Censored Lévy and related processes

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DOCTORAL THESIS

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Contents

Chapter 1

Introduction

Lévy processes constitute a rich class of space-time homogeneous Markov processes and have become increasingly important both in theory and applications. Lévy processes have independent stationary increments and can therefore be thought of as random walks in continuous time. Some of the best known examples of Lévy processes are Brownian motion, Poisson process and stable Lévy process. Their applications range from mathematical finance to biology, biomedicine, geology, hydrology, etc. Recently, a new class of discontinuous Markov processes derived from stable Lévy processes has been introduced in [BBC03]. This process is called the censored or resurrected stable process and is obtained by suppressing jumps of a stable Lévy process outside of some open set. The main goal of this thesis is to introduce censored processes corresponding to a wider class of discontinuous Lévy processes, as well as consider their boundary behavior and some results from potential theory.

1.1 Overview

In Chapter 2 we give some preliminary results and definitions regarding Markov and Lévy processes. We refer to the classical textbooks [Ber98], [Sat99], [App09] for general theory of Lévy processes, [BG68], [CW05], [CZ95], [Jac02] for potential theory of Markov processes, [FOT10], [CF12] for theory of Dirichlet forms and [Far02], [JW84], [Tri78], [Tri10], [Jac01] for theory of Besov spaces and their generalizations.

In Chapter 3 we define the censored process on an open set D corresponding to a rotationally symmetric Lévy process and analyze its behavior near the boundary ∂D. In Section 3.1 we consider three equivalent constructions of a censored process - via the corresponding Dirichlet form (i.e. restricting the jumping measure of the Lévy process to set D), through the Feynman-Kac transform of the Lévy process killed outside of set D and from the same killed process by the Ikeda-Nagasawa-Watanabe piecing together procedure. From this point on we restrict ourselves to censored process corresponding to a subordinate Brownian motion with the Laplace exponent of the subordinator $\phi \in \mathcal{CBF}$

satisfying one or both of the following scaling conditions:
(H1): There exist constants $a_1, a_2 > 0$ and $0 < \delta_1 \le \delta_2 < 1$ such that

$$a_1 \lambda^{\delta_1} \le \frac{\phi(\lambda r)}{\phi(r)} \le a_2 \lambda^{\delta_2}, \quad \lambda \ge 1, r \ge 1,$$

(H2): There exist constants $a_3, a_4 > 0$ and $0 < \delta_3 \le \delta_4 < 1$ such that

$$a_3 \lambda^{\delta_3} \le \frac{\phi(\lambda r)}{\phi(r)} \le a_4 \lambda^{\delta_4}, \quad \lambda \ge 1, r \le 1.$$

In Section 3.2 and Section 3.3 we give basic definitions and results regarding Besov spaces of generalized smoothness and prove the trace theorem for a special subclass of these spaces on a n-set D. This result gives us necessary tools to address the problem of boundary behavior of the censored subordinate Brownian motion in Section 3.4.

In Chapter 4 we prove the so called 3G inequality for transient subordinate Brownian motion X on bounded κ-fat open sets. We show that for $r > 0$ and every bounded κ-fat open set B with characteristics (R, κ) and $\text{diam}(B) \le r$ there exists a constant $c = c(r, n, R, \kappa, \phi) > 0$ such that

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \le c \frac{\Phi(|x-y|)\Phi(|y-z|)}{\Phi(|x-z|)} \frac{|x-z|^n}{|x-y|^n |y-z|^n},$$

where $\Phi(|x|) = \frac{1}{\phi(|x|^{-2})}$. A similar result was proved in [KSV16], as well as in [KL07] under stronger conditions on the Lévy exponent ϕ, but with the constant c depending on $\text{diam}(B)$. Using this result in Section 4.2 for open balls of small radius we prove the scale invariant Harnack inequality for nonnegative harmonic functions for the censored process Y. More precisely, we say that the nonnegative Borel function h is harmonic in E for Y if for any bounded open subset $B \subset \overline{B} \subset E$

$$h(x) = \mathbb{E}_x \left[h(Y_{\tau_B}) \right], \quad x \in B.$$

We show that for any $L > 0$ there exists a constant $c = c(n, \phi, L) > 1$ such that the following is true: If $x_1, x_2 \in D$ and $r \in (0,1)$ are such that $B(x_1, r) \cup B(x_2, r) \subset D$ and $|x_1 - x_2| < Lr$, then for every nonnegative function h which is harmonic with respect to Y on $B(x_1, r) \cup B(x_2, r)$, we have

$$c_{12}^{-1} h(x_1) \le h(x_2) \le c_{12} h(x_1).$$

Note that due to jumps of the censored process Y the open set $B(x_1, r) \cup B(x_2, r)$ may be disconnected.

Finally, in Chapter 5 we consider a one-dimensional subordinate Brownian motion X with 0 being regular for itself and two related processes - the censored process Y on

$(0, \infty)$ and process Z which is equal to the absolute value of X killed at 0. In Section 5.1 we prove several properties of the first exit time of process Z from a finite interval in terms of the harmonic function h,

$$h(x) = \frac{1}{\pi} \int_0^\infty \frac{1 - \cos(\lambda x)}{\phi(\lambda^2)} d\lambda.$$

In the following section we introduce several Kato classes of functions for the killed processes $X^{(a,b)}$ and $Y^{(a,b)}$, where $0 < a < b < \infty$. Using the conditional gauge theorems from [Che02] for continuous and discontinuous Feynman-Kac transforms we prove that the Green functions of processes $X^{(a,b)}$, $Y^{(a,b)}$ and $Z^{(a,b)}$ are comparable, i.e.

$$G^Z_{(a,b)} \asymp G^Y_{(a,b)} \asymp G^X_{(a,b)}.$$

Applying these results in the last section we obtain the Harnack inequality and the boundary Harnack principle for the killed process $Z^{(a,b)}$.

1.2 Notation

For $n \in \mathbb{N}$ denote by $\mathcal{B}(\mathbb{R}^n)$ the Borel σ-algebra, i.e. the smallest σ-algebra containing all open sets in $\mathbb{R}^n$. The inner product on $\mathbb{R}^n$ is denoted by $x \cdot y = \sum_{i=1}^n x_i y_i$. The diameter of a set $D \subset \mathbb{R}^n$ and distance of a point to set D are defined by

$$\begin{aligned} \text{diam}(D) &= \sup\{|x-y| : x, y \in D\}, \\ d(x, D) &= \inf\{|x-y| : y \in D\}, \; x \in \mathbb{R}^n \end{aligned}$$

respectively. Denote by $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.

For a measure space $(E, \mathcal{E}, \mu)$ and $p \in [1, \infty]$ let

$$L^p(E, \mu) = \{f : E \to \mathbb{R} : f \text{ is } (\mathcal{E}, \mathcal{B}(\mathbb{R})) - \text{measurable and } ||f||_{L^p(E,\mu)} < \infty\}$$

where

$$\begin{aligned} ||f||_{L^p(E,\mu)} &= \left(\int_E |f(x)|^p \mu(dx) \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty \\ ||f||_{L^\infty(E,\mu)} &= \text{ess sup} f = \inf\{x \in \mathbb{R} : \mu(f^{-1}(x, \infty)) = 0\}. \end{aligned}$$

Here we use the convention that two functions in $L^p(E, \mu)$ are equal if they are equal μ-almost everywhere. Spaces $(L^p(E, \mu), || \cdot ||_{L^p(E,\mu)})$ are Banach spaces. For Lebesgue measure λ on $\mathbb{R}^n$ and $(E, \mathcal{E}, \mu) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda)$ we use a shorter notation $L^p(\mathbb{R}^n)$.

We say that functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are comparable and denote

$f \asymp g$ if there exists a constant $c > 1$ such that for all x

$$c^{-1} \leq \frac{f(x)}{g(x)} \leq c.$$

Denote by $C(\mathbb{R}^n)$, $C_0(\mathbb{R}^n)$ and $C_c(\mathbb{R}^n)$ the spaces of continuous functions, continuous functions vanishing at infinity and continuous functions with compact support, respectively. The space $(C_0(\mathbb{R}^n), ||\cdot||_\infty)$ with uniform norm

$$||f||_\infty = \sup_{x\in\mathbb{R}^n} |f(x)|$$

is a Banach space. For a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ let

$$D^\alpha f(x) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} ... \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f(x)$$

and $|\alpha| = \alpha_1 + ... + \alpha_n$. For $k \in \mathbb{N} \cup \{\infty\}$ denote by $C^k(\mathbb{R}^n)$ the space of k times differentiable functions and

$$C_c^k(\mathbb{R}^n) = \{f \in C^k(\mathbb{R}^n) : D^\alpha f \in C_c(\mathbb{R}^n),\ |\alpha| \in \{0, 1, ..., k\}\}.$$

Also for $k \in \mathbb{N}_0$, a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ and $f \in C^\infty(\mathbb{R}^n)$ set

$$||f||_{k,\alpha} = \sup_{x\in\mathbb{R}^n} (1 + |x|^2)^{\frac{k}{2}} |D^\alpha f(x)|.$$

The Schwartz space or the space of rapidly decreasing functions on $\mathbb{R}^n$

$$S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : ||f||_{k,\alpha} < \infty \text{ for all } k \in \mathbb{N}_0 \text{ and all multi-indices } \alpha \in \mathbb{N}_0^n\}.$$

is a Fréchet space space whose topology is defined by the countable family of semi-norms $||\cdot||_{k,\alpha}$. The family of tempered distributions $S'(\mathbb{R}^n)$ is a collection of all complex-valued linear continuous functionals T over $S(\mathbb{R}^n)$.

For a function $f \in L^1(\mathbb{R}^n)$ the Fourier transform $\mathcal{F}f$ of f is defined as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x) dx, \quad \xi \in \mathbb{R}^n.$$

We use the same notation to denote the continuous extension of the Fourier transform $\mathcal{F} : S \to S$ to a unitary map from $L^2(\mathbb{R}^n)$ to itself.

Chapter 2

Preliminaries

2.1 Markov processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *stochastic process* with values in $\mathbb{R}^n$ is a family $X = (X_t)_{t\geq 0}$ of $(\mathcal{F}, \mathcal{B}(\mathbb{R}^n))$-measurable functions $X_t : \Omega \to \mathbb{R}^n$, $t \geq 0$. The family $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ of σ-algebras is a *filtration* on $(\Omega, \mathcal{F})$ if for every $0 \leq s \leq t < \infty$

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}.$$

A stochastic process X is $\mathbb{F}$-*adapted* if the random variables X_t are $(\mathcal{F}_t, \mathcal{B}(\mathbb{R}^n))$-measurable, for all $t \geq 0$. For a stochastic process X we define the *natural filtration* $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ as

$$\mathcal{F}_t = \sigma\{X_s : s \leq t\}, \quad t \geq 0.$$

It is easy to see that every process is adapted with respect to the corresponding natural filtration.

Definition 2.1 A family $p = (p_{s,t} : 0 \leq s < t)$ of functions $p_{s,t} : \mathbb{R}^n \times \mathcal{B}(\mathbb{R}^n) \to [0,1]$ is called a *Markov kernel* (or a Markov transition function) on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ if

(i) $x \mapsto p_{s,t}(x, B)$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}^n))$-measurable for all $B \in \mathcal{B}(\mathbb{R}^n)$ and all s, t such that $0 \leq s < t$,

(ii) $B \mapsto p_{s,t}(x, B)$ is a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ for all $x \in \mathbb{R}^n$ and all s, t such that $0 \leq s < t$,

(iii) the *Chapman-Kolmogorov identity* holds, i.e. for all $x \in \mathbb{R}^n$, $B \in \mathcal{B}(\mathbb{R}^n)$ and all s, t, u such that $0 \leq s < t < u$

$$p_{s,u}(x, B) = \int_{\mathbb{R}^n} p_{t,u}(y, B) p_{s,t}(x, dy),$$

(iv) $p_t(x, \mathbb{R}^n) = 1$.

Additionally, for $t \geq 0$ and $x \in \mathbb{R}^n$ we set $p_{t,t}(x, \cdot) := \delta_x(\cdot)$. The Markov kernel p is *temporally homogeneous* if for all $x \in \mathbb{R}^n$, $B \in \mathcal{B}(\mathbb{R}^n)$ and all s, t such that $0 \leq s < t$

$$p_{s,t}(x, B) = p_{0,t-s}(x, B) =: p_{t-s}(x, B).$$

Remark 2.2 In general, if instead of (iv) $p_t(x, \mathbb{R}^n) < 1$ holds for some $x \in \mathbb{R}^n$ and $t \geq 0$ we call the kernel *sub-Markovian.* By introducing the *cemetery* $\partial \notin \mathbb{R}^n$ and redefining p_t to a function on $(\mathbb{R}^n \cup \{\partial\}) \times \sigma\left(\mathcal{B}(\mathbb{R}^n) \cup \{\partial\}\right)$ every sub-Markov kernel can be considered as a Markov kernel on the extended domain. From now on we will always implicitly consider Markov kernels on the extended domain.

Definition 2.3 Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be a filtration, $X = (X_t)_{t \geq 0}$ a $\mathbb{F}$-adapted stochastic process and $p = (p_{s,t} : 0 \leq s < t)$ a Markov kernel. The structure $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, p, X)$ is called a *Markov process* if the the *Markov property* holds, i.e. for every $B \in \mathcal{B}(\mathbb{R}^n)$ and $0 \leq s < t$

$$\mathbb{P}(X_t \in B | \mathcal{F}_s) = p_{s,t}(X_s, B), \quad \mathbb{P}\text{-a.s.} \tag{2.1}$$

From now on we only look at temporally homogeneous Markov processes X, i.e. Markov processes with temporally homogeneous Markov kernels.

Given a Markov kernel p and using the Kolmogorov extension theorem we can construct the *canonical Markov process* starting from x with Markov kernel p in the following way. Let $\Omega = (\mathbb{R}^n)^{[0,\infty)}$ and $\mathcal{F} = (\mathcal{B}(\mathbb{R}^n))^{[0,\infty)}$. For $t \geq 0$ define the function $X_t : \Omega \to \mathbb{R}^n$ as

$$X_t(\omega) = \omega(t),\, \omega \in \Omega$$

and let $\mathbb{F}$ be the natural filtration of the process X. For $x \in \mathbb{R}^n$, $k \in \mathbb{N}$ and $0 < t_1 < t_2 < ... < t_k$ we define a probability measure on $(\mathcal{B}(\mathbb{R}^n))^k$ by

$$P_{x,t_1,...,t_k}(B_1, ..., B_k) = \int_{B_1} p_{t_1}(x, dx_1) \int_{B_2} p_{t_2-t_1}(x_1, dx_2) ... \int_{B_k} p_{t_k-t_{k-1}}(x_{k-1}, dx_k)$$

for $B_1, ..., B_k \in \mathcal{B}(\mathbb{R}^n)$. The Kolmogorov extension theorem implies that there exists a unique probability measure $\mathbb{P}_x$ on $(\Omega, \mathcal{F})$ such that

$$\mathbb{P}_x(X_{t_1} \in B_1, ..., X_{t_k} \in B_k) = P_{x,t_1,...,t_k}(B_1, ..., B_k),$$

for every $k \in \mathbb{N}$, $0 < t_1 < t_2 < ... < t_k$ and $B_1, ..., B_k \in \mathcal{B}(\mathbb{R}^n)$. Therefore every Markov process starting from x with Markov kernel p has the same finite-dimensional distribution as the corresponding canonical Markov process X starting from x. Since

$$\mathbb{P}_x(X_t \in B) = p_t(x, B)$$

for all $t > 0$, $x \in \mathbb{R}^n$ and $B \in \mathcal{B}(\mathbb{R}^n)$ and $x \mapsto \mathbb{P}_x$ is a Borel function we can rewrite the Markov property (2.1) as

$$\mathbb{P}_x(X_{t+s} \in B|\mathcal{F}_s) = \mathbb{P}_{X_s}(X_t \in B), \quad \mathbb{P}_x\text{-a.s.} \tag{2.2}$$

for all $s, t \geq 0$, $x \in \mathbb{R}^n$ and $B \in \mathcal{B}(\mathbb{R}^n)$. For a Markov process X define the *shift operators* $(\theta_t)_{t\geq 0}$ as $(\mathcal{F}, \mathcal{F})$-measurable functions $\theta_t : \Omega \to \Omega$, $t \geq 0$ such that

$$X_t \circ \theta_s = X_{t+s}$$

for all $s, t \geq 0$. Using the shift operator for a Markov process X the identity (2.2) is equivalent to

$$\mathbb{E}_x \left[f(X_t) \circ \theta_s|\mathcal{F}_s\right] = \mathbb{E}_{X_s} \left[f(X_t)\right], \quad \mathbb{P}_x\text{-a.s.}$$

for all $s, t \geq 0$, $x \in \mathbb{R}^n$ and every bounded $(\mathcal{B}(\mathbb{R}^n), \mathcal{B}(\mathbb{R}))$-measurable function f. From now on we can denote the Markov process as $X = ((X_t)_{t\geq 0}, \mathbb{F}, (\mathbb{P}_x)_{x\in\mathbb{R}^n})$. If the filtration $\mathbb{F}$ is omitted in the notation, we consider the natural filtration for X.

If the Markov process X satisfies the condition (iv) from Definition 2.1 we say that the Markov process X is *conservative*. Note that this is equivalent to

$$\mathbb{P}_x(\zeta < \infty) = 0, \quad \forall x \in \mathbb{R}^n,$$

where $\zeta = \inf\{t \geq 0 : X_t = \partial\}$ is the *lifetime* of the process X.

A function $T : \Omega \to [0, \infty]$ is called a *stopping time* with respect to the filtration $\mathbb{F}$ if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. For a stopping time T define the family $\mathcal{F}_T$ as

$$\mathcal{F}_T = \{F \in \mathcal{F} : \ F \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$$

and let $\mathcal{G}_T = \{F \in \mathcal{F} : F \subset \{T < \infty\}\}$. Note that both $\mathcal{F}_T$ and $\mathcal{G}_T$ are σ-algebras. We can think of $\mathcal{F}_T$ as information up to random time T and $\mathcal{G}_T$ as that information conditioned on $\{T < \infty\}$.

Definition 2.4 A Markov process X is a *strong Markov process* if for every stopping time T

(i) X_T is $(\mathcal{G}_T, \mathcal{B}(\mathbb{R}^n))$-measurable

(ii) the *strong Markov property* holds, i.e. for all $t \geq 0$, $x \in \mathbb{R}^n$ and $B \in \mathcal{B}(\mathbb{R}^n)$

$$\mathbb{P}_x(X_{T+t} \in B|\mathcal{G}_T) = \mathbb{P}_{X_T}(X_t \in B), \quad \mathbb{P}_x\text{-a.s. on } \{T < \infty\}.$$

Again, using the shift operator, we can rewrite the strong Markov property as

$$\mathbb{E}_x\left[f(X_t)\circ\theta_T|\mathcal{F}_T\right]=\mathbb{E}_{X_T}\left[f(X_t)\right],\quad \mathbb{P}_x\text{-a.s. on } \{T<\infty\}$$

for every stopping time T, $t\geq 0$, $x\in\mathbb{R}^n$ and every bounded $(\mathcal{B}(\mathbb{R}^n),\mathcal{B}(\mathbb{R}))$-measurable function f.

For $t\geq 0$ and $x\in\mathbb{R}^n$ we define the *augmentation* of the σ-algebra $\mathcal{F}_t$ with respect to $\mathbb{P}_x$ as the smallest σ-algebra containing $\mathcal{F}_t$ and the family $\mathcal{N}_x$ of all $\mathbb{P}_x$-null sets,

$$\mathcal{F}_t^x=\sigma(\mathcal{F}_t\cup\mathcal{N}_x).$$

Definition 2.5 A Markov process X is a *Hunt process* if

(i) it is right-continuous $\mathbb{P}_x$-a.s. for all $x\in\mathbb{R}^n$,

(ii) it is a strong Markov process,

(iii) it is *quasi left-continuous*, i.e. for every $x\in\mathbb{R}^n$ and every sequence of increasing stopping times (T_n) such that $\lim_n T_n=T$ $\mathbb{P}_x$-a.s.

$$\lim_{n\to\infty}X_{T_n}=X_T,\quad \mathbb{P}_x\text{-a.s. on } \{T<\infty\},$$

(iv) the filtration $\mathbb{F}$ is *right-continuous*, i.e.

$$\mathcal{F}_t=\bigcap_{s>t}\mathcal{F}_s,\quad \forall t\geq 0$$

and

$$\mathcal{F}_t=\bigcap_{x\in\mathbb{R}^n}\mathcal{F}_t^x,\quad \forall t\geq 0.$$

For a Hunt process X and $B\in\mathcal{B}(\mathbb{R}^n)$ we define the *first exit time* from B as

$$\tau_B=\inf\{t>0: X_t\notin B\},\tag{2.3}$$

and *first hitting time* of B as

$$\sigma_B=\inf\{t>0: X_t\in B\}.\tag{2.4}$$

By [BG68, Theorem I.10.7] it follows that τ_B and σ_B are stopping times with respect to the augmented filtration $\mathbb{F}$.

Definition 2.6 Let B be a Banach space with norm $||\cdot||$. A family of operators $(T_t)_{t\geq 0}$ on B is called a *normal contraction semigroup* if

(i) $T_0 u = u$, for all $u \in B$,

(ii) it satisfies the *semigroup property*, i.e. $T_t T_s = T_{t+s}$ for all $t, s \geq 0$,

(iii) it satisfies the *contraction property*, i.e. $||T_t u|| \leq ||u||$ for all $t \geq 0$ and $u \in B$.

The contraction semigroup is *strongly continuous* if $||T_t u - u|| \to 0$ when $t \downarrow 0$, for every $u \in B$.

The *infinitesimal generator* A with domain $\mathcal{D}(A)$ of a strongly continuous normal contraction semigroup $(T_t)_{t\geq 0}$ is a linear operator $A : \mathcal{D}(A) \to B$ defined by

$$\begin{aligned} &Au = \lim_{t\to 0} \frac{T_t u - u}{t} \\ &\mathcal{D}(A) = \{u \in B : Au \text{ exists as a strong limit in } B\}. \end{aligned}$$

For a Markov process X we define a family of operators $(P_t)_{t\geq 0}$ on $L^\infty(\mathbb{R}^n)$ as

$$P_t f(x) = \mathbb{E}_x \left[f(X_t) \right] = \int_{\mathbb{R}^n} f(y) p_t(x, dy),\ t \geq 0,\ x \in \mathbb{R}^n.$$

The family of linear operators $(P_t)_{t\geq 0}$ is a normal contraction semigroup and additionally preserves positivity, i.e. $P_t f \geq 0$ for all $t \geq 0$ and all positive functions $f \in L^\infty(\mathbb{R}^n)$. Note that the Markov process X is conservative if and only if $P_t 1 = 1$. We say that the process X is *symmetric* if the corresponding transition semigroup satisfies the condition

$$\int_{\mathbb{R}^n} P_t u(x) v(x) dx = \int_{\mathbb{R}^n} u(x) P_t v(x) dx,$$

for all $t \geq 0$ and all non-negative $(\mathcal{B}(\mathbb{R}^n), \mathcal{B}(\mathbb{R}))$-measurable functions u and v.
In general, every semigroup $(T_t)_{t\geq 0}$ satisfying the condition

$$u \in L^2(\mathbb{R}^n),\ 0 \leq u \leq 1 \ \Rightarrow\ 0 \leq T_t u \leq 1,\ \forall t \geq 0$$

is called a *Markovian semigroup*.

Definition 2.7 A Markovian semigroup (T_t) is said to have the *Feller property* if

(i) the C_0-*Feller property* holds, i.e. $T_t f \in C_0(\mathbb{R}^n)$ for every $t \geq 0$ and $f \in C_0(\mathbb{R}^n)$,

(ii) it is strongly continuous on $C_0(\mathbb{R}^n)$, i.e. $\lim_{t\to\infty} ||T_t f - f||_\infty = 0$ for all $f \in C_0(\mathbb{R}^n)$.

Markov process whose corresponding semigroup has the Feller property is called the *Feller process*. By [CW05, Chapter 2] every Feller process is a Hunt process.

Definition 2.8 A Markov process is said to be *irreducible* if

$$\mathbb{E}_x\left[\int_0^\infty 1_B(X_t)dt\right] > 0$$

for every $x \in \mathbb{R}^n$ and $B \in \mathcal{B}(\mathbb{R}^n)$ with positive Lebesgue measure. An irreducible Markov process is *recurrent* if for all $B \in \mathcal{B}(\mathbb{R}^n)$, $\lambda(B) > 0$ and $x \in \mathbb{R}^n$

$$\mathbb{E}_x\left[\int_0^\infty 1_B(X_t)dt\right] = \infty,$$

otherwise it is *transient*.

2.2 Dirichlet forms

A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is an analytic object that can be used to construct and study certain Markov processes. Dirichlet forms use a quasi-sure analysis, meaning that we are permitted to ignore certain exceptional sets which are not visited by the process, which can sometimes have certain advantages.

Definition 2.9 A *symmetric form* on $L^2(\mathbb{R}^n)$ is a function $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ such that

(i) $\mathcal{D}(\mathcal{E})$ is dense in $L^2(\mathbb{R}^n)$,

(ii) $\mathcal{E}(u, v) = \mathcal{E}(v, u)$ for all $u, v \in \mathcal{D}(\mathcal{E})$,

(iii) $\mathcal{E}(au + v, w) = a\mathcal{E}(u, w) + \mathcal{E}(v, w)$ for all $u, v, w \in \mathcal{D}(\mathcal{E})$ and $a \in \mathbb{R}$,

(iv) $\mathcal{E}(u, u) \geq 0$ for all $u \in \mathcal{D}(\mathcal{E})$.

For $\alpha > 0$ denote by $\mathcal{E}_\alpha$ a new symmetric form on $L^2(\mathbb{R}^n)$ with domain $\mathcal{D}(\mathcal{E})$

$$\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha(u, v)_{L^2(\mathbb{R}^n)}, \quad u, v \in \mathcal{D}(\mathcal{E})$$

and note that forms $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are comparable for different $\alpha, \beta > 0$. Then the space $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$ is a pre-Hilbert space with inner product $\mathcal{E}_1$. A symmetric form $\mathcal{E}$ is said to be *closed* if $\mathcal{D}(\mathcal{E})$ is complete with respect to the norm induced by $\mathcal{E}_1$. The space $\mathcal{D}(\mathcal{E})$ is then a Hilbert space with inner product $\mathcal{E}_\alpha$ for every $\alpha > 0$.

Definition 2.10 A closed symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mathbb{R}^n)$ is a *Dirichlet form* if it is a *unit contraction*, i.e.

$$u \in \mathcal{D}(\mathcal{E}),\ v = (u \vee 0) \wedge 1 \Rightarrow v \in \mathcal{D}(\mathcal{E}), \mathcal{E}(v, v) \leq \mathcal{E}(u, u).$$

A Dirichlet form is *regular* if it possesses a *core*, i.e. if there exists a subset $\mathcal{C}$ of $\mathcal{D}(\mathcal{E}) \cap C_c(\mathbb{R}^n)$ such that

(i) $\mathcal{C}$ is dense in $\mathcal{D}(\mathcal{E})$ with respect to the $\mathcal{E}_1$-norm,

(ii) $\mathcal{C}$ is dense in $C_c(\mathbb{R}^n)$ with respect to the uniform norm.

A core $\mathcal{C}$ of $\mathcal{E}$ is said to be *standard* if it is a dense linear subspace of $C_c(\mathbb{R}^n)$.

A general representation theorem of regular Dirichlet forms is due to Beurling and Deny, [FOT10, Section 3.2]. Any regular Dirichlet form $\mathcal{E}$ on $L^2(\mathbb{R}^n)$ can be expressed as

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \int_{\mathbb{R}^n\times\mathbb{R}^n\setminus d} (u(x)-u(y))(v(x)-v(y))J(dx,dy) + \int_{\mathbb{R}^n} u(x)v(x)\kappa(dx), \tag{2.5}$$

for $u, v \in \mathcal{D}(\mathcal{E})$. Here

(i) $\mathcal{E}^{(c)}$ is the *local part* of $\mathcal{E}$, i.e. a symmetric form with domain $\mathcal{D}(\mathcal{E}^{(c)}) = \mathcal{D}(\mathcal{E}) \cap C_c(\mathbb{R}^n)$ which satisfies the strong local property:

$$\mathcal{E}^{(c)}(u,v) = 0 \text{ for all } u,v \in \mathcal{D}(\mathcal{E}^{(c)}) \text{ such that } v \text{ is constant on } U \subset \mathbb{R}^n,\ \operatorname{supp}[u] \subset U,$$

(ii) J is a symmetric positive Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ off the diagonal d, called the *jumping measure*,

(iii) κ is a positive Radon measure on $\mathbb{R}^n$ called the *killing measure*.

Such $\mathcal{E}^{(c)}$, J and κ are uniquely determined by $\mathcal{E}$.

By [FOT10, Theorem 1.3.1] there is a one-to-one correspondence between the family of closed symmetric forms $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mathbb{R}^n)$ and the family of non-positive definite self-adjoint operators $(A, \mathcal{D}(A))$ on $L^2(\mathbb{R}^n)$. The correspondence is determined by:

$$\begin{aligned} &\mathcal{D}(\mathcal{E}) = \mathcal{D}(\sqrt{-A}) \\ &\mathcal{E}(u,v) = (\sqrt{-A}u, \sqrt{-A}v),\ u,v \in \mathcal{D}(\mathcal{E}). \end{aligned} \tag{2.6}$$

This correspondence can be also characterized by

$$\mathcal{E}(u,v) = (-Au, v),\ u \in \mathcal{D}(A),\ v \in \mathcal{D}(\mathcal{E}),\ \mathcal{D}(A) \subset \mathcal{D}(\mathcal{E}).$$

Given (2.6), the closed symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ can be directly described in terms of the strongly continuous semigroup T_t corresponding to A. Define the approximation forms $\mathcal{E}^{(t)}$ determined by T_t as

$$\mathcal{E}^{(t)}(u,v) = \frac{1}{t}(u - T_t u, v), \quad u, v \in L^2(\mathbb{R}^n).$$

By [FOT10, Lemma 1.3.4] it follows that the closed symmetric form $\mathcal{E}$ corresponding to A can be defined as

$$\begin{aligned} \mathcal{D}(\mathcal{E}) &= \{u \in L^2(\mathbb{R}^n) : \lim_{t\downarrow 0} \mathcal{E}^{(t)}(u,u) < \infty\} \\ \mathcal{E}(u,v) &= \lim_{t\downarrow 0} \mathcal{E}^{(t)}(u,v), \quad u,v \in \mathcal{D}(\mathcal{E}). \end{aligned} \tag{2.7}$$

Furthermore, by [FOT10, Theorem 1.4.1.] the strongly continuous semigroup T_t is Markovian if and only if the closed symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is Markovian, that is $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form.

Therefore, given a symmetric Hunt process X there exists a unique Dirichlet form $\mathcal{E}$ in $L^2(\mathbb{R}^n)$ associated with X. By [FOT10, Theorem 4.2.8] two symmetric Hunt processes $X^{(1)}$ and $X^{(2)}$ possessing a common regular Dirichlet form are *equivalent* in the sense that their transition functions $p^{(1)}$ and $p^{(2)}$ coincide outside of a common *properly exceptional set* N, i.e. a set N of Lebesgue measure zero such that

$$P_t^{(i)}(u1_{N^c}) = 1_{N^c} P_t^{(i)} u \quad \text{a.e.}$$

for any $u \in L^2(\mathbb{R}^n)$ and $i = 1, 2$. In general, given a Dirichlet form on $L^2(\mathbb{R}^n)$ it is not possible to construct a Feller transition kernel such that (2.7) holds. But with regularity we are able to ignore sets of $\mathcal{E}$-capacity zero and construct a Hunt process outside of some set of zero $\mathcal{E}$-capacity.

Theorem 2.11 [FOT10, Theorem 7.2.1] Given a regular Dirichlet form $\mathcal{E}$ there exists a symmetric Hunt process X with Dirichlet form $\mathcal{E}$.

2.3 Killed Hunt processes

Let $X = (X_t)_{t\geq 0}$ be a symmetric Hunt process and D an open set in $\mathbb{R}^n$. The process X^D obtained by killing X upon exiting D is defined by

$$X_t^D(\omega) = \begin{cases} X_t(\omega), & t < \tau_D(\omega) \\ \partial, & t \geq \tau_D(\omega) \end{cases}, \quad \omega \in \Omega,$$

where τ_D is the first exit time of X from D defined by (2.3). The corresponding transition semigroup P_t^D is given by

$$P_t^D u(x) = \mathbb{E}_x[u(X_t^D)] = \mathbb{E}_x[u(X_t) : t < \tau_D]$$

for $t \geq 0$, $x \in D$ and $u \in L^\infty(\mathbb{R}^n)$. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the Dirichlet form corresponding to X. By [FOT10, Theorem 4.4.3] the Dirichlet form $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$ corresponding to X^D is actually the *part of the Dirichlet form $\mathcal{E}$ on D*, i.e.

$$\begin{aligned} \mathcal{D}(\mathcal{E}_D) &= \{u \in \mathcal{D}(\mathcal{E}) : u = 0 \ \text{ q.e. on } D^c\} \\ \mathcal{E}_D(u,v) &= \mathcal{E}(u,v), \quad u,v \in \mathcal{D}(\mathcal{E}_D). \end{aligned} \tag{2.8}$$

By [FOT10, Theorem 4.4.3] the Dirichlet form $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$ is regular and if $\mathcal{C}$ is a special standard core for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ then

$$\mathcal{C}_D = \{u \in \mathcal{C} : u = 0 \ \text{ q.e. on } D^c\}$$

is a special standard core for $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$.

Suppose that the transition kernel of X is absolutely continuous with respect to the Lebesgue measure. Then the corresponding transition density $p_t(x,y)$ is symmetric, i.e.

$$p_t(x,y) = p_t(y,x) \ \text{ for a.e. } x,y \in \mathbb{R}^n.$$

By a version of [CZ95, Theorem 2.4] the killed process X^D also has an absolutely continuous transition kernel with a symmetric density

$$p_t^D(x,y) = p_t(x,y) - \mathbb{E}^x[p_{t-\tau_D}(X_{\tau_D}, y) : \tau_D < t], \ x,y \in D, \ t > 0. \tag{2.9}$$

2.4 Capacity and polar sets

In this section we will recall several definitions of capacity and discuss their relations.

Definition 2.12 Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form on $L^2(\mathbb{R}^n)$.

(i) *$\mathcal{E}$-capacity* (1-*capacity*) of a set is defined in the following way; for an open set $U \subset \mathbb{R}^n$

$$\mathrm{Cap}_{\mathcal{E}}(U) = \inf\{\mathcal{E}_1(u,u) : u \in \mathcal{D}(\mathcal{E}),\, u \geq 1 \text{ a.e. on } U\},$$

and for $A \subset \mathbb{R}^n$ arbitrary set

$$\mathrm{Cap}_{\mathcal{E}}(A) = \inf\{\mathrm{Cap}_{\mathcal{E}}(U) : A \subset U \text{ open }\}.$$

If X is the symmetric Hunt process associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, we will sometimes use the notation Cap_X instead of $\mathrm{Cap}_{\mathcal{E}}$.

(ii) We say that a statement depending on $x \in A$ holds *$\mathcal{E}$-quasi-everywhere* (q.e.) on A if there exists a set $N \subset A$ of zero $\mathcal{E}$-capacity such that the statement is true for every $x \in A \setminus N$.

By [FOT10, (2.1.6)] capacity of any Borel set A can be calculated as

$$\mathrm{Cap}_{\mathcal{E}}(A) = \sup\{\mathrm{Cap}_{\mathcal{E}}(K) : K \subset A,\ K \text{ is compact}\}. \tag{2.10}$$

Also for $\mathcal{E}_D$ from (2.8), by [FOT10, Theorem 4.4.3] a set $B \subset D$ is of $\mathcal{E}_D$-capacity zero if and only if it is $\mathcal{E}$-capacity zero.

Definition 2.13 We say that $u \in \mathcal{D}(\mathcal{E})$ is *quasi continuous* if for every $\varepsilon > 0$ there exists an open set U such that $\mathrm{Cap}_{\mathcal{E}}(U) < \varepsilon$ and $u_{|U^c}$ is continuous.

By [FOT10, Theorem 2.1.3] every function $u \in \mathcal{D}(\mathcal{E})$ admits a *quasi-continuous modification* $\tilde{u}$, i.e. there exists a quasi-continuous function $\tilde{u} \in \mathcal{D}(\mathcal{E})$ such that $u = \tilde{u}$ a.e.

Definition 2.14 Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form on $L^2(\mathbb{R}^n)$ corresponding to the Hunt process X.

(i) A set A is called *$\mathcal{E}$-polar* if there is a Borel measurable set $B \supset A$ such that

$$\int_{\mathbb{R}^n} \mathbb{P}_x(\sigma_B^X < \infty) dx = 0.$$

(ii) A set A is called *polar* for the process X if there is a Borel measurable set $B \supset A$ such that

$$\mathbb{P}_x(\sigma_B^X < \infty) = 0, \quad \forall x \in \mathbb{R}^n.$$

Remark 2.15 If the symmetric Hunt process X has a continuous transition density then by [FOT10, Theorem 4.1.2] two definitions of polarity coincide.

2.5 Lévy processes

Definition 2.16 A stochastic process $X = (X_t)_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a *Lévy process* if

(i) $\mathbb{P}(X_0 = 0) = 1$,

(ii) it has *independent increments*, i.e. for any $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \ldots < t_n$,

$$X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}} \quad \text{are independent,}$$

(iii) it has *stationary increments*, i.e. for any $0 \leq s < t$,

$$X_t - X_s \stackrel{\mathrm{d}}{=} X_{t-s},$$

(iv) the function $t \mapsto X_t$ is $\mathbb{P}$-a.s. càdlàg, i.e. right continuous with left limits.

Conditions (i), (iii) and (iv) together imply that the Lévy process X is also *stochastically continuous*, i.e.

$$X_s \xrightarrow{(\mathbb{P})} X_t, \quad s \to t.$$

The primary tool in the analysis of distributions of Lévy processes are the *characteristic functions*, that is Fourier transforms of the distributions. The characteristic function of X_t is equal to

$$\mathbb{E}[e^{i\xi\cdot X_t}] = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^n,$$

where the function ψ is called the *characteristic exponent* of the process X. By the *Lévy-Khintchine formula* the characteristic exponent is of the form

$$\psi(\xi) = i\xi\cdot\gamma + \frac{1}{2}A\xi\cdot\xi + \int_{\mathbb{R}^n\setminus\{0\}} (1 - e^{i\xi\cdot x} + i\xi\cdot x 1_{\{|x|\leq 1\}})\nu(dx), \tag{2.11}$$

where $\gamma \in \mathbb{R}^n$, $A \in M_n(\mathbb{R})$ is a symmetric and nonnegative-definite matrix and ν is a measure on $\mathcal{B}(\mathbb{R}^n)$, called the *Lévy measure*, such that

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^n\setminus\{0\}} (1 \wedge |x|^2)\nu(dx) < \infty.$$

This means that the distribution of the process X is characterized by the *generating triplet* (γ, A, ν) and formula (2.11).

By [Ber98, Proposition I.6 and Proposition I.7] Lévy processes are Hunt processes with respect to the augmented natural filtration $\mathbb{F}$ and family of probability measures $(\mathbb{P}_x)_{x\in\mathbb{R}^n}$, where

$$\mathbb{P}_x(X_{t_1} \in B_1, ..., X_{t_k} \in B_k) = \mathbb{P}(x + X_{t_1} \in B_1, ..., x + X_{t_k} \in B_k)$$

for every $k \in \mathbb{N}$, $t_1, ..., t_k \geq 0$, $x \in \mathbb{R}^n$ and $B_1, ..., B_k \in \mathcal{B}(\mathbb{R}^n)$. By [Ber98, Proposition 16] a Lévy process is either transient or recurrent. From [Sat99, Section 35] we get the following characterization of recurrence and transience of Lévy processes.

Proposition 2.17 A Lévy process is

(a) recurrent if and only if

$$\liminf_{t\to\infty} |X_t| = 0 \ \text{ a.s.},$$

(b) transient if and only if

$$\lim_{t\to\infty} |X_t| = \infty \ \text{ a.s.}$$

For a symmetric Lévy process X we also have a recurrence criterion of Chung-Fuchs type, i.e. the process X is recurrent if and only if for some $r > 0$

$$\int\limits_{B(0,r)} \frac{1}{\psi(\xi)} d\xi = \infty. \tag{2.12}$$

Note that if a Lévy processes possesses a strictly positive transition density function with respect to the Lebesgue measure, it is also irreducible.

Let $(P_t)_{t\geq 0}$ be the transition semigroup of the Lévy process X, i.e.

$$P_t f(x) = \mathbb{E}_x[f(X_t)] = \mathbb{E}[f(x + X_t)], \quad f \in L^\infty(\mathbb{R}^n)$$

and $(L, \mathcal{D}(L))$ the corresponding infinitesimal generator. From [App09, Theorem 3.3.3] (also [Sat99, Theorem 21.5]) it follows that $S(\mathbb{R}^n) \subset \mathcal{D}(L)$ and that for $t \geq 0$ and $u \in S(\mathbb{R}^n)$

$$\begin{aligned} P_t u(x) &= \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(\xi)} \hat{u}(\xi) d\xi \qquad \text{and} \\ Lu(x) &= -\int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi(\xi) \hat{u}(\xi) d\xi, \end{aligned}$$

so the generator L is a pseudo-differential operator and

$$Lu(x) = i\gamma \cdot \nabla u(x) + A\nabla u(x) \cdot \nabla u(x) + \int_{\mathbb{R}^n \setminus \{0\}} (u(x+y) - u(x) + i\nabla u(x) \cdot y 1_{|y|<1}) \nu(dy).$$

If the Lévy process is additionally symmetric with generating triplet $(0, A, \nu)$ there exists a unique regular Dirichlet form $\mathcal{E}$ corresponding to X. Using the approximation method from (2.7) and the Parseval formula it follows that for $u \in L^2(\mathbb{R}^n)$ and $t \geq 0$

$$\begin{aligned} \mathcal{E}^{(t)}(u,u) &= \frac{1}{t}(u - P_t u, u) = \frac{1}{t}(\widehat{u} - \widehat{P_t u}, \widehat{u}) = \frac{1}{t} \int_{\mathbb{R}^n} (\widehat{u}(\xi) - e^{-t\psi(\xi)} \widehat{u}(\xi)) \overline{\widehat{u}}(\xi) \, d\xi \\ &= \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \frac{1 - e^{-t\psi(\xi)}}{t} \, d\xi. \end{aligned}$$

For given $u \in L^2(\mathbb{R}^n)$ the function $t \mapsto \mathcal{E}^{(t)}(u,u)$ is increasing so by the Lebesgue monotone convergence theorem,

$$\begin{aligned} \mathcal{E}(u,u) &= \lim_{t\downarrow 0} \mathcal{E}^{(t)}(u,u) = \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \psi(\xi) \, d\xi \\ \mathcal{D}(\mathcal{E}) &= \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \psi(\xi) \, d\xi < \infty \right\}. \end{aligned} \tag{2.13}$$

Example 2.18 (i) When $A = \mathbb{I}$ and $\nu = 0$ the corresponding Lévy process is called the *Brownian motion* and has the characteristic exponent $\psi(\xi) = |\xi|^2$ and transition density

$$p_t(x,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

Then the Dirichlet form (2.13) reduces to the form $(\mathcal{D}, H^1(\mathbb{R}^n))$ where $\mathcal{D}$ is the Dirichlet integral,

$$\mathcal{D}(u,v) = \sum_{i=1}^{n} \int_{\mathbb{R}^n} \partial_i u(x) \partial_i v(x) dy$$

and $H^1(D)$ is the Sobolev space of order 1,

$$H^1(D) = \{u \in L^2(\mathbb{R}^n) : \partial_i u \in L^2(\mathbb{R}^n),\ i = 1, ..., n\}.$$

(ii) When X is a purely discontinuous symmetric Lévy process with generating triplet $(0,0,\nu)$ the Dirichlet form (2.13) can be rewritten as

$$\mathcal{E}(u,v) = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus \{0\}} (u(x+y) - u(x))(v(x+y) - v(x)) \nu(dy)\, dx,$$
$$\mathcal{D}(\mathcal{E}) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus \{0\}} (u(x+y) - u(x))^2 \nu(dy)\, dx < \infty \right\}.$$

To show this, note that for $u \in L^2(\mathbb{R}^n)$, $y \in \mathbb{R}^n$ and $v_y(x) := u(x+y) - u(x)$ the Fourier transform of function v_y is equal to $\widehat{v_y}(\xi) = \widehat{u}(\xi)(e^{-i\xi \cdot y} - 1)$. Since the Lévy measure ν is symmetric, by Parseval formula (2.13) reduces to

$$\begin{aligned}
\mathcal{E}(u,u) &= \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(\xi \cdot y)) \nu(dy)\, d\xi \\
&= \frac{1}{2} \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \int_{\mathbb{R}^n \setminus \{0\}} |e^{-i\xi \cdot y} - 1|^2 \nu(dy)\, d\xi \\
&= \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus \{0\}} (u(x+y) - u(x))^2 \nu(dy)\, d\xi.
\end{aligned}$$

In the following chapters we will concentrate on purely discontinuous *rotationally symmetric* Lévy processes in $\mathbb{R}^n$ with generating triplet $(0,0,\nu)$, where the measure ν has a radial density j. The characteristic exponent ψ of such a processes is equal to

$$\begin{aligned}
\psi(\xi) &= \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{ix \cdot \xi} + ix \cdot \xi 1_{|x|<1}\right) \nu(dx) \\
&= \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(x\xi))\, j(|x|) dx, \quad \xi \in \mathbb{R}^n.
\end{aligned}$$

A special example of such a process is the subordinate Brownian motion which we will address in the following section.

2.6 Subordinate Brownian motion

Definition 2.19 A *subordinator* is a Lévy process S taking values in $[0, \infty)$, which implies that its sample paths are $\mathbb{P}$-a.s. nondecreasing.

Since the subordinator is almost surely nonnegative, we can consider the Laplace transform of the transition probability of S, which is of the form

$$\mathbb{E}[e^{-\lambda X_t}] = e^{-t\phi(\lambda)}, \quad \lambda \geq 0.$$

Here ϕ is called the *Laplace exponent* and is given by

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\nu(dx),$$

where $b \geq 0$ is called the *drift* coefficient and ν is the Lévy measure satisfying

$$\nu((-\infty, 0]) = 0 \text{ and } \int_{(0,\infty)} (1 \wedge x)\nu(dx) < \infty. \tag{2.14}$$

Definition 2.20 A function $f : (0, \infty) \to \mathbb{R}$ is

(i) a *completely monotone function* if f is of class C^∞ and for all $\lambda > 0$ and $n \in \mathbb{N}_0$

$$(-1)^n f^{(n)}(\lambda) \geq 0.$$

We will denote the family of completely monotone functions by $\mathcal{CM}$.

(ii) a *Bernstein function* if $f \geq 0$ and $f' \in \mathcal{CM}$. Denote by $\mathcal{BF}$ the collection of Bernstein functions.

By [SSV09, Theorem 3.2] $f \in \mathcal{BF}$ if and only if it admits a representation of the form

$$f(\lambda) = a + b\lambda + \int\limits_{(0,\infty)} (1 - e^{\lambda y})\nu(dy), \tag{2.15}$$

where $a, b \geq 0$ and the measure ν satisfies (2.14). Therefore, a function $\phi : (0, \infty) \to (0, \infty)$ is the Laplace exponent of the subordinator if and only if $\phi \in \mathcal{BF}$ and $a = 0$.

Definition 2.21 A Bernstein function is *complete* if its Lévy measure ν in (2.15) has a completely monotone density $\nu(t)$. We will use $\mathcal{CBF}$ to denote the collection of all complete Bernstein functions.

Let $S = (S_t)_{t \geq 0}$ be the subordinator with Laplace exponent $\phi \in \mathcal{CBF}$ defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Let $B = (B_t)_{t \geq 0}$ be the standard Brownian motion in $\mathbb{R}^n$

defined on the same probability space, independent of S. A process $X = (X_t)_{t\geq 0}$ defined by

$$X_t(\omega) = B_{S_t(\omega)}(\omega), \quad t \geq 0,\, \omega \in \Omega$$

is called a *subordinate Brownian motion*. It is easy to see that X is again a Markov process with the associated transition semigroup

$$P_t^\phi u(x) = \mathbb{E}_x\left[u(B_{S_t})\right] = \int\limits_{(0,\infty)} \mathbb{E}_x[u(B_s)]\mathbb{P}(S_t \in ds) = \int\limits_{(0,\infty)} P_s^B u(x)\mathbb{P}(S_t \in ds).$$

The semigroup $\mathbb{P}_t^\phi$ is said to be *subordinate in the sense of Bochner* to the semigroup P_t^B with respect to the complete Bernstein function ϕ. It follows that

$$\mathbb{E}_x\left[e^{i\xi\cdot X_t}\right] = \int\limits_{(0,\infty)} \mathbb{E}_x\left[e^{i\xi\cdot B_s}\right]\mathbb{P}(S_t \in ds) = \int\limits_{(0,\infty)} e^{-s|\xi|^2}\mathbb{P}(S_t \in ds) = e^{-t\phi(|\xi|^2)}$$

so X is a Lévy process with the generating triplet $(0, A, j(|x|)dx)$, where $A = b\mathbb{I}$ and

$$j(r) = \int_0^\infty (4\pi s)^{-n/2} e^{-\frac{r^2}{4s^2}} \nu(s)ds, \quad r > 0.$$

Note that the density j is a decreasing function.

2.7 Green function and harmonic functions

Let X be a symmetric Hunt process in $\mathbb{R}^n$.

Definition 2.22 For every $x \in \mathbb{R}^n$ we define a *potential measure* $G(x, \cdot)$ for X by

$$G(x, B) = \mathbb{E}_x\left[\int_0^\infty 1_{\{X_t\in B\}}dt\right], \quad B \in \mathcal{B}(\mathbb{R}^n).$$

If the potential measure $G(x, \cdot)$ is absolutely continuous for all $x \in \mathbb{R}^n$ then we call the corresponding density $G(x, y)$ the *Green function* for X.

Suppose that the Green measure is finite and that X has a transition density $p_t(x, y)$. By Fubini's theorem

$$G(x, B) = \int_0^\infty \mathbb{P}_x(X_t \in B)dt = \int_B\int_0^\infty p_t(x, y)dt\, dy, \quad B \in \mathcal{B}(\mathbb{R}^n),$$

so the potential measure is absolutely continuous for all $x \in \mathbb{R}^n$ and the Green function is equal to

$$G(x, y) = \int_0^\infty p_t(x, y)dt, \quad x, y \in \mathbb{R}^n.$$

Let D be an open set in $\mathbb{R}^n$ and X^D the corresponding killed process. The potential measure of X^D is defined in the same way:

$$G_D(x,B) = \mathbb{E}_x\left[\int_0^\infty 1_{\{X_t^D \in B\}} dt\right] = \mathbb{E}_x\left[\int_0^{\tau_D} 1_{\{X_t \in B\}}\, dt\right] = \int_0^\infty \mathbb{P}_x(X_t \in B :\, t < \tau_D)\, dt.$$

Since the killed process X^D has a transition density $p_t^D(x,y)$ given by (2.9), $G_D(x,\cdot)$ is absolutely continuous and the corresponding Green function $G_D(x,y)$ is given by

$$\begin{aligned} G_D(x,y) &= \int_0^\infty p^D(t,x,y)\, dt = \int_0^\infty p(t,x,y)\, dt - \int_0^\infty \mathbb{E}_x[p_{t-\tau_D}(X_{\tau_D},y) : \tau_D < t]\, dt \\ &= G(x,y) - \mathbb{E}_x[G(X_{\tau_D},y) : \tau_D < \infty],\ \ x,y \in D. \end{aligned} \tag{2.16}$$

We call $G_D(x,y)$ the *Green function of the set D*.

Recall the representation of Beurling-Deny and LeJan from (2.5). If the Green function for X on B exists then the jumping measure J for X has the following representation. By [FOT10, Theorem 4.5.2, Lemma 4.5.5] the jumping measure is a unique symmetric positive Radon measure such that

$$\mathbb{E}_x[f(X_{\tau_B-})g(X_{\tau_B}) : \tau_B < \infty] = 2\int_{\overline{B}^c}\int_B f(y)g(z)G_B(x,y)J(dy,dz) \tag{2.17}$$

for every $x \in B$, f and g bounded nonnegative Borel measurable functions such that $\operatorname{supp} f \subset B$ and $\operatorname{supp} g \subset \overline{B}^c$. Also by [FOT10, Lemma 4.5.2] the killing measure κ of X^B is a unique positive Radon measure such that

$$\mathbb{E}_x[f(X_{\tau_B-}) : \tau_B < \infty] = \int_B f(y)G_B(x,y)\kappa(dy). \tag{2.18}$$

for every positive Borel function f.

By Definition 2.8 the potential measure of a Lévy process X is finite for all $x \in \mathbb{R}^n$ if and only if X is transient. Let X be a rotationally symmetric pure jump Lévy process with the generating triplet $(0,0,j(|x|)dx)$ and D an open set in $\mathbb{R}^n$. For an open set $B \subset \overline{B} \subset D$ by Remark 2.18(ii) and (2.17) the joint distribution of $(X_{\tau_B-}, X_{\tau_B})$ restricted to the event $\{X_{\tau_B-} \neq X_{\tau_B},\, \tau_B < \infty\}$ is given by the *Ikeda-Watanabe formula*

$$\mathbb{E}_x[f(X_{\tau_B-})g(X_{\tau_B})] = \int_{B^c}\int_B f(y)g(z)G_B(x,y)j(|y-z|)dydz \tag{2.19}$$

for all nonnegative Borel measurable functions f and g on $\mathbb{R}^n$. If B is a Lipschitz domain (for example a ball) then by [Szt00, Theorem 1]

$$\mathbb{P}_x(X_{\tau_B} \in \partial B) = 0, \quad \forall x \in B.$$

Then the density function of the $\mathbb{P}_x$-distribution of X_{τ_B} is determined by the *Poisson kernel* K_B,

$$K_B(x,z) = \int_B G_B(x,y)j(|y-z|)dy, \quad x \in B,\, z \in \overline{B}^c. \tag{2.20}$$

Furthermore, let X be a transient subordinate Brownian motion and let $\phi \in \mathcal{CBF}$ be the Laplace exponent of the subordinator S. By the Chung-Fuchs-type criteria (2.12) X is transient if and only if

$$\int_0^a \frac{\lambda^{\frac{n}{2}-1}}{\phi(\lambda)} d\lambda < \infty$$

for all $a > 0$. This is always true for $n \geq 3$ and depending on the subordinator, may be true for $n = 1$ or $n = 2$. The potential measure of the subordinator S is given by

$$U(A) = \mathbb{E}\left[\int_0^\infty 1_{\{S_t \in A\}} dt\right] = \int_0^\infty p_t^S(A)dt,\ A \in \mathcal{B}([0,\infty))$$

with Laplace transform

$$\mathcal{L}U(\lambda) = \int_0^\infty e^{-\lambda t} U(dt) = \mathbb{E}\left[\int_0^\infty e^{-\lambda S_t}\, dt\right] = \int_0^\infty e^{-t\phi(\lambda)}\, dt = \frac{1}{\phi(\lambda)}.$$

By [KSV12, Corollary 2.3] if $\phi \in \mathcal{CBF}$ has a generating triplet $(0, b, \nu)$ such that

$$b > 0 \quad \text{or} \quad \nu(0,\infty) = \infty$$

then the potential measure U has a completely monotone density u. Since X is transient the Green measure is finite on all bounded sets and

$$G(x,A) = \mathbb{E}_x\left[\int_0^\infty 1_{\{X_t \in A\}}\right] = \int_A \int_0^\infty p_t(x,y)U(dt)dy,$$

for $x \in \mathbb{R}^n$, $A \in \mathcal{B}(\mathbb{R}^n)$ and p the transition density of the Brownian motion. Then the Green function of X is given by

$$G(x,y) = G(|x-y|),$$

where

$$G(r) = (4\pi)^{-\frac{n}{2}} \int_0^\infty t^{-\frac{n}{2}} e^{-\frac{r^2}{4t}} U(dt).$$

Note that the function G is a positive nonincreasing function.

Definition 2.23 Let D be an open subset of $\mathbb{R}^n$. A Borel function $h : \mathbb{R}^n \to [0,\infty)$ is said to be *harmonic* in D for X if for any bounded open subset $B \subset \overline{B} \subset D$

$$h(x) = \mathbb{E}_x\left[h(X_{\tau_B})\right], \tag{2.21}$$

for all $x \in B$. If the previous equality additionally holds for $B = D$ then we say that h is *regular harmonic* for X in D.

Here we use the convention that $X_\infty = \partial$. We will always make a tacit assumption about all functions that they take value 0 at the cemetery point ∂ and that the expectation in (2.21) is absolutely convergent and so finite.

Remark 2.24 Suppose that the Green function $G(x, y)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n \setminus d$, where d is the diagonal in $\mathbb{R}^n$. This is true, for example, when the subordinator has a potential density u. From the strong Markov property and formula (2.16) it follows that the function $u(x) = G_D(x, y)$ is harmonic in $D \setminus \{y\}$.

Definition 2.25 A nonnegative function h is said to be *excessive* if

(i) $\mathbb{E}_x[h(X_t)] \le h(x)$ for all $x \in \mathbb{R}^n$ and $t \ge 0$,

(ii) $\lim_{t \downarrow 0} \mathbb{E}_x[h(X_t)] = h(x)$ for all $x \in \mathbb{R}^n$.

2.8 Feynman-Kac transforms

Let X be a symmetric Hunt process with respect to the filtration $\mathbb{F}$ and lifetime ζ. Let p be the corresponding transition density and suppose that the corresponding Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular.

Definition 2.26 $A = (A_t)_{t \ge 0}$ is a *positive continuous additive functional* (PCAF) for X if

(i) A_t is $\mathcal{F}_t$-measurable,

(ii) A is $[0, +\infty]$-valued,

(iii) $t \mapsto A_t$ is continuous on $[0, \zeta)$,

(iv) $A_{t+s} = A_t \circ \theta_s + A_s$, where θ_t is the time shift operator for X.

By [FOT10, Theorem 5.1.4] for every PCAF A there exists a measure μ, called the *Revuz measure* corresponding to A such that

$$\lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}^n} \mathbb{E}_x \left[\int_0^t f(X_s) d\, A_s \right] h(x) dx = \int_{\mathbb{R}^n} h(x) f(x) \mu(dx),$$

for all excessive functions h and positive Borel functions f.

Let $\kappa : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative and continuous function. Define a functional $A = (A_t)_{t \ge 0}$ of X by

$$A_t = \int_0^t \kappa(X_s) ds, \quad t \ge 0. \tag{2.22}$$

It is easy to see that A is a positive continuous additive functional (PCAF) for X with Revuz measure $\kappa(x)dx$. The function κ is sometimes called the *potential* of the PCAF A. Furthermore, $(e_\kappa(t))_{t\geq 0}$

$$e_\kappa(t) = e^{A_t}$$

is a multiplicative functional for Y, i.e. $e_\kappa(t)$ is $\mathcal{F}_t$-measurable and

$$e_\kappa(t+s,\omega) = e_\kappa(t, \theta_s \circ \omega) e_\kappa(s,\omega)$$

for all $\omega \in \Omega$, and $t, s \geq 0$. Next we state a well-known result, known as Khasminskii's lemma, which will play an important role in the following sections.

Lemma 2.27 [CZ95, Lemma 3.7] Let τ be a stopping time for X, $\kappa \geq 0$ and suppose that $\mathbb{E}_x[A_\tau] < \infty$ for all x. Then for every $n \in \mathbb{N}_0$

$$\sup_x \mathbb{E}_x[A_\tau^n] \leq n! \sup_x \left(\mathbb{E}_x[A_\tau]\right)^n .$$

Furthermore if

$$\sup_x \mathbb{E}_x[A_\tau] = \alpha < 1$$

then

$$\sup_x \mathbb{E}_x \left[e^{A_\tau}\right] \leq \frac{1}{1-\alpha}.$$

Proposition 2.28 [CZ95, Proposition 3.8] If

$$\lim_{t\to 0} \sup_x \mathbb{E}_x[A_t] = 0 \tag{2.23}$$

then

$$\lim_{t\to 0} \sup_x \mathbb{E}_x[e_\kappa(t)] = 1$$

and there exist positive constants C_0 and C_1 such that for all $t > 0$

$$\sup_x \mathbb{E}_x[e_\kappa(t)] \leq C_0 e^{C_1 t}. \tag{2.24}$$

If (2.23) holds we can define the *Feynman-Kac semigroup* $(T_t)_{t\geq 0}$ on $L^2(\mathbb{R}^n)$ as

$$T_t u(x) = \mathbb{E}_x[e_\kappa(t) u(X_t)],$$

for $t \geq 0$, $x \in \mathbb{R}^n$ and $u \in L^2(\mathbb{R}^n)$. By [CZ95, Theorem 3.10 and Proposition 3.12] T_t is a strongly continuous symmetric Markov semigroup and if X is a Feller process then the semigroup T_t also has the Feller property. Furthermore, if the transition probability of the process X has a density then so does the Feynman-Kac semigroup T_t.

We use the approximation principle (2.7) to calculate the Dirichlet form corresponding to the Feynman-Kac semigroup. Note that

$$\int_0^t e^{A_s}\kappa(X_s)ds = e^{A_t} - 1, \quad \text{a.s.} \tag{2.25}$$

For $u, v \in L^2(\mathbb{R}^n)$

$$\begin{aligned}
\mathcal{E}^{\kappa,(t)}(u,v) &= \frac{1}{t}(u - T_t u, v)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \mathbb{E}_x\left[\frac{u(x) - e^{A_t}u(X_t)}{t}\right] v(x)dx \\
&= \int_{\mathbb{R}^n} \mathbb{E}_x\left[\frac{u(x) - e^{A_t}u(x)}{t}\right] v(x)dx + \int_{\mathbb{R}^n} \mathbb{E}_x\left[e^{A_t}\frac{u(x) - u(X_t)}{t}\right] v(x)dx \\
&\overset{(2.25)}{=} -\int_{\mathbb{R}^n} \mathbb{E}_x\left[\frac{1}{t}\int_0^t e^{A_s}\kappa(X_s)ds\right] u(x)v(x)dx + \int_{\mathbb{R}^n} \mathbb{E}_x\left[e^{A_t}\frac{u(x) - u(X_t)}{t}\right] v(x)dx.
\end{aligned}$$

From (2.24) by the dominated convergence theorem it follows that

$$\begin{aligned}
\mathcal{E}^{\kappa}(u,v) &= -\lim_{t\downarrow 0}\int_{\mathbb{R}^n} \mathbb{E}_x\left[\frac{1}{t}\int_0^t e^{A_s}\kappa(X_s)ds\right] u(x)v(x)dx + \lim_{t\downarrow 0}\int_{\mathbb{R}^n} \mathbb{E}_x\left[e^{A_t}\frac{u(x) - u(X_t)}{t}\right] v(x)dx \\
&= -\int_{\mathbb{R}^n} \kappa(x)u(x)v(x)dx + \mathcal{E}(u,v) \qquad (2.26)
\end{aligned}$$

and

$$\mathcal{D}(\mathcal{E}^{\kappa}) = \mathcal{D}(\mathcal{E}) \cap L^2(\mathbb{R}^n, \kappa(x)dx).$$

By [FOT10, Theorem 6.1.2] the Dirichlet form $(\mathcal{E}^{\kappa}, \mathcal{D}(\mathcal{E}^{\kappa}))$ is regular and every special standard core for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is also a special standard core for $(\mathcal{E}^{\kappa}, \mathcal{D}(\mathcal{E}^{\kappa}))$. We say that the corresponding symmetric Hunt process X^{κ} is obtained by *resurrecting* X at a rate κ. Analogously, if we consider the semigroup generated through the bounded multiplicative functional e^{-A_t} the corresponding symmetric Hunt process X^{κ} is a subprocess of X, i.e.

$$\mathbb{E}_x[f(X_t^{\kappa})] \le \mathbb{E}_x[f(X_t)], \quad t \ge 0,\ x \in \mathbb{R}^n,\ u \in L^{\infty}(\mathbb{R}^n),$$

and we say that X^{κ} is obtained by *killing* X at a rate κ.

Let p be the transition measure of the rotationally invariant purely discontinuous Lévy process X with the generating triplet $(0, 0, j(|x|)dx)$ and D an open set in $\mathbb{R}^n$. Let h be a harmonic function for X^D and $E_h = \{x \in D : 0 < h(x) < \infty\}$. Define

$$p_t^{D,h}(x,y) = \frac{h(y)}{h(x)} p_t^D(x,y), \quad t > 0,\ x, y \in E_h.$$

Clearly $p^{D,h}$ is Borel measurable and satisfies the Chapman-Kolmogorov identity. Furthermore, for an increasing sequence $D_n \subset \overline{D}_n \subset D$ of bounded open sets such that

$D = \bigcup_n D_n$ it follows that

$$\begin{aligned}\mathbb{E}_x[h(X_t) : t < \tau_D] &= \lim_{n\to\infty} \mathbb{E}_x[h(X_t) : t < \tau_{D_n}] = \lim_{n\to\infty} \mathbb{E}_x[\mathbb{E}_{X_t}[h(X_{\tau_{D_n}}) : t < \tau_{D_n}]] \\ &\leq \lim_{n\to\infty} \mathbb{E}_x[h(X_{\tau_{D_n}}) : t < \tau_{D_n}] = h(x).\end{aligned}$$

Therefore, $p^{D,h}$ is a sub-Markovian kernel, i.e.

$$\int_{E_h} p_t^{D,h}(x,y)dy = \frac{1}{h(x)}\mathbb{E}_x[h(X_t) : t < \tau_D] \leq \frac{1}{h(x)}h(x) = 1.$$

The corresponding process X^h on E_h is called the *Doob's h-transformed process* of X or the *h-conditioned* process. For $x \in D$ we denote by $\mathbb{P}_x^h$ and $\mathbb{E}_x^h$ the probability and expectation for the h-conditioned process starting from x respectively. By the monotone class argument, as in [CZ95, Proposition 5.2] it follows that

$$\mathbb{E}_x^h[\Phi : t < \tau_D] = \frac{1}{h(x)}\mathbb{E}_x[\Phi h(X_t) : t < \tau_D] \tag{2.27}$$

for every $\mathcal{F}_t$-measurable function $\Phi : \Omega \to [0,\infty)$. Suppose X is transient and the Green function G_B is continuous for some open set $B \subset \overline{B} \subset D$. Let

$$h(\cdot) = G_B(\cdot, y)$$

for some $y \in B$ and denote the corresponding probability and expectation as $\mathbb{P}_x^y$ and $\mathbb{E}_x^y$. For the PCAF A from (2.22) with Revuz potential

$$\kappa(x) = \int_{D^c} j(|x-z|)dz$$

we define the *conditional gauge function* $u : \mathbb{R}^n \times \mathbb{R}^n \to [-\infty, \infty]$ as

$$u(x,y) = \mathbb{E}_x^y\left[e_\kappa(\tau_B)\right]. \tag{2.28}$$

Proposition 2.29 For $\Phi \geq 0$ measurable with respect to $\mathcal{F}_{\tau_D -}$ and any nonnegative Borel function f

$$\mathbb{E}_x[f(X_{\tau_D})\Phi : X_{\tau_D -} \neq X_{\tau_D}] = \mathbb{E}_x[f(X_{\tau_D})\mathbb{E}_x^{X_{\tau_D -}}[\Phi] : X_{\tau_D -} \neq X_{\tau_D}], \quad x \in D. \tag{2.29}$$

Proof. Using the monotone class argument it is enough to prove (2.29) for Φ of the form $\Phi = \Phi_t 1_{\{t<\tau_D\}}$, for some $\mathcal{F}_t$-measurable nonnegative function Φ_t. By (2.27) the function $y \mapsto \mathbb{E}_x^y[\Phi]$ is Borel measurable. Let κ be the killing measure for the killed symmetric

Hunt process X^D. For $A \in \mathcal{B}(D)$ it follows that

$$\begin{aligned}
\mathbb{E}_x[\Phi : X_{\tau_D -} \in A] &= \mathbb{E}_x[\Phi_t 1_{\{t<\tau_D\}} : X_{\tau_D -} \in A] = \mathbb{E}_x[\Phi_t \mathbb{E}_{X_t}[1_{\{X_{\tau_D -} \in A\}}] : t < \tau_D] \\
&\overset{(2.18)}{=} \mathbb{E}_x\left[\Phi_t \int_A G_D(X_t, y)\kappa(y)dy : t < \tau_D\right] \\
&= \int_A \mathbb{E}_x[\Phi_t G_D(X_t, y)\kappa(y) : t < \tau_D]dy \\
&\overset{(2.27)}{=} \int_A \mathbb{E}_x^y[\Phi] G_D(x, y)\kappa(y)dy \\
&\overset{(2.18)}{=} \mathbb{E}_x\left[\mathbb{E}_x^{X_{\tau_D -}}[\Phi] : X_{\tau_D -} \in A\right]
\end{aligned}$$

an therefore

$$\mathbb{E}_x[\Phi | X_{\tau_D -}] = \mathbb{E}_x^{X_{\tau_D -}}[\Phi], \quad \mathbb{P}_x\text{-a.s.}$$

Finally, we have

$$\begin{aligned}
\mathbb{E}_x[f(X_{\tau_D})\Phi : X_{\tau_D -} \neq X_{\tau_D}] &= \mathbb{E}_x[f(X_{\tau_D})\mathbb{E}_x[\Phi | X_{\tau_D -}] : X_{\tau_D -} \neq X_{\tau_D}] \\
&= \mathbb{E}_x[f(X_{\tau_D})\mathbb{E}_x^{X_{\tau_D -}}[\Phi] : X_{\tau_D -} \neq X_{\tau_D}].
\end{aligned}$$

□

From Proposition 2.29 and (2.17) (also by [Che02, Lemma 3.5.]) it follows that the Green function for X^κ on B is equal to

$$G_B^\kappa(x, y) = G_B(x, y)u(x, y). \tag{2.30}$$

So the function u can be interpreted as the conditional expectation of the Feynman-Kac transform of X by κ and it is also the ratio of the Green functions of X^κ and X. The *conditional gauge theorem* which we will introduce in the following sections says that under suitable conditions on X and κ, either u is identically infinite or u is bounded between two positive numbers.

Chapter 3

Construction and boundary behavior of the censored process

In the first part of this chapter we will define the censored rotationally symmetric Lévy process Y on an open set D and discuss three equivalent construction procedures for such a process.

In the following sections we will present results regarding boundary behavior of the censored subordinate Brownian motion Y. In order to do so, we introduce a new process called the *reflected process* through its Dirichlet form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_{\mathrm{a}}^{\mathrm{ref}})$. Section 2.2 is devoted to the theory of Besov spaces of generalized smoothness which are closely related to Dirichlet spaces corresponding to Lévy processes and their censored counterparts. The main result of Section 2.3 is the trace theorem for a certain type of Besov spaces of generalized smoothness. Using this result in Section 2.4 we prove that, under certain conditions on the subordinator, the Dirichlet form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_{\mathrm{a}}^{\mathrm{ref}})$ is actually the *active reflected Dirichlet form* corresponding to the censored process Y. When D is an open n-set in $\mathbb{R}^n$, using this connection between Y and the reflected process, we can determine under which conditions the process Y approaches the boundary ∂D in finite time.

3.1 Construction

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space and $X = (X_t)_{t\geq 0}$ be a rotationally symmetric Lévy process in $\mathbb{R}^n$ with the generating triplet $(0, 0, \nu)$, where $\nu(dx) = j(|x|)dx$. The Fourier transform of the transition probability of X is characterized by the characteristic exponent ψ;

$$\mathbb{E}\left[e^{i\xi\cdot X_t}\right] = \int_{\mathbb{R}^n} e^{i\xi x} p_t(dx) = e^{-t\psi(\xi)},$$
$$\psi(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{ix\cdot\xi} + ix\cdot\xi 1_{|x|<1}\right)\nu(dx)$$

$$= \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos (x \cdot \xi))\, j(|x|)dx, \quad \xi \in \mathbb{R}^n.$$

The regular Dirichlet form $(\mathcal{C}, \mathcal{F}^{\mathbb{R}^n})$ associated with X (see Example 2.18) is given by

$$\begin{aligned}\mathcal{C}(u, v) &= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus\{0\}} (u(x + y) - u(x))(v(x + y) - v(x))j(|y|)dy\, dx \\ \mathcal{F}^{\mathbb{R}^n} &= \left\{u \in L^2(\mathbb{R}^n) : \mathcal{C}(u, u) < \infty\right\}.\end{aligned}$$

Note that $j(x, y) = 2j(|x - y|)$ is the density of the jumping measure from the Beurling-Deny representation (2.5) of $\mathcal{C}$. Let $D \subset \mathbb{R}^n$ be an open set and X^D the process X killed upon exiting D. Recall from Section 2.3 that

$$X_t(\omega) = \begin{cases} X_t(\omega), & t < \tau_D(\omega) \\ \partial, & t \geq \tau_D(\omega) \end{cases}$$

for $\tau_D(\omega) = \inf\{t > 0 : X_t(\omega) \notin D\}$ the first exit time of X from D. The Dirichlet form for X^D is $(\mathcal{C}, \mathcal{F}^D)$, where

$$\mathcal{F}^D = \{u \in \mathcal{F}^{\mathbb{R}^n} : u = 0 \text{ q.e. on } D^c\}.$$

Since $C_c^\infty(\mathbb{R}^n)$ is a special standard core for $(\mathcal{C}, \mathcal{F}^{\mathbb{R}^n})$ it follows that $C_c^\infty(D)$ is a special standard core for $(\mathcal{C}, \mathcal{F}^D)$. So $\mathcal{F}^D$ is the closure of $C_c^\infty(D)$ under the norm generated by $\mathcal{C}_1 = \mathcal{C} + (\cdot, \cdot)_{L^2(D)}$. For $u, v \in \mathcal{F}^D$ we can write

$$\begin{aligned}\mathcal{C}(u, v) &= \frac{1}{2}\int_D\int_D (u(x) - u(y))(v(x) - v(y))j(|x - y|)dxdy \\ &+ \frac{1}{2}\int_D\int_{D^c} (u(x) - u(y))(v(x) - v(y))j(|x - y|)dxdy \\ &+ \frac{1}{2}\int_{D^c}\int_D (u(x) - u(y))(v(x) - v(y))j(|x - y|)dxdy \\ &+ \frac{1}{2}\int_{D^c}\int_{D^c} (u(x) - u(y))(v(x) - v(y))j(|x - y|)dxdy \\ &= \frac{1}{2}\int_D\int_D (u(x) - u(y))(v(x) - v(y))j(|x - y|)dxdy + \int_D u(x)v(x)\kappa_D(x)dx,\end{aligned}$$

where $\kappa_D(x) = \int_{D^c} j(|x - y|)dy$ is called the *killing density* of X^D. It is also the density of the killing measure from the Beurling-Deny representation (2.5) of a Dirichlet form.

By removing the killing part from the Dirichlet form $(\mathcal{C}, \mathcal{F}^D)$ we obtain a new Dirichlet form: for every $u, v \in C_c^\infty(D)$ let

$$\mathcal{E}(u, v) = \frac{1}{2}\int_D\int_D (u(x) - u(y))(v(x) - v(y))j(|x - y|)dx\, dy.$$

By Fatou's lemma the symmetric form $(\mathcal{E}, C_c^\infty(D))$ is closable in $L^2(D)$, i.e. for every sequence $u_n \in C_c^\infty(D)$ such that $u_n \xrightarrow{L^2} 0$

$$\mathcal{E}(u_n - u_m, u_n - u_m) \xrightarrow{n,m\to\infty} 0 \;\Rightarrow\; \mathcal{E}(u_n, u_n) \xrightarrow{n\to\infty} 0,$$

so we take $\mathcal{F}$ to be the closure of $C_c^\infty(D)$ under the inner product $\mathcal{E}_1 = \mathcal{E} + (\cdot,\cdot)_{L^2(D)}$. From [FOT10, Section 1.4] it follows that the closed symmetric form $(\mathcal{E}, \mathcal{F})$ is Markovian since it operates on a normal contraction, i.e. for $u \in \mathcal{F}$ and $v \in L^2(\mathbb{R}^n)$,

$$|v(x) - v(y)| \leq |u(x) - u(y)|,\; |v(x)| \leq |u(x)|,\; \forall x, y \in \mathbb{R}^n \;\Rightarrow\; \mathcal{E}(v, v) \leq \mathcal{E}(u, u).$$

Therefore, the form $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form. By Theorem 2.11 there exists a symmetric Hunt process Y associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, taking values in D with lifetime ζ. We call Y the *censored (resurrected) process* associated with X. So the censored process Y can be interpreted as the process obtained from the Lévy process X by restricting its jumping measure to D.

The following theorem gives us two alternative constructions for the process Y; by using the *Ikeda-Nagasawa-Watanabe piecing together procedure* and the *Feynman-Kac transform.*

Theorem 3.1 The following processes have the same distribution

(i) The symmetric Hunt process Y associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D)$.

(ii) The strong Markov process obtained from the symmetric Levy process X^D in D through the Ikeda-Nagasawa-Watanabe piecing together procedure.

(iii) The process obtained from X^D through the Feynman-Kac transform $e^{\int_0^t \kappa_D(X_s^D)ds}$.

Proof. First we prove that (i) is equivalent to (ii). Define a functional $A = (A_t)_{t\geq 0}$ of Y by

$$A_t = \int_0^t \kappa_D(Y_s)ds.$$

From Section 2.8 it follows that A is a positive continuous additive functional for Y with Revuz measure $\kappa_D(x)dx$. So e^{-A_t} is a decreasing multiplicative functional for Y and it uniquely determines a probability measure $\widehat{\mathbb{P}}_x$ on Ω for $\mathcal{E}$-q.e. $x \in D$, such that

$$\widehat{\mathbb{E}}_x[f(Y_t)] = \mathbb{E}_x\left[e^{-A_t} f(Y_t)\right] \tag{3.1}$$

for every function $f \in L^\infty(D)$. Let Y^κ be the process with distribution $\widehat{\mathbb{P}}_x$ and lifetime ζ^κ. Then Y^κ is a symmetric Hunt process obtained from Y by killing with rate κ_D. Recall from (2.26) that the associated Dirichlet form on $L^2(D)$ is given by

$$\mathcal{E}^\kappa(u,v) = \mathcal{E}(u,v) + \int_D u(x)v(x)\kappa_D(x)dx$$
$$\mathcal{F}^\kappa = \mathcal{F} \cap L^2(D, \kappa_D(x)dx).$$

The Dirichlet form $(\mathcal{E}^\kappa, \mathcal{F}^\kappa)$ is regular on $L^2(D)$ with special standard core $C_c^\infty(D)$. Since

$$\mathcal{E}^\kappa = \mathcal{C} \text{ on } \mathcal{F}^\kappa \cap \mathcal{F}^D \text{ and } C_c^\infty(D) \subset \mathcal{F}^\kappa \cap \mathcal{F}^D$$

it follows that

$$(\mathcal{E}^\kappa, \mathcal{F}^\kappa) = (\mathcal{C}, \mathcal{F}^D). \tag{3.2}$$

This implies that the processes Y^κ and X^D are equivalent, i.e. they have the same distribution q.e. Since Y^κ is a subprocess of Y, by [BG68, Section III.3.] we can alternatively obtain the process Y^κ by killing Y at the random time ζ^κ. Actually, ζ^κ is the lifetime of Y^κ and

$$\zeta^\kappa \le \zeta \text{ a.s.},$$
$$\mathbb{P}_x(\zeta^\kappa > 0) = 1, \, \forall x \in D$$
$$t + \zeta^\kappa \circ \theta_t = \zeta^\kappa \text{ on } \{\zeta^\kappa > t\}, \text{ for all } t \ge 0.$$

Let $(Y^{\kappa,j})_{j\in\mathbb{N}}$ be a sequence of independent copies of the process Y^κ and let $\zeta^{\kappa,j}$ be the lifetime of $Y^{\kappa,j}$. Define the sequence of random times $(\tau_j)_{j\in\mathbb{N}}$ as

$$\tau_1 = \zeta^{\kappa,1},$$
$$\tau_{j+1} = \begin{cases} \tau_j + \zeta^{\kappa,j+1} \circ \theta_{\tau_j}, & \tau_j < \zeta \\ \zeta, & \text{otherwise} \end{cases}, \quad j \in \mathbb{N}.$$

We will show that this increasing sequence converges almost surely to ζ. Denote by $\eta = \lim_{j\to\infty} \tau_j$ and note that $\eta \le \zeta$ a.s. Define a subprocess Z of Y by

$$Z_t(\omega) = \begin{cases} Y_t(\omega), & t < \eta(\omega) \\ \partial, & t \ge \eta(\omega) \end{cases}, \quad t \ge 0, \, \omega \in \Omega.$$

Process Z is again a symmetric Hunt process so by its quasi-left continuity

$$\mathbb{P}(\eta < \zeta) = \mathbb{P}(Z_{\eta-} \in D, \, \eta < \infty) = \mathbb{P}\left(\lim_{j\to\infty} Z_{\tau_j} \in D, \, \eta < \infty\right) = \mathbb{P}(Z_\eta \in D, \eta < \infty) = 0.$$

Therefore $\eta = \zeta$ a.s. Next Using the Ikeda-Nagasawa-Watanabe piecing together procedure from [INW66] we define a new process $Y^{(1)}$ in the following way. Let $x \in D$ be an arbitrary starting point for Y. For a given $\omega \in \Omega$ we start the processes $Y^{\kappa,j}$ at the following points:

$$\begin{aligned} Y_0^{\kappa,1}(\omega) &= x \\ Y_0^{\kappa,j+1}(\omega) &= Y^{\kappa,j}_{\zeta^{\kappa,j}(\omega)-}(\omega). \end{aligned}$$

Define

$$Y_t^{(1)}(\omega) = \begin{cases} Y_t^{\kappa,1}(\omega), & 0 \le t < \tau_1(\omega) \\ Y^{\kappa,j+1}_{t-\tau_j(\omega)}(\omega), & \tau_j(\omega) \le t < \tau_{j+1}(\omega),\ \tau_j(\omega) < \zeta(\omega) \\ \partial, & t \ge \zeta(\omega) \end{cases}.$$

Note that the *piecing together* procedure is repeated countably many times. By [INW66, Proposition 4.2] process $Y^{(1)}$ is a symmetric Hunt process on D with lifetime ζ. From the construction of Y^{κ} it follows that for $j \in \mathbb{N}_0$

$$Y_t \stackrel{d}{=} Y^{\kappa,j+1}_{t-\tau_j} \quad \text{on} \quad \{\tau_j \le t \le \tau_j + \zeta^{\kappa,j+1} \circ \tau_j,\, Y_{\tau_j} = Y_0^{\kappa,j}\},$$

so by the strong Markov property it follows that $Y^{(1)}$ is a version of the process Y corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F})$

Next we show the equivalence of (i) and (iii). Let $Y^{(2)}$ be the Feynman-Kac transform of X^D through the positive continuous additive functional $B_t = \int_0^t \kappa_D(X_s^D)ds$. Then for any function $f \in L^{\infty}(D)$

$$\begin{aligned} \mathbb{E}_x[f(Y_t^{(2)})] &= \mathbb{E}_x\left[e^{\int_0^t \kappa_D(X_s^D)ds} f(X_t^D)\right] \stackrel{(3.2)}{=} \mathbb{E}_x\left[e^{\int_0^t \kappa_D(Y_s^{\kappa})ds} f(Y_t^{\kappa})\right] \\ &\stackrel{(3.1)}{=} \widehat{\mathbb{E}}_x\left[e^{\int_0^t \kappa_D(Y_s)ds} f(Y_t)\right] = \mathbb{E}_x\left[e^{-\int_0^t \kappa_D(Y_s)ds} e^{\int_0^t \kappa_D(Y_s)ds} f(Y_t)\right] \\ &= \mathbb{E}_x[f(Y_t)], \end{aligned}$$

that is $Y^{(2)} \stackrel{D}{=} Y$. Therefore Y can also be obtained from X^D by creation at the rate κ_D through the Feynman-Kac transform with PCAF B_t. □

From the construction of the censored process Y through the Ikeda-Nagasawa-Watanabe piecing together procedure it follows that the censored process Y can be obtained from the symmetric Lévy process X by suppressing its jumps from D to the complement D^c. Several useful properties of the censored process follow directly from Theorem 3.1.

Remark 3.2 If the Levy process X has a transition density then by Section 2.3 so does the process X^D. By [CZ95, Theorem 3.10.] and Theorem 3.1(iii) it follows that the corresponding censored process Y also has an absolutely continuous transition measure. The censored process Y is also irreducible.

Let X be the subordinate Brownian motion. In order to investigate the boundary behavior of the corresponding censored process we introduce a new type of process through its Dirichlet form. Let $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_{\mathrm{a}}^{\mathrm{ref}})$ be a Dirichlet form on $L^2(D)$ defined by

$$\mathcal{F}_{\mathrm{a}}^{\mathrm{ref}} = \left\{ u \in L^2(D) : \frac{1}{2} \int_D \int_D (u(x) - u(y))^2 j(|x-y|) dx\, dy < \infty \right\}$$
$$\mathcal{E}^{\mathrm{ref}}(u,v) = \frac{1}{2} \int_D \int_D (u(x) - u(y))(v(x) - v(y)) j(|x-y|) dx\, dy, \quad u, v \in \mathcal{F}_{\mathrm{a}}^{\mathrm{ref}}.$$

The Dirichlet form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_{\mathrm{a}}^{\mathrm{ref}})$ is not necessarily regular for every open set D. By an analogue of [BBC03, Theorem 2.2], [CF12, Theorem 6.2.13] we will show that, under certain conditions, $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_{\mathrm{a}}^{\mathrm{ref}})$ is the *active reflected Dirichlet form* for $(\mathcal{E}, \mathcal{F})$ in the sense of Silverstein and [CF12]. This implies that there exists a compactification D^* of D such that $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_{\mathrm{a}}^{\mathrm{ref}})$ is regular on $L^2(D^*)$ and we refer to the corresponding process Y^* as the *reflected process* related to Y. When D is an open n-set, the process Y can be represented as the process Y^* killed upon hitting the boundary ∂D. To show this we relate the domains $\mathcal{F}^{\mathbb{R}^n}$, $\mathcal{F}_{\mathrm{a}}^{\mathrm{ref}}$ and $\mathcal{F}$ with the corresponding *ψ-Bessel potential spaces* $H^{\psi,1}$ of order 1 studied in [Jac01] and prove the trace theorem for these spaces.

3.2 Besov spaces of generalized smoothness

The domain $\mathcal{F}^{\mathbb{R}^n}$ of the Dirichlet form $\mathcal{C}$ is a type of a much more general class of function spaces called *Besov spaces of generalized smoothness*. These spaces were introduced in the seventies by M.L. Goldman and G.A. Kalyabin as a generalization of the classical Sobolev and Besov spaces. Since then they have been studied by many authors from various points of view. Here we adopt the standpoint of a Fourier analytic characterization considered by Farkas and Leopold in [Far02] and [FL06]. First we introduce these spaces in their generalized form.

Definition 3.3 A sequence $(\gamma_j)_{j \in \mathbb{N}_0}$ of positive real numbers is called

(i) *almost increasing* if there exists $d_0 > 0$ such that

$$d_0 \gamma_j \leq \gamma_k, \quad \forall j \leq k;$$

(ii) *strongly increasing* if it is almost increasing and in addition there exists a $\kappa_0 \in \mathbb{N}$ such that

$$2\gamma_j \leq \gamma_k, \quad j + \kappa_0 \leq k;$$

(iii) of *bounded growth* if there are positive constants d_1 and $J_0 \in \mathbb{N}_0$ such that

$$\gamma_{j+1} \leq d_1 \gamma_j, \quad \forall j \geq J_0;$$

(iv) an *admissible sequence* if both $(\gamma_j)_{j\in\mathbb{N}_0}$ and $(\gamma_j^{-1})_{j\in\mathbb{N}_0}$ are of bounded growth and $J_0 = 0$, i.e. there exist positive constants d_0 and d_1 such that

$$d_0\gamma_j \le \gamma_{j+1} \le d_1\gamma_j, \quad \forall j \in \mathbb{N}_0.$$

Definition 3.4 Let $N = (N_j)_{j\in\mathbb{N}_0}$ be a strongly increasing sequence. Define

$$\begin{aligned} \Omega_0^N &= \{x \in \mathbb{R}^n : |x| \le N_0\} \\ \Omega_j^N &= \{x \in \mathbb{R}^n : N_{j-1} \le |x| \le N_{j+1}\}, \quad j \in \mathbb{N}. \end{aligned}$$

Let Φ^N be a collection of all partitions of unity of $C_c^\infty(\mathbb{R}^n)$ functions associated with this decomposition.

Definition 3.5 Let $N = (N_j)_{j\in\mathbb{N}_0}$ and $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be a strongly increasing and admissible sequence respectively and $(\varphi_j^N)_{j\in\mathbb{N}_0} \in \Phi^N$. The *Besov space of generalized smoothness* associated with N and σ is

$$B_2^{\sigma,N} = \{g \in S'(\mathbb{R}^n) : ||g||_{B,\sigma,N} := ||(\sigma_j\varphi_j^N(D)g)_{j\in\mathbb{N}_0}|l_2(L_2(\mathbb{R}^n))|| < \infty\},$$

where $\varphi(D)g(x) = (\varphi(\cdot)\hat{g})^\vee(x)$ and

$$||(f_j)_{j\in\mathbb{N}_0}|l^2(L^2(\mathbb{R}^n))|| = \left(\sum_{j=0}^\infty ||f_j||^2_{L^2(\mathbb{R}^n)}\right)^{\frac{1}{2}}.$$

By [Far02, Remark 10.1.2.] the space $B_2^{\sigma,N}$ is independent of the choice of system $(\varphi_j^N)_{j\in\mathbb{N}_0}$ in the sense of equivalent norms. This is why we omit in our notation the subscript $(\varphi_j^N)_{j\in\mathbb{N}_0}$. We will restrict ourselves to a special subclass of spaces $B_2^{\sigma,N}$ associated to an *admissible symbol.*

Definition 3.6 A non-negative function $a \in C^\infty(\mathbb{R}^n)$ is an *admissible symbol* if the following hold

(i) $\lim_{|x|\to\infty} a(x) = \infty$,

(ii) a is almost increasing in $|x|$, i.e. there exist constants $\delta_0 \ge 1$ and $R > 0$ such that $a(x) \le \delta_0 a(y)$ if $R \le |x| \le |y|$,

(iii) there exists an $m > 0$ such that $x \to \frac{a(x)}{|x|^m}$ is almost decreasing in $|x|$,

(iv) for every multi-index $\alpha \in \mathbb{N}_0^n$ there exist constants $c_\alpha > 0$ and $R > 0$ such that

$$|D^\alpha a(x)| \le c_\alpha \frac{a(x)}{(1+|x|^2)^{|\alpha|/2}}, \quad \forall |x| \ge R.$$

The family of all admissible functions will be denoted by $\mathcal{A}$.

Lemma 3.7 [FL06, Lemma 3.1.17, Remark 3.1.18]
For a function $a \in \mathcal{A}$ and $r > 0$ the sequence $(N_j^{a,r})_{j\in\mathbb{N}_0}$,

$$N_j^{a,r} = \sup\{|x| : a(x) \le 2^{jr}\}, \quad j \in \mathbb{N}_0,$$

is strongly increasing.

Therefore, for $a \in \mathcal{A}$ we can define the *Besov space of generalized smoothness associated with* a as

$$H^{a,1}(\mathbb{R}^n) := B_2^{\sigma, N^{a,2}}(\mathbb{R}^n),$$

where $\sigma = \{2^j\}_{j\in\mathbb{N}_0}$ is an admissible sequence. These spaces have two useful representations in the sense of equivalent norms; one given by the Littlewood-Paley-type theorem and the other by means of differences.

Proposition 3.8 [FL06, Theorem 3.1.20, Corollary 3.1.21]
Let $a \in \mathcal{A}$, $N = N^{a,2}$ the strongly increasing sequence associated with a, $\alpha > 0$ and $\sigma^\alpha = \{2^{\alpha j}\}_{j\in\mathbb{N}_0}$ an admissible sequence. Then the norm $||\cdot||_{a,\alpha}$,

$$||u||_{a,\alpha} := ||(id + a(D))^{\alpha/2} u||_{L_2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1 + a(\xi))^\alpha |\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}}, \tag{3.3}$$

is equivalent to $||\cdot||_{B,\sigma^\alpha,N^{a,2}}$ on $H^{a,\alpha}(\mathbb{R}^n) = B_2^{\sigma^\alpha, N^{a,2}}(\mathbb{R}^n)$.

Definition 3.9 For a function f on $\mathbb{R}^n$, $h \in \mathbb{R}^n$ and $k \in \mathbb{N}$ we define the *k-th difference* of the function f as

$$(\Delta_h^k f)(x) := \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(x + jh) = \Delta_h^1(\Delta_h^{k-1} f)(x), \quad x \in \mathbb{R}^n.$$

The *k-th modulus of continuity* of a function $f \in L^2(\mathbb{R}^n)$ is defined as

$$\omega_k(f, t) = \sup_{|h|<t} ||\Delta_h^k f||_{L^2(\mathbb{R}^n)}, \quad t > 0.$$

Also, for an admissible sequence $(\gamma_j)_{j\in\mathbb{N}_0}$ let

$$\overline{\gamma}_j = \sup_{k\ge 0} \frac{\gamma_{j+k}}{\gamma_k} \text{ and } \underline{\gamma}_j = \inf_{k\ge 0} \frac{\gamma_{j+k}}{\gamma_k}$$

we define the lower and upper Boyd index respectively,

$$\underline{s}(\gamma) := \lim_{j\to\infty} \frac{\log \underline{\gamma}_j}{j} \text{ and } \overline{s}(\gamma) := \lim_{j\to\infty} \frac{\log \overline{\gamma}_j}{j}.$$

Since $\gamma_{j+i+k} \leq \overline{\gamma}_j \gamma_{i+k}$ for all $i, j, k \in \mathbb{N}_0$ it follows that

$$\overline{\gamma}_{j+i} \leq \overline{\gamma}_j \overline{\gamma}_i$$

so the sequence $\log \overline{\gamma}_j$ is subadditive. By Fekete's subadditive lemma the sequence $\left(\frac{\log \overline{\gamma}_j}{j}\right)_j$ converges to $\inf\limits_j \frac{\log \overline{\gamma}_j}{j}$, so the upper index $\overline{s}(\gamma)$ is well defined. The analogous conclusion follows for the lower index $\underline{s}(\gamma)$, since $\log \underline{\gamma}_j = -\log \left(\overline{\gamma^{-1}}_j\right)$.

Theorem 3.10 [Mou07, Theorem 4.1]
Let σ and N be admissible sequences and $\underline{N}_1 = \inf\limits_{k \geq 0} \frac{N_{k+1}}{N_k} > 1$ and $\dfrac{\underline{s}(\sigma)}{\overline{s}(N)} > 0$. Let k be an integer such that $k > \dfrac{\overline{s}(\sigma)}{\underline{s}(N)}$. Then the norm $||\cdot||_{B,\sigma,N}$ on $B_2^{\sigma,N}$ is equivalent to

$$||u||_{L^2(\mathbb{R}^n)} + \left(\sum_{j=0}^{\infty} \sigma_j^2 \omega_k(u, N_j^{-1})^2\right)^{\frac{1}{2}}.$$

Remark 3.11 Note that for every $a \in \mathcal{A}$ the sequence $N^{a,r}$ satisfies the assumption $\underline{N}_1^{a,r} > 1$.

We want to generalize the trace theorem for an important subclass of continuous negative definite functions of the form

$$\psi(\xi) = \phi(|\xi|^2),$$

where $\phi \in \mathcal{CBF}$, Section 2.6. Also, suppose that the killing term and drift of ϕ are zero, that is

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\nu(t)dt.$$

The corresponding process X is the subordinate Brownian motion with the subordinator having the Laplace exponent ϕ. The density of the Levy measure is given by

$$j(x) = \int_0^\infty (4\pi t)^{-n/2} e^{-x^2/4t} \nu(t)dt.$$

Note that j is continuous and decreasing on $(0, \infty)$. Also, by [KSV15, Lemma 2.1] for every $\lambda, r > 0$

$$1 \wedge \lambda \leq \frac{\phi(\lambda r)}{\phi(r)} \leq 1 \vee \lambda. \tag{3.4}$$

By [SSV09, Theorem 7.13] function $\tilde{\phi}$,

$$\tilde{\phi}(\lambda) = \phi^{\frac{1}{2}}(\lambda)\lambda^{\frac{1}{4}}, \quad \lambda > 0 \tag{3.5}$$

is also a complete Bernstein function. Define a function $a : \mathbb{R}^n \to \mathbb{R}$ as

$$a(x) := \phi(|x|^2)|x| = \tilde{\phi}(|x|^2)^2. \tag{3.6}$$

Lemma 3.12 Let ϕ be a Bernstein function such that $\lim_{x\to\infty} \phi(x) = \infty$. Functions $\psi(\cdot) = \phi(|\cdot|^2)$ and a from (3.6) are admissible symbols.

Proof. That ψ is an admissible symbol follows from [FL06, Lemma 3.1.13]. Properties (i)-(iii) for the function a follow directly. For a multi-index $\alpha \in \mathbb{N}_0^n$ by the generalized Leibniz rule

$$D^\alpha a(x) = \sum_{\{\beta:\beta\leq\alpha\}} \binom{\alpha}{\beta} D^\beta \tilde{\phi}(|x|^2) D^{\alpha-\beta}\tilde{\phi}(|x|^2).$$

Since $\tilde{\phi}(|\cdot|^2) \in \mathcal{A}$ there exist $R > 0$ and constants $c_\beta > 0$, $\beta \in \mathbb{N}_0^n$, such that

$$\begin{aligned} |D^\alpha a(x)| &\leq \sum_{\{\beta:\beta\leq\alpha\}} \binom{\alpha}{\beta} c_\beta \frac{\tilde{\phi}(|x|^2)}{(1+|x|^2)^{|\beta|/2}} \cdot c_{\alpha-\beta} \frac{\tilde{\phi}(|x|^2)}{(1+|x|^2)^{|\alpha-\beta|/2}} \\ &\leq \sum_{\{\beta:\beta\leq\alpha\}} \binom{\alpha}{\beta} c_\beta c_{\alpha-\beta} \frac{a(x)}{(1+|x|^2)^{|\alpha|/2}}. \end{aligned}$$

□

By Proposition 3.8 Besov spaces of generalized smoothness associated with ψ and a can be characterized as

$$H^{\psi,1}(\mathbb{R}^n) = \left\{ u \in S'(\mathbb{R}^n) : \exists f \in L^2(\mathbb{R}^n) \text{ such that } \hat{u} = \frac{1}{\sqrt{1+\psi}}\hat{f} \right\}$$
$$H^{a,1}(\mathbb{R}^n) = \left\{ u \in S'(\mathbb{R}^n) : \exists f \in L^2(\mathbb{R}^n) \text{ such that } \hat{u} = \frac{1}{\sqrt{1+a}}\hat{f} \right\}.$$

Since the function $x \mapsto (1+x)^{-\alpha}$ is completely monotone for every $\alpha > 0$, by [SSV09, Theorem 3.7] functions $(1+\phi)^{-\alpha}$ and $(1+\tilde{\phi})^{-\alpha}$ are also completely monotone. By Schoenberg's theorem, [Sch38, Theorem 2], functions $(1+\psi)^{-\alpha}$ and $(1+\sqrt{a})^{-\alpha}$ are positive definite functions and therefore Fourier transforms of integrable functions, [SSV09, Theorem 4.14]. Denote

$$\hat{K}_\psi(\xi) = \frac{1}{\sqrt{1+\psi(\xi)}}$$

and

$$\hat{K}_a(\xi) = \frac{1}{1+\sqrt{a(\xi)}}.$$

Since for every $\alpha > 0$

$$(1+a(x))^\alpha \asymp (1+\sqrt{a(x)})^{2\alpha},$$

spaces $H^{\psi,1}(\mathbb{R}^n)$ and $H^{a,1}(\mathbb{R}^n)$ can be characterized as convolution spaces via the ψ-Bessel convolution kernel K_ψ and a-Bessel convolution kernel K_a respectively, i.e.

$$\begin{aligned} H^{\psi,1}(\mathbb{R}^n) &= \{K_\psi * f : f \in L^2(\mathbb{R}^n)\}, \quad ||K_\psi * f||_{\psi,1} := ||f||_{L^2(\mathbb{R}^n)}, \\ H^{a,1}(\mathbb{R}^n) &= \{K_a * f : f \in L^2(\mathbb{R}^n)\}, \quad ||K_a * f||_{a,1} := ||f||_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{3.7}$$

Furthermore, if we assume additional conditions on the complete Bernstein function ϕ we can obtain estimates for the kernels K_ψ and K_a. These conditions also imply a useful characterization of the spaces $H^{\psi,1}(\mathbb{R}^n)$ and $H^{a,1}(\mathbb{R}^n)$ via differences, as well as estimates of the kernels K_ψ and K_a. From now on we impose the following two conditions:
(H1): There exist constants $a_1, a_2 > 0$ and $0 < \delta_1 \le \delta_2 < 1$ such that

$$a_1\lambda^{\delta_1} \le \frac{\phi(\lambda r)}{\phi(r)} \le a_2\lambda^{\delta_2}, \quad \lambda \ge 1, r \ge 1,$$

(H2): There exist constants $a_3, a_4 > 0$ and $0 < \delta_3 \le \delta_4 < 1$ such that

$$a_3\lambda^{\delta_3} \le \frac{\phi(\lambda r)}{\phi(r)} \le a_4\lambda^{\delta_4}, \quad \lambda \ge 1, r \le 1.$$

Conditions **(H1)** and **(H2)** are called the upper and lower scaling condition respectively and were used in [KSV14]. It is easily shown that together **(H1)** and **(H2)** are equivalent to the global scaling condition **(H)**,
(H): There exist constants $a_5, a_6 > 0$ such that

$$a_5\lambda^{\delta_1\wedge\delta_3} \le \frac{\phi(\lambda r)}{\phi(r)} \le a_6\lambda^{\delta_2\vee\delta_4}, \quad \lambda \ge 1, r > 0.$$

Also, without loss of generality let $\phi(1) = 1$. Since a is a radial function we will abuse the notation by using $a(x) = a(|x|)$. By **(H1)** and **(H2)** it follows that

$$a_1\lambda^{2\delta_1+1} \le \frac{a(\lambda r)}{a(r)} \le a_2\lambda^{2\delta_2+1}, \quad \lambda \ge 1, r \ge 1, \tag{3.8}$$

and

$$a_3\lambda^{2\delta_3+1} \le \frac{a(\lambda r)}{a(r)} \le a_4\lambda^{2\delta_4+1}, \quad \lambda \ge 1, r \le 1. \tag{3.9}$$

The following estimates for the kernel K_ψ were obtained in [KZ06, Remark 33, Remark 34].

Lemma 3.13 Let $\alpha > 0$ and $\hat{K}_{\psi,\alpha} = (\hat{K}_\psi)^\alpha$. If $\alpha\delta_2 < n$ then for every $R > 0$ there exist constants $c_i = c_i(\phi, \alpha, n, R) > 0$, $i = 1, 2$, such that for all $x \in B(0, R) \subset \mathbb{R}^n$ and

$0 \le j \le n$

$$|K_{\psi,\alpha}(x)| \le \frac{c_1}{|x|^n \phi(|x|^{-2})^{\alpha/2}},$$
$$|(K_{\psi,\alpha}(x))'_{x_j}| \le \frac{c_2}{|x|^{n+1}\phi(|x|^{-2})^{\alpha/2}}.$$

Proof. For $x \in \mathbb{R}^n$ let $g_n(|x|) = K_{\psi,\alpha}(x)$ and $B_n(|x|) = \hat{K}_{\psi,\alpha}(x)$. The function $\hat{K}_{\psi,\alpha}$ is a positive definite radial function on $\mathbb{R}^n$ so by [Gra08, Section B.5]

$$B_n(r) = \int_0^\infty \frac{(2\pi)^{\frac{n}{2}}}{\lambda^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(\lambda r) r^{\frac{n}{2}} g_n(r) dr = \int_0^\infty Y_{\frac{n}{2}-1}(\lambda r) G_n(dr),$$

where $J_{\frac{n}{2}-1}$ and $Y_{\frac{n}{2}-1}$ are the Bessel and spherical Bessel function respectively and

$$G_n(\lambda) = \int_0^\lambda \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} r^{n-1} g_n(r) dr.$$

By [Leo99, Lemma 1.4.11] for $y > 0$

$$\hat{G}_n(iy) = A(n) y \int_0^\infty \frac{u^{n-1}}{(u^2+y^2)^{\frac{n+1}{2}}} B_n(u) du$$

for some constant $A(n)$. Let $L : (0,\infty) \to (0,\infty)$,

$$L(\lambda) = \frac{1}{\phi^{\alpha/2}(\lambda^2)}$$

and note that $B_n(\lambda) \sim L(\lambda)$ as $\lambda \to \infty$ and $B_n(\lambda) \le L(\lambda)$ for all $\lambda > 0$. We will show that this implies that $G'_n(\lambda) \sim \frac{1}{\lambda} L(\frac{1}{\lambda})$, $\lambda \to 0$. First note that

$$\begin{aligned}\frac{\hat{G}_n(iy)}{L(y)} &= A(n) \int_0^\infty \frac{t^{n-1}}{(t^2+1)^{\frac{n+1}{2}}} \frac{B_n(ty)}{L(y)} dt \le A(n) \int_0^\infty \frac{t^{n-1}}{(t^2+1)^{\frac{n+1}{2}}} \frac{L(ty)}{L(y)} dt \\ &\overset{(\mathbf{H})}{\le} A(n) a_6^{\alpha/2} \int_0^1 \frac{t^{n-1-\alpha(\delta_2 \vee \delta_4)}}{(t^2+1)^{\frac{n+1}{2}}} dt + \frac{A(n)}{a_5^{\alpha/2}} \int_1^\infty \frac{t^{n-1-\alpha(\delta_1 \wedge \delta_3)}}{(t^2+1)^{\frac{n+1}{2}}} dt = \tilde{c}_1\end{aligned}$$

where $\tilde{c}_1$ is a positive finite constant since $\alpha\delta_1 < n$. For $R > 1$ let $c_R > 0$ be such that $c_R L(y) \le B_n(y)$ for all $y \ge R$. It follows that

$$\begin{aligned}\frac{\hat{G}_n(iy)}{L(y)} &\ge A(n) \int_1^\infty \frac{t^{n-1}}{(t^2+1)^{\frac{n+1}{2}}} \frac{B_n(ty)}{L(y)} dt \ge A(n) c_R \int_1^\infty \frac{t^{n-1}}{(t^2+1)^{\frac{n+1}{2}}} \frac{L(ty)}{L(y)} dt \\ &\ge A(n) c_R \frac{1}{a_2^{\alpha/2}} \int_1^\infty \frac{t^{n-1-\alpha\delta_2}}{(t^2+1)^{\frac{n+1}{2}}} dt = \tilde{c}_2\end{aligned}$$

where $\tilde{c}_2$ is a positive finite constant since $\alpha\delta_2 < n$. By a variation of the Karamata Tauberian theorem for O-regularly varying functions, [BGT87, Theorem 2.10.2, de Haan-Stadtmüller Theorem], since for all $\lambda \geq 1$

$$0 < \liminf_{t\to\infty} \frac{G_n(\lambda t)}{G_n(t)} \leq \limsup_{t\to\infty} \frac{G_n(\lambda t)}{G_n(t)} < \infty$$

it follows that $\mathcal{L}(G_n')(\frac{1}{\cdot})$ and $\mathcal{L}(G_n'')(\frac{1}{\cdot})$ are also O-regularly varying functions. Furthermore,

$$G_n'(\lambda) \sim \mathcal{L}(G_n')\left(\frac{1}{\lambda}\right) \sim \frac{1}{\lambda} L\left(\frac{1}{\lambda}\right), \quad \lambda \to 0$$
$$G_n''(\lambda) \sim \mathcal{L}(G_n'')\left(\frac{1}{\lambda}\right) \sim \frac{1}{\lambda^2} L\left(\frac{1}{\lambda}\right), \quad \lambda \to 0,$$

which implies that

$$|g_n(r)| = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \left|\frac{G_n'(r)}{r^{n-1}}\right| \leq \tilde{c}_3 \frac{L(\frac{1}{r})}{r^n}$$

and

$$|g_n'(r)| \leq \frac{\Gamma(\frac{n}{2})n}{2\pi^{\frac{n}{2}}} \left(\left|\frac{G_n''(r)}{r^{n-1}}\right| + \left|\frac{G_n'(r)}{r^n}\right|\right) \leq \tilde{c}_4 \frac{L(\frac{1}{r})}{r^{n+1}}.$$

□

Remark 3.14 Lemma 3.13 applied to the function $\tilde{\phi}$ from (3.5) gives us estimates for the Bessel kernel $K_{a,\alpha}$. For $\alpha > 0$ such that $\alpha(\delta_2 + \frac{1}{2}) < n$ and $R > 0$ it follows that

$$|K_{a,\alpha}(x)| \leq \frac{c_1}{|x|^n a(|x|^{-1})^{\alpha/2}},$$
$$|(K_{a,\alpha}(x))'_{x_j}| \leq \frac{c_2}{|x|^{n+1} a(|x|^{-1})^{\alpha/2}},$$

for some $c_1, c_2 > 0$ and all $x \in B(0,R)$.

Next we will consider the characterization of spaces $H^{\psi,1}(\mathbb{R}^n)$ and $H^{a,1}(\mathbb{R}^n)$ via differences. By [Jac01, Thm 3.10.4.] space $H^{\psi,1}(\mathbb{R}^n)$ is continuously embedded in $L^2(\mathbb{R}^n)$ and it is a Hilbert space with the inner product

$$(u,v)_{\psi,1} = \int_{\mathbb{R}^n} (1+\psi(\xi))\hat{u}(\xi)\hat{v}(\xi)d\xi. \tag{3.10}$$

As in Example 2.18(ii) it follows that for $u \in H^{\psi,1}(\mathbb{R}^n)$

$$\begin{aligned}||u||_{\psi,1} &\leq ||u||_{L^2(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \psi(\xi) d\xi\right)^{\frac{1}{2}} \\ &= ||u||_{L^2(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 (1-\cos(\xi y)) j(y) dy\, d\xi\right)^{\frac{1}{2}} \\ &= ||u||_{L^2(\mathbb{R}^n)} + \left(\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 |1-e^{i\xi y}|^2 j(y) dy\, d\xi\right)^{\frac{1}{2}}\end{aligned}$$

which is by Parseval's identity equal to

$$||u||_{L^2(\mathbb{R}^n)} + \left(\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(x+y)-u(x))^2 j(y) dy dx\right)^{\frac{1}{2}}.$$

This implies that the ψ-Bessel potential space $(H^{\psi,1}(\mathbb{R}^n), ||\cdot||_{\psi,1})$ is equivalent to the Dirichlet space $(\mathcal{F}^{\mathbb{R}^n}, \sqrt{\mathcal{C}_1})$. We also introduce an equivalent norm on $H^{\psi,1}(\mathbb{R}^n)$ which we will later use in the proof of the trace theorem. For $u \in H^{\psi,1}(\mathbb{R}^n)$ let

$$||u||_{(1)} = ||u||_{L^2(\mathbb{R}^n)} + \left(\iint\limits_{|x-y|<1} |u(x)-u(y)|^2 \frac{\phi\left(|x-y|^{-2}\right)}{|x-y|^n} dx dy\right)^{\frac{1}{2}}.$$

Lemma 3.15 Norms $||\cdot||_{\psi,1}$ and $||\cdot||_{(1)}$ on $H^{\psi,1}(\mathbb{R}^n)$ are equivalent.

Proof. By [KSV12b, Theorem 2.3.] for every $R > 0$ there exists a constant $\tilde{c}\,(R) > 1$ such that for all $|x| < R$

$$\tilde{c}(R)^{-1}\frac{\phi(|x|^{-2})}{|x|^n} < j(|x|) < \tilde{c}(R)\frac{\phi(|x|^{-2})}{|x|^n}$$

and therefore for $\tilde{c} = \tilde{c}(1)$

$$\begin{aligned}||u||_{\psi,1} \leq ||u||_{L^2(\mathbb{R}^n)} &+ \left(\frac{\tilde{c}}{2}\iint\limits_{|x-y|<1} |u(x)-u(y)|^2 \frac{\phi\left(|x-y|^{-2}\right)}{|x-y|^n} dx dy\right)^{\frac{1}{2}} \\ &+ \left(\frac{1}{2}\iint\limits_{|x-y|\geq 1} |u(x)-u(y)|^2 j(|x-y|) dx dy\right)^{\frac{1}{2}}.\end{aligned}$$

Since

$$\iint\limits_{|x-y|\geq 1} |u(x)-u(y)|^2 j(|x-y|) dx dy \leq \int\limits_{|z|\geq 1} \left(\int_{\mathbb{R}^n} 2\left(u(y+z)^2 + u(y)^2\right) dy\right) j(|z|) dz$$

$$\leq 4||u||^2_{L^2(\mathbb{R}^n)} \int\limits_{|z|\geq 1} j(|z|)dz,$$

it follows that

$$||u||_{\psi,1} \leq \left(\left(\frac{\tilde{c}}{2} \vee 4 \int\limits_{|z|\geq 1} j(|z|)dz \right)^{\frac{1}{2}} + 1 \right) ||u||_{(1)}.$$

For the other inequality we get

$$\begin{aligned} ||u||_{\psi,1} &\geq 2^{-\frac{1}{2}}||u||_{L^2(\mathbb{R}^n)} + \left(\frac{\tilde{c}^{-1}}{4} \iint\limits_{|x-y|<1} |u(x)-u(y)|^2 \frac{\phi\left(|x-y|^{-2}\right)}{|x-y|^n} dxdy \right)^{\frac{1}{2}} \\ &\geq \frac{(\tilde{c}^{-1}\wedge 2)^{\frac{1}{2}}}{2} ||u||_{(1)}. \end{aligned}$$

□

Recall that $\sigma = (2^j)_{j\in\mathbb{N}_0}$ is an admissible sequence and by Lemma 3.7 the sequence $N^{a,2} = \left(a^{-1}(2^{2j})\right)_{j\in\mathbb{N}_0}$ is strongly increasing. Since a is strictly increasing when considered as a radial function and

$$a_1 \left(\frac{a^{-1}(\lambda x)}{a^{-1}(x)} \right)^{2\delta_1+1} \leq \lambda = \frac{a(a^{-1}(\lambda x))}{a(a^{-1}(x))} \leq a_2 \left(\frac{a^{-1}(\lambda x)}{a^{-1}(x)} \right)^{2\delta_2+1}, \quad \lambda \geq 1$$

it follows that for $j \in \mathbb{N}$ and $k \in \mathbb{N}_0$

$$\left(\frac{1}{a_2} 2^{2j} \right)^{\frac{1}{2\delta_2+1}} \leq \frac{N^{a,2}_{j+k}}{N^{a,2}_k} \leq \left(\frac{1}{a_1} 2^{2j} \right)^{\frac{1}{2\delta_1+1}} \tag{3.11}$$

so the sequence $N^{a,2}$ is also admissible. Furthermore,

$$\frac{\underline{s}(\sigma)}{\overline{s}(N^{a,2})} \geq \frac{\log 2}{\frac{2}{2\delta_1+1}\log 2} = \frac{2\delta_1+1}{2} > 0 \text{ and } \frac{\overline{s}(\sigma)}{\underline{s}(N^{a,2})} \leq \frac{\log 2}{\frac{2}{2\delta_2+1}\log 2} = \frac{2\delta_2+1}{2} < 2$$

so Theorem 3.10 holds for $k = 2$. For simpler notation, denote $N_j = N^{a,2}_j$. Moreover,

$$\begin{aligned} &||u||_{L^2(\mathbb{R}^n)} + \left(\sum_{j=0}^{\infty} 2^{2j} \sup_{|H|<N_j^{-1}} ||\Delta^2_H u||^2_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{2}} \\ &= ||u||_{L^2(\mathbb{R}^n)} + \left(\frac{2}{3} \sum_{j=0}^{\infty} \int\limits_{2^{-(j+1)}\leq t<2^{-j}} \frac{1}{t^3} \sup_{|H|<N_j^{-1}} ||\Delta^2_H u||^2_{L^2(\mathbb{R}^n)} dt \right)^{\frac{1}{2}} \end{aligned}$$

$$\asymp ||u||_{L^2(\mathbb{R}^n)} + \left(\int_0^1 \frac{1}{t^3} \sup_{|H|<1/a^{-1}(t^{-2})} ||\Delta_H^2 u||_{L^2(\mathbb{R}^n)}^2 dt\right)^{\frac{1}{2}}, \tag{3.12}$$

since by (3.11)

$$2^{-(j+1)} \leq t < 2^{-j} \Rightarrow \left(\frac{a_1}{4}\right)^{\frac{1}{2\delta_1+2}} N_j^{-1} \leq \frac{1}{a^{-1}(t^{-2})} < N_j^{-1}.$$

By change of variable $t^{-2} = a(|h|^{-1})$ it follows that (3.12) is comparable to

$$||u||_{L^2(\mathbb{R}^n)} + \left(\frac{1}{2} \int\limits_{|h|<1} \frac{a'(|h|^{-1})}{|h|^{n+1}} \sup_{|H|<|h|} ||\Delta_H^2 u||_{L^2(\mathbb{R}^n)}^2 dh\right)^{\frac{1}{2}}.$$

Since $a'(t) = \phi'(t)\sqrt{t} + \frac{\phi(t)}{2\sqrt{t}}$ and $\phi'(t) \leq \frac{\phi(t)}{t}$ it follows that $\frac{a(t)}{2t} \leq a'(t) \leq \frac{3a(t)}{2t}$, so the last line is comparable to

$$||u||_{L^2(\mathbb{R}^n)} + \left(\int\limits_{|h|<1} \frac{a(|h|^{-1})}{|h|^n} \sup_{|H|<|h|} ||\Delta_H^2 u||_{L^2(\mathbb{R}^n)}^2 dh\right)^{\frac{1}{2}}, \tag{3.13}$$

which is by the generalization of [Tri10, Theorem 2.6.1] equivalent to

$$||u||_{(1),a} := ||u||_{L^2(\mathbb{R}^n)} + \left(\int\limits_{|h|<1} \frac{a(|h|^{-1})}{|h|^n} ||\Delta_h^2 u||_{L^2(\mathbb{R}^n)}^2 dh\right)^{\frac{1}{2}}. \tag{3.14}$$

Before we prove this assertion we note the following Remark.

Remark 3.16 (i) By Theorem 3.10 and calculation above norms of the form

$$||u||_{L^2(\mathbb{R}^n)} + \left(\int\limits_{|h|<1} \frac{a(|h|^{-1})}{|h|^n} \sup_{|H|<|h|} ||\Delta_H^k u||_{L^2(\mathbb{R}^n)}^2 dh\right)^{\frac{1}{2}}$$

are equivalent for all $k \geq 2$.

(ii) Since the function $\frac{a(|\cdot|^{-1})}{|\cdot|^n}$ is continuous and $||\Delta_h^2 u||_{L^2(\mathbb{R}^n)} \leq 4||u||_{L^2(\mathbb{R}^n)}$ the norms $||\cdot||_{(1),a}^{h_0}$,

$$||u||_{(1),a}^{h_0} := ||u||_{L^2(\mathbb{R}^n)} + \left(\int\limits_{|h|<h_0} \frac{a(|h|^{-1})}{|h|^n} ||\Delta_h^2 u||_{L^2(\mathbb{R}^n)}^2 dh\right)^{\frac{1}{2}},$$

are equivalent for all $h_0 > 0$.

Lemma 3.17 The norms $||\cdot||_{a,1}$ and $||\cdot||_{(1),a}$ are equivalent on $H^{a,1}(\mathbb{R}^n)$.

Proof. By calculation above it is enough to prove that norms in (3.13) and (3.14) are equivalent. Obviously, the norm (3.14) is dominated by (3.13). For the other inequality note that

$$\begin{aligned} I &= \int\limits_{|h|<1} \frac{a(|h|^{-1})}{|h|^n} \sup_{|H|<|h|} ||\Delta_H^2 u||^2_{L^2(\mathbb{R}^n)} dh \\ &\le \int\limits_{|h|<1} \frac{a(|h|^{-1})}{|h|^n} \sup_{\frac{|h|}{2}<|H|<|h|} ||\Delta_H^2 u||^2_{L^2(\mathbb{R}^n)} dh + \int\limits_{|h|<1} \frac{a(|h|^{-1})}{|h|^n} \sup_{|H|<\frac{|h|}{2}} ||\Delta_H^2 u||^2_{L^2(\mathbb{R}^n)} dh \\ &= \int\limits_{|h|<1} \frac{a(|h|^{-1})}{|h|^n} \sup_{\frac{|h|}{2}<|H|<|h|} ||\Delta_H^2 u||^2_{L^2(\mathbb{R}^n)} dh + 2 \int\limits_{|h|<\frac{1}{2}} \frac{a(2^{-1}|h|^{-1})}{2^n|h|^n} \sup_{|H|<|h|} ||\Delta_H^2 u||^2_{L^2(\mathbb{R}^n)} dh \\ &\le \int\limits_{|h|<1} \frac{a(|h|^{-1})}{|h|^n} \sup_{\frac{|h|}{2}<|H|<|h|} ||\Delta_H^2 u||^2_{L^2(\mathbb{R}^n)} dh + \frac{I}{2^n}. \end{aligned} \tag{3.15}$$

By [CL09, (3.3.8)] there exists a constant $\tilde{c}_1 > 0$ such that

$$||\Delta_H^{2k} u||^2_{L^2(\mathbb{R}^n)} \le \tilde{c}_1 \left(||\Delta_{H_0}^k u||^2_{L^2(\mathbb{R}^n)} + ||\Delta_{H_1}^k u||^2_{L^2(\mathbb{R}^n)} \right)$$

for every $H > 0$ and $H_0 + H_1 = H$. For $\frac{|h|}{2} \le |H| \le |h|$ it follow that

$$\begin{aligned} \int\limits_{\substack{\frac{|h|}{8} \le |x| \le \frac{|h|}{4} \\ |H-x|<|h|}} ||\Delta_H^4 u||^2_{L^2(\mathbb{R}^n)} dx &\le \tilde{c}_1 \int\limits_{\substack{\frac{|h|}{8} \le |x| \le \frac{|h|}{4} \\ |H-x|<|h|}} \left(||\Delta_x^2 u||^2_{L^2(\mathbb{R}^n)} + ||\Delta_{H-x}^2 u||^2_{L^2(\mathbb{R}^n)} \right) dx \\ &\le 2\tilde{c}_1 \int\limits_{\frac{|h|}{8} \le |x| \le |h|} ||\Delta_x^2 u||^2_{L^2(\mathbb{R}^n)} dx, \end{aligned}$$

so for some $\tilde{c}_2, \tilde{c}_3 > 0$

$$\sup_{\frac{|h|}{2} \le |H| \le |h|} ||\Delta_H^4 u||^2_{L^2(\mathbb{R}^n)} \le \tilde{c}_2 |h|^{-n} \int\limits_{\frac{|h|}{8} \le |x| \le |h|} ||\Delta_x^2 u||^2_{L^2(\mathbb{R}^n)} dx \le \tilde{c}_3 \int_{\frac{1}{8}}^1 \int\limits_{B(0,1)} ||\Delta_{|h|y\omega}^2 u||^2_{L^2(\mathbb{R}^n)} d\omega dy. \tag{3.16}$$

By Remark 3.16(i) there exists a constant $\tilde{c}_4 > 0$ such that

$$I \overset{(3.15)}{\le} \frac{2^n}{2^n - 1} \int\limits_{|h|<1} \frac{a(|h|^{-1})}{|h|^n} \sup_{\frac{|h|}{2}<|H|<|h|} ||\Delta_H^2 u||^2_{L^2(\mathbb{R}^n)} dh$$

$$
\begin{aligned}
&\le \frac{2^n}{2^n-1}\tilde{c}_4 \int\limits_{|h|<1} \frac{a(|h|^{-1})}{|h|^n} \sup_{\frac{|h|}{2}<|H|<|h|} ||\Delta_H^4 u||^2_{L^2(\mathbb{R}^n)} dh \\
&\overset{(3.16)}{\le} \frac{2^n}{2^n-1}\tilde{c}_3\tilde{c}_4 \int\limits_{|h|<1} \int_{\frac{1}{8}}^{1} \int\limits_{B(0,1)} \frac{a(|h|^{-1})}{|h|^n} ||\Delta^2_{|h|y\omega} u||^2_{L^2(\mathbb{R}^n)} d\omega dy dh \\
&= \frac{2^n}{2^n-1}\tilde{c}_3\tilde{c}_4 \int_0^1 \int_{\frac{1}{8}}^{1} \int\limits_{B(0,1)} \frac{a(h^{-1})}{h} ||\Delta^2_{hy\omega} u||^2_{L^2(\mathbb{R}^n)} d\omega dy dh \\
&\le \tilde{c}_5 \int\limits_{|h|<1} \frac{a(|h|^{-1})}{|h|^n} ||\Delta_h^2 u||^2_{L^2(\mathbb{R}^n)} dh,
\end{aligned}
$$

for some $\tilde{c}_5 > 0$. □

3.3 The trace theorem

In the previous section we identified the space $(\mathcal{F}, \sqrt{\mathcal{C}_1})$ with the ψ-Bessel potential space $(H^{\psi,1}(\mathbb{R}^n), ||\cdot||_{\psi,1})$. We will show that the space $(\mathcal{F}_{\mathrm{a}}^{\mathrm{ref}}, \sqrt{\mathcal{E}_1^{\mathrm{ref}}})$ is the trace space of $(\mathcal{F}, \sqrt{\mathcal{C}_1})$ on D when D is an open n-set on $\mathbb{R}^n$. First we introduce the notion of the trace space.

Definition 3.18 Let μ be a positive Radon measure on $D \subset \mathbb{R}^n$. For $f \in S(\mathbb{R}^n)$ we denote the pointwise trace of f on D by $tr_D f$. If there is a constant $c > 0$ such that

$$||tr_D f||_{L^2(D,\mu)} \le c||f||_{\psi,1}, \quad \forall f \in S(\mathbb{R}^n)$$

then we call the continuous extension tr_D of this mapping to $H^{\psi,1}(\mathbb{R}^n)$ the *trace operator* and the *trace space* of $H^{\psi,1}(\mathbb{R}^n)$ on D is given by

$$
\begin{aligned}
&H^{\psi,1}(D,\mu) = \left\{u \in L^2(D,\mu) : u = tr_D f \ \mu\text{-a.e. on } D \text{ for some } f \in H^{\psi,1}(\mathbb{R}^n)\right\} \\
&||u||_{\psi,1,D,\mu} = \inf\left\{||f||_{\psi,1} : f \in H^{\psi,1}(\mathbb{R}^n),\ u = tr_D f \ \mu\text{-a.e. on } D\right\}.
\end{aligned}
$$

Remark 3.19 Since $C_c^\infty(\mathbb{R}^n) \subset H^{\psi,1}(\mathbb{R}^n)$, $H^{\psi,1}(D,\mu)$ is a Banach space containing $C_c^\infty(D)$. Furthermore, $||u||_{\psi,1,D,\mu}$ satisfies the parallelogram identity,

$$
\begin{aligned}
2||u||^2_{\psi,1,D,\mu} + 2||v||^2_{\psi,1,D,\mu} &= \inf_{\substack{u=tr_D f\\ v=tr_D g}} \left(2||f||^2_{\psi,1} + 2||g||^2_{\psi,1}\right) = \inf_{\substack{u=tr_D f\\ v=tr_D g}} \left(||f+g||^2_{\psi,1} + ||f-g||^2_{\psi,1}\right) \\
&\ge \inf_{\substack{u=tr_D f\\ v=tr_D g}} ||f+g||^2_{\psi,1} + \inf_{\substack{u=tr_D f\\ v=tr_D g}} ||f-g||^2_{\psi,1} \ge \inf_{u+v=tr_D h} ||h||^2_{\psi,1} + \inf_{u-v=tr_D h} ||h||^2_{\psi,1} \\
&= ||u+v||^2_{\psi,1,D,\mu} + ||u-v||^2_{\psi,1,D,\mu},
\end{aligned}
$$

where the other inequality follows by taking $\tilde{u} = \frac{u+v}{2}$ and $\tilde{v} = \frac{u-v}{2}$. Therefore, $H^{\psi,1}(D,\mu)$ is also a Hilbert space.

We will limit ourselves to a special class of open sets D called *d-sets*.

Definition 3.20 Let D be a non-empty Borel subset of $\mathbb{R}^n$ and d such that $0 < d \leq n$. A positive Borel measure μ on D is called a d-measure if there exist positive constants c_1 and c_2 such that for all $x \in D$ and $r \in (0,1]$,

$$c_1 r^d \leq \mu(D \cap B(x,r)) \leq c_2 r^d.$$

A non-empty Borel set D is called a d-set if there exists a d-measure μ on D. Note that by definition all d-measures on D are equivalent to the restriction of the d-dimensional Hausdorff measure to D.

For a d-set D in $\mathbb{R}^n$ with d-measure μ let

$$H(D,\mu) = \{u \in L^2(D,\mu) : ||u||_{(1),D,\mu} < \infty\},$$
$$||u||_{(1),D,\mu} = ||u||_{L^2(D,\mu)} + \left(\iint\limits_{|x-y|<1} |u(x)-u(y)|^2 \frac{\phi\left(|x-y|^{-2}\right)}{|x-y|^{2d-n}} \mu(dx)\mu(dy) \right)^{\frac{1}{2}}.$$

and

$$H_a(D,\mu) = \{u \in L^2(D,\mu) : ||u||_{(1),a,D,\mu} < \infty\},$$
$$||u||_{(1),a,D,\mu} = ||u||_{L^2(D,\mu)} + \left(\iint\limits_{|x-y|<1} |u(x)-u(y)|^2 \frac{a\left(|x-y|^{-1}\right)}{|x-y|^{2d-n}} \mu(dx)\mu(dy) \right)^{\frac{1}{2}}.$$

When μ is the Lebesgue measure λ_D on a n-set D by similar calculations as in Lemma 3.15 it follows that the space $(H(D,\lambda_D), ||\cdot||_{(1),D,\lambda_D})$ is equivalent to $(\mathcal{F}_{\mathrm{a}}^{\mathrm{ref}}, \sqrt{\mathcal{E}_1^{\mathrm{ref}}})$.

Example 3.21 For a rotationally symmetric α-stable Lévy process X, i.e. $\psi(\xi) = |\xi|^\alpha$, $\alpha \in (0,2)$, space $H^{\psi,1}(\mathbb{R}^n)$ is the fractional Sobolev space $W^{\alpha/2,2}(\mathbb{R}^n)$. By the *trace theorem* for Besov spaces, [JW84, Theorem V.1.1.], for any open n-set D the trace space of $W^{\alpha/2,2}(\mathbb{R}^n)$ on D is equal to $\mathcal{F}^{\mathrm{ref}}$.

Before we state the trace theorem for ψ-Bessel potential space $H^{\psi,1}(\mathbb{R}^n)$ we prove the following useful lemma.

Lemma 3.22 Let $c > 0$ and $N \in \mathbb{Z}$. The norm $||\cdot||^{c,N}_{(2),D,\mu}$ on $H(D,\mu)$ defined by

$$||u||^{c,N}_{(2),D,\mu} = ||u||_{L^2(D,\mu)} + \left(\sum_{j=N}^{\infty} \phi\left(2^{2j}\right) 2^{(2d-n)j} \iint\limits_{|x-y|<c2^{-j}} |u(x)-u(y)|^2 \mu(dx)\mu(dy) \right)^{\frac{1}{2}} \tag{3.17}$$

is equivalent to the norm $||\cdot||_{(1),D,\mu}$. The same statement is true for the corresponding norms $||\cdot||_{(1)}$ and $||\cdot||^{c,N}_{(2),\mathbb{R}^n,\lambda}$ on $H^{\psi,1}(\mathbb{R}^n)$, as well as for $||\cdot||_{(1),a,D,\mu}$ and $||\cdot||^{c,N}_{(2),a,D,\mu}$ on $H_a(D,\mu)$,

$$||u||^{c,N}_{(2),a,D,\mu} = ||u||_{L^2(D,\mu)} + \left(\sum_{j=N}^{\infty} a\left(2^{j}\right) 2^{(2d-n)j} \iint\limits_{|x-y|<c2^{-j}} |u(x)-u(y)|^2 \mu(dx)\mu(dy) \right)^{\frac{1}{2}}$$

Proof. First note that for all $c > 0$ and $N \in \mathbb{N}$ by similar calculations as in Lemma 3.15

$$||\cdot||_{(1),D,\mu} \asymp ||u||_{L^2(D,\mu)} + \iint\limits_{|x-y|<c2^{-N}} |u(x)-u(y)|^2 \frac{\phi\left(|x-y|^{-2}\right)}{|x-y|^{2d-n}} \mu(dx)\mu(dy).$$

Let $2d > n$. Since ϕ is nondecreasing it follows that

$$\begin{aligned}
&\iint\limits_{|x-y|<c2^{-N}} |u(x)-u(y)|^2 \frac{\phi\left(|x-y|^{-2}\right)}{|x-y|^{2d-n}} \mu(dx)\mu(dy) = \\
&= \sum_{j=N}^{\infty} \iint\limits_{c2^{-j-1}\le|x-y|<c2^{-j}} |u(x)-u(y)|^2 \frac{\phi\left(|x-y|^{-2}\right)}{|x-y|^{2d-n}} \mu(dx)\mu(dy) \\
&\le \sum_{j=N}^{\infty} \phi\left(c^{-2}2^{2(j+1)}\right) c^{-(2d-n)} 2^{(2d-n)(j+1)} \iint\limits_{c2^{-j-1}\le|x-y|<c2^{-j}} |u(x)-u(y)|^2 \mu(dx)\mu(dy) \\
&\underset{(3.4)}{\le} \left(1 \vee \frac{4}{c^2}\right) \frac{2^{(2d-n)}}{c^{(2d-n)}} \sum_{j=N}^{\infty} \phi\left(2^{2j}\right) 2^{(2d-n)j} \iint\limits_{c2^{-j-1}\le|x-y|<c2^{-j}} |u(x)-u(y)|^2 \mu(dx)\mu(dy) \\
&\le \left(1 \vee \frac{4}{c^2}\right) \frac{2^{(2d-n)}}{c^{(2d-n)}} \sum_{j=N}^{\infty} \phi\left(2^{2j}\right) 2^{(2d-n)j} \iint\limits_{|x-y|<c2^{-j}} |u(x)-u(y)|^2 \mu(dx)\mu(dy).
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\sum_{j=N}^{\infty} \phi\left(2^{2j}\right) 2^{(2d-n)j} \iint\limits_{|x-y|<c2^{-j}} |u(x)-u(y)|^2 \mu(dx)\mu(dy) = \\
&= \sum_{j=N}^{\infty} \phi\left(2^{2j}\right) 2^{(2d-n)j} \sum_{i=j}^{\infty} \iint\limits_{c2^{-i-1}\le|x-y|<c2^{-i}} |u(x)-u(y)|^2 \mu(dx)\mu(dy) =
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=N}^{\infty}\sum_{j=N}^{i} \phi\left(2^{2j}\right) 2^{(2d-n)j} \iint\limits_{c2^{-i-1}\le|x-y|<c2^{-i}} |u(x)-u(y)|^2\mu(dx)\mu(dy)\\
&\le \sum_{i=N}^{\infty} \phi\left(2^{2i}\right) \frac{2^{(2d-n)(i+1)}-1}{2^{2d-n}-1} \iint\limits_{c2^{-i-1}\le|x-y|<c2^{-i}} |u(x)-u(y)|^2\mu(dx)\mu(dy)\\
&\le 2^{2d-n}\sum_{i=N}^{\infty} \phi\left(2^{2i}\right) 2^{(2d-n)i} \iint\limits_{c2^{-i-1}\le|x-y|<c2^{-i}} |u(x)-u(y)|^2\mu(dx)\mu(dy)\\
&\underset{(3.4)}{\le} 2^{2d-n}c^{2d-n}(1\vee c^2)\sum_{i=N}^{\infty} \phi\left((c2^{-i})^{-2}\right)(c2^{-i})^{-(2d-n)} \iint\limits_{c2^{-i-1}\le|x-y|<c2^{-i}} |u(x)-u(y)|^2\mu(dx)\mu(dy)\\
&\le 2^{2d-n}c^{2d-n}(1\vee c^2)\sum_{i=N}^{\infty} \iint\limits_{c2^{-i-1}\le|x-y|<c2^{-i}} |u(x)-u(y)|^2\frac{\phi\left(|x-y|^{-2}\right)}{|x-y|^{2d-n}}\mu(dx)\mu(dy) =\\
&= 2^{2d-n}c^{2d-n}(1\vee c^2) \iint\limits_{|x-y|<c2^{-N}} |u(x)-u(y)|^2\frac{\phi\left(|x-y|^{-2}\right)}{|x-y|^{2d-n}}\mu(dx)\mu(dy).
\end{aligned}
$$

The similar calculation follows through for $2d \le n$. $\square$

Theorem 3.23 Trace theorem
Let D be a n-set in $\mathbb{R}^n$, λ_D the Lebesgue measure on D and ϕ a complete Bernstein function such that **(H1)** and **(H2)** hold. Then the trace space $(H^{\psi,1}(D,\lambda_D), ||\cdot||_{\psi,1,D,\lambda_D})$ of $(H^{\psi,1}(\mathbb{R}^n), ||\cdot||_{\psi,1})$ on D is equivalent to the space $(H(D,\lambda_D), ||\cdot||_{(1),D,\lambda_D})$.

Remark 3.24 (i) To prove that $H(D,\lambda_D)$ is truly the trace space of $H^{\psi,1}(\mathbb{R}^n)$ on D we will define operators $R : H^{\psi,1}(\mathbb{R}^n) \to H(D,\lambda_D)$ and $E : H(D,\lambda_D) \to H^{\psi,1}(\mathbb{R}^n)$ such that

$$Ru = u \text{ a.e. on } D \text{ and } ||Ru||_{(1),D,\lambda_D} \le C_1||u||_{\psi,1}, \quad \forall u \in H^{\psi,1}(\mathbb{R}^n) \tag{3.18}$$

$$Eu = u \text{ a.e. on } D \text{ and } ||Eu||_{\psi,1} \le C_2||u||_{(1),D,\lambda_D}, \quad \forall u \in H(D,\lambda_D) \tag{3.19}$$

for some constants $C_1, C_2 > 0$ and that

$$REu = u \text{ a.e. on } D \text{ for all } u \in H(D,\lambda_D). \tag{3.20}$$

Operator R satisfying 3.18 is called the continuous *restriction* operator and operator E satisfying 3.19 and 3.20 is called the continuous *extension* operator.
(ii) Note that $\tilde{D} = D \times \{0\}$ is a n-set in $\mathbb{R}^{n+1}$ and that every function $u \in H(D,\lambda_D)$ can be represented as a function $\tilde{u}$ in $H_a(\tilde{D},\mu)$ such that

$$\tilde{u}(\tilde{x}) = u(x),\ \tilde{x} = (x,0),\ x \in D$$

and

$$||\tilde{u}||_{(1),a,\tilde{D},\mu} = ||\tilde{u}||_{(1),D,\lambda_D} \tag{3.21}$$

where μ is the restriction of the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$ to $\tilde{D}$ and a is defined by (3.6). Analogously, the space $H^{\psi,1}(\mathbb{R}^n)$ can be represented as $H_a(\mathbb{R}^n \times \{0\}, \bar{\mu})$, where $\bar{\mu}$ is the restriction of the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$ to $\mathbb{R}^n \times \{0\}$.
(iii) The proof of the trace theorem consists of four parts and follows the proof of the trace theorem for Besov spaces, [JW84]. First we define the restriction operator R and prove its continuity. In the same way, we prove the continuity of restriction operators $\tilde{R}$ and $\bar{R}$ from $H^{a,1}(\mathbb{R}^{n+1})$ to $H_a(\tilde{D}, \mu)$ and $H_a(\mathbb{R}^n \times \{0\}, \bar{\mu})$ respectively, which we later use in the definition of the extension operator E. Here $\bar{\mu}$ is the restriction of the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$ to $\mathbb{R}^n \times \{0\}$. Using the approach in [JW84] we can directly prove the continuity of the extension operator only in the case when D is a d-set of order strictly less then the dimension of the space, that is $d < n$. This is why we first prove the continuity of the operator $\tilde{E}$ from $H_a(\tilde{D}, \mu)$ to $H^{a,1}(\mathbb{R}^{n+1})$ and later define the operator E using $\tilde{E}$ and restriction operators $\tilde{R}$ and $\bar{R}$.

Assuming the conditions from Theorem 3.23 and notation from Remark 3.24 we state the *restriction theorem*.

Theorem 3.25 There exist continuous restriction operators $R : H^{\psi,1}(\mathbb{R}^n) \to H(D, \lambda_D)$, $\tilde{R} : H^{a,1}(\mathbb{R}^{n+1}) \to H_a(\tilde{D}, \mu)$ and $\bar{R} : H^{a,1}(\mathbb{R}^{n+1}) \to H_a(\mathbb{R}^n \times \{0\}, \bar{\mu})$.

First we prove the following useful Lemma.

Lemma 3.26 Let $d \leq n$, D a d-set in $\mathbb{R}^n$ and μ the restriction of the d-dimensional Hausdorff measure on D. Let ϕ be a complete Bernstein function such that **(H)** holds and $\alpha > 0$ such that

$$\frac{n-d}{2} < \alpha\delta_1 \leq \alpha(\delta_2 \vee \delta_4) < \frac{n-d}{2} + 1. \tag{3.22}$$

Then there exists a constant $c > 0$ such that for all $r \leq 1$ and $f \in L^2(\mathbb{R}^n)$

$$\iint_{|x-y|<r} (K_{\psi,\alpha} * f(x) - K_{\psi,\alpha} * f(y))^2 \mu(dx)\mu(dy) \leq c\frac{r^{2d-n}}{\phi^\alpha(r^{-2})}||f||^2_{L^2(\mathbb{R}^n)}$$

Proof. Without loss of generality, we will consider the measure μ as a measure on $\mathbb{R}^n$ with support on D and assume that $\mu(B(0,1)) = \frac{1}{d}$. Note that

$$\begin{aligned}
&\iint_{|x-y|<r} (K_{\psi,\alpha} * f(x) - K_{\psi,\alpha} * f(y))^2 \mu(dx)\mu(dy) \\
&= \iint_{|x-y|<r} \left(\int (K_{\psi,\alpha}(x-t) - K_{\psi,\alpha}(y-t)) f(t)dt \right)^2 \mu(dx)\mu(dy)
\end{aligned}$$

$$
\begin{aligned}
&\leq 2 \iint\limits_{|x-y|<r} \left(\int\limits_{|y-t|<2r} (K_{\psi,\alpha}(x-t) - K_{\psi,\alpha}(y-t)) f(t)dt \right)^2 \mu(dx)\mu(dy) \\
&+ 2 \iint\limits_{|x-y|<r} \left(\int\limits_{2r\leq|y-t|} (K_{\psi,\alpha}(x-t) - K_{\psi,\alpha}(y-t)) f(t)dt \right)^2 \mu(dx)\mu(dy) \\
&= 2(A+B) \qquad (3.23)
\end{aligned}
$$

Also, by the Cauchy-Schwartz inequality, for every $0 < a < 1$

$$
\begin{aligned}
\left(\int (K_{\psi,\alpha}(x-t) - K_{\psi,\alpha}(y-t)) f(t)dt \right)^2 &\leq \int |K_{\psi,\alpha}(x-t) - K_{\psi,\alpha}(y-t)|^{2a} f^2(t)dt \\
&\cdot \int |K_{\psi,\alpha}(x-t) - K_{\psi,\alpha}(y-t)|^{2(1-a)}dt.
\end{aligned}
$$

First we estimate the integral A. Let $|x-y| < r$ and $c_1 = c_1(\phi, \alpha, n, 3)$ be the constant from Lemma 3.13. It follows that

$$
\begin{aligned}
&\int_{|y-t|<2r} |K_{\psi,\alpha}(x-t) - K_{\psi,\alpha}(y-t)|^{2(1-a)}dt \\
&\leq \left(1 \vee 2^{2(1-a)-1}\right) \left(\int_{|x-t|<3r} |K_{\psi,\alpha}(x-t)|^{2(1-a)}dt + \int_{|y-t|<2r} |K_{\psi,\alpha}(y-t)|^{2(1-a)}dt \right) \\
&\leq 2\left(1 \vee 2^{2(1-a)-1}\right) \int_{|z|<3r} |K_{\psi,\alpha}(z)|^{2(1-a)}dz \\
&\leq 2\left(1 \vee 2^{2(1-a)-1}\right) c_1^{2(1-a)} \int_{|z|<3r} \left(\frac{1}{|z|^n \phi^{\frac{\alpha}{2}}(|z|^{-2})} \right)^{2(1-a)} dz, \\
&\overset{\textbf{(H1)}}{\leq} \frac{2\left(1 \vee 2^{2(1-a)-1}\right) c_1^{2(1-a)}}{a_1^{\alpha(1-a)}} \frac{(3r)^{-2\alpha\delta_1(1-a)}}{\left(\phi^{\frac{\alpha}{2}}((3r)^{-2})\right)^{2(1-a)}} \int_{|z|<3r} \left(\frac{1}{|z|^{n-\alpha\delta_1}} \right)^{2(1-a)} dz \\
&\overset{(3.4)}{\leq} \frac{2\left(1 \vee 2^{2(1-a)-1}\right) c_1^{2(1-a)}}{a_1^{\alpha(1-a)}} \frac{(3r)^{-2\alpha\delta_1(1-a)}}{\left(3^{-\alpha}\phi^{\frac{\alpha}{2}}(r^{-2})\right)^{2(1-a)}} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^{3r} z^{2(1-a)(\alpha\delta_1-n)+n-1}dz \\
&\leq \frac{2\left(1 \vee 2^{2(1-a)-1}\right) c_1^{2(1-a)} 3^{2(1-a)(\alpha-n)+n} 2\pi^{\frac{n}{2}}}{a_1^{\alpha(1-a)}(2(1-a)(\alpha\delta_1-n)+n)\Gamma(\frac{n}{2})} \frac{r^n}{\left(r^n\phi^{\frac{\alpha}{2}}(r^{-2})\right)^{2(1-a)}},
\end{aligned}
$$

for a such that

$$
2(1-a)(n-\alpha\delta_1) < n. \qquad (3.24)
$$

Analogously, if

$$
2a(n-\alpha\delta_1) < d \qquad (3.25)
$$

it follows that for all $t \in \mathbb{R}^n$

$$\begin{aligned}
&\iint\limits_{\substack{|x-y|<r\\|y-t|<2r}} |K_{\psi,\alpha}(x-t)-K_{\psi,\alpha}(y-t)|^{2a}\mu(dx)\mu(dy)\\
&\leq (1\vee 2^{2a-1})\left(\iint\limits_{\substack{|x-y|<r\\|y-t|<2r}} |K_{\psi,\alpha}(x-t)|^{2a}\mu(dx)\mu(dy)+\iint\limits_{\substack{|x-y|<r\\|y-t|<2r}} |K_{\psi,\alpha}(y-t)|^{2a}\mu(dx)\mu(dy)\right)\\
&\leq 2(1\vee 2^{2a-1})\mu(B(0,r))\int\limits_{|z|<3r} |K_{\psi,\alpha}(z)|^{2a}\mu(dz)\\
&\overset{\text{Lem3.13}}{\leq} 2(1\vee 2^{2a-1})\mu(B(0,r))c_1^{2a}\int\limits_{|z|<3r}\left(\frac{1}{|z|^n\phi^{\frac{\alpha}{2}}(|z|^{-2})}\right)^{2a}\mu(dz)\\
&\overset{\textbf{(H1)}}{\leq} \frac{2(1\vee 2^{2a-1})c_1^{2a}}{a_1^{\alpha a}}\frac{\mu(B(0,r))(3r)^{-2\alpha a\delta_1}}{\left(\phi^{\frac{\alpha}{2}}((3r)^{-2})\right)^{2a}}\int\limits_{|z|<3r}\frac{1}{|z|^{2a(n-\alpha\delta_1)}}\mu(dz)\\
&\overset{(3.4)}{\leq} \frac{2(1\vee 2^{2a-1})c_1^{2a}}{a_1^{\alpha a}}\frac{\mu(B(0,r))(3r)^{-2\alpha a\delta_1}}{\left(3^{-\alpha}\phi^{\frac{\alpha}{2}}(r^{-2})\right)^{2a}}\int_0^{3r} z^{2a(\alpha\delta_1-n)+d-1}dz\\
&\leq \frac{2(1\vee 2^{2a-1})c_1^{2a}3^{2a(\alpha-n)+d}}{a_1^{\alpha a}(2a(\alpha\delta_1-n)+d)}\frac{\mu(B(0,r))r^d}{\left(r^n\phi^{\frac{\alpha}{2}}(r^{-2})\right)^{2a}}.
\end{aligned}$$

It follows that for some $\tilde{c}_1>0$

$$\begin{aligned}
A &\leq \tilde{c}_1\frac{r^n}{\left(r^n\phi^{\frac{\alpha}{2}}(r^{-2})\right)^{2(1-a)}}\frac{r^{2d}}{\left(r^n\phi^{\frac{\alpha}{2}}(r^{-2})\right)^{2a}}\int f^2(t)dt\\
&= \tilde{c}_1\frac{r^{2d-n}}{\phi^{\alpha}(r^{-2})}||f||^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

For the second part, integral B, by the mean value theorem and Lemma 3.13 for $c_2=c_2(\phi,\alpha,n,3)$ it follows that

$$\begin{aligned}
&\iint\limits_{\substack{|x-y|<r\\2r<|y-t|}} |K_{\psi,\alpha}(x-t)-K_{\psi,\alpha}(y-t)|^{2a}\mu(dx)\mu(dy)\\
&\leq c_2^{2a}r^{2a}\iint\limits_{\substack{|x-y|<r\\2r<|y-t|}}\left(\frac{1}{|z_{x,y}|^{n+1}\phi^{\frac{\alpha}{2}}(|z_{x,y}|^{-2})}\right)^{2a}\mu(dx)\mu(dy),\\
&\overset{\textbf{(H)}}{\leq} c_2^{2a}r^{2a}\frac{\tilde{a}_6^{\alpha a}}{\left(r^{\alpha\delta}\phi^{\frac{\alpha}{2}}(r^{-2})\right)^{2a}}\iint\limits_{\substack{|x-y|<r\\2r<|y-t|}}\left(\frac{1}{|z_{x,y}|^{n+1-\alpha\delta}}\right)^{2a}\mu(dx)\mu(dy) \qquad (3.26)
\end{aligned}$$

where $z_{x,y} = y-t+\theta_{x,y}(x-y)$ for some $\theta_{x,y} \in (0,1)$ and $\delta = \delta_2 \vee \delta_4$. Since $|z_{x,y}| \leq 2+r < 3$ and

$$|z_{x,y}| \geq |y-t| - |x-y| \geq |y-t| - r \geq \frac{|y-t|}{2}.$$

it follows that (3.26) is less than

$$\begin{aligned}
&\leq c_2^{2a} \frac{a_6^{\alpha a} 2^{2a(n+1-\alpha\delta)}}{\left(\phi^{\frac{\alpha}{2}}(r^{-2})\right)^{2a}} r^{2a(1-\alpha\delta)} \iint\limits_{\substack{|x-y|<r \\ 2r<|y-t|}} \left(\frac{1}{|y-t|^{n+1-\alpha\delta}}\right)^{2a} \mu(dx)\mu(dy) \\
&\leq c_2^{2a} \frac{a_6^{\alpha a} 2^{2a(n+1-\alpha\delta)}}{\left(\phi^{\frac{\alpha}{2}}(r^{-2})\right)^{2a}} r^{2a(1-\alpha\delta)} \mu(B(0,r)) \int_{2r}^{\infty} z^{d-1-2a(n+1-\alpha\delta)} dz \\
&= c_2^{2a} \frac{a_6^{\alpha a} 2^{2a(n+1-\alpha\delta)}}{\left(\phi^{\frac{\alpha}{2}}(r^{-2})\right)^{2a}} r^{2a(1-\alpha\delta)} \mu(B(0,r)) \frac{(2r)^{d-1-2a(n+1-\alpha\delta)}}{2a(n+1-\alpha\delta)-d} \\
&\leq c_2^{2a} \frac{a_6^{\alpha a} 2^d}{2a(n+1-\alpha\delta)-d} \frac{\mu(B(0,r)) r^d}{\left(r^n \phi^{\frac{\alpha}{2}}(r^{-2})\right)^{2a}}
\end{aligned}$$

if

$$2a(n+1-\alpha\delta) > d. \tag{3.27}$$

Similarly, if

$$2(1-a)(n+1-\alpha\delta) > n \tag{3.28}$$

then for x and y such that $|x-y| < r$

$$\int\limits_{2r<|y-t|} |K_{\psi,\alpha}(y-t) - K_{\psi,\alpha}(x-t)|^{2(1-a)} dt \leq \frac{c_2^{2(1-a)} a_6^{\alpha(1-a)} 2^n}{n - 2(1-a)(n+1-\alpha\delta)} \frac{r^n}{\left(r^n \phi^{\frac{\alpha}{2}}(r^{-2})\right)^{2(1-a)}}.$$

In the same way as in the estimate of A, this implies that there exists a constant $\tilde{c}_2 > 0$ such that

$$B \leq \tilde{c}_2 \frac{r^{2d-n}}{\phi(r^{-2})} ||f||^2_{L^2(\mathbb{R}^n)}.$$

Since $\frac{n-d}{2} < \alpha\delta_1 \leq \alpha\delta < \frac{n-d}{2} + 1$ it follows that

$$\left(\frac{d}{2(n+1-\alpha\delta)}, 1 - \frac{n}{2(n+1-\alpha\delta)}\right) \cap \left(1 - \frac{n}{2(n-\alpha\delta_1)}, \frac{d}{2(n-\alpha\delta_1)}\right) \neq \emptyset,$$

so we can choose a such that (3.24), (3.25), (3.27) and (3.28) hold. Combining the bounds for A and B we get the statement of the Lemma. □

Proof of Theorem 3.25: Let Ru be the pointwise restriction on D of the *strictly defined* function corresponding to $u \in H^{\psi,1}(\mathbb{R}^n)$, i.e.

$$Ru(x) = \lim_{r \to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} u(y)dy,\ x \in D.$$

Since $u \in L^1_{\text{loc}}$, by the Lebesgue differentiation theorem it holds that $Ru = u$ a.e. on D. Using the classical interpolation theorem for a special class of spaces associated with $H^{\psi,1}(\mathbb{R}^n)$ we will show that the restriction operator R from $H^{\psi,1}(\mathbb{R}^n)$ to $H(D, \lambda_D)$ is continuous, i.e. that there exists a constant $\tilde{c}_1 > 0$ such that

$$||Ru||^{1,1}_{(2),D,\lambda_D} \leq \tilde{c}_1 ||u||_{\psi,1},\ \forall u \in H^{\psi,1}(\mathbb{R}^n), \tag{3.29}$$

where $\|\cdot\|^{1,1}_{(2),D,\lambda_D}$ is defined in (3.17). Denote by $a_j(x,y) = |Ru(x) - Ru(y)| 1_{|x-y|<2^{-j}}$, $j \in \mathbb{N}_0$ and $L = L^2(D \times D, \lambda_D \times \lambda_D)$. Since $Ru = u$ a.e. on D for every $u \in L^2(\mathbb{R}^n)$, by Lemma 3.26 there exists a constant $\tilde{c}_2 > 0$ such that

$$\sup_{j \in \mathbb{N}_0} \left(\phi^\alpha \left(2^{2j}\right) 2^{j(2d-n)} \iint\limits_{|x-y|<2^{-j}} |Ru(x) - Ru(y)|^2 \mu(dx)\mu(dy) \right) \leq \tilde{c}_2 ||f||^2_{L^2(\mathbb{R}^n)}$$

for all $u = K_{\psi,\alpha} * f$, $f \in L^2(\mathbb{R}^n)$ and $\alpha > 0$ satisfying condition (3.22), i.e.

$$0 < \alpha\delta_1 \leq \alpha(\delta_2 \vee \delta_4) < 1.$$

For such α it follows that

$$(a_j)_j \in l^{\phi;\alpha}_\infty(L) = \{(\xi_j)_{j \in \mathbb{N}_0} : \xi_j \in L, ||\xi||_{l^{\phi;\alpha}_\infty(L)} = \sup_{j \in \mathbb{N}_0} \phi^\alpha \left(2^{2j}\right) 2^{nj} ||\xi_j||^2_L < \infty\}$$

and that the operator T

$$Tu = (a_j)_{j \in \mathbb{N}_0}$$

is bounded from $H^{\psi,\alpha}(\mathbb{R}^n) = \{K_{\psi,\alpha} * f : f \in L^2(\mathbb{R}^n)\}$ to $l^{\phi,\alpha}_\infty(L)$, i.e.

$$||(a_j)_{j \in \mathbb{N}_0}||_{l^{\phi;\alpha}_\infty(L)} \leq \tilde{c}_2 ||K_{\psi,\alpha} * f||_{\psi,\alpha} := \tilde{c}_2 ||f||_{L^2(\mathbb{R}^n)}.$$

We can choose $\alpha_0 < 1 < \alpha_1$ such that

$$0 < \delta_1\alpha_0 \leq (\delta_2 \vee \delta_4)\alpha_1 < 1,$$

for which the operator T is bounded from $H^{\psi,\alpha_i}(\mathbb{R}^n)$ to $l^{\phi,\alpha_i}_\infty(L)$, $i = 0, 1$.

As in [Tri78, Section 1.3, p.23] we define the K-interpolation space $(X_1, X_2)_{\theta,p}$ of Banach spaces X_1 and X_2 for some $\theta \in (0,1)$ and $1 \leq p \leq \infty$ as

$$\begin{aligned}
(X_1, X_2)_{\theta,p} &= \{a : \, a \in X_1 + X_2, \, ||a||_{(X_1,X_2)_{\theta,p}} < \infty\} \\
||a||_{(X_1,X_2)_{\theta,p}} &= \left(\int_0^\infty (t^{-\theta} K(t,a))^q \frac{dt}{t} \right)^{\frac{1}{q}}, \quad 1 \leq p < \infty \\
||a||_{(X_1,X_2)_{\theta,p}} &= \sup_{0<t<\infty} t^{-\theta} K(t,a), \quad p = \infty
\end{aligned}$$

where the Peetre K-functional is defined by

$$K(t,a) = \inf_{a=a_1+a_2} (||a_1||_{X_1} + t||a_2||_{X_2}).$$

Let $\theta = \frac{\alpha_1 - 1}{\alpha_1 - \alpha_0} \in (0,1)$. By [Tri78, Theorem 1.3.3(a)] (also [AC10, Lemma 4.1]), operator T is bounded from $(H^{\psi,\alpha_0}(\mathbb{R}^n), H^{\psi,\alpha_1}(\mathbb{R}^n))_{\theta,2}$ to $(l_\infty^{\phi,\alpha_0}(L), l_\infty^{\phi,\alpha_1}(L))_{\theta,2}$. By a version of [Tri78, Theorem 1.18.2],

$$(l_\infty^{\phi,\alpha_0}(L), l_\infty^{\phi,\alpha_1}(L))_{\theta,2} = l_2^{\phi,1}(L)$$

and

$$(l_2^{\phi,\alpha_0}(L^2(\mathbb{R}^n)), l_2^{\phi,\alpha_1}(L^2(\mathbb{R}^n)))_{\theta,2} = l_2^{\phi,1}(L^2(\mathbb{R}^n)), \tag{3.30}$$

where

$$l_2^{\phi,\alpha}(L) = \left\{ (\xi_j)_{j\in\mathbb{N}_0} : \xi_j \in L, ||\xi||_{l_2^{\phi}(L)} = \left(\sum_{j\in\mathbb{N}_0} \phi^\alpha \left(2^{2j}\right) 2^{nj} ||\xi_j||_L^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

This result has been also proved in [CF88, Theorem 5.2] in a more general setting. Furthermore, by [CF88, Theorem 2.5 and Theorem 3.4] it follows that the space $H^{\psi,\alpha}(\mathbb{R}^n)$ is a retract of the space $l_2^{\phi,\alpha}(L^2(\mathbb{R}^n))$ and therefore by [CF88, Theorem 5.3] the interpolation identity

$$(H^{\psi,\alpha_0}(\mathbb{R}^n), H^{\psi,\alpha_1}(\mathbb{R}^n))_{\theta,2} = H^{\psi,1}(\mathbb{R}^n)$$

follows from (3.30). This result was shown using the so-called *retraction* and *co-retraction method* and [CF88, Theorem 5.2].

Therefore, there exists a constant $\tilde{c}_3 > 0$ such that

$$\sum_{j=0}^{\infty} \phi\left(2^{2j}\right) 2^{j(2d-n)} \iint\limits_{|x-y|<2^{-j}} |Ru(x) - Ru(y)|^2 \lambda_D(dx)\lambda_D(dy) \leq \tilde{c}_3 ||u||_{\psi,1}^2$$

which implies (3.29).

Next we define the operator $\tilde{R}$ on $H^{a,1}(\mathbb{R}^{n+1})$ as

$$\tilde{R}u(x) = \lim_{r\to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} u(y)dy,\ x \in \tilde{D}.$$

By calculations as in Lemma 3.26, there exists a constant $\tilde{c}_4 > 0$ such that

$$||u||_{L^2(\tilde{D},\mu)} \leq \tilde{c}_4 ||f||_{L^2(\mathbb{R}^{n+1})}$$

for every $u = K_a * f$, $f \in L^2(\mathbb{R}^{n+1})$. By [FOT10, Theorem 2.1.3] every function $u \in H^{a,1}(\mathbb{R}^{n+1})$ has a quasi continuous modification. That means that there exists a function $\tilde{u} \in H^{a,1}(\mathbb{R}^{n+1})$ such that for every $\varepsilon > 0$ there exists an open set N such that

$$\mathrm{Cap}_{a,1}(N) = \inf\{||v||_{a,1} : v \in H^{a,1}(\mathbb{R}^{n+1}), v \geq 1 \text{ a.e. on } N\} < \varepsilon,$$

$\tilde{u} = u$ a.e. and the set of Lebesgue points for $\tilde{u}$ is of capacity zero. This means that outside of some set N, $\mathrm{Cap}_{a,1}(N) = 0$, function u can be strictly defined and that $\tilde{R}u = u$. We will show that this implies $\tilde{R}u = u$ μ-a.e. It follows that for every $\varepsilon > 0$ there exists a function $v \in C_c^\infty(\mathbb{R}^{n+1})$ such that $v \geq 1$ on N and $||v||_{a,1} < \frac{\varepsilon}{\tilde{c}_4}$. Therefore,

$$\mu(N) \leq \left(\int_N |v(x)|^2 \mu(dy) \right)^{\frac{1}{2}} \leq \tilde{c}_4 ||v||_{a,1} < \varepsilon,$$

that is $\mu(N) = 0$. Next we show that $\tilde{R}$ is continuous, i.e. there exists a constant $\tilde{c}_4 > 0$ such that

$$||\tilde{R}u||^{1,1}_{(2),a,\tilde{D},\mu} \leq \tilde{c}_4 ||u||_{a,1},\ \forall u \in H^{a,1}(\mathbb{R}^{n+1}). \tag{3.31}$$

By applying Lemma 3.26 to the Bernstein function $\tilde{\phi}$ from (3.5) and 2α instead of α there exists a constant $\tilde{c}_5 > 0$ such that

$$\sup_{j\in\mathbb{N}_0} \left(a^\alpha(2^j) 2^{j(n-1)} \iint\limits_{|x-y|<2^{-j}} |\tilde{R}u(x) - \tilde{R}u(y)|^2 \mu(dx)\mu(dy) \right) \leq \tilde{c}_5 ||f||^2_{L^2(\mathbb{R}^{n+1})}$$

for all $u = K_{a,\alpha} * f$, $f \in L^2(\mathbb{R}^{n+1})$ and $\alpha > 0$ satisfying condition (3.22),

$$\frac{1}{2} < \alpha \left(\delta_1 + \frac{1}{2} \right) \leq \alpha \left(\delta_2 \vee \delta_4 + \frac{1}{2} \right) < \frac{3}{2}.$$

The continuity of $\tilde{R}$ follows using the same interpolation argument as in the case of the restriction operator R.

Analogously, since $\mathbb{R}^n \times \{0\}$ is a n-set in $\mathbb{R}^{n+1}$ the restriction operator $\bar{R}$ from $H^{a,1}(\mathbb{R}^{n+1})$ to $H_a(\mathbb{R}^n \times \{0\}, \bar{\mu})$ is also continuous. □

Next we prove the first part of the *extension theorem*. Again, we assume the conditions from Theorem 3.23 and notation from Remark 3.24.

Theorem 3.27 There exists a continuous operator $\tilde{E}$ from $H_a(\tilde{D},\mu)$ to $H^{a,1}(\mathbb{R}^{n+1})$ such that for all $u \in H_a(\tilde{D},\mu)$

$$\tilde{E}u = u, \ \mu\text{-a.e. on } \tilde{D}.$$

Proof. If B is a d-set in $\mathbb{R}^{n+1}$ then by [JW84, Proposition VIII.1.1] the closure $\overline{B}$ of B is also a d-set and $\mu(\overline{B} \setminus B) = 0$ for every d-measure μ. Therefore, it is enough to prove the theorem for a closed n-set $\tilde{D}$.

We define the operator $\tilde{E}$ from $H_a(\tilde{D},\mu)$ to $H^{a,1}(\mathbb{R}^{n+1})$ using the Whitney decomposition of $\tilde{D}^c$ with some additional properties. Denote by x_i the center of the cube Q_i and by l_i and s_i its diameter and the side length. Let $\{Q_i\}_{i\in\mathbb{N}}$ be a collection of closed cubes, with disjoint interiors and sides parallel to the axes such that $\tilde{D}^c = \cup Q_i$, $s_i = 2^{-M_i}$ for some $M_i \in \mathbb{Z}$ and

$$l_i \le d(Q_i, \tilde{D}) \le 4l_i.$$

Let $\varepsilon \in (0, \frac{1}{4})$ and denote by $Q_i^* = (1+\varepsilon)Q_i$ the cube with the same center as Q_i expanded by factor $1+\varepsilon$. If $x \in Q_k \cap Q_i^*$ then

$$1/4 s_k \le s_i \le 4 s_k \tag{3.32}$$

and Q_i and Q_k touch each other. This implies that every point in $\tilde{D}^c$ is covered by N_0 cubes Q_i^*, where $N_0 \in \mathbb{N}$ depends only on n.

By [JW84, Section I.2.3] we can associate with decomposition $\{Q_i^*\}$ a partition of unity $\{\varphi_i\}_{i\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$, i.e. a family of nonnegative functions with the following properties:

$$\begin{aligned}
&\operatorname{supp} \varphi_i \subset Q_i^*,\\
&\sum \varphi_i = 1 \text{ on } \tilde{D}^c,\\
&|D^j \varphi_i| \le \tilde{c} l_i^{-|j|} \text{ for some } \tilde{c} > 0.
\end{aligned} \tag{3.33}$$

Let $\omega_i = \mu(B(x_i, 6l_i))^{-1}$ and $I = \{i \in \mathbb{N} : s_i \le 1\}$. Note that $c_2^{-1}(6l_i)^{-n} \le \omega_i \le c_1^{-1}(6l_i)^{-n}$, $i \in \mathbb{N}$, where c_1 and c_2 are constants from Definition 3.20. For $u \in H_a(\tilde{D},\mu)$ define

$$\tilde{E}u(x) = \begin{cases} u(x), & x \in \tilde{D} \\ \displaystyle\sum_{i\in I} \varphi_i(x)\omega_i \int_{|y-x_i|<6l_i} u(y)\mu(dy), & x \notin \tilde{D}. \end{cases}$$

Note that $\operatorname{supp} \tilde{E}u$ is bounded for every function u with bounded support. Since $\varphi_i \in C_c^\infty(\mathbb{R}^n)$, it follows that for $u \in C_c^\infty(\tilde{D})$

$$\tilde{E}u \in C_c^\infty(\mathbb{R}^n). \tag{3.34}$$

We will show that $\tilde{E}$ is a continuous operator from $(H_a(\tilde{D},\mu), ||\cdot||^{c,N}_{(2),a,\tilde{D},\mu})$ for some c and N to $(H^{a,1}(\mathbb{R}^{n+1}), ||\cdot||^{h_0}_{(1),a})$ for $h_0 = 2^{-6}$, i.e. that there exists a constant $\tilde{c}_1$ such that

$$||\tilde{E}u||^{h_0}_{(1),a} \leq \tilde{c}_1 ||u||^{c,N}_{(2),a,\tilde{D},\mu}, \quad \forall u \in H_a(\tilde{D},\mu). \tag{3.35}$$

Recall that by Lemma 3.22, norms of the form $||\cdot||^{c,N}_{(2),a,\tilde{D},\mu}$ are equivalent to the norm $||\cdot||_{(1),a,\tilde{D},\mu}$. Since $\tilde{D}$ is of Lebesgue measure zero in $\mathbb{R}^{n+1}$ it is enough to prove (3.35) for $\tilde{E}u 1_{\tilde{D}^c}$.

For every $x \in \tilde{D}^c$ there exists a k such that $x \in Q_k$. If $s_k > 4$ then by (3.33)

$$x \notin Q_i^* \text{ for all } i \in I \text{ and } \tilde{E}u(x) = 0.$$

Therefore it is enough to consider the case when $s_k \leq 4$. Also for $s_k < 1/4$,

$$\sum_i \varphi_i(x) = \sum_{i\in I} \varphi_i(x).$$

Let $x \in Q_k$ and let $i \in I$ be such that $\phi_i(x) \neq 0$. Then for all $y \in B(x_i, 6l_i)$ we have

$$|y - x_k| \leq |y - x_i| + |x_i - x| + |x - x_k| \leq 6l_i + l_i + l_k \leq 29 l_k, \tag{3.36}$$

which implies that

$$\begin{aligned}
|\tilde{E}u(x)| &\leq \sum_{i\in I} \varphi_i(x)\omega_i \int_{|y-x_i|<6l_i} |u(y)|\mu(dy) \\
&\leq \sum_{i\in I} \varphi_i(x) c_1^{-1} 6^{-n} l_i^{-n} \int_{|y-x_k|<29l_k} |u(y)|\mu(dy) \\
&\underset{(3.32)}{\leq} \sum_{i\in I} \varphi_i(x) c_1^{-1} 6^{-n} 4^n l_k^{-n} \int_{|y-x_k|<29l_k} |u(y)|\mu(dy) \\
&\leq c_1^{-1} 6^{-n} 4^n l_k^{-n} \int_{|y-x_k|<29l_k} |u(y)|\mu(dy) \\
&\leq c_1^{-1} 6^{-n} 4^n (c_2 29^n)^{\frac{1}{2}} \left(l_k^{-n} \int_{|y-x_k|<29l_k} u^2(y)\mu(dy) \right)^{1/2}.
\end{aligned}$$

Let $\Delta_j = \bigcup_{\{k: s_k = 2^{-j}\}} Q_k$. Note that there exists an integer N_1 depending only on n such that every point $y \in \tilde{D}^c$ is covered by at most N_1 balls $B(x_k, 29l_k)$ where $Q_k \subset \Delta_j$. This follows from the fact that $|x_k - x_{k'}| \geq 2^{-j}$ and $l_k = \sqrt{n+1}2^{-j}$, for all $Q_k, Q_{k'} \subset \Delta_j$. By previous calculations it follows that

$$\begin{aligned}
\int_{\tilde{D}^c} |\tilde{E}u(x)|^2 dx &= \sum_{j=-2}^{\infty} \sum_{Q_k \subset \Delta_j} \int_{Q_k} |\tilde{E}u(x)|^2 dx \\
&\leq \sum_{j=-2}^{\infty} \sum_{Q_k \subset \Delta_j} \int_{Q_k} \left(\frac{c_2}{c_1^2} \left(\frac{116}{9}\right)^n l_k^{-n} \int_{|y-x_k|<29l_k} u^2(y)\mu(dy) \right) dx \\
&= \sum_{j=-2}^{\infty} \sum_{Q_k \subset \Delta_j} \frac{c_2}{c_1^2} \left(\frac{116}{9}\right)^n l_k^{-n} s_k^{n+1} \int_{|y-x_k|<29l_k} u^2(y)\mu(dy) \\
&\leq \sum_{j=-2}^{\infty} \frac{c_2}{c_1^2} \left(\frac{116}{9\sqrt{n+1}}\right)^n 2^{nj} 2^{-(n+1)j} \sum_{Q_k \subset \Delta_j} \int_{|y-x_k|<29l_k} u^2(y)\mu(dy) \\
&\leq \frac{c_2}{c_1^2} \left(\frac{116}{9\sqrt{n+1}}\right)^n N_1 ||u||^2_{L^2(\tilde{D},\mu)} \sum_{j=-2}^{\infty} 2^{-j} \\
&\leq \frac{c_2 2^3 N_1}{c_1^2} \left(\frac{116}{9\sqrt{n+1}}\right)^n ||u||^2_{L^2(\tilde{D},\mu)} \quad ,
\end{aligned}$$

that is

$$||\tilde{E}u 1_{\tilde{D}^c}||^2_{L^2(\mathbb{R}^{n+1})} \leq \left(\frac{c_2 2^3 N_1}{c_1^2} \left(\frac{116}{9\sqrt{n+1}}\right)^n + 1 \right) ||u||^2_{L^2(\tilde{D},\mu)}. \tag{3.37}$$

Next, for $x \in \Delta_i$, $y \in \Delta_j$ and $|x-y| < 2^{-i}/2$ we have

$$\begin{aligned}
2^{-j}\sqrt{n+1} &\leq d(\Delta_j, \tilde{D}) \leq d(y, \tilde{D}) \leq d(x,y) + d(x, \tilde{D}) \\
&\leq d(x,y) + 5\sqrt{n+1}2^{-i} \leq \frac{11}{2}\sqrt{n+1}2^{-i},
\end{aligned}$$

so $j \geq i-2$. Analogously, $\sqrt{n+1}2^{-i} \leq 5\sqrt{n+1}2^{-j} + \frac{1}{2}2^{-i}$ so

$$2^{-i} \leq 5\frac{\sqrt{2}}{\sqrt{2}-\frac{1}{2}} 2^{-j} < 2^{-j+3},$$

that is $j \leq i+2$. Therefore,

$$x \in \Delta_i,\ |x-y| < 2^{-i}/2 \Rightarrow y \in \bigcup_{j=i-2}^{i+2} \Delta_j. \tag{3.38}$$

Since $\tilde{E}u(x) = 0$ for for $x \in \Delta_i$, $i \leq -3$ it follows that for $x \in \Delta_i$, $i \leq -5$,

$$\tilde{E}u(y) = 0 \text{ if } |x-y| < 2^{-i}/2$$

and

$$\Delta_h^2(\tilde{E}u)(x) = 0 \text{ if } |h| < 2^5/4.$$

Therefore,

$$\begin{aligned}
&\int\limits_{\tilde{D}^c}\int\limits_{|h|<h_0} |\Delta_h^2(\tilde{E}u)(x)|^2 \frac{a\left(|h|^{-1}\right)}{|h|^{n+1}} dh\, dx \\
&\leq \sum_{i=-4}^{\infty} \iint\limits_{\substack{x\in\Delta_i\\ |h|<2^{-4}/4}} |\Delta_h^2(\tilde{E}u)(x)|^2 \frac{a\left(|h|^{-1}\right)}{|h|^{n+1}} dh\, dx \\
&\leq \sum_{i=-4}^{\infty} \iint\limits_{\substack{x\in\Delta_i\\ |h|<2^{-i}/4}} |\Delta_h^2(\tilde{E}u)(x)|^2 \frac{a\left(|h|^{-1}\right)}{|h|^{n+1}} dh\, dx \\
&+ \sum_{i=5}^{\infty} \iint\limits_{\substack{x\in\Delta_i\\ 2^{-i}/4\leq|h|<2^{-4}/4}} |\Delta_h^2(\tilde{E}u)(x)|^2 \frac{a\left(|h|^{-1}\right)}{|h|^{n+1}} dh\, dx \\
&= A + B.
\end{aligned}$$

Let $h_i = 2^{-i}/4$. Then

$$\begin{aligned}
B &= \sum_{i=5}^{\infty}\sum_{m=4}^{i-1} \int\limits_{h_{m+1}\leq|h|<h_m} \frac{a\left(|h|^{-1}\right)}{|h|^{n+1}} \int\limits_{x\in\Delta_i} |\Delta_h^2(\tilde{E}u)(x)|^2 dx\, dh \\
&= \sum_{m=4}^{\infty}\sum_{i=m+1}^{\infty} \int\limits_{h_{m+1}\leq|h|<h_m} \frac{a\left(|h|^{-1}\right)}{|h|^{n+1}} \int\limits_{x\in\Delta_i} |\Delta_h^2(\tilde{E}u)(x)|^2 dx\, dh \\
&= \sum_{m=4}^{\infty} \int\limits_{h_{m+1}\leq|h|<h_m} \frac{a\left(|h|^{-1}\right)}{|h|^{n+1}} \int\limits_{x\in F_{m+1}} |\Delta_h^2(\tilde{E}u)(x)|^2 dx\, dh \\
&\overset{(3.4)}{\leq} 2^{3n+9} \sum_{m=4}^{\infty} a\left(2^m\right) 2^{m(n+1)} \iint\limits_{\substack{x\in F_{m+1}\\ h_{m+1}\leq|h|<h_m}} |\Delta_h^2(\tilde{E}u)(x)|^2 dx\, dh.
\end{aligned}$$

Similarly as in (3.38), for $x \in F_{i+1} = \bigcup_{j=i+1}^{\infty} \Delta_j$ and $|h| < h_i$ it follows that $x, x+h, x+2h \in F_{i-2}$. Since

$$|\Delta_h^2(\tilde{E}u)(x)|^2 \leq 2\left(|\tilde{E}u(x) - \tilde{E}u(x+h)|^2 + |\tilde{E}u(x+h) - \tilde{E}u(x+2h)|^2\right),$$

$$B \leq 2^{3n+11} \sum_{m=4}^{\infty} a\left(2^m\right) 2^{m(n+1)} \iint\limits_{\substack{x,y\in F_{m-2}\\ |x-y|<h_m}} (\tilde{E}u(x) - \tilde{E}u(y))^2 dx\, dy.$$

For $x \in \Delta_k$ and $y \in \Delta_m$, $k, m \geq 2$, since $\sum_{i\in I} \varphi_i(x) = \sum_i \varphi_i(x) = 1$ and $\sum_{i\in I} \varphi_i(y) = \sum_i \varphi_i(y) = 1$ it follows that

$$|\tilde{E}u(x) - \tilde{E}u(y)| = \left| \sum_i \varphi_i(x)\omega_i \int\limits_{|s-x_i|<6l_i} u(s)\mu(ds) - \sum_j \varphi_j(y)\omega_j \int\limits_{|t-x_j|<6l_j} u(t)\mu(dt) \right|$$

$$= \left| \sum_i \sum_j \varphi_i(x)\varphi_j(y)\omega_i\omega_j \iint\limits_{\substack{|s-x_i|<6l_i \\ |t-x_j|<6l_j}} u(s)\mu(ds)\mu(dt) - \sum_j \sum_i \varphi_j(y)\varphi_i(x)\omega_j\omega_i \iint\limits_{\substack{|t-x_j|<6l_j \\ |s-x_i|<6l_i}} u(t)\mu(dt)\mu(ds) \right|$$

$$\leq \sum_i \sum_j \varphi_i(x)\varphi_j(y)\omega_i\omega_j \iint\limits_{\substack{|s-x_i|<6l_i \\ |t-x_j|<6l_j}} |u(s) - u(t)|\mu(ds)\mu(dt)$$

Since

$$x \in \Delta_k,\ \varphi_i(x) \neq 0 \Rightarrow \frac{1}{8} l_k \leq l_i \leq 64 l_k \tag{3.39}$$

there exist $\tilde{c}_2, \tilde{c}_3 > 0$ such that

$$\begin{aligned}
|\tilde{E}u(x) - \tilde{E}u(y)| &\overset{(3.36)}{\leq} \tilde{c}_2 \sum_i \sum_j \varphi_i(x)\varphi_j(y) l_k^{-n} l_m^{-n} \iint\limits_{\substack{|s-x_k|<29l_k \\ |t-x_m|<29l_m}} |u(s) - u(t)|\mu(ds)\mu(dt) \\
&= \tilde{c}_2 l_k^{-n} l_m^{-n} \iint\limits_{\substack{|s-x_k|<29l_k \\ |t-x_m|<29l_m}} |u(s) - u(t)|\mu(ds)\mu(dt) \\
&\leq \tilde{c}_3 \left(l_k^{-n} l_m^{-n} \iint\limits_{\substack{|s-x_k|<29l_k \\ |t-x_m|<29l_m}} (u(t) - u(s))^2 \mu(dt)\mu(ds) \right)^{1/2}.
\end{aligned}$$

Here x_k, x_m are the centers and l_k, l_m diameters of cubes $Q_p \subset \Delta_k$ and $Q_r \subset \Delta_m$ containing x and y respectively. Now it follows that for $i \in \mathbb{N}$, $y \in \Delta_m$ and $k, m \geq 2$

$$\begin{aligned}
\int_{\substack{x\in\Delta_k, \\ |x-y|<2^{-i}}} |\tilde{E}u(x) - \tilde{E}u(y)|^2 dx &\leq \tilde{c}_3^2 \int_{\substack{x\in\Delta_k \\ |x-y|<2^{-i}}} l_k^{-n} l_m^{-n} \iint\limits_{\substack{|s-x_k|<29l_k \\ |t-x_m|<29l_m}} (u(t) - u(s))^2 \mu(dt)\mu(ds) dx \\
&\leq \tilde{c}_3^2 l_k^{-n} l_m^{-n} N_0 \int_{x\in Q_p} dx \iint\limits_{\substack{|s-y|<30\sqrt{n+1}2^{-k}+2^{-i} \\ |t-x_m|<29l_m}} (u(t) - u(s))^2 \mu(dt)\mu(ds) \\
&\leq \tilde{c}_3^2 (n+1)^{-n/2} N_0 2^{k(n-(n+1))} l_m^{-n} \iint\limits_{\substack{|s-y|<c2^{-k}+2^{-i} \\ |t-x_m|<29l_m}} (u(t) - u(s))^2 \mu(ds)\mu(dt),
\end{aligned}$$

where $c = 30\sqrt{n+1}$. Analogously, we get

$$\iint\limits_{\substack{x\in\Delta_k,\, y\in\Delta_m\\ |x-y|<2^{-i}}} |\tilde{E}u(x)-\tilde{E}u(y)|^2 dxdy \le \tilde{c}_3^2(n+1)^{-n}N_0^2\, 2^{-k}2^{-m} \iint\limits_{|t-s|<2^{-i}+c2^{-k}+c2^{-m}} (u(t)-u(s))^2 d\mu(s)d\mu(t).$$

This implies that for $i \ge 4$

$$\begin{aligned}
\iint\limits_{\substack{x,y\in F_{i-2}\\ |x-y|<2^{-i}}} |\tilde{E}u(x)-\tilde{E}u(y)|^2 dx\,dy &= \sum_{k,m=i-2}^{\infty} \iint\limits_{\substack{x\in\Delta_k,\, y\in\Delta_m\\ |x-y|<2^{-i}}} |\tilde{E}u(x)-\tilde{E}u(y)|^2 dxdy \\
&\le \sum_{k,m=i-2}^{\infty} \tilde{c}_3^2(n+1)^{-n}N_0^2\, 2^{-k}2^{-m} \iint\limits_{|t-s|<2^{-i}+c2^{-k}+c2^{-m}} (u(t)-u(s))^2\mu(ds)\mu(dt) \\
&\le \tilde{c}_3^2(n+1)^{-n}N_0^2 \left(\sum_{k,m=i-2}^{\infty} 2^{-k}2^{-m}\right) \iint\limits_{|t-s|<(8c+1)2^{-i}} (u(t)-u(s))^2\mu(ds)\mu(dt) \\
&\le \tilde{c}_3^2(n+1)^{-n}N_0^2 2^{-2(i-2)+2} \iint\limits_{|t-s|<(8c+1)2^{-i}} (u(t)-u(s))^2\mu(ds)\mu(dt)
\end{aligned}$$

and therefore

$$\begin{aligned}
B &\le 2^{3n+11} \sum_{i=4}^{\infty} a\left(2^i\right) 2^{i(n+1)} \iint\limits_{\substack{x,y\in F_{i-2}\\ |x-y|<2^{-i}}} |\tilde{E}u(x)-\tilde{E}u(y)|^2 dx\,dy \\
&\le \tilde{c}_3^2(n+1)^{-n}N_0^2\, 2^{3n+17} \sum_{i=4}^{\infty} a\left(2^i\right) 2^{i(n-1)} \iint\limits_{|t-s|<(8c+1)2^{-i}} (u(t)-u(s))^2\mu(ds)\mu(dt) \\
&\le \tilde{c}_3^2(n+1)^{-n}N_0^2\, 2^{3n+17} \sum_{i=4}^{\infty} \phi\left(2^{2i}\right) 2^{in} \iint\limits_{|t-s|<(8c+1)2^{-i}} (u(t)-u(s))^2\mu(ds)\mu(dt).
\end{aligned} \tag{3.40}$$

Next, by the mean value theorem there exists a constant $\tilde{c}_4 > 0$ such that

$$\begin{aligned}
A &= \sum_{i=-4}^{\infty} \iint\limits_{\substack{x\in\Delta_i\\ |h|<2^{-i}}} |\Delta_h^2(\tilde{E}u)(x)|^2 \frac{a\left(|h|^{-1}\right)}{|h|^{n+1}} dx\,dh \\
&\le \tilde{c}_4 \sum_{i=-4}^{\infty} \int\limits_{|h|<2^{-i}} \left(\sum_{|j|=2} \int\limits_{\Delta_i} \int_0^1 \int_0^1 |h|^4 |D^j(\tilde{E}u)(x+(t_1+t_2)h)|^2 dt_1\,dt_2\,dx\right) \frac{a\left(|h|^{-1}\right)}{|h|^{n+1}} dh \\
&\overset{(3.38)}{\le} \tilde{c}_4 \sum_{i=-4}^{\infty} \int\limits_{|h|<2^{-i}} |h|^4 \left(\sum_{|j|=2} \int_0^1 \int_0^1 \int\limits_{F_{i-2}\setminus F_{i+3}} |D^j(\tilde{E}u)(z)|^2 dz\,dt_1\,dt_2\right) \frac{a\left(|h|^{-1}\right)}{|h|^{n+1}} dh
\end{aligned}$$

$$= \tilde{c}_4 \sum_{i=-4}^{\infty} \int\limits_{|h|<2^{-i}} |h|^4 \left(\sum_{|j|=2} \int\limits_{F_{i-2}\setminus F_{i+3}} |D^j(\tilde{E}u)(z)|^2 dz \right) \frac{a\left(|h|^{-1}\right)}{|h|^{n+1}} dh.$$

Since $D^j(\tilde{E}u)(z) = 0$ if $z \in \Delta_i$ and $i \leq -3$, we get

$$A \leq 5\tilde{c}_4 \sum_{i=-2}^{\infty} \int_{|h|<2^{-i}} \frac{a\left(|h|^{-1}\right)}{|h|^{n-3}} dh \sum_{|j|=2} \int_{\Delta_i} |D^j(\tilde{E}u)(z)|^2 dz.$$

By (3.8) there exists a constant $\tilde{a}_2 > 0$ such that

$$\frac{a(\lambda r)}{a(r)} \leq \tilde{a}_2 \lambda^{2\delta_2+1}, \quad \lambda \geq 1, r \geq \frac{1}{4}$$

so

$$\begin{aligned} A &\leq 5\tilde{c}_4\tilde{a}_2 \sum_{i=-2}^{\infty} a(2^i)2^{-i(2\delta_2+1)} \int_{|h|<2^{-i}} \frac{1}{|h|^{n-2+2\delta_2}} dh \sum_{|j|=2} \int_{\Delta_i} |D^j(\tilde{E}u)(z)|^2 dz \\ &\leq 5\tilde{c}_4\tilde{a}_2 \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})(3-2\delta_2)} \sum_{i=-2}^{\infty} a(2^i)2^{-4i} \sum_{|j|=2} \int_{\Delta_i} |D^j(\tilde{E}u)(z)|^2 dz. \end{aligned}$$

Take $z \in \Delta_k$ and $y \in \Delta_l$, $k, l \geq 2$ and $|j| = 2$. Since $\sum\limits_i D^j\varphi_i(z) = 0$ it follows that

$$\begin{aligned} |D^j(\tilde{E}u)(z)| &= \left| \sum_i D^j\varphi_i(z)\omega_i \int\limits_{|s-x_i|<6l_i} u(s)\mu(ds) \right| \\ &= \left| \sum_i D^j\varphi_i(z)\omega_i \int\limits_{|s-x_i|<6l_i} u(s)\mu(ds) - \sum_i D^j\varphi_i(z)\tilde{E}u(y) \right| \\ &= \left| \sum_i D^j\varphi_i(z)\omega_i \int\limits_{|s-x_i|<6l_i} (u(s) - \tilde{E}u(y))\mu(ds) \right| \\ &\leq \sum_i \sum_m |D^j\varphi_i(z)|\varphi_m(y)\omega_i\omega_m \iint\limits_{\substack{|s-x_i|<6l_i \\ |t-x_m|<6l_m}} |u(s)-u(t)|\mu(ds)\mu(dt) \\ &\leq \sum_i \sum_m |D^j\varphi_i(z)|\varphi_m(y) \left(\omega_i\omega_m \iint\limits_{\substack{|s-x_i|<6l_i \\ |t-x_m|<6l_m}} |u(s)-u(t)|^2\mu(ds)\mu(dt) \right)^{\frac{1}{2}}. \end{aligned}$$

There are at most N_0 indices i for which $z \in Q_i^*$ and $D^j\varphi_i(z) \neq 0$. By (3.32) and (3.33) $z \in Q_i^*$ implies $\omega_i \asymp l_k^{-n}$ and $|D^j\varphi_i(z)| \leq \tilde{c} l_i^{-|j|} \leq \tilde{c}\left(\frac{1}{4}l_k\right)^{-|j|}$. Also, by (3.39) $\omega_m \asymp l_l^{-n}$ for m such that $\varphi_m(y) \neq 0$. Therefore, there exists a constant $\tilde{c}_5 > 0$ such that

$$|D^j(\tilde{E}u)(z)| \overset{(3.36)}{\leq} \tilde{c}_5 N_0 \tilde{c} 4^2 l_k^{-2} \sum_m \varphi_m(y) \left(l_k^{-n} l_l^{-n} \iint\limits_{\substack{|s-x_k|<29l_k \\ |t-x_l|<29l_l}} |u(s)-u(t)|^2 \mu(ds)\mu(dt) \right)^{\frac{1}{2}}$$

$$\leq 16\tilde{c}_5 N_0 \tilde{c} l_k^{-2} \left(l_k^{-n} l_l^{-n} \iint\limits_{\substack{|s-x_k|<29l_k \\ |t-x_l|<29l_l}} |u(s)-u(t)|^2 \mu(ds)\mu(dt) \right)^{\frac{1}{2}}.$$

Let x_m and l_m be the center and the diameter of cube $Q_m \subset \Delta_i$, $i \geq 2$. Then there exists a constant $\tilde{c}_6 > 0$ such that for $z \in Q_m$

$$|D^j(\tilde{E}u)(z)|^2 \leq \frac{\tilde{c}_6}{(n+1)^{n+2}} 2^{4i+2in} \iint\limits_{\substack{|s-x_m|<29l_m \\ |t-x_m|<29l_m}} |u(s)-u(t)|^2 \mu(ds)\mu(dt).$$

and

$$\sum_{i=2}^{\infty} a(2^i) 2^{-4i} \sum_{|j|=2} \int_{\Delta_i} |D^j(\tilde{E}u)(z)|^2 dz$$

$$\leq \frac{\tilde{c}_6}{(n+1)^{n+2}} \sum_{i=2}^{\infty} a(2^i) 2^{-4i} \sum_{|j|=2} \sum_{Q_m \subset \Delta_i} \int_{Q_m} 2^{4i+2in} \iint\limits_{\substack{|s-x_m|<29l_m \\ |t-x_m|<29l_m}} |u(s)-u(t)|^2 \mu(ds)\mu(dt) dz$$

$$= \frac{\tilde{c}_6}{(n+1)^{n+2}} \sum_{i=2}^{\infty} a(2^i) 2^{-4i} \frac{(n+1)(n+2)}{2} \sum_{Q_m \subset \Delta_i} \int_{Q_m} 2^{4i+2in} \iint\limits_{\substack{|s-x_m|<29l_m \\ |t-x_m|<29l_m}} |u(s)-u(t)|^2 \mu(ds)\mu(dt) dz.$$

Since every $s \in \tilde{D}^c$ is covered by at most N_1 balls $B(x_m, 29l_m)$ the last line is less than

$$\frac{\tilde{c}_6(n+2)}{2(n+1)^{n+1}} N_1 \sum_{i=2}^{\infty} a(2^i) 2^{-4i} 2^{4i+2in} 2^{-i(n+1)} \iint\limits_{|s-t|<60\sqrt{n+1}2^{-i}} |u(s)-u(t)|^2 \mu(ds)\mu(dt)$$

$$= \frac{\tilde{c}_6(n+2)}{2(n+1)^{n+1}} N_1 \sum_{i=2}^{\infty} a(2^i) 2^{(2n-(n+1))i} \iint\limits_{|s-t|<60\sqrt{n+1}2^{-i}} |u(s)-u(t)|^2 \mu(ds)\mu(dt)$$

$$= \frac{\tilde{c}_6(n+2)}{2(n+1)^{n+1}} N_1 \sum_{i=2}^{\infty} \phi(2^{2i}) 2^{in} \iint\limits_{|s-t|<60\sqrt{n+1}2^{-i}} |u(s)-u(t)|^2 \mu(ds)\mu(dt). \tag{3.41}$$

For the remaining part in A, take $z \in \Delta_k$, $k \geq -2$. By the same arguments as before, there exists a constant $\tilde{c}_7 > 0$ such that

$$|D^j(\tilde{E}u)(z)| \leq \sum_i |D^j\varphi_i(z)|\omega_i \int\limits_{|s-x_i|<6l_i} |u(s)|\mu(ds) \overset{(3.33)}{\leq} \tilde{c} \sum_i l_i^{-2} \left(\omega_i \int\limits_{|s-x_i|<6l_i} |u(s)|^2\mu(ds) \right)^{\frac{1}{2}}$$
$$\leq \tilde{c}_7 2^{2k} \left(2^{nk} \int\limits_{|s-x_k|<29l_k} |u(s)|^2\mu(ds) \right)^{\frac{1}{2}}$$

and

$$\sum_{i=-2}^{1} \int_{\Delta_i} |D(\tilde{E}u)(z)|^2 dz \leq \tilde{c}_8 ||u||_{L^2(\tilde{D},\mu)}$$

for some $\tilde{c}_8 > 0$. This inequality together with (3.37), (3.40) and (3.41) implies (3.35) for $h_0 = 2^{-6}$, $N = 2$ and $c = 240\sqrt{n+1} + 1$. □

Next we have to prove that $\tilde{E}$ is truly the extension operator for $\tilde{R}$.

Theorem 3.28 For every $u \in H_a(\tilde{D}, \mu)$

$$\tilde{R}\tilde{E}u = u, \ \mu\text{-a.e.}$$

Proof. Take $t_0 \in \tilde{D}$, $u \in H_a(\tilde{D}, \mu)$ and $r > 0$ small enough. Similarly as in the proof of Theorem 3.27 it follows that there exist constants $\tilde{c}_1, \tilde{c}_2 > 0$ such that for $k \geq 2$

$$\int\limits_{\substack{x\in\Delta_k \\ |x-t_0|\leq r}} (\tilde{E}u(x) - u(t_0))^2 dx = \int\limits_{\substack{x\in\Delta_k \\ |x-t_0|\leq r}} \left(\sum_i \varphi_i(x)\omega_i \int_{|t-x_i|<6l_i} u(t)\mu(dt) - u(t_0) \right)^2 dx$$
$$\leq \tilde{c}_1 \int\limits_{\substack{x\in\Delta_k \\ |x-t_0|\leq r}} N_0^2 2^{2kn} \left(\int_{|t-x_k|<29l_k} |u(t) - u(t_0)|\mu(dt) \right)^2 dx$$
$$\leq \tilde{c}_2 2^{-k(n+1)} 2^{kn} \int_{|t-t_0|<r+29l_k} (u(t) - u(t_0))^2 \mu(dt).$$

Let $i_0 \geq 2$ be such that

$$l_{i_0} \leq r < l_{i_0-1}.$$

Since $t_0 \in \tilde{D}$, $\{x \in \tilde{D}^c : |x - t_0| < r\} \subset \bigcup_{i=i_0}^{\infty} \Delta_i$, and therefore

$$\int_{|x-t_0|\leq r} (\tilde{E}u(x) - u(t_0))^2 dx \leq \sum_{i=i_0}^{\infty} \tilde{c}_2 2^{-i} \int_{|t-t_0|<r+29l_i} (u(t) - u(t_0))^2 \mu(dt)$$

$$
\begin{aligned}
&\leq \sum_{i=i_0}^{\infty} \tilde{c}_2 2^{-i} \int_{|t-t_0|<30r} (u(t)-u(t_0))^2 \mu(dt) \\
&\leq \tilde{c}_2 2^{-i_0+1} \int_{|t-t_0|<30r} (u(t)-u(t_0))^2 \mu(dt) \\
&\leq \frac{2\tilde{c}_2}{\sqrt{n+1}} r \int_{|t-t_0|<30r} (u(t)-u(t_0))^2 \mu(dt) \\
&\overset{(3.4)}{\leq} \frac{2\tilde{c}_8 30^{n+2}}{\sqrt{n+1}} \frac{r^{n+1}}{\phi(r^{-2})} \int_{|t-t_0|<30r} (u(t)-u(t_0))^2 \frac{a(|t-t_0|^{-1})}{|t-t_0|^{n-1}} \mu(dt).
\end{aligned}
$$

Since $u \in H_a(\tilde{D}, \mu)$ the last integral is finite for μ-almost all t_0. Also, this integral is decreasing as r goes to 0. Therefore, since $\lim_{r\to 0} \frac{1}{\phi(r^{-2})} = 0$

$$
\begin{aligned}
|R\tilde{E}u(t_0) - u(t_0)| &= \lim_{r\to 0} \left| \frac{1}{\lambda(B(t_0,r))} \int_{B(t_0,r)} \tilde{E}u(x)dx - u(t_0) \right| \\
&\leq \lim_{r\to 0} \frac{\Gamma(\frac{n+1}{2}+1)}{\pi^{\frac{n+1}{2}}} \left(r^{-(n+1)} \int_{|x-t_0|\leq r} (\tilde{E}u(x) - u(t_0))^2 dx \right)^{1/2} = 0
\end{aligned}
$$

for μ almost all t_0. □

Finally, we can define the extension operator E and prove it is bounded.

Theorem 3.29 There exists a continuous extension operator E from $H(D, \lambda_D)$ to $H^{\psi,1}(\mathbb{R}^n)$.

Proof. Take $u \in H(D, \lambda_D)$ and let $\tilde{u}$ be the corresponding function in $H_a(\tilde{D}, \mu)$, Remark 3.24(ii). By Theorem 3.27 function $\tilde{u}$ can be extended to a function $\tilde{E}\tilde{u} \in H^{a,1}(\mathbb{R}^{n+1})$, which can then be restricted to a function in $H_a(\mathbb{R}^n \times \{0\}, \bar{\mu})$ applying the continuous restriction operator $\bar{R}$, Theorem 3.25. Since the function space $H_a(\mathbb{R}^n \times \{0\}, \bar{\mu})$ can be considered as $H^{\psi,1}(\mathbb{R}^n)$, we define the extension operator E as

$$
(Eu)(x) = (\bar{R}\tilde{E}\tilde{u})(x, 0),\ x \in \mathbb{R}^n.
$$

Note that by (3.34),

$$
x \in C_c^{\infty}(D) \Rightarrow Eu \in C_c^{\infty}(\mathbb{R}^n). \tag{3.42}
$$

Also the continuity of E follows from the continuity of the extension and restriction operators $\tilde{E}$ and $\bar{R}$.

Finally, we show that E is truly the extension operator for R. Since

$$
R\bar{R}u = \tilde{R}u
$$

for all $u \in C_c^\infty(\mathbb{R}^{n+1})$ and $C_c^\infty(\mathbb{R}^{n+1})$ is dense in $H^{a,1}(\mathbb{R}^{n+1})$ it follows that

$$REu = u$$

almost everywhere for all $u \in H^{\psi,1}(\mathbb{R}^n)$. $\square$

Proof of Theorem 3.23: The proof follows directly from Theorem 3.25 and Theorem 3.29 combined with Remark 3.24(i). $\square$

Finally, recall that $\mathcal{F}$ is the closure of $C_c^\infty(D)$ under the inner product $\mathcal{E}_1$. Therefore, the Dirichlet space $(\mathcal{F}, \sqrt{\mathcal{E}_1})$ is equivalent to $(H_0(D, \lambda_D), ||\cdot||_{(1),D,\lambda_D})$, where $H_0(D, \mu)$ is the closure of $C_c^\infty(D)$ in $(H^{\psi,1}(D, \mu), ||u||_{\psi,1,D,\mu})$.

3.4 The active reflected Dirichlet form and the boundary behavior of the censored process

Let $\psi(\xi) = \phi(|\xi|^2)$, where ϕ is a complete Bernstein function satisfying **(H1)** and **(H2)**. First we show that $(\mathcal{E}^{\text{ref}}, \mathcal{F}_{\text{a}}^{\text{ref}})$ is the active reflected Dirichlet form associated with $(\mathcal{E}, \mathcal{F})$ in the sense of [CF12, Theorem 6.2.13 and Section 6.3], when D is an arbitrary open set. The corresponding result in the stable case was proven in [BBC03, Theorem 2.2].

Definition 3.30 We say that the function f is *locally* in $\mathcal{F}$, $f \in \mathcal{F}_{\text{loc}}$ if for every relatively compact open set D_0 in D there exists a function $f_0 \in \mathcal{F}$ such that $f = f_0$ a.e. on D_0.

Theorem 3.31 Let D be an open set in $\mathbb{R}^n$. The Dirichlet form $(\mathcal{E}^{\text{ref}}, \mathcal{F}_{\text{a}}^{\text{ref}})$ is the active reflected Dirichlet form associated with $(\mathcal{E}, \mathcal{F})$, i.e.

$$\mathcal{F}_a^{\text{ref}} = \{u \in L^2(D) : u_k = ((-k) \vee u) \wedge k \in \mathcal{F}_{\text{loc}} \text{ and } \sup_k \mathcal{E}^{\text{ref}}(u_k.u_k) < \infty\}$$
$$\mathcal{E}^{\text{ref}}(u, u) = \lim_{k\to\infty} \mathcal{E}^{\text{ref}}(u_k, u_k).$$

Proof. Since

$$\{u \in L^2(D) : u_k = ((-k) \vee u) \wedge k \in \mathcal{F}_{\text{loc}}, \sup_k \mathcal{E}^{\text{ref}}(u_k.u_k) < \infty\} \subset \mathcal{F}_a^{\text{ref}}$$

it is enough to show that $u \in \mathcal{F}_a^{\text{ref}} \cap L^\infty(D) \subset \mathcal{F}_{\text{loc}}$. For any relatively compact open subset D_0 of D, there exists a relatively compact smooth open n-set U_0 such that $\overline{D}_0 \subset U_0 \subset D$ and a function $\varphi \in C_c^\infty(D)$ such that $\varphi \leq 1$, $\varphi = 1$ on D_0 and supp$[\varphi] \subset U_0$. Since U_0 is a smooth set and therefore a n-set on $\mathbb{R}^n$, by Theorem 3.23 we can extend the function $u1_{U_0}$ to a function $v \in H^{\psi,1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $u = v$ a.e. on U_0. Since $C_c^\infty(\mathbb{R}^n)$

is $\mathcal{C}_1$-dense in $H^{\psi,1}(\mathbb{R}^n)$, there is a sequence $\{v_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ $\mathcal{C}_1$-convergent to v. This implies that for some $k_0 \in \mathbb{N}$

$$\mathcal{C}_1(v_k, v_k) < \mathcal{C}_1(v, v) + 1, \quad \forall k \geq k_0$$

that is

$$\sup_k \mathcal{E}_1(\varphi v_k, \varphi v_k) \leq \sup_k \mathcal{C}_1(v_k, v_k) < \infty.$$

Hence by the Banach-Saks theorem there is a subsequence $\{\varphi v_{k_m}\}_{m\in\mathbb{N}}$ in $C_c^\infty(D)$ such that Cesàro means $(\frac{1}{m}\sum_{i=1}^{m} \varphi v_{k_m})_{m\in\mathbb{N}}$ are $\mathcal{E}_1$-convergent. Therefore the limit function f is in $\mathcal{F}$ and

$$f = \varphi v = u \text{ a.e. on } D_0.$$

This implies that $u \in \mathcal{F}_{\text{loc}}$. Lastly, by the Lebesgue dominated convergence theorem it follows that

$$\mathcal{E}^{\text{ref}}(u, u) = \lim_{k\to\infty} \mathcal{E}^{\text{ref}}(u_k, u_k).$$

□

By [CF12, Theorem 6.6.3] the active reflected Dirichlet form is a Silverstein extension of the corresponding regular Dirichlet form. This means that $\mathcal{F}_b = \mathcal{F} \cap L^\infty(\mathbb{R}^n)$ is an ideal in $\mathcal{F}^{\text{ref}}_{a,b} = \mathcal{F}^{\text{ref}}_a \cap L^\infty(\mathbb{R}^n)$, i.e.

$$\mathcal{F}_b \subset \mathcal{F}^{\text{ref}}_{a,b} \text{ and } fg \in \mathcal{F}_b \text{ for every } f \in \mathcal{F}_b,\ g \in \mathcal{F}^{\text{ref}}_{a,b}.$$

Furthermore, by [CF12, Theorem 6.6.5, Remark 6.6.7] a Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ is a Silverstein extension of a quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D)$ if and only if there exists a symmetric Hunt process Y^* associated with the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ that extends Y to some state space D^* which contains D as an $\mathcal{E}^*$-quasi-open subset of D^* up to an $\mathcal{E}$-polar set. Therefore, there exists a compactification D^* of D such that the active reflected Dirichlet form $(\mathcal{E}^{\text{ref}}, \mathcal{F}^{\text{ref}}_{\text{a}})$ is regular on $L^2(D^*)$ and we call the corresponding process Y^* the *reflected process* associated with the process Y. The set $D^* \setminus D$ is Lebesgue negligible, but not necessarily of zero $\mathcal{E}^{\text{ref}}$ capacity. Note that $\mathcal{F}$ is the $\mathcal{E}_1$-closure of $C_c^\infty(D)$, that is $\mathcal{F} = H_0^{\psi,1}(D)$. Therefore, the process Y^* killed upon leaving D has the same distribution as Y. We will use this relation to study the boundary behavior of the process Y, when D is an open n-set.

Remark 3.32 (i) Since every compact set is of finite capacity, by [FOT10, Theorem 4.2.1] a set A is $\mathcal{E}$-polar if and only if $\text{Cap}_Y(A) = 0$. This justifies the usage of the term *q.e.* instead of $\mathcal{E}$-q.e. The same is true for $\mathcal{E}^{\text{ref}}$-polar and $\mathcal{C}$-polar sets.

(ii) The two notions of polarity are also related in this case. Since X has an absolutely continuous transition density, so does the process Y (Remark 3.2) so by Remark 2.15 every $\mathcal{E}$-polar ($\mathcal{C}$-polar) is polar for the process Y (X).

First we state a results which is the analogue of [BBC03, Theorem 2.4], proved for the stable case.

Theorem 3.33 Let D be an open set in $\mathbb{R}^n$ with finite Lebesgue measure and ζ the lifetime of process Y. The following statements are equivalent

(i) $\mathbb{P}_x(\zeta < \infty) > 0$ for some (and hence for all) $x \in D$;

(ii) $\mathbb{P}_x(\zeta < \infty) = 1$ for some (and hence for all) $x \in D$;

(iii) $1 \notin \mathcal{F}$;

(iv) $\mathcal{F} \neq \mathcal{F}_a^{\mathrm{ref}}$.

Proof. Trivially, (ii) implies (i). If $1 \in \mathcal{F}$ then $\mathcal{E}(1,1) = 0$ so by [FOT10, Theorem 1.6.3] Y is recurrent and therefore conservative. This shows that (i) implies (iii). Statement (iii) implies (iv) since D has finite Lebesgue measure so $1 \in \mathcal{F}_a^{\mathrm{ref}}$ and $\mathcal{E}^{\mathrm{ref}}(1,1) = 0$. Also, process Y^* is irreducible, recurrent and therefore conservative. For the last implication, note that $D^* \setminus D$ is polar for Y^* if and only if

$$\mathbb{P}_x(\zeta < \infty) = \mathbb{P}_x(\sigma_{D^* \setminus D}^{Y^*} < \infty) = 0,$$

i.e. Y and Y^* are equivalent processes. Therefore, if $\mathcal{F} \neq \mathcal{F}_a^{\mathrm{ref}}$, $D^* \setminus D$ is non-polar for Y^* so by [FOT10, Theorem 4.7.1(iii)] $D^* \setminus D$ is visited by Y^* infinitely many times almost surely, so (iv) implies (ii). □

When D is an open n-set in $\mathbb{R}^n$, Theorem 3.23 says that the reflected Dirichlet form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_a^{\mathrm{ref}})$ is actually the trace Dirichlet form of $(\mathcal{C}, \mathcal{F}^{\mathbb{R}^n})$ on $\overline{D}$, see [FOT10, (6.2.4)]. Since $C_c(\mathbb{R}^n)$ is the special standard core in $(\mathcal{C}, \mathcal{F}^{\mathbb{R}^n})$, by [FOT10, Theorem 6.2.1] $C_c^{\infty}(\overline{D})$ is a special standard core for $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_a^{\mathrm{ref}})$, and therefore $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_a^{\mathrm{ref}})$ is a regular Dirichlet form on $\overline{D}$. This means that we can take $D^* = \overline{D}$ and that there exists a Hunt process Y^* on $\overline{D}$ such that Y can be represented as the process Y^* killed upon leaving D.

Since X is irreducible, by the construction of the censored process Y it follows that the processes Y and Y^* are also irreducible.

Remark 3.34 If $\mathcal{F} \subsetneq \mathcal{F}_a^{\mathrm{ref}}$ then Y is a proper subprocess of Y^* and ∂D is not polar for Y^*. This implies that

$$\mathbb{P}_x(Y_{\zeta-} \in \partial D, \zeta < \infty) > 0, \quad \forall x \in D.$$

Additionally, if D has finite Lebesgue measure, Y^* is recurrent and therefore ζ is finite almost surely and

$$\mathbb{P}_x(Y_{\zeta-} \in \partial D, \zeta < \infty) = 1, \quad \forall x \in D.$$

So we see that the question of boundary behavior of the censored process Y is related to $\mathcal{E}^{\mathrm{ref}}$-polarity of the boundary ∂D. The following theorem gives us the characterization of $\mathcal{E}^{\mathrm{ref}}$-polar sets. The corresponding result for the stable case was proven in [BBC03, Theorem 2.5].

Theorem 3.35 Let D be an open n-set in $\mathbb{R}^n$.

(i) A set $A \subset \overline{D}$ is $\mathcal{E}^{\text{ref}}$-polar if and only if it is polar for the process X.

(ii) A set $A \subset D$ is polar for the process Y if and only if it is polar for the process X.

(iii) If $A \subset \partial D$ is polar for the process X then

$$\mathbb{P}_x(Y_{\zeta-} \in A) = 0, \quad \forall x \in D.$$

Proof.

(i) Let $A \subset \overline{D}$ and R and E the restriction and extension operator from Remark 3.24. We will show that $\text{Cap}_{Y^*}(A) \asymp \text{Cap}_X(A)$ for every set $A \subset \overline{D}$. By [FOT10, Lemma 2.1.5] if

$$\mathcal{L}_U = \{u \in \mathcal{F}^{\mathbb{R}^n} : u \geq 1 \text{ a.e. on } U\} \neq \emptyset$$

then there exists a unique function $u_0 \in \mathcal{L}_U$ such that $\text{Cap}_X(U) = \mathcal{C}_1(u_0, u_0)$. Therefore,

$$\begin{aligned} \text{Cap}_X(U) &= \mathcal{C}_1(u_0, u_0) \geq \mathcal{E}_1^{\text{ref}}(Ru_0, Ru_0) \\ &\geq \inf\{\mathcal{E}_1^{\text{ref}}(u, u) : u \in \mathcal{L}_U^{\text{ref}}\} \\ &= \text{Cap}_{Y^*}(U). \end{aligned}$$

and

$$\begin{aligned} \text{Cap}_{Y^*}(A) &= \inf\{\text{Cap}_{Y^*}(U) : \ A \subset U, U \text{ is a relatively open set in } \overline{D}\} \\ &\leq \inf\{\text{Cap}_X(U) : \ A \subset U, U \text{ is a relatively open set in } \overline{D}\} \\ &\leq \inf\{\text{Cap}_X(U) : \ A \subset U \subset \mathbb{R}^n, U \text{ is open}\} \\ &= \text{Cap}_X(A). \end{aligned}$$

For the other inequality, take a compact subset K of $\overline{D}$. Since $C_c^\infty(\overline{D})$ is $\mathcal{E}_1^{\text{ref}}$-dense in $\mathcal{F}_a^{\text{ref}}$, by [FOT10, Lemma 2.2.7] it follows that

$$\text{Cap}_{Y^*}(K) = \inf\{\mathcal{E}_1^{\text{ref}}(u, u) : \ u \in C_c^\infty(\overline{D}),\ u \geq 1 \text{ on } K\}.$$

By Theorem 3.23 the extension operator E is continuous so there exists a constant $\tilde{c}_1 > 0$ such that

$$\mathcal{C}_1(Eu, Eu) \leq \tilde{c}_1 \mathcal{E}_1^{\text{ref}}(u, u).$$

Recall that by (3.34) and (3.42) $Eu \in C_c^\infty(\mathbb{R}^n)$ for every $u \in C_c^\infty(\overline{D})$. Therefore

$$\begin{aligned} \text{Cap}_{Y^*}(K) &\geq \tilde{c}_1^{-1} \inf\{\mathcal{C}_1(Eu, Eu) : \ u \in C_c^\infty(\overline{D}),\ u \geq 1 \text{ on } K\} \\ &\geq \tilde{c}_1^{-1} \inf\{\mathcal{C}_1(u, u) : \ u \in C_c^\infty(\mathbb{R}^n),\ u \geq 1 \text{ on } K\} \\ &= \tilde{c}_1^{-1} \text{Cap}_X(K). \end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{Cap}_{Y^*}(A) &\overset{(2.10)}{=} \sup\{\mathrm{Cap}_{Y^*}(K) : K \subset A,\, K \text{ is compact}\}\\ &\geq \tilde{c}_1^{-1} \sup\{\mathrm{Cap}_X(K) : K \subset A,\, K \text{ is compact}\}\\ &= \tilde{c}_1^{-1}\mathrm{Cap}_X(A).\end{aligned}$$

(ii) Since Y is the subprocess of Y^* killed upon leaving D by [FOT10, Theorem 4.4.3] a subset A in D is $\mathcal{E}$-polar if and only if it is $\mathcal{E}^{\mathrm{ref}}$-polar. Hence by (i) $A \subset D$ is polar for the process Y if and only if it is polar for X.

(iii) Let $A \subset \partial D$ be a polar set for X. By (i) it is $\mathcal{E}^{\mathrm{ref}}$-polar and therefore there exists a nearly measurable set B containing A such that

$$\mathbb{P}_x(\sigma_B^{Y^*} < \infty) = 0 \quad \text{for a.e. } x \in D.$$

Since $\sigma_B^{Y^*} \leq \inf\{t > 0 : Y^*_{t-} \in B\}$ almost surely, it follows that

$$\mathbb{P}_x(\text{ there exists a } t > 0 \text{ such that } Y^*_t \in A \text{ or } Y^*_{t-} \in A) = 0 \quad \text{for a.e. } x \in D$$

and therefore

$$\mathbb{P}_x(Y_{\zeta-} \in A) = 0 \quad \text{for a.e. } x \in D.$$

Since Y has a transition density p^Y this statement holds for every $x \in D$, that is

$$\begin{aligned}\mathbb{P}_x(Y_{\zeta-} \in A) &= \lim_{t\downarrow 0} \mathbb{P}_x(Y_{\zeta-} \in A, \zeta > t)\\ &= \lim_{t\downarrow 0} \int_D \mathbb{P}_y(Y_{\zeta-} \in A) p^Y(t,x,y)dy = 0.\end{aligned}$$

Therefore A is polar for Y.

□

Remark 3.36 The converse of Theorem 3.35(iii) is not true, [BBC03, Remark 2.2]. Take, for example, D to be the unit ball in $\mathbb{R}^2$ centered at x_0, $\psi(\xi) = |\xi|^\alpha$ and $\alpha \in (1, 2)$. Since D has positive and finite $(n-1)$-dimensional Hausdorff measure it follows that $\mathcal{H}_h(\partial D) = \infty$, where the gauge function is equal $h(x) = x^{n-\alpha}$. By [BBC03, Remark 2.2], $\mathrm{Cap}_X(\partial D) > 0$ and therefore Y is transient and

$$\mathbb{P}_x(Y_{\zeta-} \in \partial D, \zeta < \infty) = 1, \quad \forall x \in D.$$

By the rotation invariance of Y, it is easy to see that the distribution of Y_ζ under P_{x_0} is the normalized surface measure on ∂D. It follows from the Harnack inequality [BBC03, Theorem 3.2] that the distribution of Y_ζ under $\mathbb{P}_x$ is absolutely continuous with respect to the surface measure on ∂D for every $x \in D$. Let A be a Cantor set embedded into the circle ∂D. It is well known that A has Hausdorff dimension $\log 2/\log 3$ so $\mathbb{P}_x(Y_\zeta \in A) = 0$ for every $x \in D$. However when $\alpha > 2 - \log 2/\log 3$, the set A will be visited by the symmetric α-stable process X.

Combining the results presented in this section we get the following corollary which gives the final answer to the question of boundary behavior of the censored subordinate Brownian motion Y.

Corollary 3.37 Let D be an open n-set in $\mathbb{R}^n$ and ζ lifetime of the censored process Y. Then the following statements are equivalent.

(i) $Y \neq Y^*$;

(ii) $H_0^{\psi,1}(D, \lambda_D) \subsetneq H^{\psi,1}(D, \lambda_D)$;

(iii) ∂D is not polar for process X;

(iv) $\mathbb{P}_x\left(\lim_{t\uparrow\zeta} Y_t \in \partial D,\, \zeta < \infty\right) > 0$ for every $x \in D$;

(v) $\mathbb{P}_x\left(\lim_{t\uparrow\zeta} Y_t \in \partial D,\, \zeta < \infty\right) > 0$ for some $x \in D$.

Proof. The equivalence (i)-(iv) follows from Theorem 3.35 and Remark 3.34. Similarly as before, since Y has a strictly positive transition density p^Y statements (iv) and (v) are equivalent. Let (v) hold. Since for some $x \in D$

$$\lim_{t\downarrow 0} \int_D \mathbb{P}_y(Y_{\zeta-} \in \partial D) p^Y(t,x,y)dy = \mathbb{P}_x(Y_{\zeta-} \in \partial D) > 0$$

it follows that for all $w \in D$

$$\mathbb{P}_w(Y_{\zeta-} \in \partial D) = \lim_{t\downarrow 0} \int_D \mathbb{P}_y(Y_{\zeta-} \in \partial D) p^Y(t,w,y)dy > 0$$

so (v) implies (iv). □

Chapter 4

Harmonic functions for the censored process

Let $\phi \in \mathcal{CBF}$ be the Laplace exponent of the subordinator S with killing term and drift zero, i.e.

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\nu(t)dt,$$

such that the following conditions hold:

(H1): There exist constants $0 < \delta_1 \leq \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq a_2 \lambda^{\delta_2} \phi(t), \quad \lambda \geq 1,\, t \geq 1$$

and

$$\int_0^r \frac{\lambda^{\frac{n}{2}-1}}{\phi(\lambda)} d\lambda < \infty, \text{ for some } r > 0.$$

Let X be a subordinate Brownian motion with subordinator S and characteristic exponent $\psi(\xi) = \phi(|\xi|^2)$. The condition **(H1)** is the upper scaling condition introduced in Section 3.2 which is responsible for the small time and small space behavior of the process X. The second condition is by (2.12) equivalent to the transience property of the subordinate Brownian motion X. Also, for easier notation we define the function $\Phi : (0, \infty) \to (0, \infty)$ as

$$\Phi(\lambda) = \frac{1}{\phi(\lambda^{-2})}.$$

Let D be an open set in $\mathbb{R}^n$ and Y the censored process on D corresponding to X. Denote by d the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ and $\delta_B(x) = d(x, B^c)$ for a bounded open set $B \subset \mathbb{R}^n$. Also, let G and G_B be the Green function and the Green function of the set B for X respectively, $B \in \mathcal{B}(\mathbb{R}^n)$.

Recall from Theorem 3.1 that the censored process Y can be obtained from the killed process X^D through the Feynman-Kac transform with the PCAF $A_t = \int_0^t \kappa_D(X_s^D)ds$. The key ingredient in proving the Harnack inequality for harmonic functions for the censored process is to relate the Green functions of processes Y and X^D on an open Borel set $B \subset D$ through the *conditional gauge function* u, Section 2.8. Let

$$u(v,w) := \mathbb{E}_v^w[e_\kappa(\tau_B)],$$

where e_κ is the multiplicative functional $e_\kappa(t) = e^{A_t}$ and $\mathbb{E}_v^w$ is the expectation of the $G_B(\cdot,w)$-conditioned process of X^D. Recall from (2.30) that the Green function of the censored process Y on $B \subset D$ is of the form

$$G_B^Y(x,y) = G_B(x,y)u(x,y). \tag{4.1}$$

To show that u is bounded, i.e. that the Green functions G_B^Y and G_B are comparable, first we show the boundedness of the conditional expectation

$$\begin{aligned}\mathbb{E}_v^w[A(\tau_B)] = \mathbb{E}_v^w\left[\int_0^{\tau_B}\kappa_D(X_t)dt\right] &= \int_0^\infty \mathbb{E}_v\left[\kappa_D(X_t)\frac{G_B(X_t,w)}{G_B(v,w)} : t < \tau_B\right]dt \\ &= \int_B \kappa_D(y)\frac{G_B(v,y)G_B(y,w)}{G_B(v,w)}dy\end{aligned}$$

and then use Khasminskii's lemma.

First we state a couple of results regarding the Green function and harmonic functions for the subordinate Brownian motion X proved in [KSV15].

Theorem 4.1 [KSV15, Theorem 2.4]
For every $R \geq 1$ there exists a constant $c_1(R) = c_1(R,\phi,n) > 1$ such that for all $x \in B(0,R)$

$$c_1(R)^{-1}\frac{\Phi(|x|)}{|x|^n} \leq G(x) \leq c_1(R)\frac{\Phi(|x|)}{|x|^n}.$$

The following lemma is also true in the recurrent case.

Lemma 4.2 [KSV15, Lemma 2.7, Lemma 2.8]
Let $R \in (0,1)$ and B be a bounded open set such that $\mathrm{diam}(B) \leq R$. The Green function $G_B(x,y)$ is finite and continuous on $B \times B \setminus d$ and

(i) there exists a constant $c_2 = c_2(R,\phi,n)$ such that for all $x,y \in B$

$$G_B(x,y) \leq c_2\frac{\Phi(|x-y|)}{|x-y|^n},$$

(ii) for every $L > 0$ there exists a constant $c_3 = c_3(L,R,\phi,n) > 0$ such that for all $x,y \in B$ with $|x-y| \leq L(\delta_B(x) \wedge \delta_B(y))$,

$$G_B(x,y) \geq c_3\frac{\Phi(|x-y|)}{|x-y|^n}.$$

Theorem 4.3 Scale invariant Harnack inequality [KSV15, Theorem 2.2]
Let $L > 0$. There exists a positive constant $c_4 = c(L,\phi,n) > 1$ such that the following is true: If $x_1, x_2 \in R^n$ and $r \in (0,1)$ are such that $|x_1 - x_2| < Lr$, then for every nonnegative function

h which is harmonic with respect to X in $B(x_1, r) \cup B(x_2, r)$, we have

$$c_4^{-1} h(x_2) \le h(x_1) \le c_4 h(x_2).$$

Theorem 4.4 Boundary Harnack principle, [KSV15, Theorem 2.3(ii)]
Let $R \in (0,1)$. There exists a positive constant $c_5 = c_5(\phi, R, n) > 0$ such that for every $x_0 \in \mathbb{R}^n$, every open set $B \subset \mathbb{R}^n$, every $r \in (0, R)$ and all nonnegative functions h, v in $\mathbb{R}^n$ which are regular harmonic in $B \cap B(x_0, r)$ with respect to X and vanish a.e. in $B^c \cap B(x_0, r)$, we have

$$\frac{h(x)}{v(x)} \le c_5 \frac{h(y)}{v(y)}, \quad x, y \in B \cap B\left(x_0, \frac{r}{2}\right).$$

In the following section we will use several results proven for a special family of sets called κ-fat open sets.

Definition 4.5 An open set $D \subset R^n$ is said to be *κ-fat* if there exist some $R > 0$ and $\kappa \in \left(0, \frac{1}{2}\right]$ such that for every $Q \in \partial D$ and $r \in (0, R)$ there exists a ball $B(A_r(Q), \kappa r) \subset D \cap \mathcal{B}(Q, r)$. The pair (R, κ) is called the characteristics of the κ-fat open set D.

Note that the ball of radius $r > 0$ is a κ-fat open set with characteristics $\left(2r, \frac{1}{2}\right)$. Let B be a bounded κ-fat open set with characteristics (R, κ) and $\operatorname{diam}(B) \le r$, for some $r > 0$. Fix $z_0 \in B$ such that $\kappa R < \delta_B(z_0) \le R$. By Lemma 4.2(i) and (3.4) it follows that

$$G_B(x, z_0) \le c_r \frac{\Phi(\delta_B(z_0))}{\delta_B(z_0)^n}, \quad x \in B \setminus B\left(z_0, \frac{\delta_B(z_0)}{2}\right)$$

where $c_r = 2^n c_2$ and c_2 is the constant from Lemma 4.2(i) depending only on r, ϕ and n. Now we define a function g_B on B by

$$g_B(x) = G_B(x, z_0) \wedge c_r \frac{\Phi(\delta_B(z_0))}{\delta_B(z_0)^n} \tag{4.2}$$

and note that if $|x - z_0| > \frac{\delta_B(z_0)}{2}$ then $g_B(x) = G_B(x, z_0)$. Let $\varepsilon_1 = \frac{\kappa R}{24}$ and for $x, y \in B$ define $r(x, y) = \delta_B(x) \vee \delta_B(y) \vee |x - y|$ and

$$\mathcal{B}(x, y) = \begin{cases} \left\{A \in B : \delta_B(A) > \frac{\kappa}{2} r(x, y),\ |x - A| \vee |y - A| < 5 r(x, y)\right\}, & \text{if } r(x, y) < \varepsilon_1 \\ \{z_0\}, & \text{if } r(x, y) \ge \varepsilon_1. \end{cases} \tag{4.3}$$

4.1 3G inequality for subordinate Brownian motion

The following 3G theorem for X will play an important role in proving the Harnack inequality for the censored process Y, but it is also an interesting result by itself. Note that for $r > 0$ the constant c_6 appearing in the theorem is a uniform constant for all balls of radius smaller than r.

Theorem 4.6 (3G Theorem)
Let $r > 0$, $a > 0$ and $\kappa \in \left(0, \frac{1}{2}\right]$. There exists a constant $c_6 = c_6(r, a, \kappa, \phi, n) > 0$ such that

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \leq c_6 \frac{\Phi(|x-y|)\Phi(|y-z|)}{\Phi(|x-z|)} \frac{|x-z|^n}{|x-y|^n|y-z|^n} \asymp \frac{G(x,y)G(y,z)}{G(x,z)} \tag{4.4}$$

for every bounded κ-fat open set B with characteristics (R, κ) such that $\text{diam}(B) \leq r$ and $\frac{R}{\text{diam}(B)} \geq a$.

The proof of Theorem 4.6 is divided into several parts. The first theorem is a version of [KSV16, Theorem 2.10] and we follow the proof of [KSV12a, Theorem 1.2] and [Han05, Theorem 2.4].

Theorem 4.7 There exists a constant $c_7 = c_7(r, a, \kappa, \phi, n) > 1$ such that for every bounded κ-fat open set B with characteristics (R, κ) such that $\text{diam}(B) \leq r$ and $\frac{R}{\text{diam}(B)} \geq a$ and every $x, y \in B$ and $A \in \mathcal{B}(x, y)$,

$$c_7^{-1} \frac{g(x)g(y)\Phi(|x-y|)}{g(A)^2|x-y|^n} \leq G_B(x,y) \leq c_7 \frac{g(x)g(y)\Phi(|x-y|)}{g(A)^2|x-y|^n}, \tag{4.5}$$

where $g = g_B$ and $\mathcal{B}(x, y)$ are defined by (4.2) and (4.3) respectively.

Proof. Without loss of generality we can assume $\varepsilon_1 \leq 1$. Let

$$r_0 := \frac{1}{2}(|x-y| \wedge \varepsilon_1).$$

We only consider the case $\delta_B(x) \leq \delta_B(y) \leq \frac{\kappa r_0}{2}$, case (g) in [Han05], which implies $r(x, y) = |x-y|$. The remaining cases follow analogously.

Choose $Q_x, Q_y \in \partial B$ with $|Q_x - x| = \delta_B(x)$ and $|Q_y - y| = \delta_B(y)$ and let $x_1 = A_{\frac{\kappa r_0}{2}}(Q_x)$ and $y_1 = A_{\frac{\kappa r_0}{2}}(Q_y)$. This means that $x, x_1 \in B \cap B(Q_x, \frac{\kappa r_0}{2})$ and $y, y_1 \in B \cap B(Q_y, \frac{\kappa r_0}{2})$. Since

$$|z_0 - Q_x| \geq \delta_B(z_0) \geq \kappa R = 24\varepsilon_1 > r_0$$

and

$$|y - Q_x| \geq |x-y| - \delta_B(x) \geq \left(2 - \frac{\kappa}{2}\right) r_0 > r_0$$

functions $G_B(\cdot, y)$ and $G_B(\cdot, z_0)$ are regular harmonic in $B \cap B(Q_x, \kappa r_0)$ and vanish outside B. Recall from (4.2) that

$$\delta_B(z) < \frac{\delta_B(z_0)}{2} \quad \Rightarrow \quad g(z) = G_B(z, z_0). \tag{4.6}$$

Since $\delta_B(x_1) \vee \delta_B(y_1) < \frac{\kappa r_0}{2}$ by the boundary Harnack principle, Theorem 4.4 we get

$$c_5^{-1} \frac{G_B(x_1,y)}{g(x_1)} \leq \frac{G_B(x,y)}{g(x)} \leq c_5 \frac{G_B(x_1,y)}{g(x_1)}.$$

On the other hand, $|z_0 - Q_y| > r_0$ and

$$|x_1 - Q_y| \geq |x - Q_y| - |x_1 - Q_x| - \delta_B(x) \geq \left(2 - \frac{\kappa}{2}\right) r_0 - \frac{\kappa r_0}{2} - \frac{\kappa r_0}{2} > r_0,$$

so functions $G_B(x_1,\cdot)$ and $G_B(\cdot,z_0)$ are regular harmonic on $B\cap B(Q_y,\kappa r_0)$. Applying the boundary Harnack principle as before we get

$$c_5^{-1}\frac{G_B(x_1,y_1)}{g(y_1)}\le\frac{G_B(x_1,y)}{g(y)}\le c_5\frac{G_B(x_1,y_1)}{g(y_1)}.$$

Putting the two inequalities above together we get

$$c_5^{-2}\frac{G_B(x_1,y_1)}{g(x_1)g(y_1)}\le\frac{G_B(x,y)}{g(x)g(y)}\le c_5^2\frac{G_B(x_1,y_1)}{g(x_1)g(y_1)}.$$

Since $\delta_B(x_1)\wedge\delta_B(y_1)\ge\frac{\kappa^2 r_0}{2}$, $\varepsilon_1|x-y|\le 2r_0\mathrm{diam}(B)$ and

$$|x_1-y_1|\le|x_1-x|+|x-y|+|y-y_1|<\kappa r_0+|x-y|+\kappa r_0\le(1+\kappa)|x-y| \tag{4.7}$$

it follows that

$$|x_1-y_1|\le\frac{4(1+\kappa)\mathrm{diam}(B)}{\kappa^2\varepsilon_1}(\delta_B(x_1)\wedge\delta_B(y_1))\le\frac{96(1+\kappa)}{a\kappa^3}(\delta_B(x_1)\wedge\delta_B(y_1))$$

and we can apply Lemma 4.2 on $G_B(x_1,y_1)$. Therefore, there exist positive constants c_2 and c_3 depending only on κ, R, ϕ and n such that

$$\frac{c_3c_5^{-2}}{g(x_1)g(y_1)|x_1-y_1|^n\phi(|x_1-y_1|^{-2})}\le\frac{G_B(x,y)}{g(x)g(y)}\le\frac{c_2c_5^2}{g(x_1)g(y_1)|x_1-y_1|^n\phi(|x_1-y_1|^{-2})}.$$

Applying (3.4), (4.7) and

$$|x_1-y_1|\ge|x-y|-|x_1-x|-|y_1-y|\ge|x-y|-2\kappa r_0\ge(1-\kappa)\,|x-y|$$

the previous inequality transforms to

$$\frac{c_3c_5^{-2}(1+\kappa)^{-n}(1-\kappa)^2}{g(x_1)g(y_1)|x-y|^n\phi(|x-y|^{-2})}\le\frac{G_B(x,y)}{g(x)g(y)}\le\frac{c_2c_5^2(1-\kappa)^{-n}(1+\kappa)^2}{g(x_1)g(y_1)|x-y|^n\phi(|x-y|^{-2})}.$$

Lastly, we have to show that for all $A\in\mathcal{B}(x,y)$

$$g(A)^2\asymp g(x_1)g(y_1). \tag{4.8}$$

Consider two cases, $r_0<\frac{\varepsilon_1}{2}$ and $r_0=\frac{\varepsilon_1}{2}$. If $r_0<\frac{\varepsilon_1}{2}$ then

$$r(x,y)=|x-y|<\varepsilon_1,\ r_0=\frac{1}{2}r(x,y)\text{ and }\delta_B(x_1)\wedge\delta_B(y_1)\ge\frac{\kappa^2r_0}{2}=\frac{\kappa^2r(x,y)}{4}.$$

Since $G_B(\cdot,z_0)$ is harmonic on $B(x_1,\delta_B(x_1))\cup B(A,\delta_B(A))$ and

$$|x_1-A|\le|x_1-x|+|x-A|\le\kappa r_0+5r(x,y)\le\frac{4}{\kappa^2}\left(\frac{\kappa}{2}+5\right)(\delta_B(x_1)\wedge\delta_B(A))$$

by (4.6) and the scale invariant Harnack inequality, Theorem 4.3,

$$c_4^{-1} g(x_1) \le G_B(A, z_0) \le c_4 g(x_1)$$

and therefore

$$c_4^{-1} g(x_1) \le g(A) \le c_4 g(x_1).$$

The analogous inequality follows for y_1 in place of x_1 and therefore

$$c_4^{-2} g(x_1) g(y_1) \le g^2(A) \le c_4^2 g(x_1) g(y_1).$$

On the other hand, if $r_0 = \frac{\varepsilon_1}{2}$ then $r(x,y) = |x-y| \ge \varepsilon_1$, so by (4.3) and (4.2) it follows that

$$g(A) = g(z_0) = c_r \frac{\Phi(\delta_B(z_0))}{\delta_B(z_0)^n}.$$

Let $v \in \{x_1, y_1\}$ and $z \in B$ such that $|z - z_0| = \frac{\delta_B(z_0)}{2} = \delta_B(z)$. Since $\delta_B(v) \ge \frac{\kappa^2 r_0}{2} = \frac{\kappa^2 \varepsilon_1}{4}$ it follows that

$$|v - z| \le \operatorname{diam}(B) \le \frac{4\operatorname{diam}(B)}{\kappa^2 \varepsilon_1} (\delta_B(v) \wedge \delta_B(z)) \le \frac{96}{a\kappa^3} (\delta_B(v) \wedge \delta_B(z))$$

and by applying Theorem 4.3 we get

$$c_4^{-1} G_B(z, z_0) \le g(v) \le c_4^{-1} G_B(z, z_0).$$

Therefore, by Lemma 4.2

$$\tilde{c}^{-1} \frac{\Phi(\delta_B(z_0))}{\delta_B(z_0)^n} \le g(v) \le \tilde{c} \frac{\Phi(\delta_B(z_0))}{\delta_B(z_0)^n}$$

for some $\tilde{c} = \tilde{c}(r, \kappa, \phi, n) > 1$, which implies (4.8). □

We will also need the following result from [KSV16, Lemma 2.7].

Lemma 4.8 Carleson's estimate
Let $a > 0$, $r > 0$ and $\kappa \in \left(0, \frac{1}{2}\right]$. There exists a constant $c_8 = c_8(r, \kappa, \phi, n) > 0$ such that for every bounded open κ-fat set B with characteristics (R, κ) and $\operatorname{diam}(B) \le r$, $z \in \partial B$, $r_0 \in (0, \frac{\kappa R}{4})$ and $y \in B \setminus \overline{B(z, 3r_0)}$

$$G_B(x, y) \le c_8 G_B(A_{r_0}(z), y), \quad x \in B \cap B(z, r_0).$$

Proof. Let $y \in B \setminus \overline{B(z, 3r_0)}$ and $x \in B \cap B(z, r_0)$. Note that the functions $G_B(\cdot, y)$ and $G_B(\cdot, A_{4r_0/\kappa}(z))$ are regular harmonic in $B \cap B(z, 3r_0)$ and $B \setminus \overline{B(A_{4r_0/\kappa}(z), 2r_0)}$ respectively. Since

$$\left|A_{4r_0/\kappa}(z) - x\right| \ge \delta_B\left(A_{4r_0/\kappa}(z)\right) - \delta_B(x) \ge 4r_0 - r_0 = 3r_0 \tag{4.9}$$

and

$$\left|A_{4r_0/\kappa}(z) - A_{r_0}(z)\right| \ge \delta_B\left(A_{4r_0/\kappa}(z)\right) - \delta_B\left(A_{r_0}(z)\right) \ge 4r_0 - \kappa r_0 > 3r_0, \tag{4.10}$$

by the boundary Harnack principle, Theorem 4.4, it follows that

$$\frac{G_B(x,y)}{G_B(A_{r_0}(z),y)} \le c_5 \frac{G_B(x, A_{4r_0/\kappa}(z))}{G_B(A_{r_0}(z), A_{4r_0/\kappa}(z))}.$$

It is enough to show that

$$\frac{G_B(x, A_{4r_0/\kappa}(z))}{G_B(A_{r_0}(z), A_{4r_0/\kappa}(z))} \le \tilde{c} \tag{4.11}$$

for some $\tilde{c} = \tilde{c}(r, \kappa, \phi, n) > 0$. By Lemma 4.2(i)

$$G_B(x, A_{4r_0/\kappa}(z)) \le c_2 \frac{\Phi(|x - A_{4r_0/\kappa}(z)|)}{|x - A_{4r_0/\kappa}(z)|^n},$$

and from (3.4), (4.9) and

$$|x - A_{4r_0/\kappa}(z)| \le |x - z| + |z - A_{4r_0/\kappa}(z)| \le 5r_0$$

it follows that

$$\frac{\Phi(|x - A_{4r_0/\kappa}(z)|)}{|x - A_{4r_0/\kappa}(z)|^n} \le 5^2 3^{-n} \frac{\Phi(r_0)}{r_0^n}. \tag{4.12}$$

On the other hand, since

$$|A_{r_0}(z) - A_{4r_0/\kappa}(z)| \le 8r_0/\kappa \le \frac{8}{\kappa^2}\delta_B(A_{r_0}(z)) = \frac{8}{\kappa^2}\left(\delta_B(A_{r_0}(z)) \wedge \delta_B(A_{4r_0/\kappa}(z))\right)$$

by Lemma 4.2(ii), (4.10) and (3.4) it follows that

$$G_B(A_{r_0}(z), A_{4r_0/\kappa}(z)) \ge c_3 \frac{\Phi(|A_{r_0}(z) - A_{4r_0/\kappa}(z)|)}{|A_{r_0}(z) - A_{4r_0/\kappa}(z)|^n} \ge c_3 \left(\frac{\kappa}{4}\right)^n \frac{\Phi(r_0)}{r_0^n},$$

which together with (4.12) implies (4.11). □

Applying the Carleson's estimate, the Harnack inequality and Lemma 4.2 the proofs of the following lemmas follow entirely as in [KL07, Lemma 3.8-3.11]. Let B be a bounded κ-fat open set with $\mathrm{diam}(B) \le r$ and characteristics (R, κ) such that $\frac{R}{\mathrm{diam}(B)} \ge a$. As in the proof of Theorem 4.7, for $x \in B$ let $Q_x \in \partial B$ be such that $|x - Q_x| = \delta_B(x)$.

Lemma 4.9 There exists a constant $c_9 = c_9(r, a, \kappa, \phi, n) > 0$ such that for every $x, y \in B$ with $r(x,y) < \varepsilon_1$,

$$g(z) < c_9 g(A_{r(x,y)}(Q_x)), \quad z \in B \cap B(Q_x, r(x,y)). \tag{4.13}$$

Lemma 4.10 There exists a constant $c_{10} = c_{10}(r, a, \kappa, \phi, n) > 0$ such that for every $x, y \in B$

$$g(x) \vee g(y) < c_{10} g(A), \quad A \in \mathcal{B}(x,y). \tag{4.14}$$

Lemma 4.11 If $x, y, z \in B$ satisfy $r(x,z) \le r(x,y)$, then there exists a constant $c_{11} =$

$c_{11}(r, a, \kappa, \phi, n) > 0$ such that

$$g(A_{x,y}) < c_{11} g(A_{y,z}), \quad \text{for every } (A_{x,y}, A_{y,z}) \in \mathcal{B}(x,y) \times \mathcal{B}(y,z). \tag{4.15}$$

Lemma 4.12 There exists a constant $c_{12} = c_{12}(r, a, \kappa, \phi, n) > 0$ such that for every $x, y, z, w \in B$ and $(A_{x,y}, A_{y,z}, A_{x,z}) \in \mathcal{B}(x,y) \times \mathcal{B}(y,z) \times B(x,z)$,

$$g(A_{x,z})^2 < c_{12} \left(g(A_{x,y})^2 + g(A_{y,z})^2 \right) \tag{4.16}$$

Proof of Theorem 4.6: Applying Theorem 4.7 we get

$$\frac{G_B(x,y) G_B(y,z)}{G_B(x,z)} \leq c_7^3 \frac{g(y)^2 g(A_{xz})^2}{g(A_{xy})^2 g(A_{yz})^2} \frac{\Phi(|x-y|)\Phi(|y-z|)}{\Phi(|x-z|)} \frac{|x-z|^n}{|x-y|^n |y-z|^n}.$$

By (4.16) and (4.14),

$$\frac{g(y)^2 g(A_{xz})^2}{g(A_{xy})^2 g(A_{yz})^2} \leq c_{12} \left(\frac{g(y)^2}{g(A_{xy})^2} + \frac{g(y)^2}{g(A_{yz})^2} \right) \leq 2 c_{12} c_{10}^2,$$

which proves the 3G inequality (4.4) with $c_6 = 2 c_7^3 c_{12} c_{10}^2$ depending only on r, a, κ, ϕ and n.

□

4.2 Harnack inequality for censored subordinate Brownian motion

As a consequence of the 3G Theorem from the previous section, we first obtain the uniform boundedness of the conditional expectation $\mathbb{E}_v^w[A(\tau_B)]$ for small balls.

Lemma 4.13 There is a constant $r_1 = r_1(n, \phi) \in (0, \frac{1}{3})$, independent of D, such that for every $r \in (0,1)$ and every ball $B = B(x, r_1 r) \subset B(x, r) \subset D$,

$$\int_B \frac{G_B(v,y) G_B(y,w)}{G_B(v,w)} \kappa_D(y) dy \leq \frac{1}{2}, \quad \forall v, w \in B.$$

Proof. Let $r_1 \leq \frac{1}{3}$ and $r \in (0,1)$. Since $r r_1 < \frac{1}{3}$ by Theorem 4.6

$$\frac{G_B(v,y) G_B(y,w)}{G_B(v,w)} \leq c_6 \frac{\phi(|v-w|^{-2})}{\phi(|v-y|^{-2}) \phi(|y-w|^{-2})} \frac{|v-w|^n}{|v-y|^n |y-w|^n}, \quad \forall v, y, w \in B.$$

First we will show that there exists a constant $\tilde{c} = \tilde{c}(n, \phi) > 0$ such that

$$\phi(|v-w|^{-2}) |v-w|^n \leq \tilde{c} \left(\phi(|v-y|^{-2}) |v-y|^n + \phi(|y-w|^{-2}) |y-w|^n \right).$$

From (3.4) it follows that

$$\phi(s^{-2})s^n \leq \frac{r^2}{s^2}\phi(r^{-2})s^n \leq \phi(r^{-2})r^n, \quad \forall s < r \leq 1.$$

Without loss of generality we can assume $|v-y| \leq |y-w|$, so

$$|v-w| \leq |v-y| + |y-w| \leq 2|y-w|.$$

Since $|y-w| \leq 2rr_1 < 1$ it follows that

$$\begin{aligned}\phi(|v-w|^{-2})|v-w|^n &\leq 2^n \phi\left(\left(\frac{|v-w|}{2}\right)^{-2}\right)\left(\frac{|v-w|}{2}\right)^n \\ &\leq 2^n \phi(|y-w|^{-2})|y-w|^n \\ &\leq 2^n \left(\phi(|v-y|^{-2})|v-y|^n + \phi(|y-w|^{-2})|y-w|^n\right).\end{aligned}$$

Therefore for every $v, w \in B$

$$\begin{aligned}&\int_B \frac{G_B(v,y)G_B(y,w)}{G_B(v,w)}dy \leq c_6 2^n \int_B \left(\frac{1}{\phi(|v-y|^{-2})|v-y|^n} + \frac{1}{\phi(|y-w|^{-2})|y-w|^n}\right)dy \\ &\leq c_6 2^n \left(\int\limits_{B(v,2rr_1)} \frac{1}{\phi(|v-y|^{-2})|v-y|^n}dy + \int\limits_{B(w,2rr_1)} \frac{1}{\phi(|y-w|^{-2})|y-w|^n}dy\right) \\ &\leq \tilde{c}_1 \int_0^{2rr_1} \frac{1}{\phi(s^{-2})s^n} s^{n-1} ds \overset{\textbf{(H1)}}{\leq} \frac{\tilde{c}_1}{2a_1\delta_1}\phi((2r_1 r)^{-2})^{-1},\end{aligned}$$

for some $\tilde{c}_1 = \tilde{c}_1(\phi, n) > 0$. Furthermore, for every $y \in B = B(x, rr_1) \subset D$ it follows that

$$r \leq \delta_D(x) = \delta_D(y) + |x-y| \leq \delta_D(y) + rr_1,$$

so $B(y, r(1-r_1)) \subset D$ and

$$\kappa_D(y) = \int_{D^c} j(|y-z|)dz \leq \int\limits_{B(y,r(1-r_1))^c} j(|y-z|)dz = \tilde{c}_2 \int_{r(1-r_1)}^{\infty} s^{n-1} j(s) ds$$

for some $\tilde{c}_2 = \tilde{c}_2(n) > 0$. By [KSV15, Lemma 2.2] there exists a constant $\tilde{c}_3 = \tilde{c}_3(n) > 0$ such that for all $r > 0$

$$\kappa_D(y) \leq \tilde{c}_3 \phi(r^{-2}(1-r_1)^{-2}).$$

Finally, for r_1 small enough we have

$$\int_B \frac{G_B(v,y)G_B(y,w)}{G_B(v,w)}\kappa_D(y)dy \leq \frac{\tilde{c}_1\tilde{c}_3}{2a_1\delta_1}\frac{\phi(r^{-2}(1-r_1)^{-2})}{\phi((2r_1 r)^{-2})} \overset{\textbf{(H1)}}{\leq} \frac{\tilde{c}_1\tilde{c}_3}{2a_1^2\delta_1}\left(\frac{2r_1}{1-r_1}\right)^{2\delta_1} \leq \frac{1}{2}.$$

□

By Lemma 2.27 it follow that for every $r < 1$, every ball $B = B(x, rr_1) \subset D$ and $v, w \in B$

$$\mathbb{E}_v^w[A(\tau_B)] = \int_B \kappa_D(y) \frac{G_B(v,y)G_B(y,w)}{G_B(v,w)} dy \le \frac{1}{2},$$

so by Khasminskii's lemma, Lemma 2.27

$$1 \le u(v,w) = \mathbb{E}_v^w[e^{A(\tau_B)}] \le \frac{1}{1-\frac{1}{2}} = 2. \tag{4.17}$$

Recall from (2.19) that the density of the joint $\mathbb{P}_x$-distribution of $(X_{\tau_B -}, X_{\tau_B})$ restricted to the event $\{X_{\tau_B -} \neq X_{\tau_B}, \tau_B < \infty\}$ is given by

$$g_x(y,z) = G_B(x,y) j(|y-z|), \quad (y,z) \in B \times B^c.$$

From (2.17) and (2.30) we get an analogous formula for the joint distribution of $(Y_{\tau_B -}, Y_{\tau_B})$ restricted to the event $\{Y_{\tau_B -} \neq Y_{\tau_B},\, \tau_B < \infty\}$, i.e. for all nonnegative Borel measurable functions f and g on D and open Borel sets $B \subset \overline{B} \subset D$

$$\mathbb{E}_x[f(Y_{\tau_B -})g(Y_{\tau_B})] = \int_{B^c}\int_B f(y)g(z)G_B(x,y)u(x,y)j(|y-z|)dydz. \tag{4.18}$$

Also, since the subordinator of X has zero drift, by [Szt00, Theorem 1] it follows that for every ball $B \subset \overline{B} \subset D$ and all $y \in B$

$$\mathbb{P}_y(X_{\tau_B} \in \partial B) = \mathbb{P}_y\left(X_{\tau_B -} = X_{\tau_B}\right) = 0.$$

From Theorem 3.1(iii) it follows that

$$\mathbb{P}_y\left(Y_{\tau_B -} = Y_{\tau_B}\right) = 0, \quad \forall y \in B \subset \overline{B} \subset D.$$

Using (4.17) and (4.18) we are able to prove the scale invariant Harnack inequality for harmonic functions with respect to the censored process Y.

Theorem 4.14 For any $L > 0$, there exists a constant $c_{12} = c_{12}(n, \phi, L) > 1$ such that the following is true: If $x_1, x_2 \in D$ and $r \in (0,1)$ are such that $B(x_1, r) \cup B(x_2, r) \subset D$ and $|x_1 - x_2| < Lr$, then for every nonnegative function h which is harmonic with respect to Y on $B(x_1, r) \cup B(x_2, r)$, we have

$$c^{-1}h(x_1) \le h(x_2) \le ch(x_1).$$

Proof. Let $r_1 \in (0, \frac{1}{3})$ be the constant from Lemma 4.13 and $B_i = B(x_i, r_1 r)$, $i = 1, 2$. Let G_{B_i} and $G_{B_i}^Y$ be the Green function of X and Y on B_i respectively. Since $\overline{B_1} \subset D$ it follows that for $y \in B_1$

$$\begin{aligned} h(y) = \mathbb{E}_y\left[h(Y_{\tau_{B_1}})\right] &\overset{(4.18)}{=} \int_{D\setminus\overline{B}_1}\int_{B_1} h(w)G_{B_1}^Y(y,v)j(|v-w|)dv\,dw \\ &\overset{(4.1)}{=} \int_{D\setminus\overline{B}_1}\int_{B_1} h(w)G_{B_1}(y,v)u(y,v)j(|v-w|)dv\,dw \end{aligned}$$

$$= \mathbb{E}_y \left[h(X_{\tau_{B_1}}) u(y, X_{\tau_{B_1}-}) \right].$$

Here we implicitly assume $h = 0$ on D^c. Define

$$w(y) = \mathbb{E}_y \left[h(X_{\tau_{B_1}}) \right], \quad y \in B_1,$$

and note that w is harmonic in B_1 with respect to X. From (4.17) it follows that

$$w(y) \leq h(y) \leq 2w(y), \quad \forall y \in B_1 \tag{4.19}$$

and analogously

$$\mathbb{E}_y \left[h(X_{\tau_{B_2}}) \right] \leq h(y) \leq 2\mathbb{E}_y \left[h(X_{\tau_{B_2}}) \right], \quad \forall y \in B_2. \tag{4.20}$$

By [KSV15, Proposition 2.3] there exists a constant $\tilde{c}_1 = \tilde{c}_1(n, \phi) > 0$ such that for any $y \in B\left(x_1, \frac{rr_1}{2}\right)$ and almost every $z \in \overline{B(x_1, rr_1)}^c$,

$$K_{B(x_1, rr_1)}(y, z) \geq \tilde{c}_1 K_{B(x_1, rr_1)}(x_1, z),$$

where K_B is the Poisson kernel of the process X on $B \times \overline{B}^c$ defined in (2.20). This implies that for any $y \in B(x_1, \frac{rr_1}{2})$

$$w(y) = \int_{D \setminus \overline{B}_1} h(z) K_{B_1}(y, z) dz \geq \tilde{c}_1 \int_{D \setminus \overline{B}_1} h(z) K_{B_1}(x_1, z) dz = \tilde{c}_1 w(x_1) \geq \frac{\tilde{c}_1}{2} h(x_1). \tag{4.21}$$

First we consider the case when $r \leq |x_1 - x_2| < Lr$. It follows that $B_2 \cap B(x_1, r_1 r/2) = \emptyset$ and therefore

$$\begin{aligned} h(x_2) &\overset{(4.20)}{\geq} \mathbb{E}_{x_2} \left[h(X_{\tau_{B_2}}) \right] \geq \mathbb{E}_{x_2} \left[h(X_{\tau_{B_2}}); X_{\tau_{B_2}} \in B(x_1, r_1 r/2) \right] \\ &\overset{(4.19)}{\geq} \mathbb{E}_{x_2} \left[w(X_{\tau_{B_2}}); X_{\tau_{B_2}} \in B(x_1, r_1 r/2) \right] \\ &\overset{(4.21)}{\geq} \frac{\tilde{c}_1}{2} h(x_1) \mathbb{P}_{x_2} \left(X_{\tau_{B_2}} \in B(x_1, r_1 r/2) \right) \\ &= \frac{\tilde{c}_1}{2} h(x_1) \int_{B(x_1, r_1 r/2)} K_{B_2}(x_2, z) dz. \end{aligned} \tag{4.22}$$

By [KSV15, Lemma 2.6] there exists a constant $\tilde{c}_2 = \tilde{c}_2(\phi, n) > 0$ such that for all $z \in \overline{B}_2^c$

$$K_{B_2}(x_2, z) \geq \tilde{c}_2 \frac{j(|z - x_2|)}{\phi((r_1 r)^{-2})}. \tag{4.23}$$

Also, for $z \in B(x_1, r_1 r/2)$,

$$|z - x_2| \leq r(r_1/2 + L) < r_1/2 + L$$

so by [KSV14, Lemma 3.2] there exists a constant $\tilde{c}_3 = \tilde{c}_3(\phi, n, L) > 0$ such that

$$j(|z - x_2|) \geq j(r(r_1/2 + L)) \geq \tilde{c}_3 \frac{\phi(r^{-2}(r_1/2 + L)^{-2})}{r^n(r_1/2 + L)^n}. \tag{4.24}$$

Combining (4.22), (4.23) and (4.24) we get

$$\begin{aligned} h(x_2) &\geq \frac{\tilde{c}_1\tilde{c}_2\tilde{c}_3}{2} \frac{|B(x_1, \frac{rr_1}{2})|}{r^n(r_1/2 + L)^n} \frac{\phi(r^{-2}(r_1/2 + L)^{-2})}{\phi((r_1 r)^{-2})} h(x_1) \\ &\overset{(3.4)}{\geq} \frac{\tilde{c}_1\tilde{c}_2\tilde{c}_3}{2} \frac{|B(x_1, \frac{rr_1}{2})|}{r^n(r_1/2 + L)^n} \left(1 \wedge \left(\frac{r_1}{\frac{r_1}{2} + L}\right)^2\right) h(x_1) = c_{12}(n, \phi, L) h(x_1). \end{aligned}$$

On the other hand, if $|x_1 - x_2| < r$ take $r' = |x_1 - x_2|$ and $L' = 1$. Since $r' \leq |x_1 - x_2| < L'r'$ the proof follows in the same way as in the previous case. □

Remark 4.15 If for a Lipschitz domain $B \subset \overline{B} \subset D$

$$\inf_{y \in B} \int_{D \setminus B} j(|z - y|)dz \geq c$$

for some constant $c > 0$, then by (4.18) it follows that

$$1 = \int_{D \setminus \overline{B}} \int_B G_B^Y(x, y) j(|z - y|) dy\, dz \geq c \int_B G_B^Y(x, y) dy$$

and therefore

$$\mathbb{E}_x[\tau_B^Y] = \int_B G_B^Y(x, y) dy < \infty, \quad \forall x \in B. \tag{4.25}$$

Furthermore, (4.25) holds for all $x \in D$ and implies that

$$\mathbb{P}_x(\tau_B^Y < \infty) = 1, \quad \text{for all } x \in D.$$

4.3 Generator of the censored subordinate Brownian motion

By [Sat99, Theorem 31.5] the generator of X is a non-local operator of the form

$$\begin{aligned} Au(x) &= P.V. \int_{\mathbb{R}^n} (u(x + y) - u(x)) j(|y|) dy \\ &= \lim_{\varepsilon \downarrow 0} \int_{\{|y| > \varepsilon\}} (u(x + y) - u(x)) j(|y|) dy \\ &= \int_{\mathbb{R}^n} (u(x + y) - u(x) - \nabla u(y) \cdot y 1_{|y| \leq r}) j(|y|) dy \end{aligned}$$

for $u \in C_c^2(\mathbb{R}^n)$ and all $r > 0$. The restriction of the generator of the censored process Y on $C_c^2(D)$ is analogously equal to

$$\begin{aligned}
A^Y u(x) &= P.V. \int_D (u(y) - u(x)) j(|x-y|) dy \\
&= \lim_{\varepsilon \downarrow 0} \int_{\{y \in D: |y-x| > \varepsilon\}} (u(y) - u(x)) j(|x-y|) dy \\
&= P.V. \int_{\mathbb{R}^n} (u(x+y) - u(x)) j(|y|) dy + \int_{D^c} u(x) j(|x-y|) dy \\
&= P.V. \int_{\mathbb{R}^n} (u(x+y) - u(x)) j(|y|) dy + u(x)\kappa_D(x).
\end{aligned}$$

For a C^2 function u on $\mathbb{R}^n$ we write

$$||u||_{C^2} = \sum_{|j| \le 2} ||D^j u||_\infty,$$

where j ranges over multi-indices.

Lemma 4.16 Let $\delta_1 > \frac{1}{2}$. There exists a constant $c_{13} = c_{13}(n, \phi) > 0$ such that for every $u \in C^2(\mathbb{R}^n)$

$$|A^Y u(x)| \le c_{13} ||u||_{C^2} \left[1 + \phi(\delta_D(x)^{-2}) \delta_D(x)\right].$$

Proof. By the Taylor's expansion theorem

$$\begin{aligned}
|Au(x)| &= \left| P.V. \int_{\mathbb{R}^n} (u(x+y) - u(x)) j(|y|) dy \right| \\
&= \left| \int_{\mathbb{R}^n} (u(x+y) - u(x) - \nabla u(x) y 1_{|y|<1}) j(|y|) dy \right| \\
&\le ||u||_{C^2} \int_{\mathbb{R}^n} (1 \wedge |y|^2) j(|y|) dy = \tilde{c} ||u||_{C^2},
\end{aligned}$$

for some $\tilde{c} > 1$. Therefore,

$$\begin{aligned}
|A^Y u(x)| &\le |Au(x)| + \left| \int_{D^c} (u(y) - u(x)) j(|y-x|) dy \right| \\
&\le \tilde{c} ||u||_{C^2} + \int_{D^c} |u(y) - u(x)| j(|y-x|) dy \\
&\le \tilde{c} ||u||_{C^2} + ||u||_{C^2} \int_{D^c} (2 \wedge |y-x|) j(|y-x|) dy \\
&\le \tilde{c} ||u||_{C^2} \left(1 + \int_{|z| > \delta_D(x)} (2 \wedge |z|) j(|z|) dz \right).
\end{aligned}$$

If $\delta_D(x) \ge 2$ then there exists a constant $\tilde{c}_1 > 1$ such that

$$|A^Y u(x)| \le \tilde{c} ||u||_{C^2} \left(1 + 2 \int_{|z|>2} j(|z|) dz \right) = \tilde{c}_1 ||u||_{C^2}.$$

By (**??**) for $\delta_D(x) < 2$ it follows that

$$
\begin{aligned}
|A^Y u(x)| &\leq \tilde{c}||u||_{C^2} \left(1 + \int_{\delta_D(x)}^{2} \phi(z^{-2})dz + 2 \int\limits_{|z|>2} j(|z|)dz \right) \\
&\leq \tilde{c}_1||u||_{C^2} \left(1 + \int_{\delta_D(x)}^{2} \phi(z^{-2})dz \right).
\end{aligned}
$$

By **(H1)** there exists a constant $\tilde{a}_1 > 0$ such that

$$
\begin{aligned}
|A^Y u(x)| &\leq \tilde{c}_1||u||_{C^2} \left(1 + \tilde{a}_1^{-1}\phi(\delta_D(x)^{-2})\delta_D(x)^{2\delta_1} \int_{\delta_D(x)}^{2} z^{-2\delta_1}dz \right) \\
&\leq \tilde{c}_1||u||_{C^2} \left(1 + \frac{1}{\tilde{a}_1(2\delta_1 - 1)}\phi(\delta_D(x)^{-2})\delta_D(x)^{2\delta_1}(\delta_D(x)^{1-2\delta_1} + 2^{1-2\delta_1}) \right) \\
&\leq \tilde{c}_1||u||_{C^2} \left(1 + \frac{2}{\tilde{a}_1(2\delta_1 - 1)}\phi(\delta_D(x)^{-2})\delta_D(x) \right),
\end{aligned}
$$

which concludes the proof. □

Chapter 5

Potential theory of absolute value of one-dimensional subordinate Brownian motion killed at zero

In this chapter we consider the potential theory of two processes associated with a one-dimensional subordinate Brownian motion X. First process is the absolute value of X killed at zero, which we denote by $Z = (Z_t)_{t\geq 0}$ and the second is the process Y obtained by censoring X on $(0,\infty)$. The goal is to establish the Harnack inequality and boundary Harnack principle for Z on finite intervals. To do so, we examine a function h called the *compensated resolvent kernel* and prove in Section 5.1 several properties of the first exit time of Z from a finite interval. In Section 5.2 we prove that process Z killed outside of a finite interval (a,b), $0<a<b$, can be obtained from Y by a combination of a discontinuous and continuous Feynman-Kac transform and show that the corresponding Green functions are comparable. Finally, in the last section we give the proof of the Harnack inequality and boundary Harnack principle for $Z^{(a,b)}$.

Let X be a 1-dimensional recurrent subordinate Brownian motion with the characteristic exponent $\psi(t) = \phi(t^2)$, $t \in \mathbb{R}$, where ϕ is a complete Bernstein function. By (2.12) process X is recurrent if and only if the Laplace exponent ϕ of the subordinator satisfies the condition

$$\int_0^r \frac{1}{\phi(\lambda^2)} d\lambda = \infty, \tag{5.1}$$

for some $r>0$. Let X^0 be the process X killed at 0 and Z the absolute value of that process, i.e.

$$Z_t(\omega) = \begin{cases} |X_t(\omega)|, & t < \sigma_{\{0\}}(\omega) \\ \partial, & t \geq \sigma_{\{0\}}(\omega) \end{cases}, \quad t \geq 0,\, \omega \in \Omega,$$

where $\sigma_B = \inf\{t>0 : X_t \in B\}$ is the first hitting time of $B \in \mathcal{B}(\mathbb{R})$. As in Section 3.2 we will impose the upper and lower scaling condition on the Laplace exponent ϕ,

(H1): There exist constants $a_1, a_2 > 0$ and $0 < \delta_1 \le \delta_2 < 1$ such that

$$a_1 \lambda^{\delta_1} \le \frac{\phi(\lambda r)}{\phi(r)} \le a_2 \lambda^{\delta_2}, \quad \lambda \ge 1, r \ge 1.$$

(H2): There exist constants $a_3, a_4 > 0$ and $0 < \delta_3 \le \delta_4 < 1$ such that

$$a_3 \lambda^{\delta_3} \le \frac{\phi(\lambda r)}{\phi(r)} \le a_4 \lambda^{\delta_4}, \quad \lambda \ge 1, r \le 1.$$

and note that they are equivalent to the *global scaling condition*

(H): There exist constants $a_5, a_6 > 0$ such that for $\delta_5 = \delta_1 \wedge \delta_3$ and $\delta_6 = \delta_2 \vee \delta_4$

$$a_5 \lambda^{\delta_5} \le \frac{\phi(\lambda r)}{\phi(r)} \le a_6 \lambda^{\delta_6}, \quad \lambda \ge 1, r > 0.$$

Note that for $\delta_3 \ge \frac{1}{2}$ the condition (5.1) holds and X is recurrent. Furthermore, we will only consider the case when the point 0 is regular for itself.

Definition 5.1 Let $B \in \mathcal{B}(\mathbb{R})$ and $x \in \mathbb{R}$. We say that the point x is *regular* for B if

$$\mathbb{P}_x(\sigma_B = 0) = 1. \tag{5.2}$$

The point x is *regular for itself* if it is regular for $\{x\}$. If the probability in (5.2) is 0 then the point x is *irregular* for B.

By the Blumenthal 0-1 law every point is either regular or irregular for $B \in \mathcal{B}(\mathbb{R})$. By [Ber98, Corollary II.20] 0 is regular for itself if

$$\int_1^\infty \frac{1}{\phi(\lambda^2)} d\lambda < \infty. \tag{5.3}$$

This condition is known as the Kesten-Bretagnolle condition and in our setting is actually equivalent to point regularity, [Yan10, Lemma 3.1]. Also, note that this condition is satisfied when $\delta_1 > \frac{1}{2}$. This regularity condition implies that 0 is not polar, that is

$$\mathbb{P}_x(\sigma_0 < \infty) > 0, \quad \forall x \in \mathbb{R}.$$

so X^0 is a proper subprocess of X. If X is also recurrent then by [Yan10, Theorem 3.1]

$$\mathbb{P}_x(\sigma_0 < \infty) = 1, \quad \forall x \in \mathbb{R}.$$

From now on we will assume that both

$$\delta_1 > \frac{1}{2} \quad \text{and} \quad \delta_3 \ge \frac{1}{2}.$$

Let $G^{X^0}(x, dy)$ and $G^Z(x, dy)$ be Green measures for X^0 and Z respectively. Note that for every $x > 0$ and $A \in \mathcal{B}((0,\infty))$

$$\begin{aligned} G^Z(x,A) &= \int_0^\infty (\mathbb{P}_x(X_t^0 \in A) + \mathbb{P}_x(-X_t^0 \in A))dt \\ &= \int_A \left(G^{X^0}(x,y) + G^{X^0}(x,-y) \right) dy \end{aligned}$$

and thus the Green function of Z is equal to

$$G^Z(x,y) = G^{X^0}(x,y) + G^{X^0}(x,-y). \tag{5.4}$$

Define the *local time at 0* as

$$L(0,t) = \lim_{\varepsilon \downarrow 0} \int_0^t 1_{\{|X_s| < \varepsilon\}} ds. \tag{5.5}$$

Note that $L(0,t)$ can be interpreted as time spent in 0 by the process X up to time t. By [Ber98, Proposition V.2.] L is well defined and a.s. a continuous function. Let $h : \mathbb{R} \to [0,\infty)$ be a function defined by

$$h(x) = \frac{1}{2}\mathbb{E}[L(0, \sigma_{\{x\}})].$$

Under our assumptions there exists a bounded and continuous density u^q of the q-resolvent, i.e.

$$U^q f(x) = \int_0^\infty e^{-qt}\mathbb{E}_x[f(X_t)]dt = \int_{\mathbb{R}} f(x)u^q(x)dx.$$

Note that

$$u^q(x) = \int_0^\infty e^{-qt}p_t(x)dt = \int_0^\infty e^{-qt}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda x}e^{-t\phi(\lambda^2)}d\lambda\ dt = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\cos(\lambda x)}{q + \phi(\lambda^2)}d\lambda.$$

Since the transition density $p_t(x)$ is deacresing in x it follows that u^q is decreasing as well. By [Ber98, Lemma V.11] h is of the form

$$h(x) = \lim_{q \downarrow 0}(u^q(0) - u^q(x)) = \frac{1}{2\pi}\int_0^\infty \frac{1 - \cos(\lambda x)}{\phi(\lambda^2)}d\lambda. \tag{5.6}$$

The function h is symmetric and since u^q is decreasing, h is also increasing on $[0,\infty)$, see also [SV06, Proposition 1.1]. By [Yan10, Theorem 1.1] h is harmonic for the process X^0 on $\mathbb{R}\backslash\{0\}$ and, since it is symmetric, it is also harmonic for Z on $(0,\infty)$. Furthermore, by [Yan10, Lemma 4.5] the Green function of the processes X^0 and Z can be represented using the function h.

Lemma 5.2 Let X be a symmetric recurrent Lévy process with infinite Lévy measure and let 0 be regular for itself. Then the following equalities hold:

$$\begin{aligned} &G^{X^0}(x,y) = h(x) + h(y) - h(y-x), \quad x,y \in \mathbb{R} \setminus \{0\} \\ &G^Z(x,y) = 2h(x) + 2h(y) - h(y-x) - h(y+x), \quad x,y > 0. \end{aligned} \tag{5.7}$$

Proof. Let U^q and U_0^q be the q-resolvent measures of the processes X and X^0 respectively. By the strong Markov property it follows that

$$\begin{aligned}
U^q(x,B) &= \mathbb{E}_x\left[\int_0^\infty e^{-qt}1_B(X_t)dt\right] \\
&= \mathbb{E}_x\left[\int_0^{\sigma_{\{0\}}} e^{-qt}1_B(X_t^0)dt\right] + \mathbb{E}_x\left[\mathbb{E}_x\left[\int_{\sigma_{\{0\}}}^\infty e^{-qt}1_B(X_t)dt|\mathcal{F}_{\sigma_{\{0\}}}\right]\right] \\
&= U_0^q(x,B) + \mathbb{E}_x\left[e^{-q\sigma_{\{0\}}}\mathbb{E}_0\left[\int_0^\infty e^{-qt}1_B(X_t)dt\right]\right] \\
&= U_0^q(x,B) + \mathbb{E}_x\left[e^{-q\sigma_{\{0\}}}\right]U^q(0,B), \quad B \in \mathcal{B}(0,\infty).
\end{aligned}$$

Since $\mathbb{P}_x(\sigma_{\{0\}} < \infty) = 1$, for all $x, y > 0$

$$G^{X^0}(x,y) = \lim_{q\downarrow 0} u_0^q(x,y) = \lim_{q\downarrow 0}\left(u^q(x,y) - \mathbb{E}_x\left[e^{-q\sigma_{\{0\}}}\right]u^q(0,y)\right) \overset{(5.6)}{=} -h(y-x) + h(y) + h(y).$$

By symmetry of h and (5.4) it follows that for all $x, y > 0$

$$G^Z(x,y) = G^{X^0}(x,y) + G^{X^0}(x,-y) = 2h(x) + 2h(y) - h(y-x) - h(y+x).$$

□

We will use the following asymptotic behavior of h near zero several times in this chapter. This statement was also proven in [GR15, Lemma 2.14].

Lemma 5.3 For every $x > 0$

$$h(x) \asymp \frac{1}{x\psi\left(\frac{1}{x}\right)}.$$

Proof. For every $x \in \mathbb{R}$ it follows that

$$\begin{aligned}
h(x) &\le \frac{1}{\pi}\int_0^\infty \left(\frac{\xi^2x^2}{2}\wedge 2\right)\frac{1}{\psi(\xi)}d\xi = \frac{x^2}{2\pi}\int_0^{\frac{2}{x}}\frac{\xi^2}{\phi(\xi^2)}d\xi + \frac{2}{\pi}\int_{\frac{2}{x}}^\infty \frac{1}{\phi(\xi^2)}d\xi \\
&\overset{(\mathbf{H})}{\le} \frac{x^2}{2\pi}\frac{a_6}{\phi(4x^{-2})}\left(\frac{2}{x}\right)^{2\delta_6}\int_0^{\frac{2}{x}}\xi^{2-2\delta_6}d\xi + \frac{2x^{-2\delta_5}}{a_5\pi\phi\left(4x^{-2}\right)}\int_{\frac{2}{x}}^\infty x^{-2\delta_5}d\xi \\
&\le \tilde{c}_1\frac{1}{x\phi\left(x^{-2}\right)}.
\end{aligned}$$

On the other hand, by [JK15, Lemma 2.4] we have

$$\begin{aligned}
h(x) &= \frac{1}{2\pi}(\mathcal{F}\frac{1}{\psi}(0) - \mathcal{F}\frac{1}{\psi}(x)) \ge \frac{1}{4\pi}\int_0^\infty\left(\frac{\xi^2x^2}{4}\wedge 1\right)\frac{1}{\psi(\xi)}d\xi \\
&= \frac{x^2}{\pi}\int_0^{\frac{2}{x}}\frac{\xi^2}{\phi(\xi^2)}d\xi + \frac{1}{\pi}\int_{\frac{2}{x}}^\infty\frac{1}{\phi(\xi^2)}d\xi \\
&\overset{(\mathbf{H})}{\ge} \frac{x^2}{4\pi}\frac{a_5}{\phi(4x^{-2})}\left(\frac{2}{x}\right)^{2\delta_5}\int_0^{\frac{2}{x}}\xi^{2-2\delta_5}d\xi + \frac{2x^{-2\delta_6}}{a_6\pi\phi\left(4x^{-2}\right)}\int_{\frac{2}{x}}^\infty x^{-2\delta_6}d\xi \\
&\ge \tilde{c}_2\frac{1}{x\phi\left(x^{-2}\right)}.
\end{aligned}$$

$\square$

From the previous lemma and **(H)** it follows that h also satisfies the global scaling conditions, i.e. there exist constants $d_1, d_2 > 0$ such that

$$d_1 \lambda^{2\delta_5 - 1} \leq \frac{h(\lambda t)}{h(t)} \leq d_2 \lambda^{2\delta_6 - 1}, \quad \forall \lambda \geq 1,\, t > 0. \tag{5.8}$$

The following two lemmas were also proven in [GR15, Proposition 2.2, Proposition 2.4, Lemma 4.2].

Lemma 5.4 For every $x, y > 0$

$$h(x \wedge y) \leq G^Z(x, y) \leq 4h(x \wedge y).$$

Proof. First we show that h is a subadditive function on $\mathbb{R}$. By Lemma 5.2 and symmetry of h it follows that

$$h(x) + h(y) - h(x+y) = h(-x) + h(y) - h(x+y) = G^{X^0}(-x, y) \geq 0.$$

By (5.7) and subadditivity of h for $0 < x < y$ we get

$$\begin{aligned} G^Z(x, y) &= 2h(x) + 2h(y) - h(y-x) - h(y+x) \\ &\leq 2h(x) + h(x) + h(y-x) + h(-x) + h(y+x) - h(y-x) - h(y+x) = 4h(x). \end{aligned}$$

Since h is increasing,

$$G^Z(x, y) \geq h(x) + h(y) - h(y-x) \geq h(x).$$

$\square$

Lemma 5.5 There exist $\lambda_1 \in \left(0, \frac{1}{2}\right)$ and $\lambda_2 > 0$ such that for every $R > 0$

$$G^Z_{(0,R)}(x, y) \geq \lambda_2 h(R),\ x, y \in (0, \lambda_1 R)$$

Proof. Take $\lambda < \frac{1}{2}$ and $x, y \in (0, \lambda R)$. Let $\tau_{(0,R)}$ be the first exit time of Z from the interval $(0, R)$. By (2.16) it follows that

$$\begin{aligned} G^Z_{(0,R)}(x, y) &= G^Z(x, y) - \mathbb{E}_x[G^Z(Z_{\tau_{(0,R)}}, y)] \\ &= 2h(x) + 2h(y) - h(x+y) - h(y-x) \\ &\quad - \mathbb{E}_x[2h(Z_{\tau_{(0,R)}}) + 2h(y) - h(Z_{\tau_{(0,R)}} + y) - h(y - Z_{\tau_{(0,R)}})]. \end{aligned}$$

For $\varepsilon > 0$ by harmonicity of h in $(0, \infty)$ it follows that

$$h(x) = \mathbb{E}_x \left[h\left(Z_{\tau_{(\varepsilon,R)}} \right) \right] = \mathbb{E}_x \left[h\left(Z_{\tau_{(\varepsilon,R)}} \right) : \tau_{(\varepsilon,R)} < \sigma_{\{0\}} \right].$$

Since h is continuous and $h(0) = 0$ by the dominated convergence theorem and quasi-left continuity of Z we get

$$h(x) = \lim_{\varepsilon \to 0} \mathbb{E}_x \left[h\left(Z_{\tau_{(\varepsilon,R)}} \right) : \tau_{(\varepsilon,R)} < \sigma_{\{0\}} \right] = \mathbb{E}_x \left[h\left(Z_{\tau_{(0,R)}} \right) : \tau_{(0,R)} < \sigma_{\{0\}} \right].$$

so h is regular harmonic for Z in $(0, R)$.

$$G^Z_{(0,R)}(x, y) = -\, h(x + y) - h(y - x) + \mathbb{E}_x[h(Z_{\tau_{(0,R)}} + y)] + \mathbb{E}_x[h(y - Z_{\tau_{(0,R)}})].$$

Furthermore, by the monotonicity of h it follows that

$$\begin{aligned} &h(Z_{\tau_{(0,R)}} - y) \geq h(R - \lambda R) \text{ a.s.,} \\ &h(y - x) \leq h(\lambda R) \quad \text{and} \\ &h(Z_{\tau_{(0,R)}} + y) > h(x + y) \text{ a.s.} \end{aligned}$$

Therefore,

$$G^Z_{(0,R)}(x, y) \geq h((1 - \lambda)R) - h(\lambda R) \overset{(5.8)}{\geq} \left(d_2^{-1}(1 - \lambda)^{2\delta_6 - 1} - d_1^{-1}\lambda^{2\delta_5 - 1} \right) h(R) = \lambda_2 h(R),$$

where $\lambda_2 > 0$ for λ small enough. □

5.1 Properties of the exit time of Z from the interval

Let $\sigma_0 := \sigma_{\{0\}}$ be the lifetime of Z and $\tau_{(0,R)}$ the first exit time of Z from $(0, R)$.

Lemma 5.6 For every $R > 0$ and $x \in (0, R)$

$$\frac{1}{8} \frac{h(x)}{h(R)} \leq \mathbb{P}_x \left(\tau_{(0,R)} < \tau \right) \leq \frac{h(x)}{h(R)}.$$

Proof. First we prove the right inequality. Recall from the proof of Lemma 5.5 that the function h is regular harmonic in $(0, R)$ for Z so

$$h(x) = \mathbb{E}_x \left[h\left(Z_{\tau_{(0,R)}} \right) : \tau_{(0,R)} < \sigma_0 \right].$$

Since h is increasing it follows that

$$\begin{aligned} h(x) &= \int_R^\infty h(y) \mathbb{P}_x \left(Z_{\tau_{(0,R)}} \in dy : \tau_{(0,R)} < \tau \right) \\ &\geq h(R) \int_R^\infty \mathbb{P}_x \left(Z_{\tau_{(0,R)}} \in dy : \tau_{(0,R)} < \tau \right) \end{aligned}$$

$$= h(R)\mathbb{P}_x\left(\tau_{(0,R)} < \tau\right).$$

For the other inequality, by continuity and harmonicity of the Green function $G^Z(\cdot, 2R)$ on (ε, R) and Lemma 5.4, it follows that

$$\begin{aligned} h(x) \le G^Z(x,2R) &= \lim_{\varepsilon\to 0}\mathbb{E}_x\left[G^Z(Z_{\tau_{(\varepsilon,R)}}, 2R)\right] = \int_R^\infty G^Z(z,2R)\mathbb{P}_x(Z_{\tau_{(0,R)}} \in dz) \\ &\le 4h(2R)\mathbb{P}_x(\tau_{(0,R)} < \tau) \overset{(3.4)}{\le} 8h(R)\mathbb{P}_x(\tau_{(0,R)} < \tau). \end{aligned}$$

□

The previous lemma was also proven in [GR15, Proposition 2.7].

Next we consider estimates for the tail distribution function of the lifetime of Z. Under additional assumptions it is also possible to obtain estimates of the derivatives of the tail distribution with respect to the time component. For more detail see [JK15].

Lemma 5.7 [GR15, Corollary 3.5.] If

$$\frac{\phi(\lambda t)}{\phi(t)} \ge a_1\lambda^{\delta_1}, \quad \forall \lambda \ge 1,\, t > 0$$

holds for some $\delta_1 \in (0,1]$ then there exist positive constants c_1, c_2 such that

$$c_1\frac{h(x)}{h\left(1/\psi^{-1}\left(\frac{1}{t}\right)\right)} \le \mathbb{P}_x(\tau > t) \le c_2\frac{h(x)}{h\left(1/\psi^{-1}\left(\frac{1}{t}\right)\right)} \tag{5.9}$$

for every $x \neq 0$ and $t > 0$ such that $t\psi(\frac{1}{x}) \ge 1$.

Using this estimate we can easily derive estimates for the expected exit time from interval $(0,R)$ in terms of the function h.

Lemma 5.8 Then there exists constant $c_3 = c_3(R,\phi) > 0$ such that

$$\begin{aligned} &\text{(i)} \qquad \mathbb{E}_x\left[\tau_{(0,R)}\right] \le 4Rh(x), \quad 0 < x < R \\ &\text{(ii)} \qquad \mathbb{E}_x\left[\tau_{(0,R)}\right] \ge c_3 h(x), \quad \text{for } x \text{ small enough} \end{aligned}$$

Proof. (i) By Lemma 5.4

$$\mathbb{E}_x\left[\tau_{(0,R)}\right] = \int_0^R G^Z_{(0,R)}(x,y)dy \le \int_0^R 4h(x)dy = 4Rh(x)$$

(ii) For the other inequality note that for all $t > 0$

$$\begin{aligned} \mathbb{P}_x(\tau > t) &= \mathbb{P}_x(\tau > t, \tau_{(0,R)} \ge \tau) + \mathbb{P}_x(\tau > t, \tau_{(0,R)} < \tau) \\ &\le \mathbb{P}_x(\tau_{(0,R)} > t) + \mathbb{P}_x(\tau_{(0,R)} < \tau) \\ &\le \frac{\mathbb{E}_x\left[\tau_{(0,R)}\right]}{t} + \mathbb{P}_x(\tau_{(0,R)} < \tau), \end{aligned}$$

where the last line follows from the Markov's inequality. Hence, by Lemma 5.6, Lemma 5.7 and Lemma5.3, if $t\psi\left(\frac{1}{x}\right) > 1$ there exists a constant $\tilde{c}_1 > 0$ such that

$$\mathbb{E}_x\left[\tau_{(0,R)}\right] \geq t\left(\mathbb{P}_x(\tau > t) - \mathbb{P}_x(\tau_{(0,R)} < \tau)\right) \geq c_1 t\frac{h(x)}{h\left(1/\psi^{-1}\left(\frac{1}{t}\right)\right)} - t\frac{h(x)}{h(R)}$$
$$\geq \left(\frac{c_1\tilde{c}_1}{\psi^{-1}\left(\frac{1}{t}\right)} - \frac{t}{h(R)}\right)h(x) = f_R(t)h(x). \tag{5.10}$$

Note that by **(H1)** there exist constants $\tilde{c}_2, \tilde{c}_3 > 0$ such that for all $t \geq \psi(1)$ and $\lambda \geq 1$

$$\tilde{c}_2\lambda^{\frac{1}{2\delta_2}} \leq \frac{\psi^{-1}(\lambda t)}{\psi^{-1}(t)} \leq \tilde{c}_3\lambda^{\frac{1}{2\delta_1}}.$$

Therefore, for all $t \leq 1$

$$f_R(t) \geq c_1\tilde{c}_1\tilde{c}_2\psi^{-1}(1)t^{\frac{-1}{2\delta_1}} - \frac{t}{h(R)},$$

so there exists $t_0 = t_0(\phi, R) \in (0,1)$ such that $f_R(t) > 0$ for all $t < t_0$. Therefore,

$$\mathbb{E}_x[\tau_{(0,R)}] \geq f_R(t_0)h(x), \text{ for all } x < \frac{1}{\psi^{-1}(\frac{1}{t_0})}.$$

□

5.2 Green function for $Z^{(a,b)}$

Let Y be the censored process of a subordinate Brownian motion X on $(0,\infty)$ and $X^{(a,b)}$, $Y^{(a,b)}$ and $Z^{(a,b)}$ processes X, Y and Z killed outside of interval (a,b), $0 < a < b$. In this section we show that the Green functions of processes $X^{(a,b)}$, $Y^{(a,b)}$ and $Z^{(a,b)}$ are comparable.

From the representation of Beurling-Deny and LeJan, (2.5) and (2.17), the jumping measure i associated with the Dirichlet form $(\mathcal{E}^Z, \mathcal{F}^Z)$ is of the form

$$i(x,y) = j(|x-y|) + j(|x+y|). \tag{5.11}$$

Furthermore,

$$\mathcal{E}^Z(u,v) = \frac{1}{2}\int_0^\infty\int_0^\infty (u(x)-u(y))(v(x)-v(y))i(x,y)dx\,dy.$$

Then the Dirichlet forms corresponding to processes $X^{(a,b)}$, $Y^{(a,b)}$ and $Z^{(a,b)}$ are equal to

$$\begin{aligned}
\mathcal{E}^{X^{(a,b)}}(u,u) &= \frac{1}{2}\int_a^b\int_a^b (u(x)-u(y))^2 j(|x-y|)dydx + \int_a^b u(x)^2\kappa_1(x)dx \\
\mathcal{E}^{Y^{(a,b)}}(u,u) &= \frac{1}{2}\int_a^b\int_a^b (u(x)-u(y))^2 j(|x-y|)dydx + \int_a^b u(x)^2\kappa_2(x)dx \\
\mathcal{E}^{Z^{(a,b)}}(u,u) &= \frac{1}{2}\int_a^b\int_a^b (u(x)-u(y))^2 i(x,y)dydx + \int_a^b u(x)^2\kappa_3(x)dx,
\end{aligned} \tag{5.12}$$

where κ_1, κ_2 and κ_3 are the densities of the corresponding killing measures of $X^{(a,b)}$, $Y^{(a,b)}$ and $Z^{(a,b)}$,

$$\begin{aligned}\kappa_1(x) &= \int_{(a,b)^c} j(|x-y|)dy \\ \kappa_2(x) &= \int_{(0,\infty)\setminus(a,b)} j(|x-y|)dy \\ \kappa_3(x) &= \int_{(0,\infty)\setminus(a,b)} i(x,y)dy.\end{aligned}$$

Since $\mathcal{E}^{X^{(a,b)}} = \mathcal{E}^{Y^{(a,b)}} + \kappa_{(0,\infty)}1_{(a,b)}$, where

$$\kappa_{(0,\infty)}(x) = \int_{-\infty}^{0} j(|x-y|)dy,$$

$Y^{(a,b)}$ can be obtained from $X^{(a,b)}$ by creation through the Feynman-Kac transform at the rate $\kappa_{(0,\infty)}$, i.e. for every nonnegative Borel function f

$$P_t^{Y^{(a,b)}} f(x) = \mathbb{E}_x\left[f\left(Y_t^{(a,b)}\right)\right] = \mathbb{E}_x\left[f\left(X_t^{(a,b)}\right)e^{\int_0^t \kappa_{(0,\infty)}\left(X_s^{(a,b)}\right)ds}\right] = \mathbb{E}_x\left[f\left(X_t^{(a,b)}\right)e_\kappa(t)\right].$$

By (2.30) we can relate the Green functions of processes $X^{(a,b)}$ and $Y^{(a,b)}$ through a conditional gauge function u,

$$G_{(a,b)}^{Y}(x,y) = u(x,y)G_{(a,b)}^{X}(x,y),$$

where

$$u(x,y) = \mathbb{E}_x^y\left[e_\kappa(\tau_{(a,b)})\right].$$

Recall that $\mathbb{P}_x^y$ denotes the probability measure of the $G_{(a,b)}^X(\cdot,y)$-conditioned process starting from x, i.e. the process with transition probability

$$p_t^y(x,z) = \frac{G_{(a,b)}^X(z,y)}{G_{(a,b)}^X(x,y)} p_t^{X^{(a,b)}}(x,z).$$

Similarly as in Section 4.1, we want to show that the conditional gauge function u is bounded, i.e. that the Green functions $G_{(a,b)}^X$ and $G_{(a,b)}^Y$ are comparable. Since the interval (a,b) can be arbitrary large, it is not possible to obtain a result equivalent to Lemma 4.13. Nevertheless, by obtaining a somewhat weaker result and applying the conditional gauge theorem it is possible to prove the boundedness of function u. First we introduce a special Kato class of Revuz measures for the process X.

Definition 5.9 Let X be a transient Hunt process with the Green function G. A nonnegative Borel function κ is said to be of the Kato class $S_\infty(X)$ if for any $\varepsilon > 0$ there is a Borel set K of finite measure and a constant $\delta > 0$ such that

$$\sup_{x,z\in\mathbb{R}^n} \int_{K^c\cup B} \frac{G^X(x,y)G^X(y,z)}{G^X(x,z)}\kappa(y)dy < \varepsilon \tag{5.13}$$

for all measurable sets $B \subset K$ such that $\lambda(B) < \delta$.

The following *conditional gauge theorem* was proved in [Che02, Theorem 3.3].

Theorem 5.10 Let X be a transient Hunt process and κ a nonnegative Borel function. Let X^y be the $G(\cdot, y)$- conditioned process with lifetime ζ^y. If $\kappa \in S_\infty(X)$ and the conditional gauge function u,

$$u(x,y) = \mathbb{E}_x^y \left[e_\kappa(\zeta^y) \right],$$

is finite for some $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^n$ then it is bounded on $\mathbb{R}^n \times \mathbb{R}^n \setminus d$. We say that the pair (X, κ) is *conditionally gaugeable.*

We will also need the following Green function estimates from [CKS14, Corollary 7.4 (ii)].

Theorem 5.11 Suppose that X is a one-dimensional subordinate Brownian motion with Lévy exponent $\psi(\xi) = \phi(|\xi|^2)$ with ϕ being a complete Bernstein function satisfying condition **(H1)**. Let D be a bounded $C^{1,1}$ open subset of $\mathbb{R}$ with characteristics (R_2, λ), $a(x,y) = \Phi(\delta_D(x))^{1/2}\Phi(\delta_D(y))^{1/2}$ and $\Phi(x) = \frac{1}{\phi(x^{-2})}$. Suppose that for every $T > 0$, there is a constant $c_4 = c_4(T, \phi) > 0$ such that

$$\int_0^r \frac{\Phi(s)}{s^2} ds \leq c_4 \frac{\Phi(r)}{r} \tag{5.14}$$

for every $r \in (0, T]$. Then for all $(x,y) \in D \times D$,

$$G_D^X(x,y) \asymp \frac{a(x,y)}{\Phi^{-1}(a(x,y))} \wedge \frac{a(x,y)}{|x-y|}. \tag{5.15}$$

Remark 5.12 Note that the condition (5.14) is satisfied for $\delta_1 > \frac{1}{2}$.

From **(H1)** one can easily see that Φ^{-1} satisfies the following scaling condition: for all $T > 0$ there exists a constant $c_T = c_T(T, \phi) > 0$ such that for all $0 < r \leq R \leq T$

$$c_T^{-1} \left(\frac{r}{R} \right)^{1/(2\delta_1)} \leq \frac{\Phi^{-1}(r)}{\Phi^{-1}(R)} \leq c_T \left(\frac{r}{R} \right)^{1/(2\delta_2)}. \tag{5.16}$$

Theorem 5.13 Let X be a recurrent subordinate Brownian motion with Laplace exponent of the subordinator $\phi \in \mathcal{CBF}$ satisfying **(H1)**. The function $\kappa_{(0,\infty)}$ is in Kato class $S_\infty(X^{(a,b)})$. Therefore, the pair $(X^{(a,b)}, \kappa_{(0,\infty)})$ is conditionally gaugeable and consequently the Green functions $G_{(a,b)}^X$ and $G_{(a,b)}^Y$ are comparable.

Proof. Let $\delta(x) := \delta_{(a,b)}(x)$. From (5.15) we get the following 3G inequality,

$$\frac{G_{(a,b)}^X(x,y) G_{(a,b)}^X(y,z)}{G_{(a,b)}^X(x,z)} \leq \tilde{c}_1 \Phi(\delta(y)) \frac{|x-z| \vee \Phi^{-1}(a(x,z))}{(|x-y| \vee \Phi^{-1}(a(x,y)))(|y-z| \vee \Phi^{-1}(a(y,z)))} \tag{5.17}$$

for some $\tilde{c}_1 > 0$. We will show that for every $\varepsilon > 0$ there exists $0 < \delta < 1$ and K such that for measurable sets $B \subset K$, $\lambda(B) < \delta$

$$\sup_{x,z \in (a,b)} \int_{K^c \cup B} \frac{G_{(a,b)}^X(x,y) G_{(a,b)}^X(y,z)}{G_{(a,b)}^X(x,z)} dy \leq \varepsilon.$$

First note that for $\delta(y) \le 2\delta(x)$,

$$\begin{aligned}\Phi^{-1}(a(x,y)) &\ge \Phi^{-1}\left(\Phi\left(\frac{1}{2}\delta(y)\right)^{1/2}\Phi(\delta(y))^{1/2}\right)\\ &\overset{(3.4)}{\ge} \Phi^{-1}\left(\frac{1}{4}\Phi\left(\delta(y)\right)^{1/2}\Phi(\delta(y))^{1/2}\right)\\ &\overset{(5.16)}{\ge} c_T^{-1}2^{-\frac{1}{\delta_1}}\delta(y).\end{aligned}$$

Since

$$\delta(y) \le \delta(x) + |x-y| \le 2(\delta(x) \vee |x-y|)$$

it follows that

$$|x-y| \vee \Phi^{-1}(a(x,y)) \ge \left(\frac{1}{2} \wedge c_T^{-1}2^{-\frac{1}{\delta_1}}\right)\delta(y).$$

This implies that

$$\begin{aligned}\frac{G^X_{(a,b)}(x,y)G^X_{(a,b)}(y,z)}{G^X_{(a,b)}(x,z)} &\le \tilde{c}_1\left(4 \vee c_T^2 2^{\frac{2}{\delta_1}}\right)\frac{\Phi(\delta(y))}{\delta(y)^2}(|x-z| \vee \Phi^{-1}(a(x,z)))\\ &\le \tilde{c}_1\left(4 \vee c_T^2 2^{\frac{2}{\delta_1}}\right)(b-a)\frac{\Phi(\delta(y))}{\delta(y)^2}.\end{aligned}$$

Let $\tilde{c}_2 = \tilde{c}_1\left(4 \vee c_T^2 2^{\frac{2}{\delta_1}}\right)(b-a)$ and $A = [a, a+\eta] \cup [b-\eta, b]$, for some $\eta < 1$. It follows that

$$\begin{aligned}\sup_{x,z\in(a,b)}\int_A \frac{G^X_{(a,b)}(x,y)G^X_{(a,b)}(y,z)}{G^X_{(a,b)}(x,z)}dy &\le \tilde{c}_2\int_A \frac{\Phi(\delta(y))}{\delta(y)^2}dy = 2\tilde{c}_2\int_0^\eta \frac{\Phi(s)}{s^2}ds\\ &\overset{(\mathbf{H1})}{\le} \frac{2\tilde{c}_2}{a_1\phi(1)}\int_0^\eta \frac{s^{2\delta_1}}{s^2}ds = \frac{2\tilde{c}_2}{a_1\phi(1)(2\delta_1-1)}\eta^{2\delta_1-1}.\end{aligned}$$

Therefore, for $\eta \in \left(0, \left(\frac{a_1\phi(1)(2\delta_1-1)}{2\tilde{c}_2}\varepsilon\right)^{\frac{1}{2\delta_1-1}}\right)$ and $K = [a+\eta, b-\eta]$ we get

$$\sup_{x,z\in(a,b)}\int_{K^c}\frac{G^X_{(a,b)}(x,y)G^X_{(a,b)}(y,z)}{G^X_{(a,b)}(x,z)}dy < \frac{\varepsilon}{2}.$$

Since the function $s \mapsto \frac{\Phi(s)}{s^2}$ is continuous on $[\eta, \frac{b-a}{2}]$ it is uniformly bounded, i.e. there exists a constant $M > 0$ such that for all $B \subset K$, $\lambda(B) < \delta = \frac{\varepsilon}{2\tilde{c}_1 M}$

$$\sup_{x,z\in(a,b)}\int_B \frac{G^X_{(a,b)}(x,y)G^X_{(a,b)}(y,z)}{G^X_{(a,b)}(x,z)}dy \le \tilde{c}_1\int_B\frac{\Phi(\delta(y))}{\delta(y)^2}dy \le \tilde{c}_1 M\delta < \frac{\varepsilon}{2}.$$

Since $\kappa_{(0,\infty)}$ is bounded on (a,b) this is enough to conclude that $\kappa_{(0,\infty)} \in S_\infty(X^{(a,b)})$, so by Theorem 5.10 the pair $(X^{(a,b)}, \kappa_{(0,\infty)})$ is gaugeable. □

Next, we want to associate the Green functions for processes $Y^{(a,b)}$ and $Z^{(a,b)}$. Since

$$\mathcal{E}^{Z^{(a,b)}}(u,u) = \mathcal{E}^{Y^{(a,b)}}(u,u) + \int_a^b \int_a^b (u(x)-u(y))^2 F(x,y) j(|x-y|) dy dx + \int_a^b u(x)^2 q(x) dx,$$

where $F(x,y) = \frac{j(|x+y|)}{j(|x-y|)}$ and $q = \kappa_3 - \kappa_2$, $Z^{(a,b)}$ can be obtained from $Y^{(a,b)}$ through the Feynman-Kac transform driven by a discontinuous additive functional

$$A_{q+F}(t) = \int_0^t q(Y_s^{(a,b)}) ds + \sum_{s \le t} F(Y_{s-}^{(a,b)}, Y_s^{(a,b)}). \tag{5.18}$$

Therefore for every Borel function $f \ge 0$

$$P_t^{Z^{(a,b)}} f(x) = \mathbb{E}_x \left[f(Z_t^{(a,b)}) \right] = \mathbb{E}_x \left[f(Y_t^{(a,b)}) e^{A_{q+F}(t)} \right] = \mathbb{E}_x \left[f(Y_t^{(a,b)}) e_{q+F}(t) \right]$$

Let $\tau_{(a,b)}$ be the first exit time of $Y^{(0,\infty)}$ from (a,b). Now by [Che02, Lemma 3.9]

$$\begin{aligned}\mathbb{E}_x \left[\int_0^\infty f(Z_t^{(a,b)}) dt \right] &= \mathbb{E}_x \left[\int_0^\infty f(Y_t^{(a,b)}) e_{q+F}(t) dt \right] = \int_a^b G_{(a,b)}^Y(x,y) \mathbb{E}_x^y \left[e_{q+F}(\tau_{(a,b)}) \right] f(y) dy \\ &=: \int_a^b G_{(a,b)}^Y(x,y) u(x,y) f(y) dy\end{aligned}$$

and therefore

$$G_{(a,b)}^Z(x,y) = u(x,y) G_{(a,b)}^Y(x,y).$$

Definition 5.14 Let X be a transient Hunt process with values in $E \in \mathcal{B}(\mathbb{R})$ with Green function G and Lévy system (J,H), where $H_s \equiv s$. A bounded nonnegative function F on $E \times E$ vanishing on the diagonal is said to be in the Kato class $A_\infty(X)$ if for any $\varepsilon > 0$ there is a Borel subset $K = K(\varepsilon)$ of finite measure and a constant $\delta = \delta(\varepsilon) > 0$

$$\sup_{x,w \in E} \int_A \frac{G(x,y)G(z,w)}{G(x,w)} F(y,z) J(x,dy) dz < \varepsilon \tag{5.19}$$

for all measurable sets $B \subset K$ such that

$$\int_B \left(\int_E F(x,y) J(x,dy) \right) dx < \delta$$

and $A = (K \times K)^c \cup (B \times E) \cup (E \times B)$.

The following theorem from [Che02, Theorem 3.8] is the analogue of the conditional gauge theorem, Theorem 5.10 for non-local perturbations corresponding to the discontinuous additive functional A_{q+F}.

Theorem 5.15 Let X be a transient Hunt process with values in $E \in \mathcal{B}(\mathbb{R})$, κ a nonnegative Borel function on E and F a nonnegative bounded function vanishing on the diagonal. Suppose that $\kappa \in S_\infty(X)$ and $F \in A_\infty(X)$. Then the conditional gauge function u,

$$\mathbb{E}_x^y[e_{\mu+F}(\zeta^y)]$$

is bounded on $(E \times E) \setminus d$, i.e. we say that the pair $(X, A_{\kappa+F})$ is gaugeable.

□

Theorem 5.16 Let A_{q+F} be the discontinuous additive functional for $Y^{(a,b)}$ from (5.18). Then $q \in S_\infty(Y^{(a,b)})$ and $F \in A_\infty(Y^{(a,b)})$ and consequently the Green functions of the processes $Y^{(a,b)}$ and $Z^{(a,b)}$ are comparable,

$$G^Z_{(a,b)}(x,y) \asymp G^Y_{(a,b)}(x,y).$$

Proof. Since the density q is bounded on (a,b) by Theorem 5.13 it follows that $q \in S_\infty(Y^{(a,b)})$. Furthermore, since $(y,z) \mapsto F(y,z)j(|y-z|)$ is bounded on $(a,b)\times(a,b)$ it is enough to show that for any $\varepsilon > 0$ there is a Borel subset $K = K(\varepsilon)$ of finite measure and a constant $\delta = \delta(\varepsilon) > 0$ such that

$$\sup_{x,w\in(a,b)} \int_A \frac{G^Y_{(a,b)}(x,y)G^Y_{(a,b)}(z,w)}{G^Y_{(a,b)}(x,w)} dydz < \varepsilon \tag{5.20}$$

for all measurable sets $B \subset K$ such that $\lambda(B) < \delta$ and $A = (K \times K)^c \cup (B \times (a,b)) \cup ((a,b) \times B)$. This is shown similarly as in Theorem 5.13 using the following generalized 3G inequality derived from (5.15)

$$\begin{aligned}\frac{G^Y_{(a,b)}(x,y)G^Y_{(a,b)}(z,w)}{G^Y_{(a,b)}(x,w)} &\le \tilde{c}_1 \frac{G^X_{(a,b)}(x,y)G^X_{(a,b)}(z,w)}{G^X_{(a,b)}(x,w)} \\ \le \tilde{c}_2 \Phi^{\frac{1}{2}}(\delta(y))\Phi^{\frac{1}{2}}(\delta(z)) &\frac{|x-w| \vee \Phi^{-1}(a(x,w))}{(|x-y| \vee \Phi^{-1}(a(x,y)))(|z-w| \vee \Phi^{-1}(a(z,w)))}.\end{aligned}$$

This implies $F \in A_\infty(Y^{(a,b)})$. □

5.3 Boundary Harnack principle for Z

Let $\tau_{(a,b)}$ be the first exit time of Z from the interval (a,b) and the $G^Z_{(a,b)}$ the Green function of $Z^{(a,b)}$. The process Z can exit the interval (a,b) only by jumping out, since by [Szt00, Theorem 1]

$$\mathbb{P}_x\left(X_{\tau_{(a_1,a_2)}} = a_i\right) = 0, \quad i = 1, 2$$

for all $x \in (a_1, a_2) \subset \mathbb{R}$. Recall from (2.17) and (5.12) that the transition probability of $Z_{\tau_{(a,b)}}$ starting from x is equal to

$$\mathbb{P}_x\left(Z_{\tau_{(a,b)}} \in B\right) = \int_B P^Z_{(a,b)}(x,z)dz, \quad x \in (a,b),\, B \in \mathcal{B}((0,\infty) \setminus [a,b]),$$

where $P^Z_{(a,b)}$ is the Poisson kernel of Z in (a,b) given by

$$P^Z_{(a,b)}(x,z) = \int_a^b G^Z_{(a,b)}(x,y)i(y,z)dy, \quad x \in (a,b),\, z \in (0,\infty) \setminus [a,b].$$

Using the results from the previous sections we can similarly as in [KSV10, Section 4] prove the Harnack inequality and boundary Harnack principle for harmonic functions of process $Z^{(a,b)}$.

Theorem 5.17 Harnack inequality
Let $R > 0$ and $a \in (0,1)$. There exists a constant $c_5 = c_5(R,a,\phi) > 0$ such that for all $r \in (0,R)$ and every nonnegative function u on $\mathbb{R}$ which is harmonic with respect to Z in $(0,3r)$,

$$u(x) \le c_5 u(y), \text{ for all } x,y \in (ar,(3-a)r).$$

Proof. Let $b_1 = ar/2$, $b_2 = ar$, $b_3 = (3-a)r$ and $b_4 = (3-a/2)r$. By Theorem 5.13, Theorem 5.16 and (5.15) the exists a $\tilde{c}_1 = \tilde{c}_1(\phi,R) > 1$ such that

$$\tilde{c}_1^{-1}\frac{a(x_i,y)}{\Phi^{-1}(a(x_i,y)) \vee |x-y|} \le G^Z_{(b_1,b_4)}(x_i,y) \le \tilde{c}_1\frac{a(x_i,y)}{\Phi^{-1}(a(x_i,y)) \vee |x-y|}, \quad i=1,2,$$

for all $x_1, x_2 \in (b_2,b_3)$ and $y \in (b_1,b_4)$. Furthermore, note that for $i = 1,2$

$$\frac{ar}{2} \le \delta_{(a,b)}(x_i) \le \frac{(3-a)r}{2} \quad \text{ and } \quad \delta_{(a,b)}(y) \le \frac{ar}{4} \Leftrightarrow |x_i - y| \ge \frac{ar}{4}.$$

Therefore $\Phi^{-1}(a(x_i,y)) \vee |x_i - y| \asymp r$, $i = 1,2$ so by **(H1)** and (5.16) there exists a constant $\tilde{c}_2 = \tilde{c}_2(R,a,\phi) > 0$ such that

$$G^Z_{(b_1,b_4)}(x_1,y) \le \tilde{c}_2 G^Z_{(b_1,b_4)}(x_1,y)$$

for all $x_1, x_2 \in (b_2,b_3)$ and $y \in (b_1,b_4)$. Consequently, we have

$$\begin{aligned}
P^Z_{(b_1,b_4)}(x_1,z) &= \int_{(b_1,b_4)} G^Z_{(b_1,b_4)}(x_1,y)i(y,z)dy \\
&\le \tilde{c}_2 \int_{(b_1,b_4)} G^Z_{(b_1,b_4)}(x_2,y)i(y,z)dy \\
&= \tilde{c}_2 P^Z_{(b_1,b_4)}(x_2,z)
\end{aligned}$$

for all $x_1, x_2 \in (b_2,b_3)$, $z \in [b_1,b_4]^c$. It follows that

$$u(x_1) = \mathbb{E}_{x_1}\left[u\left(X_{\tau_{(b_1,b_4)}}\right)\right] = \int_{(b_1,b_4)^c} u(z) P_Z^{(b_1,b_4)}(x_1,z)$$

$$\leq \tilde{c}_2 \int_{(b_1,b_4)^c} u(z) P_Z^{(b_1,b_4)}(x_2,z) = \tilde{c}_2 u(x_2)$$

for all $x_1, x_2 \in (ar, (3-a)r)$. □

Theorem 5.18 Boundary Harnack principle
Let $R > 0$. There exists a constant $c_6 = c_6(R,\phi) > 0$ such that for all $r \in (0,R)$, and every nonnegative function u which is harmonic for Z in $(0,3r)$ and continuously vanishes at 0 it holds that

$$\frac{u(x)}{u(y)} \leq c_6 \frac{h(x)}{h(y)}$$

for all $x, y \in (0, \lambda_1 r)$, where λ_1 is the constant from Lemma 5.5.

Proof. Let $x \in (0, \lambda_1 r)$. Since u is harmonic in $(0,3r)$ and vanishes continuously at 0 we have

$$\begin{aligned} u(x) = \lim_{\varepsilon \to 0} \mathbb{E}_x \left[u\left(Z_{\tau_{(\varepsilon,r)}} \right) \right] &= \mathbb{E}_x \left[u\left(Z_{\tau_{(0,r)}} \right) \right] = \mathbb{E}_x \left[u\left(Z_{\tau_{(0,r)}} \right) : Z_{\tau_{(0,r)}} \in (r,2r) \right] \\ &+ \mathbb{E}_x \left[u\left(Z_{\tau_{(0,r)}} \right) : Z_{\tau_{(0,r)}} \geq 2r \right] = u_1(x) + u_2(x). \end{aligned}$$

First note that

$$\frac{u(x)}{u(\lambda_1 r)} \leq \frac{u_1(x)}{u(\lambda_1 r)} + \frac{u_2(x)}{u_2(\lambda_1 r)}$$

and we estimate each term separately. By the Harnack inequality (Proposition 5.17) for $a = \frac{\lambda_1}{2}$ and Lemma 5.6

$$\begin{aligned} u_1(x) &\leq c_5 \mathbb{E}_x \left[u(\lambda_1 r) : Z_{\tau_{(0,r)}} \in (r,2r) \right] \leq c_5 u(\lambda_1 r) \mathbb{P}_x \left(Z_{\tau_{(0,r)}} \geq r \right) \\ &= c_5 u(\lambda_1 r) \mathbb{P}_x \left(\tau_{(0,r)} < \tau \right) \leq c_5 u(\lambda_1 r) \frac{h(x)}{h(r)} \leq c_5 u(\lambda_1 r) \frac{h(x)}{h(\lambda_1 r)}. \end{aligned}$$

For the second term, since j is decreasing

$$\begin{aligned} u_2(x) &= \int_0^r \int_{2r}^\infty u(z) G_{(0,r)}^Z(x,y) i(y,z) dz dy \\ &= \int_0^r \int_{2r}^\infty u(z) G_{(0,r)}^Z(x,y) (j(z-y) + j(y+z)) dz dy \\ &\leq \int_0^r G_{(0,r)}^Z(x,y) dy \int_{2r}^\infty u(z)(j(z-r) + j(z)) dz \\ &= \mathbb{E}_x[\tau_{(0,r)}] \int_{2r}^\infty u(z)(j(z-r) + j(z)) dz \\ &\leq 4rh(x) \int_{2r}^\infty u(z)(j(z-r) + j(z)) dz \end{aligned}$$

where the last line follows from Lemma 5.8. By [KSV14, Theorem 3.4] there exists a constant $\tilde{c}_1 = \tilde{c}_1(\phi) > 0$ such that

$$\tilde{c}_1^{-1} \frac{\phi(z^{-2})}{z} \leq j(z) \leq \tilde{c}_1 \frac{\phi(z^{-2})}{z}, \quad z > 0,$$

so by (3.4) it follows that

$$j(z-r) \le \tilde{c}_1 2^3 j(z), \quad z \ge 2r$$

and

$$u_2(x) \le \tilde{c}_1 2^6 h(x) \int_{2r}^{\infty} u(z) j(z) dz.$$

On the other hand, by Lemma 5.5

$$\begin{aligned} u_2(x) &\ge \int_0^{\lambda_1 r} G^Z_{(0,r)}(x,y)dy \int_{2r}^{\infty} u(z)(j(z)+j(z+r))dz \\ &\ge \lambda_2 h(\lambda_1 r)\lambda_1 r \int_{2r}^{\infty} u(z)j(z)dz. \end{aligned}$$

Therefore, it follows that

$$\frac{u(x)}{u(\lambda_1 r)} \le c_5 \frac{h(x)}{h(\lambda_1 r)} + \frac{\tilde{c}_1 2^6}{\lambda_1 \lambda_2} \frac{h(x)}{h(\lambda_1 r)} = \tilde{c}_2 \frac{h(x)}{h(\lambda_1 r)}. \tag{5.21}$$

On the other hand

$$\frac{u(x)}{u(\lambda_1 r)} \ge \frac{u_2(x)}{u(\lambda_1 r)}.$$

For the other inequality, from [KSV12b, Lemma 5.1] for $p = \frac{1}{3}$ it follows that there exists a constant $\tilde{c}_3 = \tilde{c}_3(\phi, R) > 0$ such that for all $x \in (0,r)$ and $y \in (2r, 3r)$

$$\begin{aligned} \int_{2r}^{y} P^Z_{(0,s)}(x,y)ds &\le \int_{2r}^{y} (P^X_{(-s,s)}(x,y) + P^X_{(-s,s)}(x,-y))ds \\ &= \int_{3r(1+1/3)/2}^{y} (P^X_{(-s,s)}(x,y) + P^X_{(-s,s)}(x,-y))ds \\ &\le \tilde{c}_3 \frac{3r}{\phi((3r)^{-2})} j(y) \le 27\tilde{c}_3 \frac{r}{\phi(r^{-2})} j(y). \end{aligned}$$

Now by applying [KSV12b, Lemma 5.2 and Lemma 5.3] for $U = B(0,2r)$ and $p = \frac{1}{3}$ it follows that

$$u(x) \le \frac{\tilde{c}_4}{\phi(r^{-2})} \int_{2r}^{\infty} u(y)j(y)dy$$

for some constant $\tilde{c}_4 = \tilde{c}_4(\phi) > 0$ and all $x \in (0,r)$. Furthermore by Lemma 5.4

$$u_2(x) \ge \int_0^{\lambda_1 r} G^Z_{(0,r)}(x,y)dy \int_{2r}^{\infty} u(z)(j(z)+j(z+r))dz \ge \lambda_2 h(x)\lambda_1 r \int_{2r}^{\infty} u(z)j(z)dz.$$

By the last two displays, (3.4) and Lemma 5.3 we get the required inequality, i.e.

$$\frac{u_2(x)}{u(\lambda_1 r)} \ge \frac{\lambda_1 \lambda_2 r h(x)}{\frac{\tilde{c}_4}{\phi(r^{-2})}} \ge \tilde{c}_5 \frac{h(x)}{h(\lambda_1 r)}. \tag{5.22}$$

Combining (5.21) and (5.22) we get the statement of the theorem. □

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Abstract

We examine three equivalent constructions of a censored rotationally symmetric Lévy process on an open set D - via the corresponding Dirichlet form, through the Feynman-Kac transform of the Lévy process killed outside of the set D and from the same killed process by the Ikeda-Nagasawa-Watanabe piecing together procedure.

For a complete Bernstein function ϕ satisfying condition **(H)**:

$$a_1\lambda^{\delta_1} \leq \frac{\phi(\lambda r)}{\phi(r)} \leq a_2\lambda^{\delta_2}, \quad \lambda \geq 1, r > 0$$

for some constants $a_1, a_2 > 0$ and $\delta_1, \delta_2 \in (0,1)$, we prove the trace theorem for the Besov space of generalized smoothness $H^{\phi(|\cdot|^2),1}(\mathbb{R}^n)$ on n-sets. We analyze the behavior of the corresponding censored Brownian motion near the boundary ∂D and determine conditions under which the process approaches the boundary of the set D in finite time.

Under a weaker condition **(H1)**, i.e. **(H)** for $\lambda, r \geq 1$, on the Laplace exponent ϕ of the subordinator we prove the 3G inequality for Green functions of the subordinate Brownian motion on κ-fat open sets. Using this result we obtain the scale invariant Harnack inequality for the corresponding censored process.

Finally, we consider a subordinate Brownian motion such that **(H)** holds and 0 is regular for itself. We establish a connection between this process and two related processes - censored process on the positive half-line and the absolute value of the subordinate Brownian motion killed at zero. We show that the corresponding Green functions on finite intervals away from 0 are comparable. Furthermore, we prove the Harnack inequality and the boundary Harnack principle for the absolute value of the subordinate Brownian motion killed at zero.

Sažetak

Cenzurirani Lévyjev proces na otvorenom skupu D dobije se suzbijanjem skokova Lévyjevog procesa izvan skupa D restrikcijom pripadne Lévyjeve mjere na taj skup. U radu promotramo tri ekvivalentna pristupa u konstrukciji takvih procesa - preko pripadne Dirichletove forme, Feynman-Kacovom transformacijom Lévyjevog procesa ubijenog izvan skupa D te Ikeda-Nagasawa-Watanabe procedurom spajanja nezavisnih kopija Lévyjevog procesa ubijenog izvan skupa D.

Dokazan je teorem o tragu na n-skupovima za generalizirane Besovljeve prostore $H^{\psi,1}(\mathbb{R}^n)$ i to za karakteristične funkcije oblika

$$\psi(x) = \phi(|x|^2), \quad x \in \mathbb{R}^n$$

gdje je ϕ potpuna Bernsteinova funkcija koja zadovoljova svojstvo **(H)**:

$$a_1 \lambda^{\delta_1} \leq \frac{\phi(\lambda r)}{\phi(r)} \leq a_2 \lambda^{\delta_2}, \quad \lambda \geq 1, r > 0$$

za neke konstante $a_1, a_2 > 0$ i $\delta_1, \delta_2 \in (0,1)$. Također, promatran je problem graničnog ponašanja cenzuriranog subordiniranog Brownovog gibanja s Laplaceovim eksponentom subordinatora ϕ, te su dani uvjeti pod kojima se proces približava rubu skupa D u konačnom vremenu.

Uz pretpostavku da uvjet **(H)** vrijedi samo za $\lambda, r \geq 1$ dokazana je 3G nejednakost za Greenovu funkciju tranzijentnog subordiniranog Brownovog gibanja na κ-debelim otvorenim skupovima. Korištenjem ovog rezultata pokazana je Harnackova nejednakost za pripadni cenzurirani proces.

Promatramo subordinirano Brownovo gibanje za koje je 0 regularna točka za sebe te Laplaceov ekponent subordinatora zadovoljava uvjet **(H)**. Uspostavlja se veza između ovog procesa i dva vezana procesa - cenzuriranog procesa na $(0,\infty)$ i apsolutne vrijednosti pripadnog procesa ubijenog u nuli. Pokazano je da su pripadne Greenove funkcije procesa ubijenih izvan konačnog intervala (a,b), za $0 < a < b$, usporedive. Nadalje, dokazana je Harnackova nejednakost i granični Harnackov princip za apsolutnu vrijednost subordiniranog Brownovog gibanja ubijenog u 0.

Curriculum Vitae

Personal information

- date of birth: January 5th, 1986
- place of birth: Sarajevo, BIH

Education

- 12/2009- Ph.D. student in mathematics at the University of Zagreb, Department of Mathematics, Croatia - supervisor: prof. dr. sc. Zoran Vondraček
- 9/2004-12/2009 Graduated in Financial and Business Mathematics at the University of Zagreb, Department of Mathematics, Croatia - diploma thesis *Point processes* under supervision of prof. dr. sc. Zoran Vondraček
- 9/2000-6/2004 V. Gymnasium, Zagreb, Croatia

Professional experience

- 09/2010- Research and Teaching Assistant at the University of Zagreb, Department of Mathematics, Croatia

Scholarships

- 2006-2008 The City of Zagreb Student Scholarship
- 2004-2006 The Republic of Croatia Student Scholarship

Grant participation

- 2014- Croatian Science Foundation Grant no. 3526 *Stochastic Methods in Analytical and Applied Problems*
- 2010-2014 Croatian Ministry of Science, Education and Sports Grant no. 037-0372790-2801 *Random processes with jumps*

Publications

- Radas, S., Anić I.D., Tafro A., Wagner,V.: *Effects of Public Support Schemes on Small and Medium Enterprises*, Technovation, (38) 15 - 30, 2014

Conferences, Workshops and Visits

- 5/2015 The Mathematical Research and Conference Center of the Institute of Mathematics of the Polish Academy of Sciences, Probability and Analysis, Bedlewo, Poland
- 11-12/2014 Visit to the University of Illinois at Urbana-Champaign, USA
- 5/2014 7th International Conference on Lévy Processes: Theory and Applications, CIMI Toulouse, France
- 7/2013 Non local operators, PDEs and Lévy processes, Wroclaw University of Technology, Poland
- 11-12/2012 Visit to Institut für Mathematische Stochastik, Technische Universität Dresden,Germany, DAAD supported visit
- 7/2012 Nonlocal Operators: Analysis, Probability, Geometry and Applications, Universität Bielefeld in collaboration with ZiF, Germany
- 4/2012 Spring School in Probability, IUC, Dubrovnik
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- 8/2010 Third SMAI European Summer School in Financial Mathematics, Paris, France

Talks

- 2/2016 *Censored Lévy and related processes*, Symposium of PhD students of Faculty of science, University of Zagreb, Croatia

Other professional activities

- 2012- Member of the Executive Committee of the Croatian Mathematical Society
- 2010- Member of the National committee for organizing and implementing mathematical competitions

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Odobni podaci

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Obrazovanje

- 12/2009- Doktorski studij matematike, PMF - Matematički odsjek, Sveučilište u Zagrebu - mentor: prof. dr. sc. Zoran Vondraček
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Profesionalno iskustvo

- 09/2010- Znanstveni novak na PMF - Matematičkom odsjeku, Sveučilište u Zagrebu

Stipendije

- 2006-2008 Stipendija Grada Zagreba
- 2004-2006 Stipendija Republike Hrvatske

Sudjelovanje na projektima

- 2014- Hrvatska zaklada za znanost, projekt br. 3526, *Stochastic Methods in Analytical and Applied Problems*
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Znanstveni radovi

- Radas, S., Anić I.D., Tafro A., Wagner,V.: *Effects of Public Support Schemes on Small and Medium Enterprises*, Technovation, (38) 15 - 30, 2014

Konferencije, radionice i posjeti sveučilištima

- 5/2015 The Mathematical Research and Conference Center of the Institute of Mathematics of the Polish Academy of Sciences, Probability and Analysis, Bedlewo, Poljska
- 11-12/2014 boravak na University of Illinois at Urbana-Champaign, SAD
- 5/2014 7th International Conference on Lévy Processes: Theory and Applications, CIMI Toulouse, France
- 7/2013 Non local operators, PDEs and Lévy processes, Wroclaw University of Technology, Poljska
- 11-12/2012 boravak na Institut für Mathematische Stochastik, Technische Universität Dresden, Njemačka
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- 4/2012 Spring School in Probability, IUC, Dubrovnik
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- 8/2010 Third SMAI European Summer School in Financial Mathematics, Paris, Francuska

Javna izlaganja

- 2/2016 *Cenzurirani Lévyjevi i njima srodni procesi*, Simpozij doktoranada Prirodoslovno-matematičkog fakulteta u Zagrebu

Ostale profesionalne aktivnosti

- 2012- Član Izvršnog odbora Hrvatskog matematičkog društva
- 2010- Član Državnog povjerenstva za organizaciju i provedbu natjecanja iz matematike

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