manner as it is done for bounded subsets in $\mathbb{R}^N$. For a short exposition of this theory and its main results see Chapter 2 and [LapRaŽu1, LapRaŽu4–5] for details.

[5] There is also another mild technical condition on the pair (A,Ω) but we will leave out the details until Section 2.1.

In Chapter 3 we incorporate the most recent results (announced in [LapRaŽu7] and fully exposed in [LapRaŽu8]) concerning the problem of obtaining *fractal tube formulas*, for a class of relative fractal drums in terms of sums over the residues of their relative distance or tube zeta functions. By a fractal tube formula of the relative fractal drum (A, Ω) we mean an exact or asymptotic expansion of the relative tube function $t \mapsto |A_t \cap \Omega|$ when $t \to 0^+$. These formulas will hold pointwise or distributionally, depending on the growth properties of the corresponding relative zeta function. These results extend the corresponding ones obtained in [Lap–vFr1–3] for fractal strings. We refer to [Lap–vFr3, §13.1] for many additional references on tube formulas in various settings, including, [DeKÖÜ, Gra, HuLaWe, Schn, Zäh, LapPe3, LapPeWi1, LapPeWi2].

In the rest of the thesis we will mainly be concerned in extending the theory of complex dimensions (with appropriate definitions) to the case of unbounded subsets of $\mathbb{R}^N$. There are two different (but, in a way, related) approaches to this problem. One of them was explored in the paper [RaŽuŽup], where for an unbounded subset $A \subseteq \mathbb{R}^N$ the fractal properties of its image $\Phi(A)$ under the geometric inversion $\Phi(x) := x/|x|^2$ in $\mathbb{R}^N$ were applied in investigating bifurcations of some polynomial vector fields in $\mathbb{R}^2$. There it was also shown that this approach is equivalent to studying the fractal properties of the image of $\Psi(A)$ on the Riemann sphere $\mathbb{S}^2$ under the stereographic projection $\Psi\colon \mathbb{R}^2 \to \mathbb{S}^2$. The contents of [RaŽuŽup] is incorporated into Chapter 5 of this dissertation. We point out that Section 5.2 is expanded, in comparison to the original paper, with results concerning the stereographic projection in $\mathbb{R}^N$ and put into the context of some of the results of the previous chapters of the thesis. Furthermore, Section 5.3 contains completely new results and relates the two different approaches in analyzing fractal properties of unbounded sets.

The second approach, covered in Chapter 4, deals with the notion of a "fractal set at infinity". More precisely, D. Žubrinić suggested to try to analyze an unbounded Lebesgue measurable set Ω of finite N-dimensional Lebesgue measure by means of its *tube function at infinity* which is defined as

$$t \mapsto |B_t(0)^c \cap \Omega|, \tag{1.1.6}$$

where $B_t(0)^c$ denotes the complement of the open ball in $\mathbb{R}^N$ of radius t with center at the origin. Also, a suggestion by D. Žubrinić was to define a *Lapidus* (*distance*) *zeta function of* Ω *at infinity* by replacing the integrand in (1.1.5) by some suitably chosen power of $|x|$. As it turned out, the right way to define the Lapidus zeta function of Ω at infinity was

$$\zeta_\infty(s, \Omega) := \int_\Omega |x|^{-s-N}\,\mathrm{d}x, \tag{1.1.7}$$

for $\operatorname{Re} s$ sufficiently large. Furthermore, as will be shown in Chapter 4, this definition is perfectly in accordance with the (also new) notion of the *r-dimensional Minkowski content* *of* Ω *at infinity* defined by

$$\mathcal{M}^r(\infty,\Omega) := \lim_{t\to+\infty}\frac{|B_t(0)^c\cap\Omega|}{t^{N+r}} \tag{1.1.8}$$

for $r\in\mathbb{R}$ whenever it exists and also with the notion of *box dimension of* Ω *at infinity* which it induces. Using these definitions, we will extend the theory of [LapRaŽu1] to the case of unbounded Lebesgue measurable subsets Ω of $\mathbb{R}^N$.

The notation of (1.1.8) suggests that a fractal set Ω at infinity may be understood as a special case of a relative fractal drum (A,Ω) where the set A has degenerated to a point at infinity. This is indeed the case and the fractal properties of this relative fractal drum will be closely related to the fractal properties of its 'inverted relative fractal drum'; that is, of $(\{\mathbf{0}\},\Phi(\Omega))$. Hence, therein lies the relation with the first approach to fractal analysis of unbounded sets mentioned above and exposed in Chapter 5. In light of this, it will be no surprise that the box dimensions of unbounded sets at infinity will always be nonpositive[6] or, more precisely, less than or equal to $-N$.

The examples of relative fractal drums of type (∞,Ω) presented in this thesis will provide some interesting insights into the notion of 'fractality' or rather, 'relative fractality'.[7] Namely, although the 'fractal set' A has degenerated to a point at infinity (and thus, one would not expect it to be fractal in any way; that is, to have nontrivial fractal properties), we will show that the set Ω will be the source of fractality in this case. This will be fully demonstrated in Section 4.6 by constructing quasiperiodic sets at infinity and even a set Ω that is *maximally hyperfractal at infinity*.[8] The idea of this construction can also be applied to the case of ordinary relative fractal drums of form $(\{\mathbf{0}\},\Omega)$ in order to demonstrate the existence of such a complicated objects even though the set A consists only of a single point and thus, again, the source of fractality of $(\{\mathbf{0}\},\Omega)$ in this case is the set Ω. To demonstrate this, one could also apply the geometric inversion to quasiperiodic sets Ω at infinity constructed in Section 4.6 to get a quasiperiodic relative fractal drum $(\{\mathbf{0}\},\Phi(\Omega))$, although in this approach there are some loose ends that still need to be proved.

By looking at Equation (1.1.7); that is, at the definition of the distance zeta function at infinity, one can see that for it to make sense, it is not necessary for the set Ω to have finite Lebesgue measure, but rather, to be just Lebesgue measurable. This turned out to be nicely related to a new notion of a parametric Minkowski content introduced in Section 4.7. As he was unable to find a similar notion in the literature, the author suggests to call it the *r-dimensional ϕ-shell Minkowski content at infinity*, where, $\phi>1$

[6]The box dimension of a relative fractal drum of type $(\{\mathbf{0}\},\Omega)$ is at most equal to 0 since the set A here consists of a single point.

[7]For a discussion of the notion of fractality see [Lap–vFr3] and the relevant references therein.

[8]The notion of a (maximally) hyperfractal set was introduced in [LapRaŽu1] in terms of the corresponding fractal zeta function associated to that set. In a way, such sets exhibit the most complicated geometrical nature.

is a parameter and $r \in \mathbb{R}$. For a Lebesgue measurable subset $\Omega \in \mathbb{R}^N$ we define it as

$$\mathcal{M}_\phi^r(\infty, \Omega) := \lim_{t \to +\infty} \frac{|B_t(0)^c \cap B_{\phi t}(0) \cap \Omega|}{t^{N+r}} \tag{1.1.9}$$

whenever this limit exists. The idea to introduce this notion originally came into existence as a side effect of studying the connection between fractal properties of the relative fractal drum (∞, Ω) and the fractal properties of its image on the N-dimensional Riemann sphere under the stereographic projection Ψ. Furthermore, the notion of *ϕ-shell box dimension of Ω at infinity*, which the ϕ-shell Minkowski content at infinity induces, generalizes the already introduced definition of box dimension at infinity for sets of finite Lebesgue measure. Moreover, the sets of infinite Lebesgue measure will have their ϕ-shell box dimension (if it exists) always in the interval $[-N, 0]$ which fills out the 'dimensional gap' left over by the sets of finite Lebesgue measure.[9]

We point out that one can also define an analog of the ϕ-shell Minkowski content for relative fractal drums and study its properties. In particular, it would be interesting to fully relate this notion to the notion of *surface Minkowski content* studied in [RatWi1] and [RatWi2]. Some preliminary results about this problem (in the case of fractal sets at infinity) can be found in Section 4.8 but we leave out the rest for future work.

The motivation to study the fractal properties of unbounded sets comes from a variety of sources. In particular, the notion of "unbounded" or "divergent" oscillations appears in problems in oscillation theory (see, e.g. [Džu, Karp]), automotive industry (see, e.g., [SBOPQD]), civil engineering (see, e.g, [Pou]) and mathematical applications in biology (see, e.g., [May]). Unbounded (divergent) oscillations are oscillations the amplitude of which increases with time. For instance, the oscillations of an airplane that has positive static stability but negative dynamic stability is an example of divergent oscillations that appears in aerodynamics (see, e.g. [Dol]).

Furthermore, unbounded domains themselves are also interesting in the theory of elliptic partial differential equations. More precisely, the question of solvability of the Dirichlet problem for quasilinear equations in unbounded domains is addressed in [Maz1] and [Maz2, Section 15.8.1]. Also, unbounded domains can be found in other aspects of the theory of partial differential equations; see, for instance [An,Hur,Lan,Rab] and [VoGoLat]. Research dealing with unbounded domains of infinite volume can be found in [GeWe], and connected with that is the research dealing with cusp-shaped domains (see, e.g., [ExBa1–2]), which also appear in examples in this thesis. Furthermore, the new notion of the ϕ-shell Minkowski content could possibly have a connection to certain comparison principles for the p-Laplacian (see, e.g., [Ag,MarMizPin,PolSha] and the relevant references therein). Fractal properties of unbounded domains, studied here, could therefore have a future

[9] Recall that the sets of finite Lebesgue measure always have (if it exists) their box dimension at infinity less than or equal to $-N$.

impact and lead to a new approach to these problems.

1.2 The Thesis Overview

This section contains a brief overview of the thesis with the emphasis on the main results. The rest of the introductory chapter contains Section 1.3 where basic definitions and notation that will be used in the rest of the thesis is introduced. In Sections 1.4 and 1.5 we recall the well-known notions of Dirichlet-type integrals and almost periodic functions and distributions, respectively. The main results about these notions, which will be needed later, are listed and references to the literature are given.

Chapter 2 represents a brief overview of the main definitions and results from [LapRaŽu1] in order to give the reader an idea about the theory of complex dimensions of relative fractal drums and their fractal zeta functions. Most of the theory of Section 2.1 will be needed in Chapter 3. Furthermore, it also serves to put into perspective the generalization of the theory of complex dimensions and fractal zeta functions to the case of 'fractal sets at infinity' given in Chapter 4.

Chapter 3 represents the first main contribution of this thesis to the higher dimensional theory of complex dimensions and fractal zeta functions. In this chapter we derive 'fractal tube formulas' for a class of relative fractal drums. More precisely, for a relative fractal drum (A,Ω) we derive formulas that express the relative tube function $t\mapsto|A_t\cap\Omega|$ as an appropriate sum over the residues of its corresponding fractal zeta function. In a way, these formulas justify the term 'complex dimension' since we show that these complex dimensions are closely connected to the oscillatory nature of the geometry of the given relative fractal drum. This is analogous to the situation encountered in the one-dimensional case of fractal strings which is extensively studied in [Lap-vFr1–3]. (We draw the attention of the reader to [Lap–vFr3, Chapter 5 and 8] for the corresponding one-dimensional analog.)

The main result of Section 3.2 is Theorem 3.14, which provides a pointwise fractal tube formula with and without an error term depending on the growth properties of the corresponding relative tube zeta function. In order to weaken the growth conditions imposed in Theorem 3.14, we use the distributional approach in Section 3.3 and derive a fractal tube formula that holds distributionally (on an appropriate space of test functions). Thus, the main results of Section 3.3 are Theorem 3.20 which is the distributional analog of Theorem 3.14 and Theorem 3.22 which provides an estimate on the corresponding distributional error term.

In Section 3.4 we "translate" the results of Sections 3.2 and 3.3 in terms of the much more operable relative distance zeta function. In order to do so, we introduce a new type of fractal zeta function called the *relative shell zeta function* in Definition 3.23. The shell zeta function firstly appeared in Section 4.9 in order to deal with 'fractal sets at infinity' of infinite Lebesgue measure but proved very useful in Section 3.4. Namely, it enabled us to obtain the results with analogous growth conditions imposed on the relative distance zeta function as it was done in Sections 3.2 and 3.3 for the relative tube zeta function. We point out that this was not possible to do directly from the functional equation connecting the relative tube and distance zeta functions (Theorem 2.5). Furthermore, we derive the main properties of the relative shell zeta function as well as functional equations that connect it to the relative tube and distance zeta functions; see Theorems 3.24, 3.25 and 3.27. Finally, the main results of Section 3.4 are the pointwise and distributional tube formulas of Theorems 3.37 and 3.40, respectively.

In Section 3.5 we derive a criterion for Minkowski measurability of a large class of relative fractal drums in terms of their fractal zeta functions. The sufficiency part (Theorem 3.42) is a consequence of the Wiener–Pitt Tauberian theorem. In short, it states that an RFD is Minkowski measurable if the only pole of its corresponding fractal zeta function contained in the critical line is real and simple; that is, the pole is then equal to its relative box dimension. Moreover, Theorem 3.42 gives then a nice connection between the Minkowski content of the given RFD and the residue of the corresponding fractal zeta function computed at this pole. On the other hand, if, in addition, there are other poles on the critical line, the Wiener–Pitt Tauberian theorem gives an upper bound on the upper Minkowski content of the RFD under consideration (see Theorem 3.44).

In order to prove the other direction of the Minkowski measurability characterization we introduce a new fractal zeta function in Definition 3.46 which we call the *Mellin zeta function*. Its basic properties are expressed in Theorems 3.47, 3.49 and 3.50. This new zeta function is needed in order to extend the distributional tube formula of Theorem 3.40 to a larger space of test function which allows one to use the Uniqueness theorem for almost periodic distributions in the proof of Theorem 3.56. Finally, combining Theorem 3.42 and 3.56, the Minkowski measurability criterion is obtained in Theorem 3.58.

In Section 3.6 we give several interesting examples and applications of the theory developed so far. Notable is the example of an RFD based on the Cantor's function graph (Example 3.69). As it is well-known, the box dimension of the Cantor's function graph is trivial; that is, equal to one and the graph is rectifiable. One the other hand, one intuitively would still like to call this graph fractal for obvious reasons and Example 3.69 gives a sort of a justification for that. Namely, it shows that its relative zeta function has nonreal poles located to the left of the critical line $\{\mathrm{Re}\, s = 1\}$ and having real part equal exactly to the box dimension of the middle-third Cantor set; that is, to $\log_3 2$. From the theory developed in Chapter 3 we then deduce that these poles generate lower order oscillations of the relative tube function of the RFD associated to the graph of the Cantor function.

Alongside other examples of Section 3.6 we also analyze fractal nests (Example 3.70) and unbounded geometric chirps (Example 3.75) which are not self-similar. The example of a fractal nest depending on a real parameter exhibits an interesting phenomena of two simple complex dimensions (i.e., simple poles of the associated fractal zeta function) that "merge" for a particular value of the parameter and form a single complex dimension of second order (i.e., a pole of second order). This second order complex dimension generates then logarithmic terms in the asymptotic expansion of the associated relative tube zeta function. A general result about a class of RFDs that have a single complex dimension of higher order located on the critical line is given in Theorems 3.73 and 3.74. In short, such an RFD is then Minkowski degenerate with Minkowski content equal to $+\infty$ but it is h-Minkowski measurable where h is an appropriate gauge function. More precisely, $h(t) = (\log t^{-1})^{m-1}$ for $t \in (0, 1)$ and m is the order of the associated complex dimension having maximal real part. Furthermore, an explicit formula for the h-Minkowski content is also given.

Towards the end of Section 3.6 we show how some already established results about complex dimensions of *self-similar sprays* (see [LapPe3, DeKÖÜ]) can be recovered from the results of Chapter 3. Finally, Example 3.76 provides an explicit construction of a relative fractal drum of $\mathbb{R}$ that possesses an infinite set of poles of arbitrary order or even essential singularities located on the critical line in arithmetic progression. The example is based on an "iterated Cantor spray" but it is clear that a similar iterated construction can be applied to any RFD of $\mathbb{R}^N$.

In Section 3.7 we show that the complex dimensions of RFDs are preserved by embedding them into higher-dimensional spaces. This represents a generalization of [Res1] where the independence of the normalized Minkowski content on the dimension of the ambient space was established. Theorem 3.84 establishes a connection between the relative tube zeta function of the original RFD and the relative tube zeta function of its embedding in higher-dimensional spaces. As a consequence of the results about embeddings of RFDs we are able to extend the Minkowski measurability criterion of Theorem 3.58 to the special case when the box dimension of the RFD is equal to the dimension of the ambient space. This result is stated in Theorem 3.86. Another application of the results of Section 3.7 is in the fact that one can use them to determine the possible complex dimensions of special cases of higher-dimensional relative fractal drums without explicitly computing their distance (or tube) zeta function. This application is nicely demonstrated in Example 3.91 where the (possible) complex dimensions of the Cantor dust are determined.

Chapter 4 represents the second main contribution of this thesis to the higher-dimensional theory of complex dimensions and fractal zeta functions. In Section 4.1 we introduce the notions of Minkowski content and box dimension of unbounded sets, of finite Lebesgue measure, *at infinity*, derive the basic properties of these notions and give a number of examples.

In Section 4.2 we introduce the *Lapidus* (*or distance*) *zeta function at infinity* and derive its basic properties as well as results that connect it to the notions introduced in

Section 4.1. Theorem 4.21 establishes a connection between the distance zeta function of an unbounded set Ω at infinity and the distance zeta function of a relative fractal drum which arises as the image of the set Ω under the geometric inversion in $\mathbb{R}^N$. The main result of Section 4.2 is Theorem 4.24 which establishes the half-plane of absolute convergence of the distance zeta function at infinity and is an analog of Theorem 2.6.

In Section 4.3 we introduce the tube zeta function at infinity and derive a functional equation which connects it to the distance zeta function at infinity; see Theorem 4.31. This result enables us to establish the analogs of Theorems 2.8 and 2.9 given in Theorems 4.32 and 4.33, respectively. Finally, we apply these results to establish Theorem 4.34 which connects the notion of Minkowski measurability of an unbounded set Ω at infinity and the Minkowski measurability of the relative fractal drum which arises as the image of the set Ω under the geometric inversion in $\mathbb{R}^N$.

Section 4.4 is dedicated to establishing sufficient conditions on unbounded sets of finite Lebesgue measure that will ensure that their fractal zeta functions at infinity have a meromorphic extension to a neighborhood of their critical line. Two results are given: Theorem 4.36 for a Minkowski measurable case and Theorem 4.49 for a Minkowski nonmeasurable case. Both of the theorems are analogs of the corresponding ones for relative fractal drums (see [LapRaŽu1]). Furthermore, a sufficient condition for Minkowski measurability at infinity is given in Theorem 4.38 by means of the Wiener–Pitt Tauberian theorem as well as a corresponding upper bound on the upper Minkowski content at infinity in Theorem 4.40.

In Section 4.5 we derive some useful properties of fractal zeta functions at infinity. One of them is the scaling property given in Proposition 4.51. The other property is a result given in Theorem 4.55 which enables us to replace the norm appearing in the integrand of the distance zeta function at infinity with a another norm while preserving the complex dimension up to an open right half-plane that is strictly larger than the half-plane of absolute convergence. The norms have to be equivalent in a stronger Hölder-type form than the usual norm equivalence (see Definition 4.54). This result proved to be very useful in calculating the distance zeta function at infinity for various unbounded sets and determining their complex dimensions. An analogous result can be also derived for the case of relative fractal drums.

In Section 4.6 we provide a construction of quasiperiodic sets at infinity based on a two parameter unbounded set introduced in Definition 4.59. The construction provides quasiperiodic sets at infinity with finite or infinite number of quasiperiods. Moreover, the case of infinite number of quasiperiods gives also an example of a *maximally hyperfractal set at infinity*; that is, an unbounded set such that its fractal zeta functions at infinity have the critical line as their natural boundary (see Theorem 4.64).

Furthermore, we distinguish cases of algebraically and transcendentally quasiperiodic sets at infinity and show that both families are infinite for any number (including infinite) of quasiperiods; see Theorems 4.75 and 4.81. The transcendental case is a consequence of the well-known Baker's theorem from number theory (recalled in Theorem 4.72) and the algebraical case is a consequence of Besicovitch's theorem about the rational independence of roots of prime numbers (recalled in Theorem 4.73).

In Section 4.7 we explore the connection between the fractal properties of unbounded sets at infinity and their images on the Riemann sphere; that is, we explore the effect of the one-point compactification of $\mathbb{R}^N$ on the fractal properties of unbounded sets. We introduce the notions of *spherical Minkowski content* and *spherical box dimension* and provide a general result in Theorem 4.84 that connects them to the notions of Minkowski content at infinity and box dimension at infinity, respectively. Inspired by the fact that the spherical Minkowski content is well defined even for unbounded sets of infinite Lebesgue measure we introduce a new notion of a parametric Minkowski content in Definition 4.86 which we call the *ϕ-shell Minkowski content* and which depends on a real parameter $\phi > 1$. A corresponding notion of a parametric (upper and lower) ϕ-shell box dimension is also introduced in Definition 4.92 and various properties of these notions are given in the rest of Section 4.7. We point out that analogous notions of parametric Minkowski content and parametric box dimension can be introduced for relative fractal drums (see Definition 4.96 and Proposition 4.98).

In Section 4.8 we introduce new notions of *surface Minkowski content at infinity* and *surface box dimension at infinity* for an unbounded Lebesgue measurable set and establish a preliminary connection between these notions and the notions of its ϕ-shell Minkowski content at infinity and its ϕ-shell box dimension at infinity (see Theorem 4.108).

In Section 4.9 we show that the distance zeta function at infinity retains good properties even for unbounded sets of infinite Lebesgue measure. Of course, in case of sets of infinite Lebesgue measure, we use the notion of the ϕ-shell Minkowski content at infinity instead of the usual Minkowski content at infinity since it is not defined. The main results of this section are the holomorphicity Theorem 4.117 which generalizes Theorem 4.24 and Theorems 4.121 and 4.122. We also introduce here the *shell zeta function at infinity* in Definition 4.118 to replace the tube zeta function at infinity which is not defined for sets of infinite Lebesgue measure and derive its basic properties in Theorems 4.119, 4.123 and 4.124.

Chapter 5 is basically the incorporation of the research paper [RaŽuŽup] into the broader context of the thesis. In Section 5.2 we analyze the effect of the geometric inversion on the fractal properties of sets at infinity. The section contains a new result not appearing in the paper given in Theorem 5.5 which connects the upper and lower Minkowski contents of a set near the origin with its upper and lower spherical Minkowski contents. In the rest of the section we analyze the basic properties of the box dimension of unbounded sets defined via the geometric inversion.

Section 5.3 is also a new addition not appearing in the original paper. Here, we give in

Theorem 5.19 a connection between the two approaches to fractal analysis of unbounded sets; that is, between the approach via the geometric inversion of Chapter 5 and the approach via 'fractality at infinity' of Chapter 4. In Section 5.4 we examine the effect of the Poincaré compactification on the fractal properties of a focus type spiral.

In Section 5.5 it is shown that every polynomial vector field P on $\mathbb{R}^N$ can be geometrically inverted; that is, there exist a polynomial vector field Q such that its phase portrait is equal to the geometric inversion of the phase portrait of P (see Lemma 5.23).

In Section 5.6 we analyze several examples of vector fields having a weak focus at infinity, including the example of the classical Hopf bifurcation (see Theorem 5.34), while in Section 5.7 we analyze the Hopf-Takens bifurcation at infinity (see Theorems 5.35 and 5.36).

Finally, in Appendix A we provide a list of open problems scattered throughout the thesis and indicate possible directions for further research.

1.3 Basic Definitions and Notation

Let us recall the definitions of Minkowski content and box dimension. By $|\Omega|$ we denote the N-dimensional Lebesgue measure of an open subset Ω of $\mathbb{R}^N$. Let A be a nonempty bounded subset of $\mathbb{R}^N$, A_δ the δ-neighborhood of A in the Euclidean metric and $r \in \mathbb{R}$. The *upper r-dimensional Minkowski content* of A is defined by

$$\overline{\mathcal{M}}^r(A) = \limsup_{\delta \to 0^+} \frac{|A_\delta|}{\delta^{N-r}}, \tag{1.3.1}$$

and we define analogously the *lower r-dimensional Minkowski content* of A, denoted by $\underline{\mathcal{M}}^r(A)$. The *upper box* (or *Minkowski*) *dimension* of A is defined by

$$\overline{\dim}_B A = \inf\{r \in \mathbb{R} \,:\, \overline{\mathcal{M}}^r(A) = 0\}; \tag{1.3.2}$$

it is easy to see that we also have

$$\overline{\dim}_B A = \sup\{r \in \mathbb{R} \,:\, \overline{\mathcal{M}}^r(A) = \infty\}. \tag{1.3.3}$$

The *lower box* (or *Minkowski*) *dimension* of A, denoted by $\underline{\dim}_B A$, is defined analogously, using $\underline{\mathcal{M}}^r(A)$ instead of $\overline{\mathcal{M}}^r(A)$ in (1.3.2) (and in (1.3.3)). If both dimensions $\overline{\dim}_B A$ and $\underline{\dim}_B A$ are equal, the common value is denoted by $\dim_B A$, and is called the *box dimension* of A (also known as Minkowski-Bouligand dimension, or limit capacity).

If there exists a $D \in \mathbb{R}$ such that $0 < \underline{\mathcal{M}}^D(A) \leq \overline{\mathcal{M}}^D(A) < \infty$, we say that A is *Minkowski nondegenerate*, and *Minkowski degenerate* otherwise. Note that if A is nondegenerate, it is easy to see that such a D is necessarily nonnegative and it then follows from (1.3.2)–(1.3.3) and their counterpart for $\underline{\mathcal{M}}^r(A)$ that $\dim_B A$ exists and is equal to D. Furthermore, if $\underline{\mathcal{M}}^D(A) = \overline{\mathcal{M}}^D(A)$, the common value is denoted by $\mathcal{M}^D(A)$, and called the *Minkowski content* of A. If moreover $\mathcal{M}^D(A) \in (0, \infty)$, then A is said to be *Minkowski measurable*.[10] For more information about these notions and their generalizations see [Fal1], [ŽuŽup1], [PaŽuŽup2] and [MaResŽup].

In the sequel we will use the following notation. If $f, g : \mathbb{R} \to (0, \infty)$ are two functions such that $f(t) \to 0$ and $g(t) \to 0$ as $t \to t_0$ (t_0 can be ∞ as well), we write $f(t) \sim g(t)$ as $t \to t_0$ if $\lim_{t \to t_0} \frac{f(t)}{g(t)} = 1$. We write $f(t) \asymp g(t)$ and say that f and g are *comparable* as $t \to t_0$ if there exist positive constants c_1 and c_2 such that $c_1 g(t) \leq f(t) \leq c_2 g(t)$ for all t in a neighborhood of t_0. A function $f : V \to \mathbb{R}^N$, $V \subseteq \mathbb{R}^N$, is said to be bi-Lipschitzian if $|f(a) - f(b)| \asymp |a - b|$ for all $a, b \in V$.

1.4 Dirichlet-type Integrals

The results and proofs of this subsection are reproduced and adapted from [LapRaŽu1] as they will be needed in the later sections. We recall that *the abscissa of convergence* $D(\zeta_\varphi) \in [-\infty, +\infty]$ of the *Dirichlet integral* (or rather, of the *Dirichlet-type integral*)

$$\zeta_\varphi(s) := \int_0^{+\infty} \varphi(x)^s \, \mathrm{d}x, \tag{1.4.1}$$

where φ is a suitable bounded and positive function on $(0, +\infty)$, is defined by

$$D(\zeta_\varphi) = \inf \left\{ \alpha \in \mathbb{R} : \int_0^{+\infty} \varphi(x)^\alpha \, \mathrm{d}x < +\infty \right\}. \tag{1.4.2}$$

It is then well-known that $\{\operatorname{Re} s > D(\zeta_\varphi)\}$[11] is the largest open right half-plane on which the Dirichlet integral converges absolutely. Hence, in this half-plane ζ_φ is holomorphic; see, e.g., [Pos]. Furthermore, we will call this half-plane the *half-plane of absolute convergence* of the Dirichlet-type integral ζ_φ and denote it as $\Pi(\zeta_\varphi)$. On the other hand, if φ is unbounded from above, and bounded from below by a positive constant, then we must replace the exponent s by $-s$, and α by $-\alpha$ in order to define the abscissa of convergence $D(\zeta_\varphi)$ and get a right half-plane of convergence $\{\operatorname{Re} s > D(\zeta_\varphi)\}$. We will call the set $\{\operatorname{Re} s = D(\zeta_\varphi)\}$ the *critical line* of the Dirichlet-type integral ζ_φ.

It is easy to extend this definition (and statement) to the case of *Dirichlet-type inte-*

[10] Minkowski measurability is easily seen to be equivalent to $|A_t| \sim C\, t^{N-D}$ as $t \to 0^+$, where $C \in (0, +\infty)$; then, we must have $\mathcal{M}^D(A) = C$. The notation $\sim$ is explained just below.

[11] Here, and in the remainder of this dissertation, we will slightly abuse notation in order to save space in the sense that $\{\operatorname{Re} s > D(\zeta_\varphi)\} := \{s \in \mathbb{C} : \operatorname{Re} s > D(\zeta_\varphi)\}$. Similar abuses for denoting right or left half-planes will also be made as well as for denoting vertical lines of type $\{s \in \mathbb{C} : \operatorname{Re} s = D\}$.

grals, or as we will call them in short, *DTIs*

$$\zeta_\varphi(s) = \int_E \varphi(x)^s \, \mathrm{d}\mu(x), \tag{1.4.3}$$

where (E, μ) is a measurable space, φ is a suitable positive (or more generally, under a suitable assumption, nonnegative, see Remark 1.4) measurable function on E which is $|\mu|$-essentially bounded, and μ is a positive or a complex measure on E.[12] Recall that if μ is positive, then $|\mu| = \mu$. Furthermore, if μ is a complex measure, then the total variation measure $|\mu|$ is defined as

$$|\mu|(E) := \sup \sum_i |\mu(E_i)| \tag{1.4.4}$$

where the supremum is taken over all partitions $\{E_i\}_{i\in I}$ of E into measurable subsets E_i. (Here, I is finite or countably infinite.) In particular, $|\mu|$ is a positive and bounded measure; see e.g. [Coh]. Moreover, if μ is absolutely continuous with respect to the Lebesgue measure; that is, if

$$\mu(E) = \int_E f(x) \, \mathrm{d}x \tag{1.4.5}$$

for some Lebesgue integrable function f and all measurable sets E, then we have

$$|\mu|(E) = \int_E |f(x)| \, \mathrm{d}x. \tag{1.4.6}$$

In this, more general setting, one defines the abscissa of (absolute) convergence of ζ_φ as

$$D(\zeta_\varphi) = \inf \left\{ \alpha \in \mathbb{R} : \int_0^{+\infty} \varphi(x)^\alpha \, \mathrm{d}|\mu|(x) < +\infty \right\}. \tag{1.4.7}$$

The statement about the convergence of the Dirichlet-type integral is a consequence of the following general result.

Theorem 1.1 (see, e.g., [LapRaŽu1])**.** *Let μ be a positive or complex measure on a measure space E, with total variation measure denoted by $|\mu|$, and let $\varphi : E \to (0, +\infty)$ be a measurable function.*[13] *Then:*

(*a*) *If φ is essentially bounded* (*that is, if there exists $C > 0$ such that $\varphi(t) \leq C$ for $|\mu|$-a.e. $t \in E$*)*, and if there exists $\sigma \in \mathbb{R}$ such that $\int_E \varphi(t)^\sigma \, \mathrm{d}|\mu|(t) < \infty$, then*

$$F(s) := \int_E \varphi(t)^s \, \mathrm{d}\mu(t) \tag{1.4.8}$$

[12]When $\varphi(x) = 0$, we let $\varphi(x)^s := 0$. (This is quite reasonable, at least for $\operatorname{Re} s > 0$.)

[13]See Remark 1.4 (and the text following it) for the case where $\varphi \geq 0$ μ-almost everywhere.

is holomorphic on the right half-plane $\{\operatorname{Re} s > \sigma\}$, *and*

$$F'(s) = \int_E \varphi(t)^s \log \varphi(t) \, \mathrm{d}\mu(t) \tag{1.4.9}$$

in that region;

(*b*) *if there exists* $C > 0$ *such that* $\varphi(t) \geq C$ *for* $|\mu|$*-a.e.* $t \in E$, *and if there exists* $\sigma \in \mathbb{R}$ *such that* $\int_E \varphi(t)^{-\sigma} \, \mathrm{d}|\mu|(t) < \infty$, *then*

$$G(s) := \int_E \varphi(t)^{-s} \, \mathrm{d}\mu(t) \tag{1.4.10}$$

is holomorphic on $\{\operatorname{Re} s > \sigma\}$, *and*

$$G'(s) = -\int_E \varphi(t)^{-s} \log \varphi(t) \, \mathrm{d}\mu(t) \tag{1.4.11}$$

in that region;

(*c*) *if there exist positive constants* C_1 *and* C_2 *such that* $C_1 \leq \varphi(t) \leq C_2$ *for* $|\mu|$*-a.e.* $t \in E$ *and there exists* $\sigma \in \mathbb{R}$ *such that* $\int_E \varphi(t)^{\sigma} \, \mathrm{d}|\mu|(t) < \infty$, *then* (1.4.8) *and* (1.4.10) *are entire functions.*

The above result follows from a more general and well-known result (see, e.g., [Carl, pp. 295–296] or [CarMi, pp. 152–153]), dealing with the holomorphicity of integrals depending on a parameter, of the form

$$H(s) = \int_E f(s,t) \, \mathrm{d}\mu(t). \tag{1.4.12}$$

We will state it here in a form that appears to be little known and is more convenient than the results of this type that are usually given in textbooks on complex analysis.

Theorem 1.2 (Cited from [Mattn]). *Let* $(E, \mathcal{E}, \mu)$ *be a measurable space where* μ *is a positive or complex measure. Furthermore, let* $U \subseteq \mathbb{C}$ *be open and let* $f \colon U \times E \to \mathbb{C}$ *be a function that satisfies the following assumptions*:

(*D*1) $f(\,\cdot\,, x)$ *is* μ*-measurable for every* $s \in U$,

(*D*2) $f(s, \cdot\,)$ *is holomorphic for* $|\mu|$*-a.e.* $x \in E$,

(*D*3) $\int_E |f(\,\cdot\,, x)| \, \mathrm{d}|\mu|(x)$ *is locally bounded, that is, for* $s_0 \in U$ *there exists* $\delta > 0$ *such that*

$$\sup_{s \in U, |s - s_0| < \delta} \int_E |f(s,x)| \, \mathrm{d}|\mu|(x) < \infty. \tag{1.4.13}$$

Then

$$F(s) := \int_E f(s,x) \, \mathrm{d}\mu(x) \tag{1.4.14}$$

is holomorphic on U and can be differentiated under the integral, i.e.,

$$F^{(n)}(s) = \int_E \frac{\partial^n}{\partial s^n} f(s,x)\,\mathrm{d}\mu(x) \tag{1.4.15}$$

Remark 1.3. The convenience in the above theorem is in the fact that condition $(D3)$ is usually easier to check than the standard condition given in literature, that for any $s_0 \in U$ there exists $\delta > 0$ such that

$$\int_E \sup_{s\in U, |s-s_0|<\delta} |f(s,x)|\,\mathrm{d}|\mu|(x) < \infty. \tag{1.4.16}$$

Furthermore, under hypothesis $(D1)$ and $(D2)$ the conditions $(D3)$ and (1.4.16) are, in fact, equivalent. Also, condition $(D3)$ can be weakened, again in the sense of greater convenience in checking it. Namely it is enough that the function $\int_E |f(\,\cdot\,,x)|\,\mathrm{d}|\mu|(x)$ is locally integrable with respect to the Lebesgue measure on U. The proofs of all of these statements can be found in [Mattn]. Although the results there are stated and proved in the case of a positive measure μ, it is clear that they extend to the case of a complex measure by means of the Hahn–Jordan decomposition of its real and imaginary parts.

Note that conditions $(D1)$ and $(D2)$, appearing in Theorem 1.2, imply that the complex-valued function $f(s,x)$ satisfies the well-known *Carathéodory conditions*, that is, $f(s,x)$ is continuous with respect to $s \in V$ for $|\mu|$-a.e. $x \in E$, and μ-measurable with respect to $x \in E$ for all $s \in V$.

Proof of Theorem 1.1. Note that in our case, $f(s,t) := \varphi(t)^s$ and $U := \{\operatorname{Re} s > \sigma\}$ and it is clear that f satisfies conditions $(D1)$ and $(D2)$ of Theorem 1.2. Furthermore, note that for any $s \in \mathbb{C}$ such that $\operatorname{Re} s > \sigma$,

$$\int_E |\varphi(t)^s|\,\mathrm{d}|\mu|(t) = \int_E \varphi(t)^{\operatorname{Re} s}\,\mathrm{d}|\mu|(t) \le \|\varphi\|_\infty^{\operatorname{Re} s-\sigma} \int_E \varphi(t)^\sigma\,\mathrm{d}|\mu|(t).$$

The right-hand side above is obviously locally bounded on the set U. More precisely, for a compact set $K \subseteq U$ we have that $\max_{s\in K} \|\varphi\|_\infty^{\operatorname{Re} s-\sigma} < \infty$ and, by hypothesis, $\int_E \varphi(t)^\sigma\,\mathrm{d}|\mu|(t) < \infty$. Since the condition $(D3)$ of Theorem 1.2 is satisfied we have proved case (a). Case (b) follows from (a) applied to $\varphi(t)^{-1}$. Finally, case (c) follows analogously as case (a) the only difference being that this time we have

$$\varphi(t)^{\operatorname{Re} s} \le \max\{C_1^{\operatorname{Re} s-\sigma}, C_2^{\operatorname{Re} s-\sigma}\}\varphi(t)^\sigma$$

for every $s \in \mathbb{C}$ and $|\mu|$-a.e. t, which yields that

$$\int_E |\varphi(t)^s|\,\mathrm{d}|\mu|(t) \le \max\{C_1^{\operatorname{Re} s-\sigma}, C_2^{\operatorname{Re} s-\sigma}\} \int_E \varphi(t)^\sigma\,\mathrm{d}|\mu|(t)$$

and the right-hand side is locally bounded on $\mathbb{C}$. Finally, for the claim about the function $G(s)$ defined by (1.4.10), we apply the same reasoning as above to the function $\varphi(t)^{-1}$. $\square$

Remark 1.4. Theorem 1.1 extends without any difficulties if $\varphi(t) \geq 0$ for μ-a.e. $t \in E$ and $|\mu|(\{t \in E : \varphi(t) = 0\}) = 0$.

For a DTI ζ_φ it makes sense to introduce the following definitions (as it was done in [LapRaŽu1]).

Definition 1.5. Let ζ_φ be a DTI. We define its *abscissa of holomorphic convergence* by

$$D_{\rm hol}(\zeta_\varphi) := \inf\{\alpha : \zeta_\varphi \text{ is holomorphic on } \{\operatorname{Re} s > \alpha\}\}, \tag{1.4.17}$$

and the *half-plane of holomorphic convergence* $\mathscr{H}(\zeta_\varphi) = \{\operatorname{Re} s > D_{\rm hol}(\zeta_\varphi)\}$ as the largest open half-plane on which ζ_φ is holomorphic, or more precisely, to which ζ_φ possesses a (necessarily unique) analytic continuation. Furthermore, we define its *abscissa of meromorphic convergence* by

$$D_{\rm mer}(\zeta_\varphi) := \inf\{\alpha : \zeta_\varphi \text{ is meromorphic on } \{\operatorname{Re} s > \alpha\}\}. \tag{1.4.18}$$

Associated with ζ_φ is also the *half-plane of meromorphic convergence* $\operatorname{Mer}(\zeta_\varphi) = \{\operatorname{Re} s > D_{\rm mer}(\zeta_\varphi)\}$; that is, the largest open right-half plane to which ζ_φ possesses a (necessarily unique) meromorphic continuation.

It is clear that for every DTI ζ_φ we have

$$D_{\rm mer}(\zeta_\varphi) \leq D_{\rm hol}(\zeta_\varphi) \leq D(\zeta_\varphi) \tag{1.4.19}$$

or, equivalently,

$$\Pi(\zeta_\varphi) \subseteq \mathscr{H}(\zeta_\varphi) \subseteq \operatorname{Mer}(\zeta_\varphi). \tag{1.4.20}$$

Furthermore, these inequalities are sharp. For this result and other properties, as well as generalizations of DTIs see [LapRaŽu1, Appendix A].

The following lemma is a simple consequence of the principle of analytic continuation. We state it here as it will be often used for proving the forthcoming results. Furthermore, we point out that from now on, we will refer to the set of poles of a meromorphic function as a multiset of poles whenever we count their multiplicities (or orders).

Lemma 1.6 (Cited from [LapRaŽu1]). *Assume that $\zeta_1(s)$ is a Dirichlet-type integral with abscissa of convergence equal to $D(\zeta_1)$, such that it possesses a meromorphic extension to $\{\operatorname{Re} s > a_1\}$, where $a_1 \in [-\infty, D(\zeta_1))$. Assume that $\zeta_2(s)$ is a function holomorphic on the right half-plane $\{\operatorname{Re} s > a_2\}$ such that $a_1 \leq a_2 < D(\zeta_1)$. Then, $\zeta_{\rm pert}(s) := \zeta_1(s) +$*

$\zeta_2(s)$ possesses a unique meromorphic extension (at least) to the half-plane $\{\operatorname{Re} s > a_2\}$. Furthermore, the multisets of poles of $\zeta_{\rm pert}(s)$ and $\zeta_1(s)$ coincide in this half-plane.

Proof. The function $\zeta_1(s)$ is meromorphic in $\{\operatorname{Re} s > a_2\}$, while $\zeta_2(s)$ is holomorphic in this same half-plane. Hence, their sum, $\zeta_{\rm pert}(s)$, is meromorphic in this half-plane. As it is well-known, the uniqueness of the meromorphic extension of $\zeta_{\rm pert}(s)$ follows from the principle of analytic continuation since any two meromorphic extensions must coincide on $\{\operatorname{Re} s > D(\zeta_1)\}$. The poles of $\zeta_1(s)$ in the half-plane $\{\operatorname{Re} s > D(\zeta_2)\}$, as well as their corresponding multiplicities (or orders), do not change after adding the holomorphic function $\zeta_2(s)$. □

1.5 Almost periodic functions and distributions

In this section we recall the notions of almost periodic functions and distributions and state some basic results about them. We will need these results in order to obtain a lemma which will be crucial for proving one direction of the Minkowski measurability criterion of Section 3.5; that is, Theorem 3.56. There are several (not all mutually equivalent) definitions of almost periodic functions introduced by Bohr [Boh], Stepanov [Ste], Weyl [We] and Besicovitch [Bes1]. Here, we will use an equivalent definition given by Bochner in [Boc] of Bohr almost periodic functions since it was a basis for introducing almost periodic distributions.

Almost periodic distributions were introduced by Schwartz in [Schw] as a generalization of Bochner almost periodic functions. Further investigations of distributions of this type were made, among others, by [Ron], [FavRas], [BouKha], [Kha]. The following exposition of notions and results about almost periodicity follows mainly the survey given in [Kha, Chapter 2].

Let us denote with $C_b(\mathbb{R})$ the space of bounded and continuous complex valued functions defined on $\mathbb{R}$ endowed with the $\|\cdot\|_\infty$ norm of uniform convergence. Note that the space $(C_b(\mathbb{R}), \|\cdot\|_\infty)$ is a Banach algebra. H. Bohr introduced in 1923 almost periodic functions in order to generalize the idea of a periodic function. Indeed, observe that if f is a T-periodic function; that is, $f(x+nT)-f(x)=0$ for all $x\in\mathbb{R}$ and $n\in\mathbb{Z}$, then the discrete set of periods $\{nT : n\in\mathbb{Z}\}$ has the following property:

$$(\forall\alpha\in\mathbb{R})\,(\exists n_0\in\mathbb{Z})\,(n_0T\in[\alpha,\alpha+T]). \tag{1.5.1}$$

Definition 1.7. A set $E\subseteq\mathbb{R}$ is said to be *relatively dense in* $\mathbb{R}$ if there exists a positive number l such that, any interval of length l contains at least one number of E.

Definition 1.8. Let ε be a positive real number. A real number τ is called an ε*-almost* *period* of f if

$$\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| < \varepsilon. \tag{1.5.2}$$

We denote by $E\{\varepsilon, f\}$ the set of ε-almost periods of f, i.e.,

$$E\{\varepsilon, f\} := \left\{\tau \in \mathbb{R} : \sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| < \varepsilon\right\}. \tag{1.5.3}$$

Definition 1.9 (H. Bohr)**.** A continuous complex valued function f defined on $\mathbb{R}$ is said to be *almost periodic* if for any $\varepsilon > 0$ the set $E\{\varepsilon, f\}$ of its ε-almost periods is relatively dense in $\mathbb{R}$.

We note that the set $E\{\varepsilon, f\}$ is not discrete in general. In the following proposition we list some of the elementary properties of almost periodic functions.

Proposition 1.10. (a) Any almost periodic function is bounded on $\mathbb{R}$.
(b) Any almost periodic function is uniformly continuous on $\mathbb{R}$.
(c) Any continuous periodic function is an almost periodic function.

Note that there exist almost periodic functions which are not periodic, for instance, it can be shown that the quasiperiodic functions introduced in Definition 4.79 below are a subset of almost periodic functions (see [Vin]).

Theorem 1.11. *The set $C_{ap}(\mathbb{R})$ of almost periodic functions is a closed subalgebra of $C_b(\mathbb{R})$.*

Theorem 1.12. *If the derivative f' of an almost periodic function f is uniformly continuous on $\mathbb{R}$, then it is almost periodic.*

Theorem 1.13 (Bohl-Bohr)**.** *If a primitive of an almost periodic function is bounded, then it is almost periodic.*

It can be shown that for a periodic function f, the set

$$H(f) := \{f_h : f_h(x) := f(x+h), h \in \mathbb{R}\} \tag{1.5.4}$$

is a compact subset of $(C_b(\mathbb{R}), \|\cdot\|_\infty)$. Bochner gave the following definition of almost periodicity by replacing the condition of compactness of $H(f)$ with relative compactness.

Definition 1.14 (S. Bochner)**.** A continuous complex valued function f defined on $\mathbb{R}$ is said to be *almost periodic* if for any sequence of real numbers $(h_n)_{n\geq 1}$ there exists a subsequence $(h_{n_k})_{k\geq 1}$ such that the sequence of functions $(f_{h_{n_k}})_{k\geq 1}$ is uniformly convergent on $\mathbb{R}$.

It turns out that almost periodic functions are exactly the functions which can be approximated by *trigonometric polynomials*; that is, by functions of the form

$$P(x) = \sum_{k=1}^{n} c_k \mathrm{e}^{\mathbb{i}\lambda_k x}, \quad \text{where } c_k \in \mathbb{C} \text{ and } \lambda_k \in \mathbb{R}. \tag{1.5.5}$$

Here, and else in this thesis, we use the notation $\mathbb{i}$ for the imaginary unit; that is, $\mathbb{i} := \sqrt{-1}$. Hence, we have the third definition of almost periodicity.

Definition 1.15 (Polynomial approximation)**.** A continuous complex valued function f defined on $\mathbb{R}$ is said to be *almost periodic* if there exists a sequence of trigonometric polynomials P_j such that $\|f - P_j\|_\infty \to 0, \quad \text{as } j \to \infty$.

Theorem 1.16. *All of the three definitions of almost periodicity given above are mutually equivalent.*

There exist several proofs of the the above fundamental theorem, see [Bes3] or [Cor]. The fact that almost periodic functions may be approximated by trigonometric polynomials suggests that we can associate a Fourier series to these type of functions and this is indeed the case.

Definition 1.17. We define the *mean value* of a function f as

$$M(f) := \lim_{T\to+\infty} \frac{1}{T} \int_0^T f(x)\,\mathrm{d}x. \tag{1.5.6}$$

Theorem 1.18. *The mean value exists for any almost periodic function.*

Theorem 1.19. *The mean value* $M\colon C_{ap}(\mathbb{R}) \to \mathbb{C}$ *is a continuous linear functional. Furthermore, let* $f \in C_{ap}(\mathbb{R})$, *then:*
(i) $M(\overline{f}) = \overline{M(f)}$.
(ii) If $f \geq 0$, *then* $M(f) \geq 0$.
(iii) $M(f) = \lim_{T\to+\infty} \frac{1}{T}\int_a^{a+T} f(x)\,\mathrm{d}x$ *for all* $a \in \mathbb{R}$.

The simplest examples of almost periodic function are the periodic functions, such as $\mathrm{e}^{\mathbb{i}\lambda x}$ for $\lambda \in \mathbb{R}$. Note that

$$M(\mathrm{e}^{\mathbb{i}\lambda x}) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T \mathrm{e}^{\mathbb{i}\lambda x}\,\mathrm{d}x = \delta_{\lambda,0}, \tag{1.5.7}$$

where δ denotes the Kronecker delta. Furthermore, if f is almost periodic, then so is $\mathrm{e}^{-\mathbb{i}\lambda x} f(x)$ for any $\lambda \in \mathbb{R}$, and, consequently, its mean value exists. Let us denote it with $a_f(\lambda)$; that is,

$$a_f(\lambda) := M(\mathrm{e}^{-\mathbb{i}\lambda x} f(x)). \tag{1.5.8}$$

Theorem 1.20. *For any almost periodic function, the set* $\{a_f(\lambda) : \lambda \in \mathbb{R}$ *and* $a_f(\lambda) \neq 0\}$ *is at most countable.*

Definition 1.21. The numbers $\lambda_n \neq 0$, $n \in \mathbb{N}$ and for which $a_f(\lambda_n) \neq 0$ are called the *Fourier exponents* of f. Furthermore, the numbers $a_f(\lambda_n)$ for $n \in \mathbb{N}$, are called the *Fourier coefficients* of f. The formal series

$$f(x) \sim \sum_{n=1}^{\infty} a_f(\lambda_n) \mathrm{e}^{\mathrm{i}\lambda_n x} \tag{1.5.9}$$

is called the *Fourier series associated with* f.

Theorem 1.22 (Parseval's equality)**.** *Let* f *be an almost periodic function such that* $f(x) \sim \sum_{n=1}^{\infty} a_f(\lambda_n)\mathrm{e}^{\mathrm{i}\lambda_n x}$. *Then the following equality holds:*

$$\sum_{n=1}^{\infty} |a_f(\lambda_n)|^2 = M(|f|^2). \tag{1.5.10}$$

Theorem 1.23. *If the mean value of a nonnegative almost periodic function* f *is zero, then* $f \equiv 0$.

A consequence of the above two theorems is the following uniqueness result.

Theorem 1.24 (Uniqueness Theorem for almost periodic functions)**.** *If two almost periodic functions have the same Fourier series, then they are identical.*

Note that in the hypothesis of the uniqueness theorem above there is no assumption on the convergence of the Fourier series associated to f.

Theorem 1.25. *If the Fourier series of an almost periodic function* f *converges uniformly, then the function* f *coincides with it.*

Lemma 1.26. *A nonconstant almost periodic function does not have a limit at* $\pm\infty$.

Proof. Choose $a, b \in \mathbb{R}$ such that $f(a) \neq (b)$ and fix $\varepsilon > 0$ such that $|f(a) - f(b)| > 3\varepsilon$. Since the set $E\{\varepsilon, f\}$ is relatively dense in $\mathbb{R}$ we can choose a number $l(\varepsilon)$ such that for every integer n there exists $\tau_n \in [n, n + l(\varepsilon)]$ such that

$$|f(a + \tau_n) - f(a)| < \varepsilon \quad \text{and} \quad |f(b + \tau_n) - f(b)| < \varepsilon.$$

But now we have

$$|f(a + \tau_n) - f(b + \tau_n)| > \big||f(a) - f(b)| - |f(a + \tau_n) - f(a) + f(b) - f(b + \tau_n)|\big| > \varepsilon.$$

On the other hand, $a + \tau_n \to \pm\infty$ and $b + \tau_n \to \pm\infty$ as $n \to \pm\infty$ from which it follows that the limit $\lim_{x\to\pm\infty} f(x)$ does not exist. $\square$

The idea to generalize the above notions to distributions is originally due to L. Schwartz (see [Schw]), and it is based on Bochner's topological definition of almost periodic functions.

For $p \in [1, +\infty]$ we let

$$\mathcal{D}_{L^p}(\mathbb{R}) := \{\varphi \in C^\infty(\mathbb{R}) : \varphi^{(j)} \in L^p(\mathbb{R}),\ \forall j \in \mathbb{N}_0\}. \tag{1.5.11}$$

The space $\mathcal{D}_{L^p}(\mathbb{R})$ equipped with the topology defined by the countable family of seminorms

$$|\varphi|_{k,p} := \sum_{j \le k} \|\varphi^{(p)}\|_{L^p}, \tag{1.5.12}$$

for $k \in \mathbb{N}_0$, is a differential Fréchet subalgebra of $C^\infty(\mathbb{R})$. We call the topological dual of $\mathcal{D}_{L^1}(\mathbb{R})$, the *space of bounded distributions* and denote it with $\mathcal{D}'_{L^\infty}(\mathbb{R})$.

For $h \in \mathbb{R}$ and a distribution $\mathcal{T} \in \mathcal{D}'(\mathbb{R})$ we define its *h-translate* denoted by $\tau_h \mathcal{T}$ as

$$\langle \tau_h \mathcal{T}, \varphi \rangle := \langle \mathcal{T}, \tau_{-h}\varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}), \tag{1.5.13}$$

where $\tau_{-h}\varphi(x) := \varphi(x+h)$. We have the following fundamental theorem about almost periodic distributions (see [Schw] or the exposition in [Ché, Chapter 4]).

Theorem 1.27. *For any $\mathcal{T} \in \mathcal{D}'(\mathbb{R})$, the following statements are equivalent*:

(i) The set $\{\tau_h \mathcal{T} : h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^\infty}(\mathbb{R})$.

(ii) $\mathcal{T} * \varphi \in C_{ap}(\mathbb{R})$ for all φ in $\mathcal{D}(\mathbb{R})$.

(iii) $\mathcal{T}$ is a finite sum of distributional derivatives of almost periodic functions; *that is, there exist $f_j \in C_{ap}(\mathbb{R})$, $j \le k$ such that $\mathcal{T} = \sum_{j \le k} f_j^{(j)}$.*

Here, for a distribution $\mathcal{T} \in \mathcal{D}'(\mathbb{R})$ and a test function $\varphi \in \mathcal{D}(\mathbb{R})$, we define

$$(\mathcal{T} * \varphi)(t) := \langle \mathcal{T}, \tau_t \check{\varphi} \rangle \quad \text{and} \quad \check{\varphi}(x) := \varphi(-x). \tag{1.5.14}$$

Definition 1.28 (Almost periodic distribution)**.** A distribution $\mathcal{T} \in \mathcal{D}'_{L^\infty}(\mathbb{R})$ is said to be *almost periodic* if it satisfies any (hence, all) of the conditions of the above theorem. We denote the space of almost periodic distributions by $\mathcal{B}'_{ap}(\mathbb{R})$.

Proposition 1.29. Any almost periodic function $f \in C_{ap}(\mathbb{R})$ generates a unique almost periodic distribution $\mathcal{T}_f$ defined by

$$\langle \mathcal{T}_f, \varphi \rangle := \int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x \quad \text{for} \quad \varphi \in \mathcal{D}_{L^1}(\mathbb{R}). \tag{1.5.15}$$

Similarly as in the case of almost periodic functions, we can define a mean value for almost periodic distributions.

Definition 1.30. Let $\varphi \in \mathcal{D}_{L^1}(\mathbb{R})$ be such that $\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x \neq 0$ and $\mathcal{T} \in \mathcal{D}'_{L^\infty}(\mathbb{R})$. Then, the φ-*mean* of $\mathcal{T}$ is defined by

$$M^\varphi(\mathcal{T}) := \frac{M(\mathcal{T} * \varphi)}{\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x}. \tag{1.5.16}$$

Proposition 1.31. If $\mathcal{T} \in \mathcal{B}'_{ap}(\mathbb{R})$, then $M^\varphi(\mathcal{T})$ does not depend on the choice of φ and, hence, we denote it by $M(\mathcal{T})$.

It can be shown that for $\lambda \in \mathbb{R}$ and $\mathcal{T} \in \mathcal{B}'_{ap}(\mathbb{R})$ we have that $\mathrm{e}^{-\mathrm{i}\lambda x}\mathcal{T} \in \mathcal{B}'_{ap}(\mathbb{R})$. Consequently, the mean of $\mathrm{e}^{-\mathrm{i}\lambda x}\mathcal{T}$ exists and we define

$$a_{\mathcal{T}}(\lambda) := M(\mathrm{e}^{-\mathrm{i}\lambda x}\mathcal{T}). \tag{1.5.17}$$

Definition 1.32. For $\mathcal{T} \in \mathcal{B}'_{ap}(\mathbb{R})$ we define its *spectra* as

$$\Lambda_{\mathcal{T}} := \{\lambda \in \mathbb{R} : a_{\mathcal{T}}(\lambda) \neq 0\}. \tag{1.5.18}$$

Proposition 1.33. For $\mathcal{T} \in \mathcal{B}'_{ap}(\mathbb{R})$ the set $\Lambda_{\mathcal{T}}$ is at most countable.

Theorem 1.34 (Approximation theorem for almost periodic distributions)**.** *Let $\mathcal{T} \in \mathcal{D}'_{L^\infty}(\mathbb{R})$. Then, $\mathcal{T}$ is almost periodic if and only if there exists a sequence of trigonometric polynomials $(P_n)_{n\geq 1}$ such that $\lim_{n\to\infty} P_n = \mathcal{T}$ in $\mathcal{D}'_{L^\infty}(\mathbb{R})$.*

Definition 1.35. For $\mathcal{T} \in \mathcal{D}'_{L^\infty}(\mathbb{R})$ we define its *Fourier series* as the following formal series:

$$f(x) \sim \sum_{\lambda\in\Lambda_{\mathcal{T}}} a_{\mathcal{T}}(\lambda)\mathrm{e}^{\mathrm{i}\lambda x}. \tag{1.5.19}$$

We call the numbers $a_{\mathcal{T}}(\lambda)$ the *Fourier coefficients of* $\mathcal{T}$.

Similarly as in the case of almost periodic functions, we have the following uniqueness theorem.

Theorem 1.36 (Uniqueness Theorem for almost periodic distributions)**.** *If two almost periodic distributions have the same Fourier series, then they are identical.*

The theorem follows from the corresponding one for almost periodic functions (Theorem 1.24) and Equation (1.5.16).

Let us now introduce the notion of the distributional order of growth (see [EsKa, PiStVi] and also [Lap–vFr3]). For a test function $\varphi \in \mathcal{D}(0,\infty)$ and $a > 0$ we denote

$$\varphi_a(t) := \frac{1}{a}\varphi\left(\frac{t}{a}\right). \tag{1.5.20}$$

Note that $\int_0^{+\infty} \varphi_a(t)\,\mathrm{d}t = \int_0^{+\infty} \varphi(t)\ \mathrm{d}t$ for every $a > 0$. Furthermore, the support of φ_a becomes 'more narrow' and 'closer' to zero, while the amplitude tends to infinity in absolute value when $a \to 0^+$. On the other hand, when $a \to +\infty$, then the support of φ_a becomes 'wider' and 'escapes' to infinity while the amplitude tends to zero in absolute value.

Definition 1.37. Let $\mathcal{R}$ be a distribution in $\mathcal{D}'(0,\delta)$. We will say that $\mathcal{R}$ is of *asymptotic order* at most t^α (respectively, less than t^α) as $t \to 0^+$ if applied to a test function φ, we have that[14]

$$\langle \mathcal{R}, \varphi_a \rangle = O(a^\alpha) \quad \text{(respectively, } \langle \mathcal{R}, \varphi_a \rangle = o(a^\alpha)), \quad \text{as } a \to 0^+. \tag{1.5.21}$$

We will denote this with $\mathcal{R}(t) = O(t^\alpha)$ (respectively, $\mathcal{R}(t) = o(t^\alpha)$) as $a \to 0^+$.

Remark 1.38. We point out that if f is a continuous function such that $f(t) = O(t^\alpha)$ or $f(t) = o(t^\alpha)$ when $t \to 0^+$ for some α, then f also satisfies the same asymptotics in the distributional sense of Definition 1.37. Namely, by taking $\varphi \in \mathcal{D}(0,\delta)$ we have

$$\langle f, \varphi_a \rangle = \int_0^{+\infty} f(t)\varphi_a(t)\,\mathrm{d}t = \int_0^{+\infty} f(a\tau)\varphi(\tau)\,\mathrm{d}\tau. \tag{1.5.22}$$

If $f = O(t^\alpha)$ then, since φ has compact support, we can take a sufficiently small so that in the above integral $|f(a\tau)| \le C a^\alpha \tau^\alpha$ for some positive constant C. In other words, for a sufficiently small we have

$$|\langle f, \varphi_a \rangle| \le C a^\alpha \int_0^{+\infty} \tau^\alpha \varphi(\tau)\,\mathrm{d}\tau = K_\varphi\, a^\alpha, \tag{1.5.23}$$

where the constant K_φ depends only on the test function φ. One shows analogously the case when $f(t) = o(t^\alpha)$ when $t \to 0^+$. The same comment can be also made about the asymptotics when $t \to +\infty$. On the other hand, distributional asymptotics does not in general imply the usual one; see [PiStVi].

The following two lemmas will be needed for deriving a Minkowski measurability criterion in Section 3.5; more precisely, in the proof of Theorem 3.56.

Lemma 1.39. *Let $\mathcal{R} \in \mathcal{D}'(0,+\infty)$ such that $\mathcal{R}(t) = o(t^\alpha)$ as $t \to 0^+$ for some $\alpha \in \mathbb{R}$ and let $\beta \in \mathbb{R}$. Then, $t^\beta \mathcal{R}(t) = o(t^{\alpha\beta})$ as $t \to 0^+$. Similarly, if $\mathcal{R}(t) = O(t^\alpha)$, then $t^\beta \mathcal{R}(t) = O(t^{\alpha\beta})$ as $t \to 0^+$.*

Proof. Let $\varphi \in \mathcal{D}(0,+\infty)$ and $a > 0$. Then, we have

$$\langle t^\beta \mathcal{R}(t), \varphi_a(t) \rangle = \langle \mathcal{R}(t), t^\beta a^{-1} \varphi(a^{-1}t) \rangle = a^\beta \langle \mathcal{R}(t), \psi_a(t) \rangle, \tag{1.5.24}$$

where $\psi(t) := t^\beta \varphi(t)$. Let now $\varepsilon > 0$, choose a sufficiently small so that $|\langle \mathcal{R}(t), \psi_a(t) \rangle| \le$

[14]In this formula, the implicit constant depends on the test function φ.

εa^α and use (1.5.24) to derive the conclusion. The other part of the lemma is proven analogously. $\square$

Lemma 1.40. *Let $\mathcal{T} \in \mathcal{D}'(0,\infty)$ be a distribution such that*

$$\mathcal{T} = \sum_{n=1}^{\infty} a_n t^{-\mathrm{i}\lambda_n}, \tag{1.5.25}$$

where $a_n \in \mathbb{C}$ and $\lambda_n \in \mathbb{R}$ for all $n \geq 1$. If $\mathcal{T} = o(1)$ as $t \to 0^+$, then $a_n = 0$ for all $n \in \mathbb{N}$.

Proof. A change of variables yields that

$$\langle t^{-\mathrm{i}\lambda}, \varphi\rangle = \int_0^{+\infty} t^{-\mathrm{i}\lambda}\varphi(t)\,\mathrm{d}t = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\lambda x}\mathrm{e}^{-x}\varphi(\mathrm{e}^{-x})\,\mathrm{d}x, \tag{1.5.26}$$

for every $\varphi \in \mathcal{D}(0,\infty)$.

We note that $\Xi\colon \mathcal{D}(0,\infty) \to \mathcal{D}(\mathbb{R})$ defined by $\{\Xi\varphi\}(x) := \mathrm{e}^{-x}\varphi(\mathrm{e}^{-x})$ is an isomorphism of differential Fréchet algebras. Indeed, it is clearly a bijection with the inverse given by $\{\Xi^{-1}\psi\}(t) = t^{-1}\psi(\log t^{-1})$. Let now $\varphi_n \to 0$ in $\mathcal{D}(0,\infty)$. Then there exists a compact $K \subseteq (0,\infty)$ such that $\operatorname{supp}\varphi_n \subseteq K$ for all $n \geq 1$. Since $0 \notin K$ we have that $\log K^{-1} := \{\log t^{-1} : t \in K\}$ is a compact subset of $\mathbb{R}$ and $\operatorname{supp}\{\Xi\varphi_n\} \subseteq \log K^{-1}$. Furthermore,

$$\|\{\Xi\varphi_n\}\|_\infty = \sup_{x\in\mathbb{R}} \|\mathrm{e}^{-x}\varphi_n(\mathrm{e}^{-x})\| \leq \|\varphi_n\|_\infty \max_{x\in\log K^{-1}} \mathrm{e}^{-x} \to 0, \quad \text{as } n \to \infty. \tag{1.5.27}$$

Furthermore, for $\varphi \in \mathcal{D}(0,\infty)$ we have

$$\begin{aligned}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\{\Xi\varphi\}(x) &= \frac{\mathrm{d}^j}{\mathrm{d}x^j}(\mathrm{e}^{-x}\varphi(\mathrm{e}^{-x})) = \sum_{k=0}^{j}\binom{j}{k}(-1)^{j-k}\mathrm{e}^{-x}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\varphi(\mathrm{e}^{-x})\\ &= (-1)^j\sum_{k=0}^{j}\binom{j}{k}\mathrm{e}^{-x}\varphi^{(k)}(\mathrm{e}^{-x}) = (-1)^j\sum_{k=0}^{j}\binom{j}{k}\{\Xi\varphi^{(k)}\}(x).\end{aligned} \tag{1.5.28}$$

Since $\varphi_n \to 0$ in $\mathcal{D}(0,\infty)$ implies that $\|\varphi_n^{(j)}\|_\infty \to 0$ for every $j \geq 0$, we conclude from (1.5.27) and (1.5.28) that $\{\Xi\varphi_n\} \to 0$ in $\mathcal{D}(\mathbb{R})$; that is, Ξ is continuous.

On the other hand, let now $\psi_j \to 0$ in $\mathcal{D}(\mathbb{R})$ and $H \subseteq \mathbb{R}$ compact such that $\operatorname{supp}\psi_j \subseteq H$. Then, $\mathrm{e}^{-H} := \{\mathrm{e}^{-x} : x \in H\}$ is a compact subset of $(0,\infty)$ and $\operatorname{supp}\{\Xi^{-1}\psi_j\} \subseteq \mathrm{e}^{-H}$ for every $n \geq 1$. Furthermore, for every $j \in \mathbb{N}$ we have that

$$\begin{aligned}\|t^{-j}\{\Xi^{-1}\varphi_n\}(t)\|_\infty &= \sup_{t\in(0,\infty)} \|t^{-j-1}\psi_n(\log t^{-1})\|\\ &\leq \|\psi_n\|_\infty \max_{t\in\mathrm{e}^{-H}} t^{-j-1} \to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{1.5.29}$$

Similarly as before, for $\psi \in \mathcal{D}(\mathbb{R})$ we now have that

$$\begin{aligned}\frac{\mathrm{d}^j}{\mathrm{d}t^j}\{\Xi^{-1}\psi\}(t) &= \frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(t^{-1}\psi(\log t^{-1})\right) = \sum_{k=0}^{j}\binom{j}{k}(-1)^{j-k}t^{-j+k-1}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\psi(\log t^{-1})\\ &= \sum_{k=0}^{j}\binom{j}{k}(-1)^j t^{-j-1}\psi^{(k)}(\log t^{-1}) = (-1)^j\sum_{k=0}^{j}\binom{j}{k}t^{-j}\{\Xi^{-1}\psi^{(k)}\}(t).\end{aligned}$$

We now conclude, similarly as for Ξ, that Ξ^{-1} is also continuous.

The isomorphism Ξ now induces an isomorphism of the duals $\mathcal{D}'(0,\infty)$ and $\mathcal{D}'(\mathbb{R})$ defined by

$$\langle \Xi\{\mathcal{T}\}, \psi\rangle := \langle \mathcal{T}, \{\Xi^{-1}\psi\}\rangle, \tag{1.5.30}$$

for $\mathcal{T} \in \mathcal{D}'(0,\infty)$. Let now $\mathcal{T}$ be given by (1.5.25). Then, by (1.5.26) and the fact that Ξ is an isomorphism we have that

$$\Xi\{\mathcal{T}\} = \sum_{n=1}^{\infty} a_n \mathrm{e}^{\mathbb{i}\lambda_n x}, \tag{1.5.31}$$

as a distribution on $\mathcal{D}(\mathbb{R})$. Moreover, $\Xi\{\mathcal{T}\} * \psi$ is an almost periodic function for any $\psi \in \mathcal{D}(\mathbb{R})$. To see this, consider that

$$\begin{aligned}(\Xi\{\mathcal{T}\} * \psi)(t) = \langle \Xi\{\mathcal{T}\}, \tau_t\check{\psi}\rangle &= \sum_{n=1}^{\infty} a_n \int_{-\infty}^{+\infty} \mathrm{e}^{\mathbb{i}\lambda_n x}\psi(t-x)\,\mathrm{d}x\\ &= \sum_{n=1}^{\infty}\left(a_n\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathbb{i}\lambda_n y}\psi(y)\,\mathrm{d}y\right)\mathrm{e}^{\mathbb{i}\lambda_n t},\end{aligned}$$

and the above sum converges uniformly on $\mathbb{R}$. By Theorem 1.25 we conclude that $\Xi\{\mathcal{T}\}*\psi$ is almost periodic. This, in turn, implies that $\Xi(\mathcal{T})$ is an almost periodic distribution by Theorem 1.27(*ii*) and a_n, for $n \geq 1$, are its Fourier coefficients by (1.5.16). From the asymptotics of $\mathcal{T}$, we have that for $a > 0$

$$\begin{aligned}\langle \mathcal{T}, \varphi_a\rangle = \langle \Xi\{\mathcal{T}\}, \Xi\{\varphi_a\}\rangle &= \sum_{n=1}^{\infty} a_n \int_{-\infty}^{+\infty} \mathrm{e}^{\mathbb{i}\lambda_n x} a^{-1}\mathrm{e}^{-x}\varphi(a^{-1}\mathrm{e}^{-x})\,\mathrm{d}x\\ &= \sum_{n=1}^{\infty} a_n \int_{-\infty}^{+\infty}\mathrm{e}^{\mathbb{i}\lambda_n x}\,\Xi\{\varphi\}(x - \log a^{-1})\,\mathrm{d}x\\ &= (\Xi\{\mathcal{T}\} * \Xi\{\check{\varphi}\})(\log a^{-1}) \to 0, \quad \text{as} \quad a \to 0^+.\end{aligned} \tag{1.5.32}$$

But since $\Xi\{\mathcal{T}\} * \Xi\{\check{\varphi}\}$ is an almost periodic function, the above is possible only if $\Xi\{\mathcal{T}\} * \Xi\{\check{\varphi}\} \equiv 0$ by Lemma 1.26. This, in turn, implies that all the Fourier coefficients of $\Xi\{\mathcal{T}\} * \Xi\{\check{\varphi}\}$ are zero. Finally, by choosing φ such that $\int_{\mathbb{R}} \Xi\{\check{\varphi}\}(x)\,\mathrm{d}x \neq 0$ we conclude that all the Fourier coefficients of $\Xi\{\mathcal{T}\}$ are equal to zero; that is, $a_n = 0$ for all $n \geq 1$. $\square$

Chapter 2

Lapidus Zeta Functions of Relative Fractal Drums

In this chapter, we introduce the notion of relative fractal drums. They represent a simple and natural extension of two fundamental objects of fractal analysis, simultaneously: that of bounded sets in $\mathbb{R}^N$ (i.e., of fractals) and that of bounded fractal strings (introduced by M. L. Lapidus and C. Pomerance in the early 1990s). Furthermore, there is a natural way to define their associated Minkowski contents and relative distance zeta functions. We stress a new phenomenon exhibited by relative fractal drums: namely, their box dimensions can be negative as well (and even equal to $-\infty$). This can be viewed as a property of their 'flatness', since it is related to the loss of the cone property (see Proposition 2.17). The content of this chapter is reproduced from [LapRaŽu1] and provides an overview of the main results about the notions mentioned above.

In short, a relative fractal drum consists of an ordered pair (A, Ω), where A is an arbitrary (possibly unbounded) subset of $\mathbb{R}^N$ and Ω is an open subset of $\mathbb{R}^N$ of finite volume with another mild technical condition. (See Definition 2.1.) This notion generalizes the the notion of a bounded (fractal) and the additional flexibility it provides enables us to account for a broader range of situations and phenomena, including the case of unbounded geometric chirps.

2.1 Relative Minkowski Content, Box Dimension, and Zeta Functions

In this subsection, we introduce the notion of a relative zeta function, associated to an appropriate ordered pair (A, Ω) of two suitable subsets of $\mathbb{R}^N$, which may be unbounded. The relative distance zeta function (see (2.1.1)), is a natural generalization of the standard distance zeta function for bounded subsets of $\mathbb{R}^N$ defined in [LapRaŽu1].

In order to exclude dealing with trivial cases and shorten the statements of the results, we will always assume throughout this thesis that all the sets A and Ω are nonempty. First of all, for a subset A of $\mathbb{R}^N$, we denote its *δ-neighborhood* (or *δ-parallel set*) by $A_\delta := \{x \in \mathbb{R}^N : d(x, A) < \delta\}$. Here, $d(x, A) := \inf\{|x - y| : y \in A\}$ is the Euclidean distance between the point x and the set A.

Definition 2.1 (Cited from [LapRaŽu1]). Let Ω be Lebesgue measurable subset of $\mathbb{R}^N$, not necessarily bounded, but of finite N-dimensional Lebesgue measure (or "volume"). Furthermore, let $A \subseteq \mathbb{R}^N$, also possibly unbounded, be such that Ω is contained in A_δ for some $\delta > 0$. The *distance zeta function $\zeta_A(\,\cdot\,, \Omega)$ of A relative to Ω* (or the *relative distance zeta function*) is defined by the following Lebesgue integral:

$$\zeta_A(s, \Omega) := \int_\Omega d(x, A)^{s-N}\, \mathrm{d}x, \tag{2.1.1}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large. The ordered pair (A, Ω), appearing in Definition 2.1 is called a *relative fractal drum* or RFD in short. In light of this, we will also use the phrase *zeta functions of relative fractal drums* instead of relative zeta functions.

Remark 2.2. If we replace the domain of integration Ω in (2.1.1) with $A_\delta \cap \Omega$ for some $\delta > 0$; that is, if we let

$$\zeta_A(s, \Omega; \delta) := \int_{A_\delta \cap \Omega} d(x, A)^{s-N}\, \mathrm{d}x, \tag{2.1.2}$$

then difference $\zeta_A(s, \Omega) - \zeta_A(s, \Omega; \delta)$ is an entire function (see [LapRaŽu1]). Therefore, we can alternatively define the relative distance function of (A, Ω) by (2.1.2), since in the theory of complex dimensions we are mostly interested in poles of meromorphic extensions of (various) fractal zeta functions. Then, in light of the principle of analytic continuation, the dependence of $\zeta_A(\,\cdot\,, \Omega; \delta)$ on δ is inessential.

The condition that $\Omega \subseteq A_\delta$ for some $\delta > 0$ is of technical nature and ensures that $x \mapsto d(x, A)$ is bounded for $x \in \Omega$. If Ω does not satisfy this condition we can still use the alternative definition by Equality (2.1.2).[1]

Remark 2.3. We point out that the notion of a relative fractal drum generalizes the notion of a bounded subset. Indeed, to apply any of the results about RFDs to a bounded subset A of $\mathbb{R}^N$ one only has to identify it with a relative fractal drum (A, A_δ) for some $\delta > 0$.

An entirely analogous comment can be made about the tube zeta function of a relative fractal drum which we now introduce.

[1]Since then $\Omega \setminus A_\delta$ and A are a positive distance apart, this replacement will not affect the relative box dimension of (A, Ω) introduced just below.

Definition 2.4 (Cited from [LapRaŽu1]). Let (A,Ω) be an RFD in $\mathbb{R}^N$ and fix $\delta > 0$. We define the *tube zeta function* $\widetilde{\zeta}_A(\,\cdot\,,\Omega)$ *of* A *relative to* Ω (or the *relative tube zeta function*) by

$$\widetilde{\zeta}_A(s,\Omega;\delta) := \int_0^\delta t^{s-N-1}|A_t\cap\Omega|\,\mathrm{d}t, \tag{2.1.3}$$

for all $s\in\mathbb{C}$ with $\operatorname{Re} s$ sufficiently large and the integral is taken in the Lebesgue sense.

The distance and tube zeta functions of relative fractal drums are a special case of Dirichlet-type integrals (or, in short, DTIs; see Subsection 1.4), and as such, have well defined *abscissa of* (*absolute*) *convergence*. Furthermore, the relative distance and tube zeta functions are connected by a functional equation which is stated in the following theorem.

Theorem 2.5 (Cited from [LapRaŽu1]). *Let* (A,Ω) *be a relative fractal drum in* $\mathbb{R}^N$. *Then,*

$$\zeta_A(s,\Omega;\delta) = \delta^{s-N}|A_\delta\cap\Omega| + (N-s)\widetilde{\zeta}_A(s,\Omega;\delta), \tag{2.1.4}$$

is valid on any open connected set U *containing* $\{\operatorname{Re} s > \overline{\dim}_B(A,\Omega)\}$ *to which any of these two zeta functions has a meromorphic continuation.*

This result is very useful since the distance zeta function is much more practical to calculate in concrete examples as opposed to the tube zeta function. On the other hand, the tube zeta function has an important theoretical value and many results in [LapRaŽu1] are proven in terms of the tube zeta function and then rewritten in terms of the distance zeta function.

We now proceed by introducing the notions of Minkowski content and box dimension of a relative fractal drum and relating them to its distance (and tube) zeta functions. For any real number r, we define the *upper* r*-dimensional Minkowski content of* A *relative to* Ω (or *the upper relative Minkowski content*, or *the upper Minkowski content of the relative fractal drum* (A,Ω)) by

$$\overline{\mathcal{M}}^r(A,\Omega) = \limsup_{t\to0^+}\frac{|A_t\cap\Omega|}{t^{N-r}}, \tag{2.1.5}$$

and then we proceed in the usual way:

$$\begin{aligned}\overline{\dim}_B(A,\Omega) &= \inf\{r\in\mathbb{R} : \overline{\mathcal{M}}^r(A,\Omega)=0\}\\ &= \sup\{r\in\mathbb{R} : \overline{\mathcal{M}}^r(A,\Omega)=+\infty\}.\end{aligned} \tag{2.1.6}$$

We call it the *relative upper box dimension* (or *relative Minkowski dimension*) of A with respect to Ω (or else the *relative upper box dimension of* (A,Ω)). Note that $\overline{\dim}_B(A,\Omega)\in[-\infty,N]$, and the values can indeed be negative, even equal to $-\infty$; see [LapRaŽu1]. Also note that for these definitions to make sense it is sufficient that $|A_\delta\cap\Omega|<\infty$ for some $\delta>0$.

The value $\underline{\mathcal{M}}^r(A,\Omega)$ of the *lower r-dimensional Minkowski content* of (A,Ω), is defined as in (2.1.5), except for a lower instead of an upper limit. Analogously as in (2.1.6), we define the *relative lower box* (or *Minkowski*) *dimension* of (A,Ω):

$$\begin{aligned}\underline{\dim}_B(A,\Omega) &= \inf\{r\in\mathbb{R} : \underline{\mathcal{M}}^r(A,\Omega)=0\}\\ &= \sup\{r\in\mathbb{R} : \underline{\mathcal{M}}^r(A,\Omega)=+\infty\}.\end{aligned} \tag{2.1.7}$$

Furthermore, in the case when $\underline{\dim}_B(A,\Omega)=\overline{\dim}_B(A,\Omega)$, we denote by $\dim_B(A,\Omega)$ this common value and call it the *relative box* (or *Minkowski*) *dimension*. If $0<\underline{\mathcal{M}}^D(A,\Omega)\le\overline{\mathcal{M}}^D(A,\Omega)<\infty$, we say that the relative fractal drum (A,Ω) is *Minkowski nondegenerate*. It then follows that $\dim_B(A,\Omega)$ exists and is equal to D.

If $\underline{\mathcal{M}}^D(A,\Omega)=\overline{\mathcal{M}}^D(A,\Omega)$, we denote this common value by $\mathcal{M}^D(A,\Omega)$ and call it the *relative Minkowski content* of (A,Ω). If $\mathcal{M}^D(A,\Omega)$ exists and is different from 0 and ∞ (in which case $\dim_B(A,\Omega)$ exists and then necessarily $D=\dim_B(A,\Omega)$), we say that the relative fractal drum (A,Ω) is *Minkowski measurable*. Various examples and properties of relative box dimensions can be found in [Lap1–3], [LapPo1–3], [HeLap], [Lap–vFr1–3], [Žu4], [LaPe2–3], [LapPeWi1–2] and [LapRaŽu1–7].

In the following three theorems we recall some basic results about zeta functions of relative fractal drums.

Theorem 2.6 (Cited from [LapRaŽu1])**.** *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$. Then the following properties hold*:

(*a*) *The relative distance zeta function $\zeta_A(s,\Omega)$ is holomorphic in the half-plane $\{\operatorname{Re} s>\overline{\dim}_B(A,\Omega)\}$. More precisely,*

$$D(\zeta_A(\,\cdot\,,\Omega))=\overline{\dim}_B(A,\Omega) \tag{2.1.8}$$

(*b*) *If the relative box* (*or Minkowski*) *dimension $D:=\dim_B(A,\Omega)$ exists, $D<N$, and $\underline{\mathcal{M}}^D(A,\Omega)>0$, then $\zeta_A(s,\Omega)\to+\infty$ as $s\in\mathbb{R}$ converges to D from the right.*

Remark 2.7. For a general relative fractal drum (A,Ω) in $\mathbb{R}^N$ the right half-plane $\{\operatorname{Re} s>\overline{\dim}_B(A,\Omega)\}$ is not necessarily the maximal open right half-plane to which its relative distance zeta function has an analytic continuation. For instance, for the segment $I:=[0,1]\subset\mathbb{R}$, understood as a relative fractal drum (I,I_δ), we clearly have $D(\zeta_I(\,\cdot\,,I_\delta))=\dim_B(I,I_\delta)=\dim_B I=1$. Furthermore, a simple calculation yields that its distance zeta function has a meromorphic continuation $\zeta_I(s)=2\delta^s/s$ to the whole complex plane and, in particular, it is holomorphic on $\{\operatorname{Re} s>0\}$.[2] This situation cannot happen if (A,Ω) satisfies the hypotheses of part (*b*) of Theorem 2.6.

[2]We would like to thank E. P. J. Pearse for this example.

On the other hand, one can easily calculate the tube zeta function of (I, I_δ) and get that $\widetilde{\zeta}_I(s, I_\delta) = 2\delta^s/s + \delta^{s-1}/(s-1)$, i.e., it is meromorphic on $\mathbb{C}$ and $\{\operatorname{Re} s > 1\}$ is the maximal right half-plane on which it is holomorphic. This is in accordance with (2.1.4), since it implies that a (possible) simple pole at $s = N$ of the relative tube zeta function will never be a pole of the relative distance zeta function since the factor $(N - s)$ cancels it. This is also the reason why when working with meromorphic extensions of the relative distance zeta function one must always additionally assume that $\overline{\dim}_B(A, \Omega) < N$ as opposed to the situation with the relative tube zeta function.

It would be interesting to determine general conditions under which the maximal half-plane of holomorphicity coincides with the half-plane of absolute convergence, or at least, find such an example of an RFD (A, Ω) in $\mathbb{R}^N$ for which $\dim_B(A, \Omega) < N$ and this two half-planes are not equal.

Furthermore, if $\overline{\dim}_B(A, \Omega) < N$, in light of the functional equation (2.1.4), Theorem 2.6 is also valid if we interchange the relative distance zeta function with the relative tube zeta function in its statement. Moreover, it can be shown directly; that is, without the use of the functional equation, that in case of the tube zeta function, Theorem 2.6 is also valid in the special case when $\overline{\dim}_B(A, \Omega) = N$.

Theorem 2.8 (Cited from [LapRaŽu1])**.** *Assume that (A, Ω) is a nondegenerate RFD in $\mathbb{R}^N$, that is, $0 < \underline{\mathcal{M}}^D(A, \Omega) \le \overline{\mathcal{M}}^D(A, \Omega) < \infty$ (in particular, $\dim_B(A, \Omega) = D$), and $D < N$. If $\zeta_A(s, \Omega)$ can be extended meromorphically to a neighborhood of $s = D$, then D is necessarily a simple pole of $\zeta_A(s, \Omega)$, and*

$$(N - D)\underline{\mathcal{M}}^D(A, \Omega) \le \operatorname{res}(\zeta_A(\,\cdot\,, \Omega), D) \le (N - D)\overline{\mathcal{M}}^D(A, \Omega). \tag{2.1.9}$$

Furthermore, if (A, Ω) is Minkowski measurable, then

$$\operatorname{res}(\zeta_A(\,\cdot\,, \Omega), D) = (N - D)\mathcal{M}^D(A, \Omega). \tag{2.1.10}$$

The above theorem can also be reformulated in terms of the relative tube zeta function and in that case we can remove the condition $\dim_B(A, \Omega) < N$.

Theorem 2.9 (Cited from [LapRaŽu1])**.** *Assume that (A, Ω) is a nondegenerate RFD in $\mathbb{R}^N$ (so that $D := \dim_B(A, \Omega)$ exists), and that for some $\delta > 0$ there exists a meromorphic extension of $\widetilde{\zeta}_A(\,\cdot\,, \Omega; \delta)$ to a neighborhood of D. Then, D is a simple pole, and $\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,, \Omega), D)$ is independent of δ. Furthermore, we have*

$$\underline{\mathcal{M}}^D(A, \Omega) \le \operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,, \Omega), D) \le \overline{\mathcal{M}}^D(A, \Omega). \tag{2.1.11}$$

In particular, if (A, Ω) is Minkowski measurable, then

$$\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,, \Omega), D) = \mathcal{M}^D(A, \Omega). \tag{2.1.12}$$

Theorem 2.10 (Scaling property, cited from [LapRaŽu1, Chapter 4])**.** *Let $\zeta_A(s,\Omega)$ be the relative distance zeta function of (A,Ω). Then, for any positive real number λ, we have that $D(\zeta_{\lambda A}(\,\cdot\,,\lambda\Omega;\lambda\delta)) = D(\zeta_A(\,\cdot\,,\Omega;\delta)) = \overline{\dim}_B(A,\Omega)$ and*

$$\zeta_{\lambda A}(s,\lambda\Omega;\lambda\delta) = \lambda^s\zeta_A(s,\Omega;\delta), \tag{2.1.13}$$

for $\operatorname{Re} s > \overline{\dim}_B(A,\Omega)$ and any $\lambda > 0$.

Furthermore, assume that $\zeta_A(\,\cdot\,,\Omega;\delta)$ admits a meromorphic continuation to some open connected neighborhood U of the open half-plane $\{\operatorname{Re} s > \overline{\dim}_B(A,\Omega)\}$. Then, so is the case for $\zeta_{\lambda A}(\,\cdot\,,\lambda\Omega;\lambda\delta)$ and the identity (2.1.13) *continues to hold for every $s \in U$ which is not a pole of $\zeta_A(\,\cdot\,,\Omega;\delta)$* (*and hence, of $\zeta_{\lambda A}(\,\cdot\,,\lambda\Omega;\lambda\delta)$ as well*)*.*

Moreover, if we assume, for simplicity,[3] *that ω is a simple pole of $\zeta_A(\,\cdot\,,\Omega;\delta)$* (*and hence also, of $\zeta_{\lambda A}(\,\cdot\,,\lambda\Omega;\lambda\delta)$*)*, then the following identity holds*:

$$\operatorname{res}(\zeta_{\lambda A}(\,\cdot\,,\lambda\Omega),\omega) = \lambda^\omega \operatorname{res}(\zeta_A(\,\cdot\,,\Omega),\omega). \tag{2.1.14}$$

Remark 2.11. An entirely analogous theorem to the one above is also valid for the relative tube zeta function of (A,Ω) Furthermore, note that by taking in the above theorem $\delta > 0$ such that $\Omega \subseteq A_\delta$ we have that $\zeta_{\lambda A}(s,\lambda\Omega) = \lambda^s\zeta_A(s,\Omega)$.

We shall also state the following simple scaling property of the tube functions and Minkowski contents of relative fractal drums. We note that Relation (2.1.16) below yields a partial extension of [Žu4, Proposition 4.4.].

Lemma 2.12 (Cited from [LapRaŽu1])**.** *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$. Then for any fixed $\lambda > 0$, and for all $t > 0$, we have*

$$(\lambda A)_t \cap \lambda\Omega = \lambda(A_{t/\lambda} \cap \Omega), \quad |(\lambda A)_t \cap \lambda\Omega| = \lambda^N |A_{t/\lambda} \cap \Omega|. \tag{2.1.15}$$

Furthermore, for any real parameter $r \in \mathbb{R}$, we have

$$\overline{\mathcal{M}}^r(\lambda A,\lambda\Omega) = \lambda^r\overline{\mathcal{M}}^r(A,\Omega), \quad \underline{\mathcal{M}}^r(\lambda A,\lambda\Omega) = \lambda^r\underline{\mathcal{M}}^r(A,\Omega). \tag{2.1.16}$$

2.2 Cone Property of Relative Fractal Drums

In this section we introduce the cone property of a relative fractal drum (A,Ω) at a point, in order to ensure that the abscissa of convergence of the associated relative zeta function $\zeta_{(A,\Omega)}$ be nonnegative. We also construct a simple class of relative fractal drums for which the relative box dimension is negative; see Proposition 2.18.

[3] If s is a multiple pole, then an analogous statement can be made about the principal parts (instead of the residues) of the zeta functions involved, as the reader can easily verify.

Definition 2.13 (Cited from [LapRaŽu1]). Let $B_r(a)$ be a given ball in $\mathbb{R}^N$ of radius r. Let ∂B be the boundary of the ball, which is an $(N-1)$-dimensional sphere, and assume that G is a closed connected subset contained in a hemisphere of ∂B.[4] We assume that G is open with respect to the relative topology of ∂B. The *cone* $K = K_r(a, G)$ *with vertex at* a, and of radius r, is defined as the interior of the convex hull of the union of $\{a\}$ and G.

Definition 2.14 (Cited from [LapRaŽu1]). Let (A, Ω) be a relative fractal drum in $\mathbb{R}^N$. We say that a *relative fractal drum* (A, Ω) *has the cone property at a point* $a \in \overline{A} \cap \overline{\Omega}$ if there exists $r > 0$ such that Ω contains a cone $K_r(a, G)$ with vertex at a (and of radius r).

Remark 2.15 (Cited from [LapRaŽu1]). If $a \in \overline{A} \cap \Omega$ (hence, a is an inner point of Ω), then the cone property of the relative fractal drum (A, Ω) is obviously satisfied at this point. So, the cone property is actually interesting only on the boundary of Ω, that is, at $a \in \overline{A} \cap \partial\Omega$.

It is not difficult to construct a domain Ω such that its boundary $\partial\Omega$ (more precisely, the relative fractal drum $(\partial\Omega, \Omega)$) does not satisfy the cone property at any of its points. For example, it suffices to consider a bounded domain Ω in the plane the boundary $\partial\Omega$ of which is locally representable as the graph of the Weierstrass function. Furthermore, we note that the relative fractal drum (A, Ω) described in Example 2.16 just below does not satisfy the cone property.

Example 2.16 (Cited from [LapRaŽu1]). Given $\alpha > 0$, let (A, Ω_α) be the relative fractal drum in $\mathbb{R}^2$ defined by $A = \{(0, 0)\}$ and $\Omega_\alpha = \{(x, y) \in \mathbb{R}^2 : 0 < y < x^\alpha,\ x \in (0, 1)\}$. If $0 < \alpha \leq 1$, then the cone property of (A, Ω) is fulfilled at $a = (0, 0)$, while it is not satisfied (at $a = (0, 0)$) for $\alpha > 1$. Using these domains, we can construct a one-parameter family of relative fractal drums with negative box dimension; see Proposition 2.18 below.

Proposition 2.17 (Cited from [LapRaŽu1]). Let (A, Ω) be a relative fractal drum in $\mathbb{R}^N$.

(*a*) If the sets A and Ω are a positive distance apart (i.e., if $d(A, \Omega) > 0$), then $D(\zeta_A(\,\cdot\,, \Omega)) = -\infty$; that is, $\zeta_A(\,\cdot\,, \Omega)$ is an entire function. Furthermore, $\dim_B(A, \Omega) = -\infty$.

(*b*) Assume that there exists at least one point $a \in \overline{A} \cap \overline{\Omega}$ at which the relative fractal drum (A, Ω) satisfies the cone property. Then $D(\zeta_A(\,\cdot\,, \Omega)) \geq 0$.[5]

The following proposition (building on Example 2.16 above) shows that the box dimension of a relative fractal drum can be negative.

[4]Intuitively, G is a disk-like subset ('calotte') of a hemisphere contained in the sphere ∂B.

[5]The cone condition can be replaced by a much weaker condition; see [LapRaŽu1] for details.

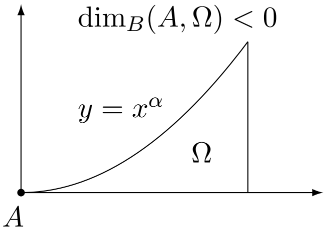

Figure 2.1: A relative fractal drum (A,Ω) with negative box dimension $\dim_B(A,\Omega)=1-\alpha<0$ (here $\alpha>1$), due to the 'flatness' of the open set Ω at A; see Proposition 2.18.

Proposition 2.18 (Cited from [LapRaŽu1])**.** Let $A=\{(0,0)\}$ and

$$\Omega=\{(x,y)\in\mathbb{R}^2:0<y<x^\alpha,\ x\in(0,1)\},\tag{2.2.1}$$

where $\alpha>1$; see Figure 2.1. Then the relative fractal drum (A,Ω) has a negative box dimension. More specifically, $\dim_B(A,\Omega)$ exists, the relative fractal drum (A,Ω) is Minkowski measurable and

$$\begin{aligned}\dim_B(A,\Omega)=D(\zeta_A(\,\cdot\,,\Omega))&=1-\alpha<0,\\ \mathcal{M}^{1-\alpha}(A,\Omega)&=\frac{1}{1+\alpha},\\ D_{\rm mer}(\zeta_A(\,\cdot\,,\Omega))&\le 3(1-\alpha).\end{aligned}\tag{2.2.2}$$

Furthermore, $s=1-\alpha$ is a simple pole of $\zeta_A(\,\cdot\,,\Omega)$.

Example 2.19. Let (A,Ω) be the relative fractal drum in $\mathbb{R}^2$ defined by $A=\{(0,0)\}$ and $\Omega=\{(x,y)\in\mathbb{R}^2:0<y<x^2,\ x\in(0,1)\}$. This relative fractal drum does not satisfy the cone property.[6] According to Proposition 2.18, its relative box dimension is equal to -1. We will show directly that the relative distance zeta function $\zeta_A(s,\Omega)$ is well defined at $s=0$, and equal to Catalan's constant. First, using polar coordinates, for $s>0$ we have

$$\begin{aligned}\zeta_A(s,\Omega)&=\int_\Omega d((x,y),A)^{s-2}\,\mathrm{d}x\,\mathrm{d}y=\int_0^1\mathrm{d}x\int_0^{x^2}(x^2+y^2)^{s-2}\,\mathrm{d}y\\ &=\int_0^{\pi/4}\mathrm{d}\varphi\int_{\tan\varphi/\cos\varphi}^{1/\cos\varphi}r^{s-1}\,\mathrm{d}r=\frac{1}{s}\int_0^{\pi/4}\frac{1-\tan^s\varphi}{\cos^s\varphi}\,\mathrm{d}\varphi.\end{aligned}$$

The function under the integral sign is dominated by a constant (independent of s), so we conclude from the Lebesgue dominated convergence theorem that the integral in the last expression above converges to zero. We can now apply l'Hospital's rule and differentiation

[6]Note that since $\overline{A}\cap\overline{\Omega}=\{(0,0)\}$, it suffices to check that (A,Ω) does not have the cone property at $a=(0,0)$, which is the case since $2>1$; see Remark 2.15 and Example 2.16.

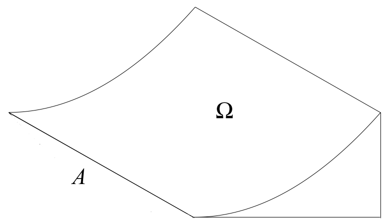

Figure 2.2: A relative fractal drum (A,Ω) with infinite flatness, described in Remark 2.21. In other words, Ω has infinite flatness near A.

under the integral sign in order to compute the limit at $s=0$:

$$\begin{aligned}
\lim_{s\to 0^+}\zeta_A(s,\Omega) &= \lim_{s\to 0^+}\int_0^{\pi/4}\frac{\partial}{\partial s}\Big(\frac{1-\tan^s\varphi}{\cos^s\varphi}\Big)\,\mathrm{d}\varphi\\
&= \lim_{s\to 0^+}\int_0^{\pi/4}\bigg[\Big(\frac{\tan\varphi}{\cos\varphi}\Big)^s\log(\cot\varphi)+\frac{\log(\cos\varphi)}{\cos^s\varphi}(\tan^s\varphi-1)\bigg]\,\mathrm{d}\varphi\\
&= \int_0^{\pi/4}\log(\cot\varphi)\,\mathrm{d}\varphi=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}.
\end{aligned}$$

The next-to-last equality again follows from an application of Lebesgue's dominated convergence theorem, while the last sum is Catalan's constant which is approximately equal to 0.915.

The following result provides an example of a nontrivial relative fractal drum (A,Ω) such that $\dim_B(A,\Omega)=-\infty$. It suffices to construct a domain Ω in $\mathbb{R}^2$ which is *flat* in a neighborhood of one of its boundary points.

Proposition 2.20 (Cited from [LapRaŽu1])**.** *Let* $A=\{(0,0)\}$ *and*

$$\Omega'=\{(x,y)\in\mathbb{R}^2 : 0<y<\mathrm{e}^{-1/x},\ 0<x<1\}. \tag{2.2.3}$$

Then $\dim_B(A,\Omega')$ *exists and*

$$\dim_B(A,\Omega')=D(\zeta_A(\,\cdot\,,\Omega'))=-\infty. \tag{2.2.4}$$

Remark 2.21. It is easy to see that Proposition 2.20 can be significantly generalized. For example, it suffices to assume that A is a point on the boundary of Ω such that Ω has the *flatness property of* A *relative to* Ω. This can even be formulated in terms of subsets A. We can imagine a bounded open set $\Omega\subset\mathbb{R}^3$ with a Lipschitz boundary $\partial\Omega$, except on a subset $A\subset\partial\Omega$, which may be a line segment, near which Ω is flat. A simple construction of such a set is $\Omega=\Omega'\times(0,1)$, where Ω' is given as in Corollary 2.20, and $A=(0,0)\}\times(0,1)$; see Equation (2.2.3). Note that this domain is not Lipschitz near the points of A, and not even Hölderian; see Figure 2.2. The *flatness of a relative fractal* *drum* (A,Ω) can be defined by

$$\mathrm{fl}(A,\Omega)=\left(\overline{\dim}_B(A,\Omega)\right)^-,$$

where $(r)^- := \max\{0,-r\}$ is the negative part of a real number r. *We say that the flatness of* (A,Ω) *is nontrivial if* $\mathrm{fl}(A,\Omega)>0$*, that is, if* $\overline{\dim}_B(A,\Omega)<0$. In the example just mentioned above, we have a relative fractal drum (A,Ω) with infinite flatness, i.e., with $\mathrm{fl}(A,\Omega)=+\infty$. Intuitively, it can be viewed as an 'ax' with an 'infinitely sharp' blade.

Chapter 3

Fractal Tube Formulas for Relative Fractal Drums

3.1 Introduction

The core of this chapter is in the fact that the tube zeta function equal to the Mellin transform of a modification of the tube function $t \mapsto |A_t \cap \Omega|$. More precisely, one has that

$$\widetilde{\zeta}_A(s, \Omega; \delta) = \int_0^{+\infty} t^{s-1} \left(\chi_{(0,\delta)}(t) t^{-N} |A_t \cap \Omega| \right) \mathrm{d}t =: \{\mathfrak{M} f\}(s), \tag{3.1.1}$$

where $\chi_{(0,\delta)}$ denotes the characteristic function of the set $(0, \delta)$ and $\{\mathfrak{M} f\}$ the Mellin transform of the function f. (Here, $f(t) = \chi_{(0,\delta)}(t) t^{-N} |A_t \cap \Omega|$.)

We continue by stating a few definitions that are already introduced in [Lap–vFr3] in the setting of generalized fractal strings and adapt them to the setting of relative fractal drums in $\mathbb{R}^N$.

Definition 3.1. The *screen* S is a graph of a bounded, real-valued Lipschitz continuous function $S(\tau)$, with the horizontal and vertical axes interchanged:

$$S := \{S(\tau) + \mathrm{i}\tau \, : \, \tau \in \mathbb{R}\}. \tag{3.1.2}$$

The Lipschitz constant will be denoted by $\|S\|_{\mathrm{Lip}}$, i.e.,

$$|S(x) - S(y)| \leq \|S\|_{\mathrm{Lip}} |x - y|, \qquad \text{for all } x, y, \in \mathbb{R}.$$

Furthermore, with the screen the following finite quantities will be associated:

$$\inf S := \inf_{\tau \in \mathbb{R}} S(\tau) \qquad \text{and} \qquad \sup S := \sup_{\tau \in \mathbb{R}} S(\tau).$$

From now on for an RFD (A, Ω) in $\mathbb{R}^N$ we will denote its upper relative box dimension by $\overline{D} := \overline{\dim}_B(A,\Omega) \leq N$. We will assume, additionally, that the screen S lies always to the left of the *critical line* $\{\operatorname{Re} s = \overline{D}\}$, i.e., that $\sup S \leq \overline{D}$. Furthermore the *window* W is defined as the part of the complex plane to the right of S; that is,

$$W := \{s \in \mathbb{C} : \operatorname{Re} s \geq S(\operatorname{Im} s)\}. \tag{3.1.3}$$

We will say that the relative fractal drum (A,Ω) is *admissible* if its relative tube (or distance) zeta function can be meromorphically extended to an open connected neighborhood of some window W.

The next definition adapts [Lap–vFr3, Definition 5.2] to the case of relative fractal drums in $\mathbb{R}^N$ (and, in particular, to the case of bounded subsets of $\mathbb{R}^N$):

Definition 3.2 (Languidity adapted from [Lap–vFr3])**.** An admissible relative fractal drum (A,Ω) in $\mathbb{R}^N$ is said to be *languid* if for some $\delta > 0$ its tube zeta function $\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta)$ satisfies the following growth conditions: There exist real constants κ and $C > 0$ and a two-sided sequence $(T_n)_{n\in\mathbb{Z}}$ of real numbers such that $T_{-n} < 0 < T_n$ for $n \geq 1$ and

$$\lim_{n\to\infty} T_n = +\infty, \qquad \lim_{n\to\infty} T_{-n} = -\infty \tag{3.1.4}$$

satisfying the following two hypotheses[1]

L1 For all $n \in \mathbb{Z}$ and all $\sigma \in (S(T_n), c)$,

$$|\widetilde{\zeta}_A(\sigma + \mathbb{i}T_n,\Omega;\delta)| \leq C(|T_n|+1)^{\kappa}, \tag{3.1.5}$$

where $c > \overline{\dim}_B(A,\Omega)$ is some constant.[2]

L2 For all $\tau \in \mathbb{R}$, $|\tau| \geq 1$,

$$|\widetilde{\zeta}_A(S(\tau) + \mathbb{i}\tau,\Omega;\delta)| \leq C|\tau|^{\kappa}. \tag{3.1.6}$$

Note that hypothesis **L1** is a polynomial growth condition along horizontal segments (necessarily not passing through any singularities of $\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta)$), while hypothesis **L2** is a polynomial growth condition along the vertical direction of the screen. These hypotheses will be needed to establish the pointwise and distributional tube formulas with error term.

In order to obtain the pointwise and distributional tube formulas without an error term, we will need a stronger notion of languidity and hence we introduce the next definition.

[1]Here we do not need the assumption that $\lim_{n\to+\infty} T_n/|T_{-n}| = 1$ as opposed to the definition in [Lap–vFr3].

[2]This is a slight modification of the original definition of languidity from [Lap–vFr3] where c was replaced by $+\infty$.

Definition 3.3 (Strong languidity adapted from [Lap–vFr3]). We say that a relative fractal drum (A,Ω) in $\mathbb{R}^N$ is *strongly languid* if for some $\delta>0$ its tube zeta function satisfies condition **L1** with $S(T_n)\equiv-\infty$ in (3.1.5), i.e., for every $\sigma<c$ and, additionally, there exists a sequence of screens $S_m(\tau)\colon\tau\mapsto S_m(\tau)+\mathbb{i}\tau$ for $m\geq 1$, $\tau\in\mathbb{R}$ with $\sup S_m\to-\infty$ as $m\to\infty$ and with a uniform Lipschitz bound $\sup_{m\geq1}\|S_m\|_{\mathrm{Lip}}<\infty$, such that

L2' there exist constants $B,C>0$ such that for all $\tau\in\mathbb{R}$ and $m\geq 1$,

$$|\widetilde{\zeta}_A(S_m(\tau)+\mathbb{i}\tau,\Omega;\delta)|\leq CB^{|S_m(\tau)|}(|\tau|+1)^{\kappa}. \tag{3.1.7}$$

It is obvious that hypothesis **L2'** implies hypothesis **L2**, so that a strongly languid relative fractal drum is languid. We also note that if a relative fractal drum is languid for some κ, then it is also languid for any $\kappa_1>\kappa$. We will also use the notions of languid and strongly languid relative tube zeta function in the obvious sense.

Recall from (3.1.1) that the tube zeta function of a relative fractal drum (A,Ω) with a fixed $\delta>0$ satisfies

$$\widetilde{\zeta}_A(s,\Omega;\delta)=\{\mathfrak{M}f\}(s), \tag{3.1.8}$$

where $f(t)=\chi_{(0,\delta)}(t)t^{-N}|A_t\cap\Omega|$ is locally of bounded variation since $t\mapsto|A_t\cap\Omega|$ is nondecreasing. Furthermore, observe that f is piecewise continuous on $(0,+\infty)$ and we have from Theorem 2.6 and Remark 2.7 that the tube zeta function $\widetilde{\zeta}_A(\,\cdot\,,\Omega)$ is holomorphic on the half-plane $\{\operatorname{Re}s>\overline{\dim}_B(A,\Omega)\}$. This ensures that we can recover the relative tube function from the relative tube zeta function by the Mellin inversion:

$$\chi_{(0,\delta)}(t)t^{-N}|A_t\cap\Omega|=\frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty}t^{-s}\widetilde{\zeta}_A(s,\Omega;\delta)\,\mathrm{d}s; \tag{3.1.9}$$

where $c>\overline{\dim}_B(A,\Omega)$ is arbitrary, which leads to the following theorem.

Theorem 3.4. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ and fix $\delta>0$. Then for every $t\in(0,\delta)$ and $c>\overline{\dim}_B(A,\Omega)$ we have*

$$|A_t\cap\Omega|=\frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty}t^{N-s}\widetilde{\zeta}_A(s,\Omega;\delta)\,\mathrm{d}s. \tag{3.1.10}$$

This fact follows directly from the Mellin inversion theorem which we state here for completeness (see, e.g. [Ti2, Theorem 28]).

Theorem 3.5 (Mellin inversion theorem). *Let $y^{c-1}f(y)$ belong to $L^1(0,\infty)$ where $f\colon(0,\infty)\to\mathbb{R}$ and c is a real number. Furthermore, let $f(y)$ be of bounded variation in a neighborhood of the point $y=t$. Let*

$$\{\mathfrak{M}f\}(s):=\int_0^{+\infty}t^{s-1}f(t)\,\mathrm{d}t\quad(s=c+\mathbb{i}\tau). \tag{3.1.11}$$

Then

$$\frac{1}{2}(f(t+0)+f(t-0)) = \frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} t^{-s}\{\mathfrak{M}f\}(s)\,\mathrm{d}s. \tag{3.1.12}$$

Conversely, let $\{\mathfrak{M}f\}(c+\mathbb{i}u)$ *belong to* $L^1(-\infty,\infty)$*, and let it be of bounded variation in a neighborhood of the point* $u=\tau$*. Let*

$$f(t) := \frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} t^{-s}\{\mathfrak{M}f\}(s)\,\mathrm{d}s. \tag{3.1.13}$$

Then

$$\frac{1}{2}\Big(\{\mathfrak{M}f\}\big(c+\mathbb{i}(\tau+0)\big)+\{\mathfrak{M}f\}\big(c+\mathbb{i}(\tau-0)\big)\Big) = \int_0^{+\infty} t^{c+\mathbb{i}\tau-1}f(t)\,\mathrm{d}t. \tag{3.1.14}$$

Let us now introduce the notion of complex dimensions of a relative fractal drum.

Definition 3.6 (Complex dimensions of a relative fractal drum [LapRaŽu1])**.** Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$. Assume that the relative tube zeta function $\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta)$ has a meromorphic extension to a connected neighborhood U of the critical line $\{\mathrm{Re}\,s = \overline{\dim}_B(A,\Omega)\}$. Then, the set of *visible complex dimensions of* (A,Ω) (*with respect to* U) is the set of poles of $\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta)$ that are contained in U and we denote it with

$$\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta),U) := \{\omega\in U : \omega \text{ is a pole of } \widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta)\}. \tag{3.1.15}$$

If $U=\mathbb{C}$, we say that $\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta),\mathbb{C})$ is the set of *complex dimensions* of (A,Ω) and denote it by $\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta))$.

Furthermore, we call the set of poles located on the critical line $\{\mathrm{Re}\,s = \overline{\dim}_B(A,\Omega)\}$ the set of *principal complex dimensions of* (A,Ω) and denote it by

$$\dim_{PC}(A,\Omega) := \{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta),U) : \mathrm{Re}\,\omega = \overline{\dim}_B(A,\Omega)\} \tag{3.1.16}$$

Remark 3.7. In light of the functional equation (2.1.4) and the relevant discussion about it, the above definition can also be made in terms of the relative distance zeta function; that is, we always have $\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta),U) = \mathcal{P}(\zeta_A(\,\cdot\,,\Omega;\delta),U)$ whenever one of the above zeta functions has a meromorphic extension to the domain U containing the critical line $\{\mathrm{Re}\,s = \overline{\dim}_B(A,\Omega)\}$ and if $N\notin\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta),U)$.[3] Furthermore, according to Remark 2.2, the set of (visible) complex dimensions $\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta),U)$ of a relative fractal drum (A,Ω) does not depend on δ.

For deriving the relative tube formula in terms of the complex dimensions of the relative fractal drum (A,Ω) we will need the k-th primitive function of the relative tube

[3]In that case, the other one also has a meromorphic continuation to U.

function and, hence, we let

$$V(t) = V^{[0]}(t) := |A_t \cap \Omega| \tag{3.1.17}$$

and

$$V^{[k]}(t) := \int_0^t V^{[k-1]}(\tau)\,\mathrm{d}\tau, \qquad \text{for } k \in \mathbb{N}. \tag{3.1.18}$$

Furthermore, for any $s \in \mathbb{C}$ we recall that the *Pochammer* symbol is defined by

$$(s)_0 := 1, \quad (s)_k := s(s+1)\cdots(s+k-1) \tag{3.1.19}$$

for any nonnegative integer k.

Proposition 3.8. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ and $\delta > 0$ fixed. Then for $t \in (0,\delta)$, $c \in (\overline{\dim}_B(A,\Omega), N+1)$ and $k \in \mathbb{N}_0$ we have*

$$V^{[k]}(t) = \frac{1}{2\pi\mathbb{i}} \int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} \frac{t^{N-s+k}}{(N-s+1)_k} \widetilde{\zeta}_A(s,\Omega;\delta)\,\mathrm{d}s. \tag{3.1.20}$$

Proof. For $t \in (0,\delta)$ we calculate

$$\begin{aligned} V^{[1]}(t) = \int_0^t V(\tau)\,\mathrm{d}\tau &= \frac{1}{2\pi\mathbb{i}} \int_0^t \int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} \tau^{N-s}\widetilde{\zeta}_A(s,\Omega;\delta)\,\mathrm{d}s\,\mathrm{d}\tau \\ &= \frac{1}{2\pi\mathbb{i}} \int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} \int_0^t \tau^{N-s}\widetilde{\zeta}_A(s,\Omega;\delta)\,\mathrm{d}\tau\,\mathrm{d}s \\ &= \frac{1}{2\pi\mathbb{i}} \int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} \frac{t^{N-s+1}}{N-s+1}\widetilde{\zeta}_A(s,\Omega;\delta)\,\mathrm{d}s, \end{aligned}$$

since $N - c + 1 > 0$. The change of the order of integration is justified by combining Lebesgue's dominated convergence theorem and the Fubini–Tonelli theorem. Iterating this calculation $k-1$ more times we prove the statement of the proposition. ☐

We will adapt the following definition of the truncated screen and window from [Lap–vFr3] where it was stated for languid generalized fractal strings and can be used in the same form in the case of relative fractal drums in $\mathbb{R}^N$. Note that here the definition of the truncated window is lightly modified in contrast to [Lap–vFr3,], and hence, we will use a slightly different notation.

Definition 3.9 (The truncated screen and window)**.** Given an integer $n \geq 1$ and a (strongly) languid relative fractal drum in $\mathbb{R}^N$, the *truncated screen* $S_{|n}$ is the part of the screen S restricted to the interval $[T_{-n}, T_n]$, and the *truncated window* $W_{|n}$ is the window W intersected with the horizontal strip between T_{-n} and T_n, i.e.,

$$W_{|n} := W \cap \{s \in \mathbb{C} : T_{-n} \leq \operatorname{Im} s \leq T_n\}. \tag{3.1.21}$$

We then call $\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega), W_{|n})$ the set of *truncated visible complex dimensions*, i.e., it is the set of visible complex dimensions of (A,Ω) with imaginary part between T_{-n} and T_n.

3.2 Pointwise Tube Formula

In this section we will derive the relative pointwise tube formula of a relative fractal drum (A,Ω) in $\mathbb{R}^N$ in terms of its complex dimensions. The technique used is very similar to the one developed in [Lap–vFr3] in the case of the geometric zeta functions of fractal strings for finding the pointwise and distributional explicit formulas and follows basically the same steps but in the new context of relative fractal drums. Furthermore, we stress that from now on the phrase "let (A,Ω) be a languid (or strongly languid) relative fractal drum" implicitly means that (A,Ω) is admissible for some window W and for some $\delta>0$ the relative tube zeta function $\widetilde{\zeta}_A(s,\Omega;\delta)$ of (A,Ω) satisfies the languidity conditions of Definition 3.2 (or Definition 3.3, respectively). First, let us derive a 'truncated pointwise formula' from which the general theorem will follow.

Lemma 3.10 (Truncated pointwise formula)**.** *Let $k\geq 0$ be an integer and (A,Ω) a languid relative fractal drum in $\mathbb{R}^N$. Furthermore, fix a constant $c\in(\overline{\dim}_B(A,\Omega),N+1)$. Then for all $t\in(0,\delta)$ and $n\geq 1$ we have*

$$\begin{aligned}I_n &:= \frac{1}{2\pi\mathrm{i}}\int_{c+\mathrm{i}T_{-n}}^{c+\mathrm{i}T_n}\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}_A(s,\Omega;\delta)\,\mathrm{d}s\\ &= \sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W_{|n})}\operatorname{res}\left(\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}_A(s,\Omega;\delta),\omega\right)\\ &\quad+\frac{1}{2\pi\mathrm{i}}\int_{S_{|n}}\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}_A(s,\Omega;\delta)\,\mathrm{d}s+E_n(t).\end{aligned}\tag{3.2.1}$$

Moreover, assuming that hypothesis **L1** *is fulfilled, we have the following remainder estimate*

$$|E_n(t)|\leq t^{N+k}K_\kappa\max\{T_n^{\kappa-k},|T_{-n}|^{\kappa-k}\}(c-\inf S)\max\{t^{-c},t^{-\inf S}\}\tag{3.2.2}$$

where K_κ is a constant depending only on κ.

Finally, for each point $s=S(\tau)+\mathrm{i}\tau$ where $\tau\in\mathbb{R}$ such that $|\tau|>1$ and for all $t\in(0,\delta)$ the integrand over the truncated screen appearing in (3.2.1) *is bounded in absolute value by:*

$$Ct^{N+k}\max\{t^{-\sup S},t^{-\inf S}\}|\tau|^{\kappa-k},\tag{3.2.3}$$

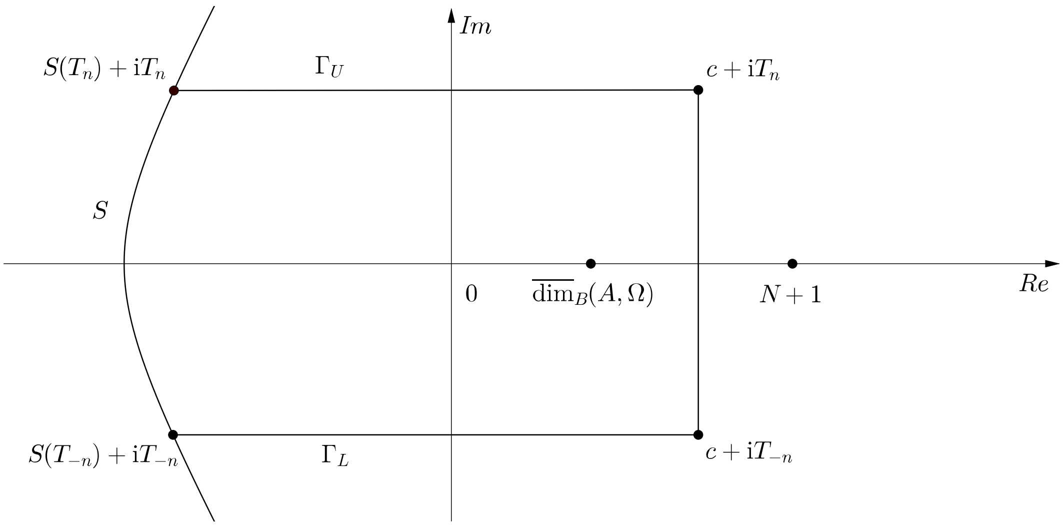

Figure 3.1: The truncated window $W_{|n}$ and the contour Γ which we use to estimate the integral I_n in Lemma 3.10.

when hypothesis **L2** *holds, and by*

$$C_\kappa t^{N+k}\max\{B^{|\inf S|},B^{|\sup S|}\}\max\{t^{-\sup S},t^{-\inf S}\}|\tau|^{\kappa-k}, \tag{3.2.4}$$

when hypothesis **L2'** *holds with the constant C_κ depending only on κ.*

Proof. Let $\overline{D}:=\overline{\dim}_B(A,\Omega)$ and for the sake of brevity we will write $\widetilde{\zeta}(s)$ instead of $\widetilde{\zeta}_A(s,\Omega;\delta)$ throughout the proof. Now, we replace the integral over the segment $[c+\mathbb{i}T_{-n},c+\mathbb{i}T_n]$ with the integral over the contour Γ consisting of this segment, the truncated screen $S_{|n}$ and the two horizontal segments joining $S(T_{\pm n})+\mathbb{i}T_{\pm n}$ and $c+\mathbb{i}T_{\pm n}$ (see Figure 3.1). In other words, we have:

$$\begin{aligned}I_n&=\frac{1}{2\pi\mathbb{i}}\int_{c+\mathbb{i}T_{-n}}^{c+\mathbb{i}T_n}\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}(s)\,\mathrm{d}s\\&=\frac{1}{2\pi\mathbb{i}}\oint_\Gamma\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}(s)\,\mathrm{d}s\\&\quad+\frac{1}{2\pi\mathbb{i}}\int_{S_{|n}}\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}(s)\,\mathrm{d}s+E_n(t),\end{aligned}$$

where

$$E_n(t):=\frac{1}{2\pi\mathbb{i}}\int_{\Gamma_L\cup\Gamma_U}\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}(s)\,\mathrm{d}s$$

Furthermore, the integrand appearing above is meromorphic on the bounded domain having Γ as its boundary and its poles are exactly the poles of the relative tube zeta function since $c\in(\overline{\dim}_B(A,\Omega),N+1)$ ensures that there are no zeroes of $(N-s+1)_k$

inside of Γ. Consequently, from the residue theorem we get

$$\begin{aligned} I_n = &\sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W_{|n})} \operatorname{res}\left(\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}(s),\omega\right) \\ &+\frac{1}{2\pi\mathbb{i}}\int_{S_{|n}}\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}(s)\,\mathrm{d}s+E_n(t). \end{aligned}$$

To get the bound on $E_n(t)$, firstly we observe that for $s=\sigma+\mathbb{i}T_n$ we have $|(N-s+1)_k|\geq T_n^k$ and we estimate the integrals over the upper segment Γ_U and the lower segment Γ_L under the hypothesis **L1**:

$$\begin{aligned} \left|\int_{\Gamma_U}\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}(s)\,\mathrm{d}s\right| &= \left|\int_{S(T_n)}^{c}\frac{t^{N+k-\sigma-\mathbb{i}T_n}}{(N+1-(\sigma+\mathbb{i}T_n))_k}\widetilde{\zeta}(\sigma+\mathbb{i}T_n)\,\mathrm{d}\sigma\right| \\ &\leq t^{N+k}C(T_n+1)^{\kappa}T_n^{-k}\int_{S(T_n)}^{c}t^{-\sigma}\,\mathrm{d}\sigma \\ &\leq t^{N+k}K_\kappa T_n^{\kappa-k}(c-S(T_n))\max\{t^{-c},t^{-S(T_n)}\}, \end{aligned}$$

where K_κ is a constant such that $C(|T_n|+1)^\kappa\leq K_\kappa|T_n|^\kappa$ for all $n\in\mathbb{Z}$. Furthermore, since $\inf S\leq S(t)$ we have

$$\left|\int_{\Gamma_U}\frac{t^{N-s+k}\widetilde{\zeta}(s)\,\mathrm{d}s}{(N-s+1)_k}\right|\leq t^{N+k}K_\kappa T_n^{\kappa-k}(c-\inf S)\max\{t^{-c},t^{-\inf S}\} \tag{3.2.5}$$

A similar calculation for the integral over the lower line segment yields

$$\left|\int_{\Gamma_U}\frac{t^{N-s+k}\widetilde{\zeta}(s)\,\mathrm{d}s}{(N-s+1)_k}\right|\leq t^{N+k}K_\kappa |T_{-n}|^{\kappa-k}(c-\inf S)\max\{t^{-c},t^{-\inf S}\} \tag{3.2.6}$$

and putting (3.2.5) and (3.2.6) together we get (3.2.2).[4]

To estimate the integrand over the truncated screen $S_{|n}$ we observe that for $s=S(\tau)+\mathbb{i}\tau$ with $|\tau|>1$ we have

$$\begin{aligned} \left|\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}(s)\right| &\leq Ct^{N-S(\tau)+k}|\tau|^{\kappa-k} \\ &\leq Ct^{N+k}\max\{t^{-\sup S},t^{-\inf S}\}|\tau|^{\kappa-k} \end{aligned} \tag{3.2.7}$$

under hypothesis **L2** and similarly under hypothesis **L2'**. (C_κ is a constant chosen so that $C(|\tau|+1)^\kappa\leq C_\kappa|\tau|^\kappa$ holds for $|\tau|>1$.) This completes the proof of the lemma. □

We can now state and prove the announced theorem.

[4]The constant K_κ in (3.2.2) is actually equal to the present K_κ divided by π.

Theorem 3.11. *Let* (A,Ω) *be a relative fractal drum in* $\mathbb{R}^N$ *that is languid for some* κ *and let* $k>\kappa+1$ *be a nonnegative integer. Then for every* $t\in(0,\delta)$ *the following pointwise formula is valid*

$$V^{[k]}(t)=\sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W)}\operatorname{res}\left(\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}_A(s,\Omega;\delta),\omega\right)+R^{[k]}(t). \tag{3.2.8}$$

Here, for $t\in(0,\delta)$ *the error term* $R^{[k]}$ *is given by the absolutely convergent integral*

$$R^{[k]}(t)=\frac{1}{2\pi\mathbb{i}}\int_S\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}_A(s,\Omega;\delta)\,\mathrm{d}s. \tag{3.2.9}$$

Furthermore, for $t\in(0,\delta)$ *we have*

$$|R^{[k]}(t)|\leq\frac{C}{2\pi}(1+\|S\|_{\mathrm{Lip}})\frac{t^{N+k}\max\{t^{-\sup S},t^{-\inf S}\}}{k-\kappa-1}+C', \tag{3.2.10}$$

where C *is the constant appearing in* **L1** *and* **L2** *and* C' *is some suitable positive constant. These constants depend only on the relative fractal drum* (A,Ω) *and the screen, but not on* k*.*

In particular, we have the following pointwise error estimate

$$R^{[k]}(t)=O(t^{N-\sup S+k})\quad \textit{as } t\to 0^+. \tag{3.2.11}$$

Moreover, if $S(\tau)<\sup S$*, i.e., if the screen lies strictly left of the vertical line* $\{\operatorname{Re}s=\sup S\}$*, then we have*

$$R^{[k]}(t)=o(t^{N-\sup S+k})\quad \textit{as } t\to 0^+. \tag{3.2.12}$$

Remark 3.12. The sum appearing in (3.2.8) in the above theorem is to be understood as the limit

$$\lim_{n\to\infty}\sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W_{|n})}\operatorname{res}\left(\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}_A(s,\Omega;\delta),\omega\right),$$

where $W_{|n}$ is the truncated window given by Definition 3.9. Furthermore, the existence of this limit follows from the proof of the theorem but we do not know the nature of its convergence.

Furthermore, the sum over the set $\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W)$ in Theorem 3.11 does not depend on δ since changing the parameter δ has no effect on the residues appearing in (3.2.8) (see Remark 2.2). In other words, when applying Theorem 3.11, one has to determine that (A,Ω) is languid for some $\delta>0$, but when calculating the sum one can take any $\delta>0$; that is, the most convenient one in the particular example one is interested in. The same comment also applies in all other theorems below in which a sum over the complex dimensions appears.

Proof. Let $c \in (\overline{\dim}_B(A,\Omega), N+1)$ be the constant from the languidity condition **L1** of Definition 3.2. We will prove the theorem by using Lemma 3.10 to get (3.2.1) and letting $n \to \infty$. We note that $E_n(t)$ tends to zero for $k > \kappa$ at the rate of some negative power of $\min\{T_n, |T_{-n}|\}$. Furthermore, for $k > \kappa+1$, the error term $R^{[k]}(t)$ is absolutely convergent. Indeed, note that, since $S(\tau)$ is Lipschitz continuous, it is differentiable almost everywhere and, consequently, the derivative of $\tau \mapsto S(\tau)+\mathbb{i}\tau$ is bounded with $(1+\|S\|_{\mathrm{Lip}})$ for almost all $\tau \in \mathbb{R}$. Moreover, since

$$\int_1^{+\infty} \tau^{\kappa-k}\,\mathrm{d}\tau = \frac{1}{k-\kappa-1}$$

for $k > \kappa+1$, the estimate of the error term now follows from (3.2.3). Here C' is the constant which amounts to the integral over the part of the screen for which $|\tau| < 1$.

In the case when the screen stays to the left of the line $\{\operatorname{Re} s = \sup S\}$ we can obtain the better estimate (3.2.12) by using a well-known method; see, e.g. [In, pp. 33–34].[5] Namely, for any given $\varepsilon > 0$ we have to show that (3.2.9) is bounded with $\varepsilon t^{N-\sup S+k}$. For a $T > 0$ we can split the integral (3.2.9) into the following two parts. The first one, is the integral over the part of the screen for which $|\operatorname{Im} S| > T$ and the second one is the integral over the part of the screen for which $|\operatorname{Im} S| \le T$. Since the first integral is absolutely convergent, we can choose T sufficiently large so that it is bounded with $\frac{1}{2}\varepsilon t^{N-\sup S+k}$. For the second integral we observe that the maximum of $S(\tau)$ for $\tau \in [-T,T]$ is strictly less than $\sup S$, i.e., we can choose $\alpha > 0$ such that $S(\tau) < \sup S - \alpha$ for $\tau \in [-T,T]$. This implies that the integral over the part of the screen for which $|\operatorname{Im} S| \le T$ is of order $O(t^{N-\sup S+k+\alpha})$ as $t \to 0^+$.[6] Hence, for sufficiently small t it is bounded with $\frac{1}{2}\varepsilon t^{N-\sup S+k}$ and this proves that $R^{[k]}(t) = o(t^{N-\sup S+k})$ as $t \to 0^+$. □

In the case of a strongly languid relative fractal drum we get a pointwise formula without an error term.

Theorem 3.13. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ that is strongly languid for some κ and let $k > \kappa$ be a nonnegative integer. Then for every $t \in (0, \min\{1,\delta,B^{-1}\})$ the pointwise formula is given by*

$$V^{[k]}(t) = \sum_{\omega \in \mathcal{P}(\tilde{\zeta}_A(\,\cdot\,,\Omega),\mathbb{C})} \operatorname{res}\left(\frac{t^{N-s+k}}{(N-s+1)_k}\tilde{\zeta}_A(s,\Omega;\delta), \omega\right). \tag{3.2.13}$$

Here B is the constant appearing in **L2'** *and κ is the exponent occurring in the statement of hypotheses* **L1** *and* **L2'**.

Proof. For a fixed integer $n \ge 1$ we apply Lemma 3.10 with the screen S_m given by hypothesis **L2'**. We first let $m \to \infty$ while keeping n fixed. Since the screens S_m have

[5]One proves the estimate (3.2.11) in an analogous way by the same method.

[6]Note that since the screen S avoids the poles of the relative tube zeta function, we have that $\tilde{\zeta}_A(s,\Omega;\delta)$ is bounded for s in the part of the screen S for which $|\operatorname{Im} S| \le T$.

a uniform Lipschitz bound, if we take $t < \min\{1, B^{-1}\}$, then the sequence of integrals over the truncated screens $(S_m)_{|n}$ converges to 0 as $m \to \infty$. Indeed, let us take m_0 large enough so that $\sup S_m < 0$ for every $m \geq m_0$.[7] Furthermore, the integral over $(S_m)_{|n}$ is equal to

$$I_{n,m} := \frac{1}{2\pi\mathbb{i}} \int_{(S_m)_{|n}} \frac{t^{N-s+k}}{(N-s+1)_k} \widetilde{\zeta}_A(s, \Omega; \delta)\,\mathrm{d}s \tag{3.2.14}$$

and, similarly as in the proof of Lemma 3.10, we have that the integrand is bounded in absolute value by

$$C_\kappa |\sup S_m|^{\kappa-k} \max\left\{B^{|\inf(S_m)_{|n}|}, B^{|\sup(S_m)_{|n}|}\right\} t^{N+|\sup(S_m)_{|n}|+k}, \tag{3.2.15}$$

where C_κ is a suitable constant depending only on κ. Here, we denote

$$\inf(S_m)_{|n} := \inf_{\tau\in[T_{-n},T_n]} S_m(\tau) \quad \text{and} \quad \sup(S_m)_{|n} := \sup_{\tau\in[T_{-n},T_n]} S_m(\tau). \tag{3.2.16}$$

We now let $L := \sup_{m\geq 1} \|S_m\|$ be the uniform Lipschitz bound for the sequence of screens S_m. Then, the derivative of $\tau \mapsto S_m(\tau) + \mathbb{i}\tau$ is bounded for almost every $\tau \in [T_{-n}, T_n]$ with $(1 + L)$. Now we have two cases; firstly, if $B < 1$, we then have that

$$|I_{n,m}| \leq \frac{C_\kappa(1 + L)B^{|\sup(S_m)_{|n}|}}{2\pi|\sup(S_m)_{|n}|^{k-\kappa}}(T_n - T_{-n})t^{N+|\sup(S_m)_{|n}|+k},$$

and, since $t < 1$, we have that $I_{n,m} \to 0$ as $m \to \infty$. Secondly, if $B \geq 1$ we observe that from the Lipschitz condition on S_m we have

$$|\inf(S_m)_{|n}| \leq |\sup(S_m)_{|n}| + L(T_n - T_{-n}),$$

from which we get

$$|I_{n,m}| \leq \frac{C_\kappa(1 + L)B^{L(T_n-T_{-n})}}{2\pi|\sup(S_m)_{|n}|^{k-\kappa}}(T_n - T_{-n})(Bt)^{|\sup(S_m)_{|n}|}t^{N+k},$$

so that $I_{n,m} \to 0$ as $m \to \infty$ since $Bt < 1$.

We now let $E_{n,m}(t)$ be the error function appearing in (3.2.1) for the truncated screen $(S_m)_{|n}$ and we will complete the proof by showing that its iterated limit converges to zero pointwise. For $c \in (\overline{\dim}_B(A, \Omega), N + 1)$ and since $(0 < t < 1)$ we have, analogously as in

[7]This is possible, since $\sup S_m \to -\infty$ as $m \to \infty$.

the proof of Lemma 3.10, that

$$\begin{aligned}\left|\int_{\Gamma_{U_m}}\frac{t^{N-s+k}\widetilde{\zeta}(s)\,\mathrm{d}s}{(N-s+1)_k}\right|&\leq t^{N+k}C(T_n+1)^{\kappa}T_n^{-k}\int_{-\infty}^{c}t^{-\sigma}\,\mathrm{d}\sigma\\&\leq t^{N+k}K_\kappa T_n^{\kappa-k}\frac{t^{-c}}{\log(1/t)}.\end{aligned}\tag{3.2.17}$$

Here, Γ_{U_m} is the segment connecting $S_m(T_n)+\mathbb{i}T_n$ and $c+\mathbb{i}T_n$. A similar reasoning for the corresponding integral over the lower segment gives us the following bound on $|E_{n,m}(t)|$ independent of m:

$$|E_{n,m}(t)|\leq\frac{t^{N-c+k}}{\pi\log(1/t)}K_\kappa\max\{T_n^{\kappa-k},|T_{-n}|^{\kappa-k}\}.$$

Finally, this implies that for $k>\kappa$ the iterated limit of $E_{n,m}(t)$ tends to 0 when $m\to\infty$ and $n\to\infty$, i.e., we have

$$\lim_{n\to\infty}\big(\lim_{m\to\infty}E_{n,m}(t)\big)=0,$$

which concludes the proof of the theorem. □

Theorems 3.11 and 3.13 are of most interest in the case when $k=0$, i.e., when we get a pointwise formula for the volume of the relative t-neighborhood $|A_t\cap\Omega|$ in terms of the complex dimensions of (A,Ω). We will state this case as a separate theorem.

Theorem 3.14. *Under the same hypothesis as in Theorem 3.11 with $\kappa<-1$ (respectively, Theorem 3.13 with $\kappa<0$), we have the following pointwise formula for the relative tube function of the relative fractal drum (A,Ω) in $\mathbb{R}^N$:*

$$|A_t\cap\Omega|=\sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W)}\operatorname{res}\Big(t^{N-s}\widetilde{\zeta}_A(s,\Omega),\omega\Big)+R^{[0]}(t),\tag{3.2.18}$$

where $R^{[0]}(t)$ is the error term given by formula (3.2.9) with $k=0$. Furthermore, we have the following error estimate:

$$R^{[0]}(t)=O(t^{N-\sup S})\qquad as\ t\to0^+.\tag{3.2.19}$$

Moreover, if $S(\tau)<\sup S$ for every $\tau\in\mathbb{R}$, we then have

$$R^{[0]}(t)=o(t^{N-\sup S})\qquad as\ t\to0^+.\tag{3.2.20}$$

(Respectively, if (A,Ω) is strongly languid, then $R^{[0]}(t)\equiv0$ and $W=\mathbb{C}$).

3.3 Distributional Tube Formula

In this section our goal is to get the distributional analogs of Theorem 3.11 and 3.13 in order to derive the distributional formula for $V^{[k]}(t)$ for any integer $k \in \mathbb{Z}$. This will provide us with information (in the distributional sense) about the tube function of a relative fractal drum (A, Ω) no matter for which coefficient κ the relative fractal drum (A, Ω) is languid. (See Definition 3.2.) More precisely, let $\delta > 0$ and $\mathcal{D}(0, \delta) := C_c^\infty(0, \delta)$ be the space of infinitely differentiable (complex valued) test functions with compact support. In fact, we will start with a larger space of test function for which the formulas obtained here will be valid. Namely, let $\mathcal{K}(0, \delta)$ be the set of test functions φ in the class $C^\infty(0, \delta)$, such that for all $m \in \mathbb{Z}$ and $q \in \mathbb{N}$ we have $t^m \varphi^{(q)}(t) \to 0$, as $t \to 0^+$ and $(t - \delta)^m \varphi^{(q)}(t) \to 0$ as $t \to \delta^-$. Note that $\mathcal{D}(0, \delta) \subseteq \mathcal{K}(0, \delta)$.

Definition 3.15. Let (A, Ω) be a relative fractal drum in $\mathbb{R}^N$ and $k \in \mathbb{Z}$ an integer. We define the distribution $\mathcal{V}^{[k]}$ on $\mathcal{K}(0, \delta)$ to be the $|k|$-th distributional derivative of $V(t) = |A_t \cap \Omega|$ in case $k < 0$ and the k-th primitive function (considered as a distribution on $\mathcal{K}(0, \delta)$) of $V(t)$ if $k > 0$. For $k = 0$ this is the distribution generated by $V(t)$. More precisely, for a function $\varphi \in \mathcal{K}(0, \delta)$ we have

$$\langle \mathcal{V}^{[k]}, \varphi \rangle = \int_0^{+\infty} V^{[k]}(t) \varphi(t) \, \mathrm{d}t, \qquad \text{for } k \geq 0, \tag{3.3.1}$$

and

$$\langle \mathcal{V}^{[k]}, \varphi \rangle = (-1)^{|k|} \int_0^{+\infty} V(t) \varphi^{(|k|)}(t) \, \mathrm{d}t, \qquad \text{for } k < 0. \tag{3.3.2}$$

Here, and from now on, for convenience, we always extended the function $\varphi \in \mathcal{K}(0, \delta)$ to the interval $[\delta, +\infty)$ by letting $\varphi_{|[\delta, +\infty)} \equiv 0$.

We will also need the extended definition of the Pochammer symbol $(s)_k$ defined with (3.1.19) for the case when $k \in \mathbb{Z}$:

$$(s)_k := \frac{\Gamma(s + k)}{\Gamma(s)}, \tag{3.3.3}$$

with Γ being the gamma function.

Suppose now that $\varphi \in \mathcal{K}(0, \delta)$ is a test function. The decay conditions on φ imply that $t^s \varphi(t)$ is integrable on $(0, \delta)$ for every $s \in \mathbb{C}$ and that its Mellin transform $\{\mathfrak{M}\varphi\}$ is an entire function (see [Ti2, Theorem 31]). Furthermore, let $g(s)$ be a meromorphic function. Then the residue $\operatorname{res}(g(s), \omega)$ vanishes unless ω is a pole of g. Moreover, for

$k \in \mathbb{Z}$, $N \in \mathbb{N}$ and by choosing a suitable contour Γ around ω, we have

$$\begin{aligned}\int_0^{+\infty} \varphi(t)\,\mathrm{res}(t^{N-s+k}g(s),\omega)\,\mathrm{d}t &= \int_0^{+\infty} \varphi(t)\frac{1}{2\pi\mathbb{i}}\oint_\Gamma t^{N-s+k}g(s)\,\mathrm{d}s\,\mathrm{d}t\\ &= \frac{1}{2\pi\mathbb{i}}\oint_\Gamma g(s)\int_0^{+\infty} t^{N-s+k}\varphi(t)\,\mathrm{d}t\,\mathrm{d}s\\ &= \mathrm{res}\big(\{\mathfrak{M}\varphi\}(N-s+k+1)\,g(s),\omega\big).\end{aligned}$$

The change of the order of integration is justified by the Fubini–Tonelli theorem since the last integral above is absolutely convergent. In short, for $\varphi \in \mathcal{K}(0,\delta)$ we have

$$\big\langle\,\mathrm{res}(t^{N-s+k}g(s),\omega),\varphi\big\rangle = \mathrm{res}\big(\{\mathfrak{M}\varphi\}(N-s+1+k)\,g(s),\omega\big), \tag{3.3.4}$$

where $g(s)$ is a meromorphic function on a neighborhood of $\omega \in \mathbb{C}$, $k \in \mathbb{Z}$ and $N \in \mathbb{N}$.

Now we can state the distributional analog of Theorem 3.11.

Theorem 3.16. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ that is languid for some κ and $\delta > 0$. Then, for every $k \in \mathbb{Z}$ the distribution $\mathcal{V}^{[k]}$ is given by*

$$\mathcal{V}^{[k]}(t) = \sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W)} \mathrm{res}\left(\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}_A(s,\Omega;\delta),\omega\right) + \mathcal{R}^{[k]}(t). \tag{3.3.5}$$

That is, the action of $\mathcal{V}^{[k]}(t)$ on a test function $\varphi \in \mathcal{K}(0,\delta)$ is given by

$$\begin{aligned}\big\langle\mathcal{V}^{[k]},\varphi\big\rangle = &\sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W)} \mathrm{res}\left(\frac{\{\mathfrak{M}\varphi\}(N-s+1+k)\,\widetilde{\zeta}_A(s,\Omega;\delta)}{(N-s+1)_k},\omega\right)\\ &+\big\langle\mathcal{R}^{[k]},\varphi\big\rangle.\end{aligned} \tag{3.3.6}$$

Here, the distribution $\mathcal{R}^{[k]}(t)$ is the error term given by

$$\big\langle\mathcal{R}^{[k]},\varphi\big\rangle = \frac{1}{2\pi\mathbb{i}}\int_S \frac{\{\mathfrak{M}\varphi\}(N-s+1+k)\,\widetilde{\zeta}_A(s,\Omega;\delta)}{(N-s+1)_k}\,\mathrm{d}s. \tag{3.3.7}$$

Proof. We start the proof by fixing $k \in \mathbb{N}_0$ such that $k > \kappa + 1$ and a constant $c \in$

$(\overline{\dim}_B(A,\Omega), N+1)$.[8] Then, for every test function $\varphi \in \mathcal{K}(0,\delta)$, we have

$$\begin{aligned}\left\langle V^{[k]}, \varphi \right\rangle &= \int_0^{+\infty} V^{[k]}(t)\varphi(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty}\int_0^{+\infty}\varphi(t)\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}_A(s)\,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty}\frac{\{\mathfrak{M}\varphi\}(N-s+1+k)\,\widetilde{\zeta}_A(s)}{(N-s+1)_k}\,\mathrm{d}s\end{aligned}$$

Here, the change of the order of integration in the second equality is justified by the Fubini-Tonelli theorem since the first integral above is absolutely convergent.[9] One can now approximate the last integral above in the same way as in Lemma 3.10; that is, by the following expression:

$$\begin{aligned}&\sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W_{|n})}\operatorname{res}\left(\frac{\{\mathfrak{M}\varphi\}(N-s+1+k)\,\widetilde{\zeta}_A(s)}{(N-s+1)_k}\right) \\ &+\frac{1}{2\pi\mathbb{i}}\int_{S_{|n}}\frac{\{\mathfrak{M}\varphi\}(N-s+1+k)\,\widetilde{\zeta}_A(s)}{(N-s+1)_k}\,\mathrm{d}s \\ &+\frac{1}{2\pi\mathbb{i}}\int_{\Gamma_L\cup\Gamma_U}\frac{\{\mathfrak{M}\varphi\}(N-s+1+k)\,\widetilde{\zeta}_A(s)}{(N-s+1)_k}\,\mathrm{d}s\end{aligned}$$

Furthermore, by (3.3.4) the above is equal to

$$\begin{aligned}&\sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W_{|n})}\left\langle\operatorname{res}\left(\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}_A(s),\omega\right),\varphi\right\rangle \\ &+\frac{1}{2\pi\mathbb{i}}\int_{S_{|n}}\frac{\{\mathfrak{M}\varphi\}(N-s+1+k)\,\widetilde{\zeta}_A(s)}{(N-s+1)_k}\,\mathrm{d}s \\ &+\int_0^{+\infty}E_n(t)\varphi(t)\,\mathrm{d}t.\end{aligned}$$

Now, by letting $n\to\infty$ the integral of the error function $E_n(t)\varphi(t)$ tends to zero by the same argument as in Theorem 3.11 and also by the same argument, the integral over the truncated screen converges. Thus, we have

$$\begin{aligned}\left\langle V^{[k]},\varphi\right\rangle = &\sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W)}\operatorname{res}\left(\frac{\{\mathfrak{M}\varphi\}(N-s+1+k)\,\widetilde{\zeta}_A(s)}{(N-s+1)_k},\omega\right) \\ &+\langle\mathcal{R}^{[k]},\varphi\rangle.\end{aligned} \tag{3.3.8}$$

[8]Note, that by fixing $c\in(\overline{\dim}_B(A,\Omega),N+1)$ we have ensured that no poles of $(N-s+1)_k^{-1}$ will be contained in the window W. Indeed, since $(N-s+1)_k^{-1}=\Gamma(N-s+1)/\Gamma(N-s+1+k)$ we can see that the set of its poles is a subset of $\{N+n : n\in\mathbb{N}\}$.

[9]It is easy to see that $|V^{[k]}(t)|\le|A_t|t^k$ for $t\in(0,+\infty)$ and $k\ge 0$.

and the expression on the right defines a distribution on $\mathcal{K}(0,\delta)$ (since $V^{[k]}$ is locally integrable).

In the case when $k \le \kappa + 1$ we choose an integer q such that $k+q > \max\{\kappa+1,-1\}$ and note that

$$\big\langle \mathcal{V}^{[k]}, \varphi \big\rangle = (-1)^q \big\langle \mathcal{V}^{[k+q]}, \varphi^{(q)} \big\rangle. \tag{3.3.9}$$

To finish the proof we use the above equality together with (3.3.8) applied to the level $k+q$ and the well-known fact about the Mellin transform, namely,

$$\{\mathfrak{M}\varphi\}(s) = \frac{(-1)^q}{(s)_q}\big\{\mathfrak{M}\varphi^{(q)}\big\}(s+q). \tag{3.3.10}$$

$\square$

Remark 3.17. Note that in Theorem 3.16 we have proved that the sum over the complex dimensions appearing in (3.3.5) defines a distribution in $\mathcal{K}'(0,\delta)$ and hence in $\mathcal{D}'(0,\delta)$. This, in turn, implies that both terms on the right-hand side of (3.3.5) are, on their own, distributions in $\mathcal{K}'(0,\delta)$. Namely, this is a consequence of the well-known fact about the convergence of distributions (see, for example [Hö, Theorem 2.1.8, p. 39]). More precisely, let $(\mathcal{T}_n)_{n\ge1}$ be a sequence of distributions such that

$$\langle \mathcal{T}, \varphi \rangle := \lim_{n\to\infty} \langle \mathcal{T}_n, \varphi \rangle$$

exists for every test function $\varphi \in \mathcal{D}(0,\delta)$. Then, it is known that $\mathcal{T}$ is a distribution in $\mathcal{D}'(0,\delta)$. This result applied to the appropriate sequence of partial sums over the set $\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W)$ implies that the sum over the visible complex dimensions in (3.3.5) is indeed a distribution in $\mathcal{D}'(0,\delta)$ and hence, each term taken separately on the right-hand side of (3.3.5) defines a distribution in $\mathcal{D}'(0,\delta)$.

An entirely analogous comment applies to Theorem 3.18 below with test functions in $\mathcal{D}(0,\delta_0)$.

In the next theorem we will obtain a distributional analog of the pointwise relative tube formula without the error term. This will be an asymptotic distributional formula, i.e., it will be valid for test functions in $\mathcal{K}(0,\delta)$ that are supported on the left of B^{-1} where $B>0$ is the constant appearing in hypothesis **L2'**.

Theorem 3.18. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ that is strongly languid for some κ. Furthermore, let $\delta_0 := \min\{1,\delta,B^{-1}\}$. Then, for every $k\in\mathbb{Z}$ and for a test function in $\varphi\in\mathcal{D}(0,\delta_0)$ the distribution $\mathcal{V}^{[k]}\in\mathcal{D}'(0,\delta_0)$ is given by*

$$\mathcal{V}^{[k]}(t) = \sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),\mathbb{C})} \operatorname{res}\left(\frac{t^{N-s+k}}{(N-s+1)_k}\widetilde{\zeta}_A(s,\Omega;\delta),\omega\right). \tag{3.3.11}$$

That is, the action of $\mathcal{V}^{[k]}$ on a test function $\varphi \in \mathcal{D}(0,\delta_0)$ is given by

$$\big\langle \mathcal{V}^{[k]}(t),\varphi\big\rangle = \sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),\mathbb{C})} \operatorname{res}\left(\frac{\{\mathfrak{M}\varphi\}(N-s+1+k)\,\widetilde{\zeta}_A(s,\Omega;\delta)}{(N-s+1)_k},\omega\right). \tag{3.3.12}$$

Proof. We will prove the theorem by applying Theorem 3.16 to the sequence of screens S_m and showing that the corresponding error term tends to zero as $m\to\infty$. By choosing $q\in\mathbb{N}$ such that $k+q>\kappa+1$ and $m\in\mathbb{N}$ such that $\sup S_m<0$ we have from (3.3.5) the next distributional equality

$$\begin{aligned}\mathcal{V}^{[k+q]}(t) = \sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W_m)} &\operatorname{res}\left(\frac{t^{N-s+k+q}}{(N-s+1)_{k+q}}\widetilde{\zeta}_A(s),\omega\right)\\ &+\mathcal{R}_m^{[k+q]}(t).\end{aligned} \tag{3.3.13}$$

We now take a test function $\varphi\in\mathcal{D}(0,\delta_0)$, and since it has compact support, there exists $\nu\in(0,1)$ such that its support is contained in $(0,\nu B^{-1}]$. Using this fact, we can estimate its Mellin transform in the following way for $s\in\mathbb{C}$ such that $\operatorname{Re}s<0$:

$$\big|\{\mathfrak{M}\varphi\}(N-s+1+k+q)\big| \le (\nu B^{-1})^{-\operatorname{Re}s}\int_0^{+\infty} t^{N+k+q}|\varphi(t)|\,\mathrm{d}t. \tag{3.3.14}$$

Using this, hypothesis **L2'** and the fact that

$$|(N-S_m(\tau)-\mathrm{i}\tau+1)_{k+q}| \ge (\sqrt{1+\tau^2})^{k+q},$$

we estimate the distributional error $\mathcal{R}_m^{[k+q]}$:[10]

$$\begin{aligned}\big|\langle\mathcal{R}_m^{[k+q]},\varphi\rangle\big| &\le \int_{S_m}\big|\{\mathfrak{M}\varphi\}(N-s+1+k+q)\big|\frac{|\widetilde{\zeta}_A(s)|}{|(N-s+1)_{k+q}|}\,|\mathrm{d}s|\\ &\le \widetilde{K}(1+\|S_m\|_{\mathrm{Lip}})\int_{-\infty}^{+\infty}(B\nu B^{-1})^{|S_m(\tau)|}\frac{(1+|\tau|)^\kappa}{(\sqrt{1+\tau^2})^{k+q}}\,\mathrm{d}\tau\\ &\le K\nu^{|\sup S_m|}\int_{-\infty}^{+\infty}\frac{(1+|\tau|)^\kappa}{(\sqrt{1+\tau^2})^{k+q}}\,\mathrm{d}\tau,\end{aligned}$$

with K being a suitable positive constant. The last inequality follows since the sequence of screens $(S_m)_{m\ge1}$ has a uniform Lipschitz bound. Furthermore, the last integral in the above calculation is convergent since $k+q>\kappa+1$. Now, by letting $m\to\infty$, we get $\big\langle\mathcal{R}_m^{[k+q]},\varphi\big\rangle\to0$ since $|\sup S_m|\to\infty$, and we conclude that $\mathcal{R}_m^{[k+q]}\to0$. Finally, from (3.3.13) we get the statement of the theorem for the distribution $\mathcal{V}^{[k+q]}$ and to get the statement for $\mathcal{V}^{[k]}$ we use the same argument as in the proof of Theorem 3.16. □

Remark 3.19. From the proof of Theorem 3.18 one can see that the distributional

[10] We denote $|\mathrm{d}s| := |s'(\tau)|\,\mathrm{d}\tau$.

formula (3.3.12) is actually valid for a larger class of test functions. More precisely, for $\varphi \in \mathcal{K}(0, \delta_0)$ that have support in $(0, \nu B^{-1}]$ for some $\nu \in (0,1)$.

We state the most interesting special cases when $k = 0$ of the distributional relative tube formula (with and without an error term) as a separate theorem.

Theorem 3.20 (Distributional relative tube formula)**.** *Under the same hypothesis as in Theorem 3.16 (respectively, Theorem 3.18), we have the following distributional equality for the relative tube function $t \mapsto |A_t \cap \Omega|$ of the relative fractal drum (A, Ω) in $\mathbb{R}^N$:*

$$|A_t \cap \Omega| = \sum_{\omega \in \mathcal{P}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),W)} \operatorname{res}\left(t^{N-s}\widetilde{\zeta}_A(s,\Omega;\delta),\omega\right) + \mathcal{R}^{[0]}(t), \tag{3.3.15}$$

where $\mathcal{R}^{[0]}(t)$ is the distribution given by formula (3.3.7) with $k = 0$. (Respectively, if (A, Ω) is strongly languid, then $\mathcal{R}^{[0]}(t) \equiv 0$ and $W = \mathbb{C}$).

Remark 3.21. Note that when the expression on the right of (3.3.15) defines a locally integrable function, we have an equality a.e. between the tube function and this expression.

Now we would like to give a distributional estimate of the error term appearing in Theorem 3.16 in the same sense as was done in [Lap–vFr3] in the case of the distributional explicit formula for the generalized fractal string (see Definition 1.37 and the discussion around it). Recall that for a test function φ and $a > 0$ we denote $\varphi_a(t) := \frac{1}{a}\varphi\left(\frac{t}{a}\right)$ and that for the Mellin transform of φ_a we have

$$\{\mathfrak{M}\varphi_a\}(s) = a^{s-1}\{\mathfrak{M}\varphi\}(s). \tag{3.3.16}$$

Theorem 3.22. *For a fixed integer k assume that the hypotheses of Theorem 3.16 are satisfied. Then the distribution $\mathcal{R}^{[k]}(t)$ given by (3.3.7) is of asymptotic order at most $t^{N-\sup S+k}$ as $t \to 0^+$, i.e,*

$$\mathcal{R}^{[k]}(t) = O(t^{N-\sup S+k}) \quad \text{as } t \to 0^+ \tag{3.3.17}$$

in the sense of Definition 1.37.

Moreover, if $S(\tau) < \sup S$ for all $\tau \in \mathbb{R}$ (that is, if the screen lies strictly to the left of the line $\operatorname{Re} s = \sup S$), then $\mathcal{R}^{[k]}(t)$ is of asymptotic order less than $t^{N-\sup S+k}$, i.e.,

$$\mathcal{R}^{[k]}(t) = o(t^{N-\sup S+k}) \quad \text{as } t \to 0^+. \tag{3.3.18}$$

Proof. We have that for a test function φ the integral $\left\langle \mathcal{R}^{[k]}, \varphi \right\rangle$ converges absolutely.

Furthermore, for $a \in (0,1)$ and by using (3.3.16) we have

$$\begin{aligned}\big|\langle \mathcal{R}^{[k]}, \varphi_a\rangle\big| &\leq \frac{1}{2\pi}\int_S \frac{\big|\{\mathfrak{M}\varphi_a\}(N-s+1+k)\big|}{|(N-s+1)_k|}|\widetilde{\zeta}_A(s)|\,|\,\mathrm{d}s| \\ &= \int_S a^{N-\operatorname{Re}s+k}\frac{\big|\{\mathfrak{M}\varphi\}(N-s+1+k)\big|}{|(N-s+1)_k|}|\widetilde{\zeta}_A(s)|\,|\,\mathrm{d}s| \\ &\leq \mathrm{const}\cdot a^{N-\sup S+k}\end{aligned}$$

which proves the first part of the theorem.

For the second part of the theorem we use a similar argument as in the proof of the estimate (3.2.12) of Theorem 3.11. □

3.4 Tube Formulas in Terms of the Relative Distance Zeta Function

In this section we will translate the results from the previous sections in terms of the relative distance zeta functions. This is extremely useful in applications since the relative distance zeta function of an RFD can be calculated without knowing its relative tube function. Of course, the results will follow from the functional equation (2.1.4) which connects these two zeta functions. More precisely, to derive the analogous results in terms of the distance zeta function we will introduce a new fractal zeta function which satisfies a more direct functional equation than (2.1.4). For $A \subseteq \mathbb{R}^N$ let us denote by

$$A_{t,\delta} := A_\delta \setminus \overline{A_t}. \tag{3.4.1}$$

Stachó proved in [Sta] that for any bounded set $A \subset \mathbb{R}^N$ and $t > 0$ we have that $|\partial A_t| = 0$ and since any unbounded set may be partitioned into a countable union of bounded subsets this is also true for unbounded subsets of $\mathbb{R}^N$. Consequently, for any relative fractal drum (A,Ω) in $\mathbb{R}^N$ we have

$$|A_{t,\delta}\cap\Omega| = |A_\delta\cap\Omega| - |\overline{A_t}\cap\Omega| = |A_\delta\cap\Omega| - |A_t\cap\Omega| \tag{3.4.2}$$

Let $\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta)$ be the tube zeta function of the relative fractal drum (A,Ω) in $\mathbb{R}^N$ and assume that $\operatorname{Re}s > N$, then we have

$$\begin{aligned}\widetilde{\zeta}_A(s,\Omega;\delta) &= \int_0^\delta t^{s-N-1}|A_t\cap\Omega|\,\mathrm{d}t \\ &= \int_0^\delta t^{s-N-1}(|A_\delta\cap\Omega| - |A_{t,\delta}\cap\Omega|)\,\mathrm{d}t \\ &= \frac{\delta^{s-N}|A_\delta\cap\Omega|}{s-N} - \int_0^\delta t^{s-N-1}|A_{t,\delta}\cap\Omega|\,\mathrm{d}t.\end{aligned} \tag{3.4.3}$$

Definition 3.23. Let (A, Ω) be an RFD in $\mathbb{R}^N$ and fix $\delta > 0$. We define the *shell zeta function* $\breve{\zeta}_A(\,\cdot\,, \Omega)$ *of* A *relative to* Ω (or the *relative shell zeta function*) by

$$\breve{\zeta}_A(s, \Omega; \delta) := -\int_0^\delta t^{s-N-1}|A_{t,\delta} \cap \Omega|\,\mathrm{d}t, \tag{3.4.4}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large and the integral is taken in the Lebesgue sense.

In light of (3.4.3) the following theorem is almost immediate.

Theorem 3.24. *Let* (A, Ω) *be an RFD in* $\mathbb{R}^N$ *and fix* $\delta > 0$. *Then the shell zeta function* $\breve{\zeta}_A(\,\cdot\,, \Omega; \delta)$ *of* (A, Ω) *is holomorphic on the open right half-plane* $\{\operatorname{Re} s > N\}$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}s}\breve{\zeta}_A(s, \Omega; \delta) = -\int_0^\delta t^{s-N-1}|A_{t,\delta} \cap \Omega| \log t\,\mathrm{d}t, \tag{3.4.5}$$

for all s *with* $\operatorname{Re} s > N$.

Furthermore, for $\operatorname{Re} s > N$, $\breve{\zeta}_A(\,\cdot\,, \Omega; \delta)$ *satisfies the following functional equations*

$$\widetilde{\zeta}_A(s, \Omega; \delta) = \frac{\delta^{s-N}|A_\delta \cap \Omega|}{s-N} + \breve{\zeta}_A(s, \Omega; \delta); \tag{3.4.6}$$

$$\zeta_A(s, \Omega; \delta) = (N-s)\breve{\zeta}_A(s, \Omega; \delta); \tag{3.4.7}$$

Proof. To prove the holomorphicity of $\breve{\zeta}_A(\,\cdot\,, \Omega; \delta)$ one observes that for $\sigma > N$ we have

$$|\breve{\zeta}_A(\sigma, \Omega; \delta)| \leq |A_\delta \cap \Omega| \int_0^\delta t^{\sigma-N-1}\,\mathrm{d}t < \infty,$$

and uses Theorem 1.1 which also gives the formula (3.4.5) for the derivative. Formula (3.4.6) is a rewriting of (3.4.3) and by combining it with the functional equation (2.1.4), that connects the relative distance and tube zeta functions, we derive (3.4.7). $\square$

The principle of analytic continuations and equations (3.4.6) (or (3.4.7)) now immediately yield the following properties of the relative shell zeta function.

Theorem 3.25. *Let* (A, Ω) *be a relative fractal drum in* $\mathbb{R}^N$ *and fix* $\delta > 0$. *Then the following properties hold*:

(*a*) *The relative shell zeta function* $\breve{\zeta}_A(s, \Omega; \delta)$ *is meromorphic in the half-plane* $\{\operatorname{Re} s > \overline{\dim}_B(A, \Omega)\}$ *with a single simple pole at* $s = N$. *Furthermore,*

$$\operatorname{res}(\breve{\zeta}_A(\,\cdot\,, \Omega; \delta), N) = -|A_\delta \cap \Omega| \tag{3.4.8}$$

(*b*) *If the relative box* (*or Minkowski*) *dimension* $D := \dim_B(A, \Omega)$ *exists,* $D < N$, *and* $\underline{\mathcal{M}}^D(A, \Omega) > 0$, *then* $\breve{\zeta}_A(s, \Omega) \to +\infty$ *as* $s \in \mathbb{R}$ *converges to* D *from the right.*

Proof. By the principle of analytic continuation we conclude that the functional equalities (3.4.6) and (3.4.7) continue to hold on any open connected domain $U \subseteq \mathbb{C}$ to which any of the three relative zeta functions has a holomorphic continuation. In light of this, part (a) follows from the counterpart of Theorem 2.6 for the relative tube zeta function and (3.4.6) while part (b) follows from Theorem 2.6 and (3.4.7). $\square$

The following corollary is an immediate consequence of the above theorem.

Corollary 3.26. *Let (A,Ω) be an RFD in $\mathbb{R}^N$ and fix $\delta_1, \delta_2 > 0$ such that $\delta_1 < \delta_2$. Then, the difference $\breve{\zeta}_A(s,\Omega;\delta_1) - \breve{\zeta}_A(s,\Omega;\delta_2)$ is meromorphic on $\mathbb{C}$ with a single simple pole at $s = N$ of residue $|A_{\delta_1,\delta_2} \cap \Omega|$.*

Furthermore, in light of Theorem 2.9 and (3.4.6) one has the following result.

Theorem 3.27. *Assume that (A,Ω) is a nondegenerate RFD in $\mathbb{R}^N$, that is, $0 < \underline{\mathcal{M}}^D(A,\Omega) \le \overline{\mathcal{M}}^D(A,\Omega) < \infty$ (in particular, $\dim_B(A,\Omega) = D$), and $D < N$. If $\breve{\zeta}_A(s,\Omega)$ can be extended meromorphically to a neighborhood of $s = D$, then D is necessarily a simple pole of $\breve{\zeta}_A(s,\Omega)$, and*

$$\underline{\mathcal{M}}^D(A,\Omega) \le \operatorname{res}(\breve{\zeta}_A(\,\cdot\,,\Omega), D) \le \overline{\mathcal{M}}^D(A,\Omega). \tag{3.4.9}$$

Furthermore, if (A,Ω) is Minkowski measurable, then

$$\operatorname{res}(\breve{\zeta}_A(\,\cdot\,,\Omega), D) = \mathcal{M}^D(A,\Omega). \tag{3.4.10}$$

The most useful fact about the relative shell zeta function is that the residues of its meromorphic extension at any poles contained in the open half-plane $\{\operatorname{Re} s < N\}$ have a simple connection to the residues of the relative tube or distance zeta functions.

Lemma 3.28. *Assume that (A,Ω) is an RFD in $\mathbb{R}^N$ such that its tube or distance or shell zeta function is meromorphic on some open connected domain $U \subseteq \{\operatorname{Re} s < N\}$. Then the multisets of poles located in U of all of the three zeta functions coincide. Moreover, if $\omega \in U$ is a simple pole of one of the three zeta functions, then*

$$\operatorname{res}(\breve{\zeta}_A(\,\cdot\,,\Omega),\omega) = \operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),\omega) = \frac{\operatorname{res}(\zeta_A(\,\cdot\,,\Omega),\omega)}{N-\omega}. \tag{3.4.11}$$

Although the shell zeta function seems rather artificial in the present context of relative fractal drums it will prove quite useful as a "translation tool" for deriving the tube formulas in terms of the much more operable distance zeta function. On the other hand, the shell zeta function will arise naturally in Chapter 4 when dealing with fractal sets at infinity of infinite Lebesgue measure. It was used there to generalize the theory of complex dimensions to the special case of *unbounded sets at infinity* of infinite Lebesgue measure.

Similarly as in the case of the relative tube zeta function of (A,Ω) we observe that $\breve{\zeta}_A(s,\Omega)=\{\mathfrak{M}f\}(s)$ where $f(s)=-t^{-N}\chi_{(0,\delta)}(t)|A_{t,\delta}\cap\Omega|$. Note that f is continuous and of bounded variation on $(0,\infty)$ so that we can apply the Mellin inversion theorem and conclude that

$$|A_{t,\delta}\cap\Omega|=-\frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty}t^{N-s}\breve{\zeta}_A(s,\Omega;\delta)\,\mathrm{d}s, \tag{3.4.12}$$

where $c>N$ is arbitrary and $t\in(0,\delta)$. In light of (3.4.2) the following theorem is an immediate consequence.

Theorem 3.29. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ and fix $\delta>0$. Then for every $t\in(0,\delta)$ and $c>N$ we have*

$$|A_t\cap\Omega|=|A_\delta\cap\Omega|+\frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty}t^{N-s}\breve{\zeta}_A(s,\Omega;\delta)\,\mathrm{d}s. \tag{3.4.13}$$

It is now clear that if the shell zeta function of (A,Ω) satisfies the languidity conditions of Definition 3.2, with the constant $c>N$ in the condition **L1**, or the strong languidity conditions of Definition 3.3, that we can rewrite the results of Sections 3.2 and 3.3 verbatim in terms of the shell zeta function. Note that for this to work, it was crucial that in the truncated pointwise formula of Lemma 3.10 we had the freedom to choose any $c\in(\overline{\dim}_B(A,\Omega),N+1)$. The additional pole of the shell zeta function at $s=N$ will cancel out the term $|A_\delta\cap\Omega|$ in (3.4.13) above. More precisely, in the analog of the pointwise formula of Theorem 3.11 for the relative shell zeta function we get

$$\begin{aligned}V^{[k]}(t)=&\sum_{\omega\in\mathcal{P}(\breve{\zeta}_A(\,\cdot\,,\Omega),W)}\operatorname{res}\left(\frac{t^{N-s+k}}{(N-s+1)_k}\breve{\zeta}_A(s,\Omega;\delta),\omega\right)\\&+|A_\delta\cap\Omega|\frac{t^k}{(1)_k}+R^{[k]}(t).\end{aligned} \tag{3.4.14}$$

Furthermore, by singling out the residue at $s=N$ from the above sum and using Lemma 3.28 and Theorem 3.25(a) we can rewrite the above equation as

$$V^{[k]}(t)=\sum_{\omega\in\mathcal{P}(\zeta_A(\,\cdot\,,\Omega),W)}\operatorname{res}\left(\frac{t^{N-s+k}}{(N-s)_{k+1}}\zeta_A(s,\Omega;\delta),\omega\right)+R^{[k]}(t). \tag{3.4.15}$$

Let us now define the analogs of the languidity conditions of a relative fractal drum in terms of its relative distance zeta function.

Definition 3.30 ((Strongly) d-languid)**.** We will say that a relative fractal drum (A,Ω) in $\mathbb{R}^N$ is *d-languid* if it is languid in the sense of Definition 3.2 but with the relative tube zeta function $\widetilde{\zeta}_A(\,\cdot\,,\Omega)$ replaced by the relative distance zeta function $\zeta_A(\,\cdot\,,\Omega)$ and with a constant $c>N$ appearing in **L1**.

Analogously, we will say that a relative fractal drum (A,Ω) in $\mathbb{R}^N$ is *strongly d-languid.* (See Definition 3.3.)

The following lemma is an immediate consequence of the functional equation (3.4.7).

Lemma 3.31. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ with $\overline{\dim}_B(A,\Omega) < N$ such that it is d-languid for some exponent $\kappa_d \in \mathbb{R}$. Then the shell zeta function $\breve{\zeta}_A(\,\cdot\,,\Omega)$ of (A,Ω) satisfies the languidity conditions of Definition 3.2 with the exponent $\kappa = \kappa_d - 1$.*

Furthermore, if (A,Ω) is strongly d-languid with the corresponding constant $B > 0$ and for some exponent $\kappa_d \in \mathbb{R}$, then the shell zeta function $\breve{\zeta}_A(\,\cdot\,,\Omega)$ of (A,Ω) satisfies the strong languidity conditions of Definition 3.3 with the exponent $\kappa = \kappa_d - 1$ and with the same constant B.

Theorem 3.32. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ that is d-languid for some $\delta > 0$ and with the exponent $\kappa_d \in \mathbb{R}$. Furthermore, assume also that $\overline{\dim}_B(A,\Omega) < N$ and let $k > \kappa_d$ be a nonnegative integer. Then, for every $t \in (0,\delta)$ the following pointwise formula is valid*

$$V^{[k]}(t) = \sum_{\omega \in \mathcal{P}(\zeta_A(\,\cdot\,,\Omega),W)} \operatorname{res}\left(\frac{t^{N-s+k}}{(N-s)_{k+1}}\zeta_A(s,\Omega;\delta),\omega\right) + R^{[k]}(t). \tag{3.4.16}$$

Here, for $t \in (0,\delta)$ the error term $R^{[k]}$ is given by the absolutely convergent integral

$$R^{[k]}(t) = \frac{1}{2\pi\mathbb{i}}\int_S \frac{t^{N-s+k}}{(N-s)_{k+1}}\zeta_A(s,\Omega;\delta)\,\mathrm{d}s. \tag{3.4.17}$$

Furthermore, for $t \in (0,\delta)$ we have

$$|R^{[k]}(t)| \leq \frac{C}{2\pi}(1+\|S\|_{\mathrm{Lip}})\frac{t^{N+k}\max\{t^{-\sup S}, t^{-\inf S}\}}{k-\kappa_d} + C', \tag{3.4.18}$$

where C is the constant appearing in **L1** *and* **L2** *and C' is some suitable positive constant. These constants depend only on the relative fractal drum (A,Ω) and the screen, but not on k.*

In particular, we have the following pointwise error estimate

$$R^{[k]}(t) = O(t^{N-\sup S+k}) \quad \text{as } t \to 0^+. \tag{3.4.19}$$

Moreover, if $S(\tau) < \sup S$, i.e., if the screen lies strictly left of the vertical line $\{\mathrm{Re}\, s = \sup S\}$, then we have

$$R^{[k]}(t) = o(t^{N-\sup S+k}) \quad \text{as } t \to 0^+. \tag{3.4.20}$$

Proof. In light of Lemma 3.31 we have that the shell zeta function of (A,Ω) also satisfies the appropriate languidity conditions with $\kappa = \kappa_d - 1$. The Theorem now follows analogously as in the case of the relative tube zeta function (see Theorem 3.11 and the discussion after Theorem 3.29). □

Remark 3.33. In the above theorem we have the additional assumption that $\overline{\dim}_B(A,\Omega) < N$ in order to avoid the situation when $s = N$ is a pole of $\widetilde{\zeta}_A(s,\Omega)$. We will assume this also for all other theorems involving the relative distance zeta function.

Theorem 3.34. *Let (A,Ω) be a strongly d-languid relative fractal drum in $\mathbb{R}^N$ for some $\delta > 0$, $\kappa_d \in \mathbb{R}$ and let $k > \kappa_d - 1$ be a nonnegative integer. Furthermore, assume, additionally that $\overline{\dim}_B(A,\Omega) < N$. Then, for every $t \in (0, \min\{1, \delta, B^{-1}\})$ the pointwise tube formula is given by*

$$V^{[k]}(t) = \sum_{\omega \in \mathcal{P}(\zeta_A(\,\cdot\,,\Omega),\mathbb{C})} \operatorname{res}\left(\frac{t^{N-s+k}}{(N-s)_{k+1}}\zeta_A(s,\Omega),\omega\right). \tag{3.4.21}$$

Here, B is the constant appearing in **L2'** *and κ_d is the exponent occurring in the statement of hypotheses* **L1** *and* **L2'**.

Proof. In light of Lemma 3.31 and the functional equation (3.4.7) the theorem follows analogously as Theorem 3.32 with the tube zeta function $\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta)$ replaced by the shell zeta function $\check{\zeta}_A(\,\cdot\,,\Omega;\delta)$. □

In some cases we will have a relative fractal drum (A,Ω) that is 'almost' strongly d-languid but not exactly. More precisely, (A,Ω) will satisfy all of the conditions of strong d-languidity except the condition that **L1** is satisfied for all $\sigma < c$. For example, let A be the middle-third Cantor set constructed in $[0,1]$ and let $\Omega = [0,1]$. Then, the relative distance zeta function $\zeta_A(\,\cdot\,,\Omega)$ is meromorphic on all of $\mathbb{C}$ and given by (see [LapRaŽu1]):

$$\zeta_A(s,\Omega) = \frac{2}{2^s s(3^s-2)}. \tag{3.4.22}$$

As one can easily check, it almost satisfies the strong languidity conditions with $\kappa_d = -1$ where the sequence of screens S_m can be taken as the sequence of vertical lines $\{\operatorname{Re} s = -m\}$ for $m \in \mathbb{N}$. The problem here is the factor 2^{-s} which tends exponentially to $+\infty$ as $\operatorname{Re} s \to -\infty$ so that **L1** cannot be fulfilled for all $\sigma < c$. In order to get a pointwise formula in this and similar cases we can multiply $\zeta_A(s,\Omega)$ by 2^s and the resulting function will be strongly d-languid. On the other hand, by the scaling property of the relative distance zeta function (see Theorem 2.10), we have that $2^s\,\zeta_A(s,\Omega) = \zeta_{2A}(s,2\Omega)$. Hence we state the following corollary dealing with this problem.

Corollary 3.35. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $\overline{\dim}_B(A,\Omega) < N$. Furthermore, assume that there exists a $\lambda > 0$ such that $(\lambda A, \lambda\Omega)$ is a strongly d-languid relative fractal drum for some $\delta > 0$, $\kappa_d \in \mathbb{R}$ and let $k > \kappa_d - 1$ be a nonnegative integer.*

Then, for every $t \in (0, \lambda^{-1}\min\{1, \delta, B^{-1}\})$ the pointwise formula is given by

$$V^{[k]}(t) = \sum_{\omega \in \mathcal{P}(\zeta_A(\,\cdot\,,\Omega),\mathbb{C})} \operatorname{res}\left(\frac{t^{N-s+k}}{(N-s)_{k+1}}\zeta_A(s,\Omega),\omega\right). \tag{3.4.23}$$

Here, B is the constant appearing in **L2'** *(for the function $s \mapsto \zeta_{\lambda A}(s, \lambda\Omega; \delta) = \lambda^s\zeta_A(s, \Omega; \delta\lambda^{-1})$) and κ_d is the exponent occurring in the statement of hypotheses* **L1** *and* **L2'**.

Proof. Let us denote by $V_\lambda^{[k]}(\tau)$ the k-th primitive of the function

$$\tau \mapsto |(\lambda A)_\tau \cap \lambda\Omega|.$$

Since, from Lemma 2.12 we have that $V_\lambda^{[0]}(\tau) = \lambda^N V^{[0]}(\tau/\lambda)$ we deduce that

$$V_\lambda^{[1]}(\tau) = \int_0^\tau V_\lambda^{[0]}(t)\,\mathrm{d}t = \lambda^N \int_0^\tau V^{[0]}(t/\lambda)\,\mathrm{d}t = \lambda^{N+1}\int_0^{\tau/\lambda} V^{[0]}(\xi)\,\mathrm{d}\xi, \tag{3.4.24}$$

or, in other words, $V_\lambda^{[1]}(\tau) = \lambda^{N+1}V^{[1]}(\tau/\lambda)$ and hence, by induction,

$$V_\lambda^{[k]}(\tau) = \lambda^{N+k}V^{[k]}(\tau/\lambda), \tag{3.4.25}$$

for all $k \geq 0$. We now apply Theorem 3.34 to the relative fractal drum $(\lambda A, \lambda\Omega)$ and get that

$$V_\lambda^{[k]}(\tau) = \sum_{\omega \in \mathcal{P}(\zeta_{\lambda A}(\,\cdot\,,\lambda\Omega),\mathbb{C})} \operatorname{res}\left(\frac{\tau^{N-s+k}}{(N-s)_{k+1}}\zeta_{\lambda A}(s,\lambda\Omega,\delta),\omega\right), \tag{3.4.26}$$

valid pointwise for $t \in (0, \min\{1, \delta, B^{-1}\})$. Combining now (3.4.25) with (3.4.26) and the scaling property of the relative distance zeta function, yields

$$\lambda^{N+k}V^{[k]}(\tau/\lambda) = \sum_{\omega \in \mathcal{P}(\zeta_A(\,\cdot\,,\Omega),\mathbb{C})} \operatorname{res}\left(\frac{\tau^{N-s+k}\lambda^s}{(N-s)_{k+1}}\zeta_A(s,\Omega,\delta\lambda^{-1}),\omega\right). \tag{3.4.27}$$

Finally, multiplying the above by λ^{-N-k} and introducing a new variable $t = \tau/\lambda$ finalizes the proof of the corollary. □

Remark 3.36. We point out that an analogous corollary can be stated in terms of the relative tube zeta function and the exact pointwise tube formula of Theorem 3.13.

The most interesting situation is, of course, when we can apply Theorems 3.32 and 3.34 at the level $k = 0$ and we state now these corollaries as a separate theorem.

Theorem 3.37. *Under the same hypothesis as in Theorem 3.32 with $\kappa_d < 0$ (respectively, Theorem 3.34 or Corollary 3.35 with $\kappa_d < 1$), we have the following pointwise formula* *for the relative tube function of the RFD (A,Ω) in $\mathbb{R}^N$:*

$$|A_t\cap\Omega| = \sum_{\omega\in\mathcal{P}(\zeta_A(\,\cdot\,,\Omega),W)} \operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_A(s,\Omega),\omega\right) + R^{[0]}(t), \tag{3.4.28}$$

where $R^{[0]}(t)$ is the error term given by formula (3.4.17) with $k=0$. Furthermore, we have the following error estimate:

$$R^{[0]}(t) = O(t^{N-\sup S}) \qquad \text{as } t\to 0^+. \tag{3.4.29}$$

Moreover, if $S(\tau)<\sup S$ for every $\tau\in\mathbb{R}$, we then have

$$R^{[0]}(t) = o(t^{N-\sup S}) \qquad \text{as } t\to 0^+. \tag{3.4.30}$$

(*Respectively, if (A,Ω) is strongly languid, then $R^{[0]}(t)\equiv 0$ and $W=\mathbb{C}$*).

Let us now state the distributional analogs of the above results in terms of the relative distance zeta function. The proofs are completely analogous to the ones from Section 3.3 for the case of the relative tube zeta function. Again, we use the relative shell zeta function and the same technique of scaling as in the proof of Corollary 3.32 above to state the results under the hypothesis of (strong) d-languidity.

Theorem 3.38. *Let (A,Ω) be a d-languid relative fractal drum in $\mathbb{R}^N$ for some $\delta>0$ and $\kappa_d\in\mathbb{R}$. Furthermore, assume also that $\overline{\dim}_B(A,\Omega)<N$. Then, for every $k\in\mathbb{Z}$ the distribution $\mathcal{V}^{[k]}$ is given by*

$$\mathcal{V}^{[k]}(t) = \sum_{\omega\in\mathcal{P}(\zeta_A(\,\cdot\,,\Omega),W)} \operatorname{res}\left(\frac{t^{N-s+k}}{(N-s)_{k+1}}\zeta_A(s,\Omega),\omega\right) + \mathcal{R}^{[k]}(t). \tag{3.4.31}$$

That is, the action of $\mathcal{V}^{[k]}(t)$ on a test function $\varphi\in\mathcal{K}(0,\delta)$ is given by

$$\begin{aligned}\big\langle\mathcal{V}^{[k]},\varphi\big\rangle = &\sum_{\omega\in\mathcal{P}(\zeta_A(\,\cdot\,,\Omega),W)} \operatorname{res}\left(\frac{\{\mathfrak{M}\varphi\}(N-s+1+k)\,\zeta_A(s,\Omega)}{(N-s)_{k+1}},\omega\right)\\ &+\big\langle\mathcal{R}^{[k]},\varphi\big\rangle.\end{aligned} \tag{3.4.32}$$

Here, the distribution $\mathcal{R}^{[k]}(t)$ is the error term given by

$$\big\langle\mathcal{R}^{[k]},\varphi\big\rangle = \frac{1}{2\pi\mathbb{i}}\int_S \frac{\{\mathfrak{M}\varphi\}(N-s+1+k)\,\zeta_A(s,\Omega;\delta)}{(N-s)_{k+1}}\,\mathrm{d}s. \tag{3.4.33}$$

Furthermore, the distribution $\mathcal{R}^{[k]}(t)$ is of asymptotic order at most $t^{N-\sup S+k}$ as $t\to 0^+$, i.e,

$$\mathcal{R}^{[k]}(t) = O(t^{N-\sup S+k}) \quad \text{as } t\to 0^+ \tag{3.4.34}$$

in the sense of Definition 1.37.

Moreover, if $S(\tau) < \sup S$ *for all* $\tau \in \mathbb{R}$ (*that is, if the screen lies strictly to the left of the line* $\operatorname{Re} s = \sup S$), *then* $\mathcal{R}^{[k]}(t)$ *is of asymptotic order less than* $t^{N-\sup S+k}$, *i.e.,*

$$\mathcal{R}^{[k]}(t) = o(t^{N-\sup S+k}) \quad \text{as } t \to 0^+. \tag{3.4.35}$$

In the case of a (possibly scaled) strongly d-languid relative fractal drum, as before, we have a distributional formula without an error term.

Theorem 3.39. *Let* (A,Ω) *be a relative fractal drum in* $\mathbb{R}^N$ *and assume also that* $\overline{\dim}_B(A,\Omega) < N$. *Furthermore, assume that there exists a* $\lambda > 0$ *such that* $(\lambda A, \lambda\Omega)$ *is strongly d-languid for some* $\delta > 0$, $\kappa_d \in \mathbb{R}$ *and let* $\delta_0 := \lambda^{-1}\min\{1,\delta,B^{-1}\}$.[11] *Then, for every* $k \in \mathbb{Z}$ *and for a test function in* $\varphi \in \mathcal{D}(0,\delta_0)$ *the distribution* $\mathcal{V}^{[k]} \in \mathcal{D}'(0,\delta_0)$ *is given by*

$$\mathcal{V}^{[k]}(t) = \sum_{\omega\in\mathcal{P}(\zeta_A(\,\cdot\,,\Omega),\mathbb{C})} \operatorname{res}\left(\frac{t^{N-s+k}}{(N-s)_{k+1}}\zeta_A(s,\Omega),\omega\right). \tag{3.4.36}$$

That is, the action of $\mathcal{V}^{[k]}$ *on a test function* $\varphi \in \mathcal{D}(0,\delta_0)$ *is given by*

$$\left\langle \mathcal{V}^{[k]}(t),\varphi\right\rangle = \sum_{\omega\in\mathcal{P}(\zeta_A(\,\cdot\,,\Omega),\mathbb{C})} \operatorname{res}\left(\frac{\{\mathfrak{M}\varphi\}(N-s+1+k)\,\zeta_A(s,\Omega)}{(N-s)_{k+1}},\omega\right). \tag{3.4.37}$$

We will finalize this section by stating the most interesting special case when $k=0$ of the above results as a separate theorem.

Theorem 3.40. *Under the same hypothesis as in Theorem 3.38* (*respectively, Theorem 3.39*), *we have the following distributional equality for the relative tube function* $t \mapsto |A_t \cap \Omega|$ *of the relative fractal drum* (A,Ω) *in* $\mathbb{R}^N$:

$$|A_t \cap \Omega| = \sum_{\omega\in\mathcal{P}(\zeta_A(\,\cdot\,,\Omega),W)} \operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_A(s,\Omega),\omega\right) + \mathcal{R}^{[0]}(t), \tag{3.4.38}$$

where $\mathcal{R}^{[0]}(t)$ *is given by* (3.4.33) *for* $k=0$ *and* $\mathcal{R}^{[0]}(t) = O(t^{N-\sup S})$ *as* $t \mapsto 0^+$ *or, if* $S(\tau) < \sup S$ *for all* $\tau \in \mathbb{R}$, *then* $\mathcal{R}^{[0]}(t) = o(t^{N-\sup S})$ *as* $t \mapsto 0^+$. (*Respectively, if* $(\lambda A, \lambda\Omega)$ *is strongly d-languid for some* $\lambda > 0$, *then* $\mathcal{R}^{[0]}(t) \equiv 0$ *and* $W = \mathbb{C}$).

3.5 A Criterion for Minkowski Measurability

In this section we will show that a sufficient condition for Minkowski measurability of a relative fractal drum (A,Ω) can be given in terms of its relative tube (or distance) zeta function. This will be a consequence of a well-known Tauberian theorem due to Wiener

[11] Here, B is the constant appearing in condition **L2'** for the function $\zeta_{\lambda A}(s,\lambda\Omega;\delta)$.

and Pitt which generalizes the famous Ikehara's Tauberian theorem. Its proof can be found in [Kor, Chapter III, Lemma 9.1 and Proposition 4.3] or in [Pitt, Section 6.1] and in [Dia] where a different proof using a technique of Bochner is given. We state it here for the sake of completeness.

Theorem 3.41 (Wiener–Pitt, cited from [Kor])**.** *Let $\sigma\colon\mathbb{R}\to\mathbb{R}$ be such that $\sigma(t)$ vanishes for $t<0$, is nonnegative for $t\geq 0$ and such that the Laplace transform*

$$F(s):=\{\mathfrak{L}\sigma\}(s):=\int_0^{+\infty}\mathrm{e}^{-st}\sigma(t)\,\mathrm{d}t \tag{3.5.1}$$

exists for $s\in\mathbb{C}$ such that $\operatorname{Re}s>0$. Furthermore, suppose that for some constants $A>0$ and $\lambda>0$ the function

$$G(s)=F(s)-\frac{A}{s},\quad s=x+\mathbb{i}y, \tag{3.5.2}$$

converges to a boundary function $G(\mathbb{i}y)$ in $L^1(-\lambda,\lambda)$ when $x\to 0^+$. Then for every number $h\geq 2\pi/\lambda$ we have that

$$\sigma_h(u):=\frac{1}{h}\int_u^{u+h}\sigma(t)\,\mathrm{d}t\leq CA+o(1)\quad\text{as}\quad u\to+\infty, \tag{3.5.3}$$

for some positive constant $C<3$. Moreover, if λ can be taken arbitrarily large, then for every $h>0$,

$$\sigma_h(u)\to A\quad\text{as}\quad u\to+\infty. \tag{3.5.4}$$

By using the above theorem let us prove the announced result.

Theorem 3.42 (Sufficient condition for Minkowski measurability)**.** *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ and let $\overline{D}:=\overline{\dim}_B(A,\Omega)$. Furthermore, suppose that the relative tube zeta function $\widetilde{\zeta}_A(\,\cdot\,,\Omega)$ of (A,Ω) can be meromorphically extended to a neighborhood U of the critical line $\{\operatorname{Re}s=\overline{D}\}$. Let $\overline{D}$ be its only pole in U and assume that it is simple. Then $D:=\dim_B(A,\Omega)$ exists, $D=\overline{D}$ and (A,Ω) is Minkowski measurable with*

$$\mathcal{M}^D(A,\Omega)=\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),D). \tag{3.5.5}$$

Furthermore, if we additionally assume that $\overline{D}<N$ then the theorem is also valid if we replace the relative tube zeta function $\widetilde{\zeta}_A(\,\cdot\,,\Omega)$ with the relative distance zeta function $\zeta_A(\,\cdot\,,\Omega)$ of (A,Ω) and in that case we have

$$\mathcal{M}^D(A,\Omega)=\frac{\operatorname{res}(\zeta_A(\,\cdot\,,\Omega),D)}{N-D}. \tag{3.5.6}$$

Proof. Without loss of generality, for the tube zeta function $\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta)$ we may choose

$\delta = 1$ and change the variable of integration by $u = 1/t$:

$$\begin{aligned}\widetilde{\zeta}_A(s+\overline{D},\Omega) &= \int_0^1 t^{s+\overline{D}-1-N}|A_t\cap\Omega|\,\mathrm{d}t\\ &= \int_1^{+\infty} u^{-s-1-\overline{D}+N}|A_{1/u}\cap\Omega|\,\mathrm{d}u \\ &= \int_0^{+\infty} \mathrm{e}^{-sv}\mathrm{e}^{v(N-\overline{D})}|A_{\mathrm{e}^{-v}}\cap\Omega|\,\mathrm{d}v = \{\mathfrak{L}\sigma\}(s).\end{aligned} \tag{3.5.7}$$

where we have made another change of variables in the second last equality, namely, $v = \log u$ and $\sigma(v) := \mathrm{e}^{v(N-\overline{D})}|A_{\mathrm{e}^{-v}}\cap\Omega|$. The definition of the relative tube zeta function of (A,Ω) implies that its residue at $s = \overline{D}$ must be real and positive. Furthermore, since $s = \overline{D}$ is the only pole of $\widetilde{\zeta}_A(\,\cdot\,,\Omega)$ in U, we conclude that

$$G(s) := \widetilde{\zeta}_A(s+\overline{D},\Omega) - \frac{\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),\overline{D})}{s} \tag{3.5.8}$$

is holomorphic on the neighborhood $U_{\overline{D}} := \{s\in\mathbb{C} \,:\, s+\overline{D}\in U\}$ of the vertical line $\{\operatorname{Re} s = 0\}$. In other words, we can apply Theorem 3.41 (for arbitrarily large $\lambda > 0$ in the notation of that theorem) and conclude that

$$\sigma_h(u) = \frac{1}{h}\int_u^{u+h}\sigma(v)\,\mathrm{d}v \to \operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),\overline{D}) \quad \text{as} \quad u\to+\infty, \tag{3.5.9}$$

for every $h > 0$. In particular, since $v\mapsto|A_{\mathrm{e}^{-v}}\cap\Omega|$ is nonincreasing we consider now the following two cases.

Case (a). We assume that $\overline{D} < N$ and have

$$\begin{aligned}\frac{1}{h}\int_u^{u+h}\mathrm{e}^{v(N-\overline{D})}|A_{\mathrm{e}^{-v}}\cap\Omega|\,\mathrm{d}v &\le \frac{|A_{\mathrm{e}^{-u}}\cap\Omega|}{h}\int_u^{u+h}\mathrm{e}^{v(N-\overline{D})}\,\mathrm{d}v\\ &= \frac{|A_{\mathrm{e}^{-u}}\cap\Omega|}{\mathrm{e}^{-u(N-\overline{D})}}\,\frac{\mathrm{e}^{h(N-\overline{D})}-1}{(N-\overline{D})h}.\end{aligned}$$

By taking the lower limit of both sides as $u\to+\infty$ we get

$$\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),\overline{D}) \le \underline{\mathcal{M}}^{\overline{D}}(A,\Omega)\frac{\mathrm{e}^{h(N-\overline{D})}-1}{(N-\overline{D})h}. \tag{3.5.10}$$

Since this is true for every $h > 0$, by letting $h\to0^+$ we get that

$$\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),\overline{D}) \le \underline{\mathcal{M}}^{\overline{D}}(A,\Omega). \tag{3.5.11}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{h}\int_u^{u+h} \mathrm{e}^{v(N-\overline{D})}|A_{\mathrm{e}^{-v}}\cap\Omega|\,\mathrm{d}v &\geq \frac{|A_{\mathrm{e}^{-(u+h)}}\cap\Omega|}{h}\int_u^{u+h}\mathrm{e}^{v(N-\overline{D})}\,\mathrm{d}v\\ &= \frac{|A_{\mathrm{e}^{-(u+h)}}\cap\Omega|}{\mathrm{e}^{-(u+h)(N-\overline{D})}}\,\frac{1-\mathrm{e}^{-h(N-\overline{D})}}{(N-\overline{D})h}\end{aligned}\tag{3.5.12}$$

and, similarly as before, by taking the upper limit of both sides as $u\to+\infty$ we get

$$\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),\overline{D}) \geq \overline{\mathcal{M}}^{\overline{D}}(A,\Omega)\frac{1-\mathrm{e}^{-h(N-\overline{D})}}{(N-\overline{D})h}.\tag{3.5.13}$$

Since this is true for every $h>0$, we let $h\to0^+$ and conclude that

$$\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),\overline{D}) \geq \overline{\mathcal{M}}^{\overline{D}}(A,\Omega).\tag{3.5.14}$$

This, together with (3.5.11), implies that (A,Ω) is $\overline{D}$-Minkowski measurable which, a fortiori, implies that $D=\dim_B(A,\Omega)=\overline{D}$. Furthermore we have that $\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),D)=\mathcal{M}^D(A,\Omega)$.

Case (b). We will now assume that $\overline{D}=N$, so that in this case we have

$$\frac{|A_{\mathrm{e}^{-(u+h)}}\cap\Omega|}{\mathrm{e}^{-(u+h)(N-N)}} \leq \frac{1}{h}\int_u^{u+h}|A_{\mathrm{e}^{-v}}\cap\Omega|\,\mathrm{d}v \leq \frac{|A_{\mathrm{e}^{-u}}\cap\Omega|}{\mathrm{e}^{-u(N-N)}}$$

and by taking the lower and upper limits above as $u\to+\infty$ yields

$$\overline{\mathcal{M}}^N(A,\Omega)\leq\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),N)\leq\underline{\mathcal{M}}^N(A,\Omega).\tag{3.5.15}$$

Finally, if $D<N$ then, in light of the functional equation (2.1.4), i.e., the relation between the residues at $s=D$ of the two zeta functions which follows from it, namely, $\operatorname{res}(\zeta_A(\,\cdot\,,\Omega),D)=(N-D)\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),D)$, the part of the theorem dealing with the distance zeta function follows immediately. $\square$

Remark 3.43. In light of Theorem 3.41 the assumptions of Theorem 3.42 can be weakened. More precisely, it suffices to assume that

$$\widetilde{\zeta}_A(s,\Omega)-\frac{\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),\overline{D})}{s-\overline{D}}\tag{3.5.16}$$

converges to a boundary function $G(\operatorname{Im}s)$ as $\operatorname{Re}s\to\overline{D}^+$ such that

$$\int_{-\lambda}^{\lambda}|G(\tau)|\,\mathrm{d}\tau<\infty\tag{3.5.17}$$

for every $\lambda>0$.

In the case when, besides $\overline{D}$, there are other singularities on the critical line $\{\mathrm{Re}\, s = \overline{D}\}$ of the relative fractal drum (A, Ω), we can use Theorem 3.41 to derive a bound for the upper $\overline{D}$-dimensional Minkowski content of (A, Ω) in terms of the residue of its relative tube (or distance) zeta function at $s = \overline{D}$.

Theorem 3.44 (Bound for the upper Minkowski content)**.** *Let (A, Ω) be a relative fractal drum in $\mathbb{R}^N$ and let $\overline{D} := \overline{\dim}_B(A, \Omega)$. Furthermore, assume that the relative tube zeta function $\widetilde{\zeta}_A(\,\cdot\,, \Omega)$ of (A, Ω) can be meromorphically extended to a neighborhood U of the critical line $\{\mathrm{Re}\, s = \overline{D}\}$ and that $\overline{D}$ is its simple pole. Assume also that $\{\mathrm{Re}\, s = \overline{D}\}$ contains another pole of $\widetilde{\zeta}_A(\,\cdot\,, \Omega)$ different from $\overline{D}$. Furthermore, let*

$$\lambda_{(A,\Omega)} := \inf\left\{|\overline{D} - \omega| \,:\, \omega \in \dim_{PC}(A, \Omega) \setminus \left\{\overline{D}\right\}\right\} \tag{3.5.18}$$

Then, if $\overline{D} < N$ we have the following bound for the upper $\overline{D}$-dimensional Minkowski content of (A, Ω) :

$$\overline{\mathcal{M}}^{\overline{D}}(A, \Omega) \leq \frac{C\lambda_{(A,\Omega)}(N - \overline{D})}{2\pi\left(1 - \mathrm{e}^{-2\pi(N-\overline{D})/\lambda_{(A,\Omega)}}\right)} \operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,, \Omega), \overline{D}) \tag{3.5.19}$$

and in the case when $\overline{D} = N$ we have

$$\overline{\mathcal{M}}^{N}(A, \Omega) \leq C \operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,, \Omega), N), \tag{3.5.20}$$

where C is a positive constant such that $C < 3$. Furthermore, if $\overline{D} < N$ we have that the residue of the relative distance zeta function of (A, Ω) satisfies the following:

$$\overline{\mathcal{M}}^{\overline{D}}(A, \Omega) \leq \frac{C\lambda_{(A,\Omega)}}{2\pi\left(1 - \mathrm{e}^{-2\pi(N-\overline{D})/\lambda_{(A,\Omega)}}\right)} \operatorname{res}(\zeta_A(\,\cdot\,, \Omega), \overline{D}). \tag{3.5.21}$$

Proof. We use the same reasoning as in the proof of Theorem 3.42 with the only difference being in the fact that now we can only use the weaker statement (3.5.3) of Theorem 3.41 since we have another pole on the critical line $\{\mathrm{Re}\, s = \overline{D}\}$. More precisely, if $\overline{D} < N$ and $\lambda < \lambda_{(A,\Omega)}$, then for every $h \geq 2\pi/\lambda$ by using (3.5.12) and (3.5.3) we have that

$$C \operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,, \Omega), \overline{D}) \geq \overline{\mathcal{M}}^{\overline{D}}(A, \Omega) \frac{1 - \mathrm{e}^{-h(N-\overline{D})}}{(N - \overline{D})h}. \tag{3.5.22}$$

Since the right-hand side above is decreasing in h we get the best estimate for $h = 2\pi/\lambda$. Moreover, since this is true for every $\lambda < \lambda_{(A,\Omega)}$ we get (3.5.19) by letting $\lambda \to \lambda_{(A,\Omega)}^-$. Furthermore, (3.5.22) is also valid if $\overline{D} = N$ but without the factor that depends on h by a similar argument as in case (b) of the proof of Theorem 3.42. Finally, the statement about the relative distance zeta function follows by the same argument as in Theorem 3.42. $\square$

Remark 3.45. Much as in the case of Theorem 3.42 (see Remark 3.43) the hypotheses of Theorem 3.44 can be weakened but we state it here in this form since this is the most common case we encounter in our examples of RFDs. For instance, to bound the upper $\overline{D}$-dimensional Minkowski content of (A,Ω) it is sufficient that the relative tube zeta function can be holomorphically continued to a pointed disc $B_r(\overline{D})$. In that case (3.5.19) is valid with $\lambda_{(A,\Omega)}$ replaced with the radius r. Of course, the bigger the radius of the disc, the better the bound. All one actually needs is the L^1-convergence of the relative tube or distance zeta function of (A,Ω) to a boundary function defined on a symmetric vertical interval $(\overline{D}-r\mathbb{i},\overline{D}+r\mathbb{i})$ as $\operatorname{Re}s\to\overline{D}^+$, similarly as in Remark 3.43.

In order to prove a criterion for Minkowski measurability, we will need to extend the distributional tube formulas derived in the previous sections to a larger space of test functions. It will suffice to extend them to $\mathcal{K}(0,+\infty)$. We now observe that in Definition 2.1 we have assumed that an RFD (A,Ω) has the property that there exists a $\delta>0$ such that $\Omega\subseteq A_\delta$. Also, we have commented in Remark 2.2 that if this is not fulfilled, we can always replace Ω by $\widetilde{\Omega}:=A_\delta\cap\Omega$ and operate with the new RFD $(A,\widetilde{\Omega})$. Furthermore, this property implies that $A_\delta\cap\Omega=\Omega$ for δ sufficiently large and, consequently, $|A_\delta\cap\Omega|=|\Omega|$ in light of which we can actually redefine the tube zeta function in a way which will be more suitable. More precisely, let $\overline{D}:=\overline{\dim}_B(A,\Omega)<N$ and recall the functional equation (2.1.4) written in the integral form:

$$\int_{A_\delta\cap\Omega} d(x,A)^{s-N}\,\mathrm{d}x=\delta^{s-N}|A_\delta\cap\Omega|+(N-s)\int_0^\delta t^{s-N-1}|A_t\cap\Omega|\,\mathrm{d}t, \tag{3.5.23}$$

valid for $\operatorname{Re}s>\overline{D}$. Furthermore, by taking s in the vertical strip $\{\operatorname{Re}s>\overline{D}\}\cap\{\operatorname{Re}s<N\}$ and letting $\delta\to+\infty$ we get the following equality

$$\int_\Omega d(x,A)^{s-N}\,\mathrm{d}x=(N-s)\int_0^{+\infty} t^{s-N-1}|A_t\cap\Omega|\,\mathrm{d}t. \tag{3.5.24}$$

As we can see, on the right-hand side we have the Mellin transform of the function $t^{-N}|A_t\cap\Omega|$ and this integral is absolutely convergent inside the vertical strip $\{\operatorname{Re}s>\overline{D}\}\cap\{\operatorname{Re}s<N\}$. Indeed, we have that

$$\int_0^{+\infty} t^{s-N-1}|A_t\cap\Omega|\,\mathrm{d}t=\int_0^1 t^{s-N-1}|A_t\cap\Omega|\,\mathrm{d}t+\int_1^{+\infty} t^{s-N-1}|A_t\cap\Omega|\,\mathrm{d}t, \tag{3.5.25}$$

and the integral over $(0,1)$ is equal to $\widetilde{\zeta}_A(s,\Omega;1)$, i.e., is absolutely convergent on $\operatorname{Re}s>\overline{D}$, while for the integral over $(1, +\infty)$ we have

$$\begin{aligned}\left|\int_1^{+\infty} t^{s-N-1}|A_t \cap \Omega| \,\mathrm{d}t\right| &\leq \int_1^{+\infty} t^{\operatorname{Re} s-N-1}|A_t \cap \Omega| \,\mathrm{d}t \\ &\leq |\Omega| \int_1^{+\infty} t^{\operatorname{Re} s-N-1} \,\mathrm{d}t = \frac{|\Omega|}{N - \operatorname{Re} s}.\end{aligned} \tag{3.5.26}$$

We conclude from Theorem 1.2 that the integral on the right-hand side of (3.5.24) defines a holomorphic function on the vertical strip $\{\overline{D} < \operatorname{Re} s < N\}$ and that the whole right-hand side of (3.5.24) coincides with the relative distance zeta function $\zeta_A(s, \Omega)$.

Definition 3.46. Let (A, Ω) be an RFD in $\mathbb{R}^N$ such that $\overline{\dim}_B(A, \Omega) < N$. We define the *Mellin zeta function* $\zeta_A^{\mathfrak{M}}(\,\cdot\,, \Omega)$ *of* (A, Ω) by

$$\zeta_A^{\mathfrak{M}}(s, \Omega) := \int_0^{+\infty} t^{s-N-1}|A_t \cap \Omega| \,\mathrm{d}t, \tag{3.5.27}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s \in (\overline{\dim}_B(A, \Omega), N)$ and the integral is taken in the Lebesgue sense.

We have proved the following theorem.

Theorem 3.47. *Let* (A, Ω) *be an RFD in* $\mathbb{R}^N$ *such that* $\overline{\dim}_B(A, \Omega) < N$*. Then the Mellin zeta function* $\zeta_A^{\mathfrak{M}}(\,\cdot\,, \Omega)$ *is holomorphic on the open vertical strip* $\{\overline{\dim}_B(A, \Omega) < \operatorname{Re} s < N\}$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}s}\zeta_A^{\mathfrak{M}}(s, \Omega) = \int_0^{+\infty} t^{s-N-1}|A_t \cap \Omega| \log t \,\mathrm{d}t, \tag{3.5.28}$$

for all s *with* $\{\overline{\dim}_B(A, \Omega) < \operatorname{Re} s < N\}$ *and this is the largest vertical strip on which the integral* (3.5.27) *absolutely converges.*

Furthermore, for $s \in \mathbb{C}$ *such that* $\overline{\dim}_B(A, \Omega) < \operatorname{Re} s < N$ *and a fixed* $\delta > 0$ *such that* $\Omega \subseteq A_\delta$, $\zeta_A^{\mathfrak{M}}(\,\cdot\,, \Omega)$ *satisfies the following functional equations*

$$\zeta_A^{\mathfrak{M}}(s, \Omega) = \widetilde{\zeta}_A(s, \Omega; \delta) + \frac{\delta^{s-N}|\Omega|}{N - s}; \tag{3.5.29}$$

$$\zeta_A^{\mathfrak{M}}(s, \Omega) = \frac{\zeta_A(s, \Omega; \delta)}{N - s}; \tag{3.5.30}$$

Remark 3.48. We point out that similar functional equations to (3.5.29) and (3.5.30) are also satisfied for $\delta > 0$ such that $\Omega \not\subseteq A_\delta$ but one has to add to the right-hand side a suitable function f meromorphic on $\mathbb{C}$ with a single simple pole at $s = N$.

Proof of Theorem 3.47. We have already proved the first part of the theorem. The optimality of the vertical strip follows directly from (3.5.24). Namely, the lower bound $\overline{\dim}_B(A, \Omega)$ is a consequence of the first integral on the right-hand side of (3.5.24) since it is equal to $\widetilde{\zeta}_A(s, \Omega; 1)$ and the upper bound is a consequence of the second integral on the right-hand side of (3.5.24), since it is clearly divergent for real $s > N$.

The functional equation (3.5.30) is already proven, while (3.5.29) can be proven directly by splitting the integral defining $\zeta_A^{\mathfrak{M}}(\,\cdot\,,\Omega)$ over the intervals $(0,\delta)$ and $(\delta,+\infty)$ or from the functional equation connecting the relative tube and distance zeta functions. $\square$

As a consequence of the functional equations (3.5.30), (3.5.29) and the principle of analytic continuation we have the following theorems which follow from the corresponding ones for the relative distance and tube zeta functions.

Theorem 3.49. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $\overline{\dim}_B(A,\Omega) < N$. Then the following properties hold:*

(*a*) *The Mellin zeta function $\zeta_A^{\mathfrak{M}}(\,\cdot\,,\Omega)$ is meromorphic in the half-plane $\{\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)\}$ with a single simple pole at $s = N$. Furthermore,*

$$\mathrm{res}(\zeta_A^{\mathfrak{M}}(\,\cdot\,,\Omega), N) = -|\Omega| \tag{3.5.31}$$

(*b*) *If the relative box (or Minkowski) dimension $D := \dim_B(A,\Omega)$ exists, and $\underline{\mathcal{M}}^D(A,\Omega) > 0$, then $\zeta_A^{\mathfrak{M}}(s,\Omega) \to +\infty$ as $s \in \mathbb{R}$ converges to D from the right.*

Proof. By the principle of analytic continuation we conclude that the functional equalities (3.5.29) and (3.5.30) continue to hold on any open connected domain $U \subseteq \mathbb{C}$ to which any of the three relative zeta functions has a holomorphic continuation. In light of this part (*a*) follows from the counterpart of Theorem 2.6 for the relative tube zeta function and (3.5.29) while part (*b*) follows from Theorem 2.6 and (3.5.30). $\square$

Furthermore, in light of Theorem 2.9 and (3.5.29) one has the following result.

Theorem 3.50. *Assume that (A,Ω) is a nondegenerate RFD in $\mathbb{R}^N$, that is, $0 < \underline{\mathcal{M}}^D(A,\Omega) \le \overline{\mathcal{M}}^D(A,\Omega) < \infty$ (in particular, $\dim_B(A,\Omega) = D$), and $D < N$. If $\zeta_A^{\mathfrak{M}}(\,\cdot\,,\Omega)$ can be extended meromorphically to a neighborhood of $s = D$, then D is necessarily a simple pole of $\zeta_A^{\mathfrak{M}}(\,\cdot\,,\Omega)$, and*

$$\underline{\mathcal{M}}^D(A,\Omega) \le \mathrm{res}(\zeta_A^{\mathfrak{M}}(\,\cdot\,,\Omega), D) \le \overline{\mathcal{M}}^D(A,\Omega). \tag{3.5.32}$$

Furthermore, if (A,Ω) is Minkowski measurable, then

$$\mathrm{res}(\zeta_A^{\mathfrak{M}}(\,\cdot\,,\Omega), D) = \mathcal{M}^D(A,\Omega). \tag{3.5.33}$$

Lemma 3.51. *Assume that (A,Ω) is an RFD in $\mathbb{R}^N$ with $\overline{\dim}_B(A,\Omega) < N$ such that its tube or distance or Mellin zeta function is meromorphic on some open connected domain $U \subseteq \{\mathrm{Re}\, s < N\}$. Then the multisets of poles located in U of all of the three zeta functions* *coincide. Moreover, if $\omega \in U$ is a pole of any of these three zeta functions, then*

$$\operatorname{res}(\zeta_A^{\mathfrak{M}}(\,\cdot\,,\Omega),\omega) = \operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,,\Omega),\omega) = \frac{\operatorname{res}(\zeta_A(\,\cdot\,,\Omega),\omega)}{N-\omega}. \tag{3.5.34}$$

We can now use the Mellin inversion theorem to derive the following inversion formula for the Mellin zeta function.

Theorem 3.52. *Let (A,Ω) be an RFD in $\mathbb{R}^N$ such that $\overline{\dim}_B(A,\Omega) < N$. Then, for any $c \in (\overline{\dim}_B(A,\Omega), N)$ and $t > 0$ the following formula is valid pointwise*:

$$|A_t \cap \Omega| = \frac{1}{2\pi\mathbb{i}} \int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} t^{N-s} \zeta_A^{\mathfrak{M}}(s,\Omega)\,\mathrm{d}s. \tag{3.5.35}$$

Proof. The conclusion follows directly from Theorem 3.5, the fact that $t^{-N}|A_t \cap \Omega|$ is continuous and of bounded variation on $(0,\infty)$ and $t^{c-N-1}|A_t \cap \Omega|$ is in $L^1(0,\infty)$ for all $c \in (\overline{\dim}_B(A,\Omega), N)$. □

Note that in the above theorem it is crucial that we choose $c \in (\overline{\dim}_B(A,\Omega), N)$ for the hypothesis of the Mellin inversion theorem to be satisfied. In other words, the theorem will not work in the case when $\overline{\dim}_B(A,\Omega) = N$ since then we cannot define the Mellin zeta function. Note that this is in contrast with the situation from sections 3.2 and 3.3 where we have worked with the relative tube zeta function. One can now impose languidity conditions on the Mellin zeta function and rewrite sections 3.2 and 3.3 in terms of it since the fact that we have to choose $c \in (\overline{\dim}_B(A,\Omega), N)$ is of no hindrance. Indeed, we had originally the freedom to choose any $c \in (\overline{\dim}_B(A,\Omega), N+1)$ in Proposition 3.8. Furthermore, this will ensure that although $s = N$ is always a pole of the Mellin zeta function it will never be a part of the sum over the residues of $\zeta_A^{\mathfrak{M}}(\,\cdot\,,\Omega)$ since it is always to the right of the vertical line over which we integrate.

Moreover, one can also derive the corresponding results about the distance zeta function directly from the Mellin zeta function and without the use of the shell zeta function. However, one has to be careful and always choose δ sufficiently large such that $\Omega \subseteq A_\delta$ in order for (3.5.30) to be fulfilled. One other thing that is not clear is whether the restriction of choosing δ large enough for $\Omega \subseteq A_\delta$ to hold could increase the 'languidity exponent' κ_d of $\zeta_A(s,\Omega)$. This is not the case in all of the examples we will consider but a general result has yet to be obtained.

Proposition 3.53. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$. If the relative tube zeta function $\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta)$ satisfies the languidity conditions* **L1** *and* **L2** *for some $\delta > 0$ and $\kappa, \in \mathbb{R}$ then so does $\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta_1)$ for $\kappa_1 = \max\{\kappa, -1\}$.*

Proof. Without loss of generality, we may assume that $\delta < \delta_1$. Then, the conclusion follows from the fact that $\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta_1)=\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta)+f(s)$ where f is holomorphic and

$$|f(s)|\leq\int_{\delta}^{\delta_1}t^{s-N-1}|A_t\cap\Omega|\,\mathrm{d}t\leq|\Omega|\frac{\delta_1^{s-N}-\delta^{s-N}}{s-N}.$$

□

Proposition 3.54. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$. If the relative distance zeta function $\zeta_A(\,\cdot\,,\Omega;\delta)$ satisfies the languidity conditions* **L1** *and* **L2** *for some $\delta>0$ and $\kappa_d\in\mathbb{R}$ then so does $\zeta_A(\,\cdot\,,\Omega;\delta_1)$ for $(\kappa_d)_1=\max\{\kappa_d,0\}$.*

Proof. Similarly as above we assume that $\delta<\delta_1$ and have $\zeta_A(\,\cdot\,,\Omega;\delta_1)=\zeta_A(\,\cdot\,,\Omega;\delta)+g(s)$ with

$$|g(s)|\leq\int_{(A_{\delta_1}\setminus A_{\delta_2})\cap\Omega}d(x,A)^{s-N}\,\mathrm{d}x\leq|\Omega|\max\{\delta_1^{\mathrm{Re}\,s-N},\delta_2^{\mathrm{Re}\,s-N}\}.$$

□

We will not restate all of the theorems of Sections 3.2 and 3.3 in terms of the Mellin zeta function but only the distributional tube formula with error term since it will be needed for establishing a Minkowski measurability criterion. Recall that the motivation for introducing the Mellin zeta function in the first place was to obtain a distributional tube formula valid on a larger space of test functions. More precisely, on the space $\mathcal{K}(0,+\infty)$; that is, the space of test functions φ in the class $C^\infty(0,\delta)$, such that for all $m\in\mathbb{Z}$ and $q\in\mathbb{N}$ we have $t^m\varphi^{(q)}(t)\to 0$, as $t\to 0^+$ and $t\to+\infty$. Also note that $\mathcal{D}(0,+\infty)\subseteq\mathcal{K}(0,+\infty)$.

Theorem 3.55. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ with $\overline{\dim}_B(A,\Omega)<N$. Furthermore, assume that $\zeta_A^{\mathfrak{M}}(\,\cdot\,,\Omega)$ satisfies the languidity conditions for some $\kappa\in\mathbb{R}$. Then, the distribution $\mathcal{V}^{[0]}$ on $\mathcal{K}(0,+\infty)$ is given by*

$$\mathcal{V}^{[0]}(t)=\sum_{\omega\in\mathcal{P}(\zeta_A^{\mathfrak{M}}(\,\cdot\,,\Omega),W)}\operatorname{res}\left(t^{N-s}\zeta_A^{\mathfrak{M}}(s,\Omega),\omega\right)+\mathcal{R}^{[0]}(t).\tag{3.5.36}$$

That is, the action of $\mathcal{V}^{[0]}(t)$ on a test function $\varphi\in\mathcal{K}(0,+\infty)$ is given by

$$\left\langle\mathcal{V}^{[0]},\varphi\right\rangle=\sum_{\omega\in\mathcal{P}(\zeta_A^{\mathfrak{M}}(\,\cdot\,,\Omega),W)}\operatorname{res}\left(\{\mathfrak{M}\varphi\}(N-s+1)\,\zeta_A^{\mathfrak{M}}(s,\Omega),\omega\right)+\left\langle\mathcal{R}^{[0]},\varphi\right\rangle.\tag{3.5.37}$$

Here, the distribution $\mathcal{R}^{[0]}(t)$ is the error term given by

$$\left\langle\mathcal{R}^{[0]},\varphi\right\rangle=\frac{1}{2\pi\mathbb{i}}\int_S\{\mathfrak{M}\varphi\}(N-s+1)\,\zeta_A^{\mathfrak{M}}(s,\Omega)\,\mathrm{d}s.\tag{3.5.38}$$

Furthermore, the distribution $\mathcal{R}^{[0]}(t)$ is of asymptotic order at most $t^{N-\sup S}$ as $t\to 0^+$, *i.e.,*

$$\mathcal{R}^{[0]}(t) = O(t^{N-\sup S}) \quad \text{as } t \to 0^+ \tag{3.5.39}$$

in the sense of Definition 1.37.

Moreover, if $S(\tau) < \sup S$ for all $\tau \in \mathbb{R}$ (that is, if the screen lies strictly to the left of the line $\operatorname{Re} s = \sup S$), then $\mathcal{R}^{[0]}(t)$ is of asymptotic order less than $t^{N-\sup S}$, i.e.,

$$\mathcal{R}^{[0]}(t) = o(t^{N-\sup S}) \quad \text{as } t \to 0^+. \tag{3.5.40}$$

Having now expanded the space of test functions for which the distributional tube formula is valid we can now derive a necessary condition for Minkowski measurability of a languid relative fractal drum.

Theorem 3.56 (Necessary condition for Minkowski measurability). *Let (A, Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $D = \dim_B(A, \Omega)$ exists, $D < N$ and (A, Ω) is Minkowski measurable. Furthermore, assume also that its Mellin zeta function $\zeta_A^{\mathfrak{M}}(\,\cdot\,, \Omega)$ is languid for some screen S passing between the critical line $\{\operatorname{Re} s = D\}$ and all the complex dimensions of (A, Ω) with real part strictly less than D. Then D is the only pole of $\zeta_A^{\mathfrak{M}}(\,\cdot\,, \Omega)$ located on the critical line $\{\operatorname{Re} s = D\}$ and it is simple.*

Proof. Since (A, Ω) is languid, the hypothesis of Theorem 3.50 is satisfied and, therefore, $s = D$ is a simple pole of $\zeta_A^{\mathfrak{M}}(\,\cdot\,, \Omega)$. Furthermore, we have that $\mathcal{M} := \mathcal{M}^D(A, \Omega) = \operatorname{res}(\zeta_A^{\mathfrak{M}}(\,\cdot\,, \Omega), D)$. It remains to show that this is the only pole located on the critical line. Firstly, directly from the definition of the Mellin zeta function we have that $|\zeta_A^{\mathfrak{M}}(s, \Omega)| \leq \zeta_A^{\mathfrak{M}}(\operatorname{Re} s, \Omega)$ for all $s \in \{D < \operatorname{Re} s < N\}$. From this we conclude that if ξ is another pole of $\zeta_A^{\mathfrak{M}}(\,\cdot\,, \Omega)$ with $\operatorname{Re} \xi = D$ then it is also simple.

We reason by contradiction; that is, let us assume that there exist other simple poles $\xi_n = D + \mathbb{i}\gamma_n$ of $\zeta_A^{\mathfrak{M}}(\,\cdot\,, \Omega)$ for $\gamma_n \in \mathbb{R}$, $n \in \mathbb{N}$ and let $a_n := \operatorname{res}(\zeta_A^{\mathfrak{M}}(\,\cdot\,, \Omega), \xi_n)$.[12] Then, by Theorem 3.55 we have that

$$|A_t \cap \Omega| = \mathcal{M} t^{N-D} + t^{N-D} \sum_{n=1}^{\infty} a_n t^{-\mathbb{i}\gamma_n} + o(t^{N-D}), \text{ as } t \to 0^+ \tag{3.5.41}$$

in the distributional sense since the screen is strictly to the left of the critical line $\{\operatorname{Re} s = D\}$.

On the other hand, since (A, Ω) is Minkowski measurable, we know that its relative tube function satisfies

$$|A_t \cap \Omega| = \mathcal{M} t^{N-D} + o(t^{N-D}), \quad \text{as} \quad t \to 0^+ \tag{3.5.42}$$

[12] A meromorphic function on an open domain can only have a countable number of poles. This follows from the fact that any compact subset $K \subseteq \mathbb{C}$ may contain only finitely many poles (since otherwise there would be a limit point of poles in K) and any open domain is contained in a countable union of compact sets.

in the usual sense. Combining (3.5.41) with (3.5.42) together with Lemma 1.39 yields that

$$\sum_{n=1}^{\infty} a_n t^{-\mathbb{i}\gamma_n} = o(1), \quad \text{as} \quad t \to 0^+ \tag{3.5.43}$$

in the distributional sense. Since $\mathcal{D}(0,\infty) \subseteq \mathcal{K}(0,\infty)$, we may now use Lemma 1.40 to conclude that this can only be true if $a_n = 0$ for all $n \in \mathbb{N}$; that is, if there are no other poles on the critical line except $s = D$. $\square$

Remark 3.57. The above theorem can also be stated in terms of the relative tube and distance zeta functions of (A, Ω). This follows from the fact that the functional equations (3.5.29) and (3.5.30) that connect the relative tube zeta function, the relative distance zeta function and the Mellin zeta function of (A, Ω) together with Propositions 3.53 and 3.54 imply that if the languidity conditions **L1** and **L2** are satisfied by the tube or distance zeta functions, then they are also satisfied by the Mellin zeta function with a possibly different exponent but we can still apply Theorem 3.55.

Finally, combining Theorems 3.42 and 3.56 we can state the announced Minkowski measurability criterion.

Theorem 3.58 (Minkowski measurability criterion)**.** *Let (A, Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $D := \dim_B(A, \Omega)$ exists and $D < N$. Furthermore, assume that (A, Ω) is d-languid for a screen passing between the critical line $\{\operatorname{Re} s = D\}$ and all the complex dimensions of (A, Ω) with real part strictly less than D. Then the following is equivalent*:

(*a*) *(A, Ω) is Minkowski measurable.*

(*b*) *D is the only pole of the relative distance zeta function $\zeta_A(\,\cdot\,, \Omega)$ located on the critical line $\{\operatorname{Re} s = D\}$ and it is simple.*

Remark 3.59. The above criterion is also valid if in (*b*) we replace $\zeta_A(\,\cdot\,, \Omega)$ with the relative tube zeta function $\widetilde{\zeta}_A(\,\cdot\,, \Omega)$, the Mellin zeta function $\zeta_A^{\mathfrak{M}}(\,\cdot\,, \Omega)$ or the relative shell zeta function $\breve{\zeta}_A(\,\cdot\,, \Omega)$. In this case, it is enough to assume that the chosen fractal zeta function satisfies the languidity conditions.

Remark 3.60. Although we cannot apply directly Theorem 3.58 in the case when $\dim_B(A, \Omega) = N$, we will show that this problem can be solved by appropriately embedding the relative fractal drum (A, Ω) into $\mathbb{R}^{N+1}$. See Section 3.7 for more details.

3.6 Examples and Applications

In this section we will demonstrate the theory developed so far on a few examples. We will begin with the trivial example of the unit interval in $\mathbb{R}$ which demonstrates the case when we cannot use the distance zeta function to recover the tube formula since $D = N = 1$.

Example 3.61. Let $I = [0,1]$ be the unit interval in $\mathbb{R}$. Then the meromorphic continuations to $\mathbb{C}$ of its distance and tube zeta functions are given by

$$\zeta_I(s) = \frac{2\delta^s}{s} \quad \text{and} \quad \widetilde{\zeta}_I(s) = \frac{2\delta^s}{s} + \frac{\delta^{s-1}}{s-1}, \tag{3.6.1}$$

respectively. As we have already alluded in Remark 2.7, the distance zeta function fails to give information about the Minkowski content in this case since the pole at $s = 1$ is being canceled by means of the functional equation (2.1.4). On the other hand, it is clear that $\widetilde{\zeta}_I$ is strongly languid if we choose $\delta > 1$ for $\kappa = -1$ and a sequence of screens consisting of vertical lines $\{\operatorname{Re} s = -m\}$, where $m \in \mathbb{N}$. From Theorem 3.14 we recover the pointwise tube formula:

$$|I_t| = t^{N-0}\operatorname{res}(\widetilde{\zeta}_I, 0) + t^{N-1}\operatorname{res}(\widetilde{\zeta}_I, 1) = 2t + 1, \tag{3.6.2}$$

valid for all $t \in (0, \delta)$ and since $\delta > 1$ may be taken arbitrary large the formula is actually valid for all $t > 0$ (which is also obvious).

It is noteworthy to observe that (as opposed to the case when $\overline{\dim}_B(A,\Omega) < N$) in this and similar cases (when the RFD (A,Ω) in $\mathbb{R}^N$ has upper box dimension equal to N) it is not true that $\zeta_A(s, N; \delta) = |A_\delta \cap \Omega|$ as one could wrongly deduce by substituting $s = N$ in (2.2) and evaluating the integral. This can be explained by the fact that we are integrating the indeterminate form 0^0 over a set of positive Lebesgue measure. More precisely, if there exists a meromorphic continuation of the relative tube zeta function to a neighborhood of $s = N$, one generally has, from the functional equation (2.1.4), that

$$\zeta_A(N, \Omega; \delta) = |A_\delta \cap \Omega| - \operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,, \Omega; \delta), N), \tag{3.6.3}$$

observing that $\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,, \Omega; \delta), N) = 0$ if $\widetilde{\zeta}_A(\,\cdot\,, \Omega; \delta)$ is holomorphic at $s = N$.[13] Additionally if we assume that $\dim_B(A,\Omega) = N$ and (A,Ω) is Minkowski measurable, then by Theorem 2.9 we get that

$$\zeta_A(N, \Omega; \delta) = |A_\delta \cap \Omega| - \mathcal{M}^N(A,\Omega) = |A_\delta \cap \Omega| - |\overline{A} \cap \Omega|. \tag{3.6.4}$$

Let us look at the example of the $(N-1)$-dimensional sphere in $\mathbb{R}^N$ for which the tube zeta function has been explicitly calculated in [LapRaŽu1].

Example 3.62. Let $B_R(0)$ be the ball of $\mathbb{R}^N$ centered at the origin with radius $R > 1$, and let $A := \partial B_R(0)$ be its boundary, i.e., the $(N-1)$-dimensional sphere of radius R.

[13] This is exactly what happens in the case when $\overline{\dim}_B(A,\Omega) < N$.

Then, for a fixed $\delta \in (0,R)$, its tube zeta function is meromorphic on $\mathbb{C}$ and given by[14]

$$\widetilde{\zeta}_A(s) = \omega_N \sum_{k=0}^{N} (1-(-1)^k) R^{N-k} \binom{N}{k} \frac{\delta^{s-N+k}}{s-(N-k)}. \tag{3.6.5}$$

Here, ω_N is the N-dimensional Lebesgue measure (or volume) of the unit ball of $\mathbb{R}^N$.[15] It then follows that $\dim_B A$ exists and

$$\dim_B A = D(\widetilde{\zeta}_A) = N-1 \tag{3.6.6}$$

and moreover, the set of complex dimensions of A (i.e., the set of poles of $\widetilde{\zeta}_A$ or, equivalently, of ζ_A), is given by (with $\lfloor x \rfloor$ denoting the integer part of $x \in \mathbb{R}$)

$$\begin{aligned} \mathcal{P}(\widetilde{\zeta}_A) &= \left\{ N-(2j+1) : j = 0,1,2,\ldots, \left\lfloor \frac{N-1}{2} \right\rfloor \right\} \\ &= \left\{ N-1, N-3, \ldots, N - \left(2 \left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right) \right\}. \end{aligned} \tag{3.6.7}$$

For an odd N, the last number (on the right) in this set is equal to 0, while for an even N, it is equal to 1. Furthermore, the residue of the tube zeta function $\widetilde{\zeta}_A$ at any of its poles $N-k \in \mathcal{P}(\widetilde{\zeta}_A)$ is equal to

$$\operatorname{res}(\widetilde{\zeta}_A, N-k) = 2\omega_N \binom{N}{k} R^{N-k}. \tag{3.6.8}$$

Since $\binom{N}{k} = \binom{N}{N-k}$, we can write this result in an even more 'symmetric' form:

$$\operatorname{res}(\widetilde{\zeta}_A, m) = 2\omega_N \binom{N}{m} R^m, \quad \text{for all} \quad m \in \mathcal{P}(\widetilde{\zeta}_A). \tag{3.6.9}$$

Clearly, in light of (3.6.6) and (3.6.7), we have $\dim_{PC} A = \{N-1\}$ and according to (3.6.8) or (3.6.9), we have

$$\operatorname{res}(\widetilde{\zeta}_A, N-1) = 2N\omega_N R^{N-1}. \tag{3.6.10}$$

Observe that, by choosing $\delta = 1$ we have that $\widetilde{\zeta}_A$ is strongly languid with $\kappa = -1$. Namely, we may take the sequence of screens S_m as the sequence of vertical lines $\{\operatorname{Re} s = -m\}$. Then, from Theorem 3.14 (strongly languid case) and equations (3.6.7) and (3.6.8)

[14]Here, the numbers $\binom{N}{k}$ stand for the usual binomial coefficients.

[15]Hence, $\omega_N = \pi^{N/2}/(N/2)!$, where $x! := \Gamma(x+1)$ and with Γ denoting the classic gamma function; so that $x!$ is is the usual factorial function when $x \in \mathbb{N}$.

we can recover the well-known tube formula of A, i.e., for $t < 1$ we have:

$$\begin{aligned} |A_t| &= \sum_{\omega\in\mathcal{P}(\widetilde{\zeta}_A)} t^{N-\omega}\operatorname{res}(\widetilde{\zeta}_A,\omega) \\ &= 2\omega_N \sum_{j=0}^{\lfloor\frac{N-1}{2}\rfloor} \binom{N}{2j+1} t^{2j+1} R^{N-(2j+1)} \\ &= \omega_N \sum_{k=0}^{N} \binom{N}{k}(1-(-1)^k)t^k R^{N-k} \\ &= \omega_N\left((R+t)^N-(R-t)^N\right) \end{aligned} \tag{3.6.11}$$

Example 3.63 (The standard ternary Cantor set)**.** Let C be the standard ternary Cantor set in $[0,1]$ and fix $\delta \geq 1/6$. Then, from [LapRaŽu1] we have that the 'absolute' distance zeta function of C is given by

$$\zeta_C(s,C_\delta) = \frac{2}{2^s s(3^s-2)} + \frac{2\delta^s}{s} \tag{3.6.12}$$

where the part $2\delta^s/s$ amounts to the integral over the 'outer' neighborhood of the two endpoints 0 and 1. Consequently the relative distance zeta function of $(C,[0,1])$ is then given by

$$\zeta_C(s,[0,1]) = \frac{2}{2^s s(3^s-2)}. \tag{3.6.13}$$

Furthermore, the sets of complex dimensions of C and $(C,[0,1])$ coincide:

$$\mathcal{P}(\zeta_C) = \mathcal{P}(\zeta_C(\,\cdot\,,[0,1])) = \{0\}\cup\left(\log_3 2 + \frac{2\pi\mathbb{i}}{\log 3}\mathbb{Z}\right). \tag{3.6.14}$$

It is clear that $(\lambda C,\lambda[0,1])$ is strongly d-languid for $\kappa_d = -1$, any $\lambda \geq 2$ and a sequence of screens consisting of vertical lines $\{\operatorname{Re} s = -m\}$ with the constant $B_\lambda = 2/\lambda$.[16] Theorem 3.37 enables us to recover the exact pointwise formula for the 'inner' t-neighborhood of C valid for $t \in (0,\min\{1/\lambda,1/2\}) = (0,1/2)$:

$$\begin{aligned} |C_t\cap[0,1]| &= \sum_{\omega\in\mathcal{P}(\zeta_C(\,\cdot\,,[0,1]))} \operatorname{res}\left(\frac{t^{1-s}}{1-s}\zeta_C(s,[0,1]),\omega\right) \\ &= \frac{t^{1-\log_3 2}}{\log 3}\sum_{k=-\infty}^{\infty}\frac{2^{-\omega_k}t^{-\mathbf{p}k\mathbb{i}}}{(1-\omega_k)\omega_k} - 2t, \end{aligned} \tag{3.6.15}$$

where $\omega_k := \log_3 2 + \mathbf{p}k\mathbb{i}$ and $\mathbf{p} := 2\pi/\log 3$. Of course, the above formula coincides with the one obtained by direct computation (see [Lap–vFr3, Subsection 1.1.2]). Note that the 'absolute' tube function $|C_t|$ has the same expression as above but without the term $-2t$

[16] We can fix here $\delta \geq 1$ without loss of generality.

which is in accordance with (3.6.12).

The above example demonstrates how the theory of this chapter generalizes the corresponding one for fractal strings developed in [Lap–vFr3]. More generally, the following result from [LapRaŽu1] gives a general connection between the geometric zeta function of a nontrivial fractal string $\mathcal{L} = (l_j)_{j\geq 1}$[17] and the distance zeta function of the set

$$A_{\mathcal{L}} := \{a_k := \sum_{j\geq k} l_j : k \geq 1\}. \tag{3.6.16}$$

Recall that the geometric zeta function of $\mathcal{L}$ is defined by

$$\zeta_{\mathcal{L}}(s) := \sum_{k=1}^{\infty} l_j{}^s, \tag{3.6.17}$$

for $s \in \mathbb{C}$ such that $\operatorname{Re} s$ is sufficiently large.

Proposition 3.64. *Let $\mathcal{L} = (l_j)_{j\geq 1}$ be a nontrivial fractal string and $l := \zeta_{\mathcal{L}}(1)$ its total length. Then, for $\delta \geq l_1/2$ we have the following functional equation for the distance zeta function of the relative fractal drum $(A_{\mathcal{L}}, [0, l])$:*

$$\zeta_{A_{\mathcal{L}}}(s, [0, l]; \delta) = \frac{\zeta_{\mathcal{L}}(s)}{2^{s-1}s}, \tag{3.6.18}$$

valid on any connected open set $U \subseteq \mathbb{C}$ to which any of the two zeta functions possesses a meromorphic continuation.[18]

Furthermore, if $\zeta_{\mathcal{L}}$ is languid for some $\kappa_{\mathcal{L}} \in \mathbb{R}$, then $\zeta_{A_{\mathcal{L}}}(\,\cdot\,;\delta)$ is d-languid for $\kappa_d = \kappa_{\mathcal{L}} - 1$, with any $\delta \geq l_1/2$.

Moreover, if $\zeta_{\mathcal{L}}$ is strongly languid, then so is $\zeta_{\lambda A_{\mathcal{L}}}(s, [0, \lambda l]; \delta\lambda)$ for any $\lambda \geq 2$ and any $\delta \geq l_1/2$.

Proof. For the proof of the functional equation (3.6.18) see [LapRaŽu1] and the statements about the languidity follow directly from the definition. □

Let us now apply Proposition 3.64 in order to recover the formula of the tubular volume of the boundary of a well-known fractal string studied in [Lap–vFr3, Subsection 2.3.2].

Example 3.65 (The Fibonacci string)**.** Let Fib be the Fibonacci string (with total length 4) where the sequence of lengths is given by $l_j := 2^{-j}$ and each length has multiplicity F_{n+1} where F_n is the n-th Fibonacci number defined by the following recursive formula:

[17] Here, $(l_j)_{j\geq 1}$ denotes the nonincreasing sequence of lengths of $\mathcal{L}$.

[18] If we do not require that $\delta \geq l_1/2$, then we have that $\zeta_{A_{\mathcal{L}}}(s;\delta) = 2^{1-s}s^{-1}\zeta_{\mathcal{L}}(s) + v(s)$, where v is holomorphic on $\{\operatorname{Re} s > 0\}$. On the other hand, for applying the theory we may restrict ourselves to the case when $\delta \geq l_1/2$.

bonacci number defined by the following recursive formula:

$$F_{n+1} = F_n + F_{n-1}, \quad \text{and} \quad F_0 = 0,\ F_1 = 1. \tag{3.6.19}$$

Then for the geometric zeta function we have

$$\zeta_{\mathrm{Fib}}(s) = \frac{1}{1 - 2^{-s} - 4^{-s}}, \tag{3.6.20}$$

and, from Proposition 3.64 we get that

$$\zeta_{A_{\mathrm{Fib}}}(s, [0,4]; 1) = \frac{1}{2^{s-1}s(1 - 2^{-s} - 4^{-s})} = \frac{2^{s+1}}{s(4^s - 2^s - 1)}. \tag{3.6.21}$$

One can easily check that the set of complex dimensions of A_{Fib} is given by

$$\mathcal{P}(\zeta_{A_{\mathrm{Fib}}}) = \Big(-D + \frac{\mathbf{p}\mathbb{i}}{2} + \mathbf{p}\mathbb{i}\mathbb{Z}\Big) \cup \{0\} \cup (D + \mathbf{p}\mathbb{i}\mathbb{Z})\,, \tag{3.6.22}$$

where $D = \log_2 \phi$ with $\phi = (1 + \sqrt{5})/2$ is the golden mean and $\mathbf{p} = 2\pi/\log 2$. Similarly as in Example 3.63 one checks that we may apply Theorem 3.37 with any $\lambda \geq 1/2$ and a corresponding $B_\lambda = 1/(2\lambda)$ to recover the pointwise tube formula valid for all $t \in (0,2)$:

$$\begin{aligned}
|(A_{\mathrm{Fib}})_t \cap [0,4]| &= \sum_{\omega \in \mathcal{P}(\zeta_{A_{\mathrm{Fib}}})} \operatorname{res}\left(\frac{t^{1-s}}{1-s} \zeta_{A_{\mathrm{Fib}}}(s; 1/2), \omega \right) \\
&= \frac{2^{1-D}\phi\, t^{1-D}}{\sqrt{5}\log 2} \sum_{k=-\infty}^{\infty} \frac{(2t)^{-\mathbf{p}k\mathbb{i}}}{(1 - D - \mathbf{p}k\mathbb{i})(D + \mathbf{p}k\mathbb{i})} - 2t \\
&\quad + \frac{2^{1+D}(\phi - 1)\, t^{1+D}}{\sqrt{5}\log 2} \sum_{k=-\infty}^{\infty} \frac{(2t)^{-\mathbf{p}\mathbb{i}/2 - \mathbf{p}k\mathbb{i}}}{(1 + D - \mathbf{p}\mathbb{i}/2 - \mathbf{p}k\mathbb{i})(-D + \mathbf{p}\mathbb{i}/2 + \mathbf{p}k\mathbb{i})}.
\end{aligned}$$

Of course, the above formula coincides with the formula derived in [Lap–vFr3, Subsection 2.3.2].

Example 3.66 (The a-string)**.** For a given $a > 0$ the a-string $\mathcal{L}$ can be realized as the open set $\Omega \subset \mathbb{R}$ obtained by removing the points j^{-a} for $j \in \mathbb{N}$ from the interval $(0,1)$; that is,

$$\Omega = \bigcup_{j=1}^{\infty} \big((j+1)^{-a}, j^{-a}\big), \tag{3.6.23}$$

so that the sequence of lengths of $\mathcal{L}$ is given by

$$l_j = j^{-a} - (j+1)^{-a}, \quad j = 1, 2, \ldots. \tag{3.6.24}$$

Its geometric zeta function is then given by

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s = \sum_{j=1}^{\infty} \left(j^{-a} - (j+1)^{-a}\right)^s$$

and from Proposition 3.64 its distance zeta function for $\delta > (1-2^{-a})/2$ is then given by

$$\zeta_{A_{\mathcal{L}}}(s,[0,1];\delta) = \frac{\zeta_{\mathcal{L}}(s)}{2^{s-1}s} = \frac{1}{2^{s-1}s}\sum_{j=1}^{\infty} \left(j^{-a} - (j+1)^{-a}\right)^s. \tag{3.6.25}$$

Furthermore, the properties of the geometric zeta function $\zeta_{\mathcal{L}}$ of the a-string are well-known (see [Lap–vFr3, Theorem 6.21]). Namely, it has a meromorphic continuation to the whole of $\mathbb{C}$ and its poles are located at $D := \frac{1}{a+1}$ and at (a subset of) $\{-\frac{m}{a+1} : m \in \mathbb{N}\}$. Furthermore, all of its poles are simple and $\operatorname{res}(\zeta_{\mathcal{L}}, D) = Da^D$.[19] Moreover, for any screen S not passing through a pole, the function $\zeta_{\mathcal{L}}$ satisfies **L1** and **L2** with $\kappa = \frac{1}{2} - (a+1)\inf S$ if $\inf S \leq 0$ and $\kappa = \frac{1}{2}$ if $\inf S \geq 0$. From these facts and Equation (3.6.25) we conclude that the set $A_{\mathcal{L}}$ is d-languid with $\kappa_d = -\frac{1}{2} - (a+1)\inf S$ if $\inf S \leq 0$ and with $\kappa_d = -\frac{1}{2}$ if $\inf S \geq 0$. For $M \in \mathbb{N} \cup \{0\}$ we can now choose the screen S_M to be some vertical line between $-\frac{M+1}{1+a}$ and $-\frac{M+2}{1+a}$ and let W_M be the corresponding window. Applying Theorem 3.40 we now obtain the following asymptotic distributional formula for the tube function $t \mapsto |(A_{\mathcal{L}})_t \cap [0,1]|$ when $t \to 0^+$:

$$|(A_{\mathcal{L}})_t \cap [0,1]| = \sum_{\omega \in \mathcal{P}(\zeta_{A_{\mathcal{L}}}, W_M)} \operatorname{res}\left(\frac{t^{1-s}}{1-s}\zeta_{A_{\mathcal{L}}}(s;\delta), \omega\right) + O(t^{1-\sup S_M}) \tag{3.6.26}$$

or, more precisely, since we know that all the poles are simple and that $\zeta_{\mathcal{L}}(0) = -1/2$ (see [Lap–vFr3, p. 205]), we have that

$$\begin{aligned}\operatorname{res}(\zeta_{A_{\mathcal{L}}}, D) &= 2^{1-D}D^{-1}\operatorname{res}(\zeta_{\mathcal{L}}, D) = 2^{1-D}a^D,\\ \operatorname{res}(\zeta_{A_{\mathcal{L}}}, 0) &= 2\zeta_{\mathcal{L}}(0) = -1,\end{aligned}$$

and, consequently,

$$\begin{aligned}|(A_{\mathcal{L}})_t \cap [0,1]| = {}& \frac{2^{1-D}a^D}{1-D}t^{1-D} - t - \sum_{m=1}^{M} \frac{\operatorname{res}\left(\zeta_{\mathcal{L}}, -mD\right)(2t)^{1+mD}}{(1+mD)mD}\\ &+ O\big(t^{1+(M+1)D}\big), \quad \text{as} \quad t \to 0^+,\end{aligned} \tag{3.6.27}$$

where the sum is interpreted as 0 if $M = 0$. In particular, $\dim_B A_{\mathcal{L}} = D$ and the a-string is Minkowski measurable with $\mathcal{M}^D(A_{\mathcal{L}}) = 2^{1-D}a^D/(1-D)$. We point out that (3.6.27)

[19] In [Lap–vFr3, Theorem 6.21] it is stated that $\operatorname{res}(\zeta_{\mathcal{L}}, D) = a^D$ which is a misprint. Namely, in the proof of that theorem the source of the misprint is the fact that the residue of $\zeta((a+1)s)$ at $s = 1/(a+1)$ is equal to $1/(a+1)$ and not to 1. Here, ζ is the Riemann zeta function.

coincides with the 'inner' tube formula of the a-string (see [Lap–vFr3, Subsection 8.1.2]).[20] Furthermore, by choosing a screen to the right of $-D/2$ we conclude that (3.6.27) is valid pointwise since then $\kappa_d < 0$ (see Theorem 3.40).

Example 3.67 (The Sierpiński gasket)**.** Let A be the Sierpiński gasket in $\mathbb{R}^2$, constructed in the usual way inside the unit triangle. Furthermore, we assume without loss of generality that $\delta > 1/4\sqrt{3}$, so that the set A_δ is connected. Then the distance zeta function ζ_A of the Sierpinski gasket is equal to

$$\zeta_A(s;\delta) = \frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(2^s-3)} + 2\pi\frac{\delta^s}{s} + 3\frac{\delta^{s-1}}{s-1}, \tag{3.6.28}$$

which is meromorphic on the whole complex plane (see [LapRaŽu1]). In particular, the set of complex dimensions of the Sierpiński gasket is given by

$$\mathcal{P}(\zeta_A) = \{0,1\} \cup \left(\log_2 3 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}\right). \tag{3.6.29}$$

By letting $\omega_k := \log_2 3 + \mathbf{p}k\mathbb{i}$ and $\mathbf{p} := 2\pi/\log 2$ we have that

$$\operatorname{res}(\zeta_A, \omega_k) = \frac{6(\sqrt{3})^{1-\omega_k}}{4^{\omega_k}(\log 2)\omega_k(\omega_k - 1)},$$

$$\operatorname{res}(\zeta_A, 0) = 3\sqrt{3} + 2\pi, \quad \text{and} \quad \operatorname{res}(\zeta_A, 1) = 0.$$

Similarly as in the above examples one can check that $\zeta_{\lambda A}(\,\cdot\,;\delta\lambda)$ is strongly languid with $\kappa_d = -1$ for $\delta \geq 1/2\sqrt{3}$ and any $\lambda \geq 2\sqrt{3}$ so that we can apply Theorem 3.37 to get the pointwise tube formula:

$$\begin{aligned} |A_t| &= \sum_{\omega \in \mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s;\delta), \omega\right) \\ &= t^{2-\log_2 3}\,\frac{6\sqrt{3}}{\log 2}\sum_{k=-\infty}^{\infty}\frac{(4\sqrt{3})^{-\omega_k}t^{-\mathbf{p}k\mathbb{i}}}{(2-\omega_k)(\omega_k - 1)\omega_k} + \left(\frac{3\sqrt{3}}{2} + \pi\right)t^2, \end{aligned}$$

valid for all $t \in (0, 1/2\sqrt{3})$. Note that this formula coincides with the one obtained in [LapPe3] and, more recently, in [DeKÖÜ].

Example 3.68. Let A be the three-dimensional analog of the Sierpiński carpet. More precisely, we construct A by dividing the closed unit cube of $\mathbb{R}^3$ into 27 congruent cubes and remove the open middle cube, then we iterate this step with each of the 26 remaining smaller closed cubes; and so on, ad infinitum. By choosing $\delta > 1/6$, we have that A_δ

[20]More precisely, the two expressions coincide after we take into account the misprint mentioned in footnote 19 and add the term $2\zeta_{\mathcal{L}}(0)$ which seems to be forgotten in [Lap–vFr3].

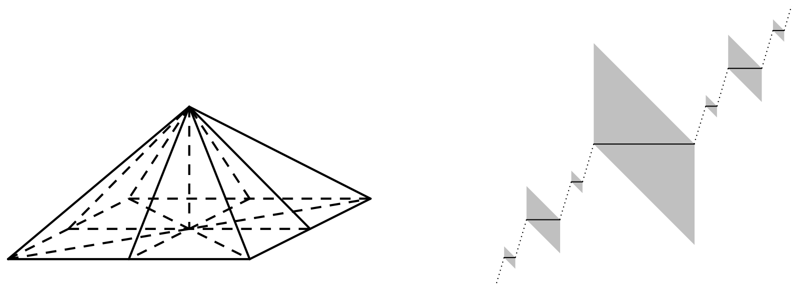

Figure 3.2: Left: the mutually congruent pyramids into which we subdivide the cube A_1 from Example 3.68. Eight of them that correspond to one face of A_1 are shown. Right: the third step in the construction of the Cantor function graph relative fractal drum (A, Ω) from Example 3.69. One can see the sets A_k, Δ_k and $\widetilde{\Delta}_k$ for $k = 1, 2, 3$.

is simply connected and let us calculate its distance zeta function. Note that $\zeta_A(s;\delta) = \zeta_A(s, I) + \zeta_A(s, A_\delta \setminus I)$ where I denotes the closed unit cube in $\mathbb{R}^3$. Let us denote with A_1 the open unit cube of side $1/3$ removed in the first step of the construction so that we have the following:

$$\zeta_A(s, I) = \zeta_A(s, A_1) + \zeta_A(s, I \setminus A_1) = \zeta_{\partial A_1}(s, A_1) + 26\,\zeta_{3^{-1}A}(s, 3^{-1}I).$$

The first equality is obvious and the second equality follows from the self-similarity of A. More precisely, it follows since the relative fractal drum $(A, I \setminus A_1)$ consists of 26 copies of (A, I) scaled down by 3^{-1}. Hence, by the scaling property of the relative distance zeta function (see Theorem 4.5.4), we have that

$$\zeta_A(s, I) = \zeta_{\partial A_1}(s, A_1) + 26 \cdot 3^{-s}\zeta_A(s, I),$$

which yields

$$\zeta_A(s, I) = \frac{\zeta_{\partial A_1}(s, A_1)}{1 - 26 \cdot 3^{-s}}. \tag{3.6.30}$$

The zeta function $\zeta_{\partial A_1}(\,\cdot\,, A_1)$ can be easily calculated by dividing the cube A_1 into 48 mutually congruent pyramids (see Figure 3.2, left) and integrating in local Cartesian coordinates over each pyramid:

$$\zeta_{\partial A_1}(s, A_1) = 48 \int_0^{1/6} \mathrm{d}x \int_0^x \mathrm{d}y \int_0^y z^{s-3}\,\mathrm{d}z = \frac{48 \cdot 6^{-s}}{s(s-1)(s-2)}. \tag{3.6.31}$$

On the other hand, the zeta function $\zeta_A(s, A_\delta \setminus I)$ corresponding to the 'outside' of the unit cube is easy to calculate once we subdivide the parts that correspond to the faces, edges and vertices of the unit cube and use local Cartesian, cylindrical and spherical coordinates in $\mathbb{R}^3$, respectively:

$$
\begin{aligned}
\zeta_A(s, A_\delta \setminus I) &= 6\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^\delta z^{s-3}\,\mathrm{d}z + 12\int_0^{\pi/2} \mathrm{d}\varphi \int_0^\delta r^{s-2}\,\mathrm{d}r \int_0^1 \mathrm{d}z \\
&\quad + 8\int_0^{\pi/2} \sin\theta\,\mathrm{d}\theta \int_0^{\pi/2} \mathrm{d}\varphi \int_0^\delta r^{s-1}\,\mathrm{d}r \\
&= \frac{6\delta^{s-2}}{s-2} + \frac{6\pi\delta^{s-1}}{s-1} + \frac{4\pi\delta^s}{s}.
\end{aligned}
$$

From the above calculation and from (3.6.30) together with (3.6.31) we deduce that ζ_A is meromorphic on $\mathbb{C}$ and given for all $s \in \mathbb{C}$ by

$$
\zeta_A(s, \delta) = \frac{48 \cdot 2^{-s}}{s(s-1)(s-2)(3^s - 26)} + \frac{4\pi\delta^s}{s} + \frac{6\pi\delta^{s-1}}{s-1} + \frac{6\delta^{s-2}}{s-2}.
$$

In particular, the complex dimensions of A are given by

$$
\mathcal{P}(\zeta_A, \mathbb{C}) = \{0, 1, 2\} \cup \left(\log_3 26 + \mathbf{p}\mathbb{i}\mathbb{Z}\right),
$$

where $\mathbf{p} := 2\pi/\log 3$. Furthermore, it is easy to determine that $\operatorname{res}(\zeta_A, 0) = 4\pi - 24/25$, $\operatorname{res}(\zeta_A, 1) = 6\pi + 24/23$, $\operatorname{res}(\zeta_A, 2) = 96/17$ and, by letting $\omega_k := \log_3 26 + \mathbf{p}k\mathbb{i}$, (for all $k \in \mathbb{Z}$),

$$
\operatorname{res}(\zeta_A, \omega_k) = \frac{24}{13 \cdot 2^{\omega_k}\omega_k(\omega_k - 1)(\omega_k - 2)\log 3}.
$$

One easily checks that the hypotheses of Theorem 3.37 are satisfied for $\delta \geq 1/2$ and any $\lambda \geq 2$, and thus we obtain the following exact pointwise tube formula, valid for all $t \in (0, 1/2)$:

$$
\begin{aligned}
|A_t| &= \frac{24\,t^{3-\log_3 26}}{13\log 3} \sum_{k=-\infty}^{\infty} \frac{2^{-\omega_k} t^{-\mathbf{p}k\mathbb{i}}}{(3-\omega_k)(\omega_k - 1)(\omega_k - 2)\omega_k} \\
&\quad + \left(6 - \frac{6}{17}\right)t + \left(3\pi + \frac{12}{23}\right)t^2 + \left(\frac{4\pi}{3} - \frac{8}{25}\right)t^3.
\end{aligned}
$$

In particular, we conclude that $\dim_B A = \log_3 26$ and, by Theorem 3.58, that the three-dimensional Sierpiński carpet is not Minkowski measurable which is expected. We point out also that the part $6t + 3\pi t^2 + 4\pi t^3/3$ from the above equation is exactly equal to $|I_t| - |I|$, where I is the closed unit cube of $\mathbb{R}^3$.

Example 3.69 (The Cantor function RFD)**.** In this example, we will compute the distance zeta function of (A, Ω) where A is the graph of the Cantor function and Ω is the union of triangles Δ_k that lie above and the triangles $\widetilde{\Delta}_k$ that lie below each of the straight parts of the graph denoted by A_k. (At each step of the construction there are 2^{k-1} mutually congruent triangles Δ_k and $\widetilde{\Delta}_k$.) Each of these triangles is isosceles, has for one of its sides a straight part of the Cantor function graph and has a right angle at the left end of A_k in case of Δ_k or at the right end of A_k in case of $\widetilde{\Delta}_k$ (see Figure 3.2, right).

For obvious geometrical reasons and by using the scaling property of the relative distance zeta function, we then have the following:

$$\begin{aligned}\zeta_A(s,\Omega)&=\sum_{k=1}^{\infty}2^k\zeta_{A_k}(s,\Delta_k)\\&=\sum_{k=1}^{\infty}2^k\zeta_{3^{-k}A_1}(s,3^{-k}\Delta_1)\\&=\zeta_{A_1}(s,\Delta_1)\sum_{k=1}^{\infty}\frac{2^k}{3^{ks}}=\frac{2\zeta_{A_1}(s,\Delta_1)}{3^s-2}.\end{aligned}\tag{3.6.32}$$

Here, (A_1,Δ_1) is the relative fractal drum described above with two perpendicular sides of length equal to 1. It is straightforward to compute its relative distance zeta function.

$$\zeta_{A_1}(s,\Delta_1)=\int_0^1\mathrm{d}x\int_0^x y^{s-2}\,\mathrm{d}y=\frac{1}{s(s-1)}.\tag{3.6.33}$$

This gives us the zeta function of (A,Ω) which is clearly meromorphic on $\mathbb{C}$:

$$\zeta_A(s,\Omega)=\frac{2}{s(3^s-2)(s-1)}\tag{3.6.34}$$

and one has that

$$\mathcal{P}(\zeta_A(\,\cdot\,,\Omega))=\{0,1\}\cup\left(\log_3 2+\frac{2\pi\mathbb{i}}{\log 3}\mathbb{Z}\right).\tag{3.6.35}$$

From Theorem 3.42 we conclude that $\dim_B(A,\Omega)=1$ and it is Minkowski measurable. Moreover, one also has from Theorem 3.42 that

$$\mathcal{M}^1(A,\Omega)=\frac{\operatorname{res}(\zeta_A(\,\cdot\,,\Omega),1)}{2-1}=2,\tag{3.6.36}$$

which coincides with the length of the Cantor function graph.

We do not know if (3.6.35) coincides with the complex dimensions of the 'full' graph of the Cantor function, but we do expect that this is the case since (A,Ω) is a 'relative fractal subdrum' of $(A,A_{1/3})$. Moreover, it is obvious that for the distance zeta function of the Cantor function graph one has

$$\zeta_A(s,A_{1/3})=\zeta_A(s,\Omega)+\zeta_A(s,A_{1/3}\setminus\Omega).\tag{3.6.37}$$

In order to prove that (3.6.35) is a subset of the complex dimensions of the 'full' Cantor function graph it remains to show that $\zeta_A(s,A_{1/3}\setminus\Omega)$ has a meromorphic continuation to some domain containing (3.6.35) and that there is no pole-pole cancellation in the right-hand side of (3.6.37). One checks easily that $\lambda^s\zeta_A(s,\Omega;1/3)$ is strongly d-languid for any $\lambda\geq 1$ with $\kappa_d=-2$ and we can apply Theorem 3.37 to recover the pointwise tube formula valid for all $t \in (0,1)$:

$$
\begin{aligned}
|A_t \cap \Omega| &= \sum_{\omega \in \mathcal{P}(\zeta_A(\,\cdot\,,\Omega))} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s,\Omega),\omega\right) \\
&= 2t + \frac{t^{2-\log_3 2}}{\log 3}\sum_{k=-\infty}^{\infty}\frac{t^{-\mathbf{p}k\mathbb{i}}}{(2-\omega_k)(\omega_k-1)\omega_k} + t^2,
\end{aligned}
\tag{3.6.38}
$$

where $\omega_k := \log_3 2 + \mathbf{p}k\mathbb{i}$ and $\mathbf{p} := 2\pi/\log 3$.

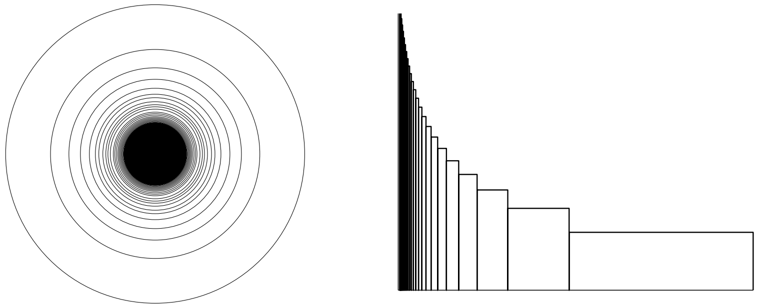

Figure 3.3: Left: the fractal nest generated by the a-string with $a = -1/2$. Right: the unbounded geometric $(-1/2,1)$-chirp. (The axes are not in scale.)

Example 3.70 (Fractal nests). We let $\mathcal{L} = (l_j)_{j\geq 1}$ be a bounded fractal string and as before let $A_{\mathcal{L}} = \{a_k : k \in \mathbb{N}\} \subset \mathbb{R}$ with $a_k := \sum_{j\geq k} l_j$ for each $k \geq 1$. Furthermore, consider now $A_{\mathcal{L}}$ as a subset of the x_1-axis in $\mathbb{R}^2$ and let A be a planar set obtained by rotating $A_{\mathcal{L}}$ around the origin, i.e., a union of concentric circles of radii a_k (see Figure 3.3, left). For $\delta > l_1/2$ the distance zeta function of A is given by

$$
\zeta_A(s) = \frac{2^{2-s}\pi}{s-1}\sum_{j=1}^{\infty} l_j^{s-1}(a_j + a_{j+1}) + \frac{2\pi\delta^s}{s} + \frac{2\pi a_1 \delta^{s-1}}{s-1}, \tag{3.6.39}
$$

(see [LapRaŽu1, Chapter 3]). The last two terms in the above formula correspond to the annulus $a_1 < r < a_1+\delta$ and we will neglect them; that is, we will consider only the relative distance zeta function $\zeta_A(\,\cdot\,,\Omega)$ with $\Omega := B_{a_1}(0)$.[21] Furthermore, since $a_{j+1} = a_j - l_j$ we have

$$
\begin{aligned}
\zeta_A(s,\Omega) &= \frac{2^{2-s}\pi}{s-1}\sum_{k=1}^{\infty} l_j^{s-1}(2a_j - l_j) \\
&= \frac{2^{3-s}\pi}{s-1}\sum_{j=1}^{\infty} a_j l_j^{s-1} - \frac{2^{2-s}\pi}{s-1}\sum_{j=1}^{\infty} l_j^s \\
&= \frac{2^{3-s}\pi}{s-1}\zeta_1(s) - \frac{2^{2-s}\pi}{s-1}\zeta_{\mathcal{L}}(s),
\end{aligned}
\tag{3.6.40}
$$

[21] Here, $B_r(x)$ denotes the open ball of radius r with center at x.

where we have denoted the first of the two sums after the second equality as ζ_1 and $\zeta_{\mathcal{L}}$ is the geometric zeta function of the fractal string $\mathcal{L}$. Let us now consider a special case of the fractal nest above; that is, the relative fractal drum (A_a,Ω) corresponding to the a-string with $a>0$. In this case we have that

$$\zeta_{A_a}(s,\Omega)=\frac{2^{3-s}\pi}{s-1}\sum_{j=1}^{\infty}j^{-a}l_j^{s-1}-\frac{2^{2-s}\pi}{s-1}\zeta_{\mathcal{L}}(s). \tag{3.6.41}$$

Since the geometric zeta function has been already analyzed in Example 3.66, we will now do the same for the zeta function ζ_1 by an analogous technique as in [Lap–vFr3, Theorem 6.21].

Theorem 3.71. *Let $a>0$, $b\in\mathbb{R}$, and let $\mathcal{L}$ be the a-string with lengths l_j given by (3.6.24). Then $\zeta_{\mathcal{L},b}(s):=\sum_{j=1}^{\infty}j^bl_j^s$ has a meromorphic continuation to all of $\mathbb{C}$. The poles of $\zeta_{\mathcal{L},b}$ are located at $\frac{b+1}{a+1}$ and in (a subset of) $\{\frac{b-m}{a+1}:m\in\mathbb{N}_0\}\setminus\{0\}$ and they are all simple.*[22] *The residue of $\zeta_{\mathcal{L},b}$ at $C:=\frac{b+1}{a+1}$ is equal to $\frac{a^C}{a+1}$.*

Furthermore, for any screen S_σ equal to a vertical line $\{\operatorname{Re}s=\sigma\}$ with $\sigma\in\mathbb{R}$ and not passing through a pole, $\zeta_{\mathcal{L},b}$ satisfies the languidity conditions **L1** *and* **L2** *with $\kappa=\frac12+b-(a+1)\sigma$ if $\sigma\le\frac{b}{a+1}$ and $\kappa=\frac12(1+b-(a+1)\sigma)$ if $\sigma\in\left[\frac{b}{a+1},\frac{b+1}{a+1}\right]$.*

Moreover, we have that $\zeta_{\mathcal{L},b}(0)=\zeta(-b)$ for all $b\in\mathbb{R}\setminus\{1\}$, where ζ is the Riemann zeta function.

Proof. We begin by computing the first term of an asymptotic expansion of l_j

$$l_j=j^{-a}-(j+1)^{-a}=a\int_j^{j+1}x^{-a-1}\,\mathrm{d}x=aj^{-a-1}+H(j), \tag{3.6.42}$$

where $H(j)=a\int_j^{j+1}(x^{-a-1}-j^{-a-1})\,\mathrm{d}x$. We introduce now a new variable $t=x/j-1$ and let

$$h_j:=a^{-1}j^{a+1}H(j)=j\int_0^{1/j}\big((1+t)^{-a-1}-1\big)\,\mathrm{d}t. \tag{3.6.43}$$

Note that $h_j=O(1/j)$, as $j\to+\infty$. By choosing now an integer $M\ge0$ we have

$$\begin{aligned}j^bl_j^s&=j^b\big(aj^{-a-1}(1+h_j)\big)^s\\&=a^sj^{b-s(a+1)}\left(\sum_{n=0}^{M}\binom{s}{n}h_j^n+O\left(\frac{(|s|+1)^{M+1}}{j^{M+1}}\right)\right)\quad\text{as } j\to\infty,\end{aligned} \tag{3.6.44}$$

where we have set

$$\binom{s}{n}:=\frac{(s-n+1)_n}{n!},\quad\text{for } s\in\mathbb{C}. \tag{3.6.45}$$

[22]Here, and from now on, we let $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$.

We thus obtain the following

$$\zeta_{\mathcal{L},b}(s) = \sum_{n=0}^{M} a^s \binom{s}{n} \sum_{j=0}^{\infty} h_j^n j^{b-s(a+1)} + f(s), \tag{3.6.46}$$

where $f(s)$ is defined and holomorphic on the open half-plane $\{\operatorname{Re} s > \frac{b-M}{a+1}\}$. Furthermore, the first term, i.e., for $n = 0$, in the above sum is equal to $a^s\zeta((a+1)s - b)$ where ζ is the Riemann zeta function and thus has a single simple pole at $s = C := \frac{1+b}{1+a}$.[23] To compute the residue of $a^s\zeta((a+1)s-b)$ at $s = \frac{1+b}{1+a}$ we use the fact that the principal part of the Riemann zeta function at $s = 1$ is equal to $1/(s-1)$ and consequently,

$$\lim_{s\to C}(s-C)a^s\zeta((a+1)s-b) = \lim_{s\to C} a^s \frac{s-C}{(a+1)s-b-1} = \frac{a^{\frac{1+b}{1+a}}}{a+1}. \tag{3.6.47}$$

A well-known result about the growth of the Riemann zeta function along vertical lines (see, e.g., [Edw, Section 9.2]) implies that the first term in (3.6.46) grows as $(|t|+1)^{\frac{1}{2}+b-\sigma(a+1)}$ on vertical lines $\{\operatorname{Re} s = \sigma\}$ with $\sigma < \frac{b}{a+1}$, as $(|t|+1)^{\frac{1}{2}(1+b-(a+1)\sigma)}$ for $\sigma \in \left[\frac{b}{a+1}, \frac{b+1}{a+1}\right]$ and is bounded by a constant if $\sigma > \frac{b+1}{a+1}$.

It remains now to analyze the functions

$$\sum_{j=1}^{\infty} h_j^n j^{b-(a+1)s}, \tag{3.6.48}$$

for $n \geq 1$. The asymptotic expansion $(1+t)^{-a-1} = \sum_{m=0}^{M}\binom{-a-1}{m}t^m + O(t^{M+1})$ as $t \to 0^+$ together with (3.6.43) yields

$$\begin{aligned} h_j &= j\int_0^{1/j} \sum_{m=1}^{M} \binom{-a-1}{m} t^m \,\mathrm{d}t + O(j^{-M-1}) \\ &= -\frac{1}{a}\sum_{m=1}^{M}\binom{-a}{m+1} j^{-m} + O(j^{-M-1}), \quad \text{as } j \to +\infty. \end{aligned} \tag{3.6.49}$$

We proceed by taking the n-th power of the above expansion to get an asymptotic expansion for h_j^n and substitute this into (3.6.48). This enables us to express each of the functions in (3.6.48) as a sum of constant multiples of $\zeta(m+(a+1)s-b)$ for $n \leq m \leq M$ and a remainder term of order $O(j^{-M-1})$. Since $\zeta(m+(a+1)s-b)$ has a simple pole at $s = \frac{1+b-m}{a+1}$ and in view of (3.6.46), we conclude that $\zeta_{\mathcal{L},b}(s)$ has a meromorphic continuation to $\{\operatorname{Re} s > \frac{1+b-M}{1+a}\}$ with simple poles at $s = \frac{1+b-m}{1+a}$ for $m = 0, 1, 2, \ldots, M$. Some of these poles may vanish depending on the choice of parameters a and b. Moreover, 0 is never a pole of $\zeta_{\mathcal{L},b}$, since by looking at (3.6.46) we can see that it gets canceled by the factor $\binom{s}{m}$ for $m \geq 1$. Furthermore, since M is arbitrary, we conclude that $\zeta_{\mathcal{L},b}$ has a

[23]See, e.g., [Ti3] or [Edw] for the properties of the Riemann zeta function.

meromorphic continuation to all of $\mathbb{C}$. Finally, for $m \geq 1$ the growth of $\zeta(m+(a+1)s-b)$ is superseded by the growth of the first term $a^s\zeta((a+1)s-b)$ and thus we have proved the statement about the languidity of $\zeta_{\mathcal{L},b}$.

The last statement of the theorem follows by the principle of analytic continuation since we have directly from the definition that $\zeta_{\mathcal{L},b}(0) = \zeta(-b)$ for all $b \in \{\operatorname{Re} s > 1\}$. $\square$

To complete the example of the fractal nest and for the example of the unbounded geometric chirp below, we will need a simple consequence of the above theorem.

Corollary 3.72. *Let $a > 0$, $b \in \mathbb{R}$, $\tau \in \mathbb{R}$ and let $\mathcal{L}$ be the a-string with lengths l_j given by (3.6.24). Then $\zeta_{\mathcal{L},b,\tau}(s) := \sum_{j=1}^{\infty} j^b l_j^{s-\tau}$ has a meromorphic continuation to all of $\mathbb{C}$. The poles of $\zeta_{\mathcal{L},b,\tau}$ are located at $\frac{b+1}{a+1} + \tau$ and in (a subset of) points $\{\frac{b-m}{a+1} + \tau : m \in \mathbb{N}_0\} \setminus \{\tau\}$ and they are all simple. The residue of $\zeta_{\mathcal{L},b,\tau}$ at $\frac{b+1}{a+1} + \tau$ is equal to $\frac{a^{(b+1)/(a+1)}}{a+1}$.*

Furthermore, for any screen S_σ equal to a vertical line $\{\operatorname{Re} s = \sigma\}$ with $\sigma \in \mathbb{R}$ and not passing through a pole, $\zeta_{\mathcal{L},b}$ satisfies the languidity conditions **L1** *and* **L2** *with $\kappa = \frac{1}{2} + b - (a+1)\sigma$ if $\sigma \leq \frac{b}{a+1} + \tau$ and $\kappa = \frac{1}{2}(1 + b - (a+1)\sigma)$ if $\sigma \in \left[\frac{b}{a+1} + \tau, \frac{b+1}{a+1} + \tau\right]$.*

Moreover, we have that $\zeta_{\mathcal{L},b,\tau}(\tau) = \zeta(-b)$ for all $b \in \mathbb{R} \setminus \{1\}$.

Proof. Since $\zeta_{\mathcal{L},b,\tau}(s) = \zeta_{\mathcal{L},b}(s-\tau)$, this an immediate consequence of Theorem 3.71. $\square$

Let us now go back to Example 3.70 where the distance zeta function of (A_a, Ω) is given by (3.6.41); that is,

$$\zeta_{A_a}(s,\Omega) = \frac{2^{3-s}\pi}{s-1}\zeta_{\mathcal{L},-a,1}(s) - \frac{2^{2-s}\pi}{s-1}\zeta_{\mathcal{L}}(s). \tag{3.6.50}$$

It is meromorphic on $\mathbb{C}$ and by Corollary 3.72 and Example 3.70, for the complex dimensions of (A_a, Ω) we have that

$$\mathcal{P}(\zeta_{A_a}(\,\cdot\,,\Omega)) \subseteq \left\{1, \frac{2}{a+1}, \frac{1}{a+1}\right\} \cup \left\{-\frac{m}{a+1} : m \in \mathbb{N}\right\}. \tag{3.6.51}$$

Furthermore, we are certain that $\frac{2}{a+1}$ is always a complex dimension of (A_a, Ω) since it is never canceled. Namely, by letting $D := \frac{2}{a+1}$, we have for $a \neq 1$ that

$$\operatorname{res}\left(\zeta_{A_a}(\,\cdot\,,\Omega), D\right) = \frac{2^{2-D}D\pi}{D-1}a^D. \tag{3.6.52}$$

We now conclude by Theorem 3.42 that $\dim_B(A_a, \Omega) = D$ if $a \in (0,1)$ and

$$\mathcal{M}^D(A_a,\Omega) = \frac{2^{2-D}D\pi}{(2-D)(D-1)}a^D. \tag{3.6.53}$$

Furthermore, if $a > 1$, we have that $\dim_B(A_a, \Omega) = 1$ and for the residue that

$$\operatorname{res}(\zeta_{A_a}(\,\cdot\,,\Omega), 1) = 4\pi\zeta_{\mathcal{L},-a,1}(1) - 2\pi\zeta_{\mathcal{L}}(1) = 4\pi\zeta(a) - 2\pi \tag{3.6.54}$$

Moreover, we have that

$$\mathcal{M}^1(A_a,\Omega)=4\pi\zeta(a)-2\pi \tag{3.6.55}$$

and it is nonzero since $\zeta(a)>1$ for $a>1$. In the special case when $a=1$, we have that $s=1$ is a pole of second order of $\zeta_{A_1}(s,\Omega)$ and since it is a simple pole of $\zeta_{\mathcal{L},-1,1}$, by looking at (3.6.50) we deduce that

$$\operatorname{res}(\zeta_{A_1}(\,\cdot\,,\Omega),1)=4\pi\zeta_{\mathcal{L},-1,1}[1]_0-2\pi, \tag{3.6.56}$$

where $\zeta_{\mathcal{L},-1,1}[\omega]_m$ indicates the m-th coefficient, (for $m\in\mathbb{Z}$), in the Laurent expansion of $\zeta_{\mathcal{L},-1,1}$ around $s=\omega$. We conclude that in this case, by Theorem 2.8, (A_1,Ω) must be Minkowski degenerate with $\dim_B(A_1,\Omega)=1$.[24] We may also compute the coefficient corresponding to $(s-1)^{-2}$ in the Laurent expansion of $\zeta_{A_1}(\,\cdot\,,\Omega)$ around $s=1$ by using Corollary 3.72:

$$\zeta_{A_1}(\,\cdot\,,\Omega)[1]_{-2}=4\pi\operatorname{res}(\zeta_{\mathcal{L},-1,1},1)=2\pi. \tag{3.6.57}$$

Assume now that $a\neq 1$. For $M\in\mathbb{N}\cup\{0\}$, as before, we choose the screen S_M to be some vertical line between $-\frac{M+1}{1+a}$ and $-\frac{M+2}{1+a}$ and let W_M be the corresponding window. Applying Theorem 3.40 we now obtain the following asymptotic distributional formula for the tube function $V(t):=|(A_a)_t\cap\Omega|$ when $t\to 0^+$:

$$\begin{aligned}V(t)=&\frac{2^{2-D}D\pi}{(2-D)(D-1)}a^Dt^{2-D}+\big(4\pi\zeta(a)-2\pi\big)t\\&+\frac{\operatorname{res}\big(\zeta_{A_a}(\,\cdot\,,\Omega),\frac{1}{a+1}\big)\,t^{2-\frac{1}{a+1}}}{2-\frac{1}{a+1}}\\&+\sum_{m=1}^{M}\frac{\operatorname{res}\big(\zeta_{A_a}(\,\cdot\,,\Omega),-\frac{m}{a+1}\big)\,t^{2+\frac{m}{a+1}}}{2+\frac{m}{a+1}}+O\big(t^{2+\frac{M+1}{a+1}}\big),\quad\text{as}\quad t\to 0^+,\end{aligned}$$

where the sum is interpreted as 0 if $M=1$. By choosing a vertical line $\{\operatorname{Re}s=\sigma\}$ for $\sigma>-\frac{1}{2(a+1)}$ as a screen we get a pointwise formula with an error term since then we have that $\kappa_d<0$ by Corollary 3.72.

Let us look now into the special case when $a=1$ and choose a screen such that $\sigma\in(-3/4,-1/2)$ so that we get a pointwise tube formula with an error term:

$$\begin{aligned}V(t)=&\operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_{A_1}(s,\Omega),1\right)+\frac{2}{3}\operatorname{res}\left(\zeta_{A_1}(\,\cdot\,,\Omega),\frac{1}{2}\right)t^{\frac{3}{2}}\\&+\frac{2}{5}\operatorname{res}\left(\zeta_{A_1}(\,\cdot\,,\Omega),-\frac{1}{2}\right)t^{\frac{5}{2}}+O(t^{2-\sigma})\text{ as }t\to 0^+.\end{aligned} \tag{3.6.58}$$

[24]It can be shown directly that $\mathcal{M}^1(A_1,\Omega)$ exists in this case and is equal to $+\infty$, see [LapRaŽu1, Chapter 3].

We expand the function $t^{2-s}/(2-s)$ into a Taylor series around $s=1$ which is given by

$$\frac{t^{2-s}}{2-s}=t\sum_{n=0}^{\infty}(s-1)^n\sum_{k=0}^{n}\frac{(-1)^{n-k}(\log t^{-1})^k}{k!(n-k)!}. \tag{3.6.59}$$

Now, from (3.6.56) and (3.6.57) we conclude that

$$\operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_{A_1}(s,\Omega),1\right)=2\pi t\log t^{-1}+4\pi t(\zeta_{\mathcal{L},-1,1}[1]_0-1), \tag{3.6.60}$$

so that

$$V(t)=2\pi t\log t^{-1}+4\pi t(\zeta_{\mathcal{L},-1,1}[1]_0-1)+o(t)\quad\text{as } t\to 0^+. \tag{3.6.61}$$

The above tube formula is in accordance with the fact that (A_1,Ω) is Minkowski degenerate but it is also clear that one can choose the function $h(t):=\log t^{-1}$, $t\in(0,1)$ for an appropriate gauge function. More precisely, one then has that the *gauge relative Minkowski content* of (A_1,Ω) is well defined and

$$\mathcal{M}^1(A_1,\Omega,h)=\lim_{t\to 0^+}\frac{|(A_1)_t\cap\Omega|}{t\,h(t)}=2\pi. \tag{3.6.62}$$

The relative gauge Minkowski content was introduced in [Žu4], motivated by [HeLap]. See also [LapRaŽu1] for more on this topic.

In fact, from the pointwise tube formula (Theorem 3.13) we can deduce the following general result.

Theorem 3.73. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that it is languid with $\kappa<-1$ or such that $(\lambda A,\lambda\Omega)$ is strongly languid for some $\lambda>0$ with $\kappa<0$ for a screen passing between the critical line $\{\operatorname{Re}s=\overline{\dim}_B(A,\Omega)\}$ and all the complex dimensions of (A,Ω) with real part strictly less than $\overline{D}:=\overline{\dim}_B(A,\Omega)$. Furthermore, suppose also that $\overline{D}$ is the only pole of its relative tube zeta function with real part equal to $\overline{D}$ and of order $m\geq 2$. Then $\dim_B(A,\Omega)$ exists and is equal to $D:=\overline{D}$. Furthermore, $\mathcal{M}^D(A,\Omega)$ exists and is equal to $+\infty$; that is (A,Ω), is Minkowski degenerate.*

Moreover, an appropriate gauge function for (A,Ω) is $h(t):=(\log t^{-1})^{m-1}$ for $t\in(0,1)$ and we have that

$$\mathcal{M}^D(A,\Omega,h)=\frac{\widetilde{\zeta}_A(\,\cdot\,,\Omega)[D]_{-m}}{(m-1)!}. \tag{3.6.63}$$

Proof. By Theorem 3.13 and since the screen S is to the right of all the other complex dimensions of (A,Ω) and to the left of the critical line, we have a pointwise tube formula for $|A_t\cap\Omega|$ with (or without) an error term that is of strictly higher asymptotic order as

$t \to 0^+$ than the term corresponding to the residue at $s = \overline{D}$; that is, we have

$$|A_t \cap \Omega| = \operatorname{res}(t^{N-s}\widetilde{\zeta}_A(s,\Omega), \overline{D}) + O(t^{N-\sup S}), \quad \text{as } t \to 0^+. \tag{3.6.64}$$

We now take the Taylor expansion of t^{N-s} around $s = \overline{D}$:

$$t^{N-s} = t^{N-\overline{D}} \sum_{n=0}^{\infty} \frac{(\log t^{-1})^n}{n!} (s - \overline{D})^n, \tag{3.6.65}$$

multiply it by the Laurent expansion of $\widetilde{\zeta}_A(s,\Omega)$ around $s = \overline{D}$ and extract the residue of this product to deduce that

$$\operatorname{res}(t^{N-s}\widetilde{\zeta}_A(s,\Omega), \overline{D}) = t^{N-\overline{D}} \sum_{n=0}^{m-1} \frac{(\log t^{-1})^n}{n!} \widetilde{\zeta}_A(\,\cdot\,,\Omega)[\overline{D}]_{-n-1}. \tag{3.6.66}$$

In light of this and (3.6.64) we conclude that $\dim_B(A,\Omega)$ exists and is equal to $\overline{D}$. Furthermore, the statements about the Minkowski content and the gauge Minkowski content also follow now from (3.6.64). $\square$

It would be interesting to try to expand the above result and obtain a kind of a gauge Minkowski measurability criterion in the likes of Theorem 3.58. See [LapRaŽu1, Chapter 4] for a partial converse of the above theorem in the case when the relative tube function satisfies the following asymptotics:

$$|A_t \cap \Omega| = t^{N-D} (\log t^{-1})^{m-1} (\mathcal{M} + O(t^{\alpha})) \quad \text{as} \quad t \to 0^+, \tag{3.6.67}$$

where $m \geq 2$ and $\alpha > 0$. Like always, we can reformulate the above theorem in terms of the distance zeta function.

Theorem 3.74. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $\overline{\dim}_B(A,\Omega) < N$ and that it is d-languid with $\kappa < 0$ or is such that $(\lambda A, \lambda\Omega)$ is strongly d-languid for some $\lambda > 0$ with $\kappa < 1$ for a screen passing between the critical line $\{\operatorname{Re} s = \overline{\dim}_B(A,\Omega)\}$ and all the complex dimensions of (A,Ω) with real part strictly less than $\overline{D} := \overline{\dim}_B(A,\Omega)$. Furthermore, suppose also that $\overline{D}$ is the only pole of its relative tube zeta function with real part equal to $\overline{D}$ and of order $m \geq 2$. Then, $\dim_B(A,\Omega)$ exists and is equal to $D := \overline{D}$. Furthermore, $\mathcal{M}^D(A,\Omega)$ exists and is equal to $+\infty$; that is (A,Ω), is Minkowski degenerate.*

Moreover, an appropriate gauge function for (A,Ω) is $h(t) := (\log t^{-1})^{m-1}$ for $t \in (0,1)$ and we have that

$$\mathcal{M}^D(A,\Omega,h) = \frac{\zeta_A(\,\cdot\,,\Omega)[D]_{-m}}{(N-D)(m-1)!}. \tag{3.6.68}$$

Proof. By Theorem 3.37 we have an asymptotic pointwise tube formula

$$|A_t \cap \Omega| = \operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_A(s,\Omega), \overline{D}\right) + O(t^{N-\sup S}), \quad \text{as } t \to 0^+. \tag{3.6.69}$$

Furthermore, we expand $(N-s)^{-1}$ into a Taylor series around $s = \overline{D}$:

$$\frac{1}{N-s} = \sum_{n=0}^{\infty} \frac{(-1)^n (s-\overline{D})^n}{n!(N-\overline{D})^{n+1}} \tag{3.6.70}$$

and multiply it by (3.6.65) to get a Taylor expansion of $t^{N-a}/(N-s)$:

$$\frac{t^{N-s}}{N-s} = \sum_{n=0}^{\infty}(s-\overline{D})^n \sum_{k=0}^{n} \frac{(-1)^{n-k}(\log t^{-1})^k}{k!(n-k)!(N-\overline{D})^{n-k+1}}. \tag{3.6.71}$$

We now multiply the above with the Laurent expansion of $\zeta_A(s,\Omega)$ around $s = \overline{D}$ and extract the residue of this product to get

$$\operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_A(s,\Omega), \overline{D}\right) = t^{N-\overline{D}} \sum_{n=0}^{m-1}\sum_{k=0}^{n} \frac{(-1)^{n-k}(\log t^{-1})^k \zeta_A(\,\cdot\,,\Omega)[\overline{D}]_{-n-1}}{k!(n-k)!(N-\overline{D})^{n-k+1}}.$$

We complete the proof now by reasoning analogously as in the proof of Theorem 3.73. □

Example 3.75 (Unbounded geometric chirps)**.** In this example we will examine a type of unbounded geometric chirp. A standard *geometric* (α,β)*-chirp* with positive parameters α and β is a simple geometric approximation of the graph of the function $f(x) = x^{\alpha}\sin(\pi x^{-\beta})$.

By choosing parameters $-1 < \alpha < 0 < \beta$ we get an example of an unbounded chirp function f which we approximate by the unbounded geometric (α,β)-chirp. More precisely, let $A_{\alpha,\beta}$ be the union of vertical segments with abscissae $x = j^{-1/\beta}$ and of length $j^{-\alpha/\beta}$ for $j \in \mathbb{N}$. Furthermore, define Ω as a union of the rectangles R_j for $j \in \mathbb{N}$ where R_j has a base of length $j^{-1/\beta} - (j+1)^{-1/\beta}$ and height $j^{-\alpha/\beta}$; see Figure 3.3, right. The relative distance zeta function of (A,Ω) is computed in [LapRaŽu1, Chapter 4] and is given by

$$\begin{aligned} \zeta_{A_{\alpha,\beta}}(s,\Omega) &= \frac{2^{2-s}}{(s-1)}\sum_{j=1}^{\infty} j^{-\alpha/\beta}\left(j^{-1/\beta} - (j+1)^{-1/\beta}\right)^{s-1} \\ &= \frac{2^{2-s}}{(s-1)}\zeta_{\mathcal{L},-\alpha/\beta,1}(s), \end{aligned} \tag{3.6.72}$$

where $\mathcal{L}$ is the β^{-1}-string. By Corollary 3.72 we conclude that $\zeta_{A_{\alpha,\beta}}(s,\Omega)$ has a meromorphic continuation to all of $\mathbb{C}$ and

$$\mathcal{P}(\zeta_{A_{\alpha,\beta}}(\,\cdot\,,\Omega)) \subseteq \left\{1, 2 - \frac{1+\alpha}{1+\beta}\right\} \cup \{D_m : m \in \mathbb{N}\}, \tag{3.6.73}$$

where $D_m := 2 - \frac{1+\alpha+m\beta}{1+\beta}$. By letting $D := 2 - \frac{1+\alpha}{1+\beta}$ we have that $D > 1$ and, consequently, $\dim_B(A_{\alpha,\beta}, \Omega) = D$ with

$$\begin{aligned}\mathcal{M}^D(A_{\alpha,\beta}, \Omega) &= \frac{\operatorname{res}(\zeta_{A_{\alpha,\beta}}(s,\Omega), D)}{2-D} = \frac{2^{2-D}}{(2-D)(D-1)} \frac{\beta^{\frac{1+\alpha}{1+\beta}}}{1+\beta} \\ &= \frac{(2\beta)^{2-D}}{(2-D)(D-1)(1+\beta)}.\end{aligned} \tag{3.6.74}$$

Furthermore, for the residue at $s = 1$ we have

$$\operatorname{res}(\zeta_{A_{\alpha,\beta}}(s,\Omega), 1) = 2\zeta_{\mathcal{L},-\alpha/\beta,1}(1) = 2\,\zeta\Big(\frac{\alpha}{\beta}\Big). \tag{3.6.75}$$

Similarly as in previous examples, for $M \in \mathbb{N} \cup \{0\}$, we choose the screen S_M to be some vertical line between $2 - \frac{1+\alpha+(M+1)\beta}{1+\beta}$ and $2 - \frac{1+\alpha+(M+2)\beta}{1+\beta}$ and let W_M be the corresponding window. From Theorem 3.40 we obtain the following asymptotic distributional formula for the tube function $V(t) := |(A_{\alpha,\beta})_t \cap \Omega|$:

$$\begin{aligned}V(t) &= \frac{(2\beta t)^{2-D}}{(2-D)(D-1)(1+\beta)} + \frac{t^{2-D_1}\operatorname{res}(\zeta_{A_{\alpha,\beta}}(\,\cdot\,,\Omega), D_1)}{2-D_1} \\ &\quad + 2t\,\zeta\Big(\frac{\alpha}{\beta}\Big) + \sum_{m=2}^{M} \frac{t^{2-D_m}\operatorname{res}(\zeta_{A_{\alpha,\beta}}(\,\cdot\,,\Omega), D_m)}{2-D_m} \\ &\quad + O(t^{2-D_{M+1}}) \quad \text{as} \quad t \to 0^+.\end{aligned} \tag{3.6.76}$$

Note that the second noninteger complex dimension $D_1 = 1 - \frac{\alpha}{1+\beta}$ is also greater than 1. Finally, by choosing as a screen a vertical line to the right of $-\frac{2\alpha+\beta}{1+\beta}$ we actually get a pointwise formula above since then $\kappa_d < 0$.

We conclude this section by briefly demonstrating how the results of this chapter may also be applied to recover the tube formulas for *self-similar sprays* generated by an arbitrary open set $G \subset \mathbb{R}^N$. A self-similar spray is defined as a collection $(G_k)_{k\in\mathbb{N}}$ of pairwise disjoint open sets $G_k \subset \mathbb{R}^N$ with $G_0 := G$ such that G_k is a scaled copy of G by some factor $\lambda_k > 0$. The sequence $(\lambda_k)_{k\in\mathbb{N}}$ is called the associated scaling sequence of the spray and is obtained from a "ratio list" $\{r_1, r_2, \ldots, r_J\}$ with $0 < r_j < 1$, by building all possible words of multiples of the ratios r_j.

Let us assume now that (A, Ω) is the relative fractal drum defined as $A := \partial(\cup G_k)$ and $\Omega := \cup G_k$ with $\overline{\dim}_B(\partial G, G) < N$. Then, since $(A, \Omega) = (\partial G, G) \cup \bigcup_{j=1}^{J}(r_j A, r_j \Omega)$, it is clear that its relative distance zeta function satisfies the following functional equation:

$$\zeta_A(s, \Omega) = \zeta_{\partial G}(s, G) + \zeta_{r_1 A}(s, r_1\Omega) + \cdots + \zeta_{r_J A}(s, r_J \Omega), \tag{3.6.77}$$

where $(r_j A, r_j \Omega)$ denotes the relative fractal drum (A, Ω) scaled by the factor r_j. Furthermore, by using the scaling property of the relative distance zeta function (Theorem

4.5.4) the above equation becomes

$$\zeta_A(s,\Omega) = \zeta_{\partial G}(s,G) + r_1^s\zeta_A(s,\Omega) + \cdots + r_J^s\zeta_A(s,\Omega), \tag{3.6.78}$$

which yields that

$$\zeta_A(s,\Omega) = \frac{\zeta_{\partial G}(s,G)}{1-\sum_{j=1}^J r_j^s}. \tag{3.6.79}$$

It is now enough to assume that the relative distance zeta function $\zeta_{\partial G}(s,G)$ of the generating relative fractal drum $(\partial G, G)$ satisfies suitable languidity conditions to obtain a pointwise or distributional formula for the 'inner' volume of $\cup G_k$:

$$|A_t\cap\Omega| = \sum_{\omega\in\mathfrak{D}\cup\mathcal{P}(\zeta_{\partial G}(\,\cdot\,,G),W)} \operatorname{res}\left(\frac{t^{N-s}\zeta_{\partial G}(s,G)}{(N-s)\Big(1-\sum_{j=1}^J r_j^s\Big)},\omega\right) + \mathcal{R}(t), \tag{3.6.80}$$

where $\mathfrak{D}$ denotes the set of solutions of $\sum_{j=1}^J r_j^s = 1$ and $\mathcal{R}$ is the pointwise or distributional error term.

Assume now that the generator G is monophase; that is, the volume of its 'inner' t-neighborhood is given by a polynomial $\sum_{i=0}^{N-1}\kappa_i t^{N-i}$ for $t<g$. Here g is the *inradius* of G, i.e., the supremum of the radii of all the balls that are contained in G. Since then

$$|(\partial G)_t\cap G| = \sum_{i=0}^{N-1}\kappa_i t^{N-i}$$

for $t<g$, we can then explicitly calculate its relative tube function

$$\widetilde{\zeta}_{\partial G}(s,G;g) = \int_0^g t^{s-N-1}\sum_{i=0}^{N-1}\kappa_i t^{N-i}\,\mathrm{d}t = \sum_{i=0}^{N-1}\frac{\kappa_i g^{s-i}}{s-i} \tag{3.6.81}$$

and it is obviously meromorphic on $\mathbb{C}$ and, moreover, after appropriate scaling, strongly languid with $\kappa=-1$ for a sequence of vertical lines. We conclude that the tube formula (3.6.80) is valid pointwise and without an error term in this case for t sufficiently small; that is, we recover a well-known result obtained in [LapPe3], and more recently via a different technique in [DeKÖÜ]:

$$|A_t\cap\Omega| = \sum_{\omega\in\mathfrak{D}\cup\{1,2,\dots,N-1\}} \operatorname{res}\left(t^{N-s}\frac{\sum_{i=0}^{N-1}\kappa_i\frac{g^{s-i}}{s-i}}{\Big(1-\sum_{j=1}^J r_j^s\Big)},\omega\right). \tag{3.6.82}$$

A completely analogous reasoning can be made for the case of *pluriphase* generators G for which the 'inner' tubular tubular volume is given as a piecewise polynomial. In a future work we plan to investigate for which classes of generators the tube formula (3.6.80) can be applied pointwise or distributionally.

The next example shows how one can effectively construct fractal sets (even fractal strings) which can have poles of any order on the critical line and even essential singularities.

Example 3.76. We will provide an example of a relative fractal drum of $\mathbb{R}$ such that its distance zeta function has an infinite set of poles of order m in arithmetic progression located on the critical axis. The construction is based on an 'iterated' Cantor set. Let C be the standard middle-third Cantor set contained in $[0,1]$ and $\Omega := [0,1]$. Let $\Omega \setminus C$ be our generator for the fractal spray defined with scaling ratios $r_1 = r_2 = 1/3$. We will denote this fractal spray by (C_2, Ω_2). Then, by the scaling property of the relative distance zeta function, we have

$$\zeta_{C_2}(s,\Omega_2) = \zeta_C(s,\Omega) + 2\,\zeta_{3^{-1}C_2}(s,3^{-1}\Omega_2) = \zeta_C(s,\Omega) + 2\cdot 3^{-s}\,\zeta_{C_2}(s,\Omega_2),$$

or, in other words,[25]

$$\zeta_{C_2}(s,\Omega_2) = \frac{3^s}{3^s-2}\zeta_C(s,\Omega) = \frac{2\cdot 3^s}{2^s s(3^s-2)^2}.$$

It is clear that $\zeta_{C_2}(\,\cdot\,,\Omega_2)$ is meromorphic on $\mathbb{C}$ and

$$\mathcal{P}(\zeta_{C_2}(\,\cdot\,\Omega_2)) = \{0\} \cup \left(\log_3 2 + \frac{2\pi\mathbb{i}}{\log 3}\mathbb{Z}\right). \tag{3.6.83}$$

Furthermore, the poles $\omega_k := \log_3 2 + \frac{2k\pi\mathbb{i}}{\log 3}$ for $k \in \mathbb{N}$ are all of second order. We conclude that $\overline{\dim}_B(C_2,\Omega_2) = \log_3 2$. Actually, by Theorem 3.74 it follows that $D := \dim_B(C_2,\Omega_2) = \log_3 2$, and $\mathcal{M}^D = +\infty$. We conjecture that for $h(t) := \log t^{-1}$, we have

$$0 \le \underline{\mathcal{M}}^D((C_2,\Omega_2),h) \le \frac{\zeta_{C_2}(\,\cdot\,,\Omega_2)[D]_{-2}}{(1-D)} \le \overline{\mathcal{M}}^D((C_2,\Omega_2),h) \le +\infty. \tag{3.6.84}$$

Note that it in order to prove the above conjecture one needs to analyze in more detail the pointwise tube formula of (C_2,Ω_2) given by Theorem 3.37. We leave that for future work; see Problem A.2.

We can now repeat the above process inductively, that is, define the relative fractal drum (C_n,Ω_n) as a fractal spray generated by (C_{n-1},Ω_{n-1}) and the same scaling ratios $r_1 = r_2 = 1/3$ for $n \ge 2$. Similarly as before, we have

$$\zeta_{C_n}(s,\Omega_n) = \frac{2\cdot 3^{(n-1)s}}{2^s s(3^s-2)^n}.$$

The poles at ω_k are of order n, $D := \dim_B(C_2,\Omega_2) = \log_3 2$, and for the gauge function

[25] ζ_C is given by (3.6.13).

$h_n(t) := (\log t^{-1})^{n-1}$ we conjecture that

$$0 \le \underline{\mathcal{M}}^D((C_n,\Omega_n),h_n) \le \frac{\zeta_{C_n}(\,\cdot\,,\Omega_n)[D]_{-n}}{(1-D)(n-1)!} \le \overline{\mathcal{M}}^D((C_n,\Omega_n),h_n) \le +\infty. \tag{3.6.85}$$

Finally, we can now use the relative fractal drums (C_n,Ω_n) to construct an RFD that will have essential singularities on the critical axis. We let $(C_1,\Omega_1) := (C,\Omega)$, we scale down every RFD (C_n,Ω_n) by the factor $3^{-n}/n!$ and we define (A,Ω) as a disjoint union of the resulting RFDs, that is

$$(A,\Omega) := \bigcup_{n=1}^{\infty} \frac{3^{-n}}{n!}(C_n,\Omega_n).$$

Then we have

$$\begin{aligned}\zeta_A(s,\Omega) &= \sum_{n=1}^{\infty} \zeta_{3^{-n}(n!)^{-1}C_n}(s,(n!)^{-1}\Omega_n) = \sum_{n=1}^{\infty} \frac{3^{-ns}}{(n!)^s}\zeta_{C_n}(s,\Omega_n)\\ &= \frac{2}{6^s s}\sum_{n=1}^{\infty}\frac{1}{(n!)^s(3^s-2)^n}.\end{aligned}$$

By the Weierstrass M-test, $\zeta_A(s,\Omega)$ is holomorphic on $\{\operatorname{Re} s > 0\} \setminus \Big(\log_3 2 + \frac{2\pi\mathbb{i}}{\log 3}\mathbb{Z}\Big)$. More precisely, it has essential singularities at $\log_3 2 + \frac{2\pi\mathbb{i}}{\log 3}\mathbb{Z}$.

The above construction can be generalized verbatim for any fractal string $\mathcal{L}$ or self-similar fractal spray in $\mathbb{R}^N$. This suggests that the definition of complex dimensions should be updated to also include essential singularities of the fractal zeta functions.

3.7 Embeddings in Higher Dimensions

In this section we will obtain potentially useful results concerning relative fractal drums and bounded subsets of $\mathbb{R}^N$ embedded into higher dimensional spaces. We will apply these results to obtain the complex dimensions of the Cantor dust.

Proposition 3.77. Let $A \subseteq \mathbb{R}^N$ be a bounded set with $\overline{\dim}_B A = \overline{D}$. Then for the tube zeta functions of A and $A \times \{0\} \subseteq \mathbb{R}^{N+1}$ the following equality holds

$$\widetilde{\zeta}_{A\times\{0\}}(s;\delta) = 2\int_0^{\pi/2} \frac{\widetilde{\zeta}_A(s;\delta\sin\tau)}{\sin^{s-N-1}\tau}\,\mathrm{d}\tau \tag{3.7.1}$$

for all $s \in \{\operatorname{Re} s > \overline{D}\}$.

Proof. First of all, it is well-known[26] that $\overline{\dim}_B(A\times\{0\}) = \overline{\dim}_B A$ from which we conclude that the tube zeta functions of A and $A\times\{0\}$ are both holomorphic on the right half-plane

[26] See [Res1].

$\{\operatorname{Re} s > \overline{D}\}$. Furthermore, we will use the fact, (see [Res1, Proposition 6]) that for $t > 0$ we have

$$|(A \times \{0\})_t|_{N+1} = 2\int_0^t |A_{\sqrt{t^2-u^2}}|_N \,\mathrm{d}u \tag{3.7.2}$$

where $|\cdot|_N$ denotes the N-dimensional Lebesgue measure. After changing the variable of integration by $u = t\cos v$ this yields

$$|(A \times \{0\})_t|_{N+1} = 2t\int_0^{\pi/2} |A_{t\sin v}|_N \sin v \,\mathrm{d}v. \tag{3.7.3}$$

Finally, we have the following for the tube zeta function of $A \times \{0\}$:

$$\begin{aligned}
\widetilde{\zeta}_{A\times\{0\}}(s;\delta) &= \int_0^\delta t^{s-N-2}|(A\times\{0\})_t|_{N+1}\,\mathrm{d}t \\
&= 2\int_0^\delta t^{s-N-1}\,\mathrm{d}t \int_0^{\pi/2} |A_{t\sin v}|_N \sin v\,\mathrm{d}v \\
&= 2\int_0^{\pi/2} \sin v\,\mathrm{d}v \int_0^\delta t^{s-N-1}|A_{t\sin v}|_N\,\mathrm{d}t \\
&= 2\int_0^{\pi/2} \sin^{N+1-s} v\,\mathrm{d}v \int_0^{\delta\sin v} \tau^{s-N-1}|A_\tau|_N\,\mathrm{d}\tau \\
&= 2\int_0^{\pi/2} \frac{\widetilde{\zeta}_A(s;\delta\sin v)}{\sin^{s-N-1} v}\,\mathrm{d}v
\end{aligned}$$

where we have used the Fubini–Tonelli theorem and another change of variable of integration, namely, $\tau = t\sin v$. ☐

Theorem 3.78. *Let $A \subseteq \mathbb{R}^N$ be a bounded set with $\overline{\dim}_B A = \overline{D}$. Then, for $s \in \mathbb{C}$ with $\operatorname{Re} s > \overline{D}$ we have the following equality between the tube zeta function of A and the tube zeta function of $A_M := A \times \{0\} \cdots \times \{0\} \subseteq \mathbb{R}^{N+M}$:*

$$\widetilde{\zeta}_{A_M}(s;\delta) = \frac{(\sqrt{\pi})^M\,\Gamma\left(\frac{N-s}{2}+1\right)}{\Gamma\left(\frac{N+M-s}{2}+1\right)}\widetilde{\zeta}_A(s;\delta) + E(s;\delta), \tag{3.7.4}$$

where $E(s)$ is meromorphic on $\mathbb{C}$. The poles of $E(s;\delta)$ are located at $s_k := N+2+2k$ for $k \in \mathbb{N}\cup\{0\}$ and all of them are simple. Moreover, we have that

$$\operatorname{res}(E(\,\cdot\,;\delta), s_k) = \frac{(-1)^{k+1}\,(\sqrt{\pi})^M}{k!\,\Gamma\left(\frac{M}{2}-k\right)}\widetilde{\zeta}_A(s_k;\delta). \tag{3.7.5}$$

Proof. We will prove the theorem in the case when $M = 1$. The general case follows then immediately by induction. From Proposition 3.77 we have that for $\operatorname{Re} s > \overline{\dim}_B A$

formula (3.7.1) holds which, in turn, can be written as

$$
\begin{aligned}
\widetilde{\zeta}_{A\times\{0\}}(s;\delta) &= 2\widetilde{\zeta}_A(s;\delta)\int_0^{\pi/2}\frac{\mathrm{d}\tau}{\sin^{s-N-1}\tau} \\
&\quad -2\int_0^{\pi/2}\frac{\mathrm{d}v}{\sin^{s-N-1}v}\int_{\delta\sin v}^{\delta}\tau^{s-N-1}|A_\tau|_N\,\mathrm{d}\tau \\
&= \widetilde{\zeta}_A(s;\delta)\cdot \mathrm{B}\left(\frac{N-s}{2}+1,\frac{1}{2}\right)+E(s;\delta)
\end{aligned}
\tag{3.7.6}
$$

where B denotes the beta function and

$$
E(s;\delta) := -2\int_0^{\pi/2}\frac{\mathrm{d}v}{\sin^{s-N-1}v}\int_{\delta\sin v}^{\delta}\tau^{s-N-1}|A_\tau|_N\,\mathrm{d}\tau. \tag{3.7.7}
$$

By using the functional equation which links the beta function with the gamma function,[27] we get that (3.7.4) (with $M=1$) holds for all $s\in\mathbb{C}$ such that $\operatorname{Re}s>\overline{\dim}_B A$.

By looking at $E(s;\delta)$ we see that the integrand is holomorphic for every $v\in(0,\pi/2)$ since the integral $\int_{\delta\sin v}^{\delta}\tau^{s-N-1}|A_\tau|_N\,\mathrm{d}\tau$ is equal to $\widetilde{\zeta}_A(s;\delta)-\widetilde{\zeta}_A(s;\delta\sin v)$ which is an entire function. Furthermore, if we assume that $\operatorname{Re}s<N+1$, then since $\tau\mapsto\tau^{\operatorname{Re}s-N-1}$ is decreasing we have the following estimate

$$
\begin{aligned}
|E(s;\delta)| &\leq 2\int_0^{\pi/2}\sin^{N+1-\operatorname{Re}s}v\,\mathrm{d}v\int_{\delta\sin v}^{\delta}\tau^{\operatorname{Re}s-N-1}|A_\tau|_N\,\mathrm{d}\tau \\
&\leq 2|A_\delta|_N\int_0^{\pi/2}\sin^{N+1-\operatorname{Re}s}v\,\mathrm{d}v\int_{\delta\sin v}^{\delta}\tau^{\operatorname{Re}s-N-1}\,\mathrm{d}\tau \\
&\leq 2\delta^{\operatorname{Re}s-N-1}|A_\delta|_N\int_0^{\pi/2}\sin^{N+1-\operatorname{Re}s}v\,\sin^{\operatorname{Re}s-N-1}v\int_{\delta\sin v}^{\delta}\mathrm{d}\tau \\
&= 2\delta^{\operatorname{Re}s-N}|A_\delta|_N\int_0^{\pi/2}(1-\sin v)\,\mathrm{d}v \\
&= 2\delta^{\operatorname{Re}s-N}|A_\delta|_N\left(\frac{\pi}{2}-1\right).
\end{aligned}
\tag{3.7.8}
$$

From this we conclude that for $s_0\in\{\operatorname{Re}s<N+1\}$ the condition $(D3)$ of Theorem 1.2 is satisfied which implies, by Theorem 1.2, that $E(s;\delta)$ is holomorphic on $\{\operatorname{Re}s<N+1\}$.

On the other hand, we know that both of the tube zeta functions are holomorphic on $\{\operatorname{Re}s>\overline{\dim}_B A\}\supseteq\{\operatorname{Re}s>N\}$. The fact that $E(s;\delta)$ is meromorphic on $\mathbb{C}$, as well as the statement about its poles, follows now from equation (3.7.4) (with $M=1$) and the fact that the gamma function is always nonzero.[28] More precisely, the locations of the poles of $E(s;\delta)$ must coincide with the locations of the poles of $\Gamma((N-s)/2+1)$ since the left-hand side of (3.7.4) is holomorphic on $\{\operatorname{Re}s>\overline{\dim}_B A\}$ and because $\widetilde{\zeta}_A(s_k)>0$ (since it is defined as an integral of a positive function).

[27] Namely, $\mathrm{B}(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$.

[28] In fact, $1/\Gamma(s)$ is an entire function with zeros at the nonpositive integers.

Finally, by multiplying (3.7.4) with $(s - s_k)$, taking the limit as $s \to s_k$ and using the fact that the residue of the gamma function at $-k$ is equal to $(-1)^k/k!$ we derive (3.7.5). $\square$

Theorem 3.78 has as an important consequence the fact that the notion of complex dimensions does not depend on the dimension of the ambient space.

Theorem 3.79. *Let $A \subseteq \mathbb{R}^N$ be a bounded set and A_M be its embedding in $\mathbb{R}^{N+M}$. Then the tube zeta function $\widetilde{\zeta}_A$ of A has a meromorphic extension to a neighborhood of the critical line $\{\operatorname{Re} s = \overline{\dim}_B A\}$ if and only if this is true for the tube zeta function $\widetilde{\zeta}_{A_M}$ of A_M. Furthermore the multisets[29] of their poles contained in U coincide.*

Proof. This is a direct consequence of Theorem 3.78 and the principle of analytic continuation. More precisely, identity (3.7.4) is valid for $s \in \mathbb{C}$ such that $\operatorname{Re} s > \overline{\dim}_B A$ and the function $E(s;\delta)$ is meromorphic on $\mathbb{C}$. Furthermore, the poles of $E(s;\delta)$ according to Theorem 3.78 are contained in $\{\operatorname{Re} s \geq N+2\}$ which implies that it is holomorphic on $\{\operatorname{Re} s < N+2\}$. Identity (3.7.4) then remains valid if any of the two zeta functions involved has a meromorphic continuation to some neighborhood of the critical line and this completes the proof. $\square$

Corollary 3.80. Let $A \subseteq \mathbb{R}^N$ be a bounded set such that its tube zeta function has a meromorphic continuation to a neighborhood U of the critical line $\operatorname{Re} s = \overline{\dim}_B A$ and suppose that $s = \overline{D}$ is its simple pole. Let $A_M \subseteq \mathbb{R}^{N+M}$ be the embedding of A in $\mathbb{R}^{N+M}$ as in Theorem 3.78. Then

$$\operatorname{res}(\widetilde{\zeta}_{A_M}, \overline{D}) = \frac{(\sqrt{\pi})^M \, \Gamma\left(\frac{N-\overline{D}}{2}+1\right)}{\Gamma\left(\frac{N+M-\overline{D}}{2}+1\right)} \operatorname{res}(\widetilde{\zeta}_A, \overline{D}) \tag{3.7.9}$$

We point out here that the above Corollary is, of course, in accord with the dimensional invariance of the normalized Minkowski content, as it is stated in [Res1]. More precisely, if we have that in the above Corollary $\overline{D}$ is the only pole of the tube zeta function of A on the critical line $\{\operatorname{Re} s = \overline{D}\}$, then, according to Theorem 3.42, A and $A \times \{0\}$ are D-Minkowski measurable with $D := \overline{D}$ and

$$\frac{\mathcal{M}^D(A)}{\pi^{\frac{D-N}{2}}\,\Gamma\left(\frac{N-D}{2}+1\right)} = \frac{\mathcal{M}^D(A \times \{0\})}{\pi^{\frac{D-N-1}{2}}\,\Gamma\left(\frac{N+1-D}{2}+1\right)}. \tag{3.7.10}$$

The above observations can also be made in the context of relative fractal drums. More precisely, let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ and let $(A \times \{0\}, \Omega \times [-1,1])$ be its embedding in $\mathbb{R}^{N+1}$. We want to link the relative tube zeta functions of these two RFDs and the following lemma will be needed for that.

[29]In this multiset the multiplicity of each pole is equal to its order.

Lemma 3.81. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ and $0<\delta<1$. Then we have*

$$\big|(A\times\{0\})_\delta\cap(\Omega\times[-1,1])\big|_{N+1}=2\int_0^\delta|A_{\sqrt{\delta^2-u^2}}\cap\Omega|_N\,\mathrm{d}u. \tag{3.7.11}$$

Proof. We proceed analogously as in [Res1, Proposition 6]. Namely, if we let $(x,y)\in\mathbb{R}^N\times\mathbb{R}\equiv\mathbb{R}^{N+1}$, and define

$$V:=\{(x,y)\,:\,d_{N+1}((x,y),A\times\{0\})\le\delta\}\cap\{(x,y)\,:\,x\in\Omega,|y|\le1\} \tag{3.7.12}$$

where d_N denotes the Euclidean distance in $\mathbb{R}^N$. It is obvious that the following equality holds: $d_{N+1}((x,y),A\times\{0\})=\sqrt{d_N(x,A)^2+y^2}$. This implies that for a fixed $y\in[-\delta,\delta]$[30] we have

$$\begin{aligned}V_y:&=\left\{x\in\mathbb{R}^N\,:\,d_{N+1}((x,y),A\times\{0\})\le\delta\right\}\\&=\left\{x\,:\,d_N(x,A)\le\sqrt{\delta^2-y^2}\right\}.\end{aligned} \tag{3.7.13}$$

Finally, Fubini's theorem implies that

$$\begin{aligned}\big|(A\times\{0\})_\delta\cap(\Omega\times[-1,1])\big|_{N+1}&=\int_V\mathrm{d}x\,\mathrm{d}y\\&=\int_{-\delta}^{\delta}\mathrm{d}y\int_{V_y\cap\{x\in\mathbb{R}^N\,:\,x\in\Omega\}}\mathrm{d}x\\&=2\int_0^\delta|A_{\sqrt{\delta^2-y^2}}\cap\Omega|_N\,\mathrm{d}y\end{aligned}$$

which completes the proof. □

The above lemma will give an RFD analog of Proposition 3.77 but first we will show that the upper and lower relative box dimension is independent of the ambient space dimension.

Proposition 3.82. Let (A,Ω) be an RFD in $\mathbb{R}^N$ and let

$$(A,\Omega)_M:=(A_M,\Omega\times[-1,1]^M) \tag{3.7.14}$$

its embedding in $\mathbb{R}^{N+M}$. Then we have that

$$\overline{\dim}_B(A,\Omega)=\overline{\dim}_B(A,\Omega)_M \tag{3.7.15}$$

and

$$\underline{\dim}_B(A,\Omega)=\underline{\dim}_B(A,\Omega)_M \tag{3.7.16}$$

Proof. We will prove the proposition in the case when $M=1$ from which the general

[30] If $|y|>\delta$, then the corresponding set V_y is empty.

result follows by induction. It is evident that for $\delta < 1$ we have

$$\begin{aligned}(A \times \{0\})_\delta \cap (\Omega \times [-1,1]) &\subseteq (A \times \{0\})_\delta \cap (\Omega \times [-\delta,\delta]) \\ &\subseteq (A_\delta \cap \Omega) \times [-\delta,\delta]\end{aligned}$$

so that

$$|(A \times \{0\})_\delta \cap (\Omega \times [-1,1])|_{N+1} \leq 2\delta |A_\delta \cap \Omega|_N. \tag{3.7.17}$$

This, in turn implies that for $r \in \mathbb{R}$ we have

$$\frac{|(A \times \{0\})_\delta \cap (\Omega \times [-1,1])|_{N+1}}{\delta^{N+1-r}} \leq \frac{2|A_\delta \cap \Omega|_N}{\delta^{N-r}}. \tag{3.7.18}$$

Furthermore, by taking the upper and lower limit above as $\delta \to 0^+$ we get the following inequalities involving the r-dimensional Minkowski contents:

$$\overline{\mathcal{M}}^r(A,\Omega)_1 \leq 2\overline{\mathcal{M}}^r(A,\Omega) \quad \text{and} \quad \underline{\mathcal{M}}^r(A,\Omega)_1 \leq 2\underline{\mathcal{M}}^r(A,\Omega) \tag{3.7.19}$$

which implies that

$$\overline{\dim}_B(A,\Omega)_1 \leq \overline{\dim}_B(A,\Omega) \quad \text{and} \quad \underline{\dim}_B(A,\Omega)_1 \leq \underline{\dim}_B(A,\Omega) \tag{3.7.20}$$

On the other hand, by geometrical reasons, we have that

$$(A_{\delta/2} \cap \Omega) \times \left[-\frac{\delta\sqrt{3}}{2}, \frac{\delta\sqrt{3}}{2}\right] \subseteq (A \times \{0\})_\delta \cap (\Omega \times [-1,1])$$

so that

$$\delta\sqrt{3}|A_{\delta/2} \cap \Omega|_N \leq |(A \times \{0\})_\delta \cap (\Omega \times [-1,1])|_{N+1}\,. \tag{3.7.21}$$

Similarly as before, this implies that for $r \in \mathbb{R}$ we have

$$\frac{\sqrt{3}|A_{\delta/2} \cap \Omega|_N}{2^{N-r}(\delta/2)^{N-r}} \leq \frac{|(A \times \{0\})_\delta \cap (\Omega \times [-1,1])|_{N+1}}{\delta^{N+1-r}} \tag{3.7.22}$$

and by taking the upper and lower limit as $\delta \to 0^+$ we get

$$\frac{\sqrt{3}\overline{\mathcal{M}}^r(A,\Omega)}{2^{N-r}} \leq \overline{\mathcal{M}}^r(A,\Omega)_1 \quad \text{and} \quad \frac{\sqrt{3}\underline{\mathcal{M}}^r(A,\Omega)}{2^{N-r}} \leq \underline{\mathcal{M}}^r(A,\Omega)_1. \tag{3.7.23}$$

Finally, this completes the proof as it implies the reverse inequalities for the relative box dimensions in (3.7.20). $\square$

We can now state the results for embedded RFDs and their relative zeta functions. Due to Lemma 3.81 and Proposition 3.82 the proofs follow the same steps as in the corresponding results about bounded subsets of $\mathbb{R}^N$ and for this reason we will omit them.

Proposition 3.83. Let (A,Ω) be a RFD in $\mathbb{R}^N$ with $\overline{\dim}_B(A,\Omega)=\overline{D}$ and fix $\delta\in(0,1)$. Then for the relative tube zeta functions of (A,Ω) and $(A,\Omega)_1=(A\times\{0\},\Omega\times[-1,1])$ the following equality holds

$$\widetilde{\zeta}_{A\times\{0\}}(s,\Omega\times[-1,1];\delta)=2\int_0^{\pi/2}\frac{\widetilde{\zeta}_A(s,\Omega;\delta\sin\tau)}{\sin^{s-N-1}\tau}\,\mathrm{d}\tau \tag{3.7.24}$$

for all $s\in\{\operatorname{Re}s>\overline{D}\}$.

Theorem 3.84. *Let (A,Ω) be an RFD in $\mathbb{R}^N$ with $\overline{\dim}_B(A,\Omega)=\overline{D}$ and fix $\delta\in(0,1)$. Then, for $s\in\mathbb{C}$ with $\operatorname{Re}s>\overline{D}$ we have the following equality between the tube zeta function of (A,Ω) and the tube zeta function of the relative fractal drum $(A,\Omega)_M=(A_M,\Omega\times[-1,1]^M)$ in $\mathbb{R}^{N+M}$:*

$$\widetilde{\zeta}_{A_M}(s,\Omega\times[-1,1]^M;\delta)=\frac{(\sqrt{\pi})^M\,\Gamma\left(\frac{N-s}{2}+1\right)}{\Gamma\left(\frac{N+M-s}{2}+1\right)}\widetilde{\zeta}_A(s,\Omega;\delta)+E(s;\delta), \tag{3.7.25}$$

where $E(s;\delta)$ is meromorphic on $\mathbb{C}$. The poles of $E(s;\delta)$ are located at $s_k:=N+2+2k$ for $k\in\mathbb{N}\cup\{0\}$ and all of them are simple. Moreover, we have that

$$\operatorname{res}(E(\,\cdot\,;\delta),s_k)=\frac{(-1)^{k+1}\,(\sqrt{\pi})^M}{k!\,\Gamma\left(\frac{M}{2}-k\right)}\widetilde{\zeta}_A(s_k,\Omega;\delta). \tag{3.7.26}$$

We immediately obtain the following result about the invariance of the complex dimensions of a relative fractal drum on the dimension of the ambient space which complements Theorem 3.79.

Theorem 3.85. *Let (A,Ω) be an RFD in $\mathbb{R}^N$ and $(A,\Omega)_M$ be its embedding in $\mathbb{R}^{N+M}$. Then the tube zeta function $\widetilde{\zeta}_A(\,\cdot\,,\Omega)$ of (A,Ω) has a meromorphic extension to a neighborhood of the critical line $\{\operatorname{Re}s=\overline{\dim}_B(A,\Omega)\}$ if and only if this is true for the tube zeta function $\widetilde{\zeta}_{A_M}(\,\cdot\,,\Omega\times[-1,1]^M)$ of $(A,\Omega)_M$. Furthermore the multisets of their poles contained in U coincide.*

One of the important consequences of Theorem 3.84 is that we can now extend the Minkowski measurability criterion given in Theorem 3.58 of Section 3.5 to the case when $\dim_B(A,\Omega)=N$.

Theorem 3.86 (Minkowski measurability criterion, special case)**.** *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $D:=\dim_B(A,\Omega)$ exists and $D=N$. Furthermore, assume that (A,Ω) is languid for a screen passing between the critical line $\{\operatorname{Re}s=N\}$ and all the complex dimensions of (A,Ω) with real part strictly less than N. Then the following is equivalent:*

(a) *(A, Ω) is Minkowski measurable.*

(b) *N is the only pole of the relative tube zeta function $\widetilde{\zeta}_A(\,\cdot\,, \Omega)$ located on the critical line $\{\operatorname{Re} s = N\}$ and it is simple.*

Proof. First of all, the fact that (b) implies (a) is a consequence of Theorem 3.42 since there are no restrictions of the type $\dim_B(A, \Omega) < N$ in the hypothesis of that theorem. Actually, it follows directly from the definition of the relative Minkowski content that $\dim_B(A, \Omega) = N$ implies that $\mathcal{M}^N(A, \Omega)$ exists and $\mathcal{M}^N(A, \Omega) = |\overline{A} \cap \Omega|$.

To prove that (a) implies (b) we embed (A, Ω) into $\mathbb{R}^{N+1}$ as $(A \times \{0\}, \Omega \times [-\delta, \delta])$ and conclude by Theorem 3.84 that their relative tube zeta functions are connected by the functional equation (3.7.25) (for $M = 1$); that is,

$$\widetilde{\zeta}_{A\times\{0\}}(s, \Omega \times [-\delta, \delta]; \delta) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2} + 1\right)}{\Gamma\left(\frac{N+1-s}{2} + 1\right)} \widetilde{\zeta}_A(s, \Omega; \delta) + E(s; \delta). \tag{3.7.27}$$

Here, $\delta > 0$ is chosen such that $\widetilde{\zeta}_A(s, \Omega; \delta)$ satisfies the languidity hypothesis of the theorem. The error function $E(\,\cdot\,; \delta)$ is holomorphic on $\{\operatorname{Re} s < N + 1\}$ and bounded by $2\delta^{\operatorname{Re} s - N}|A_\delta \cap \Omega|_N \left(\frac{\pi}{2} - 1\right)$ (see the proof of Theorem 3.78 and Equation (3.7.8)). In other words, $E(\,\cdot\,; \delta)$ is languid (with the languidity exponent equal to 0). Furthermore, for $a, b \in \mathbb{C}$ such that $\operatorname{Re}(b - a) > 0$ we have the following asymptotic expansion:

$$\frac{\Gamma(z + a)}{\Gamma(z + b)} \sim z^{a-b} \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(a-b+1)}(a)}{n!} \frac{\Gamma(b - a + n)}{\Gamma(b - a)} \frac{1}{z^n}, \quad \text{as} \quad |z| \to +\infty, \tag{3.7.28}$$

in the sector $|\arg z| < \pi$.[31] Substituting $z = \frac{N-s}{2} + 1$, $a = 0$ and $b = 1/2$ we obtain that

$$\frac{\Gamma\left(\frac{N-s}{2}+1\right)}{\Gamma\left(\frac{N+1-s}{2}+1\right)} \sim (N - s + 2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(2n)!\sqrt{2}(-1)^n B_n^{(1/2)}(0)}{2^n (n!)^2 (N - s + 2)^n}, \quad \text{as} \quad |s| \to +\infty, \tag{3.7.29}$$

for all $s \in \mathbb{C} \setminus [N + 2, +\infty)$.[32] In particular,

$$\frac{\Gamma\left(\frac{N-s}{2} + 1\right)}{\Gamma\left(\frac{N+1-s}{2} + 1\right)} = O(|s|^{-1/2}), \quad \text{as} \quad |s| \to +\infty, \tag{3.7.30}$$

for all $s \in \mathbb{C} \setminus [N+2, +\infty)$ from which we conclude that the product of this ratio of gamma functions with the relative tube zeta function $\widetilde{\zeta}_A(\,\cdot\,, \Omega; \delta)$ is languid with the exponent not greater than $\kappa - 1/2$ where κ is the languidity exponent of $\widetilde{\zeta}_A(\,\cdot\,, \Omega; \delta)$. This, together with the languidity of $E(\,\cdot\,; \delta)$ and Equation (3.7.27) implies that $\widetilde{\zeta}_{A\times\{0\}}(\,\cdot\,, \Omega \times [-\delta, \delta]; \delta)$

[31]Here, $B_n^{(\sigma)}(x)$ is the generalized Bernoulli polynomial (see, e.g., [SriTod] for the exact definition and an explicit formula). See also [Tem, Subsection 3.6.2] for this result on asymptotics of the ratio of gamma functions.

[32]We have used here the identities $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1/2 + n) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$.

is languid with the same choice of the double sequence $(T_n)_{n\in\mathbb{Z}\setminus\{0\}}$ and the screen S as for $\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta)$ and with the languidity exponent not greater than $\max\{\kappa-1/2,0\}$.

On the other hand, if (A,Ω) is Minkowski measurable then this is also true for the embedded RFD $(A\times\{0\},\Omega\times[-\delta,\delta])$. In light of Lemma 3.81, this fact follows in a completely analogous way as in the case of bounded subsets of $\mathbb{R}^N$ which was proven in [Res1]. We conclude now the proof by invoking Theorem 3.56; that is, its analog in terms of the relative tube zeta function (see Remark 3.57). $\square$

Remark 3.87. In the above discussion about embedding RFDs in higher-dimensional spaces we can also make similar observations if embed (A,Ω) as a 'one-sided' RFD $(A\times\{0\},\Omega\times[0,1])$ which can be more useful when decomposing a relative fractal drum into a union of relative fractal subdrums in order to compute its distance (or tube) zeta function. This follows immediately since by symmetry

$$\widetilde{\zeta}_{A\times\{0\}}(s,\Omega\times[-1,1])=2\,\widetilde{\zeta}_{A\times\{0\}}(s,\Omega\times[0,1]). \tag{3.7.31}$$

One only has to be careful when using the above formulas to take into account the factor 2. Furthermore, we can also embed (A,Ω) as $(A\times\{0\},\Omega\times[-\alpha,\alpha])$ or $(A\times\{0\},\Omega\times[0,\alpha])$ for some $\alpha>0$ but in that case the corresponding formulas will be valid for $\delta\in(0,\alpha)$.

We can now use the functional equation (2.1.4) which connects the tube and distance zeta functions to translate the above results in terms of the (relative) distance zeta function. We will make another approach which gives a little bit more information on the error function. Namely, recall the Mellin zeta function of a relative fractal drum introduced in Section 3.5:

$$\zeta_A^{\mathfrak{M}}(s,\Omega)=\int_0^{+\infty}t^{s-N-1}|A_t\cap\Omega|\,\mathrm{d}t \tag{3.7.32}$$

and recall that the above Lebesgue integral is absolutely convergent for $s\in\mathbb{C}$ such that $\operatorname{Re}s\in(\overline{\dim}_B(A,\Omega),N)$. Moreover, the relative distance and Mellin zeta functions of (A,Ω) are connected by

$$\zeta_A(s,\Omega)=(N-s)\zeta_A^{\mathfrak{M}}(s,\Omega), \tag{3.7.33}$$

on every open connected domain U to which any of the two zeta functions has a holomorphic continuation. Note that in (3.7.33) there is no parameter δ since the equation is valid only for δ such that $\Omega\subseteq A_\delta$; that is, when $\zeta_A(s,\Omega;\delta)=\int_\Omega d(x,A)^{s-N}\,\mathrm{d}x$.

We will now embed the relative fractal drum (A,Ω) of $\mathbb{R}^N$ into $\mathbb{R}^{N+1}$ as $(A\times\{0\},\Omega\times\mathbb{R})$. Of course, this is not technically a relative fractal drum in $\mathbb{R}^{N+1}$ since there does not exist a $\delta>0$ such that $\Omega\times\mathbb{R}\subseteq(A\times\{0\})_\delta$. On the other hand, observe that now Lemma 3.81 is valid for every $\delta>0$; that is,

$$\big|(A\times\{0\})_\delta\cap(\Omega\times\mathbb{R})\big|_{N+1}=2\int_0^\delta|A_{\sqrt{\delta^2-u^2}}\cap\Omega|_N\,\mathrm{d}u. \tag{3.7.34}$$

Proposition 3.88. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $\overline{\dim}_B(A,\Omega)<N$. Then the integral*

$$F(s):=\int_0^{+\infty}t^{s-N-2}\big|(A\times\{0\})_t\cap(\Omega\times\mathbb{R})\big|_{N+1}\,\mathrm{d}t \tag{3.7.35}$$

is holomorphic inside the vertical strip $\{\overline{\dim}_B(A,\Omega)<\operatorname{Re}s<N\}$.

Proof. We split the integral into two parts: $F(s)=\int_0^1+\int_1^{+\infty}$. The first part

$$\int_0^1 t^{s-N-2}\big|(A\times\{0\})_t\cap(\Omega\times\mathbb{R})\big|_{N+1}\,\mathrm{d}t=\int_0^1 t^{s-N-2}\big|(A\times\{0\})_t\cap(\Omega\times[-1,1])\big|_{N+1}\,\mathrm{d}t$$

defines a holomorphic function on the right half-plane $\{\operatorname{Re}s>\overline{\dim}_B(A,\Omega)\}$ according to Proposition 3.82. For the second part we observe that

$$\big|(A\times\{0\})_t\cap(\Omega\times\mathbb{R})\big|_{N+1}\le 2t|\Omega|_N,$$

and, consequently,

$$\left|\int_1^{+\infty}t^{s-N-2}\big|(A\times\{0\})_t\cap(\Omega\times\mathbb{R})\big|_{N+1}\,\mathrm{d}t\right|\le 2|\Omega|_N\int_1^{+\infty}t^{\operatorname{Re}s-N-1}\,\mathrm{d}t=\frac{2|\Omega|_N}{N-\operatorname{Re}s},$$

for all $s\in\mathbb{C}$ such that $\operatorname{Re}s<N$ which, according to Theorem 1.2, implies that the integral over $(1,\infty)$ defines a holomorphic function on the left half-plane $\{\operatorname{Re}s<N\}$ and thus, we complete the proof of the proposition. $\square$

In light of the above proposition we will still use the notation $\zeta^{\mathfrak{M}}_{A\times\{0\}}(\,\cdot\,,\Omega\times\mathbb{R})$ for the integral (3.7.35) although $(A\times\{0\},\Omega\times\mathbb{R})$ is not technically a relative fractal drum.

Theorem 3.89. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $\overline{D}:=\overline{\dim}_B(A,\Omega)<N$. Then, the following functional equation is valid*

$$\zeta_{A\times\{0\}}(s,\Omega\times[-a,a])=\frac{\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2}\right)}{\Gamma\left(\frac{N+1-s}{2}\right)}\zeta_A(s,\Omega)+E(s;a). \tag{3.7.36}$$

$E(s;a)$ is meromorphic on $\mathbb{C}$ and

$$E(s;a):=(s-N-1)\int_a^{+\infty}t^{s-N-2}|(A\times\{0\})_t\cap\Omega\times(\mathbb{R}\setminus[-a,a])|_{N+1}\,\mathrm{d}t, \tag{3.7.37}$$

with a set of simple poles contained in $\{N+2k:k\in\mathbb{N}_0\}$.

Proof. In a completely analogous way as in Theorem 3.78 we obtain that

$$\widetilde{\zeta}_{A\times\{0\}}(s,\Omega\times\mathbb{R};\delta)=\frac{\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2}+1\right)}{\Gamma\left(\frac{N+1-s}{2}+1\right)}\widetilde{\zeta}_A(s,\Omega;\delta)+\widetilde{E}(s;\delta), \tag{3.7.38}$$

valid now for all $\delta > 0$ (see (3.7.34) and the discussion preceding it). Furthermore, the error function $\widetilde{E}(\,\cdot\,;\delta)$ is holomorphic on $\{\mathrm{Re}\, s < N+1\}$ and

$$|\widetilde{E}(s,\delta)| \leq 2\delta^{\mathrm{Re}\, s - N}|A_\delta \cap \Omega|_N \left(\frac{\pi}{2} - 1\right) \tag{3.7.39}$$

for all s such that $\mathrm{Re}\, s < N+1$. See the proof of Theorem 3.78 and (3.7.8) to derive the above estimate. The estimate (3.7.39) now implies that the sequence of holomorphic functions $\widetilde{E}(\,\cdot\,;n) \to 0$ as $n \to +\infty$ uniformly on compact subsets of $\{\mathrm{Re}\, s < N\}$ since $|A_n \cap \Omega| = |\Omega|$ for n sufficiently large. Furthermore, we also have that $\widetilde{\zeta}_A(\,\cdot\,,\Omega;n) \to \zeta_A^{\mathfrak{M}}(\,\cdot\,,\Omega)$ and $\widetilde{\zeta}_{A\times\{0\}}(s,\Omega\times\mathbb{R};n) \to \zeta_{A\times\{0\}}^{\mathfrak{M}}(\,\cdot\,,\Omega\times\mathbb{R})$ as $n \to +\infty$, uniformly on compact subsets of $\{\overline{D} < \mathrm{Re}\, s < N\}$. This implies that by taking the limit in (3.7.36) as $\delta \to +\infty$ we get the following functional equality of holomorphic functions

$$\zeta_{A\times\{0\}}^{\mathfrak{M}}(s,\Omega\times\mathbb{R}) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2}+1\right)}{\Gamma\left(\frac{N+1-s}{2}+1\right)}\zeta_A^{\mathfrak{M}}(s,\Omega), \tag{3.7.40}$$

valid in the vertical strip $\{\overline{D} < \mathrm{Re}\, s < N\}$.[33] Furthermore, according to (3.7.33), we have

$$\zeta_{A\times\{0\}}^{\mathfrak{M}}(s,\Omega\times\mathbb{R}) = \frac{2\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2}\right)}{\Gamma\left(\frac{N+1-s}{2}+1\right)}\zeta_A(s,\Omega), \tag{3.7.41}$$

from which we conclude that the right-hand side is meromorphic on $\{\mathrm{Re}\, s > \overline{D}\}$ with simple poles located at the simple poles of $\Gamma((N-s)/2)$; that is, at $s_k := N + 2k$ for $k \in \mathbb{N}_0$. From this we conclude that by the principle of analytic continuation, this is also true for the left-hand side of (3.7.41) and, moreover, the left-hand side has a meromorphic continuation to any open connected domain U to which the right hand side can be meromorphically continued. To conclude the proof of the theorem we now observe that for any $a > 0$, since

$$\big|(A\times\{0\})_t \cap (\Omega\times\mathbb{R})\big| = \big|(A\times\{0\})_t \cap (\Omega\times[-a,a])\big| + \big|(A\times\{0\})_t \cap (\Omega\times(\mathbb{R}\setminus[-a,a]))\big|$$

the right-hand side of (3.7.41) can be split into two parts:

$$\begin{aligned}\zeta_{A\times\{0\}}^{\mathfrak{M}}(s,\Omega\times\mathbb{R}) &= \zeta_{A\times\{0\}}^{\mathfrak{M}}(s,\Omega\times[-a,a]) \\ &\quad + \int_a^{+\infty} t^{s-N-2}\big|(A\times\{0\})_t \cap (\Omega\times(\mathbb{R}\setminus[-a,a]))\big|\,\mathrm{d}t \\ &= \frac{\zeta_{A\times\{0\}}(s,\Omega\times[-a,a])}{N+1-s} - \frac{E(s;a)}{N+1-s}\end{aligned}$$

and combine this with (3.7.41) to obtain (3.7.36). From the theory, we know that

[33] We can get this equality even more directly by Lebesgue's dominated convergence theorem applied to a counterpart of (3.7.24).

$\zeta_{A\times\{0\}}(s,\Omega\times[-a,a])$ is holomorphic on $\{\operatorname{Re} s > \overline{D}\}$ and it we can show by a similar argument as in Proposition 3.88 that $E(s;a)$ defines a holomorphic function on $\{\operatorname{Re} s < N\}$. This, together with the functional equation (3.7.36) now ensures that $E(s;a)$ is meromorphic on all of $\mathbb{C}$ with poles contained in $\{N+2k : k\in\mathbb{N}_0\}$. $\square$

Since in the following example, we actually want to embed (A,Ω) into $\mathbb{R}^{N+1}$ as $(A\times\{0\},\Omega\times[0,a])$ for some $a>0$, by looking at the proof of the above theorem and using symmetry we can obtain the following result dealing with this kind of embeddings.

Theorem 3.90. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $\overline{D} := \overline{\dim}_B(A,\Omega) < N$. Then, the following functional equation is valid*

$$\zeta_{A\times\{0\}}(s,\Omega\times[0,a]) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2}\right)}{2\Gamma\left(\frac{N+1-s}{2}\right)}\zeta_A(s,\Omega) + E(s;a). \tag{3.7.42}$$

$E(s;a)$ is meromorphic on $\mathbb{C}$ and

$$E(s;a) := (s-N-1)\int_a^{+\infty} t^{s-N-2}|(A\times\{0\})_t\cap\Omega\times(\mathbb{R}\setminus[0,a])|_{N+1}\,\mathrm{d}t, \tag{3.7.43}$$

with a set of simple poles contained in $\{N+2k : k\in\mathbb{N}_0\}$.

Example 3.91. In this example we will consider the relative fractal drum consisting of the Cantor dust contained in $[0,1]^2$ and compute its distance zeta function. More precisely, let $A := C^{(1/3)}\times C^{(1/3)}$ be the Cantor dust and $\Omega := [0,1]^2$. We will not get an explicit formula in a closed form but we will use Theorem 3.78 in order to deduce that the distance zeta function of the Cantor dust has a meromorphic continuation to the whole of $\mathbb{C}$. More interestingly, we will also show that the set of complex dimensions of the Cantor dust is equal to a subset of the union of a periodic set contained in the critical line $\operatorname{Re} s = \log_3 4$ and the set of complex dimensions of the Cantor set (which is a periodic set contained in the critical line $\{\operatorname{Re} s = \log_3 2\}$). This is interesting because it shows that the distance (or tube) zeta function in this case also detects the 'lower dimensional' fractal nature of the Cantor dust.

Constructing (A,Ω) can be made by beginning with the unit square and removing the open middle-third 'cross' and then iterating this procedure ad infinitum. This procedure implies that we can subdivide the Cantor dust into a countable union of RFDs which are scaled down versions of two base RFDs. The first one is defined with $\Omega_1 := [0,1/3]^2$ and A_1 being the union of the four vertices of Ω_1. Furthermore, the second one is defined with $\Omega_2 := [0,1/3]\times[0,1/6]$ and A_2 being the Cantor set contained in $[0,1/3]\times\{0\}$.

At the n-th level of the iteration we have exactly 4^{n-1} RFDs of the type $(a_nA_1,a_n\Omega_1)$ and $8\cdot 4^{n-1}$ RFDs of the type $(a_nA_2,a_n\Omega_2)$ where $a_n := 3^{-n}$. This, together with the scaling property of the relative distance zeta function yields

$$
\begin{aligned}
\zeta_A(s,\Omega) &= \sum_{n=1}^{\infty} 4^{n-1}\zeta_{a_nA_1}(s,a_n\Omega_1) + 8\sum_{n=1}^{\infty} 4^{n-1}\zeta_{a_nA_2}(s,a_n\Omega_2) \\
&= \left(\zeta_{A_1}(s,\Omega_1) + 8\zeta_{A_2}(s,\Omega_2)\right)\sum_{n=1}^{\infty} 4^{n-1}\cdot 3^{-ns} \\
&= \frac{1}{3^s-4}\left(\zeta_{A_1}(s,\Omega_1) + 8\zeta_{A_2}(s,\Omega_2)\right).
\end{aligned}
\tag{3.7.44}
$$

Furthermore, for the relative distance zeta function of (A_1,Ω_1) we have

$$
\begin{aligned}
\zeta_{A_1}(s,\Omega_1) &= 8\int_0^{1/6}\mathrm{d}x\int_0^x\left(\sqrt{x^2+y^2}\right)^{s-2}\mathrm{d}y \\
&= 8\int_0^{\pi/4}\mathrm{d}\varphi\int_0^{1/6\cos\varphi} r^{s-1}\,\mathrm{d}r \\
&= \frac{8}{6^s s}\int_0^{\pi/4}\cos^{-s}\varphi\,\mathrm{d}\varphi = \frac{8I(s)}{6^s s}
\end{aligned}
\tag{3.7.45}
$$

where $I(s) := \int_0^{\pi/4}\cos^{-s}\varphi\,\mathrm{d}\varphi$ and it is easy to see that it is entire by using Theorem 1.2.[34] In other words, we have

$$
\zeta_A(s,\Omega) = \frac{8}{3^s-4}\left(\frac{I(s)}{6^s s} + \zeta_{A_2}(s,\Omega_2)\right).
\tag{3.7.46}
$$

Furthermore, let $\zeta_C(s,[0,1])$ be the relative zeta function of the Cantor middle-third set constructed inside $[0,1]$; see Example 3.63. From Theorem 3.90 and the scaling property of the relative distance zeta function (Theorem 2.10), we now have that

$$
\begin{aligned}
\zeta_{A_2}(s,\Omega_2) &= \frac{\sqrt{\pi}\,\Gamma\left(\frac{1-s}{2}\right)}{2\Gamma\left(\frac{2-s}{2}\right)}\zeta_{3^{-1}C}(s,3^{-1}[0,1]) + E(s;6^{-1}) \\
&= \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}\frac{\sqrt{\pi}}{6^s s(3^s-2)} + E(s;6^{-1})
\end{aligned}
\tag{3.7.47}
$$

where $E(s;6^{-1})$ is meromorphic on all of $\mathbb{C}$ with simple poles contained in $\{2k+1 : k\in\mathbb{N}_0\}$ so that

$$
\zeta_A(s,\Omega) = \frac{8}{s(3^s-4)}\left(\frac{I(s)}{6^s} + \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}\frac{\sqrt{\pi}}{6^s s(3^s-2)} + E(s;6^{-1})\right).
\tag{3.7.48}
$$

Formula (3.7.48) implies that the set of all complex dimensions $\mathcal{P}(\zeta_A(\,\cdot\,,\Omega))$ of the ‘relative’

[34] In fact, $I(s) = 2^{-1}\mathrm{B}_{1/2}\left(1/2,(1-s)/2\right)$ where $\mathrm{B}_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$ is the incomplete beta function.

Cantor dust is a subset of

$$\left(\log_3 4 + \frac{2\pi\mathbb{i}}{\log 3}\mathbb{Z}\right) \cup \left(\log_3 2 + \frac{2\pi\mathbb{i}}{\log 3}\mathbb{Z}\right) \cup \{0\}. \tag{3.7.49}$$

Of course, we do know that $\log_3 4 \in \mathcal{P}(\zeta_A(\,\cdot\,,\Omega))$, but we can only conjecture that the other poles on the critical line $\{\operatorname{Re} s = \log_3 4\}$ are in $\mathcal{P}(\zeta_A(\,\cdot\,,\Omega)$ since it may happen that we have a zero - pole cancellation in (3.7.48). On the other hand, since it is known that the Cantor dust is not Minkowski measurable (see [FaZe]), we can deduce from Theorem 3.42 that there must exist at least two other poles $s_{\pm k} = \log_3 4 \pm \frac{2k\pi\mathbb{i}}{\log 3}$ of $\zeta_A(\,\cdot\,,\Omega)$ for some $k \in \mathbb{N}$. From (3.7.48) we cannot even claim that $0 \in \mathcal{P}(\zeta_A(\,\cdot\,,\Omega)$ for certain, but we do see that all of the principal complex dimensions of the Cantor set are elements of $\mathcal{P}(\zeta_A(\,\cdot\,,\Omega)$, i.e., $\log_3 2 + \frac{2\pi\mathbb{i}}{\log 3}\mathbb{Z} \subseteq \mathcal{P}(\zeta_A(\,\cdot\,,\Omega)$.

The above example can be easily generalized to the case of Cartesian products of generalized Cantor sets in which case we would get that the complex dimensions of the product contain the complex dimensions of the each of the factors. In light of this and other similar examples it would be interesting to obtain some results about zero-free regions of fractal zeta functions. We leave this as a possible direction of further investigation (see Problem A.1).

Chapter 4

Lapidus Zeta Functions of Unbounded Sets at Infinity

In this chapter we are interested in relative fractal drums (A,Ω) in which the set A has degenerated to infinity. From now on, we will denote this new kind of relative fractal drums with (∞,Ω). In this case it is clear that the fractal properties of such a relative fractal drum will depend only on the set Ω. We will extend the notions of Minkowski content and box dimension for such relative fractal drums and define a new class of Lapidus zeta functions associated to them. Furthermore, it will be shown that this new class of Lapidus zeta functions has analogous properties as in the case of ordinary relative fractal drums studied in [LapRaŽu1] and recalled in Chapter 2.

4.1 Minkowski Content and Box Dimension of Unbounded Sets at Infinity

Let Ω be a Lebesgue measurable subset of the N-dimensional Euclidean space $\mathbb{R}^N$ of finite Lebesgue measure, i.e., $|\Omega|<\infty$. Firstly, we will introduce a new notation for the sake of brevity, namely,

$$ {}_t\Omega := B_t(0)^c\cap\Omega, \tag{4.1.1}$$

where $t>0$. We introduce the *tube function of* Ω *at infinity* by $t\mapsto|{}_t\Omega|$ for $t>0$ and we will be interested in the asymptotic properties of this function when $t\to+\infty$. Furthermore, for any real number r we define the *upper* r*-dimensional Minkowski content* of Ω *at infinity*

$$\overline{\mathcal{M}}^r(\infty,\Omega) := \limsup_{t\to+\infty}\frac{|{}_t\Omega|}{t^{N+r}}, \tag{4.1.2}$$

and, analogously, we define the *lower* r*-dimensional Minkowski content* of Ω *at infinity*,

$$\underline{\mathcal{M}}^r(\infty,\Omega) := \liminf_{t\to+\infty}\frac{|{}_t\Omega|}{t^{N+r}}. \tag{4.1.3}$$

It is easy to see that the above definition implies the existence of a unique $D \in \mathbb{R}$ such that $\overline{\mathcal{M}}^r(\infty, \Omega) = +\infty$ for $r < \overline{D}$ and $\overline{\mathcal{M}}^r(\infty, \Omega) = 0$ for $r > \overline{D}$ and similarly for the lower Minkowski content (see Figure 4.1). The value $\overline{D}$ is called the *upper box dimension of* Ω *at infinity* and we denote it with $\overline{\dim}_B(\infty, \Omega)$. Similarly as in the case of ordinary relative fractal drums, we have

$$\overline{\dim}_B(\infty, \Omega) := \sup\{r \in \mathbb{R} : \overline{\mathcal{M}}^r(\infty, \Omega) = +\infty\} = \inf\{r \in \mathbb{R} : \overline{\mathcal{M}}^r(\infty, \Omega) = 0\}. \quad (4.1.4)$$

Analogously we define the *lower box dimension of* Ω *at infinity* denoted with $\underline{\dim}_B(\infty, \Omega)$ by using the lower Minkowski content of Ω at infinity and have

$$\underline{\dim}_B(\infty, \Omega) := \sup\{r \in \mathbb{R} : \underline{\mathcal{M}}^r(\infty, \Omega) = +\infty\} = \inf\{r \in \mathbb{R} : \underline{\mathcal{M}}^r(\infty, \Omega) = 0\}. \quad (4.1.5)$$

Of course, if the upper and lower box dimensions coincide, we define the *box dimension* of Ω *at infinity* and denote it with $\dim_B(\infty, \Omega)$.

In the case when the upper and lower limits in (4.1.2) and (4.1.3) coincide we define the *r-dimensional Minkowski content* of Ω *at infinity* and denote it with $\mathcal{M}^r(\infty, \Omega)$. Clearly, in this case we have

$$\mathcal{M}^r(\infty, \Omega) = \lim_{t \to +\infty} \frac{|{}_t\Omega|}{t^{N+r}}. \quad (4.1.6)$$

Definition 4.1. Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ of finite Lebesgue measure. We say that it is *Minkowski nondegenerate at infinity* if there exists $D \in \mathbb{R}$ such that

$$0 < \underline{\mathcal{M}}^D(\infty, \Omega) \leq \overline{\mathcal{M}}^D(\infty, \Omega) < +\infty.^1 \quad (4.1.7)$$

We say that Ω is *Minkowski measurable at infinity* if in (4.1.7) equality holds. We will also say that Ω is *Minkowski degenerate* if (4.1.7) is not satisfied. (See Figure 4.1.)

Proposition 4.2. *Let* Ω *be a Lebesgue measurable subset of* $\mathbb{R}^N$ *of finite Lebesgue measure. Then* $\underline{\dim}_B(\infty, \Omega) \leq \overline{\dim}_B(\infty, \Omega) \leq -N$, *i.e., the upper and lower box dimensions of* Ω *at infinity are always negative, and moreover, less than or equal to* $-N$.

Proof. From the definitions (4.1.2) and (4.1.4) and the fact that $|\Omega| < \infty$ we have that $|{}_t\Omega| \to 0$ as $t \to +\infty$. From this it is clear that if $N + r > 0$ then $\overline{\mathcal{M}}^r(\infty, \Omega) = 0$. ☐

Remark 4.3. Intuitively the conclusion of Proposition 4.2 is expected, as Ω to have finite Lebesgue measure implies that it must have a certain flatness property relative to infinity. (Compare with the notion of flatness introduced in [LapRaŽu1] and recalled in Section 2.2.) Furthermore, if $\overline{\dim}_B(\infty, \Omega) = -N$, then it follows from the definition that $\overline{\mathcal{M}}_\infty^{-N}(\Omega) = 0$ and, consequently, Ω must be Minkowski degenerate at infinity.

[1] This implies that $D = \dim_B(\infty, \Omega)$.

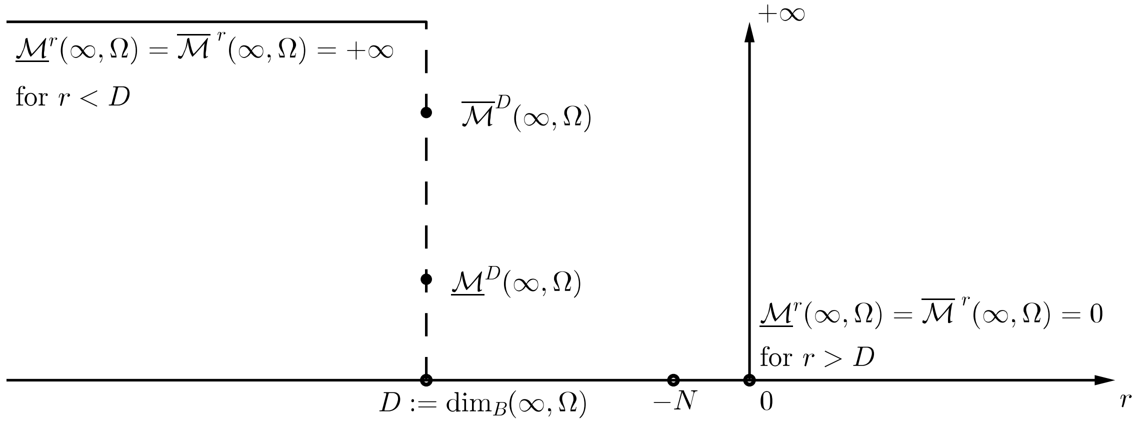

Figure 4.1: The graphs of the functions $r \mapsto \overline{\mathcal{M}}^r(\infty,\Omega)$ and $r \mapsto \underline{\mathcal{M}}^r(\infty,\Omega)$, assuming that Ω is Minkowski nondegenerate and nonmeasurable at infinity, that is, $D := \dim_B(\infty,\Omega)$ exists and $0 < \underline{\mathcal{M}}^r(\infty,\Omega) < \overline{\mathcal{M}}^r(\infty,\Omega) < \infty$.

The next two results about the monotonicity are simple consequences of the definitions involved.

Lemma 4.4. *Let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^N$ be two Lebesgue measurable sets and $|\Omega_2| < \infty$. Then for any real number r we have that*

$$\overline{\mathcal{M}}^r(\infty,\Omega_1) \leq \overline{\mathcal{M}}^r(\infty,\Omega_2) \tag{4.1.8}$$

and also

$$\underline{\mathcal{M}}^r(\infty,\Omega_1) \leq \underline{\mathcal{M}}^r(\infty,\Omega_2) \tag{4.1.9}$$

Proof. This is a consequence of the fact that for every $t > 0$ we have $|{}_t\Omega_1| \leq |{}_t\Omega_2|$. $\square$

Corollary 4.5. Let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^N$ be two Lebesgue measurable sets with $|\Omega_2| < \infty$. Then

$$\overline{\dim}_B(\infty,\Omega_1) \leq \overline{\dim}_B(\infty,\Omega_2) \quad \text{and} \quad \underline{\dim}_B(\infty,\Omega_1) \leq \underline{\dim}_B(\infty,\Omega_2).$$

Let us now take a look at a few examples.

Definition 4.6. Let $\alpha > 0$ and $\beta > 1$ be fixed and define

$$a_j := j^{\alpha},\ l_j := j^{-\beta} \quad \text{and} \quad b_j := a_j + l_j. \tag{4.1.10}$$

We define

$$\Omega(\alpha,\beta) := \bigcup_{j=1}^{\infty} I_j \subseteq \mathbb{R}, \tag{4.1.11}$$

that is, as a union of countably many intervals $I_j := (a_j, b_j)$.

Proposition 4.7. *For the set* $\Omega(\alpha,\beta)$ *defined by* (4.1.11) *we have that*

$$D := \dim_B(\infty,\Omega(\alpha,\beta)) = \frac{1-(\alpha+\beta)}{\alpha} \quad \textit{and} \quad \mathcal{M}^D_\infty(\Omega(\alpha,\beta)) = \frac{1}{\beta-1}. \tag{4.1.12}$$

Proof. Firstly, we observe that for j large enough the intervals I_j become disjoint, i.e., $j^{-\beta} < (j+1)^\alpha - j^\alpha$. As we see, $\Omega(\alpha,\beta)$ is a union of intervals that "escape" to infinity and $|\Omega(\alpha,\beta)| \leq \sum_{j=1}^\infty j^{-\beta} < \infty$. Let us compute the box dimension and Minkowski content of $\Omega(\alpha,\beta)$ at infinity. For $t>0$ let j_0 be such that for every $j>j_0$ it holds that $a_j>t$, that is, $j_0 = \lfloor t^{1/\alpha} \rfloor$. Now we fix t large enough so that the intervals I_j are disjoint for $j \geq j_0$. From this, we have

$$|{}_t\Omega(\alpha,\beta)| = \sum_{j>j_0} j^{-\beta} + \chi_\Omega(t)(b_{j_0}-t), \tag{4.1.13}$$

with χ_Ω being the characteristic function of Ω. This implies the following estimate

$$\sum_{j>j_0} j^{-\beta} \leq |{}_t\Omega(\alpha,\beta)| \leq \sum_{j>j_0} j^{-\beta} + j_0^{-\beta}. \tag{4.1.14}$$

Furthermore, using the integral criterion

$$\int_{j_0+1}^{+\infty} x^{-\beta}\,\mathrm{d}x \leq \sum_{j>j_0} j^{-\beta} \leq (j_0+1)^{-\beta} + \int_{j_0+1}^{+\infty} x^{-\beta}\,\mathrm{d}x$$

for estimating the sum, we have

$$\frac{1}{\beta-1}(j_0+1)^{1-\beta} \leq |{}_t\Omega(\alpha,\beta)| \leq \frac{1}{\beta-1}(j_0+1)^{1-\beta} + (j_0+1)^{-\beta} + j_0^{-\beta}.$$

Finally, by using the fact that $t^{1/\alpha} - 1 \leq j_0 + 1 \leq t^{1/\alpha} + 1$, we conclude that

$$\frac{1}{\beta-1}(t^{\frac{1}{\alpha}}+1)^{1-\beta} \leq |{}_t\Omega(\alpha,\beta)| \leq \frac{1}{\beta-1}(t^{\frac{1}{\alpha}}-1)^{1-\beta} + 2(t^{\frac{1}{\alpha}}-2)^{-\beta}$$

which implies that $\mathcal{M}^r(\infty,\Omega(\alpha,\beta))$ is different from 0 and $+\infty$ if and only if $r+1 = (1-\beta)/\alpha$, i.e.,

$$D := \dim_B(\infty,\Omega(\alpha,\beta)) = \frac{1-(\alpha+\beta)}{\alpha} \quad \text{and} \quad \mathcal{M}^D(\infty,\Omega(\alpha,\beta)) = \frac{1}{\beta-1}.$$

In particular, Ω is Minkowski measurable at infinity. $\square$

Remark 4.8. As we can see, the Minkowski content in the above case depends only on the parameter β, i.e., the rate at which $\Omega(\alpha,\beta)$ "escapes" to infinity is not relevant for it. Furthermore, by changing the values of parameters α and β, we can obtain any prescribed value in $(-\infty,-1)$ for $\dim_B(\infty,\Omega(\alpha,\beta))$. Moreover, we have that $\dim_B(\infty,\Omega(\alpha,\beta)) \to -\infty$ and $\mathcal{M}^D(\infty,\Omega(\alpha,\beta)) \to 0$ as $\beta \to +\infty$.

Proposition 4.9. *For $\alpha > 1$ let $\Omega := \{(x,y) \in \mathbb{R}^2 \,:\, x > 1,\ 0 < y < x^{-\alpha}\}$. Then we have that*

$$D := \dim_B(\infty, \Omega) = -1 - \alpha \quad \text{and} \quad \mathcal{M}^D(\infty, \Omega) = \frac{1}{\alpha - 1}. \tag{4.1.15}$$

Proof. Let $t > 1$ and let $x(t)$ be such that

$$x(t)^2 + x(t)^{-2\alpha} = t^2. \tag{4.1.16}$$

Then we have

$$\int_t^{+\infty} x^{-\alpha}\,\mathrm{d}x \le |{}_t\Omega| \le \int_{x(t)}^{+\infty} x^{-\alpha}\,\mathrm{d}x$$

which implies that

$$\frac{1}{1-\alpha} \le \frac{|{}_t\Omega|}{t^{1-\alpha}} \le \frac{1}{1-\alpha}\left(\frac{x(t)}{t}\right)^{1-\alpha}.$$

Furthermore, from (4.1.16) we have that

$$\frac{x(t)}{t} = (1 + x(t)^{-2(\alpha+1)})^{-\frac{1}{2}} \to 1, \quad \text{as} \quad t \to +\infty,$$

and we conclude that (4.1.15) holds. $\square$

Remark 4.10. Note that $\dim_B(\infty, \Omega) \to -\infty$ and $\mathcal{M}^D(\infty, \Omega) \to 0$ as $\alpha \to +\infty$.

Next we will prove a useful lemma which states that the box dimension and Minkowski measurability are independent on the choice of the norm on $\mathbb{R}^N$ in a sense that we can replace the ball $B_t(0)$ in the definition of the Minkowski content at infinity with a ball in any other norm on $\mathbb{R}^N$. More precisely, let $\|\cdot\|$ be another norm on $\mathbb{R}^N$.[2] First, we denote by $K_t(0)$ the open ball of radius t around 0 in the new norm. Next, we define the associated Minkowski content

$$\overline{\mathcal{N}}^r(\infty, \Omega) := \limsup_{t \to +\infty} \frac{|K_t(0)^c \cap \Omega|}{t^{N+r}}$$

and analogously, $\overline{\mathcal{N}}^r(\infty, \Omega)$ and $\mathcal{N}^r(\infty, \Omega)$.

Lemma 4.11. *Let $\Omega \subseteq \mathbb{R}^N$ with $|\Omega| < \infty$ and assume that two norms, $|\cdot|$ and $\|\cdot\|$, are given on $\mathbb{R}^N$, i.e., there are $a, b > 0$ such that $a|\cdot| \le \|\cdot\| \le b|\cdot|$. Then, for any $r \in \mathbb{R}$ we have*

$$a^{-(N+r)}\overline{\mathcal{M}}^r(\infty, \Omega) \le \overline{\mathcal{N}}^r(\infty, \Omega) \le b^{-(N+r)}\overline{\mathcal{M}}^r(\infty, \Omega), \tag{4.1.17}$$

and

$$a^{-(N+r)}\underline{\mathcal{M}}^r(\infty, \Omega) \le \underline{\mathcal{N}}^r(\infty, \Omega) \le b^{-(N+r)}\underline{\mathcal{M}}^r(\infty, \Omega). \tag{4.1.18}$$

[2]Of course, it is equivalent to the Euclidean norm.

Proof. From $a|x| \leq \|x\| \leq b|x|$ we have that $B_{t/b}(0) \subseteq K_t(0) \subseteq B_{t/a}(0)$ for any $t > 0$ and, consequently,

$$a^{-(N+r)}\frac{|B_{t/a}(0)^c \cap \Omega|}{\left(\frac{t}{a}\right)^{N+r}} \leq \frac{|K_t(0)^c \cap \Omega|}{t^{N+r}} \leq b^{-(N+r)}\frac{|B_{t/b}(0)^c \cap \Omega|}{\left(\frac{t}{b}\right)^{N+r}}.$$

Taking the upper limit as $t \to +\infty$, we obtain the first statement of the lemma. The second one is obtained by taking the lower limit instead of the upper. $\square$

Corollary 4.12. *Let Ω be an arbitrary Lebesgue measurable subset of $\mathbb{R}^N$ with finite N-dimensional Lebesgue measure. Then*

(*a*) *The upper and lower box dimensions of Ω at infinity do not depend on the choice of the norm on $\mathbb{R}^N$ in which we measure the neighborhood of infinity.*

(*b*) *The Minkowski nondegeneracy of Ω is independent of the choice of the norm on $\mathbb{R}^N$ in which we measure the neighborhood of infinity.*

A similar result holds also in the standard setting where we have a relative fractal drum (A, Ω) with $|\Omega| < \infty$. More precisely, in order to calculate $\overline{\dim}_B(A, \Omega)$ and $\underline{\dim}_B(A, \Omega)$ we can measure the δ-neighborhood of A in any metric on $\mathbb{R}^N$ that is equivalent to the Euclidean one.

There are special cases when we even get the same values for the Minkowski contents for different norms on $\mathbb{R}^N$. One of these cases is addressed in the next lemma which will prove to be useful in some of the future calculations. It can easily be generalized to the N-dimensional case but we will need it only in the case of $\mathbb{R}^2$.

Lemma 4.13. *Let $\Omega \subseteq \mathbb{R}^2$ with $|\Omega| < \infty$ such that Ω is a subset of a horizontal* (*vertical*) *strip of finite width. Let $K_t(0)$ be an open ball in the $|\cdot|$-norm of radius $t > 0$ with center at the origin and r a real number. Then, we have that*

$$\overline{\mathcal{M}}^r(\infty, \Omega) = \overline{\mathcal{N}}^r(\infty, \Omega) \quad \textit{and} \quad \underline{\mathcal{M}}^r(\infty, \Omega) = \underline{\mathcal{N}}^r(\infty, \Omega). \tag{4.1.19}$$

Proof. Without loss of generality we will assume that the set Ω is contained in the horizontal half-strip $\{(x, y) : x \geq 0,\ 0 \leq y \leq d\}$. Then, for $t \geq d$ we have that $|K^c_{\sqrt{t^2-d^2}} \cap \Omega| \leq |{}_t\Omega| \leq |K^c_t \cap \Omega|$ and consequently for $r \in \mathbb{R}$

$$\frac{(\sqrt{t^2-d^2})^{N+r}}{t^{N+r}}\frac{|K^c_{\sqrt{t^2-d^2}} \cap \Omega|}{(\sqrt{t^2-d^2})^{N+r}} \leq \frac{|{}_t\Omega|}{t^{N+r}} \leq \frac{|K^c_t \cap \Omega|}{t^{N+r}}.$$

Taking the upper and lower limits as $t \to +\infty$ completes the proof. $\square$

In the next example we will show that the value $\dim_B(\infty, \Omega) = -\infty$ can be achieved.

Example 4.14. Let $\Omega := \{(x,y) \in \mathbb{R}^2 \,:\, x > 1,\ 0 < y < \mathrm{e}^{-x}\}$ and let us calculate the box dimension of Ω at infinity using the $|\cdot|_\infty$-ball in $\mathbb{R}^2$.

$$|K_t(0)^c \cap \Omega| = \int_t^{+\infty} \mathrm{e}^{-x}\,\mathrm{d}x = \mathrm{e}^{-t}.$$

Consequently, we have that

$$\frac{|K_t(0)^c \cap \Omega|}{t^{2+r}} = \frac{\mathrm{e}^{-t}}{t^{2+r}} \to 0$$

when $t \to +\infty$ for every $r \in \mathbb{R}$ and therefore $\dim_B(\infty, \Omega) = -\infty$.

Remark 4.15. From now on, we will always implicitly assume that $\overline{\dim}_B(\infty, \Omega) > -\infty$ when dealing with relative fractal drums of the type (∞, Ω) (unless stated otherwise). We will do this, because most of the results that will be stated in the rest of this dissertation fail or require a slightly different proof in the special case when $\overline{\dim}_B(\infty, \Omega) = -\infty$. We leave this special case for the interested reader or future work (see Problem A.4).

As we have shown in Proposition 4.2, the upper box dimension of any subset of the plane of finite Lebesgue measure does not exceed -2. The next proposition will show that the value -2 can be achieved and it can be easily adapted for constructing a subset Ω of $\mathbb{R}^N$ with finite Lebesgue measure such that $\overline{\dim}_B(\infty, \Omega) = -N$.

Proposition 4.16. *There exists a Lebesgue measurable subset $\Omega \subseteq \mathbb{R}^2$ with $|\Omega| < \infty$ such that*

$$\dim_B(\infty, \Omega) = -2 \quad \text{and} \quad \mathcal{M}^{-2}(\infty, \Omega) = 0. \tag{4.1.20}$$

Proof. Let $\alpha_k := 1 + 1/k$ for $k \geq 1$ and we define

$$\widetilde{\Omega}_k := \{(x,y) \in \mathbb{R}^2 \,:\, x > 1,\ 0 < y < \frac{2^{-k}}{k} x^{-\alpha_k}\}.$$

We will "stack" the sets $\widetilde{\Omega}_k$ on top of each other. In order to do so, we define Ω_k to be an S_k-translated image of $\widetilde{\Omega}_k$ along the y-axis where

$$S_k := \sum_{j=1}^{k} \frac{2^{-j}}{j}$$

and define $\Omega := \cup_{k \geq 1} \Omega_k$. We observe that Ω is contained in the horizontal strip of finite height

$$\{(x,y) \in \mathbb{R}^2 \,:\, 1/2 \leq y \leq S\},$$

where $S := \lim_{k \to \infty} S_k = \log 2$. Furthermore, we have that

$$|\Omega_k| = |\widetilde{\Omega}_k| = \frac{2^{-k}}{k} \int_1^{+\infty} x^{-1-\frac{1}{k}}\,\mathrm{d}x = \frac{2^{-k}}{k} \cdot k = 2^{-k}$$

so that $|\Omega| = \sum_{k=1}^{\infty} 2^{-k} = 1$. Using the same calculation as in Proposition 4.9 yields that

$$D_k := \dim_B(\infty,\Omega)_k = -1 - \alpha_k = -2 - \frac{1}{k} \quad \text{and} \quad \mathcal{M}_\infty^{D_k}(\Omega_k) = 2^{-k}.$$

Finally, by using Corollary 4.5 we have that $-2 \geq \dim_B(\infty,\Omega) \geq D_k$ for every $k \geq 1$ which implies (4.1.20). $\square$

Remark 4.17. Note that rotating the set Ω from the above proposition by $\pi/2$ in the positive direction around the origin we get a set that can be identified with an epigraph of a Lebesgue integrable function.

Let us state another result which relates the tube functions at infinity $t \mapsto |B_t(0)^c \cap \Omega|$ and $t \mapsto |K_t(0)^c \cap \Omega|$. This result will be needed when constructing quasiperiodic sets at infinity in Section 4.6 below. Recall that the volume of the N-dimensional unit ball is given by

$$\omega_N := \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}+1\right)} \tag{4.1.21}$$

where Γ denotes the Gamma function. Furthermore, the volume of the N-dimensional ball of radius R is then given by

$$|B_R(0)| = \omega_N R^N. \tag{4.1.22}$$

On the other hand, note that the surface area; that is, the $(N-1)$-dimensional Hausdorff measure (denoted by H^{N-1}), of the $(N-1)$-dimensional sphere $S_R(0)$ of radius R is then given by

$$H^{N-1}(S_R(0)) = \frac{\mathrm{d}}{\mathrm{d}R}|B_R(0)| = N\omega_N R^{N-1}. \tag{4.1.23}$$

Lemma 4.18. *Let $\Omega \subseteq \mathbb{R}^N$ with $|\Omega| < \infty$ be such that it is contained in a cylinder*

$$x_2^2 + x_3^2 + \cdots + x_N^2 \leq C \tag{4.1.24}$$

for some constant $C > 0$ where $x = (x_1,\ldots,x_N)$. Then

$$|B_t(0)^c \cap \Omega| = |K_t(0)^c \cap \Omega| + O(t^{-1}) \quad \text{as} \quad t \to +\infty. \tag{4.1.25}$$

Proof. We note that for t sufficiently large the difference $|B_t(0)^c \cap \Omega| - |K_t(0)^c \cap \Omega|$ is less than the volume of the N-dimensional cylinder of height $h := t - \sqrt{t^2 - C^2}$ with base of radius C. In other words, we have that

$$\big||B_t(0)^c \cap \Omega| - |K_t(0)^c \cap \Omega|\big| \leq h\omega_{N-1}C^{N-1} = \frac{\omega_{N-1}C^{N+1}}{t + \sqrt{t^2 + C^2}} = O(t^{-1}).$$

$\square$

4.2 Holomorphicity of Lapidus Zeta Functions at Infinity

Let $\Omega \subseteq \mathbb{R}^N$ be a measurable set with $|\Omega| < \infty$. We define the *Lapidus zeta function* of Ω *at infinity* by

$$\zeta_\infty(s, \Omega) := \int_{T\Omega} |x|^{-s-N}\,\mathrm{d}x, \tag{4.2.1}$$

for a fixed $T > 0$ and s in $\mathbb{C}$ with $\operatorname{Re} s$ sufficiently large. We will also call this zeta function the *distance zeta function of* Ω *at infinity* and use the two notions interchangeably. From now on, our main goal will be to show that this new zeta function has analogous properties as the distance zeta function for relative fractal drums. First of all, the dependence of the distance zeta function at infinity on $T > 0$ is inessential in the sense that for $0 < T_1 < T_2$ the difference

$$\zeta_\infty(s, \Omega; T_1) - \zeta_\infty(s, \Omega; T_2) = \int_{B_{T_1,T_2}(0)\cap\Omega} |x|^{-s-N}\,\mathrm{d}x,$$

with

$$B_{a,b}(0) := \{x \in \mathbb{R}^N \,:\, a < |x| < b\}, \tag{4.2.2}$$

is an entire function of s. Indeed, this follows from Theorem 1.1(c) with $E := B_{T_1,T_2}(0)\cap\Omega$, $\varphi(x) := |x|$ and $\mathrm{d}\mu(x) := |x|^{-N}\,\mathrm{d}x$.[3] Therefore, from now on, we will emphasize the dependence of the Lapidus zeta function of Ω at infinity on T and write $\zeta_\infty(s, \Omega; T)$ only when it is explicitly needed. Also note that if Ω is bounded, then for T sufficiently large, we have that $\zeta_\infty(s, \Omega; T) \equiv 0$.[4]

The definition of the Lapidus zeta function of Ω at infinity is, as we will demonstrate immediately, closely related to the distance zeta function of a certain relative fractal drum. This relative fractal drum is actually the image of (∞, Ω) under the geometric inversion in $\mathbb{R}^N$, i.e., it is equal to $(\mathbf{0}, \Phi(\Omega))$[5], where

$$\Phi(x) := \frac{x}{|x|^2} \tag{4.2.3}$$

and $\mathbf{0}$ is the origin. To derive the mentioned relation we will need to compute the Jacobian of the geometric inversion and use the change of variables formula for the Lebesgue integral. To compute the Jacobian we will use the well-known Matrix determinant lemma (see, e.g., [Har]) which we state here for the sake of exposition.

Lemma 4.19 (Matrix determinant lemma)**.** *Let* $\mathbf{A}$ *be an invertible matrix and* $\mathbf{u}$*,* $\mathbf{v}$

[3] Note that $T_1 \le \varphi(x) \le T_2$ for $x \in E$.

[4] We will see in a moment that this will be in accord with the fact that $\dim_B(\infty, \Omega) = -\infty$ for a bounded Lebesgue measurable set Ω.

[5] We should actually write $(\{\mathbf{0}\}, \Phi(\Omega))$ here, but we will always abuse notation in this way for a relative fractal drum (A, Ω) when the set A consists of a single point.

column vectors. Then we have that

$$\det(\mathbf{A} + \mathbf{u} \otimes \mathbf{v}) = (1 + \mathbf{v}^\tau \mathbf{A}^{-1}\mathbf{u}) \det \mathbf{A}, \tag{4.2.4}$$

where

$$\mathbf{u} \otimes \mathbf{v} := \mathbf{u}\mathbf{v}^\tau \tag{4.2.5}$$

and τ denotes the transpose operator.

Lemma 4.20. *Let $\Phi(x) := x/|x|^2$ be the geometric inversion on $\mathbb{R}^N$. Then for the Jacobian of Φ we have:*

$$\det \frac{\partial \Phi}{\partial x} = -|x|^{-2N}. \tag{4.2.6}$$

Proof. With $x = (x_1, \dots, x_N)$ and δ_{ij} the Kronecker delta we have that

$$\left(\frac{\partial \Phi}{\partial x}\right)_{ij} = \frac{\partial \Phi_i}{\partial x_j} = \frac{\delta_{ij}}{|x|^2} - \frac{2x_i x_j}{|x|^4} \tag{4.2.7}$$

and consequently

$$\frac{\partial \Phi}{\partial x} = \frac{1}{|x|^4}(|x|^2 \mathbf{I} - 2\,\mathbf{x} \otimes \mathbf{x}), \tag{4.2.8}$$

where $\mathbf{x} := [x_1, \dots, x_N]^\tau$ and $\mathbf{I}$ is the identity matrix. Now we can apply the matrix determinant lemma with $\mathbf{A} := |x|^2\mathbf{I}$, $\mathbf{u} := -2\mathbf{x}$ and $\mathbf{v} := \mathbf{x}$ from which we get

$$\begin{aligned}\det \frac{\partial \Phi}{\partial x} &= \frac{1}{|x|^{4N}}(1 - 2\mathbf{x}^\tau(|x|^2\mathbf{I})^{-1}\mathbf{x}) \det(|x|^2\mathbf{I}) \\ &= \frac{1}{|x|^{4N}}(1 - 2|x|^{-2}\mathbf{x}^\tau\mathbf{x})|x|^{2N} = -\frac{1}{|x|^{2N}}.\end{aligned}$$

□

The next theorem will show that, from the point of view of the distance zeta function, there is no difference between the unbounded relative fractal drum (∞, Ω) and the relative fractal drum $(\mathbf{0}, \Phi(\Omega))$ obtained from it by geometric inversion.

Theorem 4.21. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ of finite measure, $\mathbf{0}$ the origin and fix $T > 0$. Then we have*[6]

$$\zeta_\infty(s, \Omega; T) = \zeta_{\mathbf{0}}(s, \Phi(\Omega); 1/T). \tag{4.2.9}$$

Proof. Defining $y = \Phi^{-1}(x)$ and using Lemma 4.20 this is a consequence of the change of

[6]Here, and in the remainder of this thesis, we will abuse notation and write $\zeta_{\mathbf{0}}(s, \Phi(\Omega); 1/T)$ instead of $\zeta_{\{\mathbf{0}\}}(s, \Phi(\Omega); \cdot\,)$.

variables formula once we observe the fact that $|y| = 1/|x|$:

$$\begin{aligned}\zeta_\infty(s,\Omega;T) &= \int_{B_T(0)^c\cap\Omega} |x|^{-s-N}\,\mathrm{d}x\\ &= \int_{\Phi(B_T(0)^c\cap\Omega)} |y|^{s+N}|y|^{-2N}\,\mathrm{d}y\\ &= \int_{B_{1/T}(0)\cap\Phi(\Omega)} |y|^{s-N}\,\mathrm{d}y = \zeta_{\{\mathbf{0}\}}(s,\Phi(\Omega);1/T).\end{aligned}$$

□

This result suggests that we can analyze fractal properties of $\Omega\subseteq\mathbb{R}^N$ at infinity by analyzing the fractal properties of the 'inverted' relative fractal drum $(\mathbf{0},\Phi(\Omega))$. A similar approach (in the context of unbounded subsets of $\mathbb{R}^N$) was made in [RaŽuŽup] and will be exposed in Chapter 5. Of course, in that approach, we can use results of [LapRaŽu1] about relative fractal drums and relative distance (and tube) zeta functions. On the other hand, we stress that in that case we are dealing with the usual relative box dimension of the inverted relative fractal drum, i.e., with $\dim_B(\mathbf{0},\Phi(\Omega))$ which is defined via the r-dimensional relative Minkowski content, namely, $\mathcal{M}^r(\mathbf{0},\Phi(\Omega))$. However, it is not immediately evident what are the relations between the "classical" relative box dimension and Minkowski content of the inverted relative fractal drum with the notions of box dimension and Minkowski content at infinity introduced in Section 4.1. Among others, we will try to answer this question in the remainder of this dissertation.

To prove the holomorphicity theorem, we will need the following proposition which complements [Žu3, Lemma 3].

Proposition 4.22. *Let $\Omega\subseteq\mathbb{R}^N$ be a Lebesgue measurable set with $|\Omega|<\infty$, $T>0$ and let $u\colon(T,+\infty)\to[0,+\infty)$ be a strictly monotone C^1 function. Then the following equality holds*

$$\int_{{}_T\Omega} u(|x|)\,\mathrm{d}x = u(T)|{}_T\Omega| + \int_T^{+\infty} |{}_t\Omega|u'(t)\,\mathrm{d}t. \tag{4.2.10}$$

Proof. We will use a well-known fact (see, e.g., [Mat, Theorem 1.15]) that for a nonnegative Borel function f on a separable metric space X the following identity holds

$$\int_X f(x)\,\mathrm{d}x = \int_0^\infty |\{x\in X\,:\,f(x)\geq t\}|\,\mathrm{d}t. \tag{4.2.11}$$

We let $f(x):=u(|x|)$, $X:={}_T\Omega$ and consider separately the cases of strictly decreasing and strictly increasing function u.

(a) Let u be strictly decreasing and $u(+\infty):=\lim_{\tau\to+\infty}u(\tau)$. For the set appearing on the right-hand side of (4.2.11) we have

$$A(t):=\{x\in{}_T\Omega\,:\,u(|x|)\geq t\}=\{x\in{}_T\Omega\,:\,|x|\leq u^{-1}(t)\}.$$

For $0 \le t \le u(+\infty)$ it is true that $u(|x|) \ge t$ for any $x \in \mathbb{R}^N$ because $u(+\infty) = \min_{\tau \ge 0} u(\tau)$ and we have $A(t) = {}_T\Omega$. Furthermore, if $u(+\infty) < t \le u(T)$, it is clear that

$$A(t) = ({}_T\Omega) \setminus (B_{u^{-1}(t)}(0)^c \cap \Omega) = B_{T,u^{-1}(t)}(0) \cap \Omega.$$

Finally, for $t > u(T)$ we have that $A(t) = \emptyset$ because $u(T) = \max_{\tau \ge 0} u(\tau)$ and using (4.2.11) we get

$$\begin{aligned}\int_{{}_T\Omega} u(|x|)\,\mathrm{d}x &= \int_0^{u(+\infty)} |{}_T\Omega|\,\mathrm{d}t + \int_{u(+\infty)}^{u(T)} |B_{T,u^{-1}(t)}(0) \cap \Omega|\,\mathrm{d}t \\ &= u(+\infty)|{}_T\Omega| + \int_{u(+\infty)}^{u(T)} |B_T(0)^c \cap \Omega|\,\mathrm{d}t \\ &\quad - \int_{u(+\infty)}^{u(T)} |B_{u^{-1}(t)}(0)^c \cap \Omega|\,\mathrm{d}t \\ &= u(T)|{}_T\Omega| + \int_T^{+\infty} |B_s(0)^c \cap \Omega| u'(s)\,\mathrm{d}s,\end{aligned}$$

where we have introduced the new variable $s = u^{-1}(t)$ in the last equality.

(*b*) We now let u be a strictly increasing function and denote $u(+\infty) := \lim_{\tau \to +\infty} u(\tau) = \sup_{\tau \ge 0} u(\tau) \in (0, +\infty]$. In this case we have

$$A(t) := \{x \in {}_T\Omega \,:\, u(|x|) \ge t\} = \{x \in {}_T\Omega \,:\, |x| \ge u^{-1}(t)\}.$$

For $0 \le t \le u(T)$ we have that $u(|x|) \ge t$ for any $x \in \mathbb{R}^N$ because $u(T) = \min_{\tau \ge 0} u(\tau)$ and we have $A(t) = {}_T\Omega$. Furthermore, if $u(T) < t < u(+\infty)$ it is clear that $A(t) = B_{u^{-1}(t)}(0)^c \cap \Omega$, and for $t \ge u(+\infty)$ the set $A(t)$ is an empty set. Altogether, we have

$$\begin{aligned}\int_{{}_T\Omega} u(|x|)\,\mathrm{d}x &= \int_0^{u(T)} |{}_T\Omega|\,\mathrm{d}t + \int_{u(T)}^{u(+\infty)} |B_{u^{-1}(t)}(0)^c \cap \Omega|\,\mathrm{d}t \\ &= u(T)|{}_T\Omega| + \int_T^{+\infty} |B_s(0)^c \cap \Omega| u'(s)\,\mathrm{d}s\end{aligned}$$

where, again, we have introduced the new variable $s = u^{-1}(t)$ in the last equality. This concludes the proof of the proposition. □

The following proposition is analogous to the one from [LapRaŽu1] dealing with relative fractal drums.

Proposition 4.23. *Let $\Omega \subseteq \mathbb{R}^N$ be a measurable set with $|\Omega| < \infty$, $T > 0$. Then for every $\sigma \in (\overline{\dim}_B(\infty, \Omega), +\infty)$, the following identity holds*:

$$\int_{{}_T\Omega} |x|^{-\sigma-N}\,\mathrm{d}x = T^{-\sigma-N}|{}_T\Omega| - (\sigma+N)\int_T^{+\infty} t^{-\sigma-N-1}|{}_t\Omega|\,\mathrm{d}t. \tag{4.2.12}$$

Furthermore, the above integrals are finite for such σ.

Proof. The proposition is a direct consequence of Proposition 4.22 with $u(t) := t^{-\sigma-N}$ when $\sigma \neq -N$ and for $\sigma = -N$ the equation (4.2.12) is trivially fulfilled. Let us fix $\sigma_1 \in (\overline{\dim}_B(\infty,\Omega),\sigma)$. Then for T large enough we have that for a constant $M > 0$ we have

$$|{}_t\Omega| \leq M t^{\sigma_1+N}$$

for every $t > T$. From this we get that

$$\int_T^{+\infty} t^{-\sigma-N-1}|{}_t\Omega|\,\mathrm{d}t \leq M\int_T^{+\infty} t^{-\sigma-N-1}t^{\sigma_1+N}\,\mathrm{d}t = M\int_T^{+\infty} t^{\sigma_1-\sigma-1}\,\mathrm{d}t < +\infty$$

because $\sigma_1 - \sigma - 1 < -1$. $\square$

Now we can state and prove the holomorphicity theorem for the Lapidus zeta function at infinity, but firstly we will introduce a new notation for the sake of brevity, namely,

$${}_{a,b}\Omega := B_{a,b}(0) \cap \Omega. \tag{4.2.13}$$

Theorem 4.24. *Let Ω be any Lebesgue measurable subset of $\mathbb{R}^N$ of finite N-dimensional Lebesgue measure. Assume that T is a fixed positive number. Then the following conclusions hold.*
(a) The abscissa of convergence of the Lapidus zeta function at infinity

$$\zeta_\infty(s,\Omega) = \int_{{}_T\Omega} |x|^{-s-N}\,\mathrm{d}x \tag{4.2.14}$$

is equal to the upper box dimension of Ω at infinity, i.e.,

$$D(\zeta_\infty(\,\cdot\,,\Omega)) = \overline{\dim}_B(\infty,\Omega). \tag{4.2.15}$$

Consequently, $\zeta_\infty(\,\cdot\,,\Omega)$ is holomorphic on the half-plane $\{\operatorname{Re} s > \overline{\dim}_B(\infty,\Omega)\}$ and for every complex number s in that half-plane we have that

$$\zeta_\infty'(s,\Omega) = -\int_{{}_T\Omega} |x|^{-s-N}\log|x|\,\mathrm{d}x. \tag{4.2.16}$$

(b) If $D = \dim_B(\infty,\Omega)$ exists and $\underline{\mathcal{M}}^D(\infty,\Omega) > 0$, then $\zeta_\infty(s,\Omega) \to +\infty$ for $s \in \mathbb{R}$ as $s \to D^+$.

Proof. (a) If we let $\overline{D} := \overline{\dim}_B(\infty,\Omega)$, then from the definitions of the upper Minkowski content and of the upper box dimension at infinity we have that $\limsup_{t\to+\infty} \frac{|{}_t\Omega|}{t^{N+\sigma}} = 0$ for every $\sigma > \overline{D}$. Now, let us fix σ_1 such that $\overline{D} < \sigma_1 < \sigma$ and take $T > 1$ large enough, such that for a constant $M > 0$ it holds that

$$|{}_t\Omega| \leq M t^{\sigma_1+N} \quad \text{for every} \quad t > T.$$

Furthermore, we estimate $\zeta_\infty(\sigma,\Omega)$ in the following way

$$\begin{aligned}
\zeta_\infty(\sigma,\Omega) &= \int_{{}_T\Omega} |x|^{-\sigma-N}\,\mathrm{d}x = \sum_{k=1}^{\infty} \int_{T^k,T^{k+1}\Omega} |x|^{-\sigma-N}\,\mathrm{d}x \\
&\leq \sum_{k=1}^{\infty} \max\left\{(T^k)^{-\sigma-N},(T^{k+1})^{-\sigma-N}\right\} |{}_{T^k,T^{k+1}}\Omega| \\
&\leq \max\left\{1,T^{-\sigma-N}\right\} \sum_{k=1}^{\infty} (T^k)^{-\sigma-N} M (T^k)^{\sigma_1+N} \\
&= M \max\left\{1,T^{-\sigma-N}\right\} \sum_{k=1}^{\infty} (T^{\sigma_1-\sigma})^k < \infty.
\end{aligned}$$

The last inequality follows from the fact that $T > 1$ and $\sigma_1 - \sigma < 0$. We let now $E := {}_T\Omega$, $\varphi(x) := |x|$ and $\mathrm{d}\mu(x) := |x|^{-N}\,\mathrm{d}x$ and note that $\varphi(x) \geq T > 1$ for $x \in E$. Part (a) follows now from Theorem 1.1(b).

To conclude the proof that $\overline{D}$ is the abscissa of convergence of $\zeta_\infty(\,\cdot\,,\Omega)$ we take $s \in (-\infty,\overline{D})$ and use Proposition 4.23:

$$\begin{aligned}
I_T := \int_{{}_T\Omega} |x|^{-s-N}\,\mathrm{d}x &= T^{-s-N}|{}_T\Omega| - (s+N)\int_T^{+\infty} t^{s-N-1}|{}_t\Omega|\,\mathrm{d}t \\
&\geq T^{-s-N}|{}_T\Omega|.
\end{aligned} \tag{4.2.17}$$

Now, we fix σ such that $s < \sigma < \overline{D}$. From $\overline{\mathcal{M}}^{\sigma}_{\infty}(\Omega) = +\infty$ we conclude that there exists a sequence $(t_k)_{k\geq 1}$ such that

$$C_k := \frac{|{}_{t_k}\Omega|}{t_k^{N+\sigma}} \to +\infty \quad \text{when} \quad t_k \to +\infty.$$

It is clear that the function $T \to I_T$ is nonincreasing and we have

$$I_T \geq I_{t_k} \geq t_k^{-s-N}|{}_{t_k}\Omega| = t_k^{-s-N} t_k^{N+\sigma} C_k = C_k t_k^{\sigma-s} \to +\infty. \tag{4.2.18}$$

Therefore, $I_T = +\infty$ for every $s < \overline{D}$ which proves that $D(\zeta_\infty(\,\cdot\,,\Omega)) = \overline{D}$.

(b) Let us assume now that $D = \dim_B(\infty,\Omega)$ exists, and $\underline{\mathcal{M}}^D(\infty,\Omega) > 0$. From Proposition 4.2 we have that $D \leq -N$. On the other hand, the condition $\underline{\mathcal{M}}^D(\infty,\Omega) > 0$ and Remark 4.3 imply that $D \neq -N$. Consequently, we may assume that $D < -N$. Furthermore, $\underline{\mathcal{M}}^D(\infty,\Omega) > 0$ implies that there exists a constant $C > 0$ such that for a sufficiently large T we have that $|{}_t\Omega| \geq C t^{N+D}$ for every $t > T$. Hence, for $D < s < -N$

we have the following:

$$
\begin{aligned}
\zeta_\infty(s,\Omega) = \int_{{}_T\Omega} |x|^{-s-N}\,\mathrm{d}x &= T^{-s-N}|{}_T\Omega| - (s+N)\int_T^{+\infty} t^{-s-N-1}|{}_t\Omega|\,\mathrm{d}t \\
&\geq -(s+N)\int_T^{+\infty} t^{-s-N-1}|{}_t\Omega|\,\mathrm{d}t \\
&\geq -(s+N)C\int_T^{+\infty} t^{-s-N-1+N+D}\,\mathrm{d}t \\
&= -(s+N)C\frac{T^{D-s}}{s-D} \to +\infty,
\end{aligned}
\tag{4.2.19}
$$

when $s \to D^+$, and this proves part (b). $\square$

Remark 4.25. In the special case when $\overline{\dim}_B(\infty,\Omega) = -N$ we have from the definition of the upper Minkowski content at infinity that

$$
\overline{\mathcal{M}}^{-N}(\infty,\Omega) = 0 \quad \text{and} \quad \zeta_\infty(-N,\Omega) = |{}_T\Omega|.
$$

This shows that the condition $\underline{\mathcal{M}}^D(\infty,\Omega) > 0$ from part (b) of Theorem 4.24 cannot be omitted in the general case.

Remark 4.26. Similarly as in the case of standard relative fractal drums (see [LapRaŽu1]), it is easy to see that Theorem 4.24 is still true if we replace the norm appearing in the definition of the distance zeta function at infinity with any other norm on $\mathbb{R}^N$.

Let us now revisit Propositions 4.7 and 4.9 from the previous section and compute the corresponding distance zeta functions at infinity.

Proposition 4.27. *Let $\Omega := \Omega(\alpha,\beta)$ be the set from Definition 4.6. Then, for $T := a_{j_0}$ large enough so that ${}_T\Omega$ is a countable union of disjoint intervals we have that*

$$
\zeta_\infty(s,\Omega;T) = \frac{1}{s}\sum_{j=j_0}^{\infty}\left(j^{-\alpha s} - (j^\alpha + j^{-\beta})^{-s}\right). \tag{4.2.20}
$$

Furthermore, we have that

$$
D(\zeta_\infty(\,\cdot\,,\Omega;T)) = \frac{1-(\alpha+\beta)}{\alpha} = \dim_B(\infty,\Omega)
$$

and $s = 0$ is a removable singularity of $\zeta_\infty(\,\cdot\,,\Omega;T)$.

Proof. For the distance zeta function of Ω at infinity we have:

$$
\zeta_\infty(s,\Omega;T) = \int_{{}_T\Omega} x^{-s-1}\,\mathrm{d}x = \sum_{j=j_0}^{\infty}\int_{a_j}^{b_j} x^{-s-1}\,\mathrm{d}x
$$

from which follows (4.2.20) after integrating. By setting $\sigma := \operatorname{Re} s$ and using the mean value theorem for integrals, we estimate

$$|\zeta_\infty(s,\Omega;T)| \leq \sum_{j=j_0}^{\infty} \int_{a_j}^{b_j} x^{-\sigma-1}\,\mathrm{d}x = \sum_{j=j_0}^{\infty} c_j^{-\sigma-1}(b_j - a_j)$$

for some $c_j \in (a_j, b_j)$ so that $c_j \asymp j^\alpha$ as $j \to +\infty$ which, in turn, implies that

$$\sum_{j=j_0}^{\infty} c_j^{-\sigma-1}(b_j - a_j) \asymp \sum_{j=j_0}^{\infty} j^{-\alpha(\sigma+1)} j^{-\beta}.$$

The right-hand side is convergent if and only if $\sigma > \frac{1-(\alpha+\beta)}{\alpha}$ from which we conclude by using (4.1.12) that $D(\zeta_\infty(\,\cdot\,,\Omega;T)) = \frac{1-(\alpha+\beta)}{\alpha} = \dim_B(\infty,\Omega)$, which is in accord with Theorem 4.24.

$\square$

Proposition 4.28. *Let $\Omega := \{(x,y) \in \mathbb{R}^2 \,:\, x > 1,\ 0 < y < x^{-\alpha}\}$ for $\alpha > 1$. Then for the distance zeta function of Ω at infinity calculated using the $|\cdot|_\infty$ norm on $\mathbb{R}^2$ we have*

$$\zeta_\infty(s,\Omega;1;|\cdot|_\infty) = \frac{1}{s+\alpha+1}.$$

It is meromorphic on $\mathbb{C}$ with a single simple pole at $s = -1-\alpha$. In particular, $\dim_B(\infty,\Omega) = -1-\alpha$.

Proof. Let us compute the distance zeta function of Ω at infinity:

$$\zeta_\infty(s,\Omega;1;|\cdot|_\infty) = \int_{{}_1\Omega} |(x,y)|_\infty^{-s-2}\,\mathrm{d}x\mathrm{d}y = \int_1^{+\infty} \mathrm{d}x \int_0^{x^{-\alpha}} x^{-s-2}\,\mathrm{d}y = \frac{1}{s+\alpha+1}.$$

The last equation holds if and only if $\operatorname{Re} s > -1-\alpha$. From this and (4.1.15), we conclude that $D(\zeta_\infty(\,\cdot\,,\Omega);|\cdot|_\infty) = -1-\alpha = \dim_B(\infty,\Omega)$ which is, of course, in accord with Theorem 4.24. Moreover, the distance zeta function $\zeta_\infty(\,\cdot\,,\Omega;|\cdot|_\infty)$ of Ω at infinity can be meromorphically extended to the whole complex plane with a single simple pole at $s = D$.

$\square$

Revisiting Proposition 4.16 will show that the conditions of Theorem 4.24 cannot be relaxed.

Proposition 4.29. *Let Ω be as in Proposition 4.16, i.e., a union of sets Ω_k for $k \geq 1$ (contained in the horizontal strip $\{1/2 \leq y \leq \log 2\}$) that are vertically translated images of the sets*

$$\widetilde{\Omega}_k = \{(x,y) \in \mathbb{R}^2 \,:\, x > 1,\ 0 < y < \frac{2^{-k}}{k} x^{-\alpha_k}\}$$

with $\alpha_k = 1+1/k$. *Then for the corresponding Lapidus zeta function at infinity calculated via the* $|\cdot|_\infty$*-norm on* $\mathbb{R}^2$ *we have*

$$\zeta_\infty(s,\Omega;|\cdot|_\infty) = \sum_{k=1}^{\infty} \frac{2^{-k}}{k(s+2+\frac{1}{k})}. \tag{4.2.21}$$

Furthermore, we also have that

$$D(\zeta_\infty(\,\cdot\,,\Omega;|\cdot|_\infty)) = D_{\mathrm{mer}}(\zeta_\infty(\,\cdot\,,\Omega;|\cdot|_\infty)) = \dim_B(\infty,\Omega) = -2 \tag{4.2.22}$$

and

$$\zeta_\infty(-2,\Omega;|\cdot|_\infty) = |\Omega| = 1. \tag{4.2.23}$$

Moreover, $\zeta_\infty(\,\cdot\,,\Omega;|\cdot|_\infty)$ *is holomorphic on the set*

$$\mathbb{C}\setminus(\{-2\}\cup\{-2-1/k\,:\,k\geq 1\}) \tag{4.2.24}$$

and $s=-2$ *is an accumulation point of its simple poles. Finally, for the residues of* $\zeta_\infty(\,\cdot\,,\Omega;|\cdot|_\infty)$ *we have that*

$$\operatorname{res}\left(\zeta_\infty\left(\,\cdot\,,\Omega;|\cdot|_\infty\right), -2-\frac{1}{k}\right) = \frac{2^{-k}}{k} \tag{4.2.25}$$

for every $k\geq 1$.

Proof. Let us calculate the distance zeta function at infinity using the $|\cdot|_\infty$ norm on $\mathbb{R}^N$. For $T=1>\log 2$ we have that $|(x,y)|_\infty = x$ for $(x,y)\in {}_1\Omega$ and consequently

$$\begin{aligned}
\zeta_\infty(s,\Omega;1;|\cdot|_\infty) &= \int_\Omega |(x,y)|_\infty^{-s-2}\,\mathrm{d}x\mathrm{d}y = \sum_{k=1}^{\infty}\int_{\Omega_k} |(x,y)|_\infty^{-s-2}\,\mathrm{d}x\mathrm{d}y \\
&= \sum_{k=1}^{\infty}\int_{\Omega_k} x^{-s-2}\,\mathrm{d}x\mathrm{d}y = \sum_{k=1}^{\infty}\int_{\widetilde{\Omega}_k} x^{-s-2}\,\mathrm{d}x\mathrm{d}y \\
&= \sum_{k=1}^{\infty}\int_1^{+\infty}\mathrm{d}x\int_0^{\frac{2^{-k}}{k}x^{-\alpha_k}} x^{-s-2}\,\mathrm{d}y = \sum_{k=1}^{\infty}\frac{2^{-k}}{k}\int_1^{+\infty} x^{-s-3-\frac{1}{k}}\,\mathrm{d}x \\
&= \sum_{k=1}^{\infty}\frac{2^{-k}}{k(s+2+\frac{1}{k})}.
\end{aligned}$$

The last equation above is valid if and only if $\operatorname{Re}s > -2-1/k$ for every $k\geq 1$. Furthermore, by using the Weierstrass M-test we have that the last sum appearing above defines a holomorphic function on $\mathbb{C}\setminus(\{-2\}\cup\{-2-1/k\,:\,k\geq 1\})$, which implies that $D(\zeta_\infty(\,\cdot\,,\Omega;|\cdot|_\infty)) = -2$. On the other hand, by direct computation we have that $\zeta_\infty(-2,\Omega;|\cdot|_\infty) = |\Omega| = 1$, but the zeta function cannot be even meromorphically extended to a neighborhood of $s = -2$. This follows from the fact that for $\operatorname{Re} s > -2$ we have that

$$\zeta_\infty(s, \Omega; |\cdot|_\infty) = \sum_{k=1}^{\infty} \frac{2^{-k}}{k} z_k(s),$$

where the functions z_k are meromorphic on $\mathbb{C}$ with simple poles at $s_k = -2 - 1/k$. Furthermore, the above sum converges uniformly on compact subsets of $\mathbb{C} \setminus \{s_k : k \geq 1\}$, i.e., it defines a holomorphic function on that set, but it has an accumulation of simple poles at $s = -2$, and by the principle of analytic continuation, the same is true for $\zeta_\infty(\,\cdot\,, \Omega; |\cdot|_\infty)$. In other words, $D(\zeta_\infty(\,\cdot\,, \Omega; |\cdot|_\infty)) = D_{\mathrm{mer}}(\zeta_\infty(\,\cdot\,, \Omega; |\cdot|_\infty)) = -2$ and this, in turn, is equal to $\dim_B(\infty, \Omega)$ according to (4.1.20). $\square$

Remark 4.30. Although Proposition 4.29 is stated in terms of the distance zeta function calculated via the $|\cdot|_\infty$-norm, Proposition 4.58 below will guarantee that the difference $\zeta_\infty(\,\cdot\,, \Omega; |\cdot|_\infty) - \zeta_\infty(\,\cdot\,, \Omega)$ is holomorphic at least on the half-plane $\{\operatorname{Re} s > -4\}$. From this we conclude that (4.2.22) is also true for $\zeta_\infty(\,\cdot\,, \Omega)$, $\zeta_\infty(-2, \Omega) = 1$, and $\zeta_\infty(\,\cdot\,, \Omega)$ is holomorphic (at least) on the set

$$\{\operatorname{Re} s > -4\} \setminus (\{-2\} \cup \{-2 - 1/k : k \geq 1\})$$

with $s = -2$ being an accumulation point of its simple poles. We also have that the residues of $\zeta_\infty(\,\cdot\,, \Omega)$ satisfy (4.2.25) for every $k \geq 1$.

4.3 Residues of Lapidus Zeta Functions at Infinity

In this section we will derive results which relate the the upper and lower Minkowski content of (∞, Ω) with the residue of the distance and tube zeta functions at infinity at $s = \dim_B(\infty, \Omega)$.

Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ and $|\Omega| < \infty$. Similarly as in the case of standard relative fractal drums and inspired by Corollary 4.23 we define the *tube zeta function* of Ω *at infinity* and denote it with $\widetilde{\zeta}_\infty(\,\cdot\,, \Omega)$:

$$\widetilde{\zeta}_\infty(s, \Omega; T) := \int_T^{+\infty} t^{-s-N-1} |{}_t\Omega| \,\mathrm{d}t, \tag{4.3.1}$$

where $T > 0$ is fixed. The next theorem will establish a connection between the tube zeta function and the Lapidus zeta function at infinity, from which the analyticity of the tube zeta function will follow.

Theorem 4.31. *Let $\Omega \subseteq \mathbb{R}^N$ with $|\Omega| < \infty$ and let $T > 0$ be fixed. Then for every $s \in \mathbb{C}$*

such that $\operatorname{Re} s > \overline{\dim}_B(\infty,\Omega)$ *it holds that*

$$\int_{{}_T\Omega} |x|^{-s-N}\,\mathrm{d}x = T^{-s-N}|{}_T\Omega| - (s+N)\int_T^{+\infty} t^{-s-N-1}|{}_t\Omega|\,\mathrm{d}t, \tag{4.3.2}$$

or, in short,

$$\zeta_\infty(s,\Omega;T) = T^{-s-N}|{}_T\Omega| - (s+N)\widetilde{\zeta}_\infty(s,\Omega;T). \tag{4.3.3}$$

Proof. Firstly, from Corollary 4.23 we have that (4.3.2) is valid for a real number s such that $s > \overline{\dim}_B(\infty,\Omega)$. To show that the equality holds in the half-plane $\{\operatorname{Re} s > \overline{\dim}_B(\infty,\Omega)\}$ it suffices to prove that both sides of Equation (4.31) are holomorphic functions on that domain.[7] The left-hand side of (4.31) is holomorphic on the set $\{\operatorname{Re} s > \overline{\dim}_B(\infty,\Omega)\}$ according to Theorem 4.24. Let us show that the same is valid for the right-hand side of (4.31), i.e., that it is valid for $\widetilde{\zeta}_\infty(s,\Omega;T)$. This is a Dirichlet type integral with $\varphi(t) = t^{-s}$ and $\mathrm{d}\mu(t) = t^{-N-1}|{}_t\Omega|\,\mathrm{d}t$, and according to Theorem 1.1 it is sufficient to show that the integral on the right hand side of (4.31) is convergent for $\operatorname{Re} s > \overline{\dim}_B(\infty,\Omega)$.

For $\overline{D} := \overline{\dim}_B(\infty,\Omega)$ and $s \in \mathbb{C}$ such that $\operatorname{Re} s > \overline{D}$, let us choose $\varepsilon > 0$ sufficiently small such that $\operatorname{Re} s > \overline{D} + \varepsilon$. Since $\overline{\mathcal{M}}_\infty^{\overline{D}+\varepsilon}(\Omega) = 0$, there exists a constant $C_T > 0$ such that $|{}_t\Omega| \le C_T t^{N+\overline{D}+\varepsilon}$ for every $t \in (T,+\infty)$. Now we have the following estimate:

$$\begin{aligned} |\widetilde{\zeta}_\infty(s,\Omega;T)| &\le \int_T^{+\infty} t^{-\operatorname{Re} s-N-1}|{}_t\Omega|\,\mathrm{d}t \le C_T\int_T^{+\infty} t^{-\operatorname{Re} s-N-1}t^{N+\overline{D}+\varepsilon}\,\mathrm{d}t \\ &= C_T\int_T^{+\infty} t^{\overline{D}+\varepsilon-\operatorname{Re} s-1}\,\mathrm{d}t = C_T\frac{T^{\overline{D}+\varepsilon-\operatorname{Re} s}}{\operatorname{Re} s-(\overline{D}+\varepsilon)} < +\infty. \end{aligned} \tag{4.3.4}$$

This completes the proof of the theorem. □

Theorem 4.32. *Let* $\Omega \subseteq \mathbb{R}^N$ *be such that* $|\Omega| < \infty$ *and* $\dim_B(\infty,\Omega) = D < -N$, $0 < \underline{\mathcal{M}}^D(\infty,\Omega) \le \overline{\mathcal{M}}^D(\infty,\Omega) < \infty$. *If* $\zeta_\infty(\,\cdot\,,\Omega)$ *has a meromorphic continuation to a neighborhood of* $s = D$*, then* D *is a simple pole and it holds that*

$$-(N+D)\underline{\mathcal{M}}^D(\infty,\Omega) \le \operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D) \le -(N+D)\overline{\mathcal{M}}^D(\infty,\Omega). \tag{4.3.5}$$

Moreover, if Ω *is Minkowski measurable at infinity, then we have*

$$\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D) = -(N+D)\mathcal{M}^D(\infty,\Omega). \tag{4.3.6}$$

Proof. Firstly, using the fact that $\underline{\mathcal{M}}^D(\infty,\Omega) > 0$ we can apply part (c) of Theorem 4.24 to get that $\zeta_\infty(s,\Omega) \to +\infty$ as $\mathbb{R} \ni s \to D^+$. In fact, by looking at the proof of part (c) of Theorem 4.24 we can see that $s = D$ is a singularity of $\zeta_\infty(\,\cdot\,,\Omega)$ that is at least a simple

[7]The equality follows from the fact that two holomorphic functions that coincide on a set that has an accumulation point in their common domain coincide then on the whole common domain.

pole. It remains to show that the order of this pole is not greater than one. Let us define

$$C_T := \sup_{t \geq T} \frac{|{}_t\Omega|}{t^{N+D}}.$$

From $\overline{\mathcal{M}}^D(\infty, \Omega) < +\infty$ we have that $C_T < +\infty$ for T large enough. Now, for $s \in \mathbb{R}$ such that $D < s < -N$ by using Theorem 4.31 we have

$$\begin{aligned}
\zeta_\infty(s, \Omega) &= T^{-s-N} |{}_T\Omega| - (s+N) \int_T^{+\infty} t^{-s-N-1} |{}_t\Omega| \, \mathrm{d}t \\
&\leq T^{-s-N} C_T T^{N+D} - (s+N) \int_T^{+\infty} t^{-s-N-1} C_T t^{N+D} \, \mathrm{d}t \\
&= C_T T^{D-s} - C_T (s+N) \int_T^{+\infty} t^{D-s-1} \, \mathrm{d}t \\
&= C_T T^{D-s} - C_T (s+N) \frac{T^{D-s}}{s-D} = -(N+D) C_T \frac{T^{D-s}}{s-D}.
\end{aligned} \tag{4.3.7}$$

This implies that $0 \leq \zeta_\infty(s, \Omega) \leq C_1 (s-D)^{-1}$ where $C_1 > 0$ is a constant independent of s and T and from this we conclude that $s = D$ is a pole of at most order one, i.e., it is a simple pole. To compute the residue at $s = D$ we observe that its value is independent of T because the difference $\zeta_\infty(s, \Omega; T_2) - \zeta_\infty(s, \Omega; T_1)$ is an entire function. Furthermore, from (4.3.7) we have

$$(s-D)\zeta_\infty(s, \Omega) \leq -(N+D) C_T T^{D-s}$$

and taking limits on both sides as $s \to D^+$ yields

$$\operatorname{res}(\zeta_\infty(\,\cdot\,, \Omega), D) \leq -(N+D) C_T.$$

Finally, by taking the limit as $T \to +\infty$ we get $\operatorname{res}(\zeta_\infty(\,\cdot\,, \Omega), D) \leq -(N+D)\overline{\mathcal{M}}^D(\infty, \Omega)$. The proof of the inequality involving the lower Minkowski content is completely analogous and this completes the proof. $\square$

The next theorem is a consequence and analog of Theorem 4.32 concerning the tube zeta function of Ω at infinity, its residue at $s = \dim_B(\infty, \Omega)$ and the upper and lower Minkowski contents of Ω at infinity.

Theorem 4.33. *Let $\Omega \subseteq \mathbb{R}^N$ be such that $|\Omega| < \infty$, $\dim_B(\infty, \Omega) = D < -N$ and $0 < \underline{\mathcal{M}}^D(\infty, \Omega) \leq \overline{\mathcal{M}}^D(\infty, \Omega) < \infty$. If $\widetilde{\zeta}_\infty(\,\cdot\,, \Omega)$ has a meromorphic continuation to a neighborhood of $s = D$, then D is a simple pole and it holds that*

$$\underline{\mathcal{M}}^D(\infty, \Omega) \leq \operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,, \Omega), D) \leq \overline{\mathcal{M}}^D(\infty, \Omega). \tag{4.3.8}$$

Moreover, if Ω is Minkowski measurable at infinity, then we have

$$\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),D)=\mathcal{M}^D(\infty,\Omega). \tag{4.3.9}$$

Proof. Using the fact that $\zeta_\infty(s,\Omega)=T^{-s-N}|_T\Omega|-(s+N)\widetilde{\zeta}_\infty(s,\Omega)$ for every $s\in\mathbb{C}$ such that $\operatorname{Re}s>D$ (proved in Theorem 4.31) and Theorem 4.32, we immediately have

$$\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D)=\lim_{s\to D}(s-D)\left[T^{-s-N}|_T\Omega|-(s+N)\widetilde{\zeta}_\infty(s,\Omega)\right],$$

i.e.,

$$\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D)=-(N+D)\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),D).$$

□

In light of Theorem 4.21 we deduce the following conclusion in the case when (∞,Ω) is Minkowski measurable at infinity and its zeta function has a meromorphic continuation to a neighborhood of $D=\dim_B(\infty,\Omega)$.

Theorem 4.34. *Let $\Omega\subseteq\mathbb{R}^N$ be such that $|\Omega|<\infty$ and $\dim_B(\infty,\Omega)=D<-N$ such that it is Minkowski measurable at infinity and assume that $\zeta_\infty(\,\cdot\,,\Omega)$ has a meromorphic continuation to a neighborhood of $s=D$. Furthermore, assume also that the inverted relative fractal drum $(\mathbf{0},\Phi(\Omega))$ is Minkowski measurable. Then, we have*:

$$\mathcal{M}^D(\mathbf{0},\Phi(\Omega))=-\frac{N+D}{N-D}\mathcal{M}^D(\infty,\Omega). \tag{4.3.10}$$

Proof. Since, for a fixed $T>1$ from Theorem 4.21 we have the equality

$$\zeta_\infty(s,\Omega;T)=\zeta_{\mathbf{0}}(s,\Phi(\Omega);1/T), \tag{4.3.11}$$

it is obvious that the relative distance zeta function of $(\mathbf{0},\Phi(\Omega))$ satisfies the analog of Theorem 4.32 for relative fractal drums (see Theorem 2.9) and we have that

$$\dim_B(\mathbf{0},\Phi(\Omega))=D(\zeta_{\mathbf{0}}(\,\cdot\,,\Phi(\Omega)))=D(\zeta_\infty(\,\cdot\,,\Omega))=\dim_B(\infty,\Omega)=D.$$

Furthermore, D is a simple pole and its residue is independent of T which together with Theorem 4.32 yields

$$\begin{aligned}(N-D)\mathcal{M}^D(\mathbf{0},\Phi(\Omega))&=\operatorname{res}(\zeta_{\mathbf{0}}(\,\cdot\,,\Phi(\Omega)),D)\\&=\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D)=-(N+D)\mathcal{M}^D(\infty,\Omega)\end{aligned} \tag{4.3.12}$$

□

Remark 4.35. In the above theorem we have to assume that the inverted relative fractal drum $(\mathbf{0},\Phi(\Omega))$ is also Minkowski measurable, since in general, the equation (4.3.11) only implies that $D(\zeta_{\mathbf{0}}(\,\cdot\,,\Psi(\Omega))) = D(\zeta_\infty(\,\cdot\,,\Omega))$ from which we can only conclude that $\overline{\dim}_B(\infty,\Omega) = \overline{\dim}_B(\mathbf{0},\Phi(\Omega))$.

4.4 Meromorphic Extensions of Lapidus Zeta Functions at Infinity and Complex Dimensions

In this section we will give sufficient conditions on the Lebesgue measurable set $\Omega \subseteq \mathbb{R}^N$ of finite Lebesgue measure which will ensure that the Lapidus zeta function of Ω at infinity has a meromorphic continuation to a neighborhood of its critical line. Firstly, we will state and prove the theorems in terms of the tube zeta function at infinity and then, by using the functional equation between the Lapidus and the tube zeta function at infinity (see Theorem 4.31), we will get the statements in terms of the Lapidus zeta function at infinity. We will also define the complex dimensions of Ω at infinity much in the same way as it was done in Definition 3.6 for relative fractal drums.

Furthermore, we will give a sufficient condition for a relative fractal drum (∞,Ω) to be Minkowski measurable at infinity in terms of its distance or tube zeta function at infinity.

Theorem 4.36. *Let $\Omega \subseteq \mathbb{R}^N$ be a Lebesgue measurable set of finite Lebesgue measure such that there exist $\alpha > 0$, $\mathcal{M} \in (0,+\infty)$ and $D < -N$ satisfying*

$$|{}_t\Omega| = t^{N+D}(\mathcal{M} + O(t^{-\alpha})) \quad \text{as } t \to +\infty. \tag{4.4.1}$$

Then, $\dim_B(\infty,\Omega)$ exists and $\dim_B(\infty,\Omega) = D$. Furthermore, Ω is Minkowski measurable at infinity with Minkowski content $\mathcal{M}^D(\infty,\Omega) = \mathcal{M}$. Moreover, the tube zeta function $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$ has for abscissa of convergence $D(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)) = \dim_B(\infty,\Omega) = D$ and possesses a unique meromorphic continuation (still denoted by $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$) to (at least) the open half-plane $\{\operatorname{Re} s > D - \alpha\}$; that is,

$$D_{\text{mer}}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)) \le D - \alpha.$$

The only pole of $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$ in this half-plane is $s = D$; it is simple, and

$$\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega), D) = \mathcal{M}.$$

Proof. For a fixed $T > 0$ we have

$$
\begin{aligned}
\widetilde{\zeta}_\infty(s,\Omega) &= \int_T^{+\infty} t^{-s-N-1}|{}_t\Omega|\,\mathrm{d}t = \int_T^{+\infty} t^{-s-N-1}t^{N+D}(\mathcal{M}+O(t^{-\alpha}))\,\mathrm{d}t \\
&= \mathcal{M}\int_T^{+\infty} t^{D-s-1}\,\mathrm{d}t + \int_T^{+\infty} t^{-s}O(t^{D-\alpha-1})\,\mathrm{d}t \\
&= \underbrace{\frac{\mathcal{M}T^{D-s}}{s-D}}_{\zeta_1(s)} + \underbrace{\int_T^{+\infty} t^{-s}O(t^{D-\alpha-1})\,\mathrm{d}t}_{\zeta_2(s)}
\end{aligned}
$$

provided $\operatorname{Re} s > D$. The function ζ_1 is meromorphic in the entire complex plane and $D(\zeta_1) = D$. Furthermore, for the function ζ_2 we have that

$$
|\zeta_2(s)| \leq K\int_T^{+\infty} t^{D-\operatorname{Re} s-\alpha-1}\,\mathrm{d}t < \infty
$$

for $\operatorname{Re} s > D-\alpha$ and K a positive constant. Therefore, $D(\zeta_2) \leq D-\alpha < D = D(\zeta_1)$ and the claim of the theorem now follows from Lemma 1.6 with $a_1 = -\infty$ in the notation of that lemma. $\square$

Remark 4.37. We point out that, in general, if there exists a holomorphic extension of $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$ to an open domain $U \subseteq \mathbb{C}$ which is symmetric with respect to the real axis, then any isolated singularities of $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$ in $\overline{U}$ come in complex conjugate pairs. Namely, it is clear that for real s, the values of $\widetilde{\zeta}_\infty(s,\Omega)$ are also real. Furthermore, by using *the principle of reflection* (see, e.g., [Ti1, p. 155]), we deduce that for all complex numbers s such that $\operatorname{Re} s > \overline{\dim}_B(\infty,\Omega)$, we have $\overline{\widetilde{\zeta}_\infty(s,\Omega)} = \widetilde{\zeta}_\infty(\overline{s},\Omega)$. Naturally, this identity remains valid upon holomorphic continuation (in whichever domain $U \subseteq \mathbb{C}$ the tube zeta function $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$ can holomorphically extended). An analogous comment can be made about the distance zeta function at infinity and all of the other fractal zeta functions appearing in this thesis.

We would like to show that Minkowski measurability of Ω at infinity can be characterized by its tube (or distance) zeta function at infinity similarly as it was done in Chapter 3 for relative fractal drums. One direction of this result will be again a consequence of the Wiener–Pitt Tauberian theorem (see Theorem 3.41). The other direction is partially resolved by Theorem 4.36, where we have the additional assumption on the asymptotics of the tube formula of the set Ω. For the general case we will need to express the tube formula of Ω in terms of its tube or distance zeta function at infinity. This will be done in a future work by using the inverse Mellin transform applied to the tube zeta function of Ω at infinity, i.e., by the technique of Chapter 3.

We can now state and prove the announced sufficient condition for Minkowski measurability at infinity.

Theorem 4.38 (Sufficient condition for Minkowski measurability at infinity)**.** *Let Ω be a subset of $\mathbb{R}^N$ of finite Lebesgue measure such that and let $\overline{\dim}_B(\infty,\Omega) = \overline{D} < -N$. Furthermore, assume that the relative tube zeta function $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$ of (∞,Ω) can be meromorphically extended to a neighborhood U of the critical line $\{\operatorname{Re} s = \overline{D}\}$. Let $\overline{D}$ be its only pole in U and assume that it is simple. Then $D := \dim_B(\infty,\Omega)$ exists, $D = \overline{D}$ and (∞,Ω) is Minkowski measurable with*

$$\mathcal{M}^D(\infty,\Omega) = \operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega), D). \tag{4.4.2}$$

Furthermore, the theorem is also valid if we replace the relative tube zeta function with the relative distance zeta function $\zeta_\infty(\,\cdot\,,\Omega)$ of (∞,Ω) and in that case we have

$$\mathcal{M}^D(\infty,\Omega) = \frac{\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega), D)}{-(N+D)}. \tag{4.4.3}$$

Proof. We start with the tube zeta function $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega;T)$ (choosing $T=1$ without loss of generality) and change the variable of integration by $v = \log t$.

$$\begin{aligned}\widetilde{\zeta}_\infty(s+\overline{D},\Omega) &= \int_1^{+\infty} t^{-s-D-1-N}|_t\Omega|\,\mathrm{d}t\\ &= \int_0^{+\infty} \mathrm{e}^{-sv}\mathrm{e}^{-v(\overline{D}+N)}|_{\mathrm{e}^v}\Omega|\,\mathrm{d}v\\ &= \{\mathfrak{L}\sigma\}(s).\end{aligned} \tag{4.4.4}$$

where $\sigma(v) := \mathrm{e}^{-v(\overline{D}+N)}|_{\mathrm{e}^v}\Omega|$. Furthermore, from the definition of the tube zeta function of Ω at infinity it is clear that its residue at $s=\overline{D}$ is real and positive. Since $s=\overline{D}$ is the only pole of $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$ in U, we conclude that

$$G(s) := \widetilde{\zeta}_\infty(s+\overline{D},\Omega) - \frac{\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),\overline{D})}{s} \tag{4.4.5}$$

is holomorphic on the neighborhood $\widetilde{U}$ of the critical line $\{\operatorname{Re} s \geq 0\}$ so that we can apply Theorem 3.41 in its stronger form; that is, for arbitrary large $\lambda > 0$ (in the notation of Theorem 3.41 so that (3.5.4) is valid) and conclude that

$$\sigma_h(u) = \frac{1}{h}\int_u^{u+h} \sigma(v)\,\mathrm{d}v \to \operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),\overline{D}) \quad \text{as} \quad u \to +\infty, \tag{4.4.6}$$

for every $h>0$. In particular, since $v\mapsto|_{\mathrm{e}^v}\Omega|$ is nonincreasing, we have that

$$\begin{aligned}\frac{1}{h}\int_u^{u+h}\mathrm{e}^{-v(\overline{D}+N)}|_{\mathrm{e}^v}\Omega|\,\mathrm{d}v&\le\frac{|_{\mathrm{e}^u}\Omega|}{h}\int_u^{u+h}\mathrm{e}^{-v(\overline{D}+N)}\,\mathrm{d}v\\&=\frac{|_{\mathrm{e}^u}\Omega|}{\mathrm{e}^{u(\overline{D}+N)}}\,\frac{\mathrm{e}^{-h(\overline{D}+N)}-1}{-(\overline{D}+N)h}\end{aligned}\tag{4.4.7}$$

and by taking the lower limit of both sides as $u\to+\infty$ we get

$$\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),\overline{D})\le\underline{\mathcal{M}}^{\overline{D}}(\infty,\Omega)\frac{\mathrm{e}^{-h(\overline{D}+N)}-1}{-(\overline{D}+N)h}.\tag{4.4.8}$$

Since this is true for every $h>0$, by letting $h\to0^+$ we get that

$$\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),\overline{D})\le\underline{\mathcal{M}}^{\overline{D}}(\infty,\Omega).\tag{4.4.9}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{h}\int_u^{u+h}\mathrm{e}^{-v(\overline{D}+N)}|_{\mathrm{e}^v}\Omega|\,\mathrm{d}v&\ge\frac{|_{\mathrm{e}^{u+h}}\Omega|}{h}\int_u^{u+h}\mathrm{e}^{-v(\overline{D}+N)}\,\mathrm{d}v\\&=\frac{|_{\mathrm{e}^{u+h}}\Omega|}{\mathrm{e}^{(u+h)(\overline{D}+N)}}\,\frac{1-\mathrm{e}^{h(\overline{D}+N)}}{-(\overline{D}+N)h}\end{aligned}\tag{4.4.10}$$

and, similarly as before, by taking the upper limit of both sides as $u\to+\infty$ we get

$$\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),\overline{D})\ge\overline{\mathcal{M}}^{\overline{D}}(\infty,\Omega)\frac{1-\mathrm{e}^{h(\overline{D}+N)}}{-(\overline{D}+N)h}.\tag{4.4.11}$$

Finally, since this is true for every $h>0$, we let $h\to0^+$ and conclude that

$$\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),\overline{D})\ge\overline{\mathcal{M}}^{\overline{D}}(\infty,\Omega).\tag{4.4.12}$$

From this, together with (4.4.9), we have that Ω is $\overline{D}$-Minkowski measurable at infinity and, a fortiori, that $\dim_B(\infty,\Omega)=D=\overline{D}$. Furthermore, $\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),D)=\mathcal{M}^D(\infty,\Omega)$. The part of the theorem dealing with the distance zeta function at infinity follows now from Theorem 4.31 and the relation $\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D)=-(N+D)\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),D)$. $\square$

Remark 4.39. The analog of Remark 3.43 is also valid in the context of fractal sets at infinity; that is, the assumptions of Theorem 4.38 can be weakened. More precisely, it suffices to assume that

$$\widetilde{\zeta}_\infty(s,\Omega)-\frac{\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),\overline{D})}{s-\overline{D}}\tag{4.4.13}$$

converges to a boundary function $G(\operatorname{Im} s)$ as $\operatorname{Re} s \to \overline{D}^+$ such that

$$\int_{-\lambda}^{\lambda} |G(\tau)|\,\mathrm{d}\tau < \infty \tag{4.4.14}$$

for every $\lambda > 0$.

Similarly as in the case of relative fractal drums in Section 3.5; the case when, besides $\overline{D}$, there are other singularities on the critical line $\{\operatorname{Re} s = \overline{D}\}$ of the relative fractal drum (∞, Ω), Theorem 3.41 can be used to derive an upper bound for the upper $\overline{D}$-Minkowski content of (∞, Ω). This is stated precisely in the next theorem.

Theorem 4.40 (Bound for the upper Minkowski content at infinity)**.** *Let Ω be a subset of $\mathbb{R}^N$ of finite Lebesgue measure and let $\overline{D} := \overline{\dim}_B(\infty, \Omega) < -N$. Furthermore, assume that the relative tube zeta function $\widetilde{\zeta}_\infty(\,\cdot\,, \Omega)$ of (∞, Ω) can be meromorphically extended to a neighborhood U of the critical line $\{\operatorname{Re} s = \overline{D}\}$ and that $\overline{D}$ is its simple pole. Assume also that $\{\operatorname{Re} s = \overline{D}\}$ contains another pole of $\widetilde{\zeta}_\infty(\,\cdot\,, \Omega)$ different from $\overline{D}$. Furthermore, let*

$$\lambda_{(\infty,\Omega)} := \inf \left\{ |\overline{D} - \omega| \,:\, \omega \in \dim_{PC}(\infty, \Omega) \setminus \{\overline{D}\} \right\} \tag{4.4.15}$$

Then, we have the following bound for the upper $\overline{D}$-dimensional Minkowski content of (∞, Ω) :

$$\overline{\mathcal{M}}^{\overline{D}}(\infty, \Omega) \le \frac{-(N + \overline{D}) C \lambda_{(\infty,\Omega)}}{2\pi \left(1 - \mathrm{e}^{2\pi(N+\overline{D})/\lambda_{(\infty,\Omega)}}\right)} \operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,, \Omega), \overline{D}), \tag{4.4.16}$$

where C is a positive constant such that $C < 3$. Furthermore, we have that the residue of the relative distance zeta function of (∞, Ω) satisfies the following:

$$\overline{\mathcal{M}}^{\overline{D}}(\infty, \Omega) \le \frac{C \lambda_{(\infty,\Omega)}}{2\pi \left(1 - \mathrm{e}^{2\pi(N+\overline{D})/\lambda_{(\infty,\Omega)}}\right)} \operatorname{res}(\zeta_\infty(\,\cdot\,, \Omega), \overline{D}). \tag{4.4.17}$$

Proof. We use the same reasoning as in the proof of Theorem 4.38 with the only difference being in the fact that now we can only use the weaker statement (3.5.3) of Theorem 3.41 since we have another pole on the critical line $\{\operatorname{Re} s = \overline{D}\}$. More precisely, if $\lambda < \lambda_{(\infty,\Omega)}$, then for every $h \ge 2\pi/\lambda$ by using (4.4.10) and (3.5.3) we have

$$C \operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,, \Omega), \overline{D}) \ge \overline{\mathcal{M}}^{\overline{D}}(A, \Omega) \frac{1 - \mathrm{e}^{h(N+\overline{D})}}{-(N + \overline{D})h}, \tag{4.4.18}$$

where C is a positive constant such that $C < 3$. Since the right-hand side above is decreasing in h, we get the best estimate for $h = 2\pi/\lambda$. Moreover, since this is true for every $\lambda < \lambda_{(\infty,\Omega)}$, we get (4.4.16) by letting $\lambda \to \lambda_{(\infty,\Omega)}^-$. Finally, the statement about the relative distance zeta function follows by the same argument as in Theorem 4.38. $\square$

We note that a completely analogous comment can be made about weakening the hypothesis of Theorem 4.40 as it was made in Remark 3.45 for the case of relative fractal drums.

Let us introduce the notion of complex dimensions at infinity which is already introduced for the case of bounded sets and relative fractal drums in [LapRaŽu1] and recalled in Chapter 3.

Definition 4.41 (The screen and the window)**.** For a Lebesgue measurable subset Ω of $\mathbb{R}^N$ such that $|\Omega| < \infty$, we define the *window* as the closed subset

$$W = \{s \in \mathbb{C} : \operatorname{Re} s \geq S(\operatorname{Im} s)\} \tag{4.4.19}$$

associated to the function $S : \mathbb{R} \to (-\infty, \overline{\dim}_B(\infty, \Omega)]$ which is assumed to be Lipschitz continuous. The *screen* $\boldsymbol{S}$ is then defined as the graph of $S(t)$, with the horizontal and vertical axes interchanged:

$$\boldsymbol{S} := \{S(t) + \mathbb{i}t \,:\, t \in \mathbb{R}\}. \tag{4.4.20}$$

We will denote the Lipschitz constant by $\|S\|_{\mathrm{Lip}}$, i.e.,

$$|S(x) - S(y)| \leq \|S\|_{\mathrm{Lip}}|x - y|, \qquad \text{for all } x, y, \in \mathbb{R}.$$

Furthermore, if we additionally assume that $S(t)$ is bounded from below, the following two finite quantities will be associated with the screen $\boldsymbol{S}$:

$$\inf S := \inf_{t\in\mathbb{R}} S(t) \qquad \text{and} \qquad \sup S := \sup_{t\in\mathbb{R}} S(t). \tag{4.4.21}$$

Note that the window W contains the critical line $\{\operatorname{Re} s = D(\zeta_\infty(\,\cdot\,, \Omega))\}$; and, in fact, it also contains the closed half-plane $\{\operatorname{Re} s \geq D(\zeta_\infty(\,\cdot\,, \Omega))\}$.

Similarly as in [Lap–vFr3, Sections 1.2.1 and 5.1], we will introduce the next definition.

Definition 4.42 (Admissible set at infinity)**.** We will call the set Ω of finite N-dimensional Lebesgue measure *admissible* if Ω has the property that $\zeta_\infty(\,\cdot\,, \Omega)$ can be meromorphically extended to an open and connected neighborhood $G \subseteq \mathbb{C}$ of the *window* W associated to some screen $\boldsymbol{S}$. In other words, Ω is such that its distance zeta function at infinity can be extended meromorphically to an open domain G containing the closed half-plane $\{\operatorname{Re} s \geq D(\zeta_A)\}$.[8]

There exist nonadmissible sets at infinity. One example is the set Ω constructed in Theorem 4.64 below. Furthermore, as Theorems 4.38 and 4.40 indicate, of great interest

[8]As it is usual, we will still denote by $\zeta_\infty(\,\cdot\,, \Omega)$ the meromorphic continuation of $\zeta_\infty(\,\cdot\,, \Omega)$ to G, which is necessarily unique due to the principle of analytic continuation. Furthermore, as in [Lap–vFr3], we will assume that the screen does not contain any poles of $\zeta_\infty(\,\cdot\,, \Omega)$.

are the poles of $\zeta_\infty(\,\cdot\,,\Omega)$ of an admissible set Ω that are located on the critical line $\{\operatorname{Re} s = D(\zeta_\infty(\,\cdot\,,\Omega))\}$.

Definition 4.43. Let $\Omega \subseteq \mathbb{R}^N$ be of finite N-dimensional Lebesgue measure and admissible. We define the *set of principal complex dimensions of* Ω *at infinity* as the set of poles ω of $\zeta_\infty(\,\cdot\,,\Omega)$ with real part equal to $D(\zeta_\infty(\,\cdot\,,\Omega))$ and denote it with

$$\dim_{PC}(\infty,\Omega) := \{\omega \in \{\operatorname{Re} s = D(\zeta_\infty(\,\cdot\,,\Omega))\} : \omega \text{ is a pole of } \zeta_\infty(\,\cdot\,,\Omega)\}. \tag{4.4.22}$$

Furthermore, we will also denote this set by $\mathcal{P}_c(\zeta_\infty(\,\cdot\,,\Omega))$.

Remark 4.44. We point out that if Ω is admissible, we have that $\overline{\dim}_B(\infty,\Omega) \in \dim_{PC}(\infty,\Omega)$ but we do not know if an admissible Ω has equal upper and lower box dimensions at infinity (see Problem A.5). On the other hand, Theorem 4.38 implies that this is the case if $\dim_{PC}(\infty,\Omega) = \{\overline{\dim}_B(\infty,\Omega)\}$; that is, if the only pole of $\zeta_\infty(\,\cdot\,,\Omega)$ located at the critical line is $\omega = \overline{\dim}_B(\infty,\Omega)$.

Definition 4.45. Let $\Omega \subseteq \mathbb{R}^N$ be of finite N-dimensional Lebesgue measure and admissible for some window W. We define the *set of visible complex dimensions of* (∞,Ω) *through the window* W as the set of poles of the distance zeta function $\zeta_\infty(\,\cdot\,,\Omega)$ that are contained in W and denote it by

$$\mathcal{P}(\zeta_\infty(\,\cdot\,,\Omega),W) := \{\omega \in W : \omega \text{ is a pole of } \zeta_\infty(\,\cdot\,,\Omega)\} \tag{4.4.23}$$

which we will abbreviate to $\mathcal{P}(\zeta_\infty(\,\cdot\,,\Omega))$ when there is no ambiguity concerning the choice of the window W (or when $W = \mathbb{C}$).

Furthermore, if $\zeta_\infty(\,\cdot\,,\Omega)$ possesses a meromorphic continuation to the whole of $\mathbb{C}$, we will call the set $\mathcal{P}(\zeta_\infty(\,\cdot\,,\Omega),\mathbb{C})$ the *set of* (*all*) *complex dimensions of* (∞,Ω).

Remark 4.46. Note that, while the set $\mathcal{P}(\zeta_\infty(\,\cdot\,,\Omega),W)$ obviously depends on the choice of the window W, this is not the case with the set $\dim_{PC}(\infty,\Omega)$ since, by definition, W contains the critical line $\{\operatorname{Re} s = D(\zeta_\infty(\,\cdot\,,\Omega))\}$. Moreover, we have that $\dim_{PC}(\infty,\Omega) \subseteq \mathcal{P}(\zeta_\infty(\,\cdot\,,\Omega),W)$ for every window W.

We will also use the notation $\mathcal{P}(\zeta_\infty(\,\cdot\,,\Omega),U)$ in the case when the set U is not necessarily a window but only a subset of $\mathbb{C}$; that is, when we want to denote only the poles of $\zeta_\infty(\,\cdot\,,\Omega))$ that are contained in U.

Remark 4.47. In light of the functional equation between the distance and tube zeta functions of Ω at infinity stated in Theorem 4.31, it is clear that the Definitions 4.43 and 4.45 can be stated with the distance zeta function $\zeta_\infty(\,\cdot\,,\Omega)$ at infinity interchanged with the tube zeta function $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$ at infinity. Moreover, we have that

$$\dim_{PC}(\infty,\Omega) = \mathcal{P}_c(\zeta_\infty(\,\cdot\,,\Omega)) = \mathcal{P}_c(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)) \tag{4.4.24}$$

and

$$\mathcal{P}(\zeta_\infty(\,\cdot\,,\Omega),W)=\mathcal{P}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),W). \tag{4.4.25}$$

We will now state a version of Theorem 4.36 dealing with a relative fractal drum (∞,Ω) that is not Minkowski measurable, but its tube function satisfies a log-periodic asymptotic formula. The theorem will demonstrate how the relative tube zeta function of (∞,Ω) detects its 'inner geometric oscillations' in terms of the principal complex dimensions of (∞,Ω). Let us first introduce some new notation and state a lemma which will be needed for the proof of the theorem. For a periodic function $G\colon\mathbb{R}\to\mathbb{R}$ with minimal period $T>0$, we define

$$G_0(\tau):=\chi_{[0,T]}(\tau)G(\tau) \tag{4.4.26}$$

where χ_A is the characteristic function of a set A. Furthermore we denote the Fourier transform of G with $\{\mathfrak{F}G\}$ or $\hat{G}$, i.e.,

$$\{\mathfrak{F}G\}(s)=\hat{G}(s):=\int_{-\infty}^{+\infty}\mathrm{e}^{-2\pi\mathbbm{i}s\tau}G(\tau)\,\mathrm{d}\tau. \tag{4.4.27}$$

The following lemma together with its proof is cited from [LapRaŽu1].

Lemma 4.48. *Let $P>0$, $F:(P,+\infty)\to\mathbb{R}$ a continuous function, and assume that $G:\mathbb{R}\to\mathbb{R}$ is a T-periodic function, for some $T>0$. If $F(t)=G(\log t)+o(1)$ as $t\to+\infty$, then G is continuous.*

Proof. Since G is periodic, it is enough to show that G is continuous on $(0,+\infty)$. Let us assume that this is not the case, i.e., let suppose that G is not continuous at some $\tau_0>0$. Then, by periodicity, for every $k\geq 1$, we have that G is not continuous at $\tau_k=kT+\tau_0$. Furthermore, recall that *the oscillation of G at a point $x\in\mathbb{R}$* is given by

$$\operatorname*{osc}_x G:=\lim_{\varepsilon\to 0^+}\left(\sup_{(x-\varepsilon,x+\varepsilon)}G-\inf_{(x-\varepsilon,x+\varepsilon)}G\right).$$

By letting $t_k=\mathrm{e}^{\tau_k}$, we have that $\operatorname{osc}_{t_k}G(\log t)=\operatorname{osc}_{\tau_k}G=c>0$, where c does not depend on k. (Here and in the sequel, we choose k sufficiently large so that $t_k\in(P,+\infty)$.) Since $t_k\to+\infty$ as $k\to\infty$, we may take k large enough and fixed, such that $|o(1)|\leq c/2$ for $t=t_k$, where $o(1)$ is the function of t given in the statement of the lemma. (In particular, $o(1)\to 0$ as $t\to+\infty$.) Therefore,[9]

$$\begin{aligned}\operatorname*{osc}_{t_k}F&=\operatorname*{osc}_{t_k}(G(\log t)+o(1))\\&\geq\operatorname*{osc}_{t_k}(G(\log t)-\operatorname*{osc}_{t_k}|o(1)|\geq c-\frac{1}{2}c=\frac{1}{2}c>0.\end{aligned}$$

[9]We use the fact that $\sup(f+g)\geq\sup f-\sup(-g)=\sup f+\inf g$ and $\inf(f+g)\leq\inf f-\inf(-g)=\inf f+\sup g$ for every pair of functions $f,g\colon\mathbb{R}\to\mathbb{R}$.

On the other hand, since F is continuous on $(P,+\infty)$, we must have $\operatorname{osc}_{t_k} F = 0$, which is a contradiction. Hence, G must be continuous everywhere. $\square$

Theorem 4.49. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ such that there exist $D < -N$, $\alpha > 0$, and let $G\colon \mathbb{R} \to (0,+\infty)$ be a nonconstant periodic function with period $T > 0$, satisfying*

$$|{}_t\Omega| = t^{N+D}(G(\log t) + O(t^{-\alpha})) \quad \text{as } t \to +\infty. \tag{4.4.28}$$

Then G is continuous, $\dim_B(\infty,\Omega)$ exists and $\dim_B(\infty,\Omega) = D$. Furthermore, Ω is Minkowski nondegenerate at infinity with upper and lower Minkowski contents at infinity respectively given by

$$\underline{\mathcal{M}}^D(\infty,\Omega) = \min G, \quad \overline{\mathcal{M}}^D(\infty,\Omega) = \max G. \tag{4.4.29}$$

(Hence, the range of $G|_{[0,T]}$ is equal to the compact interval $[\underline{\mathcal{M}}^D(\infty,\Omega), \overline{\mathcal{M}}^D(\infty,\Omega)]$.) Moreover, the tube zeta function $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$ has for abscissa of convergence $D(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)) = D$ and possesses a unique meromorphic extension (still denoted by $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$) to (at least) the half-plane $\{\operatorname{Re} s > D - \alpha\}$; that is,

$$D_{\rm mer}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)) \le D - \alpha.$$

In addition, the set of all the poles of $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$ located in this half-plane is given by

$$\mathcal{P}_\alpha(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)) = \left\{ s_k = D + \frac{2\pi}{T}\mathbb{i}k \;:\; \hat{G}_0\left(\frac{k}{T}\right) \neq 0,\ k \in \mathbb{Z} \right\}; \tag{4.4.30}$$

they are all simple, and the residue at each $s_k \in \mathcal{P}_\alpha(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega))$, $k \in \mathbb{Z}$, is given by

$$\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega), s_k) = \frac{1}{T}\hat{G}_0\left(\frac{k}{T}\right). \tag{4.4.31}$$

If $s_k \in \mathcal{P}_\alpha(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega))$, then $s_{-k} \in \mathcal{P}_\alpha(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega))$ by the reality principle and

$$|\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega), s_k)| \le \frac{1}{T}\int_0^T G(\tau)\,\mathrm{d}\tau, \quad \lim_{k\to\pm\infty} \operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega), s_k) = 0. \tag{4.4.32}$$

Moreover, the set of poles $\mathcal{P}_\alpha(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega))$ contains $s_0 = D$, and

$$\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega), D) = \frac{1}{T}\int_0^T G(\tau)\,\mathrm{d}\tau. \tag{4.4.33}$$

In particular, Ω is not Minkowski measurable at infinity and

$$\underline{\mathcal{M}}^D(\infty,\Omega) < \operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),D) < \overline{\mathcal{M}}^D(\infty,\Omega) < \infty. \tag{4.4.34}$$

If, in addition, $G \in C^m(\mathbb{R})$ (i.e., G is m times continuously differentiable on $\mathbb{R}$) for some integer $m \geq 1$, and G has an extremum t_0 such that

$$G'(t_0) = G''(t_0) = \ldots = G^{(m)}(t_0) = 0, \tag{4.4.35}$$

then there exists $C_m > 0$ such that for all $k \in \mathbb{Z}$ and $s_k \in \mathcal{P}_\alpha(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega))$ we have

$$|\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),s_k)| \leq C_m|k|^{-m}. \tag{4.4.36}$$

Proof. The fact that G is continuous follows from Lemma 4.48 by applying it to $F(t) := |{}_t\Omega|t^{N+D}$ which is defined and continuous for $t > 0$. We have that

$$\begin{aligned}\widetilde{\zeta}_\infty(s,\Omega) &= \int_P^{+\infty} t^{-s-N-1}|{}_t\Omega|\,\mathrm{d}t = \int_P^{+\infty} t^{-s-N-1}t^{N+D}(G(\log t) + O(t^{-\alpha}))\,\mathrm{d}t \\ &= \underbrace{\int_P^{+\infty} t^{D-s-1}G(\log t)\,\mathrm{d}t}_{\zeta_1(s)} + \underbrace{\int_P^{+\infty} t^{-s}O(t^{D-\alpha-1})\,\mathrm{d}t}_{\zeta_2(s)}\end{aligned}$$

for some fixed $P > 0$. As in the proof of Theorem 4.49 we have that $D(\zeta_2) = D - \alpha$ and it suffices to prove that ζ_1 can be meromorphically extended to the whole complex plane. This will be shown by finding a closed form for ζ_1. Since G is T-periodic, we have that

$$\zeta_1(s) = \int_P^{+\infty} t^{D-s-1}G(\log t + T)\,\mathrm{d}t.$$

Let us introduce a new variable u such that $\log u = \log t + T$, i.e., $u = \mathrm{e}^T t$ and we have

$$\begin{aligned}\zeta_1(s) &= \int_{\mathrm{e}^T P}^{+\infty} \mathrm{e}^{-T(D-s-1)}u^{D-s-1}G(\log u)\mathrm{e}^{-T}\,\mathrm{d}u \\ &= \mathrm{e}^{-T(D-s)}\int_{\mathrm{e}^T P}^{+\infty} u^{D-s-1}G(\log u)\,\mathrm{d}u \\ &= \mathrm{e}^{-T(D-s)}\left(\int_P^{+\infty} u^{D-s-1}G(\log u)\,\mathrm{d}u + \int_{\mathrm{e}^T P}^{P} u^{D-s-1}G(\log u)\,\mathrm{d}u\right) \\ &= \mathrm{e}^{-T(D-s)}\left(\zeta_1(s) + \int_{\mathrm{e}^T P}^{P} u^{D-s-1}G(\log u)\,\mathrm{d}u\right)\end{aligned}$$

which gives us a closed form for ζ_1:

$$\begin{aligned}\zeta_1(s) &= \frac{\mathrm{e}^{-T(D-s)}}{\mathrm{e}^{-T(D-s)}-1}\int_P^{\mathrm{e}^T P} t^{D-s-1}G(\log t)\,\mathrm{d}t\\ &= \frac{\mathrm{e}^{T(s-D)}}{\mathrm{e}^{T(s-D)}-1}\underbrace{\int_{\log P}^{\log P+T}\mathrm{e}^{-\tau(s-D)}G(\tau)\,\mathrm{d}\tau,}_{I(s)}\end{aligned}$$

where in the last equality we have introduced a new variable τ such that $\tau = \log t$. The integral $I(s)$ is obviously an entire function since $P \neq 0, +\infty$.[10] From this we conclude that ζ_1 is meromorphic on $\mathbb{C}$ and the set of its poles is equal to the set of solutions s_k of $\exp(T(s-D)) = 1$ for which $I(s_k) \neq 0$. If $I(s_k) = 0$ then s_k is a removable singularity of ζ_1:

$$\lim_{s\to s_k}\zeta_1(s) = \lim_{s\to s_k}\frac{s-s_k}{\mathrm{e}^{T(s-D)}-1}\,\mathrm{e}^{T(s-D)}\,\frac{I(s)}{s-s_k} = \frac{1}{P}I'(s_k)$$

where I' denotes the derivative of I. Furthermore, since $\exp(T(s_k-D)) = 1$ we have that $\exp(-\tau(s_k-D)) = \exp(-2\pi\mathbb{i}k\tau/T)$ and

$$\begin{aligned}I(s_k) &= \int_{\log P}^{\log P+T}\mathrm{e}^{\frac{-2\pi\mathbb{i}k}{T}\tau}G(\tau)\,\mathrm{d}\tau\\ &= \int_0^T \mathrm{e}^{\frac{-2\pi\mathbb{i}k}{T}\tau}G(\tau)\,\mathrm{d}\tau = \hat{G}_0\left(\frac{k}{T}\right),\end{aligned} \tag{4.4.37}$$

where we have used the fact that both $\tau \mapsto G(\tau)$ and $\tau \mapsto \exp(-2\pi\mathbb{i}k\tau/T)$ are T-periodic. This proves that the description of the poles of the tube zeta function of Ω at infinity that are contained in $\{\mathrm{Re}\, s > D-\alpha\}$ is given by (4.4.30). Moreover, we observe that this set contains D since

$$I(D) = I(s_0) = \int_0^T G(\tau)\,\mathrm{d}\tau > 0. \tag{4.4.38}$$

Indeed, we have that the range of $G|_{[0,T]}$ is equal to the interval $[\underline{\mathcal{M}}^D(\infty,\Omega), \overline{\mathcal{M}}^D(\infty,\Omega)]$ and since G is nonconstant, we deduce from (4.4.28) that $0 < \underline{\mathcal{M}}^D(\infty,\Omega) < \overline{\mathcal{M}}^D(\infty,\Omega) < \infty$. From this we conclude that $D(\zeta_1) = D > D-\alpha = D(\zeta_2)$ and from Lemma 1.6 (with $a_1 = -\infty$ in the notation of that lemma) we conclude that $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$ possesses a (unique) meromorphic extension to (at least) the half-plane $\{\mathrm{Re}\, s > D-\alpha\}$, i.e., $D_{\mathrm{mer}}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)) \leq D-\alpha$.

Let us now compute the residues of $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$ at $s_k = D + \frac{2\pi\mathbb{i}k}{T}$ for an arbitrary $k \in \mathbb{Z}$,

[10]See Theorem 1.1.

using (4.4.37) and L'Hospital's rule:

$$\begin{aligned}\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),s_k) &= \operatorname{res}(\zeta_1,s_k)\\ &= \lim_{s\to s_k}\frac{s-s_k}{\mathrm{e}^{T(s-D)}-1}\,\mathrm{e}^{T(s-D)}I(s)=\frac{1}{T}\hat{G}_0\left(\frac{k}{T}\right).\end{aligned}\tag{4.4.39}$$

Substituting $k=0$ in the above expression we get (4.4.33) which, in turn, implies the inequalities in (4.4.34).

Furthermore, as it is well-known, since $G_0\in L^1(\mathbb{R})$, we have $|\hat{G}_0(\tau)|\le\|G_0\|_{L^1(\mathbb{R})}=\|G\|_{L^1(0,T)}$ and $\lim_{t\to\infty}\hat{G}_0(t)=0$ (by the Riemann–Lebesgue lemma; see, e.g., [Ru] or [MiŽu, p. 101]), so that (4.4.32) follows immediately from (4.4.39).

If the function G is of class C^m, this does not imply that G_0 is of the same class. However, we can define $G_1:\mathbb{R}\to\mathbb{R}$ by

$$G_1(\tau)=\begin{cases}G(\tau)-\underline{\mathcal{M}}^D(\infty,\Omega) & \text{if } \tau\in[0,T],\\ 0 & \text{if } \tau\notin[0,T].\end{cases}\tag{4.4.40}$$

Since the value of $\underline{\mathcal{M}}^D(\infty,\Omega)$ is in the range of G, we may assume without loss of generality that $t_0=0$ is a minimum of G; namely, $G(0)=G(T)=\underline{\mathcal{M}}^D(\infty,\Omega)$. If that is not the case, we can translate the graph of G in the horizontal direction in order to achieve this. Furthermore, $\underline{\mathcal{M}}^D(\infty,\Omega)$ is equal to the minimal value of G; hence, $G_1(0)=G_1(T)=0$. This implies that G_1 is continuous on $\mathbb{R}$, and moreover, from (4.4.35), we have that G_1 has the same regularity as G; that is, $G_1\in C^m(\mathbb{R})$. A direct computation shows that for each $t\in\mathbb{R}$,

$$\hat{G}_1(t)=\hat{G}_0(t)-\underline{\mathcal{M}}^D(\infty,\Omega)\frac{1-\mathrm{e}^{-2\pi\mathbb{i}t\cdot T}}{2\pi\mathbb{i}t},\tag{4.4.41}$$

from which it follows that

$$\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),s_k)=\frac{1}{T}\hat{G}_0\left(\frac{k}{T}\right)=\frac{1}{T}\hat{G}_1\left(\frac{k}{T}\right).\tag{4.4.42}$$

Since $G_1\in C^m(\mathbb{R})$, by a standard result from Fourier analysis (see e.g. [MiŽu, p. 103]) we know that there exists a constant $C_m>0$ such that $|\hat{G}_1(t)|\le C_m t^{-m}$ for all $t\in\mathbb{R}$, which proves (4.4.36). We observe that the same conclusion can be achieved by defining $G_1(\tau)=G(\tau)-\overline{\mathcal{M}}^D(\infty,\Omega)$. $\square$

4.5 Properties of Fractal Zeta Functions at Infinity

In this section we will prove some useful properties of the distance and tube zeta functions at infinity. We will start the section with a useful lemma from which we will derive the scaling property of fractal zeta functions at infinity. Recall that for a parameter

$\lambda > 0$ and a subset Ω of $\mathbb{R}^N$ we define

$$\lambda\Omega := \{\lambda x \,:\, x \in \Omega\}. \tag{4.5.1}$$

Lemma 4.50. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ of finite Lebesgue measure. For any $\lambda > 0$ and $t > 0$ we have*:

$$|B_t(0)^c \cap \lambda\Omega)| = \lambda^N |B_{t/\lambda}(0)^c \cap \Omega| \tag{4.5.2}$$

and

$$\overline{\mathcal{M}}^r(\infty, \lambda\Omega) = \lambda^{-r}\overline{\mathcal{M}}^r(\infty, \Omega), \quad \underline{\mathcal{M}}^r(\infty, \lambda\Omega) = \lambda^{-r}\underline{\mathcal{M}}^r(\infty, \Omega), \tag{4.5.3}$$

for every real number r.

Proof. We have that $\lambda(B_{t/\lambda}(0)^c \cap \Omega) = B_t(0)^c \cap \lambda\Omega$ from which the first part of the lemma follows directly. For the second part, we observe that

$$\overline{\mathcal{M}}^r(\infty, \lambda\Omega) = \limsup_{t\to+\infty} \frac{|B_t(0)^c \cap \lambda\Omega|}{t^{N+r}} = \limsup_{t\to+\infty} \frac{\lambda^{-r}|B_{t/\lambda}(0)^c \cap \Omega|}{(t/\lambda)^{N+r}} = \lambda^{-r}\overline{\mathcal{M}}^r(\infty, \Omega)$$

and similarly for the lower limit which concludes the proof of the lemma. □

The next result is a scaling property of the distance zeta function at infinity which will prove useful in examples, and in the construction of quasiperiodic sets at infinity.

Proposition 4.51 (Scaling property of the distance zeta function at infinity)**.** *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ with finite Lebesgue measure, $T > 0$ and $\lambda > 0$. Then we have $D(\zeta_\infty(\,\cdot\,, \lambda\Omega)) = D(\zeta_\infty(\,\cdot\,, \Omega)) = \overline{\dim}_B(\infty, \Omega)$ and*

$$\zeta_\infty(s, \lambda\Omega; \lambda T) = \lambda^{-s}\zeta_\infty(s, \Omega; T) \tag{4.5.4}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \overline{\dim}_B(\infty, \Omega)$. Furthermore, if ω is a simple pole of a meromorphic extension of $\zeta_\infty(\,\cdot\,, \Omega)$ to some open connected neighborhood of the critical line $\{\operatorname{Re} s = \overline{\dim}_B(\infty, \Omega)\}$, then

$$\operatorname{res}(\zeta_\infty(\,\cdot\,, \lambda\Omega), \omega) = \lambda^{-\omega} \operatorname{res}(\zeta_\infty(\,\cdot\,, \Omega), \omega). \tag{4.5.5}$$

Proof. From Lemma 4.50 we know that $\overline{\dim}_B(\infty, \lambda\Omega) = \overline{\dim}_B(\infty, \Omega)$. We will prove the scaling formula (4.5.4) by introducing a new variable $y = x/\lambda$ and using the change of variables formula for the Lebesgue integral. Noting that $\mathrm{d}x = \lambda^N \,\mathrm{d}y$, we have

$$\begin{aligned}\zeta_\infty(s, \lambda\Omega; \lambda T) &= \int_{B_{\lambda T}(0)\cap\lambda\Omega} |x|^{-s-N}\,\mathrm{d}x \\ &= \int_{B_T(0)\cap\Omega} |\lambda y|^{-s-N}\lambda^N\,\mathrm{d}y \\ &= \lambda^{-s}\int_{B_T(0)\cap\Omega} |y|^{-s-N}\,\mathrm{d}y = \lambda^{-s}\zeta_\infty(s, \Omega; T)\end{aligned} \tag{4.5.6}$$

for $s \in \mathbb{C}$ such that $\operatorname{Re} s > \overline{\dim}_B(\infty, \Omega)$.

Note that by the principle of analytic continuation, if one of the two zeta functions in (4.5.4) has a meromorphic extension to some open connected neighborhood U of the critical line, then so does the other and (4.5.4) is still valid for $s \in U$. Furthermore, if that is the case and $\omega \in U$ is a simple pole of $\zeta_\infty(\,\cdot\,, \Omega)$, then we have that

$$\lambda^{-s}(s-\omega)\zeta_\infty(s, \Omega; T) = (s-\omega)\zeta_\infty(s, \lambda\Omega; \lambda T)$$

holds on a pointed neighborhood of ω. Finally, since the value of the residue of the distance zeta function at infinity does not depend on T we get (4.5.5) by letting $s \to \omega$, $s \neq \omega$. $\square$

We will now prove a result which will be very useful for almost all examples that we will look at. Namely, as before, for a set Ω in $\mathbb{R}^N$ with finite Lebesgue measure it will be easier to calculate (as we have already done several times) a closed form for the corresponding distance zeta function at infinity by using the infinity norm in $\mathbb{R}^N$ instead of the usual Euclidean norm. From Theorem 4.31 we know that these two zeta functions have the same abscissa of convergence. Theorem 4.58 below will give us a stronger result for a special case of (∞, Ω), but let us first state a complex version of the mean value theorem which we will be using in our proof.

Theorem 4.52 (Complex mean value Theorem [EvJa, Theorem 2.2])**.** *Let f be a holomorphic function defined on an open convex subset U_f of $\mathbb{C}$. Furthermore let a and b be two distinct points in U_f. Then there exist $s_1, s_2 \in (a, b)$ such that*[11]

$$\operatorname{Re}(f'(s_1)) = \operatorname{Re}\left(\frac{f(b)-f(a)}{b-a}\right) \tag{4.5.7}$$

and

$$\operatorname{Im}(f'(s_2)) = \operatorname{Im}\left(\frac{f(b)-f(a)}{b-a}\right). \tag{4.5.8}$$

We also state here a simple and useful corollary of the above theorem.

[11] Here, (a, b) denotes the open interval in $\mathbb{C}$ in the usual sense; that is, all the complex numbers that lie on the straight line between a and b with the endpoints excluded.

Corollary 4.53. Let f be a holomorphic function defined on an open convex subset U_f of $\mathbb{C}$. Furthermore let a and b be two distinct points in U_f.

$$|f(b)-f(a)| \leq \sqrt{2}|b-a| \max_{s\in[a,b]} |f'(s)|. \tag{4.5.9}$$

Proof. From Theorem 4.52, we have that there are $s_1, s_2 \in (a,b)$ such that[12]

$$\begin{aligned}\left|\frac{f(b)-f(a)}{b-a}\right|^2 &= |\operatorname{Re}(f'(s_1))|^2 + |\operatorname{Im}(f'(s_2))|^2\\ &\leq |f'(s_1)|^2 + |f'(s_2)|^2\\ &\leq 2\max_{s\in[a,b]}|f'(s)|^2.\end{aligned}$$

Taking the square root of both sides and multiplying by $|b-a|$ completes the proof of the corollary. □

Definition 4.54. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two (necessarily equivalent) norms on $\mathbb{R}^N$ and let $\Omega \subseteq \mathbb{R}^N$. We will say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent of order* $\alpha \in \mathbb{R}$ *for* (∞, Ω) if

$$\|x\|_1 = \|x\|_2 + O\left(\|x\|_1^\alpha\right), \quad \text{as} \quad \|x\|_1 \to +\infty,\ x\in\Omega. \tag{4.5.10}$$

In this case we will write

$$\|\cdot\|_1 \underset{(\infty,\Omega)}{\overset{\alpha}{\sim}} \|\cdot\|_2. \tag{4.5.11}$$

This equivalence is well defined since the two norms are equivalent in the standard sense. More precisely, since there exist $m, M > 0$ such that $m\|\cdot\|_1 \leq \|\cdot\|_2 \leq M\|\cdot\|_1$ we have that $O\left(\|x\|_1^\alpha\right) = O\left(\|x\|_2^\alpha\right)$ for every $\alpha\in\mathbb{R}$ when $\|x\|_1 \to +\infty$ or $\|x\|_2 \to +\infty$. From this, one gets symmetry and transitivity easily.

Theorem 4.55. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ with finite Lebesgue measure and assume $\overline{D} := \overline{\dim}_B(\infty,\Omega) < -N$. Furthermore, assume that $\|\cdot\|$ is a norm in $\mathbb{R}^N$ such that for some $\alpha \in (-\infty, 1)$ we have*

$$|x| \underset{(\infty,\Omega)}{\overset{\alpha}{\sim}} \|x\|. \tag{4.5.12}$$

Then, the difference

$$\zeta_\infty(\,\cdot\,,\Omega) - \zeta_\infty(\,\cdot\,,\Omega;\|\cdot\|) \tag{4.5.13}$$

is holomorphic on (*at least*) *the half-plane*

$$\{\operatorname{Re} s > (\overline{D} - (1-\alpha))\}. \tag{4.5.14}$$

[12]Here, we also use the obvious inequalities $|\operatorname{Re} s|, |\operatorname{Im} s| \leq |s|$ for $s\in\mathbb{C}$.

Proof. We observe that for every $s \in \mathbb{C}$ the function $f_s(z) := z^{-s-N}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and define $F(s,x) := f_s(|x|) - f_s(\|x\|)$. Then, from Corollary 4.53 applied to f_s, we conclude that there exists a function $r\colon \mathbb{C} \times (\Omega \setminus \{0\}) \to (0,+\infty)$ such that

$$|F(s,x)| = \big||x|^{-s-N} - \|x\|^{-s-N}\big| \leq \sqrt{2}|s+N| r(s,x)^{-\operatorname{Re} s - N - 1}\big||x| - \|x\|\big|. \tag{4.5.15}$$

Let m and M be the positive constants such that $m|x| \leq \|x\| \leq M|x|$ for $x \in \mathbb{R}^N$ and denote

$$C_m := \min\{1, m\}, \quad C_M := \max\{1, M\}. \tag{4.5.16}$$

Furthermore, since

$$|x| < r(s,x) < \|x\| \quad \text{or} \quad \|x\| < r(s,x) < |x| \tag{4.5.17}$$

we have that

$$C_m|x| \leq r(s,x) \leq C_M|x|, \tag{4.5.18}$$

which implies that

$$r(s,x)^{-\operatorname{Re} s - N - 1} \leq |x|^{-\operatorname{Re} s - N - 1} \max\{C_m^{-\operatorname{Re} s - N - 1}, C_M^{-\operatorname{Re} s - N - 1}\}. \tag{4.5.19}$$

Furthermore, by taking $T > 1$ sufficiently large, we can assume that $\big||x| - \|x\|\big| \leq c|x|^\alpha$ which together with (4.5.19) and (4.5.15) yields

$$|F(s,x)| \leq c\sqrt{2}|s+N| \max\{C_m^{-\operatorname{Re} s - N - 1}, C_M^{-\operatorname{Re} s - N - 1}\}|x|^{-\operatorname{Re} s - N - 1 + \alpha}. \tag{4.5.20}$$

Suppose now that K is a compact subset in $\{\operatorname{Re} s > \overline{D} - (1-\alpha)\}$, and let

$$C_K := \max_{s \in K} \big\{c\sqrt{2}|s+N| \max\{C_m^{-\operatorname{Re} s - N - 1}, C_M^{-\operatorname{Re} s - N - 1}\}\big\} \tag{4.5.21}$$

and define the function g_K as follows:

$$g_K(x) := C_K|x|^{-(\min\{\operatorname{Re} s\,:\, s \in K\} - \alpha + 1) - N} \tag{4.5.22}$$

so that we have $|F(s,x)_{|K}| \leq g_K(x)$ for $x \in {}_T\Omega$. We observe that g_K is in $L^1({}_T\Omega)$, since if $s \in K$, then $\operatorname{Re} s > \overline{D} - (1-\alpha)$ so that

$$\min\{\operatorname{Re} s \,:\, s \in K\} - \alpha + 1 > \overline{D} - (1-\alpha) - \alpha + 1 = \overline{D},$$

which, in turn, implies that

$$\int_{{}_T\Omega} g_K(x)\,\mathrm{d}x = C_K\,\zeta_\infty(\min\{\operatorname{Re} s \,:\, s \in K\} - \alpha + 1; \Omega) < \infty.$$

Finally, we conclude that $F(s,x)$ satisfies the hypotheses of Theorem 1.2 (see also Remark 1.3) and therefore

$$\int_{{}_T\Omega} F(s,x)\,\mathrm{d}x = \zeta_\infty(\,\cdot\,,\Omega) - \zeta_\infty(\,\cdot\,,\Omega;\|\cdot\|)$$

is holomorphic on $\{\operatorname{Re} s > \overline{D} - (1-\alpha)\}$ which completes the proof of the theorem. $\square$

Corollary 4.56. *Let Ω be a measurable subset in $\mathbb{R}^N$ with $|\Omega| < \infty$ such that $\dim_B(\infty,\Omega) = D$ exists. Furthermore, assume that the distance zeta function of Ω at infinity can be meromorphically extended to an open connected neighborhood U of the closed half-plane $\{\operatorname{Re} s \geq D\}$. Let $\|\,.\,\|$ be another norm in $\mathbb{R}^N$ such that $|x| \underset{(\infty,\Omega)}{\overset{\alpha}{\sim}} \|x\|$ for some $\alpha \in (-\infty, 1)$. Then $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega;\|\,.\,\|)$ can be meromorphically extended to (at least) $V := U \cap \{\operatorname{Re} s > D - (1-\alpha)\}$. Furthermore, the sets of poles in V of the two zeta functions, together with their multiplicities, coincide. Moreover, the principal parts of the Laurent expansion around each pole in V also coincide. In particular, if ω is a simple pole in V, then*

$$\operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega),\omega) = \operatorname{res}(\widetilde{\zeta}_\infty(\,\cdot\,,\Omega;\|\,.\,\|),\omega). \tag{4.5.23}$$

Proof. Since, by hypothesis, $\widetilde{\zeta}_\infty(\,\cdot\,,\Omega)$ is meromorphic on $V = U \cap \{\operatorname{Re} s > D - (1-\alpha)\}$ the corollary follows directly from Theorem 4.55 which states that the difference of these two distance zeta functions is holomorphic on V. $\square$

Remark 4.57. It is clear that the above corollary is still valid if we interchange the roles of the two distance zeta functions.

An important special case of the above theorem, which we will be using in almost all examples considered, is when the set $\Omega \subseteq \mathbb{R}^N$ is contained in a cylinder of finite radius.

Proposition 4.58. *Let $\Omega \subseteq \mathbb{R}^N$ with $|\Omega| < \infty$ be such that it is contained in a cylinder*

$$x_2^2 + x_3^2 + \cdots + x_N^2 \leq C \tag{4.5.24}$$

for some constant $C > 0$ where $x = (x_1,\ldots,x_N)$. Furthermore, let $\overline{D} := \overline{\dim}_B(\infty,\Omega)$ and $T > 0$. Then

$$\zeta_\infty(s,\Omega;T) - \int_{{}_T\Omega} |x|_\infty^{-s-N}\,\mathrm{d}x \tag{4.5.25}$$

is holomorphic on (at least) the half-plane $\{\operatorname{Re} s > \overline{D} - 2\}$.

Furthermore, if any of the two distance zeta functions possesses a meromorphic extension to some open connected neighborhood U of the critical line $\{\operatorname{Re} s = \overline{D}\}$, then the other one possesses a meromorphic extension to (at least) $V := U \cap \{\operatorname{Re} s > \overline{D} - 2\}$.

Moreover, their multisets of poles in $U \cap \{\operatorname{Re} s > \overline{D} - 2\}$ coincide.

Proof. We observe that for $T > 0$ sufficiently large we have

$$|x| - |x|_\infty = |x| - |x_1| = \frac{\sum_{i=2}^N x_i^2}{|x| + |x_1|} \leq C|x|^{-1}, \quad x \in {}_T\Omega.$$

In other words $|x| \underset{(\infty,\Omega)}{\overset{-1}{\sim}} \|x\|$ and the conclusion now follows by applying Theorem 4.55. $\square$

4.6 Quasiperiodic Sets at Infinity

In this section we will construct quasiperiodic subsets of $\mathbb{R}^2$ with prescribed box dimension at infinity. We will start by defining a two parameter set $\Omega_\infty^{(a,b)}$ in Definition 4.59 which will be our building block for the construction of a *maximally hyperfractal set at infinity*; that is, according to the terminology of [LapRaŽu1], a set Ω with its distance zeta function at infinity having the critical line $\{\operatorname{Re} s = \overline{\dim}_B(\infty, \Omega)\}$ as a natural boundary. This construction will also give examples of algebraically and transcendentally quasiperiodic sets at infinity by using some classical results from transcendental number theory.

One of the open problems in [LapRaŽu1] was the question of existence of algebraically quasiperiodic bounded sets and relative fractal drums. The results of this section give a positive answer in the case of relative fractal drums of type (∞, Ω). Furthermore, a similar construction can be performed in the context of standard relative fractal drums. More precisely, one can take the inverted relative fractal drum $(\mathbf{0}, \Phi(\Omega))$ where Ω is the quasiperiodic set at infinity constructed here. The distance zeta functions of (∞, Ω) and $(\mathbf{0}, \Phi(\Omega))$ are essentially the same by Theorem 4.21. On the other hand, one should check directly the condition of quasiperiodicity of the relative fractal drum $(\mathbf{0}, \Phi(\Omega))$ since we do not have a direct way of doing it via the geometric inversion. The reason for this is in the fact that we would need an asymptotic formula which relates the tube function $t \mapsto |B_t(0)^c \cap \Omega|$ of Ω at infinity and the relative tube function $t \mapsto |B_{1/t}(0) \cap \Phi(\Omega)|$ as $t \to +\infty$. We do not know if such formula can be derived in the general case (see Problem A.6), but we still conjecture that $(\mathbf{0}, \Phi(\Omega))$ will be a quasiperiodic relative fractal drum with the same quasiperiods as (∞, Ω).

Another idea to construct an algebraically quasiperiodic relative fractal drum (A, Ω) is to use the geometric inversion in one coordinate; that is, $\Phi_1(x, y) := (1/x, y)$ and apply it to the quasiperiodic relative fractal drum (∞, Ω) constructed here. We leave this, as well as other properties of the 'partial geometric inversion' for future work (see Problem A.11).

Definition 4.59. For $a \in (0, 1/2)$ and $b \in (1 + \log_{1/a} 2, +\infty)$ we define a *two parameter unbounded set* denoted by $\Omega_\infty^{(a,b)}$. We start with the countable family of sets

$$\Omega_m^{(a,b)} := \{(x, y) \in \mathbb{R}^2 \,:\, x > a^{-m},\ 0 < y < x^{-b}\}, \quad m \geq 1.$$

Now, we will construct the set $\Omega_\infty^{(a,b)}$ by "stacking" the translated images of the sets $\Omega_m^{(a,b)}$

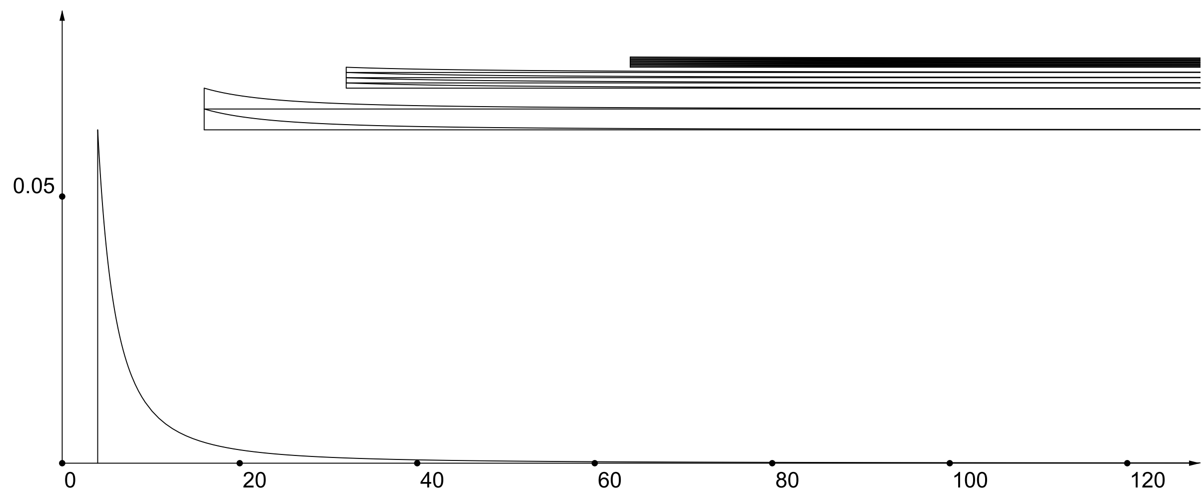

Figure 4.2: An example of the two parameter set $\Omega_\infty^{(a,b)}$ from Definition 4.59. Here, $a = 1/4$ and $b = 2$. Note that the axes are not in the same scale and only the first four steps of the set $\Omega_\infty^{(1/4,2)}$ are shown; that is, for $m = 1, 2, 3, 4$.

along the y-axis on top of each other. More precisely, for each $m \geq 1$ we take 2^{m-1} copies of $\Omega_m^{(a,b)}$ and arrange all of these sets by vertical translations so that they are pairwise disjoint and lie in the strip $\{0 \leq y \leq S\}$. Here, S is the sum of widths of all of these sets, i.e.,[13]

$$S = \sum_{m=1}^{\infty} 2^{m-1} \cdot (a^{-m})^{-b} = \frac{a^b}{1 - 2a^b}.$$

Moreover, without loss of generality, we can arrange them in an "increasing fashion", i.e., stacking them from bottom to top as m increases (see Figure 4.2). Finally, we define $\Omega_\infty^{(a,b)}$ as the disjoint union of all of these sets.

Remark 4.60. The condition $b > 1 + \log_{1/a} 2$ ensures that $\Omega_\infty^{(a,b)}$ has finite Lebesgue measure:[14]

$$\begin{aligned} |\Omega_\infty^{(a,b)}| &= \sum_{m=1}^{\infty} 2^{m-1} |\Omega_m^{(a,b)}| = \sum_{m=1}^{\infty} 2^{m-1} \int_{a^{-m}}^{+\infty} x^{-b}\,\mathrm{d}x \\ &= \frac{1}{b-1} \sum_{m=1}^{\infty} 2^{m-1} (a^{-m})^{1-b} = \frac{1}{2(b-1)} \sum_{m=1}^{\infty} (2a^{b-1})^m \\ &= \frac{a^{b-1}}{(b-1)(1-2a^{b-1})}. \end{aligned} \tag{4.6.1}$$

Proposition 4.61. *The distance zeta function of the two parameter unbounded set $\Omega_\infty^{(a,b)}$ calculated via the $|\cdot|_\infty$-norm on $\mathbb{R}^2$ is given by*

$$\zeta_\infty(s, \Omega_\infty^{(a,b)}; |\cdot|_\infty) = \frac{1}{s+b+1} \cdot \frac{1}{a^{-(s+b+1)} - 2}. \tag{4.6.2}$$

[13] For $b > 1 + \log_{1/a} 2 > \log_{1/a} 2$ we have that $2a^b < 1$ and the sum is convergent.

[14] For $b > 1 + \log_{1/a} 2$ we have that $2a^{b-1} < 1$ and the last sum in (4.6.1) is convergent.

It is meromorphic on $\mathbb{C}$ where the set of complex dimensions of $\Omega_\infty^{(a,b)}$ at infinity visible through $W := \{\operatorname{Re} s > \log_{1/a} - b - 3\}$ is given by

$$\{-(b+1)\} \cup \left(\log_{1/a} 2 - (b+1) + \frac{2\pi}{\log(1/a)}\mathbb{i}\mathbb{Z}\right). \tag{4.6.3}$$

Furthermore, we also have that

$$\overline{\dim}_B(\infty, \Omega_\infty^{(a,b)}) = \log_{1/a} 2 - (b+1). \tag{4.6.4}$$

Proof. Let us choose $T = 1$ and calculate:

$$\begin{aligned}
\zeta_\infty(s, \Omega_\infty^{(a,b)}; 1; |\cdot|_\infty) &= \int_{\Omega_\infty^{(a,b)}} |(x,y)|_\infty^{-s-2}\,\mathrm{d}x\mathrm{d}y = \sum_{m=1}^{\infty} 2^{m-1} \int_{\Omega_m^{(a,b)}} x^{-s-2}\,\mathrm{d}x\mathrm{d}y \\
&= \sum_{m=1}^{\infty} 2^{m-1} \int_{a^{-m}}^{+\infty} \mathrm{d}x \int_0^{x^{-b}} x^{-s-2}\,\mathrm{d}y \\
&= \sum_{m=1}^{\infty} 2^{m-1} \int_{a^{-m}}^{+\infty} x^{-s-2-b}\,\mathrm{d}x \\
[\text{for } \operatorname{Re} s > -(b+1)] &= \frac{1}{2(s+b+1)} \sum_{m=1}^{\infty} (2a^{s+b+1})^m \\
[\text{for } \operatorname{Re} s > \log_{1/a} 2 - (b+1)] &= \frac{1}{s+b+1} \cdot \frac{1}{a^{-(s+b+1)} - 2}.
\end{aligned}$$

From this we see that $D(\zeta_\infty(\,\cdot\,, \Omega_\infty^{(a,b)}); |\cdot|_\infty)) = \log_{1/a} 2 - (b+1)$ and the zeta function has a (unique) meromorphic extension to all of $\mathbb{C}$ defined by

$$\zeta_\infty(s, \Omega_\infty^{(a,b)}; |\cdot|_\infty) = \frac{1}{s+b+1} \cdot \frac{1}{a^{-(s+b+1)} - 2}.$$

Furthermore, we have that

$$\overline{\dim}_B(\infty, \Omega_\infty^{(a,b)}) = \log_{1/a} 2 - (b+1).$$

Since $(\infty, \Omega_\infty^{(a,b)})$ is contained in a strip of finite width, we can apply Proposition 4.58 to conclude that the difference $\zeta_\infty(\,\cdot\,, \Omega_\infty^{(a,b)}; |\cdot|_\infty) - \zeta_\infty(\,\cdot\,, \Omega_\infty^{(a,b)})$ is holomorphic on the half-plane $\{\operatorname{Re} s > \log_{1/a} 2 - (b+1) - 2\} = \{\operatorname{Re} s > \log_{1/a} 2 - b - 3\}$ from which we conclude that the complex dimensions of $(\infty, \Omega_\infty^{(a,b)})$ visible through the window $W = \{\operatorname{Re} s > \log_{1/a} 2 - b - 3\}$ are given by

$$\{-(b+1)\} \cup \left\{\log_{1/a} 2 - (b+1) + \frac{2\pi k \mathbb{i}}{\log(1/a)} \,:\, k \in \mathbb{Z}\right\}.$$

$\square$

Remark 4.62. As we can see, the *oscillatory period* of $\Omega_\infty^{(a,b)}$ is equal to $\mathbf{p}(a) = 2\pi/\log(1/a)$. Note that $\mathbf{p}(a) \to 0$ as $a \to 0^+$.

Example 4.63. We will compute the box dimension of $\Omega_\infty^{(a,b)}$ at infinity directly. For the calculation we will measure the neighborhoods of infinity in the $|\cdot|_\infty$ norm. As $\Omega_\infty^{(a,b)}$ is contained in a horizontal strip of finite width, according to Lemma 4.13, this will not affect the value of the Minkowski content of $\Omega_\infty^{(a,b)}$ at infinity. Now, for $t > 1/a$ we have

$$\begin{aligned}|K_t(0)^c \cap \Omega_\infty^{(a,b)}| &= \sum_{n=1}^{\lfloor \log_{1/a} t\rfloor} 2^{n-1}\int_t^{+\infty} x^{-b}\,\mathrm{d}x + \sum_{n>\lfloor \log_{1/a} t\rfloor}^{\infty} 2^{n-1}\int_{a^{-n}}^{+\infty} x^{-b}\,\mathrm{d}x \\ &= \frac{1}{b-1}\left[t^{1-b}\sum_{n=1}^{\lfloor \log_{1/a} t\rfloor} 2^{n-1} + \sum_{n>\lfloor \log_{1/a} t\rfloor}^{\infty} 2^{n-1}(a^{b-1})^n\right] \\ &= \frac{1}{b-1}\left[t^{1-b}\left(2^{\lfloor \log_{1/a} t\rfloor}-1\right) + \frac{1}{a^{1-b}-2}\cdot 2^{\lfloor \log_{1/a} t\rfloor}a^{(b-1)\lfloor \log_{1/a} t\rfloor}\right].\end{aligned}$$

Using the fact that $\lfloor \log_{1/a} t\rfloor = \log_{1/a} t - \{\log_{1/a} t\}$ and $2^{\log_{1/a} t} = t^{\log_{1/a} 2}$, we then have that

$$|K_t(0)^c \cap \Omega_\infty^{(a,b)}| = \frac{t^{1-b+\log_{1/a} 2}}{b-1}\left[2^{-\{\log_{1/a} t\}} + \frac{1}{a^{1-b}-2}\cdot\left(\frac{a^{1-b}}{2}\right)^{\{\log_{1/a} t\}}\right] - \frac{t^{1-b}}{b-1}.$$

From this we deduce that for $D := \log_{1/a} 2 - (b+1)$ we have

$$|K_t(0)^c \cap \Omega_\infty^{(a,b)}| = t^{2+D}\left(G(\log t) - \frac{t^{-\log_{1/a} 2}}{b-1}\right) \quad \text{as } t \to +\infty \tag{4.6.5}$$

with G being the T-periodic function

$$G(\tau) := \frac{2^{-\left\{\frac{\tau}{\log(1/a)}\right\}}}{b-1}\left(1 + \frac{(a^{1-b})^{\left\{\frac{\tau}{\log(1/a)}\right\}}}{a^{1-b}-2}\right), \tag{4.6.6}$$

where $T := \log(1/a)$. Furthermore, this result implies that

$$\dim_B(\infty, \Omega_\infty^{(a,b)}) = \log_{1/a} 2 - (b+1).$$

Note that $\dim_B(\infty, \Omega_\infty^{(a,b)}) \to -\infty$ as $b \to +\infty$ and $\dim_B(\infty, \Omega_\infty^{(a,b)}) \to -(b+1) < -2$ as $a \to 0^+$ but can be made as close to -2 as desirable. Moreover, $\Omega_\infty^{(a,b)}$ is not Minkowski measurable at infinity but it is Minkowski nondegenerate with

$$\overline{\mathcal{M}}^D(\infty, \Omega_\infty^{(a,b)}) = \max G = G(0) = \frac{1}{b-1}\cdot\frac{a^{1-b}-1}{a^{1-b}-2} \tag{4.6.7}$$

and

$$\underline{\mathcal{M}}^D(\infty, \Omega_\infty^{(a,b)}) = \min G = G(\tau_{\min}), \tag{4.6.8}$$

where $\tau_{\min}$ is the unique point of the global minimum of the function G on the interval $[0,1]$ which can be explicitly computed:

$$\tau_{\min} = \frac{\log(1 + (b-1)\log_2 a) - \log(2 - a^{1-b})}{(b-1)\log a}.$$

In the next theorem we will construct a maximal hyperfractal set Ω at infinity, that is a set whose zeta function at infinity has its abscissa of convergence as a natural barrier and cannot be meromorphically extended to any neighborhood of it. More precisely, we will now construct a set with a prescribed box dimension $D < -2$ at infinity such that every point on the abscissa of convergence $\{\operatorname{Re} s = D\}$ is a nonremovable singularity of its zeta function at infinity. In accordance with the definitions introduced in [LapRaŽu1] in the case of relative fractal drums, will call such sets maximally hyperfractal at infinity.

Theorem 4.64. *For $D < -2$ there exists a set $\Omega \subseteq \mathbb{R}^2$ of finite Lebesgue measure such that it is maximally hyperfractal with $\dim_B(\infty, \Omega) = D$ and Minkowski nondegenerate at infinity.*

Proof. Let us fix $D < -2$ and choose a nonincreasing sequence $(a_n)_{n\geq 1}$ such that $0 < a_n < 1/2$ for every $n \in \mathbb{N}$ and $a_n \to 0^+$ as $n \to +\infty$. Furthermore, we define the sequence $b_n := \log_{1/a_n} 2 - D - 1$ and observe that for $D < -2$ the condition $b_n > 1 + \log_{1/a_n} 2$ is fulfilled. For the two parameter unbounded set $\Omega_\infty^{(a_n,b_n)}$ we have that $\dim_B(\infty, \Omega_\infty^{(a_n,b_n)}) = D$. The next step is to scale every one of this sets with a suitable parameter, namely we define the sets $\widetilde{\Omega}_n$ for every $n \in \mathbb{N}$ as follows:

$$\widetilde{\Omega}_n := \frac{1}{2^n}\Omega_\infty^{(a_n,b_n)}.$$

Finally we construct the sets Ω_n by translating each set $\widetilde{\Omega}_n$ vertically for the amount l_n which is equal to the sum of the heights of each $\widetilde{\Omega}_k$ for $k < n$, i.e., $l_1 := 0$ and

$$l_n := \sum_{k=1}^{n-1} \frac{1}{2^k} \frac{a_k^{b_k}}{1 - 2a_k^{b_k}}$$

for $n > 1$ and define the set Ω to be the disjoint union of the sets Ω_n. Now we observe that the scaling factor in the definition of the sets $\widetilde{\Omega}$ ensures that the set Ω has finite Lebesgue measure and that it lies in a horizontal strip of finite width.

Similarly as before, this ensures us that calculating the tube formula of Ω using the $|\cdot|_\infty$-norm on $\mathbb{R}^2$ will not affect the values of the upper and lower Minkowski contents of

Ω at infinity. For $t > 1$ we have that

$$
\begin{aligned}
|K_t(0)^c \cap \Omega| &= \sum_{n=1}^{\infty} |K_t(0)^c \cap \Omega_n| = \sum_{n=1}^{\infty} |K_t(0)^c \cap 2^{-n}\Omega_\infty^{(a_n,b_n)}| \\
&= \sum_{n=1}^{\infty} 2^{-2n} |K_{2^n t}(0)^c \cap \Omega_\infty^{(a_n,b_n)}| \\
&= \sum_{n=1}^{\infty} \frac{t^{2+D}}{2^{-Dn}} \left(G_n\left(\log(2^n t)\right) - \frac{t^{-\log_{1/a_n} 2}}{2^{n\log_{1/a_n} 2}(b_n - 1)} \right)
\end{aligned}
$$

where we have used (4.5.2) with $N = 2$ and G_n is the $\log(1/a_n)$-periodic function defined by (4.6.6) with a and b replaced by a_n and b_n respectively. In other words, we have:

$$
|K_t(0)^c \cap \Omega| = t^{2+D} \left(G\left(\log t\right) - \sum_{n=1}^{\infty} \frac{t^{-\log_{1/a_n} 2}}{(b_n - 1)2^{n(-D+\log_{1/a_n} 2)}} \right), \tag{4.6.9}
$$

where

$$
G(\tau) := \sum_{n=1}^{\infty} 2^{nD} G_n\left(\tau + n\log 2\right). \tag{4.6.10}
$$

The convergence of the sum for every $t > 1$ in (4.6.9) follows from the facts that $\log_{1/a_n} 2 \in (0,1)$, $-D > 2$ and $b_n - 1 > -D - 2 > 0$ for all $n \in \mathbb{N}$, i.e.,

$$
\sum_{n=1}^{\infty} \frac{t^{-\log_{1/a_n} 2}}{(b_n - 1)2^{n(-D+\log_{1/a_n} 2)}} \leq -\frac{1}{D+2} \sum_{n=1}^{\infty} \frac{1}{(2^{-D+\log_{1/a_n} 2})^n} < \infty.
$$

Furthermore, the series defining the function G is also convergent for $\tau > 0$. To see this, we observe that from (4.6.7) we have:

$$
G_n(\tau) \leq \overline{\mathcal{M}}_\infty^D(\Omega_\infty^{(a_n,b_n)}) = \frac{1}{b_n - 1} \frac{a_n^{1-b_n} - 1}{a_n^{1-b_n} - 2} \leq 1
$$

for all $n \in \mathbb{N}$. The last inequality above can be easily shown from the conditions on a_n and b_n. Furthermore, from this we conclude that

$$
G(\tau) = \sum_{n=1}^{\infty} 2^{nD} G_n\left(\tau + n\log 2\right) \leq \sum_{n=1}^{\infty} \frac{1}{(2^{-D})^n} < \infty.
$$

In particular,

$$
\overline{\mathcal{M}}^D(\infty, \Omega) \leq \sum_{n=1}^{\infty} \frac{1}{(2^{-D})^n} < \infty.
$$

On the other hand, for the lower Minkowski content of Ω at infinity we can use the fact that $\Omega \supseteq \Omega_1$ and therefore

$$\underline{\mathcal{M}}^D(\infty,\Omega) \geq \underline{\mathcal{M}}^D(\infty,\Omega_1) = \underline{\mathcal{M}}^D(\infty, 2^{-1}\Omega_\infty^{(a_1,b_1)}) = 2^D\underline{\mathcal{M}}^D(\infty,\Omega_\infty^{(a_1,b_1)}) > 0.$$

The last equality above is a consequence of Lemma 4.50 with $r = D$, while the conclusion of positivity follows from (4.6.8).

Let us now show that for the distance zeta function of Ω at infinity the critical line $\{\operatorname{Re} s = D\}$ is a natural boundary. Using the scaling property of the distance zeta function at infinity from Proposition 4.51 we have that

$$\zeta_\infty(s,\Omega) = \sum_{n=1}^{\infty} \zeta_\infty(s, 2^{-n}\Omega_\infty^{(a_n,b_n)}; 1) = \sum_{n=1}^{\infty} 2^{ns}\zeta_\infty(s,\Omega_\infty^{(a_n,b_n)}; 2^n)$$

and it is holomorphic on $\{\operatorname{Re} s > D\}$. Furthermore, according to Proposition 4.61, for every $n \in \mathbb{N}$ the zeta function $\zeta_\infty(s,\Omega_\infty^{(a_n,b_n)}; 2^n; |\cdot|_\infty)$ is meromorphic on $\mathbb{C}$ and has simple poles at $D + \frac{2\pi\mathbb{i}}{\log(1/a_n)}\mathbb{Z}$. Since $\Omega_\infty^{(a_n,b_n)}$ is contained in a strip of finite height, according to Proposition 4.58, we have that $\zeta_\infty(s,\Omega;2^n)$ is meromorphic at least on $\{\operatorname{Re} s > D-2\}$ and its poles in that half-plain coincide with that of $\zeta_\infty(s,\Omega_\infty^{(a_n,b_n)}; 2^n; |\cdot|_\infty)$. From this we conclude that the set of poles of $\zeta_\infty(s,\Omega)$ is dense in the critical line $\{\operatorname{Re} s = D\}$ since $\log(1/a_n) \to +\infty$ as $n \to +\infty$. This, in turn, implies that every point of the critical line is a nonremovable singularity of $\zeta_\infty(s,\Omega)$, i.e., Ω is maximally hyperfractal at infinity. ☐

We recall here that the *field of algebraic numbers* (often denoted by $\overline{\mathbb{Q}}$ in the literature) can be viewed (up to isomorphism) as the algebraic closure of $\mathbb{Q}$ (the field of rational numbers) and is obtained by adjoining to $\mathbb{Q}$ the roots of the polynomial equations with coefficients in $\mathbb{Q}$ (or, equivalently, in $\mathbb{Z}$). Note that, as a result, it is a countable set.

Definition 4.65. A finite set of real numbers is said to be *rationally* (*resp., algebraically*) *linearly independent* or simply, *rationally* (*resp., algebraically*) *independent*, if it is linearly independent over the field of rational (resp., algebraic) real numbers.

In order to define quasiperiodic sets at infinity, it will be convenient to introduce the following definition of quasiperiodic functions.[15]

Definition 4.66 (Cited from [LapRaŽu1]). A function $G = G(\tau)\colon \mathbb{R} \to \mathbb{R}$ is said to be *n-quasiperiodic* (or *quasiperiodic, of order of quasiperiodicity equal to n*) if it is of the form

$$G(\tau) = H(\tau,\dots,\tau), \tag{4.6.11}$$

[15]There is a wide variety of different definitions of quasiperiodic and almost periodic functions (and sets) in the existing literature on dynamical systems, mathematical physics and harmonic analysis. See, for example, [WaMLI], [Sen], [Boh], [Kat], [Lap–vFr3], [Lap6, Appendix F], along with the relevant references therein. Definition 4.66 is the most suitable for our purposes in is adapted from the one in [Vin].

where for some $n \geq 2$, $H\colon \mathbb{R}^n \to \mathbb{R}$ is a function which is nonconstant and T_k-periodic in its k-th component, for each $k = 1, \dots, n$, and the corresponding periods $T_1, \dots, T_n$ are rationally independent; that is linearly independent over the field of rational numbers. The values T_k are called the *quasiperiods of* G.

In addition, we say that a function $G = G(\tau)$ is

(a) *transcendentally n-quasiperiodic* if the periods $T_1, \dots, T_n$ are algebraically independent;[16]

(b) *algebraically n-quasiperiodic* if the corresponding periods $T_1, \dots, T_n$ are rationally independent and algebraically dependent.[17]

One can clearly see from Definition 4.66 that every quasiperiodic function is either transcendentally quasiperiodic or algebraically quasiperiodic. More precisely, the set $\mathcal{F}_{qp}$ of quasiperiodic functions is equal to the disjoint union of the set $\mathcal{F}_{tqp}$ of transcendentally quasiperiodic functions and the set $\mathcal{F}_{aqp}$ of algebraically quasiperiodic functions:

$$\mathcal{F}_{qp} = \mathcal{F}_{tqp} \cup \mathcal{F}_{aqp}.$$

Example 4.67. Let $\lambda_i, \nu_i \in \mathbb{R}$ for $i = 1, 2$. If $G(\tau) = \lambda_1 G_1(\tau+\nu_1)+\lambda_2 G_2(\tau+\nu_2)$, where the functions G_i are nonconstant and T_i-periodic for $i = 1, 2$, such that T_1/T_2 is an irrational algebraic number, then G is algebraically 2-quasiperiodic. In this case and in the notation of Definition 4.66, we have $H(\tau_1, \tau_2) = G_1(\tau_1) + G_2(\tau_2)$. If T_1/T_2 is transcendental, then G is transcendentally 2-quasiperiodic (in the sense of Definition 4.66).

We now define quasiperiodic sets at infinity which complements the analogous definition of quasiperiodic bounded sets from [LapRaŽu1].

Definition 4.68. Assume $\Omega \subseteq \mathbb{R}^N$ is of finite Lebesgue measure and such that it has the following tube formula at infinity:

$$|{}_t\Omega| = t^{N+D}(G(\log t) + o(1)) \quad \text{as} \quad t \to +\infty, \tag{4.6.12}$$

such that G is nonnegative, $0 < \liminf_{\tau\to+\infty} G(\tau) \leq \limsup_{\tau\to+\infty} G(\tau) < +\infty$ and $D \in (-\infty, -N]$ is a given constant.[18]

We say that Ω is an *n-quasiperiodic set* (of *order of quasiperiodicity equal to n*) *at infinity* if the corresponding function $G = G(\tau)$ is n-quasiperiodic.

In addition, the set Ω is said to be

[16]If the periods are algebraically independent, then all of the quotients T_i/T_j, for $i \neq j$, are transcendental (and hence, irrational) numbers.

[17]More precisely, we assume here that the set of quasiperiods $\{T_1, \dots, T_n\}$ is algebraically dependent; in other words, there exist algebraic numbers $\lambda_1, \dots, \lambda_n$, not all of them zero, such that $\lambda_1 T_1 + \cdots + \lambda_n T_n = 0$.

[18]Note that it then follows that $\dim_B(\infty, \Omega)$ exists and is equal to D. Moreover, $\underline{\mathcal{M}}^D(\infty, \Omega) = \liminf_{\tau\to+\infty} G(\tau)$ and $\overline{\mathcal{M}}^D(\infty, \Omega) = \limsup_{\tau\to+\infty} G(\tau)$.

(a) *transcendentally n-quasiperiodic at infinity* if the corresponding function G is transcendentally n-quasiperiodic;

(b) *algebraically n-quasiperiodic at infinity* if the corresponding function G is algebraically n-quasiperiodic.

In light of Definition 4.68 and the comment following Definition 4.66, one can see that each n-quasiperiodic set at infinity is either transcendentally n-quasiperiodic at infinity or n-algebraically quasiperiodic at infinity. In other words, the family $\mathscr{D}_{qp}^{\infty}(n)$ of n-quasiperiodic sets at infinity is equal to the disjoint union of the family $\mathscr{D}_{tqp}^{\infty}(n)$ of transcendentally n-quasiperiodic sets at infinity and the family $\mathscr{D}_{aqp}^{\infty}(n)$ of algebraically n-quasiperiodic sets at infinity:

$$\mathscr{D}_{qp}^{\infty}(n) = \mathscr{D}_{tqp}^{\infty}(n) \cup \mathscr{D}_{aqp}^{\infty}(n).$$

Note that the family $(\mathscr{D}_{qp}^{\infty}(n))_{n\geq 2}$ is *disjoint*, as well as the family $(\mathscr{D}_{tqp}^{\infty}(n))_{n\geq 2}$ and the family $(\mathscr{D}_{aqp}^{\infty}(n))_{n\geq 2}$. Denoting

$$\mathscr{D}_{qp}^{\infty} := \bigcup_{n\geq 2} \mathscr{D}_{qp}^{\infty}(n), \quad \mathscr{D}_{tqp}^{\infty} := \bigcup_{n\geq 2} \mathscr{D}_{tqp}^{\infty}(n), \quad \mathscr{D}_{aqp}^{\infty} := \bigcup_{n\geq 2} \mathscr{D}_{aqp}^{\infty}(n), \tag{4.6.13}$$

we have

$$\mathscr{D}_{qp}^{\infty} = \mathscr{D}_{tqp}^{\infty} \cup \mathscr{D}_{aqp}^{\infty}.$$

Theorem 4.64, or, more precisely, the construction in its proof will show that the families $\mathscr{D}_{tqp}^{\infty}(2)$ and $\mathscr{D}_{aqp}^{\infty}(2)$ are infinite. We first mention a classical result from transcendental number theory.

Theorem 4.69 (Gel'fond–Schneider, [Gel])**.** *Let m be a positive algebraic number, and let x be an irrational algebraic number. Then m^x is transcendental.*

From the above theorem it is easy to derive the fact that the number $\log_k 3$ is transcendental (see [LapRaŽu1, Example 3.1.8 of Section 3.1.2]) where $k > 3$ is an integer that is not a power of 3. More generally, we mention that $\log \alpha$ is transcendental for all algebraic numbers $\alpha \neq 0, 1$ which is a result going back to F. von Lindemann and K. Weierstrass; see [Ba, p. 4].

Theorem 4.70. *The families $\mathscr{D}_{tqp}^{\infty}(2)$ and $\mathscr{D}_{aqp}^{\infty}(2)$ are infinite.*

Proof. We note that in the construction of the set Ω in the proof of Theorem 4.64 if we only take two sets $\Omega_\infty^{(a_1,b_1)}$ and $\Omega_\infty^{(a_2,b_2)}$ instead of infinitely many, we can construct an algebraically or a transcendentally 2-quasiperiodic unbounded set at infinity with prescribed box dimension at infinity equal to $D < -2$. We point out here that the set Ω constructed from sets $\Omega_\infty^{(a_1,b_1)}$ and $\Omega_\infty^{(a_2,b_2)}$ has the following tube formula at infinity

$$|K_t \cap \Omega| = t^{2+D}(G(\log t) + O(t^{-\log_{1/a_1} 2})) \quad \text{as} \quad t \to +\infty, \tag{4.6.14}$$

where

$$G(\tau) = 2^D G_1(\tau + \log 2) + 2^{2D} G_2(\tau + 2\log 2) \tag{4.6.15}$$

is a 2-quasiperiodic function with

$$G_i(\tau) = \frac{2^{-\left\{\frac{\tau}{\log(1/a_i)}\right\}}}{b_i - 1}\left(1 + \frac{(a_i^{1-b_i})^{\left\{\frac{\tau}{\log(1/a_i)}\right\}}}{a_i^{1-b_i} - 2}\right) \tag{4.6.16}$$

for $i = 1, 2$. As we can see the set Ω is then 2-quasiperiodic at infinity but in the sense of the 'cube' tube function at infinity $t \mapsto |K_t(0)^c \cap \Omega|$. To get a 'proper' 2-quasiperiodic set at infinity one should mimic this construction in a radial way, i.e., use an analog of sets $\Omega_\infty^{(a_i,b_i)}$ that are "arranged" around radial rays emanating from the origin. We will not get into the details of this construction, but on the other hand, we can use Lemma 4.18 to deduce that if we choose $D \in (-3, -2)$ we do get "proper" 2-quasiperiodic sets at infinity even in the present construction. More precisely, since Ω is contained in a strip of finite height, by Lemma 4.18 we have that

$$\begin{aligned}|{}_t\Omega| &= |K_t(0)^c \cap \Omega| + O(t^{-1})\\ &= t^{2+D}(G(\log t) + O(t^{-\log_{1/a_1} 2}) + O(t^{-2-D-1}))\\ &= t^{2+D}(G(\log t) + o(1))\end{aligned}$$

as $t \to +\infty$; that is, Ω is 2-quasiperiodic at infinity.

Now, for the algebraical case it suffices to choose $a_1 \in (0, 1/2)$ and define, for instance, $a_2 := a_1^{\sqrt{m}}$ where $m \geq 2$ is an integer that is not a perfect square. Then we have that $b_1 = \log_{1/a_1} 2 - D - 1$ and $b_2 = \log_{1/a_2} 2 - D - 1$. Furthermore for the periods we have that

$$T_1 = \log(1/a_1) \quad \text{and} \quad T_2 = \log(1/a_2) = \sqrt{m}\log(1/a_1),$$

i.e., $T_2/T_1 = \sqrt{m}$.

On the other hand, if we choose, for instance, $a_1 = 1/3$ and $a_2 = 1/k$ where $k > 3$ is an integer that is not a power of 3, we have that

$$\frac{T_1}{T_2} = \frac{\log 3}{\log k} = \log_k 3$$

which is a transcendental number, a fact that follows from the Gel'fond–Schneider theorem recalled in Theorem 4.69. $\square$

Remark 4.71. As a consequence of (4.6.3) of Proposition 4.61, we have that the complex dimensions at infinity of the set Ω from Theorem 4.70 visible through $W := \{\operatorname{Re} s > D-2\}$ are given by

$$\{D - \log_{1/a_i} 2 : i = 1, 2\} \cup (D + \mathbf{p}(a_1)\mathbb{i}\mathbb{Z}) \cup (D + \mathbf{p}(a_2)\mathbb{i}\mathbb{Z}) \tag{4.6.17}$$

where $\mathbf{p}(a_i) = 2\pi/\log(1/a_i)$ for $i = 1, 2$ are the *oscillatory quasiperiods* of Ω.

We can extend Theorem 4.70 to the case of $\mathscr{D}^{\infty}_{tqp}(n)$ and $\mathscr{D}^{\infty}_{aqp}(n)$ but for the transcendental case we will need a nontrivial extension of the Gel'fond–Schneider theorem which is due to Baker and we state it here.

Theorem 4.72 (Baker, [Ba, Theorem 2.1])**.** *Let $n \in \mathbb{N}$ with $n \geq 2$. If $m_1, \ldots, m_n$ are positive algebraic numbers such that $\log m_1, \ldots, \log m_n$ are linearly independent over the rationals, then*

$$1, \log m_1, \ldots, \log m_n$$

are linearly independent over the field of all algebraic numbers (or algebraically independent, in short).[19]

Furthermore, for the algebraical case we will need a result about the rational independence of roots of prime numbers. This result follows from a more general result due to Besicovitch which we also state here.

Theorem 4.73 (Besicovitch [Bes2])**.** *Let*

$$a_1 = b_1 p_1, \;\; a_2 = b_2 p_2, \ldots, \;\; a_k = b_k p_k, \tag{4.6.18}$$

where $p_1, p_2, \ldots, p_k$ are different primes and $b_1, b_2, \ldots, b_k$ are positive integers not divisible by any of these primes. Then, if $x_1, x_2, \ldots, x_k$ are positive real roots of the equations

$$x^{n_1} - a_1 = 0, \;\; x^{n_2} - a_2 = 0, \ldots, \;\; x^{n_k} - a_k = 0, \tag{4.6.19}$$

and $P(x_1, x_2, \ldots, x_k)$ is a polynomial with rational coefficients of degree less than or equal to $n_1 - 1$ with respect to x_1, less than or equal to $n_2 - 1$ with respect to x_2, etc., then $P(x_1, x_2, \ldots, x_k)$ can vanish only if all its coefficients are equal to zero.

A special case of the above theorem is the following corollary.

Corollary 4.74 (Besicovitch [Bes2])**.** Let $n_1, \ldots, n_k \in \mathbb{N} \setminus \{1\}$ and $p_1, \ldots, p_k$ different prime numbers. A polynomial $P(p_1^{1/n_1}, \ldots, p_k^{1/n_k})$ with rational coefficients not all equal to zero, cannot vanish. In other words, the numbers $p_1^{1/n_1}, \ldots, p_k^{1/n_k}$ are rationally independent.

Theorem 4.75. *The families $\mathscr{D}^{\infty}_{tqp}(n)$ and $\mathscr{D}^{\infty}_{aqp}(n)$ are infinite for every integer $n \geq 2$.*

Proof. The proof is analogous to the proof of Theorem 4.70 the difference being in the fact that we take n sets $\Omega_{\infty}^{(a_i, b_i)}$, for $i = 1, \ldots, n$ instead of only two. In that way we construct a set Ω with n quasiperiods at infinity which will be 'proper' n-quasiperiodic

[19] In particular, the numbers $\log m_1, \ldots, \log m_n$ are transcendental, as well as their pairwise quotients.

if we additionally restrict ourselves to $D \in (-3, -2)$. (See the discussion in the proof of Theorem 4.70.) For the algebraically n-quasiperiodic case we may choose $a_1 \in (0, 1/2)$ and define $a_{i+1} := a_1^{\sqrt{p_i}}$ where p_i is the i-th prime number for $i \geq 1$. Then for the quasiperiods of Ω we have that

$$T_1 = \log(1/a_1) \quad \text{and} \quad T_{i+1} = \log(1/a_1^{\sqrt{p_i}}) = T_1\sqrt{p_i}$$

for $i \geq 1$. It is obvious that the quasiperiods $T_1, \ldots, T_n$ are algebraically dependent. On the other hand, they are rationally independent. Namely suppose that there are $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}$ such that

$$\lambda_1 T_1 + \lambda_2 T_2 + \ldots + \lambda_n T_n = 0.$$

This is equivalent to

$$\lambda_1 + \lambda_2\sqrt{2} + \cdots + \lambda_n\sqrt{p_{n-1}} = 0$$

which is possible only if $\lambda_1 = \cdots = \lambda_n = 0$ according to Corollary 4.74. This proves that the set Ω indeed is algebraically n-quasiperiodic at infinity.

Let us now construct a transcendentally n-quasiperiodic set at infinity. We choose now $a_i := 1/p_{i+1}$ with p_i being the i-th prime number for $i \geq 1$. Note that now $T_i = \log(1/a_i) = \log p_{i+1}$ and these numbers are rationally independent. Indeed, if we assume the contrary; that is, that there exist rational numbers $\lambda_1, \ldots, \lambda_n$ such that $\sum_{i=1}^n \lambda_i \log p_{i+1} = 0$, then this implies that $\prod_{i=1}^n p_{i+1}^{\lambda_i} = 1$ which is in contradiction with the Fundamental theorem of algebra. Now, Baker's theorem (recalled in Theorem 4.72) implies that the numbers $T_1, \ldots, T_n$ are also algebraically independent; that is, the set Ω is transcendentally n-quasiperiodic. $\square$

Remark 4.76. Similarly as in Remark 4.71, the set Ω constructed in Theorem 4.75 will have the following set of complex dimensions visible through $W = \{\operatorname{Re} s > D - 2\}$:

$$\bigcup_{i=1}^{n} \big(\{D - \log_{1/a_i} 2\} \cup (D + \mathbf{p}(a_i)\mathbb{i}\mathbb{Z})\big) \tag{4.6.20}$$

where $\mathbf{p}(a_i) = 2\pi/\log(1/a_i)$ for $i = 1, \ldots, n$ are the oscillatory quasiperiods of Ω at infinity.

Remark 4.77. It is clear that one can construct somewhat more general examples of n-quasiperiodic sets at infinity (by using Theorem 4.73 for the algebraical case and Theorem 4.72 for the transcendental case) than the ones from the proof of Theorem 4.75 by choosing other admissible values for the parameters a_i.

Let us conclude this section by defining the notion of ∞-quasiperiodic sets at infinity and showing that the maximally hyperfractal set Ω at infinity from Theorem 4.64 gives an example of such a set. Moreover, by carefully choosing the parameters a_i we can construct an infinite number of algebraically and transcendentally ∞-quasiperiodic sets at infinity.

Definition 4.78. A sequence $(T_i)_{i\geq 1}$ of real numbers is said to be *rationally* (*resp., algebraically*) *linearly independent*, if any of its finite subsets is rationally (resp., algebraically) independent.

In the following two definitions, Definition 4.79 and Definition 4.80, we will refine and extend the definition of an n-quasiperiodic function and set (Definition 4.66 and Definition 4.68, respectively).

Definition 4.79 (Cited from [LapRaŽu1])**.** A function $G : \mathbb{R} \to \mathbb{R}$ is said to be ∞-*quasiperiodic*, if it is of the form

$$G(\tau) = H(\tau, \tau, \dots),$$

where $H : \mathbb{R}_b^\infty \to \mathbb{R}$,[20] $H = H(\tau_1, \tau_2, \dots)$ is a function which is T_j-periodic in its j-th component, for each $j \in \mathbb{N}$, with $T_j > 0$ as minimal periods, and such that the set of periods

$$\{T_j : j \geq 1\} \tag{4.6.21}$$

is *rationally* independent. We say that the *order of quasiperiodicity* of the function G is equal to infinity (or that the function G is ∞-*quasiperiodic*).

In addition, we say that G is

(a) *transcendentally quasiperiodic of infinite order* (or *transcendentally* ∞-*quasiperiodic*) if the periods in (4.6.21) are *algebraically* independent;

(b) *algebraically quasiperiodic of infinite order* (or *algebraically* ∞-*quasiperiodic*) of infinite order if the periods in (4.6.21) are *rationally* independent and algebraically dependent.[21]

Definition 4.80. Let Ω be a measurable subset of $\mathbb{R}^N$ with $|\Omega| < \infty$ such that (∞, Ω) has the following tube formula at infinity:

$$|{}_t\Omega| = t^{N+D}(G(\log t) + o(1)) \quad \text{as} \quad t \to +\infty, \tag{4.6.22}$$

where $D < -N$, and G is nonnegative such that

$$0 < \liminf_{\tau \to +\infty} G(\tau) \leq \limsup_{\tau \to +\infty} G(\tau) < \infty.^{22}$$

[20] $\mathbb{R}_b^\infty$ stands here for the usual Banach space of bounded sequences $(\tau_j)_{j\geq 1}$ of real numbers, endowed with the norm $\|(\tau_j)_{j\geq 1}\|_\infty := \sup_{j\geq 1} |\tau_j|$.

[21] We say that a sequence $(T_i)_{i\geq 1}$ of real numbers is *algebraically dependent* of infinite order if there exists a finite subset J of $\mathbb{N}$ such that $(T_i)_{i\in J}$ is algebraically dependent. Recall that a finite set of real numbers $\{T_1, \dots, T_k\}$ is said to be *algebraically dependent* if there exist k algebraic real numbers $\lambda_1, \dots, \lambda_k$, not all of them equal to zero, such that $\lambda_1 T_1 + \dots + \lambda_k T_k = 0$.

[22] Note that it then follows that $\dim_B(\infty, \Omega)$ exists and is equal to D. Moreover, $\underline{\mathcal{M}}^D(\infty, \Omega) =$

We say that the set Ω *is quasiperiodic at infinity* and of *infinite order of quasiperiodicity* (or, in short, ∞*-quasiperiodic*) if the function $G = G(\tau)$ is ∞-quasiperiodic; see Definition 4.79.

In addition, Ω is said to be

(a) *transcendentally* ∞*-quasiperiodic at infinity* if the corresponding function G is transcendentally ∞-quasiperiodic;

(b) *algebraically* ∞*-quasiperiodic at infinity* if the corresponding function G is algebraically ∞-quasiperiodic.

In much the same way as before, if we denote with $\mathscr{D}^{\infty}_{qp}(\infty)$ the family of all ∞-quasiperiodic sets at infinity, then it is clear that this family is a disjoint union of $\mathscr{D}^{\infty}_{aqp}(\infty)$ and $\mathscr{D}^{\infty}_{tqp}(\infty)$; that is, the algebraically ∞-quasiperiodic subfamily and the transcendentally ∞-quasiperiodic subfamily, respectively.

Theorem 4.81. *The families $\mathscr{D}^{\infty}_{aqp}(\infty)$ and $\mathscr{D}^{\infty}_{tqp}(\infty)$ are infinite.*

Proof. For a fixed $D < -2$ a member of each subfamily is the maximal hyperfractal Ω at infinity constructed in Theorem 4.64 for a specifically chosen sequence of parameters a_i. More precisely, to get a 'proper' ∞-quasiperiodic set at infinity, we have to choose $D \in (-3, -2)$. (See the discussion in the proof of Theorem 4.70.) We proceed analogously as in the proof of Theorem 4.75; that is, let $(p_i)_{i\geq 1}$ be the increasing sequence of all prime numbers. For the algebraically ∞-quasiperiodic set at infinity we may choose $a_1 \in (0, 1/2)$ and define $a_{i+1} := a_1^{\sqrt{p_i}}$ for $i \geq 1$. Again, Corollary 4.74 assures that the sequence of quasiperiods $T_i = \log(1/a_i)$, $i \geq 1$ is rationally independent.

On the other hand, for the transcendentally ∞-quasiperiodic set at infinity we may choose $a_i := 1/p_{i+1}$ for $i \geq 1$ and, again, Baker's theorem (Theorem 4.72) assures that the the sequence of quasiperiods $T_i = \log(1/a_i)$, $i \geq 1$ is algebraically independent. $\square$

4.7 One-point Compactification and the ϕ-shell Minkowski Content

In this section we are interested in the natural question which arises when dealing with unbounded sets in $\mathbb{R}^N$ and their fractal dimensions. Namely, we would like to study the connection between the fractal properties of unbounded sets in $\mathbb{R}^N$ studied so far and the fractal properties of their images under the the stereographic projection Ψ to the N-dimensional Riemann sphere $\mathbb{S}^N \subseteq \mathbb{R}^{N+1}$. In other words, how does the one-point compactification of $\mathbb{R}^N$ affect the fractal properties of unbounded sets? This question is also closely related with the connection between the fractal properties of unbounded sets

$\liminf_{\tau\to+\infty} G(\tau)$ and $\overline{\mathcal{M}}^D(\infty,\Omega) = \limsup_{\tau\to+\infty} G(\tau)$.

and the fractal properties of their images under the geometric inversion $\Phi(x) := x/|x|^2$ on $\mathbb{R}^N$ with $|\cdot|$ being the Euclidean norm. Moreover, if we choose the Riemann sphere $\mathbb{S}^N$ to be the unit sphere in $\mathbb{R}^{N+1}$ with $\mathbb{R}^N$ considered as the equatorial hyper-plane $\{y_{N+1} = 0\}$, then the composition $\Psi\circ\Phi\circ\Psi^{-1}$ is an isometry of $\mathbb{S}^N$ which is in fact equal to the reflection of the upper half-sphere to the lower half-sphere over the equatorial hyper-plane.

Let

$$\mathbb{S}^N := \{(y_1, \dots, y_{N+1}) \, : \, \sum_{i=1}^{N+1} y_i^2 = 1\}$$

be the unit sphere in $\mathbb{R}^{N+1}$ and identify the equatorial hyper-plane $\{y_{N+1} = 0\}$ with $\mathbb{R}^N$. Then the stereographic projection $\Psi\colon \mathbb{R}^N \to \mathbb{S}^N$ is defined by

$$\Psi(x_1, \dots, x_N) := \left(\frac{2x_1}{|x|^2+1}, \dots, \frac{2x_N}{|x|^2+1}, \frac{|x|^2-1}{|x|^2+1}\right). \tag{4.7.1}$$

It maps a point $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ to the point $y = (y_1 \dots, y_{N+1}) \in \mathbb{S}^N$ at which the line that passes through the north pole $\mathbf{N} := (0, \dots, 0, 1)$ and the point $(x, 0) \in \mathbb{R}^N \times \{0\} \subseteq \mathbb{R}^{N+1}$ intersects the unit sphere in $\mathbb{R}^{N+1}$. Furthermore, we also extend Ψ to $\mathbb{R}^N \cup \{\infty\}$ with $\Psi(\infty) := \mathbf{N}$, the north pole. The inverse mapping is then given by

$$\Psi^{-1}(y_1, \dots, y_{N+1}) := \left(\frac{y_1}{1-y_{N+1}}, \dots, \frac{y_N}{1-y_{N+1}}\right). \tag{4.7.2}$$

It is easy to see that $\Psi \in C^1(\mathbb{R}^N, \mathbb{R}^{N+1})$ and its differential is injective on $\mathbb{R}^N$, i.e., $\Psi(\mathbb{R}^N) = \mathbb{S}^N \setminus \{\mathbf{N}\}$ is an immersed submanifold of $\mathbb{R}^{N+1}$.

This gives us a way to calculate the distance on $\mathbb{S}^N$ and the Lebesgue measure, that is the N-dimensional volume of subsets of $\mathbb{S}^N$. More precisely, the Euclidean metric generated by the standard scalar product on $\mathbb{R}^{N+1}$, when restricted to vectors tangent to $\mathbb{S}^N$, gives a means for calculating the dot product of these tangent vectors. This is called the *induced metric* on $\mathbb{S}^N$ and (with $\mathbb{S}^N \setminus \{\mathbf{N}\}$ being an immersed submanifold of $\mathbb{R}^{N+1}$ via Ψ) it can be calculated using the pushforward of vectors in $\mathbb{R}^N$ along Ψ. More precisely, let $T_x\mathbb{R}^N$ be the *tangent space at* $x \in \mathbb{R}^N$ and $T\mathbb{R}^N = \{T_x \, : \, x \in \mathbb{R}^N\}$ be the *tangent bundle* of $\mathbb{R}^N$.[23] Similarly, let $T_y\mathbb{S}^N$ be the tangent space at $y \in \mathbb{S}^{\mathbb{N}}$ and $T\mathbb{S}^N = \{T_y\mathbb{S}^N \, : \, y \in \mathbb{S}^N\}$ the tangent bundle of $\mathbb{S}^N$. Furthermore, the differential $D\Psi(x)\colon T_x\mathbb{R}^N \to T_{\Psi(x)}\mathbb{S}^N$ is a linear map and the metric tensor of the induced metric on $\mathbb{S}^N$ in the standard basis of $\mathbb{R}^N$ is then equal to $(D\Psi)^\tau D\Psi$. This metric tensor can be then used to calculate the length of curves on $\mathbb{S}^N$ and the N-dimensional volume of subsets of $\mathbb{S}^N$. More precisely, if we take $\Omega \subseteq \mathbb{R}^N$ and want to calculate the N-dimensional volume of its image $\Psi(\Omega) \subseteq \mathbb{S}^N$, we have that

$$|\Psi(\Omega)|_{\mathbb{S}} = \int_\Omega \sqrt{\det\left((D\Psi)^\tau D\Psi\right)}\, \mathrm{d}x_1 \cdots \mathrm{d}x_N. \tag{4.7.3}$$

[23]We can identify $T_x\mathbb{R}^N$ with $\{x\} \times \mathbb{R}^N$ and $T\mathbb{R}^N$ with $\mathbb{R}^N \times \mathbb{R}^N$.

(See, e.g., [Mich] or [DoPo] for more details about this subject.)

We will call $|\Psi(\Omega)|_{\mathbb{S}}$ the *spherical N-dimensional volume* of Ω and note that for every Lebesgue measurable $\Omega \subseteq \mathbb{R}^N$ its spherical volume is finite. Now, we can analogously as in the case of subsets of $\mathbb{R}^N$ of finite volume define the notions of *spherical* (*upper, lower*) *Minkowski content* and *spherical* (*upper, lower*) *box dimension*. Of course, in the definitions, we will use the *spherical* δ-neighborhood for subsets of $\mathbb{S}^N$:

$$A_{\delta,\mathbb{S}} := \{y \in \mathbb{S}^N \,:\, d_{\mathbb{S}}(y, A) < \delta\}, \tag{4.7.4}$$

where $A \subseteq \mathbb{S}^N$, $\delta > 0$ and $d_{\mathbb{S}}$ is the induced metric on $\mathbb{S}^N$.

Definition 4.82. The *upper and lower spherical Minkowski contents* for $A \subseteq \mathbb{S}^N$ are defined in the usual way:

$$\overline{\mathcal{M}}^r_{\mathbb{S}}(A) := \limsup_{\delta\to 0^+} \frac{|A_{\delta,\mathbb{S}}|_{\mathbb{S}}}{\delta^{N-r}}, \tag{4.7.5}$$

$$\underline{\mathcal{M}}^r_{\mathbb{S}}(A) := \liminf_{\delta\to 0^+} \frac{|A_{\delta,\mathbb{S}}|_{\mathbb{S}}}{\delta^{N-r}}, \tag{4.7.6}$$

and so are the upper and lower box dimensions of a set $A \subseteq \mathbb{S}^N$:

$$\overline{\dim}_{\mathbb{S}} A := \sup\{r \in \mathbb{R} \,:\, \overline{\mathcal{M}}^r_{\mathbb{S}}(A) = +\infty\} = \inf\{r \in \mathbb{R} \,:\, \overline{\mathcal{M}}^r_{\mathbb{S}}(A) = 0\}; \tag{4.7.7}$$

$$\underline{\dim}_{\mathbb{S}} A := \sup\{r \in \mathbb{R} \,:\, \underline{\mathcal{M}}^r_{\mathbb{S}}(A) = +\infty\} = \inf\{r \in \mathbb{R} \,:\, \underline{\mathcal{M}}^r_{\mathbb{S}}(A) = 0\}. \tag{4.7.8}$$

Analogously we define these notions in the case of relative fractal drums (A, Ω) with A and Ω being subsets of $\mathbb{S}^N$ with the only difference being in replacing $|A_{\delta,\mathbb{S}}|_{\mathbb{S}}$ by $|A_{\delta,\mathbb{S}} \cap \Omega|_{\mathbb{S}}$ in Definition 4.82 above. In order to obtain results which will connect these new notions of spherical Minkowski contents and spherical box dimensions with the old ones we will need the following proposition.

Proposition 4.83. *Let $\Omega \subseteq \mathbb{R}^N$ be a Lebesgue measurable set. Then, for the spherical N-dimensional volume of Ω we have*:

$$|\Psi(\Omega)|_{\mathbb{S}} = \int_{\Omega} \frac{2^N}{(1 + |x|^2)^N}\, \mathrm{d}x_1 \cdots \mathrm{d}x_N. \tag{4.7.9}$$

Proof. From Equation (4.7.3) we see that we have to calculate $\sqrt{\det\left((D\Psi)^\tau D\Psi\right)}$. The differential $D\Psi$ is represented by the $(N+1) \times N$ matrix $\left[\left(\frac{\partial \Psi_i}{\partial x_j}\right)_{ij}\right]$ and for $i \leq N$ we have

$$\frac{\partial \Psi_i}{\partial x_j} = \frac{2\delta_{ij}}{\Delta + 1} - \frac{4x_i x_j}{(\Delta + 1)^2},$$

with δ_{ij} being the Kronecker delta and $\Delta := |x|^2$. Furthermore, we also have

$$\frac{\partial \Psi_{N+1}}{\partial x_j} = \frac{4x_j}{(\Delta+1)^2}.$$

Now, we have

$$\begin{aligned}
((D\Psi)^\tau D\Psi)_{ij} =& \sum_{k=1}^{N+1} \frac{\partial \Psi_k}{\partial x_i}\frac{\partial \Psi_k}{\partial x_j} \\
=& \sum_{k=1}^{N} \left[\left(\frac{2\delta_{ki}}{\Delta+1} - \frac{4x_k x_i}{(\Delta+1)^2}\right)\left(\frac{2\delta_{kj}}{\Delta+1} - \frac{4x_k x_j}{(\Delta+1)^2}\right)\right] + \frac{16x_i x_j}{(\Delta+1)^4} \\
=& \frac{4}{(\Delta+1)^4}\sum_{k=1}^{N}\left[\left((\Delta+1)\delta_{ki} - 2x_k x_i\right)\left((\Delta+1)\delta_{kj} - 2x_k x_j\right)\right] + \frac{16x_i x_j}{(\Delta+1)^4} \\
=& \frac{4}{(\Delta+1)^4}\sum_{k=1}^{N}\left[(\Delta+1)^2\delta_{ki}\delta_{kj} + 4x_k^2 x_i x_j - 2x_k(\Delta+1)(x_i\delta_{kj} + x_j\delta_{ki})\right] \\
&+ \frac{16x_i x_j}{(\Delta+1)^4}
\end{aligned}$$

and, by summing the terms in brackets separately, we get

$$\begin{aligned}
((D\Psi)^\tau D\Psi)_{ij} =& \frac{4}{(\Delta+1)^2}\sum_{k=1}^{N}\delta_{ki}\delta_{kj} + \frac{16x_i x_j}{(\Delta+1)^4}\sum_{k=1}^{N} x_k^2 - \frac{8x_i}{(\Delta+1)^3}\sum_{k=1}^{N} x_k\delta_{kj} \\
&- \frac{8x_j}{(\Delta+1)^3}\sum_{k=1}^{N} x_k\delta_{ki} + \frac{16x_i x_j}{(\Delta+1)^4} \\
=& \frac{4\delta_{ij}}{(\Delta+1)^2} + \frac{16x_i x_j \Delta}{(\Delta+1)^4} - \frac{8x_i x_j}{(\Delta+1)^3} - \frac{8x_i x_j}{(\Delta+1)^3} + \frac{16x_i x_j}{(\Delta+1)^4} \\
=& \frac{4\delta_{ij}}{(\Delta+1)^2}
\end{aligned}$$

and we see that the matrix $(D\Psi)^\tau D\Psi$ is diagonal. Calculating the square root of its determinant completes the proof of the proposition. □

Theorem 4.84. *Let $\Omega \subseteq \mathbb{R}^N$ be a Lebesgue measurable set of finite measure and $\mathbf{N}$ the north pole of $\mathbb{S}^N$. Then, for every $\phi > 1$ and $r \leq -N$ we have*

$$\frac{2^r(1-\phi^{N+r})}{\phi^{2N}}\overline{\mathcal{M}}^r(\infty,\Omega) \leq \overline{\mathcal{M}}^r_{\mathbb{S}}(\mathbf{N},\Psi(\Omega)) \leq 2^r\overline{\mathcal{M}}^r(\infty,\Omega) \tag{4.7.10}$$

and

$$\underline{\mathcal{M}}^r_{\mathbb{S}}(\mathbf{N},\Psi(\Omega)) \leq 2^r\underline{\mathcal{M}}^r(\infty,\Omega). \tag{4.7.11}$$

Furthermore, if $r<-N$, then for

$$\phi=\left(\frac{2N}{N-r}\right)^{\frac{1}{N+r}}$$

we get the optimal left hand side inequality in (4.7.10):

$$-\frac{N+r}{N-r}\left(\frac{2N}{N-r}\right)^{-\frac{N+r}{N-r}}2^r\overline{\mathcal{M}}^r(\infty,\Omega)\leq\overline{\mathcal{M}}^r_{\mathbb{S}}(\mathbf{N},\Psi(\Omega))\tag{4.7.12}$$

To prove the theorem we will need the following lemma.

Lemma 4.85. *Let $A\subseteq\mathbb{S}^N$ and $\mathbf{N}\in\mathbb{S}^N$ be the north pole. Then, for every $\delta\in(0,\pi)$ we have*

$$\Psi^{-1}(\{\mathbf{N}\}_{\delta,\mathbb{S}}\cap A)=B_{\cot\frac{\delta}{2}}(0)^c\cap\Psi^{-1}(A).$$

Proof. Firstly, with Ψ being a bijection we have $\Psi^{-1}(\{\mathbf{N}\}_{\delta,\mathbb{S}}\cap A)=\Psi^{-1}(\{\mathbf{N}\}_{\delta,\mathbb{S}})\cap\Psi^{-1}(A)$. Secondly, by using elementary trigonometry we get that $\Psi^{-1}(\{\mathbf{N}\}_{\delta,\mathbb{S}})=B_{\frac{\sin\delta}{1-\cos\delta}}(0)^c=B_{\cot\frac{\delta}{2}}(0)^c$, which proves the lemma. $\square$

Proof of Theorem 4.84. By Proposition 4.83 and Lemma 4.85 we have for $\delta\in(0,\pi)$ that

$$\begin{aligned}|\{\mathbf{N}\}_{\delta,\mathbb{S}}\cap\Psi(\Omega)|_{\mathbb{S}}=&\int_{B_{\cot\frac{\delta}{2}}(0)^c\cap\Omega}\frac{2^N}{(1+|x|^2)^N}\,\mathrm{d}x_1\cdots\mathrm{d}x_N\\ \leq&\frac{2^N}{\left(1+\cot^2\frac{\delta}{2}\right)^N}\int_{B_{\cot\frac{\delta}{2}}(0)^c\cap\Omega}\mathrm{d}x_1\cdots\mathrm{d}x_N\\ =&\frac{2^N}{\left(1+\cot^2\frac{\delta}{2}\right)^N}|B_{\cot\frac{\delta}{2}}(0)^c\cap\Omega|.\end{aligned}$$

Next, we introduce a new variable $t:=\cot(\delta/2)$ and observe that $\delta\to0^+$ if and only if $t\to+\infty$. Furthermore, for $r\in\mathbb{R}$ from the above inequality we have

$$\begin{aligned}\frac{|\{\mathbf{N}\}_{\delta,\mathbb{S}}\cap\Psi(\Omega)|_{\mathbb{S}}}{\delta^{N-r}}\leq&\frac{2^N}{\left(1+\cot^2\frac{\delta}{2}\right)^N}\frac{|B_{\cot\frac{\delta}{2}}(0)^c\cap\Omega|}{\delta^{N-r}}\\ =&\frac{2^N}{(1+t^2)^N}\frac{|B_t(0)^c\cap\Omega|}{2^{N-r}(\operatorname{arccot}t)^{N-r}}\\ =&2^r\frac{t^{2N}}{(1+t^2)^N}\frac{1}{(t\operatorname{arccot}t)^{N-r}}\frac{|B_t(0)^c\cap\Omega|}{t^{N+r}}.\end{aligned}$$

Now, since $t\operatorname{arccot}t\to1$ when $t\to+\infty$ we prove the right-hand side inequalities (4.7.10) and (4.7.11) by taking the upper and lower limit as $\delta\to0^+$.

To prove the left-hand side inequality (4.7.10) we fix $\phi>1$ and similarly as before we have

$$
\begin{aligned}
|\{\mathbf{N}\}_{\delta,\mathbb{S}} \cap \Psi(\Omega)|_{\mathbb{S}} &= \int_{B_{\cot\frac{\delta}{2}}(0)^c\cap\Omega} \frac{2^N}{(1+|x|^2)^N}\,\mathrm{d}x_1\cdots\mathrm{d}x_N \\
&\geq \int_{B_{\cot\frac{\delta}{2}}(0)^c\cap B_{\phi\cot\frac{\delta}{2}}(0)\cap\Omega} \frac{2^N}{(1+|x|^2)^N}\,\mathrm{d}x_1\cdots\mathrm{d}x_N \\
&\geq \frac{2^N}{(1+\phi^2\cot^2\frac{\delta}{2})^N}\int_{B_{\cot\frac{\delta}{2}}(0)^c\cap B_{\phi\cot\frac{\delta}{2}}(0)\cap\Omega} \mathrm{d}x_1\cdots\mathrm{d}x_N \\
&= \frac{2^N}{(1+\phi^2t^2)^N}|B_{t,\phi t}(0)\cap\Omega|,
\end{aligned}
$$

where we have again introduced the variable $t := \cot(\delta/2)$. This implies that for $r \leq -N$ we have

$$
\begin{aligned}
\frac{|\{\mathbf{N}\}_{\delta,\mathbb{S}} \cap \Psi(\Omega)|_{\mathbb{S}}}{\delta^{N-r}} &\geq \frac{2^r}{(1+\phi^2t^2)^N}\,\frac{|B_{t,\phi t}(0)\cap\Omega|}{(\operatorname{arccot} t)^{N-r}} \\
&= 2^r\frac{t^{2N}}{(1+\phi^2t^2)^N}\,\frac{1}{(t\operatorname{arccot} t)^{N-r}}\,\frac{|B_{t,\phi t}(0)\cap\Omega|}{t^{N+r}}.
\end{aligned}
$$

Let us now introduce a new notation:

$$
\overline{\mathcal{M}}^r_\phi(\infty,\Omega) := \limsup_{t\to+\infty}\frac{|B_{t,\phi t}(0)\cap\Omega|}{t^{N+r}} \tag{4.7.13}
$$

and by taking the upper limit as $\delta \to 0^+$ we get

$$
\overline{\mathcal{M}}^r_{\mathbb{S}}(\mathbf{N},\Psi(\Omega)) \geq \frac{2^r}{\phi^{2N}}\,\overline{\mathcal{M}}^r_\phi(\infty,\Omega). \tag{4.7.14}
$$

To complete the proof all we need to show is that

$$
\overline{\mathcal{M}}^r_\phi(\infty,\Omega) \geq (1-\phi^{N+r})\overline{\mathcal{M}}^r(\infty,\Omega)
$$

is satisfied and this will be a statement of Proposition 4.87 below.

The last part of the theorem is easily proved by analyzing the real function $\phi \mapsto \phi^{-2N}(1-\phi^{N+r})$ on the interval $(1,\infty)$. □

Before stating the aforementioned proposition let us first define new notions inspired by the proof of Theorem 4.84.

Definition 4.86. Let $\Omega \subseteq \mathbb{R}^N$ be a Lebesgue measurable set,[24] $\phi > 1$ and $r \in \mathbb{R}$. We define the *upper ϕ-shell Minkowski content* of Ω *at infinity* as

$$
\overline{\mathcal{M}}^r_\phi(\infty,\Omega) := \limsup_{t\to+\infty}\frac{|B_{t,\phi t}(0)\cap\Omega|}{t^{N+r}} \tag{4.7.15}
$$

[24] Note that here we do not require that Ω has finite Lebesgue measure.

and analogously the *lower ϕ-shell Minkowski content* of Ω *at infinity* as

$$\underline{\mathcal{M}}_\phi^r(\infty,\Omega) := \liminf_{t\to+\infty} \frac{|B_{t,\phi t}(0)\cap\Omega|}{t^{N+r}}. \tag{4.7.16}$$

If for some $r\in\mathbb{R}$ the upper and lower limits above coincide, we call this value the *r-dimensional ϕ-shell Minkowski content of Ω at infinity* and denote it with $\mathcal{M}_\phi^r(\infty,\Omega)$. Furthermore, we will call the function $t\mapsto|B_{t,\phi t}(0)\cap\Omega|$ the *ϕ-shell function* of Ω.

Proposition 4.87. *Let $\Omega\subseteq\mathbb{R}^N$ be a Lebesgue measurable set with $|\Omega|<\infty$. Then, for every $\phi>1$ and $r<-N$ we have*

$$\overline{\mathcal{M}}_\phi^r(\infty,\Omega) \le \overline{\mathcal{M}}^r(\infty,\Omega) \le \frac{1}{1-\phi^{N+r}}\,\overline{\mathcal{M}}_\phi^r(\infty,\Omega) \tag{4.7.17}$$

and

$$\frac{1}{1-\phi^{N+r}}\,\underline{\mathcal{M}}_\phi^r(\infty,\Omega) \le \underline{\mathcal{M}}^r(\infty,\Omega). \tag{4.7.18}$$

Proof. The left-hand side of inequality (4.7.17) is a simple consequence of the fact that $|B_{t,\phi t}(0)\cap\Omega|\le|{}_t\Omega|$. To prove the rest of the proposition, we observe that

$$|B_t(0)^c\cap\Omega| = \sum_{n=0}^{\infty}|B_{\phi^n t,\phi^{n+1}t}(0)\cap\Omega|,$$

which, in turn, implies that for $r<-N$ we have

$$\frac{|B_t(0)^c\cap\Omega|}{t^{N+r}} = \sum_{n=0}^{\infty}\phi^{n(N+r)}\frac{|B_{\phi^n t,\phi^{n+1}t}(0)\cap\Omega|}{(\phi^n t)^{N+r}}.$$

We now apply Fatou's lemma[25] and the fact that $\phi^{N+r}<1$ to get

$$\begin{aligned}\limsup_{t\to+\infty}\frac{|B_t(0)^c\cap\Omega|}{t^{N+r}} &\le \sum_{n=0}^{\infty}\phi^{n(N+r)}\limsup_{t\to+\infty}\frac{|B_{\phi^n t,\phi^{n+1}t}(0)\cap\Omega|}{(\phi^n t)^{N+r}}\\ \overline{\mathcal{M}}^r(\infty,\Omega) &\le \sum_{n=0}^{\infty}\phi^{n(N+r)}\overline{\mathcal{M}}^r(\infty,\Omega) = \frac{1}{1-\phi^{N+r}}\,\overline{\mathcal{M}}^r(\infty,\Omega).\end{aligned}$$

Finally, by the same reasoning applied to the lower limit we get (4.7.18) and this concludes the proof of the proposition. $\square$

Since $\phi\mapsto\underline{\mathcal{M}}_\phi^r(\infty,\Omega)$ and $\phi\mapsto\overline{\mathcal{M}}_\phi^r(\infty,\Omega)$ are nondecreasing functions with values in $[0,+\infty]$, the next corollary follows immediately from the above proposition.

[25] More precisely, first we choose an arbitrary sequence of positive numbers $(t_k)_{k\ge1}$ such that $t_k\to+\infty$ and apply Fatou's lemma on the counting measure in this case. From that we get the conclusion in the general case when $t\to+\infty$.

Corollary 4.88. *Let $\Omega \subseteq \mathbb{R}^N$ be a Lebesgue measurable set with $|\Omega| < \infty$. Then, for $r < -N$ we have that*

$$\lim_{\phi \to +\infty} \overline{\mathcal{M}}^r_\phi(\infty, \Omega) = \overline{\mathcal{M}}^r(\infty, \Omega) \tag{4.7.19}$$

and

$$\lim_{\phi \to +\infty} \underline{\mathcal{M}}^r_\phi(\infty, \Omega) \leq \underline{\mathcal{M}}^r(\infty, \Omega) \tag{4.7.20}$$

Remark 4.89. In light of the above corollary, a valid question is to interpret the meaning of $\lim_{\phi \to 1^+} \underline{\mathcal{M}}^r_\phi(\infty, \Omega)$ and $\lim_{\phi \to 1^+} \overline{\mathcal{M}}^r_\phi(\infty, \Omega)$. Moreover, if $\underline{\mathcal{M}}^r(\infty, \Omega) < \infty$ we have from Proposition 4.87 that $\lim_{\phi \to 1^+} \underline{\mathcal{M}}^r_\phi(\infty, \Omega) = 0$. One would expect that these limits are somehow related to the notion of the *surface Minkowski content* that was investigated by Winter and Rataj in [RatWi1] and [RatWi2]. More on this subject will be said in Section 4.8 below.

From Theorem 4.84 we have the next two corollaries which establish a connection between the box dimensions of a set at infinity and the box dimension of its image on the Riemann sphere.

Corollary 4.90. *Let $\Omega \subseteq \mathbb{R}^N$ be a Lebesgue measurable set of finite measure. Then it holds that*

$$\overline{\dim}_{\mathbb{S}}(\mathbf{N}, \Psi(\Omega)) = \overline{\dim}_B(\infty, \Omega)$$

and

$$\underline{\dim}_{\mathbb{S}}(\mathbf{N}, \Psi(\Omega)) \leq \underline{\dim}_B(\infty, \Omega).$$

Furthermore, the upper Minkowski content of $(\mathbf{N}, \Psi(\Omega))$ is in $(0, \infty)$ if and only if the upper Minkowski content of Ω at infinity is in $(0, \infty)$.

Remark 4.91. We do not know if the inequalities in Theorem 4.84 and Corollary 4.90 are sharp (see Problem A.7).

Let us now go back for a moment to the notion of ϕ-shell Minkowski content at infinity and introduce a new definition.

Definition 4.92. Let $\Omega \subseteq \mathbb{R}^N$ be a Lebesgue measurable set and let $\phi > 1$. Now, we can define the *upper* and *lower ϕ-shell box dimension* of Ω *at infinity*:

$$\begin{aligned} \overline{\dim}^\phi_B(\infty, \Omega) &:= \sup\{r \in \mathbb{R} : \overline{\mathcal{M}}^r_\phi(\infty, \Omega) = +\infty\} \\ &= \inf\{r \in \mathbb{R} : \overline{\mathcal{M}}^r_\phi(\infty, \Omega) = 0\}; \end{aligned} \tag{4.7.21}$$

$$\begin{aligned} \underline{\dim}^\phi_B(\infty, \Omega) &:= \sup\{r \in \mathbb{R} : \underline{\mathcal{M}}^r_\phi(\infty, \Omega) = +\infty\} \\ &= \inf\{r \in \mathbb{R} : \underline{\mathcal{M}}^r_\phi(\infty, \Omega) = 0\}. \end{aligned} \tag{4.7.22}$$

As we can see, this gives us a way to analyze fractal properties of unbounded sets at infinity that do not have to be of finite Lebesgue measure. From Proposition 4.87 we immediately have the next result concerning the case of sets Ω of finite Lebesgue measure and the new definition of the ϕ-shell Minkowski dimensions.

Corollary 4.93. *Let $\Omega \subseteq \mathbb{R}^N$ be of finite Lebesgue measure such that $\overline{\dim}_B(\infty, \Omega) < -N$. Then for every $\phi > 1$ we have that*

$$\overline{\dim}_B^{\phi}(\infty, \Omega) = \overline{\dim}_B(\infty, \Omega)$$

and

$$\underline{\dim}_B^{\phi}(\infty, \Omega) \leq \underline{\dim}_B(\infty, \Omega).$$

Furthermore, if $D := \dim_B^{\phi}(\infty, \Omega)$ exists, then $\dim_B(\infty, \Omega)$ exists and in that case we have

$$D = \dim_B(\infty, \Omega) = \dim_B^{\phi}(\infty, \Omega).$$

Moreover, if Ω is ϕ-shell Minkowski measurable at infinity, then it is Minkowski measurable at infinity and in that case we have

$$\mathcal{M}^D(\infty, \Omega) = \frac{1}{1 - \phi^{N+r}} \mathcal{M}_{\phi}^D(\infty, \Omega).$$

The analog of Corollary 4.93 for the general case when we do not require Ω to be of finite Lebesgue measure still holds. This is the statement of the next proposition that will show that this new notion of the upper ϕ-shell box dimension at infinity is essentially independent of the choice of $\phi > 1$. This is not true for its lower counterpart as we will see in the example provided after the proposition.

Proposition 4.94. *Let $\Omega \subseteq \mathbb{R}^N$ be a Lebesgue measurable set and $\phi_1, \phi_2 \in \mathbb{R}$ such that $1 < \phi_1 < \phi_2$. Then, for $r \in \mathbb{R} \setminus \{-N\}$ we have:*

$$\overline{\mathcal{M}}_{\phi_1}^r(\infty, \Omega) \leq \overline{\mathcal{M}}_{\phi_2}^r(\infty, \Omega) \leq \frac{1 - \phi_1^{(N+r)(\lfloor \log_{\phi_1} \phi_2 \rfloor + 1)}}{1 - \phi_1^{N+r}} \overline{\mathcal{M}}_{\phi_1}^r(\infty, \Omega) \tag{4.7.23}$$

and

$$\frac{1 - \phi_1^{(N+r)\lfloor \log_{\phi_1} \phi_2 \rfloor}}{1 - \phi_1^{N+r}} \underline{\mathcal{M}}_{\phi_1}^r(\infty, \Omega) \leq \underline{\mathcal{M}}_{\phi_2}^r(\infty, \Omega). \tag{4.7.24}$$

In the case when $r = -N$ we have:

$$\overline{\mathcal{M}}_{\phi_1}^{-N}(\infty, \Omega) \leq \overline{\mathcal{M}}_{\phi_2}^{-N}(\infty, \Omega) \leq (\lfloor \log_{\phi_1} \phi_2 \rfloor + 1) \overline{\mathcal{M}}_{\phi_1}^{-N}(\infty, \Omega) \tag{4.7.25}$$

and

$$\lfloor \log_{\phi_1} \phi_2 \rfloor \underline{\mathcal{M}}_{\phi_1}^{-N}(\infty, \Omega) \leq \underline{\mathcal{M}}_{\phi_2}^{-N}(\infty, \Omega). \tag{4.7.26}$$

Moreover, for the ϕ-shell Minkowski box dimensions at infinity we have:

$$\overline{\dim}_B^{\phi_1}(\infty,\Omega) = \overline{\dim}_B^{\phi_2}(\infty,\Omega) \tag{4.7.27}$$

and

$$\underline{\dim}_B^{\phi_1}(\infty,\Omega) \leq \underline{\dim}_B^{\phi_2}(\infty,\Omega). \tag{4.7.28}$$

Furthermore, if $D := \dim_B^{\phi_1}(\infty,\Omega)$ exists, then $\dim_B^{\phi_2}(\infty,\Omega)$ exists as well and in that case we have

$$D = \dim_B^{\phi_1}(\infty,\Omega) = \dim_B^{\phi_2}(\infty,\Omega).$$

Moreover, if Ω is ϕ_1-shell Minkowski measurable at infinity, then it is ϕ_2-shell Minkowski measurable at infinity.

Proof. Firstly, we observe that the left-hand part of (4.7.23) is a simple consequence of the fact that $|B_{t,\phi_1 t}(0)\cap\Omega| \leq |B_{t,\phi_2 t}(0)\cap\Omega|$. Secondly, it is easy to see that

$$\frac{1-\phi_1^{(N+r)\lfloor\log_{\phi_1}\phi_2\rfloor}}{1-\phi_1^{N+r}} > 1$$

is fulfilled regardless of the sign of $N+r\neq 0$. Consequently, this factor gives us a better estimate in (4.7.24) than using the same argument as for (4.7.23). Now, we let $k := \lfloor\log_{\phi_1}\phi_2\rfloor$ and observe that

$$\sum_{n=0}^{k-1}|B_{\phi_1^n t,\phi_1^{n+1}t}(0)\cap\Omega| \leq |B_{t,\phi_2 t}(0)\cap\Omega| \leq \sum_{n=0}^{k}|B_{\phi_1^n t,\phi_1^{n+1}t}(0)\cap\Omega|.$$

Furthermore, from this we get that

$$\sum_{n=0}^{k-1}\phi_1^{n(N+r)}\frac{|B_{\phi_1^n t,\phi_1^{n+1}t}(0)\cap\Omega|}{(\phi_1^n t)^{N+r}} \leq \frac{|B_{t,\phi_2 t}(0)\cap\Omega|}{t^{N+r}} \leq \sum_{n=0}^{k}\phi_1^{n(N+r)}\frac{|B_{\phi_1^n t,\phi_1^{n+1}t}(0)\cap\Omega|}{(\phi_1^n t)^{N+r}}.$$

Finally, taking the upper and lower limits when $t\to+\infty$ gives us

$$\overline{\mathcal{M}}_{\phi_2}^r(\infty,\Omega) \leq \overline{\mathcal{M}}_{\phi_1}^r(\infty,\Omega)\sum_{n=0}^{k}\phi_1^{n(N+r)} = \begin{cases}\frac{1-\phi_1^{(N+r)(\lfloor\log_{\phi_1}\phi_2\rfloor+1)}}{1-\phi_1^{N+r}}\overline{\mathcal{M}}_{\phi_1}^r(\infty,\Omega), & r\neq -N\\ (\lfloor\log_{\phi_1}\phi_2\rfloor+1)\overline{\mathcal{M}}_{\phi_1}^{-N}(\infty,\Omega), & r=-N\end{cases}$$

and

$$\underline{\mathcal{M}}_{\phi_2}^r(\infty,\Omega) \geq \underline{\mathcal{M}}_{\phi_1}^r(\infty,\Omega)\sum_{n=0}^{k-1}\phi_1^{n(N+r)} = \begin{cases}\frac{1-\phi_1^{(N+r)\lfloor\log_{\phi_1}\phi_2\rfloor}}{1-\phi_1^{N+r}}\underline{\mathcal{M}}_{\phi_1}^r(\infty,\Omega), & r\neq -N\\ \lfloor\log_{\phi_1}\phi_2\rfloor\underline{\mathcal{M}}_{\phi_1}^{-N}(\infty,\Omega), & r=-N.\end{cases}$$

$\square$

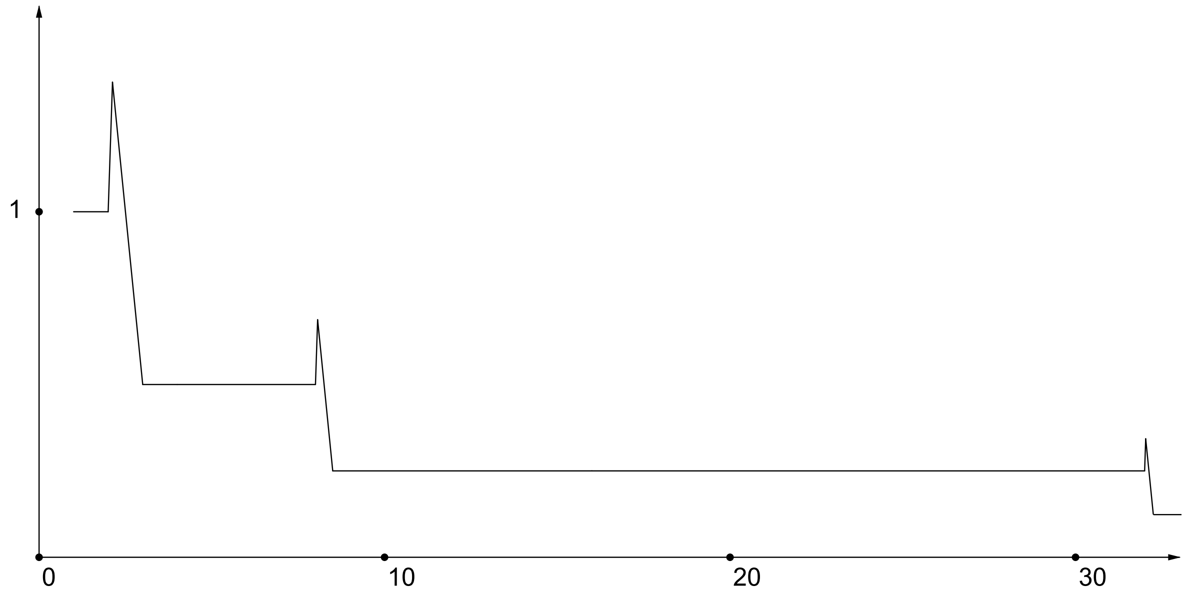

Figure 4.3: A plot of the function $t \mapsto |B_{t,4t}(0) \cap \Omega|$ from Example 4.95. Here, the parameter q is equal to 1/2. Note that the axes are not in the same scale.

Example 4.95. Let us fix a number $q > 0$ and define $\Omega \subseteq \mathbb{R}$ as a disjoint union of intervals:

$$\Omega := \bigcup_{n=0}^{\infty} \left(2^{2n+1}, 2^{2n+1} + \frac{1}{2^{2nq}} \right).$$

Note that $|\Omega| = \sum_{n=0}^{\infty} 2^{-2nq} = 4^q/(4^q - 1)$. We take $\phi = 2$ and observe that for the sequence $t_n := 2^{2n}$, where $n \geq 0$ we have that $|B_{t_n,2t_n}(0) \cap \Omega| = 0$. This implies that $\underline{\mathcal{M}}_2^r(\infty, \Omega) = 0$ for every $r \in \mathbb{R}$, and, consequently, $\underline{\dim}_B^2(\infty, \Omega) = -\infty$. On the other hand, if we take $\phi = 4$, we have for $n \in \mathbb{N} \cup \{0\}$ that

$$|B_{t,4t}(0) \cap \Omega| = \begin{cases} 4^{-nq}, & t \in [2^{2n}, 2^{2n+1}] \\ 4^{-nq} + 3(t - 2^{2n+1}), & t \in [2^{2n+1}, 2^{2n+1} + 4^{-(n+1)q-1}] \\ 4^{-nq} + 4^{-(n+1)q} + 2^{2n+1} - t, & t \in [2^{2n+1} + 4^{-(n+1)q-1}, 2^{2n+1} + 4^{-nq}] \\ 4^{-(n+1)q}, & t \in [2^{2n+1} + 4^{-nq}, 2^{2(n+1)}]. \end{cases}$$

As we can see, the 4-shell function is constant on the intervals of the first and fourth type above, and linear on the intervals of the second and third type. In other words, it is a kind of a step function with 'tents' between every two steps (see Figure 4.3):

$$|B_{t,4t}(0) \cap \Omega| = \begin{cases} 2^{q\{\log_2 t\}} t^{-q}, & t \in [2^{2n}, 2^{2n+1}] \\ 2^{q(1-\{\log_2 t\})} t^{-q} + 3(t - 2^{2n+1}), & t \in [2^{2n+1}, 2^{2n+1} + 4^{-(n+1)q-1}] \\ 2^{-q\{\log_2 t\}} (2^q + 2^{-q}) t^{-q} + 2^{2n+1} - t, & t \in [2^{2n+1} + 4^{-(n+1)q-1}, 2^{2n+1} + 4^{-nq}] \\ 2^{q(-1-\{\log_2 t\})} t^{-q}, & t \in [2^{2n+1} + 4^{-(n+1)q-1}, 2^{2(n+1)}]. \end{cases}$$

From this we have
$$\frac{1}{t^q} \leq |B_{t,4t}(0) \cap \Omega| \leq \frac{2^q(1+3\cdot 4^{-q-1})}{t^q}$$
which, in turn, implies that $\dim_B^4(\infty,\Omega) = -1-q$ and

$$1 \leq \underline{\mathcal{M}}_4^{-1-q}(\infty,\Omega) \leq \overline{\mathcal{M}}_4^{-1-q}(\infty,\Omega) \leq 2^q(1+3\cdot 4^{-q-1}).$$

This demonstrates that the conclusions of Proposition 4.94 concerning the lower ϕ-shell Minkowski content and ϕ-shell box dimension at infinity cannot be improved in general[26] since we have
$$0 = \underline{\mathcal{M}}_2^{-1-q}(\infty,\Omega) < \underline{\mathcal{M}}_4^{-1-q}(\infty,\Omega)$$
and
$$-\infty = \underline{\dim}_B^2(\infty,\Omega) < \overline{\dim}_B^2(\infty,\Omega) = \dim_B^4(\infty,\Omega) = -1-q.$$

Similarly as before, for $\phi > 1$ we can define the analogous notions of ϕ-shell Minkowski contents and box dimensions in the standard case of relative fractal drums. Furthermore, analogs of Proposition 4.87 and Corollary 4.93 are valid and as they are proved essentially in the same way as before, we will state them here without proof. Firstly, for $A \subseteq \mathbb{R}^N$, $0 < a < b$ and $\phi > 1$ we introduce the following notation:

$$A_{a,b} := \{x \in \mathbb{R}^N \,:\, a < d(x,A) < b\}. \tag{4.7.29}$$

Similarly, for $A \subseteq \mathbb{S}^N$ we define

$$A_{a,b,\mathbb{S}} := \{x \in \mathbb{S}^N \,:\, a < d_{\mathbb{S}}(x,A) < b\}. \tag{4.7.30}$$

Definition 4.96. Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$, $r \in \mathbb{R}$ and $\phi > 1$. We define the *upper* and *lower ϕ-shell Minkowski content* of the relative fractal drum (A,Ω) as
$$\overline{\mathcal{M}}_\phi^r(A,\Omega) := \limsup_{\delta\to 0^+} \frac{|A_{\delta/\phi,\delta} \cap \Omega|}{\delta^{N-r}} \tag{4.7.31}$$
and
$$\underline{\mathcal{M}}_\phi^r(A,\Omega) := \liminf_{\delta\to 0^+} \frac{|A_{\delta/\phi,\delta} \cap \Omega|}{\delta^{N-r}}. \tag{4.7.32}$$
Furthermore, we define the (*upper, lower*) *ϕ-shell dimension* of the relative fractal drum (A,Ω) in the usual way:

$$\begin{aligned}\overline{\dim}_B^\phi(A,\Omega) &:= \sup\{r \in \mathbb{R} \,:\, \overline{\mathcal{M}}_\phi^r(A,\Omega) = +\infty\} \\ &= \inf\{r \in \mathbb{R} \,:\, \overline{\mathcal{M}}_\phi^r(A,\Omega) = 0\};\end{aligned} \tag{4.7.33}$$

[26] A similar example can be constructed in $\mathbb{R}^N$ by using the shells of appropriate radii of the N-dimensional ball centered at the origin.

$$\begin{aligned}\underline{\dim}_B^\phi(A,\Omega) &:= \sup\{r\in\mathbb{R}\,:\,\underline{\mathcal{M}}_\phi^r(A,\Omega)=+\infty\}\\ &= \inf\{r\in\mathbb{R}\,:\,\underline{\mathcal{M}}_\phi^r(A,\Omega)=0\}.\end{aligned}\tag{4.7.34}$$

As usual, if for some $D\in\mathbb{R}$ the upper and lower limits above coincide, we denote by $\mathcal{M}_\phi^D(A,\Omega)$ the *ϕ-shell Minkowski content* of the relative fractal drum (A,Ω). In this case, D is equal to the *ϕ-shell dimension* of the relative fractal drum (A,Ω) denoted by $\dim_B^\phi(A,\Omega)$.

Remark 4.97. We can also introduce analogous definitions in the context of relative fractal drums on the Riemann sphere $\mathbb{S}^N$. Namely, for a relative fractal drum (A,Ω) in $\mathbb{S}^N$ we replace the appropriate elements of Definition 4.96 with their spherical counterparts; that is, we replace $|A_{\delta/\phi,\delta}\cap\Omega|$ by $|A_{\delta/\phi,\delta,\mathbb{S}}\cap\Omega|_{\mathbb{S}}$. We will denote the corresponding notions with $\mathcal{M}_{\phi,\mathbb{S}}^r(A,\Omega)$ and $\dim_{\mathbb{S}}^\phi(A,\Omega)$. Of course, the upper and lower counterparts are denoted, as usual, with an overline and an underline, respectively.

Furthermore, it is straightforward to extend notions like (relative) Minkowski degeneracy and (relative) Minkowski measurability in the ϕ-shell sense, both, in $\mathbb{R}^N$ and on the Riemann sphere $\mathbb{S}^N$.

Proposition 4.98. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$. Then, for every $\phi>1$ and $r<N$ we have*

$$\overline{\mathcal{M}}_\phi^r(A,\Omega)\le\overline{\mathcal{M}}^r(A,\Omega)\le\frac{1}{1-\phi^{r-N}}\,\overline{\mathcal{M}}_\phi^r(A,\Omega)\tag{4.7.35}$$

and

$$\frac{1}{1-\phi^{r-N}}\,\underline{\mathcal{M}}_\phi^r(A,\Omega)\le\underline{\mathcal{M}}^r(A,\Omega).\tag{4.7.36}$$

Corollary 4.99. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $\overline{\dim}_B(A,\Omega)<N$. Then for every $\phi>1$ we have that*

$$\overline{\dim}_B^\phi(A,\Omega)=\overline{\dim}_B(A,\Omega)$$

and

$$\underline{\dim}_B^\phi(A,\Omega)\le\underline{\dim}_B(A,\Omega).$$

Furthermore, if $D:=\dim_B^\phi(A,\Omega)$ exists, then $\dim_B(A,\Omega)$ exists and in that case we have

$$D=\dim_B(A,\Omega)=\dim_B^\phi(A,\Omega).$$

Moreover, if (A,Ω) is ϕ-shell Minkowski measurable, then it is Minkowski measurable and in that case we have

$$\mathcal{M}^D(A,\Omega)=\frac{1}{1-\phi^{r-N}}\,\mathcal{M}_\phi^D(A,\Omega).$$

After this digression, let us go back to the case of sets at infinity. The next example will show that for every Lebesgue measurable set $\Omega \subseteq \mathbb{R}^N$ and for every $\phi > 1$ we have that $\underline{\dim}_B^\phi(\infty,\Omega) \le \overline{\dim}_B^\phi(\infty,\Omega) \le 0$. Furthermore, we will see that analyzing the fractal properties of Ω at infinity in the ϕ-shell sense is closely related to analyzing the fractal properties of the relative fractal drum $(\mathbf{N},\Psi(\Omega))$ in the ϕ-shell sense on the Riemann sphere $\mathbb{S}^N$. Moreover, the fact that $\overline{\dim}_B^\phi(\infty,\Omega) \le 0$ is in accord with the fact that the one-point set $\{\mathbf{N}\}$ has spherical upper ϕ-shell box dimension relative to any subset of $\mathbb{S}^N$ maximally equal to 0.

Example 4.100. Let $\Omega = \mathbb{R}^N$ and $\phi > 1$. Then we have that $\dim_B^\phi(\infty,\mathbb{R}^N) = 0$. This follows from

$$|B_{t,\phi t}(0)\cap\mathbb{R}^N| = |B_{\phi t}(0)| - |B_t(0)| = \frac{\pi^{\frac{N}{2}}(\phi^N-1)t^N}{\Gamma\left(\frac{N}{2}+1\right)}.$$

Moreover, we have that

$$\mathcal{M}_\phi^0(\infty,\mathbb{R}^N) = \frac{\pi^{\frac{N}{2}}(\phi^N-1)}{\Gamma\left(\frac{N}{2}+1\right)}. \tag{4.7.37}$$

As a consequence of the above example we immediately get the following proposition.

Proposition 4.101. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ and $\phi > 1$. Then the upper and lower ϕ-shell box dimensions of Ω at infinity are always nonpositive, i.e.,*

$$\underline{\dim}_B^\phi(\infty,\Omega) \le \overline{\dim}_B^\phi(\infty,\Omega) \le 0.$$

In the next proposition we will establish a connection between the ϕ-shell Minkowski contents of (∞,Ω) and the ϕ-shell Minkowski content of its image $(\mathbf{N},\Psi(\Omega))$ on the Riemann sphere $\mathbb{S}^N$.

Proposition 4.102. *Let $\Omega \subseteq \mathbb{R}^N$ be a Lebesgue measurable set. Then for $r \in \mathbb{R}$, $\phi > 1$ and for every $\varepsilon > 0$ we have the following inequalities*:

$$\frac{2^r}{\phi^{2N}}\,\overline{\mathcal{M}}_\phi^r(\infty,\Omega) \le \overline{\mathcal{M}}_{\phi,\mathbb{S}}^r(\mathbf{N},\Psi(\Omega)) \le 2^r\overline{\mathcal{M}}_{\phi+\varepsilon}^r(\infty,\Omega); \tag{4.7.38}$$

$$\frac{2^r}{\phi^{2N}}\,\underline{\mathcal{M}}_\phi^r(\infty,\Omega) \le \underline{\mathcal{M}}_{\phi,\mathbb{S}}^r(\mathbf{N},\Psi(\Omega)) \le 2^r\underline{\mathcal{M}}_{\phi+\varepsilon}^r(\infty,\Omega). \tag{4.7.39}$$

Proof. Similarly as in the proof of Theorem 4.84 we have

$$|\{\mathbf{N}\}_{(\phi^{-1}\delta,\delta),\mathbb{S}}\cap\Psi(\Omega)|_{\mathbb{S}} = \int_{B_{\cot\frac{\delta}{2},\cot\frac{\delta}{2\phi}}(0)\cap\Omega}\frac{2^N}{(1+|x|^2)^N}\,\mathrm{d}x_1\cdots\mathrm{d}x_N.$$

From this we get the following estimates:

$$|\{\mathbf{N}\}_{(\phi^{-1}\delta,\delta),\mathbb{S}}\cap\Psi(\Omega)|_{\mathbb{S}} \le \frac{2^N}{(1+\cot^2\frac{\delta}{2})^N}|B_{\cot\frac{\delta}{2},\cot\frac{\delta}{2\phi}}(0)\cap\Omega| \tag{4.7.40}$$

and

$$|\{\mathbf{N}\}_{(\phi^{-1}\delta,\delta),\mathbb{S}}\cap\Psi(\Omega)|_{\mathbb{S}}\geq\frac{2^N}{(1+\cot^2\frac{\delta}{2\phi})^N}|B_{\cot\frac{\delta}{2},\cot\frac{\delta}{2\phi}}(0)\cap\Omega|. \tag{4.7.41}$$

Let us now fix $\varepsilon>0$. Then, for such ε there exists $\delta_\varepsilon>0$ such that for every $\delta\in(0,\delta_\varepsilon)$ we have

$$\phi\cot\frac{\delta}{2}\leq\cot\frac{\delta}{2\phi}\leq(\phi+\varepsilon)\cot\frac{\delta}{2}.$$

This can be easily seen, for instance, from the following series expansion of the cotangent function:[27]

$$\cot x=\frac{1}{x}+2x\sum_{k=1}^{\infty}\frac{1}{x^2-k^2\pi^2},\quad x\notin\pi\mathbb{Z}$$

Furthermore, by applying this to inequalities (4.7.40) and (4.7.41) we get

$$|\{\mathbf{N}\}_{(\phi^{-1}\delta,\delta),\mathbb{S}}\cap\Psi(\Omega)|_{\mathbb{S}}\leq\frac{2^N}{(1+\cot^2\frac{\delta}{2})^N}|B_{\cot\frac{\delta}{2},(\phi+\varepsilon)\cot\frac{\delta}{2}}(0)\cap\Omega| \tag{4.7.42}$$

and

$$|\{\mathbf{N}\}_{(\phi^{-1}\delta,\delta),\mathbb{S}}\cap\Psi(\Omega)|_{\mathbb{S}}\geq\frac{2^N}{(1+\cot^2\frac{\delta}{2\phi})^N}|B_{\cot\frac{\delta}{2},\phi\cot\frac{\delta}{2}}(0)\cap\Omega|. \tag{4.7.43}$$

Now, we introduce a new variable $t:=\cot(\delta/2)$ and observe that $\delta\to0^+$ if and only if $t\to+\infty$. Next, for $r\in\mathbb{R}$ we get that

$$\begin{aligned}\frac{|\{\mathbf{N}\}_{(\phi^{-1}\delta,\delta),\mathbb{S}}\cap\Psi(\Omega)|_{\mathbb{S}}}{\delta^{N-r}}&\leq\frac{2^N}{(1+t^2)^N}\frac{|B_{t,(\phi+\varepsilon)t}(0)\cap\Omega|}{2^{N-r}(\operatorname{arccot}t)^{N-r}}\\&=\frac{2^rt^{N+r}(\operatorname{arccot}t)^{r-N}}{(1+t^2)^N}\frac{|B_{t,(\phi+\varepsilon)t}(0)\cap\Omega|}{t^{N+r}}.\end{aligned}$$

Taking the upper limit when $\delta\to0^+$ we have

$$\overline{\mathcal{M}}^r_{\phi,\mathbb{S}}(\mathbf{N},\Psi(\Omega))\leq2^r\,\overline{\mathcal{M}}^r_{\phi+\varepsilon}(\infty,\Omega).$$

Analogously we get the right-hand side inequality in (4.7.39).

To get the other reversed inequalities in (4.7.38) and (4.7.39) we observe that from (4.7.43) we have

$$\begin{aligned}\frac{|\{\mathbf{N}\}_{(\phi^{-1}\delta,\delta),\mathbb{S}}\cap\Psi(\Omega)|_{\mathbb{S}}}{\delta^{N-r}}&\geq\frac{2^N}{\left(1+\cot^2\left(\frac{1}{\phi}\operatorname{arccot}t\right)\right)^N}\frac{|B_{t,\phi t}(0)\cap\Omega|}{2^{N-r}(\operatorname{arccot}t)^{N-r}}\\&=\frac{2^rt^{N+r}(\operatorname{arccot}t)^{r-N}}{\left(1+\cot^2\left(\frac{1}{\phi}\operatorname{arccot}t\right)\right)^N}\frac{|B_{t,\phi t}(0)\cap\Omega|}{t^{N+r}}\end{aligned}$$

[27]The function represented by the series is actually meromorphic on $\mathbb{C}$ and coincides with the complex cotangent. This result is due to Euler and its proof can be seen, for instance, in [Car].

and taking the upper limit when $\delta \to 0^+$ yields

$$\overline{\mathcal{M}}^r_{\phi,\mathbb{S}}(\mathbf{N}, \Psi(\Omega)) \geq \frac{2^r}{\phi^{2N}} \overline{\mathcal{M}}^r_\phi(\infty, \Omega).$$

The proof is completed by making the same observations in the case of the lower limit when $\delta \to 0^+$. □

In case of the upper Minkowski contents, i.e., in inequalities (4.7.38) above, we can get rid of the ε. This is a consequence of Proposition 4.94 and is stated in the following corollary.

Corollary 4.103. *Let $\Omega \subseteq \mathbb{R}^N$ be a Lebesgue measurable set. Then for $r \in \mathbb{R}$ and $\phi > 1$ we have the following inequalities for the upper Minkowski contents*:

$$\frac{2^r}{\phi^{2N}} \overline{\mathcal{M}}^r_\phi(\infty, \Omega) \leq \overline{\mathcal{M}}^r_{\phi,\mathbb{S}}(\mathbf{N}, \Psi(\Omega)) \leq 2^r(1 + \phi^{N+r})\overline{\mathcal{M}}^r_\phi(\infty, \Omega). \tag{4.7.44}$$

Proof. We fix $\varepsilon > 0$ and use (4.7.38) together with Proposition 4.94 to get

$$2^r \frac{1 - \phi^{(N+r)(\lfloor \log_\phi(\phi+\varepsilon) \rfloor + 1)}}{1 - \phi^{N+r}} \overline{\mathcal{M}}^r_\phi(\infty, \Omega) \geq \overline{\mathcal{M}}^r_{\phi,\mathbb{S}}(\mathbf{N}, \Psi(\Omega))$$

in case when $r \neq -N$. Furthermore, in the case when $r = -N$ we get

$$2^{-N}(\lfloor \log_\phi(\phi + \varepsilon) \rfloor + 1)\overline{\mathcal{M}}^{-N}_\phi(\infty, \Omega) \geq \overline{\mathcal{M}}^{-N}_{\phi,\mathbb{S}}(\mathbf{N}, \Psi(\Omega)).$$

Finally, by letting $\varepsilon \to 0^+$ in both cases we prove the corollary. □

Remark 4.104. We do not know at this moment if the analog of Corollary 4.103 is valid for the lower Minkowski contents due to not having the analog of inequalities that are involved in the proof (see Problem A.8).

Furthermore, one could let $\varepsilon \to 0^+$ in (4.7.38) (or (4.7.39)) and the limit on the right-hand side exists since the corresponding function is nondecreasing but it could possibly be strictly greater than the upper (or lower) r-dimensional ϕ-shell Minkowski content of Ω at infinity.

4.8 Surface Minkowski Content at Infinity

In this section we will take a closer look into the connection between the notion of the ϕ-shell Minkowski content at infinity and a new notion of *surface Minkowski content at infinity* introduced just below. Inspired by Remark 4.89 we now introduce the following definition.

Definition 4.105. Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$, and denote with H^{N-1} the $(N-1)$-dimensional Hausdorff measure. Then, for $r \in \mathbb{R}$, we define the *r-dimensional upper surface Minkowski content of Ω at infinity* as

$$\overline{\mathcal{S}}^r(\infty, \Omega) := \limsup_{t \to +\infty} \frac{H^{N-1}(S_t(0) \cap \Omega)}{t^{N-1+r}} \tag{4.8.1}$$

where $S_t(0)$ denotes the $(N-1)$-dimensional sphere of radius t with center at 0. Analogously, we define the *r-dimensional lower surface Minkowski content of Ω at infinity* as

$$\underline{\mathcal{S}}^r(\infty, \Omega) := \liminf_{t \to +\infty} \frac{H^{N-1}(S_t(0) \cap \Omega)}{t^{N-1+r}}. \tag{4.8.2}$$

Furthermore, if for some $r \in \mathbb{R}$ the upper and lower limits above coincide we call this value the *r-dimensional surface Minkowski content of Ω at infinity* and denote it with $\mathcal{S}^r(\infty, \Omega)$.

Proposition 4.106. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$. Then, for a.e. $t > 0$ we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t}|B_t(0) \cap \Omega| = H^{N-1}(S_t(0) \cap \Omega). \tag{4.8.3}$$

Furthermore, if $|\Omega| < \infty$ then we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}|B_t(0)^c \cap \Omega| = -H^{N-1}(S_t(0) \cap \Omega). \tag{4.8.4}$$

Proof. We will use [Žu2, Proposition 2.10]. In short, this result states that for a closed subset A of $\mathbb{R}^N$ with Lebesgue measure equal to zero and a Lebesgue measurable[28] subset Ω of $\mathbb{R}^N$ we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}|A_t \cap \Omega| = H^{N-1}(\partial A_t \cap \Omega) \tag{4.8.5}$$

for a.e. $t > 0$. This proves (4.8.3) if we let $A := \{0\}$. Furthermore, since for Ω of finite Lebesgue measure we have that $|B_t(0)^c \cap \Omega| = |\Omega| - |B_t(0) \cap \Omega|$, (4.8.3) implies (4.8.4) in this case. ☐

Proposition 4.107. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$. Then, we have*

$$\lim_{\phi \to 1^+} \frac{|B_{t,\phi t}(0) \cap \Omega|}{\log \phi} = t\, H^{N-1}(S_t(0) \cap \Omega) \tag{4.8.6}$$

for a.e. $t > 0$.

Proof. We observe that $|B_{t,\phi t}(0) \cap \Omega| = |B_{\phi t}(0) \cap \Omega| - |B_t(0) \cap \Omega|$ and by letting $h = \log \phi$ we have

$$\lim_{\phi \to 1^+} \frac{|B_{t,\phi t}(0) \cap \Omega|}{\log \phi} = \lim_{h \to 0^+} \frac{|B_{\mathrm{e}^h t}(0) \cap \Omega| - |B_t(0) \cap \Omega|}{h}. \tag{4.8.7}$$

[28]The original assumption in [Žu2, Proposition 2.10] was that Ω is open, but by looking at the proof, it is clear that it is enough to assume that Ω is Lebesgue measurable.

Furthermore, if we define

$$f(\tau) := |B_{e^\tau}(0) \cap \Omega|, \tag{4.8.8}$$

we can rewrite (4.8.7) as

$$\lim_{\phi \to 1^+} \frac{|B_{t,\phi t}(0) \cap \Omega|}{\log \phi} = \lim_{h \to 0^+} \frac{f(\log t + h) - f(\log t)}{h} = f'(\log t). \tag{4.8.9}$$

On the other hand, by the chain rule and from (4.8.3), we have that

$$f'(\tau) = e^\tau\, H^{N-1}(S_{e^\tau}(0) \cap \Omega) \tag{4.8.10}$$

for a.e. $\tau \in \mathbb{R}$. Finally, combining this with (4.8.9) we prove the theorem. $\square$

Now, for a relative fractal drum (∞, Ω), if we could justify the interchange of the order of taking the limit as $\phi \to 1^+$ and the upper limit as $t \to +\infty$ we would get that

$$\overline{\mathcal{S}}^r(\infty, \Omega) = \limsup_{t \to +\infty} \frac{H^{N-1}(S_t(0) \cap \Omega)}{t^{N-1+r}} = \lim_{\phi \to 1^+} \frac{\overline{\mathcal{M}}^r_\phi(\infty, \Omega)}{\log \phi}, \tag{4.8.11}$$

and an analogous equality for the lower surface Minkowski content of (∞, Ω). Of course, the interchange above is not justified and the conditions when it can be made need to be investigated in future work (see Problem A.9). Also, note that, a priori, the limit $\lim_{\phi \to 1^+} \overline{\mathcal{M}}^r_\phi(\infty, \Omega)/\log \phi$ does not have to even exist.

We will get an interesting result concerning the distance zeta function of Ω at infinity in the case when (∞, Ω) is ϕ-shell Minkowski measurable for every $\phi \in (0, \delta)$, where δ is some positive constant and the limit $\lim_{\phi \to 1^+} \mathcal{M}^r_\phi(\infty, \Omega)/\log \phi$ exists. We will not even need the assumption that $|\Omega| < \infty$ as it will turn out that the new notions of ϕ-shell Minkowski dimensions and contents are well connected with the distance zeta function of (∞, Ω) even in the case when $|\Omega| = \infty$. (See Corollary 4.122 below.) We point out that for the definition of the distance zeta function of Ω at infinity to make sense it is not actually needed that $|\Omega| < \infty$, but more on this will be presented in Section 4.9 below.

For now, we can state the following theorem which is a consequence of the Moore–Osgood theorem about the interchange of two limits. (See, e.g., [KaMi].)

Theorem 4.108. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ such that $D = \dim_B^\phi(\infty, \Omega)$ exists for some $\phi > 1$. Furthermore, assume that the following limits exist*

$$f(\psi) := \lim_{t \to +\infty} \frac{|B_{t,\psi t}(0) \cap \Omega|}{t^{N+D} \log \psi}, \tag{4.8.12}$$

$$g(t) := \lim_{\psi \to 1^+} \frac{|B_{t,\psi t}(0) \cap \Omega|}{t^{N+D} \log \psi}. \tag{4.8.13}$$

Then, if (4.8.12) *exists uniformly in* ψ,[29] *or, on the other hand, if* (4.8.13) *exists uniformly in* t*, we have the following equality*:

$$\mathcal{S}^D(\infty,\Omega)=\lim_{\psi\to1^+}\frac{\mathcal{M}_\psi^D(\infty,\Omega)}{\log\psi}. \tag{4.8.14}$$

It is clear that the notions of upper and lower surface Minkowski contents of Ω at infinity introduced in Definition 4.105 are also well defined when $|\Omega|=\infty$ as it is the case with the ϕ-shell Minkowski content at infinity. Furthermore, we can also introduce the notion of the *upper* and *lower surface Minkowski* (*or box*) *dimension of* (∞,Ω) in the standard way.

Definition 4.109. Let $\Omega\subseteq\mathbb{R}^N$ be a Lebesgue measurable set. We define the *upper* and *lower surface box dimension* of (∞,Ω) *at infinity*:

$$\begin{aligned}\overline{\dim}_{\mathcal{S}}(\infty,\Omega)&:=\sup\{r\in\mathbb{R}\,:\,\overline{\mathcal{S}}^r(\infty,\Omega)=+\infty\}\\&=\inf\{r\in\mathbb{R}\,:\,\overline{\mathcal{S}}^r(\infty,\Omega)=0\};\end{aligned} \tag{4.8.15}$$

$$\begin{aligned}\underline{\dim}_{\mathcal{S}}(\infty,\Omega)&:=\sup\{r\in\mathbb{R}\,:\,\underline{\mathcal{S}}^r(\infty,\Omega)=+\infty\}\\&=\inf\{r\in\mathbb{R}\,:\,\underline{\mathcal{S}}^r(\infty,\Omega)=0\}.\end{aligned} \tag{4.8.16}$$

Let us now revisit Example 4.100 where we have obtained the ϕ-shell Minkowski content of $(\infty,\mathbb{R}^N)$.

Example 4.110. Recall that $\mathcal{M}_\phi^0(\infty,\mathbb{R}^N)=\pi^{\frac N2}(\phi^N-1)/\Gamma\left(\frac N2+1\right)$ and note that in this example we have

$$\lim_{\phi\to1^+}\frac{\mathcal{M}_\phi^0(\infty,\mathbb{R}^N)}{\log\phi}=\frac{N\pi^{\frac N2}}{\Gamma\left(\frac N2+1\right)}. \tag{4.8.17}$$

On the other hand, for the 0-dimensional surface Minkowski content of $(\infty,\mathbb{R}^N)$ we have

$$\mathcal{S}^0(\infty,\mathbb{R}^N)=\lim_{t\to+\infty}\frac{H^{N-1}(S_t(0))}{t^{N-1}}=\frac{N\pi^{\frac N2}}{\Gamma\left(\frac N2+1\right)} \tag{4.8.18}$$

so that in the conclusion of Theorem 4.108 holds which is not surprising due to the simple nature of the relative fractal drum $(\infty,\mathbb{R}^N)$. One can also easily check that the hypothesis of Theorem 4.108 are satisfied in this example.

We would like to point out here that one would like to establish analogous relations between the relative Minkowski content and the corresponding relative surface Minkowski content as was done in [RatWi2] for the nonrelative case. One of the problems that arises is in the fact that for a relative fractal drum (A,Ω) in $\mathbb{R}^N$, its relative tube function $t\mapsto$

[29]Here, uniformly in ψ means that for every $\varepsilon>0$ there is a $T>0$ such that for all $\psi\in(1,\phi)$ and $t>T$ we have $\left|f(\psi)-\frac{|B_{t,\psi t}(0)\cap\Omega|}{t^{N+D}\log\psi}\right|<\varepsilon$.

$|A_t \cap \Omega|$ need not be a Kneser function of order N. This is demonstrated in Example 4.111 below. Let us recall that a function $f\colon (0,\infty) \to (0,\infty)$ is called a *Kneser function of order* $r \geq 1$, if for all $0 < a \leq b < \infty$ and $\phi \geq 1$ we have

$$f(\phi b) - f(\phi a) \leq \phi^r (f(b) - f(a)). \tag{4.8.19}$$

This property played a crucial part of proving that a bounded subset A of $\mathbb{R}^N$ is Minkowski nondegenerate if and only if it is surface Minkowski nondegenerate in [RatWi2]. More precisely, the fact that for a bounded subset $A \subseteq \mathbb{R}^N$ its tube function $t \mapsto |A_t|$ is a Kneser function of order N (see [Kne]). We also point out that the same problem arises in the context of relative fractal drums of type (∞, Ω). (See Problem A.10.)

We conclude this section with an example that shows that there exists a relative fractal drum (A, Ω) in $\mathbb{R}^2$ such that its relative tube function $t \mapsto |A_t \cap \Omega|$ is not a Kneser function of order 2.

Example 4.111. Let (A, Ω) be a relative fractal drum in $\mathbb{R}^2$ such that $A := \{(0,0)\}$ and Ω is the the closed quarter-disc $\overline{B_2(0,0)} \cap \{y \geq 0\} \cap \{x \geq 0\}$ intersected with $B_1(0,1)^c$. We can calculate its relative tube function for $t \leq 2$ by using polar coordinates in $\mathbb{R}^2$:

$$\begin{aligned} |A_t \cap \Omega| &= \int_0^t r\,\mathrm{d}r \int_0^{\arcsin(r/2)} \mathrm{d}\varphi \\ &= \int_0^t r \arcsin\frac{r}{2}\,\mathrm{d}r \\ &= \left(\frac{t^2}{2} - 1\right) \arcsin\frac{t}{2} + \frac{t}{4}\sqrt{4 - t^2}. \end{aligned} \tag{4.8.20}$$

From this we can see that $V(t) := |A_t \cap \Omega|$ is not a Kneser function of order 2 since for $a = 1$, $b = \sqrt{2}$ and $\phi = \sqrt{2}$ we have

$$V(\phi b) - V(\phi a) = V(2) - V(\sqrt{2}) = \frac{\pi - 1}{2} \approx 1.0708,$$

which is greater than

$$\phi^2 (V(b) - V(a)) = 2(V(\sqrt{2}) - V(1)) = 1 + \frac{\pi}{6} - \frac{\sqrt{3}}{2} \approx 0.65755,$$

so that (4.8.19) is not satisfied.

Note that from (4.8.20) we have that $\lim_{t\to 0^+} \frac{V(t)}{t^3} = \frac{1}{6}$ and, consequently,

$$\dim_B(A, \Omega) = -1 \quad \text{and} \quad \mathcal{M}^{-1}(A, \Omega) = \frac{1}{6}. \tag{4.8.21}$$

There should be no problem to generalize the above example to $\mathbb{R}^N$ by mimicking the above construction but we will not get into the details. Also, one can think of a similar example constructed at infinity; that is a relative fractal drum (∞, Ω) such that its relative tube function at infinity is not a Kneser function of order N.

4.9 Lapidus Zeta Functions at Infinity and the ϕ-shell Minkowski Content

In this section we will show that the results about the Lapidus zeta functions of subsets of finite Lebesgue measure at infinity studied so far can be generalized to the case of subsets that do not have finite Lebesgue measure. The generalization will be made by using the notions of the ϕ-shell Minkowski contents and box dimensions at infinity. To this end we will need the following result which complements Proposition 4.22.

Proposition 4.112. *Let $\Omega \subseteq \mathbb{R}^N$ be a Lebesgue measurable set, $T > 0$ and $u\colon (T, +\infty) \to [0, +\infty)$ a strictly decreasing C^1 function such that $u(t) \to 0$ as $t \to +\infty$. Then, the following equality holds*

$$\int_{{}_T\Omega} u(|x|)\,\mathrm{d}x = \int_T^{+\infty} |B_{T,t}(0) \cap \Omega||u'(t)|\,\mathrm{d}t. \tag{4.9.1}$$

Proof. Much as in Proposition 4.22, we will use (4.2.11) which we recall here:

$$\int_X f(x)\,\mathrm{d}x = \int_0^{\infty} |\{x \in X \,:\, f(x) \geq t\}|\,\mathrm{d}t, \tag{4.9.2}$$

where f is a nonnegative Borel function on a separable metric space X. Again, we let $f(x) := u(|x|)$ and $X := {}_T\Omega$. By assumption u is strictly decreasing and $u(+\infty) := \lim_{\tau\to+\infty} u(\tau) = 0$. For the set appearing on the right side of (4.9.2) we have

$$A(t) := \{x \in {}_T\Omega \,:\, u(|x|) \geq t\} = \{x \in {}_T\Omega \,:\, |x| \leq u^{-1}(t)\}.$$

For $0 = u(+\infty) < t < u(T)$ it is clear that

$$A(t) = ({}_T\Omega) \setminus (B_{u^{-1}(t)}(0)^c \cap \Omega) = B_{T,u^{-1}(t)}(0) \cap \Omega.$$

Furthermore, for $t \geq u(T)$ we have that $A(t) = \emptyset$ because $u(T) = \max_{\tau\geq 0} u(\tau)$ and using (4.9.2) we get

$$\begin{aligned}
\int_{{}_T\Omega} u(|x|)\,\mathrm{d}x &= \int_{u(+\infty)}^{u(T)} |B_{T,u^{-1}(t)}(0) \cap \Omega|\,\mathrm{d}t \\
&= \int_{+\infty}^{T} |B_{T,s}(0) \cap \Omega| u'(s)\,\mathrm{d}s \\
&= \int_T^{+\infty} |B_{T,s}(0) \cap \Omega||u'(s)|\,\mathrm{d}s,
\end{aligned}$$

where we have introduced the new variable $s = u^{-1}(t)$ in the second to last equality and this concludes the proof of the proposition. $\square$

Proposition 4.113. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ with $|\Omega| = \infty$. Then for every $\phi > 1$ we have*:

$$-N \leq \overline{\dim}_B^{\phi}(\infty, \Omega) \leq 0.$$

Proof. We reason by contradiction, i.e., we assume that there exists $\phi > 0$ such that $\overline{\dim}_B^{\phi}(\infty, \Omega) < -N$. Then we fix $\sigma \in (\overline{\dim}_B^{\phi}(\infty, \Omega), -N)$ and take T large enough such that there exists a constant $M > 0$ and

$$|B_{t,\phi t}(0) \cap \Omega| \leq M t^{\sigma+N}$$

for every $t > T$. Now we have

$$|{}_T\Omega| = \sum_{n=0}^{\infty} |B_{\phi^n T, \phi^{n+1} T}| \leq M \sum_{n=0}^{\infty} (\phi^n T)^{\sigma+N} = \frac{M T^{\sigma+N}}{1 - \phi^{\sigma+N}} < \infty,$$

since $\sigma + N < 0$ which contradicts $|\Omega| = \infty$. $\square$

The statement of the above proposition is optimal, i.e., there are sets of infinite volume with upper ϕ-shell box dimension equal to $-N$. This illustrates the next example in $\mathbb{R}^2$ and can be easily adapted in the case of $\mathbb{R}^N$.

Example 4.114. Let $\Omega := \{(x, y) \in \mathbb{R}^2 : x > 1,\ 0 < y < x^{-1}\}$ Then, for any $\phi > 1$ and $t > 1$ we have

$$|K_{t,\phi t}(0) \cap \Omega| = \int_t^{\phi t} \frac{1}{t}\,\mathrm{d}t = \log(\phi t) - \log t = \log \phi.$$

From this we see that $\dim_B^{\phi}(\infty, \Omega) = -2$ and $\mathcal{M}_{\phi}^{-2}(\infty, \Omega) = \log \phi$.

Furthermore, we also have that

$$\lim_{\phi \to 1^+} \frac{\mathcal{M}_{\phi}^{-2}(\infty, \Omega)}{\log \phi} = 1 = \mathcal{S}^{-2}(\infty, \Omega), \tag{4.9.3}$$

since it is clear that in this case, the hypotheses of Theorem 4.108 are satisfied. We can check this also by direct computation, since

$$H^1(S_t(0) \cap \Omega) = \frac{1}{2}\sqrt{1 + t^2}\ \arcsin \frac{2}{t^2} \tag{4.9.4}$$

which yields

$$\mathcal{S}^{-2}(\infty, \Omega) = \lim_{t \to +\infty} \frac{H^1(S_t(0) \cap \Omega)}{t^{2-1-2}} = 1. \tag{4.9.5}$$

Example 4.115. Let Ω be a horizontal strip of finite height, i.e., let $h > 0$ and

$$\Omega := \left\{(x, y) \in \mathbb{R}^2 : 0 < y < h\right\}.$$

Then, for any $\phi > 1$ and $t > h$ it is clear that we have

$$2h(\sqrt{\phi^2 t^2 - h^2} - t) \leq |B_{t,\phi t}(0) \cap \Omega| \leq 2h(\phi t - \sqrt{t^2 - h^2}),$$

which implies that $\dim_B^\phi(\infty, \Omega) = -1$ and $\mathcal{M}_\phi^{-1}(\infty, \Omega) = 2h(\phi - 1)$.

Furthermore, we have that

$$\lim_{\phi \to 1^+} \frac{\mathcal{M}_\phi^{-1}(\infty, \Omega)}{\log \phi} = 2h, \tag{4.9.6}$$

and, since $H^1(S_t(0) \cap \Omega) = \sqrt{1 + t^2} \arcsin(h/t)$, we also have that the above coincides with the (-1)-dimensional surface Minkowski content of (∞, Ω):

$$\mathcal{S}^{-1}(\infty, \Omega) = \lim_{t \to +\infty} \frac{H^1(S_t(0) \cap \Omega)}{t^{2-1-1}} = 2h \tag{4.9.7}$$

so that in this case these two quantities coincide.

Moreover, if we consider a modified $\widetilde{\Omega} := \{(x, y) \in \mathbb{R}^2 : x \geq 1/2, 0 \leq y \leq 1/2\}$ then this is exactly the image of the relative fractal drum (A, Ω) from Example 4.111 under the geometric inversion; that is, $(\infty, \widetilde{\Omega}) = (\Phi(A), \Phi(\Omega))$. Note that,

$$\dim_B^\phi(\infty, \widetilde{\Omega}) = \dim_B(A, \Omega) = -1 \tag{4.9.8}$$

and, moreover, since $\mathcal{M}^{-1}(A, \Omega) = 1/6$ we also have that[30]

$$\lim_{\phi \to 1^+} \frac{\mathcal{M}_\phi^{-1}(\infty, \widetilde{\Omega})}{\log \phi} = \frac{1}{2} = (N - D)\mathcal{M}^{-1}(A, \Omega). \tag{4.9.9}$$

This will be in accordance with a general result of Theorem 4.122.

The following proposition complements Proposition 4.23.

Proposition 4.116. *Let $\Omega \subseteq \mathbb{R}^N$ be a measurable set with $|\Omega| = \infty$, $T > 0$ and $\phi > 1$. Then for every $\sigma \in (\overline{\dim}_B^\phi(\infty, \Omega), +\infty)$, the following identity holds*:

$$\int_{{}_T\Omega} |x|^{-\sigma - N} \, \mathrm{d}x = (\sigma + N) \int_T^{+\infty} t^{-\sigma - N - 1} |B_{T,t}(0) \cap \Omega| \, \mathrm{d}t. \tag{4.9.10}$$

Furthermore, the above integrals are finite for such σ.

Proof. First we observe that the condition $|\Omega| = \infty$ implies that $\overline{\dim}_B^\phi(\infty, \Omega) \geq -N$ From this we have that for $\sigma \in (\overline{\dim}_B^\phi(\infty, \Omega), +\infty)$ the function $u(t) := t^{-\sigma - N}$ satisfies the conditions of Proposition 4.112 and from that we get (4.9.10). Let us now fix $\sigma_1 \in$

[30]Here, $N = 2$ and $D = -1 = \dim_B^\phi(\infty, \widetilde{\Omega}) = \dim_B(A, \Omega)$.

$(\overline{\dim}_B^{\phi}(\infty,\Omega),\sigma)$. Then for T large enough we have that for a constant $M>0$ we have

$$|B_{t,\phi t}\cap\Omega|\leq Mt^{\sigma_1+N}$$

for every $t\geq T$, which, in turn, implies that

$$|B_{\phi^n T,\phi^{n+1}T}\cap\Omega|\leq MT^{\sigma_1+N}\phi^{n(\sigma_1+N)}, \tag{4.9.11}$$

for every $n\in\mathbb{N}$. Let us now denote

$$I_T:=\int_T^{+\infty}t^{-\sigma-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t$$

and calculate

$$\begin{aligned}
I_T&=\sum_{n=0}^{\infty}\int_{\phi^n T}^{\phi^{n+1}T}t^{-\sigma-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t\\
&\leq\sum_{n=0}^{\infty}\int_{\phi^n T}^{\phi^{n+1}T}t^{-\sigma-N-1}|B_{T,\phi^{n+1}T}(0)\cap\Omega|\,\mathrm{d}t\\
&=\sum_{n=0}^{\infty}\int_{\phi^n T}^{\phi^{n+1}T}t^{-\sigma-N-1}\sum_{k=0}^{n}|B_{\phi^k T,\phi^{k+1}T}(0)\cap\Omega|\,\mathrm{d}t.
\end{aligned}$$

Then, by using (4.9.11), we have

$$\begin{aligned}
I_T&\leq MT^{\sigma_1+N}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\phi^{k(\sigma_1+N)}\int_{\phi^n T}^{\phi^{n+1}T}t^{-\sigma-N-1}\,\mathrm{d}t\\
&=MT^{\sigma_1+N}\sum_{n=0}^{\infty}\frac{\phi^{(n+1)(\sigma_1+N)}-1}{\phi^{\sigma_1+N}-1}\,\frac{(\phi^n T)^{-\sigma-N}-(\phi^{n+1}T)^{-\sigma-N}}{\sigma+N}\\
&=\underbrace{\frac{MT^{\sigma_1-\sigma}(1-\phi^{-\sigma-N})}{(\sigma+N)(\phi^{\sigma_1+N}-1)}}_{K}\sum_{n=0}^{\infty}(\phi^{(n+1)(\sigma_1+N)}-1)\phi^{n(-\sigma-N)}\\
&\leq K\sum_{n=0}^{\infty}\phi^{(n+1)(\sigma_1+N)+n(-\sigma-N)}=K\phi^{\sigma_1+N}\sum_{n=0}^{\infty}(\phi^{\sigma_1-\sigma})^n<+\infty,
\end{aligned}$$

since $\sigma_1-\sigma<0$ and $\phi>1$. $\square$

Now we can state and prove the holomorphicity theorem for the Lapidus zeta functions at infinity that extends Theorem 4.24 and also holds in the case of Lebesgue measurable sets of infinite measure.

Theorem 4.117. *Let Ω be any Lebesgue measurable subset of $\mathbb{R}^N$. Assume that T is a fixed positive number and $\phi>1$. Then the following conclusions hold.*

(a) *The distance zeta function at infinity*

$$\zeta_\infty(s,\Omega)=\int_{{}_T\Omega}|x|^{-s-N}\,\mathrm{d}x \tag{4.9.12}$$

is holomorphic on the half-plane $\{\mathrm{Re}\, s>\overline{\dim}_B^\phi(\infty,\Omega)\}$ *and for every complex number* s *in that half-plane*

$$\zeta_\infty'(s,\Omega)=-\int_{{}_T\Omega}|x|^{-s-N}\log|x|\,\mathrm{d}x \tag{4.9.13}$$

(b) *The half-plane from* (a) *is optimal.*[31]
(c) *If* $D=\dim_B^\phi(\infty,\Omega)$ *exists and* $\underline{\mathcal{M}}_\phi^D(\infty,\Omega)>0$*, then* $\zeta_\infty(s,\Omega)\to+\infty$ *for* $s\in\mathbb{R}$ *and* $s\to D^+$.

Proof. Firstly we note that if $|\Omega|<\infty$, then in light of the fact that in this case the upper box dimension of Ω at infinity coincides with its upper ϕ-shell box dimension at infinity (see Corollary 4.93) the statements (a) and (b) of the theorem follow immediately from Theorem 4.24. Additionally, if $\overline{\dim}_B^\phi(\infty,\Omega)<-N$, then part ($c$) also follows from Theorem 4.24 by using the fact that $\underline{\mathcal{M}}_\phi^D(\infty,\Omega)\leq(1-\phi^{N+D})\underline{\mathcal{M}}^D(\infty,\Omega)$ (see Proposition 4.87), so that $\underline{\mathcal{M}}_\phi^D(\infty,\Omega)>0$ implies that $\underline{\mathcal{M}}^D(\infty,\Omega)>0$.

It remains to prove the theorem in the case when $|\Omega|=\infty$. First, we fix $\phi>1$ and observe that in light of Proposition 4.113 and Proposition 4.101 we have that $\overline{\dim}_B^\phi(\infty,\Omega)\in[-N,0]$.

(a) If we let $\overline{D}:=\overline{\dim}_B^\phi(\infty,\Omega)$, then from the definitions of the upper ϕ-shell Minkowski content and of the upper ϕ-shell box dimension at infinity we have that $\limsup_{t\to+\infty}\frac{|B_{t,\phi t}(0)\cap\Omega|}{t^{N+\sigma}}=0$ for every $\sigma>\overline{D}$. Now, let us fix σ_1 such that $\overline{D}<\sigma_1<\sigma$ and take $T>1$ large enough, such that for a constant $M>0$ it holds that

$$|B_{t,\phi t}(0)\cap\Omega|\leq Mt^{\sigma_1+N}\quad\text{for every}\quad t\geq T,$$

which implies that

$$|B_{\phi^nT,\phi^{n+1}T}(0)\cap\Omega|\leq MT^{\sigma_1+N}\phi^{n(\sigma_1+N)}\quad\text{for every}\quad n\in\mathbb{N}.$$

Furthermore, since $\sigma>D\geq-N$ we have that $-\sigma-N<0$ and we estimate $\zeta_\infty(\sigma,\Omega)$ in

[31] Optimal in the sense that the integral appearing in (4.2.14) is divergent for $s\in(-\infty,\overline{\dim}_B^\phi(\infty,\Omega))$.

the following way

$$
\begin{aligned}
\zeta_\infty(\sigma,\Omega) &= \int_{{}_T\Omega} |x|^{-\sigma-N}\,\mathrm{d}x = \sum_{n=0}^{\infty}\int_{B_{\phi^n T,\phi^{n+1}T}\cap\Omega} |x|^{-\sigma-N}\,\mathrm{d}x \\
&\le T^{-\sigma-N}\sum_{n=0}^{\infty}\phi^{n(-\sigma-N)}|B_{\phi^n T,\phi^{n+1}T}\cap\Omega| \\
&\le T^{-\sigma-N}\sum_{n=0}^{\infty}\phi^{n(-\sigma-N)}MT^{\sigma_1+N}\phi^{n(\sigma_1+N)} \\
&= MT^{\sigma_1-\sigma}\sum_{n=0}^{\infty}(\phi^{\sigma_1-\sigma})^n < \infty.
\end{aligned}
$$

The last inequality follows from the fact that $\phi > 1$ and $\sigma_1 - \sigma < 0$. Similarly as in the proof of Theorem 4.24, we let now $E := {}_T\Omega$, $\varphi(x) := |x|$ and $\mathrm{d}\mu(x) := |x|^{-N}\,\mathrm{d}x$ and note that $\varphi(x) \ge T > 1$ for $x \in E$. Part (a) follows now from Theorem 1.1(b).

To prove part (b) of the theorem we denote $\overline{D} := \overline{\dim}_B^{\phi}(\infty,\Omega) \in [-N, 0]$. In case $s \le -N$ we have

$$
\int_{{}_T\Omega} |x|^{-s-N}\,\mathrm{d}x \ge T^{-s-N}\int_{{}_T\Omega}\mathrm{d}x = +\infty.
$$

It remains to prove the case when $s \in (-N, \overline{D})$. By using Proposition 4.116, we have

$$
\begin{aligned}
I_T := \int_{{}_T\Omega} |x|^{-s-N}\,\mathrm{d}x &= (s+N)\int_T^{+\infty} t^{-s-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t \\
&\ge (s+N)\int_{\phi T}^{\phi^2 T} t^{-s-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t \\
&\ge (s+N)|B_{T,\phi T}(0)\cap\Omega|\int_{\phi T}^{\phi^2 T} t^{-s-N-1}\,\mathrm{d}t \\
&= \phi^{-s-N}(1-\phi^{-s-N})T^{-s-N}|B_{T,\phi T}(0)\cap\Omega|.
\end{aligned}
\tag{4.9.14}
$$

Now, we fix σ such that $s < \sigma < \overline{D}$. From $\overline{\mathcal{M}}_\phi^{\sigma}(\infty,\Omega) = +\infty$ we conclude that there exists a sequence $(t_k)_{k\ge1}$ such that

$$
C_k := \frac{|B_{t_k,\phi t_k}(0)\cap\Omega|}{t_k^{N+\sigma}} \to +\infty \quad \text{when} \quad t_k \to +\infty.
$$

It is clear that the function $T \to I_T$ is nonincreasing and we have

$$
\begin{aligned}
I_T \ge I_{t_k} &\ge \phi^{-s-N}(1-\phi^{-s-N})t_k^{-s-N}|B_{t_k,\phi t_k}(0)\cap\Omega| \\
&= \phi^{-s-N}(1-\phi^{-s-N})C_k t_k^{N+\sigma} \\
&= \phi^{-s-N}(1-\phi^{-s-N})C_k t_k^{\sigma-s} \to +\infty.
\end{aligned}
\tag{4.9.15}
$$

Therefore, $I_T = +\infty$ for every $s < \overline{D}$ which proves part (b).

For part (c) we assume that $D = \dim_B^{\phi}(\infty, \Omega)$ exists, $D \in [-N, 0]$ and $\underline{\mathcal{M}}_{\phi}^D(\infty, \Omega) > 0$. This implies that there exists a constant $C > 0$ such that for a sufficiently large T we have that $|B_{t,\phi t}(0) \cap \Omega| \geq C t^{N+D}$ for every $t \geq T$. Now, for $-N < D < s$ we have the following:

$$
\begin{aligned}
\zeta_\infty(s,\Omega) &= \int_{{}_T\Omega} |x|^{-s-N}\,\mathrm{d}x = (s+N)\int_T^{+\infty} t^{-s-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t \\
&\geq (s+N)\int_{\phi T}^{+\infty} t^{-s-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t \\
&\geq (s+N)\int_{\phi T}^{+\infty} t^{-s-N-1}|B_{\phi^{-1}t,t}(0)\cap\Omega|\,\mathrm{d}t \\
&\geq C(s+N)\int_{\phi T}^{+\infty} t^{-s-N-1}(\phi^{-1}t)^{N+D}\,\mathrm{d}t \\
&= C(s+N)\phi^{-N-D}\frac{T^{D-s}}{s-D} \to +\infty,
\end{aligned}
\tag{4.9.16}
$$

when $s \to D^+$, and this proves part (c) in the case when $D \in (-N, 0]$. For the special case when $D = -N < s$ we proceed in a slightly different manner:

$$
\begin{aligned}
\zeta_\infty(s,\Omega) &= \int_{{}_T\Omega} |x|^{D-s}\,\mathrm{d}x = (s-D)\int_T^{+\infty} t^{D-s-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t \\
&\geq (s-D)\int_T^{+\infty} t^{D-s-1}|B_{T,\phi^{\lfloor\log_\phi(t/T)\rfloor}T}(0)\cap\Omega|\,\mathrm{d}t \\
&\geq (s-D)\int_T^{+\infty} t^{D-s-1}\sum_{k=0}^{\lfloor\log_\phi(t/T)\rfloor-1}|B_{\phi^k T,\phi^{k+1}T}(0)\cap\Omega|\,\mathrm{d}t \\
&\geq C(s-D)\int_T^{+\infty} t^{D-s-1}\lfloor\log_\phi(t/T)\rfloor\,\mathrm{d}t \\
&\geq C(s-D)\int_T^{+\infty} t^{D-s-1}(\log_\phi t - \log_\phi T - 1)\,\mathrm{d}t =: I.
\end{aligned}
$$

The last integral appearing above (denoted by I) can be explicitly calculated:

$$
\begin{aligned}
I &= \frac{C(s-D)}{\log\phi}\left(\frac{t^{D-s}\log t}{D-s} - \frac{t^{D-s}}{(D-s)^2}\right)\Bigg|_T^{+\infty} - C(s-D)\frac{(\log_\phi T+1)t^{D-s}}{D-s}\Bigg|_T^{+\infty} \\
&= C(s-D)\left(\frac{T^{D-s}}{(s-D)^2\log\phi} + \frac{T^{D-s}\log_\phi T}{s-D} - \frac{T^{D-s}(\log_\phi T+1)}{s-D}\right) \\
&= \frac{C}{\log\phi}\frac{T^{D-s}}{s-D} - CT^{D-s} \to +\infty,
\end{aligned}
$$

when $s \to D^+$, which concludes the proof of the theorem. $\square$

Inspired by Proposition 4.116, let us introduce a new zeta function of Lebesgue measurable sets at infinity.

Definition 4.118. Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ and $T > 0$ a fixed positive number. We define the *shell zeta function of Ω at infinity* as

$$\breve{\zeta}_\infty(s,\Omega;T) := -\int_T^{+\infty} t^{-s-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t. \tag{4.9.17}$$

The next theorem will generalize Theorem 4.31 for the case when we do not require that Ω is of finite Lebesgue measure.

Theorem 4.119. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$, $T > 0$ be fixed and $\phi > 1$. Then for every $s \in \mathbb{C}$ we have*

$$\int_{{}_T\Omega} |x|^{-s-N}\,\mathrm{d}x = (s+N)\int_T^{+\infty} t^{-s-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t, \tag{4.9.18}$$

for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > \max\{-N, \overline{\dim}_B^\phi(\infty,\Omega)\}$ *or, in short,*

$$\zeta_\infty(s,\Omega;T) = -(s+N)\breve{\zeta}_\infty(s,\Omega;T), \tag{4.9.19}$$

on the open right half-plane $\Pi := \{\operatorname{Re} s > \max\{-N, \overline{\dim}_B^\phi(\infty,\Omega)\}\}$.

In particular, the shell zeta function $\breve{\zeta}_\infty(\,\cdot\,,\Omega;T)$ is holomorphic on Π and the functional equality (4.9.19) *is valid on any open connected neighborhood of Π to which any of the two zeta functions in* (4.9.19) *has an analytic continuation.*

Furthermore, for all $s \in \Pi$ we have that

$$\frac{\mathrm{d}}{\mathrm{d}s}\breve{\zeta}_\infty(s,\Omega;T) = \int_T^{+\infty} t^{-s-N-1}|B_{T,t}(0)\cap\Omega|\log t\,\mathrm{d}t. \tag{4.9.20}$$

Proof. First, we will show that in the case when $|\Omega| < \infty$, this is actually a rewriting of Theorem 4.31. Namely, $|\Omega| < \infty$ implies that $\overline{\dim}_B^\phi(\infty,\Omega) = \overline{\dim}_B(\infty,\Omega) \le -N$ for every $\phi > 1$ (see Proposition 4.2). Furthermore, for $\operatorname{Re} s > -N$ we note that $-\operatorname{Re} s - N < 0$ and from Theorem 4.31, we have

$$\begin{aligned}
\int_{{}_T\Omega} |x|^{-s-N}\,\mathrm{d}x &= T^{-s-N}|{}_T\Omega| - (s+N)\int_T^{+\infty} t^{-s-N-1}|{}_t\Omega|\,\mathrm{d}t \\
&= T^{-s-N}|{}_T\Omega| - (s+N)\int_T^{+\infty} t^{-s-N-1}(|{}_T\Omega| - |B_{T,t}(0)\cap\Omega|)\,\mathrm{d}t \\
&= T^{-s-N}|{}_T\Omega| - |{}_T\Omega|t^{-s-N}\Big|_T^{+\infty} + (s+N)\int_T^{+\infty} t^{-s-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t \\
&= (s+N)\int_T^{+\infty} t^{-s-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t.
\end{aligned}$$

Again, from Theorem 4.31 we have that the left-hand side above is holomorphic on

$\operatorname{Re} s > -N$. On the other hand, we have that

$$\left|\int_T^{+\infty} t^{-s-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t\right| \le |\Omega|\int_T^{+\infty} t^{-\operatorname{Re} s-N-1}\,\mathrm{d}t = \frac{|\Omega|T^{-\operatorname{Re} s-N}}{\operatorname{Re} s+N}$$

and Theorem 1.2 implies that the above integral defines a holomorphic function on $\{\operatorname{Re} s > -N\}$.

It remains to prove the theorem in the case when $|\Omega| = \infty$ and, in light of Proposition 4.113, we then have that $\overline{\dim}_B^{\phi}(\infty,\Omega) \ge -N$ for every $\phi > 1$.

From Proposition 4.116 we have that (4.9.18) is valid for $\mathbb{R} \ni \sigma > \overline{\dim}_B^{\phi}(\infty,\Omega)$ and both integrals are finite. (See Theorem 4.117 for the left-hand side integral.) Furthermore, to show that the equality holds in the half-plane $\{\operatorname{Re} s > \overline{\dim}_B^{\phi}(\infty,\Omega)\}$, it suffices to prove that both sides of Equation (4.9.18) are holomorphic functions on that domain.[32] We already have that the left-hand side of (4.9.18) is holomorphic on the set $\{\operatorname{Re} s > \overline{\dim}_B^{\phi}(\infty,\Omega)\}$ according to Theorem 4.117. Furthermore, the right-hand side of (4.9.18) is a Dirichlet type integral with $\varphi(t) = t^{-s}$ and $\mathrm{d}\mu(t) = t^{-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t$, and according to Theorem 1.1 it is sufficient to show that the integral on the right hand side of (4.9.18) is convergent for $\operatorname{Re} s > \overline{\dim}_B^{\phi}(\infty,\Omega)$.

For $\overline{D} := \overline{\dim}_B^{\phi}(\infty,\Omega)$ and $s \in \mathbb{C}$ such that $\operatorname{Re} s > \overline{D}$, let us choose $\varepsilon > 0$ sufficiently small such that $\operatorname{Re} s > \overline{D}+\varepsilon$. Since $\overline{\mathcal{M}}_{\phi}^{\overline{D}+\varepsilon}(\infty,\Omega) = 0$, there exists a constant $C_T > 0$ such that $|B_{t,\phi t}(0)\cap\Omega| \le C_T t^{N+\overline{D}+\varepsilon}$ for every $t \in [T,+\infty)$. Now we have the following estimate exactly in the same way as in the proof of the second part of Proposition 4.116 (by letting $\sigma = \operatorname{Re} s$ and $\sigma_1 = \overline{D}+\varepsilon$ in the notation of that proof, and K being a positive constant):

$$\begin{aligned} |\breve{\zeta}_\infty(s,\Omega;T)| &\le \int_T^{+\infty} t^{-\operatorname{Re} s-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t \\ &\le K\phi^{\sigma_1+N}\sum_{n=0}^{\infty}(\phi^{\overline{D}+\varepsilon-\operatorname{Re} s})^n < +\infty. \end{aligned} \tag{4.9.21}$$

This, together with the principle of analytic continuation, completes the proof of the theorem. (The derivative formula follows by differentiating under the integral sign; see Theorem 1.1.) □

Corollary 4.120. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ such that $|\Omega| < \infty$ and $T > 0$ be fixed. Additionally, assume that $\overline{\dim}_B(\infty,\Omega) < -N$. Then, $s = -N$ is a simple pole of $\breve{\zeta}_\infty(\,\cdot\,,\Omega;T)$ and*

$$\operatorname{res}(\breve{\zeta}_\infty(\,\cdot\,,\Omega;T),-N) = -|{}_T\Omega|. \tag{4.9.22}$$

Proof. According to Theorem 4.24 we have that $\zeta_\infty(\,\cdot\,,\Omega;T)$ is holomorphic on the right half-plane $\{\operatorname{Re} s > \overline{\dim}_B(\infty,\Omega)\}$. Furthermore, Theorem 4.119; that is, (4.9.19) implies

[32] The equality follows from the fact that two holomorphic functions that coincide on a set that has an accumulation point in their common domain coincide then on the whole common domain.

then that $\breve{\zeta}_\infty(\,\cdot\,,\Omega;T)$ is meromorphic in that half plane and

$$\begin{aligned}\operatorname{res}(\breve{\zeta}_\infty(\,\cdot\,,\Omega;T),-N) &= \lim_{s\to -N}(s+N)\breve{\zeta}_\infty(s,\Omega;T)\\ &= -\lim_{s\to -N}\zeta_\infty(s,\Omega;T) = -\zeta_\infty(-N,\Omega;T) = -|_T\Omega|.\end{aligned}$$

□

Now we can state a theorem that expands Theorem 4.32 to the case of unbounded sets of infinite Lebesgue measure.

Theorem 4.121. *Let Ω be a Lebesgue measurable set and $\phi>1$ such that $\dim_B^\phi(\infty,\Omega)=D>-\infty$ exists. Furthermore, let $0<\underline{\mathcal{M}}_\phi^D(\infty,\Omega)\le\overline{\mathcal{M}}_\phi^D(\infty,\Omega)<\infty$. If $\zeta_\infty(\,\cdot\,,\Omega)$ has a meromorphic continuation to a neighborhood of $s=D$, then D is a simple pole. Furthermore in the case when $D\in[-N,0]$ we have that*

$$\frac{1}{\phi^{N+D}\log\phi}\underline{\mathcal{M}}_\phi^D(\infty,\Omega)\le\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D)\le\frac{1}{\log\phi}\overline{\mathcal{M}}_\phi^D(\infty,\Omega) \tag{4.9.23}$$

and in the case when $D\in(-\infty,-N)$ we have that

$$-\frac{N+D}{1-\phi^{N+D}}\underline{\mathcal{M}}_\phi^D(\infty,\Omega)\le\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D)\le-\frac{N+D}{1-\phi^{N+D}}\overline{\mathcal{M}}_\phi^D(\infty,\Omega). \tag{4.9.24}$$

Proof. Firstly, by looking at the proof of part (c) of Theorem 4.117 we can conclude that $s=D$ is a singularity of $\zeta_\infty(\,\cdot\,,\Omega)$ which is at least a simple pole. It remains to prove that the order of this pole is not larger than one and we proceed in a similar way as in the proof of Theorem 4.32. We define

$$C_T:=\sup_{t\ge T}\frac{|B_{t,\phi t}(0)\cap\Omega|}{t^{N+D}}$$

and conclude from $\overline{\mathcal{M}}_\phi^D(\infty,\Omega)<+\infty$ that we have $C_T<+\infty$ for T large enough. Let us first assume that $-N<D\le 0$ and take $s\in\mathbb{R}$ such that $s>D$. From Theorem 4.119 we then have

$$\begin{aligned}\zeta_\infty(s,\Omega;T) &= (s+N)\int_T^{+\infty}t^{-s-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t\\ &= (s+N)\sum_{n=0}^{\infty}\int_{\phi^nT}^{\phi^{n+1}T}t^{-s-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t\\ &\le (s+N)\sum_{n=0}^{\infty}\int_{\phi^nT}^{\phi^{n+1}T}t^{-s-N-1}|B_{T,\phi^{n+1}T}(0)\cap\Omega|\,\mathrm{d}t.\end{aligned}$$

Furthermore, from the definition of C_T we have

$$
\begin{aligned}
\zeta_\infty(s,\Omega;T) &\leq (s+N)\sum_{n=0}^{\infty}\int_{\phi^n T}^{\phi^{n+1}T} t^{-s-N-1}\sum_{k=0}^{n}|B_{\phi^k T,\phi^{k+1}T}(0)\cap\Omega|\,\mathrm{d}t\\
&\leq (s+N)C_T T^{N+D}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\phi^{k(N+D)}\int_{\phi^n T}^{\phi^{n+1}T} t^{-s-N-1}\,\mathrm{d}t\\
&= (s+N)C_T T^{N+D}\sum_{n=0}^{\infty}\frac{\phi^{(n+1)(N+D)}-1}{\phi^{N+D}-1}\,\frac{(\phi^n T)^{-s-N}-(\phi^{n+1}T)^{-s-N}}{s+N}\\
&= \frac{C_T T^{D-s}(1-\phi^{-s-N})}{\phi^{N+D}-1}\sum_{n=0}^{\infty}(\phi^{(n+1)(N+D)}-1)\phi^{n(-s-N)}
\end{aligned}
$$

which can be further bounded by neglecting the -1 from the braces above to get

$$
\begin{aligned}
\zeta_\infty(s,\Omega;T) &\leq \frac{C_T T^{D-s}(1-\phi^{-s-N})}{\phi^{N+D}-1}\sum_{n=0}^{\infty}\phi^{(n+1)(N+D)+n(-s-N)}\\
&= \frac{C_T T^{D-s}(1-\phi^{-s-N})\phi^{N+D}}{\phi^{N+D}-1}\sum_{n=0}^{\infty}\phi^{n(D-s)}\\
&= \frac{C_T T^{D-s}(\phi^{N+D}-\phi^{D-s})}{(\phi^{N+D}-1)(1-\phi^{D-s})}.
\end{aligned}
$$

From this we conclude that $|\zeta_\infty(s,\Omega)|\leq C(1-\phi^{D-s})^{-1}$ where $C>0$ is a positive constant independent of s and T which implies that $s=D$ is a pole of order at most one, i.e., a simple pole. By the same reasoning as in the proof of Theorem 4.32 we conclude that the residue at $s=D$ of $\zeta_\infty(\,\cdot\,,\Omega;T)$ is independent of T and for $s>D$ we have that

$$
(s-D)\zeta_\infty(s,\Omega)\leq \frac{C_T T^{D-s}(\phi^{N+D}-\phi^{D-s})}{\phi^{N+D}-1}\,\frac{s-D}{1-\phi^{D-s}}.
$$

Furthermore, by letting $s\to D^+$ we get

$$
\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D)\leq\frac{C_T}{\log\phi}.
$$

Finally, by taking the limit as $T\to\ +\infty$ we get

$$
\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D)\leq\frac{1}{\log\phi}\overline{\mathcal{M}}_\phi^D(\infty,\Omega).
$$

For the inequality involving the lower ϕ-shell Minkowski content at infinity, we define

$$
K_T := \inf_{t\geq T}\frac{|B_{t,\phi t}(0)\cap\Omega|}{t^{N+D}}
$$

and conclude from $\underline{\mathcal{M}}_\phi^D(\infty,\Omega)>0$ that we have $K_T>0$ for T large enough. Furthermore, we take $s > D$ and proceed in a similar manner as before:

$$\begin{aligned}
\zeta_\infty(s,\Omega;T) &= (s+N)\sum_{n=0}^{\infty}\int_{\phi^n T}^{\phi^{n+1}T} t^{-s-N-1}|B_{T,t}(0)\cap\Omega|\,\mathrm{d}t\\
&\geq (s+N)\sum_{n=0}^{\infty}\int_{\phi^n T}^{\phi^{n+1}T} t^{-s-N-1}|B_{T,\phi^n T}(0)\cap\Omega|\,\mathrm{d}t\\
&= (s+N)\sum_{n=0}^{\infty}\int_{\phi^n T}^{\phi^{n+1}T} t^{-s-N-1}\sum_{k=0}^{n-1}|B_{\phi^k T,\phi^{k+1}T}(0)\cap\Omega|\,\mathrm{d}t.
\end{aligned}$$

Similarly as before, from the definition of K_T we get:

$$\begin{aligned}
\zeta_\infty(s,\Omega;T) &\geq (s+N)K_T T^{N+D}\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\phi^{k(N+D)}\int_{\phi^n T}^{\phi^{n+1}T} t^{-s-N-1}\,\mathrm{d}t\\
&= (s+N)K_T T^{N+D}\sum_{n=0}^{\infty}\frac{\phi^{n(N+D)}-1}{\phi^{N+D}-1}\,\frac{(\phi^n T)^{-s-N}-(\phi^{n+1}T)^{-s-N}}{s+N}\\
&= \frac{K_T T^{D-s}(1-\phi^{-s-N})}{\phi^{N+D}-1}\sum_{n=0}^{\infty}(\phi^{n(N+D)}-1)\phi^{n(-s-N)}.
\end{aligned}$$

Interchanging summation and subtraction above yields

$$\begin{aligned}
\zeta_\infty(s,\Omega;T) &\geq \frac{K_T T^{D-s}(1-\phi^{-s-N})}{\phi^{N+D}-1}\left(\sum_{n=0}^{\infty}\phi^{n(D-s)}-\frac{1}{1-\phi^{-s-N}}\right)\\
&= \frac{K_T T^{D-s}(1-\phi^{-s-N})}{(\phi^{N+D}-1)(1-\phi^{D-s})}-\frac{K_T T^{D-s}}{(\phi^{N+D}-1)}.
\end{aligned}$$

This implies that

$$(s-D)\zeta_\infty(s,\Omega) \geq \frac{K_T T^{D-s}(1-\phi^{-s-N})}{\phi^{N+D}-1}\,\frac{s-D}{1-\phi^{D-s}}-\frac{K_T T^{D-s}(s-D)}{(\phi^{N+D}-1)},$$

and by letting $s\to D^+$ we have

$$\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega)) \geq \frac{K_T}{\phi^{N+D}\log\phi}.$$

Finally, we let $T\to+\infty$ to get

$$\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D) \geq \frac{1}{\phi^{N+D}\log\phi}\underline{\mathcal{M}}_\phi^D(\infty,\Omega).$$

Let us now treat the special case when $D=-N$. We take $s\in\mathbb{R}$ such that $s > D$

and, similarly as before, we have

$$
\begin{aligned}
\zeta_\infty(s,\Omega) &\le (s+N)\int_T^{+\infty} t^{-s-N-1}\sum_{n=0}^{\lfloor\log_\phi(t/T)\rfloor}|B_{\phi^nT,\phi^{n+1}T}\cap\Omega|\,\mathrm{d}t\\
&\le C_T(s+N)\int_T^{+\infty} t^{-s-N-1}\sum_{n=0}^{\lfloor\log_\phi(t/T)\rfloor}\phi^{n(N+D)}\,\mathrm{d}t\\
&= C_T(s+N)\int_T^{+\infty} t^{-s-N-1}(\lfloor\log_\phi(t/T)\rfloor+1)\,\mathrm{d}t\\
&\le \frac{C_T(s+N)}{\log\phi}\int_T^{+\infty} t^{-s-N-1}\log t\,\mathrm{d}t\\
&= \frac{C_TT^{-s-N}}{(s+N)\log\phi}(N\log T+s\log T+1).
\end{aligned}
$$

From this, we conclude that

$$
\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),-N)\le\frac{C_T}{\log\phi},
$$

and, by letting $T\to+\infty$ we get the desired inequality. For the other inequality we have the following estimates:

$$
\begin{aligned}
\zeta_\infty(s,\Omega) &\ge (s+N)\int_T^{+\infty} t^{-s-N-1}\sum_{n=0}^{\lfloor\log_\phi(t/T)\rfloor-1}|B_{\phi^nT,\phi^{n+1}T}\cap\Omega|\,\mathrm{d}t\\
&\ge K_T(s+N)\int_T^{+\infty} t^{-s-N-1}\sum_{n=0}^{\lfloor\log_\phi(t/T)\rfloor-1}\phi^{n(N+D)}\,\mathrm{d}t\\
&\ge K_T(s+N)\int_T^{+\infty} t^{-s-N-1}(\log_\phi(t/T)-1)\,\mathrm{d}t\\
&\ge \frac{K_T(s+N)}{\log\phi}\int_T^{+\infty} t^{-s-N-1}\log t\,\mathrm{d}t-K_T(s+N)(1+\log_\phi T)\int_T^{+\infty} t^{-s-N-1}\,\mathrm{d}t\\
&= \frac{K_TT^{-s-N}}{(s+N)\log\phi}(N\log T+s\log T+1)-K_T(1+\log_\phi T)T^{-s-N}.
\end{aligned}
$$

Finally, as before, we first multiply both sides by $(s+N)$, let $s\to -N^+$ and then let $T\to+\infty$.

It remains to prove the theorem in the case when $D\in(-\infty,-N)$. The argumentation will be similar as before but we will assume that $D < s < -N$ and use (4.9.18) to get

$$\begin{aligned}
\zeta_\infty(s,\Omega) &\leq T^{-s-N}|{}_T\Omega| - (s+N)\int_T^{+\infty} t^{-s-N-1}|B_t(0)^c \cap \Omega|\,\mathrm{d}t \\
&\leq -(s+N)\int_T^{+\infty} t^{-s-N-1}\sum_{n=0}^{\infty}|B_{\phi^n t,\phi^{n+1}t}(0)\cap\Omega|\,\mathrm{d}t \\
&\leq -C_T(s+N)\int_T^{+\infty} t^{-s-N-1}\sum_{n=0}^{\infty} t^{N+D}\phi^{n(N+D)}\,\mathrm{d}t \\
&= -\frac{C_T(s+N)}{1-\phi^{N+D}}\int_T^{+\infty} t^{D-s-1}\,\mathrm{d}t \\
&= -\frac{C_T(s+N)}{1-\phi^{N+D}}\,\frac{T^{D-s}}{s-D}.
\end{aligned}$$

Exactly as before, we multiply both sides by $(s-D)$, let $s \to D^+$ and, after that, let $T \to +\infty$. To get the other inequality, and conclude the proof, we proceed in a similar manner by using the lower ϕ-shell Minkowski content of Ω at infinity. $\square$

Theorem 4.122. *Let Ω be a Lebesgue measurable set and $\phi > 1$ such that $\dim_B^\phi(\infty,\Omega) = D > -\infty$ exists. Furthermore, let $0 < \mathcal{M}_\phi^D(\infty,\Omega) < \infty$ and assume that $\zeta_\infty(\,\cdot\,,\Omega)$ has a meromorphic continuation to a neighborhood of $s = D$. Then, in the case when $D \in [-N, 0]$ we have that*

$$\frac{1}{\phi^{N+D}\log\phi}\mathcal{M}_\phi^D(\infty,\Omega) \leq \operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D) \leq \frac{1}{\log\phi}\mathcal{M}_\phi^D(\infty,\Omega) \tag{4.9.25}$$

and in the case when $D \in (-\infty,-N)$ we have that

$$\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D) = -\frac{N+D}{1-\phi^{N+D}}\mathcal{M}_\phi^D(\infty,\Omega). \tag{4.9.26}$$

Moreover, if Ω is ψ-shell Minkowski measurable at infinity for every $\psi \in (1,\phi)$, we have that

$$\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D) = \lim_{\psi\to1^+}\frac{\mathcal{M}_\psi^D(\infty,\Omega)}{\log\psi}. \tag{4.9.27}$$

Proof. Everything is evident from Theorem 4.121 with additional assumptions on the set Ω that are stated in the corollary. Note also that (4.9.23) implies that the limit $\lim_{\psi\to1^+}\mathcal{M}_\psi^D(\infty,\Omega)/\log\psi$ exists. It remains only to see that (4.9.27) holds even in the case when $D \in (-\infty,-N)$, but this is a simple consequence of L'Hospital's rule and Equation (4.9.26):

$$\begin{aligned}
\lim_{\psi\to1^+}\frac{\mathcal{M}_\psi^D(\infty,\Omega)}{\log\psi} &= \lim_{\psi\to1^+}\frac{1-\psi^{N+D}}{-(N+D)}\,\frac{\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D)}{\log\psi} \\
&= \operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D)\lim_{\psi\to1^+}\frac{-(N+D)\psi^{N+D-1}}{-(N+D)\psi^{-1}} = \operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega),D).
\end{aligned}$$

□

We can also express Theorems 4.121 and 4.122 in terms of the shell zeta function at infinity by means of Theorem 4.119 and the functional equation (4.9.19).

Theorem 4.123. *Let Ω be a Lebesgue measurable set and $\phi > 1$ such that $\dim_B^\phi(\infty, \Omega) = D > -\infty$ exists. Furthermore, let $0 < \underline{\mathcal{M}}_\phi^D(\infty, \Omega) \leq \overline{\mathcal{M}}_\phi^D(\infty, \Omega) < \infty$. If $\breve{\zeta}_\infty(\,\cdot\,, \Omega)$ has a meromorphic continuation to a neighborhood of $s = D$, then D is a simple pole. Furthermore in the case when $D \in [-N, 0]$ we have that*

$$\frac{N+D}{\phi^{N+D}\log\phi}\underline{\mathcal{M}}_\phi^D(\infty, \Omega) \leq -\operatorname{res}(\breve{\zeta}_\infty(\,\cdot\,, \Omega), D) \leq \frac{N+D}{\log\phi}\overline{\mathcal{M}}_\phi^D(\infty, \Omega) \tag{4.9.28}$$

and in the case when $D \in (-\infty, -N)$ we have that

$$\frac{1}{1-\phi^{N+D}}\underline{\mathcal{M}}_\phi^D(\infty, \Omega) \leq \operatorname{res}(\breve{\zeta}_\infty(\,\cdot\,, \Omega), D) \leq \frac{1}{1-\phi^{N+D}}\overline{\mathcal{M}}_\phi^D(\infty, \Omega). \tag{4.9.29}$$

Proof. The conclusion of the theorem follows directly from Theorem 4.121 and the functional equation 4.9.19 since we then have that

$$\operatorname{res}(\breve{\zeta}_\infty(\,\cdot\,, \Omega), D) = -(N+D)\operatorname{res}(\zeta_\infty(\,\cdot\,, \Omega), D).$$

□

Theorem 4.124. *Let Ω be a Lebesgue measurable set and $\phi > 1$ such that $\dim_B^\phi(\infty, \Omega) = D > -\infty$ exists. Furthermore, let $0 < \mathcal{M}_\phi^D(\infty, \Omega) < \infty$ and assume that $\breve{\zeta}_\infty(\,\cdot\,, \Omega)$ has a meromorphic continuation to a neighborhood of $s = D$. Then, in the case when $D \in [-N, 0]$ we have that*

$$\frac{N+D}{\phi^{N+D}\log\phi}\mathcal{M}_\phi^D(\infty, \Omega) \leq -\operatorname{res}(\breve{\zeta}_\infty(\,\cdot\,, \Omega), D) \leq \frac{N+D}{\log\phi}\mathcal{M}_\phi^D(\infty, \Omega) \tag{4.9.30}$$

and in the case when $D \in (-\infty, -N)$ we have that

$$\operatorname{res}(\breve{\zeta}_\infty(\,\cdot\,, \Omega), D) = \frac{1}{1-\phi^{N+D}}\mathcal{M}_\phi^D(\infty, \Omega). \tag{4.9.31}$$

Moreover, if Ω is ψ-shell Minkowski measurable at infinity for every $\psi \in (1, \phi)$, we have that

$$\operatorname{res}(\breve{\zeta}_\infty(\,\cdot\,, \Omega), D) = -\lim_{\psi\to1^+}\frac{\mathcal{M}_\psi^D(\infty, \Omega)}{(N+D)\log\psi}. \tag{4.9.32}$$

Recall the two parameter set $\Omega_\infty^{(a,b)}$ from Definition 4.59 where we had $a \in (0, 1/2)$ and $b \in (1+\log_{1/a}2, +\infty)$. Furthermore, if we now allow $b \in (\log_{1/a}2, 1+\log_{1/a}2]$ the set $\Omega_\infty^{(a,b)}$ will be of infinite Lebesgue measure (see Remark 4.60) but it will be contained in a strip

$\{0 \le y \le S\}$ of finite width $S = \frac{a^b}{1-2a^b}$. By going through the proof of Proposition 4.61 one can see that the condition on $\Omega_\infty^{(a,b)}$ being of finite Lebesgue measure is not needed and thus we have the following result.

Proposition 4.125. Let $\Omega_\infty^{(a,b)}$ be the two parameter set from Definition 4.59 of infinite Lebesgue measure; that is, with $a \in (0, 1/2)$ and $b \in (\log_{1/a} 2, 1 + \log_{1/a} 2]$. Then, its distance zeta function at infinity calculated via the $|\cdot|_\infty$-norm on $\mathbb{R}^2$ is given by

$$\zeta_\infty(s, \Omega_\infty^{(a,b)}; |\cdot|_\infty) = \frac{1}{s+b+1} \cdot \frac{1}{a^{-(s+b+1)} - 2}. \tag{4.9.33}$$

It is meromorphic on $\mathbb{C}$ where the set of complex dimensions of $\Omega_\infty^{(a,b)}$ at infinity visible through $W := \{\operatorname{Re} s > \log_{1/a} -b - 3\}$ is given by[33]

$$\{-(b+1)\} \cup \left(\log_{1/a} 2 - (b+1) + \frac{2\pi}{\log(1/a)}\mathbb{i}\mathbb{Z}\right). \tag{4.9.34}$$

Furthermore, we also have that for any $\phi > 1$,

$$\overline{\dim}_B^\phi(\infty, \Omega_\infty^{(a,b)}) = \log_{1/a} 2 - (b+1). \tag{4.9.35}$$

Proof. We have already commented the first part of the proposition. Furthermore, by analyzing Definition 4.54 and Theorem 4.55 one can see that there is no need to assume there that the set Ω has finite Lebesgue measure and thus we conclude that (4.9.34) holds in the same way as in Proposition 4.61. Finally, (4.9.35) follows from the fact that the upper ϕ-shell box dimension does not depend on the choice of ϕ (see Proposition 4.94) and from Theorem 4.117(b). $\square$

[33] We define the complex dimensions of sets at infinity with infinite Lebesgue measure in a completely the same way as for sets of finite Lebesgue measure.

Chapter 5

Fractal Properties of Hopf Bifurcation at Infinity

5.1 Introduction

One of the applications of fractal geometry can be seen in dynamics where the complexity of invariant sets and measures can be analyzed by using fractal dimensions. An illustrative example of this can be seen in [PiaHaPo] where the Hausdorff dimension of a particular case of the Hénon attractor is estimated and compared to its box dimension. It is clear that in many cases, the dynamics appearing in various problems in physics, engineering, chemistry, medicine, etc. can be better understood by using tools of fractal geometry.

For instance, the box dimension was used in [ŽuŽup1] to analyze spiral trajectories of some planar vector fields. Among else, the Hopf bifurcation was studied, which is a well-known bifurcation of 1-parameter families of vector fields. More precisely, the Hopf bifurcation gives birth to a limit cycle, that is, an isolated periodic orbit, from a singular point when the parameter passes through some critical value. Its generalization is the Hopf–Takens bifurcation in which multiple limit cycles are born from a singular point. In particular, it was shown that the box dimension of spiral trajectories can detect the moment at which the bifurcation occurs. Namely, it becomes nontrivial, that is, greater than 1 near singular points or nonhyperbolic (multiple) limit cycles, precisely at the moment at which the corresponding dynamical system undergoes the bifurcation. Even more interesting is the fact that the box dimension of these spiral trajectories can only take values in a discrete set and, moreover, it depends on the multiplicity of the corresponding singular point or limit cycle. This property could be useful in finding the multiplicity of limit cycles and singular points which relates it to the famous 16^{th} Hilbert problem of finding an upper uniform bound for the number of limit cycles in dependence on the degree of the polynomial vector field.

Generalization of these results to spiral trajectories of some vector fields in $\mathbb{R}^3$ can be found in [ŽuŽup2]. Further studies using the asymptotic behavior of the analytic Poincaré map associated to the spiral trajectories near singular points and periodic orbits can be seen in [ŽuŽup3].

The multiplicity of the Poincaré map is related to the notion of cyclicity, that is the number of limit cycles that can be born after a small perturbation of the system. Furthermore, the Poincaré map of a planar vector field generates a 1-dimensional discrete dynamical system. In [EleŽuŽup] it was shown that the box dimension of the 1-dimensional trajectory is related to this discrete dynamical system. There the box dimension of the corresponding orbit is studied for the classical saddle-node and period doubling bifurcations. Using these results, classical theorems about these bifurcations were extended. Further extensions to 2-dimensional discrete dynamical systems and applications to continuous dynamical systems were obtained in [Hor1].

It is known that limit cycles can also be generated from a polycycle, which is an ordered collection of singular points (vertices) and bi-asymptotic trajectories (edges) connecting them in a specified order. Remark that an isolated singular point is a special case of a polycycle. The next simplest case is a saddle-loop, that is a polycycle with only one vertex and one edge. The Poincaré map near a saddle loop, although it is not analytic, shows its cyclicity (see [Rou], [ZhaWa]). In [MaResŽup] this was investigated from the point of view of fractal geometry. The classical box dimension was not fine enough to distinguish between all the cases which could appear, so a generalization called the critical Minkowski order has been introduced.

As limit cycles can also be born from a point or a polycycle at infinity, it makes sense to generalize the previous results to this case. It is also interesting to study the problem of Hopf-Takens bifurcation of polynomial vector fields at infinity from the fractal point of view. Related problems have been studied in [CaLliTo], [BloRou] and [Gin].

In this chapter we deal with vector fields possessing spiral trajectories tending to infinity. The provided visualizations clearly show that in the case of a weak focus at infinity such trajectories exhibit an almost "planar" nature. We measure this phenomenon using the box dimension of trajectories. Since the trajectories tending to infinity are unbounded, we have adapted the definition of box dimension to this case, since the usual box dimension is defined for bounded sets only. We do this using the geometric inversion, see Definition 5.7 below.

5.2 The Geometric Inversion and Fractal Properties of Sets at Infinity

Up to now we were concerned with relative fractal drums of the type (∞, Ω) where Ω is an unbounded Lebesgue measurable subset of $\mathbb{R}^N$. In this section we will be concerned with a different situation. Namely, if A is an unbounded subset of $\mathbb{R}^N$ we cannot define its upper and lower box dimensions in a direct way since $|A_\delta| = \infty$ for every $\delta > 0$. It can be done for relative fractal drums (A, Ω) if there exists $\delta > 0$ such that $|A_\delta \cap \Omega| < \infty$ but in the general case we can use the geometric inversion Φ and analyze the image of A (or (A, Ω)) under it (under the hypothesis that $0 \notin \overline{A}$). This approach has been done in [RaŽuŽup] where unbounded trajectories of polynomial dynamical systems in $\mathbb{R}^2$ have been studied and it is presented in this chapter. In the present section we will refine and generalize some of the results from [RaŽuŽup] concerning the geometric inversion of unbounded sets and its effect on their fractal properties.

Remark 5.1. Lemma 4.20 is, of course, in accord with the fact that the composition $\Psi \circ \Phi \circ \Psi^{-1}$ is an isometry of the Riemann sphere $\mathbb{S}^N$ which, in turn, preserves the spherical volume of subsets of $\mathbb{S}^N$. Indeed, for $A \subseteq \mathbb{S}^N$ and using $y = \Phi^{-1}(x)$ as a change of variables in $\mathbb{R}^N$ we have:

$$\begin{aligned}|A|_{\mathbb{S}} &= \int_{\Psi^{-1}(A)} \frac{2^N}{(1+|x|^2)^N}\,\mathrm{d}x = \int_{\Phi\circ\Psi^{-1}(A)} \frac{2^N}{(1+|y|^{-2})^N}|y|^{-2N}\,\mathrm{d}y \\ &= \int_{\Phi\circ\Psi^{-1}(A)} \frac{2^N}{(1+|y|^2)^N}\,\mathrm{d}y = |\Psi\circ\Phi\circ\Psi^{-1}(A)|_{\mathbb{S}}.\end{aligned}$$

From the next results, we will see that analyzing the fractal properties of an unbounded set via the geometric inversion essentially corresponds to analyzing the fractal properties of its image on the Riemann sphere via the stereographic projection. This will be a consequence of the fact that the stereographic projection restricted to a ball of radius $R < 1$ around the origin is a bi-Lipschitz mapping. The first step is to derive a formula which expresses the spherical distance of two points in terms of their Euclidean distance.

Lemma 5.2. *Let $x, y \in \mathbb{R}^N$, then we have*:

$$d_{\mathbb{S}}(\Psi(x), \Psi(y)) = 2\arcsin\frac{d(x,y)}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}. \tag{5.2.1}$$

Proof. On $\mathbb{S}^N$ there is another metric that is equivalent to $d_{\mathbb{S}}$, namely, the *chordal metric* which measures the distance between two points of $\mathbb{S}^N$ as the Euclidean distance between these two points in $\mathbb{R}^{N+1}$. In other words, the chordal distance between two points of $\mathbb{S}^N$ is equal to the length of the chord having them as endpoints. Furthermore, denoting the chordal metric on $\mathbb{S}^N$ with ρ, we have for $x, y \in \mathbb{R}^N$ that

$$\rho(\Psi(x), \Psi(y)) = \frac{2d(x,y)}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}. \tag{5.2.2}$$

Let us prove the above equation for completeness. Since the stereographic projection is realized in $\mathbb{R}^{N+1}$ as projecting from the equatorial hyper-plane to the unit sphere it is clear that if we identify $x \in \mathbb{R}^N$ with $(x, 0) \in \mathbb{R}^{N+1}$, we have that $\rho(\Psi(x), \Psi(y)) = |\Psi(x) - \Psi(y)|$ and $d(x, y) = |x - y|$ where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{N+1}$. Let us denote the origin with $\mathbf{0}$ and the orthogonal projection of $\Psi(x)$ to the y_{N+1}-axis with $\Psi(x)_{N+1}$. Firstly, for $x \in \mathbb{R}^N$ from the similarity of triangles $\triangle \mathbf{N}\mathbf{0}x$ and $\triangle \mathbf{N}\Psi(x)_{N+1}\Psi(x)$ we have that

$$\frac{|\mathbf{N} - x|}{1} = \frac{|\mathbf{N} - \Psi(x)|}{1 - |\Psi(x)_{N+1}|} = \frac{|\mathbf{N} - \Psi(x)|}{1 - \frac{|x|^2-1}{|x|^2+1}},$$

and, since $|\mathbf{N} - x| = \sqrt{1+|x|^2}$, we have that

$$|\mathbf{N} - \Psi(x)| = \frac{2|\mathbf{N} - x|}{1+|x|^2} = \frac{2}{\sqrt{1+|x|^2}}$$

which implies that

$$|\mathbf{N} - \Psi(x)||\mathbf{N} - x| = 2.$$

Secondly, from this we have for $x, y \in \mathbb{R}^N$ that $|\mathbf{N} - \Psi(x)||\mathbf{N} - x| = |\mathbf{N} - \Psi(y)||\mathbf{N} - y|$ which implies that the triangles $\triangle \mathbf{N}xy$ and $\triangle \mathbf{N}\Psi(y)\Psi(x)$ are similar and that infers

$$|\Psi(x) - \Psi(y)| = \frac{2|x - y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}.$$

Finally, $d_{\mathbb{S}}(\Psi(x), \Psi(y))$ is the length of the unit circle arc having chord length equal to $|\Psi(x) - \Psi(y)|$, i.e.,

$$d_{\mathbb{S}}(\Psi(x), \Psi(y)) = 2 \arcsin \frac{|\Psi(x) - \Psi(y)|}{2}$$

which completes the proof of the lemma. □

We will now use the result of Lemma 5.2 in order to show that the stereographic projection Ψ is a bi-Lipschitz mapping on every ball of radius $R < 1$ centered at the origin.

Lemma 5.3. *Fix $R \in (0, 1)$. Then the stereographic projection $\Psi\colon (B_R(0), d) \to (\mathbb{S}^N, d_{\mathbb{S}})$ is a bi-Lipschitz mapping. More precisely, for $x, y \in B_R(0)$ we have:*

$$\frac{2}{1+R^2} d(x,y) \leq d_{\mathbb{S}}(\Psi(x), \Psi(y)) \leq \frac{2}{\sqrt{1-R^2}} d(x,y). \tag{5.2.3}$$

Proof. We have that $\arcsin t \geq t$ for $t \in (0,1)$ which together with (5.2.1) for $x, y \in B_R(0)$ yields

$$d_{\mathbb{S}}(\Psi(x), \Psi(y)) = 2 \arcsin \frac{d(x,y)}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \geq \frac{2d(x,y)}{1+R^2}.$$

On the other hand, for $0 \leq t \leq R < 1$ we have that $\arcsin t \leq t/\sqrt{1-R^2}$ which leads to

$$\begin{aligned} d_{\mathbb{S}}(\Psi(x), \Psi(y)) &= 2 \arcsin \frac{d(x,y)}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \\ &\leq \frac{2d(x,y)}{\sqrt{1-R^2}\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \leq \frac{2d(x,y)}{\sqrt{1-R^2}} \end{aligned}$$

and this completes the proof of the lemma.[1] $\square$

If $A \subseteq \mathbb{R}^N \cap B_R(0)$ for some $R < 1$, the above result gives a way to make the following bounds on the volumes of the corresponding δ-neighborhood of the set A and of its image $\Psi(A)$.

Lemma 5.4. *Let A be a subset of $\mathbb{R}^N$ such that $A \subseteq B_R(0)$ for some $R < 1$. Then for every $\delta > 0$, such that $A_\delta \subseteq B_R(0)$ we have that*

$$\frac{1}{2^N} |\Psi(A)_{\frac{2}{1+R^2}\delta, \mathbb{S}}|_{\mathbb{S}} \leq |A_\delta| \leq \frac{(1+R^2)^N}{2^N} |\Psi(A)_{\frac{2}{\sqrt{1-R^2}}\delta, \mathbb{S}}|_{\mathbb{S}}. \tag{5.2.4}$$

Proof. Using the equivalence of the metrics d and d_Ψ from Lemma 5.3 we have that

$$\Psi(A)_{\frac{2}{1+R^2}\delta, \mathbb{S}} \subseteq \Psi(A_\delta) \subseteq \Psi(A)_{\frac{2}{\sqrt{1-R^2}}\delta, \mathbb{S}}$$

and this infers

$$|\Psi(A)_{\frac{2}{1+R^2}\delta, \mathbb{S}}|_{\mathbb{S}} \leq \int_{A_\delta} \frac{2^N}{(1+|x|^2)^N} \, \mathrm{d}x \leq |\Psi(A)_{\frac{2}{\sqrt{1-R^2}}\delta, \mathbb{S}}|_{\mathbb{S}}.$$

Finally, using the fact that

$$\frac{2^N}{(1+R^2)^N} |A_\delta| \leq \int_{A_\delta} \frac{2^N}{(1+|x|^2)^N} \, \mathrm{d}x \leq 2^N |A_\delta|$$

we conclude the proof. $\square$

We can now easily derive the main theorem of this section which gives the relations between the upper and lower Minkowski contents of sets $A \subseteq \mathbb{R}^N$ such that $\overline{A} \subseteq B_1(0)$

[1] The fact that $x, y \in B_R(0)$ ensures that the argument of the arcsin function in the above expression is less than 1. This follows from the fact that this expression represents half of the length of the chord that connects $\Psi(x)$ and $\Psi(y)$. As both of these points are contained in the southern hemisphere of $\mathbb{S}^N$ the length of the chord connecting them is at most 2.

and the upper and lower spherical Minkowski contents of their images on the Riemann sphere $\mathbb{S}^N$.

Theorem 5.5. *Let $A \subseteq \mathbb{R}^N \cap B_R(0)$ for some $R < 1$ and $r \in \mathbb{R}$. Then we have*

$$\frac{1}{2^r(1+R^2)^{N-r}} \overline{\mathcal{M}}^r_{\mathbb{S}}(\Psi(A)) \leq \overline{\mathcal{M}}^r(A) \leq \frac{(1+R^2)^N}{2^r(\sqrt{1-R^2})^{N-r}} \overline{\mathcal{M}}^r_{\mathbb{S}}(\Psi(A)) \tag{5.2.5}$$

and

$$\frac{1}{2^r(1+R^2)^{N-r}} \underline{\mathcal{M}}^r_{\mathbb{S}}(\Psi(A)) \leq \underline{\mathcal{M}}^r(A) \leq \frac{(1+R^2)^N}{2^r(\sqrt{1-R^2})^{N-r}} \underline{\mathcal{M}}^r_{\mathbb{S}}(\Psi(A)). \tag{5.2.6}$$

In particular, we have $\overline{\dim}_{\mathbb{S}}(\Psi(A)) = \overline{\dim}_B A$ and $\underline{\dim}_{\mathbb{S}}(\Psi(A)) = \underline{\dim}_B A$. Moreover if $D := \dim_B A = \dim_{\mathbb{S}}(\Psi(A))$ exists, then A is Minkowski nondegenerate if and only if $\Psi(A)$ is spherically Minkowski nondegenerate.

Proof. This is a simple consequence of the previous lemma. Namely, for $r \in \mathbb{R}$ we have

$$\frac{1}{2^N} \frac{\left|\Psi(A)_{\frac{2}{1+R^2}\delta,\mathbb{S}}\right|_{\mathbb{S}}}{\delta^{N-r}} \leq \frac{|A_\delta|}{\delta^{N-r}} \leq \frac{(1+R^2)^N}{2^N} \frac{\left|\Psi(A)_{\frac{2}{\sqrt{1-R^2}}\delta,\mathbb{S}}\right|_{\mathbb{S}}}{\delta^{N-r}}$$

and taking the upper and lower limit as $\delta \to 0^+$ completes the proof. □

We point out that the above theorem can also be derived from Lemma 5.3 and a more general theorem concerning the Minkowski content and bi-Lipschitz mappings that can be found in [ŽuŽup2, Theorem 1].

Suppose now that $A \subseteq \mathbb{R}^N$ is an unbounded set such that $\overline{A} \subseteq (\overline{B_1(0)})^c$. Then $\Psi(A)$ is a subset of the northern hemisphere of $\mathbb{S}^N$. On the other hand, $\overline{\Phi(A)}$ is a subset of the unit ball in $\mathbb{R}^N$ and is bi-Lipschitz equivalent to its image under Ψ; that is, to $\Psi \circ \Phi(A)$ Furthermore, this set is an isometric image of $\Psi(A)$ under the reflection over the equatorial hyperplane. Having all of this in mind, Theorem 5.5 shows that $\Phi(A)$ and $\Psi(A)$ have very closely related fractal properties. In particular, their corresponding upper and lower box dimensions coincide and the property of Minkowski nondegeneracy is preserved.

In light of the discussion above it makes sense to define the upper and lower box dimensions of unbounded sets as the usual upper and lower box dimensions of their images under the geometric inversion.

Remark 5.6. The condition that $\overline{A} \subseteq (\overline{B_1(0)})^c$ is actually only technical and can be relaxed into assuming only that $0 \notin \overline{A}$ in the following sense. Namely, if this is the case, we then have that there exists $\lambda > 0$ such that the scaled set λA satisfies $\overline{\lambda A} \subseteq (\overline{B_1(0)})^c$ and we can apply the above reasoning to this set. Since the fractal properties of A behave in a good way under scaling we can analyze the scaled set λA and from that conclude about the fractal properties of A. More precisely the box dimensions of $\Phi(A)$ are preserved by scaling and the corresponding Minkowski contents are scaled by a factor of λ^{-D}.

Hence, in light of the above remark, we can introduce the following definition.

Definition 5.7. Let A be an unbounded set in $\mathbb{R}^N$, which is away from the origin, that is, $d(A,\{0\}) = \inf\{|a| : a \in A\} > 0$. Then clearly $\Phi(A)$ is bounded, and we define the upper box dimension of A by

$$\overline{\dim}_B A = \overline{\dim}_B \Phi(A).$$

Analogously we define the lower box dimension. If both the upper and the lower box dimensions of A coincide, we call it just the box dimension of A, and denote it by $\dim_B A$.

First of all, let us show that this definition of the box dimension of an unbounded set A does not depend on the choice of the origin.

Proposition 5.8. Assume that A is a given subset of $\mathbb{R}^N$, and $0 \notin \overline{A}$. Assume that also $w \notin \overline{A}$. Let Φ be the geometric inversion with respect to the origin, and Ξ the geometric reflection with respect to the point $w \in \mathbb{R}^N$, that is, $\Xi(x) = \frac{x-w}{|x-w|^2}$. Then $f = \Xi \circ \Phi : \Phi(A) \to \Xi(A)$ is a bi-Lipschitz mapping. In particular, see [Fal1], we have $\overline{\dim}_B \Phi(A) = \overline{\dim}_B \Xi(A)$, and similarly for the lower box dimension.

Proof. Let us first show that f is Lipschitzian. Denoting the Jacobian matrix $f'(x) := \frac{\partial f}{\partial x}$, it suffices to show that

$$\sup_{x\in\Phi(A)} \|f'(x)\| < \infty. \tag{5.2.7}$$

The matrix norm $\|\cdot\|$ can be taken as any operator norm, for instance, the ∞-norm. First, by Lemma 4.20; that is, by Equation (4.2.8), we have that

$$\Phi'(x) = \frac{|x|^2 I - 2x \otimes x}{|x|^4},$$

where I is the identity matrix and x is understood as a column vector.[2] Now, denoting $a = \Phi(x)$ we have:

$$\begin{aligned}
f'(x) &= \Psi'(\Phi(x)) \cdot \Phi'(x) \\
&= \frac{|a-w|^2 I - 2(a-w)\otimes(a-w)}{|a-w|^4} \cdot \frac{|a|^{-2} I - 2\frac{a}{|a|^2} \otimes \frac{a}{|a|^2}}{|a|^{-4}} \\
&= \frac{|a|^2}{|a-w|^2}\left(I - 2\frac{a-w}{|a-w|} \otimes \frac{a-w}{|a-w|}\right) \cdot \left(I - 2\frac{a}{|a|} \otimes \frac{a}{|a|}\right).
\end{aligned}$$

Therefore, since $\|v \otimes v\|_\infty \le \sqrt{N}$ for any unit vector v, we have that $\|f'(x)\| \le C\frac{|a|^2}{|a-w|^2}$,

[2] As before, $x \otimes x := x\, x^\tau$.

where C is a positive constant and $a = \Phi(x)$. Since $w \notin \overline{A}$ and $\frac{|a|^2}{|a-w|^2} \to 1$ as $a \to \infty$, the expression $\frac{|a|^2}{|a-w|^2}$ is bounded by a constant independent of a. This proves (5.2.7).

On the other hand, $f^{-1} = (\Psi \circ \Phi)^{-1} = \Phi^{-1} \circ \Psi^{-1} = \Phi \circ \Psi$, and we can show in the similar way that f^{-1} is Lipschitzian. Hence, f is bi-Lipschitzian. $\square$

We can also define the upper and lower r-dimensional Minkowski contents of A as the corresponding upper and lower Minkowski contents of $\Phi(A)$ for $r \in \mathbb{R}$. We say that A is Minkowski nondegenerate (Minkowski measurable) if $\Phi(A)$ is nondegenerate (Minkowski measurable).

Remark 5.9. If we wish, we can easily get rid of the condition $0 \notin \overline{A}$. Indeed, if A is any set in $\mathbb{R}^N$, we can proceed as follows. Define $A_1 = A \cap B_1(0)$ and $A_2 = A \setminus A_1$, and define

$$\overline{\dim}_B A = \max\{\overline{\dim}_B A_1, \overline{\dim}_B \Phi(A_2)\}.$$

It is easy to see that the upper box dimension so defined for unbounded sets satisfies the property of monotonicity (indeed, if $A \subseteq B$ then $\Phi(A) \subseteq \Phi(B)$, hence $\dim_B A = \dim_B \Phi(A) \leq \dim_B \Phi(B) = \dim_B B$), and the property of finite stability. See [Fal1].

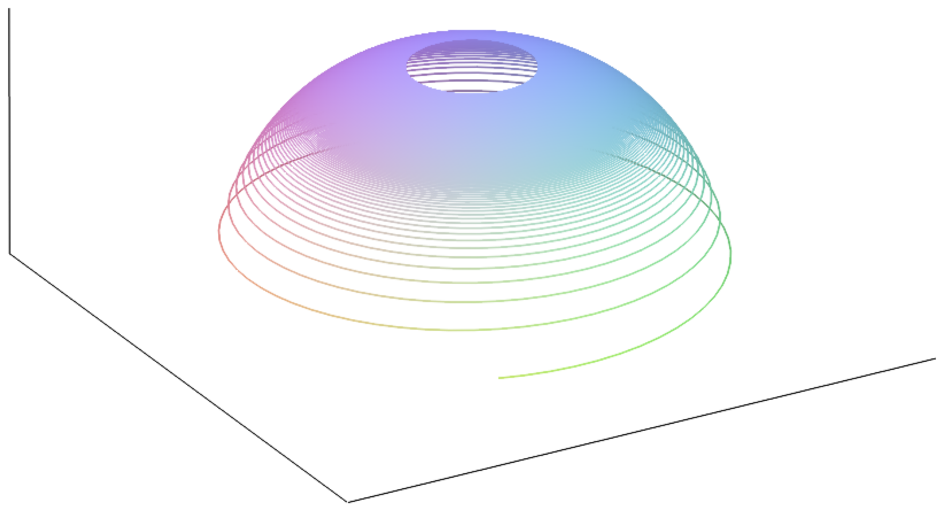

Figure 5.1: The unbounded spiral $f(\varphi) = \varphi^{1/4}$ projected to the Riemann sphere.

Another basic property, as expected, is that the box dimension is preserved for unbounded sets with positive distance from the origin that are bi-Lipschitz equivalent.

Theorem 5.10. *Let $V_{1,2}$ be two neighborhoods of ∞ in $\mathbb{R}^N$ such that $0 \notin \overline{V}_1$ and $0 \notin \overline{V}_2$. Let $f : V_1 \to V_2$ be a bi-Lipschitz map, where $0 \notin \overline{V}_1$ and $0 \notin \overline{V}_2$. If $A \subset V_1$ is unbounded, then $\overline{\dim}_B A = \overline{\dim}_B f(A)$, and analogously, $\underline{\dim}_B A = \underline{\dim}_B f(A)$.*

Theorem 5.10 follows immediately from the following proposition, the proof of which we postpone.

Proposition 5.11. *Let $V_{1,2}$ be two neighborhoods of ∞ in $\mathbb{R}^N$ such that $0 \notin \overline{V}_1$ and $0 \notin \overline{V}_2$. The mapping $f : V_1 \to V_2$ is bi-Lipschitzian if and only if the mapping $g : \Phi(V_1) \to \Phi(V_2)$ defined by*

$$g(x) = (\Phi \circ f \circ \Phi)(x) \tag{5.2.8}$$

is bi-Lipschitzian.

Proof of Theorem 5.10. We have that $\Phi(A) \subset \Phi(V_1)$, and using Proposition 5.11 we obtain that $\Phi(A)$ is a bi-Lipschitz equivalent to $g(\Phi(A))$, with g defined by (5.2.8). Hence,

$$\begin{aligned}\overline{\dim}_B A &= \overline{\dim}_B \Phi(A) = \overline{\dim}_B g(\Phi(A)) \\ &= \overline{\dim}_B (\Phi \circ f \circ \Phi^2)(A) \\ &= \overline{\dim}_B \Phi(f(A)) = \overline{\dim}_B f(A).\end{aligned}$$

The last equality follows from the bi-Lipschitz invariance of the upper box dimension for bounded sets. Analogously, we get the equality of the lower box dimensions. $\square$

To prove Proposition 5.11, we start with the following two lemmas.

Lemma 5.12. *For any $a, b \in \mathbb{R}^N \setminus \{0\}$ we have*

$$|\Phi(a) - \Phi(b)| = \frac{|a-b|}{|a|\,|b|},$$

where $|\cdot|$ is the Euclidean norm.

Proof. Multiplying (5.2.9) by $|a|\,|b|$ we see that the claim is equivalent with $\big|\,a_0|b| - b_0|a|\,\big| = |a-b|$, where $a_0 = a/|a|$ and $b_0 = b/|b|$ are unit vectors. Furthermore, taking the scalar product in $\mathbb{R}^N$, denoted by $(\cdot\,|\,\cdot)$, we have $\big|\,a_0|b| - b_0|a|\,\big|^2 = \big(a_0|b| - b_0|a|\,\big|\,a_0|b| - b_0|a|\big) = |a|^2 + |b|^2 - 2ab = (a-b\,|\,a-b) = |a-b|^2$, and the claim follows. $\square$

Lemma 5.13. *Let $f : V_1 \to V_2$ be a bi-Lipschitz map, where $0 \notin \overline{V}_1$ and $0 \notin \overline{V}_2$. Then there exist two positive constants C_1 and C_2 such that for all $a \in V_1$,*

$$C_1 \le \frac{|f(a)|}{|a|} \le C_2.$$

Proof. To prove the upper bound in (5.2.9), let us fix any $b \in V_1$. For any $a \in V_1$ we have $|f(a)| - |f(b)| \le |f(a) - f(b)| \le C|a-b| \le C(|a| + |b|)$, and therefore

$$\frac{|f(a)|}{|a|} \le C\left(1 + \frac{|b|}{|a|}\right) + \frac{|f(b)|}{|a|} \le C\left(1 + \frac{|b|}{d(0,V_1)}\right) + \frac{|f(b)|}{d(0,V_1)} = C_2, \tag{5.2.9}$$

where $d(0,V_1)$ is the distance from V_1 to the origin.

To prove the lower bound in (5.2.9), let D be the closed shell centered at the origin, of inner radius $d(0,V_1)$ and the outer radius $|b|+1$, where $b\in V_1$ is fixed again. Let $A=V_1\cap D$. Then

$$\inf_{a\in A}\frac{|f(a)|}{|a|}\ge\frac{\inf_{a\in A}|f(a)|}{|b|+1}\ge\frac{d(0,V_2)}{|b|+1}>0. \tag{5.2.10}$$

Now take any $a\in V_1\setminus A$. Using $|a|>|b|+1$ we have:

$$\frac{|a-b|}{|a|}\ge\frac{|a|-|b|}{|a|}=1-\frac{|b|}{|a|}\ge 1-\frac{|b|}{|b|+1}=\frac{1}{|b|+1}>0.$$

Also,

$$\frac{|f(a)|}{|f(a)-f(b)|}\ge\frac{|f(a)|}{|f(a)|+|f(b)|}=\frac{1}{1+\frac{|f(a)|}{|f(b)|}}\ge\frac{1}{1+\frac{|f(b)|}{d(0,V_2)}}>0.$$

Therefore,

$$\frac{|f(a)|}{|a|}=\frac{|f(a)|}{|f(a)-f(b)|}\cdot\frac{|f(a)-f(b)|}{|a-b|}\cdot\frac{|a-b|}{|a|} \tag{5.2.11}$$

is bounded below by a positive constant. This together with (5.2.10) proves the lower bound in (5.2.9). □

Now we proceed to the proof of the proposition.

Proof of Proposition 5.11. Assume that f is bi-Lipschitzian, i.e. $|f(a)-f(b)|\asymp|a-b|$ for all $a,b\in V_1$. Let $x=\Phi(a)$ and $y=\Phi(b)$ be any two elements from $\Phi(V_1)$. Using Lemma 5.12 we have

$$|f(\Phi(x))-f(\Phi(y))|\asymp|\Phi(x)-\Phi(y)|=\frac{|x-y|}{|x|\,|y|}=|x-y|\,|\Phi(x)|\,|\Phi(y)|.$$

Therefore,

$$\frac{|f(\Phi(x))-f(\Phi(y))|}{|f(\Phi(x))|\,|f(\Phi(y))|}\asymp|x-y|\frac{|\Phi(x)|\,|\Phi(y)|}{|f(\Phi(x))|\,|f(\Phi(y))|}.$$

Applying Lemma 5.12 on the left-hand side, and Lemma 5.13 on the right-hand side, we obtain

$$|\Phi(f(\Phi(x)))-\Phi(f(\Phi(y)))|\asymp|x-y|,$$

i.e. $|g(x)-g(y)|\asymp|x-y|$ for all $x,y\in\Phi(V_1)$.

The proof of the converse implication is similar, and therefore we omit it. □

Example 5.14. If Γ is a smooth curve (typically, an unbounded spiral) in $\mathbb{R}^N$ converging to infinity, which does not pass through the origin. Assume that $D=\dim_B\Gamma$ is well defined. Then its D-dimensional Minkowski content is defined as $\mathcal{M}^D(\Gamma)=\mathcal{M}^D(\Phi(\Gamma))$ provided the right-hand side exists. Note that the right-hand side is well defined, since the set $\Phi(\Gamma)$ is bounded. Furthermore, if we remove from Γ a portion of finite length, then the remaining part Γ_r has the same D-dimensional Minkowski content as Γ. This is due to the excision lemma, see [Žu4, Lemma 5.6].

Example 5.15. Let α be a given positive real number, and $A = \{k^\alpha : k \in \mathbb{N}\}$. It is well-known that the box dimension of $\Phi(A) = \{k^{-\alpha} : k \in \mathbb{N}\}$ is equal to $1/(1+\alpha)$, see e.g. [Lap–vFr3]. Therefore,

$$\dim_B A = \frac{1}{1+\alpha}.$$

The above example is a special case of the following result dealing with monotone strings $\mathcal{L} = (l_j)$ of infinite length, i.e. sequences of positive real numbers such that $\sum_{j=1}^{\infty} l_j = \infty$ and (l_j) is nonincreasing. We do not require that $l_j \to 0$ as $j \to \infty$. This string is associated with an unbounded sequence of real numbers $A = (a_k)$ defined by $a_k = \sum_{j=1}^{k} l_j$. Conversely, it is clear that a nondecreasing, unbounded sequence of real numbers $A = (a_k)$ defines the string $\mathcal{L} = (l_j)_j$, where $l_j = a_{j+1} - a_j$, and the string $\mathcal{L} = (l_j)$ is monotone if we require that l_j is nonincreasing. Note that here the set $\Phi(A) = \{a_k^{-1} : k \in \mathbb{N}\}$ is bounded, so that the classical box dimension makes sense. If we denote

$$\mu_k = a_k^{-1} - a_{k+1}^{-1}, \quad \mathcal{L}' = (\mu_k) \tag{5.2.12}$$

then

$$\overline{\dim}_B \Phi(A) = \overline{\dim}_B \mathcal{L}' := \inf \left\{ \gamma > 0 : \sum_{j=1}^{\infty} \mu_j^\gamma < \infty \right\}. \tag{5.2.13}$$

See [Lap–vFr3], where the right-hand side of (5.2.13) is taken as the definition of the upper box dimension of a general bounded, monotone string $\mathcal{L}' = (\mu_k)$, denoted by $D_{\mathcal{L}'}$ in this reference. Note that since $\sum_{j=1}^{\infty} \mu_j < \infty$, then $\overline{\dim}_B \Phi(A) \le 1$. The following simple lemma provides a sufficient condition for a string associated with the geometric inverse of an unbounded set to be monotone.

Lemma 5.16. *Let $A = (a_k)$ be an unbounded, monotonically nondecreasing sequence of positive numbers. The string $\mathcal{L}' = (\mu_k)$, defined by (5.2.12), is monotone if and only if for each $k \ge 1$,*

$$\frac{a_{k+1}}{a_k} + \frac{a_{k+1}}{a_{k+2}} \ge 2. \tag{5.2.14}$$

Furthermore,

$$\overline{\dim}_B A = \overline{\dim}_B \mathcal{L}'.$$

Proof. It is easy to check that $\mu_{k+1} \le \mu_k$ is equivalent with (5.2.14). □

Example 5.17. For $a_k = k^\alpha$, where α is a fixed positive number, the condition (5.2.14) is fulfilled, since

$$\frac{b_{k+1}}{b_k} + \frac{b_{k+1}}{b_{k+2}} = (1 + k^{-1})^\alpha + (1 + (k+1)^{-1})^{-\alpha} > (1 + k^{-1})^\alpha + (1 + k^{-1})^{-\alpha} > 2,$$

where the last inequality follows from the elementary inequality $t + t^{-1} > 2$ for $t > 1$. Therefore the conclusion of Example 5.15 is a special case of Lemma 5.16, since for α-strings $\mathcal{L}' = (k^{-\alpha} - (k+1)^{-\alpha})_{k\geq 1}$ we have $\overline{\dim}_B \mathcal{L}' = 1/(1+\alpha)$, see [Lap–vFr3].

5.3 Inverted Relative Fractal Drums

By using the geometric inversion and its connection with the one-point compactification presented in the previous section one can also consider relative fractal drums (A, Ω) that are unbounded, even if there does not exist a $\delta > 0$ such that $|A_\delta \cap \Omega| < \infty$. In that case we can work with the inverted relative fractal drum $(\Phi(A), \Phi(\Omega))$ which is, in light of the above theorem, related to analyzing the 'compactified' (A, Ω); that is, its image under the stereographic projection. It is clear that the analog of Theorem 5.5, is also valid for relative fractal drums and we state it here for the sake of completeness.

Theorem 5.18. *Let (A, Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $A \subseteq B_R(0)$ for some $R < 1$ and $r \in \mathbb{R}$. Then we have*

$$\frac{\overline{\mathcal{M}}^r_{\mathbb{S}}(\Psi(A), \Psi(\Omega))}{2^r(1+R^2)^{N-r}} \leq \overline{\mathcal{M}}^r(A, \Omega) \leq \frac{(1+R^2)^N}{2^r(\sqrt{1-R^2})^{N-r}} \overline{\mathcal{M}}^r_{\mathbb{S}}(\Psi(A), \Psi(\Omega)) \tag{5.3.1}$$

and

$$\frac{\underline{\mathcal{M}}^r_{\mathbb{S}}(\Psi(A), \Psi(\Omega))}{2^r(1+R^2)^{N-r}} \leq \underline{\mathcal{M}}^r(A, \Omega) \leq \frac{(1+R^2)^N}{2^r(\sqrt{1-R^2})^{N-r}} \underline{\mathcal{M}}^r_{\mathbb{S}}(\Psi(A), \Psi(\Omega)). \tag{5.3.2}$$

In particular, we have $\overline{\dim}_{\mathbb{S}}(\Psi(A), \Psi(\Omega)) = \overline{\dim}_B(A, \Omega)$ and $\underline{\dim}_{\mathbb{S}}(\Psi(A), \Psi(\Omega)) = \underline{\dim}_B(A, \Omega)$. Moreover if $D := \dim_B(A, \Omega) = \dim_{\mathbb{S}}(\Psi(A), \Psi(\Omega))$ exists, then (A, Ω) is Minkowski nondegenerate if and only if $(\Psi(A), \Psi(\Omega))$ is spherically Minkowski nondegenerate.

Now, if we additionally assume that the unbounded relative fractal drum (A, Ω) is such that $\overline{A} \subseteq (\overline{B_1(0)})^c$, we have that the inverted RFD $(\Phi(A), \Phi(\Omega))$ satisfies the conditions of Theorem 5.18. In light of Remark 5.6 this condition can also be relaxed if needed.

Furthermore, it is natural to ask what happens in the case of RFDs (A, Ω) for which exists a $\delta > 0$ such that $|A_\delta \cap \Omega| < \infty$. More precisely, what are the relations between the upper and lower relative box dimensions of (A, Ω) and that of $(\Phi(A), \Phi(\Omega))$. Although one would expect that we have an equality at least for the upper relative box dimension, we have not yet been able to prove this and leave it for future work. The same question could be asked for the relations between the upper and lower Minkowski contents of (A, Ω) and that of $(\Phi(A), \Phi(\Omega))$, as well as for their corresponding relative distance (and tube) zeta functions (see Problem A.12).

In the special case of relative fractal drums (∞, Ω), where Ω is a Lebesgue measurable subset of $\mathbb{R}^N$ we already know that the distance zeta functions of (A, Ω) and the inverted fractal drum $(\mathbf{0}, \Phi(\Omega))$ are essentially the same (see Theorem 4.21) which automatically implies that their respective complex dimensions also coincide. Furthermore, we have a connection between their corresponding Minkowski contents given in Theorem 4.34.

Theorem 5.19. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ with finite Lebesgue measure. Then the following relations are valid for every $\phi > 1$ and $r \leq -N$:*

$$\frac{1-\phi^{N+r}}{\phi^{2N}}\overline{\mathcal{M}}^r(\infty,\Omega) \leq \overline{\mathcal{M}}^r(\mathbf{0},\Phi(\Omega)) \leq \overline{\mathcal{M}}^r(\infty,\Omega) \tag{5.3.3}$$

and

$$\underline{\mathcal{M}}^r(\mathbf{0},\Phi(\Omega)) \leq \underline{\mathcal{M}}^r(\infty,\Omega). \tag{5.3.4}$$

Furthermore, if $r < -N$, then for

$$\phi = \left(\frac{2N}{N-r}\right)^{\frac{1}{N+r}}$$

we get the optimal left hand side inequality in (4.7.10):

$$-\frac{N+r}{N-r}\left(\frac{2N}{N-r}\right)^{-\frac{N+r}{N-r}}\overline{\mathcal{M}}^r(\infty,\Omega) \leq \overline{\mathcal{M}}^r(\mathbf{0},\Phi(\Omega)) \tag{5.3.5}$$

Proof. The theorem is a consequence of Theorems 4.84 and 5.18 combined with the fact that $\Psi^{-1}\circ\Phi\circ\Psi$ is an isometry of $\mathbb{S}^N$. More precisely, let $\mathbf{S}$ be the south pole of the Riemann sphere $\mathbb{S}^N$. Then the relative fractal drums $(\mathbf{N},\Psi(\Omega))$ and $(\mathbf{S},\Psi\circ\Phi(\Omega))$ are isometric images of each other. Let $0 < R < 1$ and combine the right-hand side of (5.3.1) (Theorem 5.18) with the right-hand side of (4.7.10) (Theorem 4.84) to get that

$$\frac{2^r(\sqrt{1-R^2})^{N-r}}{(1+R^2)^N}\overline{\mathcal{M}}^r(\mathbf{0},\Phi(\Omega)) \leq \overline{\mathcal{M}}^r_{\mathbb{S}}(\mathbf{N},\Psi(\Omega)) \leq 2^r\overline{\mathcal{M}}^r(\infty,\Omega). \tag{5.3.6}$$

Since this is true for every $R \in (0,1)$, by letting $R \to 0^+$ we get the right-hand side of (5.3.3). The same argument, applied to the lower contents yields (5.3.4). Similarly, to get the left-hand side of (5.3.3) we combine the left-hand side of (5.3.1) from Theorem 5.18 and the left hand side of (4.7.10) from Theorem 4.84 to get

$$\frac{2^r(1-\phi^{N+r})}{\phi^{2N}}\overline{\mathcal{M}}^r(\infty,\Omega) \leq \overline{\mathcal{M}}^r_{\mathbb{S}}(\mathbf{N},\Psi(\Omega)) \leq 2^r(1+R^2)^N\overline{\mathcal{M}}^r(\mathbf{0},\Phi(\Omega)). \tag{5.3.7}$$

Again, we let $R \to 0^+$ to get the left-hand side of (5.3.3). Finally, the claim about the optimal choice of ϕ is a direct consequence of the analogous claim in Theorem 4.84.

□

5.4 The Poincaré Compactification

In the theory of dynamical systems the Poincaré compactification is usually more useful than the one-point compactification so that in the fractal analysis approach to the problems in dynamics it is a valid question what can be said about the relation of fractal properties of sets and their projections to the Poincaré sphere. Note that in case of the Poincaré compactification we no longer have a point at infinity, but rather an '$(N-1)$-dimensional sphere at infinity'. In the special case of $\mathbb{S}^2_P$ this is equal to the equator of the unit sphere in $\mathbb{R}^3$, i.e., the unit circle that lies in the xy-plane.

Let $\mathbb{S}^N_P$ be the Poincaré sphere in $\mathbb{R}^{N+1}$ of radius 1, i.e.

$$\mathbb{S}^N_P := \{(y_1,\dots,y_{N+1}) : \sum_{i=1}^{N+1} y_i^2 = 1\}. \tag{5.4.1}$$

We shall project onto $\mathbb{S}^N_P$ from the hyperplane placed tangentially at the north pole $\mathbf{N}$. More precisely, let $\Psi_P\colon \mathbb{R}^N \times \{1\} \equiv \mathbb{R}^N \to \mathbb{S}^N_P$ be the Poincaré projection.

The following discussion concerning a focus type spiral will show that the fractal properties of a set are not preserved under the Poincaré compactification. We will show how the box dimension of a focus type spiral is affected by a composition of geometric inversion and projection onto the Poincaré sphere. (See Problem A.13.)

Proposition 5.20. Let $\Gamma_1 \dots r = f(\varphi)$ be a spiral of focus type given in polar coordinates in $\mathbb{R}^2$ such that

$$f(\varphi) \asymp \varphi^{-\alpha}, \quad |f'(\varphi)| \asymp \varphi^{-\alpha-1}, \quad |f''(\varphi)| \le M\varphi^{-\alpha}$$

as $\varphi \to \infty$, for some positive constants α and M. Let $\Gamma_2 := \Psi_P \circ \Phi(\Gamma_1)$ be a spiral in $\mathbb{S}^2_P$.[3] Then

$$\dim_B \Gamma_2 = \frac{2+\alpha}{1+\alpha}. \tag{5.4.2}$$

Proof. In cylindrical coordinates in $\mathbb{R}^3$ the above described map of the spiral Γ_1 is given by

$$(f(\varphi),\varphi) \mapsto \left(\frac{1}{\sqrt{1+f(\varphi)^2}}, \varphi, \frac{f(\varphi)}{\sqrt{1+f(\varphi)^2}}\right).$$

We will use [ŽuŽup2, Theorem 5(b)]. In our case we have $r = 1 - F(\varphi)$, where

$$F(\varphi) = 1 - \frac{1}{\sqrt{1+f(\varphi)^2}},$$

and $Z = \sqrt{1-r^2} = g(|1-r|)$, for $g(t) = \sqrt{2t-t^2}$.

[3]In other words, firstly we geometrically invert Γ_1, and then project it onto $\mathbb{S}^2_P$.

Now we have to check the conditions from [ŽuŽup2, Theorem 5(b)]:

$$g(t) \sim t^{1/2}\sqrt{2},\ g'(t) \sim \frac{1}{\sqrt{2}}t^{-1/2},\ g''(t) \sim -\frac{1}{2\sqrt{2}}t^{-3/2}.$$

We can see that g meets the conditions for $\beta = 1/2$. Now we check the conditions on F:

$$F(\varphi) \sim \frac{1}{2}f(\varphi)^2 \asymp \varphi^{-2\alpha};$$

$$|F'(\varphi)| = \frac{f(\varphi)|f'(\varphi)|}{(1+f(\varphi)^2)^{3/2}} \asymp \varphi^{-2\alpha-1};$$

$$|F''(\varphi)| \le \frac{1}{(1+f^2)^{5/2}}\left(|f'|^2 + f|f''| + f^3|f''| + 2f^2|f'|^2\right) \le c\varphi^{-2\alpha}$$

for φ sufficiently large. So we can see that F meets the conditions of the [ŽuŽup2, Theorem 5(b)] for $\alpha_2 = 2\alpha$ and $\beta = 1/2$ and we have the conclusion

$$\dim_B \Gamma_2 = \frac{2+\alpha_2\beta}{1+\alpha_2\beta} = \frac{2+\alpha}{1+\alpha}.$$

□

Remark 5.21. For $\alpha \ge 1$ the box dimension of the spiral Γ_1 is one. This shows that by projecting a spiral of trivial box dimension to the Poincaré sphere we can get a spiral of nontrivial box dimension.

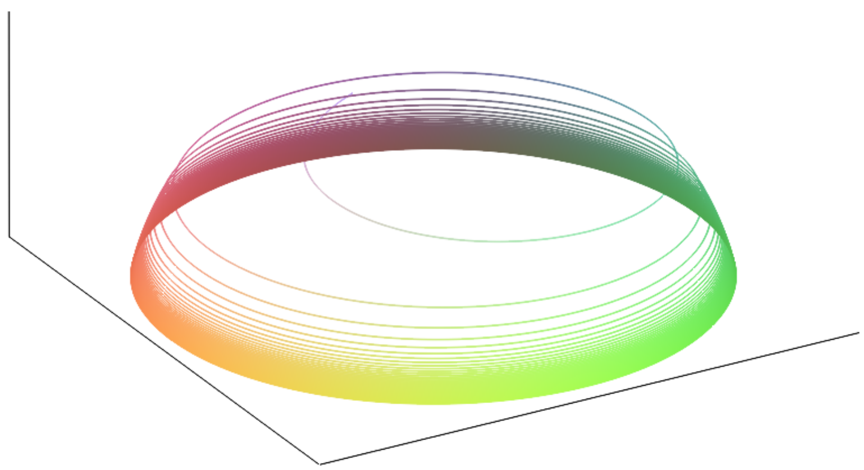

Figure 5.2: The unbounded spiral $f(\varphi) = \varphi^{1/4}$ projected to the Poincaré sphere of radius 1.

Remark 5.22. As the Poincaré sphere is usually represented by projecting it orthogonally on the xy-plane, this will further affect the box dimension of a spiral defined in Proposition

5.20. After projecting it, we will have a limit cycle type spiral $\Gamma_3 \ldots r = 1 - F(\varphi)$ in the plane and its box dimension will be reduced to

$$\dim_B \Gamma_3 = \frac{2+2\alpha}{1+2\alpha}.$$

For a focus type spiral Γ_1 of nontrivial box dimension i.e. for $\alpha \in (0,1)$ we have the next relations between box dimensions:

$$\begin{aligned}
\dim_B \Gamma_2 &= 1 + \frac{1}{2}\dim_B \Gamma_1,\\
\dim_B \Gamma_3 &= \frac{2}{3-\dim_B \Gamma_2},\\
\dim_B \Gamma_3 &= \frac{4}{4-\dim_B \Gamma_1}.
\end{aligned}$$

It remains to see if any general results can be obtained concerning the effect of the Poincaré projection on fractal properties of unbounded sets.

5.5 The Inverted Polynomial Vector Field

We start with a polynomial system

$$\dot{x} = P(x) \tag{5.5.1}$$

defined on $\mathbb{R}^N$. Applying the change of variables $u = \Phi(x) = x/|x|^2$, after a short computation, we arrive at the following system:

$$\dot{u} = |u|^2 \widetilde{P}(u) - 2u(u \cdot \widetilde{P}(u)), \tag{5.5.2}$$

defined on $\mathbb{R}^N \setminus \{0\}$, where

$$\widetilde{P}(u) = P\left(\frac{u}{|u|^2}\right).$$

Since the geometric inversion is clearly involutive, we have that $x = u/|u|^2$. Although the right-hand side of (5.5.2) is not necessarily a polynomial field in dependence of $u_1, \ldots, u_n$, where $u = (u_1, \ldots, u_n)$, we can easily transform it into one in a way that preserves its phase portrait. Namely, observe that the largest exponent of $|u|^{-2}$ appearing within the component functions of $\widetilde{P}(u)$ is equal to $k = \deg P := \max_i \deg P_i$. Hence, we have that

$$\dot{u} = |u|^{2k}\left(|u|^2 \widetilde{P}(u) - 2u(u \cdot \widetilde{P}(u))\right), \tag{5.5.3}$$

is a polynomial vector field.

If we denote the phase portrait of (5.5.1) by $\mathcal{P} = \{\Gamma_i : i \in I\}$ (the family of trajectories

Γ_i), it will be convenient to define $\Phi(\mathcal{P})$ by

$$\Phi(\mathcal{P}) = \{\Phi(\Gamma_i) : i \in I\}. \tag{5.5.4}$$

It is clear that $\Phi(\mathcal{P})$ is the phase portrait of (5.5.2) on $\mathbb{R}^N \backslash \{0\}$. All this can be summarized in the following lemma.

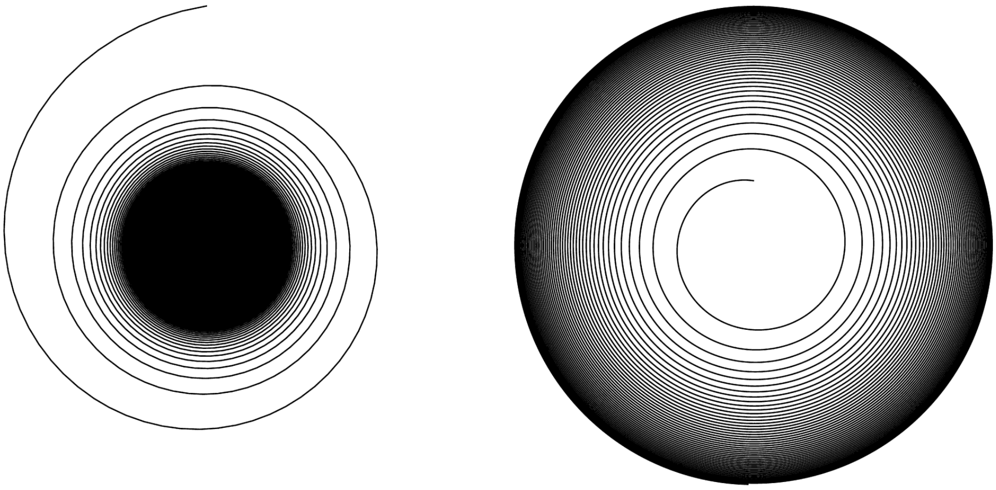

Figure 5.3: The bounded spiral $r = \varphi^{-1/4}$ (left) and the unbounded spiral $r = \varphi^{1/4}$ (right) both have the same box dimension equal to 8/5. The nucleus of the bounded spiral is at the origin whereas the nucleus of the unbounded spiral is at infinity.

Lemma 5.23. *Let P be a given polynomial vector field in $\mathbb{R}^N$, and let $\mathcal{P}$ be the phase portrait of* (5.5.1)*. Then there exists an explicit polynomial vector field in $\mathbb{R}^N$, given by* (5.5.3)*, such that its phase portrait is equal to $\Phi(\mathcal{P})$.*

In particular, if $\{C_i : i \in I\}$ is the collection of all limit cycles of a polynomial vector field, then there exists a polynomial vector field in $\mathbb{R}^N$ such that $\{\Phi(C_i) : i \in I\}$ is the collection of all limit cycles of the new vector field.

As we can see, if C is a limit cycle of a polynomial system, then its geometric inverse $\Phi(C)$ is also a limit cycle of a polynomial system. The following definition is also inspired by Lemma 5.23.

Definition 5.24. We say that the point at infinity is a weak focus of a dynamical system in $\mathbb{R}^N$ if the origin is a weak focus of the system obtained by its geometric inversion.

Example 5.25. Let Γ be a spiral defined in polar coordinates by $r = \varphi^{-\alpha}$, where $\varphi \geq \varphi_0 > 0$, and α is a given positive constant. Then $\Phi(\Gamma)$ is an unbounded spiral defined by $r = \varphi^{\alpha}$, where $\varphi \geq \varphi_0 > 0$. We have

$$\dim_B \Phi(\Gamma) = \max\left\{1, \frac{2}{1+\alpha}\right\},$$

see [Tri, p. 121]. Note that the nucleus of the spiral Γ is concentrated near the origin, so that the nucleus of $\Phi(\Gamma)$ is concentrated at infinity. For a strict definition of the nucleus see [Tri]. Intuitively, it is the part where the ε-neighborhood of the spiral selfintersects, see Figure 5.3.

Example 5.26. Let α and β be two given positive constants. Let A be defined as the union of two spirals Γ_1 and Γ_2, defined in polar coordinates as follows: $\Gamma_1 \ldots r = \varphi^{-\alpha}$ when $\varphi > 1$ (bounded spiral tending to the origin), while $\Gamma_2 \ldots r = \varphi^{\beta}$ when $\varphi > 1$ (unbounded spiral, away from the origin). It is easy to see, using finite stability of the box dimension, that

$$\dim_B A = \max\left\{1, \frac{2}{1+\min\{\alpha,\beta\}}\right\}.$$

Starting from $\dot{x} = P(x)$, see (5.5.1), using geometric inversion we arrive at $\dot{u} = P^*(u)$ where

$$P^*(u) = |u|^2 \widetilde{P}(u) - 2u(u \cdot \widetilde{P}(u)). \tag{5.5.5}$$

It is clear that $P^{**} = P$ for each vector field P, since the geometric inversion with respect to the origin is involutive. It is easy to see that

$$^* : C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^N) \to C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$$

is a linear operator with real coefficients: for any $\lambda, \mu \in \mathbb{R}$ and $F, G \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$ we have $(\lambda F + \mu G)^* = \lambda F^* + \mu G^*$.

Remark 5.27. If in (5.5.1) $P(x)$ is a rational function (that is, the component functions are rational functions of x_j, $j = 1, \ldots, n$), then from (5.5.5) we see that $P^*(u)$ is also a rational function. The phase portrait of the system (5.5.1) is the same (outside the origin) as for the polynomial system corresponding to $d(x)P(x)$, where $d(x)$ is the common denominator of all $P_j(x)$. Analogously for the system (5.5.2).

The following lemma deals with a special class of right-hand sides $P(x)$ of (5.5.1) for which $P^*(u)$ can be easily computed.

Lemma 5.28. *Let us consider the system* (5.5.1) *with* $P(x) = Rx - \gamma x g(|x|)$, $x \in \mathbb{R}^N$, *where* γ *is a real constant and* $g : (0,\infty) \to \mathbb{R}$ *a continuous function, and* R *is an* $N \times N$ *real antisymmetric matrix:* $R^\tau = -R$. *Then*

$$P^*(u) = Ru + \gamma u g(|u|^{-1}).$$

Proof. The matrix R is antisymmetric if and only if $Rx \cdot x = 0$ for all x. The claim follows from (5.5.5) and $\widetilde{P}u = |u|^{-2}(Ru - \gamma ug(|u|^{-1}))$ after a short computation. □

Example 5.29. In particular, if $P(x) = Rx$, where R is a real antisymmetric matrix, then $P^*(u) = Ru$, that is, $P = P^*$. If $P(x) = cx$, where c is a real constant, then $P^*(u) = -cu$.

A typical example of a real matrix R satisfying the condition $Rx \cdot x = 0$ for all $x \in \mathbb{R}^N$ is any diagonal block matrix containing either matrices of the form

$$\lambda_j \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

on the diagonal (here $\lambda_j \in \mathbb{R}$), or zeros.

We shall also often need the following technical lemma, dealing with planar systems of weak focus type near the origin:

$$\begin{aligned} \dot{x} &= -y + p(x, y) \\ \dot{y} &= \quad x + q(x, y). \end{aligned} \tag{5.5.6}$$

A typical situation is when p and q are analytic functions with McLaurin series containing quadratic or higher order terms only. It is an extension of Lemma 5.28 in the case of $N = 2$.

Lemma 5.30. *The system obtained from* (5.5.6) *by geometric inversion is equal to*

$$\begin{aligned} \dot{u} &= -v + (v^2 - u^2)\widetilde{p} - 2uv\,\widetilde{q} \\ \dot{v} &= \quad u + (u^2 - v^2)\widetilde{q} - 2uv\,\widetilde{p}, \end{aligned} \tag{5.5.7}$$

where $\widetilde{p} = p\left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$ *and* $\widetilde{q} = q\left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$.

The corresponding general result concerning the box dimension of spiral trajectories of (5.5.7) can be seen in Theorem 5.33.

5.6 Weak Focus at Infinity

In this section we apply fractal analysis to several examples of polynomial systems having weak foci at infinity.

5.6.1 Weakly Damped Oscillator

Let us consider a weakly damped oscillator

$$\ddot{y} + Cy^\alpha(\dot{y})^\beta + y = 0,$$

where α is an even and β an odd positive integer, and C is any positive constant. It is well-known that it is globally stable. It is equivalent to the following planar system:

$$\begin{aligned}\dot{x} &= -y - Cx^\beta y^\alpha \\ \dot{y} &= \quad x.\end{aligned}$$

All nontrivial trajectories Γ, corresponding to $t \geq 0$, are spirals converging clockwise to the origin, and the origin is the weak focus. Using Lemma 5.30 we conclude that the corresponding system obtained by geometric inversion is

$$\begin{aligned}\dot{u} &= -v + C\frac{u^\beta u^\alpha(u^2 - v^2)}{(u^2+v^2)^{\alpha+\beta}} \\ \dot{v} &= \quad u + 2C\frac{u^{\beta+1}v^{\alpha+1}}{(u^2+v^2)^{\alpha+\beta}}.\end{aligned} \tag{5.6.1}$$

All trajectories corresponding to $t \geq 0$, starting outside the origin, are of the form $\Phi(\Gamma)$ for some Γ as above. The spirals $\Phi(\Gamma)$ are converging clockwise to infinity, and the infinity is the weak focus. The system (5.6.1) becomes polynomial after multiplying the right-hand sides by $(u^2+v^2)^{\alpha+\beta}$:

$$\begin{aligned}\dot{u} &= -v(u^2+v^2)^{\alpha+\beta} + Cu^\beta u^\alpha(u^2 - v^2) \\ \dot{v} &= \quad u(u^2+v^2)^{\alpha+\beta} + 2Cu^{\beta+1}v^{\alpha+1}.\end{aligned} \tag{5.6.2}$$

Theorem 5.31. *Any nontrivial trajectory* Γ *of the system* (5.6.2)*, corresponding to* $t \geq 0$*, is a spiral converging to infinity, and*

$$\dim_B \Gamma = 2\left(1 - \frac{1}{\alpha+\beta}\right).$$

Furthermore, the spirals are Minkowski nondegenerate.

This follows immediately from [PaŽuŽup1, Theorem 7].

5.6.2 Liénard Systems

Let us consider the following Liénard system:

$$\begin{aligned}\dot{x} &= -y + \sum_{i=1}^{N} a_{2i}x^{2i} + \sum_{i=k}^{N} a_{2i+1}x^{2i+1} \\ \dot{y} &= \quad x.\end{aligned}$$

Here we assume that $a_{2k+1} \neq 0$, which means that this is the first nontrivial coefficient on the right-hand side having an odd index. The system obtained by geometric inversion is

$$\begin{aligned} \dot{u} &= -v + (v^2 - u^2)\widetilde{p}(u) \\ \dot{v} &= \quad u - 2uv\,\widetilde{p}(u), \end{aligned} \tag{5.6.3}$$

where

$$\widetilde{p}(u) = \sum_{i=1}^{N} a_{2i} \frac{u^{2i}}{(u^2+v^2)^{2i}} + \sum_{i=k}^{N} a_{2i+1} \frac{u^{2i+1}}{(u^2+v^2)^{2i+1}}.$$

Multiplying the right-hand sides of (5.6.3) by $(u^2 + v^2)^N$, the system becomes polynomial, retaining the same phase portrait outside the origin. An immediate consequence of [ŽuŽup3, Theorem 6] is the following result.

Theorem 5.32. *If in* (5.6.3) *we have $a_{2k+1} \neq 0$, then for any initial point $(u_0, v_0) \neq (0,0)$ we have that the corresponding trajectory $\Gamma = \Gamma(u_0, v_0)$ starting from that point is a spiral tending to infinity, and*

$$\dim_B \Gamma = 2\left(1 - \frac{1}{2k+1}\right).$$

Furthermore, Γ is Minkowski nondegenerate.

As we see, unbounded spiral trajectories (in fact, semitrajectories, i.e. starting from initial point) of Liénard systems can achieve box dimensions with values from the following set only:

$$D_0 = \left\{\frac{4k}{2k+1} : k \in \mathbb{N}\right\} = \left\{\frac{4}{3}, \frac{8}{5}, \frac{12}{7}, \frac{16}{9}, \frac{20}{11}, \dots\right\}.$$

For analytic systems, these are the only values of box dimensions that unbounded spiral trajectories can achieve, see [ŽuŽup3].

A more general result can be stated in terms of the Poincaré map at infinity. We deal with the system (5.5.6) such that $p(x,y)$ and $q(x,y)$ are analytic functions of the form

$$p(x,y) = \sum_{k=2}^{\infty} p_k(x,y), \quad q(x,y) = \sum_{k=2}^{\infty} q_k(x,y) \tag{5.6.4}$$

where p_k and q_k are homogeneous polynomials of k-th degree. The Lyapunov coefficient of a system at infinity is defined as the Lyapunov coefficient at the origin of the system obtained by geometric inversion. The *Lyapunov coefficient* (also known as *the Lyapunov number*) near the weak focus is defined as the coefficient of the leading term of the Taylor expansion of the displacement function (see, e.g., [Per, Section 3.4, Theorem 3]). The following result follows immediately from [ŽuŽup3, Theorem 6].

Theorem 5.33. *Let Γ be an unbounded spiral trajectory, away from the origin, associated to the system* (5.5.7)*. Assume that $p(x,y)$ and $q(x,y)$ are analytic functions as in (5.6.4).*

If V_{2k+1} *is the first nonzero Lyapunov coefficient of* (5.5.7) *at infinity, then*

$$\dim_B \Gamma = 2\left(1 - \frac{1}{2k+1}\right).$$

Furthermore, Γ *is Minkowski nondegenerate.*

5.6.3 The Classical Hopf Bifurcation

The classical Hopf bifurcation is defined by the following system for $k = 1$:

$$\begin{aligned} \dot{x} &= -y - x\left((x^2+y^2)^k + a\right) \\ \dot{y} &= \quad x - y\left((x^2+y^2)^k + a\right), \end{aligned} \tag{5.6.5}$$

where a is the bifurcation parameter. The corresponding spiral trajectories Γ are converging clockwise to the origin. Using Lemma 5.28 (here R is the symplectic 2×2 matrix and $g(r) = r^{2k} + a$) we have that the related system obtained from (5.6.5) by geometric inversion is

$$\begin{aligned} \dot{u} &= -v + u\left((u^2+v^2)^{-k} + a\right) \\ \dot{v} &= \quad u + v\left((u^2+v^2)^{-k} + a\right), \end{aligned} \tag{5.6.6}$$

and the corresponding spirals are converging clockwise to infinity, which is a weak focus. The corresponding polynomial system

$$\begin{aligned} \dot{u} &= -v(u^2+v^2)^k + u\left(1 + a(u^2+v^2)^k\right) \\ \dot{v} &= \quad u(u^2+v^2)^k + v\left(1 + a(u^2+v^2)^k\right), \end{aligned} \tag{5.6.7}$$

has the same phase portrait as (5.6.6) outside the origin.

In polar coordinates (r, φ) system (5.6.5) has the form

$$\begin{aligned} \dot{r} &= -r(r^{2k} + a) \\ \dot{\varphi} &= 1. \end{aligned} \tag{5.6.8}$$

For $a < 0$ the limit cycle is born at the origin, $r = (-a)^{1/k}$, while system (5.6.6) in polar coordinates (ρ, φ) has the form

$$\begin{aligned} \dot{\rho} &= \rho(\rho^{-2k} + a) \\ \dot{\varphi} &= 1. \end{aligned} \tag{5.6.9}$$

In this case, for $a < 0$ the limit cycle is born at infinity. Here $r = (-a)^{-1/k}$ and $r \to \infty$ as $a \to 0_-$. The system (5.6.9) is clearly the same as the one obtained from (5.6.5) by introducing the coordinates (ρ, φ) defined via $x = \frac{\cos\varphi}{\rho}$, $y = \frac{\sin\varphi}{\rho}$. The following result shows that the box dimension 'recognizes' the Hopf bifurcation.

Theorem 5.34. *Let $a = 0$ in the bifurcation problem* (5.6.6) *or* (5.6.7)*. Then any unbounded spiral trajectory Γ, away from the origin, has its box dimension equal to*

$$\dim_B \Gamma = \frac{4k}{2k+1},$$

and is Minkowski measurable. For all the other values of a the box dimension is trivial, i.e. equal to 1*.*

This is an immediate consequence of [ŽuŽup1, Theorem 7].

5.7 The Hopf–Takens Bifurcation at Infinity

Using geometric inversion and results from [ŽuŽup1], we shall study fractal properties of the Hopf-Takens bifurcation occurring at infinity. For a standard generic Hopf–Takens bifurcation we have the following normal form:

$$\begin{aligned} X_{\pm}^{(l)} := &\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \\ &\pm \left((x^2+y^2)^l + a_{l-1}(x^2+y^2)^{l-1} + \cdots + a_0\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \end{aligned}$$

where $(a_0, \ldots, a_{l-1}) \in \mathbb{R}^l$ is fixed. In the sequel we will consider $X_+^{(l)}$ only, since the case $X_-^{(l)}$ is treated similarly. Furthermore, in the case of $X_+^{(l)}$, the normal form in polar coordinates is given by

$$\begin{aligned} \dot r &= r\left(r^{2l} + \sum_{j=0}^{l-1} a_j r^{2j}\right) \\ \dot\varphi &= 1. \end{aligned} \tag{5.7.1}$$

Geometric inversion with $\rho := 1/r$ yields a new system of differential equations

$$\begin{aligned} \dot\rho &= -\rho\left(\rho^{-2l} + \sum_{j=0}^{l-1} a_j \rho^{-2j}\right) \\ \dot\varphi &= 1. \end{aligned} \tag{5.7.2}$$

Now it is easy to see that the following analogous versions of [ŽuŽup1, Theorems 9 and 10] are valid.

Theorem 5.35 (The focus case)**.** *Let Γ be a part of a trajectory of* (5.7.2) *near infinity.*

(*a*) *Assume that $a_0 \neq 0$. Then the spiral Γ is of exponential type, that is, comparable with $\rho = e^{-a_0\varphi}$, and hence $\dim_B \Gamma = 1$.*

(*b*) *Let k be fixed, $1 \le k \le l$, $a_0 = \cdots = a_{k-1} = 0$, $a_k \neq 0$. Then Γ is comparable with* *the spiral* $\rho = \varphi^{1/2k}$, *and*

$$\dim_B \Gamma = \frac{4k}{2k+1}.$$

Theorem 5.36 (The limit cycle case)**.** *Let the system* (5.7.2) *have a limit cycle* $\rho = a$ *of multiplicity* m, $1 \le m \le l$. *By* Γ_1 *and* Γ_2 *we denote the parts of two trajectories of* (5.7.2) *near the limit cycle from outside and inside respectively. Then the trajectories* Γ_1 *and* Γ_2 *are comparable with*

(*a*) *exponential spirals* $\rho = a \pm e^{-\beta\varphi}$ *of limit cycle type when* $m = 1$, *for some constants* $\beta \neq 0$ (*depending only on the coefficients* a_i, $0 \le i \le l-1$),

(*b*) *power spirals* $\rho = a \pm \varphi^{-1/(m-1)}$ *when* $m > 1$.

In both cases we have

$$\dim_B \Gamma_i = 2 - \frac{1}{m}, \quad i = 1, 2.$$

Remark 5.37. In Theorem 5.36 the parts of trajectories we are observing are contained in an open annulus containing the limit cycle which is a bounded set that does not contain the origin. As geometric inversion Φ is bi-Lipschitzian on such sets, Theorem 5.36 is a direct consequence of Theorem 10 from [ŽuŽup1].

Remark 5.38. From Theorem 5.36 we know that for (5.7.2) each spiral trajectory of limit cycle type has box dimension from the set

$$D_1 = \left\{2 - \frac{1}{m} : m \in \mathbb{N}\right\} = \left\{1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \dots\right\}.$$

See [ŽuŽup3, p. 958].

Let us have a look at the inversion of a standard Hopf bifurcation in polar coordinates, i.e. system (5.7.2) for $l = 1$:

$$\begin{aligned} \dot{\rho} &= -\rho(\rho^{-2} + a_0) \\ \dot{\varphi} &= 1. \end{aligned} \tag{5.7.3}$$

Viewing a_0 as a bifurcation parameter, we have the following three possibilities.

(1) For $a_0 < 0$ the trajectories of (5.7.3) are given by

$$\rho(\varphi) = -\frac{1}{a_0}\sqrt{-a_0 + a_0^2 C e^{-2a_0\varphi}}, \quad C \ge 0, \ \varphi \in \mathbb{R} \text{ or } C < 0, \ \varphi \le \frac{\log(a_0 C)}{2a_0}.$$

We can see that we have a strong focus at infinity and the circle $\rho = (-a_0)^{-1/2}$ is the limit cycle for trajectories from inside and outside near the circle. The corresponding spirals near infinity are comparable with $\rho = e^{-a_0\varphi}$, while the spirals near the circle are comparable with $\rho = (-a_0)^{-1/2} \pm e^{-a_0\varphi}$. All these spiral trajectories are of exponential type and hence of box dimension equal to 1. See Figure 5.4, left.

(2) For $a_0 = 0$ the trajectories of (5.7.3) are given by

$$\rho(\varphi) = \sqrt{-2\varphi + C}, \quad C \in \mathbb{R}, \ \varphi \leq \frac{C}{2}.$$

and infinity is a weak focus with $\dim_B \Gamma = 4/3$ where by Γ we denote a part of the trajectory near infinity. See Figure 5.4, middle.

(3) For $a_0 > 0$ the trajectories of (5.7.3) are given by

$$\rho(\varphi) = \frac{1}{a_0}\sqrt{-a_0 + a_0^2 C e^{-2a_0\varphi}}, \quad C > 0, \ \varphi \leq \frac{\log(a_0 C)}{2a_0}.$$

Infinity is a strong focus and all the trajectories near infinity are comparable with the spiral $\rho = e^{-a_0\varphi}$ of exponential type, and hence have box dimension equal to 1. See Figure 5.4, right.

Let us now consider the case $l = 2$ in (5.7.2):

$$\begin{aligned} \dot{\rho} &= -\rho(\rho^{-4} + a_0 + a_1\rho^{-2}) \\ \dot{\varphi} &= 1. \end{aligned} \tag{5.7.4}$$

Let us fix the value $a_1 = -2$ and consider a_0 as a bifurcation parameter. Since it is clearer to see what is happening, the phase portraits will be drawn on the Poincaré disc.

(a) When $a_0 < 0$, all box dimensions are equal to 1 because all the trajectories are of exponential type, see Figure 5.6, left.

(b) For $a_0 = 0$ we have a weak focus at infinity and any part of a trajectory Γ near infinity has box dimension equal to $D = 4/3$ (power case), whereas the part near the limit cycle $r = 1/\sqrt{2}$ has box dimension equal to 1 (exponential case), see Figure 5.6, middle. Actually, because the process of projecting onto the Poincaré disc affects the box dimension (see Remark 5.22), the trajectories on the figure near the equator have box

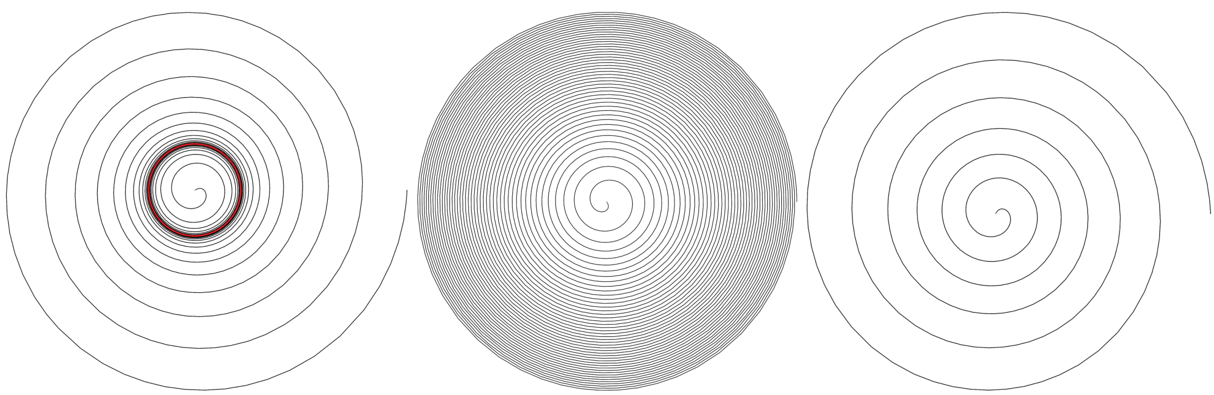

Figure 5.4: Trajectories of the system (5.7.3). Left: $a_0 = -1/25$ with a limit cycle born from infinity; Middle: $a_0 = 0$ with a weak focus at infinity; Right: $a_0 = 1/25$ with an exponential spiral at infinity.

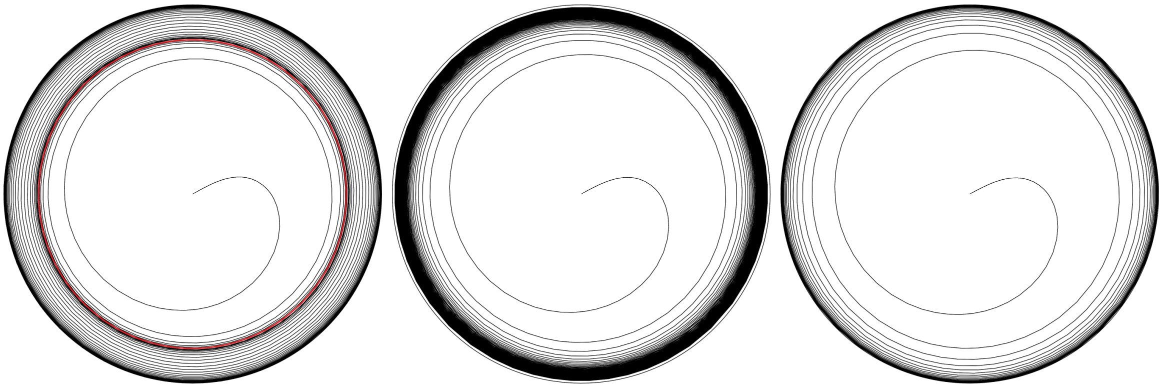

Figure 5.5: Trajectories of the system (5.7.3) drawn on the Poincaré disc. For $a_0 = 0$ (middle) the box dimension around the equator would be $1+\frac{1}{2}\frac{4}{3} = 5/3$ but after projecting the half sphere onto the disc the box dimension is reduced to $\frac{2}{3-5/3} = \frac{3}{2}$.

dimension equal to $\frac{4}{4-D} = \frac{3}{2}$.

(c) For $a_0 \in (0, 1)$ we have two limit cycles of multiplicity one, and all box dimensions are equal to 1 (exponential case), see Figure 5.6, right.

(d) For $a_0 = 1$ we have a limit cycle $r = 1$ of multiplicity two, and all trajectories near the limit cycle (either inside or outside) have box dimension equal to 3/2 (power case), see Figure 5.7, left. On the other hand, trajectories near the equator have box dimension equal to one (exponential case).

(e) For $a_0 > 1$ box dimensions of all trajectories are equal to one (exponential case), see Figure 5.7, right.

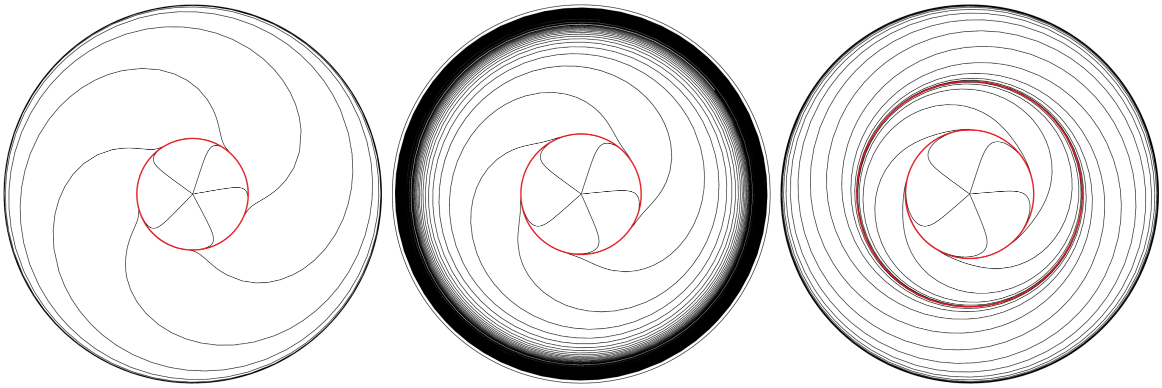

Figure 5.6: Trajectories of the system (5.7.4) drawn on the Poincaré disc with the fixed parameter $a_1 = -2$. Left: for $a_0 < 0$ there is one limit cycle. Middle: for $a_0 = 0$ in addition to the limit cycle, a weak focus forms at infinity and any corresponding unbounded spiral trajectory has a nontrivial box dimension. Right: for $a_0 \in (0, 1)$, from the weak focus at infinity another limit cycle is born.

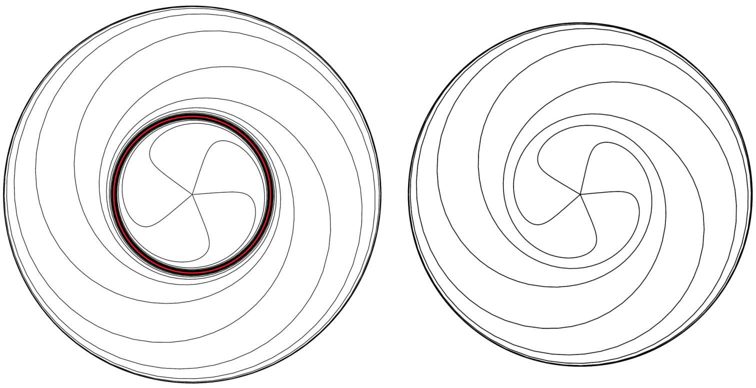

Figure 5.7: Trajectories of the system (5.7.4) drawn on the Poincaré disc with the fixed parameter $a_1 = -2$. Left: for $a_0 = 1$ the two limit cycles merge and form a single limit cycle of multiplicity 2. This is detected by the nontriviality of the box dimension of any spiral trajectory near this limit cycle. Right: for $a_0 > 1$ the single limit cycle of multiplicity 2 disappears.

Chapter 6

Conclusion and Perspectives

This thesis is a continuation of work on the theory of complex dimensions, fractal zeta functions, fractal geometry and fractal analysis of differential equations and dynamical systems. The theory of fractal strings, their geometric zeta functions and the complex dimensions which these zeta functions generate has been a topic of extensive research in the last few decades and has, in its own, a wide variety of applications. The foundations for generalizing this theory to higher dimensions are laid in the research monograph [LapRaŽu1] which is close to completion and will also contain a part of the material from this thesis. (See [LapRaŽu5] for a survey of some of the material from [LapRaŽu1].)

The fractal tube formulas obtained in Chapter 3 and the Minkowski measurability criterion given as their application are important results that generalize the corresponding ones for the case of one-dimensional fractal strings obtained in [Lap–vFr1–3] and give a justification of the notion of *complex dimensions* as a new tool to measure fractal properties of subsets of Euclidean spaces and, more generally, of relative fractal drums. Although it seems that a fairly large class of sets satisfy the *languidity* conditions of Chapter 3 and, hence, the theory may be applied to them, it remains to investigate this in detail and obtain some general results.

Furthermore, the results given here about embedded relative fractal drums give a strategy of computing complex dimensions of a class of higher-dimensional fractal sets by decomposing them into their lower-dimensional 'relative fractal subdrums' like it was shown in the example of the Cantor dust. Since it is not, in general, easy to compute the distance zeta function of a given relative fractal drum and, hence, its complex dimensions, we propose further investigation into other types of embeddings of relative fractal drums in higher dimensions and their fractal zeta functions. For instance, one could consider relative fractal drums $(\partial\Omega, \Omega)$ where the boundary $\partial\Omega$ is a subset of a piecewise smooth curve but also has lower-dimensional fractal properties. This situation appears, for instance in the well-known fractal sets such as the von Koch snowflake and the Menger sponge. Furthermore, also concerning the computation of fractal zeta functions, it would be of interest to obtain zero-free regions for these zeta functions as well as general results about stability of complex dimensions under perturbations of the integrand appearing in the definition of the distance zeta function.

The theory developed in Chapter 4 generalizes the idea of complex dimensions and fractal zeta functions to the case of unbounded sets at infinity. This gives means of applying fractal analysis to unbounded regions of finite or infinite Lebesgue measure in Euclidean spaces and the examples provided demonstrate that such regions can have a very complex fractal structure at infinity exhibiting quasiperiodicity and even maximal hyperfractality. Since unbounded regions are of interest in the theory of partial differential equations we propose to study if our approach of fractal analysis can be applied to any problems in this area of research.

Our effort to apply fractal analysis to unbounded sets of infinite Lebesgue measure led to the introduction of new notions, at least to our knowledge, of parametric ϕ-shell Minkowski content and the corresponding parametric ϕ-shell Minkowski (or box) dimension. Although introduced in the context of unbounded sets at infinity these notions are also well defined for bounded subsets and relative fractal drums. Preliminary results obtained in this thesis show that these new notions are connected with notions of the (usual) Minkowski content and the *surface Minkowski content* studied by Rataj and Winter in [RatWi1–2]. We suggest to study this connection in detail in a future work and obtain general results as well as to investigate possible applications.

Chapter 5 demonstrates how fractal analysis of unbounded sets may be applied to investigate dynamical systems and their bifurcations at infinity. Fractal analysis of dynamical systems and differential equations has been an ongoing investigation for the past decade by our research group resulting in the publication of a number of articles (see, e.g., [Hor1–2, MaResŽup, PaŽuŽup1–2, Res2–3, Vl, ŽuŽup1–3, ŽupŽu]). We propose to continue this research by applying the new theory developed in this thesis and in the research monograph [LapRaŽu1].

Appendix A

Open Problems

In this appendix we will list some of the open problems that have emerged during the writing of this thesis. We also mention some possible further research directions connected with these problems.

Problem A.1. Obtain results about zero-free regions for various types of fractal zeta functions investigated in this thesis and throughout [LapRaŽu1].

Preliminary investigation by the author of this thesis into the above problem suggests that for a relative fractal drum (A,Ω) of $\mathbb{R}^N$ the relative zeta function $\zeta_A(\,\cdot\,,\Omega;1)$ has a strictly positive real part and a strictly negative imaginary part on the half-plane $\{\operatorname{Re} s > N\}$. Furthermore, this actually implies that the function $f(s) := \zeta_A(s+N,\Omega;1)$ belongs to the class of *positive real functions* introduced by O. Brune in [Bru]. Further investigation into this problematic is required.

Problem A.2. Generalize the results of Section 3.5 to the case of gauge functions corresponding to RFDs having poles of higher orders on the critical line; that is, obtain, if possible, a gauge Minkowski measurability criterion in the spirit of Theorem 3.58.

Problem A.3. Construct a Lebesgue measurable set $\Omega \subseteq \mathbb{R}^N$ of finite Lebesgue measure such that $\underline{\dim}_B(\infty,\Omega) < \overline{\dim}_B(\infty,\Omega)$.

Problem A.4. Construct a Lebesgue measurable set $\Omega \subseteq \mathbb{R}^N$ of finite Lebesgue measure such that $-\infty = \underline{\dim}_B(\infty,\Omega) < \overline{\dim}_B(\infty,\Omega)$. Furthermore, check which of the results of this dissertation are valid for relative fractal drums of type (∞,Ω) such that $\overline{\dim}_B(\infty,\Omega) = -\infty$.

Problem A.5. Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$ with $|\Omega| < \infty$. Furthermore, assume that Ω is admissible (see Definition 4.42). Prove or disprove that $\underline{\dim}_B(\infty,\Omega) = \overline{\dim}_B(\infty,\Omega)$.

Problem A.6. For a Lebesgue measurable subset Ω of $\mathbb{R}^N$ with $|\Omega| < \infty$ find an asymptotic formula which relates the tube functions $t \mapsto |B_t(0)^c \cap \Omega|$ and $t \mapsto |B_{1/t}(0) \cap \Phi(\Omega)|$ as $t \to +\infty$ where $\Phi(x) := x/|x|^2$ is the geometric inversion in $\mathbb{R}^N$.

Problem A.7. Determine if the inequalities in Theorem 4.84 and Corollary 4.90 are sharp.

Problem A.8. Determine if the inequalities in Proposition 4.102 and Corollary 4.103 are sharp.

Problem A.9. Determine under which general conditions is the conclusion of Theorem 4.108 valid. More generally, determine under which general conditions are the following equalities valid:

$$\overline{\mathcal{S}}^r(\infty,\Omega)=\lim_{\phi\to 1^+}\frac{\overline{\mathcal{M}}^r_\phi(\infty,\Omega)}{\log\phi},\qquad \underline{\mathcal{S}}^r(\infty,\Omega)=\lim_{\phi\to 1^+}\frac{\underline{\mathcal{M}}^r_\phi(\infty,\Omega)}{\log\phi}.$$

Problem A.10. Generalize the results about the surface Minkowski content from [RatWi1] and [RatWi2] to the case of relative fractal drums and unbounded sets at infinity. (See the discussion after Example 4.110 of Section 4.8.)

Problem A.11. For a relative fractal drum (∞,Ω) where Ω is a Lebesgue measurable subset of $\mathbb{R}^N$ with finite Lebesgue measure, study the effect of the *partial geometric inversion* in $\mathbb{R}^N$ on the fractal properties of Ω. More precisely, for $k\le N$, let Φ_k be the *partial geometric inversion in the first k-coordinates*:

$$\Phi_k(x_1,\dots,x_N):=\frac{(x_1,\dots,x_N)}{\sum_{i=1}^k x_i^2}. \tag{A.0.1}$$

What are the relations between the fractal properties of (∞,Ω) and $(\Phi_k(\infty),\Phi_k(\Omega))$? Note that Φ_k is not a bijection of the one-point compactification of $\mathbb{R}^N$ (as opposed to Φ). For instance, $\Phi_1(x,y)=(1/x,y)$ and $\Phi_1(\infty)=\{(x,y)\,:\,x=0\}$.

Problem A.12. Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that Ω is Lebesgue measurable, unbounded and of finite Lebesgue measure. Study the relations between the fractal properties of (A,Ω) and the inverted relative fractal drum $(\Phi(A),\Phi(\Omega))$ where Φ is the geometric inversion in $\mathbb{R}^N$. Find connections (if any exist) between the corresponding distance and tube zeta functions.

Problem A.13. Study the effect of the Poincaré compactification on the fractal properties of unbounded sets.

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Curriculum Vitae

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Education

- **November 2008 –** Ph.D. student of mathematics at the University of Zagreb, Faculty of Science, Department of Mathematics, Croatia. Ph.D. thesis under supervision of prof. dr. sc. Michel L. Lapidus, University of California, Riverside and prof. dr. sc. Darko Žubrinić, University of Zagreb.
- **September 2003 – October 2008** Mathematics graduate at the University of Zagreb, Faculty of Science, Department of Mathematics, Croatia. Diploma-Thesis: *Spherical functions on* $SL_2(\mathbb{R})$ under supervision of prof. dr. sc. Marko Tadić
- **September 1999 – June 2003** The 3rd Gymnasium, Osijek, Croatia
- **October 1992 – September 1999** Elementary school "Višnjevac", Višnjevac, Croatia
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Professional Experience

- **2009 –** Research and Teaching Assistant, University of Zagreb, Faculty of Electrical Engineering and Computing, Croatia. Teaching courses: Mathematics 1, 3, Statistics and probability, Linear Algebra, Fourier Analysis

Scholarships and awards

- The Republic of Croatia scholarship during the undergraduate studies
- The Dean's award on the 3rd year of undergraduate studies for excellence in academic performance

Research publications

1. G. Radunović, D. Žubrinić and V. Županović, Fractal analysis of Hopf bifurcation at infinity, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **22** (2012), 1230043-1–1230043-15.
2. M. L. Lapidus, G. Radunović and D. Žubrinić, Fractal zeta functions and complex dimensions of relative fractal drums, appeared in the Festschrift volume of the *Journal of Fixed Point Theory and Applications* **15** (2014), 321–378 in honor of Haim Brezis' 70th birthday (DOI 10.1007/s11784-014-0207-y) (Also: e-print, `arXiv:1407.8094v3 [math-ph]`, 2014, 60 pp.)

Submitted work

- M. L. Lapidus, G. Radunović and D. Žubrinić, *Fractal Zeta Functions and Fractal Drums*: *Higher-Dimensional Theory of Complex Dimensions*, research monograph, Springer, New York, 2016, to appear, approx. 450 pages.

Grant participation

- **2014**: University of Zagreb, Croatia, *Multiplicity of fixed and singular points of dynamical systems and mappings*
- **2009 – 2014**: Croatian Ministry of Science, Education and Sports Grant no. 036-0361621-1291 *Nonlinear analysis of differential equations and dynamical systems*
- **2011 – 2012** : Bilateral scientific-research project with France, "Cogito", *Analyse fractale de l'application de Poincaré aux voisinages des polycycles*

Conferences, Workshops and Summer Schools

- 5th *Croatian Mathematical Congress*, CroMC, Rijeka, Croatia, June 18–21, 2012
- *School and Workshop on Dynamical Systems*, International Centre for Theoretical Physics (ICTP), Trieste, Italy, May 21–June 8, 2012
- *PISRS Conference 2011*, Messina, Italy, November 8–12, 2011
- *Equadiff Conference 2011*, Loughborough, England, August 1–5, 2011
- *RTNS* 8th *Winter School on Dynamical Systems of the DANCE Spanish network*, Vilanova i la Geltrú, Spain, January 24–28, 2011

Talks and Posters

- 5th Croatian Mathematical Congress, Rijeka, Croatia, June 2012, *Relative zeta functions of fractal sets in Euclidean spaces*, a short talk about the joint work with Michel L. Lapidus and Darko Žubrinić
- PISRS Conference 2011, Messina, Italy, November 2011, *Relative Zeta Functions of Lapidus Type*, an invited short talk about the joint work with Michel L. Lapidus and Darko Žubrinić
- Equadiff Conference 2011, Loughborough, England, August 2011, *Fractal analysis of Hopf bifurcation at infinity*, short communication about the joint work with Vesna Županović and Darko Žubrinić

Other talks

- Seminar talks *Fractal analysis of unbounded sets in Euclidean spaces and Lapidus zeta functions, I, II and III* at the Department of Mathematics, University of California, Riverside, USA, May 2013

Short term visits

- Department of Mathematics, University of California, Riverside, USA, April 23–May 16, 2014
- Institut de Mathématiques de Bourgogne, Université de Bourgogne, UFR Sciences et Techniques, Dijon, France, October 16–25, 2011

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Osobni podaci

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- **rujan 2003. – listopad 2008.** dodiplomski studij, Sveučilište u Zagrebu, Prirodoslovno-matematički fakultet, Matematički odsjek, Hrvatska, diplomski rad *Sferičke funkcije na* $SL_2(\mathbb{R})$ pod vodstvom prof. dr. sc. Marka Tadića
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Profesionalno iskustvo

- **2009. –** asistent i znanstveni novak, Sveučilište u Zagrebu, Fakultet elektrotehnike i računarstva, Hrvatska. Nastavu držao iz predmeta: Matematika 1, 3, Vjerojatnost i statistika, Linearna algebra, Fourierova analiza

Školarine i nagrade

- Državna stipendija Republike Hrvatske za vrijeme dodiplomskog studija
- Dekanova pohvalnica na 3. godini dodiplomskog studija za izuzetan uspjeh u studiju

Objavljeni znanstveni radovi

1. G. Radunović, D. Žubrinić and V. Županović, Fractal analysis of Hopf bifurcation at infinity, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **22** (2012), 1230043-1–1230043-15.
2. M. L. Lapidus, G. Radunović and D. Žubrinić, Fractal zeta functions and complex dimensions of relative fractal drums, appeared in the Festschrift volume of the *Journal of Fixed Point Theory and Applications* **15** (2014), 321–378 in honor of Haim Brezis' 70th birthday (DOI 10.1007/s11784-014-0207-y) (Also: e-print, `arXiv:1407.8094v3 [math-ph]`, 2014, 60 pp.)

Znanstveni radovi poslani za objavu

- M. L. Lapidus, G. Radunović and D. Žubrinić, *Fractal Zeta Functions and Fractal Drums*: *Higher-Dimensional Theory of Complex Dimensions*, research monograph, Springer, New York, 2016, to appear, approx. 450 pages.

Sudjelovanje u projektima

- **2014.** Sveučilište u Zagrebu, Hrvatska, *Multiplicitet fiksnih i singularnih točaka dinamičkih sustava i preslikavanja*
- **2009. – 2014.** Ministarstvo znanosti, obrazovanja i športa Republike Hrvatske, projekt br. 036-0361621-1291 *Nelinearna analiza diferencijalnih jednadžbi i dinamičkih sustava*
- **2011. – 2012.** Bilateralni znanstveno-istraživački projekt s Francuskom, "Cogito", *Analyse fractale de l'application de Poincaré aux voisinages des polycycles*

Konferencije, radionice i ljetne škole

- 5$^{\text{th}}$ *Croatian Mathematical Congress*, CroMC, Rijeka, Hrvatska, 18.–21. lipanj 2012.
- *School and Workshop on Dynamical Systems*, International Centre for Theoretical Physics (ICTP), Trst, Itaija, 21. svibanj–8. lipanj, 2012.
- *PISRS Conference 2011*, Messina, Italija, 8.–12. studeni, 2011.
- *Equadiff Conference 2011*, Loughborough, Engleska, 1.–5. Kolovoz, 2011.
- *RTNS* 8$^{\text{th}}$ *Winter School on Dynamical Systems of the DANCE Spanish network*, Vilanova i la Geltrú, Španjolska, 24.–28. Siječanj, 2011.

Javna izlaganja i posteri

- 5$^{\text{th}}$ Croatian Mathematical Congress, Rijeka, Hrvatska, lipanj 2012. *Relative zeta functions of fractal sets in Euclidean spaces*, kratko izlaganje o zajedničkom radu s Michel L. Lapidusom i Darkom Žubrinićem
- PISRS Conference 2011, Messina, Italija, studeni 2011. *Relative Zeta Functions of Lapidus Type*, pozvano kratko izlaganjeo zajedničkom radu s Michel L. Lapidusom i Darkom Žubrinićem
- Equadiff Conference 2011, Loughborough, England, kolovoz 2011. *Fractal analysis of Hopf bifurcation at infinity*, kratko izlaganje o zajedničkom radu s Vesnom Županović i Darkom Žubrinićem

Ostala izlaganja

- Seminarska izlaganja *Fractal analysis of unbounded sets in Euclidean spaces and Lapidus zeta functions, I, II and III* na Odjelu za matematiku, Sveučilište u Kaliforniji, Riverside, SAD, Svibanj 2013.

Kratki studijski boravci

- Odjel za matematiku, Sveučilište u Kaliforniji, Riverside, SAD, 23. travanj–16. svibanj, 2014.
- Institut de Mathématiques de Bourgogne, Université de Bourgogne, UFR Sciences et Techniques, Dijon, Francuska, 16.–25. listopad, 2011.